Mastering Mathematics

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For Teaching
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2nd Edition

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Preface

Dear colleagues , Dear students.

The new national curricula have encouraged us to conduct a work that seeks to provide effective assistance to students and colleagues alike. Our main objective is to help students think rigorously, coherently and critically.
From this perspective, we have followed a systematic recourse that involves:

- Incessant course review;
- Typical detailed solved problems;
- Supplementary problems with indications.
- Sample Tests.

We were keen to ensure that every problem has a well-defined importance and a special reach. It is imperative, therefore, to solve all these problems.

We advise students to solve the problems by themselves before referring to their solutions. The simple reading of the solution is not sufficient to achieve mathematical reasoning. Thus, students are advised to reflect on the problems with their simple indications.

We hope that this work will assure success to our students in the official exams and aid those who have the desire to develop their mathematical knowledge and skills outside class.

We welcome any comments and suggestions that would enhance this work.

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CHAPTER 1

Continuous and Inverse Functions

Chapter Review

1) Continuous Function over an Interval.

- If f is a continuous and strictly monotonic function over an interval I then, $f(I)$, the set of images of I by f, is an interval of the same nature.
- If f is a continuous function over the closed interval $I = [a\,;b]$, then:
  - * f is bounded, there exist two real numbers m and M such that: $m \leq f(x) \leq M$.
  - * f attains its bounds, there exist two real numbers α and β such that: $m = f(\alpha)$ and $M = f(\beta)$.
  - * f takes all the values between m and M.
    If $\lambda \in [m\,;M]$, then there exists at least one real number $c \in [a\,;b]$ such that $c = f(\lambda)$.

Ex:

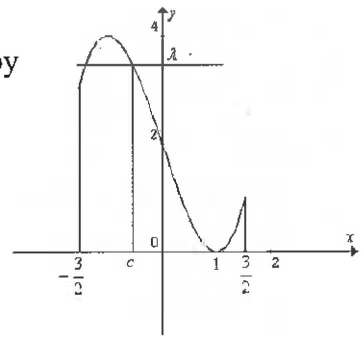
The function f defined over $\left[-\dfrac{3}{2};\dfrac{3}{2}\right]$ by

$f(x) = x^3 - 3x + 2$, verifies:

$0 \leq f(x) \leq 4$

$0 = f(1)$ and $4 = f(-1)$.

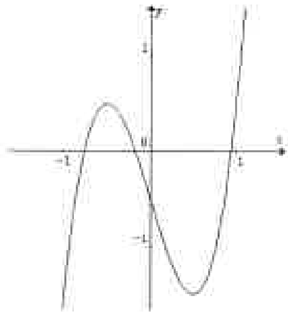
If $\lambda \in [0;4]$, then $\lambda = f(c)$.



- If f is a continuous function over a closed interval $I = [a\,;b]$ and if $f(a) \times f(b) \leq 0$, then the equation $f(x) = 0$ has at least one root in the interval $I = [a\,;b]$.

  Ex :

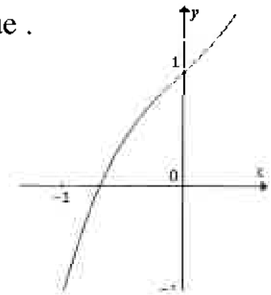
  The function f defined over $[-1\,;1]$ by $f(x) = 4x^3 - 3x - \dfrac{1}{2}$, admits at least one root in the interval $[-1\,;1]$.



  If f is a continuous and strictly monotonic function over the interval $I = [a\,;b]$, then this root is unique.

  Ex :

  The function f defined over $[-2\,;1]$ by $f(x) = x^3 + x + 1$, admits only one root in the interval $[-2\,;1]$.



2) Inverse Functions:

- If f is a continuous and strictly monotonic function over an interval I, then it admits an inverse function f^{-1} defined and continuous over $f(I)$.
- If (C) and (C') are the representative curves of f and f^{-1} in an orthonormal system, then (C) and (C') are symmetric to each other with respect to the first bisector of equation $y = x$.
- $\left(f^{-1}\right)'(y) = \dfrac{1}{f'(x)}$.

  Ex:

  The function f defined over $[0\,;+\infty[$ by $f(x) = x^2 - 1$ admits an Inverse function f^{-1} defined over $[-1\,;+\infty[$ by $f(x) = \sqrt{x+1}$.

  $\left(f^{-1}\right)'(0) = \dfrac{1}{f'(1)} = \dfrac{1}{2}$.

3) Derivatives.

- f admits a point of inflection at $x = a$, if $f''(a) = 0$ and if $f''(x)$ vanishes and changes its sign at a.

  Ex :

  The function f defined over IR by $f(x) = x^3 - 1$ admits the point $A(0;-1)$ as a point of inflection .

  The function f defined over IR by $f(x) = x^4$, has no point of inflection since $f''(x) = 12x^2$ vanishes at 0, but it does not change sign .

- L'Hopital's Rule :

  $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$, if $f(a) = g(a) = 0$ and $f'(x) \neq 0$ for all $x \in IR$.

Solved Problems

N° 1.

Consider the function f defined over IR by $f(x) = x^3 + 2x - 1$.

1) Study the variations of f and draw its table of variations.
2) Show that the equation $f(x) = 0$ admits a unique root α and verify that $0.4 < \alpha < 0.5$

N° 2.

Consider the function f defined over $\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ by $f(x) = \tan x - x + 1$ .Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f.
2) Show that, the equation $\tan x = x - 1$ admits only one solution between $-\frac{\pi}{2}$ and $-\frac{\pi}{3}$.
3) Write an equation of the tangent (T) to (C) at the point of abscissa $\frac{\pi}{4}$.

N° 3.

Show that the equation $\sin x = 2x - 2$ admits only one solution α between $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

N° 4.

1) Let f be a continuous function over $[0;1]$ whose image is $[0;1]$.
   Show that, the equation $f(x) = x$ admits at least one root in $[0;1]$.

2) Let g be the function defined over $[0;1]$ by $g(x) = \sin\left(\frac{\pi}{1+x}\right)$.

Show that $0 \leq g(x) \leq 1$ and deduce that the equation $g(x) = x$ admits at least one root in $[0;1]$.

N°5.

Show that the equation $x - \cos x = 0$ admits only one root α in the interval $\left[0, \dfrac{\pi}{2}\right]$.

N°6.

Consider the function f defined over IR by $f(x) = 2x^3 + 5x - 4$.
Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and set up its table of variations .
2) Show that f admits an inflection point to be determined .
3) Draw (C).
4) Show that the equation $f(x) = 1$ admits only one solution $\alpha \in]0;1[$.

N°7.

Consider the function f defined over $I = \left[0;\dfrac{\pi}{2}\right]$ by $f(x) = x + \sin x$.

(C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $f(I)$.
2) a- Show that f admits an inverse function g over $I = \left[0;\dfrac{\pi}{2}\right]$.

   b- Determine the domain of definition of g.
3) Consider the point $A'\left(\dfrac{2\sqrt{2}+\pi}{4};\dfrac{\pi}{4}\right)$.

   a- Show that A' belongs to the curve (Γ) representative of g.

   b- Write an equation of the tangent (T') at A' to (Γ)

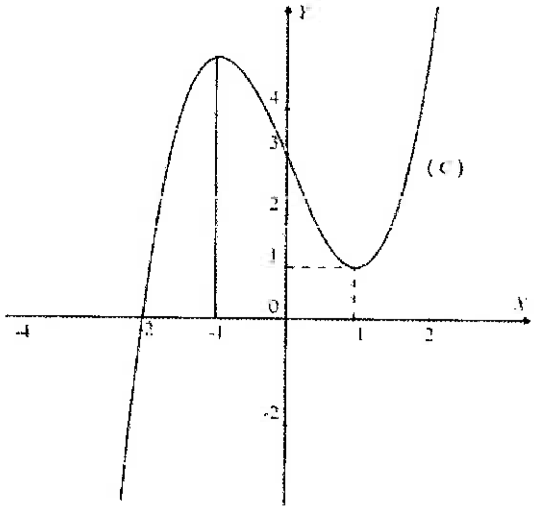
N°8.

Let f be the function defined over IR by $f(x) = x^4 + 2x^2 + 2$.
(C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that f admits an inverse function f^{-1} for $x \in [0;+\infty[$.
2) a- Write an equation of the tangent (T) to (C) at point $A(1;5)$.
   b- Deduce an equation of the tangent (T') to (C') at $B(5;1)$.
3) Determine $f^{-1}(x)$.

N°9

The curve below is that of a function f defined over IR by:
$f(x) = ax^3 + bx^2 + cx + d$, where a , b , c and d are real numbers.



1) Determine a , b , c and d .
2) If $I = [-1;1]$, determine $J = f(I)$.
3) Give three intervals over which f admits an inverse function to be determined, as well as their domains and their image sets.
4) Designate by g the inverse function of f for $x \in [1;+\infty[$.
   a- Designate by (C') the curve of g in the same figure of f. (C) and (C') intersect in two points $A(1;1)$ *and* B, determine the abscissa of B .
   b- Calculate the slope of the tangent at B to (C) and the slope of the tangent to (C') at B.

N° 10.

Consider the function f defined over $]0;1]$ by $f(x)=\sqrt{\frac{1-x}{x}}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and set up its table of variations .
2) Show that f admits an inverse function f^{-1} over $]0;1]$ and determine $f^{-1}(x)$.
3) Calculate $f\left(\frac{1}{2}\right)$ and draw the curves (C) and (C') ; representative of f^{-1}.
4) Let (δ) be the representative curve of the function g defined over $[0;+\infty[$ by: $g(x)=x^3+x-1$.
   a- Show that the equation $g(x)=0$ admits only one solution α such that $\alpha\in]0;1[$.
   b- Show that (C) and (C') intersect at a point A of abscissa α

N° 11.

Let f be the function defined over $[1;+\infty[$ by $f(x)=x^2-2x-4$.
Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that f admits an inverse function f^{-1}.
2) Trace in the system $(O;\vec{i},\vec{j})$, the curve (C) and deduce the curve (C') representative of f^{-1}.
3) (C) and (C') intersect at a point A, find the coordinates of A.
4) Calculate the slope of the tangent at A to (C) and deduce the slope of the tangent at A to (C') .
5) g is the function defined over $[-5;+\infty[$ by $g(x)=1+\sqrt{x+5}$.
   Calculate $g\circ f(x)$ and deduce the expression of f^{-1}.

N° 12.

Let f be the function defined over IR by $f(x)=\dfrac{x^2+2}{x^2+1}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and set up its table of variations.
2) Show that f admits an inverse function f^{-1} for $x\in[0;+\infty[$. Determine $f^{-1}(x)$.
3) Calculate the derivative of f^{-1} at the point of abscissa $\dfrac{3}{2}$.

N° 13.

Consider the function f defined over $[0;+\infty[$ by $f(x)=x+2\sqrt{x}$.

1) Show that f admits over $[0;+\infty[$ an inverse function f^{-1} and determine the domain of definition of f^{-1}.
2) Consider the function defined over $[0;+\infty[$ by $g(x)=\left(\sqrt{x+1}-1\right)^2$. Show that $g=f^{-1}$.

N° 14.

Consider the function f defined over IR by $f(x)=x^3-6x^2+9x-2$. Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

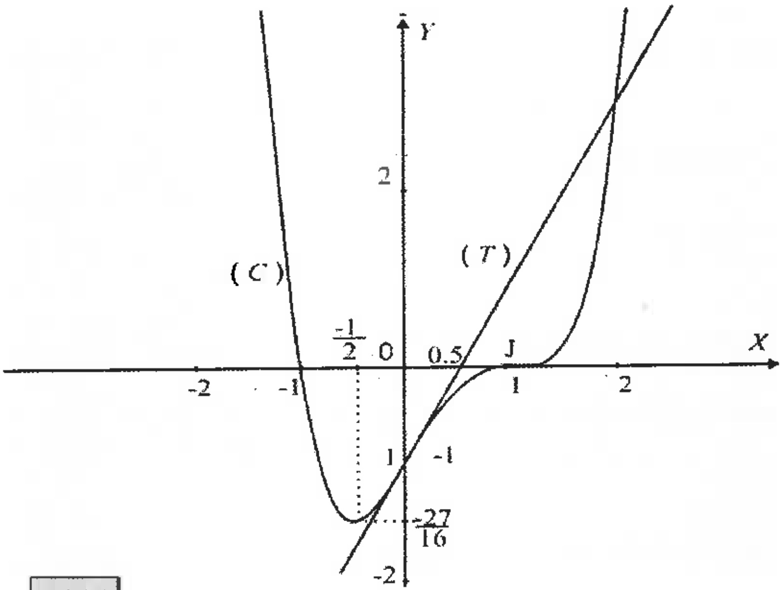
1) Show that f admits an inflection point I to be determined.
2) Show that I is a center of symmetry of (C).
3) Given the intervals $I=[1;3]$ and $J=[0;3]$, determine $f(I)$ and $f(J)$.

N° 15.

The curve (C) below is that of a function f defined over IR.

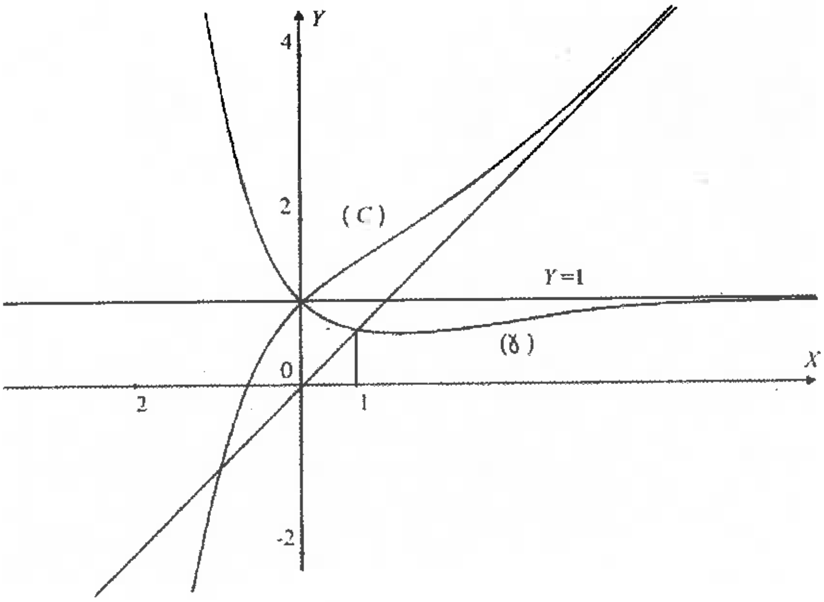
1) Find the inflection points of f.
2) Write an equation of the tangent (T) at point I to (C).
3) Determine $(f)'\left(-\dfrac{1}{2}\right)$ and $f'(1)$.
4) Solve the inequality $f(x)\leq 0$.

5) Set up the table of variations of f.
6) Set up the table of variations of f'.



N° 16.

In the figure below, the curves (C) and (γ) are the representative curves, in an orthonormal system $(O;\vec{i},\vec{j})$, of the functions f and its derivative f'.



1) Show that (γ) is the representative curve of f'.
2) Justify, why the function f admits an inflection point I .
3) a- Show that f admits an inverse function g.
   b- Calculate $g'(1)$.

N° 17.

Let f be the function defined over $IR - \{-1\}$ by $f(x) = \dfrac{x-1}{x+1}$.

Define $f \circ f$ and f^4 .

N° 18.

Let f be the function defined over $[1\,;+\infty[$ by $f(x) = \sqrt{x-1}$

and g the function defined over $IR - \{2\}$ by $g(x) = \dfrac{x+1}{x-2}$.

Define $g \circ f$.

N° 19.

Show that, for $x \in \left]0\,;+\infty\right[$ we have: $x - \dfrac{x^3}{6} < \sin x < x$.

Supplementary Problems

N° 1.

Let f be the function defined over IR by $f(x)=4x^3-3x+2$.

1) Show that f admits an inverse function over for $\left[\frac{1}{2};+\infty\right[$.

2) Let $I=\left[-\frac{3}{2};-\frac{1}{2}\right]$, determine $J=f(I)$.

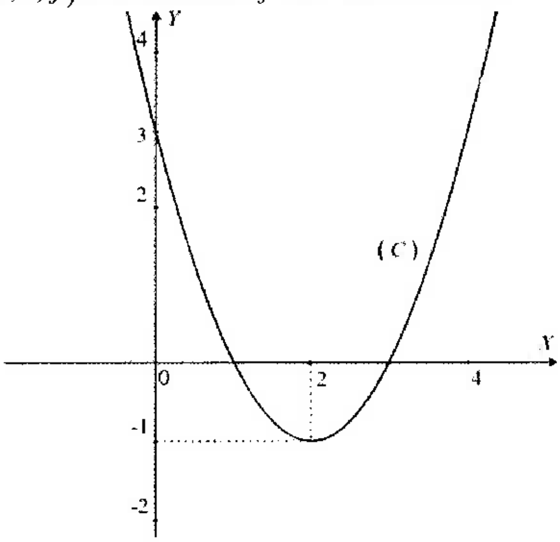
3) Show that the equation $f(x)=0$ admits only one solution in I.

4) Let $\lambda\in[0;1]$.

   Discuss, according to the values of λ, the existence of the roots of the equation $f(x)=\lambda$.

N° 2.

The curve below is the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ of a function f defined over IR.



1) Give two intervals over which f admits an inverse function to be determined. Find, also, the domains and the image sets of the inverse functions.
2) Suppose that $f(x)=ax^2+bx+c$. Find a, b and c.

3) Designate by g the inverse function of f for $x \in [2\,;+\infty[$.
   a- Trace the curve (C') representative of g in the same figure of f .
   b- (C) and (C') intersect at a point A, calculate the abscissa of A .
   c- Calculate the slope of the tangent at A to (C) and the slope of the tangent at A to (C') .
   d- Determine $g(x)$.

N° 3.

Consider the function f defined over IR by $f(x) = x^2 + 4x + 4$.
Designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1) Show that there exist two intervals I_1 and I_2 over which f admits an inverse function .
2) Designate by f_1 the inverse function of f in I_1 and by f_2 the inverse function of f in I_2.
   Determine $f_1(x)$ and $f_2(x)$.
3) Trace (C) and the curves (C_1) and (C_2) representative of f_1 and f_2 respectively .

N° 4.

Consider the function f defined over $IR - \{0\}$ by $f(x) = x - \dfrac{1}{x}$.

Designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1) Show that the straight line (d) of equation $y = x$ is an asymptote to (C).
2) Study the variations of f and trace (C) .
3) a- Show that f admits an inverse function f^{-1} for $x \in [1\,;+\infty[$.
   b- Determine the domain of definition of f^{-1}.
   c- Trace the curve (C') representative of f^{-1}.

d- Determine $f^{-1}(x)$.

4) Let (δ) be the straight line of equation $y=-x+m$, m is a real parameter. (δ) cuts (C) in a point A and cuts (C') in a point B. Determine m when $AB=\frac{\sqrt{2}}{2}$.

N° 5.

Consider the function f defined over IR by $f(x)=-x^3+6x^2-9x+1$ .Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and set up its table of variations .
2) Show that f admits an inflection point w to be determined .
3) Show that w is a center of symmetry of (C) .
4) Write an equation of the tangent (T) at w to (C) .
5) Trace (C) and (T) .
6) Show that the equation $f(x)=0$ admits three solutions .
7) Discuss ,graphically , according to the values of the real parameter m , the number of solutions of the equation $f(x)=m$.

N° 6.

Consider the function f defined over $I=[\,0\,;+\infty[$ by :

$f(x)=\frac{1}{1+\sqrt{x}}$. Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and determine $f(I)$.
2) Show that f has an inverse function f^{-1} over I and find $f^{-1}(x)$.
3) Trace the curve (C) and the curve (C') representative of f^{-1} .
4) Let (Γ) be the representative curve of the function g defined over $[\,0\,;+\infty[$ by $g(x)=x^3-x^2+2x-1$.

a- Show that g is continuous and strictly increasing over $[0;+\infty[$ and determine $g([0;+\infty[)$.

b- Show that the equation $g(x)=0$ admits only one solution $\alpha \in]0;1[$.

5) a- Show that (C) and (C') intersect in a point A of abscissa α

b- g^{-1} is the inverse function of g, calculate the derivative of g^{-1} at -1.

$\boxed{N°7.}$

Consider the function f defined over $IR-\{-1\}$ by $f(x)=\dfrac{x+2}{x+1}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and trace (C).

2) a- Show that f admits an inverse function f^{-1} for $x \in]-1;+\infty[$.

b- Determine the domain of definition of f^{-1}.

c- Determine $f^{-1}(x)$.

d- Compare $f'(2)$ and $(f^{-1})'\left(\dfrac{4}{3}\right)$.

3) Write an equation of the tangent (T) to (C) at point $A\left(2;\dfrac{4}{3}\right)$ and an equation of the tangent (T') to (C'), the representative curve of f^{-1}, at $A'\left(\dfrac{4}{3},2\right)$.

Solutions

N° 1.

1) $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} x^3\left(1+\dfrac{2}{x^2}-\dfrac{1}{x^3}\right) = +\infty$.

$\lim\limits_{x \to -\infty} f(x) = \lim\limits_{x \to -\infty} x^3\left(1+\dfrac{2}{x^2}-\dfrac{1}{x^3}\right) = -\infty$.

$f'(x) = 3x^2 + 2 > 0$, for all x is IR, therefore the table of variations of f is as follows:

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	$-\infty$	$\nearrow$	$+\infty$

2) The function f is continuous and strictly increasing from $-\infty$ to $+\infty$, so, $f(x)$ takes the value 0 only once, then $f(x)=0$ has only one root α.
Note that $f(0.4) \approx -0.136$ and $f(0.5) \approx 0.125$, which gives $f(0.4)\times f(0.5) < 0$ then $0.4 < \alpha < 0.5$

N° 2.

1) $f'(x) = 1+\tan^2 x - 1 = \tan^2 x \geq 0$ for any $x \in \left]-\dfrac{\pi}{2},\dfrac{\pi}{2}\right[$.

So, f is strictly increasing for $x \in \left]-\dfrac{\pi}{2},\dfrac{\pi}{2}\right[$.

2) $\lim\limits_{x \to -\frac{\pi}{2}} f(x) = -\infty$ and $\lim\limits_{x \to +\frac{\pi}{2}} f(x) = +\infty$.

The function f is continuous and strictly increasing for $x \in \left]-\dfrac{\pi}{2},\dfrac{\pi}{2}\right[$ and since

$f\left(-\dfrac{\pi}{3}\right) = \tan\left(-\dfrac{\pi}{3}\right) - \left(-\dfrac{\pi}{3}\right) + 1 = -\sqrt{3} + \dfrac{\pi}{3} + 1 \approx 0.31 > 0$

and $\lim\limits_{x \to -\frac{\pi}{2}} f(\alpha) = -\infty$; so, $f(x)$ changes sign from negative to

positive as x varies from $-\infty$ to $-\frac{\pi}{3}$, hence $f(x) = 0$ admits only

one solution α between $-\frac{\pi}{2}$ and $-\frac{\pi}{3}$. Which means that

$\tan\alpha - \alpha + 1 = 0$ or $\tan x = x - 1$ has only one root α

in $\left]-\frac{\pi}{2}; -\frac{\pi}{3}\right[$.

3) $f\left(\frac{\pi}{4}\right) = \tan\frac{\pi}{4} - \frac{\pi}{4} + 1 = 2 - \frac{\pi}{4}$ and $f'\left(\frac{\pi}{4}\right) = \tan^2\frac{\pi}{4} = 1$.

Therefore ,an equation of the tangent (T) is $y = x + 2 - \frac{\pi}{2}$

N°3.

Consider the function f defined over IR by $f(x) = \sin x - 2x + 2$.
$f'(x) = \cos x - 2 < 0$ for all x , since $\cos x \leq 1$.
Then f is a continuous, strictly decreasing function for all x .
$f\left(\frac{\pi}{3}\right) \approx 0.7$ and $f\left(\frac{\pi}{2}\right) \approx -0.14$, then $f\left(\frac{\pi}{3}\right) \times f\left(\frac{\pi}{2}\right) < 0$, thus

$f(x) = 0$ has only one root $\alpha \in \left]\frac{\pi}{3}; \frac{\pi}{2}\right[$, consequently , the equation

$\sin x = 2x - 2$ admits only one root $\alpha \in \left]\frac{\pi}{3}; \frac{\pi}{2}\right[$.

N°4.

1) Let φ be a function defined over $[0;1]$ by $\varphi(x) = f(x) - x$.
   The function φ is continuous since it is the difference of two continuous functions .
   But $f(x) \in [0, 1]$ gives $0 \leq f(x) \leq 1$,then :
   $\varphi(1) = f(1) - 1 \leq 0$ and $\varphi(0) = f(0) - 0 \geq 0$, which gives $\varphi(1) \times \varphi(0) \leq 0$.
   So, $\varphi(x) = 0$ has at least one root in $[0;1]$.
   Therefore the equation $f(x) = x$ admits at least one root in $[0;1]$.
2) The function g is defined and continuous over $[0;1]$.

$0 \le x \le 1$ gives $1 \le x+1 \le 2$, so $\frac{1}{2} \le \frac{1}{x+1} \le 1$, consequently $\frac{\pi}{2} \le \frac{\pi}{x+1} \le \pi$. Then $0 \le g(x) = \sin\left(\frac{\pi}{1+x}\right) \le 1$.

From part 1) , the equation $g(x) = x$ admits at least one root in $[0;1]$.

N° 5.

Consider the function f defined over $\left[0;\frac{\pi}{2}\right]$ by $f(x) = x - \cos x$.

$f'(x) = 1 + \sin x \ge 0$ since $\sin x \ge 0$ for $x \in \left[0;\frac{\pi}{2}\right]$.

Then f is continuous and strictly increasing for $x \in \left[0;\frac{\pi}{2}\right]$.

Since $f(0) = 0 - \cos 0 = -1 < 0$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} - \cos\frac{\pi}{2} = \frac{\pi}{2} > 0$,

Then $f(0) \times f\left(\frac{\pi}{2}\right) < 0$. Finally, $x - \cos x = 0$ has only

one root $\alpha \in \left[0;\frac{\pi}{2}\right]$.

N° 6.

1) $f'(x) = 6x^2 + 5 > 0$ for any x in IR .

$$\lim_{x \to +\infty} f(x) = \lim_{x \to +\infty} x^3\left(2 + \frac{5}{x^2} - \frac{4}{x^3}\right) = +\infty .$$

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} x^3\left(2 + \frac{5}{x^2} - \frac{4}{x^3}\right) = -\infty .$$

Therefore the table of variations of f is as follows:

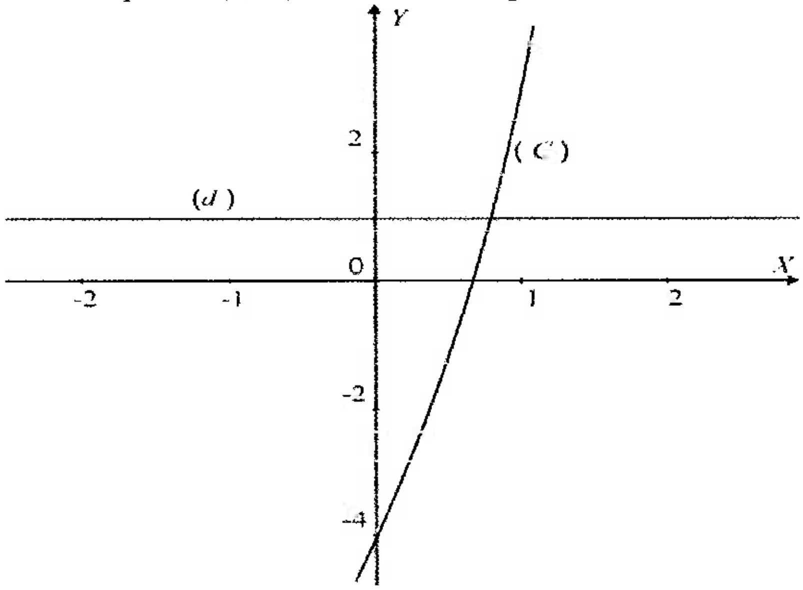
x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	$-\infty$	$\nearrow$	$+\infty$

2) $f''(x) = 12x$, then : $f''(x) = 0$ for $x = 0$.

$f''(x) > 0$ for $x > 0$, $f''(x) < 0$ for $x < 0$.

Then the point $I(0;-4)$ is an inflection point of f .

3)



4) Since f is continuous and strictly increasing from $-\infty$ to $+\infty$ then the representative curve (C) of f cuts $y = 1$ in exactly one point .

$f(0) = -4$, $f(\alpha) = 1$ and $f(1) = 3$, then $f(0) < f(\alpha) < f(1)$.

Since f is strictly increasing , we get $0 < \alpha < 1$.

$\boxed{N°7.}$

1) $f'(x) = \cos x + 1 > 0$ for any x in $\left[0;\frac{\pi}{2}\right]$, then f is continuous and strictly increasing over $\left[0;\frac{\pi}{2}\right]$, then :

$$f\left(\left[0;\frac{\pi}{2}\right]\right) = \left[f(0);f\left(\frac{\pi}{2}\right)\right] = \left[0;1+\frac{\pi}{2}\right].$$

2) a- Since f is continuous and strictly increasing over $\left[0;\frac{\pi}{2}\right]$ then f admits an inverse function g .

b- The domain of definition of g *is* $f\left(\left[0;\frac{\pi}{2}\right]\right) = \left[0;1+\frac{\pi}{2}\right]$.

3) a- $A'\left(\dfrac{2\sqrt{2}+\pi}{4};\dfrac{\pi}{4}\right)$ belongs to (Γ) if and only if point

$A\left(\dfrac{\pi}{4};\dfrac{2\sqrt{2}+\pi}{4}\right)$ belongs to (C).

Since $f\left(\dfrac{\pi}{4}\right)=\dfrac{\pi}{4}+\dfrac{\sqrt{2}}{2}=\dfrac{2\sqrt{2}+\pi}{4}$, then A belongs to (C).

Therefore A' belongs to (Γ).

b- An equation of the tangent at A' to (Γ) is

$y-y_{A'}=g'(x_{A'})(x-x_{A'})$.

But $g'\left(\dfrac{2\sqrt{2}+\pi}{4}\right)=\dfrac{1}{f'\left(\dfrac{\pi}{4}\right)}=2-\sqrt{2}$, consequently an

equation of (T') is $y=\left(2-\sqrt{2}\right)x-\dfrac{\left(\sqrt{2}-1\right)(4-\pi)}{4}$

N°8.

1) $f'(x)=4x^3+4x=4x\left(x^2+1\right)\geq 0$ for any $x\in[0;+\infty[$.

Then f is continuous and strictly increasing and it admits an inverse function f^{-1} over $[0;+\infty[$.

2) a- An equation of (T) is $y=f(1)+f'(1)(x-1)$.

Then $y=5+8(x-1)$ or $y=8x-3$.

b- B is the symmetric of A with respect to the straight line of equation $y=x$, then, (T') is symmetric of (T) with respect to the straight line of equation $y=x$.

Then an equation of (T') is $x=8y-3$, which is equivalent to

$y=\dfrac{1}{8}x+\dfrac{3}{8}$.

3) $y=x^4+2x^2+2=x^4+2x^2+1+1=(x^2+1)^2+1$ same as

$\left(x^2+1\right)^2=y-1$ which gives $x^2+1=\sqrt{y-1}$ and $x^2=-1+\sqrt{y-1}$

therefore $x=\sqrt{-1+\sqrt{y-1}}$ since $x\geq 0$, consequently

$f^{-1}(x)=\sqrt{-1+\sqrt{x-1}}$.

N° 9.

1) From the figure, we have $f(0)=3$; $f(1)=1$; $f'(1)=0$ and $f'(-1)=0$.

Since $f'(x)=3ax^2+2bx+c$ then we get the system:

$\begin{cases} d=3 \\ a+b+c+d=1 \\ 3a+2b+c=0 \\ 3a-2b+c=0 \end{cases}$ that has as a solution $a=1$; $b=0$; $c=-3$

and $d=3$, which gives $f(x)=x^3-3x+3$.

2) Over $I=[-1;1]$, f is continuous and strictly decreasing, then $f(I)=[f(1);f(-1)]=[1;5]=J$.

3) For $x\in]-\infty;-1]$, f is continuous and strictly increasing, it admits an inverse function f_1^{-1} defined over $]-\infty;5]$ whose image set is $]-\infty;-1]$.

For $x\in[-1;1]$, f is continuous and strictly decreasing, f admits an inverse function f_2^{-1} defined over $[1;5]$ whose image set is $[-1;1]$.

Finally, for $x\in[1;+\infty[$, f is also continuous and strictly increasing, it admits an inverse function f_3^{-1} defined over $[1;+\infty[$ whose image set is $[1;+\infty[$.

4) a- The common points between (C') and (C) are the common points between (C) and the line of equation $y=x$.

Then, $x^3-3x+3=x$ or $x^3-4x+3=0$, 1 is clearly a root of the last equation, consequently $(x-1)$ is a factor of $x^3-4x+3=0$ or: $x^3-4x+3=(x-1)(x^2+x-3)$

Solving: $x^2+x-3=0$ gives: $x=\dfrac{-1-\sqrt{13}}{2}<1$ rejected and $x=\dfrac{-1+\sqrt{13}}{2}>1$ accepted so, $B\left(\dfrac{-1+\sqrt{13}}{2};\dfrac{-1+\sqrt{13}}{2}\right)$.

b- The slope of the tangent to (C) at B is

$$f'(x_B) = 3x_B^{\,2} - 3 = \frac{30 - 6\sqrt{13}}{4}.$$

The slope of the tangent to (C') at B is $\dfrac{1}{f'(x_B)} = \dfrac{5+\sqrt{13}}{18}$.

N° 10.

1) $f'(x) = \dfrac{1}{2\sqrt{\dfrac{1-x}{x}}} \times \dfrac{-x-1+x}{x^2} = \dfrac{-1}{2x^2\sqrt{\dfrac{1-x}{x}}} < 0$ for any $x \in \left]0;1\right[$.

$\lim\limits_{x\to 0} f(x) = +\infty$, then (C) has $y'oy$ as a vertical asymptote.

Therefore the table of variations of f is as follows:

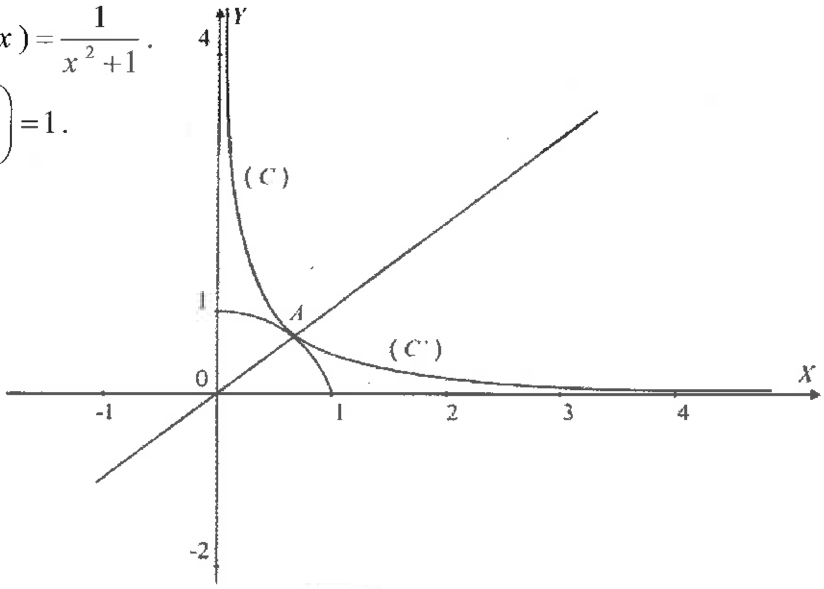
x	0		1
$f'(x)$	‖	$-$	
$f(x)$	‖ $+\infty$	$\searrow$	0

2) Since f is continuous and strictly decreasing over $\left]0;1\right]$, then f admits an inverse function f^{-1} defined over $\left[0;+\infty\right[$.

$y = \sqrt{\dfrac{1-x}{x}}$ then $y^2 = \dfrac{1-x}{x}$ and $x = \dfrac{1}{y^2+1}$, therefore

$f^{-1}(x) = \dfrac{1}{x^2+1}$.

3) $f\left(\dfrac{1}{2}\right) = 1$.



4) a- $g'(x) = 3x^2 + 1 > 0$ for all real x .

Since g is continuous and strictly increasing for $x \in]0;1[$ and $g(0) \times g(1) = (-1) \times (1) < 0$, then $g(x) = 0$ has only one root α such that $\alpha \in]0;1[$.

b- (C) and (C') are symmetric with respect to the first bisector of equation $y = x$.

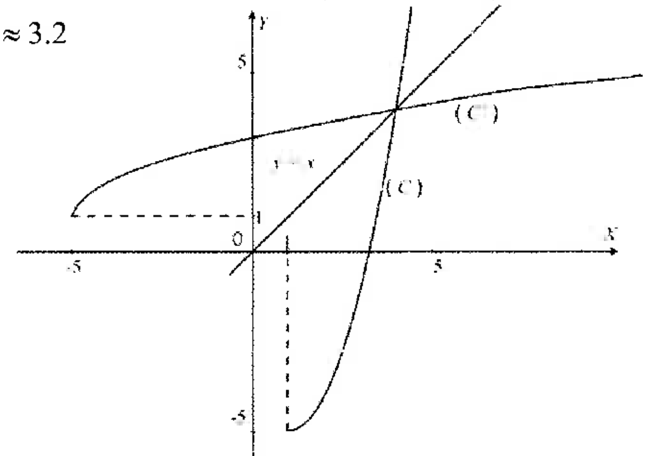
The abscissa of their common point A is a root of the equation $f(x) = x$, then $\sqrt{\frac{1-x}{x}} = x$ which gives $x^3 + x - 1 = 0$, which has only one root α over $]0;1[$, therefore (C) and (C') intersect at point A of abscissa α .

N° 11.

1) $f'(x) = 2x - 2 \geq 0$ then f is continuous and strictly increasing for $x \in [1;+\infty[$, so, f admits an inverse function f^{-1} .

The domain of definition of f^{-1} is $\left[f(1); \lim\limits_{x \to +\infty} f(x)\right[= [-5;+\infty[$.

2) f admits a minimum -5 and (C) cuts $x'x$ at point of abscissa $1+\sqrt{5} \approx 3.2$



3) The abscissa of the common point A of (C) and (C') is a root of the equation $f(x) = x$, then $x^2 - 2x - 4 = x$, which gives $x^2 - 3x - 4 = 0$, that has as solution $x' = -1 < 1$ rejected and $x'' = 4 > 1$ accepted , then $A(4;4)$.

4) The slope of the tangent at A to (C) is $f'(4) = 6$, then the slope

of the tangent at A to (C') is $\left(f^{-1}\right)'(4)=\dfrac{1}{f'(4)}=\dfrac{1}{6}$.

5) $g\circ f(x)=g\left(f(x)\right)=g\left(x^2-2x-4\right)$

$$=1+\sqrt{x^2-2x-4+5}=1+\sqrt{x^2-2x+1}$$

$$=1+\sqrt{(x-1)^2}=1+|x-1| .$$

Since $x\in[1;+\infty[$, then $x\geq 1$, so $|x-1|=x-1$, therefore $g\circ f(x)=1+x-1=x$.

Which means that g and f are inverse functions, therefore $f^{-1}(x)=g(x)=1+\sqrt{x+5}$.

N° 12.

1) $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\dfrac{x^2}{x^2}=1$, similarly $\lim\limits_{x\to+\infty} f(x)=1$.

Then the straight line (d) of equation $y=1$ is a horizontal asymptote to (C).

$$f'(x)=\frac{2x\left(x^2+1\right)-2x\left(x^2+2\right)}{\left(x^2+1\right)^2}=\frac{-2x}{\left(x^2+1\right)^2} .$$

Therefore the table of variations of f is as follows :

x	$-\infty$		0		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	1	$\nearrow$	2	$\searrow$	1

2) Over $[0,+\infty[$, the function f is continuous and strictly decreasing, so, f admits an inverse function f^{-1}.

The domain of definition of f^{-1} is $\left]\lim\limits_{x\to+\infty} f(x);f(0)\right]=]1;2]$.

To find $f^{-1}(x)$, means to express x in terms of y within the restricted conditions for the values of x and y .

$y=\dfrac{x^2+2}{x^2+1}$ implies $x^2y-x^2=2-y$ which gives $x^2=\dfrac{2-y}{y-1}$

But $x \geq 0$ then $x = \sqrt{\dfrac{2-y}{y-1}}$, finally $f^{-1}(x) = \sqrt{\dfrac{2-x}{x-1}}$.

3) We know that $\left(f^{-1}\right)'(y) = \dfrac{1}{f'(x)}$. But if $y = \dfrac{3}{2}$ we get

$\dfrac{3}{2} = \dfrac{x^2+2}{x^2+1}$ which gives $x = 1$. Then

$\left(f^{-1}\right)'\left(\dfrac{3}{2}\right) = \dfrac{1}{f'(1)} = \dfrac{1}{-\frac{1}{2}} = -2$.

N° 13.

1) The function f is continuous over $[\,0\,;+\infty\,[$ and since

$f'(x) = 1 + \dfrac{2}{2\sqrt{x}} > 0$ for $x \in]0;+\infty[$, then f is strictly increasing

consequently f admits an inverse function f^{-1} over $[\,0\,;+\infty\,[$.

$\lim\limits_{x \to +\infty} f(x) = +\infty$ and since $f(0) = 0$, then $D(f^{-1}) = [0\,;+\infty[$.

2) $y = x + 2\sqrt{x} = \left(\sqrt{x}\right)^2 + 2\sqrt{x} + 1 - 1 = \left(\sqrt{x}+1\right)^2 - 1$, then

$y + 1 = \left(\sqrt{x}+1\right)^2$ which gives $\sqrt{x} = -1 + \sqrt{y+1}$, then

$x = \left(-1+\sqrt{y+1}\right)^2$. Finally $f^{-1}(x) = \left(-1+\sqrt{x+1}\right)^2$.

Then $g = f^{-1}$ since they have same domain $[\,0\,;+\infty\,[$, and same range $[\,0\,;+\infty\,[$ and $g(x) = f^{-1}(x)$ for every $x \in [0;+\infty[$.

N° 14.

1) The function f is defined ,continuous and differentiable over IR.

$f'(x) = 3x^2 - 12x + 9$ and $f''(x) = 6x - 12$, therefore the table of signs of f'' is as follows :

x	$-\infty$		2		$+\infty$
$f''(x)$		$-$	0	$+$	

Since at $I\,(2\,;\,0)$; $f''(2) = 0$ and f''changes sign then $I\,(2\,;\,0)$ is an inflection point of f .

2) $f(4-x) = (4-x)^3 - 6(4-x)^2 + 9(4-x) - 2 = -x^3 + 6x^2 - 9x + 2$.

So $f(4-x)+f(x)=0$, therefore the point $I(2;0)$ is center of symmetry of (C).

3) The table of variations of f is as follows :

x	$-\infty$		1		3		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	2	$\searrow$	-2	$\nearrow$	$+\infty$

From the table of variations of f we have $f(I)=[-2;2]$.
Since $f(0)=-2$, then $f(J)=[-2;2]$.

N° 15.

1) Inflection points are points of the curve where concavity changes its direction. Using (C) we notice that :
Before I , (C) is concaved upwards , between I and J, (C) is concaved downwards, after J ,(C) is again concaved upwards .
Then $J(1;0)$ and $I(0;-1)$ are inflection points of f.

2) (T) is passing through $I(0;-1)$ and $E\left(\frac{1}{2};0\right)$, so its equation is $y=2x-1$.

3) At the point J, the tangent to (C) is parallel to $x'x$,then $f'(1)=0$.
For $x=-\frac{1}{2}$, (C) takes a minimum, then $f'\left(-\frac{1}{2}\right)=0$.

4) From the curve , $f(x)\leq 0$ for $x\in[-1;1]$.

5) The table of variations of f is as follows :

x	$-\infty$		$-\frac{1}{2}$		1		$+\infty$
$f'(x)$		$-$	0	$+$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$-\frac{27}{16}$	$\nearrow$	0	$\nearrow$	$+\infty$

6) The table of variations of f' is as follows :

x	$-\infty$		0		1		$+\infty$
$f''(x)$		$+$	0	$-$	0	$+$	
$f'(x)$		$\nearrow$	2	$\searrow$	0	$\nearrow$	

Notice that $f'(0) = 2$ is the slope of the tangent (T).

N°16.

1) The function that represents (C) is strictly increasing and the function that represents (γ) is positive since the curve (γ) is above $x'x$, we deduce that (γ) is the representative curve of f' and (C) is that of f.
2) Notice that the function f' has a minimum at point of abscissa 1 since for $x < 1$, f' is strictly decreasing and for $x > 1$ f' is strictly increasing , then $f''(x) < 0$ for $x < 1$ and $f''(x) > 0$ for $x > 1$.
   Then $f''(1) = 0$ and f'' changes sign at $x = 1$ then the point I of abscissa 1 is an inflection point of f.
3) a- f is continuous, strictly increasing over IR , then it admits an inverse function g .
   b- Notice that $f'(0) = 1$ since the curve (γ) passes through the point $(0;1)$, then $g'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{1} = 1$.

N°17.

f is defined over $IR - \{-1\}$ and of range $IR - \{1\}$ by $f(x) = \dfrac{x-1}{x+1}$.

$$(f \circ f)(x) = f(f(x)) = f\left(\frac{x-1}{x+1}\right) = \frac{\dfrac{x-1}{x+1} - 1}{\dfrac{x-1}{x+1} + 1} = \frac{-1}{x}.$$

$f \circ f$ is defined if $\dfrac{x-1}{x+1} \neq -1$ which gives: $x \neq 0$.

Then $f \circ f$ is defined over $IR - \{-1;0\}$ by $f \circ f(x) = \frac{-1}{x}$.

$f^4 = f^2 \circ f^2$ then $f^4(x) = f^2(f^2(x)) = f^2\left(-\frac{1}{x}\right) = \frac{-1}{-\frac{1}{x}} = x$,

so f^4 is defined if $\frac{-1}{x} \neq -1$, then $x \neq 1$, therefore f^4 is defined over $IR - \{-1;0;1\}$ by $f^4(x) = x$.

$\boxed{N^\circ 18.}$

$(g \circ f)(x) = g(f(x)) = g(\sqrt{x-1}) = \frac{\sqrt{x-1}+1}{\sqrt{x-1}-2}$.

$g \circ f$ is defined if $\sqrt{x-1} \neq 2$ and $x-1 \geq 0$,
which gives $x \neq 5$ and $x \geq 1$, then $g \circ f$ is defined over
$I = [1;+\infty[- \{5\}$ by: $g \circ f(x) = \frac{\sqrt{x-1}+1}{\sqrt{x-1}-2}$.

$\boxed{N^\circ 19.}$

.Consider the function f defined by $f(x) = x - \sin x$ for $x > 0$.
$f'(x) = 1 - \cos x \geq 0$ then f is increasing but $x > 0$,
then $f(x) > f(0)$, so $x - \sin x > 0$ then $\sin x < x$ (as required).

Similarly, take $g(x) = \sin x - x + \frac{x^3}{6}$.

$g'(x) = \cos x - 1 + \frac{x^2}{2}$ and $g''(x) = -\sin x + x > 0$,

which makes g' increasing over $]0;+\infty[$.

Then $x > 0$ gives $g'(x) > g'(0)$ but $g'(0) = 0$ then $g'(x) > 0$,
consequently g is increasing over $]0;+\infty[$.
Then $x > 0$ gives $g(x) > g(0)$, and since $g(0) = 0$ we get

$g(x) > 0$, then $g(x) = \sin x > x - \frac{x^3}{6}$ as required.

Indications

N° 2.

3) b- (C) and (C') intersect at point $A\left(\frac{5+\sqrt{13}}{2};\frac{5+\sqrt{13}}{2}\right)$.

d- $g(x)=2+\sqrt{x+1}$.

N° 4.

3) d- $f^{-1}(x)=\frac{1}{2}\left(x+\sqrt{x^2+4}\right)$, $D(f^{-1})=[0;+\infty[$.

4) $A(\alpha;\beta)\in(C)$ and $B(\beta;\alpha)\in(C')$, we have $\beta=\alpha-\frac{1}{\alpha}$ and $\alpha\geq 1$.

$AB^2=\frac{2}{\alpha^2}=\left(\frac{\sqrt{2}}{2}\right)^2$, which gives $\alpha=2$, then $A\left(2;\frac{3}{2}\right)$ and

$m=\alpha+\beta=\frac{7}{2}$.

N° 5.

2) $w(2;-1)$ is an inflection point of and center of symmetry of (C).

N° 6.

2) $f^{-1}(x)=\frac{(x-1)^2}{x^2}$.

5) b- $\left(g^{-1}\right)'(-1)=\frac{1}{g'(0)}=\frac{1}{2}$.

N° 7.

2) c- $f^{-1}(x)=\frac{-x+2}{x-1}$. d- $\left(f^{-1}\right)'\left(\frac{4}{3}\right)=\frac{1}{f'(2)}$.

3) $y=-\frac{1}{9}x+\frac{14}{9}$ is an equation of (T) and $y=-9x+14$ is an equation of (T').

CHAPTER 2

Complex Numbers

Chapter Review

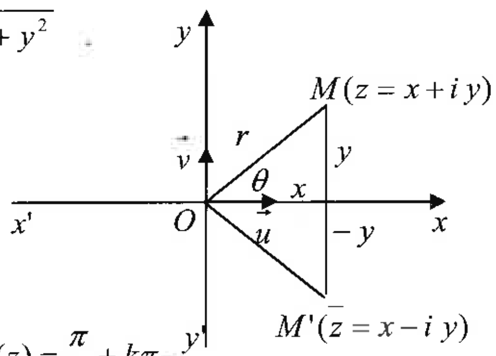
Modulus and Argument of a Complex Number:

Modulus of z is : $r = |z| = \sqrt{x^2 + y^2}$

Argument of z is :

$\arg(z) = \theta = (\vec{u}; \overrightarrow{OM})(2\pi)$.

$z = r(\cos\theta + i\sin\theta)$.



N.B.

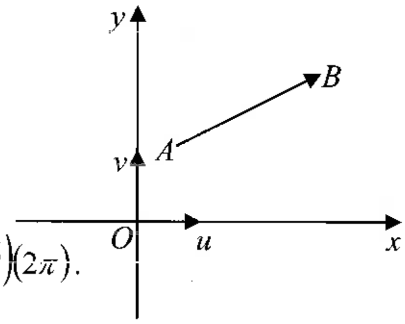
If z is real then, $\arg(z) = k\pi$.

If z is pure imaginary then, $\arg(z) = \dfrac{\pi}{2} + k\pi$.

Properties :

- Trigonometric form of a complex number: $z = r(\cos\theta + i\sin\theta)$, we can also denote it by $z = r\,e^{i\theta}$ which is said to be the exponential form of z.
  If $z = r\,e^{i\theta}$ and $z' = \rho\,e^{i\alpha}$, then:
  $$z z' = r\rho e^{i(\theta+\alpha)} \quad ; \quad \frac{z}{z'} = \frac{r}{\rho} e^{i(\theta-\alpha)} \quad ; \quad z^n = r^n\, e^{i(n\theta)}$$
- $|z| = |\bar{z}|$ and $\arg(\bar{z}) = -\arg(z)\,(2\pi)$ then $\bar{z} = r\,e^{-i\theta}$.
- $(\cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha)$.
- $z + \bar{z} = 2\operatorname{Re}(z)$; $z - \bar{z} = 2i\operatorname{Im}(z)$.

- z is real , if and only if $\bar{z} = z$
- z is pure imaginary , if and only if $\bar{z} = -z$.
- $z\bar{z} = |z|^2$.
- If A is a point of affix a and
  B is a point of affix b , then :
  $b - a = z_B - z_A = z_{\overrightarrow{AB}}$.
  $|b-a| = |z_{\overrightarrow{AB}}| = \|\overrightarrow{AB}\|$.
  $\arg(z_{\overrightarrow{AB}}) = \arg(b-a) = (\vec{u}\,;\overrightarrow{AB})(2\pi)$.



- If A , B, C and D are four points of respective affixes:
  a , b , c and d , then $\dfrac{b-a}{d-c} = \dfrac{z_{\overrightarrow{AB}}}{z_{\overrightarrow{CD}}} = \dfrac{AB}{CD} e^{i\,(\overrightarrow{CD}\,;\,\overrightarrow{AB})}$.

N.B.

If $\dfrac{b-a}{d-c}$ is real , then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear .

If $\dfrac{b-a}{d-c}$ is pure imaginary , then $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are orthogonal .

- $\cos\phi = \dfrac{1}{2}\left(e^{i\phi} + e^{-i\phi}\right)$; $\sin\phi = \dfrac{1}{2i}\left(e^{i\phi} - e^{-i\phi}\right)$.

Solved Problems

N° 1.

In the table below, only one among the proposed answers to each question is correct. Write down the number of each question and give, with justification, the answer corresponding to it.

N°	Questions	Answers		
		a	b	c
1	If $z=1-e^{-i\frac{\pi}{3}}$ then $\arg(\bar{z})=$	$\frac{\pi}{3}$	$-\frac{\pi}{3}$	$\frac{2\pi}{3}$
2	If $z=(1+i)^n$ and n is a natural number, then z is a real positive number if :	n is even	n is odd	n is a multiple of 8
3	If an argument of z is $\frac{\pi}{6}$ then an argument of $-\frac{2}{z}$ is :	$-2\times\frac{6}{\pi}$	$-\frac{\pi}{6}$	$\frac{5\pi}{6}$
4	If $z=\frac{1+it}{1-it}$ where t is real then $\lvert z\rvert=$	1	$\sqrt{t}$	$2t$
5	The exponential form of the complex number $z=-2(\sin\theta+i\cos\theta)$ is	$2e^{i\theta}$	$2e^{i\left(\frac{\pi}{2}+\theta\right)}$	$2e^{i\left(\frac{3\pi}{2}-\theta\right)}$
6	If z is a complex number such that $\lvert z\rvert=\sqrt{2}$, then $\lvert\bar{z}+i\bar{z}\rvert=$	$2\sqrt{2}$	2	$\sqrt{2}$
7	If $z=e^{i\frac{\pi}{2}}+e^{-i\frac{\pi}{6}}$ then :	$\arg(z)=\frac{\pi}{2}$	$\arg(z)=\frac{\pi}{2}+\frac{\pi}{6}$	$\arg(z)=\frac{\pi}{6}$
8	If $\frac{z_{\vec{t}}}{z_{\vec{s}}}$ is a real number then the two vectors $\vec{t}$ and $\vec{s}$ are :	equal	collinear	orthogonal

N°2.

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Let M be a point of affix z . Let $Z=\dfrac{z-i}{z-1}$ where $z\neq 1$.

Answer by true of false and justify:

1) The set of points M, such that Z is real is a straight line deprived of a point.
2) The set of points M, such that $|Z|=1$ is a circle .
3) The set of points M, such that $\arg(Z)=0(2\pi)$ is a circle.
4) The set of points M, such that $Z+\overline{Z}=0$ is a straight line

N°3.

Consider in the complex plane of an orthonormal system $\left(O;\vec{u},\vec{v}\right)$ the variable point M fo affix z.

Determine the set of points M in each of the following cases:

1) $|z-1+2i|=2$.
2) $|z-2i|=|z+4|$
3) $z+\overline{z}=|z|$
4) $z-i=\dfrac{4}{z+i}$
5) $|\overline{z}+5-i|=|z-4i|$
6) $|z+1+i|\times|\overline{z}+1-i|=4$
7) $z\overline{z}+2z-4i\overline{z}-4+2i$ is pure imaginary.
8) The points $M(z)$, $N\left(z^2\right)$ and $P\left(z^3\right)$ are collinear. $(z\neq 1$ and $z\neq 0)$.

N°4.

Given the complex numbers $z_1=2\sqrt{3}+2i$ and

$z_2=\left(1+\sqrt{3}\right)+i\left(1-\sqrt{3}\right)$ and let $z=\dfrac{z_1}{z_2}$.

1) Write z_1 in exponential form .
2) Find the algebraic form of z then deduce the modulus and an argument of z.
3) Deduce the modulus and an argument z_2 , then find the exact value of $\cos\dfrac{\pi}{12}$ and $\sin\dfrac{\pi}{12}$.

N°5.

Given the complex numbers: $z_1 = 1 + i\sqrt{3}$, $z_2 = 1 + i$, $Z = z_1 z_2$.

1) a- Write z_1 and z_2 in exponential form.

   b- Deduce the exponential form of Z .

2) a- Write the algebraic form of Z.

   b- Deduce the exact value of $\cos\frac{7\pi}{12}$ and $\sin\frac{7\pi}{12}$.

3) Determine the possible values of the natural number n such that z_1^n is real.

4) Determine the possible values of the natural number p such that z_2^p is pure imaginary.

N°6.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

For all points M of affix z we associate the point M' of affix z' such that $z' = \frac{z+2}{z-i}$.

1) a- Find the algebraic form of z' when $z = -\frac{3}{5} + \frac{1}{5}i$.

   b- Show that in this case, $(z')^{40}$ is a real positive number.

2) Let $z = x + iy$ and $z' = x' + iy'$.

   a- Express x' and y' in terms of x and y .

   b- Deduce the set of points M when z' is pure imaginary

3) a- Show that for any z , we have $|z-i| \times |z'-1| = \sqrt{5}$.

   b- Suppose that M describes the circle (C) of center A of affix i and radius $R = 1$, determine the set of points M' .

N°7.

Consider in the complex plane reffered to an orthonormal system $(O;\vec{u},\vec{v})$ the point A of $z_A = -2i$.

For each point M , distinct from A , we associate the point M' of affix z' such that $z' = -2\bar{z} + 2i$, θ is an argument of $z + 2i$.

1) Show that $(z+2i)(z'+2i)$ is a real non zero negative number.

2) Deduce an argument of $z'+2i$ in terms of θ.
3) What can we say about the two semi straight lines $[AM)$ and $[AM')$?

N° 8.

Given the complex number $z = 1+\sqrt{3}+i\left(\sqrt{3}-1\right)$.

1) Write z^2 in exponential form.
2) Deduce the modulus and an argument of z.
3) Deduce the exact values of $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$.

N° 9.

Consider in the complex plane of an orthonormal system $\left(O;\vec{u},\vec{v}\right)$ the points $A\left(-\frac{1}{2}\right)$, $M(z)$, $N(1+3z)$ and $P(1+z)$ where $z \neq 0$.

Determine the set of points M when the triangle MNP is isoceles of principal vertex M.

N° 10.

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Consider the points A, B, M and M' of respective affixes 1, 5, z and z' such that $z' = \frac{z-5}{z-1}$ where $z \neq 1$.

1) Show that if z' is pure imaginary then $z\bar{z}-3\left(z+\bar{z}\right)+5=0$ and show, in this case, that M varies on a circle to be determined.
2) Interpret, geometrically $|z-5|$, $|z-1|$, $|z'|$ and $\arg(z')$.
3) Deduce :
   a- The set of points M such that z' is pure imaginary.
   b- The set of points M such that z' is real.

N° 11.

Consider the complex number $z = 1+\cos\theta+i\sin\theta$ where $\pi < \theta < 2\pi$.

1) a- Expand $e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}}+e^{-i\frac{\theta}{2}}\right)$.

   b- Determine the modulus and an argument of z in terms of $\frac{\theta}{2}$.

2) Deduce an argument of the complex number $z = 1 - \frac{\sqrt{3}}{2} - \frac{1}{2}i$.

N° 12.

Consider in the complex plane of an orthonormal system $(O;\vec{u},\vec{v})$ the points $M_1(z_1)$, $M_2(z_2)$ and $M_3(z_3)$ such that $(1-i)z_1 + iz_2 - z_3 = 0$.

1) Show that $\frac{z_3 - z_1}{z_2 - z_1} = i$ and deduce the nature of the triangle $M_1M_2M_3$.
2) Given the two points A and B of affixes $z_A = -1+2i$ and $z_B = 2+i$.
   Determine the affixes of the two points E and F such that the two triangles ABE and AFB are both right direct isosceles of the same principal vertex A.

N° 13.

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points A and B of respective affixes $a = 2+2i$ and $b = 1-\sqrt{3}+i\left(1+\sqrt{3}\right)$.

1) a- Write a in trigonometric form.
   b- Calculate the modulus of b and deduce that $OA = OB$.
2) a- Let $z = b - a$, calculate $|z|$ and deduce that triangle OAB is equilateral.
   b- Using the figure calculate an argument of b, and deduce the exact values of $\cos\left(\frac{7\pi}{12}\right)$ and $\sin\left(\frac{7\pi}{12}\right)$.

N° 14.

Given the complex number $z = \cos\varphi + i\sin\varphi = e^{i\varphi}$.

1) Write $\bar{z}$ in trigonometric and exponential form.
2) Show that $\cos\varphi = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right)$ and $\sin\varphi = \frac{1}{2i}\left(e^{i\varphi} - e^{-i\varphi}\right)$.
3) Using the relations obtained in part 2) linearize $\cos^2 x \sin^4 x$ and $\cos x \sin^4 x$.

4) θ and θ' are two real numbers, show that $a=\dfrac{e^{i\theta}+e^{i\theta'}}{1+e^{i\theta}.e^{i\theta'}}$ is real.

N° 15.

Let $\left(O;\vec{u},\vec{v}\right)$ be a direct orthonormal system of a complex plane, and let z_1 and z_2 be any two complex numbers.

1) Show that $|z_1+z_2|^2+|z_1-z_2|^2=2\left(|z_1|^2+|z_2|^2\right)$.

2) Let M_1 and M_2 be two points of respective affixes $1+i$ and $1-i\sqrt{3}$.

   a- Calculate the affix of point M such that OM_1MM_2 is a parallelogram.

   b- Calculate: OM_1, OM_2, OM and M_1M_2.

   c- Verify that: $M_1M_2^2+OM^2=2\left(OM_1^2+OM_2^2\right)$.

N° 16.

$\left(O;\vec{u},\vec{v}\right)$ is an orthonormal system of a complex plane , where $j=e^{2i\frac{\pi}{3}}$ is a complex number.

1) Show that $j^3=1$, $1+j+j^2=0$ and $e^{i\frac{\pi}{3}}+j^2=0$.

2) ABC is an equilateral triangle such that $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\dfrac{\pi}{3}(2\pi)$, where a,b and c are the respective affixes of A,B and C.

   a- Show that $\dfrac{c-a}{b-a}=e^{i\frac{\pi}{3}}$.

   b- Deduce that: $a+bj+cj^2=0$.

3) For any complex number $z\neq1$, we associate the points R,M and M' of respective affixes 1, z and $\bar{z}$.

   a- Find the values of z when M and M' are distinct points.

   b- Suppose that the triangle RMM' is equilateral such that $\left(\overrightarrow{RM};\overrightarrow{RM'}\right)=\dfrac{\pi}{3}(2\pi)$.

   Show that M varies on a straight line (Δ) to be determined.

N° 17.

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points A , B , C and D of respective affixes $z_A = -1+2i$, $z_B = 7-2i$, $z_C = 6+i$ and $z_D = 2+3i$.

1) Write $\dfrac{z_B - z_A}{z_C - z_D}$ in exponential form and deduce that the two lines (AB) and (CD) are parallel.

2) Write $\dfrac{z_C - z_B}{z_A - z_D}$ in exponential form and deduce that $ABCD$ is an isosceles trapezoid.

3) The two lines (BC) and (AD) intersect in a point E.

   a- Interpret , geometrically $\left|\dfrac{z_B - z_E}{z_A - z_E}\right|$ and $\arg\left(\dfrac{z_B - z_E}{z_A - z_E}\right)$

   b- Deduce the affix of point E.

4) Let M be a variable point of affix z.

   a- Determine the set (T) of points M if $|z-5-4i| = \sqrt{10}$

   b- Precise the position of C with respect to (T).

N° 18.

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points $A(-2)$, $M(z)$ and $M'(z')$ such that

$z' = \dfrac{4-2\bar{z}}{\bar{z}}$, $z \neq 0$.

1) Write z' in algebraic form when $z = \sqrt{2}e^{i\frac{\pi}{4}}$.

2) a- Verify that $(z'+2)\bar{z} = 4$.

   b- Deduce that $AM' \times OM = 4$ and the vectors $\overrightarrow{AM'}$ and $\overrightarrow{OM}$ are parallel with same sense.

   c- Determine the set (T) of points M' when M moves on the circle (C) with center O and radius 1.

N° 19.

Let $A(i)$ and $B(-i)$ be two points in a complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, (C) is a circle of center O and radius $R=1$, M is a variable point of the plane of affix z.

1) Suppose that $z = r e^{i\theta}$ where $\frac{\pi}{2} < \theta < \pi$.

   M_1 is a point of affix $z_1 = (z + \bar{z})z$.

   Calculate an argument of z_1 in terms of θ and deduce that the point O belongs to $[MM_1]$.

2) For any point M of affix $z \neq -i$, associate a point M' of affix z' such that $z' = i\,\frac{z-i}{z+i}$.

   a- Show that: $OM' = \frac{AM}{BM}$ and $\arg(z') = \frac{\pi}{2} + (\overrightarrow{BM};\overrightarrow{AM})(2\pi)$

   b- Deduce that if M moves on the circle (C), then M' varies on the $x-$axis.

N° 20.

Consider in the complex plane, the points A, B and C of respective affixes $z_A = -1$, $z_B = 3i$ and $z_C = 2-i$.

Let M be a point of affix z and M' a point of affix z' such that $z' = \frac{iz+3}{z+1}$, where $z \neq -1$.

1) Calculate $\frac{z_B - z_A}{z_C - z_A}$ and deduce the nature of the triangle ABC.

2) a- Verify that $z' = i\frac{z-3i}{z+1}$.

   b- Show that :

   $OM' = \frac{BM}{AM}$ and $(\vec{u};\overrightarrow{OM'}) = \frac{\pi}{2} + (\overrightarrow{AM};\overrightarrow{BM})(\text{mod}\,2\pi)$.

   c- Deduce :

   - The set of M when M' moves on a circle of center O and radius 1.
   - The set of points M when z' is real.

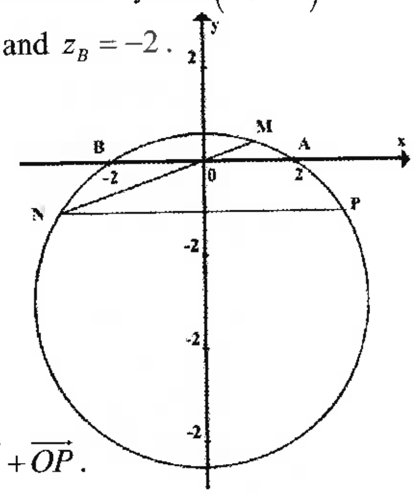
N°21.

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider a point M, distinct from O of affix z and a point M' of affix z' such that $z'\bar{z}=1$, $z\neq 0$.

1) Determine the algebraic form of z' in each of the following cases:
   $z=2e^{i\frac{\pi}{6}}$; $z=\sqrt{2}e^{i\left(-\frac{\pi}{4}\right)}$.
2) a- Show that $OM\times OM'=1$.
   b- Compare $\left(\vec{u};\overrightarrow{OM}\right)$ and $\left(\vec{u};\overrightarrow{OM'}\right)$.
   Deduce that the points O, M and M' are collinear.
3) Show that O, $E\left(\bar{z}\right)$ and $F\left(\frac{1}{z}\right)$ are collinear.
4) Prove that $\overline{z'-1}=-1+\frac{1}{z}$.
5) Suppose now that M moves on a circle of center $E\ (1;0)$ and radius $R=1$, M is distinct from O.
   a- Verify that $|z-1|=1$.
   b- Prove that $|z'-1|=|z'|$.
   c- Deduce the set of points M'.

N°22. ***For the students of the G.S. section***

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points A and B of affixes $z_A=2$ and $z_B=-2$.



Let (C) be a variable circle passing through A and B and M a variable point on (C) of affix $z=re^{i\theta}$.
(OM) recuts (C) in N, designate by P the symmetric of N with respect to $y'Oy$.

1) Determine the affixes z_N and z_P of the points N and P in terms of r and θ.
2) Let Q be the point defined by $\overrightarrow{OQ}=\overrightarrow{OM}+\overrightarrow{OP}$.

a- Show that $x_Q = \left(r + \frac{4}{r}\right)\cos\theta$ and $y_Q = \left(r - \frac{4}{r}\right)\sin\theta$.

b- Suppose that $r = 4$, show that when M varies on (C) then Q moves on the ellipse of equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

N°23. ***For the students of the G.S. section.***

In the complex plane consider the mapping f that associates for every point M of affix z the point M' of affix z' such that $z' = z^2 - 4z$.

Part A.

1) Let A and B be the points of affixes $z_A = 1 - i$ and $z_B = 3 + i$.
   a- Show that the two points A and B have the same image by f.
   b- Two points have the same image by f, show that either these two points are confounded or they are symmetric to each other with respect to a point to be determined.
2) Let I be a point of affix -3.
   a- Show that $OMIM'$ is a parallelogram if and only if $z^2 - 3z + 3 = 0$.
   b- Solve the equation $z^2 - 3z + 3 = 0$.
3) a- Express $z' + 4$ in terms of $z - 2$.
   Deduce the relation between $|z' + 4|$ and $|z - 2|$ and also for $\arg(z' + 4)$ and $\arg(z - 2)$.
   b- Consider the points J and K of affixes $z_J = 2$ and $z_K = -4$.
   Let (C) be the circle with center J and radius 2. Show that any point M on (C) has an image M' on a circle (C').
   c- Let E be the point of affix $z_E = -4 - 3i$.
   Write the exponential form of $z_E + 4$.

Partie B.

Let $z = x + iy$ and $z' = x' + iy'$.

1) Express x and y in terms of x' and y'.
2) a- Show that when z' is pure imaginary then M describes a hyperbole (H).
   b- Determine the vertices, the asymptotes of (H) and draw (H).

3) Suppose that M moves on the straight line of equation $y = x - 3$.
   Show that M' moves on the parabola (P) of equation
   $(x'+4)^2 = 2\left(y'+\frac{1}{2}\right)$.
   Determine the vertex, the focus of (P) and draw (P).

N°24

In the complex plane consider the points A and B of affixes $z_A = 2$ and $z_B = 3$.

Part A .

1) Designate by M_1 and M_2 the points of respective affixes $z_1 = 2 + i\sqrt{2}$ and $z_2 = 2 - i\sqrt{2}$.
   a- Determine the algebraic form of the complex number $\frac{z_1 - 3}{z_1}$.
   b- Deduce that the triangle OBM_1 is right angled.
2) Prove geometrically that the points O, B, M_1 and M_2 belong to the same circle (T) to be determined.

Part B .

Let f be the mapping that associates for every point M of affix z the point M' of affix z' such that $z' = z^2 - 4z + 6$.
Let (Γ) be the circle of center A and radius $\sqrt{2}$.
M is a point on (Γ) such that $(\vec{u};\overrightarrow{AM}) = \theta$ where $-\pi < \theta \leq \pi$.

1) Verify that the affix of M is $z = 2 + \sqrt{2}e^{i\theta}$.
2) Verify that $z' = 2 + 2e^{i\theta}$ and M' that belongs to a circle (Γ').
3) Let D be the point of affix $d = 2 + \frac{\sqrt{2} + i\sqrt{6}}{2}$ and let D' be the image of D by f.
   a- Write in exponential form the complex number $d - 2$ and deduce that D belongs to (Γ).
   b- Give the measure of the angle $(\vec{u};\overrightarrow{AD'})$ and show that the triangle OAD' is equilateral.

Supplementary Problems

N°1.

In the table below, only one among the proposed answers to each question is correct. Write down the number of each question and give, with justification, the answer corresponding to it.

$N°$	Questions	Answers		
		a	b	c
1	An argument of $z=-4e^{-i\frac{\pi}{3}}$ is :	$\frac{\pi}{3}$	$-\frac{\pi}{3}$	$\frac{2\pi}{3}$
2	An argument of $z=-2i\left(\cos\frac{\pi}{3}-i\sin\frac{\pi}{3}\right)$ is :	$\frac{5\pi}{6}$	$\frac{-5\pi}{6}$	$\frac{\pi}{6}$
3	Given $z'=\frac{\bar{z}-1}{z-1}$ where $(z\neq 1)$ then $\lvert z'\rvert=$	$\lvert z\rvert$	1	$2\lvert z\rvert$
4	If $z=\sin\frac{\pi}{3}+i\cos\frac{\pi}{3}$ then $z^{15}=$	i	1	$-i$
5	M is a point of affix $z\neq 0$ such that $\bar{z}-\frac{4}{z}=0$, Then M moves on :	The axis of abscissas	The circle of center O and radius 2	The axis of ordinates
6	If $z=e^{i\frac{\pi}{6}}-i$ thens $\bar{z}=$	$e^{-i\frac{\pi}{6}}$	$e^{i\frac{\pi}{6}}$	$e^{i\frac{\pi}{3}}$

N°2.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

Consider the points A and B of respective affixes 1 and $2i$

Designate by :

(E) the set of points M of affix z such that $\lvert z-2i\rvert=\lvert z-1\rvert$.

(F) the set of points M, distinct from A and B, of affix z such that $\arg\left(\dfrac{z-2i}{z-1}\right)=\dfrac{\pi}{2}(2\pi)$.

Answer by true or false and justify:

1) (E) is a circle.
2) The points M of (F) move on a semi-circle deprived of two points.
3) The point C of affix $-\dfrac{1}{2}+\dfrac{1}{2}i$ belongs to (E) and to (F).
4) (F) is the set of points M such that the complex number $Z=\dfrac{z-2i}{z-1}$ is pure imaginary.

N°3.

Consider in the complex plane of an orthonormal system $(O;\vec{u},\vec{v})$ the variable point M of affix z.
Determine the set of points M in each of the following cases:

1) $\left|\dfrac{2z-4-2i}{z-i}\right|=2$ 2) $z+\bar{z}=|z|^2$
3) $|\bar{z}+i|=2$ 4) $z=\sqrt{3}e^{i\theta}$ where $0\le\theta\le\dfrac{\pi}{2}$.
5) The points $A(i)$, $M(z)$ and $M'(iz)$ are collinear.

N°4.

Given the complex number $z=\left(1+i\sqrt{3}\right)e^{i\theta}$.

1) Write z in exponential form and in algebraic form.
2) Deduce the expression of $\cos\left(\dfrac{\pi}{3}+\theta\right)$ and that of $\sin\left(\dfrac{\pi}{3}+\theta\right)$ in terms of $\cos\theta$ and $\sin\theta$.
3) Deduce the exact values of $\cos 15^\circ$ and $\sin 105^\circ$.

N° 5.

Consider in the complex plane of an orthonormal system $(O;\vec{u},\vec{v})$

The points $M(z)$, $N(2z)$, $I(1)$ and $J(i)$.

Determine the set of points M when the two straight lines (IM) and (JN) are parallel.

N° 6.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

Consider the points A and E of affixes $z_A = 2$ and $z_E = \sqrt{3}+i$.

1) Write z_A and z_E in exponential form.
2) Let C be the point defined by $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OE}$.
   a- What is the nature of quadrilateral $OACE$?
   b- Calculate OC then write z_C in exponential form.
   c- Deduce the exact values of $\sin 15^\circ$ and $\cos 15^\circ$.

N° 7.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

1) Determine the set of points M of affix z such that $z\bar{z} = i\bar{z} - iz$.
2) Determine the set of points M of affix $z \neq 0$ such that the distinct points of affixes 1, z^2 and $\dfrac{1}{z^2}$ are collinear.

N° 8.

Consider in the complex plane of an orthonormal system $(O;\vec{u},\vec{v})$ the complex number $a = 2+2i$ and let M be a variable point of affix z such that $\arg(z+a) = \arg z + \arg a$.

Find set of points M.

N° 9.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

Let A be the point of affix $-2i$. For any point M of affix z we associate the point M' of affix z' such that $z' = -2\bar{z} + 2i$.

1) Let B be the point of affix $b = 3-2i$.

Determine the algebraic form of the affixes a' and b' of the points A' and B' associated to points A and B respectively and place the points A' and B' on a figure.

2) Show that if M belongs to the straight line (d) of equation $y=-2$ then M' belongs to (d).
3) Show that for any point M of affix z, $|z'+2i| = 2|z+2i|$.
   Interpret geometrically this equality.
4) For any point M distinct from A, designate by θ an argument of $z+2i$.
   a- Prove that $(z+2i)(z'+2i)$ is a non zero real negative number.
   b- Deduce an argument of $z'+2i$ in terms of θ.
   c- What can we say about the two semi straight lines $[AM)$ and $[AM')$?

N° 10

In the complex plane referred to an orthonormal system $(O,\vec{u},\vec{v})$, designate by I the point of affix 1 and by (C) the circle of diameter $[OI]$ and of center Ω.

Part A:

Suppose $a_0 = \frac{1}{2}+\frac{1}{2}i$ and denote by A_0 its image.

1) Show that A_0 belongs to (C).
2) Let B be the point of affix $b=-1+2i$ and B' the point of affix b' such that $b' = a_0 b$.
   a- Calculate b'.
   b- Prove that triangle OBB' is right at B'.

Part B:

Let a be a non-zero complex number different from 1 and let A be its image in the complex plane.

For all points M of affix z, $z \neq 0$, we associate the point M' of affix z' such that $z' = az$.

1) a- Interpret, geometrically, $\arg\left(\frac{a-1}{a}\right)$.
   b- Show that $\left(\overrightarrow{M'O},\overrightarrow{M'M}\right) = \arg\left(\frac{a-1}{a}\right) + 2k\pi$ where $k \in Z$.

c- Show that if A belongs to circle (C) deprived of O and I, then triangle OMM' is right at M' .

2) In this question, suppose that M is a point of the axis of abscissas different from O and denote by x its affix.
Choose x in such a way that A is a point of (C) different from I and O.
Show that the point M' belongs to the straight line (OA) .
Deduce that M' is the orthogonal projection of M on this straight line .

N° 11.

In the complex plane referred to a direct orthonormal system $(O,\vec{u},\vec{v})$, consider the points A, B and C of respective affixes $a=-1$, $b=2i$ and $c=-i$.
For all points M, of affix z , $z \neq -1$, we associate the point M' of affix $z'=\dfrac{-iz-2}{z+1}$.

1) Determine the affix c' of point C' associated to C, and write c' in exponential form.
2) a- Show that $|z+1| \times |z'+i| = \sqrt{5}$.
   b- Deduce that if the point M belongs to circle (C) of center A and radius 2, then M' belongs to a circle whose center and radius are to be determined .

N° 12.

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$. Consider the points A, B and C of respective affixes $a=3+i$, $b=2i$ and $c=2-2i$.

1) Calculate $\dfrac{c-a}{b-a}$ and deduce the nature of triangle ABC.
2) Let M be a point of affix z and M' a point of affix z' such that $\overrightarrow{MM'}=\overrightarrow{AC}$.
   a- Express z' in terms of z .
   b- Calculate the affix of point D so that $ABDC$ is a rhombus .
3) a- Write c in the exponential form.
   b- For what values of the natural integer n , is c^n real ?
   For what values of the natural integer n, is c^n pure imaginary?

c- Calculate c^{2005}.

N° 13.

In the complex plane referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$, designate by A the point of affix 1, by B the point of affix -1, by M the point of affix $z \neq 1$ and consider the point M' of affix z' such that $z' = \dfrac{z-1}{1-\bar{z}}$.

1) Prove that $|z'| = 1$ and that $(z-1)(1-\bar{z})$ is real.

2) Show that $\dfrac{z'-1}{z-1}$ is real and deduce that the points M, A, M' are collinear.

3) a- Prove that $\dfrac{z'+1}{z-1}$ is pure imaginary.

   b- Deduce that the two straight lines (BM') and (AM) are perpendicular.

N° 14.

In the complex plane referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points $A(1+i)$ and $B(2)$.
Let M be a point of affix $z \neq 2$ and let M' be the point of affix z' such that $z' = \dfrac{z-(1+i)}{z-2}$.
Show that if M moves on the perpendicular bisector of $[AB]$ then M' moves on a circle to be determined.

N° 15.

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points A and B of respective affixes i and $2-i$.
Let M be a point of affix z, ($z \neq 0$), and M' a point of affix z' such that : $z' = \dfrac{-2}{z}$.

1) Express OM' in terms of OM.

2) a- Express $\left(\vec{u};\overrightarrow{OM'}\right)$ in terms of $\left(\vec{u};\overrightarrow{OM}\right)$.

   b- What can you say about $[OM)$ and $[OM')$?

3) Show that if M belongs to circle (C) of center O and radius $\sqrt{2}$ then M' is also on (C) .
4) Let (D) be the straight line of equation $y=1$.
   a- Show that if M belongs to (D) then its affix verifies the relation: $|z-2i|=|z|$ and also the affix z' of point M' verifies the relation $|z'-i|=1$.
   b- Deduce that if M moves on (D) then, point M' belongs to a circle whose center and radius are to be determined.

N° 16.

In the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$, consider the points E, F and G of respective affixes:

$z_E = i$, $z_F = 2$ and $Z_G = \left(1+\frac{\sqrt{3}}{2}\right)+i\left(\frac{1}{2}+\sqrt{3}\right)$.

1) Express, in algebraic form, the complex number $z' = \frac{z_G - z_E}{z_F - z_E}$ and verify that $z' = e^{i\frac{\pi}{3}}$.
2) Prove that triangle EFG is equilateral.
3) Let M be a variable point of affix z ,determine the set (T) of points M such that $|z-z_E|=\sqrt{5}$ and verify that F belongs to (T) .

N° 17.

In the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$, consider the points A and B such that $z_A = 1$ and $z_B = \frac{3}{2}+i\frac{\sqrt{3}}{2}$.

Let (C) be the circle of center A and radius 1.

1) a- Write $z_B - z_A$ in exponential form.
   b- Determine a measure of the angle $(\vec{u};\overrightarrow{AB})$.
   c- Show that the point B belongs to circle (C).
2) For all points M of affix z , $z \neq 0$, we associate the point M' of affix z' such that $z' = \frac{\bar{z}+2}{\bar{z}}$.

a- Prove that $\bar{z}(z'-1)=2$.

b- Deduce that, as M' describes circle (C), M describes a circle (T) to be determined.

N° 18.

Given the complex number $z=\frac{\sqrt{3}+1}{4}-i\frac{\sqrt{3}-1}{4}$.

1) Calculate z^2 and write z^2 in trigonometric form .

2) a- Determine the modulus of z and verify that $-\frac{\pi}{12}$ is an argument of z .

b- Deduce $\cos\frac{\pi}{12}$ and $\sin\frac{\pi}{12}$.

N° 19.

Consider in the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$, the points A , B and C of respective affixes:

$z_A=4+\frac{5}{2}i$; $z_B=4-\frac{5}{2}i$ and $z_C=2+\frac{3}{2}i$.

1) Locate the points A , B and C in the plane.

2) a- Write down the expression: $\frac{z_B-z_C}{z_A-z_C}$ in the exponential form.

b- Deduce the nature of triangle ABC .

3) Let (E) be the set of points M of affix z verifying the relation $|z-4|=\frac{5}{2}$.

a- Do the points A , B and C belong to (E) ? Why?

b- Determine the nature of the set (E) .

c- Find the points of (E) having real affixes.

N° 20.

$(O;\vec{u},\vec{v})$ is an orthonormal system of a complex plane.

1) Locate the points A, B and D of respective affixes $z_A=-2-2i$; $z_B=2$ and $z_D=-2+2i$.

2) Calculate the affix of point C when $ABCD$ is a parallelogram and locate C .

3) E is a vertex of the right isosceles triangle CBE with $\left(\overrightarrow{BC};\overrightarrow{BE}\right)=-\frac{\pi}{2}(2\pi)$.

Show that $\frac{z_E - z_B}{z_C - z_B} = -i$ and deduce the affix of E.

4) Calculate the affix of point F if CDF is right isosceles at D with $\left(\overrightarrow{DC};\overrightarrow{DF}\right)=\frac{\pi}{2}(2\pi)$.

5) Show that: $\frac{z_F - z_A}{z_E - z_A} = i$ .Deduce the nature of triangle AEF.

N°21.

$\left(O;\vec{u},\vec{v}\right)$ is an orthonormal system of a complex plane.

A is a point of affix $a = 5 - i\sqrt{3}$, and B is a point such that triangle OAB is equilateral, with $\left(\overrightarrow{OA},\overrightarrow{OB}\right)=\frac{\pi}{3}(2\pi)$, let Q be the midpoint of $[OB]$.

1) a- Show that the affix of B is $b = 4 + 2i\sqrt{3}$ and deduce the affix of Q.

   b- Determine the affix z_K of point K where $ABQK$ is a parallelogram.

   c- Show that: $\frac{z_K - a}{z_K}$ is pure imaginary and deduce the nature of triangle OKA, and that of quadrilateral $OQAK$.
   Locate the points A, B, Q and K in the same plane.

2) Let C be a point of affix $c = \frac{2a}{3}$. Calculate $\frac{z_K - b}{z_K - c}$ and deduce the position of points B, C and K.

N°22. ***For the students of the G.S. section.***
The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Part A .
Consider the points A , B and C of respective affixes $a=3+i$, $b=-1+3i$ and $c=-\sqrt{5}-i\sqrt{5}$.

1) Show that the points A , B and C belong to the same circle (C) of center O.
2) Locate the points A , B and C and the point H of affix $a+b+c$ then verify that H is the orthocenter of the triangle ABC.

Part B .
Let ABC be a triangle inscribed in a circle of center O and designate by a , b and c the respective affixes of the points A , B and C.

1) Justify that $a\bar{a}=b\bar{b}=c\bar{c}$.
2) Let $w=\bar{b}c-b\bar{c}$.
   a- Prove that w is pure imaginary.
   b- Verify that $(b+c)\left(\bar{b}-\bar{c}\right)=w$ and justify that $\dfrac{b+c}{b-c}=\dfrac{w}{|b-c|^2}$
   c- Deduce that $\dfrac{b+c}{b-c}$ is pure imaginary.
3) Let H be a point of affix $a+b+c$.
   a- Express in terms of a , b and c the affixes of the vectors $\overrightarrow{AH}$ and $\overrightarrow{CB}$.
   b- Prove that H is the orthocenter of triangle ABC.

N°23.
In the complex plane referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$, 1 unit = 2 cm, consider the point A of affix $z_A=1$ and the circle (C) of center A and radius $R=1$.

1) Let F the point of affix 2, B the point of affix $z_B=1+e^{i\frac{\pi}{3}}$ and E the point of affix $z_E=1+z_B^2$.
   a- Prove that B belongs to (C).

b- Determine a measure of the angle $\left(\overrightarrow{AF};\overrightarrow{AB}\right)$ and locate B.

2) a- Determine the exponential form of $z_E - z_A$.

b- Deduce that the points A, B and E are collinear.

3) Consider the points M and M' of respective affixes z and z' where $z' = 1 + z^2$, $z \neq 1$.

a- For $z \neq 0$ and $z \neq 1$, give a geometric interpretation of an argument of the complex number $\frac{z'-1}{z-1}$.

b- Suppose that the points A, M and M' are collinear.

Prove that $\frac{z^2}{z-1}$ is real.

N°24.

In the complex plane referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points A, B and C of respective affixes $a = \sqrt{3} - i$, $b = \sqrt{3} + i$ and $c = 2i$.

1) Show that the points A, B and C belong to the same circle of center O.

2) Write $\frac{c-b}{a-b}$ in algebraic form and in exponential form.

3) Let M be a point of (P) distinct from O, of affix $z = x + iy$ (x and y are real numbers). Let $Z = \frac{z-b}{z}$.

a- Determine the set (E) of points M such that $|Z| = 1$.

b- Verify that A and C belong to (E).

c- Determine the set (F) of points M such that Z is pure imaginary.

N°25.

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points E, F, G of respective affixes $z_E = 2i$, $z_F = -2i$, $z_G = -1 + i$ and let M be a point of affix z.

1) a- Find the set (T) of points M such that $|z - 2i| = \sqrt{2}$.

b- Show that the point G belongs to (T) .

2) a- Find the line (L) on which point M moves when $\left|\dfrac{z-2i}{z+2i}\right|=1$.

b- Determine the affix z_0 of a point W on (L) such that $|z_0-2i|=3$.

3) Let A and B be the points of respective affixes z_A and z_B such that $z_A=z_F+z_G$ and $z_B=z_F\times z_G$.

a- Write the complex numbers z_A and z_B in the exponential form.

b- Prove that the points O, A and B are collinear.

$\boxed{N°26.}$

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points A and B of affixes 1 and -1 respectively.
Let (C) be the circle of center A and of radius 1.The exponential form of the affix z of a point M on (C), other than O, is given by $z=re^{i\theta}$. Let M' be the point of affix z' such that $z'=\dfrac{1}{r}e^{i(\pi+\theta)}$.

1) Show that $z'\times\bar{z}=-1$
2) Show that the points O, M and M' are collinear.
3) a- Justify that $|z-1|=1$.

b- Prove that $|z'+1|=|z'|$, and deduce that M' moves on a line (d) to be determined.

4) Determine the points M on (C) for which $z'=-z$.

Solutions

N° 1.

1) $z = 1 - e^{-i\frac{\pi}{3}} = 1 - \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$ so

$\bar{z} = \frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-i\frac{\pi}{3}}$, then $\arg(\bar{z}) = -\frac{\pi}{3}(2\pi)$. (b)

2) If $z = (1+i)^n = \left(\sqrt{2}\right)^n e^{i\frac{n\pi}{4}}$, z is a real positive number then

$\frac{n\pi}{4} = 0 + 2k\pi$, so $n = 8k$ with $k \in IN$ then n is a multiple of 8.(c)

3) $\arg\left(-\frac{2}{z}\right) = \arg(-2) - \arg(z) = \pi - \frac{\pi}{6} = \frac{5\pi}{6}(2\pi)$. (c)

4) $|z| = \frac{|1+it|}{|1-it|} = \frac{\sqrt{1+t^2}}{\sqrt{1+t^2}} = 1$. (a)

5) $z = -2(\sin\theta + i\cos\theta) = 2i^2(\sin\theta + i\cos\theta)$

$= 2i(-\cos\theta + i\sin\theta) = 2e^{i\frac{\pi}{2}}e^{i(\pi-\theta)} = 2e^{i\left(\frac{3\pi}{2}-\theta\right)}$ (c)

6) $\left|\bar{z} + i\bar{z}\right| = |\bar{z}| \times |1+i| = \sqrt{2} \times \sqrt{2} = 2$. (b)

7) $z = e^{i\frac{\pi}{2}} + e^{-i\frac{\pi}{6}} = i + \frac{\sqrt{3}}{2} - \frac{1}{2}i = \frac{\sqrt{3}}{2} + \frac{1}{2}i = e^{i\frac{\pi}{6}}$, then $\arg(z) = \frac{\pi}{6}$.(c)

8) If $\frac{z_{\vec{t}}}{z_{\vec{s}}}$ is real then the two vectors $\vec{t}$ and $\vec{s}$ are collinear.(b)

N° 2.

Let A be the point of affix i and B the point of affix 1,

$Z = \frac{z_M - z_A}{z_M - z_B} = \frac{z_{\overrightarrow{AM}}}{z_{\overrightarrow{BM}}} = \frac{AM}{BM} e^{i\left(\overrightarrow{BM};\overrightarrow{AM}\right)}$

1) Z is real then $\left(\overrightarrow{BM};\overrightarrow{AM}\right) = k\pi$, then the two vectors

$\overrightarrow{AM}$ and $\overrightarrow{BM}$ are collinear , therefore M moves on the straight line (AB) deprived of point B .

2) $|Z|=1$ gives $\dfrac{AM}{BM}=1$, then M moves on the perpendicular bisector of $[AB]$.

3) $\arg(Z)=0(2\pi)$ gives $(\overrightarrow{BM};\overrightarrow{AM})=0(2\pi)$, then the set of points M is the straight line (AB) deprived segment $[AB]$.

4) $Z+\overline{Z}=0$ is equivalent to $\overline{Z}=-Z$ then Z is pure imaginary which gives $(\overrightarrow{BM};\overrightarrow{AM})=\dfrac{\pi}{2}+k\pi$,then the set of points M is the circle of diameter $[AB]$ deprived of the points A and B.

$\boxed{N°3.}$

1) Let A be the point of affix $1-2i$, $|z-1+2i|=2$ is equivalent to $|z_M-z_A|=2$ and to $|z_{\overrightarrow{AM}}|=2$, which gives $AM=2$ then M moves on the circle of center A and radius 2.
2) Let A be the point of affix $2i$ and B the point of affix -4, $|z-2i|=|z+4|$ is equivalent to $|z_M-z_A|=|z_M-z_B|$ and to $|z_{\overrightarrow{AM}}|=|z_{\overrightarrow{BM}}|$, which gives $AM=BM$ then M moves on the perpendicular bisector of $[AB]$.
3) Let $z=x+iy$, $z+\overline{z}=|z|$ is equivalent to $2x=x^2+y^2$ that is $(x-1)^2+y^2=1$ then M moves on the circle of center (1 ;0) and radius 1.
4) $z-i=\dfrac{4}{\overline{z}+i}$ is equivalent to $(z-i)(\overline{z}+i)=4$, that is $(z-i)(\overline{z-i})=4$, then $|z-i|^2=4$.

If A is the point of affix i then $|z-i|^2=4$ is equivalent to $AM^2=4$ then $AM=2$, therefore M moves on the circle of center A and radius 2.

5) $|\overline{z}+5-i|=|z-4i|$ is equivalent to $|\overline{z-(-5-i)}|=|z-4i|$ and to $|z-(-5-i)|=|z-4i|$.

Let A be the point of affix $-5-i$ and B the point of affix $4i$.
$|z-(-5-i)|=|z-4i|$ is equivalent to $AM = BM$ then M varies on the perpendicular bisector of $[AB]$.

6) $|z+1+i|\times|\overline{z}+1-i|=4$ is equivalent to $|z+1+i|\times|\overline{z+1+i}|=4$ and to $|z+1+i|^2=4$. Let A be the point of affix $-1-i$, $|z+1+i|^2=4$ is equivalent to $AM^2=4$, so $AM=2$ then M moves on the circle with center A and radius 2.

7) Let $z=x+iy$.
$z\overline{z}+2z-4i\overline{z}-4+2i=\left(x^2+y^2+2x-4y-4\right)+i(-4x+2y+2)$
and since it is a pure imaginary then $x^2+y^2+2x-4y-4=0$
so $(x+1)^2+(y-2)^2=1$ therefore M on the circle with center (-1 ; 2) and of radius 1.

8) The points $M(z)$, $N\left(z^2\right)$ and $P\left(z^3\right)$ are collinear, then $\overrightarrow{NP}$ and $\overrightarrow{MN}$ are collinear then $\dfrac{z_{\overrightarrow{NP}}}{z_{\overrightarrow{MN}}}=\dfrac{z_P-z_N}{z_N-z_M}=\dfrac{z^3-z^2}{z^2-z}=\dfrac{z^2(z-1)}{z(z-1)}=z$ is real which gives $y=0$, then M moves on the x-axis.

N°4.

1) $|z_1|=\sqrt{12+4}=4$, then $z_1=4\left(\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}i\right)=4e^{i\frac{\pi}{6}}$.

2) $z=\dfrac{2\sqrt{3}+2i}{\left(1+\sqrt{3}\right)+\left(1-\sqrt{3}\right)i}\times\dfrac{\left(1+\sqrt{3}\right)-\left(1-\sqrt{3}\right)i}{\left(1+\sqrt{3}\right)-\left(1-\sqrt{3}\right)i}=\dfrac{8+8i}{8}=1+i$.
$z=\sqrt{2}e^{i\frac{\pi}{4}}$.

3) $z=\dfrac{z_1}{z_2}$ gives $z_2=\dfrac{z_1}{z}$, then $|z_2|=\dfrac{|z_1|}{|z|}=\dfrac{4}{\sqrt{2}}=2\sqrt{2}$ and
$\arg z_2=\arg z_1-\arg z=\dfrac{\pi}{6}-\dfrac{\pi}{4}=-\dfrac{\pi}{12}(2\pi)$, thus:
$z_2=2\sqrt{2}\left(\cos\dfrac{\pi}{12}-i\sin\dfrac{\pi}{12}\right)=\left(1+\sqrt{3}\right)+\left(1-\sqrt{3}\right)i$

Therefore $2\sqrt{2}\cos\frac{\pi}{12}=1+\sqrt{3}$ and $-2\sqrt{2}\sin\frac{\pi}{12}=1-\sqrt{3}$, then

$\cos\frac{\pi}{12}=\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{2}+\sqrt{6}}{4}$ and $\sin\frac{\pi}{12}=\frac{1-\sqrt{3}}{-2\sqrt{2}}=\frac{\sqrt{6}-\sqrt{2}}{4}$.

N° 5.

1) a- $z_1=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2e^{i\frac{\pi}{3}}$, $z_2=\sqrt{2}e^{i\frac{\pi}{4}}$.

b- $Z=z_1z_2=2e^{i\frac{\pi}{3}}\times\sqrt{2}e^{i\frac{\pi}{4}}=2\sqrt{2}e^{i\left(\frac{\pi}{3}+\frac{\pi}{4}\right)}=2\sqrt{2}e^{i\frac{7\pi}{12}}$

2) a- $Z=z_1z_2=\left(1+i\sqrt{3}\right)\left(1+i\right)=1-\sqrt{3}+i\left(1+\sqrt{3}\right)$.

b- $Z=2\sqrt{2}e^{i\frac{7\pi}{12}}=2\sqrt{2}\left(\cos\frac{7\pi}{12}+i\sin\frac{7\pi}{12}\right)=1-\sqrt{3}+i\left(1+\sqrt{3}\right)$

So $2\sqrt{2}\cos\frac{7\pi}{12}=1-\sqrt{3}$ and $2\sqrt{2}\sin\frac{7\pi}{12}=1+\sqrt{3}$ then:

$\cos\frac{7\pi}{12}=\frac{1-\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{2}-\sqrt{6}}{4}$, $\sin\frac{7\pi}{12}=\frac{1+\sqrt{3}}{2\sqrt{2}}=\frac{\sqrt{2}+\sqrt{6}}{4}$.

3) $z_1^{\,n}=2^ne^{in\frac{\pi}{3}}$, z_1^n is real if $n\frac{\pi}{3}=k\pi$ then $n=3k$ with $k\in IN$.

4) $z_2^{\,p}=\left(\sqrt{2}\right)^pe^{i\frac{p\pi}{4}}$, z_2^p is pure imaginary if $\frac{p\pi}{4}=\frac{\pi}{2}+k\pi$

which gives $p=2+4k$, $k\in IN$.

N° 6.

1) a- $z'=\dfrac{-\frac{3}{5}+\frac{1}{5}i+2}{-\frac{3}{5}+\frac{1}{5}i-i}=\frac{7+i}{-3-4i}\times\frac{-3+4i}{-3+4i}=\frac{-25+25i}{25}=-1+i$.

b- $z' = -1+i = \sqrt{2}e^{i\frac{3\pi}{4}}$, so $(z')^{40} = \left(\sqrt{2}\right)^{40} e^{i(30\pi)}$, then

$(z')^{40} = 2^{20}$, real positive number.

2) a- $z' = \dfrac{x+iy+2}{x+i(y-1)} \times \dfrac{x-i(y-1)}{x-i(y-1)} = \dfrac{x^2+y^2+2x-y+i(x-2y+2)}{x^2+(y-1)^2}$

Then $x' = \dfrac{x^2+y^2+2x-y}{x^2+(y-1)^2}$ and $y' = \dfrac{x-2y+2}{x^2+(y-1)^2}$

b- z' is pure imaginary then $x' = \dfrac{x^2+y^2+2x-y}{x^2+(y-1)^2} = 0$.

Which gives $x^2+y^2+2x-y=0$, then M moves on the circle of center $I\left(-1;\dfrac{1}{2}\right)$, radius $R = \dfrac{\sqrt{5}}{2}$ deprived of point $(0;1)$.

3) a- $(z-i)(z'-1) = (z-i)\left(\dfrac{z+2}{z-i}-1\right) = (z-i)\left(\dfrac{z+2-z+i}{z-i}\right) = 2+i$

Which gives $|(z-i)(z'-1)| = |2+i|$ then $|z-i| \times |z'-1| = \sqrt{5}$,

b- M moves on the circle (C) of center A of affix i and of radius $R=1$, then $AM=1$, which is equivalent to $|z-i|=1$ so $|z'-1| = \sqrt{5}$ then M' varies on the circle of center $I(1;0)$ and of radius $\sqrt{5}$.

N° 7.

1) $(z+2i)(z'+2i) = (z+2i)\left(-2\bar{z}+2i+2i\right) =$
$-2(z+2i)\left(\bar{z}-2i\right) = -2(z+2i)\overline{(z+2i)} = -2|z+2i|^2$, non zero real negative number.

2) $(z+2i)(z'+2i)$ real negative number, then
$\arg(z+2i)+\arg(z'+2i) = \pi$ so:
$\arg(z'+2i) = \pi - \arg(z+2i)(2\pi) = \pi - \theta(2\pi)$.

3) The two semi straight lines $[AM)$ and $[AM')$ are symmetrical with respect to the y-axis.

N°8.

1) $z^2 = \left(1+\sqrt{3}\right)^2 + 2i\left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right) - \left(\sqrt{3}-1\right)^2$.

$z^2 = 4 + 2\sqrt{3} + 4i - 4 + 2\sqrt{3} = 4\sqrt{3} + 4i$

$\left|z^2\right| = 8$, then $z^2 = 8\left(\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i\right) = 8e^{i\frac{\pi}{6}}$

2) If $z = re^{i\alpha}$ then $z^2 = r^2e^{2i\alpha}$ so $r^2e^{2i\alpha} = 8e^{i\frac{\pi}{6}}$

Which gives $r^2 = 8$ and $2\alpha = \dfrac{\pi}{6} + 2k\pi$, therefore $r = 2\sqrt{2}$ and

$\alpha = \dfrac{\pi}{12} + k\pi$ then $\alpha = \dfrac{\pi}{12}$ or $\alpha = \dfrac{13\pi}{12}$, but $\mathrm{Re}(z) > 0$ and

$\mathrm{Im}(z) > 0$ then $\arg(z) = \dfrac{\pi}{12}$ therefore $z = 2\sqrt{2}e^{i\frac{\pi}{12}}$

3) $z = 2\sqrt{2}e^{i\frac{\pi}{12}} = 2\sqrt{2}\left(\cos\dfrac{\pi}{12} + i\sin\dfrac{\pi}{12}\right) = 1 + \sqrt{3} + i\left(\sqrt{3}-1\right)$

Then $2\sqrt{2}\cos\dfrac{\pi}{12} = 1 + \sqrt{3}$ and $2\sqrt{2}\sin\dfrac{\pi}{12} = \sqrt{3} - 1$

Therefore $\cos\dfrac{\pi}{12} = \dfrac{1+\sqrt{3}}{2\sqrt{2}} = \dfrac{\sqrt{2}+\sqrt{6}}{4}$ and $\sin\dfrac{\pi}{12} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$.

N°9.

MNP is a an isosceles triangle of principal vertex M then $MN = MP$,

So $\left|z_{\overrightarrow{MN}}\right| = \left|z_{\overrightarrow{MP}}\right|$, which is equivalent to $\left|z_N - z_M\right| = \left|z_P - z_M\right|$.

Which gives $|1 + 3z - z| = |1 + z - z|$, so $|1 + 2z| = 1$ or $\left|z + \dfrac{1}{2}\right| = \dfrac{1}{2}$.

Let A be the point of affix $-\dfrac{1}{2}$, $\left|z + \dfrac{1}{2}\right| = \dfrac{1}{2}$ is equivalent to

$\left|z_M - z_A\right| = \dfrac{1}{2}$ and to $\left|z_{\overrightarrow{AM}}\right| = \dfrac{1}{2}$, then $AM = \dfrac{1}{2}$, therefore M moves on

the circle of center A and radius $\dfrac{1}{2}$.

N° 10.

1) z' is pure imaginary then $\overline{z'}=-z'$ so $\overline{\left(\frac{z-5}{z-1}\right)}=-\frac{z-5}{z-1}$

which gives $\frac{\overline{z}-5}{\overline{z}-1}=-\frac{z-5}{z-1}$, then $z\overline{z}-\overline{z}-5z+5=-z\overline{z}+5\overline{z}+z-5$

so $2z\overline{z}-6\overline{z}-6z+10=0$ and therefore $z\overline{z}-3\left(z+\overline{z}\right)+5=0$.

z' is pure imaginary then $z\overline{z}-3\left(z+\overline{z}\right)+5=0$, if $z=x+iy$ then $x^2+y^2-6x+5=0$ or $\left(x-3\right)^2+y^2=4$ then M moves on the circle of center $\left(3;0\right)$ and radius 2.

2) $\left|z-5\right|=\left|z_M-z_B\right|=\left|z_{\overrightarrow{BM}}\right|=BM$,

$\left|z-1\right|=\left|z_M-z_A\right|=\left|z_{\overrightarrow{AM}}\right|=AM$, $\left|z'\right|=OM'=\frac{BM}{AM}$ and

$\arg\left(z'\right)=\arg\left(\frac{z_{\overrightarrow{BM}}}{z_{\overrightarrow{AM}}}\right)=\left(\overrightarrow{AM};\overrightarrow{BM}\right)\left(2\pi\right)$.

3) a- z' is pure imaginary then $\arg\left(z'\right)=\left(\overrightarrow{AM};\overrightarrow{BM}\right)=\frac{\pi}{2}+k\pi$,
M varies on the circle of diameter [AB] deprived of the points A and B.

b- z' is a real number then $\arg\left(z'\right)=\left(\overrightarrow{AM};\overrightarrow{BM}\right)=k\pi$, M moves on the straight line $\left(AB\right)$ deprived of the point A.

N° 11.

1) a- $e^{i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}}+e^{-i\frac{\theta}{2}}\right)=e^{i\theta}+1=z$.

b- $e^{i\frac{\theta}{2}}+e^{-i\frac{\theta}{2}}=2\,\text{Re}\left(e^{i\frac{\theta}{2}}\right)=2\cos\frac{\theta}{2}$, then $z=2\cos\frac{\theta}{2}e^{i\frac{\theta}{2}}$

But $\pi<\theta<2\pi$, then $\frac{\pi}{2}<\frac{\theta}{2}<\pi$, which gives $\cos\frac{\theta}{2}<0$.

$$z=-2\cos\frac{\theta}{2}e^{i\frac{\theta}{2}}\times e^{i\pi}=-2\cos\frac{\theta}{2}e^{i\left(\pi+\frac{\theta}{2}\right)}$$, therefore

$|z|=-2\cos\frac{\theta}{2}$ and $\arg(z)=\left(\pi+\frac{\theta}{2}\right)(2\pi)$.

2) $z=1-\frac{\sqrt{3}}{2}-\frac{1}{2}i=1+e^{i\frac{7\pi}{6}}$, then:

$$z=-2\cos\frac{7\pi}{12}e^{i\left(\pi+\frac{7\pi}{12}\right)}=-2\cos\frac{7\pi}{12}e^{i\frac{19\pi}{12}}.$$

N° 12.

1) $\frac{z_3-z_1}{z_2-z_1}=\frac{(1-i)z_1+iz_2-z_1}{z_2-z_1}=\frac{i(z_2-z_1)}{z_2-z_1}=i$.

$\left|\frac{z_3-z_1}{z_2-z_1}\right|=|i|$, then $\frac{M_1M_3}{M_2M_1}=1$ and $\arg\left(\frac{z_3-z_1}{z_2-z_1}\right)=\arg(i)$, so

$\left(\overrightarrow{M_1M_2};\overrightarrow{M_1M_3}\right)=\frac{\pi}{2}(2\pi)$. Then the triangle $M_1M_2M_3$ is a direct right isosceles triangle of principal vertex M_1.

2) Let z_E be the affix of E, since ABE is a direct right isosceles triangle of principal vertex A, then $(1-i)z_A+iz_B-z_E=0$ which gives $z_E=(1-i)z_A+iz_B=(1-i)(-1+2i)+i(2+i)=5i$.

