TOP PHYSICS

Grade 12
Life sciences section
General sciences section

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PREFACE

Physics is the most fundamental of the sciences. Its principles underlie phenomena studied in all other technical fields, which is one reason why its study is required by so many academic programs. Physics has many real-life applications, and the principles of physics are at work any time someone drives a car, rides a rollercoaster, lifts something heavy, turns on the radio, switches on a lamp, uses the headphone, or tries to stay warm in the winter…

As the authors of this guide, it is our intent to facilitate both student and instructor in the process of learning, emphasizing the conceptual simplicity and power of these ideas which make up our understanding of the physical world. Although the ability to solve exercises is the ultimate test of learning physics, in this guide we emphasize on the development of conceptual reasoning and mathematical manipulation. An overriding goal of this guide is to impart to the student an understanding of the process of problem analysis and solution. It is that understanding that empowers the student and will lead him or her to a greater degree of confidence and a true mastery of the subject.

Studying physics is exciting because it can help you answer many questions about how and why our world works. Our guide is designed to take some "real-life" situations and examine them with the use of equations.

This guide is not meant to stand alone. It is not meant to replace your physics text, the laboratory work that you do, or your physics teacher. Its purpose is to reinforce the concepts that you have already learned in class and to give you the opportunity to try plenty of practice working with both fundamental physical concepts and problem-solving skills.

This guide is designed for grade twelve (general sciences and life sciences sections) students. The guide is organized into four units:

- Mechanics.
- Electricity.
- Aspects of light.
- Atom and nucleus.

The four units are divided into eleven chapters. Each chapter contains three parts:

- Course.
- Exercises and problems.
- Solutions of exercises and problems.

The course section provides a review of the concepts and equations your teacher has discussed in class. It is presumed that you have already learned everything in the course section before beginning the exercises. This review is simply a reminder and a place to find all the equations you need.

Each course starts with an opening question to introduce the concepts of the chapter. This question is followed by introduction that relates these concepts with real life. The course also includes activities that help students to grasp the concepts.

Following the course, there is a section called exercises where the course is applied to exercises. In some chapters, the exercises' section is divided into domains where each domain is related to an objective in the chapter.
The Exercises are provided with detailed solutions. The section of solutions is organized to make it easy to follow a calculation from beginning to end. The solutions of the exercises marked with (*) are not provided. Solving physics exercises is much like baking a cake. The first time you try to do it, you must read the recipe very carefully and use exactly the ingredients listed. The next time, you are a little less nervous about how well the cake will turn out. Pretty soon you can make the cake without having to read the recipe at all. You eventually become so comfortable making cakes that you are able to experiment by adding ingredients in a different order or changing the recipe slightly to make the cake even better. When solving physics exercises, you will find it easy to follow the prescribed "recipe" shown in the solution of the exercises. After trying a few exercises, you will have started to develop a strategy for constructing your solution that you can retain throughout the entire guide. As you get better and better at doing calculations and you develop a greater conceptual understanding of the physics involved, you may even come up with an alternative method of solving an exercise that is different from the one used in this guide. If so, congratulations! You have done just what the physicist does when he or she tries to find a solution. Be sure to show your teacher and classmates your alternative approach. It is valuable to look at many different solutions to the same exercise.

When solving numerical exercises, it is always important to include the proper units with any number you are using. Not only will this help you determine the units in the final answer, but it will also help you with your numerical solution as well. If the units in an exercise do not combine to give the correct units in your final answer, then you may have made a mistake in setting up the original equation. In other words, using the correct units is a way of double-checking all of your work.

Although we have made our best efforts while planning the lay-out of the text and the subject matter, we cannot guess as to how far we have come up to the expectations of esteemed readers. We request them to judge our work critically and pass their constructive criticisms to us so that any conceptual mistakes and typographical errors, which might have escaped our attention, may be eliminated in the next edition.

We are thankful to our colleagues, family members and the publishers for their cooperation during the preparation of the text.

List of some of the action verbs and their requirements

1. **Analyze:** Decompose a whole into its constituent elements to make evident the variations.
2. **Calculate**: (Compute) Perform mathematical operations.
3. **State:** Express without explaining.
4. **Compare:** Indicate the similarities and/or differences between two or more entities.
5. **Complete**: Add what is missing.
6. **Conclude:** Reach to a decision.
7. **Determine:** Reach to a decision or a result through logical reasoning, calculation, ...
8. **Describe:** Express, using scientific language, to give the details of an observation, an experiment, a schema, an apparatus, ...
9. **Show:** Prove something is evident by logical reasoning, experimenting, calculating,...
10. **Deduce:** Draw using logical reasoning new information from given or existing information.
11. **Draw out:** Draw from a set of given and without reasoning a relation, a role, a law,...
12. **Distinguish:** Recognize or discern one thing from another according to particular traits.
13. **Explain**: Clarify, make understandable a phenomenon, a result,...
14. **Identify:** Recognize something based on its characteristics or its properties.
15. **Interpret:** Analyze and give significance to the result.
16. **Indicate:** Designate something without justification.
17. **Justify:** Prove something as true and real.
18. **Specify:** Indicate and justify.
19. **Pick out (Extract):** Select one or more information from a document.
20. **Verify:** Confirm using arguments, logical reasoning,... whether something is true or false.

TABLE OF CONTENTS

CHAPTER 1 – WORK AND ENERGY
COURSE

Energy may exist in potential, kinetic, thermal, electrical, chemical, nuclear or other various forms. In this chapter we introduce several forms of energy including kinetic, potential and thermal energy. In solving problems, we will use a key fact about energy; energy is neither created nor destroyed; if one form of energy in a system decreases, it must appear in an equal amount in another form. Every system in nature is associated with a quantity we call it total energy. The total energy is the sum of different kinds of energy at one time. An exchange of energy between a system and environment (everything that is not a part of the system) is called an energy transfer. There are two primary energy-transfer processes: work, the mechanical transfer of energy to or from a system by pushing or pulling on it, and heat, the non-mechanical transfer of energy from the environment to the system (or vice versa) because of a temperature difference between the two. In this chapter we will consider energy transfer by means of work.

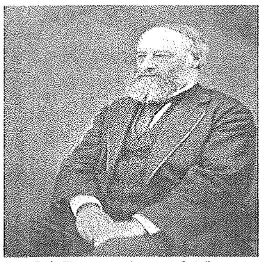
1.1- INTRODUCTION

The terms work, energy and power are frequently used in everyday language. Any activity that requires muscular or mental effort agrees with the everyday meaning of work. A man, carrying water from a well to his house, is said to be working. If he can do so, he is said to have energy. Energy is thus the capacity to do work. The term power is usually associated with speed.

Work is said to be done when a force applied on a body displaces the body through a certain distance in the direction of force.

Work is the measure of the amount of energy transferred when a force acts over a given displacement. But just what is energy? The concept of energy has grown and changed overtime, and it is not easy to define in a general way just what energy is.

James Prescott Joule



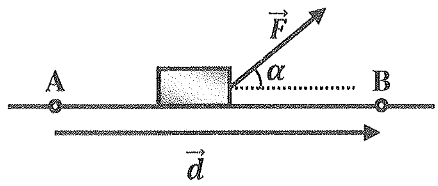
Born: 24 December 1818
Salford, Lancashire, England, UK
Died: 11 October 1889 (aged 70)
Sale, Cheshire, England, UK
Citizenship: British
Field: Physics

1.2- WORK DONE BY A CONSTANT FORCE

Work is the transfer of energy to or from a system by application of forces exerted on the system by the environment. Thus, work is done on a system by forces outside the system, external forces. Only external forces can change the energy of a system. Internal forces, forces between objects within the system, cause energy transformations within the system but do not change the system's total energy. In order for energy to be transferred as work, the system must undergo displacement during the time that the force is applied.

Rectilinear trajectory

A constant force $\vec{F}$ is applied on a body such that it makes an angle α with the horizontal and the body is displaced through a distance d from A to B.



By resolving force $\vec{F}$ into two components:

- $F\cos\alpha$ in the direction of displacement of the body.
- $F\sin\alpha$ in the perpendicular direction of displacement of the body.

The body is being displaced in the direction of $F\cos\alpha$; therefore, work done by the force in displacing the body through a distance d is given by: $W_{\vec{F}} = (F\cos\alpha)d = Fd\cos\alpha = FAB\cos\alpha$.

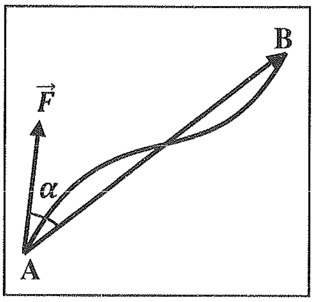
Or $W_{\vec{F}} = \vec{F}.\overrightarrow{AB} = \vec{F}.\vec{d} = F\times d\times\cos(\vec{F};\vec{d}) = F\times d\times\cos\alpha$.

ATTENTION

The work done by a variable force in one dimensional motion is:
$$W = \int_{x_i}^{x_f} F(x)dx$$

Thus, work done by a force is equal to the scalar or dot product of the force and the displacement of the body. Note that work is a scalar quantity; it has an algebraic value but not a direction.

Curvilinear trajectory



The work done by a constant and conservative force on a body is independent of the path followed while moving from A to B:

$$W_{\vec{F}} = \vec{F}.\overrightarrow{AB} = F \times AB \times \cos(\vec{F}, \overrightarrow{AB})$$

General expression of work done by a constant force along any path from A to B:

$W_{\vec{F}} = \vec{F}.\overrightarrow{AB} = F \times AB \times cos(\vec{F}, \overrightarrow{AB}) = F \times d \times cos\,\alpha$

SI unit: The SI unit of work is Joule [J] or [Nm] where 1J = 1Nm.

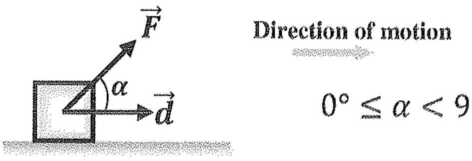
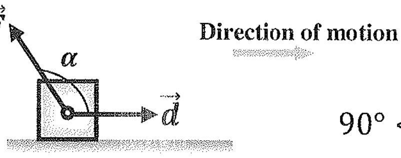
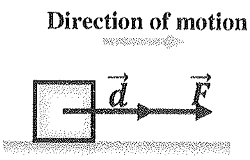
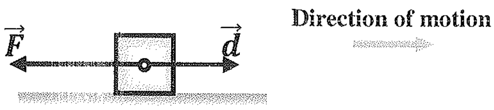
One Joule: one newton force displaces the body through 1 meter in its own direction.

Because work is simply energy being transferred, the joule is the unit of all forms of energy.

Net work: if several forces act on an object that undergoes a displacement, each does work on the object. The total (net) work W_{total} is the sum of the work done by each force. The total work represents the total energy transfer to the system from environment (if $W_{total} > 0$) or from the system to the environment (if $W_{total} < 0$).

$$\sum W = W_1 + W_2 + \dots + W_n$$

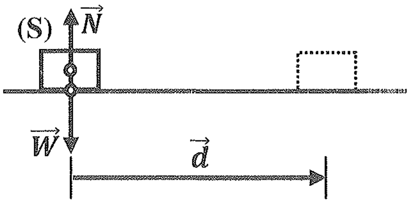
The quantities F and d are always positive, so the sign of W is determined by the angle α between the force and the displacement.

Positive/motive work	Negative/resistive work
Motive work means that force (or its component) is parallel to displacement.  $0° \leq \alpha < 90°$ The motive work signifies that the external force favors the motion of the body. Energy is transferred into the system.	Resistive work means that force (or its component) is opposite to displacement.  $90° < \alpha \leq 180°$ The resistive work signifies that the external force opposes the motion of the body. Energy is transferred out of the system.
Maximum work: $W_{max} = F \times d$ When $\cos\alpha = maximum = 1$ $i.e.$ $\alpha = 0$.  It means force does maximum work when angle between force and displacement is zero. **Maximum energy transfer into the system.**	Minimum work: $W_{min} = -F \times d$ When $\cos\alpha = minimum = -1$ $i.e.$ $\alpha = 180$.  It means force does minimum [maximum negative] work when angle between force and displacement is 180°. **Example:** work done by sliding friction. $W_{\vec{f}} = -f \times d$ **Maximum energy transfer out of the system.**

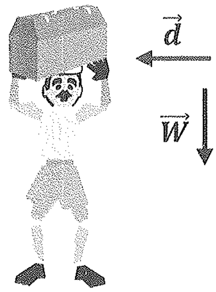
Zero work

Under three conditions, work done becomes zero: $W = Fd\cos\alpha = 0$.

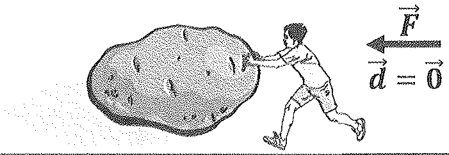
1- If the force is perpendicular to the displacement: $\vec{F} \perp \vec{d}$
$\alpha = 90° \Rightarrow \cos\alpha = 0 \Rightarrow W = 0$.
Example: along a horizontal plane, the work done by the normal reaction of a support and the weight are zero.



$W_{\vec{W}} = \vec{W}.\vec{d} = 0$
$W_{\vec{N}} = \vec{N}.\vec{d} = 0$



2- If there is no displacement $(\vec{d} = \vec{0})$.



3- If there is no force acting on the body $(\vec{F} = \vec{0})$.

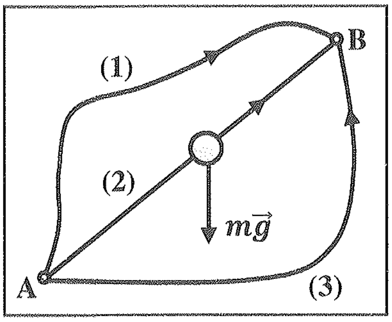
Work done by conservative and non-conservative forces

Conservative force

A conservative force is one for which the work done by the force on an object as it goes from point A to point B doesn't depend on the path taken. The work done by a conservative force agrees with these conditions:

- The work depends on the initial and final positions only.
- The work is reversible.
- Along a closed path, the work is zero.

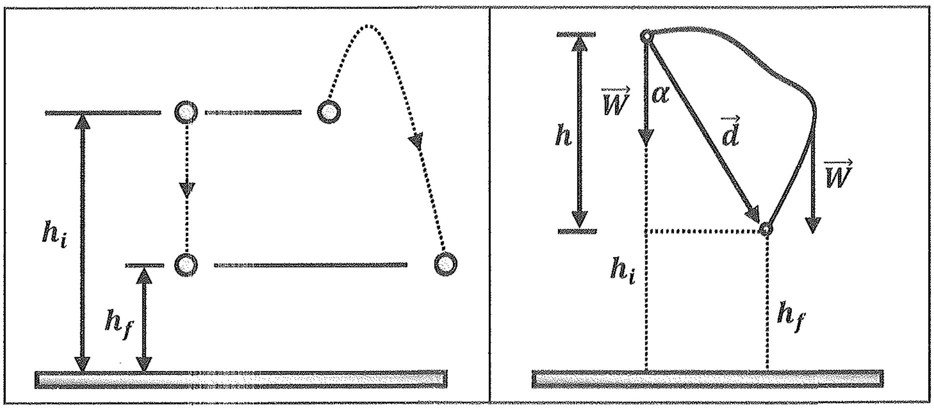
Examples: electrostatic forces, gravitational forces, elastic spring forces, magnetic forces etc.



$$W_{\substack{A\to B \\ Path\,1}} = W_{\substack{A\to B \\ Path\,2}} = W_{\substack{A\to B \\ Path\,3}}$$

The work done by weight along any path:

The work done by gravity, for example, doesn't depend on path since any horizontal motion is perpendicular to the force and doesn't contribute to the work.



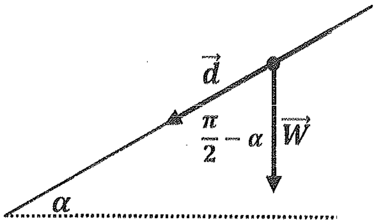
When an object falls from an initial height h_i to a final height h_f, the equation used to calculate the work done by gravity is: $W_{\vec{W}} = \vec{W}.\vec{d} = W \times d \times \cos\alpha = mgh = mg(h_i - h_f)$

If an object initially goes up and then comes down, we need only consider the difference in initial height and final height.

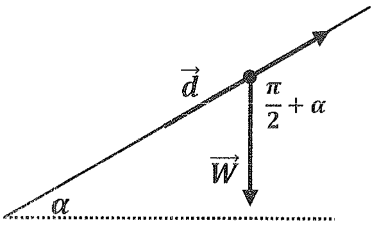
$$W_{\overrightarrow{W}} = mg(h_i - h_f)$$

This expression shows that the work of the weight of a body between two positions is independent of the path followed between these positions. It depends only on the difference in initial height and final height.
If the body moves up: $h_i < h_f \Longrightarrow W_{\overrightarrow{W}} = -mgh$ then the weight resists the motion.
If the body moves down: $h_i > h_f \Longrightarrow W_{\overrightarrow{W}} = mgh$ then the weight helps the motion.

Work done by weight along an inclined plane
First method: $W_{\overrightarrow{W}} = \overrightarrow{W}.\vec{d} = W \times d \times \cos(\overrightarrow{W};\vec{d}) = mg \times d \times \cos(\overrightarrow{W};\vec{d})$

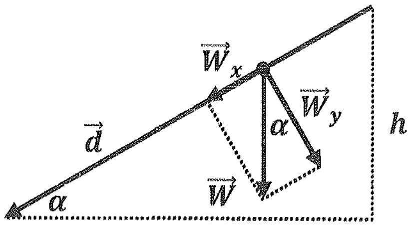


Downwards motion:
$W_{\overrightarrow{W}} = mg \times d \times \cos\left(\frac{\pi}{2} - \alpha\right) = mgd\sin\alpha$

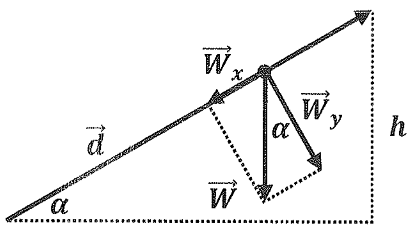


Upwards motion:
$W_{\overrightarrow{W}} = mg \times d \times \cos\left(\frac{\pi}{2} + \alpha\right) = -mgd\sin\alpha$

Second method: by resolving force $\overrightarrow{W}$ into two components.
The magnitudes of the components of weight are $W_x = mg\sin\alpha$ and $W_y = mg\cos\alpha$
$W_{\overrightarrow{W}} = W_{\overrightarrow{W}_x} + W_{\overrightarrow{W}_y} = \overrightarrow{W}_x.\vec{d} + \overrightarrow{W}_y.\vec{d} = W_x \times d \times \cos(\overrightarrow{W}_x;\ \vec{d}) + 0$



Downwards motion:
$(\overrightarrow{W}_x;\ \vec{d}) = 0° \Longrightarrow W_{\overrightarrow{W}} = +mgd\sin\alpha = +mgh$



Upwards motion:
$(\overrightarrow{W}_x;\ \vec{d}) = 180° \Longrightarrow W_{\overrightarrow{W}} = -mgd\sin\alpha = -mgh$

Non-conservative force
A non-conservative force is one for which the work done by or against the force in moving a body from one position to another, depends on the path as well as the initial and final positions.
The work done by non-conservative forces agrees with these conditions:

- The work is irreversible.
- Along a closed path, the work is not zero.

Examples: frictional force, viscous force, air resistance, braking force, traction force of engine, tension in an inextensible strings etc.

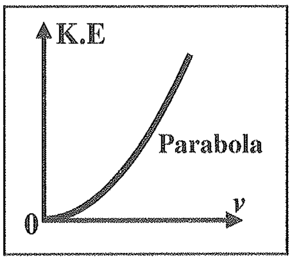
If a body is moved from position A to another position B on a rough table, work done by frictional force $\vec{f}$ depends on the length of the path between A and B and not only on the positions A and B.

$$W_{\vec{f}\,(A\to B)} = -f \times \widehat{AB}$$

Rotational work: the rotational work W_R done by a constant moment $\mathcal{M}$ in turning an object through an angular displacement $\Delta\theta$ is: $W_R = \mathcal{M}\Delta\theta$ which is the analogue of $W = \vec{F}.\vec{d}$ (translational motion).
In SI units, W_R is expressed in joule [J], $\mathcal{M}$ in [Nm] and $\Delta\theta$ in radians [rad].

1.3- KINETIC ENERGY

Kinetic energy of an object is energy of motion. An object in motion along a rectilinear trajectory has **translational kinetic energy**; on the other hand, an object, like the blade of a fixed wind turbine, rotating about an axis has **rotational kinetic energy**.



Translational kinetic energy

A particle of mass m moving with a speed v has kinetic energy:

$$K.E = \frac{1}{2}mv^2$$

In SI units, K.E is expressed in joule [J], m in [kg] and v in [m/s].

For a system of N particles, the total kinetic energy of the system is the sum of the kinetic energy of each particle.

$$K.E_S = K.E_1 + K.E_2 + \cdots + K.E_N = \sum_{i=1}^{N} K.E_i = \sum_{i=1}^{N} \frac{1}{2}m_i v_i^2$$

The kinetic energy of a rigid system/solid (all particles have the same speed V) is:

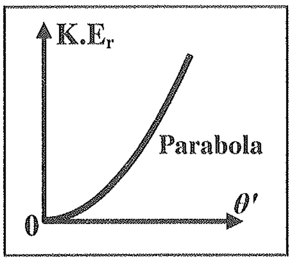
$$K.E_S = K.E_1 + K.E_2 + \cdots + K.E_N = \frac{1}{2}m_1V^2 + \frac{1}{2}m_2V^2 + \cdots + \frac{1}{2}m_NV_N^2$$

$$K.E_S = \frac{1}{2}(m_1 + m_2 + \cdots + m_N)V^2 = \frac{1}{2}MV^2$$

Where $M = m_1 + m_2 + \cdots + m_N$ is the mass of the solid

Conclusion: the kinetic energy of a solid of mass M and speed V is:

$$K.E = \frac{1}{2}MV^2$$

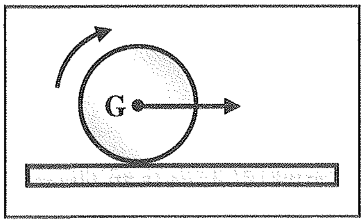


Rotational kinetic energy

The rotational kinetic energy $K.E_r$ of a solid (rigid body) rotating with an angular speed θ' about a fixed axis and having a moment of inertia I is:

$$K.E_r = \frac{1}{2}I\theta'^2$$

In SI units, $K.E_r$ is expressed in joule [J], I in [kgm^2] and θ' in [rad/s]



Note: a rolling object, such as a wheel, is undergoing both translational and rotational motion (combined motion). Consequently, its total kinetic energy is: $K.E = K.E_t + K.E_r = \frac{1}{2}mv^2 + \frac{1}{2}I\theta'^2$ with v is speed of the center of mass G, I is the moment of inertia about an axis passing through G and θ' is the angular speed.
In the case of smooth rolling (without sliding): $v = R\theta'$ with R is the radius.

Work-kinetic energy theorem

The work- kinetic energy theorem states that the algebraic sum of the work done by the forces (external and internal) acting on a system results in a variation in the kinetic energy of the system between two instants t_1 and t_2.

$$\sum W_{\vec{F}} = \Delta K.E \text{ (Internal and external forces)}$$

In case of a rigid system (solid), the algebraic sum of the work done by the internal forces is zero. Then,

$\Sigma W_{\vec{F}_{ext}} = \Delta KE = KE_f - KE_i = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = \frac{1}{2}m(v_f^2 - v_i^2)$ [solid in translation].

If the kinetic energy of the solid increases, work is positive *i.e.* the solid moves in the direction of the force and if the kinetic energy decreases, work will be negative.

By analogy, $\Sigma W_{\vec{F}_{ext}} = \Delta KE = KE_f - KE_i = \frac{1}{2}I\theta_f'^2 - \frac{1}{2}I\theta_i'^2$ [Solid in rotation]

$$F_{net} = ma$$
$$F_{net} = m\frac{dv}{dt} = m\frac{dv}{dx}\frac{dx}{dt}$$
$$F_{net} = m\frac{dv}{dx}v$$
$$F_{net}\,dx = mvdv$$
$$\int_{x_i}^{x_f} F_{net}\,dx = m\int_{v_i}^{v_f} vdv$$
$$W_{net} = m\left[\frac{v^2}{2}\right]_{v_i}^{v_f}$$
$$W_{net} = \Delta K.E$$

1.4- POTENTIAL ENERGY

When two or more objects in a system interact, it is possible to store energy in the system such that the energy can be easily recovered. The Earth and a ball interact by the gravitational force between them. If the ball is lifted up into air, energy is stored in the [ball; Earth] system, energy that can later be recovered as kinetic energy when the ball is released and falls.

A spring is a system made of countless atoms that interact by their atomic "springs" ... This form of stored energy is called potential energy, since it has the potential to be converted into other forms of energy ...

TIP

Potential energy is a property of a system and not a single body alone. The forces due to gravity and springs (conservative forces) are special since they allow for the storage of energy.

Gravitational potential energy

The gravitational potential energy is the hidden energy of a body due to its position relative to the surface of Earth.

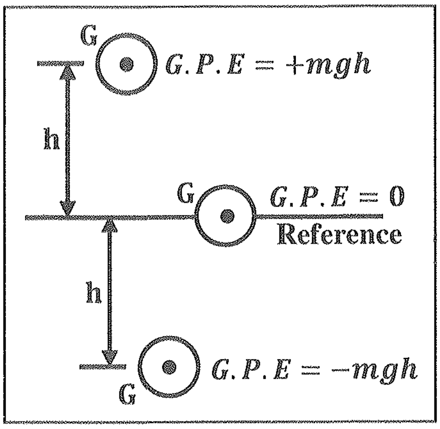
As a result of the gravitational interaction between the body and the Earth, the system [body, Earth] stores a form of energy due to its position relative to Earth called gravitational potential energy.

For a body of mass *m* raised to a height *h* above the Earth's surface, the change in gravitational potential energy is mgh. The system here is the body plus the Earth, and properties of both are involved: body (*m*) and Earth (*g*).

The gravitational potential energy depends on the height or the vertical position of the center of mass of the body relative to some chosen reference level, known as reference level for zero gravitational potential energy. As long as the body experiences the constant gravitational force $m\vec{g}$, the gravitational potential energy can be written:

$$G.P.E = mgh \text{ where } \begin{cases} \text{G.P.E is the gravitational potential energy in [J]} \\ \text{m is the mass of the body in [kg]} \\ \text{g is the gravitational strength in } [m/s^2] \\ \text{h is the height of the center of mass of the body} \\ \text{relative to a chosen reference level in [m]} \end{cases}$$

G.P.E can be positive (center of mass of the body is above the reference level, negative (center of mass of the body is below the reference level or zero (center of mass of the body is at the reference level h = 0).



Variation in gravitational potential energy

Changes in G.P.E do not depend on the path by which a body changes position, but only on its initial and final vertical positions.

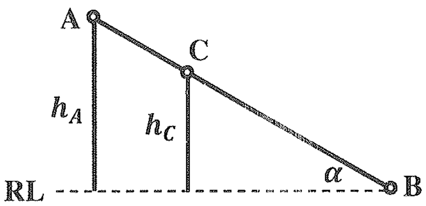
$\Delta G.P.E = G.P.E_f - G.P.E_i = mgh_f - mgh_i = mg(h_f - h_i)$.

Relation between work done by weight and variation in gravitational potential energy

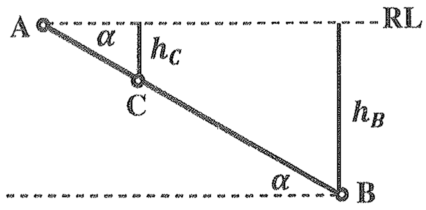
$\Delta G.P.E = mg(h_f - h_i) = -mg(h_i - h_f) = -W_{\overrightarrow{W}}$.

Particular heights

Motion of a particle along an inclined plane

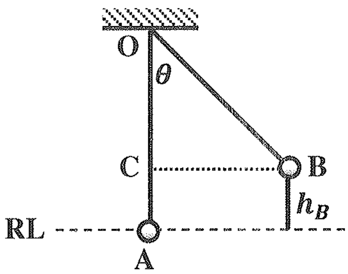


$$AB = L, AC = x \text{ and } CB = L - x$$
$$\sin\alpha = \frac{h_A}{AB} \Rightarrow h_A = AB\sin\alpha = L\sin\alpha$$
$$\sin\alpha = \frac{h_C}{CB} \Rightarrow h_C = CB\sin\alpha = (L - x)\sin\alpha$$
$$h_B = 0m$$

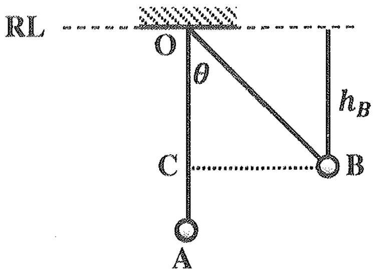


$$AB = L, AC = x \text{ and } CB = L - x$$
$$h_A = 0m$$
$$h_C = -x\sin\alpha$$
$$h_B = -L\sin\alpha$$

Simple Pendulum

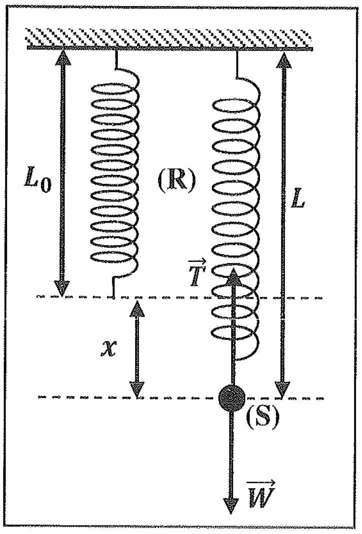


$$OA = OB = L$$
$$\cos\theta = \frac{OC}{OB} \Rightarrow OC = OB\cos\theta = L\cos\theta$$
$$h_A = 0m$$
$$h_B = OA - OC = L - L\cos\theta = L(1 - \cos\theta)$$



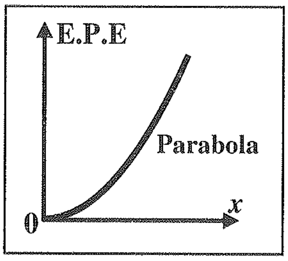
$$h_A = -OA = -L$$
$$h_B = -OC = -L\cos\theta$$

Elastic potential energy
Energy can also be stored in a compressed or stretched spring as elastic potential energy.



Hooke's law: the magnitude of the tension in a spring, of negligible mass and stiffness k, elongated or compressed by $|x|$ is: $T = k|x|$ with $x = \Delta L = L - L_0$ where L_0 and L are the natural (original) and the final lengths of the spring respectively.
In SI units, T is expressed in [N], k in [N/m] and x in [m].
The external forces acting on (S) are weight $\vec{W}$ and tension $\vec{T}$.
(S) is at equilibrium: $\sum \vec{F}_{ext} = \vec{0}$.
$\vec{T} + \vec{W} = \vec{0} \Rightarrow \vec{T} = -\vec{W} \Rightarrow T = W$.

The work done to stretch (or compress) a spring, by a distance x from its equilibrium position, results in storing elastic potential energy (E.P.E) in the spring. The elastic potential energy stored in the spring (elastic bodies) is expressed by:



$$E.P.E = \frac{1}{2}kx^2$$

Relation between work done by tension in an ideal spring and variation in elastic potential energy:
$\Delta E.P.E = E.P.E_f - E.P.E_i = \frac{1}{2}kx_f^2 - \frac{1}{2}kx_i^2 = -W_{\vec{T}}$

TIP

$$W_{\vec{T}} = \int_{x_i}^{x_f} T dx = \int_{x_i}^{x_f} -kx dx$$

$$W_{\vec{T}} = \left[-\frac{kx^2}{2}\right]_{x_i}^{x_f} = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2$$

1.5- MECHANICAL ENERGY OF A SYSTEM

The mechanical energy of a system is the sum of its kinetic energy and its potential energy.
At the macroscopic scale, the mechanical energy of a system [body, Earth] is given by:

$$M.E = K.E + G.P.E + E.P.E$$

Form of energy	symbol
Kinetic energy	K.E or E_k
Gravitational potential energy	G.P.E or $P.E_g$
Elastic energy	E.P.E or $P.E_e$
Mechanical energy	M.E or E_m
Internal energy	U

1.6- INTERNAL ENERGY (MICROSCOPIC ENERGY)

At the microscopic level, however, even if the system is at rest, its particles possess a microscopic kinetic energy due to their thermal agitation and a microscopic potential energy due to their interaction. The sum of these energies is called microscopic mechanical energy or internal energy of the system.

$$U = K.E_{mic} + P.E_{mic}$$

When a system exchanges heat with the surrounding, the variation of its internal energy appears as a variation of its temperature or as a change in its phase.

ATTENTION

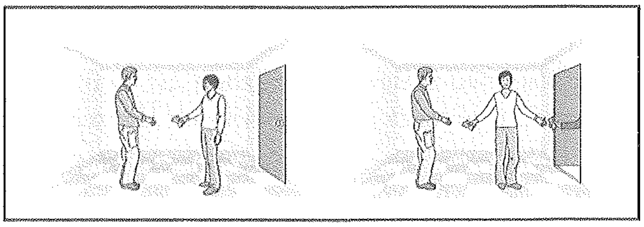
The work done by conservative forces results in modifying the potential energy P.E of the system:
$W_C = -\Delta P.E$

1.7- TOTAL ENERGY OF A SYSTEM AND ITS CONSERVATION

The total energy E of a system is the sum of its internal energy U and its mechanical energy M.E.

$$E = M.E + U$$

Energy-isolated and non-isolated systems



Imagine two people are in an isolated (sealed) room. They may complete as many money transfers as they like but the total amount of money in the room before and after each transfer will be the same. We can say that the total amount of money in this system is conserved, in that it does not change during transactions.

Now imagine that the room is not isolated. In this case, money may be taken out of (or put into) the room so that the total amount of money in the room is not necessarily constant. In this system, it cannot be said that money is conserved. It would be much more complex to keep track of the money transfers that occur in this non-isolated room compared with those occurring in the isolated room.

In physics, when the energy interactions of a group of objects need to be analyzed, we often assume that these objects are isolated from all other objects in the universe. Such a group is called an isolated system.

An isolated system (energy-isolated system) cannot exchange energy or matter with its surroundings. Thus, the total energy is conserved:

$$E = M.E + U = constant$$

$$E_i = E_f \Longrightarrow \Delta E = \Delta M.E + \Delta U = 0 \Longrightarrow \Delta M.E = -\Delta U$$

$\Delta M.E$ and ΔU represent the variations of the mechanical and internal energies respectively.

A non-isolated system is a system in which there is an energy exchange with the surroundings.

1.8- ENERGY ISOLATED SYSTEM WITH CONSERVED MECHANICAL ENERGY

For an energy-isolated system, in presence of conservative forces and absence of non-conservative forces, the sum of kinetic and potential energies at any point remains constant throughout the motion; i.e. the mechanical energy at any point remains constant. This is known as the law of conservation of mechanical energy. Moreover, the internal energy of the system doesn't vary, $\Delta U = 0$:

$\Delta E = \Delta M.E + \Delta U = 0$

$$\text{But } \Delta U = 0 \Longrightarrow \Delta M.E = 0 \Longrightarrow M.E_i = M.E_f$$

TIP

$M.E = constant$

$\frac{dM.E}{dt} = 0$

Relationship between kinetic energy and potential energy in an energy-isolated system of conserved mechanical energy

$M.E = K.E + P.E \Longrightarrow \Delta M.E = \Delta K.E + \Delta P.E = 0 \Longrightarrow \Delta K.E = -\Delta P.E$

The variation of the kinetic energy is equal to the opposite of the variation of the potential energy i.e. if the kinetic energy of the body increases its potential energy will decrease by an equal amount and vice-versa).

TIP

Conservative forces in an isolated system do not affect the M.E of the system

Conclusion: for an energy-isolated system, where non-conservative forces (friction, air resistance, traction…) and chemical (or nuclear) reactions are absent, the mechanical energy is conserved.

1.9- ENERGY ISOLATED SYSTEM WITH VARYING MECHANICAL ENERGY

If a system is subjected to non-conservative forces (internal or external), the mechanical energy is no more constant. It changes by the amount of work done by the non-conservative forces W_{NC}.

$$\Delta K.E + \Delta P.E = \Delta M.E = W_{NC}$$

Energy-isolated system with friction as internal force

If a system is subjected to friction, the mechanical energy is no more constant and the internal energy of the energy isolated system varies, then $\Delta U \neq 0$.

The amount of work done by friction $W_{\vec{f}}$ will cause the mechanical energy of the system to change so that:

$$\Delta M.E = W_{\vec{f}} \Longrightarrow M.E_f - M.E_i = W_{\vec{f}} \Longrightarrow M.E_i + W_{\vec{f}} = M.E_f$$

Friction always reduces the mechanical energy of a system which in turn leads to an increase in the internal energy of the system.

The lost energy is transformed into heat and the thermal energy developed is exactly equal to loss in mechanical energy: $Q = |\Delta M.E|$.

The total energy of an energy-isolated system is conserved:

$\Delta E = 0 \Longrightarrow \Delta M.E + \Delta U = 0 \Longrightarrow \Delta M.E = -\Delta U$

If frictional forces exist within an energy-isolated system, then the variation of the M.E is equal and opposite to the variation of the internal energy of the system.

$\Delta M.E = W_{\vec{f}}$ and $\Delta M.E = -\Delta U$ so $\Delta M.E = -\Delta U = W_{\vec{f}}$.

TIP

$\sum W_{ext} = W_C + W_{NC}$

W_C: work done by conservative forces

W_{NC}: work done by non-conservative forces

Work-kinetic energy theorem for a solid:

$$\Delta K.E = \sum W_{ext}$$
$$\Delta K.E = W_C + W_{NC}$$
$$\Delta K.E = -\Delta P.E + W_{NC}$$
$$\Delta K.E + \Delta P.E = W_{NC}$$
$$\Delta M.E = W_{NC}$$

If $W_{NC} = 0 \Longrightarrow \Delta M.E = 0 \Longrightarrow M.E = constant$

If $W_{NC} \neq 0 \Longrightarrow \Delta M.E \neq 0$ and $\Delta M.E = W_{NC}$

But $\Delta M.E = -\Delta U \Longrightarrow \Delta U = -W_{NC}$

Notes:

- Even though friction may seem like an internal force, its effect is to allow energy to escape from a system as heat. The system is, by definition, a non-isolated system. Thus, an isolated system must also be frictionless.
  To be compatible with the curriculum in Lebanon, we will identify a system as energy-isolated system if it is subjected to conservative forces or if it is also subjected to frictional forces on the condition that these frictional forces are considered as internal forces.
- Not all external forces remove mechanical energy from a system. Motors, in general, are used to add mechanical energy to a system. More generally, if several external forces (A, B, C, . . .), as well as friction, act on a system, then the total work done by all of these forces produces the change in mechanical energy.

$$\Delta M.E = W_A + W_B + W_C + \cdots = W_{NC}$$

1.10- MECHANICAL POWER

We studied how energy can be transformed from one kind to into another and how it can be transferred between the environment and the system as work. How quickly is the energy transformed or transferred? So, when we raise the issue of how fast the energy is transformed, we are talking about the rate of transformation of energy.

Power is defined as the rate at which work is done by a force. In a more general sense, power P is the rate at which energy is transferred by a force from one type to another. If an amount of energy ΔE is transferred in an interval of time Δt, the average power due to the force is:

$$P_{av} = \frac{\Delta E}{\Delta t} = \frac{W_{\vec{F}}}{\Delta t} = \frac{\vec{F}.\vec{d}}{\Delta t} = \vec{F}.\vec{V}_{av}$$

The instantaneous power due to the force is:

$$P = \lim_{\Delta t \to 0} P_{av} = \lim_{\Delta t \to 0} \frac{W}{\Delta t} = \frac{dW}{dt} \left(P = \frac{dE}{dt}\right)$$

$$P = \lim_{\Delta t \to 0} P_{av} = \lim_{\Delta t \to 0} \vec{F}.\vec{V}_{av} = \vec{F}.\vec{V} = F \times V \times \cos(\vec{F};\vec{V})$$

The SI unit of power is expressed in watt [W] where 1W = 1J/s.

The English unit of power is the horsepower [hp] where 1hp = 746W.

$$\mathbf{\Delta M.E = W_{NC} \Longrightarrow \frac{\Delta M.E}{\Delta t} = \frac{W_{NC}}{\Delta t} = P_{NC(av)}}$$
$$\mathbf{P_{NC(inst)} = \frac{dM.E}{dt}}$$

Rotational motion:

Average power: $P_{av} = \mathcal{M}\theta'_{av}$

Instantaneous power: $P = \mathcal{M}\theta'$

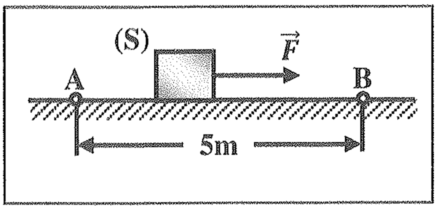
θ' is the angular velocity in [rad/s]

CHAPTER 1 – WORK AND MECHANICAL ENERGY
EXERCISES AND PROBLEMS

Whenever needed, take the magnitude of the gravitational acceleration $g = 10 m/s^2$ and neglect the effect of air resistance except as otherwise indicated.

Exercise 1:

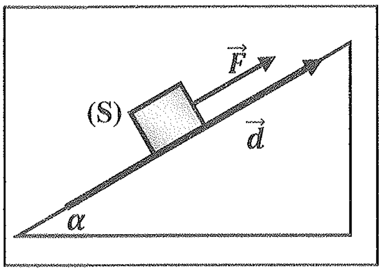
A solid (S), supposed as a particle, is pulled along a horizontal rough surface by means of a horizontal force $\vec{F}$ of magnitude F = 10N as shown in the adjacent document. The force of friction between the surface and (S) opposes its motion and is assumed constant of magnitude $f = 2N$.



1- Name and represent the external forces acting on (S).
2- Calculate the work done by each force along the displacement $\overrightarrow{AB}$; then, deduce the net work on (S).

Exercise 2*:

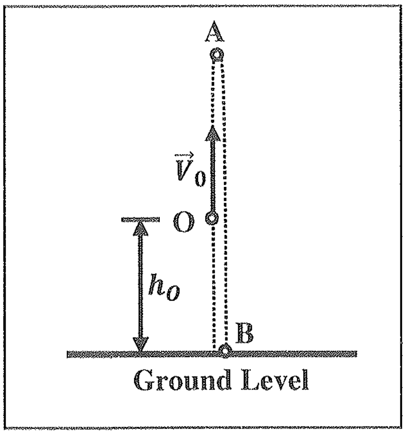
A solid (S), supposed as a particle of mass m = 2kg, is pulled up an inclined plane of inclination $\alpha = 30°$ by means of a force $\vec{F}$ parallel to the inclined pane and of magnitude $F = 20N$ as shown in the adjacent document.
The force of friction between the inclined plane and (S) opposes its motion and is assumed constant of magnitude $f = 5N$.



1- Name and represent the external forces acting on (S).
2- Calculate the work done by each force along a displacement $\vec{d}$ of magnitude $d = 2m$; then, deduce the net work on (S).

Exercise 3*:

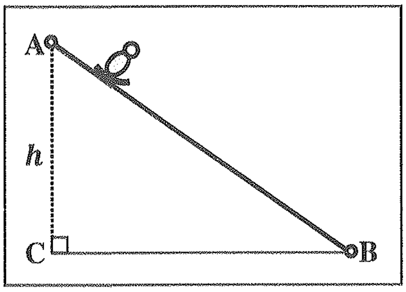
A solid (S), taken as a particle of mass $m = 2$kg, is launched vertically upwards from point O with a velocity vector $\vec{V}_0$ of magnitude $V_0 = 20m/s$ as shown in the adjacent document.
Point O is at a height $h_O = 25m$ above the ground level which is taken as a gravitational potential energy reference.



1- Calculate at point O:
   1.1- the gravitational potential energy of the system [(S); Earth],
   1.2- the kinetic energy of (S),
   1.3- the mechanical energy of the system [(S); Earth].
2- Is the mechanical energy of the system [(S); Earth] conserved? Justify.
3- (S) reaches its maximum position at point A. Determine, at point A, the gravitational potential energy of the system [(S); Earth]; then, deduce the height relative to the ground level.
4- Determine the speed of (S) at point B.

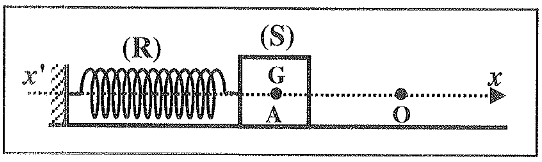
Exercise 4:

A child, of mass M = 45kg, starts from rest from the point A of an inclined track AB as shown in the adjacent document (AC = h = 80m).
The horizontal plane through BC is taken as a gravitational potential energy reference.



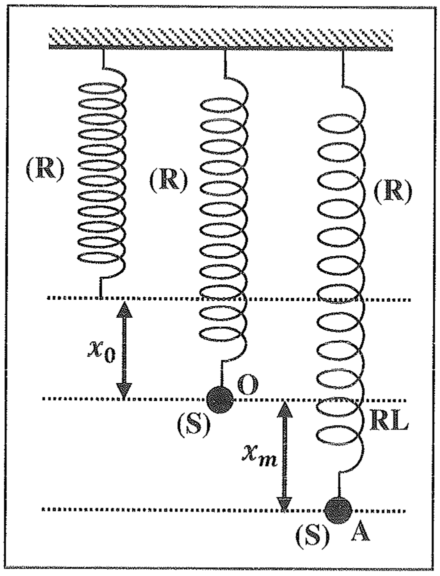
1- **Calculate at A:**
 1.1- the gravitational potential energy of the system (S) formed of the child and the Earth,
 1.2- the mechanical energy of the system (S).
2- **Neglect the forces of friction.**
 2.1- Determine the value of the mechanical energy of the system (S) at B.
 2.2- Deduce that the speed of the child at B is $V_B = 40m/s$.
3- **In reality, the child reaches B with the speed $V'_B = 35m/s$.**
 3.1- Calculate then the mechanical energy of the system (S) at B.
 3.2- Determine the variation in the mechanical energy of (S) when it passes from A to B.
 3.3- Deduce the value of the energy dissipated due to the forces of friction between A and B.
 3.4- In what form does this dissipated energy appear?
 3.5- Consider the system [(S); inclined track; atmosphere].
  3.5.1- Specify whether the system [(S); inclined track; atmosphere] is energy-isolated or not.
  3.5.2- Calculate the variation of the internal energy of the system [(S); inclined track; atmosphere] between A and B.

Exercise 5*:



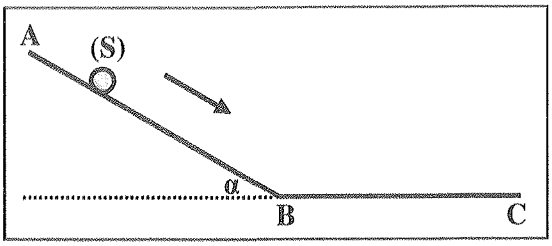
A solid (S), of mass m = 250g and center of mass G, is connected to the extremity of a horizontal spring (R) of negligible mass and stiffness k = 100N/m; the other extremity of the spring is fixed to a support. The solid (S) is placed on frictionless horizontal track. At equilibrium, G coincides with O the origin of the axis $x'x$. we shift (S) to the left starting from O; G occupies the position A such that $x_A = \overline{OA} = -10cm$. At the instant $t_0 = 0s$, (S) is released without initial velocity.
At the instant t, the abscissa of G is $x = \overline{OG}$ and the algebraic value of its velocity is $v = \frac{dx}{dt}$.
The horizontal plane passing through G is taken as a gravitational potential energy reference.
1- Calculate the mechanical energy of the system [(S); (R); Earth] at the instant $t_0 = 0s$.
2- Determine the speed of (S) at O
3- Write, at the instant t, the expression of the mechanical energy of the system [(S); (R); Earth]; then, deduce the expression of the acceleration a of (S) in terms of m, k and x.

Exercise 6:



A spring (R) of stiffness k = 5N/m and of negligible mass is connected from its upper end to a fixed support while its other end carries a particle (S) of mass m = 20g. At equilibrium, the spring elongates by x_0 as shown in the adjacent document.
The spring is stretched by pulling (S) vertically downwards by $x_m = 6cm$ from its equilibrium position O; then, (S) is released without initial velocity from A at the instant $t_0 = 0s$.
The horizontal plane passing through O is taken as a gravitational potential energy reference. All forces of friction are neglected.
1- Name and represent the external forces acting on (S) at equilibrium.
2- Determine the value of x_0.
3- Is the mechanical energy of the system [(S); (R); Earth] conserved? Justify.
4- Determine the speed of (S) as it passes by its equilibrium position O.

Exercise 7*:



A solid (S), taken as a particle of mass m = 1kg, is released from point A without initial speed. The solid slides down a frictionless inclined plane of length AB = 40m that makes an angle $\alpha = 30°$ with the horizontal. (S) continues its motion along a horizontal rough plane of length BC = 25m as shown in the adjacent document.

The horizontal plane passing through BC is taken as a gravitational potential energy reference.

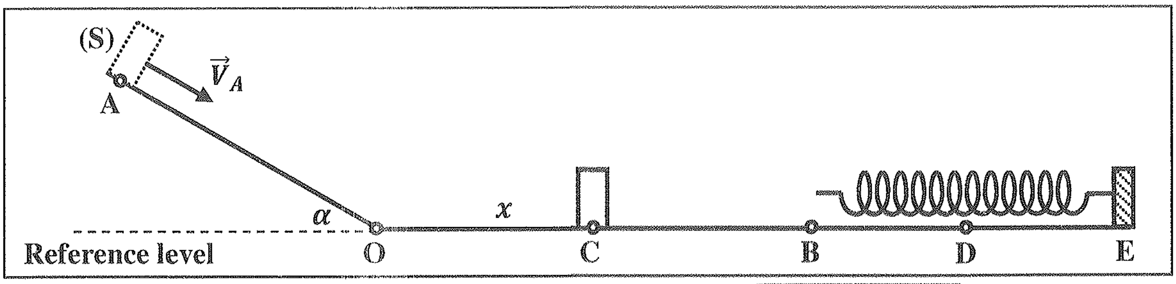
1- Calculate at point A:
   - **1.1-** the gravitational potential energy of the system [(S); Earth],
   - **1.2-** the mechanical energy of the system [(S); Earth].
2- Is the mechanical energy conserved along AB? Justify.
3- Determine the speed of (S) as it reaches B.
4- Is the mechanical energy conserved along the path BC? Justify.
5- Knowing that (S) stops at point C.
   - **5.1-** Calculate the variation in the mechanical energy between B and C.
   - **5.2-** What is the reason behind the loss of energy?
   - **5.3-** In what form does this energy appear? Determine its value.
   - **5.4-** Calculate the magnitude of the friction force f between BC and (S).
6- Consider the system [(S); inclined plane; Earth; atmosphere]. Calculate the variation of the internal energy of the system [(S); inclined plane; Earth; atmosphere].

Exercise 8:

A particle (S), of mass $m = 2$kg, is laucnhed from the top A of a frictionless inclined plane AO, of length $L = 1.6m$ and inclination $\alpha = 30°$, with an initial veloctiy vector $\vec{V}_A$ parallel to AO and of magnitude $V_A = 3m/s$. As (S) reaches O, it continues its motion along a horizontal rough plane OB of 2m length. At an instant t, (S) passes by the position C between O and B. The position of (S) at C is $x = \overline{OC}$ relative to O and the algebraic value of its velocity is $V = \frac{dx}{dt}$.

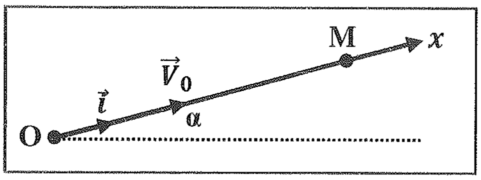
The force of friction between OB and (S) opposes its motion and is assumed constant of magnitude $f = 8N$.

The horizontal plane passing through OB is taken as a gravitational potential energy reference.



1- Show that the expression of the speed V_0 at O is given by: $V_0 = \sqrt{V_A^2 + 2gL\sin\alpha}$; then, deduce its numerical value.
2- Apply the work kinetic energy theorem to establish the following relation: $V^2 + 8x - 25 = 0$.
3- Deduce the value of the acceleration of (S) along OB.
4- Caculate the speed of (S) at B.
5- As (S) reaches B, it is attached to the free end of an un-streteched spring of neglible mass and stiffness $k = 50N/m$. The other end of the spring is fixed to a support at E. Neglect friction along BE. Determine the maximum compression $x_m = BD$ of the spring knowing that (S) stops at D.

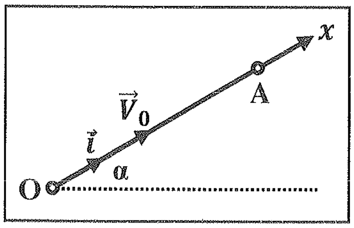
Exercise 9:



Consider an inclined plane that makes an angle α with the horizontal ($\sin\alpha = 0.25$) and a particle (S) of mass $m = 200$g.
We intend to study the energy exchange between the system [(S); Earth] and the environment.
To do that, (S) is given, at the instant $t_0 = 0$, the velocity $\vec{V}_0 = V_0\vec{i}$ along the line of greatest slope Ox where $V_0 = 5\text{m.s}^{-1}$
The horizontal plane passing through point O is taken as the gravitational potential energy reference.

1- The forces of friction are supposed negligible.
   1.1- Determine the value of the mechanical energy M.E of the system [(S); Earth] at O.
   1.2- At the instant t, (S) passes through a point M of abscissa OM = x. Determine, as a function of x, the expressions of the mechanical energy M.E, and the gravitational potential energy $P.E_g$ of the system [(S); Earth] when (S) passes through M.
   1.3- Determine the expression of the kinetic energy K.E of (S) at M by using two methods.
   1.4-
      1.4.1- Trace, on the same system of axes, the curves representing the variations of the energies M.E, $P.E_g$ and K.E as a function of x.
      Scale: - on the axis of abscissas: 1div represents 1m;
      - on the axis of energy: 1div represents 0.5J.
      1.4.2- Determine, using the graph, the speed of (S) for x = 3m.
      1.4.3- Determine, using the graph, the value of x_m of x for which the speed of (S) is zero.
2-
   2.1- In reality, the speed of (S) becomes zero at a point of abscissa x = 3m. The forces of friction are no longer negligible. Calculate then the work done by the forces of friction between x = 0 and x = 3m.
   2.2- The system [(S); Earth] thus exchanges energy with its environment. In what form and by how much?
3- Consider the system [(S); inclined plane; Earth; atmosphere]. Calculate the variation of the internal energy of the system [(S); inclined plane; Earth; atmosphere] between x = 0 and x = 3m.

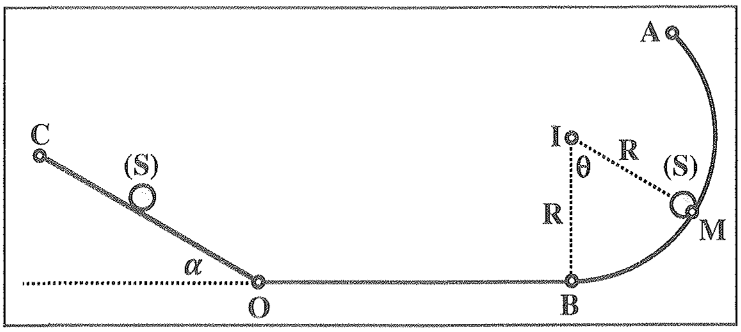
Exercise 10*:



A solid (S), taken as a particle of mass m = 0.1kg, is launched at t_0 = 0s from point O with an initial velocity $\vec{V}_0 = V_0\vec{i}$ ($V_0 = 10m/s$) along the line of greatest slope Ox of an inclined plane that makes an angle α = 30° with the horizontal.
The horizontal plane passing through O is taken as a gravitational potential energy reference.
The force of friction $\vec{f}$ between the inclined plane and (S) opposes its motion and is assumed constant of magnitude $f = 2N$.
At the instant t, (S) passes by the point A such that OA = x.

1- The system [(S); Earth] gains gravitational potential energy. Why?
2- Calculate the value of the mechanical energy of the system [(S); Earth] at O.
3- Determine at A and as a function of x the expression of the:
   3.1- gravitational potential energy G.P.E of the system [(S); Earth],
   3.2- kinetic energy K.E of (S) by applying the work energy theorem,
   3.3- mechanical energy M.E of the system [(S); Earth] by using two methods.
4- Trace, on the same system of axes, the curves representing the variations of the energies M.E, G.P.E and K.E of (S) as a function of x.

Exercise 11:



We construct, in the vertical plane, a track COBA formed of three rails:
- OC: a frictionless inclined rail of length CO = 1.6m and making an angle $\alpha = 30°$ with the horizontal.
- OB: a horizontal rough rail of length OB = 2m.
- BA: a smooth circular rail of center I and radius R = 1m.

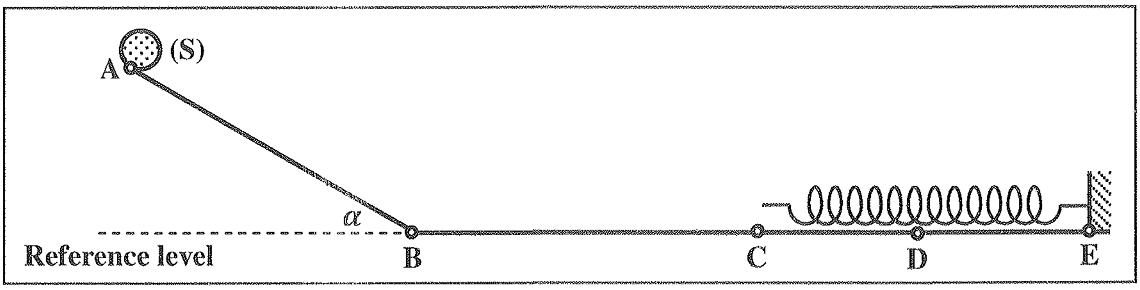
An object (S), taken as a particle of mass $m = 500$g, is released from C without initial speed.
The horizontal plane containing O and B is taken as a gravitational potential energy reference.
The frictional force $\vec{f}$ between OB and (S) opposes its motion and is assumed constant of magnitude f.

1- Verify that the speed of (S) at O is $v_0 = 4m/s$.
2- (S) moves along the rail OB and reaches B with a speed $v_B = 2m/s$.
Determine, by applying the work-kinetic energy theorem, the magnitude of the force of friction f.
3- As (S) moves along the circular rail BA, it passes through the position M where it makes an angle θ with respect to the vertical as shown in the above document.
3.1- Calculate the mechanical energy of (S) at B.
3.2- Determine the expression of the gravitational potential energy of (S) at M as a function of m, g, R, and θ.
3.3- Determine, by applying the principle of conservation of mechanical energy, θ knowing that M is the highest position reached by (S).

Exercise 12*:

Consider a track ABE situated in the vertical plane and formed of two rails:
- AB: a frictionless inclined rail of length L = AB = 2.5m and making an angle $\alpha = 30°$ with the horizontal.
- BE: a rough horizontal rail where the force of friction $\vec{f}$ between BE and (S) opposes its motion and is assumed constant of magnitude $f = 2N$.

A solid (S), taken as a particle of mass m = 500g, is released without initial speed from the point A.
The horizontal plane passing through BE is taken as a gravitational potential energy reference.



1- Apply the principle of conservation of mechanical energy to show that the expression of the speed of (S) as its passes by the point B is $V_B = \sqrt{2gL\sin\alpha}$; then, deduce its value.
2- The solid (S) reaches point C of the horizontal rail BE with a speed $V_C = 3m/s$.
2.1- Name and represent the external forces acting on (S) along BC.
2.2- Determine, by applying the work-kinetic energy theorem, the distance BC.
3- At C, (S) is attached to the free end of an un-stretched spring (R) of negligible mass and stiffness k = 10N/m. The other end of the spring is fixed to a support. (S) reaches point D where it stops.
Determine the maximum compression $x_m = CD$.

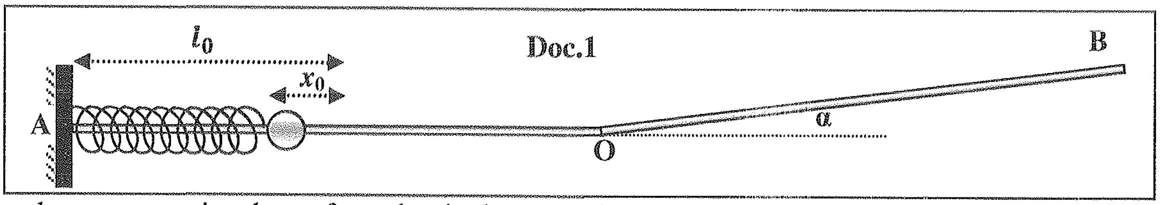
Exercise 13:

A solid (S), taken as a particle of mass m = 200g, is free to move on a track AOB lying in a vertical plane. In document 1, the track is formed of two parts: the first one AO is straight and horizontal and the other OB is straight and inclined by an angle α with respect to the horizontal (sin α = 0.1). Along the part AO, (S) moves without friction, and along the part OB, (S) is acted upon by a force of friction $\vec{f}$ that is assumed constant and parallel to the path.

The object of this exercise is to determine the magnitude f of the force $\vec{f}$ of friction.

Part A: Launching the solid

In order to launch this solid on the part AO, we use a spring of constant k = 320N/m and of free length l_0; one end of the spring is fixed at A to a support. We compress the spring by x_0; we place the solid next to the free end of the spring and then we release them. When the spring attains its free length l_0, the solid leaves the spring with the speed $V_0 = 8m/s$; it thus slides along the horizontal part and then rises up at O the inclined part OB.



1- Apply the conservation law of mechanical energy between the instant of maximum compression of the spring and the instant the solid leaves the spring to show that the value of x_0 is 20cm.

2- The solid reaches O with the speed V_0 = 8m/s. Justify.

Part B: Motion of the solid along the inclined part OB

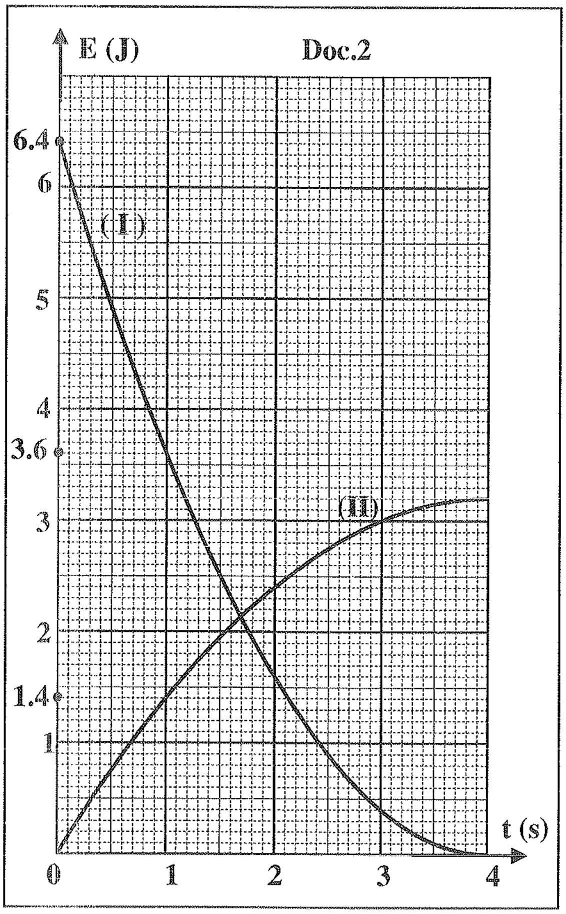
(S) moves, at O, up the inclined part OB with the speed V_0 = 8m/s at the instant t_0 = 0. A convenient apparatus is used to trace, as a function of time, the curves representing the variations of the kinetic energy K.E of the solid and the gravitational potential energy $P.E_g$ of the system (solid - Earth). These curves are represented in document 2 between the instants t_0 = 0 and t_4 = 4s, according to the scale:

1 division on the time axis corresponds to 1s

1 division on the energy axis corresponds to 1J.

The horizontal plane through point O is taken as gravitational potential energy reference.

Take g = 10m/s^2.



1- The curve (I) represents the variation of the kinetic energy K.E of (S) as a function of time. Why?

2- Using the curves:

2.1- Specify the form of the energy of the system at the instant t_4 = 4s.

2.2- Determine the maximum distance covered by the solid along the part OB.

2.3-

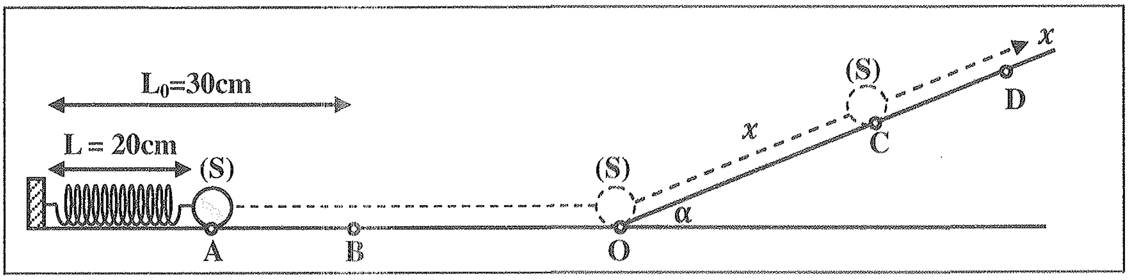
2.3.1- Refer to the graph to complete the table with the values of the mechanical energy M.E of the system at each instant t.

t (s)	0	1	2	3	4
M.E (J)		5			

2.3.2- Justify the existence of a force of friction $\vec{f}$.
2.3.3- Calculate the variation in the mechanical energy of the system between the instants $t_0 = 0$ and $t_4 = 4s$.
2.3.4- Determine f.

Exercise 14:

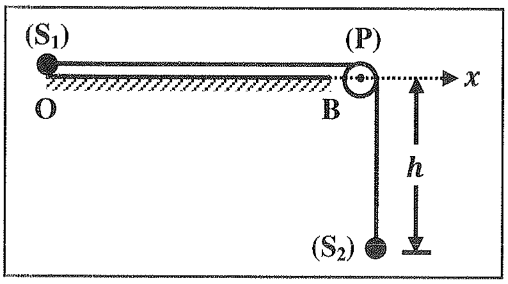
A solid (S), taken as a particle of mass $m = 0.2kg$, is connected to the free end of a horizontal spring of negligible mass, stiffness $k = 180N/m$ and natural length $L_0 = 30cm$. The other end of the spring is fixed to a vertical support. We shift (S) from its equilibrium position B, it occupies the position A such that the length of the spring becomes L = 20cm. At the instant $t_0 = 0s$, (S) is released without initial velocity from A; thus, its moves along the frictionless rail AB and reaches B with a velocity $\vec{V}_B$. (S) continues its path along a rough horizontal rail BO of 50cm length. As (S) reaches O with a velocity $\vec{V}_0$, it climbs a smooth inclined plane OD making an angle $\alpha = 30°$ with the horizontal.
At C, the position of (S) relative to O is $x = \overline{OC}$ and the algebraic value of its velocity is v.
The force of friction $\vec{f}$ between (S) and the rail BO is supposed to be constant of magnitude $f = 1N$.
The horizontal plane containing AB is taken as a reference level for the gravitational potential energy of the system [(S); Earth]. Take $g = 10m/s^2$.



1- **Theoretical Study**
  1.1- Determine the speed V_B of (S) as it passes through the position B.
  1.2- Verify that the speed of (S) as it reaches O is $V_0 = 2m/s$.

2- **Graphical Study of the motion between O and D**
  2.1- Calculate the mechanical energy of the system [(S); Earth] as it moves along the inclined plane.
  2.2- Determine, at C, the expressions of the gravitational potential energy and the kinetic energy of (S) in terms of x.
  2.3- Trace, on the same system of axes, the curves representing the variations of the energies M.E, G.P.E and K.E of (S) as a function of x.
  2.4- Determine using two methods:
    2.4.1- the distance covered by (S),
    2.4.2- the distance where the gravitational potential energy is equal to the kinetic energy of (S).

Exercise 15:



A particle (S_1), of mass m_1, is placed on a horizontal plane OB. (S_1) is connected to a particle (S_2), of mass m_2, by means of a light and inextensible string that passes over the groove of a light pulley (P) as shown in the adjacent document.
(S_1) is at the origin O of the space reference system Ox confounded with OB and (S_2) hangs vertically at a distance h relative to OB.
At the instant $t_0 = 0s$, the system (S), formed of [(S_1); (S_2); the string; and (P)], is released from rest. Thus, (S_1) moves along OB and (S_2) descends vertically.
The position of (S_1), at the instant t, is given by $x = \overline{OS_1}$ and the algebraic value of its velocity is $v = \frac{dx}{dt}$.
The horizontal plane passing through OB is taken as a gravitational potential energy reference for the system [S; Earth] and all forces of friction are neglected.

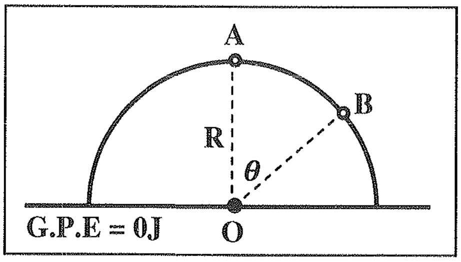
Part I: Energetic study

1- Write, at the instant $t_0 = 0s$, the expression of the mechanical energy of the system [S; Earth] as a function of $m_2, g,$ and h.
2- Determine, at the instant t, the expression of the mechanical energy of the system [(S); Earth] as a function of $m_1, m_2, g, h, x,$ and v.
3- Applying the principle of conservation of mechanical energy, show that: $v^2 = \frac{2m_2 g}{m_1+m_2}x$.
4- Deduce the expression of the common acceleration a of (S_1) and (S_2).

Part II: Dynamical study

1- Redraw a diagram of the above document and show on it the external forces acting on (S_1) and (S_2). (The tension in the string acting on (S_1) is denoted by $\vec{T_1}$ of magnitude T_1 and that acting on (S_2) is denoted by $\vec{T_2}$ of magnitude T_2).
2- Applying the theorem of the center of mass on each particle, determine the expressions of T_1 and T_2 in terms of $m_1, m_2, g,$ and a.
3- Knowing that $T_1 = T_2$, deduce the expression of a.

Exercise 16:

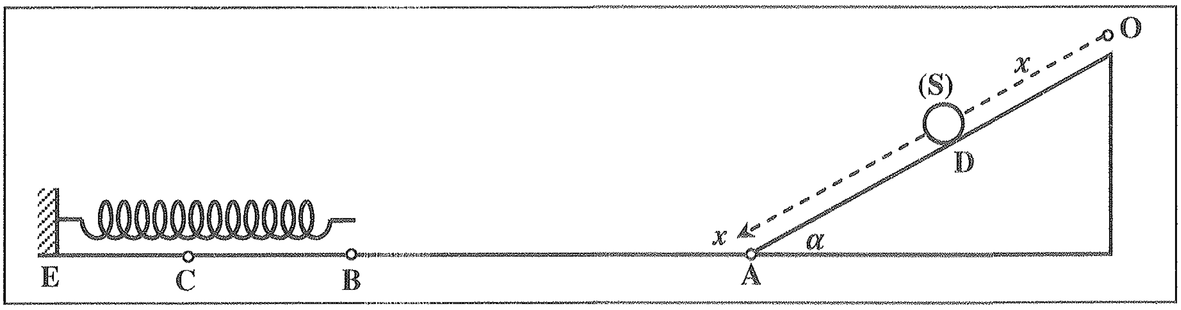


Adam, taken as a particle of mass m, is initially seated on the top A of a hemispherical ice mound of radius $R = 15m$ as shown in the adjacent document. Starting from rest, he begins to slide down the ice which is assumed frictionless.
Take the horizontal plane containing the center O of the hemispherical ice mound as a reference level for the gravitational potential energy.

1- Apply the principle of conservation of mechanical energy to determine the expression of V_B the speed of Adam at B in terms of $R, g,$ and θ.
2- Apply Newton's 2nd law of translational motion to determine the expression of N, the magnitude of the normal reaction exerted by ice on Adam at B in terms of $m, g,$ and θ.
3- At what height does Adam lose contact with the ice?

Exercise 17*:

At the instant $t_0 = 0s$, a solid (S), taken as a particle of mass $m = 0.4kg$, is released without initial velocity from the top O of a frictionless inclined plane OA of inclination $\alpha = 30°$ and length $l = 40cm$.
At the instant t, (S) reaches D at a position x relative to O which is considered as an origin for the space reference system Ox.
The horizontal plane containing AB is taken as a gravitational potential energy reference for the system [(S), Earth].



1- Motion along OA

1.1- The mechanical energy M.E of the system [(S); Earth] is conserved along OA. Justify.
1.2- Calculate the mechanical energy of the system [(S); Earth] at O; then, deduce its value at D.
1.3- Show that the expression of the gravitational potential energy of the system [(S); Earth] at D is $G.P.E_D = 0.8 - 2x$.
1.4- Deduce the expression of the kinetic energy K.E of (S) at D.
1.5- Copy and complete the table below.

$x\,[m]$	0	0.1	0.2	0.3	0.4
$K.E\,[J]$					
$G.P.E\,[J]$					
$M.E\,[J]$					

2.5- Trace, on the same system of axes, the curves representing the variations of the energies M.E, G.P.E and K.E of (S) as a function of x.

Scale: $\begin{cases} \text{Horizontal axis: 1div} \rightarrow 0.1\text{m} \\ \text{Vertical axis: 1div} \rightarrow 0.2\text{J} \end{cases}$

1.6- Calculate the speed of (S) as it passes by A.

2- Motion along AB

As (S) reaches A, it moves along a rough horizontal plane AB of 50cm length. The magnitude of the force of friction between A and B is supposed to be constant of magnitude $f = 1.2N$.

2.1- Name and represent the external forces acting on (S).
2.2- Verify, by applying the work kinetic energy theorem, that the speed of (S) at B is $v_B = 1m/s$.

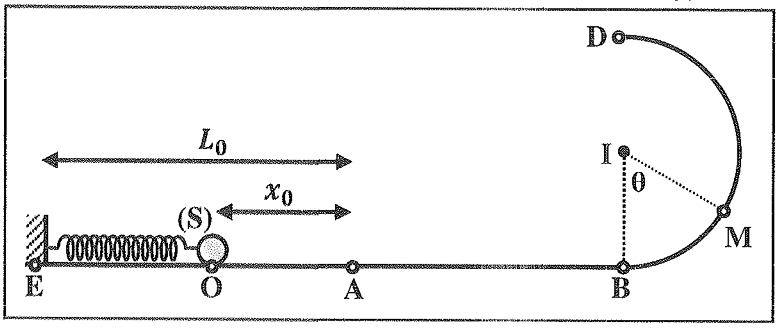
3- Motion along BE

As (S) reaches B, it is attached to an un-stretched spring of stiffness k = 10N/m and negligible mass. (S) continues its motion along the smooth rail BE where it stops at C.
Determine the maximum compression $x_m = BC$ of the spring.

Exercise 18:

A solid (S), taken as a particle of mass m = 250g, is free to move on a frictionless track EBD in a vertical plane. In the document below, the track is formed of two parts: the first one EB is straight and horizontal and the other BD is circular of radius OB = R = 40cm. in order to launch (S) on the track EB, we use a massless spring of stiffness k = 100N/m and free length L_0.

The horizontal plane passing through EB is taken as a gravitational potential energy reference.

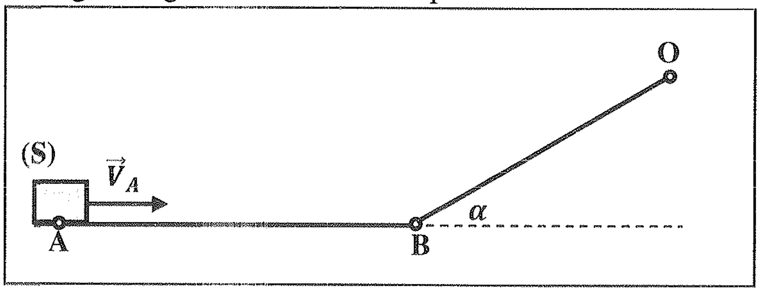


1- One end of the spring is fixed at E to a support. We compress the spring by x_0 = 10cm; we place (S) next to the free end of the spring and then we release them from the position O at the instant t_0 = 0s.
   1.1- Calculate the mechanical energy M.E of the system [(S); spring; Earth] at the instant t_0 = 0s.
   1.2- Determine the speed of (S) at the position A.
2- The solid (S) leaves the spring and moves along AB.
   2.1- What is the speed of (S) at the position B? Justify.
   2.2- (S) then moves on the circular part. (S) stops at a point C on BD where it turns back. Determine the height of point C.
   2.3- Deduce the maximum angular displacement θ_m of (S).
3- After reaching C, (S) moves down along CB.
   3.1- Determine, at a point M between C and B, the expression of gravitational potential energy G.P.E of the system [(S); Earth] in terms of the angle θ.
   3.2- Deduce the expression of the kinetic energy K.E of (S).
   3.3- Determine the value of θ where the gravitational potential energy is equal to the kinetic energy of the system.
4- Trace, on the same system of axes, the curves representing the variations of the energies M.E, G.P.E and K.E of (S) as a function of x.

Exercise 19*:

The adjacent document represents a track ABO formed of a horizontal rough rail of length AB = 4m and a frictionless inclined rail of length BO = 3.9m and making an angle $\alpha = 30°$ with respect to the horizontal.

A solid (S), taken a particle of mass m = 0.4kg, is launched at A with a horizontal velocity of magnitude $V_A = 10m/s$.



Given:
- The horizontal plane passing through A and B is taken as a gravitational potential energy reference.
- The magnitude of the gravitational acceleration is $g = 10m/s^2$.
- The force of friction between AB and (S) opposes it motion and is assumed constant of magnitude $f = 1.8N$.

1- **Motion between A and B**

1.1- Name and represent the external forces acting on (S).

1.2- Apply the work-kinetic energy theorem to show that the speed of (S) at B is: $V_B = 8m/s$.

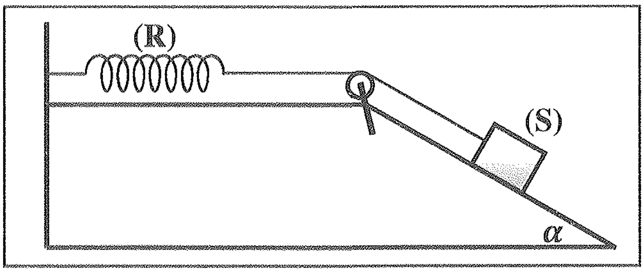
2- **Motion between B and O**

2.1- Is the mechanical energy of the system [(S); Earth] conserved between B and O? Justify.

2.2- Determine the speed of (S) at the position O.

2.3- In reality, friction exists between B and O. Determine the magnitude of the force of friction between BO and (S) knowing that it stops at O.

Exercise 20:



A block (S), taken as a particle of mass m = 4kg, is placed on a rough inclined plane that makes an angle $\alpha = 30°$ with the horizontal. (S) is connected to a light and inextensible string which passes over a light and frictionless pulley fixed at the top of the inclined plane. The other end of the string is connected to a vertical wall by means of a horizontal spring (R) of negligible mass and stiffness k = 100N/m.

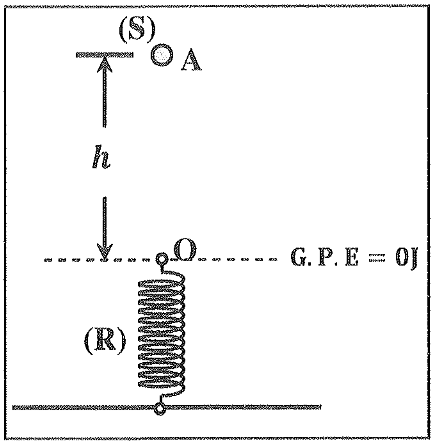
At $t_0 = 0s$, (S) is released from rest while the spring is un-stretched. The block moves a distance d = 20cm on the inclined plane before stopping.

The force of friction between the inclined plane and (S) opposes its motion and is assumed constant of magnitude f.

1- Name and represent the external forces acting on (S).

2- Apply the work-energy theorem to determine f.

Exercise 21:



A solid (S), taken as a particle of mass m, is released without initial velocity from a point A at a height h relative to the upper extremity O of a vertical un-stretched spring (R) of negligible mass and stiffness k. The lower extremity of (R) is fixed to a horizontal support as shown in the adjacent document.

As (S) reaches point O, it compresses (R) by a distance y $(y > 0)$ where it stops at point B.

The horizontal plane passing through O is taken as a gravitational potential energy reference.

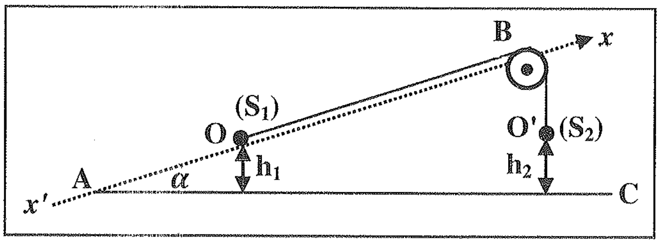
1- Determine the expression of the speed of (S) as it reaches O.

2- Write, in terms of m, g, k and y, the expression of the mechanical energy of the system [(S); (R); Earth] at B.

3- Establish the following relation: $\frac{1}{2}ky^2 - mgy - mgh = 0$.

4- Verify that the maximum compression of the spring is given by: $y = \frac{mg}{k}\left(1 + \sqrt{1 + \frac{2kh}{mg}}\right)$.

Exercise 22:

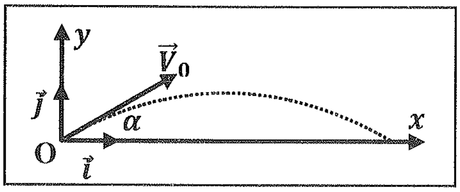


The aim of this exercise is to determine the expression of the magnitude of the acceleration of a particle using two methods. The apparatus used is formed of two particles (S_1) and (S_2) of respective masses m_1 and m_2, fixed at the extremities of an inextensible and massless string passing over the groove of a light pulley.
(S_1), (S_2), the string and the pulley form a mechanical system (S).
(S_1) may move on the line of greatest slope AB of an inclined plane that makes an angle α with the horizontal AC and (S_2) hangs vertically. At rest, (S_1) is found at point O at a height h_1 above AC and (S_2) is found at O' at a height h_2 as shown in the adjacent document.
At the instant $t_0 = 0$, we release the system (S) from rest. (S_1) ascends on AB and (S_2) descends vertically.
At an instant t, the position of (S_1) is defined by its abscissa $x = \overline{OS_1}$ on an axis $x'Ox$ confounded with AB, directed from A to B. Take the horizontal plane containing AC as a gravitational potential energy reference and neglect all the forces of friction.

1- **Energetic method**
   - **1.1-** Write down, at the instant $t_0 = 0$, the expression of the mechanical energy of the system [(S), Earth].
   - **1.2-** At the instant t, the abscissa of (S_1) is x and the algebraic value of its velocity is v.
     Determine, at that instant t, the expression of the mechanical energy of the system [(S), Earth].
   - **1.3-** Applying the principle of conservation of mechanical energy, show that: $v^2 = \frac{2(m_2 - m_1 \sin\alpha)gx}{(m_1+m_2)}$.
   - **1.4-** Deduce the expression of the value a of the acceleration of (S_1).

2- **Dynamical method**
   Applying the theorem of the center of mass $\sum \vec{F}_{ext} = m\vec{a}$, on each particle, determine the expression of a.

Exercise 23*:

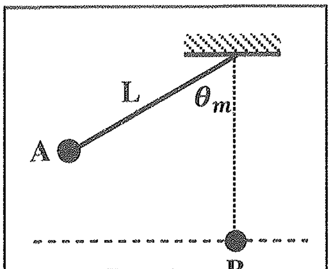


At the instant t = 0s, a particle (P) of mass m = 2kg is launched from O with an initial velocity vector $\vec{V}_0$ of magnitude V_0 and making an angle α with the horizontal.
The parametric equations of motion of (P) in the space reference system $(O, \vec{i}, \vec{j})$ are given by:

$$\begin{cases} x = 15\sqrt{3}t \\ y = -5t^2 + 15t \end{cases} \quad [SI]$$

The horizontal plane passing through O is taken as a gravitational potential energy reference.
1- Determine V_x and V_y the components of the velocity vector of (P) at any time t.
2- Deduce the values of V_0 and α.
3- Determine at any time t:
   - **3.1-** the gravitational potential energy of the system [(P), Earth],
   - **3.2-** the kinetic energy of (P).
4- Show that the resistive forces are neglected.
5- Determine the maximum height reached by (P).
6- Determine the range of (P).

Exercise 24:

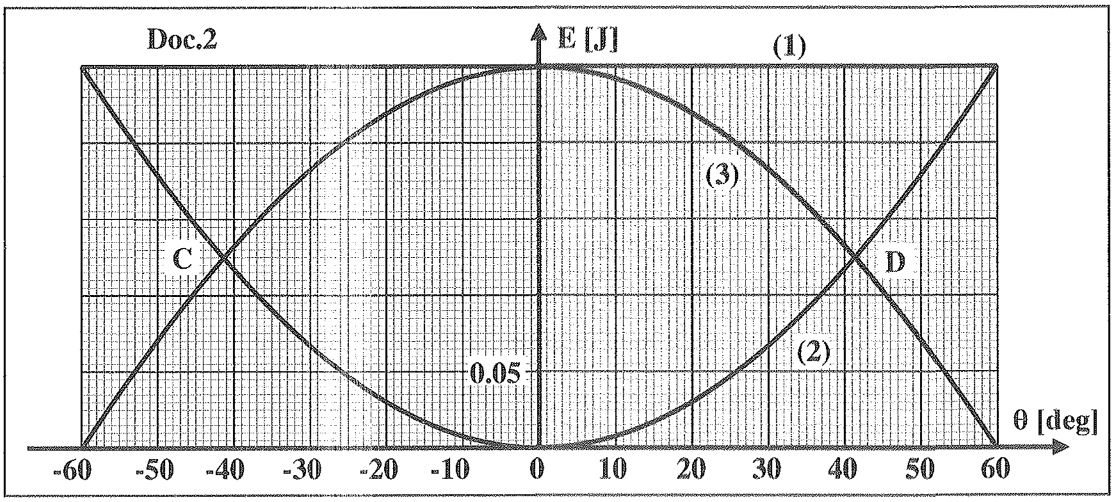


Doc.1

A ball, taken as a particle of mass m = 100g, is suspended from a massless and inextensible string of length L = 0.5m. The system (string, ball) is called a simple pendulum. The pendulum is shifted by an angle $\theta_m = 60^\circ$ from its equilibrium position, then released without initial speed from point A and swings down to point B (doc.1). The horizontal plane passing through B is taken a gravitational potential energy reference. All frictional forces are neglected.

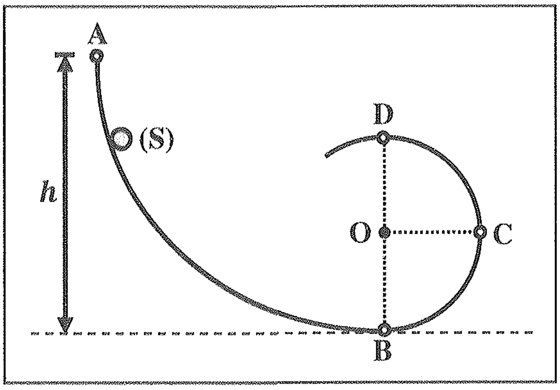
1- Is the mechanical energy of the system [pendulum; Earth] conserved? Justify.
2- Determine using two methods the speed of the ball as it passes by its equilibrium position B; then, deduce the value of the tension in the string at point B.
3- Determine the speed of the ball as the pendulum makes an angle $\theta = 22^\circ$ with the vertical.
4- Document 2 shows the variation of the energy as a function of the angle of deviation of the pendulum. The graphs (1), (2) and (3) of document 2 represent the energies of the system. What kind of energy (potential, kinetic or mechanical) does each represent? Justify.
5- What do the intersection points C and D of graphs 2 and 3 represent? Determine, at these points, the corresponding angles of deviation of the pendulum from its equilibrium position.

Doc.2



Exercise 25:

A frictionless track ABCD, situated in the vertical plane, is formed of a ramp AB of height h and a circular rail BCD of center O and radius R. A solid (S), taken as a particle of mass m, is released from rest from the top A of the ramp.
The horizontal plane passing through the bottom B of the track ABCD is taken as a gravitational potential energy reference.



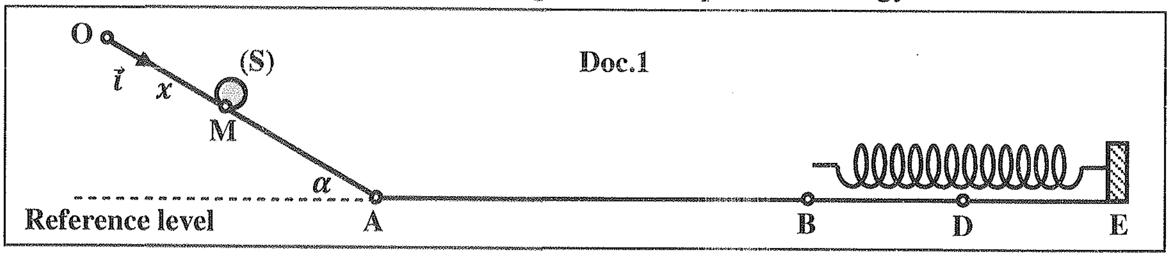
1- Determine the expression of the speed of (S) at the points B, C and D.
2- Determine the minim height for which the marble remain on the track at the point D.

Exercise 26:

At the instant $t_0 = 0s$, a particle (S), of mass m = 0.4kg, is launched with initial velocity $\vec{V_0} = V_0\vec{i}$ from the top O of an inclined plane of length $L = OA = 1.2m$ and making an angle $\alpha = 30°$ with the horizontal.

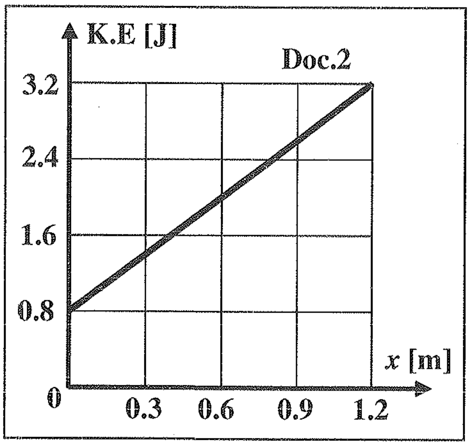
At the instant t, (S) passes by the point M and its position in the space reference system $(O; \vec{i})$ is $x = \overline{OM}$ as shown in document 1.

The horizontal plane passing through A is taken as a gravitational potential energy reference.



Part I: Motion along OA

Document 2 represents the variation of the kinetic energy of (S) as function of its abscissa x.



1- Referring to document 2:
   1.1- Give the value of the kinetic enegy of (S) at $t_0 = 0s$; then, deduce the value of V_0.
   1.2- Show that the expression of the kinetic energy as a function of x is: $K.E = 2x + 0.8$.
   1.3- Calculate the speed of (S) as it passes by the point A.
2- Determine, at the instant t, the expression of the gravitational potential energy of the system [(S); Earth] as a function of x
3- Show that the mechanical energy of the system [(S); Earth] is conserved.
4- Trace, on the same graph, the variation of the gravitational potential energy and the mechanical energy of the system [(S); Earth] as a function of x.

Part II: Motion along AB

As (S) reaches the point A with a speed $V_A = 4m/s$, it continues its motion along a horizontal rough surface AB of 5m length.

1- Name and represent the external forces acting on (S) along AB.
2- Apply the work-kinetic energy theorem to determine the magnitude of the force of friction, supposed constant, between AB and (S) knowing that it reaches B with a speed of $V_B = 2m/s$.

Part III: Motion along BE

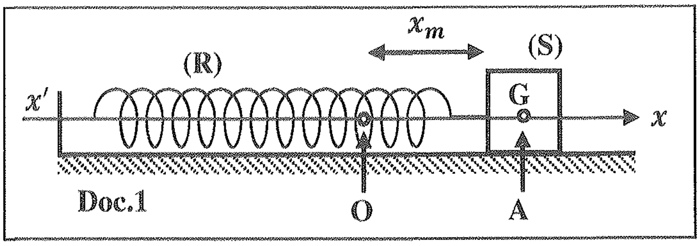
At B, (S) is attached to the free end of an un-stretched spring (R) of negligible mass and stiffness k = 20N/m. The other end of the spring is fixed to a support. (S) reaches point D where it stops. The resistive forces along BE are neglected.

Determine the maximum compression $x_m = BD$ of the spring.

Exercise 27*:

Consider a horizontal elastic pendulum formed of a massless spring (R) of stiffness k and a solid (S) of mass m = 0.4kg.

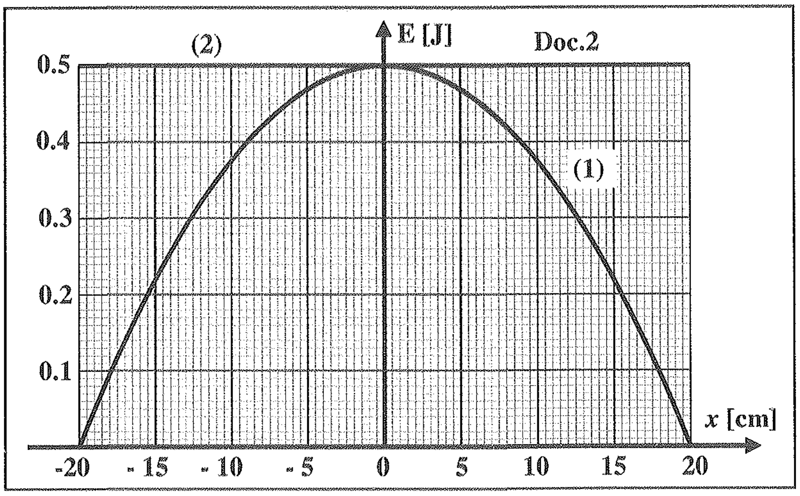
The spring (R) is fixed from one of its extremities to a fixed support and the other extremity is attached to (S). (S) may slide without friction on a horizontal rail and its center of inertia G can move along a horizontal axis $x'x$. At equilibrium, G coincides with the origin O of the axis $x'x$. The horizontal plane passing through G is taken as a gravitational potential energy reference.



The solid (S) is shifted from its equilibrium position O to the right by a distance x_m and then released from position A without initial speed as shown in document 1.

1- Apply the principle of conservation of mechanical energy to show that the expression of the speed of (S) as it passes by its equilibrium position O is $V_0 = \omega_0 x_m$ where $\omega_0 = \sqrt{\frac{k}{m}}$.

2- Document 2 represents the variation of the kinetic energy of (S) and the mechanical energy of the system [(S); (R); Earth] as a function of the abscissa x of G.



2.1- Curve (2) corresponds to the mechanical energy of the system [(S); (R); Earth]. Why?

2.2- Determine the values of x_m and V_0; then, deduce the value of k.

2.3- Trace the curve that represents the variation of the elastic potential energy of the system [(S); (R); Earth] as a function of x.

2.4- Determine the position where the kinetic energy of (S) and the elastic potential energy of the system [(S); (R); Earth] are equal.

Exercise 28:

At the instant $t_0 = 0s$, a ball (B), taken as a particle of mass m = 400g, is dropped without initial velocity from a height $H_0 = 1.2m$ above the ground level that is taken as a gravitational potential energy reference. When (B) hits the ground, it rebounds and reaches a maximum height H_1 and then it hits the ground again then rebounds back to a height H_2 and so on ... Each time (B) hits the ground, it loses 10% from its maximum height.

Let H_n be the n^{th} height reached by (B) after it hits the ground n times.

1- Calculate the mechanical energy of the system [(B), Earth] at the instant t_0.

2- Verify that $H_n = 1.2 \times 0.9^n$. Deduce the expression of the mechanical energy of the system [(B); Earth] after reaching the n^{th} height.

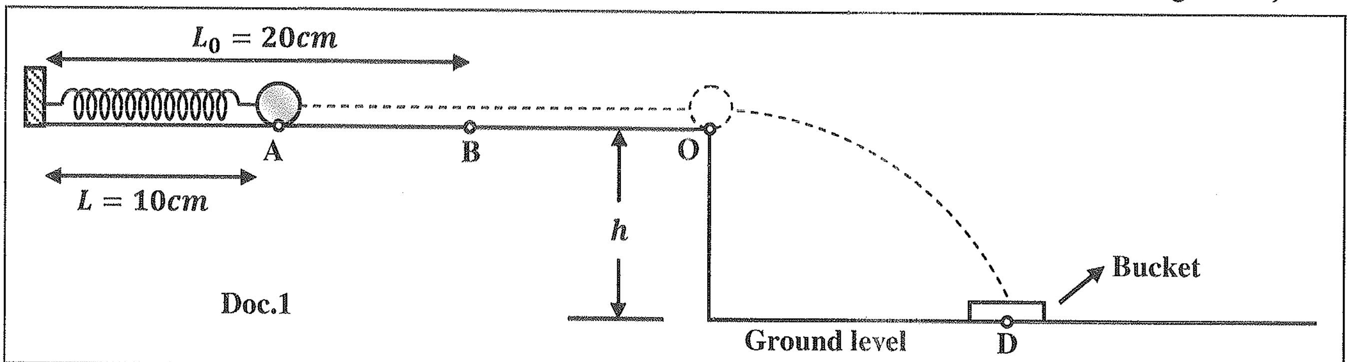
3- Determine the speed of (B) just before it hits the ground for the fifth time.

4- Calculate the variation in the mechanical energy of the system [(B), Earth] after the fifth bounce with the ground.

5- How much times will (B) hit the ground in order to lose half its initial maximum height for the first time?

Exercise 29*:

The objective of this exercise is to study the motion of a ball (S) until it enters a bucket designed to capture it. The ball (S), taken as particle of mass $m = 20g$, is shot out a spring cannon that is placed at a platform of height h relative to the ground level. The spring cannon of negligible mass, stiffness $k = 20N/m$ and natural length $L_0 = 20cm$, is aimed to fire (S) in a horizontal direction as shown in document 1.
In order to score (S) in the bucket, we compress the spring until its length becomes $L = 10cm$. At $t_0 = 0$, (S) is released without initial velocity from position A and moves on the frictionless track AB until it leaves the spring at position B with a speed V_B. (S) then moves along the rough horizontal rail BO of 18cm length.
Upon reaching position O, (S) describes a parabolic trajectory until it hits the bucket at point D.
The plane passing through the positions A, B and O is taken as a gravitational potential energy reference.
The friction force $\vec{f}$ between BO and (S) opposes its motion and is assumed constant of magnitude $f = 2N$.



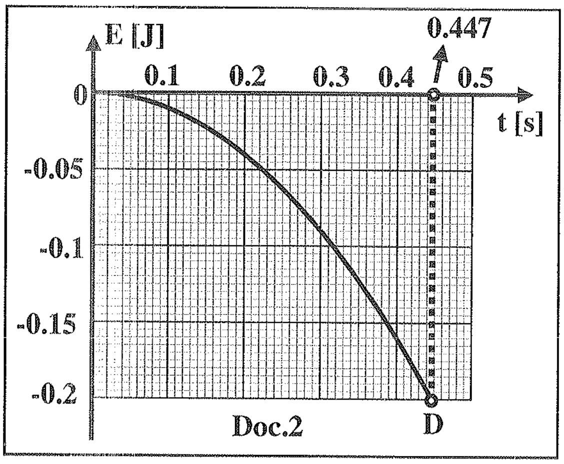
Doc.1

Part I: Motion of (S) along AO

1- Determine the speed of (S) at the position B.
2- Determine, by applying the work-kinetic energy theorem, the speed V_O of (S) at the position O.

Part II: Motion of (S) along OD

The graph of the variation of one form of energy of the system [(S); Earth] as a function of time is shown in document 2. A new origin of time $t_0 = 0$ is taken at the moment (S) leaves position O.

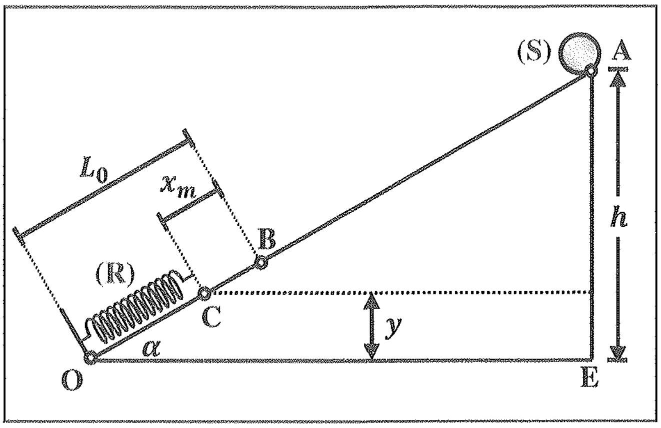


Doc.2

1- Calculate the mechanical energy of the system [(S); Earth] as it falls from position O to position D.
2- Specify the form of energy that this graph represents.
3- Referring to document 2, Give:
   3.1- The time needed by (S) to reach the position D.
   3.2- The gravitational potential energy of the system [(S); Earth] at the position D; then, deduce the value of h.
4- Determine the kinetic energy of (S) as it reaches D; then, deduce its speed at D.
5- Plot the other two forms of energy of (S) as a function of time.

Exercise 30:

A solid (S), taken as a particle of mass $m = 2kg$, is released without initial velocity from point A at a height $h = AE = 4m$ of a frictionless inclined plane AO making an angle $\alpha = 30°$ with the horizontal. A spring (R) of negligible mass, stiffness $k = 50N/m$ and natural length $L_0 = BO = 2m$ is fixed at the end O of the inclined plane as shown in the adjacent document.



The horizontal plane passing through the points O and E is taken as a gravitational potential energy reference.

1- Apply the law of conservation of mechanical energy to show that the speed of (S) as it reaches point B is $V_B = \sqrt{2g(h - L_0 \sin\alpha)}$. Calculate its numerical value.

2- As (S) reaches point B, it compresses the spring by a maximum distance $x_m = BC$ at point C.

 2.1- Express x_m in terms of y where y is the height of C relative to the gravitational potential energy reference.

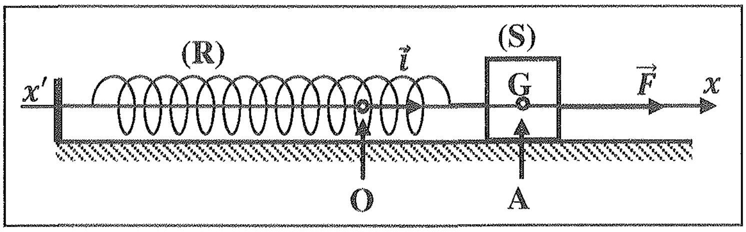
 2.2- Verify that y satisfies the following equation: $y^2 - 1.8y + 0.2 = 0$.

 2.3- Deduce the value of x_m.

Exercise 31:

Consider a horizontal elastic pendulum formed of a:

- massless spring (R) of stiffness $k = 40N/m$ and natural length $L_0 = 60cm$,
- solid (S) of mass m = 0.5kg and center of inertia G.



The spring (R) is fixed from one of its extremities to a fixed support and (S) is attached to the other extremity. (S) may slide without friction on a horizontal rail and its center of inertia G can move along a horizontal axis $x'x$. At equilibrium, G coincides with the origin O of the axis $x'x$.

At the instant t_0 = 0s, G is at rest at O. A horizontal force $\vec{F} = F\vec{i}$ $(F = 20N)$ is applied on (S) as shown in the adjacent document.

The horizontal plane passing through G is taken as a gravitational potential energy reference.

1- At the instant t, (S) reaches the position A such that $x_A = \overline{OA} = 25cm$.

 1.1- Name and represent the external forces acting on (S).

 1.2- Show that the work done by the tension force along OA is $W_{\vec{T}} = -\frac{1}{2}kx_A^2$.

 1.3- Apply the work-kinetic energy theorem to determine the speed of (S) as it reaches the position A.

2- As (S) reaches A, the force $\vec{F}$ is removed.

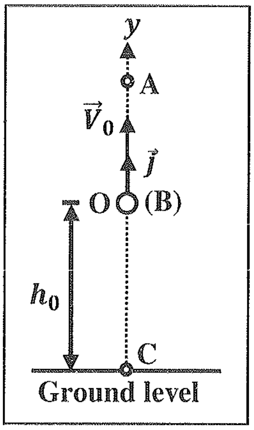
 2.1- How far from the position A to the right does (S) reach before it returns back?

 2.2- How close does (S) get to the wall where the left end of the spring is attached?

Exercise 32*:

At the instant $t_0 = 0s$, a basketball (B), taken as a particle of mass $m = 500g$, is launched vertically upwards with a velocity $\vec{V}_0 = V_0\vec{j}$ ($V_0 = 10m/s$) from a point O, at a height $h_0 = 2m$ above the ground level which is taken as a gravitational potential energy reference. (B) reverses its motion at point A and then it hits the ground at point C.

At the instant t, the position of (B) in the space reference system $(O;\vec{j})$ is defined by $y = \overline{OB}$ and the algebraic value of its velocity is $V = \frac{dy}{dt}$.

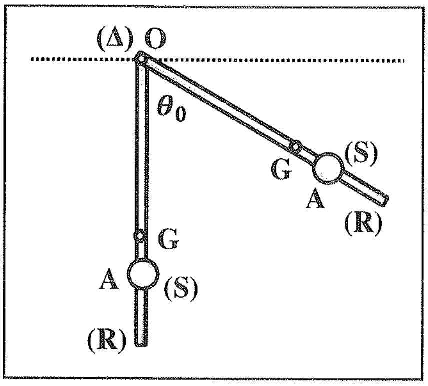


1- **Dynamical study of the motion of (B)**
  1.1- Apply Newton's 2nd law to determine the algebraic value of the acceleration a of (B) at the instant t.
  1.2- Deduce the expressions of y and V as a function of t.

2- **Energetic study of the motion of (B)**
  2.1- Calculate the mechanical energy M.E of the system [(B); Earth] at O.
  2.2- Determine, at the instant t, the expressions of the kinetic energy K.E of (B) and the gravitational potential energy G.P.E of the system [(B); Earth] in terms of y.
  2.3- Calculate t_A and t_C the instants when (B) reaches the points A and C respectively.
  2.4- Trace, on the same system of axes, the curves representing the variations of the energies M.E, G.P.E and K.E of (S) as a function of x.
  2.5- Determine the positions of (B) where its gravitational potential energy and its kinetic energy are equal.
  2.6- Calculate the work done by the weight of (B) as it moves from O to A. Compare the obtained value to the variation in G.P.E between O and A.

Exercise 33:

A compound pendulum (E) consists of:
- a homogeneous rigid rod (R), of length $L = 40cm$ and mass $m_R = m = 200g$, free to rotate without friction about an axis (Δ) passing through its upper extremity O.
- a point mass (S), of mass $m_s = m = 200g$, fixed at the point A of (R) so that $OA = d = 30cm$.



