16 May

Unit - 2 - Vector Calculus

A vector is a physical quantity which has both magnitude and direction.

Eg:- velocity

Scalar: A physical quantity which has only magnitude but no direction.

Eg:- Distance

Derivative of a Vector Function in terms of its components

$$\vec{y} = x\hat{i} + y\hat{j} + z\hat{k}$$

where x, y, z are components / scalars

$$\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\frac{dv}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

Note: ① $\frac{d(\vec{u}\cdot\vec{v})}{dt} = \vec{u}\cdot\frac{d\vec{v}}{dt} + \vec{v}\cdot\frac{d\vec{u}}{dt}$

② $\frac{d(\bar{u}\times\bar{v})}{dt} = \bar{u}\times\frac{d\bar{v}}{dt} + \bar{v}\times\frac{d\bar{u}}{dt}$

Velocity and Acceleration

Let $\bar{r} = f(t)$ be the vector function where vector r is a position vector at any time t. Then velocity is $\vec{v} = \frac{d\vec{r}}{dt}$, $\vec{a} = \frac{d\vec{v}}{dt}$

Scalar Point Function

A function ϕ which associates with each point $p(x, y, z)$ of a region R. A unique scalar is called a scalar point function. The totality of values of scalar function ϕ in the region R is called a scalar field R

$$\phi = x^2 + y^2 + z^2, \quad \phi = xy^2z^3 \text{ etc}$$

Vector Point Function

A function F which associates with each point $p(x, y, z)$ of a region R of a unique vector is a vector point function. The totality of values of vector function F in the region R is called a vector field F.

Eg:- $\vec{A} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$

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Vector Differential Operator

The operator ∇ (nabla) is a vector operator.

$$\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \nabla \text{ (nabla)}$$

Gradient

∇ applied to a scalar point function.

The Gradient of a scalar point function $\phi(x,y,z)$ is defined as

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

The gradient of a scalar point function is vector point function. It is also written as grad (ϕ)

Geometrical Meaning of Gradient

If $\phi(x,y,z)$ is a scalar function then grad (ϕ) is a vector normal to the surface $\phi(x,y,z) = C$

Unit Normal

If $\hat{n}$ = the unit vector in the direction of grad (ϕ), the vector $\hat{n}$ is called unit normal to the surface S at the point (x,y,z)

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

1) If $\phi = 3x^2 + 2y - 3z$, find grad (ϕ) and $|\text{grad}\,\phi|$ at $(1,1,1)$

Sol: Given: $\phi = 3x^2 + 2y - 3z$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = 6x\hat{i} + 2\hat{j} - 3\hat{k}$$

$$\nabla\phi\Big|_{(1,1,1)} = 6\hat{i} + 2\hat{j} - 3\hat{k}$$

$$|\nabla\phi| = \sqrt{36+4+9} = \sqrt{49} = 7$$

$$\boxed{\nabla\phi = 6\hat{i} + 2\hat{j} - 3\hat{k}}$$

$$\boxed{|\nabla\phi| = 7}$$

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② $\phi = xy^3z^2 - 4$, find grad(ϕ) and |grad(ϕ)| at (-1,-1,2). Find unit normal vector

Ans Given: $xy^3z^2 - 4 = \phi$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2zy^3x\hat{k}$$

At (-1,-1,2): $\nabla\phi = -4\hat{i} - 12\hat{j} - 4\hat{k}$

$$|\nabla\phi| = \sqrt{16+144+16} = \sqrt{65+16} = \sqrt{81}$$

$$\downarrow$$

$$\sqrt{122+16} = \sqrt{138}$$

$$|\nabla\phi| = \sqrt{16+144+16} = \sqrt{32+144} = \sqrt{176} = \boxed{4\sqrt{11}}$$

$$\hat{n} = \frac{-1}{4\sqrt{11}}(4\hat{i}+12\hat{j}+4\hat{k})$$

③ $\phi = x^2y + 2xz - 4$ at (2,-2,3)

$\nabla\phi$, $|\nabla\phi|$, $\hat{n}$

Ans $\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$$\nabla\phi = (2xy+2z)\hat{i} + x^2\hat{j} + 2x\hat{k}$$

At (2,-2,3): $\nabla\phi = (2(2)(-2) + 2(3)\hat{i}) + (2)^2\hat{j} + 2(2)\hat{k}$

$$\nabla\phi = (-8+6)\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\nabla\phi = -2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$|\nabla\phi| = \sqrt{4+16+16} = \sqrt{36} = 6$$

$$|\nabla\phi| = 6$$

$$\hat{n} = -\frac{1}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$$

④ If $\phi = \frac{1}{r}$, show that $\nabla\phi = -\frac{\vec{r}}{r^3}$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

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Ans $\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\frac{1}{r}$

$|\vec{r}| = \sqrt{x^2+y^2+z^2} = r$

$\frac{\partial r}{\partial x} = \frac{1}{2\sqrt{x^2+y^2+z^2}}\,2x = \frac{x}{r}$ $\quad \frac{\partial r}{\partial y} = \frac{y}{r}$ $\quad \frac{\partial r}{\partial z} = \frac{z}{r}$

$$\nabla\phi = -\frac{\hat{i}}{r^2}\frac{\partial r}{\partial x} - \frac{\hat{j}}{r^2}\frac{\partial r}{\partial y} - \frac{\hat{k}}{r^2}\frac{\partial r}{\partial z}$$

$$\nabla\phi = \frac{-\hat{i}}{r^2}\frac{x}{r} - \frac{\hat{j}}{r^2}\frac{y}{r} - \frac{\hat{k}}{r^2}\frac{z}{r}$$

$$= \frac{-1}{r^2}\frac{1}{r}\left(x\hat{i} + y\hat{j} + z\hat{k}\right)$$

$$\boxed{\nabla\phi = -\frac{\vec{r}}{r^3}}$$

② Prove $\nabla r^n = n r^{n-1}\hat{r}$ $\qquad r = x\hat{i} + y\hat{j} + z\hat{k}$

Directional Derivatives

The directional directive of scalar point func. ϕ in the direction of unit vector $\hat{a}$ is given by

$\hat{a}\cdot\nabla\phi$ or $\nabla\phi\cdot\hat{n}$ $\qquad \hat{a} = \frac{\vec{a}}{|\vec{a}|}$

① Find the directional derivative of

$\phi(x,y,z) = x^2 - 2y^2 + 4z^2$ at $(1,1,-1)$

$\vec{a} = 2\hat{i} + \hat{j} + \hat{k}$

Ans $\hat{a} = \frac{2\hat{i}+\hat{j}+\hat{k}}{\sqrt{4+1+1}} = \frac{1}{\sqrt{6}}\,2\hat{i}+\hat{j}+\hat{k}$

$$\nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

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$$\nabla\phi = \hat{i}(2x) + \hat{j}(-4y) + \hat{k}(8z)$$

$$\nabla\phi\Big|_{(1,1,-1)} = 2\hat{i} - 4\hat{j} - 8\hat{k}$$

Directional Derivative of ϕ in direction of given vector $\Rightarrow$ $\nabla\phi\cdot\hat{a} = (2\hat{i} - 4\hat{j} - 8\hat{k})\cdot\dfrac{(2\hat{i}+\hat{j}+\hat{k})}{\sqrt{6}}$

$$\nabla\phi\cdot\hat{a} = \frac{\cancel{4}-\cancel{4}-8}{\sqrt{6}} = \frac{-8}{\sqrt{6}}$$

$$\nabla\phi\cdot\hat{a} = \frac{-8}{\sqrt{6}}$$

) $\phi = 2x^2 - 3xy + 9z - 2$ at $(1,0,0)$ $\quad \vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$

