Date 09/04

Unit - IV

Numerical methods - I

Solution of algebraic and transcidential equation.

An equation involving terms of the form polynomial $x^2, x^3, 4x \dots$ etc are called algebraic equations.

Eg: $x^3 - 2x - 5 = 0$ $\quad$ $x^4 - x - 10 = 0$

An equation involving terms e^x, $\sin x$, $\sec x$, $\log x$, x^{x^2}, etc are called transdential equation.

Eg: $xe^x - 2 = 0$, $x\log_{10} - 12 = 0$

$3x = \cos x + 1$

Solution of $f(x) = 0$

*1) Regular falsa method

*2) Newton Raphson method useful on finding the real root of the form $f(x) = 0$.

Regular falsi method

(Method of false position) - let the given equation is of the form $f(x) = 0$ contains only one root in the interval (a, b) and we suppose that the root lies b/w $f(a) < 0$ & $f(b) > 0$ or $f(a) > 0$, $f(b) < 0$. then Regular falsi method is given by

$$x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

NOTE - If 'a' & 'b' are close enough then we can obtain the approximate root to the desired accuracy quickly. The problem are worked out by considering the

* ~~Always~~ root lies b/n the functional value opp side.

value of a & b at the difference of 0.1 to determine ~~to~~ the faster.

1. Find the real root of an equation $x^3-2x-5=0$ by regular false method correct to 3 decimal places.

A: Let $f(x)=0$;
$f(x)=x^3-2x-5=0$.
$f(0)=-5<0$, $f(1)=-6<0$.
$f(2)=-1<0$, $f(3)=16>0$

1-2-5
1-7
-6
2
27-6-5
27-11

The root lies b/w (2,3).
∴ The real roots exists b/w 2 & 2.1.

$f(2.2)=+ve$
$f(2.1)=0.061$ +ve
$f(2)=-1<0$ -ve
$f(1.9)=-ve$.

∴ ~~The real~~ By using RFM.

1st appx $x_1=\dfrac{af(b)-bf(a)}{f(b)-f(a)}$ $a=2$ $b=2.1$

$=\dfrac{2\times 0.061-2.1(-1)}{0.061+1}$

$\boxed{x_1=2.0942}$

$f(x_1)=f(2.0942)=(2.0942)^3-2\times 2.0942-5$
$=-0.00392<0$.

∴ The root lies b/n (2.0942, 2.1)
$a=2.0942$ $b=2.1$ $f(a)=-0.00392$
$f(b)=0.061$

0.000389

$x_2=\dfrac{2.0942\times 0.061-2.1\times -0.00392}{0.061+0.00392}$

$x_2=2.0945$

∴ The required root correct to three decimal places is 2.094

2. Find the real root of the eqn $x\log_{10}x=1.2$ using RFM.

A: Let $f(x)=0$. ⇒ $f(x)=x\log_{10}x=1.2$

$f(0)=-1.2<0$ $f(1)=-1.2$ $f(2)=-0.59<0$.
$f(3)=0.23>0$.
The root lies b/n 2 & 3.

-0.59 0 0.23
f(2) f(3)

$0.23<0.59$.

$f(2.7)=-0.035<0$
$f(2.8)=0.052$
$f(2.9)=0.14$
$f(3)=0.23$

∴ The root lies b/n 2.7 & 2.8
By using RFM
1st app: $\dfrac{af(b)-bf(a)}{f(b)-f(a)}$

$a=2.7$ $b=2.8$
$f(a)=-0.035$ $f(b)=0.052$.

$=\dfrac{2.7\times 0.052-2.8\times -0.035}{0.052+0.035}$

$=\dfrac{0.2384}{0.087}=2.7402=x_1$

$f(x_1)=f(2.7402)=-0.000389<0$.

$f(2.7)=-ve$
$f(2.8)=+ve$
$f(2.7402)=-ve$

2^{nd} app $= \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ $a = 2.7402$ $b = 2.8$.

$f(a) = -0.00038$ $f(b) = 0.052$

$= \dfrac{2.7402 \times 0.052 - 2.8 \times -0.00038}{0.052 + 0.00038}$

$= \dfrac{0.1435544}{0.05238}$

$= 2.7406$

$\therefore$ The required root correct 3 decimal point is 2.740

~~$f(x) = x^2 - \log_e x - 1.2$~~ $A \to 3.64604$

$f(0) = -1.2$ $f(1) = -0.2$ $f(2) = 2.4989$

The root lies b/n 1 & 2.

-0.2 2.4989 0.2 0 2.4

1 0 2

$f(1.3) =$

$f(1.2) =$ ~~0.16081~~ 0.05767

$f(1.1) =$ ~~-0.03139~~ -0.08531

$f(1.) =$

The root lies b/w 1.1 & 1.2.

By using RFM.

1^{st} approximation $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ $a = 1.1$ $b = 1.2$

$f(a) = -0.03139$ $f(b) = 0.16081$

$\dfrac{1.1 \times 0.05767 - 1.2 \times -0.08531}{0.05767 + 0.08531}$ ~~$= \dfrac{1.1 \times 0.16081 - 1.2 \times -0.03139}{0.16081 + 0.03139}$~~

$= 1.15966$ ~~$= \dfrac{0.214559}{0.1922}$~~

$f(x_1) = -0.00331$

$= 1.11633 = x_1$

$f(x_1) = -0.001599$

~~2^{nd} app $f(a) = -0.00159$ $f(b) = 0.16081$~~

~~$a = 1.11633$ $b = 1.2$~~

~~2^{nd} approximation $= \dfrac{1.11633(0.16081) - 1.2 \times -0.00159}{0.16081 - (-0.00159)}$~~

~~$= \dfrac{0.18142}{0.1624}$~~

~~$= 1.11711$~~

2^{nd} app $a = 1.15966$ $f(a) = -0.00331$ $b = 1.2$ $f(b) = 0.05767$.

2^{nd} app $= f(x_2) = \dfrac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} = \dfrac{1.15966 \times 0.05767 - 1.2 \times -0.00331}{0.05767 + 0.00331}$

$= 1.16184$

3^{rd} app

- 0.00351

Find the real root of equation $xe^x = \cos x$ using regular falsi method correct to 4 decimal places

Let $f(x)=0$

$\Rightarrow f(x) = xe^x - \cos x.$

$f(0) = e^0 - \cos 0$

$f(0) = -1. < 0.$

$f(1) = 2.718 > 0.$

The root lies b/w 0 & 1.

$f(0)$ $f(1)$

-1 0 2.718.

{1 is near to 0 compare to 2.718} the ~~to~~

The root lies b/n 1 & 0.

Radian

$f(0) = -1$

$f(0.1) =$

$f(0.2) =$

$f(0.3) =$

$f(0.4) =$

$f(0.5) = -0.0532 < 0$

$f(0.6) = 0.2679 > 0.$

The root lies b/n (0.5 , 0.6)

$a=0.5 \quad f(a) = -0.0532 \quad b = 0.6 \quad f(b) = -0.2679$

1st approxiation $= \dfrac{a.f(b) - b.f(a)}{f(b) - f(a)}$

$= \dfrac{0.5 \times -0.2679 - 0.6 \times -0.0532}{-0.2679 + 0.0532}$

$= \dfrac{-0.10203}{-0.2147} = \underline{0.5166}$

$f(x_1) = -0.0035$

$f(0.5) = -ve$

$f(0.6) + ve$

$f(0.5166) = -ve.$

Root lies b/n 0.5166 & 0.6

2nd $\quad a = 0.5166 \quad b = 0.6 \quad f(a) = \cancel{-0.0532}\ -0.0035 \quad f(b) = 0.2679$

$\dfrac{a.f(b) - b.f(a)}{f(b) - f(a)}$

$\cancel{\dfrac{0.5166 \times 0.2679 - 0.6 \times -0.0035}{0.5166 - 0.5}}$

$\dfrac{0.5166 \times 0.2679 - 0.6 \times -0.0035}{0.2679 + 0.0035}$

$x_2 = \underline{0.5176}$

$f(0.5176) = f(x_2) = -0.0004$

$f(0.5176) = -0.0004.$

$f(0.6) = +ve = 0.2679.$

$a = 0.5176 \quad b = 0.6.$

$\dfrac{a.f(b) - b.f(a)}{f(b) - f(a)} = \dfrac{0.5176 \times 0.2679 - 0.6 \times -0.0004.}{0.2679 + 0.0004}$

$x_3 = \underline{0.5177}$

$f(0.5177) = -0.0001$

$f(a) = -0.0001 \quad a = 0.5177 \quad f(b) = 0.2679 \quad b = 0.6$

$x_4 = \dfrac{a.f(b) - b.f(a)}{f(b) - f(a)} = \dfrac{0.5177 \times 0.2679 - 0.6 \times -0.0001}{0.2679 + 0.0001} = 0.5177$

~~At~~ $f(x_4) = -1.74499 \times 10^{-4} = -0.00017$

-2.706

3 DA

** Find the real root of the eqn $x\log_{10}x = 1.2$ using RFM.

$f(x) = x\log_{10}x - 1.2.$

$f(0) = -1.2$

$f(1) =$

⇒ $x^3 - 4x + 9 = 0.$

$f(0) = +ve$ $\quad f(-1) = 12$

$f(1) = 6$ $\quad f(-2) = 9 > 0$

$f(2) = 9$ $\quad f(-3) = -6 < 0$

$f(3)$

$f(4) = 57$

The root lies b/n $(-3, -2)$

$f(-3)$ $\quad$ $f(-2)$

-6 $\quad 0 \quad$ $9.$

The root is near to -3.

$f(-2.7) = 0.117 \rightarrow +ve$

$f(-2.8) = -1.752 \rightarrow -ve$

$f(-2.9) = -3.789$

$f(3)$

∴ The root lies b/w -2.7 to -2.8

$a = -2.7$ $\quad b = -2.8$

$f(a) = 0.117$ $\quad f(b) = -1.752$

$$f(x_1) = \frac{a.f(b) - b f(a)}{f(b) - f(a)}$$

$$f(x_1) = \frac{-2.7 \times -1.752 - (-2.8) \times 0.117}{-1.752 - 0.117} = -2.7062$$

$f(b) = -2.7062 \log_{10}(2.7062) - 1.2$

$a = -2.7$ $\quad b = -2.7062$

$f(a) = 0.117$ $\quad$ ~~$f(b) = -2.7062$~~ -0.00589 $\quad f(b) = -0.00589$

$$f(x_2) = \frac{a.f(b) - b f(a)}{f(b) - f(a)} = \frac{-2.7 \times \cancel{-2.7062} + 2.7062 \times 0.117}{-0.00589 - 0.117}$$

$x_2 = -2.7059$

$f(x_2) = 0.011$

$b = -2.7062$ $\quad a = -2.7059$

$f(b) = -0.00589$ $\quad f(a) = 0.011.$

$$x_3 = \frac{a.f(b) - b f(a)}{f(b) - f(a)} = \frac{-2.7059 \times -0.00589 + 2.7062 \times 0.011}{-0.00589 - 0.011}$$

$= -2.7059.$

∴ The required root correct 3 decimal point is -2.705

Ans $\frac{\cancel{0.74065}}{3.7892}$

Using RFM obtain real root of the equation $2x - \log_{10} x = 7$ lies b/n 3.5 and 4.

