Exercises on Exponential Functions for LS(sheet 3): 2020-2021

Nº1)Extra Math Consider the function f defined over IR by $f(x) = (x-1)e^x$ and designate by (C) its representative curve in an orthonormal system $(O\,;\,\vec{\imath}\,,\vec{\jmath})$.

1) Calculate $\lim\limits_{x\to -\infty} f(x)$ & $\lim\limits_{x\to +\infty} f(x)$. Deduce the equation of the asymptote.
2) Verify $f'(x) = xe^x$ and then set up the table of variations of f.
3) a. Write the equation of the tangent (T) to the curve (C) at the point of abscissa 1.
   b. Verify that $f(x) - y_{(T)} = (e^x - e)(x-1)$
   c. Deduce the relative position of (C) and (T).
4) Draw the curve (C) and tangent line (T).
5) Prove that for $x > 0$, the equation $f(x) = x$ admits a unique root α such that $1.3 < \alpha < 1.4$

Nº2)Extra Math

Part A Consider the function g defined over IR by $g(x) = a + b\,x\,e^{-x}$, where a *and* b are integers. The adjacent table is the table of variations of g.

x	$-\infty$		1		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+\infty$	$\searrow$	$1-\frac{1}{e}$	$\nearrow$	1

1) Verify that $a\,e + b = e - 1$
2) Determine $g'(x)$ in terms of b and x.
3) Calculate a then deduce the value of b.

Part B Consider the function f defined over IR by $f(x) = x + (x+1)e^{-x}$ and designate by (C) its representative curve in an orthonormal system $(O\,;\,\vec{\imath}\,,\vec{\jmath})$.

1) Calculate the limits of f at $-\infty$ and $+\infty$.
2) a. Verify that the straight line (d) with equation $y = x$ is an oblique asymptote to (C) at $+\infty$.
   b. Study the relative position of (C) and (d).
3) Prove that $f'(x) = g(x)$ and then set up the table of variations of f.
4) Determine the coordinates of the point E of (C) for which the tangent (T) to (C) at E is parallel to the asymptote (d).Write the equation of (T).
5) Trace $(C), (d)$ and (T).

Nº3) Mastering Mathematics

Part A Consider the function g defined over IR by $g(x) = 2\,e^x - x - 2$ and designate by (C) its representative curve in an orthonormal system $(O\,;\,\vec{\imath}\,,\vec{\jmath})$.

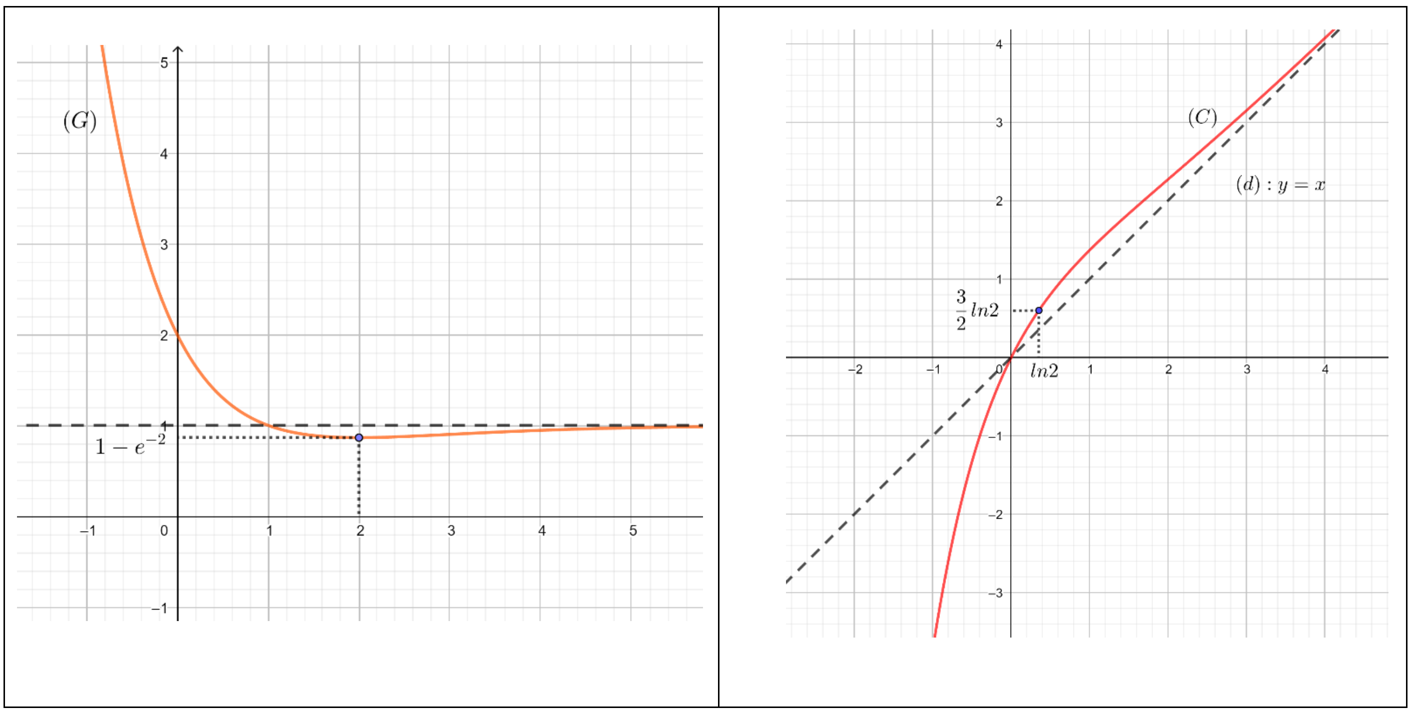
1) Calculate $\lim\limits_{x\to -\infty} g(x)$ & $\lim\limits_{x\to +\infty} g(x)$.
2) Verify that the straight line (d) with equation $y = -x - 2$ is an oblique asymptote to (C) as $x \to -\infty$, and that (C) is above (d).