$|\vec{a}| = \sqrt{4+4+1} = \sqrt{9} = 3$

$\hat{a} = \frac{1}{3}(2\hat{i} - 2\hat{j} + \hat{k})$

$$\nabla\phi = \hat{i}(4x - 3y) + \hat{j}(-3\underset{y}{\cancel{y}}x) + \hat{k}(9)$$

$$= (4x - 3y)\hat{i} - 3x\hat{j} + 9\hat{k}$$

$$\nabla\phi\Big|_{(1,0,0)} = 4\hat{i} + 9\hat{k} - 3\hat{j}$$

$$\text{D.D} = \nabla\phi\cdot\hat{a} = (4\hat{i} + 9\hat{k} - 3\hat{j})\cdot\frac{(2\hat{i} - 2\hat{j} + \hat{k})}{3}$$

$$= \frac{8 + \overset{6+9}{\cancel{18}}}{3} = +\frac{10}{3} + 13$$

$$\nabla\phi\cdot\hat{a} = +\frac{10}{3} + 13$$

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③ $\phi = 3x^2y^{2-1} - y^3z^2$ at $(1,-2,-1)$

Ans $\nabla\phi = \hat{i}\,6xy + \hat{j}\,(3x^2 - 3y^2z^2) + \hat{k}\,(-2y^3z)$

$$\nabla\phi\Big|_{(1,-2,-1)} = \hat{i}\,(6(1)(-2)) + \hat{j}\,(3(1) - 3(4)(1)) + \hat{k}\,(-2(-8)(-1))$$

$$= -12\hat{i} + \hat{j}\,(3-12) + \hat{k}\,(-16)$$

$$= -12\hat{i} - 9\hat{j} - 16\hat{k}$$

$$|\nabla\phi| = \sqrt{144+81+256}$$

$$= \sqrt{225+256}$$

$$|\nabla\phi| = \sqrt{481}$$

④ A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = t+5$. Find components of velocity and acceleration at $t=1$ in direction $\hat{i}+3\hat{j}+2\hat{k} = \bar{b}$

Ans Let $\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$= (t^3+1)\hat{i} + t^2\hat{j} + (t+5)\hat{k}$$

$$\bar{v} = \frac{d\hat{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + \hat{k}$$

$$\frac{d\hat{r}}{dt}\Big|_{t=1} = 3\hat{i} + 2\hat{j} + \hat{k}$$

Vector $\bar{v}$ in direction of $\hat{i}+3\hat{j}+2\hat{k}$ is

$$\bar{v}\cdot\hat{b} = \bar{v}\cdot\frac{\bar{b}}{|\bar{b}|} = (3\hat{i}+2\hat{j}+\hat{k})\cdot\frac{(\hat{i}+3\hat{j}+2\hat{k})}{\sqrt{14}}$$

$$\bar{v}\cdot\hat{b} = \frac{3+6+2}{\sqrt{14}} = \frac{11}{\sqrt{14}}$$

$$\bar{v}\cdot\hat{b} = \frac{11}{\sqrt{14}}$$

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$$\bar{v} = \frac{d\bar{r}}{dt} = 3t^2\hat{i} + 2t\hat{j} + \hat{k}$$

$$\bar{a} = \frac{d^2\bar{r}}{dt^2} = 6t\,\hat{i} + 2\hat{j}$$

$$\left.\frac{d^2\bar{r}}{dt^2}\right|_{t=1} = 6\hat{i} + 2\hat{j} = \bar{a}$$

$$\bar{a}\cdot\hat{b} = (6\hat{i} + 2\hat{j})\cdot\frac{(\hat{i} + 3\hat{j} + 2\hat{k})}{\sqrt{14}}$$

$$= \frac{6+6}{\sqrt{14}}$$

$$\bar{a}\cdot\hat{b} = \frac{12}{\sqrt{14}}$$

5) A particle moves along curve with equations $x = t - \frac{t^3}{3}$, $y = t^2$, $z = t + \frac{t^3}{3}$ where t is time, find velocity and acceleration at any time t and also find magnitude at $t=3$

Ans $\frac{dr}{dt} = \frac{d}{dt}\left(\left(t - \frac{t^3}{3}\right)\hat{i} + t^2\hat{j} + \left(t + \frac{t^3}{3}\right)\hat{k}\right)$

$$\bar{v} = \left(1 - \frac{3t^2}{3}\right)\hat{i} + 2t\hat{j} + \left(1 + \frac{3t^2}{3}\right)\hat{k}$$

$$\bar{v} = (1-t^2)\hat{i} + 2t\hat{j} + (1+t^2)\hat{k}$$

$$\bar{v}\big|_{t=3} = -8\hat{i} + 6\hat{j} + 10\hat{k}$$

$$|\bar{v}| = \sqrt{64+36+100} = \sqrt{200} = 10\sqrt{2}$$

$$|\bar{v}| = 10\sqrt{2}$$

$$\frac{d^2r}{dt^2} = -2t\hat{i} + 2\hat{j} + 2t\hat{k} = \bar{a}$$

$$|\bar{a}| = 2\sqrt{19}$$

$$\bar{a}\big|_{t=3} = -6\hat{i} + 2\hat{j} + 6\hat{k}$$

$$|\bar{a}| = \sqrt{36+36+4} = \sqrt{76} = 2\sqrt{19}$$

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Divergence

Let $f(x, y, z)$ be a differentiable vector point function. Divergence of $\vec{f} = \nabla \cdot \vec{f}$ or $\text{div}(\vec{f})$

$$\nabla \cdot \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{f}$$

$$\nabla \cdot \vec{f} = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z} = \Sigma \hat{i} \cdot \frac{\partial f}{\partial x}$$

If vector $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$

$$\nabla \cdot \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(f_1\hat{i} + f_2\hat{j} + f_3\hat{k}\right)$$

$$\nabla \cdot \vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Divergence of a Vector point Function is always a scalar function.