(E) is shifted from its equilibrium position by an angle $\theta_0 = 60°$ and then released from rest at the instant t = 0s as shown in the adjacent document. The pendulum oscillates then, without friction, around its equilibrium position.

The position of the center of inertia G of (E) is given by $a = OG$.

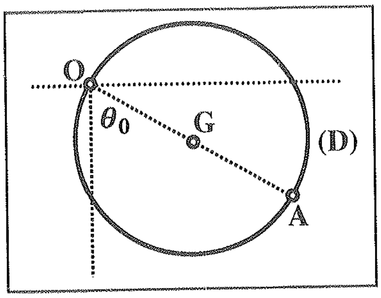
Denote by I the moment of inertia of (E) about (Δ).

The horizontal plane passing through O is taken as a gravitational potential energy reference.

The moment of inertia of (R) about (Δ) is $I_R = \frac{1}{3}mL^2$.

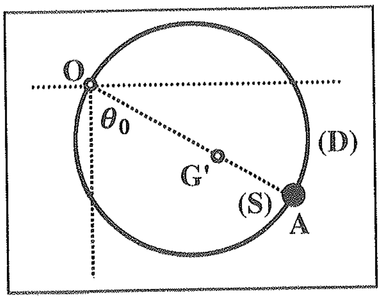
1- Show that $a = \frac{L+2d}{4}$. Calculate its numerical value.
2- Determine the expression of the moment of inertia I in terms of m, L and d; then, calculate its numerical value.
3- Calculate the mechanical energy of the system [(E); Earth] at the instant t = 0s.
4- Determine the angular speed of (E) when it passes by its equilibrium position.

Exercise 34:



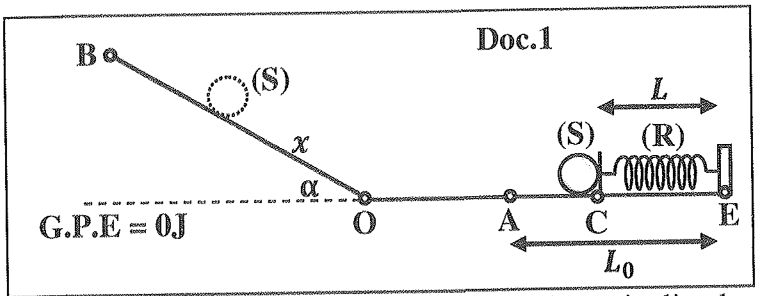
A homogeneous disk (D) has a radius R = 10cm, mass M = 300g and center of mass G. (D) is free to rotate without friction about a horizontal axis (Δ) perpendicular to its plane through a point O of its periphery. The moment of inertia of (D) about (Δ) is $I = \frac{3}{2}MR^2$. The horizontal plane passing through the point O is taken as a gravitational potential energy reference.

1- (D) is shifted by an angle $\theta_0 = 60°$ from its equilibrium position and then released from rest at the instant t_0 = 0s.
   1.1- Calculate the mechanical energy of the system [(D); Earth] at the instant t_0 = 0s.
   1.2- Determine the angular speed θ_1' of (D) when it passes through its equilibrium position.



2- A particle (S), of mass m = 100g, is fixed at point A which is diametrically opposite to point O.
   2.1- Calculate the new moment of inertia I' of the system [(D); (S)] with respect to (Δ)
   2.2- Determine the new position $a = OG'$ of the center of mass of the system [(D), (S)].
   2.3- If the system [(D); (S)] is shifted again by $\theta_0 = 60°$ from its equilibrium position and then released from rest. Determine the new angular speed θ_2' of the system [(D), (S)] when it passes through its equilibrium position.

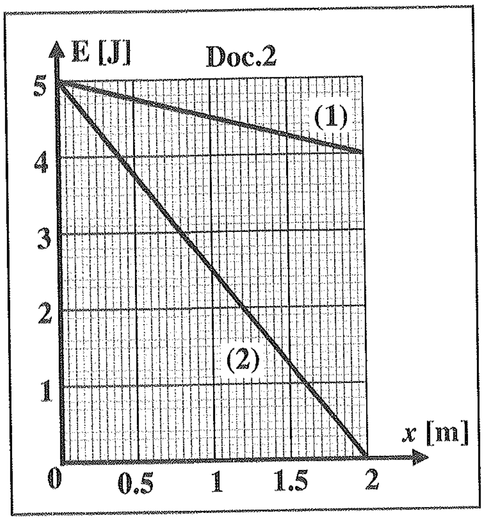
Exercise 35*:



In document 1, (S) is a particle of mass $m = 0.4kg$ and (R) is a massless spring of natural length $L_0 = 40cm$ and stiffness K. (R) is fixed from one end to a support at point E while the other end is compressed by (S) to a final length $L = 20cm$. At the instant t_0 = 0s, (S) is released from rest from point C.

As (S) leaves (R) at A, it continues its motion along the horizontal plane EO and then it climbs an inclined plane OB of length d and making an angle α with the horizontal. (S) stops at point B.

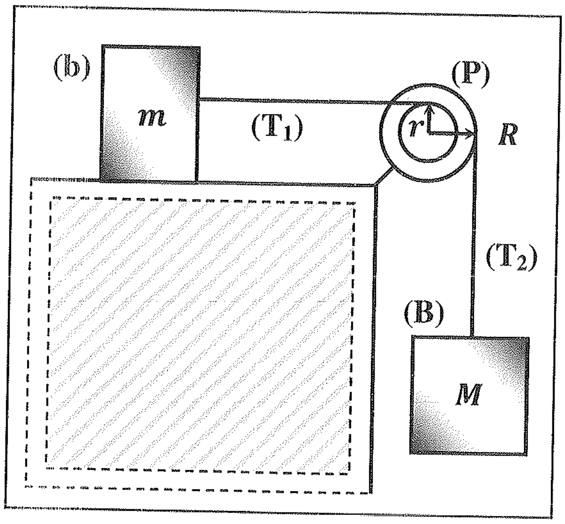
During the motion of (S) along OB, it is subjected to a frictional force $\vec{f}$ of constant magnitude f and opposing its motion. The horizontal plane passing through O is taken as a gravitational potential energy reference.



Along OB, the position of (S) is defined by its abscissa $x = \overline{OS}$.
The graph of document 2 represents the variations of the kinetic energy K.E of (S) and the mechanical energy M.E of the [(S); Earth] as a function of x.

1- Specify the form of energy represented by the curves (1) and (2).
2- Use the graph to determine:
   2.1- the value of d,
   2.2- the speed of (S) at O,
   2.3- the value of α,
   2.4- the value of f.
3- The forces of friction are neglected along the horizontal plane OE.
   3.1- Show that $V_O = V_A$.
   3.2- Determine the value of K.

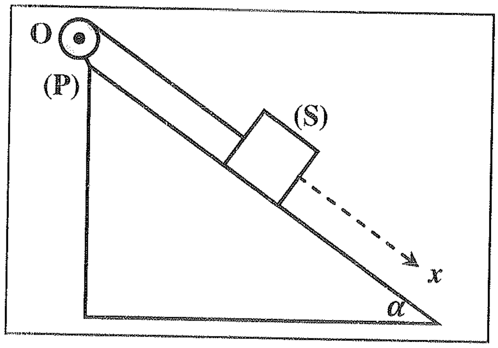
Exercise 36:



A pulley (P), with moment of inertia I about the horizontal axis passing through its center, has two grooves of radii r and R with $r < R$. (P) is fixed at the edge of a frictionless horizontal table.
A block (b), taken as a particle of mass m, slides on the horizontal table and it is connected to a massless and inextensible string (T$_1$) wound around the radius r of the pulley. Another block (B), taken as a particle of mass M, is connected to another massless and inextensible string (T$_2$) wound around the radius R of the pulley.
(b), (B), (T$_1$), (T$_2$) and (P) form a mechanical system (S).
At the instant $t = 0s$, (S) is released from rest.
At the instant t, (B) falls down by a distance X and attains a speed V, while (b) slides by a distance x and attains a speed v.

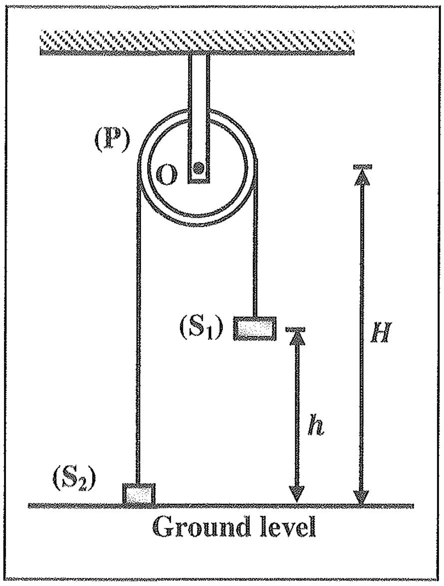
1- Express x in terms of X, r and R.
2- Determine the angular acceleration θ'' of the pulley in terms of M, m, I, R and r.
3- Deduce a_1 and a_2 the accelerations of (b) and (B) respectively in terms of M, m, I, R and r.
4- Deduce that $T_1 = \frac{mMgRr}{MR^2+mr^2+I}$ and $T_2 = Mg\left(1 - \frac{MR^2}{MR^2+mr^2+I}\right)$, where T_1 and T_2 are respectively the tensions in strings (T$_1$) and (T$_2$).
5- Show that the kinetic energy of the system (S) is given by $K.E = \frac{1}{2}\left(m + \frac{MR^2}{r^2} + \frac{I}{r^2}\right)v^2$.

Exercise 37:



A pulley (P), considered as a homogenous disk of mass $m = 2kg$ and radius $R = 10cm$, is free to rotate around a horizontal axis (Δ) passing through its center O. (P) is fixed on the top of a rough inclined plane that makes an angle $\alpha = 30°$ with the horizontal.
A block (S), taken as particle of mass $M = 3kg$, can slide on the inclined plane. (S) is connected to a massless and inextensible string whose other end passes over the groove of (P).
During the motion of (S) on the inclined plane, it is subjected to a constant frictional force $\vec{f}$ of magnitude $f = 12N$.
At the instant $t = 0s$, the system [(S); (P); string] is released from rest. After a time t, (S) covers a distance x and attains a speed v.
The moment of inertia of (P) with respect to (Δ) is $I = \frac{1}{2}mR^2$.

1- Use the work-kinetic energy theorem in order to establish the following relation: $v^2 = \frac{4(Mg\sin\alpha - f)}{m+2M}x$.
2- Deduce expression of the acceleration of (S) at the instant t; then, deduce its numerical value.



Exercise 38:

Two solids (S_1) and (S_2), taken as particles of respective masses m_1 and m_2 ($m_1 > m_2$), are connected by an inextensible string of negligible mass that passes over a homogenous pulley (P), of mass μ radius R. The moment of inertia of (P) about an axis (Δ) passing through its center O is $I = \frac{1}{2}\mu R^2$.

(S_1) is initially at an altitude h from the ground level and (S_2) is just above the ground. Point O is at an altitude H from the ground level.

At the instant $t_0 = 0s$, the system (S) = [(S_1); (S_2); (P); string] is released from rest.

Take the ground level as a reference for the gravitational potential energy. All forces of friction are neglected.

1- Write the expression of the mechanical energy of the system [(S); Earth] at the instant $t_0 = 0s$.

2- After a time t, (S_1) falls a distance x and attains a speed V.

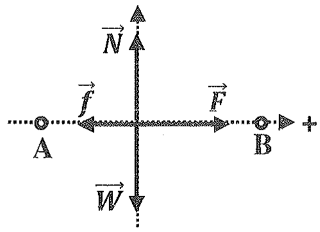
2.1- Write the expression of the mechanical energy of the system [(S); Earth] at the instant t.

2.2- Determine, at the instant t, the expression of V.

2.3- Deduce the acceleration of (S_1).

CHAPTER 1 – WORK AND MECHANICAL ENERGY
SOLUTION OF EXERCISES AND PROBLEMS

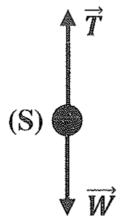
Exercise 1:

Part	Answer	
1	The forces acting on (S) are: weight $\vec{W}$, normal reaction of a support $\vec{N}$, the applied force $\vec{F}$ and friction $\vec{f}$.	
2	$W_{\vec{F}} = \vec{F}.\overrightarrow{AB} = F \times AB \times \cos(\vec{F};\overrightarrow{AB}) = 10 \times 5 \times \cos 0° = 50J$. $W_{\vec{f}} = \vec{f}.\overrightarrow{AB} = f \times AB \times \cos(\vec{f};\overrightarrow{AB}) = 2 \times 5 \times \cos(180°) = -10J$. $W_{\vec{W}} = \vec{W}.\overrightarrow{AB} = 0J$ and $W_{\vec{N}} = \vec{N}.\overrightarrow{AB} = 0J$. $\sum W_{ext} = W_{\vec{F}} + W_{\vec{f}} + W_{\vec{W}} + W_{\vec{N}} = 50J - 10J + 0J + 0J = 40J$.	

Exercise 4:

Part	Answer key
1.1	$G.P.E_A = mgh = 45 \times 10 \times 80 = 36{,}000J$.
1.2	$M.E_A = K.E_A + G.P.E_A = 0J + 36{,}000J = 36{,}000J$ ($V_A = 0 \Longrightarrow K.E_A = 0J$).
2.1	Since forces of friction are negligible: $M.E_A = M.E_B = 36{,}000J$.
2.2	$h_B = 0m \Longrightarrow G.P.E_B = 0J$ (B is on the reference level). $K.E_B = M.E_B - G.P.E_B = 36{,}000J - 0J = 36{,}000J$. $K.E_B = \frac{1}{2}mV_B^2 \Longrightarrow V_B = \sqrt{\frac{2K.E_B}{m}} = \sqrt{\frac{2\times 36{,}000}{45}} = 40m/s$.
3.1	$M.E_B^{'} = K.E_B^{'} + G.P.E_B^{'} = \frac{1}{2}mV_B^{'2} + 0 = \frac{1}{2} \times 45 \times 35^2 = 27{,}562.5J$.
3.2	$\Delta M.E = M.E_B^{'} - M.E_B = 2{,}7562.5J - 36{,}000J = -8{,}437.5J$.
3.3	Energy dissipated by forces of friction $E_f = -\Delta ME = 8{,}437.5J$.
3.4	In the form of heat.
3.5.1	The system is energy isolated since there is no exchange of energy or matter with its surroundings.
3.5.2	$E = M.E + U \Longrightarrow \Delta E = \Delta M.E + \Delta U = 0 \Longrightarrow \Delta U = -\Delta M.E = 8{,}437.5J$.

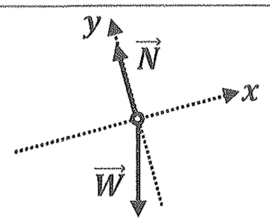
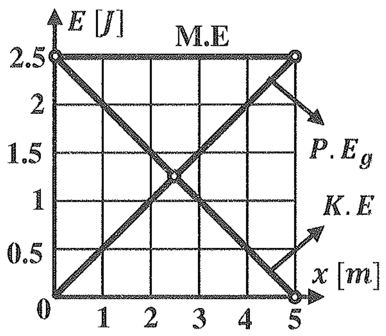
Exercise 6:

Part	Answer key	
1	The forces acting on (S) are: weight $\vec{W}$ and tension $\vec{T}$.	
2	Newton's 1st law: $\sum \vec{F}_{ext} = \vec{0} \Longrightarrow \vec{T} + \vec{W} = \vec{0} \Longrightarrow \vec{T} = -\vec{W}$. $T = W \Longrightarrow kx_0 = mg \Longrightarrow x_0 = \frac{mg}{k} = \frac{0.02\times 10}{5} = 0.04m = 4cm$.	
3	The forces of friction are neglected; then, the mechanical energy is conserved.	
4	Principle of conservation of mechanical energy: $M.E_A = M.E_O$. $K.E_A + G.P.E_A + E.P.E_A = K.E_O + G.P.E_O + E.P.E_O$. $0 - mgx_m + \frac{1}{2}k(x_0 + x_m)^2 = \frac{1}{2}mV_O^2 + 0 + \frac{1}{2}kx_0^2$. $-0.02 \times 10 \times 0.06 + \frac{1}{2} \times 5 \times 0.1^2 = \frac{1}{2} \times 0.02 \times V_O^2 + \frac{1}{2} \times 5 \times 0.04^2$. $-0.012 + 0.025 = 0.01V_O^2 + 0.004 \Longrightarrow V_O = 0.95m/s$.	

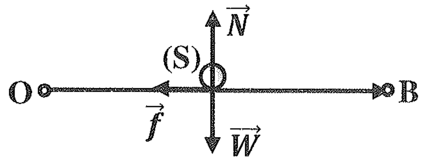
Exercise 8:

Part	Answer key
1	The non-conservative force (friction) is negleted along AO; then, the mechanical energy of the system [(S); Earth] is conserved. $M.E_A = M.E_O \Longrightarrow K.E_A + G.P.E_A = K.E_O + G.P.E_O$. $\frac{1}{2}mV_A^2 + mgh_A = \frac{1}{2}mV_0^2 + mgh_0$ with $h_A = L\sin\alpha$ and $h_0 = 0$. $\frac{1}{2}mV_A^2 + mgL\sin\alpha = \frac{1}{2}mV_0^2 \Longrightarrow V_0^2 = V_A^2 + 2gL\sin\alpha$. $V_0 = \sqrt{V_A^2 + 2gL\sin\alpha} = \sqrt{3^2 + 2\times10\times1.6\times0.5} = 5m/s$.
2	$\Delta K.E = \sum W_{ext} \Longrightarrow K.E_C - K.E_0 = W_{\vec{W}} + W_{\vec{N}} + W_{\vec{f}}$. $\frac{1}{2}mV^2 - \frac{1}{2}mV_0^2 = 0 + 0 - fx$ ($W_{\vec{W}} = W_{\vec{N}} = 0J$ since $\vec{W} \perp \overrightarrow{OC}$ and $\vec{N} \perp \overrightarrow{OC}$). $V^2 - 25 = -8x \Longrightarrow V^2 + 8x - 25 = 0$.
3	Derive both sides with respect to time: $2VV' + 8x' = 0$ with $V = x'$ and $V = a$. $2Va + 8V = 0 \Longrightarrow 2V(a+4) = 0 \Longrightarrow a = -4m/s^2$ ($V = 0$ is not ture $\forall\, t$).
4	For $x = 2m$. $V_B^2 + 8\times2 - 25 = 0 \Longrightarrow V_B^2 = 9m^2/s^2 \Longrightarrow V_B = 3m/s$.
5	Principle of conservation of mechanical energy: $M.E_B = M.E_D$. $K.E_B + G.P.E_B + E.P.E_B = K.E_D + G.P.E_D + E.P.E_D$. $\frac{1}{2}mV_B^2 + 0 + 0 = 0 + 0 + \frac{1}{2}kx_m^2$. $mV_B^2 = kx_m^2 \Longrightarrow x_m = \sqrt{\frac{m}{k}}V_B = \sqrt{\frac{2}{50}}\times3 = 0.6m$.

Exercise 9:

Part	Answer key
1.1	$M.E_O = K.E_O + P.E_{gO} = \frac{1}{2}mV_0^2 + mgh_0 = \frac{1}{2}(0.2)(5)^2 + 0 = 2.5J$
1.2	$M.E = M.E_O = 2.5J$ (principle of conservation of mechanical energy since friction is neglected). $P.E_g = mgh = mgOM\sin\alpha = (0.2)(10)(x)(0.25) = 0.5x$
1.3	**1st method:** $K.E = M.E - P.E_g = 2.5 - 0.5x$ **2nd method:** by applying the kinetic energy theorem $\Delta K.E = \sum W_{ext}$ $K.E - K.E_O = W_{\vec{W}} + W_{\vec{N}}$ $K.E - K.E_O = -mg\sin\alpha\, OM + 0$ $K.E - 2.5 = -(0.2)(10)(0.25)(x) \Longrightarrow K.E = 2.5 - 0.5x$ 
1.4.1	
1.4.2	For $x = 3m; K.E = 1J; v = \sqrt{\frac{2K.E}{m}} = \sqrt{\frac{2\times1}{0.2}} = 3.16m/s$.
1.4.3	$K.E = 0; x = x_m = 5m$.
2.1	$M.E_{x=3} = K.E_{x=3} + P.E_{g(x=3)}$. $M.E_{x=3} = 0 + 1.5 = 1.5J$. Law of non-conservation of mechanical energy: $\Delta M.E = W_{\vec{f}}$. $W_{\vec{f}} = M.E_{x=0} - M.E_{x=0} = 1.5J - 2.5J = -1J$.
2.2	Thermal energy or heat. $Q = \|\Delta M.E\| = 1J$.
3	$E = M.E + U \Longrightarrow \Delta E = \Delta M.E + \Delta U = 0 \Longrightarrow \Delta U = -\Delta M.E = 1J$.

Exercise 11:

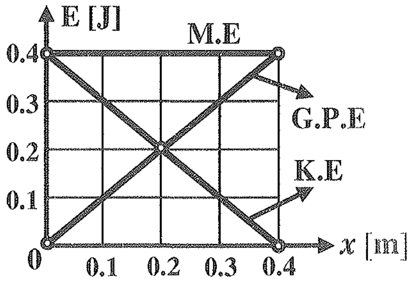
<table>
<tr><th>Part</th><th>Answer key</th></tr>
<tr><td>1</td><td>Since friction is neglected, the mechanical energy is conserved.
By applying the law of conservation of mechanical energy between C and O:
$M.E_C = M.E_O \Longrightarrow K.E_C + G.P.E_C = K.E_O + G.P.E_O$.
$0 + mgh_C = \frac{1}{2}mv_0^2 + 0$ with $\sin\alpha = \frac{h_C}{CO} \Longrightarrow h_C = CO\sin\alpha$ and $v_C = 0$.
$mgCO\sin\alpha = \frac{1}{2}mv_0^2 \Longrightarrow v_O = \sqrt{2gCO\sin\alpha} = \sqrt{2\times 10\times 1.6\times \sin 30°} = 4m/s$.</td></tr>
<tr><td>2</td><td>The forces acting on (S) along OB are: weight $\vec{W}$, normal reaction of a support $\vec{N}$ and friction $\vec{f}$.


$\Delta K.E = \sum W_{ext}$ (Work-kinetic energy theorem).
$K.E_B - K.E_0 = W_{\vec{W}} + W_{\vec{N}} + W_{\vec{f}}$.
$W_{\vec{W}} = \vec{W}.\overrightarrow{OB} = 0J\ (\vec{W} \perp \overrightarrow{OB})$.
$W_{\vec{N}} = \vec{N}.\overrightarrow{OB} = 0J\ (\vec{N} \perp \overrightarrow{OB})$.
$W_{\vec{f}} = \vec{f}.\overrightarrow{OB} = f\times OB\times\cos(\vec{f};\overrightarrow{OB}) = f\times OB\times\cos(180°) = -f\times OB$.
$\frac{1}{2}mv_B^2 - \frac{1}{2}mv_0^2 = 0 + 0 - f\times OB \Longrightarrow \frac{1}{2}\times 0.5\times 2^2 - \frac{1}{2}\times 0.5\times 4^2 = -f\times 2 \Longrightarrow f = 1.5N$.</td></tr>
<tr><td>3.1</td><td>$M.E_B = K.E_B + G.P.E_B = \frac{1}{2}mv_B^2 + 0 = \frac{1}{2}\times 0.5\times 2^2 = 1J$.</td></tr>
<tr><td>3.2</td><td>$G.P.E_M = mgh_M = mgR(1-\cos\theta)$.</td></tr>
<tr><td>3.3</td><td>$M.E_B = M.E_M$ (principle of conservation of mechanical energy).
$M.E_B = K.E_M + G.P.E_M \Longrightarrow 1 = 0 + mgR(1-\cos\theta)$ with $v_M = 0 \Longrightarrow K.E_M = 0$.
$1 = 0.5\times 10\times 1\times(1-\cos\theta) \Longrightarrow \theta = 36.8°$.</td></tr>
</table>

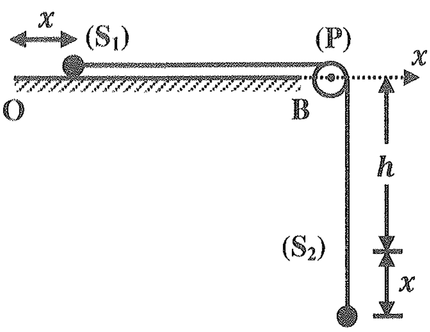
Exercise 13:

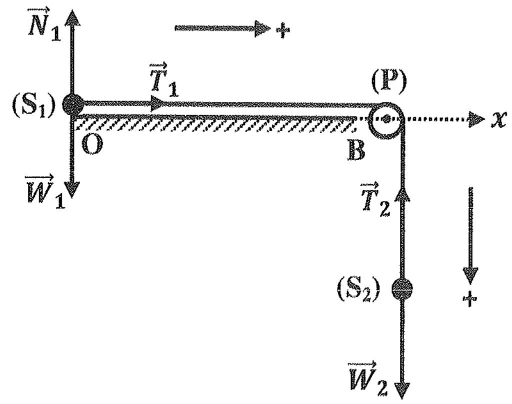
<table>
<tr><th>Part</th><th>Answer key</th></tr>
<tr><td>1.1</td><td>By applying the principle of conservation of mechanical energy:
$M.E_i = M.E_f \Longrightarrow K.E_i + G.P.E_i + E.P.E_i = K.E_f + G.P.E_f + E.P.E_f$.
$G.P.E_i = G.P.E_f$ since $h_i = h_f$.
$\frac{1}{2}kx_0^2 + 0 = \frac{1}{2}mV_0^2 + 0 \Longrightarrow x_0 = \sqrt{\frac{m}{k}}V_0 = \sqrt{\frac{0.2}{320}}\times 8 = 0.2m = 20cm$.</td></tr>
<tr><td>1.2</td><td>$\sum\vec{F}_{ext} = \vec{W} + \vec{N} = \vec{0}$ (S) is in equilibrium (URM).</td></tr>
<tr><td>2.1</td><td>At t = 0; $K.E = \frac{1}{2}mV_0^2 = \frac{1}{2}\times 0.2\times 8^2 = 6.4J$.
As (S) goes up the incline its speed decreases so its kinetic energy will decrease.</td></tr>
<tr><td>2.2.1</td><td>At t = 4s; K.E = 0 (The energy is totally gravitational potential).</td></tr>
<tr><td>2.2.2</td><td>At t = 4s; $G.P.E = 3.2J$; $h = \frac{G.P.E}{mg} = \frac{3.2}{0.2\times 10} = 1.6m$.
$\sin\alpha = \frac{h}{d} \Longrightarrow d = \frac{h}{\sin\alpha} = \frac{1.6}{0.1} = 16m$.</td></tr>
<tr><td>2.2.3.1</td><td><table><tr><td>t (s)</td><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td></tr><tr><td>M.E (J)</td><td>6.4</td><td>5</td><td>4</td><td>3.4</td><td>3.2</td></tr></table></td></tr>
<tr><td>2.2.3.2</td><td>The mechanical energy decreases as a function of time so friction exists.</td></tr>
<tr><td>2.2.3.3</td><td>$\Delta M.E = M.E_4 - M.E_0 = 3.2 - 6.4 = -3.2J$.</td></tr>
<tr><td>2.2.3.4</td><td>$\Delta M.E = W_{\vec{f}} = -f\times d \Longrightarrow -3.2 = -16f \Longrightarrow f = 0.2N$.</td></tr>
</table>

Exercise 14:

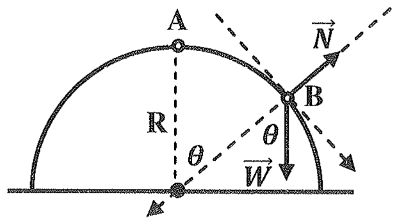
Part	Answer key
1.1	The forces of friction are neglected, so Mechanical energy of the system is conserved. By applying the law of conservation of mechanical energy between A and B: $M.E_A = M.E_B \Rightarrow K.E_A + G.P.E_A + E.P.E_A = K.E_B + G.P.E_B + E.P.E_B$. $0+\frac{1}{2}kx^2+0=\frac{1}{2}mv_B^2+0+0 \Rightarrow v_B=\sqrt{\frac{k}{m}}x=\sqrt{\frac{k}{m}}(L-L_0)=\sqrt{\frac{180}{0.2}}\times(0.1)=3m/s$.
1.2	$\Delta K.E=\sum W_{ext} \Rightarrow K.E_O - K.E_B = W_{\vec{f}}+W_{\vec{W}}+W_{\vec{N}}$. $\frac{1}{2}mv_O^2-\frac{1}{2}mv_B^2=-f\times BO \Rightarrow \frac{1}{2}\times 0.2\times v_O^2-\frac{1}{2}\times 0.2\times 3^2=-1\times 0.5 \Rightarrow v_O=2m/s$.
2.1	$M.E_O = K.E_O + G.P.E_O = \frac{1}{2}mv_O^2+0=\frac{1}{2}\times 0.2\times 4 = 0.4J$. At any position, $M.E = M.E_O = 0.4J$ (law of conservation of mechanical energy).
2.2	$G.P.E = mgh = mgx\sin\alpha = 0.2\times 10\times x\times \sin 30° = x$. $K.E = M.E - G.P.E = 0.4 - x$.
2.3	
2.4.1	Graphically: $x = 0.4m$. By calculation: for $K.E = 0 \Rightarrow 0.4 - x = 0$. $x = 0.4m$.
2.4.2	Graphically: $x = 0.2m$. By calculation: $K.E = G.P.E \Rightarrow 0.4 - x = x$. $2x = 0.4 \Rightarrow x = 0.2m$.

Exercise 15:

Part	Answer key
I.1	$M.E_0 = K.E_{1_0} + G.P.E_{1_0} + K.E_{2_0} + G.P.E_{2_0} + K.E_P + G.P.E_P$. $M.E_0 = 0+0+0+0+0-m_2gh = -m_2gh$.
I.2	$M.E_t = K.E_{1_t} + G.P.E_{1_t} + K.E_{2_t} + G.P.E_{2_t}$. $M.E_t = \frac{1}{2}m_1v^2 + 0 + \frac{1}{2}m_2v^2 - m_2g(h+x)$. $M.E_t = \frac{1}{2}(m_1+m_2)v^2 - m_2g(h+x)$. 
I.3	The non-conservative force (friction) is neglected; then, the mechanical energy of the system [S; Earth] is conserved. $M.E_0 = M.E_t$. $-m_2gh = \frac{1}{2}(m_1+m_2)v^2 - m_2g(h+x)$. $-m_2gh = \frac{1}{2}(m_1+m_2)v^2 - m_2gh - m_2gx$. $\frac{1}{2}(m_1+m_2)v^2 = m_2gx \Rightarrow v^2 = \frac{2m_2gx}{m_1+m_2}$.
I.4	Derive both sides with respect to time: $2vv' = \frac{2m_2gx'}{m_1+m_2}$ with $x' = v$ and $v' = a$. $va = \frac{m_2gv}{m_1+m_2} \Rightarrow a = \frac{m_2g}{m_1+m_2}$.

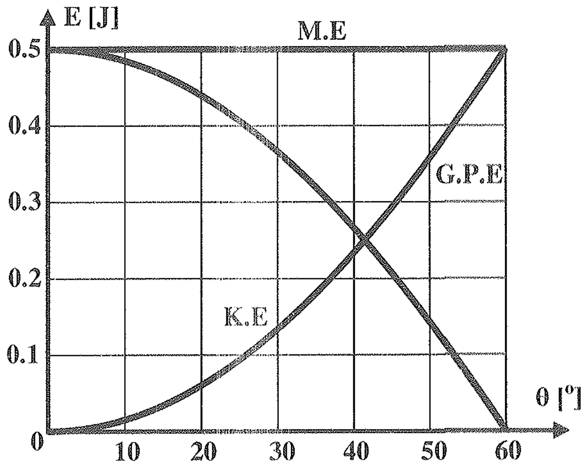
II.1		Inextensible string: $x_1 = x_2 = x$. $v_1 = v_2 = v$. $a_1 = a_2 = a$. Massless pulley: $T_1 = T_2$.
II.2	Isolate (S_1) $\sum \vec{F}_{ext/S_1} = m_1\vec{a}$ $\vec{N} + m_1\vec{g} + \vec{T}_1 = m_1\vec{a}$ Projection along the direction of motion: $T_1 = m_1 a$	Isolate (S_2) $\sum \vec{F}_{ext/S_2} = m_2\vec{a}$ $\vec{T}_2 + m_2\vec{g} = m_2\vec{a}$ Projection along the direction of motion: $-T_2 + m_2 g = m_2 a \Longrightarrow T_2 = m_2 g - m_2 a$
II.3	$T_1 = T_2 \Longrightarrow m_1 a = m_2 g - m_2 a$ $m_1 a + m_2 a = m_2 g \Longrightarrow (m_1 + m_2)a = m_2 g \Longrightarrow a = \frac{m_2 g}{m_1+m_2}$	

Exercise 16:

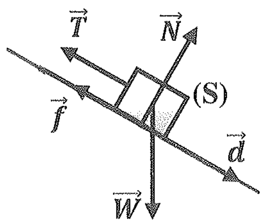
Part	Answer
1	The non-conservative force (friction) is neglected; then, mechanical energy of the system [Adam; Earth] is conserved. By applying the law of conservation of mechanical energy: $M.E_A = M.E_B$. $K.E_A + G.P.E_A = K.E_B + G.P.E_B \Longrightarrow mgR + 0 = mgR\cos\theta + \frac{1}{2}mv_B^2$. $v_B^2 = 2gR(1-\cos\theta) \Longrightarrow v_B = \sqrt{2gR(1-\cos\theta)}$.
2	By applying Newton's 2nd law for translational motion: $\sum \vec{F}_{ext} = m\vec{a} \Longrightarrow \vec{N} + \vec{W} = m\vec{a}$ with $\vec{a} = \vec{a}_t + \vec{a}_N$. Projection along the normal axis: $-N + mg\cos\theta = m\frac{v_B^2}{R}$. $N = mg\cos\theta - 2mg(1-\cos\theta)$. $N = mg(\cos\theta - 2 + 2\cos\theta) \Longrightarrow N = mg(3\cos\theta - 2)$. 
3	$N = 0 \Longrightarrow \cos\theta = \frac{2}{3} \Longrightarrow h = R\cos\theta = 15 \times \frac{2}{3} = 10m$.

Exercise 18:

Part	Answer key
1.1	$M.E_O = K.E_O + G.P.E_O + E.P.E_O = \frac{1}{2}kx_0^2 = \frac{1}{2}(100)(0.1)^2 = 0.5J$
1.2	The non-conservative force (friction) is neglected, then, the mechanical energy is conserved. $M.E_O = M.E_A \Longrightarrow M.E_O = K.E_A + G.P.E_A + E.P.E_A$ $M.E_O = \frac{1}{2}mV_A^2 \Longrightarrow 0.5 = \frac{1}{2}(0.25)V_A^2 \Longrightarrow V_A^2 = 4 \Longrightarrow V_A = 2m/s$
2.1	**1st method:** $\sum \vec{F}_{ext} = \vec{N} + \vec{W} = \vec{0}$ (URM) $V_A = V_B = 2m/s$ **2nd method:** $M.E_A = M.E_B$ (law of conservation of mechanical energy. $K.E_A + G.P.E_A = K.E_B + G.P.E_B \Longrightarrow K.E_A = K.E_B \Longrightarrow V_A = V_B = 2m/s$

2.2	$M.E_C = M.E_B \Longrightarrow G.P.E_C + K.E_C = G.P.E_B + K.E_B$ $mgh_C = \frac{1}{2}mV_B^2 \Longrightarrow h_C = \frac{V_B^2}{2g} = \frac{4}{20} = 0.2m = 20cm$
2.3	$h_C = R - R\cos\theta_m \Longrightarrow \cos\theta_m = \frac{R-h_C}{R} = \frac{0.4-0.2}{0.4} = 0.5 \Longrightarrow \theta_m = 60°$
3.1	$G.P.E_M = mgh_M = mgR(1-\cos\theta) = 0.25 \times 10 \times 0.4 \times (1-\cos\theta)$ $G.P.E_M = 1 - \cos\theta$
3.2	$K.E_M = M.E_M - G.P.E_M = 0.5 - 1 + \cos\theta = \cos\theta - 0.5$
3.3	$G.P.E_M = K.E_M \Longrightarrow 1-\cos\theta = \cos\theta - 0.5 \Longrightarrow \cos\theta = 0.75 \Longrightarrow \theta = 41.4°.$
3.4	

Exercise 20:

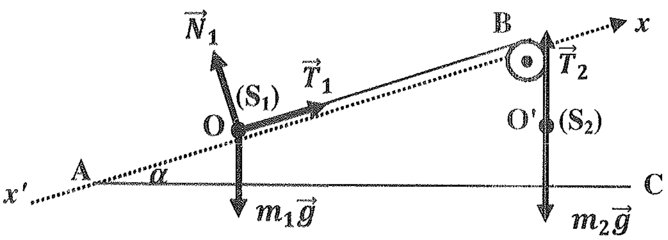
Part	Answer key
1	$\vec{W}$: Weight $\vec{N}$: Normal reaction of a support $\vec{T}$: Tension $\vec{f}$: Friction 
2	Work-energy theorem: $\Delta K.E = \sum W_{ext}$. $K.E_f - K.E_i = W_{\vec{W}} + W_{\vec{N}} + W_{\vec{T}} + W_{\vec{f}}$ with $W_{\vec{W}} = W_{\vec{W_x}} + W_{\vec{W_y}}$. $W_{\vec{W_x}} = \vec{W_x}.\vec{d} = W_x \times d \times \cos(\vec{W_x};\vec{d}) = mg\sin\alpha \times d \times \cos 0° = mg\sin\alpha\, d.$ $W_{\vec{W_y}} = \vec{W_y}.\vec{d} = 0J$ and $W_{\vec{N}} = \vec{N}.\vec{d} = 0J.$ $W_{\vec{f}} = \vec{f}.\vec{d} = f \times d \times \cos(\vec{f};\vec{d}) = f \times d \times \cos 180° = -fd.$ $W_{\vec{T}} = -\Delta E.P.E = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2 = 0 - \frac{1}{2}kd^2 = -\frac{1}{2}kd^2.$ $0 - 0 = mg\sin\alpha\, d + 0 - \frac{1}{2}kd^2 - fd.$ $f = mg\sin\alpha - \frac{1}{2}kd = (4)(10)\left(\frac{1}{2}\right) - \left(\frac{1}{2}\right)(100)(0.2) = 10N.$

Exercise 21:

Part	Answer key
1	The non-conservative force (friction) is neglected, so, the mechanical energy is conserved. $M.E_A = M.E_O \Longrightarrow K.E_A + G.P.E_A = K.E_O + G.P.E_O.$ $\frac{1}{2}mV_A^2 + mgh_A = \frac{1}{2}mV_O^2 + mgh_O$ with $V_A = 0,\ h_A = h$ and $h_O = 0.$

	$mgh = \frac{1}{2}mV_O^2 \Longrightarrow v_O = \sqrt{2gh}$.
2	$M.E_B = K.E_B + G.P.E_B + E.P.E_B = \frac{1}{2}mV_B^2 + mgh_B + \frac{1}{2}kx_B^2$ with $V_B = 0, h_B = -y$ and $x_B = y \Longrightarrow M.E_B = -mgy + \frac{1}{2}ky^2$.
3	The non-conservative force (friction) is neglected, so, the mechanical energy is conserved. $M.E_A = M.E_B \Longrightarrow mgh = \frac{1}{2}ky^2 - mgy \Longrightarrow \frac{1}{2}ky^2 - mgy - mgh = 0$.
4	$\Delta = b^2 - 4ac = (mg)^2 + 2kmgh = (mg)^2\left(1 + \frac{2kh}{mg}\right) > 0$. $y_1 = \frac{-b+\sqrt{\Delta}}{2a} = \frac{mg + mg\sqrt{1+\frac{2kh}{mg}}}{k} = \frac{mg}{k}\left(1 + \sqrt{1 + \frac{2kh}{mg}}\right)$ accepted. $y_2 = \frac{-b-\sqrt{\Delta}}{2a} = \frac{mg - mg\sqrt{1+\frac{2kh}{mg}}}{k} = \frac{mg}{k}\left(1 - \sqrt{1 + \frac{2kh}{mg}}\right)$ rejected.

Exercise 22:

Part	Answer key
1.1	$M.E_0 = M.E_{1_0} + M.E_{2_0} = K.E_{1_0} + G.P.E_{1_0} + K.E_{2_0} + G.P.E_{2_0}$. $M.E_0 = 0 + m_1gh_1 + 0 + m_2gh_2 = m_1gh_1 + m_2gh_2$.
1.2	$M.E_t = M.E_{1_t} + M.E_{2_t} = K.E_{1_0} + G.P.E_{1_0} + K.E_{2_0} + G.P.E_{2_0}$. $M.E_t = \frac{1}{2}m_1v^2 + m_1g(h_1 + x\sin\alpha) + \frac{1}{2}m_2v^2 + m_2g(h_2 - x)$. $M.E_t = \frac{1}{2}(m_1 + m_2)v^2 + m_1g(h_1 + x\sin\alpha) + m_2g(h_2 - x)$.
1.3	$M.E_0 = M.E_t \Longrightarrow m_1gh_1 + m_2gh_2 = \frac{1}{2}(m_1 + m_2)v^2 + m_1g(h_1 + x\sin\alpha) + m_2g(h_2 - x)$. $\frac{1}{2}(m_1 + m_2)v^2 = m_1gh_1 + m_2gh_2 - m_1gh_1 - m_1gx\sin\alpha - m_2gh_2 + m_2gx$. $v^2 = \frac{2(m_2 - m_1\sin\alpha)gx}{m_1+m_2}$.
1.4	Derive both sides with respect to time: $2vv' = \frac{2(m_2-m_1\sin\alpha)gx'}{(m_1+m_2)}$ with $v = x'$ amd $v' = x'' = a$. $2va = \frac{2(m_2-m_1\sin\alpha)gv}{(m_1+m_2)} \Longrightarrow a = \frac{(m_2-m_1\sin\alpha)g}{m_1+m_2}$.
2	
	Isolate (S_1): $\sum\vec{F}_{ext/S_1} = m_1\vec{a} \Longrightarrow \vec{N}_1 + m_1\vec{g} + \vec{T}_1 = m_1\vec{a}$. Projection along the direction of motion: $-m_1g\sin\alpha + T_1 = m_1a \Longrightarrow T_1 = m_1g\sin\alpha + m_1a$. Isolate ($S_2$): $\sum\vec{F}_{ext/S_2} = m_2\vec{a} \Longrightarrow \vec{T}_2 + m_2\vec{g} = m_2\vec{a}$ Projection along the direction of motion:$-T_2 + m_2g = m_2a \Longrightarrow T_2 = m_2g - m_2a$. $T_1 = T_2 \Longrightarrow m_1g\sin\alpha + m_1a = m_2g - m_2a$. $m_1a + m_2a = m_2g - m_1g\sin\alpha \Longrightarrow (m_1 + m_2)a = (m_2 - m_1\sin\alpha)g \Longrightarrow a = \frac{(m_2-m_1\sin\alpha)g}{m_1+m_2}$.

Exercise 24:

Part	Answer
1	The non-conservative force (friction) is neglected; then, the mechanical energy is conserved.
2	1st method: principle of conservation of mechanical energy. $M.E_A = M.E_B \Rightarrow K.E_A + G.P.E_A = K.E_B + G.P.E_B$. $\frac{1}{2}mV_A^2 + mgh_A = \frac{1}{2}mV_B^2 + mgh_B$ with $V_A = 0$, $h_A = L(1-\cos\theta_m)$ and $h_B = 0$. $0 + mgL(1-\cos\theta_m) = \frac{1}{2}mV_B^2$. $V_B = \sqrt{2gL(1-\cos\theta_m)} = \sqrt{0.1 \times 10 \times 0.5 \times (1-\cos 60°)} = \sqrt{5}m/s$. 2nd method: work-kinetic energy theorem. $\Delta K.E = \sum W_{ext} \Rightarrow K.E_B - K.E_A = W_{\vec{T}} + W_{\vec{W}}$. $W_{\vec{W}} = mg(h_A - h_B) = mg[L(1-\cos\theta_m) - 0] = mgL(1-\cos\theta_m)$ and $W_{\vec{T}} = 0J$. $\frac{1}{2}mV_B^2 - 0 = mgL(1-\cos\theta_m)$. $V_B = \sqrt{2gL(1-\cos\theta_m)} = \sqrt{0.1 \times 10 \times 0.5 \times (1-\cos 60°)} = \sqrt{5}m/s$. By applying Newton's 2nd law: $\sum \vec{F}_{ext} = m\vec{a} \Rightarrow \vec{T} + \vec{W} = m\vec{a}$. Projection along the normal axis: $-W + T = ma_n$ with $a_n = \frac{V_B^2}{L}$. $T = W + ma_n = mg + ma_n = m\left(g + \frac{V_B^2}{L}\right) = 0.1\left(10 + \frac{5}{0.5}\right) = 2N$.
3	$M.E_A = M.E \Rightarrow K.E_A + G.P.E_A = K.E + G.P.E$. $0 + mgL(1-\cos\theta_m) = \frac{1}{2}mV^2 + mgL(1-\cos\theta)$. $V^2 = 2gL(\cos\theta - \cos\theta_m) \Rightarrow V = \sqrt{2gL(\cos\theta - \cos\theta_m)}$. $V = \sqrt{2 \times 10 \times 0.5 \times (\cos 22° - \cos 60°)} = 2m/s$.
4	Curve 1: M.E since it is constant with respect to time. Curve 2: G.P.E since at $\theta = 60°; K.E = 0J \text{ and } G.P.E = M.E = 0.25J$. Curve 3: K.E since at $\theta = 0°; G.P.E = 0 \text{ and } K.E = M.E = 0.25J$.
5	At these positions $M.E_C = M.E_D = 0.25J$ since the mechanical energy is conserved. C and D correspond to the position where the kinetic energy and the gravitational potential energy of the ball are equal. $G.P.E_C = K.E_C = \frac{M.E_C}{2} = 0.125J$ (Similarly for D). First method: graphically $\theta_D = 41.5°$ and $\theta_C = -41.5°$. Second method: $G.P.E = mgL(1-\cos\theta)$ $\cos\theta = 1 - \frac{G.P.E}{mgL} = 1 - \frac{0.125}{0.1\times10\times0.5} \Rightarrow \cos\theta = 0.75$. $\theta_D = 41.4°$ and $\theta_C = -41.4°$.

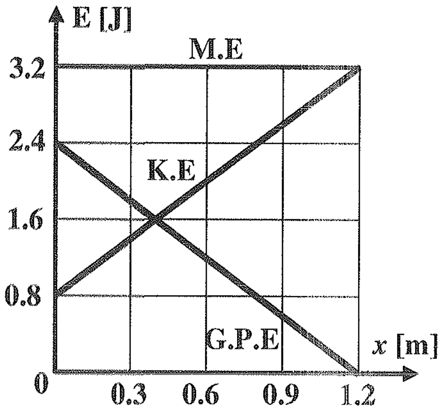
Exercise 25:

Part	Answer
1	The non-conservative force (friction) is neglected; then, the mechanical energy is conserved. $M.E_A = M.E_B \Rightarrow K.E_A + G.P.E_A = K.E_B + G.P.E_B$. $0 + mgh = \frac{1}{2}mV_B^2 + 0 \Rightarrow V_B = \sqrt{2gh}$. $M.E_A = M.E_C \Rightarrow K.E_A + G.P.E_A = K.E_C + G.P.E_C$. $0 + mgh = \frac{1}{2}mV_C^2 + mgR \Rightarrow V_C = \sqrt{2g(h-R)}$. $M.E_A = M.E_D \Rightarrow K.E_A + G.P.E_A = K.E_D + G.P.E_D$.

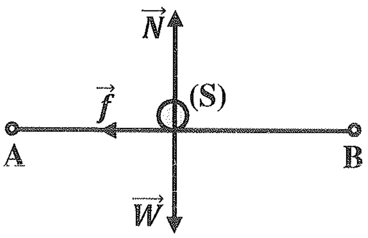
	$0 + mgh = \frac{1}{2}mV_D^2 + mg(2R) \Longrightarrow V_D = \sqrt{2g(h-2R)}$.
2	The external forces acting on (S) at D are: weight $\vec{W}$ and normal reaction of a support $\vec{N}$. Apply Newton's 2nd law: $\sum \vec{F}_{ext} = m\vec{a}$ with $\vec{a} = \vec{a}_t + \vec{a}_n$. $\vec{N} + \vec{W} = m(\vec{a}_t + \vec{a}_n)$. Projection along the normal axis: $N + W = ma_n \Longrightarrow N + mg = \frac{mV_D^2}{R} \Longrightarrow N = \frac{2mg\,(h-2R)}{R} - mg$. For $N = 0 \Longrightarrow \frac{2mg\,(h-2R)}{R} - mg = 0 \Longrightarrow \frac{2mg\,(h-2R)}{R} = mg \Longrightarrow \frac{2h-4R}{R} = 1 \Longrightarrow 2h - 4R = R$ $h = \frac{5}{2}R$.

Exercise 26:

Part	Answer key
I.1.1	At $t_0 = 0s$; $x = 0m \Longrightarrow K.E_0 = 0.8J$. $K.E_0 = \frac{1}{2}mV_0^2 \Longrightarrow V_0 = \sqrt{\frac{2K.E}{m}} = \sqrt{\frac{2\times0.8}{0.4}} = 2m/s$.
I.1.2	The general equation of a straight line is: $K.E = ax + b$. $a = \frac{\Delta K.E}{\Delta x} = \frac{3.2-0.8}{1.2-0} = 2J/m$. For $x = 0$; $K.E = b = 0.8$. Therefore, $K.E = 2x + 0.8$.
I.1.3	For $x_A = 1.2m$; $K.E_A = 3.2J \Longrightarrow V_A = \sqrt{\frac{2K.E_A}{m}} = \sqrt{\frac{2\times3.2}{0.4}} = 4m/s$.
I.2	$G.P.E = mgh = mg(L-x)\sin\alpha = 0.4 \times 10 \times (1.2-x) \times 0.5 = 2.4 - 2x$.
I.3	$M.E = K.E + G.P.E = 2x + 0.8 + 2.4 - 2x = 3.2J$.
I.4	



Part	Answer key
II.1	The external forces acting on (S) are: $\vec{W}$: Weight $\vec{N}$: Normal reaction of a support $\vec{f}$: Friction
II.2	$\Delta K.E = \sum W_{ext} \Longrightarrow K.E_B - K.E_A = W_{\vec{W}} + W_{\vec{N}} + W_{\vec{f}}$. $\frac{1}{2}mV_B^2 - \frac{1}{2}mV_A^2 = -f \times AB + 0 + 0$. $\frac{1}{2}(0.4)(2)^2 - \frac{1}{2}(0.4)(4)^2 = -f \times 5$. $f = 0.48N$.



Part	Answer key
III	$M.E_B = M.E_D \Longrightarrow K.E_B + G.P.E_B + E.P.E_B = K.E_D + G.P.E_D + E.P.E_D$. $\frac{1}{2}mV_B^2 + 0 + 0 = 0 + 0 + \frac{1}{2}kx_m^2 \Longrightarrow mV_B^2 = kx_m^2 \Longrightarrow x_m = \sqrt{\frac{mV_B^2}{k}} = \sqrt{\frac{0.4\times4}{20}} = 0.28m = 28cm$.

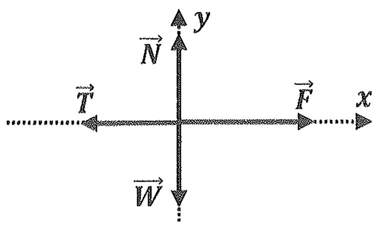
Exercise 28:

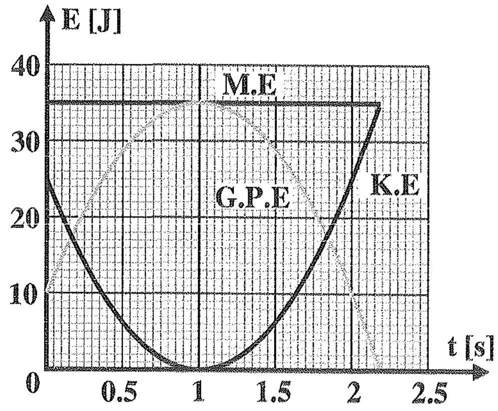
Part	Answer
1	$M.E_0 = K.E_0 + G.P.E_0 = 0 + mgH_0 = 0.4 \times 10 \times 1.2 = 4.8J.$
2	$H_1 = 0.9H_0.$ $H_2 = 0.9H_1 = 0.9^2H_0.$ $H_3 = 0.9H_2 = 0.9^3H_0.$ $H_n = 0.9^nH_0 = 1.2 \times 0.9^n.$ $M.E_n = K.E_n + G.P.E_n = 0 + mgH_n = 0.4 \times 10 \times 1.2 \times 0.9^n = 4.8 \times 0.9^n.$
3	$M.E_4 = 4.8 \times 0.9^4 = 3.15J.$ The non-conservative force (friction) is neglected; then, the mechanical energy is conserved. $M.E_4 = K.E + G.P.E.$ $M.E_4 = \frac{1}{2}mV^2 + mgh$ with $h = 0.$ $3.15 = \frac{1}{2}(0.4)V^2 \Longrightarrow V = 3.96m/s.$
4	$M.E_5 = 4.8 \times 0.9^5 = 2.83J.$ $\Delta M.E = M.E_5 - M.E_0 = 2.83J - 4.8 = -1.97J.$
5	$H_n = \frac{H_0}{2} \Longrightarrow \frac{1.2}{2} = 1.2 \times 0.9^n \Longrightarrow 0.9^n = 0.5 \Longrightarrow n = \frac{\ln 0.5}{\ln 0.9} = 6.$

Exercise 30:

Part	Answer
1	By applying the law of conservation of mechanical energy: $M.E_A = M.E_B.$ $K.E_A + G.P.E_A = K.E_B + G.P.E_B \Longrightarrow 0 + mgh = \frac{1}{2}mV_B^2 + mgL_0 \sin\alpha.$ $V_B = \sqrt{2g(h - l_0 \sin\alpha)} = \sqrt{2 \times 10 \times (4 - 2 \times \sin 30°)} = \sqrt{60}m/s.$
2.1	$\sin\alpha = \frac{y}{L_0 - x_m} \Longrightarrow \sin 30° = \frac{y}{2 - x_m} \Longrightarrow \frac{1}{2} = \frac{y}{2 - x_m} \Longrightarrow 2 - x_m = 2y \Longrightarrow x_m = 2 - 2y.$
2.2	The non-conservative force (friction) is neglected; then, the mechanical energy is conserved. By applying the law of conservation of mechanical energy: $M.E_A = M.E_C \Longrightarrow K.E_A + G.P.E_A + E.P.E_A = K.E_C + G.P.E_C + E.P.E_C.$ $0 + mgh + 0 = 0 + mgy + \frac{1}{2}kx_m^2 \Longrightarrow mgh = mgy + \frac{1}{2}k(2 - 2y)^2.$ $2 \times 10 \times 4 = 2 \times 10 \times y + \frac{1}{2} \times 50 \times (4 + 4y^2 - 8y).$ $80 = 20y + 100 + 100y^2 - 200y \Longrightarrow 100y^2 - 180y + 20 = 0 \Longrightarrow y^2 - 1.8y + 0.2 = 0.$
2.3	$y_1 = 1.68m$ and $y_2 = 0.118m.$ $x_{m1} = 2 - 2y_1 = 2 - 2 \times 1.68 = -1.36m$ (rejected). $x_{m2} = 2 - 2y_2 = 2 - 2 \times 0.118 = 1.764m$ (accepted).

Exercise 31:

Part	Answer
1.1	$\vec{W}$: Weight. $\vec{N}$: Normal reaction of a support. $\vec{F}$: Applied force. $\vec{T}$: Tension. 
1.2	$W_{\vec{T}} = \int_{x_O}^{x_A} Tdx = \int_{x_O}^{x_A} -kxdx = \left[-\frac{1}{2}kx^2\right]_{x_O}^{x_A} = -\frac{1}{2}kx_A^2 + \frac{1}{2}kx_O^2 = -\frac{1}{2}kx_A^2$ with $x_O = 0m.$

1.3	Apply the work-kinetic energy theorem: $\Delta K.E = \sum W_{ext}$. $K.E_A - K.E_O = W_{\vec{N}} + W_{\vec{W}} + W_{\vec{T}} + W_{\vec{F}} \Longrightarrow \frac{1}{2}mV_A^2 - \frac{1}{2}mV_O^2 = W_{\vec{N}} + W_{\vec{W}} + W_{\vec{F}} + W_{\vec{T}}$. $W_{\vec{T}} = -\frac{1}{2}kx_A^2 = -\frac{1}{2}\times 40 \times 0.25^2 = -1.25J$. $W_{\vec{F}} = \vec{F}.\overrightarrow{OA} = F \times AB \times \cos(\vec{F}; \overrightarrow{OA}) = 20 \times 0.25 \times \cos 0° = 5J$. $W_{\vec{N}} = W_{\vec{W}} = 0J$ (the two forces are perpendicular to the displacement; thus, they do no work) $\frac{1}{2}(0.5)V_A^2 - 0 = 0 + 0 + 5 - 1.25 \Longrightarrow V_A = \sqrt{15}m/s$.
2.1	The non-conservative force (friction) is neglected; then, the mechanical energy is conserved. Denote by B the position of (S) where it stops and returns back. $M.E_A = M.E_B \Longrightarrow K.E_A + G.P.E_A + E.P.E_A = K.E_B + G.P.E_B + E.P.E_B$. $\frac{1}{2}mV_A^2 + mgh_A + \frac{1}{2}kx_A^2 = \frac{1}{2}mV_B^2 + mgh_B + \frac{1}{2}kx_B^2$ with $h_A = h_B = 0m$. $\frac{1}{2}(0.5)(15) + 0 + \frac{1}{2}(40)(0.25)^2 = 0 + 0 + \frac{1}{2}(40)x_B^2$. $3.75 + 1.25 = 20x_B^2 \Longrightarrow x_B = \overline{OB} = 0.5m = 50m$. $AB = x_B - x_A = 50cm - 25cm = 25cm$.
2.2	(S) reaches the position D such that $x_D = \overline{OD} = -50cm$. $d = 60cm - 50cm = 10cm$.
2.4	
2.5	$K.E = G.P.E$. $25t^2 - 50t + 25 = -25t^2 + 50t + 10$. $50t^2 - 100t + 15 = 0$. $t_1 = 1.836s$ and $t_2 = 0.16s$.
2.6	$W_{\vec{W}} = mg(h_0 - h_A) = 0.5 \times 10 \times (2 - 5) = -15J$ with $h_A = y_A = -5(1)^2 + 10(1) = 5m$. $\Delta G.P.E = G.P.E_A - G.P.E_0 = mgh_A - mgh_0 = mg(h_A - h_0) = 0.5 \times 10 \times (5 - 2) = 15J$. Therefore, $\Delta G.P.E = -W_{\vec{W}}$.

Exercise 33:

Part	Answer
1	$a = \frac{m_R x_R + m_S x_S}{m_R + m_S} = \frac{m\left(\frac{L}{2}\right) + md}{m + m} = \frac{\frac{L}{2} + d}{2} = \frac{L + 2d}{4} = \frac{0.4 + 2 \times 0.3}{4} = 0.25m = 25cm$.
2	$I = I_R + I_s = \frac{1}{3}mL^2 + md^2 = \frac{1}{2}(0.2)(0.4)^2 + (0.2)(0.3)^2 = 0.034kgm^2$.
3	$M.E_0 = K.E_0 + G.P.E_0 = 0 + (m_S + m_R)gh = -(m_R + m_S)ga\cos\theta_0$. $M.E_0 = -0.4 \times 10 \times 0.25 \times \cos 60° = -0.5J$.
4	The forces of friction are neglected; then, the mechanical energy is conserved. By applying the law of conservation of mechanical energy: $M.E = M.E_0$. $K.E + G.P.E = M.E_0 \Longrightarrow \frac{1}{2}I\theta'^2 - (m_S + m_R)ga = M.E_0$. $\frac{1}{2}(0.034)\theta'^2 - 0.4 \times 10 \times 0.25 = -0.5 \Longrightarrow \theta' = 5.4rd/s$.

Exercise 34:

Part	Answer
1.1	$M.E_0 = K.E_O + G.P.E_0 = \frac{1}{2} I\theta_0'^2 + Mgh_0$ with $\theta' = 0$ and $h_0 = -OG\cos\theta_0 = -R\cos\theta$. $M.E_0 = 0 - MgR\cos\theta_0 = -0.3 \times 10 \times 0.1 \times \cos 60° = -0.15J$.
1.2	The non-conservative force (friction) is neglected; then, the mechanical energy is conserved. By applying the principle of conservation of mechanical energy: $M.E_0 = M.E_1$. $M.E_0 = K.E_1 + G.P.E_1 \Longrightarrow M.E_0 = \frac{1}{2} I\theta_1'^2 + Mgh_1$ with $h_1 = -R$. $M.E_0 = \frac{3}{4} MR^2\theta'^2 - MgR \Longrightarrow -0.15 = \left(\frac{3}{4}\right)(0.3)(0.1)^2\theta_1'^2 - (0.3)(10)(0.1)$. $-0.15 = 2.25 \times 10^{-3}\theta_1'^2 - 0.3 \Longrightarrow \theta_1' = 8.16 rad/s$.
2.1	$I' = I_D + I_S = \frac{3}{2} MR^2 + m(2R)^2 = \frac{3}{2} \times 0.3 \times 0.1^2 + 0.1 \times 0.2^2 = 8.5 \times 10^{-3} kgm^2$.
2.2	$a = OG' = \frac{m_D x_D + m_S x_S}{m_D + m_S} = \frac{M \times R + m \times 2R}{M+m} = \frac{0.3\times0.1+0.1\times0.2}{0.3+0.1} = 0.125m = 12.5cm$.
2.3	By applying the principle of conservation of mechanical energy: $M.E_2 = M.E_0$. $K.E_2 + G.P.E_2 = K.E_0 + G.P.E_0 \Longrightarrow \frac{1}{2} I'\theta_2'^2 - (m+M)ga = 0 - (m+M)ga\cos\theta_0$. $\left(\frac{1}{2}\right)(8.5 \times 10^{-3})\theta_2'^2 - 0.4 \times 10 \times 0.125 = -0.4 \times 10 \times 0.125 \times \cos 60°$. $4.25 \times 10^{-3}\theta_2'^2 - 0.5 = -0.25 \Longrightarrow \theta_2' = 7.67 rad/s$.

Exercise 36:

Part	Answer
1	At the instant t, the two grooves have the same angular abscissa: $\theta_1 = \theta_2 \Longrightarrow \frac{x}{r} = \frac{X}{R} \Longrightarrow x = \frac{r}{R} X$.
2	**Isolate (b):** Apply Newton's 2nd law for translational motion: $\sum \vec{F}_{ext/(b)} = m\vec{a}_1 \Longrightarrow \vec{T}_1 + \vec{W}_1 + \vec{N} = m\vec{a}_1$. Projection along the direction of motion: $T_1 = ma_1 = mr\theta_1''$. **Isolate (B):** Apply Newton's 2nd law for translational motion: $\sum \vec{F}_{ext/M} = M\vec{a}_2 \Longrightarrow \vec{T}_2 + \vec{W}_2 = M\vec{a}_2$. Projection along the direction of motion: $Mg - T_2 = Ma_2 \Longrightarrow T_2 = Mg - Ma_2 = Mg - MR\theta_2''$. **Isolate (P):** Apply Newton's 2nd law for rotational motion: $\sum \mathcal{M}_{ext} = I\theta''$. $\mathcal{M}_{\vec{T}_1'} + \mathcal{M}_{\vec{T}_2'} + \mathcal{M}_{\vec{W}_P} + \mathcal{M}_{\vec{R}} = I\theta'' \Longrightarrow T_2'R - T_1'r + 0 + 0 = I\theta'' \Longrightarrow T_2'R - T_1'r = I\theta''$. The strings ($T_1$) and ($T_2$) are massless: $T_1 = T_1'$ and $T_2 = T_2'$ At the instant t, the two grooves have the same angular acceleration since: $\theta_1 = \theta_2 \Longrightarrow \theta_1'' = \theta_2'' = \theta''$. We get the following system: $T_1 = mr\theta''$ ... (1) $T_2 = Mg - MR\theta''$ ... (2) $T_2R - T_1r = I\theta''$ ... (3) Replace (1) and (2) in (3): $MgR - MR^2\theta'' - mr^2\theta'' = I\theta'' \Longrightarrow \theta'' = \frac{MgR}{MR^2+mr^2+I}$.
3	$a_1 = r\theta'' = \frac{MgRr}{MR^2+mr^2+I}$ and $a_2 = R\theta'' = \frac{MgR^2}{MR^2+mr^2+I}$.
4	$T_1 = ma_1 = \frac{MmgRr}{MR^2+mr^2+I}$ and $T_2 = Mg - Ma_2 = Mg - \frac{M^2R^2g}{MR^2+mr^2+I} = Mg\left(1 - \frac{MR^2}{MR^2+mr^2+I}\right)$.
5	At the instant t, the two grooves have the same angular speed: $\theta_1' = \theta_2' = \theta'$. $\frac{v}{r} = \frac{V}{R} \Longrightarrow V = \frac{R}{r} v$. $K.E = \frac{1}{2} mv^2 + \frac{1}{2} MV^2 + \frac{1}{2} I\theta'^2 = \frac{1}{2} mv^2 + \frac{1}{2}\frac{MR^2}{r^2} v^2 + \frac{1}{2} I \frac{v^2}{r^2} = \frac{1}{2}\left(m + \frac{MR^2}{r^2} + \frac{I}{r^2}\right) v^2$.

Exercise 37:

Part	Answer
1	Apply the work kinetic energy theorem on the system [(P); (S); String]: $\Delta K.E = \sum W \Longrightarrow K.E_f - K.E_i = W_{\overrightarrow{W_S}} + W_{\vec{N}} + W_{\vec{f}} + W_{\overrightarrow{W_P}} + W_{\vec{R}} + W_{\vec{T}}$. **Note:** the string is inextensible; then, the net work done by all tensions forces along any displacement is zero. $W_{\vec{N}} = 0$ (perpendicular to the displacement). $W_{\vec{R}} = W_{\overrightarrow{W_P}} = 0$ since $\overrightarrow{W_P}$ and $\vec{R}$ pass through the center of rotation $\Longrightarrow \mathcal{M}_{\overrightarrow{W_P}} = \mathcal{M}_{\vec{R}} = 0$. $\left(\frac{1}{2}Mv^2 + \frac{1}{2}I\theta'^2\right) - 0 = +Mgx\sin\alpha + 0 - fx + 0 + 0$ with $v = R\theta'$. $\frac{1}{2}Mv^2 + \frac{1}{4}mr^2\frac{v^2}{r^2} = (Mg\sin\alpha - f)x \Longrightarrow 2Mv^2 + mv^2 = 4(Mg\sin\alpha - f)x$ $v^2 = \frac{4(Mg\sin\alpha - f)}{m+2M}x$
2	Derive both sides with respect to time: $2vv' = \frac{4(Mg\sin\alpha - f)}{m+2M}x'$ with $v = x'$ and $a = v' = x''$. $a = \frac{2(Mg\sin\alpha - f)}{m+2M} = \frac{2(3\times10\times\sin30^\circ - 12)}{2+2\times3} = 0.75\text{m/s}^2$.

Exercise 38:

Part	Answer
1	$M.E_0 = M.E_{S_1} + M.E_{S_2} + M.E_P$. $M.E_0 = K.E_{S_1} + G.P.E_{S_1} + K.E_{S_2} + G.P.E_{S_2} + K.E_P + G.P.E_P$. $M.E_0 = 0 + m_1gh + 0 + 0 + 0 + \mu gH = m_1gh + \mu gH$.
2.1	$M.E_t = M.E_{S_1} + M.E_{S_2} + M.E_P$. $M.E_t = K.E_{S_1} + G.P.E_{S_1} + K.E_{S_2} + G.P.E_{S_2} + K.E_P + G.P.E_P$. $M.E_t = \frac{1}{2}m_1V_1^2 + m_1g(h-x) + \frac{1}{2}m_2V_2^2 + m_2gx + \frac{1}{2}I\theta'^2 + \mu gH$. $V_1 = V_2 = V$ and $V = R\theta' \Longrightarrow \theta' = \frac{V}{R}$. $M.E_t = \frac{1}{2}m_1V^2 + m_1g(h-x) + \frac{1}{2}m_2V^2 + m_2gx + \frac{1}{2}\frac{I}{R^2}V^2 + \mu gH$ with $\frac{I}{R^2} = \frac{\mu}{2}$. $M.E_t = \frac{1}{2}\left(m_1 + m_2 + \frac{\mu}{2}\right)V^2 + m_1g(h-x) + m_2gx + \mu gH$.
2.2	The non-conservative force (friction) is neglected; then, the mechanical energy is conserved. $M.E_0 = M.E_t \Longrightarrow m_1gh + \mu gH = \frac{1}{2}\left(m_1 + m_2 + \frac{\mu}{2}\right)V^2 + m_1g(h-x) + m_2gx + \mu gH$. $\frac{1}{2}\left(m_1 + m_2 + \frac{\mu}{2}\right)V^2 = m_1gx - m_2gx \Longrightarrow V^2 = \frac{2(m_1-m_2)gx}{m_1+m_2+\frac{\mu}{2}} \Longrightarrow V = \sqrt{\frac{2(m_1-m_2)gx}{m_1+m_2+\frac{\mu}{2}}}$.
2.3	Derive both sides with respect to time: $2VV' = \frac{2(m_1-m_2)gx'}{m_1+m_2+\frac{\mu}{2}}$ with $V = x'$ and $a = V'$. Therefore, $a = \frac{(m_2-m_1)g}{m_1+m_2+\frac{\mu}{2}}$.

CHAPTER 2 – LINEAR MOMENTUM
COURSE

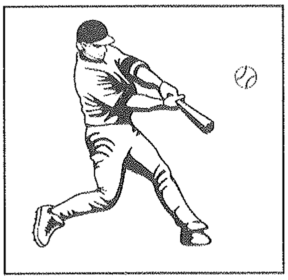
Have you ever seen a car accident? Was it a head-on collision? Was one car travelling faster than the other? Would you have reacted differently if the object in your path was a base ball as opposed to a parked car? In either case, the masses of the objects involved and their velocities play a role in a collision. In this chapter, we will look at the effects of mass and velocity on objects involved in collisions and how these quantities are related in momentum. We will also study the conservation of linear momentum in one dimensional and two dimensional systems. The concept of center of mass in relation to linear momentum will also be discussed.

2.1- PREREQUISITES

Position, velocity and acceleration in the Cartesian coordinate system

Case of a particle		
	One dimensional motion	**Two dimensional motion**
Position	$\vec{x} = x\vec{i}$ $v = \frac{dx}{dt} \Rightarrow x = \int v dt$	$\vec{r} = x\vec{i} + y\vec{j}$ $\vec{v} = \frac{d\vec{r}}{dt} \Rightarrow \vec{r} = \int \vec{v} dt$
Velocity	$v = \frac{dx}{dt} = x'$ $a = \frac{dv}{dt} \Rightarrow v = \int a dt$	$\vec{v} = \frac{d\vec{r}}{dt} = \vec{r}' = v_x\vec{i} + v_y\vec{j}$ $\vec{a} = \frac{d\vec{v}}{dt} \Rightarrow \vec{v} = \int \vec{a} dt$
Acceleration	$a = \frac{dv}{dt} = v'$	$\vec{a} = \frac{d\vec{v}}{dt} = \vec{v}' = a_x\vec{i} + a_y\vec{j}$
Case of a system of particles		
	One dimensional motion	**Two dimensional motion**
Position	$X_G = \frac{m_1x_1+m_2x_2+m_3x_3+\cdots}{m_1+m_2+m_3+\cdots}$	$\vec{r}_G = \frac{m_1\vec{r}_1+m_2\vec{r}_2+m_3\vec{r}_3+\cdots}{m_1+m_2+m_3+\cdots}$
Velocity	$V_G = \frac{dx_G}{dt} = x_G'$	$\vec{V}_G = \frac{d\vec{r}}{dt} = \vec{r}_G'$
Acceleration	$a_G = \frac{dV_G}{dt} = V_G'$	$\vec{a}_G = \frac{d\vec{V}_G}{dt} = \vec{V}_G'$

2.2- NOTION OF LINEAR MOMENTUM



Momentum is the key to success in many sporting events, including baseball, football, tennis …

A fast moving car has more linear momentum than a slow moving car of the same mass.

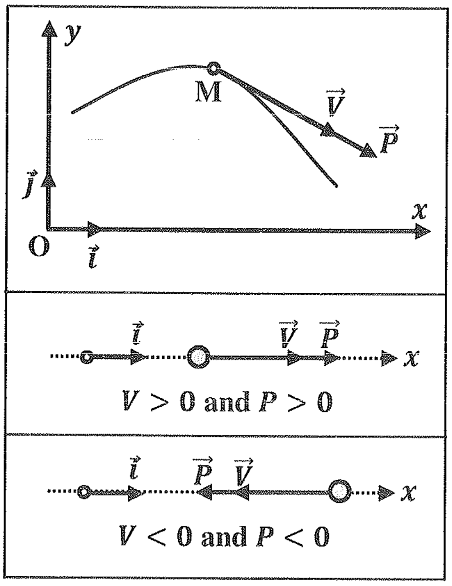
A heavy truck has more linear momentum than a small car moving at the same speed.

$m \longrightarrow \vec{V}_1$ $m \longrightarrow \vec{V}_2$	$m_1 \longrightarrow \vec{V}$ $m_2 \longrightarrow \vec{V}$
$V_2 > V_1$ **More speed ⟹ harder to stop**	$m_2 > m_1$ **More mass ⟹ harder to stop**

The more momentum an object has, the harder it is to stop and the greater effect it will have if it is brought to rest by striking another object.

The concept of linear momentum was first developed by Sir Isaac Newton, who thought that a change in momentum was caused by a force. He called linear momentum "the quantity of motion" and combined a moving object's mass and velocity in the following way: $\vec{P} = m\vec{V}$ where $\vec{P}$ is the object's linear momentum. [kgm/s] is the SI unit of linear momentum, m is the mass of the object in [kg] and $\vec{V}$ is the velocity of the object in [m/s].

Linear momentum is a vector quantity that has the same line of action and direction as the velocity of the object since the mass of the object is always positive.



- In the case of one dimensional motion (rectilinear motion) in the system $(O; \vec{i})$, the linear momentum of a particle can be written in the form: $P = mV$ where P and V are the algebraic values of the linear momentum and the velocity of the moving particle respectively.
- In case of two dimensional motion in the Cartesian coordinate system $(O; \vec{i}; \vec{j})$, the velocity, at any instant t, can be written in the form: $\vec{V} = \vec{V}_x + \vec{V}_y = V_x\vec{i} + V_y\vec{j}$.

  The instantaneous speed of the particle is: $V = \|\vec{V}\| = \sqrt{V_x^2 + V_y^2}$.

  The linear momentum of the particle at any instant t is:

  $$\vec{P} = m\vec{V} = mV_x\vec{i} + mV_y\vec{j} = P_x\vec{i} + P_y\vec{j} = \vec{P}_x + \vec{P}_y$$

  The magnitude of the linear momentum is: $P = \|\vec{P}\| = \sqrt{P_x^2 + P_y^2}$.

2.3- LINEAR MOMENTUM AND IMPULSE

In order to change an object's momentum, we must change either its velocity or its mass. To change velocity, we need to apply a net force $\sum \vec{F}_{ext}$ on the object. For example, a car changes its speed as a force is applied during a duration Δt, thus changing the linear momentum of the car. Newton suggested that the rate of change of momentum in an object is directly proportional to the net force applied on it. Mathematically, this change in momentum with respect to time can be written as:

$$\sum \vec{F}_{ext} = \frac{\Delta\vec{P}}{\Delta t} \text{ where } \Delta\vec{P} = \vec{P}_{final} - \vec{P}_{initial}$$

The direction of the net force is the same as the direction of the change in momentum.

The change in momentum is called impulse $\vec{J}$. Mathematically, $\vec{J} = \Delta\vec{P}$ where $\Delta\vec{P} = \sum \vec{F}_{ext}\,\Delta t$.

Then, $\vec{J} = \sum \vec{F}_{ext}\,\Delta t = \Delta\vec{P} = \vec{P}_{final} - \vec{P}_{initial}$, where $\vec{J}$ is measured in [Ns], $\sum \vec{F}_{ext}$ in [N] and Δt in [s].

$\sum \vec{F}_{ext}$ is constant or considered as the average net force.

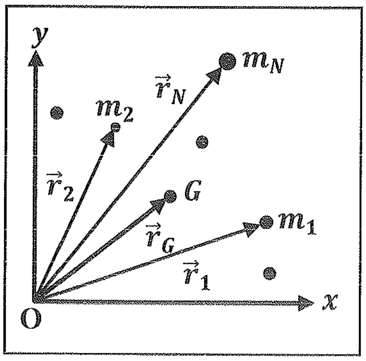
The impulse $\vec{J}$ is calculated by the area under the force-time graph if $\sum \vec{F}_{ext}$ is constant or not.

2.4- LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

The linear momentum $\vec{P}_S$ of a system of particles is the sum of the linear momenta of its particles:

$$\vec{P}_S = \vec{P}_1 + \vec{P}_2 + \cdots + \vec{P}_N = m_1\vec{V}_1 + m_2\vec{V}_2 + \cdots + m_N\vec{V}_N$$

$$\vec{P}_S = \textstyle\sum_{i=1}^{n} \vec{p}_i = \sum_{i=1}^{n} m_i\vec{v}_i$$



Linear momentum of the center of mass of a system of particles

The position vector of the center of mass G of a system of particles is:

$\vec{r}_G = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n}{M} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{M}$ with $M = m_1 + m_2 + \cdots + m_N = \sum_{i=1}^{n} m_i$.

$M\vec{r}_G = m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_n\vec{r}_n$ (Derive both sides with respect to time).
$M\vec{V}_G = m_1\vec{V}_1 + m_2\vec{V}_2 + \cdots + m_n\vec{V}_n = \sum_{i=1}^{n} m_i\vec{V}_i$.
$\vec{P}_G = \vec{P}_1 + \vec{P}_2 + \cdots + \vec{P}_N = \vec{P}_S$.
The linear momentum $\vec{P}_S$ of a system of particles of constant mass is equal to that of its center of mass G where the total mass is assumed to be concentrated. Then $\vec{P}_S = \vec{P}_G = M\vec{V}_G$.

2.5- NEWTON'S SECOND LAW

General expression

A force applied to an object changes its velocity, and consequently its linear momentum. The more rapid the variation of this linear momentum, the stronger the applied force causing this variation is. This variation is governed by the general expression of Newton's 2nd law:

$$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$$

This law is stated as follows:
The time derivative of the linear momentum of a system of particles, in an inertial frame, is equal to the resultant of all external forces applied on this system.

Newton's 2nd law: $\vec{F}_{net}.\Delta t = M.\Delta\vec{V}$

$\vec{F}_{net}.\Delta t$ (cause)	$M.\Delta\vec{V}$ (effect)
$\vec{J}$ (impulse)	$\Delta\vec{P}$ (change in momentum)
[Ns]	[kgm/s]

The linear momentum of a system of particles is equal to the linear momentum of its center of mass:

$$\vec{P}_S = \vec{P}_G = \vec{P} = M\vec{V}_G$$

Then, $\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} = \frac{d(M\vec{V}_G)}{dt} = \frac{dM}{dt}\vec{V}_G + M\frac{d\vec{V}_G}{dt}$.

$$\sum \vec{F}_{ext} = \frac{dM}{dt}\vec{V}_G + M\vec{a}_G$$

Case 1: system of constant mass

$M = constant \Longrightarrow \frac{dM}{dt} = 0$; then, $\sum \vec{F}_{ext} = M\vec{a}_G$ (theorem of center of mass or inertia).
The theorem of center of mass can be stated as:
In an inertial frame, the sum of external forces applied on a system of constant mass is equal to the product of the mass of this system by the acceleration of its center of mass.

Case 2: system with varying mass

A system of particles whose mass M is variable (moving rocket as an example) is moving under the action of external forces $\sum \vec{F}_{ext}$.
Then, $\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} = \frac{dM}{dt}\vec{V}_G + M\vec{a}_G$ where $\frac{dM}{dt} \neq 0$.
The expression of Newton's second law given by $\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$ is more general than $\sum \vec{F}_{ext} = M\vec{a}_G$ since it may include cases where mass varies.

TIP

An inertial frame of reference is the one in which the principle of inertia is applied. Inertial frames move with respect to each other with a uniform rectilinear motion, the surface of the Earth is assumed, for a short period, to be an inertial frame.

2.6- CONSERVATION OF LINEAR MOMENTUM IN ONE DIMENSION

The law of conservation of energy states that energy cannot be created nor destroyed; it can only change from one form to another. The same concept holds true in linear momentum. In the 17th century, Newton recognized that linear momentum is conserved in a collision. The total linear momentum of a system before a collision is equal to the total linear momentum of the system after the collision. It applies to all collisions as long the net external force acting on the system is zero.

Isolated (mechanically isolated) system
A system is said to be isolated if the net external force acting on the system is zero.

$$\sum \vec{F}_{ext} = \vec{0}$$

Statement of the law of conservation of linear momentum
The linear momentum of an isolated system remains constant as time varies.

$$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} = \vec{0} \Longrightarrow \vec{P} = constant\ at\ any\ time$$

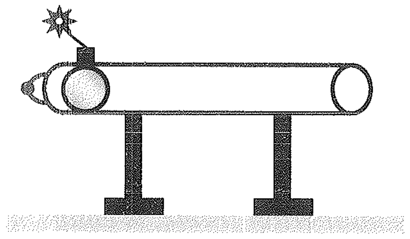
TIP

$\frac{d\vec{P}}{dt} \simeq \frac{\Delta\vec{P}}{\Delta t}$ *if*:
- **Linear momentum varies linearly with time.**
- **Δt is very small ($\Delta t \to 0$).**
- **$\sum \vec{F}_{ext}$ is constant or taken as average net force.**

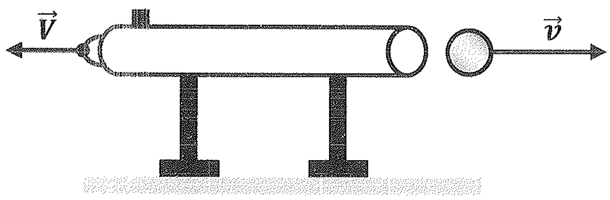
Applications on the conservation of linear momentum

1- Recoil of a cannon

A cannon consists of two parts, the gun and the shell.
The system (gun; shell), initially at rest, is subjected to its weight and the force due to the support. The two forces cancel each other, and the system is isolated.
Before shooting, the system is at rest and its linear momentum is zero: $\vec{P}_i = \vec{0}$.
At the moment of being fired, the explosion of the powder produces a gas and the shell is ejected. The forces exerted by the gas on both gun and shell are internal ones with respect to the (gun, shell) system. The sum of external forces remains zero and the system is always isolated.
Therefore its linear momentum is conserved.
Immediately after being fired, the shell of mass m leaves the gun with a velocity $\vec{v}$ and the gun of mass M recoils at a velocity $\vec{V}$.



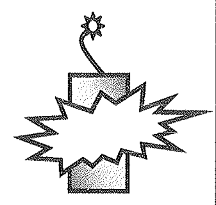
$\vec{P}_i = \vec{0}$



$\vec{P}_f = M\vec{V} + m\vec{v}$

The linear momentum of the system is then: $\vec{P}_f = m\vec{v} + M\vec{V}$.
We neglect the linear momentum of gas produced by the explosion of the powder relative to that of the shell.
$\vec{P}_i = \vec{P}_f \Longrightarrow \vec{0} = m\vec{v} + M\vec{V} \Longrightarrow \vec{V} = -\frac{m}{M}\vec{v}$.
This relation shows that $\vec{V}$ and $\vec{v}$ are of opposite directions.

2- Explosions



If the explosion of a solid, which is torn into pieces, lasts for a very short time, the external forces are neglected relative to the internal forces due to explosion. The solid can be considered as isolated during explosion. Then, $\vec{P}_i = \vec{P}_f$.

3- Collisions

In physics, a collision is an event in which two or more bodies exert forces on each other for a relatively short time. Collisions usually last for a very short time, during which external forces are neglected with respect to the huge internal forces. Consequently, the system of the colliding objects can be considered as isolated.
If the net external force on all objects that are involved in a collision is zero, the system is an isolated system (mechanically isolated). The conservation of linear momentum is written as: $\vec{P}_i = \vec{P}_f$
For two solid objects (particles) colliding, we write:

$$\vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2' \Longrightarrow m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_1' + m_2\vec{V}_2'$$

m_1 and m_2 are the masses of the colliding objects.
$\vec{V}_1$ and $\vec{V}_1'$ are the velocities of the center of mass of the first object just before and after collision respectively.
$\vec{V}_2$ and $\vec{V}_2'$ are the velocities of the center of mass of the second object just before and after collision respectively.

In physics, a collision can be classified by the change in the total kinetic energy of the system before and after the collision.

- If most or all the total kinetic energy is lost (dissipated as heat, sound, deformation, ... etc or absorbed by the objects themselves), the collision is said to be **inelastic or non-elastic collision**. An example on such a collision is a car crash.
  In **a perfectly inelastic collision**, the colliding objects stick together and move with the same velocity after collision.
- If most of the kinetic energy is conserved where the objects continue moving afterwards, the collision is said to be **elastic collision**. An example of such a collision is a baseball bat hitting a baseball.
  The sound of the bat hitting the ball represents the loss of energy.
- If all of the total kinetic energy is conserved where no energy is released as sound, heat, etc, the collision is said to be **perfectly elastic collision**. Such a system is an idealization and cannot occur in nature.

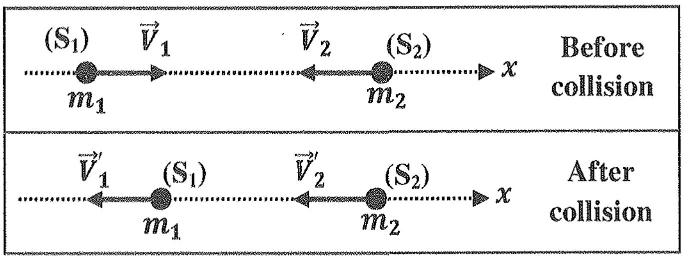
Types of collisions:
There are two types of collision between two objects:

1- **Head-on collision or one-dimensional collision:** the velocity of each object just before collision is along the line of collision (collinear velocities).
2- **Non-head-on collision, oblique collision or two dimensional collision:** the velocity of each object just before collision is not along the line of collision (non-collinear velocities).

In conclusion, the linear momentum of a system is conserved during collision, whether the collision is elastic or non-elastic. However, the kinetic energy of the system is only conserved during perfectly elastic collision.

2.7- PERFECTLY ELASTIC HEAD ON COLLISION

Two solids (S_1) and (S_2), of masses m_1 and m_2 respectively and moving horizontally with respective velocities $\vec{V}_1$ and $\vec{V}_2$, enter into a perfectly elastic head-on collision and then move with respective velocities $\vec{V}_1'$ and $\vec{V}_2'$ after collision.



Note: the velocities are drawn in arbitrary directions

During collision, the system (S): [(S_1); (S_2)] is isolated:
$\sum \vec{F}_{ext} = \vec{0} \Longrightarrow \frac{d\vec{P}_S}{dt} = \vec{0} \Longrightarrow \vec{P}_S = constant.$
Principle of conservation of linear momentum
$\vec{P}_i = \vec{P}_f$.
$\vec{P}_i$: linear momentum of (S) just before collision.
$\vec{P}_f$: linear momentum of (S) just after collision.
$\vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2'$.
$m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_1' + m_2\vec{V}_2'$.

The collision is head-on and the velocities are collinear; then, they can be expressed by their algebraic values: $m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2' \Longrightarrow m_1(V_1 - V_1') = m_2(V_2' - V_2)$ ... (1)
The collision is perfectly elastic; then, the kinetic energy of (S) is conserved: $K.E_i = K.E_f$.
$K.E_1 + K.E_2 = K.E_1' + K.E_2'$.
$\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 \Longrightarrow m_1(V_1^2 - V_1'^2) = m_2(V_2'^2 - V_2^2)$.
$m_1(V_1 - V_1')(V_1 + V_1') = m_2(V_2' - V_2)(V_2' + V_2)$ ... (2)
Divide (2) by (1), we get: $V_1 + V_1' = V_2' + V_2 \Longrightarrow V_2' = V_1 + V_1' - V_2$ ... (3)
Substitute (3) in (1): $m_1(V_1 - V_1') = m_2(V_1 + V_1' - V_2 - V_2)$.
$m_1V_1 - m_1V_1' = m_2V_1 + m_2V_1' - 2m_2V_2$.
$(m_1 + m_2)V_1' = (m_1 - m_2)V_1 + 2m_2V_2$.
$V_1' = \frac{m_1-m_2}{m_1+m_2}V_1 + \frac{2m_2}{m_1+m_2}V_2$.
Similarly we get: $V_2' = \frac{m_2-m_1}{m_1+m_2}V_2 + \frac{2m_1}{m_1+m_2}V_1$.

TIP

Perfectly in-elastic collision
$V_1' = V_2' = V$
$m_1V_1 + m_2V_2 = (m_1 + m_2)V$
$V = \frac{m_1V_1+m_2V_2}{m_1+m_2}$

Interaction between (S₁) and (S₂)

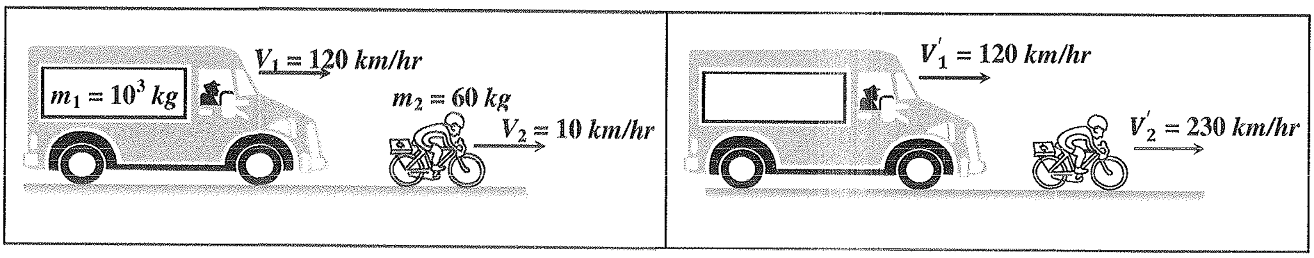
During collision, the external forces acting on m_1 and m_2 are neglected relative to the internal forces.
$\vec{F}_{1/2} = -\vec{F}_{2/1}$ **(Newton's 3rd law)**
$\sum\vec{F}_{ext/S_1} = \vec{F}_{2/1} = \frac{d\vec{P}_1}{dt}$
$\sum\vec{F}_{ext/S_2} = \vec{F}_{1/2} = \frac{d\vec{P}_2}{dt}$
Therefore, $\frac{d\vec{P}_1}{dt} = -\frac{d\vec{P}_2}{dt}$.

Special cases of perfectly elastic head-on collision

1- (S_2) is initially at rest, then $V_2 = 0$.
$V_1' = \frac{m_1-m_2}{m_1+m_2}V_1$ and $V_2' = \frac{2m_1}{m_1+m_2}V_1$.
  1.1- For $m_1 > m_2$, then V_1' and V_2' have the same sign, consequently, (S_1) and (S_2) move in the same direction as $\vec{V}_1$ after collision.
  1.2- For $m_1 < m_2$, then V_1' and V_2' have opposite signs where (S_1) bounces back (rebounds) with a velocity $\vec{V}_1'$. On the other hand, V_2' and V_1 have the same sign where (S_2) moves along the direction of $\vec{V}_1$ after collision.
  1.3- $m_1 << m_2$, then (S_1) rebounds with same speed but in opposite direction.
  $V_1' = -V_1$ and $V_2' = 0$.
  Example: collision between an object and the Earth.

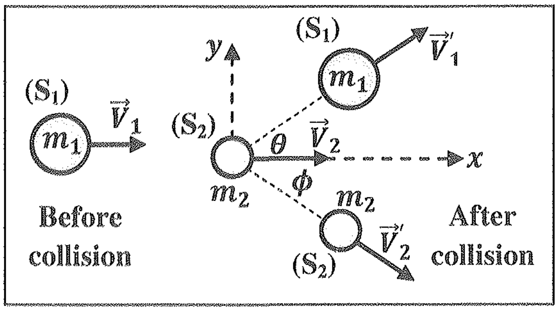
2- For $m_1 = m_2$, then: $V_1' = V_2$ and $V_2' = V_1 \Longrightarrow$ (S_1) and (S_2) exchange their velocities.

3- For $m_1 >> m_2$, then $V_1' = V_1$ and $V_2' = -V_2 + 2V_1$.



2.8- PERFECTLY ELASTIC OBLIQUE COLLISION

Two solids (S_1) and (S_2), of masses m_1 and m_2 respectively and moving horizontally with respective velocities $\vec{V_1}$ and $\vec{V_2}$, enter into a perfectly elastic oblique collision and then move with respective velocities $\vec{V_1'}$ and $\vec{V_2'}$ after collision.



During short duration collision, the system (S): [(S_1); (S_2)] is isolated.

$\sum \vec{F}_{ext} = \vec{0} \Longrightarrow \frac{d\vec{P}_S}{dt} = \vec{0} \Longrightarrow \vec{P}_S = constant.$

Law of conservation of momentum:

$\vec{P}_i = \vec{P}_f \Longrightarrow \vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2'.$

Along x-axis: $m_1V_1 + m_2V_2 = m_1V_1' \cos\theta + m_2V_2' \cos\phi$ ... (1)

Along y-axis: $0 = m_1V_1' \sin\theta - m_2V_2' \sin\phi$ ... (2)

Law of conservation of kinetic energy:

$\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_1{V_1'}^2 + \frac{1}{2}m_2{V_2'}^2$ ... (3)

2.9- PROJECTILE MOTION

A projectile is an object that moves through space is acted upon only by Earth's gravity.

Initial velocity vector

$\vec{V_0} \begin{vmatrix} V_{0x} = V_0 \cos\alpha \\ V_{0y} = V_0 \sin\alpha \end{vmatrix}$

Initial linear momentum vector

$\vec{P_0} \begin{vmatrix} P_{0x} = mV_{0x} = mV_0 \cos\alpha \\ P_{0y} = mV_{0y} = mV_0 \sin\alpha \end{vmatrix}$

Newton's 2nd law:

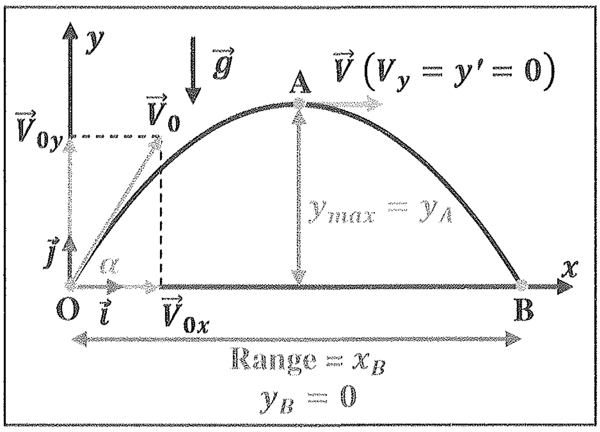
$\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt}$ with $\vec{P} = \vec{P}_x + \vec{P}_y = P_x\vec{i} + P_y\vec{j}.$

$m\vec{g} = \frac{d\vec{P}_x}{dt} + \frac{d\vec{P}_y}{dt}$ with $\vec{g} = -g\vec{j}.$

$-mg\vec{j} = \frac{dP_x}{dt}\vec{i} + \frac{dP_y}{dt}\vec{j}.$

$\frac{dP_x}{dt} = 0 \Longrightarrow P_x = constant = P_{0x} = mV_0 \cos\alpha.$

$\frac{dP_y}{dt} = -mg \Longrightarrow P_y = -mgt + P_{0y} = -mgt + mV_0 \sin\alpha.$



Acceleration vector at any time t:

$\sum \vec{F}_{ext} = m\vec{a} \Longrightarrow m\vec{g} = m\vec{a} \Longrightarrow \vec{a} = \vec{g} = -g\vec{j}.$

$\vec{a} \begin{vmatrix} a_x = 0\ (URM) \\ a_y = -g\ (UVRM) \end{vmatrix}$

If y is directed vertically downwards:
$\vec{a} = \vec{g} = g\vec{j}$

Velocity vector at any instant t:

$\vec{V} \begin{vmatrix} V_x = \frac{P_x}{m} = V_0 \cos\alpha \\ V_y = \frac{P_y}{m} = -gt + V_0 \sin\alpha \end{vmatrix}$ or $\vec{V} \begin{vmatrix} V_x = \int a_x dt = V_{0x} = V_0 \cos\alpha \\ V_y = \int a_y dt = -gt + V_{0y} = -gt + V_0 \sin\alpha \end{vmatrix}$

Position vector at any time t:

$$\vec{r}\left|\begin{array}{l} x = \int V_x dt = V_0 \cos\alpha\, t + x_0 = V_0 \cos\alpha\, t \\ y = \int V_y\, dt = -\frac{1}{2}gt^2 + V_0 \sin\alpha\, t + y_0 = -\frac{1}{2}gt^2 + V_0 \sin\alpha\, t \end{array}\right. \quad \text{with } x_0 = y_0 = 0$$

Trajectory equation: is a relation between x and y independent of t.

$$x = V_0 \cos\alpha\, t \Longrightarrow t = \frac{x}{V_0 \cos\alpha}$$

$$y = -\frac{1}{2}g\left(\frac{x}{V_0\cos\alpha}\right)^2 + V_0 \sin\alpha\left(\frac{x}{V_0\cos\alpha}\right) = -\frac{g}{2V_0^2\cos^2\alpha}x^2 + \tan\alpha\, x$$

Time needed to reach the maximum height at A: $V_y = 0 \Longrightarrow -gt_A + V_0\sin\alpha = 0 \Longrightarrow t_A = \frac{V_0\sin\alpha}{g}$

Maximum height: $t_A = \frac{V_0\sin\alpha}{g} \Rightarrow y_{max} = y_A = -\frac{1}{2}g\left(\frac{V_0\sin\alpha}{g}\right)^2 + V_0\sin\alpha\left(\frac{V_0\sin\alpha}{g}\right) = \frac{V_0^2\sin^2\alpha}{2g}$

Range: is the horizontal distance covered by the projectile between the shooting point and the point of intersection with the horizontal line passing through the shooting point.

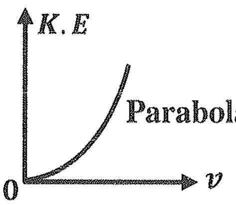
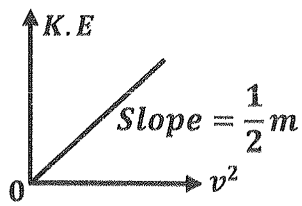
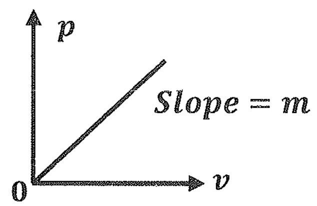
$$y_B = y_O = 0 \Longrightarrow -\frac{g}{2V_0^2\cos^2\alpha}x_B + \tan\alpha\, x_B = 0 \Longrightarrow x_B = \frac{V_0^2\sin 2\alpha}{g}$$

Time needed to reach the range: $y_B = 0 \Longrightarrow -\frac{1}{2}gt_B^2 + V_0\sin\alpha\, t_B = 0 \Longrightarrow t_B = \frac{2V_0\sin\alpha}{g}$.

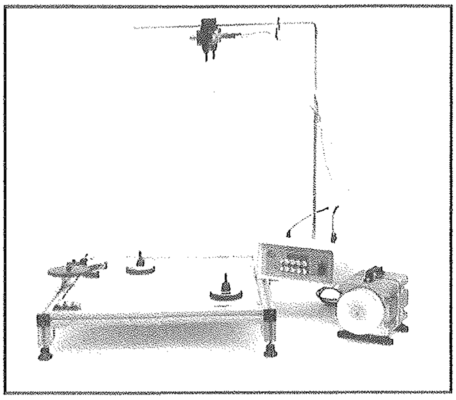
Speed of the projectile as it reaches B:

$$\vec{V}_B\left|\begin{array}{l} V_{Bx} = V_0\cos\alpha \\ V_{By} = -g\left(\frac{2V_0\sin\alpha}{g}\right) + V_0\sin\alpha = -2V_0\sin\alpha + V_0\sin\alpha = -V_0\sin\alpha = -V_{0y} \end{array}\right.$$

2.10- KINETIC ENERGY VERSUS LINEAR MOMENTUM

Kinetic Energy	Linear Momentum
Scalar quantity.	Vector quantity.
SI unit is [kgm^2/s^2] or [J].	SI unit is [kgm/s].
Kinetic energy is proportional to the square of the speed.  	The magnitude of the linear momentum is proportional to the speed. 
$\Delta K.E = \sum W_{ext} = \sum \vec{F}_{ext}\,.\vec{d}$ Work kinetic energy theorem for a rigid system	$d\vec{P} = \sum \vec{F}_{ext}\,.dt$ Newton's second law

2.11- MOTION OF A PUCK ALONG AN AIR TABLE



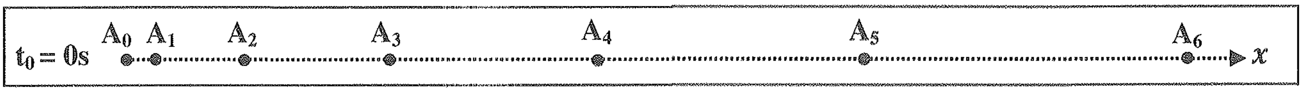
An air table is an apparatus used to study the motion of an object (puck) in the absence of frictional forces.

Mechanism

- The puck carries a pump that blows air from its bottom against the surface of the table.
- The puck is connected to a recording system. During its motion, it leaves regular traces of black points on recording paper.
- The time interval between two consecutive traces (time constant τ) remains constant during the experiment.
- Tracing is produced by an electric spark between a sharp edged rod placed at the bottom of the puck and a carbon paper placed below the recording paper.

Kinematical study of the motion of the puck

The document below shows the recording of the successive positions of the tracer A of a puck on an inclined air table.



$cm \xrightarrow{\times 10^{-2}} m$

$ms \xrightarrow{\times 10^{-3}} s$

1- Time needed to reach the dot A_i is $t_i = i\tau$.
Example: the time needed to reach the dot A_2 is $t_2 = 2\tau$.

2- The position (abscissa) of the dot A_i is $x_i = \overline{A_0A_i}$.
Example: the position (abscissa) of the dot A_3 is $x_3 = \overline{A_0A_3}$.

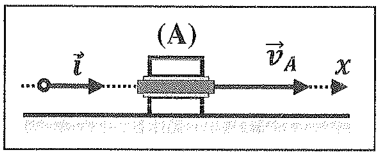
3- The velocity of the puck at the dot A_i is $V_i = V_{av(i-1,i+1)} = \frac{\Delta x}{\Delta t} = \frac{x_{i+1}-x_{i-1}}{t_{i+1}-t_{i-1}} = \frac{\overline{A_0A_{i+1}}-\overline{A_0A_{i-1}}}{2\tau}$.
Example: the velocity of the puck at the dot A_2 is:
$V_2 = V_{av(1,3)} = \frac{\Delta x}{\Delta t} = \frac{x_3-x_1}{t_3-t_1} = \frac{\overline{A_0A_3}-\overline{A_0A_1}}{t_3-t_1} = \frac{\overline{A_0A_3}-\overline{A_0A_1}}{3\tau-\tau} = \frac{\overline{A_1A_3}}{2\tau} (A_1A_3\ in\ [m] and\ \tau\ in\ [s])$.

4- The acceleration of the puck at the dot A_i is: $a_i = a_{av(i-1,i+1)} = \frac{\Delta V}{\Delta t} = \frac{V_{i+1}-V_{i-1}}{t_{i+1}-t_{i-1}} = \frac{V_{i+1}-V_{i-1}}{2\tau}$.
Example: the acceleration at the dot A_3 is $a_3 = a_{av(2,4)} = \frac{\Delta V}{\Delta t} = \frac{V_4-V_2}{t_4-t_2} = \frac{V_4-V_2}{2\tau}$.

CHAPTER 2 – LINEAR MOMENTUM
EXERCISES AND PROBLEMS

Whenever needed, take the magnitude of the gravitational acceleration $g = 10m/s^2$ and neglect the effect of air resistance except as otherwise indicated

Exercise 1*:



A puck (A), of mass $m_A = 400g$, is launched with a velocity $\vec{v}_A = 0.5\vec{i}\,[m/s]$ on a horizontal air table. Calculate the linear momentum $\vec{p}_A$ of puck (A).

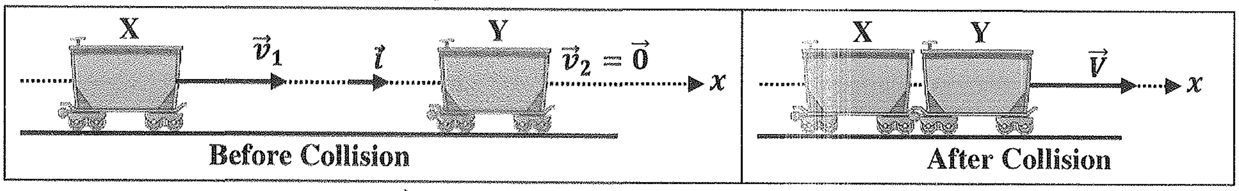
Exercise 2:

A tennis ball, of mass m = 60g, may leave the racket on a serve with a speed $v_2 = 198km/h$. The ball is in contact with the racket for $\Delta t = 4ms$.

1- Calculate the magnitude of the change in linear momentum of the ball during $\Delta t = 4ms$.
2- Calculate the magnitude of the average net force on the ball.

Exercise 3:

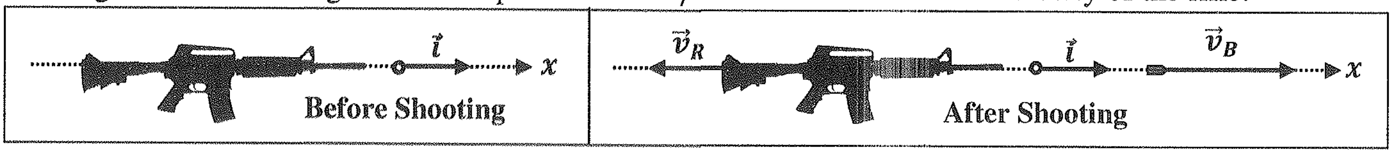
A railroad cart X has a mass m = 2 tons and is travelling at a speed $v_1 = 24m/s$. Cart X strikes an identical stationary cart Y. The two carts lock together as a result of the collision as shown in the document below.



1- Calculate the linear momentum $\vec{P}$ of the system [X; Y] before the collision.
2- Determine the common speed V of the two railroad carts just after the collision.
3- Specify the nature of the collision.
4- How much of the initial kinetic energy is transformed to thermal energy or other forms of energy?

