Combinatorial aspects of τ-tilting theory

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1 Introduction

In these notes, $A = k\Gamma/I$ will denote the quotient of a path algebra over an admissible ideal. In particular, A is finite dimensional. We now recall some important definitions from tau-tilting theory.

2 Duality on suppport τ-tilting modules

There is a theory of duality for support τ-tilting modules which we now describe.

For a support τ-tilting A-module (M, P), let M_p be the projective part of M. We then have

$$(M, P)^\dagger = (\mathrm{Tr}M \oplus P^*, M_p^*)$$

and $(M, P)^\dagger$ is a support τ-tilting A^{op}-module.

The following observation will be particularly useful when studying the mutation graphs of symmetric algebras.

Observation 2.1. *An ordering of (M, P) induces an ordering on $(M, P)^\dagger$. With this induced ordering,*

$$G_{(M,P)^\dagger} = -G_{(M,P)}$$

Proof. Any indecomposable summand X in (M, P) appears as exactly one indecomposable summand of $(M, P)^\dagger$, either in the tau-rigid part or in the second part.

Also note that the canonical bijection between the vertices of A and A^{op} induces a canonical isomorphism between the Groethendieck groups $K_0(A)$ and $K_0(A^{\mathrm{op}})$.

Let now X be a nonprojective summand of M. Then we have a projective presentation $P_i \to P_j \to X \to 0$, inducing a projetive presentation of the transpose $\mathrm{Tr}X$

$$P_j^* \to P_i^* \to \mathrm{Tr}X \to 0$$

Then it directly follows that $g^{\mathrm{Tr}X} = -g^X$.

If X is a projective summand of M, $(M, P)^\dagger = (N, Q \oplus X^*)$ for some appropriate N and Q. Then by the definition of g-matrices for support τ-tilting modules, the column corresponding to the g-vector of X^* is given a negative sign, as it is placed in the second part.

Lastly, if X is a summand of P, it is moved from the second part to the τ-rigid part. This means that the column of the G-matrix $(M, P)^\dagger$ corresponding to this summand will be sign-inverted.

As all columns are thus sign-inverted and we have proved the statement.

$\square$

3 Mutation quivers of symmetric algebras

For a symmetric algebra A, any A^{op}-module M may be viewed a an A-module.

In particular, for a support τ-tilting A-module (M, P), $(M, P)^\dagger$ may also be viewed as a support τ-tilting module over A. This implies that the duality of support τ-tilting modules induces an automorphism on the set of (not necessarily ordered) support τ-tilting modules of symmetric algebras.