I – (5 points)

For each of the following statements, tell, with justification, whether it is true or false:

1) Let g be the function defined by $g(x)=\dfrac{\ln(x+3)}{x-2}$.

Statement:
The domain of definition of g is $]-3\ ;\ +\infty[$.

2) Consider the equation $(E):(\ln x-1)(2e^{x}-1)=0$.

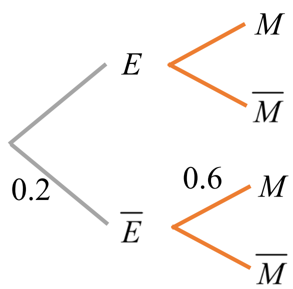
Statement:
The equation (E) admits in $\mathbb{R}$ two distinct solutions.

3) Consider the function h defined over $]0\ ;+\infty[$ by $h(x)=\dfrac{x+\ln(x)}{x}-1$.

Denote by (H) its representative curve in an orthonormal system $(O\ ;\ \vec{i}\ ;\vec{j})$.

Statement:
The curve (H) admits the $x-$axis as a horizontal asymptote.

4) E and M are two events of a sample space Ω of a random experiment.
The situation is modeled by the adjacent weighted probability tree.
Given that $P(M)=0.4$.



Statement:
The probability of the event $M\cap E$ is $P(M\cap E)=0.28$.

5) Consider the function f defined over $]0\ ;\ +\infty[$ by $f(x)=x(\ln x)^{2}+1$

and the expression $A=f'(e^{-1})+\ln(7-4\sqrt{3})+\ln(7+4\sqrt{3})$.

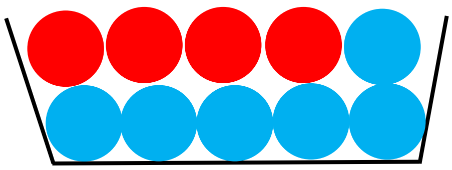
Statement:
The value of the expression A is -1.

II – (5 points)

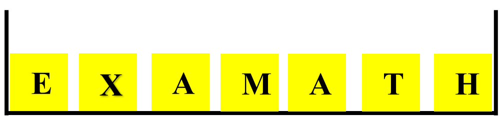
When Probability falls in Love with ExaMath Groups:

U and V are two urns such that:

- U contains 10 balls: four red balls and six blue balls.
- V contains 7 cards holding the letters E, X, A, M, A, T, and H.



U



V

Part A

Two balls are selected successively with replacement from urn U.
Consider the following events:
A : « The two selected balls have the same color ».
B : « The two selected balls have different colors ».

1) Calculate the number of possible outcomes.

2) Show that the probability $P(A)=\dfrac{13}{25}$ and deduce $P(B)$.

<u>**Part B**</u>

In this part, a card is selected from urn V:

- If the card shows a vowel letter, then two balls are selected randomly and simultaneously from U.
- If the card doesn't show vowel letter, then two balls are selected randomly and successively without replacement from U.

Consider the following events:

E : « The selected card shows a vowel letter ».

F : « The selected balls from U have the same color ».

1) Calculate $P(F/E)$ and verify that $P(F\cap E)=\frac{1}{5}$.

2) Verify that $P(F\cap \overline{E})=\frac{4}{15}$ and deduce $P(F)$.

3) Knowing that that the selected balls from U have the same color, calculate the probability that the selected card doesn't show a vowel letter.

III – (10 points)

Consider the function f defined over $\mathbb{R}$ by $f(x)=x+xe^{-x}$ and let (C) be its representative curve in an orthonormal system $(O\ ;\ \vec{i}\ ;\ \vec{j})$ **(unit: 2 cm).**

1) **a)** Calculate $\lim\limits_{x\to+\infty} f(x)$ and show that the line (d) of equation $y=x$ is an asymptote to (C).

b) Study the relative position of (C) and (d).

c) Calculate $\lim\limits_{x\to-\infty} f(x)$ and give the value of $f(-1)$ to the nearest 10^{-2}.