Exercise 4:

A 3.4kg rifle shoots a 55g bullet at a speed of $960m/s$. Determine the recoil velocity of the rifle.



Exercise 5:

A gun that does not recoil ejects gases towards the rear when the shell leaves at the front. Explain how the gun functions.

Exercise 6:

An object (S), considered as a particle and initially at rest on a horizontal Ox axis, explodes into two pieces. One piece moves, just after explosion, along the axis Ox with a speed of 6m/s. What are the direction and the magnitude of the velocity of the second piece whose mass is three times that of the first?

Exercise 7*:

A stationary skater (S_1), taken as a particle, has a mass $m_1 = 60kg$. Another skater (S_2), taken as a particle of mass $m_2 = 40kg$, rushes and holds the first at a speed $V_2 = 12km/h$. Both skaters hold together and move along a rectilinear path as one system (S) with a speed V.

1- Determine the value of V.
2- Calculate the kinetic energy of (S) before and after collision. Is the collision elastic? Justify your answer.

Exercise 8*:

The position-time equation of a particle (M), of mass m = 0.5kg and moving in the space reference system $(O; \vec{i})$, is: $x = 2t^2 - 4t + 1$ $[SI]$

1- Determine the expressions of the velocity, acceleration and the linear momentum of (M) at the instant t.
2- Calculate the net external force acting on (M).

Exercise 9:

In an inertial frame of reference $(O; \vec{i}; \vec{j})$, the coordinates and masses of two particles (S_1) and (S_2) are given as a function of time t as follows:

$$(S_1)\left|\begin{array}{l} x_1 = 2t^2 + 5 \\ y_1 = t - 1 \\ m_1 = 1kg \end{array}\right. \qquad (S_2)\left|\begin{array}{l} x_2 = 2t + 4 \\ y_2 = t^2 + t \\ m_2 = 2kg \end{array}\right. \qquad [SI]$$

Denote by G the center of mass of the system (S): [(S_1); (S_2)].

1- Determine $\vec{P}_1$ and $\vec{P}_2$ the respective linear momenta of (S_1) and (S_2) at any instant t.
2- Calculate $\vec{P}_S$ the linear momentum of the system (S) at any instant t.
3- Determine $\vec{r}_G$ the position vector of (S) at any instant t.
4- Calculate $\vec{P}_G$ the linear momentum of G at any instant t. Draw out a conclusion.
5- Determine $\vec{a}_1$ and $\vec{a}_2$ the respective acceleration vectors of (S_1) and (S_2) at any instant t. Deduce $\vec{F}_1$ and $\vec{F}_2$ the net forces acting on (S_1) and (S_2) respectively.
6- Verify that $\sum \vec{F}_{ext/(s)} = \frac{d\vec{P}}{dt}$.
7- Specify whether the system (S) is isolated or not.

Exercise 10*:

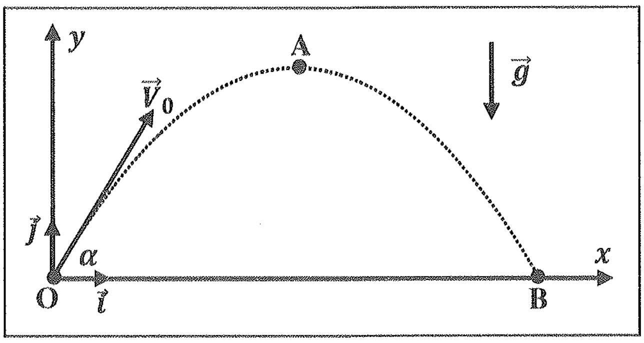
A billiard ball (C) of mass m moving with a speed v collides head-on with a ball (D) of equal mass at rest. The collision is perfectly elastic.

Apply the laws of conservation of linear momentum and kinetic energy to determine the speeds of the two balls after collision.

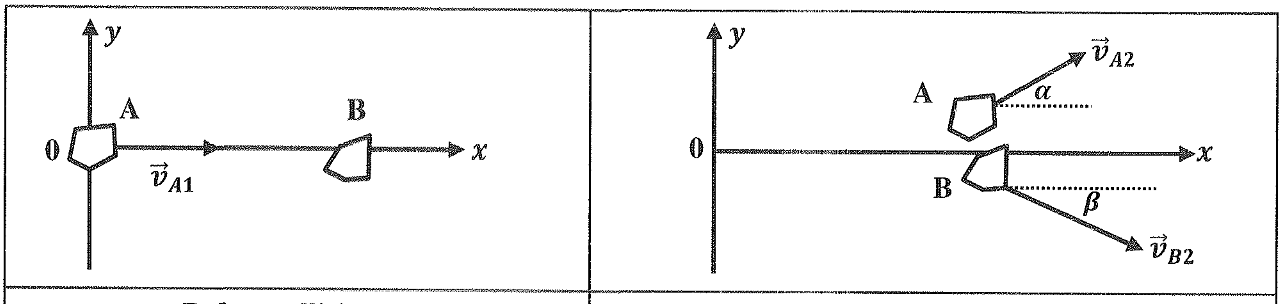
Exercise 11:

At the instant t_0 = 0s, a particle (S), of mass m = 2kg, is launched from point O with an initial velocity vector $\vec{V}_0$ making an angle α with the horizontal. The position vector of (S) in the space reference system $(O; \vec{i}; \vec{j})$ is given by: $\vec{r} = (5\sqrt{3}t)\vec{i} + (-5t^2 + 5t)\vec{j}$ $[SI]$

1- Determine the equation of the trajectory described by (S); then, deduce its shape.
2- Determine, at the instant t, the:
   2.1- Velocity vector of (S),
   2.2- Acceleration vector of (S),
   2.3- Linear momentum of (S).
3- Show that $\sum \vec{F}_{ext} = m\vec{a} = \frac{d\vec{P}}{dt}$.
4- Determine the instant t_1 when (S) reaches the highest point of the trajectory.
5- Calculate the range of (S).
6- Calculate the value of the shooting angle α.



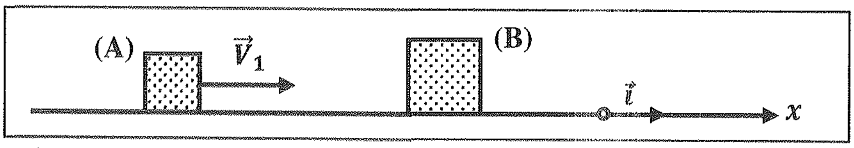
Exercise 12:
The document below shows two chucks of ice sliding on a frictionless frozen pond. Chunk A, with mass $m_A = 5kg$, moves with a velocity $v_{A1} = 2m/s$ parallel to the x-axis. Chunk A collides with chunk B, which has mass $m_B = 3kg$ and is initially at rest.
After collision, the velocity of chunk A is $v_{A2} = 1m/s$ in a direction making an angle $\alpha = 30°$ with the initial direction. Determine the final velocity of chunk B.



Before collision	After collision

Exercise 13*:
What happens to a driver when a crash suddenly stops a car?

Exercise 14:
In order to study the collision between two bodies, we consider a horizontal air table equipped with a launcher and two pucks (A) and (B) of respective masses $m_A = 0.4kg$ and $m_B = 0.6kg$.
(A), launched with the velocity $\vec{V}_1 = 0.5\vec{i}$, collides with (B) initially at rest.
(A) rebounds with the velocity $\vec{V}_2 = -0.1\vec{i}$ and (B) moves with the velocity $\vec{V}_3 = 0.4\vec{i}$ (V_1, V_2 and V_3 are expressed in m/s). Neglect all frictional forces.



Part A: Linear momentum

1-
- 1.1- Determine the linear momentums:
  - 1.1.1- $\vec{P}_1$ and $\vec{P}_2$ of (A), before and after collision respectively;
  - 1.1.2- $\vec{P}_3$ of (B) after collision.
- 1.2- Deduce the linear momentums $\vec{P}$ and $\vec{P}'$ of the system [(A), (B)] before and after collision respectively.
- 1.3- Compare $\vec{P}$ and $\vec{P}'$. Conclude.

2-
- 2.1- Name the external forces acting on the system [(A), (B)].
- 2.2- Give the value of the resultant of these forces.
- 2.3- Is this resultant compatible with the conclusion in question (1.3)? Why?

Part B: Type of collision
1- Determine the kinetic energy of the system [(A), (B)] before and after collision.
2- Deduce the type of the collision.

Part C: Principle of interaction

The duration of collision is $\Delta t = 0.04$ s; we can consider that $\frac{\Delta\vec{P}}{\Delta t} \simeq \frac{d\vec{p}}{dt}$.

1- Determine during Δt:

1.1- the variations $\Delta\vec{P}_A$ and $\Delta\vec{P}_B$ in the linear momentums of the pucks (A) and (B) respectively;

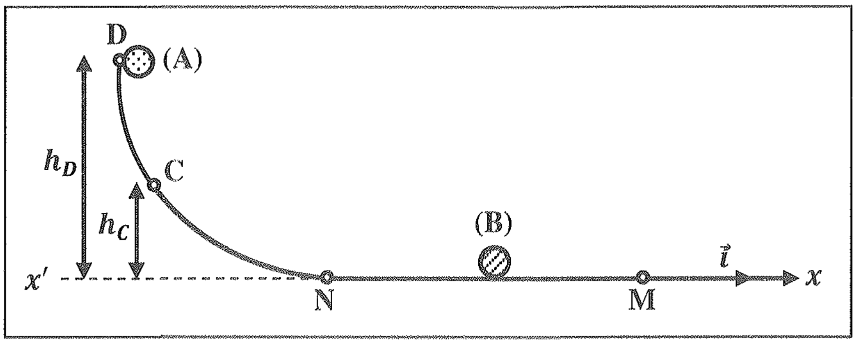
1.2- the forces $\vec{F}_{A/B}$ exerted by (A) on (B) and $\vec{F}_{B/A}$ exerted by (B) on (A).

2- Deduce that the principle of interaction is verified.

Exercise 15:

The aim of this exercise is to determine the nature of a collision between two objects.

For this aim, an object (A), considered as a particle of mass $m_A = 2kg$, can slide without friction on a path situated in a vertical plane and formed of two parts: a circular part DN and a horizontal rectilinear part NM.



(A) is released, without initial velocity, from the point D situated at a height $h_D = 0.45m$ above the horizontal part NM as shown in the adjacent document.

The horizontal plane passing through MN is taken as the reference level of gravitational potential energy.

1- Calculate the mechanical energy of the system [(A), Earth] at the point D.

2- Deduce the speed V_{A1} of (A) when it reaches the point N.

3- (A) reaches N and moves along NM with the same velocity $\vec{V}_{A1} = V_{A1}\vec{\imath}$. Another object (B), considered as a particle of mass $m_B = 4kg$, moves along the horizontal path from M toward N with the velocity $\vec{V}_{1B} = -1\vec{\imath}$ (V_{1B} in m/s).

3.1- Determine the linear momentum $\vec{P}_s$ of the system [(A), (B)] before collision.

3.2- Deduce the velocity $\vec{V}_G$ of the center of inertia G of the system [(A), (B)].

4- After collision, (A) rebounds and attains a maximum height $h_C = 0.27m$.

4.1- Determine the mechanical energy of the system [(A), Earth] at the point C.

4.2- Deduce the speed V_{A2} of (A) just after collision.

5- Determine, by applying the principle of the conservation of the linear momentum of the system [(A), (B)], the velocity V_{B2} of (B) just after collision.

6- Specify the nature of the collision.

Exercise 16:

The ballistic pendulum is a device used to measure the speed of a projectile, such as a bullet.

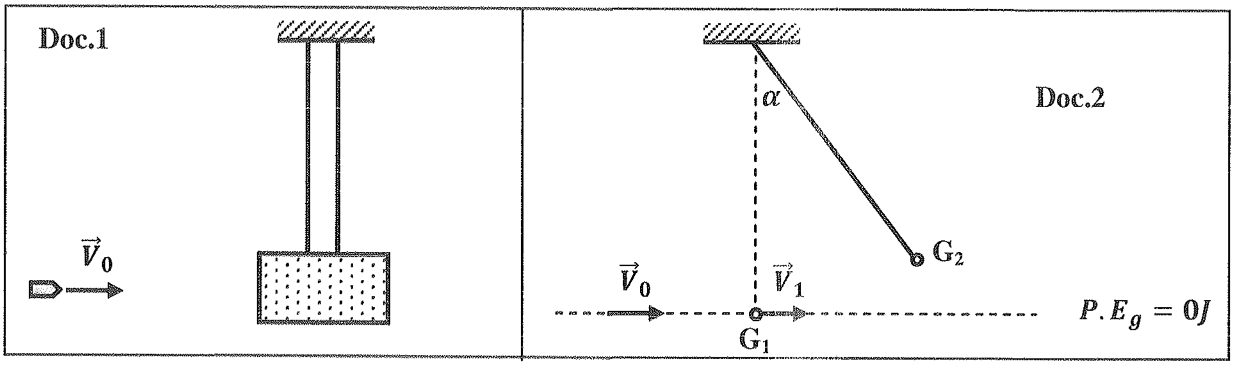
A gun is used to shoot bullets, each of mass m = 20 g, with a horizontal velocity $\vec{V}_0$ of value V_0.

In order to determine V_0, we consider a setup formed of a wooden block of mass M = 1kg, suspended from the ends of two inextensible strings of negligible mass and of the same length (doc.1).

This setup can be taken as a block of wood suspended from the free end a string of length l = 1m, initially at rest in the equilibrium position at G_1.

A bullet having the velocity $\vec{V}_0$ hits the block and is embedded in at the level of the center of mass G of the block.

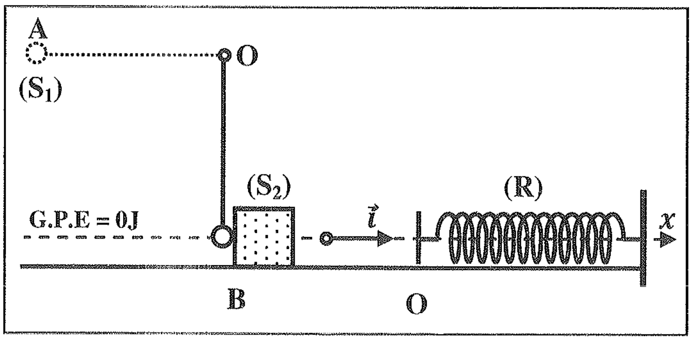
Just after impact, the system (block, bullet) moves with a horizontal velocity $\vec{V}_1$. The pendulum thus attains a maximum angular deviation $\alpha = 37°$.
G_1 and G_2 are the respective positions of G in the equilibrium position and in the highest position.
Take the horizontal plane passing through G_1 as a gravitational potential energy reference (doc.2).
Take $g = 9.8\ m/s^2$.



1- During a collision, which one of the two physical quantities, the linear momentum or the kinetic energy of the system does not remain always conserved?
2- Determine the expression of the value of V_1 of the velocity V_1 in terms of M, m and V_0.
3-
   3.1- Determine, just after impact, the mechanical energy of the system (pendulum, Earth) in terms of V_0, M, and m.
   3.2- Determine, in terms of M, m, g, l and α, the mechanical energy of the system (pendulum, Earth) at point G_2.
   3.3- Deduce the value of V_0.
4- Verify the answer of question (1).

Exercise 17:

A steel ball (S_1), taken as a particle of mass $m_1 = 1kg$, is connected to one extremity of a massless and inextensible string of length $L = 80cm$. The other extremity of the string is fixed at the point O. (S_1) is shifted from its equilibrium position and then released without initial velocity from point A where the string is horizontal as shown in the adjacent document. At the bottom B of its path, (S_1) enters in a head-on collision with a block (S_2), taken as a particle of mass $m_2 = 3kg$, and is initially at rest on a frictionless surface.



After collision, (S_2) moves along BO and hits the free end of an ideal spring (R) of negligible mass and stiffness $k = 120N/m$. The other end of (R) is fixed to a support. (S_2) reaches point C where it compresses the spring by a maximum distance $x_m = 31.7cm$.

Given:

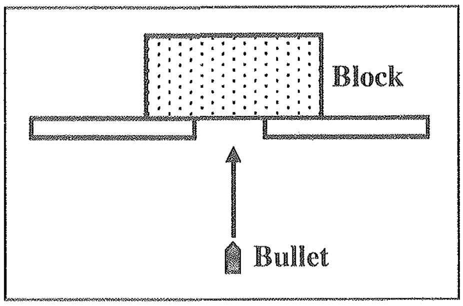
- During collision, the center of masses G_1 and G_2 of (S_1) and (S_2) respectively lie on the same horizontal level.
- The horizontal plane passing through the center of mass G_2 of (S_2) is taken as a gravitational potential energy reference.

1- Apply the work-kinetic energy theorem to determine the speed V_1 of (S_1) at B just before it strikes (S_2).

2- Apply the principle of conservation of mechanical energy to verify that the speed of (S_2) at B just after collision is $V_2' = 2m/s$.
3- Determine V_1' the speed of (S_1) just after collision.
4- After collision, (S_1) reaches its maximum position D. Determine, at D, the angle that the string makes with the vertical.
5- Specify the type of collision that took place between (S_1) and (S_2).

Exercise 18*:

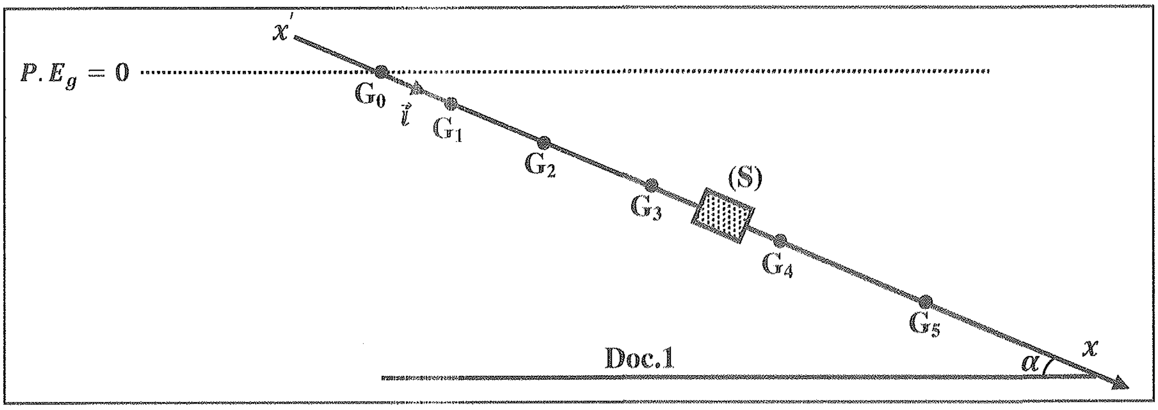
A 10g bullet, moving vertically upward at 1000 m/s, strikes and passes through the center of mass of a 5kg stationary block taken as a particle. The bullet emerges from the block and continues its motion vertically upwards at a speed of 400 m/s.
The horizontal level passing through the center of mass of the block in its equilibrium position is taken as a gravitational potential energy reference.



1- Determine the speed of the block when the bullet emerges from it.
2- Determine the maximum height that the center of mass of the block can reach above its initial position.
3- Specify if the collision is elastic.

Exercise 19:

A puck (S) of mass M = 100g and of center of mass G, may slide along an inclined track that makes an angle α with the horizontal so that $\sin\alpha = 0.40$. Thus G moves along an axis $x'x$ parallel to the track as shown in document 1.



Doc.1

We release (S) without initial velocity at the instant $t_0 = 0$ and at the end of each interval of time $\tau = 50ms$, some positions $G_0, G_1, G_2, \ldots G_5$ of G are recorded at the instants $t_0 = 0, t_1, t_2, \ldots t_5$ respectively.
The values of the abscissa x of G $\left(x = \overline{G_0G}\right)$ are given in the table below.

t	0	τ	2τ	3τ	4τ	5τ
x (cm)	0	G_0G_1= 0.50	G_0G_2= 2.00	G_0G_3 = 4.50	G_0G_4 = 8.00	G_0G_5 =12.50

1- Verify that the speed of the puck at the instants $t_2 = 2\tau$ and $t_4 = 4\tau$ are $V_2 = 0.40$ m/s and $V_4 = 0.80$ m/s respectively.
2-
  2.1- Calculate the mechanical energy of the system (puck-Earth) at the instants t_0, t_2 and t_4 knowing that the horizontal plane through G_0 is taken as a gravitational potential energy reference.

2.2- Why can we suppose that the puck moves without friction along the rail?

3- Determine the variation in the linear momentum $\Delta\vec{P} = \vec{P}_4 - \vec{P}_2$ of (S) during $\Delta t = t_4 - t_2$.

4-

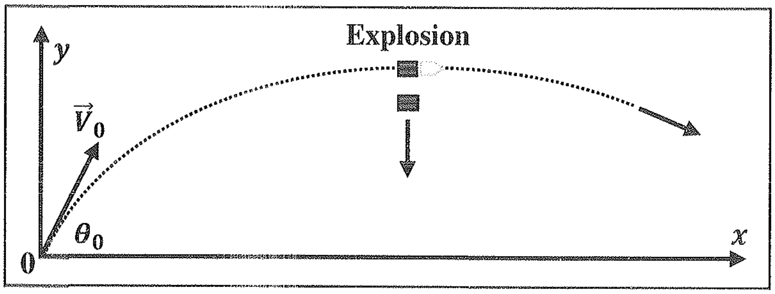
4.1- Name the forces acting on (S) during its motion.

4.2- Show that the resultant $\sum\vec{F}$ of these forces may be written as $\sum\vec{F} = (Mg\sin\alpha)\vec{\imath}$.

5- Assuming that Δt is very small, $\frac{\Delta\vec{P}}{\Delta t}$ may be considered equal to $\frac{d\vec{P}}{dt}$. Show that Newton's second law is verified between the instants t_2 and t_4.

Exercise 20:

At the instant $t_0 = 0s$, a shell taken as a particle of mass m = 2kg, is shot with an initial velocity $\vec{v}_0$ of magnitude $V_0 = 20m/s$ at an angle $\theta_0 = 60°$ with the horizontal.



1- Apply Newton's second law on the shell to determine, at an instant t, the horizontal and vertical components P_x and P_y of the linear momentum of the shell.

2- Verify that the parametric equations of the motion of the shell are: $\begin{cases} x = 10t \\ y = -5t^2 + 10\sqrt{3}t \end{cases}$ $[SI]$

3- Determine the coordinates of the vertex of the trajectory described by the shell.

4- At the vertex of the trajectory, the shell explodes, at a new origin of time $t_0 = 0s$, into two fragments of equal masses. One fragment, whose speed immediately after the explosion is zero, falls vertically.

4.1- Determine the velocity of the second fragment just after the explosion.

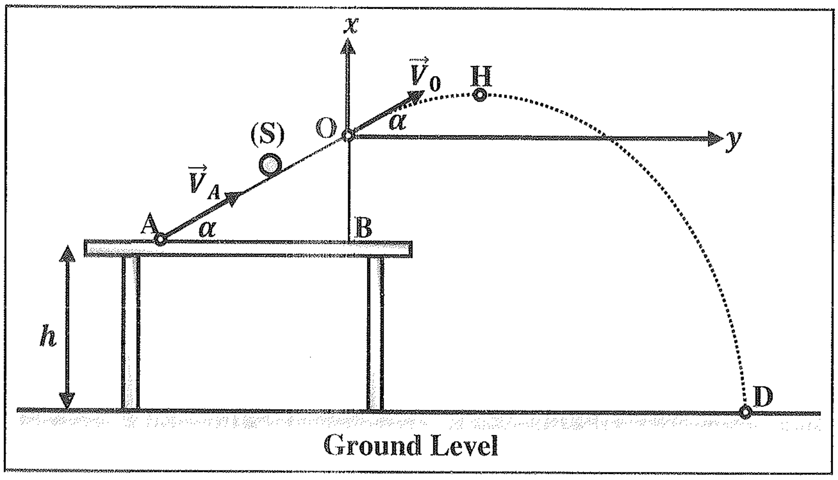
4.2- How far from the gun does the other fragment land?

Exercise 21*:

In the adjacent document, AO is a frictionless inclined plane of 2.8m length and making an angle $\alpha = 30°$ with the horizontal level.

AO is held by a horizontal table of height $h = 1m$ relative to the ground level which is taken as a gravitational potential energy reference.

At the instant $t_0 = 0s$, a solid (S), taken as a particle of mass $m = 2kg$, is launched at A with a velocity $\vec{V}_A$ of magnitude $V_A = 8m/s$ and parallel to AO. (S) reaches O with a speed V_0.



Part I: Motion of (S) along AO

1- Name and represent the external forces acting on (S) along AO.

2- Apply Newton's second law to express the acceleration a of (S) along AO in terms of g and α. Calculate its numerical value.

3- Calculate V_0.

Part II: Motion of (S) along OD

(S) reaches the position O with a speed V_0. At a new origin of time $t_0 = 0s$, it describes a parabolic trajectory and hits the ground level at D.

1- Calculate the horizontal and the vertical components P_{0x} and P_{0y} of the linear momentum vector $\vec{P}_0$ of (S) at O.

2- Show that, at an instant t, $P_x = P_{0x}$ where P_x is the horizontal component of the linear momentum vector $\vec{P}$ of (S) at an instant t.

3- Determine the vertical component P_y of $\vec{P}$ at the instant t.

4- At an instant t_H, (S) reaches the highest position H of its trajectory.

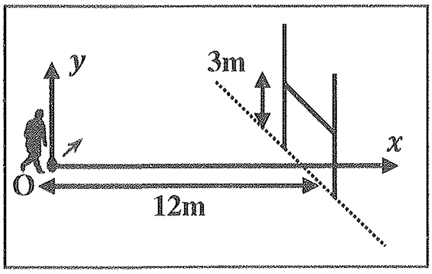
  4.1- Determine the linear momentum $\vec{P}_H$ of (S) at H.

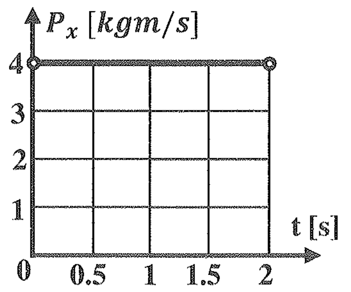
  4.2- Calculate the value of t_H.

5- Determine the speed of (S) when it reaches D; then, deduce the time t_D taken by (S) to travel from O to D.

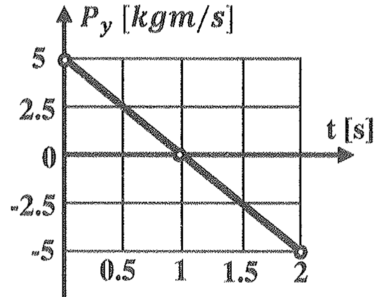
Exercise 22:

In the training of the rugby team of our city, Firas kicked the ball, considered as a particle of mass m = 0.5kg, with an initial speed V_0 [m/s] at an angle α to the horizontal. The objective of this exercise is to know if the ball passes the horizontal bar of the post whose height is 3m and whose base is 12m far from the ball as shown in the adjacent document. Graphs (1) and (2) show the components $P_x(t)$ and $P_y(t)$ of the linear momentum $\vec{P}$ of the ball as a function of time.





Graph 1



Graph 2

1- Give the values of the initial horizontal and vertical components of $\vec{P}$.

2- Deduce V_0 and α.

3- Determine graphically the time taken by the ball to reach the maximum height and the time it takes to reach the range of motion. Deduce a relation between the two instants.

4- Determine the expressions of P_x and P_y as a function of time.

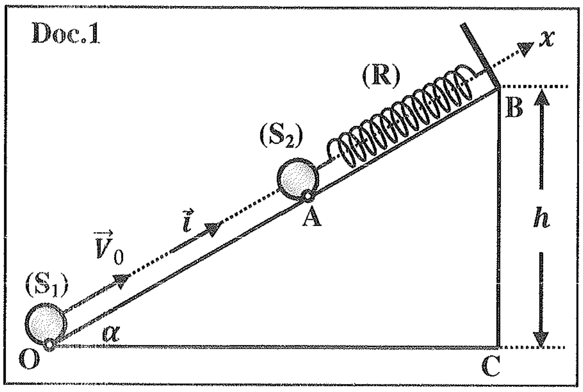
5- Write the time equations $x(t)$, $y(t)$, $V_x(t)$ and $V_y(t)$ of the ball.

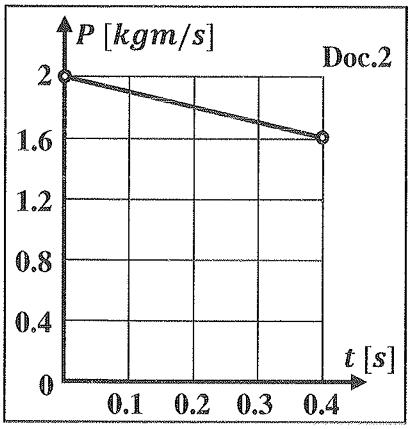
6- Specify, along each axis, the nature of motion of the ball in the 2s time interval.

7- The ball passes the horizontal bar. Show complete work to verify this result.

Exercise 23:

At an instant $t_0 = 0s$, a solid (S_1), taken as a particle of mass $m_1 = 200g$, is launched with an initial velocity $\vec{V_0} = V_0\vec{i}$ from the bottom O of an inclined plane that makes an angle α with the horizontal. At the top B of the inclined plane, we fix a spring (R) of stiffness $k = 50N/m$ and of free length $L_0 = 41cm$. A solid (S_2), taken as a particle of mass $m_2 = 900g$, is suspended from the free end of (R) as shown in document 1.





Given:
- The horizontal plane passing through A is taken as a gravitational potential energy reference,
- Friction forces are neglected,
- $h = 2.05m$.

1- The graph of document 2 shows the variation of the linear momentum P of (S_1) along OA as a function of time. Use the graph to answer the following questions.
 1.1- Determine V_0.
 1.2- Determine the expression of P as function of time.

2- Apply Newton's second law on (S_1) to verify that $\alpha = 30°$.

3- Show that, at equilibrium, the elongation of the spring is $\Delta L = \frac{m_2 g \sin\alpha}{k}$; then, calculate its value.

4- Calculate the distance OA; then, find the speed V_1 of (S_1) at A.

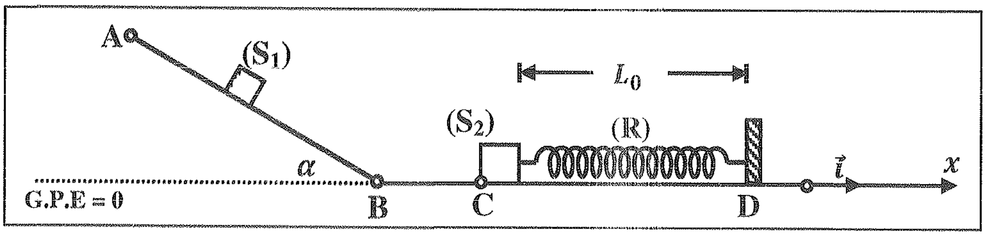
5- As (S_1) reaches A with a speed $\vec{V_1} = V_1\vec{i}$, it enters in a head-on collision with (S_2). (S_1) rebounds with a velocity $\vec{V_1'} = -3.25\vec{i}\ [m/s]$.
 5.1- Determine the velocity V_2' of (S_2) just after collision.
 5.2- Is the collision between (S_1) and (S_2) perfectly elastic? Justify your answer.

6- After the collision, (S_2) compresses (R) to a maximum distance AD. Calculate AD.

Exercise 24:

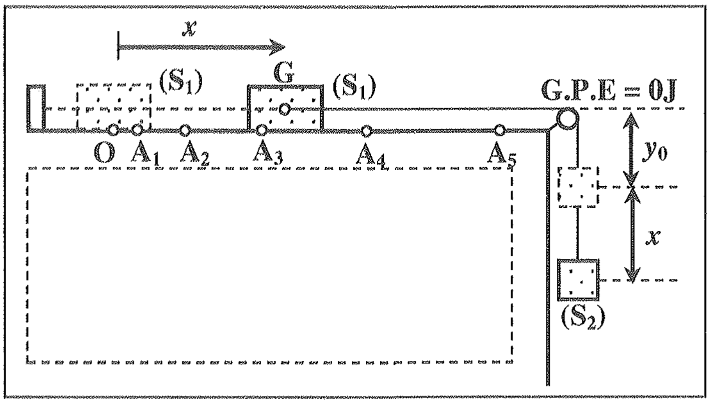
In the document below:
- AB is a frictionless inclined plane of 90cm length and making an angle $\alpha = 30°$ with the horizontal.
- BD is a horizontal frictionless plane.
- (R) is a spring of negligible mass, stiffness $k = 50N/m$ and free length $L_0 = 50cm$.
- (S_1) and (S_2) are two particles of respective masses m_1 and m_2. (S_2) is attached to the free end of (R) while the other end is fixed to a support D.

At an instant $t_0 = 0s$, (S_1) is released, without initial velocity, from the top A of the inclined plane. The horizontal plane passing through B is taken as a gravitational potential energy reference.



1- Determine the speed V_1 of (S_1) as it reaches point B.
2- (S_1) reaches B and moves along BC with the same velocity $\vec{V_1} = V_1\vec{i}$. At C, it enters in a head-on collision with (S_2). After collision, (S_1) and (S_2) are stuck together and form one body (S) of mass $M = 2kg$. The velocity of (S) just after the collision is $\vec{V} = 1.2\vec{i}\ (m/s)$.
   2.1- Determine the masses m_1 and m_2.
   2.2- Establish the relation $v^2 + 25x^2 = 1.44$ where x is the abscissa of (S) at any instant t and $v = \frac{dx}{dt}$.
   2.3- Verify that the acceleration of (S) is proportional to x.
   2.4- Find the maximum compression of the spring.

Exercise 25*:



The aim of this exercise is to determine the expression of the magnitude of the acceleration of a particle. The used apparatus is formed of two particles (S_1) and (S_2) of respective masses $m_1 = 1kg$ and $m_2 = 0.25kg$, fixed at the extremities of a massless and inextensible string passing over the groove of a light pulley. The string does not slide over the groove of the pulley.
(S_1), (S_2), the string and the pulley form a mechanical system (S).
(S_1) may move on a horizontal frictionless surface and (S_2) hangs vertically.
At rest, (S_1) is found at point O and (S_2) is found at a distance y_0 ($y_0 > 0$) with respect to the horizontal plane passing through the center of inertia of (S_1) that is taken as a gravitational potential energy reference.
At the instant $t_0 = 0s$, we release the system (S) from rest. (S_1) moves to the right and (S_2) descends vertically. At an instant t, the position of (S_1) is defined by its abscissa $x = \overline{OG}$ and the algebraic value of its velocity is $v = \frac{dx}{dt}$.
Neglect all the forces of friction.

1- **Energetic Study**
   1.1- Write down, at the instant $t_0 = 0s$, the expression of the mechanical energy of the system [(S), Earth] in terms of m_2, y_0 and g.
   1.2- Determine, at the instant t, the expression of the mechanical energy of the system [(S), Earth] in terms of m_1, m_2, y_0, x, v and g.
   1.3- Show that: $V^2 = \frac{2m_2gx}{m_1+m_2}$.
   1.4- Deduce the expression of the acceleration of (S); then, calculate its value.

2- **Dynamical Study**
   2.1- Represent the external forces acting on (S_1) and on (S_2). The tension in the string acting on (S_1) is denoted by $\vec{T_1}$ of magnitude T_1 and that acting on (S_2) is denoted by $\vec{T_2}$ of magnitude T_2.
   2.2- Apply the theorem of the center of mass, $\sum \vec{F}_{ext} = m\vec{a}$, on each particle to determine the expression of the acceleration of (S) in terms of m_1, m_2, and g.
   2.3- Calculate the magnitude of the tension in the string.

3- **Kinematical study**

An apparatus is used to register the positions of G at a successive instants separated by a constant time interval $\tau = 40$ms.

3.1- Complete the adjacent table.

3.2- Determine the variation in the linear momentum $\Delta\vec{P} = \vec{P}_3 - \vec{P}_1$ of (S) during $\Delta t = t_3 - t_1$.

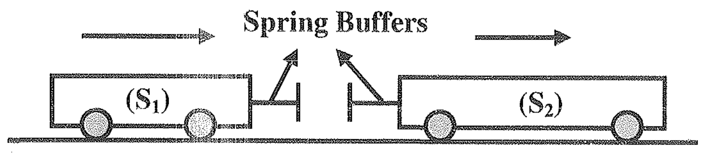
Point	A_0	A_1	A_2	A_3	A_4	A_5
x [mm]	0	1.6	6.4	14.4	25.6	40
V [m/s]	0					
P [kgm/s]	0					

3.3- Assuming that Δt is very small, $\frac{\Delta\vec{P}}{\Delta t}$ may be considered equal to $\frac{d\vec{P}}{dt}$. Show that Newton's second law is verified between the instants t_1 and t_3.

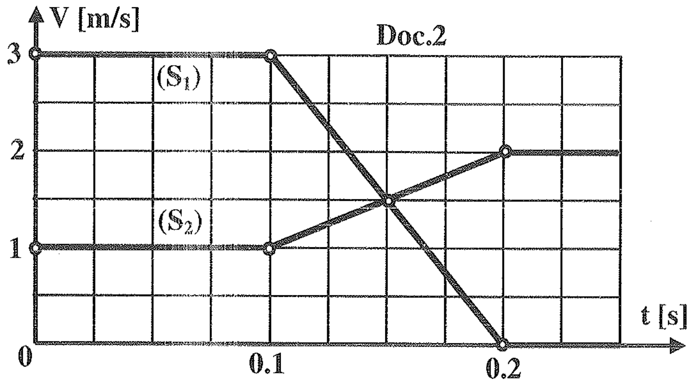
Exercise 26*:

A railway vehicle (S_1), considered as a particle of mass m = 22 tons, moves toward and collides with another vehicle (S_2), considered as a particle of mass M = 66 tons, moving in the same direction as shown in the adjacent document.

The graph in document 2 shows the variations in the speeds of (S_1) and (S_2) as a function of time.



Doc.1



1- Use the graph to answer the following questions:

1.1- Give the initial speeds of (S_1) and (S_2).

1.2- Choose the best answer:
The graph shows the variations in the speeds of (S_1) and (S_2).
(a) before collision,
(b) after collision,
(c) during collision,
(d) all of the above,

1.3- State the principle of conservation of linear momentum. Verify that the principle is valid in this case.

1.4- Show that the collision is perfectly elastic.

1.5- Give the duration through which the collision is taking place.

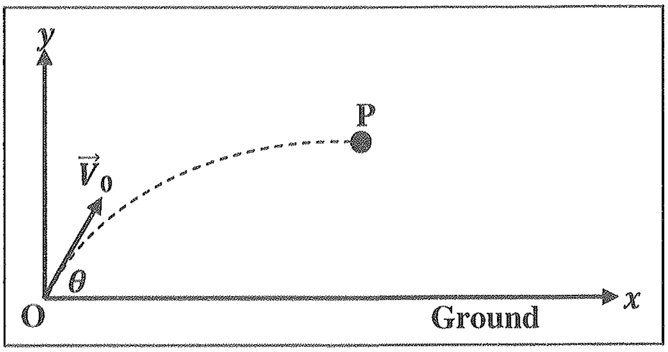
1.6- Show that the variation in the kinetic energy between the instants $t_1 = 0.15$s and $t_2 = 0.2$s is 33000J. In what form of energy does this variation appear?

1.7- Trace the graph that shows the variation of linear momentum as function of time.

2- Calculate the value of the impulse exerted by (S_1) on (S_2).

3- Specify if the magnitude of the impulse vary if the elastic collision between (S_1) and (S_2) takes double the time. If no, what factor changes then?

Exercise 27:



The adjacent document shows the trajectory followed by a firecracker (S) from the instant of launching at O to its vertex at P.

At P, (S) explodes internally into two fragments (S_1) and (S_2). Just after explosion, (S_1) is instantly at rest and (S_2) moves with a horizontal velocity $\vec{V}_2$. Consequently, (S_1) and (S_2) follow different trajectories to reach the ground.

Given:

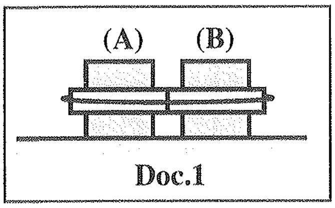
(S) is considered as a particle of mass $m = 0.3kg$ that is launched, at the instant $t_0 = 0s$, from O with an initial velocity $\vec{V}_0$ of magnitude $V_0 = 8m/s$ and making an angle $\theta = 60°$ with the ground level.

(S_1) and (S_2) are considered as particles of respective masses $m_1 = 0.2kg$ and $m_2 = 0.1kg$.

Take $g = 9.8m/s^2$.

1- Calculate the velocity and the linear momentum of (S) just before the explosion.
2- Determine the velocity of (S_2) just after the explosion.
3- How muck kinetic energy is added to the fragments (S_1) and (S_2) by the explosion?
4- Sketch the trajectories followed by (S_1) and (S_2) after the explosion.
5- Determine the time taken by (S_1) to reach the ground. Suggest if (S_2) takes less, same or more time to reach the ground than (S_1). Calculate the distance separating them when they reach the ground.

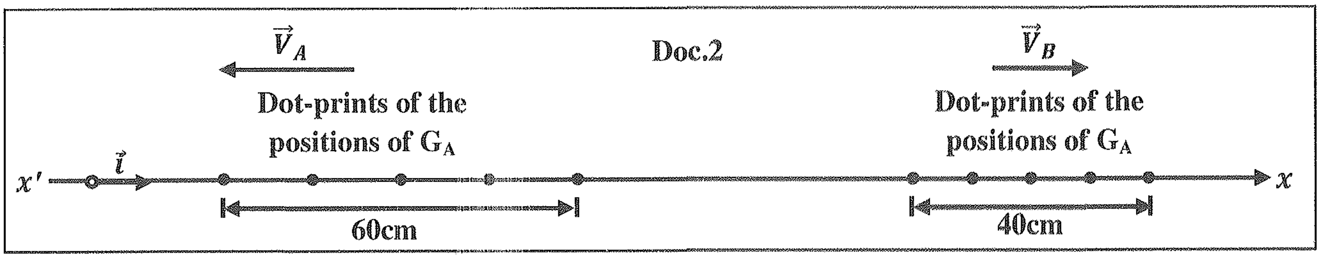
Exercise 28*:



The object of this exercise is to study some physical quantities of a system whose parts are in mechanical interaction. For this, we use two pucks (A) and (B), taken as particles of respective masses m_A and m_B, that are placed on a horizontal smooth plane. The two pucks are connected to each other with a light and inextensible string, thus compressing the steel shock rings as shown in document 1.

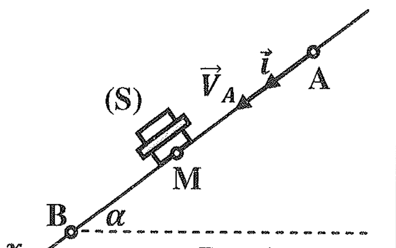
At the instant $t_0 = 0s$, the string is cut. The positions of the center of mass of each puck are registered at successive instants separated by a constant time interval $\tau = 50ms$.

Document 2 represents, on the axis $x'x$, the dot-prints of the positions of the centers of masses G_A and G_B of the two pucks after cutting the string.



1- Use the above document to show that:
   1.1- the motion of each puck is uniform,
   1.2- The speeds of (A) and (B) are $V_A = 3m/s$ and $V_B = 2m/s$ respectively.
2- Apply the principle of conservation of linear momentum to determine m_B knowing that $m_A = 2kg$.

Exercise 29:



Doc.1

The objective of this exercise is to determine, by two methods, the magnitude f of the frictional force $\vec{f}$, assumed constant, which acts on a moving puck (S). The puck (S), taken as a particle of mass $m = 400g$, is released without initial speed from the top of an inclined table of inclination $\alpha = 37°$ ($\sin\alpha = 0.6$) with respect to the horizontal.

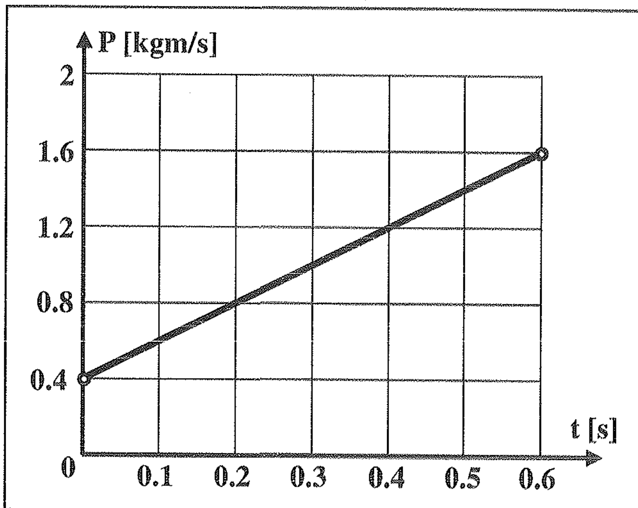
At the instant $t_0 = 0$s, (S) passes by the point A with a velocity $\vec{V}_A = V_A\vec{i}$ as shown in document 1.

At the instant t, the position of (S) in the space reference system $(A; \vec{i})$ is defined by $x = \overline{AM}$ and the algebraic value of its velocity is $V = \frac{dx}{dt}$.

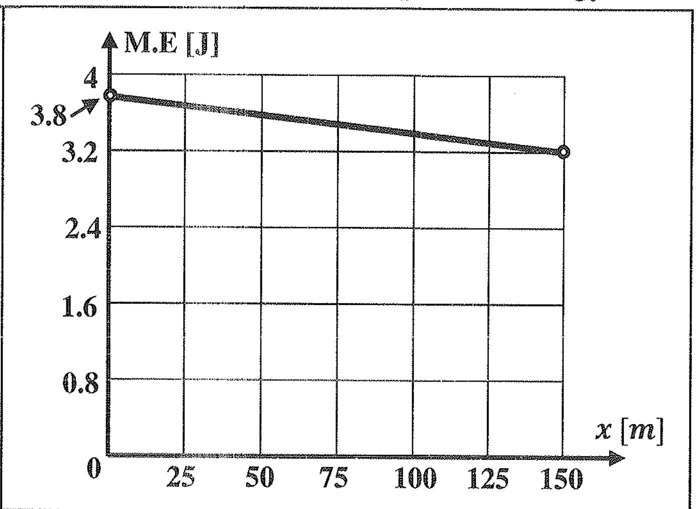
Two convenient devices enable us to trace the variation of the:

- linear momentum P of (S) as a function of time as shown in document 2,
- the mechanical energy M.E of the system [(S); Earth] as a function of x as (S) moves from A to B.

The horizontal plane passing by B is taken as a reference level for the gravitational potential energy.



Doc.2



Doc.3

Part I: Law related to the variation of the linear momentum

1- Determine the expression of P as a function of time.
2- Name and represent the external forces acting on (S)
3- Verify that the sum of the external forces acting on (S) is written as $\sum\vec{F}_{ext} = (mg\sin\alpha - f)\vec{i}$.
4- Apply Newton's 2nd law to determine f.

Part II: Law related to the variation of the energy of the system

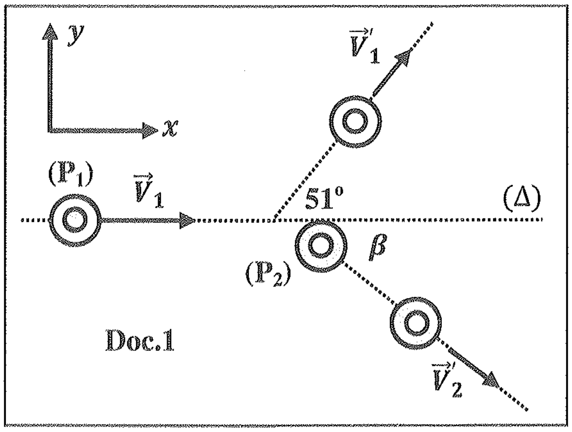
1- Determine the distance AB.
2- Determine the time needed by (S) to reach the point B.
3- Determine, by referring to document 2, the value of f.

Part III:

Compare the two results.

Exercise 30:

The aim of this exercise is to study the collisions, on a horizontal air table, between two pucks (P_1) and (P_2) taken as particles, of masses $m_1 = 0.8kg$ and $m_2 = 1kg$ respectively.



Doc.1

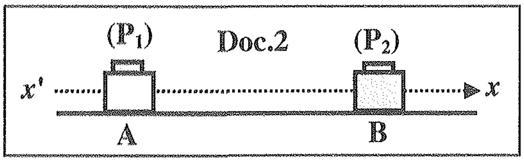
Part I: Two dimensional collision

Puck (P_1), moving with a velocity $\vec{V}_1 = V_1\vec{i}$ ($V_1 = 5m/s$) along a direction (Δ), collides with puck (P_2) that is initially at rest. Consequently, puck (P_1) deviates from (Δ) by an angle $\alpha = 51°$ and acquires a speed $V_1' = 2m/s$. On the other hand, the speed of (P_2) becomes V_2' in a direction making an angle β with respect to (Δ) as shown in document 1

1- The linear momentum $\vec{P}$ of the system [(P_1); (P_2)] remains conserved during the collision. Justify your answer.
2- Determine the direction and the magnitude of the velocity vector $\vec{V}_2'$.

Part II: Head-on collision

Puck (P_1) is launched along the axis x'x from the point A which is 50cm away from (P_2) that is initially at rest at point B as shown in document 2



(P_1) reaches (P_2) with a velocity $\vec{V}_1 = V_1\vec{i}$ and collides with it.
After the collision, (P_2) moves in the direction of x'x with a velocity $\vec{V}_2' = V_2'\vec{i}$ ($V_2' = 3.2m/s$), while (P_1) rebounds backwards with a velocity $\vec{V}_1' = V_1'\vec{i}$.
The collision between (P_1) and (P_2) is perfectly elastic.

1- Determine the algebraic values of the velocities V_1 and V_1' of (P_1) before and after the collision respectively.
2- In reality, a constant frictional force of magnitude $f = 1.8N$ opposes the motion of the puck between its launching at A and its collision with (P_2). With what speed was the puck launched at A?

Exercise 31*:

A proton (p) of mass $m_p = 1.01u$ travelling with a speed $v_p = 3.6 \times 10^4 m/s$ enters in elastic head-on collision with a helium nucleus (He) of mass $m_{He} = 4u$ initially at rest. The collision takes place in nearly empty space.

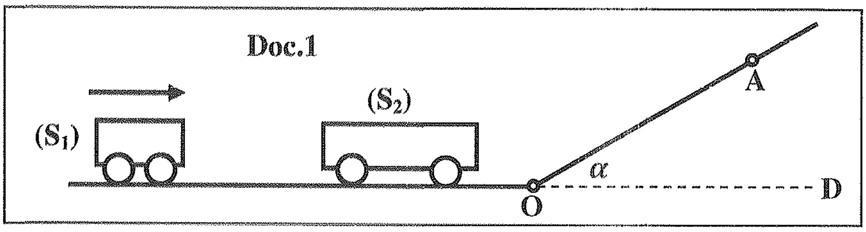
Given:
- $1u = 1.66 \times 10^{-27} kg$.
- The only external force acting on the system [p; He] is the gravitational force which is negligible with respect to the strong force during collision.

Determine the velocities of the proton and the helium nucleus after collision.

Exercise 32*:

In a lab experiment, a student launches a cart (S_1), considered as a particle of mass $m_1 = 1kg$, along a horizontal frictionless track with initial velocity $\vec{V_1}$ of magnitude $V_1 = 10m/s$ and directed to the right, as shown in the document 1. Cart (S_1) collides with a stationary cart (S_2), considered as a particle of mass $m_2 = 3kg$, which is initially at rest. After the collision, cart (S_2) moves up a smooth inclined plane that makes an angle $\alpha = 30°$ with respect to the horizontal.

The horizontal plane passing through OD is taken as a gravitational potential energy reference.



Part I: The collision between (S_1) and (S_2) is head-on and perfectly elastic

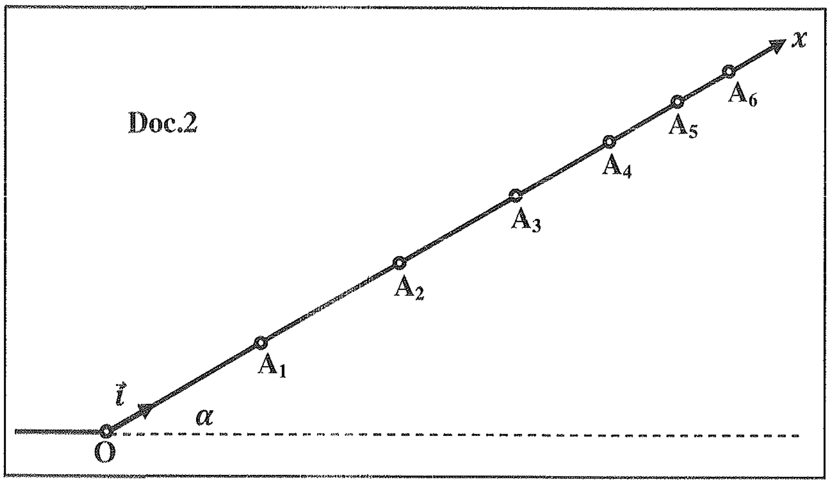
1- Determine V_1' and V_2' the respective speeds of (S_1) and (S_2) right after collision.

2- Apply the principle of conservation of mechanical energy to determine the distance OA covered by (S_2) knowing that it stops at A.

Part II: Motion along an inclined plane

In reality, friction exists between (S_2) and the inclined plane. Suppose now that the speed of cart (S_2) at O is 5m/s. (S_2) passes by O at the instant $t_0 = 0s$. A suitable registration system allows us to produce the print dot diagram as shown in document 2 for the positions of the cart at equal and successive intervals of time $\tau = 100ms$.

The values of the abscissa x of G $x = \overline{OA}$ are given in the table below.



t [ms]	0	τ	2τ	3τ	4τ	5τ	6τ
x [cm]	0	$OA_1 = 47$	$OA_2 = 88$	$OA_3 = 123$	$OA_4 = 152$	$OA_5 = 175$	$OA_6 = 192$

1- Calculate V_2 and V_4 the speeds of (S_2) at $t_2 = 2\tau$ and $t_4 = 4\tau$ respectively.

2-

2.1- Calculate the mechanical energy of the system [(S_2); Earth] at instants t_0, t_2 and t_4.

2.2- Verify that friction exists along the inclined plane.

3- Determine the variation in the linear momentum $\Delta\vec{P} = \vec{P_4} - \vec{P_2}$ of cart (S_2) during $\Delta t = t_4 - t_2$.

4-

4.1- Name and represent the external forces acting on cart (S_2) during its motion.

4.2- Show that the resultant force acting on (S_2) is $\sum \vec{F}_{ext} = (-m_2 g \sin\alpha - f)\vec{i}$

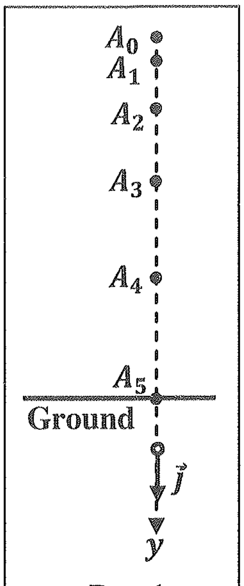
5- Assuming that Δt is very small, $\frac{\Delta\vec{P}}{\Delta t}$ may be considered equal to $\frac{d\vec{P}}{dt}$. Determine the value of the frictional force on the inclined plane.

Exercise 33:

At the instant $t_0 = 0s$, a ball (A), considered as a particle of mass $m_A = 100g$, is released without initial velocity from A_0 at height h above ground. A digital camera registers the positions of (A) while falling at equal intervals of time each of $\tau = 0.2s$.

Document 1 shows a diagram of the positions of the ball. The table below represents the distances (in cm) separating each two consecutive positions.

A_0A_1	A_1A_2	A_2A_3	A_3A_4	A_4A_5
20	60	100	140	180



Doc.1

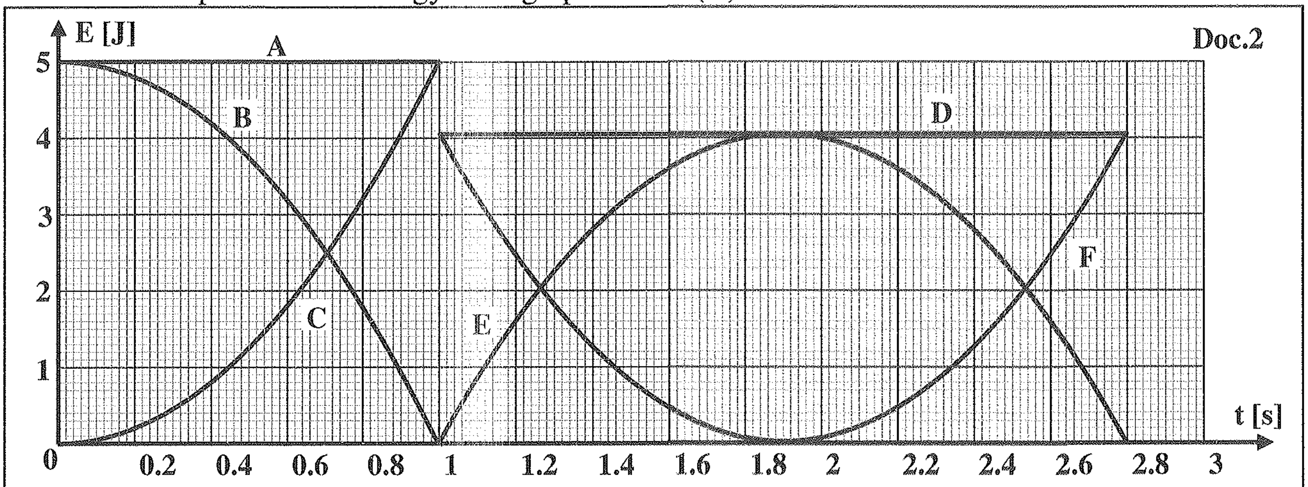
Part I: Determination of the type of motion of the ball

1- Calculate the speed of (A) at positions A_2 and A_3.
2- Calculate the change in linear momentum between A_2 and A_3.
3- Apply Newton's second law to determine the acceleration of the ball; then, deduce the type of motion of the ball.
4- Calculate the speed of (A) as it hits the ground level at the position A_5.
5- Calculate, using two methods, the value of h.

Part II: Energetic study and exploitation of graph

Ball (A) reaches the ground at a speed of 10 m/s; consequently, it rebounds and reaches a maximum height of $h' = 4.05m$ above the ground. The ground level is taken as a reference for the gravitational potential energy.

1- Use the conservation law of mechanical energy to determine the speed at which the ball leaves the level ground just after impact.
2- Determine the amount of energy lost during collision.
3- Determine the change in linear momentum of (A) during collision.
4- Deduce the magnitude of the force exerted by the ground on the ball if the collision lasted 0.01s.
5- Document 2 represents the energy-time graph of ball (A).



Doc.2

5.1- Give the name of energies that A, B, C, D, E and F represent.
5.2- At what instant does (A) collide with the ground for the second time?

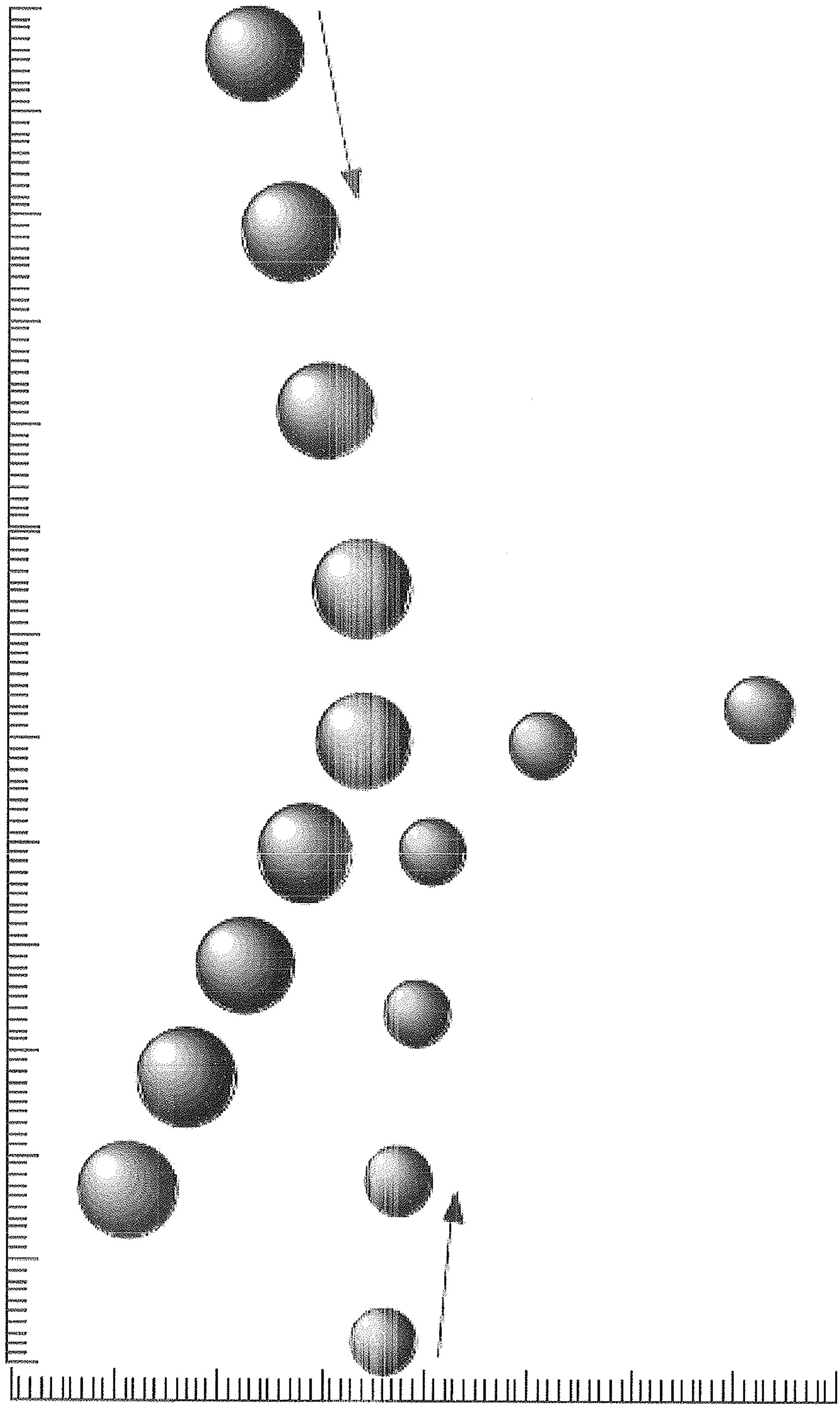
Part III: Elastic collision

Ball (A), after many collisions, moved on the smooth ground along a horizontal track at a speed of 0.8m/s. Another ball (B), considered as a particle of mass $m_B = 200g$ and moving in the opposite direction to that of (A), enters into a head-on perfectly elastic collision with (A). The speed of (B) before collision is 1m/s, and both balls rebounded along the same track after collision. Determine the velocities of both balls after collision.

Exercise 34*:

Two pucks A_1 and A_2, of respective masses $M_1 = 0.3$ kg and $M_2 = 0.6$ kg are placed on a horizontal air table. The adjacent figure shows, at a **real scale,** the collision process between the two pucks where the arrows show the initial direction of the two pucks before collision. The time interval between the images of each ball is 0.1s.

1- Determine graphically the linear momentums of the pucks before and after collision.
2- Represent these momentums on the diagram relative to a scale of your choice.
3- Is the linear momentum conserved?
4- Is the collision between the two pucks elastic?
5- Determine the magnitude of the force exerted on the small puck if the collision lasted 0.1s.



CHAPTER 2 – LINEAR MOMENTUM
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 2:

Part	Answer
1	$\Delta P = m_2 v_2 - m_1 v_1 = 0.06 \times 55 - 0 = 3.3 kgm/s.$
2	$F_{av} = \frac{\Delta P}{\Delta t} = \frac{3.3}{0.004} = 825N.$

Exercise 3:

Part	Answer
1	$\vec{P}_{bc} = \vec{P}_X + \vec{P}_y = m_1\vec{v}_1 + m_2\vec{v}_2 = 2000 \times 24\vec{i} + \vec{0} = 48{,}000\vec{i}\,[kgm/s].$
2	During short duration collision, the system [(X); (Y)] is isolated. $\sum \vec{F}_{ext} = \frac{d\vec{P}_S}{dt} = \vec{0} \Longrightarrow \vec{P}_s = constant.$ Law of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac}.$ $\vec{P}_{bc} = 2m\vec{V} \Longrightarrow 48{,}000\vec{i} = 4000\vec{V} \Longrightarrow \vec{V} = 12\vec{i}\,[m/s].$
3	$K.E_{bc} = K.E_X + K.E_Y = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(2000)(24)^2 + 0 = 576{,}000J.$ $K.E_{ac} = \frac{1}{2}(m_1 + m_2)V^2 = \frac{1}{2}(4000)(12)^2 = 288{,}000J.$ $K.E \neq K.E'$ (the collision is perfectly inelastic).
4	The energy transformed to other forms is: $E = \lvert\Delta K.E\rvert = \lvert 288{,}000 - 576{,}000\rvert = 288{,}000J.$

Exercise 4:

Part	Answer
	Initially, the rifle and the bullet are art rest before shooting. During the very short time interval of explosion, we can assume that the external forces are small compared to the force exerted by the exploding gun powder. During explosion, the system [rifle; bullet] is isolated. $\sum \vec{F}_{ext} = \frac{d\vec{P}_s}{dt} = \vec{0} \Longrightarrow \vec{P}_S = constant.$ Principle of conservation of linear momentum: $\vec{P}_i = \vec{P}_f \Longrightarrow \vec{0} = m_R\vec{v}_R + m_B\vec{v}_B.$ $\vec{v}_R = -\frac{m_B}{m_R}\vec{v}_B = -\frac{55\times10^{-3}}{3.4} \times 960\vec{i} = -15.53\vec{i}\,[m/s].$ The minus sign indicates that the velocity of the rifle is opposite to that of the bullet.

Exercise 5:

Part	Answer
	Before explosion, the system is isolated $\Longrightarrow \sum \vec{F}_{ext} = \frac{d\vec{P}_s}{dt} = \vec{0} \Longrightarrow \vec{P}_S = constant.$ Principle of conservation of linear momentum: $\vec{P}_i = \vec{P}_f \Longrightarrow \vec{0} = \vec{P}_{shell} + \vec{P}_{gas} \Longrightarrow \vec{P}_{shell} = -\vec{P}_{gas}.$ The gas ejects in the opposite direction to that of the shell.

Exercise 6:

Part	Answer
	During explosion, the system (S) is isolated $\Longrightarrow \sum \vec{F}_{ext} = \frac{d\vec{P}_s}{dt} = \vec{0} \Longrightarrow \vec{P}_S = constant.$ Principle of conservation of linear momentum: $\vec{P}_i = \vec{P}_f.$ $\vec{0} = m_1\vec{v}_1 + m_2\vec{v}_2 \Longrightarrow \vec{v}_2 = -\frac{m_1}{m_2}\vec{v}_1 = -\frac{m}{3m}\vec{v}_1 = -\frac{1}{3}\vec{v}_1 = -\frac{1}{3}(6\vec{i}) = -2\vec{i}\,[m/s].$

Exercise 9:

Part	Answer key	
1	$\vec{r}_1 = x_1\vec{i} + y_1\vec{j} = (2t^2 + 5)\vec{i} + (t-1)\vec{j}\,[m].$	$\vec{r}_2 = x_2\vec{i} + y_2\vec{j} = (2t + 4)\vec{i} + (t^2 + t)\vec{j}\,[m].$

	$\vec{V}_1 = \frac{d\vec{r}_1}{dt} = \vec{r}_1' = 4t\vec{\imath} + \vec{\jmath}\ [m/s]$. $\vec{P}_1 = m_1\vec{V}_1 = 4t\vec{\imath} + \vec{\jmath}\ [kgm/s]$.	$\vec{V}_2 = \frac{d\vec{r}_2}{dt} = \vec{r}_2' = 2\vec{\imath} + (2t+1)\vec{\jmath}\ [m/s]$. $\vec{P}_2 = m_2\vec{V}_2 = 4\vec{\imath} + (4t+2)\vec{\jmath}\ [kgm/s]$.
2	$\vec{P}_S = \vec{P}_1 + \vec{P}_2 = (4t+4)\vec{\imath} + (4t+3)\vec{\jmath}\ [kgm/s]$	
3	$X_G = \frac{m_1x_1+m_2x_2}{m_1+m_2} = \frac{(1)(2t^2+5)+(2)(2t+4)}{3} = \frac{2t^2+4t+13}{3}\ [m]$. $Y_G = \frac{m_1y_1+m_2y_2}{m_1+m_2} = \frac{(1)(t-1)+(2)(t^2+t)}{3} = \frac{2t^2+3t-1}{3}\ [m]$. $\vec{r}_G = X_G\vec{\imath} + Y_G\vec{\jmath} = \left(\frac{2t^2+4t+13}{3}\right)\vec{\imath} + \left(\frac{2t^2+3t-1}{3}\right)\vec{\jmath}\ [m]$.	
4	$\vec{V}_G = \frac{d\vec{r}_G}{dt} = \vec{r}_G' = \left(\frac{4t+4}{3}\right)\vec{\imath} + \left(\frac{4t+3}{3}\right)\vec{\jmath}\ [m/s]$. $\vec{P}_G = (m_1+m_2)\vec{V}_G = (4t+4)\vec{\imath} + (4t+3)\vec{\jmath}\ [kgm/s]$. $\vec{P}_S = \vec{P}_G$.	
5	$\vec{a}_1 = \frac{d\vec{V}_1}{dt} = \vec{V}_1' = 4\vec{\imath}\ [m/s^2]$. $\vec{a}_2 = \frac{d\vec{V}_2}{dt} = \vec{V}_2' = 2\vec{\jmath}\ [m/s^2]$.	$\vec{F}_1 = m_1\vec{a}_1 = 4\vec{\imath}\ [N]$. $\vec{F}_2 = m_2\vec{a}_2 = 4\vec{\jmath}\ [N]$.
6	$\sum\vec{F}_{ext/(S)} = \vec{F}_1 + \vec{F}_2 = 4\vec{\imath} + 4\vec{\jmath}\ [N]$.	$\frac{d\vec{P}}{dt} = 4\vec{\imath} + 4\vec{\jmath}\ [N]$.
7	The system (S) is not isolated since $\sum\vec{F}_{ext/(S)} \neq \vec{0}$.	

Exercise 11:

Part	Answer key
1	$t = \frac{x}{5\sqrt{3}}$; then, $y = -5\left(\frac{x}{5\sqrt{3}}\right)^2 + 5\left(\frac{x}{5\sqrt{3}}\right) = -\frac{1}{15}x^2 + \frac{1}{\sqrt{3}}x$ (Parabolic).
2.1	$\vec{V} = \frac{d\vec{r}}{dt} = \vec{r}' = 5\sqrt{3}\vec{\imath} + (-10t+5)\vec{\jmath}\ [m/s]$.
2.2	$\vec{a} = \frac{d\vec{V}}{dt} = \vec{V}' = -10\vec{\jmath}\ [m/s^2]$.
2.3	$\vec{P} = m\vec{V} = 10\sqrt{3}\vec{\imath} + (-20t+10)\vec{\jmath}\ [kgm/s]$.
3	The only force acting on a projectile is its weight $(\vec{W} = m\vec{g})$. $\sum\vec{F}_{ext} = m\vec{g} = 2\times(-10\vec{\jmath}) = -20\vec{\jmath}\ [N]$. $m\vec{a} = 2\times(-10\vec{\jmath}) = -20\vec{\jmath}\ [kgm/s^2]$. $\frac{d\vec{P}}{dt} = -20\vec{\jmath}\ [kgm/s^2]$.
4	$V_y = 0 \Longrightarrow -10t + 5 = 0 \Longrightarrow t = 0.5s$.
5	$y = 0 \Longrightarrow -\frac{1}{15}x^2 + \frac{1}{\sqrt{3}}x = 0 \Longrightarrow x\left(-\frac{1}{15}x + \frac{1}{\sqrt{3}}\right) = 0$. $x = 0$ (launching position) and $-\frac{1}{15}x + \frac{1}{\sqrt{3}} = 0 \Longrightarrow x = 5\sqrt{3}m$.
6	At $t_0 = 0s$, $\vec{V}_0 = 5\sqrt{3}\vec{\imath} + 5\vec{\jmath}\ [m/s]$. $\tan\alpha = \frac{V_{0y}}{V_{0x}} \Longrightarrow \alpha = \tan^{-1}\frac{V_{0y}}{V_{0x}} = \tan^{-1}\frac{5}{5\sqrt{3}} = 30°$.

Exercise 12:

Part	Answer
1	During short duration collision, the system [(A); (B)] is isolated. $\sum\vec{F}_{ext} = \frac{d\vec{P}_S}{dt} = \vec{0} \Longrightarrow \vec{P}_s = constant$. Principle of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac}$.

	$m_A\vec{v}_{A1} + m_B\vec{v}_{B1} = m_A\vec{v}_{A2} + m_B\vec{v}_{B2}$. Projection along x-axis: $m_A v_{A1x} + m_B v_{B1x} = m_A v_{A2x} + m_B v_{B2x}$. $m_A v_{A1} + 0 = m_A v_{A2}\cos\alpha + m_B v_{B2x} \Longrightarrow v_{B2x} = \frac{m_A(v_{A1}-v_{A2}\cos\alpha)}{m_B}$. $v_{B2x} = \frac{5(2-1\times\cos 30°)}{3} = 1.89m/s$. Projection along y-axis: $m_A v_{A1y} + m_B v_{B1y} = m_A v_{A2y} + m_B v_{B2y}$. $0 + 0 = m_A v_{A2}\sin\alpha + m_B v_{B2y} \Longrightarrow v_{B2y} = -\frac{m_A v_{A2}\sin\alpha}{m_B} = -\frac{5\times1\times\sin 30°}{3} = -0.83m/s$. $v_{B2} = \sqrt{v_{B2x}^2 + v_{B2y}^2} = 2.1m/s$ and $\tan\beta = \frac{v_{B2y}}{v_{B2x}} \Longrightarrow \beta = -24°$.

Exercise 14:

Part	Answer
A.1.1.1	$\vec{P}_1 = m_A\vec{V}_1 = 0.4\times0.5\vec{\imath} = 0.2\vec{\imath}\ [kgm/s]$. $\vec{P}_2 = m_A\vec{V}_2 = 0.4\times(-0.1\vec{\imath}) = -0.04\vec{\imath}\ [kgm/s]$.
A.1.1.2	$\vec{P}_3 = m_B\vec{V}_3 = 0.6\times0.4\vec{\imath} = 0.24\vec{\imath}\ [kgm/s]$.
A.1.2	$\vec{P} = \vec{P}_A + \vec{P}_B = \vec{P}_1 + \vec{0} = 0.2\vec{\imath} + \vec{0} = 0.2\vec{\imath}\ [kgm/s]$. $\vec{P}' = \vec{P}_A' + \vec{P}_B' = \vec{P}_2 + \vec{P}_3 = -0.04\vec{\imath} + 0.24\vec{\imath} = 0.2\vec{\imath}\ [kgm/s]$.
A.1.3	$\vec{P} = \vec{P}' = 0.2\vec{\imath}\ [kgm/s]$. During collision between A and B the linear momentum of the system [A; B] is conserved.
A.2.1	$\vec{W}_A = m_A\vec{g}$: Weight of A. $\vec{N}_A$: Normal reaction of the table acting on A. $\vec{W}_B = m_B\vec{g}$: Weight of B. $\vec{N}_B$: Normal reaction of the table acting on B.
A.2.2	$\sum\vec{F}_{ext} = \vec{W}_A + \vec{N}_A + \vec{W}_B + \vec{N}_B = \vec{0}$. The sum of the external forces acting on the system (A, B) is thus zero.
A.2.3	Yes, Since the system [(A),(B)] is isolated. From part A.1.3: $\vec{P} = constant \Longrightarrow \frac{d\vec{P}}{dt} = \vec{0}$. Newton's 2nd law for translational motion: $\sum\vec{F}_{ext} = \frac{d\vec{P}}{dt} = \vec{0}$.
B.1	The kinetic energy of the system [A; B] before collision is: $K.E = K.E_A + K.E_B = \frac{1}{2}m_A V_1^2 + 0 = \frac{1}{2}(0.4)(0.5)^2 = 0.05J$. The kinetic energy of the system [A; B] after collision is: $K.E' = K.E_A' + K.E_B' = \frac{1}{2}m_A V_2^2 + \frac{1}{2}m_B V_3^2 = \frac{1}{2}(0.4)(0.1)^2 + \frac{1}{2}(0.6)(0.4)^2 = 0.05J$.
B.2	$K.E = K.E'$ (the kinetic energy of the system [A; B] is conserved). Therefore, the collision is elastic.
C.1.1	$\Delta\vec{P}_A = \vec{P}_2 - \vec{P}_1 = -0.04\vec{\imath} - 0.2\vec{\imath} = -0.24\vec{\imath}\ [kgm/s]$. $\Delta\vec{P}_B = \vec{P}_3 - \vec{0} = 0.24\vec{\imath}\ [kgm/s]$.
C.1.2	$\vec{F}_{B/A} = \frac{\Delta\vec{P}_A}{\Delta t} = \frac{-0.24\vec{\imath}}{0.04} = -6\vec{\imath}\ [N]$. $\vec{F}_{A/B} = \frac{\Delta\vec{P}_B}{\Delta t} = \frac{0.24\vec{\imath}}{0.04} = 6\vec{\imath}\ [N]$.
C.2	$\vec{F}_{A/B} = -\vec{F}_{B/A} = 6\vec{\imath}\ [N] \Longrightarrow$ the principle of interaction is thus verified.

Exercise 15:

Part	Answer
1	$M.E_D = K.E_D + G.P.E_D = 0 + m_A g h_D = 9J.$
2	No friction $\Longrightarrow$ the mechanical energy of the system [(A); Earth] is conserved: $M.E_D = M.E_N \Longrightarrow K.E_D + G.P.E_D = K.E_N + G.P.E_N.$ $0 + m_A g h_D = \frac{1}{2} m_A V_{A1}^2 + 0 \Longrightarrow V_{A1} = \sqrt{2gh_D} = \sqrt{2 \times 10 \times 0.45} = 3m/s.$
3.1	Linear momentum of the system [(A), (B)] before collision: $\vec{P_S} = \vec{P_A} + \vec{P_B} = m_A \vec{V}_{A1} + m_B \vec{V}_{B1} = 2 \times 3\vec{\imath} + 4 \times (-\vec{\imath}) = 2\vec{\imath}\ [kgm/s].$
3.2	$\vec{P_S} = \vec{P_G} = (m_A + m_B)\vec{V}_G \Longrightarrow \vec{V}_G = \frac{\vec{P}_S}{m_A + m_B} = \frac{2\vec{\imath}}{2+4} = \frac{1}{3}\vec{\imath} = 0.33\vec{\imath}\ [m/s].$
4.1	$M.E_C = K.E_C + G.P.E_C = 0 + m_A g h_C = 2 \times 10 \times 0.27 = 5.4J.$
4.2	Conservation of the mechanical energy of the system [(A), Earth]: $0 + m_A g h_C = \frac{1}{2} m V_{A2}^2 + 0 \Longrightarrow V_{A2} = \sqrt{2gh_C} = \sqrt{2 \times 10 \times 0.27} = 2.33m/s.$
5	Conservation of the linear momentum of the system [(A), (B)]: $\vec{P_S} = \vec{P'_S} \Longrightarrow m_A \vec{V}_{A1} + m_B \vec{V}_{B1} = m_A \vec{V}_{A2} + m_B \vec{V}_{B2}.$ $2(3\vec{\imath}) + 4(-\vec{\imath}) = 2(-2.33\vec{\imath}) + 4\vec{V}_{B2} \Longrightarrow \vec{V}_{B2} = 1.66\vec{\imath}\ [m/s].$
6	The kinetic energy of the system [(A), (B)] before collision: $K.E_{B.C} = K.E_A + K.E_B = \frac{1}{2} m_A V_{A1}^2 + \frac{1}{2} m_B V_{B1}^2 = \frac{1}{2}(2)(3)^2 + \frac{1}{2}(4)(1)^2 = 11J.$ The kinetic energy of the system [(A), (B)] before collision: $K.E_{AC} = K.E'_A + K.E'_B = \frac{1}{2} m_A V_{A2}^2 + \frac{1}{2} m_B V_{B2}^2 = \frac{1}{2}(2)(2.33)^2 + \frac{1}{2}(4)(1.66)^2 = 11J.$ $K.E_{BC} = K.E_{AC}$ (the kinetic energy of the system [(A); (B)] is conserved. Therefore, the collision is elastic.

Exercise 16:

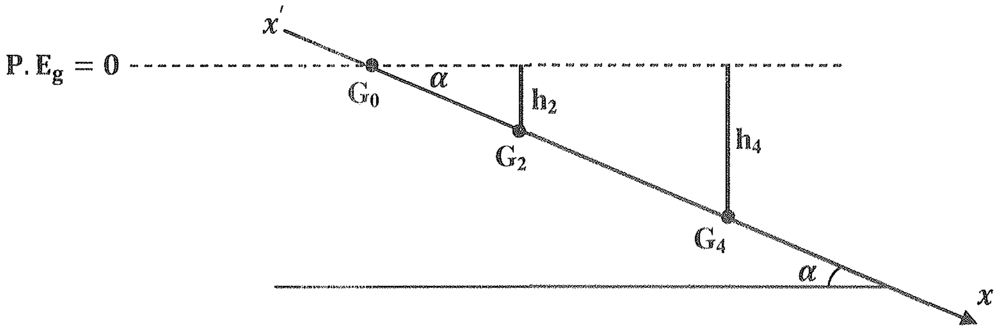
Part	Answer
1	The kinetic energy of the system [bullet, block].
2	During collision, the system [bullet, block] is isolated $\Longrightarrow \sum \vec{F}_{ext} = \frac{d\vec{P}_S}{dt} = \vec{0} \Longrightarrow \vec{P}_S = constant.$ Principle of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac} \Longrightarrow m\vec{V}_0 + \vec{0} = (m + M)\vec{V}_1.$ The collision is head-on, so, the velocity vectors are collinear. Then the above equation can be written in algebraic form: $mV_0 = (m + M)V_1 \Longrightarrow V_1 = \frac{m}{m+M} V_0.$
3.1	$M.E_1 = K.E_1 + G.P.E_1 = \frac{1}{2}(m + M)V_1^2 + 0 = \frac{1}{2}\frac{m^2}{m+M} V_0^2.$
3.2	$M.E_2 = K.E_2 + G.P.E_2 = 0 + (m + M)gh_2 = (m + M)gl(1 - \cos\alpha).$
3.3	The non-conservative force (friction) is neglected; then, the mechanical energy of the system [bullet, block] is conserved. $M.E_1 = M.E_2 \Longrightarrow \frac{1}{2}\frac{m^2}{m+M} V_0^2 = (m + M)gl(1 - \cos\alpha) \Longrightarrow V_0^2 = \frac{2(m+M)^2}{m^2} g(1 - \cos\alpha).$ $V_0 = \frac{m+M}{m}\sqrt{2gl(1 - \cos\alpha)} = \frac{0.02+1}{0.02}\sqrt{2 \times 9.8 \times 1 \times (1 - \cos 37°)} = 101.3m/s.$
4	The kinetic energy of the system [bullet, block] before collision: $K.E = \frac{1}{2} m V_0^2 + 0 = \frac{1}{2}(0.02)(101.3)^2 = 102.6J.$ The kinetic energy of the system [bullet, block] after collision: $K.E' = \frac{1}{2}(m + M)V_1^2 = \frac{1}{2}\frac{m^2}{m+M} V_0^2 = \frac{1}{2}\frac{0.02^2}{0.02+1} \times 101.3^2 = 2J.$

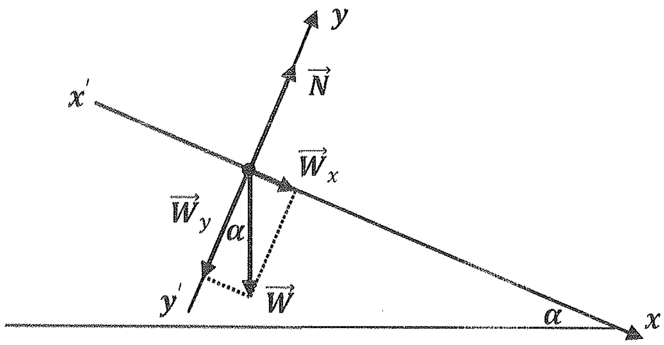
	$K.E \neq K.E'$ (the kinetic energy of the system [bullet, block] is not conserved.

Exercise 17:

Part	Answer
1	Work-kinetic energy theorem: $\Delta K.E = \sum W_{ext} \Longrightarrow K.E_B - K.E_A = W_{\vec{W}} + W_{\vec{T}}$. $\frac{1}{2}m_1V_1^2 - 0 = m_1gL + 0 \Longrightarrow V_1 = \sqrt{2gL} = \sqrt{2 \times 10 \times 0.8} = 4m/s$.
2	The non-conservative force (friction) is neglected, so, the mechanical energy of the system [(S_2); Earth] is conserved. $M.E_B = M.E_C \Longrightarrow K.E_B + G.P.E_B + E.P.E_B = K.E_C + G.P.E_C + E.P.E_D$. $\frac{1}{2}m_2{V_2'}^2 + 0 + 0 = 0 + 0 + \frac{1}{2}kx_m^2 \Longrightarrow V_2' = x_m\sqrt{\frac{k}{m_2}} = 0.317 \times \sqrt{\frac{120}{3}} = 2m/s$.
3	During collision, the system [(S_1); (S_2)] is isolated. $\sum \vec{F}_{ext} = \frac{d\vec{P}_S}{dt} = \vec{0} \Longrightarrow \vec{P}_S = constant$. Principle of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac}$. $\vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2' \Longrightarrow m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_1' + m_2\vec{V}_2'$. The collision is head-on, so, the velocities are collinear. Thus, the above equation can be written in algebraic form: $m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2' \Longrightarrow (1)(4) + 0 = (1)V_1' + (3)(2) \Longrightarrow V_1' = -2m/s$.
4	$M.E_B = M.E_D \Longrightarrow K.E_B + G.P.E_B = K.E_D + G.P.E_D$. $\frac{1}{2}m_1{V_2'}^2 + 0 = 0 + m_1gL(1 - \cos\theta) \Longrightarrow \frac{{V_2'}^2}{2} = gL(1 - \cos\theta)$. $\cos\theta = 1 - \frac{{V_1'}^2}{2gL} = 1 - \frac{2^2}{2\times10\times0.8} \Longrightarrow \theta = 41.1°$.
5	$K.E_{bc} = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}(1)(4)^2 = 8J$ $K.E_{ac} = \frac{1}{2}m_1{V_1'}^2 + \frac{1}{2}m_2{V_2'}^2 = \frac{1}{2}(1)(2)^2 + \frac{1}{2}(3)(2)^2 = 8J$ $K.E_{bc} = K.E_{ac} \Longrightarrow$ The kinetic energy is conserved and the collision is perfectly elastic.

Exercise 19:

Part	Answer
1	$V_2 = \frac{G_1G_3}{2\tau} = \frac{G_0G_3 - G_0G_1}{2\tau} = \frac{x_3 - x_1}{2\tau} = \frac{(4.5-0.5)\times10^{-2}}{0.1} = 0.4m/s$. $V_4 = \frac{G_3G_5}{2\tau} = \frac{G_0G_5 - G_0G_3}{2\tau} = \frac{x_5 - x_3}{2\tau} = \frac{(12.5-4.5)\times10^{-2}}{0.1} = 0.8m/s$.
2	$M.E = K.E + P.E_g$; $M.E_0 = K.E_0 + P.E_{g_0} = 0 + 0 = 0$ with $V_0 = 0$ and $h_0 = 0$. $M.E_2 = K.E_2 + P.E_{g_2} = \frac{1}{2}MV_2^2 - Mgh_2$;
	

	$h_2 = G_0G_2 \times \sin\alpha = 2 \times 0.4 = 0.8cm = 0.008m \Rightarrow M.E_2 = 0J.$ $M.E_4 = K.E_4 + P.E_{g_4} = \frac{1}{2}MV_4^2 - Mgh_4;$ $h_4 = G_0G_4 \times \sin\alpha = 8 \times 0.4 = 3.2cm = 0.032m \Rightarrow M.E_4 = 0J.$
3.1	$M.E_0 = M.E_2 = M.E_4 \Rightarrow$ The mechanical energy is conserved during motion $\Rightarrow$ No friction.
3.2	$\Delta\vec{P} = \vec{P}_4 - \vec{P}_2 = M\vec{V}_4 - M\vec{V}_2 = M(\vec{V}_4 - \vec{V}_2) = 0.1(0.8\vec{\imath} - 0.4\vec{\imath}) = 0.04\vec{\imath}\ [kgm/s].$
4.1	The forces acting on (S): The weight $\vec{W}$ of (S) and the normal reaction $\vec{N}$ of the path.
4.2	$\sum\vec{F} = \vec{W} + \vec{N} = \vec{W}_x + \vec{W}_y + \vec{N}$ where: $\vec{W}_x = Mg\sin\alpha\,\vec{\imath}$, $\vec{W}_y = -Mg\cos\alpha\,\vec{\jmath}$ and $\vec{N} = N\vec{\jmath}$. $\vec{W}_y + \vec{N} = \vec{0}$ (no motion along $y'y$). $\Rightarrow \sum\vec{F} = \vec{W}_1 = Mg\sin\alpha\,\vec{\imath}.$
	
5	The 2[nd] Law of is given by: $\sum\vec{F} = \frac{d\vec{P}}{dt} = \frac{\Delta\vec{P}}{\Delta t}$. We have: $\sum\vec{F} = Mg\sin\alpha\,\vec{\imath} = 0.4\vec{\imath}$ and $\frac{\Delta\vec{P}}{\Delta t} = \frac{0.04\vec{\imath}}{0.1} = 0.4\vec{\imath}\ [N].$ $\Rightarrow$ The 2[nd] law of Newton is thus verified.

Exercise 20:

Part	Answer
1	The only external force acting on the shell is its weight. Apply Newton's 2[nd] law for translational motion: $\sum\vec{F}_{ext} = \frac{d\vec{P}}{dt}$. $m\vec{g} = \frac{d\vec{P}}{dt}$ with $\vec{P} = \vec{P}_x + \vec{P}_y = P_x\vec{\imath} + P_y\vec{\jmath}.$ $-mg\vec{\jmath} = \frac{dP_x}{dt}\vec{\imath} + \frac{dP_y}{dt}\vec{\jmath}.$ $\frac{dP_x}{dt} = 0 \Rightarrow P_x = P_{0x} = mV_{0x} = mV_0\cos\theta_0 = 2 \times 20 \times \cos 60° = 20kgm/s.$ $\frac{dP_y}{dt} = -mg \Rightarrow P_y = -mgt + P_{0y} = -mgt + mV_{0y} = -mgt + mV_0\sin\theta_0.$ $P_y = -2 \times 10 \times t + 2 \times 20 \times \sin 60° = -20t + 20\sqrt{3}\ [kgm/s].$
2	$V_x = \frac{P_x}{m} = \frac{20}{2} = 10m/s.$ $V_y = \frac{P_y}{m} = \frac{-20t+20\sqrt{3}}{2} = -10t + 10\sqrt{3}\ [m/s].$ $x = \int V_x dt = 10t + x_0 = 10t$ with $x_0 = 0.$ $y = \int V_y dt = -5t^2 + 10\sqrt{3}t + y_0 = -5t^2 + 10\sqrt{3}t$ with $y_0 = 0.$
3	$V_y = 0 \Rightarrow t = \frac{10\sqrt{3}}{10} = \sqrt{3}s.$

	$x = 10\sqrt{3} = 17.32m$. $y = -5 \times 3 + 10 \times 3 = 15m$.
4.1	During the very short time interval of explosion, we can assume that the external forces acting on the shell are small compared to the internal forces. The linear momentum of the shell is conserved: $\frac{d\vec{P}}{dt} = 0 \Rightarrow \vec{P} = constant$. The linear momentum of the shell is conserved along the x-axis: $mV_x = m_1 v_{1x} + m_2 v_{2x}$ with $m_1 = m_2 = \frac{m}{2}$. $mV_x = 0 + \frac{m}{2} v_{2x}$. $v_{2x} = 2V_x = 2 \times 10 = 20m/s$. The linear momentum of the shell is conserved along the y-axis: $mV_y = m_1 v_{1y} + m_2 v_{2y} \Rightarrow v_{2y} = 0$ since $V_y = 0$ and $v_{1y} = 0$.
4.2	At $t_0 = 0s$, the coordinates of the second fragment are $x_{2_0} = 17.3m$ and $y_{2_0} = 15m$. $x_2 = v_{2x}t + x_{2_0} = 20t + 17.3$. $y_2 = \frac{1}{2}a_{2y}t^2 + v_{2y_0}t + y_{2_0} = -5t^2 + 15$. For $y_2 = 0 \Rightarrow t = \sqrt{3}s$. $x_2 = 20(\sqrt{3}) + 17.3 = 52m$.

Exercise 22:

Part	Answer
1	$P_{0x} = 4kgm/s$ and $P_{0y} = 5kgm/s$.
2	$P_0 = \sqrt{P_{0x}^2 + P_{0y}^2} = \sqrt{4^2 + 5^2} = \sqrt{41}kgm/s = 6.4m/s$. $V_0 = \frac{P_0}{m} = \frac{6.4}{0.5} = 12.8m/s$. $\tan\alpha = \frac{P_{0y}}{P_{0x}} = \frac{5}{4} \Rightarrow \alpha = 51.34°$.
3	For $P_y = 0; t_{max} = 1s$. For $\lvert P_y \rvert = \lvert P_{0y} \rvert = 5kgm/s;\ t_R = 2s$. Therefore, $t_R = 2t_{max}$.
4	$P_x = 4m/s$. $P_y = kt + P_{0y}$. $k = \frac{\Delta P_y}{\Delta t} = \frac{0-5}{1-0} = -5kgm/s^2$. Therefore, $P_y = -5t + 5$.
5	$V_x = \frac{P_x}{m} = \frac{4}{0.5} = 8m/s$ and $V_y = \frac{P_y}{m} = \frac{-5t+5}{0.5} = -10t + 10$. $x = \int V_x dt = 8t + x_0 = 8t$ with $x_0 = 0$. $y = \int V_y dt = -5t^2 + 10t + y_0 = -10t^2 + 10t$ with $y_0 = 0$.
6	Between 0s and 2s, the motion along x-axis is uniform rectilinear since $V_x = constant$. Between 0s and 1s, UDRM along y-axis since $a_y \times V_y < 0$ with $a_y = -10m/s^2$. Between 1s and 2s, UARM along y-axis since $a_y \times V_y > 0$.
7	For $x = 12m$. $t = \frac{x}{8} = \frac{12}{8} = 1.5s$. $y = -5(1.5)^2 + 10(1.5) = 3.75m > 3m$.

Exercise 23:

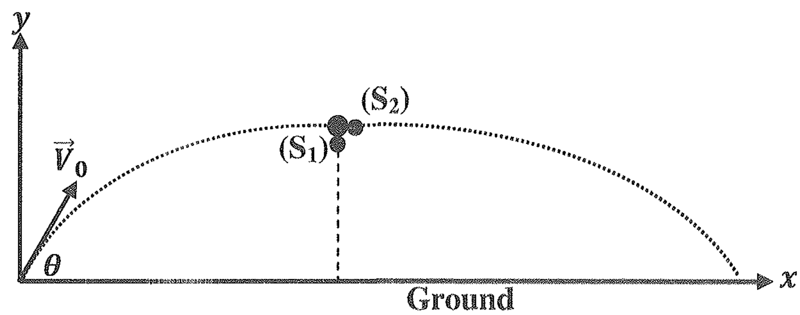
Part	Answer
1.1	At $t_0 = 0s; P_0 = 2kgm/s \Longrightarrow P_0 = m_1V_0 \Longrightarrow V_0 = \frac{P_0}{m_1} = \frac{2}{0.2} = 10m/s.$
1.2	The general equation of a straight line is: $P = at + b$ with $a = \frac{\Delta P}{\Delta t} = \frac{1.6-2}{0.4-0} = -1kgm/s^2.$ At $t = 0; P = b = 2kgm/s$. Therefore, $P = -t + 2.$
2	Apply Newton's 2nd law: $\sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} \Longrightarrow \vec{W}_1 + \vec{N}_1 = \frac{dP}{dt}\vec{i}.$ $-m_1g\sin\alpha\,\vec{i} - m_1g\cos\alpha\,\vec{j} + N_1\vec{j} = \frac{dP}{dt}\vec{i}$ with $-m_1g\cos\alpha\,\vec{j} + N_1\vec{j} = \vec{0}.$ $-m_1g\sin\alpha = \frac{dP}{dt} \Longrightarrow -0.2 \times 10 \times \sin\alpha = -1 \Longrightarrow \alpha = 30°.$
3	Apply Newton's 1st law: $\sum \vec{F}_{ext} = \vec{0} \Longrightarrow \vec{T}_2 + \vec{W}_2 + \vec{N}_2 = \vec{0}.$ Projection along x-axis: $-m_2g\sin\alpha + T_2 = 0 \Longrightarrow k\Delta L = m_2g\sin\alpha.$ $\Delta L = \frac{m_2g\sin\alpha}{k} = \frac{0.9\times10\times0.5}{50} = 0.09m = 9cm.$
4	$\sin\alpha = \frac{h}{OB} \Longrightarrow OB = \frac{h}{\sin\alpha} = \frac{2.05}{0.5} = 4.1m.$ $OA = OB - AB = OB - (L_0 + \Delta L) = 4.1 - (0.41 + 0.09) = 3.6m.$ By applying the work-kinetic energy theorem: $\Delta K.E = \sum W_{ext}.$ $K.E_A - K.E_O = W_{\vec{W}_1} + W_{\vec{N}_1} \Longrightarrow \frac{1}{2}m_1V_1^2 - \frac{1}{2}m_1V_0^2 = -m_1g\sin\alpha \times OA + 0.$ $\frac{V_1^2}{2} - \frac{V_0^2}{2} = -gOA\sin\alpha \Longrightarrow \frac{V_1^2}{2} - \frac{10^2}{2} = -10 \times 0.5 \times 3.6 \Longrightarrow V_1 = \sqrt{64} = 8m/s.$
5.1	During collision, the system [(S1); (S2)] is isolated $\Longrightarrow \sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} = \vec{0} \Longrightarrow \vec{P} = constant.$ Principle of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac}.$ $\vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2' \Longrightarrow m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_1' + m_2\vec{V}_2'.$ The collision is head-on; then, the above equation can be written in the algebraic form: $m_1V_1 + m_2V_2 = m_1V_1' + m_2V_2'.$ $(0.2)(8) + 0 = (0.2)(-3.25) + (0.9)V_2' \Longrightarrow V_2' = 2.5m/s.$
5.2	$K.E_{bc} = K.E_1 + K.E_2 = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2} \times 0.2 \times 8^2 = 6.4J.$ $K.E_{ac} = K.E_1' + K.E_2' = \frac{1}{2}m_1V_1'^2 + \frac{1}{2}m_2V_2'^2 = \frac{1}{2} \times 0.2 \times 3.25^2 + \frac{1}{2} \times 0.9 \times 2.5^2 = 3.86875J.$ $K.E_{bc} \neq K.E_{ac} \Longrightarrow$ The kinetic energy of the system [(S1); (S2)] is not conserved. Therefore, the collision is not elastic.
6	The non-conservative force (friction) is neglected; then, the mechanical energy of the system [(S2); Earth] is conserved. $M.E_A = M.E_D \Longrightarrow K.E_A + G.P.E_A + E.P.E_A = K.E_D + G.P.E_D + E.P.E_D.$ $\frac{1}{2}m_2V_2'^2 + 0 + \frac{1}{2}k\Delta L^2 = 0 + m_2gAD\sin\alpha + \frac{1}{2}k(\Delta L - AD)^2.$ $\frac{1}{2}(0.9)(2.5)^2 + \frac{1}{2}(50)(0.09)^2 = (0.9)(10)(AD)(\sin 30°) + \frac{1}{2}(50)(0.09 - AD)^2.$ $3.015 = 4.5AD + 25(0.0081 + AD^2 - 0.18AD).$ $3.41 = 4.5AD + 0.2025 + 25AD^2 - 4.5AD.$ $25AD^2 = 3.2075 \Longrightarrow AD = 0.36m = 36cm.$

Exercise 24:

Part	Answer
1	The non-conservative force (friction) is neglected along AB; then, the mechanical energy of the system [(S1); Earth] is conserved.

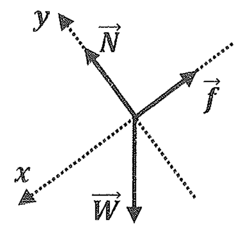
	Principle of conservation of mechanical energy: $M.E_A = M.E_B$. $K.E_A + G.P.E_A = K.E_B + G.P.E_B \Longrightarrow 0 + m_1 gAB\sin\alpha = \frac{1}{2}m_1V_1^2 + 0$. $V_1 = \sqrt{2gAB\sin\alpha} = \sqrt{2\times10\times0.9\times0.5} = \sqrt{9} = 3m/s$.
2.1	During collision, the system [(S₁); (S₂)] is isolated $\Longrightarrow \sum\vec{F}_{ext} = \frac{d\vec{P}}{dt} = \vec{0} \Longrightarrow \vec{P} = constant.$ Principle of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac} \Longrightarrow \vec{P}_1 + \vec{P}_2 = \vec{P} \Longrightarrow m_1\vec{V}_1 + \vec{0} = M\vec{V}$. The collision is head-on; then, the velocities are collinear and the above equation can be written in the algebraic form: $m_1V_1 = MV \Longrightarrow m_1 = \frac{MV}{V_1} = \frac{2\times1.2}{3} = 0.8kg$. $M = m_1 + m_2 \Longrightarrow m_2 = M - m_1 = 2 - 0.8 = 1.2kg$.
2.2	The non-conservative force (friction) is neglected along CD; then, the mechanical energy of the system [(S); (R); Earth] is conserved. $M.E_C = M.E_t \Longrightarrow K.E_C + G.P.E_C + E.P.E_C = K.E_t + G.P.E_t + E.P.E_t$. $\frac{1}{2}MV^2 + 0 + 0 = \frac{1}{2}Mv^2 + 0 + \frac{1}{2}kx^2$. $\frac{1}{2}(2)(1.2)^2 = \frac{1}{2}(2)v^2 + \frac{1}{2}(50)x^2 \Longrightarrow v^2 + 25x^2 = 1.44$.
2.3	Derive the above equation with respect to time: $2vv' + 50xx' = 0$ with $v = x'$ and $a = v' = x''$. Therefore, $a = -25x$.
2.4	For $v = 0 \Longrightarrow 25x_m^2 = 1.44 \Longrightarrow x_m = 0.24m = 24cm$.

Exercise 27:

Part	Answer
1	$V_x = V_{0x} = V_0\cos\theta = 8\times0.5 = 4m/s$ and $V_y = 0$. $\vec{V} = \vec{V}_x + \vec{V}_y = V_x\vec{i} + V_y\vec{j} = 4\vec{i}\ [m/s]$. $\vec{P} = m\vec{V} = 0.3\times4\vec{i} = 1.2\vec{i}\ [kgm/s]$.
2	During the very short time interval of explosion, we can assume that the external forces acting on (S) are small compared to the internal forces. The linear momentum of the system (S) = [(S₁); (S₂)] is conserved $\Longrightarrow \vec{P} = constant.$ Principle of conservation of linear momentum: $\vec{P}_{be} = \vec{P}_{ae}$. $\vec{P} = \vec{P}_1 + \vec{P}_2$ with $\vec{P}_1 = \vec{0} \Longrightarrow \vec{P}_2 = \vec{P} = 1.2\vec{i}$. $\vec{V}_2 = \frac{\vec{P}_2}{m_2} = \frac{1.2\vec{i}}{0.1} = 12\vec{i}\ [m/s]$.
3	$K.E_{be} = \frac{1}{2}mV^2 = \frac{1}{2}\times0.3\times4^2 = 2.4J$. $K.E_{ae} = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = 0 + \frac{1}{2}\times0.1\times12^2 = 7.2J$. the increase in the kinetic energy is: $E = \Delta K.E = 7.2 - 2.4 = 4.8J$.
4	
5	The time needed by (S) to reach its maximum position:

	$V_y = a_y t + V_{0y} = -gt + V_0 \sin\theta = -9.8t + 6.93$. $V_y = 0 \Longrightarrow t = 0.7s$. The coordinates of (S) at its maximum position are: $x = \int V_x dt = V_0 \cos\theta\, t + x_0 = 8 \times 0.5 \times 0.7 = 2.8m$ with $x_0 = 0$. $y = \int V_y dt = -4.9t^2 + 6.93t + y_0 = -4.9(0.7)^2 + 6.93(0.7) = 2.45m$ with $y_0 = 0$. The time equation of (S_1) is: $y_1 = \frac{1}{2}a_1 t^2 + V_{01}t + y_{01} = -4.9t^2 + 2.45$. For $y_1 = 0 \Longrightarrow -4.9t^2 + 2.45 = 0$. $t_1 = 0.7s$ (accepted) and $t_2 = -0.7s$ (rejected). (S_1) and (S_2) take the same time to reach the ground (same g, zero initial vertical velocity and same height relative to the ground). $x_2 = V_{2x}t + x_{02} = 12 \times 0.7 + 2.8 = 11.2m$. $d = x_2 - x_1 = 11.2m - 2.8 = 8.4m$.

Exercise 29:

Part	Answer	
I.1	The general equation of a straight line is: $P = kt + P_0$. At t = 0s; $P = P_0 = 0.4kgm/s$. $k = \frac{\Delta P}{\Delta t} = \frac{1.6-0.4}{0.6-0} = 2kgm/s^2$. Therefore, $P = 2t + 0.4\ [SI]$.	
I.2	The external forces acting on (S) are: weight $\vec{W}$, normal reaction of a support $\vec{N}$ and friction $\vec{f}$.	
I.3	$\sum \vec{F}_{ext} = \vec{W} + \vec{N} + \vec{f} = \vec{W}_x + \vec{W}_y + \vec{N} + \vec{f} = mg\sin\alpha\,\vec{i} - mg\cos\alpha\,\vec{j} + N\vec{j} - f\vec{i}$. But $-mg\cos\alpha\,\vec{j} + N\vec{j} = \vec{0}$ (no motion along y-axis). Therefore, $\sum \vec{F}_{ext} = (mg\sin\alpha - f)\vec{i}$.	
I.4	Newton's 2nd law: $\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt} \Longrightarrow (mg\sin\alpha - f)\vec{i} = \frac{dp}{dt}\vec{i}$. $mg\sin\alpha - f = \frac{dp}{dt} \Longrightarrow 0.4 \times 10 \times 0.6 - f = 2 \Longrightarrow f = 0.4N$.	
II.1	At t = 0s; $P_A = 0.4kgm/s \Longrightarrow V_A = \frac{P_A}{m} = \frac{0.4}{0.4} = 1m/s$. $K.E_A = \frac{1}{2}mV_A^2 = \frac{1}{2} \times 0.4 \times 1^2 = 0.2J$. For $x = 0$; $M.E_A = 3.8J \Longrightarrow G.P.E_A = M.E_A - K.E_A = 3.8 - 0.2 = 3.6J$. $G.P.E_A = mgh_A = mgAB\sin\alpha \Longrightarrow AB = \frac{G.P.E_A}{mg\sin\alpha} = \frac{3.6}{0.4\times10\times0.6} = 1.5m = 150cm$.	
II.2	For $x = 150cm$; $M.E_B = 3.2J$ and $G.P.E_B = 0J$ (B is on the reference level). $K.E_B = M.E_B - G.P.E_B = 3.2J - 0J = 3.2J$. $K.E_B = \frac{1}{2}mV_B^2 \Longrightarrow V_B = \sqrt{\frac{2K.E_B}{m}} = \sqrt{\frac{2\times3.2}{0.4}} = 4m/s$ $P_B = mV_B = 0.4 \times 4 = 1.6kgm/s$. Using document 1, $t_B = 0.6s$.	
I.3	By applying the law of non-conservation of mechanical energy: $\Delta M.E = W_{NC} = W_{\vec{f}}$. $M.E_B - M.E_A = -f \times AB \Longrightarrow 3.2 - 3.8 = -1.5f \Longrightarrow f = 0.4N$.	
III	The results are the same.	

