Problem set 7 CSCI 467 Teresa Liu

1) a) $C_1 = \left\{ x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right.$ $C_2 = \left\{ x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ $C_3 = \left\{ x_3 = \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$

$w(0) = 0_{2\times 3}$ $\quad w(i+1) = w(i) + \alpha x(i) e^T$

epoch 1

$w(1) = w(0) + \alpha x(0) e^T$

x_1: $y_1 = f(w^T x_1) = f\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix}$

$w(1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} [0 \; -2 \; -2] = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$

x_2: $y_2 = f(w^T x_2) = f\left(\begin{bmatrix} 0 & 0 \\ -1 & -1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$e_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}$

$w(1) = \begin{bmatrix} 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ -1 \end{bmatrix} [-2 \; 0 \; -2] = \begin{bmatrix} -1 & -1 & -2 \\ 1 & -1 & 0 \end{bmatrix}$

$x_3 = y_3 = f(w^T x_3) = f\left(\begin{bmatrix} -1 & 1 \\ -1 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$e_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -2 \\ 0 \end{bmatrix}$

$w(1) = \begin{bmatrix} -1 & -1 & -2 \\ 1 & -1 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} -1 \\ -1 \end{bmatrix} [-2 \; -2 \; 0] = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 0 & 0 \end{bmatrix}$

epoch 2:

$w(2) = w(1) + \alpha x(1) e^T$

x_1: $y_1 = f(w^T x_1) = f\left(\begin{bmatrix} 0 & 2 \\ 0 & 0 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

$w(2) = \begin{bmatrix} 0 & 0 & -2 \\ 2 & 0 & 0 \end{bmatrix} + 0.5 \begin{bmatrix} 1 \\ 1 \end{bmatrix} [0 \; -2 \; 0] = \begin{bmatrix} 0 & -1 & -2 \\ 2 & -1 & 0 \end{bmatrix}$

x_2: $y_2 = f(w^T x_2) = f\left(\begin{bmatrix} 0 & 2 \\ -1 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = f\left(\begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$e_2 = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ no update

x_3: $y_3 = f(w^T x) = f\left(\begin{bmatrix} 0 & 2 \\ -1 & -1 \\ -2 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = f\left(\begin{bmatrix} -2 \\ 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$e_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

$w(2) = \begin{bmatrix} 0 & -1 & -2 \\ 2 & -1 & 0 \end{bmatrix} + 0.5\begin{bmatrix} -1 \\ -1 \end{bmatrix}\begin{bmatrix} 0 & -2 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & -2 \\ 2 & 0 & 0 \end{bmatrix}}$

epoch 3:

$w(3) = w(2) + \alpha x(2) e^T$

x_1: $y_1 = f(w^T x) = f\left(\begin{bmatrix} 1 & 3 \\ 1 & 1 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = f\left(\begin{bmatrix} 4 \\ 2 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$e_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

$w(3) = \begin{bmatrix} 1 & 1 & -3 \\ 3 & 1 & -1 \end{bmatrix} + 0.5\begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 0 & -1 \end{bmatrix}$

x_2: $y_2 = f(w^T x) = f\left(\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = f\left(\begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

$e_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ no update

x_3: $y_3 = f(w^T x) = f\left(\begin{bmatrix} 1 & 3 \\ 0 & 0 \\ -3 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix}\right) = f\left(\begin{bmatrix} -4 \\ 0 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$

$e_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$

$w(3) = \begin{bmatrix} 1 & 0 & -3 \\ 3 & 0 & -1 \end{bmatrix} + 0.5\begin{bmatrix} -1 \\ -1 \end{bmatrix}\begin{bmatrix} 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -3 \\ 3 & 1 & -1 \end{bmatrix}$

since $w(1) = w(2)$ at the ends of epochs 1 and 2 respectively, the updates to the weights will continue to cycle and end up at the same weights vector at the end of each epoch, every time failing to fully classify all the inputs correctly.