Solenoidal Vector

A vector $\vec{f}$ is solenoidal if divergence $(\nabla \cdot \vec{f}) = 0$

$\text{div}(\vec{f}) = 0$

Curl

A vector $\vec{f}$ function, then curl of vector point function is denoted as $\nabla \times \vec{f}$ or $(\text{curl}(\vec{f}))$ and defined as:

$$\nabla \times \vec{f} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \vec{f}$$

$$= \hat{i} \times \frac{\partial f}{\partial x} + \hat{j} \times \frac{\partial f}{\partial y} + \hat{k} \times \frac{\partial f}{\partial z} = \Sigma \hat{i} \times \frac{\partial f}{\partial x}$$

If $\vec{f} = f_1\hat{i} + f_2\hat{j} + f_3\hat{k}$, then

$$\nabla \times \vec{f} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

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Irrotational Vector

A vector $\vec{f}$ is said to be irrotational if

$$\nabla \times \vec{f} = 0$$

Laplacian

$$\nabla^2 = \nabla \cdot \nabla = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

If $\phi(x,y,z)$ is a scalar point function then

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

① Find $\nabla\cdot\vec{f}$ if vector $\vec{f} = x^2z\hat{i} - 2y^3z^2\hat{j} + xy^2z\hat{k}$

Ans $\nabla\cdot\vec{f} = \frac{\partial(x^2z)}{\partial x} + \frac{\partial(-2y^3z^2)}{\partial y} + \frac{\partial(xy^2z)}{\partial z}$

$$\nabla\cdot\vec{f} = 2xz + (-6y^2z^2) + xy^2$$

② Find $\nabla\cdot\vec{f}$ if $\vec{f} = xy^2\hat{i} + 2x^2yz\hat{j} - 3yz^2\hat{k}$ at $(1,-1,1)$

Ans $\nabla\cdot\vec{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$= y^2 + 2x^2z - 6yz$$

$$\nabla\cdot\vec{f}\Big|_{(1,-1,1)} = \underline{\underline{9}}$$

③ Show that $\vec{F} = 3y^4z^2\hat{i} + 4x^3z^2\hat{j} - 3x^2y^2\hat{k}$ is solenoidal

Ans $\nabla F = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$

$$= 0 + 0 + 0 = \underline{\underline{0}}$$

$\therefore$ Solenoidal vector

④ If $\vec{F} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$, find curl F

Ans $\nabla\vec{F}$ (curl F) $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & -2xz & 2yz \end{vmatrix}$

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$= \hat{i}\,(2z+2x) - \hat{j}\,(0-0) + \hat{k}\,(-2z - x^2)$

$= (2z+2x)\hat{i} - (2z+x^2)\hat{k}$

) Show that $\vec{F} = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$ irrotational

$$\nabla\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x-y \end{vmatrix}$$

$$= \hat{i}\,(\underset{+1}{-1}) - \hat{j}[(1)-(1)] + \hat{k}\,(\cos y - \cos y) = 0$$

) If $\vec{F} = \nabla(xy^3z^2)$. Find $\nabla\vec{F}$ and $\nabla\times\vec{F}$ at $(1,-1,1)$

$\nabla\phi = \hat{i}\,y^3z^2 + \hat{j}\,3xy^2z^2 + \hat{k}\,2xy^3z$

$F = \nabla\phi = (y^3z^2)\hat{i} + (3xy^2z^2)\hat{j} + (2xy^3z)\hat{k}$

$\nabla\vec{F} = 0\hat{i} + 6xyz^2\hat{j} + 2xy^3\hat{k}$

$\nabla\vec{F}\Big|_{(1,-1,1)} = -6 + (-2) = -8$

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^3z^2 & 3xy^2z^2 & 2xy^3z \end{vmatrix} = 0$$

$\nabla\times\vec{F} = 0$ $\quad$ $\nabla\vec{F} = -8$

) Find value of a such that vector $\vec{F} = (axy - z^3)\hat{i} + (a-2)x^2\hat{j} + (1-a)xz^2\hat{k}$ is irrotational

$$\nabla\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy - z^3 & (a-2)x^2 & (1-a)xz^2 \end{vmatrix} = \hat{i}\,(0-0) - \hat{j}\,((1-a)z^2 - 3z^2) + \hat{k}\,((2a-4)x - ax)$$

$= -\hat{j}\,(z^2a - 4az^2) + \hat{k}\,(ax - 4x)$ $\quad$ $a = 4$

$= -z^2\hat{j}\,(a - 4a) + x\hat{k}\,(a-4)$

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⑧ Find constants a and b such that

$\vec{F} = (axy + z^3)\hat{i} + (3x^2 - z)\hat{j} + (bxz^2 - y)\hat{k}$

is irrotational.

Ans

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ axy + z^3 & 3x^2 - z & bxz^2 - y \end{vmatrix}$$

$$= \hat{i}(-1-(-1)) - \hat{j}(bz^2 - 3z^2) + \hat{k}(6x - ax)$$

For $a = 6$ $b = 3$ $\nabla \times \vec{F} = 0$

$a = 6$ $b = 3$

⑨ Find div(F) and curl(F) if $\vec{F} = \nabla(x^3 + y^3 + z^3 - 3xyz)$ at $(1,0,0)$

Ans $\text{div}F = \nabla(\nabla\phi)$

Let $\phi = x^3 + y^3 + z^3 - 3xyz$

$\nabla\phi = (3x^2 - 3yz)\hat{i} + (3y^2 - 3xz)\hat{j} + (3z^2 - 3xy)\hat{k}$

$\nabla\phi|_{(1,0,0)} = \cancel{3} - \cancel{3} - 3$

$\nabla\phi|_{(1,0,0)} = -3$

~~$\text{div}F = \nabla(3x^2 + 3y^2 + 3z^2 - 3xy - 3yz - 3xz)$~~

~~$= \nabla$~~

$\hat{i}(-3\cancel{y}^{x}x + 3x) - \hat{j}(-3y + 3y)$

$+ \hat{k}(-3z + 3z)$

$\text{div}F = 6x + 6y + 6z$

$\text{div}F|_{(1,0,0)} = 6$

$$\text{curl}F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2 - 3yz & 3y^2 - 3xz & 3z^2 - 3xy \end{vmatrix} = \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$\text{Curl}F = [0, 0, 0]$

⑩ If $\vec{f} = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x - y)\hat{k}$

show that $\nabla \times \vec{F} = 0$ and find a function ϕ such that $\vec{F} = \nabla\phi$

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Ans $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$

$= (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$

$\frac{\partial\phi}{\partial x} = \sin y + z \qquad \frac{\partial\phi}{\partial y} = x\cos y - z \qquad \frac{\partial\phi}{\partial z} = x - y$

Integrating w.r.t x, y and z

$\phi = (\sin y + z)x + f_1(y,z)$ —— ①

$\phi = (x\sin y) - (z)y + f_2(x,z)$ —— ②

$\phi = (x-y)z + f_3(x,y)$ —— ③

From ①, ② and ③,

$f_1(y,z) = -zy$

$f_2(x,z) = xz$

$f_3(x,y) = x\sin y$

$$\boxed{\phi = x\sin y + xz - zy}$$

(11) Show that $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$ is irrotational and find ϕ such that $\vec{F} = \nabla\phi$

Ans $\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$

$\frac{\partial\phi}{\partial x} = x^2 - yz \qquad \frac{\partial\phi}{\partial y} = y^2 - zx \qquad \frac{\partial\phi}{\partial z} = z^2 - xy$

$\phi = \frac{x^3}{3} - xyz + f_1(y,z)$ —— ①

$\phi = \frac{y^3}{3} - xyz + f_2(x,z)$ —— ②

$\phi = \frac{z^3}{3} - xyz + f_3(x,y)$ —— ③

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - xy \end{vmatrix}$$

$= \hat{i}(-x + x) - \hat{j}(-y + y) + \hat{k}(-x + x)$

$= 0$

From ①, ② and ③, there are no functions satisfying $f_1(y,z)$, $f_2(x,z)$ and $f_3(x,y)$

