Homework 5

Due: January 9, 2019 in class

Note: No late homework will be accepted. You may discuss with your classmates but you may not plagiarize. You need to turn in your analysis and also your code (printout) written in Octave or Matlab.

Part A. (30%)

A.1 Plot the stability diagram for the second-order Runge-Kutta method. Where does the neutral curve intersect with $\mathrm{Re}(\lambda h)$?

A.2 Plot the stability diagram for the fourth-order Runge-Kutta method. Where does the neutral curve intersect with $\mathrm{Re}(\lambda h)$ and with $\mathrm{Im}(\lambda h)$?

Part B. (30%)

Consider the equation

$$\frac{dy}{dx} + (2 + 0.01x^2)y = 0,$$

subject to $y(0) = 4$ in the range $0 \leq x \leq 15$.

B.1 Solve this equation using the following numerical schemes: i) Euler, ii) backward Euler, iii) trapezoidal, iv) second-order Runge-Kutta and v) fourth-order Runge-Kutta. Use $\Delta x = 0.1, 0.5, 1.0$ and compare to the exact solution.

B.2 For each scheme, estimate the maximum Δx for stable solution over the given domain and discuss your estimate in terms of results of B.1.

Part C. (40%)

A laminar boundary layer on a flat plate is self-similar and is governed by

$$f''' + ff'' = 0$$

where $f = f(\eta)$ and η is the similarity variable. In the boundary layer, velocity profile is represented by $f'(\eta)$ and shear stress is represented by $f''(\eta)$. Boundary conditions are summarized as

$$f(0) = 0, \quad f'(0) = 0, \quad \text{and} \quad f'(\infty) = 1.$$

We wish to solve for $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ throughout the boundary layer. Since one of the boundary conditions is prescribed at $\eta = \infty$ we are required to solve a non-linear boundary value problem. Taking $f_1 = f''$, $f_2 = f'$ and $f_3 = f$ gives the following set of ordinary differential equations for the solution:

$$f_1' = -f_1 f_3, \quad f_2' = f_1, \quad \text{and} \quad f_3' = f_2.$$

Solutions have been found to converge very quickly for large η and marching from $\eta = 0$ to $\eta = 10$ has been shown to be sufficient for accurate solution. Two conditions are specified at the wall ($\eta = 0$): $f_2 = 0$ and $f_3 = 0$. We must repeatedly solve the whole system and iterate to find the value of $f_1(0)$ that gives the required condition, $f_2 = 1$ at $\eta = 10$. In this problem, please use the fourth-order Runge-Kutta to march the solution from $\eta = 0$ to $\eta = 10$ with a step of $\Delta\eta = 0.01$.

C.1 In order to find the correct value of $f_1(0)$ that gives the required condition, $f_2(10) = 1$, we have to try many different values of $f_1(0)$. Please use the values of $f_1(0)$ in increments of 0.005 in the range $0.1 \le f_1(0) \le 1.0$ and solve the whole system repeatedly to find the values of $f_2(10)$ that correspond to $f_1(0)$ in the range $0.1 \le f_1(0) \le 1.0$. Plot your results in the form of $f_2(10)$ against $f_1(0)$ in the range $0.1 \le f_1(0) \le 1.0$. Based on your results, what is the closest value of $f_1(0)$ that approximately satisfies $f_2(10) = 1$?

C.2 We may use secant method to iterate toward an arbitrarily accurate value for $f_1(0)$ based on two initial guesses for $f_1(0)$: $f_1^{(0)}(0) = 1.0$ and $f_1^{(1)}(0) = 0.5$, two solutions for $f_2(10)$: $f_2^{(0)}(10)$ and $f_2^{(1)}(10)$ corresponding to the initial guesses, and the secant method formula

$$f_1^{(\alpha+1)}(0) = f_1^{(\alpha)}(0) + \frac{f_1^{(\alpha)}(0) - f_1^{(\alpha-1)}(0)}{f_2^{(\alpha)}(10) - f_2^{(\alpha-1)}(10)}(1 - f_2^{(\alpha)}(10)),$$

where $\alpha = 1, 2, 3, ...$ How many iterations ($\alpha =$?) are used to find an 'accepted' value of $f_1(0)$ which is good to 5 decimal points? What is this 'accepted' value of $f_1(0)$? Plot $f(\eta)$, $f'(\eta)$ and $f''(\eta)$ against $0 \le \eta \le 10$ corresponding to this 'accepted' value of $f_1(0)$.