$\Rightarrow f(x) = 2x - \log_{10} x - 7$

$f(3.5) = -0.5440$

$f(3.6) = -0.3563$

$f(3.7) = -0.1682$

$f(3.8) = 0.0202$

The root lies b/n 3.7 and 3.8

$a = 3.7 \quad b = 3.8 \quad f(a) = -0.1682 \quad f(b) = 0.0202$

$$x_1 = \frac{a f(b) - b \cdot f(a)}{f(b) - f(a)} = \frac{3.7 \times 0.0202 - 3.8 \times -0.1682}{0.0202 + 0.1682}$$

$x_1 = 3.78927 \qquad f(x_1) = -0.000015$

x_2 :, $b = 3.8 \quad f(a) = -0.000015 \quad a = 3.78927 \quad f(b) = 0.0202$

$$x_2 = \frac{3.78927 \times 0.0202 - 3.8 \times -0.000015}{0.0202 + 0.000015}$$

$\boxed{x_2 = 3.78927}$

Convert to Rad

0.607

$\cos x = 3x - 1 \Rightarrow f(x) = \cos x - 3x + 1$

0 & 1
0.6 & 0.7

$f(x) = \cos x - 3x + 1$

$f(0) = 2$

$f(1) = -1.4596$

The root lies b/n 1 & 0

1.4596   0   2

1   0

~~The root lies b/~~

$f(1.$

$f(1.2) = -2.237$

$f(1.1) = -1.8464$

$f(0.6) = 0.0253$

$f(0.7) = -0.3351$

$f(0.8) = -0.7032$

$f(0.9) = -1.07839$

$\therefore$ The root lies b/n 0.7 & 0.6

$a = 0.6 \quad b = 0.7 \quad f(a) = 0.0253 \quad f(b) = -0.3351$

$$x_1 = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} = \frac{0.6 \times -0.3351 - 0.7 \times 0.0253}{-0.3351 - 0.0253}$$

$x_1 = \cancel{-0.5087}\ 0.6070$

$f(x_1) = 0.0003$

$a = 0.6070 \quad f(a) = 0.0003 \quad b = 0.7 \quad f(b) = -0.3351$

$$x_2 = \frac{a \cdot f(b) - b \cdot f(a)}{f(b) - f(a)} = \frac{0.6070 \times -0.3351 - 0.7 \times 0.0003}{-0.3351 - 0.0003}$$

$x_2 = 0.6070$

Newton Raphson Method (NRM)

let $f(x)=0$ be the given eqn, the general newton raphson formula is given by $\boxed{x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}}$

put $n=0$; $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$n=1$; $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$n=2$; $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$

...... so on.

→ Find the real root of the eqn $3x = \cos x + 1$ correct to 4 decimal places using NRM.

let $f(x) = 3x - \cos x - 1$

$f(0) = -2$

$f(1) = 1.4597$

∴ Root lies b/w 0 & 1.

$\dfrac{0+1}{2} \Rightarrow \boxed{x_0 = 0.5}$

$f(0)$ $f(1)$

-2 0 1.4597

∴ The app root $\boxed{x_0 = 0.5}$

$f(x_0) = f(0.5) = -0.3775$

$f'(x) = f'(0.5) = 3 + \sin x = 3.4794$

from NRM,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \left(\frac{3x - \cos x_n - 1}{3 + \sin x}\right)$$

$$x_{n+1} = \frac{\cancel{3x_n} + x_n \sin x_n - \cancel{3x_n} + \cos x_n + 1}{3 + \sin x_n}$$

$$\boxed{x_{n+1} = \frac{x_n \sin x_n + \cos x_n + 1}{3 + \sin x_n}} \quad (2)$$

1st approx ⇒ $n = 0$.

$$x_1 = \frac{x_0 \sin x_0 + \cos x_0 + 1}{3 + \sin x_0}$$

$x_1 = 0.6085$

$$x_2 = \frac{x_1 \sin x_1 + \cos x_1 + 1}{3 + \sin x_1} \qquad x_3 = \frac{x_2 \sin x_2 + \cos x_2 + 1}{3 + \sin x_2}$$

$x_2 = 0.6071$ $\qquad$ $x_3 = 0.6070$

Using NR method. $x \log_{10} x = 1.2$.

$f(x) = x \log_{10} x - 1.2$

$f(0) = -1.2 < 0$, $f(1) = -1.2 < 0$ $f(2) = -0.597 < 0$,

$f(3) = 0.2313 > 0$.

The root lies in $(2, 3)$

Approx root is $x_0 = 2.5$

$$\left\{ \log_{10} x = \frac{\log_e x}{\log_{10} x} \right\}$$

$f(x) = x \log_{10} x - 1.2$

$$f(x) = x \frac{\log x}{\log 10} - 1.2$$

$$f'(x) = \frac{1}{\log 10}\left[1.\log x + \frac{1}{x}.x\right] - 0.$$

$$= \frac{\log x}{\log 10} + \frac{1}{\log 10} = \frac{1 + \log x}{\log 10}.$$

By NRM; $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{x_n\left(\frac{\log x_n}{\log 10} - 1.2\right)}{\frac{1+\log x_n}{\log 10}}$$

$$= x_n - \frac{\frac{x_n \log x_n - 1.2 \log 10}{\log 10}}{\frac{1+\log x_n}{\log 10}}$$

$$= x_n - \left(\frac{x_n \log x_n - 1.2 \log 10}{1+\log x_n}\right)$$

$$= \frac{x_n + x_n \log x_n - x_n \log x_n - 1.2 \log 10)}{1+\log x_n}$$

$$= \frac{x_n + \cancel{x_n \log x_n} - \cancel{x_n \log x_n} + 1.2 \log 10}{1+\log x_n}$$

$$x_{n+1} = \frac{x_n + 1.2 \log 10}{1+\log x_n} \quad (2)$$

1st iteration $x_{0+1} = \frac{x_0 + 1.2\log 10}{1+\log x_0} = 2.7465$

$$x_2 = \frac{x_1 + 1.2 \log 10}{1+\log x_1} = 2.7406$$

$$x_3 = \frac{x_2 + 1.2 \log 10}{1+\log x_2} = 2.7406.$$

$\therefore$ The required root is 2.7406.

4.493409.

$x_3 =$

$x_2 = 4.493409$

$x_1 = 4.49361$

Use NRM; Find the real root near to 4.5 of the eqn $\tan x = x$ correct to 4 d. places.

let $f(x) = \tan x - x$

$f'(x) = \sec^2 x - 1.$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{(\tan x_n - x_n)}{\sec^2 x_n - 1}.$$

$$= \frac{x_n \sec^2 x_n - \cancel{x_n} - \tan x_n + \cancel{x_n}}{\sec^2 x_n - 1}$$

$$= \frac{x_n \sec^2 x_n - \tan x_n}{\sec^2 x_n - 1}$$

$$= \frac{x_n \frac{1}{\cos^2 x_n} - \tan x_n}{\frac{1}{\cos^2 x_n} - 1}$$

$$= \frac{\frac{x_n - \tan x_n \cos^2 x_n}{\cos^2 x_n}}{\frac{1-\cos^2 x_n}{\cos^2 x_n}}$$

$$= \frac{x_n - \tan x_n \cdot \cos^2 x_n}{1-\cos^2 x_n}.$$

$$= \frac{x_n - \sin x_n \cos x_n}{1-\cos^2 x_n}.$$

$$= \frac{x_n - \sin x_n \cos x_n}{\sin^2 x_n}.$$

~~$x_1 = \frac{x_0 - \sin 0 \cos 0}{\sin^2 0}.$~~

x

$x_0 = 2.5$ $\Rightarrow 2.0945$

$f(x) = x^3 - 2x - 5$

$f'(x) = 3x^2 - 2.$

$f(0) = -5$

$f(1) = -6$

$f(2) = -1$

$f(3) = 16$

The root lies b/n 2 & 3.

$$\frac{2+3}{2} = 2.5$$

$$x_{n+1} = x_n - \frac{x^3 - 2x - 5}{3x^2 - 2}$$

$$= \frac{x_n 3x_n^2 - \cancel{2x_n} - x_n^3 + \cancel{2x_n} + 5}{3x_n^2 - 2}$$

$$\cancel{x_{n+1} = \frac{2x_n^2 + 5}{3x_n^2 - 2}} \quad x_n = \frac{2x_n^3 + 5}{3x_n^2 - 2}$$

$$x_1 = \frac{2(2.5)^3 + 5}{3(2.5)^2 - 2.} = 2.1641$$

$$x_2 = \frac{2x_1^3 + 5}{3x_1^2 - 2} = 2.0971$$

$x_3 = 2.09455$

$x_4 = 2.09455$

The required root is 2.0945

$f'(x) = 4x^3 - 1$

$x_0 = 1.5 \Rightarrow 1.855$

$x^4 - 4x = 10. \Rightarrow$

$f(x) = x^4 - 4x - 10.$

$f'(x) = 4x^3 - 1$

$f(0) = -10$

$f(1) = -13$

$f(2) = -2$

$f(3) = 59.$

The root lies b/n 2 & 3.

$$\frac{2+3}{2} = \frac{5}{2} = 2.5$$

$$x_{n+1} = x_n - \frac{(x_n^4 - 4x_n - 10)}{4x_n^3 - 1}$$

$$= \frac{4x_n^3 (x_n) - x_n - x_n^4 + 4x_n + 10}{4x_n^3 - 1}$$

$$= \frac{\cancel{4x_n^4} - x_n - \cancel{x_n^4} + 4x_n + 10}{4x_n^3 - 1.}$$

$$x_{n+1} = \frac{3x_n^4 + 3x_n + 10}{4x_n^3 - 1}$$

~~$x_1 = 0.28455$~~

~~$x_2 = -11.95544$~~

~~$x_3 = -0.003783$~~

~~$x_4 = -9.98864$~~

~~$x_5 =$~~

$x_1 = 2.19004$

$x_2 = 2.08656$

$x_3 = 2.06933$

$x_4 = 2.06760$

$x_5 = 2.06745$

$x_6 = 2.0644.$

The required root is 2.0644.