The matrix does not converge, no weights vector will be able to classify all 3 inputs correctly.

b) $W(0) = [w_1(0)\ w_2(0)\ w_3(0)]$

$w_1(1) = w_1(0) + \alpha e_1 x(1)$

$w_2(1) = w_2(0) + \alpha e_2 x(1)$

$w_3(1) = w_3(0) + \alpha e_3 x(1)$

$e_i = y_{i1} - \text{sign}(w_i(0)^T x(1))$

epoch 1 $\quad W(1) = [w_1(1)\ w_2(1)\ w_3(1)]$

x_1: $y_{11} = f(w_1(0)^T x_1) = f([0\ 0]\begin{bmatrix}1\\1\end{bmatrix}) = f[0] = [1]$

$e_1 = 1 - 1 = 0 \Rightarrow$ no update $w_1(1)$

$y_{12} = f(w_2(0)^T x_1) = f([0\ 0]\begin{bmatrix}1\\1\end{bmatrix}) = f[0] = [1]$

$e_{12} = -1 - 1 = -2$

$w_2(1) = w_2(0) + \alpha e_2 x(1)$

$w_2(1) = \begin{bmatrix}0\\0\end{bmatrix} + 0.5(-2)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$

$y_{13} = f(w_3(0)^T x_1) = f([0\ 0]\begin{bmatrix}1\\1\end{bmatrix}) = f[0] = [1]$

$e_{13} = -1 - 1 = -2$

$w_3(1) = \begin{bmatrix}0\\0\end{bmatrix} + 0.5(-2)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$

$w(1) = \begin{bmatrix}0 & -1 & -1\\0 & -1 & -1\end{bmatrix}$

x_2: $y_{21} = f(w_1(1)^T x_2) = f([0\ 0]\begin{bmatrix}1\\-1\end{bmatrix}) = f[0] = [1]$

$e = -1 - 1 = -2$

$w_1(1) = \begin{bmatrix}0\\0\end{bmatrix} + 0.5(-2)\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$

$y_{22} = f(w_2(1)^T x_2) = f([-1\ -1]\begin{bmatrix}1\\-1\end{bmatrix}) = f[0] = [1]$

$e = 1 - 1 = 0 \Rightarrow$ no update $w_2(1)$

$y_{23} = f(w_3(1)^T x_2) = f([-1\ -1]\begin{bmatrix}1\\-1\end{bmatrix}) = f[2] = [1]$

$e = -1 - 1 = -2$

$w_3(1) = \begin{bmatrix}-1\\-1\end{bmatrix} + 0.5(-2)\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix} + \begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-2\\0\end{bmatrix}$

$w(1) = \begin{bmatrix}-1 & -1 & -2\\1 & -1 & 0\end{bmatrix}$

x_3: $y_{31} = f(w_1(1)^T x_3) = f([-1\ 1]\begin{bmatrix}-1\\-1\end{bmatrix}) = f[0] = [1]$

$e = -1 - 1 = -2$

$w_1(1) = \begin{bmatrix}-1\\1\end{bmatrix} + 0.5(-2)\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix}$

$y_{32} = f(w_2(1)^T x_3) = f([-1\ -1]\begin{bmatrix}-1\\-1\end{bmatrix}) = f[2] = [1]$

$e = -1 - 1 = -2$

$w_2(1) = \begin{bmatrix}-1\\-1\end{bmatrix} + 0.5(-2)\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix} + \begin{bmatrix}+1\\+1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$y_{33} = f(w_3(1)^T x_3) = f([-2\ 0]\begin{bmatrix}-1\\-1\end{bmatrix}) = f[2] = [1]$

$e = 1 - 1 = 0 \Rightarrow$ no update

$$\boxed{w(1) = \begin{bmatrix}0 & 0 & -2\\ 2 & 0 & 0\end{bmatrix}}$$

epoch 2

$x_1: y_{11} = f(w_1(2)^T x_1) = f([0\ 2]\begin{bmatrix}1\\1\end{bmatrix}) = f[2] = [1]$