Let z_F be the affix of F, since AFB is a direct right isosceles triangle of principal vertex A, then $(1-i)z_A+iz_F-z_B=0$ which gives $iz_F=z_B-(1-i)z_A=2+i-(1-i)(-1+2i)=1-2i$.

Therefore $z_F=-2-i$.

N° 13.

1) a- $|a|=\sqrt{4+4}=2\sqrt{2}$, then $a=2\sqrt{2}\left(\frac{1}{\sqrt{2}}+i\frac{1}{\sqrt{2}}\right)$.

Thus, $a=2\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$.

b- We have : $|b|=\sqrt{\left(1-\sqrt{3}\right)^2+\left(1+\sqrt{3}\right)^2}=\sqrt{8}=2\sqrt{2}$.

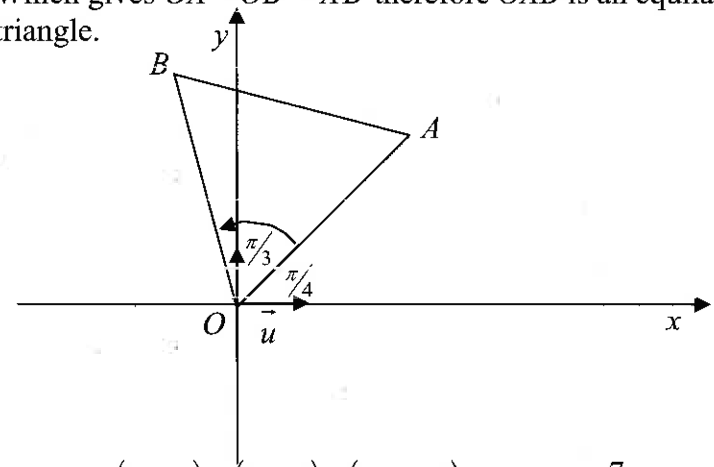
So $|a|=|b|$, and consequently $OA = OB$.

2) a- $z = b - a = -1-\sqrt{3}+i\left(-1+\sqrt{3}\right)$.

Therefore, $|z| = \sqrt{1+3+2\sqrt{3}+1+3-2\sqrt{3}} = 2\sqrt{2}$.

But $|z| = |b-a| = |z_B - z_A| = |z_{\overrightarrow{AB}}| = AB$.

Which gives $OA = OB = AB$ therefore OAB is an equilateral triangle.



b- $\arg(b) = \left(\vec{u};\overrightarrow{OB}\right) = \left(\vec{u};\overrightarrow{OA}\right) + \left(\overrightarrow{OA};\overrightarrow{OB}\right) = \frac{\pi}{4}+\frac{\pi}{3} = \frac{7\pi}{12}(2\pi)$,so

$\arg(b) = \frac{7\pi}{12}(2\pi)$.

Therefore $b = 2\sqrt{2}e^{i\frac{7\pi}{12}} = 2\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$.

And since $b = 1-\sqrt{3}+i\left(1+\sqrt{3}\right)$ we will get:

$2\sqrt{2}\cos\frac{7\pi}{12} = 1-\sqrt{3}$ and $2\sqrt{2}\sin\frac{7\pi}{12} = 1+\sqrt{3}$.

Which gives:

$\cos\frac{7\pi}{12} = \frac{1-\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{4}$ and $\sin\frac{7\pi}{12} = \frac{1+\sqrt{3}}{2\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$.

N° 14.

1) $\bar{z} = \cos(-\varphi) + i\sin(-\varphi) = e^{-i\varphi}$.

2) We have: $\cos\varphi + i\sin\varphi = e^{i\varphi}$ and $\cos\varphi - i\sin\varphi = e^{-i\varphi}$

By adding, we get $\cos\varphi = \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi}\right)$

and by subtracting, we get $\sin\varphi = \frac{1}{2i}\left(e^{i\varphi} - e^{-i\varphi}\right)$.

3) $\cos^2 x \sin^4 x = \frac{1}{2^2}\left(e^{ix} + e^{-ix}\right)^2 \cdot \frac{1}{2^4 \cdot i^4}\left(e^{ix} - e^{-ix}\right)^4$

$$= \frac{1}{2^6}\left(e^{ix} + e^{-ix}\right)^2\left(e^{ix} - e^{-ix}\right)^2\left(e^{ix} - e^{-ix}\right)^2$$

$$= \frac{1}{64}\left(e^{2ix} - e^{-2ix}\right)^2\left(e^{ix} - e^{-ix}\right)^2$$

$$= \frac{1}{64}\left(e^{4ix} + e^{-4ix} - 2\right)\left(e^{2ix} + e^{-2ix} - 2\right)$$

$$= \frac{1}{64}\left[e^{6ix} + e^{-6ix} - 2\left(e^{4ix} + e^{-4ix}\right) - \left(e^{2ix} + e^{-2ix}\right) + 4\right]$$

$$= \frac{1}{32}\left(\cos 6x - 2\cos 4x - \cos 2x + 2\right).$$

By using the same reasoning we obtain:

$\cos x \sin^4 x = \frac{1}{16}\left(\cos 5x - 3\cos 3x + 2\cos x\right)$.

4) In order to prove that a is a real number, it is sufficient to demonstrate that $\overline{a} = a$.

$$\overline{a} = \overline{\left(\frac{e^{i\theta} + e^{i\theta'}}{1 + e^{i\theta} \cdot e^{i\theta'}}\right)} = \frac{\overline{\left(e^{i\theta}\right)} + \overline{\left(e^{i\theta'}\right)}}{1 + \overline{\left(e^{i\theta}\right)} \cdot \overline{\left(e^{i\theta'}\right)}} = \frac{e^{-i\theta} + e^{-i\theta'}}{1 + e^{-i\theta} \cdot e^{-i\theta'}}.$$

So, $\overline{a} = \dfrac{\frac{1}{e^{i\theta}} + \frac{1}{e^{i\theta'}}}{1 + \frac{1}{e^{i\theta}} \cdot \frac{1}{e^{i\theta'}}} = \dfrac{e^{i\theta} + e^{i\theta'}}{1 + e^{i\theta} \cdot e^{i\theta'}} = a$ .Thus, a is real.

N° 15.

1) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)} + (z_1 - z_2)\overline{(z_1 - z_2)}$

$$= (z_1 + z_2)\left(\overline{z_1} + \overline{z_2}\right) + (z_1 - z_2)\left(\overline{z_1} - \overline{z_2}\right)$$

$$= 2z_1\overline{z_1} + 2z_2\overline{z_2} = 2|z_1|^2 + 2|z_2|^2 = 2\left(|z_1|^2 + |z_2|^2\right).$$

2) M_1 has an affix of $1 + i$ so $M_1(1,1)$.

M_2 has an affix of $1 - i\sqrt{3}$ so $M_2\left(1, -\sqrt{3}\right)$.

a- $\overrightarrow{OM} = \overrightarrow{OM_1} + \overrightarrow{OM_2}$ therefore $z_M = z_{M_1} + z_{M_2} = 2 + i(1-\sqrt{3})$.

b- $OM_1 = |1+i| = \sqrt{2}$; $OM_2 = |1-i\sqrt{3}| = 2$.

$OM = \left|2+i(1-\sqrt{3})\right| = \sqrt{4+(1-\sqrt{3})^2} = \sqrt{8-2\sqrt{3}}$.

Also, $z_{\overrightarrow{M_1M_2}} = z_{M_2} - z_{M_1} = 1 - i\sqrt{3} - 1 - i = i(-1-\sqrt{3})$.

So $M_1M_2 = \left|i(-1-\sqrt{3})\right| = 1+\sqrt{3}$.

c- $M_1M_2^2 + OM^2 = (1+\sqrt{3})^2 + \left(\sqrt{8-2\sqrt{3}}\right)^2 = 12$.

$2\left(OM_1^2 + OM_2^2\right) = 2(2+4) = 12$.

Therefore $M_1M_2^2 + OM^2 = 2\left(OM_1^2 + OM_2^2\right)$.

N° 16.

1) $j^3 = \left(e^{2i\frac{\pi}{3}}\right)^3 = e^{2i\pi} = 1$.

$$j = e^{2i\frac{\pi}{3}} = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$j^2 = e^{4i\frac{\pi}{3}} = e^{i\left(\frac{4\pi}{3}-2\pi\right)} = e^{-2i\frac{\pi}{3}} = \bar{j} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

Then $1 + j + j^2 = 1 - \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2} = 0$.

Similarly $e^{i\frac{\pi}{3}} + j^2 = \frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - \frac{i\sqrt{3}}{2} = 0$.

2) a- $\dfrac{c-a}{b-a} = \dfrac{Z_C - Z_A}{Z_B - Z_A} = \dfrac{Z_{\overrightarrow{AC}}}{Z_{\overrightarrow{AB}}} = \dfrac{AC}{AB} e^{i\left(\overrightarrow{AB};\overrightarrow{AC}\right)} = e^{i\frac{\pi}{3}}$.

Therefore $\dfrac{c-a}{b-a} = e^{i\frac{\pi}{3}}$.

b- $e^{i\frac{\pi}{3}} = -j^2$ therefore $\dfrac{c-a}{b-a} = -j^2$ which gives

$c - a = -bj^2 + aj^2$.

So $(1+j^2)a - bj^2 - c = 0$ or $-ja - bj^2 - c = 0$.

Thus $ja+bj^2+c=0$, by multiplying by j^2, we get:
$j^3a+bj^4+cj^2=0$ therefore $a+bj+cj^2=0 \quad (1)$.

3) a- M and M' are distinct if $z \neq \bar{z}$ thus $z-\bar{z} \neq 0$ which gives $2iy \neq 0$ thus $y \neq 0$.

b- RMM' being equilateral then $z_R+jz_M+j^2z_{M'}=0$ according to (1). Which gives $1+jz+j^2\bar{z}=0$,

so $1+\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)(x+iy)+\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)(x-iy)=0$,

therefore, $1-x-y\sqrt{3}=0$ hence M describes the line of equation $x+y\sqrt{3}-1=0$.

N° 17.

1) $\dfrac{z_B-z_A}{z_C-z_D}=\dfrac{7-2i+1-2i}{6+i-2-3i}=\dfrac{8-4i}{4-2i}=\dfrac{4-2i}{2-i}=2=2e^{0i}$

Then $\dfrac{z_B-z_A}{z_C-z_D}=\dfrac{z_{\overrightarrow{AB}}}{z_{\overrightarrow{DC}}}=\dfrac{AB}{DC}e^{i\left(\overrightarrow{DC};\overrightarrow{AB}\right)}=2e^{0i}$.

Which gives $\dfrac{AB}{DC}=2$ and $\left(\overrightarrow{DC};\overrightarrow{AB}\right)=0(2\pi)$, therefore the straight lines (AB) and (CD) are parallel.

2) $\dfrac{z_C-z_B}{z_A-z_D}=\dfrac{-1+3i}{-3-i}\times\dfrac{-3+i}{-3+i}=-i$, then

$\dfrac{z_C-z_B}{z_A-z_D}=\dfrac{z_{\overrightarrow{BC}}}{z_{\overrightarrow{DA}}}=\dfrac{BC}{DA}e^{i\left(\overrightarrow{DA};\overrightarrow{BC}\right)}=1e^{i\frac{\pi}{2}}$, which gives that

$\dfrac{BC}{DA}=1$ $\left(\overrightarrow{DA};\overrightarrow{BC}\right)=\dfrac{\pi}{2}(2\pi)$, therefore $ABCD$ is an isosceles trapezoid.

3) a- $\left|\dfrac{z_B-z_E}{z_A-z_E}\right|=\left|\dfrac{z_{\overrightarrow{EB}}}{z_{\overrightarrow{EA}}}\right|=\dfrac{EB}{EA}$.

$ABCD$ is an isosceles trapezoid then $AD=BC$.

On the other hand since $D\hat{A}B=C\hat{B}A$ then $E\hat{D}C=E\hat{C}D$

therefore EDC is an isosceles triangle of principal vertex E, then $ED = EC$ which implies that $EA = EB$.

Then $\left|\dfrac{z_B - z_E}{z_A - z_E}\right| = \left|\dfrac{z_{\overrightarrow{EB}}}{z_{\overrightarrow{EA}}}\right| = \dfrac{EB}{EA} = 1$

$\arg\left(\dfrac{z_B - z_E}{z_A - z_E}\right) = \arg\dfrac{z_{\overrightarrow{EB}}}{z_{\overrightarrow{EA}}} = \left(\overrightarrow{EA};\overrightarrow{EB}\right) = \dfrac{\pi}{2}(2\pi)$.

b- $\dfrac{z_B - z_E}{z_A - z_E} = i$, then $\dfrac{7-2i-z_E}{-1+2i-z_E} = i$ which gives $z_E = 5+4i$.

4) a- $|z_M - z_E| = \sqrt{10}$ is equivalent to $|z_{\overrightarrow{EM}}| = \sqrt{10}$ so $EM = \sqrt{10}$

then (T) is the circle of center E and of radius $\sqrt{10}$

b- $EC = \sqrt{10}$ then C belongs to (T).

N° 18.

1) $z = \sqrt{2}e^{i\frac{\pi}{4}} = 1+i$ then:

$z' = \dfrac{4-2\bar{z}}{\bar{z}} = \dfrac{4}{\bar{z}} - 2 = \dfrac{4}{1-i} - 2 = \dfrac{4(1+i)}{2} - 2 = 2i$.

2) a- $z' = \dfrac{4-2\bar{z}}{\bar{z}}$ gives $z'\bar{z} = 4 - 2\bar{z}$, so $z'\bar{z} + 2\bar{z} = 4$ therefore

$(z'+2)\bar{z} = 4$.

b- $(z'+2)\bar{z} = 4$ is equivalent to $(z_{M'} - z_A)\bar{z} = 4$ and to

$z_{\overrightarrow{AM'}}\bar{z} = 4$ which gives $|z_{\overrightarrow{AM'}}||\bar{z}| = 4$ and

$\arg(z_{\overrightarrow{AM'}}) + \arg(\bar{z}) = 0 + 2k\pi$

But $|\bar{z}| = |z| = OM$ and $|z_{\overrightarrow{AM'}}| = AM'$ then $|z_{\overrightarrow{AM'}}| \times |\bar{z}| = 4$ which is equivalent to $AM' \times OM = 4$.

$\arg(z_{\overrightarrow{AM'}}) + \arg(\bar{z}) = 0 + 2k\pi$ gives

$\arg(z_{\overrightarrow{AM'}}) - \arg(z) = 0 + 2k\pi$ or $\left(\vec{u};\overrightarrow{AM'}\right) - \left(\vec{u};\overrightarrow{OM}\right) = 0(2\pi)$ so

$\left(\overrightarrow{OM};\overrightarrow{AM'}\right) = 0(2\pi)$ then the vectors $\overrightarrow{AM'}$ and $\overrightarrow{OM}$ are collinear of same sense.

c- M moves on the circle (C) of center O and radius 1 , so $OM = 1$, but $AM' \times OM = 4$ then $AM' = 4$, then (T) is the circle of center A and radius 4.

N° 19.

1) $z = re^{i\theta}$ therefore $z + \bar{z} = 2r\cos\theta$.

However, $|z_1| = |z + \bar{z}||z| = |2r\cos\theta||z|$.

Since $\frac{\pi}{2} < \theta < \pi$, then $\cos\theta < 0$, therefore $|2r\cos\theta| = -2r\cos\theta$.

Since $|z| = r$ we get: $|z_1| = -2r^2\cos\theta$.

$\arg(z_1) = \arg(z + \bar{z}) + \arg(z)$.

Since $z + \bar{z}$ is a negative real number then $\arg(z + \bar{z}) = \pi(2\pi)$.

Therefore, $\arg(z_1) = \pi + \arg(z) = \pi + \theta(2\pi)$.

We found that $\arg(z_1) = \pi + \arg(z)$ thus $(\vec{u};\overrightarrow{OM_1}) - (\vec{u};\overrightarrow{OM}) = \pi\ (2\pi)$ hence $(\overrightarrow{OM};\overrightarrow{OM_1}) = \pi\ (2\pi)$, therefore points O, M and M_1 are collinear and O belongs to segment $[MM_1]$.

2) a- $z' = i\dfrac{z_M - z_A}{z_M - z_B} = i\dfrac{z_{\overrightarrow{AM}}}{z_{\overrightarrow{BM}}}$ therefore

$$OM' = |z'| = |i| \times \left|\frac{z_{\overrightarrow{AM}}}{z_{\overrightarrow{BM}}}\right| = 1 \times \frac{AM}{BM} = \frac{AM}{BM}.$$

$$\arg(z') = \arg(i) + \arg\left(\frac{z_{\overrightarrow{AM}}}{z_{\overrightarrow{BM}}}\right) = \frac{\pi}{2} + (\overrightarrow{BM};\overrightarrow{AM})(2\pi).$$

b- If M describes (C), then $(\overrightarrow{BM};\overrightarrow{AM}) = \pm\frac{\pi}{2}(2\pi)$ since $[AB]$ is a diameter, hence $\arg(z') = 0(\pi)$, thus z' is real and so M' varies on $x'x$.

N° 20.

1) $\dfrac{z_B - z_A}{z_C - z_A} = \dfrac{3i+1}{2-i+1} = i = e^{i\frac{\pi}{2}}$, but $\dfrac{z_B - z_A}{z_C - z_A} = \dfrac{z_{\overrightarrow{AB}}}{z_{\overrightarrow{AC}}} = \dfrac{AB}{AC}e^{i(\overrightarrow{AC},\overrightarrow{AB})}$

then, $\frac{AB}{AC}=1$ and $\left(\overrightarrow{AC},\overrightarrow{AB}\right)=\frac{\pi}{2}(2\pi)$. Consequently, triangle ABC is right isosceles at A.

2) a- $z'=\frac{iz+3}{z+1}=i\frac{\left(z+\frac{3}{i}\right)}{z+1}=i\frac{z-3i}{z+1}$

b- $z'=i\frac{z_M-z_B}{z_M-z_A}=i\frac{z_{\overrightarrow{BM}}}{z_{\overrightarrow{AM}}}=i\frac{BM}{AM}e^{i\left(\overrightarrow{AM},\overrightarrow{BM}\right)}$

Then, $OM'=|z'|=|i|\frac{BM}{AM}=\frac{BM}{AM}$ and

$\left(\vec{u},\overrightarrow{OM'}\right)=\arg(z')=\arg(i)+\left(\overrightarrow{AM},\overrightarrow{BM}\right)(2\pi)$

$=\frac{\pi}{2}+\left(\overrightarrow{AM};\overrightarrow{BM}\right)(2\pi)$.

c- • M moves on the circle of center O and radius 1 then, $OM'=1$ and consequently, $AM=BM$.
Hence, M moves on the perpendicular bisector of $[AB]$.

• z' is real, then $\arg(z')=k\pi$ which gives
$\frac{\pi}{2}+\left(\overrightarrow{AM},\overrightarrow{BM}\right)=k\pi$. Consequently, $\left(\overrightarrow{AM},\overrightarrow{BM}\right)=-\frac{\pi}{2}+k\pi$.
Therefore, M moves on the circle of diameter $[AB]$ deprived of point A.

$N°21$.

1) $z=2e^{i\frac{\pi}{6}}$ so $z'=\frac{1}{z}=\frac{1}{2}e^{i\frac{\pi}{6}}$, thus $z'=\frac{1}{2}\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)=\frac{\sqrt{3}}{4}+\frac{1}{4}i$.

For $z=\sqrt{2}e^{-i\frac{\pi}{4}}$, we get $z'=\frac{1}{z}=\frac{1}{\sqrt{2}}e^{-i\frac{\pi}{4}}=\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)$,

then $z'=\frac{1}{2}-\frac{1}{2}i$.

2) a- $z'=\frac{1}{\bar{z}}$, so $|z'|=\frac{1}{|\bar{z}|}=\frac{1}{|z|}$, therefore $OM'=\frac{1}{OM}$ and so $OM\times OM'=1$.

b- $\left(\vec{u};\overrightarrow{OM'}\right) = \arg(z') = \arg\left(\dfrac{1}{\bar{z}}\right) = -\arg(\bar{z}) = \arg(z) = \left(\vec{u};\overrightarrow{OM}\right)$.

Therefore, points O, M and M' are collinear.

3) $\dfrac{z_{\overrightarrow{OF}}}{z_{\overrightarrow{OE}}} = \dfrac{1}{z\bar{z}} = \dfrac{1}{|z|^2}$, so:

$\arg\left(\dfrac{z_{\overrightarrow{OF}}}{z_{\overrightarrow{OE}}}\right) = \arg\left(\dfrac{1}{|z|^2}\right) = 0(2\pi)$, so $\left(\overrightarrow{OE};\overrightarrow{OF}\right) = 0(2\pi)$, then

O, E and F are collinear.

4) $z'-1 = \dfrac{1}{\bar{z}} - 1 = \dfrac{1-\bar{z}}{\bar{z}}$, so:

$\overline{z'-1} = \overline{\left(\dfrac{1-\bar{z}}{\bar{z}}\right)} = \dfrac{\overline{1-\bar{z}}}{\bar{\bar{z}}} = \dfrac{1-z}{z} = \dfrac{1}{z} - 1$.

5) a- $|z-1| = |z_M - z_E| = |z_{\overrightarrow{EM}}| = EM$.

But, radius of circle (C) is $EM = 1$, thus $|z-1| = 1$.

b- $|z'-1| = \left|\dfrac{1}{\bar{z}} - 1\right| = \left|\dfrac{1-\bar{z}}{\bar{z}}\right| = \dfrac{|1-\bar{z}|}{|\bar{z}|}$,

Then, $|z'-1| = \dfrac{|\overline{1-z}|}{|\bar{z}|} = \dfrac{|1-z|}{|\bar{z}|} = \dfrac{|z-1|}{|\bar{z}|} = \dfrac{1}{|\bar{z}|} = |z'|$.

c- $|z'-1| = |z'|$, gives $|z_{M'} - z_E| = |z_{M'}|$,

thus $|z_{\overrightarrow{EM'}}| = |z_{\overrightarrow{OM'}}|$, therefore $EM' = OM'$. Hence the point M' describes the perpendicular bisector of $[OE]$ of equation $x = \dfrac{1}{2}$.

$\boxed{N^\circ 22}$

1) $[AB]$ and $[MN]$ are two intersecting cords of the circle then

$OM \times ON = OA \times OB$, so $r \times ON = 2 \times 2$, which gives $ON = \dfrac{4}{r}$.

On the other hand :

$\arg(z_N) = (\vec{u};\overrightarrow{ON}) = (\vec{u};\overrightarrow{OM}) + (\overrightarrow{OM};\overrightarrow{ON}) = \theta + \pi \quad (2\pi)$

Then $z_N = \frac{4}{r}e^{i(\theta+\pi)} = \frac{4}{r}e^{i\theta}e^{i\pi} = -\frac{4}{r}e^{i\theta}$.

P is the symmetric of N with respect to the y-axis , then

$z_P = -\overline{z_N} = \frac{4}{r}e^{-i\theta}$.

2) a- $\overrightarrow{OQ} = \overrightarrow{OM} + \overrightarrow{OP}$ then :

$$z_Q = z_M + z_P = re^{i\theta} + \frac{4}{r}e^{-i\theta} = \left(r + \frac{4}{r}\right)\cos\theta + i\left(r - \frac{4}{r}\right)\sin\theta$$

Then $x_Q = \left(r + \frac{4}{r}\right)\cos\theta$ and $y_Q = \left(r - \frac{4}{r}\right)\sin\theta$.

b- $r = 4$ gives $x_Q = 5\cos\theta$ and $y_Q = 3\sin\theta$ witch gives

$\cos\theta = \frac{x_Q}{5}$ and $\sin\theta = \frac{y_Q}{3}$ .But $\cos^2\theta + \sin^2\theta = 1$, then

$\frac{x_Q{}^2}{25} + \frac{y_Q{}^2}{9} = 1$, consequently if M varies on (C) then Q moves

on the ellipse of equation $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

N° 23.

Part A.

1) a- Let $A' = f(A)$, $z_{A'} = z_A^2 - 4z_A = (1-i)^2 - 4(1-i) = -4 + 2i$.

Let $B' = f(B)$, $z_{B'} = z_B^2 - 4z_B = (3+i)^2 - 4(3+i) = -4 + 2i$.

Then A and B have the same image by f.

b- If C and D have the same image by f, then

$z_C^2 - 4z_C = z_D^2 - 4z_D$, so $z_C^2 - z_D^2 - 4z_C + 4z_D = 0$, that is

$(z_C - z_D)(z_C + z_D) - 4(z_C - z_D) = 0$, or

$(z_C - z_D)(z_C + z_D - 4) = 0$, which gives $z_C - z_D = 0$ or

$z_C + z_D - 4 = 0$, then $z_C = z_D$ or $\frac{z_C + z_D}{2} = 2$.

Then these two points are confounded or they are symmetric with respect to the point $J(2)$.

2) a- $OMIM'$ is a parallelogram if and only if $\overrightarrow{OM'} = \overrightarrow{MI}$.
then $z_{\overrightarrow{OM'}} = z_{\overrightarrow{MI}}$ which gives
$z' = -3 - z$ therefore $z^2 - 4z = -3 - z$, then $z^2 - 3z + 3 = 0$.

b- $\omega^2 = 9 - 12 = -3 = 3i^2$, so $z_1 = \frac{3}{2} + i\frac{\sqrt{3}}{2}$ and $z_2 = \frac{3}{2} - i\frac{\sqrt{3}}{2}$.

3) a- $z' + 4 = z^2 - 4z + 4 = (z-2)^2$, $|z'+4| = \left|(z-2)^2\right| = |z-2|^2$,
$\arg(z'+4) = \arg(z-2)^2 = 2\arg(z-2)(2\pi)$.

b- $JM = 2$ gives $|z-2| = 2$ so $|z'+4| = 4$ then
$KM' = 4$ consequently M' varies on the circle (C') of center K and radius 4.

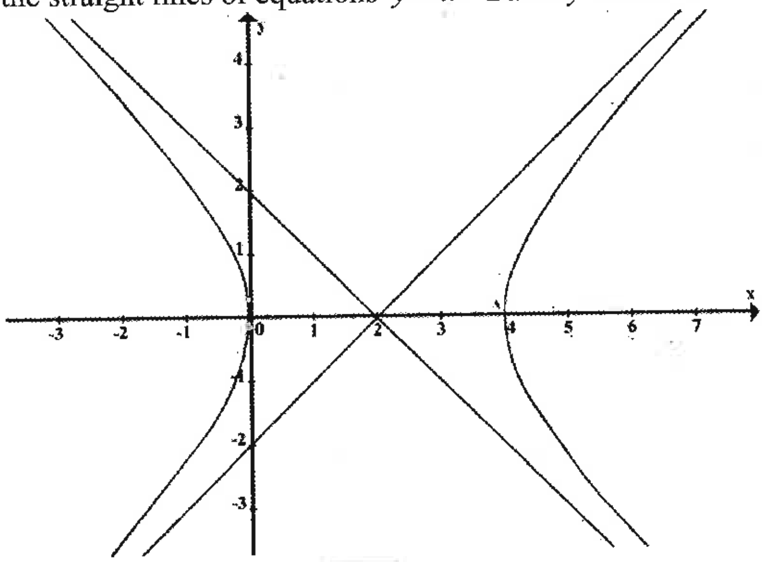
c- $z_E + 4 = -3i = 3e^{-i\frac{\pi}{2}}$.

***Part B*.**

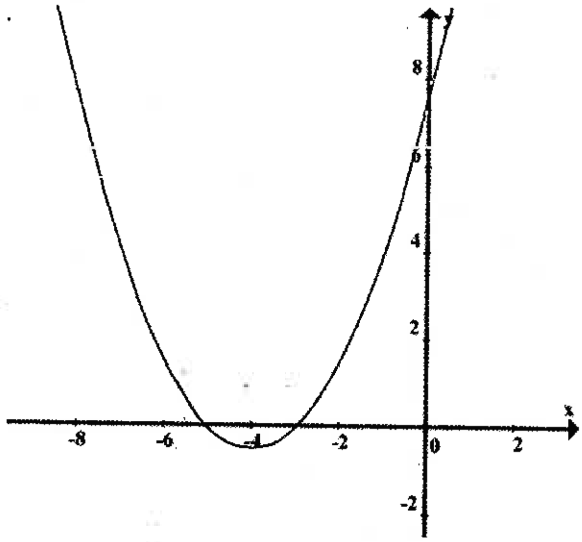
1) $x' + iy' = (x+iy)^2 - 4(x+iy) = x^2 - y^2 + 2ixy - 4x - 4iy$
Then $x' = x^2 - y^2 - 4x$ and $y' = 2xy - 4y$.

2) a- z' pure imaginary then $x' = 0$ which gives $x^2 - y^2 - 4x = 0$ as an equation which is equivalent to $(x-2)^2 - y^2 = 4$, which is the equation of a rectangular hyperbola.

b- The vertices are $A(4;0)$ and O and the asymptotes of (H) are the straight lines of equations $y = x - 2$ and $y = -x + 2$.



3) $y = x-3$ then $x' = x^2 - (x-3)^2 - 4x = 2x - 9$ and
$y' = 2x(x-3) - 4(x-3) = 2x^2 - 10x + 12$.
$x' = 2x - 9$ gives $x = \dfrac{x'+9}{2}$ and by substitution in y' we get:
$2y' = x'^2 + 8x' + 15$ or $(x'+4)^2 = 2\left(y' + \dfrac{1}{2}\right)$
which is the equation of a parabola of vertex $S\left(-4; -\dfrac{1}{2}\right)$ and of focus $F(-4;0)$.



N°24.

Part A .

1) a- $\dfrac{z_1 - 3}{z_1} = \dfrac{2+i\sqrt{2}-3}{2+i\sqrt{2}} = \dfrac{-1+i\sqrt{2}}{2+i\sqrt{2}} \times \dfrac{2-i\sqrt{2}}{2-i\sqrt{2}} = \dfrac{1}{2} i\sqrt{2}$.

b- $\dfrac{z_{\overrightarrow{BM_1}}}{z_{\overrightarrow{OM_1}}} = \dfrac{1}{2} i\sqrt{2}$, pure imaginary, then $\overrightarrow{BM_1}$ and $\overrightarrow{OM_1}$ are perpendicular, therefore OBM_1 is a right angled triangle at M_1.

2) The triangle OM_2B is the symmetric of the triangle OM_1B with respect to $x'x$ then OM_2B is right at M_2 then the points O, B, M_1 and M_2 belong to the same circle (T) of diameter $[OB]$.

The points M_1 and M_2 are the points of intersection of (T) with the straight line of equation $x = 2$.

***Part B*.**

1) $z_{\overrightarrow{AM}} = z_M - z_A$, so $\sqrt{2}e^{i\theta} = z - 2$ then $z = 2 + \sqrt{2}e^{i\theta}$.

2) $z' = z^2 - 4z + 6 = (z-2)^2 + 2$, then $z' - 2 = (z-2)^2$ which gives

$z' = 2 + \left(\sqrt{2}e^{i\theta}\right)^2 = 2 + 2e^{2i\theta}$.

$z' - 2 = 2e^{2i\theta}$ then $|z' - 2| = 2$ then $AM' = 2$ therefore M' varies on the circle (Γ') of center A and radius 2.

3) a- $d - 2 = \dfrac{\sqrt{2} + i\sqrt{6}}{2} = \sqrt{2}e^{i\frac{\pi}{3}}$.

$|d-2| = \sqrt{2}$ gives $AD = \sqrt{2}$ then D belongs to the circle (Γ).

b- $z' - 2 = (z-2)^2$ gives:

$\arg(z'-2) = \arg(z-2)^2 = 2\arg(z-2)(2\pi)$.

Then $\left(\vec{u}; \overrightarrow{AD'}\right) = 2\left(\vec{u}; \overrightarrow{AD}\right)(2\pi) = \dfrac{2\pi}{3}(2\pi)$.

$\dfrac{z_{\overrightarrow{OD'}}}{z_{\overrightarrow{OA}}} = \dfrac{d'}{2} = \dfrac{d^2 - 4d + 6}{2} = \dfrac{(d-2)^2 + 2}{2} = 1 + e^{2i\frac{\pi}{3}}$

$= \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2} = e^{i\frac{\pi}{3}}$, so:

$\dfrac{OD'}{OD} = \left|e^{i\frac{\pi}{3}}\right| = 1$ and $\left(\overrightarrow{OA}; \overrightarrow{OD'}\right) = \arg\left(e^{i\frac{\pi}{3}}\right) = \dfrac{\pi}{3}(2\pi)$, then the triangle OAD' is equilateral.

Indications

N°2.

1) (E) is the perpendicular bisector of $[AB]$.

2) $\left(\overrightarrow{AM};\overrightarrow{BM}\right)=\frac{\pi}{2}(2\pi)$.

N°4.

1) $z=2e^{i\left(\theta+\frac{\pi}{3}\right)}$

$z=\left(1+i\sqrt{3}\right)e^{i\theta}=\cos\theta-\sqrt{3}\sin\theta+i\left(\sqrt{3}\cos\theta+\sin\theta\right)$.

N°5.

$\frac{z_{\overrightarrow{IM}}}{z_{\overrightarrow{JN}}}=\frac{z-1}{2z-i}=\frac{z-1}{2\left(z-\frac{1}{2}i\right)}$ real number, if $C\left(\frac{1}{2}i\right)$ then $\overline{IM}$ and $\overline{CM}$ are collinear.

N°7.

2) $A(1)$, $B\left(z^2\right)$ and $C\left(\frac{1}{z^2}\right)$, $\frac{z_{\overrightarrow{AB}}}{z_{\overrightarrow{AC}}}=\frac{z^2-1}{\frac{1}{z^2}-1}=\frac{z^2\left(z^2-1\right)}{z^2-1}=z^2$ real number , and if $z=x+iy$ then $xy=0$.

N°8.

$\arg(z+a)=\arg z+\arg a$, gives $\arg(z+a)=\arg za$

and $\arg(z+a)-\arg za=0$, then $\arg\left(\frac{z+a}{za}\right)=0(\pi)$, therefore

$\frac{z+a}{za}$ is a real number.

N°9.

1) $a'=-2i$ and $b'=-2\bar{b}+2i=-2(3+2i)+2i=-6-2i$.

2) Take $z=x-2i$, we get $z'=-2x-2i$.

4) a- $(z+2i)(z'+2i)=(z+2i)\left(-2\bar{z}+4i\right)=-2(z+2i)(\bar{z}-2i)$.

$(z+2i)(z'+2i)=-2(z+2i)\overline{(z+2i)}=-2|z+2i|^2\leq 0$.

N° 10.

Part B.

1) a- $\arg\left(\dfrac{a-1}{a}\right)=\left(\overrightarrow{OA};\overrightarrow{IA}\right)(2\pi)$.

b- $\dfrac{z_{\overrightarrow{M'M}}}{z_{\overrightarrow{M'O}}}=\dfrac{z-z'}{-z'}=\dfrac{a-1}{a}$, then $\arg\left(\dfrac{z_{\overrightarrow{M'M}}}{z_{\overrightarrow{M'O}}}\right)=\arg\left(\dfrac{a-1}{a}\right)(2\pi)$

$\left(\overrightarrow{M'O},\overrightarrow{M'M}\right)=\arg\left(\dfrac{a-1}{a}\right)+2k\pi$.

$\arg\left(\dfrac{z_A-z_I}{z_A-z_O}\right)=\arg\left(\dfrac{z_{\overrightarrow{IA}}}{z_{\overrightarrow{OA}}}\right)=\left(\overrightarrow{OA};\overrightarrow{IA}\right)$.

c- If A belongs to the circle (C) then $\left(\overrightarrow{OA};\overrightarrow{IA}\right)=\dfrac{\pi}{2}(2\pi)$.

2) $\dfrac{z_{\overrightarrow{OM'}}}{z_{\overrightarrow{OA}}}=\dfrac{a\,z_M}{z_A}=\dfrac{a\,x}{a}=x$.

N° 11.

2) a- $|z+1|\times|z'+i|=|z+1|\times\left|\dfrac{-iz-2}{z+1}+i\right|=|-2+i|=\sqrt{5}$.

b- $AM=2$ gives $|z+1|=1$ then $|z'+i|=\sqrt{5}$.

N° 13.

1) $|z'|=\left|\dfrac{z-1}{1-\bar{z}}\right|=\dfrac{|z-1|}{|z-1|}=1$, $(z-1)(1-\bar{z})=z+\bar{z}-z\bar{z}-1$

Which is a real number because $z+\bar{z}$ and $z\bar{z}$ are real numbers.

N° 17.

2) a- $\bar{z}(z'-1)=\bar{z}\left(\dfrac{\bar{z}+2}{\bar{z}}-1\right)=\bar{z}\left(\dfrac{2}{\bar{z}}\right)=2$.

b- M' moves on the circle (C) then $AM'=1$, but $|\bar{z}(z'-1)|=2$,
then $|\bar{z}|\times|z'-1|=|z|\times AM'=2$, which gives $OM=2$.

Indications .

N° 23.

1) b- $\dfrac{z_{\overrightarrow{AB}}}{z_{\overrightarrow{AF}}}=\dfrac{z_B-z_A}{z_F-z_A}=\dfrac{1+e^{i\frac{\pi}{3}}-1}{2-1}=e^{i\frac{\pi}{3}}$. Then

$\arg\left(\dfrac{z_{\overrightarrow{AB}}}{z_{\overrightarrow{AF}}}\right)=\arg\left(e^{i\frac{\pi}{3}}\right)$, which gives $\left(\overrightarrow{AF};\overrightarrow{AB}\right)=\dfrac{\pi}{3}(2\pi)$.

2) a- $z_E-z_A=1+z_B^2-1=z_B^2=\left(1+e^{i\frac{\pi}{3}}\right)^2$

$$z_E-z_A=\left(\frac{3}{2}+i\,\frac{\sqrt{3}}{2}\right)^2=\left(\sqrt{3}e^{i\frac{\pi}{6}}\right)^2=3e^{i\frac{\pi}{3}}\ .$$

N° 26.

1) $z'\times\overline{z}=-\dfrac{1}{r}e^{i(\pi+\theta)}\times re^{-i\theta}=e^{i\pi}=-1$.

3) b- $|z'+1|=\left|\dfrac{-1}{\overline{z}}+1\right|=\left|\dfrac{-1+\overline{z}}{\overline{z}}\right|=\left|\dfrac{\overline{z-1}}{\overline{z}}\right|=\dfrac{|z-1|}{|\overline{z}|}=\dfrac{1}{|\overline{z}|}=|z'|$.

CHAPTER 3

Integral

Chapter Review

1) ***Definition*** :

Given a continuous function f over an interval I of IR and F an antiderivative of f over I ; a and b are two real numbers of I.
We call the integral of f from a to b, the real number $F(b)-F(a)$.

We denote $\int_a^b f(x)\,dx = F(b)-F(a)$.

2). ***Properties***:

- If f is a continuous function over the interval I and if a , b and c are three real numbers of I, then :

$$\int_a^a f(x)\,dx = 0 \ ; \ \int_a^b f(x)dx = -\int_b^a f(x)\,dx \ ;$$

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

$$\int_a^b [\alpha\, f(x) + \beta\, g(x)]dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx \ .$$

- ***Integrals and Inequalities*** :

f and g are two continuous functions over $[a\,;b]$ with $a \le b$:

If $f \ge 0$ over $[a\,;b]$ then $\int_a^b f(x)\,dx \ge 0$.

If $f \le 0$ over $[a\,;b]$ then $\int_a^b f(x)\,dx \le 0$.

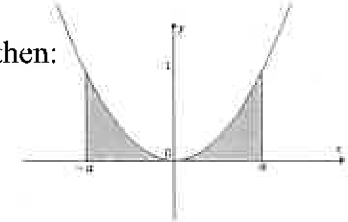
If $f \leq g$ over $[a;b]$ then $\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$.

- ***Integrals of Even and Odd Functions:***.

  Given f a continuous function over the interval $[-a;+a]$ with $a > 0$.

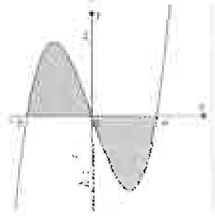
  If f is even over $[-a;+a]$ then:

  $$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$



  If f is odd over $[-a;+a]$ then:

  $$\int_{-a}^{a} f(x)\,dx = 0.$$



- ***Integrals of Periodic Functions:***

  N.B : A period of a function f is said to be the least positive real number T verifying the relation $f(x+T) = f(x)$.

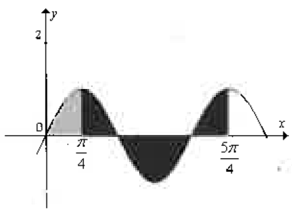
  Ex : $\sin(x+2\pi) = \sin x$ and $\sin(x+4\pi) = \sin x$.

  The period is $T = 2\pi$.

  If f is a continuous function over IR and periodic of period T, then for all real numbers a we have:

  $$\int_a^{a+T} f(x)\,dx = \int_0^T f(x)\,dx.$$

  Ex: $\int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} \sin 2x\,dx = \int_0^{\pi} \sin 2x\,dx = 0$.



- ***Integration by Parts*** .

  u and v are two differentiable functions over I with their derivatives u' and v' being continuous over I.

  For all real numbers a and b of I we have:

  $$\int_a^b u(x)\,v'(x)\,dx = [u(x)\,v(x)]_a^b - \int_a^b u'(x)\,v(x)\,dx.$$

- ***Fundamental Theorem of Integration .***
  Let f be a continuous function over an interval I .
  a and x are two real numbers of I with a being constant and x variable .

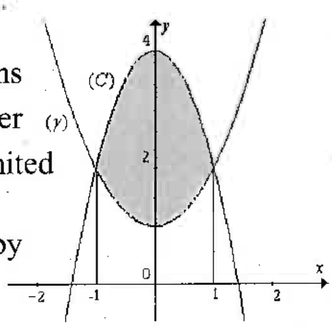
  If $F(x) = \int_a^x f(t)\,dt$ then $F'(x) = f(x)$.

  Ex : If $F(x) = \int_a^x t^2 dt$ then $F'(x) = x^2$.

3) ***Applications of Integral Calculations .***

- If f and g are two continuous functions over an interval $[a\,;b]$ and if $g < f$ over $[a\,;b]$ then the area A of the domain limited by the two representative curves (C) and (γ) of f and g is calculated by

  $A = \int_a^b (f(x) - g(x))\,dx$ square units .



$Ex : (C)$: $f(x) = -2x^2 + 4$, $(\gamma) : g(x) = x^2 + 1$.

$A = \int_{-1}^{1} [(-2x^2 + 4) - (x^2 + 1)]dx = -x^3 + 3x\Big|_{-1}^{1} = 4$ square units .

- ***Volume .***
  Let (D) be the domain limited by the representative curve (C), of a continuous function f ,the axis of abscissas and the two straight lines of equations $x = a$ and $x = b$.
  The volume of the solid obtained by rotating (D) around the axis $x'x$ is given by the formula $V = \pi \int_a^b [f(x)]^2 dx$ cubic units .

Solved Problems

N° 1.

Calculate each of the following integrals:

1) $\int_1^4 (\sqrt{x}+\frac{3}{\sqrt{x}})dx$ 2) $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}dx$ 3) $\int_0^4 \sqrt{2x+1}\,dx$

4) $\int_{-1}^0 (3x+2)^4 dx$ 5) $\int_2^6 \frac{dx}{\sqrt{2x-3}}$ 6) $\int_0^2 x(x^2-1)^4 dx$

7) $\int_{-2}^{-1} \left(x^2+\frac{3}{x^2}\right)dx$ 8) $\int_1^2 \frac{3x^4-2x^3+5}{x^2}dx$ 9) $\int_{-3}^{-2} \frac{dx}{(x+1)^3}$

10) $\int_0^1 x\sqrt{x^2+1}\,dx$ 11) $\int_0^2 \frac{xdx}{\sqrt{2x^2+1}}$ 12) $\int_0^3 |2x-4|\,dx$

13) $\int_{-2}^0 |x^2-x-2|\,dx$ 14) $\int_{-3}^1 (3x^2-|2x+4|)dx$

N° 2.

Calculate each of the following integrals:

1) $\int_{\pi/4}^{\pi/2} \cos^3 x \sin x dx$ 2) $\int_0^{\frac{\pi}{4}} \cos 2x \sin^3 2x\, dx$ 3) $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}dx$

4) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cot x}{\sin^2 x}dx$ 5) $\int_0^{\frac{\pi}{4}} (\tan^2 x+3)dx$ 6) $\int_0^{\frac{\pi}{8}} \frac{dx}{\cos^2 2x}$

7) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin^2 2x}$ 8) $\int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \cot^2(2x)dx$ 9) $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{1+\cos^2 x}}dx$

10) $\int_0^{\frac{\pi}{4}} \frac{\sin(2t)}{(1+\cos 2t)^3}dt$ 11) $\int_0^{\frac{\pi}{2}} \sin 2x.\sqrt{1+3\cos^2 x}dx$

12) $\int_0^{\frac{\pi}{4}} \frac{1}{\cos^4 x}dx$

N°3.

After linearization, calculate each of the following integrals:

1) $\int_{0}^{\frac{\pi}{4}} \cos 3x \cos x dx$ 2) $\int_{\pi/6}^{\pi/3} \cos 2x \sin 4x dx$

3) $\int_{0}^{\frac{\pi}{4}} \sin 3x \sin x\, dx$ 4) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2x.dx$

5) $\int_{0}^{\pi} \sin^2 3x.dx$ 6) $\int_{0}^{\frac{\pi}{2}} \cos^3 x dx$

7) $\int_{0}^{\frac{\pi}{2}} \sin^3 x \cos^2 x dx$ 8) $\int_{\frac{\pi}{2}}^{3\frac{\pi}{4}} \cos^4 x dx$

9) $\int_{0}^{\frac{\pi}{2}} \sin^3 x dx$

N°4.

Calculate each of the following integrals:

1) $\int_{-3}^{3} x\sqrt{x^2+1}\, dx$ 2) $\int_{-2}^{2} \frac{5x}{x^2+1} dx$ 3) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sin^3 x + \tan x + x^5\right) dx$

N°5.

1) Calculate the following integrals using integration by parts:

a- $\int_{0}^{\pi} x \cos x\, dx$ b- $\int_{0}^{\pi} x^2 \cos x\, dx$ c- $\int_{0}^{\pi} x \cos^2 x dx$

2) Deduce the integral $I = \int_{0}^{\pi} (x^2 - 2x) \cos x\, dx$.

3) Calculate $\int_{1}^{2} x\sqrt{-x+3}\, dx$.

N° 6.

Given the two integrals: $I = \int_0^{\pi} x\cos^2 x\,dx$ and $J = \int_0^{\pi} x\sin^2 x\,dx$.

1) a- Calculate $I+J$.
   b- Calculate $I-J$.
2) Deduce the values of I and J.

N° 7.

Consider the two integrals: $I = \int_0^{\pi} \cos^4(x)\,dx$ and $J = \int_0^{\pi} \sin^4(x)\,dx$.

1) a- Show that $I = \int_0^{\pi} \cos x\,(\cos x - \cos x \sin^2 x)dx$.
   b- Show that:
   $I = \dfrac{-1}{3}J + \int_0^{\pi} \sin^2 x\,dx$. (You may use integration by parts).
   c- Similarly prove that $J = \dfrac{-1}{3}I + \int_0^{\pi} \cos^2 x\,dx$.
2) a- Show that $I + J = \dfrac{3\pi}{4}$ and $I - J = 0$.
   b- Deduce I and J.

N° 8.

Given: $I_n = \int_0^1 (1 - x^n)\sqrt{1 - x^2}\,dx$, $J_n = \int_0^1 x^n \sqrt{1 - x^2}\,dx$.

Let $J_0 = \int_0^1 \sqrt{1 - x^2}\,dx$ where $n \in IN^*$.

1) Justify that $J_0 = \dfrac{\pi}{4}$.
2) Calculate J_1 and deduce the value of I_1.
3) Show that, for any $n \in IN^*$ we have: $J_n \le \int_0^1 x^n dx$.

   Deduce $\lim\limits_{n \to +\infty} J_n$.

N° 9.

Consider the function f defined over $[-1;+\infty[$ by $f(x)=\sqrt{x+1}$. Designate by (C), its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.(1 unit = 2 cm).

1) Calculate, in cm^2, the area of the region (D), limited by (C), $x'Ox$ and the two straight lines of equations $x=0$ and $x=3$.
2) Calculate, in cm^3, the volume of the solid generated by rotating (D) about $x'Ox$.

N° 10.

Consider the function f defined over IR by $f(x)=\dfrac{x}{\sqrt{x^2+1}}$.

(C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to+\infty} f(x)$, and deduce that (C) has an asymptote (d).
2) Study the relative positions of (C) and (d).
3) Show that f is odd and deduce the asymptote of (C) at $-\infty$.
4) a- Calculate the area of the region limited by (C), (d) and the straight lines of equations $x=0$ and $x=3$.
   b- Deduce the area of the domain limited by (C), and the straight lines of equations $y=-1$, $x=0$ and $x=-3$.

N° 11.

Consider the function f defined over $IR-\{0\}$ by $f(x)=x+\dfrac{4}{x^2}$.

(C) is the curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that the straight line (d) of equation $y=x$ is an asymptote to (C), and study the relative positions of (C) and (d) .
2) Calculate the area of the region bounded by (C), (d) and the straight lines of equations $x=1$ and $x=2$.

N° 12.

Consider the integrals $I_n=\displaystyle\int_0^{\frac{\pi}{2}} x^n\cos x\,dx$ and $J_n=\displaystyle\int_0^{\frac{\pi}{2}} x^n\sin x\,dx$, where n is a non zero natural number .

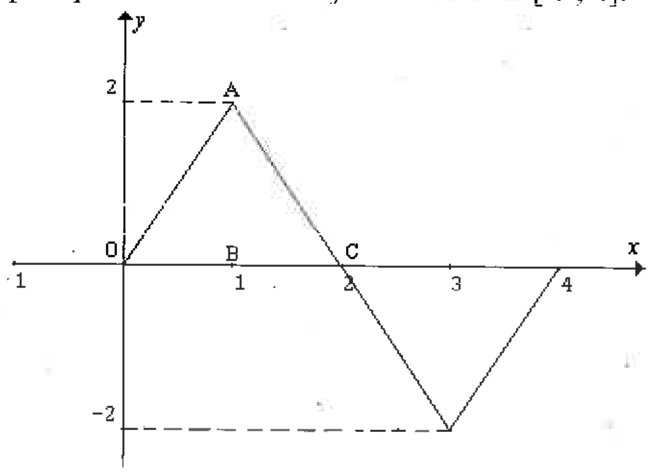
1) a- Using integration by parts , find a relation between I_n and J_{n-1}.

b- Using integration by parts , find a relation between J_n and I_{n-1}.

2) Deduce I_n in terms of I_{n-2}.

N° 13.

The given graph represents a function f defined over [0 ; 4].



Let $F(x) = \int_0^x f(t)\,dt$.

1) Calculate $F(0)$, $F(1)$, $F(2)$, $F(3)$ and $F(4)$.
2) Calculate $F'(x)$, draw the table of variations of F on [0 ; 4].
3) Show that F is defined over [0 ; 4] by

$$F(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ -x^2 + 4x - 2 & 1 \le x \le 3 \\ x^2 - 8x + 16 & 3 \le x \le 4 \end{cases}$$

N° 14.

Calculate the derivative function f' of each of the functions f defined as:

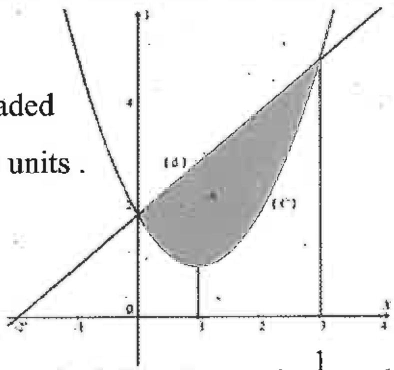
1) $f(x) = \int_x^1 \sqrt{2t+6}\,dt$ 2) $f(x) = \int_1^{x^2} \frac{t^3+2}{t+1}\,dt$

N° 15.

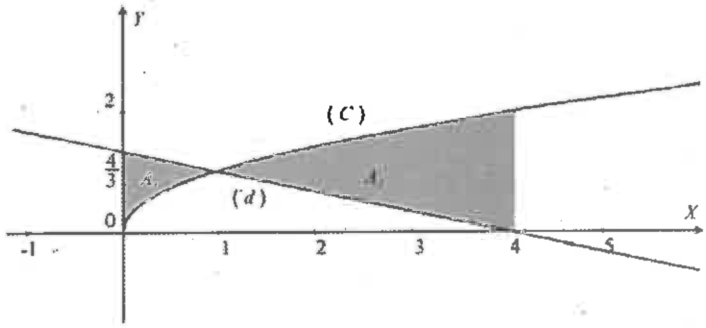
The figures below represent the graphs of a straight line (d) and a curve (C) of a function f.

1) In this figure the area of the shaded region A is equal to $\frac{9}{2}$ square units.

   Calculate $\int_0^3 f(x)\,dx$.



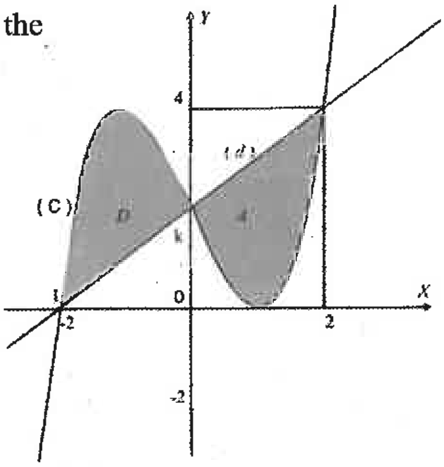
2) In the figure below the area of the shaded region A_1 is $\frac{1}{2}$ and the area of the shaded region A_2 is $\frac{19}{6}$, calculate $\int_0^4 f(x)\,dx$.



3) In the adjacent figure, the area of the shaded region A is 4, and $w(0;2)$ is a center of symmetry of (C).

   Calculate $\int_0^2 f(x)\,dx$

   and deduce $\int_{-2}^2 f(x)\,dx$.



N° 16.

Consider the function f defined over $[1;3]$ by $f(x)=\frac{3}{x}$.

Designate by (H) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$, (d) is the straight line of equation $y=-x+4$.

1) Prove that (H) is below (d) in the interval $[1;3]$.
2) Let (D) be the region limited by (H) and (d).
   Calculate the volume of the solid generated by revolving (D) about $x'x$.

N° 17.

Calculate the area of each figure limited by the following lines:

1) $y=x(x-1)(x-2)$	and	$x'ox$.
2) $y=x^2(x-1)$	and	$x'ox$.
3) $y=-x^2+2x$	and	$y=-x$.
4) $y=(x-1)^2$	and	$y=-x^2+6x-5$.

N° 18.

Let (D) be the region limited by the straight lines (d_1), (d_2) and (d_3) of respective equations $y=2x$, $y=\frac{1}{2}x$ and $x=2$.

A designates the point of intersection of (d_1) and (d_3), B is the common point of (d_2) and (d_3).

1) Calculate the area of the domain (D).
2) Calculate ,by two methods, the volume of the solid generated by rotating (D) about $x'ox$.

N° 19.

Consider the function f defined over $[2;+\infty[$ by $f(x)=\sqrt{3x^2-4}$.

(H) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and draw its table of variations.
2) Let (D) be the domain limited by (H), the axis $x'x$ and the two straight lines of equations $x=2$ and $x=4$.
   Calculate the volume of the solid generated by revolving (D) about $x'x$.

Supplementary Problems

N° 1.

Given the two integrals:

$$I = \int_0^{\frac{\pi}{4}} (2x+1)\cos^2 x\, dx \text{ and } J = \int_0^{\frac{\pi}{4}} (2x+1)\sin^2 x\, dx$$

1) Calculate $I+J$, and $I-J$.
2) Deduce the values of I and J.

N° 2.

Consider the function g defined over $[\,0\,;2\,]$ by:

$$g(x) = \begin{cases} x & 0 \le x \le 1 \\ -x+2 & 1 \le x \le 2 \end{cases}$$

1) Trace the line representing g in an orthonormal system $(O;\vec{i},\vec{j})$.
2) Let G be the function defined over $[\,0\,;2\,]$ by $G(x) = \int_0^x g(t)\,dt$.
   a- Study the variations of G over $[\,0\,;2\,]$ and set up the table of variations of G.
   b- Deduce the expression of $G(x)$ and trace its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

N° 3.

Given the two integrals $I = \int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x}$ and $J = \int_0^{\frac{\pi}{4}} \frac{dx}{\cos^4 x}$.

1) Calculate I.
2) Let f be the function defined over $\left[0;\frac{\pi}{4}\right]$ by $f(x) = \frac{\sin x}{\cos^3 x}$.
   a- Show that $f'(x) = \frac{3}{\cos^4 x} - \frac{2}{\cos^2 x}$.
   b- Find a relation between I and J and deduce values of I and J.

Supplementary Problems

N° 4.

Let (C) be the curve representing, in an orthonormal system $(O;\vec{i},\vec{j})$, of the function f defined over $[0,\pi]$ by $f(x)=\cos^2 x$.

Let (D) be the domain limited by (C) , $x'x$ and the straight lines of equations $x=0$ and $x=\pi$.

1) Calculate the area of (D)
2) Calculate the volume of the solid generated by rotating (D) about $x'x$..

N° 5.

Consider the function f defined over $[0;3]$ by $f(x)=x\sqrt{3-x}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and set up its table of variations .
2) Let (D) be the domain limited by (C) and $x'x$.
   Calculate the volume of the solid generated by rotating (D) about $x'x$.

Solutions

N° 1.

1) $\int_1^4 (\sqrt{x}+\frac{3}{\sqrt{x}})dx = [\frac{2}{3}x\sqrt{x}+6\sqrt{x}]_1^4 = \frac{16}{3}+12-\frac{2}{3}-6=\frac{32}{3}.$

2) $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}dx = \frac{2}{3}\int_0^2 \frac{3x^2}{2\sqrt{x^3+1}}dx$

$$= \frac{2}{3}\sqrt{x^3+1}\Big]_0^2 = \frac{2}{3}\left(\sqrt{9}-\sqrt{1}\right) = \frac{4}{3}.$$

3) $\int_0^4 \sqrt{2x+1}.dx = \int_0^4 \frac{1}{2}(2x+1)^{\frac{1}{2}}(2x+1)'dx = \frac{1}{2}\times\frac{2}{3}(2x+1)^{\frac{3}{2}}\Big|_0^4 = \frac{26}{3}.$

4) $\int_{-1}^0 (3x+2)^4 dx = \int_{-1}^0 \frac{1}{3}(3x+2)^4(3x+2)'dx = \frac{1}{15}(3x+2)^5\Big|_{-1}^3 = \frac{11}{5}.$

5) $\int_2^6 \frac{dx}{\sqrt{2x-3}} = .\int_2^6 \frac{(2x-3)'}{2\sqrt{2x-3}}dx = [\sqrt{2x-3}]_2^6 = 2.$

6) $\int_0^2 x(x^2-1)^4 dx = \int_0^2 \frac{1}{2}(x^2-1)^4(x^2-1)'dx = \frac{1}{10}(x^2-1)^5\Big|_0^2 = \frac{122}{5}.$

7) $\int_{-2}^{-1}\left(x^2+\frac{3}{x^2}\right)dx = \frac{x^3}{3}-\frac{3}{x}\Big]_{-2}^{-1} = \left(-\frac{1}{3}+3\right)-\left(-\frac{8}{3}+\frac{3}{2}\right)=\frac{23}{6}.$

8) $\int_1^2 \frac{3x^4-2x^3+5}{x^2}dx = \int_1^2 (3x^2-2x+\frac{5}{x^2})dx = \left[x^3-x^2-\frac{5}{x}\right]_1^2 = \frac{13}{2}.$

9) $\int_{-3}^{-2} \frac{dx}{(x+1)^3} = \int_{-3}^{-2} (x+1)^{-3}.(x+1)'.dx = -\frac{1}{2}(x+1)^{-2}\Big|_{-3}^{-2} = \frac{-3}{8}.$

10) $\int_0^1 x\sqrt{x^2+1}dx = \frac{1}{2}\int_0^1 \left(x^2+1\right)^{\frac{1}{2}}(x^2+1)'dx = \frac{1}{2}\left(x^2+1\right)^{\frac{3}{2}}\times\frac{2}{3}\Big]_0^1$

$$= \frac{\left(x^2+1\right)\sqrt{x^2+1}}{3}\Big]_0^1 = \frac{2\sqrt{2}-1}{3}$$

11) $\int_0^2 \frac{xdx}{\sqrt{2x^2+1}} = \frac{1}{2}\int_0^2 \frac{4x}{2\sqrt{2x^2+1}}dx = \frac{1}{2}\sqrt{2x^2+1}\Big]_0^2 = \frac{3-1}{2} = 1$.

12) Let I = $\int_0^3 |2x-4|dx$, we know $|2x-4| = \begin{cases} 2x-4 & if\ x \geq 2 \\ -2x+4 & if\ x \leq 2 \end{cases}$

So

$$I = \int_0^2 (-2x+4)\,dx + \int_2^3 (2x-4)\,dx = (-x^2+4x)\Big|_0^2 + (x^2-4x)\Big|_2^3 = 5.$$

13) Let J = $\int_{-2}^0 |x^2-x-2|dx$, we know that :

$$|x^2-x-2| = \begin{cases} x^2-x-2 & x \leq -1\ or\ x \geq 2 \\ -x^2+x+2 & -1 \leq x \leq 2 \end{cases}$$

So $J = \int_{-2}^{-1} (x^2-x-2)\,dx + \int_{-1}^{0} -(x^2-x-2)dx$

Then $J = \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-2}^{-1} - \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^{0} = 3$

14) Let $K = \int_{-3}^{1} (3x^2 - |2x+4|)\,dx$, we know that :

$$|2x+4| = \begin{cases} 2x+4 & if\ x \geq -2 \\ -2x-4 & if\ x \leq -2 \end{cases}$$, so:

$$K = \int_{-3}^{-2} (3x^2+2x+4)\,dx + \int_{-2}^{1} (3x^2-2x-4)\,dx$$

$$K = x^3 + x^2 + 4x\Big|_{-3}^{-2} + x^3 - x^2 - 4x\Big|_{-2}^{1} = 18$$

N°2.

1) Let $\cos x = u(x)$ then $u'(x) = -\sin x$, so

$$\int_{\pi/4}^{\pi/2} \cos^3 x \sin x dx = -\int_{\pi/4}^{\pi/2} u^3(x).u'(x)dx = -\frac{\cos^4 x}{4}\Big]_{\pi/4}^{\pi/2} = \frac{1}{16}.$$

2) $\int_0^{\frac{\pi}{4}} \cos 2x \sin^3 2x \, dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} (\sin 2x)'(\sin 2x)^3 dx$

$$= \frac{1}{2} \times \frac{1}{4}(\sin 2x)^4 \Big|_0^{\frac{\pi}{4}} = \frac{1}{8}.$$

3) $\int_0^{\pi/4} \frac{\sin x}{\cos^3 x} dx = -\int_0^{\pi/4} (\cos x)^{-3} (\cos x)' dx.$

$$= -\frac{(\cos x)^{-2}}{-2}\Bigg]_0^{\frac{\pi}{4}} = \frac{1}{2\cos^2 x}\Bigg]_0^{\frac{\pi}{4}} = \frac{1}{2}$$

4) $\int_{\pi/4}^{\pi/2} \frac{\cot x}{\sin^2 x} dx = -\int_{\pi/4}^{\pi/2} (\cot x).(\cot x)' dx = -\frac{(\cot x)^2}{2}\Bigg]_{\pi/4}^{\pi/2} = \frac{1}{2}.$

5) $\int_0^{\frac{\pi}{4}} (\tan^2 x + 3) dx = \int_0^{\frac{\pi}{4}} (\tan^2 x + 1 + 2) dx = [\tan x + 2x]_0^{\frac{\pi}{4}} = 1 + \frac{\pi}{2}.$

6) $\int_0^{\frac{\pi}{8}} \frac{dx}{\cos^2 2x} = [\frac{1}{2}\tan 2x]_0^{\frac{\pi}{8}} = \frac{1}{2}\tan\frac{\pi}{4} - \frac{1}{2}\tan 0 = \frac{1}{2}.$

7) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{\sin^2 2x} = [-\frac{1}{2}\cot 2x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = (-\frac{1}{2}\cot\frac{2\pi}{3}) - (-\frac{1}{2}\cot\frac{\pi}{3}) = \frac{\sqrt{3}}{3}$

8) $\int_{\pi/8}^{\pi/4} \cot^2 2x dx = \int_{\pi/8}^{\pi/4} \left[\left(1 + \cot^2 2x\right) - 1\right] dx = -\frac{\cot 2x}{2} - x\Bigg]_{\pi/8}^{\pi/4} = \frac{4-\pi}{8}$

9) $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\sqrt{1+\cos^2 x}} dx = -2\int_0^{\frac{\pi}{2}} \frac{1}{2.\sqrt{1+\cos^2 x}}.\left(1+\cos^2 x\right)' dx$

$$= -2\sqrt{1+\cos^2 x}\,\Big]_0^{\frac{\pi}{2}} = 2\sqrt{2} - 2\ .$$

10) $\displaystyle\int_0^{\pi/4}\frac{\sin 2t}{(1+\cos 2t)^3}dt=-\frac{1}{2}\int_0^{\pi/4}(1+\cos 2t)^{-3}(1+\cos 2t)'dt$

$$=-\frac{1}{2}\cdot\frac{(1+\cos 2t)^{-2}}{-2}\Bigg]_0^{\pi/4}=\frac{1}{4}\cdot\frac{1}{(1+\cos 2t)^2}\Bigg]_0^{\pi/4}$$

$$=\frac{1}{4}\left(1-\frac{1}{4}\right)=\frac{3}{16}.$$

11) $\displaystyle\int_0^{\frac{\pi}{2}}\sin 2x.\sqrt{1+3\cos^2 x}.dx=-\frac{1}{3}\int_0^{\frac{\pi}{2}}\left(1+3\cos^2 x\right)^{\frac{1}{2}}\left(1+3\cos^2 x\right)'dx$

$$=-\frac{1}{3}\left(1+3\cos^2 x\right)^{\frac{3}{2}}\times\frac{2}{3}\Bigg]_0^{\frac{\pi}{2}}=-\frac{2}{9}\left[1-\sqrt{4^3}\right]=-\frac{2}{9}[1-8]=\frac{14}{9}.$$

12) $\displaystyle\int_0^{\frac{\pi}{4}}\frac{1}{\cos^4 x}dx=\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2 x}\cdot\frac{1}{\cos^2 x}dx=\int_0^{\frac{\pi}{4}}\left(1+\tan^2 x\right)\left(\tan x\right)'dx$

$$=\tan x+\frac{\tan^3 x}{3}\Bigg]_0^{\frac{\pi}{4}}=1+\frac{1}{3}=\frac{4}{3}.$$

N° 3.

1) We have $\cos 3x\cos x=\frac{1}{2}(\cos 4x+\cos 2x)$, then

$$\int_0^{\frac{\pi}{4}}\cos 3x\cos x dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}(\cos 4x+\cos 2x)dx=\frac{\sin 4x}{8}+\frac{\sin 2x}{4}\Bigg|_0^{\frac{\pi}{4}}=\frac{1}{4}.$$

2) $\cos 2x\sin 4x=\frac{1}{2}[\sin 6x+\sin 2x]$, then :

$$\int_{\pi/6}^{\pi/3}\cos 2x\sin 4x dx=-\frac{\cos 6x}{12}-\frac{\cos 2x}{4}\Bigg]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$$

$$=\left(-\frac{1}{12}-\frac{\cos\frac{2\pi}{3}}{4}\right)-\left(\frac{1}{12}-\frac{\cos\frac{\pi}{3}}{4}\right)=\frac{1}{12}.$$

3) $\sin 3x\sin x=\frac{1}{2}[\cos 2x-\cos 4x]$, then

$$\int_0^{\frac{\pi}{4}} \sin 3x \sin x dx = \left.\frac{\sin 2x}{4} - \frac{\sin 4x}{8}\right]_0^{\frac{\pi}{4}} = \frac{1}{4}.$$

4) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos^2 2x dx = 2\int_0^{\frac{\pi}{4}} \cos^2 2x dx = \int_0^{\frac{\pi}{4}} (1+\cos 4x)\, dx =$

$$\left[x + \frac{1}{4}\sin 4x\right]_0^{\frac{\pi}{4}} = \frac{\pi}{4}.$$

Notice that the cosine function is even.

5) $\int_0^{\pi} \sin^2 3x dx = \frac{1}{2}\int_0^{\pi}(1-\cos 6x)dx = \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\pi} = \frac{\pi}{2}.$

6) $\int_0^{\frac{\pi}{2}} \cos^3 x dx = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)\cos x dx = \int_0^{\frac{\pi}{2}} (1-\sin^2 x)(\sin x)' dx.$

Then $\int_0^{\frac{\pi}{2}} \cos^3 x\, dx = \left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{2}} = \frac{2}{3}.$

7) Let $I = \int_0^{\frac{\pi}{2}} \sin^3 x \cos^2 x dx$, we have:

$$I = \int_0^{\frac{\pi}{2}} \sin x(1-\cos^2 x)\cos^2 x\, dx = \int_0^{\frac{\pi}{2}} (\cos x)'(\cos^2 x - \cos^4 x)dx.$$

Then $I = -\left[\frac{\cos^3 x}{3} - \frac{\cos^5 x}{5}\right]_0^{\frac{\pi}{2}} = \frac{2}{15}$

8) $\cos^4 x = \left(\frac{1+\cos 2x}{2}\right)^2 = \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x$

$$= \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1}{8} + \frac{\cos 4x}{8} = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.$$

Then $\int_{\pi/2}^{3\pi/4}\cos^4 x dx = \left[\frac{3}{8}x+\frac{\sin 2x}{4}+\frac{\sin 4x}{32}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{4}} = \frac{3\pi-8}{32}.$

9) $\int_0^{\frac{\pi}{2}}\sin^3 xdx = \int_0^{\frac{\pi}{2}}\sin^2 x.\sin xdx = \int_0^{\frac{\pi}{2}}(1-\cos^2 x)\sin xdx$

$$= \int_0^{\frac{\pi}{2}}\sin xdx + \int_0^{\frac{\pi}{2}}\cos^2 x(\cos x)'dx = \left[-\cos x+\frac{\cos^3 x}{3}\right]_0^{\frac{\pi}{2}} = \frac{2}{3}.$$

$N^\circ 4.$

1) The function f defined over $[-3;3]$ by $f(x)=x\sqrt{x^2+1}$ is an odd function then $\int_{-3}^{3}x\sqrt{x^2+1}\,dx=0$.

2) The function f defined over $[-2;2]$ by $f(x)=\frac{5x}{x^2+1}$ is odd then $\int_{-2}^{2}\frac{5x}{x^2+1}dx=0$.

3) The function f defined over $\left[-\frac{\pi}{4};\frac{\pi}{4}\right]$ by

$f(x)=\sin^3 x+\tan x+x^5$ is odd ,then $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\sin^3 x+\tan x+x^5)dx=0$.

$N^\circ 5.$

1) a- $\int_0^{\pi}x\cos x dx$. Let $u(x)=x$ and $v'(x)=\cos x$, which gives:

$u'(x)=1$ and $v(x)=\sin x$, then

$$\int_0^{\pi}x\cos x\,dx = [x\sin x]_0^{\pi} - \int_0^{\pi}\sin x\,dx = [x\sin x]_0^{\pi} + [\cos x]_0^{\pi}$$

Therefore $\int_0^{\pi}x\cos x\,dx=-2$.

b- $\int_0^{\pi} x^2 \cos x\, dx$. Let $u = x^2$ and $v' = \cos x$ which gives:

$u' = 2x$ and $v = \sin x$, then :

Therefore $\int_0^{\pi} x^2 \cos x\, dx = x^2 \sin x\Big|_0^{\pi} - \int_0^{\pi} 2x \sin x\, dx$.

Again let $u(x) = x$ and $v'(x) = \sin x$ which gives :
$u'(x) = 1$ and $v(x) = -\cos x$, then :

$$\int_0^{\pi} x \sin x\, dx = -[x\cos x]_0^{\pi} + \int_0^{\pi} \cos x\, dx = -[x\cos x]_0^{\pi} + [\sin x]_0^{\pi}$$

Therefore

$$\int_0^{\pi} x^2 \cos x\, dx = x^2 \sin x\Big|_0^{\pi} + 2x\cos x\Big|_0^{\pi} - 2\sin x\Big|_0^{\pi} = -2\pi .$$

c) $\int_0^{\pi} x\cos^2 x dx$. Let $u(x) = x$ and $v'(x) = \cos^2 x = \dfrac{1+\cos 2x}{2}$ which

gives : $u'(x) = 1$ and $v(x) = \dfrac{2x + \sin 2x}{4}$

Therefore $\int_0^{\pi} x\cos^2 x dx = \left.\dfrac{x(2x+\sin 2x)}{4}\right]_0^{\pi} - \dfrac{1}{4}\int_0^{\pi}(2x + \sin 2x)dx$

$$= \frac{\pi^2}{2} - \frac{1}{4}\left[x^2 - \frac{\cos 2x}{2}\right]_0^{\pi} = \frac{\pi^2}{2} - \frac{1}{4}\left[\left(\pi^2 - \frac{1}{2}\right) + \frac{1}{2}\right] = \frac{\pi^2}{4}.$$

2) $\int_0^{\pi}(x^2 - 2x)\cos x\, dx = \int_0^{\pi} x^2 \cos x\, dx - 2\int_0^{\pi} x\cos x\, dx = -2\pi + 4$.

3) $\int_1^2 x\sqrt{-x+3}dx$.

Let $u(x) = x$ and $v'(x) = \sqrt{-x+3}$, which gives :

$u'(x) = 1$ and $v(x) = -\dfrac{2}{3}(-x+3)^{\frac{3}{2}}$

Therefore $\int_1^2 x\sqrt{-x+3}dx = \left.-\dfrac{2x}{3}(-x+3)^{\frac{3}{2}}\right]_1^2 + \dfrac{2}{3}\int_1^2(-x+3)^{\frac{3}{2}}dx$

$$= \frac{4(\sqrt{2}-1)}{3} + \frac{2}{3}\left[-(-x+3)^{\frac{5}{2}} \times \frac{2}{5}\right]_1^2$$

$$= \frac{4(\sqrt{2}-1)}{3} + \frac{4}{15}\left[-1+4\sqrt{2}\right] = \frac{12\sqrt{2}}{5} - \frac{8}{5}.$$

N° 6.

1) a- $I + J = \int_0^{\pi} x(\cos^2 x + \sin^2 x)\,dx = \int_0^{\pi} x\,dx = \frac{1}{2}x^2\Big|_0^{\pi} = \frac{\pi^2}{2}.$

b- $I - J = \int_0^{\pi} x(\cos^2 x - \sin^2 x)\,dx = \int_0^{\pi} x\cos(2x)\,dx$

Integrate by parts:

Let $u = x$ and $v' = \cos(2x)$, then $u' = 1$ and $v = \frac{1}{2}\sin(2x)$.

Then, we will have: $I - J = \frac{1}{2}x\sin(2x)\Big|_0^{\pi} - \frac{1}{2}\int_0^{\pi}\sin(2x)\,dx$

$$= \frac{1}{2}x\sin(2x)\Big|_0^{\pi} + \frac{1}{4}\cos(2x)\Big|_0^{\pi} = 0$$

2) We have the following system $I + J = \frac{\pi^2}{2}$ and $I - J = 0$ which gives

$\mathrm{I} = \mathrm{J} = \frac{\pi^2}{4}$.

N° 7.

1) a- $I = \int_0^{\pi} \cos^4 x\,dx = \int_0^{\pi} \cos^2 x\cos^2 x\,dx = \int_0^{\pi} \cos^2 x(1-\sin^2 x)\;dx.$

Therefore $I = \int_0^{\pi} \cos x(\cos x - \cos x\sin^2 x)dx$.

b- Let $u(x) = \cos x$ and $v'(x) = \cos x - \cos x\sin^2 x$ which gives

$u'(x) = -\sin x$ and $v(x) = \sin x - \frac{1}{3}\sin^3 x$.

Therefore:

$$I = \cos x\left(\sin x - \frac{1}{3}\sin^3 x\right)\Big]_0^{\pi} + \int_0^{\pi}\left(\sin^2 x - \frac{1}{3}\sin^4 x\right)dx.$$

Then $I = \int_0^{\pi}\sin^2 x\,dx - \frac{1}{3}\int_0^{\pi}\sin^4 x\,dx = \int_0^{\pi}\sin^2 x\,dx - \frac{1}{3}J.$

c- $J = \int_0^{\pi} \sin^4 x\,dx = \int_0^{\pi} \sin^2 x(1-\cos^2 x)dx$

$= \int_0^{\pi}\sin x\,(\sin x - \sin x\,\cos^2 x)\,dx$

Integrate by parts: Let $u = \sin x$ and $v' = \sin x - \sin x\cos^2 x$

Then $u'(x) = \cos x$ and $v(x) = -\cos x + \frac{1}{3}\cos^3 x$, thus:

$$J = \sin x(-\cos x + \frac{1}{3}\cos^3 x)\Big|_0^{\pi} + \int_0^{\pi}(\cos^2 x - \frac{1}{3}\cos^4 x)dx.$$

Then $J = \int_0^{\pi} \cos^2 x\,dx - \frac{1}{3}I$.

2) a- $I + J = \int_0^{\pi}\sin^2 x\,dx + \int_0^{\pi}\cos^2 x\,dx - \frac{1}{3}I - \frac{1}{3}J.$

Then $I + J = \int_0^{\pi} 1\,dx - \frac{1}{3}(I+J)$ which gives

$\frac{4}{3}(I+J) = x\,\Big]_0^{\pi} = \pi$, therefore $I + J = \frac{3}{4}\pi$.

Similarly, $I - J = \int_0^{\pi}\cos^2 x\,dx - \int_0^{\pi}\sin^2 x\,dx - \frac{1}{3}J + \frac{1}{3}I.$

Then $I - J = \frac{1}{3}(I-J) + \int_0^{\pi}\cos(2x)dx.$

Therefore $\frac{2}{3}(I-J) = \frac{1}{2}\sin 2x\,\Big]_0^{\pi} = 0$ then $I - J = 0$.

b- We have $I + J = \frac{3}{4}\pi$ and $I = J$ which gives $I = J = \frac{3}{8}\pi$.

N° 8.

1) $J_0 = \int_0^1 \sqrt{1-x^2}\,dx$, J_0 is equal to the area of the domain bounded by $x'x$, the curve (Γ) of equation $y=\sqrt{1-x^2}$ and the two straight lines of equations $x=0$ and $x=1$, however, $y=\sqrt{1-x^2}$ gives $x^2+y^2=1$, then (Γ) is the semicircle of equation $x^2+y^2=1$ for $y \geq 0$. Then $J_0 = \dfrac{\pi R^2}{4} = \dfrac{\pi}{4}$.

2) $J_1 = \int_0^1 x\sqrt{1-x^2}\,dx = -\dfrac{1}{2}\int_0^1 \left(1-x^2\right)^{\frac{1}{2}}(-2x)\,dx$.

Then $J_1 = -\dfrac{1}{2}\left.\dfrac{\left(1-x^2\right)^{\frac{3}{2}}}{3/2}\right|_0^1 = \dfrac{1}{3}$.

$I_1 = \int_0^1 (1-x)\sqrt{1-x^2}\,dx = \int_0^1 \sqrt{1-x^2}\,dx - \int_0^1 x\sqrt{1-x^2}\,dx$.

Then $I_1 = J_0 - J_1$, therefore $I_1 = \dfrac{\pi}{4} - \dfrac{1}{3}$.

3) For all x in $[0;1]$, $0 \leq \sqrt{1-x^2} \leq 1$ then: $0 \leq x^n\sqrt{1-x^2} \leq x^n$,

therefore: $0 \leq \int_0^1 x^n\sqrt{1-x^2}\,dx \leq \int_0^1 x^n\,dx$.

Then $0 \leq J_n \leq \int_0^1 x^n\,dx = \left[\dfrac{x^{n+1}}{n+1}\right]_0^1 = \dfrac{1}{n+1}$, thus $0 \leq J_n \leq \dfrac{1}{n+1}$.

$0 \leq \lim\limits_{n\to+\infty} J_n \leq \lim\limits_{x\to+\infty} \dfrac{1}{n+1}$, and since $\lim\limits_{x\to+\infty} \dfrac{1}{n+1} = 0$.

then $\lim\limits_{n\to+\infty} J_n = 0$.

N° 9.

1) The curve (C) is above $x'x$, therefore the area is given by:

$A = \int_0^3 f(x)\,dx = \int_0^3 \sqrt{x+1}\,dx = \left.\dfrac{2}{3}(x+1)\sqrt{x+1}\right|_0^3 = \dfrac{14}{3}$ square units, or $A = \frac{14}{3} \times (2\text{ cm})^2 = \frac{56}{3}\text{ cm}^2$.

2) The volume V is:

$$V = \pi\int_0^3 [f(x)]^2\,dx = \pi\int_0^3 (x+1)\,dx = \pi\left(\frac{x^2}{2}+x\right)\Big|_0^3 = \frac{15\pi}{2} \text{ cubic units},$$

or $V = 60\,\pi\text{ cm}^3$.

N° 10.

1) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \dfrac{x}{x\sqrt{1+\dfrac{1}{x^2}}} = 1$, then the straight line (d) of equation $y = 1$ is an asymptote to (C) at $+\infty$.

2) $f(x) - y = \dfrac{x}{\sqrt{x^2+1}} - 1$, since $x^2 < x^2+1$, then $\sqrt{x^2} < \sqrt{x^2+1}$, which gives $x < \sqrt{x^2+1}$, because $x > 0$, then $\dfrac{x}{\sqrt{x^2+1}} < 1$ therefore $f(x) - 1 < 0$, then (C) is below (d).

3) For all $x \in IR$, $-x \in IR$ and $f(-x) = \dfrac{-x}{\sqrt{x^2+1}} = -f(x)$, then f is an odd function and O is a center of symmetry of (C).
The line (d') of equation $y = -1$ which is the symmetric of (d) with respect to O is an asymptote to (C) at $-\infty$.

4) a- (C) is below (d) then the required area is:

$$A = \int_0^3 \left(1 - \frac{x}{\sqrt{x^2+1}}\right)dx = \int_0^3 \left[1 - \frac{2x}{2\sqrt{x^2+1}}\right]dx$$

$$= x - \sqrt{x^2+1}\Big|_0^3 = 4 - \sqrt{10} \text{ square units.}$$

b- Since O is a center of symmetry of (C) then the required area A' is equal to A, then $A' = 4 - \sqrt{10}$ square units.

N° 11.

1) $\lim\limits_{x\to+\infty} (f(x) - y) = \lim\limits_{x\to+\infty} \dfrac{4}{x^2} = 0$, then the line (d) is an asymptote to (C) at $+\infty$.

Similarly, the line (d) is an asymptote to (C) at $-\infty$.

Since $f(x)-y=\dfrac{4}{x^2}>0$ then (C) is above (d) for any non-zero real number.

2) The required area is:

$$A=\int_1^2[f(x)-y_D]\,dx=\int_1^2\frac{4}{x^2}dx=-\frac{4}{x}\Bigg|_1^2=2 \text{ square units.}$$

Then $A=2\times(0.5\text{ cm})^2=0.5\text{ cm}^2$.

N° 12.

1) a- Let $u=x^n$ and $v'=\cos x$, which gives:

$u'=nx^{n-1}$ et $v=\sin x$, then:

$$I_n=\int_0^{\frac{\pi}{2}}x^n\cos x\,dx=\left[x^n\sin x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}nx^{n-1}\sin x\,dx.$$

Therefore $I_n=\left(\dfrac{\pi}{2}\right)^n-n\,J_{n-1}$ (1).

b- Let $u=x^n$ and $v'=\sin x$, which gives:

$u'=nx^{n-1}$ and $v=-\cos x$, then:

$$J_n=\int_0^{\frac{\pi}{2}}x^n\sin x\,dx=\left[-x^n\cos x\right]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}nx^{n-1}(-\cos x)\,dx.$$

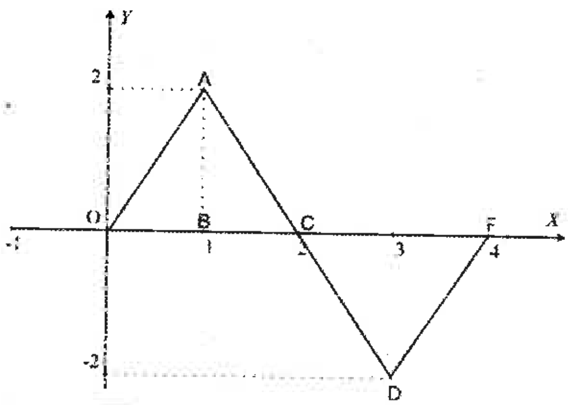
Therefore $J_n=n\,I_{n-1}$ (2)

2) The relations (1) and (2) give:

$$I_n=\left(\frac{\pi}{2}\right)^n-n\,J_{n-1}=\left(\frac{\pi}{2}\right)^n-n(n-1)I_{n-2}.$$

N° 13.

1) $F(0)=\displaystyle\int_0^0 f(t)\,dt=0$.



$F(1)=\int_0^1 f(t)\,dt$, represents the area of the right triangle OAB

which is equal to $\dfrac{OB\times AB}{2}=1$, then $F(1)=\int_0^1 f(t)\,dt=1$.

$F(2)=\int_0^2 f(t)\,dt$, represents the area of the triangle OAC which

is equal to $\dfrac{OC\times AB}{2}=2$, then $F(2)=\int_0^2 f(t)\,dt=2$.

$\int_2^3 f(t)dt=-\int_1^2 f(t)dt$ since in $[2;3]$, $f(t)<0$, then

$$F(3)=\int_0^3 f(t)\,dt=\int_0^2 f(t)dt+\int_2^3 f(t)dt=2-1=1.$$

$$F(4)=\int_0^4 f(t)\,dt=\int_0^2 f(t)dt+\int_2^4 f(t)dt=2-2=0.$$

2) We know that $F'(x)=f(x)$, then:
$F'(x)>0$ when $f(x)>0$ hence for $0<x<2$
$F'(x)<0$ when $f(x)<0$ hence for $2<x<4$,
Therefore the table of variations of F is as follows :

x	0		2		4
$F'(x)$		$+$	0	$-$	
$F(x)$	0	$\nearrow$	2	$\searrow$	0

3) For $0\le x\le 1$, the equation of the straight line (OA) is $y=2x$.
Therefore, $F(x)=x^2+C$ and since $F(1)=1$, then $1=1+C$, which gives $C=0$, and consequently, $F(x)=x^2$.
For $1\le x\le 3$ the equation of the straight line (AD) is $y=-2x+4$, therefore $F(x)=-x^2+4x+C$ and since $F(2)=2$, then $2=-4+8+C$, which gives $C=-2$, and consequently, $F(x)=-x^2+4x-2$.

For $3 \le x \le 4$ the equation of the straight line (DF) is $y = 2x - 8$, therefore $F(x) = x^2 - 8x + C$ and since $F(4) = 0$, then $0 = 16 - 32 + C$, which gives $C = 16$, and consequently, $F(x) = x^2 - 8x + 16$.

$$\text{Therefore } F(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ -x^2 + 4x - 2 & 1 \le x \le 3 \\ x^2 - 8x + 16 & 3 \le x \le 4 \end{cases}$$

N° 14.

Let $F(x) = \int_a^x f(t)\,dt$, we know that if f is a continuous function over an interval I and if a and x are two real numbers of I with a constant and x variable then: $F'(x) = f(x)$.

Remark: if $F(x) = \int_a^{u(x)} f(t)\,dt$, then $F'(x) = f(u(x)) \times u'(x)$.

1) We have $f(x) = -\int_1^x \sqrt{2t+6}\,dt$, the function defined by $u(t) = -\sqrt{2t+6}$ is defined for $t \ge -3$ as the lower boundary is the number $1 \in [-3;+\infty[$, then x has to belong to $[-3;+\infty[$ therefore $D_u = [-3;+\infty[$ and consequently $f'(x) = -\sqrt{2x+6}$.

2) The function define by $s(t) = \dfrac{t^3+2}{t+1}$ is defined over $IR - \{-1\}$.

$f'(x) = s(x) \times 2x$, then $f'(x) = \dfrac{x^3+2}{x+1} \times 2x = \dfrac{2x\left(x^3+2\right)}{x+1}$.

N° 15.

1) The equation of the line (d) is $y = x + 2$.

The area A is given by:

$\int_0^3 [y_d - f(x)]dx = \int_0^3 (x+2)\,dx - \int_0^3 f(x)dx = \dfrac{9}{2}$, which gives :

$$\int_0^3 f(x)\,dx = \int_0^3 (x+2)dx - \frac{9}{2} = \left[\frac{x^2}{2} + 2x\right]_0^3 - \frac{9}{2} = \frac{9}{2} + 6 - \frac{9}{2} = 6.$$

2) The equation of the line (d) is $y = \frac{-x}{3} + \frac{4}{3}$.

The area A_1 is given by:

$$\frac{1}{2} = \int_0^1 \left[\frac{-x}{3} + \frac{4}{3} - f(x)\right] dx = \left[\frac{-x^2}{6} + \frac{4}{3}x\right]_0^1 - \int_0^1 f(x)\,dx,$$

which gives $\int_0^1 f(x)dx = \frac{2}{3}$. The area A_2 is given by:

$$\frac{19}{6} = \int_1^4 \left[f(x) + \frac{x}{3} - \frac{4}{3}\right] dx = \int_1^4 f(x)dx + \left[\frac{x^2}{6} - \frac{4}{3}x\right]_1^4$$

Which gives $\int_1^4 f(x)dx = \frac{14}{3}$.

Therefore $\int_0^4 f(x).dx = \int_0^1 f(x).dx + \int_1^4 f(x).dx = \frac{2}{3} + \frac{14}{3} = \frac{16}{3}$

3) The equation of the line (d) is $y = x + 2$.

The area A is given by:

$4 = \int_0^2 [(x+2) - f(x)]\,dx = \int_0^2 (x+2)\,dx - \int_0^2 f(x)\,dx$, which gives:

$$\int_0^2 f(x)\,dx = \int_0^2 (x+2)\,dx - 4 = \left[\frac{x^2}{2} + 2x\right]_0^2 - 4 = (2+4) - 4 = 2.$$

The point $w(0;2)$ is a center of symmetry of (C) then the area of the shaded region D is equal 4 and since the area of the triangle IOK is equal 2, then $\int_{-2}^{2} f(x)\,dx = 4 + 2 + 2 = 8$.

N° 16.

1) $f(x) - y = \frac{3}{x} + x - 4 = \frac{x^2 - 4x + 3}{x}$

The trinomial $T(x) = x^2 - 4x + 3$ admits two roots $x' = 1$ and $x'' = 3$, in addition we know that $T(x) < 0$ for $1 < x < 3$, then for $x \in [1;3]$, the curve (H) is below (d).

2) (D) is limited by (H) and (d), so the volume generated by rotating

the region (D) about $x'x$ is :

$$V=\pi\int_1^3\left[(-x+4)^2-\left(\frac{3}{x}\right)^2\right]dx=\pi\left[-\frac{(-x+4)^3}{3}+\frac{9}{x}\right]_1^3.$$

hence $V=\frac{8}{3}\pi$ cubic units .

N° 17.

1) The curve (C) cuts $x'x$ at the points of abscissas 0, 1 and 2.

$$\int_0^1 x(x-1)(x-2)dx=\int_0^1(x^3-3x^2+2x)dx=\left[\frac{1}{4}x^4-x^3+x^2\right]_0^1=\frac{1}{4}.$$

Therefore the area of domain D is: $A_1=\frac{1}{4}$ square units.

Since $\int_0^1 f(x)dx>0$ we deduce that (C) is above $x'x$ between 0 and 1 and since f is a polynomial function , then (C) is below $x'x$ between 1 and 2 therefore:

$$A_2=-\int_1^2(x^3-3x^2+2x)dx=-\left[\frac{1}{4}x^4-x^3+x^2\right]_1^2=\frac{1}{4}\text{ square units}$$

And consequently the area A limited by (C) and $x'x$ is:

$A=A_1+A_2=\frac{1}{2}$ square units .

2) The curve (C) cuts $x'x$ at the points $x=0$ and $x=1$ therefore:

$$\int_0^1 x^2(x-1)dx=\int_0^1(x^3-x^2)dx=\left[\frac{1}{4}x^4-\frac{1}{3}x^3\right]_0^1=-\frac{1}{12}$$

And consequently: $A=\frac{1}{12}$ square units.`sghd

3) The abscissas of the points of intersection of (C) and (D) are the square roots of the equation: $-x^2+2x=-x$ or $x(-x+3)=0$ then $x=0$ or $x=3$.

$$\int_0^3[(-x^2+2x)-(-x)]dx=\int_0^3(-x^2+3x)dx=\left[-\frac{x^3}{3}+\frac{3x^2}{2}\right]_0^3=\frac{9}{2}$$

Therefore $A=\frac{9}{2}$ square units.

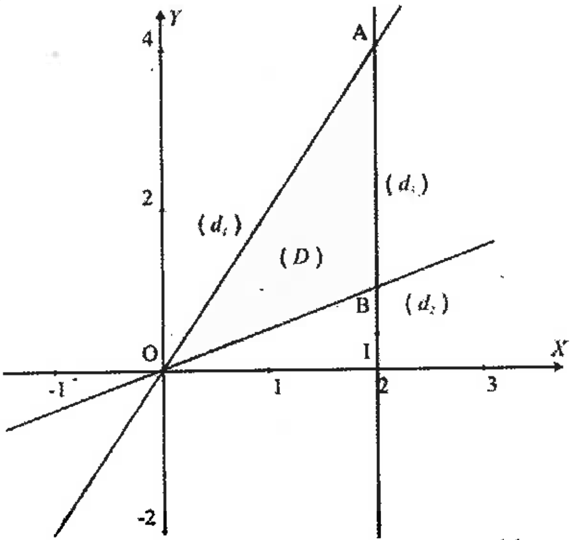
4) The two curves intersect at the points of abscissas $x=1$ and $x=3$ therefore:

$$\int_1^3\left[(x-1)^2-\left(-x^2+6x-5\right)\right]dx=\int_1^3\left(2x^2-8x+6\right)dx$$

$$=\left[\frac{2x^3}{3}-4x^2+6x\right]_1^3=-\frac{8}{3}$$,therefore $A=\frac{8}{3}$ square units.

N° 18.

1) $A=\int_0^2\left(2x-\frac{1}{2}x\right)dx=\int_0^2\frac{3}{2}x\,dx=\left[\frac{3}{4}x^2\right]_0^2=3$ square units.



2) Let V_1 be the volume of the solid generated by rotating (d_1) about $x'x$ between $x=0$ and $x=2$, we get:

$$V_1=\int_0^2\pi(2x)^2\,dx=\left[4\pi\frac{x^3}{3}\right]_0^2=\frac{32\pi}{3}$$ cubic units.

V_2, the volume of the solid generated by rotating (d_2) about $x'x$ between $x=0$ and $x=2$, we get:

$$V_2 = \int_0^2 \pi\left(\frac{1}{2}x\right)^2 dx = \left[\frac{\pi}{12}x^3\right]_0^2 = \frac{2\pi}{3} \text{ cubic units .}$$

The required volume is $V = V_1 - V_2 = \dfrac{32\pi}{3} - \dfrac{2\pi}{3} = 10\pi$ cubic units.

Second method :

The region OIA rotated about $x'x$ generates a cone whose base has a radius $AI = 4$, consequently the volume of the cone is $V_1 = \dfrac{\pi \times AI^2 \times OI}{3} = \dfrac{32\pi}{3}$.

Simiiarly the region OIB rotated about $x'x$ generates a cone whose base has a radius $BI = 1$, consequently the volume of the cone is $V_2 = \dfrac{\pi \times BI^2 \times OI}{3} = \dfrac{2\pi}{3}$.

Then $V = V_1 - V_2 = \dfrac{32\pi}{3} - \dfrac{2\pi}{3} = 10\pi$ cubic units.

N° 19.

1) $f'(x) = \dfrac{6x}{2\sqrt{3x^2-4}} > 0$ for $x \in [2;+\infty[$, therefore the table of variations of f is as follows :

x	2		$+\infty$
$f'(x)$		$+$	
$f(x)$	$2\sqrt{2}$	$\nearrow$	$+\infty$

Notice that $\lim\limits_{x\to+\infty} f(x) = +\infty$.

2) $V = \int_2^4 \pi y^2 dx = \pi \int_2^4 (3x^2-4)dx = \pi(x^3-4x)\Big|_2^4 = 48\pi$ cubic units.

Indications

N° 1.

1) $I+J=\int_0^{\frac{\pi}{4}}(2x+1)\,dx=\frac{\pi^2}{16}+\frac{\pi}{4}$.

2) Using integration by parts : $I-J=\int_0^{\frac{\pi}{4}}(2x+1)\cos 2x\,dx=\frac{\pi}{4}$.

N° 2.

2) $G'(x)=g(x)\geq 0$, $G(0)=0$, $G(1)=\frac{1}{2}$, $G(2)=1$.

$$G(x)=\begin{cases}\frac{x^2}{2} & 0\leq x\leq 1\\ -\frac{x^2}{2}+2x-1 & 1\leq x\leq 2\end{cases}$$

N° 3.

2) b- $f'(x)=\frac{3}{\cos^4 x}-\frac{2}{\cos^2 x}$, then :

$$\int_0^{\frac{\pi}{4}}\left(\frac{3}{\cos^4 x}-\frac{2}{\cos^2 x}\right)dx=\int_0^{\frac{\pi}{4}}f'(x)dx=f(x)\Big|_0^{\frac{\pi}{4}}=2.$$

Then $3J-2I=2$.

N° 5.

1) $f'(x)=\frac{6-3x}{2\sqrt{3-x}}$.

2) $V=\pi\int_0^3\left(x\sqrt{3-x}\right)^2dx=\pi\int_0^3(3x^2-x^3)\,dx=\frac{27\pi}{4}$ cubic units.

CHAPTER 4

Natural Logarithm

Chapter Review

1) **Definition.**
The natural logarithm is the antiderivative of the function
$x \mapsto \frac{1}{x}$ over the interval $]0;+\infty[$ that vanishes at 1.
The natural logarithm is denoted by $\ln$.

2) **Properties .**
- $\ln 1 = 0$ $\quad \ln e = 1$, $\quad 2.718 < e < 2.719$.
- For all real numbers : $a > 0$, $b > 0$ and for every real number r :

$$\ln(ab) = \ln a + \ln b \quad ; \quad \ln\frac{a}{b} = \ln a - \ln b \quad ; \quad \ln(a^r) = r\ln a$$

N.B
If $ab > 0$, then :

$$\ln(ab) = \ln|a| + \ln|b| \quad ; \quad \ln\frac{a}{b} = \ln|a| - \ln|b| .$$

- For $x > 0$, $\ln'(x) = \frac{1}{x}$ and for $u(x) > 0$, $\ln(u)'(x) = \frac{u'(x)}{u(x)}$.

3) **Consequences .**
The $\ln$ function is a strictly increasing function over $]0;+\infty[$.
If $x > 1$ then $\ln x > 0$
If $0 < x < 1$ then $\ln x < 0$.
In general :
$x > \alpha$ is equivalent to $\ln x > \ln\alpha$.
$x < \alpha$ is equivalent to $\ln x < \ln\alpha$.

$$\int \frac{1}{x}dx = \ln|x| + k \quad ; \quad \int \frac{u'(x)}{u(x)}dx = \ln|u(x)| + k .$$

4) **Limits .**

$$\lim_{\substack{x\to 0 \\ x>0}} \ln x = -\infty \ , \ \lim_{x\to+\infty} \ln x = +\infty \ , \lim_{\substack{x\to 0 \\ x>0}} x\ln x = 0^- , \ \lim_{x\to+\infty} \frac{\ln x}{x} = 0$$

Solved Problems

N° 1.

Answer by True or False with justification :

1) The solution of the inequality $\ln|1-x| \leq 0$ is the interval $[-1;0]$.
2) The function f defined over $IR-\{-1;1\}$ by $f(x)=\ln\left|1-x^2\right|$ is increasing over $[0;1[$.
3) The equation $x\ln x - x - 1 = 0$ has a unique solution for $x>0$.
4) $\int \frac{1}{x(x+1)}dx = \ln\left|\frac{x}{x+1}\right| + C$.
5) For all real numbers x, $\ln x(x+1) = \ln x + \ln(x+1)$.
6) For all real numbers x, $\ln x^2 = 2\ln x$.

N° 2.

Solve , in IR, the following equations :

1) $\ln(x+2)+\ln(x-1)=2\ln x$
2) $\ln(2x+4) - \ln(x-1) = \ln(x)$
3) $\ln(10-x^2) = 2\ln 3 - \ln x^2$
4) $2\ln\left|\ln\left(x^2\right)\right| = \ln 9$

N° 3.

Solve, in IR, the following inequalities:

1) $\ln\left(x^2-3\right)-\ln 2 > \ln x$
2) $\ln\left(x^2-2x\right) > \ln(4x-5)$

N° 4.

Calculate the following integrals :

1) $\int_0^1 \frac{x+1}{x^2+2x+3}dx$
2) $\int_e^{e^2} \frac{dx}{x\ln x}$
3) $\int_1^e \frac{\ln x}{x}dx$
4) $\int_1^e \ln x\, dx$
5) $\int_1^e x\ln x\, dx$
6) $\int_2^3 \frac{-3x^2+4x-3}{x-1}dx$
7) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{2-\cos x}dx$
8) $\int_0^{\frac{\pi}{6}} \tan(2x)dx$

N° 5.

1) Consider the function f defined over $\mathbb{R} - \{1, 2\}$ by $f(x) = \dfrac{2x-1}{(x-1)(x-2)}$.

a- Write $f(x)$ in the form: $f(x) = \dfrac{a}{x-1} + \dfrac{b}{x-2}$.

b- Calculate $\int_3^4 f(x)dx$.

2) Decompose $\dfrac{1}{x^2-6x+5}$ into a sum of two rational fractions, then calculate: $\int_2^3 \dfrac{dx}{x^2-6x+5}$.

3) Calculate the following integrals:

a- $\int_1^2 \dfrac{1}{x(x+1)}dx$ b- $\int_3^4 \dfrac{1}{x^2+x-2}dx$ c- $\int_1^2 \dfrac{dx}{x(x^2+1)}$

N° 6.

Consider the two integrals $I_1 = \int_0^1 \dfrac{x}{1+x^2}dx$ and $I_2 = \int_0^1 \dfrac{x^3}{1+x^2}dx$.

1) Calculate I_1.

2) Calculate $I_1 + I_2$ and deduce the value of I_2.

N° 7.

Calculate the following limits:

1) $\lim\limits_{x\to+\infty}(x - \ln x)$

2) $\lim\limits_{x\to+\infty}\left(x - \dfrac{\ln x}{x}\right)$

3) $\lim\limits_{x\to+\infty} x\ln\left(1+\dfrac{1}{x}\right)$

4) $\lim\limits_{x\to+\infty}\dfrac{2-\ln x}{1-\ln x}$

5) $\lim\limits_{x\to+\infty}\left(\ln^2 x - 2\ln x + 2\right)$

6) $\lim\limits_{x\to0^+} 2x(1-\ln x)$

7) $\lim\limits_{x\to0^+}\left(\ln x + \dfrac{1}{x}\right)$

8) $\lim\limits_{x\to e}\dfrac{\ln x - 1}{x - e}$

N° 8.

Solve the following systems of equations:

1) $\begin{cases} 2\ln x + \ln y = 1 \\ 5\ln x + 3\ln y = 4 \end{cases}$ 2) $\begin{cases} (\ln x)(\ln y) = -15 \\ \ln(xy) = -2 \end{cases}$

3) $\begin{cases} x^2 + y^2 = 10 \\ \ln x + \ln y = \ln 3 \end{cases}$ 4) $\begin{cases} \ln x + \ln y^2 = 4 \\ \ln^2 x - 3\ln xy = -\dfrac{13}{2} \end{cases}$

N° 9.

Consider the function f defined over $]0;+\infty[$ by $f(x) = \dfrac{\ln x}{x}$, designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to 0^+} f(x)$.
2) Deduce the asymptotes to (C).
3) Study the variations of f and set up its table of variations.
4) Write an equation of the tangent (T) to (C) at the point of abscissa 1.
5) Draw (C) and (T).
6) Discuss according to the values of the real number m the number of solutions of the equation $x = e^{mx}$.
7) Deduce a comparison of the two numbers 2007^{2008} and 2008^{2007}.
8) Calculate the area of the region limited by (C), the axis $x'x$ and the two Straight lines of equations $x = 1$ and $x = e$.

N° 10.

Consider the function f defined over $]1;+\infty[$ by $f(x) = \dfrac{x}{\ln x}$, designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

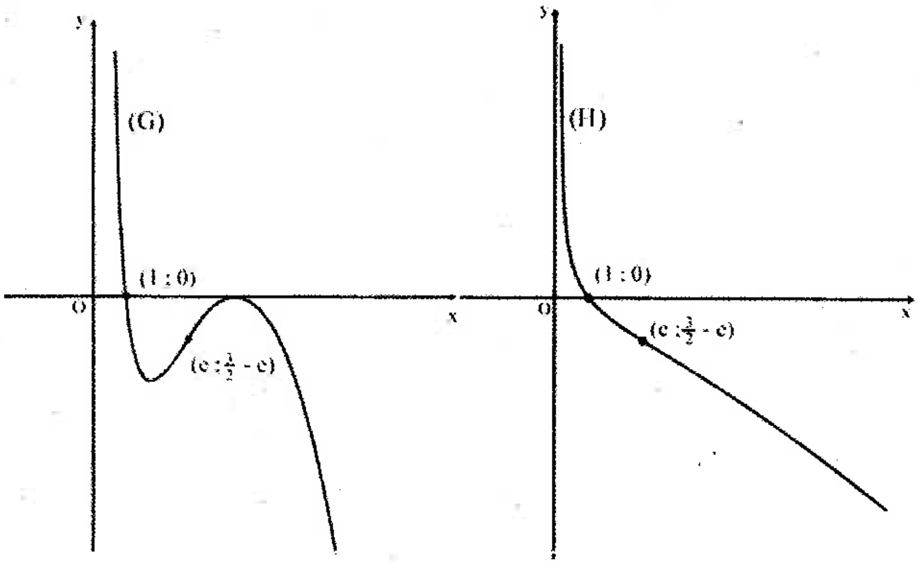
1) Determine $\lim\limits_{x \to +\infty} f(x)$.
2) Show that (C) admits an asymptote parallel to the axis $y'y$.
3) Study the variations of f and set up its table of variations.
4) Deduce a comparison of the two numbers a^b and b^a for $e < a < b$
5) Calculate $f(e^2)$ and draw (C).

N° 11.

Consider the function f defined over the interval $I =]0;+\infty[$ by

$f(x) = \dfrac{\ln x - x}{x}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. *(unit : 2cm)*.

1) a- Calculate the limits of f at the boundaries of the domain I.
   b- Determine the asymptotes of (C).
2) Calculate $f'(x)$ and set up its table of variations of f.
3) Verify that the tangent (d) to (C), at the point A (1; − 1), has an equation $y = x - 2$.
4) Draw the straight line (d) and the curve (C).
5) One of the two curves (G) and (H) drawn below is the representative curve of an antiderivative F of the function f.



a- Which of the curves above is that of the function F?

b- Without finding the expression of $F(x)$, calculate in cm^2, the area of region limited by the curve (C), the axis of abscissas and the two straight lines of equations $x = 1$ and $x = e$. Give your answer to the nearest 10^{-2}.

N° 12.

Consider the function f defined over $]0;e[\cup]e;+\infty[$ by

$f(x)=\dfrac{1}{x(1-\ln x)}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine the limits of at the boundaries of its domain of definition.
   b- Deduce the asymptotes to (C).
2) a- Study the variations of f and set up its table of variatións.
   b- Draw (C).
3) Calculate the area of the region bounded by (C), the axis $x'x$ and the two straight lines of equations $x=\dfrac{1}{e^2}$ and $x=\dfrac{1}{e}$.