$xe^x - 2 = 0$

$f(x) = xe^x - 2$

$f'(x) = xe^x + e^x$

$x_0 = 0.5 \Rightarrow 0.852$

$f(x) = xe^x - 2$

$f'(x) = xe^x + e^x$

$$x_{n+1} = x_n - \frac{xe^x - 2}{xe^x + e^x}$$

$$= \frac{x_n^2 e^x + \cancel{x_n e^x} - \cancel{xe^x} + 2}{xe^x + e^x}$$

$$x_{n+1} = \frac{x_n^2 e^x + 2}{x_n e^x + e^x}$$

$$x_{0+1} = \frac{(0.5)^2 \times e^{0.5} + 2}{0.5 \times e^{0.5} + e^{0.5}} = \cancel{0.6420}\ 0.9753$$

$$x_2 = \frac{x_1^2 e^1 + 2}{x_1 e^{x_1} + e^{x_1}} = 0.8633.$$

$x_3 = 0.8526 \quad - \quad x_4 = 0.8526.$

$\underline{0.852}$

$x_0 = 1.5$

1.5106

$x^3 + 5x - 11 = 0 \qquad f(x) = x^3 + 5x - 11 \quad f'(x) = 3x^2 + 5$

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)},$$

$$= x_n - \frac{x_n^3 + 5x_n - 11}{3x_n^2 + 5}$$

$$= \frac{3x_n^3 + \cancel{5x_n} - x_n^3 - \cancel{5x_n} + 11}{3x_n^2 + 5}$$

$$x_{n+1} = \frac{2x_n^3 + 11}{3x_n^2 + 5}$$

$$x_1 = \frac{2x_0^3 + 11}{3x_0^2 + 5} = \frac{2 \times (1.5)^3 + 11}{3 \times (1.5)^2 + 5} = \underline{1.51063}$$

$x_2 = \underline{1.5106}$

2.79838

$f(x) = x\sin x + \cos x$ Near to $x_0 = \pi$

$f'(x) = 1.\sin x + x\cos x - \sin x$

$x_{n+1} = x_n - \dfrac{f(x)}{f'(x)}$

$= x_n - \dfrac{(x\sin x + \cos x)}{\cancel{\sin x} + x\cos x - \cancel{\sin x}}$

$x_{n+1} = x_n^2 \cdot \cos x_n - x_n \sin x_n - \cos x_n$

$x_{0+1} = x_0^2 \cos x_0 - x_0 \sin x_0 - \cos x_0$

$x_1 = \pi^2 \cos\pi - \pi\sin\pi - \cos\pi$

$x_1 = \cancel{-8.8693}$ 8.683533

$x_2 = x_1^2 \cos x_1 - x_1 \sin x_1 - \cos x_1$

72.23986

$f(\pi) = \pi\sin\pi + \cos\pi$

$= 0 + (-1)$

$= -1$

$f'(\pi) = \sin\pi + \pi\cos\pi - \sin\pi$

$= 0 + \pi(-1) - 0$

$= -\pi$

$x_{n+1} = \pi - \dfrac{(-1)}{-\pi}$

$= \dfrac{-\pi^2 + 1}{-\pi}$

$= 2.8231$

Date 16/04/25

Finite difference

Consider a function $y = f(x)$ and the values of y will be given at various of 'x' as given in the table.

$x: x_0\ x_1\ x_2\ x_3 \ldots\ldots x_n$

$y: y_0\ y_1\ y_2\ y_3 \ldots\ldots y_n$

for each values of x and y is called tabulated values associated with the function $y = f(x)$.

Interpolation - It is a process of determining the value of y in a tabulated value of (x) without the knowledge of relation b/w x and y within a given interval.

Extrapolation - Out of the given interval.

Types of Interpolation

Interpolation with equal interval ('x' values are equally spaced with the steplength 'h')

*1 Newton Gregory forward interpolation formula.
*2 Newton Gregory backward interpolation formula.

Interpolation with unequal intervals (where in x values are not equally spaced)

*1 Newton's divided difference formula
*2 Lagrange's and inverse lagrange's formula.

Forward difference table: $\Delta y_n = y_{n+1} - y_n$

$\Delta \rightarrow$ forward diff. operator

x	y	Δy (1st diff)	$\Delta^2 y$ (2nd diff)	$\Delta^3 y$ (3rd)	$\Delta^4 y$ (4th)	5th
x_0	y_0	$y_1 - y_0 = \Delta y_0$	$\Delta^2 y_1 - \Delta y_0 = \Delta^2 y_0$	$\Delta^2 y_1 - \Delta^2 y_0 = \Delta^3 y_0$	$\Delta y_1^3 - \Delta^3 y_0 = \Delta^4 y_0$	$\Delta y_1^4 - \Delta^4 y_0 = \Delta^5 y_0$
x_1	y_1	$y_2 - y_1 = \Delta y_1$	$\Delta y_2 - \Delta y_1 = \Delta^2 y_1$	$\Delta^2 y_2 - \Delta^2 y_1 = \Delta^3 y_1$		
x_2	y_2	$y_3 - y_2 = \Delta y_2$	$\Delta y_3 - \Delta y_2 = \Delta^2 y_2$			
x_3	y_3	$y_4 - y_3 = \Delta y_3$	$\Delta y_4 - \Delta y_3 = \Delta^2 y_3$			
x_4	y_4	$y_5 - y_4 = \Delta y_4$				
x_5	y_5	$y_6 - y_5 = \Delta y_5$				
x_6	y_6					

1st entries in each column refers to leading diagonal element.

Newton forward interpolation formula is given by

$$y = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-3)}{4!}\Delta^4 y_0 + \ldots$$

where $x = x_0 + ph \Rightarrow \boxed{p = \frac{x - x_0}{h}}$

$x \rightarrow$ pt at which y is required.
$x_0 \rightarrow$ Initial value
$h \rightarrow$ step length of x.
$p \rightarrow$ any real number.

Backward difference table

$\nabla y_{n+1} = y_{n+1} - y_n$

x	y	Δy (1st diff)	$\nabla^2 y$ (2nd diff)	$\nabla^3 y$ (3rd diff)	$\nabla^4 y$ (4th)
x_0	y_0	$y_1 - y_0 = \nabla y_1$	$\nabla y_2 - \nabla y_1 = \nabla^2 y_2$	$\nabla^2 y_3 - \nabla^2 y_2 = \nabla^3 y_3$	$\nabla^2 y_4 - \nabla^3 y_3 = \nabla^4 y$
x_1	y_1	$y_2 - y_1 = \nabla y_2$	$\nabla y_3 - \nabla y_2 = \nabla^3 y_3$	$\nabla^2 y_4 - \nabla^2 y_3 = \nabla^3 y_4$	
x_2	y_2	$y_3 - y_2 = \nabla y_3$	$\nabla y_4 - \nabla y_3 = \nabla^2 y_4$		
x_3	y_3	$y_4 - y_3 = \nabla y_4$			
x_4	y_5				

Last entries in each column refers to indiagonal elements

so NBIF,

$$y = y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^2 y_n + \dots$$

where $x = x_n + ph \Rightarrow p = \frac{x - x_n}{h}$.

The table gives a distance in ~~notical~~ nautical miles after a visible ~~origin of~~ horizon of for the given heights in feet above the earth surface

x: Height	100	150	200	250	300	350	400
y: Distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the value when $x = 218$ ft, $y = 410$ ft.

x	y	1st diff ∇y	2nd diff $\nabla^2 y$	3rd diff $\nabla^3 y$	4th diff $\nabla^4 y$	5th diff $\nabla^5 y$	$\Delta^6 y_0$ $\nabla^6 y$
100	10.63						
		2.4					
150	13.03		-0.39				
		2.01		+0.15			
200	15.04		-0.24		-0.07		
		1.77		+0.08		0.02	
250	16.81		-0.16		-0.05		0.02
		1.61		0.03		0.04	
300	18.42		-0.13		-0.01		
		1.48		0.02			
350	19.90		-0.11				
		1.37					
400	21.27						

$y_{218} = 15.7$ miles.

$y_{410} = 21.5$ miles.

NFIF;

$$y = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \dots$$

6th diff

where $x = x_0 + ph \Rightarrow p = \frac{x - x_0}{h}$;

here $x = 218$ $x_0 = 100$ $h = 50$.

$$p = \frac{218 - 100}{50} = 2.36$$

$y_0 = 10.63$ $\Delta y_0 = 2.4$ $\Delta^2 y_0 = -0.39$ $\Delta^3 y_0 = 0.15$ $\Delta^4 y_0 = -0.07$

$\Delta^5 y_0 = 0.02$ $\Delta^6 y_0 = 0.02$

$$y_{218} = 10.63 + \frac{2.36}{1} \times 2.4 + \frac{2.36(2.36-1)}{2} \times -0.39 + \frac{2.36(2.36-1)(2.36-2)}{6} \times 0.15 + \frac{2.36(2.36-1)(2.36-2)(2.36-3)}{24} \times -0.07 + \frac{2.36(2.36-1)(2.36-2)(2.36-3)(2.36-4)}{120} \times 0.02 + \frac{2.36(2.36-1)(2.36-2)(2.36-3)(2.36-4)(2.36-5)}{720} \times 0.02$$

$10.63 + 5.664 - 0.625872 + 0.0288864 + 0.002156851 - 0.00020212727696 ~~- 0.0044465109~~ - 0.00008893621862$

$= 15.698$ miles

$= 15.7$ miles

NBIF:

$$y = y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y_n + \frac{p(p+1)(p+2)(p+3)(p+4)(p+5)}{6!}\nabla^6 y_n$$

$p = \frac{x-x_n}{h}$

$x=410$ $y_n = 21.27$ $\nabla^2 y_n = -0.11$ $\nabla y_n = 1.37$ $\nabla^3 y_n = 0.02$