3) Determine $g'(x)$ then set up the table of variations of g.
4) Show that the equation $g(x) = 0$ admits two roots 0 and α and that $-1.6 < \alpha < -1.5$.
5) Trace (C) and (d).

Part B f is a function defined over IR by $f(x) = e^{2x} - (x+1)e^{x}$, and designate by (Γ) its representative curve in an orthonormal system $(O\,;\,\vec{\imath},\vec{\jmath})$.

1) Calculate $\lim\limits_{x \to -\infty} f(x)$ & $\lim\limits_{x \to +\infty} f(x)$.
2) Show that $f'(x) = e^{x} g(x)$ then draw the table of variations of f.
3) Show that $f(\alpha) = \frac{-\alpha^2 - 2\alpha}{4}$, then give an approximate value of $f(\alpha)$ to the nearest 10^{-2} knowing that $\alpha = -1.55$.
4) Trace (Γ).
5) Show that the equation $f(x) = 1$ admits one and only one root β and that $0.813 < \beta < 0.815$.

Nº4) In the figure below , one of the two graphs is for a function f defined over IR and the other one is that of the derivative function f'.



1) Assume the first curve (G) is that of a function g , and the second curve is that of a function h.
   a. Set up the table of variations of both functions.
   b. Verify that $g(x)$ and $h'(x)$ have the same sign.
   c. Deduce the curve associated to f and the curve associated to f'.
2) Knowing that $f(x) = xe^{-x} + a\,x + b$, where a and b are two integers.
   a. Show that $b = 0$.
   b. Verify that $f(x) = xe^{-x} + x$.
3) Show that for every real number m, the equation $x = \ln\left(\frac{x}{m-x}\right)$ admits a unique root.

Answers of Exercises on exponential functions (Sheet 3):

Nº1) $f(x) = (x-1)e^{x}$ defined over IR

1) $\lim_{x\to -\infty} f(x) = (-\infty)(0)$ $indeterminate\ form$

$$= \lim_{x\to -\infty} \frac{x-1}{e^{-x}} = \frac{-\infty}{+\infty} \quad indeterminate\ \ (L'HopitalRule)$$

$$= \lim_{x\to -\infty} \frac{1}{-e^{-x}}$$

$$= \frac{1}{-\infty}$$

$$= 0$$

Then the straight line with equation $y = 0$ is a H.A as $x \to -\infty$

$\lim_{x\to +\infty} f(x) = (+\infty)(+\infty) = +\infty$ P.O.A

2) $f'(x) = u'v + u\,v'$

$= (1)(e^{x}) + (x-1)(e^{x})$

$= (1 + x - 1)e^{x}$

$= xe^{x}$

Assume $f'(x) \geq 0$ then $xe^{x} \geq 0$ but $e^{x} > 0$ for any $x \in IR$

$x \geq 0$

x	$-\infty$	0	$+\infty$
$f'(x)$	$-$	0	$+$
$f(x)$	0 ↘	-1	↗ $+\infty$

3) a. The equation of the tangent (T) to the curve (C) at the point of abscissa 1:

