Unit - I
Introduction to Mathematics II involving Integral calculus in CSE

1. Multiple Integrals.

In this topic, we discuss the repeated process of integration of a function of 2 & 3 variables referred as double integral and triple integral.

Double integrals : $\iint f(x,y)\,dx\,dy$

Note - The principle of partial differentiation is adopted in the process of double or triple integration.

Illustration:- Evaluate $\int_0^1 \int_2^3 x^2 y\,dx\,dy$

Direct method - We first see the order of integration, here we integrate w.r.t 'x' first then w.r.t 'y' keeping the limits for 'x' from 2 to 3 & y from 0 to 1.

$$I = \int_0^1 \left[\int_2^3 x^2 y\,dx \right] dy$$

$$I = \int_{y=0}^1 \left[\int_{2=x}^3 x^2 y\,dx \right] dy$$

$$I = \int_0^1 \left[\frac{x^3 y}{3} \right]_2^3 dy$$

$$= \frac{1}{3} \int_0^1 (27y - 8y)\,dy$$

$$= \frac{1}{3} \int_0^1 19y\,dy.$$

$$= \frac{19}{3} \int_0^1 y\,dy$$

$$= \frac{19}{3} \left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{19}{6} \left[1 - 0\right]$$

$$= \frac{19}{6}$$

2 Method 2 - Changing the order,

While changing the order of integration, the corresponding integrals should also be interchanged & in case of multiple integrals involving constant limit, there will be no changes in new limit for the corresponding integrals.

$$I = \int_{x=2}^{3}\int_{y=0}^{y=1} x^2 y\,dy\,dx$$

$$= \int_2^3 \left[\int_0^1 x^2 y\,dy\right] dx$$

$$= \int_2^3 \left[\frac{x^2y^2}{2}\right]_0^1 dx$$

$$= \int_2^3 \frac{x^2}{2}\,dx.$$

$$= \frac{1}{2}\int_2^3 x^2\,dx$$

$$= \frac{1}{2} \left.\frac{x^3}{3}\right|_2^3$$

$$= \frac{1}{2}\left[\frac{27}{3} - \frac{8}{3}\right]$$

$$= \frac{1}{2}\left[\frac{19}{3}\right]$$

$$= \frac{19}{6}$$

Evaluation of double integral ~~with~~ & triple integral

1. $\int_1^2\int_1^3 xy^2\,dx\,dy.$

$$I = \int_{y=1}^{2}\left[\int_{x=1}^{3} xy^2\,dx\right] dy$$

$$= \int_1^2 \left.\frac{x^2y^2}{2}\right|_1^3 dy.$$

$$= \frac{1}{2}\int_1^2 \left[9y^2 - y^2\right] dy.$$

$$= \frac{1}{2}\int_1^2 8y^2\,dy.$$

$$= \frac{8}{2} \left.\frac{y^3}{3}\right|_1^2$$

$$= \cancel{4}\ \frac{8}{3} - \frac{1}{3}$$

$$= 4\left(\frac{7}{3}\right)$$

$$= \frac{28}{3}.$$

$$= \frac{19}{3} \int_0^1 y\,dy$$

$$= \frac{19}{3}\left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{19}{6}\left[1 - 0\right]$$

$$= \frac{19}{6}$$

2. Method 2 - Changing the order,

While changing the order of integration, the corresp-onding integrals should also be interchanged & in case of multiple integrals involving constant limit, there will be no changes in new limit for the corresponding integrals.

$$I = \int_{x=2}^{3}\int_{y=0}^{y=1} x^2 y\,dy\,dx$$

$$= \int_2^3 \left[\int_0^1 x^2 y\,dy\right] dx$$

$$= \int_2^3 \left[\frac{x^2 y^2}{2}\right]_0^1 dx$$

$$= \int_2^3 \frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\int_2^3 x^2\,dx$$

$$= \frac{1}{2}\left.\frac{x^3}{3}\right|_2^3$$

$$= \frac{1}{2}\left[\frac{27}{3} - \frac{8}{3}\right]$$

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Evaluation of double integral ~~with~~ & triple integral

1. $\int_1^2\int_1^3 xy^2\,dx\,dy$.

$$I = \int_{y=1}^{2}\left[\int_{x=1}^{3} xy^2\,dx\right] dy$$

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$$= 4\;\frac{8}{3} - \frac{1}{3}$$

$$= 4\left(\frac{7}{3}\right)$$

$$= \frac{28}{3}.$$

MCQ *

2 Evaluate $\int_0^1 \int_0^1 \frac{dx\,dy}{\sqrt{1-x^2}\sqrt{1-y^2}}$

$$I = \int_{y=0}^{y=1} \int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}} \, dx\,dy$$

$$= \int_{y=0}^{1} \left[\int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}} dx\right] \frac{1}{\sqrt{1-y^2}} dy$$

$$= \int_{y=0}^{1} \left[\sin^{-1}x\right]_0^1 \frac{1}{\sqrt{1-y^2}} dy$$

$$= \int_{y=0}^{1} \left(\frac{\pi}{2} - 0\right) \frac{1}{\sqrt{1-y^2}} dy$$

$$= \frac{\pi}{2} \int_0^1 \frac{1}{\sqrt{1-y^2}} dy$$

$$= \frac{\pi}{2} \left(\sin^{-1} y\right)_0^1$$

$$= \frac{\pi}{2} \left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi^2}{4}$$

3 Evaluate $\int_0^1 \int_0^2 \int_1^2 x^2 yz \, dx\,dy\,dz$

$$I = \int_{z=0}^{1} \int_{y=0}^{2} \left[\int_{x=1}^{2} x^2 yz \, dx\right] dy\,dz$$

$$I = \int_{z=0}^{1} \int_{y=0}^{2} \left[\frac{x^3 yz}{3}\right]_1^2 dy\,dz$$

$$= \int_0^1 \int_0^2 \left[\frac{8yz}{3} - \frac{yz}{3}\right] dy\,dz$$

$$= \int_0^1 \int_0^2 \frac{7yz}{3} \, dy\,dz$$

$$= \frac{7}{3} \int_0^1 \left[\int_0^2 yz \, dy\right] dz$$

$$= \frac{7}{3} \int_0^1 \left[\frac{y^2 z}{2}\right]_0^2 dz$$

$$= \frac{7}{3} \int_0^1 \left(\frac{4z}{2} - 0\right) dz$$

$$= \frac{7}{3} \times 2 \int_0^1 z \, dz$$

$$= \frac{14}{3} \left[\frac{z^2}{2}\right]_0^1$$

$$= \frac{14}{3} \left(\frac{1}{2}\right)$$

$$= \frac{14}{6}$$

$$= \frac{7}{3}$$

4. $\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} (x^2+y^2+z^2)\,dx\,dy\,dz.$