$e = 1 - 1 = 0 \Rightarrow$ no update

$y_{12} = f(w_2(2)^T x_1) = f([0\ 0]\begin{bmatrix}1\\1\end{bmatrix}) = f[0] = [1]$

$e = -1 - 1 = -2$

$w_2(2) = \begin{bmatrix}0\\0\end{bmatrix} + 0.5(-2)\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix}$

$y_{13} = f(w_3(2)^T x_1) = f([-2\ 0]\begin{bmatrix}1\\1\end{bmatrix}) = f([-2]) = [-1]$

$e = -1 + 1 = 0 \Rightarrow$ no update

$$w(2) = \begin{bmatrix}0 & -1 & -2\\ 2 & -1 & 0\end{bmatrix}$$

$x_2: y_{21} = f(w_1(2)^T x_2) = f([0\ 2]\begin{bmatrix}-1\\-1\end{bmatrix}) = f([-2]) = [-1]$

$e = -1 + 1 = 0 \Rightarrow$ no update

$y_{22}: f(w_2(2)^T x_2) = f([-1\ -1]\begin{bmatrix}-1\\-1\end{bmatrix}) = f([0]) = [1]$

$e = 1 - 1 = 0 \Rightarrow$ no update

$y_{23}: f(w_3(2)^T x_2) = f([-2\ 0]\begin{bmatrix}-1\\-1\end{bmatrix}) = f([-2]) = [-1]$

$e = -1 + 1 = 0 \Rightarrow$ no update

$x_3: y_{31} = f(w_1(2)^T x_3) = f([0\ 2]\begin{bmatrix}-1\\-1\end{bmatrix}) = f([-2]) = [-1]$

$e = -1 + 1 = 0 \Rightarrow$ no update

$y_{32} = f(w_2(2)^T x_3) = f([-1\ -1]\begin{bmatrix}-1\\-1\end{bmatrix}) = f([-2]) = [-1]$

$e = -1 - 2 = -2$

$w_2(2) = \begin{bmatrix}-1\\-1\end{bmatrix} + 0.5(-2)\begin{bmatrix}-1\\-1\end{bmatrix} = \begin{bmatrix}-1\\-1\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$

$y_{33} = f(w_3(2)^T x_3) = f([-2\ 0]\begin{bmatrix}-1\\-1\end{bmatrix}) = f([2]) = [1]$

$e = 1 - 1 = 0 \Rightarrow$ no update

$$\boxed{w(2) = \begin{bmatrix}0 & 0 & -2\\ 2 & 0 & 0\end{bmatrix}}$$

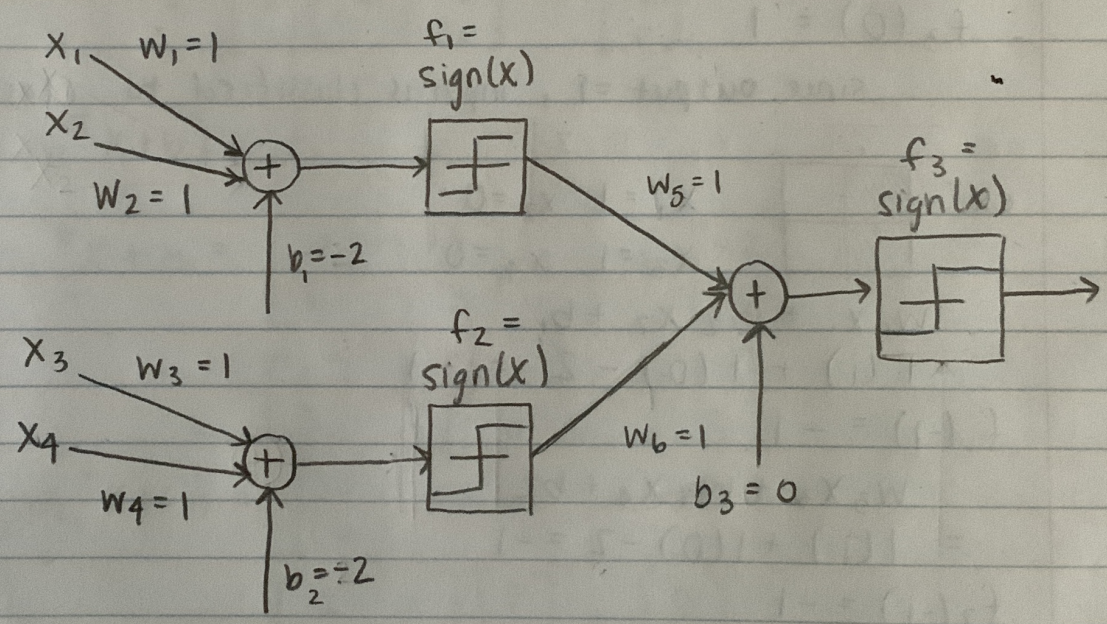
2)