$x_n = 400$ $\nabla^4 y_n = -0.01$ $\nabla^5 y_n = 0.04$ $\nabla^6 y_n = 0.02$

$h = 50$

$p = 0.2$

$$y = 21.27 + \frac{0.2}{1} \times 1.37 + \frac{0.2(0.2+1)}{2} \times -0.11 + \frac{0.2(0.2+1)(0.2+2)}{6} \times 0.02 +$$

$$\frac{0.2(0.2+1)(0.2+2)(0.2+3)}{24} \times -0.01 + \frac{0.2(0.2+1)(0.2+2)(0.2+3)(0.2+4)}{120} \times 0.04$$

$$+ \frac{0.2(0.2+1)(0.2+2)(0.2+3)(0.2+4)(0.2+5)}{720} \times 0.02$$

$$y = 21.27 + 0.274 - 0.0132 + 0.00176 - 0.000704 + 0.00236544$$
$$+ 0.001025024$$

$y = 21.5$ miles

17/04

→ In a table given the value of y are consecutive type of a series of which 23.6 is 6th term. Find the first and 10th term from the given data.

x	y	1st diff	2nd di	3rd	4th
3	4.8	3.6	2.5	0.5	0
4	8.4	6.1	3	0.5	0
5	14.5	9.1	3.5	0.5	0
6	23.6	12.6	4	0.5	
7	36.2	16.6	4.5		
8	52.8	21.1			
9	73.9				

To find $x=1$; NFIF

$$y = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0.$$

$x=1$ $x_0 = 3$ $h=1$ $y_0 = 4.8$ $\Delta y_0 = 3.6$ $\Delta^2 y_0 = 2.5$

$\Delta^3 y_0 = 0.5$ $\Delta^4 y_0 = 4.$

$$p = \frac{x-x_0}{h} = \frac{1-3}{1} = -2.$$

$$= 4.8 + \frac{-2}{1} \times 3.6 + \frac{-2(-2-1)}{2} \times 2.5 + \frac{-2(-2-1)(-2-2)}{6} \times 0.5$$

$$-\frac{2(-2-1)(-2-2)(-2-3)}{24} \times 0$$

$= 4.8 - 7.2 + 7.5 - 2 + 0$

$= 3.1$

To find $x=10$, NBIF.

$$y = y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$x = 10$ $x_n = 9$ $y_n = 73.9$ $h=1$ $\nabla y_n = 21.1$ $\nabla^2 y_n = 4.5$

$\nabla^3 y_n = 0.5$ $\nabla^4 y_n = 0.$

$$p = \frac{x-x_n}{h} = \frac{10-9}{1} = 1.$$

$= 21.1 + 73.9 + 4.5 + \frac{6}{6} \times 0.5 + 1.$

$= 100$

NFIF
$p=0.4$
$y=27.885$

NBIF
$p=-0.6$
$y=84.303$

The population of town is given by Calculate the increase in population from 1955 to 1985

Year	1951	1961	1971	1981	1991
Pop in thousand	19.96	39.65	57.81	77.21	94.61

x	y	$y=f(x)$	Δy_0	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1951	19.96	19.96	19.69	-0.53	-0.23	-0.01
1961	39.65	39.65	19.16	-0.76	-0.24	
1971	58.81	58.81	18.4	-1		
1981	77.21	77.21	17.4			
1991	94.61	94.61				

Applying NFIF,

$$y=y_0+\frac{p}{1!}\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0+\frac{p(p-1)(p-2)}{3!}\Delta^3 y_0+\frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0$$

$$p=\frac{x-x_0}{h}=\frac{1955-1951}{10}=\frac{4}{10}=0.4$$

$$y=19.96+\frac{0.4\times 19.69}{1}+\frac{0.4(0.4-1)}{2!}\times -0.53+\frac{(0.4)(0.4-1)(0.4-2)}{6}\times -0.23$$

$$+\frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}\times -0.01 = -25.07072$$

$$= -25.070304$$

Applying NBIF,

$$y=y_n+\frac{p}{1!}\nabla y_n+\frac{p(p+1)}{2!}\nabla^2 y_n+\frac{p(p+1)(p+2)}{3!}\nabla^3 y_n+\frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$p=\frac{x-x_0}{h}=\frac{1985-1991}{10}=\frac{-6}{10}=-0.6$$

$$=94.61+(-0.6)\times 17.4+\frac{(-0.6)(-0.6+1)}{2}\times -1+\frac{(-0.6)(-0.6+1)(0.6+2)}{6}\times -0.24$$

$$+\frac{(-0.6)(-0.6+1)(-0.6+2)(-0.6+3)}{24}\times -0.01$$

$$=83.303$$

From the following table estimate, the no of students who obtained marks b/w i) 40 & 45 ii) below 55

Marks	30-40	40-50	50-60	60-70	70-80
No of stu	31	42	51	35	31

Let $y=f(x)$ denotes the no of students who obtained less than or equal to x marks. then using the concept of cumulative frequency table can be evaluated.

x	y	$y=f(x)$	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
40	31	31	31	42	9	-25	37
50	42	31+42=73	73	51	-16	12	
60	51	73+51=124	124	35	-4		
70	35	124+35=159	159	31			
80	31	159+31=190	190				

$$y=y_0+\frac{p}{1!}$$

$x=45$ $h=10$ $x_0=40$ $y_0=31$ $\Delta y_0=42$ $\Delta^2 y_0=9$

$\Delta^3 y_0=-25$ $\Delta^4 y_0=37$

$$=31+0.5\times 42+\frac{0.5(0.5-1)}{2}\times 9+\frac{0.5(0.5-1)(0.5-2)}{6}\times -25$$

$$+\frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}\times 37$$

$$=31+21-1.125-1.5625-1.4453125$$

$$=47.86$$

$$=48$$

Marks obtained b/w 40 & 45 is given by $48-31=17$ students.
$45-40$.

$y_5 = 100$ here NFIF, $y = y_0 + \frac{p}{1!}\Delta y + \frac{p-1}{2!}\Delta^2 y + \frac{p(p-1)(p-2)}{3!}\Delta^3 y + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y.$

$y_0 = 31 \quad x = 55 \quad x_0 = 40 \quad h = 10. \quad \Delta y_0 = 42 \quad \Delta^2 y_0 = 9 \quad \Delta^3 y_0 = -25 \quad \Delta^4 y = 37$

$p = \frac{x - x_0}{h} = \frac{55-40}{10} = \frac{15}{10} = 1.5$

$y = 31 + 1.5 \times 42 + \frac{1.5-1}{2} \times 9 + \frac{1.5(1.5-1)(1.5-2)}{6} \times -25 + \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{24} \times 37$

$y = 31 + 63 + 2.25 + 1.\cancel{041666667}8625 + 0.8671875$

98.679

Find the cubic $\boxed{\text{polynomial}}$ which takes the following values hence find y at $x = 0$. — do not substitute the value of x.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	5			
		1		
2	6		4	
		5		6
3	11		10	
		15		
4	26.			

$x^3 - 6x^2 + 11x - 6$

40
-33
7

$y = y_0 + \frac{p}{1!}\Delta y + \frac{p-1}{2!}\Delta^2 y + \frac{p(p-1)(p-2)}{3!}\Delta^3 y$

$x = 0 \quad x_0 = 1 \quad y_0 = 5 \quad \Delta y_0 = 1 \quad \Delta^2 y_0 = 4 \quad \Delta^3 y_0 = 6.$

$h = 1$

$p = \frac{x - x_0}{h} = \frac{x-1}{1} = x - 1$

$x(x-2) - 3(x-2)$

$x^2 - 2x - 3x + 6$

$(x^2 - 5x + 6) \times 5$

$5x^2 - 25x + 30$

$x^2 - 5x + 6(x-1)$

$x^3 - 5x^2 + 6x - x^2 + 5x$

$y = 5 + x - 1 \times 1 + \frac{(x-1-1)}{2} \times \cancel{4}2 + \frac{(x-1)(x-2)(x-3)}{\cancel{6}} \times \cancel{6}$

$\cancel{5 + x - 1 + (x-2)2 +}$

$= 5 + x - 1 + \frac{(x-1)(x-2)}{2} \times 4 + \frac{(x-1)(x-2)(x-3)(6)}{6.}$

$= 5 + x - 1 + 2(x-1)(x-2) + (x-1)(x-2)(x-3)$

$y = x^3 - 4x^2 + 6x + 2$

At $x = 0$

$y = 2 \Rightarrow \boxed{y_{at}\ x = 0 = 2}$

$y : 1\ 2\ 1\ 10$
$x : 0\ 1\ 2\ 3$

$x-3$ / $y!$

Find the interpolating polynomial which takes the following value hence evaluate $f(4)$. or Find the curve $y = f(x)$ which passes through the point $(0,1)\ (1,2)\ (2,1)\ \&\ (3,10)$ Evalue $f(4)$.

x	y	1st diff	2nd diff	3rd diff
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

NBIF, $y = y_n + \frac{p}{1!}\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$

$x = x \quad x_0 = 3 \quad y_n = 10 \quad \nabla y_n = 9 \quad \nabla^2 y_n = 10 \quad \nabla^3 y_n = 12.$

$h = 1.$

$p = \frac{x - x_0}{h} = \frac{x-3}{1} = x - 3.$

$y = 10 + (x-3) \times 9 + \frac{(x-3)(x-2)}{2}10 + \frac{(x-3)(x-2)(x-1)}{6} \times 12$

$= 10 + 9x - 27 + 5x^2 - 25x + 30 + x^3 - 6x^2 + 11x - 6.$

$= x^3 - 11x^2 - 5x + 7$

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0	1			
		1		
1	2		-2	
		-1		12
2	1		+10	
		9		
3	10			

NBIF,

$\cancel{y = y_0 + \nabla y \frac{P(P+1)}{1!} + P}$

$$y = y_0 + \frac{P}{1!}\nabla y + \frac{P(P+1)}{2!}\nabla^2 y + \frac{P(P+1)(P+2)}{3!}\nabla^3 y$$

$$P = \frac{x - x_0}{h} = \frac{4-3}{1} = 1$$

$$= 10 + 1 \times 9 + \frac{1(1+1)}{2} \times 10 + \frac{1(2)(3)}{6} \times 12$$

$$= 10 + 9 + 10 + 12$$

$$= \underline{\underline{41}}$$

1. Using Newton's forward formula, find the value of $f(1.6)$, if

x	$y = f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1	3.49			
		1.33		
1.4	4.82		-0.19	
		1.14		-0.41
1.8	5.96		-0.6	
		0.54		
2.2	6.5			

$$y = y_0 + \frac{P}{1!}\Delta y_0 + \frac{P(P-1)}{2!} \times \Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0$$