2) The following table represents the table of variations of the function f' the derivative of f.

x	$-\infty$		1		2		$+\infty$
$f''(x)$				$-$	0	$+$	
$f'(x)$	$+\infty$	$\searrow$	1	$\searrow$	$1-e^{-2}$	$\nearrow$	1

a) Use the above table to prove that f is strictly increasing over $\mathbb{R}$.

b) Set up the table of variations of f.

c) Show that the curve (C) has an inflection point whose coordinates are to be determined.

3) Determine the coordinates of the point A on the curve (C) at which the tangent (T) is parallel to the line (d).

4) Show that the equation $f(x)=1$ has a unique root α and verify that $0.65<\alpha<0.66$.

5) Draw (d), (T) and (C).

6) Consider the function h defined by $h(x)=\dfrac{1}{\ln[f(x)]}$.

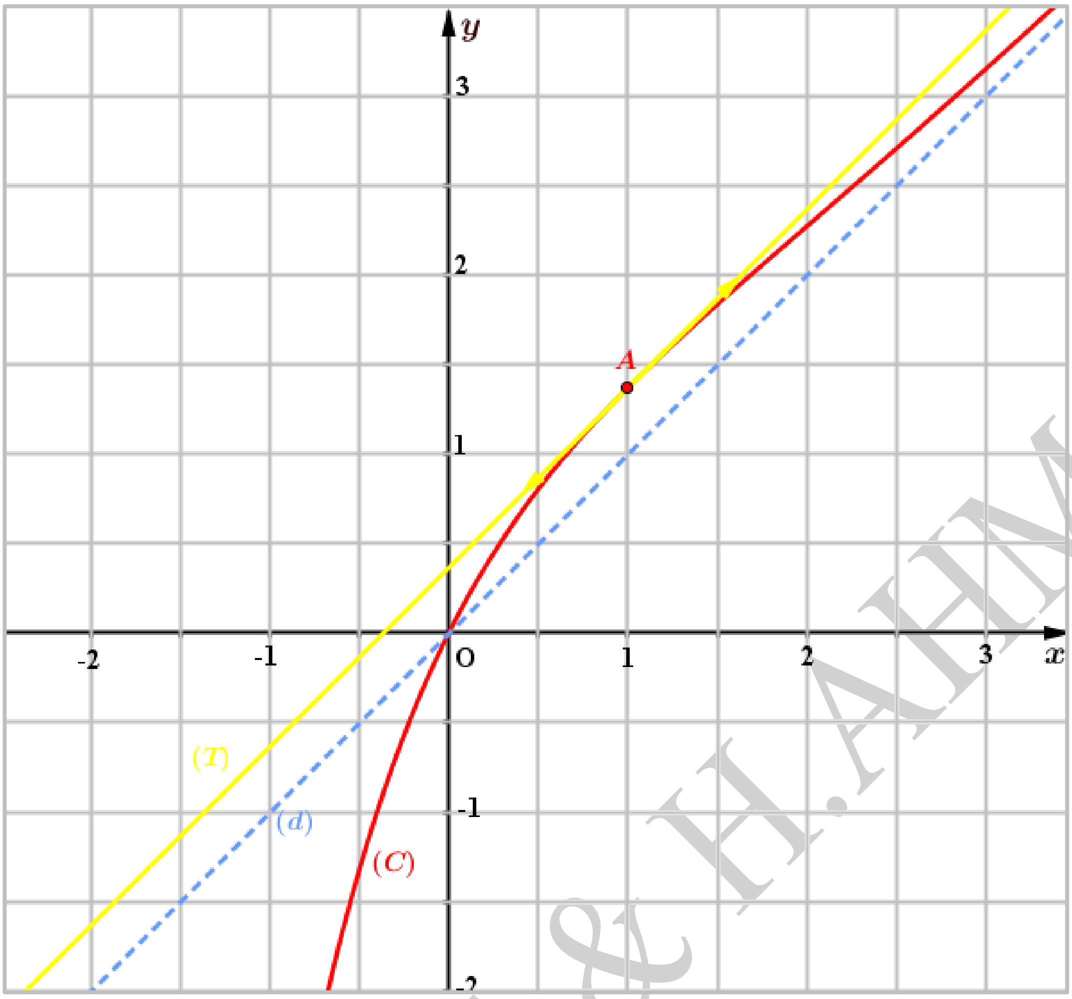
a) Prove that the domain of definition of h is $D_h=]0\ ;\ \alpha[\cup]\alpha\ ;\ +\infty[$.

b) Determine the number of solutions of the equation $h(x)=1$.