N° 13.

Consider the function f defined over $]0;2[$ by $f(x)=\ln\left(\dfrac{x}{2-x}\right)$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine the limits of f at the boundaries of its domain of definition and deduce the asymptotes to (C).
   b- Study the variations of f and set up its table of variations.
2) a- Show that the point $A(1;0)$ is a center of symmetry of (C).
   b- Write an equation of the tangent (T) at A to (C).
3) Draw (C).
4) Show that the equation $f(x)=x$ has a unique root α and that $1.6<\alpha<1.7$.
5) a- Show that f has an inverse function f^{-1} over $]0;2[$.
   b- Determine the domain of definition of f^{-1} and draw its curve (C'), representative of f^{-1} in the same system.
   c- Determine $f^{-1}(x)$.
   d- Solve the equation $f(x)=f^{-1}(x)$.

N° 14.

Part A .

Consider the function g defined over $]0;+\infty[$ by $g(x)=x+(x-2)\ln x$.

1) Show that $g'(x) = 2\frac{x-1}{x} + \ln x$.
2) Deduce that :
if $x > 1$ then $g'(x) > 0$ and if $0 < x < 1$ then $g'(x) < 0$.
3) Show that $g(x) \geq 1$.

Part B.

Consider the function f defined over $]0;+\infty[$ by $f(x) = 1 + x\ln x - \ln^2 x$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$
Graphical unit: 2 cm.

1) a- Verify that $f'(x) = \frac{g(x)}{x}$, study the limits of f at the boundaries of its domain of definition and set up the table of variations of f.
   b- Deduce that f has an inverse function f^{-1} defined over the interval J to be determined.
2) a- Write an equation of the tangent (T) to (C) at the point of abscissa1.
   b- Study the variations of the function h defined over $]0;+\infty[$ by $h(x) = x - 1 - \ln x$ and deduce the sign of $h(x)$.
   c- Show that $f(x) - x = (\ln x - 1)h(x)$ and deduce the relative positions of (C) with respect to (T).
   d- Determine the abscissa of the point B of (C) such that the tangent (T') at B to (C) is parallel to (T)
3) a- Draw (C) and (T).
   b- f has an inverse function f^{-1} over $]0;+\infty[$, draw, in the same system, the curve (C') representative of the function f^{-1}.
4) a- Calculate the integrals $I_1 = \int_1^e x\ln x dx$ and $I = \int_1^e (\ln x)^2 dx$.
   b- Designate by $\mathscr{A}$ the area, in cm^2 , of the region limited by the curves (C) and (C'). Calculate to the nearest 10^{-3} the value of $\mathscr{A}$.

N° 15.

Part A.

Consider the function u defined over $]-\infty;0[\cup]0;+\infty[$ by
$u(x) = 2x^3 - 1 + 2\ln|x|$.

1) Calculate $\lim_{x\to 0} u(x)$ and $\lim_{x\to +\infty} u(x)$.
2) Study the variations of u and set up its table of variations.
3) Show that the equation $u(x)=0$ admits over $[0.8\,;0.9]$ a unique root α .
4) Deduce the sign of $u(x)$ for $x \in]-\infty\,;0[\cup]0\,;+\infty[$.

Part B.

Consider the function f defined over $]-\infty\,;0[\cup]0\,;+\infty[$ by $f(x)=2x-\dfrac{\ln|x|}{x^2}$.

Designate by (C) its representative curve in an orthonormal system $(O\,;\vec{i},\vec{j})$.

1) Determine the limits of f at the boundaries of its domain of definition. Deduce an asymptote to (C).
2) Calculate $f'(x)$ and set up the table of variations of f .
3) Show that the straight line (d) of equation $y=2x$ is an asymptote to (C) at $+\infty$ and at $-\infty$.
4) Study the position of (C) with respect to (d).
5) Taking $\alpha=0.85$, calculate $f(\alpha)$ to the nearest 10^{-2} and draw (C).

N° 16.

1) Consider the function f defined over $]0,+\infty[$ by $f(x)=x+\ln x-\ln(x+1)$, and designate by (C) its representative curve in an orthonormal system $(O\,;\vec{i},\vec{j})$.
   a- Show that the axis $y'y$ is an asymptote to (C).
   b- Show that the straight line (d) of equation $y=x$ is an asymptote to (C) in the neighborhood of $+\infty$.
   c- Study the variations of f and draw (C).
2) Consider the function g defined over $]-\infty,-1[\cup]0,+\infty[$ by $g(x)=x+\ln\left(\dfrac{x}{x+1}\right)$, and designate by (γ) its representative curve in an orthonormal system $(O\,;\vec{i},\vec{j})$.

   Show that the point $I\left(-\dfrac{1}{2};-\dfrac{1}{2}\right)$ is a center of symmetry (γ) and deduce the drawing of (γ) .
3) a- Let $F(x)=(ax+b)\ln(ax+b)$, calculate $F'(x)$.

b- Deduce the area of the region limited by (C) , (d) and the two straight lines of equations $x = 1$ and $x = e$.

N° 17.

Consider the function f defined over $]0;+\infty[$ by $f(x) = \frac{(\ln x)^2}{x}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine the limits of f at $+\infty$ and at 0 and deduce the asymptotes to (C).

b- Show that $f'(x)$ has the same sign as $(2-\ln x)\ln x$.

c- Study the variations of f and set up its table of variations.

d- Draw (C) .

2) For $p \geq 1$, let $I_p = \int_1^{e^2} \frac{(\ln x)^p}{x^2} dx$

a- Using integration by parts, calculate $I_1 = \int_1^{e^2} \frac{\ln x}{x^2} dx$.

b- Show that $I_{p+1} = -\frac{2^{p+1}}{e^2} + (p+1)I_p$.

c- Deduce I_2, I_3 *and* I_4 .

3) Let (D) be the region limited by (C) , $x'x$ and the two straight lines of equations $x = 1$ and $x = e^2$.

Calculate the volume of the solid generated by rotating (D) about $x'x$.

N° 18.

Consider the function f defined over $]0;+\infty[$ by $f(x) = x - \frac{\ln x}{x}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim_{x \to 0} f(x)$, $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to +\infty} (f(x) - x)$.

Deduce the asymptotes to (C) .

2) The table below is the table of variations of the function f' derivative of f .

x	0		1		$e^{\frac{3}{2}}$		$+\infty$
$f''(x)$	||	$+$		$+$	0	$-$	
$f'(x)$	|| $-\infty$	$\nearrow$	0	$\nearrow$	$1+\frac{1}{2}e^{-3}$	$\searrow$	1

a- Study the variations of f and set up its table.

b- Prove that f has only one inflection point I.

Determine the coordinates of I to the nearest 10^{-2}.

c- Draw (C).

3) a- Prove that f has an inverse function f^{-1} over the interval $[1\,;+\infty[$ and determine the domain of definition of f^{-1}.

b- Draw the curve (C') representative of f^{-1} in the same system as that of (C).

c- Let $M(t;f(t))$ be a variable point of (C) with $t \geq 1$.

The straight line (d) passing through M and perpendicular to the first bisector of axes of equation $y = x$ cuts (C') at N.

Calculate MN in terms of t.

N° 19.

Part A.

Consider the function f defined over $]0,+\infty[$ by $f(x)=\left(1-\frac{1}{x}\right)(-2+\ln x)$

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$, then deduce an asymptote to (C).
2) Calculate $f'(x)$.
3) Consider the function u defined over $]0,+\infty[$ by $u(x)=x-3+\ln x$.

a- Study the variations of u.

b- Show that the equation $u(x)=0$ has a unique solution α such that $2.20<\alpha<2.21$ and deduce the sign of $u(x)$.

4) Remarking that $f'(x)=\dfrac{u(x)}{x^2}$ study the variations of f

and set up its table .

5) Express $\ln(\alpha)$ in terms of α and show that $f(\alpha)=-\frac{(\alpha-1)^2}{\alpha}$.

6) Suppose that $\alpha=2.205$, calculate $f(\alpha)$ and draw (C) .

Part B.

Consider the function F the antiderivative f over $]0,+\infty[$ that vanishes at $x=1$.

1) Without calculating $F(x)$, study the variations of F over $]0,+\infty[$.
2) What can you say about the tangents to (C'), the representative curve of F at the the points of abscissas 1 and e^2 ?
3) Show that $f(x)=\ln(x)-\frac{\ln(x)}{x}+\frac{2}{x}-2$ and deduce the expression of $F(x)$
4) Calculate the area of the region limited by (C) , the axis $x'x$ and the two straight lines of equations $x=1$ and $x=e^2$.

N° 20.

Consider the function f defined over $]0;+\infty[$ by $f(x)=-3-\ln x+2\ln^2 x$

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$ and deduce an asymptote to (C).
2) Solve the inequality $f(x)>0$.
3) Calculate $f'(x)$, study the variations of f and set up its table of variations.
4) Write an equation of the tangent (T) to (C) at the point of abscissa $e^{\frac{5}{4}}$.
5) Consider the function g defined over $]0;+\infty[$ by

$$g(x)=f(x)-\left(4e^{-\frac{5}{4}}x-\frac{41}{8}\right).$$

a- Show that $g'(x)=\frac{4\ln x-1}{x}-4e^{-\frac{5}{4}}$ then calculate $g''(x)$.

b- Study the variations of g' over $]0;+\infty[$ and set up its table of variations.

c- Deduce that $g'(x)\le 0$.

6) a- Calculate $g\left(e^{\frac{5}{4}}\right)$, then determine the sign of $g(x)$.

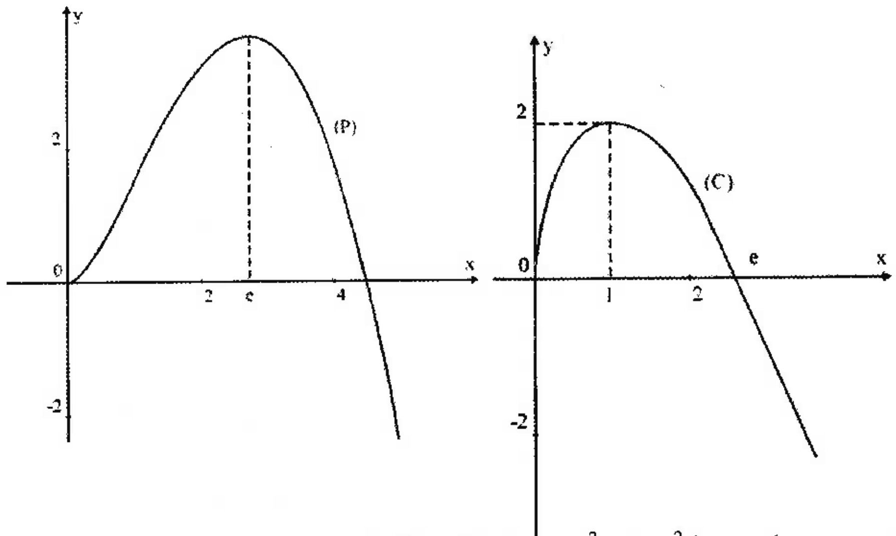
   b- Deduce the position of (C) with respect to (T).

7) Draw (C) and (T).

8) Calculate the area of the domain limited by (C), the axis $x'x$ and the two straight lines of equations $x=\frac{1}{e}$ and $x=e^{\frac{3}{2}}$.

N° 21.

1) The curves shown below are the representative curves of the functions f defined over $]0;+\infty[$ and its antiderivative F.
   Determine the representative curve of f and that of F.



2) Suppose that the equation of F is $F(x)=ax^2+bx^2\ln x$ where a and b are two real numbers. Calculate a and b and deduce the expression of $f(x)$.

3) Discuss according to the values of the real parameter m the existence of the roots of the equation $x=e^{1-\frac{m}{2x}}$.

4) The function f has an inverse function f^{-1} for $x>1$.

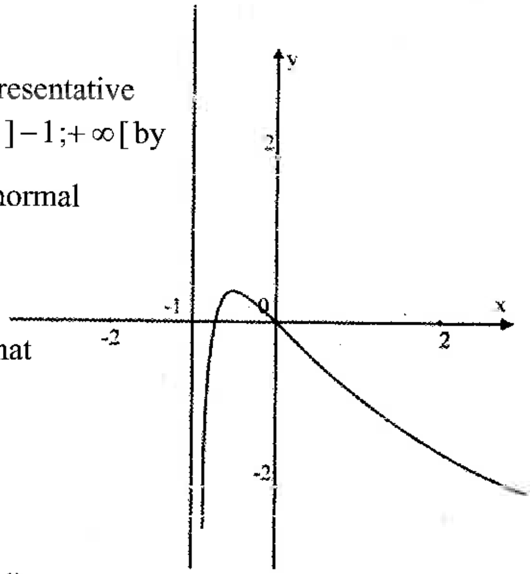
   a- Calculate the coordinates of the point A, the intersection of (C) and the curve (C'), representative of f^{-1}.

b- Show that (C) and (C') have the same tangent (T) at A.

N° 22.

Part A.

The curve (C) to the right is the representative curve of the function f defined over $]-1;+\infty[$ by $f(x)=\dfrac{x}{x+1}-2\ln(x+1)$, in an orthonormal system $(O;\vec{i},\vec{j})$.



1) Show that the equation $f(x)=0$ admits two roots 0 and α such that $\alpha\in]-0.8;-0.7[$.

2) Determine $\lim\limits_{x\to -1} f(x)$.

Part B.

Consider the function g defined over $]-1;0[\cup]0;+\infty[$ by : $g(x)=\dfrac{\ln(x+1)}{x^2}$ and designate by (γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. unit : 2 cm.

1) Calculate $\lim\limits_{\substack{x\to 0\\ x>0}} g(x)$, $\lim\limits_{\substack{x\to 0\\ x<0}} g(x)$, $\lim\limits_{x\to -1} g(x)$ and $\lim\limits_{x\to +\infty} g(x)$.

2) a- Verify that $g'(x)=\dfrac{f(x)}{x^3}$ and deduce that the sign of g'(x) is given by the following table :

x	-1		α				$+\infty$
$g'(x)$	‖	+	0	−	‖	−	

b- Set up the table of variations of g.

3) a- Show that $g(\alpha)=\dfrac{1}{2\alpha(\alpha+1)}$.

b- Taking $\alpha=-0.75$, find a value of $g(\alpha)$ to the nearest 10^{-1}.

c- Draw (γ).

Part C.

1) Remarking that $\dfrac{1}{x(x+1)}=\dfrac{1}{x}-\dfrac{1}{x+1}$, calculate the integral $\displaystyle\int_1^{\lambda}\frac{1}{x(x+1)}dx$, with $\lambda>1$.
2) a- Calculate, in cm^2, the area A_λ of the region limited by (γ) , the axis $x'x$ and the two straight lines of equations $x=1$ and $x=\lambda$.
   b- Calculate $\lim\limits_{\lambda\to+\infty}A_\lambda$.

N° 23. ***For the students of the G.S. section .***

Consider the function f defined over $]-1;+\infty[$ by $f(x)=x\ln(x+1)$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

Part A .

1) Consider the function h defined over $]-1;+\infty[$ by $h(x)=\dfrac{1}{x+1}$.

   Show that the derivative of order n of h is given by $h^{(n)}(x)=\dfrac{(-1)^n\,n!}{(x+1)^{n+1}}$.

2) a- Show that $f'(x)=\ln(x+1)+1-\dfrac{1}{x+1}$.
   b- Deduce the derivative n of f .

Part B .

1) a- Calculate $f''(x)$ for $x\in]-1;+\infty[$.
   b- Determine $\lim\limits_{x\to-1}f'(x)$ and $\lim\limits_{x\to+\infty}f'(x)$.
   c- Study the variations of f' and set up its table of variations.
   d- Show that the equation $f'(x)=0$ admits 0 as a unique solution.
   e- Deduce the sign of $f'(x)$.
2) a- Calculate $\lim\limits_{x\to-1}f(x)$, $\lim\limits_{x\to+\infty}f(x)$ and $\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}$.
   b- Set up the table of variations of f.
   c- Draw (C).
3) a- Show that f has an inverse function g over $[0;+\infty[$ and give the domain of definition of g .
   b- Find the coordinates of the points of intersection of (C) and (C') which is the representative curve of g.

c- Draw (C'), the representative curve of g, in the given system.

4) a- Calculate a, b and c so that $\frac{x^2}{x+1} = ax + b + \frac{c}{x+1}$.

b- Using integration by parts, calculate the area of the region limited by (C) and (C').

Part C.

Let $I = [0; e-1]$, consider the sequence (u_n) defined by $u_0 = \frac{3}{2}$ and $u_{n+1} = f(u_n)$ for all $n \geq 1$.

1) Show that $f(I) = I$.
2) Prove by induction, that $u_n \in I$ for any n.
3) a- Prove that the sequence (u_n) is decreasing for any n.

b- Deduce that (u_n) is convergent and determine its limit.

$\boxed{N^{\circ}24}$ * ***For the students of the G.S. section.***

Part A.

Consider the function h defined over $]0;+\infty[$ by $h(x) = x^2 - \ln x$.

Study the variations of h and deduce that $h(x) > 0$ over $]0;+\infty[$.

Part B.

Consider the function f defined over $]0;+\infty[$ by $f(x) = \frac{1+\ln x}{x} + x - 1$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Verify that $\lim\limits_{x \to +\infty} \frac{1+\ln x}{x} = 0$ and deduce that the straight line (d) of equation $y = x - 1$ is an asymptote to (C).
2) Study the relative positions of (C) and (d).
3) Show that the axis $y'y$ is an asymptote to (C).
4) Verify that $f'(x) = \frac{h(x)}{x^2}$ and set up the table of variations of f.
5) Let F be the point of (C) of abscissa 1.
   Show that the tangent (D) at F to (C) is parallel to (d).
6) Show that the equation $f(x) = 0$ admits a unique solution $\alpha \in]0.4; 0.5[$.
7) Draw (C), (d) and (D).

8) a- Show that f has an inverse function f^{-1} for $x \in]0;+\infty[$.

b- Determine the domain of definition of f^{-1} and draw (C'), the representative curve of f^{-1} in the same system.

9) Calculate, in cm^2, the area of the region limited by (C), (d) and the two straight lines of equations $x=1$ and $x=e$.

Part C.

Consider the numerical sequence (x_n) defined by $x_n = e^{\frac{n-2}{2}}$ for all natural numbers n.

1) Show that (x_n) is a geometrical sequence whose first term and ratio are to be determined.
2) Show that (x_n) is increasing.
3) For all n, let $a_n = 4\int_{x_n}^{x_{n+1}}\left(f(x)-(x-1)\right)dx$.

   Show that (a_n) is an arithmetic sequence.

$\boxed{N°25.}$ ***For the students of the G.S. section.***

Part A.

Consider the two functions f and g defined over $[-1;+\infty[$ by $f(x)=-1+\sqrt{x+1}$ and $g(x)=-1-\sqrt{x+1}$.

Designate by (C) and (C') the representative curves of f and g respectively in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of f and g and draw (C) and (C') in the same system $(O;\vec{i},\vec{j})$.
2) Let $(\gamma)=(C)\cup(C')$.

   a- Show that $(y+1)^2 = x+1$ is an equation of (γ).

   b- Determine the nature of (γ) as well as its elements.

   c- Calculate the area of the region D limited by (γ) and the straight line of equation $x=0$.

Part B .

Define a function h by $h(x)=\ln\left(-1+\sqrt{x+1}\right)$ and designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Verify that the domain of definition of h is $]0;+\infty[$.
   b- Study the variations of h and set up its table of variations.

2) Let r be the rotation of center O and angle $\frac{\pi}{2}$.
   For all points M of affix z associate the point M' of affix z' image of M by r.
   a- Express z' in terms of z.
   b- Let A be the point of (Γ) of abscissa 3 and B is the point of (Γ) of abscissa $\frac{5}{4}$. Designate by P the orthogonal projection of B over $x'x$ and by H the orthogonal projection of B over $y'y$.
   Determine the affix of B' the image of B by r and the affix of the point A' image of A by r.
   c- Let $z=x+iy$ and $z'=x'+iy'$, express x and y in terms of x' and y'.
   d- Show that when M describes (Γ) then the point M' describes the curve (Γ') of equation $y=e^{-2x}+2e^{-x}$.
   e- Place on the preceding graph the points A, B A' and B' and draw (Γ) and (Γ')in the same system.

3) a- Calculate the integral $\int_0^{\ln 2}\left(e^{-2x}+2e^{-x}\right)dx$ and interpret this integral graphically.
   b- Determine , in square units , the area α of the region limited by the segments $[OA]$,$[OH]$ and $[HB]$ and the arc limited by B and A.
   c- Let $I=\int_{\frac{5}{4}}^{3}\ln\left(-1+\sqrt{1+x}\right)dx$.
   Find a relation between α and I then deduce the exact value of the integral I.

Supplementary Problems

N° 1.

Consider the function f defined over $]1;+\infty[$ by $f(x)=x-\dfrac{1}{x\ln x}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$

1) Calculate $\lim\limits_{x\to 1} f(x)$ and deduce an asymptote to (C).
2) Calculate $\lim\limits_{x\to +\infty} f(x)$. Prove that the straight line (d) of equation $y=x$ is an asymptote to (C) and study the position of (C) with respect to (d).
3) Calculate $f'(x)$ and show that f is strictly increasing.
   Set up the table of variations of f.
4) Show that the equation $f(x)=0$ admits a unique root α and verify that $1.5<\alpha<1.6$.
5) Draw (d) and (C).
6) a- Calculate the area $A(t)$ of the region limited by the curve (C), The straight line (d) and the two straight lines of equations $x=e$ and $x=t$ where $t>e$.
   b- Show that for all $t>e$, $A(t)<t$.

N° 2.

1) Let g be the function defined over $]0;+\infty[$ by $g(x)=1-x^2-\ln x$ and (C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
   a- Calculate $g(1)$, $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to 0} g(x)$.
   b- Study the variations of g and draw its table of variations.
   c- Trace (C) .
   d- Deduce the sign of $g(x)$.
2) f is the function defined over $]0;+\infty[$ by $f(x)=-x+2+\dfrac{\ln x}{x}$.
   Let (Γ) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
   a- Show that (d) of equation $y=-x+2$ is an asymptote to (C) and study the relative positions of (C) with respect to (d) .

b- Show that $f'(x)$ and $g(x)$ have the same sign and draw the table of variations of f.

c- Show that the equation $f(x)=0$ admits two roots α and β and that $0.4<\alpha<0.5$ and $2.3<\beta<2.4$

d- Determine the coordinates of the point A of (C) where the tangent (T) to (C) is parallel to (d).

e- Trace (T) and (C).

f- Calculate the area of the domain limited by (C), (d) and the two straight lines of equations $x=1$ and $x=2$.

N°3.

f is the function defined over $]0;+\infty[$ by $f(x)=\frac{1}{2}x^2(3-2\ln x)+1$, and (C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Unit: 2 cm.

1) a- Determine $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.

b- Calculate $f'(x)$ and draw the table of variations of f.

c- Show that the equation $f(x)=0$ admits a unique root α such that $4.6<\alpha<4.7$

2) Write an equation of the tangent (T) to (C) at the point of abscissa 1.

3) Let g be the function defined over $]0;+\infty[$ by

$g(x)=f(x)-2x-\frac{1}{2}$.

a- Study the variations of g' and deduce that $g'(x)\le 0$.

b- Study the variations of g and deduce the position of (C) with respect to (T).

c- Trace (C) and (T).

4) Calculate, in cm^2, the area of the domain limited by (C), (T) and the two straight lines of equations $x=1$ and $x=2$.

N°4.

Part A

Let g be the function defined over $]0;+\infty[$ by $g(x)=x^3-x+1-2\ln x$.

1) Show that $g'(x)=\dfrac{(x-1)\left(3x^2+3x+2\right)}{x}$.
2) Study the variations of g.
3) Deduce the sign of $g(x)$ according to the values of x.

Part B:

Consider the function f defined over $]0;+\infty[$ by $f(x)=x+1+\dfrac{x+\ln x}{x^2}$.

Designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$. Ggraphical unit = 2 cm.

1) Determine $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.
2) Show that the straight line (D) of equation $y=x+1$ is an asymptote to (C).
3) a- Show that the function h defined by $h(x)=x+\ln x$ is strictly increasing and that the function takes positive and negative values.
   b- Deduce that the straight line (D) cuts (C) at a unique point of abscissa α and that $0.56<\alpha<0.57$
4) Study the relative positions of (C) and (D) .
5) Verify that $f'(x)=\dfrac{g(x)}{x^2}$ and draw the table of variations of f.
6) Study the variations of f and draw its table of variations.
7) Show that the equation $f(x)=0$ admits a unique root β such that $0.46<\beta<0.47$
8) Trace (C) and (D).

$\boxed{N^\circ 5.}$

Part A.

g is the function defined over $]0;+\infty[$ by $g(x)=\dfrac{x-2\ln x}{x}$, designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1) a- Determine $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to 0} g(x)$.
   b- Deduce the asymptotes to (C).
   c- Study the variations of g and trace (C).

2) Calculate the area of the domain limited by (C), the horizontal asymptote and the two straight lines of equations $x=1$ and $x=e$.

Part B:

f is the function defined over $]0;+\infty[$ by $f(x)=x-\ln^2 x$, designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $f'(x)>0$ for $x>0$.
2) Determine an equation of the tangent (T) to (Γ) at the point of abscissa 1 and study the relative positions of (Γ) and (T).
3) Find the coordinates of the point of inflection I of f.
4) Trace (Γ) and (T).
5) a- Show that f admits an inverse function f^{-1} for $x>0$.
   b- Trace the curve (Γ') of the function f^{-1} in the same system as that of f.
   c- A straight line (d) of equation $y=-x+m$, $m\in IR$,cuts (Γ) at A and (Γ') at B. Determine m if $AB=\sqrt{2}$.

N°6.

Let f be the function defined over $I=]-\infty;-1[\cup]0;+\infty[$ by $f(x)=x-1+2\ln\left(\dfrac{x+1}{x}\right)$. Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that the straight line (d) of equation $y=x-1$ is an asymptote to (C) at $-\infty$ and $+\infty$.
2) Determine $\lim\limits_{x\to-1} f(x)$ and $\lim\limits_{x\to 0} f(x)$.
3) Calculate $f'(x)$ and draw the table of variations of f.
4) Show that the point $I\left(-\dfrac{1}{2};-\dfrac{3}{2}\right)$ is a center of symmetry of (C).
5) Trace (C).

N° 7.

Part A:

Let h be the function defined over $]0;+\infty[$ by $h(x)=\ln x$ .Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine an equation of the tangent (d) to (C) at the point of abscissa 1.
2) Study the variations of the function f defined over $]0;+\infty[$ by $f(x)=x-1-\ln x$ and deduce the position of (C) with respect to (d) .
3) Deduce the minimum value taken by $x-\ln x$ over the interval $]0;+\infty[$.
4) Let M be a point of (C) of abscissa x and N a point of the straight line (D) of equation $y=x$ of abscissa x also. Determine the least value taken by the distance MN as x varies in $]0;+\infty[$.

Part B:

1) Let M be a point of abscissa x of the curve (C),express the distance OM in terms of x.
2) Let u be the function defined over $]0;+\infty[$ by $u(x)=x^2+\ln x$.
   a- Determine $\lim\limits_{x\to 0} u(x)$ and $\lim\limits_{x\to+\infty} u(x)$.
   b- Show that the equation $u(x)=0$ admits a unique root α and that $0.5<\alpha<1$
   c- Deduce the sign of $u(x)$ according to the values of x.
3) Let g be the function defined over $]0;+\infty[$ by $g(x)=x^2+\ln^2 x$. Designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
   a- Calculate $\lim\limits_{x\to 0} g(x)$ and $\lim\limits_{x\to+\infty} g(x)$.
   b- Show that $g'(x)=\dfrac{2}{x}u(x)$ and set up the table of variations of g.

c- Trace (Γ).

4) Deduce from the previous questions the exact value of the shortest distance from the origin to the points of (C).

5) A being the point of abscissa α of (C), prove that the tangent at A to (C) is perpendicular to the straight line (OA).

$N°8.$

Consider the function f defined over $]-\infty;0[\cup]0;+\infty[$ by $f(x)=x-\ln|x|$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine the limits of f at the boundaries of its domain of definition and deduce an asymptote to (C).

2) Calculate $f'(x)$ and set up the table of variations of f.

3) Find the points of intersection of (C) with the straight line (d) of equation $y=x$.

4) Draw (C) and (d).

5) Show that the equation $x=\ln|x|$ has a unique root α such that $-0.568<\alpha<-0.566$.

6) Let (D) be the straight line of equation $y=x+m$ where m is a real parameter, show that (D) cuts (C) in two points M_1 and M_2 for all real numbers m.

7) write the equations of the tangents (T_1) and (T_2) to (C) at the points M_1 and M_2.

8) Show that (T_1) andt (T_2) intersect on $y'y$ at the point $I(0;m+1)$.

9) Calculate the area of the region limited by (C), the axis $x'x$ and the two straight lines of equations $x=1$ and $x=e$.

$N°9.$

Consider the function f defined over $]0;+\infty[$ by $f(x)=1-\dfrac{1}{x}+\ln x$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$

1) a- Determine $\lim\limits_{x\to 0} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$ and deduce an

asymptote to (C) .

b- Study the variations of f and set up its table of variations.

c- Write an equation of the tangent (T) to (C) at the point A of abscissa 1.

d- Calculate $f(4)$ and draw (C) and (T).

2) a- Show that f has an inverse function g over $]0;+\infty[$ and find the domain of g .

b- Set up the table of variations of g and deduce the sign of $g(x)$.

c- Calculate $g(0)$ and $g'(0)$.

d- Draw the curve (C'), representative of g , in the same system .

3) a- Calculate the area of the region limited by (C), the axis $x'x$ And the two straight lines of equations $x=1$ and $x=e$.

b- Deduce the area of the region bounded by (C'), the axis $y'y$ and the two straight lines of equations $y=1$ and $y=e$.

N° 10.

Consider the function f defined over $]0:+\infty[$ by $f(x)=\dfrac{2\ln x-1}{x^2}$ and

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate the limits of f at the boundaries of its domain of definition.

2) Show that $f'(x)=\dfrac{4(1-\ln x)}{x^3}$ and set up the table of variations of f.

3) a- Show that (C) cuts the axis of abscissas at a unique point A to be determined and study the sign of $f(x)$.

b- Write an equation of the tangent at A to (C).

4) Draw (C).

5) a- Calculate $\displaystyle\int_{\sqrt{e}}^{x}\frac{\ln t}{t^2}dt$ and deduce $\displaystyle\int_{\sqrt{e}}^{x}f(t)dt$.

b- Calculate the area of the region limited by (C), the axis $x'x$ and the two straight lines of equations $x=\sqrt{e}$ and $x=e$.

N° 11.

Part A .

Consider the function g defined over $]0;+\infty[$ by $g(x)=\dfrac{x+1}{2x+1}-\ln x$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to 0} g(x)$.

2) Study the variations of g and set up its table of variations.

3) Show that the equation $g(x)=0$ admits a unique root α such that $1.8<\alpha<1.9$ and deduce the sign of $g(x)$.

4) Calculate the area of the region limited by (C), the x-axis and the two straight lines of equations $x=\dfrac{1}{e}$ and $x=1$.

Part B .

Consider the function f defined over $]0;+\infty[$ by $f(x)=\dfrac{2\ln x}{x^2+x}$ and designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
Graphical unit : 2 cm

1) Calculate $\lim\limits_{x\to+\infty} f(x)$, $\lim\limits_{x\to 0} f(x)$ and interpret graphically the result.

2) a- Show that $f'(x)=\dfrac{2(2x+1)}{(x^2+x)^2}\times g(x)$.

   b- Set up the table of variations of f.

3) Show that $f(\alpha)=\dfrac{2}{\alpha(2\alpha+1)}$.

4) Taking $\alpha=1.85$, draw (Γ).

N° 12. ***For the students of the G.S. section .***

Consider the function f defined over $]0;+\infty[$ by $f(x)=\begin{cases}\dfrac{\ln x}{x-\ln x} & \text{if } x>0\\ -1 & \text{if } x=0\end{cases}$.

and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

Part A.

Show that $x-\ln x\geq 1$.

Part B .

1) Study the continuity and the differentiability of f at 0.
2) Calculate $\lim\limits_{x\to+\infty} f(x)$.
3) Study the variations of f and set up its table of variations.
4) Draw (C).

Part C .

Let $F(x)=\int_1^x f(t)dt$ where $x\geq 0$.

1) Study the variations of F over $[0;+\infty[$.
2) Using part A , show that $-1\leq f(t)\leq t-1$ for $t\in]0;1]$.
3) a- Show that for $t\geq 1$, $f(t)\geq \dfrac{\ln t}{t}$.

   b- Calculate $\int_1^x \dfrac{\ln t}{t}dt$ and deduce that $\lim\limits_{x\to+\infty} F(x)$.
4) Let $u_n=\int_n^{n+1} f(t)dt$.

   a- Show that for $n\geq 3$, $f(n+1)\leq u_n\leq f(n)$.

   b- Show that the sequence (u_n) converges to 0.
5) Let $S_n=\sum\limits_{p=1}^{n-1} u_p=u_1+u_2+............+u_{n-1}$.

   a- Express S_n in terms of F.

   b- Deduce $\lim\limits_{n\to+\infty} S_n$.

Solutions

N° 1.

1) $\ln|1-x| \le 0$ gives $|1-x| \le 1$ then $(1-x)^2 \le 1$ that is equivalent to $x(x-2) \le 0$ which is verified for $0 \le x \le 2$, consequently the solution of the inequality is the interval $[0;2]$.

2) $f'(x) = \dfrac{-2x}{1-x^2}$. for $x \in [0;1[$, $-2x < 0$ and $1-x^2 > 0$, then $f'(x) < 0$ and consequently f is decreasing over $[0;1[$.

3) Let $f(x) = x\ln x - x - 1$, $f'(x) = \ln x + 1 - 1 = \ln x$.

   For $0 < x < 1$, f decreases from $-1 = \lim\limits_{x\to 0^+} f(x)$ to $f(1) = -2$.

   For $x > 1$, f increases from $f(1) = -2$ to $+\infty = \lim\limits_{x\to+\infty} f(x)$, then its curve cuts the x-axis at a unique point . Consequently, the equation $x\ln x - x - 1 = 0$ has a unique root for $x > 0$.

4) $\left[\ln\left|\dfrac{x}{x+1}\right| + C\right]' = \left[\ln|x| - \ln|x+1| + C\right]' = \dfrac{1}{x} - \dfrac{1}{x+1} = \dfrac{1}{x(x+1)}$ then

   $\displaystyle\int \frac{1}{x(x+1)}\,dx = \ln\left|\frac{x}{x+1}\right| + C$.

5) $\ln x(x+1) = \ln x + \ln(x+1)$ for $x > 0$.

6) For all real numbers x , $\ln x^2 = 2\ln x$ for $x > 0$.

N° 2.

1) $\ln(x+2) + \ln(x-1) = 2\ln x$ gives $\ln(x+2)(x-1) = \ln x^2$.

   Then, $(x+2)(x-1) = x^2$, which gives $x = 2$.

   For $x = 2$, the equation becomes $\ln 4 + \ln 1 = 2\ln 2$, true then $x = 2$ is acceptable.

2) $\ln(2x + 4) - \ln(x - 1) = \ln(x)$ gives $\ln\dfrac{2x+4}{x-1} = \ln x$,

   Then, $\dfrac{2x+4}{x-1} = x$, so $x^2 - 3x - 4 = 0$, which gives $x = -1$ or $x = 4$.

For $x=-1$, the equation becomes $\ln 2-\ln(-2)=\ln(-1)$, so $x=-1$ is inacceptable.
For $x=4$, the equation becomes $\ln 12-\ln 3=\ln 4$, true then $x=4$ is acceptable.

3) $\ln(10-x^2)=2\ln 3-\ln x^2$ gives $\ln\left(10-x^2\right)=\ln\frac{9}{x^2}$, then

$10-x^2=\frac{9}{x^2}$ consequently $x^4-10x^2+9=0$, which gives

$x=1$, $x=-1$, $x=3$ or $x=-3$, all accepted.

4) $2\ln\left|\ln\left(x^2\right)\right|=\ln 9$ gives $\ln\left[\ln\left(x^2\right)\right]^2=\ln 9$, then $\left[\ln\left(x^2\right)\right]^2=9$

Which gives $\ln\left(x^2\right)=3$ or $\ln\left(x^2\right)=-3$, then $x^2=e^3$, $x^2=e^{-3}$

$x=\sqrt{e^3}$ or $x=-\sqrt{e^3}$, $x=\sqrt{e^{-3}}$ or $x=-\sqrt{e^{-3}}$, all accepted.

$\boxed{N°3.}$

1) $\ln\left(x^2-3\right)-\ln 2>\ln x$ is defined for $\begin{cases}x^2-3>0\\x>0\end{cases}$ that is for

$x>\sqrt{3}$.

$\ln\left(x^2-3\right)-\ln 2>\ln x$ gives $\ln\left(x^2-3\right)>\ln 2x$, that is $x^2-3>2x$

consequently $x^2-2x-3>0$ which is verified for $x<-1$ or $x>3$.

Hence, the solution of the inequality is $S=]3;+\infty[$.

2) $\ln\left(x^2-2x\right)>\ln(4x-5)$ is defined for $\begin{cases}x^2-2x>0\\4x-5>0\end{cases}$ that is

for $\begin{cases}x<0 \text{ or } x>2\\x>\frac{5}{4}\end{cases}$, which gives $x>2$.

$\ln\left(x^2-2x\right)>\ln(4x-5)$ gives $x^2-2x>4x-5$, so

$x^2-6x+5>0$ that has as a solution $x<1$ or $x>5$.

Hence, the solution of this inequality is $S=]5;+\infty[$.

N°4.

1) $\int_0^1 \frac{x+1}{x^2+2x+3}dx = \frac{1}{2}\int_0^1 \frac{(x^2+2x+3)'}{x^2+2x+3}dx = \frac{1}{2}\ln(x^2+2x+3)\Big|_0^1$

$= \frac{1}{2}\ln(2)$.

2) $\int_e^{e^2} \frac{dx}{x\ln x} = \left[\ln|\ln x|\right]_e^{e^2} = \ln 2$.

3) $\int_1^e \frac{\ln x}{x}dx = \int_1^e (\ln x)'\ln x\, dx = \left[\frac{(\ln x)^2}{2}\right]_1^e = \frac{1}{2}$.

4) $\int_1^e \ln x\, dx = \left[x\ln x - x\right]_1^e = 1$.

5) $\int_1^e x\ln x\, dx$. Integrating by parts.

Let $u = \ln x$ and $v' = x$ then $u' = \frac{1}{x}$ and $v = \frac{1}{2}x^2$, so we get :

$\int_1^e x\ln x dx = \left[\frac{1}{2}x^2\ln x\right]_1^e - \int_1^e \frac{1}{2}x dx = \frac{1}{2}e^2 - \left[\frac{1}{4}x^2\right]_1^e = \frac{1}{4}e^2 + \frac{1}{4}$.

6) $\int_2^3 \frac{-3x^2+4x-3}{x-1}dx$

$\frac{-3x^2+4x-3}{x-1} = -3x+1-\frac{2}{x-1}$

$\int_2^3 \frac{-3x^2+4x-3}{x-1}dx = \int_2^3 (-3x+1-\frac{2}{x-1})dx =$

$\left[-\frac{3}{2}x^2 + x - 2\ln(x-1)\right]_2^3 = -\frac{13}{2} - 2\ln 2$.

7) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{2-\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{(2-\cos x)'}{2-\cos x}dx = \ln(2-\cos x)\Big|_0^{\frac{\pi}{2}} = \ln 2$

8) $\tan 2x = \frac{\sin 2x}{\cos 2x} = -\frac{1}{2}\frac{(\cos 2x)'}{(\cos 2x)}$, then :

$$I = -\frac{1}{2}\ln\left|\cos 2x\right|\Big]_0^{\pi/6} = -\frac{1}{2}\left(\ln\frac{1}{2} - \ln 1\right) = +\frac{\ln 2}{2}.$$

N° 5.

1) a- $f(x) = \dfrac{2x-1}{(x-1)(x-2)} = \dfrac{a}{x-1} + \dfrac{b}{x-2} = \dfrac{(a+b)x - 2a - b}{(x-1)(x-2)}$,

then: $\begin{cases} a+b=2 \\ -2a-b=-1 \end{cases}$ that has as a solution $a = -1$ and $b = 3$

which gives $f(x) = \dfrac{-1}{x-1} + \dfrac{3}{x-2}$.

b- $\displaystyle\int_3^4 f(x)dx = -\ln|x-1| + 3\ln|x-2|\Big]_3^4 = 4\ln 2 - \ln 3$.

2) $x^2 - 6x + 5$ is a trinomial having $x' = 1$ and $x'' = 5$ as roots.

Then, $x^2 - 6x + 5 = (x-1)(x-5)$.

$\dfrac{1}{x^2-6x+5} = \dfrac{a}{x-1} + \dfrac{b}{x-5} = \dfrac{(a+b)x - 5a - b}{(x-1)(x-5)}$

Which gives $\begin{cases} a+b=0 \\ -5a-b=1 \end{cases}$ that has as a solution $a = -\dfrac{1}{4}$ and $b = \dfrac{1}{4}$.

Consequently, $\dfrac{1}{x^2-6x+5} = \dfrac{-\frac{1}{4}}{x-1} + \dfrac{\frac{1}{4}}{x-5}$.

$\displaystyle\int_2^3 \frac{dx}{x^2-6x+5} = -\frac{1}{4}\ln|x-1| + \frac{1}{4}\ln|x-5|\Big]_2^3 = -\frac{\ln 3}{4}$.

3) a- $\dfrac{1}{x(x+1)} = \dfrac{a}{x} + \dfrac{b}{x+1} = \dfrac{(a+b)x + a}{x(x+1)}$ then $a = 1$ and

$a + b = 0$ which gives $b = -1$ then $\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$

consequently $\displaystyle\int_1^2 \frac{1}{x(x+1)}dx = \left[\ln x\right]_1^2 - \left[\ln(x+1)\right]_1^2 = 2\ln 2 - \ln 3$

b- $\dfrac{1}{x^2+x-2}=\dfrac{1}{(x-1)(x+2)}=\dfrac{a}{x-1}+\dfrac{b}{x+2}=\dfrac{(a+b)x+2a-b}{(x-1)(x+2)}$

which gives $a+b=0$ then $2a-b=1$, which has as a solution $a=\dfrac{1}{3}$ and $b=-\dfrac{1}{3}$.

$$\int_3^4 \frac{1}{x^2+x-2}dx=\frac{1}{3}\left[\int_3^4 \frac{1}{x-1}dx-\int_3^4 \frac{1}{x+2}dx\right]$$

$$=\frac{1}{3}\left[\ln(x-1)-\ln(x+2)\right]_3^4=\frac{1}{3}\ln\frac{5}{4}$$

c- $\dfrac{1}{x(x^2+1)}=\dfrac{a}{x}+\dfrac{bx+c}{x^2+1}=\dfrac{(a+b)x^2+cx+a}{x(x^2+1)}$

which gives $a=1$, $c=0$ and $a+b=0$ then $b=-1$

So, $\dfrac{1}{x(x^2+1)}=\dfrac{1}{x}+\dfrac{-x}{x^2+1}$ and consequently

$$\int_1^2 \frac{dx}{x(x^2+1)}=\int_1^2\left(\frac{1}{x}+\frac{-x}{x^2+1}\right)dx=\left[\ln x-\frac{1}{2}\ln(x^2+1)\right]_1^2$$

$$=\frac{3}{2}\ln 2-\frac{1}{2}\ln 5.$$

N°6.

1) $I_1=\displaystyle\int_0^1 \frac{x}{1+x^2}dx=\frac{1}{2}\ln(x^2+1)\Big|_0^1=\frac{1}{2}\ln 2$.

2) $I_1+I_2=\displaystyle\int_0^1 \frac{x}{x^2+1}dx+\int_0^1 \frac{x^3}{x^2+1}dx=\int_0^1 \frac{x+x^3}{x^2+1}dx=\int_0^1 x\,dx$.

Then, $I_1+I_2=\dfrac{1}{2}x^2\Big|_0^1=\dfrac{1}{2}$, which gives $I_2=\dfrac{1}{2}-\dfrac{1}{2}\ln 2$.

N°7.

1) $\lim\limits_{x\to+\infty}(x-\ln x)=+\infty-\infty$, indeterminate form.

$\lim\limits_{x\to+\infty}(x-\ln x)=\lim\limits_{x\to+\infty}x\left(1-\dfrac{\ln x}{x}\right)=+\infty(1-0)=+\infty$, since

$\lim\limits_{x\to+\infty}\dfrac{\ln x}{x}=0$.

2) $\lim\limits_{x\to+\infty}\left(x-\dfrac{\ln x}{x}\right)=+\infty-0=+\infty$.

3) $\lim\limits_{x\to+\infty}x\ln\left(1+\dfrac{1}{x}\right)=+\infty\times\ln 1=+\infty\times 0$, indeterminate form.

$\lim\limits_{x\to+\infty}x\ln\left(1+\dfrac{1}{x}\right)=\lim\limits_{x\to+\infty}\dfrac{\ln\left(1+\dfrac{1}{x}\right)}{\dfrac{1}{x}}$ and applying L'hopital's rule, we

get: $\lim\limits_{x\to+\infty}x\ln\left(1+\dfrac{1}{x}\right)=\lim\limits_{x\to+\infty}\dfrac{\ln\left(1+x^{-1}\right)}{x^{-1}}=\lim\limits_{x\to+\infty}\dfrac{\dfrac{-x^{-2}}{1+x^{-1}}}{-x^{-2}}$

$=\lim\limits_{x\to+\infty}\dfrac{1}{1+x^{-1}}=1$

4) $\lim\limits_{x\to+\infty}\dfrac{2-\ln x}{1-\ln x}=\dfrac{-\infty}{-\infty}$, indeterminate form and applying L'hopital's

rule, we get: $\lim\limits_{x\to+\infty}\dfrac{-\dfrac{1}{x}}{-\dfrac{1}{x}}=1$.

5) $\lim\limits_{x\to+\infty}\left(\ln^2 x-2\ln x+2\right)=+\infty-\infty$, indeterminate form.

$\lim\limits_{x\to+\infty}\left(\ln^2 x-2\ln x+2\right)=\lim\limits_{x\to+\infty}\left[\ln x(\ln x-2)+2\right]=+\infty$.

6) $\lim\limits_{x\to 0^+}2x(1-\ln x)=0\times(+\infty)$, indeterminate form.

$\lim\limits_{x\to 0^+}2x(1-\ln x)=\lim\limits_{x\to 0^+}\dfrac{2(1-\ln x)}{\dfrac{1}{x}}=\lim\limits_{x\to 0^+}\dfrac{-\dfrac{2}{x}}{-\dfrac{1}{x^2}}$

$=\lim\limits_{x\to 0^+}\dfrac{-2}{x}\times\dfrac{-x^2}{1}=\lim\limits_{x\to 0^+}-2x=0^-$.

7) $\lim\limits_{x\to 0^+}\left(\ln x+\dfrac{1}{x}\right)=-\infty+\infty$ indeterminate form.

$\lim\limits_{x\to 0^+}\dfrac{x\ln x+1}{x}=\dfrac{0^-+1}{0^+}=+\infty$ since $\lim\limits_{x\to 0^+}x\ln x=0^-$

8) $\lim\limits_{x\to e}\dfrac{\ln x-1}{x-e}=\dfrac{0}{0}=\lim\limits_{x\to e}\dfrac{\frac{1}{x}}{1}=\dfrac{1}{e}$.

N° 8.

1) Multiplying the first equation of the system by -3 , the system becomes $\begin{cases}-6\ln x-3\ln y=-3\\ 5\ln x+3\ln y=4\end{cases}$. Adding, we get $-\ln x=1$ which gives $\ln x=-1$ and consequently, $x=e^{-1}$.
Replacing $\ln x$ by -1 we get $\ln y=3$ which gives $y=e^{3}$, x and y are accepted.

2) For $x>0$ and $y>0$, the system becomes $\begin{cases}(\ln x)(\ln y)=-15\\ \ln x+\ln y=-2\end{cases}$
$\ln x$ and $\ln y$ are solutions of the equation $t^2-S\,t+P=0$ so $t^2+2t-15=0$ which has as a roots $t'=-5$ and $t''=3$, then $x=e^{-5}$ and $y=e^{3}$ or $x=e^{3}$ and $y=e^{-5}$, all accepted.

3) For $x>0$ and $y>0$, the equation $\ln x+\ln y=\ln 3$ gives $xy=3$.
On the other hand, the equation $x^2+y^2=10$ can be written as :
$(x+y)^2-2xy=10$ which gives $(x+y)^2=16$ and consequently $x+y=4$.
So, we have $x+y=4$ and $xy=3$.
x and y are then the solutions of the equation $t^2-S\,t+P=0$ so $t^2-4t+3=0$ that are $t'=1$ and $t''=3$, which gives $x=1$ and $y=3$ or $x=3$ and $y=1$, all accepted.

4) For $x>0$ and $y>0$, the given system is equivalent to the system $\begin{cases}\ln x+2\ln y=4\\ \ln^2 x-3\ln x-3\ln y=-\dfrac{13}{2}\end{cases}$

The equation $\ln x + 2\ln y = 4$ gives $\ln y = \dfrac{4-\ln x}{2}$ and replacing $\ln y$ by its value in the second equation, we get:

$\ln^2 x - 3\ln x - 3\left(\dfrac{4-\ln x}{2}\right) = -\dfrac{13}{2}$ which is equivalent to the equation

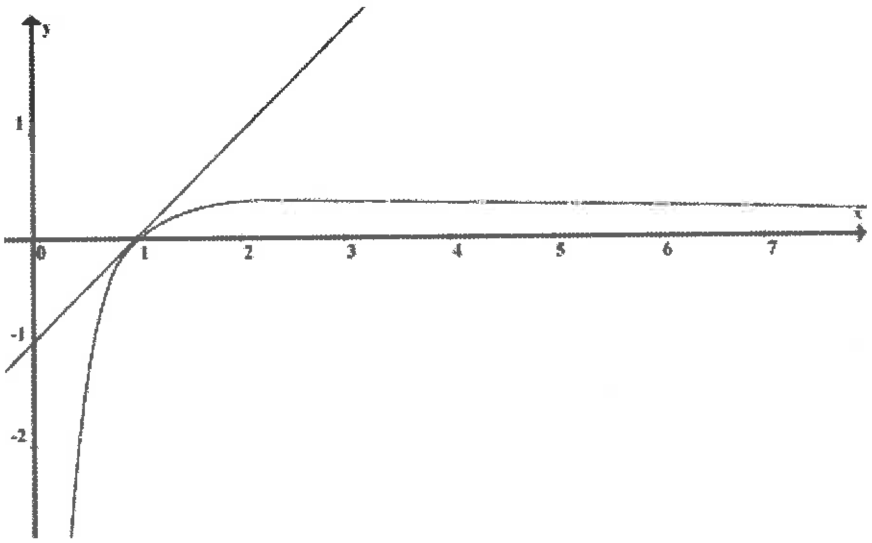
$2\ln^2 x - 3\ln x + 1 = 0$ whose roots are $\ln x = 1$ or $\ln x = \dfrac{1}{2}$ then :

$x = e$ which gives $y = e^{\frac{3}{2}}$ or $x = e^{\frac{1}{2}}$ which gives $y = e^{\frac{7}{4}}$, all accepted.

N°9

1) $\lim\limits_{x\to+\infty} f(x) = 0^+$ and $\lim\limits_{x\to 0^+} f(x) = -\infty$.
2) The axis $x'x$ and the axis $y'y$ are asymptotes to (C).
3) $f'(x) = \dfrac{1-\ln x}{x^2}$, so the table of variations of f is:

x	0		e		$+\infty$
$f'(x)$	‖	$+$	0	$-$	
$f(x)$	‖ $-\infty$	$\nearrow$	$1/e$	$\searrow$	0

4) $f(1) = 0$ and $f'(1) = 1$, an equation of the tangent (T) to (C) at the point of abscissa 1 is $y = x - 1$.
5) 

6) The equation $x=e^{mx}$ is equivalent to $\ln x=\ln e^{mx}$, that is $\ln x = mx$, which gives $\frac{\ln x}{x}=m$.

So, it is sufficient to study the intersection of (C) with the straight line (d) of equation $y=m$ parallel to the axis $x'x$.

For $m\le 0$, (d) cuts (C) in one point only then the equation $x=e^{mx}$ has a unique root.

For $0<m<\frac{1}{e}$, (d) cuts (C) in two distinct points then the equation $x=e^{mx}$ has two distinct roots.

For $m=\frac{1}{e}$, (d) cuts (C) in one point only then $x=e^{mx}$ has a unique root $x'=x''=e$.

For $m>\frac{1}{e}$, (d) does not cut (C) so the equation $x=e^{mx}$ has no roots.

7) For $x>e$, f is decreasing and since $2007<2008$ then $f(2007)>f(2008)$ so we get $\frac{\ln 2007}{2007}>\frac{\ln 2008}{2008}$ which gives $2008\ln 2007>2007\ln 2008$, then $\ln 2007^{2008}>\ln 2008^{2007}$ and consequently $2007^{2008}>2008^{2007}$.

8) $A=\int_1^e f(x)\,dx=\int_1^e\frac{\ln x}{x}dx=\frac{1}{2}\left[\ln^2 x\right]_1^e=\frac{1}{2}$ square units.

$\boxed{N^\circ 10.}$

1) $\lim\limits_{x\to+\infty} f(x)=\frac{+\infty}{+\infty}=\lim\limits_{x\to+\infty}\frac{1}{\frac{1}{x}}=\lim\limits_{x\to+\infty} x=+\infty$.

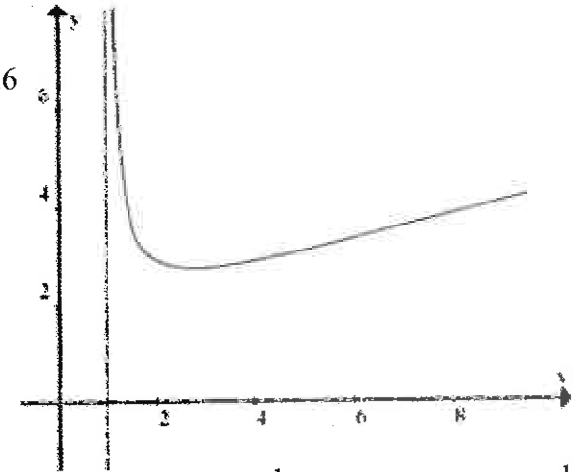
2) $\lim\limits_{x\to 1^+} f(x)=\frac{1}{0^+}=+\infty$, then the straight line $x=1$ is an asymptote to (C).

3) $f'(x)=\frac{\ln x-1}{\ln^2 x}$, then the table of variations of f is:

x	1		e		$+\infty$
$f'(x)$	||	$-$	0	$+$	
$f(x)$	|| $+\infty$	$\searrow$	e	$\nearrow$	$+\infty$

4) For $e < a < b$, f is increasing and since $a < b$ then $f(a) < f(b)$ so we get $\dfrac{a}{\ln a} < \dfrac{b}{\ln b}$ which gives $a\ln b < b\ln a$ and $\ln b^a < \ln a^b$ consequently $b^a < a^b$.

5) $f\left(e^2\right) = \dfrac{e^2}{2} \approx 3.6$



N° 11.

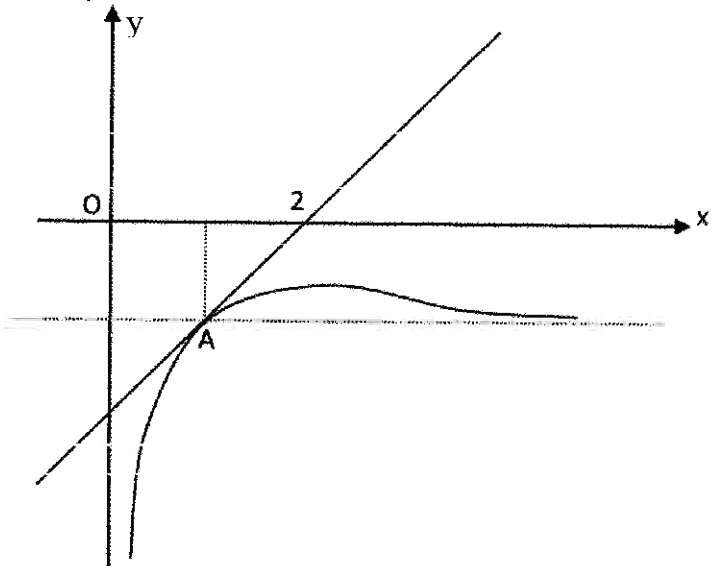
1) a- $\lim\limits_{x\to0^+} f(x) = -\infty$, $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \dfrac{\ln x}{x} - 1 = -1$ since $\lim\limits_{x\to+\infty} \dfrac{\ln x}{x} = 0$.

b- The axis $y'y$ and the straight line of equation $y = -1$ are asymptotes to (C).

2) $f'(x) = \dfrac{1-\ln x}{x^2}$, then the table of variations of f is the following:

x	0		e		$+\infty$
$f'(x)$	||	$+$	0	$-$	
$f(x)$	|| $-\infty$	$\nearrow$	$\dfrac{1}{e} - 1$	$\searrow$	-1

3) $f'(1)=1$, an equation of (d) is $y+1=1(x-1)$.
That is $y = x-2$.

4)


5) a- $f(x)<0$ in $]0;+\infty[$ then F is strictly decreasing, consequently (H) is the representative curve of F.

b- The area of the region is :

$$A = |F(e)-F(1)| = \left(e-\frac{3}{2}\right)u^2 = \left(e-\frac{3}{2}\right)\times 4\,cm^2 = 4.87\ cm^2.$$

N° 12.

1) a- $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0}\frac{1}{x-x\ln x} = +\infty$, since $\lim\limits_{x\to 0}[x\ln x]=0$.

$\lim\limits_{\substack{x\to e\\ x<e}} f(x) = +\infty$; $\lim\limits_{\substack{x\to e\\ x>e}} f(x) = -\infty$.

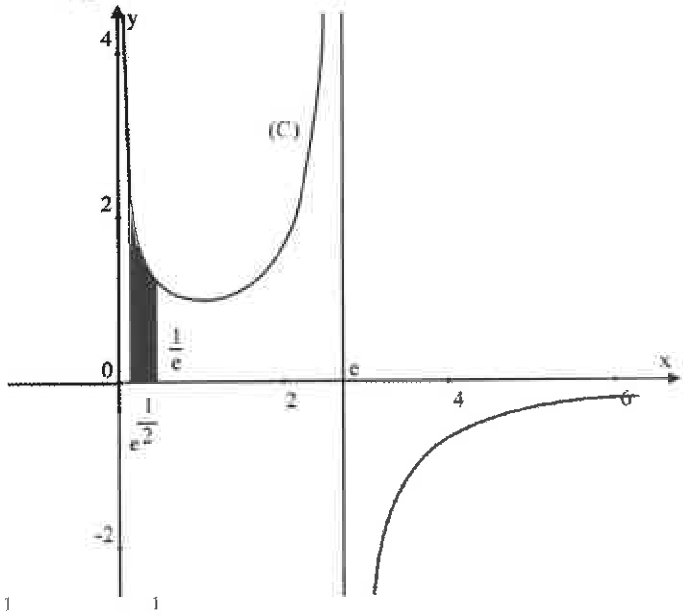
$$\lim_{x\to+\infty}\frac{1}{x(1-\ln x)} = \frac{1}{+\infty(1-\infty)} = \frac{1}{-\infty} = 0.$$

b- The straight lines of equations $x=0$; $x=e$ and $y=0$ are the asymptotes to (C).

2) a- $f'(x) = \frac{\ln x}{(x-x\ln x)^2}$, then the table of variations of f is the following:

x	0		1			e		$+\infty$
$f'(x)$	‖	$-$	0	$+$		‖	$+$	
$f(x)$	‖ $+\infty$	$\searrow$	1	$\nearrow$	$+\infty$	‖ $-\infty$	$\nearrow$	0

b-



3) $A = \int_{\frac{1}{e^2}}^{\frac{1}{e}} f(x)\,dx = \int_{\frac{1}{e^2}}^{\frac{1}{e}} \frac{1}{x(1-\ln x)}dx$ if $u(x) = 1 - \ln x$ then $u'(x) = -\frac{1}{x}$,

$$A = \int_{\frac{1}{e^2}}^{\frac{1}{e}} \frac{-u'(x)}{u(x)}dx = -\ln(1-\ln x)\Big|_{\frac{1}{e^2}}^{\frac{1}{e}} = \ln 3 - \ln 2 = \ln\left(\frac{3}{2}\right) \text{ square units.}$$

N° 13.

1) a- $\lim\limits_{x \to 0} f(x) = -\infty$ et $\lim\limits_{x \to 2} f(x) = +\infty$ .then the straight lines of equations $x = 0$ and $x = 2$ are asymptotes to (C).

b- $f(x)$ can be written in the form $f(x) = \ln x - \ln(2-x)$.

Therefore, $f'(x) = \frac{1}{x} + \frac{1}{2-x} = \frac{2}{x(2-x)} > 0$ for $x \in]0;2[$.

Then the table of variations of f is the following:

x	0		2
$f'(x)$	‖	$+$	‖
$f(x)$	‖ $-\infty$	$\nearrow$	$+\infty$ ‖

2) a- $f(2-x)+f(x)=\ln\left(\dfrac{2-x}{2-(2-x)}\right)+\ln\left(\dfrac{x}{2-x}\right)$.

Then, $f(2-x)+f(x)=\ln\left(\dfrac{2-x}{x}\times\dfrac{x}{2-x}\right)=\ln 1=0$.

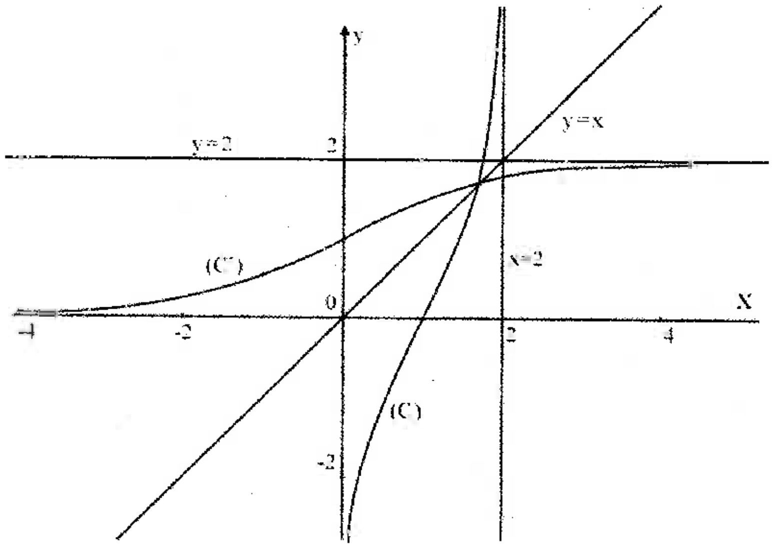
So, the point $A(1;0)$ is a center of symmetry for (C).

b- An equation of the tangent (T) is:

$y-y_A=f'(x_A)(x-x_A)$.

$f'(x_A)=f'(1)=2$, so an equation of (T) is $y=2x-2$.

3)



4) The straight line (d) of equation $y=x$ cuts the curve (C) in one point only, then the equation $f(x)=x$ has a unique root α that is the abscissa of the point of intersection.

$f(1.6)\approx 1.3<1.6$ and $f(1.7)\approx 1.73>1.7$, then $1.6<\alpha<1.7$.

5) a- f is continuous and strictly increasing over $]0;2[$ then it admits an inverse function f^{-1}.

b- The domain of definition of f^{-1} is $]-\infty;+\infty[$.

The curve (C') is the symmetric of (C) with respect to the straight line of equation $y = x$.

c- $y = \ln\left(\dfrac{x}{2-x}\right)$ then, $\dfrac{x}{2-x} = e^y$, which gives $x + x\,e^y = 2\,e^y$,

so $x(1+e^y) = 2e^y$, consequently $x = \dfrac{2\,e^y}{1+e^y}$,

then $f^{-1}(x) = \dfrac{2\,e^x}{1+e^x}$.

d- (C') and (C) intersect at a unique point of abscissa α , then α is the only solution of the equation $f(x) = f^{-1}(x)$.

N° 14.

Part A .

1) $g'(x) = 1 + \ln x + \dfrac{x-2}{x} = \dfrac{x+x-2}{x} + \ln x = 2\dfrac{x-1}{x} + \ln x$.

2) If $x > 1$ then $\dfrac{x-1}{x} > 0$ and $\ln x > 0$ then $g'(x) > 0$

If $0 < x < 1$ then $\dfrac{x-1}{x} < 0$ and $\ln x < 0$ then $g'(x) < 0$.

3) The function g has at the point of abcissa 1 a minimum equal to $g(1) = 1 + (1-2)\ln 1 = 1$ then $g(x) \geq 1$.

Part B.

1) a- $f'(x) = \ln x + 1 - 2\ln x \times \dfrac{1}{x} = \dfrac{x\ln x + x - 2\ln x}{x}$

$$= \frac{(x-2)\ln x + x}{x} = \frac{g(x)}{x}.$$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\left(1 + x\ln x - \ln^2 x\right) = 1 + 0 - \infty = -\infty$ since $\lim\limits_{x\to 0^+} x\ln x = 0$.

$\lim\limits_{x\to +\infty} f(x) = \lim\limits_{x\to +\infty}\left(1 + x\ln x - \ln^2 x\right) = +\infty - \infty$

$$= 1 + \lim_{x\to+\infty} x\ln x\left(1 - \frac{\ln x}{x}\right) = 1 + \infty(1-0) = +\infty$$

since $\lim\limits_{x \to +\infty} \frac{\ln x}{x} = 0$.

$f'(x) = \frac{g(x)}{x} > 0$, then the table of variations of f is:

x	0		$+\infty$
$f'(x)$	‖	$+$	
$f(x)$	‖ $-\infty$	$\nearrow$	$+\infty$

b- f is continuous and strictly increasing over $]0;+\infty[$ then it admits an inverse function f^{-1} defined over $J =]-\infty;+\infty[$.

2) a- $f(1) = 1$, $f'(1) = \frac{g(1)}{1} = 1$, then an equation of (T) is :

$y - 1 = 1(x-1)$, that is $y = x$.

b- $h'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$, then the table of variations of h is:

x	0		1		$+\infty$
$h'(x)$	‖	$-$	0	$+$	
$h(x)$	‖	$\searrow$	0	$\nearrow$	

The minimum of h is 0 then $h(x) > 0$ over $]0;+\infty[$.

c- $f(x) - x = 1 + x\ln x - \ln^2 x - x = x(\ln x - 1) + 1 - \ln^2 x$

$= x(\ln x - 1) + (1 - \ln x)(1 + \ln x) = (\ln x - 1)(x - 1 - \ln x)$

$= (\ln x - 1)h(x)$.

Since $h(x) > 0$ then for $\ln x - 1 > 0$ then for $x > e$, the curve (C) is above (T).

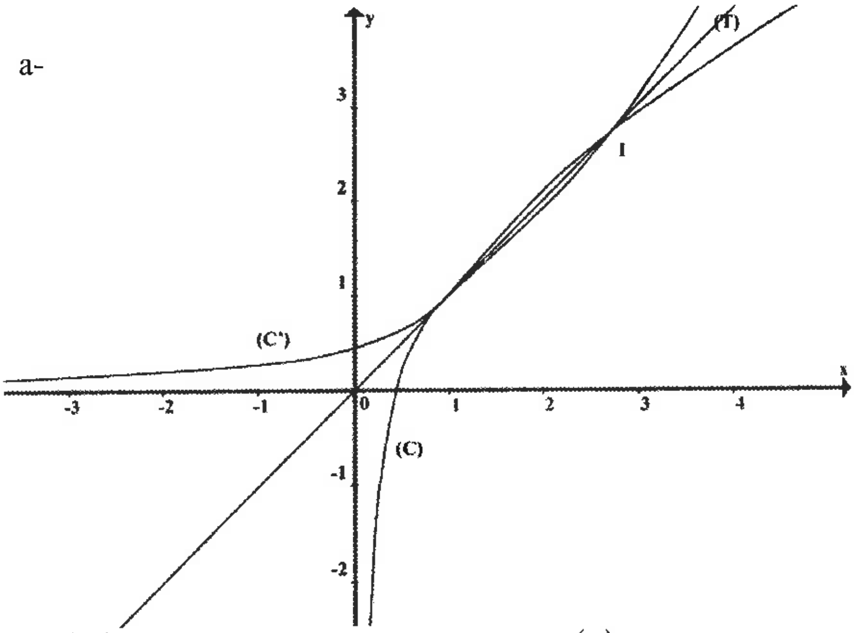
For $\ln x - 1 < 0$ that is for $x < e$, the curve (C) is above (T).

(C) and (T) intersect at the point of tangency $I(e;e)$.

d- (T') is parallel to (T) then they have same slope so

$f'(x) = 1$ which gives $x + (x-2)\ln x = x$ then $x = 2$.

3) a-



b- (C') is symmetric to (C) with respect to (T), see figure.

4) a- Let $u=\ln x$ and $v'=x$, then $u'=\dfrac{1}{x}$ and $v=\dfrac{1}{2}x^2$

$$I_1=\int_1^e x\ln x\,dx=\frac{1}{2}\left[x^2\ln x\right]_1^e-\frac{1}{2}\int_1^e x\,dx$$

$$=\frac{1}{2}\left[x^2\ln x\right]_1^e-\frac{1}{4}\left[x^2\right]_1^e=\frac{1}{4}e^2+\frac{1}{4}$$

Let $u=\ln^2 x$ and $v'=1$, then $u'=2\dfrac{\ln x}{x}$ and $v=x$

$$I=\left[x\ln^2 x\right]_1^e-2\int_1^e \ln x\,dx=\left[x\ln^2 x\right]_1^e-2\left[x\ln x-x\right]_1^e=e-2.$$

b- $$\mathscr{A}=2\int_1^e\left[x-f(x)\right]dx=2\int_1^e\left[x-1-x\ln x+\ln^2 x\right]dx$$

$$=2\left[\frac{1}{2}x^2-x\right]_1^e-2I_1+2I=\frac{e^2}{2}-\frac{7}{2}\text{u}^2$$

$$=\left(\frac{e^2}{2}-\frac{7}{2}\right)\times 4=2\left(e^2-7\right)\text{cm}^2=0.778\text{cm}^2$$

N° 15.

Part A

1) $\lim\limits_{x\to 0} u(x) = -\infty$ and $\lim\limits_{x\to +\infty} u(x) = +\infty$.

2) $u'(x) = 2\times\dfrac{3x^3+1}{x}$, then the table of variations of u is:

x	$-\infty$		$-\sqrt[3]{\dfrac{1}{3}}$		0		$+\infty$
$u'(x)$		$+$	0	$-$	||	$+$	
$u(x)$		$\nearrow$	$-\dfrac{1}{3}(5+2\ln 3)$	$\searrow$	$-\infty$ || $-\infty$	$\nearrow$	$+\infty$

3) u is continuous and strictly decreasing over$[0.8\,;0.9]$, and since $u(0.8)\approx -0.42<0$ and $u(0.9)\approx 0.24>0$, then the equation $u(x)=0$ admits a unique root α such that $0.8<\alpha<0.9$.

4) From the table of variations of u , we remark that u is strictly negative over $]-\infty;0[$ and over $]0;+\infty[$, u is strictly increasing then : If $0<x<\alpha$, then $u(x)<u(\alpha)$, so $u(x)<0$
If $x>\alpha$ then $u(x)>0$.

Part B.

1) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}\left(2x-\dfrac{\ln|x|}{x}\times\dfrac{1}{x}\right) = -\infty$, $\lim\limits_{x\to+\infty} f(x) = +\infty$.

$\lim\limits_{x\to 0} f(x) = +\infty$ then the axis $y'y$ is an asymptote to (C).

2) $f'(x) = \dfrac{u(x)}{x^3}$, then the table of variations of f is:

x	$-\infty$		0		α		$+\infty$
x^3		$-$	||	$+$		$+$	
$u(x)$		$-$	||	$-$	0	$+$	
$f'(x)$		$+$	||	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	$+\infty$ || $+\infty$	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

3) $\lim_{x\to\pm\infty}[f(x)-2x]=\lim_{x\to\pm\infty}\left[-\frac{\ln|x|}{x}\right]=0$, then the straight line of equation $y=2x$ is an asymptote to (C) at $+\infty$ and at $-\infty$.

4) $f(x)-2x=-\frac{\ln|x|}{x^2}$.

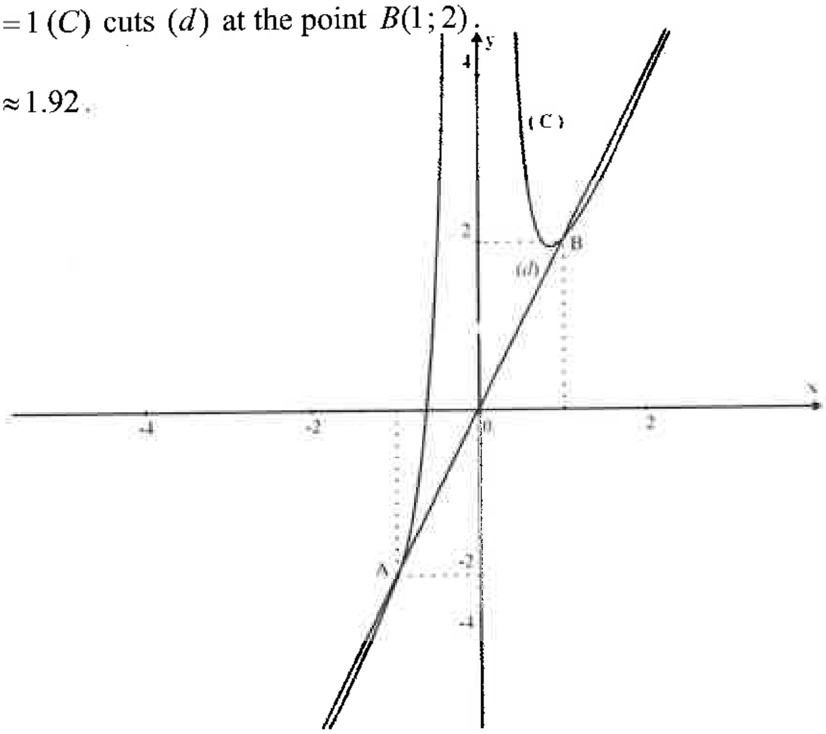
$f(x)-2x>0$, for $\ln|x|<0$, then for $|x|<1$, which gives $-1<x<1$ with $x\neq 0$. In this case (C) is above (d).

For $x<-1$ or $x>1$ the curve (C) is below (d).

For $x=-1$ (C) cuts (d) at the point $A(-1;-2)$

For $x=1$ (C) cuts (d) at the point $B(1;2)$.

5) $f(\alpha)\approx 1.92$.



N° 16.

1) a- $\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}[x+\ln x-\ln(x+1)]=-\infty$.

then $y'y$ is an asymptote to (C).

b- $\lim_{x\to+\infty}[f(x)-x]=\lim_{x\to+\infty}[\ln x-\ln(x+1)]=\lim_{x\to+\infty}\ln\left(\frac{x}{x+1}\right)$

$$=\lim_{x\to+\infty}\ln\left(1-\frac{1}{x+1}\right)=0.$$

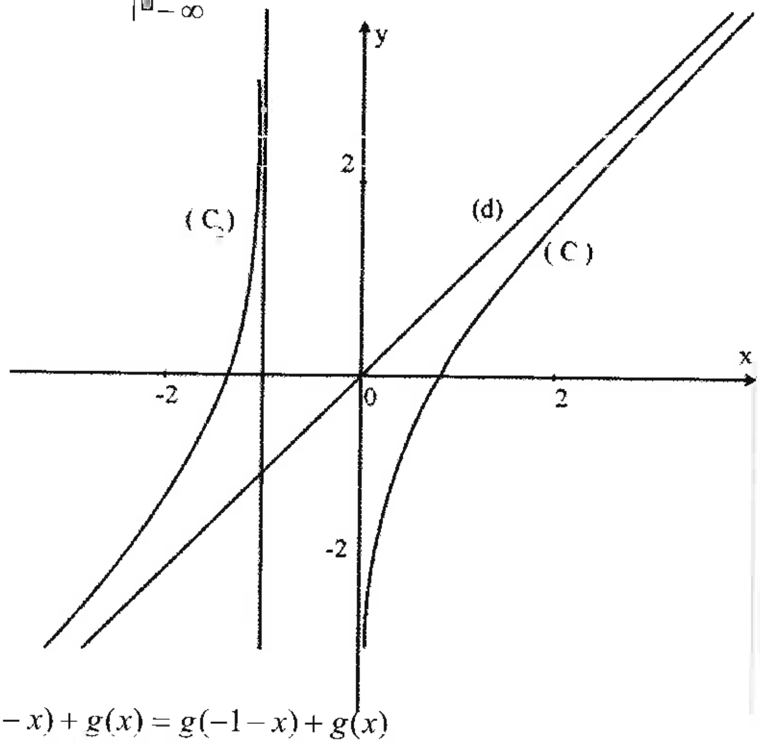
Then, (d) is an asymptote to (C) at $+\infty$.

c- $f'(x)=1+\frac{1}{x}-\frac{1}{x+1}=\frac{x^2+x+x+1-x}{x(x+1)}=\frac{x^2+x+1}{x(x+1)}$.

Then, $f'(x)>0$ for all $x\in\left]0;+\infty\right[$.

Therefore, the table of variations of f is:

x	0		$+\infty$
$f'(x)$	$\|$	$+$	
$f(x)$	$\|$ $-\infty$	$\nearrow$	$+\infty$



2) $g(2a-x)+g(x)=g(-1-x)+g(x)$

$$=-1-x+\ln\left(\frac{-1-x}{-x}\right)+x+\ln\left(\frac{x}{x+1}\right)$$

$$=-1+\ln\left(\frac{-1-x}{-x}\times\frac{x}{x+1}\right)=-1+\ln 1=-1=2b.$$

Then, $I\left(-\frac{1}{2};-\frac{1}{2}\right)$ is a center of symmetry of (γ).

The curve (γ) is the union of the two parts (C_1) and (C_2) with $(C_1)$and (C_2) being symmetric with respect to the point I .
For $x > 0$, $g(x) = x + \ln x - \ln(x+1) = f(x)$.
Then, (C_1) is itself (C).
For $x < -1$, (C_2) is symmetric to (C_1) with respect to I (see figure).

3) a- $F'(x) = a\ln(ax+b) + \dfrac{a}{ax+b}(ax+b) = a\ln(ax+b) + a$.

b- $A = \int_1^e [x - f(x)]dx = \int_1^e [-\ln x + \ln(x+1)]dx$.

For a =1 and b = 1, $F(x) = (x+1)\ln(x+1)$,
therefore $F'(x) = \ln(x+1) + 1$, which gives :

$$\int_1^e \ln(x+1)\,dx = \int_1^e (F'(x)-1)dx = F(x) - x\Big]_1^e.$$

$$\int_1^e \ln(x+1)\,dx = (x+1)\ln(x+1) - x\Big]_1^e.$$

$$= (e+1)\ln(e+1) - e - 2\ln 2 + 1.$$

For $a = 1$ and $b = 0$, $F(x) = x\ln x$ and $F'(x) = \ln x + 1$.

Therefore, $\int_1^e \ln x\,dx = \int_1^e (F'(x)-1)\,dx = F(x) - x\Big]_1^e$

$$= x\ln x - x\Big]_1^e = e\ln e - e + 1.$$

Which gives $A = (e+1)\ln(e+1) - e - 2\ln 2 + 1 - e\ln e + e - 1$.
Hence, $A = (e+1)\ln(e+1) - e - 2\ln 2$ square units.

N° 17

1) a- $\lim\limits_{x\to 0} f(x) = +\infty$, $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \dfrac{2\ln x}{x} = 0$

Then, the axis $x'x$ and the axis $y'y$ are asymptotes to (C).

b- $f'(x) = \dfrac{2\ln x - \ln^2 x}{x^2} = \dfrac{(2-\ln x)\ln x}{x^2}$.

Then $f'(x)$ has the same sign as $(2-\ln x)\ln x$.

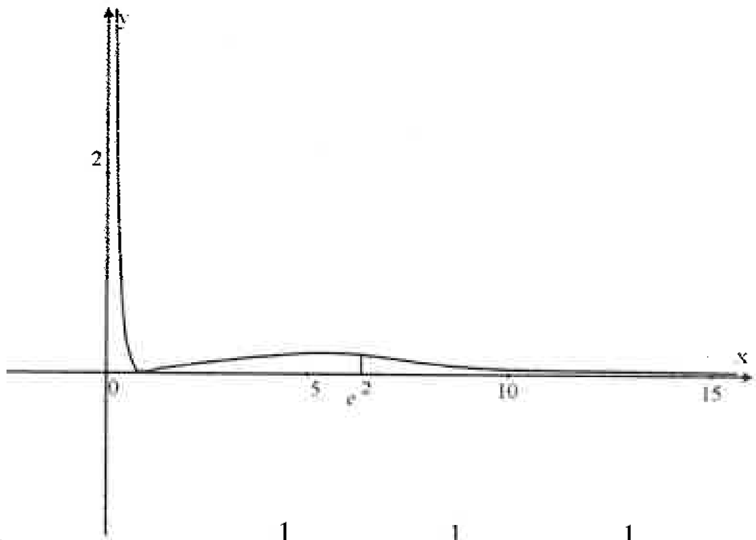
c- $f'(x) = 0$ for $\ln x = 0$ or $\ln x = 2$, which gives $x = 1$ or $x = e^2$.

$f'(x) > 0$ for $1 < x < e^2$, $f'(x) < 0$ for $x < 1$ or $x > e^2$.

The table of variations of f is the following :

x	0		1		e^2		$+\infty$
$f'(x)$	$\Vert$	$-$	0	$+$	0	$-$	
$f(x)$	$\Vert$	$+\infty$ ↘	0	↗	$\frac{4}{e^2}$	↘	0

d-



2) a- Let $u = \ln x$ and $v' = \frac{1}{x^2}$, then $u' = \frac{1}{x}$ and $v = -\frac{1}{x}$.

So we get $I_1 = -\frac{\ln x}{x}\Big|_1^{e^2} - \int_1^{e^2} -\frac{1}{x^2}dx = -\frac{\ln x}{x}\Big|_1^{e^2} - \frac{1}{x}\Big|_1^{e^2}$.

Which gives $I_1 = 1 - \frac{3}{e^2}$.

b- $I_{p+1} = \int_1^{e^2} \frac{(\ln x)^{p+1}}{x^2} dx$.

Let $u = (\ln x)^{p+1}$ and $v' = \frac{1}{x^2}$, we get:

$u' = (p+1)\frac{(\ln x)^p}{x}$ and $v = -\frac{1}{x}$, then :

$$I_{p+1} = -\frac{(\ln x)^{p+1}}{x}\Bigg|_1^{e^2} - \int_1^{e^2} -(p+1)\frac{(\ln x)^p}{x^2}dx$$

$$I_{p+1} = -\frac{(\ln x)^{p+1}}{x}\Bigg|_1^{e^2} + (p+1)\int_1^{e^2}\frac{(\ln x)^p}{x^2}dx \text{, then:}$$

$$I_{p+1} = -\frac{2^{p+1}}{e^2} + (p+1)I_p .$$

c- For $p = 1$, the preceding relation gives

$$I_2 = -\frac{2^2}{e^2} + 2I_1 = 2 - \frac{10}{e^2}.$$

For $p = 2$, the preceding relation gives $I_3 = -\frac{2^3}{e^2} + 3I_2 = 6 - \frac{38}{e^2}$.

For $p = 3$, the preceding relation gives

$$I_4 = -\frac{2^4}{e^2} + 4I_3 = 24 - \frac{168}{e^2}.$$

3) $V = \int_1^{e^2} \pi\, y^2\, dx = \pi \int_1^{e^2} \frac{(\ln x)^4}{x^2} dx = \pi\, I_4 = \pi(24 - \frac{168}{e^2})$ u^3

N° 18.

1) $\lim\limits_{x\to 0} f(x) = +\infty$. $\lim\limits_{x\to+\infty} f(x) = +\infty - 0 = +\infty$.

$\lim\limits_{x\to+\infty} [f(x) - x] = \lim\limits_{x\to+\infty} \frac{\ln x}{x} = 0$.

Then, the axis $y'y$ is an asymptote (C) and the straight line (d) of equation $y = x$ is an oblique asymptote to (C).

2) a- From the table of variations, we note that:

$f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$.

Then, the table of variations of f is:

x	0		1		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖ $+\infty$	↘	1	↗	$+\infty$

b- From the table of variations of f', note that :

For $x < e^{\frac{3}{2}}$, $f''(x) > 0$, for $x > e^{\frac{3}{2}}$, $f''(x) < 0$

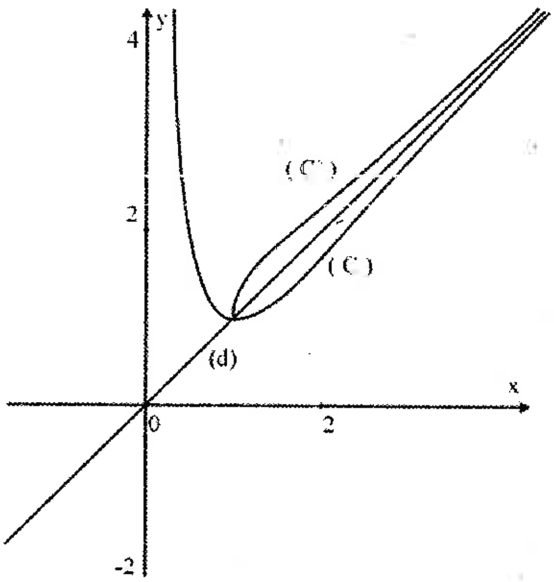
For $x = e^{\frac{3}{2}}$, $f''(x) = 0$.

Then , the point $I\left(e^{\frac{3}{2}}; f\left(e^{\frac{3}{2}}\right)\right)$ is a point of inflection for f , but

$f\left(e^{\frac{3}{2}}\right) = e^{\frac{3}{2}} - \dfrac{\ln e^{\frac{3}{2}}}{e^{\frac{3}{2}}} = e^{\frac{3}{2}} - \dfrac{3}{2}e^{-\frac{3}{2}} \approx 4.14$ et $e^{\frac{3}{2}} \approx 4.48$, then

$I(4.48; 4.14)$.

c-



3) a- f is continuous and strictly increasing for $x \in [1; +\infty[$, then it admits an inverse function f^{-1}, the domain of definition of f^{-1} is : $\left[f(1); \lim\limits_{x \to +\infty} f(x)\right[= [1; +\infty[$.

b- (C') is the symmetric of (C) with respect to $y = x$.

c- Remark that N is the symmetric of M with respect to $y = x$, then $N(f(t); t)$.

$$MN = \sqrt{(f(t) - t)^2 + (t - f(t))^2} = \sqrt{2\left(\frac{\ln t}{t}\right)^2} = \sqrt{2}\frac{\ln t}{t}.$$

N° 19.

Part A.

1) $\lim\limits_{x\to 0} f(x)=(-\infty)(-\infty)=+\infty$. $\lim\limits_{x\to +\infty} f(x)=(1-0)(-2+\infty)=+\infty$.

Then, the axis $y'y$, is an asymptote to (C).

2) $f'(x)=\dfrac{1}{x^2}(-2+\ln x)+\dfrac{1}{x}\left(1-\dfrac{1}{x}\right)=\dfrac{x-3+\ln x}{x^2}$.

3) a- $u'(x)=1+\dfrac{1}{x}>0$ pour $x\in\left]0;+\infty\right[$, then u is Strictly increasing for $x\in\left]0;+\infty\right[$.

b- The function u is continuous and strictly increasing over $\left]2.20\,;2.21\right[$.

$u(2.20)\times u(2.21)\approx -0.01\times 0.002\approx -0.00002<0$.

Then, the equation $u(x)=0$ has a unique root $\alpha\in\left]2.20\,;2.21\right[$.

u is strictly increasing then : If $0<x<\alpha$ then $u(x)<u(\alpha)=0$.

If $x>\alpha$ then $u(x)>u(\alpha)=0$.

4) $f'(x)=\dfrac{u(x)}{x^2}$, then:

If $0<x<\alpha$, then $f'(x)<0$.

If $x>\alpha$, then $f'(x)>0$.

Then, the table of variations of f is:

x	0		α		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

5) α is a solution of the equation $u(x)=0$, then

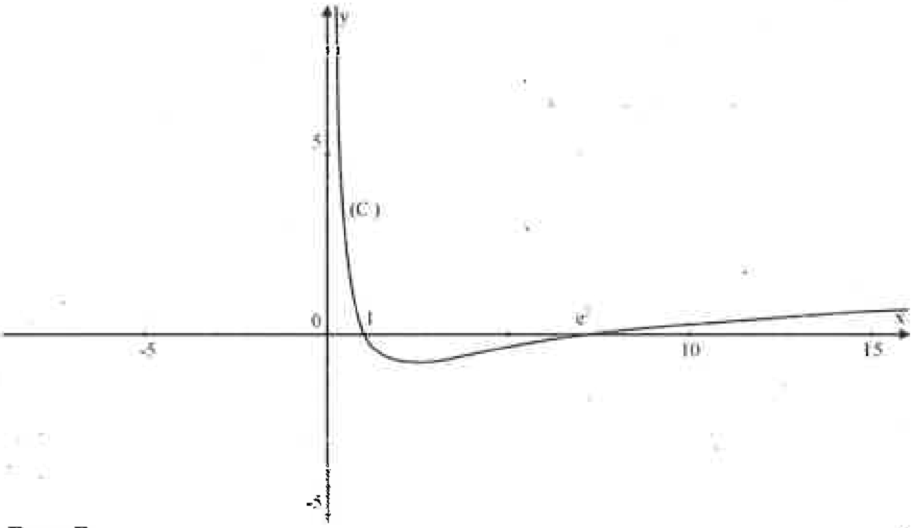
$u(\alpha)=\alpha-3+\ln\alpha=0$, which gives $\ln\alpha=-\alpha+3$.

$$f(\alpha)=\left(1-\frac{1}{\alpha}\right)(-2+\ln\alpha)=\left(1-\frac{1}{\alpha}\right)(-2-\alpha+3)$$

$$=\left(1-\frac{1}{\alpha}\right)(-\alpha+1).\ \text{Then } f(\alpha)=\frac{\alpha-1}{\alpha}.(-\alpha+1)=-\frac{(\alpha-1)^2}{\alpha}.$$

6) For $\alpha = 2.205$, $f(\alpha) \approx -0.6$.

Remark that $f(x) = 0$ for $x = 1$ and for $x = e^2$.



Part B.

1) F is an antiderivative of f, then $F'(x) = f(x)$.

$F'(x) < 0$ for $f(x) < 0$, then for $1 < x < e^2$ and consequently F is strictly decreasing for $1 < x < e^2$.

$F'(x) > 0$ for $f(x) > 0$, that is for $0 < x < 1$ or $x > e^2$ and consequently F is strictly increasing.

2) $F'(1) = f(1) = 0$ and $F'(e^2) = f(e^2) = 0$.

Then, the tangents to (C') at the points of abscissas 1 and e^2 Are parallel to the axis $x'x$.

3) $f(x) = \left(1 - \frac{1}{x}\right)(-2 + \ln x) = -2 + \ln x + \frac{2}{x} - \frac{\ln x}{x}$.

So $f(x) = \ln x - \frac{\ln x}{x} + \frac{2}{x} - 2$.

$F(x) = \int f(x)\,dx = \int (\ln x - \frac{\ln x}{x} + \frac{2}{x} - 2)\,dx$.

But $\int \ln x\,dx = x\ln x - x$, then

$F(x) = x\ln x - x - \frac{1}{2}\ln^2 x + 2\ln x - 2x + k$.

And since $F(1) = 0$, we get: $-1 - 2 + k = 0$ so $k = 3$.

Hence, $F(x) = x\ln x - \frac{1}{2}\ln^2 x + 2\ln x - 3x + 3$.

4) $A = \int_1^{e^2} - f(x)\,dx = \left(-F(x)\right]_1^{e^2} = -F\left(e^2\right) + F(1)$.

But $F(1) = 0$ et $F(e^2) = e^2 - 1$, therefore $A = e^2 - 5$ square units .

$\boxed{N^\circ 20.}$

1) $\lim\limits_{x\to 0} f(x) = +\infty + \infty = +\infty$,

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left[-3 - \ln x(1 - 2\ln x)\right] = +\infty$.

Then, the axis $y'y$ is an asymptote to (C).

2) $f(x) > 0$ for $2\ln^2 x - \ln x - 3 > 0$.

The roots of the equation $2\ln^2 x - \ln x - 3 = 0$ are

$\ln x = -1$ or $\ln x = \frac{3}{2}$ .which gives $x = e^{-1}$ or $x = e^{\frac{3}{2}}$.

Therefore, $2\ln^2 x - \ln x - 3 > 0$ for $\ln x < -1$ or $\ln x > \frac{3}{2}$

And consequently, $2\ln^2 x - \ln x - 3 > 0$ for $x < e^{-1}$ or $x > e^{\frac{3}{2}}$.

3) $f'(x) = \frac{4\ln x - 1}{x}$. $f'(x) > 0$ for $4\ln x - 1 > 0$, which gives $x > e^{\frac{1}{4}}$.

$f\left(e^{\frac{1}{4}}\right) = -3 - \frac{1}{4} + \frac{2}{16} = -\frac{25}{8}$.

Then the table of variations of f is the following:

x	0		$e^{\frac{1}{4}}$		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖ $+\infty$	$\searrow$	$-\frac{25}{8}$	$\nearrow$	$+\infty$

4) An equation of (T) is : $y = f(e^{\frac{5}{4}}) + f'(e^{\frac{5}{4}})(x - e^{\frac{5}{4}})$.

(T) is $y = 4e^{-\frac{5}{4}}x - \frac{41}{8}$.

5) a- $g'(x) = f'(x) - 4e^{-\frac{5}{4}} = \dfrac{4\ln x - 1}{x} - 4e^{-\frac{5}{4}}$,

$g''(x) = \dfrac{5 - 4\ln x}{x^2}$.

b- The table of variations of g' is the following :

x	0		$e^{\frac{5}{4}}$		$+\infty$
$g''(x)$	‖	$+$	0	$-$	
$g'(x)$	‖	$\nearrow$	0	$\searrow$	

c- The function g' admits a maximum equal to 0 , then $g'(x) \le 0$, consequently g is strictly decreasing over $]0;+\infty[$.

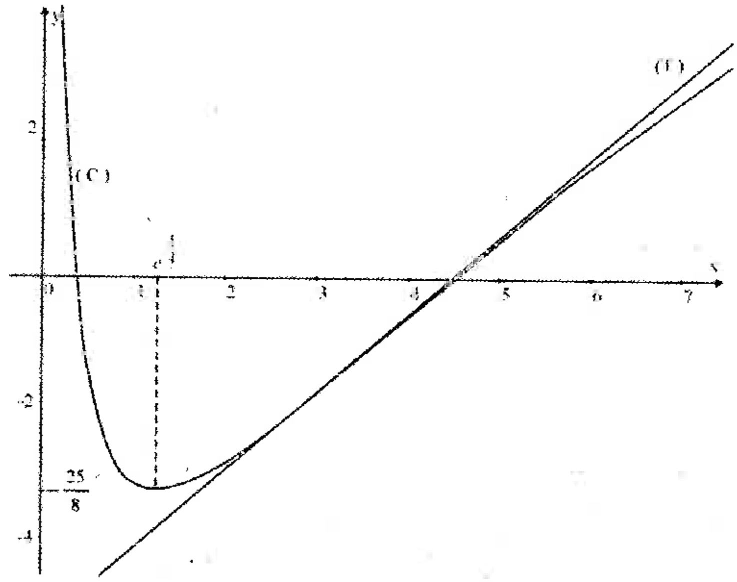
6) a- $g(e^{\frac{5}{4}}) = 0$ and g is decreasing over $]0;+\infty[$, then :

If $0 < x \le e^{\frac{5}{4}}$ then $g(x) \ge 0$. If $x \ge e^{\frac{5}{4}}$ then $g(x) \le 0$.

b- If $0 < x \le e^{\frac{5}{4}}$, then (C) is above (T) .

If $x \ge e^{\frac{5}{4}}$ then (C) is below (T) .

7)



8) $A = \int_{\frac{1}{e}}^{e^{\frac{3}{2}}} -f(x)\,dx = \int_{\frac{1}{e}}^{e^{\frac{3}{2}}} (3 + \ln x - 2\ln^2 x)\,dx$

We know that $\int \ln x\,dx = x\ln x - x$.

To calculate $\int \ln^2 x\,dx$, let $u = \ln^2 x$ and $v' = 1$

then $u' = \dfrac{2\ln x}{x}$ and $v = x$, which means that :

$\int \ln^2 x\,dx = x\ln^2 x - \int 2\ln x\,dx = x\ln^2 x - 2(x\ln x - x)$.

Finally, we get:

$$A = \int_{\frac{1}{e}}^{e^{\frac{3}{2}}} -f(x)\,dx = 3x + x\ln x - x - 2x\ln^2 x + 4(x\ln x - x)\Big|_{\frac{1}{e}}^{e^{\frac{3}{2}}}$$

So $A = e^{\frac{3}{2}} + \dfrac{9}{e}$ square units .

N° 21.

1) For $0 < x < e$, the curve (C) is above $x'x$, then the function represented by (C) is positive, then its antiderivative function is increasing.
For $x > e$, the curve (C) is below $x'x$, then the function represented by (C) is negative so the function that represents one of its antiderivatives is decreasing.
The function that represents (P) is increasing over $0 < x < e$ and decreasing for $x > e$.
Then, (P) is the representative curve of F and (C) that of f.
2) $F'(e) = 0$ and since $F'(x) = 2ax + 2bx\ln x + bx$ we get
$2ae + 2be + be = 0$, which gives $2a + 3b = 0$.
$f(1) = 2$ and since $F'(1) = f(1)$, we get:
$2a + b = 2$ so we get the following system:
$\begin{cases} 2a + 3b = 0 \\ 2a + b = 2 \end{cases}$ that has as a solution $a = \dfrac{3}{2}$ and $b = -1$.

Therefore $F(x)=\frac{3}{2}x^2-x^2\ln x$ and consequently

$f(x)=F'(x)=2x-2x\ln x$.

3) $x=e^{1-\frac{m}{2x}}$, then $\ln x=1-\frac{m}{2x}$, which gives $m=2x-2x\ln x$,

Then we should study the intersection of (C) and a straight line (d) parallel to $x'x$ of equation $y=m$.

If $m<0$, (d) cuts (C) at one point only, then (E) has a unique root.

If $m=0$, (d) cuts (C) at the point of abscissa e , then (E) has e as a unique root.

If $0<m<2$, (d) cuts (C) in two distinct points, then (E) has two distinct roots.

If $m=2$, (E) has a double root $x'=x''=1$.

If $m>2$, (d) does not cut (C) and (E) has no roots.

4) a- The point of intersection of (C) and (C') is the point of intersection of (C) with the first bisector $y=x$, then:

$2x-2x\ln x=x$, which gives $x=e^{\frac{1}{2}}$ and consequently $A\left(e^{\frac{1}{2}};e^{\frac{1}{2}}\right)$.

b- The slope of the tangent at A to (C) is $f'\left(e^{\frac{1}{2}}\right)$.

But $f'(x)=-2\ln x$, then $f'\left(e^{\frac{1}{2}}\right)=-2\ln e^{\frac{1}{2}}=-1$.

The slope of the tangent at A to (C') is:

$$\left(f^{-1}\right)'\left(e^{\frac{1}{2}}\right)=\frac{1}{f'\left(e^{\frac{1}{2}}\right)}=\frac{1}{-1}=-1 .$$

Then the two curves have the same tangent (T) at A .

N°22.

Part A.

1) The representative curve of f passes through the origin and cuts the

axis $x'x$ at another point of abscissa α.

Since $f(-0.8)\times f(-0.7)\approx -0.05<0$, we deduce that $-0.8<\alpha<-0.7$.

2) From the curve of f , we get $\lim\limits_{x\to -1} f(x)=-\infty$.

Part B.

1) $\lim\limits_{\substack{x\to 0\\ x>0}} g(x)=\dfrac{0}{0}=\lim\limits_{\substack{x\to 0\\ x>0}}\dfrac{\frac{1}{1+x}}{2x}=\lim\limits_{\substack{x\to 0\\ x>0}}\dfrac{1}{2x(x+1)}=+\infty$ (L'hopital's rule)

$\lim\limits_{\substack{x\to 0\\ x<0}} g(x)=-\infty$, $\lim\limits_{x\to -1} g(x)=-\infty$.

$\lim\limits_{x\to +\infty} g(x)=\lim\limits_{x\to +\infty}\dfrac{1}{2x(x+1)}=0$.

2) a- $g'(x)=\dfrac{\frac{1}{1+x}x^2-2x\ln(x+1)}{x^4}=\dfrac{\frac{x}{x+1}-2\ln(x+1)}{x^3}$

Then, $g'(x)=\dfrac{f(x)}{x^3}$ so the table of signs of $g'(x)$ is.

x	-1		α		0	$+\infty$
$f(x)$	‖	$-$	0	$+$	‖	$-$
x^3	‖	$-$		$-$	‖	$+$
$g'(x)$	‖	$+$	0	$-$	‖	$-$

b- The table of variations of g is the following :

x	-1		α		0		$+\infty$
$g'(x)$	‖	$+$	0	$-$	‖	$-$	
$g(x)$	‖ $-\infty$	↗	$g(\alpha)$	↘	$-\infty$ ‖ $+\infty$	↘	0

3) a- $g(\alpha)=\dfrac{\ln(1+\alpha)}{\alpha^2}$, but α is a root of the equation

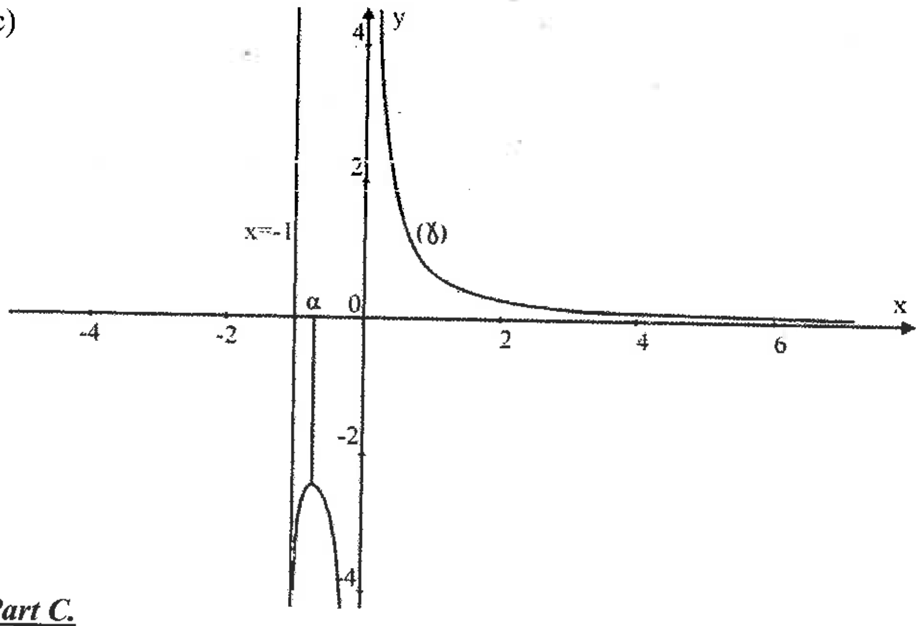
$f(x)=0$, then $f(\alpha)=\frac{\alpha}{\alpha+1}-2\ln(\alpha+1)=0$, which gives

$\ln(\alpha+1)=\frac{\alpha}{2(\alpha+1)}$ and

$$\text{consequently, } g(\alpha)=\frac{\frac{\alpha}{2(\alpha+1)}}{\alpha^2}=\frac{1}{2\alpha(\alpha+1)}.$$

b- For $\alpha=-0.75$, $g(\alpha)\approx-2.6$.

c)



Part C.

1) Remark that: $\frac{1}{x}-\frac{1}{x+1}=\frac{x+1-x}{x(x+1)}=\frac{1}{x(x+1)}$.

$$\int_1^{\lambda}\frac{1}{x(x+1)}dx=\int_1^{\lambda}\left(\frac{1}{x}-\frac{1}{x+1}\right)dx=\ln x-\ln(x+1)]_1^{\lambda}$$

$$=\ln(\lambda)-\ln(\lambda+1)-\ln 1+\ln 2=\ln\left(\frac{\lambda}{\lambda+1}\right)+\ln 2.$$

2) a- $A_{\lambda}=\int_1^{\lambda}g(x)dx=\int_1^{\lambda}\frac{\ln(x+1)}{x^2}dx$

Let $u=\ln(x+1)$ and $v'=\frac{1}{x^2}$ then $u'=\frac{1}{x+1}$ and $v=-\frac{1}{x}$

$$A_\lambda = -\frac{\ln(x+1)}{x}\Bigg]_1^\lambda - \int_1^\lambda \frac{-1}{x(x+1)}dx$$

$$= \frac{-\ln(\lambda+1)}{\lambda} + \ln 2 + \ln\left(\frac{\lambda}{\lambda+1}\right) + \ln 2 \text{ square units.}$$

Then. $A_\lambda = \left[\ln\left(\frac{\lambda}{\lambda+1}\right) - \frac{\ln(\lambda+1)}{\lambda} + 2\ln 2\right] \times 4\,cm^2$.

b- $\lim\limits_{\lambda\to+\infty} A_\lambda = 8\ln 2\,cm^2$, car $\lim\limits_{\lambda\to+\infty} \ln\left(\frac{\lambda}{\lambda+1}\right) = \lim\limits_{\lambda\to+\infty} \ln\left(\frac{\lambda}{\lambda}\right) = 0$

and $\lim\limits_{\lambda\to+\infty} \frac{\ln(\lambda+1)}{\lambda} = \lim\limits_{\lambda\to+\infty} \frac{1}{\lambda+1} = 0$ (L'hopital's rule).

$\boxed{N^\circ 23.}$

***Part A*.**

1) $h'(x) = \frac{-1}{(x+1)^2} = \frac{(-1)^1 1!}{(x+1)^{1+1}}$, then the formula is true for $n=1$.

Suppose the formula is true at the nth order, we can verify that it is true in the (n+1)th order.

$$h^{(n+1)}(x) = \left(h^{(n)}(x)\right)' = \left(\frac{(-1)^n n!}{(x+1)^{n+1}}\right)' = \left((-1)^n n!(x+1)^{-n-1}\right)'$$

$$= (-n-1)(-1)^n n!(x+1)^{-n-2} = \frac{-1\times(-1)^n (n+1)\times n!}{(x+1)^{n+2}}$$

$$= \frac{(-1)^{n+1}(n+1)!}{(x+1)^{n+2}}.$$

2) a- $f'(x) = \ln(x+1) + \frac{x}{x+1} = \ln(x+1) + \frac{x+1-1}{x+1}$

$$= \ln(x+1) + 1 - \frac{1}{x+1}.$$

b- $f''(x) = \frac{1}{x+1} - \left(\frac{1}{x+1}\right)' = h(x) - h'(x)$

$f^{(n)}(x) = h^{(n-2)}(x) - h^{(n-1)}(x)$, then:

$$f^{(n)}(x)=\frac{(-1)^{n-2}(n-2)!}{(x+1)^{n-1}}-\frac{(-1)^{n-1}(n-1)!}{(x+1)^{n}}$$

Part B .