$= 3.49 +$ here $x = 1.6$ $x_0 = 1$ $y = 3.49$ $\Delta y_0 = 1.33$

$\Delta^2 y_0 = -0.19$ $\Delta^3 y_0 = -0.41$ $P = x - x_0 / h = 1.6 - 1 / 0.4 = 1.5$

$$y = 3.49 + 1.5 \times 1.33 + \frac{1.5((1.5)-1)}{2} \times -0.19 + \frac{1.5(1.5-1)(1.5-2)}{6} \times -0.41$$

$$= 3.49 + 1.995 - 0.07125 + 0.025625$$

$$= 5.439375$$

$$= 110 + (-0.022\ldots)$$

$$= \underline{\underline{109.977152}}$$

2. If $f(1.15) = 1.0723$, $f(1.20) = 1.0954$, $f(1.25) = 1.1180$ and $f(1.30) = 1.1401$, find 1.28.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
1.15	1.0723			
		0.0231		
1.20	1.0954		-0.0005	
		0.0226		0.
1.25	1.1180		-0.0005	
		0.0221		
1.30	1.1401			

$$y = y_n + \frac{P}{1!}\nabla y_n + \frac{P(P+1)}{2!}\nabla^2 y_n + \frac{P(P+1)(P+2)}{3!}\nabla^3 y_n$$

$$= 1.1401 + (-0.4 \times 0.0221) + \frac{(-0.4+1)}{2} \times -0.0005 + \frac{-0.4(-0.4+1)}{6} \times 0.$$

$$P = \frac{x - x_0}{h} = \frac{1.28 - 1.30}{0.05} = -0.4$$

$$y = \underline{\underline{1.13148}}$$

3. From the following table of half-yearly premium for policies

Age (x)	Premium (₹) y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
45	114.84				
		-18.68			
50	96.16		5.84		
		-12.84		-1.84	
55	83.32		4		0.68
		-8.84		-1.16	
60	74.48		2.84		
		-6			
65	68.48				

$$y = y_0 + \frac{P}{1!}\Delta y + \frac{P(P-1)}{2!}\Delta^2 y + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$$

$x = \cancel{44}\ 46$ $x_0 = 45$ $y_0 = 114.84$ $\Delta y = -18.68$ $\Delta^2 y = 5.84$

$\Delta^3 y = -1.84$ $\Delta^4 y = 0.68$ $P = x - x_0 / h = 46 - 45 / 5 = 0.2$

$$y = y_0 + \frac{P}{1!}\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0.$$

$$= 114.84 + 0.2 \times -18.68 + \frac{0.2(0.2-1)}{2} \times 5.84 + \frac{0.2(0.2-1)(0.2-2)}{6} \times -1.84$$

$$+ \frac{0.2(0.2-1)(0.2-2)(0.2-3)}{} \times 0.68.$$

→ The area A of a circle of diameter d is given for the following values: Calculate the area of a circle of diameter 105

d(x)	A(y)	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$
80	5026				
		48			
85	5074		1240		
		1288		-1802	
90	6362		-562		2404
		726		602	
95	7088		40		
		766			
100	7854				

$$\therefore y = y_n + \frac{p}{1!}(\nabla y_n) + \frac{p(p+1)}{2!}\nabla^2 y + \frac{p(p+1)(p+2)}{3!}\nabla^3 y + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y$$

$$= \cancel{7854+} \quad p = \frac{x-x_0}{h} = \frac{105-100}{5} = 1$$

$$= 7854 + 1\times 766 + \frac{1(2)}{2}\times 40 + \frac{1(2)(3)}{6}\times 602 + \frac{1(2)(3)(4)}{24}\times 2404$$

$$y = 11666$$

→ Estimate the value of f(22) & f(42) from the following available data.

x	f(x)=y	1st diff	2nd diff	3rd diff	4th diff	5th diff
20	354					
		-22				
25	332		-19			
		-41		29		
30	291		10		-37	
		-31		-8		45
35	260		2		8	
		-29		0		
40	231		2			
		-27				
45	204					

$$y_{22} = y_0 + \frac{p}{1!}(\Delta y_0) + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\times\Delta^4 y_0$$
$$+ \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\times\Delta^5 y_0.$$

$y_0 = 354$, $x = 22$, $x_0 = 20$, $\Delta y = -22$, $\Delta^2 y = -19$, $\Delta^3 y = 29$
$\Delta^4 y = -37$, $\Delta^5 y = 45$, $p = \frac{x-x_0}{h} = \frac{2}{5} = 0.4$

$$= 354 + 0.4\times -22 + \frac{0.4(0.4-1)}{2}\times -19 + \frac{0.4(0.4-1)(0.4-2)}{6}\times 29 +$$
$$\frac{0.4(0.4-1)(0.4-2)(0.4-3)}{24}\times -37 + \frac{0.4(0.4-1)(0.4-2)(0.4-3)(0.4-4)}{120}\times 45$$

$$= 347.48 + 4.74304$$
$$= 352.22304$$

$$y_{42} = y_n + \frac{p}{1!}(\nabla y_n) + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\times\nabla^4 y_n$$
$$+ \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\times\nabla^5 y_n$$

$p = \frac{42-45}{5} = -0.6$

$$= 204 + (-0.6\times -27) + \frac{-0.6(-0.6+1)}{2}\times 2 + \frac{-0.6(-0.6+1)(-0.6+2)}{6}\times 0 +$$
$$\frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{24}\times 8 + \frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)(-0.6+4)}{120}\times 45$$

$$= 219.96 - 1.29696$$
$$= 218.66304$$

From the following table:

$x°$:-	10	20	30	40	50	60	70
cos x :	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420

Calculate cos 25° & cos 73° using Gregory Newton formula.

x	cos x = y
10	0.9848
20	
30	

x	y	1st diff	2nd diff	3rd diff	4th diff	5th diff	6th diff
10	0.9848	-0.0451	-0.0286	0.0023	0.0008	0.0001	-0.001
20	0.9397	-0.0737	-0.0263	0.0031	0.0009	-0.0009	
30	0.8660	-0.1	-0.0232	0.004	0		
40	0.7660	-0.1232	-0.0192	0.004			
50	0.6428	-0.1428	-0.0152				
60	0.5000	-0.158					
70	0.3420						

Calculate cos 25° & cos 73° using

To find cos 25°:

Applying NFIF:-

$$y = y_0 + \frac{p}{1!}\Delta y_0 + \frac{p(p-1)}{2!}\Delta y_0^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y$$

$$+ \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 y + \frac{p(p-1)(p-2)(p-3)(p-4)(p-5)}{6!}\Delta^6 y$$

$\Rightarrow 0.9848$ $\quad p = \frac{x-x_0}{h} = \frac{25-10}{10} = \frac{15}{10} = 1.5$

$$y = 0.9848 + \frac{1.5}{1}(-0.0451) + \frac{1.5(0.5)}{2}\times(-0.0286) + \frac{1.5(1.5-1)(1.5-2)}{6}\times 0.0023$$

$$+ \frac{1.5(1.5-1)(1.5-2)(1.5-3)}{24}\times 0.0008 + \frac{1.5(1.5-1)(1.5-2)(1.5-3)(1.5-4)}{120}\times 0.0001$$

$$+ \frac{1.5(1.5-1)(1.5-2)(1.5-3)(1.5-4)(1.5-5)}{720}\times -0.001$$

$y = 0.9848 - 0.06765 - 0.010725 - 0.00014375 - 0.00001875$
$- 0.0000171875 - 0.00000683593375$

$y = 0.9062539453$

B NBIF:

$$y_0 = y_n + \frac{p}{1!}\nabla y + \frac{p(p+1)}{2!}\nabla y^2 + \frac{p(p+1)(p+2)}{6}\times\nabla^3 y + \frac{p(p+1)(p+2)(p+3)}{24}\times\nabla^4 y$$

$$+ \frac{p(p+1)(p+2)(p+3)(p+4)}{5!}\nabla^5 y + \frac{p(p+1)(p+2)(p+3)(p+4)(p+5)}{6!}\times\nabla^6 y$$

$$p = \frac{x-x_0}{h} = \frac{73-70}{10} = \frac{3}{10} = 0.3.$$

$$y = 0.3420 + 0.3\times -0.158 + \frac{0.3(0.3+1)}{2}\times -0.0152 + \frac{0.3(0.3+1)(2.3)}{6}\times 0.004$$

$$+0 + \frac{0.3(1.3)(2.3)(3.3)(4.3)}{120}\times -0.0009 +$$

$$\frac{0.3(1.3)(2.3)(3.3)(4.3)(5.3)}{720}\times -0.001$$

$y = 0.3420 - 0.0474 + 0.2185 - 0.002964 + 0.000598 - 0.0000954$
$- 0.000093695 3875$ $\quad 63225$

$y = 0.2920498414$

Interpolation with unequal intervals: (Not necessarily of equal width)

Newton divided difference formula or Newton General interpolation formula:

Let $y_0, y_1, y_2 \dots y_n$ be the values of $y=f(x)$ corresponding to the values of $x_0, x_1, x_2 \dots x_n$ then Newton divided diff. formula is given by

$$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + \dots \text{so on.}$$

Divided diff. table is given by

x	y	1^{st} D.D	2^{nd} D.D	3^{rd} D.D	4^{th} D.D	5^{th} DD
x_0	y_0	$\frac{y_1-y_0}{x_1-x_0}=\Delta y_0$	$\frac{\Delta y_1-\Delta y_0}{x_2-x_0}=\Delta^2 y_0$	$\frac{\Delta^2 y_1-\Delta^2 y_0}{x_3-x_0}=\Delta^3 y_0$	$\frac{\Delta y_1^3-\Delta^3 y_0}{x_4-x_0}=\Delta^4 y_0$	$\frac{\Delta^4 y_1-\Delta^4 y_0}{x_5-x_0}=\Delta^5 y_0$
x_1	y_1	$\frac{y_2-y_1}{x_2-x_1}=\Delta y_1$	$\frac{\Delta y_2-\Delta y_1}{x_3-x_1}=\Delta^2 y_1$	$\frac{\Delta^2 y_2-\Delta^2 y_1}{x_4-x_1}=\Delta^3 y_1$	$\frac{\Delta^3 y_2-\Delta^3 y_1}{x_5-x_1}=\Delta^4 y_1$	
x_2	y_2	$\frac{y_3-y_2}{x_3-x_2}=\Delta y_2$	$\frac{\Delta y_3-\Delta y_2}{x_4-x_2}=\Delta^2 y_2$	$\frac{\Delta^2 y_3-\Delta^2 y_2}{x_5-x_2}=\Delta^3 y_2$		
x_3	y_3	$\frac{y_4-y_3}{x_4-x_3}=\Delta y_3$	$\frac{\Delta y_4-\Delta y_3}{x_5-x_3}=\Delta^2 y_3$			
x_4	y_4	$\frac{y_5-y_4}{x_5-x_4}=\Delta y_4$				
x_5	y_5					

Q → Evaluate $f(9)$ using NDDF.

x	$y=f(x)$	1^{st} DD.	2^{nd} DD.	3^{rd} DD	4^{th} DD.
5	150	121	$\frac{265-121}{11-5}=24$	$\frac{32-24}{13-5}=1$	0.
7	392	$\frac{1452-392}{11-7}=265$	$\frac{457-265}{13-7}=32$	$\frac{42-32}{17-7}=1$	
11	1452	$\frac{2366-1452}{13-11}=457$	$\frac{709-457}{17-11}=42$		
13	2366	$\frac{5202-2366}{17-13}=709$			
17	5202				

NDDF.