$$I = \int_{z=-c}^{c}\int_{y=-b}^{b}\left[\int_{x=-a}^{a}(x^2+y^2+z^2)\,dx\right]dy\,dz$$

$$= \int_{-c}^{c}\int_{-b}^{b}\left[\frac{x^3}{3}+xy^2+xz^2\right]_{-a}^{a} dy\,dz$$

$$= \int_{-c}^{c}\int_{-b}^{b}\left[\frac{a^3}{3}+ay^2+az^2-\left(-\frac{a^3}{3}-ay^2-az^2\right)dy\,dz\right.$$

$$= \int_{-c}^{c}\left\{\int_{-b}^{b}\left(\frac{2a^3}{3}+2ay^2+2az^2\right)dy\right\}dz$$

$$= \int_{-c}^{c}\left(\frac{2a^3}{3}y+\frac{2ay^3}{3}+2az^2y\right)_{-b}^{b}dz$$

$$= \int_{-c}^{c}\left[\left(\frac{2a^3b}{3}+\frac{2ab^3}{3}+2az^2b\right)-\left(-\frac{2a^3b}{3}-\frac{2ab^3}{3}-2az^2b\right)\right]$$

$$= \int_{-c}^{c}\left(\frac{4a^3b}{3}+\frac{4ab^3}{3}+4abz^2\right)dz$$

$$= \left[\frac{4a^3bz}{3}+\frac{4ab^3z}{3}+\frac{4abz^3}{3}\right]_{-c}^{c}$$

$$= \frac{4a^3bc}{3}+\frac{4ab^3c}{3}+\frac{4abc^3}{3}-\left(-\frac{4a^3bc}{3}-\frac{4ab^3c}{3}-\frac{4abc^3}{3}\right]$$

$$= \frac{8a^3bc}{3}+\frac{8ab^3c}{3}+\frac{8abc^3}{3}$$

$$= \frac{8abc}{3}\left[a^2+b^2+c^2\right]$$

5. $\int_{0}^{a}\int_{0}^{b}\int_{0}^{c} (x^2+y^2+z^2)\,dx\,dy\,dz.$

$$I = \int_{z=0}^{a}\int_{y=0}^{b}\left[\int_{x=0}^{c}(x^2+y^2+z^2)\,dx\right]dy\,dz.$$

$$= \int_{0}^{a}\int_{0}^{b}\left[\frac{x^3}{3}+xy^2+xz^2\right]_{0}^{c} dy\,dz.$$

$$= \int_{0}^{a}\left[\int_{0}^{b}\left(\frac{c^3}{3}+cy^2+cz^2\right)dy\right]dz$$

$$= \int_{0}^{a}\left[\frac{yc^3}{3}+\frac{cy^3}{3}+cyz^2\right]_{0}^{b} dz$$

$$= \int_{0}^{a}\left(\frac{bc^3}{3}+\frac{cb^3}{3}+cbz^2-0\right)dz$$

$$= \left[\frac{bc^3z}{3}+\frac{cb^3z}{3}+\frac{cbz^3}{3}\right]_{0}^{a}$$

$$= \frac{abc^3}{3}+\frac{acb^3}{3}+\frac{a^3bc}{3}$$

$$= \frac{abc}{3}\left[a^2+b^2+c^2\right]$$

6. $$\int_0^a\int_0^a\int_0^a (x^2+y^2+z^2)\,dx\,dy\,dz$$

$$I = \int_0^a\int_0^a\left[\int_0^a (x^2+y^2+z^2)\,dx\right] dy\,dz$$

$$= \int_0^a\int_0^a \left[\frac{x^3}{3} + xy^2 + xz^2\right]_0^a dy\,dz$$

$$= \int_0^a\int_0^a \left(\frac{a^3}{3} + ay^2 + az^2 - 0\right) dy\,dz$$

$$= \int_0^a \left[\int_0^a \left(\frac{a^3}{3} + ay^2 + az^2\right) dy\right] dz$$

$$= \int_0^a \left[\frac{ya^3}{3} + \frac{ay^3}{3} + ayz^2\right]_0^a dz$$

$$= \int_0^a \left(\frac{a^4}{3} + \frac{a^4}{3} + a^4\right) dz$$

$$\cancel{= \int_0^a \left[\frac{a^5}{15} + \frac{a^5}{15} + \frac{a^5}{5}\right] dz}$$

$$= \left[\frac{a^4 z}{3} + \frac{a^4 z}{3} + \frac{a^4 z}{1}\right]_0^a$$

$$= \frac{a^5}{3} + \frac{a^5}{3} + a^5$$

$$= \frac{a^5 + a^5 + 3a^5}{3}$$

$$= \frac{5a^5}{3}$$

Valuation of double and triple integral involving variable limit.

7. Evaluate $$\int_0^1\int_x^{\sqrt{x}} xy\,dy\,dx$$

$$I = \int_{x=0}^{1}\left[\int_{y=x}^{\sqrt{x}} xy\,dy\right] dx$$

$$= \int_0^1 \left[\frac{y^2 x}{2}\right]_x^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\frac{x^2}{2} - \frac{x^3}{2}\right] dx$$

$$= \frac{1}{2}\int_0^1 (x^2 - x^3)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{1}{3} - \frac{1}{4}\right]$$

$$= \frac{1}{2}\left[\frac{4-3}{12}\right]$$

$$= \frac{1}{2}\left[\frac{1}{12}\right]$$

$$= \frac{1}{24}$$

8. $$\int_0^1\int_0^{\sqrt{1-y^2}} x^3 y\,dx\,dy.$$

$$I = \int_0^1 \left[\int_0^{\sqrt{1-y^2}} x^3 y\,dx\right] dy$$

$$= \int_0^1 \left[\frac{x^4 y}{4}\right]_0^{\sqrt{1-y^2}} dy.$$

$$= \int_0^1 \left[\frac{(1-y^2)^2}{4} y - 0\right] dy$$

$$= \int_0^1 \frac{(1+y^4-2y^2)}{4} y \, dy$$

$$= \int_0^1 \frac{y+y^5-2y^3}{4} dy$$

$$= \frac{1}{4}\int_0^1 (y+y^5-2y^3)\, dy$$

$$= \frac{1}{4}\left[\frac{y^2}{2}+\frac{y^6}{6}-\frac{2y^4}{4}\right]_0^1$$

$$= \frac{1}{4}\left[\frac{y^2}{2}+\frac{y^6}{6}-\frac{y^4}{2}\right]_0^1$$

$$= \frac{1}{4}\left[\cancel{\frac{1}{2}}+\frac{1}{6}-\cancel{\frac{1}{2}}\right]$$

