

Computing τ -rigid modules

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Abstract

In this thesis, we give a brief introduction to τ -tilting theory which was introduced by Adachi, Iyama and Reiten in [AIR13]. We have particular focus on applying wall and chamber structures to understand the τ -tilting theory of an algebra as in [BST19].

Using the machinery from [BST19] and some combinatorial arguments, we give examples of algebras with more than 2 components in their mutation quiver of support τ -tilting objects. We also classify the τ -tilting modules for some specific classes of algebras with two simple modules using a technique inspired by [DIJ17].

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1 Introduction

Representation theory of algebras may be thought of as understanding the ways a finite set of linear transformations could act together on vector spaces given some constraints. This can be formulated using the language of module theory, where systems of linear transformations corresponds to modules over a finite dimensional algebra depending on the network one studies.

In these terms, representation theory of algebras seeks to classify modules over finite dimensional algebras. To give an example, we may consider the quiver Γ below, where we work over the field \mathbb{R} . A representation of this quiver is simply a linear transformation between two finite dimensional real vector spaces. Such a representation corresponds to a finitely generated module over the algebra $k\Gamma \cong \begin{bmatrix} k & 0 \\ k & k \end{bmatrix}$.

$$\Gamma : 1 \longrightarrow 2$$

A representation of Γ is in other words given by an arbitrary $n \times m$ real matrix. When dealing with a single linear transformation, classical linear algebra is sufficient, and computing the rank, nullity and corank of the matrix gives us everything we need to understand the transformation (corank for an $n \times m$ matrix is m minus the rank of the matrix). Thus, there is no surprise that there are three indecomposable representations of this quiver, as seen in fig. 1. Any representation of Γ given as a matrix M can be written uniquely up to isomorphism as the sum $M_1^r \oplus M_2^n \oplus M_3^c$, where r, n and c correspond to rank, nullity and corank of M .

$$M_1 : k \xrightarrow{1} k$$

$$M_2 : k \xrightarrow{0} 0$$

$$M_3 : 0 \xrightarrow{0} k$$

Figure 1: The three indecomposable representations of the quiver with one arrow.

For more complicated algebras, finding all indecomposable modules can be challenging. However, Auslander-Reiten theory offers some useful tools. In particular, for algebras of finite representation type (i.e with only finitely many indecomposable modules up to isomorphism), one can compute all indecomposable modules algorithmically. Indeed, all modules fit into a so-called AR (Auslander-Reiten) quiver. For algebras of finite representation type, it is well-known that this quiver must be connected. Since we know all projective modules, we can compute all projectives and simply traverse the AR-quiver of our algebra. These steps are of algorithmic nature and are implemented in [QPA].

More general algebras can have an infinite amount of indecomposable modules up to isomorphism and also uncountably many components in their AR-quiver. Thus it is no surprise that one in general have trouble describing all indecomposable modules. Other tools such as tilting theory therefore come into play, where the general idea is to use module categories of relatively well-understood algebras to say something about module categories over different algebras.

The topic of this thesis is to understand support τ -tilting modules, which unify some of the different "tilting-theories" that has been constructed. However, in this thesis we are not interested so much in the "tilting" aspect, but instead seek to classify support τ -tilting modules as they themselves have become objects of independent interest.

Like indecomposable modules in an AR-quiver, maximal τ -rigid modules fit into a combinatorial structure given by a quiver. Like AR-quivers, these objects may be computed. And for algebras with only finitely many τ -rigid objects up to isomorphism, the underlying graph of the quiver must be connected. This gives one example of how algebras with finitely many τ -rigid objects play a similar role in τ -tilting theory as representation finite algebras play in classical representation theory.

In this thesis, our main results concern algebras with infinitely many τ -rigid modules. For such algebras, many of the tools developed for τ -tilting finite algebras fail and therefore new techniques are needed.

2 Preliminaries

We refer the reader to the two books [ASS06] and [ARS95] for general representation theory of algebras. We here collect some prerequisites for this thesis not found in these two texts.

2.1 Notation and setting

We closely follow the notation of [ASS06]. By algebra A we mean a finite dimensional algebra over a field k , where k need not be algebraically closed. All algebras encountered in this thesis will be on the form $k\Gamma/I$ for some finite quiver Γ and admissible ideal I . We denote by r the ideal of $k\Gamma/I$ generated by all paths of length 1 in Γ . The number n is, unless otherwise stated, reserved for the number of indecomposable projective summands of A . By $\text{mod } A$ we denote the category of finite dimensional left A -modules (in contrast to [ASS06], who employ right modules). We implicitly identify kQ/I -modules with its representation over the bound quiver Q with relations I . For the algebra $k\Gamma/I$, there is for all vertices i in Γ an associated indecomposable projective module $P(i)$ generated by all finite paths starting in that vertex. Similarly, $I(i)$ is the associated injective module for vertex i .

We denote by D the duality on modules, and by Tr the transpose. The Auslander-Reiten translate $D\text{Tr}$ is denoted τ .

A module M over A is called basic if no two nonzero summands are isomorphic as A -modules, that is $M \cong A \oplus X \oplus X$ implies $X \cong 0$. We denote by $|M|$ the number of indecomposable summands of M .

We denote by $\dim_A(M, N)$ the dimension of the k -vector space $\text{Hom}_A(M, N)$.

Given a subcategory \mathcal{C} of $\text{mod } A$, the right-perpendicular category of \mathcal{C} , denoted \mathcal{C}^\perp is defined as the subcategory of all modules M such that $\text{Hom}_A(C, M) = 0$ for all $C \in \mathcal{C}$. Dually, the left perpendicular of \mathcal{C} , denoted ${}^\perp\mathcal{C}$ is defined as the subcategory of all modules M such that $\text{Hom}_A(M, C) = 0$ for all $C \in \mathcal{C}$.

2.2 Approximations

Approximations are very often used in τ -tilting theory, for example to compute mutations which are central to this thesis.

Definition 2.1. Let \mathcal{C} be a subcategory of $\text{mod } A$. Let M be an A -module, and $X \in \mathcal{C}$. A map $f : X \rightarrow M$ is called a *right \mathcal{C} -approximation* of M if it is the case that for any $Y \in \mathcal{C}$ and map $g : Y \rightarrow M$, there is a map $h : Y \rightarrow X$ such that $g = f \circ h$.

Let M be an A -module, and $X \in \mathcal{C}$. A map $f : M \rightarrow X$ is called a *left \mathcal{C} -approximation* of M if it is the case that for any $Y \in \mathcal{C}$ and map $g : M \rightarrow Y$, there is a map $h : X \rightarrow Y$ such that $g = h \circ f$.

A left (right) approximation which is also left (right) minimal as a map is called a minimal left (right) approximation.

If $\mathcal{C} = \text{add } M$ for a module M , we shorten add M -approximation to M -approximation.

We say that a subcategory \mathcal{C} of $\text{mod } A$ is *contravariantly finite* (resp. *covariantly finite*) if every A -module M has a right (resp. left) \mathcal{C} -approximation. If \mathcal{C} is both contravariantly and covariantly finite, we call it *functorially finite*.

Example 2.1. Let M, N be modules in $\text{mod } A$. We may always find a left M -approximation of N as follows. $\text{Hom}(N, M)$ is finite dimensional as a k -vector space, so pick a basis $F = (f_1, f_2, \dots, f_i)$. Then we have a map $f : N \rightarrow$

$$\bigoplus_{j=1}^i M \text{ given by } f = \begin{bmatrix} f_1 \\ \vdots \\ f_i \end{bmatrix}. \text{ Given any map } g : N \rightarrow \bigoplus_{j=1}^k M_j \text{ where } M_j \text{ is a}$$

summand of M , this map will naturally factor through a map $g : N \rightarrow \bigoplus_{j=1}^k M$, which we may describe as $g = [g_1, g_2, \dots, g_k]$. Since F is a basis for $\text{Hom}(M, N)$, we may write

$$g = [a_{1,1}f_1 + \dots + a_{1,i}f_i, \dots, a_{k,1}f_1 + \dots + a_{k,i}f_i]$$

Interpreting $c \in k$ as a map $M \rightarrow M$ given by $c \cdot 1$ where 1 is the identity map, we can write g as the composition

$$g = \begin{bmatrix} a_{1,1} & \dots & a_{1,i} \\ \vdots & \dots & \vdots \\ a_{k,1} & \dots & a_{k,i} \end{bmatrix} \times \begin{bmatrix} f_1 \\ \vdots \\ f_i \end{bmatrix}$$

and thus g_r also factors through as wanted.

We now give an example of a non-trivial, non-minimal left approximation.

Example 2.2. Let A be the algebra defined by the quiver

$$\begin{array}{ccc} & \alpha & \\ 1 & \xrightarrow{\quad} & 2 \\ & \xleftarrow{\beta} & \end{array}$$

modulo the relations $r^4 = 0$. Then $\text{Hom}(P(1), P(2))$ has dimension 2, with maps $f_1(x) = x\beta$ and $f_2(x) = x\beta\alpha\beta$. Clearly, $f_2 = (\beta\alpha) \circ f_1$.

Using the construction in the above example, the map $f = \begin{bmatrix} \beta \\ \beta\alpha\beta \end{bmatrix}$ is a left $P(2)$ -approximation of $P(1)$. It is not left minimal, however. To see this, consider the endomorphism on $P(2) \oplus P(2)$ given by

$$\phi = \begin{bmatrix} 1 & 0 \\ \alpha\beta & 0 \end{bmatrix}$$

Then $\phi \circ f = f$, but ϕ is not an isomorphism. In fact, f_1 is a minimal left approximation.