X_1	X_2
X_3	X_4

$\boxed{X_i} \Rightarrow x_i = 1$ $\quad$ $\boxed{X_i} \Rightarrow x_i = 0$

Class I = (X_1 AND X_3) OR (X_2 AND X_4)
Class II = (X_1 AND X_2) OR (X_3 AND X_4)

assuming that the 4 patterns provided are the only 4 possible inputs, and that our output will exactly one class of Class I or Class II

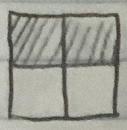


Output will either be 1 or -1.

Output 1 $\Rightarrow$ Class II

Output -1 $\Rightarrow$ Class I

$$\text{sign}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

ex. 

$x_1 = 1 \quad x_2 = 1$

$x_3 = 0 \quad x_4 = 0$

$w_1 x_1 + w_2 x_2 + b_1$

$= 1(1) + 1(1) - 2 = 0$

$f_1(0) = 1$

$w_3 x_3 + w_4 x_4 + b_2$

$= 1(0) + 1(0) - 2 = -2$

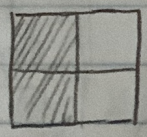
$f_2(-2) = -1$

$w_5 f_1(0) + w_6 f_2(-2) + b_3$

$= 1 - 1 = 0$

$f_3(0) = 1$

since output = 1, input is classified to class II

ex. 

$x_1 = 1 \quad x_2 = 0$

$x_3 = 1 \quad x_4 = 0$

$w_1 x_1 + w_2 x_2 + b_1$

$= 1(1) + 1(0) - 2 = -1$

$f_1(-1) = -1$

$w_3 x_3 + w_4 x_4 + b_2$

$= 1(1) + 1(0) - 2 = -1$

$f_2(-1) = -1$

$w_5 f_1(-1) + w_6 f_2(-1) + b_3$

$= -1 - 1 = -2$

$f_3(-2) = -1$

Since output = -1, input is classified to class I

3) a) 1st : 3

2nd : 2

3rd : 2

b) $f_3 = W_3^T X + b_3 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

dimension of a is $\boxed{2 \times 1}$

c) $X = [1\ 1\ 1]^T$

$$W_1^T X + b = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 3 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}.$$

$$f_1(W_1^T X + b) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$W_2^T X + b = \begin{bmatrix} 0 & -1 & 3 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

$$f_2(W_2^T X + b) = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

$$W_3^T X + b = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 6 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$f_3(W_3^T X + b) = \tanh\left(\begin{bmatrix} 12 \\ 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} \frac{e^{24}-1}{e^{24}+1} \\ \frac{e^4-1}{e^4+1} \end{bmatrix} = \boxed{\begin{bmatrix} 0.999 \\ 0.964 \end{bmatrix} = a}$$