$y - f(1) = f'(1)(x-1)$ $\qquad f(1) = 0$ & $f'(1) = e$

$y - 0 = e(x-1)$

$y = e\,x - e$

b. $f(x) - y_{(T)} = (x-1)e^{x} - (e\,x - e)$

$= (x-1)e^{x} - e(x-1)$

$= (x-1)(e^{x} - e)$

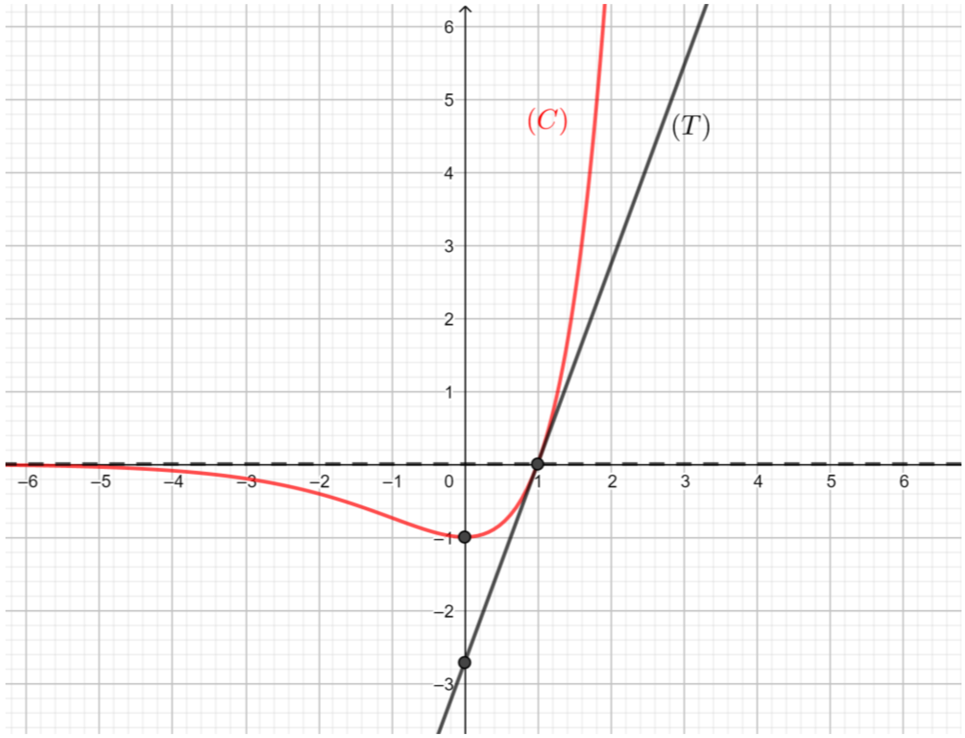
c. Assume $f(x) - y_{(T)} = 0$ then $(x-1)(e^{x} - e) = 0$

$x = 1$ $\quad or \quad$ $x = \ln e = 1$

x	$-\infty$	1	$+\infty$
$(x-1)$	$-$	0	$+$
$(e^{x} - e)$	$-$	0	$+$
$f(x) - y_{(T)}$	$+$ $(C) above\ (T)$	0	$+$ $(C) above\ (T)$

$(C) \cap (T)$

At (1,0)



5)

Using the graph , for $x > 0$,

The curve (C) intersects line $y = x$ at one point then the equation $f(x) = x$ admits a unique root α .