$\phi = \left(\frac{x^3}{3} - xyz\right) + \left(\frac{y^3}{3} - xyz\right) + \left(\frac{z^3}{3} - xyz\right)$

$$\boxed{\phi = \frac{x^3 + y^3 + z^3}{3} - 3xyz}$$

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12) $\phi = x^2y + y^2z + z^2x$, find $\nabla^2\phi$ (Laplacian)

Ans $\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$

$= \frac{\partial(2xy + z^2)}{\partial x} + 2z + 2x$

$= 2y + 2z + 2x$

$= 2(x+y+z)$

13)

Angle between two surfaces

$$\cos\theta = \frac{\nabla\phi_1 * \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

It is defined to be angle equal to between their normals.
Let $\phi_1 = c_1$ and $\phi_2(x,y,z) = c_2$ be equation of two surfaces
(x,y,z)

$$\boxed{\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}}$$

If $\theta = \frac{\pi}{2}$, $\cos\frac{\pi}{2} = 0$, $\nabla\phi_1 \cdot \nabla\phi_2 = 0$ (Condition for surfaces to intersect at right angle)

If $\theta = 0$, $\nabla\phi_1 \cdot \nabla\phi_2 = |\nabla\phi_1||\nabla\phi_2|$

① Find the angle between the surfaces
$x^2+y^2+z^2 = 9$ and x^2+y^2-3 at $(2,-1,2)$

Ans Let $\phi_1 = x^2+y^2+z^2-9$

$\nabla\phi_1 = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\nabla\phi_1|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} + 4\hat{k} \neq 6$

$|\nabla\phi_1| = \sqrt{16+4+16} = \sqrt{12+24} = 2\sqrt{9} = 6$

$\phi_2 = x^2+y^2-z$

$\nabla\phi_2 = 2x + 2y - 1$

$\nabla\phi_2|_{(2,-1,2)} = 4\hat{i} - 2\hat{j} - \hat{k} \neq 2 - 1 \neq 1$

$|\nabla\phi_2| = \sqrt{16+4+1} = \sqrt{21} = \sqrt{21}$

$\cos\theta = \frac{16+4-4}{6\sqrt{21}}$

$\cos\theta = \frac{8}{3\sqrt{21}}$

$$\boxed{\theta = \cos^{-1}\frac{8}{3\sqrt{21}}}$$

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② Find angle b/w normals to the surface at $xy = z^2$ $(4,1,2)$ and $(3,3,-3)$

Given surface $\Rightarrow xy - z^2 = 0$

Ans

$$\phi_1 = xy - z^2$$

$$\nabla\phi_1 = y\hat{i} + x\hat{j} - 2z\hat{k}$$

$$\nabla\phi_1|_{(4,1,2)} = \cancel{1+4-4=1} \quad \hat{i} + 4\hat{j} - 4\hat{k}$$

$$|\nabla\phi_1| = \sqrt{16+16+1} = \sqrt{33}$$

$$\phi_2 = xy - z^2$$

$$\nabla\phi_2 = y\hat{i} + x\hat{j} - 2z\hat{k}$$

$$\nabla\phi_2|_{(3,3,-3)} = \cancel{3+3+6 \neq 12} \quad 3\hat{i} + 3\hat{j} + 6\hat{k}$$

$$|\nabla\phi_2| = \sqrt{9+9+36} = \sqrt{9+45} = \sqrt{54}$$

$$\cos\theta = \frac{12 \times 1}{\sqrt{33}\ \sqrt{54}}$$

$$\cos\theta = \frac{(\hat{i}+4\hat{j}-4\hat{k})\cdot(3\hat{i}+3\hat{j}+6\hat{k})}{\sqrt{33}\ \sqrt{54} \times 1}$$

$$\theta = \cos^{-1}\frac{12}{\sqrt{33}\sqrt{54}}$$

$$\theta = \cos^{-1}\left(\frac{3+12-24}{\sqrt{33}\sqrt{54}}\right) = \cos^{-1}\left(\frac{-9}{3\sqrt{33}\sqrt{6}}\right)$$

$$\boxed{\theta = \cos^{-1}\left(\frac{-3}{\sqrt{198}}\right)}$$

---

③ $\phi_1 = 4x^2y + z^4 - 12$ $\quad (1,-1,2)$

$\phi_2 = 6x^2 - yz - 9x$

Ans

$$\nabla\phi_1 = 8xy\hat{i} + 4x^2\hat{j} + 4z^3\hat{k}$$

$$\nabla\phi_1|_{(1,-1,2)} = -8\hat{i} + 4\hat{j} + 32\hat{k}$$

$$|\nabla\phi_1| = \sqrt{64+16+1024} = \sqrt{80+1024} = \sqrt{1124} = 2\sqrt{281}$$

$$\nabla\phi_2 = (12x-9)\hat{i} + (-z)\hat{j} + (-y)\hat{k}$$

$$\nabla\phi_2|_{(1,-1,2)} = 3\hat{i} - 2\hat{j} + \hat{k} \qquad |\nabla\phi_2| = \sqrt{9+4+1} = \sqrt{14}$$

$$\cos\theta = \frac{(-8\hat{i}+4\hat{j}+32\hat{k})\cdot(3\hat{i}-2\hat{j}+\hat{k})}{2\sqrt{281}\ \sqrt{14}} = \frac{-24-8+32}{2\sqrt{281} * \sqrt{14}}$$

$$\cos\theta = \frac{-32+32}{2\sqrt{14}\ \sqrt{281}} = 0 \qquad \theta = \cos^{-1}(0)$$

$$\boxed{\theta = \frac{\pi}{2}}$$

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④ Find the constants a and b such that
$\phi = Ax^2y + z^3$ and $5x^2 - Byz - 9x$
intersect orthogonally at $(1,-1,2)$

Ans $\nabla\phi_1 = 2Axy\,\hat{i} + Ax^2\,\hat{j} + 3z^2\,\hat{k}$

$\nabla\phi_1|_{(1,-1,2)} = -2A\hat{i} + A\hat{j} + 12\hat{k}$

$|\nabla\phi_1| = \sqrt{4A^2 + A^2 + 144} = \sqrt{5A^2+144}$

$\nabla\phi_2 = (10x-9)\hat{i} + (-Bz)\hat{j} + (-By)\hat{k}$

$\nabla\phi_2|_{(1,-1,2)} = \hat{i} - 2B\hat{j} + B\hat{k}$

$|\nabla\phi_2| = \sqrt{1+4B^2+B^2} = \sqrt{5B^2+1}$

~~cos~~ $\theta = \frac{\pi}{2}$ $\quad \boxed{\nabla\phi_1 \cdot \nabla\phi_2 = 0}$

$(-2A\hat{i} + A\hat{j} + 12\hat{k}) \cdot (\hat{i} - 2B\hat{j} + B\hat{k}) = 0$

$-2A - 2AB + 12B = 0$

$2A + 2AB = 12B$

$A + AB = 6B$

$AB - 6B = -A$

$B = \frac{-A}{A-6}$

ϕ_1 and ϕ_2 intersect orthogonally at $(1,-1,2)$
Equate ϕ_1 or ϕ_2 to $=0$ at $(1,-1,2)$