Exercise 30:

Part	Answer
I.1	During collision, the system [(P_1); (P_2)] is isolated $\Longrightarrow \sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} = \vec{0} \Longrightarrow \vec{P} = constant.$
I.2	Principle of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac}$. $\vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2' \Longrightarrow m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_1' + m_2\vec{V}_2'$. Projection along x-axis: $m_1V_1 + 0 = m_1V_1' \cos\alpha + m_2V_{2x}'$. $V_{2x}' = \frac{m_1}{m_2}(V_1 - V_1' \cos\alpha) = \frac{0.8}{1}(5 - 2\cos 51°) = 3m/s$. Projection along y-axis: $0 + 0 = m_1V_1' \sin\alpha - m_2V_{2y}' \Longrightarrow V_{2y}' = -\frac{m_1}{m_2}V_1' \sin\alpha$. $V_{2y}' = -\frac{0.8}{1} \times 2\sin 51° = -1.24m/s$. $V_2' = \sqrt{V_{2x}'^2 + {V_{2y}'}^2} = \sqrt{3^2 + 1.24^2} = 3.24m/s$ and $\tan\beta = \frac{V_{2y}'}{V_{2x}'} = \frac{-1.24}{3} \Longrightarrow \beta = -22.4°$
II.1	During collision, the system [(P_1); (P_2)] is isolated $\Longrightarrow \sum \vec{F}_{ext} = \frac{d\vec{P}}{dt} = \vec{0} \Longrightarrow \vec{P} = constant$ Principle of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac}$. $\vec{P}_1 + \vec{P}_2 = \vec{P}_1' + \vec{P}_2' \Longrightarrow m_1\vec{V}_1 + m_2\vec{V}_2 = m_1\vec{V}_1' + m_2\vec{V}_2'$ with $\vec{V}_2 = \vec{0}$. The collision is head-on, then, the above equation can be written in its algebraic form: $m_1V_1 = m_1V_1' + m_2V_2' \Longrightarrow m_1(V_1 - V_1') = m_2V_2'$ ... (1). The collision is elastic; then, the kinetic energy of the system [(P_1); (P_2)] is conserved: $K.E_{bc} = K.E_{ac} \Longrightarrow K.E_1 + K.E_2 = K.E_1' + K.E_2'$. $\frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2 = \frac{1}{2}m_1{V_1'}^2 + \frac{1}{2}m_2{V_2'}^2 \Longrightarrow m_1(V_1 - V_1')(V_1 + V_1') = m_2V_2'$ ... (2) Divide (2) by (1): $V_1 + V_1' = V_2' \Longrightarrow V_1' = V_2' - V_1$ ... (3). Replace (3) in (1): $m_1(V_1 - V_2' + V_1) = m_2V_2'$. $2m_1V_1 - m_1V_2' = m_2V_2' \Longrightarrow V_1 = \frac{m_1+m_2}{2m_1}V_2' = \frac{0.8+1}{2\times0.8}(3.2) = 3.6m/s$. Use equation (3): $V_1' = V_2' - V_1 = 3.2 - 3.6 = -0.4m/s$.
II.2	By applying Newton's 2nd law on (P_1): $\sum \vec{F}_{ext} = m_1\vec{a} \Longrightarrow \vec{W}_1 + \vec{N}_1 + \vec{f} = m_1\vec{a}$. Projection along the direction of motion: $-f = m_1a \Longrightarrow a = -\frac{f}{m_1} = -\frac{1.8}{0.8} = -2.25m/s^2$. $a = constant > 0$ and $v > 0$ (UDRM). $V_1^2 - V_0^2 = 2ad \Longrightarrow 3.6^2 - V_0^2 = 2(-2.25)(0.5) \Longrightarrow V_0 = 3.9m/s$.

Exercise 33:

Part	Answer
I.1	$V_2 = \frac{A_1A_3}{2\tau} = \frac{160\times10^{-2}}{2\times0.2} = 4m/s$. $V_3 = \frac{A_2A_4}{2\tau} = \frac{240\times10^{-2}}{2\times0.2} = 6m/s$.
I.2	$\Delta\vec{P} = \vec{P}_3 - \vec{P}_2 = m_A\vec{V}_3 - m_A\vec{V}_2 = 0.1 \times 6\vec{j} - 0.1 \times 4\vec{j} = 0.2\vec{j}\ [kgm/s]$.
I.3	The only force acting on (A) is its weight. $\sum \vec{F}_{ext} = m_A\vec{a} \Longrightarrow m_A\vec{g} = m_A\vec{a} \Longrightarrow \vec{a} = \vec{g} = g\vec{j} = 10\vec{j}\ [m/s^2]$. $a = 10m/s^2 = constant > 0$ and $v > 0$ and $av > 0$ (UARM).
I.4	$v = at + v_0 = 10t + 0 = 10(1) = 10m/s$ with $t = 5\tau = 5 \times 0.2 = 1s$.
I.5	$h = 20 + 60 + 100 + 140 + 180 = 500cm = 5m$. $h = \frac{1}{2}at^2 + v_0t = \frac{1}{2} \times 10 \times 1^2 = 5m$.

II.1	$M.E = M.E' \Longrightarrow K.E + G.P.E = K.E' + G.P.E'$. $\frac{1}{2}m_AV^2 + 0 = 0 + m_Agh' \Longrightarrow V = \pm\sqrt{2gh} = \pm\sqrt{2\times 10\times 4.05} = \pm 9m/s$. The algebraic value of the velocity is: $V' = -9m/s$ (it moves in the negative direction). The speed of (A) is $V' = 9m/s$.
II.2	The mechanical energy before collision: $M.E = K.E + G.P.E = \frac{1}{2}m_AV^2 + 0 = \frac{1}{2}\times 0.1\times 10^2 = 5J$. The mechanical energy after collision: $M.E' = K.E' + G.P.E' = \frac{1}{2}m_AV'^2 + 0 = \frac{1}{2}\times 0.1\times 9^2 = 4.05J$. The loss in energy is: $E = \lvert\Delta M.E\rvert = 0.95J$.
II.3	The linear momentum before collision is: $\vec{P} = m_A\vec{V} = 0.1\times 10\vec{j} = \vec{j}\,[kgm/s]$. The linear momentum after collision is: $\vec{P}' = m_A\vec{V}' = 0.1\times(-9\vec{j}) = -0.9\vec{j}\,[kgm/s]$. The variation in linear momentum is: $\Delta\vec{P} = \vec{P}' - \vec{P} = -1.9\vec{j}\,[kgm/s]$.
II.4	By applying Newton's 2nd law: $\sum\vec{F}_{ext} = \frac{d\vec{P}}{dt}$ with $\frac{d\vec{P}}{dt} \simeq \frac{\Delta\vec{P}}{\Delta t}$ $(\Delta t = 0.01s)$. $\vec{W} + \vec{F} = \frac{\Delta\vec{P}}{\Delta t} \Longrightarrow m_Ag\vec{j} + \vec{F} = \frac{\Delta\vec{P}}{\Delta t} \Longrightarrow 0.1\times 10\vec{j} + \vec{F} = -\frac{1.9}{0.01}\vec{j} \Longrightarrow \vec{F} = -191\vec{j}\,[N]$. Then, $F = 191N$.
II.5.1	A and D represent the mechanical energy of the system [(A); Earth] before and after collision respectively. B and E represent the gravitational potential energy of the system [(A); Earth] before and after collision respectively. C and F represent the kinetic energy of (A) before and after collision respectively.
II.5.2	$t = 2.8s$.
III	During collision, the system [(A); (B)] is isolated. $\sum\vec{F}_{ext} = \frac{d\vec{P}}{dt} = \vec{0} \Longrightarrow \vec{P} = constant$. Principle of conservation of linear momentum: $\vec{P}_{bc} = \vec{P}_{ac} \Longrightarrow m_A\vec{V}_A + m_B\vec{V}_B = m_A\vec{V}'_A + m_B\vec{V}'_B$. The collision is head-on (collinear velocities); then, the above equation can be in its algebraic form: $m_AV_A + m_BV_B = m_AV'_A + m_BV'_B \Longrightarrow m_A(V_A - V'_A) = m_B(V'_B - V_B)$ ... (1). The collision between (A) and (B) is perfectly elastic; then, the kinetic energy is conserved: $K.E_{bc} = K.E_{ac} \Longrightarrow K.E_A + K.E_B = K.E'_A + K.E'_B$. $\frac{1}{2}m_AV_A^2 + \frac{1}{2}m_BV_B^2 = \frac{1}{2}m_AV_A'^2 + \frac{1}{2}m_BV_B'^2$. $m_A(V_A - V'_A)(V_A + V'_A) = m_B(V'_B - V_B)(V'_B + V_B)$ ... (2). Divide (2) by (1): $V_A + V'_A = V'_B + V_B \Longrightarrow V'_B = V'_A + V_A - V_B$ ... (3). Replace (3) in (1): $m_A(V_A - V'_A) = m_B(V'_A + V_A - V_B - V_B)$. $m_AV_A + 2m_BV_B - m_BV_A = m_BV'_A + m_AV'_A$. $V'_A = \frac{m_A - m_B}{m_A + m_B}V_A + \frac{2m_B}{m_A + m_B}V_B = \frac{0.1-0.2}{0.1+0.2}(0.8) + \frac{2\times 0.2}{0.1+0.2}(-1) = -1.6m/s \Longrightarrow \vec{V}'_A = -1.6\vec{i}$. Using equation (3): $V'_B = -1.6 + 0.8 + 1 = 0.2m/s \Longrightarrow \vec{V}'_B = +0.2\vec{i}$.

CHAPTER 3 – ELECTROMAGNETIC INDUCTION COURSE

3.1- PREREQUISITES

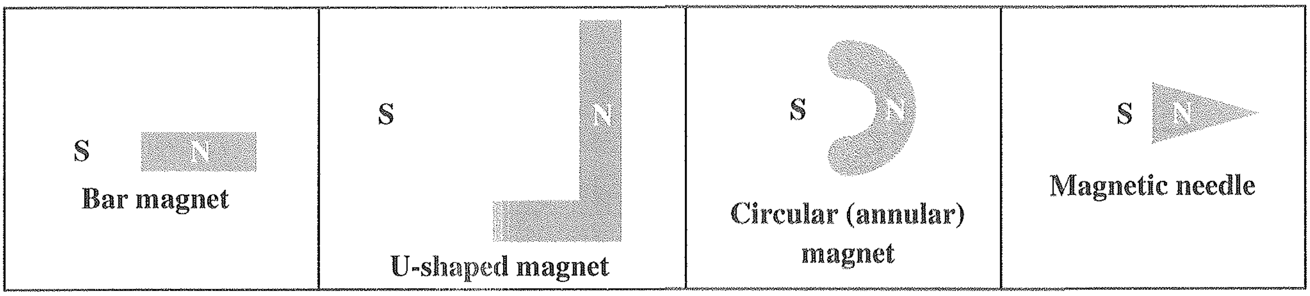
Magnets and magnetism

A **magnet** is any object that attracts the metal iron.

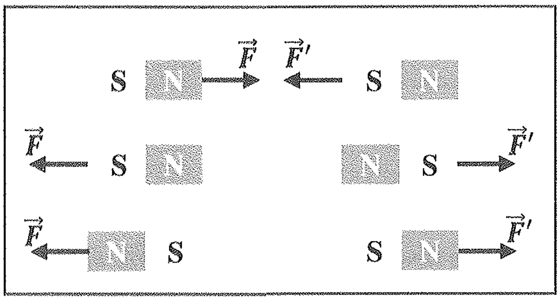
Types of magnets

Natural magnets are made of the magnetic oxide of iron (Fe_3O_4).
Artificial magnets are produced from magnetic materials. Artificial magnets are generally made from special iron or steel alloys which are usually magnetized electrically.

Some forms of artificial magnets



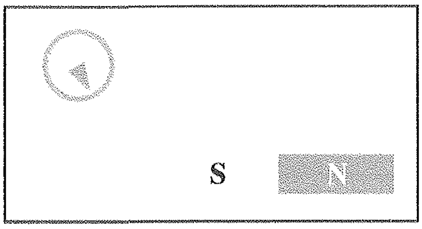
Magnetic substances are substances that can be attracted by a magnet and can be changed into a magnet like iron, cobalt, nickel and steel.



Magnetic poles: a magnetic pole is a place on a magnet where the force it applies is the strongest. There are two magnetic poles on all magnets: a north pole (red color) and a south pole (blue color).

Magnetic interactions

A magnetic force exists between the poles of any two magnets. Like magnetic poles repel each other and opposite magnetic poles attract each other as illustrated in the adjacent document.



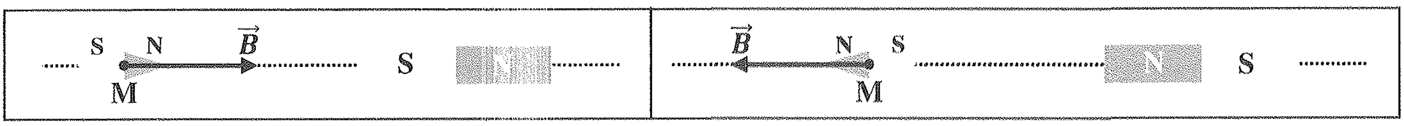
Magnetic Field

A **magnetic field** is defined as a region of space where a small test magnet (magnetic needle) experiences a turning force.

Magnetic field vector

The magnetic field at a point is characterized by a vector called magnetic field vector $\vec{B}$.

- The direction of the magnetic field vector $\vec{B}$ at any point in space is the direction indicated by the north pole of a small compass needle placed at that point. In other words, the direction of $\vec{B}$ at a point in space is along the $S \rightarrow N$ direction of a magnetic needle placed at that point.
- In S.I units, the magnitude (strength) B of the magnetic field is expressed in Tesla [T], named in honor of the Serbian-American scientist Nicola Tesla, and is measured using a Teslameter.

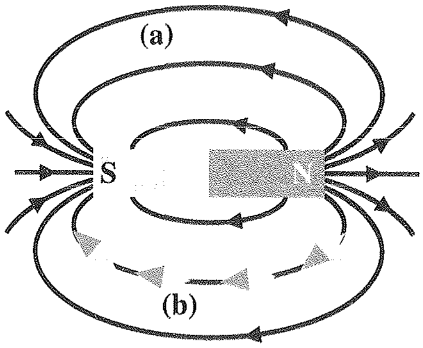


Magnetic field lines and magnetic spectrum

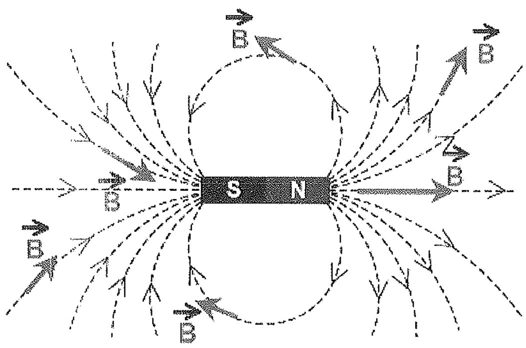
Magnetic fields exist in three dimensions surrounding a magnet and are more intense at the poles. Magnetic fields are invisible, but can be represented in diagrams with magnetic field lines. Magnetic field lines:

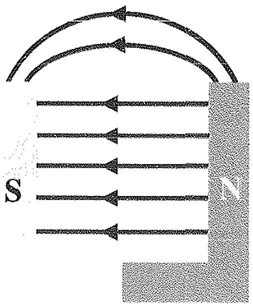
- Point away from the north pole to the south pole outside a magnet, and from the south pole to the north pole inside a magnet.
- Never cross one another.
- Are closer together where the magnetic field is stronger (at the poles) and thus the magnetic force is strongest.

The magnetic field line is an imaginary line; the tangent at any of its points represents the direction of the magnetic field $\vec{B}$ at this point. The set of field lines of a magnetic field constitutes the magnetic spectrum.

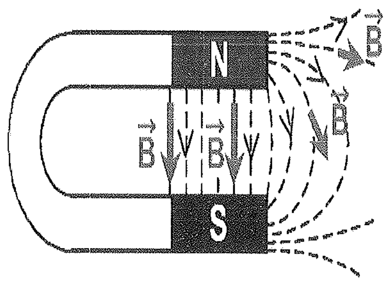


The magnetic field (a) of a bar magnet can be traced with a compass (b). Note that the north poles of the compasses point in the direction of the field lines from the magnet's North Pole to its south pole.





The magnetic field lines created by a U shaped magnetic.



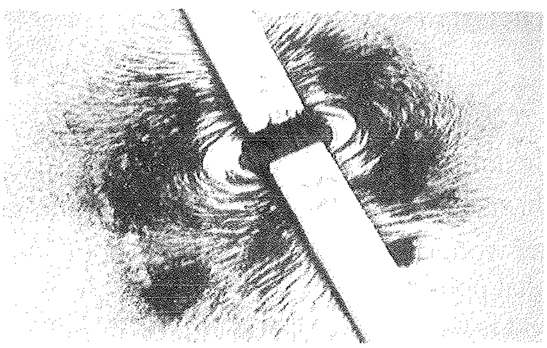


Figure (a): the magnetic field around these magnets are shown using iron filings, which line up with the field lines.

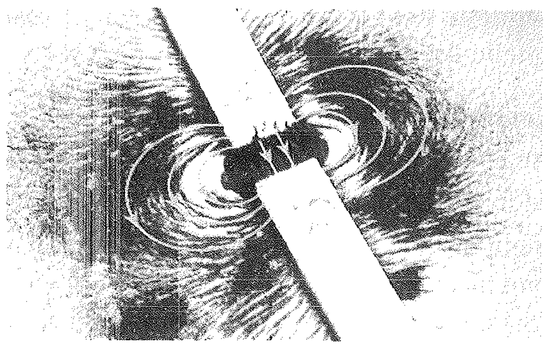
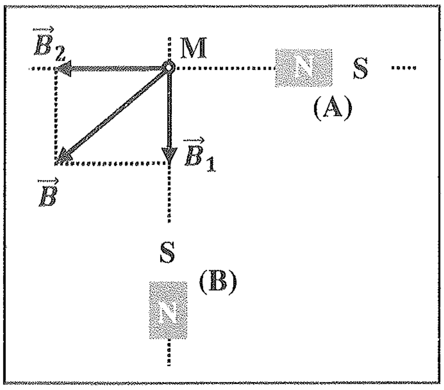


Figure (b): the direction of the magnetic field lines

Superposition of magnetic fields

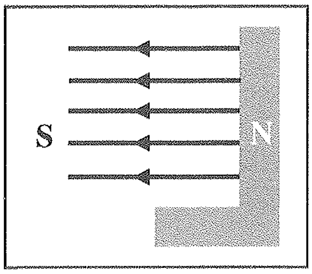


The resultant magnetic field at M is: $\vec{B} = \vec{B}_1 + \vec{B}_2$.

In general, the resultant magnetic field at a point M in space is the vector sum of the acting magnetic fields:

$$\vec{B} = \vec{B}_1 + \vec{B}_2 + \cdots + \vec{B}_n = \sum_{i=1}^{n} \vec{B}_i$$

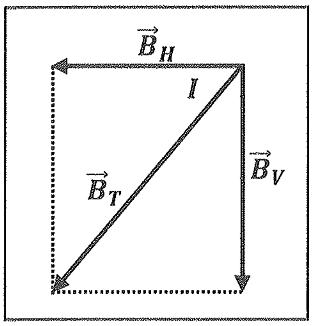
Uniform magnetic field



In a region of space, a magnetic field is said to be uniform when the magnetic field vector $\vec{B}$ has at any point in this space:
- Same line of action.
- Same direction.
- Same magnitude.

The field lines of a uniform magnetic field are parallel straight lines that are equally spaced.
Example: magnetic field inside a U-shaped magnet.

The Magnetic field of the earth (terrestrial magnetic field)



The terrestrial magnetic field $\vec{B}_T$ has two components:
- The horizontal component of the terrestrial magnetic field $\vec{B}_H$.
- The vertical component of the terrestrial magnetic field $\vec{B}_V$.

The terrestrial magnetic field is the vector sum of these two components:

$$\vec{B}_T = \vec{B}_H + \vec{B}_V$$

The magnitude (strength) of the terrestrial magnetic field can be written as

$B_T^2 = B_H^2 + B_V^2$ (since $\vec{B}_H$ is perpendicular to $\vec{B}_V$) then: $B_T = \sqrt{{B_H}^2 + {B_V}^2}$

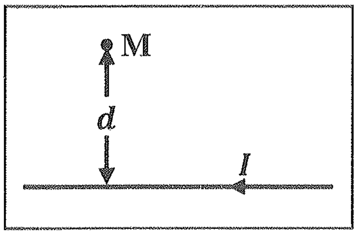
The **magnetic inclination** is the angle between the terrestrial magnetic field $\vec{B}_T$ and its horizontal component $\vec{B}_H$.

$$\tan I = \frac{B_V}{B_H} \quad ; \quad \cos I = \frac{B_H}{B_T} \quad ; \quad \sin I = \frac{B_V}{B_T} \quad \text{with} \quad I = \left(\vec{B}_H; \vec{B}_T\right)$$

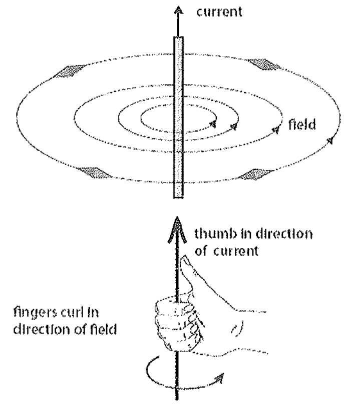
Magnetic field created by an electric current

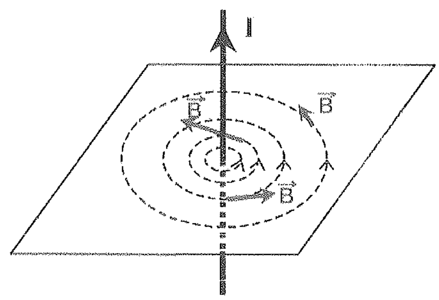
In 1820, the Danish physicist Hans Christian Oersted announced a discovery which has changed the path of civilization. He discovered that an electric current has a magnetic effect identical to that of a magnet. An electric current creates a magnetic field.

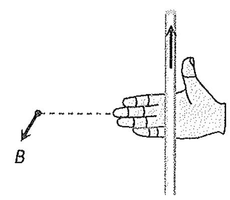
Magnetic field created by a long rectilinear wire



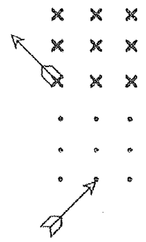
The characteristics of the magnetic field vector $\vec{B}$ created by a long rectilinear wire, carrying a current I, at a point M located at a distance d from the wire are:
Point of application: M
Line of action: perpendicular to the plane containing the wire and M.
Direction: determined by applying the right hand rule (RHR).
Magnitude: $B = \frac{\mu_0 I}{2\pi d}$ where $\mu_0 = 4\pi \times 10^{-7} SI$ is the magnetic permeability of vacuum (permeability is the measure of the ability of a material to support the formation of a magnetic field within itself).
Therefore. $B = 2 \times 10^{-7} \frac{I}{d}$. In SI units, B is expressed in [T], d in [m] and I in [A].





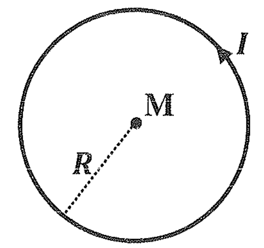


Attention:
Field into the page can be represented by crosses and field out by dots.



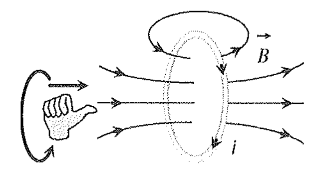
Magnetic field created by a flat circular coil

The characteristics of the magnetic field vector $\vec{B}$ created by a flat circular coil, of radius R, number of turns N and carrying a current I, at its center M are:



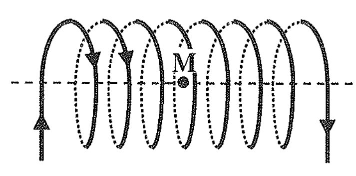
Point of application: M.
Line of action: perpendicular to the plane of the coil.
Direction: determined by applying the right hand rule.
Magnitude: $B = \frac{\mu_0 NI}{2R}$. In SI units, B is expressed in [T], R in [m] and I in [A].
Case of a circular wire: $B = \frac{\mu_0 I}{2R}$.
The thickness of a flat coil is small compared with its radius.

Right hand thumb rule of circular currents
The direction of current in circular conducting loop is in the direction of folding fingers of right hand; then, the direction of magnetic field will be in the direction of stretched thumb.

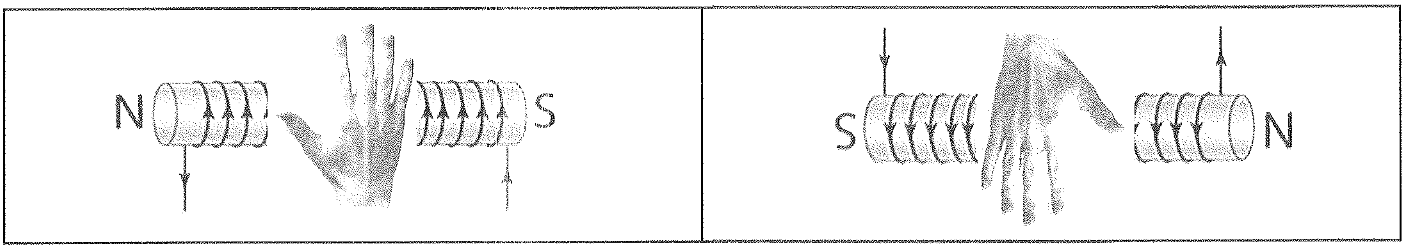


Magnetic field created by a solenoid

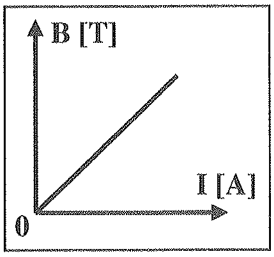
The characteristics of the magnetic field vector $\vec{B}$ created by a solenoid of length L, number of turns N and carrying a current I, at its center M are:



Point of application: M.
Line of action: along the axis of the solenoid.
Direction: determined by applying the right hand rule.
Magnitude: $B = \frac{\mu_0 NI}{L} = \mu_0 nI$ with $n = \frac{N}{L}$ is the number of turns per unit length.
In SI units, B is expressed in [T], L in [m] and I in [A].
Right hand rule:
The four fingers curls in the direction of current and the thumbs points the direction of the magnetic field.



Generalization



Long rectilinear wire: $B = \frac{\mu_0}{2\pi d} I$.

Circular wire: $B = \frac{\mu_0 N}{2R} I$.

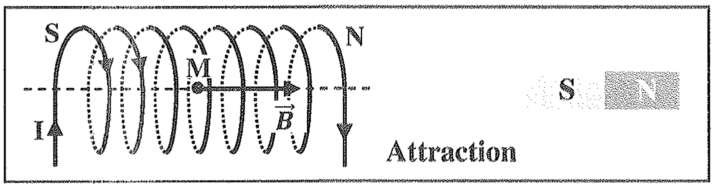
Solenoid: $B = \frac{\mu_0 N}{L} I$.

The expressions of B as a function of I has the form of the equation of a straight line passing through origin $B = kI$ with k is the slope in.

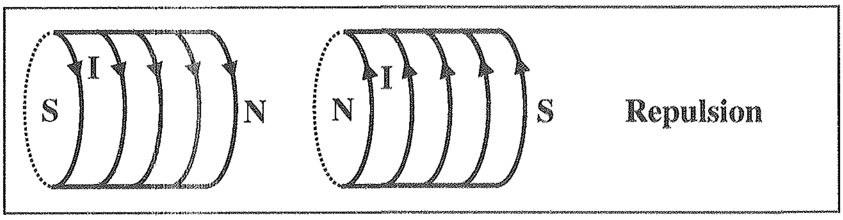
We deduce that there exists a linear relation between B and I (B and I are directly proportional).

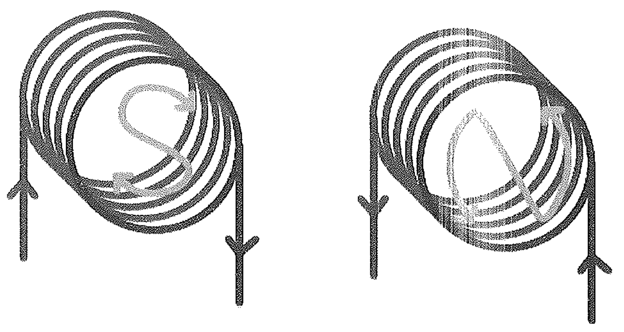
Analogy between magnets and coils

Magnet-coil interaction



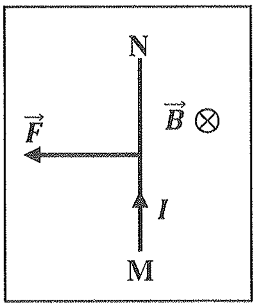
Coil -Coil Interaction





The easiest way of remembering the direction of the magnetic field in a solenoid is to note that when looking into a loop or solenoid, if the current is moving in a clockwise direction then the pole facing you is a South Pole; anti-clockwise represents a North Pole.

The electromagnetic force (Laplace's force)



A conductor MN made up of a non-magnetic substance of length L, carrying a direct electric current I and placed in a uniform magnetic field $\vec{B}$ is under the action of an electromagnetic force called Laplace's force.

The characteristics of the electromagnetic force $\vec{F}$ are:

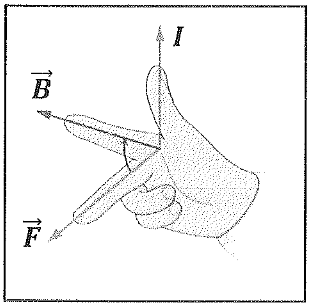
Point of application: center of mass of the conductor.

Line of action: perpendicular to the plane containing $\vec{B}$ and the conductor.

Direction: determined by applying the right hand rule.

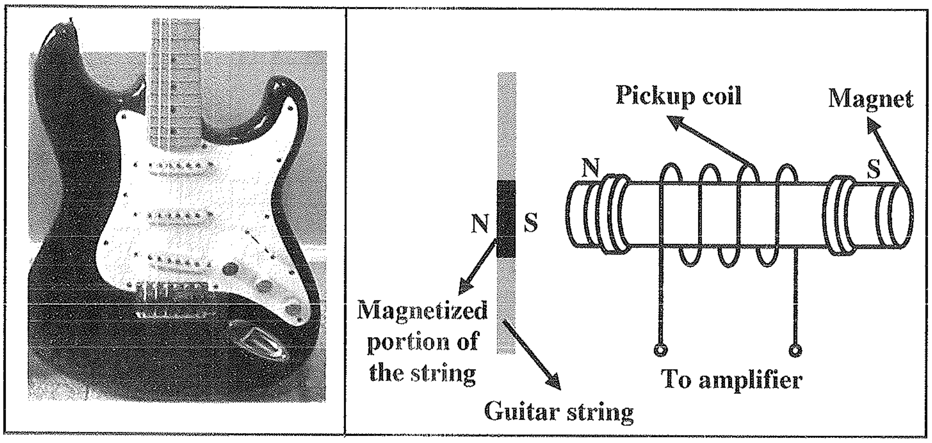
Magnitude: $F = IBL\sin(\overrightarrow{MN}, \vec{B}) = IBL\sin\alpha$ where $\alpha = (\overrightarrow{MN}; \vec{B})$.
In SI units, F is expressed in [N], L in [m], B in [T] and I in [A].

Notes:
- If $\vec{B}$ is parallel to the conductor: $\alpha = 0 \Longrightarrow \sin\alpha = 0 \Longrightarrow \vec{F} = \vec{0}$.
- If $\vec{B}$ is perpendicular to the conductor: $\alpha = 90° \Longrightarrow \sin\alpha = 1 \Longrightarrow F_{max} = IBL$.
- If we reverse the direction of I or $\vec{B}$ then the direction of $\vec{F}$ will be reversed.
- If the directions of I and $\vec{B}$ are reversed, the direction of $\vec{F}$ will remain the same.



The right hand rule
- The thumb points the direction of I.
- The pointer points the direction of $\vec{B}$.
- The middle finger points the direction of $\vec{F}$.

3.2- INTRODUCTION



The vibrations of the strings in an electric guitar change the magnetic field near a coil of wire called the pickup. In turn, this induces an electric current in the coil, which is then amplified to create the unique sound of an electric guitar.
Recall that when you were studying circuits, you were asked if it was possible to produce an electric current using only wires and no battery. So far, all electric circuits that you have studied have used a battery or an electrical power supply to create a potential difference within a circuit. The electric field associated with that potential difference causes charges to move through the circuit and to create a current.
It is also possible to *induce* a current in a circuit without the use of a battery or an electrical power supply. You have learned that a current in a circuit is the source of a magnetic field. Conversely, a current results when a closed electric circuit moves with respect to a magnetic field. **The process of inducing a current in a circuit by changing the magnetic field that passes through the circuit is called electromagnetic induction.**
The production and distribution of electrical energy would not be economically feasible if the only source of electricity was from chemical sources such as dry cells. The development of electrical engineering began with the work of Faraday and Henry. **Electromagnetic induction involves the transformation of mechanical energy into electrical energy.**
The applications of magnetic fields and magnetic forces are countless and changing rapidly every year. Here are just a few examples. For decades, the entertainment industry depended on the magnetic recording of music and images on audiotape and videotape. Although digital technology has largely replaced magnetic recording, the industry still depends on the magnets that control CD and DVD players and computer hard drives; magnets also drive the speaker cones in headphones, TVs, computers, and telephones. A modern car comes equipped with dozens of magnets because they are required in the motors for engine ignition,

automatic window control, sunroof control, and windshield wiper control. Most security alarm systems, doorbells, and automatic door latches employ magnets. In short, you are surrounded by magnets.

The science of magnetic fields is physics; the application of magnetic fields is engineering.
This link between a magnetic field and the electric field it produces (induces) is now called Faraday's law of induction.

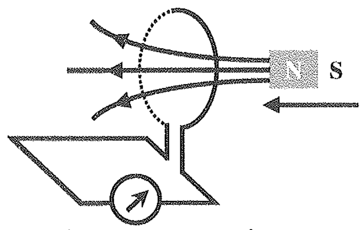
Conclusion
Electromagnetic induction is the process of using magnetic fields to produce voltage, and in a complete circuit, a current.
Michael Faraday first discovered it, using some of the works of Hans Christian Oersted. His work started at first using different combinations of wires and magnetic strengths and currents, but it wasn't until he tried moving the wires that he got any success.
It turns out that electromagnetic induction is created by just that - the moving of a conductive substance through a magnetic field.
Before we get to applications, we must examine two simple experiments about Faraday's law of induction.

3.3- EXPERIMENTAL EVIDENCE OF THE ELECTROMAGNETIC INDUCTION PHENOMENON

Let us examine two simple experiments to prepare for our discussion of Faraday's law of induction.
First Experiment: document 1 shows a conducting loop connected to a sensitive ammeter. Because there is no battery or other source of emf included, there is no current in the circuit. However, if we move a bar magnet toward the loop, a current suddenly appears in the circuit. The current disappears when the magnet stops. If we then move the magnet away, a current again suddenly appears, but now in the opposite direction.
If we experimented for a while, we would discover the following:
- A current appears only if there is relative motion between the loop and the magnet (one must move relative to the other); the current disappears when the relative motion between them ceases.
- Faster motion produces a greater current.
- If moving the magnet's north pole toward the loop causes, say, clockwise current, then moving the north pole away causes counterclockwise current.

Moving the south pole toward or away from the loop also causes currents, but in the reversed directions.
The current produced in the loop is called an **induced current;** the work done per unit charge to produce that current (to move the conduction electrons that constitute the current) is called an **induced electromotive force emf;** and the process of producing the current and emf is called electromagnetic **induction.**
Second Experiment: for this experiment we use the apparatus of document 2, with the two conducting loops close to each other but not touching. If we close switch S, to turn on a current in the right-hand loop, the ammeter suddenly and briefly registers a current - an induced current - in the left-hand loop. If we then open the switch, another sudden and brief induced current appears in the left hand loop, but in the opposite direction. We get an induced current (and thus an induced emf) only when the current

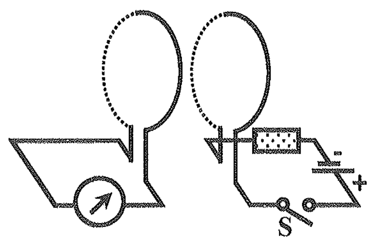
The magnet's motion creates a current in the loop.



An ammeter registers a current in the wire loop when the magnet is moving with respect to the loop.

Doc.1

Closing the switch causes a current in the left-hand loop.



An ammeter registers a current in the left-hand loop as switch S is closed (to turn on the current in the right-hand wire loop). No motion of the coils is involved.

Doc.2

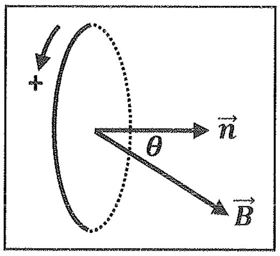
in the right-hand loop is changing (either turning on or turning off) and not when it is constant (even if it is large).
The induced emf and induced current (in a closed circuit) in these experiments are apparently caused when something changes - but what is that "something"? Faraday knew.

3.4- MAGNETIC FLUX

The **magnetic flux** is a measure of the strength of a magnetic field over a given area perpendicular to it. In other words, it is a measure of the strength of the number of magnetic field lines passing through the area.

The **magnetic flux** ϕ of a uniform magnetic field $\vec{B}$, through a loop of area S and having N turns, is given by the relation:

$$\phi = N\vec{B}.\vec{S} = N\vec{B}.S\vec{n} = NBS\cos\theta \text{ with } \vec{S} = S\vec{n} \text{ and } \theta = (\vec{B},\vec{n})$$



$\vec{n}$: a unit vector oriented along the normal to the surface of the loop.

Direction of the unit vector $\vec{n}$

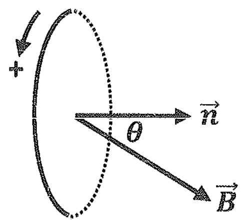
We choose a positive sense along the loop in order to determine the direction of the unit vector $\vec{n}$ by applying the right hand rule. The four fingers are curled in the positive sense and the thumb points the direction of $\vec{n}$.

The SI units, ϕ is expressed in webers [Wb], B in [T] and S in [m^2].

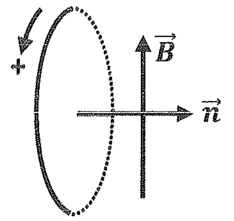
$$1\text{Wb} = 1\text{T.m}^2$$

Sign of ϕ

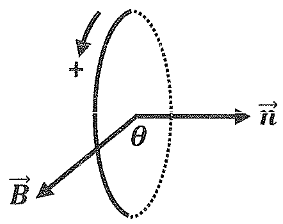
The magnetic flux is an algebraic scalar quantity.



$0 \le \theta < \frac{\pi}{2}$
$\cos\theta > 0 \Rightarrow \phi > 0$



$\theta = \frac{\pi}{2}$
$\cos\theta = 0 \Rightarrow \phi = 0$

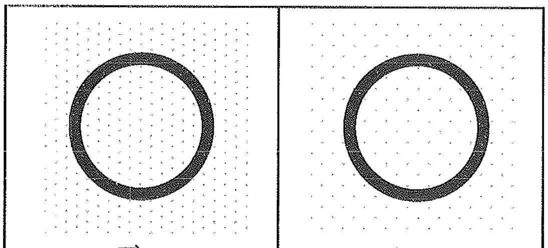


$\frac{\pi}{2} < \theta \le \pi$
$\cos\theta < 0 \Rightarrow \phi < 0$

3.5- VARIATION OF MAGNETIC FLUX

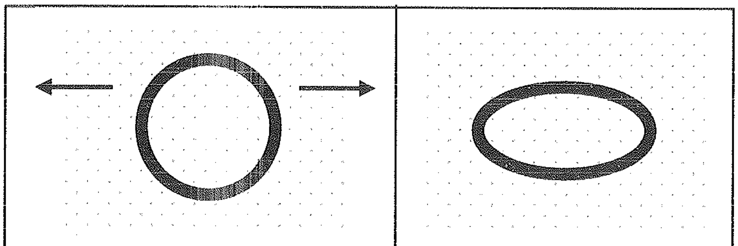
Generating an electromotive force by a variation of the magnetic flux through the surface of a wire loop can be achieved in several ways:

1- By varying the magnitude of $\vec{B}$ with time.

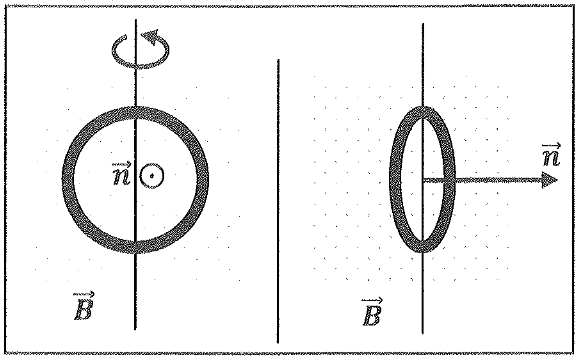


$\vec{B}$ | $B' < B$

2- By varying the magnitude of $\vec{S}$, i.e., the area enclosed by the loop with time.



$Area = S$ | $Area = S' < S$



3- By varying the angle between $\vec{B}$ and the area vector $\vec{S}$ with time (by rotation).

4- Any combination of the above.

The existence of induced electric currents, in a no-generator circuit, is associated with the variation of a physical quantity called magnetic flux. The magnetic field $\vec{B}$ is one of the factors of magnetic flux.
The phenomenon observed in the experiments, where an electric current traverses a **closed circuit** crossed by a varying magnetic flux, is called electromagnetic induction. The circuit traversed by the induced current is called an induced circuit.
The source of the magnetic field, which is at the origin of the phenomenon of induction, is called the inducing source.
A variation of the magnetic flux through a circuit gives rise to an induced e.m.f "e" whether the circuit is closed or not.
- If the circuit is closed, a current traverses it.
- If the circuit is open, the electric current does not exist.

3.6- FARADAY'S LAW OF INDUCTION

Faraday realized that an emf and a current can be induced in a loop, as in our two experiments, by changing the amount of magnetic field passing through the loop.
He further realized that the "amount of magnetic field" can be visualized in terms of the magnetic field lines passing through the loop
In general, it is found experimentally that the induced emf depends on the rate of change of magnetic flux through the coil.
Moreover, the induced current and the induced electromotive force increase when the variation of the magnetic flux is more rapid.
This leads to the statement of Faraday's law:

Statement of Faraday's law

The induced electromotive force "e" at any instant is equal to the opposite of the derivative with respect to time of the magnetic flux crossing the circuit.

Faraday's law is expressed by the relation:

$$e = -\frac{d\phi}{dt}$$

The minus sign indicates that the induced emf "e" tends to oppose the flux change.

3.7- LENZ'S LAW

Soon after Faraday proposed his law of induction, Heinrich Friedrich Lenz devised a rule for determining the direction of an induced current "i" in a closed loop.

Statement of Lenz's law

The direction of the induced current is such that its electromagnetic effects always oppose the cause that has established this current.

Furthermore, the direction of an induced emf is that of the induced current ($i.e > 0$).

To illustrate **Lenz's law**, we apply it in two different but equivalent ways where the north pole of a magnet is being moved toward a conducting loop.

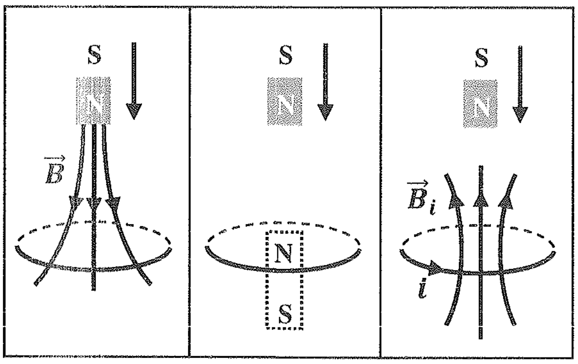
1- Opposition to Pole Movement

The approach of the magnet's north pole in the document below increases the magnetic flux through the loop and hereby induces a current in the loop. The loop then acts as a bar magnet with a south pole and a north pole and with its north pole pointing up.

Interpretation

To oppose the magnetic flux increase being caused by the approaching magnet, the loop's north pole must face toward the approaching north pole so as to repel it. Then the curled - straight right - hand rule tells us that the current induced in the loop must be counterclockwise in document.

We next pull the magnet away from the loop, a current will again be induced in the loop. Now, however, the loop will have a south pole facing the retreating north pole of the magnet, so as to oppose the retreat. Thus, the induced current will be clockwise by applying the right hand rule.



2- Opposition to Flux Change

With the magnet initially distant, no magnetic flux passes through the loop. As the north pole of the magnet then approaches the loop with its magnetic field $\vec{B}$ directed downward, the flux through the loop increases. The variation in the magnetic flux induces an emf according to Faraday's law.

To oppose this increase in flux, the induced current i must set up its own field $\vec{B}$ induced directed upward inside the loop (by Lenz's law), then the upward flux of field $\vec{B}$ induced opposes the increasing downward flux of field. The curled-straight right-hand rule then tells us that i must be counterclockwise.

> Note carefully that the flux of $\vec{B}_{in}$ always opposes the change in the flux of $\vec{B}$, but does not always means that $\vec{B}_{in}$ points opposite to $\vec{B}$. For example, if we next pull the magnet away from the loop, the flux ϕ_B from the magnet is still directed downward through the loop, but it is now decreasing. The flux of $\vec{B}_{in}$ must now be downward inside the loop, to oppose the decrease in ϕ_B. Thus, $\vec{B}_{in}$ and $\vec{B}$ are now in the same direction.
> In Figures c and d, the south pole of the magnet approaches and retreats from the loop, respectively.

The sign of induced "e" and "i"

1- Define a positive direction for the unit vector $\vec{n}$

2- Assuming that $\vec{B}$ is uniform, take the dot product of $\vec{B}$ and $\vec{S}$. This allows for the determination of the sign of the magnetic flux ϕ.

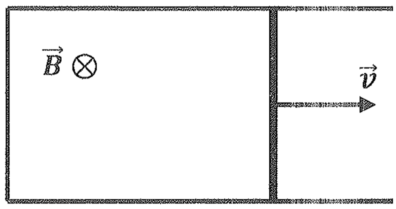
3- Obtain the rate of flux change $\frac{d\phi}{dt}$ by differentiation. There are three possibilities:

$$\frac{d\phi}{dt}\begin{cases} > 0 \Longrightarrow induced\ emf\ e < 0 \\ < 0 \Longrightarrow induced\ emf\ e > 0 \\ = 0 \Longrightarrow induced\ emf\ e = 0 \end{cases}$$

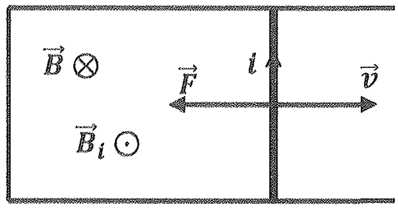
4- Determine the direction of the induced current $(i.e > 0)$.
$i > 0$ then it circulates in the chosen positive direction.
$i < 0$ then it circulates in a direction opposite to that of the chosen positive direction.

Lenz's law and the slide wire generator

Find the direction of the current induced and the force on the rod.



(a) Slide wire moving in magnetic field.



(b) Induced current, magnetic field and electromagnetic force on slide wire.

Note that the force is opposing the motion generating the current in line with Lenz's law
Note that the original field will be stronger than the induced field, so the right-hand rule application has the field into the page.

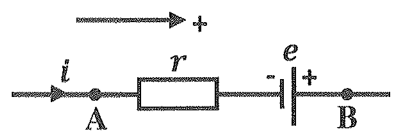
> If the magnetic flux change is due to the motion of the conductor, the direction of the induced current in the moving conductor is such that the direction of the magnetic force on the conductor is opposite in direction to its motion. Thus, the motion of the conductor, which causes the induced current, is opposed.

3.8- THE EQUIVALENT GENERATOR OF THE COIL

Characteristics: a coil, crossed by a magnetic flux ϕ which varies with respect to time t, behaves as a generator characterized by:

- An electromotive force e: $e = -\frac{d\phi}{dt}$.
- An internal resistance r which is the resistance of the wire forming the coil.

The coil is equivalent to a series combination of an ideal generator of emf e and a resistor of resistance r.

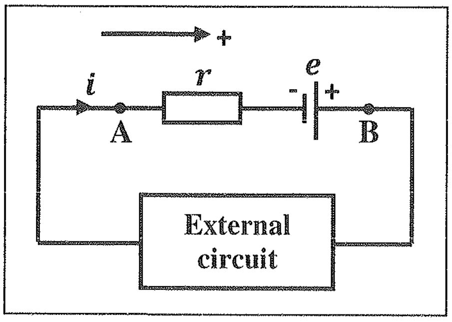


Voltage across the terminals of a coil

The positive sense is oriented from A to B; then, $u_{AB} = ri - e$.
$u_{BA} = -u_{AB} = e - ri$.
In general, if a coil is oriented positively from A to B, i.e. the chosen positive sense along the coil is from A to B then $u_{AB} = ri - e$.
If the circuit is open $\Longrightarrow i = 0$; then, $u_{AB} = -e$ and $u_{BA} = e$.
i and e are substituted in algebraic values.
The expression of u_{AB} is not altered if the direction of i is reversed.

3.9- POWER DISTRIBUTION IN A MAGNET-COIL SYSTEM

At any instant t, the total electrical power delivered to the coil by the generator is:



$$p = e \times i$$

Where i is the induced current which traverses the closed circuit of the coil at the same instant.

How is this power distributed in the circuit and what is its origin?

The potential difference across the coil, oriented from A to B, in the adjacent document is: $u_{AB} = ri - e$, then $u_{BA} = e - ri$.

$u_{BA} = e - ri \Rightarrow e = u_{BA} + ri$.

Multiplying both sides by i, one obtains:

$e.i = u_{BA}.i + r.i^2 \Rightarrow p_{total} = p_{useful} + p_{dissipated}$.

In SI units, p is expressed in watts [W], u_{BA} in volts [V], r in ohms [Ω], and i in amperes [A].

The power dissipated due to Joule's effect in the coil is: $p_{dissipated} = r.i^2$.

The electrical power transferred to the external circuit across the terminals A and B of the coil is: $p_{useful} = u_{BA}.i$.

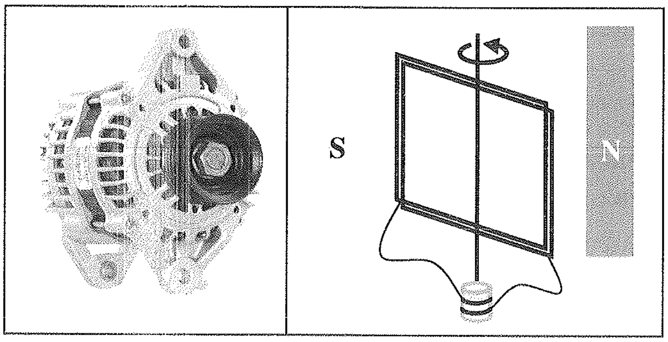
The total electric power due to the variation of magnetic flux caused by the relative motion of the magnet and the coil is: $p_{total} = e.i$.

So at any instant a part of the mechanical power supplied by the operator to produce a relative motion of the system magnet-coil system is converted to a total electric power $p_{total} = e \times i$. This power is in turn converted in the induced circuit under various forms, heat, chemical, or mechanical.

So, the magnet-coil system constitutes a converter of mechanical energy into electric energy.

3.10- APPLICATIONS OF ELECTROMAGNETIC INDUCTION

Alternators: an alternator is an electrical generator that converts mechanical energy to electrical energy in the form of alternating current.



Principle: a coil of N turns, each of area S, rotates at a constant angular speed ω in a uniform magnetic field $\vec{B}$.

At any instant t:

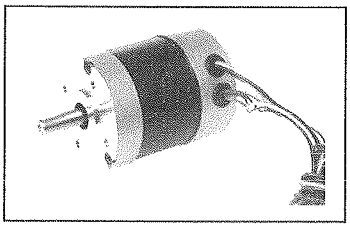
- The angular abscissa of the coil is: $\theta = \omega t + \theta_0$.
- The magnetic flux crossing the coil is:

$$\phi = NBS\cos\theta = NBS\cos(\omega t + \theta_0)$$

- The induced emf in the coil is:

$$e = -\frac{d\phi}{dt} = NBS\omega\sin(\omega t + \theta_0)$$

The induced emf is a sinusoidal function of time. The rotating coil is called the rotor and the fixed magnet is called the stator.

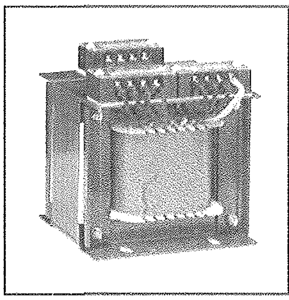


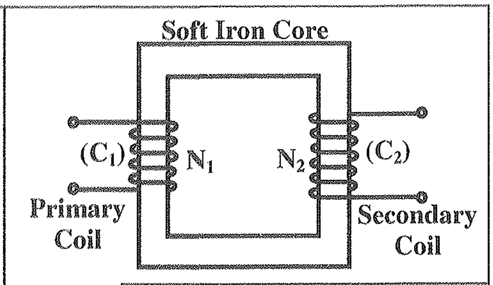
DC motors: a device that converts electric energy into mechanical energy.

Principle: the coil is traversed by an electric current of intensity I and placed in a uniform magnetic field $\vec{B}$. Thus, the coil rotates uniformly under the action of a couple of electromagnetic forces.

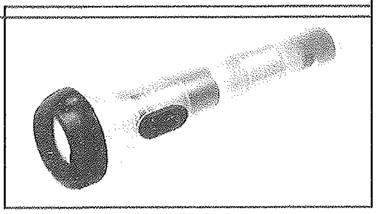
The continuous rotation of the coil produces an induced emf "e" across it. According to Lenz's law, "e" creates an induced current i, in a direction opposite to I. Consequently, the polarity of "e" is opposite of that of the external generator, and therefore it is called a back electromotive force of the motor.

Transformers: a transformer is a device that changes (transforms) an alternating potential difference (voltage) from one value to another value using the principle of electromagnetic induction.
Probably one of the greatest inventions of all time is the transformer.

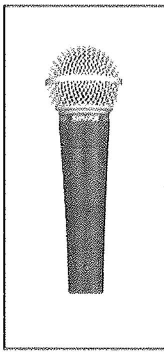


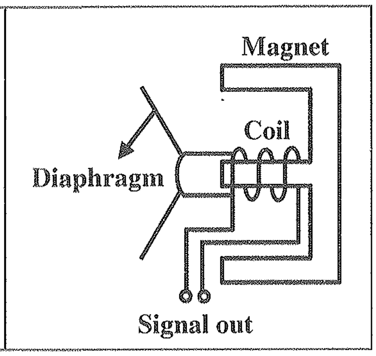


The forever flashlight: it uses the Faraday principle of electromagnetic energy to eliminate the need for batteries. The Faraday principle states that if an electric conductor, like copper wire, is moved through a magnetic field, electric current will be generated and flow into the conductor.



Microphones: a microphone works when sound waves enter the filter of a microphone. Inside the filter, a diaphragm is vibrated by the sound waves which in turn moves a coil of wire wrapped around a magnet. The movement of the wire in the magnetic field induces a current in the wire. Thus, sound waves can be turned into electronic signals and then amplified through a speaker.



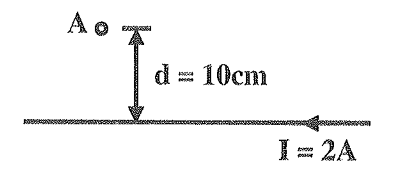
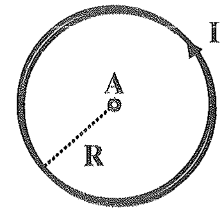
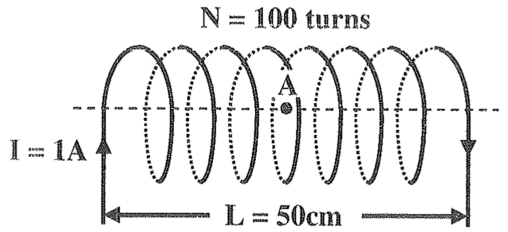


CHAPTER 3 – ELECTROMAGNETIC INDUCTION
EXERCISES AND PROBLEMS

Whenever needed, the magnetic permeability of vacuum is $\mu_0 = 4\pi \times 10^{-7} SI$

Exercise 1*:

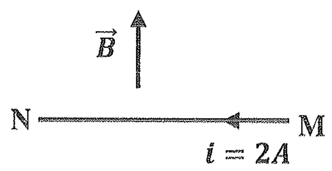
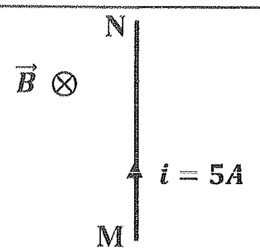
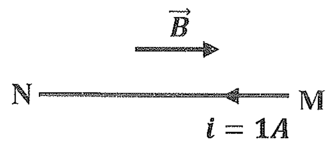
Determine, in each of the following documents, the characteristics of the magnetic field vector $\vec{B}$ created at point A.

	 R = 10cm; N = 20 turns	
Doc.1 - Rectilinear wire	**Doc.2 - Circular flat coil**	**Doc.3 - Solenoid**

Exercise 2*:

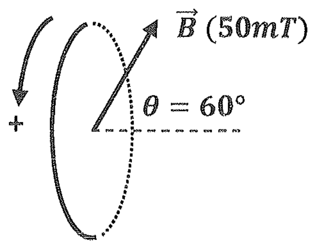
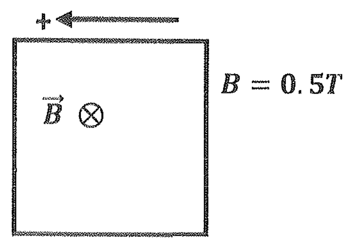
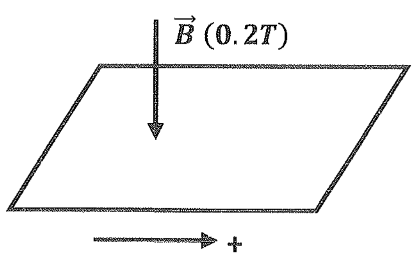
Determine, in each of the following documents, the characteristics of the electromagnetic force (Laplace's force) acting on the conductor MN.

Given: B = 0.5T and MN = L = 20cm.

		
Doc.1	**Doc.2**	**Doc.3**

Exercise 3:

Calculate, in each of the following documents, the magnetic flux crossing the loop.

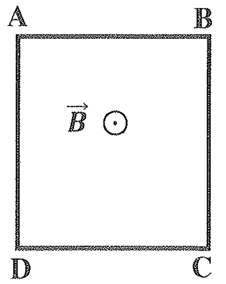
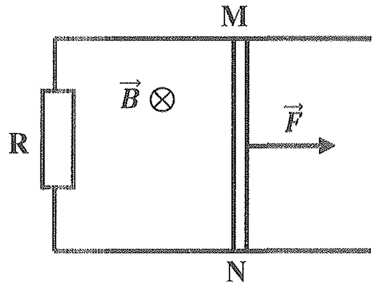
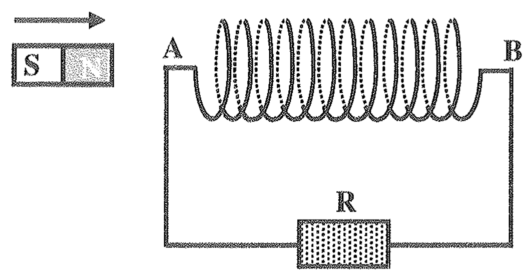
		
Doc.1: a circular loop of radius R = 10cm.	**Doc.2:** a square coil of N = 20 turns and side a = 20cm.	**Doc.3:** A rectangular loop of dimensions a = 10cm and b = 20cm.

Exercise 4*:

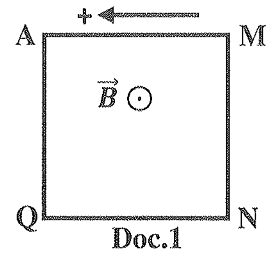
1- Define electromagnetic induction.
2- State the laws of electromagnetic induction.
3- List three sources of magnetic field.
4- On what factors does the magnetic flux crossing a coil depend on?
5- Under what condition is a conductor subjected to the action of an electromagnetic force?

Exercise 5:

1- Explain the existence of an induced e.m.f in each of the following documents.
2- Use Lenz's law to determine the direction of the induced current in each of the following documents.

		
Doc.1: a square loop is placed in a magnetic field $\vec{B}$ whose intensity decreases with time.	**Doc.2:** a conductor MN slides, under the action of a force $\vec{F}$, along two parallel rails.	**Doc.3:** a bar magnet is approached towards the axis of a solenoid.

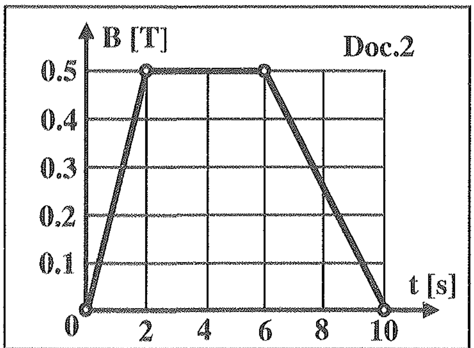
Exercise 6:



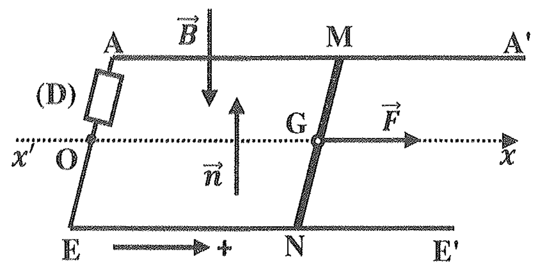
Doc.1

A conducting square loop AMNQ, of side a = 10cm and resistance R = 5Ω, is placed in a uniform magnetic field $\vec{B}$ perpendicular to its plane as shown in document 1.
Document 2 represents the variation of the intensity B of $\vec{B}$ as a function of time.

1- Give the name of the physical phenomenon that takes places in the loop.
2- State Faraday's and Lenz's laws.
3- Consider the following time intervals:
$0 \leq t \leq 2s;\ 2s \leq t \leq 6s\ and\ 6s \leq t \leq 10s$
For each time interval, find:
3.1- the expression of B as a function of time,
3.2- the expression of the magnetic flux crossing the loop as a function of time.
3.3- the value of the induced e.m.f "e".
3.4- the value of the electric current i carried by the loop; then, deduce its direction.
4- Apply Lenz's law to verify the results found in 3.4.



Exercise 7:



A conducting rod MN, of length $\ell = 50cm$ and negligible resistance, may slide without friction along two long parallel and horizontal rectilinear rails AA' and EE' of negligible resistance. During its displacement, the rod remains perpendicular to the rails. A resistor (D), of resistance R = 10Ω, is connected to the extremities A and E of the rails with connecting wires.
The whole set-up is placed in a uniform vertically downward magnetic field $\vec{B}$ of magnitude B = 0.4T as shown in the adjacent document.
At the instant $t_0 = 0$, the center of mass G of the rod is at O. The rod MN is under the action of a horizontal force $\vec{F}$ that causes G to describe, in the steady state, a uniform rectilinear motion from left to right with a constant speed V = 0.6m/s.
At an instant t, the position of G is defined by its abscissa $x = \overline{OG}$ on the axis $x'x$ and its velocity is $v = \frac{dx}{dt}$.

1- Determine, at the instant t, the expression of the magnetic flux crossing AMNE in terms of B, ℓ and x.

2-

 2.1- Explain the existence of an induced emf 'e' across the ends M and N of the rod.

 2.2- Determine, at the instant t, the expression of 'e' in terms of B, ℓ and v.

3- Determine, at the instant t, the expression of the induced current i carried by AMNE in terms of B, ℓ, v and R; then, deduce its direction.

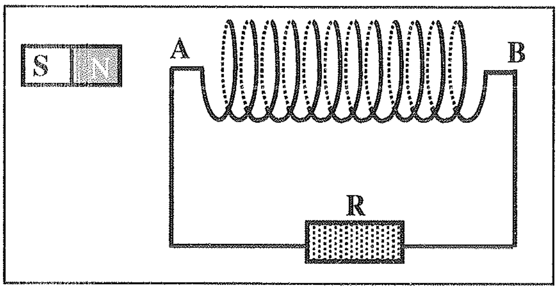
4-

 4.1- Name and represent the external forces acting on MN.

 4.2- Apply Newton's 1st law of translational motion to determine the expression of the magnitude F of $\vec{F}$ in terms of B, ℓ, V and R; then, seduce its value.

Exercise 8:

A bar magnet is displaced along the axis of a coil whose terminals A and C are connected to a resistor of resistance R.



1- Give the name of the physical phenomenon that takes place in the circuit.

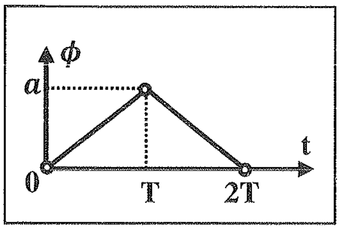
2- Explain the existence of an induced current i carried by the circuit.

3- Give, with justification, the name of each face of the coil, and the direction of the induced current i in each of the following cases:

 3.1- the magnet is displaced towards the coil,

 3.2- the magnet is displaced away from the coil.

Exercise 9*:



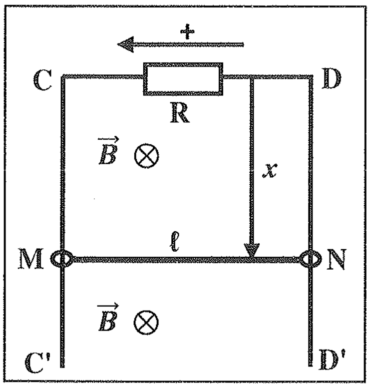
The graph in the adjacent document shows the variation of the magnetic flux ϕ in a coil as a function of time t.

1- Give three reasons that lead to the variation of the magnetic flux in the coil.

2- Explain the existence of an induced e.m.f in the coil.

3- Plot a graph that shows the variation of the induced e.m.f as a function of time.

Exercise 10*:



Two vertical rails CC' and DD' are connected by a resistor of resistance $R = 0.1\Omega$. A conducting rod MN, of mass m = 10g and length ℓ = 20cm, can slide without friction along these rails and remains horizontally in contact with these rails.

The whole set-up is placed within a uniform magnetic field $\vec{B}$, of magnitude B = 0.5T, that is perpendicular to the plane of the rails.

The rod MN, released from rest at the instant $t_0 = 0$, is found at an instant t at a distance x from CD, moving with a velocity whose algebraic value is v ($v > 0$). Take $g = 10m/s^2$.

1- Determine, at the instant t, the expression of the magnetic flux due to $\vec{B}$ through the circuit CMND in terms of B, ℓ and x, taking into consideration the arbitrary positive direction shown in the above document.

2-

 2.1- Determine the expression of:

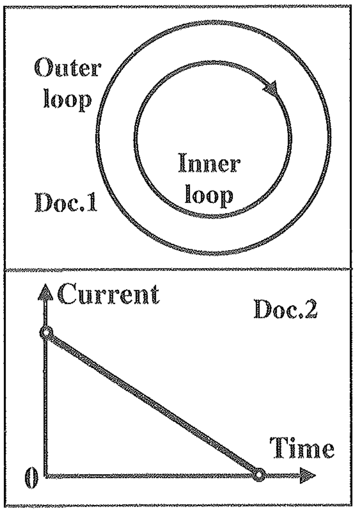
  2.1.1- The e.m.f "e" induced across the rod MN, in terms of v, B and ℓ.

2.1.2- The induced current i in terms of R, B, ℓ and v.

2.2- Indicate, with justification, the direction of the current.

3- Show that the electric power dissipated by the resistor, at the instant t, is given by: $P_{el} = \frac{B^2\ell^2}{R}v^2$.

4- The rod MN is acted upon by two forces: its weight $m\vec{g}$ and the Laplace's force $\vec{F}$ of magnitude $F = i\ell B$.
In the steady state, the rod MN describes a uniform rectilinear motion. The mechanical energy of the system (MN in the field $\vec{B}$, Earth) decreases.
4.1- Explain this decrease.
4.2- In what form is this energy dissipated?
4.3- Calculate the value of i, then deduce v.
4.4- Calculate the power dissipated.



Exercise 11:

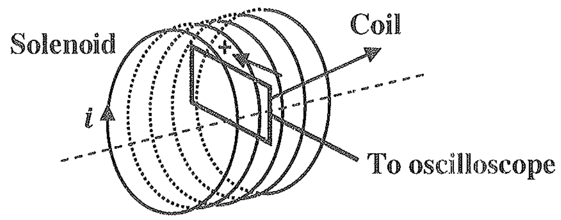
The objective of this exercise is to show the effect of a variable current traversing an inner loop on an outer loop concentric with the inner one and placed in the same plane (document 1).
The graph of document 2 shows the variation of the intensity of current in the inner loop as a function of time.

1- Explain the existence of an induced current in the outer loop.

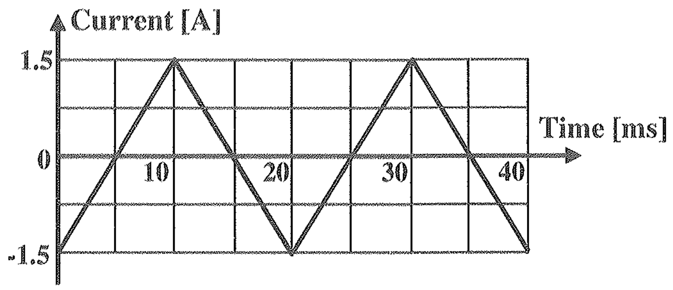
2- Determine the direction of the induced current and specify whether its magnitude is increasing, decreasing or constant.

Exercise 12:

Document 1 shows a solenoid and a coil placed at the center of the solenoid. The channels of an oscilloscope are connected to the extremities of the coil. The variation of the intensity of the alternating current in the solenoid as a function of time is shown in document 2.



Doc.1



Doc.2

Given:

	Solenoid	Coil
Number of turns	$N_S = 400$	$N_C = 8$
Cross-sectional area [m²]	$S_S = 5 \times 10^{-3}$	$S_C = 2 \times 10^{-4}$
Length [cm]	$L_S = 50$	$L_C = 2$

- The axis of the solenoid is perpendicular to the plane of the coil.
- The permeability of free space $\mu_0 = 4\pi \times 10^{-7}$ SI.

1- Explain the existence of an induced e.m.f in the coil.

2- Plot a graph that shows the variation of the induced e.m.f as a function of time.

Exercise 13:

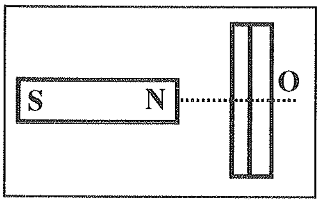
A solenoid of length L = 20cm consists of N = 300 circular turns of radius R = 2.5cm.

1- Calculate the intensity of the magnetic field at the center of the solenoid if it carries a current of 5A.

2- Calculate the magnetic flux crossing the solenoid when placed in a uniform magnetic field of intensity 500mT knowing that the unit vector and the magnetic field vector have the same direction.

Exercise 14*:

Consider a flat coil with number of turns N = 1000 of cross-sectional area $S = 6cm^2$ and of resistance $r = 10\Omega$. A magnet is placed in front of this coil and parallel to its axis. The positive direction of current is chosen arbitrarily.



1- Represent the magnetic field created by the magnet in the center of the coil.

2- The magnet is moved parallel to the axis of the coil so that the magnet field created by the magnet at its center varies uniformly from 20mT to 5mT in 0.1s.

2.1- Show on the figure the direction of the unit normal vector. Calculate the magnetic flux crossing the coil.

2.2- Calculate the induced e.m.f. created in the coil.

3- Calculate the value of the induced current in the coil supposing that it forms a closed circuit. Determine the direction of this current.

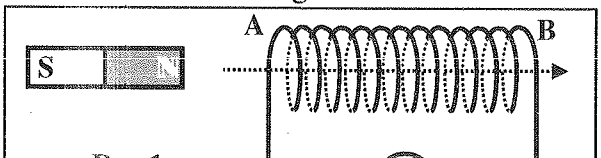
Exercise 15:

The objective of this exercise is to study the phenomenon of electromagnetic induction and the laws that explain it. For this, different cases are investigated.

1- Define magnetic flux and electromagnetic induction.

2- Name and state the two laws that basically explain the phenomenon of electromagnetic induction.

3- A bar magnet approaches a solenoid as shown in the document 1.



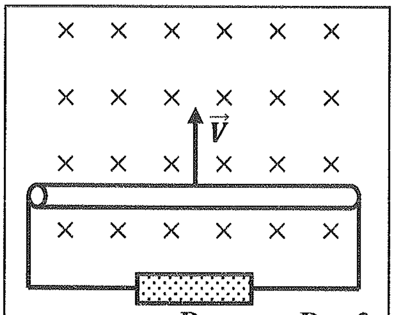
Doc.1

3.1- Explain the existence of induced current in the solenoid.

3.2- Show on the figure the direction of the induced current and the induced magnetic field.

3.3- In this experiment, the law of conservation of energy is valid. Explain.

4- In another situation, a metallic wire is connected in series with a resistor R as shown in document 2.



Doc.2

Given:

- The wire is uniform and homogeneous of length L = 0.12m and mass m = 8g.
- The wire moves with a speed V across a uniform and horizontal magnetic field of intensity B = 0.02T.

4.1- Refer to document 2 to answer the following questions:

4.1.1- Is the magnetic field vector directed into the page or out of the page?

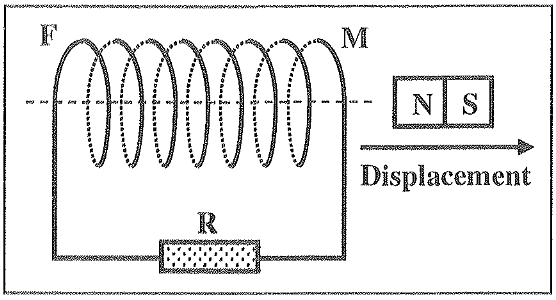
4.1.2- Is the wire moving upward or downward through the magnetic field?

4.1.3- Explain the existence of an induced e.m.f across the extremities of the wire.

4.1.4- The motion of the wire across a uniform magnetic field results in a force that is exerted on the wire. Name this force and specify its direction. Illustrate with a diagram.

4.1.5- Label the extremity of the wire that is at lower potential. Full explanation is required.

4.2- The wire is moved horizontally across the magnetic field. Does an induced e.m.f exist anymore? Explain.



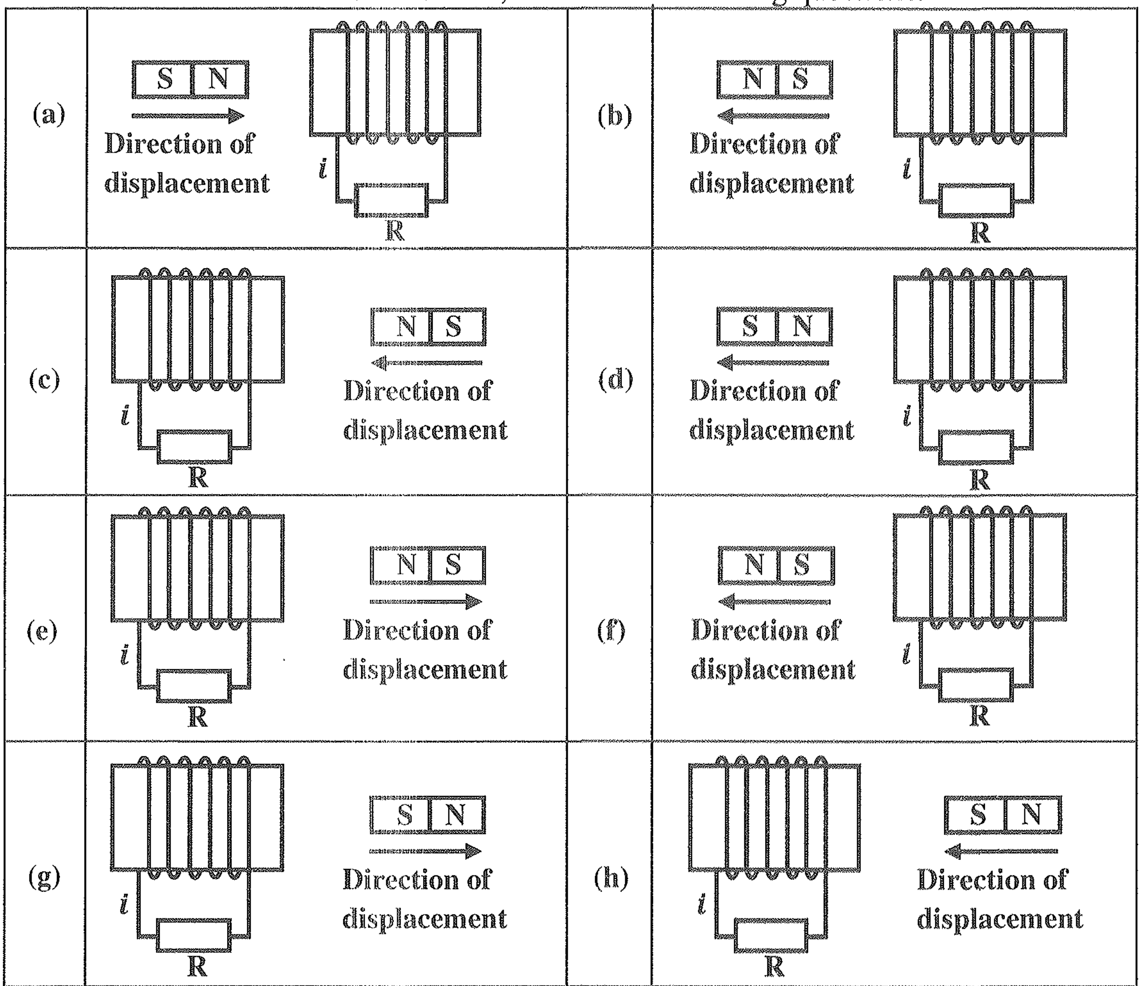
Exercise 16*:

A closed electric circuit consists of a coil and a resistor of resistance R. The north pole of a bar magnet moves away from the solenoid as indicated on the adjacent document.

1- Give the name of the phenomenon that occurs in the coil.
2- Specify the poles of the coil; then, deduce the direction of the induced current i in the coil.
3- State the two laws that explain this phenomenon.
4- The magnet is placed stationary near the solenoid. Give the value of the induced e.m.f. Justify.
5- The resistor is removed. The magnet with its south pole approaches the solenoid. Specify if any change occur corresponding to your answer in number (2).

Exercise 17*:

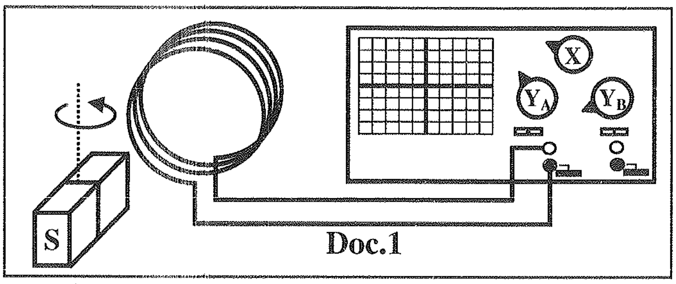
In the document below, a magnet moves near a closed electric circuit formed of an inductor and a resistor. For the different cases indicated in the document, answer the following questions:



1- Draw, at the center of the inductor, the magnetic field vector $\vec{B}$ due to the presence of the magnet.
2- Is the strength of $\vec{B}$ increasing or decreasing?
3- State Lenz's law.
4- Draw, at the center of the inductor, the induced magnetic field vector $\vec{B}_{ind}$.
5- Specify the direction of the induced current i that is passing through R.

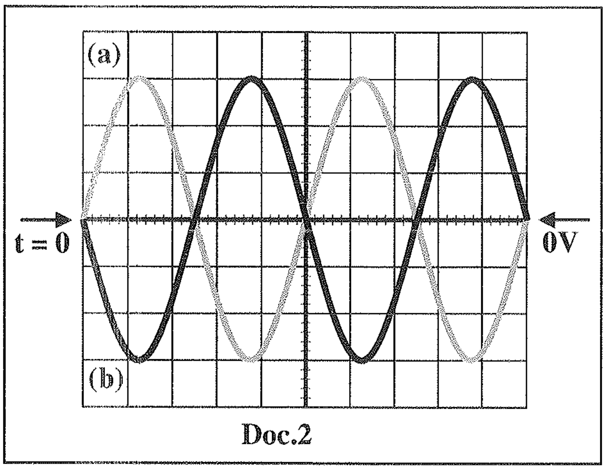
Exercise 18*:

The objective of this exercise is to study the production of alternating sinusoidal voltage using a magnet and a coil. Document 1 shows a coil connected to an oscilloscope and a bar magnet whose position corresponds to $t_0 = 0s$. At an instant t, the magnet starts rotating in the direction shown in the figure.



Doc.1

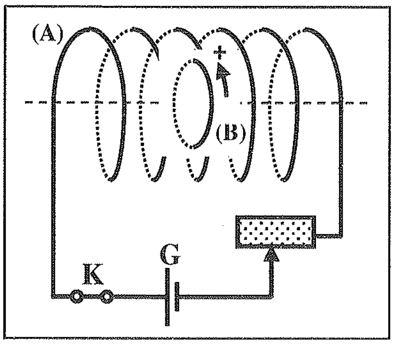
1- Draw, at the center of the coil, the magnetic field vector $\vec{B}$ due to the presence of the magnet at $t_0 = 0s$.
2- Is the strength of $\vec{B}$ increasing or decreasing in the coil?
3- State Lenz's law.
4- Draw, at the center of the coil, the induced magnetic field vector $\vec{B}_{ind}$.
5- Specify the direction of the induced current i in the coil.
6- The coil behaves as a source of induced e.m.f. just like a battery. Specify the polarity of the induced e.m.f. then label the positive terminal on the figure.
7- In document 2, two alternating sinusoidal waveforms are drawn in the oscillogram. One of the two waveforms represents correctly the variation of the induced e.m.f. as time increases.
Given: $S_h = 20ms/div$, $Sv = 10mV/div$.
   7.1- Specify the correct waveform.
   7.2- Calculate the period of rotation in the magnet. Deduce its frequency.
8- Give the name of the phenomenon discussed in this exercise.



Doc.2

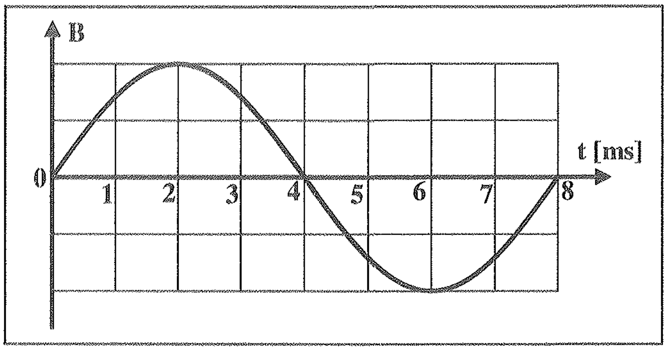
Exercise 19:

Consider a long solenoid (A) of $n_1 = 8000$ turns per meter. (A) is traversed by a current of intensity $I = 5A$. Inside (A), a coil (B) is placed of area $2cm^2$ and having $N_2 = 500$ turns. The coil is closed on itself and its resistance is $R = 2\Omega$. The setup is shown in the adjacent document.



1- Represent the direction of the current I traversing the solenoid.
2-
   2.1- Represent the magnetic field vector $\vec{B}$ created by (A) at the center of (B). Calculate its magnitude.
   2.2- Specify the north and south faces of (A).
   2.3- Sketch the orientation of a magnetic needle placed in the center of the solenoid.
3- Calculate the magnetic flux created by the magnetic field of (A) through the coil (B).
4- Switch K is opened.
   4.1- Explain why (B) will carry an electric current.
   4.2- Give the name of this current.
   4.3- Calculate the variation in the flux of $\vec{B}$ through the coil which results from the opening of the switch.
   4.4- Calculate the average induced e.m.f. which appears at the terminals of (B) if the variation of flux lasts 0.1s. Deduce the value of the current flowing in (B).
5- Determine the direction of the current which traverses the coil if the cursor of the rheostat is moved to the left (switch K is closed).
6- Specify the inducer.

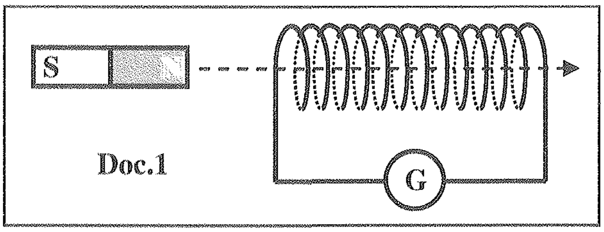
Exercise 20*:

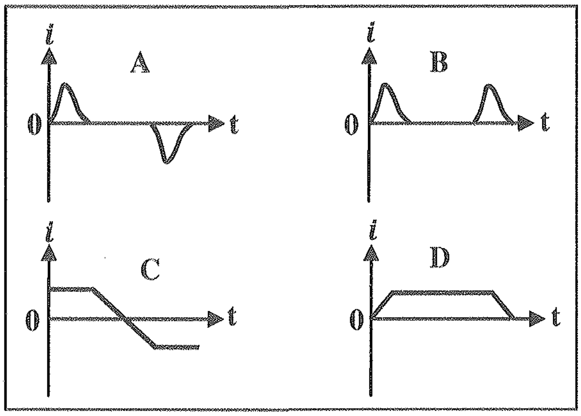


A conductor is influenced by a variable magnetic field. The graph in the adjacent document shows the variation of the intensity of the magnetic field B in the conductor as a function of time.

1- Explain the existence of an induced e.m.f in the conductor.
2- Give the value of the period of the oscillation of the magnetic field.
3- The induced e.m.f in the conductor is maximum at t = 4ms. Explain with calculations.

Exercise 21*:

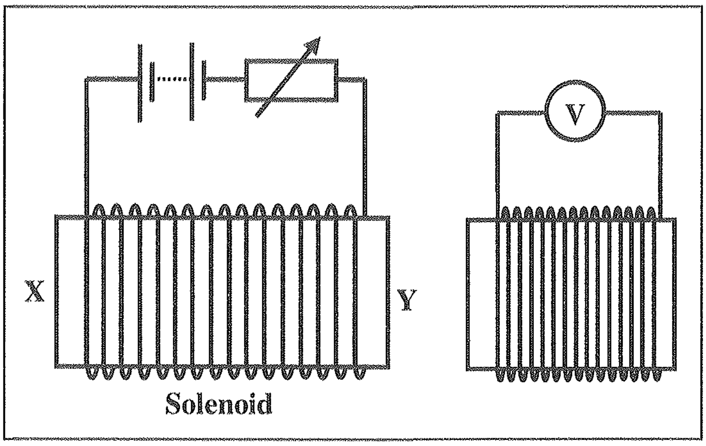




A small magnet with its N pole is moved with a constant speed into a solenoid and then continues until its S pole leaves the solenoid. A galvanometer, branched across the solenoid as shown in the adjacent document, is used to detect the existence of an induced current.

1- Explain the existence of induced current in the solenoid.
2- Specify the poles of the solenoid as the magnet with its N pole approaches the left extremity of the solenoid.
3- Choose the letter that corresponds to the correct graph of the variation of the induced current as a function of time as the magnet moves towards, into and out of the solenoid. Explain your answer.

Exercise 22*:

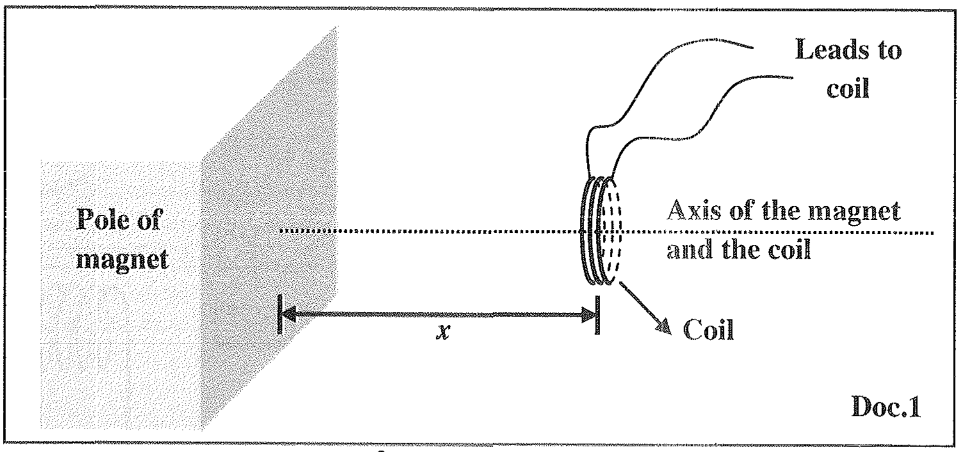


A solenoid is connected to a battery and a rheostat as shown in the adjacent document. The extremities of the solenoid are labeled X and Y. A voltmeter is branched across a coil which is placed next to the extremity Y of the solenoid.

1- The solenoid carries a current of intensity I.
   1.1- Draw the magnetic field lines inside the solenoid. Label the extremities X and Y of the solenoid as north and south.
   1.2- Specify if the voltmeter in the adjacent document registers any value.
2- The resistance of the rheostat is varied leading to an increase in the intensity of the current in the solenoid.
   2.1- Explain the existence of induced e.m.f in the solenoid.
   2.2- Does an induced current flow in the coil-voltmeter circuit. Justify.
   2.3- Specify the poles of the coil.

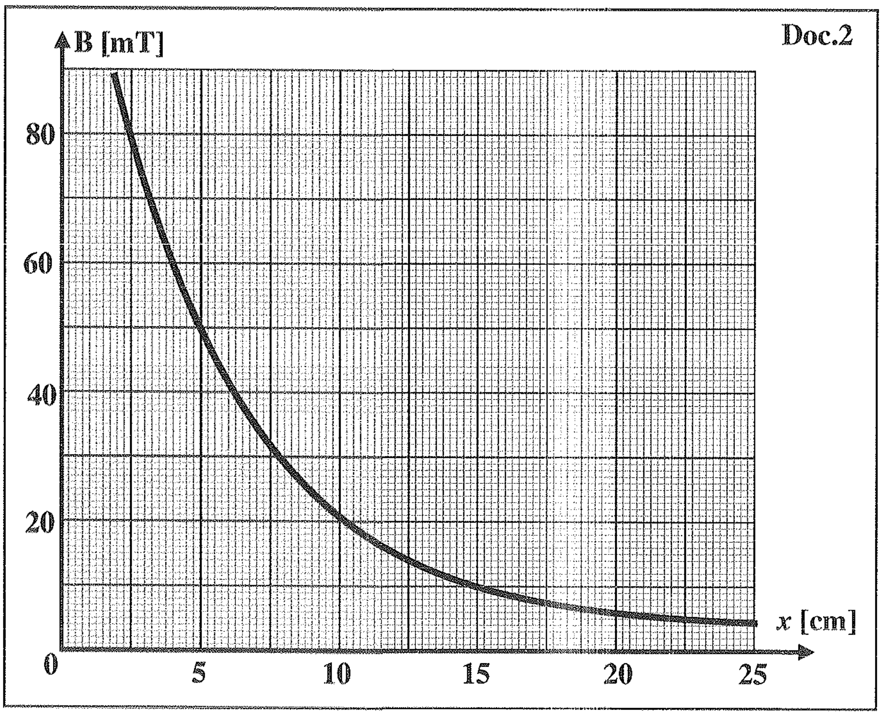
Exercise 23:

In document 1, a small coil is placed such that its axis lies along the axis of a large bar magnet.



The coil has a cross-sectional area of $0.40 cm^2$ and contains 150 turns of wire.

The average magnetic induction vector $\vec{B}$ between the face of the magnet and the plane of the coil varies, as shown in document 2.



1- The coil is 5cm from the face of the magnet. Use document 2 to determine the magnitude of the magnetic induction vector in the coil.

2- Show that the magnetic flux of the coil is 3×10^{-4}Wb.

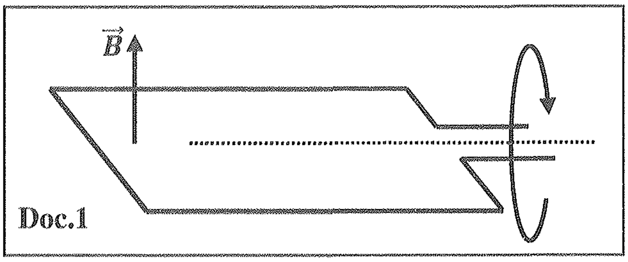
3- The coil is moved along the axis of the magnet so that the distance x changes from x = 5cm to x = 15cm in a time of 0.40s. Calculate:

3.1- the change in flux of the coil,

3.2- the average e.m.f. induced in the coil.

4- State and explain the variation, if any, of the speed of the coil so that the induced e.m.f. remains constant during the movement in part 3.

Exercise 24:

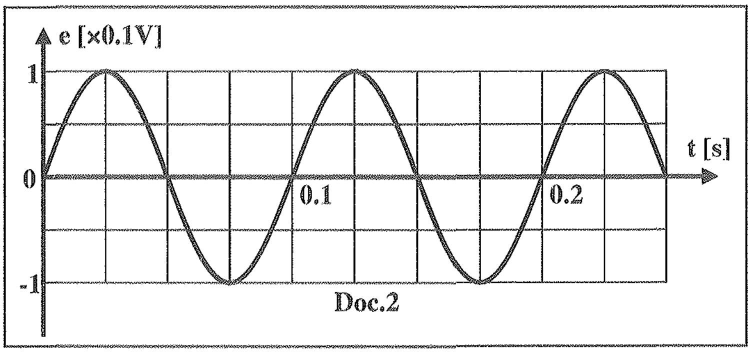


Doc.1

A rectangular loop is placed in a uniform magnetic field. The loop is free to rotate around a horizontal axis as shown in document 1.

The area of the loop is $5.5 \times 10^{-3} m^2$, and the magnetic field of magnitude 0.2T is directed vertically upwards.

As the loop rotates, with a frequency f, a suitable mechanism allows up to plot the variation of the induced electromotive force e.m.f. as a function of time (document 2).



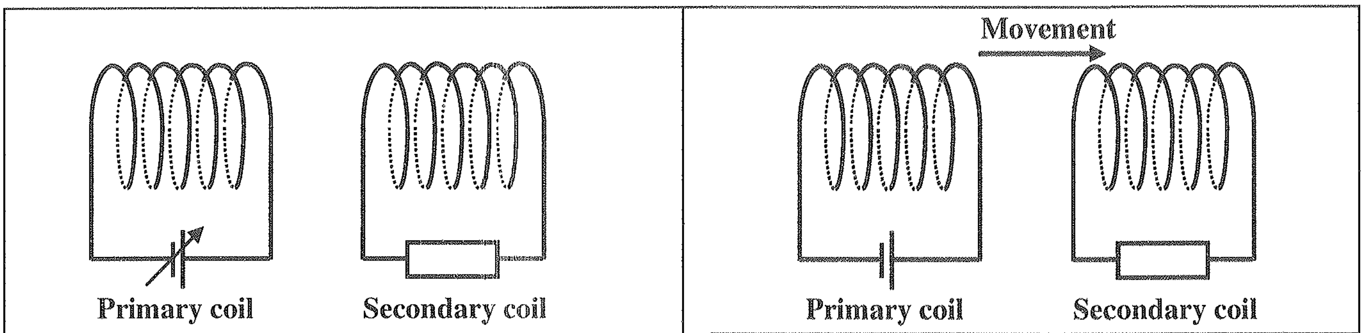
Doc.2

1- Using document 2:
   1.1- Determine the value of f.
   1.2- Give the maximum value of the induced e.m.f.
2- Interpret the sinusoidal variation of "e" versus "t".
3- Use document 2 to indicate the position of the coil relative to the normal of the coil at t = 0s.
4- What is the maximum flux traversing the coil?
5- Determine the time equation of the emf as a function of time.
6- Deduce the time equation of the variation of the flux.

Exercise 25*:

In the following questions, changes to the current in a primary coil induce a current in a secondary coil placed beside it.

In each case, show on the diagram:
1- The magnetic field along the axis of the primary coil.
2- The induced magnetic field along the axis of the secondary coil.
3- The induced current in the secondary coil.

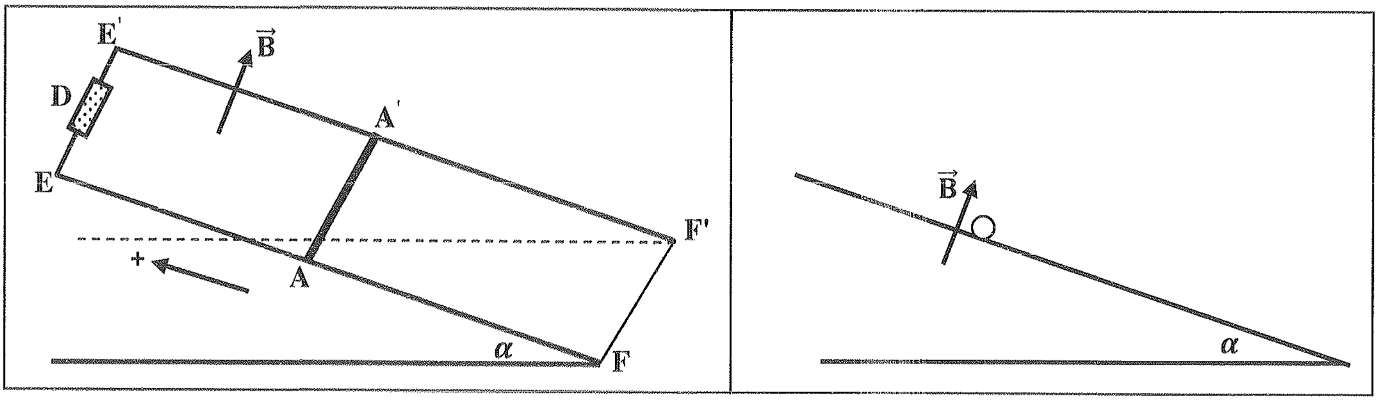


(a) The current in the primary coil is increasing	**(b) The primary coil is moving towards the secondary coil**

Exercise 26:

A rigid and homogeneous metal rod (AA'), of mass m = 30g and length AA' = ℓ = 15cm, slides without friction on two parallel and metallic rails (EF) and (E'F'). (EF) and (E'F') form a plane (Q) inclined by an angle α = 0.05rd to the horizontal. (EF) and (E'F') form two lines of greatest slope of (Q) which are located in a region of space where there is a uniform magnetic field $\vec{B}$, of intensity B = 0.5T and perpendicular to

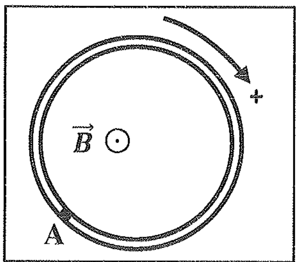
(Q). The two extremities E and E 'are connected to a resistor (D) of resistance R = 0.6Ω.
At time $t_0 = 0$, (AA') is released without initial speed from the top of (Q). (AA') moves parallel to itself while remaining perpendicular to the rails (EF) and (E'F').



1- Show that when the rod (AA') moves, the electrical circuit formed by (AA'), the rails and the (D) is the seat of an induced e.m.f.
2- Determine the expression of the induced e.m.f in terms ℓ, B and v, where v is the speed of the rod (AA') at the instant t.
3- An induced current i passes through this circuit. Explain; then, determine the direction of this current.
4- The rod (AA') is subjected to an electromagnetic force $\vec{F}$. Specify the characteristics of this force.
5- List and draw the external forces acting on (AA').
6- Prove that, by applying Newton's second law, the expression of the acceleration of the motion of the rod (AA) is written as: $a = \frac{-F+mg\sin\alpha}{m}$.
7- Deduce that, the speed v of the rod (AA') reaches a constant limit v_l. Calculate the value of v_l.

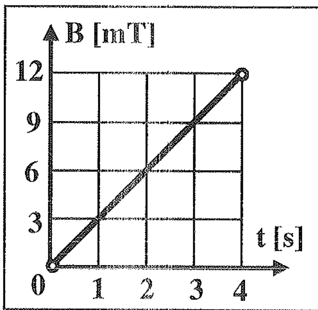
Exercise 27:

A circular conducting loop (C), of area S = 10cm^2 and resistance R, is situated in a vertical plane. (C) is placed in a horizontal magnetic field $\vec{B}$ of magnitude B and directed out of the page as shown in document 1.
Document 2 represents the variation of B as a function of time and document 3 represents the quantity of charge q passing through the section A of the loop as a function of time.

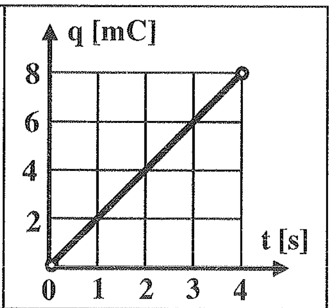


Doc.1

1- Explain the existence of an induced current i in the loop. Use document 3 to calculate its value.
2- Apply Lenz's law to specify the direction of i.
3- Use document 2 to determine the expression of B as a function of time.
4- Taking into consideration the chosen positive direction indicated on document 1, determine, as a function of time, the expression of the magnetic flux crossing the loop.
5- Deduce the value of the induced emf e.
6- Calculate the value of R.

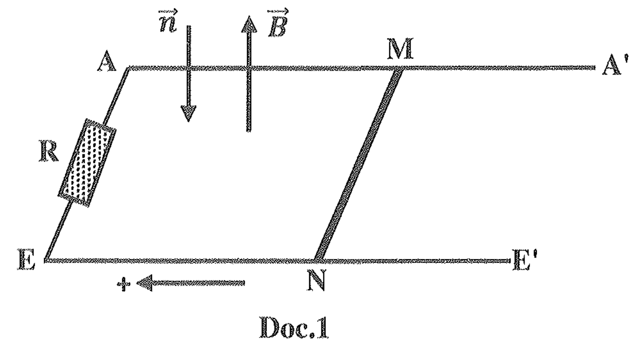


Doc.2



Doc.3

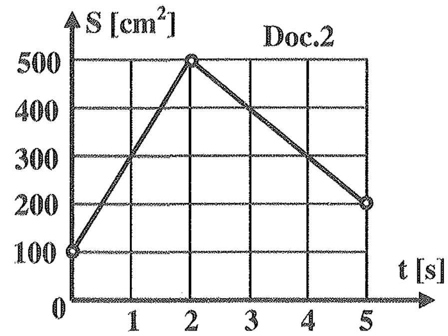
Exercise 28:



Doc.1

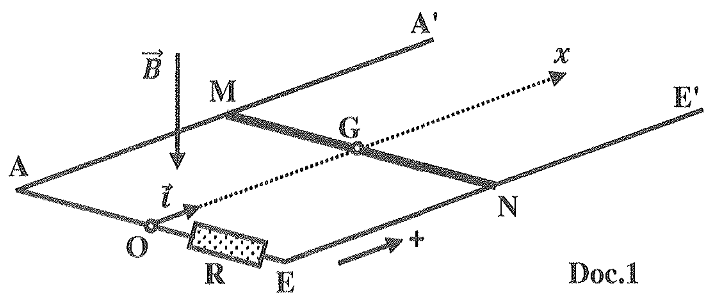
A homogeneous metallic rod MN slides on two horizontal and parallel metallic rails AA' and EE' at a constant velocity $\vec{V}$. During its sliding, the rod remains perpendicular to the rails. A resistor of resistance R = 10Ω is connected to the extremities A and E of the rod. The whole set-up is put within a uniform magnetic field $\vec{B}$ of magnitude B = 0.2T and perpendicular to the plane of the rails as shown in document 1. Neglect the resistance of the rod and the rails.

Document 2 represents the variation of the surface area S of AMNE as a function of time.



1- Taking into consideration the positive sense indicated on document 1, determine the expression of the magnetic flux crossing AMNE in terms of B and S.
2- Explain the existence of an induced current i carried by the loop AMNE; then deduce its expression in terms of B, S and R.
3- Calculate the value of i for $0 \leq t \leq 2s$ and $2s \leq t \leq 5s$. Deduce its direction.
4- Apply Lenz's law to verify the results obtained in part 3.

Exercise 29:



Doc.1

A metallic rod MN, of mass m = 20g, length ℓ = 50cm and negligible resistance, may slide without friction along two long parallel and horizontal rectilinear rails AA' and EE' of negligible resistance. During its displacement, the rod remains perpendicular to the rails. A resistor of resistance R = 10Ω is connected to the rails. The whole set-up thus described is placed in a uniform vertically downwards magnetic field $\vec{B}$ of magnitude B (document 1).

The center of mass G of the rod may move along a horizontal axis $(O;\vec{i})$ parallel to the rails.

At the instant $t_0 = 0$, G being at O, the rod is given an initial velocity $\vec{V_0} = V_0\vec{i}$.

At an instant t, the position of G is defined by its abscissa $x = \overline{OG}$ and the algebraic value of its velocity is $V = \frac{dx}{dt}$.

1- Find, at the instant t, the expression of the magnetic flux that crosses the surface AMNE in terms of B, ℓ and x taking into consideration the positive direction indicated on the document.
2-
   2.1- Explain the existence of an induced e.m.f 'e' across the ends M and N of the rod and determine its expression in terms of B, ℓ and V.
   2.2- Deduce the expression of the potential difference u_{NM}.
3-
   3.1- Determine the expression of the induced current i in terms of B, ℓ, V and R. Deduce its direction.
   3.2- Apply Lenz's law to verify the results obtained in part 3.1.

4- Show that the magnitude of the electromagnetic force acting on the rod MN is given by: $F = \frac{B^2\ell^2}{R}V$.