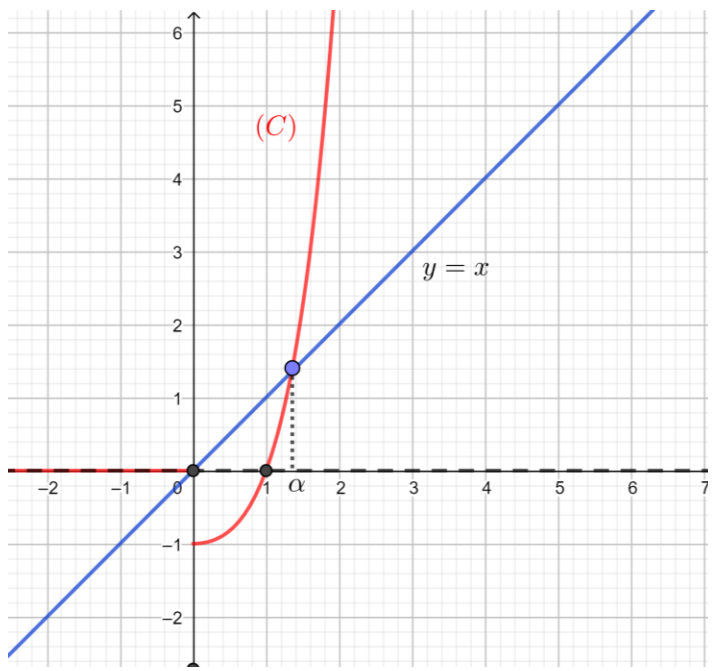
Now , $f(1.3) - 1.3 = -0.19 < 0$

(C) *below this line*

$f(1.4) - 1.4 = 0.2 > 0$

(C) *above this line*

Then $1.3 < \alpha < 1.4$



Nº2) Part A $g(x) = a + b\,x\,e^{-x}$ defined over IR

x	$-\infty$		1		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+\infty$	$\searrow$	$1-\frac{1}{e}$	$\nearrow$	1

1) Using the given $g(1) = a + be^{-1}$ and using the table $g(1) = 1 - \frac{1}{e}$

Then $a + \frac{b}{e} = 1 - \frac{1}{e}$

$\frac{a\ e\ +\ b}{e} = \frac{e\ -\ 1}{e}$

$a\,e + b = e - 1$

2) $g'(x) = 0 + (b)(e^{-x}) + (bx)(-e^{-x})$

$= (b - bx)e^{-x}$

$= b(1-x)e^{-x}$

$u = bx \quad \to u' = b$

$v = e^{-x} \quad \to \quad v'$

3) Using $\lim\limits_{x\to +\infty} g(x) = 1$ in above table of variation

$\lim\limits_{x\to +\infty} (a + b\,x\,e^{-x}) = 1$

$a + b(0) = 1$

$a = 1$

$\lim\limits_{x\to +\infty} \frac{x}{e^x} = \frac{+\infty}{+\infty}\,indeterminate$

$= \lim\limits_{x\to +\infty} \frac{1}{e^x} = 0$

Then $a\,e + b = e - 1$

$(1)e + b = e - 1$

$b = e - 1 - e$

$b = -1$ so that $g(x) = 1 - \ x\,e^{-x}$

Part B $f(x) = x + (x+1)e^{-x}$ defined over IR

1) $\lim\limits_{x\to -\infty} f(x) = \lim\limits_{x\to -\infty} [x + (x+1)e^{-x}] = -\infty + (-\infty)(+\infty) = -\infty - \infty = -\infty$

$\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty} [x + (x+1)e^{-x}] = +\infty + (+\infty)(0)$

Indeterminate

$= \lim\limits_{x\to +\infty} [x + \frac{x\ +\ 1}{e^x}]$

$= +\infty + 0$

$= +\infty$

$\lim\limits_{x\to +\infty} \frac{x+1}{e^x} = \frac{+\infty}{+\infty}\,indeterminate$

$= \lim\limits_{x\to +\infty} \frac{1}{e^x} = 0$

2) a. $\lim\limits_{x\to +\infty} [f(x) - y_{(d)}] = \lim\limits_{x\to +\infty} [x + (x+1)e^{-x} - x]$

$= \lim\limits_{x\to +\infty} (x+1)e^{-x}$

$= 0$

b. $f(x) - y_{(d)} = (x+1)e^{-x}$

Assume $f(x) - y_{(d)} \geq 0$ then $(x+1)e^{-x} \geq 0$

$(x+1) \geq 0 \quad since \quad e^{-x} > 0 \quad for \ any \ x \in IR$

$x \geq -1$

x	$-\infty$	-1	$+\infty$
$sign\ of \quad f(x) - y_{(d)}$	$-$ $(C) below\ (d)$	0 $(C) \cap (d)$ at $(-1,-1)$	$+$ $(C) above\ (d)$