$5x^2 - Byz - 9x = 0$

$5(1) - B(-2) - 9 = 0$

$2B - 4 = 0$

$\boxed{B = 2}$

$-A = 2A - 12$

$-3A = -12$

$\boxed{A = 4}$

⑤ Find the directional derivative of xyz along direction of normal to surface $xy^2 + yz^2 + zx^2 = 3$ at $(1,1,1)$

Ans $\nabla\phi = (yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k})$

$\nabla\phi|_{(1,1,1)} = \hat{i} + \hat{j} + \hat{k}$

$\psi = xy^2 + yz^2 + zx^2 - 3$

$\nabla\psi = (y^2 + 2zx)\hat{i} + (2xy + z^2)\hat{j} + (2yz + x^2)\hat{k}$

$\nabla\psi|_{(1,1,1)} = 3\hat{i} + 3\hat{j} + 3\hat{k} = \vec{a}$

$\nabla\phi \cdot \hat{a} = dd \Rightarrow \frac{\cancel{3}(\hat{i}+\hat{j}+\hat{k}) \cdot (\hat{i}+\hat{j}+\hat{k})}{\cancel{3}\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$

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⑥ Find curl (curl $\vec{A}$) if $\vec{A} = xy\hat{i} + y^2z\hat{j} + z^2y\hat{k}$

Ans: $$\nabla \times \vec{A} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & y^2z & z^2y \end{vmatrix}$$

$$= (z^2 - y^2)\hat{i} + (-x)\hat{k}$$

$$= (z^2 - y^2)\hat{i} - x\hat{k}$$

$$\nabla \times (\nabla \times \vec{A}) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - y^2 & 0 & -x \end{vmatrix}$$

$$= 0\hat{i} - (-1 - 2z)\hat{j} + (0 + 2y)\hat{k}$$

$$\boxed{\text{curl}(\text{curl } A) = (1 + 2z)\hat{j} + 2y\hat{k}.}$$

Vector Identities

① $\nabla(\nabla\phi) = \nabla^2\phi$

② $\nabla \cdot (\phi A) = \phi\, \nabla \cdot \vec{A} + \nabla\phi \cdot \vec{A}$

③ $\nabla \times (\phi A) = \phi(\nabla \times \vec{A}) + \nabla\phi \times \vec{A}$

④ $\text{curl}(\nabla\phi) = 0$

$\nabla \cdot (\nabla \times \vec{A}) = 0$

(or)

$\text{div}(\text{curl } \vec{A}) = 0$

⑤ $\nabla \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{B})$

⑥ $\nabla \times (\nabla \times \vec{F}) = \nabla(\nabla \cdot \vec{F}) - \nabla^2\vec{F}$

⑦ $\nabla \times (\vec{A} \times \vec{B}) = (\nabla \cdot \vec{B})\vec{A} - (\nabla \cdot \vec{A})\vec{B} + (\vec{B} \cdot \nabla)\vec{A} - (\vec{A} \cdot \nabla)\vec{B}$

⑧ $\nabla \cdot (\vec{A} \cdot \vec{B}) = \vec{A} \times (\nabla \times \vec{B}) + \vec{B} \times (\nabla \times \vec{A}) + (\vec{A} \cdot \nabla)\vec{B} + (\vec{B} \cdot \nabla)\vec{A}$

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① If $\vec{A} = 2x^2\hat{i} - 3yz\hat{j} + xz^2\hat{k}$

$\phi = 2z - x^3y$

Compute $\vec{A}\cdot\nabla\phi$, $\vec{A}\times\nabla\phi$ at $(1,-1,1)$

Ans $\nabla\phi = \left(\frac{\partial}{\partial x} i + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$

$\nabla\phi = -3x^2y\hat{i} - x^3\hat{j} + 2\hat{k}$

At $(1,-1,1)$: $\nabla\phi = -3(1)(-1)\hat{i} - 1\hat{j} + 2\hat{k}$

$\nabla\phi = 3\hat{i} - \hat{j} + 2\hat{k}$

(i) $\vec{A}\cdot\nabla\phi$

$\vec{A}$ at $(1,-1,1)$: $\vec{A} = 2\hat{i} + 3\hat{j} + \hat{k}$

$\vec{A}\cdot\nabla\phi = (2\hat{i} + 3\hat{j} + \hat{k})\cdot(3\hat{i} - \hat{j} + 2\hat{k})$

$\vec{A}\cdot\nabla\phi = 6 - 3 + 2$

$\vec{A}\cdot\nabla\phi = 5$

(ii) $\vec{A}\times\nabla\phi$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 1 \\ 3 & -1 & 2 \end{vmatrix} = 7\hat{i} - \hat{j} - 11\hat{k}$$

$\vec{A}\times\nabla\phi = 7\hat{i} - \hat{j} - 11\hat{k}$

② $\phi = 2x^3y^2z^4$, Find div (grad ϕ) and verify that div(grad ϕ) $= \nabla^2\phi$

Ans $\nabla\phi = 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$

$\nabla(\nabla\phi) = 12xy^2z^4 + \cancel{12x^2yz^4} + \cancel{24x^2y^2z^3}$

$+ \cancel{12x^2yz^4} + 4x^3\overset{(1)}{y}z^4 + \cancel{16x^3yz^3}$

$+ \cancel{24x^2y^2z^3} + \cancel{16x^3yz^3} + 24x^3y^2z^2$

$= \cancel{24x^2yz^4} + \cancel{48x^2y^2z^3} + \cancel{32x^3yz^3} + (4x^3z^4 + 24x^3y^2z^2$

$+ 12xy^2z^4)$ ①

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div (grad ϕ)

$$\text{div}(\nabla\phi) = 12xy^2z^4 + 12x^3yz^3 + 4$$

$$\nabla\phi = 6x^2y^2z^4\hat{i} + 4x^3yz^4\hat{j} + 8x^3y^2z^3\hat{k}$$

$$\text{div}(\nabla\phi) = 12xy^2z^4 + 4x^3z^4 + 24x^3y^2z^2 \quad (2)$$

$\therefore$ (1) and (2) are equal

(3) $\vec{A} = x^2y\hat{i} - 2xz\hat{j} + 2yz\hat{k}$, verify

$$\nabla\times(\nabla\times\vec{A}) = \nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A}$$

Ans

$$\nabla\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2xz & 2yz \end{vmatrix}$$

$$\nabla\times\vec{A} = (2z+2x)\hat{i} - (0-0)\hat{j} + (-2z-x^2)\hat{k}$$

$$\nabla\times(\nabla\times\vec{A}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z+2x & 0 & -(2z+x^2) \end{vmatrix}$$

$$= 0-(-2x+(-2))\hat{j} + (0-0)\hat{k}$$

$$= [2x+2]\hat{j} \quad (1)$$

$$\nabla\cdot\vec{A} = 2xy - 0 + 2y = 2xy + 2y$$

$$\nabla(\nabla\cdot\vec{A}) = 2y\hat{i} + 2\hat{k}$$

$$\text{div}(\nabla A) = [2y+2]$$

Hence Proved

(2)

$$\nabla(\nabla\cdot\vec{A}) = 2y\hat{i} + (2x+2)\hat{j}$$

$$\nabla^2\vec{A} = 2y\hat{i}$$

$$\nabla(\nabla\cdot\vec{A}) - \nabla^2\vec{A} = (2x+2)\hat{j}$$

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(4) $\phi = xyz \quad \vec{A} = x^2yz\,\hat{i} + y^2zx\,\hat{j} + z^2xy\,\hat{k}$