$$= \cancel{\frac{1}{4}\left[\frac{3+}{6}\right.}$$

$$= \frac{1}{24}$$

9) $$\int_1^4 \int_0^{\sqrt{4-x}} xy \, dy \, dx.$$

$$I = \int_{x=1}^{4}\left[\int_{y=0}^{\sqrt{4-x}} xy\,dy\right]dx$$

$$= \int_1^4 \left[\frac{xy^2}{2}\right]_0^{\sqrt{4-x}} dx$$

$$= \int_1^4 \left(\frac{x(4-x)}{2} - 0\right) dx$$

$$= \int_1^4 \left(\frac{4x-x^2}{2}\right) dx$$

$$= \frac{1}{2}\left[\frac{4x^2}{2}-\frac{x^3}{3}\right]_1^4$$

$$= \frac{1}{2}\left[2x^2-\frac{x^3}{3}\right]_1^4.$$

$$= \frac{1}{2}\left[2\times 16 - \frac{64}{3}\right] - \left[2-\frac{1}{3}\right]$$

$$= \frac{1}{2}\left[32-\frac{64}{3}\right] - \frac{5}{3}$$

$$= \frac{32}{2} - \frac{64}{6} - \frac{5}{3}$$

$$= 16 - \frac{32}{6} - \frac{5}{3}$$

$$= \frac{96-64-10}{6}$$

$$= 22/\ldots = 11/\ldots$$

$$\int e^{ax}dx = \frac{e^{ax}}{a}$$

$$\int_0^1 \int_0^{x^2} e^{y/x}\, dy\, dx.$$

F.

Bernoulli's formula to integrate u*v

$$I = \int_{x=0}^{1}\left[\int_{y=0}^{x^2} e^{y/x}\, dy\right] dx$$

$$= \int_0^1 \left[\frac{e^{y/x}}{1/x}\right]_0^{x^2} dx$$

$$= \int_0^1 \left[x e^{y/x}\right]_0^{x^2} dx$$

$$= \int_0^1 x\left[e^x - e^0\right] dx$$

$$= \int_0^1 x\left[e^x \cancel{dx} - 1\right] dx.$$

$$= \int_0^1 \left[xe^x - x\right] dx.$$

$$= \left[xe^x - (1)e^x - \frac{x^2}{2}\right]_0^1$$

$$= \cancel{1e^1} - \cancel{e^1} - \frac{1}{2} - e^0 + e^0$$

$$= \frac{1}{2}.$$

$$\int_0^4 \int_0^{x^2} e^{y/x}\, dy\, dx$$

$$I = \int_0^4\left[\int_0^{x^2} e^{y/x}\, dy\right] dx$$

$$= \int_0^4 \left[x e^{y/x}\right]_0^{x^2} dx$$

$$= \int_0^4 \left[x e^x - xe^0\right] dx$$

$$= \int_0^4 \left[xe^x - x\right] dx$$

$$= \int x.$$

e^{x+}

$$\int_0^a\int_0^x\int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx.$$

$$I = \int_{x=0}^a\int_{y=0}^x\left[\int_{z=0}^{z=x+y} e^{x+y}\cdot e^{z}\, dz\right] dy\, dx$$

$$= \int_{x=0}^a\int_0^x e^{x+y}\left[e^{z}\right]_0^{x+y} dy\, dx$$

$$= \int_0^a\int_0^x e^{x+y}\left[e^{x+y}-1\right] dy\, dx$$

$$= \int_0^a\int_0^x \left[e^{2x+2y} - e^{x+y}\right] dy\, dx$$

$$\cancel{= \int_0^a\left[\int_0^x e^{2x+2y} - e^{x+y}\right.}$$

$$= \int_0^a\left[\int_0^x e^{2x}\left[e^{2y}\right] - e^{x}\left[e^{y}\right] dy\right] dx$$

$$= \int_0^a e^{2x}\cdot\left[\frac{e^{2y}}{2}\right]_0^x - e^{x}\left[e^{y}\right]_0^x dx.$$

$$= \int_0^a\left[e^{2x}\frac{e^{2x}}{2} - \frac{e^{2x}}{2} - \left(e^{2x}-e^{x}\right)\right] dx.$$

$$= \int_0^a\left[\frac{e^{4x}}{2} - \frac{e^{2x}}{2} - \left(e^{2x}-e^{x}\right)\right] dx.$$

$$= \int_0^a\left[\frac{1}{2}\left(e^{4x}-e^{2x}\right) - \left(e^{2x}-e^{x}\right)\right] dx$$

$$= \left[\frac{1}{2}\left(\frac{e^{4x}}{4}-\frac{e^{2x}}{2}\right) - \left(\frac{e^{2x}}{2}-e^{x}\right)\right]_0^a.$$

$$\frac{e^{4a}}{8} - \frac{e^{2a}}{4} + e^{a} - \frac{3}{8}.$$

$$= \frac{1}{4}\left(\frac{e^{4a}}{2} - e^{2a}\right) - \frac{e^{2a}}{2} + e^{a} - \left(\frac{1}{2}\left(\frac{e^0}{4}-\frac{e^0}{2}\right) - \frac{e^0}{2} + e^0\right)$$

$$= \frac{1}{4}\left(\frac{e^{4a}-2e^{2a}}{2}\right) - \frac{e^{2a}}{2} + e^{a} - \left[\frac{1}{2}\left(\frac{1}{4}-\frac{1}{2}\right) - \frac{1}{2} + 1\right]$$

$$\cancel{\Rightarrow \frac{e^{4a}}{8} - \frac{2e^{2a}}{4}}$$

$$= \frac{e^{4a}}{8} - \frac{e^{2a}}{4} - \frac{e^{2a}}{2} + e^{a} - \frac{1}{8} - \frac{3}{2}$$

$$= \frac{e^{4a}}{8} - \frac{e^{2a}}{4} - \frac{e^{2a}}{2} + e^{a} - \frac{1-12}{8}$$