3 Tau-tilting theory

τ -tilting theory was introduced in [AIR13] as a possible generalization of tilting theory, which itself has had tremendous impact on the field of representation theory of finite dimensional algebras. Although first introduced in [AIR13], some of the main results of τ -tilting theory stem from work by Auslander and Smalø in the 80's, for example [AS81]. In this section, we recall some important definitions and results from τ -tilting theory.

We start by discussing τ -rigid modules, which may be considered the building blocks of τ -tilting theory.

Definition 3.1. A module M in $\text{mod } A$ is called τ -rigid if $\text{Hom}(M, \tau M) = 0$.

First note that the Auslander-Reiten formula gives that τ -rigid objects must be exceptional objects. Note that X, Y being τ -rigid does not imply $X \oplus Y$ being τ -rigid. In fact, studying how indecomposable τ -rigid modules together form larger τ -rigid modules is part of τ -tilting theory.

Definition 3.2. [AIR13, Definition 0.3] A pair (M, P) where M is basic τ -rigid and P basic projective such that $\text{Hom}_A(P, M) = 0$ is called a τ -rigid pair.

Along with this definition follows many conventions. For a τ -tilting module $T = (M, P)$, P will be denote the projective part of T and M the module part. T is called support τ -tilting if $|M| + |P| = n$, and almost complete support τ -tilting if $|M| + |P| = n - 1$. A support τ -tilting pair with no non-zero projective

part may also be denoted a τ -tilting module. Support τ -tilting modules really are maximal.

For T_1, T_2 both τ -rigid pairs, we consider T_1 a summand of T_2 is both the module part and the projective part of T_1 is a summand of the module part and the projective part of T_2 respectively.

Proposition 3.1. [AIR13, Proposition 1.2, Proposition 2.3] A basic τ -rigid pair over A has at most $|A|$ summands.

Example 3.1. Since $\tau P = 0$ for any projective module P , any projective module is τ -rigid. As a consequence, $A = \oplus_{i=1}^n P(i)$ is a τ -tilting module.

Example 3.2. For an hereditary algebra A , partial tilting modules coincide with τ -rigid modules. For M partial tilting, $0 = \text{Ext}_A^1(M, M) \cong D\text{Hom}_A(M, \tau M)$, where the first equality is part of the definition of tilting-modules and the second equality follows from the Auslander-Reiten formula in the hereditary case.

3.1 Torsion pairs induced by support τ -tilting modules

Given a support τ -tilting pair (M, P) , we consider the modules generated by M , $\text{Fac } M$. As in classical tilting theory, this is a torsion class.

Definition 3.3. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories of $\text{mod } A$ for an algebra A is called a torsion pair if $\mathcal{T} = {}^\perp \mathcal{F}$ and $\mathcal{F} = \mathcal{T}^\perp$.

A torsion class is then the left part of a torsion pair, as \mathcal{T} above. A torsion-free class is the right part of a torsion pair, as \mathcal{F} above.

Torsion pairs appear in classical tilting theory. Using the more general setting of τ -tilting theory we can classify all functorially finite torsion pairs, i.e. torsion pairs where both the torsion and torsion-free parts are functorially finite.

Theorem 3.1. [AIR13, Theorem 2.7] There is a bijection between the set of functorially finite torsion pairs, denoted $\text{f-tors } A$ and support τ -tilting modules, denoted $\text{s}\tau\text{-tilt } A$, given by sending a support τ -tilting pair (M, P) to $(\text{Fac } M, M^\perp)$.

We order the objects in $\text{f-tors } A$ by inclusion on the torsion classes. Thus $(X_1, Y_1) < (X_2, Y_2)$ if $X_1 \subseteq X_2$. This induces an ordering on the set of support τ -tilting modules, where $(M_1, P_1) < (M_2, P_2)$ if $\text{Fac}(M_1) \subset \text{Fac}(M_2)$.

We denote by $Q(\text{f-tors } A)$ the Hasse-quiver of the above defined ordering on $\text{f-tors } A$.

3.2 Mutation of support τ -tilting modules

An essential property of support τ -tilting modules is that one can *mutate* at each summand. More precisely, we have the following.

Theorem 3.2. For a almost complete support τ -tilting pair (M, P) , there exists exactly two distinct support τ -tilting pairs (M_0, P_0) and (M_1, P_1) containing (M, P) as a summand.

In the above theorem, (M_0, P_0) and (M_1, P_1) are said to be mutations of each other, and we may write $M_1 = \mu_X(M_0)$ where X is the indecomposable summand of M_0 that we must replace to get M_1 ; formally, we require $M_0 = X \oplus M$ or $P_0 = X \oplus P$. In this case, we *mutate* at X .

Proposition 3.2. [AIR13, Definition-Proposition 2.28] Let T_1 and T_2 be support τ -tilting modules which are mutations of each other. Then either $T_1 < T_2$ or $T_2 < T_1$. In the first case, we consider T_2 a right-mutation of T_1 and in the second case we consider T_2 a left-mutation of T_1 . Note that if T_2 is a left-mutation of T_1 if and only if T_1 is a right-mutation of T_2 .

Left-mutations may be computed using approximations.

Proposition 3.3. [AIR13, Theorem 2.30] Let $T = X \oplus U$ be τ -tilting and X indecomposable such that T is the Bongartz completion of U . Let

$$X \xrightarrow{f} U' \rightarrow Y \rightarrow 0$$

be exact, where f is a minimal left add U -approximation.

1. If $Y = 0$, then U is the module part of some support τ -tilting object.
2. If $Y \neq 0$, then Y decomposes as $Y = \bigoplus_{i=1}^k Y^i$ for some indecomposable, τ -rigid Y^i . Then $U \oplus Y'$ is τ -tilting.

The following observation is practical for computational reasons as it demonstrates that we need not compute Bongartz completions before mutating; it is sufficient to check that the resulting Y defines a new τ -tilting object.

Proposition 3.4. Let $T = X \oplus U$ be τ -tilting. Let

$$X \xrightarrow{f} U' \rightarrow Y \rightarrow 0$$

be exact, where f is a minimal left add U -approximation.

If $Y \neq 0$, and $U \oplus Y$ is τ -rigid, Y must decompose as $Y = \bigoplus_{i=1}^k Y^i$. Then $U \oplus Y'$ is τ -tilting.

Proof. If $U \oplus Y$ is τ -rigid with at least n summands, then since U is basic with $n - 1$ summands, Y must decompose as $Y = \bigoplus_{i=1}^k Y^i$ for some $k \geq 1$. $U \oplus Y'$ must then τ -tilting and a mutation of $U \oplus X$, since they differ on exactly one summand. \square

3.3 Mutation quivers

Mutation of support τ -tilting modules naturally induce a quiver structure on the set of these modules.

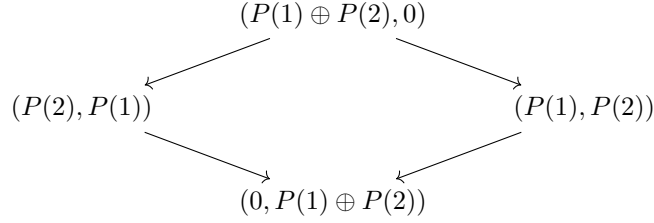
Definition 3.4. [AIR13, Definition 2.29] Fix an algebra A . The support τ -tilting quiver $Q(s\tau\text{-tilt } A)$ has a vertex for each support τ -tilting module and arrows on the form $T_1 \rightarrow T_2$, where T_2 is a left mutation of T_1 .

In the remaining of this subsection, we give examples of mutation quivers and discuss some elementary facts about them.

Example 3.3. Let A be the path algebra of the disconnected quiver with two points.



Then $Q(s\tau\text{-tilt } A)$ may easily be computed to be



A crucial example which is needed for many of the main results in this thesis is the τ -tilting theory of the Kronecker algebra K . The mutation quiver of K is given in for example [Jas14] and is considered to be well-known. For completeness, we include here a brief explanation of its τ -tilting theory. For rigorous proofs that the modules we exhibit in fact are τ -rigid (exceptional is sufficient in the hereditary case) we refer the reader to [ASS06, Chapter 9.2].

Example 3.4. Consider the Kronecker algebra K , which has quiver as drawn below.

$$1 \rightrightarrows 2$$

We will compute the mutation quiver $Q(s\tau\text{-tilt } K)$. $(K, 0)$ and $(0, K)$ are support τ -tilting modules.

Since $\text{Hom}_K(P(1), P(2)) = 0$, $(P(2), P(1))$ is a support τ -tilting module which is a left mutation of $(K, 0)$ and a right mutation of $(0, K)$.

The postprojective modules are τ -rigid, and in fact

$$T_i^1 = \tau^{-i}P(1) \oplus \tau^{-(i+1)}P(2) \quad T_i^2 = \tau^{-i}P(1) \oplus \tau^{-i}P(2)$$

define two classes of τ -tilting modules, where we let $i \in \mathbb{N}$. See fig. 2 for an illustration of the pre-projective component.

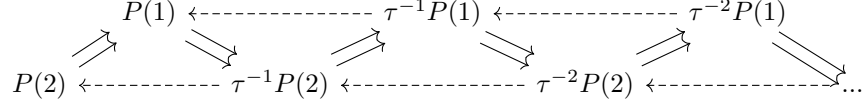


Figure 2: The postprojective component of the Auslander-Reiten quiver of K . The action of the Auslander-Reiten translate is indicated by dashed arrows.

Dually, the preinjectives also define τ -rigid modules.