$\nabla\cdot(\phi\vec{A}) = \phi(\nabla\cdot\vec{A}) + \nabla\phi\cdot\vec{A}$

Ans Given: $\phi = xyz \quad \vec{A} = x^2yz\,\hat{i} + y^2zx\,\hat{j} + z^2xy\,\hat{k}$

$\nabla\cdot\vec{A} = (2xyz + y^2z + z^2y) + (x^2z + 2xyz + z^2x)$
$+ (x^2y + y^2x + 2xyz)$

$\nabla\cdot\vec{A} = \cancel{6xyz}\ (2xyz + y^2z + z^2y) + (x^2z + 2xyz + z^2x)$
$+ (x^2y + y^2x + 2xyz)$

$\phi(\nabla\cdot\vec{A}) = \underline{2x^2y^2z^2} + xy^3z^2 + z^3y^2x + x^3yz^2 + \underline{2x^2y^2z^2} + z^3yx^3$
$+ x^3y^2z + y^3zx^2 + \underline{2x^2y^2z^2}$

$\phi(\nabla\cdot\vec{A}) = 6x^2y^2z^2$

$\nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(xyz)$

$\nabla\phi = yz\hat{i} + xz\hat{j} + xy\hat{k}$

$\nabla\phi\cdot\vec{A} = (yz\hat{i} + xz\hat{j} + xy\hat{k})\cdot(x^2yz\,\hat{i} + y^2zx\,\hat{j} + z^2xy\,\hat{k})$
$= x^2y^2z^2 + x^2y^2z^2 + x^2y^2z^2$

$\nabla\phi\cdot\vec{A} = 3x^2y^2z^2$

$\nabla\cdot\vec{A} = 2xyz + 2xyz + 2xyz = 6xyz$

$\phi(\nabla\cdot\vec{A}) = 6x^2y^2z^2$

$\phi(\nabla\cdot\vec{A}) + \nabla\phi\cdot\vec{A} = 9x^2y^2z^2 \longrightarrow$ R.H.S (2)

$\nabla(\phi\vec{A}) = \nabla(x^3y^2z^2\,\hat{i} + x^2y^3z^2\,\hat{j} + x^2y^2z^3\,\hat{k})$

$\nabla(\phi\vec{A}) = 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2$
$= 9x^2y^2z^2 \longrightarrow$ (1) L.H.S

L.H.S = R.H.S

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Orthogonal Curvilinear Co-ordinates

$\mu_1 = \mu_1(x, y, z)$
$\mu_2 = \mu_2(x, y, z)$ are three single variable functions
$\mu_3 = \mu_3(x, y, z)$ of x, y, z such that x, y, z can be expressed uniquely in the form of

$$x = x(\mu_1, \mu_2, \mu_3)$$
$$y = y(\mu_1, \mu_2, \mu_3)$$
$$z = z(\mu_1, \mu_2, \mu_3)$$

, then μ_1, μ_2, μ_3 are said to form a curvilinear co-ordinate system. A curvilinear co-ordinat system is called orthogonal if surfaces μ_1, μ_2 and μ_3 are constants through each point of space. are mutually at right angles.

o-ordinate surfaces, co-ordinate curves and asic vectors

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Orthogonal Curvilinear Co-ordinates

$\mu_1 = \mu_1(x, y, z)$
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are three single variable functions of x, y, z such that x, y, z can be expressed uniquely in the form of

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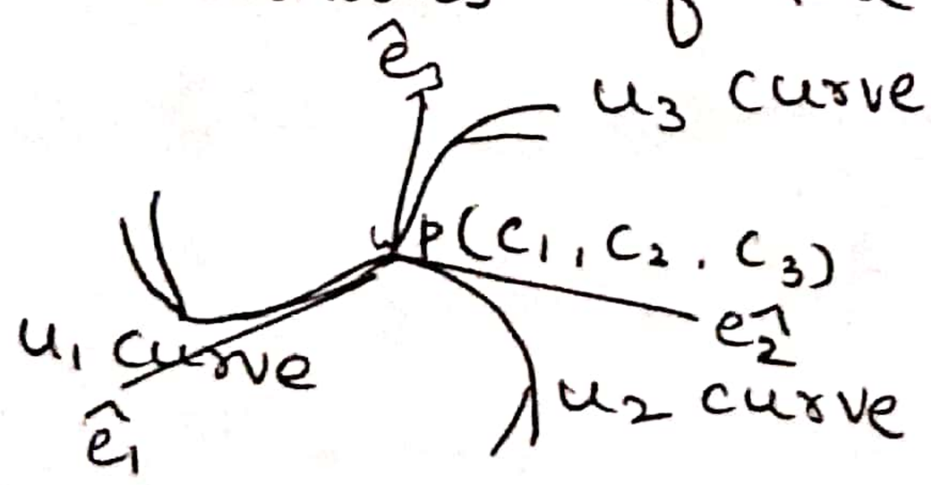
$y = y(\mu_1, \mu_2, \mu_3)$
$z = z(\mu_1, \mu_2, \mu_3)$

, then μ_1, μ_2, μ_3 are said to form a curvilinear co-ordinate system. A curvilinear co-ordinate system is called orthogonal if surfaces μ_1, μ_2 and μ_3 are constants through each point of space are mutually at right angles.

Co-ordinate surfaces, co-ordinate curves and basic vectors

The equation $\mu_1(x, y, z) = c_1$ will represent for different values of c_1 a family of surfaces, such that through each point in space, there will be one surface of family. Similarly through each point in space, there will be a member of the family surfaces, ~~the~~ $\mu_2 = c_2$, $\mu_3 = c_3$

Ordered Triplet c_1, c_2, c_3 of values of the constants will be the co-ordinates of the point



$\mu_1 = c_1$ $\mu_2 = c_2$ $\mu_3 = c_3$ at point (c_1, c_2, c_3) will intersect three curves. The surfaces $\mu_2 = c_2$ and $\mu_3 = c_3$ intersect along a curve for which μ_2 and μ_3 are both constants while μ_1 alone varies along the curve. For this reason, the curve of intersection of the surfaces $\mu_2 = c_2, \mu_3 = c_3$ is called the μ_1 curve through point (c_1, c_2, c_3)