3) $f'(x) = 1 + (1)e^{-x} + (x+1)(-e^{-x})$

$= 1 + e^{-x} - xe^{-x} - e^{-x}$

$= 1 - xe^{-x}$

$= g(x)$

Then $f'(x) \ and \ g(x)$ have the same sign but $g(x) \geq 1 - \frac{1}{e} \quad and \quad 1 - \frac{1}{e} = 0.63 > 0$

$g(x) > 0 \quad for \ any \ x \in IR$

x	$-\infty$	$+\infty$
$f'(x)$	$+$	
$f(x)$	$-\infty$	$\nearrow$ $+\infty$

4)The tagent (T) is parallel to the asymptote (d)

Then they have the same slope , $f'(x_E) = 1$

$$1 - x_E\, e^{-x_E} = 1$$

$$-x_E \ \ e^{-x_E} = 0$$

$$-x_E = 0 \quad where \quad e^{-x_E} > 0$$

$$x_E = 0$$

Then $y_E = \ ???$

We may sustitute x_E in the equation of the function f:

$y_E = f(x_E) = f(0) = 0 + e^0 = 1$

Therefore , $E(0\,,1)$

Then , equation of tangent (T) at point E:

$$y - y_E = f'(x_E)(x - x_E)$$

$y - 1 = f'(0)(x - 0)$ where $f'(0) = 1$

$$(T) : \quad y = x + 1$$

$a = \lim_{x \to -\infty} f'(x)$

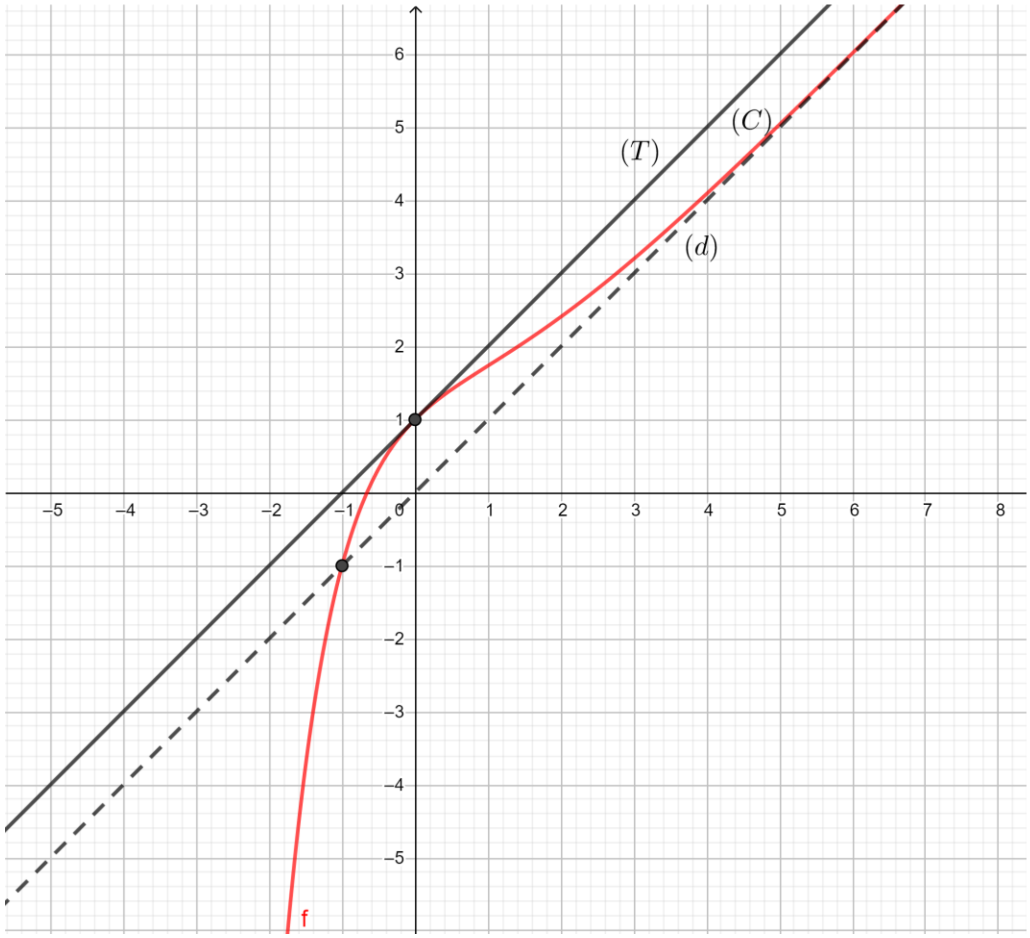
$= \lim_{x \to -\infty} (1 - xe^{-x})$

$= 1 - (-\infty)(+\infty)$

$= 1 + \infty$

$= +\infty$

The curve has asymptotic direction parallel to $y'Oy$.