QI	Answers	5 pts.
1)	g is defined for $\begin{cases} x+3>0 \\ x-2\neq 0 \end{cases}$; $\begin{cases} x>-3 \\ x\neq 2 \end{cases}$. Then the domain of definition of g is $]-3;2[\cup]2;+\infty[$. Thus the statement is ✗.	1
2)	Condition: $x>0$. $(\ln x-1)(2e^x-1)=0$; so $(\ln x-1)=0; \ln x=1; x=e$ (accepted), or $(2e^x-1)=0$; $e^x=\frac{1}{2}$; $x=\ln\left(\frac{1}{2}\right)=-\ln 2$ (rejected). Thus (E) admits only one solution. Thus the statement is ✗.	1
3)	$\lim\limits_{x\to+\infty} h(x)=\lim\limits_{x\to+\infty}\left[\frac{x+\ln(x)}{x}-1\right]=\lim\limits_{x\to+\infty}\left[1+\frac{\ln(x)}{x}-1\right]=\lim\limits_{x\to+\infty}\left[\frac{\ln(x)}{x}\right]=0$ (rule). Thus the $x-$axis is a horizontal asymptote to (H) at $(+\infty)$. Thus the statement is ✓.	1
4)	$P(M\cap E)=P(M)-P(M\cap\overline{E})=P(M)-P(M/\overline{E})\times P(\overline{E})=0.4-0.6\times 0.2=0.28$. Thus the statement is ✓.	1
5)	$f'(x)=(\ln x)^2+\frac{2x\ln x}{x}=(\ln x)^2+2\ln x$. $f'(e^{-1})=(\ln e^{-1})^2+2\ln e^{-1}=1-2=-1$; $A=f'(e^{-1})+\ln(7-4\sqrt{3})+\ln(7+4\sqrt{3})=-1+\ln\left[(7-4\sqrt{3})\times(7+4\sqrt{3})\right]$ $=-1+\ln(49-48)=-1+\ln 1=-1$. Thus the statement is ✓ .	1

QII	Answers	5 pts.
A1)	The number of possible outcomes is $10^2=100$.	½
A2)	$P(A)=\frac{4}{10}\times\frac{4}{10}+\frac{6}{10}\times\frac{6}{10}=\frac{13}{25}$. $P(B)=1-P(A)=1-\frac{13}{25}=\frac{12}{25}$.	½ ½
B1)	$P(F/E)=\frac{C_6^2+C_4^2}{C_{10}^2}=\frac{7}{15}$. $P(F\cap E)=P(F/E)\times P(E)=\frac{7}{15}\times\frac{3}{7}=\frac{1}{5}$.	½ ½
B2)	$P(F\cap\overline{E})=P(F/\overline{E})\times P(\overline{E})=\left[\frac{4}{10}\times\frac{3}{9}+\frac{6}{10}\times\frac{5}{9}\right]\times\frac{4}{7}=\frac{4}{15}$. $P(F)=P(F\cap E)+P(F\cap\overline{E})=\frac{1}{5}+\frac{4}{15}=\frac{7}{15}$.	1 ½
B3)	$P(\overline{E}/F)=\frac{P(\overline{E}\cap F)}{P(F)}=\frac{\frac{4}{15}}{\frac{7}{15}}=\frac{4}{7}$.	1