Since $\text{Hom}_A(P(2), I(1)) = 0$, $(I(1), P(2))$ is a support τ -tilting object. Also, we get two classes of τ -tilting objects,

$$U_i^1 = \tau^i I(1) \oplus \tau^i I(2) \quad U_i^2 = \tau^{i+1} I(1) \oplus \tau^i I(2)$$

Note that even without computing left or right mutations, there is no difficulties finding the direction of the arrows. Clearly all mutations from $(K, 0)$ must be left-mutations and all mutations from $(0, K)$ must be right mutations. Also, any other object must have exactly one left mutation and exactly one right mutation; thus we can by induction build the mutation quiver $Q(s\tau\text{-tilt } K)$, as shown in fig. 3. By our argument in example 4.3 and theorem 4.1, all support τ -rigid objects must lie in this quiver.

We now show how the τ -tilting theory of a disconnected algebra is determined by its components.

Definition 3.5. Let Q_1, Q_2 be quivers with vertex sets Q_1^0, Q_2^0 and arrow sets Q_1^1, Q_2^1 . The Cartesian product $Q = Q_1 \square Q_2$ is defined as the quiver with vertex set $Q_1^0 \times Q_2^0$ and edges on the form $(x_1, y_1) \rightarrow (x_2, y_2)$ such that either $x_1 = x_2$ and $y_1 \rightarrow y_2$ is an edge in Q_2^1 , or $y_1 = y_2$ and $x_1 \rightarrow x_2$ is an edge in Q_1^1 .

Lemma 3.1. Let $\Lambda = A \times B$ be a disconnected algebra. Then $Q(s\tau\text{-tilt } \Lambda) \cong Q(s\tau\text{-tilt } A) \square Q(s\tau\text{-tilt } B)$ as quivers.

Proof. $(M, P) = (M_A \oplus M_B, P_A \oplus P_B)$ is support τ -tilting over Λ if and only if (M_A, P_A) is support τ -tilting over A and (M_B, P_B) is support τ -tilting over B . This follows from the general fact that the module category of Λ is the product of the module category of A and the module category of B . More concretely, one can see that computing the Auslander-Reiten translate of M_A in A is essentially exactly the same operation as computing it in Λ , and this is the only nontrivial observation needed.

This allows us to compute $Q(s\tau\text{-tilt } \Lambda)$ as the Cartesian product of $Q(s\tau\text{-tilt } A)$ and $Q(s\tau\text{-tilt } B)$. To see this, let (X, Y) be a point in $Q(s\tau\text{-tilt } A) \square Q(s\tau\text{-tilt } B)$. Then $X \oplus Y$ is clearly a point in $Q(s\tau\text{-tilt } \Lambda)$ and vice versa. Thus we have a natural bijection between the graphs, we need to investigate the edges. Let $(X_1, Y) \rightarrow (X_2, Y)$ be an edge in $Q(s\tau\text{-tilt } A) \square Q(s\tau\text{-tilt } B)$. Then $(X_1 \oplus Y) \rightarrow (X_2 \oplus Y)$ is a valid mutation as only one summand has changed, thus corresponding to an edge in $Q(s\tau\text{-tilt } \Lambda)$, and similarly for edges on the form

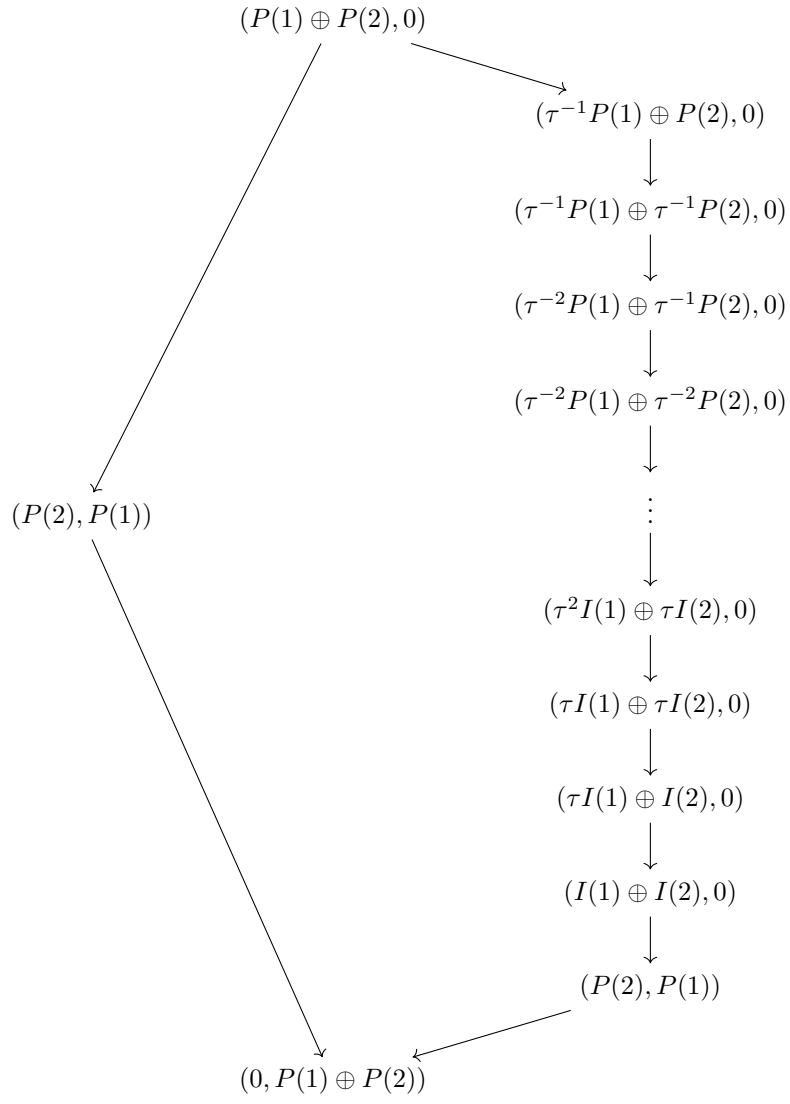


Figure 3: A finite part of the mutation quiver of the Kronecker algebra

$(X, Y_1) \rightarrow (X, Y_2)$. Since all edges in $Q(s\tau\text{-tilt}A) \sqcup Q(s\tau\text{-tilt}B)$ are on one of these forms, and clearly all edges in $Q(s\tau\text{-tilt}\Lambda)$ come from these edges, we have a quiver isomorphism. \square

One may also consider the graph-theoretical properties of $Q(s\tau\text{-tilt}A)$ by forgetting the orientation of the edges. From the properties of mutation, it follows at once that the mutation graph is n -regular. It may be finite or infinite. In the finite case the graph is connected[AIR13, Corollary 3.10], which allows one to algorithmically find all support τ -tilting objects by recursively mutating from $(A, 0)$.

3.4 g -vectors

Looking at vectors coming from projective presentations of τ -rigid modules gives a combinatorial tool which is used frequently in the literature on τ -tilting, for example in [ST20] and [DIJ17]. We here define g -vectors and G -matrices.

Definition 3.6. For a module M , let $P_1 \rightarrow P_2 \rightarrow M \rightarrow 0$ be a minimal projective presentation of M , where $P_1 \cong \bigoplus_{i=1}^n P(i)^{v_i}$ and $P_2 \cong \bigoplus_{i=1}^n P(i)^{u_i}$. The vectors (v_1, v_2, \dots, v_n) and (u_1, u_2, \dots, u_n) then describe P_1 and P_2 respectively. We define g^M to be $u - v$.

For this thesis, g -vectors appear as in the following two definitions.

Definition 3.7. For a τ -rigid pair (M, P) , its g -vector is defined as $g^M - g^P$. A τ -rigid module M can be considered a τ -rigid pair $(M, 0)$ and has g -vector g^M .

Definition 3.8. [ST20, Definition 2.4] For a support τ -tilting pair $(M, P) = (M_1 \oplus \dots \oplus M_i, P_1 \oplus \dots \oplus P_j)$, we define its G -matrix $G_{(M, P)}$ to be the $n \times n$ matrix $(g^{M_1}, g^{M_2}, \dots, g^{M_i}, -g^{P_1}, \dots, -g^{P_j})$, where we consider the g -vectors to be column vectors.

An important property of g -vectors is that they uniquely identify τ -rigid pairs[AIR13, Theorem 5.5]. This immediately gives a combinatorial proof that the set of indecomposable τ -rigid objects must be countable, as there are only countably many integer vectors of dimension n .

Further, G -matrices of support τ -tilting pairs are invertible. This follows immediately from [AIR13, Theorem 5.1].

The following result is often used.

Theorem 3.3. [AR+85][Theorem 1.4(a)] Let M, N be modules. We have

$$\langle g^M, [N] \rangle = \text{hom}_A(M, N) - \text{hom}_A(N, \tau M)$$

3.5 Duality

There is a duality between the τ -tilting of A and the τ -tilting of A^{op} , described in [AIR13].

Definition 3.9. Given a τ -rigid object (M, P) , let $M = M_{pr} \oplus M_{np}$ be the decomposition of M into its projective and non-projective parts. We define the pair

$$(M, P)^\dagger = (\text{Tr } M_{np} \oplus P^*, M_{pr}^*)$$

of A^{op} -modules as the dual of (M, P) .

Proposition 3.5. [AIR13, Theorem 2.14] Given (M, P) as above, $(M, P)^\dagger$ is a τ -rigid object over A^{op} . The map

$$(-)^\dagger : \tau\text{-rigid pairs of } A \rightarrow \tau\text{-rigid pairs of } A^{\text{op}}$$

is a bijection and in fact an involution.