5-

5.1- Applying Newton's 2^{nd} law of translation motion, show that the differential equation that governs the variation of V as a function of time is given by:

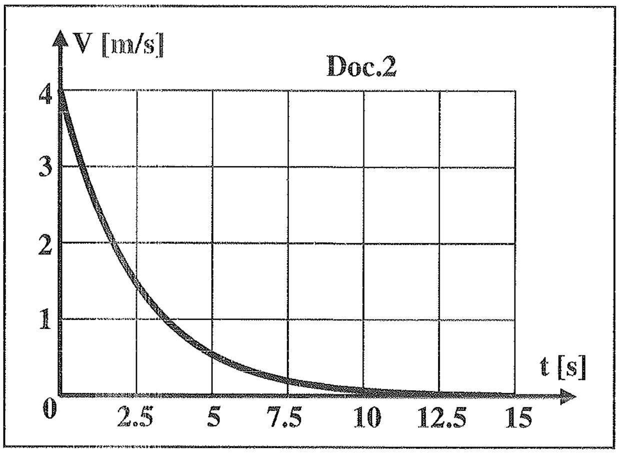
$$\frac{dV}{dt} + \frac{B^2\ell^2}{mR}V = 0$$

5.2- The solution of the differential equation has the form $V = V_0 e^{-\frac{t}{\tau}}$ where τ is a constant. Determine the expression of τ in terms of B, ℓ, m and R.

6- A convenient apparatus records the variation V as a function of time. Document 2 shows the variation of V as a function of time t.

6.1- Give the value of V_0.

6.2- Determine the value of τ, then, deduce that of B.

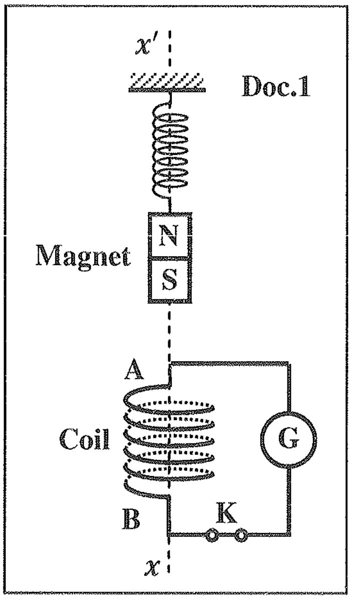


Exercise 30:

Parts I and II of this exercise are independent.

Part I: the circuit diagram shown in document 1 consists of:
- a coil whose axis $x'x$ is vertical,
- a closed switch K.
- a galvanometer G.

A bar magnet is connected to the lower extremity of a spring, while its upper extremity is fixed to a support. The magnet oscillates vertically along the axis of the coil.



1- Explain the existence of an induced current traversing the circuit.

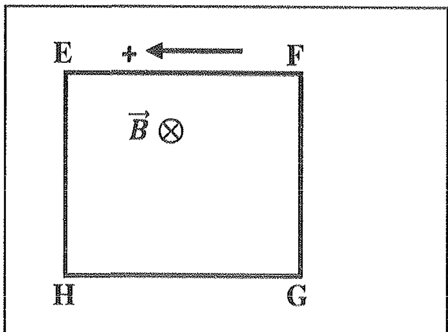
2- Give, with justification, the name of each face of the coil, and the direction of the induced current in each of the following cases:

2.1- The magnet approaches the coil.

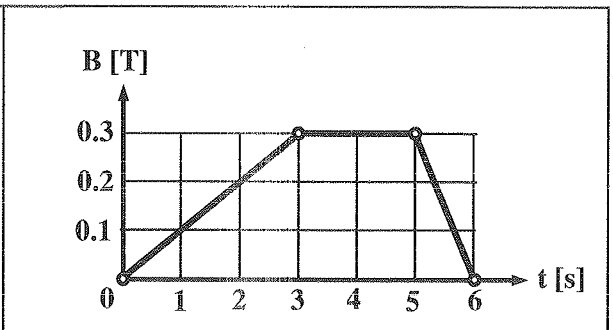
2.2- The magnet moves away from the coil.

Part II: a coil EFGH, of N = 500 turns, resistance R = 10Ω and area S = 10cm^2, is placed in a uniform magnetic field $\vec{B}$ directed into the page as shown in document 2.

Document 3 represents the variation of the intensity B of the magnetic field $\vec{B}$ as a function of time.



Doc.2



Doc.3

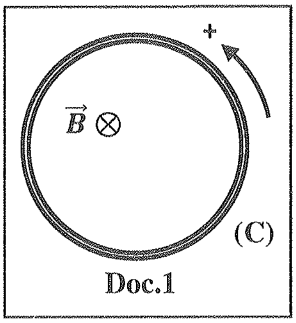
1- Show that the expression of the induced current traversing the coil is: $i = 0.05 \times \frac{\Delta B}{\Delta t}$.
2- Calculate the value of i for each of the following time intervals:

$$0s < t < 3s \qquad 3s < t < 5s \qquad 5s < t < 6s$$

3- Deduce the direction of i for each of the time intervals mentioned in part II.2.
4- Apply Lenz's law to verify the results obtained in part II.3

Exercise 31*:

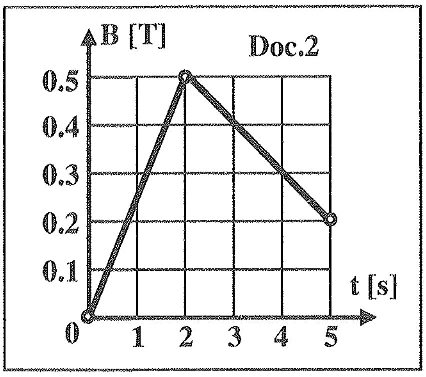
The object of this exercise is to study the electromagnetic magnetic induction phenomenon. For this, we consider a flat coil (C) of area $S = 5 \times 10^{-3} m^2$, resistance $R = 10\Omega$ and N = 50 turns.



Part I: Variation of the intensity of the magnetic field

(C) is placed in a magnetic field $\vec{B}$ perpendicular to its plane as shown in document 1.

Document 2 represents the variation of the intensity B of $\vec{B}$ as a function of time.

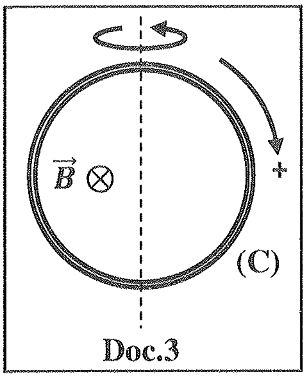


1- Explain the existence of an induced current i traversing (C).
2- Show that the expression of this induced current is $i = \frac{NS}{R} \times \frac{dB}{dt}$.
3- Calculate the values of i for $0 \le t \le 2s$ and for $2s \le t \le 5s$. Deduce its direction.
4- Apply Lenz's law to verify the results obtained in part 3.

Part II: Rotation of the coil

The intensity of the magnetic field $\vec{B}$ is adjusted at B = 0.5T.

(C) rotates at a constant angular velocity ω about its diameter as shown in document 3. Let θ be the angle between the normal $\vec{n}$ to the plane of (C) and $\vec{B}$ at an instant t.



1- Knowing that $\theta = 0$ at the instant $t_0 = 0$, show that $\theta = \omega t$.
2- Deduce that the expression of the magnetic flux crossing (C) is given by:

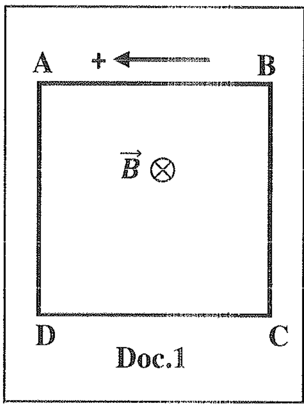
$$\phi = NBS\cos(\omega t)$$

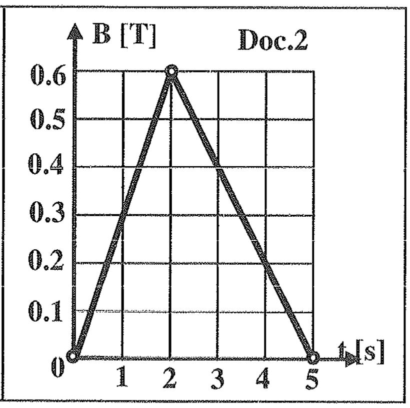
3- Justify, qualitatively, the existence of an induced e.m.f "e" during the rotation of (C).
4- Determine, in terms of N, S, B, ω and t, the expression of the induced e.m.f "e".
5- Calculate the value of ω knowing that the maximum induced emf is 0.5V.

Exercise 32*:

A conducting square loop ABCD, of side a = 20cm, resistance R = 10Ω and N = 20turns, is situated in the vertical plane and placed in a uniform magnetic field $\vec{B}$ perpendicular to its plane as shown in document 1.

Document 2 represents the variation of the intensity B of the magnetic field $\vec{B}$ as a function of time.



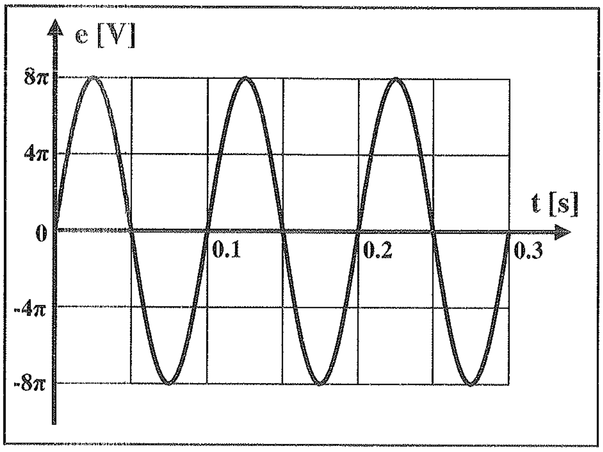


1- State Faraday's law.
2- Determine, by applying Lenz's law, the direction of the induced current for $0 < t < 2s$ and $2s < t < 5s$.
3- Taking into consideration the positive sense indicated on document 1, determine the expression of the magnetic flux crossing ABCD in terms of N, B and a.
4- Show that the expression of the induced current carried by the loop is $i = \frac{Na^2}{R} \times \frac{dB}{dt}$.
5- Calculate the values of i for $0 < t < 2s$ and $2s < t < 5s$. Deduce its direction.
6- Compare the results obtained in parts 2 and 5.

Exercise 33*:

A coil, of N = 20 turns each of surface area S = 5 × 10^{-2}m^2, rotates about its diameter, at a constant angular frequency ω, in a uniform magnetic field $\vec{B}$ of intensity B. At the instant t, the angle between $\vec{B}$ and the normal $\vec{n}$ to the plane of the coil is $\theta = \omega t$.

1- Give, at the instant t, the expression of the magnetic flux crossing the coil in terms of N, B, S, ω and t.
2- Deduce the expression of the induced e.m.f 'e' at the instant t.
3- The graph in the adjacent document shows the variation of 'e' as a function of time.
   3.1- Give the values of the maximum value e_m and the period T of 'e'.
   3.2- Deduce the values of ω and B.



CHAPTER 3 – ELECTROMAGNETIC INDUCTION
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 3:

Part	Answer key
	Document 1: $\vec{n}$ is oriented to right by RHR $\Rightarrow \theta = (\vec{B};\, \vec{n}) = 60°$. $\phi = N\vec{B}.\vec{S} = NBS\cos\theta = 1 \times 0.05 \times \pi \times 0.1^2 \times \cos 60° = 7.85 \times 10^{-4} Wb$. **Document 2:** $\vec{n}$ is oriented outwards by RHR $\Rightarrow \theta = (\vec{B};\, \vec{n}) = 180°$. $\phi = N\vec{B}.\vec{S} = NBS\cos\theta = 20 \times 0.5 \times 0.2^2 \times \cos 180° = -0.4Wb$. **Document 3:** $\vec{n}$ is oriented upwards by RHR $\Rightarrow \theta = (\vec{B};\, \vec{n}) = 180°$. $\phi = N\vec{B}.\vec{S} = NBS\cos\theta = 1 \times 0.2 \times 0.1 \times 0.2 \times \cos 180° = -4 \times 10^{-3} Wb$.

Exercise 5:

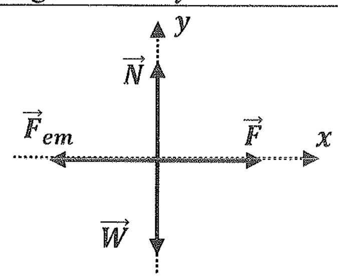
Part	Answer key
1	**Document 1:** the intensity of the magnetic field varies with time; then, the magnetic flux varies with time. Therefore, the variation in the magnetic flux gives rise to an induced e.m.f according to Faraday's law. **Document 2:** the surface area varies with time; then, the magnetic flux varies with time. Therefore, the variation in the magnetic flux gives rise to an induced e.m.f according to Faraday's law. **Document 3:** the intensity of the magnetic field varies with time; then, the magnetic flux varies with time. Therefore, the variation in the magnetic flux gives rise to an induced e.m.f according to Faraday's law.
2	**Document 1:** the intensity of the magnetic field decreases with time. Hence, the induced current i traverses the circuit in order to create $\vec{B}_{in}$ in the same direction of $\vec{B}$. Therefore, the current traverses the coil in the counter clockwise direction. **Document 2:** the induced current opposes, by its electromagnetic effect, the cause that produces it. The electromagnetic force then opposes the direction of displacement of the rod. The induced current then passes through the rod from point N to point M. **Document 3:** In order to oppose by repulsion the approach of the N-pole of the magnet (Lenz's law), A is the North face, B is the South face. The induced magnetic field $\vec{B}_i$ is directed from B to A through the solenoid. Therefore, the induced current is directed from A to B through the solenoid (or from B to A through the resistor).

Exercise 6:

Part	Answer Key
1	Electromagnetic induction.
2	**Faraday's law:** the induced electromotive force "e" at any instant is equal to the opposite of the derivative with respect to time of the magnetic flux crossing the circuit. **Lenz's law:** The direction of the induced current is such that its electromagnetic effects always oppose the cause that has established this current.
3.1	For $0s \le t \le 2s$: $B = kt$ with $k = \frac{\Delta B}{\Delta t} = \frac{0.5-0}{2-0} = 0.25T/s$. Therefore, $B = 0.25t$. For $2s \le t \le 6s$: $B = 0.5T$. For $6s \le t \le 10s$: $B = kt + c$ with $k = \frac{\Delta B}{\Delta t} = \frac{0-0.5}{10-6} = -0.125T/s$. $B = -0.125t + c \Rightarrow 0.5 = -0.125 \times 6 + c \Rightarrow c = 1.25T$. Therefore, $B = -0.125t + 1.25$.

3.2	By applying the right hand rule $\vec{n}$ is pointed outwards, so $\theta = (\vec{B};\ \vec{n}) = 0°$. $S = a^2 = 0.1^2 = 0.01m^2$. $\phi = NBS\cos\theta = 1 \times B \times 0.01 \times \cos 0° = 0.01B$. For $0s \le t \le 2s$: $\phi = 0.01(0.25t) = 2.5 \times 10^{-3}t\ [Wb]$. For $2s \le t \le 6s$: $\phi = 0.01(0.5) = 5 \times 10^{-3}Wb$. For $6s < t < 10s$: $\phi = 0.01 \times (-0.125t + 1.25) = -1.25 \times 10^{-3}t + 0.0125\ [Wb]$.
3.3	For $0s \le t \le 2s$: $e = -\frac{d\phi}{dt} = -2.5 \times 10^{-3}V$. For $2s \le t \le 6s$: $e = -\frac{d\phi}{dt} = 0V$. For $6s \le t \le 10s$: $e = -\frac{d\phi}{dt} = +1.25 \times 10^{-3}V$.
3.4	For $0s \le t \le 2s$: $i = \frac{e}{R} = \frac{-2.5\times10^{-3}}{5} = -5 \times 10^{-4}A < 0$ (clockwise direction). For $2s \le t \le 6s$: $i = \frac{e}{R} = 0A$. For $6s < t < 10s$: $i = \frac{e}{R} = \frac{-1.25\times10^{-3}}{5} = +2.5 \times 10^{-4}A > 0$ (counter clockwise direction).
4	For $0s \le t \le 2s$, B increases with time. Therefore the magnetic flux through the coil increases with time. Hence, the induced current i traverses the circuit in order to create $\vec{B}_{in}$ opposite to $\vec{B}$. Therefore, the current traverses the coil in the clockwise direction. For $2s \le t \le 6s$, B = constant $\Longrightarrow \phi$ = constant; therefore $i = 0A$. For $6s \le t \le 10s$, B decreases with time. Therefore the magnetic flux through the coil decreases with time. Hence, the induced current i traverses the circuit in order to create $\vec{B}_{in}$ in the same direction of $\vec{B}$. Therefore, the current traverses the coil in the counter clockwise direction.

Exercise 7:

Part	Answer key
1	$\phi = NBS\cos\theta = (1)(B)(\ell x)(\cos 180°) = -B\ell x$ where $\theta = (\vec{B};\vec{n}) = 180°$.
2.1	The surface area varies with time; then, the magnetic flux varies with time. The variation in the magnetic flux gives rise to an induced emf according to Faraday's law.
2.2	Faraday's law: $e = -\frac{d\phi}{dt} = -\frac{d(-B\ell x)}{dt} = B\ell\frac{dx}{dt} = B\ell v$. 
3	$i = \frac{e}{R} = \frac{B\ell v}{R} > 0$. The induced current is in the positive sense (from N to M through the rod).
4.1	$\vec{W}$: Weight $\vec{N}$: Normal reaction of a support $\vec{F}$: Applied force. $\vec{F}_{em}$: Electromagnetic or Laplace's force.
4.2	$\sum \vec{F}_{ext} = \vec{0} \Longrightarrow \vec{F} + \vec{F}_{em} + \vec{N} + \vec{W} = \vec{0}$ Projection along x-axis: $-F_{em} + F = 0 \Longrightarrow F = F_{em} = i\ell B = \frac{B\ell V}{R} \times \ell B = \frac{B^2\ell^2 V}{R} = \frac{0.4^2\times0.5^2\times0.6}{10} = 2.4 \times 10^{-3}N$.

Exercise 8:

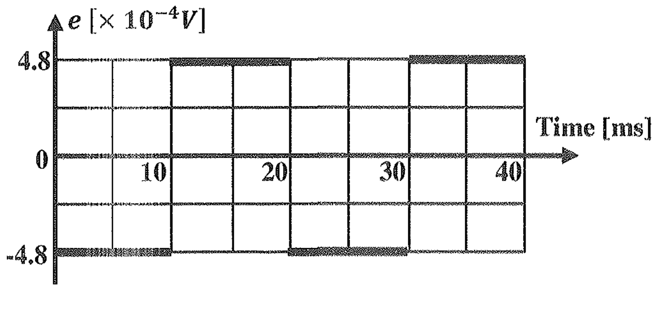
Part	Answer key
1	Electromagnetic induction.
2	The intensity of the magnetic field varies; then, the magnetic flux varies.

	Therefore, the variation in the magnetic flux gives rise to an induced e.m.f according to Faraday's law. The circuit is closed; then, it carries an induced current i.
3.1	According to Lenz's law, the coil will oppose by repulsion the incoming N-pole of the magnet. A is the north pole and C is the south pole. The induced magnetic field $\vec{B}_i$ is directed from C to A through the coil. By applying the right hand rule, the induced current i is directed from A to C through the coil.
3.2	According to Lenz's law, the coil will oppose by attraction the outgoing N-pole of the magnet. A is the south pole and C is the north pole. The induced magnetic field $\vec{B}_i$ is directed from A to C through the coil. By applying the right hand rule, the induced current i is directed from C to A through the coil.

Exercise 11:

Part	Answer key
1	The current traversing the inner loop varies with time; then, the magnetic field created by the inner loop varies with time. The outer loop is placed in a variable magnetic field created by the inner loop; then, the magnetic flux traversing the outer loop varies with time. The change in the magnetic flux gives rise to an induced e.m.f. by Faraday's law. Since the outer loop is closed, it will be traversed by an induced electric current.
2	By RHR, the magnetic field $\vec{B}$ created by the inner loop is directed inwards and its intensity decreases with time. Hence, the induced current i traverses the outer loop in order to create $\vec{B}_{in}$ in the same direction of $\vec{B}$. Therefore, the current traverses the outer loop in the clockwise direction.

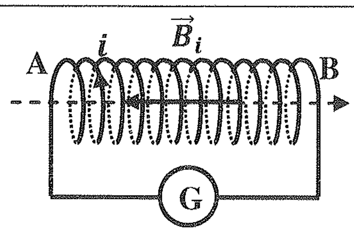
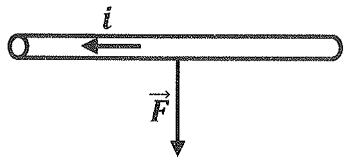
Exercise 12:

Part	Answer key
1	The current traversing the solenoid varies with time; then, the magnetic field created by the solenoid varies with time. The coil is placed in a variable magnetic field created by the solenoid; then, the magnetic flux traversing the coil varies with time. The change in the magnetic flux gives rise to an induced e.m.f. by Faraday's law.
2	$\phi = NBS\cos\theta = N_C \times \frac{\mu_0 N_S i}{L_S} \times S_C \times \cos 180° = -\frac{\mu_0 N_S N_C S_C}{L_S} i.$ By applying Faraday's law: $e = -\frac{d\phi}{dt} = -\frac{\mu_0 N_S N_C S_C}{L_S} \times \frac{di}{dt} = -\frac{4\pi\times10^{-7}\times400\times8\times2\times10^{-4}}{0.5} \times \frac{di}{dt}.$ $e = -1.6\times10^{-6}\times\frac{di}{dt} = -1.6\times10^{-6}\times\frac{\Delta i}{\Delta t}$ (linear variation). For $0s < t < 10ms$; $e = -1.6\times10^{-6}\times\frac{1.5-0}{(10-5)\times10^{-3}} = -4.8\times10^{-4}V.$ For $10ms < t < 20ms$; $e = -1.6\times10^{-6}\times\frac{0-1.5}{(15-10)\times10^{-3}} = +4.8\times10^{-4}V.$ 

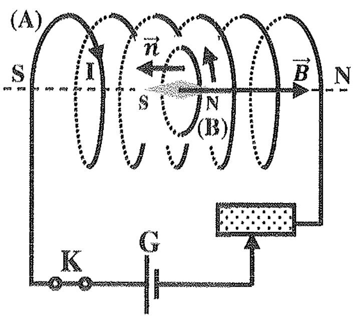
Exercise 13:

Part	Answer key
1	$B = \frac{\mu_0 NI}{L} = \frac{4\pi\times10^{-7}\times300\times5}{0.2} = 9.4\times10^{-3}T.$
2	$S = \pi R^2 = \pi\times0.025^2 = 1.96\times10^{-3}m^2.$ $\phi = N\vec{B}.\vec{S} = NBS\cos\theta = 300\times500\times10^{-3}\times1.96\times10^{-3}\times\cos 0° = 0.294Wb.$

Exercise 15:

Part	Answer key
1	**Magnetic flux:** number of magnetic field lines penetrating a given surface. **Electromagnetic induction:** generation of e.m.f when a circuit is traversed by a time varying flux.
2	**Faraday's law:** the induced e.m.f. in a circuit at any instant is equal to the opposite of the derivative of the magnetic flux crossing the circuit with respect to time. **Lenz's law:** the direction of the induced current is such that its electromagnetic effects oppose the cause that is producing it.
3.1	The intensity of the magnetic field varies with time; then, the magnetic flux varies with time. Therefore, the variation in the magnetic flux gives rise to an induced e.m.f. Since the circuit is closed, an induced current i traverses it.
3.2	In order to oppose by repulsion the approach of the N-pole of the magnet (Lenz's law), A is the North face, B is the South face. The induced magnetic field $\vec{B}_i$ is directed from B to A. 
3.3	The kinetic energy of the magnet is converted into magnetic energy stored in the coil.
4.1.1	Into the page.
4.1.2	The wire is moving upwards.
4.1.3	The surface area increases, then the magnetic flux increases. Therefore, the increase in the magnetic flux gives rise to an induced e.m.f.
4.1.4	Electromagnetic force or Laplace's force. The induced current opposes, by its electromagnetic effect, the cause that produces it. The Laplace force then opposes the direction of displacement of the wire. The induced current then passes through the wire from right to left by RHR. 
4.1.5	The current is directed from low potential to high potential through the rod; then, the right extremity of the rod is the lower potential terminal.
4.2	No, since the magnetic flux traversing the circuit is zero.

Exercise 19:

Part	Answer key
1	

2.1	$B = \mu_0 n_1 I = 4\pi \times 10^{-7} \times 8000 \times 5 = 0.05T$.
2.2	According to the right hand rule, the magnetic field $\vec{B}$ created by the solenoid (A) is directed to the right. Inside the solenoid, the magnetic field $\vec{B}$ is directed from S to N.
2.3	On document.
3	$\phi = N_2\vec{B}.\vec{S} = N_2 B S_2 \cos\theta = 500 \times 0.05 \times 2 \times 10^{-4} \times \cos 180° = -5 \times 10^{-3} Wb$.
4.1	The intensity of the electric current through (A) decreases (drops from I = 5A to zero); then, the intensity of the magnetic field $\vec{B}$ created by (A) decreases. The value of the magnetic flux crossing the coil (B) decreases. The variation in the magnetic flux traversing (B) gives rise to an induced e.m.f according to Faraday' law. Since (B) is closed, it will carry an electric current i.
4.2	Induced electric current.
4.3	$\Delta\phi = \phi_f - \phi_i = 0 - (-5 \times 10^{-3}) = 5 \times 10^{-3} Wb$.
4.4	$e_{av} = -\frac{\Delta\phi}{\Delta t} = \frac{-5\times10^{-3}}{0.1} = -5 \times 10^{-4} V$. $i = \frac{e}{R} = \frac{-5\times10^{-4}}{2} = -2.5 \times 10^{-4} A$.
5	The resistance of the rheostat increases as its cursor is moved to the left; then, the intensity of the electric current carried by (A) decreases; consequently, the intensity of the magnetic field $\vec{B}$ created by (A) decreases. According to Lenz's law, the coil (B) will oppose the decrease in $\vec{B}$ by creating an induced magnetic field $\vec{B}_i$ in the same direction. According to the right hand rule, the induced current i is in the positive sense.
6	Solenoid (A).

Exercise 23:

Part	Answer key
1	$B = 50mT$.
2	$\phi = N\vec{B}.\vec{S} = NBS\cos\theta = NBS$. Take $\theta = (\vec{B};\vec{n}) = 0$. $\phi = 150 \times 0.4 \times 10^{-4} \times 50 \times 10^{-3} \times 1 = 3 \times 10^{-4} Wb$.
3.1	For $x_2 = 15cm$; $B_2 = 8mT$. So $\phi_2 = 0.48 \times 10^{-4} Wb$. $\Delta\phi = \phi_2 - \phi_1 = 0.48 \times 10^{-4} - 3 \times 10^{-4} = -2.52 \times 10^{-4} Wb$.
3.2	$e_{av} = -\frac{\Delta\phi}{\Delta t} = \frac{2.52\times10^{-4}}{0.4} = 6.3 \times 10^{-3} V$.
4	$\phi = NBS \Longrightarrow e = -NS\frac{dB}{dt}$ and $\frac{dB}{dt} = \frac{dB}{dx} \times \frac{dx}{dt}$. e must be constant $\Longrightarrow \frac{dB}{dt} = constant \Longrightarrow \frac{dB}{dx} \times \frac{dx}{dt} = constant = k$. But $\frac{dx}{dt} = V$ (speed of the coil). So $\frac{dB}{dx}.V = k \Longrightarrow V = \frac{k}{\frac{dB}{dx}}$. So, the speed V of the coil is inversely proportional to the change of B relative to the displacement of the coil.

Exercise 24:

Part	Answer key
1	$f = \frac{1}{T} = \frac{1}{0.1} = 10Hz$.
2	$e_m = 0.1V$.

3	The change in the angle $\theta = (\vec{B}; \vec{n})$ leads to a change in the magnetic flux traversing the loop. The variation in the magnetic flux gives rise to an induced e.m.f. $\phi = NBS\cos\theta$. $e = -\frac{d\phi}{dt} = NBS\theta' \sin\theta$.
4	By applying Faraday's law: $e = -\frac{d\phi}{dt}$ with $\phi = N\vec{B}.\vec{S} = NBS\cos\theta$ where $\theta = (\vec{B}; \vec{n})$. At t = 0s; $e = 0 \Longrightarrow \frac{d\phi}{dt} = 0 \Longrightarrow \phi$ is a maximum $\Longrightarrow \cos\theta$ is a maximum. $\cos\theta = 1 \Longrightarrow \theta = 0°$.
5	$\phi = N\vec{B}.\vec{S} = NBS = 1 \times 0.55 \times 10^{-2} \times 0.2 = 1.1 \times 10^{-3} Wb$.
6	$e = e_m \sin(\omega t + \theta_0) = 0.1\sin(20\pi t)$.
7	$e = -\frac{d\phi}{dt} \Longrightarrow d\phi = -edt \Longrightarrow \phi = -\int edt = \frac{0.1}{20\pi}\cos(20\pi t)$.

Exercise 26:

Part	Answer key
1	The variation in the surface area leads to a variation in the magnetic flux. The variation in the magnetic flux gives rise to an induced e.m.f.
2	$\phi = N\vec{B}.\vec{S} = NBS\cos(\vec{B}; \vec{n}) = -B\ell x$. $e = -\frac{d\phi}{dt} = B\ell v$.
3	The circuit is closed, so it will be traversed by an induced current. $i = \frac{e}{R} = \frac{B\ell v}{R} > 0$ (the current circulates in the positive sense).
4	Point of application: midpoint of [AA']. Line of action: perpendicular to the plane containing $\overrightarrow{AA'}$ and $\vec{B}$. Direction: along the inclined plane to the left. Magnitude: $F = i\ell B\sin\alpha = \frac{B^2\ell^2 v}{R}$.
5	$\vec{W}$: Weight. $\vec{N}$: Normal reaction of a support. $\vec{F}$: The electromagnetic force.
6	Apply Newton's 2nd law : $\sum \vec{F}_{ext} = m\vec{a} \Longrightarrow \vec{W} + \vec{N} + \vec{F} = m\vec{a}$. Projection along the direction of motion : $mg\sin\alpha - F = ma \Longrightarrow a = \frac{-F + mg\sin\alpha}{m}$.
7	$a = 0 \Longrightarrow F = mg\sin\alpha$. $\frac{B^2\ell^2 v_1}{R} = mg\sin\alpha \Longrightarrow v_1 = \frac{Rmg\sin\alpha}{B^2\ell^2} = 16m/s$.

Exercise 27:

Part	Answer key
1	B varies with time; then, the magnetic flux varies with time. The variation in the magnetic flux gives rise to an induced emf e. The loop is closed; then, it carries an induced current i. $i = \frac{dq}{dt} = \frac{\Delta q}{\Delta t} = \frac{8\times10^{-3} - 0}{4 - 0} = 2 \times 10^{-3} A$.
2	B increases with time; then, the direction of the induced magnetic field $\vec{B_i}$ is opposite to $\vec{B}$. Therefore, $\vec{B_i}$ is directed into the page. By applying the right hand rule, the direction of the induced current is clockwise.

3	The general equation of a straight line passing through origin is $B = kt$ where k is the slope $k = \frac{\Delta B}{\Delta t} = \frac{12\times10^{-3}-0}{4-0} = 3\times10^{-3}T/s$. Therefore, $B = 3\times10^{-3}t$ $[SI]$.
4	$\phi = NBS\cos\theta = (1)(3\times10^{-3})(10\times10^{-4})(\cos 180°) = -3\times10^{-6}t$ $[SI]$.
5	$e = -\frac{d\phi}{dt} = 3\times10^{-6}V$.
6	$i = \frac{e}{R} \Longrightarrow R = \frac{e}{i} = \frac{3\times10^{-6}}{2\times10^{-3}} = 1.5\times10^{-3}\Omega$.

Exercise 28:

Part	Answer key
1	$\phi = NBS\cos\theta$ with $\theta = (\vec{B};\vec{n}) = 180°$. $\phi = (1)(B)(S)(\cos 180°) = -BS$.
2	The surface area AMNE varies with time; then, the magnetic flux varies with time. The variation in the magnetic flux gives rise to an induced emf e. The circuit is closed; then, it carries an induced current i. $e = -\frac{d\phi}{dt} = -\frac{d(-BS)}{dt} = B\frac{dS}{st}$. $i = \frac{e}{R} = \frac{B}{R}\frac{dS}{dt}$.
3	For $0 \leq t \leq 2s$: $i = \frac{0.2}{10}\times\frac{(500-100)\times10^{-4}}{2-0} = 4\times10^{-3}A > 0$ (positive sense). For $2s \leq t \leq 5s$: $i = \frac{0.2}{10}\times\frac{(200-500)\times10^{-4}}{5-2} = -3\times10^{-3}A < 0$ (negative sense).
4	For $0 \leq t \leq 2s$: According to Lenz's law, the circuit will oppose the increase in S by creating an induced magnetic field $\vec{B}_i$ opposite to $\vec{B}$ in direction. By applying the right hand rule, the induced current is in the positive sense. For $2s \leq t \leq 5s$: According to Lenz's law, the circuit will oppose the decrease in S by creating an induced magnetic field $\vec{B}_i$ in the same direction as $\vec{B}$. By applying the right hand rule, the induced current is the negative sense.

Exercise 29:

Part	Answer key
1	$\phi = NBS\cos\theta$ where $\theta = (\vec{B};\vec{n}) = 180°$. $\phi = (1)(B)(\ell x)(\cos 180°) = -B\ell x$.
2.1	The surface area varies with time; then, the magnetic flux varies with time. According to Faraday's law, the variation in the magnetic flux gives rise to an induced emf e. $e = -\frac{d\phi}{dt} = -\frac{d(-B\ell x)}{dt} = B\ell\frac{dx}{dt} = B\ell V$.
2.2	$u_{NM} = ri - e = -B\ell V$.
3.1	$i = \frac{e}{R} = \frac{B\ell V}{R}$. $i > 0$ (the current is directed in the positive sense).
3.2	According to Lenz's law, the electromagnetic force is opposite to the direction of the motion of the rod. By applying the right hand rule, the current is directed from N to M.

4	$F = i\ell B\sin(\overrightarrow{MN};\vec{B}) = \frac{B\ell V}{R}\ell B\sin 90° = \frac{B^2\ell^2}{R}V.$
5.1	$\sum\vec{F}_{ext} = m\vec{a} \Longrightarrow \vec{F}+\vec{N}+\vec{W} = m\frac{d\vec{V}}{dt}.$ Projection along x-axis: $-F = m\frac{dV}{dt} \Longrightarrow -\frac{B^2\ell^2}{R}V = m\frac{dV}{dt} \Longrightarrow \frac{dV}{dt}+\frac{B^2\ell^2}{mR}V = 0.$
5.2	$V = V_0e^{-\frac{t}{\tau}}$ and $\frac{dV}{dt} = -\frac{V_0}{\tau}e^{-\frac{t}{\tau}}.$ Replace V and $\frac{dV}{dt}$ in the differential equation: $-\frac{V_0}{\tau}e^{-\frac{t}{\tau}}+\frac{B^2\ell^2}{mR}V_0e^{-\frac{t}{\tau}} = 0.$ $V_0e^{-\frac{t}{\tau}}\left[-\frac{1}{\tau}+\frac{B^2\ell^2}{mR}\right] = 0.$ $-\frac{1}{\tau}+\frac{B^2\ell^2}{mR} = 0 \Longrightarrow \tau = \frac{mR}{B^2\ell^2}.$
6.1	$V_0 = 4m/s.$
6.2	At $t=\tau$; $V = 0.37V_0 = 0.37\times 4 = 1.48V.$ Graphically, $\tau = 2.5s.$ $B^2 = \frac{mR}{\tau\ell^2} = \frac{0.02\times 10}{2.5\times 0.5} = 0.16 \Longrightarrow B = 0.4T.$

Exercise 30:

Part	Answer key
1.1	The intensity of the magnetic field varies; then, the magnetic flux varies. The change in magnetic flux gives rise to an induced e.m.f. e. Since the circuit is closed, an induced current i traverses it.
1.2.1	According to Lenz's law, the coil will oppose, by repulsion, the incoming S pole of the magnet. $A \rightarrow S$ and $B \rightarrow N$. The induced magnetic field $\vec{B}_{in}$ is directed from A to B through the coil. By RHR, the induced current is directed from B to A through the coil.
1.2.2	According to Lenz's law, the coil will oppose, by attraction, the outgoing S pole of the magnet. $A \rightarrow N$ and $B \rightarrow S$. The induced magnetic field $\vec{B}_{in}$ is directed from B to A through the coil. By RHR, the induced current is directed from A to B through the coil.
2.1	$\phi = NBS\cos\theta = 500\times B\times 10\times 10^{-4}\times\cos 180° = -0.5B$ where $\theta = (\vec{B};\vec{n}).$ Faraday-Lenz Law: $e = -\frac{d\phi}{dt} = 0.5\frac{dB}{dt}$ with $\frac{dB}{dt} = \frac{\Delta B}{\Delta t}$ (linear variation). Ohm's law: $i = \frac{e}{R} = \frac{0.5}{10}\frac{dB}{dt} = 0.05\frac{\Delta B}{\Delta t}.$
2.2	$0s < t < 3s; i = 0.05\times\frac{0.3-0}{3-0} = 5\times 10^{-3}A.$ $3s < t < 5s; i = 0A.$ $5s < t < 6s; i = 0.05\times\frac{0-0.3}{6-5} = -15\times 10^{-3}A.$
2.3	$0s < t < 3s; i > 0$ (it circulates in the positive sense). $3s < t < 5s; i = 0$ (no induced current). $5s < t < 6s; i < 0$ (the current circulates in the opposite direction of the positive sense).
2.4	$0s < t < 3s$: The intensity of the magnetic field $\vec{B}$ increases. According to Lenz's law, the induced magnetic field $\vec{B}_{in}$ is opposite to $\vec{B}$.

	By RHR the current circulates in the positive sense. $3s < t < 5s$: $B = cosntant \Longrightarrow \phi = constant \Longrightarrow e = 0 \Longrightarrow i = 0A$ (no induced current). $5s < t < 6s$: The intensity of the magnetic field $\vec{B}$ decreases. According to Lenz's law, the induced magnetic field $\vec{B}_{in}$ has the same direction as $\vec{B}$. By RHR the current circulates in the opposite direction of the positive sense.

CHAPTER 4 – SELF INDUCTION
COURSE

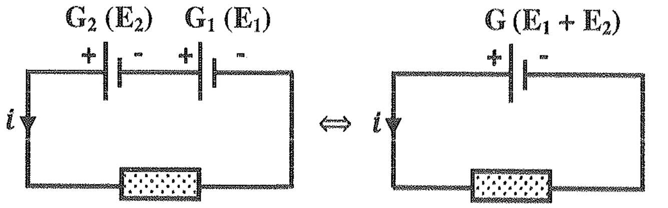
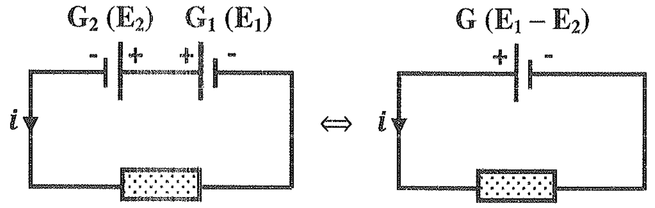
4.1- INTRODUCTION

In all examples of induced electromotive force presented so far, the magnetic field is produced by an external source such as a permanent magnet of electromagnet. However, an electromotive force can be induced in a current-carrying coil by a change in the magnetic field that the current itself produces.

4.2- PREREQUISITES

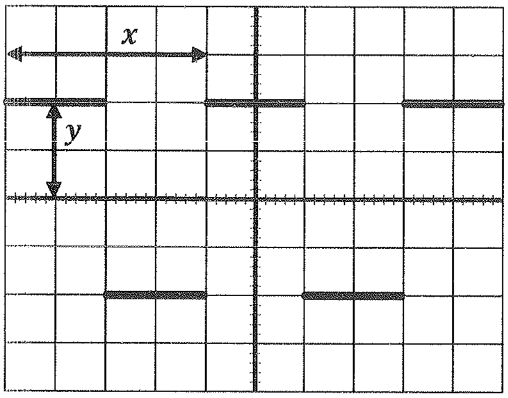
The voltage-current relation of a generator G (E; r) is $u_G = E - ri$.

Grouping of generators

Generators in series	Generators in opposition
	
G_1 and G_2 acts as generators	**G_1 acts as a generator and G_2 acts as a receiver with $E_1 > E_2$**

Square signal voltage

A square signal voltage is a non-sinusoidal periodic waveform in which the amplitude alternates at a steady frequency between fixed minimum and maximum values, with the same duration at minimum and maximum.



Characteristics of a square signal voltage

Period: $T = S_h \times x$.

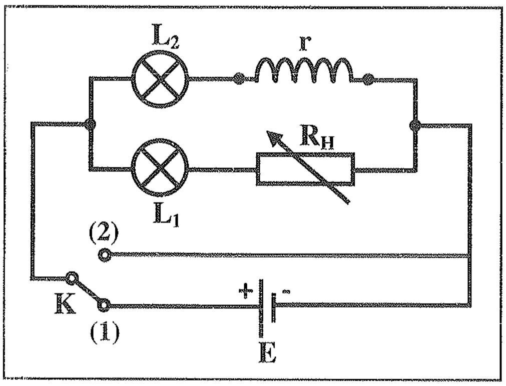
Frequency: $f = \frac{1}{T}$.

Maximum voltage: $U = S_v \times y$.

4.3- EXPERIMENTAL EVIDENCE OF THE SELF-INDUCTION PHENOMENON

Equipment

- An ideal DC generator of electromotive force E.
- Two identical light bulbs L_1 and L_2 of rated voltage slightly less than E.
- A coil of resistance r.
- A rheostat.
- A double switch K.
- Connecting wires.



Procedure and observation

- Setup the circuit diagram shown in the adjacent document.
- Adjust the rheostat to the value r.
- Turn the switch to position 1:
  Observation: L_1 glows instantly whereas L_2 glows gradually with a noticeable delay with respect to L_1. Then, L_2 glows as brightly as L_1.

- Turn the switch to position 2:
  Observation: L_1 turns off instantly, whereas L_2 turns off gradually until it turns off completely after some time.

Interpretation

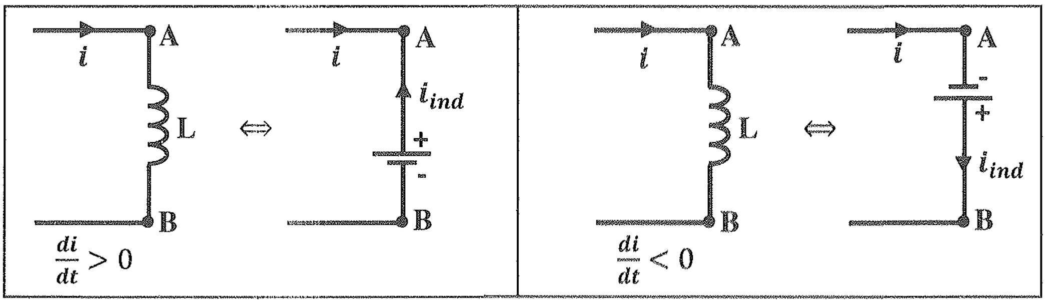
At the instant when the switch K is turned to position 1, the current rises suddenly from zero to a value "I" without any delay in the branch including the rheostat; however, in the other branch containing the coil, the current grows progressively to reach the same value "I" in the steady state.
At the instant when the switch K is switched to position 2, the current in the branch including the rheostat drops suddenly to zero without any delay; however, in the other branch containing the coil, the current decays progressively from the value "I" to zero.
In fact, during its growth or decay in the coil, a variable current creates a variable magnetic field inside the coil. Due to this current, the coil is crossed by a variable magnetic flux. This leads to an induced electromotive force, which lasts as long as the current varies, and is consistent with Lenz's law. Therefore, the induced e.m.f. (induced current) opposes the initial variation of the current and delays its establishment in the given branch.
If the circuit of the coil is closed, a self induced current exists.
Therefore, such phenomenon in which a circuit affects itself is called self-induction. The magnetic flux which crosses the coil due to its own current is called proper or the self-flux.
The electromotive force induced in the coil by the variation of its self-flux is called self-induced e.m.f.
A coil presents a systematic opposition to all **variations of the current.** It has the tendency to delay either its growth or its decay; however, a resistor opposes the current.
The phenomenon of self induction appears only in circuits including coils traversed by variable currents.
With direct currents, however, self-induction is a transient phenomenon that appears solely at the time of switching the circuit on or off.

Steady or permanent phase
Steady phase is established in a circuit when the intensity of the current traversing it remains constant, or when its variations with time are steady.

Transient phase
This is a variable phase that separates two steady phases. The duration of this phase, or the variation of the current, depends on the elements of the circuit.

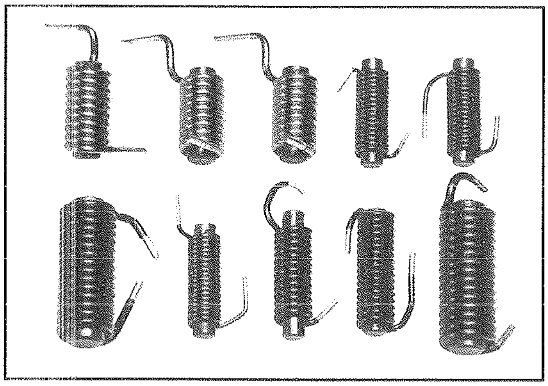
Conclusion

A variable current i passing through a coil leads to a variable magnetic flux, consequently, an induced emf is produced (Lenz's law).
If the circuit of the coil is closed, a self induced current i_{ind} is established.
The self induced current i_{ind} by self induction opposes the variation of the main current i.
- If i increases $\left(\frac{di}{dt} > 0\right)$, the induced current opposes it and the coil acts as a receiver (generator in opposition).
- If i decreases $\left(\frac{di}{dt} < 0\right)$, the induced current flows with it and the coil acts as a generator.
- If i is constant, the induced current does not exist and the coil acts as a resistor.

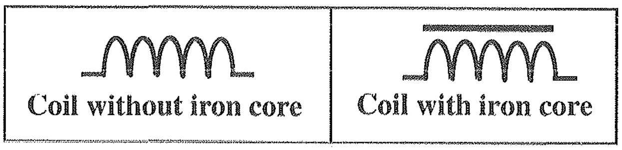


4.4- INDUCTOR OR COIL

An inductor (coil) is a device used to store magnetic energy. It consists of an electric conductor, such as a wire, that is wound into a coil.



Schematic symbol of an inductor:



Characteristics of a coil

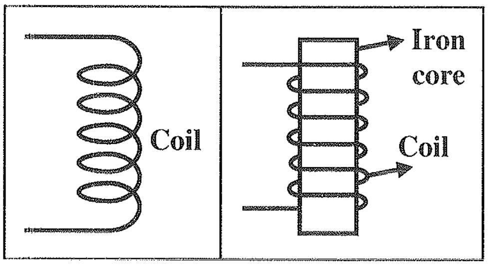
The **Internal resistance r** causes the heating of the coil while functioning due to Joule's effect.
In SI units, r is expressed in ohm [Ω].

TIP

A purely inductive coil is a coil with negligible internal resistance

The **Inductance L** is the property of the coil that causes an electromotive force to be generated by a change in the current; it depends on the nature of the coil.
In SI units, L is expressed in henry [H] where:

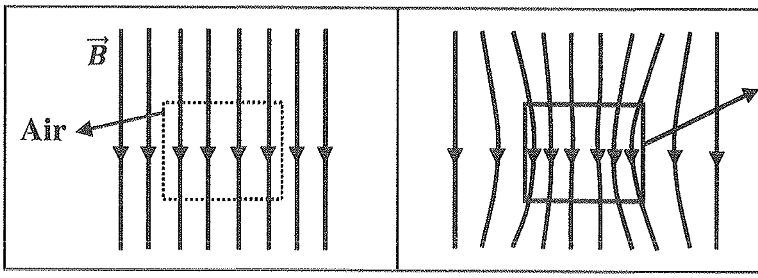
$$1H = 1Tm^2/A = 1Vs/A.$$



Inductor with soft iron core

A magnetic core is a piece of magnetic material with a high magnetic permeability used to confine and guide magnetic fields in electrical, electromechanical and magnetic devices such as electromagnets, transformers, electric motors, generators, inductors, magnetic recording heads and magnetic assemblies.

Magnetic permeability is the measure of the ability of a material to support the formation of a magnetic field within itself



Materials of high magnetic permeability generate a magnetic field $\vec{B}'$ of magnitude greater than that of $\vec{B}$

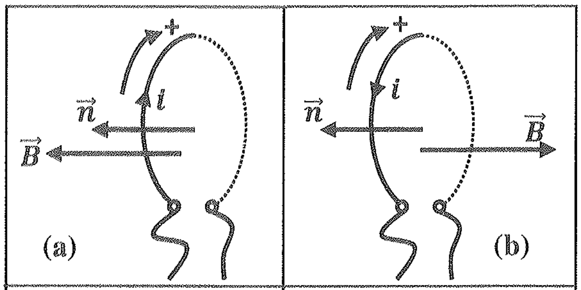
The inductance of an inductor without a soft iron core is constant and is a characteristic of the coil. The introduction of a soft iron core in a coil has the effect of increasing its inductance. In this case, the inductance is no longer a characteristic constant of the coil; it varies with the current.

4.5- SELF FLUX

The self-flux crossing a coil is proportional to the current i it carries and it is expressed by:

$$\phi = Li$$

In SI units, ϕ is expressed in webers [Wb], L in henrys [H] and i in amperes [A].
L is a positive number since ϕ and i always have the same sign as shown in the adjacent documents.



(a)	(b)
$(\vec{B};\vec{n}) = 0°$ $i > 0$ and $\phi > 0$	$(\vec{B};\vec{n}) = 180°$ $i < 0$ and $\phi < 0$

Expression of the inductance of a solenoid

A solenoid of N turns and length $\ell >> r$, where r is the cross-sectional radius of the solenoid, carries an electric current i.

The intensity of the magnetic field created inside the solenoid is: $B = \frac{\mu_0 N}{\ell} i$.

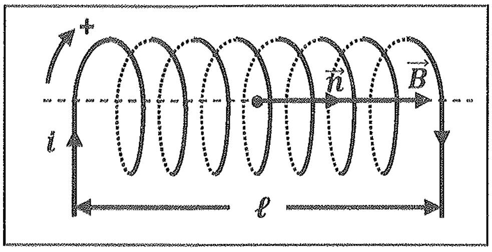
The magnetic flux crossing the solenoid is:

$\phi = N\vec{B}.\vec{S} = NBS\cos\theta$ where $\theta = (\vec{B};\vec{n}) = 0$.

$\phi = (N)\left(\frac{\mu_0 N}{\ell} i\right)(\pi r^2)(\cos 0°) = \frac{\mu_0 N^2 \pi r^2}{\ell} i.$

The magnetic self-flux through the solenoid is: $\phi = Li$.

$\frac{\mu_0 N^2 \pi r^2}{\ell} i = Li \Longrightarrow L = \frac{\mu_0 N^2 \pi r^2}{\ell}.$



4.6- EXPRESSION FOR THE SELF-INDUCED ELECTROMOTIVE FORCE

Any coil of inductance L, carrying a variable current i, is the seat of a self-induced e.m.f. e whose expression is given by Faraday's relation:

$$e = -\frac{d\phi}{dt}$$

But $\phi = Li$, therefore:

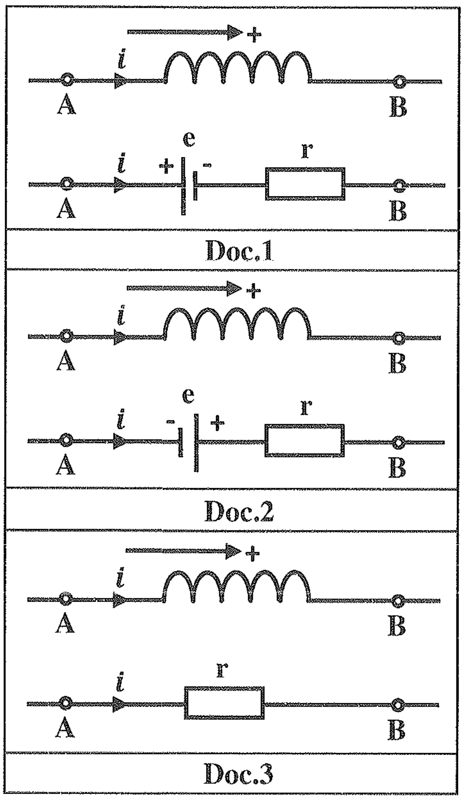
$$e = -\frac{d(Li)}{dt}$$

For a coil without a soft iron core, the inductance L is constant, therefore:

$$e = -L\frac{di}{dt}$$

In SI units, e is expressed in volts [V], L in henrys [H], i in amperes [A], and t in seconds [t].
Since the inductance L is a positive quantity, e and $\frac{di}{dt}$ have always opposite signs.
Consequently, the self-induced e.m.f. e complies with Lenz's law and always opposes the variation of i.

4.7- OHM'S LAW APPLIED TO A COIL



A coil, of internal resistance r and inductance L is placed between points A and B in a circuit and carries a variable current i.
While orienting the circuit from A to B through the coil, the potential difference cross the terminals of the coil is $u_{AB} = ri - e$ where e is the self induced emf.
But $e = -L\frac{di}{dt}$, therefore: $u_{AB} = ri + L\frac{di}{dt}$.
A coil (L; r) is equivalent to a series combination of a resistor of resistance r and a battery of electromotive force $e = -L\frac{di}{dt}$.

Role of a coil

- If the intensity of the electric current i increases with time, the coil acts as a receiver (document 1).

$$i \text{ increases} \Longrightarrow \frac{di}{dt} > 0 \Longrightarrow e < 0$$
$$\text{So } i > 0 \text{ and } e < 0 \Longrightarrow e.i < 0$$

- If the intensity of the electric current i decreases with time, the coil acts as a generator (document 2).

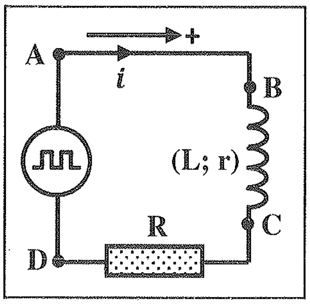
$$i \text{ decreases} \Longrightarrow \frac{di}{dt} < 0 \Longrightarrow e > 0$$
$$\text{So } i > 0 \text{ and } e > 0 \Longrightarrow e.i > 0$$

- The intensity of the electric current is constant with respect to time, the coil acts as a pure resistor (document 3).

$$i = constant \Longrightarrow \frac{di}{dt} = 0 \Longrightarrow e = 0$$

Therefore, $u_{AB} = ri$ (Ohm's law across the terminals of a resistor)

4.8- THE GROWTH AND THE DECAY OF CURRENT IN AN R-L SERIES CIRCUIT

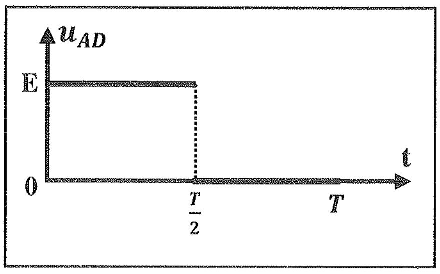


The adjacent circuit diagram consists of:
- A low frequency generator of negligible internal resistance displaying a square signal of period T, such that:

$$u_{AD} = \begin{cases} E & for\ 0 \leq t \leq \frac{T}{2} \\ 0 & for \frac{T}{2} \leq t \leq T \end{cases}$$

- A resistor of resistance R.
- A coil of inductance L and internal resistance r.
- Connecting wires of negligible resistance.

Growth of current



For $t \in [0s; \frac{T}{2}s]$, $u_{AD} = u_G = E$.
Law of addition of voltages in series connection:
$u_{AD} = u_{AB} + u_{BC} + u_{CD}$.
The coil is oriented positively from B to C. The voltage across the terminals of the coil is:
$u_{BC} = u_L = ri - e = ri + L\frac{di}{dt}$.
The voltage across the terminals of the resistor is: $u_{CD} = u_R = Ri$.

Differential equation in i

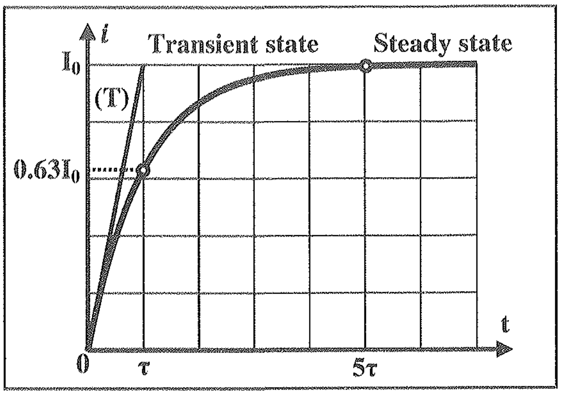
Law of addition of voltages in series connection:

$u_{AD} = u_{AB} + u_{BC} + u_{CD} \Rightarrow u_G = u_L + u_R$.

$E = ri + L\frac{di}{dt} + Ri \Rightarrow E = L\frac{di}{dt} + (R + r)i$.

$E = L\frac{di}{dt} + R_{eq}\, i$ with $R_{eq} = R + r$.

$\frac{di}{dt} + \frac{R_{eq}}{L} i = \frac{E}{L}$ (1^{st} order differential equation in "i" during its growth under square voltage)



The **time constant τ** is **the** time needed for the current to reach 63% of its maximum value during its growth. The SI unit of τ is [s].
$\tau = \frac{L}{R_{eq}} \Rightarrow \frac{[H]}{[\Omega]} = \frac{[V].[s]}{[A].[\Omega]} = \frac{[\Omega].[s]}{[\Omega]} = [s]$.

Expression of i as a function of time

The solution of the differential equation has the form:

$i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$ where the maximum value of the current is $I_0 = \frac{E}{R_{eq}}$ and the time constant is $\tau = \frac{L}{R_{eq}}$.

Graph of i versus time

t	0	τ	5τ or $t \to \infty$
i	$I_0(1 - e^0) = 0$	$I_0(1 - e^{-1}) = 0.63I_0$	$I_0(1 - e^{-5}) = 0.99I_0 \simeq I_0$ or $I_0(1 - e^{-\infty}) = I_0$

The current i increases exponentially with time.

Determination of the expressions of I_0 and τ.

$i = I_0\left(1 - e^{-\frac{t}{\tau}}\right) = I_0 - I_0 e^{-\frac{t}{\tau}}$.

$\frac{di}{dt} = 0 - I_0\left(-\frac{1}{\tau}\right)e^{-\frac{t}{\tau}} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}}$.

Replace i and $\frac{di}{dt}$ in the differential equation: $E = L\frac{di}{dt} + R_{eq}\, i$.

$E = \frac{LI_0}{\tau}e^{-\frac{t}{\tau}} + R_{eq}I_0 - R_{eq}I_0e^{-\frac{t}{\tau}} \Rightarrow E = R_{eq}I_0 + I_0e^{-\frac{t}{\tau}}\left(\frac{L}{\tau} - R_{eq}\right)$.

By identification: $E = R_{eq}I_0 \Rightarrow I_0 = \frac{E}{R_{eq}}$ and $\frac{L}{\tau} - R_{eq} = 0 \Rightarrow \tau = \frac{L}{R_{eq}}$ with $I_0e^{-\frac{t}{\tau}} = 0$ *not true* $\forall\, t$.

Steady state current

$E = L\frac{di}{dt} + R_{eq}i$.
At the steady state:
$i = I_0 = constant \Rightarrow \frac{di}{dt} = 0$.
$E = 0 + R_{eq}I_0 \Rightarrow I_0 = \frac{E}{R_{eq}}$.

Determination of the expression of τ using the tangent method

The time constant τ is the abscissa t_1 of the point of intersection F between the tangent (T) to the curve at t = 0s and the asymptote.

Equation of the asymptote: $i = I_0$.

Equation of (T): the equation of a straight line passing through origin is: $i = at$ where a is the slope.

$a = \left.\frac{di}{dt}\right|_{t=0} = \frac{I_0}{\tau}e^0 = \frac{I_0}{\tau}$.

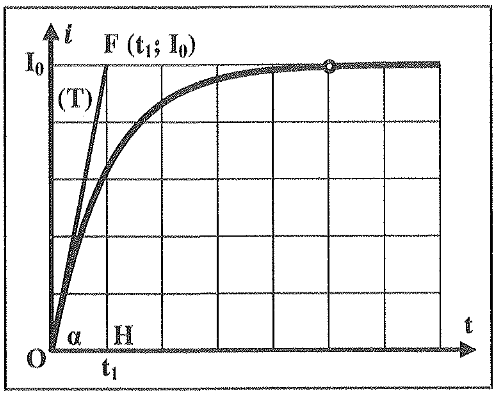
Therefore, $i = \frac{I_0}{\tau}t$.

$\frac{I_0}{\tau}t_1 = I_0 \Rightarrow t_1 = \tau$.

Or: graphically, the slope of the tangent (T) at t = 0 is:

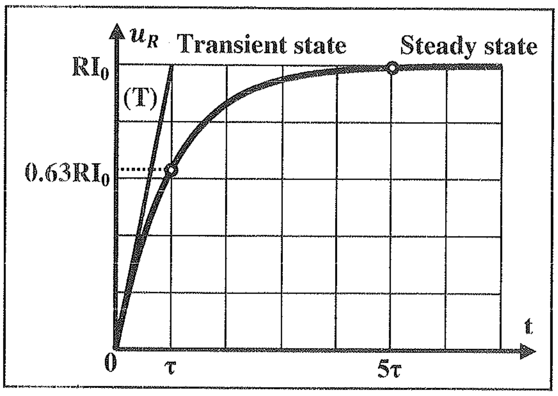
$a = \tan\alpha = \frac{HF}{OH} = \frac{I_0}{t_1}$.

$a = \tan\alpha = \left.\frac{di}{dt}\right|_{t=0} \Rightarrow \frac{I_0}{t_1} = \frac{I_0}{\tau} \Rightarrow t_1 = \tau$.



ATTENTION

In an RL series circuit, the time constant τ does not depend on E but it depends on the inductance L and the total resistance R_{eq} of the circuit



Differential equation in u_R

Law of addition of voltages in series connection:
$u_{AD} = u_{AB} + u_{BC} + u_{CD} \Longrightarrow u_G = u_L + u_R$.
$E = ri + L\frac{di}{dt} + Ri \Longrightarrow E = L\frac{di}{dt} + (R + r)i$.
$E = L\frac{di}{dt} + R_{eq}i$ with $R_{eq} = R + r$.
$E = \frac{L}{R}\frac{du_R}{dt} + \frac{R_{eq}}{R}u_R$ with $u_R = Ri \Longrightarrow i = \frac{u_R}{R} \Longrightarrow \frac{di}{dt} = \frac{1}{R}\frac{du_R}{dt}$
$\frac{RE}{L} = \frac{du_R}{dt} + \frac{R_{eq}}{L}u_R$ (First order differential equation in "u_R" during the growth of current).

Expression of u_R as a function of time

The solution of the differential equation has the form:
$u_R = U_{R_0}\left(1 - e^{-\frac{t}{\tau}}\right) = RI_0\left(1 - e^{-\frac{t}{\tau}}\right)$ where $U_{R_0} = RI_0$ is the maximum voltage across the terminals of the resistor.

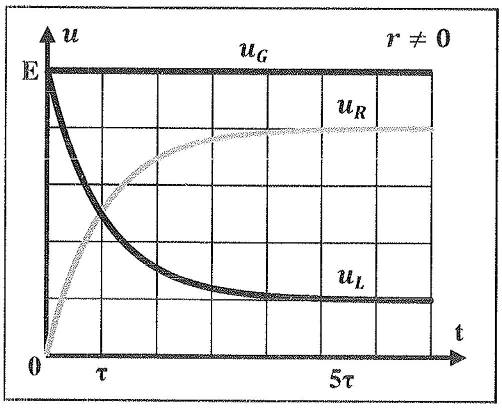
ATTENTION

In an RL series circuit, the maximum current I_0 does not depend on the inductance L of the coil but it depends on the total resistance R_{eq} of the circuit.

Graph of u_R versus time

t	0	τ	5τ or $t \to \infty$
u_R	$RI_0(1 - e^0) = 0$	$RI_0(1 - e^{-1}) = 0.63RI_0$	$RI_0(1 - e^{-5}) = 0.99RI_0 \simeq I_0$ or $RI_0(1 - e^{-\infty}) = RI_0$

The resistor's voltage u_R increases exponentially with time.



Voltage across the coil

$u_L = E - u_R = E - RI_0\left(1 - e^{-\frac{t}{\tau}}\right) = (R + r)I_0 - RI_0 + RI_0e^{-\frac{t}{\tau}}$.

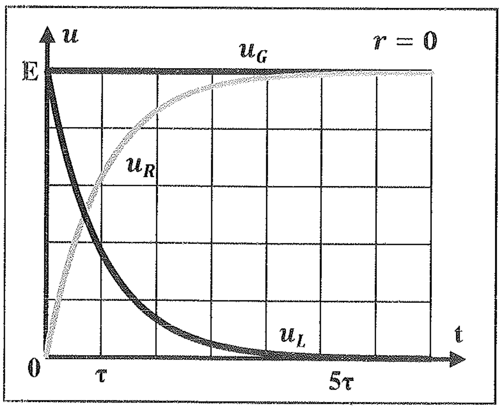
Therefore, $u_L = rI_0 + RI_0e^{-\frac{t}{\tau}}$.

Graph of u_L versus time

t	0	τ	5τ or $t \to \infty$
u_L	$E - 0 = 0$	$E - 0.63RI_0$	$E - RI_0 = rI_0$

Differential equation in u_L (case of a purely inductive coil):

Law of addition of voltages in series connection:
$u_{AD} = u_{AB} + u_{BC} + u_{CD} \Longrightarrow u_G = u_L + u_R$.
$E = u_L + Ri$ with $u_R = Ri$.
Derive both sides with respect to time:
$0 = \frac{du_L}{dt} + R\frac{di}{dt} \Longrightarrow \frac{du_L}{dt} + \frac{R}{L}u_L = 0$ (First order differential equation in "u_L" during the growth of current).



Expression of u_L as a function of time

The solution of the differential equation has the form:
$u_L = U_{L_0}e^{-\frac{t}{\tau}} = Ee^{-\frac{t}{\tau}}$ where $U_{L_0} = E$ is the maximum voltage across the terminals of the coil and $\tau = \frac{L}{R}$ is the time constant.

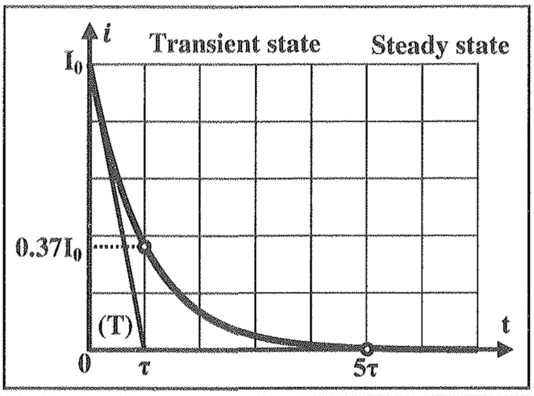
The expression of the current becomes $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$ and that of the voltage across the terminals of the resistor $u_R = E\left(1 - e^{-\frac{t}{\tau}}\right)$ where $I_0 = \frac{E}{R}$ and $\tau = \frac{L}{R}$.

Decay of current

For $t \in [\frac{T}{2}s; Ts]$, $u_{AD} = u_G = 0$.

$u_{BC} = u_L = ri + L\frac{di}{dt}$ and $u_{CD} = u_R = Ri$.

N.B: the decay of the current starts at a new origin of time t = 0.



The time constant τ is the time needed for the current to reach 37% of its initial value during the decay process

Differential equation in *i*

Law of addition of voltages in series connection:

$u_{AD} = u_{AB} + u_{BC} + u_{CD} \Rightarrow u_G = u_L + u_R$.

$0 = ri + L\frac{di}{dt} + Ri \Rightarrow 0 = L\frac{di}{dt} + (R + r)i$.

$0 = L\frac{di}{dt} + R_{eq}\, i$ with $R_{eq} = R + r$.

$\frac{di}{dt} + \frac{R_{eq}}{L}i = 0$ (First order differential equation in "i" during the decay of current).

Expression of *i* as a function of time

The solution of the differential equation has the form:

$i = I_0 e^{-\frac{t}{\tau}}$ where $I_0 = \frac{E}{R_{eq}}$ and $\tau = \frac{L}{R_{eq}}$.

Graph of *i* versus time

t	0	τ	5τ or $t \to \infty$
i	$I_0 e^0 = I_0$	$I_0 e^{-1} = 0.37I_0$	$I_0 e^{-5} \simeq 0$ or $I_0 e^{-\infty} = 0$

The current i decreases exponentially with time.

Determination of the expressions of I_0 and τ

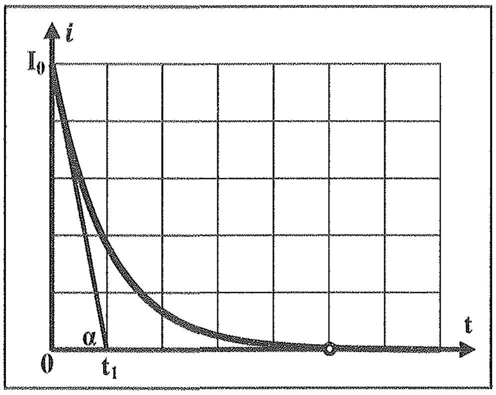
$i = I_0 e^{-\frac{t}{\tau}}$ and $\frac{di}{dt} = -\frac{I_0}{\tau}e^{-\frac{t}{\tau}}$.

Replace i and $\frac{di}{dt}$ in the differential equation: $-\frac{LI_0}{\tau}e^{-\frac{t}{\tau}} + R_{eq}I_0 e^{-\frac{t}{\tau}} = 0 \Rightarrow I_0 e^{-\frac{t}{\tau}}\left(-\frac{L}{\tau} + R_{eq}\right) = 0$.

$I_0 e^{-\frac{t}{\tau}} \neq 0\ \forall\, t \Rightarrow -\frac{L}{\tau} + R_{eq} = 0 \Rightarrow \tau = \frac{L}{R_{eq}}$.

Direction of *i*

During the current decay, the coil, according to Lenz law, produces a current in the same direction as before.



The time constant τ of the R-L series circuit is the time needed by the current through the coil to grow or decay 63% of its maximum value.

Tangent to the curve of *i* at t = 0s

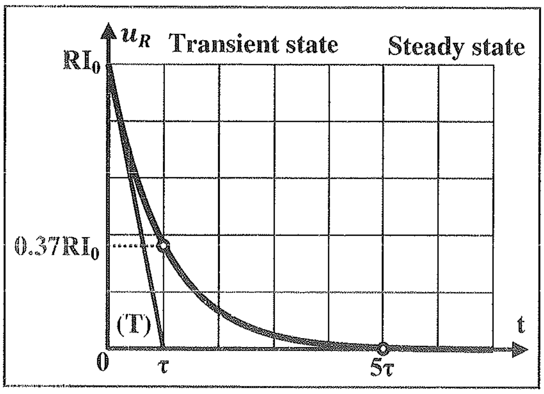
$\left.\frac{di}{dt}\right|_{t=0} = -\frac{I_0}{\tau}$ = slope to the tangent to the curve of i at t = 0s.

Graphically, slope of the tangent to the curve of i at t = 0s is $-\tan\alpha = -\frac{I_0}{t_1}$.

By comparison: $-\frac{I_0}{\tau} = -\frac{I_0}{t_1} \Rightarrow t_1 = \tau$.

The time constant τ is the abscissa of the point of intersection between the tangent to the curve of i at t = 0s and the time axis.

Diferential equation in u_R



Law of addition of voltages in series connection:
$u_{AD} = u_{AB} + u_{BC} + u_{CD} \Longrightarrow u_G = u_L + u_R$.
$0 = ri + L\frac{di}{dt} + Ri \Longrightarrow 0 = L\frac{di}{dt} + (R+r)i$.
$0 = L\frac{di}{dt} + R_{eq}\,i$ with $R_{eq} = R + r$.
$0 = \frac{L}{R}\frac{du_R}{dt} + \frac{R_{eq}}{R}u_R$ with $u_R = Ri \Longrightarrow i = \frac{u_R}{R} \Longrightarrow \frac{di}{dt} = \frac{1}{R}\frac{du_R}{dt}$
$0 = \frac{du_R}{dt} + \frac{R_{eq}}{L}u_R$ (First order differential equation in "u_R" during the decay of current).

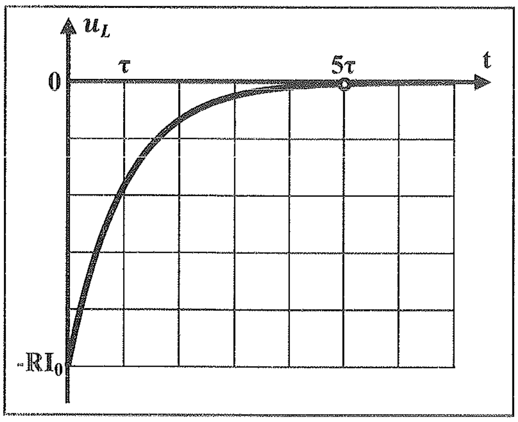
Expression of u_R as a function of time

The solution of the differential equation has the form:
$u_R = u_{R_0}e^{-\frac{t}{\tau}} = RI_0e^{-\frac{t}{\tau}}$ where $u_{R_0} = RI_0$ is the maximum voltage across the terminals of the resistor.

Graph of u_R versus time

t	0	τ	5τ or $t \to \infty$
u_R	$RI_0e^0 = RI_0$	$RI_0e^{-1} = 0.37RI_0$	$RI_0e^{-5} \simeq 0$ or $RI_0e^{-\infty} = 0$

The resistor's voltage u_R decreases exponentially with time.

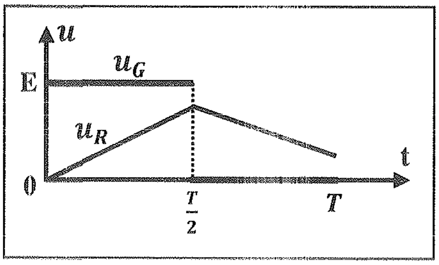


Expression of the voltage across the coil

$u_L = ri + L\frac{di}{dt} = rI_0e^{-\frac{t}{\tau}} - \frac{LI_0}{\tau}e^{-\frac{t}{\tau}} = I_0e^{-\frac{t}{\tau}}\left(r - \frac{L}{\tau}\right)$.
$u_L = I_0e^{-\frac{t}{\tau}}[r - (R+r)] = -RI_0e^{-\frac{t}{\tau}}$ where $\frac{L}{\tau} = R_{eq} = R + r$.
Or $u_G = u_L + u_R \Longrightarrow 0 = u_L + u_R \Longrightarrow u_L = -u_R = -RI_0e^{-\frac{t}{\tau}}$.

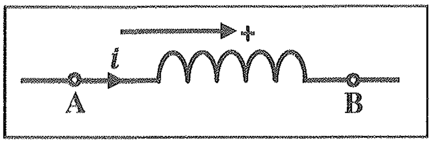
In an R-L series circuit under a square signal voltage of period T:

- If $5\tau > \frac{T}{2}$, the current neither reaches its maximum value nor its minimum value. The steady state is not reached in this case.
- If $5\tau \gg \frac{T}{2}$, the current i varies linearly with time where $i = \frac{I_0}{\tau}t$.



4.9- MAGNETIC ENERGY STORED IN A COIL

A coil, of internal resistance r and inductance L, is placed between points A and B in a circuit and carries a variable current i.



While orienting the circuit from A to B through the coil, the potential difference cross the terminals of the coil is $u_{AB} = ri - e = ri + L\frac{di}{dt}$ where e is the self-induced emf.

Multiply both sides by i: $u_{AB}i = ri^2 - ei = ri^2 + L\frac{di}{dt}i$.

The electric power received by the coil, during the growth of current, at the instant t, is: $P_{rec} = u_{AB}i$.

The power dissipated in the coil due to Joule's effect (in the form of heat), at the instant t, is: $P_{dis} = ri^2$.

The magnetic power associated with the self induction phenomenon that appear in the coil during the variation of i is: $P_{mag} = Li\frac{di}{dt} = \frac{d}{dt}\left(\frac{1}{2}Li^2\right) = -ei$.

Therefore, $P_{rec} = P_{dis} + P_{mag}$.

The magnetic power is the rate at which the magnetic energy is stored in the coil.
$$P_{mag} = \frac{d}{dt}\left(\frac{1}{2}Li^2\right)$$

The magnetic energy stored in the coil is derived as follows:

$P_{mag} = \frac{dW_{mag}}{dt} \Longrightarrow dW_{mag} = P_{mag}\,dt$.

$W_{mag} = \int \frac{d}{dt}\left(\frac{1}{2}Li^2\right)dt = \frac{1}{2}Li^2 + cst$.

For $i = 0A$; $W_{mag} = 0 \Longrightarrow cst = 0$.

Therefore, $W_{mag} = \frac{1}{2}Li^2$.

In SI units, W_{mag} is expressed in [J], L in [H] and i in [A].

During the growth of current from zero to I_0, the magnetic energy increases progressively from zero to $W_0 = \frac{1}{2}LI_0^2$ at the steady state. The coil is storing magnetic energy and acts as a receiver; consequently, it delays the establishment of the current.

During the decay of the current, the magnetic energy decreases progressively. The coil acts as a generator since it supplies (restores its energy) to the circuit; consequently, the coil prolongs the duration of traversing the current in the circuit.

4.10- VISUALIZATION OF THE GROWTH AND DECAY OF THE CURRENT IN AN R-L SERIES CIRCUIT

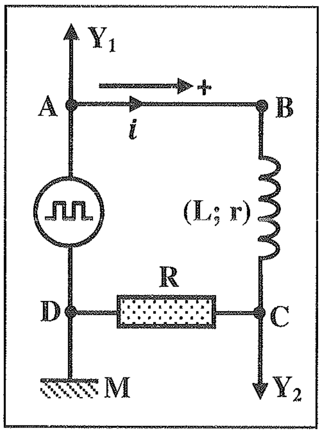
Equipment

- Low frequency generator of negligible internal resistance.
- A resistor of resistance R = 120Ω.
- A coil of inductance L = 60mH and internal resistance r = 30Ω.
- Connecting wires of negligible resistance.
- Oscilloscope.

Procedure

- Set up the circuit shown in the adjacent document.
- Adjust the low frequency generator to give a square signal of period T = 4ms, such that:

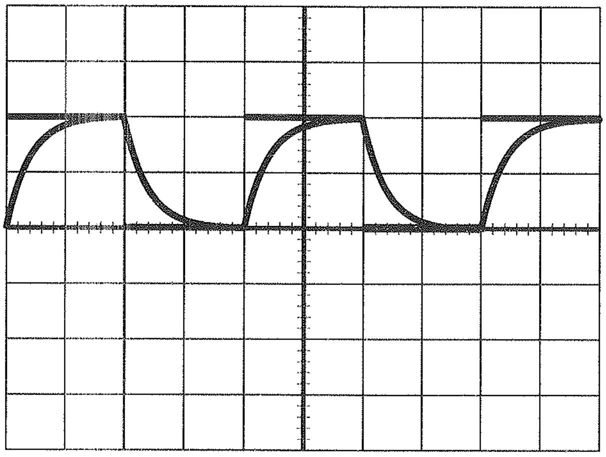
$$u_{AD} = \begin{cases} E = 2V & for\ 0 \leq t \leq \frac{T}{2} \\ 0 & for \frac{T}{2} \leq t \leq T \end{cases}$$



- Connect an oscilloscope that displays on channel Y_1 the voltage u_{AD} across the terminals of the generator and on channel Y_2 the voltage u_{CD} across the terminals of the resistor.
  Ohm's law: $u_{CD} = Ri$ with R is a positive constant; then, the graph of u_{CD} is used to represent that of the current i (u_{CD} is the image of i)
- Adjust the oscilloscope to:
  Horizontal sensitivity: $S_h = 1ms/div.$
  Vertical sensitivity on both channels: $S_v = 1V/div.$

Observation

The oscilloscope's screen is represented in the adjacent document.



Interpretation

When the voltage across the generator jumps from 0V to 2V, the electric current traverses the coil with a certain delay.

Similarly, when the voltage across the generator drops from 2V to 0V, the current in the coil decreases gradually. This leads to a self-induced emf in the coil whose effect, according to Lenz's law, opposes the reduction of the electric current in the circuit. This explains the delay in the reduction of the current.

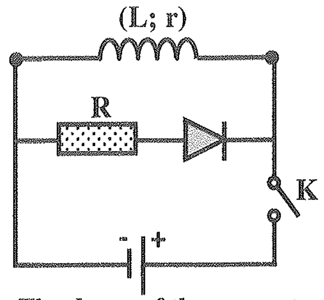
Conclusion

A coil presents a systematic opposition to all variations of the current. It has the tendency to delay either its growth or its decay.

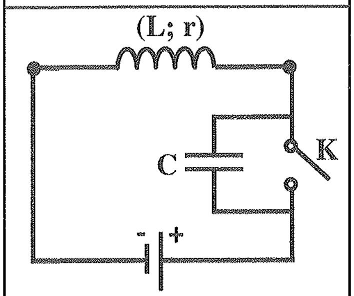
The phenomenon of self-induction appears only in the circuits including coils traversed by variable currents. With direct currents, however, self-induction is a transient phenomenon that appears slowly at the time of switching the circuit on and off.

4.11- SPARKS DUE TO "SWITCHING OFF" A CIRCUIT

When a circuit of large inductance is opened abruptly, a spark appears at the switch contacts.
Similarly, strong sparks are produced when we suddenly disconnect domestic appliances, including electric motors and transformers (those including coils), from the mains.
This phenomenon is interpreted by the presence of an excess voltage (or over-voltage) across the terminals of the switch at the instant when the circuit is opened.
The origin of this over-voltage is the self-induced emf that takes place in the circuit. This emf increases as the inductance of the coil increases, or with the variations of the current with respect to time.
The sparks produced may in the long run destroy the contacts of switches and can even destroy the brushes in generators and motors or break the insulator of the coil. We avoid such damage by connecting a resistor and a diode in parallel with the terminals of the coil, or a capacitor across the terminals of the switch.
When we carry out experiments on electric circuits, including coils, it is necessary to avoid touching the terminals of coils during transient decays. Terminals of such coils can be seat to powerful and dangerous self-induced emf's.



The decay of the current occurs through R.



There is an exchange of energy between the coil and the capacitor. The current decays through r.

CHAPTER 4 – SELF INDUCTION
EXERCISES AND PROBLEMS

Exercise 1*:

A coil C (50mH; 10Ω) is connected to an ideal battery of electromotive force E = 10V.

1- What do 50mH and 10Ω represent for the coil?
2- Calculate, at the steady state, the value of the electric current carried by the circuit.
3- Calculate the maximum energy stored in the coil. Indicate its form.

Exercise 2:

The expression of the current carried by a coil, of inductance L = 60mH and internal resistance r = 20Ω, as a function of time is:

$$i = 0.2t + 1\ [SI].$$

1- Find, as a function of time, the expression of the self-flux crossing the coil.
2- Determine the value of the self-induced electromotive force.

Exercise 3*:

Derive the expression of the inductance L of a solenoid of N turns and length $\ell \gg r$ where r is the cross-sectional radius of the coil. The magnetic permeability of vacuum is $\mu_0 = 4\pi \times 10^{-7} SI$.

Exercise 4*:

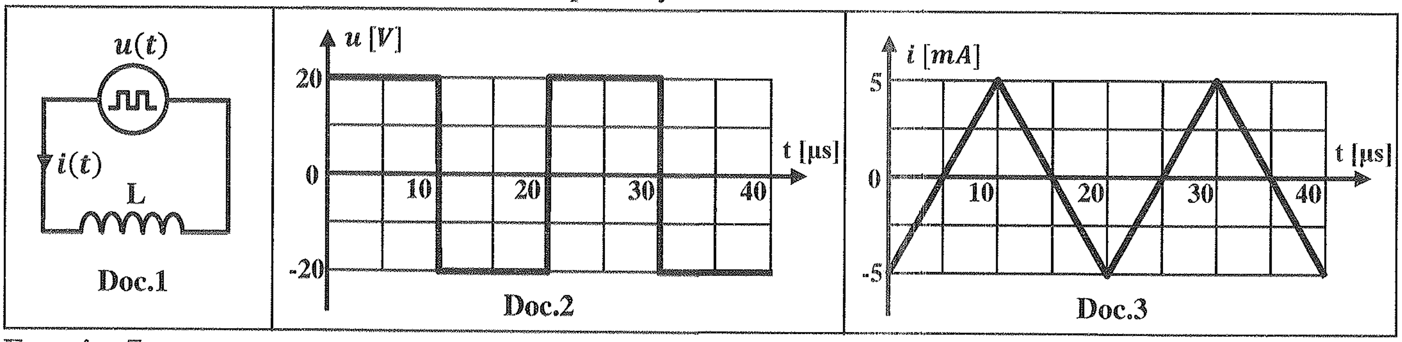
Calculate the inductance of a solenoid that consists of N = 2500 turns of radius $r = 8cm$ and its length is $\ell = 20cm$.

Exercise 5*:

A coil carries a current $i = 3A$; consequently, a magnetic flux $\phi = 0.084Wb$ is produced. Calculate the value of its inductance L.

Exercise 6:

A pure inductor is subjected to a square voltage as shown in document 1. The voltage $u(t)$ and the current $i(t)$ are shown in the documents 2 and 3 respctively. Determine the value of the inductance L of the inductor.



Doc.1 Doc.2 Doc.3

Exercise 7:

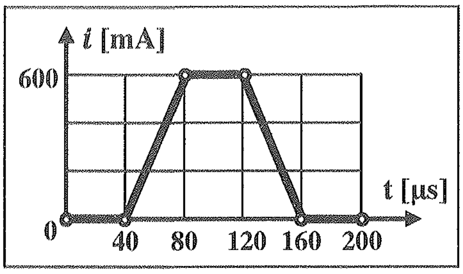
Consider a pure inductor of inductance $L = 200mH$. The inductor is connected to a switch and a generator delivering a constant voltage $U = 12V$. The switch is closed at $t_0 = 0s$. At an instant t, the inductor stores energy $W = 10\mu J$.

1- Sketch the shape of the graph that shows the variation of the current i as a function of time.
2- Calculate the value of the current i at instant t.
3- Calculate the value of t.

Exercise 8*:

Consider a purely inductive coil of inductance $L = 0.8H$. The variations of the current i traversing the coil as a function of time t are given in the adjacent document.

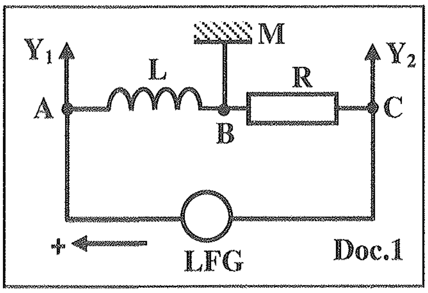
1- Calculate the voltage across the coil between 40μs and 80μs.
2- Calculate the voltage across the coil between 120μs and 160μs.
3- Trace the graph of the voltage across the coil between 0μs to 200μs.

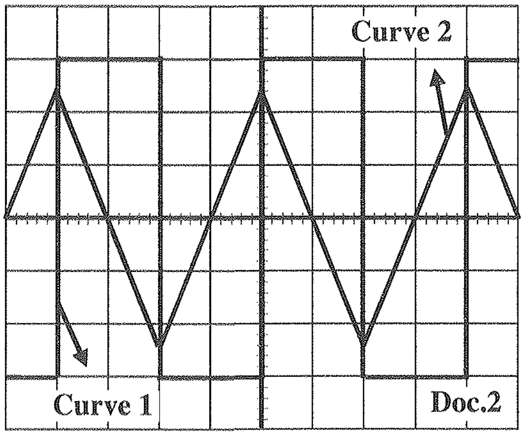


Exercise 9:

The electric circuit in document 1 consists of a purely inductive coil L, a resistor of resistance R = 500Ω and a low frequency generator (L.F.G) delivering a triangular voltage.

At an instant t, the circuit carries a triangular current i. We display, using an oscilloscope, the voltage u_{AB} on the channel Y_1 and the voltage u_{CB} on the channel Y_2 (document 1). The waveforms are represented in document 2.





Given:

- The vertical sensitivity on both channels is: 2V/div.
- The horizontal sensitivity (time base) is: 0.2ms/div.
- The horizontal center line of the oscillogram corresponds to 0V.

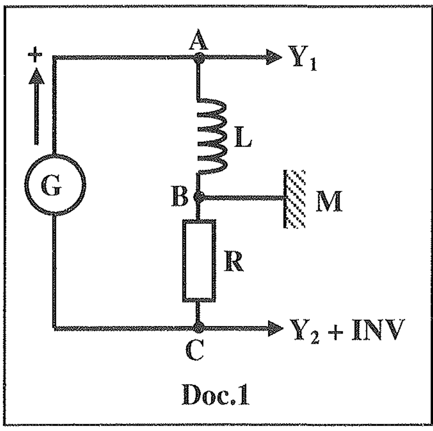
Answer by true or false. Justify your answer.

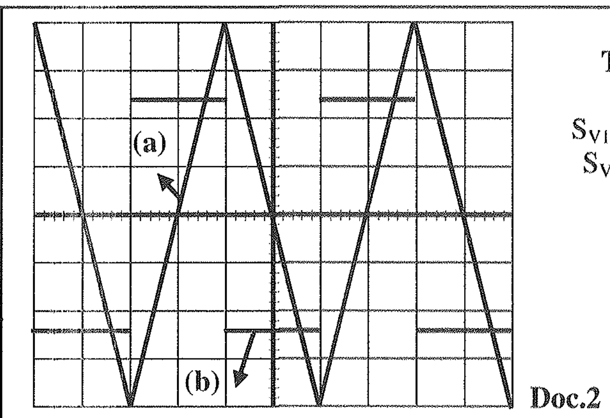
1- Curve 1 in document 2 shows the variation of the voltage across the resistor.
2- The voltage across the terminals of the coil is given by the relation: $u_{AB} = L\frac{di}{dt}$.
3- The amplitude of the voltage across the coil is 5V.
4- The value of the inductance is $L = 0.125H$.

Exercise 10*:

A coil, of inductance L and negligible internal resistance, and a resistor of resistance R = 100Ω are connected in series with a low frequency generator providing an alternating triangular voltage. The circuit thus carries a triangular current i.

An oscilloscope is used to visualize the voltage $u_{AB} = u_L$ across the coil on channel Y_1 and the voltage $u_{BC} = u_R$ across the resistor on channel Y_2.





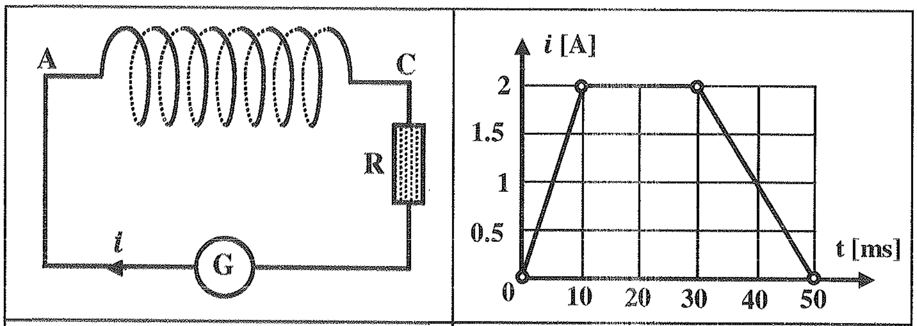
The adjustments of the oscilloscope are:

$S_{V1} = 0.1V/div$ on channel Y_1
$S_{V2} = 1V/div$ on channel Y_2
$S_h = 5ms/div$

1- Which waveform (a) or (b) corresponds to the voltage across the resistor?
2- Establish a relation between u_L and u_R.
3- Referring to document 2, determine the value of L.

Exercise 11:

A coil of inductance L = 0.02H and internal resistance r = 10Ω is connected in series with a resistor of resistance R across a generator G (Document 1). The coil thus carries a current i that varies with time as shown in document 2.



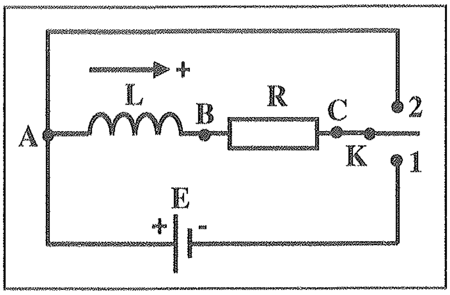
Doc.1 | Doc.2

1- Give the name of the physical phenomenon that takes place in the coil.
2- For each of the three intervals: [0; 10ms], [10ms; 30ms], and [30ms; 50ms]. Determine:
   2.1- the expression of the self flux,
   2.2- the value of the self induced electromotive force.
   2.3- the expression of the voltage u_{AC}.
3- Specify the interval when the coil:
   3.1- acts as a generator,
   3.2- acts as a receiver.

Exercise 12:

A series R-L circuit consists of a coil ($L = 20mH, r = 0\Omega$), a lamp considered as a pure resistor of resistance $R = 5\Omega$, an ideal battery of e.m.f E = 10V and a switch K.
1- Draw a schematic diagram of the circuit.
2- At t = 0s, K is closed.
   2.1- Explain what happens to the brightness of the lamp between t = 0s and the instant at which the steady state is attained.
   2.2- Determine the value of the maximum current I_0 attained in the circuit.
3- Calculate the time constant τ of the circuit.
4- Deduce the time t at the end of which the steady state is practically attained
5- Establish the differential equation that governs the growth of the current in the circuit.
6- Verify that the expression $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$ is a solution of the differential equation.
7- Calculate the current in the circuit at t = 8ms.
8- Calculate the voltage across the coil after one time constant has elapsed.
9- At an instant t, the current i in the circuit is 80 percent of I_0. Determine:
   9.1- the rate at which the current is increasing,
   9.2- the rate at which energy is being stored in the coil.

Exercise 13:



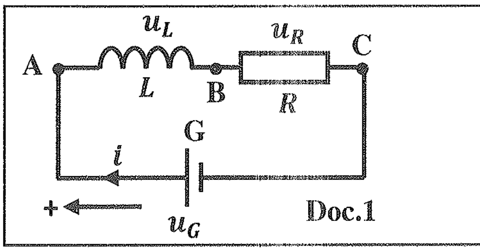
The circuit shown in the adjacent document consists of a purely inductive coil of inductance L = 0.2H, a resistor of resistance R = 100Ω and a DC generator of e.m.f E = 12V.

1- In the first experiment, the switch is in position 1.
 1.1- Calculate the current I_0 at steady state.
 1.2- Calculate the magnetic energy stored in the coil at instant $t = \tau$ and at steady state.

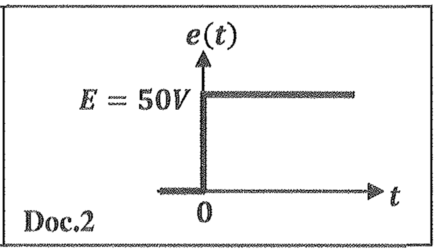
2- In the second experiment, the switch is switched suddenly to position 2.
 2.1- Establish the differential equation of the current i during its decay through R.
 2.2- Write the expression of the solution of the obtained differential equation.
 2.3- Draw the graph representing the variation of the current during its decay as a function of time.
 2.4- Determine the value of the interval of time during which the current i loses 63% of its maximum value I_0.

Exercise 14:

The objective of this exercise is to study the effect of an inductor on the form of current.
The electric circuit of document 1 is formed of a coil of inductance L and negligible resistance, a resistor of resistance R and a generator (G) that provides across its terminals a direct voltage $u_G = E = 50V$ (document 2). At an instant t, the circuit carries a current i.



Doc.1



Doc.2

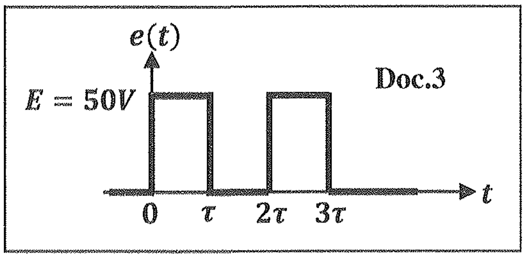
1- Derive the differential equation that governs the variation of the current i as a function of time.

2- The solution of the differential equation has the form $i = A + Be^{-\frac{t}{\tau}}$. Determine the constants A, B and the time constant τ as a function of E, R and L (Given: at t = 0s; $i = 0A$)

3- In the table below, the values of i at different instants t are given.

t [ms]	0	100	200	300	400	500	600	700	800
i [A]	0	2.8	4	4.6	4.8	4.9	5	5	5

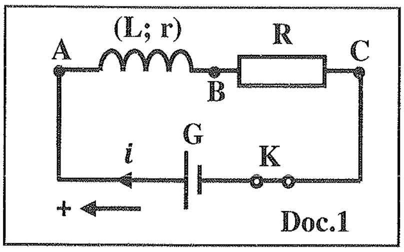
 3.1- Trace the graph of i as a function of t.
 Scale: Horizontal axis: 1div → 100ms; Vertical axis: 1div → 1A
 3.2- Use the graph to find A, B and τ.
 3.3- Show that L = 1.2H and R = 10Ω.



Doc.3

4- The DC generator is replaced by a generator that delivers a square signal as shown in document 3.
 4.1- Trace the graph of $i(t)$ between 0 and 2τ.
 4.2- We put $t' = t - \tau$ in the interval $\tau < t \leq 2\tau$.
 The equation giving the variation of i as a function of t during the decay is $i(t') = A' + B'e^{-\frac{t'}{\tau}}$.
 Determine A' and B'.

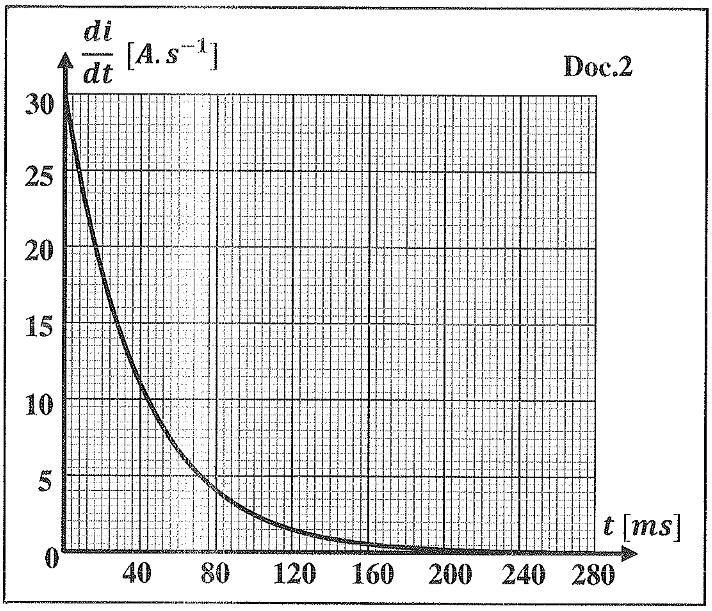
Exercise 15:



The object of this exercise is to determine the inductance L and the resistance r of a coil. For this, we connect the coil in series with a resistor of resistance $R = 8\Omega$, a switch K and a dry cell of e.m.f. E = 12V and negligible internal resistance as shown in document 1.

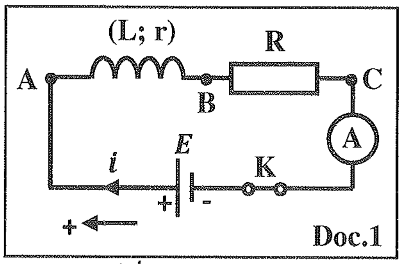
We close the switch K at the instant $t_0 = 0$. At the instant t, the circuit carries a current i.

The resistor is connected to a data logger that records the instantaneous change of current $\frac{di}{dt}$ with respect to time and the obtained graph is given in document 2.



1- Write, in the transient state, the expression of the voltage u_L across the coil.
2- Derive the differential equation that governs the variation of the current i as a function of time.
3- The solution of the differential equation has the form $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$ where I_0 and τ are constants. Determine the expression of I_0 and τ in terms of E, R, r and L.
4- Using the graph, determine:
   4.1- the value of the time constant τ.
   4.2- the value of I_0.
5- Deduce the values of L and r.
6- Calculate the time t at the end of which the steady state is practically attained. Is this value compatible with the data given in document 2?
7- Trace the graph that shows the variation of i as a function of t.
8- Calculate the maximum energy stored in the coil. What is the form of this energy?

Exercise 16:



To cause the ignition of the spark plug in a car, a spark must be created at the candles of the spark plug. The study of the ignition system in certain cars is reduced to the study of a series circuit formed of a coil (B) of inductance L and resistance r, a resistor (D) of resistance R = 2.5Ω, an ammeter (A), a switch K and a generator (G) that provides across its terminals a voltage $u_G = E = 12V$ (Document 1).

We close the switch K at the instant $t_0 = 0s$. At the instant t, the circuit carries a current i.

1- Give the expression of the voltage $u_L = u_{AB}$ across the terminals of the coil as a function of r, L and i.
2- Show that the differential equation governing the growth of i is $\frac{di}{dt} + \frac{R+r}{L}i = \frac{E}{L}$.
3- The solution of the above differential equation is given by $i = A(1 - e^{-Bt})$, where A and B are two positive non-zero constants. Determine the expressions of A and B.
4- In steady state, the current in the circuit is I_0. Show that:
   4.1- the voltage across the terminals of the coil is $u_L = rI_0$,
   4.2- the expression of I_0 is $I_0 = \frac{E}{R+r}$.

5- Show that the voltage $u_R = u_{CB}$ across the resistor satisfies the relation $\frac{RE}{L} = \frac{du_R}{dt} + \frac{R+r}{L}u_R$.

6- Deduce the expression of $\frac{du_R}{dt}$ at the instant $t_0 = 0s$.

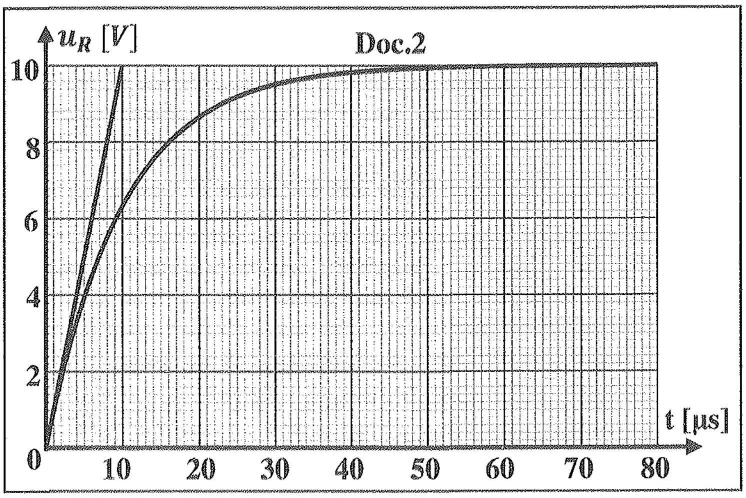
7- Determine, at the steady state, the expression of the voltage u_{R0} across the resistor.

8- Document 2 represents the variation of u_R as a function of time.

8.1- Determine the value of r.

8.2- The time constant τ is the abscissa of the intersection of the tangent at the origin to the curve of u_R and the asymptote to that curve. Show that the expression of τ is $\tau = \frac{L}{R+r}$.

8.3- Deduce the value of the inductance L of the coil.



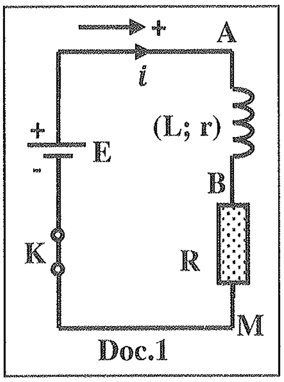
9- Identify the constants A and B.

10- Calculate W_0 the maximum energy stored in the coil (B).

11- The above circuit (ignition system) helps, through an intermediary switch, to feed the spark plugs of the car at well determined instants, with the energy needed to make the engine function normally.
We define the « rate of storage » of the coil as the ratio of the energy stored in the coil at a given instant to the maximum energy it can store. Determine the minimum duration of closure of the switch so that the rate of storage of the coil is not less than 90.3%.

Exercise 17*:

A series R-L circuit (document 1) consists of a coil (L; r), a resistor of resistance R = 100 Ω, an ideal DC generator that provides across its terminals a voltage E and a switch K.
We close the switch K at the instant $t_0 = 0s$. At the instant t, the circuit carries an instantaneous current i.
We display, using an oscilloscope, the voltage $u_{AM} = u_G$ on the channel Y_1 and the voltage $u_{BM} = u_R$ on the channel Y_2. The waveforms are represented in document 2.



1- Redraw document 1 showing on it the connections of the oscilloscope.

2- Derive the differential equation that governs the variation of the current i as a function of time.

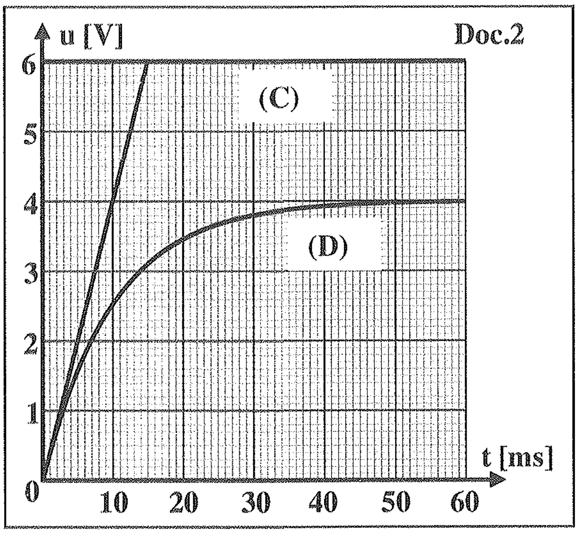
3- The solution of the differential equation has the form $i = A + Be^{-\frac{t}{\tau}}$. Determine the constants A, B and the time constant τ as a function of E, r, R and L.

4- Use document 2 to answer the following questions:

4.1- Specify the components corresponding to the two waveforms (C) and (D).

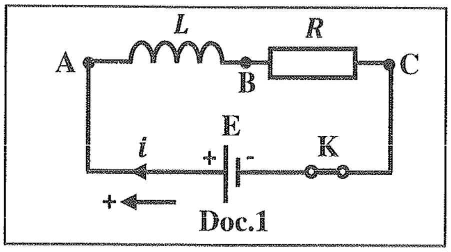
4.2- Determine E and r.

4.3- The tangent to waveform (D) is drawn at t = 0s. Determine the value of L.