Similarly we have μ_2 and μ_3 curve

unit tangent vectors at the given point drawn to u_1, u_2, u_3 curve are denoted by $\hat{e}_1$, $\hat{e}_2$ and $\hat{e}_3$

for an orthogonal curvilinear co-ordinate system, e_1, e_2 and e_3 are mutually $\perp$ vectors (unit vectors)

$$\hat{e}_1 \cdot \hat{e}_1 = 1 = \hat{e}_2 \cdot \hat{e}_2 = \hat{e}_3 \cdot \hat{e}_3$$

$$\hat{e}_1 \cdot \hat{e}_2 = 0 = \hat{e}_2 \cdot \hat{e}_3 = \hat{e}_3 \cdot \hat{e}_1$$

consider u_1 curve through point P. Let vector $\vec{r}$ is the position vector of P. Then the partial derivative of $\frac{\partial r}{\partial u_1}$ gives tangential vector to u_1 curve at P. Therefore unique tangential vector $\hat{e}_1$ to u_1 curve is

$$\hat{e}_1 = \frac{\frac{\partial r}{\partial u_1}}{\left|\frac{\partial r}{\partial u_1}\right|} = \frac{1}{h_1}\frac{\partial \vec{r}}{\partial u_1}$$

where $h_1 = \left|\frac{\partial \vec{r}}{\partial u_1}\right|$

$$\hat{e}_2 = \frac{\frac{\partial r}{\partial u_2}}{\left|\frac{\partial r}{\partial u_2}\right|} \qquad \hat{e}_3 = \frac{\frac{\partial r}{\partial u_3}}{\left|\frac{\partial r}{\partial u_3}\right|}$$

h_1, h_2, h_3 are scale factors of curvilinear co-ordinate system

$u_1 = x$, $u_2 = y$, $u_3 = z$

we get the well known rectangle co-ordinate system x, y, z

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\frac{\partial r}{\partial u_1} = \frac{\partial r}{\partial x} = \hat{i} \qquad \frac{\partial r}{\partial u_2} = \frac{\partial r}{\partial y} = \hat{j}$$

$$\frac{\partial r}{\partial u_3} = \frac{\partial r}{\partial z} = \hat{k}$$

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$$h_1 = \left|\frac{\partial \hat{r}}{\partial x}\right| = 1$$

$$h_2 = \left|\frac{\partial \hat{r}}{\partial y}\right| = 1$$

$$h_3 = \left|\frac{\partial \hat{r}}{\partial z}\right| = 1$$

$$\hat{e}_1 = \hat{i}, \quad \hat{e}_2 = \hat{j}, \quad \hat{e}_3 = \hat{k}$$

Spherical co-ordinates

Defined by $x = r\sin\theta \overset{\cos}{\cancel{\sin}}\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

$r \geq 0 \qquad 0 \leq \theta \leq \pi \qquad 0 \leq \phi \leq 2\pi$

$u_1 = r \qquad u_2 = \theta \qquad u_3 = \phi$

Show that spherical system is a orthogonal curvilinear co-ordinate system

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We know $\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$

where $x = r\sin\theta\cos\phi \qquad |\hat{r}| = r = \sqrt{x^2+y^2+z^2}$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$$\hat{e}_r \cdot \hat{e}_\theta = \hat{e}_\theta \cdot \hat{e}_\phi = \hat{e}_\phi \cdot \hat{e}_r = 0 \qquad \text{(Prove it)}$$

$$\hat{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$$

$$\hat{e}_r = \frac{\frac{\partial \vec{r}}{\partial r}}{\left|\frac{\partial \vec{r}}{\partial r}\right|} = \frac{1}{h_2}\frac{\partial \vec{r}}{\partial r} \qquad \hat{e}_\theta = \frac{1}{h_\theta}\frac{\partial \vec{r}}{\partial \theta}$$

$$h_r = \left|\frac{\partial \vec{r}}{\partial r}\right| \qquad h_\theta = \left|\frac{\partial \vec{r}}{\partial \theta}\right|$$

$$\hat{e}_\phi = \frac{1}{h_\phi}\frac{\partial \hat{r}}{\partial \phi}, \quad h_\phi = \left|\frac{\partial \hat{r}}{\partial \phi}\right|$$

$$\frac{\partial \vec{r}}{\partial r} = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$h_r = \sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}$$

$$h_r = \sqrt{\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta}$$

$$= \sqrt{\sin^2\theta + \cos^2\phi} = 1$$

$$\boxed{h_r = 1}$$

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k} \quad \text{—— (1)}$$

$$\frac{\partial \vec{r}}{\partial \theta} = r\cos\phi\cos\theta\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} + r(-\sin\theta)\,\hat{k}$$

$$\left|\frac{\partial \vec{r}}{\partial \theta}\right| = \sqrt{r^2\cos^2\phi\cos^2\theta + r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta}$$

$$= \sqrt{r^2((\sin^2\phi + \cos^2\phi)\cos^2\theta + \sin^2\theta)}$$

$$= \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = \sqrt{r^2} = r$$

$$\boxed{h_\theta = r}$$

$$\hat{e}_\theta = \cos\phi\cos\theta\,\hat{i} + \sin\phi\cos\theta\,\hat{j} - \sin\theta\,\hat{k} \quad \text{—— (2)}$$

$$\frac{\partial \vec{r}}{\partial \phi} = -r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j} + 0$$

$$\left|\frac{\partial \vec{r}}{\partial \phi}\right| = \sqrt{r^2(\sin^2\theta)(\sin^2\phi + \cos^2\phi)} = r\sin\theta$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\,\hat{k} \quad \text{—— (3)}$$

$$\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos^2\phi\cos\theta + \sin\theta\cos\theta\sin^2\phi - \sin\theta\cos\theta$$

$$\hat{e}_r \cdot \hat{e}_\theta = \sin\theta\cos\theta - \sin\theta\cos\theta = \underline{\underline{0}}$$

$$\hat{e}_r \cdot \hat{e}_\phi = -\sin\theta\cos\phi\sin\phi + \sin\theta\sin\phi\cos\phi = \underline{\underline{0}}$$

$$\hat{e}_\theta \cdot \hat{e}_\phi = -\sin\phi\cos\phi\cos\theta + \sin\phi\cos\theta\cos\phi = \underline{\underline{0}}$$

Cylindrical co-ordinates

$x = \rho \cos\phi$

$y = \rho \sin\phi$

$z = z$

$\rho \geq 0 \;,\; 0 \leq \phi \leq 2\pi$

$-\infty < z < \infty$

$u_1 = \rho \;,\; u_2 = \phi \;,\; u_3 = z$

Show that cylindrical co-ordinate system is same as orthogonal co-ordinate system.

Prove: $\hat{e}_\rho \cdot \hat{e}_\phi = \hat{e}_\phi \cdot \hat{e}_z = \hat{e}_z \cdot \hat{e}_\rho = 0$

$\hat{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\hat{k}$

$$\hat{e}_\rho = \frac{\frac{\partial \hat{r}}{\partial \rho}}{\left|\frac{\partial \hat{r}}{\partial \rho}\right|} = \frac{1}{h_\rho}\frac{\partial \hat{r}}{\partial \rho}$$

$$\frac{\partial \vec{r}}{\partial \rho} = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} + 0$$

$$\left|\frac{\partial \vec{r}}{\partial \rho}\right| = \sqrt{\cos^2\phi + \sin^2\phi} = 1$$

$$\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j} \quad \text{——} \quad (1)$$

$$\hat{e}_\phi = \frac{\frac{\partial \vec{r}}{\partial \phi}}{\left|\frac{\partial \vec{r}}{\partial \phi}\right|} = \frac{-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}}{\sqrt{\rho^2(\sin^2\phi + \cos^2\phi)}} = \frac{-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}}{\sqrt{\rho^2(1)}}$$

$$= \underline{\rho\left(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}\right)}$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} \quad \text{——} \quad (2)$$

$$\hat{e}_z = \frac{\frac{\partial \vec{r}}{\partial z}}{\left|\frac{\partial \vec{r}}{\partial z}\right|} = \frac{\hat{k}}{z} \qquad \hat{e}_z = \left(\frac{1}{z}\right)\hat{k} \quad \text{——} \quad (3)$$