This duality induces by [AIR13, Proposition 2.27 b)] an order-reversing bijection between the mutation quivers $Q(\text{stilt } A)$ and $Q(\text{stilt } A^{\text{op}})$. This gives us a way of finding right mutations. Namely, instead of explicitly computing them, we may pass to the opposite ring, do a left mutation, and go back to the original ring. The duality also gives a natural map on g -vectors.

Proposition 3.6. Let (M, P) a τ -rigid object. Given the canonical isomorphism between $K_0(A)$ and $K_0(A^{\text{op}})$, we get the equality

$$g^{(M, P)^\dagger} = -g^{(M, P)}$$

Proof. It is enough to prove this statement for indecomposable τ -rigid objects.

1. Consider the τ -rigid pair $(0, P)$. Then $(0, P)^\dagger = (P^*, 0)$. As a A^{op} -module, P^* is projective with $g^{P^*} = g^P$. Thus $g^{(P^*, 0)} = -g^{(0, P)}$ as wanted.
2. Consider the τ -rigid pair $(M, 0)$, where M is projective indecomposable. Then $\text{Tr } M = 0$, and $(M, 0)^\dagger = (0, M^*)$, and thus $g^{(M, 0)} = -g^{(M, 0)^\dagger}$ follows from the discussion above.
3. Let $(M, 0)$ be a τ -rigid pair with M non-projective indecomposable, and with minimal projective presentation

$$P_2 \rightarrow P_1 \rightarrow M$$

$\text{Tr } M$ will have a projective presentation.

$$P_1^* \rightarrow P_2^* \rightarrow \text{Tr } M$$

By [ASS06, Chapter 6.2, proposition 2.1], this presentation is in fact minimal. Then $g^{(M, 0)^\dagger} = g^{(\text{Tr } M, 0)} = -g^{(M, 0)}$ as wanted.

□

3.6 2-term silting objects

Fix an algebra A . An object P in $K^b(\text{add } A)$ is called 2-term if $P^i = 0$ for all $i \notin \{0, -1\}$.

Definition 3.10. A 2-term object P is called presilting if

$$\text{Hom}_{K^b(\text{add } A)}(P, P[1]) = 0$$

Further, a presilting object P is called silting if it generates $K^b(\text{add } A)$ as a triangulated category, in the sense that $K^b(\text{add } A) = \text{thick } P$.

The following proposition means that we can translate easily between 2-term silting objects and support τ -tilting objects, and thus need not worry too much about the homological definition of the silting property.

Proposition 3.7. [AIR13, Proposition 3.7] Let $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M$ be a projective presentation of a τ -rigid module M . Then we have

1. $P_1 \xrightarrow{d_1} P_0$ is a pre-silting complex.
2. If (M, Q) is a τ -rigid pair, then $P_1 \oplus Q \xrightarrow{\begin{bmatrix} d_0 & 0 \end{bmatrix}} P_0$ is a 2-term presilting complex which is silting if and only if (M, Q) is support τ -tilting.

4 The wall and chamber structure of finite dimensional algebras

We now introduce the wall and chamber structure of a finite dimensional algebra A , and recall how it can be used to study the τ -tilting theory of A . We closely follow the theory as developed in [BST19] and [DIJ17].

4.1 Stability conditions

Let $\langle -, - \rangle$ be the standard inner product on \mathbb{R}^n , i.e

$$\langle (v_1, v_2, \dots, v_n), (w_1, w_2, \dots, w_n) \rangle = \sum_{i=1}^n v_i w_i$$

Definition 4.1. [BST19, Definition 3.1] For $v \in \mathbb{R}^n$, a nonzero module $M \in \text{mod } A$ is called *v-stable* if $\langle v, [M] \rangle = 0$ and $\langle v, [N] \rangle < 0$ for all proper submodules N . M is called *v-semistable* if $\langle v, [M] \rangle = 0$ and $\langle v, [N] \rangle \leq 0$ for every submodule N .

We denote by $\mathcal{D}(M)$ the set of vectors $v \in \mathbb{R}^n$ such that M is *v-semistable*. We call $\mathcal{D}(M)$ the stability space of M . Stability spaces need not be linear spaces. However, they are always closed under positive scaling and induce linear spaces as their span.

We now define the wall and chamber structure of an algebra.

Definition 4.2. [BST19] The walls in the and chamber structure of A are all stability spaces $\mathcal{D}(M) \subseteq \mathbb{R}^n$ such that $\text{span}(\mathcal{D}(M))$ has dimension $n - 1$.

Let X^c denote the compliment of a set X in a topological space, and \overline{X} its closure. Let \mathcal{R} be the complement of the closure of the union of all stability spaces,

$$\mathcal{R} = (\overline{(\cup_{M \in \text{mod } A} \mathcal{D}(M))})^c$$

chamber in the wall and chamber structure of A is then a connected component of \mathcal{R} .

We now show that all walls are given by stability spaces of indecomposable modules.

Proposition 4.1. $\mathcal{D}(M)$ be a wall. Then there is an indecomposable module X such that $\mathcal{D}(M) = \mathcal{D}(X)$.

Proof. If M is indecomposable, we let $X = M$. If not, let $M = X \oplus N$, where $N, X \neq 0$. Let $v \in \mathcal{D}(M)$. Then $\langle v, [X] \rangle \leq 0$ and $\langle v, [M] \rangle \leq 0$. Since $\langle v, [M] \rangle = \langle v, [X] \rangle + \langle v, [N] \rangle = 0$, we see that $v \in \mathcal{D}(X)$. Since $[X] \neq 0$, $\mathcal{D}(X)$ must have co-dimension 1. This completes the proof. \square

Example 4.1. Let $A = k\overrightarrow{A_2}$. We compute its walls. We have three indecomposable A -modules, $S(1), S(2)$ and $P(1)$. Clearly $\mathcal{D}(S(1)) = \{(0, y) : y \in \mathbb{R}\}$ and $\mathcal{D}(S(2)) = \{(x, 0) : x \in \mathbb{R}\}$. We now compute $\mathcal{D}(P(1))$. Let $(x, y) = v \in \mathcal{D}(P(1))$. We then require $\langle v, (1, 1) \rangle = 0$ and $\langle v, (0, 1) \rangle \leq 0$. The second condition implies that $y \leq 0$. The first condition implies that $x = -y$, and these two conditions are sufficient for v to be in the stability space of $P(1)$. Thus the stability space of $P(2)$ is the ray $\{\alpha(1, -1) : \alpha \in \mathbb{R}, \alpha \geq 0\}$.

4.2 Cones, walls and chambers induced by τ -tilting pairs

We now aim to recover information about the τ -tilting theory of an algebra from its wall and chamber structure.

An important ingredient in this section is the following formula.

Proposition 4.2 (Corollary 3.10). [BST19] Let (M, P) be a τ -rigid pair. Then

$$\langle g^{(M,P)}, [N] \rangle = \text{hom}_A(M, N) - \text{hom}_A(N, \tau M) - \text{hom}_A(P, N)$$

Proof. By [AR+85, Theorem 1.4(a)], we have $\langle g^M, [N] \rangle = \text{hom}_A(M, N) - \text{hom}_A(N, \tau M)$. Since $g^{(M,P)} = g^M - g^P$, we get

$$\langle g^{(M,P)}, [N] \rangle \tag{1}$$

$$= \langle g^M, [N] \rangle - \langle g^P, [N] \rangle \tag{2}$$

$$= \text{hom}_A(M, N) - \text{hom}_A(N, \tau M) - \text{hom}_A(P, N) + \text{hom}_A(N, \tau P) \tag{3}$$

$$= \text{hom}_A(M, N) - \text{hom}_A(N, \tau M) - \text{hom}_A(P, N) + \text{hom}_A(N, 0) \tag{4}$$

$$= \text{hom}_A(M, N) - \text{hom}_A(N, \tau M) - \text{hom}_A(P, N) \tag{5}$$

\square

Definition 4.3. Let $T = (M, P)$ be a τ -rigid pair. Following [DIJ17], we define the *cone* spanned by T , as

$$C(T) = \left\{ \sum_{i=1}^j \alpha_i g^{M_i} - \sum_{i=1}^k \beta_i g^{P_i} : \alpha_i, \beta_i \geq 0 \text{ for all } i \right\}$$

The interior $C^\circ(T) \subset C(T)$ is defined by requiring all coefficients α_i, β_i to be strictly positive.

Given a vector $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $\alpha_i \geq 0$ for all i , we let $\alpha(M, P)$ denote the vector $G^{(M, P)} \times \alpha$.

The cones $C(T)$ for support τ -tilting modules and almost complete support τ -tilting modules play an important role in the wall and chamber structure of an algebra. We summarize some of the most important results below.

Proposition 4.3. [BST19, Proposition 3.15, Corollary 3.16, Corollary 3.21], [Asa20, Theorem 3.17]

1. For a support τ -tilting object $T_1 = (M, P)$, $C^\circ(T_1)$ defines a chamber in the wall and chamber structure of A .
2. For an almost complete support τ -tilting object $T_2 = (M, P)$, $C(T_2)$ is included in a wall $\mathcal{D}(N)$ in the wall and chamber structure of A . The module N may be computed given the two support τ -tilting modules containing (M, P) .
3. All chambers are on the form $C^\circ(T)$ for a support τ -tilting object T . [BST19] show this for τ -tilting finite algebra, while [Asa20] demonstrates this in general.