1) a- $f''(x)=\dfrac{1}{x+1}+\dfrac{1}{(x+1)^2}>0$ for $x\in\left]-1;+\infty\right[$.

b- $\lim\limits_{x\to-1} f'(x)=\lim\limits_{x\to-1}\left[\ln(x+1)+\dfrac{x}{x+1}\right]=-\infty-\infty=-\infty$

$\lim\limits_{x\to+\infty} f'(x)=\lim\limits_{x\to+\infty}\left[\ln(x+1)+\dfrac{x}{x+1}\right]=+\infty+1=+\infty$.

c-

x	-1		$+\infty$
$f''(x)$	‖	$+$	
$f'(x)$	‖ $-\infty$	$\nearrow$	$+\infty$

d- f' increases from $-\infty$ à $+\infty$, then its curve cuts the axis $x'x$ in one point consequently the equation $f'(x)=0$ has a unique solution. And since $f'(0)=0$ then 0 is the unique solution.

e- f' is increasing and $f'(0)=0$ then if $x>0$ so $f'(x)>f'(0)$, then $f'(x)>0$ and if $x<0$ then $f'(x)<0$.

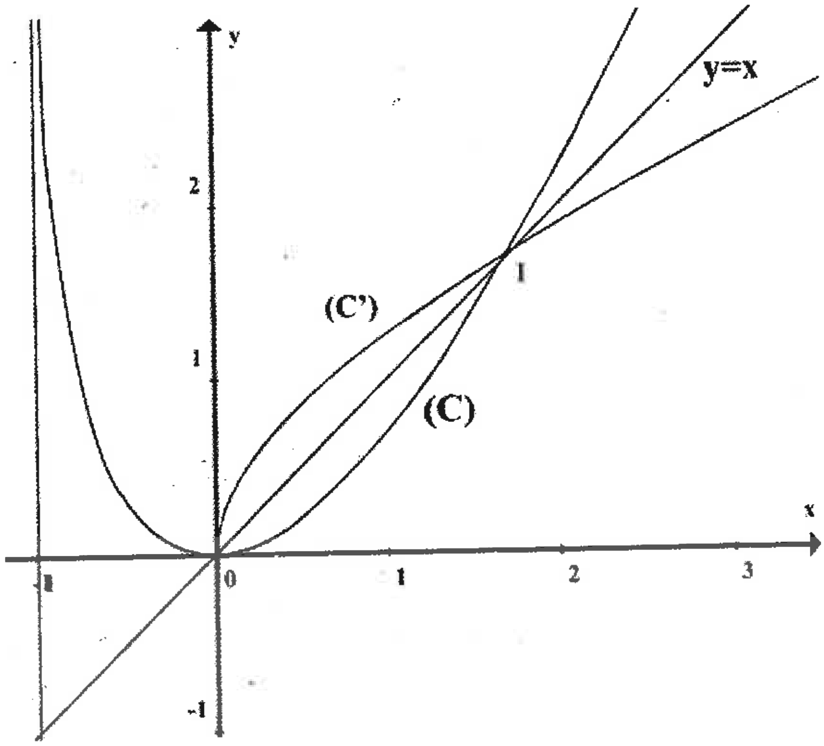
2) a- $\lim\limits_{x\to-1} f(x)=(-1)\times(-\infty)=+\infty$, $\lim\limits_{x\to+\infty} f(x)=+\infty$.

$\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=\lim\limits_{x\to+\infty}\ln(x+1)=+\infty$, then (C)admits an asymptotic direction that of the axis $y'y$.

b- The table of variations of f is the following.

x	-1		0		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖ $+\infty$	$\searrow$	0	$\nearrow$	$+\infty$

c-



3) a- For $x \in [0;+\infty[$, f is continuous and strictly increasing, then it admits an inverse function g defined over $[0;+\infty[$.

b- The point of intersection of (C) and (C') is the point of intersection of (C) and the straight line of equation $y = x$, then: $x\ln(x+1) = x$ so $x(\ln(x+1)-1) = 0$, which gives $x = 0$ or $\ln(x+1) = 1$ which gives $x+1 = e$ and consequently $x = e-1$. Hence, (C) and (C') intersect at point $O(0;0)$ and at point $I(e-1;e-1)$.

c- (C') is symmetric to (C) with respect to the line of equation $y = x$. See figure.

4) a- $ax+b+\dfrac{c}{x+1} = \dfrac{ax^2+(a+b)x+b+c}{x+1} = \dfrac{x^2}{x+1}$, then : $a = 1$, $a+b = 0$ then $b = -1$ and $b+c = 0$ so $c = 1$.

b- The area of the region limited by (C) and (C') is :

$$A=2\int_0^{e-1}(x-f(x))dx=2\int_0^{e-1}(x-x\ln(x+1))dx$$
$$=2\int_0^{e-1}x\,dx-2\int_0^{e-1}x\ln(x+1)dx$$

Let $u=\ln(x+1)$ and $v'=x$ then $u'=\frac{1}{x+1}$ and $v=\frac{1}{2}x^2$.

We get $A=\left[x^2\right]_0^{e-1}-\left[x^2\ln(x+1)\right]_0^{e-1}+\int_0^{e-1}\frac{x^2}{x+1}dx$

$$=(e-1)^2-(e-1)^2+\int_0^{e-1}\left[x-1+\frac{1}{x+1}\right]dx=$$

$\left[\frac{1}{2}x^2-x+\ln(x+1)\right]_0^{e-1}=\frac{1}{2}(e-1)^2-(e-1)+1$ square units.

***Part C*.**

1) For $x\in I$, f is strictly increasing then:
$f(I)=[f(0);f(e-1)]=[0;e-1]=I$.

2) $u_0=1.5$ and since $e-1\approx1.718$, then $0<u_0<e-1$.
Assume that $u_n\in I$ then $u_{n+1}=f(u_n)\in I$ since $f(I)=I$ then $u_n\in I$ for every natural number n.

3) a- $u_{n+1}-u_n=f(u_n)-u_n$, on the interval I, the curve (C) is below the straight line (d) of equation $y=x$ then $f(x)-x\le0$ for every $x\in I$ and since $u_n\in I$ then $f(u_n)-u_n\le0$, consequently $u_{n+1}-u_n\le0$ then (u_n) is decreasing.

b- The sequence (u_n) is decreasing and 0 is a lower bound of this sequence then it is convergent and its limit ℓ is solution of the equation $f(\ell)=\ell$, so $\ell=\ell\ln(\ell+1)$ which gives $\ell(1-\ln(\ell+1))=0$ then $\ln(\ell+1)=1$ therefore $\ell+1=e$ then $\ell=e-1$.

N°24.

Part A.

$h'(x)=2x-\frac{1}{x}=\frac{2x^2-1}{x}$ then the table of variations of h is:

x	0		$\frac{\sqrt{2}}{2}$		$+\infty$
$h'(x)$	‖	$-$	0	$+$	
$h(x)$	‖ $+\infty$	↘	$\frac{1+\ln 2}{2}$	↗	$+\infty$

The minimum of h is $h\left(\frac{\sqrt{2}}{2}\right) \approx 0.8$, then $h(x) > 0$ for $x \in]0;+\infty[$.

Part B.

1) $\lim\limits_{x \to +\infty} \frac{1+\ln x}{x} = \lim\limits_{x \to +\infty} \left(\frac{1}{x} + \frac{\ln x}{x}\right) = 0+0=0$.

   $\lim\limits_{x \to +\infty} [f(x)-(x-1)] = \lim\limits_{x \to +\infty} \frac{1+\ln x}{x} = 0$.

   Then, the straight line (d) is an asymptote to (C).

2) $f(x)-y = \frac{1+\ln x}{x}$.

   $1+\ln x > 0$ if $\ln x > -1$,which gives $x > e^{-1}$,in this case (C) is above (d).

   If $x < e^{-1}$, then (C) is below (d).

   If $x = e^{-1}$, then (C) and (d) intersect at point $A(e^{-1};e^{-1}-1)$.

3) $\lim\limits_{x \to 0} f(x) = -\infty$, then the axis $y'y$ is an asymptote to (C).

4) $f'(x) = \frac{1-(1+\ln x)}{x^2} + 1 = \frac{-\ln x + x^2}{x^2} = \frac{h(x)}{x^2}$.

   But $h(x) > 0$, then $f'(x) > 0$ for $x > 0$, so the table of variations of f is the following:

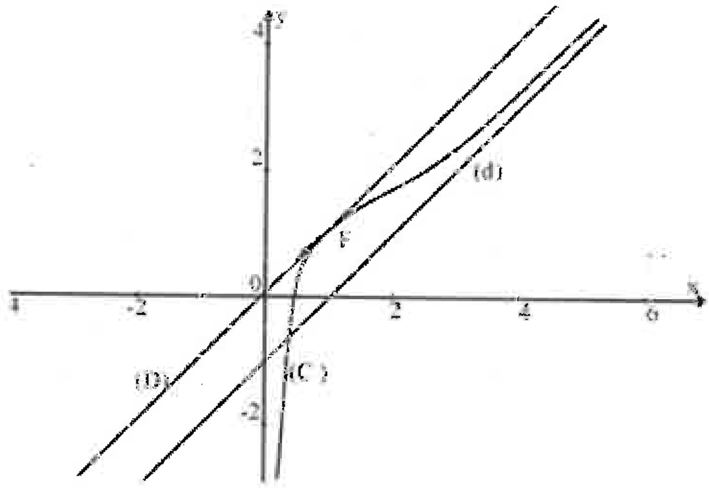
x	0		$+\infty$
$f'(x)$	‖	$+$	
$f(x)$	‖ $-\infty$	↗	$+\infty$

5) The slope of the tangent (D) at F to (C) is $f'(1)=1$, then

(D) is parallel to (d).

6) f is continuous and strictly increasing from $-\infty$ to $+\infty$.
Then, (C) cuts $x'x$ at a unique point so the equation $f(x)=0$ admits a unique root α.
$f(0.4)\approx -0.3$ and $f(0.5)\approx 0.11$, then $0.4<\alpha<0.5$.

7)



8) a- f is continuous and strictly increasing over $]0;+\infty[$, then there admits an inverse function f^{-1} for $x\in]0;+\infty[$.

b- The domain of definition of f^{-1} is $]-\infty;+\infty[$.
(C') is symmetric of (C) with respect to the straight line of equation $y=x$ (see figure).

9) $A=\int_1^e[f(x)-y]dx=\int_1^e\frac{1+\ln x}{x}dx=\int_1^e(1+\ln x)\frac{1}{x}dx$.

If $u(x)=1+\ln x$ then $u'(x)=\frac{1}{x}$, thus:

$A=\int_1^e u(x)u'(x)\,dx=\left.\frac{(1+\ln x)^2}{2}\right|_1^e=\frac{3}{2}$ square units.

Part C.

1) $\dfrac{x_{n+1}}{x_n} = \dfrac{e^{\frac{n-1}{2}}}{e^{\frac{n-2}{2}}} = e^{\frac{n-1}{2}-\frac{n-2}{2}} = e^{\frac{1}{2}}$, then $(x_n)$is a geometric sequence of

first term $x_0 = e^{-1}$ and of common ratio $e^{\frac{1}{2}}$.

2) $\dfrac{x_{n+1}}{x_n} = e^{\frac{1}{2}} > 1$ and since all the terms of this sequence are positive

then $(x_n)$is increasing.

3) $a_n = 4\displaystyle\int_{x_n}^{x_{n+1}}[f(x)-(x-1)]dx = 4\int_{x_n}^{x_{n+1}}\frac{1+\ln x}{x}dx = 4\int_{x_n}^{x_{n+1}}(1+\ln x)\frac{1}{x}dx$.

If $u(x) = 1+\ln x$ then $u'(x) = \dfrac{1}{x}$, then:

$$a_n = 4\int_{1} u(x)u'(x)\,dx = 2\left[u^2(x)\right]_{x_n}^{x_{n+1}} = \left[2(1+\ln x)^2\right]_{x_n}^{x_{n+1}}$$

$$= \left[2(1+\ln x_{n+1})^2 - 2(1+\ln x_n)^2\right].$$

$$= \left[2\left(1+\frac{n-1}{2}\right)^2 - 2\left(1+\frac{n-2}{2}\right)^2\right] = n+\frac{1}{2}.$$

$a_{n+1} - a_n = n+1+\dfrac{1}{2} - n - \dfrac{1}{2} = 1$.

Then $(a_n)$is an arithmetic sequence of common difference 1 and of

first term $\dfrac{1}{2}$.

$N^\circ 25.$

Part A.

1) $\lim\limits_{x\to+\infty} f(x) = +\infty$, (C) admits an asymptotic direction that of the

x-axis $x'x$.

$f'(x) = \dfrac{1}{2\sqrt{x+1}} > 0$, then

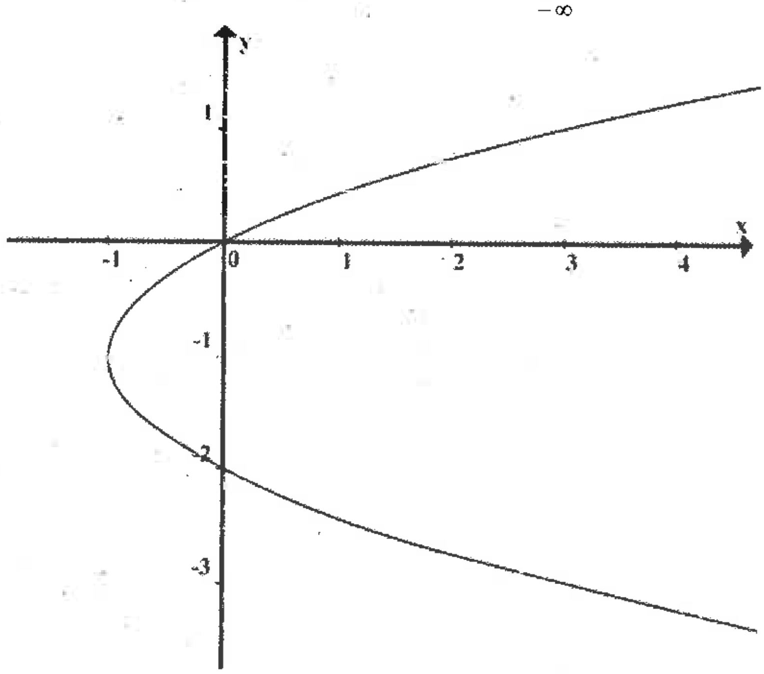
the table of variations of f is:

x	-1		$+\infty$
$f'(x)$		$+$	
$f(x)$	-1	$\nearrow$	$+\infty$

$\lim\limits_{x\to+\infty} g(x) = -\infty$, (C) has an asymptotic direction that of the axis $x'x$.

$g'(x) = \dfrac{-1}{2\sqrt{x+1}} < 0$, therefore the table of variations of g is.

x	-1		$+\infty$
$g'(x)$		$-$	
$g(x)$	-1	$\searrow$	$-\infty$



2) a- $y = -1 \pm \sqrt{x+1}$ gives $(y+1)^2 = \left(\pm\sqrt{x+1}\right)^2$ so $(y+1)^2 = x+1$.

b- (γ) is a parabola, $Y = y+1$, $X = x+1$, $Y^2 = X$

$S(-1;-1)$, $F\left(-\dfrac{3}{4};-1\right)$, directrix : $x = -\dfrac{5}{4}$.

c- $A' = 2\int_{-1}^{0}\left(\sqrt{x+1}\right)dx = 2\left[\dfrac{2}{3}(x+1)\sqrt{x+1}\right]_{-1}^{0} = \dfrac{4}{3}$ square units.

Part B .

1) a- $f(x) = -1+\sqrt{x+1} > 0$ over $]0:+\infty[$ then the domain of definition of h is $]0:+\infty[$.

b- $h'(x) = \dfrac{f'(x)}{f(x)} > 0$.

$\lim\limits_{x\to+\infty} h(x) = \ln(+\infty) = +\infty$, $\lim\limits_{x\to 0} h(x) = \ln(0) = -\infty$

x	0		$+\infty$
$h'(x)$	||	$+$	
$h(x)$	|| $-\infty$	$\nearrow$	$+\infty$

$$\lim_{x\to+\infty} \frac{h(x)}{x} = \frac{+\infty}{+\infty} = \lim_{x\to+\infty} \frac{\frac{f'(x)}{f(x)}}{1} = \lim_{x\to+\infty} \frac{1}{2\sqrt{x+1}\left(-1+\sqrt{x+1}\right)} = 0$$

then (Γ) admits an asymptotic direction that of the axis $x'x$.

2) a- The complex form of r is $z' = az+b$ where $a = 1e^{i\frac{\pi}{2}} = i$ and $b = 0$ since O is the center of r, therefore $z' = iz$.

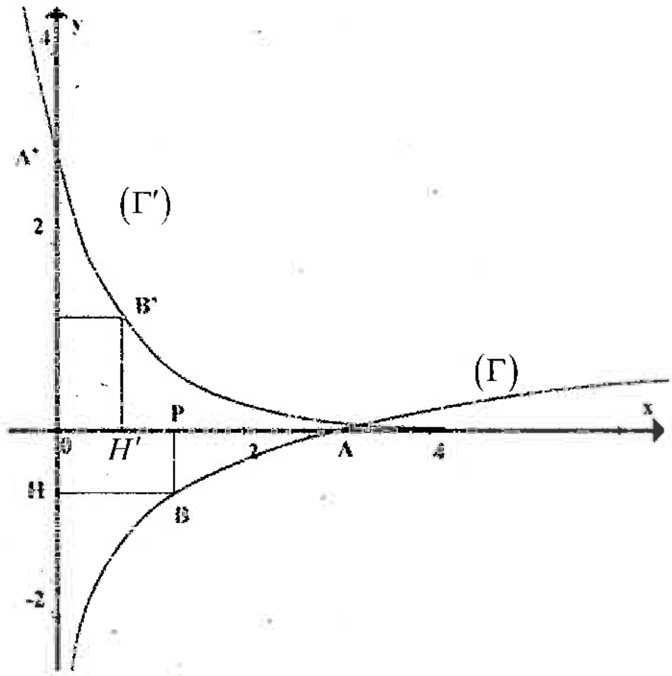
b- $z_A = 3$, $z_{A'} = 3i$, $z_B = \dfrac{5}{4} - i\ln 2$, $z_{B'} = \ln 2 + \dfrac{5}{4}i$

c- $x'+iy' = i(x+iy)$ gives $x = y'$ et $y = -x'$

d- $y = \ln\left(-1+\sqrt{x+1}\right)$ gives $e^y = -1+\sqrt{x+1}$ then $\left(e^y+1\right)^2 = \left(\sqrt{x+1}\right)^2$, therefore $x = e^{2y}+2e^y$ so $y' = e^{-2x'}+2e^{-x'}$

Then, when M describes (Γ) the point M' describes the curve (Γ') of equation $y = e^{-2x}+2e^{-x}$.

e-



(Γ') is the image of (Γ) by the rotation $r\left(O;\frac{\pi}{2}\right)$.

3) a- $\int_0^{\ln 2}\left(e^{-2x}+2e^{-x}\right)dx=\left[-\frac{1}{2}e^{-2x}-2e^{-x}\right]_0^{\ln 2}=\frac{11}{8}$.

It is the area limited by (Γ), the axis $(x'x)$ and the two straight lines of equations $x=0$ and $x=\ln 2$.

b- $r(O)=O$, $r(A)=A'$, $r(B)=B'$, $r(H)=H'$, $H'(\ln 2;0)$, and the image of the arc AB is the arc $A'B'$ and since the rotation preserves the area then $\alpha=\frac{11}{8}$.

$$\alpha=\int_0^{\ln 2}\left(e^{-2x}+2e^{-x}\right)dx=\frac{11}{8}$$

c- $|I|=\alpha-\text{area}(OPHB)=\alpha-\frac{5}{4}\ln 2=\frac{11}{8}-\frac{5}{4}\ln 2$, then $I=\frac{5}{4}\ln 2-\frac{11}{8}$.

Indications

N° 1.

6) a- $A(t)=\int_e^t \frac{1}{x\ln x}dx=\left[\ln(\ln x)\right]_e^t=\ln(\ln t)$ square units.

b- $A(t)<t$ if $\ln(\ln t)<t$ which gives $\ln t<e^t$ which is true since the representative curve of the function ln is below that of the exponential function.

N° 3.

1) a- $\lim\limits_{x\to 0} f(x)=1$ since $\lim\limits_{x\to 0} x^2\ln x=\lim\limits_{x\to 0} x(x\ln x)=0\times 0=0$.

3) $g'(x)=2x-2-2x\ln x$, $g''(x)=-2\ln x$.

N° 4.

Part A.

3) $g'(x)>0$ for $x>1$ and $g'(x)<0$ for $x<1$ and since $g(1)=1$ then $g(x)>0$ for $x>0$

N° 5.

Part B.

5) c- $A(\alpha\,;\beta)$ $B(\beta\,;\alpha)$ where $\beta=f(\alpha)=\alpha-\ln^2\alpha$.

$AB^2=2(\beta-\alpha)^2=2\ln^4\alpha=2$, which gives $\alpha=e$ or $\alpha=\frac{1}{e}$.

$m=\beta+\alpha$, then $m=2e-1$ or $m=\frac{2}{e}-1$.

N° 7.

Part A.

4) $MN=y_N-y_M=x-\ln x$, the smallest value of MN is $x-\ln x=f(x)+1$ for $x=1$, this value is $M_1N_1=1$ since $f(1)+1=1$.

N° 8.

3) $x-\ln|x|=x$, then $\ln|x|=0$, so $x=1$ or $x=-1$. $A(1\,;1)$ and $B(-1;-1)$.

N° 9.

3) b- (D') is the symmetric of (D) with respect to $y=x$ then the area of (D') is equal to that of (D)

N° 11.

Part A.

4) Note that $\dfrac{x+1}{2x+1}=\dfrac{1}{2}+\dfrac{1}{2(2x+1)}$.

Part B.

3) $f(\alpha)=\dfrac{2\ln\alpha}{\alpha^2+\alpha}$, but $\dfrac{\alpha+1}{2\alpha+1}-\ln\alpha=0$ which gives

$\ln\alpha=\dfrac{\alpha+1}{2\alpha+1}$, then $f(\alpha)=\dfrac{2}{\alpha^2+\alpha}\times\dfrac{\alpha+1}{2\alpha+1}=\dfrac{2}{\alpha(2\alpha+1)}$.

N° 12.

Part C.

2) Using the table of variations of f, for $t\in\left]0;1\right]$ we have $f(t)\geq -1$

$$f(t)-t+1=\frac{\ln t}{t-\ln t}-t+1=\frac{\ln t-t^2+t+t\ln t-\ln t}{t-\ln t}$$
$$=\frac{t(-t+1+\ln t)}{t-\ln t}\leq 0\text{, since :}$$

$t-\ln t\geq 1$ then $t-\ln t\geq 0$ and $-t+1+\ln t\leq 0$, then $f(t)\leq t-1$

4) a- For $x\geq 3$, f is increasing then for $n\leq t\leq n+1$

$f(n+1)\leq f(t)\leq f(n)$ then:

$$\int_n^{n+1} f(n+1)dt\leq\int_n^{n+1} f(t)dt\leq\int_n^{n+1} f(n)dt,$$

which gives $f(n+1)[t]_n^{n+1}\leq u_n\leq f(n)[t]_n^{n+1}$ then

$f(n+1)\leq u_n\leq f(n)$.

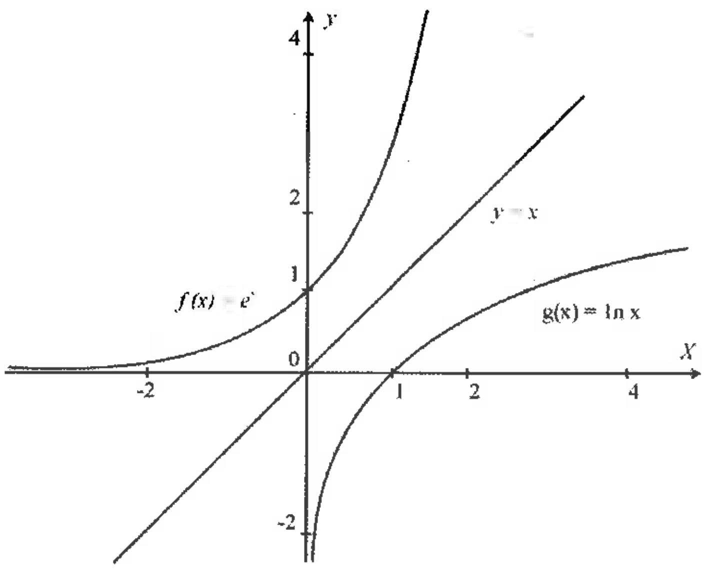
CHAPTER 5
Exponential Functions

Chapter Review

1) ***Definition***:
The exponential function is the inverse of the logarithmic function over the interval $]0,+\infty[$.

We may write : $\begin{cases} y = \ln x \\ x > 0 \end{cases}$ is equivalent to $\begin{cases} x = e^y \\ y \in IR \end{cases}$

2) ***Consequences*** :
- The graph of the function f defined over $]-\infty,+\infty[$ by $f(x) = e^x$ is the symmetric to that of the function g defined over $]0;+\infty[$ by $g(x) = \ln x$ with respect to the first bisector of equation $y = x$.



3) ***Properties.***

- For all real numbers x:

  $e^x > 0 \quad , \quad \ln(e^x) = x$

- For all strictly positive real numbers x : $e^{\ln(x)} = x$.

- For all real numbers a and for all real numbers b, we can write :

  $e^a \times e^b = e^{a+b} \quad , \quad \dfrac{e^a}{e^b} = e^{a-b} \quad , \quad (e^a)^b = e^{a\times b} \quad , \quad e^0 = 1.$

  $e^{-a} = \dfrac{1}{e^a}$

4) ***Limits.***

- $\lim\limits_{x\to-\infty} e^x = 0 \quad , \quad \lim\limits_{x\to+\infty} e^x = +\infty$.

- $\lim\limits_{x\to+\infty} \dfrac{e^x}{x} = +\infty \quad , \quad \lim\limits_{x\to-\infty} xe^x = 0.$

5) ***Derivative***:

- If $f(x) = e^x$ then $f'(x) = e^x$.

- If $f(x) = e^{u(x)}$ then $f'(x) = u'(x).e^{u(x)}$.

6) Antiderivative.

- $\int e^x dx = e^x + c$
- $\int u'(x)e^{u(x)}dx = e^{u(x)} + c$.

Solved Problems

N° 1.

Solve, in IR, the following equations and inequalities :

1) $e^{2x}+e^{x}-2=0$ 2) $e^{3x}-5e^{2x}-6e^{x}=0$

3) $e^{3x}-21\,e^{x}+20=0$ 4) $e^{2x}+3e^{x}-4\leq 0$

N° 2.

Solve the following systems :

1) $\begin{cases} e^{2x}.e^{y}=e^{-3} \\ xy=-2 \end{cases}$ 2) $\begin{cases} e^{x}+e^{y}=2 \\ e^{2x}+e^{2y}=\dfrac{5}{2} \end{cases}$

3) $\begin{cases} e^{x}=3e^{y} \\ x+y=2-\ln 3 \end{cases}$ 4) $\begin{cases} \ln(y+6)-\ln x=3\ln 2 \\ e^{5x}.e^{-y}=e^{-6} \end{cases}$

N° 3.

Consider the function f defined over IR by: $f(x)=\left(x^{2}-4\right)e^{2x}$.

Determine the real numbers a, b and c so that the function F defined over IR by : $F(x)=\left(ax^{2}+bx+c\right)e^{2x}$ is an antiderivative of f over IR.

N° 4.

Calculate the following integrals :

1) $\displaystyle\int_{0}^{1}\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$ 2) $\displaystyle\int_{0}^{\frac{\pi}{2}}\sin 2x\,e^{\sin^{2}x}dx$ 3) $\displaystyle\int_{0}^{1}xe^{x^{2}-1}dx$

4) $\displaystyle\int_{0}^{1}xe^{x}dx$ 5) $\displaystyle\int_{0}^{1}x^{2}e^{x}dx$

N° 5.

Given $I=\displaystyle\int_{0}^{\frac{\pi}{2}}e^{x}\cos x\,dx$ and $J=\displaystyle\int_{0}^{\frac{\pi}{2}}e^{x}\sin x\,dx$.

1) By integrating I by parts, prove that $I=e^{\frac{\pi}{2}}-J$.
2) By integrating J by parts, find a relation between I and J.
3) Calculate I and J.

N° 6.

Show that the equation $e^{2x}+2x-m=0$ admits a unique solution in the set IR for all real numbers m.

N° 7.

Part A .

Consider the function g defined over IR by $g(x)=2e^{x}+2x-7$.
(Γ) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to-\infty} g(x)$.
2) Show that the straight line (d) of equation $y=2x-7$ is an asymptote to (Γ).
3) Study the variations of g over IR and set up its table of variations.
4) Justify that the equation $g(x)=0$ admits a unique solution α such that $0.94<\alpha<0.941$ and deduce the sign of $g(x)$.
5) Draw (Γ) .

Part B .

Consider the function f defined over IR by $f(x)=(2x-5)(1-e^{-x})$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Unit: 2 *cm*.

1) Study the sign of f over IR.
2) Calculate $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
3) a- Verify that $f'(x)$ and $g(x)$ have the same sign.
   b- Set up the table of variations of f .
   c- Prove that the straight line (D) of equation $y=2x-5$ is an asymptote to (C) at $+\infty$ and study the relative position of (C) with respect to (D).
   d- Taking $\alpha=0.94$, draw (C).
4) Calculate, in cm^2, the area of the region bounded by (C), the axis of abscissa and the straight lines of equations $x=0$ and $x=\dfrac{5}{2}$.

N° 8.

Part A .

Consider the function g defined over IR by $g(x)=1-e^{2x}-2xe^{2x}$.

1) Calculate $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to-\infty} g(x)$.
2) Calculate $g'(x)$ and set up the table of variations of g.
3) Calculate $g(0)$ and deduce the sign of $g(x)$.

Part B.

Let (C) be the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ of the function f defined over IR by $f(x)=x+3-xe^{2x}$

1) Calculate $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
2) a- Calculate $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x}$ and deduce that the straight line (d) of equation $y=x+3$ is an asymptote to (C) at $-\infty$.
   b- Study the relative positions of (C) and (d) .
3) a- Show that f is increasing for $x<0$.
   b- Show that the equation $f(x)=0$ has two roots α and β such that $-4<\alpha<-3$ and $\dfrac{1}{2}<\beta<1$.
4) Draw (d) and (C) .
5) Let λ be a real negative number .
   a- Calculate the area S_λ of the region limited by (C) , the straight line (d) and the straight lines of equations $x=\lambda$ and $x=0$.
   b- Calculate $\lim\limits_{\lambda\to-\infty} S_\lambda$.

N° 9.

Let (C) be the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ of the function f defined over $I=]1;+\infty[$ by $f(x)=x+1-\dfrac{3e^x}{e^x-e}$.

1) a- Show that the straight line of equation $x=1$ is an asymptote to (C).
   b- Calculate $\lim\limits_{x\to+\infty} f(x)$ and show that the straight line (d) of equation $y=x-2$ is an asymptote to (C).

c- Study the relative positions of (C) and (d).

2) Show that $f'(x) > 0$ for all real numbers x of I and set up the table of variations of f.
3) Show that the equation $f(x) = 0$ admits a unique solution α and Verify that $2.6 < \alpha < 2.7$.
4) Draw (C).
5) Designate by (D) the region of the plane limited by $(C),(d)$ and the straight lines of equations $x = 3$ and $x = 4$.

   Calculate $\int_3^4 \frac{e^x}{e^x - e}\,dx$ and deduce the area of (D)
6) a- Show that f has an inverse function g in the interval I.

   b- Show that the equation $f(x) = g(x)$ has no solutions in I.

N° 10.

f is the function f defined over IR by $f(x) = 2x + 1 - x\,e^{x-1}$, designate by (C) its representative curve in a orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim_{x \to +\infty} f(x)$ and $\lim_{x \to -\infty} f(x)$.
2) Show that the straight line (d) of equation $y = 2x + 1$ is an asymptote to (C) at $-\infty$ and study the relative position of (C) and (d).
3) The table below is the table of variations of the function f' derivative of f.

x	$-\infty$		-2		1		$+\infty$
$f''(x)$		$+$	0	$-$		$-$	
$f'(x)$	2	↗	$2+e^{-3}$	↘	0	↘	$-\infty$

   a- Study the sign of f' and set up the table of variations of f

   b- Show that f admits a point of inflection L.
4) Show that the equation $f(x) = 0$ admits two roots α and β such that $-1 < \alpha < -\frac{1}{2}$ and $1 < \beta < 2$.

5) Draw (d) and (C) .

6) a- Calculate the integral $J = \int_{1}^{\beta} x\, e^{x-1} dx$.

b- Calculate the area S_β of the region limited by (C) the axis $x'x$ and the straight lines of equations $x = 1$ and $x = \beta$.

c- Prove that $S_\beta = (\beta - 1)(\beta - \frac{1}{\beta})$.

N° 11.

Part A .

h is the function defined over IR par $h(x) = (2 - x)e^x - 2$, esignate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim_{x\to+\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$ and deduce an asymptote to (C).
2) Calculate $h'(x)$ and set up the table of variations of h.
3) Draw (C).
4) The equation $h(x) = 0$ admits two roots 0 and α, show that $1.5 < \alpha < 1.6$.

Part B

Consider the function f defined by $f(x) = \dfrac{e^x - 2}{e^x - 2x}$ .Designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that the function defined over IR by $g(x) = e^x - 2x$ is positive and deduce the domain of definition of f.
2) Show that the straight lines of equations $y = 0$ and $y = 1$ are asymptotes to (Γ) .
3) Show that $f'(x)$ and $h(x)$ have the same sign and set up the table of variations of f .
4) Show that $f(\alpha) = \dfrac{1}{\alpha - 1}$.

   Deduce to the nearest 10^{-2} the value of $f(\alpha)$ for $\alpha = 1.55$.
5) Draw (Γ) , precise the position of (Γ) with respect to the straight line of equation $y = 1$.

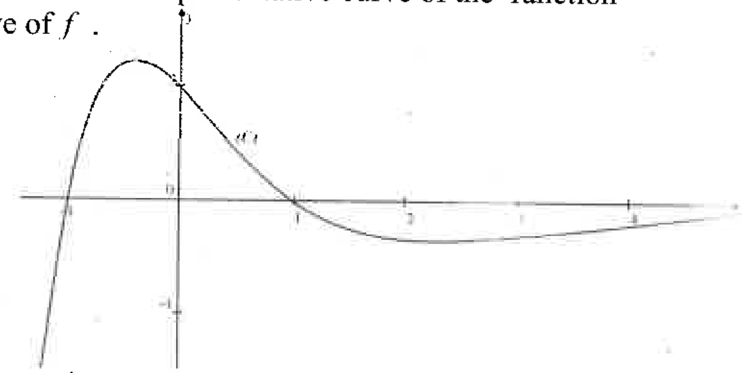
6) Let A_n be the region limited by (Γ) , the straight line of equation $y=1$ and the two straight lines of equations $x=1$ and $x=n$ with $n>1$. Calculate A_n and $\lim\limits_{n\to+\infty} A_n$.

N° 12.

Part A .

Let (Γ) be the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ of the function f defined over IR by $f(x)=(x+1)^2e^{-x}$.

1) Determine $\lim\limits_{x\to-\infty} f(x)$ andt $\lim\limits_{x\to+\infty} f(x)$ and deduce an asymptote to (Γ).
2) The curve below is the representative curve of the function f', derivative of f .



a- Set up the table of variations of f .

b- Write an equation of the tangent (T) to (Γ) at the point abscissa 0.

c- Draw (Γ) and (T) .

3) a- Show that the equation $f(x)=2$ admits a unique solution $\alpha\in[-2;-1]$.

b- Show that $\alpha=-1-\sqrt{2}e^{\frac{\alpha}{2}}$.

4) Calculate the area of the region limited by (Γ), by the axes $x'x$ and straight lines of equations $x=0$ and $x=1$.

Part B.

Consider the function g defined by $g(x)=\ln(f(x))$.

1) Show that the domain of definition of g is $]-\infty;-1[\cup]-1;+\infty[$.
2) Study the variations of g and set up its table of variations

3) Solve the equation $g(x) = -x$.

N° 13.

Part A .

1) Show that for all real numbers $x > 0$:
   $e^x \geq x+1$ and that $\ln x \leq x-1$.
2) Deduce that $e^x - \ln x > 2$ for $x > 0$.

Part B.

Consider that the function g defined over $[0;+\infty[$ by
$g(x) = e^x - \ln x - xe^x + 1$.

1) Study the variations of g and set up its table of variations.
2) Show that the equation $g(x) = 0$ admits a unique root α and that $1.23 < \alpha < 1.24$.
3) Deduce the sign of $g(x)$.

Part C .

Consider the function f defined over $[0;+\infty[$ par $f(x) = \dfrac{x}{e^x - \ln x}$.

Let (C) be its representative curve in orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine the limit of f at $+\infty$ and the limit of f at 0.
2) Show that $f'(x)$ and $g(x)$ have the same sign .
3) Find a value to the nearest 10^{-2} of $f(\alpha)$ for $\alpha = 1.23$ and draw (C) .

N° 14.

Consider the function f defined over IR par $f(x) = \ln(1+e^x) - x$.

Let (C) be its representative curve in orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that the straight line (d) of equation $y = -x$ is an asymptote to (C) at $-\infty$ and the axis $x'x$ is an asymptote to (C) at $+\infty$.
2) a- Study the variations of f and set up its table of variations.
   b- Draw (C) .
3) a- Show that the function f admits over IR an inverse function g.
   b- Draw, in the same system, the curve (C') representative of the function g.

c- Calculate $g'(\ln 2)$ and write an equation of the tangent to (C') at the point of (C') of abscissa $\ln 2$.

4) Consider the function h defined over $]0;+\infty[$ by $h(x) = -\ln(e^x - 1)$.

a- Show that $f \circ h(x) = x$.

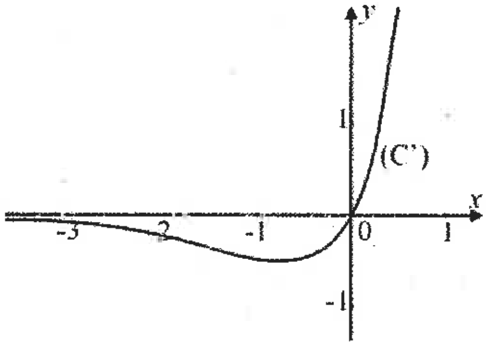
b- Deduce the expression $g(x)$ of the function g .

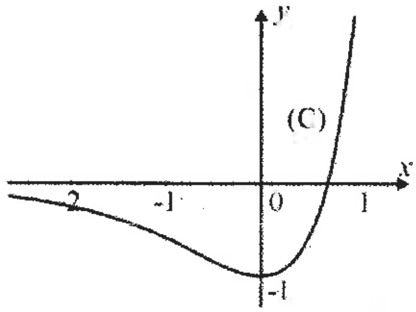
N° 15.

1) Consider the function f defined over IR by $f(x) = a\,e^{2x} + b\,e^x$ where a and b are two non-zero real numbers and designate by f' the derivative function of f .

a- The curves are the representative curves of the function f and the function f'

Show that (C) is the representative curve of f .





b- Show that $f(x) = e^{2x} - 2\,e^x$.

2) Show that the function f admits an inverse function f^{-1} for $x \geq 0$.

Determine the domain of definition of f^{-1}, find the expression f^{-1}(x).

3) Calculate the area of the region (D) bounded by the curves (C) and the semi-straight lines $[Ox)$ and the axis $y'y$.

4) Calculate the volume of the solid generated by rotating the region (D) about $x'x$.

N° 16.

Consider the function f defined over IR by $f(x) = x + \ln 4 + \dfrac{2}{e^x + 1}$

(C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.
2) Calculate for all real numbers x , $f(x) + f(-x)$, what can you say about the point $A(0;1+\ln 4)$?
3) Study the variations of f and set up its table of variations.
4) a- Show that for all real numbers x ,
   $f(x) = x + 2 + \ln 4 - \dfrac{2e^x}{e^x + 1}$.
   b- Show that the straight lines (d) and (d') of respective equations $y = x + \ln 4$ and $y = x + 2 + \ln 4$ are asymptotes to (C). Study the position of (C) with respect to (d).
5) Show that f admits an inflection point whose coordinates are to be determined.
6) Draw (C), (d) and (d').
7) Show that the equation $f(x) = 3$ admits a unique solution β and that $1.1 < \beta < 1.2$.
8) Show that the f admit over IR an inverse function f^{-1} and calculate $(f^{-1})'(\ln 4e)$.

N° 17.

Consider the function f defined over IR par $f(x) = x + xe^x$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Calculate $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.
   b- Show that the straight line (d) of equation $y = x$ is an asymptote to (C) at $-\infty$.
   c- Study the relative positions of (C) and (d).
   d- Write an equation of the tangent (T) to (C) at the point of abscissa -1.

e- Show that f admits an inflection point whose coordinates are to be determined..

f- Study the variations of f' over IR and set up its table of variations.

g- Draw (d), (T), and (C).

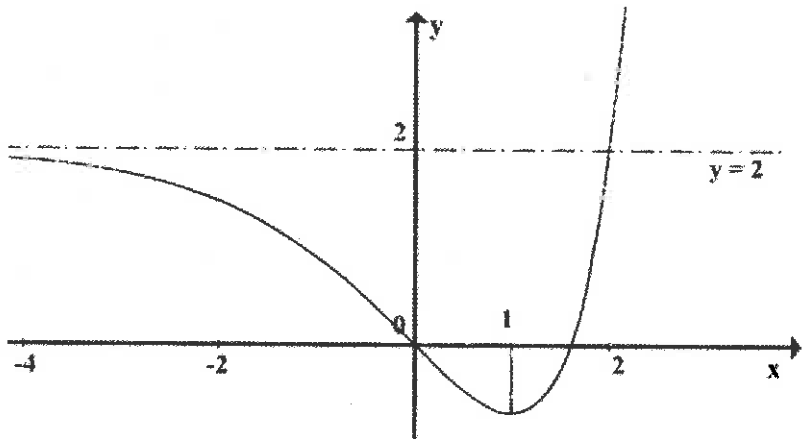
2) Calculate the area of the region limited by (C), (d) and the straight line of equations $x = 0$ and $x = -1$.

3) Study, according to the values of the real number α, the number of points of intersection of (C) and the straight line (δ) of equation $y = x + \alpha$

N° 18. **For the students of the G.S. section**

Part A.

The curve (C) below is the representative curve of a function g defined over IR by $g(x) = (ax+b)e^x + c$.



1) Show that $g(x) = (x-2)e^x + 2$.

2) Show that the equation $g(x) = 0$ admits in the interval $\left[\frac{3}{2};2\right]$ has a unique solution α.

3) Calculate the area of the region limited by (C), its horizontal asymptote $y = 2$ and the two straight lines of equations $x = 0$ and $x = 1$.

Part B .

Consider the function f defined over $]0;+\infty[$ by $f(x)=\dfrac{e^x-1}{x^2}$.

Designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{\substack{x\to 0\\ x>0}} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.

2) Justify that the table of variations of f is the following :

x	0		α		$+\infty$
$f'(x)$	$\Vert$	$-$	0	$+$	
$f(x)$	$\Vert$	$\searrow$	$f(\alpha)$	$\nearrow$	

3) Taking $\alpha=1.75$, find an approximate value of $f(\alpha)$ to the nearest 10^{-2} then draw (Γ).

Part C.

1) Show that the equation $g(x)=0$ is equivalent to $2(1-e^{-x})=x$.

2) Consider the function h defined over $\left[\dfrac{3}{2};2\right]$ by $h(x)=2(1-e^{-x})$.

   Show that $|h'(x)|\leq\dfrac{1}{2}$.

3) Consider the sequence (u_n) defined by $u_1=\dfrac{3}{2}$ and $u_{n+1}=h(u_n)$.

   a- Show that $|u_{n+1}-\alpha|\leq\dfrac{1}{2}|u_n-\alpha|$.

   b- Deduce that $|u_n-\alpha|\leq\left(\dfrac{1}{2}\right)^n$.

   c- Determine the limit of the sequence (u_n).

N° 19. ***For the students of the G.S. section***

Consider the function f_n defined over IR byr $f_n(x)=\dfrac{2e^{nx}}{1+e^x}-1$

where n is a natural number, and designate by (C_n) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Unit 2 cm.

Part A .

In this part, take $n=1$.

1) Calculate $\lim\limits_{x\to+\infty} f_1(x)$ $\lim\limits_{x\to-\infty} f_1(x)$.
2) Calculate $f_1'(x)$ and set up the table of variations of f_1
3) a- Prove that O is an inflection point of (C_1).
   b- Write an equation of the tangent (d) at O to (C_1).
4) Draw (d) and (C_1).

Part B .

Let (C_0) be the representative curve of f_0, corresponding to $n=0$, in the same system $(O;\vec{i},\vec{j})$

1) Prove that the curve (C_0) is symmetric to the curve (C_1) with respect to the axis of ordinates.
2) Prove that (C_0) is symmetric to (C_1) with respect to the axis of abscissas.
3) Calculate, in cm^2, the area of the region limited by the curves (C_1), (C_0) and the straight lines of the equations $x=0$ and $x=1$.

Part C .

Consider the sequence (u_n) defined by $u_n=\int_0^1 f_n(x)dx$.

1) Prove that $u_{n+1}+u_n=2\dfrac{e^n-n-1}{n}$.
2) Calculate $\lim\limits_{n\to+\infty}(u_{n+1}+u_n)$ and deduce that the sequence (u_n) is not convergent.

N°20. ***For the students of the G.S. section***

Part A .

Consider the function g defined, over IR, by: $g(x)=e^{x}-2x-1$,

1) Calculate $g'(x)$ and set up the table of variations of g.
2) Show that the equation $g(x)=0$ admits two solutions 0 and α such that $1<\alpha<2$.
3) Deduce the sign of $g(x)$ according to the values of x.

Part B .

Consider the function f defined over IR by

$f(x)=(2x+3)e^{-x}+x-1$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
   b- Show that the straight line (d) of equation $y=x-1$ is an asymptote to (C).
   c- Study the relative positions of (C) and (d).
2) Show that $f'(x)=g(x)e^{-x}$ and set up the table of variations of f.
3) Take $\alpha=1.5$ and draw (C).
4) Calculate the area of the region limited by the curve (C), (d) and the two straight lines of equations $x=0$ and $x=1$.
5) Determine the set of values for which the equation $(2x+3)e^{-x}+x-1+m=0$ admits three distinct roots

Part C.

Consider the function h defined by $h(x)=\ln[f(x)]$.

1) Verify that the equation $f(x)=0$ has one root β such that $-1.2<\beta<-1.1$ and deduce the domain of definition of h.
2) Calculate $\lim\limits_{x\to-\infty} h(x)$ and $\lim\limits_{x\to\beta} h(x)$.
3) Study the variations of h and set up its table of variations.

N°21 ***For the students of the G>S. section***

Part A .

Consider the function f defined over IR by $f(x)=\left(x^2-3x+1\right)e^x$.

Designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1) a- Determine the limits of f at the boundaries of its domain of definition.
   b- Deduce an asymptote (D) to (C).
2) a- Study the variations of f and set up its table of variations.
   b- Draw (C).
3) Consider $I=\int_{-3}^{0} f(x)dx$, interpret I graphically and give its exact value.

Part B.

1) The table below is the table of variations of variations of a function g defined over IR by $g(x)=e^{\left(x^2+ax+b\right)}$, where a and b are two real numbers, determine a and b.

x	$-\infty$		$3/2$		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+\infty$	$\searrow$	$e^{-5/4}$	$\nearrow$	$+\infty$

2) Consider the function h defined over IR by $h(x)=e^{\left(x^2-3x+1\right)}$.
   Designate by (Γ) its representative curve $\left(O;\vec{i},\vec{j}\right)$.
   Prove that the straight line (D_1) of equation $x=\frac{3}{2}$ is an axis of symmetry of (Γ).

Part C.

Consider the function v defined over IR by $v(x)=e^{f(x)}$.
Study the variations of v and set up its table of variations.

Supplementary Problems

N° 1.

Let f be the function defined, on IR, by $f(x)=e^{-2x}-2e^{-x}+1$ and (C) be its representative curve in the system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to+\infty} f(x)$ and deduce an asymptote (d) to (C)
2) Calculate $\lim\limits_{x\to-\infty} f(x)$
3) Find $f'(x)$ and set up the table of variations of f.
4) Prove that (C) has a point of inflection I to be determined.
5) Determine the point of intersection of (C) with its asymptote (d).
6) Draw (d) and (C).
7) Let g be the function given by $g(x)=\ln[f(x)]$ and let (G) be its representative curve.
   a- Find the domain of g and set up its table of variations.
   b- Prove that (D): $y=-2x$ is an asymptote of (G).
   c- Solve each of the equations $g(x)=0$ and $g(x)=-2x$.
   d- Draw (D) and (G) in a new system of axes.

N° 2.

Let f be the function defined on IR by $f(x)=x+2-e^{-x}$ and (C) be its representative curve in the system $(O;\vec{i},\vec{j})$.

1) a- Calculate $\lim\limits_{x\to+\infty} f(x)$ and prove that the line (d) of equation $y=x+2$ is an asymptote to (C).
   b- Calculate $\lim\limits_{x\to-\infty} f(x)$ and give a decimal value of $f(-1.5)$ and $f(-2)$.
2) Calculate $f'(x)$ and set up the table of variations of f.
3) Write an equation of the tangent (T) to (C) at the point A of abscissa 0.
4) Show that the equation $f(x)=0$ has a unique root α and verify that $-0.5<\alpha<-0.4$.
5) Draw (d), (T) and (C).
6) Let g be the inverse function of f, on IR.

a- Draw the representative curve (G) of g in the same system $(O;\vec{i},\vec{j})$

b- Designate by $A(\alpha)$ the area of the domain bounded by (C), the axis of abscissas and the two straight lines of equations $x=\alpha$ and $x=0$. Prove that $A(\alpha)=\left(-\frac{\alpha^2}{2}-3\alpha-1\right)$ square units.

$\boxed{N°3.}$

Part A .

Let f be the function defined over IR by $f(x)=e^{\frac{x}{2}}-e^{x}$.

(C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
2) Deduce an asymptote to (C).
3) Calculate $f'(x)$ and draw the table of variations of f .
4) Trace (C).

Part B

Let g be the function defined by $g(x)=\ln(e^{x}-e^{\frac{x}{2}})$.

(Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Verify that the domain of g is $]0;+\infty[$.
2) Show that $g(x)-x=\ln\left(1-e^{-\frac{x}{2}}\right)$ and deduce that the straight line (d) of equation $y=x$ is an asymptote to (Γ) .
3) Study the relative positions of (Γ) and (d) .
4) Study the variations of g and trace (Γ) .
5) Show that g admits an inverse function g^{-1} over $]0;+\infty[$ and determine $g^{-1}(x)$.
6) Show that the equation $g(x)=g^{-1}(x)$ has no solution.

$\boxed{N°4.}$

Let f be the function defined over IR by $f(x)=\frac{e^{2x}-1}{e^{2x}+1}$.

(C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the parity of f .

2) Show that, for all $x \in IR$, $-1 < f(x) < 1$.
3) a- Calculate $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to +\infty} f(x)$.
   b- Deduce the equations of the asymptotes to (C).
   c- Study the variations of f and draw its table of variations.
   d- Show that f admits, over IR, an inverse function f^{-1}.
   Determine the domain of definition of f^{-1} and calculate $f^{-1}(x)$.
4) a- Show that $f(x)$ can be written in the form $f(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.
   b- Calculate the area of the domain limited by (C), the straight line of equation $y = x$ and the two straight lines of equations $x = 0$ and $x = 1$.

N°5.

Part A.

Let g be the function defined over IR by $g(x) = \left(x^2 + 2x - 1\right)e^{-x} + 1$.

1) Study the limits of g at $+\infty$ and $-\infty$.
2) Calculate $g'(x)$ and draw the table of variations of g.
3) a- Show that the equation $g(x) = 0$ admits two solutions 0 and α Such that $-2.4 < \alpha < -2.3$
   b- Deduce the sign of $g(x)$ according to the values of x.

Part B

Consider the function f defined over IR by $f(x) = x - \left(x^2 + 4x + 3\right)e^{-x}$

(C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$

1) Determine the limits of f at $+\infty$ and $-\infty$.
2) Show that for all real numbers x, $f'(x) = g(x)$ and draw the table of variations of f.
3) Prove that the straight line (D) of equation $y = x$ is an asymptote to (C).
4) a- Show that (D) cuts (C) in two points whose coordinates are to be determined.
   b- Study the relative positions of (D) and (C).
5) Take $\alpha = -2.38$ and trace (C).

Part C

1) Let H be the function defined over IR by $H(x) = (ax^2 + bx + c)e^{-x}$

determine the real numbers a, b and c such that H is an antiderivative of the function h defined by $h(x)=(x^2+4x+3)e^{-x}$.

2) Calculate the area of the region limited by (C) and (D).

N°6.

Part A-

Let f be the function defined over IR by $f(x)=x+xe^{-x}$, (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ .Unit: 2 cm.

1) a- Calculate $\lim\limits_{x\to+\infty} f(x)$ and prove that the line (d) of equation $y=x$ is an asymptote to (C).

b- Calculate $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x}$.

2) a- Set up the table of variations of f' and deduce that $f'(x)>0$.

b- Prove that (C) has an inflection point to be determined.

c- Set up the table of variations of f.

3) Determine the coordinates of the point A of (C) where the tangent (T) is parallel to the line (d) of equation $y=x$.

4) Show that the equation $f(x)=1$ has a unique root α and verify that $0.65<\alpha<0.66$.

5) Draw (d), (T) and (C).

6) Calculate, in cm^2, the area of the domain bounded by (C), the asymptote (d) and the two straight lines of equations $x=0$ and $x=1$.

7) Designate by g the inverse function of f and let (G) be its representative curve in the system $(O;\vec{i},\vec{j})$.
Determine the asymptote and the asymptotic direction of (G) and draw (G).

Part B-

Let f_n be the function defined over IR by $f_n(x)=x+x^n e^{-x}$.
(n is a non zero natural number) and let (u_n) be the sequence defined by $u_n=\int_0^1 [f_n(x)-x]dx$.

1) Calculate u_1.
2) Prove that $0 \leq x^n e^{-x} \leq 1$ over $[0\ ;\ 1]$ and deduce that the sequence (u_n) is bounded.
3) Prove that the sequence (u_n) is decreasing. Is (u_n) convergent ?

N°7.

PartA .

Let g be the function defined over IR by $g(x) = 2e^x + 2x - 7$.
Designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Graphical unit : 2 *cm.*

1) Calculate $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to-\infty} g(x)$.
2) Show that the straight line (d) of equation $y = 2x - 7$ is an asymptote to (Γ).
3) Study the variations of g and draw its table of variations.
4) Show that the equation $g(x) = 0$ admits a unique solution α such that $0.94 < \alpha < 0.941$, then deduce the sign of $g(x)$.
5) Trace (Γ).

Part B .

Let f be the function defined, over IR, by $f(x) = (2x - 5)(1 - e^{-x})$.
(C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the sign of f over IR.
2) Calculate $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
3) a- Calculate $f'(x)$ and verify that $f'(x)$ and $g(x)$ have the same sign.
   b- Draw the table of variations of f.
   c- Prove that the straight line (D) of equation $y = 2x - 5$ is an asymptote to (C) at $+\infty$.
   d- Study the relative position of (C) and (D).
   e- Take $\alpha = 0.94$ and trace (C).

N°8.

Let f be the function defined over IR by $f(x) = x - \dfrac{e^x - 1}{e^x + 1}$.

(C) its representative curve in an othonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$.
2) Designate by (d_1) the straight line of equation $y=x+1$ and by (d_2) the straight line of equation $y=x-1$.
   a- Show that (d_1) is an asymptote to (C) at $-\infty$ and (d_2) is an asymptote to (C) at $+\infty$.
   b- Study the relative positions of (C) with respect to (d_1) and (d_2).
3) a- Show that the function f is odd.
   b- Set up the table of variations of f.
4) a- Trace the curve (C), the tangent to (C) at the point of abscissa 0 as well as (d_1) and (d_2).
   b- Show that the equation $x+m=\dfrac{e^x-1}{e^x+1}$ admits a unique solution for all m in IR.
   c- Solve the inequality $e^x-1<x(e^x+1)$.
5) Verify that $\dfrac{e^x-1}{e^x+1}=\dfrac{e^x}{e^x+1}-\dfrac{e^{-x}}{e^{-x}+1}$, then calculate the area of the domain limited by $x'x$, the curve (C) and the two straight lines of equations: $x=0$ and $x=1$.

$\boxed{N°9.}$

Part A

Let g be the function defined over IR by $g(x)=1+(1-x)e^{-x}$.

1) Calculate $\lim_{x\to-\infty} g(x)$ and $\lim_{x\to+\infty} g(x)$.
2) Calculate $g'(x)$, then draw the table of variations of g.
3) Deduce the sign of $g(x)$ over IR.

Part B

Let f be the function defined over IR by $f(x)=x+xe^{-x}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$.
2) Show that the line (d) of equation $y=x$ is an asymptote to (C) at $+\infty$.
3) a- Verify that, for all real numbers x, $f'(x)=g(x)$.

b- Draw the table of variations of f .

c- Trace (C).

N° 10.

Part A

Let f be the function defined over IR by $f(x)=(2x+1)e^{-2x}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine the limits of f at $-\infty$ and $+\infty$. Deduce an asymptote to (C).
2) Calculate $f'(x)$, then draw the table of variations of f.
3) Determine the point of intersection of (C) with the axis $x'x$ and study the sign of $f(x)$.
4) Show that f admits an inflection point B whose coordinates are to be determined.
5) Show that $y=-\frac{2}{e}x+\frac{3}{e}$ is an equation of the tangent at B to (C).

Part B

Let g be the function defined over IR by $g(x)=f(x)-\left(-\frac{2}{e}x+\frac{3}{e}\right)$.

1) Determine $g'(x)$ and $g''(x)$.
2) Study the variations of g' over IR then draw its table of variations .
3) Deduce the sign of $g'(x)$, then study the sense of variations of g over IR.
4) Determine the sign of $g(x)$ and study the relative positions of (C) and (T) .
5) Trace (C) and (T).

Part C

Let (D) be the domain limited by (C), the axis of abscissas and the two straight lines of equations $x=-\frac{1}{2}$ and $x=\frac{1}{2}$.

1) Calculate the area of the domain (D) .
2) Calculate the volume of the solid obtained by rotating (D) around $x'x$.

Solutions

N° 1.

1) Letting $e^x = t$, the equation becomes $t^2 + t - 2 = 0$ that has as roots : $t' = 1$ which gives $e^x = 1$ then $x = 0$, $t'' = -2$ rejected.
2) $e^{3x} - 5e^{2x} - 6e^x = 0$ gives $e^x\left(e^{2x} - 5e^x - 6\right) = 0$ then

   $e^{2x} - 5e^x - 6 = 0$, quadratic equation that has as roots

   $e^x = -1$ rejected, $e^x = 6$ gives $x = \ln 6$.
3) Letting $e^x = t$, the equation becomes $t^3 - 21t + 20 = 0$.

   The trinomial $p(t) = t^3 - 21t + 20$ admits 1 as a root since

   $p(1) = 0$ then it is divisible by $t - 1$.

   By division, we get $p(t) = (t-1)\left(t^2 + t - 20\right)$

   $p(t) = 0$ gives $t = 1$ or $t^2 + t - 20 = 0$ then $t = 1$ or $t = -5$, inacceptable, or $t = 4$ therefore $x = 0$ or $x = \ln 4$.
4) The roots of the equation $e^{2x} + 3e^x - 4 = 0$ are $e^x = 1$ accepted, $e^x = -4$ rejected.

   $e^{2x} + 3e^x - 4 \leq 0$ is equivalent to $\left(e^x - 1\right)\left(e^x + 4\right) \leq 0$ which gives $e^x - 1 \leq 0$, then $e^x \leq 1$ and $x \leq 0$.

N° 2.

1) The equation $e^{2x}.e^y = e^{-3}$ gives $e^{2x+y} = e^{-3}$ then $2x + y = -3$.

   The system becomes $\begin{cases} 2x + y = -3 \\ xy = -2 \end{cases}$

   $2x + y = -3$ gives $y = -3 - 2x$ then $x(-3 - 2x) = -2$, so

   $2x^2 + 3x - 2 = 0$ which has as solutions

   $x' = \dfrac{1}{2}$ and $x'' = -2$ therefore :

   $x = \dfrac{1}{2}$ and $y = -4$ or $x = -2$ and $y = 1$.
2) The equation $e^x + e^y = 2$ gives $e^y = 2 - e^x$, then the equation $e^{2x} + e^{2y} = \dfrac{5}{2}$ becomes $e^{2x} + \left(2 - e^x\right)^2 = \dfrac{5}{2}$, which

becomes $4e^{2x}-8e^{x}+3=0$ that has as roots $e^{x}=\frac{1}{2}$ or $e^{x}=\frac{3}{2}$.

$e^{x}=\frac{1}{2}$ gives $e^{y}=\frac{3}{2}$ therefore $x=\ln\frac{1}{2}$ and $y=\ln\frac{3}{2}$

$e^{x}=\frac{3}{2}$ gives $e^{y}=\frac{1}{2}$ then $x=\ln\frac{3}{2}$ and $y=\ln\frac{1}{2}$.

3) $e^{x}=3e^{y}$ gives $e^{x-y}=3$ then $x-y=\ln 3$

The system becomes $\begin{cases} x-y=\ln 3 \\ x+y=2-\ln 3 \end{cases}$ that has as a solution

$x=1$ and $y=1-\ln 3$

4) The equation $\ln(y+6)-\ln x=3\ln 2$, with $x>0$ and $y>-6$ gives

$\ln\left(\frac{y+6}{x}\right)=\ln 2^{3}$ then $\frac{y+6}{x}=8$ and consequently $y=8x-6$

The equation $e^{5x}.e^{-y}=e^{-6}$ gives $e^{5x-y}=e^{-6}$ then $5x-y=-6$

which gives $5x-(8x-6)=-6$ therefore $-3x=-12$ and consequently $x=4$ and $y=26$, acceptable.

$\boxed{N°3}$

$F(x)=(ax^{2}+bx+c)e^{2x}$ is an antiderivative of f over $I\!R$ then $F'(x)=f(x)$ for all x.

But $F'(x)=(2ax+b)e^{2x}+2e^{2x}(ax^{2}+bx+c)$

$$=e^{2x}\left(2ax^{2}+2(a+b)x+b+2c\right)$$

Then $e^{2x}\left(2ax^{2}+2(a+b)x+b+2c\right)=(x^{2}-4)e^{2x}$ for all x.

Therefore, $2a=1$, $2(a+b)=0$ and $b+2c=-4$ which gives $a=\frac{1}{2}$, $b=-\frac{1}{2}$ and $c=-\frac{7}{4}$.

N°4.

1) $\int_0^1 \frac{e^x - e^{-x}}{e^x + e^{-x}}dx = \int_0^1 \frac{\left(e^x + e^{-x}\right)'}{e^x + e^{-x}}dx = \left[\ln\left(e^x + e^{-x}\right)\right]_0^1 = \ln\left(e^1 + e^{-1}\right) - \ln 2.$

2) We know that $\left(\sin^2 x\right)' = 2\sin x \cos x = \sin 2x$

Then $\int_0^{\frac{\pi}{2}} \left(\sin^2 x\right)' e^{\sin^2 x}dx = \left[e^{\sin^2 x}\right]_0^{\frac{\pi}{2}} = e - 1.$

3) $\int_0^1 xe^{x^2-1}dx = \frac{1}{2}\int_0^1 2xe^{x^2-1}dx = \frac{1}{2}\left[e^{x^2-1}\right]_0^1 = \frac{1}{2} - \frac{1}{2}e^{-1}$

4) Letting $\begin{cases} u = x \\ v' = e^x \end{cases}$ we get $\begin{cases} u' = 1 \\ v = e^x \end{cases}$ then :

$\int_0^1 xe^x dx = \left[xe^x\right]_0^1 - \int_0^1 e^x dx = \left[xe^x\right]_0^1 - \left[e^x\right]_0^1 = 1$

5) $\int_0^1 x^2 e^x dx$

Derivative		Antiderivative
x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

Therefore $\int_0^1 x^2 e^x dx = \left[x^2e^x - 2xe^x + 2e^x\right]_0^1 = e - 2.$

N°5.

1) Let $u(x) = e^x$ and $v'(x) = \cos x$, we get :

$u'(x) = e^x$ and $v(x) = \sin x$, therefore :

$I = e^x \sin x\Big]_0^{\frac{\pi}{2}} - \int_0^{\pi/2} e^x \sin x dx$, which gives: $I = e^{\frac{\pi}{2}} - J$ (1)

2) Let $u(x) = e^x$ and $v'(x) = \sin x$, we get :

$u'(x) = e^x$ and $v(x) = -\cos x$, therefore :

$J = -e^x \cos x\Big]_0^{\pi/2} + \int_0^{\pi/2} e^x \cos x dx$, which gives $J = 1 + I$ (2)

3) The relations (1) and (2) give $I = e^{\frac{\pi}{2}} - 1 - I$, then :

$2I = e^{\frac{\pi}{2}} - 1$ which gives $I = \dfrac{e^{\frac{\pi}{2}} - 1}{2}$ and consequently $J = \dfrac{e^{\frac{\pi}{2}} + 1}{2}$.

$\boxed{N°6.}$

$f(x) = e^{2x} + 2x - m$, so $f'(x) = 2e^{2x} + 2 > 0$ then f is increasing.

Since $\lim\limits_{x \to -\infty} f(x) = -\infty$ and $\lim\limits_{x \to +\infty} f(x) = +\infty$ then the representative curve of f cuts the axis $x'x$ at a unique point and consequently the equation $e^{2x} + 2x - m = 0$ has in IR one unique solution for all real numbers m.

$\boxed{N°7.}$

Part A

1) $\lim\limits_{x \to +\infty} g(x) = \lim\limits_{x \to +\infty} 2e^x + 2x - 7 = +\infty + \infty = +\infty$.

$\lim\limits_{x \to -\infty} g(x) = \lim\limits_{x \to -\infty} 2e^x + 2x - 7 = 0 - \infty = -\infty$.

2) $\lim\limits_{x \to -\infty} [g(x) - y] = \lim\limits_{x \to -\infty} (2e^x) = 0$.

Then, (d) is an asymptote to (Γ) at $-\infty$.

3) $g'(x) = 2e^x + 2$, then $g'(x) > 0$ for all x in IR.

Therefore the table of variations of f is :

x	$-\infty$		$+\infty$
$g'(x)$		$+$	
$g(x)$	$-\infty$	$\nearrow$	$+\infty$

4) The function g is strictly increasing over $]-\infty; +\infty[$,

Then (Γ) cuts the axis $x'x$ at a unique point of abscissa α.

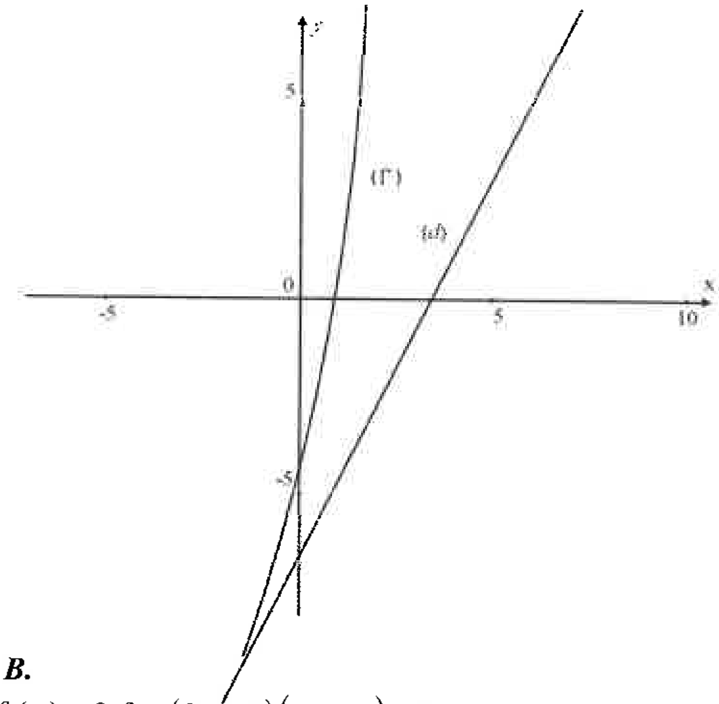
$g(0.94) \approx -0.000037$ and $g(0.941) \approx 0.0071$

Therefore, $0.94 < \alpha < 0.941$.

g is strictly increasing $g(x) = 0$ for $x = \alpha$.

Then $g(x)<0$ for $x<\alpha$, $g(x)>0$ for $x>\alpha$.

5)



Part B.

1) $f(x)=0$ for $(2x-5)(1-e^{-x})=0$, which gives ;

$2x-5=0$ so $x=\dfrac{5}{2}$ or $1-e^{-x}=0$ so $x=0$.

Therefore, the table of signs of $f(x)$ is:

x	$-\infty$		0		$\frac{5}{2}$		$+\infty$
$2x-5$		$-$		$-$	0	$+$	
$1-e^{-x}$		$-$	0	$+$		$+$	
$f(x)$		$+$	0	$-$	0	$+$	

Hence, $f(x)<0$ for $0<x<\dfrac{5}{2}$ and $f(x)>0$ for $x<0$ or

$x>\dfrac{5}{2}$.

2) $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}(2x-5)(1-e^{-x})=+\infty(1-0)=+\infty$.

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}(2x-5)(1-e^{-x})=-\infty(-\infty)=+\infty$.

3) a- $f'(x)=2\left(1-e^{-x}\right)+(2x-5)e^{-x}$, which gives

$f'(x)=e^{-x}g(x)$, then $f'(x)$ and $g(x)$ have the same sign.

b- The table of variations of f is the following:

x	$-\infty$		α		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

c- $f(x)-(2x-5)=-(2x-5)e^{-x}$, then

$$\lim_{x\to+\infty}\left[f(x)-(2x-5)\right]=\lim_{x\to+\infty}\frac{-2x+5}{e^{x}}=\lim_{x\to+\infty}\frac{-2}{e^{x}}=0.$$

Then, (D) is an asymptote to (C) at $+\infty$.

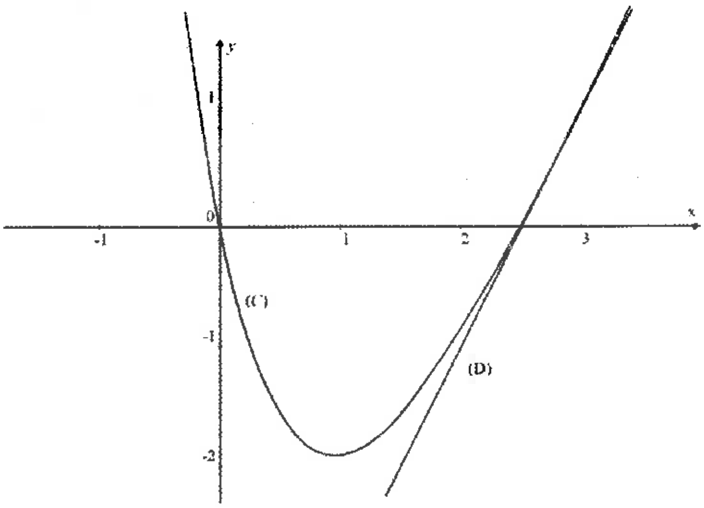
$f(x)-(2x-5)=-(2x-5)e^{-x}$

If $x<\dfrac{5}{2}$, then (C) is above (D).

If $x>\dfrac{5}{2}$, then (C) is below (D).

For $x=\dfrac{5}{2}$,(C) and (D) intersect at $A\left(\dfrac{5}{2};0\right)$.

d- For $\alpha=0.94$; $f(\alpha)=(2\times0.94-5)(1-e^{-0.94})\approx-1.9$.



4) In the interval $\left[0;\frac{5}{2}\right]$ (C) is below $x'x$, then :

The required area A is $A=\int_0^{5/2} -f(x)dx$, which is:

$$A=-\int_0^{\frac{5}{2}}\left[(2x-5)-(2x-5)e^{-x}\right]dx$$

$$A=-\int_0^{\frac{5}{2}}(2x-5)dx+\int_0^{\frac{5}{2}}(2x-5)e^{-x}dx.$$

Let $u=2x-5$ and $v'=e^{-x}$, then $u'=2$ and $v=-e^{-x}$.

So, we get $\int_0^{\frac{5}{2}}(2x-5)e^{-x}dx=-(2x-5)e^{-x}\Big|_0^{\frac{5}{2}}-\int_0^{\frac{5}{2}}-2e^{-x}dx$.

Which gives $\int_0^{\frac{5}{2}}(2x-5)e^{-x}dx=-(2x-5)e^{-x}\Big|_0^{\frac{5}{2}}-2e^{-x}\Big|_0^{\frac{5}{2}}$.

So, we get $\int_0^{\frac{5}{2}}(2x-5)e^{-x}dx=-3-2e^{-\frac{5}{2}}$.

Since $\int_0^{\frac{5}{2}}(2x-5)dx=x^2-5x\Big|_0^{\frac{5}{2}}=-\frac{25}{4}$, then :

$$A=+\frac{25}{4}-3-2e^{-\frac{5}{2}}=-2e^{-\frac{5}{2}}+\frac{13}{4} \text{ square units}.$$

Area in cm^2 is $A=\left(\frac{13}{4}-2e^{-5/2}\right)\times 4=13-8e^{-5/2}$.

$N°8.$

Part A .

1) $\lim\limits_{x\to+\infty} g(x)=\lim\limits_{x\to+\infty}\left(1-e^{2x}-2xe^{2x}\right)=-\infty-\infty=-\infty$

$\lim\limits_{x\to-\infty} g(x)=\lim\limits_{x\to-\infty}\left(1-e^{2x}-2xe^{2x}\right)=1-0+\infty\times 0$.

But $\lim\limits_{x\to-\infty} 2xe^{2x} = \lim\limits_{x\to-\infty} \dfrac{2x}{e^{-2x}} = \lim\limits_{x\to-\infty} \dfrac{2}{-2e^{-2x}} = 0$ so $\lim\limits_{x\to-\infty} g(x) = 1$

2) $g'(x) = -2e^{2x} - 2\left(e^{2x} + 2xe^{2x}\right) = -4e^{2x} - 4xe^{2x}$

$g'(x) = -4e^{2x}(1+x)$ $g'(x)$ Therefore the table of variations of g is :

x	$-\infty$		-1		$+\infty$
$g'(x)$		$+$	0	$-$	
$g(x)$	1	$\nearrow$	$1+e^{-2}$	$\searrow$	$-\infty$

3) $g(0)=0$. For $x<0$ g increases from 1 to $1+e^{-2}$ then decreases to 0 so $g(x)>0$. For $x>0$ g decreases from 0 to $-\infty$ then $g(x)<0$.

Part B .

1) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \left(x+3-xe^{2x}\right) = +\infty - \infty$ indeterminate form

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \left(3 + x\left(1-e^{2x}\right)\right) = 3+\infty(-\infty) = -\infty$.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \left(x+3-xe^{2x}\right) = -\infty-(-\infty)\times 0$ indeterminate form.

But, $\lim\limits_{x\to-\infty} xe^{2x} = \lim\limits_{x\to-\infty} \dfrac{x}{e^{-2x}} = \lim\limits_{x\to-\infty} \dfrac{1}{-2e^{-2x}} = 0$ then $\lim\limits_{x\to-\infty} f(x) = -\infty$.

2) a- $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to-\infty} \left(1+\dfrac{3}{x}-e^{2x}\right) = 1$

$\lim\limits_{x\to-\infty} f(x)-x = \lim\limits_{x\to-\infty} \left(3-xe^{2x}\right) = 3$ and we deduce that the straight line (d) of equation $y=x+3$ is an asymptote to (C) at $-\infty$.

b- $f(x)-y = -xe^{2x}$.

For $x<0$ $f(x)-y>0$ then (C) is above (d).

For $x>0$ $f(x)-y<0$ then (C) is below (d).

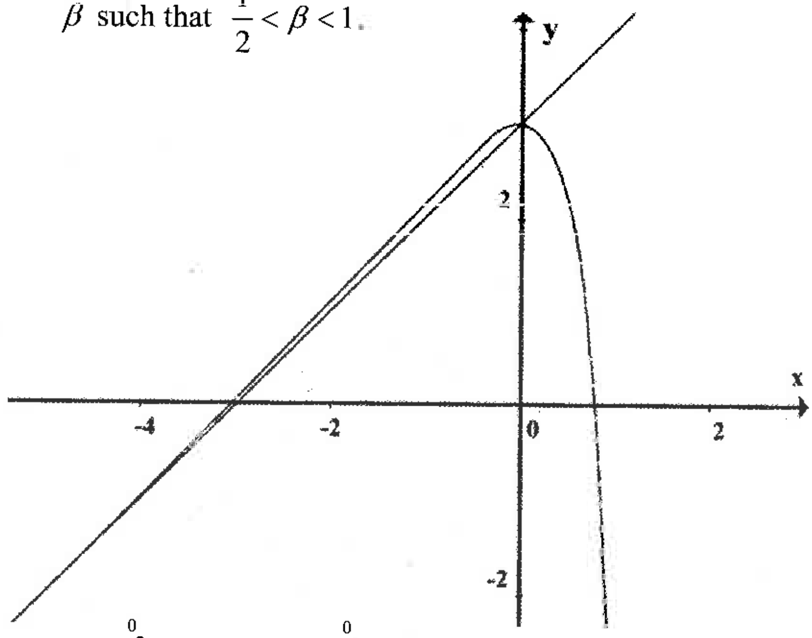
3) a- $f'(x) = 1-e^{2x}-2xe^{2x} = g(x)$

Then, for $x<0$ $f'(x)>0$ and for $x>0$ $f'(x)<0$

b- For $x<0$ f is a continuous and strictly increasing function and $f(-4)\times f(-3)<0$ then the equation $f(x)=0$ admits a unique root α such that $-4<\alpha<-3$.

For $x>0$ f is a continuous and strictly decreasing and since $f\left(\frac{1}{2}\right)\times f(1)<0$ then the equation $f(x)=0$ has a unique root β such that $\frac{1}{2}<\beta<1$.

4)



5) a- $S_\lambda=\int_\lambda^0 f(x)-y\,dx=-\int_\lambda^0 xe^{2x}dx$

Let $u=x$ and $v'=e^{2x}$, then $u'=1$ and $v=\frac{1}{2}e^{2x}$

Therefore, $\int_\lambda^0 xe^{2x}dx=\frac{1}{2}xe^{2x}\Big|_\lambda^0-\int_\lambda^0\frac{1}{2}e^{2x}dx=\frac{1}{2}xe^{2x}\Big|_\lambda^0-\frac{1}{4}e^{2x}\Big|_\lambda^0$

So $\int_\lambda^0 xe^{2x}dx=-\frac{1}{2}\lambda e^{2\lambda}-\frac{1}{4}+\frac{1}{4}e^{2\lambda}$.

Hence, $S_\lambda=\frac{\lambda}{2}e^{2\lambda}-\frac{1}{4}e^{2\lambda}+\frac{1}{4}$.

b- $\lim\limits_{\lambda\to-\infty} S_\lambda = \lim\limits_{\lambda\to-\infty}\left(\frac{\lambda}{2}e^{2\lambda}-\frac{1}{4}e^{2\lambda}+\frac{1}{4}\right)=\frac{1}{4}$.

N°9.

1) a- $\lim\limits_{\substack{x\to1\\x>1}} e^x = e$; $\lim\limits_{\substack{x\to1\\x>1}}(e^x-e)=0^+$; $\lim\limits_{\substack{x\to1\\x>1}} f(x)=-\infty$

Then the straight line of equation $x=1$ is an asymptote to (C).

b- $\lim\limits_{x\to+\infty}\frac{3e^x}{e^x-e}=3$ consequently $\lim\limits_{x\to+\infty} f(x)=+\infty$;

$\lim\limits_{x\to+\infty}[f(x)-(x-2)]=\lim\limits_{x\to+\infty}[3-\frac{3e^x}{e^x-e}]=0$

Then, (d) is an asymptote to (C).

c- $f(x)-(x-2)=3-\frac{3e^x}{e^x-e}=-\frac{3e}{e^x-e}$.

For $x>1$, $e^x>e$, $f(x)-(x-2)<0$ consequently (C) is below (d).

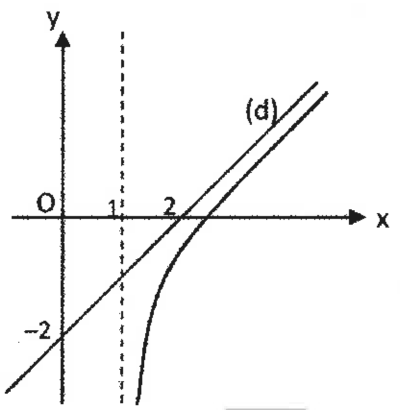
2) $f'(x)=1-3\frac{e^x(e^x-e)-e^x(e^x)}{(e^x-e)^2}=1+3\frac{e^{x+1}}{(e^x-e)^2}>0$

x	1	$+\infty$
f '(x)	‖ +	
f (x)	‖ $-\infty$ ↗	$+\infty$

3) Over I, f is continuous and strictly increasing from $-\infty$ to $+\infty$ then the equation $f(x)=0$ has a unique root α.

$f(2.6)=-0.158$ and $f(2.7)=0.0294$ then $2.6<\alpha<2.7$.

4)



5) $\int_3^4 \frac{e^x}{e^x - e}dx = [\ln(e^x - e)]_3^4 = \ln(e^4 - e) - \ln(e^3 - e) = \ln\frac{e^3 - 1}{e^2 - 1}$.

$$\mathscr{A} = \int_3^4 (x - 2 - f(x))dx = \int_3^4 (-3 + 3\frac{e^x}{e^x - e})dx = [-3x]_3^4 + 3\ln\frac{e^3 - 1}{e^2 - 1}$$

$$= [-3 + 3\ln\frac{e^3 - 1}{e^2 - 1}]\ u^2$$

6) a- Over I, f is continuous and strictly increasing, so it admits an inverse function g.

b- The equation $f(x) = g(x)$ is equivalent to $f(x) = x$ then

$1 - \frac{3e^x}{e^x - e} = 0$ which gives $2e^x = -e$ which is impossible.

➤ OR : Graphically , the curve (C) do not cut the straight line of equation $y = x$.

Then, the equation $f(x) = g(x)$ has no solutions in I.

N° 10.

1) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} [1 + x(2 - e^{x-1})] = -\infty$.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \left[2x + 1 - \frac{xe^x}{e}\right] = -\infty$, car $\lim\limits_{x\to-\infty} xe^x = 0$.

2) $\lim\limits_{x\to-\infty} [f(x) - (2x + 1)] = \lim\limits_{x\to-\infty} \left[-\frac{xe^x}{e}\right] = 0$.

Then the straight line (d) is an asymptote to (C) at $-\infty$.

$f(x) - (2x + 1) > 0$ for $x < 0$, in this case (C) is above (d).

$f(x) - (2x + 1) < 0$ for $x > 0$ and in this case (C) is below (d).

For $x = 0$, (C) and (d) intersect at $A(0;1)$.

3) a- From the table of variations of f', we remark that :

$f'(1) = 0$, $f'(x) < 0$ for $x > 1$ and $f'(x) > 0$ for $x < 1$.

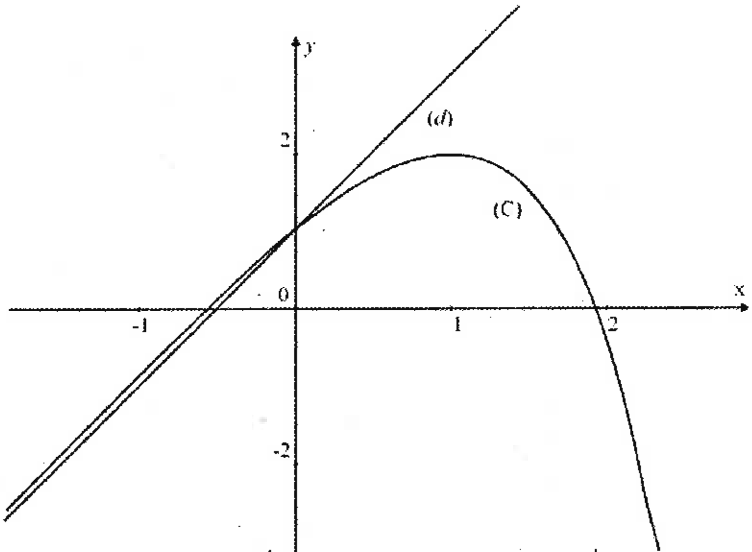
Therefore the table of variations of f is

x	$-\infty$		1		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	$-\infty$	↗	2	↘	$-\infty$

b- From the table of variations of f', we remark that :
$f''(-2)=0$, $f''(x)<0$ for $x>-2$ and $f''(x)>0$ for $x<-2$.
Then, the point L abscissa -2 is a point of inflection of f.
$y_L=-3+2e^{-3}$, then $L\left(-2\,;-3+2e^{-3}\right)$.

4) For $x<1$, f is continuous and strictly increasing from $-\infty$ to 2 then (C) cuts $x'x$ at a unique point of abscissa α and since $f(-1)\approx -0.8$ and $f\left(-\dfrac{1}{2}\right)\approx 0.1$, then $-1<\alpha<-\dfrac{1}{2}$.
Similarly for $x>1$, f continuous and strictly decreasing from 2 to $-\infty$ then (C) cuts $x'x$ at a unique point of abscissa β and since $f(1)=2$ and $f(2)\approx -0.4$, then $1<\beta<2$.

5)



6) a- Let $u=x$ and $v'=e^{x-1}$, then $u'=1$ and $v=e^{x-1}$
Which gives $\displaystyle\int_1^{\beta} xe^{x-1}dx = xe^{x-1}\Big|_1^{\beta} - \int_1^{\beta} e^{x-1}dx = xe^{x-1}\Big|_1^{\beta} - e^{x-1}\Big|_1^{\beta}$
Therefore, $J=(\beta-1)e^{\beta-1}$.

b- $\displaystyle S_\beta = \int_1^{\beta}(2x+1-xe^{x-1})dx = \int_1^{\beta}(2x+1)\,dx - J$.
$S_\beta = x^2+x\Big|_1^{\beta} - J = \beta^2+\beta-2-J$.

Then, $S_{\beta} = \beta^2 + \beta - 2 - (\beta - 1)e^{\beta - 1}$ square units.

c- β is a root of the equation $f(x) = 0$ then

$2\beta + 1 - \beta e^{\beta-1} = 0$, which gives $e^{\beta - 1} = 2 + \dfrac{1}{\beta}$, so

$$S_{\beta} = \beta^2 + \beta - 2 - (\beta - 1)\left(2 + \frac{1}{\beta}\right) = (\beta - 1)\left(\beta - \frac{1}{\beta}\right).$$

$N^{\circ}11.$

Part A.

1) $\lim\limits_{x \to +\infty} h(x) = -\infty$.

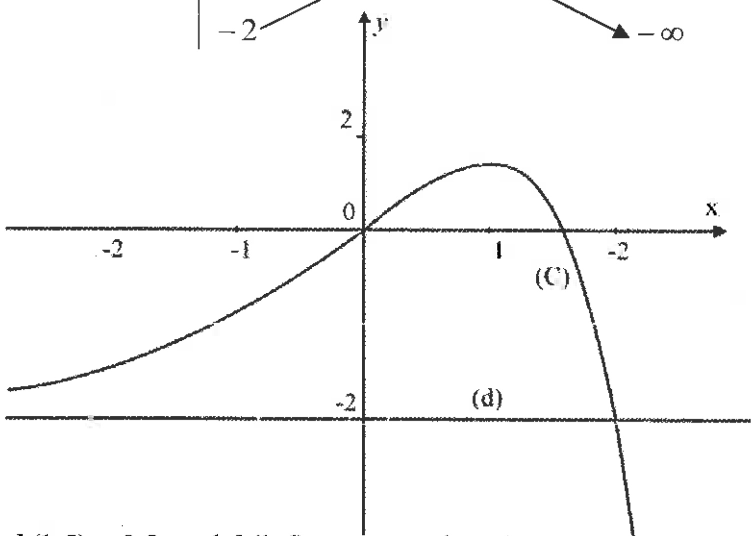
$\lim\limits_{x \to -\infty} h(x) = \lim\limits_{x \to -\infty} \left[2e^x - xe^x - 2\right] = -2$, car $\lim\limits_{x \to -\infty} xe^x = 0$.

Then, the straight line (d) of equation $y = -2$ is an asymptote to (C) at $-\infty$.

2) $h'(x) = (1 - x)e^x$, then the table of variations of h :

x	$-\infty$		1		$+\infty$
$h'(x)$		$+$	0	$-$	
$h(x)$	-2	$\nearrow$	$e - 2$	$\searrow$	$-\infty$

3)



4) $h(1.5) \approx 0.2$ and $h(1.6) \approx -0.01$, then $1.5 < \alpha < 1.6$.

Part B.

1) $g'(x) = e^x - 2$, then : $g'(x) = 0$ for $x = \ln 2$, $g'(x) > 0$

for $x > \ln 2$ and $g'(x) < 0$ for $x < \ln 2$.
Then, the function g admits for $x = \ln 2$ a minimum equal to $g(\ln 2) = 2 - 2\ln 2 \approx 0.6 > 0$, and consequently $g(x) > 0$ over IR.
Since $e^x - 2x \neq 0$ for all real numbers . then the domain of definition of f is IR.

2) $\lim\limits_{x \to -\infty} f(x) = \dfrac{-2}{+\infty} = 0$, $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{e^x}{e^x - 2} = \lim\limits_{x \to +\infty} \dfrac{e^x}{e^x} = 1$
(L'hopital's rule) .
Then, the two straight $y = 0$ and $y = 1$ are asymptotes to (Γ) .

3) $f'(x) = \dfrac{2h(x)}{(e^x - 2x)^2}$, then $f'(x)$ and $h(x)$ have the same sign
Therefore the table of variations of f is:

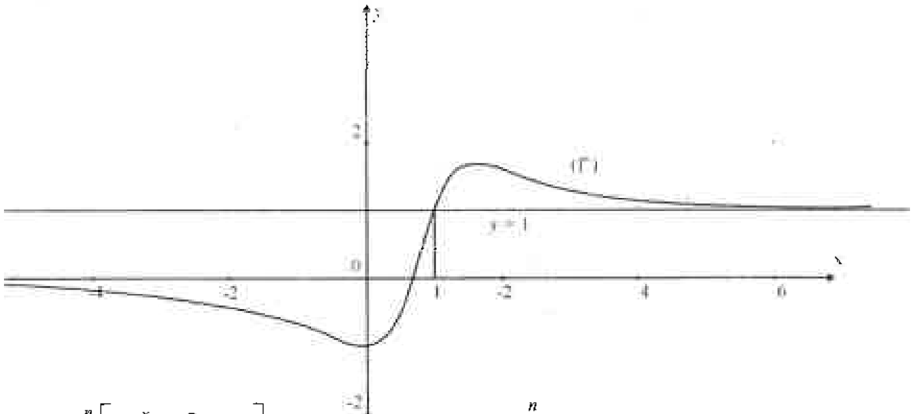
x	$-\infty$		0		α		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	0	$\searrow$	-1	$\nearrow$	$f(\alpha)$	$\searrow$	1

4) $f(\alpha) = \dfrac{e^\alpha - 2}{e^\alpha - 2\alpha}$, but α is the root of the equation $h(x) = 0$ then
$(2-\alpha)e^\alpha - 2 = 0$, which gives $e^\alpha = \dfrac{2}{2-\alpha}$ and consequently,

$$f(\alpha) = \frac{\dfrac{2}{2-\alpha} - 2}{\dfrac{2}{2-\alpha} - 2\alpha} = \frac{2(\alpha - 1)}{2(\alpha^2 - 2\alpha + 1)} = \frac{1}{\alpha - 1}.$$

For $\alpha = 1.55$, we get $f(\alpha) = \dfrac{1}{1.55 - 1} \approx 1.81$.

5) (Γ) cuts the straight line of equation $y = 1$ at the point $A(1;1)$.



6) $A_n = \int_1^n \left[\frac{e^x - 2}{e^x - 2x} - 1\right]dx = \ln(e^x - 2x) - x\Big|_1^n$.

$A_n = \ln(e^n - 2n) - n - \ln(e-2) + 1$, but $n = \ln e^n$,

then : $A_n = \ln\left(\frac{e^n - 2n}{e^n}\right) + 1 - \ln(e-2)$.

$$\lim_{n\to+\infty} A_n = \lim_{n\to+\infty}\left[\ln\left(1 - 2\frac{n}{e^n}\right) + 1 - \ln(e-2)\right] = 1 - \ln(e-2).$$

N° 12.

Part A .

1) $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x+1)^2 e^{-x} = (+\infty)\times(+\infty) = +\infty$.

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\frac{(x+1)^2}{e^x} = \lim_{x\to+\infty}\frac{2(x+1)}{e^x} = \lim_{x\to+\infty}\frac{2}{e^x} = 0.$$

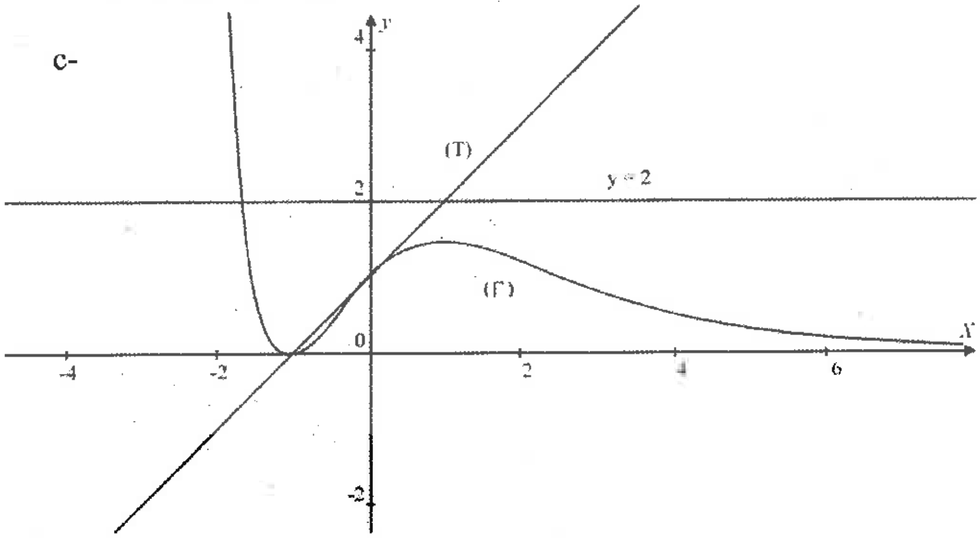
Then, the axis $x'x$ is an asymptote to (Γ) at $+\infty$.

2) a- From the representative curve (C) of f', we note that $f'(x) > 0$ for $-1 < x < 1$ and $f'(x) < 0$ for $x < -1$ or $x > 1$, then the table of variations of f is :

x	$-\infty$		-1		1		$+\infty$
$f'(x)$		$-$	0	$+$	0	$-$	
$f(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$4e^{-1}$	$\searrow$	0

b- From the representative curve of f', we note that $f'(0)=1$ and since $f(0)=1$, then an equation of (T) is : $y=f(0)+f'(0)(x-0)$, so $y=x+1$.

c-



3) a- The straight line of equation $y=2$ cuts (Γ)at a unique point, then the equation $f(x)=2$ admits a unique root α .
$f(-2)\approx 7.3$ and $f(-1)=0$ and $f(\alpha)=2$, then $f(-1)<f(\alpha)<f(-2)$ and since f is strictly decreasing over $[-2;-1]$, then $-2<\alpha<-1$.

b- $f(\alpha)=2$, then $(\alpha+1)^2e^{-\alpha}=2$, which gives $(\alpha+1)^2=2e^{\alpha}$ then $\alpha+1=-\sqrt{2e^{\alpha}}$, since $\alpha<0$ and consequently $\alpha=-1-\sqrt{2e^{\alpha}}$. Finally, $\alpha=-1-\sqrt{2}\,e^{\frac{\alpha}{2}}$.

4) $A=\int_0^1(x+1)^2e^x dx$, integrating by parts :

Let $u=(x+1)^2$ and $v'=e^x$, then $u'=2(x+1)$ and $v=e^x$.

$$A=(x+1)^2e^x\Big|_0^1-\int_0^1 2(x+1)e^x dx=4e-1-\int_0^1 2(x+1)e^x dx$$

Using integration by parts again :
Let $u=2(x+1)$ and $v'=e^x$, then $u'=2$ and $v=e^x$
So, we get:

$$\int_0^1 2(x+1)e^x dx = 2(x+1)e^x\Big|_0^1 - \int_0^1 2e^x dx = 4e-2-(2e-2) = 2e.$$

And consequently $A = 4e-1-2e = 2e-1$ square units.

Part B.

1) $f(x)=(x+1)^2e^{-x} \geq 0$, but for g to be defined then $f(x)>0$, so we should have $x \neq -1$ and consequently the domain of definition of g is $\left]-\infty;-1\right[\cup\left]-1;+\infty\right[$.

2) We know that $g'(x)=\dfrac{f'(x)}{f(x)}$, then f and g have the same sense of variations , so the table of variations of g is:

x	$-\infty$		-1		1		$+\infty$
$g'(x)$		$-$	$\|\|$	$+$	0	$-$	
$g(x)$		$\searrow$	$\|\|$	$\nearrow$	$-1+\ln 4$	$\searrow$	

3) $g(x)=-x$ gives $\ln((x+1)^2e^{-x})=\ln e^{-x}$, then $(x+1)^2e^{-x}=e^{-x}$, so $(x+1)^2=1$, which gives $x=0$ or $x=-2$

$\boxed{N^\circ 13.}$

Part A .

1) $h(x)=e^x-x-1$.

$h'(x)=e^x-1$ for, $x=0$, h admits a minimum since its derivative vanishes and changes sign from negative to positive and since $h(0)=0$ then $h(x)\geq 0$ for all values of x hence $e^x \geq x+1$ for $x>0$.

$t(x)=\ln x - x+1$, $t'(x)=\dfrac{1}{x}-1=\dfrac{1-x}{x}$ for $x=1$, t admits a maximum since its derivative vanishes and changes sign from positive to negative and since $t(1)=0$ then $t(x)\leq 0$ for all $x>0$ then $\ln x \leq x-1$ for $x>0$

2) $e^x \geq x+1$ and $-\ln x \geq -x+1$ then by adding, we get: $e^x - \ln x > 2$ for $x>0$.

Part B .

1) $g'(x) = e^x - \dfrac{1}{x} - e^x - xe^x = -\dfrac{1}{x} - xe^x < 0$, then the table of variations of g is:

x	0		$+\infty$
$g'(x)$	‖	$-$	
$g(x)$	‖ $+\infty$	$\searrow$	$-\infty$

2) g is continuous and strictly decreasing over $]0;+\infty[$ and since $g(1.23)\times g(1.24) < 0$ then the equation $g(x) = 0$ has a unique root α such that $1.23 < \alpha < 1.24$.

3) For $x < \alpha$, $g(x) > g(\alpha)$ since g is decreasing then $g(x) > 0$. For $x > \alpha$; $g(x) < g(\alpha)$ since g is decreasing, then $g(x) < 0$

Part C .

1) $\lim\limits_{x\to+\infty} \dfrac{x}{e^x - \ln x} = \dfrac{+\infty}{(+\infty)-(+\infty)} = \lim\limits_{x\to+\infty} \dfrac{1}{\dfrac{e^x}{x} - \dfrac{\ln x}{x}}$

$\lim\limits_{x\to+\infty} \dfrac{e^x}{x} = +\infty$ and $\lim\limits_{x\to+\infty} \dfrac{\ln x}{x} = 0$ then $\lim\limits_{x\to+\infty} \dfrac{x}{e^x - \ln x} = \dfrac{1}{(+\infty)-0} = 0$.

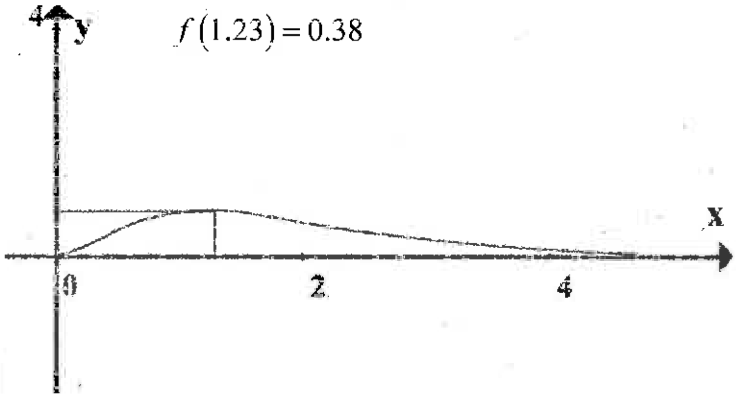
$\lim\limits_{x\to0^+} \dfrac{x}{e^x - \ln x} = \dfrac{0}{1+\infty} = 0$

2) $f'(x) = \dfrac{g(x)}{\left(e^x - \ln x\right)^2}$ then $f'(x)$ and $g(x)$ have the same sign.

Then, the table of variations of f is :

x	0		α		$+\infty$
$f'(x)$	‖	$+$	0	$-$	
$f(x)$	‖ 0	$\nearrow$	$f(\alpha)$	$\searrow$	0

3) $f(1.23) = 0.38$



N° 14.

1) $\lim\limits_{x \to -\infty} [f(x) + x] = \lim\limits_{x \to -\infty} \ln(1 + e^x) = 0$.

$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} [\ln(1 + e^x) - \ln(e^x)] = \lim\limits_{x \to +\infty} \ln\left(\dfrac{1 + e^x}{e^x}\right)$

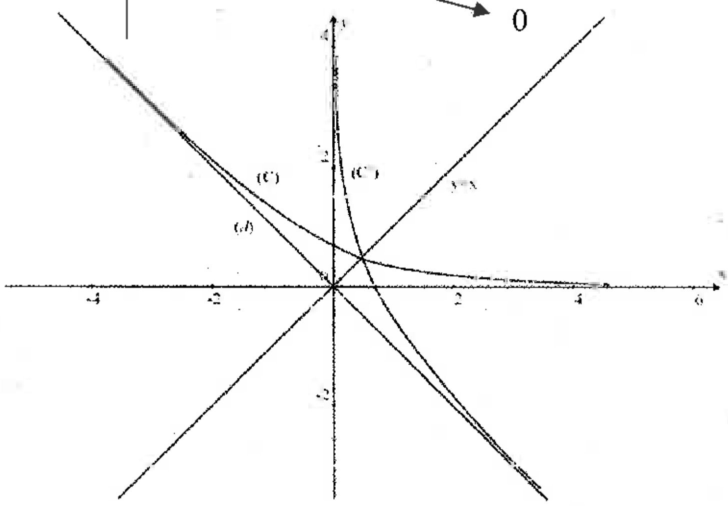
then $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \ln(1 + e^{-x}) = 0$.

Hence, the axis $x'x$ is an asymptote to (C) to $+\infty$.

2) a- $f'(x) = \dfrac{-1}{1 + e^x} < 0$, then the table of variations of f is:

x	$-\infty$		$+\infty$
$f'(x)$		$-$	
$f(x)$	$+\infty$	$\searrow$	0

b-



3) a- f is continuous and strictly decreasing in IR, then
It admits an inverse function g defined over $]0;+\infty[$.

b- (C') is symmetric to (C) with respect to $y=x$.

c- $f(0)=\ln 2$, then $g'(\ln 2)=\dfrac{1}{f'(0)}=-2$.

An equation of the tangent is $y=-2(x-\ln 2)$.

4) a- $f\circ h(x)=\ln(1+e^{h(x)})-h(x)=\ln\left(\dfrac{1+e^{h(x)}}{e^{h(x)}}\right)$,

then $f\circ h(x)=\ln(1+e^{-h(x)})=\ln\left(1+e^{\ln(e^x-1)}\right)=\ln(1+e^x-1)$.

Therefore, $f\circ h(x)=\ln(e^x)=x$..

b- Since $f\circ h(x)=x$, then $g(x)=h(x)=-\ln(e^x-1)$.

N° 15.

1) a- For $x>0$, (C') is above $x'x$, then the function that represents (C') is positive, so the function that represents an antiderivative of this function is increasing.
For $x<0$, the curve (C') is below $x'x$, then the function that represents (C') is negative, so the function that represents an antiderivative of this function is decreasing.
The function that represents (C) is increasing for $x>0$ and decreasing for $x<0$. Hence, (C) is the representative curve of f and (C') that of f'.

b- $f(0)=-1$ and $f'(0)=0$ and since $f'(x)=2ae^{2x}+be^x$ then $a+b=-1$ and $2a+b=0$, so we get the following system: $\begin{cases}a+b=-1\\2a+b=0\end{cases}$ that has as a solution $a=1$ and $b=-2$.

Therefore, $f(x)=e^{2x}-2e^x$.

2) f is continuous and strictly increasing for $x\geq 0$, then it admits an inverse function f^{-1} .
The domain of definition of f^{-1} is $[-1;+\infty[$.
$y=e^{2x}-2e^x+1-1=(e^x-1)^2-1$, which gives $e^x-1=\sqrt{1+y}$,
since $e^x>1$ for $x>0$, then $e^x=1+\sqrt{1+y}$

and consequently $f^{-1}(x)=\ln(1+\sqrt{1+x})$.

3) (C) cuts $x'x$ at the point $(\ln 2\,;0)$, therefore $A=\int_0^{\ln 2} -f(x)\,dx$.

$$A=\int_0^{\ln 2}(2e^x-e^{2x})\,dx=2e^x-\frac{1}{2}e^{2x}\Big|_0^{\ln 2}=\frac{1}{2}\text{ square units}.$$

4) $V=\int_0^{\ln 2}\pi y^2dx=\pi\int_0^{\ln 2}\left(e^{4x}+4e^{2x}-4e^{3x}\right)dx$.

$$V=\pi\left[\frac{1}{4}e^{4x}+2e^{2x}-\frac{4}{3}e^{3x}\right]_0^{\ln 2}=\frac{5}{12}\pi\text{ cubic units}.$$

N° 16

1) $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty} x+\ln 4+\dfrac{2}{e^x+1}=-\infty+\dfrac{2}{0+1}=-\infty$.

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty} x+\ln 4+\dfrac{2}{e^x+1}=+\infty+\dfrac{2}{e^{+\infty}}=+\infty$.

2) $f(-x)=-x+\ln 4+\dfrac{2}{e^{-x}+1}=-x+\ln 4+\dfrac{2e^x}{1+e^x}$.

Then $f(x)+f(-x)=x+\ln 4+\dfrac{2}{1+e^x}-x+\ln 4+\dfrac{2e^x}{1+e^x}$.

So, $f(x)+f(-x)=2\ln 4+\dfrac{2+2e^x}{1+e^x}=2(1+\ln 4)$.

Then, the point $A\,(0;1+\ln 4)$ is a center of symmetry of (C).

3) $f'(x)=1-\dfrac{2e^x}{(e^x+1)^2}=\dfrac{e^{2x}+1+2e^x-2e^x}{(e^x+1)^2}=\dfrac{e^{2x}+1}{(e^x+1)^2}$.

Then, $f'(x)>0$ for all x in IR .

Then, the table of variations of f is:

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	$-\infty$	$\nearrow$	$+\infty$

4) a- $f(x) = x + \ln 4 + \dfrac{2}{e^x + 1} - 2 + 2$

$$= x + 2 + \ln 4 + \frac{2 - 2e^x - 2}{e^x + 1} = x + 2 + \ln 4 - \frac{2e^x}{e^x + 1}.$$

b- $f(x) - y_d = \dfrac{2}{e^x + 1}$, then $\lim\limits_{x \to +\infty}[f(x) - y_d] = \lim\limits_{x \to +\infty}[\dfrac{2}{e^x + 1}] = 0$.

Hence, (d) is an asymptote to (C) at $+\infty$.

Similarly, $f(x) - y_{d'} = \dfrac{-2e^x}{e^x + 1}$, and since $\lim\limits_{x \to -\infty} \dfrac{-2e^x}{1 + e^x} = 0$,

Then (d') is an asymptote to (C) at $-\infty$.

Since $\dfrac{2}{e^x + 1} > 0$ then (C) is above (d) for all x .

