Bekaa Youth Education Center
Cycle III
Grade 12 - General Science
Name:

Test: Math
Duration: 180 minutes
Mark: /160
Date: 15/1/2024

I) (6 points)

In the table below, only one of the proposed answers to each question is correct. Write down the number of each question and give, with justification, the answer corresponding to it.

N	Questions	Answers		
		a	b	c
1	$\lim\limits_{x\to+\infty}(x^2 - xe^{2x})$	$-\infty$	0	$+\infty$
2	Given z is a complex number such that $z \neq 2i$ then $\left\lvert\frac{iz+2}{2\bar{z}+4i}\right\rvert$	1	$\frac{1}{2}$	2
3	$\arg(1+i)^{12}$	2π $\frac{\pi}{4}$	3π $\frac{\pi}{3}$	4π $\frac{\pi}{2}$
4	If A and B are two events of the same universe such that $P(B) = 0.2$ and $P(A \cap \overline{B}) = 0.5$ then $P(\overline{A} \cap \overline{B}) =$	0.2	0.7	0.3
5	In the set $\mathbb{C}$, if $\lvert z\rvert - z = 3 - i\sqrt{3}$ then $\lvert z\rvert =$	2	$2\sqrt{3}$	$3\sqrt{2}$
6	In the set $\mathbb{C}$, if $\arg(iz) = \frac{7\pi}{6}$ and $\lvert z\rvert = \sqrt{2}$ then the real part of z^3 is equal to	$2\sqrt{3}$	$3\sqrt{2}$	$2\sqrt{2}$
7	Let (V_n) be a sequence such that $V_1 + V_2 + \cdots + V_{n-1} + V_n = 2n^2 + n$ then $V_n =$	$6n-1$	$2n+1$	$4n-1$
8	$\lim\limits_{x\to+\infty}[x - \ln(e^x+1)] =$	0	e	$+\infty$

II) (5 points)

Consider the sequence (U_n) defined for every natural number $n \geq 1$ by $U_1 = e^2$ and $U_{n+1} = \sqrt{\frac{U_n}{e}}$

1) Prove by mathematical induction that for every natural number $n > 0$ we have $U_n > \frac{1}{e}$.
2) Verify that (U_n) is strictly decreasing.
3) Deduce that (U_n) is convergent to a certain limit l to be determined.
4) Let (V_n) be the sequence defined for $n \geq 1$ by $V_n = \frac{1}{2} + \frac{1}{2}\ln U_n$.
   a) Verify that (V_n) is a geometric sequence whose common ratio and first term are to be determined.
   b) Calculate V_n in terms of n then deduce U_n in terms of n.
5) Consider the sum S_n defined by $S_n = V_1 + V_2 + \cdots + V_n$
   a) Express S_n in terms of n.
   b) Deduce the product $P_n = U_1 \times U_2 \times \ldots \times U_n$ in terms of n.

III) (6 points)

A child plays with 20 marbles: 13 pink and 7 yellow.
He puts **3 pink** marbles and **4 yellow** in a cylindrical box and **10 pink** marbles and **3 yellow** in a cubic box.

Part A:
In a first game, he chooses a marble from the cylindrical box and a marble from the cubic box.
Consider the following events:
A: "The two chosen marbles are yellow".
B: "The two chosen marbles are of different colors".

Calculate P(A) and verify that $P(B)=\frac{49}{91}$.

Part B:
A second game is organized as follows: The child draws a marble from the cylindrical box.

- If it is pink, he draws successively with replacement two marbles from the cubic box.
- If it is yellow, he draws simultaneously two marbles from the cubic box.

Consider the following events:
R: "The marble drawn from the cylindrical box is pink".
C: "The two marbles drawn from the cubic box have the same color".

1) Calculate the probabilities $P(R), P(C/R)$ and show that $P(C \cap R) = \frac{327}{1183}$.
2) Calculate $P(C \cap \overline{R})$ and show that $P(C) = \frac{743}{1183}$.
3) Knowing that the two drawn marbles from the cubic box have the same color, what is the probability that the marble drawn from the cylindrical box is yellow?
4) Let X be the number of yellow marbles drawn from the cubic box.
   a) Determine the three possible values of X.
   b) Show that $P(X = 1) = \frac{440}{1183}$.
   c) Determine the probability of each value of X.

IV) (4 points)
In the complex plane referred to a direct orthonormal system $(O, \vec{\imath}, \vec{\jmath})$, consider the points A, B, M and M' with respective affixes $i, -2i, z$ and z' such that $z' = \frac{-2iz}{z-i}$ with $z \neq i$.

1) a) Determine the algebraic form of z when M and M' are confounded.
   b) Write z in exponential form in the case where $z' = 2i$.
2) a) Prove that $(z' + 2i)(z - i)$ is a real number.
   b) Deduce that $AM \times BM' = 2$ and $(\vec{U}, \overrightarrow{BM'}) = -(\vec{U}, \overrightarrow{AM}) + 2k\pi$
   c) If M moves on the circle with center A and radius 2, show that M' moves on a circle with center and radius to be determined.
   d) Prove that is M moves on the semi-line $[Ay)$ deprived the point A then M' moves on a semi-line to be determined.
3) Let $z = x + iy$ and $z' = x' + iy'$ where x, y, x' and y' are real numbers.
   a) Show that $x' = \frac{2x}{x^2+(y-1)^2}$ and $y' = \frac{-2(x^2+y^2-y)}{x^2+(y-1)^2}$
   b) If $AM = \sqrt{2}$ prove that $x' = x$.

V) (11 points)

Part A:

Let f be the function defined over $]0,+\infty[$ by $f(x) = x + \frac{1}{2} + (\frac{1}{2}\ln x - 1)\ln x$ and let (C) be its representative curve in an orthonormal system $(O, \vec{i}, \vec{j})$. (1 unit: 2 cm)

1) Calculate $\lim\limits_{x \to 0^+} f(x)$. Deduce an asymptote to (C).
2) Calculate $\lim\limits_{x \to +\infty} f(x)$ and verify that the line (d) of equation y=x is an asymptotic direction of (C) at $+\infty$.
3) a) Verify that for every $x \in]0,1]$: $(x-1) + \ln x \le 0$ and that for every $x \in [1,+\infty[$: $(x-1) + \ln x \ge 0$.
   b) Verify that $f'(x) = \frac{x-1+\ln x}{x}$ and set up the table of variations of f.
4) Show that (C) has a point of inflection W and write an equation of (T), the tangent at W to (C).
5) Show that for every $x \in]0,+\infty[$: $f(x) - x = \frac{1}{2}(\ln x - 1)^2$ then study the relative position of (C) and (d).
6) Construct (d) and (C).
7) a) Prove that the function $x \to F(x) = \frac{1}{2}x\ln^2 x - 2x\ln x + \frac{x^2}{2} + \frac{5}{2}x$ is an antiderivative of $f(x)$ over $]0,+\infty[$
   b) Deduce, in cm^2, the area of the domain limited by (C), the x-axis and the two vertical lines of equations $x = 1$ and $x = e$.

Part B:

Consider the sequence (U_n) defined for every natural number n by $\begin{cases} U_0 = 1 \\ U_{n+1} = f(U_n) \end{cases}$

1) Verify by mathematical induction that $1 \le U_n \le e$ for every $n \in \mathbb{N}$
2) Show that the sequence (U_n) is increasing.
3) Deduce that (U_n) is convergent to l. Find $l = \lim\limits_{n \to +\infty} U_n$.

Good Work!