The Kronecker algebra K , sometimes denoted K_2 in this thesis, supplies our most important example of wall and chamber structures.

Example 4.2. Consider the Kronecker algebra K . Let M be an indecomposable module with dimension vector $(1, 1)$. Then $\mathcal{D}(M) = \{(x, -x) : x \geq 0\}$. However, there is no τ -rigid module with g -vector $(i, -i)$ for any $i \in \mathbb{N}$. This shows that not all walls come from (unions of) cones of almost complete support τ -tilting objects.

We now compute the wall and chamber structure of K , which will be references frequently in this thesis.

Example 4.3. Let K be the Kronecker algebra. We wish to compute g -vectors of indecomposable τ -rigid objects over K . Since K has 2 simple modules, these modules (together with $(0, P(1))$ and $(0, P(2))$) give the almost complete partial support τ -tilting objects of K .

K has Cartan and Coxeter matrices given by

$$C = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \Phi = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix} \quad (6)$$

It follows from [ASS06, Chapter 4, corollary 2.9]) and [ASS06, Chapter 4, corollary 2.7]) that the dimension vector of $\tau^{-i}P(i)$ is $\Phi^i \times \dim(P(i))$ and that the dimension vector of $\tau^i I(i)$ is $\Phi^{-i} \times \dim(I(i))$.

Also, observe the minimal projective presentation of a module depends only on its dimension vector. A module over K with dimension vector d has g -vector $C^{-1}d$.

Then any object in the postprojective component on the form $\tau^{-i}P(1)$ will have g -vector $(1 + 2i, -2i)$ and those on the form $\tau^{-i}P(2)$ will have g -vector $(2i, 1 - 2i)$. Thus for all $j \in \mathbb{N}$ there is a postprojective module with g -vector on the form $(j + 1, -j)$.

Dually, the preinjective modules have g -vectors that are on the form $(j, -(j + 1))$. The positive span of these g -vectors gives us subsets of walls in the wall and chamber structure of K , as guaranteed by proposition 4.3. For example, the g -vectors of $(P(1), 0)$ and $(0, P(1))$ both lie in the wall $\mathcal{D}(S(2))$.

We claim that for any wall W , W is given as a union of rays in fig. 4. To see this, observe that cones of the support τ -tilting objects hit \mathbb{R}^2 minus the ray given by the positive span of $(1, -1)$. Since this ray in fact corresponds to a wall (the "limit wall"), any ray not drawn in the picture, which is also part of a wall must intersect a chamber, which is impossible.

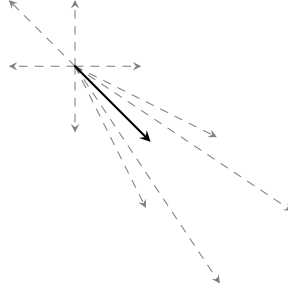


Figure 4: A finite portion of the walls of K , where the bold vector indicates the wall discussed in example 4.2, appropriately called a limit wall.

The following reduction technique is very useful.

Proposition 4.4. [BST19, Lemma 4.13] Let A be a finite dimensional algebra. Let $B = A/I$ be a quotient, where $I \subseteq r$. In particular, I does not contain any non-trivial idempotents. There is then a canonical isomorphism between $K_0(A)$ and $K_0(B)$. Using this isomorphism, walls in the wall and chamber structure of B are taken to walls in the wall and chamber structure of A .

Proof. We have a canonical map $f : A \rightarrow A/I$. A module M over B may be considered an A -module, under the action $\alpha x = f(\alpha)x$. The important observation is now that the submodules of M considered as an A -module coincide with the submodules of M considered as a B -module.

We prove one direction of this. Let $N \subset M$ be a A -submodule of M . We need to put a B -module structure on N . We attempt to define $f(\alpha)x = \alpha x$ for $\alpha \in A$ and prove that this is well-defined: $f(\alpha) = f(\beta)$ for some $\beta \in A$. Then $f(\alpha - \beta)x = 0$, so $\alpha - \beta \in I$. Thus $(\alpha - \beta)x = 0$, as $x \in M$. So the action is well-defined. The other direction is trivial.

The rest of the proof now follows from the definition of a wall. See also the proof given in [BST19]. \square

Using the third point of proposition 4.3, we may conclude that an algebra with infinitely many walls in its wall and chamber structure will be τ -tilting infinite. It is well known that algebras whose quivers have a double arrow are representation infinite. Here we show that they are τ -tilting infinite as well.

Proposition 4.5. Any algebra $A = k\Gamma/I$ where Γ has a double arrow is τ -tilting infinite.

Proof. Choose a pair of double arrows. Let J be the ideal generated by all arrows except the chosen pair of double arrows. Then A/J decomposes as an algebra to $A/J = K_2 \times k \times k \times \cdots \times k$. Since K_2 is τ -tilting infinite, A/J is also τ -tilting infinite and must then have infinitely many walls.

Thus by proposition 4.4, A has infinitely many walls and is thus τ -tilting infinite. \square

4.3 Consequences for the structure of $Q(s\tau\text{-tilt})$

This following result shows that cones of τ -rigid objects intersect in a very controlled fashion.

Proposition 4.6. [DIJ17, Corollary 6.7, b)] Let T_1, T_2 be support τ -tilting objects. Then $C(T_1) \cap C(T_2) = C(X)$ with X being the maximal τ -rigid pair such that X is a summand of T_1 and of T_2 .

This result gives a tool for classifying all support τ -tilting objects. We collect this technique as a proposition.

Proposition 4.7. [DIJ17] Let A be an algebra.

- If A is τ -tilting finite, its mutation graph is connected, thus the set of support τ -tilting modules over A are exactly the ones in this component.
- If it is τ -tilting infinite the mutation graph may be disconnected. Let X be a set of support τ -tilting modules such that $\cup_{T \in X} C(T)$ is dense in \mathbb{R}^n . Then X contains all support τ -tilting modules over A .

Proof. The τ -tilting finite case is well-known.

Let $T = (M, P)$ be a support τ -tilting object. Consider a point $x \in C(T)$ in the interior of $C(T)$. Since $\cup_{T \in X} C(T)$ is dense in \mathbb{R}^n , there is a T_2 in X such that $C(T_2) \cap C(T)$ contains x . But $C(T_2) \cap C(T) = C(Y)$ for some partial

support τ -tilting object Y . Since $T \neq T_2$, there is a summand Z in $T = (M, P)$ not in Y . Writing x as an element of $C(T)$, we get

$$x = \sum_{i=1}^j \alpha_i g^{M_i} - \sum_{i=1}^k \beta_i g^{P_i}$$

The above expression also gives a way of writing x as an element of $C(T)$, where in particular the g^Z coordinates are 0. Thus x lies on the border of $C(T)$, giving us a contradiction. \square

It is interesting to note that for tame algebras, the converse of the above theorem holds, in the sense that the closure of the union of all chambers must span \mathbb{R}^n , see [KPY20].

in [DIJ17], the authors use the above demonstrated criteria to classify the support τ -tilting objects of two specific symmetric algebras. We may however easily refine the criterion using ideas from [BST19]. We collect this refined technique as a theorem, which will be used frequently in this thesis.

Theorem 4.1. Let A be an algebra. Let W be the set of walls in the wall and chamber structure of A , and let \overline{W} be its closure.

Let X be a set of support τ -tilting modules whose cones hit all chambers of A , i.e such that

$$(\cup_{T \in X} C(T)) \cup \overline{W} = \mathbb{R}^n$$

then X contains all support τ -tilting modules over A .

Proof. The set of chambers C in the wall and chamber structure of A is defined as the connected components of $\mathbb{R}^n - \overline{W}$. If $(\cup_{T \in X} C(T)) \cup \overline{W} = \mathbb{R}^n$, then $\mathbb{R}^n - \overline{W} \subseteq (\cup_{T \in X} C(T))$. Thus every chamber $c \in C$ lies entirely inside $(\cup_{T \in X} C(T))$. Since chambers of different support τ -tilting objects only intersect on their boundary if they intersect at all, we may conclude that $c = C(T)$ for some $T \in X$. This finishes the proof. \square

4.4 Wall crossing and green sequences

We can employ the wall and chamber structure to detect the non-existence of maximal green sequences. A maximal green sequence in $Q(s\tau\text{-tilt} A)$ is a path from $(A, 0)$ to $(0, A)$ using only left mutations or equivalently respecting the orientation of the edges.

We will see that mutations in $Q(s\tau\text{-tilt} A)$ are closely linked to certain paths in the wall and chamber structure of A .

Definition 4.4. [BST19, Definition 4.1] A smooth path $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ is called a \mathcal{D} -generic path if:

1. Both $\gamma(0)$ and $\gamma(1)$ lie in the interior of some chamber.

2. If $\gamma(t)$ lies in the intersection of two walls, the dimension vectors of the modules determining these walls must be scalar multiples of each other. Note that this means that $\gamma(t) \neq 0$ for all t .
3. $\gamma(t) \in \mathcal{D}(M)$ implies $\langle \gamma'(t), [M] \rangle \neq 0$.

Further, when $\gamma(t)$ lies in a wall $\mathcal{D}(M)$, we say that the crossing is *green* if $\langle \gamma'(t), [M] \rangle > 0$ and red if $\langle \gamma'(t), [M] \rangle < 0$. A \mathcal{D} -generic path is then called green if all such crossings are green.

In [BST19, Lemma 4.2], the authors construct explicitly a \mathcal{D} -generic path moving from the chamber of some support τ -tilting pair (M, P) to the chamber induced by a mutation of (M, P) , passing exactly one wall.