Exercise 18:

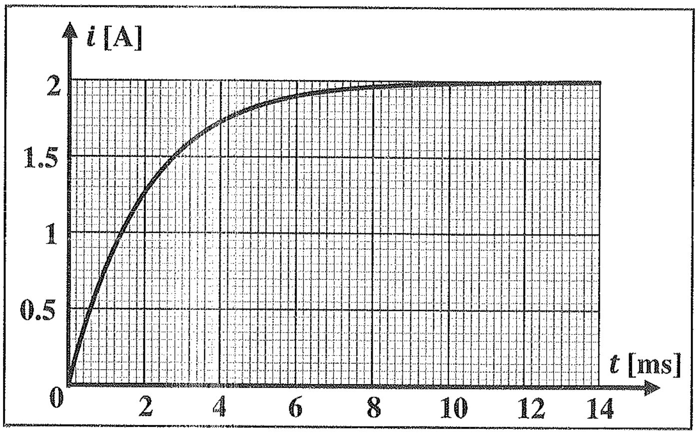
An R-L series circuit (document 1) consists of a battery of E = 100V, a purely inductive coil of inductance L, a resistor of resistance R, and a switch K which is initially opened.



A computerized oscilloscope is used to display the voltage $u_R = u_{BC}$ across the resistor R.

At instant $t_0 = 0s$, the switch K is closed. At the instant t, the circuit carries an instantaneous current i.

The screen of the oscilloscope displays a graph similar to that in document 2.



Part I: using an oscilloscope

1- Show the connections of channel 1 of the oscilloscope across R.

2- Specify if the graph in document 2 is the one observed on the screen of the oscilloscope.

Part II: determination of L and R.

1- Give a full explanation for the establishment of the self-induced e.m.f in the coil.

2- Determine the expression of the steady state current I_0 in terms of E and R.

3- Establish the differential equation that governs the variation of i as a function of t.

4- Knowing that the time constant $\tau = \frac{L}{R}$, verify that $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$ is a solution of the differential equation.

5- Referring to document 2:

5.1- Give the value of I_0; then, deduce that of r.

5.2- Define and determine the time constant τ of this RL circuit; then, deduce the value of L.

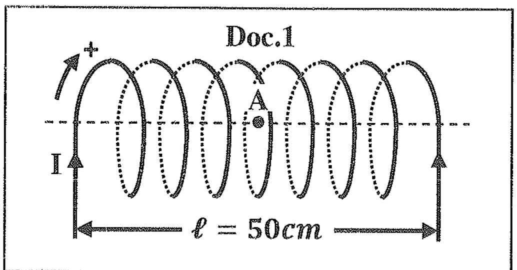
6- At what instant is the steady state attained?

7- Calculate the magnetic energy stored in the coil at the steady state.

Exercise 19*:

Document 1 represents a purely inductive solenoid of inductance L, length $\ell = 50cm$ and $N = 100turns$ each with circular shape and of radius $r = 10cm$.

Given: the magnetic permeability in vacuum is $\mu_0 = 4\pi \times 10^{-7}SI$.

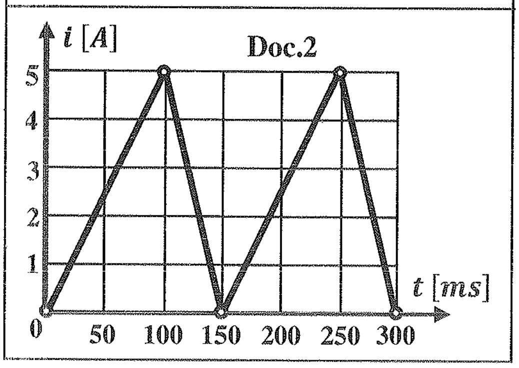


Part I: the solenoid carries a current $I = 5A$

1- Determine the characteristics of the magnetic field created by the solenoid at its center A.

2- Calculate the value of inductance L of this solenoid.

Part II: The solenoid carries now a variable current i whose variation with time is shown in document 2.



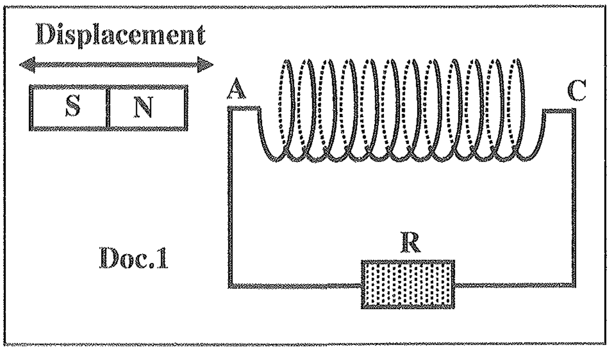
1- Give the name of the physical phenomenon that takes place.

2- Determine the e.m.f induced by the solenoid for $0 \le t \le 100ms$ and $100 \le t \le 150ms$.

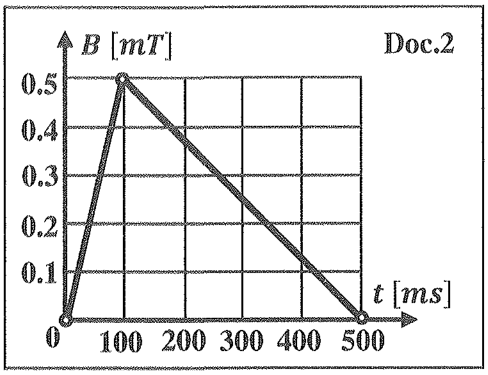
3- Draw the graph that represents the variation of the e.m.f induced solenoid as a function of time.

Exercise 20:

Consider a purely inductive solenoid of length $\ell = 60cm$ and N = 500 circular turns each of diameter d = 5cm.



Doc.1



Doc.2

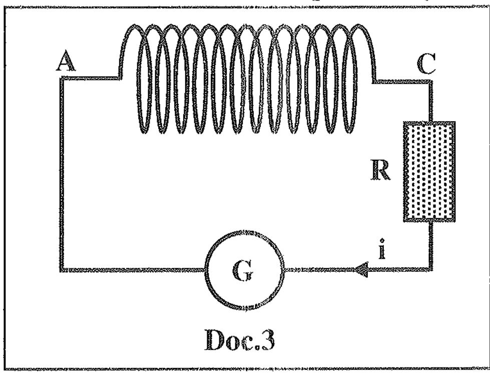
Part I: First experiment

A bar magnet may be displaced along the axis of the solenoid whose terminals A and C are connected to a resistor of resistance R = 10Ω.

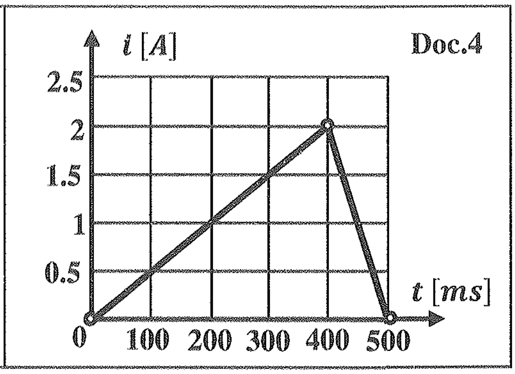
1- We approach the north pole of the magnet towards the face A of the coil (document 1). An induced current i is carried by the circuit.
   1.1- Give the name of the physical phenomenon that is responsible for the passage of this current.
   1.2- Give, with justification, the name of each face of the coil.
   1.3- Deduce direction of the electric current traversing the circuit.
   1.4- Determine the sign of the voltage u_{AC}.
2- The variation of the magnetic field crossing the solenoid as a function of time is represented in document 2. The positive sense is oriented from A to C through the coil.
   2.1- Show that the expression of the induced electromotive force can be written in the form of $e = k\frac{dB}{dt}$ where k is a constant that should be determined in terms of N and d.
   2.2- Calculate, for $0 \leq t \leq 100ms$ and $100ms \leq t \leq 500ms$, the intensity of the electric current and the voltage across the solenoid.

Part II: Second experiment

A coil of inductance L = 0.01H and of negligible resistance is connected in series with a resistor of resistance R = 10Ω and a generator G (document 3). The coil thus carries a current i that varies with time as shown in document 4. The coil is oriented positively from A to C.



Doc.3



Doc.4

1- Give the name of the physical phenomenon that takes place in the coil.
2- Determine the voltage u_{AC} in each of the two intervals: [0; 400ms] and [400ms; 500ms].
3- In which interval does the coil act as a receiver? Justify.
4- Plot the voltage across the generator u_G as a function of time.

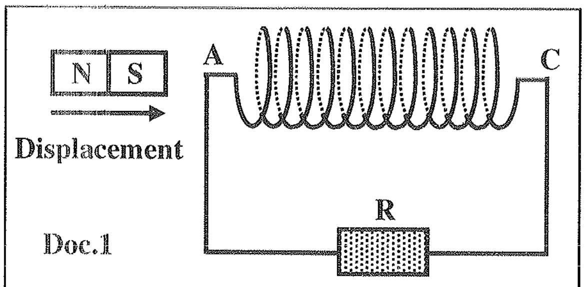
Exercise 21*:

The object of this exercise is study the electromagnetic and self induction phenomena.
For this, we conduct two experiments using the following equipment:

- a purely inductive coil of inductance L,
- a resistor of resistance R,
- a bar magnet,
- a low frequency generator G,
- an oscilloscope,
- a galvanometer,
- connecting wires.
- a switch K.

Part I: First experiment

We connect the coil in series with the resistor and the galvanometer.
The bar magnet may be displaced along the axis of the coil whose terminals A and C are connected to the resistor.
We approach the south pole of the magnet towards the face A of the coil (document 1). An induced current i is carried by the circuit.

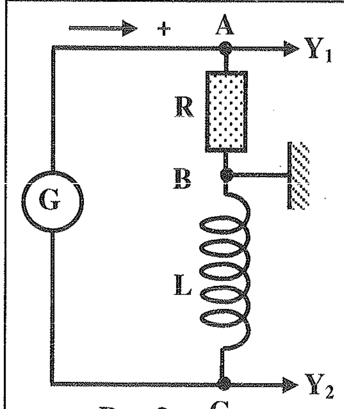


Doc.1

1- Give the name of the physical phenomenon that is responsible for the passage of this current.
2- Indicate, giving the necessary explanations, the direction of the induced current in the circuit.
3- Draw a diagram representing the generator equivalent to the coil between the points A and C.

Part II: Second experiment

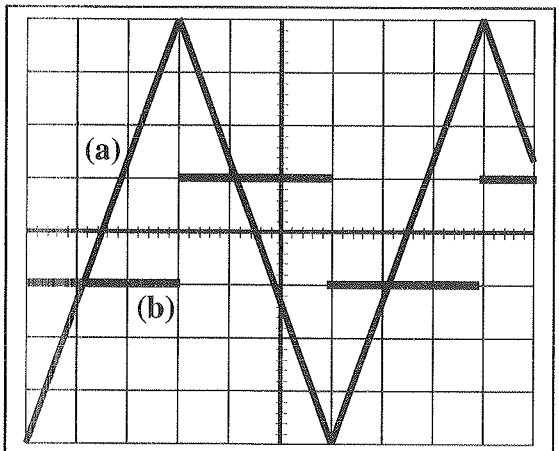
The resistor, the coil and the switch K are connected in series with the low frequency generator G that provides an alternating triangular voltage.
We close K at $t_0 = 0$s. At the instant t, the circuit carries an alternating triangular current of intensity i.
An oscilloscope is used to visualize the voltage u_{AB} on channel Y_1 and the voltage u_{CB} across the resistor on channel Y_2.



Doc.2

The settings of the oscilloscope are:
Horizontal sensitivity: $S_h = 1$ms/div
Vertical sensitivities:
- On channel (Y_1): $S_{V1} = 0.5$ V/div.
- On channel (Y_2): $S_{V2} = 0.1$ V/div.

When no voltage is displayed, the traces of the spot coincide with middle horizontal line of the screen. The oscillogram is represented in document 3.



Doc.3

1- Explain why curve (a) may represent the current i in the circuit. Deduce from this curve that the coil is the seat of a physical phenomenon. Name this phenomenon.
2- Show that the relation between the voltage u_{AB} and the voltage u_{CB} is written as:

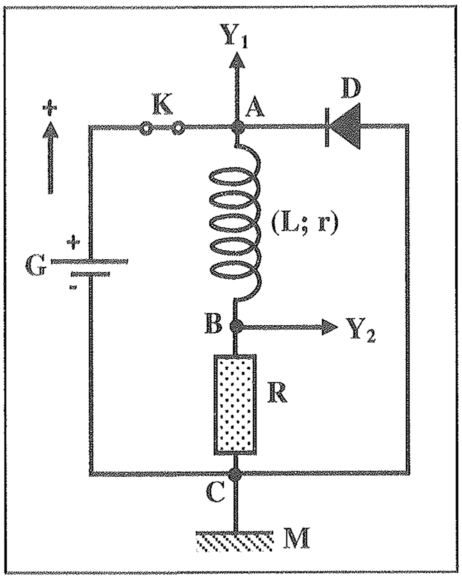
$$u_{AB} = -\frac{L}{R}\frac{du_{CB}}{dt}$$

3- Determine, referring to document 3,the value of the inductance L knowing that R = 800Ω
4- Give the difference between the two phenomena investigated in experiments I and II.

Exercise 22:

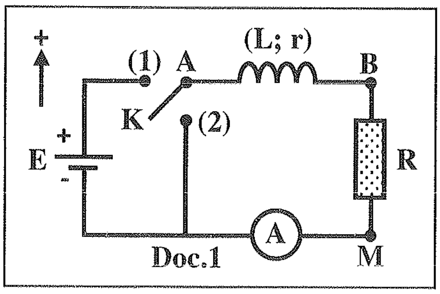
In the document below, the circuit diagram that consists of:
- an ideal generator of e.m.f E,
- a coil of inductance L and resistance r,
- a resistor of resistance R,
- an ideal diode D (when it allows the passage of current, the voltage across its terminals is zero).

An oscilloscope is used to visualize the voltage u_{AC} across the generator on channel Y_1 and the voltage u_{BC} across the resistor on channel Y_2.



1- **At t = 0s the switch K is closed.**
  1.1- In what branch of the circuit does the transient currents pass?
  1.2- What is the shape of the displayed waveforms on Y_1 and Y_2?
2- **Evolution of the electric current.**
  2.1- Express Ohm's law across the coil and the resistor R.
  2.2- Deduce a relation between L, $\frac{di}{dt}$, E and R_{eq} where $R_{eq} = R + r$.
  2.3- The solution of the obtained differential equation has the form: $i = \frac{E}{R_{eq}}(1 - e^{-kt})$. Determine the expression of the constant k.
  2.4- What is the value of the steady current I_0 inside the circuit?
  2.5- Give the value of $\frac{di}{dt}$ at t = 0sec in terms of E and L.
3- Define the term "time constant" for the above circuit.
4- Express $\frac{di}{dt}$ at t = 0 in terms I_0 and k .
5- After the steady state is achieved, the switch is opened at an instant t taken as a new origin of time.
  5.1- Give the shape of the wave from observed on Y_2.
  5.2- Derive a relation between R_{eq} ; L, $\frac{di}{dt}$, and i.
  5.3- What is the solution of the differential equation in part 5.3?
  5.4- Determine the value of $\frac{di}{dt}$ at t = 0sec as a function of I_0 and τ where $\tau = \frac{L}{R_{eq}}$.

Exercise 23*:



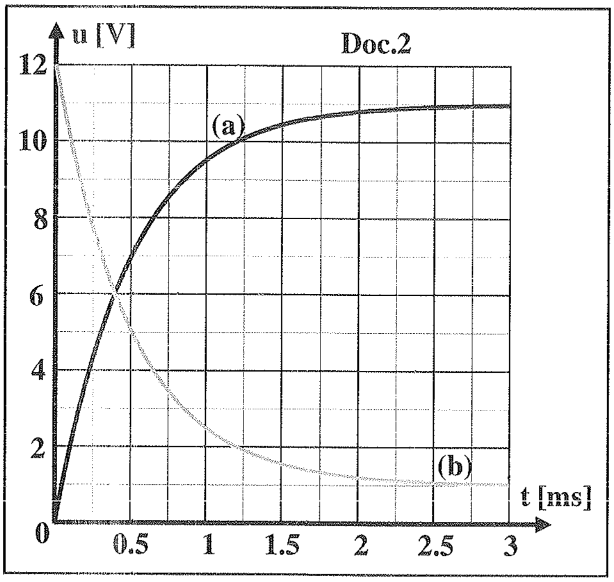
Doc.1

The circuit of document 1 consists of:
- An ideal generator of constant electromotive force E.
- A coil of inductance L and resistance r.
- A resistor of resistance R = 110Ω.
- A double switch K.
- An ammeter.
- Connecting wires.

The aim of this exercise is to determine the characteristics L and r of the coil.

I- Analytical study of the growth of the current.

At the instant $t_0 = 0$, the switch K is turned to position (1). At the instant t, the circuit carries an electric current i.
A convenient apparatus records the variations of the voltage $u_L = u_{AB}$ across the coil and the voltage $u_R = u_{BM}$ across the resistor. We obtain the waveforms of document 2.



1- Curve (a) represents the variation of u_R as a function of time. Why?
2- Applying the law of addition of voltages, derive the first order differential equation that governs the variation of the current i as a function of time.
3- Deduce the expression of the steady state current I_0 in terms of E, R and r.
4- The solution of the differential equation has the form $i = A + Be^{-\frac{t}{\tau}}$ where A, B and τ are constants. Determine the expressions of A, B and τ in terms of E, L, R and r.
5- Determine:
   5.1- the values of I_0, r and E.
   5.2- the time constant τ; then, deduce the value of L.

II- Analytical study of the decay of current

At a new origin of time $t_0 = 0$, the switch K is turned to position (2). At the instant t, the circuit carries an electric current i.
1- Draw the circuit and indicate the direction of the electric current.
2- Show that the differential equation that governs the variation of the voltage u_R across the resistor is given by:
$$\frac{du_R}{dt} + \frac{R+r}{L}u_R = 0$$
3- The solution of the differential equation has the form $u_R = De^{-\alpha t}$ where D and α re constants. Show that $D = RI_0$ and $\alpha = \frac{1}{\tau}$.
4- Deduce that $i = I_0 e^{-\frac{t}{\tau}}$.
5- Determine the magnetic energy W_m lost by the coil between $t_0 = 0$s and $t_1 = \tau$.
6- The energy dissipated due to joule's effect in the resistor between t_0 and t_1, is given by $W_h = \int_{t_0}^{t_1} Ri^2dt$.
   6.1- Determine the value of W_h.
   6.2- Deduce the energy dissipated in the coil between t_0 and t_1.

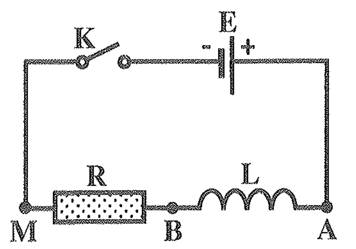
Exercise 24:

The object of this exercise is to show evidence of the analogy between an RL series circuit and the motion of a ball.

I- RL series circuit

A coil of inductance L and negligible resistance, a resistor of resistance R and a switch K are connected in series with an ideal generator of constant electromotive force E as shown in document 1.

At the instant t = 0s, we close the switch K. At the instant t, the circuit carries a current i.



Doc.1

1- Show that the differential equation that governs the variation of i as a function of time is given by: $E = Ri + L\frac{di}{dt}$ ... (1)

2- The solution of the differential equation has the form $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$ where I_0 and τ are constants. Determine the expressions of I_0 and τ in terms E, R and L.

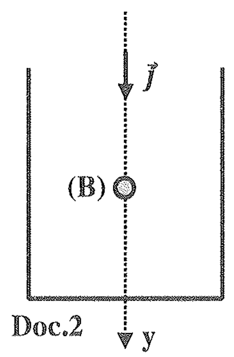
3- Determine, in terms of I_0, the expression of i at the instants t= 0s, t = τ and t = 5τ.

4- Trace the graph that represents the variation of i as a function of time.

II- Vertical fall in a liquid

Starting from rest at the instant t = 0s, a metallic ball (B), taken as a particle of mass m, falls vertically in a liquid as shown in document 2

At the instant t, (B) is submitted to a resistive force $\vec{f} = -h\vec{v}$ where $\vec{v} = v\vec{j}$ is the instantaneous velocity of (B) and h is a positive constant.



Doc.2

1- Name and represent the two external forces acting on (B).

2- Applying Newton's second law of translational motion, show that the differential equation that governs the variation of v as a function of time is given by: $mg = hv + m\frac{dv}{dt}$ ... (2)

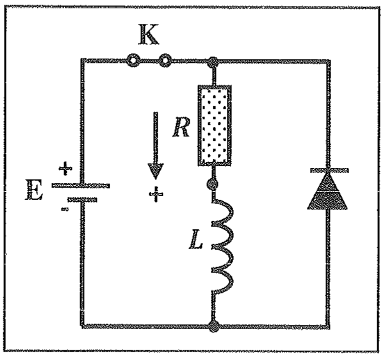
III- An analogy

1- Match each of the physical mechanical quantities mg, h, m and v with its corresponding convenient electric quantity.

2- Deduce the solution of the differential equation (2).

3- Trace the graph that represents the variation of v as a function of time.

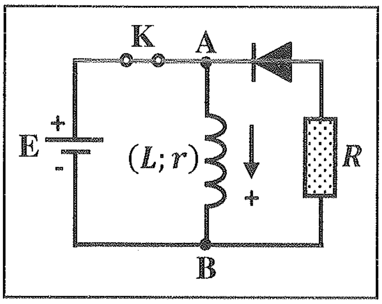
Exercise 25:



In the adjacent document the RL series circuit consists of a purely inductive coil of inductance L = 4H, a resistor of resistance R = 48Ω, an ideal battery of emf E = 12V, a diode and a switch K.
The switch is closed at t_0 = 0s.

1- What is the role of a diode?
2- Trace the shape of the graph of the current i as a function of time t.
3- Calculate $\frac{di}{dt}$ at t = 0s.
4- Derive the differential equations of the voltage across the resistor u_R and the voltage across the coil u_L.
5- The switch is opened abruptly at a new origin of time t_0 = 0s.
Explain fully what will happen in the circuit.

Exercise 26:



The circuit shown in the adjacent document consists of:
- an ideal DC generator of e.m.f E = 9V,
- a coil of inductance L = 0.4H and r = 60Ω
- an ideal diode,
- a resistor of resistance R = 40Ω.
- A switch K.

1- The switch K is closed at the instant t_0 = 0s.
  1.1- What is the role of the diode in this circuit.
  1.2- Determine, at the steady state, the expression of I_0 the intensity of the electric current traversing the circuit.
  1.3- Calculate, at the steady state, the energy stored in the coil. Indicate its form.
2- When the steady state is attained, we open the switch K at a new origin of time t_0 = 0s.
  2.1- Determine the expression of variation of the electric current with respect to time $\frac{di}{dt}$ at the instant t = 0s.
  2.2- Determine the value of the voltage u_{AB} across the coil at the instant t_0 = 0s. Conclude.
  2.3- Trace the shape of the current i traversing the circuit as a function of time.

CHAPTER 4 – SELF INDUCTION
SOLUTION OF EXERCISES AND PROBLEMS

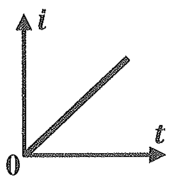
Exercise 2:

Part	Answer key
1	$\phi = Li = (0.06)(0.2t + 1) = 0.012t + 0.06$.
2	Faraday's law: $e = -\frac{d\phi}{dt} = -0.012V$.

Exercise 6:

Part	Answer key
	$u = L\frac{di}{dt} + ri$ with $r = 0$ (pure inductor) and $\frac{di}{dt} = \frac{\Delta i}{\Delta t}$ (i is linear with time). Then, $u = L\frac{\Delta i}{\Delta t} \Rightarrow 20 = L \times \frac{(5-0)\times10^{-3}}{(10-5)\times10^{-6}} \Rightarrow L = 20mH$.

Exercise 7:

Part	Answer key
1	$u_G = u_K + u_L \Rightarrow U = ri + L\frac{di}{dt}$ with $r = 0$ (pure inductor) and $u_K = 0$ (closed switch) $U = L\frac{di}{dt} \Rightarrow \frac{di}{dt} = \frac{U}{L} = constant \Rightarrow i = \frac{U}{L}t + i_0$. At t = 0s, $i = i_0 = 0A$. Therefore, $i = \frac{U}{L}t$. The expression of i has the form $i = kt$ (equation of a straight line passing through origin) where $k = slope = \frac{U}{L}$. 
2	$W_{mag} = \frac{1}{2}Li^2 \Rightarrow i = \sqrt{\frac{2W}{L}} = \sqrt{\frac{2\times10\times10^{-6}}{200\times10^{-3}}} = 0.01A = 10mA$.
3	$i = \frac{U}{L}t \Rightarrow t = \frac{iL}{U} = \frac{0.01\times0.2}{12} = 1.67 \times 10^{-4}s = 167\mu s$.

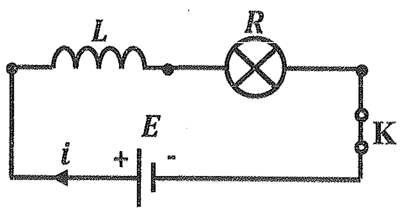
Exercise 9:

Part	Answer key
1	The voltage across the generator is triangular; then, the current carried by the circuit is triangular. The voltage across the resistor and the current are proportional according to Ohm's law: $u_R = Ri$ with R is a positive constant. Therefore, the voltage across the resistor is triangular. Curve 2 corresponds to the voltage across the resistor (false).
2	By Ohm's law for a coil: $u_L = u_{AB} = ri + L\frac{di}{dt} = L\frac{di}{dt}$ with $r = 0$ (True).
3	$U_{Lm} = S_v \times y = 2 \times 3 = 6V$ (false).
4	$u_{AB} = L\frac{di}{dt}$ and $i = \frac{u_{BC}}{R}$. $u_{AB} = \frac{L}{R}\frac{du_{BC}}{dt}$ but channel Y_2 displays u_{CB}. So $u_{AB} = -\frac{L}{R}\frac{du_{CB}}{dt} = -\frac{L}{R}\frac{\Delta u_{CB}}{\Delta t}$ with $\frac{du_{CB}}{dt} = \frac{\Delta u_{CB}}{\Delta t}$ (u_{CB} is linear). $6 = -\frac{L}{500} \times \frac{(0-4.8)}{0.2\times10^{-3}} \Rightarrow L = 0.125H$ (True).

Exercise 11:

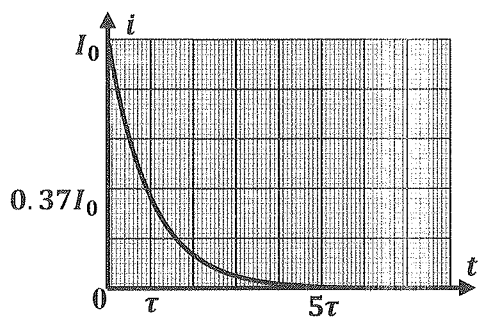
Part	Answer key		
1	Self induction.		
2.1	**For [0s; 10ms]:** $i = at + b.$ $a = \frac{\Delta i}{\Delta t} = \frac{2-0}{(10-0)\times 10^{-3}} = 200A/s.$ For $t = 0; i = b = 0.$ $i = 200t.$ $\phi = Li = 0.02 \times 200t = 4t.$	**For [10s; 30ms]:** $\phi = Li = 0.02 \times 2.$ $\phi = 0.04Wb.$	**For [30ms; 50ms]:** $i = at + b.$ $a = \frac{\Delta i}{\Delta t} = \frac{0-2}{(50-30)\times 10^{-3}} = -100A/s.$ For $t = 30ms; i = 2A.$ $2 = -100 \times 0.03 + b \Rightarrow b = 5A.$ $i = -100t + 5.$ $\phi = Li = 0.02(-100t + 5).$ $\phi = -2t + 0.1.$
2.2	**For [0s; 10ms]:** $e = -\frac{d\phi}{dt} = -4V.$	**For [10s; 30ms]:** $e = -\frac{d\phi}{dt} = 0.$	**For [30ms; 50ms]:** $e = -\frac{d\phi}{dt} = 2V.$
2.3	**For [0s; 10ms]:** $u_{AC} = ri - e = 10(200t) + 4.$ $u_{AC} = 2000t + 4.$	**For [10s; 30ms]:** $e = -\frac{d\phi}{dt} = 0.$ $u_{AC} = ri - e.$ $u_{AC} = 10 \times 2 = 20V.$	**For [30ms; 50ms]:** $u_{AC} = ri - e = 10(-100t + 5) - 2.$ $u_{AC} = -100t + 48.$
3.1	For [0s; 10ms]: $e.i < 0$ (receiver / generator in opposition).		
3.2	For [30ms; 50ms]: $e.i > 0$ (generator).		

Exercise 12:

Part	Answer key		
1		2.1	The intensity of the light emitted by the lamp increases gradually from zero to a maximum intensity at the steady state.
2.2	By applying the law of addition of voltages in series connection: $u_G = u_L + u_R.$ $E = ri + L\frac{di}{dt} + Ri$ with $r = 0.$ At the steady state: $i = I_0 = constant \Rightarrow \frac{di}{dt} = 0.$ $E = RI_0 \Rightarrow I_0 = \frac{E}{R} = \frac{10}{5} = 2A.$		
3	$\tau = \frac{L}{R} = \frac{0.02}{5} = 4 \times 10^{-3}s = 4ms.$		
4	$t = 5\tau = 20ms.$		
5	By applying the law of addition of voltages in series connection: $u_G = u_L + u_R.$ $E = ri + L\frac{di}{dt} + Ri$ with $r = 0.$Then, $E = L\frac{di}{dt} + Ri \Rightarrow \frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}.$		
6	$i = I_0\left(1 - e^{-\frac{t}{\tau}}\right) = I_0 - I_0 e^{-\frac{t}{\tau}}$ and $\frac{di}{dt} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}}.$ Replace i and $\frac{di}{dt}$ in the differential equation: $\frac{I_0}{\tau}e^{-\frac{t}{\tau}} + \frac{RI_0}{L} - \frac{RI_0}{L}e^{-\frac{t}{\tau}} = \frac{E}{L}.$ $\left(\frac{1}{\tau} - \frac{R}{L}\right)I_0 e^{-\frac{t}{\tau}} + \frac{RI_0}{L} = \frac{E}{L}$ with $\frac{1}{\tau} = \frac{R}{L}$ and $RI_0 = E.$ $\left(\frac{R}{L} - \frac{R}{L}\right)I_0 e^{-\frac{t}{\tau}} + \frac{E}{L} = \frac{E}{L} \Rightarrow \frac{E}{L} = \frac{E}{L}$ (verified).		

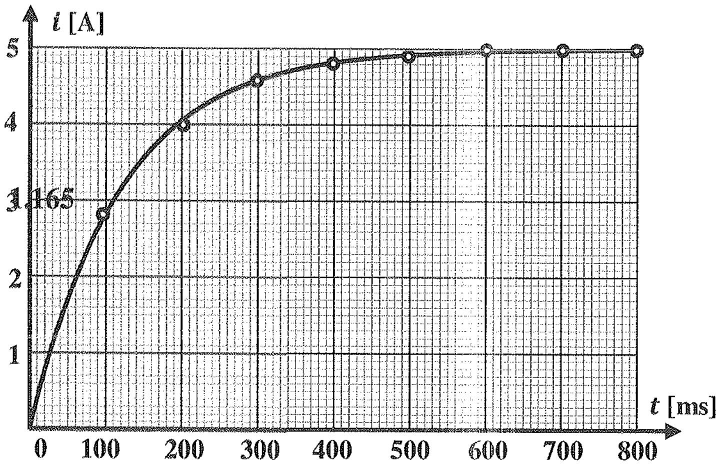
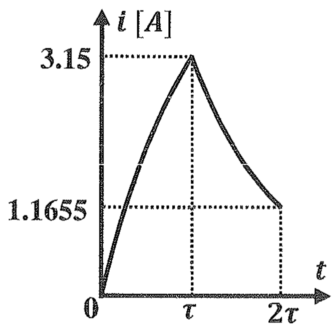
7	$i = I_0\left(1 - e^{-\frac{t}{\tau}}\right) = 2(1 - e^{-2}) = 1.729A.$
8	$\frac{di}{dt} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}} = \frac{2}{4\times10^{-3}}e^{-1} = 183.939A/s.$ $u_L = L\frac{di}{dt} = 0.02 \times 183.939 = 3.67878V.$
9.1	$i = 0.8I_0 = 1.6A \Longrightarrow 0.8I_0 = I_0\left(1 - e^{-\frac{t}{\tau}}\right) \Longrightarrow 1 - e^{-\frac{t}{\tau}} = 0.8 \Longrightarrow e^{-\frac{t}{\tau}} = 0.2.$ $-\frac{t}{\tau} = \ln 0.2 \Longrightarrow t = -\tau \ln 0.2 = -4 \times \ln 0.2 = 6.4ms.$ $\frac{di}{dt} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}} = \frac{2}{4\times10^{-3}}e^{-\frac{6.4}{4}} = 100.9A/s.$
9.2	$E = \frac{1}{2}Li^2 \Longrightarrow \frac{dE}{dt} = Li\frac{di}{di} = 0.02 \times 1.6 \times 100.9 = 3.2288J/s.$

Exercise 13:

Part	Answer key
1.1	By applying the law of addition of voltages is series connection: $u_{AC} = u_{AB} + u_{BC}$. $u_G = u_L + u_R \Longrightarrow E = ri + L\frac{di}{dt} + Ri$ with $r = 0$. Then, $E = L\frac{di}{dt} + Ri$. At the steady state $i = I_0 \Longrightarrow \frac{di}{dt} = 0$. $E = RI_0 \Longrightarrow I_0 = \frac{E}{R} = \frac{12}{100} = 0.12A.$
1.2	At $t = \tau; i = 0.63I_0 = 0.0756A.$ $W = \frac{1}{2}Li^2 = \frac{1}{2} \times 0.2 \times 0.0756^2 = 5.7 \times 10^{-4}J.$ At $t = 5\tau$. $W = \frac{1}{2}LI_0^2 = \frac{1}{2} \times 0.2 \times 0.12^2 = 1.44 \times 10^{-3}J.$
2.1	$u_{AC} = u_{AB} + u_{BC} \Longrightarrow u_L + u_R = 0 \Longrightarrow ri + L\frac{di}{dt} + Ri = 0$ with $r = 0$. $L\frac{di}{dt} + Ri = 0 \Longrightarrow \frac{di}{dt} + \frac{R}{L} = 0.$
2.2	The above equation is a first order differential equation and its solution has the form: $i = I_0e^{-\frac{t}{\tau}}$ with $I_0 = \frac{E}{R}$ and $\tau = \frac{L}{R}$.
2.3	
2.3	$\tau = \frac{L}{R} = \frac{0.2}{100} = 2 \times 10^{-3}s = 2ms.$

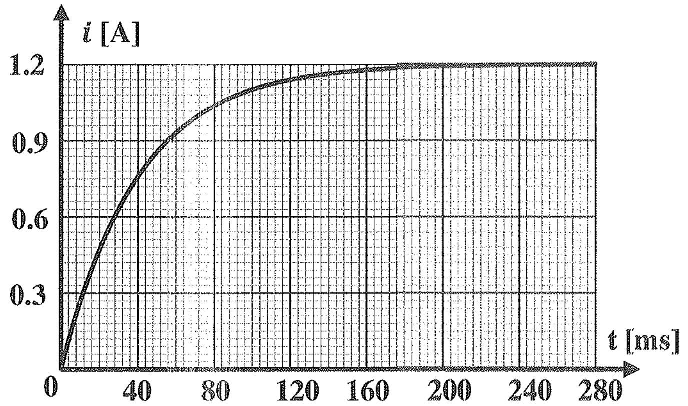
Exercise 14:

Part	Answer key
1	By applying the law of addition of voltages in series connection: $u_{AC} = u_{AB} + u_{BC}$. $u_G = u_L + u_R \Longrightarrow E = ri + L\frac{di}{dt} + Ri$ with $r = 0$. $E = L\frac{di}{dt} + Ri \Longrightarrow \frac{E}{L} = \frac{di}{dt} + \frac{R}{L}i.$
2	$i = A + Be^{-\frac{t}{\tau}}$ and $\frac{di}{dt} = -\frac{B}{\tau}e^{-\frac{t}{\tau}}$.

	Replace i and $\frac{di}{dt}$ in the differential equation: $\frac{E}{L} = -\frac{B}{\tau}e^{-\frac{t}{\tau}} + \frac{RA}{L} + \frac{RB}{L}e^{-\frac{t}{\tau}}$. $\frac{E}{L} = \frac{RA}{L} + Be^{-\frac{t}{\tau}}\left(-\frac{1}{\tau} + \frac{R}{L}\right)$. By identification: $\frac{E}{L} = \frac{RA}{L} \Longrightarrow A = \frac{E}{R}$ and $-\frac{1}{\tau} + \frac{R}{L} = 0 \Longrightarrow \tau = \frac{L}{R}$ with $Be^{-\frac{t}{\tau}} = 0$ not true $\forall\ t$. At t = 0s; $i = 0 \Longrightarrow A + Be^0 = 0 \Longrightarrow B = -A = -\frac{E}{R}$.
3.1	
3.2	At $t \to \infty; i = 5A \Longrightarrow 5 = A + Be^{-\infty} \Longrightarrow A = 5A$. At t = 0s; $i = A + Be^0 = 0 \Longrightarrow B = -A = 5A$. At $t = \tau; i = 0.63I_0 = 3.15A$. Graphically: $\tau = 120ms = 0.12s$. 
3.3	$A = \frac{E}{R} \Longrightarrow R = \frac{E}{A} = \frac{50}{5} = 10\Omega$. $\tau = \frac{L}{R} \Longrightarrow L = \tau \times R = 0.12 \times 10 = 1.2H$.
4.1	At t = τ; $i = 3.15A$. At t = 2τ; $i_1 = 0.37i = 0.37 \times 3.15 = 1.1655A$.
4.2	$i(t') = A' + B'e^{-\frac{t'}{\tau}}$. At $t' \to \infty; i = 0$. $0 = A' + B'e^{-\infty} \Longrightarrow A' = 0$. At $t = \tau; t' = 0$. $3.15 = B'e^0 \Longrightarrow B' = 3.15A$.

Exercise 15:

Part	Answer key
1	$u_L = ri + L\frac{di}{dt}$.
2	By applying the law of addition of voltages in series connection: $u_G = u_L + u_R$. $E = ri + L\frac{di}{dt} + Ri \Longrightarrow E = L\frac{di}{dt} + (R + r)i \Longrightarrow \frac{E}{L} = \frac{di}{dt} + \frac{R+r}{L}i$.
3	$i = I_0\left(1 - e^{-\frac{t}{\tau}}\right) = I_0 - I_0e^{-\frac{t}{\tau}}$. $\frac{di}{dt} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}}$. Replace i and $\frac{di}{dt}$ in the differential equation:

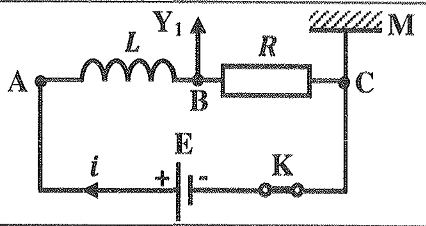
	$\frac{E}{L} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}} + \frac{R+r}{L}I_0 - \frac{R+r}{L}I_0e^{-\frac{t}{\tau}}$. $\frac{E}{L} = \frac{R+r}{L}I_0 + I_0e^{-\frac{t}{\tau}}\left(\frac{1}{\tau} - \frac{R+r}{L}\right)$. By identification: $\frac{E}{L} = \frac{R+r}{L}I_0 \Longrightarrow I_0 = \frac{E}{R+r}$ and $\frac{1}{\tau} - \frac{R+r}{L} = 0 \Longrightarrow \frac{1}{\tau} = \frac{R+r}{L} \Longrightarrow \tau = \frac{L}{R+r}$.
4.1	At $t = \tau$; $\frac{di}{dt} = 0.37\left(\frac{di}{dt}\right)_0 = 0.37 \times 30 = 11.1V$. Graphically: $\tau = 40ms$.
4.2	$\left(\frac{di}{dt}\right)_0 = \frac{I_0}{\tau} \Longrightarrow 30 = \frac{I_0}{40\times10^{-3}} \Longrightarrow I_0 = 1.2A$.
5	$I_0 = \frac{E}{R+r} \Longrightarrow 1.2 = \frac{12}{8+r} \Longrightarrow 9.6 + 1.2r = 12 \Longrightarrow r = 2\Omega$. $\tau = \frac{L}{R+r} \Longrightarrow L = \tau(R+r) = 40 \times 10^{-3} \times 10 = 0.4H$.
6	$t = 5\tau = 200ms$. Yes, it is compatible with the graph. At t = 200ms; $i = I_0 \Longrightarrow \frac{di}{dt} = 0$.
7	
8	$W = \frac{1}{2}LI_0^2 = \frac{1}{2} \times 0.4 \times 1.2^2 = 0.288J$ (magnetic energy).

Exercise 16:

Part	Answer key
1	$u_L = ri + L\frac{di}{dt}$.
2	Law of addition of voltages in series connection: $u_G = u_L + u_R \Longrightarrow E = ri + L\frac{di}{dt} + Ri$. $E = L\frac{di}{dt} + (R+r)i \Longrightarrow \frac{E}{L} = \frac{di}{dt} + \frac{R+r}{L}i$.
3	$i = A(1 - e^{-Bt}) = A - Ae^{-Bt}$ and $\frac{di}{dt} = ABe^{-Bt}$. Replace i and $\frac{di}{dt}$ in the differential equation: $\frac{E}{L} = ABe^{-Bt} + \frac{R+r}{L}A - \frac{R+r}{L}Ae^{-Bt} \Longrightarrow \frac{E}{L} = \frac{R+r}{L}A + Ae^{-Bt}\left(B - \frac{R+r}{L}\right)$. By identification: $\frac{E}{L} = \frac{R+r}{L}A \Longrightarrow A = \frac{E}{R+r}$ and $B - \frac{R+r}{L} = 0 \Longrightarrow B = \frac{R+r}{L}$. $A = \frac{E}{R+r} = \frac{12}{2.5+0.5} = \frac{12}{3} = 4A$.
4.1	At the steady state: $i = I_0 \Longrightarrow \frac{di}{dt} = \frac{dI_0}{dt} = 0$. $u_L = rI_0 + L\frac{dI_0}{dt} = rI_0$.

4.2	$E = rI_0 + L\frac{dI_0}{dt} + RI_0 \Longrightarrow E = (R+r)I_0$.
5	By applying the law of addition of voltages: $u_G = u_L + u_R$. $E = ri + L\frac{di}{dt} + u_R$ with $i = \frac{u_R}{R}$. $E = \frac{r}{R}u_R + \frac{L}{R}\frac{du_R}{dt} + u_R \Longrightarrow \frac{RE}{L} = \frac{r}{L}u_R + \frac{du_R}{dt} + \frac{R}{L}u_R \Longrightarrow \frac{RE}{L} = \frac{du_R}{dt} + \frac{R+r}{L}u_R$.
6	At t = 0s; $i = 0 \Longrightarrow u_R = 0$. $\frac{du_R}{dt} = \frac{RE}{L}$.
7	At the steady state; $i = I_0$. $u_{R0} = RI_0 = \frac{RE}{R+r}$.
8.1	$u_{R0} = \frac{RE}{R+r} \Longrightarrow 10 = \frac{2.5\times12}{2.5+r} \Longrightarrow 25 + 10r = 30 \Longrightarrow r = 0.5\Omega$.
8.2	The equation of the tangent to the curve at t_0 = 0s is: $u_R = \frac{RE}{L}t$. The equation of the asymptote is: $u_R = u_{R0} = \frac{RE}{R+r}$. $\frac{RE}{L}\tau = \frac{RE}{R+r} \Longrightarrow \tau = \frac{L}{R+r}$.
8.3	Graphically $\tau = 10\mu s$. $L = \tau(R+r) = 10\times10^{-6}\times3 = 3\times10^{-5}H$.
9	$B = \frac{1}{\tau}$ and $A = I_0$.
10	$I_0 = \frac{E}{R+r} = \frac{12}{2.5+0.5} = 4A$. $W = \frac{1}{2}LI_0^2 = \frac{1}{2}\times3\times10^{-5}\times4^2 = 2.4\times10^{-4}J$.
11	$\frac{W}{W_0} = 0.903 \Longrightarrow W = 2.1672\times10^{-4}J$. $W = \frac{1}{2}Li^2 \Longrightarrow i = \sqrt{\frac{2W}{L}} = \sqrt{\frac{2\times2.1672\times10^{-4}}{3\times10^{-5}}} = 3.8A$. $u_R = Ri = 2.5\times3.8 = 9.5V$. Graphically $t_{min} = 30\mu s$.

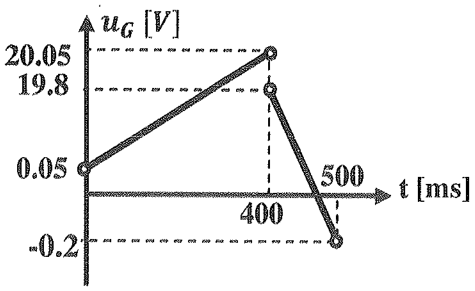
Exercise 18:

Part	**Answer key**		
I.1		**I.2**	The oscilloscope is a device used to visualize voltage not current. The obtained graph of the oscillogram corresponds to the voltage across the resistor, but this voltage is the image of the current since they are proportional according to Ohm's law: $u_R = Ri$
II.1	During the transient state, the coil is traversed by a variable current i. The self magnetic flux $\phi = Li$ traversing the coil varies with time. The change in the self magnetic flux gives rise to a self induced e.m.f according to Faraday Lenz law: $e = -\frac{d\phi}{dt}$.		
II.2	By applying the law of addition of voltages in series connection: $u_G = u_L + u_R$. $E = ri + L\frac{di}{dt} + Ri$ with $r = 0$. At the steady state, $i = I_0 \Longrightarrow \frac{di}{dt} = \frac{dI_0}{dt} = 0$.		

	$E = L\frac{dI_0}{dt} + RI_0 \Rightarrow E = RI_0 \Rightarrow I_0 = \frac{E}{R}$.
II.3	By applying the law of addition of voltages in series connection: $u_G = u_L + u_R$. $E = ri + L\frac{di}{dt} + Ri$ with $r = 0$. Then, $\frac{E}{L} = \frac{di}{dt} + \frac{R}{L}i$.
II.4	$i = I_0\left(1 - e^{-\frac{t}{\tau}}\right) = I_0 - I_0e^{-\frac{t}{\tau}}$ and $\frac{di}{dt} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}}$. Replace $\frac{di}{dt}$ and i in the differential equation: $\frac{E}{L} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}} + \frac{R}{L}I_0 - \frac{R}{L}I_0e^{-\frac{t}{\tau}}$. $\frac{E}{L} = I_0e^{-\frac{t}{\tau}}\left(\frac{1}{\tau} - \frac{R}{L}\right) + \frac{R}{L}I_0$. By identification: $\frac{E}{L} = \frac{R}{L}I_0 \Rightarrow I_0 = \frac{E}{R}$ and $\frac{1}{\tau} - \frac{R}{L} = 0 \Rightarrow \frac{1}{\tau} = \frac{R}{L} \Rightarrow \tau = \frac{L}{R}$.
II.5.1	$I_0 = 2A \Rightarrow R = \frac{E}{I_0} = \frac{100}{2} = 50\Omega$.
II.5.2	Time constant (τ): time needed for the current to reach 63% of its maximum value. At $t = \tau; i = 0.63I_0 = 0.63 \times 2 = 1.26A$. Graphically, $\tau = 2ms$. $L = \tau R = 2 \times 10^{-3} \times 500 = 1H$.
II.6	$t = 5\tau = 10ms$.
II.7	$W = \frac{1}{2}LI_0^2 = \frac{1}{2} \times 1 \times 2^2 = 2J$.

Exercise 20:

Part	Answer key
I.1.1	Electromagnetic induction.
I.1.2	According to Lenz's law, the coil will oppose by repulsion the incoming N pole of the magnet. So A acts as a north pole and C acts as a south pole.
I.1.3	The induced magnetic field is directed from S to N inside the solenoid (to the left). According to the right hand rule, the current is directed from A to C through the solenoid.
I.1.4	A is the negative pole of the equivalent generator so $u_{AC} < 0$.
I.2.1	$\phi = N\vec{B}.\vec{S} = NBS\cos\theta$ where $B = \frac{\mu_0 Ni}{\ell}$; $S = \pi\frac{d^2}{4}$ and $\theta = (\vec{B}; \vec{n}) = 180°$. $\phi = NB\left(\pi\frac{d^2}{4}\right)\cos 180° = -\frac{N\pi d^2}{4}B$. Apply Faraday-Lenz law: $e = -\frac{d\phi}{dt} = \frac{N\pi d^2}{4}\frac{dB}{dt} = k\frac{dB}{dt}$ where $k = \frac{N\pi d^2}{4}$.
I.2.2	$e = k\frac{dB}{dt} = k\frac{\Delta B}{\Delta t}$ where $\frac{dB}{dt} = \frac{\Delta B}{\Delta t}$ (B varies linearly with). For [0; 100ms]: $e = \frac{500\times\pi\times0.05^2}{4} \times \frac{(0.5-0)\times10^{-3}}{(100-0)\times10^{-3}} = 4.9 \times 10^{-3}V$. For [400ms; 500ms]: $e = \frac{500\times\pi\times0.05^2}{4} \times \frac{(0-0.5)\times10^{-3}}{(500-100)\times10^{-3}} = -1.227 \times 10^{-3}V$. $u_{AC} = ri - e = -e \Rightarrow u_{CA} = e$. $u_{CA} = Ri \Rightarrow i = \frac{u_{CA}}{R}$. For [0; 100ms]: $u_{CA} = e = 4.9 \times 10^{-3}V$. $i = \frac{4.9\times10^{-3}}{10} = 4.9 \times 10^{-4}A$. For [400ms; 500ms]: $u_{CA} = e = -1.227 \times 10^{-3}V$. $i = \frac{-1.227\times10^{-3}}{10} = -0.1227 \times 10^{-4}A$.

II.1	Self induction.
II.2	$u_{AC} = ri - e = ri + L\frac{di}{dt}$ with $r = 0$ and $e = -L\frac{di}{dt}$. $u_{AC} = L\frac{di}{dt} = L\frac{\Delta i}{\Delta t}$ (i varies linearly with time). For [0; 400ms]: $u_{AC} = 0.01 \times \frac{2-0}{(400-0)\times 10^{-3}} = 0.05V$. For [400ms; 500ms]: $u_{AC} = 0.01 \times \frac{0-2}{(500-400)\times 10^{-3}} = -0.2V$.
II.3	For [0ms; 400ms] since i increases $\Longrightarrow \frac{di}{dt} > 0 \Longrightarrow e < 0$ and $i > 0 \Longrightarrow e.i < 0$.
II.4	For [0; 400ms]: $i = at + b$. $a = \frac{\Delta i}{\Delta t} = \frac{2-0}{(400-0\times 10^{-3})} = 5A/s$. For $t = 0; i = b = 0$. Therefore, $i = 5t$. For [400ms; 500ms]: $i = at + b$. $a = \frac{\Delta i}{\Delta t} = \frac{0-2}{(500-400)\times 10^{-3}} = -20A/s$. For $t = 400ms; i = 2A$. $2 = -20 \times 400 \times 10^{-3} + b \Longrightarrow b = 10A$. Therefore, $i = -20t + 10$. By applying the law of addition of voltages: $u_G = u_L + u_R = u_L + Ri$. For [0; 400ms]: $u_G = 0.05 + 10(5t) = 50t + 0.05$. At t = 400ms; $u_G = 50 \times 0.4 + 0.05 = 20.05V$. For [400ms; 500ms]: $u_G = -0.2 + 10(-20t + 10) = -200t + 99.8$. At t = 400ms; $u_G = -200 \times 0.4 + 99.8 = 19.8V$. At t = 500ms; $u_G = -200 \times 0.5 + 99.8 = -0.2V$. 

Exercise 22:

Part	Answer key
1.1	The current sent by the generator will traverse the coil and the resistor.
1.2	Channel 1: horizontal straight line. Channel 2: increasing exponential function
2.1	$u_{AB} = u_L = ri + L\frac{di}{dt}$. $u_{BC} = u_R = Ri$.
2.2	By applying the law of addition of voltages: $u_G = u_L + u_R$. $E = ri + L\frac{di}{dt} + Ri \Longrightarrow E = L\frac{di}{dt} + (R + r)i$.

	$E = L\frac{di}{dt} + R_{eq}\, i \Longrightarrow \frac{E}{L} = \frac{di}{dt} + \frac{R_{eq}}{L}\, i.$
2.3	$i = \frac{E}{R_{eq}}(1 - e^{-kt}).$ $\frac{di}{dt} = \frac{Ek}{R_{eq}} e^{-kt}.$ Replace i and $\frac{di}{dt}$ in the differential equation: $\frac{E}{L} = \frac{Ek}{R_{eq}} e^{-kt} + \frac{E}{L} - \frac{E}{L} e^{-kt}.$ $\frac{E}{L} = \frac{E}{L} + e^{-kt}\left(\frac{k}{R_{eq}} - \frac{1}{L}\right).$ By identification: $\frac{k}{R_{eq}} = \frac{1}{L} \Longrightarrow k = \frac{R_{eq}}{L}.$
2.4	At $i \to \infty; i = I_0 = \frac{E}{R_{eq}}(1 - e^{-\infty}) = \frac{E}{R_{eq}}.$
2.5	At t = 0; $i = 0.$ $E = L\frac{di}{dt} \Longrightarrow \frac{di}{dt} = \frac{E}{L}.$
3	Time constant: time needed for the current to reach 63% of its maximum value.
4	$\frac{di}{dt} = \frac{E}{L} = \frac{R_{eq} I_0}{L} = kI_0.$
5.1	Decreasing exponential function.
5.2	Law of addition of voltages: $u_{AC} + u_{AB} + u_{BC} \Longrightarrow 0 = u_L + u_R$ where $u_{AC} = 0$ (ideal diode). $ri + L\frac{di}{dt} + Ri = 0 \Longrightarrow L\frac{di}{dt} + (R + r)i = 0.$ $\frac{di}{dt} + \frac{R_{eq}}{L} i = 0.$
5.3	$\frac{di}{dt} = -\frac{R_{eq}}{L} i.$ $\frac{di}{di} = -\frac{R_{eq}}{L} dt.$ $\ln i = -\frac{R_{eq}}{L} t + K.$ $i = K' e^{-\frac{R_{eq}}{L}t}.$ At t = 0s; $i = I_0 \Longrightarrow K' = I_0.$ Therefore, $i = I_0 e^{-\frac{R_{eq}}{L}t} = I_0 e^{-\frac{t}{\tau}}$ where $\tau = \frac{L}{R_{eq}}.$
5.4	$\frac{di}{dt} = -\frac{I_0}{\tau} e^{-\frac{t}{\tau}}.$ At t = 0s; $\frac{di}{dt} = \frac{I_0}{\tau} e^{0} = \frac{I_0}{\tau}.$

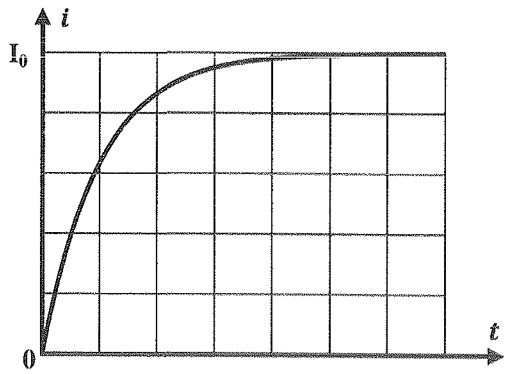
Exercise 24:

Part	Answer key
I.1	Law of addition of voltages: $E = u_R + u_L.$ $E = Ri + L\frac{di}{dt}.$
I.2	$i = I_0\left(1 - e^{-\frac{t}{\tau}}\right) = I_0 - I_0 e^{-\frac{t}{\tau}}$ and $\frac{di}{dt} = \frac{I_0}{\tau} e^{-\frac{t}{\tau}}.$ Replace i and $\frac{di}{dt}$ in the differential equation: $E = RI_0 - RI_0 e^{-\frac{t}{\tau}} + \frac{LI_0}{\tau} e^{-\frac{t}{\tau}} \Longrightarrow E = RI_0 + I_0 e^{-\frac{t}{\tau}}\left[-R + \frac{L}{\tau}\right].$ By identification: $E = RI_0 \Longrightarrow I_0 = \frac{E}{R}$ and $-R + \frac{L}{\tau} = 0 \Longrightarrow \tau = \frac{L}{R}.$

I.3	At t = 0s; $i = I_0(1 - e^0) = 0$. At t = τ; $i = I_0(1 - e^{-1}) = 0.63I_0$. At t = 5τ; $i = I_0(1 - e^{-5}) = 0.99I_0 \simeq I_0$.
I.4	Graph.
II.1	$\vec{W}$: Weight. $\vec{f}$: Resistive force.
II.2	$\sum \vec{F}_{ext} = m\vec{a} \Longrightarrow \vec{W} + \vec{f} = m\frac{d\vec{v}}{dt}$. Projection along y-axis: $mg - hv = m\frac{dv}{dt} \Longrightarrow mg = hv + m\frac{dv}{dt}$.
III.1	(see table below)
III.2	$v = V_0\left(1 - e^{-\frac{t}{\tau}}\right)$ where $V_0 = \frac{mg}{h}$ and $\tau = \frac{m}{h}$
III.3	Graph.

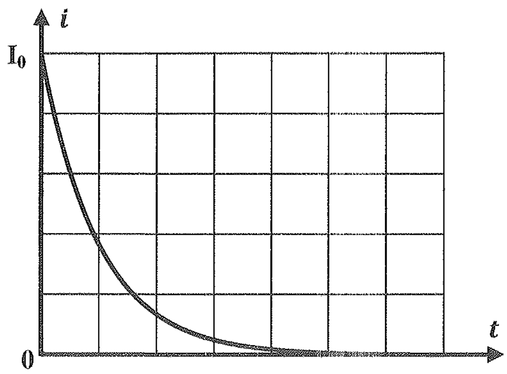
Mechanical quantity	Electric quantity
mg	E
h	R
v	i
m	L

Exercise 25:

Part	Answer key
1	The diode allows the passage of the electric current in one direction.
2	
3	Law of addition of voltages in series connection: $u_G = u_K + u_R + u_L$. $E = 0 + Ri + ri + L\frac{di}{dt}$ with $r = 0$. $E = Ri + L\frac{di}{dt} \Longrightarrow \frac{di}{dt} = \frac{E-Ri}{L} \Longrightarrow \frac{di}{dt}\Big\vert_{t=0} = \frac{E}{L} = \frac{12}{3} = 4A/s$ with $i = 0$ at t = 0s.
4	**Differential equation in u_R.** $E = u_L + u_R \Longrightarrow E = L\frac{di}{dt} + u_R$. $E = L\frac{di}{dt} + Ri$ with $u_R = Ri \Longrightarrow i = \frac{u_R}{R} \Longrightarrow \frac{di}{dt} = \frac{1}{R}\frac{du_R}{dt}$. $E = \frac{L}{R}\frac{du_R}{dt} + u_R$. **Differential equation in u_L.** $E = u_R + u_L \Longrightarrow E = Ri + u_L$ with $u_L = L\frac{di}{dt} \Longrightarrow \frac{di}{dt} = \frac{u_L}{L}$.

	Derive both sides with respect to time: $0 = R\frac{di}{dt} + \frac{du_L}{dt} \Longrightarrow 0 = \frac{R}{L}u_L + \frac{du_L}{dt}$.
5	As the switch is opened, the induced current has the direction as before according to Lenz's law. The diode is forwards biased. The current decreases gradually from a maximum value to zero.

Exercise 26:

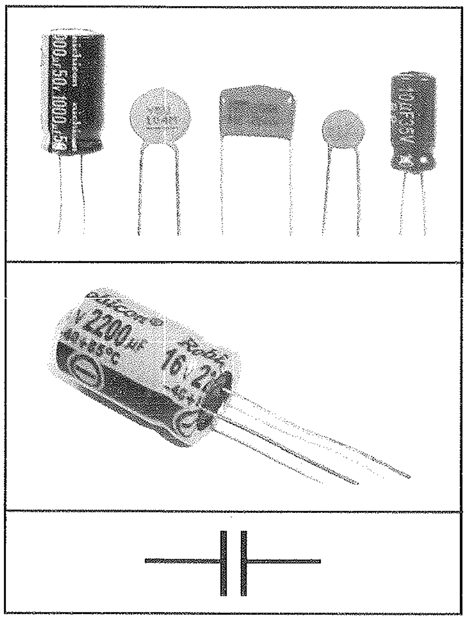
Part	Answer key
1.1	A diode allows the passage of electric current in one direction.
1.2	Law of addition of voltages in series connection: $u_G = u_K + u_L$. $E = 0 + ri + L\frac{di}{dt}$. At the steady state, $i = I_0 = constant \Longrightarrow \frac{di}{dt} = 0$. $E = rI_0 \Longrightarrow I_0 = \frac{E}{r} = \frac{9}{60} = 0.15A$.
1.3	$W = \frac{1}{2}LI_0^2 = \frac{1}{2} \times 0.4 \times 0.15^2 = 4.5 \times 10^{-3}J$ (magnetic energy).
2.1	Law of addition of voltages: $u_{AA} = u_{AB} + u_{BC} + u_{CA} \Longrightarrow 0 = ri + L\frac{di}{dt} + Ri + 0$ (ideal diode). At $t_0 = 0s$, $i = I_0 = 0.15A$. $\left.\frac{di}{dt}\right\|_{t=0} = -\frac{(R+r)I_0}{L} = -\frac{(40+60)(0.15)}{0.4} = -37.5A/s$.
2.2	$u_{AB} = u_L = rI_0 + L\left.\frac{di}{dt}\right\|_{t=0} = 60 \times 0.15 - 0.4 \times 37.5 = -6V$ (the voltage across the terminals of the coil is negative since it acts as a generator).
2.3	

CHAPTER 5 – CAPACITORS
COURSE

5.1- INTRODUCTION

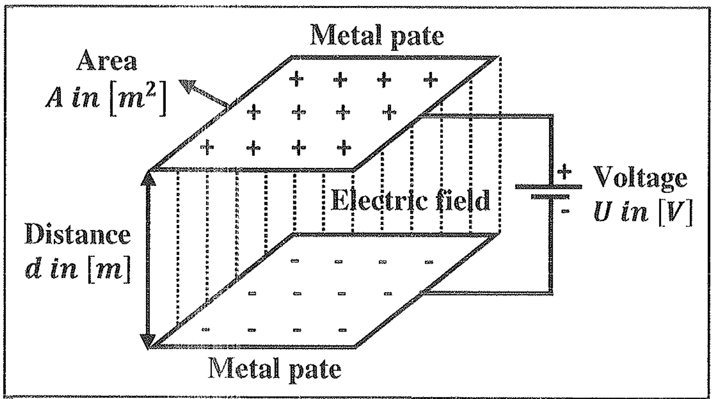
When you pull back the string of an archer's bow, you are storing mechanical energy as elastic potential energy. A capacitor is a device that stores electric potential energy and electric charge. To make a capacitor, just insulate two conductors from each other. To store energy in this device, transfer charge from one conductor to the other so that one has a negative charge and the other has an equal amount of positive charge. Work must be done to move the charges through the resulting potential difference between the conductors, and the work done is stored as electric potential energy.
Capacitors have a tremendous number of applications. Capacitors are widely used in electronic circuits. They store charge which can later be released, as in camera flash, and as energy backup in computers if the power fails. Capacitors can protect sensitive components by smoothing out variations in voltage due to power surges. Very tiny capacitors serve as memory in the random access memory (RAM) of computers. Capacitors serve many other applications as well, such as those in telephones, radio and television receivers, and defibrillator (electric shock to treat heart failure)....

5.2- CAPACITORS AND CAPACITANCE



A capacitor, sometimes referred to as a condenser, is a device that can store electric charge, and consists of two conductors (armatures) of any shape placed near each other without touching. The conductors are called plates. The plates are electrically separated either by air (vacuum) or by some form of a good insulating material such as waxed paper, mica, ceramic, plastic or some form of a liquid gel as used in electrolytic capacitors. The insulating layer between capacitors plates is commonly called the dielectric.
In circuit diagrams, a capacitor is represented by the following symbol shown in the adjacent document.

When used in a DC circuit, a capacitor charges up to its supply voltage but blocks the flow of current through it because the dielectric of a capacitor is non-conductive and basically an insulator. When a DC generator (battery) is placed across a capacitor (each plate initially has zero net charge) by the aid of connecting wires, the two plates quickly become charged: one plate acquires a negative charge, the other an equal amount of positive charge but the net on the capacitor as a whole remains zero.



Each battery terminal and the plate of the capacitor connected to it are at the same electric potential; hence, the full battery voltage appears across the capacitor. Once the capacitor reaches its steady state condition, an electrical current is unable to flow through the capacitor itself and around the circuit due to the insulating properties of the dielectric used to separate the plates.

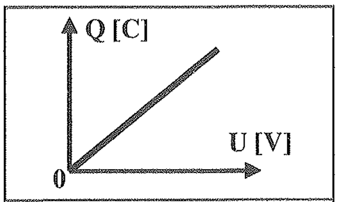
Any capacitor is characterized by its **capacitance** which is defined as the measure of a capacitor's ability to store charge and electric potential energy.

The capacitance C of a capacitor is the ratio of the magnitude of the charge Q on either conductor to the magnitude of the potential difference U between them:

$$C = \frac{Q}{U}$$

In SI units, Q is expressed in coulombs [C], U in volts [V] and C in farads [F] or [C/V], which was named in honor of the British scientists Michael Faraday.

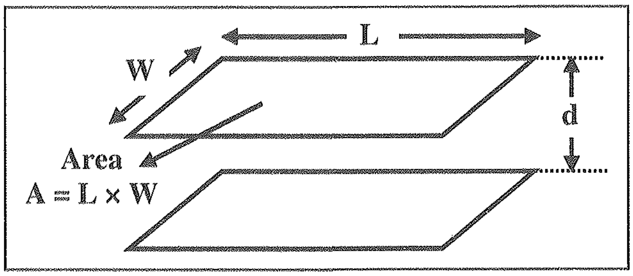
Other units: 1mF = 10^{-3}F, 1μF = 10^{-6}F, 1nF = 10^{-9}F and 1pF = 10^{-9}F.



The shape of the graph of Q versus U is a straight line passing through origin where C is the slope

5.3- CAPACITANCE OF A PARALLEL PLATE CAPACITOR

The capacitance C does not in general depend on the charge or on the potential difference between the plates. Its value depends only on the size, shape, relative positions of the two conductors and also on the material that separates them.



The capacitance C_0 of an air parallel-plate formed of two plates of common area A and separated by a distance d is:

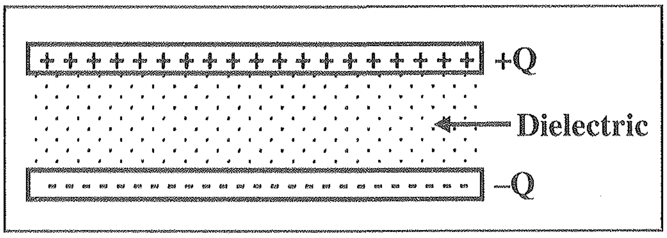
$$C_0 = \varepsilon_0 \frac{A}{d}$$

$\varepsilon_0 = 8.85 \times 10^{-12} C^2/Nm^2$ is the permittivity of free space or vacuum.

In SI units, C_0 is expressed in [F], A in [m^2] and d in [m].

Capacitors with dielectrics

If a dielectric completely fills the space between the plates, the capacitance increases by a dimensionless factor ε_r, which is called the dielectric constant or relative permittivity where the dielectric constant is a property of a material and varies from one material to another and of value greater than 1.



$$C = \varepsilon_r C_0$$

The capacitance of a parallel-plate capacitor in the presence of the dielectric is given by:

$C = \varepsilon_r C_0 = \varepsilon_r \varepsilon_0 \frac{A}{d} = \varepsilon \frac{A}{d}$ where $\varepsilon = \varepsilon_0 \varepsilon_r$ is the permittivity of the dielectric.

5.4- ENERGY STORED IN A CHARGED CAPACITOR

When a capacitor charges up from the power supply connected to it, an electrostatic field is established which stores energy in the capacitor. The electric potential energy that is stored in the charged capacitor is just equal to the amount of work required to charge it, that is, to separate opposite charges and place them on different conductors.

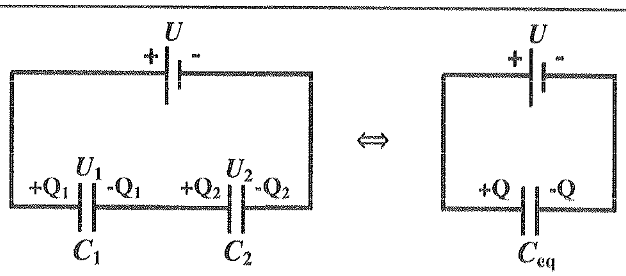
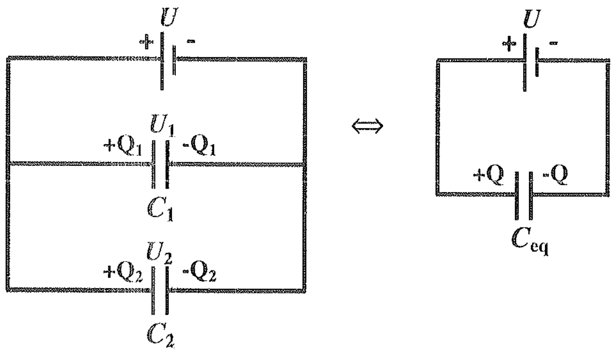
The **electric potential energy** stored in a capacitor is expressed by:

$$W = \frac{1}{2}CU^2 = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QU$$

In SI units, W is expressed in [J] , C in [F], U in [V] and Q in [C].

An electronic flash unit stores electric potential energy in a capacitor. Depressing the camera's shutter button provides the discharge of the capacitor; the stored energy is converted into a brief but intense flash of light.

5.5- Grouping of Capacitors

<table>
<tr><th>Series connection</th><th>Parallel connection</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>Voltage is additive: $U = U_1 + U_2$
Charge is unique: $Q = Q_1 = Q_2$
Equivalent capacitance: $\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$</td><td>Voltage is unique: $U = U_1 = U_2$
Charge is additive: $Q = Q_1 + Q_2$
Equivalent capacitance: $C_{eq} = C_1 + C_2$</td></tr>
</table>

5.6- THE TIME CONSTANT OF AN RC SERIES CIRCUIT

The charge on the plates of a capacitor is given as $Q = CU$. The charging (storage) and discharging (release) of a capacitor's energy is never instantaneous but takes a certain amount of time to occur.
If a resistor is connected in series with a capacitor forming an RC circuit, the capacitor will charge up gradually through the resistor until the voltage across the capacitor reaches that of the supply voltage. The time called the transient response, required for this to occur is equivalent to about 5 time constants or 5τ.
This transient response time T, is measured in terms of $\tau = R \times C$, in seconds, where R is the value of the resistor in ohms and C is the value of the capacitor in Farads. This then forms the basis of an RC charging circuit where 5 τ can also be thought of as "5 × RC".
So an RC circuit's time constant is a measure of how quickly it either charges or discharges.
The time constant τ is the time needed to complete 63% of the process (charging process or discharging process). It represents the time it takes the current to decrease to 37% of its initial value.
If R or C is increased, the time constant increases and the process of charging and discharging takes a longer time.

SI unit of the time constant τ

$R = \frac{u}{i} \Longrightarrow [\Omega] = \frac{[V]}{[A]}$.

$i = \frac{dq}{dt} \Longrightarrow [A] = \frac{[C]}{[s]} \Longrightarrow [C] = [A][s]$.

$C = \frac{q}{u} \Longrightarrow [F] = \frac{[C]}{[V]} = \frac{[A][s]}{[V]}$.

$\tau = RC \Longrightarrow$ SI unit of τ is $[\Omega].[F] = \frac{[V]}{[A]}.\frac{[A][s]}{[V]} = [s]$.

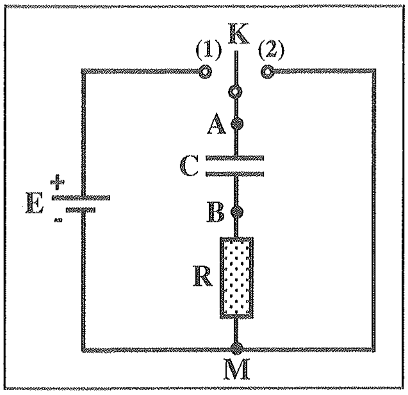
TIP

Electric current is defined as rate of flow of electric charge.
Average current: $i_{av} = \frac{\Delta q}{\Delta t}$.
Instantaneous current: $i = \lim_{\Delta t \to 0} \frac{\Delta q}{\Delta t} = \frac{dq}{dt}$.
Case of constant current:
$i = \frac{dq}{dt} = k \Longrightarrow q = kt + q_0$.

Note: sign convention of i

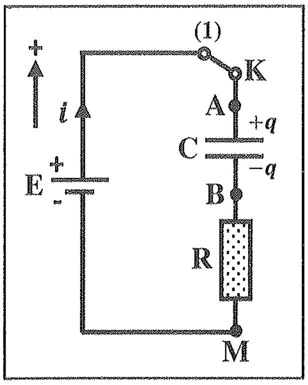
- If the positive sense enters the positive plate of the capacitor, then $i = +\frac{dq}{dt}$.
- If the positive sense leaves the positive plate of the capacitor, then $i = -\frac{dq}{dt}$.

5.7- CHARGING AND DISCHARGING OF A CAPACITOR



The circuit diagram shown in the adjacent documents consists of:
- an ideal DC generator G of emf E,
- a neutral capacitor (initially uncharged) of capacitance C,
- a resistor of resistance R,
- a double switch K,
- connecting wires of negligible resistance.

Charging phase



The switch is turned to position (1). When the switch is closed and the time begins at t = 0s, the current begins to flow and the capacitor will gradually charge up through the resistor until the voltage across it reaches the supply voltage of the battery.
The positively charged pole of the generator attracts the free electrons of the plate connected to it leaving a deficit number of electrons on it hence becomes positively charged.
Electrons flow from the negative pole of the generator to the neutral plate of the capacitor connected to it hence becomes negatively charged.
During charging, the voltage across the capacitor will attain 63% of its final value E after one time constant.

The voltage across a real generator G (E; r) is: $u_G = E - ri$
Ideal generator $\Rightarrow r = 0 \Rightarrow u_G = E$

Differential equation in u_C

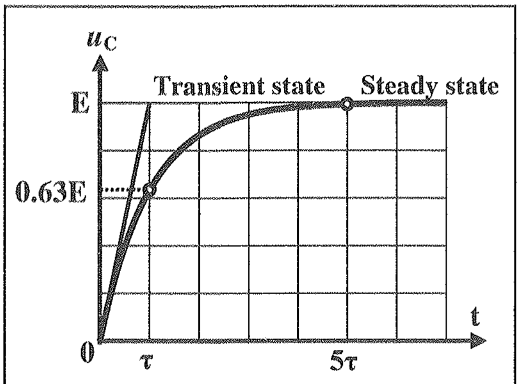
Law of addition of voltages in series connection:
$u_{AM} = u_{AB} + u_{BM} \Rightarrow u_G = u_C + u_R$.
$E = u_C + Ri$ with $u_R = Ri$.
$E = u_C + RC\frac{du_C}{dt}$ with $i = \frac{dq}{dt} = \frac{d(Cu_C)}{dt} = C\frac{du_C}{dt}$.
$\frac{du_C}{dt} + \frac{1}{RC}u_C = \frac{E}{RC}$ (First order differential equation in u_C during the charging process of the capacitor).

TIP

During the charging process, the current i enters the positive plate of the capacitor.
$i > 0A$ (is in the positive sense)
q increases with time $\Rightarrow \frac{dq}{dt} > 0$
So $i = +\frac{dq}{dt}$

Expression of u_C as a function of time

The solution of the differential equation has the form
$u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$ with $A = E$ is the maximum voltage across the terminals of the capacitor and $\tau = RC$ is the time constant of the RC circuit.
Therefore, $u_C = E\left(1 - e^{-\frac{t}{RC}}\right) = U_{Cm}\left(1 - e^{-\frac{t}{RC}}\right)$ with $u_{Cm} = E$.



u_C increases exponentially with time

The following expressions are deduced:

The charge stored in the plate A as a function of time is:
$q = Cu_C = CE\left(1 - e^{-\frac{t}{RC}}\right) = Q_m\left(1 - e^{-\frac{t}{RC}}\right)$ with $Q_m = CE$.
The current carried by the circuit as a function of time is:
$i = C\frac{du_C}{dt} = \frac{E}{R}e^{-\frac{t}{RC}} = I_m e^{-\frac{t}{RC}}$ with $I_m = \frac{E}{R}$.
The voltage across the terminals of the resistor as a function of time is: $u_R = Ri = Ee^{-\frac{t}{RC}} = U_{Rm}e^{-\frac{t}{RC}}$ with $U_{Rm} = E$.

Verification of the solution of the differential equation

$u_C = E\left(1 - e^{-\frac{t}{RC}}\right) = E - Ee^{-\frac{t}{RC}}$ and $\frac{du_C}{dt} = 0 - E\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}} = \frac{E}{RC}e^{-\frac{t}{RC}}$.

Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $\frac{E}{RC}e^{-\frac{t}{RC}} + \frac{E}{RC} - \frac{E}{RC}e^{-\frac{t}{RC}} = \frac{E}{RC} \Rightarrow \frac{E}{RC} = \frac{E}{RC}$ (verified).

Determination of the expressions of A and τ

$u_C = A\left(1 - e^{-\frac{t}{\tau}}\right) = A - Ae^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = 0 - A\left(-\frac{1}{\tau}e^{-\frac{t}{\tau}}\right) = \frac{A}{\tau}e^{-\frac{t}{\tau}}$.

Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = A - Ae^{-\frac{t}{\tau}} + RC\frac{A}{\tau}e^{-\frac{t}{\tau}} \Rightarrow E = A + Ae^{-\frac{t}{\tau}}\left(-1 + \frac{RC}{\tau}\right)$.

By identification: $E = A$ and $-1 + \frac{RC}{\tau} = 0 \Rightarrow \tau = RC$ with $Ae^{-\frac{t}{\tau}} = 0$ *is not true* $\forall\, t$.

TIP

$$i = \frac{dq}{dt} = C\frac{du_C}{dt}$$
$$\frac{du_C}{dt} = \frac{i}{C} \Rightarrow du_C = \frac{i}{C}dt$$
$$u_C = \frac{1}{C}\int i\,dt$$

Particular values of u_C

At $t = 0;\ u_C = E(1 - e^{-0}) = 0$.
At $t = \tau;\ u_C = E(1 - e^{-1}) = 0.63E$.
At $t = 5\tau;\ u_C = E(1 - e^{-5}) = 0.99E \simeq E$ or $t \to \infty;\ u_C = E(1 - e^{-\infty}) = E$.

Tangent to the curve of u_C at the instant $t_0 = 0$s

The time constant τ is the abscissa of the point of intersection F between the tangent to the curve of u_C at $t_0 = 0$s and the asymptote.

Equation of the asymptote: $u = E$.

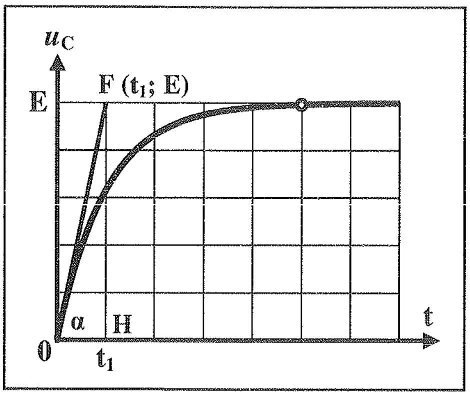
Equation of the tangent: the equation of a straight line passing through the origin is: $u = at$ where a is the slope.

$a = \left.\frac{du_C}{dt}\right|_{t_0=0} = \frac{E}{RC}e^0 = \frac{E}{RC}$.

Therefore, $u = \frac{E}{RC}t$.

$u = u \Rightarrow \frac{E}{RC}t_1 = E \Rightarrow t_1 = RC$.

Or $a = \tan\alpha = \frac{HF}{OH} = \frac{E}{t_1} \Rightarrow a = \left.\frac{du_C}{dt}\right|_{t_0=0} \Rightarrow \frac{E}{t_1} = \frac{E}{RC} \Rightarrow t_1 = RC$.



Differential equation in q

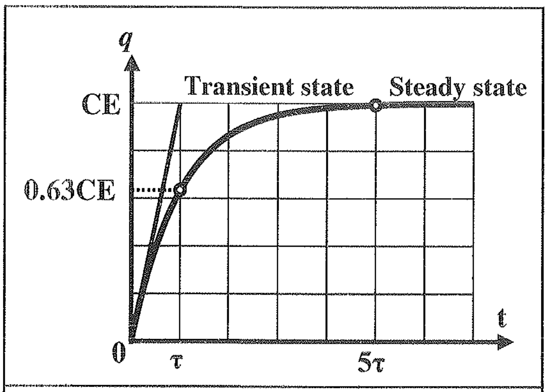
Law of addition of voltages in series connection:

$u_{AM} = u_{AB} + u_{BM} \Rightarrow u_G = u_C + u_R$.

$E = \frac{q}{C} + Ri$ with $u_C = \frac{q}{C}$ and $u_R = Ri$.

$E = \frac{q}{C} + R\frac{dq}{dt}$ with $i = \frac{dq}{dt}$.

$\frac{E}{R} = \frac{dq}{dt} + \frac{1}{RC}q$ (First order differential equation in q during the charging process of the capacitor).



q **increases exponentially with time**

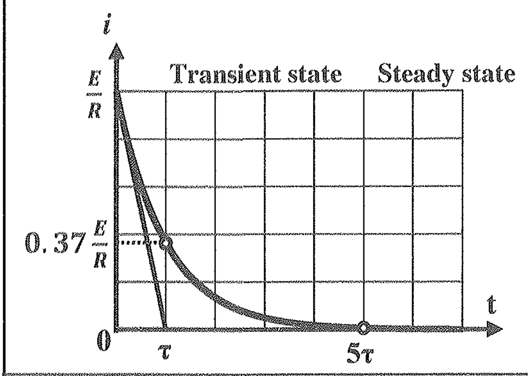
Expression of q as a function of time

The solution of the differential equation has the form $q = B\left(1 - e^{-\frac{t}{\tau}}\right)$ with $B = CE$ is the maximum charge stored the capacitor.

Therefore, $q = CE\left(1 - e^{-\frac{t}{RC}}\right) = Q_m\left(1 - e^{-\frac{t}{RC}}\right)$ with $Q_m = CE$.

Particular values of q
At $t = 0$; $q = CE(1 - e^{-0}) = 0$.
At $t = \tau$; $q = CE(1 - e^{-1}) = 0.63CE$.
At $t = 5\tau$; $q = CE(1 - e^{-5}) = 0.99CE \simeq CE$ or $t \to \infty$; $q = CE(1 - e^{-\infty}) = CE$.

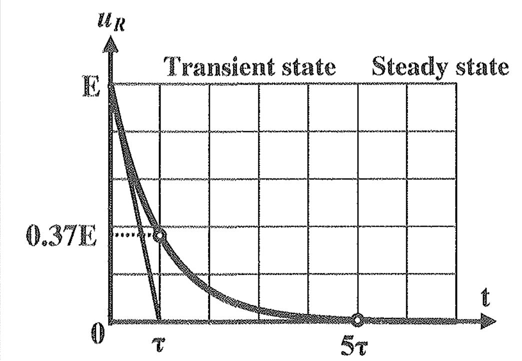
Differential equation in i
Law of addition of voltages in series connection:
$u_{AM} = u_{AB} + u_{BM} \Longrightarrow u_G = u_C + u_R$.
$E = \frac{q}{C} + Ri$ with $u_C = \frac{q}{C}$ and $u_R = Ri$.
Derive both sides with respect to time: $0 = \frac{1}{C}\frac{dq}{dt} + R\frac{di}{dt}$
$\frac{1}{C}i + R\frac{di}{dt} = 0$ with $i = \frac{dq}{dt}$.
$\frac{di}{dt} + \frac{1}{RC}i = 0$ (First order differential equation in i during the charging process of the capacitor).



i decreases exponentially with time

Expression of i as a function of time
The solution of the differential equation has the form:
$i = De^{-\frac{t}{\tau}}$ where $D = \frac{E}{R}$ is the maximum current carried by the circuit.
Therefore, $i = \frac{E}{R}e^{-\frac{t}{RC}} = I_m e^{-\frac{t}{RC}}$.

Particular values of i
At $t = 0$; $i = \frac{E}{R}e^{0} = \frac{E}{R}$.
At $t = \tau$; $i = \frac{E}{R}e^{-1} = 0.37\frac{E}{R}$.
At $t = 5\tau$; $i = \frac{E}{R}e^{-5} \simeq 0$ or $t \to \infty$; $i = \frac{E}{R}e^{-\infty} = 0$.



u_R decreases exponentially with time

Differential equation in u_R
Law of addition of voltages in series connection:
$u_{AM} = u_{AB} + u_{BM} \Longrightarrow u_G = u_C + u_R$.
$E = \frac{q}{C} + u_R$ with $u_C = \frac{q}{C}$.
Derive both sides with respect to time:
$0 = \frac{1}{C}\frac{dq}{dt} + \frac{du_R}{dt}$ with $i = \frac{u_R}{R} = \frac{dq}{dt}$.
$\frac{du_R}{dt} + \frac{1}{RC}u_R = 0$ (first order differential equation in u_R during the charging process of the capacitor)

Expression of u_R as a function of time
The solution of the differential equation has the form:
$u_R = De^{-\frac{t}{\tau}}$ where $D = E$ is th maximum voltage across the terminals of the capacitor.
Therefore, $u_R = Ee^{-\frac{t}{RC}} = U_{Rm}e^{-\frac{t}{RC}}$.

Natural logarithm of u_R
$u_R = Ee^{-\frac{t}{RC}} \Longrightarrow \ln u_R = \ln\left(Ee^{-\frac{t}{RC}}\right)$.
$\ln u_R = \ln E + \ln e^{-\frac{t}{RC}}$.
$\mathbf{\ln u_R = \ln E - \frac{t}{RC}}$.
The expression of $\ln u_R$ as a function of time has the form of the general equation of a straight line $\mathbf{\ln u_R = at + b}$ where $\mathbf{a = -\frac{1}{RC} = slope}$ and $\mathbf{b = \ln E}$

Particular values of u_R
At $t = 0; u_R = Ee^0 = E.$
At $t = \tau; u_R = Ee^{-1} = 0.37E.$
At $t = 5\tau; u_R = Ee^{-5} \simeq 0$ or $t \to \infty; i = Ee^{-\infty} = 0.$

The table below gives u_C, q, u_R and i at specific instants which are characteristics of an RC charging circuit:

	t = 0	t = τ = RC	t = 5τ or t → ∞
u_C	0	0.63E	E
q	0	0.63CE	CE
u_R	E	0.37E	0
i	$\frac{E}{R}$	$0.37\,\frac{E}{R}$	0

The current and the voltage across the resistor approach zero exponentially with time.
The curve of the voltage across the capacitor as function of time during this phase is exponentially increasing (growth) from zero to E.
The capacitor now starts to charge up as shown, with the rise in the RC charging curve steeper at the beginning because the charging rate is fastest at the start and then tapers off as the capacitor takes on additional charge at a slower rate.
As the capacitor charges up, the potential difference across its plates slowly increases with the actual time taken for the charge on the capacitor to reach 63% of its maximum possible voltage
At its final condition greater than five time constants, the capacitor is said to be fully charged;
thus, at t → ∞, $i = 0$, $u_C = E$ and $q = CE$.
After a period equivalent to 4 time constants 4τ, the capacitor in this RC charging circuit is virtually fully charged and the voltage across the capacitor is now approximately 99% of its maximum value, 0.99E. The time period taken for the capacitor to reach this 4τ point is known as the **transient period.**
After a time of 5τ, the capacitor is now fully charged and the voltage across the capacitor $u_C c$ is equal to the supply voltage E; thus, no more current flows in the circuit. The time period after this 5τ point is known as the **steady state period**.
Once fully charged, a capacitor in a DC circuit acts like an open switch in the branch in which it is placed. This property is used in many electronic circuits to remove a DC voltage component from a signal.

Methods to determine the time constant τ
Method 1: direct calculation using the relation $\tau = RC$ if R and C are given.

Method 2: the electromotive force E of a generator (assumed ideal) is given:
1- Determine the voltage across the capacitor at $t = \tau$; $u_C = 63\%E = 0.63E.$
2- Draw the horizontal straight line $u_C = 0.63E.$
3- Specify the point of intersection between this straight line and the graph.
4- Project this point to the time axis to deduce the value of τ.

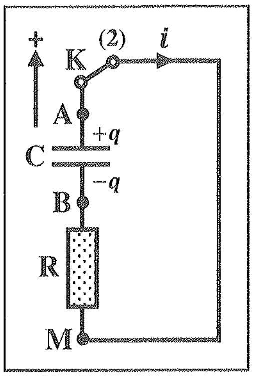
Method 3: E is unknown
1- Draw the tangent to the curve at t = 0s.
2- Draw the horizontal asymptote to the graph whose equation is $u_C = E$.
3- Specify the point of intersection between the tangent and the asymptote.
4- Project to the time axis to deduce the value of τ.
The time needed for the capacitor to complete its charging process is $t = 5\tau = 5RC.$

Discharging phase

The capacitor is fully charged. The switch is turned to position (2) at a new origin of time $t_0 = 0s$; at this instant $u_C = E$ and $q = Q_m$. The capacitor then discharges through the resistor, and its charge eventually decreases to zero. The current and the voltage approach zero exponentially with time.

During discharging:

- the direction of the current is opposite to that in the case of charging since the current in the case of discharging leaves the positive plate of the capacitor,
- the voltage across the capacitor will lose 63% of its initial value E after one time constant or the voltage across the capacitor becomes 37% of its initial value.



TIP

During the discharging process, the current i leaves the positive plate of the capacitor.
$i > 0A$ (is in the positive sense)
q decreases with time $\Longrightarrow \frac{dq}{dt} < 0$
So $i = -\frac{dq}{dt}$

Differential equation in u_C

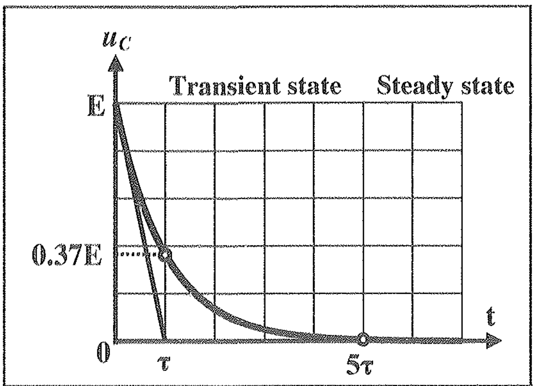
$u_{AB} = u_{AM} + u_{MB} \Longrightarrow u_C = 0 + u_R$.
$u_C = Ri$ with $u_R = Ri$.
$u_C = -RC\frac{du_C}{dt} = 0$ with $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$.
$u_C + RC\frac{du_C}{dt} = 0 \Longrightarrow \frac{du_C}{dt} + \frac{1}{RC}u_C = 0$ (First order differential equation in u_C during the discharging process of the capacitor).

Expression of u_C as a function of time

The solution of the differential equation has the form:

$u_C = Ae^{-\frac{t}{\tau}}$ with $A = E$ is the maximum voltage across the terminals of the capacitor and $\tau = RC$ is the time constant.

Therefore, $u_C = Ee^{-\frac{t}{RC}} = U_{Cm}e^{-\frac{t}{RC}}$.



u_C **decreases exponentially with time**

Verification of the solution of the differential equation

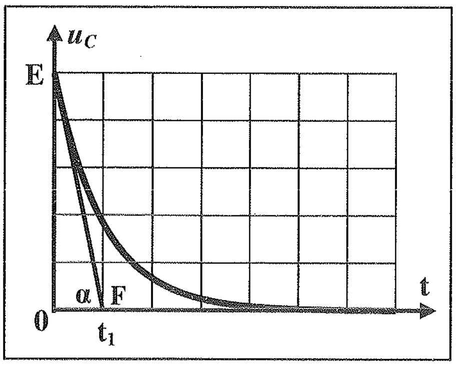
$u_C = Ee^{-\frac{t}{RC}}$ and $\frac{du_C}{dt} = -\frac{E}{RC}e^{-\frac{t}{RC}}$.
Replace u_C and $\frac{du_C}{dt}$ in the differential equation:
$-\frac{E}{RC}e^{-\frac{t}{RC}} + \frac{E}{RC}e^{-\frac{t}{RC}} = 0 \Longrightarrow 0 = 0$ (verified).

Determination of the expressions of A and τ

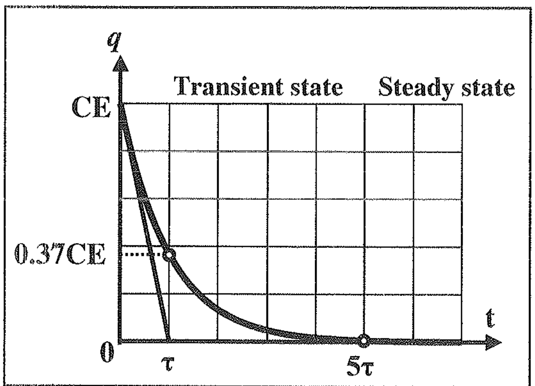
$u_C = Ae^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = -\frac{A}{\tau}e^{-\frac{t}{\tau}}$.
Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $-\frac{A}{\tau}e^{-\frac{t}{\tau}} + \frac{A}{RC}e^{-\frac{t}{\tau}} = 0 \Longrightarrow Ae^{-\frac{t}{\tau}}\left(-\frac{1}{\tau} + \frac{1}{RC}\right) = 0$.
$Ae^{-\frac{t}{\tau}} = 0$ *is not true* $\forall\, t$. Then, $-\frac{1}{\tau} + \frac{1}{RC} = 0 \Longrightarrow \tau = RC$.
At $t_0 = 0s$, $u_C = E \Longrightarrow E = Ae^0 \Longrightarrow A = E$.

Particular values of u_C

At $t = 0; u_C = Ee^0 = E$.
At $t = \tau = RC;\ u_C = Ee^{-1} = 0.37E$.
At $t = 5\tau = 5RC; u_C = Ee^{-5} \simeq 0$ or $t \to \infty; u_C = Ee^{-\infty} = 0$.



Tangent to the curve of u_C at $t_0 = 0s$
The time constant τ is the abscissa of the point of intersection F between the tangent to the curve of u_C at $t_0 = 0s$ and the time axis.
Equation of the tangent: the equation of a straight line is $u = at + b$ where a is the slope and b in the u_C intercept.
At t = 0s, $u = b = E$ and $a = \frac{du_C}{dt}\Big|_{t_0=0} = -\frac{E}{RC}e^0 = -\frac{E}{RC}t.$
Therefore, $u = -\frac{E}{RC}t + E.$
$u = 0 \Longrightarrow -\frac{E}{RC}t_1 + E = 0 \Longrightarrow t_1 = RC.$
Or $a = -\tan\alpha = -\frac{E}{t_1} \Longrightarrow -\frac{E}{t_1} = -\frac{E}{RC} \Longrightarrow t_1 = RC.$

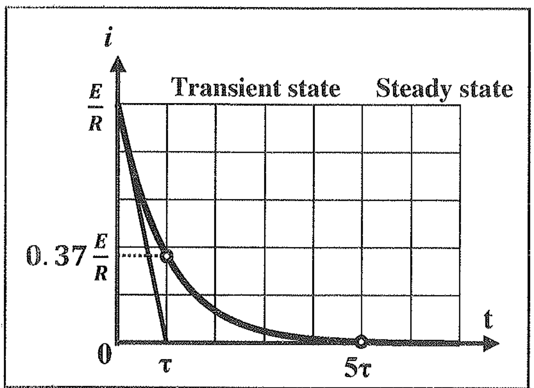


q decreases exponentially with time

Differential equation in q
$u_{AB} = u_{AM} + u_{MB} \Longrightarrow u_C = 0 + u_R.$
$\frac{q}{C} = Ri$ with $u_C = \frac{q}{C}$ and $u_R = Ri.$
$\frac{q}{C} = -R\frac{dq}{dt}$ with $i = -\frac{dq}{dt}.$
$\frac{q}{C} + R\frac{dq}{dt} = 0 \Longrightarrow \frac{dq}{dt} + \frac{1}{RC}q = 0$ (First order differential equation in q during the discharging process of the capacitor).

Expression of q as a function of time
The solution of the differential equation has the form:
$q = Be^{-\frac{t}{\tau}}$ with $B = CE$ is the maximum charge stored in the plate of the capacitor.
Therefore, $q = CEe^{-\frac{t}{RC}} = Q_m e^{-\frac{t}{\tau}}.$

Particular values of q
At $t = 0; q = CEe^0 = CE.$
At $t = \tau = RC;\ q = CEe^{-1} = 0.37CE.$
At $t = 5\tau = 5RC; q = CEe^{-5} \simeq 0$ or $t \to \infty; q = CEe^{-\infty} = 0.$



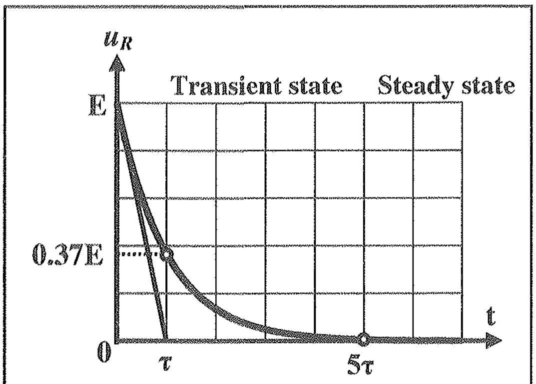
i decreases exponentially with time

Differential equation in i
$u_{AB} = u_{AM} + u_{MB} \Longrightarrow u_C = 0 + u_R.$
$\frac{q}{C} = Ri$ with $u_C = \frac{q}{C}$ and $u_R = Ri.$
Derive both sides with respect to time: $\frac{1}{C}\frac{dq}{dt} = R\frac{di}{dt}.$
$-\frac{1}{C}i = R\frac{di}{dt}$ with $i = -\frac{dq}{dt} \Longrightarrow \frac{dq}{dt} = -i.$
$R\frac{di}{dt} + \frac{1}{C}i = 0 \Longrightarrow \frac{di}{dt} + \frac{1}{RC}i = 0$ (First order differential equation in i during the discharging process of the capacitor).

Expression of i as a function of time
The solution of the differential equation has the form:
$i = De^{-\frac{t}{\tau}}$ where $D = \frac{E}{R}$ is the maximum current carried by the circuit.
Therefore, $i = \frac{E}{R}e^{-\frac{t}{RC}} = I_m e^{-\frac{t}{RC}}.$

Particular values of i
At $t = 0; i = \frac{E}{R}e^0 = \frac{E}{R}$.
At $t = \tau; i = \frac{E}{R}e^{-1} = 0.37\frac{E}{R}$.
At $t = 5\tau; i = \frac{E}{R}e^{-5} \simeq 0$ or $t \to \infty; i = \frac{E}{R}e^{-\infty} = 0$.

Differential equation in u_R
$u_{AB} = u_{AM} + u_{MB} \Rightarrow u_C = 0 + u_R$.
$\frac{q}{C} = u_R$ with $u_C = \frac{q}{C}$.
Derive both sides with respect to time: $\frac{1}{C}\frac{dq}{dt} = \frac{du_R}{dt}$.
$-\frac{1}{RC}u_R = \frac{du_R}{dt}$ with $i = \frac{u_R}{R} = -\frac{dq}{dt} \Rightarrow \frac{dq}{dt} = -\frac{u_R}{R}$.
$\frac{du_R}{dt} + \frac{1}{RC}u_R = 0$ (First order differential equation in u_R during the discharging process of the capacitor).



u_R decreases exponentially with time

Expression of u_R as a function of time
The solution of the differential equation has the form:
$u_R = De^{-\frac{t}{\tau}}$ where $D = E$ is th maximum voltage across the terminals of the capacitor.
Therefore, $u_R = Ee^{-\frac{t}{RC}} = U_{Rm}e^{-\frac{t}{RC}}$.

Particular values of u_R
At $t = 0; u_R = Ee^0 = E$.
At $t = \tau; u_R = Ee^{-1} = 0.37E$.
At $t = 5\tau; u_R = Ee^{-5} \simeq 0$ or $t \to \infty; i = Ee^{-\infty} = 0$.

The table below gives u_C, q, u_R and ia t specific instants which are characteristics of an RC discharging circuit:

	t = 0	t = τ = RC	t = 5τ or t → ∞
u_C	E	0.37E	0
q	CE	0.37CE	0
u_R	E	0.37E	0
i	$\frac{E}{R}$	$0.37\frac{E}{R}$	0

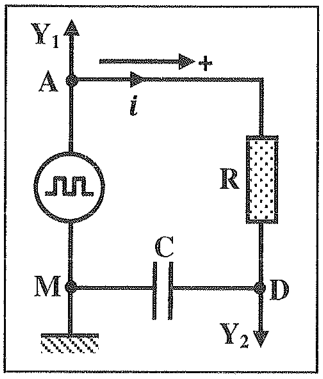
With the switch closed, the capacitor now starts to discharge, with the decay in the RC discharging curve steeper at the beginning because rate is fastest at the start and then tapers off as the capacitor loses charge at a slower rate. As the discharge continues, u_C goes down and there is less discharge current.

Attention:
If the same resistor is used for both modes of charging and discharging for the same capacitor, the duration of charging is equal to that of discharging. If the resistors are different, the longer duration is when using the larger resistor. In other words, the faster process is the one having the smaller resistor.

5.8- VISUALIZATION OF THE CHARGING AND DISCHARGING OF A CAPACITOR

Equipment

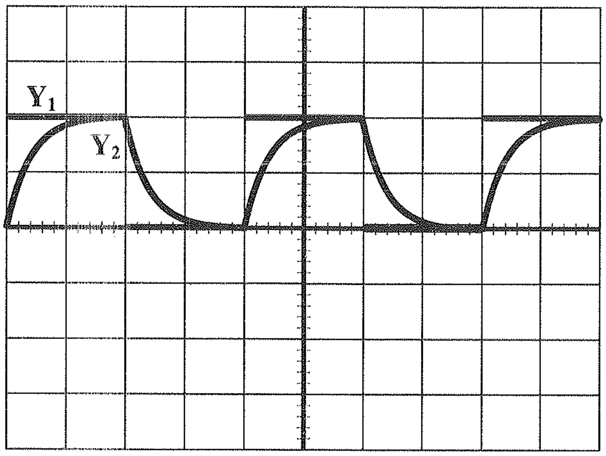
- Low frequency generator of negligible internal resistance.
- A resistor of resistance R.
- A neutral capacitor of capacitance C.
- An oscilloscope.
- Connecting wires.



Procedure

- Set up the circuit shown in the adjacent document.
- Adjust the low frequency generator to give a square signal voltage of period T such that:

$$u_{AM} = \begin{cases} E & for\ 0 \leq t \leq \frac{T}{2} \\ 0 & for\ \frac{T}{2} \leq t \leq T \end{cases}$$

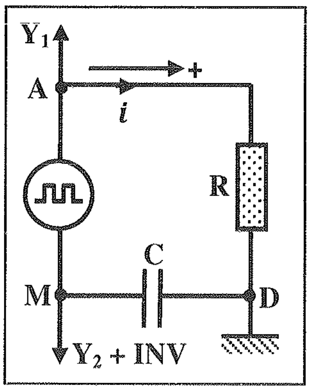


Displaying the voltages across the generator and across the capacitor

Connect the oscilloscope in order to displays on channel Y_1 the voltage $u_{AM} = u_G$ across the terminals of the generator and on channel Y_2 the voltage $u_{DM} = u_C$ across the terminals of the capacitor.

Observation

The oscilloscope's screen is represented in the adjacent document.

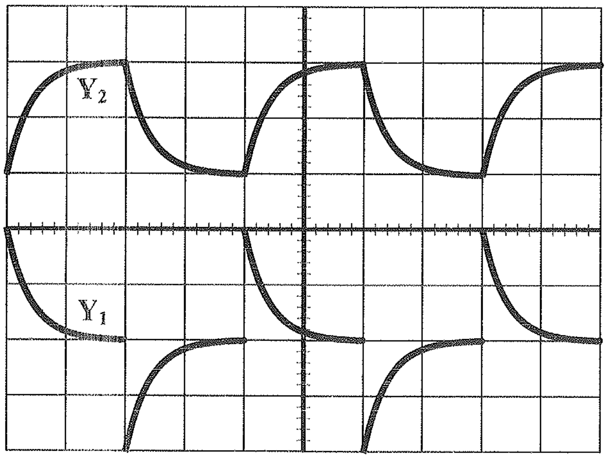


Displaying the voltages across the resistor and across the capacitor

Procedure

Connect the oscilloscope in order to display on channel Y_1 the voltage $u_{AD} = u_R$ across the terminal of the resistor and on channel Y_2 + INV the voltage $u_{DM} = u_C$ across the terminals of the capacitor.

The invert button is pressed on channel Y2 in order to display u_{DM} instead of u_{MD} and to display u_{AD} and u_{DM} in the same direction at the same time. In other words, it is required to eliminate the anti-phase effect between u_{AD} and u_{DM}.

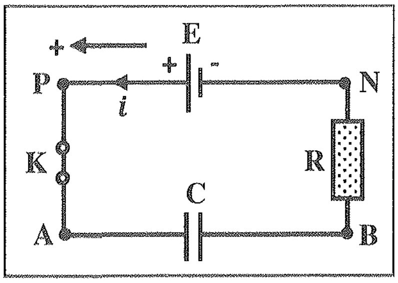


Notes:

If $5\tau > \frac{T}{2}$, the capacitor neither charges nor discharges completely.

If $5\tau \ll \frac{T}{2}$, u_C varies linearly with time such that $u_C = \frac{E}{RC}t$.

5.9- ENERGY CONSIDERATION IN RC SERIES CIRCUIT FED BY A CONSTANT VOLTAGE



A resistor of resistance R, a neutral capacitor of capacitance C and a switch K are connected in series with an ideal generator of constant emf E.
At $t_0 = 0s$, the switch K is closed.
At the instant t, the expression of the voltage across the terminals of the capacitor is $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$ and that of the current carried by the circuit is $i = \frac{E}{R}e^{-\frac{t}{\tau}}$ where $\tau = RC$ is the time constant of the circuit.
Law of addition of voltages in series connection: $u_{PN} = u_{PA} + u_{AB} + u_{BN}$
$E = u_C + u_R \Rightarrow E = u_C + Ri$ with $u_{PA} = u_K = 0$ (closed switch) and $u_R = Ri$.
Multiply both sides by i, we get: $Ei = u_C i + Ri^2 \Rightarrow P_G = P_C + P_R$
The electric power supplied by the generator is: $P_G = Ei$.
The electric power received by the capacitor is: $P_C = u_C i$.
The power dissipated by the resistor due to Joule's effect is: $P_R = Ri^2$.