$\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$\hat{e}_z = \left(\frac{1}{z}\right)\hat{k}$

$\hat{e}_\rho \cdot \hat{e}_\phi = -\sin\phi\cos\phi + \sin\phi\cos\phi = 0$

$\hat{e}_\rho \cdot \hat{e}_z = \left(\frac{\cos\phi\, z\,\hat{i}}{z} + \frac{z\sin\phi}{z}\hat{j}\right)\cdot\left(\frac{1}{z}\right)\hat{k} = 0$

$\hat{e}_\phi \cdot \hat{e}_z = (-\sin\phi\,\hat{i} + \cos\phi\,\hat{j})\cdot\left(\frac{1}{z}\right)\hat{k} = 0$

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① Represent $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ in cylindrical

Ans

$x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = z$

$\vec{r} = \rho\cos\phi\,\bar{i} + \rho\sin\phi\,\bar{j} + z\bar{k}$

$\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$ — ①

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$ — ②

$\hat{e}_z = \hat{k}$

① × $\sin\phi$ and ② by $\cos\phi$

$\hat{e}_\rho \sin\phi = \cancel{\sin\phi\cos\phi\,\hat{i}} + \sin^2\phi\,\hat{j}$

$\hat{e}_\phi \cos\phi = \cancel{-\sin\phi\cos\phi\,\hat{i}} + \cos^2\phi\,\hat{j}$

$\hat{e}_\rho \sin\phi + \hat{e}_\phi \cos\phi = j$ — ③

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① by $\cos\phi$ and ② by $\sin\phi$

$\hat{e}_\rho \cos\phi = \cos^2\phi\,\hat{i} + \sin\phi\cos\phi\,\hat{j}$

$\hat{e}_\phi \sin\phi = -\sin^2\phi\,\hat{i} + \sin\phi\cos\phi\,\hat{j}$

(−) (+) (−)

$\hat{e}_\rho \cos\phi - \hat{e}_\phi \sin\phi = \hat{i}$ — ④

Put ③ and ④ in $\vec{r}$

$\vec{r} = (\hat{e}_\rho\cos\phi - \hat{e}_\phi\sin\phi)\overset{x}{\hat{i}} + (\hat{e}_\rho\sin\phi + \hat{e}_\phi\cos\phi)y + z\hat{k}_z$

$\vec{r} = \rho(\hat{e}_\rho\cos^2\phi - \hat{e}_\phi\sin\phi\cos\phi) + (\hat{e}_\rho\sin^2\phi + \hat{e}_\phi\sin\phi\cos\phi)\rho + z\hat{e}_z$

$\vec{r} = \rho\hat{e}_\rho\cos^2\phi + \rho\hat{e}_\rho\sin^2\phi + z\hat{e}_z$

$\boxed{\vec{r} = \rho\hat{e}_\rho + z\hat{e}_z}$

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③ Represent $\vec{A} = x\hat{i} + y\hat{j}$ in spherical

Ans $x = \rho\cos\phi$

$y = \rho\sin\phi$

$z = (0)$

$\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$ — ①

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$ — ②

$\hat{e}_z = \hat{k}$

① by $\sin\phi$ and ② by $\cos\phi$

$\hat{e}_\rho\sin\phi = \cancel{\sin\phi\cos\phi\,\hat{i}} + \sin^2\phi\,\hat{j}$

$\hat{e}_\phi\cos\phi = \cancel{-\sin\phi\cos\phi\,\hat{i}} + \cos^2\phi\,\hat{j}$

$\hat{e}_\rho\sin\phi + \hat{e}_\phi\cos\phi = \hat{j}$ — ③

② by $\cos\phi$ and

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = 0$

$\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$ — ①

$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \sin\phi\cos\theta\,\hat{j} - \sin\theta\,\hat{k}$ — ②

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\hat{k}$ — ③

① × $\sin\theta$ and ② × $\cos\theta$

$\hat{e}_r\sin\theta = \sin^2\theta\cos\phi\,\hat{i} + \sin^2\theta\sin\phi\,\hat{j} + \cancel{\sin\theta\cos\theta\,\hat{k}}$

$\hat{e}_\theta\cos\theta = \cos^2\theta\cos\phi\,\hat{i} + \sin\phi\cos^2\theta\,\hat{j} - \cancel{\sin\theta\cos\theta\,\hat{k}}$

$\hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}$ — ④

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j} + 0\hat{k}$ — ⑤

④ by $\sin\phi$ and ⑤ by $\cos\phi$

$\hat{e}_r\sin\theta\sin\phi + \hat{e}_\theta\cos\theta\sin\phi = \cancel{\sin\phi\cos\phi\,\hat{i}} + \sin^2\phi\,\hat{j}$

$\hat{e}_\phi\cos\phi = \cancel{-\sin\phi\cos\phi\,\hat{i}} + \cos^2\phi\,\hat{j}$

$\hat{e}_r\sin\theta\sin\phi + \hat{e}_\theta\cos\theta\sin\phi + \hat{e}_\phi\cos\phi = \hat{j}$

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④ by $\cos\phi$ and ⑤ by $\sin\phi$

$$\hat{e}_r \sin\theta \cos\phi + \hat{e}_\theta \cos\theta \cos\phi = \cos^2\phi\,\hat{i} + \cancel{\sin\phi\cos\phi}\,\hat{j}$$

$$\hat{e}_\phi \cancel{\cos\phi} \sin\phi = -\sin^2\phi\,\hat{i} + \cancel{\cos\phi\sin\phi}\,\hat{j}$$

$$(-) \qquad + \qquad (-)$$

$$\hat{e}_r \sin\theta \cos\phi + \hat{e}_\theta \cos\theta \cos\phi - \hat{e}_\phi \sin\phi$$

$$= \hat{i}$$

$$\vec{A} = (\hat{e}_r \sin\theta\sin\phi + \hat{e}_\theta \cos\theta\sin\phi + \hat{e}_\phi \cos\phi)\, r\sin\theta\sin\phi$$

$$+ (\hat{e}_r \sin\theta\cos\phi + \hat{e}_\theta \cos\theta\cos\phi - \hat{e}_\phi \sin\phi)\,(r\cos\phi\sin\theta)$$

$$\vec{A} = r\hat{e}_r \sin^2\phi \sin^2\theta + r\hat{e}_\theta \sin\theta\cos\theta\sin^2\phi \cancel{+ r\hat{e}_\phi \sin\theta\cos\phi\sin\phi}$$

$$+ r\hat{e}_r \sin^2\theta\cos^2\phi + r\hat{e}_\theta \sin\theta\cos\theta\cos^2\phi \cancel{- r\hat{e}_\phi \sin\phi\cos\phi\sin\theta}$$

$$\boxed{\vec{A} = r\hat{e}_r \sin^2\theta + r\hat{e}_\theta \sin\theta\cos\theta}$$