★ <u>Evaluate</u> $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{1-x^2-y^2-z^2}}$

$$I = \int_{x=0}^1\int_{y=0}^{\sqrt{1-x^2}}\int_{z=0}^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\cdot dy\, dx$$

$$\sqrt{1-x^2-y^2-z^2} = \sqrt{\left(\sqrt{1-x^2-y^2}\right)^2 - z^2}$$

put $\left(\sqrt{1-x^2-y^2}\right) = k.$

$$I = \int_{x=0}^1\int_0^{\sqrt{1-x^2}}\int_0^{k} \frac{1}{\sqrt{k^2-z^2}}\, dy\, dz$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\sin^{-1}\left(\frac{z}{k}\right)\right]_0^{k} dy\, dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\sin^{-1}1 - \sin^{-1}0\right] dy\, dz$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx.$$

$$= \frac{\pi}{2} \int_0^1 \left[\int_0^{\sqrt{1-x^2}} dy \right] dx$$

$$= \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

W.K.T $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

$$= \frac{\pi}{2} \left[\sqrt{1-x^2}/2 \cdot x + \frac{1}{2} \sin^{-1}\left(\frac{x}{1}\right) \right]_0^1$$

$$= \frac{\pi}{2} \left[\frac{1}{2} \left(\sin^{-1}(1) - \sin^{-1}(0) \right) \right]$$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\,dy\,dx}{\sqrt{a^2-x^2-y^2-z^2}} \quad \Big| \; = \frac{a^2\pi^2}{8}$$

$$I = \int_{x=0}^a \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dz.dy.dx.$$

$$= \sqrt{a^2-x^2-y^2-z^2} = \sqrt{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2}$$

put $\sqrt{a^2-x^2-y^2} = k.$

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\int_0^k \frac{1}{\sqrt{k^2-z^2}}\, dz \right] dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{k}\right) \right]_0^k dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} [\sin^{-1}(1) - \sin 0]\, dy.dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2}\, dy.dx.$$

$$= \frac{\pi}{2} \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} dy \right] dx$$

$$= \frac{\pi}{2} \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2}\, dx.$$

W.K.T $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

$$= \frac{\pi}{2} \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$= \frac{\pi}{2} \cdot \left[\frac{a^2}{2} \cdot \frac{\pi}{2} - 0 \right]$$

$$= \frac{a^2\pi^2}{8}$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\pi}{2}\, dy\, dx.$$

$$= \frac{\pi}{2} \int_0^1 \left[\int_0^{\sqrt{1-x^2}} dy \right] dx$$

$$= \frac{\pi}{2} \int_0^1 [y]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^1 \sqrt{1-x^2}\, dx$$

W.K.T $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$

$$= \frac{\pi}{2} \left[\sqrt{1-x^2}/2 \cdot x + \frac{1}{2} \sin^{-1}\left(\frac{x}{1}\right) \right]_0^1$$

$$= \frac{\pi}{2} \left[\frac{1}{2} \left(\sin^{-1}(1) - \sin^{-1}(0) \right) \right]$$

$$= \frac{\pi}{2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} \frac{dz\, dy\, dx}{\sqrt{a^2-x^2-y^2-z^2}} \quad = a^2\pi^2/8$$

$$I = \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} \frac{1}{\sqrt{a^2-x^2-y^2-z^2}}\, dz.dy.dx.$$

$$= \sqrt{a^2-x^2-y^2-z^2} = \sqrt{\left(\sqrt{a^2-x^2-y^2}\right)^2 - z^2}$$

put $\sqrt{a^2-x^2-y^2} = k$.

$$I = \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\int_0^k \frac{1}{\sqrt{k^2-z^2}}\, dz \right] dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{z}{k}\right) \right]_0^k dy\, dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \left[\sin^{-1}(1) - \sin 0 \right] dy.dx$$

$$= \int_0^a \int_0^{\sqrt{a^2-x^2}} \frac{\pi}{2}\, dy.dx.$$

$$= \frac{\pi}{2} \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} dy \right] dx$$

$$= \frac{\pi}{2} \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \frac{\pi}{2} \int_0^a \sqrt{a^2-x^2}\, dx.$$

W.K.T $\int \sqrt{a^2-x^2}\, dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right).$

$$= \frac{\pi}{2} \left[\frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right) \right]_0^a$$

$$= \frac{\pi}{2} \cdot \left[\frac{a^2}{2} \cdot \frac{\pi}{2} - 0 \right]$$

$$= \frac{a^2\pi^2}{8}$$

$$\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx.$$ $\frac{1}{48}$

$$I = \int_0^1\int_0^{\sqrt{1-x^2}}\left[\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\right].dy.dx.$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\frac{xyz^2}{2}\right]_0^{\sqrt{1-x^2-y^2}} dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\frac{xy(1-x^2-y^2)}{2} - 0\right]dy\,dx$$

$$= \int_0^1\int_0^{\sqrt{1-x^2}}\left[\frac{xy - x^3y - xy^3}{2}\right]dy\,dx$$

$$= \int_0^1 \frac{1}{2}\left[\frac{xy^2}{2} - \frac{x^3y^2}{2} - \frac{xy^4}{4}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1}{2}\left[\frac{x(1-x^2)}{2} - \frac{x^3(1-x^2)}{2} - \frac{x(1-x^2)^2}{4}\right]dx.$$

$$= \int_0^1 \frac{1}{2}\left[\frac{x-x^3}{2} - \frac{x^3-x^5}{2} - \frac{x-x^3}{4}\right]dx.$$

$$= \int_0^1 \frac{1}{4}\left[(x-x^3) - (x^3-x^5) - \frac{x-x^3}{2}\right]dx$$

$$= \frac{1}{4}\left[\frac{x^2}{2} - \frac{x^4}{4} - \left(\frac{x^4}{4} - \frac{x^6}{6}\right) - \left(\frac{x^2}{4} - \frac{x^4}{8}\right)\right]_0^1 dx$$

$$= \frac{1}{4}\left[\frac{1}{2} - \frac{1}{4} - \frac{1}{4} + \frac{1}{6} - \frac{1}{4} + \frac{1}{8}\right]$$

$$= \frac{1}{48}.$$

$$(a-b)^3 = a^3 - b^3 - 3a^2b + 3ab^2$$

$$\int_1^2\int_0^{2-y} xy\,dy\,dx$$ $\frac{5}{24}$

$$I = \int_1^2\left[\int_0^{2-y} xy\,dy\right]dx.$$

$$\int_1^2\left[\frac{xy^2}{2}\right]_0^{2-y} dx$$

$$\int_1^2\left(\frac{x(2-y)^2}{2}\right)_0 dx$$

$$\int_1^2\left(\frac{x(4+y^2-4y)}{2}\right)dx$$

$$\int_1^2\left(\frac{4x+4xy^2-4xy}{2}\right)dx.$$

$$\int_1^2 (2x + 2xy^2 - 2xy)\,dx.$$

$$= \left[\frac{2x^2}{2} - \frac{2x^2y^2}{2} - \frac{2x^2y}{2}\right]_1^2$$

$$= \left[x^2 - x^2y^2 - x^2y\right]_1^2$$

$$= 4 -$$

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y^2} dz\,dy\,dx$$ $\frac{1}{4}.\ (1-x)^3$