$x_0=5$, $y_0=150$, $\Delta y_0=121$, $\Delta^2 y_0=24$, $\Delta^3 y_0=1$, $\Delta^4 y_0=0$.

$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 y_0.$

$= 150 + (9-5)\,121 + (9-5)(9-7)\,24 + (9-5)(9-7)(9-11)\times 1 + 0$

$= \underline{\underline{810}}$

→ Given $u_{20}=24.37$, $u_{22}=49.28$, $u_{29}=162.86$, $u_{32}=240.5$

Find $u_{28}=?$

x	y	Δy	$\Delta^2 y_0$	$\Delta^3 y$
20	24.37	12.455	0.418967	0.0455
22	49.28	16.225	0.9643	
29	162.86	25.88		
32	240.5			

$y = ?$ $x=28$ $x_0=20$ $\Delta y = 12.455$ $\Delta^2 y = 0.418967$

$\Delta^3 y = 0.0455$. $y_0 = 24.37$.

$y = 24.37 + (28-20)\times 12.455 + (28-20)(28-22)\times 0.418967 + (28-20)($

$= 141.938$

→ Determine $f(x)$ as a polynomial in x.

x	$f(x)$	1^{st} diff	2^{nd} diff	3^{rd} diff	4^{th} diff
-4 x_0	1245	-1212	1184	-1152	2442.
-1 x_1	33	-28	32	1290	
0 x_2	5	4	1322		
2 x_3	9	1326			
5 x_4	1335				

$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 y_0.$

$= 1245 + (x+4)\,{-1212} + (x+4)(x+1)\,1184 + (x+4)(x+1)(x-0)\,{-1152} + (x+4)(x+1)(x-0)(x-2)\,2442.$

$= 1245 - 1212x - 4848 + 1184x^2 + 5920x + 4736 - 1152x^3 - 5760x^2 - 4608x$

NDDF ⇒ Give x find y

INDDF ⇒ Given y, find x.

$f(4)=448$, $f(15)=3150$ find $f(8)$ & $f(15)$;

x	y	1st diff	2nd diff	3rd diff	4th diff
4	48	52	142	270	-2048
5	100	194	412	-1778	
7	294	606	-1366	4014	
10	900	-760	2648		
11	1210	1888			
13	2028				

x	y	1st diff	2nd diff	3rd diff	4th diff	5th diff
$4\ x_0$	$48\ y_0$	52	15	~~2.1667~~ 4.1667	0.20047	~~-0.15144~~
$5\ x_1$	$100\ y_1$	97	21	~~0.49888~~		
$7\ x_2$	$294\ y_2$	202	27			
$10\ x_3$	$900\ y_3$	310	33			
$11\ x_4$	$1210\ y_4$	409				
$13\ x_5$	$2028\ y_5$					

$1\ (8-4)(8-5)(8-7)(8-10)(8-11) \times -0.15144$

$y = y_0 + (x-x_0)\Delta y_0 + (x-x_0)(x-x_1)\Delta^2 y_0 + (x-x_0)(x-x_1)(x-x_2)\Delta^3 y_0 + (x-x_0)(x-x_1)(x-x_2)(x-x_3)\Delta^4 y_0 + (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\Delta^5 y_0$

$=448$

$= 48 + (8-4)52 + (8-4)(8-5)15 + (8-4)(8-5)(8-7)\,2.1667 + (8-4)(8-5)(8-7)(8-10)\,0.90477 +$

Inverse interpolation by using inverse interpolation formula.

→ The process of estimating the value x for given y is called Inverse Interpolation

NOTE – 1. We can interchange the role of x & y in the formula & the procedure will be same as that of Newton's divided diff

$x = x_0 + (y-y_0)\Delta x_0 + (y-y_0)(y-y_1)\Delta^2 x_0 + (y-y_0)(y-y_1)(y_2-y_2)\Delta^3 x_0 + \dots$

→ Apply NDF inversely to obtain the root of eqn $f(x)=0$, where $f(30)=-30$, $f(34)=-13$, $f(38)=3$, $f(42)=18$.

y	x	Δx	$\Delta^2 x$	$\Delta^3 x$
-30	30	0.23529	0.000448	0.0000018
-13	34	0.25	0.000529	
3	38	0.2667		
18	42			

x	y
30	-30
34	-13
38	3
42	18

~~$x = 30 + ($~~

$x = 37.228$.

→ Apply NDD inversely to obtain value of x when $y=6$.

y	x	Δx	$\Delta^2 x$
$2\ y_0$	20	10	0
$4.4\ y_1$	30	10	
$7.9\ y_2$	40		

$x = x_0 + (y-y_0)\Delta x + (y-y_0)(y-y_1)\Delta^2 x_0$.

$= 20 + (6-2)10 + (6-2)(6-4.4)\,0$

$= 20 + 24$

36.3118

y	x	Δx_0	$\Delta^2 x$
$2\ y_0$	$20\ x_0$	4.1667	-0.22196
$4.4\ y_1$	$30\ x_1$	2.8571	
$7.9\ y_2$	$40\ x_2$		

$\frac{x_1-x_0}{y_1-y_0}$

$\frac{30-20}{4.4-2}$

$x = x_0 + (y-y_0)\Delta x_0 + (y-y_0)(y-y_1)\Delta^2 x_0$

$= 20 + (6-2)4.1667 + (6-2)(6-4.4) -0.22196$

$= 35.24625$

Inverse Lagrange's method

Let $y_0, y_1, y_2, \dots, y_n$ be the values of $y=f(x)$ corresponding to the values of $x_0, x_1, x_2, \dots, x_n$ then: inverse lagrange's formula is given by

$x:$	x_0	x_1	x_2	x_3	$\dots$	x_n	Given y
Given the set of values : $y:$	y_0	y_1	y_2	y_3	$\dots$	y_n	find x.

$y=f(x) \Rightarrow$ Given x, find $y \Rightarrow$ Langrange's

Unequal
- NDD (INDD)
- LM. (IL).

$$x=f(y) = \frac{(y-y_1)(y-y_2)(y-y_3)\dots(y-y_n)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)\dots(y_0-y_n)}x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)\dots(y-y_n)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)\dots(y_1-y_n)}x_1 + \frac{(y-y_0)(y-y_1)(y-y_3)\dots(y-y_n)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)\dots(y_2-y_n)}x_2$$

→ Apply inverse lagrange's formula given $f(x)=0$ where $f(30)=-30$, $f(34)=-13$, $f(38)=3$, $f(42)=18$

Here given y find x.

x	y
30 x_0	-30 y_0
34 x_1	-13 y_1
38 x_2	3 y_2
42 x_3	18 y_3

$f(x)=0 \quad y=0.$

$$x=f(y) = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}\cdot x_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1 +$$

3706 $$\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2 + \frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}\times x_3.$$

$$= \cancel{\frac{(0+13)(0-3)(-18)}{(-30+13)(30-3)(30-18)}\,30} + \cancel{\frac{(0+30)(0-3)(0-18)}{(-13+30)(-13-3)(-13-18)}\times 34} + \cancel{\frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}\times 38} + \cancel{\frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}\times 42}$$

$$= \frac{(0+13)(0-3)(0-18)}{(-30+13)(-30-3)(-30-18)}\times 30 + \frac{(0+30)(0-3)(0-18)}{(-13+30)(-13-3)(-13-18)}\times 34 + \frac{(0+30)(0+13)(0-18)}{(3+30)(3+13)(3-18)}\times 38 + \frac{(0+30)(0+13)(0-3)}{(18+30)(18+13)(18-3)}\times 42.$$

$= 5.75017 + 31.4802 = 37.2303.$

→ The following table gives the value of x & y, find the value of x when $y=12$ using inverse lagrange's formula.

3.382

x	y
1.2 x_0	4.2 y_0
2.1 x_1	6.8 y_1
2.8 x_2	9.8 y_2
4.1 x_3	13.4 y_3
4.9 x_4	15.5 y_4
6.2 x_5	19.6 y_5

$$x=f(y) = \frac{(y-y_1)(y-y_2)(y-y_3)(y-y_4)(y-y_5)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)(y_0-y_4)(y_0-y_5)}\times 1.2 + \frac{(y-y_0)(y-y_2)(y-y_3)(y-y_4)(y-y_5)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)(y_1-y_4)(y_1-y_5)}\times 2.1$$

$$+ \frac{(y-y_0)(y-y_1)(y-y_3)(y-y_4)(y-y_5)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)(y_2-y_4)(y_2-5)}\times 2.8 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_4)(y-y_5)}{(y_3-y_0)(y_3-y_2)(y_3-y_1)(y_3-y_4)(y_3-y_5)}\times 4.1$$

$$\frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)(y-y_5)}{(y_4-y_0)(y_4-y_1)(y_4-y_2)(y_4-y_3)(y_4-y_5)}\times 4.9 + \frac{(y-y_0)(y-y_1)(y-y_2)(y-y_3)(y-y_4)}{(y_5-y_0)(y_5-y_1)(y_5-y_2)(y_5-y_3)(y_5-y_4)}\times 6.2$$

$$= \frac{(12-6.8)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(4.2-6.8)(4.2-13.4)(4.2-9.8)(4.2-15.5)(4.2-19.6)}\times 1.2 + \frac{(12-4.2)(12-9.8)(12-13.4)(12-15.5)(12-19.6)}{(6.8-4.2)(6.8-9.8)(6.8-13.4)(6.8-15.5)(6.8-19.6)}\times 4.1$$

$= 2.782$

Central diff. Interpolation formula.