Nº3) Part A $g(x) = 2\,e^{x} - x - 2$ defined over IR

1) $\lim_{x \to -\infty} g(x) = \lim_{x \to -\infty} (2\,e^{x} - x - 2) = 2e^{-\infty} + \infty - 2 = +\infty$

$\lim_{x \to +\infty} g(x) = \lim_{x \to +\infty} (2\,e^{x} - x - 2) = 2e^{+\infty} - \infty - 2 = +\infty - \infty$ indeterminate form

$= \lim_{x \to +\infty} x\left(\frac{2\,e^{x}}{x} - 1 - \frac{2}{x}\right)$

$= (+\infty)(+\infty - 1 - 0)$

$= +\infty$

$\lim_{x \to +\infty} \frac{e^{x}}{x} = +\infty$

2) $\lim_{x \to -\infty} [g(x) - y_{(d)}] = \lim_{x \to -\infty} [2\,e^{x} - x - 2 - (-x - 2)]$

$= \lim_{x \to -\infty} 2\,e^{x}$

$= 0$

Then the straight line (d) with equation $y = -x - 2$ is an oblique asymptote to (C) as $x \to -\infty$

$g(x) - y_{(d)} = 2\,e^{x}$ but $e^{x} > 0$ for any $x \in IR$

Then $g(x) - y_{(d)} > 0$ and (C) is above (d) for any $x \in IR$.

3) $g'(x) = 2\,e^{x} - 1$

Assume $g'(x) \geq 0$ then $2\,e^{x} - 1 \geq 0$

$$2\,e^{x} \geq 1$$

$$e^{x} \geq \frac{1}{2}$$

$$x \geq \ln\left(\frac{1}{2}\right)$$

$$x \geq -\ln 2$$

x	$-\infty$		α		$-\ln 2$		0		$+\infty$
$g'(x)$			$-$		0		$+$		
$g(x)$	$+\infty$	$\searrow$	0	$\searrow$	$-1+\ln 2$	$\nearrow$	0	$\nearrow$	$+\infty$

Since $g(-\ln 2) = 2e^{-ln2} + \ln 2 - 2 = 2e^{\ln\frac{1}{2}} + \ln 2 - 2 = 2\left(\frac{1}{2}\right) + \ln 2 - 2 = -1 + \ln 2 \cong -0.306$