$$I = \int_0^1\int_0^{1-x}\left[\int_0^{1-x-y^2} dz\right]dy\,dx$$

$$= \int_0^1\int_0^{1-x}\left[z\right]_0^{1-x-y^2} dy\,dx$$

$$= \int_0^1\left[\int_0^{1-x} 1-x-y^2\,dy\right]dx$$

$$= \int_0^1\left[y - xy - y^3/3\right]_0^{1-x} dx$$

$$= \int_0^1\left[1-x - x(1-x) - \frac{(1-x)^3}{3}\right]_0 dx$$

$$= \int_0^1 1-x-x+x^2 - \left[\frac{1-x^3-3x+3x^2}{3}\right] dx$$

$$= \int_0^1 \cancel{1} - 2x + x^2 - \cancel{1} + x^3 + 3x^2 - 3x^2\ dx$$

$$= \int_0^1 (x - 2x^2 + x^3)\, dx$$

$$= \left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} \right]_0^1$$

$$= \left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right]$$

$$= \frac{6 - 8 + 3}{12}$$

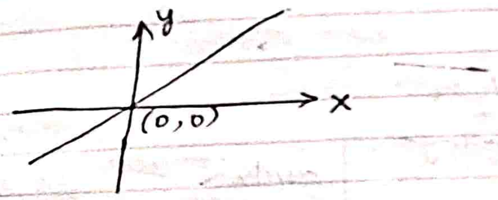
$$= \frac{1}{12}$$

2 (2, 3, 4
1, 3, 2

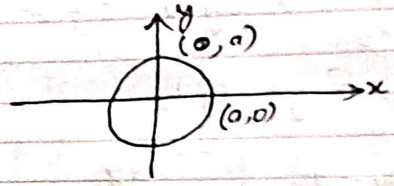
20/03/25

Sum standard curves

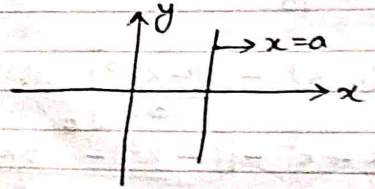
1. $y = mx$



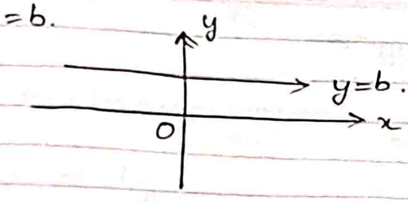
2. $x^2 + y^2 = a^2$



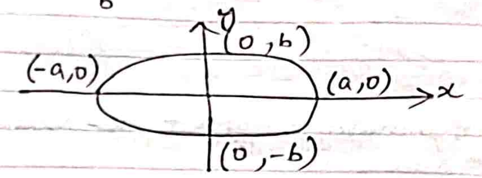
3. $y = a$



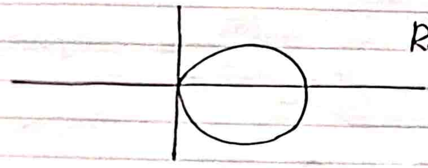
4. $y = b$.



5. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



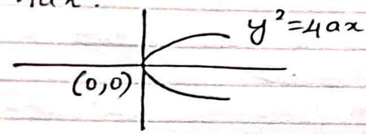
6. $x^2 + y^2 + 2gx + 2fy + c = 0$.



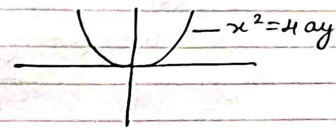
Radius $= \sqrt{g^2 + f^2 - c}$

Coordinates $(-g, -f)$.

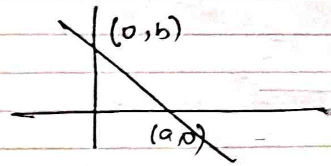
7. $y^2 = 4ax$.



8) $x^2 = 4ay$

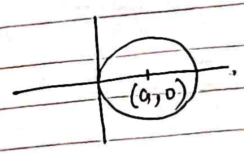


9. $\frac{x}{a} + \frac{y}{b} = 1$.



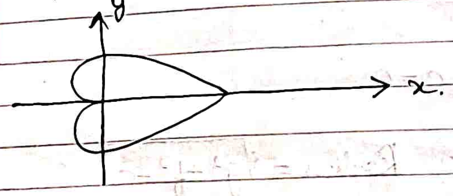
* Equation along x-axis, $y=0$
* Equation along y-axis, $x=0$.

10) $r = 2a\cos\theta$

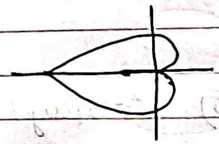


11. $r = a(1+\cos\theta)$

Represents cardiod symmetrical about x-axis.



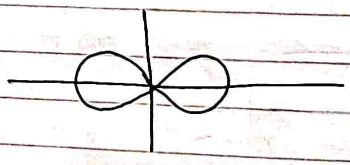
12. $r = a(1-\cos\theta)$



13. $r = a$

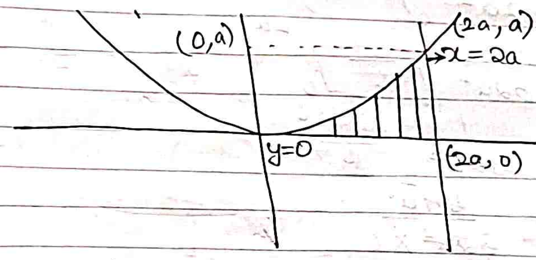
Represents a circle, centre at origin & radius $= a$.

14. Lemniscate



** Evaluation of the given integral over the given region.

1. Evaluate $\iint_R xy\,dy\,dx$ where R is a region bounded by x-axis, the ordinate $x = 2a$ & the curve $x^2 = 4ay$.



Region - closed path.

$y = ?$ point of intersection for the given curve $x^2 = 4ay$, $x = 2a$

$$(2a)^2 = 4ay$$
$$4a^2 = 4ay$$
$$\boxed{y = a}$$

$$POI = (2a, a)$$

In the given region 'x' varies from $x = 0$ to $x = 2a$. for all these values of 'x' $(\forall x)$ y varies from $y = 0$ to $y = \frac{x^2}{4a}$

$\therefore$ The given integral becomes

$$\int_0^{2a}\int_0^{x^2/4a} xy\,dy\,dx$$

$$I = \int_{x=0}^{2a}\left[\int_{y=0}^{x^2/4a} xy\,dy\right]dx$$

$$= \int_0^{2a}\left[\frac{xy^2}{2}\right]_0^{x^2/4a} dx.$$

$$A = \iint_R dx\,dy$$

$$V = \iiint dx\,dy\,dz$$

$$= \int_0^{2a} \frac{x}{2}\left(\frac{x^4}{16a^2}\right) dx$$

$$= \int_0^{2a} \frac{x^5}{32a^2}\, dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\, dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