Expression of the electric energy supplied by the generator as a function of time

$P_G = \frac{dW_G}{dt} \Rightarrow dW_G = P_G dt \Rightarrow W_G = \int_0^t P_G dt.$

$W_G = \int_0^t Eidt = \int_0^t \frac{E^2}{R}e^{-\frac{t}{\tau}}dt = \left[-\frac{\tau E^2}{R}e^{-\frac{t}{\tau}}\right]_0^t = CE^2\left[-e^{-\frac{t}{\tau}}\right]_0^t$ with $\tau = RC \Rightarrow C = \frac{\tau}{R}$.

$W_G = CE^2\left(-e^{-\frac{t}{\tau}} + e^0\right) = CE^2\left(1 - e^{-\frac{t}{\tau}}\right).$

Expression of the electric energy received by the capacitor as a function of time

$P_C = \frac{dW_C}{dt} \Rightarrow dW_C = P_C dt \Rightarrow W_C = \int_0^t P_C dt.$

$W_C = \int_0^t u_C idt = \int_0^t \frac{E^2}{R}\left(e^{-\frac{t}{\tau}} - e^{-\frac{2t}{\tau}}\right)dt = \left[\frac{E^2}{R}\left(-\tau e^{-\frac{t}{\tau}} + \frac{\tau}{2}e^{-\frac{2t}{\tau}}\right)\right]_0^t = \left[\frac{\tau E^2}{R}\left(-e^{-\frac{t}{\tau}} + \frac{1}{2}e^{-\frac{2t}{\tau}}\right)\right]_0^t.$

$W_C = CE^2\left[-e^{-\frac{t}{\tau}} + \frac{1}{2}e^{-\frac{2t}{\tau}}\right]_0^t = CE^2\left(-e^{-\frac{t}{\tau}} + \frac{1}{2}e^{-\frac{2t}{\tau}} + e^0 - \frac{1}{2}e^0\right) = CE^2\left(\frac{1}{2} + \frac{1}{2}e^{-\frac{2t}{\tau}} - e^{-\frac{t}{\tau}}\right).$

$W_C = \frac{CE^2}{2}\left(1 - 2e^{-\frac{t}{\tau}} + e^{-\frac{2t}{\tau}}\right) = \frac{CE^2}{2}\left(1 - e^{-\frac{t}{\tau}}\right)^2 = \frac{1}{2}Cu_C^2.$

Expression of the energy dissipated by a resistor as a function of time

$P_R = \frac{dW_R}{dt} \Rightarrow dW_R = P_R dt \Rightarrow W_R = \int_0^t P_R dt.$

$W_R = \int_0^t Ri^2dt = \int_0^t \frac{E^2}{R}e^{-\frac{2t}{\tau}}dt = \left[-\frac{\tau E^2}{2R}e^{-\frac{2t}{\tau}}\right]_0^t = \frac{CE^2}{2}\left[-e^{-\frac{2t}{\tau}}\right]_0^t.$

$W_R = \frac{CE^2}{2}\left(-e^{-\frac{2t}{\tau}} + e^0\right) = \frac{CE^2}{2}\left(1 - e^{-\frac{2t}{\tau}}\right).$

> **TIP**
>
> $\int e^u du = \frac{e^u}{u'}$
>
> $\int_a^b f(x)dx = [F(x)]_a^b = F(b) - F(a)$
>
> **F is the primitive of f**

Value of W_R, W_C and W_R at the instant $t = 5\tau$

$W_G = CE^2(1 - e^{-5}).$

$W_C = \frac{CE^2}{2}(1 - 2e^{-5} + e^{-10}).$

$W_R = \frac{CE^2}{2}(1 - e^{-10})$

$W_C + W_R = \frac{CE^2}{2}(1 - 2e^{-5} + e^{-10} + 1 - e^{-10}) = \frac{CE^2}{2}(2 - 2e^{-5}) = CE^2(1 - e^{-5}).$

> **TIP**
>
> $W_G = W_C + W_R$
>
> **Valid at any instant t**

CHAPTER 5 – CAPACITORS
EXERCISES AND PROBLEMS

Exercise 1*:

A capacitor stores a charge of 300μC when the voltage across its plates is 600V. Calculate:

1- the electric energy stored in the capacitor,
2- the capacitance of the capacitor.

Exercise 2:

A capacitor of capacitance 20mF is charged to 10V. Calculate:

1- the charge on the capacitor,
2- the electric energy stored by the capacitor.

Exercise 3*:

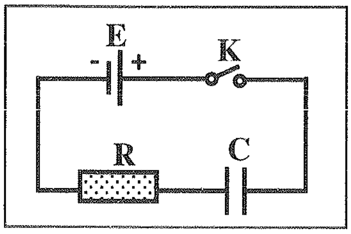
A capacitor of capacitance C = 2μF stores electric energy W = 1.44 × 10^{-4}J. Calculate:

1- the charge stored in the capacitor.
2- the voltage across its plates.

Exercise 4*:

1- Define the time constant of an RC series circuit.
2- Calculate the time constant of an RC series circuit of resistance R = 10kΩ and capacitance C = 5μF.

Exercise 5*:



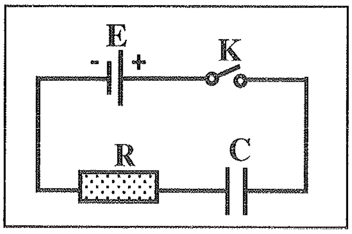
A resistor of resistance R = 1MΩ, a neutral capacitor of capacitance C = 1μF and a switch K are connected to an ideal DC generator of emf E = 10V.
At t_0 = 0s, the switch K is closed.

1- Calculate the time constant τ of the circuit.
2- Explain how the capacitor becomes charged.
3- Copy and complete the table below.

	u_C [V]	q [C]	u_R [V]	i [A]
t = 0				
t = τ				
t = 5τ				

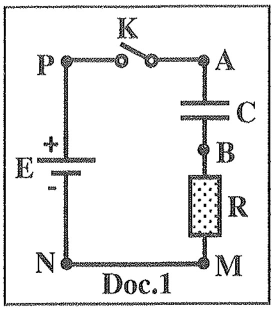
4- Trace the graph that represents the variation of u_C as a function of time.
5- Calculate the maximum electric energy stored in the capacitor.
6- Calculate the average electric power received by the capacitor between t = 0 and t = 5τ.

Exercise 6:



A resistor of resistance R = 200Ω, a capacitor of capacitance C and a switch K are connected in series with an ideal generator of constant emf E = 9V.
At t_0 = 0s, the switch K is closed. The voltage across the terminals of the capacitor at the instant t = 0.8s is $u_C = 5.67V$.

1- Determine the value of C.
2- At what instant does the capacitor complete its charging process? What is the value of the current at this instant?
3- Calculate the current at t = 0.8s; then, deduce its maximum value.
4- Calculate the maximum electric energy stored in the capacitor.



Exercise 7:

A switch K, a neutral capacitor of capacitance C and a resistor of resistance R = 500Ω are connected in series with an ideal battery of electromotive force E as shown in document 1.

At the instant $t_0 = 0s$, the switch K is closed.

Document 2 is a graph that represents the variation of the voltage $u_C = u_{AB}$ across the terminals of the capacitor as a function of time.

1- Indicate with justification the phase of the capacitor in the circuit.

2- Determine graphically:

  2.1- the value of E,

  2.2- the time constant τ of this circuit.

3- Verify that the capacitance of the capacitor is C = 20μF.

4- Calculate the time needed for the capacitor to be practically completely charged.

5- Find (using the graph if possible) at the instant t = 20ms:

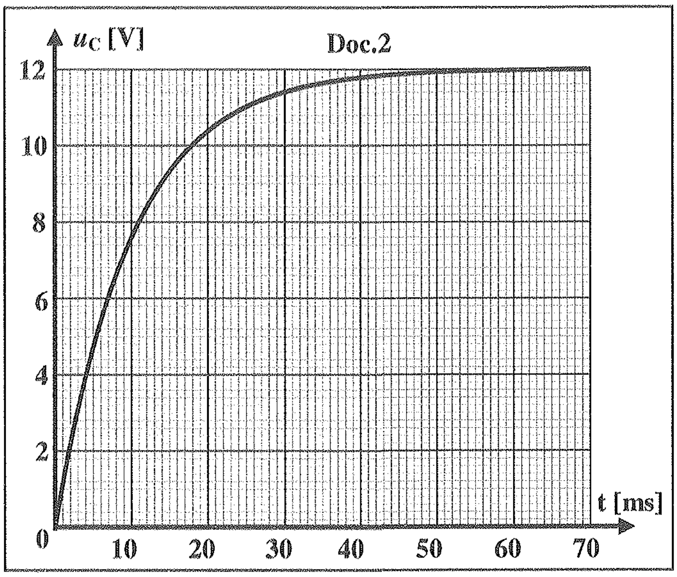
  5.1- the voltage across the terminals of the capacitor,

  5.2- the charge stored in the plate of the capacitor,

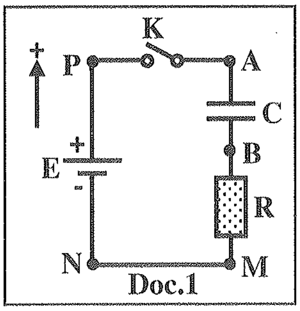
  5.3- the voltage across the terminals of the resistor,

  5.4- the current in the circuit,

  5.5- the electric energy stored in the capacitor.



Exercise 8*:



A resistor of resistance R = 10Ω, a capacitor of capacitance C = 20mF and a switch K are connected in series with an ideal generator of constant emf E = 9V. At $t_0 = 0s$, the switch K is closed. At the instant t, the armature A of the capacitor acquires a charge q and the circuit carries a current i.

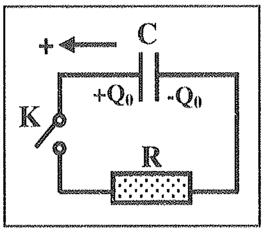
1- Establish the differential equation that governs the variation of the voltage across the terminals of the capacitor u_C as a function of time.

2- Show that $u_C = E\left(1 - e^{-\frac{t}{RC}}\right)$ is a solution of the differential equation.

3- Determine, using two methods, the voltage across the terminals of the resistor.

4- The voltage across the terminals of the capacitor at the instant t_1 is 7V. Determine t_1.

Exercise 9:



A resistor of resistance R = 5kΩ and a switch are connected to a capacitor of capacitance C = 2μF that is initially charged with $Q_0 = 24\mu C$. We close the switch K at the instant $t_0 = 0s$.

1- Calculate the voltage U_{C0} across the terminals of the capacitor at $t_0 = 0s$.

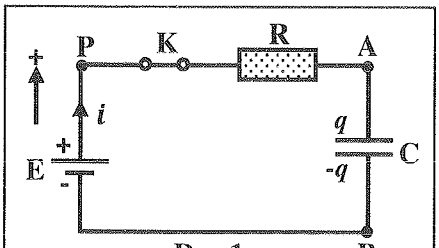
2- Taking into consideration the positive sense indicated in the adjacent document, establish the differential equation that governs the variation of the voltage u_C across the terminals of the capacitor as a function of time.

3- The solution of the differential equation has the form $u_C = Ae^{-\frac{t}{\tau}}$. Determine the values of the constants A and τ.

4- Determine the expression of the electric current i carried by the circuit.

Exercise 10:

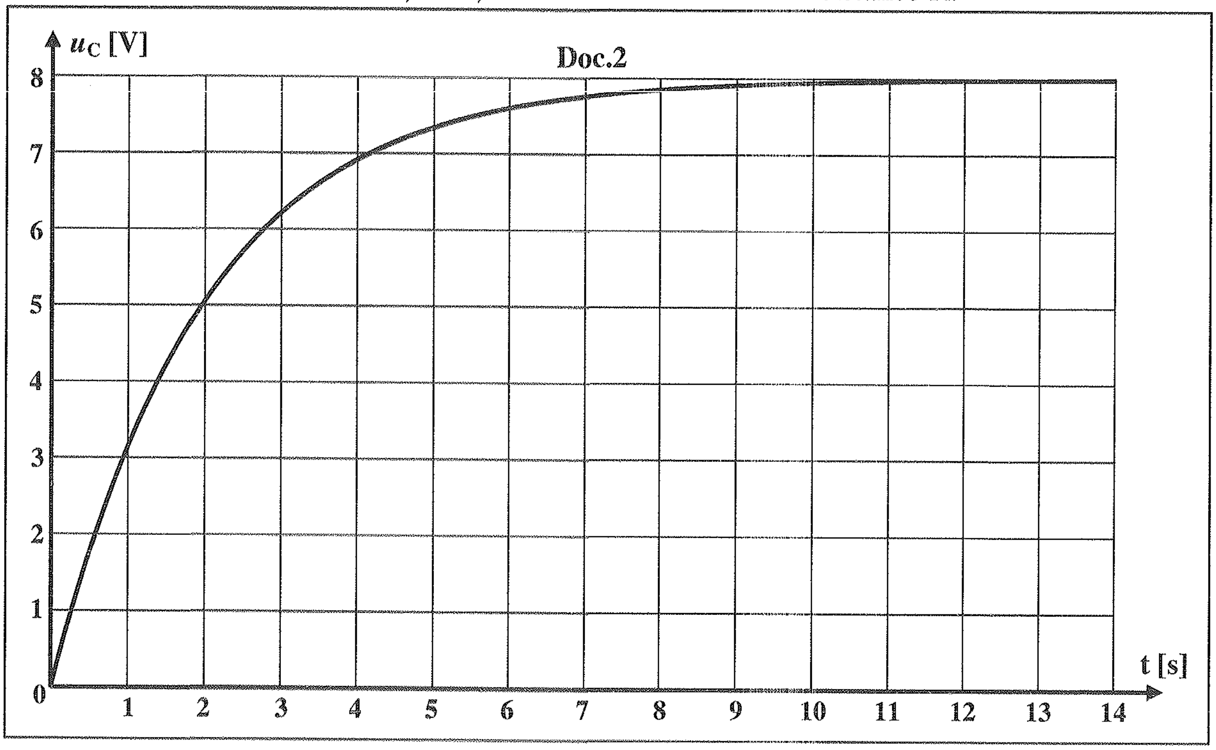
The electric circuit of document 1 consists of:
- an ideal generator of constant emf E;
- a resistor of unknown resistance R;
- a capacitor of capacitance C = 100μF and initially discharged;
- a switch K,
- connecting wires of negligible resistance.



Doc.1

At the instant t = 0, we close the switch K. At an instant t, the capacitor has a charge q and the circuit carries a current i.

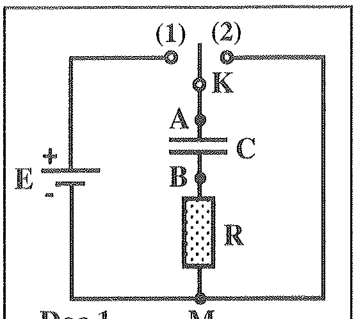
1- Redraw the circuit of document 1 and show the connections of an oscilloscope that displays the voltage $u_G = u_{PB}$ across the generator and the voltage $u_C = u_{AB}$ across the capacitor.
2- Write the expression of the current i in terms of q. Justify your answer.
3- Deduce the expression of i in terms of the capacitance C and the voltage u_C.
4- Establish the differential equation that governs the variation of u_C as a function of time.
5- The solution of this differential equation is $u_C = D\left(1 - e^{-\frac{t}{\tau}}\right)$. Determine the expressions of the positive constants D and τ in terms of E, R and C.
6- Using the graph of $u_C = f(t)$ represented in document 2:
   6.1- Specify the value of E.
   6.2- Determine the time constant τ; then, deduce the value of the resistance R.



7- At what instant does the capacitor practically complete its charging process?
8- Calculate the maximum energy stored in the capacitor.
9- Determine the expression of the current i as a function of time.
10- Find the value of i in steady state.

Exercise 11:

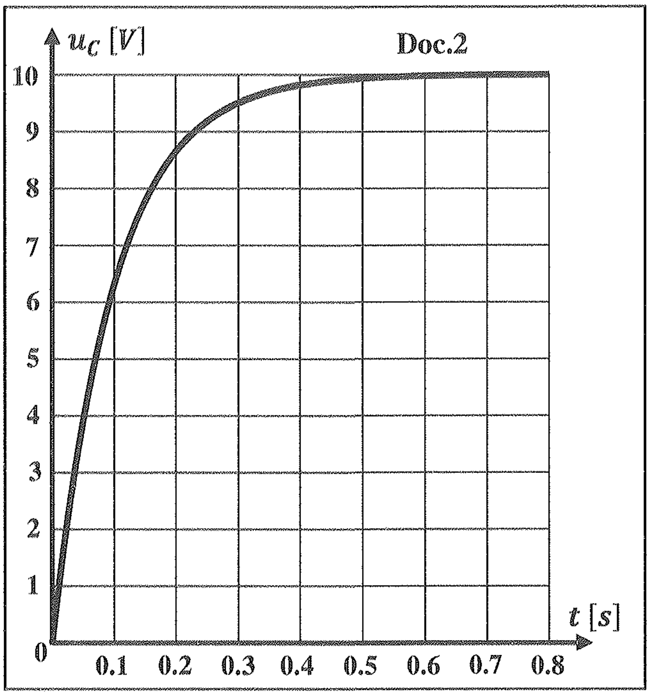
Document 1 is a circuit diagram that consists of:
- An ideal generator of constant electromotive force E.
- A capacitor of capacitance C and initially uncharged.
- A resistor of resistance R = 100Ω.
- A double switch K.
- Connecting wires.



Doc.1

Part I: at the instant $t_0 = 0s$, the switch K is placed at position (1).

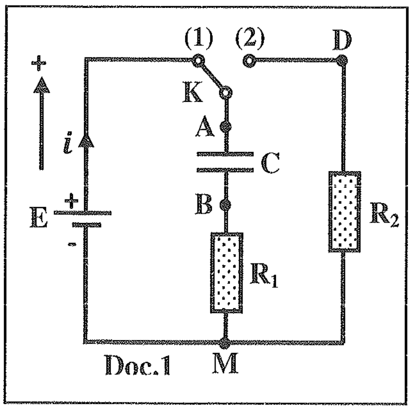
At the instant t, the armature A carries a charge q and the circuit carries an electric current i.
A convenient apparatus enables us to trace the graph of the variation of the voltage $u_C = u_{AB}$ across the capacitor as a function of time (document 2).



1- Indicate with justification the phase of the capacitor.
2- Establish the differential equation that governs the variation of u_C as a function of time.
3- The solution of the differential equation has the form $u_C = A + Be^{-\frac{t}{\tau}}$ where A, B and τ are constants. Determine the expressions of A, B and τ in terms of E, R and C.
4- Determine, by referring to document 2:
   4.1- the value of E.
   4.2- the time constant τ of the circuit.
5- Show that the capacitance of the capacitor is $C = 1mF$.

Part II: the switch K is turned to position (2) at a new origin of time $t_0 = 0s$.

At the instant t, the armature A carries a charge q and the circuit carries a current i.
1- What is the mode of the capacitor?
2- Taking the positive sense as that of the electric current, show that the differential equation that governs the variation of the voltage $u_C = u_{AB}$ across the capacitor as a function of time is:
$$RC\frac{du_C}{dt} + u_C = 0$$
3- Show that $u_C = Ee^{-\frac{t}{RC}}$ is a solution of the differential equation.
4- Draw the shape of the graph that represents the variation of $\ln u_C$ as a function of time.
5- Deduce the expression of i as a function of time.



Exercise 12:

Consider the following electric components:

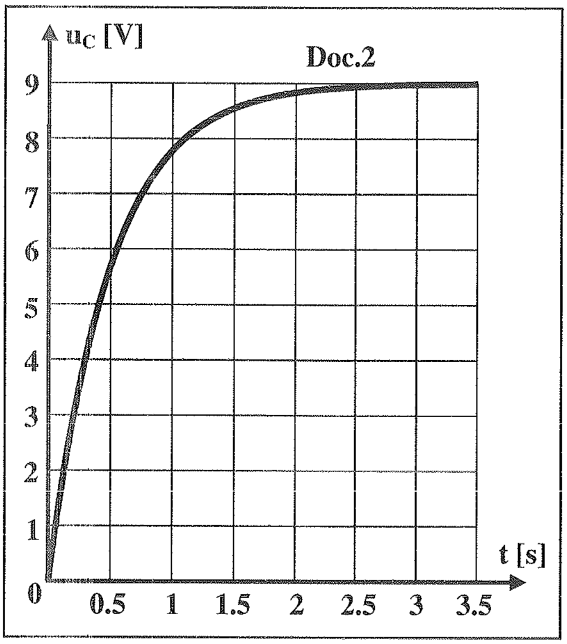
- An ideal generator G of constant electromotive force E;
- Two resistors of resistances $R_1 = 1k\Omega$ and R_2;
- A capacitor of capacitor C;
- A double switch K;
- Connecting wires.

We set up the circuit of document 1.

A- First Experiment

At the instant $t_0 = 0$, the switch K is turned to position (1). At the instant t, the armature A carries a charge q and the circuit carries a current i.

Using a convenient apparatus, we display the variation of the voltage $u_C = u_{AB}$ across the capacitor. We obtain the waveform of document 2.



1- Establish the differential equation that governs the variation of u_C as a function of time.

2- The solution of the differential equation has the form $u_C = A\left(1 - e^{-\frac{t}{\tau_1}}\right)$ where A and τ_1 are constants.
Determine the expressions of A and τ_1 in terms of E, C and R_1.

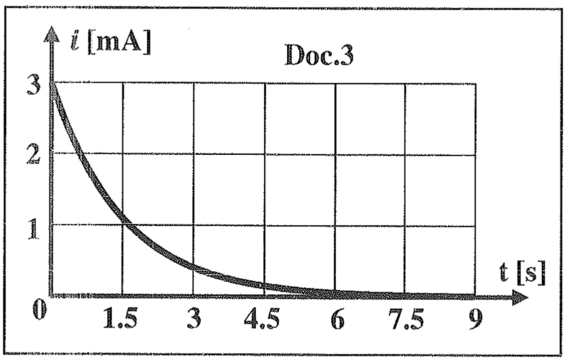
3- Determine, referring to document 2:
3.1- the value of E;
3.2- the time constant τ_1 of the circuit.

4- Show that $C = 500\mu F$.

B- Second Experiment

The switch K is turned to position (2) at a new origin of time $t_0 = 0s$. The armature A thus carries a charge q and the circuit carries a current i at the instant t.

Using a convenient apparatus, we display the variation of i as a function of time. We obtain the waveform of document 3.



1- Taking the positive direction along the circuit as that of the current, show that the differential equation that governs the variation of i as a function of time is given by:

$$\frac{di}{dt} + \frac{1}{(R_1+R_2)C}i = 0$$

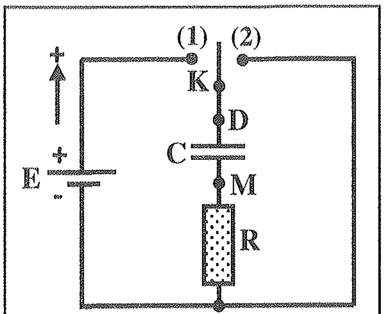
2- The solution of the differential equation has the form $i = I_0 e^{-\frac{t}{\tau_2}}$. Show that $\tau_2 = (R_1 + R_2)C$.

3- Referring to document 3:
3.1- Give the value of I_0 of i at $t_0 = 0s$.
3.2- Determine the value of τ_2; then deduce the value of R_2.

Exercise 13*:

The object of this exercise is to determine the capacitance C of a capacitor. To do this, we connect the circuit shown in document 1 that consists of:

- An ideal generator of constant emf E;
- A resistor of resistance R = 1kΩ;
- A neutral capacitor of capacitance C.
- A double switch K,
- Connecting wires.

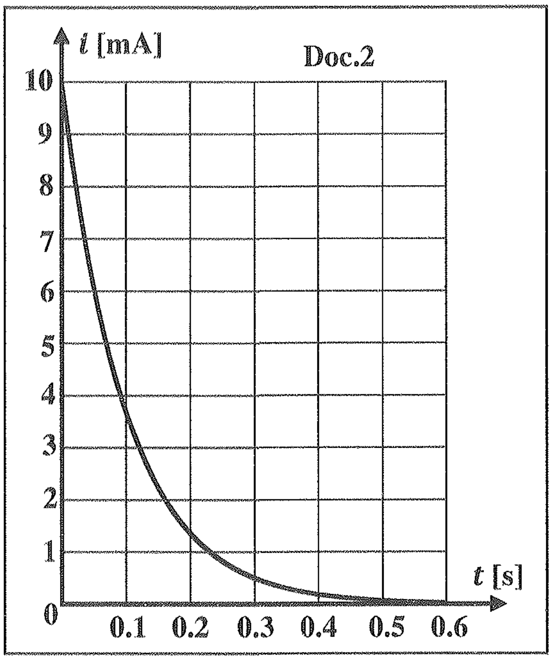


Doc.1

Part I: charging of the capacitor

At the instant t = 0s, the switch K is turned to position (1). Thus, the circuit carries a current i at the instant t.

A convenient apparatus records the variation of i as a function of time (document 2).



1- Apply the law of addition of voltages to determine, at t = 0, the expression of the current I_0 carried by the circuit in terms of E and R.

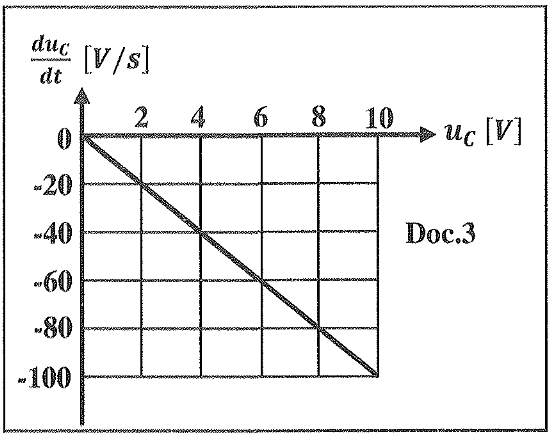
2- Show that the differential equation that governs the variation of i as a function of time is given by:
$$\frac{di}{dt} + \frac{1}{RC} i = 0$$

3- The solution of the differential equation has the form $i = I_0 e^{-\frac{t}{\tau}}$ where τ is a constant. Determine the expression of τ in terms of R and C.

4- Referring to document 2, determine:
  - **4.1-** the value of I_0; then, deduce the value of E.
  - **4.2-** the time constant τ; then, deduce the value of C.

Part II: discharging of the capacitor

The switch K is turned to position (2) at a new origin of time t = 0s. Document 3 represents the variation of $\frac{du_C}{dt}$ as a function the voltage u_C across the capacitor.



1- Show that the differential equation that describes the variation of u_C as a function of time is:
$$u_C + RC\frac{du_C}{dt} = 0$$

2- Show that the shape of the graph is in agreement with the differential equation obtained in II.1.

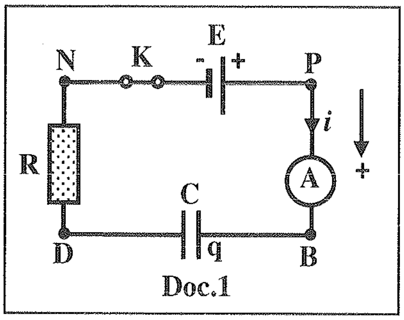
3- Determine again the value of C.

Exercise 14:

The aim of this exercise is to determine the capacitance C of a capacitor.

We set-up the series circuit of document 1 that includes:

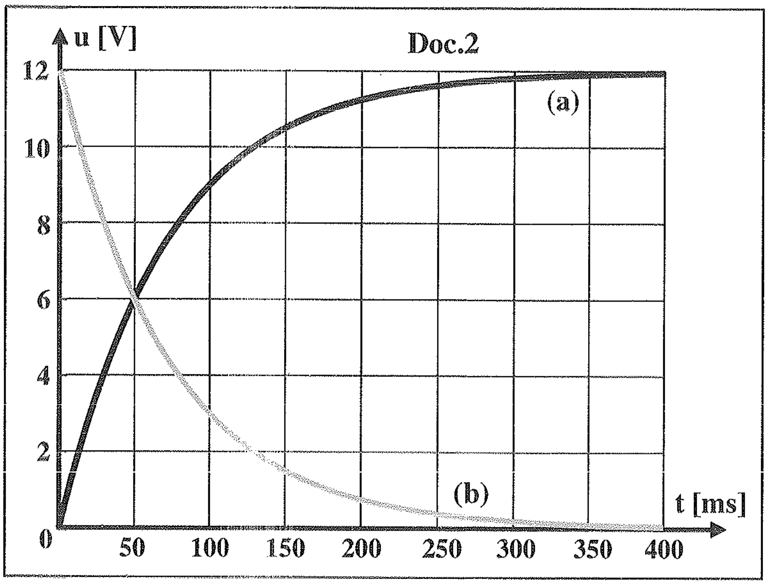
- An ideal battery of electromotive force E.
- A resistor of R.
- A capacitor of capacitance C.
- An ammeter (A) of negligible resistance.
- A switch K.



Doc.1

Initially, the capacitor is uncharged. We close the switch K at the instant $t_0 = 0$. At an instant t, plate B of the capacitor carries a charge q and the circuit carries a current i as shown in document 1.

A convenient apparatus records the variations, as a function of time, of the voltage $u_C = u_{BD}$ across the capacitor and the voltage $u_R = u_{DN}$ across the resistor. We obtain the curves shown in document 2.



1- Curve (a) represents the variation of u_C as a function of time. Justify.

2- Determine the value of E.

3- Write the expression of i in terms of C and u_C.

4- Establish the differential equation that governs the variation of u_C.

5- The solution of this differential equation is of the form: $u_C = a + be^{-\frac{t}{\tau}}$. Determine the expressions of the constants a, b and τ in terms of E, R and C.

6-

6.1- Show that the expression of the current is: $i = \frac{E}{R}e^{-\frac{t}{\tau}}$.

6.2- The ammeter (A) indicates a value $I_0 = 6mA$ at $t_0 = 0$. Deduce the value of R.

6.3- Write the expression of u_R in terms of E, R, C and t.

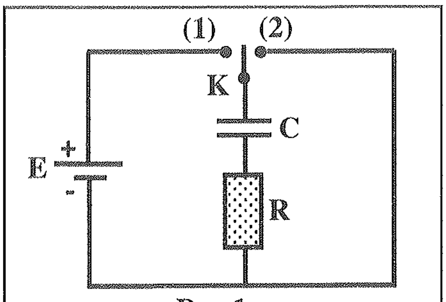
7- At an instant $t = t_1$, the voltage across the capacitor is $u_C = u_R$.

7.1- Show that $t_1 = RC \ln 2$.

7.2- Deduce the value of C.

Exercise 15:

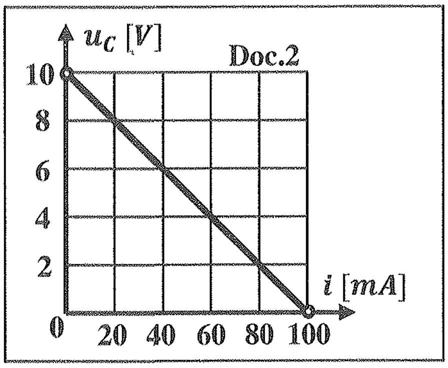
Document 1 is an electric circuit diagram that consists of:
- an ideal DC generator of electromotive force E,
- a neutral capacitor of capacitance C,
- a resistor of resistance R,
- a double switch K,
- connecting wires.



Doc.1

Part I: The switch K is placed at position (1).

The graph of document 2 shows the variation of the voltage across the terminals of the capacitor u_C as a function of the current i traversing the circuit.

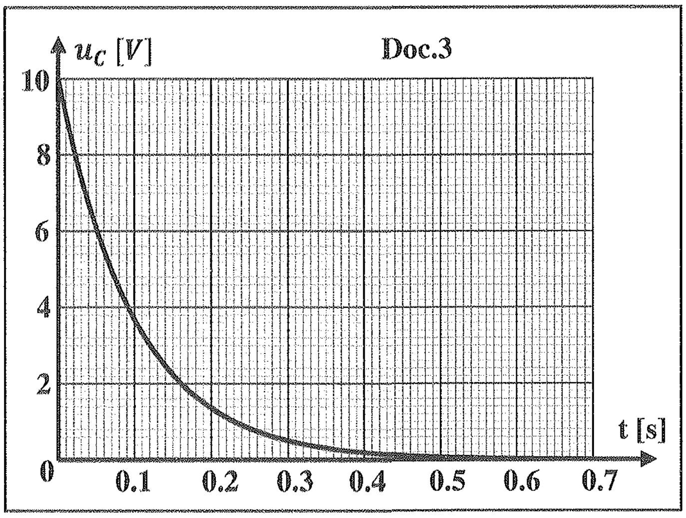


Doc.2

1- Indicate the phase of the capacitor.
2- Determine the equation of the straight line shown in document2.
3- Apply the law of addition of voltages to determine the expression of u_C in terms of E, R, and i.
4- Deduce the values of E and R.

Part II: The switch K is flipped to position (2) at a new origin of time t = 0s.

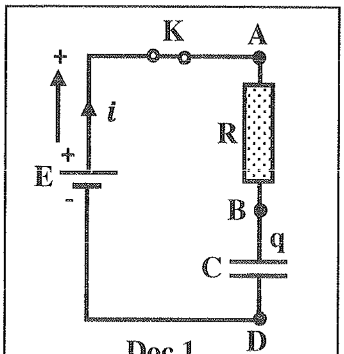
The graph below (document 3) represents the variation of the voltage across the terminals of the capacitor u_C as a function of time.



Doc.3

1- In what mode is the capacitor? Justify.
2- Establish the differential equation in u_C. Take the positive sense with the direction of the electric current traversing the circuit.
3- The solution of the differential equation can be written in the form: $u_C = Ae^{-\frac{t}{\tau}}$. Determine the expression of A and τ in terms of E, R, and C.
4- Define the time constant τ of the RC circuit in this phase. Using (document 3), determine the value of τ.
5- Deduce:
   - **5.1-** the capacitance of the capacitor,
   - **5.2-** the instant at which the capacitor is completely discharged.
6- Calculate the initial charge and the electric energy stored in the capacitor.
7- Determine, at the instant t = 0.2s, the intensity of the current traversing the circuit.

Exercise 16*:



Doc.1

The object of this exercise is to determine the expression of the time constant τ of an RC circuit using different methods.
The circuit diagram of document 1 is formed of:
- an ideal DC generator of e.m.f E = 10V,
- a resistor of resistance R = 2kΩ,
- a capacitor of capacitance C,
- a switch K.

The capacitor is initially neutral. We close the switch at the instant t_0 = 0s.

1- **Theoretical study**

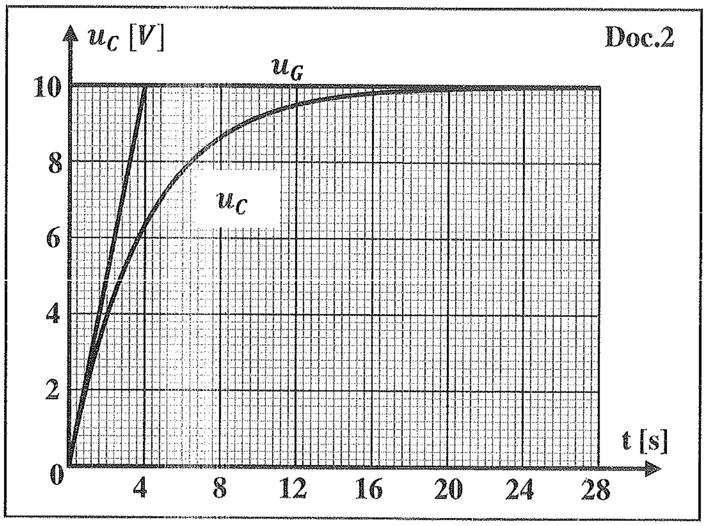
1.1- Derive the differential equation that governs the variation of the voltage $u_{BD} = u_C$ across the capacitor as a function of time.

1.2- The solution of the differential equation has the form: $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$ where A and τ are constants. Determine the expressions of A and τ as a function of E, R, and C.

1.3- Show that, at the steady state, the voltage across the capacitor is E.

2- **Experimental study**

An oscilloscope is used to visualize the voltage $u_G = u_{AC}$ on channel Y_1 and the voltage $u_{BD} = u_C$ on channel Y_2.
Document 2 represents the variations of u_G and u_C as a function of time.



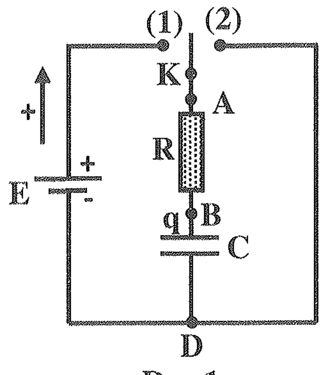
1- Show on document 1 the connections of the oscilloscope.
2- Give the maximum value of u_C.
3- One method to determine the time constant τ is to find the intersection between the tangent to the graph of u_C at t = 0s with the asymptote of the graph.
   3.1- Find, at t = 0s, the expression of $\frac{du_C}{dt}$ in terms of E, R and C.
   3.2- Show that the equation of this tangent to the curve at t = 0s is $u = \frac{E}{RC}t$.
   3.3- Verify that this tangent intersects the asymptote to the curve at the point of abscissa $\tau = RC$.
   3.4- Determine then the value of the capacitance C of the capacitor.
4- Another method to determine the time constant τ is to find the time at the end of which the voltage u_C attains 63% of its maximum value.
   4.1- Calculate the value of u_C at $t = \tau$. Deduce the value of τ.
   4.2- Deduce again the value of C.

Exercise 17:

The object of this exercise is to determine the capacitance of a capacitor. For this aim, we construct the circuit of document 1 that consists of an ideal battery of electromotive force E, a resistor of resistance R = 1kΩ, and a double switch K.

The capacitor is initially neutral. The switch K is put at position 1 at $t_0 = 0s$.

At an instant t, the circuit carries an instantaneous current i.



Doc.1

1- **Charging of the capacitor**

1.1- Derive the differential equation giving the variation of the voltage $u_{BD} = u_C$ across the capacitor as a function of time.

1.2- The solution of the above differential equation has the form:

$u_C = A + Be^{-\frac{t}{\tau}}$ where A, B, and τ are constants.

Determine the expressions of A, B, and τ in terms of E, R, and C.

1.3- Deduce the expression of the current i traversing the circuit.

1.4- Show that:

1.4.1- at the end of the charging $u_C = E$,

1.4.2- the voltage across the capacitor is one third of its maximum value at the instant $t = RC\ln\frac{3}{2}$,

1.4.3- the final energy stored in the capacitor is half the energy supplied by the battery.

Hint: the electric energy supplied by the battery is: $W = \int_0^{+\infty} Eidt$.

1.5- Determine, using two methods, the expression of the voltage across the terminals of the resistor as a function of time.

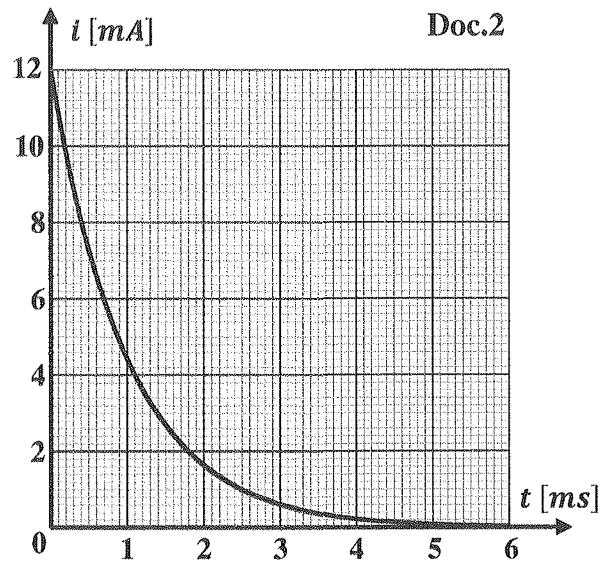
2- **Discharging of the capacitor**

K is flipped to position 2 at a new origin of time $t_0 = 0s$.

Thus, an instantaneous current i traverses the circuit.

Choose the positive direction as that of the direction of the electric current.

Document 2 represents the variation of i as a function of time.



2.1- Draw the circuit of discharging and indicate on it the real direction of the current chosen as a positive direction.

2.2- In this case, the current is written as $i = -\frac{dq}{dt}$ and not as $i = \frac{dq}{dt}$. Why?

2.3- Derive the differential equation giving the variation of i as a function of time.

2.4- Show that the solution of the differential equation is $i = \frac{E}{R}e^{-\frac{t}{\tau}}$ where $\tau = RC$.

2.5- Referring to the graph of document 2, determine:

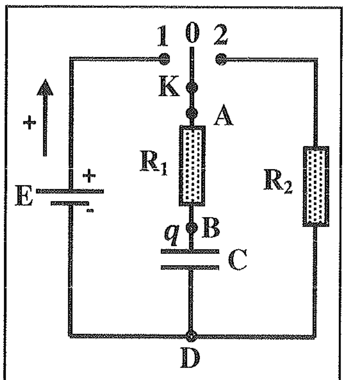
2.5.1- the value of E,

2.5.2- the time constant τ.

2.6- Deduce the capacitance of the capacitor.

Exercise 18:

The aim of this exercise is to determine, by two different methods, the value of the capacitance C of a capacitor. For this aim, we setup the circuit of document 1. This circuit is formed of an ideal DC generator of e.m.f E, a capacitor of capacitance C, two identical resistors of resistances $R_1 = R_2 = 10k\Omega$ and a double switch K.



Doc.1

Part I: Charging the capacitor

The switch K is in the position (0) and the capacitor is neutral. At the instant $t_0 = 0$, we turn K to position (1) and the charging of the capacitor starts.

1- **Theoretical study**

1.1- Show that the differential equation that governs the variation u_{R_1} the voltage across the resistor R_1 with respect to time is: $\frac{du_{R_1}}{dt} + \frac{1}{RC}u_{R_1} = 0$

1.2- The solution of the differential equation has the form: $u_{R_1} = Ae^{-\frac{t}{\tau}}$. Determine the constants A and τ in terms of E, R, and C.

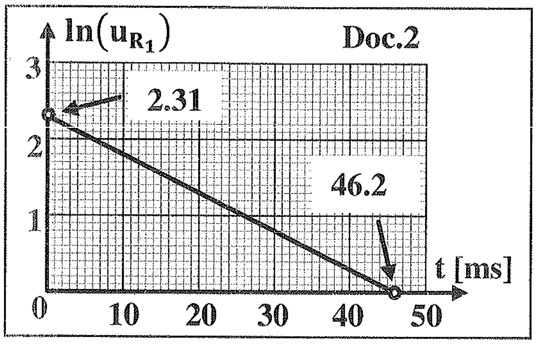
1.3- Establish, as a function of time, the expression of the natural logarithm of u_{R_1} $[\ln(u_{R_1})]$ in terms of E, R_1 and C.

2- **Graphical study**

The variation of $\ln(u_{R_1})$ as a function of time is represented by document 2.

2.1- Justify that the shape of the obtained graph agrees with the expression of $\ln(u_{R_1})$ as a function of time.

2.2- Deduce, using the graph, the values of E and C.



Part II: Discharging the capacitor

The capacitor being fully charged, we turn the switch K to position (2). At an instant $t_0 = 0$, taken as a new origin of time, the discharging of the capacitor starts.

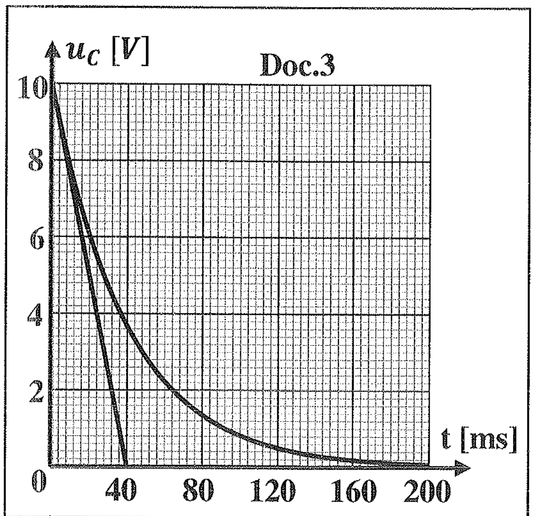
1- During discharging, the current circulates from B to A in the resistor of resistance R_1. Justify.

2- Taking the positive direction along the circuit as that of the current, show that the differential equation in the voltage u_C across the capacitor has the form: $u_C + (R_1 + R_2)C\frac{du_c}{dt} = 0$

3- The solution of the above differential equation has the form: $u_c = Ee^{-\frac{t}{\tau_2}}$ where τ_2 is the time constant of the circuit during discharging. Show that $\tau_2 = (R_1 + R_2)C$.

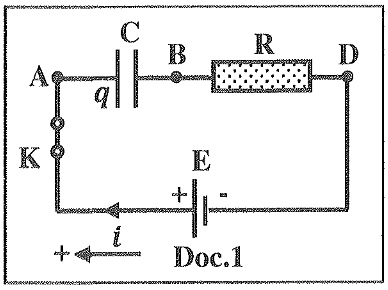
4- The variation of the voltage u_C across the capacitor and the tangent to the curve $u_C = f(t)$ at the instant $t_0 = 0$, are represented in document 3.

Deduce, from this figure, the value of the capacitance C.



Exercise 19*:

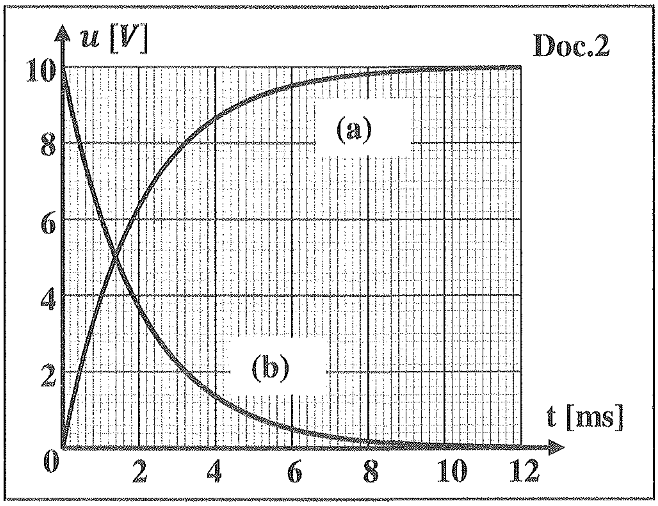
A capacitor of capacitance C, a resistor of resistance R = 200Ω and a switch K are connected in series with an ideal DC generator of e.m.f E.
The capacitor is being initially neutral, we close the switch K at the instant $t_0 = 0s$. At the instant t, the circuit carries an instantaneous current i as shown in document 1.
An oscilloscope is used to visualize the voltage $u_{AB} = u_C$ across the capacitor on channel Y_1 and the voltage $u_{BD} = u_E$ across the capacitor on channel Y_2.



Doc.1

1- **Theoretical study**
   1.1- Establish the differential equation that governs the variation of u_C as a function of time.
   1.2- The solution of the differential equation has the form: $u_C = A + Be^{\alpha t}$. Determine the expressions of the constants A, B and α in terms of E, R, and C.
   1.3- Show that, at steady state, the voltage across the capacitor is $U_0 = E$.
   1.4- Show that the expression across the terminals of the resistor is: $u_R = Ee^{-\frac{t}{RC}}$.
   1.5- What does the term RC represent for this circuit?

2- **Experimental study**
   Document 2 represents the variations of u_C and u_R as a function of time.



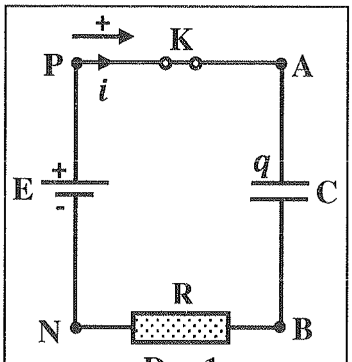
   2.1- Show on document 1 the connections of the oscilloscope.
   2.2- The "IVN" button on channel Y_2 is pressed. Explain.
   2.3- The curve (a) represents the variation of u_C as a function of time. Explain.
   2.4- Referring to document 2:
      2.4.1- Determine the value of E.
      2.4.2- Calculate, at t = 0s, the intensity of the electric current I_0 traversing the circuit.
      2.4.3- Determine the time constant τ; then, deduce the capacitance of the capacitor.
   2.5- Calculate the time at which the capacitor and the resistor have the same voltage.
   2.6- Calculate the time needed for the capacitor to be practically fully charged.

3- **Energetic study**
   3.1- Determine the expression of i as a function of time.
   3.2- Calculate, at $t = \tau$, the electric energy stored in the capacitor.
   3.3- Determine the electric energy supplied by the generator between t = 0s and $t = RC$.
   3.4- Determine, using two methods, the energy dissipated by the resistor between t = 0s and $t = RC$.

Exercise 20:

In order to charge a capacitor, we connect up the series circuit that is represented in document 1. This circuit is formed of:
- an ideal DC generator of e.m.f E = 12V;
- a resistor of resistance R;
- a capacitor of capacitance;
- a switch K.

The capacitor is initially neutral. At the instant $t_0 = 0$, we close K.
At an instant t, the armature A of the capacitor carries a charge q and the circuit is traversed by a current i whose direction is shown on the circuit.



Doc.1

1- Theoretical study

1.1- Write the relation between q, C, and $u_C = u_{AB}$.

1.2- Derive the differential equation that governs the variation of q as a function of time.

1.3- The solution of the differential equation has the form: $q = A + Be^{-\frac{t}{\tau}}$. Determine the expressions of the constants A, B and τ in terms of E, R, and C.

1.4- Shown that, at steady state, the armature A of the capacitor carries a maximum charge Q_m whose expression should be determined in terms of C and E.

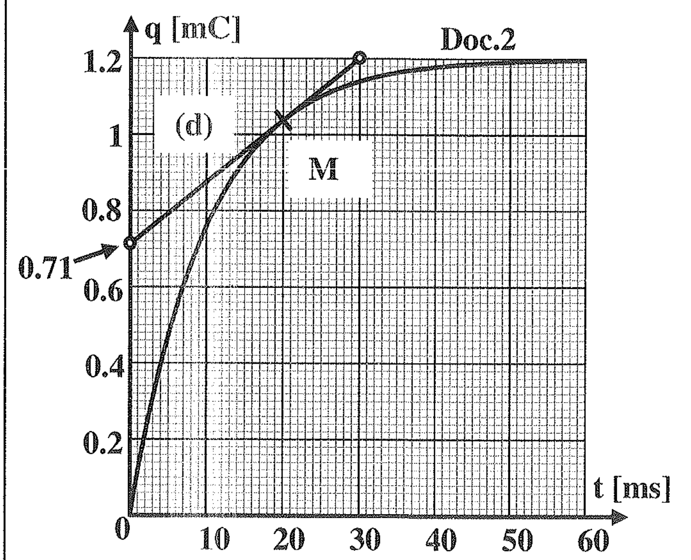
2- Exploitation of the curve

The variation of the charge q, as a function of time, is represented by the curve of document 2.
The straight line (d) represents the tangent to the curve at the instant t = 20ms.
Using document 2:

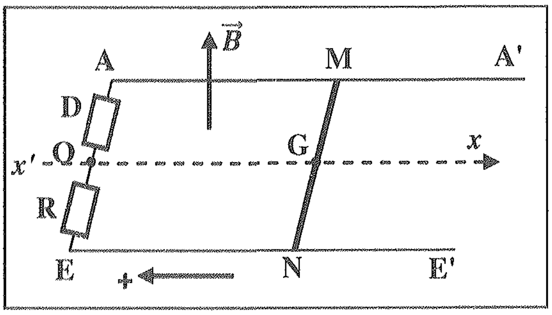
2.1- Indicate the value of Q_m; then, deduce the value of C.

2.2- Determine the time constant of this RC circuit; then, deduce the value of R.

2.3- Determine the intensity of the electric current traversing the circuit at the instant t = 20ms.



Exercise 21:



A metallic rod MN, of length ℓ =1m and of negligible resistance, may slide without friction along two long parallel and horizontal rectilinear rails AA' and EE' of negligible resistance. During its displacement, the rod remains perpendicular to the rails. An electric component (D) and a resistor of resistance R = 100 Ω are connected to the rails with connecting wires. The whole set-up thus described is placed in a uniform vertically upwards magnetic field $\vec{B}$ of magnitude B = 0.8 T (adjacent document).

At the instant $t_0 = 0$, the center of mass G of the rod is at O. A convenient apparatus causes the rod to move in a uniform translational motion from left to right with a speed v = 0.5 m/s.

At an instant t, the position of G is defined by its abscissa $x = \overline{OG}$ on the axis x'x.

1- Find, at the instant t, the expression of the magnetic flux that crosses the surface AMNE in terms of B, ℓ and x taking into consideration the positive direction indicated on the figure.

2-

2.1- Explain the existence of an induced e.m.f e across the ends M and N of the rod and show that its value is 0.4 V.

2.2- At the instant t, an induced current i passes in the circuit. Determine its direction.

2.3- Draw a diagram showing the equivalent generator between M and N and specify its positive terminal.

3- The component (D) is a capacitor of capacitance $C = 10^{-2}$ F. During the displacement of the rod, (D) undergoes the phenomenon of electric charging.

3.1- Derive the differential equation that describes the variations of $u_C = u_{OA}$ as a function of time.

3.2-

3.2.1- Calculate the value of the time constant of the circuit thus formed.

3.2.2- After how long would the capacitor be practically charged completely?

3.3- At the end of charging, the voltage across the capacitor is U and its charge is Q. Calculate U and Q.

3.4- Determine the values of i at the instants $t_0 = 0$ and $t_1 = 6$ s.

3.5- At the instant $t_1 = 6$ s, the rod is stopped. The circuit carries again a current.

3.5.1- Due to what is this current?

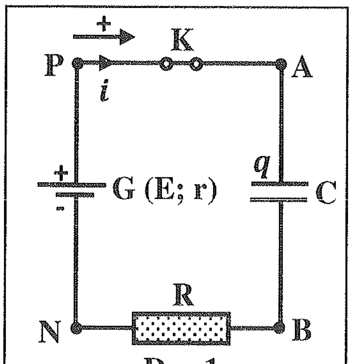
3.5.2- Specify the duration of the passage of this current.

Exercise 22:

Document 1 is a circuit diagram that consists of:

- a DC generator G of e.m.f E and internal resistance r,
- a resistor of resistance R = 8Ω,
- a capacitor of capacitance C,
- a switch K.

The capacitor is initially neutral. At the instant $t_0 = 0$, we close K.
At an instant t, the circuit is traversed by an instantaneous current i



Doc.1

1- Theoretical study

1.1- Give, at the instant t, the expression of u_G the voltage across the generator in terms of $E, r,$ and i.

1.2- Show that the differential equation that governs the variation of $u_C = u_{AB}$ the voltage across the capacitor as a function of time is: $E = (R + r)C\frac{du_C}{dt} + u_C$

1.3- The solution of the differential equation has the form: $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$. Determine the expressions of the constants A and τ in terms of E, R, r and C.

1.4- Show that the expression across the terminals of the generator is given by:

$$u_G = u_{PN} = E\left(1 - \frac{r}{R+r}e^{-\frac{t}{\tau}}\right).$$

1.5- Using the equation obtained in 1.3:

1.5.1- Give the expression of U_0 the voltage across the generator at t = 0s, in terms of E, R and r.

1.5.2- Show that, at the steady state, the voltage across the generator is $U_m = E$.

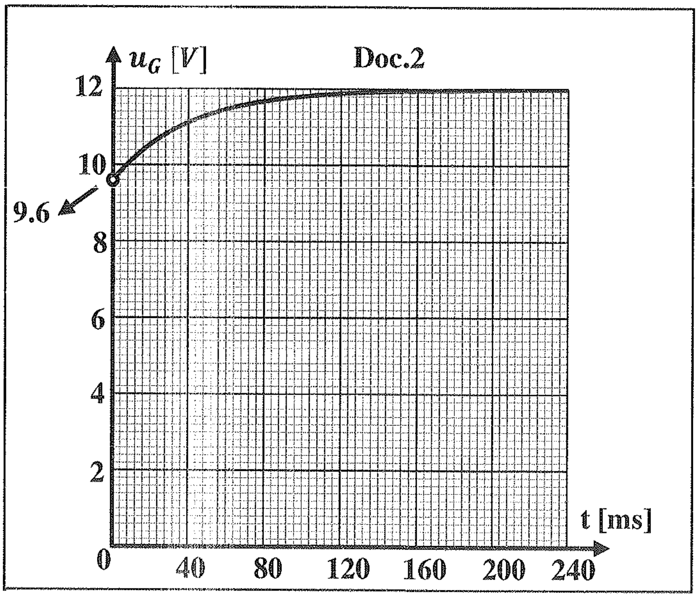
2- Experimental study

Document 2 represents the variation of u_G as a function of time.

2.1- Using document 2, determine the values of E and r.

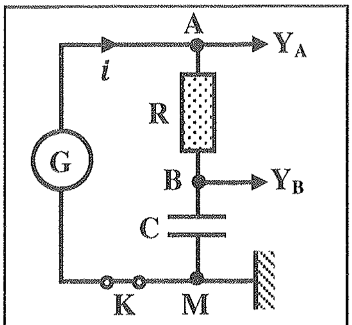
2.2- Calculate, at $t = \tau$, the voltage across the terminals of the generator.

2.3- Deduce the capacitance of the capacitor.

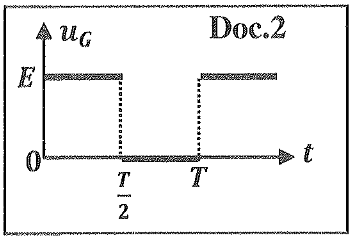


Exercise 23:

The circuit in document 1 consists of a generator (G) delivering a square signal (E, 0) of period T (document 2), a resistor of resistance R = 10kΩ and a capacitor (C) of capacitance C = 0.2μF. An oscilloscope displays the voltage $u_G = u_{AM}$ across G and the voltage $u_C = u_{BM}$ across (C).



Doc.1



Doc.2

Part I: Theoretical study

1- During the charging of (C), the voltage u_G has the value E and at an instant t, the circuit carries a current i.

   1.1- Give the expression of i in terms C and $\frac{du_C}{dt}$.

   1.2- Derive, for $0 \leq t \leq \frac{T}{2}$, the differential equation in u_C.

   1.3- The solution of this differential equation has the form:

   $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$, where A and τ are constants.

      1.3.1- Determine, in terms of E, R and C, the expressions of A and τ.

      1.3.2- Draw the shape of the graph representing the variation of u_C as a function of time and show, on this graph, the points corresponding to A and τ.

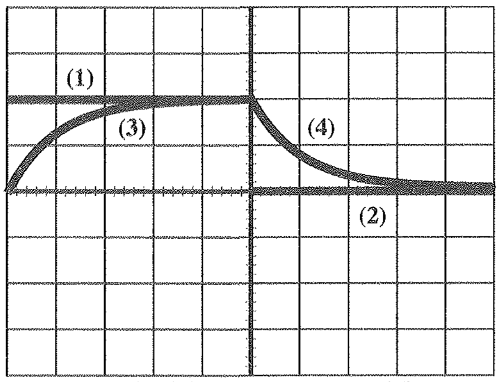
2- During discharging of (C) the voltage $u_G = 0$. We consider the instant $\frac{T}{2}$ as a new origin of time.

   Verify that $u_C = Ee^{-\frac{t}{\tau}}$.

3-

   3.1- What must the minimum duration of charging be so that u_C reaches practically the value E?

   3.2- What is then the minimum value of T?



$S_V = 5V/div$; $S_h = 2ms/div$

Doc.3

Part II: Experimental study

1- On the screen of the oscilloscope, we observe the waveforms of document 3.

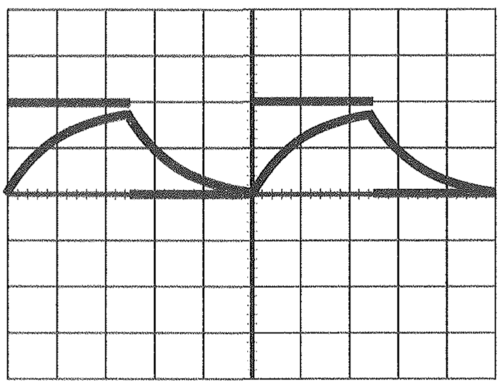
   1.1- Which curve corresponds to the charging of the capacitor? Justify the answer.

   1.2- Calculate the value of E and that of the period T of the square signal.

2- We increase the frequency of the voltage delivered by G. The waveforms obtained are shown in document 4.

   2.1- Determine the new period of the square signal. Justify the shape of the waveform of u_C.

   2.2- We keep increasing the frequency of the voltage delivered by G. The waveform becomes almost triangular. Why?

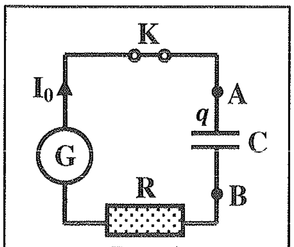


$S_V = 5V/div$; $S_h = 1ms/div$

Doc.4

Exercise 24:

Part I: charging the capacitor under a constant current source

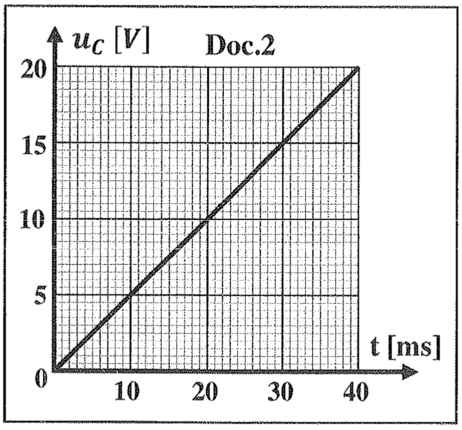


Doc.1

A resistor of resistance R, a capacitor of capacitance C and a switch K are connected to an ideal generator G.

The capacitor is being initially uncharged, we close the switch K at the instant $t_0 = 0s$. The generator delivers a constant current $I_0 = 0.5mA$ as shown in document 1.

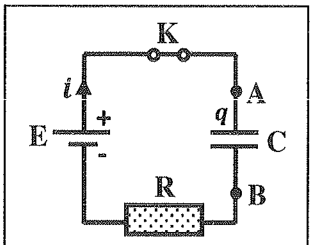
Using a convenient apparatus we display the graph that represents the variation of the voltage u_C across the capacitor as a function of time (document 2).



At the instant t, the armature A of the capacitor has a charge q.

1- Determine the expression u_C in terms of I_0, C and t.
2- Show that the expression of u_C is in agreement with the graph of document 2.
3- Deduce the value of C.
4- The breakdown potential of the capacitor is $U_m = 30V$.
   4.1- Define the breakdown potential of the capacitor.
   4.2- Determine the maximum duration t_1 of the charging process of the capacitor.
   4.3- Calculate the maximum charge and maximum electric energy stored in the capacitor.

Part II: charging the capacitor under a constant voltage source



Doc.3

The generator G is replaced by another one G' of constant emf E = 30V and negligible internal resistance.

The capacitor is being initially uncharged, we close the switch K at the instant $t_0 = 0s$. At the instant t, the armature A of the capacitor has a charge q and the circuit carries a current i.

1- Establish the differential equation that governs the variation of the voltage u_C across the capacitor as a function of time.
2- The solution of the differential equation has the form $u_C = A + Be^{-\frac{t}{\tau}}$. Determine in terms of E, R and C the expressions of the constants A, B and τ.
3- Show that the capacitor completes its charge process at the instant $t_2 = 5\tau$.
4- Determine the value of R so that the capacitor completes its charging process during the same duration calculated in part I.

CHAPTER 5 – CAPACITORS
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 2:

Part	Answer key
1	$q = Cu_C = 20 \times 10^{-3} \times 10 = 0.2C.$
2	$W = \frac{1}{2}Cu_C^2 = \frac{1}{2} \times 20 \times 10^{-3} \times 10^2 = 1J.$

Exercise 6:

Part	Answer key
1	$\frac{u_C}{E} = \frac{5.67}{9} = 0.63 \Longrightarrow u_C = 0.63E$ then $t_1 = \tau = 0.8s.$ $\tau = RC \Longrightarrow C = \frac{\tau}{R} = \frac{0.8}{200} = 4 \times 10^{-3}F = 4mF.$
2	$t = 5\tau = 4s.$ $i = 0A.$
3	Law of addition of voltages in series connection: $u_G = u_K + u_C + u_R$ $E = u_C + Ri$ with $u_R = Ri$ and $u_K = 0$ (closed switch) $i = \frac{E-u_C}{R} = \frac{9-5.67}{200} = 16.65 \times 10^{-3}A = 16.65mA.$ At $t = \tau; i = 0.37I_m \Longrightarrow I_m = \frac{i}{0.37} = \frac{16.65\times10^{-3}}{0.37} = 45 \times 10^{-3}A = 45mA.$
4	$W = \frac{1}{2}CE^2 = \frac{1}{2} \times 10^{-6} \times 9^2 = 40.5 \times 10^{-6}J.$

Exercise 7:

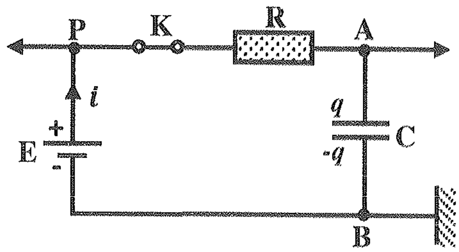
Part	Answer key
1	Charging since u_C increases exponentially with time from zero to a constant limiting value. $u_C = \frac{q}{C}$ during charging q increases $\Longrightarrow u_C$ increases.
2.1	At the end of charging $E = u_{Cmax} = 12V.$
2.2	At t = τ; $u_C = 0.63E = 0.63 \times 12 = 7.56V.$ Graphically: $\tau = 10ms.$
3	$C = \frac{\tau}{R} = \frac{10\times10^{-3}}{500} = 2 \times 10^{-5}F = 20\mu F.$
4	$t = 5\tau = 50ms.$
5.1	Graphically $u_C = 10.4V.$
5.2	$q = Cu_C = 20 \times 10^{-6} \times 10.4 = 2.08 \times 10^{-4}C.$
5.3	Law of addition of voltages in series connection: $E = u_C + u_R.$ $u_R = 12 - 10.4 = 1.6V.$
5.4	By applying Ohm's law: $i = \frac{u_R}{R} = \frac{1.6}{500} = 3.2 \times 10^{-3}A.$
5.5	$W = \frac{1}{2}Cu_C^2 = \frac{1}{2}(20 \times 10^{-6})(10.4)^2 = 1.0816 \times 10^{-4}J.$

Exercise 9:

Part	Answer key
1	$Q_0 = CU_{C0} \Longrightarrow U_{C0} = \frac{Q_0}{C} = \frac{24\times10^{-6}}{2\times10^{-6}} = 12V.$
2	Law of uniqueness of voltage: $u_C = u_R \Longrightarrow u_C = Ri$ with $u_R = Ri.$ $u_C = -RC\frac{du_C}{dt}$ with $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$. Therefore, $u_C + RC\frac{du_C}{dt} = 0.$
3	$u_C = Ae^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = A\left(-\frac{1}{\tau}\right)e^{-\frac{t}{\tau}} = -\frac{A}{\tau}e^{-\frac{t}{\tau}}.$

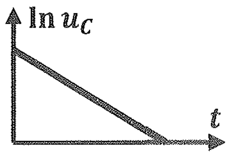
	Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $Ae^{-\frac{t}{\tau}} - \frac{RCA}{\tau}e^{-\frac{t}{\tau}} = 0$. $Ae^{-\frac{t}{\tau}}\left(1 - \frac{RC}{\tau}\right) = 0$ with $Ae^{-\frac{t}{\tau}} \neq 0 \; \forall \, t$. $1 - \frac{RC}{\tau} = 0 \Longrightarrow \tau = RC = 5 \times 10^3 \times 2 \times 10^{-6} = 10^{-2}s = 10ms$. At t = 0s; $u_C = U_{C0} \Longrightarrow U_{C0} = Ae^0 \Longrightarrow A = U_{C0} = 12V$.
4	$i = -C\frac{du_C}{dt} = -C\left(-\frac{A}{\tau}e^{-\frac{t}{\tau}}\right) = \frac{CA}{\tau}e^{-\frac{t}{\tau}} = \frac{2\times10^{-6}\times12}{10^{-2}}e^{-\frac{t}{10^{-2}}} = 2.4 \times 10^{-3}e^{-100t}$ $[SI]$.

Exercise 10:

Part	Answer key
1	
2	During the charging process, q increases; then, $\frac{dq}{dt} > 0$ and $i > 0$ (directed in the positive sense). Therefore, $i = \frac{dq}{dt}$
3	$i = \frac{dq}{dt} = \frac{d(Cu_C)}{dt} = C\frac{du_C}{dt}$.
4	Law of addition of voltages: $u_G = u_C + u_R \Longrightarrow E = u_C + Ri \Longrightarrow E = u_C + RC\frac{du_C}{dt}$.
5	$u_C = D\left(1 - e^{-\frac{t}{\tau}}\right) = D - De^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = 0 - D\left(-\frac{1}{\tau}\right)e^{-\frac{t}{\tau}} = \frac{D}{\tau}e^{-\frac{t}{\tau}}$. Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = D - De^{-\frac{t}{\tau}} + \frac{RCD}{\tau}e^{-\frac{t}{\tau}} \Longrightarrow E = D + De^{-\frac{t}{\tau}}\left(-1 + \frac{RC}{\tau}\right)$. By identification: $D = E$ and $-1 + \frac{RC}{\tau} = 0 \Longrightarrow \tau = RC$ with $De^{-\frac{t}{\tau}} \neq 0 \; \forall \, t$. Therefore, $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$
6.1	At the end of charging, $E = u_{Cm} = 8V$.
6.2	At $t = \tau$; $u_C = E(1 - e^{-1}) = 0.63E = 0.63 \times 8 = 5.04V$. Graphically, $\tau = 2s$. $\tau = RC \Longrightarrow R = \frac{\tau}{C} = \frac{2}{100\times10^{-6}} = 20 \times 10^3\Omega = 20k\Omega$.
7	$t = 5\tau = 5 \times 2 = 10s$.
8	$W = \frac{1}{2}CE^2 = \frac{1}{2} \times 100 \times 10^{-6} \times 8^2 = 3.2 \times 10^{-3}J$.
9	$i = C\frac{du_C}{dt} = C\frac{D}{\tau}e^{-\frac{t}{\tau}} = C\frac{E}{RC}e^{-\frac{t}{\tau}} = \frac{E}{R}e^{-\frac{t}{\tau}}$. Or: law of addition of voltages in series connection: $u_R = E - u_C = E - E + Ee^{-\frac{t}{\tau}} = Ee^{-\frac{t}{\tau}}$. $i = \frac{u_R}{R} = \frac{E}{R}e^{-\frac{t}{\tau}}$.
10	For $t \to \infty$; $i = \frac{E}{R}e^{-\infty} = 0$.

Exercise 11:

Part	Answer key
I.1	Charging since u_C increases exponentially with time from zero to a constant limiting value.
I.2	$i = \frac{dq}{dt} = \frac{d(Cu_C)}{dt} = C\frac{du_C}{dt}$. Law of addition of voltages: $u_G = u_C + u_R \Longrightarrow E = u_C + Ri \Longrightarrow E = u_C + RC\frac{du_C}{dt}$.
I.3	$u_C = A + Be^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = 0 + B\left(-\frac{1}{\tau}\right)e^{-\frac{t}{\tau}} = -\frac{B}{\tau}e^{-\frac{t}{\tau}}$.

	Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = A + Be^{-\frac{t}{\tau}} - \frac{RCB}{\tau}e^{-\frac{t}{\tau}} \Rightarrow E = A + Be^{-\frac{t}{\tau}}\left(1 - \frac{RC}{\tau}\right).$ By identification: $A = E$ and $1 - \frac{RC}{\tau} = 0 \Rightarrow \tau = RC.$ At $t_0 = 0s$; $u_C = 0 \Rightarrow 0 = A + Be^0 \Rightarrow A + B = 0 \Rightarrow B = -A = -E.$ Therefore, $u_C = E - Ee^{-\frac{t}{\tau}} = E\left(1 - e^{-\frac{t}{\tau}}\right).$
I.4.1	At the end of charging $t \to \infty; u_C = E(1 - e^{-\infty}) = E = 10V.$
I.4.2	At $t = \tau$; $u_C = E(1 - e^{-1}) = 0.63E = 0.63 \times 10 = 6.3V.$ Graphically, $\tau = 0.1s.$
I.5	$\tau = RC \Rightarrow C = \frac{\tau}{R} = \frac{0.1}{100} = 10^{-3}F = 1mF.$
II.1	Discharging.
II.2	$i = -\frac{dq}{dt} = -\frac{d(Cu_C)}{dt} = -C\frac{du_C}{dt}.$ $u_{AB} = u_{AM} + u_{MD} \Rightarrow u_C = u_R \Rightarrow u_C = Ri \Rightarrow u_C = -RC\frac{du_C}{dt} \Rightarrow u_C + RC\frac{du_C}{dt} = 0.$
II.3	$u_C = Ee^{-\frac{t}{RC}}$ and $\frac{du_C}{dt} = E\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}} = -\frac{E}{RC}e^{-\frac{t}{RC}}.$ Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $Ee^{-\frac{t}{RC}} + RC\left(-\frac{E}{RC}\right)e^{-\frac{t}{RC}} = 0$ $Ee^{-\frac{t}{RC}} - Ee^{-\frac{t}{RC}} = 0 \Rightarrow 0 = 0$ (Verified).
II.4	$\ln u_C = \ln Ee^{-\frac{t}{RC}} \Rightarrow \ln u_C = \ln E + \ln e^{-\frac{t}{RC}} \Rightarrow \ln u_C = \ln E - \frac{t}{RC}.$ $\ln u_C$ has the form $\ln u_C = at + b$ where $a = slope = \frac{-1}{RC}$ and $b = \ln E$ Shape: decreasing straight line with negative slope and not passing through origin. 
II.4	$i = -C\frac{du_C}{dt} = -C\left(-\frac{E}{RC}e^{-\frac{t}{RC}}\right) = \frac{E}{R}e^{-\frac{t}{RC}}.$

Exercise 12:

Mark	Answer key
A.1	Law of addition of voltages: $E = u_C + u_{R_1} \Rightarrow E = u_C + R_1 i$ with $i = \frac{dq}{dt} = C\frac{du_C}{dt}.$ $E = u_C + R_1 C\frac{du_C}{dt}.$
A.2	$u_C = A\left(1 - e^{-\frac{t}{\tau_1}}\right) = A - Ae^{-\frac{t}{\tau_1}}$ and $\frac{du_C}{dt} = \frac{A}{\tau_1}e^{-\frac{t}{\tau_1}}.$ Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = A - Ae^{-\frac{t}{\tau_1}} + R_1 C\frac{A}{\tau_1}e^{-\frac{t}{\tau_1}} \Rightarrow E = A + Ae^{-\frac{t}{\tau_1}}\left[-1 + \frac{RC}{\tau_1}\right].$ By identification: $E = A$ and $-1 + \frac{R_1 C}{\tau_1} = 0 \Rightarrow \tau_1 = R_1 C.$
A.3.1	At the end of charging $E = u_{Cmax} = 9V.$

A.3.2	At $t = \tau_1; u_C = 0.63E = 0.63 \times 9 = 5.67V$. Graphically $\tau_1 = 0.5s$.
A.4	$C = \frac{\tau_1}{R_1} = \frac{0.5}{10^3} = 5 \times 10^{-4}F = 500\mu F$.
B.1	Law of addition of voltages: $u_{AB} = u_{AD} + u_{DM} \Longrightarrow u_C = R_1 i + R_2 i \Longrightarrow \frac{q}{C} = (R_1 + R_2)i$. Derive both sides with respect to time: $\frac{1}{C}\frac{dq}{dt} = (R_1 + R_2)\frac{di}{dt}$ with $i = -\frac{dq}{dt}$. $-\frac{i}{C} = (R_1 + R_2)\frac{di}{dt} \Longrightarrow \frac{di}{dt} + \frac{1}{(R_1+R_2)C}i = 0$.
B.2	$i = I_0 e^{-\frac{t}{\tau_2}}$ and $\frac{di}{dt} = -\frac{I_0}{\tau_2}e^{-\frac{t}{\tau_2}}$. Replace i and $\frac{di}{dt}$ in the differential equation: $-\frac{I_0}{\tau_2}e^{-\frac{t}{\tau_2}} + \frac{1}{(R_1+R_2)C}I_0 e^{-\frac{t}{\tau_2}} = 0 \Longrightarrow I_0 e^{-\frac{t}{\tau_2}}\left[-\frac{1}{\tau_2} + \frac{1}{(R_1+R_2)C}\right] = 0$. By identification: $-\frac{1}{\tau_2} + \frac{1}{(R_1+R_2)C} = 0 \Longrightarrow \tau_2 = (R_1 + R_2)C$.
B.3.1	$I_0 = 3mA$.
B.3.2	At $t = \tau_2; i = 0.37I_0 = 0.37 \times 3 = 1.11mA$. Graphically, $\tau_2 = 1.5s$. $R_2 = \frac{\tau_2}{C} - R_1 = \frac{1.5}{5\times10^{-4}} - 10^3 = 2 \times 10^3\Omega = 2k\Omega$.

Exercise 14:

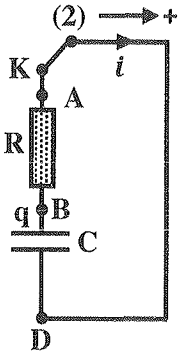
Part	Answer key
1	$u_{BD} = u_C = \frac{q}{C}$. During charging process, q increases so u_C increases.
2	At the end of charging $E = U_{Cmax} = 12V$.
3	$i = \frac{dq}{dt} = \frac{d(Cu_C)}{dt} = C\frac{du_C}{dt}$ with $i > 0$ and $\frac{dq}{dt} > 0$.
4	Law of addition of voltages: $E = u_C + u_R \Longrightarrow E = u_C + Ri \Longrightarrow E = u_C + RC\frac{du_C}{dt}$.
5	$u_C = a + be^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = -\frac{b}{\tau}e^{-\frac{t}{\tau}}$. Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = a + be^{-\frac{t}{\tau}} - \frac{RCb}{\tau}e^{-\frac{t}{\tau}}$. $E = a + be^{-\frac{t}{\tau}}\left[1 - \frac{RC}{\tau}\right]$. By identification: $E = a$ and $1 - \frac{RC}{\tau} = 0 \Longrightarrow \tau = RC$. At $t_0 = 0$; $u_C = 0 \Longrightarrow 0 = a + be^0 \Longrightarrow a + b = 0 \Longrightarrow b = -a = -E$.
6.1	$i = C\frac{du_C}{dt} = -\frac{Cb}{\tau}e^{-\frac{t}{\tau}} = \frac{CE}{RC}e^{-\frac{t}{\tau}} = \frac{E}{R}e^{-\frac{t}{\tau}}$.
6.2	At $t_0 = 0s$; $i = I_0 = \frac{E}{R}e^0 \Longrightarrow I_0 = \frac{E}{R} \Longrightarrow R = \frac{E}{I_0} = \frac{12}{6\times10^{-3}} = 2000\Omega = 2k\Omega$.
6.3	$u_R = Ri = Ee^{-\frac{t}{\tau}}$.
7.1	$u_C = u_R \Longrightarrow E - Ee^{-\frac{t_1}{\tau}} = Ee^{-\frac{t_1}{\tau}} \Longrightarrow E = 2Ee^{-\frac{t_1}{\tau}} \Longrightarrow \frac{1}{2} = e^{-\frac{t_1}{\tau}}$ $\ln\frac{1}{2} = -\frac{t_1}{\tau} \Longrightarrow -\ln 2 = -\frac{t_1}{\tau} \Longrightarrow t_1 = \tau\ln 2 = RC\ln 2$
7.2	$t_1 = 50ms$ (from graph) $C = \frac{t_1}{R\ln 2} = \frac{50\times10^{-3}}{2000\times\ln 2} = 36 \times 10^{-6}F = 36\mu F$.

Exercise 15:

Part	Answer key
I.1	Charging
I.2	The general equation of a straight line is: $u_C = ai + b$ $a = \frac{\Delta u_C}{\Delta i} = -\frac{10-0}{0-0.1} = -100V/A$ For $i = 0; u_C = b = 10V$ Therefore the equation of the graph is: $u_C = -100i + 10$
I.3	Law of addition of voltages in series connection: $E = u_C + u_R$ with $u_R = Ri$ (Ohm's law) $E = u_c + Ri \Longrightarrow u_C = E - Ri$
I.4	By comparison: $E = 10V$ and $R = 100\Omega$.
II.1	Discharging, since u_C decreases exponentially with respect to time.
II.2	By applying the law of uniqueness of voltage: $u_C = u_R$ $u_C = Ri$ with $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$ and $u_R = Ri$ (Ohm's law) $u_C = -RC\frac{du_C}{dt} \Longrightarrow RC\frac{du_C}{dt} + u_C = 0$
II.3	$u_C = Ae^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = -\frac{A}{\tau}e^{-\frac{t}{\tau}}$. Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $-\frac{RCA}{\tau}e^{-\frac{t}{\tau}} + Ae^{-\frac{t}{\tau}} = 0$. $Ae^{-\frac{t}{\tau}}\left(-\frac{RC}{\tau} + 1\right) = 0$. By identification: $-\frac{RC}{\tau} + 1 = 0 \Longrightarrow \frac{RC}{\tau} = 1 \Longrightarrow \tau = RC$.
II.4	Time constant: time needed for the capacitor to reach 37% of its maximum voltage. At $t = \tau; u_C(\tau) = 0.37 \times u_{Cmax} = 0.37 \times 10 = 3.7V$. Graphically: $\tau = 0.1s$.
II.5.1	$C = \frac{\tau}{R} = \frac{0.1}{100} = 10^{-3}F = 1mF$.
II.5.2	$t = 5\tau = 5 \times 0.1s = 0.5s$.
II.6	$q = Cu_c = 10^{-3} \times 10 = 10^{-2}C$. $W = \frac{1}{2}Cu_c^2 = \frac{1}{2}(10^{-3})(10)^2 = 0.05J$.
II.7	At t = 0.2s; $u_R = u_C = 1.35V$. $i = \frac{u_R}{R} = \frac{1.35}{100} = 0.0135A$.

Exercise 17:

Part	Answer key
1.1	By applying the law of addition of voltages: $u_{AD} = u_{AB} + u_{BD} \Longrightarrow u_G = u_C + u_R$. $E = u_C + Ri$ with $i = \frac{dq}{dt} = C\frac{du_C}{dt}$. $E = u_C + RC\frac{du_C}{dt}$.
1.2	$u_C = A + Be^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = -\frac{B}{\tau}e^{-\frac{t}{\tau}}$. Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = A + Be^{-\frac{t}{\tau}} - \frac{RCB}{\tau}e^{-\frac{t}{\tau}}$. $E = A + Be^{-\frac{t}{\tau}}\left(1 - \frac{RC}{\tau}\right)$. By identification: $E = A$ and $1 - \frac{RC}{\tau} = 0 \Longrightarrow \tau = RC$. At t = 0s; $u_C = 0 \Longrightarrow 0 = A + Be^0 \Longrightarrow A + B = 0 \Longrightarrow B = -A = -E$.

1.3	$i = C\frac{du_C}{dt} = -\frac{CB}{\tau}e^{-\frac{t}{\tau}} = \frac{E}{R}e^{-\frac{t}{\tau}}$.
1.4.1	$u_C = E - Ee^{-\frac{t}{\tau}} = E\left(1 - e^{-\frac{t}{\tau}}\right)$. At $t \to \infty$; $u_C = E(1 - e^{-\infty}) = E$ with $e^{-\infty} = 0$.
1.4.2	For $u_C = \frac{E}{3}$. $\frac{E}{3} = E\left(1 - e^{-\frac{t}{\tau}}\right) \Longrightarrow 1 - e^{-\frac{t}{\tau}} = \frac{1}{3} \Longrightarrow e^{-\frac{t}{\tau}} = 1 - \frac{1}{3} = \frac{2}{3} \Longrightarrow \ln e^{-\frac{t}{\tau}} = \ln\frac{2}{3}$. $-\frac{t}{\tau} = \ln\frac{2}{3} \Longrightarrow \frac{t}{\tau} = \ln\frac{3}{2} \Longrightarrow t = \tau\ln\frac{3}{2} = RC\ln\frac{3}{2}$.
1.4.3	The final electric energy stored by the capacitor is: $W_e = \frac{1}{2}CE^2$. $W = \int_0^{+\infty} Eidt = \int_0^{+\infty} E \times \frac{E}{R}e^{-\frac{t}{\tau}} = \frac{E^2}{R}\int_0^{+\infty} e^{-\frac{t}{\tau}}$. $W = \frac{E^2}{R}\left[-\tau e^{-\frac{t}{\tau}}\right]_0^{\infty} = \frac{E^2}{R}[0 + \tau] = \frac{\tau E^2}{R} = CE^2$.
1.5	By applying Ohm's law: $u_R = Ri = Ee^{-\frac{t}{\tau}}$. By applying the law of addition of voltages: $u_G = u_R + u_C$. $E = u_R + E - Ee^{-\frac{t}{\tau}} \Longrightarrow u_R = Ee^{-\frac{t}{\tau}}$.
2.1	
2.2	The current circulates in the positive sense $\Longrightarrow i > 0$. The capacitor undergoes discharging $\Longrightarrow q$ decreases with time $\Longrightarrow \frac{dq}{dt} < 0$. Therefore, $i = -\frac{dq}{dt}$.
2.3	By applying the law of uniqueness of voltage: $u_C = u_R$ with $u_C = \frac{q}{C}$ and $u_R = Ri$. $\frac{q}{C} = Ri$ (Derive both sides with respect to time). $\frac{1}{C}\frac{dq}{dt} = R\frac{di}{dt} \Longrightarrow -\frac{i}{C} = R\frac{di}{dt} \Longrightarrow R\frac{di}{dt} + \frac{i}{C} = 0 \Longrightarrow \frac{di}{dt} + \frac{i}{RC} = 0$.
2.4	$i = \frac{E}{R}e^{-\frac{t}{\tau}}$ and $\frac{di}{dt} = -\frac{E}{R\tau}e^{-\frac{t}{\tau}} = \frac{E}{R^2C}e^{-\frac{t}{\tau}}$. Replace i and $\frac{di}{dt}$ in the differential equation: $-\frac{E}{R^2C}e^{-\frac{t}{\tau}} + \frac{E}{R^2C}e^{-\frac{t}{\tau}} = 0 \Longrightarrow 0 = 0$ (Verified).
2.5.1	At t = 0; $i = I_0 = 12mA$. $I_0 = \frac{E}{R}e^0 \Longrightarrow E = RI_0 = 10^3 \times 12 \times 10^{-3} = 12V$.
2.5.2	At $t = \tau$; $i = 0.37I_0 = 4.44mA$. Graphically: $\tau = 1ms$.
2.6	$\tau = RC \Longrightarrow C = \frac{\tau}{R} = \frac{10^{-3}}{10^3} = 10^{-6}F = 1\mu F$.

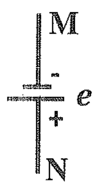
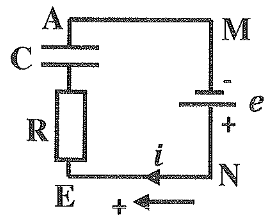
Exercise 18:

Part	Answer key
I.1.1	By applying the law of addition of voltages: $u_{AD} = u_{AB} + u_{BD} \Longrightarrow u_G = u_{R_1} + u_C$. $E = u_{R_1} + \frac{q}{C}$ with $u_C = \frac{q}{C}$. Derive both sides with respect to time: $0 = \frac{du_{R1}}{dt} + \frac{1}{C}\frac{dq}{dt}$ with $i = \frac{dq}{dt}$ and $i = \frac{u_{R1}}{R_1}$. $0 = \frac{du_{R1}}{dt} + \frac{1}{R_1C}u_{R_1}$.
I.1.2	$u_{R_1} = Ae^{-\frac{t}{\tau}}$ and $\frac{du_{R1}}{dt} = -\frac{A}{\tau}e^{-\frac{t}{\tau}}$. Replace u_{R_1} and $\frac{du_{R1}}{dt}$ in the differential equation: $-\frac{A}{\tau}e^{-\frac{t}{\tau}} + \frac{A}{R_1C}e^{-\frac{t}{\tau}} = 0 \Longrightarrow Ae^{-\frac{t}{\tau}}\left(-\frac{1}{\tau} + \frac{1}{R_1C}\right) = 0$ By identification: $-\frac{1}{\tau} + \frac{1}{R_1C} = 0 \Longrightarrow \tau = R_1C$. At t = 0s; $u_{R_1} = E - u_C = E - 0 = E$. $E = Ae^0 \Longrightarrow A = E$.
I.1.3	$u_{R_1} = Ee^{-\frac{t}{R_1C}}$. $\ln u_{R_1} = \ln Ee^{-\frac{t}{R_1C}} \Longrightarrow \ln u_{R_1} = \ln E + \ln e^{-\frac{t}{R_1C}}$. $\ln u_{R_1} = \ln E - \frac{1}{R_1C}t$.
I.2.1	The graph of $\ln u_{R_1}$ versus t shows a straight line not passing through origin with negative slope Consequently, the shape of the obtained graph agrees with the expression of $\ln u_{R_1}$ versus t. $\ln u_{R_1}$ can be written in the form of $\ln u_{R_1} = b + at$.
I.2.2	By comparison: $b = \ln E = 2.31 \Longrightarrow E = e^{2.31} = 10V$. $-\frac{1}{R_1C} = a \Longrightarrow a = \frac{\Delta \ln u_{R1}}{\Delta t} = \frac{0-2.31}{0.0462-0} = 50$. $C = -\frac{1}{R_1a} = \frac{1}{10^4\times 50} = 2\times 10^{-6}F = 2\mu F$.
II.1	Because the armature B of the capacitor is charged positively.
II.2	$u_{BD} = u_{BA} + u_{AD} \Longrightarrow u_C = u_{R1} + u_{R2}$. $u_C = (R_1 + R_2)i$ with $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$. $u_C = -(R_1 + R_2)C\frac{du_C}{dt}$. $u_C + (R_1 + R_2)C\frac{du_C}{dt} = 0$.
II.3	$u_C = Ee^{-\frac{t}{\tau_2}}$ and $\frac{du_C}{dt} = -\frac{E}{\tau_2}e^{-\frac{t}{\tau_2}}$. Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $Ee^{-\frac{t}{\tau_2}} - \frac{R_1+R_2}{\tau_2}CEe^{-\frac{t}{\tau_2}} = 0$. $Ee^{-\frac{t}{\tau_2}}\left(1 - \frac{R_1+R_2}{\tau_2}C\right) = 0$. By identification: $1 - \frac{R_1+R_2}{\tau_2}C = 0 \Longrightarrow \tau_2 = (R_1 + R_2)C$.
II.4	The tangent to the curve $u_C = f(t)$ at the instant $t_0 = 0$ meets the time axis at a point of abscissa $\tau_2 = 40ms = 0.04s$. $C = \frac{\tau_2}{R_1+R_2} = \frac{0.04}{2\times 10^4} = 2\times 10^{-6}F = 2\mu F$.

Exercise 20:

Part	Answer key
1.1	$q = Cu_C$
1.2	Law of addition of voltages in series connection: $u_{PN} = u_{PA} + u_{AB} + u_{BN} \Longrightarrow u_G = u_C + u_R$. $E = \frac{q}{C} + Ri$ with $i = \frac{dq}{dq}$. $E = \frac{q}{C} + R\frac{dq}{dt} \Longrightarrow \frac{E}{R} = \frac{dq}{dt} + \frac{q}{RC}$.
1.3	$q = A + Be^{-\frac{t}{\tau}}$ and $\frac{dq}{dt} = -\frac{B}{\tau}e^{-\frac{t}{\tau}}$. Replace q and $\frac{dq}{dt}$ in the differential equation: $\frac{E}{R} = -\frac{B}{\tau}e^{-\frac{t}{\tau}} + \frac{A}{RC} + \frac{B}{RC}e^{-\frac{t}{\tau}}$. $\frac{E}{R} = \frac{A}{RC} + Be^{-\frac{t}{\tau}}\left(-\frac{1}{\tau} + \frac{1}{RC}\right)$. By identification: $\frac{E}{R} = \frac{A}{RC} \Longrightarrow A = CE$ and $-\frac{1}{\tau} + \frac{1}{RC} = 0 \Longrightarrow \tau = RC$. At t = 0; $u_C = 0 \Longrightarrow q = 0$. $0 = A + Be^0 \Longrightarrow A + B = 0 \Longrightarrow B = -A = -CE$.
1.4	$q = CE - CEe^{-\frac{t}{\tau}} = CE\left(1 - e^{-\frac{t}{\tau}}\right)$. At $t \to \infty$; $Q_m = CE(1 - e^{-\infty}) = CE$ with $e^{-\infty} = 0$.
2.1	$Q_m = 1.2mC = 1.2 \times 10^{-3}C$. $Q_m = CE \Longrightarrow C = \frac{Q_m}{E} = \frac{1.2\times10^{-3}}{12} = 10^{-4}F = 100\mu F$.
2.2	At $t = \tau; q = 0.63Q_m = 0.63 \times 1.2 = 0.756mC$. Graphically: $\tau = 10ms$. $R = \frac{\tau}{C} = \frac{10\times10^{-3}}{10^{-4}} = 100\Omega$.
2.3	$i = \frac{dq}{dt} \Longrightarrow i(t = 20ms) = \frac{\Delta q}{\Delta t} = \frac{(1.2-0.71)\times10^{-3}}{(30-0)\times10^{-3}} = 0.163A$.

Exercise 21:

Part	Answer key
1	$\phi = N\vec{B}.\vec{S} = NBS\cos\theta$ with $\theta = (\vec{B};\vec{n}) = 180°$ (by RHR). $\phi = BS\cos 180° = -BS = -B\ell x$.
2.1	φ varies because S varies $\Longrightarrow e = -\frac{d\phi}{dt}$ exists. $e = B\ell\frac{dx}{dt} = B\ell v = 0.8 \times 1 \times 0.5 = 0.4V$.
2.2	The induced current opposes, by its electromagnetic effect, the cause that produces it. The Laplace force then opposes the direction of displacement of the rod ; The induced current then passes through the rod from point M to point N.
2.3	$u_{MN} = ri - e = -e \Longrightarrow u_{MN} = e$ with $r = 0$.
3.1	by applying the law of addition of voltages: $u_G = u_R + u_C$. $e = Ri + u_C$ with $i = \frac{dq}{dt} = C\frac{du_C}{dt}$. $e = RC\frac{du_C}{dt} + u_C$.  
3.2.1	T = RC = 100 × 10^{-2} = 1s.

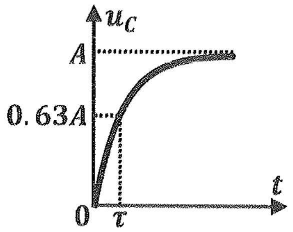
3.2.2	The complete charge is practically attained at $5\tau = 5s$.
3.3	U = e = 0.4V $Q = CU = 10^{-2} \times 0.4 = 0.004C$.
3.4	$e = Ri + u_C$. For $t_0 = 0s; u_S = 0 \Longrightarrow e = RI_0 \Longrightarrow I_0 = \frac{e}{R} = \frac{0.4}{100} = 4mA$. For t = 6s, the capacitor is charged completely $\Longrightarrow u_C = e \Longrightarrow i = 0$.
3.5.1	Is a result of discharging of the capacitor through the resistor.
3.5.2	The duration of the passage of the current of discharging is $5\tau = 5RC = 5s$.

Exercise 22:

Part	Answer key
1.1	According to Ohm's law for a generator: $u_G = E - ri$
1.2	By applying the law of addition of voltages: $u_{PN} = u_{PA} + u_{AB} + u_{BN} \Longrightarrow u_G = u_R + u_C$. $E - ri = Ri + u_C$. $E = u_C + (R + r)i \text{ with } i = \frac{dq}{dt} = C\frac{du_C}{dt}$. Therefore, $E = (R + r)C\frac{du_C}{dt} + u_C$.
1.3	$u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$ and $\frac{du_C}{dt} = \frac{A}{\tau}e^{-\frac{t}{\tau}}$. Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = \frac{(R+r)C}{\tau}Ae^{-\frac{t}{\tau}} + A - Ae^{-\frac{t}{\tau}}$. $E = A + Ae^{-\frac{t}{\tau}}\left[\frac{(R+r)C}{\tau} - 1\right]$. By identification: $E = A$ and $\frac{(R+r)C}{\tau} - 1 = 0 \Longrightarrow \tau = (R + r)C$.
1.4	$i = C\frac{du_C}{dt} = \frac{CE}{\tau}e^{-\frac{t}{\tau}} = \frac{E}{R+r}e^{-\frac{t}{\tau}}$. $u_G = E - ri = E - \frac{rE}{R+r}e^{-\frac{t}{\tau}} = E\left(1 - \frac{r}{R+r}e^{-\frac{t}{\tau}}\right)$.
1.5.1	At t = 0; $U_0 = E\left(1 - \frac{r}{R+r}e^0\right) = E\left(1 - \frac{r}{R+r}\right) = \frac{RE}{R+r}$.
1.5.2	At $t \to \infty; U_m = E\left(1 - \frac{r}{R+r}e^{-\infty}\right) = E$ with $e^{-\infty} = 0$.
2.1	$U_m = E = 12V$. $U_0 = 9.6V \Longrightarrow 9.6 = \frac{8\times12}{8+r} \Longrightarrow 76.8 + 9.6r = 96 \Longrightarrow r = 2\Omega$.
2.2	$u_G = 12\left(1 - 0.2e^{-\frac{t}{\tau}}\right)$. At $t = \tau; u_G = 12(1 - 0.2e^{-1}) = 11.1V$.
2.3	Graphically: $\tau = 40ms$. $C = \frac{\tau}{R+r} = \frac{40\times10^{-3}}{10} = 4 \times 10^{-3}F = 4mF$.

Exercise 23:

Part	Answer key
I.1.1	$i = \frac{dq}{dt}$ and $q = Cu_C$ then $i = C\frac{du_C}{dt}$.
I.1.2	Law of addition of voltages in series connection: $E = u_R + u_C$. $E = Ri + u_C$ with $u_R = Ri$. $E = RC\frac{du_C}{dt} + u_C$.

I.1.3.1	$u_C = A\left(1 - e^{-\frac{t}{\tau}}\right) = A - Ae^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = \frac{A}{\tau}e^{-\frac{t}{\tau}}$. Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = \frac{RCA}{\tau}e^{-\frac{t}{\tau}} + A - Ae^{-\frac{t}{\tau}}$. $E = A + Ae^{-\frac{t}{\tau}}\left(\frac{RC}{\tau} - 1\right) = 0$. By identification: $A = E$ and $\frac{RC}{\tau} - 1 = 0 \Longrightarrow \frac{RC}{\tau} = 1 \Longrightarrow \tau = RC$.
I.1.3.2	
I.2	$u_{AA} = u_{AB} + u_{BM} \Longrightarrow u_R + u_C = 0$. $Ri + u_C = 0$. $RC\frac{du_C}{dt} + u_C = 0$ $u_C = Ee^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = -\frac{E}{\tau}e^{-\frac{t}{\tau}}$ Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $-\frac{RCE}{\tau}e^{-\frac{t}{\tau}} + Ee^{-\frac{t}{\tau}} = 0$ which is true since $\tau = RC$.
I.3.1	The minimum duration of charging mode or discharging mode so that u_C reaches its steady state must be 5τ.
I.3.2	The minimum value of T must be 10τ.
II.1.1	The curve (3) corresponds to the charging mode of the capacitor since u_C increases exponentially with time.
II.1.2	$E = 5V/div \times 2div = 10V$. The period T of the square signal = $2ms/div \times 10div = 20ms$.
II.2.1	The period T of the square signal is now: $1ms/div \times 5div = 5ms$. The duration of the charging and of the discharging is now less than 5τ. The capacitor has no more time to be completely charged and discharged.
II.2.2	$T \ll 10\tau$, the curve becomes linear (straight line) during charging and discharging.

Exercise 24:

Part	Answer key
I.1	$i = \frac{dq}{dt} \Longrightarrow dq = idt \Longrightarrow q = \int idt$. $q = \int I_0 dt = I_0 t + q_0 = I_0 t$ with $q_0 = 0$ (the capacitor is initially uncharged). $q = Cu_C \Longrightarrow u_C = \frac{q}{C} = \frac{I_0}{C}t$.
I.2	The expression of u_C has the form of $u_C = at$ where $a = slope$ (general equation of a straight line passing through origin). The graph of u_C versus t shows a straight line passing through origin reflecting the agreement between the expression of u_C and the graph of document 2.
I.3	$a = \frac{\Delta u_C}{\Delta t} = \frac{20-0}{40\times10^{-3}-0} = 500V/s$. $a = \frac{I_0}{C} \Longrightarrow C = \frac{I_0}{a} = \frac{0.5\times10^{-3}}{500} = 10^{-6}F = 1\mu F$.

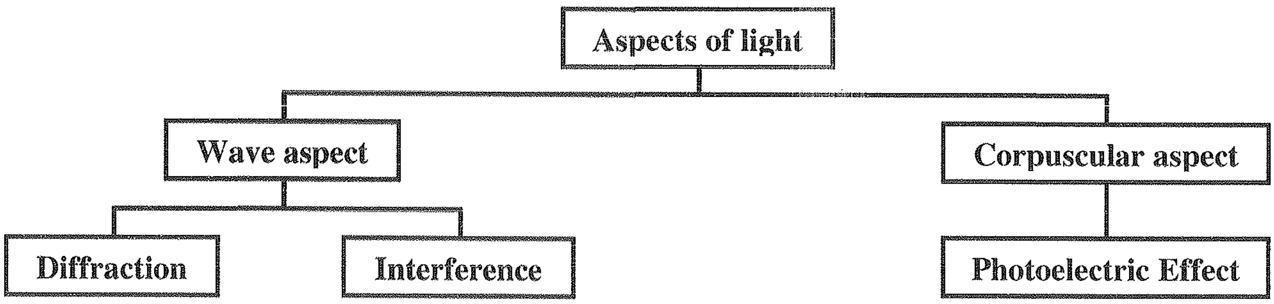
I.4.1	Breakdown potential is the maximum voltage that should not be exceeded in order not to damage the capacitor.
I.4.2	$U_m = \frac{I_0}{C}t_1 \Rightarrow t_1 = \frac{U_m C}{I_0} = \frac{30\times10^{-6}}{0.5\times10^{-3}} = 60 \times 10^{-3}s = 60ms.$
I.4.3	$Q_m = CU_m = 10^{-6} \times 30 = 3 \times 10^{-5}C.$ $W = \frac{1}{2}CU_m^2 = \frac{1}{2} \times 10^{-6} \times 30^2 = 4.5 \times 10^{-4}J.$
II.1	Law of addition of voltages in series connection: $u_{G'} = u_C + u_R$. $E = u_C + Ri$ with $u_R = Ri.$ $E = u_C + RC\frac{du_C}{dt}$ with $i = \frac{dq}{dt} = C\frac{du_C}{dt}.$
II.2	$u_C = A + Be^{-\frac{t}{\tau}}$ and $\frac{du_C}{dt} = 0 + B\left(-\frac{1}{\tau}\right)e^{-\frac{t}{\tau}} = -\frac{B}{\tau}e^{-\frac{t}{\tau}}.$ Replace u_C and $\frac{du_C}{dt}$ in the differential equation: $E = A + Be^{-\frac{t}{\tau}} - \frac{RCB}{\tau}e^{-\frac{t}{\tau}} \Rightarrow E = A + Be^{-\frac{t}{\tau}}\left(1 - \frac{RC}{\tau}\right).$ By identification: $E = A$ and $1 - \frac{RC}{\tau} = 0 \Rightarrow \tau = RC$ with $Be^{-\frac{t}{\tau}} = 0$ *not true* $\forall t$. At $t_0 = 0s$, $u_C = 0 \Rightarrow 0 = A + Be^0 \Rightarrow B = -A = -E.$ Therefore, $u_C = E - Ee^{-\frac{t}{\tau}} = E\left(1 - e^{-\frac{t}{\tau}}\right).$
II.3	At $t_2 = 5\tau$, $u_C = E(1 - e^{-5}) = 0.99E \simeq E.$
II.4	$t_2 = 5\tau = 5RC \Rightarrow R = \frac{t_2}{5C} = \frac{60\times10^{-3}}{5\times10^{-6}} = 12 \times 10^3\Omega = 12k\Omega.$

CHAPTER 6 – WAVE ASPECT OF LIGHT (DIFFRACTION) COURSE

6.1- TIMELINE OF LIGHT THEORY

Throughout history, scientists and philosophers tried to explain the nature of light. This aim was difficult because light is very complicated. Some of the theories concerning the nature of light are mentioned below:
Era of Ancient Greeks: around 300BC, Euclid believed that light rays came from the eyes of the observer.
Ibn al-Haytham (965 to 1040): around 1000CE, he believed that vision would only take place when a light ray was issued from a luminous source or was reflected from such a source before it entered the eye.
Christian Huygens (1629 to 1695): in 1678, he postulated that light is a wave.
Isaac Newton (1643-1727): in 1704, he proposed that light is comprised of tiny colored particles called corpuscles.
Thomas Young (1773 to 1829): in 1807, he gave a proof to the wave theory of light.
James Clerk Maxwell (1831 to 1879): in 18644, he predicted the existence of electromagnetic waves.
Max Planck (1858 to 1947): in 1900, he proposed the theory of quanta which gave the birth of quantum mechanics.
Albert Einstein (1879 to 1955): in 1905, he created the quantum theory of light. He proposed that light is a photon.

6.2- ASPECTS OF LIGHT



6.3- PREREQUISITES

A **wave** is a disturbance (perturbation) that transfers energy through matter or space, without mass transport.

Nature of waves according to medium
Electromagnetic waves, such as light, radio or X-ray, do not require a medium to propagate.
A **matter wave** is the wave-like behavior shown by electrons and other particles under certain conditions.
Mechanical waves, such as water waves, sound waves and the waves traveling along a spring or rope, require a medium such as water, air, springs or ropes to propagate.

Index of refraction of a transparent medium
The index of refraction n of a transparent medium is the ratio of the speed of light c in vacuum to the speed of light v in this medium:

$$n = \frac{c}{v}$$

In SI units, c an v are expressed in [m/s].

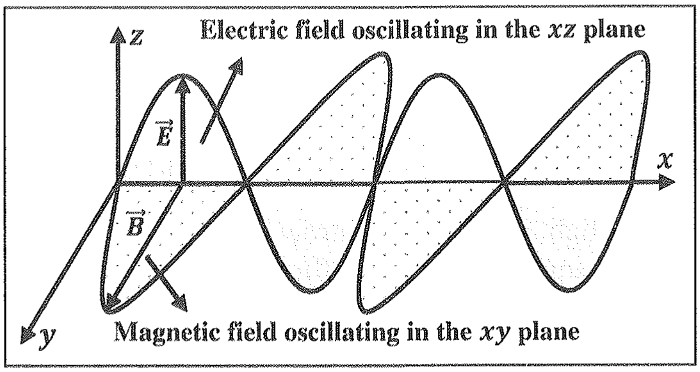
ATTENTION
The index of refraction of a transparent medium depends on the wavelength of the light.