Inductively, they show in [BST19, Theorem 4.3] that a sequence of left and right mutations from (M_1, P_1) to (M_2, P_2) induces a \mathcal{D} -generic path between $C^\circ(M_1, P_1)$ and $C^\circ(M_2, P_2)$ passing one wall for each mutation. The following corollary will be used later.

Corollary 4.1. Let $T = (M_1, P_1)$ and $T = (M_2, P_2)$ be support τ -tilting objects over A .

If there is no \mathcal{D} -generic path γ from $C^\circ(T_1)$ and $C^\circ(T_2)$ in the wall and chamber structure of A passing only finitely many walls, then T_1 and T_2 lie in different components of $Q(s\tau\text{-tilt } A)$

Proof. This is the contra-positive of the result discussed above; [BST19, Theorem 4.3]. \square

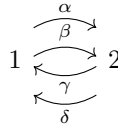
5 Main results

We now exploit the techniques discussed so far to determine properties of mutation quivers $Q(s\tau\text{-tilt } A)$ for various algebras A .

5.1 Inheriting mutation from Kronecker algebras

The idea of the following theorem is to use the τ -tilting theory of the Kronecker algebra with two arrows, K_2 , to explicitly compute mutations in a more complicated algebra.

Theorem 5.1. Let A be an algebra on the form kQ/r^2 where Q is the quiver



Then $Q(s\tau\text{-tilt } A)$ has exactly two components.

Proof. A module M over A where either the maps induced by γ and δ , or α and β are zero may naturally be considered a K_2 -module.

Then it is not difficult to see that the g -vector of such an M as a K_2 -module is the same as the g -vector of M as an A -module, up to switching coordinates (if we demand that arrows go from vertex 1 to 2 in K_2 , we need to do this swap for the case where M is an A -module where α, β are zero).

Consider the mutation of $P(1) \oplus P(2)$ at $P(1)$.

$$P(1) \xrightarrow{\begin{bmatrix} \gamma \\ \delta \end{bmatrix}} P(2) \oplus P(2)$$

Is a minimal right $P(2)$ -approximation of $P(1)$, and its cokernel X is naturally a K_2 module, as $\alpha X = 0$, and $\beta X = 0$. It is also τ -rigid as a K_2 module. We thus have a τ -tilting pair $T = (P(2), X)$ over A , which may naturally be considered a τ -tilting pair over K_2 .

This means that all iterated left mutations from T are controlled by mutation in K_2 .

A symmetrical argument shows that mutating $P(1) \oplus P(2)$ at $P(2)$ gives a τ -tilting module whose iterated left-mutations are again dictated by mutation on K_2 .

Thus we may compute the g -vectors of these τ -tilting modules, as shown in the figure.

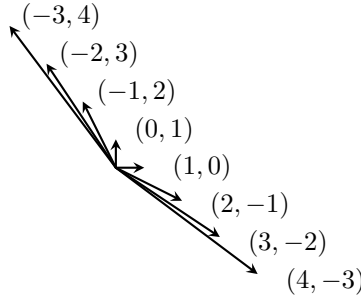


Figure 5: The g -vectors given by indecomposable summands of support τ -tilting objects at most three left mutations from $(P(1) \oplus P(2), 0)$ for the algebra A

The closure of the cones coming from the support τ -tilting objects which are left mutations of $(P(1) \oplus P(2), 0)$ clearly hit any point $(x, y) \in \mathbb{R}^2$ where $y + x \geq 0$.

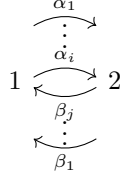
By duality on support τ -tilting objects, we may see that the closure of the cones coming from iterated right mutations of $(0, P(1) \oplus P(2))$ hit any point $(x, y) \in \mathbb{R}^2$ where $y + x \leq 0$.

Thus we see that every point $(x, y) \in \mathbb{R}^2$ lies in the closure of the cones coming from support τ -tilting objects in these two components. This concludes the proof.

□

We may generalize the theorem above slightly.

Corollary 5.1. Let $A_{i,j}$ be the algebra on the form kQ/r^2 where Q is on the form



Then $Q(s\tau\text{-tilt} A_{i,j})$ has exactly two connected components for all $i, j \geq 2$.

Proof. Entirely equivalent to the case where $i = 2 = j$, we will still have that left mutations of $P(1) \oplus P(2)$ behave as if they were left mutations appearing in the Kronecker algebra case.

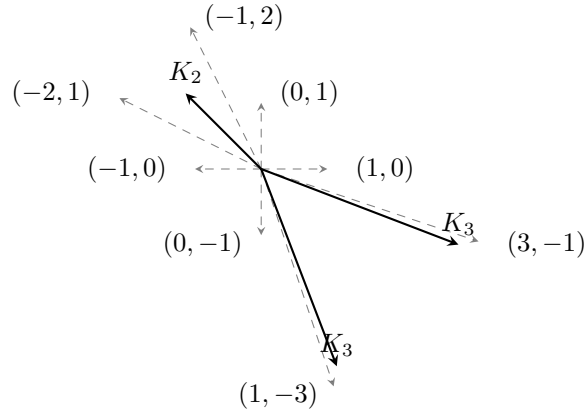


Figure 6: The g -vectors given by indecomposable summands of support τ -tilting objects at most two left mutations from $(P(1) \oplus P(2), 0)$ or two right mutations from $(0, P(1) \oplus P(2))$ for the algebra $A_{3,2}$. The area between the vectors labeled K_3 is densely filled with walls.

The g -vectors of right mutations from $(0, P(1) \oplus P(2))$ may be computed indirectly by left mutations of $(P(1)^* \oplus P(2)^*, 0)$ on A^{op} .

Note that a wall v in K_i gives a wall v in $A_{i,j}$. Also, a wall $w_1 = (x, y)$ in K_j gives a wall $w_2 = (y, x)$ in $A_{i,j}$ because we need to swap the arrows in K_j to view it as a quotient of $A_{i,j}$.

The closure cones of the identified support τ -tilting objects now span $\mathbb{R}^2 - C(W)$, where $C(W)$ is the closure of all walls. Thus by theorem 4.1, these are all support τ -tilting objects.

□

The following corollary gives an example of an algebra with more than two components in its τ -tilting mutation quiver.

Corollary 5.2. Let A be the algebra on the form kQ/r^2 with the quiver.

$$\begin{array}{ccc} 1 & & 3 \\ \left(\begin{array}{c} \downarrow \uparrow \end{array} \right) & & \left(\begin{array}{c} \downarrow \uparrow \end{array} \right) \\ 2 & & 4 \end{array}$$

Then $Q(s\tau\text{-tilt}A)$ has exactly 4 connected components.

Proof. Let $A_1 = P(1) \oplus P(2)$, $A_2 = P(3) \oplus P(4)$. Let $Q = Q(s\tau\text{-tilt}A)$.

Then $Q(s\tau\text{-tilt}A)$ is, as a directed graph, the cartesian product of $Q(s\tau\text{-tiltEnd}(A_1))$ and $Q(s\tau\text{-tiltEnd}(A_2))$. Each of these have two connected components, so its product has four.

More concretely, it is not hard to see that every support τ -tilting object lies in the same components as either $(A_1 \oplus A_2, 0)$, (A_1, A_2) , (A_2, A_1) or $(0, A_1 \oplus A_2)$ and that none of these four objects lie in the same component. \square

We now discuss one consequence of the above result. It is natural to seek an algorithm which enumerates all support τ -tilting objects. For τ -tilting finite algebras, the mutation quiver is connected, and may thus easily be traversed. More generally, for an algebra A where every support τ -tilting object may be found in the same component as $(A, 0)$ or $(0, A)$, one may enumerate all support τ -tilting objects by recursively left- and right-mutating at $(A, 0)$ and at $(0, A)$. The above corollary demonstrates that this algorithm does not in general enumerate all support τ -tilting objects.

There are also connected algebras with more than two components in its τ -tilting mutation quiver.

Corollary 5.3. Let A be the algebra on the form kQ/r^2 with Q as the quiver below.

$$\begin{array}{ccc} 1 & \longrightarrow & 3 \\ \left(\begin{array}{c} \downarrow \uparrow \end{array} \right) & & \left(\begin{array}{c} \downarrow \uparrow \end{array} \right) \\ 2 & & 4 \end{array}$$

Then $Q(s\tau\text{-tilt}A)$ has at least 4 connected components.

Proof. Note that $\text{Hom}_A(P(1) \oplus P(2), P(3) \oplus P(4)) = 0$ as there are no paths from vertices 3, 4 to vertices 1, 2. Thus

$$T_1 = (P(3) \oplus P(4), P(1) \oplus P(2))$$

is a support τ -tilting object. We intend now to show that it may not lie within a finite number of right or left mutations from $(A, 0)$ and $(0, A)$. This means that T_1 must lie in a third component of $Q(s\tau\text{-tilt}A)$, as wanted.

From [BST19][Lemma 4.13], we see that a wall in the wall and chamber structure of the algebra presented in corollary 5.2 gives a wall in the wall and chamber structure of A . Let B be the algebra in corollary 5.2.

Consider $T_{B,1} = ((P_B(3) \oplus P_B(4), P_B(1) \oplus P_B(2)))$. Then T_B and T_1 will have equal G -matrices.

$$G^{T_1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = G^{T_{B,1}}$$

This means in particular that g^{T_1} lies in the cone of $T_{B,1}$. Assume now, that there is a \mathcal{D} -generic path γ from g^{T_1} to $g^{(A,0)} = (1, 1, 1, 1)$ in $Q(s\tau\text{-tilt } A)$. In the wall and chamber structure of B , this path is also \mathcal{D} -generic and must cross infinitely many walls since $(B, 0)$ and $T_{B,1}$ lie in different components of $Q(s\tau\text{-tilt } B)$. Now, since walls in B induce walls in A , γ must pass infinitely many walls also in A .