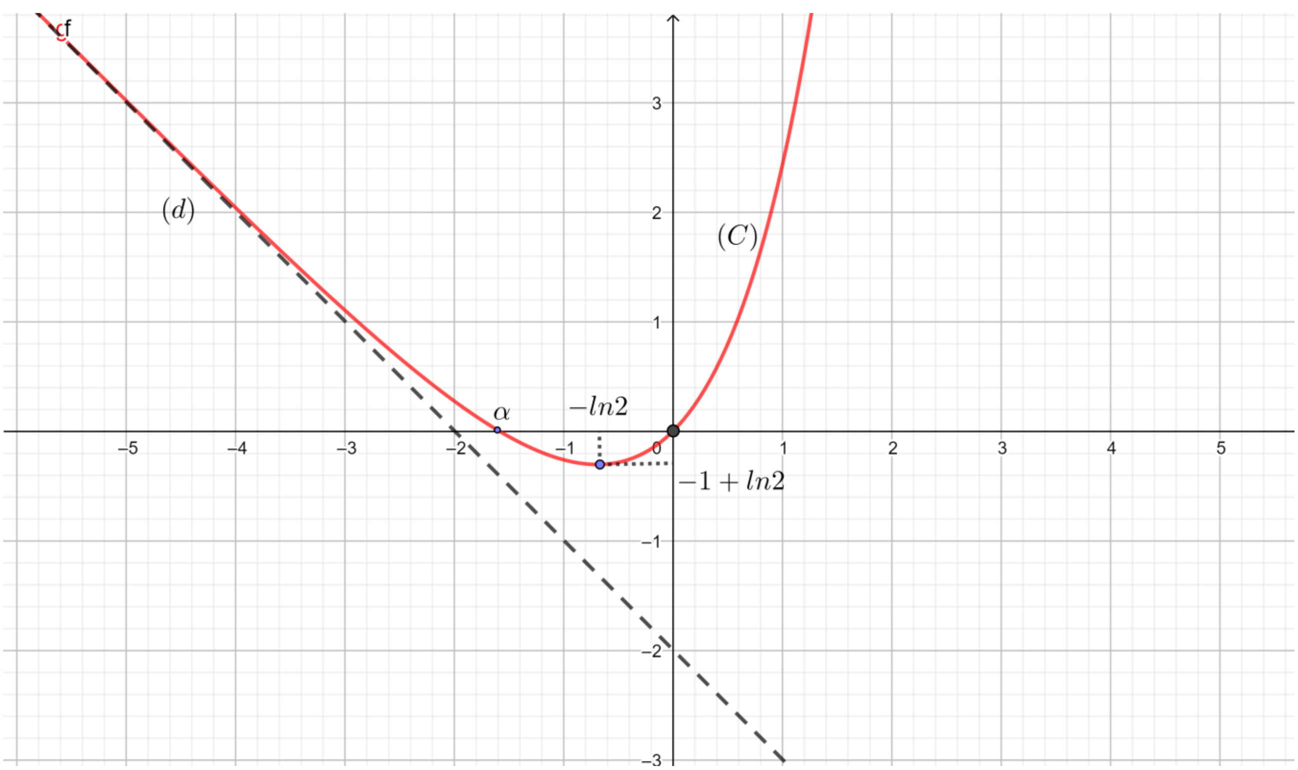
4) Now $g(0) = 2e^{0} - 0 - 2 = 0$ then $x = 0$ is the first root

For $x \in]-\infty, -\ln 2[$, the function g is continuous , strictly decreasing and change sign from $+\infty$ to $-1 + \ln 2$ then $g(x) = 0$ admits here a unique root α such that $g(\alpha) = 0$

$$g(-1.6) \times g(-1.5) = (0.00379)(-0.0537) < 0$$

Then $-1.6 < \alpha < -1.5$

5)



Part B $f(x) = e^{2x} - (x+1)e^x$ defined over IR

1) $\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} [e^{2x} - (x+1)e^x] = 0 - (-\infty)(0)$ indeterminate form

$= \lim\limits_{x \to -\infty} [e^{2x} - \frac{x+1}{e^{-x}}]$

$= 0$

$\lim\limits_{x \to -\infty} \frac{x+1}{e^{-x}} = \frac{-\infty}{+\infty}$ *indeterminate*

$(L'Hopital\ \ Rule)$

$= \lim\limits_{x \to -\infty} \frac{1}{-e^{-x}} = 0^-$

Then $y = 0$ is H.A as $x \to -\infty$

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} [e^{2x} - (x+1)e^x] = +\infty - \infty$

$= \lim\limits_{x \to +\infty} e^{2x} [1 - \frac{x+1}{e^x}]$

$= (+\infty)[1 - 0]$

$= +\infty$

$\lim\limits_{x \to +\infty} \frac{x+1}{e^x} = \frac{+\infty}{+\infty}$ *indeterminate*

$= \lim\limits_{x \to +\infty} \frac{1}{e^x} = 0$

2) $f'(x) = 2e^{2x} - [(1)e^x + (x+1)e^x]$

$= 2e^{2x} - (x+2)e^x$

$= e^x[2e^x - (x+2)]$

$= e^x[2e^x - x - 2]$

$= e^x\, g(x)$

Assume $f'(x) \geq 0$ $\quad then \quad$ $e^x g(x) \geq 0$

$g(x) \geq 0 \quad since \quad e^x > 0 \quad for\ any\ x \in IR$

Using part A :

x	$-\infty$	α		$-\ln 2$		0		$+\infty$
$g'(x)$			$-$	0	$+$			
$g(x)$	$+\infty$	0 ↘		$-1+\ln 2$		0 ↗		$+\infty$

$x \in]-\infty, \alpha] \cup [0, +\infty[$

x	$-\infty$		α		0		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	0	↗	$f(\alpha)$	↘	0	↗	$+\infty$