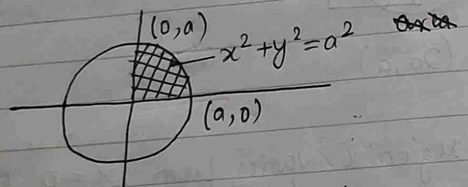
$$= \frac{1}{32a^2}\,\frac{(2a)^6}{6}$$

$$= \frac{\cancel{64}^{2}a^6}{\cancel{32}a^2 \times 6}$$

$$= \frac{a^4}{3}$$

2. Evaluate $\iint_R xy\,dx\,dy$ over the first quadrant of the circle $x^2+y^2=a^2$.

→



for the given curve 'y' varies from $y=0$ to $y=a$
~~Hlly~~ $\forall y$ 'x' varies from $x=0$ to $x=\sqrt{a^2-y^2}$

Therefore, the given integral becomes

$$I = \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\right] dy$$

$$= \int_0^a \left[\frac{x^2 y}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\frac{(a^2-y^2)\,y}{2}\right] dy$$

$$= \int_0^a \left(\frac{a^2y-y^3}{2}\right) dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2}-\frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2\times a^2}{2}-\frac{a^4}{4}\right]$$

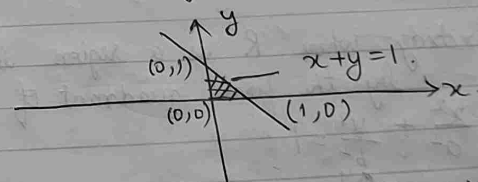
$$= \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{2a^4-a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{4}\right]$$

$$= \frac{a^4}{8}$$

3. Evaluate $\iint_R xy\,dx\,dy$, where 'R' is a region bounded by the coordinate axis & the line $x+y=1$



for the given curve 'y' varies ~~y~~ ~~to~~ $y=0$ to $y=1$ $\forall y$
'x' varies $x=0$ to $x=\cancel{y}\,1-y$

Therefore, the given integral becomes

$$I = \int_{y=0}^{1}\int_{x=0}^{1-y} xy\,dx\,dy.$$

$$V = \iiint dx\,dy\,dz$$

$$= \int_0^{2a} \frac{x}{2}\left(\frac{x^4}{16a^2}\right) dx$$

$$= \int_0^{2a} \frac{x^5}{32a^2}\, dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\, dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

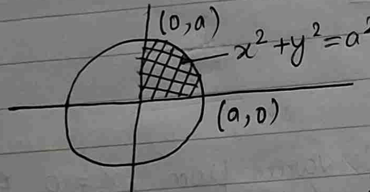
$$= \frac{1}{32a^2}\,\frac{(2a)^6}{6}$$

$$= \frac{\cancel{64}\,^{2} a^6}{\cancel{32}a^2 \times 6}$$

$$= \frac{a^4}{3}$$

2. Evaluate $\iint_R xy\,dx\,dy$ over the first quadrant of the circle $x^2+y^2=a^2$.

→



~~area~~

for the given curve 'y' varies from $y=0$ to $y=a$

~~HIIy~~ $\forall y$ 'x' varies from $x=0$ to $x=\sqrt{a^2-y^2}$

Therefore, the given integral becomes

$$I = \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\right] dy$$

$$= \int_0^a \left[\frac{x^2 y}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\frac{(a^2-y^2)\,y}{2}\right] dy$$

$$= \int_0^a \left(\frac{a^2y-y^3}{2}\right) dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2}-\frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2\times a^2}{2}-\frac{a^4}{4}\right]$$

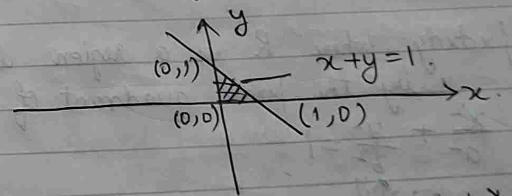
$$= \frac{1}{2}\left[\frac{a^4}{2}-\frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{2a^4-a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{4}\right]$$

$$= \frac{a^4}{8}$$

3. Evaluate $\iint_R xy\,dx\,dy$, where 'R' is a region bounded by the coordinate axis & the line $x+y=1$



for the given curve 'y' varies ~~to~~ $y=0$ to $y=1$ $\forall y$ 'x' varies $x=0$ to $x=\cancel{y-x}\ 1-y$

Therefore, the given integral becomes

$$I = \int_{y=0}^{1}\int_{x=0}^{1-y} xy\,dx\,dy.$$

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$$= \int_0^{2a} \frac{x}{2}\left(\frac{x^4}{16a^2}\right) dx$$

$$= \int_0^{2a} \frac{x^5}{32a^2}\, dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\, dx$$

$$= \frac{1}{32a^2}\left[\frac{x^6}{6}\right]_0^{2a}$$

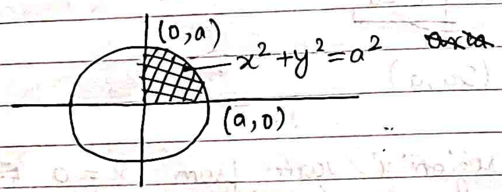
$$= \frac{1}{32a^2}\,\frac{(2a)^6}{6}$$

$$= \frac{\cancel{64}^{\,2}a^6}{\cancel{32}a^2 \times 6}$$

$$= \frac{a^4}{3}$$

2. Evaluate $\iint_R xy\,dx\,dy$ over the first quadrant of the circle $x^2+y^2=a^2$

→



for the given curve 'y' varies from $y=0$ to $y=a$
~~IIIy~~ $\forall y$ 'x' varies from $x=0$ to $x=\sqrt{a^2-y^2}$

Therefore, the given integral becomes

$$I = \int_{y=0}^{a}\left[\int_{x=0}^{\sqrt{a^2-y^2}} xy\,dx\right] dy$$

$$= \int_0^a \left[\frac{x^2 y}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[\frac{(a^2-y^2)y}{2}\right] dy$$

$$= \int_0^a \left(\frac{a^2y-y^3}{2}\right) dy$$

$$= \frac{1}{2}\left[\frac{a^2y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2\times a^2}{2} - \frac{a^4}{4}\right]$$

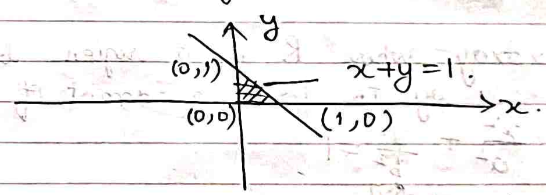
$$= \frac{1}{2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{2a^4-a^4}{4}\right]$$

$$= \frac{1}{2}\left[\frac{a^4}{4}\right]$$

$$= \frac{a^4}{8}$$

3. Evaluate $\iint_R xy\,dx\,dy$, where 'R' is a region bounded by the coordinate axis & the line $x+y=1$



for the given curve 'y' varies ~~to~~ $y=0$ to $y=1$ $\forall y$
'x' varies $x=0$ to $x=\cancel{y}\,1-y$