Then by corollary 4.1, T_1 does not lie in the same component as $(A, 0)$. A similar argument shows that it may not lie in the same component as $(0, A)$.

To identify a fourth component, observe first that $(P(1) \oplus P(2), P(3) \oplus P(4))$ is not support τ -tilting, as there is a nonzero map $P(3) \rightarrow P(1)$ induced by the path of length 1 from 1 to 3. We consider instead the mutation $\mu_{P(1)}(P(1) \oplus P(2) \oplus P(3) \oplus P(4))$.

Since $\text{Hom}(P(1), P(4)) = 0 = \text{Hom}_A(P(1), P(3))$, the mutation may be computed as a co-kernel of $P(2) \oplus P(2)$ exactly as in theorem 5.1.

Let then

$$X \oplus P(2) \oplus P(3) \oplus P(4) = \mu_{P(1)}(P(1) \oplus P(2) \oplus P(3) \oplus P(4))$$

X can easily be computed to be a module with dimension vector $(3, 2, 0, 0)$ and g -vector $(-1, 2, 0, 0)$.

Since $\text{Hom}_A(P(3) \oplus P(4), X \oplus P(2)) = 0$ we get that

$$T_2 = (X \oplus P(2), P(3) \oplus P(4))$$

is a support τ -tilting object. Consider $T_{B,2} = \mu_{P_B(1)}((P_B(1) \oplus P_B(2), P_B(3) \oplus P_B(4)))$. Then $T_{B,2}$ and T_2 will have equal G -matrices.

$$G^{T_2} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = G^{T_{B,2}}$$

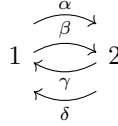
This means in particular that g^{T_2} lies in the cone of $T_{B,2}$. Again then we conclude that T_2 can not lie in the component containing $(A, 0)$ or the component containing $(0, A)$.

Lastly, T_2 and T_1 themselves must also lie in two distinct components of $Q(s\tau\text{-tilt } A)$, because their g -vectors lie in cones corresponding to support τ -tilting objects of B lying in distinct components.

This concludes our proof. \square

5.2 Further reductions

We will here further generalize theorem 5.1. Note that if we wish to study the algebra $A = kQ/r^3$ with Q being the quiver



we can no longer use the same proof as for theorem 5.1 to compute its mutation quiver. In this section, we will show that we can still use a certain reduction from K_2 to compute $Q(\text{st-tilt } A)$. This technique becomes visible not so much directly in the module category but rather in the bounded homotopy category $K^b(\text{add } A)$.

Lemma 5.1. Let A be an admissible quotient over some path algebra with a given ordering of its vertices, such that $\text{Hom}(P(2), P(1))$ is 2-dimensional as a k -vector space.

1. For any non-negative integer n there is a 2-term presilting complex in $K^b(\text{add } A)$, called $\mathbb{P}_A(n)$, on the form

$$P(2)^n \rightarrow P(1)^{n+1}$$

2. $\mathbb{P}_A(n) \oplus \mathbb{P}_A(n+1)$ will also be 2-term presilting.

Proof. (1.) The case where $n = 0$ is trivial, so assume that $n \geq 1$. Let first K be the Kronecker algebra with two arrows. Let now $\mathbb{P}_K(n)$ be the unique indecomposable 2-term presilting object on the form

$$P_K(2)^n \xrightarrow{f} P_K(1)^{n+1}$$

over $K^b(\text{add } K)$. Note that f may be viewed as an $n \times (n+1)$ matrix over $\text{Hom}_K(P(2), P(1))$ which is 2-dimensional. Pick an isomorphism ϕ between $\text{Hom}_K(P(2), P(1))$ and $\text{Hom}_A(P(2), P(1))$. Then we may let ϕ act entry-wise on f . Call this induced map $\phi(f)$. We may also consider ϕ^{-1} , which will have the symmetric property. We claim now that the complex \mathbb{P}_A given by

$$P_A(2)^n \xrightarrow{\phi(f)} P_A(1)^{n+1}$$

is rigid.

To check rigidity of \mathbb{P}_A , we must show that $\text{Hom}_{K^b(\text{add } A)}(\mathbb{P}_A, \mathbb{P}_A[1]) = 0$, meaning that all maps from \mathbb{P}_A to \mathbb{P}_A shifted should be null-homotopic. Thus, we seek for any $g : P_A(2)^n \rightarrow P_A(1)^{n+1}$ two maps h_1^A and h_2^A , such that $\phi(f) \circ h_1^A + h_2^A \circ \phi(f) = g$. This setup is drawn below.

$$\begin{array}{ccc}
& P_A(2)^n & \xrightarrow{\phi(f)} P_A(1)^{n+1} \\
& \downarrow g & \\
P_A(2)^n & \xrightarrow{\phi(f)} & P_A(1)^{n+1}
\end{array}
\begin{array}{c}
\swarrow h_1^A \quad \searrow h_2^A \\
\swarrow \quad \searrow
\end{array}$$

Certainly, $\phi^{-1}(g)$ is well-defined. Since

$$P_K(2)^n \xrightarrow{f} P_K(1)^{n+1}$$

is rigid, given the map $\phi^{-1}(g)$, there exist maps $h_1 : P_K(2)^n \rightarrow P_K(2)^n$ and $h_2 : P_K(1)^{n+1} \rightarrow P_K(1)^{n+1}$ such that $h_2 \circ f + f \circ h_1 = \phi^{-1}(g)$. This setup is drawn below.

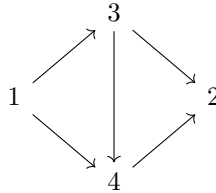
$$\begin{array}{ccc}
& P_K(2)^n & \xrightarrow{f} P_K(1)^{n+1} \\
& \downarrow \phi^{-1}(g) & \\
P_K(2)^n & \xrightarrow{\phi(f)} & P_K(1)^{n+1}
\end{array}
\begin{array}{c}
\swarrow h_1 \quad \searrow h_2 \\
\swarrow \quad \searrow
\end{array}$$

Note that h_1 and h_2 are scalar matrices, and may thus also be considered elements of $\text{End}_A(P_A(2))$ and $\text{End}_A(P_A(1))$ respectively. We may conclude that $\phi(h_2 \circ f + f \circ h_1) = g$. The left hand side might be computed to be $h_2 \circ \phi(f) + \phi(f) \circ h_1$. We then have the equality $h_2 \circ \phi(f) + \phi(f) \circ h_1 = g$ as wanted. Letting $h_1^A = h_1$ and $h_2^A = h_2$, we conclude the proof.

2. Let $\mathbb{T}_K = \mathbb{P}_K(n) \oplus \mathbb{P}_K(n+1)$. Then \mathbb{T}_K is 2-term pre-silting. We may then apply essentially the same proof as for (1.) to see that the corresponding $\mathbb{T}_A = \mathbb{P}_A(n) \oplus \mathbb{P}_A(n+1)$ must be rigid. \square

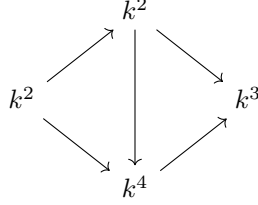
We give two examples utilizing this lemma.

Example 5.1. Consider the algebra $A = kQ/r^3$, where Q is the quiver below.

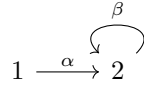


Of course, in kQ , there are three paths from 1 to 2. However, the path of length 3 is eliminated in the quotient A . Since the paths $1 \rightarrow 3 \rightarrow 2$ and $1 \rightarrow 4 \rightarrow 2$ are not identified in A , we may conclude that $\text{Hom}_A(P(2), P(1))$ is 2-dimensional. We may now employ GAP to compute $\mathbb{P}_A(1)$. The zeroth homology of this complex gives us a module, say T_1 , (with dimensions as displayed below). This module must be τ -rigid from lemma 5.1, which we may be

confirmed using GAP. In this case, the module is also indecomposable.



Example 5.2. Let $A = kQ/\beta^2$ where Q is the quiver below.



This algebra is τ -tilting finite, and $\text{Hom}_A(P(2), P(1))$ is 2-dimensional. We are still promised an infinite family of τ -tilting objects from lemma 5.1, where we let $T_i = H^0(\mathbb{P}_A(i))$. The consequence must be that almost all modules coming from this lemma must decompose. To investigate this, we first compute $Q(s\tau\text{-tilt } A)$. The mutation quiver is displayed in fig. 7, where the vertices are written as support τ -tilting objects.

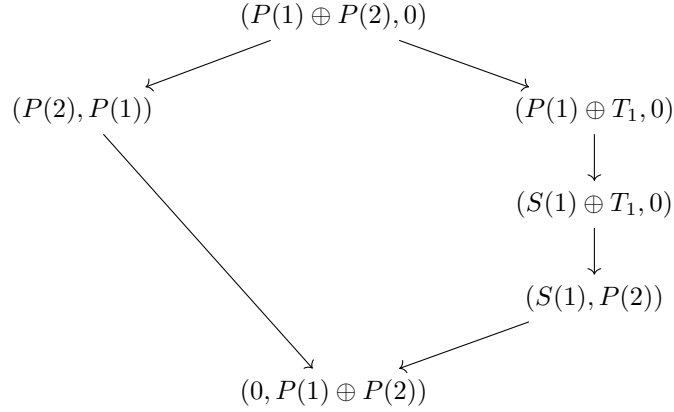


Figure 7: The mutation quiver of A from example 5.2.