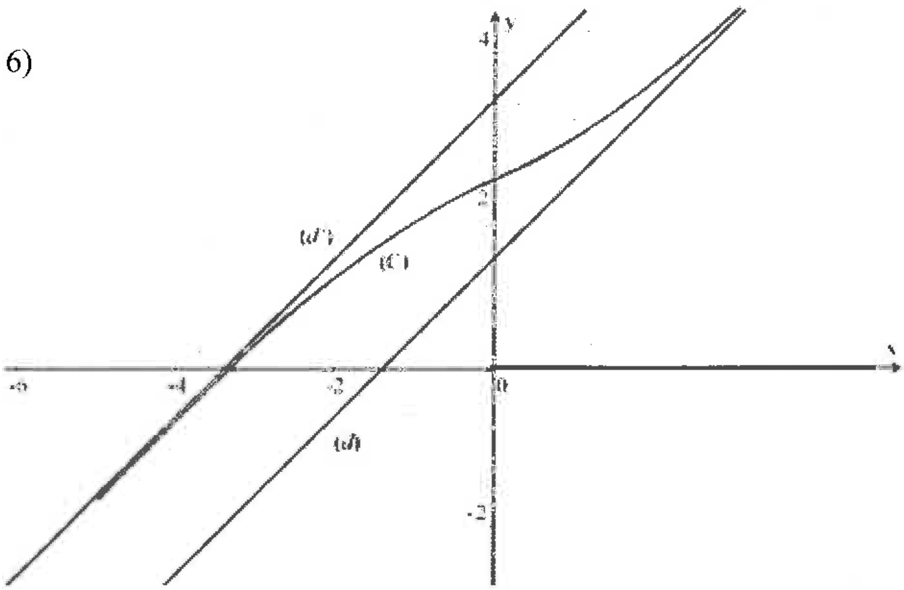
5) $f''(x) = \dfrac{2e^x(e^x + 1)(e^x - 1)}{(e^x + 1)^4}$.

$f''(x) \geq 0$ for $e^x - 1 \geq 0$ which gives $x \geq 0$.

Similarly, $f''(x) < 0$, for $x < 0$.

Then, the point $A\left(0; 1 + \ln 4\right)$ is a point of inflection of f.

6)



7) f is continuous and strictly increasing over $]-\infty;+\infty[$,
Then the straight line of equation $y=3$ cuts the curve (C) in a unique point of abscissa β.
$f(1.1)\approx 2.986$, $f(1.2)=3.049$ and $f(\beta)=3$ then $f(1.1)<f(\beta)<f(1.2)$ and since f is strictly increasing then $1.1<\beta<1.2$.

8) f is continuous and strictly increasing over $]-\infty;+\infty[$
Then it admits an inverse function f^{-1}.
$f(0)=\ln 4+1=\ln 4+\ln e=\ln 4e$, which gives that
$\left(f^{-1}\right)'(\ln 4e)=\dfrac{1}{f'(0)}=2$.

N° 17.

1) a- $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty} x\left(1+e^{x}\right)=-\infty$.
$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty} x\left(1+e^{x}\right)=+\infty$.

b- $\lim\limits_{x\to-\infty}\left[f(x)-y\right]=\lim\limits_{x\to-\infty} xe^{x}=0$.

c- $f(x)-y=xe^{x}$ then : If $x>0$ then (C) is above (d).
If $x=0$ then (C) cuts (d) at point O.
If $x<0$ then (C) is below (d).

d- An equation of (T) is $y=f(-1)+f'(-1)(x+1)$.
$f'(x)=1+e^{x}+xe^{x}$ then $f'(-1)=1$ and since $f(-1)=-1-e^{-1}$ then an equation of (T) is $y=x-e^{-1}$.

e- $f''(x)=e^{x}+e^{x}+xe^{x}=e^{x}(x+2)$, then :
$f''(x)=0$ for $x=-2$, $f''(x)>0$ for $x>-2$ and $f''(x)<0$ for $x<-2$,then the point $I\left(-2;-2-\dfrac{2}{e^{2}}\right)$ is a point of inflection of f.

f- The table of variations of f' is the following :

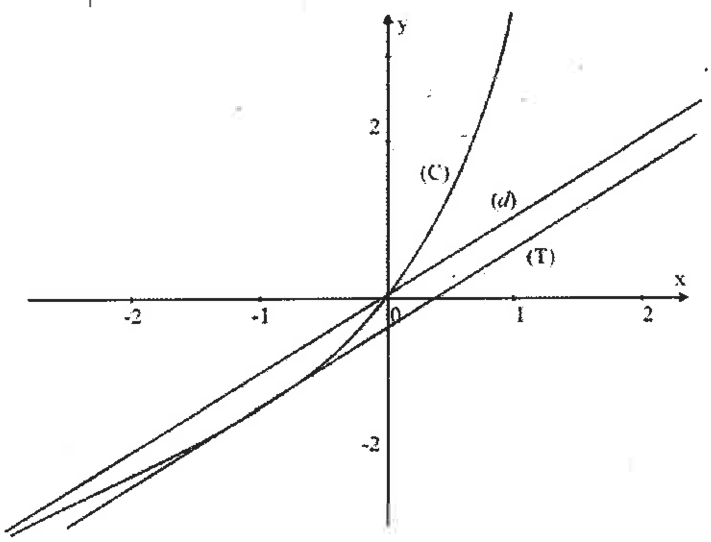
x	$-\infty$		-2		$+\infty$
$f''(x)$		$-$	0	$+$	
$f'(x)$	$\searrow$		$1-e^{-2}$		$\nearrow$

The minimum of f' is $1-e^{-2} \approx 0.8 > 0$, then $f'(x) > 0$ for all real numbers x .

Therefore, the table of variations of f is:

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	$-\infty$	$\nearrow$	$+\infty$

g-



2) $S = \int_{-1}^{0} [x - f(x)]dx = \int_{-1}^{0} -xe^x dx$.

Using integration by parts, we get :

If $u = x$ and $v' = e^x$ then $u' = 1$ and $v = e^x$, so

$$S = -xe^x\Big|_{-1}^{0} + \int_{-1}^{0} e^x dx = -xe^x\Big|_{-1}^{0} + e^x\Big|_{-1}^{0} = 1 - 2e^{-1} \text{ square units.}$$

3) (δ) is parallel to (T) and (T) cuts $y'y$ at the point $\left(0;-\dfrac{1}{e}\right)$.

Using (C) we find that :

If $\alpha < -\dfrac{1}{e}$ then (δ) does not cut (C).

If $\alpha = -\dfrac{1}{e}$ then (δ) is tangent to (C).

If $-\dfrac{1}{e} < \alpha < 0$ then (δ) cuts (C) in 2 points.

If $\alpha \geq 0$ then (δ) cuts (C) in one point.

N° 18. ***For the students of the GS section:***

Part A .

1) $g(0)=0$ then $b+c=0$. $g'(1)=0$ and since $g'(x)=ae^{x}+(ax+b)e^{x}$ then $ae+(a+b)e=0$ which gives $2a+b=0$.

The straight line of equation $y=2$ is an asymptote to (C) at $-\infty$ then $\lim\limits_{x\to-\infty} g(x)=2$ therefore $c=2$ since $\lim\limits_{x\to-\infty}(ax+b)e^{x}=0$.

Therefore, $c=2$, $b=-2$ and $a=1$ then $g(x)=(x-2)e^{x}+2$.

2) In the interval $[1;+\infty[$ g is continuous and strictly increasing

Since $g\left(\dfrac{3}{2}\right)=-\dfrac{1}{2}e^{\frac{3}{2}}+2<0$ and $g(2)=2>0$ then the equation $g(x)=0$ admits in the interval $\left[\dfrac{3}{2};2\right]$ a unique solution α.

3) $a=\int_0^1 (2-g(x))dx=\int_0^1 -(x-2)e^{x}dx$.

Let $z=-(x-2)e^{x}$, $z'=-e^{x}-(x-2)e^{x}$ then

$z'=-e^{x}-(x-2)e^{x}=-e^{x}+z$ consequently

$a=\int_0^1 (z'+e^{x})dx=\left[z+e^{x}\right]_0^1=\left[-(x-2)e^{x}+e^{x}\right]_0^1=2e-3\text{u}^2$.

Part B .

1) $\lim\limits_{\substack{x\to 0\\ x>0}} f(x) = \lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{e^x-1}{x^2} = \dfrac{0}{0} = \lim\limits_{\substack{x\to 0\\ x>0}} \dfrac{e^x}{2x} = +\infty$.

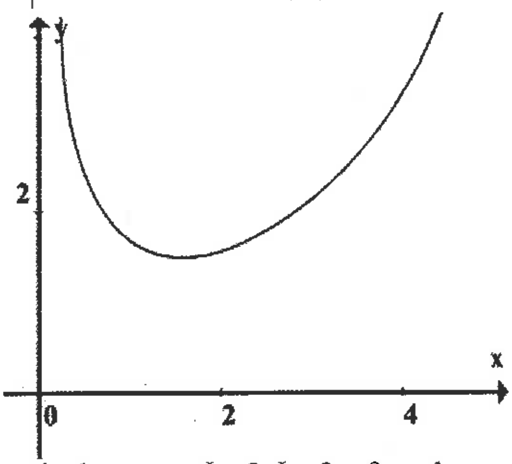
$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \dfrac{e^x-1}{x^2} = \dfrac{+\infty}{+\infty} = \lim\limits_{x\to+\infty} \dfrac{e^x}{2x} = \lim\limits_{x\to+\infty} \dfrac{e^x}{2} = +\infty$

2) $f'(x) = \dfrac{e^x\left(x^2\right) - 2x\left(e^x-1\right)}{x^4} = \dfrac{e^x\left(x^2-2x\right)+2x}{x^4}$

$f'(x) = \dfrac{x\left(e^x\left(x-2\right)+2\right)}{x^4} = \dfrac{g\left(x\right)}{x^3}$ and since $x>0$ then $f'(x)>0$ for $g(x)>0$ that is for $x>\alpha$ and $f'(x)<0$ for $g(x)<0$ that is for $x<\alpha$ .Therefore the table of variations of f :is

x	0		α		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖	$\searrow$	$f(\alpha)$	$\nearrow$	

3) $f(\alpha) \approx 1.55$



Part C.

1) $\mathrm{g(x)} = 0$ is equivalent to $xe^x - 2e^x + 2 = 0$ and to $xe^x = 2e^x - 2$ then $x = \left(2e^x - 2\right)e^{-x} = 2\left(1 - e^{-x}\right)$.

2) $h'(x) = 2e^{-x}$, if $\dfrac{3}{2} < x < 2$ then $-2 < -x < -\dfrac{3}{2}$

consequently $e^{-\frac{3}{2}} < e^{-x} < e^{-2}$ then $2e^{-\frac{3}{2}} < 2e^{-x} < 2e^{-2}$ so

$0.27 < 2e^{-x} < 0.44$ Consequently, $|h'(x)| \leq \frac{1}{2}$.

3) a- Applying the inequality of the Mean Value Theorem of h over $[\alpha; u_n]$, we get : $|h(u_n) - h(\alpha)| \leq \frac{1}{2}|u_n - \alpha|$

and since $u_{n+1} = h(u_n)$ and $h(\alpha) = \alpha$ then we get

$|u_{n+1} - \alpha| \leq \frac{1}{2}|u_n - \alpha|$.

b- $|u_2 - \alpha| \leq \frac{1}{2}|u_1 - \alpha|$

$|u_3 - \alpha| \leq \frac{1}{2}|u_2 - \alpha|$

.

$|u_n - \alpha| \leq \frac{1}{2}|u_{n-1} - \alpha|$

Multiplying these inequalities by a positive number, we get :

$|u_n - \alpha| \leq \left(\frac{1}{2}\right)^{n-1} |u_1 - \alpha|$.

But, $-2 < -\alpha < -\frac{3}{2}$ and $u_1 = \frac{3}{2}$ then $-\frac{1}{2} < u_1 - \alpha < 0$ which

gives $|u_1 - \alpha| < \frac{1}{2}$ and consequently $|u_n - \alpha| \leq \left(\frac{1}{2}\right)^n$.

c- $\lim\limits_{n\to+\infty} |u_n - \alpha| = 0$, then $\lim\limits_{n\to+\infty} u_n = \alpha$.

N° 19 ***For the students of the GS section***

Part A .

1) $f_1(x) = \frac{2e^x}{1+e^x} - 1$.

$\lim\limits_{x\to+\infty} f_1(x) = \lim\limits_{x\to+\infty} \frac{2}{e^{-x}+1} - 1 = 2 - 1 = 1$; $\lim\limits_{x\to-\infty} f_1(x) = -1$.

2) $f_1'(x) = \frac{2e^x}{(1+e^x)^2}$ then the table of variations of f_1 is:

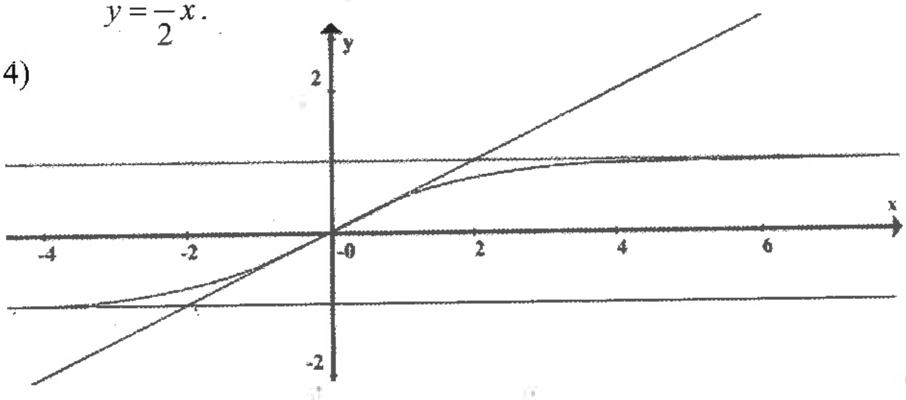
x	$-\infty$		$+\infty$
$f_1'(x)$		$+$	
$f_1(x)$	-1	$\nearrow$	1

3) a- $f_1''(x) = \dfrac{2e^x(1-e^x)}{(1+e^x)^3}$; $f_1''(x) = 0$ for $x = 0$.

$f_1''(x) < 0$ for $x > 0$ and $f_1''(x) > 0$ for $x < 0$ and since $f_1(0) = 0$ then O is a point of inflection of (C_1).

b- $f_1'(0) = \dfrac{1}{2}$ then an equation of the tangent (d) at O to (C_1) is $y = \dfrac{1}{2}x$.

4)



Part B .

1) $f_0(x) = \dfrac{2}{1+e^x} - 1$; $f_1(-x) = \dfrac{2e^{-x}}{1+e^{-x}} - 1 = \dfrac{2}{e^x+1} - 1 = f_0(x)$,

Then the curve (C_0) is symmetric to (C_1) with respect to $y'y$.

2) $f_1(x) + f_0(x) = \dfrac{2e^x}{1+e^x} - 1 + \dfrac{2}{1+e^x} - 1 = \dfrac{2(e^x+1)}{1+e^x} - 2 = 0$,

then (C_0) is symmetric to (C_1) with respect to the $x-$axis.

3) The required area is double the domain limited by (C_1), the axis of abscissas and the straight lines of equations $x = 0$ and $x = 1$.

$$\int_0^1 f_1(x)dx = \int_0^1 \left(\frac{2e^x}{1+e^x} - 1\right).dx = \left[2\ln(1+e^x) - x\right]_0^1 = 2\ln\frac{1+e}{2} - 1$$

$$A = 2\int_0^1 f_1(x).dx\ u^2 = \left(4\ln\frac{1+e}{2}-2\right)u^2 = \left(16\ln\frac{1+e}{2}-8\right)cm^2$$

Part C .

1) $u_{n+1}+u_n = \int_0^1 \left(\frac{2e^{(n+1)x}}{e^x+1}+\frac{2e^{nx}}{e^x+1}-2\right)dx = \int_0^1 \left[\frac{2e^{nx}(e^x+1)}{e^x+1}-2\right]dx$

$$= \int_0^1 [2e^{nx}-2].dx = \left[\frac{2}{n}e^{nx}-2x\right]_0^1 = 2\frac{e^n-n-1}{n}.$$

2) $\lim\limits_{n\to+\infty}(u_{n+1}+u_n) = \lim\limits_{n\to+\infty}\left(2\frac{e^n}{n}-2-\frac{2}{n}\right) = +\infty$.

If (u_n) converges to a real number ℓ then $(u_{n+1}+u_n)$ converges to 2ℓ ; which is impossible.

N°20. ***For the students of the GS section:***

Part A .

1) $g'(x) = e^x - 2$, then the table of variations of g is.

x	$-\infty$		$\ln 2$		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$		$\searrow$	$1-2\ln 2$	$\nearrow$	

2) $g(0) = 0$ and for $x > \ln 2$, g is continuous and strictly increasing from $1-2\ln 2 < 0$ to $+\infty$ then its curve cuts the axis $x'x$ at a unique point then the equation admits in this interval a unique root, hence the equation $g(x) = 0$ has two roots 0 and α.

$g(1) = e-3 < 0$ and $g(2) = e^2-5 > 0$ then $1 < \alpha < 2$.

3) For $x < 0$ or $x > \alpha$, $g(x) > 0$, for $0 < x < \alpha$, $g(x) < 0$.

Part B .

1) a- $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left[(2x+3)e^{-x}+x-1\right] = +\infty\times 0+\infty$

But $\lim\limits_{x\to+\infty}(2x+3)e^{-x} = \lim\limits_{x\to+\infty}\frac{2x+3}{e^x} = \lim\limits_{x\to+\infty}\frac{2}{e^x} = 0$ then

$\lim\limits_{x\to+\infty} f(x) = +\infty$

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}\left[(2x+3)e^{-x}+x-1\right] = -\infty-\infty=-\infty$.

b- $\lim\limits_{x\to+\infty}[f(x)-y] = \lim\limits_{x\to+\infty}(2x+3)e^{-x}=0$ then (d) is an asymptote à (C).

c- $f(x)-y=(2x+3)e^{-x}$.

For $2x+3>0$ that is $x>-\dfrac{3}{2}$ (C) is above (d).

for $2x+3<0$ that is $x<-\dfrac{3}{2}$ (C) is below (d).

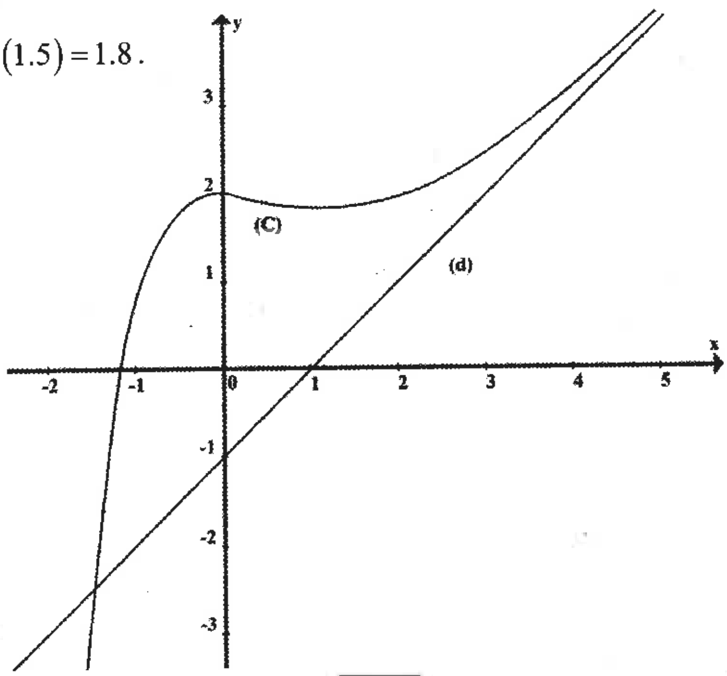
(C) and (d) intersect at point $I\left(-\dfrac{3}{2};-\dfrac{5}{2}\right)$.

2) $f'(x)=2e^{-x}-(2x+3)e^{-x}+1=1-e^{-x}(1+2x)=e^{-x}(e^{x}-2x-1)$

Then, $f'(x)=g(x)e^{-x}$ therefore the table of variations of f :

x	$-\infty$		0		α		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	2	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

3) $f(1.5)=1.8$.



4) $S=\int_0^1\left[f(x)-y\right]dx=\int_0^1(2x+3)e^{-x}\,dx$.

Letting $\begin{cases}u=2x+3\\ v'=e^{-x}\end{cases}$ we get $\begin{cases}u'=2\\ v=-e^{-x}\end{cases}$, therefore:

$$S=\left[-(2x+3)e^{-x}\right]_0^1+2\int_0^1 e^{-x}\,dx=\left[-(2x+3)e^{-x}\right]_0^1-2\left[e^{-x}\right]_0^1$$

$S=-5e^{-1}+3-2e^{-1}+2=5-7e^{-1}$ square units.

5) The equation $(2x+3)e^{-x}+x-1+m=0$ gives $-m=(2x+3)e^{-x}+x-1$. Graphically, for $f(\alpha)<-m<2$, then the straight line of equation $y=-m$ cuts (C) in three points then $-2<m<-f(\alpha)$,

The equation admits three distinct roots.

Part C.

1) (C) cuts $x'x$ in one point , then the equation $f(x)=0$ has only one root β.

$f(-1.2)=-0.2$ and $f(-1.1)=0.3$ then $-1.2<\beta<-1.1$.

h is defined for $f(x)>0$ then for $x>\beta$, $D_f=]\beta;+\infty[$.

2) $\lim\limits_{x\to\beta}h(x)=\lim\limits_{x\to\beta}\left[\ln[f(x)]\right]=-\infty$

$\lim\limits_{x\to+\infty}h(x)=\lim\limits_{x\to+\infty}\left[\ln[f(x)]\right]=+\infty$

3) $h'(x)=\dfrac{f'(x)}{f(x)}$, then h and f vary in the same sense, then the table of variations of f:

x	β		0		α		$+\infty$
$h'(x)$		$+$	0	$-$	0	$+$	
$h(x)$	$-\infty$	$\nearrow$	$\ln 2$	$\searrow$	$h(\alpha)$	$\nearrow$	$+\infty$

N°21. ***For the students of the GS section***

Part A .

1) a- $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} x^2(1-\frac{3}{x}+\frac{1}{x^2})e^x = +\infty$.

$\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} \frac{x^2-3x+1}{e^{-x}} = \lim\limits_{x\to-\infty} \frac{2x-3}{-e^{-x}} = \lim\limits_{x\to-\infty} \frac{2}{e^{-x}} = 0$.

b- The line of equation $y=0$ is an asymptote to (C) at $-\infty$.

2) a- $f'(x) = (2x-3)e^x + (x^2-3x+1)e^x = (x^2-x-2)e^x$.

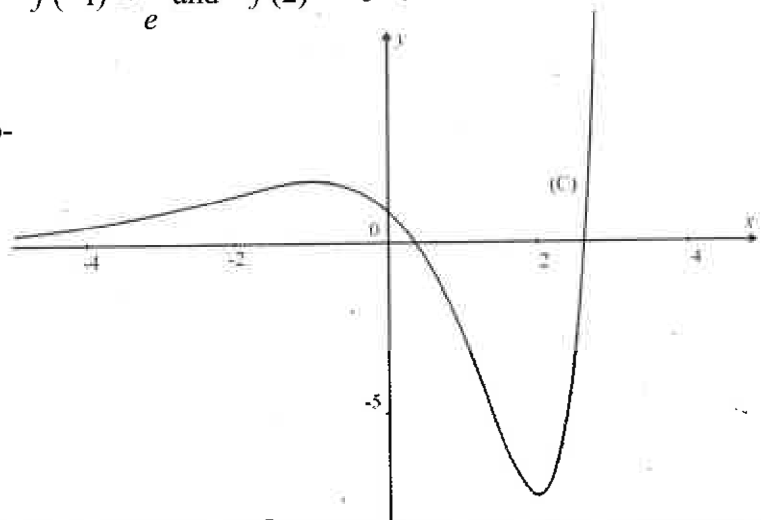
$f'(x) \geq 0$ for $x^2-x-2 \geq 0$, which gives $x \leq -1$ or $x \geq 2$.

Then, the table of variations of f is :

x	$-\infty$		-1		2		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	0	$\nearrow$	$5/e$	$\searrow$	$-e^2$	$\nearrow$	$+\infty$

$f(-1) = \frac{5}{e}$ and $f(2) = -e^2$.

b-



3) In the interval $]-\infty;0]$ (C) is above $x'x$, then I represents the area limited by (C) the axis $x'x$ and the two straight lines of equations $x=-3$ et $x=0$.
We can use tabular integration also.

Derivative		Antiderivative
x^2-3x+1	$+$	e^x
$2x-3$	$-$	e^x
2	$+$	e^x
0		e^x

$I=(x^2-3x+1)e^x-(2x-3)e^x+2e^x\,\big|_{-3}^{0}=6-30e^{-3}$ square units.

Part B.

1) $g\left(\frac{3}{2}\right)=e^{-\frac{5}{4}}$ and $g'\left(\frac{3}{2}\right)=0$

$g'(x)=(2x+a)e^{x^2+ax+b}$, then $\begin{cases} e^{\frac{9}{4}+\frac{3}{2}a+b}=e^{-\frac{5}{4}} \\ (3+a)e^{\frac{9}{4}+\frac{3}{2}a+b}=0 \end{cases}$

which gives $\frac{9}{4}+\frac{3}{2}a+b=-\frac{5}{4}$ and $3+a=0$ and consequently $a=-3$ and $b=1$.

2) $h(3-x)=e^{(3-x)^2-3(3-x)+1}=e^{9+x^2-6x-9+3x+1}=e^{x^2-3x+1}=h(x)$.

Hence, the straight line (D_1) of equation: $x=\frac{3}{2}$ is an axis of symmetry of (Γ).

Part C.

$v(x)=e^{f(x)}$, then $v'(x)=f'(x)e^{f(x)}$, so $v'(x)$ and $f'(x)$ have the same sign then $v(x)$ and $f(x)$ have the same sense of variations then the table of variations of v is:

x	$-\infty$		-1		2		$+\infty$
$v'(x)$		$+$	0	$-$	0	$+$	
$v(x)$	1	$\nearrow$	$e^{e^{\frac{5}{e}}}$	$\searrow$	e^{-e^2}	$\nearrow$	$+\infty$

$\lim\limits_{x\to-\infty} v(x)=e^0=1$ and $\lim\limits_{x\to+\infty} v(x)=e^{+\infty}=+\infty$.

Indications

N° 1.

7) a- $g(x)=\ln\left(e^{-x}-1\right)^2$, $e^{-x}-1\neq 0$ gives $x\neq 0$.

b- $\lim\limits_{x\to-\infty}[g(x)+2x]=\lim\limits_{x\to-\infty}\left[\ln\left(e^{-2x}-2e^{-x}+1\right)+\ln e^{2x}\right]=$

$\lim\limits_{x\to-\infty}\left[\ln e^{2x}\left(e^{-2x}-2e^{-x}+1\right)\right]=\lim\limits_{x\to-\infty}\left[\ln\left(1-2e^{x}+e^{2x}\right)\right]=0$

c- $g(x)=-2x$, $\ln\left(e^{-2x}-2e^{-x}+1\right)=\ln e^{-2x}$.

N° 2.

6) b- $A(\alpha)=1-\dfrac{\alpha^2}{2}-2\alpha-e^{-\alpha}$ but $f(\alpha)=0$ then $\alpha+2-e^{-\alpha}=0$

which gives $e^{-\alpha}=\alpha+2$ then $A(\alpha)=\left(-\dfrac{\alpha^2}{2}-3\alpha-1\right)$ u^2

N° 3.

***Part B*.**

2) $g(x)-x=\ln\left(e^{x}-e^{\frac{x}{2}}\right)-\ln e^{x}=\ln\left(\dfrac{e^{x}-e^{\frac{x}{2}}}{e^{x}}\right)=\ln\left(1-e^{-\frac{x}{2}}\right)$.

5) $g^{-1}(x)=2\ln\left(\dfrac{1}{2}+\sqrt{e^{x}+\dfrac{1}{4}}\right)$.

6) (Γ) and (C) do not intersect

N° 4.

***Part A*.**

3) d- $y=\dfrac{e^{2x}-1}{e^{2x}+1}$, then $ye^{2x}+y=e^{2x}-1$, which gives

$e^{2x}(y-1)=-y-1$ and $e^{2x}=\dfrac{y+1}{1-y}$, then :

$2x=\ln\left(\dfrac{y+1}{1-y}\right)$ and $x=\dfrac{1}{2}\ln\left(\dfrac{y+1}{1-y}\right)=\ln\sqrt{\dfrac{y+1}{1-y}}$.

N° 6.

A- 2) a- $f'(x) \geq 1-\dfrac{1}{e^2}$ then $f'(x) > 0$.

3) $f'(x)=1$; $(1-x)e^{-x}=0$; $x=1$; $\mathrm{A}(1,\ 1+1/\mathrm{e})$

B-

2) $0 \leq x \leq 1$; $0 \leq x^n \leq 1$; $-1 \leq -x \leq 0$; $\mathrm{e}^{-1} \leq \mathrm{e}^{-x} \leq 1$ then
$0 \leq x^n e^{-x} \leq 1$

$0 \leq \int_0^1 x^n e^{-x} dx \leq \int_0^1 1.dx$; $0 \leq u_n \leq 1$, then (u_n) is bounded.

3) $u_{n+1}-u_n=\int_0^1 (x^{n+1}e^{-x}-x^n e^{-x})dx=\int_0^1 x^n e^{-x}(x-1)dx$; but $x-1 \leq 0$ over $[0 ; 1]$ then $x^n e^{-x}(x-1) \leq 0$ over $[0 ; 1]$ and $u_{n+1}-u_n \leq 0$

N° 10.

***Part A*.**

3) $f(x)>0$ for $x>-\dfrac{1}{2}$ and $f(x)<0$ for $x<-\dfrac{1}{2}$.

***Part B*.**

2)

x	$-\infty$		$\frac{1}{2}$		$+\infty$
$g''(x)$		$-$	0	$+$	
$g'(x)$		$\searrow$	0	$\nearrow$	

3) $g'(x)>0$.

4) $g\left(\dfrac{1}{2}\right)=0$.

$g(x)>0$ for $x>\dfrac{1}{2}$ then $f(x)>y_T$ and (C) is above (T).

$g(x)<0$ for $x<\dfrac{1}{2}$ then $f(x)<y_T$ and (C) is below (T).

CHAPTER 6

Differential Equations

Chapter Review

1) ***Definition***:
A diffential equation of order n , is a relation between the variable x , the function y, and its derivatives y', y'', y''' ....., $y^{(n)}$.

N.B. To solve a differential equation is to find the function y that satisfies this equation.

2) **Types of Differential Equations**

Types of Differential Equations	Form	General Solution
Simple	$y'=f(x)$ Example : $y'=2x$	$y=\int f(x)dx$ $y=\int 2xdx=x^2+k$
Separable	$g(y)y'=f(x)$ Example: $y'-2xy=0$	$\int g(y)dy=\int f(x)dx$ $\int \frac{y'}{y}dy=\int 2xdx$ $\ln\lvert y\rvert=x^2+k$, $y=ce^{x^2}$
Reduced Linear	$y'+a\,y=0$ Example: $y'+2y=0$	$y=ce^{-ax}$ $y=ce^{-2x}$
Complete Linear	$y'+a\,y=f(x)$ Example: $y'+2y=2x+1$	$y=ce^{-ax}+Y$ Y is a particular solution $y=ce^{-2x}+x$

3) **Particular Solution of the equation** $y'+a\,y = f(x)$.

Form of $f(x)$	Form of Y
A polynomial of degree n Example : $y'+2y = 4x$	A polynomial of degree n $Y = ax+b$ $Y = 2x-1$
$f(x) = ke^{\alpha x}$ $(\alpha \neq -a)$ $y'+\frac{3}{2}y = \frac{1}{2}e^{-2x}$	$Y = \lambda\, e^{\alpha x}$ $Y = -e^{-2x}$
$f(x) = ke^{-a x}$ Example : $y'+2y = e^{-2x}$	$Y = \lambda x e^{-a x}$ $Y = xe^{-2x}$
$f(x) = k_1\sin(\alpha x) + k_2\cos(\alpha x)$ $y'-y = 5\sin 2x$	$Y = c_1\sin(\alpha x) + c_2\cos(\alpha x)$ $Y = -\sin(2x) - 2\cos(2x)$

N.B :

1) If Y is a particular solution of the equation $y'+a\,y = f(x)$ and y is the general solution of the equation $y'+a\,y = 0$, then the general solution of $y'+a\,y = f(x)$, is $y+Y$.
2) The general solution of the differential equation $y'+\varphi(x)\,y = 0$ is $y = Ce^{-\int\varphi(x)dx}$.

   Example :

   The solution of the equation $y'+xy = 0$ is $y = Ce^{-\int xdx} = Ce^{-\frac{1}{2}x^2}$

4) **Second order Differential Equations .**

Equation	General Solution
$y''+w^2y=k$	$y=c_1\cos wx+c_2\sin wx+\dfrac{k}{w^2}$
$y''=f(x)$	$y'=\int f(x)dx+c_1$ and $y=\int y'dx+c_2$ $*$Example : $y''=6x$ $y'=\int 6x\,dx=3x^2+c_1$ $y=\int(3x^2+c_1)dx=x^3+c_1x+c_2$
$ay''+by'+cy=0$ Characteristic Equation associated with this equation is $ar^2+br+c=0$	• If $\Delta=b^2-4ac>0$, then the characteristic equation admits two roots r_1 and r_2. The general solution is: $y=c_1e^{r_1x}+c_2e^{r_2x}$. $*$Example: $y''-3y'+2y=0$. The general solution is: $y=c_1e^{x}+c_2e^{2x}$. • If $\Delta=b^2-4ac=0$, then the characteristic equation has a double root $r=-\dfrac{b}{2a}$. The general solution is: $y=(c_1x+c_2)e^{rx}$. $*$Example : $y''-2y'+y=0$. The general solution is : $y=(c_1x+c_2)e^{x}$ • If $\Delta=b^2-4ac<0$, then the characteristic equation admits two complex roots: $r_1=\alpha+i\beta$ and $r_2=\alpha-i\beta$. The general solution is: $y=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$. $*$Example : $y''-2y'+5y=0$. $r_1=1+2i$ et $r_2=1-2i$. The general solution is: $y=e^{x}(c_1\cos 2x+c_2\sin 2x)$.

Solved Problems

N°1.

Determine a first order differential equation whose general solution is $y(x)$ in each of the following cases :

1) $y(x)=Ce^{x}+2x-5$. (C is a constant)
2) $y(x)=\ln x+Cx+4$. (C is a constant)
3) $y(x)=Cxe^{-x}+1$. (C is a constant)

N°2.

Determine a second order differential equation whose general solution is $y(x)$ in each of the following cases :

1) $y(x)=Ae^{-x}+Be^{2x}$. (A and B are two constants)
2) $y(x)=C_1e^{-x}+C_2e^{2x}+3x-1$. (C_1 and C_2 are two constants)
3) $y(x)=(C_1x+C_2)e^{2x}+x-2$. (C_1 and C_2 are two constants)
4) $y(x)=C_1e^{x}+C_2x+4$. (C_1 and C_2 are two constants)

N°3.

Solve each of the following differential equations:

1) $y'=\ln x$ $\quad x>0$
2) $y'=xe^{-x}$ $\quad x\in IR$
3) $y'=\dfrac{1}{x\ln x}$ $\quad 0<x<1$
4) $y'=\dfrac{1}{x(x+1)}$ $\quad x>0$

N°4.

Solve each of the following differential equations :

1) $y'+y=2e^{x}$
2) $y'+y=2x+1$
3) $y'+2y=2\cos x$
4) $y''+4y=0$

N°5.

Solve each of the following differential equations:

1) $x^2y'+y=0$
2) $x+yy'=2$
3) $(1+x)y'e^{y}=1$ with $x>-1$
4) $(1+x^2)y'-2xy=0$

N° 6.

Solve each of the following differential equations:

1) $y''-4y'+3y=0$ 2) $y''+4y'+5y=0$ 3) $y''-4y'+4y=0$

N° 7.

Consider the differential equation $(E): y'+2y=2x^2+1$.

Let $y=z+x^2-x+1$.

1) Form a differential equation (F) satisfied by z.
2) Solve (F) and deduce the general solution of (E).

N° 8.

Consider the differential equation $(E): y'+y=e^{-x}\ln x$.

Suppose that $y=ze^{-x}$.$(x>0)$

1) Form the differential equation (F) satisfied by z .
2) Solve (F) and deduce the general solution of (E).
3) Determine, from the solutions of (E), the one verifying $y(1)=\frac{1}{e}$.

N° 9.

Given the differential equation $(E): y'-(\tan x)y=\cos x$.

1) Find the particular solution of the equation $y'-(\tan x)y=0$, verifying $y(0)=1$.
2) Suppose $y=\dfrac{z}{\cos x}$.
   a- Form the differential equation (F) satisfied by z.
   b- Solve (F) and deduce the general solution of (E).

N° 10.

Consider the differential equation $(E):(x-1)y''-xy'+y=0$, $x\neq 1$.

1) Show that $y'''=y''$.
2) Deduce that $y''=Ce^x$.
3) Determine the general solution of (E).

N° 11.

Consider the differential equation $(E): y''+4y=3\cos x$.

1) Determine a and b so that $Y=a\cos x+b\sin x$ is solution of (E).

2) a- Solve the equation $y''+4y=0$ and deduce the general solution of (E).

   b- Find, among the solutions of (E), the one that satisfies the following conditions $y(0)=0$ and $y'\left(\frac{\pi}{2}\right)=0$.

N° 12.

Consider the differential equation $(E): xy'+y+\frac{1}{x}=0$ with $x>0$ and let $z=xy$.

1) Form the differential equation (F) satisfied by z.
2) Solve (F) and deduce the general solution of (E).

N° 13.

Part A .

Consider the differential equation $(E): y''+2y'+y=-2e^{-x}+1$ and let $z=y+x^2e^{-x}-1$.

1) Determiner a differential equation (F) satisfied by z.
2) Solve the equation (F) and deduce the general solution of (E).
3) Let f be the function defined over IR by $f(x)=(-x^2+ax+b)e^{-x}+1$ with $f'(0)=0$.
   a- Show that $a=b$.
   b- Suppose that $a\neq-2$, show that the function $f(x)=(-x^2+ax+a)e^{-x}+1$ admits two extrema one whose abscissa is 0 and the other M whose abscissa is $a+2$.
   c- Determine the set of points M as a varies.
   d- determine a and b such that $f'(0)=0$ and $f(0)=1$.

Part B .

Consider the function f defined over IR by $f(x)=-x^2e^{-x}+1$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ and deduce an asymptote to (C).
2) a- Calculate $f'(x)$ and set up the table of variations of f.

b- Deduce that the equation $x^2 = e^x$ has a unique solution α such that $-0.8 < \alpha < -0.7$.

3) Draw (C).

N° 14.

Consider the differential equation $(E): y' - y = 2xe^x$.

1) Let $y = ze^x$.
   a- Form the differential equation (F) satisfied by z.
   b- Solve (F) and deduce the particular solution f of (E) verifying $y(0) = 1$.

2) Let g be the function defined over IR by $g(x) = \left(x^2+1\right)e^x$.
   Designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.
   a- Determine $\lim\limits_{x \to +\infty} g(x)$ and $\lim\limits_{x \to -\infty} g(x)$.
   b- Deduce an asymptote to (C).
   c- Study the variations of g and set up its table of variations.
   d- Show that g has two inflection points .
   e- Draw (C).

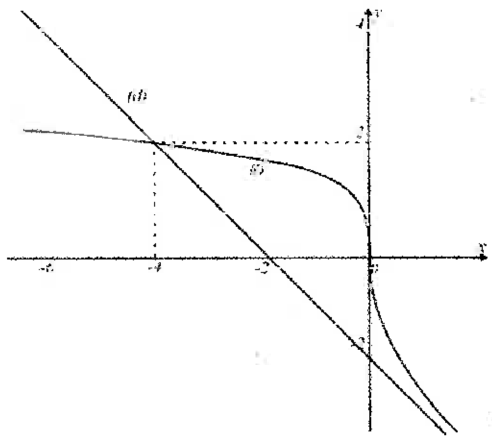
3) a- Calculate the real numbers a , b and c so that $G(x) = \left(ax^2+bx+c\right)e^x$ is an antiderivative of $g(x)$.
   b- Calculate the area of the region limited by (C), the axis $x'x$, the axis $y'y$ and the straight line of equation $x = 1$.

4) a- Show that g has an inverse function g^{-1} over IR and draw its representative curve (C').
   b- Is g^{-1} differentiable at the point of abscissa $x = 2e^{-1}$?

N° 15.

The plane is referred to an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

A- The curve (ℓ) below represents a function h defined over IR.



1) Prove that h admits, over IR, an inverse function g .
2) (γ) is the representative curve of g :
   a- Determine the tangent to (γ)at the point O and deduce $g'(0)$.
   b- Prove that (d) is an asymptote to $(\gamma)$and determine the point of intersection of (γ) and (d).
   c- Draw (γ) in another system.
3) Let g be the function defined, over IR, by $g(x)=(ax+b)(1+e^x)+c$,with a , b and c being real numbers .
   a- Calculate $g'(x)$.
   b- Using the values of $g(0)$, $g'(0)$ and $g(2)$, calculate a , b and c and verify that $g(x)=(2-x)e^x-x-2$.

B- Consider the differential equation (E): $(1+e^x)y'-y=0$.

1) Noting that $\frac{1}{1+e^x}=\frac{e^{-x}}{1+e^{-x}}$, calculate $\int\frac{dx}{1+e^x}$.
2) Solve the differential equation (E) and deduce the particular solution of (E) whose representative curve passes through the point $I(0;2)$.

N° 16.

Part A.

Consider the differential equation (E) : $y''+2y'+y=x+2$.

Let $z=y-x$.

1) Write a differential equation (E') satisfied by z.

2) Solve (E') and deduce the general solution of (E).
3) Determine the particular solution f of (E) verifying $f(0)=1$ and $f'(0)=1$.

Part B.

Consider the function f defined over IR by $f(x)=(x+1)e^{-x}+x$

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Calculate $\lim\limits_{x\to+\infty} f(x)$ and show that the straight line (d) of equation $y=x$ is an asymptote to (C).
   b- Study the position of (C) with respect to (d).
   c- Calculate $\lim\limits_{x\to-\infty} f(x)$.
2) a- Study the variations of f' and deduce that the function f is strictly increasing over IR.
   b- Set up the table of variations of f.
3) Determine the point of (C) where the tangent (T) is parallel to (d).
4) Calculate $f(-2)$ and draw $(d),(T)$ and (C).
5) a- Show that f admits an inverse function f^{-1} over IR.
   b- Draw the representative curve of f^{-1}.
6) Let (δ) be the straight line of equation $y=x+m$ where m is a real parameter. Study according to the values of m the number of points of intersection of (C) and (δ).
7) Calculate the area of the region limited by $(C),(d)$ and the straight lines of equations $x=0$ and $x=1$.

N° 17.

Part A

Consider the differential equation $(E): y'-3y=\dfrac{-3e}{(1+e^{-3x})^2}$

and suppose $z=ye^{3x}$.

1) Show that $z'-3z=y'e^{3x}$.
2) Knowing that z is a solution of (E) and that $z(0)=\dfrac{e}{2}$. Determine the particular solution of (E).

Part B

f is the function defined over IR by $f(x)=\dfrac{e^{1-3x}}{1+e^{-3x}}$.

Denote by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$ and deduce the equations of the asymptotes to (C) .
2) Study the variations of f and draw its table of variations.
3) Calculate $f(-x)+f(x)$ and deduce the center of symmetry of (C).
4) a- Calculate the area A_α of the region limited by (C), $x'x$ and the straight lines of equations $x=0$ and $x=\alpha$ $(\alpha>0)$.
   b- Calculate $\lim\limits_{\alpha\to+\infty} A_\alpha$.
5) a- Show that f admits an inverse function f^{-1} over IR .
   b- Determine an equation of the tangent (T) to (C'), the representative curve of f^{-1}, at the point A of abscissa $\dfrac{1}{2}e$.
   c- Let g be the function defined over $]0;e[$ by $g(x)=\dfrac{1}{3}\ln\left(\dfrac{e-x}{x}\right)$, calculate $g\circ f(x)$ and deduce the expression of $f^{-1}(x)$.

N° 18.

Part A .

Consider the differential equation $(E): y''-2y'+y=-x+1$.

Suppose $y=z-x-1$.

1) Form the differential equation (F) satisfied by z .
2) Solve (F) and deduce the solution f of (E) verifying $f(0)=-1$ and $f'(0)=0$.

Part B .

f is a function defined over IR by $f(x)=xe^x-x-1$.

Denote by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
2) Show that the straight line (d) of equation $y=-x-1$ is an asymptote to (C) as $x\to-\infty$.
3) Study the relative positions of (C) and (d).
4) The table below is the table of variations of the function f' the derivative of f .

x	$-\infty$		-2		0		$+\infty$
$f''(x)$		$-$	0	$+$			
$f'(x)$	-1	$\searrow$	$-1-e^{-2}$	$\nearrow$	0	$\nearrow$	$+\infty$

a- Draw the table of variations of f .

b- Show that (C) admits a point of inflection I .

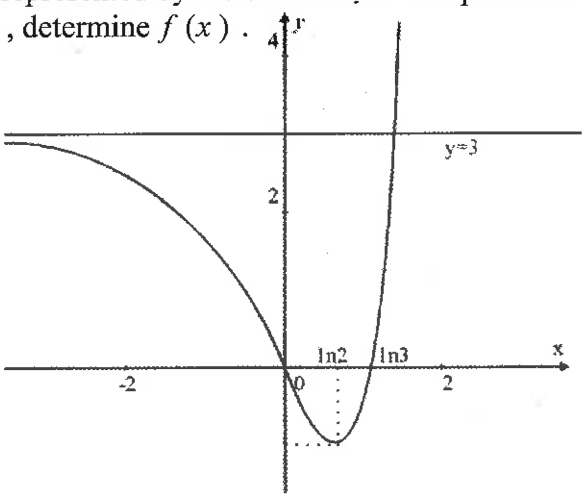
c- Trace (C) and (d).

d- Calculate the area of the region bounded by (C), (d) and the two straight lines of equations $x=-1$ and $x=0$.

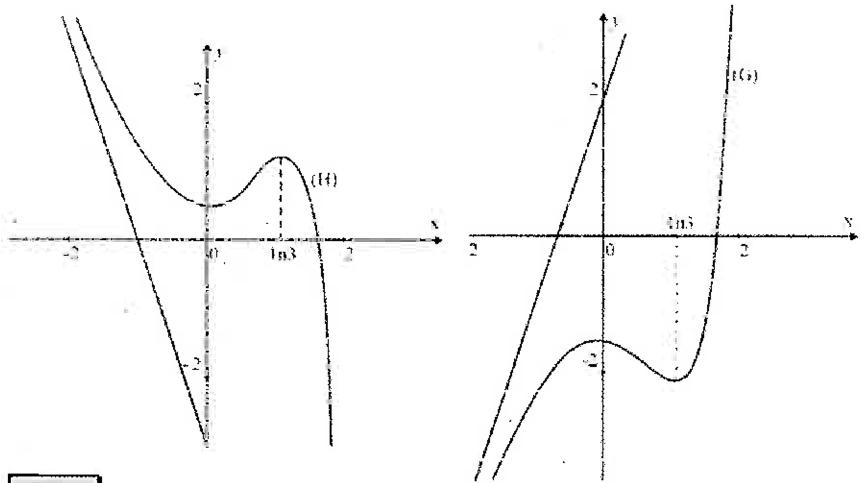
N° 19.

Given the differential equation $(E): y''-3y'+2y=2k$ where k is a real number and suppose that $y=z+k$.

1) Form the differential equation (F) satisfied by z .
2) Solve (F) and deduce the general solution of (E).
3) The function f represented by the curve below is a particular solution of (E), determine $f(x)$.



4) One of the two curves given below is the representative curve of the primitive of f .Indicate which one and justify your answer.



$\boxed{N°20.}$ ***For the students of the GS section***

Part A.

Consider the differential equation (E) : $y' + 2y^2e^x - y = 0$ where y is a function defined over IR such that , for all real numbers x ,
$y(x) \neq 0$ .Let $z = \dfrac{1}{y}$ and $u = z - e^x$ where z is a differentiable function defined over IR

1) Determine the differential equation (E') satisfied by z.
2) Solve $(E')$and deduce the general solution of (E).
3) Determine the particular solution of (E) verifying $y(0) = \dfrac{1}{2}$.

Part B.

Consider the function f defined over IR by $f(x) = \dfrac{1}{e^x + e^{-x}}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Graphical unit : 4 cm .

1) Show that f is an even function.
2) Calculate the limits of f at the boundaries of its domain of définition.
3) Calculate $f'(x)$ and set up the table of variations of f.
4) a- Set up the table of variations of the function g defined over

$[0;+\infty[$ by $g(x)=f(x)-x$.

b- Deduce that the equation $f(x)=x$ admits over $[0;+\infty[$ a unique solution α. Verify that $0.4<\alpha<0.5$.

c- Draw (C).

5) a- Show that the restriction of f over $[0;+\infty[$ admits an inverse function f^{-1}.

b- Determine the domain of definition of f^{-1} and find the expression of $f^{-1}(x)$ in terms of x.

c- Draw the curve (γ) of f^{-1} in the same system as that of (C).

Part C.

Consider the function h defined by $h(x)=\ln[f(x)]$.

1) Justify that the domain of definition of h is IR.

2) a- Verify that $h(x)+x=-\ln(1+e^{-2x})$ and deduce that the straight line (d) of equation $y=-x$ is an asymptote to the representative curve (H) of the function h in the neighborhood of $+\infty$.

b- Show that h is an even function and deduce an asymptote (d') to (H) in the neighborhood of $-\infty$.

c- Study the variations of h and draw (H).

Part D.

Consider the sequence (v_n) defined over IN by $v_n=\int_0^n f(x)dx$.

1) a- Show that, for all $x\geq 0$, $f(x)<e^{-x}$.

b- Deduce that, for all natural numbers n, $v_n\leq 1-e^{-n}$.

2) a- Verify that $v_{n+1}-v_n=\int_n^{n+1} f(x)dx$.

b- Deduce that the sequence (v_n) is strictly increasing.

c- Show that the sequence (v_n) converges towards a limit ℓ such that $0\leq\ell<1$.

3) Verify that $f(x)=\dfrac{e^x}{1+e^{2x}}$. Calculate, then, v_n in terms of n and determine ℓ.

4) Calculate, in cm^2, the area of the region bounded by (γ), $y'y$, $x'x$ and the straight line of equation $y = 2$.

N°21. ***For the students of the GS section***

Part A.

Consider the differential equation $(E): x^2y' - xy + y^2 = 0$ with $x > 0$.

Let $z = \dfrac{x}{y}$.

1) Determine a differential equation (F) satisfied by z.
2) Solve the equation (F) and deduce the general solution of (E).
3) Determine the particular solution of (E) verifying $y(1) = 1$.

Part B.

Consider the function f defined over $\left]0;\dfrac{1}{e}\right[\cup \left]\dfrac{1}{e};+\infty\right[$ by

$f(x) = \dfrac{x}{1+\ln x}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$; (Graphical unit : 2 cm).

1) a- Calculate the limits of $f(x)$ at the boundaries of its domain of definition.
   b- Calculate $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x}$ and interpret the result graphically.
2) Calculate $f'(x)$ and set up the table of variations of f.
3) a- Prove that the curve (C) admits an inflection point I.
   b- Write an equation of the tangent (d) to (C) at the point I.
4) Study, according to the values of x, the position of (C) and the straight line (D) of equation $y = x$.
5) Draw (d), (D) and (C).

Part C.

Consider the interval $I = [1;e]$.

1) a- Prove that $f(I)$ is included in I.
   b- Study the sign of $f'(x) - \dfrac{1}{4}$ and deduce that, for all x of I, $0 \le f'(x) \le \dfrac{1}{4}$.

c- Prove that, for all x of I that $|f(x)-1| \leq \frac{1}{4}|x-1|$.

2) Let (u_n)be the sequence defined by :
$u_0 = 2$ and for all $n \geq 0, u_{n+1} = f(u_n)$.
a- Prove, by mathematical induction over n, that u_n belongs to I.
b- Prove that $|u_{n+1} - 1| \leq \frac{1}{4}|u_n - 1|$.
c- Prove that $|u_n - 1| \leq \frac{1}{4^n}$ and deduce $\lim\limits_{n \to +\infty} u_n$.

Part D .

Consider the function g defined over $]0;+\infty[$ by $g(x) = \frac{1}{f(x)}$.

1) Study the variations of g and set up its table of variations .

2) Draw the curve (G) representative of g in an orthonormal system $(O;\vec{i},\vec{j})$.

3) Let (v_n) be the sequence defined by $v_n = \int_{e^n}^{e^{n+1}} g(x)dx$.
a- Calculate v_n in terms of n .
b- Show that the sequence (v_n) is an arithmetic sequence whose first term and common difference are to be determined.
c- Deduce the area a_n of the region bounded by (G), the axis $x'x$ and the straight lines of equations $x = 1$ and $x = e^n$.

N°22. *For the students of the GS section.*

Part A .

Consider the differential equation (E): $yy' - 2xy' - 2y = 0$.
Let $y = z + 2x$
1) Form the differential equation (E') satisfied by z.
2) Solve $(E')$and deduce the general solution of (E) .
3) Find the particular solution of (E) whose representative curve, In an orthonormal system $(O;\vec{i},\vec{j})$,passes through the point $(0;1)$.

Part B .

Consider the function *f* defined over *IR* by $f(x) = 2x + \sqrt{4x^2 + 1}$.
(C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim_{x\to+\infty} f(x)$ and show that the straight line (d) of equation $y = 4x$ is an asymptote to (C) in the neighborhood of $+\infty$.
2) Calculate $\lim_{x\to-\infty} f(x)$.
3) Prove that *f* is strictly increasing over IR .
4) Draw (C) .
5) a- Show that *f* admits an inverse function f^{-1}.
   b- Determine $f^{-1}(x)$.
   c- Calculate $(f^{-1})'(1)$ in two different ways.
   d- Draw the curve (C') of f^{-1} in the same system.

Supplementary Problems

N° 1.

Consider the differential equation $(E): y' - 2y = 2x + 1$.

1) Determine a and b so that $Y = ax + b$ is a solution of (E).
2) a- Solve the equation $y' - 2y = 0$.
   b- Deduce the particular solution of (E) verifying $y(0) = 0$.

N° 2.

Consider the D.E $(E): 2y' + y = 5e^{2x}$. Let $y = z + e^{2x}$.

1) Form a differential equation (F) satisfied by z.
2) Solve (F) and deduce the particular solution f of (E) verifying $f(0) = 0$.

N° 3.

Consider the D.E (E) : $y'' + 2y' + y = x^2 + 2x - 2$. Let $z = y - x^2 + 2x$.

1) Form the differential equation (F) satisfied by z.
2) Solve (F) and deduce the general solution of (E).
3) Find the particular solution $f(x)$ of (E) whose representative curve admits at the origin O a tangent parallel to $x'x$.

N° 4.

Given the differential equation (E) : $y'' - 7y' + 12y = e^{3x}$.

1) Determine the real number a if $y_0 = a\,x\,e^{3x}$ is a solution of (E).
2) Suppose that $y = z + y_0$ and use the value of a found above.
   a- Form the differential equation (F) satisfied by z.
   b- Solve (F) and deduce the general solution of (E).
   c- Determine a particular solution of (E) verifying $f(0) = -1$ and $f'(0) = -1$.

N° 5.

Consider the differential equation $(E): y'' - 2y' = 0$.

1) Determine the general solution of (E).
2) Determine the particular solution f of (E) whose

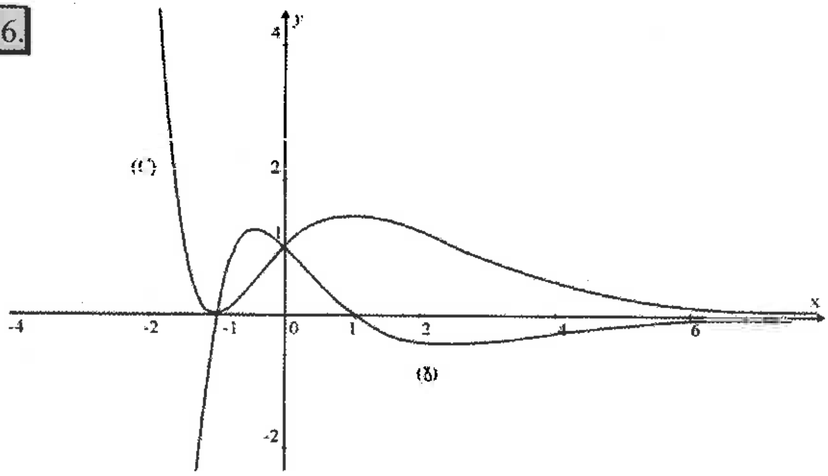
representative curve(C) passes through the point $(0;1)$ and has the line of equation $y=2$ as an asymptote at $-\infty$.

3) Designate by (C') the representative curve of g defined by $g(x)=2f(x)$.

   a- Determine the coordinates of the point of intersection of (C) and (C').

   b- Calculate the area of the region bounded by (C), (C') and the axis of ordinates.

N°6.



The curves (C) and (γ), represented above in an orthonormal system $(O;\vec{i},\vec{j})$, are defined and differentiable over IR.
One of **these** functions is the derivative of the other.
Denote them by (g) and (g').

Part A

1) Identify the graphs of the two functions g and g'
2) Determine the slope of the tangent to (C) at the point of abscissa 0.

Part B

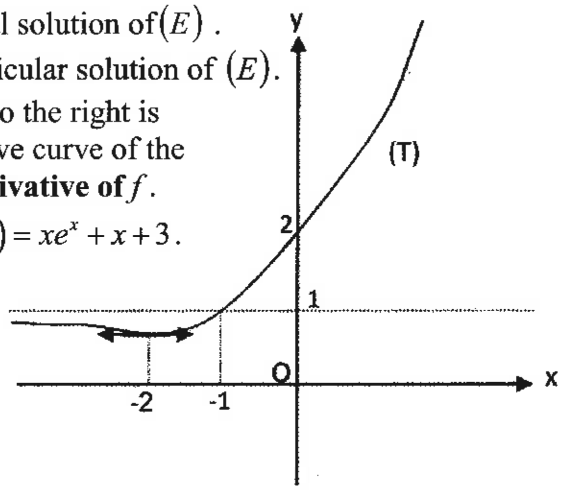
Given the differential equation $(E): y'+y=2(x+1)e^{-x}$.

1) Show that the function defined over IR by $f_0(x)=(x^2+2x)e^{-x}$ is a particular solution of (E).
2) Solve the differential equation $(E'): y'+y=0$.
3) Deduce the general solution of (E).

N°7.

Consider the D.E (E) : $y''-2y'+y=x+1$. Let $y=z+x+3$.

1) Find the general solution of (E) .
2) Let f be a particular solution of (E).
   The curve (T) to the right is the representative curve of the function f' **derivative of** f.
   Show that $f(x)=xe^{x}+x+3$.



N°8.

Part A .

Consider the D.E (E): $y''+4y'+4y=8x-4$ .Let $z=y-2x+3$.

1) Find the general solution of (E).
2) Determine the particular solution of (E) , whose representative curve (C) in an orthonormal system $(O;\vec{i},\vec{j})$, admits at the point $A(0;-3)$ a tangent parallel to $x'x$.

Part B .

Let f be the function defined over IR by $f(x)=-2x\,e^{-2x}+2x-3$.
Designate by (Γ) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1) Determine $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
2) Show that the straight line (d) of equation $y=2x-3$ is an asymptote to (Γ) at $+\infty$.
3) Study the relative positions of (Γ) and (d).
4) Study the variations of f' and the variations of f .
5) Trace (Γ) and (d).

N°9.

Consider the differential equation (E): $y''-y'+1=0$.

1) a- Determine the general solution h of the differential equation (E'): $y''-y'=0$.

b- Find a particular solution g of (E) of the form $g(x)=\lambda x$.

c- Show that $h+g$ is the general solution of (E).

2) Consider the differential equation (F): $z''+z'+e^{-x}=0$.

a- Put $z=ve^{-x}$ where v is a twicely differentiable function of x, show that v satisfies the differential equation (E). Deduce the general solution of (F).

b- Verify that the particular solution of (F) verifying

$z(0)=1$ and $z'(0)=1$ is $z(x)=\dfrac{x+e^x}{e^x}$.

3) Study the variations and trace, in an orthonormal system $(O;\vec{i},\vec{j})$, the curve representing the function f defined over IR by $f(x)=\dfrac{x+e^x}{e^x}$.

$\boxed{N^\circ 10.}$ ***For the students of the G.S.section***

Consider the differential equation $(E): y'+2y=2x-3$.

Let $y=z+x-2$.

1) Form a differential equation (F) satisfied by z.
2) Solve (F) and deduce the general solution of (E).
3) Consider the family of curves (C_m) of equation

$y=f_m(x)=me^{-2x}+x-2$.

a- Show that the tangent (T_m) to (C_m) at point A of (C_m) of abscissa 0 passes through a fixed point as m varies.

b- Determine m so that f_m is strictly increasing over IR.

c- In the case where f_m admit a minimum S, determine the set of points S as m varies.

d- Calculate m so that the tangent at A to (C_m) is parallel to the straight line of equation $y=3x$.

4) Suppose that $m=-1$ and consider the function $f_{-1}(x)=-e^{-2x}+x-2$. Designate by (C_{-1}) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

a- Calculate $\lim_{x\to+\infty} f_{-1}(x)$, $\lim_{x\to-\infty} f_{-1}(x)$ and $\lim_{x\to-\infty} \frac{f_{-1}(x)}{x}$.

b- Show that (C_{-1}) admits an oblique asymptote in the neighborhood of $+\infty$.

c- Study the variations of f_{-1} and draw (C_{-1}).

d- Show that f_{-1} admits a inverse function over IR and draw the representative curve (γ) of f_{-1} in the same system .

$N°11.$

g is the function g defined over IR by $g(x)=\frac{e^x}{1+2e^x}-\ln(1+2e^x)$.

1) Show that g is strictly decreasing over IR.

2) Determine $\lim_{x\to+\infty} g(x)$ and $\lim_{x\to-\infty} g(x)$.

3) Set up the table of variations of g and deduce the sign of g.

Part B.

Consider the function f defined over IR by $f(x)=e^{-2x}\ln(1+2e^x)$
(C) is the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $f'(x)=2e^{-2x}g(x)$.

2) Calculate $\lim_{x\to-\infty} f(x)$, $\lim_{x\to+\infty} f(x)$.

3) Set up the table of variations of f and draw (C).

Part C.

Consider the D.E (E) : $y'+2y=2\frac{e^{-x}}{1+2e^x}$. Let $y=z+f(x)$

Verify that the function f is a solution of (E) and find the general solution of (E).

Solutions

N°1.

1) $y = Ce^x + 2x - 5$ gives $y' = Ce^x + 2$ then
$y' - y = -2x + 7$. Thus, $y(x)$ is a solution of the differential equation $y' - y = -2x + 7$.

2) $y = \ln x + Cx + 4$ gives $y' = \frac{1}{x} + C$, so $C = y' - \frac{1}{x}$ and consequently
$y = \ln x + \left(y' - \frac{1}{x}\right)x + 4$. Thus, $y(x)$ is a solution of the differential equation $xy' - y = -3 - \ln x$.

3) $y = Cxe^{-x} + 1$ gives $y' = C\left(e^{-x} - xe^{-x}\right)$, so $C = \frac{y'}{e^{-x}(1-x)}$ and consequently $y = \frac{y'}{e^{-x}(1-x)}xe^{-x} + 1$ so $y = \frac{xy'}{1-x} + 1$. Thus, $y(x)$ is a solution of the differential equation $xy' - (1-x)y = x - 1$.

N°2.

1) $y(x)$ is of the form $y(x) = Ae^{r_1x} + Be^{r_2x}$ with $r_1 = -1$ and $r_2 = 2$, then $y(x)$ is a solution of the equation $y'' - y' - 2y = 0$.

2) $y(x) = C_1e^{-x} + C_2e^{2x} + 3x - 1$ is the sum of two functions $y_1 = C_1e^{-x} + C_2e^{2x}$ and $y_2 = 3x - 1$.
y_1 is the general solution of the equation $y'' - y' - 2y = 0$.
y_2 is a particular solution of the equation $y'' - y' - 2y = ax + b$.
But, $y_2 = 3x - 1$ gives $y_2' = 3$ and $y_2'' = 0$ which gives
$-6x - 1 = ax + b$ then $a = -6$ and $b = -1$ consequently y_2 is a particular solution of the differential equation $y'' - y' - 2y = -6x - 1$.
Hence, $y(x)$ is a solution of the equation $y'' - y' - 2y = -6x - 1$.

3) $y(x) = (C_1x + C_2)e^{2x} + x - 2$ is the sum of the two functions $y_1(x) = (C_1x + C_2)e^{2x}$ and $y_2(x) = x - 2$.
y_1 is the general solution of $y'' - 4y' + 4y = 0$.

y_2 is the particular solution of $y'' - 4y' + 4y = ax + b$.

But, $y_2(x) = x - 2$ gives $y_2' = 1$ and $y_2'' = 0$ which gives $-4 + 4(x-2) = ax + b$ then $a = 4$ and $b = -12$ consequently, y_2 is a particular solution of the equation $y'' - 4y' + 4y = 4x - 12$.

Hence, $y(x)$ is a solution of the equation $y'' - 4y' + 4y = 4x - 12$.

4) $y(x) = C_1e^x + C_2x + 4$ gives $y' = C_1e^x + C_2$ and $y'' = C_1e^x$.

Therefore, $y = y'' + (y' - y'')x + 4$ so $y(x)$ is a solution of the equation $(1-x)y'' + xy' - y + 4 = 0$.

N°3.

1) $y' = \ln x$ gives $y = \int \ln x dx = x\ln x - x + C$ $\qquad x > 0$

2) $y' = xe^{-x}$ gives $y = \int xe^{-x} dx$

Letting $\begin{cases} u = x \\ v' = e^{-x} \end{cases}$ we get $\begin{cases} u' = 1 \\ v = -e^{-x} \end{cases}$, then

$y = \int xe^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C = e^{-x}(-1-x) + C$.

3) $y' = \dfrac{1}{x\ln x}$ gives $y = \int \dfrac{1}{x\ln x}dx = \ln(-\ln x) + C$

4) $y' = \dfrac{1}{x(x+1)}$ gives $y = \int \dfrac{1}{x(x+1)}dx$.

But, $\dfrac{1}{x(x+1)}$ can be written in the form $\dfrac{1}{x(x+1)} = \dfrac{1}{x} - \dfrac{1}{x+1}$, then

$$y = \int \left[\frac{1}{x} - \frac{1}{(x+1)}\right]dx = \ln x - \ln(x+1) + C.$$

N°4.

1) $y' + y = 2e^x$ is of the form $y' + a\,y = f(x)$ with $f(x) = ke^{\alpha\, x}$ $(\alpha \neq -a)$, is a particular solution of the equation $Y = \lambda\, e^{\,x}$.

$Y = \lambda\, e^{\,x}$ gives $Y' = \lambda\, e^{\,x}$ then, $\lambda\, e^{\,x} + \lambda\, e^{\,x} = 2e^x$ which gives $\lambda = 1$, so, $Y = e^{\,x}$.

Since the solution of the equation $y'+y=0$ is $y=Ce^{-x}$ then the general solution of the equation is $y=Ce^{-x}+e^{x}$.

2) $y'+y=2x+1$ is of the form $y'+a\,y=f(x)$ where $f(x)=ax+b$, a particular solution of this equation is $Y=ax+b$. $Y=ax+b$ then $Y'=a$ then, $a+ax+b=2x+1$ which gives $a=2$ and $b=-1$ hence, $Y=2x-1$.

Since the solution of the equation $y'+y=0$ is $y=Ce^{-x}$ then the general solution of the equation is $y=Ce^{-x}+2x-1$.

3) A particular solution of the equation is $Y=a\cos x+b\sin x$. $Y=a\cos x+b\sin x$ gives $Y'=-a\sin x+b\cos x$ then, $-a\sin x+b\cos x+2\left(a\cos x+b\sin x\right)=2\cos x$ so

$\cos x\left(2a+b\right)+\left(-a+2b\right)\sin x+=2\cos x$ which gives $2a+b=2$ and $-a+2b=0$. Therefore, $b=\dfrac{2}{5}$ and $a=\dfrac{4}{5}$ then $Y=\dfrac{4}{5}\cos x+\dfrac{2}{5}\sin x$

Since the solution of the equation $y'+2y=0$ is $y=Ce^{-2x}$ then the general solution of the given equation is

$y=Ce^{-x}+\dfrac{4}{5}\cos x+\dfrac{2}{5}\sin x$.

4) $y''+4y=0$ is of the form $y''+\omega^2 y=0$ with $\omega^2=4$ then $\omega=\pm 2$. Taking $\omega=2$, the solution is $y=C_1\cos 2x+C_2\sin 2x$.

$\boxed{N^\circ 5.}$

1) $x^2y'+y=0$ gives $\dfrac{y'}{y}=-\dfrac{1}{x^2}$, then $\int\dfrac{y'}{y}dx=-\int\dfrac{1}{x^2}dx$ which gives

$\ln\dfrac{y}{C}=\dfrac{1}{x}$ consequently, $y=Ce^{\frac{1}{x}}$.

2) $x+yy'=2$ gives $yy'=-x+2$ then $\int yy'dx=\int(-x+2)dx$ which gives $\dfrac{1}{2}y^2=-\dfrac{1}{2}x^2+2x+C$ so $y^2=-x^2+4x+2C$. Consequently, $y=\pm\sqrt{-x^2+4x+2C}$.

3) $\left(1+x\right)y'e^{y}=1$ gives $\int y'e^{y}dx=\int\dfrac{1}{1+x}dx$ then $e^{y}=\ln\left(1+x\right)+C$ and consequently $y=\ln\left(\ln\left(1+x\right)+C\right)$.

4) $(1+x^2)y'-2xy=0$ gives $\frac{y'}{y}=\frac{2x}{1+x^2}$, then $\int\frac{y'}{y}dx=\int\frac{2x}{1+x^2}dx$

which gives $\ln\frac{y}{C}=\ln(1+x^2)$. Thus, $y=C(1+x^2)$.

N°6.

1) The characteristic equation associated with this differential equation is $r^2-4r+3=0$ that has $r_1=1$ and $r_2=3$ as roots. Thus, the solution of this equation is $y=C_1e^x+C_2e^{3x}$.

2) The characteristic equation associated with this differential equation is $r^2+4r+5=0$ that has $r_1=-2-i=\alpha-i\beta$ and $r_2=-2+i=\alpha+i\beta$ as roots, so $\alpha=-2$ and $\beta=1$ and consequently, the solution is $y=e^{-2x}(C_1\cos x+C_2\sin x)$.

3) The characteristic equation associated with this differential equation is $r^2-4r+4=0$ that has $r_1=r_2=2$ as roots and consequently, the solution is $y=(C_1x+C_2)e^{2x}$.

N°7.

1) $y=z+x^2-x+1$, then $y'=z'+2x-1$.
Replacing y and y' by their values in (E), we get:
$z'+2x-1+2z+2x^2-2x+2=2x^2+1$ which gives
$z'+2z=0 \quad (F)$.

2) The general solution of (F) is $z=ce^{-2x}$ and consequently, the general solution of (E) is $y=z+x^2-x+1=ce^{-2x}+x^2-x+1$.

N°8.

1) $y=ze^{-x}$, then $y'=z'e^{-x}-ze^{-x}$.
Replacing y and y' by their values in (E), we get:
$z'e^{-x}-ze^{-x}+ze^{-x}=e^{-x}\ln x$, which gives $z'=\ln x \quad (F)$.

2) $z'=\ln x$ then $z=\int\ln(x)\,dx$.
Integrating by parts.
Letting $u=\ln x$ and $v'=1$, then $u'=\frac{1}{x}$ and $v=x$.
We get then : $z=\int\ln(x)\,dx=x\ln(x)-\int dx=x\ln(x)-x+c$.

The general solution of (E) is $y = ze^{-x} = (x\ln x - x + c)\,e^{-x}$.

3) $y(1) = \frac{1}{e}$ gives $\frac{1}{e} = (-1+c)e^{-1}$ then $c = 2$ and consequently,

$y = (x\ \ln x - x + 2)e^{-x}$.

N°9.

1) $y' - (\tan x)y = 0$ gives $\frac{y'}{y} = \tan x$, then $\int \frac{y'}{y}dx = \int \tan x\, dx$

which gives : $\ln|y| = -\ln|\cos x| + c$, so $\ln|y\cos x| = c$

and consequently $y\cos x = \pm e^{c} = k$.

Therefore, $y = \frac{k}{\cos x}$, $y(0) = 1$ gives $1 = \frac{k}{1}$, so $k = 1$ and

consequently $y = \frac{1}{\cos x}$.

2) a- $y = \frac{z}{\cos x}$ gives $y' = \frac{z'\cos x + z\sin x}{\cos^2 x}$.

Replacing y and y' by their values in (E), we get:

$\frac{z'\cos x + z\sin x}{\cos^2 x} - \frac{z\sin x}{\cos^2 x} = \cos x$, which gives $z' = \cos^2 x$. (F)

b- $z' = \cos^2 x$ gives $z = \int \frac{1+\cos 2x}{2}dx = \frac{1}{2}(x + \frac{1}{2}\sin 2x) + c$.

Then, $y = \frac{z}{\cos x} = \frac{2x + \sin 2x + 4c}{4\cos x}$ so $y = \frac{2x + \sin 2x + k}{4\cos x}$.

N°10.

1) $(x-1)y'' - xy' + y = 0$ gives $(x-1)y'' = xy' - y$

Deriving both sides with respect to x , we get :

$y'' + (x-1)y''' = y' + xy'' - y'$ which gives $(x-1)y''' = (x-1)y''$ and since $x \neq 1$ we get $y''' = y''$.

2) Letting $y'' = z$, the equation $y''' = y''$ becomes $z' = z$ or $z' - z = 0$ that has $z = Ce^{x}$ as a solution, thus $y'' = Ce^{x}$.

3) $y'' = Ce^{x}$ gives $y' = Ce^{x} + C_1$ and consequently $y = Ce^{x} + C_1x + C_2$.

But, $(x-1)y'' - xy' + y = 0$ thus,

$(x-1)Ce^{x} - x(Ce^{x} + C_1) + Ce^{x} + C_1x + C_2 = 0$ which gives

$C_2 = 0$ and consequently, $y = Ce^x + C_1 x$.

N°11.

1) $Y = a\cos x + b\sin x$, then $Y' = -a\sin x + b\cos x$ and
$Y'' = -a\cos x - b\sin x$.
Replacing Y and Y'' by their values in (E), we get:
$-a\cos x - b\sin x + 4a\cos x + 4b\sin x = 3\cos x$,
then $3a = 3$ and $3b = 0$, which gives $a = 1$ and $b = 0$ and consequently $Y = \cos x$.

2) a- The general solution of the equation $y'' + 4y = 0$ is
$y_1 = c_1\cos 2x + c_2\sin 2x$.
The general solution of (E) is $y = y_1 + Y$, so
$y = c_1\cos 2x + c_2\sin 2x + \cos x$.

b- $y(0) = 0$ gives $0 = c_1 + 1$, then $c_1 = -1$.
$y' = -2c_1\sin 2x + 2c_2\cos 2x - \sin x$, and since
$y'\left(\frac{\pi}{2}\right) = 0$ we get $-2c_2 - 1 = 0$, so $c_2 = -\frac{1}{2}$.
Therefore, $y = -\cos 2x - \frac{1}{2}\sin 2x + \cos x$.

N°12.

1) $z = xy$ gives $y = \frac{z}{x}$ then $y' = \frac{z'x - z}{x^2}$.
(E) becomes $x\left(\frac{z'x - z}{x^2}\right) + \frac{z}{x} + \frac{1}{x} = 0$, so $xz' + 1 = 0$ (F).

2) $xz' + 1 = 0$ gives $z' = -\frac{1}{x}$ then $z = -\ln x + C$ and consequently,
$y = \frac{-\ln x + C}{x}$.

N°13.

Part A.

1) $z = y + x^2e^{-x} - 1$ gives $y = z - x^2e^{-x} + 1$ then
$y' = z' - 2xe^{-x} + x^2e^{-x} = z' + \left(x^2 - 2x\right)e^{-x}$ and
$y'' = z'' + \left(-x^2 + 4x - 2\right)e^{-x}$

Replacing y' and y'' in (E), we get : $z''+2z'+z=0$. (F)

2) The characteristic equation associated with this differential equation (F) is $r^2+2r+1=0$ that has as a solution $r'=r''=-1$, the solution of (F) is $z=(C_1x+C_2)e^{-x}$ and the general solution of (E) is $y=(C_1x+C_2)e^{-x}-x^2e^{-x}+1$.

3) a- $f(x)=(-x^2+ax+b)e^{-x}+1$ gives

$f'(x)=(-2x+a)e^{-x}-e^{-x}(-x^2+ax+b)$

$=(x^2-ax-2x+a-b)e^{-x}$ and since $f'(0)=0$

then $0=a-b$ and consequently $a=b$.

b- $f(x)=(-x^2+ax+a)e^{-x}+1$ gives

$f'(x)=(x^2-(a+2)x)e^{-x}$, $f'(x)=0$ for $x=0$ or for $x=a+2$, thus f' vanishes twice and changes sign . Thus, (C) admits two extrema of which one is 0 and the other, M , which has an abscissa $a+2$.

c- $M(a+2;f(a+2))$, $x=a+2$ gives $a=x-2$

thus, $y=(-x^2+ax+a)e^{-x}+1=(-x^2+(x-2)x+x-2)e^{-x}+1$,

so $y=(-x-2)e^{-x}+1$ .hence, M varies on the curve of equation $y=(-x-2)e^{-x}+1$

d- $f(0)=1$ gives $1=b+1$ then $b=0$.

$f'(0)=0$ gives $0=a-b$ and consequently, $a=0$ then $f(x)=-x^2e^{-x}+1$.

Part B .

1) $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}(-x^2e^{-x}+1)=-\infty\times 0+1$ inderterminate form.

But, $\lim\limits_{x\to+\infty} -x^2e^{-x}=\lim\limits_{x\to+\infty}\dfrac{-x^2}{e^x}=\lim\limits_{x\to+\infty}\dfrac{-2x}{e^x}=\lim\limits_{x\to+\infty}\dfrac{-2}{e^x}=0$ then $\lim\limits_{x\to+\infty} f(x)=1$.

$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}(-x^2e^{-x}+1)=-\infty$.

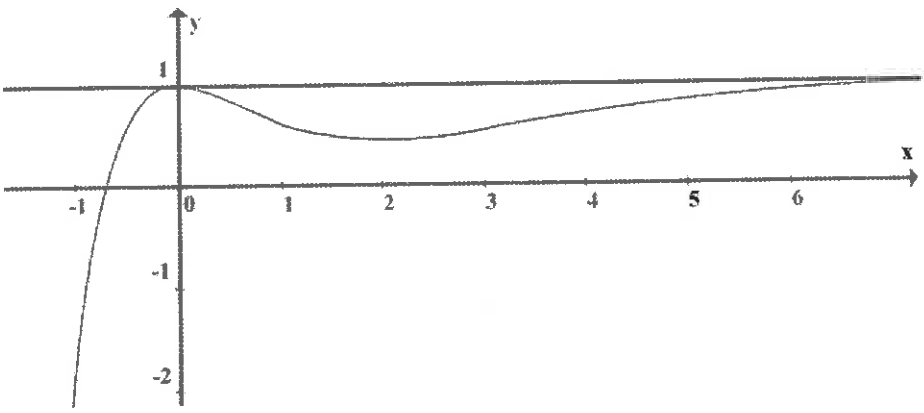
The straight line of equation $y=1$ is an asymptote to (C).

2) a- $f'(x)=(x^2-2x)e^{-x}$ thus the table of variations of f is :

x	$-\infty$		0		2		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	1	$\searrow$	$1-4e^{-2}$	$\nearrow$	1

b- $x^2 = e^x$ is equivalent to $-x^2e^{-x}+1=0$.
In the interval,]-0.8;-0.7[f is continuous and strictly increasing and since
$f(-0.8)\times f(-0.7)<0$ then the equation $x^2 = e^x$ admits a unique solution α such that $-0.8<\alpha<-0.7$.

3)



N° 14.

1) a- $y = ze^x$ gives $y' = z'e^x + ze^x$, the equation (E) becomes
$z'e^x + ze^x - ze^x = 2xe^x$ and the equation (F) is $z' = 2x$.

b- $z' = 2x$ gives $z = \int 2xdx = x^2 + k$ and consequently,
$y = \left(x^2+k\right)e^x$.
$f(0)=1$ gives $1=k$ then $f(x)=\left(x^2+1\right)e^x$.

2) a- $\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty} \left(x^2+1\right)e^x = +\infty$

$\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to+\infty} \left(x^2+1\right)e^x = +\infty\times 0$, indeterminate form.

$$\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty} \frac{x^2+1}{e^{-x}} = \lim_{x\to-\infty} \frac{2x}{-e^{-x}} = \lim_{x\to-\infty} \frac{2}{e^{-x}} = 0.$$

b- The axis $x'x$ is an asymptote to (C).

c- $g'(x) = 2xe^x + (x^2+1)e^x = (x^2+2x+1)e^x = (x+1)^2 e^x$.

x	$-\infty$		$+\infty$
$g'(x)$		$+$	
$g(x)$	0	$\nearrow$	$+\infty$

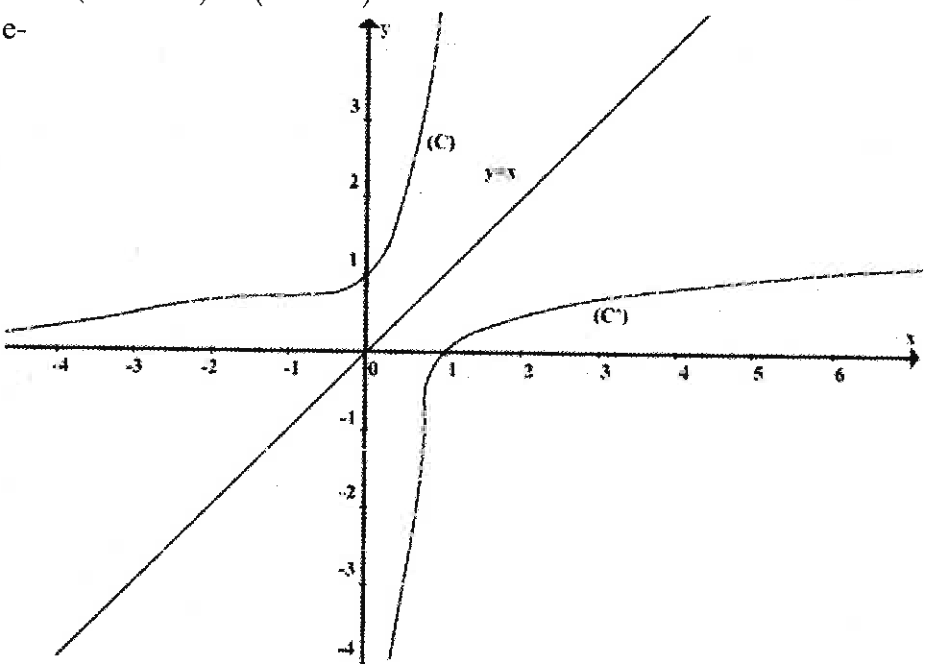
d- $g''(x) = 2(x+1)e^x + (x+1)^2 e^x = (x+1)(x+3)e^x$.

$g''(x)$ vanishes at the point of abscissa -1 and at the point of abscissa -3.

$g''(x) > 0$ for $x < -3$ or $x > -1$

$g''(x) < 0$ for $-3 < x < -1$, then g has two inflection points $I(-3;10e^{-3})$. $J(-1;2e^{-1})$

e-



3) a- $G'(x)=(2ax+b)e^x+(ax^2+bx+c)e^x$
$=(ax^2+(2a+b)x+b+c)e^x$
$G'(x)=g(x)$ for all real numbers $a=1$, $2a+b=0$ which gives $b=-2$ and $b+c=1$ so $c=3$.

b- $a=\int_0^1 g(x)dx=[G(x)]_0^1=[(x^2-2x+3)e^x]_0^1=2e-3\ u^2$.

4) a- g is continuous and strictly increasing over *IR* , then it admits an inverse function g^{-1}.
The representative curve of g^{-1} is symmetric to (C) with respect to the straight line of equation $y=x$, see figure.

b- At the point of abscissa -1, $g'(-1)=0$ then the tangent at this point to is horizontal, since $g(-1)=2e^{-1}$ consequently, the tangent to (C') at the point of abscissa $x=2e^{-1}$ is vertical.
Hence, g^{-1} is not differentiable at $x=2e^{-1}$.

$N°15.$

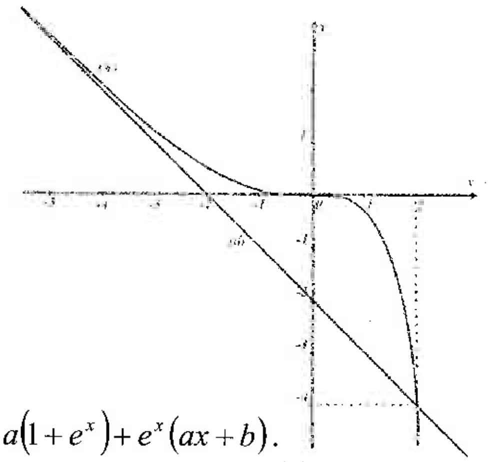
A-

1) *h* is continuous and strictly decreasing over *IR* then it admits, over *IR*, an inverse function g.
Since (ℓ) passes through *O* then g passes through O so $g(0)=0$.

2) a- The axis $y'y$ is tangent at *O* to (ℓ), so the curve (γ) is tangent at *O* to $x'x$, consequently $g'(0)=0$.

b- The straight line (d) is an asymptote to (ℓ), then (γ) admits as an oblique asymptote the straight line (d') the symmetric to (d) with respect to the straight line of equation $y=x$.
But, (d) passes through the point $(0;-2)$ so (d') passes through the point $(-2;0)$ .Also, (d) passes through the point $(-2;0)$ then (d') passes through the point $(0;-2)$ hence, (d') is itself (d) and consequently, (d) is an asymptote to (γ).

c-



3) a- $g'(x)=a(1+e^x)+e^x(ax+b)$.

b- $g(0)=0$ gives $2b+c=0$, $g'(0)=0$ $2a+b=0$

$h(-4)=2$ so $g(2)=-4$ which gives

$(2a+b)(1+e^2)+c=-4$, but $2a+b=0$ so $c=-4, 2b+c=0$

gives $b=2$, $2a+b=0$ gives $a=-1$ consequently,

$g(x)=(2-x)e^x-x-2$.

B-

1) $\int\frac{dx}{1+e^x}=\int\frac{e^{-x}}{1+e^{-x}}dx=-\ln(1+e^{-x})+k$.

2) $(E):(1+e^x)y'-y=0$ is equivalent to $\frac{y'}{y}=\frac{1}{1+e^x}$ so

$\int\frac{y'}{y}dx=\int\frac{1}{1+e^x}dx$, which gives $\ln|y|=-\ln(1+e^{-x})+k$, then

$\ln|y|+\ln(1+e^{-x})=k$ or $\ln|y(1+e^{-x})|=k$ so $y(1+e^{-x})=\pm e^k$, thus

$y=\pm\frac{e^k}{1+e^{-x}}$ and then $y=\frac{C}{1+e^{-x}}$. At the point $I(0;2)$,

$2=\frac{C}{1+1}$ which gives $C=4$ and consequently $y=\frac{4}{1+e^{-x}}=\frac{4e^x}{1+e^x}$.

$N°16$.

Part A.

1) $z=y-x$ gives $y=z+x$ then $y'=z'+1$ and $y''=z''$, the equation (E) becomes $z''+2(z'+1)+z+x=x+2$, the equation (E') is then $z''+2z'+z=0$.

2) The characteristic equation associated to (E') is $r^2+2r+1=0$
That has $r'=r''=-1$ as roots, a solution of (E') is
$z=(C_1x+C_2)e^{-x}$ and the general solution of (E) is
$y=(C_1x+C_2)e^{-x}+x$.

3) $f(0)=1$ gives $1=C_2$.
$f'(x)=C_1e^{-x}-e^{-x}(C_1x+C_2)+1$, $f'(0)=1$ gives $1=C_1-C_2+1$
Then, $C_1=1$ consequently, $f(x)=(x+1)e^{-x}+x$.

Part B.

1) a- $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\left[(x+1)e^{-x}+x\right]=+\infty\times 0+\infty$, indeterminate form.

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\left[\dfrac{x+1}{e^x}+x\right]=\lim\limits_{x\to+\infty}\left[\dfrac{1}{e^x}+x\right]=+\infty$.

$\lim\limits_{x\to+\infty}(f(x)-y)=\lim\limits_{x\to+\infty}(x+1)e^{-x}=\lim\limits_{x\to+\infty}\dfrac{x+1}{e^x}=\lim\limits_{x\to+\infty}\dfrac{1}{e^x}=0$.

Then, (d) is an asymptote to (C).

b- $f(x)-y=(x+1)e^{-x}$. For $x>-1$ (C) is above (d).
For $x<-1$ (C) is below (d).
(C) and (d) intersect at the point $I(-1;-1)$.

c- $\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\left[(x+1)e^{-x}+x\right]=-\infty$.

$\dfrac{f(x)}{x}=\left(1+\dfrac{1}{x}\right)e^{-x}+1$, then $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}=+\infty$.

2) a- $f'(x)=e^{-x}-(x+1)e^{-x}+1=-xe^{-x}+1$.
$f''(x)=-e^{-x}+xe^{-x}=e^{-x}(x-1)$, then the table of variations of f' is the following:

x	$-\infty$		1		$+\infty$
$f''(x)$		$-$	0	$+$	
$f'(x)$	$+\infty$	$\searrow$	$1-e^{-1}$	$\nearrow$	$+\infty$

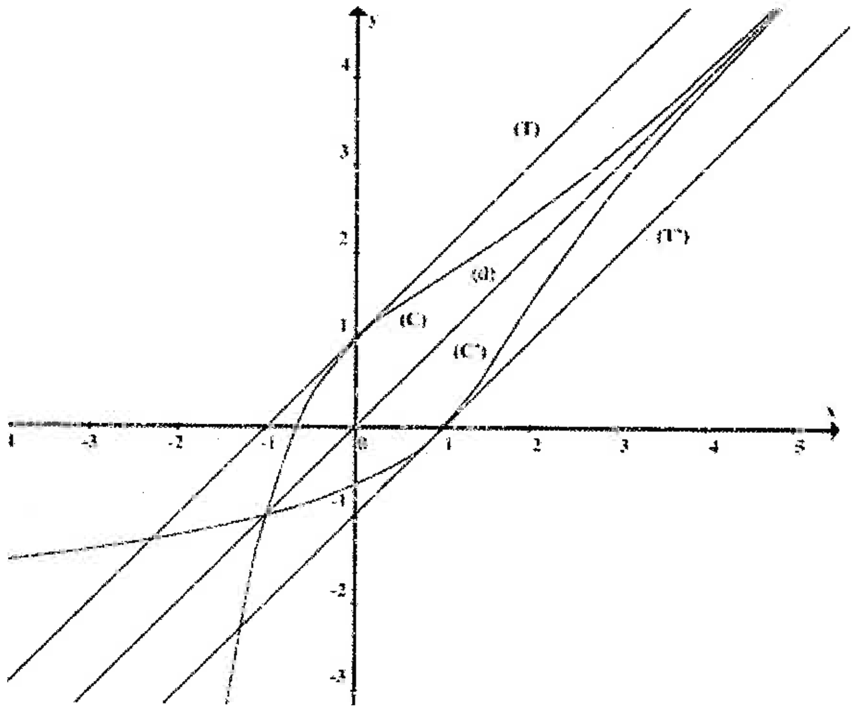
The minimum of f' is positive then $f'(x)>0$ and consequently f is strictly increasing over IR.

b-

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	$-\infty$	$\nearrow$	$+\infty$

3) (T) is parallel to (d) then they have the same slope $f'(x)=1$ which gives $-xe^{-x}+1=1$ then $-xe^{-x}=0$ consequently $x=0$ thus at point $I(0;1)$, (T) is parallel to (d).

4)



5) a- The function f is continuous and strictly increasing over IR, then it admits an inverse function f^{-1}.

b- The representative curve of f^{-1} is symmetric to (C) with respect to the straight line of equation $y=x$, see figure.

6) (δ) is parallel to (T) and (T) cuts $y'y$ at the point $(0;1)$.

Using (C), we find that :

If $m>1$, then (δ) and (C) do not intersect.

If $m=1$, then (δ) is tangent to (C).
If $0<m<1$, then (δ) cuts (C) in 2 points.
If $m\leq 0$, then (δ) cuts (C) in one point.

7) $S=\int_0^1[f(x)-x]dx=\int_0^1(x+1)e^{-x}dx$.

Using integration by parts, we get :
Let $u=x+1$ and $v'=e^{-x}$ then, $u'=1$ and $v=-e^{-x}$, so

$$S=-(x+1)e^{-x}\Big|_0^1+\int_0^1 e^{-x}dx=-(x+1)e^{-x}\Big|_0^1-\left[e^{-x}\right]_0^1$$

$S=-2e^{-1}+1-e^{-1}+1=2-3e^{-1}$ square units.

N° 17.

Part A

1) $z=ye^{3x}$ so $z'=y'e^{3x}+3ye^{3x}$.
then: $z'-3z=y'e^{3x}+3ye^{3x}-3ye^{3x}=y'e^{3x}$.

2) Since z is a solution of (E), we get:

$z'-3z=\dfrac{-3e}{(1+e^{-3x})^2}$, then $y'e^{3x}=\dfrac{-3e}{(1+e^{-3x})^2}$, and consequently

$y'=\dfrac{-3e\,e^{-3x}}{(1+e^{-3x})^2}$, which gives : $y=\int e\dfrac{-3e^{-3x}}{(1+e^{-3x})^2}dx=e\dfrac{-1}{1+e^{-3x}}+c$

But, $z(0)=\dfrac{e}{2}$, then $y(0)e^0=\dfrac{e}{2}$, so $y(0)=\dfrac{e}{2}$, which gives

$\dfrac{e}{2}=e\dfrac{-1}{1+e^0}+c$ and consequently $c=e$, then

$$y=\frac{-e}{1+e^{-3x}}+e=\frac{-e+e+e^{1-3x}}{1+e^{-3x}}=\frac{e^{1-3x}}{1+e^{-3x}}.$$

Part B

1) $\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to-\infty}\dfrac{e^{-3x}(e)}{e^{-3x}(e^{3x}+1)}=\lim\limits_{x\to-\infty}\dfrac{e}{e^{3x}+1}=e$

$\lim\limits_{x\to+\infty}f(x)=\dfrac{0}{1}=0$ so the two straight lines of equations $y=0$ and $y=e$ are asymptotes to (C).

2) $f'(x) = \dfrac{-3e^{1-3x}}{(1+e^{-3x})^2} < 0$ for all real numbers x ,then the table of variations of f is the following.

x	$-\infty$		$+\infty$
$f'(x)$		$-$	
$f(x)$	e	$\searrow$	0

3) $f(-x)+f(x) = \dfrac{e^{1+3x}}{1+e^{3x}} + \dfrac{e^{1-3x}}{1+e^{-3x}} = \dfrac{e^{1+3x}}{1+e^{3x}} + \dfrac{e^{1-3x}e^{3x}}{(1+e^{-3x})e^{3x}}$

$$= \frac{e^{1+3x}}{1+e^{3x}} + \frac{e}{e^{3x}+1} = \frac{ee^{3x}+e}{1+e^{3x}} = \frac{e(e^{3x}+1)}{1+e^{3x}} = e$$

Then, the point $I(0;\frac{1}{2}e)$ is a center of symmetry of (C).

4) a- $f(x) > 0$ for all real numbers , then :

$$A_\alpha = \int_0^\alpha f(x)\,dx = \int_0^\alpha \frac{e^{1-3x}}{1+e^{-3x}}\,dx = \int_0^\alpha e\frac{e^{-3x}}{1+e^{-3x}}\,dx = \frac{-e}{3}\int_0^\alpha \frac{-3e^{-3x}}{1+e^{-3x}}\,dx$$

$$A_\alpha = \left[\frac{-e}{3}\ln(1+e^{-3x})\right]_0^\alpha = \frac{-e}{3}\ln(1+e^{-3\alpha}) + \frac{e}{3}\ln(2) \text{ square units.}$$

b- $\lim\limits_{\alpha\to+\infty} A_\alpha = \dfrac{e}{3}\ln 2$.

5) a- The function f is continuous and strictly decreasing over IR, Then it admits an inverse function f^{-1} defined over $]0;e[$.

b- An equation of (T) is $y = f^{-1}\left(\frac{1}{2}e\right) + \left(x - \frac{1}{2}e\right)(f^{-1})'\left(\frac{1}{2}e\right)$.

$\frac{1}{2}e = f(0)$, then $f^{-1}\left(\frac{1}{2}e\right) = 0$ and

$$(f^{-1})'\left(\frac{1}{2}e\right) = \frac{1}{f'(0)} = \frac{1}{\frac{-3e}{4}} = -\frac{4}{3e}$$

An equation of (T) becomes: $y=\dfrac{-4}{3e}\left(x-\dfrac{1}{2}e\right)$.

c- $g\circ f(x)=g(f(x))=\dfrac{1}{3}\ln\left[\dfrac{e-f(x)}{f(x)}\right]$.

But, $\dfrac{e-f(x)}{f(x)}=\dfrac{e-\dfrac{e^{1-3x}}{1+e^{-3x}}}{\dfrac{e^{1-3x}}{1+e^{-3x}}}=e^{3x}$.

Therefore, $g\circ f(x)=\dfrac{1}{3}\ln(e^{3x})=\dfrac{1}{3}\times(3x)=x$, so

$f^{-1}(x)=g(x)=\dfrac{1}{3}\ln\left(\dfrac{e-x}{x}\right)$.

N° 18.

Part A

1) $y=z-x-1$, so $y'=z'-1$ and $y''=z''$.
Replacing y, y' and y'' in (E), we get the equation :
$(F): z''-2z'+z=0$.

2) The characteristic equation associated to (F) is $r^2-2r+1=0$,
That admits $r'=r''=1$ as double roots .
The general solution of (F) is then $z=e^x(c_1x+c_2)$.
The general solution of (E) is
$y=z-x-1=e^x(c_1x+c_2)-x-1$.
$f(0)=-1$ gives $-1=c_2-1$, then $c_2=0$.
$f'(x)=e^x(c_1x+c_2)+c_1e^x-1$, $f'(0)=0$ gives
$c_2+c_1-1=0$, so $c_1=1$, and consequently, $f(x)=xe^x-x-1$.

Part B

1) $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\left[x\left(e^x-1\right)-1\right]=+\infty$.
$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}(xe^x-x-1)=0+\infty-1=+\infty$.

2) $\lim\limits_{x\to-\infty}[f(x)-(-x-1)]=\lim\limits_{x\to-\infty} xe^x=0$, then (d) is an asymptote to (C) at $-\infty$.

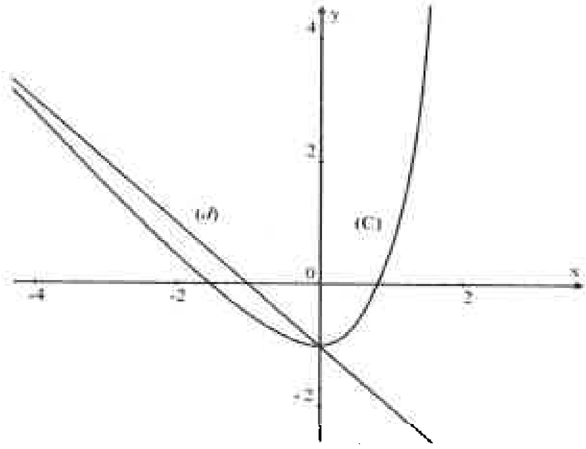
3) If $x>0$ then $f(x)>(-x-1)$, thus (C) is above (d).
If $x<0$ then $f(x)<(-x-1)$, thus (C) is below (d).
For $x=0$, (C) and (d) intersect at point $A(0;-1)$.

4) a- From the table of variations of f', we note that :
$f'(0)=0$, $f'(x)<0$ for $x<0$ and $f'(x)>0$ for $x>0$.
Then, the table of variations of f is the following:

x	$-\infty$		0		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	-1	$\nearrow$	$+\infty$

b- From the table of variations of f', we note that:
$f''(x)=0$ for $x=-2$ and changing sign from negative to positive, then the point $I(-2;1-2e^{-2})$ is an inflection point for f.

c-



d- $A=\int_{-1}^{0}\left[(-x-1)-(xe^{x}-x-1)\right]dx=\int_{-1}^{0}-x\,e^{x}dx$.

Letting $u=-x$ and $v'=e^{x}$, then $u'=-1$ and $v=e^{x}$, thus :
$A=-xe^{x}\Big|_{-1}^{0}+e^{x}\Big|_{-1}^{0}=1-2e^{-1}$ square units.

N° 19.

1) $y=z+k$, then $y'=z'$ and $y''=z''$, replacing y, y' and y'' In (E), we get (F): $z''-3z'+2z=0$.

2) The characteristic equation associated to (F) is $r^2 - 3r + 2 = 0$,
That has $r' = 1$ and $r'' = 2$ as roots.
The general solution of (F) is $z = c_1e^x + c_2e^{2x}$.
The general solution of (E) is $y = z + k = c_1e^x + c_2e^{2x} + k$.

3) The curve (C) gives the following information:
The straight line (d) of equation $y = 3$ is an asymptote to (C) at $-\infty$.
The curve (C) passes through the origin O.
A t the point of abscissa $\ln 2$, (C) admits a minimum.
Therefore:
$\lim_{x \to -\infty} f(x) = 3$ and since $\lim_{x \to -\infty} f(x) = k$, we get $k = 3$.
$f(0) = 0$ gives $0 = c_1 + c_2 + k$.
$f'(x) = c_1e^x + 2c_2e^{2x}$, therefore, $0 = f'(\ln 2) = c_1e^{\ln 2} + 2c_2e^{2\ln 2}$,
so $0 = 2c_1 + 8c_2$.
Then, we get the following system : $\begin{cases} c_1 + c_2 = -3 \\ c_1 + 4c_2 = 0 \end{cases}$, that has as a solution $c_1 = -4$ and $c_2 = 1$, and consequently
$f(x) = e^{2x} - 4e^x + 3$.

4) If F is an antiderivative of f then $F'(x) = f(x)$.
Hence, if $f(x) > 0$ then F is increasing and if $f(x) < 0$ then F is decreasing.
From the curve (C), we note that :
For $x < 0$, $f(x) > 0$, then the function F is increasing.
For $0 < x < \ln 3$, $f(x) < 0$ then the function F is decreasing.
For $x > \ln 3$, $f(x) > 0$ then the function F is increasing.
If F is the function represented by the curve (G), we note that:
F is increasing for $x < 0$, F is decreasing for $0 < x < \ln 3$, F is increasing for $x > \ln 3$, then the function F represented by the curve (G) is an antiderivative of f.

N°20.

Part A.

1) $z = \dfrac{1}{y}$ gives $y = \dfrac{1}{z}$, then $y' = -\dfrac{z'}{z^2}$

$y' + 2y^2e^x - y = 0$ becomes $-\dfrac{z'}{z^2} + 2\dfrac{1}{z^2}e^x - \dfrac{1}{z} = 0$, so

$-z' + 2e^x - z = 0$ consequently (E'): $z' + z = 2e^x$.

2) The general solution of the equation $z' + z = 0$ is $z_1 = Ce^{-x}$.

$z_2 = e^x$ is a particular solution of the equation (E') then

$z = z_1 + z_2 = Ce^{-x} + e^x$ is a general solution of (E').

Hence, $y = \dfrac{1}{z} = \dfrac{1}{Ce^{-x} + e^x}$ is a general solution of (E).

3) $y(0) = \dfrac{1}{2}$ gives $\dfrac{1}{2} = \dfrac{1}{C+1}$ then $C = 1$. Consequently $y = \dfrac{1}{e^{-x} + e^x}$ is a particular solution of (1).

Part B.

1) The domain of definition of f is centered at O, also

$f(-x) = \dfrac{1}{e^x + e^{-x}} = f(x)$, then f is an even function.

2) $\lim\limits_{x \to -\infty} f(x) = 0$ and $\lim\limits_{x \to +\infty} f(x) = 0$.

3) $f'(x) = \dfrac{e^{-x} - e^x}{e^x + e^{-x}}$, then the table of variations of f:

x	$-\infty$		0		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	0	↗	$\frac{1}{2}$	↘	0

4) a- $g'(x) = f'(x) - 1$, since for $x > 0$, $f'(x) < 0$ then

$x > 0$ $g'(x) < 0$ therefore the table of variations of g is:

x	0		$+\infty$
$g'(x)$		$-$	
$g(x)$	$\frac{1}{2}$	$\searrow$	$-\infty$

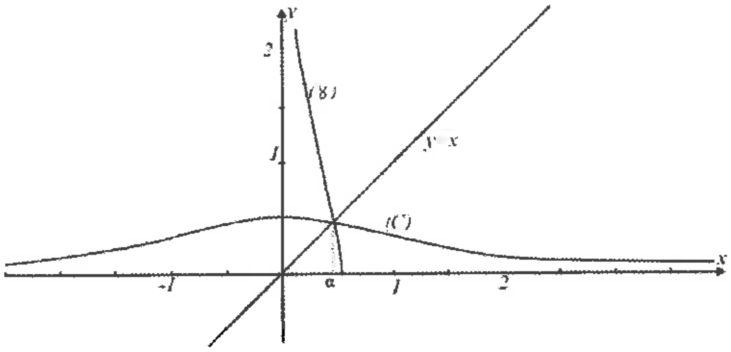
$$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty}[f(x)-x] = 0-\infty = -\infty$$

b- g is continuous and strictly decreasing over $[0;+\infty[$, it decreases from $\frac{1}{2}$ to $-\infty$ thus its representative curve cuts the axis $x'x$ at a unique point. Consequently, the equation $g(x)=0$ admits a unique root α, hence the equation $f(x)=x$ admits over $[0;+\infty[$ a unique solution α.

$g(0.4)=f(0.4)-0.4=0.0625>0$

$g(0.6)=f(0.6)-0.6=-0.056<0$, hence, $0.4<\alpha<0.5$.

c-



5) a- f is continuous and strictly decreasing over $[0;+\infty[$, then it admits an inverse function f^{-1}.

b- The domain of definition of f^{-1} is $\left]0;\frac{1}{2}\right]$.

$y=\dfrac{1}{e^{-x}+e^{x}}$ gives $y=\dfrac{e^{x}}{1+e^{2x}}$ then $ye^{2x}-e^{x}+y=0$,

A quadratic equation in e^{x}, $\Delta=b^2-4ac=1-4y^2$ then

$e^x = \dfrac{1+\sqrt{1-4y^2}}{2y}$ or $e^x = \dfrac{1-\sqrt{1-4y^2}}{2y}$, which gives

$x = \ln\left[\dfrac{1+\sqrt{1-4y^2}}{2y}\right]$ or $x = \ln\left[\dfrac{1-\sqrt{1-4y^2}}{2y}\right]$.

For $y = \dfrac{1}{4}$; $x = \ln\left[\dfrac{1+\sqrt{1-\frac{1}{4}}}{\frac{1}{2}}\right] = \ln\left(2+\sqrt{3}\right) > 0$ or

$x = \ln\left[\dfrac{1-\sqrt{1-\frac{1}{4}}}{\frac{1}{2}}\right] = \ln\left(2-\sqrt{3}\right) < 0$ then the accepted solution is

$x = \ln\left[\dfrac{1+\sqrt{1-4y^2}}{2y}\right]$, consequently $f^{-1}(x) = \ln\left[\dfrac{1+\sqrt{1-4x^2}}{2x}\right]$.

c- The curve (γ) of f^{-1} is symmetric to that of (C) with respect to the straight line of equation $y = x$, (C) and (γ) intersect at the point $E(\alpha;\alpha)$.