Newton FIF & NBIF which are applicable for interpolation near to the beginning and the end of the tabulated values. Now the consider the central diff. formula which are usefull to interpolate the middle value.

1. Gauss forward interpolation formula.

The diff table is given by;

x_{-3}	y_{-3}	Δy_{-3}	$\Delta^2 y_{-3}$				
x_{-2}	y_{-2}	Δy_{-2}	$\Delta^2 y_{-2}$	$\Delta^3 y_{-3}$			
x_{-1}	y_{-1}				$\Delta^4 y_{-3}$	$\Delta^5 y_{-3}$	
x_0	y_0	Δy_{-1}, Δy_0	$\Delta^2 y_{-1}$	$\Delta^3 y_{-2}$, $\Delta^3 y_{-1}$	$\Delta^4 y_{-2}$	$\Delta^5 y_{-2}$	$\Delta^6 y_{-3}$
x_1	y_1		$\Delta^2 y_0$		$\Delta^4 y_{-1}$		
x_2	y_2	Δy_1		$\Delta^3 y_0$			
x_3	y_3	Δy_2	$\Delta^2 y_1$				

$$y = y_0 + P\Delta y_0 + \frac{(P-1)P}{2!}\Delta^2 y_{-1} + \frac{(P-1)P(P+1)}{3!}\Delta^3 y_{-1} + \frac{(P-2)(P-1)P(P+1)}{4!}\Delta^4 y_{-2}$$
$$+ \frac{(P-2)(P-1)(P)(P+1)(P+2)}{5!}\Delta^5 y_{-2} + \frac{(P-3)(P-2)(P-1)P(P+1)(P+2)}{6!}\Delta^6 y_{3}$$

GBIF;

$$y = y_0 + P\Delta y_{-1} + \frac{P(P+1)}{2!}\Delta^2 y_{-1} + \frac{(P-1)P(P+1)}{3!}\Delta^3 y_{-2} + \frac{(P-1)P(P+1)(P+2)}{4!}\Delta^4 y_{-2}$$
$$+ \frac{(P-2)(P-1)P(P+1)(P+2)}{5!}\Delta^5 y_{3} + \frac{(P-2)(P-1)P(P+1)(P+2)(P+3)}{6!}\Delta^6 y_{-3}$$

Stirling's formula

It is a avg (mean) of Gauss forward interpolation formula & Gauss forward intepolation formula.

$$y = y_0 + \frac{P}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{P^2}{2!}\Delta^2 y_{-1} + \frac{P(P^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+ \frac{P^2(P^2-1)}{4!}\Delta^4_{-2} + \cdots$$

Interpolate by using Gauss FIF, Find the value of $e^{1.17}$

x	$y = e^x$						
x_{-3} 1.00	2.7183 y_{-3}	0.1394 Δy_{-3}	0.0071 $\Delta^2 y_{-3}$				
x_{-2} 1.05	2.8577 y_{-2}	0.146 Δy_{-2}	0.0075 $\Delta^2 y_{-2}$	0.0004 $\Delta^3 y_{-3}$	0		
x_{-1} 1.10	3.0042 y_{-1}	0.154 Δy_{-1}	0.0079 $\Delta^2 y_{-1}$	0.0004 $\Delta^3 y_{-2}$		0	
x_0 1.15	3.1582 y_0	0.1619 Δy_0	0.0083 $\Delta^2 y_1$	0.0004 $\Delta^3 y_{-1}$	0	0.0001	0.0001
x_1 1.20	3.3201 y_1	0.1702 Δy_1	0.0088 $\Delta^2 y_2$	0.0005 $\Delta^3 y_0$	0.0001		
x_2 1.25	3.4903 y_2	0.179 Δy_2					
x_3 1.30	3.6693 y_3						

$$y = y_0 + \frac{P\Delta y_0}{1!} + \frac{(P-1)P}{2!}\Delta^2 y_{-1} + \frac{(P-1)P(P+1)}{3!}\Delta^3 y_{-1}$$

$$P = \frac{x - x_0}{h} = \frac{1.17 - 1.15}{0.05} = 0.4$$

$$= 3.1582 + \frac{0.4 \times 0.1619}{1} + \frac{(0.4-1)\times 0.4 \times 0.0079}{2} + \frac{(0.4-1)\times 0.4 \times (0.4+1)\times 0.0004}{6}$$

$$\boxed{y_{at\,1.17} = 3.2219}$$

27 to 39.
32.34

For the given data interpolate using ~~NBIF~~ GBIF, 1974.

Year	Pop in thou					
1939	12	3	2	0	3	
1949	15	5	2	3	-7	-10
1959	20	7	5	-4		
1969 x_0	27 y_0	12	1			
1979	39	13				
1989	52					

GBIF,

$$y = y_0 + P\Delta y_{-1} + \frac{P(P+1)}{2!}\Delta^2 y_{-1} + \frac{(P-1)P(P+1)}{3!}\Delta^3 y_{-2} + \frac{(P-1)P(P+1)(P+2)}{4!}\Delta^4 y_{-2}$$

$P = \frac{x - x_0}{h} = \frac{1974-1969}{10} = \frac{5}{10} = 0.5$

$$y = 27 + 0.5\times 7 + \frac{0.5(1.5)}{2}\times 5 + \frac{(-0.5)(0.5)(1.5)}{6}\times 3 + \frac{(-0.5)(0.5)(1.5)(2.5)}{24}\times -7$$

$\boxed{y = 32.4609}$

→ Find $f(3.74)$ GFIF & GBIF.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
2.5	24.145	-2.102	0.284	-0.047	-0.091	-0.003
3.0	22.043	-1.818	0.237	-0.038	0.006	
3.5	20.225	-1.581	0.199	-0.032		
4.0	18.644	-1.382	0.167			
4.5	17.262	-1.215				
5.0	16.047					

GFIF 12.7696

GBIF 12.7696

$$y = y_0 + \frac{P}{1!}\Delta y_0 + \frac{(P-1)P}{2!}\Delta^2 y_{-1} + \frac{(P-1)P(P+1)}{3!}\Delta^3 y_{-1} + \frac{(P-2)(P-1)P(P+1)}{4!}\Delta^4 y$$

$P = \frac{x - x_0}{h} = \frac{3.75 - 3.5}{5} = 0.5$

$$y = 20.225 + 0.5\times -1.581 + \frac{(0.5-1)(0.5)}{2}\times 0.237 + \frac{(0.5-1)(0.5)(0.5+1)}{6}\times -0.038 + \frac{(0.5-2)(0.5-1)(0.5)(0.5+1)}{24}\times -0.091 = \cancel{19.40722}$$

$+ (0.5-2)(0.5-1)(0.5)(0.5+1)(0.5+2)/120\times -0.003.$

$\underline{19.40725}$

$$y = y_0 + P\Delta y_{-1} + \frac{P(P+1)}{2}\times 0.237 + \frac{(P-1)P(P+1)}{6}\times\Delta^3 y_{-2} + \frac{(P-1)P(P+1)(P+2)}{24}\times\Delta^4 y_{-2}$$

$$= 20.225 + (0.5\times -1.818) + 0.5(1.5)\times 0.237 + \frac{(-0.5)0.5(1.5)}{6}\times -0.047 + \frac{(-0.5)(0.5)(1.5)(2.5)}{24}\times -0.091 = \underline{19.411}$$

Find $f(22)$ from Gauss forward and Gauss backward.

GFIF = 347.48

GBIF = 345.2

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
20	354	-22	-19	29	-37	45
25	332	-41	10	-8	8	
30	291	-31	2	0		
35	260	-29	2			
40	231	-27				
45	204					

$P = \frac{22-25}{5} = -0.6$

GFIF, $y = y_0 + \frac{P}{1!}\Delta y_0 + \frac{(P-1)P}{2!}\Delta^2 y_{-1} = 332 + (-0.6)\times -41 + \frac{(-0.6)(-0.6-1)}{2}\times -19$

$y = 347.48$

GBIF,

$$y = y_0 + P\Delta y_{-1} + \frac{P(P+1)}{2!}\times\Delta^2 y_{-1} = 332 + (-0.6\times -22) + \frac{(-0.6)(-0.6+1)}{2}\times -19$$

$\underline{347.48}$

$$y = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2}$$

The following table gives the value of e^x when $x = 0.644$ using stirling's formula.

x	$y = e^x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.61	y_{-3} 1.8404				
		0.0185 Δy_{-3}			
0.62	y_{-2} 1.8589		0.0002 $\Delta^2 y_{-3}$		
		0.0187 Δy_{-2}		0 $\Delta^3 y_{-3}$	
0.63	y_{-1} 1.8776		0.0002 $\Delta^2 y_{-2}$		-0.0001 $\Delta^4 y_{-3}$
		0.0189 Δy_{-1}		-0.0001 $\Delta^3 y_{-2}$	
0.64	y_0 1.8965		0.0001 $\Delta^2 y_{-1}$		0.0003 $\Delta^4 y_{-2}$
		0.0190 Δy_0		0.0002 $\Delta^3 y_{-1}$	
0.65	y_1 1.9155		0.0003 $\Delta^2 y_0$		-0.0004 $\Delta^4 y_{-1}$
		0.0193 Δy_1		-0.0002 $\Delta^3 y_0$	
0.66	y_2 1.9348		0.0001 $\Delta^2 y_1$		
		0.0194 Δy_2			
0.67	y_3 1.9542				

$$P = \frac{x - x_0}{h} = \frac{0.644 - 0.64}{0.01} = 0.4$$

$$y = 1.8965 + 0.4\left(\frac{0.0189 + 0.0190}{2}\right) + \frac{0.4^2}{2} \times 0.0001 + \frac{0.4(0.4^2-1)}{6} \times \left(\frac{+0.0002 - 0.0001}{2}\right) + \frac{0.4^2(0.4^2-1)}{24} \times 0.0003$$