$T_0 = P(1)$ and is thus clearly indecomposable. T_1 shows up as a summand in the mutation quiver and is indecomposable. Since any τ -rigid module must show up in the mutation quiver as a summand of some support τ -tilting object, all T_i for $i > 1$ must decompose into other τ -rigid objects. We know that τ -rigid modules are uniquely identified by their g -vectors. We also know that $S(1)$ has g -vector $(1, -1)$, and that $S(1) \oplus T_1$ is τ -rigid. Thus we have the following equation

$$g^{T_i} = (i + 1, -i) = g^{T_1^{i-1} \oplus S(1)}$$

Thus it follows that in fact $T_i \cong T_1^{i-1} \oplus S(1)$ for all $i > 1$.

The above example shows that to fully utilize lemma 5.1, we need a criterion for indecomposability of the modules produced by the lemma.

In the next lemma, we give an elementary example of such a condition for the case where A has two simple modules. We dare hope that there exists a similar criterion which may be applied for algebras with more than two simple modules, but for our purposes the following tool will be sufficient.

Lemma 5.2. Let A be as in lemma 5.1, where A has two simple modules. Assume that $(1, -1)$ is not a g -vector of A .

Then $T(i)$ is indecomposable for all $i \geq 0$.

Proof. By the second part of lemma 5.1, we know that $T_k \oplus T_{k+1}$ is τ -rigid for all $k \geq 0$.

To seek a contradiction, assume that $k \geq 1$ is the smallest integer such that T_k decomposes (T_0 is projective indecomposable). Recall that the g -vector of T_i is $(i+1, -i)$, and that a basic τ -rigid module over A may have at most two summands.

A has 2 simple modules, so any basic τ -tilting object must have at most 2 summands. If T_k decomposes, $T_{k-1} \oplus T_k$ has three summands, and must therefore not be basic. Since T_{k-1} is indecomposable, this means that T_k must decompose in one of the following ways.

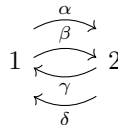
1. $T_k \cong X^n$, where $n \geq 2$.
2. $T_k = T_{k-1} \oplus Y$

Indeed, if T_k decomposes into an object with at least two non-isomorphic summands, then one of them must be isomorphic to T_{k-1} . Otherwise the sum $T_{k-1} \oplus T_k$ could not be τ -rigid.

The first case is not possible, because $-k$ and $k+1$ are relatively prime; thus $g^{T_k} = (k+1, -k)$ cannot be written as mv for some other g -vector v and integer $m > 1$. In the second case, Y would be τ -rigid with g -vector $(1, -1)$ by additivity, contradicting our assumptions. \square

We may now work with a class of algebras including the one mentioned in the introduction to this section.

Theorem 5.2. Let $A = kQ/I$ be an algebra where Q is as drawn below and I is an admissible ideal with $r^3 \subseteq I$, where r is the radical of kQ . Then $Q(s\tau\text{-tilt } A)$ has exactly two components.



Proof. First, note that for any choice of I as given above, paths of length 3 are killed by I , so $\text{Hom}(P(1), P(2))$ and $\text{Hom}(P(2), P(1))$ are both 2-dimensional as k -vector spaces. We may thus apply lemma 5.1 and the symmetrical version of it.

Let T_i be the τ -rigid module with g -vector $(i+1, -i)$ coming from lemma 5.1. We wish to apply lemma 5.2. Of course A has two simples.

We need to show that $(1, -1)$ may not be a g -vector of any partial τ -tilting module over A .

Observe that any open neighborhood around the point $(1, -1)$ as a point in \mathbb{R}^2 intersects infinitely many walls in the wall and chamber structure of A (see fig. 8). Thus $(1, -1)$ cannot lie inside a chamber. Also, given any $v \neq 0$ not parallel to $(1, -1)$, we see that there exists for any real $\epsilon > 0$ an $\alpha < \epsilon$ such that $(1, -1) + \alpha v$ crosses a wall. Thus there cannot exist a chamber having $(1, -1)$ as a wall.

We have thus infinitely many chambers on the form $C(T_i \oplus T_{i+1})$ for $i \geq 0$. Using the fact that $\text{Hom}(P(1), P(2))$ and $\text{Hom}(P(2), P(1))$ are both 2-dimensional, we see that the closure of the chambers of the component of $Q(s\tau\text{-tilt } A)$ containing $(P(1) \oplus P(2), 0)$ contains all points $(x, y) \in \mathbb{R}^2$ where $x + y \geq 0$. Dually, since A^{op} satisfies the same assumptions as we needed about A , the closure of the chambers of the component containing $(0, P(1) \oplus P(2))$ must contain all points (x, y) where $x + y \leq 0$.

We may then conclude that $Q(s\tau\text{-tilt } A)$ has exactly two components in its mutation quiver. □

5.3 Applying homotopy equivalence

The main idea needed to prove theorem 5.2 is that the 2-term silting objects of K_2 can say something about 2-term silting objects over richer module categories. However, the proof above also requires some non-trivial technicalities because the category of 2-term objects of projective K_2 -modules are not equivalent to the category of 2-term objects of projective modules in the more complicated category.

In the setting where we require this to hold, stronger reduction results hold.

Proposition 5.1. Let A be an algebra with $P = \bigoplus_{i \in I} P(k)$ be some direct sum of projective indecomposable modules. Let B be an algebra such that $\text{add } B \cong \text{add } P$ holds. Then

$$K^b(\text{add } B) \cong K^b(\text{add } P)$$

and there is a bijection between indecomposable 2-term pre-silting complexes over B and indecomposable 2-term pre-silting objects over A having g -vectors with support on I .

Proof. The proposition follows immediately from noting that $K^b(\text{add } P)$ embeds naturally into $K^b(\text{add } A)$, where the 2-term silting complexes over A live. □

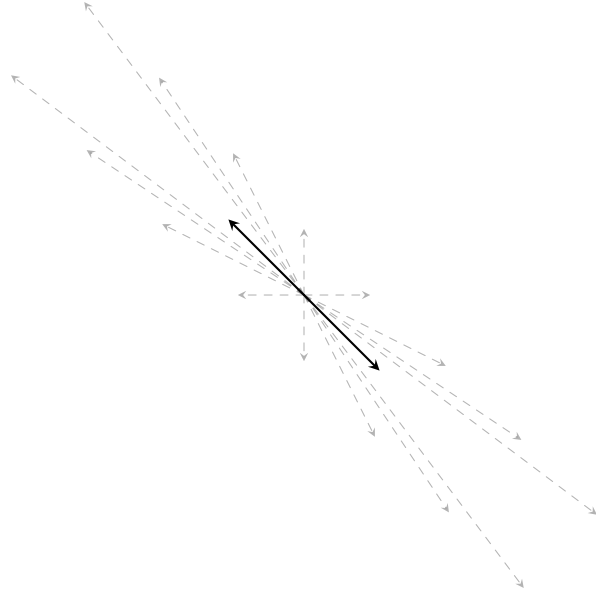


Figure 8: Some of the walls inherited from K_2 in two different ways into A . The bold arrows are contained in the closure of these walls, and point in directions $(1, -1)$ and $(-1, 1)$.

Proposition 5.2. Let Q be an acyclic quiver, and let $A = kQ/I$ with I an admissible ideal. If there exists indecomposable projective modules P and Q such that $\text{Hom}_A(Q, P)$ is i -dimensional as a k -vector space for $i \geq 2$, then A is τ -tilting infinite.

Proof. We need only add $P \oplus Q \cong \text{add } K_i$.

We define a functor

$$F : \text{add } K_i \rightarrow \text{add } P \oplus Q$$

as follows. Let $F(P(1)) = Q$, and $F(P(2)) = P$.

Then, we pick any isomorphism ϕ between $\text{Hom}_{K_2}(P(2), P(1))$ and $\text{Hom}_A(Q, P)$, and let $F(f) = \phi(f)$ for any $f \in \text{Hom}_{K_2}(P(2), P(1))$. Since the endomorphisms of any indecomposable object in either additive categories are just scalar multiples of the identity, we can see that F is an equivalence of categories.

Then the proposition follows from proposition 5.1. \square

The following corollary follows immediately.

Corollary 5.4. If kQ/I is τ -tilting finite and Q is acyclic, then $\text{Hom}(P(i), P(j))$ must be 1 or 0 dimensional for every $i \in Q_0$.

Proof. The proof follows directly from the above discussion. \square

We remark that this criterion does not classify τ -tilting finite algebras, as for example the algebra $\mathbb{C}Q$ with Q drawn in fig. 9 is τ -tilting infinite.

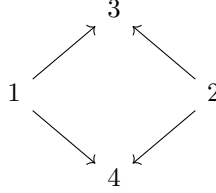
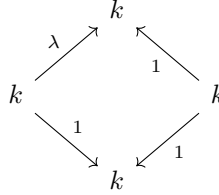


Figure 9: An acyclic quiver inducing a τ -tilting infinite algebra satisfying $\text{Hom}(P(i), P(j)) \leq 1$ for all points $i, j \in Q_0$.

To see that this particular algebra is τ -tilting infinite, observe that letting $\lambda \in \mathbb{C}$ vary, we get an infinite family of pairwise non-isomorphic bricks, with representations drawn below.



By [DIJ17, Theorem 1.4] this shows that the algebra is τ -tilting infinite.

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