Exploring Negami's Conjecture

Håvard Utne Terland

May 24, 2019

A short introduction

Negami's conjecture is a conjecture in **topological graph theory**. In this presentation, all graphs are assummed to be finite and simple.

Topological graph theory

Topological graph theory is (more or less) the study of graphs drawn on surfaces. An interesting combination of discrete mathematics and topology.

Example

For a graph G, what's the smallest number $g \geq 0$ such that G can be drawn on the orientable surface of genus g? NP-Complete, but computable. One can say interesting things about given classes of graphs, e.g K_n.

Graph covers

Definition

A cover of a graph G is a graph H such that there is a surjective map $p : H \to G$ which is an isomorphism when restricted to any set of the form $N_H(v) \cup v$ for $v \in H$. ("local isomorphism")

Theorem

For connected graphs, any cover of it has a well-defined fold-number

Proof.

Let G be connected, let u, v be two vertices in $V(G)$ and let H be a cover of G with projection p. Pick a path between them $P = (u, v_1, v_2, v_3, \ldots, v_n, v)$ in G and consider $P \subseteq G$ a subgraph of the graph G. $p^{-1}(P)$ is a disjoint set of paths, and give bijection between $p^{-1}(u)$ and $p^{-1}(v)$. $\square$

Planar covers

Some non-planar graphs have covers that are planar! K_5 and $K_{3,3}$, for example.

Remark

Not all graphs have planar covers, since planar graphs must have a vertex of degree maximum 5. Thus K_n cannot have planar cover for $n > 6$.

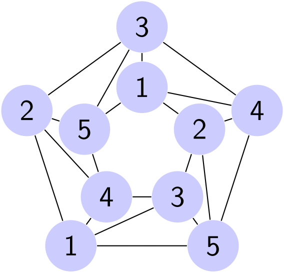

Figure: A 2-fold planar cover of K_5

The conjecture

Given a graph G, if we can draw it on $\mathbb{R}P^2$ we call it projective-planar. A projective-planar graph has a 2-fold planar cover of G. Simply lift via the 2-1 map $p : S^2 \to \mathbb{R}P^2$.

Theorem (Negami 1986)

A connected graph has a 2-fold planar cover iff it is projective-planar.

Negami's Conjecture

A connected graph has a planar cover iff it is projective-planar.

Generalizing the conjecture

Observation

Negami's conjecture $\iff$ A graph has a projective-planar cover iff it can be drawn on projective plane.

Conjecture (Hliněný)

A graph can be drawn on the Klein bottle iff it has a Klein-cover.

Example

K_7 does not have a klein cover (Hliněný 1999)

A question

Which graph families cover themself?

$\mathcal{C}(\mathcal{F}) = \mathcal{F}$ for which graph families $\mathcal{F}$? Can we say anything non-trivial about this in general?