Part C.

1) $f(x) > 0$ for all x, then h is defined over IR.

2) a- $h(x)+x = \ln[f(x)] + \ln e^x = \ln[e^x f(x)] = \ln\dfrac{e^x}{e^x+e^{-x}}$

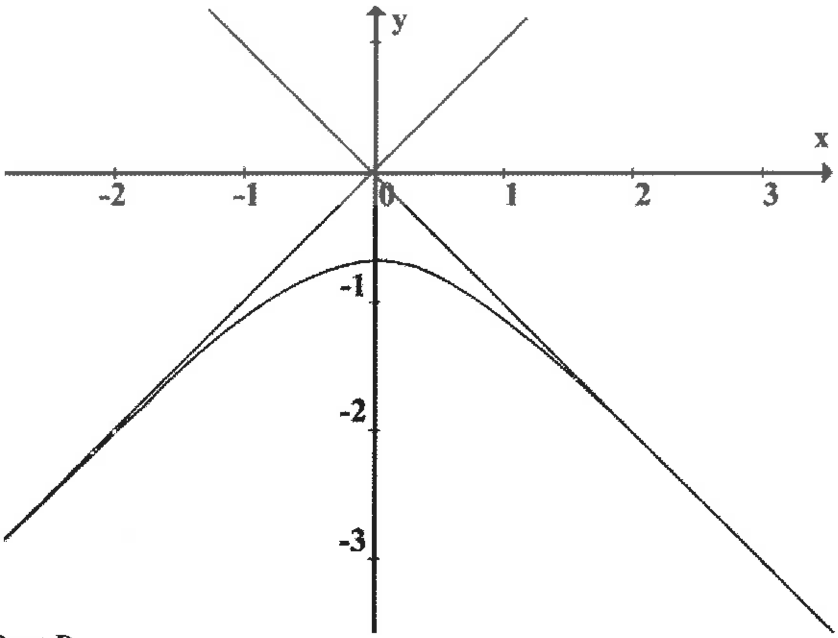
$= \ln\dfrac{1}{1+e^{-2x}} = -\ln[1+e^{-2x}]$.

$\lim\limits_{x\to+\infty}[h(x)+x] = \lim\limits_{x\to+\infty}\left[-\ln\left(1+e^{-2x}\right)\right] = 0$. Then the straight line (d) of equation $y = -x$ is an asymptote to (H).

b- The domain of definition of h is centered at O, also $h(-x) = \ln[f(-x)] = \ln f(x) = h(x)$
then h is an even function then the straight line of equation $y = x$, symmetric of (d) with respect to $y'y$ is an asymptote to (H).

c- $h'(x)=\dfrac{f'(x)}{f(x)}$, so the table of variations of h :

x	$-\infty$	0	$+\infty$
$h'(x)$	$+$	0	$-$
$h(x)$	$-\infty$ ↗	$\ln\frac{1}{2}$	↘ $-\infty$



Part D.

1) a- $f(x)-e^{-x}=\dfrac{1}{e^{x}+e^{-x}}-e^{-x}=\dfrac{1-1-e^{-2x}}{e^{x}+e^{-x}}=\dfrac{-e^{-2x}}{e^{x}+e^{-x}}<0$ then $f(x)<e^{-x}$ for all x and in particular for $x\geq 0$.

b- $f(x)<e^{-x}$ then $\int_0^n f(x)dx<\int_0^n e^{-x}dx$, which gives $\int_0^n f(x)dx<\left[-e^{-x}\right]_0^n$. Consequently, $\int_0^n f(x)dx<1-e^{-n}$.

Hence, $v_n\leq 1-e^{-n}$.

2) a- $v_{n+1}-v_n=\int_0^{n+1} f(x)dx-\int_0^n f(x)dx=\int_0^{n+1} f(x)dx+\int_n^0 f(x)dx$.

$$=\int_n^{n+1} f(x)dx.$$

b- Since $f(x)>0$ then $\int_n^{n+1} f(x)dx>0$, so $v_{n+1}-v_n>0$, that is $v_{n+1}>v_n$ consequently the sequence (v_n) is strictly increasing.

c- The sequence (v_n) is increasing and bounded above by 1 since $v_n\le 1-e^{-n}<1$ then it is convergent to a limit ℓ.

Since, $0<v_n<1$ then $0\le \ell<1$.

3) $f(x)=\dfrac{1}{e^x+e^{-x}}=\dfrac{e^x}{e^x\left(e^x+e^{-x}\right)}=\dfrac{e^x}{e^{2x}+1}$

$$v_n=\int_0^n \frac{e^x}{1+e^{2x}}dx=\left[\arctan e^x\right]_0^n$$

$$=\arctan e^n-\arctan 1=\arctan e^n-\frac{\pi}{4}.$$

$\lim\limits_{n\to+\infty} v_n=\lim\limits_{n\to+\infty}\arctan e^n-\dfrac{\pi}{4}=\dfrac{\pi}{2}-\dfrac{\pi}{4}=\dfrac{\pi}{4}$, since

$\lim\limits_{n\to+\infty}\arctan e^n=\arctan(+\infty)=\dfrac{\pi}{2}$ .Then $\ell=\dfrac{\pi}{4}$.

4) The required area is:

$\int_0^2 f(x)dx=v_2=\arctan e^2-\dfrac{\pi}{4}$ square units

$$=\left(\arctan e^2-\frac{\pi}{4}\right)\times 16\ \text{cm}^2=0.004739=4.739\times10^{-3}.$$

$\boxed{N^\circ 21.}$

Part A.

1) $z=\dfrac{x}{y}$ gives $y=\dfrac{x}{z}$ then $y'=\dfrac{z-xz'}{z^2}$, the equation (E) becomes :

$$x^2\left(\frac{z-xz'}{z^2}\right)-x\left(\frac{x}{z}\right)+\left(\frac{x}{z}\right)^2=0\text{, so } xz'-1=0:(F).$$

2) $xz'-1=0$ gives $z'=\frac{1}{x}$ then $z=\int\frac{1}{x}dx=\ln x+C$ and consequently

$y=\frac{x}{\ln x+C}$.

3) $y(1)=1$ gives $1=\frac{1}{C}$ then $C=1$ and consequently $y=\frac{x}{\ln x+1}$.

***Part B*.**

1) a- $\lim\limits_{x\to0^+} f(x)=0^-$, $\lim\limits_{x\to\frac{1}{e}^+} f(x)=+\infty$, $\lim\limits_{x\to\frac{1}{e}^-} f(x)=-\infty$

$\lim\limits_{x\to+\infty} f(x)=\frac{+\infty}{+\infty}=\lim\limits_{x\to+\infty}\frac{1}{\frac{1}{x}}=+\infty$.

b- $\lim\limits_{x\to+\infty}\frac{f(x)}{x}=\lim\limits_{x\to+\infty}\frac{1}{1+\ln x}=0$ then it admits the axis $x'x$ as an asymptotic direction

2) $f'(x)=\frac{\ln x}{(1+\ln x)^2}$, then the table of variations of f is the following .

x	0		$\frac{1}{e}$		1		$+\infty$
$f'(x)$	‖	$-$	‖	$-$	0	$+$	
$f(x)$	‖ $+\infty$	$\searrow$	$-\infty$ ‖ $+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

3) a- $f''(x)=\frac{1-\ln x}{x(1+\ln x)^3}$, the sign of f'' is that of $\frac{1-\ln x}{1+\ln x}$

Then we get the following table :

x	0		$\frac{1}{e}$		e	
$f''(x)$	‖	$-$	‖	$+$	0	$-$

Hence, the point $I\left(e;\frac{e}{2}\right)$ is an inflection point of (C).

b- $f'(e)=\frac{1}{4}$, then an equation of (d) is :

$$y-\frac{e}{2}=\frac{1}{4}(x-e) \text{ that is } y=\frac{1}{4}x+\frac{e}{4}.$$

4) $f(x)-x=\frac{-x\ln x}{1+\ln x}$, its sign is that of $\frac{-\ln x}{1+\ln x}$ which is expressed in the following table :

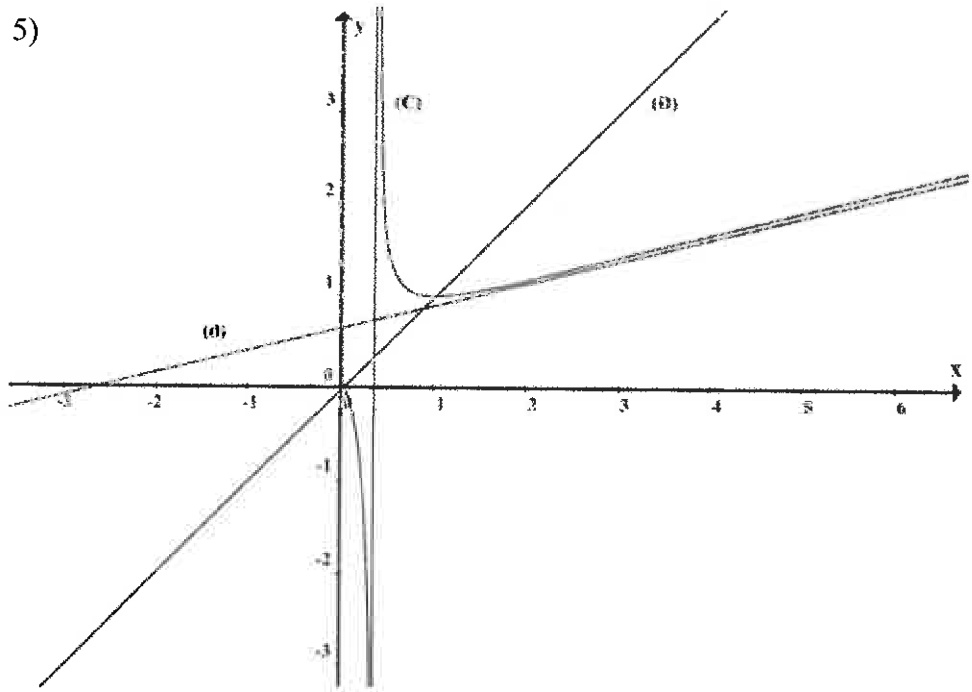
x	0		$\frac{1}{e}$		1	
$f(x)-x$	$\Vert$	$-$	$\Vert$	$+$	0	$-$

For $0<x<\frac{1}{e}$ or $x>e$, (C) is below (D).

For $\frac{1}{e}<x<1$, (C) is above (D).

For $x=1$, (C) and (D) intersect at point $(1;1)$.

5)



***Part C*.**

1) a- f is strictly increasing over I ; then

$$f(I)=\left[f(1);f(e)\right]=\left[1;\frac{e}{2}\right] \text{ thus } f(I)\subset I.$$

b- $f'(x)-\dfrac{1}{4}=\dfrac{\ln x}{(1+\ln x)^2}-\dfrac{1}{4}=\dfrac{-(1-\ln x)^2}{4(1+\ln x)^2}$;

$f'(x)-\dfrac{1}{4}\leq 0$ thus $f'(x)\leq\dfrac{1}{4}$.

But $f'(x)\geq 0$ over [1 ; e] , therefore $0\leq f'(x)\leq\dfrac{1}{4}$.

c- The inequality of the mean value theorem gives:

$|f(x)-f(1)|\leq k|x-1|$

where K is the maximum of $|f'(x)|$ over [1 ; e], therefore,

$|f(x)-1|\leq\dfrac{1}{4}|x-1|$.

2) a- $u_0=2$ then $u_0\in[1;e]$; for $n>0$ if $u_n\in[1;e]$ then $f(u_n)\in[1;e]$ that is to say. $u_{n+1}\in[1;e]$.

b- $|f(u_n)-1|\leq\dfrac{1}{4}|u_n-1|$ then; $|u_{n+1}-1|\leq\dfrac{1}{4}|u_n-1|$.

c- Using mathematical induction :

$|u_0-1|=1\leq\dfrac{1}{4^0}$.

Suppose that $|u_{n-1}-1|\leq\dfrac{1}{4^{n-1}}$, we have to prove that $|u_n-1|\leq\dfrac{1}{4^n}$.

$|u_n-1|\leq\dfrac{1}{4}|u_{n-1}-1|\leq\dfrac{1}{4}\times\dfrac{1}{4^{n-1}}$ then $|u_n-1|\leq\dfrac{1}{4^n}$.

The sequence (u_n) is increasing and bounded above by $\dfrac{1}{4}$ then it is convergent .

$$0 \leq \lim_{n\to+\infty} |u_n - 1| \leq \lim_{n\to+\infty} \frac{1}{4^n}$$ and since

$$\lim_{n\to+\infty} \frac{1}{4^n} = 0$$ then $\lim_{n\to+\infty} |u_n - 1| = 0$ and consequently $\lim_{n\to+\infty} u_n = 1$.

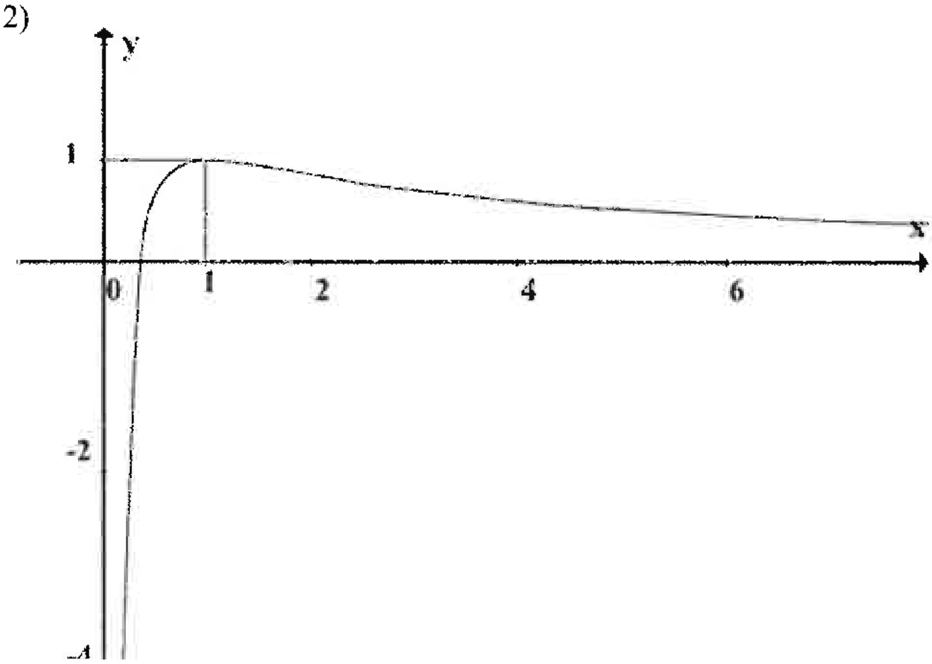
***Part D*.**

1) $\lim_{x\to 0^+} g(x) = -\infty$, $\lim_{x\to+\infty} g(x) = \frac{+\infty}{+\infty} = \lim_{x\to+\infty} \frac{\frac{1}{x}}{1} = 0$ then the axis $x'x$ and the axis
$y'y$ are asymptotes to the representative curve of g.

$g'(x) = \dfrac{-f'(x)}{(f(x))^2}$, then the table of variations of g is the following

x	0		1		$+\infty$
$g'(x)$	‖	+	0	−	
$g(x)$	‖ $-\infty$	↗	1	↘	0

2)



3) a- $v_n = \int_{e^n}^{e^{n+1}} g(x)dx = \int_{e^n}^{e^{n+1}} \frac{1+\ln x}{x}dx = \frac{1}{2}\left[(1+\ln x)^2\right]_{e^n}^{e^{n+1}}$.

$v_n = \frac{1}{2}\left[(1+n+1)^2 - (1+n)^2\right] = n + \frac{3}{2}$

b- $v_{n+1} - v_n = n+1+\frac{3}{2} - n - \frac{3}{2} = 1$, then (v_n) is an arithmetic sequence of first term $v_0 = \frac{3}{2}$ and common difference 1.

c- $a_n = \int_1^{e^n} g(x)dx = \int_1^{e} f(x)dx + \int_{e^1}^{e^2} f(x)dx + ... + \int_{e^{n-1}}^{e^n} f(x)dx$

$= v_0 + v_1 + ... + v_{n-1} = \frac{n}{2}(v_0 + v_{n-1}) = \frac{n}{2}(n+2)$ square units.

$\boxed{N°22.}$

Part A .

1) $y = z + 2x$ gives $y' = z' + 2$, (E) becomes :
$(z+2x)(z'+2) - 2x(z'+2) - 2(z+2x) = 0$ which gives
$zz' + 2z + 2xz' + 4x - 2xz' - 4x - 2z - 4x = 0$ then $(E'): zz' - 4x = 0$.

2) $zz' - 4x = 0$ gives $\int zz'dx = \int 4xdx$, then $\frac{1}{2}z^2 = 2x^2 + K$, so
$z^2 = 4x^2 + 2K$ and consequently $z = \pm\sqrt{4x^2 + 2K}$.
$z = \pm\sqrt{4x^2 + 2K}$ gives $y = \pm\sqrt{4x^2 + 2K} + 2x$.

3) The curve passes through the point $(0;1)$ then $1 = \pm\sqrt{2K}$ which gives $K = \frac{1}{2}$ consequently $y = \sqrt{4x^2 + 1} + 2x$.

Part B .

1) $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left(2x+\sqrt{4x^2+1}\right) = +\infty$.

$\lim\limits_{x\to+\infty}\left(f(x)-y\right) = \lim\limits_{x\to+\infty}\left(\sqrt{4x^2+1}-2x\right) = +\infty-\infty$, indeterminate form.

$$\lim_{x\to+\infty}\left(\sqrt{4x^2+1}-2x\right) = \lim_{x\to+\infty}\frac{\left(\sqrt{4x^2+1}-2x\right)\left(\sqrt{4x^2+1}+2x\right)}{\sqrt{4x^2+1}+2x}$$

$= \lim\limits_{x\to+\infty}\dfrac{1}{\sqrt{4x^2+1}+2x} = 0$,then the straight line (d) is an asymptote to (C) at $+\infty$.

2) $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty}\left(2x+\sqrt{4x^2+1}\right) = -\infty+\infty$ indeterminate form,

$$= \lim_{x\to+\infty}\frac{\left(2x+\sqrt{4x^2+1}\right)\left(2x-\sqrt{4x^2+1}\right)}{2x-\sqrt{4x^2+1}} = \lim_{x\to+\infty}\frac{-1}{2x-\sqrt{4x^2+1}}$$

$= \dfrac{-1}{-\infty-\infty} = 0$ then the axis $x'x$ is an asymptote to (C).

3) $f'(x) = 2+\dfrac{8x}{2\sqrt{4x^2+1}} = \dfrac{4\sqrt{4x^2+1}+8x}{2\sqrt{4x^2+1}}$

If $x>0$ then $f'(x)>0$.

If $x<0$ then $f'(x) = \dfrac{4\sqrt{4x^2+1}+8x}{2\sqrt{4x^2+1}}\times\dfrac{4\sqrt{4x^2+1}-8x}{4\sqrt{4x^2+1}-8x}$

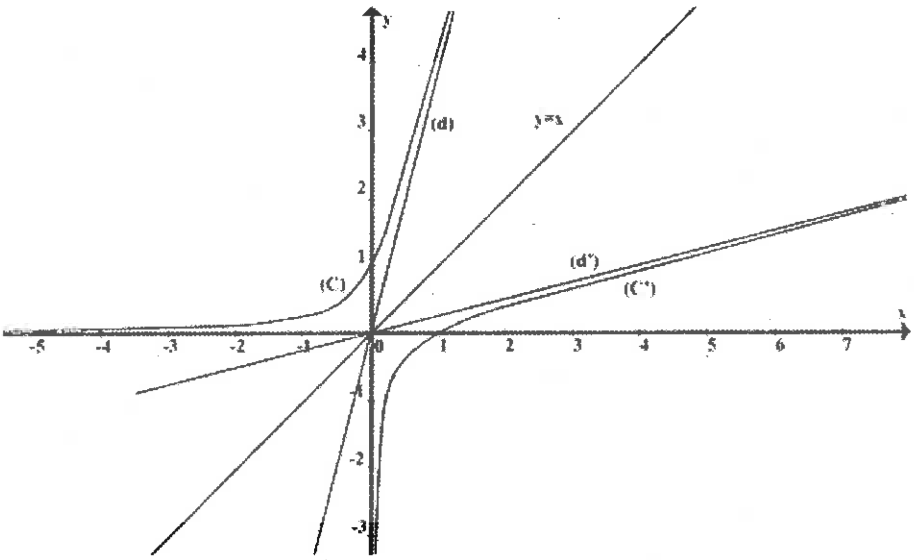
$$= \frac{16x^2+16-16x^2}{\sqrt{4x^2+1}\left(4\sqrt{4x^2+1}-8x\right)} = \frac{16}{\sqrt{4x^2+1}\left(4\sqrt{4x^2+1}-8x\right)} > 0$$

because $4\sqrt{4x^2+1}-8x>0$.

Then f is strictly increasing over IR .

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)$	0	$\nearrow$	$+\infty$

4)



5) a- f is continuous and strictly increasing over IR then it admits an inverse function f^{-1}.

b- $y = 2x + \sqrt{4x^2 + 1}$ gives $y - 2x = \sqrt{4x^2 + 1}$ then $(y - 2x)^2 = 4x^2 + 1$, so $y^2 - 4xy + 4x^2 = 4x^2 + 1$ thus $4xy = y^2 - 1$ and consequently, $x = \dfrac{y^2 - 1}{4y}$ then $f^{-1}(x) = \dfrac{x^2 - 1}{4x}$.

c- $\left(f^{-1}\right)'(1) = \dfrac{1}{f'(0)} = \dfrac{1}{2}$.

$\left(f^{-1}\right)'(x) = \dfrac{2x(4x) - 4\left(x^2 - 1\right)}{16x^2} = \dfrac{4x^2 + 4}{16x^2}$ then

$\left(f^{-1}\right)'(1) = \dfrac{4 + 4}{16} = \dfrac{1}{2}$.

d- (C') is symmetric to (C) with respect to the straight line of equation $y = x$, see figure.

Indications

N° 5.

1) $y = C_1 + C_2 e^{2x}$.
2) $\lim\limits_{x \to -\infty} f(x) = 2 = C_1$, $f(0) = C_1 + C_2 = 1$.

N° 6.

Part A:

1) Remark that the function associated to the curve (C) is positive over IR; if this function is the derivative function then the other becomes increasing over IR which is not the case.
   Then the representative of g' is the curve (γ)
   And we verify that:
   Over $]-\infty;-1[$, $g'(x) < 0$ then g is strictly decreasing.
   Over $]-1;1[$, $g'(x) > 0$ then g is strictly increasing.
   Over $]1;+\infty[$, $g'(x) < 0$ then g is strictly decreasing.

N° 7.

2) Remark that $f'(0) = 2$ and $f'(-1) = 1$

N° 8.

Part B .

4) $f'(0) = 0$, $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$.

N° 10.

3) a- $(T_m): y = (-2m+1)x + m - 2$ or $(-2x+1)m + x - 2 - y = 0$, which gives $-2x+1 = 0$ and $x - 2 - y = 0$.
   b- $f'_m(x) = -2me^{-2x} + 1$, $f'_m(x) > 0$ for any x then $-2me^{-2x} + 1 > 0$ which gives $2m < e^{2x}$ for any x, then $m \leq 0$.
   c- For $m > 0$, $f'_m(x) = -2me^{-2x} + 1 = 0$ gives $me^{-2x} = \dfrac{1}{2}$ and since $y = me^{-2x} + x - 2$ we get $y = \dfrac{1}{2} + x - 2 = x - \dfrac{3}{2}$.

CHAPTER 7
Planes and Lines

Chapter Review

1) **Vector product and mixed product.**

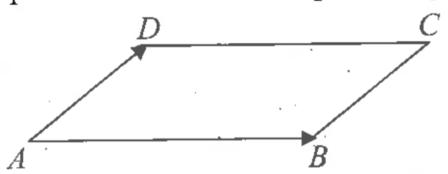
The space is referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.

- $\vec{u}$ and $\vec{v}$ are two given vectors in the space.

The vector product of $\vec{u}$ and $\vec{v}$ is the vector defined by:

$$\vec{u}\times\vec{v}=\vec{w} \text{ such that : } \begin{cases} *\ \vec{w} \quad \textit{orthogonal to } \vec{u} \\ *\ \vec{w} \quad \textit{orthogonal to } \vec{v} \\ *\ \|\vec{w}\| = \|\vec{u}\|\times\|\vec{v}\|\times\left|\sin(\vec{u};\vec{v})\right| \end{cases}$$

$\|\vec{u}\times\vec{v}\|$ represents the area of the parallelogram $ABCD$.



The area of triangle ABC is equal to $\frac{1}{2}\|\overrightarrow{AB}\times\overrightarrow{AC}\|$.

- $\vec{u}$,$\vec{v}$ and $\vec{w}$ are three vectors of the space.

The triple scalar product of $\vec{u}$, $\vec{v}$ and $\vec{w}$ is the real number defined by : $\vec{u}.(\vec{v}\times\vec{w})$.

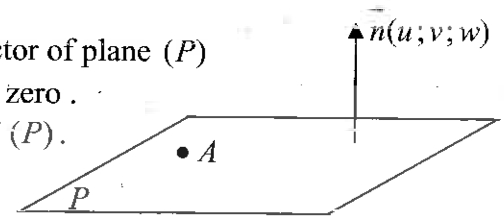
$\vec{u}.(\vec{v}\times\vec{w})=\det(\vec{u};\vec{v};\vec{w})$.

The triple scalar product is zero in the three vectors are coplanar or if one of them is the zero vector.

The absolute value of the triple scalar product represents the volume of the parallelepiped constructed on the three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in the case the three vectors are not coplanar .

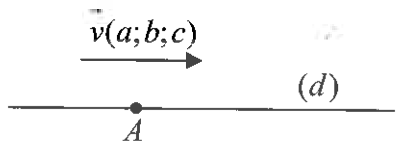
The volume of a tetrahedron is equal to $\frac{1}{6}|\vec{u}.(\vec{v}\times\vec{w})|$.

2) **Planes and lines .**



- $\vec{n}(u;v;w)$ is a normal vector of plane (P) with u , v and w not all zero .
  $A(x_0;y_0;z_0)$ is a point of (P).
  An equation of (P) is :
  $ux+vy+wz+r=0$.
- **Particular planes:**
  An equation of plane (xOy) is $z=0$, that of (xOz) is $y=0$ and equation of (yOz) is $x=0$.
- $\vec{v}(a;b;c)$ is a directing vector of (d) with a , b and c not all zero, $A(x_0;y_0;z_0)$ is a point of (d), then a system of parametric equations of (d) is :

$$\begin{cases} x = am + x_0 \\ y = bm + y_0 \\ z = cm + z_0 \end{cases}$$



- **Relative position of two planes:**
  Two planes are parallel if their normal vectors are collinear and if one point of them does not belong to the other plane .

  Two planes are perpendicular if their normal vectors are perpendicular.
  Intersection of Two Planes :
  Let (P) and (Q) be two planes of respective equations:
  $ux+vy+wz+r=0$ and $u'x+v'y+w'z+r'=0$, to find the straight line of intersection of the two planes:

  If $\frac{u}{u'} \neq \frac{v}{v'}$, we take $z=m$ then we solve the system for x and y .

  If $\frac{u}{u'} \neq \frac{w}{w'}$, we take $y=m$ then we solve the system for x and z .

  If $\frac{v}{v'} \neq \frac{w}{w'}$, we take $x=m$ then we solve the system for y and z .

Ex :
Given (P) and (Q) two planes of respective equations $2x+y-z-1=0$ and $x+2y+z-2=0$ and let (d) be the line of intersection of (P) and (Q).

Since $\frac{u}{u'} \neq \frac{w}{w'}$, we take $y=m$, so we get $\begin{cases} 2x-z=-m+1 \\ x+z=-2m+2 \end{cases}$

which gives $x=-m+1$ and $z=-m+1$, hence a system of parametric equations of (d) is:

$$\begin{cases} x=-m+1 \\ y=m \\ z=-m+1 \end{cases}$$

- **Relative positions of two straight lines (d) and (d').**

Two straight lines are parallel if their directing vectors are parallel and a point on one of them does not belong to the other.
Two straight lines are said to be skew or non-coplanar if they do not belong to the same plane .
To prove that two straight lines are non-coplanar:
We prove that these straight lines are neither parallel nor intersecting.
Or we take a point A on (d) and a point B on (d') and then we prove that: $\overrightarrow{AB}.\left(\overrightarrow{v_d} \times \overrightarrow{v_{d'}}\right) \neq 0$.

Ex :

The two straight lines $(d)\begin{cases} x=2\lambda+1 \\ y=2\lambda+2 \\ z=3\lambda \end{cases}$ and $(d')\begin{cases} x=2t \\ y=t \\ z=2t+2 \end{cases}$

are skew .

To prove this we take the directing vectors $\overrightarrow{v_d}(2;2;3)$ and $\overrightarrow{v_{d'}}(2;1;2)$ of (d) and (d') respectively. We also take $A(1;2;0)$ and $B(0;0;2)$ two points belonging to (d) and (d') respectively then, $\overrightarrow{AB}.\left(\vec{v} \times \overrightarrow{v_{d'}}\right) = \begin{vmatrix} -1 & -2 & 2 \\ 2 & 2 & 3 \\ 2 & 1 & 2 \end{vmatrix} = -9 \neq 0$.

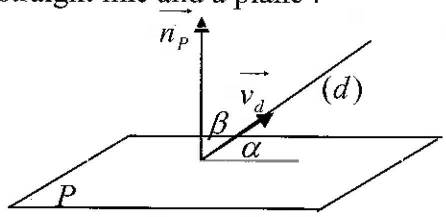
3) **Angles.**

The acute angle of two planes is equal to the acute angle of its normal vectors: $\cos\left(\overrightarrow{n_P};\overrightarrow{n_Q}\right)=\dfrac{\left|\overrightarrow{n_P}.\overrightarrow{n_Q}\right|}{\left\|\overrightarrow{n_P}\right\|.\left\|\overrightarrow{n_Q}\right\|}$

The acute angle of two straight lines is equal to the acute angle of its directing vectors .

The acute angle between a straight line and a plane :

The angle is: $\alpha=\dfrac{\pi}{2}-\beta$.



We calculate β then we find α .

Or we find $\left\|\overrightarrow{v_d}\times\overrightarrow{n_P}\right\|$, so we obtain directly $\sin\beta=\cos\alpha$.

Ex :

Let (P) $2x+y-z-1=0$ and $(d)\begin{cases}x=m-1\\ y=2m\\ z=m\end{cases}$.

$\cos\left(\overrightarrow{v_d};\overrightarrow{n_P}\right)=\dfrac{\left|\overrightarrow{v_d}.\overrightarrow{n_P}\right|}{\left\|\overrightarrow{v_d}\right\|.\left\|\overrightarrow{n_P}\right\|}=\dfrac{1}{2}$, then $\beta=\dfrac{\pi}{3}$ and consequently $\alpha=\dfrac{\pi}{6}$.

4) **Distance .**

- Distance from a point $A(x_0;y_0;z_0)$ to the plane of equation $ux+vy+wz+r=0$ is $d=\dfrac{\left|ux_0+vy_0+wz_0+r\right|}{\sqrt{u^2+v^2+w^2}}$
- The distance between two parallel planes is the distance from a point of one of the two planes to the other.

  Ex : Let (P) and (Q) be two planes of respective equations $x+y+z=0$ and $2x+2y+2z-1=0$, then the two planes are parallel .

The distance between these two planes is the distance from the point O of (P) to the plane (Q) $d = \dfrac{|-1|}{\sqrt{4+4+4}} = \dfrac{\sqrt{3}}{6}$.

- Distance from the point $A(x_0;y_0;z_0)$ to a straight line :
  We take any point B of (d), the distance is given by the formula:
  $$d = \frac{\left\|\overrightarrow{AB}\times\overrightarrow{v_d}\right\|}{\left\|\overrightarrow{v_d}\right\|}.$$
  Ex: Let (d) $\begin{cases} x = m-1 \\ y = m \\ z = m+1 \end{cases}$, $\vec{v}(1;1;1)$ a directing vector of (d)
  and $B(-1;0;1)$ a point of (d) , then the distance of O to (d) is:
  $$d = \frac{\left\|\overrightarrow{OB}\times\overrightarrow{v_d}\right\|}{\left\|\overrightarrow{v_d}\right\|} = \frac{\sqrt{6}}{\sqrt{3}} = \sqrt{2}.$$
- The distance between two parallel straight lines is the distance from a point of one of them to the other.
  Ex :
  Let (d) $\begin{cases} x = m-1 \\ y = m \\ z = m+1 \end{cases}$ (d') $\begin{cases} x = 2t \\ y = 2t \\ z = 2t \end{cases}$, the distance between these two parallel lines is equal to the distance of the point O of (d') to (d) , it is equal to $d(O;(d)) = \sqrt{2}$.

5) **Mediator plane** .
   Vector $\overrightarrow{AB}$ is a normal vector of the mediator plane of $[AB]$ and this plane passes through the midpoint of I of $[AB]$.
   Ex :
   $A(1;2;0)$ and $B(-1;0;2)$ are two points of the space , then $\overrightarrow{AB} = (-2;-2;2)$ and $I(0;1;1)$, hence an equation of the mediator plane is: $-2x-2y+2z=0$ or $x+y-z=0$.

Solved Problems

N° 1.

1) Write an equation of the plane (P) passing through the three points $A(1;1;2)$, $B(2;-1;-1)$ and $C(0;-3;-1)$.
2) Write an equation of the plane (P) determined by the point $A(1;1;2)$ and the straight line $(d): x = 2m-1,\ y = m,\ z = -m$.
3) Write an equation of the plane (P) containing the straight line $(d): x = 2m-1,\ y = m,\ z = -m$ and parallel to the straight line $(d'): x = -t,\ y = t+1,\ z = t$.
4) The two straight lines $(d): x = m-1,\ y = m,\ z = m+1$ and $(d'): x = 2t,\ y = t,\ z = -t-1$ intersect at point $A(-2;-1;0)$. Write an equation of the plane (P) containing the lines (d) and (d').
5) Write an equation of the plane (P) containing the two straight lines $(d): x = m-1,\ y = m,\ z = m+1$ and $(d'): x = 2t,\ y = 2t,\ z = 2t$.
6) Write an equation of the mediator plane of the segment $[AB]$ where $A(-1;0;1)$ and $B(1;2;3)$.

N° 2.

1) Calculate the distance of point $A(-1;0;1)$ to the plane (P) of equation $x+y+z-1=0$.
2) Calculate the distance of point $A(1;1;1)$ to the straight line $(d): x = m-1,\ y = m,\ z = m+1$.
3) Calculate the distance between the two straight lines $(d): x = \lambda-1,\ y = \lambda-1,\ z = \lambda+1$ and $(d'): x = 2t,\ y = 2t,\ z = 2t$.
4) Calculate the distance between the two planes $(P): x+y+z=0$ and $(Q): 2x+2y+2z+1=0$

N°3.

1) Calculate the acute angle of the two planes $(P): 2x+y-z-1=0$ and $(Q): x+2y-z-2=0$.
2) Calculate the cosine of the acute angle determined by the plane $(P): 2x+y-z-1=0$ and the straight line $(d): x=m-1,\ y=m,\ z=m+1$.
3) Calculate the cosine of the acute angle determined by the two straight lines $(d): x=m-1,\ y=2m,\ z=m+1$ and $(d'): x=2t,\ y=-t,\ z=t$.

N°4.

1) Write a system of parametric equations of the straight line (d) passing through the point $A(-1;0;1)$ and perpendicular to the plane $(P): 2x+y-z-1=0$.
2) Write a system of parametric equations of the straight line (d) passing through the point $A(1;-1;2)$ and parallel to the two planes $(P): 2x+y-z-1=0$ and $(Q): x+2y-z-2=0$.
3) Write a system of parametric equations of the straight line (d), intersection of the two planes $(P): 2x+y-z-1=0$ and $(Q): x+2y-z-2=0$.
4) Write a system of parametric equations of one of the bisectors of the two angles formed by $(d): x=-m,\ y=2m,\ z=m+1$ and $(d'): x=2t,\ y=-t,\ z=t+1$..
5) Write a system of parametric equations of one of the bisectors of the two angles formed by $(d): x=-m,\ y=2m,\ z=m+1$ and $(d'): x=t,\ y=-t,\ z=t+1$.

N°5.

1) Calculate the coordinates of point I, intersection of the two straight lines $(d): x=t,\ y=-t,\ z=t+1$ and $(d'): x=m,\ y=-m,\ z=-m+1$.

2) Calculate the coordinates of point I, intersection of the straight line $(d): x=t$, $y=-t$, $z=t+1$ and the plane $(Q): x+2y-z-1=0$.
3) Calculate the coordinates of point H, the orthogonal projection of $A(-1;0;1)$ on the straight line $(d): x=t$, $y=-t$, $z=t+1$.
4) a- Calculate the coordinates of H, the orthogonal projection of point $A(-1;0;1)$ on the plane $(P): x-y-z-1=0$.

   b- Calculate the coordinates of point A' the symmetric of point $A(-1;0;1)$ with respect to the plane $(P): x-y-z-1=0$.

N° 6.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the straight line $(d): x=m-1$, $y=m$, $z=m+1$ and the point $E(0;0;2)$.

1) Let (P) be the plane passing through (d) and E, show that an equation of (P) is $x-z+2=0$.
2) In the plane (P), consider the circle (C) of center E and of radius $R=1$.

   a- Show that (C) cuts (d) in two points A and B.

   b- Calculate the coordinates of the points A and B.

N° 7.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the plane (P) of equation $x+2y+3z=4$, the point $w(4;1;4)$ and the straight line (d) defined by $(d): x=\lambda+4$, $y=\lambda+1$, $z=-\lambda+4$.

1) Verify that w belongs to (d) and that (d) is parallel to (P).
2) a- Determine an equation of plane (Q) containing the straight line (d) and perpendicular to plane (P).

   b- Deduce a system of parametric equations of the straight line (d'), orthogonal projection of (d) on (P).
3) Show that the point $\omega'(3;-1;1)$ is the orthogonal projection of ω on (P).

4) Let (C) be the circle of the plane (P) of center ω' and of radius $R=\sqrt{3}$.

   a- Determine the points of intersection A and B of (C) and (d').

   b- Write the equations of the tangents at A and B to (C).

5) Let F be a point of (C) such that $A\hat{B}F=30^\circ$ and M a variable point of the straight line (d).
   Show that the volume of tetrahedron $MABF$ is a constant to be determined.

N°8.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$,
Consider the two planes (P): $x+2y+2z-1=0$ and
(Q): $2x+y-2z+1=0$.

1) Show that (P) and (Q) are perpendicular.
2) Let $A(2;1;-1)$ be a point of the space.

   a- Calculate the distance from the point A to the plane (P) and the distance from the point A to the plane (Q).

   b- Deduce the distance of point A to the straight line $(d)=(P)\cap(Q)$.
3) Determine a system of parametric equations of (d).
4) a- Write an equation of plane (R) passing through A and perpendicular to the two planes (P) and (Q).

   b- Deduce the coordinates of the point I orthogonal projection of A on (d).

N°9.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$,
consider the points $A(1;1;0)$, $B(2;0;0)$, $C(1;3;-1)$, $E(2;2;2)$ and the plane (P) of equation $x+y+2z-2=0$.

1) a- Verify that (P) is the plane determined by A, B and C.

   b- Show that the straight line (AE) is perpendicular to plane (P).

   c- Calculate the area of triangle ABC and the volume of tetrahedron $EABC$.

2) Designate by L the midpoint of $[AB]$ and by (Q) the plane passing through L and parallel to the two straight lines (AE) and (BC).

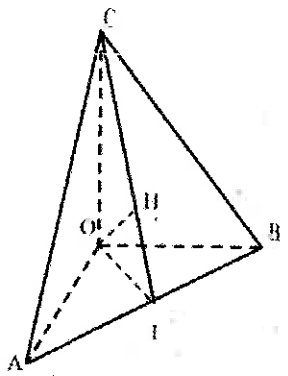
   a- Write an equation of plane (Q).

   b- Prove that the planes (P) and (Q) are perpendicular.

   c- Prove that the straight (d), intersection of planes (P) and (Q) is parallel to (BC).

N° 10.

The figure to the right is considered in a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ where : $\overrightarrow{OA}=\vec{i}$, $\overrightarrow{OB}=\vec{j}$ and $\overrightarrow{OC}=2\vec{k}$.



Let I be the midpoint of $[AB]$.

1) Justify that an equation of plane (ABC) is : $2x+2y+z-2=0$.

2) Consider the point $H\left(\frac{4}{9};\frac{4}{9};\frac{2}{9}\right)$.

   a- Show that C, H and I are collinear.

   b- Prove that (OH) is perpendicular to plane (ABC) .

   c- Prove that the two planes (OIC) and (ABC) are perpendicular.

3) a- Write a system of parametric equations of the straight line (Δ) passing through C and parallel to (OB).

   b- Let F be a variable point of (Δ).
   Prove that tetrahedron $FOAB$ has a constant volume to be calculated.

N° 11.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$; consider the point $E(-1;0;-1)$ and the straight lines (d) and (d') defined by :

$(d): x=t+1,\ y=-t,\ z=-t+1$ and $(d'): x=-m,\ y=m+2,\ z=m+1$

Let $A(1;0;1)$ be a point of (d) .

1) Prove that (d) and (d') are parallel and that plane (P) determined

by the two straight lines has as equation $2x+y+z-3=0$.
Verify that E does not belong to (P).

2) a- Write an equation of plane (Q) determined by E and (d).
   b- Calculate the distance of point E to the straight line (d) and the distance from E to (P).
   c- Deduce that the planes (P) and (Q) are not perpendicular.
3) a- Verify that the point $E'(3;2;1)$ is the symmetric of E with respect to (P).
   b- Deduce a system of parametric equations of the straight line (δ) the symmetric of the straight line (AE) with respect to (P).

$N°$ 12.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the points $A(1;-2;1)$, $B(2;-1;3)$, $C(1;1;4)$ and $H(0;0;2)$.

1) Find an equation of plane (P) determined by the points A, B and C.
2) a- Determine an equation of plane (Q) passing through A and perpendicular to the straight line (BC).
   b- Deduce a system of parametric equations of the height (δ) issued from A in triangle ABC.
3) Let (d) be the straight line defined by $x=t$, $y=t$, $z=-t+2$
   a- Prove that the straight line (d) is perpendicular to plane (P) at H.
   b- Prove that H is the center of the circle circumscribed about triangle ABC.
   c- Write a system of parametric equations of one of the bisectors of the dihedral angle $A\hat{H}B$.
   d- Let M be a variable point of (d) and $E(2;2;0)$ is a fixed point of (d). For what values of t is the volume of tetrahedron $MABC$ double that of tetrahedron $EABC$?

$N°$ 13.

Consider in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, the points $A(-1;0;0)$; $B(1;1;-1)$; $C(0;1;2)$ and $D(2;2;1)$.

1) Show that an equation of plane (ABC) is: $3x-5y+z+3=0$.
2) a- Show that the point D belongs to plane (ABC) and prove that $ABDC$ is rhombus.
   b- Determine a system of parametric equations of the interior bisector of angle $B\hat{A}C$.
   c- Calculate the distance from A to the straight line (BC).
3) (d) is the straight line passing through A and perpendicular to plane (ABC). Designate by (R) the plane determined by B and (d) and by (Q) the plane determined by C and (d).
   a- Determine a system of parametric equations of (d).
   b- Find an equation of the plane (R).
   c- Find an equation of one of the bisector planes of the dihedral determined by (R) and (Q).

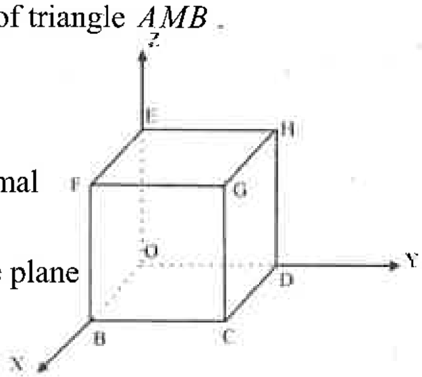
N° 14.

In the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the straight line (d) of parametric equations: $x=-t+1$; $y=t-1$; $z=t$ and the points $A(1;0;2)$ and $B(-1;-1;1)$.

1) Show that an equation of the mediator plane (P) of $[AB]$ is $2x+y+z-1=0$.
2) Find an equation of the plane (Q) containing the point A and the straight line (d).
3) Show that, the two planes (P) and (Q) intersect along (d).
4) Let (Q') be the plane of equation $-x+y-2z+2=0$.
   Show that (Q') is the symmetric of (Q) with respect to (P).
5) Let $M(1;-1;0)$ be a point of (d), calculate the distance from M to (AB) and deduce the area of triangle AMB.

N° 15.

Consider the cube $OBCDEFGH$ (as shown in the adjacent figure)
The space is referred to the orthonormal system $(O;\overrightarrow{OB},\overrightarrow{OD},\overrightarrow{OE})$.



1) a- Verify that an equation of the plane (OFH) is $x+y-z=0$.

b- Verify that (CE) is the axis of circle circumscribed about triangle OFH .

2) Consider the point $K(-1;-1;1)$.
   a- Calculate the distance from K to the plane (OFH).
   b- Calculate the distance from K to the line (OF).
   c- Show that the plane (KOF) is perpendicular to the plane (OFH).

3) Let I be the midpoint of $[EF]$ and $J\left(0;\frac{1}{4};1\right)$.
   a- Calculate the area of triangle OIJ .
   b- Calculate the volume of the tetrahedron $BOIJ$ and deduce the distance from B to plane (OIJ).

N° 16.

The space is referred to an orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$.
Given the 2 straight lines (d) and (d') defined as:
$(d): x=m-1,\ y=2m,\ z=m+1$ and $(d'): x=2t,\ y=t+2,\ z=t+2$.

1) Show that the lines (d) and (d') intersect in $A(0;2;2)$.
2) Find an equation of the plane (P) determined by (d) and (d').
3) In the plane (P) we are given a straight line (D) defined by $x=3\lambda,\ y=3\lambda+2,\ z=2\lambda+2$ (λ is a real parameter).
   a- Show that all the points of (D) are equidistant from (d) and (d').
   b- $I(-3;-1;0)$ is a point of (D), designate by (C) the circle of plane (P), center I, tangent to (d) at T and to (d') at S. Calculate the radius of (C) and the area of quadrilateral $ATIS$.
   c- Let (δ) be a straight line passing through A and perpendicular to plane (P). Find a system of parametric equations of (δ) and deduce an equation of the mediator plane of the segment $[TS]$.
   d- Let $B(1;1;-1)$ be a point of (δ), find an equation of the plane (Q) passing through B and parallel to (P).

e- N is a variable point in (Q). Show that the volume of the tetrahedron $NITS$ remains constant as N varies in Q .

N° 17.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the plane (P_m) of equation $x+y+z+m(x-y-3z)=0$ and the point $A(1\,;-1\,;3)$.

1) Show that the plane (P_m) contains the straight line (d) of parametric equations: $(d): x=t\,,\ y=-2t\,,\ z=t$.
2) Show that the two planes (P_0) and (P_1) are perpendicular.
3) Find a system of parametric equations of the straight line (D) passing through A and perpendicular to the plane (P_0) and deduce the coordinates of point H, the orthogonal projection of point A on (P_0).
4) Find a system of parametric equations of the straight line (D') orthogonal projection of (D) on (P_1).
5) Deduce a system of parametric equations of the straight line (Δ), the common perpendicular of the two straight lines (d) and (D).
6) Calculate the distance of the point A to the plane (P_0) and the distance of point A to the plane (P_1) and deduce the distance of the point A to the straight line (d) .

N° 18.

The space is referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.
Given the two straight lines (d) and (d') defined as:
$(d): x=t-2\,,\ y=2t+1\,,\ z=-2t+2$ and
$(d'): x=-2m\,,\ y=-4m+1\,,\ z=4m$.

1) Show that (d) and (d') are parallel .
2) Determine an equation of the plane (P) determined by (d) and (d').
3) a- $A(-2\,;1\,;2)$ is a point of (d), calculate the coordinates of the point H, the orthogonal projection of A on (d').

b- Determine an equation of the mediator plane of segment $[AH]$.

c- Deduce a system of parametric equations of the straight line (δ) in (P) equidistant from (d) and (d').

N° 19.

In the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the straight line (d) defined by $(d): x = 2\lambda - 1,\ y = 5\lambda + 2,\ z = \lambda - 1$.
Let $C(-1;2;-1)$ and $B(-3;-3;-2)$ be two points on (d) and $A(-1;-1;1)$ a point in space .

1) Show that: $13x - 4y - 6z + 15 = 0$ is an equation of a plane (P) containing A and (d).
2) In (P), consider the circle (γ) of center A and tangent to (d). Calculate the area of triangle ABC , and deduce the radius of (γ).
3) (δ) is a straight line passing through A and perpendicular to (P), write a system of parametric equations of (δ).
4) Let $E(12;-5;-5)$ be a point of (δ) and H the orthogonal projection of A on (BC).
   a- Show that triangle EHC is right.
   b- Calculate the tangent of the acute angle formed by the two planes (EBC) and (ABC).

N° 20.

In the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the points $A(0;1;0)$, $B(-1;2;-1)$ and $C(-1;1;-2)$.

1) Verify that an equation of the plane (P) containing the points A , B and C is: $2x + y - z - 1 = 0$.
2) Let (Q) be the plane of equation $x + 2y + z - 2 = 0$.
   a- Show that (P) and (Q) intersect along the line (AB).
   b- Show that the point $C'(0;3;-1)$ is the symmetric of C with respect to (Q). Deduce an equation of the plane (P'), the symmetric of (P) with respect to (Q).
3) Let $(\delta): x = m,\ y = m + 3,\ z = -1,$ m is a real parameter . Show that (δ) is the symmetric of (BC) with respect to (Q).

4) Find a system of parametric equations of the interior bisector of angle $C\hat{B}C'$.

N°21.

In the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the points $A(3;1;0)$, $B(1;3;0)$, $C(3;2;1)$, and $D(0;0;2)$.

1) Find an equation of plane (ABC).
2) a- Show that the point D is equidistant from A, B, and C.
   b- Find a system of parametric equations of the axis (d) of circle (C) circumscribed about triangle ABC.
   c- Find the coordinates of the center w of (C) and calculate its radius.
   d- Find a system of parametric equations of the tangent (T) drawn through A to (C).

N°22.

In the space of a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the plane (P) of equation $2x+y-z-1=0$ and the plane (Q) of equation $x+2y+z-2=0$. Designate by $(d)=(P)\cap(Q)$.

1) Calculate the acute angle between (P) and (Q).
2) Show that a system of parametric equations of (d) are:
   $(d): x=-m+1,\ y=m,\ z=-m+1,\ m$ is a real parameter.
3) Let M be a variable point on (d) and $B(0;2;1)$ a point on (P). Calculate BM^2 in terms of m and deduce the distance from B to (d).
4) Let $C(0;0;2)$ be a point on (Q). Find a point A on (d) such that ABC is an isosceles triangle of vertex A.
5) $A\left(\frac{1}{2};\frac{1}{2};\frac{1}{2}\right)$ is a given point, write a system of parametric equations of the interior bisector (δ) of the angle $B\hat{A}C$.
6) a- Let K be the midpoint of $[BC]$, show that $AK=\dfrac{\left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\|}{BC}$.
   b- Determine a point E on (δ) such that the area of triangle EBC is twice the area of triangle ABC.

N°23.

In the space of a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the straight lines (Δ) and (Δ') of parametric equations:
$(\Delta): x = 2\lambda+1,\ y = 2\lambda+2,\ z = 3\lambda$ and $(\Delta'): x = 2t,\ y = t,$
$z = 2t+2$, λ and t are two real parameters.

1) Show that (Δ) and (Δ') are skew .
2) Determine an equation of the plane (P) containing (Δ') and parallel to (Δ) .
3) Determine an equation of the plane (Q) containing (Δ) and perpendicular to (P) .
4) Let (d) be the common perpendicular to (Δ) and (Δ') and designate by A the point of intersection of (d) and (Δ) , and B the point of intersection of (d) and (Δ') .
   a- Write a system of parametric equations of (d) and calculate AB .
   b- Calculate the cosine of the acute angle between (d) and plane (xOy) and calculate the distance from O to (d) .

N°24.

In the space of a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the points $A(1,1,1)$, $B(-1;1;3)$ and $C(-1;3;1)$.

1) Show that triangle ABC is equilateral.
2) a- Show that the point $W\left(-\frac{1}{3};\frac{5}{3};\frac{5}{3}\right)$ is the center of the circle circumscribed about triangle ABC .
   b- Let W' be the symmetric of W with respect to (BC), calculate the area of quadrilateral $ABW'C$.
   c- Let D be a point on (d) , the perpendicular through W to (P) . Show that (DC) and (AB) are orthogonal.

N°25.

In the space of a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the plane (P) of equation $2x + y - 2z + 6 = 0$ and the points on (P) $A(1,-6,1)$, $B(2,-4,3)$ and $C(0,0,3)$.

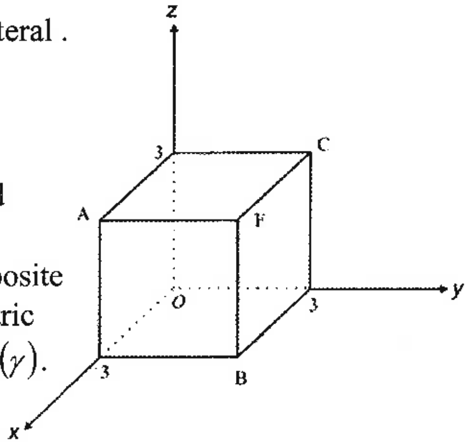
1) S is a point of the space of positive abscissa and of orthogonal projection A on plane (P).
Determine S knowing that $AS = 3$.
2) a- Determine a system of parametric equations of the straight line (d) passing through B and perpendicular to the straight lines (AB) and (AS).
b- Verify that (d) lies in (P).
3) In the plane (P) , consider the circle (C), of center B and radius 3 .
a- Verify that A belongs to (C) and that C is outside (C).
b- Let $[EF]$ be a diameter of (C) perpendicular to (AB) . Determine a system of parametric equations of (EF) .
c- Determine a system of parametric equations of the tangent drawn through E to (C).

$N^\circ 26.$

Consider, in an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, the two planes (P) $x+2y-z-1=0$ and (Q) $2x+y+z-2=0$.
Designate by (d) the intersection line of (P) and (Q).
1) Calculate the acute angle of the two planes (P) and (Q).
2) Let (C) be the circle of center $A(0;0;-1)$, radius $R=\sqrt{5}$ and lying in P .
a- Calculate the distance from A to Q .
b- Deduce that (C) intersects Q in two points E and F.
c- Calculate the area of the triangle AEF .
3) Show that a system of parametric equations of (d) is: $x=-m+1$, $y=m$, $z=m$.
4) Let (R) be the mediator plane of $[EF]$.
a- Determine an equation of (R).
b- Deduce the coordinates of I midpoint of $[EF]$.

$N^\circ 27.$

The adjacent figure is that of a cube in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.
Suppose that $A(3;0;3)$, $B(3;3;0)$, $C(0;3;3)$ and $F(3;3;3)$.



1) Show that the triangle ABC is equilateral .
2) Determine an equation of plane (P) determined by A , B and C .
3) Show that the straight line (OF) is the axis of circle (γ) circumscribed about triangle ABC .
4) Let B' be the point diametrically opposite to B on (γ), write a system of parametric equations of the tangent (T) at B' to (γ).
5) The line (T) cuts the straight lines (AB) and (BC) respectively at G and L.
   a- Calculate GL.
   b- Calculate the area of quadrilateral $ACLG$.

N°28.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, Consider the plane (P) of equation $3x - 4y + 5z = 0$ and the plane (Q) of equation $4x + 3y + 5z = 25$.

The straight line (d) is the intersection of (P) and (Q).

Let $w(-3;4;0)$ be a point of the space.

1) Calculate the acute angle of the two planes (P) and (Q).
2) Prove that a system of parametric equations of the line (d) is: $x = -7m + 4$, $y = m+3$, $z = 5m$ m being a real parameter .
3) a- Show that w is equidistant from (P) and (Q).
   b- Deduce an equation of one of the bisector planes of the dihedral formed by (P) and (Q).
4) Let w' be the orthogonal projection of w on (P), and w'' the orthogonal projection of w on (Q).
   a- Without finding w' and w'', determine an equation of plane (H) passing through w, w' and w''.
   b- Find w' and w'', then, find again, the equation of (H).
   c- Calculate the distance from w to (d).

Supplementary Problems

N° 1.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, Consider the point $A(1;1;2)$ and the straight line (d) of parametric equations $x=m-1$, $y=2m$, $z=-m+2$.

1) Write an equation of plane (P) determined by A and (d).
2) Calculate the distance of point A to (d).
3) Let (C) be the circle of plane (P), of center A and radius $\sqrt{3}$.
   Calculate the coordinates of the common points of (C) and (d).
4) Consider the plane (Q) of equation $3x-y+z+1=0$.
   a- Show that (P) and (Q) intersect along the straight line (d).
   b- Let B be the orthogonal projection of A on (Q), calculate the coordinates of B.
   c- Let I be the orthogonal projection of A on (d), calculate $\cos(A\hat{I}B)$.

N° 2.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the straight lines (d) and (d') defined by :

$(d): x=t+1$, $y=2t$, $z=t-1$ and $(d'): x=2m$, $y=-m+1$, $z=m+1$.

1) Prove that (d) and (d') are skew.
2) a- Show that $x-y+z=0$ is an equation of the plane (P) determined by O and (d).
   b- Determine the coordinates of the point E, the intersection of (P) and (d').
   c- Prove that the straight line (OE) intersects (d).
3) a- Calculate the distance of the point O to the straight line (d).
   b- Deduce that the circle, in plane (P), of center O and passing through E, is tangent to (d).

N°3.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the points $A(1;1;2)$, $B(2;-1;-1)$, $C(0;-3;-1)$ and $E(-3;2;1)$.

1) Verify that an equation of plane (P) containing the three points A , B , C is $x-y+z-2=0$.
2) Prove that the point $F(1;-2;5)$ is the symmetric of E with respect to plane (P).
3) Write a system of parametric equations of the interior bisector of angle $E\hat{A}F$.
4) Let (d) be the straight line defined by $x=1$, $y=m$, $z=-m+3$ m is a real parameter.
   Show that (d) is the symmetric of (AE) with respect to (P).
5) Let (δ) be the straight line perpendicular at A to plane (P) and let M be any point of (δ). Prove that M is equidistant of (d) and (AE).
6) Determine the points M of (δ) so that the volume of the tetrahedron $MABC$ is double that of $EABC$.

N°4.

n the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, Consider the two straight lines:

(D) : $x=2m+3$, $y=m-1$, $z=3m$ and

(D') : $x=3t+5$, $y=2t-1$, $z=t$

1) Show that (D) and (D') are skew.
2) Calculate the cosine of the acute angle formed by (D) and (D') .
3) Find the parametric equations of the straight line (AB) which is the common perpendicular of (D) and (D').
   $(A\in(D)$ and $B\in(D'))$.

N°5.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, Consider the two planes $(P): 2x-y+2z-5=0$ and $(Q): 2x+2y-z-4=0$.

1) Show that (P) and (Q) are perpendicular .
2) Let $A(1,2,-1)$ be a point in space.
   a- Calculate the distance of point A to the plane (P) and the distance of point A to the plane (Q).
   b- Deduce the distance of point A to the straight line $(d)=(P)\cap(Q)$.

N°6.

In the space referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, Consider the points $A(2;1;1)$ and $B(0;-1;3)$, the plane (P) of equation $x+y-z+1=0$, and the straight line (d) of parametric equations : $x=\lambda+1$, $y=2\lambda-3$, $z=3\lambda-1$.

1) Show that (d) lies in (P) .
2) Find an equation of the plane (Q) containing (d) and A.
3) Find an equation of the plane (R) containing (d) and B.
4) Prove that (Q) and (R) are symmetric with respect to plane (P) .
5) Let M be a variable point of (d) and K the midpoint of $[AB]$.
   a- Calculate , in terms of λ , the area S of triangle ABM.
   b- Calculate λ so that S is minimal and deduce the distance of point K to the straight line (d) .

N°7.

Consider in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ the straight lines (d) and (d') of parametric equations :

(d) : $x=m-1$, $y=2m$, $z=m-1$ and

(d') : $x=-t$, $y=t+2$, $z=-2t$, m and t are real parameters .

1) Find the coordinates of the point I, the intersection of (d) and (d').
2) Prove that an equation of the plane (P) determined by (d) and (d') is: $-5x+y+3z-2=0$.
3) Let (D) be the line passing through I and perpendicular to (P), find a system of parametric equations of (D).
4) Let $\vec{v}$ and $\vec{v'}$ be the respective directing vectors of (d) and (d').

a- Determine the vector $\vec{v}+\vec{v'}$.

b- Deduce a system of parametric equations of one of the bisectors of the angles determined by (d) and (d').

N° 8.

Consider in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ a straight line (D), A and B are any two points of (D) and M is a point in space. The orthogonal projection of M on (D) is H.

1) a- Show that : $\overrightarrow{MA}\times\overrightarrow{AB}=\overrightarrow{MH}\times\overrightarrow{AB}$.

b- Deduce that, $MH=\dfrac{\|\overrightarrow{MA}\times\overrightarrow{AB}\|}{\|\overrightarrow{AB}\|}$

2) Calculate the distance from $M(0,1,1)$ to the straight line (D) defined by : (D) : $x=m-1$, $y=m$, $z=m+1$.

N° 9.

A, B and C are respectively the three points of intersection of a plane (P) of equation $x+2y+3z-6=0$ with the coordinates axes in an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.

1) Determine the coordinates of A, B and C.
2) Let H be the orthogonal projection of O on the line (BC).

a- Show that (BC) is orthogonal to the plane (AOH).

b- Deduce an equation of the plane (AOH) and find the coordinates of H.

3) Determine a system of parametric equations of the straight line (d) passing through O and perpendicular to plane (P).

N° 10.

Consider in the space of orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ the points $A(1;1;1)$, $B(-1;2;2)$ and $C(0;-1;4)$.

1) Find an equation of the plane (P) determined by A, B and C.
2) a- Determine an equation of the plane (Q) passing through A and perpendicular to line (BC).

b- Deduce a system of parametric equations of the height issued from A in triangle ABC.

3) a- Determine a system of parametric equations of the line (AC).

b- Determine the coordinates of the point H, the orthogonal projection of B on (AC).

c- Deduce again a system of parametric equations of the height issued from B in triangle ABC.

N° 11.

Consider in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ the points $A(1;-1;-2)$, $B(0;1;2)$ and the plane (P) of equation $x-2y+z-1=0$.

1) a- Determine a system of parametric equations of the line (AB).

b- Show that (AB) cuts (P) in A.

c- Calculate the cosine of the acute angle formed by (AB) and (P).

2) a- Determine the coordinates of the point H, the orthogonal projection of B on (P).

b- Deduce the distance from B to (P).

c- Find an equation of the mediator plane (Q) of $[AB]$.

d- The plane (Q) cuts (AH) in a point J, determine a system of parametric equations of the line (d) in (P) which is perpendicular to (AH) at J.

N° 12.

Consider in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ the plane (P) of equation $x-3y+2z+6=0$ and the plane (Q) of equation $3x-y+2z+2=0$.

1) Show that (P) and (Q) intersect, and determine a system of parametric equations of their line of intersection (d).

2) Determine a system of parametric equations of the line (δ) passing through $A(1;0;1)$ and parallel to the two planes (P) and (Q).

3) Calculate the distance between the lines (d) and (δ).

N° 13.

Consider in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ the points $A(1;0;1)$, $B(-1;1;0)$ and $C(0;2;2)$.

1) Determine an equation of the plane (P) determined by the points

A , B and C.

2) Let $I(2;1;0)$ be a point in the space .
   a- Show that A is the orthogonal projection of I on (P).
   b- Calculate the volume of tetrahedron $IABC$ and deduce the area of the triangle ABC.
3) a- Show that triangle ABC is isosceles of vertex A .
   b- Deduce a system of parametric equations of the height issued A in the triangle ABC.

N° 14.

Consider in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ the two straight lines (D_1) and (D_2) defined by :

(D_1) : $x = m-1$, $y = m$, $z = m+1$ et

(D_2) $x = 2t$, $y = t$, $z = t+2$. m and t are real parameters.

1) Show that (D_1) and (D_2) are skew .
2) Show that an equation of the plane (P) containing (D_1) and parallel to (D_2) is: $y - z + 1 = 0$.
3) $A(-2;-1;1)$ is a point on (D_2).
   a- Determine the coordinates of the point H the orthogonal projection of A on (P) :
   b- Deduce a system of parametric equations of the line (d) the orthogonal projection of (D_2) on (P) .

N° 15.

Consider in the space of an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ the two straight lines (d) and (d') defined by :

(d) : $x = -m$, $y = 5m+6$, $z = 4m+2$ and

(d') $x = 4\lambda+1$, $y = \lambda+1$, $z = 5\lambda-2$.

t and m are two real parameters .

1) Determine the coordinates of the point A the intersection of (d) and (d').
2) Determine an equation of the plane (P) determined by (d) and (d').

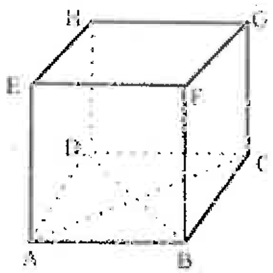
3) Let (δ) be a straight line defined by (δ) : $x = 5t + 2$, $y = -4t + 3$, $z = t + 1$.

a- Show that (δ) lies in (P).

b- Determine an equation of the plane (Q) containing (δ) and perpendicular to plane (P).

c- Show that (Q) cuts (d) and (d') in two points B and C ,whose coordinates are to be determined , verify that the triangle ABC is equilateral .

$\boxed{N^\circ 16.}$

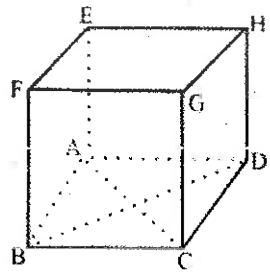


$ABCDEFGH$ is a cube, where $\left(A, \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AE}\right)$ is the system of reference.

Let I be the midpoint of $[BD]$.

1) Show that the two planes (DBG) and (AEI) are perpendicular .
2) Determine an equation of the plane (EDB).
3) Let J be the orthogonal projection of A on the plane (EDB).
   a- Find the coordinates of J .
   b- Show that the point J is the orthocenter of the triangle EDB.
   c- Show that the points A , J , G are collinear .
4) a- Calculate the area of the triangle EDB.
   b- Calculate the volume of the tetrahedron $FEDB$ and deduce the distance from F to the plane (EDB) .

$\boxed{N^\circ 17.}$



$ABCDEFGH$ is a cube of edge 1 cm.

O is the center of the square $ABCD$.

1) Show that $\overrightarrow{DA} \wedge \overrightarrow{AC} = \overrightarrow{DO} \wedge \overrightarrow{AC}$.
2) Determine the set of points M of the space such that $\left(\overrightarrow{AB} \wedge \overrightarrow{AD}\right) \wedge \overrightarrow{AM} = \vec{0}$.
3) Suppose that the system is referred to the orthonormal system $\left(A; \overrightarrow{AB}, \overrightarrow{AD}, \overrightarrow{AE}\right)$.
   Let I be the midpoint of $[DH]$ and J that of $[BF]$.

a- Show that a system of parametric equations of the straight line (IJ) is : $x = t$; $y = -t + 1$; $z = \frac{1}{2}$.

b- Find an equation of the plane (AIJ).

c- Calculate the cosine of the acute angle of the two planes (AIJ) and (ABD).

d- Let M be a point of the straight line (IJ).
Calculate t so that S_t is minimum.
Precise in this case the position of point M on $[IJ]$ and on $[AG]$.

N° 18.

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the plane (P) of equation $2x + y - 2z - 2 = 0$ and the points $A(-1;1;3)$, $B(1;2;1)$ and $C(0;4;1)$.

1) Show that the straight line (AB) is perpendicular to the plane (P) at B
2) Let (T) be the circle of (P) of center B and radius $\sqrt{5}$.
Show that the point C belongs to (T).
3) Write an equation of plane (Q) determined by A, B and C.
4) Designate by (d) the straight line perpendicular to (Q) at C.
   a- Write a system of parametric equations of (d).
   b- Calculate the distance of A to (d).
   c- Prove that the straight line (d) is tangent to circle (T).

Solutions

N° 1.

1) $\overrightarrow{AB} \wedge \overrightarrow{AC} = -6\vec{i} + 6\vec{j} - 6\vec{k}$ is a normal vector of plane (P), then $-6x + 6y - 6z + r = 0$. C belongs to (P), so $-18 + 6 + r = 0$ which gives $r = 12$, thus an equation of plane (P) is
$-6x + 6y - 6z + 12 = 0$ or $-x + y - z + 2 = 0$.
2) $B(-1;0;0)$ is a point of (d), $\overrightarrow{AB} \wedge \vec{v}_d = 3\vec{i} - 6\vec{j}$ is a normal vector of (P), then $x - 2y + r = 0$. B belongs to (P), so $-1 + r = 0$ which gives $r = 1$, thus an equation of plane (P) is $x - 2y + 1 = 0$.
3) $\overrightarrow{v_d} \wedge \overrightarrow{v_{d'}} = 2\vec{i} - \vec{j} + 3\vec{k}$ is a normal vector of plane (P), then $2x - y + 3z + r = 0$. The point $I(-1;0;0)$ of (d) belongs to (P), so $r = 2$, thus an equation of (P) is $2x - y + 3z + 2 = 0$.
4) $\overrightarrow{v_d} \wedge \overrightarrow{v_{d'}} = -2\vec{i} + 3\vec{j} - \vec{k}$ is a normal vector of (P), then $-2x + 3y - z + r = 0$. The point $I(-1;0;1)$ of (d) belongs to (P), thus $r = -1$, an equation of (P) is $-2x + 3y - z - 1 = 0$.
5) The two straight lines are parallel.
$A(-1;0;1)$ is a point of (d) and O a point of (d').
$\overrightarrow{OA} \wedge \vec{v}_d = -\vec{i} + 2\vec{j} - \vec{k}$ is a normal vector of plane (P), so $-x + 2y - z + r = 0$. O belongs to (P), Thus $r = 0$, consequently $-x + 2y - z = 0$ is an equation of plane (P).
6) The vector $\overrightarrow{AB}(2;2;2)$ is a normal vector to this plane , so $2x + 2y + 2z + r = 0$, the midpoint $I(0;1;2)$ of $[AB]$ belongs to (P), thus $r = -6$, consequently $x + y + z - 3 = 0$.

N° 2.

1) $d(A;(P)) = \dfrac{|-1+1-1|}{\sqrt{1+1+1}} = \dfrac{\sqrt{3}}{3}$.

2) $B(-1;0;1)$ is a point of (d), $d\left(A;(d)\right)=\dfrac{\left\|\overrightarrow{AB}\wedge\vec{v}_d\right\|}{\left\|\vec{v}_d\right\|}$

But, $\overrightarrow{AB}\wedge\vec{v}_d=-\vec{i}+2\vec{j}-\vec{k}$, then

$d\left(A;(d)\right)=\dfrac{\sqrt{1+4+1}}{\sqrt{1+1+1}}=\sqrt{2}$.

3) The two straight lines are parallel.
$I(-1;-1;1)$ is a point of (d) and O a point of (d'),

$$d\left((d);(d')\right)=d\left(O;(d)\right)=\frac{\left\|\overrightarrow{OI}\wedge\vec{v}_d\right\|}{\left\|\vec{v}_d\right\|}=\frac{\sqrt{4+4}}{\sqrt{1+1+1}}=\frac{2\sqrt{6}}{3}$$

But, $\overrightarrow{OI}\wedge\vec{v}_d=-2\vec{i}+2\vec{j}$ then $d\left((d);(d')\right)=\dfrac{\sqrt{4+4}}{\sqrt{1+1+1}}=\dfrac{2\sqrt{6}}{3}$.

4) The two planes are parallel.
Since O is a point of (P) then the distance between the two planes is : $d\left((P);(Q)\right)=d\left(O;(Q)\right)=\dfrac{1}{\sqrt{4+4+4}}=\dfrac{\sqrt{3}}{6}$.

N° 3.

1) The acute angle of the two planes (P) and (Q) is equal to the acute angle α of the two normal vectors.

But, $\overrightarrow{n_P}(2;1;-1)$ and $\overrightarrow{n_Q}(1;2;-1)$, then $\cos\alpha=\dfrac{\left|\overrightarrow{n_P}.\overrightarrow{n_Q}\right|}{\left\|\overrightarrow{n_P}\right\|\left\|\overrightarrow{n_Q}\right\|}=\dfrac{5}{6}$.

2) Let β be the acute angle of the two vectors $\overrightarrow{n_P}(2;1;-1)$ and $\overrightarrow{v_d}(1;1;1)$,

$\cos\beta=\dfrac{2+1-1}{\sqrt{6}\sqrt{3}}=\dfrac{\sqrt{2}}{3}$.

The acute angle α between (P) and (d) is equal to $\dfrac{\pi}{2}-\beta$.

Hence, $\cos\alpha=\cos\left(\dfrac{\pi}{2}-\beta\right)=\sin\beta=\sqrt{1-\cos^2\beta}$

$=\sqrt{1-\frac{2}{9}}=\frac{\sqrt{7}}{3}$.

3) The acute angle of the two straight lines (d) and (d') is equal to the acute angle α of the two vectors 2 $\overrightarrow{v_d}(1;2;1)$ and $\overrightarrow{v_{d'}}(2;-1;1)$

thus, $\cos\alpha=\frac{\left|\overrightarrow{v_d}.\overrightarrow{v_{d'}}\right|}{\left\|\overrightarrow{v_d}\right\|\left\|\overrightarrow{v_{d'}}\right\|}=\frac{|2-2+1|}{\sqrt{6}\sqrt{6}}=\frac{1}{6}$.

N°4.

1) $\overrightarrow{n_P}(2;1;-1)$ is a directing vector of (d) and A is a point of (d),
Thus, : $(d): x=2m-1,\ y=m,\ z=-m+1$.

2) (d) is parallel to the line of intersection of the two planes (P) and (Q), then $\overrightarrow{n_P}\wedge\vec{n}_Q=\vec{i}+\vec{j}+3\vec{k}$ is a directing vector of (d).
Thus, $(d): x=m+1,\ y=m-1,\ z=3m+2$.

3) Letting $y=t$, we get the system $\begin{cases}2x-z=-t+1\\x-z=-2t+2\end{cases}$ and by subtraction, we get $x=t-1$. Replacing x by its value in one of the two equations, we get $z=3t-3$, consequently $(d): x=t-1,\ y=t$, $z=3t-3$.

4) The two straight lines (d) and (d') intersect at point $I(0;0;1)$.
Note that $\|\vec{v}_d\|=\|\vec{v}_{d'}\|=\sqrt{6}$ so the vector $\vec{S}=\vec{v}_d+\vec{v}_{d'}=\vec{i}+\vec{j}+2\vec{k}$ is a directing vector of the bisector (δ) of
one of the two angles formed by the two straight lines, thus (δ) :
$x=\lambda,\ y=\lambda,\ z=2\lambda+1$.

5) The two straight lines (d) and (d') intersect at point $I(0;0;1)$
$\vec{u}(-1;2;1)$ is a directing vector of (d) and $\vec{v}(1;-1;1)$ is a directing vector of (d') Note that $\|\vec{u}\|=\sqrt{6}$ and $\|\vec{v}\|=\sqrt{3}$.
Since $\|\vec{u}\|\neq\|\vec{v}\|$, we consider the two vectors:
$\vec{u}'=\sqrt{3}\vec{u}=\left(-\sqrt{3};2\sqrt{3};\sqrt{3}\right)$ and $\vec{v}'=\sqrt{6}\vec{v}=\left(\sqrt{6};-\sqrt{6};\sqrt{6}\right)$.
$\|\vec{u}'\|=\|\vec{v}'\|=\sqrt{18}$ then $\vec{S}=\vec{u}'+\vec{v}'=\left(\sqrt{6}-\sqrt{3};-\sqrt{6}+2\sqrt{3};\sqrt{6}+\sqrt{3}\right)$

is a directing vector of the bisector (δ) of one of the angles formed by the two straight lines. Thus, (δ) : $x=\left(\sqrt{6}-\sqrt{3}\right)\lambda$,

$y=\left(-\sqrt{6}+2\sqrt{3}\right)\lambda$, $z=\left(\sqrt{6}+\sqrt{3}\right)\lambda+1$.

N° 5.

1) Comparing the last two equations of the system $\begin{cases} m=t \\ -m=-t \\ -m+1=t+1 \end{cases}$ we get $m=0$ which gives $t=0$ and the first equation $m=t$ is verified. Thus, the two straight lines intersect at point $I(0;0;1)$.

2) Let J be the point of intersection of (d) and (P).
J belongs to (d) then $J(t;-t;t+1)$ and since J belongs to (P) we get $t-2t-t-1-1=0$, then $t=-1$ thus $J(-1;1;0)$.

3) H belongs to (d) then $H(t;-t;t+1)$ so $\overrightarrow{AH}(t+1;-t;t)$.
$\overrightarrow{AH}.\vec{v}_d=t+1+t+t=0$ gives $t=-\frac{1}{3}$, consequently $H\left(-\frac{1}{3};\frac{1}{3};\frac{2}{3}\right)$.

4) a- Let (d) be the straight line passing through A and perpendicular to (P): $x=t-1$, $y=-t$, $z=-t+1$, H is the point of intersection of (d) and (P), then $t-1+t+t-1-1=0$ so $t=1$. Consequently, $H(0;-1;0)$.

b- H is the midpoint of $[AA']$, so $x_H=\frac{x_A+x_{A'}}{2}$, then $x_{A'}=2x_H-x_A=1$, similarly $y_{A'}=2y_H-y_A=-2$ and $z_{A'}=2z_H-z_A=-1$, hence $A'(1;-2;-1)$.

N° 6.

1) Let $I(-1;0;1)$ be a point of (d), $\overrightarrow{EI}\wedge\overrightarrow{v_d}(1;0;-1)$ is a normal to plane (P), then $x-z+r=0$ is an equation of (P) and since E belongs to (P) then $r=2$ consequently $x-z+2=0$ is an equation of (P).

2) a- $d(E;(d)) = \dfrac{\left\| \overrightarrow{EI} \wedge \overrightarrow{V_d} \right\|}{\left\| \overrightarrow{V_d} \right\|} = \dfrac{\left\| \vec{i} - \vec{j} \right\|}{\sqrt{3}} = \dfrac{\sqrt{2}}{\sqrt{3}} = \dfrac{\sqrt{6}}{3}$.

$d(E;(d)) < R$; then (C) cuts (d) in two points A and B.

b- A is a point of (d), then $A(m-1;m;m+1)$, $EA = 1$

gives $(m-1)^2 + m^2 + (m-1)^2 = 1$ which gives $m' = 1$, $m'' = \dfrac{1}{3}$.

Thus, $A(0\,;1\,;2)$ and $B\left(-\dfrac{2}{3};\dfrac{1}{3};\dfrac{4}{3}\right)$.

N°7.

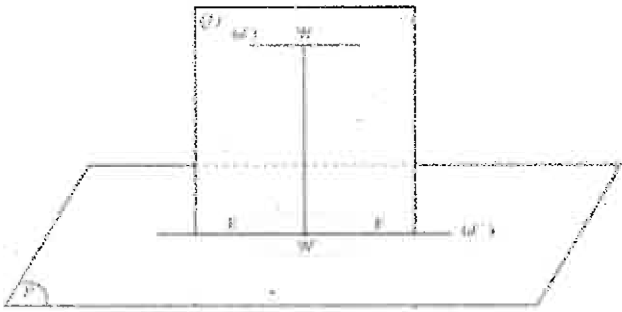
1) For $\lambda = 0$ the point $w\ (4;1;4)$ belongs to (d).

$\overrightarrow{v_d}\,(1;1;-1)$ is a directing vector of (d) $\overrightarrow{n_p}\,(1;2;3)$ is a normal vector of (P). They are perpendicular since $\overrightarrow{v_d}.\overrightarrow{n_p} = 1+2-3 = 0$, and since the point $w\ (4;1;4)$ of (d) does not belong to (P) then (d) is parallel to (P).

Another method : We can find the intersection of (d) and (P). If M is a point of 'intersection of (d) and (P) then $M(\lambda+4;\lambda+1;-\lambda+4)$ and replacing the coordinates of M in the equation of (P), we get $\lambda + 4 + 2(\lambda+1) + 3(-\lambda+4) = 4$, which gives $18 = 4$, which is impossible hence (d) is parallel to (P).

2) a- The vector $\overrightarrow{n_P} \wedge \overrightarrow{v_d}$ is normal to (Q).

But, $\overrightarrow{n_P} \wedge \overrightarrow{v_d} = -5\vec{i} + 4\vec{j} - \vec{k}$, then $-5x + 4y - z + r = 0$ is an equation of (Q). The point $w\ (4;1;4)$ of (d) belongs to (Q) then $-20 + 4 - 4 + r = 0$, so $r = 20$ and consequently an equation of (Q) is: $-5x + 4y - z + 20 = 0$.



b- The straight line (d') is the intersection of the two planes (P) and (Q)

$(d')\begin{cases} x+2y+3z-4=0 \\ -5x+4y-z+20=0 \end{cases}$, letting $z=t$, the system becomes

$\begin{cases} x+2y=-3t+4 \\ -5x+4y=t-20 \end{cases}$ which is equivalent to the system

$\begin{cases} 5x+10y=-15t+20 \\ -5x+4y=t-20 \end{cases}$

Adding, we get $14y=-14t$, so $y=-t$ and consequently $x=-t+4$ thus:

(d'): $x=-t+4,\ y=-t,\ z=t$.

3) $x_{w'}+2y_{w'}+3z_{w'}=3-2+3=4$ then w' belongs to (P) and on the other hand $\overrightarrow{ww'}(-1;-2;-3)$ is parallel to the vector $\overrightarrow{n_P}(1;2;3)$.Thus, w' is the orthogonal projection of w on (P).
Note that w' belongs to (d').

4) a- The points A and B are the points of intersection of the straight line (d') with (C).

Then, $A(-t+4;-t;t)$, since $w'A=R=\sqrt{3}$, we get:

$(-t+4-3)^2+(-t+1)^2+(t-1)^2=3$, so $3(t-1)^2=3$,which gives $t-1=+1$ or $t-1=-1$, so $t=2$ or $t=0$ and consequently, $A(2;-2;2)$ and $B(4;0;0)$.

b- Let (T_1) and (T_2) be the tangents at A and B to (C).

(T_1) is perpendicular to the two straight lines (AB) and (ww') then (T_1) is perpendicular to plane (Q) thus $\vec{n}_Q(5;-4;1)$ is a directing vector of (T_1) consequently (T_1): $x=5\lambda+2$, $y=-4\lambda-2$, $z=\lambda+2$ similarly (T_2) $x=5k+4$,, $y=-4k$, $z=k$.

5) F belongs to (C), $[AB]$ is a diameter of (C) and

$A\hat{B}F = 30^\circ$ then ABF is a semi- equilateral triangle, with $B\hat{A}F = 60^\circ$ thus $AF = R = \sqrt{3}$ and $BF = R\sqrt{3} = 3$.

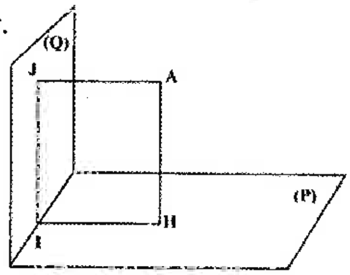
Area of$(ABF) = \frac{1}{2} AF \times BF = \frac{1}{2} \times \sqrt{3} \times 3 = \frac{3\sqrt{3}}{2}$ square units.

M belongs to (d) and (d) is parallel to (P) then the distance of M to plane (P) is constant and it is equal to $ww' = \sqrt{1+4+9} = \sqrt{14}$,

$\text{V}(MABF) = \frac{1}{3}\text{Area of}\left(ABF\right) \times ww' = \frac{1}{3} \times \frac{3\sqrt{3}}{2} \times \sqrt{14} = \frac{\sqrt{42}}{2}\ u^3$.

$\boxed{N^\circ 8.}$

1) The normal vectors $\overrightarrow{n_P}(1;2;2)$ and $\overrightarrow{n_Q}(2;1;-2)$ of these two planes are orthogonal since , then $\overrightarrow{n_P}.\overrightarrow{n_Q} = 2+2-4 = 0$, hence (P) and (Q) are perpendicular.



2) a- $d_1\left(A;(P)\right) = \frac{|2+2-2-1|}{\sqrt{1+4+4}} = \frac{1}{3}$

$d_2\left(A;(Q)\right) = \frac{|4+1+2+1|}{\sqrt{4+1+4}} = \frac{8}{3}$.

b- $AHIJ$ is a rectangle then $AI^2 = d_1^{\,2} + d_2^{\,2} = \frac{1}{9} + \frac{64}{9} = \frac{65}{9}$

consequently, $AI = \frac{\sqrt{65}}{3}$.

3) Letting $z = t$, we get the system :
$\begin{cases} x+2y = -2t+1 \\ 2x+y = 2t-1 \end{cases}$ that has as a solution $x = 2t-1$ and $y = -2t+1$.

Hence, $(d): x = 2t-1,\ y = -2t+1,\ z = t$.

4) a- (R) is perpendicular to the two planes (P) and (Q) then (R) is perpendicular to (d) thus $\overrightarrow{v_d}(2;-2;1)$ is normal to (R).

An equation of plane (R) is $2x-2y+z+r=0$ and since A belongs to (R) then $r = -1$ consequently $2x-2y+z-1=0$ is an equation of plane (R).

b- The point I is the point of intersection of (R) and (d).

Then : $2(2t-1)-2(-2t+1)+t-1=0$ which gives $t=\frac{5}{9}$

Consequently, $J\left(\frac{1}{9};-\frac{1}{9};\frac{5}{9}\right)$.

N°9.

1) a- $x_A+y_A+2z_A-2=1+1-2=0$, then the point A belongs to plane (P).
Similarly, $x_B+y_B+2z_B-2=2-2=0$ then the point B belongs to the plane (P).
Similarly, for C so it belongs to (P).
Hence, $x+y+2z-2=0$ is an equation of the plane (P) determined by A, B and C.

b- $\overrightarrow{AE}(1;1;2)$ and $\vec{n}_P(1;1;2)$ are equal then (AE) is perpendicular to plane (P) .

c- $S=\frac{1}{2}\|\overrightarrow{AB}\wedge\overrightarrow{AC}\|=\frac{\sqrt{1+1+4}}{2}=\frac{\sqrt{6}}{2}$ square units.

$\text{V}(EABC)=\frac{1}{3}\text{Area of}(ABC)\times AE=\frac{1}{3}\times\frac{\sqrt{6}}{2}\times\sqrt{6}=1\text{ u}^3$.

2) a- Let $M(x;y;z)$ be a point of (Q), $\overrightarrow{LM}\cdot(\overrightarrow{AE}\wedge\overrightarrow{BC})=0$

with $L\left(\frac{3}{2};\frac{1}{2};0\right)$; then $\begin{vmatrix} x-\frac{3}{2} & y-\frac{1}{2} & z \\ 1 & 1 & 2 \\ -1 & 3 & -1 \end{vmatrix}=0$

consequently, $(Q): 7x+y-4z-11=0$.

b- $\vec{n}_P.\vec{n}_Q=-7-1+8=0$. Then, the planes (P) and (Q) are perpendicular.

c- (BC) // (Q) and (BC) is a straight line of (P) , then (BC) is parallel to the straight line of intersection of (P) and (Q).

OR :

$$(d) = (P) \cap (Q) : \begin{cases} x + y + 2z - 2 = 0 \\ 7x + y - 4z - 11 = 0 \end{cases} ;$$

$(d): x = t + \frac{3}{2},\ y = -3t + \frac{1}{2},\ z = t$, note that $\overrightarrow{BC} = -\vec{v}_d$, thus (BC) is parallel to the line of intersection of (P) and (Q).

N° 10.

1) The coordinates of A, B and C satisfy the given equation since:
$2x_A + 2y_A + z_A - 2 = 2 + 0 + 0 - 2 = 0$,
$2x_B + 2y_B + z_B - 2 = 0 + 2 + 0 - 2 = 0$
$2x_C + 2y_C + z_C - 2 = 0 + 2 + 0 - 2 = 0$

2) a- $\overrightarrow{CH}\left(\frac{4}{9};\frac{4}{9};-\frac{16}{9}\right)$ and $\overrightarrow{CI}\left(\frac{1}{2};\frac{1}{2};-2\right)$ then $\overrightarrow{CH} = \frac{8}{9}\overrightarrow{CI}$, consequently C, H and I are collinear.

b- $\vec{n}(2;2;1)$ is a normal vector of (ABC) and since $\overrightarrow{OH}\left(\frac{4}{9};\frac{4}{9};\frac{2}{9}\right) = \frac{2}{9}\vec{n}$ then (OH) is perpendicular to plane (ABC)

c- (OH) is perpendicular to plane (ABC) and $(OH) \subset (OCI)$ then the plane (OCI) is perpendicular to plane (ABC).

Another method :
The vector $\vec{n}' = \overrightarrow{OI} \wedge \overrightarrow{OC} = \vec{i} - \vec{j}$ is normal to the plane (OCI) and since $\vec{n}'.\vec{n} = 0$ then the two planes (OIC) and (ABC) are perpendicular.

3) a- The vector $\overrightarrow{OB}(0;1;0)$ is a directing vector of (Δ), and C is a point of (Δ) ,consequently, $(\Delta): x = 0$, $y = t$ $z = 2$.

b- (Δ) is parallel to plane (OAB), then the distance of F to (OAB) is constant consequently the volume of the tetrahedron is constant.

Area of triangle $OAB=\dfrac{OA\times OB}{2}=0.5\,\text{u}^2$ and

$d(F;OAB)=OC=2$, consequently, $\text{V}=\dfrac{0.5\times 2}{3}=\dfrac{1}{3}\text{u}^3$.

Another Method :

$\overrightarrow{OF}.\left(\overrightarrow{OA}\wedge\overrightarrow{OB}\right)=2$ (independent of t).

$V=\dfrac{1}{6}\left|\overrightarrow{OF}.\left(\overrightarrow{OA}\wedge\overrightarrow{OB}\right)\right|=\dfrac{2}{6}=\dfrac{1}{3}\text{u}^2$.

N° 11.

1) $\vec{v}_d(1;-1;-1)$ and $\vec{v}_{d'}(-1;1;1)$ are the directing vectors of (d) and (d'), since $\vec{v}_{d'}=-\vec{v}_d$ then the two straight lines are parallel.

$B(0;2;1)$ is a point of (d').

$\overrightarrow{AB}\wedge\vec{v}_d=-2\vec{i}-\vec{j}-\vec{k}$ is a normal vector of plane (P),

then $-2x-y-z+r=0$. A belongs to (P), so $r=3$, consequently

$-2x-y-z+3=0$ or $2x+y+z-3=0$ is an equation of (P).

$2x_E+y_E+z_E-3=-2+0-1-3=-6$, then E does not belong to (P).

2) a- $\overrightarrow{AE}\wedge\vec{v}_d=-2\vec{i}-4\vec{j}+2\vec{k}$ is a normal vector of (Q), then $-2x-4y+2z+r=0$. A belongs to (Q), so $r=0$, consequently $x+2y-z=0$ is an equation of plane (Q).

b- $d(E;(d))=\dfrac{\left\|\overrightarrow{AE}\wedge\vec{v}_d\right\|}{\left\|\vec{v}_d\right\|}=\dfrac{\sqrt{4+16+4}}{\sqrt{3}}=2\sqrt{2}$

$d(E;(P))=\dfrac{|-2+0-1-3|}{\sqrt{4+1+1}}=\sqrt{6}$.

c- (d) is the line of intersection of (P) and (Q), if (P) is perpendicular to (Q) then $d(E;(P))=d(E;(d))$ and since $d(E;(P))\neq d(E;(d))$ then the two planes are not perpendicular.

3) a- $\overrightarrow{EE'}(4;2;2)$ then $\overrightarrow{EE'}=2\vec{n}_P$ consequently, (EE') is perpendicular to (P).

The midpoint $I(1;1;0)$ of $[EE']$ belongs to (P) since $2x_I + y_I + z_I - 3 = 2+1+0-3 = 0$, then E' is the symmetric of E with respect to (P).

b- (δ) is the straight line (AE'), but $\overrightarrow{AE'}(2;2;0)$, hence : (AE') : $x = 2\lambda + 1$, $y = 2\lambda$, $z = 1$.

N° 12.

1) $\overrightarrow{AB}(1;1;2)$, $\overrightarrow{AC}(0;3;3)$, $\overrightarrow{AB} \wedge \overrightarrow{AC}(-3;-3;3)$ is normal to the plane (P), then $x+y-z+r=0$, A belongs to (P) so $r=2$. Consequently $x+y-z+2=0$ is an equation of (P).

2) a- $\overrightarrow{BC}(-1;2;1)$ is normal to plane (Q) and since A belongs to (Q), an equation of (Q) is then $-x+2y+z+4=0$.

b- This height (δ) is the intersection of (P) and (Q).
$(P): x+y-z+2=0$; (Q) $-x+2y+z+4=0$
Adding the two equations, we get : $3y+6=0$, so $y=-2$
We get $x-z=0$, letting $x=t$ we get $z=t$ consequently :
(δ): $x=t$, $y=-2$, $z=t$.

3) a- H belongs to (d) for $t=0$.

Since $\vec{v}_d = \vec{n}_P$ then (d) is perpendicular to plane (P) at H.

b- $HA = HB = HC = \sqrt{6}$ then H is the center of the circle circumscribed about triangle ABC.

c- Triangle AHB is isosceles of vertex H, then the median (HI) is a bisector of the angle $A\hat{H}B$.

But $I\left(\frac{3}{2};-\frac{3}{2};2\right)$ then $\overrightarrow{HI}\left(\frac{3}{2};-\frac{3}{2};0\right)$ and cosequently, (HI) : $x=k$, $y=-k$, $z=2$

d- $V(MABC) = 2V(EABC)$, then

$\frac{1}{3}$area of $(ABC) \times MH = 2 \times \frac{1}{3}$area of $(ABC) \times EH$

So we get $MH = 2EH$, but $M(t;t;-t+2)$, thus :

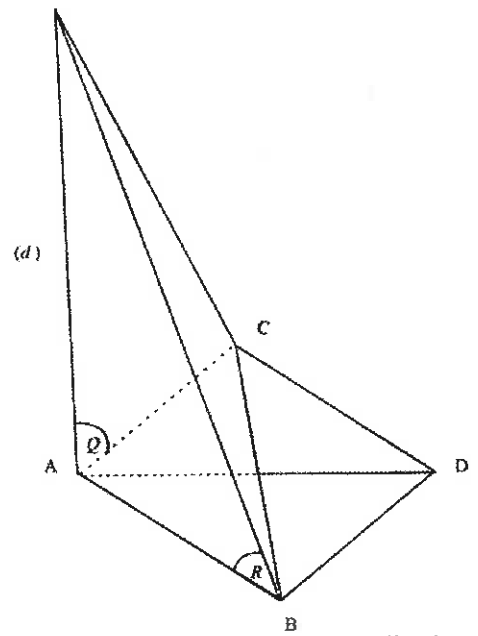
$\sqrt{t^2+t^2+t^2} = 2\sqrt{4+4+4}$, which gives $t=4$ or $t=-4$.

N° 13.

1) The three points A , B and C are not collinear since the vectors $\overrightarrow{AB}(2;1;-1)$ and $\overrightarrow{AC}(1;1;2)$ are not parallel , moreover, $3x_A - 5y_A + z_A + 3 = -3 + 3 = 0$, then the point A belongs (P).
   Similarly, $3x_B - 5y_B + z_B + 3 = 3 - 5 - 1 + 3 = 0$
   and $3x_C - 5y_C + z_C + 3 = 0$.Then, B and C belong to (P) and consequently, $3x - 5y + z + 3 = 0$ is an equation of plane (ABC).

2) a- $3x_D - 5y_D + z_D + 3 = 6 - 10 + 1 + 3 = 0$, then D belongs to plane (ABC).
   $AB = BD = DC = CA = \sqrt{6}$, then $ABDC$ is a rhombus.

   b- Since $ABDC$ is a rhombus then (AD) is a bisector of the angle $B\hat{A}C$.
   But, $\overrightarrow{AD} = 3\vec{i} + 2\vec{j} + \vec{k}$ hence, $(AD): x = 3m - 1,\ y = 2m$, $z = m$.



   c- The diagonals in a rhombus are perpendicular and bisect each other, then the distance of point A to the straight line (BC) is equal to $\frac{1}{2}AD = \frac{1}{2}\sqrt{9+4+1}$ so it is

$d = \frac{1}{2}\sqrt{14}$ length units.

3) a- The vector $\overrightarrow{n_p}(3;-5;1)$ is a directing vector of (d) and since A belongs to (d), we get: (d) : $x = 3t-1$ $y = -5t$ $z = t$, t is a real parameter.

b- The vector $\overrightarrow{v_d} \wedge \overrightarrow{AB}$ is a normal vector to plane (R) but $\overrightarrow{v_d} \wedge \overrightarrow{AB} = 4\vec{i} + 5\vec{j} + 13\vec{k}$, then an equation of (R) is $4x + 5y + 13z + r = 0$ and since A belongs to (R) then $-4 + r = 0$, so $r = 4$, consequently an equation of (R) is $4x + 5y + 13z + 4 = 0$.

c- The plane determined by the straight line (d) and the straight line (AD) is the bisector plane of the dihedral formed by (Q) and (R).

But, $\overrightarrow{v_d} \wedge \overrightarrow{AD} = -7\vec{i} + 21\vec{k}$, then an equation of the plane becomes $-7x + 21z + r = 0$, and since A belongs to the plane, then $7 + r = 0$, so $r = -7$ and consequently an equation of the plane is $-7x + 21z - 7 = 0$ or $-x + 3z - 1 = 0$.

N° 14.

1) The vector $\overrightarrow{AB}(-2;-1;-1)$ is normal to (P), an equation of (P) is then $-2x - y - z + r = 0$.

The midpoint $I\left(0;-\frac{1}{2};\frac{3}{2}\right)$ of $[AB]$ belongs to (P) then $\frac{1}{2} - \frac{3}{2} + r = 0$, which gives $r = 1$ and consequently an equation of (P) is $2x + y + z - 1 = 0$.

2) $C(1;-1;0)$ is a point of (d), the vector $\overrightarrow{AC} \wedge \overrightarrow{v_d}(1;2;-1)$ is normal to (Q).

An equation of (Q) is $x + 2y - z + r = 0$ but A belongs to (Q) then $1 - 2 + r = 0$, so $r = 1$, consequently an equation of (Q) is $: x + 2y - z + 1 = 0$.

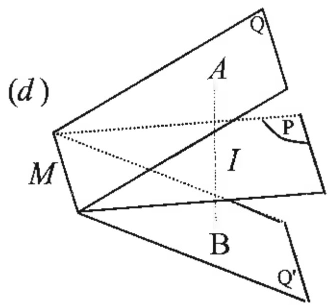
3) Let $M(-t+1; t-1; t)$ be a variable point of (d),

$2x_M + y_M + z_M - 1 = -2t + 2 + t - 1 + t - 1 = 0$ for all real numbers

t . Then, the straight line (d) lies in (P) ,and since (d) lies in (Q) then the 2 planes (P) and (Q) intersect along (d) .

4) The plane (Q') contains the straight line (d) since $t-1+t-1-2t+2=0$ and $B \in (Q')$ since $-x_B+y_B-2z_B+2=1-1-2+2=0$.
Then, the plane (Q') is the symmetric of (Q) with respect to plane (P).

5) $(AB)\perp(P)$, then $(AB)\perp(IM)$ so the distance of M to (AB) is equal to MI .
But, $\overrightarrow{MI}=-\vec{i}+\frac{1}{2}\vec{j}+\frac{3}{2}\vec{k}$ so $MI=\sqrt{1+\frac{1}{4}+\frac{9}{4}}=\frac{\sqrt{14}}{2}$.
Area of triangle MAB is equal to
$\frac{1}{2}IM\times AB=\frac{1}{2}\times\frac{\sqrt{14}}{2}\times\sqrt{6}=\frac{\sqrt{21}}{2}$ square units.



N° 15.

1) a- $O(0;0;0)$; $B(1;0;0)$; $D(0;1;0)$; $C(1;1;0)$; $E(0;0;1)$; $F(1;0;1)$, $H(0;1;1)$ and $G(1;1;1)$.
The three points O, F and H are not collinear and
$x_O+y_O-z_O=0$; $x_F+y_F-z_F=1-1=0$;
$x_H+y_H-z_H=1-1=0$.
Then, the coordinates of the points O, F and H satisfy the equation $x+y-z=0$ and consequently $x+y-z=0$ is an equation of plane (OFH).

b- The vector $\overrightarrow{EC}(1;1;-1)$ is normal to plane (OFH) and since $CO=CF=CH=\sqrt{2}$ then C is equidistant from the three points O, F and H , thus (CE) is the axis of the circle circumscribed about triangle OFH.

2) a- The distance of point K to the plane (OFH) is :

$$d = \frac{|x_K + y_K - z_K|}{\sqrt{1+1+1}} = \frac{3}{\sqrt{3}} = \sqrt{3}.$$

b- The distance of point K to the straight line (OF) is given by:

$$d' = \frac{\left\|\overrightarrow{KO} \wedge \overrightarrow{OF}\right\|}{\left\|\overrightarrow{OF}\right\|}, \text{ but } \overrightarrow{KO} \wedge \overrightarrow{OF} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix} = \vec{i} - 2\vec{j} - \vec{k}$$

then $d' = \dfrac{\sqrt{1+4+1}}{\sqrt{2}} = \sqrt{3}$.

c- The two planes (KOF) and (OFH) intersect along the straight line (OF).

Since $d(K;(OF)) = d(K;(OFH))$ then the two planes are perpendicular.

3) a- I is the midpoint on $[EF]$ then $I\left(\frac{1}{2};0;1\right)$.

Area of the triangle OIJ is equal to $\frac{1}{2}\left\|\overrightarrow{OI} \wedge \overrightarrow{OJ}\right\|$.

But, $\overrightarrow{OI} \wedge \overrightarrow{OJ} = -\frac{1}{4}\vec{i} - \frac{1}{2}\vec{j} + \frac{1}{8}\vec{k}$ then :

$$\text{Area of}(OIJ) = \frac{1}{2}\sqrt{\frac{1}{16} + \frac{1}{4} + \frac{1}{64}} = \frac{\sqrt{21}}{16} \text{ square units.}$$

b- The volume of tetrahedron $BOIJ$ is equal to $\frac{1}{6}\left|\overrightarrow{OB}.\left(\overrightarrow{OI} \wedge \overrightarrow{OJ}\right)\right|$

$$\text{But, } \overrightarrow{OB}.\left(\overrightarrow{OI} \wedge \overrightarrow{OJ}\right) = \begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 0 & 1 \\ 0 & \frac{1}{4} & 1 \end{vmatrix} = -\frac{1}{4}, \text{ then}$$

$$V = \frac{1}{6}\left|-\frac{1}{4}\right| = \frac{1}{24} \text{ cubic units.}$$

Let h be the distance of point B to the plane (OIJ),

$V_T = \dfrac{\text{Area of}(OIJ)\times h}{3}$ then: $\dfrac{1}{24} = \dfrac{\frac{\sqrt{21}}{16}\times h}{3}$, which gives

$h = \dfrac{2\sqrt{21}}{21}$.

$\boxed{N^\circ 16.}$

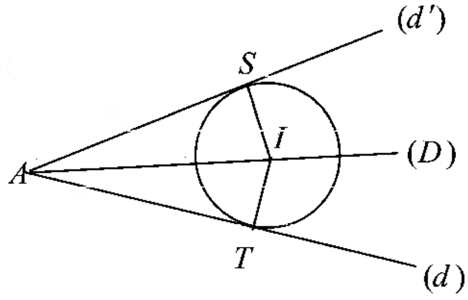
1) For $m = 1$, $x = 0$; $y = 2$ and $z = 2$,then the point $A \in (d)$.
   For $t = 0$, $x = 0$; $y = 2$ and $z = 2$, then the point $A \in (d')$.
   On the other hand, the two straight lines are not confounded since $\overrightarrow{v_d}(1;2;1)$ and $\overrightarrow{v_{d'}}(2;1;1)$ are not parallel.
   Then the straight lines (d) and (d') intersect at point A .
2) The vector $\overrightarrow{n_p} = \overrightarrow{v_d} \wedge \overrightarrow{v_{d'}} = \vec{i} + \vec{j} - 3\vec{k}$ is normal to (P) then $x + y - 3z + r = 0$ is an equation of (P) and since A belongs to (P) we get $2-6+r = 0$, so $r = 4$.
   Therefore, $x + y - 3z + 4 = 0$ is an equation de (P).
3) a- For the points of (D) to be equidistant of (d) and (d'), we should prove that (D) is a bisector of one of the angles formed by (d) and (d').
   The straight line (D) passes through the point A since for $\lambda = 0$, on a $x = 0$; $y = 2$ and $z = 2$.
   $\left\|\overrightarrow{v_d}\right\| = \left\|\overrightarrow{v_{d'}}\right\| = \sqrt{6}$, then $\overrightarrow{v_d} + \overrightarrow{v_{d'}}$ is a directing vector of one of the bisectors of the angles formed by (d) and (d'), and since $\overrightarrow{v_D} = \overrightarrow{v_d} + \overrightarrow{v_{d'}}$ then (D) is one of the bisectors of the angles formed by (d) and (d').

   b- The radius of (C) is equal to the distance of point I to the straight line (d) or (d').
   This distance is equal to $\dfrac{\left\|\overrightarrow{IA} \wedge \overrightarrow{v_d}\right\|}{\left\|\overrightarrow{v_d}\right\|}$ but $\overrightarrow{IA} \wedge \overrightarrow{v_d} = -\vec{i} - \vec{j} + 3\vec{k}$

   then $r = \dfrac{\sqrt{1+1+9}}{\sqrt{6}} = \dfrac{\sqrt{66}}{6}$.

$\overrightarrow{AI}(-3;-3;-2)$ then $AI=\sqrt{22}$.

$$AT^2=AI^2-IT^2=22-\frac{66}{36}=\frac{121}{36}\text{, so } AT=\frac{11\sqrt{6}}{6}.$$



Area of $(ATIS)=2\times$ Area of $(ATI)=2\left(\frac{IT\times AT}{2}\right)=\frac{11\sqrt{11}}{6}\text{u}^2$.

c- The vector $\overrightarrow{n_p}(1;1;-3)$ normal to (P) is a directing vector (δ) and since (δ) passes par A then the system of parametric equations of (δ) is $x=k$, $y=k+2$, $z=-3k+2$.
The mediator plane of $[TS]$ passes through the point I since $IT=IS=r$ and passes through the point A since $AT=AS$, Two tangents issued from the same point to the same circle.
This mediator plane contains the straight line (δ).
Hence, the vector $\overrightarrow{AI}\wedge\overrightarrow{v_\delta}$ is normal to the mediator plane.
$\overrightarrow{AI}\wedge\overrightarrow{v_\delta}=11\vec{i}-11\vec{j}$ then $11x-11y+r=0$ is an equation of the mediator plane, and since A belongs to this plane, then we get: $-22+r=0$, so $r=22$, consequently an equation of this plane is: $x-y+2=0$.