Therefore, the given integral becomes

$$I = \int_{y=0}^{1}\int_{x=0}^{1-y} xy\,dx\,dy.$$

$$= \int_0^1 \left[\int_0^{1-y} xy\,dx \right] dy$$

$$= \int_0^1 \left[\frac{x^2 y}{2} \right]_0^{1-y} dy$$

$$= \int_0^1 \frac{(1-y)^2 y}{2}\, dy$$

$$= \int_0^1 \frac{(1+y^2-2y)\,y}{2}\, dy$$

$$= \int_0^1 \frac{y+y^3-2y^2}{2}\, dy$$

$$= \frac{1}{2}\left[\frac{y^2}{2} + \frac{y^4}{4} - \frac{2y^3}{3} \right]_0^1$$

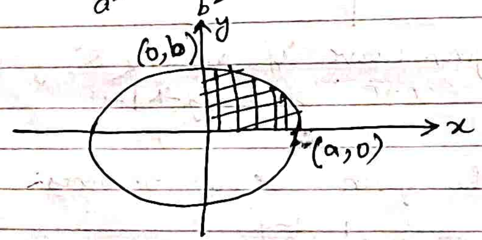
$$= \frac{1}{2}\left[\frac{1}{2} + \frac{1}{4} - \frac{2}{3} \right]$$

$$= \frac{1}{2}\left[\frac{6+3-8}{12} \right]$$

$$= \frac{1}{2}\left[\frac{1}{12} \right]$$

$$= \frac{1}{24}$$

Evaluate $\iint_R x\,dx\,dy$ where R is a region bounded by the first quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



y varies from $y=0$ to $y=b$ $\forall y$, x varies from $x=0$ to $x = \frac{a}{b}\sqrt{b^2-y^2}$.

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{x^2}{a^2} = 1 - \frac{y^2}{b^2}$$

$$\frac{x^2}{a^2} = \frac{b^2-y^2}{b^2}$$

$$x^2 = \frac{a^2}{b^2}(b^2-y^2)$$

$$x = \frac{a}{b}\sqrt{b^2-y^2}$$

$\left(\frac{3b^2a^2 - a^3}{3}\right)$

The given integral becomes

$$I = \int_{y=0}^{a} \left[\int_{x=0}^{a/b\sqrt{b^2-y^2}} x\,dx \right] dy$$

$$= \int_0^a \left[\frac{x^2}{2} \right]_0^{a/b\sqrt{b^2-y^2}} dy$$

$$= \frac{1}{2}\int_0^a \frac{a^2}{b^2}(b^2-y^2)\, dy.$$

$$= \frac{a^2}{2b^2}\int_0^a b^2-y^2\, dy$$

$$= \frac{a^2}{2b^2}\left[b^2 y - \frac{y^3}{3} \right]_0^a$$

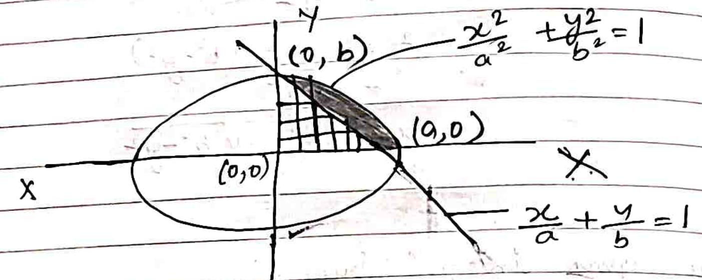
$$= \frac{a^2}{2b^2}\left[b^2 a - \frac{a^3}{3} \right] = \frac{ab^2}{3}.$$

Consider the

Note: Evaluate double integral over the region $\iint y\,dx\,dy\,dx$ over the region bounded by first quadrant of ellipse $= \frac{ab^2}{3}$

$x = 0$ to a $\quad y = 0$ to $b/a\sqrt{a^2-x^2}$

→ Evaluate $\iint_R xy\,dy\,dx$ when R is a region bounded by $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ & $\frac{x}{a}+\frac{y}{b}=1$



Here x varies from $x=0$ to $x=a$ & for x
y varies from $y=\frac{b}{a}(a-x)$ to $y=\frac{b}{a}(\sqrt{a^2-x^2})$

$\frac{x}{a}+\frac{y}{b}=1$ $\qquad$ $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$y=\frac{b}{a}(a-x)$ $\qquad$ $y=\frac{b}{a}\sqrt{a^2-x^2}$

$$I = \int_{x=0}^{a}\left[\int_{y=b/a(a-x)}^{b/a\sqrt{a^2-x^2}} xy\,dy\right]dx$$

$$= \int_0^a \left[\frac{xy^2}{2}\right]_{b/a(a-x)}^{b/a\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{x}{2}\left[\frac{b^2}{a^2}(a^2-x^2) - \frac{b^2}{a^2}(a-x)^2\right]dx.$$

$$\int_0^a \frac{x}{2}\left[\frac{b^2}{a^2}(\cancel{a^2}+x^2+2ax-\cancel{a^2}+x^2)\right]dx$$

$$\int_0^a \frac{x}{2}\left[2ax-2x^2\right]\frac{b^2}{a^2}\,dx.$$

$$\int_0^a \frac{b^2}{2a^2}\left[2ax^2-2x^3\right]dx.$$

$$\int_0^a \frac{b^2}{a^2}\left[ax^2-x^3\right]dx.$$

$$\frac{b^2}{a^2}\left[\frac{ax^3}{3}-\frac{x^4}{4}\right]_0^a$$

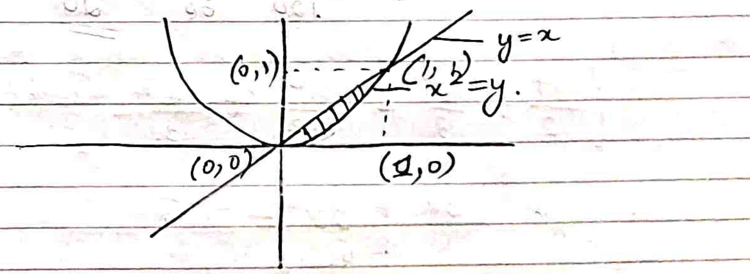
$$\frac{b^2}{a^2}\left[\frac{a^4}{3}-\frac{a^4}{4}\right]$$

$$\frac{b^2}{a^2}\left[\frac{4a^4-3a^4}{12}\right].$$

$$\frac{b^2a^4}{12a^2}$$

$$=\frac{a^2b^2}{12}$$

→ Evaluate $\iint_R xy(x+y)\,dy\,dx$ taken over the region bounded by $y=x^2$ & $y=x$.