QIII	Answers	10 pts.
1) a)	$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\left(x+xe^{-x}\right)=+\infty+0=+\infty$ since $\lim\limits_{x\to+\infty}\left(xe^{-x}\right)=\lim\limits_{x\to+\infty}\dfrac{x}{e^{x}}\overset{HR}{=}\lim\limits_{x\to+\infty}\dfrac{1}{e^{x}}=\dfrac{1}{+\infty}=0$. $\lim\limits_{x\to+\infty}\left[f(x)-y_{(d)}\right]=\lim\limits_{x\to+\infty}\left(xe^{-x}\right)=0$ (proved), then the line (d) of equation $y=x$ is an asymptote to (C) at $+\infty$.	1
1) b)	$f(x)-y_{(d)}=xe^{-x}$ has same sign of x since $e^{-x}>0$ for every $x\in\mathbb{R}$; • $f(x)-y_{(d)}>0$ if $x>0$ then (C) is above (d) if $x\in\left]0\ ;\ +\infty\right[$. • $f(x)-y_{(d)}<0$ if $x<0$ then (C) is below (d) if $x\in\left]-\infty\ ;\ 0\right[$. • $f(x)-y_{(d)}=0$ if $x=0$ then (C) cuts (d) at the point of coordinates $(0\ ;\ 0)$.	¾
1) c)	$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\left(x+xe^{-x}\right)=-\infty+(-\infty)(+\infty)=-\infty-\infty=-\infty$. $f(-1)\approx-3.72$ rounded to the nearest 10^{-2}.	¾
2) a)	Using the table of variations of f', f' admits an absolute minimum of value $1-e^{-2}$ over $\mathbb{R}$, then $f'(x)\geq 1-e^{-2}>0$ for every $x\in\mathbb{R}$, then the function f is strictly increasing over $\mathbb{R}$.	½
2) b)	**Table of variations of f:** $\begin{array}{c\|ccc} x & -\infty & & +\infty \\ \hline f'(x) & & + & \\ \hline f(x) & -\infty & \nearrow & +\infty \end{array}$	1
2) c)	Using the table of variations of f', $f''(x)=0$ when $x=2$ and changes its sign then the curve (C) of f admits an inflection point of coordinates $(2\ ;f(2))$ then $\left(2\ ;\ 2+2e^{-2}\right)$.	½
3)	(T) is parallel to (d) then slope of (T) = slope of (d) then $f'(x_A)=1$, and using table of variations of f' we have $f'(1)=1$ then $x_A=1$ and $y_A=f(1)=1+e^{-1}$, then $A\left(1\ ;1+e^{-1}\right)$.	1
4)	Consider the function g defined over $\mathbb{R}$ by $g(x)=f(x)-1$. The equation $f(x)=1$ is equivalent to $g(x)=0$. • g is continuous over $\mathbb{R}$ since f is continuous over $\mathbb{R}$. • $g'(x)=f'(x)>0$ over $\mathbb{R}$, then g is strictly increasing over $\mathbb{R}$. • $\lim\limits_{x\to-\infty} g(x)=\lim\limits_{x\to-\infty}\left[f(x)-1\right]=-\infty$ and $\lim\limits_{x\to+\infty} g(x)=\lim\limits_{x\to+\infty}\left[f(x)-1\right]=+\infty$, then g changes its sign one time only over $\mathbb{R}$ then the equation $g(x)=0$ admits a unique solution α over $\mathbb{R}$, thus the equation $f(x)=1$ admits a unique solution α over $\mathbb{R}$. In addition, $g(0.65)\approx-0.01<0$ and $g(0.66)\approx 0.001>0$ then $0.65<\alpha<0.66$.	1

5)		1 ½
6) a)	h is defined if $\begin{cases} f(x) > 0 \\ \ln[f(x)] \neq 0 \end{cases}$. • $f(x) > 0$ if $x \in]0\ ;\ +\infty[$; • $\ln[f(x)] \neq 0$; $f(x) \neq 1$; $x \neq \alpha$ then $x \in]-\infty\ ;\ \alpha[\cup]\alpha\ ;\ +\infty[$. Then $D_h = (]0\ ;\ +\infty[) \cap (]-\infty\ ;\ \alpha[\cup]\alpha\ ;\ +\infty[) =]0\ ;\ \alpha[\cup]\alpha\ ;\ +\infty[$.	1
6) b)	$h(x) = 1$; $\dfrac{1}{\ln[f(x)]} = 1$; $\ln[f(x)] = 1$; $f(x) = e$. Graphically, the curve (C) of f cuts the line of equation $y = e$ at a single point then the equation $h(x) = 1$ admits over D_h one solution only.	1