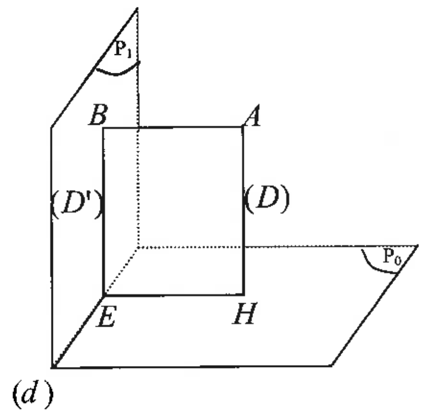
d- (Q) is parallel to (P), then $\overrightarrow{n_p}$ is normal to (Q), and since B belongs to (Q), an equation of (Q) is $x+y-3z-7=0$.

e- The volume of tetrahedron $NITS$ is given by
$$V=\frac{Area\ of(ITS)\times d(N;(P))}{3}$$

But, ITS is a fixed triangle, then the area of this triangle is constant.
Since the 2 planes (P) and (Q) are parallel then the distance of N to the plane (P) is equal to the distance between the two planes which is constant. Consequently, the volume remains constant as N varies in (Q).

$N°17.$

1) Let $M\left(t;-2t;t\right)$ be a variable point of (d), then:
$x_M + y_M + z_M + m\left(x_M - y_M - 3z_M\right) = t - 2t + t + m\left(t + 2t - 3t\right) = 0$
then, M belongs to (P_m) for all m and consequently, (d) lies in (P_m) for all m.
2) An equation of (P_0) is: $x + y + z = 0$.
An equation of (P_1) is: $x - z = 0$.
$\overrightarrow{n_{P_0}}.\overrightarrow{n_{P_1}} = 1 - 1 = 0$, then $\overrightarrow{n_{P_0}} \perp \overrightarrow{n_{P_1}}$ and consequently (P_0) and (P_1) are perpendicular.
3) The vector $\overrightarrow{n_{P_0}}(1;1;1)$ is a directing vector of (D) and (D) passes through A then: (D) : $x = \lambda + 1$, $y = \lambda - 1$, $z = \lambda + 3$
H is the point of intersection of (D) and the plane (P_0), since H belongs to (D) then $H\left(\lambda+1;\lambda-1;\lambda+3\right)$.
Since H belongs to (P_0), then $\lambda+1+\lambda-1+\lambda+3=0$ which gives $\lambda=-1$, consequently $H\left(0;-2;2\right)$.



4) B is the orthogonal projection of A on (P_1).
A system of parametric equations of (AB) is: $x = n+1$, $y = -1$ $z = -n+3$.
The point B is the intersection of (AB) and of plane (P_1) then: $n+1+n-3=0$, so $n=1$ and consequently $B(2;-1;2)$.
The straight line (D) is perpendicular to (P_0). (P_0) and (P_1) are perpendicular then (D) is parallel to (P_1) and consequently (D) is parallel to its projection (D'), thus $\overrightarrow{v_D}(1;1;1)$ is a directing vector of (D') and since (D') passes through B ,then :
(D'): $x = m+2$, $y = m-1$, $z = m+2$.

5) $(AB) \perp (P_1)$ then $(AB) \perp (d)$.
$(AH) \perp (P_0)$ then $(AH) \perp (d)$.
So the straight line (d) is perpendicular to the plane (ABH) which is the plane $(ABEH)$ thus (d) is perpendicular to (EH) at E , and since (D) is perpendicular to (EH) at H then (EH) the common perpendicular of the two straight lines (d) and (D).
Since (EH) is parallel to (AB) then the vector $\overrightarrow{AB}(1;0;-1)$ is the directing vector of (Δ) , thus (Δ): $x = p$, $y = -2$, $z = -p+2$
p is a real parameter .

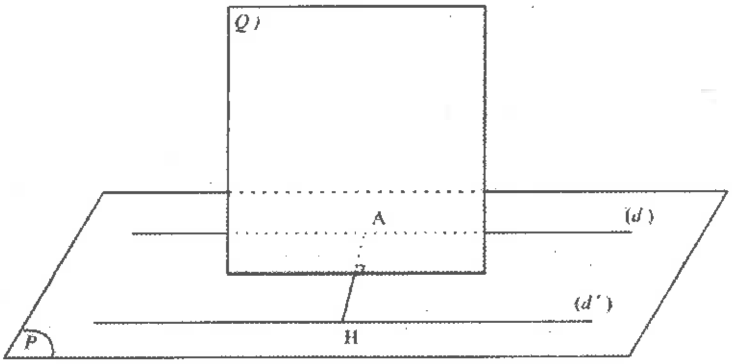
6) $d(A,P_0) = AH = \sqrt{3}$ and $d(A,P_1) = AB = \sqrt{2}$.
But, $ABEH$ is a rectangle since $\hat{A} = \hat{B} = \hat{H} = 90^\circ$.
Then the distance of A to (d) is $AE = \sqrt{AH^2 + HE^2} = \sqrt{5}$.

$\boxed{N^\circ 18.}$

1) $\overrightarrow{v_d}(1;2;-2)$ and $\overrightarrow{v_{d'}}(-2;-4;4)$ are the directing vectors of (d) and (d'), $\overrightarrow{v_{d'}} = -2\overrightarrow{v_d}$ and since the point $A(-2;1;2)$ of (d) does not belong to (d') then (d) and (d') are parallel .

2) $A(-2;1;2)$ is a point of (d) and $B(0;1;0)$ is a point of (d').
The vector $\overrightarrow{AB} \wedge \overrightarrow{v_d} = 4\vec{i} + 2\vec{j} + 4\vec{k}$ is normal to plane (P).
Thus, $4x+2y+4z+r=0$ is an equation of (P), B belongs to (P) so r = - 2 ,which gives $4x+2y+4z-2=0$, or

$2x+y+2z-1=0$.



3) a- H is a point of (d') then $H(-2m;-4m+1;4m)$.

$\overrightarrow{AH}.\overrightarrow{v_{d'}}=0$, which gives $m=\frac{1}{3}$ and consequently

$H\left(-\frac{2}{3};-\frac{1}{3};\frac{4}{3}\right)$.

b- (Q) is the mediator plane of $[AH]$, the vector

$\overrightarrow{AH}\left(\frac{4}{3};-\frac{4}{3};-\frac{2}{3}\right)$ is normal to plane (Q), then

$\frac{4}{3}x-\frac{4}{3}y-\frac{2}{3}z+r=0$, the point $I\left(-\frac{4}{3};\frac{1}{3};\frac{5}{3}\right)$,

the midpoint of [AH] belongs to (Q), thus $r=\frac{10}{3}$ and

consequently $2x-2y-z+5=0$ is an equation of

(Q), the mediator plane of [AH].

c- The straight line (δ) is the intersection of the two planes (P) and (Q).

Leting $z=\lambda$, we get $\begin{cases}2x+y=-2\lambda+1\\2x-2y=\lambda-5\end{cases}$,

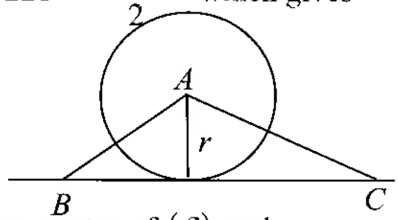
Which gives $y=-\lambda+2$ and $x=-\frac{1}{2}\lambda-\frac{1}{2}$,

And consequently, (δ): $x=-\frac{1}{2}\lambda-\frac{1}{2}$, $y=-\lambda+2$, $z=\lambda$.

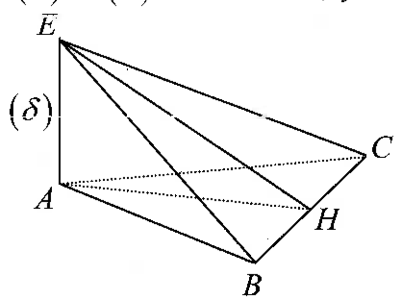
N° 19.

1) $13x_A - 4y_A - 6z_A + 15 = -13 + 4 - 6 + 15 = 0$, then A belongs to (P).
Let $M\left(2\lambda - 1; 5\lambda + 2; \lambda - 1\right)$ be a variable point of (d), then:
$13x_M - 4y_M - 6z_M + 15 = 26\lambda - 13 - 20\lambda - 8 - 6\lambda + 6 + 15 = 0$,
so M belongs to (d), which is sufficient to prove that (d) lies in (P) and consequently $13x - 4y - 6z + 15 = 0$ is an equation of (P).

2) Area of ABC is equal to $\frac{1}{2}\left\|\overrightarrow{AB} \wedge \overrightarrow{AC}\right\|$.
But, $\overrightarrow{AB}\left(-2;-2;-3\right)$ and $\overrightarrow{AC}\left(0;+3;-2\right)$ then
$\overrightarrow{AB} \wedge \overrightarrow{AC} = 13\vec{i} - 4\vec{j} - 6\vec{k}$ which gives
$Area\ of\left(ABC\right) = \frac{1}{2}\sqrt{169 + 16 + 36} = \frac{1}{2}\sqrt{221}$ square units.
The radius of (γ) is the distance of point A to the straight line (BC).
But, Area of $\left(ABC\right) = \frac{BC \times r}{2}$ so $\frac{1}{2}\sqrt{221} = \frac{\sqrt{30} \times r}{2}$ which gives
$r = \frac{\sqrt{221}}{\sqrt{30}} = \frac{\sqrt{6630}}{30}$ length units.



3) The vector $\overrightarrow{n_p}\left(13;-4;-6\right)$ is a directing vector of (δ) and A belongs to (δ) so (δ): $x = 13m - 1,\ y = -4m - 1,\ z = -6m + 1$.



4) a- $\left(AE\right) \perp \left(ABC\right)$, then $\left(AE\right) \perp \left(BC\right)$ and

$(AH) \perp (BC)$.

The straight line (BC) is perpendicular to two intersecting straight lines of plane (EAH), then (BC) is perpendicular to plane (EAH) and consequently $(BC) \perp (EH)$, thus triangle EHC is right at H.

b- The straight line (BC) is the intersection of the two planes (EBC) and (ABC).

Since $(AH) \perp (BC)$ and $(EH) \perp (BC)$, then the angle $A\hat{H}E$ is the acute angle of the two planes (EBC) and (ABC).

$\tan\left(A\hat{H}E\right) = \dfrac{AE}{AH}$, but $AE = \sqrt{221}$ and $AH = r = \dfrac{\sqrt{221}}{\sqrt{30}}$

Therefore, $\tan\left(A\hat{H}E\right) = \dfrac{\sqrt{221}}{\dfrac{\sqrt{221}}{\sqrt{30}}} = \sqrt{30}$.

N° 20.

1) $\overrightarrow{AB} \wedge \overrightarrow{AC} = -2\vec{i} - \vec{j} + \vec{k}$, then $-2x - y + z + r = 0$ is an equation of (P).

A belongs to (P) gives $r = 1$, so $2x + y - z - 1 = 0$ is an equation of plane (ABC).

2) a- A belongs to (Q) since : $0 + 2(1) + 0 - 2 = 0$.

B belongs (Q) since : $-1 + 2(2) - 1 - 2 = 0$.

(P) and (Q) are not confounded since $\overrightarrow{n_P}(2;1;-1)$ and $\overrightarrow{n_Q}(1;2;1)$ are not parallel, then the two planes (P) and (Q) intersect along the straight line (AB).

b- $\overrightarrow{CC'}(1;2;1)$, then $\overrightarrow{CC'} = \overrightarrow{n_Q}$ and consequently the straight line (CC') is perpendicular to (Q).

The midpoint $I\left(-\frac{1}{2};2;-\frac{3}{2}\right)$ of $[CC']$ belongs to (Q) since $x_I+2y_I+z_I-2=-\frac{1}{2}+4-\frac{3}{2}-2=0$, then $C'$is the symmetric of C with respect to (Q).

The plane (P') passes through the straight line (AB) and contains the point C', then $\overrightarrow{n_{P'}}=\overrightarrow{C'A}\wedge\overrightarrow{AB}=\vec{i}-\vec{j}-2\vec{k}$.

Hence, an equation of (P') is $x-y-2z+r=0$.

$C'\in(P')$ then $-3+2+r=0$, so $r=1$ and consequently, an equation of (P') is: $x-y-2z+1=0$.

3) The line symmetric of (BC) with respect to (Q) is the straight line (BC'), so we should prove that the straight line (δ) is the straight line (BC').

For $m=-1$, $x=-1$; $y=2$ and $z=-1$, then the point B belongs to (δ).

For $m=0$, $x=0$; $y=3$ and $z=-1$, then C' belongs to (δ).

Consequently the straight line (δ) is the symmetric of (BC) with respect to (Q).

4) Triangle CBC' is isosceles at B, then the straight line (BI) is a bisector of the angle $C\hat{B}C'$, but $\overrightarrow{BI}=\frac{1}{2}\vec{i}-\frac{1}{2}\vec{k}$

therefore: (BI): $x=\frac{1}{2}t-1$, $y=2$, $z=-\frac{1}{2}t-1$.

N°21.

1) The vector $\vec{N}=\overrightarrow{AB}\wedge\overrightarrow{AC}$ is normal to (P), an equation of (P) is : $2x+2y-2z+r=0$, but A belongs to (ABC), then $6+2+r=0$, so $r=-8$ and consequently, an equation of (ABC) is $2x+2y-2z-8=0$.

2) a- $\overrightarrow{AD}(-3;-1;2)$; $\overrightarrow{BD}(-1;-3;2)$ and $\overrightarrow{CD}(-3;-2;1)$.

Then, $AD = BD = CD = \sqrt{14}$, thus the point D is equidistant of the points A, B and C.

b- The axis of the circle circumscribed about triangle (ABC) passes through the point D and it is perpendicular to plane (ABC) thus the vector $\vec{n}(1;1;-1)$ is a directing vector of (d). As a result, (d): $x = m$, $y = m$, $z = -m+2$.

c- The point w is the intersection of (d) and plane (ABC).
w belongs to (d), then $w\ (m;m;-m+2)$, w belongs to (ABC) so $m+m+m-2-4=0$, which gives $m=2$ and Consequently, $w\ (2;2;0), R = Aw = \sqrt{1+1} = \sqrt{2}$.

d- (T) is perpendicular to (wA) and (T) is orthogonal to (wD) then (T) is perpendicular to plane (AwD), thus The vector $\vec{s} = \overrightarrow{wD} \wedge \overrightarrow{wA}$ is a directing vector of (T).
But, $\vec{s} = 2\vec{i} + 2\vec{j} + 4\vec{k}$ and (T) passes through A.
Therefore, (T): $x = 2\lambda+3$, $y = 2\lambda+1$, $z = 4\lambda$.

N° 22.

1) $\overrightarrow{n_p}(2;1;-1)$ and $\overrightarrow{n_Q}(1;2;1)$ are the normal vectors of (P) and (Q).
If α is the acute angle of the two planes (P) and (Q) then

$$\cos\alpha = \frac{\left|\overrightarrow{n_p}.\overrightarrow{n_Q}\right|}{\left\|\overrightarrow{n_p}\right\|\cdot\left\|\overrightarrow{n_Q}\right\|} = \frac{3}{6} = \frac{1}{2}, \text{ so } \alpha = 60^\circ.$$

2) Let $M\ (-m+1;m;-m+1)$ be a variable point of (d),
$2x_M + y_M - z_M - 1 = -2m+2+m+m-1-1 = 0$,
and $x_M + 2y_M + z_M - 2 = -m+1+2m-m+1-2 = 0$.
Then, the coordinates of M satisfy the equation of (P) and the equation of (Q) for all real numbers m, hence $x = -m+1$; $y = m$; $z = -m+1$ is a system of parametric equations of (d).

3) $M\ (-m+1;m;-m+1)$ then :
$BM^2 = (-m+1)^2 + (m-2)^2 + (-m)^2 = 3m^2 - 6m + 5$.

$f(m)=3m^2-6m+5$, which is a function of the form $f(m)=am^2+bm+c$ with $a>0$.

For $m=-\dfrac{b}{2a}=1$, f admits a minimum equal to $f(1)=2$.

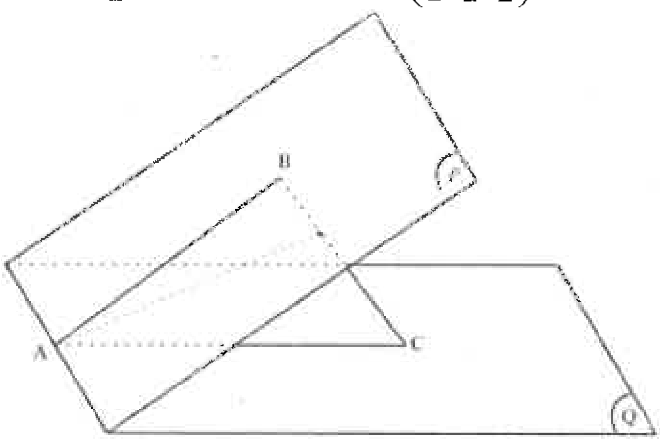
Then, the distance of point B to the straight line (d) is $d=\sqrt{f(1)}=\sqrt{2}$.

4) $A(-m+1;m;-m+1)$.

ABC is isosceles of vertex A, then $AB^2=AC^2$,

so $(-m+1)^2+(m-2)^2+(-m+1-1)^2=(-m+1)^2+m^2+(-m+1-2)^2$

hence $m=\dfrac{1}{2}$ and consequently, $A\left(\dfrac{1}{2};\dfrac{1}{2};\dfrac{1}{2}\right)$.



5) Since ABC is isosceles of vertex A, then the interior bisector of angle $B\hat{A}C$ is the median issued from A in this triangle, if we designate by K the midpoint of $[BC]$, then we get $K\left(0;1;\dfrac{3}{2}\right)$ and $\overrightarrow{AK}\left(-\dfrac{1}{2};\dfrac{1}{2};1\right)$, Thus, (δ): $x=-\dfrac{1}{2}\lambda$, $y=\dfrac{1}{2}\lambda+1$, $z=\lambda+\dfrac{3}{2}$.

6) a- Area of triangle ABC is equal to $\dfrac{\left\|\overrightarrow{AB}\wedge\overrightarrow{AC}\right\|}{2}$.

On the other hand, area of ABC is equal to $\dfrac{BC \times AK}{2}$

thus: $BC \times AK = \left\|\overrightarrow{AB} \wedge \overrightarrow{AC}\right\|$, which gives

$AK = \dfrac{\left\|\overrightarrow{AB} \wedge \overrightarrow{AC}\right\|}{BC}$.

b- Area of $(EBC) = 2$ Area of (ABC) , which gives $EK = 2AK$ then $\overrightarrow{EK}^2 = 4\overrightarrow{AK}^2$ and since E belongs to (δ) then $E\left(-\dfrac{1}{2}\lambda; \dfrac{1}{2}\lambda + 1; \lambda + \dfrac{3}{2}\right)$, so $\lambda^2 = 4$ which gives $\lambda = 2$ or $\lambda = -2$, hence, there exist two points $E\left(-1; 2; \dfrac{7}{2}\right)$ and $E\left(1; 0; -\dfrac{1}{2}\right)$.

$N°23.$

1) The two straight lines (Δ) and (Δ') are not parallel since the two directing vectors $\overrightarrow{v_\Delta}(2;2;3)$ of (Δ) and $\overrightarrow{v_{\Delta'}}(2;1;2)$ of (Δ') are not parallel.

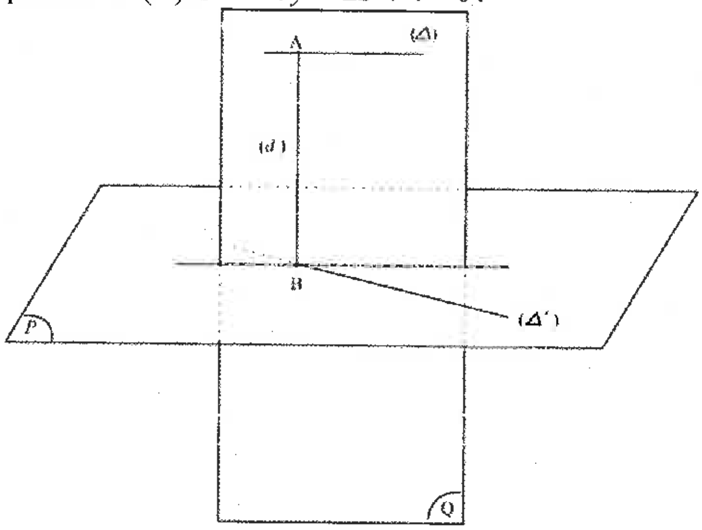
If (Δ) and (Δ') intersect at point A then the coordinates of A are the solutions of the system $\begin{cases} 2\lambda + 1 = 2t \\ 2\lambda + 2 = t \\ 3\lambda = 2t + 2 \end{cases}$

Solving the first two equations, we get: $-1 = t$ which gives $\lambda = -\dfrac{3}{2}$. Replacing t and λ by their values in the third equation, we get: $-\dfrac{9}{2} = 0$ which is impossible.

Hence, the two straight lines (Δ) and (Δ') are skew.

2) The vector $\overrightarrow{v_\Delta} \wedge \overrightarrow{v_{\Delta'}}$ is normal to (P), but $\overrightarrow{v_\Delta} \wedge \overrightarrow{v_{\Delta'}} = \vec{i} + 2\vec{j} - 2\vec{k}$,

then: $x + 2y - 2z + r = 0$, the point $I\,(0;0;2)$ is a point of (Δ'), so it is a point of (P), thus $-4 + r = 0$, so $r = 4$ and consequently an equation of (P) is $x + 2y - 2z + 4 = 0$.



3) A normal vector of (Q) is $\overrightarrow{n_Q} = \overrightarrow{n_P} \wedge \overrightarrow{v_\Delta} = 10\vec{i} - 7\vec{j} - 2\vec{k}$,
then $10x - 7y - 2z + r = 0$ is an equation of (Q).
$J\,(1;2;0)$ is a point of (Δ), J belongs to (Q) then: $10 - 14 + r = 0$, so $r = 4$ and consequently an equation of (Q) is: $10x - 7y - 2z + 4 = 0$.

4) a- A belongs to (Δ), then $A\,(2\lambda+1;2\lambda+2;3\lambda)$.
B belongs to (Δ') then $B\,(2t;t;2t+2)$.
Hence $\overrightarrow{AB}(2t-2\lambda-1;t-2\lambda-2;2t-3\lambda+2)$.
$\overrightarrow{AB}.\overrightarrow{v_\Delta} = 0$, then $12t - 17\lambda = 0$ and $\overrightarrow{AB}.\overrightarrow{v_{\Delta'}} = 0$,
Therefore $9t - 12\lambda = 0$

The solution of the system $\begin{cases} 12t - 17\lambda = 0 \\ 9t - 12\lambda = 0 \end{cases}$ is $t = 0 = \lambda$

And consequently $A\,(1;2;0)$ and $B\,(0;0;2)$.
The vector $\overrightarrow{AB}\,(-1;-2;2)$ is a directing vector of (d) and A belongs to (d) then :
(d): $x = -m + 1$, $y = -2m + 2$, $z = 2m$.

$AB=\sqrt{1+4+4}=3$.

b- $\overrightarrow{v_d}(-1;-2;2)$ is a directing vector (d), and $\vec{k}(0;0;1)$ is a normal vector of the plane (xoy).

If α designates the angle between $\overrightarrow{v_d}$ and $\vec{k}$, then:

$$\cos\alpha=\frac{\overrightarrow{v_d}.\vec{k}}{\left\|\overrightarrow{v_d}\right\|.\left\|\vec{k}\right\|}=\frac{2}{3}.$$

But the angle of (d) and (xoy) is $\beta=\frac{\pi}{2}-\alpha$, then

$$\cos\beta=\cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha=\sqrt{1-\frac{4}{9}}=\frac{\sqrt{5}}{3}.$$

The distance of O to (d) is equal to $\dfrac{\left\|\overrightarrow{OA}\wedge\overrightarrow{v_d}\right\|}{\left\|\overrightarrow{v_d}\right\|}$.

$\overrightarrow{OA}\wedge\overrightarrow{v_d}=4\vec{i}-2\vec{j}$ then the distance is equal to $\dfrac{\sqrt{20}}{3}=\dfrac{2\sqrt{5}}{3}$.

N°24.

1) $AB=2\sqrt{2}$, $AC=2\sqrt{2}$ and $BC=2\sqrt{2}$ then triangle ABC is equilateral.

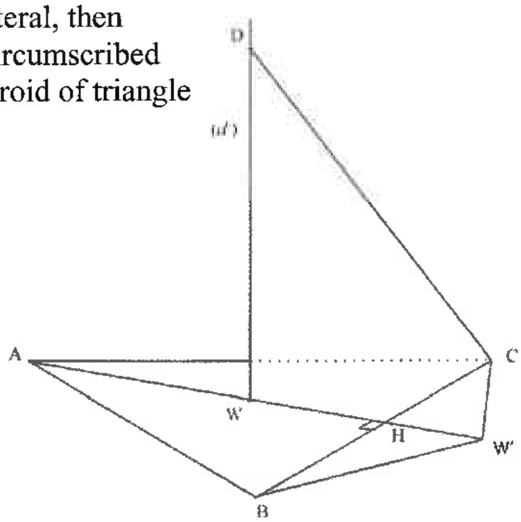
2) a- Triangle ABC is equilateral, then the center of the circle circumscribed about triangle is the centroid of triangle that has the coordinates:

$$\frac{x_A+x_B+x_C}{3}=-\frac{1}{3};$$

$$\frac{y_A+y_B+y_C}{3}=\frac{5}{3};$$

$$\frac{z_A+z_B+z_C}{3}=\frac{5}{3}$$

so it is the point W.



b- The triangle ABC is equilateral then $AB = 2\sqrt{2}$, so
$AH = 2\sqrt{2}\times\frac{\sqrt{3}}{2} = \sqrt{6}$, $WH = \frac{1}{3}AH = \frac{\sqrt{6}}{3}$ and since W' is the symmetric of W with respect to (BC) then $W'H = \frac{\sqrt{6}}{3}$.
Hence, the area of triangle $BW'C$ is:

$$\frac{BC\times W'H}{2} = \frac{2\sqrt{2}\times\frac{\sqrt{6}}{3}}{2} = \frac{2\sqrt{3}}{3} \text{ square units.}$$

Area of triangle ABC is $\frac{AH\times BC}{2} = \frac{\sqrt{6}\times 2\sqrt{2}}{2} = 2\sqrt{3}$ square units.

Then, the area of triangle $ABW'C$ is $\frac{2\sqrt{3}}{3} + 2\sqrt{3} = \frac{8\sqrt{3}}{3}$ u^2.

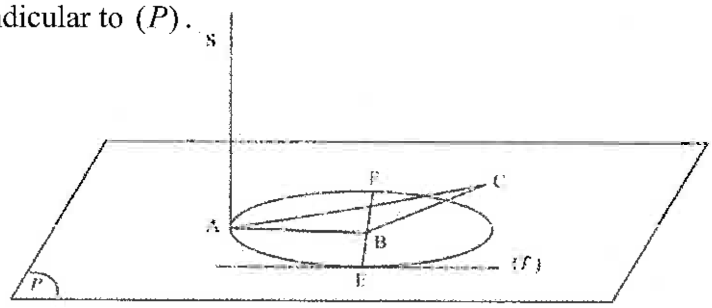
c- The straight line (WD) is perpendicular to plane (ABC), then $(WD)\perp(AB)$ and since $(WC)\perp(AB)$ then (AB) is perpendicular to plane (WDC) and consequently, (AB) and (DC) are orthogonal.

Another Method:.

A system of parametric equations of (d) is : $x = n - \frac{1}{3}$, $y = n + \frac{5}{3}$, $z = n + \frac{5}{3}$. We can verify that for a variable point $D\left(n-\frac{1}{3};n+\frac{5}{3};n+\frac{5}{3}\right)$ of (d) , $\overrightarrow{DC}.\overrightarrow{AB} = 0$.

$N°25.$

1) S belongs to the straight line (δ) passing through A and perpendicular to (P).



A system of parametric equations of (δ) is :
$x=2m+1$, $y=m-6$, $z=-2m+1$
S belongs to (δ) , then $S(2m+1;m-6;-2m+1)$
And since $AS=3$; $AS^2=4m^2+m^2+4m^2=9$,
Which gives $m=1$ or $m=-1$.
The point S corresponds to $m=1$, so $S(3;-5;-1)$.

2) a- $(d)\perp(AB)$ and $(d)\perp(AS)$, then
$\overrightarrow{AB}\wedge\overrightarrow{AS}=-6\vec{i}+6\vec{j}-3\vec{k}$ is a directing vector of (d) , a system of parametric equations of (d) is :
$x=-2\lambda+2$, $y=2\lambda-4$, $z=-\lambda+3$, λ is a real parameter .

b- For all points $M(-2\lambda+2;2\lambda-4;-\lambda+3)$ of (d) ,
$2(-2\lambda+2)+(2\lambda-4)-2(-\lambda+3)+6=0$, then (d) lies in (P).

3) a- $AB^2=1+4+4=9$, then $AB=r=3$ and consequently A belongs to (C).
$BC^2=4+16=20$, then $BC=2\sqrt{5}>r$
consequently C is exterior to (C).

b- The points E and F belong to (d), then
$E(-2\lambda+2;2\lambda-4;-\lambda+3)$
since $BE=3$ we deduce that $BE^2=4\lambda^2+4\lambda^2+\lambda^2=9$,
which gives $\lambda=1$ or $\lambda=-1$, the two points E and F are then $E(0;-2;2)$, $F(4;-6;4)$.
$\overrightarrow{EF}(4;-4;2)$, then (EF) : $x=4p$, $y=-4p-2$, $z=2p+2$.

c- The tangent (T) at E to (C) is perpendicular to (EF), then it is parallel to (AB) , Hence, (T): $x=k$, $y=2k-2$, $z=2k+2$.

$\boxed{N°26}$

1) $\overrightarrow{n_p}(1;2;-1)$ is a normal vector of (P) and $\overrightarrow{n_Q}(2;1;1)$ is a normal vector of (Q) . Let α be the acute angle of (P) and (Q),

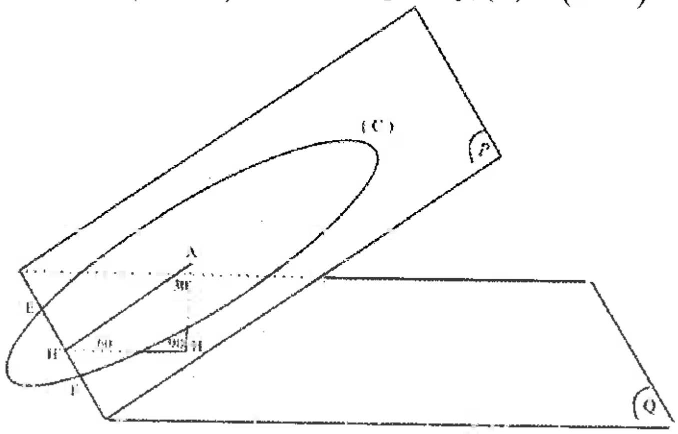
$$\cos\alpha=\frac{\left|\overrightarrow{n_P}.\overrightarrow{n_Q}\right|}{\left\|\overrightarrow{n_P}\right\|.\left\|\overrightarrow{n_Q}\right\|}=\frac{3}{6}=\frac{1}{2}$$ then $\alpha=60°$.

2) a- $d(A;Q)=\dfrac{|2x_A+y_A+z_A-2|}{\sqrt{4+1+1}}=\dfrac{3}{\sqrt{6}}=\dfrac{\sqrt{6}}{2}$.

b- H' is the orthogonal projection of A on the straight line $(d)=(P)\cap(Q)$ and H is the orthogonal projection of A on (Q).

$(AH')\perp(d)$ and $(AH)\perp(d)$ since $(AH)\perp Q$.

Then, $(d)\perp(AHH')$ and consequently, $(d)\perp(HH')$.



Then, the angle $A\hat{H}'H$ is the acute angle of the two planes (P) and (Q) then $A\hat{H}'H=60^\circ$ and triangle AHH'
Is semi-equilateral.

But, $AH=AH'\dfrac{\sqrt{3}}{2}$, which gives $AH'=\dfrac{2AH}{\sqrt{3}}=\dfrac{\sqrt{6}}{\sqrt{3}}=\sqrt{2}$.

$AH'<R$ then the circle (C) cuts (d) in 2 points E and F then (C) cuts (Q) in two points E and F.

c- Triangle $AH'F$ is right at H' then :

$AF^2=AH'^2+H'F^2$, so $H'F=\sqrt{AF^2-AH'^2}=\sqrt{3}$ then $EF=2\sqrt{3}$ and consequently the area of AEF is

$\dfrac{EF\times AH'}{2}=\dfrac{2\sqrt{3}\times\sqrt{2}}{2}=\sqrt{6}$ square units.

3) For all points $M(-m+1;m;m)$ of (d):
$-m+1+2m-m-1=0$, then the straight line (d) lies in (P).

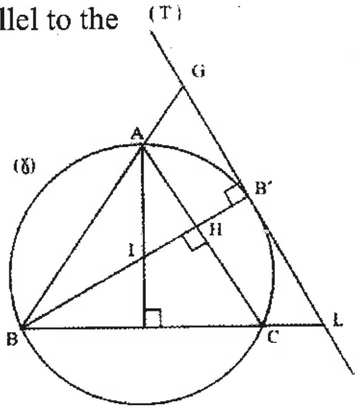
Also, $-2m+2+m+m-2=0$ then (d) lies in (Q) and consequently the system of parametric equations of (d) is :
$x=-m+1;\ y=m\ ;\ z=m$

4) a- The plane (R) is perpendicular to (d) then the directing vector $\overrightarrow{v_d}(-1;1;1)$ of (d) is normal to (R) thus $-x+y+z+r=0$ is an equation of (R).
The plane (R) passes through A since A is equidistant of E and F, $AE=AF=R$ then $0+0-1+r=0$, so $r=1$ and consequently, an equation of (R) is $-x+y+z+1=0$.

b- I is the intersection of (R) with the straight line (d) so:
$m-1+m+m+1=0$, then $m=0$ and conseqently $I\left(1;0;0\right)$.
(H' and I are confounded) .

N° 27.

1) $\overrightarrow{AB}(0;3;-3)$, $\overrightarrow{AC}(-3;3;0)$ and $\overrightarrow{BC}(-3;0;3)$, which gives $AB=AC=BC=3\sqrt{2}$, then ABC is an equilateral triangle.
2) Let $M(x;y;z)$ be a variable point of (P) , then $\overrightarrow{AM}.(\overrightarrow{AB}\wedge\overrightarrow{AC})=0$, which gives $x+y+z-6=0$.
3) The vector $\overrightarrow{OF}=3\vec{i}+3\vec{j}+3\vec{k}$ is parallel to the normal vector of (P), then the straight line (OF) is perpendicular to (P).
Since $OA=OB=OC=3\sqrt{2}$ then (OF) is the axis of the circle circumscribed about triangle ABC.
4) I is the center of the circle (γ) , $I(2;2;2)$, then the point I is the midpoint of $[BB']$ which gives $B'(1;1;4)$, the tangent at B' to the circle is parallel to the straight line (AC), then the vector $\overrightarrow{AC}=-3\vec{i}+3\vec{j}$ is a directing vector of (T) , hence (T):
$x=-\lambda+1,\ y=\lambda+1,\ z=4$.



5) a- The radius of (γ) is $IB=\sqrt{6}$.

We know that $BH=AB\dfrac{\sqrt{3}}{2}=\dfrac{3\sqrt{6}}{2}$, $IH=\dfrac{1}{3}BH=\dfrac{\sqrt{6}}{2}$

and $HB'=IH=\dfrac{\sqrt{6}}{2}$.

The two straight lines (AC) and (T) are parallel then

$\dfrac{AC}{GL}=\dfrac{BH}{BB'}=\dfrac{3}{4}$, which gives $GL=\dfrac{4AC}{3}=4\sqrt{2}$.

b- $ACLG$ is a trapezoid , its area is given by :

$A=\dfrac{(GL+AC)\times HB'}{2}=\dfrac{7\sqrt{3}}{2}$ square units .

N°28.

1) $\overrightarrow{n_p}(3;-4;5)$ and $\overrightarrow{n_Q}(4;3;5)$ are the normal vectors of (P) and (Q).

If α is the acute angle of the two planes (P) and (Q) then

$\cos\alpha=\dfrac{\left|\overrightarrow{n_p}.\overrightarrow{n_Q}\right|}{\left\|\overrightarrow{n_p}\right\|.\left\|\overrightarrow{n_Q}\right\|}=\dfrac{1}{2}$, therefore $\alpha=60^\circ$.

2) Let $M(-7m+4;m+3;5m)$ be a variable point of (d),

$3x_M-4y_M+5z_M=3(-7m+4)-4(m+3)+5(5m)=0$

$4x_M+3y_M+5z_M-25=4(-7m+4)+3(m+3)+5(5m)-25=0$.

Then, the point M belongs to (P) and to (Q) for all real numbers m , consequently the planes (P) and (Q) intersect along (d).

3) a- $d(w,P)=\dfrac{|-9-16+0|}{\sqrt{9+16+25}}=\dfrac{25}{\sqrt{50}}=\dfrac{5\sqrt{2}}{2}$.

$d(w,Q)=\dfrac{|-12+12-25|}{\sqrt{16+9+25}}=\dfrac{25}{\sqrt{50}}=\dfrac{5\sqrt{2}}{2}$.

Then, $d(w;P)=d(w;Q)$ and consequently w is equidistant from (P) and (Q).

b- A bisector plane of the dihedral (P) and (Q) contains the straight line (d) and the point w .

$A(4;3;0)$ is a point of (d), $\overrightarrow{wA} \wedge \overrightarrow{v_d}$ is normal to this plane .

But, $\overrightarrow{wA} \wedge \overrightarrow{v_d} = -5\vec{i} - 35\vec{j}$ so $-5x - 35y + r = 0$ is an equation of the required plane , A belongs to this plane then $-20 - 105 + r = 0$, so $r = 125$ consequently, an equation of the bisector plane is: $x + 7y - 25 = 0$.

4) a- The vector $\overrightarrow{n_p}$ is parallel to (ww'), then $\overrightarrow{n_p}$ is parallel to plane $(ww'w'')$.

Similarly, the vector $\overrightarrow{n_Q}$ is parallel to the plane $(ww'w'')$.

Then, $\overrightarrow{n_p} \wedge \overrightarrow{n_Q} = \overrightarrow{v_d}$ is normal to plane $(ww'w'')$,

then $-7x + y + 5z + r = 0$ is an equation of $(ww'w'')$ and since w belongs to this plane then: $21 + 4 + r = 0$, so $r = -25$,consequently an equation of plane $(ww'w'')$ is $-7x + y + 5z - 25 = 0$

b- The vector $\overrightarrow{n_p}$ is parallel to the straight line (ww') then a system of parametric equations of (ww') is:

$x = 3t - 3$; $y = -4t + 4$; $z = 5t$

w' is the point of intersection of (ww') and (P) then:

$3(3t - 3) - 4(-4t + 4) + 5(5t) = 0$, so $t = \dfrac{1}{2}$, and consequently

$w'\left(-\dfrac{3}{2}; 2; \dfrac{5}{2}\right)$. A similar reasoning gives $w''\left(-1; \dfrac{11}{2}; \dfrac{5}{2}\right)$.

The vector $\overrightarrow{ww'} \wedge \overrightarrow{ww''}$ is normal to the plane (H).

But, $\overrightarrow{ww'} \wedge \overrightarrow{ww''} = -\dfrac{35}{4}\vec{i} + \dfrac{5}{4}\vec{j} + \dfrac{25}{4}\vec{k}$, so $\overrightarrow{n_H} = -7\vec{i} + \vec{j} + 5\vec{k}$.

An equation of (H) is then $-7x + y + 5z - 25 = 0$.

c- The distance of w to (d) is:

$\dfrac{\left\|\overrightarrow{wA} \wedge \overrightarrow{v_d}\right\|}{\left\|\overrightarrow{v_d}\right\|}$; but $\overrightarrow{wA} \wedge \overrightarrow{v_d} = -5\vec{i} - 35\vec{j}$, therefore

$$\dfrac{\left\|\overrightarrow{wA} \wedge \overrightarrow{v_d}\right\|}{\left\|\overrightarrow{v_d}\right\|} = \dfrac{\sqrt{1250}}{\sqrt{75}} = \dfrac{5\sqrt{6}}{3}.$$

Indications

N°3.

2) $\overrightarrow{EF}(4;-4;4)$, (EF) is perpendicular to (P) and the point $I(-1,0,3)$, midpoint of $[EF]$, belongs to (P) .

4) (AF) is the straight line symmetric of (AE) with respect to (P).

Show that A and F belong to (d) .

5) AEF is an isoceles triangle of principal vertex A , (δ) passes through A and it is parallel to (EF), then it is a bisector of $\hat{EAF}$ and any point on (δ) is equidistant from (AE) and (AF) .

N°4.

3) Take $A(2m+3;m-1;3m)$, $B(3t+5;2t-1;t)$ and use $\overrightarrow{AB}.\overrightarrow{v_d}=0$ and $\overrightarrow{AB}.\overrightarrow{v_{d'}}=0$.

N°6.

4) The two planes (P) and (Q) intersect along the straight line (d).

$A\in(Q)$, $B\in(R)$ and the two points are symmetric with respect to plane (P).

N°8

1) a- $\overrightarrow{MA}\wedge\overrightarrow{AB}=\left(\overrightarrow{MH}+\overrightarrow{HA}\right)\wedge\overrightarrow{AB}=\overrightarrow{MH}\wedge\overrightarrow{AB}+\overrightarrow{HA}\wedge\overrightarrow{AB}$.

$\overrightarrow{HA}\wedge\overrightarrow{AB}=\vec{0}$, since points H, A and B are collinear.

b- $\left\|\overrightarrow{MA}\wedge\overrightarrow{AB}\right\|=MH\times AB\times\left|\sin\left(\overrightarrow{MH};\overrightarrow{AB}\right)\right|=MH\times AB$.

N°9.

1) $A(6;0;0)$, $B(0;3;0)$ and $C(0;0;2)$.

2) $(OH)\perp(BC)$ and $(OA)\perp(BC)$, then $(BC)\perp(AOH)$.

3) (d): $x=t$, $y=2t$, $z=3t$

N°10.

1) (P) : $x+y+z-3=0$.

2) a- $\overrightarrow{BC}$ is a normal vector of plane (Q) : $x-3y+2z=0$.

b- $(AH)=(P)\cap(Q)$.

N° 11.

2) c- $\overrightarrow{AB}$ is a normal vector of plane (Q) and the midpoint of $[AB]$

Belongs to (Q). An equation of (Q) is $-x+2y+4z+\frac{1}{2}=0$.

d- $(d)=(P)\cap(Q)$.

N° 12.

1) $x=-m+2$, $y=m$, $z=2m-4$
2) (δ) is parallel to the line of intersection (d).
3) The distance between the two straight lines (d) and (δ) is equal to the distance of A to (d).

N° 13.

1) (P) : $x+y-z=0$.
2) a- $(IA)\perp(P)$.
3) c- The height (AH) is a median in triangle ABC .

N° 14.

3) a- (AH): $x=-2$, $y=\lambda-1$, $z=-\lambda+1$, $H\left(-2;-\frac{1}{2};\frac{1}{2}\right)$.

N° 15.

1) $A(1;1;-2)$.
2) $x+y-z-4=0$.
3) b- $x+2y+3z-4=0$, $B\left(\frac{1}{3};\frac{13}{3};\frac{2}{3}\right)$, $C\left(\frac{11}{3};\frac{5}{3};\frac{4}{3}\right)$.

N° 16.

1) $(AI)\perp(BD)$ et $(AE)\perp(BD)$, donc $(BD)\perp(AEI)$, then the two planes are perpendicular .
2) $x+y+z-1=0$.
3) a- $J\left(\frac{1}{3};\frac{1}{3};\frac{1}{3}\right)$.

b- Show that $(EJ)\perp(BD)$ and $(DJ)\perp(EB)$.

CHAPTER 8
Probability

Chapter Review

1) **Permutations .**
Given a finite set $E = \{a_1; a_2; a_3 \ldots\ldots\ldots\ldots; a_n\}$.
n is the cardinal of E and is denoted by $card(E) = n$.

- A permutation or an arrangement of p elements of a set E of n elements is every ordered sequence of p elements of E.
- Number of permutations is given by the formula : $P_n^p = \dfrac{n!}{(n-p)!}$.
- A p-list of n elements is every ordered sequence of p elements of E that are not necessarily different .
Notation of a p-list is: n^p

2) **Combinations .**

- A combination of p elements out of a finite set E of n elements, is a subset of E containing p elements.
Number of combinations . is given by the formula :
$C_n^p = \dfrac{n!}{p!(n-p)!}$

* **Properties :**
$C_n^0 = C_n^n = 1$; $C_n^1 = C_n^{n-1} = n$; $C_n^p = C_n^{n-p}$; $C_n^p = C_{n-1}^{p-1} + C_{n-1}^p$.

* **Newton's formula:**
$(a+b)^n = C_n^0 a^n b^0 + C_n^1 a^{n-1} b^1 + C_n^2 a^{n-2} b^2 + \ldots\ldots\ldots\ldots + C_n^n a^0 b^n$

3) **Probability .**

- Let A be an event of a sample space Ω .
$p(A) = \dfrac{card(A)}{card(\Omega)}$ or $p(A) = \dfrac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$.
- $p(\Omega) = 1$ and $p(\phi) = 0$.

- If $\overline{A}$ is the complementary event of A then $p(\overline{A}) = 1 - p(A)$.
- If A and B are mutually exclusive events , $A \cap B = \phi$, then
  $p(A \cup B) = p(A) + p(B)$.
  For any given events A and B of a sample space Ω:
  $p(A \cup B) = p(A) + p(B) - p(A \cap B)$.
- For any given events A and B of a sample space Ω:
  $A \cap B$ and $A \cap \overline{B}$ are mutually exclusive events :
  $(A \cap B) \cup (A \cap \overline{B}) = A$.

- **Conditional Probability:**
  The probability that event B will occur, given that event A has occurred is called conditional probability , it is denoted by $p(B / A)$.
  - $p(A / B) = \dfrac{p(A \cap B)}{p(A)}$.
  - $p(A \cap B) = p(A) p(B / A) = p(B) p(A / B)$
  - A and B are independent when :
    $p(A / B) = p(A)$ or $p(A \cap B) = p(A) \times p(B)$.
- Rule of total probability:
  $p(A) = p(A \cap B) + p(A \cap \overline{B})$
- Expected value (Mean) : $E(X) = \sum_{i=1}^{n} p_i x_i$
- The variance of X : $V(X) = \sum_{i=1}^{n} p_i x_i^2 - [E(X)]^2$.
- The standard deviation of X : $\sigma(X) = \sqrt{V(X)}$
- The distribution function of X :
  $F: \ IR \to [0;1]$
  $x \to F(x) = p(X \le x)$

Solved Problems

N° 1.

Solve in $I\!N$ each of the following equations:

1) $C_n^2 = 3$ 2) $P_n^3 = 6n$ 3) $C_{2n}^1 + C_{2n}^2 + C_{2n}^3 = 35n$.

N° 2.

Calculate each of the following sums:

1) $S_1 = C_n^0 + C_n^1 + C_n^2 + ... + C_n^n$ and deduce the sum of

$$S'_1 = \frac{1}{0!\,n!} + \frac{1}{1!(n-1)!} + ... + \frac{1}{n!\,0!}$$

2) $S_1 = 1 + 3C_n^1 + 3^2 C_n^2 + ... + 3^n$

3) $S_3 = C_n^1 + 2C_n^2 + 3C_n^3 + ... + nC_n^n$

N° 3.

Consider n points $A_1, A_2, \ldots, A_n$ of the plane where no three are collinear.

1) How many straight lines can be drawn using these points?

2) How many triangles can be formed using these points?

N° 4.

Calculate the number of diagonals in a convex polygon of n sides.

N° 5.

An urn contains 6 identical balls:

Four green balls (G) and two yellow balls (Y).

1) We draw, at random and simultaneously, two balls from the urn.
   We note by A, B, C the following events:
   A : " The two drawn balls are yellow"
   B : " The two drawn balls have different colors"
   C : " The two drawn balls are green"
   Calculate $p(A)$, $p(B)$ and $p(C)$.

2) The two drawn balls are not replaced back in the urn, we draw again two balls from the urn and we designate by D the event:
   D : A green ball and a yellow ball are drawn .
   a- Calculate $p(D/A)$; $p(D/B)$ and $p(D/C)$.
   b- Deduce $p(D \cap A)$; $p(D \cap B)$ and $p(D \cap C)$.
   c- Calculate $p(D)$.

N° 6.

An urn contains 5 identical balls, two white and three black.

1) We draw, at random and simultaneously, two balls from the urn.
   Calculate the probability of drawing:
   a- Two white balls .
   b- Two balls of the same color.
2) In this part, we draw the two balls according to the following strategy:
   We draw the first ball from the urn and we note its color; we then replace it back in the urn and we add another ball having the same color of the drawn ball.
   (Hence, there are six balls in the urn before the second draw).
   We, then, draw another ball. Consider the following events:
   w_1 : Getting a white ball in the first draw.
   B_1 : Getting a black ball in the first draw.
   w_2 : Getting a white ball in the second draw.
   a- Calculate $p(w_2 / w_1)$ *and* $p(w_2 / B_1)$.
   b- Calculate $p(w_2)$.

N° 7.

In a factory, there are three machines M_1, M_2 and M_3.

Machine M_1 assures 20% of the total production of the factory of which 5% are defective.

Machine M_2 assures 30% of the total production of the factory of which 4% are defective.

Machine M_3 assures 50% of the total production of the factory of which 5% are defective.

We choose, at random, an article from the factory.

1) Draw the tree diagram showing all the probabilities.
2) Calculate the probability of choosing a defective article produced by M_1.
3) Calculate the probability of choosing a defective article.
4) The article chosen is defective, calculate the probability that it is produced by M_1.

N° 8.

An urn contains six identical balls :

Three black, two green and one white.
A player draws one ball from the urn.
If the ball drawn is white, he gains 3 points.
If the ball drawn is green, he gains 2 points.
If the ball drawn is black, he replaces the ball in the urn and draws a new ball from the urn.
In the second drawing :
If the ball drawn is white, he gains 1 point, if the ball drawn is green he loses 1 point and if the ball drawn is black, he loses 2 points.
Calculate the probabilities of each of the following events :
E : The player gains 2 points. F : The player gains 1 point.
G : The player loses 1 point. H : The player loses 2 points.

N° 9.

An urn contains 9 balls:
Three white balls numbered 1 to 3.
Three black balls numbered 1 to 3.
Three red balls numbered 1 to 3.
We draw, simultaneously and at random, two balls from the urn.
Consider the following events:
A : " The two drawn balls hold an odd number"
B : " The two drawn balls have the same color"
C : " The two drawn balls have different colors"
D : " The two drawn balls have different colors and hold odd numbers"

1) Calculate the following probabilities:
   $p(A)$, $p(B)$, $p(A \cap B)$ and $p(A/B)$.
   Are the events A and B independent?
2) a- Calculate $p(C)$ and prove that $p(D) = \frac{1}{3}$.
   b- The two drawn balls have different colors, what is the probability that they hold odd numbers?

N° 10.

For an oral exam, two teachers M_1 and M_2 prepare math exercises.
M_1 prepares 7 exercises distributed as follows :
2 exercises of probability, 1 exercise of statistics and 4 exercises of functions
M_2 prepares 7 exercises distributed as follows :

3 exercises of probability, 2 exercises of statistics and 2 exercises of functions.
Each exercise is written on a paper and placed in an envelope.
The exercises prepared by M_1 are placed in a box B_1 and the exercises prepared by M_2 are placed in a box B_2.

1) The candidate chooses, at random, a box and draws an envelope from this box. Calculate the probabilities of the following events:
   E : The candidate has drawn an exercise of probability knowing that he has chosen box B_1.
   F : The candidate has chosen an exercise of probability of box B_1.
   G : The candidate has chosen an exercise of probability.
2) In this question the candidate draws at random an envelope of each box. Calculate the probability of drawing two exercises of statistics.
3) In this question the candidate draws successively and without replacement three envelopes from box B_2.
   What is the probability that the third envelope drawn is the only exercise about functions?
4) In this question the candidate draws simultaneously 3 questions of box B_1.
   Let X be the random variable equal to the number of exercises of probability drawn. Determine the probability distribution of X.
5) In this question the candidate draws simultaneously 2 questions of box B_1 .and 1 question from box B_2
   Let X be the random variable equal to the number of exercises of functions drawn. Determine the probability distribution of X.

$\boxed{N^\circ 11.}$

An urn contains 12 white balls and 8 black balls.

1) We draw successively and with replacement 3 balls of this urn.
   Calculate the probability of each of the following events :
   D: the drawn balls are 2 white and 1 black in this order.
   E : the drawn balls are 2 white and 1 black in any order.

2) We draw successively 2 balls from the urn with respecting the following rule:
   If the ball drawn is white, we replace it back in the urn and we draw another ball. If not, we leave it outside and we draw another ball.

Calculate the probability of drawing:

a- Exactly one white ball. b- At least one white ball.

N° 12.

In a big store, a television and a *DVD* machine are on sale for a week. A person suggests that:

- The probability that he buys a television is $\frac{3}{5}$.
- The probability that he buys a *DVD* machine if he buys a television is $\frac{7}{10}$.
- The probability that he buys the *DVD* machine if he doesn't buy the television is $\frac{1}{10}$.

Designate by T and L the following events :

T : « The person buys the television »

D : «The person buys the *DVD* machine »

1) Use a tree diagram to translate the given.
2) Calculate the probabilities of each of the following events:
   a- E : « The person buys the two machines »
   b- F : « The person buys the *DVD* machine »
   c- G : « The person doesn't buy any of the two machines »
3) Before the sale, the television cost 500 000 LL and the *DVD* machine cost 200 000 LL.
   During the sale week, the store made an offer of 15 % discount for buying each of the two machines alone and 25 % for buying both machines together.
   Designate by X the possible expenses of this person(in LL).
   a- Determine the possible values of X.
   b- Determine the probability distribution of X and calculate $E(X)$.
4) On Monday morning, 30 clients visited the big store.
   Randomly we select three clients , What is the probability that the three clients bought only a television.

N° 13.

Consider two urns U and V such that :

U contains 4 **white balls and 3 black balls.**

V contains 2 **white balls and 1 black ball.**

We draw one ball from U and we place it in V then we draw one ball of V and we place it in U.

The set of these two drawings is called a « trial ».

1) Consider the event E: « After the trial, the two urns contain the same balls as in their initial state ».

   Show that $p(E)=\frac{9}{14}$.
2) Consider the event F: After the trial, urn V contains only one white ball. Calculate $p(F)$.
3) Consider the event H: After the trial, urn V contains 3 white balls. Calculate $p(H)$.
4) If after the trial, urn V contains only one white ball, we gain 3 points. If after the trial, urn V contains two white balls, we gain 1 point. If after the trial, urn V contains three white balls, we lose 9 points. Let X be the random variable equal to the algebraic gain of the player..
   a- Determine the probability distribution of X .
   b- Calculate the expected value $E(X)$. Is this game fair?

N° 14.

The 80 students of the third secondary classes of a school are distributed in three sections : GS , LS and ES according to the following table :

	GS	*LS*	*ES*
Girls	8	18	10
Boys	12	14	18

Part A.

The direction of the school chooses simultaneously and at random a group formed of 3 students of these classes to participate in a TV program.

1) What is the number of possible groups ?
2) Calculate the probability of choosing three students of the LS section.
3) The three chosen students are from the *SE* section , what is the probability that they are all boys ?
4) The chosen group is formed of 3 girls, what is the probability that they are from the same section.?
5) Show that the probability of choosing a group containing a girl of each section is $\frac{18}{1027}$.
6) Designate by X the random variable equal to the number of boys chosen. Determine the probability distribution of X.

Part B.

The direction of the school chooses one student from each section..

1) Calculate the probability of choosing three girls.
2) Consider the two events :

   B : The student chosen of the LS class is a boy.

   H : The group chosen is formed of two girls and one boy.

   G : The student chosen of the LS class is a girl.

   Calculate $p(H/B)$ and $p(H/G)$.

N° 15.

For an exam, each of ten teachers prepare 2 questions.
The 20 questions are placed in 20 identical envelopes.
Two candidates sit for the exam.
Each one chooses, at random, two questions.
The questions chosen by the first candidate are not offered for the second.
Denote by A_1 the event: The two questions obtained by the first candidate are written by the same teacher.
Denote by A_2 the event: The two questions obtained by the second candidate are written by the same teacher.

1) Show that $p(A_1) = \dfrac{1}{19}$.
2) a- Calculate $p(A_2 / A_1)$.

   b- Calculate the probability that each of the two candidates get two questions written by the same teacher.
3) a- Calculate $p(A_2 / \overline{A_1})$.

   b- Deduce $p(A_2)$ and $p(A_1 \cup A_2)$.

N° 16.

An urn U_1 contains one ball holding the number 100 and three balls holding each the number 20 .
An urn U_2 contains three balls holding each the number 50 and two balls holding each the number 20 .

1) We draw at random one ball of each of the two urns.
   Let X be the random variable equal to the sum of the numbers on the balls thus drawn.
   Determine the probability distribution of X .

2) In this part, we choose one of these two urns and we draw simultaneously two balls at random. Consider the following events:
   E : We choose urn U_1.
   F : We choose urn U_2 .
   S : The sum obtained is less than 90 .
   Suppose that $p(E)=\frac{2}{3}$ and $p(F)=\frac{1}{3}$.
   a- Calculate the probability $p(S/E)$.
   b- Calculate the probability of each of the following events: $S\cap E$, $S\cap F$ and S.

N° 17.

In a school, there are two sections: English and French.
To participate in a marathon, the director of activities in the school chooses:
From the French section, a group formed of two students of the G.S. class and 3 students of the L.S. class.
From the English section, a group formed of three students of the G.S. class and 2 students from the L.S. class.
From the 10 representatives of the school chosen above, 3 members are chosen at random to represent the school in a conference.

1) Consider the following events:
   E : The 3 members chosen represent the French section.
   F : The three members chosen represent the English section.
   G : The three members chosen are two students of the G.S. class and 1 student of the L.S. class
   a- Show that $p(G)=\frac{5}{12}$.
   b- Calculate the following probabilities: $p(E)$, $p(G/E)$ and $p(G/F)$.
   c- What is the probability of the event H : The 3 chosen members are 2 students of the G.S. class and 1 student of the L.S. class all representing the French section?
2) Let X be the random variable equal to the number of students of the G.S. class chosen.
   a- Determine the probability distribution of X .
   b- Determine and trace the distribution function of X .

N° 18.

In a multiple choice questionnaire, for each question given, two answers are proposed of which one is correct.

The questionnaire is constituted of three questions.
A candidate answers, at random, each of the three questions.

1) Consider the two following events:
   E : The candidate gives correct answers to the three questions.
   F : Among the three answers of the candidate, there are exactly two correct answers.
   Calculate $p(E)$ and $p(F)$.
2) The marking scheme gives $+5$ points for each correct answer and -3 points for each wrong answer.
   Let X be the random variable equal to the final grade of the candidate in this questionnaire.
   a- Determine the probability distribution of X.
   b- Calculate the expected value $E(X)$.
   c- Determine and trace the distribution function of X.

N° 19.

A university proposes to its students three options only: A, B and C.
Each student has to follow one and only one of the proposed options.
The students who chose option A are double those who have chosen B.
The students who have chosen B are triple those of C.
We know also that :
20% of the students who followed option A are girls.
30% of the students who followed option B are girls.
40% of the students who followed option C are girls.
We choose at random a student from this university.
Denote by :
A the event : the student follows option A.
B the event : the student follows option B.
C the event : the student follows option C.
G the event : the student is a girl.

1) Calculate the probabilities of each of the events A, B and C.
2) a- Calculate the probability that the student is a girl who has chosen option A.
   b- Calculate $p(G)$.
3) Calculate the probability that the student has chosen option A knowing that she is a girl
4) The student did not choose option A, calculate the probability that she is a girl.

N°20.

For the maintenance of a central heating system, a company controls its radiators as follows:

20 % of the radiators are guaranteed.

From the radiators under guarantee, the probability that a radiator is defective is $\frac{1}{100}$.

From the non- guaranteed radiators, the probability that a radiator is defective is $\frac{1}{10}$.

We define by G the following event: The radiator is under guarantee.

1) Calculate the probabilities of the following events:
   A : The radiator is under guarantee and is defective.
   D : The radiator is defective.
2) The radiator is defective.
   Show that the probability that it is under guarantee is $\frac{1}{41}$.
3) The control is for free if the radiator is under guarantee.
   It costs 20 000 L.P if the radiator is not under guarantee and is not defective.
   It costs 50 000 L.P if the radiator is not under guarantee and is defective.
   Denote by X the random variable that represents the cost of control of the radiator.
   a) Determine the probability distribution of X as well as its expected value.
   b) Estimate the amount of money paid by the company that owns 50 radiators.

N°21.

Part A .

An urn contains n black balls ($n \geq 1$) and 2 white balls.

We draw at random, successively and without replacement, two balls from the urn.

1) What is the probability of drawing two white balls?
2) Denote by u_n the probability of drawing two balls of the same color.
   Calculate u_n in terms of n and calculate $\lim\limits_{n \to +\infty} u_n$.

Part B .

In this part, let $n = 4$. A player draws successively and without replacement, two balls from the urn.

If the two balls are black, the player gains x Euros $(x > 0)$ and the game stops.

If the two balls are white, the player gains $6x$ Euros and the game stops.

If the two balls drawn are of different colors, then he doesn't replace them in the urn and proceeds to a third drawing:

- If the third ball is black, he gains y Euros $(y > 0)$ and the game stops
- If not he loses 3 Euros and the game stops.

Designate by G the random variable equal to the algebraic gain of the player.

1) What is the probability that the game stops after the first two drawings?
2) Determine the probability distribution of G.

N°22. ***For the students of the GS section***

Judy has a cellular for which she has subscribed in a monthly package of two hours.

Being concerned of regulating her expenses, she studies the succession of her calls.

- If during one month, she exceeds her subscription, the probability that she exceeds her subscription in the following month is $\frac{1}{5}$.
- If during one month, she doesn't exceed her subscription, the probability that she exceeds her subscription in the next month is $\frac{2}{5}$.

n being a natural number, designate by:

A_n the event: « Judy exceeds her subscription in month n »

$\overline{A_n}$ the contrary event.

Let $p_n = p(A_n)$ and suppose that $p_1 = \frac{1}{2}$.

1) Calculate the probability p_2
2) a- Find the probabilities of A_{n+1} given that A_n is realized and of the probability of A_{n+1} given that $\overline{A_n}$ is realized.

   b- Show that for all non zero natural numbers n, we have: $p(A_{n+1} \cap A_n) = \frac{1}{5} p_n$ and express $p(A_{n+1} \cap \overline{A_n})$ in terms of p_n

c- Show that $p_{n+1} = \frac{2}{5} - \frac{1}{5} p_n$.

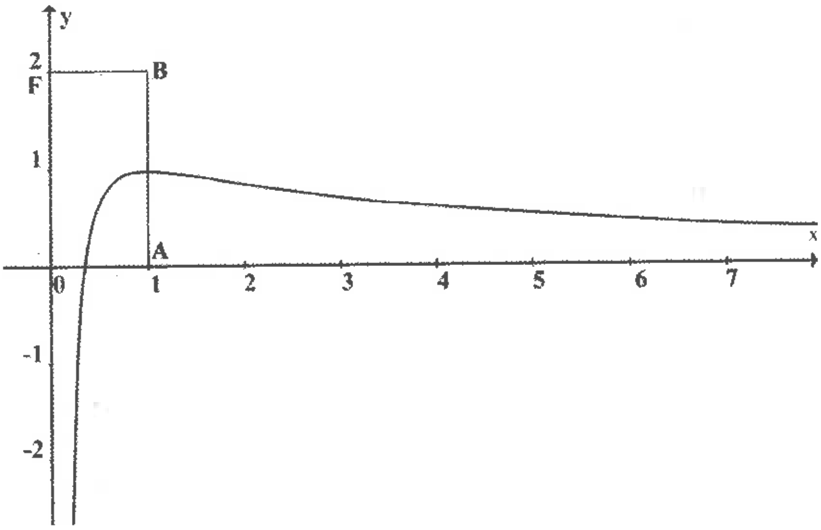
3) For all non zero natural numbers n, let $u_n = p_n - \frac{1}{3}$.

a- Show that (u_n) is a geometric sequence whose ratio and first term are to be determined.

b- Express u_n then p_n in terms of n, determine $\lim\limits_{n\to+\infty} p_n$.

N°23. ***For the students of the G.S. section***

The curve (C) below is that, in an orthonormal system $(O;\vec{i},\vec{j})$, of the function f defined over $]0;+\infty[$ by $f(x) = \dfrac{1+\ln x}{x}$.



***Part A*.**

1) Calculate the area A_1 of the region D_1 limited by par (C), the axis $x'x$ and the two straight lines of equations $x = \frac{1}{e}$ and $x = 1$.

2) Consider the rectangle $OABF$ with $A(1;0)$, $B(1;2)$ and $F(0;2)$.
D_2 is the domain inside the rectangle $OABF$ and excluding D_1.
Calculate the area A_2 of the region D_2.

Part B .
Reine throws an arrow on the rectangle $OABF$.
We admit that the probability of reaching the rectangle is 0.8 and that the probabilities of reaching D_1 and D_2 are proportional respectively to the two area of these domains.
Consider the two urns U and V.
U contains 3 red balls and 4 white balls.
V contains4 red balls and 3 white balls .
If the arrow reaches region D_1, Reine draws two balls simultaneously and a random from U .
If the arrow reaches region D_2, Reine draws two balls simultaneously and at random of V.

1) Show that the probabilities of reaching regions D_1 and D_2 are respectively $\frac{1}{5}$ and $\frac{3}{5}$.
2) Calculate the probabilities of each of the following events:
   R : « the drawn balls are red »
   w: « the drawn balls are white »
   M : « the drawn balls are of the same color »
   D : « the drawn balls are of different colors »
3) The drawn balls are white , what is the probability that they come from U.
4) Reine loses3000 LL if the arrow hits the region outside the rectangle

   $OABF$.
   Similarly, she loses 1000 LL for each white ball drawn, and gains 2000 LL for each red ball drawn .
   Let X be the random variable equal to the algebraic gain of Reine .
   a- Determine the probability distribution of X.
   b- Is the game fair ?

N° 24.
Consider :
- two urns U and V such that :
  U contains 3 red balls, 2 white balls and 1 black ball.
  V contains 2 red balls and 4 white balls.
- a bag S containing 6 counters holding the letters :
  « a ; b ; b ; c ; c ; c »

A game rolls in the following manner :
A player draws a counter of sac S.
If he obtains a counter holding the letter a then he draws simultaneously 2 balls of U.
If he obtains a counter holding the letter b then he draws simultaneously 2 balls of V.
If he obtains a counter holding the letter c then he draws one ball from U and one ball from V .
Consider the following events :
A : The player obtains a counter holding the letter a
B : The player obtains a counter holding the letter b
C : The player obtains a counter holding the letter c
R : The two drawn balls are red.
M : The two drawn balls are of the same color.

1) a- Calculate the probabilities $p(A)$, $p(B)$ and $p(C)$
   b- Calculate the probabilities $p(R/A), p(R/B)$ and $p(R/C)$.
   c- Deduce $p(R)$.
   d- Calculate $p(M)$.
2) Calculate the probability of the event B:
   « One and only one ball out of the drawn balls is black »
3) Consider the event L :
   « The two drawn balls are one red and one white »
   a- Calculate the probabilities $p(A\cap L), p(B\cap L)$ and $p(C\cap L)$.
   b- Deduce the probability $p(L)$
4) The player gains 5 $ for each red ball drawn , loses 2 $ for each white ball drawn and loses 1 $ for each black ball drawn.
   Let X be the random variable equal to the algebraic gain of the player.
   a- Give the possible values of X.
   b- Determine the probability distribution of X.

$N°25.$

An urn U_1 contains 3 green balls , 4 red balls and n yellow balls where n is a natural number verifying $1\leq n\leq 10$.
An urn U_2 contains 3 green balls , 4 red balls and $n-1$ yellow balls.

1) A player draws simultaneously and at random two balls from the urn U_1
Calculate, in terms of n, the probabilities of each of the following events :
R : the two drawn balls are red.
D : the two drawn balls are one red and one green.
2) The player decides to play a game, for this he draws one ball from urn U_1 :
   - If the ball drawn is green, he gains 16 000 LL.
   - If the ball drawn is red, he loses 12 000 LL.
   - If the ball drawn is yellow, he puts it in urn U_2 and then draws one ball from U_2.
     * If the ball drawn from U_2 is green he gains 8 000 LL
     * If the ball drawn of U_2 is red, he loses 2 000 LL
     * If the ball drawn from U_2 is yellow he neither gains or loses.

   At the beginning of the game, the player has 12 000 LL.
   Let X be the random variable equal to the amount that the player has at the end of the game.
   a- Give the possible values of the random variable X.
   b- Determine theprobability distribution of X.
   c- Prove that the expected value of X is $E(X)=4000\left[3+\dfrac{4n}{(n+7)^2}\right]$.
3) Consider the function defined over $[0;10]$ by $f(x)=\dfrac{x}{(x+7)^2}$.
   a- Set up the table of variations of f.
   b- Deduce the value of n for which $E(X)$ is maximal .

N°26.

A game consists of throwing an arrow on a target divided into three parts numbered 1, 2, 3.
Two players A and B are present.
We admit that ach of the throws hits one part only and that the throws are independent.
For the player A, the probabilities of hitting
1, 2, 3 are in this order $\dfrac{1}{12};\dfrac{1}{3};\dfrac{7}{12}$.
For player B the three eventualities are equiprobable..

Part A .

Player A throws the target three times.

Designate by E , F and G the following events :

E : Player A reaches part 3 .

F : Player A reaches parts 1, 2, 3 in this order .

G : Player A reaches parts 1, 2, 3.

Calculate $p(E)$, $p(F)$ and $p(G)$.

Part B .

We choose at random one of the two players.

The probability of choosing A is two times the probability of choosing B.

1) What is the probability of choosing A?
2) Only one throw is made .
   What is the probability that part 3 is hit?
3) Only one throw is made, and part 3 is hit.
   What is the probability that player A threw the arrow?

Supplementary Problems

N° 1.

An urn contains 4 white balls and 2 black balls, we draw simultaneously and at random two balls from the urn..

1) a- Calculate the probability of getting two white balls.
   b- Calculate the probability of getting two balls of the same color.
   c- Calculate the probability of getting at least one black ball.
2) After the 1st drawing, we replace the white balls in the urn and we keep the black balls outside and we draw again, simultaneously and at random two balls of the urn.
   Denote by A , B , C , D the following events:
   A : The two balls are white in the first drawing of two balls
   B : One ball is white and one ball is black in the first drawing of the two balls.
   C : Two black balls are drawn in the first drawing of two balls.
   D: One ball is white and one is black in the second drawing of the two balls.
   a- Calculate : $p(D/A)$, $p(D/B)$ and $p(D/C)$.
   b- Deduce $p(D\cap A)$, $p(D\cap B)$, $p(D\cap C)$ and $p(D)$.
3) We draw successively 2 balls from the urn respecting the following rule:
   If the ball drawn is white, we replace it back in the urn and we draw another ball. If not, we leave it outside and we draw another ball.
   a- Calculate the probability of drawing exactly one white ball.
   b- Let X be the random variable that is equal to the number of white balls drawn . Determine the probability distribution of X

N° 2.

We aim at testing the vaccine on a given population.
One quarter of the population has been vaccinated.
We notice that there is one vaccinated man in 10 sick people.

On the other hand , among the vaccinated people, $\frac{1}{9}$ are sick.

We choose at random one person from this population.
Denote by :
M the event : « The person is sick »
V the event : « the person is vaccinated ».

1) Calculate the probability $p(M \cap V)$ and deduce that $p(M) = \frac{5}{18}$.
2) Calculate $p(M \cap \overline{V})$ and deduce $p(M / \overline{V})$.

N°3.

An urn contains three black balls and one white ball.
We throw a fair token having one white face and one black face.
If the token shows a white face , we add one white ball to the urn and if the token shows a black face, we add one black ball to the urn.
Then , we draw, simultaneously and at random three balls from the urn.

1) Consider the two events:
   E_0 : The three drawn balls are black.
   B : The token shows a white face .
   a- Calculate : $p(B)$; $p(E_0 / B)$ and $p(E_0 / \overline{B})$.
   b- Calculate $p(E_0)$.
2) We call E_1 the event :
   One white ball is drawn among the three drawn balls.
   Calculate $p(E_1)$.

N°4.

In a store, there are two sets of pants:
Set A is formed of 50 black pants and 40 white pants.
Set B is formed of 30 black pants and 80 white pants.

1) A customer buys one pant of set A and one pant of set B.
   Consider the following events:
   E : The two pants are black
   F : One pant is black and the other is white.
   Prove that $p(E) = \frac{5}{33}$ and calculate $p(F)$.
2) The prices of the pants are given as shown in the following table:

	Black Pant	White Pant
Set A	60 000 L.P.	50 000 L.P.
Set B	40 000 L.P.	25 000 L.P.

We designate by X the random variable equal to the sum paid by this customer for the price of the two pants.

a- Find the four values of X .

b- Determine the probability distribution of X .

$N°5.$

In a fun fair, a lottery is organized with many lottery "booklets" .
Each booklet consists of 10 tickets out of which 3 win and 7 lose.

1) A person chooses at random 3 tickets of one complete booklet.
   a- Calculate the probability so that this person does not get any winning ticket
   b- Calculate the probability so that this person gets one winning ticket.
2) A second person takes one ticket from each of 3 complete booklets.
   a- Calculate the probability that he obtains three winning tickets.
   b- Calculate the probability that this person gets at least one winning ticket.

$N°6.$

Consider two urns :

- U and V such that :
  U contains 2 red balls and $n-2$ white balls.
  V contains 3 red balls and $n-3$ white balls.
- Two perfect dice, in the shape of a tetrahedron having each four faces numbered from 1 to 4 .

A game is played in the following manner:
A player throws the two dice.
If he obtains two different numbers whose sum $S \leq 5$ then he draws at random a ball from U.
If he obtains two different numbers whose sum $S \geq 6$ then he draws at random a ball from V.
If he obtains two equal numbers then he empties urns U and V in one urn W then he draws at random one ball from W.
Consider the following events:
A: The numbers obtained are different and their sum $S \leq 5$
B : The numbers obtained are different and their sum $S \geq 6$
C : The two numbers are equal.
R : The drawn ball is red.

1) Show that $p(A)=\frac{1}{2}$ and calculate $p(B)$ and $p(C)$.
2) a- Calculate $p(R/A)$, $p(R/B)$ and $p(R/C)$ in terms of n.

b- Calculate $p(R)$.

c- Determine n so that $p(R) \geq 0.475$

3) a- Calculate the probability that the dice show two different numbers knowing that the drawn ball is white.

b- Calculate the probability that the dice show two different numbers knowing that the drawn ball is red.

4) Let $\lambda = f(n) = \dfrac{p(\overline{R})}{p(R)}$ and consider the points $M_n(n;\lambda)$

a- Show that the points M_n are collinear.

b- Determine n so that the probability of the event « the drawn ball is white » is less than double that of event R.

N°7.

1) An urn U_1 contains three white , one black ball and one green ball .
An urn U_2 contains two white balls, one black ball and three green balls.
We choose at random one urn and we draw simultaneously three balls of this urn.
Consider the following events :
E : « the balls are drawn from U_2».
F : « the balls are of three different colors of U_2».
G : « the three balls have different colors ».
Calculate the probability of F and the probability of G.

2) We place the balls from the two urns in one urn W and we draw simultaneously three balls of this urn.
Let X be the random variable representing the number of color obtained. Determine the probability distribution of X .

N°8.

A doorkeeper carries 5 keys, of which 3 only open a door.
He wanted to test them all by trying one key after the other.
Note that the choice of keys is done at random and without replacement.
We call the key numbered n, the key used in the n^{th} drawing.

1) We call D_1 the event: Key number 1 does not open the door .
Calculate $p(D_1)$.

2) We call D_2 the event: Key number 2 does not open the door.

Calculate $p(D_2 / D_1)$ then deduce $p(D_1 \cap D_2)$.

3) What is the probability of event E:
Keys numbers 1 and 2 open the door and key 3 does not open it?

N°9.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$.
(1unit = 3 cm) , consider the points $I(1;0)$, $J(0;1)$ and $K(1;1)$ and construct the square $OIKJ$.
Consider the functions f and g defined over [0;1] by:
$f(x) = x^3$ and $g(x) = \sqrt{x}$ .Designate by (C_1) and (C_2) the representative curves of the functions f and g.
The curves (C_1) and (C_2) determine in the square $OIKJ$ three regions .
D_1 limited by $[OI]$; $[IK]$ and (C_1).
D_2 limited by (C_1) and (C_2).
D_3 limited by $[OJ]$; $[JK]$ and (C_2).

1) Calculate the areas of the three regions D_1, D_2 and D_3.
2) Judy throws an arrow to the square $OIKJ$.
The probability that she hits the square is 0.6
We admit that the probability that she hits regions D_1, D_2 and D_3 is proportional to their areas.
Prove that the probabilities for Nada to hit regions D_1, D_2 and D_3 are respectively $\frac{3}{20}$, $\frac{1}{4}$ and $\frac{1}{5}$.
3) Judy gets 1 point if she hits domain D_1, 2 points if she hits D_2 and 3 points for D_3.
She hits two arrows successively, knowing that the two successive throws are independent.
a- What is the probability that she gets at least 1 point?
b- What is the probability that she gets exactly 3 points?

N°10.

A bag contains 75 chocolate bars: 50 milk chocolate bars and 25 black chocolate bars.
Among the milk chocolate bars , 10 % contain a gift voucher,
And among the black chocolate bars 20 % contain a gift voucher.
We draw at random one bar from the bag.

Consider the following events:
M :" The bar obtained is a milk chocolate bar" .
B :" The bar obtained is a black chocolate bar"
G :" The tablet contains a gift voucher"

1) Calculate $p(M \cap V)$, $p(B \cap G)$.Deduce $p(G)$.
2) The chocolate bar obtained contains a gift voucher.
What is then the probability that it is a milk chocolate bar?
3) Select from the bag and simultaneously three bars and suppose that the price of a milk chocolate bar is 2000 LL and that of a black chocolate bar is 1500 LL.
Designate by X the random variable equal to the sum of three chosen bars .
   a- Determine the probability distribution of X.
   b- Calculate the expected value of X.

$\boxed{N° 11.}$

Part A .

An urn U_1 contains:
3 counters numbered 0 and two counters numbered 1 .
Another urn U_2 contains 5 counters numbered 1 to 5 .
We select at random one counter from each urn and we designate by X the random variable equal to the product of the numbers on the counters selected.

1) Prove that $p(X = 0) = \frac{3}{5}$.
2) Determine the probability distribution of X .

Part B .

We empty the contents of the two previous urns in an urn W , which will now contain 10 counters.
We draw ,simultaneously and at random 2 counters of W .
Designate by Y the product of the numbers that show on the drawn counters .

1) Prove that $p(Y = 0) = \frac{8}{15}$.
2) Calculate the probability $p(Y < 4)$.

$\boxed{N° 12.}$

An U_1 contains 4 red balls and 6 black balls.
Another U_2 contains 1 red ball and 9 black balls.

A player throws a perfect die. if the face shows the number 1 , he draws one ball from U_1 and if not he draws one ball from U_2.
Consider the following events :
A : the face shows 1 R : the drawn ball is red.

1) a- Calculate the probabilities $p(A)$ and $p(R/A)$.

   b- Show that $p(R)=0.15$.

2) The player repeats the game twice with replacing the balls in the urn.

   After the two drawings , the players gains 3 points for each red ball and gets -2 points for each black ball obtained. Designate by X the random variable equal to the algebraic sum of points of the player at the end of the game.

   a- Verify that the possible values of X are : 6 ; +1 ; -4

   b- Determine the probability distribution of X et and calculate $E(X)$.

$N°13.$

An urn contains :
Three green balls holding the number 0 .
Two red balls holding the number 5.
One ball holding the number α .
A player draws simultaneously and randomly three balls of the urn.

1) What is the probability of each of the following events :

   A : Getting three balls of the same color.

   B : Getting three balls of different color.

   C :Getting two and only two balls of the same color.

2) The player receives, in euros , the sum of the numbers on the drawn balls.

   Let X be the random variable equal to the gain of the player.

   a- Verify that the possible values of X are:

   0 , 5 , 10 , α , $5+\alpha$, $10+\alpha$.

   b- Determine the probability distribution of X .

   c- Calculate the expected value $E(X)$ in terms of a .

   d- Calculate α so that $E(X)=20$ .Calculate, in this case , the variance $V(X)$ and the standard deviation $\sigma(X)$.

Solutions

N° 1.

1) $C_n^2 = 3$ is equivalent to $\dfrac{n!}{2!(n-2)!} = 3$ and to $\dfrac{n(n-1)}{2} = 3$ which gives $n^2 - n - 6 = 0$, then $n' = -2$, rejected and $n'' = 3$ accepted.

2) $P_n^3 = 6n$ is equivalent to $\dfrac{n!}{(n-3)!} = 6n$ and to $(n-1)(n-2)n = 6n$.

   But $n \neq 0$ so $(n-2)(n-1) = 6$ which gives $n^2 - 3n - 4 = 0$.
   Then $n' = -1$ rejected and $n'' = 4$ accepted.

3) $C_{2n}^1 + C_{2n}^2 + C_{2n}^3 = 35n$ is equivalent to :

   $\dfrac{(2n)!}{(2n-1)!} + \dfrac{(2n)!}{2!(2n-2)!} + \dfrac{(2n)!}{3!(2n-3)!} = 35n.$

   Then $2n + \dfrac{(2n-1)(2n)}{2} + \dfrac{(2n-2)(2n-1)(2n)}{6} = 35n$

   Which gives $4n^2 = 100$ that is $n^2 = 25$, then $n = 5$.

N° 2.

1) Using the binomial formula:

   $(1+x)^n = C_n^0 x^0 + C_n^1 x^1 + C_n^2 x^2 + \dots\dots\dots\dots + C_n^n x^n \qquad (1)$

   and for $x = 1$ we get :

   $2^n = C_n^0 + C_n^1 + C_n^2 + \dots\dots\dots\dots + C_n^n$ then $S_1 = 2^n$.

   On the other hand we have:

   $2^n = C_n^0 + C_n^1 + C_n^2 + \dots\dots\dots\dots + C_n^n$

   $= \dfrac{n!}{0!n!} + \dfrac{n!}{1!(n-1)!} + \dots + \dfrac{n!}{n!0!}$. Then $S'_1 = \dfrac{2^n}{n!}$.

2) For $x = 3$ the equation (1) becomes:

   $(1+3)^n = C_n^0 + C_n^1 3 + C_n^2 3^2 + \dots\dots\dots\dots + C_n^n 3^n$. Then $S_2 = 4^n$

3) Derive (1) with respect to x we obtain:

   $n(1+x)^{n-1} = C_n^1 + 2C_n^2 x + \dots\dots\dots\dots + nC_n^n x^{n-1}$

   for $x = 1$ we get:

$n2^{n-1} = C_n^1 + 2C_n^2 + \dots\dots\dots\dots\dots + nC_n^n$, then $S_3 = n \cdot 2^{n-1}$.

N° 3.

1) The n points A_1, A_2, ……, A_n are given such that no three are collinear and we know that every two points A_i and A_j determine only one straight line.

   The total number of straight lines is $C_n^2 = \dfrac{n!}{2!(n-2)!} = \dfrac{n(n-1)}{2}$.

2) The total number of triangles is $C_n^3 = \dfrac{n(n-1)(n-2)}{6}$.

N° 4.

A diagonal an a convex polygon is the straight line segment joining two non adjacent vertices.

The total number of straight line segments is C_n^2 .

The number of sides in a polygon is n , then the number of diagonals is $C_n^2 - n = \dfrac{n(n-1)}{2} - n = \dfrac{n(n-3)}{2}$.

N° 5.

1) $p(A) = \dfrac{C_2^2}{C_6^2} = \dfrac{1}{15}$, $p(B) = \dfrac{C_4^1 \times C_2^1}{C_6^2} = \dfrac{8}{15}$, $p(C) = \dfrac{C_4^2}{C_6^2} = \dfrac{6}{15} = \dfrac{2}{5}$.

2) a- $p(D/A) = 0$, since the two yellow balls were selected in the first drawing, it is impossible to draw a yellow ball in the second drawing.

   $p(D/B) = \dfrac{C_3^1 \times C_1^1}{C_4^2} = \dfrac{3}{6} = \dfrac{1}{2}$, since one green ball and one yellow ball were selected in the first drawing, there remains 3 green balls and one yellow ball in the urn.

   $p(D/C) = \dfrac{C_2^1 \times C_2^1}{C_4^2} = \dfrac{4}{6} = \dfrac{2}{3}$, since two green balls were selected in the first drawing, there remains 2 green balls and 2 yellow balls .

   b- $p(D \cap A) = p(D/A) \times p(A) = 0$.

   $p(D \cap B) = p(D/B) \times p(B) = \dfrac{1}{2} \times \dfrac{8}{15} = \dfrac{4}{15}$.

$$p(D \cap C) = p(D/C) \times p(C) = \frac{2}{3} \times \frac{2}{5} = \frac{4}{15}.$$

c- $p(D) = p(D \cap A) + p(D \cap B) + p(D \cap C) = \frac{4}{15} + \frac{4}{15} = \frac{8}{15}.$

N°6.

1) a- $p(\text{white balls}) = \frac{C_2^2}{C_5^2} = \frac{1}{10}.$

b- Getting two balls of the same color means getting two white balls or two black balls. Therefore,

$$p(\text{two balls of the same color}) = \frac{C_2^2}{C_5^2} + \frac{C_3^2}{C_5^2} = \frac{1}{10} + \frac{3}{10} = \frac{2}{5}.$$

2) a- Event B_1 is realized , that is we have added a white ball to the urn therefore: $p(w_2 / w_1) = \frac{3}{6} = \frac{1}{2}.$

$p(w_2 / B_1) = \frac{2}{6} = \frac{1}{3}$ since in the first drawing, we have drawn a black ball, that is to say that we have added one black.

b- $p(w_2) = p(w_2 \cap w_1) + p(w_2 \cap B_1).$

$p(w_2 \cap w_1) = p(w_2 / w_1) \times p(w_1)$

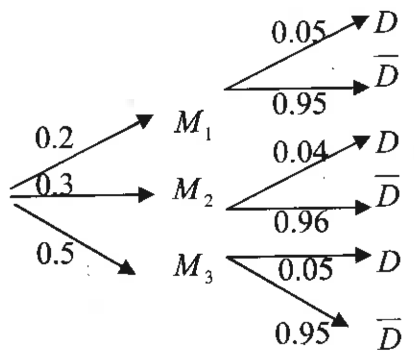
$p(w_2 \cap B_1) = p(w_2 / B_1) \times p(B_1)$

Since $p(w_1) = \frac{2}{5}$ and $p(B_1) = \frac{3}{5}$, then $p(w_2 \cap w_1) = \frac{1}{2} \times \frac{2}{5} = \frac{1}{5}$

$p(w_2 \cap B_1) = \frac{1}{3} \times \frac{3}{5} = \frac{1}{5}$, consequently, $p(w_2) = \frac{1}{5} + \frac{1}{5} = \frac{2}{5}.$

N°7.

1)



2) $p(M_1 \cap D) = p(D / M_1) \times p(M_1) = 5\% \times 20\% = 0.01$.

3) $p(D) = p(D \cap M_1) + p(D \cap M_2) + p(D \cap M_3)$.

But, $p(D \cap M_2) = p(D / M_2) \times p(M_2) = 4\% \times 30\% = 0.012$.

$p(D \cap M_3) = p(D / M_3) \times p(M_3) = 5\% \times 50\% = 0.025$.

Therefore, $p(D) = 0.01 + 0.012 + 0.025 = 0.047 = 4.7\%$.

4) $p(M_1 / D) = \dfrac{p(M_1 \cap D)}{p(D)} = \dfrac{0.01}{0.047} \approx 0.21$.

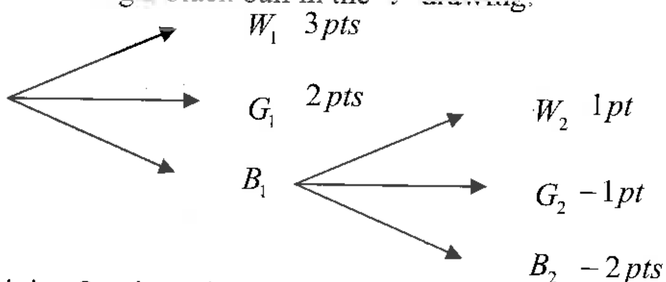
$\boxed{N^{\circ} 8.}$

Let W_i, G_i and B_i be the following events :

W_i : Drawing a white ball in the i^{th} drawing.

G_i : Drawing a green ball in the i^{th} drawing.

B_i : Drawing a black ball in the i^{th} drawing.



Gaining 2 points, that is getting one green ball in the first drawing.

Then, $p(E) = p(G_1) = \dfrac{2}{6} = \dfrac{1}{3}$.

Gaining 1 point , that is getting one black ball in the first drawing and one white ball in the second drawing ,

then : $p(F) = p(B_1 \cap W_2) = p(B_1) \times p(W_2 / B_1) = \dfrac{3}{6} \times \dfrac{1}{6} = \dfrac{1}{12}$.

Losing 1 point , that is getting one black ball in the first drawing and one green ball in the second drawing, then :

$p(G_2) = p(B_1 \cap G_2) = p(B_1) \times p(G_2 / B_1) = \dfrac{3}{6} \times \dfrac{2}{6} = \dfrac{1}{6}$.

Losing 2 points , that is getting one black ball in the first drawing and one black ball in the second drawing, then::

$p(H) = p(B_1 \cap B_2) = p(B_1) \times p(B_2 / B_1) = \dfrac{3}{6} \times \dfrac{3}{6} = \dfrac{1}{4}$.

N°9.

1) The number of possible cases is : $C_9^2 = 36$.

Drawing two balls holding odd numbers means drawing the balls :

$w_1, w_3 \quad, b_1, b_3 \quad, r_1, r_3$ then : $p(A) = \dfrac{C_6^2}{C_9^2} = \dfrac{15}{36} = \dfrac{5}{12}$.

$p(B) = p(2w) + p(2b) + p(2r) = \dfrac{3C_3^2}{C_9^2} = \dfrac{9}{36} = \dfrac{1}{4}$.

$A \cap B = \{w_1w_3 \,;\, b_1b_3 \,;\, r_1r_3\}$, then $p(A \cap B) = \dfrac{3}{36} = \dfrac{1}{12}$.

$p(A/B) = \dfrac{p(A \cap B)}{p(B)} = \dfrac{1}{3}$.

Since $p(A/B) \neq p(A)$, then the two events A and B are not independent.

2) a- Event C is opposite to the event B , $C = \overline{B}$ then

$p(C) = p(\overline{B}) = 1 - p(B) = \dfrac{3}{4}$.

Realizing the event D means drawing one ball from w_1, w_3 and one ball from b_1, b_3 or one ball from w_1, w_3 and one from r_1, r_3 or one from b_1, b_3 and one from r_1, r_3.

$p(D) = 3 \times \dfrac{C_2^1 \times C_2^1}{C_9^2} = 3 \times \dfrac{4}{36} = \dfrac{1}{3}$.

b- This event is A/C, then $p(A/C) = \dfrac{p(A \cap C)}{p(C)} = \dfrac{p(D)}{p(C)} = \dfrac{4}{9}$.

N°10.

1) $p(E) = p(P/B_1) = \dfrac{2}{7}$.

$p(F) = p(P \cap B_1) = p(P/B_1) \times p(B_1) = \dfrac{2}{7} \times \dfrac{1}{2} = \dfrac{1}{7}$

$p(G) = p(F) + p(P \cap B_2) = \dfrac{1}{7} + p(P/B_2) \times p(B_2)$

$= \dfrac{1}{7} + \dfrac{3}{7} \times \dfrac{1}{2} = \dfrac{5}{14}$

2) $p(\text{two statistics exercises}) = \frac{1}{7} \times \frac{2}{7} = \frac{2}{49}$.

3) $p(\text{the third is an exercise in functions }) = p\left(\overline{F};\overline{F};F\right) =$

$p\left(\overline{F}\right) \times p\left(\overline{F}\right) \times p(F) = \frac{5}{7} \times \frac{4}{6} \times \frac{2}{5} = \frac{4}{21}$.

4) $X(\Omega) = \{0;1;2\}$. $\quad p(X=0) = \frac{C_5^3}{C_7^3} = \frac{2}{7}$

$p(X=1) = \frac{C_2^1 C_5^2}{C_7^3} = \frac{4}{7} \qquad p(X=2) = \frac{C_2^2 C_5^1}{C_7^3} = \frac{1}{7}$

5) $X(\Omega) = \{0;1;2;3\}$, $p(X=0) = \frac{C_3^2}{C_7^2} \times \frac{C_5^1}{C_7^1} = \frac{5}{49}$,

$p(X=1) = \frac{C_4^1 \times C_3^1}{C_7^2} \times \frac{C_5^1}{C_7^1} + \frac{C_3^2}{C_7^2} \times \frac{C_2^1}{C_7^1} = \frac{22}{49}$

$p(X=2) = \frac{C_4^2}{C_7^2} \times \frac{C_5^1}{C_7^1} + \frac{C_4^1 \times C_3^1}{C_7^2} \times \frac{C_2^1}{C_7^1} = \frac{18}{49}$

$p(X=3) = \frac{C_4^2}{C_7^2} \times \frac{C_2^1}{C_7^1} = \frac{4}{49}$.

N° 11.

1) $p(D) = \left(\frac{12}{20}\right)^2 \times \left(\frac{8}{20}\right)^1 = \frac{18}{125}$.

Realizing event E means drawing in this order :
$W\,W\,B$ or $WB\,W$ or $B\,W\,W$

Then $p(E) = \frac{12}{20} \times \frac{12}{20} \times \frac{8}{20} + \frac{12}{20} \times \frac{8}{20} \times \frac{12}{20} + \frac{8}{20} \times \frac{12}{20} \times \frac{12}{20}$

$p(E) = 3 \times \frac{12}{20} \times \frac{12}{20} \times \frac{8}{20} = \frac{54}{125}$.

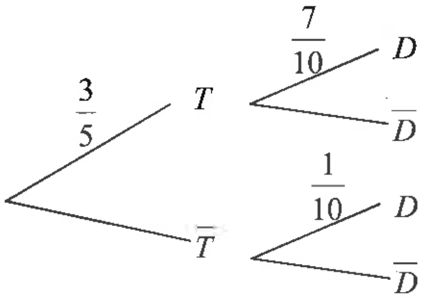
2) a- Realizing the event: : Exactly one white ball
Means drawing in this order : WB or $B\,W$, therefore,

$p = \frac{12}{20} \times \frac{8}{20} + \frac{8}{20} \times \frac{12}{19} = \frac{234}{475}$

b- Getting at least one white ball is the opposite of getting 2 black balls, then: $p=1-\frac{8}{20}\times\frac{7}{19}=\frac{81}{95}$.

N° 12.

1)



2) a- $p(E)=p\left(T\cap D\right)=p\left(D/T\right)\times p\left(T\right)=\frac{7}{10}\times\frac{3}{5}=\frac{21}{50}$.

b- $p(F)=p\left(D\right)=p\left(T\cap D\right)+p\left(\overline{T}\cap D\right)=\frac{21}{50}+p\left(D/\overline{T}\right)\times p\left(\overline{T}\right)$

$=\frac{21}{50}+\frac{1}{10}\times\frac{2}{5}=\frac{23}{50}$.

c- $p(G)=p\left(\overline{T}\cap\overline{D}\right)=p\left(\overline{D}/\overline{T}\right)\times p\left(\overline{T}\right)=\frac{9}{10}\times\frac{2}{5}=\frac{9}{25}$.

3) a- $X(\Omega)=\{0;170000;425000\,;\,525000\}$.

b- $p(X=0)=p(\overline{D}\cap\overline{T})=p(G)=\frac{9}{25}$

$p(X=170000)=p(D\cap\overline{T})=\frac{2}{50}$.

$p(X=425000)=p(T\cap\overline{D})=\frac{3}{5}\times\frac{3}{10}=\frac{9}{50}$.

$p(X=525000)=p(T\cap D)=\frac{3}{5}\times\frac{7}{10}=\frac{21}{50}$.

$E(X)=303800$ LL.

4) $\frac{3}{5}=60\,\%$ which corresponds to 18 clients who bought a

television. Then $p=\frac{C_{18}^{3}}{C_{30}^{3}}=\frac{204}{1015}$.

N° 13.

1) Event E occurs when :
   We draw one white ball from U and one white ball from V
   Or
   We draw one black ball from U and one black ball from .
   $p(E)=\frac{4}{7}\times\frac{3}{4}+\frac{3}{7}\times\frac{2}{4}=\frac{3}{7}+\frac{3}{14}=\frac{9}{14}$.
2) Event F occurs when :
   We draw one black from U and one white ball from V
   $p(F)=\frac{3}{7}\times\frac{2}{4}=\frac{3}{14}$.
3) Event H occurs when :
   We draw one white ball from U and one black ball from V
   therefore $p(H)=\frac{4}{7}\times\frac{1}{4}=\frac{1}{7}$
4) a-

x_i	-9	1	3	
Event	H	E	F	Ω
p_i	$\frac{2}{14}$	$\frac{9}{14}$	$\frac{3}{14}$	1

b- $E(X)=-9\times\frac{2}{14}+1\times\frac{9}{14}+3\times\frac{3}{14}=0$

The game is fair since $E(X)=0$

N° 14.

Part A.

1) The number of possible groups is $C_{80}^{3}=82160$.

2) p(of getting three students from the L.S. section) $=\frac{C_{32}^{3}}{C_{80}^{3}}=\frac{62}{1027}$.

3) $p=\frac{C_{18}^{3}}{C_{28}^{3}}=\frac{68}{273}$

4) $p=\frac{C_8^3}{C_{36}^3}+\frac{C_{18}^3}{C_{36}^3}+\frac{C_{10}^3}{C_{36}^3}=\frac{248}{1785}$.

5) $p(\text{a girl of each section})=\frac{C_8^1C_{18}^1C_{10}^1}{C_{80}^3}=\frac{18}{1027}$

6) $X(\Omega)=\{0\,;1\,;2;\,3\ \}$.

$p(X=0)=\frac{C_{36}^3}{C_{80}^3}=\frac{357}{4108}$.　　　$p(X=1)=\frac{C_{44}^1C_{36}^2}{C_9^2}=\frac{693}{2054}$.

$p(X=2)=\frac{C_{44}^2C_{36}^1}{C_{80}^3}=\frac{4257}{10270}$, $p(X=3)=\frac{C_{44}^3}{C_{80}^3}=\frac{3311}{20540}$

Part B.

1) $p(\text{a girl of each section})=\frac{C_8^1}{C_{20}^1}\times\frac{C_{18}^1}{C_{32}^1}\times\frac{C_{10}^1}{C_{28}^1}=\frac{9}{112}$.

2) Event *H/B* is equivalent to event :
Getting one girl from the GS section and one girl from the ES section. Therefore: $p(H/B)=\frac{C_8^1}{C_{20}^1}\times\frac{C_{10}^1}{C_{28}^1}=\frac{1}{7}$.

Event *H/G* is equivalent to the event :
Getting a girl of the GS section and a boy of the ES section
Or
Getting a boy of the GS section and a girl of the ES section.
Therefore: $p(H/G)=\frac{8}{20}\times\frac{18}{28}+\frac{12}{20}\times\frac{10}{28}=\frac{33}{70}$.

$\boxed{N°\,15}$

1) There are $C_{20}^2=190$ possible cases.

The number of favorable cases is 10 (drawing of 2 envelopes Containing questions written by the same teacher), therefore

$p(A_1)=\frac{10}{190}=\frac{1}{19}$.

2) a- If the event A_1 occurs, then there remains 18 questions , hence

there are: $C_{18}^{2}=153$ possible cases and there are 9 favorable drawings A_2 , so $p(A_2 / A_1)=\frac{9}{153}=\frac{1}{17}$.

b- $p(A_1 \cap A_2)=p(A_1)\times p(A_2 / A_1)=\frac{1}{19}\times\frac{1}{17}=\frac{1}{323}$.

3) a- If $\overline{A_1}$ occurs , the first candidate has chosen two questions written by the same teacher, then there remains 18 envelopes of which16 contain questions written by the 8 teachers ,

The two other envelopes contain each one question written by a different teacher.

$A_2 / \overline{A_1}$ is realized when the second candidate chooses two questions written by the same teacher .

There are 8 favorable cases and $C_{18}^{2}=153$ possible cases , then $p(A_2 / \overline{A_1})=\frac{8}{153}$.

b- $A_2=(A_2 \cap A_1)\cup(A_2 \cap \overline{A_1})$ and since

$(A_2 \cap A_1)$ *and* $(A_2 \cap \overline{A_1})$ are incompatible, we deduce that

$p(A_2)=p(A_2 \cap A_1)+p(A_2 \cap \overline{A_1})$.

But, $p(A_2 \cap \overline{A_1})=p(\overline{A_1})\times p(A_2 / \overline{A_1})=\frac{18}{19}\times\frac{8}{153}=\frac{16}{323}$.

then $p(A_2)=\frac{1}{323}+\frac{16}{323}=\frac{17}{323}=\frac{1}{19}$.

$p(A_2 \cup A_1)=p(A_1)+p(A_2)-p(A_1 \cap A_2)=\frac{1}{19}+\frac{1}{19}-\frac{1}{323}$.

So $p(A_2 \cup A_1)=\frac{33}{323}$.

N° 16.

1) The possible values of values of X are :

40 : (20 of U_1 and 20 of U_2). 70 : (20 of U_1 and 50 of U_2).
120 : (100 of U_1 and 20 of U_2). 150 : (100 of U_1 and 50 of U_2)

Therefore : $p(X=40)=p(20)\times p(20)=\frac{3}{4}\times\frac{2}{5}=\frac{6}{20}$.

$p(X=70)=p(20)\times p(50)=\frac{3}{4}\times\frac{3}{5}=\frac{9}{20}$.

$p(X=120)=p(100)\times p(20)=\frac{1}{4}\times\frac{2}{5}=\frac{2}{20}$.

$p(X=150)=p(100)\times p(50)=\frac{1}{4}\times\frac{3}{5}=\frac{3}{20}$.

2) a- The universe being urn U_1 , so drawing two balls having the sum less than 90 means drawing 2 balls numbered 20 from the 3 balls numbered 20 therefore: $p(S/E)=\frac{C_3^2}{C_4^2}=\frac{3}{6}=\frac{1}{2}$.

b- $p(S\cap E)=p(S/E)\times p(E)=\frac{1}{2}\times\frac{2}{3}=\frac{1}{3}$.

We know that $p(S\cap F)=p(F)\times p(S/F)$.

S/F means that we have got :

2 balls holding the number 20 of U_2

or one ball holding the number 50 and one ball holding the number 20 of U_2, therefore :

$p(S/F)=\frac{C_2^2}{C_5^2}+\frac{C_3^1\times C_2^1}{C_5^2}=\frac{1}{10}+\frac{6}{10}=\frac{7}{10}$.

$P(S\cap F)=\frac{1}{3}\times\frac{7}{10}=\frac{7}{30}$.

$P(S)=P(S\cap E)+P(S\cap F)=\frac{1}{3}+\frac{7}{30}=\frac{17}{30}$.

N° 17.

1) a- We have to choose a group of 3 members from the 10 , so the number of possible cases is C_{10}^3.

The number of favorable cases is the number of groups containing two GS students from 5 and one LS student from the 5 hence there are: $C_5^2\times C_5^1$. Therefore

$$p(G)=\frac{C_5^2\times C_5^1}{C_{10}^3}=\frac{50}{120}=\frac{5}{12}.$$

b- The number of groups formed of 3 members of the French section is C_5^3 , then $p(E)=\frac{C_5^3}{C_{10}^3}=\frac{1}{12}$.

The event G/E, means that we have to form a group formed of 2 GS students and one LS student so the number of possible cases is C_5^3. The number of favorable cases is $C_2^2\times C_3^1$.

Then $p(G/E)=\frac{C_2^2\times C_3^1}{C_5^3}=\frac{3}{10}$.

Similarly, we have $p(G/F)=\frac{C_3^2\times C_2^1}{C_5^3}=\frac{3}{5}$.

c- The event H is the event $E\cap G$, then

$$p(H)=p(E\cap G)=p(E)\times p(G/E)=\frac{1}{12}\times\frac{3}{10}=\frac{1}{40}.$$

2) a- The possible values of X are : 0 ; 1 ; 2 and 3 .

$$p(X=0)=\frac{C_5^0\times C_5^3}{C_{10}^3}=\frac{1}{12}\ ,\ p(X=1)=\frac{C_5^1\times C_5^2}{C_{10}^3}=\frac{5\times 10}{120}=\frac{5}{12}.$$

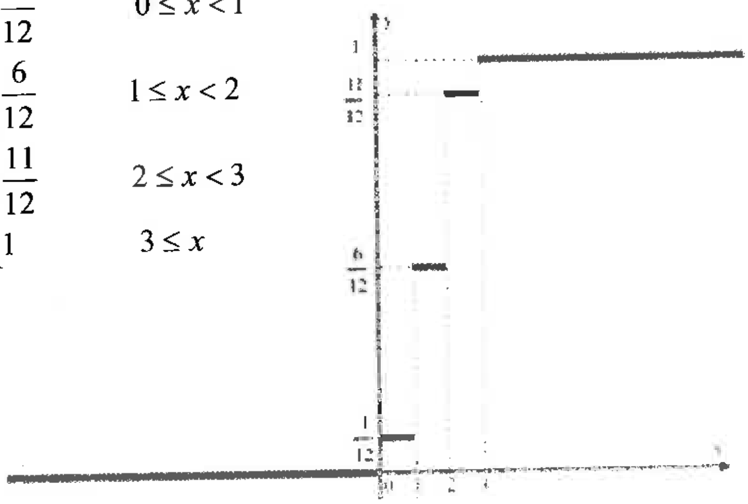
$$p(X=2)=p(G)=\frac{5}{12}\ ,\ p(X=3)=\frac{C_5^3\times C_5^0}{C_{10}^3}=\frac{20}{120}=\frac{1}{12}.$$

So we get the following table :

$X=x_i$	0	1	2	3
p_i	$\frac{1}{12}$	$\frac{5}{12}$	$\frac{5}{12}$	$\frac{1}{12}$
	$\frac{1}{12}$	$\frac{6}{12}$	$\frac{11}{12}$	1

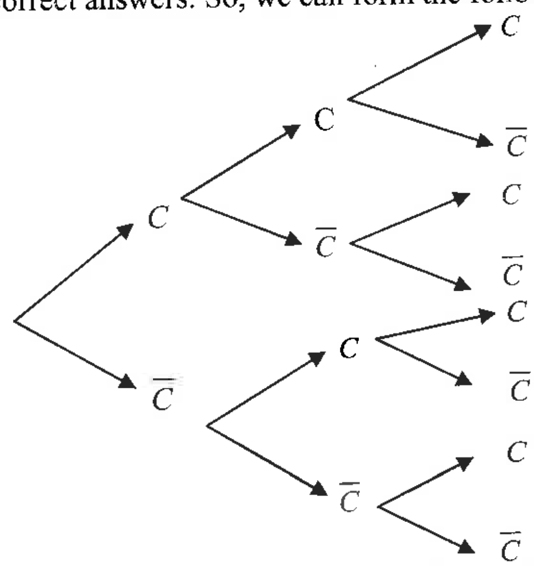
b- The distribution function is defined by :

$$F(x)=\begin{cases}0 & x<0\\ \dfrac{1}{12} & 0\le x<1\\ \dfrac{6}{12} & 1\le x<2\\ \dfrac{11}{12} & 2\le x<3\\ 1 & 3\le x\end{cases}$$



$N°$ 18.

1) We repeat the experiment three times and we either get correct or incorrect answers. So, we can form the following tree diagram:



From the two proposed answers , only one is correct then,

$p(C)=\dfrac{1}{2}$.

$$p(E) = p(C \text{ and } C \text{ and } C) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}.$$

$$p(F) = p(CC\overline{C}) + p(C\overline{C}C) + p(\overline{C}CC) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}.$$

2) a- The four possible values of the variable are :

-9 if the three answers are incorrect.