Point of intersection ⟹ $y=x^2$ & $y=x$

$x=x^2$

$(x^2-x)=0$

$x(x-1)=0$

$x=1 \quad x=0$

Since $y=x$

$y=0 \quad y=1$

3/56. Here, x varies from $x=0$ to $x=1$ for all the values of x, y varies from $y=x^2$ to $y=x$. The given integral becomes

$$I = \int_{x=0}^{1}\left[\int_{y=x^2}^{x} x^2y + xy^2\, dy\right] dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2} + \frac{xy^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left(\frac{x^2x^2}{2} + \frac{x.x^6}{3}\right) dx$$

$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^7}{3}\right] dx$$

$$= \left[\frac{x^5}{10} + \frac{x^8}{24}\right]_0^1$$

$$= \frac{1}{10} + \frac{1}{24}$$

$$= \frac{12+6}{120}$$

$$= \frac{18}{120} = \frac{9}{60} = \frac{3}{20}$$

$2\,|\,24, 10$
$2\,|\,12, 5$
$2\,|\,6, 5$
$3, 5$
$2\times 2\times 2\times 3\times 5$
120

$x^2.x^4$

$x.x$

$$= \int_0^1\left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} + \frac{x^7}{3}\right]_0^B dx$$

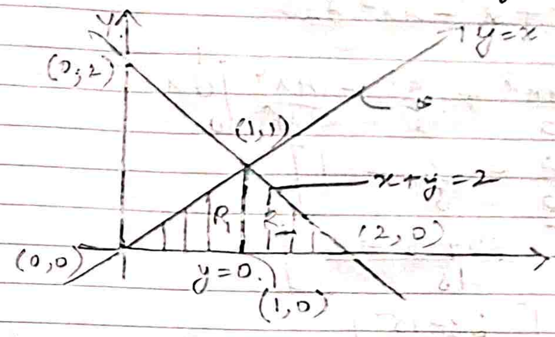
$$= \frac{3x^4 + 2x^4 - 3x^6 - 2x^7}{6}$$

$$= \frac{1}{6}\left(5x^4 - 3x^6 - 2x^7\right)$$

$$= \frac{1}{6}\; x^5 - \frac{3x^7}{7} - \frac{x^8}{4}$$

$\frac{2x^8}{8}$

→ Evaluate $\iint_R x^2y\, dy\,dx$ where R is a region bounded by lines $y=x$, $x+y=2$ & $y=0$.



$x+y=2$
$2y=2$
$y=1$
$x=1$

In the region R_1; x varies from $x=0$ to $x=1$ $\forall x$, y varies from $y=0$ to $y=x$.

Similarly In the region R_2; x varies from $x=1$ to $2=x$ $\forall x$, y varies as $y=0$ to $y=2-x$.

∴ The given integral becomes

$$I = I_1 + I_2$$

$$I_1 = \int_{x=0}^{1}\left[\int_{y=0}^{x} x^2y\,dy\right] dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2}\right]_0^x dx$$

$$= \frac{1}{2}\int_0^1 x^4\, dx$$

$$= \frac{1}{2}\left[\frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left(\frac{1}{5}\right) = \frac{1}{10}$$

$$I_2 = \int_0^1\left[\int_0^{2-x} x^2y\,dy\right] dx$$

$$= \int_0^1 \left[\frac{x^2y^2}{2}\right]_0^{2-x} dx$$

$$= \int_0^1 \frac{x^2(2-x)^2}{2}\, dx$$

$$= \int_0^1 2x^2$$

$$\frac{1}{2}\int_0^1 x^2(4+x^2-4x)\,dx$$

$$\frac{1}{2}\int_0^1 4x^2+x^4-4x^3\,dx$$

$$\frac{1}{2}\left[\frac{4x^3}{3}+\frac{x^5}{5}-\frac{4x^4}{4}\right]_0^1 dx$$

$$\frac{1}{2}\left[\frac{4}{3}+\frac{1}{5}-1\right]$$

$$\frac{1}{2}\left[\frac{20+3-15}{15}\right]$$

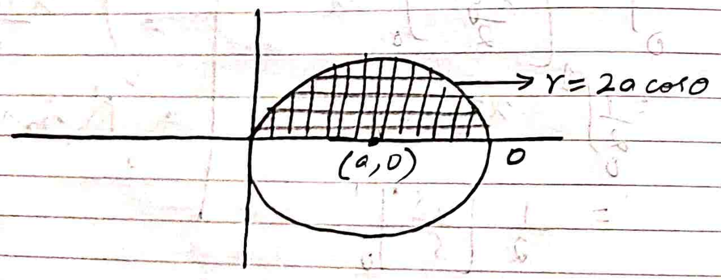
$$\frac{1}{2}\left[\frac{23-15}{15}\right]$$

$$\frac{1}{2}\left[\frac{8}{15}\right]$$

$$\frac{8}{30}=\frac{4}{15}$$

$$I=\frac{1}{10}+\frac{4}{15}=\frac{15+40}{150}=\frac{55}{150}=\underline{\underline{\frac{11}{30}}}$$

→ Evaluate $\iint_R r^2\sin\theta\,dr\,d\theta$ where R is a semicircle of $r=2a\cos\theta$ above the initial line



Here θ varies from θ = 0 to θ = π/2 + 0

r varies from r = 0 to r = 2a cosθ.

$$I=\int_{\theta=0}^{\pi/2}\left[\int_{r=0}^{2a\cos\theta} r^2\sin\theta\,dr\right]d\theta$$

$$=\int_0^{\pi/2}\sin\theta\left[\frac{r^3}{3}\right]_0^{2a\cos\theta}d\theta$$

$$\frac{1}{3}\int_0^{\pi/2}\sin\theta\left[8a^3\cos^3\theta\right]d\theta$$

$$=\frac{8a^3}{3}\int_0^{\pi/2}\sin\theta\cdot\cos^3\theta\,d\theta.$$

put $\cos\theta=t$

$-\sin\theta\,d\theta=dt.$

$$=-\frac{8a^3}{3}\int_1^0 t^3\,dt$$

$$=\frac{8a^3}{3}\int_0^1 t^3\,dt.$$

$$=\frac{8a^3}{3}\left[\frac{t^4}{4}\right]_0^1$$

$$=\frac{8a^3}{3}\left[\frac{1}{4}\right]$$

$$=\frac{2a^3}{3}$$