3) $f(\alpha) = e^{2\alpha} - (\alpha+1)e^{\alpha}$

$$= e^{\alpha}(e^{\alpha} - \alpha - 1)$$

$$= \left(\frac{\alpha+2}{2}\right)\left(\frac{\alpha+2}{2} - \alpha - 1\right)$$

$$= \left(\frac{\alpha+2}{2}\right)\left(\frac{\alpha+2-2\alpha-2}{2}\right)$$

$$= \left(\frac{\alpha+2}{2}\right)\left(\frac{-\alpha}{2}\right)$$

$$= \frac{-\alpha^2 - 2\alpha}{4}$$

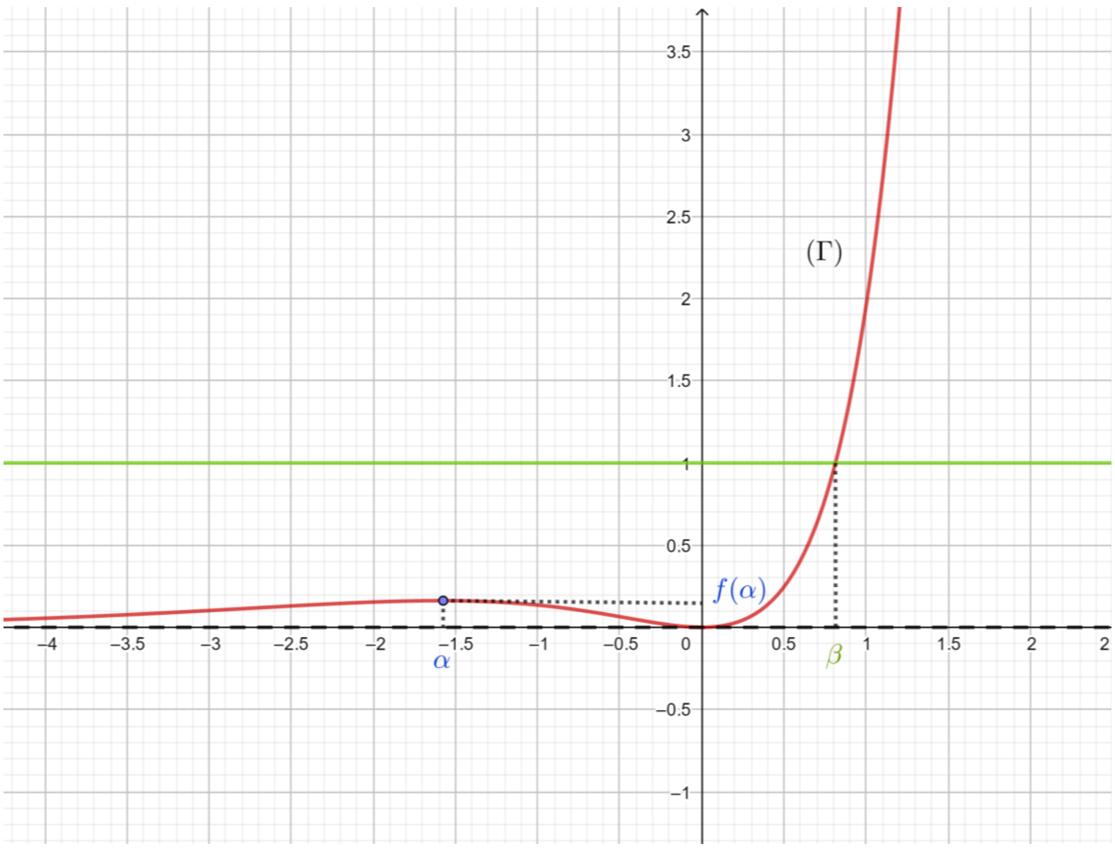
but $g(\alpha) = 0$

$$2e^{\alpha} - \alpha - 2 = 0$$

$$e^{\alpha} = \frac{\alpha+2}{2}$$

Then $f(-1.55) = \frac{-(-1.55)^2 - 2\ (-1.55)}{4} \cong 0.17$

4)



5) For $x \in]0, +\infty[$, f is continuous and strictly increasing from 0 to $+\infty$, then straight line $y = 1$ intersects (C) at one point and $f(x) = 1$ admits one and only one root β

Now , $f(0.813) - 1 = -0.0042 < 0$ where (C) *below* this line

& $f(0.815) - 1 = 0.0034 > 0$ where (C) above this line

Then $0.813 < \beta < 0.815.$