-1 if only one answer is correct.

$+7$ if only two answers are correct.

$+15$ if the three answers are correct.

The probability distribution is given by the following table :

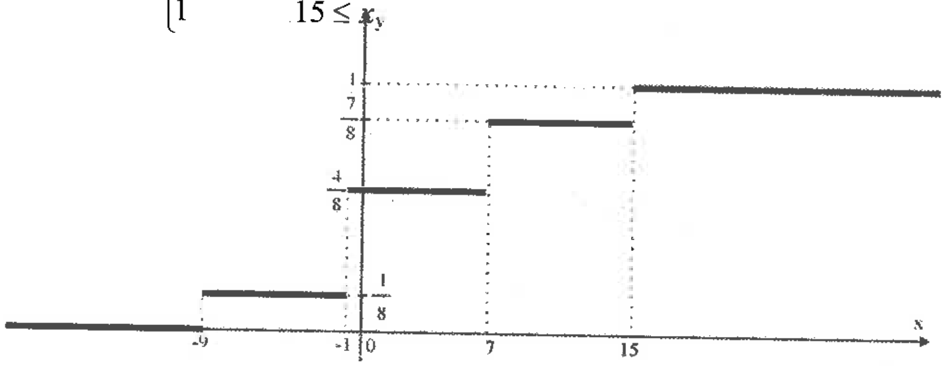
x_i	-9	-1	7	15
p_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

b- The expected value of X is :

$$E(X) = \sum p_i x_i = \frac{-9}{8} - \frac{3}{8} + \frac{21}{8} + \frac{15}{8} = 3.$$

c- The distribution function is given by :

$$F(x) = \begin{cases} 0 & x < -9 \\ \frac{1}{8} & -9 \le x < -1 \\ \frac{4}{8} & -1 \le x < 7 \\ \frac{7}{8} & 7 \le x < 15 \\ 1 & 15 \le x \end{cases}$$



N° 19.

Part A

1) Suppose that $p(C)=x$, $p(B)=3x$ and $p(A)=2p(B)=6x$

$p(A)+p(B)+p(C)=1$, therefore $10x=1$ then $x=\frac{1}{10}$

Consequently, $p(A)=\frac{6}{10}=\frac{3}{5}$, $p(B)=\frac{3}{10}$ and $p(C)=\frac{1}{10}$

2) a- $p(G\cap A)=p(G/A)\times p(A)=\frac{20}{100}\times\frac{3}{5}=\frac{3}{25}$

b- $p(G)=p(G\cap A)+p(G\cap B)+p(G\cap C)$

$$=\frac{12}{100}+\frac{30}{100}\times\frac{3}{10}+\frac{40}{100}\times\frac{1}{10}=\frac{25}{100}=\frac{1}{4}$$

3) $p(A/G)=\dfrac{p(G\cap A)}{p(G)}=\dfrac{12}{25}$

4) $p(G/\overline{A})=\dfrac{p(G\cap\overline{A})}{p(\overline{A})}$, note that $p(G)=p(G\cap A)+p(G\cap\overline{A})$

$$p(G\cap\overline{A})=p(G)-p(G\cap A)=\frac{25}{100}-\frac{12}{100}=\frac{13}{100}$$

$$p(G/\overline{A})=\frac{p(G\cap\overline{A})}{p(\overline{A})}=\frac{p(G)-p(G\cap A)}{1-p(A)}=\frac{13}{40}$$

N° 20.

1) $p(G)=\frac{10}{50}=\frac{1}{5}$ and $p(\overline{G})=\frac{4}{5}$.

$p(A)=p(D\cap G)=p(D/G)\times p(G)=\frac{1}{100}\times\frac{1}{5}=\frac{1}{500}=0.002$.

$p(D)=p(D\cap G)+p(D\cap\overline{G})$, but

$p(D\cap\overline{G})=p(D/\overline{G})\times p(\overline{G})=\frac{1}{10}\times\frac{4}{5}=\frac{4}{50}=0.08$.

Therefore, $p(D)=0.002+0.08=0.082=8.2\%$.

2) $p(G/D)=\dfrac{p(D\cap G)}{p(D)}=\dfrac{0.002}{0.082}=\dfrac{1}{41}$

3) a- The values taken by the random variable X are 0 ; $20\,000$ and $50\,000$.

$p(X=0)=p(G)=\dfrac{1}{5}$.

$p(X=50\,000)=p(\overline{G}\cap D)=\dfrac{8}{100}=0.08$.

$p(X=20\,000)=1-(0.08+0.2)=0.72$.

$E(X)=0\times 0.2+50\,000\times 0.08+20\,000\times 0.72=18\,400\ L.L$

b- The amount the company pays for 50 radiators is: $18400\times 50=920\,000\ L.L$.

N° 21

Part A .

1) $p(\text{drawing two white balls})=\dfrac{2}{n+2}\times\dfrac{1}{n+1}=\dfrac{2}{(n+1)(n+2)}$

2) $u_n=\dfrac{2}{(n+1)(n+2)}+\dfrac{n}{n+2}\times\dfrac{n-1}{n+1}=\dfrac{n^2-n+2}{(n+1)(n+2)}$

$\lim\limits_{n\to+\infty} u_n=1$.

Part B .

1) $p(\text{the game ends after the two drawings})=p(\text{2w or 2b})=$

$\dfrac{2}{6}\times\dfrac{1}{5}+\dfrac{4}{6}\times\dfrac{3}{5}=\dfrac{7}{15}$

2) $X(\Omega)=\{x;6x;y;-3\}$. $p(G=x)=\dfrac{4}{6}\times\dfrac{3}{5}=\dfrac{2}{5}$

$p(G=6x)=\dfrac{2}{6}\times\dfrac{1}{5}=\dfrac{1}{15}$ $p(G=y)=\dfrac{4}{6}\times\dfrac{2}{5}\times\dfrac{3}{4}+\dfrac{2}{6}\times\dfrac{4}{5}\times\dfrac{3}{4}=\dfrac{2}{5}$

$p(G=-3)=\dfrac{4}{6}\times\dfrac{2}{5}\times\dfrac{1}{4}+\dfrac{2}{6}\times\dfrac{4}{5}\times\dfrac{1}{4}=\dfrac{2}{15}$

N° 22.

1) $p_2=p(A_2)=p(A_2\cap A_1)+p(A_2\cap\overline{A_1})$

$$= p\left(A_2 / A_1\right)\times p\left(A_1\right)+p\left(A_2 \cap \overline{A_1}\right)\times p\left(\overline{A_1}\right)$$

$$=\frac{1}{5}\times\frac{1}{2}+\frac{2}{5}\times\left(1-\frac{1}{2}\right)=\frac{3}{10}$$

2) a- $p_n = p\left(A_{n+1} / A_n\right)=\frac{1}{5}$, $p_n = p\left(A_{n+1} / \overline{A_n}\right)=\frac{2}{5}$.

b- $p\left(A_{n+1} \cap A_n\right)= p\left(A_{n+1} / A_n\right)\times p\left(A_n\right)=\frac{1}{5}\times p_n=\frac{1}{5}p_n$

$p\left(A_{n+1} \cap \overline{A_n}\right)= p\left(A_{n+1} / \overline{A_n}\right)\times p\left(\overline{A_n}\right)=\frac{2}{5}\times\left(1-p_n\right)$

c- $p_{n+1} = p\left(A_{n+1}\right)= p\left(A_{n+1} \cap A_n\right)+p\left(A_{n+1} \cap \overline{A_n}\right)$

$= p\left(A_{n+1} / A_n\right)\times p\left(A_n\right)+p\left(A_{n+1} \cap \overline{A_n}\right)\times p\left(\overline{A_n}\right)$

$=\frac{1}{5}p_n+\frac{2}{5}\left(1-p_n\right)=-\frac{1}{5}p_n+\frac{2}{5}$.

3) a- $u_{n+1} = p_{n+1}-\frac{1}{3}=-\frac{1}{5}p_n+\frac{2}{5}-\frac{1}{3}=-\frac{1}{5}p_n+\frac{1}{15}=-\frac{1}{5}\left(p_n-\frac{1}{3}\right)$

Therefore, $u_{n+1}=-\frac{1}{5}u_n$ and consequently (u_n) is a geometric sequence of common ratio $-\frac{1}{5}$ and of first term:

$u_1 = p_1-\frac{1}{3}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$.

b- $u_n = u_1\times r^{n-1}=\frac{1}{6}\times\left(-\frac{1}{5}\right)^{n-1}$ and

$p_n = u_n+\frac{1}{3}=\frac{1}{6}\times\left(-\frac{1}{5}\right)^{n-1}+\frac{1}{3}$. $\lim\limits_{n\to+\infty} p_n=\frac{1}{3}$.

N°23.

Part A.

1) $A_1 = \int_{e^{-1}}^{1}\frac{1+\ln x}{x}dx=\frac{1}{2}\left[\left(1+\ln x\right)^2\right]_{e^{-1}}^{1}=\frac{1}{2}$ square units.

2) $A_2 =$ (area of $OABF$) $- A_1 = OA\times OF - A_1 = 2-\frac{1}{2}=\frac{3}{2}$ u^2.

Part B .

1) $\dfrac{p(D_1)}{A_1}=\dfrac{p(D_2)}{A_2} \Rightarrow \dfrac{p(D_1)}{0.5}=\dfrac{p(D_2)}{1.5}$ then $p(D_2)=3p(D_1)$ and since $p(D_2)+p(D_1)=0.8$ we get $3p(D_1)+p(D_1)=0.8$ then $p(D_1)=\dfrac{1}{5}$ and $p(D_2)=\dfrac{3}{5}$.

2) $p(R)=p(D_1\cap R)+p(D_2\cap R)$

$$=p(R/D_1)\times p(D_1)+p(R/D_2)\times p(D_2)$$

$$=\frac{1}{5}\times\frac{C_3^2}{C_7^2}+\frac{3}{5}\times\frac{C_4^2}{C_7^2}=\frac{7}{35}=\frac{1}{5}$$

$p(w)=p(D_1\cap w)+p(D_2\cap w)$

$$=p(w/D_1)\times p(D_1)+p(w/D_2)\times p(D_2)$$

$$=\frac{1}{5}\times\frac{C_4^2}{C_7^2}+\frac{3}{5}\times\frac{C_3^2}{C_7^2}=\frac{5}{35}=\frac{1}{7}$$

The two events R and w are disjoint and $M=R\cup w$

$$p(M)=p(R)+p(w)=\frac{12}{35}$$

$$p(D)=0.8-p(M)=0.8-\frac{12}{35}=\frac{16}{35}.$$

3) $p(D_1/w)=\dfrac{p(D_1\cap w)}{p(w)}=\dfrac{\frac{1}{5}\times\frac{2}{7}}{\frac{1}{7}}=\dfrac{2}{5}.$

4) a- If the arrow hits the region outside the rectangle (event H)

Then $p(H)=0.2=\dfrac{1}{5}$ and in this case $X=-3000$ LL.

If Reine draws two red balls, then $X=4000$ LL.
If Reine draws two white balls, then $X=-2000$ LL.
If Reine draws two balls of different colors, then $X=1000$ LL.

Hence, $X(\Omega)=\{-3000;-2000;1000;4000\}$.

$p(X=-3000)=p(H)=\dfrac{1}{5}$, $p(X=-2000)=p(w)=\dfrac{1}{7}$

$$p(X=1000)=p(D)=\frac{16}{35};\quad p(X=4000)=p(R)=\frac{1}{5}.$$

b- $E(X)=371.4$, Since $E(X)\neq 0$ then the game is not fair.

N°24.

1) a- $p(A)=\frac{1}{6}$, $p(B)=\frac{2}{6}=\frac{1}{3}$, and $p(C)=\frac{3}{6}=\frac{1}{2}$.

b- $p(R/A)=\frac{C_3^2}{C_6^2}=\frac{3}{15}=\frac{1}{5}$, $p(R/B)=\frac{C_2^2}{C_6^2}=\frac{1}{15}$ and

$p(R/C)=\frac{3}{6}\times\frac{2}{6}=\frac{6}{36}=\frac{1}{6}$.

c- $p(R)=p(A\cap R)+p(B\cap R)+p(C\cap R)=$

a- $p(R/A)\times p(A)+p(R/B)\times p(B)+p(R/C)\times p(C)=$

$\frac{1}{6}\times\frac{1}{5}+\frac{1}{3}\times\frac{1}{15}+\frac{1}{2}\times\frac{1}{6}=\frac{1}{30}+\frac{1}{45}+\frac{1}{12}=\frac{5}{36}$

d- $p(\text{drawing two white balls})=\frac{1}{6}\times\frac{C_2^2}{C_6^2}+\frac{1}{3}\times\frac{C_4^2}{C_6^2}+\frac{1}{2}\times\frac{2}{6}\times\frac{4}{6}$

$=\frac{1}{6}\times\frac{1}{15}+\frac{1}{3}\times\frac{6}{15}+\frac{1}{9}=\frac{23}{90}$

$p(M)=p(2\text{ red balls or 2 white balls})=\frac{5}{36}+\frac{23}{90}=\frac{71}{180}$

2) $p(B)=\frac{1}{6}\times\frac{C_1^1C_5^1}{C_6^2}+\frac{1}{2}\times\frac{1}{6}\times\frac{6}{6}=\frac{5}{36}$

3) a- $p(A\cap L)=p(A)\times p(L/A)=\frac{1}{6}\times\frac{C_3^1C_2^1}{C_6^2}=\frac{1}{15}$,

$p(B\cap L)=p(B)\times p(L/B)=\frac{1}{3}\times\frac{C_2^1C_4^1}{C_6^2}=\frac{8}{45}$

$p(C\cap L)=p(C)\times p(L/C)=\frac{1}{6}\left(\frac{3}{6}\times\frac{4}{6}+\frac{2}{6}\times\frac{2}{6}\right)=\frac{2}{27}$

b- $p(L)=p(A\cap L)+p(B\cap L)+p(C\cap L)=\frac{43}{135}$

4) a- $X(\Omega)=\{-4;-3;3;4;10\}$

b-

x_i	-4	-3	3	4	10
Event	2w	1 w and 1 B	1 R and 1 w	1 B and 1R	2R
p_i	$\frac{46}{180}$	$\frac{14}{180}$	$\frac{84}{180}$	$\frac{11}{180}$	$\frac{25}{180}$

N° 25.

1) $$p(R)=\frac{C_4^2}{C_{n+7}^2}=\frac{6}{\frac{(n+7)!}{2!(n+5)!}}=\frac{12}{(n+6)(n+7)}$$

$$p(D)=\frac{C_3^1C_4^1}{C_{n+7}^2}=\frac{12}{\frac{(n+7)!}{2!(n+5)!}}=\frac{24}{(n+6)(n+7)}$$

2) a- $X(\Omega)=\{28000;0;20000;10000;12000\}$.

b- $p(X=28000)=\frac{3}{n+7}$, $p(X=0)=\frac{4}{n+7}$

$p(X=20000)=\frac{n}{n+7}\times\frac{3}{n+7}$, $p(X=10000)=\frac{n}{n+7}\times\frac{4}{n+7}$

$p(X=12000)=\frac{n}{n+7}\times\frac{n}{n+7}$

c- $$E(X)=28000\times\frac{3}{n+7}+0\times\frac{4}{n+7}+.....+12000\times\frac{n^2}{(n+7)^2}$$

$$E(X)=4000\left[3+\frac{4n}{(n+7)^2}\right]$$

3) a- $$f'(x)=\frac{(x+7)(-x+7)}{(x+7)^4}$$

x	0		7		10
$f'(x)$		+	0	−	
$f(x)$	0	$\nearrow$	$\frac{1}{28}$	$\searrow$	$\frac{10}{289}$

b- $E(X)$ is maximal for $n = 7$.

N°26.

Part A .

The 3 throws are independent , then : $p(E) = \frac{7}{12} \times \frac{7}{12} \times \frac{7}{12} = \frac{343}{1728}$.

$p(F) = \frac{1}{12} \times \frac{1}{3} \times \frac{7}{12} = \frac{7}{432}$.

The parts are hit in the order 1, 2, 3 or in the order 1, 3, 2 or in the order 2, 1, 3 or etc (there are six possible orders).

Then, $p(G) = 6 \times \frac{7}{432} = \frac{7}{72}$.

Part B .

1) If Let $p = p(B)$ then $2p = p(A)$, so $p + 2p = 1$ then $p = \frac{1}{3}$, so the probability of choosing A is $\frac{2}{3}$.
2) Part 3 can be hit either by A or by B.
   Naming T the event "hitting part 3", then
   $p(T) = p(T \cap A) + p(T \cap B)$, and since the probability that B hits part 3 is $\frac{1}{3}$, then we get :
   $$p(T) = p(A) \times p(T/A) + p(B) \times p(T/B) = \frac{2}{3} \times \frac{7}{12} + \frac{1}{3} \times \frac{1}{3} = \frac{1}{2}.$$
3) We know that $p(A/T) = \frac{p(A \cap T)}{p(T)} = \frac{p(A) \times p(T/A)}{p(T)}$
   Therefore, $p(A/T) = \frac{\frac{2}{3} \times \frac{7}{12}}{\frac{1}{2}} = \frac{7}{9}$.

Indications

N° 1.

2) a- $p(D/A)=\dfrac{C_2^1\times C_2^1}{C_4^2}$, $p(D/B)=\dfrac{C_3^1\times C_1^1}{C_4^2}$, $p(D/C)=0$

3) a- $p(wB)+p(Bw)=\dfrac{4}{6}\times\dfrac{2}{6}+\dfrac{2}{6}\times\dfrac{4}{5}$.

N° 3.

1) a- $p(E_0/B)=\dfrac{C_3^3}{C_5^3}=\dfrac{1}{10}$. $p(E_0/\overline{B})=\dfrac{C_4^3}{C_5^3}=\dfrac{4}{10}=\dfrac{2}{5}$.

$p(E_0)=\dfrac{1}{10}\times\dfrac{1}{2}+\dfrac{2}{5}\times\dfrac{1}{2}=\dfrac{1}{4}$.

2) $p(E_1)=p(E_1\cap B)+p(E_1\cap\overline{B})$

$=p(E_1/B)\times p(B)+p(E_1/\overline{B})\times p(\overline{B})$

$p(E_1/\overline{B})=\dfrac{C_1^1\times C_4^2}{C_5^3}=\dfrac{6}{10}=\dfrac{3}{5}$, $p(E_1)=\dfrac{3}{5}\times\dfrac{1}{2}+\dfrac{3}{5}\times\dfrac{1}{2}=\dfrac{6}{10}=\dfrac{3}{5}$.

N° 4.

1) $p(E)=\dfrac{50}{90}\times\dfrac{30}{110}=\dfrac{5}{33}$, $p(F)=\dfrac{50}{90}\times\dfrac{80}{110}+\dfrac{40}{90}\times\dfrac{30}{110}=\dfrac{52}{99}$.

N° 5.

1) a- $p=\dfrac{C_7^3}{C_{10}^3}=\dfrac{7}{24}$ b- $p=\dfrac{C_3^1C_7^2}{C_{10}^3}=\dfrac{21}{40}$.

2) a- $p=\dfrac{3}{10}\times\dfrac{3}{10}\times\dfrac{3}{10}=\dfrac{27}{1000}$

b- $p=1\text{-}p(\text{no ticket win})=1-\dfrac{7}{10}\times\dfrac{7}{10}\times\dfrac{7}{10}=\dfrac{657}{1000}$.

N° 6.

1) $p(A)=\dfrac{8}{16}$, $p(B)=\dfrac{4}{16}$, $p(C)=\dfrac{4}{16}$

2) a- $p(R/A)=\dfrac{2}{n}$, $p(R/B)=\dfrac{3}{n}$

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8

b- $p(R)=p(R/A)\times p(A)+p(R/B)\times p(B)+p(R/C)\times p(C)$.

$p(R)=\dfrac{19}{8n}$

3) b- $p(\overline{C}/R)=\dfrac{p(\overline{C}\cap R)}{p(R)}$.

$p(R)=p(C\cap R)+p(\overline{C}\cap R)$.

4) $\lambda=f(n)=\dfrac{p(\overline{R})}{p(R)}=\dfrac{8}{19}n-1$.

N°7.

2) $X(\Omega)=\{1;2;3\}$, $p(X=1)=\dfrac{C_5^3}{C_{11}^3}+\dfrac{C_4^3}{C_{11}^3}$,

$p(X=3)=\dfrac{C_5^1\times C_2^1\times C_4^1}{C_{11}^3}$, $p(X=2)=1-\left[p(X=1)+p(X=3)\right]$

N°8.

1) Among the 5 keys , there are 2 keys that do not open the door , therefore $p(D_1)=\dfrac{2}{5}$.

2) D_1 has occurred , then among the 4 keys , there is 1 key that does not open the door , therefore $p(D_2/D_1)=\dfrac{1}{4}$.

$p(D_1\cap D_2)=p(D_2/D_1)\times p(D_1)=\dfrac{1}{4}\times\dfrac{2}{5}=\dfrac{1}{10}$.

3) E is $\overline{D_1}\cap\overline{D_2}\cap D_3$, then:

$p(E)=p(\overline{D_1})\times p(\overline{D_2}/\overline{D_1})\times p(D_3/(\overline{D_1}\cap\overline{D_2}))$.

N°9.

1) Area of D_1 is $a_1=\displaystyle\int_0^1 x^3dx=\left.\frac{x^4}{4}\right|_0^1=\frac{1}{4}$

$a_2=\displaystyle\int_0^1(\sqrt{x}-x^3)dx=\left[\frac{2}{3}x\sqrt{x}-\frac{x^4}{4}\right]_0^1=\frac{5}{12}$.

Indications.

$$a_3 = \int_0^1 (1 - \sqrt{x})dx = \left[x - \frac{2}{3} x\sqrt{x} \right]_0^1 = \frac{1}{3}$$

2) Let p_1, p_2 and p_3 be the respective probabilities of D_1, D_2 and D_3.

$\dfrac{p_1}{\frac{1}{4}} = \dfrac{p_2}{\frac{5}{12}} = \dfrac{p_3}{\frac{1}{3}} = k$, but $p_1 + p_2 + p_3 = 0.6$, then

$\frac{1}{4}k + \frac{5}{12}k + \frac{1}{3}k = 0.6$, $k = 0.6$, $p_1 = \frac{3}{20}$, $p_2 = \frac{1}{4}$ and $p_3 = \frac{1}{5}$.

N° 12.

1) a- $p(A) = \frac{1}{6}$ $p(R/A) = \frac{4}{10}$

b- $p(R) = p(R/A) \times p(A) + p(R/\overline{A}) \times p(\overline{A})$

$$= \frac{4}{10} \times \frac{1}{6} + \frac{1}{10} \times \frac{5}{6} = 0.15$$

2) b- $p(X=6) = 0.15 \times 0.15$, $p(X=-4) = (1-0.15) \times (1-0.15)$

$p(X=1) = 1 - (0.15)^2 - (0.85)^2$.

CHAPTER 9

Circular Functions

Chapter Review

1) The Function under study is: $f(x) = a\sin(bx + c)$, $a \neq 0$ and $b \neq 0$.
   (C) is the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.
   - The period of f is $T = \dfrac{2\pi}{|b|}$.
   - The point $w\left(-\dfrac{c}{b};0\right)$ is a center of symmetry of (C).
   - The straight line (d) of equation $x = -\dfrac{c}{b} + \dfrac{T}{4}$ is an axis of symmetry of (C).

N.B

It is advisable:

* To study the variations of the function f in the interval $\left[-\dfrac{c}{b};-\dfrac{c}{b}+\dfrac{T}{4}\right]$.
* Draw the part of (C) corresponding to this interval , then complete (C) by symmetry with respect to (d) and then with respect to w .

2) If the function under study is : $f(x) = a\cos(bx + c)$; $a \neq 0$ and $b \neq 0$.
It is better to:

* Write this function in the form $f(x) = a\sin\left(bx + c + \frac{\pi}{2}\right)$.
* Follow the same steps of study of the function $f(x) = a\sin(bx + c)$.

3) Translation :
Given (C) , (C') and (γ) ,the representative curves of the functions f , g and h defined over IR by : $f(x) = a\sin(bx)$, $g(x) = a\sin(bx + c)$ and $h(x) = a\sin(bx) + d$ in an orthonormal system $(O;\vec{i},\vec{j})$

* The vector of translation that maps (C) to (C') is the vector $\vec{v} = -\frac{c}{b}\vec{i}$.
* The vector of translation that maps (C) to (γ) is the vector $\vec{v} = d\,\vec{j}$.

4) Vibratory Motion :
If $f(t) = a\sin(bt + c)$ where t represents the time in a vibratory motion then:

* a is the amplitude.
* $|b|$ is the pulse .
* $bt + c$ is the phase angle .
* c is the phase at the origin .
* $N = \frac{1}{T}$ is the frequency .

Solved Problems

N° 1.

Given the two functions f and g defined over IR by :
$f(x) = 3\sin 2x$ and $g(x) = 3\sin(2x-4)$.
Designate by (C) and (C') the representative curves of f and g respectively in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine the amplitude of f and show that $T = \pi$ is the period of the function f.
   b- Prove that f is an odd function and that the straight line (d) of equation $x = \dfrac{\pi}{4}$ is an axis of symmetry of (C).
   c- Determine a vector of translation that maps (C) to (C').
   d- Draw (C) in the interval $\left[-\dfrac{\pi}{2};\dfrac{\pi}{2}\right]$ and deduce (C').
2) Let h be the function defined over IR by $h(x) = 3\cos 2x$.
   Designate by (γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
   Determine a vector that translates (C) to (γ) and deduce the graph of (γ).

N° 2.

Consider the function f defined over IR by $f(t) = -3\sin\left(-2t+\dfrac{\pi}{3}\right)$.

1) Determine , the period, the pulse, the phase angle, the phase at the origin and the frequency of f knowing that t is the time in a vibratory motion.
2) Determine a center and an axis of symmetry of (C).

N° 3.

Given the function f defined over IR by $f(x) = 2\sin(3x-6)$.
Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that the point $w(2;0)$ is a center of symmetry of (C).

2) Designate by T the period of f.
   a- Study the variations of f and trace (C) in the interval $\left[2-\frac{T}{2};2+\frac{T}{2}\right]$.
   b- Calculate the area of the domain (D) limited by (C), $x'x$ and the two straight lines of equations $x=2$ and $x=2+\frac{T}{2}$.
   c- Calculate the volume of the solid obtained by rotating (D) around the axis $x'x$.

N° 4.

f is the function defined over IR by $f(x)=-3\cos(2x)$. Designate by (C) its representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine the period T of f then a center and an axis of symmetry of (C).
2) Study the variations of f and trace (C) in an interval of amplitude T.
3) Deduce the representative curve of the function h defined over IR by $h(x)=-3\cos(2x)+1$.

N° 5.

Given the two functions f and g defined over IR by :

$f(x)=2\sin(\pi x)$ and $g(x)=2\sin(\pi x-\pi^2)$.

Designate by (C) and by (C') the representative curves of f and of g respectively in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine the period of f and that of g.
2) Show that the straight line (d) of equation $x=\frac{1}{2}$ is an axis of symmetry of (C) and the straight line (d') of equation $x=\frac{2\pi+1}{2}$ is an axis of symmetry of (C').
3) Study the variations of f and trace (C).
4) Determine a vector that translates (C) to (C'). Deduce the graph of (C').

N° 6.

1) Determine two real numbers A and δ satisfying the relation: $A.\sin(x+\delta)=\sqrt{3}\cos x+\sin x$.

2) Consider the function f defined over IR by $f(x)=2\sin\left(x+\frac{\pi}{3}\right)$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

   a- Determine the period T of f and calculate $f'(x)$.

   b- Solve in the interval $[0;T]$ the equation $f'(x)=0$.

   c- Draw the table of variations of f over $[0;T]$.

   d- Show that the straight line (d) of equation $x=\frac{\pi}{6}$ is an axis of symmetry of (C).

   e- Trace (C) in the interval $[0;T]$.

   f- Calculate the area of the domain limited by (C), the axis of abscissas and the two straight lines of equations: $x=0$ and $x=\frac{\pi}{3}$.

N°7.

1) Consider the differential equation $(E): y''+4y-4=0$.

   a- Determine the general solution of (E).

   b- Determine the particular solution y of (E) verifying $y(0)=2$ and $y'(0)=2$.

2) Consider the functions f and g defined over IR by $f(x)=\cos 2x+\sin 2x+1$ and $g(x)=\sqrt{2}\cos\left(2x-\frac{\pi}{4}\right)$ and designate by (C) and (γ) their representative curves respectively in an orthonormal system $(O;\vec{i},\vec{j})$.

   a- Verify that $f(x)=\sqrt{2}.\cos\left(2x-\frac{\pi}{4}\right)+1$.

   b- Show that the period of g is $T=\pi$.

   c- Determine a center of symmetry and an axis of symmetry of

(γ) in the interval $\left[-\frac{5\pi}{8};\frac{3\pi}{8}\right]$.

d) Study the variations of g and draw (γ) in the interval $\left[-\frac{5\pi}{8};\frac{3\pi}{8}\right]$.

e- Deduce the graph of (C).

f- Calculate the area of the domain limited by (γ), the axis of abscissas, and the two straight lines of equations $x = 0$ and $x = \frac{\pi}{8}$.

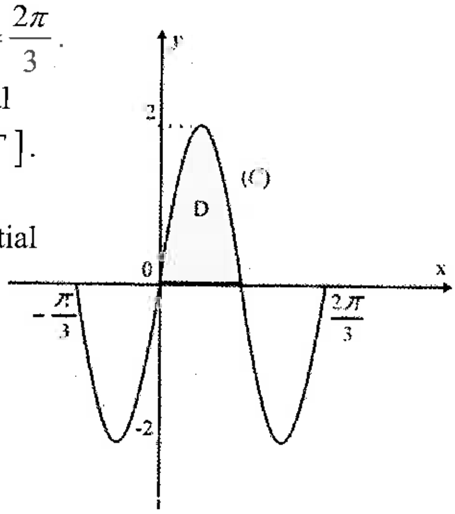
N°8.

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.
Consider the function f defined over IR by $f(x) = a\sin(wx + b)$, a , b and w are real numbers with $a \neq 0$.

Suppose that the period T of f is $T = \frac{2\pi}{3}$.

The adjacent curve (C) is the graphical representation of f in the interval $[0;T]$.

1) Determine $f(x)$.
2) Verify that f satisfies the differential equation $y'' + 9y = 0$.
3) Calculate the area of the shaded part in the figure below.



N°9.

Given the differential equation $(E): y'' + 16y = 64$.

1) a- Solve (E).

b- Determine the particular solution f of (E) whose representative curve (C) is tangent to the straight line of equation $y = 4x + 4$ at the point $A(0;4)$.

2) Consider the function g defined over $\left[0\,;\frac{\pi}{2}\right]$ by $g(x)=-\sin 4x$ and designate by (γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
   a- Study the variations of g in the interval $\left[0;\frac{\pi}{4}\right]$ and draw its table of variations.
   b- Show that the point $I\left(\frac{\pi}{4};0\right)$ is a center of symmetry of (γ).
   c- Draw (γ) in the interval $\left[0\,;\frac{\pi}{2}\right]$.
   d- Deduce the representative curve (H) of the function h defined over IR by $h(x)=-\sin\left(4x+\frac{4\pi}{3}\right)$.
   e- Show that g admits an inverse function g^{-1} over $\left[\frac{\pi}{8};\frac{3\pi}{8}\right]$.
   Determine the domain of definition of g^{-1}.
   f- Calculate $g\left(\frac{\pi}{6}\right)$ and deduce $\left(g^{-1}\right)'\left(-\frac{\sqrt{3}}{2}\right)$.
3) Draw the graph (Γ) of the function φ defined over $\left[0;\frac{\pi}{2}\right]$ by $\varphi(x)=\sin 4x$ and determine a vector that maps (Γ) to (C).
4) a- Using (Γ), draw the table of variations of f in the interval $\left[0;\frac{\pi}{2}\right]$.
   b- Let k be the function defined over IR by $k(x)=e^{4}e^{\sin 4x}$.
   Study the variations of k in the interval $\left[0;\frac{\pi}{2}\right]$ and draw its table of variations.

Solutions

N° 1.

1) a- The amplitude of f is 3 .

$f(x+\pi)=3\sin(2(x+\pi))=3\sin(2x+2\pi)=3\sin 2x=f(x)$,
therefore, $T=\pi$ is the period of the function f .

b- $f(-x)=3\sin 2(-x)=-3\sin 2x=-f(x)$, then f is odd .

$f\left(\frac{\pi}{2}-x\right)=3\sin(\pi-2x)=3\sin(2x)=f(x)$, then the straight line of equation $x=\frac{\pi}{4}$ is an axis of symmetry of (C) .

c- The translation of vector $\vec{v}=-\frac{c}{b}\vec{i}$ transforms the graph of the function f defined by $f(x)=a\sin(bx)$ to the graph of the function g defined by $g(x)=a\sin(bx+c)$.

Hence, the vector that transforms (C) onto (C') is the vector of translation $\vec{v}=-\frac{c}{b}\vec{i}=2\vec{i}$.

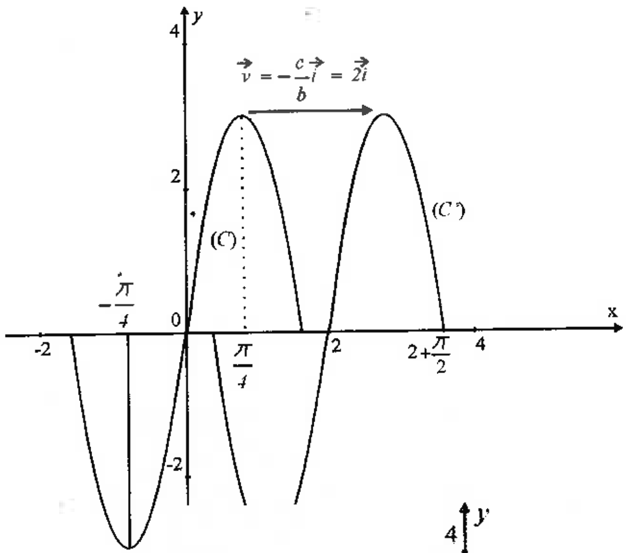
d- We first study the variations of f and draw its representative curve in the interval $\left[0;\frac{\pi}{4}\right]$, then we deduce the rest by symmetry with respect to (d) then with respect to O.

Table of variations of f :

$f'(x)=6\cos 2x$, $0\le x\le\frac{\pi}{4}$, then $0\le 2x\le\frac{\pi}{2}$,
then $f'(x)\ge 0$.

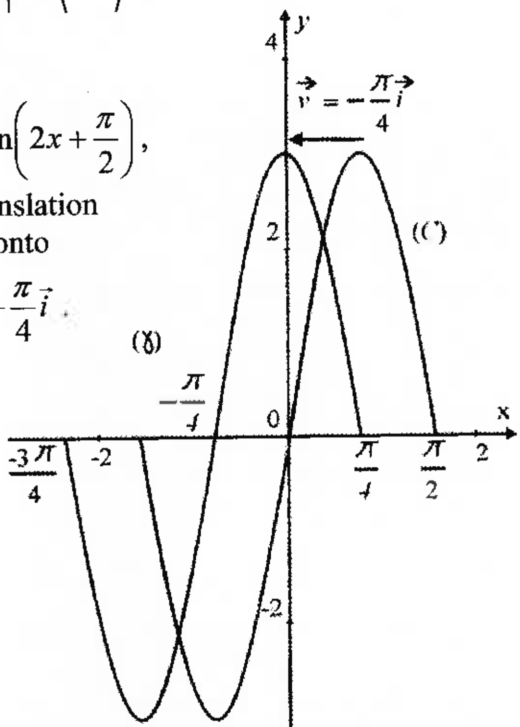
Therefore the table of variations of f :

x	0		$\frac{\pi}{4}$
$f'(x)$		$+$	
$f(x)$	0	$\nearrow$	3



2) $h(x) = 3\cos 2x = 3\sin\left(2x + \frac{\pi}{2}\right)$,

then the vector of translation that transforms (C) onto (γ) is the vector $\vec{v}' = -\frac{\pi}{4}\vec{i}$.



$\boxed{N°2.}$

1) The function f can be written in the form $f(t) = 3\sin\left(2t - \frac{\pi}{3}\right)$.

If the variable t is the time in a vibratory motion, then,:

The period of f is $T = \frac{2\pi}{|b|} = \pi$.

$|b| = 2$ is the pulse.

$bt + c = 2t - \frac{\pi}{3}$ is the phase angle.

$c = -\frac{\pi}{3}$ is the phase at the origin and $N = \frac{1}{T} = \frac{1}{\pi}$ is the frequency.

2) A center of symmetry of (C) is the point $w\left(-\frac{c}{b};0\right)$, which gives $w\left(\frac{\pi}{6};0\right)$.

An axis of symmetry of (C) is the straight line of equation $x = -\frac{c}{b} + \frac{T}{4} = \frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$.

N° 3.

1) $f(4-x) = 2\sin(3(4-x)-6) = 2\sin(-3x+6)$,
$f(4-x) = 2\sin(-(3x-6)) = -2\sin(3x-6) = -f(x)$
then $f(4-x) + f(x) = 0$ and consequently the point $w(2\,;0)$ is a center of symmetry of (C).

2) a- The period of f is $T = \frac{2\pi}{3}$, then :

$$\left[2-\frac{T}{2}:2+\frac{T}{2}\right] = \left[2-\frac{\pi}{3};2+\frac{\pi}{3}\right].$$

The straight line of equation $x = 2 + \frac{\pi}{6}$ is an axis of symmetry of (C).

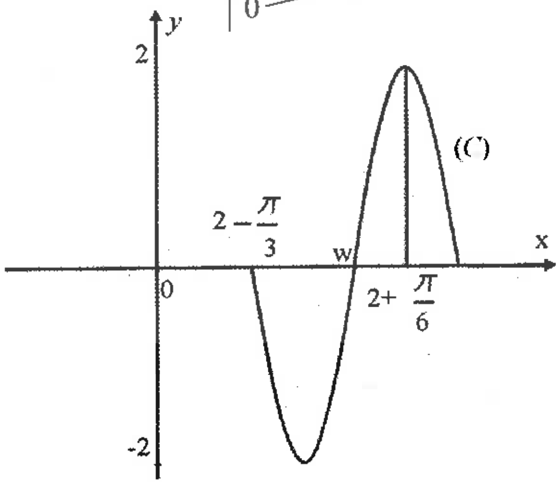
So, we study the variations of f and we draw (C) in the interval $\left[2\,;2+\frac{\pi}{3}\right]$, and we deduce the rest of the curve by symmetry with respect to the axis and then to the point $w(2\,;0)$.

$f'(x) = 6\cos(3x-6)$ .But, $2 \le x \le 2 + \frac{\pi}{6}$, then

$0 \le 3x - 6 \le \frac{\pi}{2}$, hence, $f'(x) \ge 0$.

The table of variations of f is as follows:

x	2	$2+\frac{\pi}{6}$
$f'(x)$	$+$	
$f(x)$	$0 \nearrow$	2



b- The area is given by ::

$$A = \int_{2}^{2+\frac{\pi}{3}} 2\sin(3x-6)\,dx = -\frac{2}{3}\cos(3x-6)\Big|_{2}^{2+\frac{\pi}{3}} = \frac{4}{3} \text{ square units}.$$

c- The volume is given by :

$$V = \pi \int_{2}^{2+\frac{\pi}{3}} 4\sin^2(3x-6)\,dx = 2\pi \int_{2}^{2+\frac{\pi}{3}} (1-\cos(6x-12))dx$$

$$= 2\pi\left[x-\frac{1}{6}\sin(6x-12)\right]_{2}^{2+\frac{\pi}{3}} = \frac{2\pi^2}{3} \text{ cubic units.}$$

N°4.

1) The function f can be written in the form $f(x) = -3\sin\left(2x+\frac{\pi}{2}\right)$.

The period of f is $T = \frac{2\pi}{|b|} = \pi$.

A center of symmetry of (C) is the point $w\left(-\frac{\pi}{4};0\right)$.

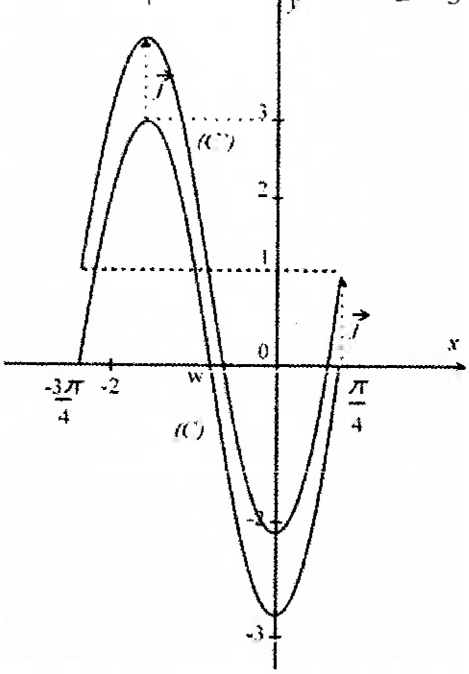
An axis of symmetry of (C) is the straight line of equation

$x=-\frac{c}{b}+\frac{T}{4}=-\frac{\pi}{4}+\frac{\pi}{4}=0$.

2) We study the variations of f and we draw (C) in the interval $\left[-\frac{\pi}{4};0\right]$, we deduce the rest of the curve by symmetry with respect to the axis and then with respect to w.

$f'(x)=-6\cos(2x+\frac{\pi}{2})$, but $-\frac{\pi}{4}\le x\le 0$, then $0\le 2x+\frac{\pi}{2}\le\frac{\pi}{2}$, hence $f'(x)\le 0$, therefore the table of f is as follows:

x	$-\frac{\pi}{4}$		0
$f'(x)$		$-$	
$f(x)$	0	↘	-3



3) We deduce the graph of the function defined by $h(x) = a\cos(bx) + c$ from the graph of the function defined by $f(x) = a\cos(bx)$ by the translation of vector $\vec{v} = c\,\vec{j}$.

In this case it is the vector $\vec{v} = \vec{j}$.

N°5.

1) The period of the function f is $T = \dfrac{2\pi}{|b|} = 2$.

The period of the function g is $T' = \dfrac{2\pi}{|b|} = 2$.

2) $f(1-x) = 2\sin(\pi(1-x)) = 2\sin(\pi - \pi x) = 2\sin \pi x = f(x)$.

Then the straight line (d) is an axis of symmetry of (C) .

$g(2\pi + 1 - x) = 2\sin(\pi(2\pi + 1 - x) - \pi^2) = 2\sin(\pi^2 + \pi - \pi x)$

$g(2\pi + 1 - x) = 2\sin(\pi - (\pi x - \pi^2)) = 2\sin(\pi x - \pi^2) = g(x)$.

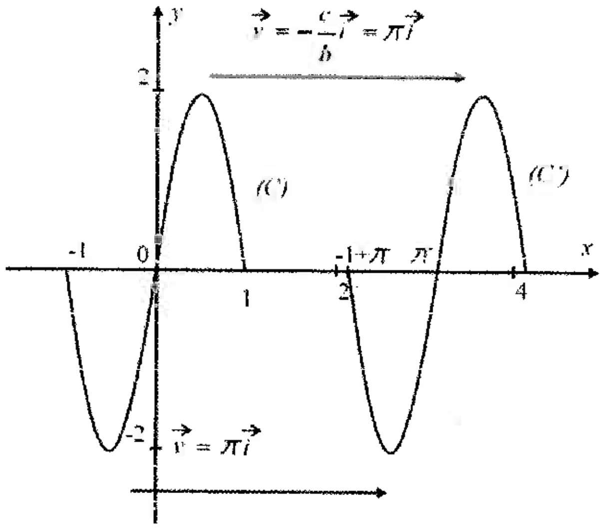
Then the straight line (d') is an axis of symmetry of (C') .

3) O is a center of symmetry of (C), so we study the variations of f and we draw (C) in the interval $\left[0;\dfrac{1}{2}\right]$, and then we deduce the rest of the graph by symmetry with respect to (d) then with O.

$f'(x) = 2\pi\cos \pi x$, but $0 \le x \le \dfrac{1}{2}$, then $0 \le \pi x \le \dfrac{\pi}{2}$,

consequently $f'(x) \ge 0$,therefore the table of variations of f is as follows:

x	0		$\frac{1}{2}$
$f'(x)$		$+$	
$f(x)$	0	$\nearrow$	2



4) The vector of translation that maps (C) to (C') is the vector $\vec{v} = -\frac{c}{b}\vec{i} = \pi\vec{i}$, refer to the figure.

N°6.

1) $A.\sin(x+\delta) = \sqrt{3}\cos x + \sin x = 2\left(\frac{1}{2}\sin x + \frac{\sqrt{3}}{2}\cos x\right)$

$$= 2\left(\sin x.\cos\frac{\pi}{3} + \cos x.\sin\frac{\pi}{3}\right) = 2\sin\left(x+\frac{\pi}{3}\right)$$

therefore $A = 2$ and $\delta = \frac{\pi}{3}$.

2) a- The period of f is $T = \frac{2\pi}{|b|} = \frac{2\pi}{1} = 2\pi$.

$f'(x) = 2\cos\left(x+\frac{\pi}{3}\right)$.

b- $f'(x) = 0$ gives $\cos\left(x+\frac{\pi}{3}\right) = 0$. Therefore

$\begin{cases} x+\frac{\pi}{3} = \frac{\pi}{2} + 2k\pi \\ x+\frac{\pi}{3} = -\frac{\pi}{2} + 2k\pi \end{cases}$ then : $x = \frac{\pi}{6} + 2k\pi$, $x = \frac{-5\pi}{6} + 2k'\pi$

$0 \le x \le 2\pi$, then $0 \le \frac{\pi}{6} + 2k\pi \le 2\pi$, which gives

$-\frac{1}{12} \le k \le \frac{11}{12}$, then $k = 0$, consequently $x = \frac{\pi}{6}$.

Similarly $0 \le x \le 2\pi$, gives $\frac{5}{12} \le k \le \frac{17}{12}$, then $k = 1$ and

consequently $x = \frac{7\pi}{6}$.

$f'(x) < 0$ for $\frac{\pi}{2} < x + \frac{\pi}{3} < \frac{3\pi}{2}$ then for $\frac{\pi}{6} < x < \frac{7\pi}{6}$.

c- Therefore the table of variations of f is as follows:

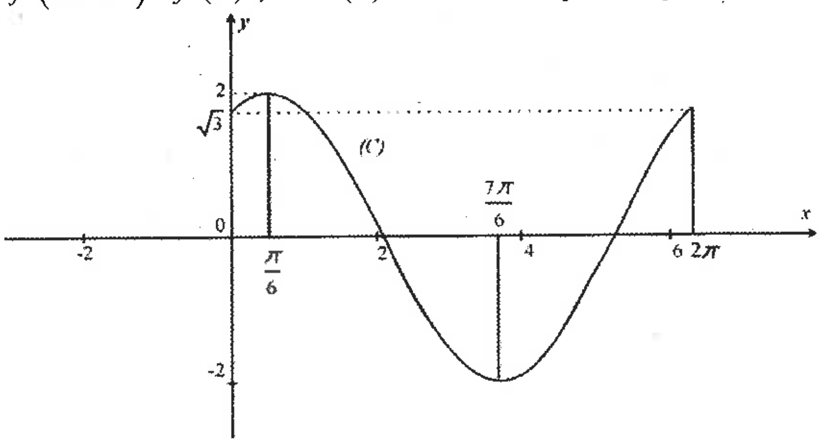
x	0		$\frac{\pi}{6}$		$\frac{7\pi}{6}$		2π
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$\sqrt{3}$	$\nearrow$	2	$\searrow$	-2	$\nearrow$	$\sqrt{3}$

d- $f(2a - x) = f\left(\frac{\pi}{3} - x\right) = 2\sin\left(\frac{\pi}{3} - x + \frac{\pi}{3}\right)$

$$= 2\sin\left[\pi - \left(\frac{\pi}{3} + x\right)\right] = 2\sin\left(x + \frac{\pi}{3}\right) = f(x).$$

$f(2a - x) = f(x)$, then (d) is an axis of symmetry of (C).

e-



f- $f(x)$ is positive over the interval $\left[0;\frac{\pi}{3}\right]$, then:

$$A=\int_0^{\pi/3} f(x)dx=\int_0^{\pi/3} 2\sin\left(x+\frac{\pi}{3}\right)dx=-2\cos\left(x+\frac{\pi}{3}\right)\Big]_0^{\pi/3}$$

$$=-2\cos\left(\frac{2\pi}{3}\right)+2\cos\left(\frac{\pi}{3}\right)=2 \text{ square units.}$$

N°7.

1) a- The differential equation (E) is of the form $y''+w^2y=k$ with $w^2=4$ and $k=4$, its general solution is:

$y=C_1\cos wx+C_2\sin wx+\frac{k}{w^2}$, then:

$y=C_1\cos 2x+C_2\sin 2x+1$.

b- $y(0)=2$ gives $C_1\cos 0+C_2\sin 0+1=2$ then $C_1=1$.

$y'=-2C_1\sin 2x+2C_2\cos 2x$.

$y'(0)=2$ gives $-2C_1\sin 0+2C_2\cos 0=2$ then $C_2=1$.

Then $y=\cos 2x+\sin 2x+1$.

2) a- $\sqrt{2}\cos\left(2x-\frac{\pi}{4}\right)+1=\sqrt{2}\left(\cos 2x.\cos\frac{\pi}{4}+\sin 2x.\sin\frac{\pi}{4}\right)+1$

$$=\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos 2x+\frac{\sqrt{2}}{2}\sin 2x\right)+1$$

$$=\cos 2x+\sin 2x+1=f(x).$$

b- $g(x+\pi)=\sqrt{2}\cos\left(2(x+\pi)-\frac{\pi}{4}\right)=\sqrt{2}\cos\left(2x+2\pi-\frac{\pi}{4}\right)$

$g(x+\pi)=\sqrt{2}\cos\left(2x-\frac{\pi}{4}\right)=g(x)$, then the period of g is $T=\pi$.

c- $g(x)=\sqrt{2}\sin\left(2x-\frac{\pi}{4}+\frac{\pi}{2}\right)=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)$.

A center of symmetry of (γ) is the point $w\left(-\frac{\pi}{8};0\right)$.

An axis of symmetry of (γ) is the straight line of equation

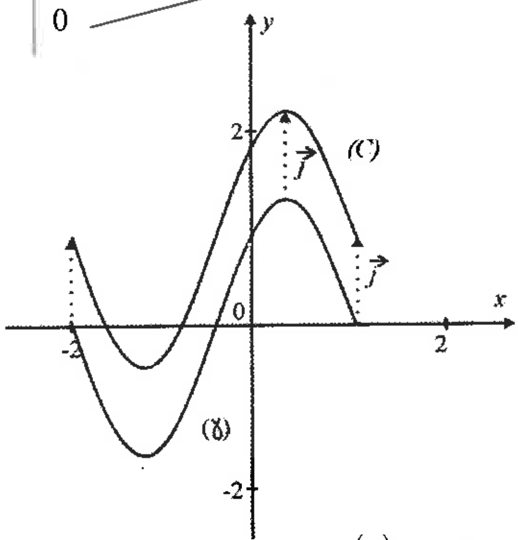
$x=-\frac{c}{b}+\frac{T}{4}=-\frac{\pi}{8}+\frac{\pi}{4}=\frac{\pi}{8}$.

d- We study the variations of g and we draw (γ) in the interval $\left[-\frac{\pi}{8};\frac{\pi}{8}\right]$, then we deduce the rest of the graph by symmetry with respect to the axis and to w.

$g'(x)=2\sqrt{2}\cos\left(2x+\frac{\pi}{4}\right)$, but $-\frac{\pi}{8}\leq x\leq\frac{\pi}{8}$, then

$0\leq 2x+\frac{\pi}{4}\leq\frac{\pi}{2}$ consequently $g'(x)\geq 0$:

x	$-\frac{\pi}{8}$		$\frac{\pi}{8}$
$g'(x)$		+	
$g(x)$	0	$\nearrow$	$\sqrt{2}$



e- The vector of translation that maps (γ) to (C) is the vector $\vec{v}=\vec{j}$.

f- The required area equals :

$$A=\sqrt{2}\int_0^{\frac{\pi}{8}}\cos\left(2x-\frac{\pi}{4}\right)dx=\left[\frac{\sqrt{2}}{2}\sin\left(2x-\frac{\pi}{4}\right)\right]_0^{\frac{\pi}{8}}$$

$A = \frac{1}{2}$ square units .

N°8.

1) Consider the part of (C), in the interval $\left[-\frac{\pi}{3};\frac{\pi}{3}\right]$.

The amplitude of f is $a = 2$ and since (C) passes through the origin O , then $f(0) = 0$, which gives $b = 0$.

The period of f is $T = \frac{2\pi}{|b|}$ and since f is strictly increasing in the interval $\left[0;\frac{\pi}{6}\right]$ then $b > 0$, consequently $T = \frac{2\pi}{|b|} = \frac{2\pi}{b} = \frac{2\pi}{3}$.

So $b = 3$, therefore $f(x) = 2\sin(3x)$.

2) $f(x) = 2\sin 3x$, then $f'(x) = 6\cos 3x$ and $f''(x) = -18\sin 3x$.
$y''+9y = -18\sin 3x + 18\sin 3x = 0$,then $y''+9y = 0$.

3) $V = \pi\int_0^{\frac{\pi}{3}} 4\sin^2(3x)\,dx = 2\pi\int_0^{\frac{\pi}{3}}(1-\cos 6x)\,dx$.

$V = 2\pi\left[x - \frac{1}{6}\sin 6x\right]_0^{\frac{\pi}{3}} = \frac{2\pi^2}{3}$ cubic units.

N°9.

1) a- This equation is of the form $y''+w^2y = k$, with $w^2 = 16$ and $k = 64$, its general solution is of the form

$y = c_1\cos wx + c_2\sin wx + \frac{k}{w^2}$, we also have

that $y = c_1\cos 4x + c_2\sin 4x + 4$.

b- $f(0) = 1$ gives $A = 1$.

The slope of the tangent at A to (C) is equal to 4 , then $f'(0) = 4$, and since $f'(x) = -4c_1\sin 4x + 4c_2\cos 4x$, we get $c_2 = 1$, Consequently $f(x) = \sin(4x) + 4$.

2) a- $g'(x) = -4\cos 4x$, but $0 \le x \le \frac{\pi}{4}$, then $0 \le 4x \le \pi$,

consequently $f'(x) \le 0$,for $0 \le 4x \le \frac{\pi}{2}$ then for $0 \le x \le \frac{\pi}{8}$,

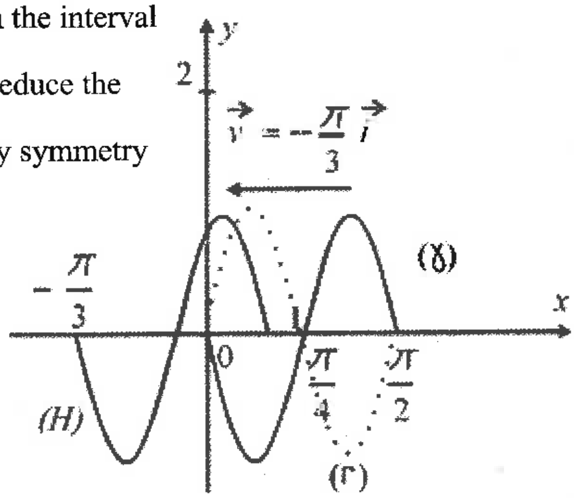
and $f'(x) \ge 0$ for $\frac{\pi}{8} \le x \le \frac{\pi}{4}$, therefore the table of variations of f is as follows:

x	0		$\frac{\pi}{8}$		$\frac{\pi}{4}$
$g'(x)$		$-$	0	$+$	
$g(x)$	0	$\searrow$	-1	$\nearrow$	0

b- $g\left(\frac{\pi}{2} - x\right) = -\sin 4\left(\frac{\pi}{2} - x\right) = -\sin(2\pi - 4x) = \sin 4x$.

$g\left(\frac{\pi}{2} - x\right) = -g(x)$, consequently $g\left(\frac{\pi}{2} - x\right) + g(x) = 0$, then the point $I\left(\frac{\pi}{4};0\right)$ is a center of symmetry of (γ).

c- I is a center of symmetry , we study the variations of g and we draw (γ)in the interval $\left[0;\frac{\pi}{4}\right]$, then we deduce the rest of the graph by symmetry with respect to I .



d- The vector of translation that maps (γ) to (H) is $\vec{v}=-\frac{\pi}{3}\vec{i}$.

e- The function g is continuous and strictly increasing over $\left[\frac{\pi}{8};\frac{3\pi}{8}\right]$ hence g admits an inverse function g^{-1} whose domain is: $g\left(\left[\frac{\pi}{8};\frac{3\pi}{8}\right]\right)=\left[g\left(\frac{\pi}{8}\right);g\left(\frac{3\pi}{8}\right)\right]=[-1;+1]$.

f- $g\left(\frac{\pi}{6}\right)=-\sin\left(\frac{2\pi}{3}\right)=-\frac{\sqrt{3}}{2}$

then $\left(g^{-1}\right)'\left(-\frac{\sqrt{3}}{2}\right)$ $\left(g^{-1}\right)'\left(-\frac{\sqrt{3}}{2}\right)=\frac{1}{g'\left(\frac{\pi}{6}\right)}=\frac{1}{2}$.

3) The representative curve of the function φ defined over $\left[0;\frac{\pi}{2}\right]$ by $\varphi(x)=\sin 4x$ is the symmetric of (γ) with respect to $x'x$.

The vector of translation that maps (Γ) to (C) is $\vec{v}=4\vec{j}$.

4) a- Using the graph of (Γ) we get:

x	0		$\frac{\pi}{8}$		$\frac{3\pi}{8}$		$\frac{\pi}{2}$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	4	$\nearrow$	5	$\searrow$	3	$\nearrow$	4

b- $k(x)=e^{f(x)}$, then $k'(x)=f'(x)e^{f(x)}$, consequently k and f vary in the same manner ,therefore the table of k over $\left[0;\frac{\pi}{2}\right]$ is as follows :

x	0		$\frac{\pi}{8}$		$\frac{3\pi}{8}$		$\frac{\pi}{2}$
$k'(x)$		$+$	0	$-$	0	$+$	
$k(x)$	e^4	$\nearrow$	e^5	$\searrow$	e^3	$\nearrow$	e^4

Solved Problems

$N° 1.$

The heights of 50 students of a class are given in the table below :

Classes	Frequency
[150;155[	8
[155;160[	6
[160 ; 165[	12
[165 ; 170[	15
[170 ; 175[	7
[175 ; 180]	2

1) Determine, for the data given:
   the population , the variable under study and its nature.
2) Determine the modal class and estimate the mode of the given data.
3) Calculate the average heights of the students of this class as well as the standard deviation and the variance of this distribution .
4) a- Construct the polygon (P) of the increasing cumulative frequency.
   b- The straight line of equation $y = 25$ cuts (P) at a point M . Calculate the abscissa of M and give a statistical interpretation of the value found.
5) What is the percentage of students whose heights belong to the interval [$\bar{x} - 2\sigma$; $\bar{x} + 2\sigma$] ?

N°2.

The table below shows the average number of study hours per week of 50 students of a certain class:

Number of study hours	Frequency
[0 ; 5[	2
[5 ; 10[	8
[10 ; 15[	20
[15 ; 20[	10
[20 ; 25[	6
[25 ; 30]	4

1) Construct the histogram of frequencies of the given data.
2) Determine the modal class and estimate the mode of the given data.
3) Calculate the mean $\bar{x}$ as well as the standard deviation σ of this data.
4) Redraw the table above and complete it by the increasing cumulative frequency and the decreasing cumulative frequency .
5) a- Construct the increasing cumulative frequency polygon and the decreasing cumulative frequency polygon on the same figure.
   b- Determine the median graphically then calculate it.
6) Consider the series x_i, the class marks (centers) of the given classes and introduce the series $y_i = 2x_i$.
   Calculate the mean $\bar{y}$ of the series y_i and compare it to $\bar{x}$.
   Calculate the standard deviation σ' of the series y_i and compare it to σ.

N° 3.

The tables below show the grades of two classes on a math exam

Class A.

Classes	$[0;4[$	$[4;8[$	$[8;12[$	$[12;16[$	$[16;20]$	Total
Frequency	5	13	11	8	3	40

Class B.

Classes	$[0;4[$	$[4;8[$	$[8;12[$	$[12;16[$	$[16;20]$	Total
Frequency	7	10	6	13	4	40

Designate by $\overline{x}$ and $\overline{y}$ the means of the two sets of data (A) and (B).

1) a- Calculate the means $\overline{x}$ and $\overline{y}$ to the nearest 10^{-1}.
   b- What is the average grade of the students of both classes ?
2) a- Calculate the standard deviation of both sets of data .
   b- In which class are the grades less dispersed around the mean ?
3) Determine for class (A), the median class and calculate the median. .
4) The teacher selects two students, at random, from the 80 students in the two classes.
   Consider the following events:
   E : The two selected students are from class A.
   F : The two selected students have a grade greater or equal to 12
   a- Calculate the probabilities $p(F/E)$, $p(E)$ and $p(F \cap E)$.
   b- Calculate $p(F \cap \overline{E})$ and $p(F)$.

N° 4.

The distribution of the yearly expenses , in millions of LP , of 50 families is given in the following table:

Expenses	[6 ; 12[	[12 ; 18[	[18 ; 24[	[24 ; 30[	[30 ; 36]
Number of families	10	5	3	20	12

1) Calculate the mean $\bar{x}$ of this distribution, and give a meaning for the value obtained.
2) Calculate the variance and the standard deviation of the given distribution.
3) We make the following change in the variable $y_i = x_i - 21$ where x_i designates the class mark (center) of the given classes.
   Calculate the mean and the standard deviation of the new statistical set (y_i) and deduce the mean and the standard deviation of the set (x_i) .

N° 5.

The following table shows the distribution of the weights of 75 rams:

Weight in kg	Frequency
[10 , 20 [	1
[20 , 30 [	7
[30 , 40 [	8
[40 , 50 [	11
[50 , 60 [	19
[60 , 70 [	10
[70 , 80 [	7
[80 , 90 [	5
[90 ,100[	4
[100,110]	3

1) Determine the modal class and estimate the mode of the given data.
2) Determine the median class and calculate the median.
3) We make a change in the variable in the following form

$y_i = x_i - 55$ where x_i designates the center (class mark) of the classes given.
Calculate the mean and the standard deviation of the new set of data obtained and deduce the mean and the standard deviation of the set (x_i) .

N°6.

90 employees work in two factories A and B of the same industrial community. The following table shows the distribution of their monthly salaries (in 100 000 L.P) .

Salaries in 100 000 L.P		$[4;8[$	$[8;12[$	$[12;16[$	$[16;20[$	$[20;24]$
Factory A	Number of employees	10	9	12	7	2
Factory B	Number of employees	9	16	14	10	1

1) Calculate the means and the standard deviations of the two sets of data A and B.
2) Which one of the two sets is less dispersed around the mean?
3) What is the mean salary of the employees of this community?
4) The community chooses, at random, an employee.
   Consider the following events :
   E : The chosen employee earns at least 1 200 000 L.P monthly .
   F : " The chosen employee works in factory A ".
   G : " The chosen employee works in factory B ".
   a- Calculate the following probabilities:
      $p(E/F)$, $p(F)$ and $p(E \cap F)$.
   b- Calculate $p(E \cap G)$ and deduce $p(E)$.

N°7.

The following table shows the distribution of the ages (in years) of 40 citizens in a country.

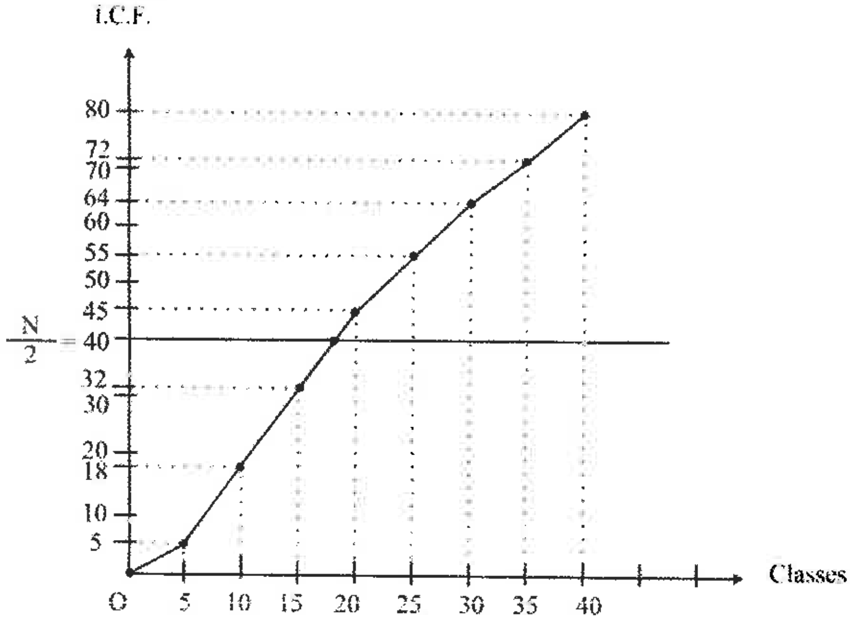
Age in years	$[10;20[$	$[20;30[$	$[30;40[$	$[40;50[$	$[50;60]$
Number of people	8	6	10	9	7

1) Calculate the mean $\overline{x}$.
2) We add 10 to this set from which we have x citizens in class $[40;50[$ and y citizens in class $[50;60]$.

   The new mean becomes $\overline{x'} = 38.6$, calculate x and y.
3) Redraw the given table and complete it by the class marks and the increasing cumulative frequency.

N°8.

Given, below, the polygon of increasing cumulative frequency of a given set of data.

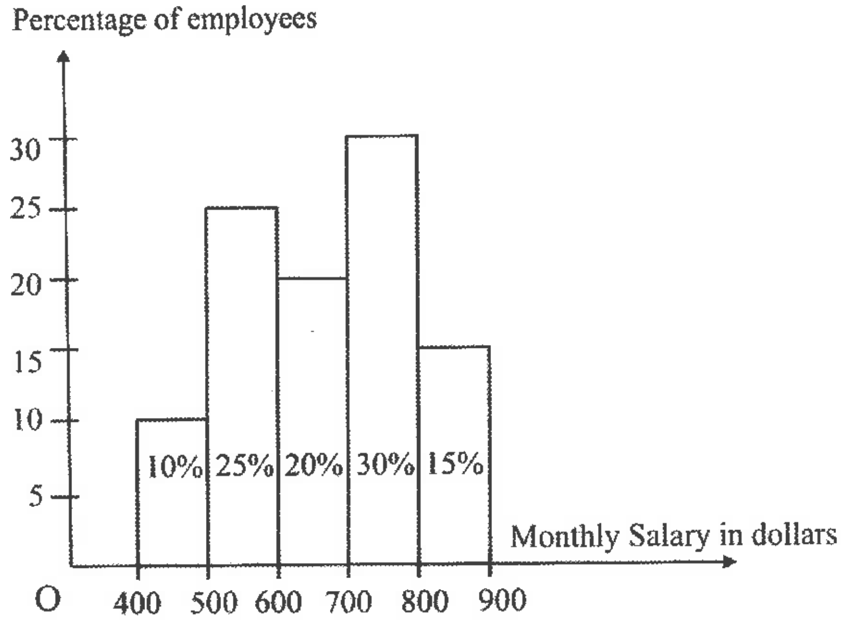


1) Draw the table associated with this data showing the classes, the frequency and the increasing cumulative frequency.
2) Calculate the mean $\overline{x}$ and the standard deviation σ of this set.
3) Calculate the median of this set and give a meaning for the value found.

N°9.

The employees of a factory are divided in 5 classes according to their salaries (in \$), as the following polygon of percentage frequencies shows.

18 employees are in the first class $[400;500[$.



Calculate the number of employees of each class.

Solutions

N° 1.

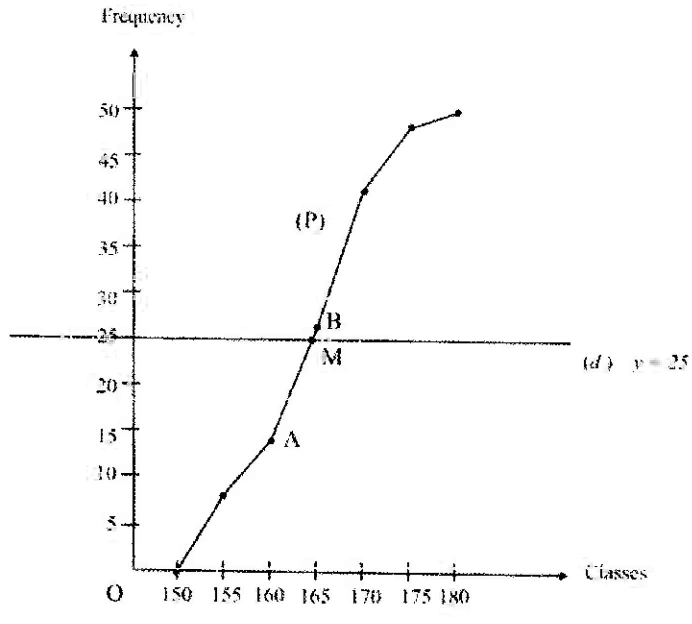
1) The population is the set of 50 students of the class, the variable under study is the heights of these students, this variable is quantitative continuous.
2) The modal class is $[165\,;\,170[$, since it has the highest frequency. The mode is estimated by the following formula:

$$M_O = L + \frac{e_{m_O} - e_i}{(e_{m_O} - e_i) + (e_{m_O} - e_s)} \times c\,.$$

where L designates the lower bound of the modal class, e_{m_O} is the frequency of the modal class, e_i is the frequency of the adjacent class inferior to the modal class, e_s is the frequency of the adjacent class superior to the modal class, and c the amplitude of each of the classes.

Then, $M_O = 165 + \dfrac{15-12}{(15-12)+(15-7)} \times 5 \approx 166.36$

3) Using a calculator, we can find $\bar{x} = 163.8$, $\sigma_X \approx 6.91$ and $v_X = \sigma_X{}^2 = 47.81$
4) a-



b- Given $A(160;14)$, $B(165;26)$ and $M(x;25)$, the two vectors $\overrightarrow{AM}(x-160;11)$ and $\overrightarrow{AB}(5;12)$ are collinear , then

$\frac{x-160}{5}=\frac{11}{12}$, which gives $x \approx 164.58$

The median M_e divides the set of data into two equal sets of data, $50\% \leq M_e$ et $50\% \geq M_e$.

5) $[\ \bar{x}-2\sigma;\ \bar{x}+2\sigma\ ]=[149.98\,;177.62]$.

We assume that the frequency is regularly divided in the classes.
To calculate the frequency between 177.62 and 175 :

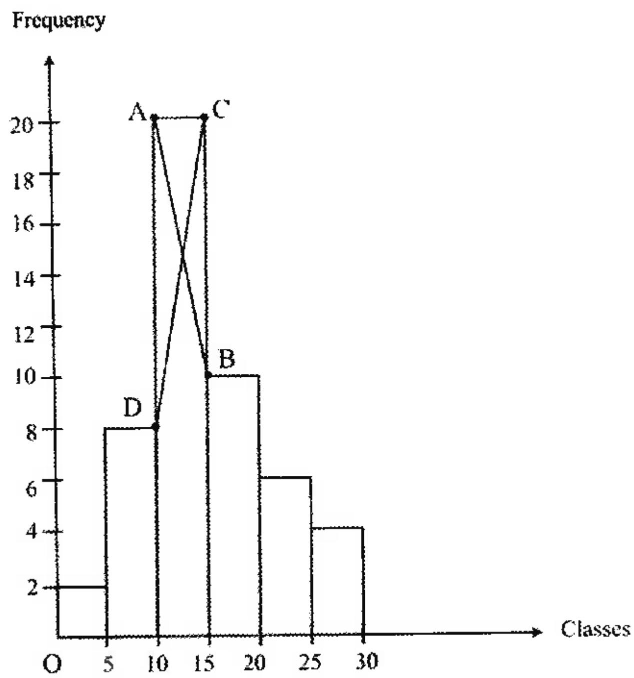
5	2
177.62-175	?

$\frac{2.62 \times 2}{5}=1.048$

Hence, the required frequency in the given interval is:
$n=8+6+12+15+7+1.048=49.48$, hence the percentage heights of students belonging to the interval $[\ \bar{x}-2\sigma;\ \bar{x}+2\sigma\ ]$ is around 98.096%.

N° 2.

1)



2) $B(15\,;10)$, $A(10\,;20)$, $C(15\,;20)$ and $D(10\,;8)$.

An equation of the straight line (AB) is $y=-2x+40$.

An equation of the straight line (CD) is $y=\dfrac{12}{5}x-16$.

The mode is the point of intersection of these two straight lines.

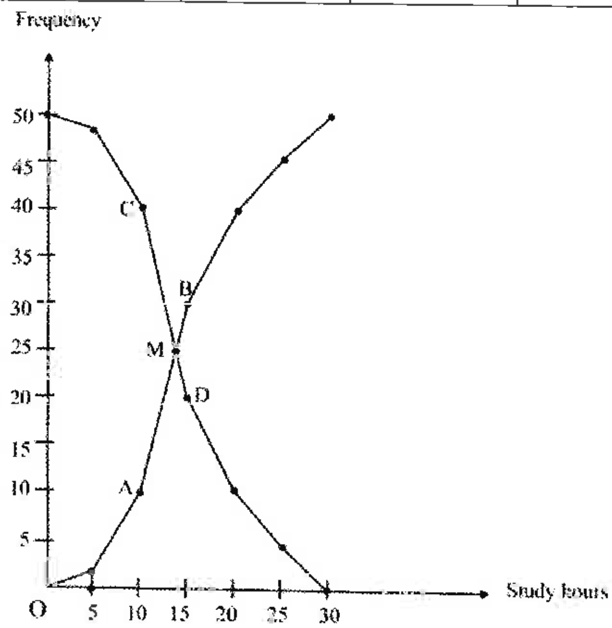
$\dfrac{12}{5}x-16=-2x+40$ gives $M_O=12.72$

3) Using a calculator, we find that $\bar{x}=14.7$ et $\sigma_X \approx 6.17$

4)

Number of Study Hours	Frequency	*I.C.F*	*D.C.F*
[0 ; 5[	2	2	50
[5 ; 10[	8	10	48
[10 ; 15[	20	30	40
[15 ; 20[	10	40	20
[20 ; 25[	6	46	10
[25 ; 30]	4	50	4

5) a-



b- The median is the abscissa of the point of intersection of the increasing cumulative frequency polygon and the decreasing cumulative frequency polygon.

The two vectors $\overrightarrow{AB}(5\,;20)$ and $\overrightarrow{AM}(x-10\,;y-10)$ are collinear , therefore $\dfrac{x-10}{5}=\dfrac{y-10}{20}$.

Similarly, the two vectors $\overrightarrow{CD}(5\,;-20)$ and $\overrightarrow{CM}(x-10\,;y-40)$ are collinear , then $\dfrac{x-10}{5}=\dfrac{y-40}{-20}$.

The two equations give $\dfrac{y-10}{20}=\dfrac{y-40}{-20}$, which gives $y=25$.

Consequently, $x=13.75$, therefore, the median is 13.75

6) The centers of the classes x_i are:

$x_1=2.5$; $x_2=7.5$; $x_3=12.5$; $x_4=17.5$; $x_5=22.5$;

$x_6=27.5$,therefore :

$y_1=5$; $y_2=15$; $y_3=25$; $y_4=35$; $y_5=45$; $y_6=55$.

Using a calculator, we can find $\overline{y}=29.4$ and $\sigma_Y\approx 12.35$,

hence $\overline{y}=2\overline{x}$ and $\sigma_Y=2\sigma_X$.

$\boxed{N°3.}$

1) a- Using a calculator, we find that $\overline{x}=9.1$ and $\overline{y}=9.7$

b- The average grade of the two classes is $\dfrac{\overline{x}+\overline{y}}{2}=\dfrac{9.1+9.7}{2}=9.4$

2) a- Using a calculator , we can find $\sigma_X\approx 4.51$ and $\sigma_Y\approx 5.16$

b- The set A is less dispersed around the mean since $\sigma_X<\sigma_Y$.

3)

Classes	$[0\,;4[$	$[4\,;8[$	$[8\,;12[$	$[12\,;16[$	$[16\,;20]$
Frequency	5	13	11	8	3
I.C.F	5	18	29	37	40

$\dfrac{N}{2}=\dfrac{40}{2}=20$, between 18 and 29, then the median class is $[8\,;12[$.

The median can be calculated by the following formula:

$M_e = L + \dfrac{\frac{N}{2} - N_i}{n_M} \times c$. Where L designates the lower bound of the median class , N is the total frequency, N_i is the cumulative frequency for the classes just preceding the median class , n_M is the frequency of the median class , and c is the amplitude of each class . Therefore, $M_e = 8 + \dfrac{20-18}{11} \times 4 \approx 8.72$

4) a- $p(F/E) = \dfrac{C_{11}^2}{C_{40}^2} = \dfrac{55}{780} = \dfrac{11}{156}$, $p(E) = \dfrac{C_{40}^2}{C_{80}^2} = \dfrac{780}{3160} = \dfrac{39}{158}$,

$$p(F \cap E) = p(F/E) \times p(E) = \frac{11}{156} \times \frac{39}{158} = \frac{11}{632}.$$

b- $\overline{E}$: means that the two students are from the class B or one student from A and one student from B

$$p(F \cap \overline{E}) = \frac{C_{17}^2}{C_{80}^2} + \frac{C_{11}^1 \times C_{17}^1}{C_{80}^2} = \frac{323}{3160}. \quad p(F) = \frac{C_{28}^2}{C_{80}^2} = 0{,}119.$$

N°4.

1) Using a calculator, we find $\overline{x} = 23.28$, then the average annual expenses of each family is 23 280 000 L.P .
2) Using a calculator, we find $\sigma_X \approx 8.72$ and $v_X \approx 76.16$
3) The centers of the classes are :

$x_1 = 9$; $x_2 = 15$; $x_3 = 21$; $x_4 = 27$; $x_5 = 33$,then :

$y_1 = -12$; $y_2 = -6$; $y_3 = 0$; $y_4 = 6$; $y_5 = 12$.

Using a calculator, we find $\overline{y} = 2.28$ and $\sigma_Y \approx 8.72$

Since $y_i = x_i - 21$ then $\overline{y} = \overline{x} - 21$ therefore

$\overline{x} = \overline{y} + 21 = 2.28 + 21 = 23.28$ and $\sigma_Y = \sigma_X$.

N°5.

1) The modal class is [50,60[, the mode is estimated by

$$M_o = 50 + \frac{19-11}{(19-11)+(19-10)} \times 10 = 54.7$$

2) $\dfrac{N}{2} = \dfrac{75}{2} = 37.5$, is between the increasing cumulative frequencies

27 and 46 then the median class is $[50;60[$.

The median is then $M_e = 50 + \frac{37.5-27}{19} \times 10 \approx 55.5$.

3) $y_1 = -40$; $y_2 = -30$; $y_3 = -20$; $y_4 = -10$; $y_5 = 0$.
$y_6 = 10$; $y_7 = 20$; $y_8 = 30$; $y_9 = 40$; $y_{10} = 50$.
Using a calculator, we find that $\overline{y} = 2.4$ and $\sigma_y \approx 21.28$.
Since $y_i = x_i - 55$, then $\overline{y} = \overline{x} - 55$ therefore $\overline{x} = \overline{y} + 55 = 57.5$
and $\sigma_X = \sigma_Y = 21.28$

N° 6.

1) Using a calculator, we find that:
$\overline{x_A} = 12.2$ and $\sigma_A \approx 4.72$; $\overline{x_B} = 12.24$ and $\sigma_B \approx 4.24$
2) Since $\sigma_B < \sigma_A$ then the set B is less dispersed around the mean .
3) The average salary of the two sets is :
$$\frac{N_A \times \overline{x_A} + N_B \times \overline{x_B}}{N_A + N_B} = \frac{40 \times 12.2 + 50 \times 12.24}{40+50} = 12.22.......$$
Then the average salary of an employee 1 222 220 L.P .
4) a- $p(E/F) = \frac{12+7+2}{40} = \frac{21}{40}$, $p(F) = \frac{40}{90} = \frac{4}{9}$,
$$p(E \cap F) = p(E/F) \times p(F) = \frac{21}{40} \times \frac{4}{9} = \frac{7}{30}.$$
b- $p(E \cap G) = p(G) \times p(E/G) = \frac{50}{90} \times \frac{25}{50} = \frac{5}{18}$.
$$p(E) = p(E \cap F) + p(E \cap G) = \frac{7}{30} + \frac{5}{18} = \frac{23}{45}.$$

N° 7.

1) The average is $\overline{x} = 35.25$
2) The mean is $\overline{x'} = 38.6$, then:
$$\frac{15 \times 8 + 25 \times 6 + 35 \times 10 + 45 \times (9+x) + 55 \times (7+y)}{50} = 38.6$$
Which gives $9x + 11y = 104$ and since $x + y = 10$, we get $x = 3$ and $y = 7$.
3)

Age in years	[10;20[	[20;30[	[30;40[	[40;50[	[50;60]
Centers x_i	15	25	35	45	55
$F.\ n_i$.	8	6	10	9	7
$I.C.F$	8	14	24	33	40

N° 8.

1)

Classes	Centers x_i	Frequency n_i	$I.C.F$
[0 ; 5[	2,5	5	5
[5 ;10[	7,5	13	18
[10 ;15[	12,5	14	32
[15;20[	17,5	13	45
[20;25[	22,5	10	55
[25;30[	27,5	9	64
[30;35[	32,5	8	72
[35;40]	37,5	8	80

2) The mean is $\bar{x} = 19.3125$, The standard deviation is $\sigma \approx 10.4$

3) $\frac{N}{2} = 40$ is between $\alpha = 32$ and $\beta = 45$ then the median class is $[15;20[$. Then, the median is $M_e = 15 + \frac{40-32}{13} \times 5 = 18.0769$

N° 9.

x, y, z, u and v are the respective frequencies of the 5 given classes of percentages: 10%, 25%, 20%, 30% and 15% then:

$\frac{x}{10} = \frac{y}{25} = \frac{z}{20} = \frac{u}{30} = \frac{v}{15}$, but $x = 18$, therefore $y = 45$, $z = 36$, $u = 54$ and $v = 27$.Hence, the total number of employees is $18 + 45 + 36 + 54 + 27 = 180$.

Sample Test 1

EXERCISE - I

In the space referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, given the points $A(1;2;3)$, $B(3;2;1)$ and $C(1;1;2)$ and the straight line (d):
$x = 4m-3$; $y = 3m-2$; $z = 2$.

1) Write an equation of the plane (P) passing through the points A , B , et C.
2) Write an equation of the plane (Q) containing (d) and parallel to (AB).
3) Verify that C belongs to (d) and write a system of parametric equations of the line (d') intersection of (P) and (Q).
4) Calculate the cosine of the acute angle of (P) and (Q) and the distance between (Q) and (AB).
5) Let (R) be the plane passing through O and perpendicular to (d).
   a- Write an equation of (R).
   b- Calculate the coordinates of the point I , the orthogonal projection of O on (d) then find the distance from O to (d) .
   c- Let A_1 and A_2 be the symmetric of A with respect to (R) and (Q) respectively.
   Calculate A_1A_2 and deduce the distance from the point A to the straight line $(\Delta) = (R) \cap (Q)$.

EXERCISE - II

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points $A(4), B(4+2i), E(-2)$, $M(z)$ and $M'(z')$ such that $z' = \dfrac{-2z+8+4i}{z-4}$ where $z \neq 4$.

1) a- Interpret geometrically $|z-4|$; $|-2z+8+4i|$ and $|z'|$.
   b- Determine the set of points M such that $|z'| = 2$.

2) a- Show that for any $z \neq 4$, $(z'+2)(z-4)=4i$.
   b- Show that $EM' \times AM = 4$ and that
   $\left(\vec{u};\overrightarrow{EM'}\right)=\frac{\pi}{2}-\left(\vec{u};\overrightarrow{AM}\right)(\bmod 2\pi)$.
   c- Suppose that point M moves on the circle (C) of center A and radius $R=4$, determine the set of points M'.
3) a- Express z in terms of z'.
   b- Let $z=x+iy$ and $z'=x'+iy'$.
   Express x and y in terms of x' and y'
   c- Deduce the set of points M' when M moves on the straight line (d) of equation $y=2$.

EXERCISE - III

A store sells only tables, chairs and boards.
120 clients enter this store during a certain week.
90 buy each a table while each of 30 buy a set of chairs.
From the clients who bought tables, 60 % bought a board.
From the clients who bought a set of chairs, 70 % bought a board.

1) We choose at random a client and we consider the following events.
   E : The client bought a table.
   F : The client bought a set of chairs.
   G : The client bought a board.
   a- Calculate $p(E \cap G)$.
   b- Calculate $p(F \cap \overline{G})$ and $p(G)$.
2) The price of a table is 35000LL, the price of a board is 15000 LL and the price of a set of chairs 40000 LL.
   Consider the random variable X equal to the sum paid by the client.
   a- Determine the probability distribution of X.
   b- Calculate $E(X)$.
   c- Give an estimation of the amount of money received by the store this week

EXERCISE - IV

Consider the differential equation $(E): y' - 2y = xe^x$.

1) Suppose $y = z - (x+1)e^x$.
   - a- Form the differential equation (F) satisfied by z.
   - b- Solve (F) and deduce the particular solution f of (E) verifying $f(0) = 0$.
2) Let g be the function defined over IR by $g(x) = 2e^x - x - 2$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
   - a- Determine $\lim\limits_{x \to +\infty} g(x)$ and $\lim\limits_{x \to -\infty} g(x)$.
   - b- Show that the straight line (d) of equation $y = -x - 2$ is an asymptote to (C) as $x \to -\infty$ and that (C) is above (d).
   - c- Study the variations of g and draw its table of variations.
   - d- Show that the equation $g(x) = 0$ admits two roots 0 and α and that $-1.6 < \alpha < -1.5$
   - e- Trace (C) and (d).
3) h is a function defined over IR by $h(x) = e^{2x} - (x+1)e^x$. (Γ) is the representative curve of h in an orthonormal system $(O;\vec{i},\vec{j})$.
   - a- Determine $\lim\limits_{x \to +\infty} h(x)$ and $\lim\limits_{x \to -\infty} h(x)$.
   - b- Show that $h'(x) = e^x g(x)$ and draw the table of variations of h.
   - c- Show that $h(\alpha) = \dfrac{-\alpha^2 - 2\alpha}{4}$ then, give an approximate value of $h(\alpha)$ to the nearest 10^{-2} knowing that $\alpha = -1.55$
   - d- Trace (Γ).
   - e- Show that the equation $h(x) = 1$ admits one and only one root β and that $0.813 < \beta < 0.815$
   - f- Calculate the area of the region bounded by (Γ), $x'x$ and the two straight lines of equations $x = 0$ and $x = 1$.

Sample Test 2

EXERCISE – I

Part A .

Let g be the function defined over IR by $g(x)=a+(b-x)e^x$ where a and b are real numbers.

Designate by (G) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

Determine a and b such that (G) has at the point $A(0;3)$ a tangent parallel to the straight line (D) of equation $y=x$.

Part B.

Let f be the function defined over IR by $f(x)=(2-x)e^x+1$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Calculate $\lim\limits_{x\to-\infty} f(x)$ and deduce an asymptote (d) to (C)

   b- Study the relative position of (C) and (d).

   c- Calculate $\lim\limits_{x\to+\infty} f(x)$.

2) Calculate $f'(x)$ and set up the table of variations of f.

3) Show that (C) has an inflection point W .

4) Show that the equation $f(x)=0$ has a unique root α such that

   $2.1<\alpha<2.2$

4) a- Draw (d) and (C)

   b- Discus graphically, according to the values of the real number m, the number of solutions of the equation $(m-1)e^{-x}=2-x$.

5) Designate by $A(\alpha)$ the area of the domain bounded by (C), the x-axis and the y-axis.

   Show that $A(\alpha)=\dfrac{(3-\alpha)^2}{\alpha-2}$.

EXERCISE – II

Consider in the space of an orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$ the straight lines (d) and (d') of parametric equations :

(d) : $x=-t+1$, $y=5t+1$, $z=4t-2$ and

(d') : $x=4m-3$, $y=m$, $z=5m-7$, m and t are real parameters.

1) Find the coordinates of the point A, the intersection of (d) and (d').
2) Write an equation of the plane (P) determined by (d) and (d') and verify that the point $B(2;3;1)$ belongs to (P).
3) Let (Q) be the plane of equation $x+2y+3z-11=0$.
   a- Verify that (P) and (Q) are perpendicular.
   b- Let $(\delta)=(P)\cap(Q)$, determine the coordinates of points E and F intersection of (δ) with (d) and (d') respectively.
   c- Is AEF an equilateral triangle?
4) Let (D) be the line passing through I and perpendicular to (P), find a system of parametric equations of (D).
5) Let $\vec{v}$ and $\vec{v'}$ be the respective directing vectors of (d) and (d').
   a- Determine the vector $\vec{v}+\vec{v'}$.
   b- Deduce a system of parametric equations of one of the bisectors of the angles determined by (d) and (d').

EXERCISE – III

An urn U_1 contains 5 black balls and 4 white balls.

Another urn U_2 contains 4 black balls and 5 white balls.

Part A.

We draw two balls of urn U_1 in the following manner :

We draw one ball :

If it is white, we replace it back in the urn and we draw a second ball.

If it is black, we leave it outside and we draw a second ball..

1) Calculate the probability of getting two black balls.
2) Calculate the probability of getting two balls of different colors.

Part B

We choose at random one of the two urns and we draw simultaneously two balls of this urn.

Consider the following events :

E : The urn chosen is U_1.

F : The two balls chosen are white.

1) Calculate $p(F/E)$ and $p(F)$.
2) The two drawn balls are white, what is the probability that they come from U_1?

Part C .

We draw simultaneously two balls from U_1 and one ball from U_2.

1) Designate by A the following event :
   « The two drawn balls are two white balls from U_1 and one black ball from U_2 ».
   Calculate $p(A)$.
2) Designate by B the following event:
   " From the three drawn balls, there are exactly two white balls "
   Calculate $p(B)$.
3) Let X be the random variable that to each drawing associates the number of white balls obtained.
   Determine the probability distribution of X.

EXERCISE – IV

Given the differential equation $(E): y'' + 4y = 0$.

1) Solve (E) and find the particular solution f verifying
   $f(0) = 1$ and $f'\left(\frac{\pi}{2}\right) = -2\sqrt{3}$.
2) a- Show that $f(x)$ can be written in the form
   $f(x) = 2\sin(2x + \frac{\pi}{6})$.
   b- Let T be the period of f.
   Study the variations of f and trace its representative curve (C) in an interval of amplitude T in an orthonormal system $(O;\vec{i},\vec{j})$.

Sample Test 3

EXERCISE - I

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the straight lines (d_1) and (d_2) defined by : (d_1) : $\begin{cases} x = m \\ y = m-1 \\ z = 1 \end{cases}$ and (d_2) : $\begin{cases} x = -t+1 \\ y = t \\ z = -2t+4 \end{cases}$ (m and t are two real numbers).

1) Prove that (d_1) and (d_2) are orthogonal and skew.
2) Verify that the vector $\vec{n}(-1;1;1)$ is orthogonal to (d_1) and (d_2).
3) Prove that an equation of the plane (P) containing (d_1) and parallel to $\vec{n}$ is $x-y+2z-3=0$.
4) The straight line (d_2) cuts the plane (P) at B. Determine the coordinates of B.
5) a- Prove that the straight line (D) passing through B and of directing vector $\vec{n}$ cuts the straight line (d_1) at point $A(1;0;1)$.
   b- Verify that (AB) is a common perpendicular to (d_1) and (d_2).
6) Let M be a variable point of (d_1) and (M') a variable point of (d_2).
   Show that $\overline{MM'}^2 = \overline{MA}^2 + \overline{AB}^2 + \overline{M'B}^2$. Deduce the minimal value of MM'.

EXERCISE - II

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

For all points M of affix z we associate the point M' of affix z' such that $z' = \sqrt{2}z + \sqrt{6}\bar{z}$.

1) Let $z = 1+i$.
   a- Find the algebraic form of z'.
   b- Show that $z'^2 = 16e^{-i\frac{\pi}{6}}$.
   c- Deduce $\sin\frac{\pi}{12}$ and $\cos\frac{\pi}{12}$.

2) Let $z = x + iy$ and $z' = x' + iy'$.
   a- Express x' and y' in terms of x and y.
   b- Deduce the if M moves on the line of equation $y = x$ then M' moves on a straight line (d') of equation $y = (-2+\sqrt{3})x$ z' is pure imaginary
   c- Deduce the set of points M' when $\arg(z) = \frac{\pi}{4}(\pi)$.

EXERCISE - III

Part A.

Consider the function g defined over $]0;+\infty[$ by $g(x) = x^2 + \ln x$.

1) Show that g is strictly increasing over $]0;+\infty[$.
2) Calculate $g(1)$.
3) Deduce that :
   If $x \geq 1$, then $x^2 + \ln x \geq 1$ and if $x \leq 1$, then $x^2 + \ln x \leq 1$.
4) Determine the sign of the expression $x^2 + \ln x - 1$ for $x \in]0;+\infty[$.

Part B

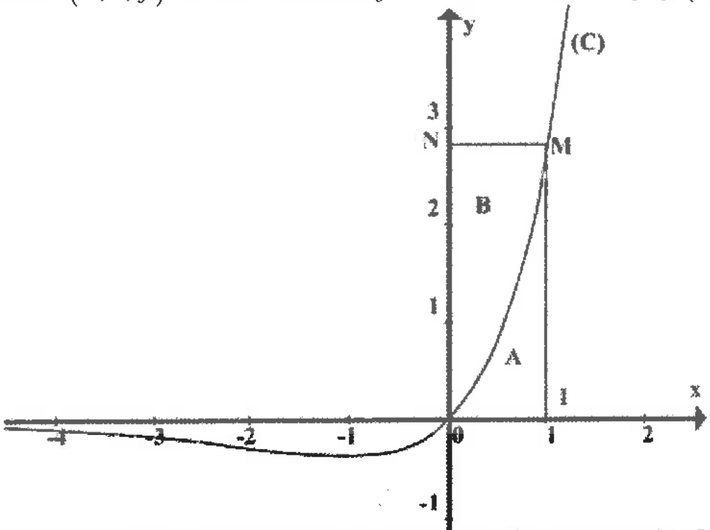
Let f be the function defined over $]0;+\infty[$ by $g(x) = x + 1 - \frac{\ln x}{x}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to 0} f(x)$.
2) Calculate $f'(x)$ and draw the table of variations of f.
3) a- Show that the straight line (D) of equation $y = x + 1$ is an asymptote of (C).
   b- Study the relative positions of (C) with respect to (D).
   c- Determine the coordinates of the point of (C), where the tangent (T) to (C) is parallel to (D).
   d- Trace (C) and (T).
   e- Calculate the area A of the domain limited by (C) , (D) and the two straight lines of equations $x = 1$ and $x = e$.

EXERCISE - IV

The curve (C) below is the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ of the function f defined over IR by $f(x)=xe^{x}$.



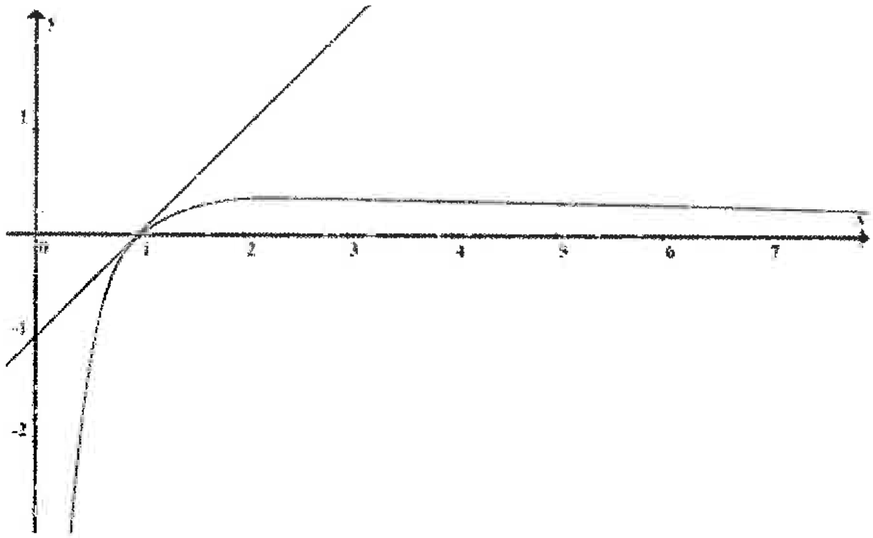
A game consists of throwing an arrow on the rectangle $OIMN$.
We admit that the probability that it hits the square is 0,5 and that the probability that it hits the regions A and B are respectively proportional to the areas of these regions

1) Show that the probability that it hits region A is $\frac{1}{2e}$.
2) We throw three arrows on the rectangle.
   Let X be the random variable equal to the number of arrows that hit region A .
   a- Show that $p(X=0)=\left(1-\frac{1}{2e}\right)^{3}$.
   b- Designate by E be the event : exactly one arrow hits A , calculate $p(E)$.
   c- Determine the probability distribution of X.
3) We throw n arrows on the rectangle.
   Designate by G the event : at least one arrow hits region A.
   a- Calculate $p(G)$.
   b- Determine the smallest integer n so that $p_n \geq 0.99$.

Sample Test 4

EXERCISE - I

The curve (C') below is the representative curve of the function f defined over $]0;+\infty[$ by $f(x)=\dfrac{a\ln x}{b+x}$.



1) Find a and b knowing that $x=0$ is an asymptote to (C') and that the area of the domain limited by (C'), $x'x$ and the straight line of equation $x=2$ is $A=\dfrac{(\ln 2)^2}{2}\text{u}^2$.

Part B.

Consider the function f defined over $]0;+\infty[$ by $f(x)=x\ln x-x$.

(C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to 0^+} f(x)$.
2) Study the variations of f and set up its table of variations.
3) Solve the equation $f(x)=0$.
4) Show that the equation $f(x)=x$ has a unique root α.
5) Draw (C).
6) Show that f admits an inverse function h for $x\geq 1$.

Draw the graphe (C_1) of h in the same system.

7) a- Show that $F(x)=\frac{x^2}{2}\ln x-\frac{3}{4}x^2$ is an antiderivative of f and

b- Calculate the area of the domain limited by (C), the axis $x'x$ and the straight lines of equations $x=0$ and $y=e^2$.

c- Deduce the area of the domain bounded by (C) and (C_1).

EXERCISE - II

In the space referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the points $A(2;1;-1)$, $B(3;0;1)$ and $C(2;-1;3)$

Part A.

1) Let $D(0;y;0)$ be a point on the $y-$axis such that $y<0$.
Calculate y such that the volume of the tetrahedron $ABCD$ is 5 u^3.
2) Calculate the area of the triangle ABC.
3) Deduce the distance from the point D to the plane (ABC).
4) Find an equation of the plane (ABC) and deduce the distance from D to the plane (ABC)

Part B.

1) Write a system of parametric equations of (AB).
2) Let $E(2;1;-3)$ be a point in the space.
   a- Calculate the distance from the point E to (AB).
   b- Show that $x+y-3=0$ is an equation of the plane (P) determined by E and (AB).
   c- Prove that C does not belong to (P).
3) Let (R) be the plane passing through C and perpendicular to (AB)
   a- Write an equation of (R).
   b- (AB) cuts (R) at a point I, find the coordinate of I.
   c- Determine the coordinates of the point F symmetric of C with respect to (AB).

EXERCISE - III

Consider three urns U_1, U_2 and U_3 such that:

U_1 contains a red ball and 4 white balls, U_2 contains 4 red balls and 4 white balls and U_3 contains 7 red balls and 3 white balls.

Designate by:

p_1 the probability of choosing urn U_1.

p_2 the probability of choosing urn U_2

p_3 the probability of choosing urn U_3.

Suppose that $p_1 = \frac{1}{6}$, $p_2 = \frac{1}{3}$ and $p_3 = \frac{1}{2}$.

1) We choose an urn of which we choose at random one ball.
   a- Calculate the probability of choosing a red ball that comes from U_1.
   b- Calculate the probability of the event :
      the ball chosen is red and comes from U_1.
2) Calculate the probability of the event: the chosen ball is red.
3) We know that the chosen ball is red, what is the probability that it comes from U_1 ?

Sample Test 5

EXERCISE - I

The 45 employees of a factory are distributed as shown in the following table :

	Engineer	Technician	Worker
Men	9	10	8
Women	6	5	7

1) The director of this factory chooses randomly and simultaneously three employees from the 45 employees of this factory.
   a- Calculate the probability to get three engineers.
   b- The chosen employees are workers , calculate the probability that they are men.
   c- Let X be the Radom variable that represents the number of technician in this group.
   Determine the set of values of X as well as its probability distribution.
2) The director of this factory chooses randomly one of each category .
   a- Calculate the probability to get three women.
   b- Consider the following events :
   E : The chosen employee among the technicians is a woman.
   F : The chosen group contains two women and one man .
   Calculate $p(F/E)$.

EXERCISE - II

In the space referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider :

The plane (P) of equation $2x+y-3z-1=0$;

The plane (Q) of equation $x+4y+2z+1=0$;

The line (d) defined by : $\begin{cases} x=2t+1 \\ y=-t-1 \\ z=t \end{cases}$ (t is a real parameter).

1) Prove that the line (d) is included in the plane (P) .
2) Find an equation of the plane (S) that is determined by the point

O and the line (d).

3) Consider the point $E\left(0;-\frac{1}{2};-\frac{1}{2}\right)$.

   Prove that E is the orthogonal projection of the point O on the line (d).

4) a- Show that the planes (P) and (Q) are perpendicular.

   b- Let (D) be the line of intersection of (P) and (Q). Calculate the distance from E to (D).

EXERCISE - III

Consider in the complex plane referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$, the points A, B C and D of respective affixes $z_A=-2i$; $z_B=4-2i$; $z_C=4+2i$ and $z_D=1$.

1) Let M be a point of affix $z\neq-2i$ and M' the point of affix z' such that $z'=\dfrac{z-(4+2i)}{z+2i}$.

   a- Interpret geometrically $|z-(4+2i)|$; $|z+2i|$ and $|z'|$.

   b- Determine the set of points M such that $|z'|=1$.

2) a- Show that for any $z\neq-2i$, $(z'-1)(z+2i)=-4-4i$.

   b- Suppose that M moves on the circle (C) with center A and radius $R=\sqrt{2}$, determine the set of points M'.

EXERCISE - IV

Consider the function f define over IR^* by $f(x)=x-2-2\ln|x|$ and designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1) Prove that the straight line (d) of equation $y=x-2$ cuts (C) in two points A and B.

2) a- Let $M(a;b)$ be a point of (C), prove that

   $y=\dfrac{a-2}{a}x-2\ln|a|$ is an equation of the tangent (T) to (C) at M.

b- Deduce the equations of the tangents to (C) that pass through the origin O.

3) Find the points of intersection of (C) and the straight line (δ) of equation $y = x$.
4) Study the variations of f and set up its table of variations.
5) Draw (C), (δ) and (d) and the tangents at A and B to (C).
6) a- Prove that the straight line (D) of equation $y = x + m$ cuts (C) in two points E and F where m is a real parameter .

   b- What is the set of points I midpoint of $[EF]$ as m varies
7) Calculate the area of the region limited by (C), the axis $x'x$ and the two straight lines of equations $x = 1$ and $x = 2$.

Sample Test 6

EXERCISE - I

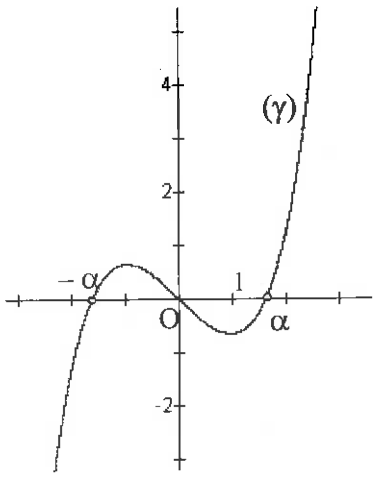
Choose the correct answer .Justify .

$N°$	Question	Answer		
		a	b	c
1	An argument of $z=-2e^{-i\frac{\pi}{12}}$ is :	$\frac{\pi}{12}$	$\frac{11\pi}{12}$	$-\frac{\pi}{12}$
2	$\left(1+i\sqrt{3}\right)^{-6}=$	$\frac{1}{64}i$	$\frac{1}{64}$	$-\frac{1}{64}i$
3	$\lim\limits_{x\to 1}\frac{\int_1^{x^2}\sqrt{1+t^2}\,dt}{x-1}=$	$\frac{2x^2}{\sqrt{1+x^4}}$	$\sqrt{1+x^4}$	$2x\sqrt{1+x^4}$
4	The equation $e^{2x}-2x-2=0$, has in the set IR :	1 root	2 roots	3 roots
5	The coefficient of a^2b^5 in the expansion of $(a+b)^7$ is:	C_7^1	C_7^3	C_7^4

EXERCISE - II

Consider the function g defined over IR by $g(x)=e^x-e^{-x}$ and let f be the function defined over IR by $f(x)=g(x)-2x$. Designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1) a- Show that f is odd.
   b- Calculate $\lim\limits_{x\to+\infty} f(x)$ and deduce $\lim\limits_{x\to-\infty} f(x)$.
   c- Verify that $f'(x)=e^{-x}\left(e^x-1\right)^2$ and set up the table of variations of f .
2) The figure below shows the representative curve (γ) of the function h defined over IR by $h(x)=g(x)-3x$.



a- Show that $1.62 < \alpha < 1.63$.

b- Using (γ), show that the curve (C) of f cuts the straight line (d) of equation $y = x$ in three points whose abscissas are to be determined.

3) Calculate $f(1)$ and $f(2)$ and draw (C) and (d).

4) a- f admits an inverse function f^{-1}, determine the values of x verifying $1 < f^{-1}(x) < 2$.

b- Draw the representative curve (C') of f^{-1} in the same system.

5) Calculate, in terms of α, the area of the region bounded by (C) and (C').

EXERCISE - III

Consider in an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$ the points $A(3;0;10)$, $B(0;0;15)$ and $C(0;20;0)$.

1) Determine a system of parametric equations of line (AB).
2) Show that (AB) cuts $x'x$ at point $E(9;0;0)$.
3) Show that the points A, B and C are not collinear.
4) Let H be the orthogonal projection of O on $[BC]$ in triangle OBC.

   a- Prove that (BC) is perpendicular to plane (OEH).

   b- Deduce that (EH) is a height in triangle EBC.

   c- Determine an equation of plane (OEH).

d- Verify that $20x + 9y + 12z - 180 = 0$ is an equation of the plane (ABC).

e- Show that the system $\begin{cases} x = 0 \\ 4y - 3z = 0 \\ 20x + 9y + 12z - 180 = 0 \end{cases}$

admits a unique solution. What does this solution represent?

f- Calculate OH then deduce EH and the area of triangle EBC.

5) Find the volume of $OEBC$ in two different ways and deduce the distance from O to the plane (ABC) .

EXERCISE - IV

A machine measures the length of a sample of 100 steel rods in mm. The result is shown in the table below:

Length in mm	Frequency
[120 ;125[	10
[125 ; 130[	20
[130 ; 135[	38
[135 ;140[	25
[140 ; 145]	7

1) Determine the modal class and estimate the mode of the given data.
2) Determine the median class and calculate the median.
3) a- Construct the polygon of increasing cumulative frequency.
   b- Deduce, using another method, the median of this distribution
4) Calculate the mean $\bar{x}$ and the standard deviation σ of this set of data .
5) The machine is considered to function properly if the three following conditions are satisfied :

   $\bar{x} \in [132;133]$.

   $\sigma < 6$.

   At least 90 % of the total frequency is in the interval: $[\ \bar{x} - 2\sigma;\ \bar{x} + 2\sigma\ ]$.

   Does the machine function properly ?