~~1.90408527~~ $= 1.90408352$

→ Given the values $y_{12.2}$ in terms of $y_x = 1 + \log_{10}(\sin x)$

x	$y = 10^{-5} y_x$
10	23967
11	28060
12	31788
13	35209
14	38368

x	y	$y = 10^{-5} y_x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	23967	0.23967 y_{-2}				
			0.04093 Δy_{-2}			
11	28060	0.28060 y_{-1}		-0.00365 $\Delta^2 y_{-2}$		
			0.03728 Δy_{-1}		0.00058 $\Delta^3 y_{-2}$	
12 x_0	31788	0.31788 y_0		-0.00307 $\Delta^2 y_{-1}$		-0.00013 $\Delta^4 y_{-2}$
			0.03421 Δy_0		-0.00045 $\Delta^3 y_{-1}$	
13	35209	0.35209 y_1		-0.00062 $\Delta^2 y_0$		
			0.03158 Δy_1			
14	38368	0.38368 y_2				

0.32497

$$P = \frac{x - x_0}{h} = \frac{12.2 - 12}{1} = 0.2$$

~~$y = y_0 + P\Delta y_0 + (P-1)$~~

$$y = y_0 + \frac{p}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2}$$

$$= 0.3178 + 0.2\left(\frac{0.03421 + 0.03728}{2}\right) + \frac{(0.2)^2}{2} \times 0.00307 + \frac{0.2(0.2^2-1)}{6} \times \left(\frac{0.00058 - 0.00045}{2}\right) + \frac{(0.2)^2(0.2^2-1)}{24} \times -0.00013$$

$$= 0.32500832 + 0.000000208$$

$y = 0.32500$

$y = 10^{-5} y_x$

$y_x = 10^5 \times 0.32500$

$\boxed{y_x = 32500}$

Bino expⁿ $= 1+t+t^2+t^3+\dots$

$(1-t)^{1} = 1-t+t^2-t^3+$

0.302322

0.286706

Using stirling formula estimate the value of tan 16°.

θ°	tanθ					
0	0	0.0875				
5	0.0875	0.0888	0.0013	0.0059		
10	0.1763	0.096	0.0072	-0.0062	-0.0121	-0.8414
15	0.2679		0.001		-0.8535	
20	0.3640	0.0961	-0.8587	-0.8597	1.7272	2.5807
25	0.4663	0.1023		0.8675		
30	0.5774	0.1111	0.0088			

$$y = y_0 + \frac{p}{1!}\frac{(\Delta y_0+\Delta y_{-1})}{2} + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\frac{(\Delta^3 y_{-1}+\Delta^3 y_{-2})}{2} + \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2}$$

$$P = \frac{x-x_0}{h} = \frac{16-15}{5} = 0.2$$

$$= 0.2679 + 0.2\frac{(0.0961+0.096)}{2} + \frac{0.2^2}{2}\times 0.001 + \frac{0.2(0.2^2-1)}{6}\frac{(-0.8597-0.0062)}{2}$$

$$+ \frac{0.2^2(0.2^2-1)}{24}\times -0.8535 = 0.286706.$$

Atleast 3 decimal accuracy.

Ramanujan's Method

Srinivasa Ramanujan described an iterative method which can be used to determine the smallest real root of the equation $f(x) = 0$ where $f(x) = 1-[a_1x+a_2x^2+a_3x^3+\dots]$

For smaller values of x rewrite $f(x)$ as as

$$[1-(a_1x+a_2x^2+a_3x^3+\dots)]^{-1} = b_1+b_2x+b_3x^2+\dots$$

and the LHS can be rewritten by using the binomial expⁿ in the form of

$$b_1+b_2x+b_3x^2+\dots = 1+(a_1x+a_2x^2+a_3x^3+\dots)+(a_1x+a_2x^2+a_3x^3\dots)^2+\dots$$

$$b_1+b_2x+b_3x^2+\dots = 1+a_1x+a_2x^2+a_3x^3+\dots+(a_1^2x^2+a_2^2x^4+a_3^2x^6+\dots+2a_1a_2x^3+2a_2a_3x^5+2a_1a_3x^4\dots)$$

x^0; $b_1 = 1$

x^1; $b_2 = a_1 = a_1\cdot 1 = a_1b_1$

x^2; $b_3 = a_2+a_1^2 = a_2\cdot 1 + a_1a_1 = a_2b_1+a_1b_2 = a_1b_2+a_2b_1$

x^3; $b_4 = a_1b_3+a_2b_2+a_3b_1$

x^4; $b_5 = a_1b_4+a_2b_3+a_3b_2+a_4b_1$

... So on.

Ratio $\frac{b_{n-1}}{b_n}$ is called as convergent values which approaches the root called smallest root of the equation.

Find the real root of $xe^x = 1$.

$\Rightarrow f(x) = 1-xe^x$

$$e^x = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots$$

$$f(x) = 1-x[1+x+x^2/2!+x^3/3!+\dots]$$

$$= 1-x-x^2-\frac{x^3}{2!}+\frac{x^4}{3!}+\frac{x^5}{4!}+\dots$$

For smaller values of x write $f(x)$ in terms of

$$\left[1-\left(x+x^2+\frac{x^3}{2!}+\frac{x^4}{3!}+\frac{x^5}{4!}+\dots\right)\right]^{-1} = b_1+b_2x+b_3x^2+b_4x^3+\dots$$

Comparing with;

$$[1-(a_1x+a_2x^2+a_3x^3+a_4x^4+\dots)]^{-1} = b_1+b_2x+b_3x^2+b_4x^3+\dots$$

Here $a_1 = 1$, $a_2 = 1$, $a_3 = \frac{1}{2}$, $a_4 = \frac{1}{6}$, $a_5 = \frac{1}{24}$.

Now, $b_1 = 1$, $b_2 = a_1b_1 = 1$, $b_3 = a_1b_2+a_2b_1 = 1+1 = 2$.

$$b_4 = a_1b_3+a_2b_2+a_3b_1 = 2+1+\tfrac{1}{2} = \frac{4+2+1}{2} = \frac{7}{2}$$

$$b_5 = a_1b_4+a_2b_3+a_3b_2+a_4b_1 = \frac{7}{2}+2+\frac{1}{2}+\frac{1}{6} = \frac{21+12+3+1}{6} = \frac{37}{6}.$$

$$b_6 = a_1b_5+a_2b_4+a_3b_3+a_4b_2+a_5b_1 = \frac{37}{6}+\frac{7}{2}+1+\frac{1}{6}+\cancel{3x}\frac{1}{24} = \frac{261}{24}.$$

$\frac{7}{2}$ 3.5

$\frac{b_1}{b_2}$ $\frac{b_2}{b_3}$ $\frac{b_3}{b_4}$ $\frac{b_4}{b_5}$ $\frac{b_5}{b_6}$

$= 1$ $= \frac{1}{2}$ $= 0.5714$ 0.5675 0.5670

$\therefore$ The smallest root is $\underline{\underline{0.567}}$

→ $\sin x = 1 - x$

$f(x) = 1 - x - \sin x$

$= 1 - (x + \sin x)$

$$\boxed{\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots}$$

$$f(x) = 1 - \left(x + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)$$

0.5108

$$= 1 - 2x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

for smaller values of x express $\frac{1}{f(x)}$ in terms of

$$\left[1 - \left(2x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}\right)\right]^{-1} = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \dots$$

Comparing with,

$$[1 - (a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots)]^{-1} = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \dots$$

$a_1 = 2$ $a_2 = 0$ $a_3 = \frac{-1}{6}$ $a_4 = 0$ $a_5 = +\frac{1}{120}$ $a_6 = 0$ $\boxed{a_7 = 5040}$

we have, $b_1 = 1$ $b_2 = a_1 b_1 = 2 \times 1 = 2$

$b_3 = a_2 b_1 + b_2 a_1$ $b_4 = b_3 a_1 + b_2 a_2 + a_3 b_1$

$= 0 + 2 \times 2 = 4$

1.444

$$1 - x + \frac{x^2}{(2!)^2} - \frac{x^3}{(3!)^2} + \frac{x^4}{(4!)^2}$$

$$1 - \left(x - \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} - \frac{x^4}{(4!)^2}\right)$$

$$= \left[1 - \left(x - \frac{x^2}{4} + \frac{x^3}{36} - \frac{x^4}{576}\right)\right]^{-1} = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \dots$$

Comparing with,

$$[1 - (a_1 x) + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots)]^{-1} = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \dots$$

$a_1 = 1$ $a_2 = -1/4$ $a_3 = 1/36$ $a_4 = -1/576$

we have, $b_1 = 1$ $b_2 = a_1 b_1 = 1 \times 1 = 1$ $b_3 = a_2 b_1 + a_1 b_2$

$= -\frac{1}{4} + 1 \times 1$

$= \frac{3}{4}$

$b_4 = b_3 a_1 + b_2 a_2 + b_1 a_3$

$= \frac{3}{4} \times 1 + \left(\frac{-1}{4}\right) \times 1 + 1 \times \frac{1}{36}$

$= \frac{19}{36}$

$b_5 = b_4 a_1 + b_3 a_2 + b_2 a_3 + b_1 a_4$

$= \frac{19}{36} \times 1 + \frac{3}{4} \times \frac{-1}{4} + 1 \times \frac{1}{36} + 1 \times \frac{-1}{576}$

$= \frac{211}{576}$.

$\frac{b_1}{b_2} = \frac{1}{1} = 1$ $\frac{b_2}{b_3} = \frac{4}{3} = 1.333$ $\frac{b_3}{b_4} = \frac{3/4}{19/36} = \frac{1}{912}$

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$x^3 - 6x^2 + 11x - 1$

$\Rightarrow 1 + 11x - 6x^2 + x^3$.

$f(x) = 1 - (11x + 6x^2 - x^3)$

$\cancel{b_1 = 1}$

$[1 - (11x + 6x^2 - x^3)]^{-1} = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \ldots$

Comparing with

$[1 - (a_1 x + a_2 x^2 + a_3 x^3 + \ldots)]^{-1} = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \ldots$

$a_1 = 11 \quad a_2 = 6 \quad a_3 = -1 \qquad b_1 = 1 \quad b_2 = a_1 b_1 = 11 \times 1 = 11 \quad b_3 = a_2 b_1 + b_2 a_1 = 127$

$b_4 = b_3 a_1 + b_2 a_2 + b_1 a_3$

$= 127 \times 11 + 11 \times 6 + 1 \times -1$

$= 1462$.

$\frac{b_1}{b_2} \quad \frac{b_2}{b_3} \quad \frac{b_3}{b_4} = 0.08686$.

$= 0.09090 \quad 0.086614$

$\therefore$ the smallest root $= \underline{\underline{0.086}}$