Electromagnetic waves

An **electromagnetic wave** is a wave that consists of oscillating electric and magnetic fields, which radiate outward from the source at the speed of light.
Examples: light, radiowaves, X-rays and microwaves …

Attention:
- When a charge oscillates, an electromagnetic wave is produced; thus, it is an electromagnetic perturbation.
- Electromagnetic waves can be produced by: (i) an electron that falls from an energy level of an atom to a lower one, (ii) an accelerating charge and (iii) an oscillating magnetic or electric field.

Characteristics of electromagnetic waves:



- An electromagnetic wave is made up of an electric field and a magnetic field vibrating sinusoidally at right angles to each other, having the same direction of propagation, are inphase and of the same frequency.
- An electromagnetic wave is a wave that doesn't need a material medium to propagate through, it can propagate in vacuum.
- Electromagnetic waves are transverse i.e. the direction of propagation of the waves is perpendicular to the direction of vibration of the particles of the medium.
- The energy carried by an electromagnetic wave is directly proportional to the frequency.
- In vacuum, electromagnetic waves propagate with the speed $c = 3 \times 10^8 m/s$.
- In a homogenous, transparent and isotropic medium, electromagnetic waves propagate with a speed $v = \frac{c}{n}$ where n is the index of refraction (optical density) of the medium.
- The frequency of the electromagnetic wave remains constant as the wave passes from one medium to another.
- All electromagnetic radiations are characterized by their frequencies.
- The speed of an electromagnetic radiation of wavelength λ and frequency f is $v = \lambda \times f$.
  In SI units, v is expressed in [m/s], λ in [m] and f in [Hz].

- **Passage of an electromagnetic wave from vacuum to a medium of index n.**

$f_0 = f \Rightarrow \frac{c}{\lambda_0} = \frac{V}{\lambda}$

$\frac{c}{\lambda_0} = \frac{c}{n\lambda} \Rightarrow \lambda_0 = n\lambda \Rightarrow \lambda = \frac{\lambda_0}{n}$

Consequently, $n = \frac{\lambda_{vacuum}}{\lambda_{medium}}$

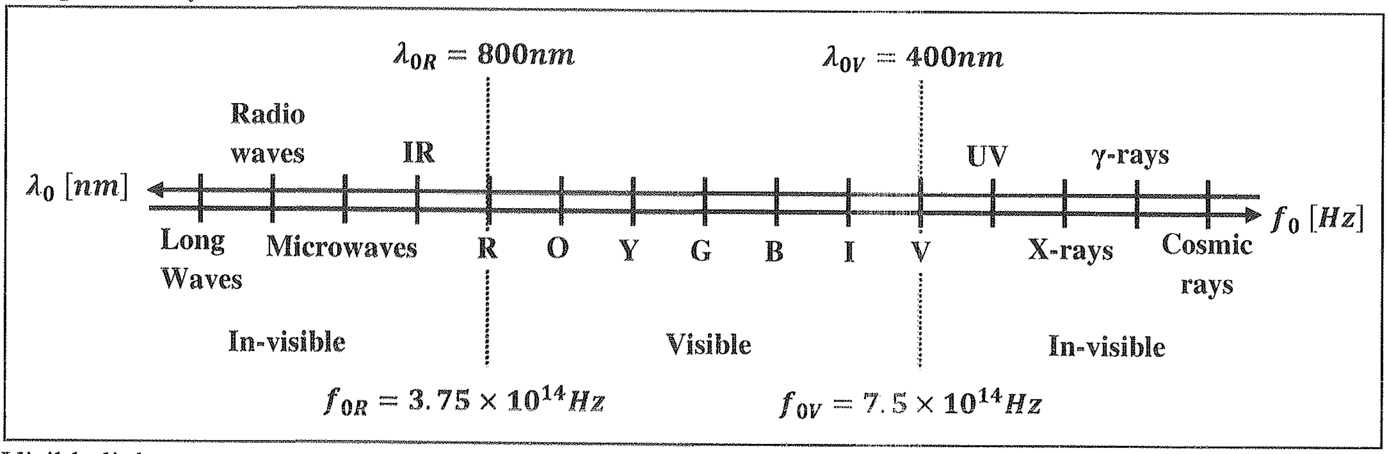
	Vacuum or still air	Medium of index n
Speed	c	$V = \frac{c}{n}$
Frequency	f_0	f
Wavelength	λ_0	λ

- **Passage of an electromagnetic wave from medium 1 to medium 2:**

$f_1 = f_2 \Rightarrow \frac{V_1}{\lambda_1} = \frac{V_2}{\lambda_2}$

	Medium of index n_1	Medium of index n_2
Speed	V_1	V_2
Frequency	f_1	f_2
Wavelength	λ_1	λ_2

The **electromagnetic (EM) spectrum** is the range of all types of EM radiations. Radiation is energy that travels and spreads out as it goes – the visible light that comes from a lamp in your house and the radio waves that come from a radio station are two types of electromagnetic radiations. The other types of EM radiation that make up the electromagnetic spectrum are microwaves, infrared light, ultraviolet light, X-rays and gamma-rays.



Visible light represents a narrow range of the electromagnetic spectrum.
Visible range in vacuum: $400nm \leq \lambda_0 \leq 800nm$

Monochromatic and polychromatic light

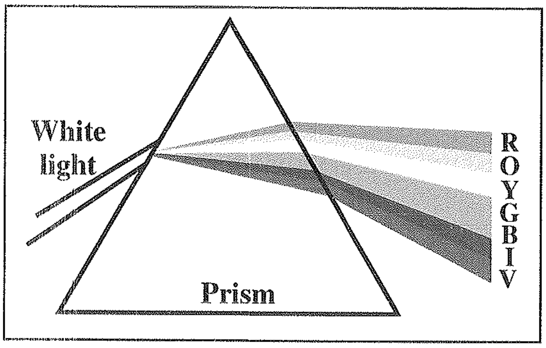
A **monochromatic light** consists of single radiation of specific color, wavelength and frequency.
Example: laser.
A **polychromatic light** consists of multiple radiations of multiple colors, wavelengths and frequencies.
Example: sunlight

> **TIP**
> Chrom is a Greek words means "color"

A **Spectroscope** is an optical apparatus that separates light into colors. It uses dispersion system like a prism

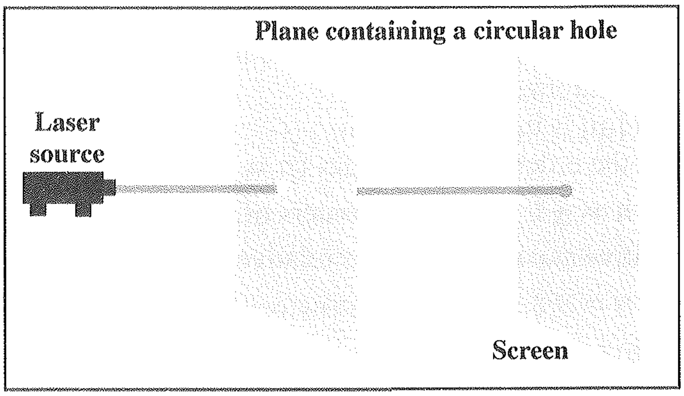
Dispersion of white light

The colors shown in the adjacent document are often observed as light passes through a triangular prism. Upon passage through the prism, the white light is separated into its component colors: red, orange, yellow, green, blue, indigo, and violet. The separation of visible light into its different colors is known as dispersion.



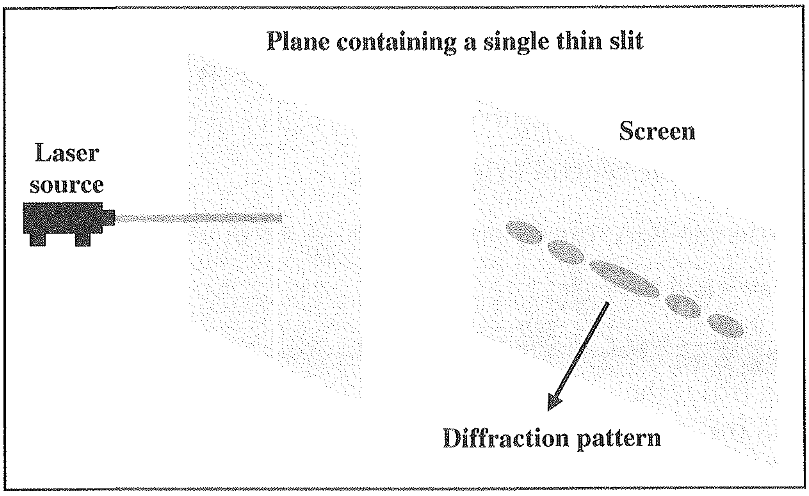
6.4- DIFFRACTION OF LIGHT BY A PLANE SLIT

Diffraction is the process by which a beam of light or other system of waves spreads out as a result of passing through a narrow aperture (slit) or across a sharp edge.
Diffraction is a phenomenon which applies to all types of waves (mechanical and electromagnetic). It keeps the frequency and the wavelength but modifies the form of the wavefronts.
Diffraction of light is used in diffraction gratings, X-ray diffraction, holograms (credit card), sunset pattern, colors in sky, satellite dish …



Experimental study of the diffraction phenomenon
First Experiment
Equipment
- A laser light source.
- An observation screen.
- A plane containing a circular hole of diameter a = 1cm (a >> 1mm).

Observation on the screen
We observe on the screen a circular spot of diameter a = 1cm (no diffraction phenomenon), where the path of light obeys the principle of rectilinear propagation.



Second Experiment
Equipment
- A laser light source.
- An observation screen.
- A plane containing a thin slit of width a < 1mm.

Observation on the screen
- Alternating bright and dark fringes on both sides of the central bright fringe.
- The width of the central bright fringe is double that of any other bright fringe.
- The direction of the pattern of fringes is perpendicular to that of the slit.
- Central bright fringe has maximum intensity.

Formation of bright and dark fringes:
Bright fringes (maximum intensity) are formed due to constructive interference (superposition of waves vibrating in-phase).
Dark fringes (minimum intensity) are formed due to destructive interference (superposition of waves vibrating in opposite phase).

Attention:
- Diffraction increases as the width of the slit decreases. Consequently, the fringes becomes wider and brighter.
- Diffraction of light is the bending of light as it passes through a slit (a < 1mm), a sharp edge … Thus, a ray of light cannot be isolated from a beam of light.

Analytical study of diffraction by a thin slit

Angles of diffraction of the centers of dark fringes

$$\sin\theta_n = \frac{n\lambda}{a}$$

For small angles: $\sin\theta_n \simeq \theta_n$, so $\theta_n = \frac{n\lambda}{a}$ $\left(\text{multiples of } \frac{\lambda}{a}\right)$.

θ_n in [rd] is the angle of diffraction of the center of the n^{th} D.F.

λ in [m] is the wave length of the incident monochromatic light.

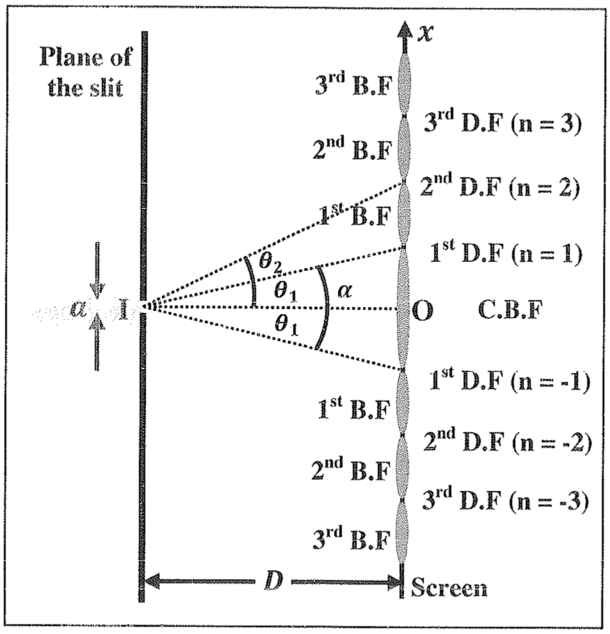
a in [m] is the width of slit.

n is the order of dark fringes ($n = \pm1;\ \pm2;\ \pm3 \ldots$).

For $n = \pm1$; $\theta_1 = \pm\frac{\lambda}{a}$ (centers of the 1st D.F on the positive and negative sides relative to the C.B.F).

For $n = \pm2$; $\theta_2 = \pm\frac{2\lambda}{a}$ (centers of the 2nd D.F on the positive and negative sides relative to the C.B.F).

Therefore, $\theta_n = n\theta_1$ (multiples of θ_1)



Angular width of the central bright fringe

For the center of the 1st D.F: $n = \pm1 \Rightarrow \sin\theta_1 = \pm\frac{\lambda}{a} \Rightarrow \theta_1 = \pm\frac{\lambda}{a}$.

$\alpha = 2|\theta_1| = \frac{2\lambda}{a}$ (the angular width α of the C.B.F is the angle through which the C.B.F is observed form the center of the slit).

ABBREVIATIONS

C.B.F: central bright fringe
B.F: bright fringe
D.F: dark fringe

Width of the central bright fringe

In triangle IOA: $\tan\theta_1 = \frac{OA}{IO} = \frac{\frac{L}{2}}{D} = \frac{L}{2D} \Rightarrow \theta_1 = \frac{L}{2D}$.

$\theta_1 = \frac{\lambda}{a}$ and $\theta_1 = \frac{L}{2D}$.

So, $\frac{\lambda}{a} = \frac{L}{2D} \Rightarrow L = \frac{2\lambda D}{a} \Rightarrow L = \alpha D$.

α in [rd] is the angular width of the C.B.F.

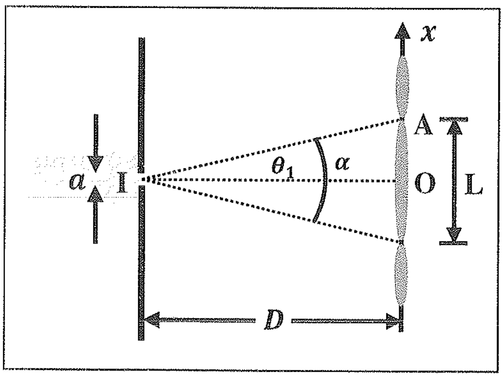
D in [m] is the distance between the plane of the slit and the screen.

L in [m] is the linear width of the C.B.F.

Remark: as λ increases or/and a decreases $\Rightarrow L$ increases.

TIP

Small angles approximation:
$\theta \le 10°$ or $\theta \le 0.17rad$
$\sin\theta \simeq \tan\theta \simeq \theta_{rad}$



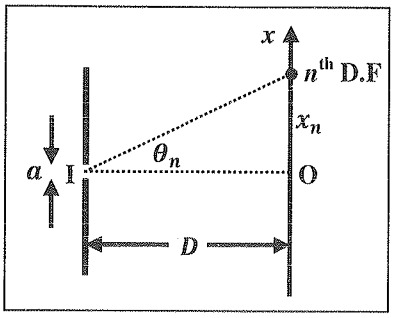
Positions of the centers of the dark fringes:

$\sin\theta_n = n\frac{\lambda}{a} \Rightarrow \theta_n = n\frac{\lambda}{a}$ (for small angles).

$\tan\theta_n = \frac{x_n}{D} \Rightarrow \theta_n = \frac{x_n}{D}$ (for small angles).

$n\frac{\lambda}{a} = \frac{x_n}{D} \Rightarrow x_n = n\frac{\lambda D}{a} = n\frac{L}{2}$.

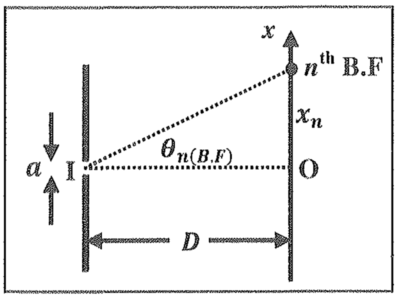
For $n = \pm1; x_n = \pm\frac{L}{2}$ (positions of the centers of the 1st D.F on the positive and negative sides relative to the C.B.F)



Angular positions of the centers of the bright fringes:

$\theta_{n(B.F)} = \theta_{n(D.F)} + \frac{\theta_1}{2} = \frac{n\lambda}{a} + \frac{\lambda}{2a}$.

$\theta_{n(B.F)} = (2n+1)\frac{\lambda}{2a} = (2n+1)\frac{\theta_1}{2}$ $\left(\text{odd multiples of } \frac{\lambda}{2a} \text{ or } \frac{\theta_1}{2}\right)$.



Positions of the centers of the bright fringe:

$\tan\theta_{n(B.F)} = \frac{x_{n(B.F)}}{D} \Rightarrow \theta_{n(B.F)} = \frac{x_{n(B.F)}}{D} \Rightarrow x_{n(B.F)} = D\theta_{n(B.F)}$.

Therefore, $x_{n(B.F)} = (2n+1)\frac{\lambda D}{2a}$.

$x_{n(B.F)} = x_{n(D.F)} + \frac{L}{4} \Rightarrow x_{n(B.F)} = n\frac{\lambda D}{a} + \frac{\lambda D}{2a} = (2n+1)\frac{\lambda D}{2a}$.

The C.B.F is twice as broad as the other B.Fs and of width L.

Consequently, the width of a B.F is $\frac{L}{2}$.

The center of a B.F is midway between the centers of two consecutive D.Fs; then, the distance between the center of D.F and the center of the next B.F is $\frac{L}{4}$.

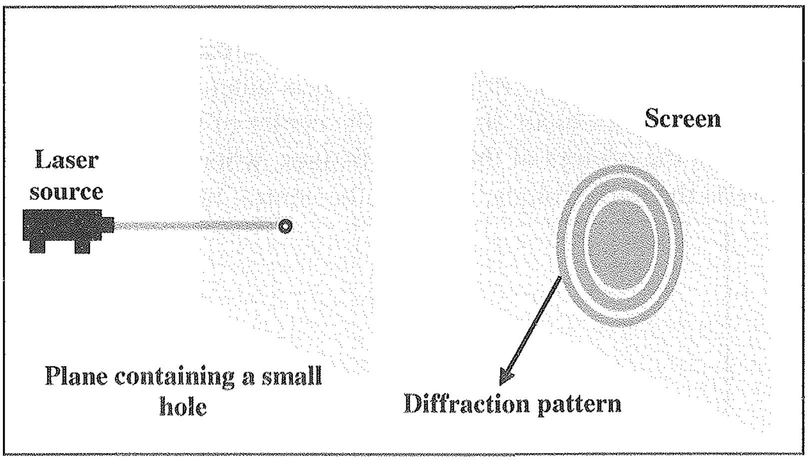
Remark: for diffraction by a hair or thin thread of diameter d, the positions of the dark fringes are given by:

$$sin\,\theta_n = \theta_n = \frac{n\lambda}{d}$$

6.5- DIFFRACTION BY A SMALL CIRCULAR HOLE

Third Experiment

In the following experiment, the thin slit is replaced by a small circular hole of diameter a < 1mm.



Observation on the screen

We observe on the screen a diffraction pattern made up of central bright fringe surrounded by concentric alternating bright and dark fringes.

These rings becomes wider and brighter as the diameter of the hole decreases.

These diffraction rings have been observed and studied for the first time by Joseph von Fraunhofer in 1823.

Conclusion

The path followed by light when it crosses small holes of diameter of the order of millimeter contradicts the light-ray model. Light undergoes the diffraction phenomenon.

Angles of diffraction of the centers of the dark fringes

$\theta_1 = 1.22\frac{\lambda}{d}$ (1st D.F)

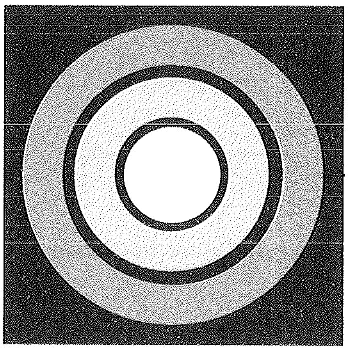
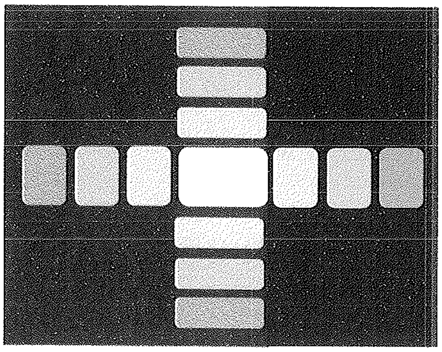
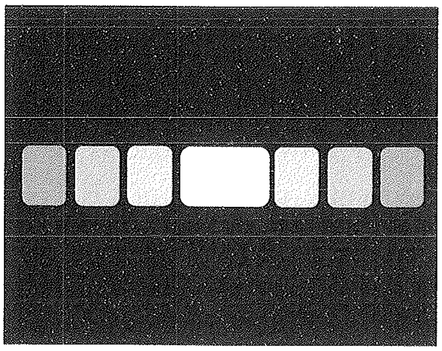
$\theta_2 = 2.23\frac{\lambda}{d}$ (2nd D.F)

$\theta_3 = 3.24\frac{\lambda}{d}$ (3rd D.F)

Angular width of the center bright fringe

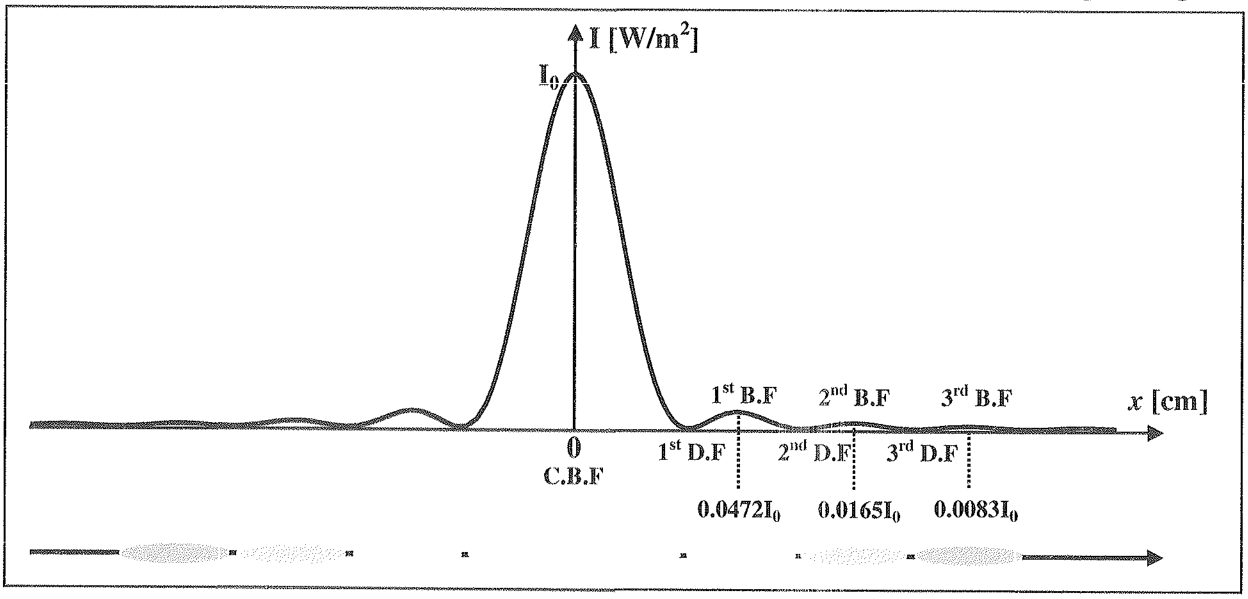
$\alpha = 2\theta_1 \Rightarrow \alpha = 2\left(1.22\frac{\lambda}{d}\right) = 2.44\frac{\lambda}{d}$

Other diffraction patterns

		
Diffraction by a small circular hole	**Diffraction by a rectangular aperture**	**Diffraction by a thin slit**

6.6- DISTRIBUTION OF LUMINOUS INTENSITY (CASE OF A SLIT)

The light intensity is maximum, of value I_0, at the center of the central bright fringe. It decreases as one moves from the center in both direction, till vanishing completely at the centers of the first dark fringes corresponding to $n = \pm 1$. This intensity increases again if we move further till we reach a new maximum I_1 at the centers of the 1st bright fringe. I_1 is much smaller I_0 ($I_1 = 0.0472 I_0 \Rightarrow I_1 \simeq 5\% I_0$).
Then it decreases again to zero at the center of the second dark fringes corresponding to $n = \pm 2$ and so on.
Note that the centers of the bright fringes, where light intensity is maximum, are approximately half way between the dark fringes, and that the central bright fringe is twice as wide as the other bright fringes.



CHAPTER 6 – WAVE ASPECT OF LIGHT (DIFFRACTION) EXERCISES AND PROBLEMS

Notes:
- The angles of diffraction of the fringes in the following exercises are small unless proved or given otherwise.
- Central bright fringe: C.B.F; bright fringe: B.F; dark fringe: D.F.

Exercise 1*:

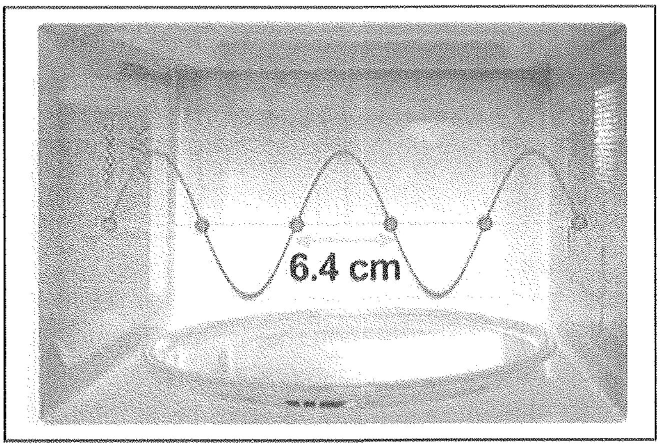
1- Define diffraction of light. Give two applications.
2- Describe the diffraction pattern.
3- Interpret the formation of bright and dark fringes.
4- Sketch the graph of the intensity of the diffracted light as a function of the angle of diffraction. Explain briefly.
5- The width of the slit producing diffraction is decreased. Specify the modifications, if any, that occur to the diffraction pattern.
6- Naya and Lamar are on opposite sides of a building. They can communicate with walkie-talkies but not with flashlights. Explain.
7- A ray of light cannot be isolated. Comment on this statement.

Exercise 2:

Helium-neon laser light of $\lambda = 632.8nm$ is sent through a vertical slit of 0.3mm width. The diffraction pattern is observed on a screen that is at $D = 1m$ from the slit.

1- The laser light is coherent and monochromatic. Explain.
2- Describe the diffraction pattern.
3- Calculate the width of the C.B.F.
4- Deduce the angular width of the C.B.F.
5- Determine the position of the center of the second D.F relative to the C.B.F.
6- The position of the screen is varied to D_1. The center of the second D.F is at 10 mm from the center of the C.B.F. Calculate D_1.

Exercise 3*:



We use radio and microwaves to communicate without wires. This is great, as we can move around and live our lives while still being in touch. Radio waves are good at broadcasting and that is how we get to listen to radio and TV broadcasts.
Radio waves are good at bending around buildings and hills by ________________.
Microwaves heat food like the sun heats your face by radiation and they propagate at the speed of light (300,000km/s).

1- Is the microwave mechanical or electromagnetic? Justify
2- In the text, complete the missing word.
3- Microwave ovens use waves of wavelength in the order of centimeters that make water molecules vibrate fast and heat up.

3.1- Using the figure, calculate the wavelength λof the microwave.
3.2- Deduce the frequency and the period of vibrations inside the microwave oven.

4- Coherent microwaves of wavelength 5 cm falls normally on a long, narrow window in a building of width 36cm.
4.1- Explain the meaning of coherent.
4.2- Calculate the diffraction angles of the centers of the first dark fringes.
4.3- Calculate the distance between the C.B.F and the center of the first D.F along a wall 6.5m away from the window.

Exercise 4:

A 690nm light is incident normally on a vertical slit of width a. The distance between the slit and the screen is 50cm. The distance between the centers of the first and third dark fringes on the same side of the C.B.F in the diffraction pattern is 3 mm.

1- Sketch a diagram that shows the diffraction pattern.
2- Determine a.
3- The 690nm light is replaced by another of higher wavelength. Specify the modifications that occur to diffraction pattern.

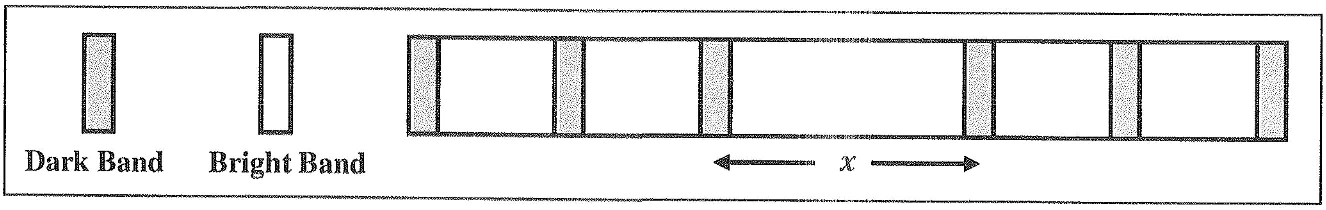
Exercise 5:

A beam of green light is diffracted by a slit of width 0.55mm. The diffraction pattern forms on a screen that is 2 m beyond the slit. The distance between the positions of the centers of the fringes of zero intensity on both sides of the C.B.F is 4.1mm.

1- Define diffraction. Use Huygens's principle to interpret this phenomenon.
2- Calculate the wavelength of the green light.
3- Indicate, with justification, the value of the width of the C.B.F; then, deduce its the angular width.
4- Calculate the distance between the centers of the second dark fringe and the third bright fringe on opposite sides of the C.B.F.
5- Determine the angular position of the center of the fourth bright fringe.
6- The green light is replaced by a violet light. In which case is the width of the C.B.F wider? Justify.

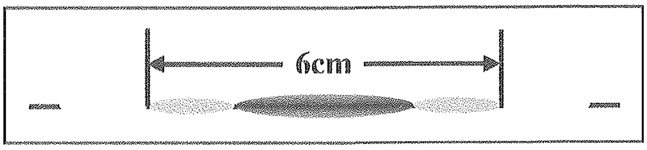
Exercise 6*:

Students conduct an experiment with monochromatic light of wavelength λ to understand the wave aspect of light. The light falls normally on a slit of width a = 0.025mm and the observed pattern on a screen is shown below. The distance between the screen and the slit is D = 2m. The center of the second dark fringe, observed on the screen, is at 4° from the centre of the slit.



1- Name the phenomenon that is investigated in the experiment.
2- Label the fringes and give their order.
3- Calculate λ. Specify if the used radiation has a green or red color.
4- Determine x. What does it represent?
5- Calculate the angular width of the C.B.F.
6- The students want to reduce x under the same incident light. Suggest two factors that can be modified in the experiment.

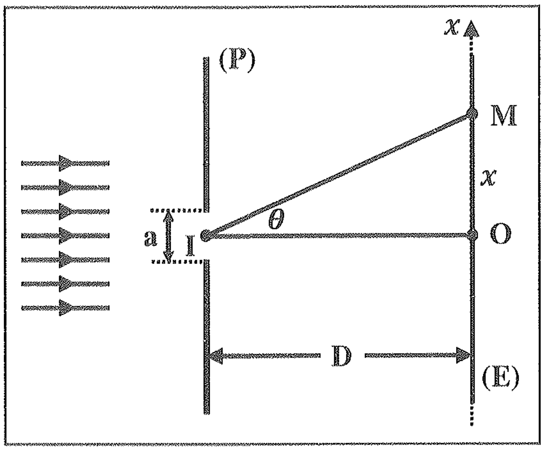
Exercise 7*:



A 589nm light is incident normally on a vertical slit of width a. The diffraction pattern, observed on a screen 2m beyond the slit, is given in the adjacent document.

1- Label the fringes and give their order.
2- Using the figure, calculate the width of the central bright fringe L.
3- Calculate a.
4- The diffraction phenomenon observed show evidence of an aspect of light. Give the name of this aspect.

Exercise 8:

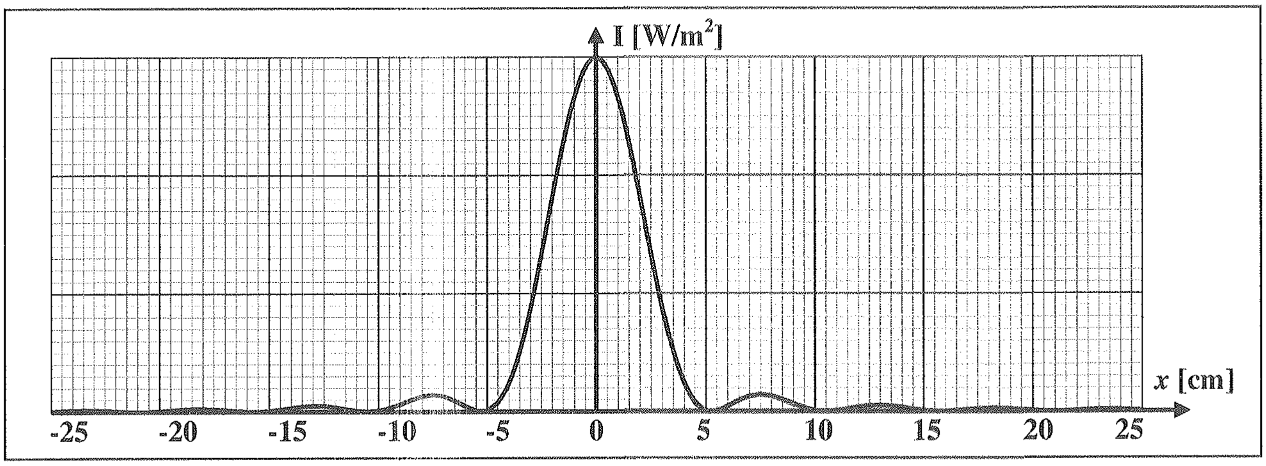


A laser source emits a monochromatic cylindrical beam of light of wavelength $\lambda = 640$nm in air. This beam falls normally on a vertical screen (P) having a horizontal slit F of width a. The phenomenon of diffraction is observed on a screen (E) parallel to (P) and situated at a distance D = 4m from (P). Consider on (E) a point M so that M coincides with the second dark fringe counted from O, the center of the central bright fringe. OIM = θ (θ is very small) is the angle of diffraction corresponding to the second dark fringe.

1- Write the expression of θ in terms of a and λ.
2- Determine the expression of OM = x in terms of a, D and λ.
3- Determine the value of a if OM = 1.28 cm.
4- We replace the slit F by another F_1 of width 100 times larger than that of F_1. What do we observe on the screen (E)?

Exercise 9:

A beam of green light of wavelength $\lambda_{air} = 555nm$ is diffracted by a slit of width 'a' and the diffraction pattern is formed on a white wall 15m away from the slit.
The graph of the intensity of the diffracted light as a function of the position x form the center of the C.B.F. is shown in the document below.

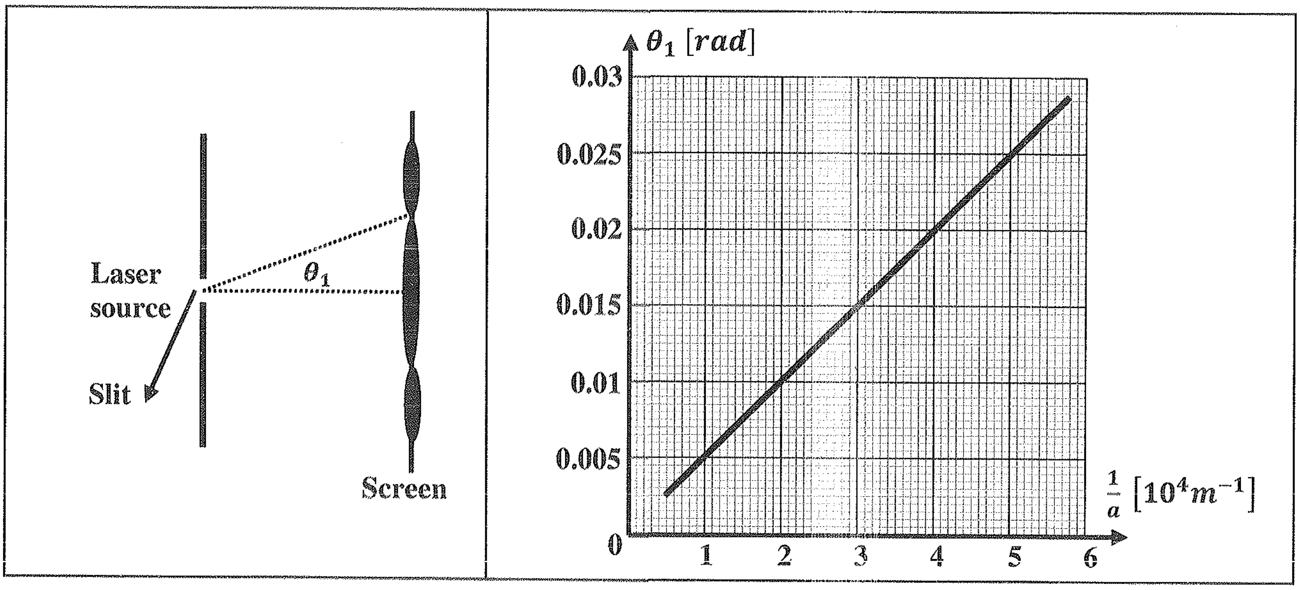


1- Interpret the given graph.
2- Label the order of the bright and dark fringes on the above graph. Give the positions of the centers of the third dark fringes and third bright fringes.
3- Determine a.

4- Determine the angular position of the center of the second bright fringe.
5- The experiment is repeated in water of index of refraction 1.33.
Calculate the new wavelength of the green light in water and the new positions of the dark fringes.

Exercise 10:

A laser source emits a monochromatic cylindrical beam of light of wavelength λ in air. This beam falls normally on slit of width a. For different slit widths, the angle of diffraction of the center of the first dark fringe on the positive side θ_1 relative to the center of the C.B.F was measured. The graph of θ_1 versus $\frac{1}{a}$ is given below. The distance between the screen and the slit is D = 2m.



1- Determine the wavelength of the laser.
2- Determine the width L of the C.B.F at different slit widths. Draw a conclusion.
3- Trace the graph of L versus $\frac{1}{a}$.

Exercise 11:

The discovery of diffraction is attributed to Francesco Grimaldi (1618-1663).

Part I: Measurement of the diameter of a hair

Based on inquiry based learning, the teacher conducted an experiment to measure the diameter of a hair using the phenomenon of diffraction. A laser beam of light, of wavelength in vacuum $\lambda = 632.8nm$, falls normally on a hair of width a. The diffraction pattern is observed on a screen placed perpendicularly to the laser beam at a distance D = 1.5m from the hair.
Let L be the linear width of the central fringe.
1- Describe the aspect of the diffraction pattern observed on the screen.
2- Establish the relation among a, λ, L and D.
3- Knowing that L = 18mm, calculate the width 'a' of the used hair.

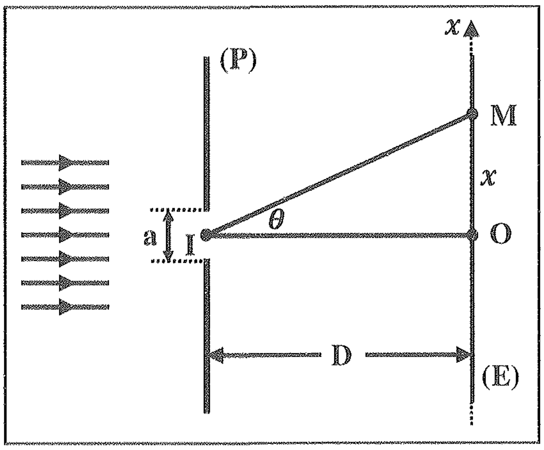
Part II: Controlling the thickness of thin wire

A manufacturer of thin wires wishes to control the diameter of his product. He uses the same set-up mentioned in part (A) but he replaces the hair by a thin vertical wire. He observes on the screen the phenomenon of diffraction
For D = 2.60m, he obtains a central fringe of constant linear width L_1 = 3.4mm.

1- Calculate the value of the diameter 'a_1' of the wire at the illuminated point.
2- The manufacturer illuminates the wire at different positions under the same precedent conditions. Specify the indicator that permits the manufacturer to check that the diameter of the wire is constant.

Part III: Measurement of the index of water
We place the whole set-up of part I in water of index of refraction n_{water} . We obtain a new diffraction pattern.
We find that for D = 1.5m and a = 0.3mm, the linear width of the central fringe is L_2 = 4.7mm.
1- Calculate the wavelength λ' of the laser light in water.
2-
 2.1- Determine the relation among λ, λ' and n_{water} .
 2.2- Deduce the value of n_{water} .

Exercise 12:
A source of white light emits a polychromatic beam of light in air. This beam falls normally on a vertical screen (P) having a horizontal slit of width a. The phenomenon of diffraction is observed on a screen (E) parallel to (P) and situated at a distance D = 2m from (P). Consider on (E) a point M such that OM = 8cm. O is the center of the C.B.F.



1- Explain the term polychromatic.
2- How many radiations does white light consist of?
3- Specify the color of the radiation that shows the widest C.B.F on (E).
4- Specify the color that is observed on the screen at O.
5- Determine the wavelengths of the missing radiations at M.

Exercise 13:
The aim of the exercise is to study a practical application of diffraction: the determination of the average size of cocoa powder by granulometry. Granulometry is the measurement of the size distribution in a collection of grains.

Introduction
The consumer success of chocolate is due to its recognized appeal to the taste buds, but also to the particle size of each of the ingredients. This is an important factor in the manufacturing process as particles that have been ground too finely will make the chocolate sticky whereas particles that are too large will make it too coarse to the eye and to the palate. It is therefore essential to have total control over the particle size of cocoa powders to provide consumers with a product that is pleasing to the eye and to the taste buds.

The various stages in the manufacture of chocolate
Manufacturing chocolate entails several essential steps starting with grinding the cocoa beans considered spherical to obtain a paste called cocoa liquor. Adding pre-ground sugar, flavorings or additives will result in pure cocoa powder or a dry mixture. Finally, a granulation phase obtains a mixture used for the production of powdered drinks. At every stage in the manufacturing process, the size of the incorporated powders is controlled in order to obtain a final mixture with the required particle size.
Measuring particle size by laser diffraction is a quick and simple technique that is suitable for determining the particle size distribution of all types of chocolate dispersed in dry mode after the dispersion protocol

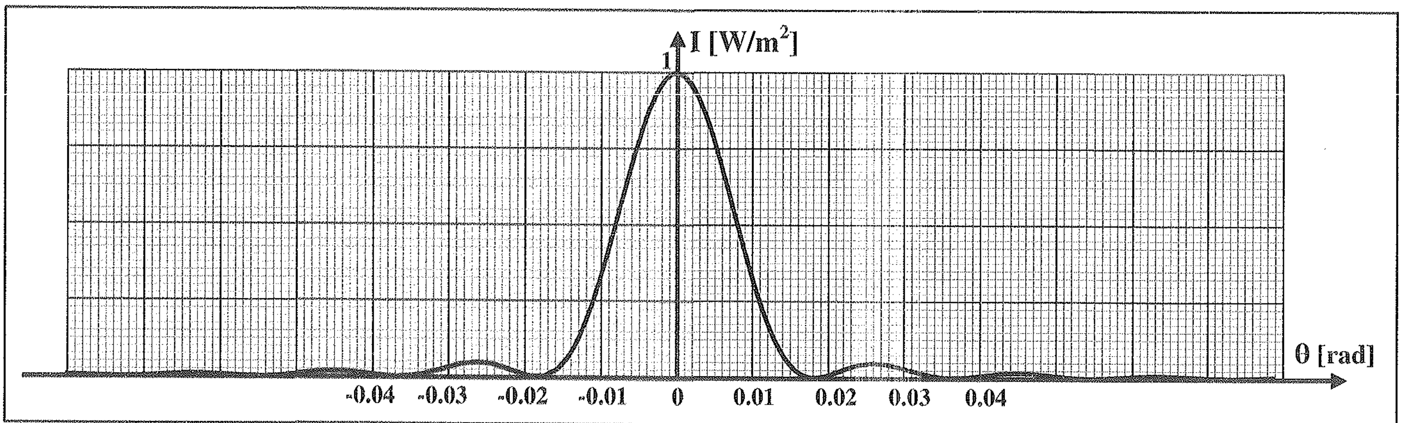
optimization. The measurements are taken in air, with an air pressure that is suitable for the friability of powders, or in liquid with a carried liquid based on vegetable oil. Particle size measurement is possible with various chocolates such as couverture chocolate (average diameter a = 10μm), milk chocolate (average diameter a = 30 μm), or agglomerated chocolate (average diameter a = 50μm) used for powdered drinks. Light of wavelength λ from a laser is shone on a cloud of particles, which are suspended in a dispersant (gas or liquid).The particles scatter the light, the larger the particles the smaller the scattering angles. The scattered light is measured by a series of photo-detectors placed at different angles and the diffraction pattern for the sample is observed on a screen placed at a distance D = 2m away from the particles.

1- List the properties of a laser source.
2- Derive the expression of the width L of a C.B.F as a function of the average diameter "a".
3- To measure the wavelength of the laser source, the width L of the C.B.F is measured for different single slit widths "a". The obtained values are tabulated below.

L [cm]	2	2.5	6	12.5
1/a [m^{-1}]	0.8×10^4	1×10^4	2.5×10^4	5×10^4

3.1- Trace the graph of L versus 1/a.
3.2- Determine the wavelength λ of the laser source.

4- The diffraction pattern obtained by a cocoa grains consists of alternately bright and dark concentric circles with $\sin\theta_1 = \frac{1.22\lambda}{a}$ where θ_1 is the angle of diffraction of the center of the first dark fringe on the positive side.
The graph of the intensity of the diffracted light as a function of θ (radian) is shown below.



I suggest that the examined sample is for couverture chocolate. Am I right?

Exercise 14*:

A white light source illuminates a slit of width a = 0.5mm. Diffraction pattern is observed on a screen located 2.6m from the slit. Upon passing through three different filters, only certain radiations of the white light are transmitted of corresponding wavelengths in vacuum: λ_1 = 541nm for filter 1, λ_2 = 433nm for filter 2 and λ_3 = 616nm for the filter 3. The C.B.F of each radiation has the same center O.

1- Give the color of these radiations.
2- Calculate the width of the C.B.F. for each radiation. Represent the C.B.F.'s on the same figure.

Exercise 15:

Part I: Measurement of the width of a slit

A laser beam of light, of wavelength in vacuum $\lambda = 632.8$ nm, falls normally on a vertical slit of width «a». The diffraction pattern is observed on a screen placed perpendicularly to the laser beam at a distance D = 1.5 m from the slit.

Let «L» be the linear width of the central fringe (Fig. 1).

The angle of diffraction θ corresponding to a dark fringe of order n is given by $\sin\theta = n\frac{\lambda}{a}$ where n = ±1, ±2, ±3…

For small angles, take $\tan\theta \approx \sin\theta \approx \theta$ in radian.

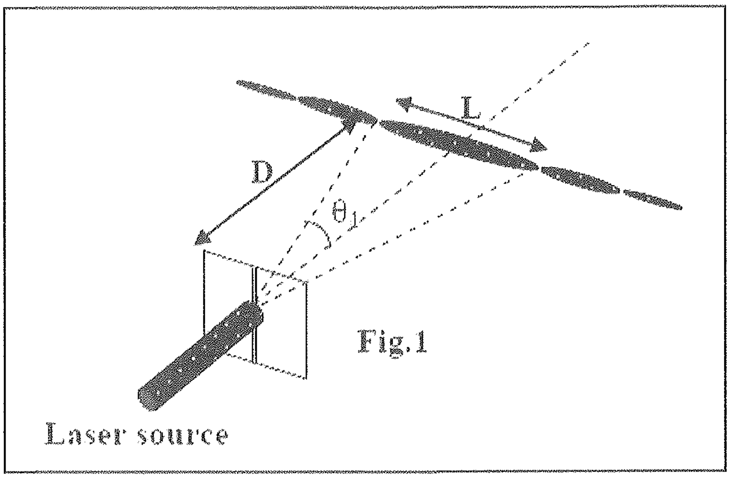


Fig.1

1- Describe the aspect of the diffraction pattern observed on the screen.
2- Write the relation among a, θ_1 and λ.
3- Establish the relation among a, λ, L and D.
4- Knowing that L = 6.3 mm, calculate the width «a» of the used slit.

Part II: Controlling the thickness of thin wire

A manufacturer of thin wires wishes to control the diameter of his product. He uses the same set-up mentioned in part (A) but he replaces the slit by a thin vertical wire. He observes on the screen the phenomenon of diffraction (figure 2).

For D = 2.60 m, he obtains a central fringe of constant linear width L_1 = 3.4 mm.

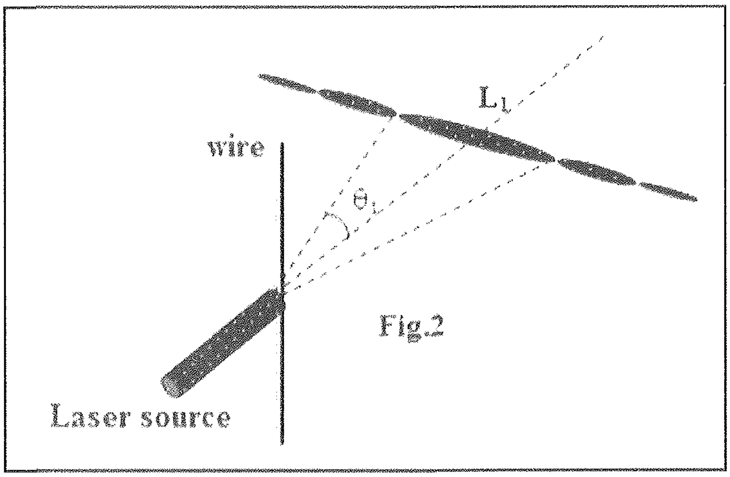


Fig.2

1- Calculate the value of the diameter «a_1» of the wire at the illuminated point.
2- The manufacturer illuminates the wire at different positions under the same precedent conditions. Specify the indicator that permits the manufacturer to check that the diameter of the wire is constant.

Part III: Measurement of the index of water

We place the whole set-up of part (A) in water of index of refraction n_{water}. We obtain a new diffraction pattern.

We find that for D = 1.5 m and a = 0.3 mm, the linear width of the central fringe is L_2 = 4.7 mm.

1- Calculate the wavelength λ' of the laser light in water.
2-
  2.1- Determine the relation among λ, λ' and n_{water}.
  2.2- Deduce the value of n_{water}.

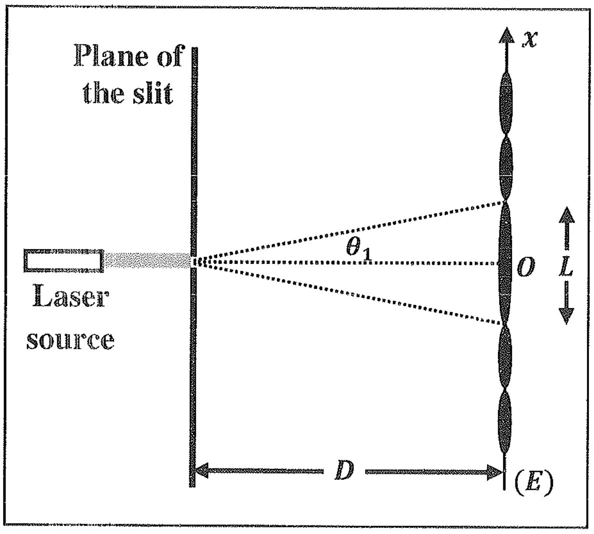
Exercise 16*:

A laser beam of light, of wavelength in vacuum $\lambda = 650nm$, falls normally on a horizontal slit of width a. The diffraction pattern is observed on a screen (E) placed perpendicularly to the laser beam at a distance $D = 2m$ from the slit.

Let L be the linear width of the central fringe.

The angle of diffraction θ_k corresponding to a dark fringe of order k is given by $\sin\theta_k = \frac{k\lambda}{a}$ where $k = \pm1, \pm2, \pm3$ ...

For small angles, take $\tan\theta \simeq \sin\theta \simeq \theta$ in radian.



1- Describe the aspect of the diffraction pattern observed on the screen (E).
2- Establish the relation among a, λ, L and D.
3- Derive the expression of the position x_k of the dark fringe of order k in terms of k, λ, D, and a.
4- Determine a knowing that the distance between the 1st and the 4th dark fringes on the same side with respect to the C.B.F is $d = 19.5mm$.
5- Deduce the value of L.
6- We place the whole set-up in water of index of refraction n_w. We obtain a new diffraction pattern. We find that for $D = 2m$ and $a = 0.2mm$, the linear width of the central fringe is $L' = 9.77mm$.
   6.1- Calculate the wavelength λ' of the laser light in water.
   6.2- Determine the relation among λ, λ' and n_w.
   6.3- Deduce the value of n_w.

Exercise 17*:

A laser light of wavelength λ is sent through a vertical slit of 12.5μm width. The diffraction pattern is observed on a screen that is at D = 1.52m from the slit. The distance between the centers of the C.B.F and the first dark fringe is 65mm. Calculate λ.

Exercise 18*:

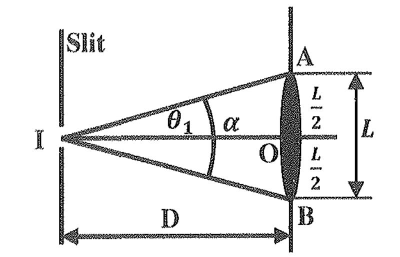
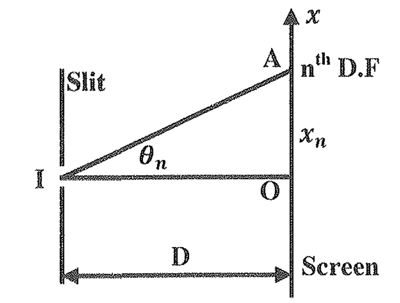
A 511nm light is incident normally on a vertical slit of width a. The distance between the slit and the screen is 1.76m. The angular width of the central bright fringe is 5.3°. Calculate the distance between the centers of the central bright fringe and the second dark fringe.

Exercise 19*:

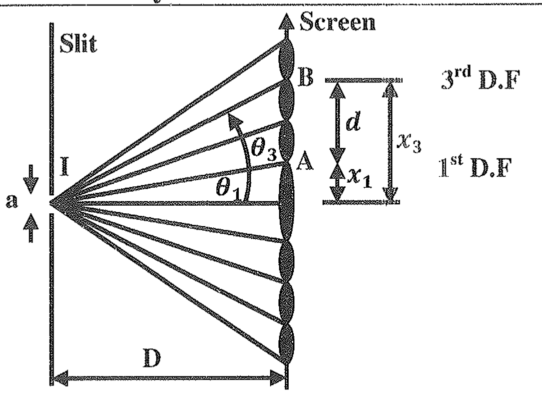
A 631nm light is incident normally on a vertical slit of width a = 2.94μm. The distance between the slit and the screen is 3m. Calculate the maximum number of dark fringes on the positive side of the central bright fringe.

CHAPTER 6 – WAVE ASPECT OF LIGHT (DIFFRACTION)
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 2:

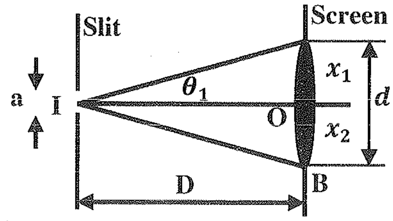
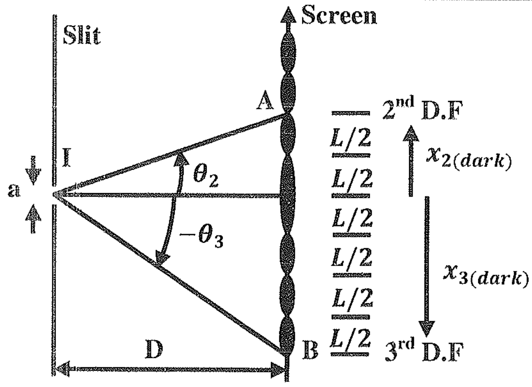
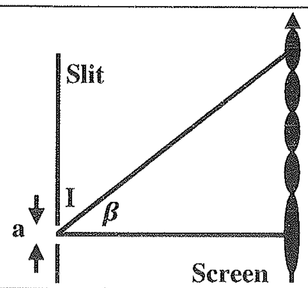
Part	Answer key
1	Coherent means that the phase difference between the emitted waves is constant. A monochromatic light consists of single radiation of specific color, wavelength and frequency.
2	- Alternating bright and dark fringes on both sides of the central bright fringe. - The width of the central bright fringe is double that of any other bright fringe. - The direction of the pattern of fringes is perpendicular to that of the slit. Central bright fringe has maximum intensity.
3	$\sin\theta_n = n\frac{\lambda}{a}$. For $n = 1$; $\sin\theta_1 = \frac{\lambda}{a}$. Using triangle IOA: $\tan\theta_1 = \frac{\frac{L}{2}}{D} \Longrightarrow \tan\theta_1 = \frac{L}{2D}$. For small angles: $\tan\theta \simeq \theta_{(rad)}$ and $\sin\theta \simeq \theta_{(rad)}$. $\theta_1 = \frac{L}{2D}$ and $\theta_1 = \frac{\lambda}{a} \Longrightarrow \frac{L}{2D} = \frac{\lambda}{a}$. $L = \frac{2\lambda D}{a} \Longrightarrow L = \frac{2\times632.8\times10^{-9}\times1}{0.3\times10^{-3}} = 4.218\times10^{-3}m = 4.218\ mm.$ 
4	The angular width is $\alpha = 2\theta_1 = \frac{2\lambda}{a} = \frac{2\times632.8\times10^{-9}}{0.3\times10^{-3}} = 4.218\ rad.$
5	LIn the right OIA: $\tan\theta_n = \frac{X_n}{D}$ and $\sin\theta_n = \frac{n\lambda}{a}$. For small angles: $\tan\theta \simeq \theta_{(rad)}$ and $\sin\theta \simeq \theta_{(rad)}$. $\frac{x_n}{D} = \frac{n\lambda}{a} \Longrightarrow x_n = \frac{n\lambda D}{a}$ (the position of the nth dark fringe relative to the C.B.F). For n = 2, we have: $x_2 = \frac{2\lambda D}{a} = \frac{2\times632.8\times10^{-9}\times1}{0.3\times10^{-3}} = 4.218\ mm.$ 
6	$x_2' = \frac{2\lambda D_1}{a} \Rightarrow D_1 = \frac{ax_2'}{2\lambda} = \frac{0.3\times10^{-3}\times10\times10^{-3}}{2\times632.8\times10^{-9}} = 2.37m.$

Exercise 4:

Part	Answer key		
1		2	$x_1 = \frac{\lambda D}{a}$ and $x_3 = \frac{3\lambda D}{a}$ and $d = x_3 - x_1 = AB$. $AB = \frac{3\lambda D}{a} - \frac{\lambda D}{a} = \frac{2\lambda D}{a}$. $a = \frac{2\lambda D}{AB} = \frac{2\times690\times10^{-9}\times0.5}{3\times10^{-3}} = 2.3\times10^{-4}m.$ $a = 0.23\ mm.$
		3	$\lambda' > \lambda \Longrightarrow A'B' = \frac{2\lambda' D}{AB} > AB.$ Then, the fringes become wider.

Exercise 5:

Part	Answer key
1	The diffraction of light is the deviation of light from its path, without being reflected or refracted when it falls on a small aperture (less than 1 mm), a sharp edge, or a sharp obstacle. Interpretation of the diffraction phenomenon using Huygens's principle:

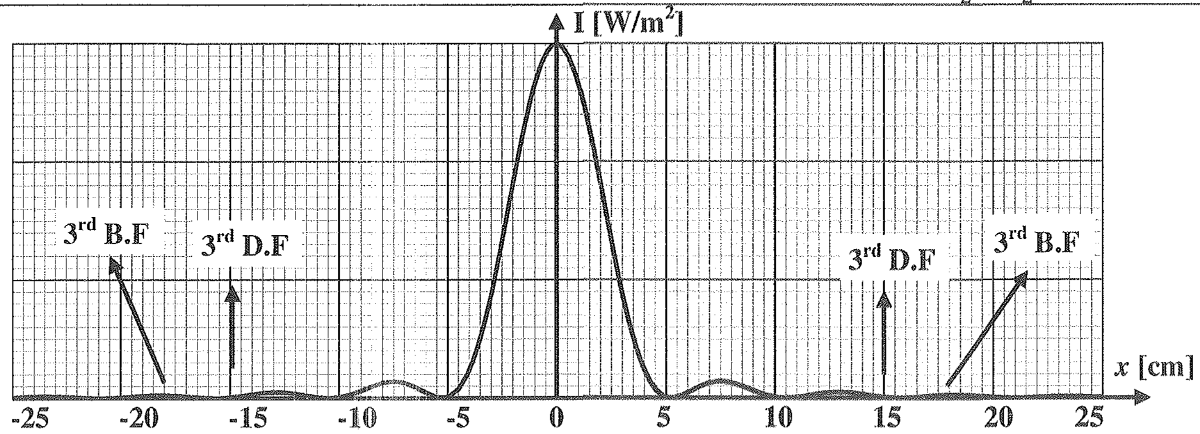
	- Each point of the slit or the aperture acts as a secondary source of wavelets. - Diffraction fringes are formed due to the super position of the wavelets.
2	$d = x_1 + \lvert -x_1 \rvert = 2x_1$ $d = 2\frac{\lambda D}{a} \Longrightarrow \lambda = \frac{a\times d}{2D}$ $\lambda = \frac{0.55\times10^{-3}\times4.1\times10^{-3}}{2\times2}$ $\lambda = 5.6375\times10^{-7}m = 563.75\ nm.$ 
3	$\theta_1 = \frac{\lambda}{a}$ and $\theta_1 = \frac{L}{2D} \Longrightarrow \frac{L}{2D} = \frac{\lambda}{a} \Longrightarrow$ the width L of the C.B.F is: $L = \frac{2\lambda D}{a} = d = 4.1\ mm.$ The angular with $\alpha = \frac{L}{D} = \frac{4.1\times10^{-3}}{2} \Longrightarrow \alpha = 2.05rad.$
4	$sin\theta_2 = \frac{2\lambda}{a}$; $\tan\theta_2 = \frac{x_2(dark)}{D}$. $\theta_2 = \frac{2\lambda}{a}$; $\theta_2 = \frac{x_2}{D}\ (dark).$ Then $\frac{x_2}{D} = \frac{2\lambda}{a} \Longrightarrow x_{2(dark)} = \frac{2\lambda}{a} = L.$ $\sin\theta_3 = \frac{3\lambda}{a}$; $\tan\theta_3 = \frac{x_{3(dark)}}{D}$. $\theta_3 = \frac{3\lambda}{a}$; $\theta_3 = \frac{x_{3(dark)}}{D}$. Then $\frac{x_3}{D} = \frac{3\lambda}{a}$. $\Longrightarrow x_{3(daek)} = \frac{3\lambda D}{a} = \frac{2\lambda D}{a} + \frac{\lambda D}{a} = L + \frac{L}{2} = \frac{3L}{2}$.  L is the width of $C.B.F$ and since its twice of the other lateral fringes then the width of a B.F is $\frac{L}{2}$. So, the distance between two consecutive dark and bright center is $\frac{L}{4}$. Therefore $d = AB = x_{2_{dark}} + \lvert x_{3_{dark}} \rvert + \frac{L}{4} = L + \frac{3}{2}L + \frac{L}{4} = \frac{11}{4}L = \frac{11}{4}\times 4.1 = 11.275\ mm.$
5	The angular position of the center of the 4th brigth fringe is: $\beta = 4\frac{\lambda}{a} + \frac{\lambda}{2a} = \frac{9\lambda}{2a} = \frac{9\times563.75\times10^{-9}}{2\times0.55\times10^{-3}} = 4.6125\times10^{-3}\ rad.$ 
6	$L_{green} = \frac{2\lambda_{green}\times D}{a}$ and $L_{violet} = \frac{2\lambda_{violet}\times D}{a}$. $\frac{L_{violet}}{L_{green}} = \frac{\lambda_{violet}}{\lambda_{green}} < 1$ since $\lambda_{violet} < \lambda_{green}$ (400 nm < 563 nm). Then $L_{violet} < L_{green}$ therefore the C.B.F is wider in case of green light.

Exercise 8:

Part	Answer key
1	$\sin\theta = \frac{2\lambda}{a}$, for small angles then $\theta = \frac{2\lambda}{a}$.
2	In the right triangle OIM: $\tan\theta = \frac{x}{D} \Longrightarrow \theta = \frac{x}{D}$ but $\theta = \frac{2\lambda}{a}$. Then $\frac{x}{D} = \frac{2\lambda}{a} \Rightarrow x = \frac{2\lambda D}{a}$.
3	$OM = x = 1.28cm$; $x = \frac{2\lambda D}{a} \Rightarrow a = \frac{2\lambda D}{x}$.

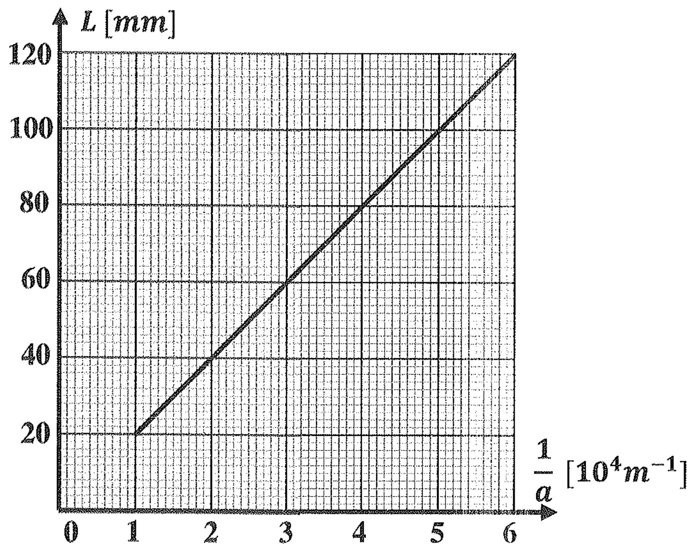
	$a = \frac{2\times640\times10^{-9}\times4}{1.28\times10^{-2}} \Rightarrow a = 4\times10^{-4}\,m = 0.4mm.$
4	$\frac{a_1}{a} = 100 \Longrightarrow a_1 = 4cm$ In this case light passes through a wide aperture of large dimensions. So, light obeys the principle of rectilinear propagation of light and we observe on the screen a luminous spot having the shape of the slit. The phenomenon of diffraction disappears.

Exercise 9:

Part	Answer key
1	The intensity I is maximum at the position $x = 0$. It decreases as one moves from $x = 0$ in both directions on the x-axis until it vanishes completely at positions $x = 5cm\ and\ x = -5\ cm$. The intensity increases again until it reaches a new maximum at $x = 7.5\ cm\ and\ x = -7.5\ cm$ and so on. The intensities at $x = \pm5\ cm$, $x = \pm10\ cm$; $x = \pm15\ cm$; $x = \pm20\ cm\ and$ $x = \pm25\ cm$ represent the center of the dark fringes and have zero intensities. The intensities at $x = \pm7.5$; $x = \pm12.5\ cm$; $x = \pm17.5\ and\ x = \pm22.5\ cm$ represent the center of the bright fringes and have maximum intensities. At $x = 0$, the intensity is the greatest (greater than all other maximum intensities), so it represents the position of the center of the C.B.F. The width of the C.B.F is twice as broad as others. Width of C.B.F is $L = 10\ cm$, while the width of the other bright fringes is $\frac{L}{2} = \frac{10}{2} = 5\ cm.$
2	 $x_{3_{dark}} = 15\ cm$ on positive side ; $x_{3_{(brig\ ht)}} = 17.5\ cm$ on positive side. $x_{3_{dark}} = -15\ cm$ on negative side $x_{3_{(brig\ ht)}} = -17.5\ cm$ on negative side.
3	$L = \frac{2\lambda D}{a} \Rightarrow a = \frac{2\lambda D}{L} = \frac{2\times555\times10^{-9}\times15}{10\times10^{-2}} \Rightarrow a = 1.665\times10^{-4}\,m = 0.1665mm.$
4	The angular position of the center of the 2nd B.F is: $\theta_{2_{(brig\ ht)}} = \frac{x_{2_{(birg\ ht)}}}{D} = \frac{12.5\times10^{-2}}{15} = 8.333\times10^{-3}rad.$ or $\theta_{2_{(brig\ ht)}} = \frac{2\lambda}{a} + \frac{\lambda}{2a} = \frac{5\lambda}{2a} = \frac{5\times55\times10^{-3}}{2\times1.665\times10^{-4}} = 8.333\times10^{-3}rad.$
5	$\lambda_{air} = \frac{c}{\nu}$ and $\lambda_{water} = \frac{v}{\nu}$ where ν is the frequency of green light in [Hz]. Then $\frac{\lambda_{air}}{\lambda_{water}} = \frac{c}{v} = n_{water} \Longrightarrow \lambda_{water} = \frac{\lambda_{air}}{n_{water}} = \frac{555\times10^{-9}}{4/3} = 4.1625\times10^{-7}m = 416.25nm.$ The position of the dark fringes can be deducted from the following formula: $x_n = \frac{n\lambda D}{a}$ where n is the rank of the D.F.

	For $n = 1$, $x_1 = \frac{\lambda_{water}\ D}{a} = \frac{4.1625 \times 10^{-7} \times 15}{1.665 \times 10^{-4}} = 0.0375\ m = 3.75cm$ (1st D.F) For $n = 2$, $x_2 = \frac{2\lambda_{water}\ D}{a} 2 \times 0.0375 = 0.075\ m = 7.5\ cm$ (2nd D.F) For $n = 3$, $x_3 = \frac{3\lambda_{water}\ D}{a} = 3 \times 0.0375 = 0.1125\ m = 11.25cm$ (3rd D.F) For $n = 4$, $x_4 = \frac{4\lambda_{water}\ D}{a} = 4 \times 0.0375 = 0.15\ m = 15cm$ (4th D.F) For $n = 5 \Longrightarrow 5^{th}D.F \Longrightarrow x_5 = \frac{5\lambda_{water}\ D}{a} = 5 \times 0.0375 = 0.1875\ m = 18.75cm.$

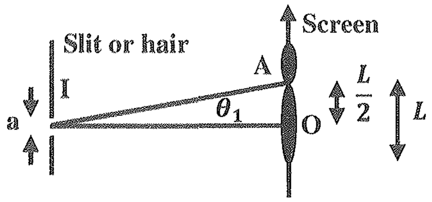
Exercise 10:

Part	Answer key
1	The graph of θ_1 versus $\frac{1}{a}$ is a straight line passing through the origin of equation $\theta_1 = k\frac{1}{a}$ where k is the slope. $\sin\theta_n = n\frac{\lambda}{a}$. For $n = 1, \sin\theta_1 = \frac{\lambda}{a} \Longrightarrow \theta_1 = \frac{\lambda}{a} \Longrightarrow \theta_1 = \lambda\frac{1}{a}$ with $\sin\theta_1 = \theta_1$ (small angle approximation). By comparing the two relations $\theta_1 = k\frac{1}{a}$ and $\theta_1 = \lambda\frac{1}{a}$, we find that $\lambda = k$. $\lambda = k = \frac{\Delta\theta_1}{\Delta\left(\frac{1}{a}\right)} = \frac{0.025-0.005}{5\times10^4-1\times10^4} = 5 \times 10^{-7}m = 500nm.$
2	$\tan\theta_1 = \frac{L}{2D} \Longrightarrow \theta_1 = \frac{L}{2D}$ with $\tan\theta_1 = \theta_1$ (small angle approximation). $\theta_1 = \frac{L}{2D} \Longrightarrow \frac{\lambda}{a} = \frac{L}{2D} \Longrightarrow L = \frac{2\lambda D}{a} = 2\lambda D\frac{1}{a}.$ (table below) Conclusion: L and $\frac{1}{a}$ are directly proportional.
3	

$\frac{1}{a}$ $[10^4 m]$	1	2	3	4	5	6
L [mm]	20	40	60	80	100	120

Exercise 11:

Part	Answer key
I.1	The diffraction pattern of a hair is similar to that of a thin slit, the diffraction pattern shows: - Alternate bright and dark spots or fringes on both sides of a central bright fringe. - The central fringe is bright and has a width double that of the other bright fringes. - The fringes are aligned in a direction perpendicular to that of the slit.

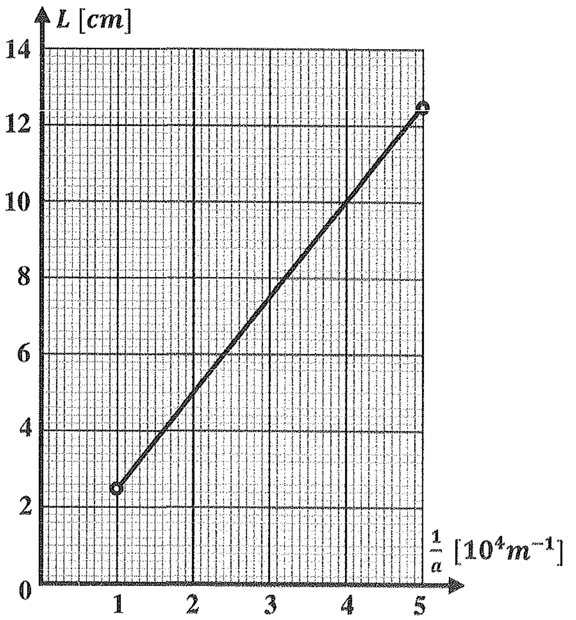
I.2	In the right triangle OIA: $\sin\theta_1 = \frac{\lambda}{a}$ *and* $\tan\theta_1 = \frac{L}{2D}$. For small angles $\sin\theta_1 \simeq \theta_{1(rad)}$ and $\tan\theta_1 \simeq \theta_{1(rad)}$. Then $\frac{\lambda}{a} = \frac{L}{2D} \Rightarrow L = \frac{2\lambda D}{a}$. 
I.3	$a = \frac{2\lambda D}{L}$, for $L = 18mm \Rightarrow a = \frac{2\times632.8\times10^{-9}\times1.5}{18\times10^{-3}} = 1.054666\times10^{-4}m = 0.10546mm.$
II.1	$a_1 = \frac{2\lambda D}{L_1} = \frac{2\times632.8\times10^{-9}\times2.6}{3.4\times\times10^{-3}} = 9.6781\times10^{-4}m = 0.96781mm.$
II.2	We have the same laser beam then same wavelength λ, also D is constant then the indicator is the linear width of C.B.F (L) since if L is constant then (a) is constant.
III.1	We apply the same relation obtained in part (I-2). So we can write: $\frac{\lambda'}{a} = \frac{L_2}{2D} \Rightarrow \lambda' = \frac{a\times L_2}{2D} = \frac{0.3\times10^{-3}\times4.7\times10^{-3}}{2\times1.5} = 4.7\times10^{-7}m \Rightarrow \lambda' = 470\ nm.$
III.2.1	In the air, we have $\lambda = \frac{c}{\nu}$ where ν is the frequency of light in [Hz]. In water, we have $\lambda' = \frac{V}{\nu}$ where V is the speed of light in water in m/s. Then $\frac{\lambda}{\lambda'} = \frac{c/_\nu}{V/_\nu} = \frac{c}{V} = n_{water} \Rightarrow \lambda' = \frac{\lambda}{n_{water}}$.
III.2.2	$n_{water} = \frac{\lambda}{\lambda'} = \frac{632.8}{470} = 1.34638.$

Exercise 12:

Part	Answer key
1	The term polychromatic means: showing a variety or a change of colors or relating to radiation that is composed of more than one wavelength.
2	White light is a polychromatic light composed of a mixture of all the wavelength of the visible spectrum (visible monochromatic radiations). It composed of 7 colors or radiations such each radiation has its own wavelength. The wavelength of visible light in vacuum range between 400 nm and 800 nm.
3	We know that $\sin\theta_1 = \frac{\lambda}{a}$ and $\tan\theta_1 = \frac{L}{2D}$. For small angles ssin $\theta_1 \simeq \theta_{1(rad)}$ *and* $\tan\theta_1 \simeq \theta_{1(rad)}$. $\theta_1 = \frac{\lambda}{a}$ *and* $\theta_1 = \frac{L}{2D} \Longrightarrow \frac{L}{2D} = \frac{\lambda}{a} \Longrightarrow L = \frac{2\lambda D}{a}$. Where L is the linear width of C.B.F. So as λ increases then L increases. Therefore, the radiation of the longest wavelength has the widest C.B.F. Consequently, the radiation of wavelength 800 nm corresponds to the broadest C.B.F. This radiation is red.
4	Each monochromatic light undergoes diffraction when it crosses the slit, and the center of the C.B.F of each of these lights is at 0. Constructive super position of all lights takes place at 0, and then the color at 0 is white.
5	$\sin\theta_n = \frac{n\lambda}{a}$ and $\tan\theta_n = \frac{x}{D}$ then for smaller angles we have: $\sin\theta_n \simeq \tan\theta_n \simeq \theta_{n(rad)} \Longrightarrow \frac{n\lambda}{a} = \frac{x}{D} \Longrightarrow n = \frac{ax}{\lambda D}$. For $\lambda = 400\ nm \Rightarrow n = \frac{0.05\times10^{-3}\times8\times10^{-2}}{400\times10^{-9}\times2} = 5.$ For $\lambda = 800\ nm \Rightarrow n = \frac{0.05\times10^{-3}\times8\times10^{-2}}{800\times10^{-9}\times2} = 2.5$ $\Longrightarrow 2.5 < n \leq 5 \Longrightarrow n = 3, 4$ and 5.

	For $n = 3$ and $\frac{ax}{D} = \frac{0.05\times10^{-3}\times0.08}{2} = 2\times10^{-6}nm.$ $n\lambda = 2\times10^{-6} \Rightarrow \lambda = \frac{2\times10^{-6}}{3} = 666.667nm.$ Then the 3rd dark fringe of the pattern whose wavelength $\lambda = 666.667nm$ is missed. For $n = 4 \Rightarrow \lambda = \frac{2\times10^{-6}}{4} = 0.5\times10^{-6}m = 500nm$, then the 4th dark fringe of the pattern whose wavelength $\lambda = 500nm$ is missed. For $n = 5 \Rightarrow \lambda = \frac{2\times10^{-6}}{5} = 0.4\times10^{-6}m = 400nm$, then the 5th dark fringe of the pattern whose wavelength $\lambda = 400nm$ is missed.

Exercise 13:

Part	Answer key
1	The properties of a laser source are: - Highly coherent light. - Highly directional nature. - Nearly monochromatic light. - A power which ranges from 1mW up to 10^{14}W.
2	$\sin\theta_1 = \frac{\lambda}{a}$ and $\tan\theta_1 = \frac{L}{2D}$. For small angles we have: $\sin\theta_1 \simeq \theta_{1(rad)}$ and $\tan\theta_1 \simeq \theta_{1(rad)} \Rightarrow \theta_1 = \frac{\lambda}{a} = \frac{L}{2D}$. Then $\frac{\lambda}{a} = \frac{L}{2D} \Rightarrow L = \frac{2\lambda D}{a}$.
3.1	
3.2	The obtained graph is a straight line passing thriught the origin; then, L is directly propotional to $\frac{1}{a}$,thus $L = k\frac{1}{a}$ where k is the slope of the straight line . $k = \frac{\Delta l}{\Delta\frac{1}{a}} = \frac{(12.5-2)\times10^{-2}}{(5-0.8\times10^4} = 25\times10^{-7}m^2$. But $L = 2\lambda D \times \frac{1}{a}$ then by comparison we obtain: $2\lambda D = k \Rightarrow \lambda = \frac{k}{2D} = \frac{25\times10^{-7}}{2\times2}$ $\lambda = 625\times10^{-9}m \Rightarrow \lambda = 625nm$
4	Graphically, the angle of diffraction of the 1st D.F is $\theta_1 = 0.018rad$. $\theta_1 = \frac{1.22\lambda}{a} \Rightarrow a = \frac{1.22\lambda}{\theta_1} = \frac{1.22\times625\times10^{-9}}{0.018}$. $a = 42\times10^{-6}m = 42\mu m$. $a = 42\mu m > 10\mu m$. Then the examined sample is not for couverture chocolate.

Exercise 15:

Part	Answer key
I.1	Alternating bright and dark fringes. The direction of the pattern of fringes is double that of any other bright fringe. The direction of the pattern of fringes is perpendicular to that of the slit.
I.2	$\sin\theta_1 \approx \theta_1 \Rightarrow \theta_1 = \frac{\lambda}{a}$ with $n = 1$.

I.3	$\tan\theta_1 \approx \theta_1 = \frac{L}{2D} \Longrightarrow \frac{\lambda}{a} = \frac{L}{2D}$.
I.4	$a = \frac{2\lambda D}{L} = \frac{2\times632.8\times10^{-9}\times1.5}{6.3\times10^{-3}} = 0.3mm.$
II.1	$a_1 = \frac{2\lambda D}{L_1} = \frac{2\times632.8\times10^{-9}\times2.6}{3.4\times10^{-3}} = 0.967mm.$
II.2	The linear width of the central fringe. Because if L = constant $\Longrightarrow$ a = constant.
III.1	$\lambda' = \frac{aL}{2D} = \frac{0.3\times10^{-3}\times4.7\times10^{-3}}{2\times1.5} = 470\times10^{-9}m = 470nm$
III.2.1	$f = \frac{V_{water}}{\lambda_{water}} \Longrightarrow f = \frac{C}{n_{water}\,\lambda'}$ and $f = \frac{C}{\lambda}$. Then, $\frac{C}{n_{water}\,\lambda'} = \frac{C}{\lambda} \Longrightarrow n_{water}\,\lambda' = \lambda \Longrightarrow \lambda' = \frac{\lambda}{n_{water}}$.
III.2.2	$n_{water} = \frac{\lambda}{\lambda'} = \frac{632.8}{470} = 1.346.$

CHAPTER 7 – CORPUSCULAR ASPECT OF LIGHT (PHOTOELECTRIC EFFECT) COURSE

7.1- INTRODUCTION

As early as 1873, it was known that if certain metals were heated, electrons would be ejected off the metals once the metals achieved a specific temperature. This was called thermionic emission.

In 1887, Heinrich Hertz discovered that certain metals emit electrons when light is incident on them. This was the first instance of light interacting with matter, so it was very mysterious. This phenomenon was called the photoelectric effect.

In 1905, Albert Einstein published the *annus mirabilis (miracle year)* papers. The first paper explained the photoelectric effect which was the only specific discovery mentioned in the citation awarding Einstein the Nobel Prize in Physics.

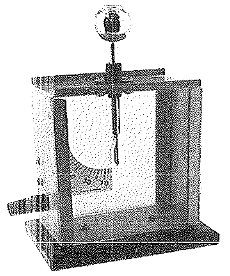
7.2- THE PHOTOELECTRIC EFFECT

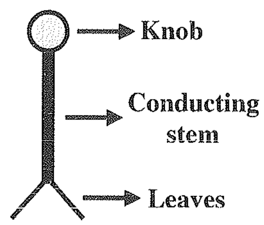
Photoelectric effect is the phenomenon of the emission of electrons from the surface of a body (generally a metal) when illuminated by suitable electromagnetic radiation. The liberated electrons are known as photoelectrons.

Electroscope

An electroscope is a device used to detect the presence of an electric charge.

A metal-leaf electroscope consists of two thin metallic leaves connected to a conducting knob by means of a conducting stem. The leaves are enclosed in a glass box and are generally made of gold or aluminum. If we touch the knob with a charged body, part of the charge of the body is distributed over the knob, the stem, and the leaves. The leaves acquire the same kind of charge as the body, repel each other, and separate.





Metal leaf electroscope	Basic Parts of an electroscope

Experimental evidence

A fundamental experiment set up by Hallwachs in 1888 verified the photoelectric effect experimentally.

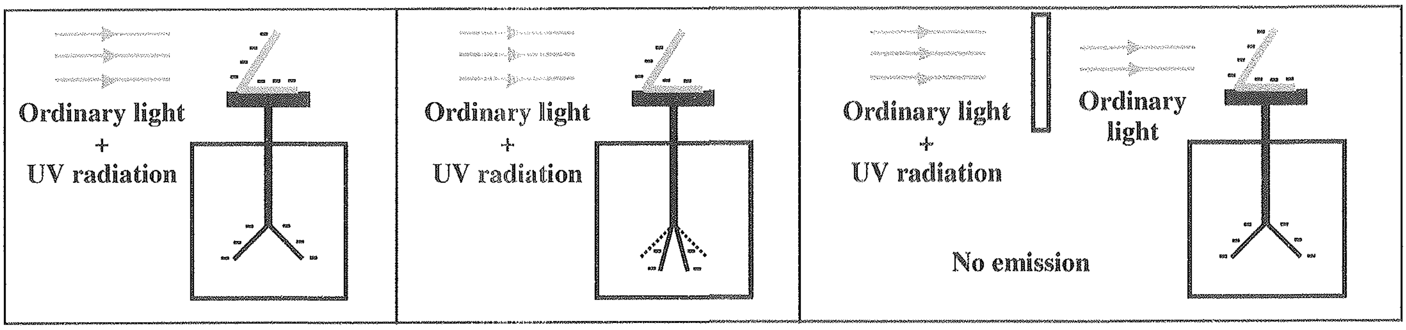
Equipment

- Electroscope.
- Source of ordinary light and ultraviolet radiation.
- A perfectly cleaned zinc plate.

Procedure and observation

The zinc plate is mounted on the knob of the electroscope and the system [plate; electroscope] is negatively charged where the leaves diverge.

The zinc plate is exposed to light rich in ultraviolet radiations. The leaves of the electroscope collapse which means that they lose negative charge, and this in turn means that the zinc plate emits electrons. If this experiment is repeated, after inserting a glass plate that does not allow ultraviolet radiations to pass and placed between the source of light and the zinc plate, the leaves of the electroscope are not affected indicating that the zinc plate does not emit electrons. We deduce that the emission of electrons by zinc is due to the ultraviolet radiations and not to ordinary light.



The experimental study of photoelectric effect leads to the following results

- For a pure metal, the photoelectric effect takes place only if the frequency ν of the incident radiation is larger than or equal to a certain limiting frequency ν_0, a characteristic of the metal, called threshold frequency of photoelectric emission of the considered metal $(\nu \geq \nu_0)$.

The threshold frequency of a pure metal is the minimum frequency of the incident electromagnetic radiation which is able to extract electrons from the surface of this metal

- To this threshold frequency ν_0 corresponds a wavelength in vacuum, called threshold wavelength λ_0 where $\lambda_0 = \frac{c}{\nu_0}$, and c is the speed of light in vacuum.

$\nu \geq \nu_0 \Rightarrow \frac{c}{\lambda} \geq \frac{c}{\lambda_0} \Rightarrow \lambda \leq \lambda_0$.

The threshold wavelength of a pure metal is the maximum wavelength of the incident electromagnetic radiation which is able to extract electrons from the surface of this metal.

- The photoelectric emission is quasi-instantaneous whatever the intensity of the incident radiation. The delay of emission is about 10^{-9}s.

The table below gives the threshold wavelength of some pure metals.

Metal	Cs	K	Ca	Zn	Cu	Ag	Pt
$\lambda_0\ in\ [\mu m]$	0.66	0.55	0.45	0.37	0.29	0.27	0.19

7.3- INTERPRETATION OF THE PHOTOELECTRIC EFFECT

Work function

In the photoelectric effect, the work function is the minimum amount of energy (per photon) needed to eject an electron from the surface of a metal. The work function depends on the kind of the metal and is of order of few electron-volts.

The table below gives the values of the work function W_0 for some materials.

Metal	Li	Cs	Rb	K	Na	Zn
$W_0\ in\ [eV]$	2.39	1.89	2.13	2.15	2.27	4.31

Inadequacy of the classical wave theory

At that time, scientists felt very comfortable with te notion that light was a wave. Light reflected, refracted, diffracted and interfered. It behaved as a wave, thus it must be a wave. Given that preconceived notion, scientists anticipated specific outcomes regarding their experiments with the photoelectric effect.

Light as a wave anticipated outcome 1: if you shine any light on a metal surface for a long enough period of time the electrons are gradually heated, the minimum energy (work function) is eventually achieved and electrons would be ejected.

Light as a wave anticipated outcome 2: the intensity (amplitude) of a wave is proportional to the energy of a wave. So if you increase the intensity of a light wave, the kinetic energy of the ejected electrons would also increase.

Experiments done with the photoelectric effect completely violated these anticipated outcomes causing great concern for scientists who were forced to rethink their notion of light and electromagnetic radiation.

The results of the experiments concerning the photoelectric effect were the following:

1- Increasing the light intensity enlarged the number of ejected electrons, but had no effect on the kinetic energy of the electrons.
2- Increasing the frequency of the light increased the kinetic energy of the ejected electrons.
3- Each metal had some specific threshold frequency, below which no electrons would be ejected. As long as the light shining upon the metal was below the threshold frequency, no electrons would be ejected, regardless of the intensity of the light or the duration of the metal's exposure to the light.
4- As long as the frequency of the light was above the threshold frequency of the metal, no time lag existed between shining the light and ejection of the electrons.

Planck's postulate: energy exchanges between matter and electromagnetic waves take place only by multiple quantities of a fundamental quantity called "quantum of energy". Energy exchanges are therefore "quantized".
Planck's-Einstein hypothesis or photons theory: the energy of an electromagnetic radiation of frequency ν is quantized where it can take only specific values that are whole multiples of a certain value called "quantum" that depends on the wavelength of the electromagnetic radiation. An electromagnetic radiation of frequency ν and wavelength λ in vacuum is composed of bundles of energy having no mass and no charge called photons that propagate in vacuum at the speed of light c. The energy of a photon is given by:

$$\boldsymbol{E_{ph} = h\nu = \frac{hc}{\lambda}}$$

Planck's constant $h = 6.626 \times 10^{-34} J.s$ and $c = 3 \times 10^8 m/s$ is the speed of light in vacuum.
In SI units, E_{ph} is expressed in [J], ν in [Hz] and λ in [m].
The photon energy E_{ph} is usually expressed in electron-volts [eV] where $1eV = 1.6 \times 10^{-19} J$.

7.4- EINSTEIN'S FORMULA

In photoelectric effect, the interaction between the incident photons and the electrons on the surface of the metal is a one to one interaction (inelastic collision). One incident photon may interact with one electron.
When a photon of energy $E_{photon} = h\nu = \frac{hc}{\lambda}$ hits the surface of a metal of work function $W_0 = h\nu_0 = \frac{hc}{\lambda_0}$, this photon may by either reflected or absorbed.
If it is reflected, no exchange of energy takes place and no photoelectric emission takes place as well.
If the photon is absorbed, its quantized energy E_{photon} is transferred to one electron on the surface of the metal. Many cases arise:

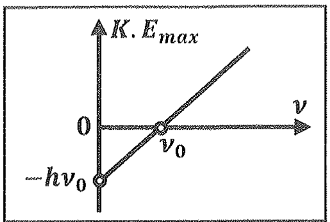
1- $E_{ph} < W_0, \nu < \nu_0$ or $\lambda > \lambda_0$: the electron is not extracted from the surface of metal, it is reflected.
2- $E_{ph} = W_0, \nu = \nu_0$ or $\lambda = \lambda_0$: the photon is absorbed where the electron is extracted from the surface of the metal at a zero speed.
3- $E_{ph} > W_0, \nu > \nu_0$ or $\lambda < \lambda_0$: the photon is absorbed where the electron is extracted from the surface of the metal with a maximum kinetic energy $K.E_{max} = \frac{1}{2}mv^2$ where:

$$E_{ph} = W_0 + K.E_{max} \Longrightarrow E_{ph} = W_0 + \frac{1}{2}mv^2 \text{ (Einstein's relation)}$$
$$h\nu = h\nu_0 + K.E_{max} \Longrightarrow h\nu = h\nu_0 + \frac{1}{2}mv^2$$
$$\frac{hc}{\lambda} = \frac{hc}{\lambda_0} + K.E_{max} \Longrightarrow \frac{hc}{\lambda} = \frac{hc}{\lambda_0} + \frac{1}{2}mv^2$$

TIP
Quantized: specific or discrete values.

Graphical representation

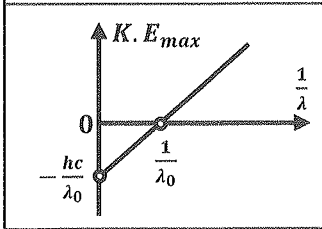
Graph of $K.E_{max}$ versus ν.



$h\nu = h\nu_0 + K.E_{max} \Longrightarrow K.E_{max} = h\nu - h\nu_0$.

The expression of $K.E_{max}$ as a function of ν has the form of the general equation of a straight line $K.E_{max} = a\nu + b$ where $a = slope = h$ and $b = -h\nu_0$.

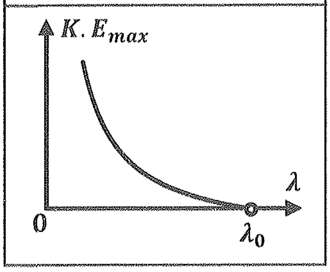
Graph of $K.E_{max}$ versus $\frac{1}{\lambda}$.



$\frac{hc}{\lambda} = \frac{hc}{\lambda_0} + K.E_{max} \Longrightarrow K.E_{max} = \frac{hc}{\lambda} - \frac{hc}{\lambda_0}$.

The expression of $K.E_{max}$ as a function of $\frac{1}{\lambda}$ has the form of the general equation of a straight line $K.E_{max} = a\frac{1}{\lambda} + b$ where $a = slope = hc$ and $b = -\frac{hc}{\lambda_0}$.

Graph of $K.E_{max}$ as a function of λ



$K.E_{max}$ decreases as λ increases and it becomes zero for $\lambda = \lambda_0$ (the exctracted electron is at rest).

7.5- NUMBER OF PHOTONS EMITTED FROM A SOURCE AND QUANTUM EFFICIENCY

The power P, assumed constant, of an electromagnetic radiation source is the energy emitted per unit time.
The total energy of an electromagnetic radiation source during a time interval Δt is:

$$E_{total} = N \times E_{ph}$$

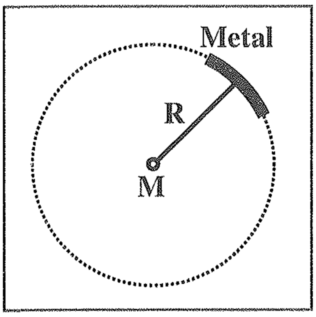
where N is the number of photons emitted by the electromagnetic radiation source.

$$E_{total} = P \times \Delta t \Longrightarrow N \times E_{ph} = P \times \Delta t \Longrightarrow N = \frac{P \times \Delta t}{E_{ph}}$$

If the source emits N photons during $\Delta t = 1s$; then: $N \times E_{ph} = P \Longrightarrow P = Nh\nu$.

Note: the number of photons emitted by an electromagnetic radiation source increases if:
- The power of the electromagnetic radiation source increases.
- The time interval Δt increases.

Number of photons received by a metal



A point source M of monochromatic light of frequency ν in vacuum emits radiation with a power P in all direction in a homogenous and non-absorbing medium. M illuminates a metal of work function W_0 and area A considered as a portion of a sphere of center M and radius R.
The power is uniformly distributed over the surface of S of the sphere of radius R.
The power of light received by the metal is:

$$P_{rec} = P \times \frac{A}{S} = P \times \frac{A}{4\pi R^2}$$

The number of the received photons duration a time interval Δt is:

$$N_{rec} = \frac{P_{rec} \times \Delta t}{h\nu}$$

TIP

Effective photons are photons that cause photoelectric emission in the metal.

The **quantum efficient r** is defined as the ratio of the effective number of photons to the number of received photons during a time interval Δt.

$$r = \frac{effective\ photons}{received\ photons} = \frac{N_{eff}}{N}$$

The number of effective photons is equal to the number of extracted electrons.

The average photo-current is defined as charge over time and it is given by: $I = \frac{\Delta q}{\Delta t} = \frac{N_{eff} \times e}{\Delta t}$.

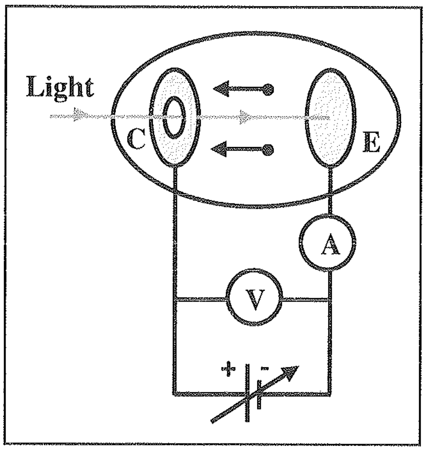
Notes

- Intensity is a measure of how many photons are incident on the surface in a given amount of time. If the frequency is large enough ($\nu \geq \nu_0$) and you increase the intensity (brightness), the current increases because there will be more ejected electrons.
- If light shining on the metal has too low frequency, nothing will happen, no matter how bright it is.
- The maximum kinetic energy ($K.E_{max}$) of the emitted electrons increases with the frequency of the incident radiation and it is independent of the intensity of the radiation

7.6- THE PHOTOELECTRIC CELL

The photoelectric cell is an apparatus used to study the photoelectric effect. An evacuated glass chamber contains a metallic plate E (the emitter) connected to the negative terminal of a DC power supply and another metallic plate C (the collector) that is connected to the positive terminal of the DC power supply.

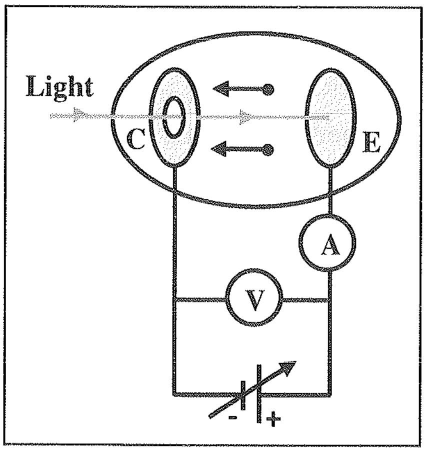
When plate E is illuminated by a monochromatic light of wavelength λ smaller than the threshold wavelength λ_0 of the metallic plate E, a current is detected by the ammeter, indicating a flow of charges across the gap between plates E and C. This current arises from photoelectrons emitted from plate E and collected at plate C.



At large values of the potential difference ΔV applied between plates E and C, the current reaches a maximum value; all the electrons emitted from E are collected at C, and the current cannot increase further. In addition, the maximum current increases as the intensity of the incident light increases, as you might expect, because more electrons are ejected by the higher-intensity light.

Cutoff voltage or stopping potential

Finally, when ΔV is negative - that is, when the terminals of the DC power supply in the circuit are reversed to make plate E positive and plate C negative - the current drops because many of the photoelectrons emitted from E are repelled by the negative plate C. In this situation, only those photoelectrons having a kinetic energy greater than $e|\Delta V|$ reach plate C, where e is the magnitude of the charge on the electron. When ΔV is equal to or more negative than $-\Delta V_S$, where ΔV_S is the **stopping potential,** no photoelectrons reach C and the current is zero.



Let's model the combination of the electric field between the plates and an electron ejected from plate E as an isolated system. Suppose this electron stops just as it reaches plate C. Because the system is isolated, we can write:

$$V_{CE} = \Delta V$$
$$\Delta E.P.E = -W_{\vec{F}} = qV_{CE}$$
$$\Delta E.P.E = -e\Delta V$$

$$\Delta K.E + \Delta E.P.E = 0$$
$$\Delta K.E = -\Delta E.P.E$$
$$0 - K.E_i = -e\Delta V \Rightarrow \Delta V = \frac{K.E_i}{e}$$

where the initial configuration is at the instant the electron leaves the metal with kinetic energy $K.E_i = 0$ and the final configuration is when the electron stops just before touching plate C.

Now suppose the potential difference ΔV is increased in the negative direction just until the current is zero at $\Delta V = \Delta V_S$. In this case, the electron that stops immediately before reaching plate C has the maximum possible kinetic energy upon leaving the metal surface. The previous equation can then be written as:

$$K.E_{max} = e\Delta V_S$$

This equation allows us to measure $K.E_{max}$ experimentally by determining the magnitude of the voltage ΔV_S at which the current drops to zero.

7.7- DUAL NATURE OF LIGHT

Photoelectric effect phenomenon confirms the corpuscular aspect of light.
Isaac Newton laid the foundations for the corpuscular theory of light which states that light is made up of tiny particles called corpuscles (from Latin corpusculum). This theory was used to explain reflection, refraction and the formation of shadows.
Christian Huygens laid the foundations for the wave theory of light which explained diffraction and interference of light.

CHAPTER 7 – CORPUSCULAR ASPECT OF LIGHT (PHOTOELECTRIC EFFECT)
EXERCISES AND PROBLEMS

Note: In all exercises, the metal plates are exposed to electromagnetic radiations in air or vacuum unless stated otherwise.

Given: Planck's constant: $h = 6.62 \times 10^{-34} Js$; elementary charge: $e = 1.6 \times 10^{-19} C$; speed of light in vacuum: $c = 3 \times 10^{8} m/s$; mass of an electron: $m_e = 9.1 \times 10^{-31} kg$.

Exercise 1*:

Define the following:

1- Photoelectric effect.
2- Threshold frequency
3- Threshold wavelength
4- Work function of a pure metal.

Exercise 2*:

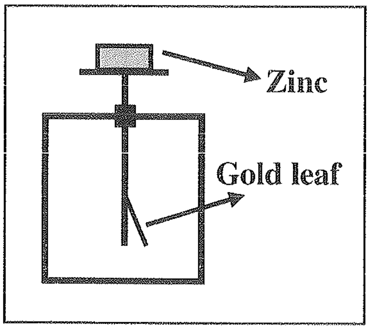
What theory of light does the photoelectric effect support? Explain why the other theory of light is inadequate to interpret this phenomenon.

Exercise 3:

In photoelectric effect, the liberated electrons from the surface of a metal may acquire kinetic energy. Explain why this kinetic energy is maximum?

Exercise 4*:

Given a red laser source, an ultraviolet source, zinc plate mounted on the knob of an electroscope where the system [plate-electroscope] is negatively charged.



1- The leaf of the electroscope diverges when the zinc plate is negatively charged. Explain
2- Carry out an experiment to investigate the emission of photoelectrons from the surface of metal under suitable electromagnetic radiation.
3- Give two applications of the photoelectric effect.
4- Outline Planck-Einstein hypothesis of the photoelectric effect.

Exercise 5:

The emission of photoelectrons from the surface of metal is quasi-instantaneous. Comment on this statement.

Exercise 6:

The work function of zinc is $W_0 = 4.31 eV$.

To extract electrons from the surface of a zinc plate, it should be exposed to radiations of wavelength λ. What are the possible values of λ.

Exercise 7*:

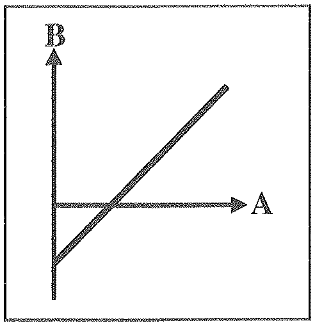
A photon carries energy of 2eV.

1- Convert this energy into Joules.
2- Calculate the wavelength of this photon.

Exercise 8*:

The surface of a metal is exposed to UV radiation of frequency 6.3×10^{14} Hz. Calculate, in [J] then in [eV], the work function of the metal if the maximum kinetic energy of the liberated electrons is 1.19×10^{-19}J.

Exercise 9*:



1- Write the expression of Einstein's relation. Identify the symbols.
2- Rearrange Einstein's relation: $K.E_{max} =$ ____________.
3- The graph of the maximum kinetic energy of the extracted electrons from the surface of metal as a function of the frequency of the incident radiation is given.
   3.1- Identify A and B?
   3.2- What does the slope represent?

Exercise 10*:

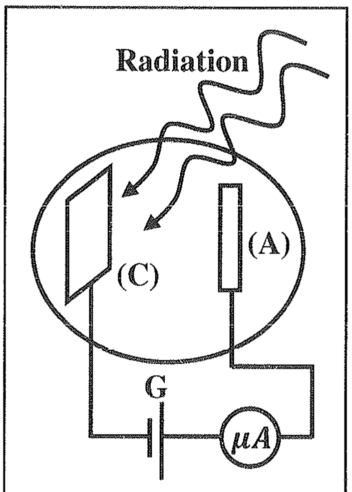
The maximum wavelength for which electrons are extracted from the surface of sodium (metal) is 650nm.
1- Give the value of the wavelength in the SI unit.
2- Specify the value of the threshold wavelength of the sodium.
3- Deduce the value of the threshold frequency
4- Calculate the work function of sodium.
5- The surface of sodium emits electrons when exposed to suitable incident radiation. Supposing that all incident photons are effective, specify if any changes occur to the number of emitted electrons if the intensity of the incident radiation is tripled.

Exercise 11:

A metallic plate of work function 1.1eV is exposed to suitable electromagnetic radiation of 2×10^{16} photons/s where a single photon acquires energy of 2.7eV. N photoelectrons are ejected from the surface of metal.
1- Give the value of the speed of light in vacuum.
2- State three properties of a photon.
3- Define the work function of a metal.
4- Convert the above given values into joules.
5- Calculate the wavelength of the incident radiation and the threshold wavelength of the metal plate. Draw a conclusion.
6- Photoelectrons are ejected from the surface of metal.
   6.1- Calculate the maximum kinetic energy of an extracted electron; then, deduce its maximum speed.
   6.2- Does the value of the maximum speed change if the intensity of the incident radiation is tripled? Justify your answer.
   6.3- State one factor that must be modified in order to change the maximum speed of the extracted electron.
7- Calculate the energy of the incident radiation received by the metallic plate during one second.
8- The quantum efficiency of the surface of the plate is 1%. Calculate N

Exercise 12:

A photocell is usually a vacuum tube with two electrodes. One is a photosensitive cathode which emits electrons when exposed to light and the other is an anode which is maintained at a positive voltage with respect to the cathode. Thus when light shines on the cathode, electrons are attracted to the anode and an electron current flows in the tube from cathode to anode. The current can be used to operate a relay, which might turn a motor on to open a door or ring a bell in an alarm system.



Doc.1

A blue light source of wavelength $\lambda = 486.13nm$ and power $P_S = 2W$, emits radiations uniformly and in all directions in a homogeneous and non- absorbing medium. This source illuminates a potassium cathode C of a photoelectric cell of work function $W_0 = 2.20eV$ and of a surface area $s = 2cm^2$.

1- Calculate the threshold wavelength of the potassium λ_0.
2- The generator G is adjusted so that the anode (A) captures all the emitted electrons by the cathode of quantum efficiency r = 0.875%.
   2.1- Show that the received power of the radiation P_0 of the cell is $2.04 \times 10^{-5}W$.
   2.2- Determine the number N_0 of the incident photons received by the cathode C in one second.
   2.3- Determine the average current I in the circuit.

Exercise 13:

The photoelectric effect has many practical applications which include the photocell, photoconductive devices and solar cells.

To determine the work function W_0 and then the nature of the metal, we illuminate the cathode of a photo cell successively and separately each time with radiations of different frequencies and we determine the maximum kinetic energy (KE_{max}) of the emitted photoelectrons for each radiation of frequency ν. We obtain the results shown in table 1.

ν [$\times 10^{14} Hz$]	5.5	6.2	6.9	7.5
KE_{max} [eV]	0.2	0.49	0.79	1.03

Table 1

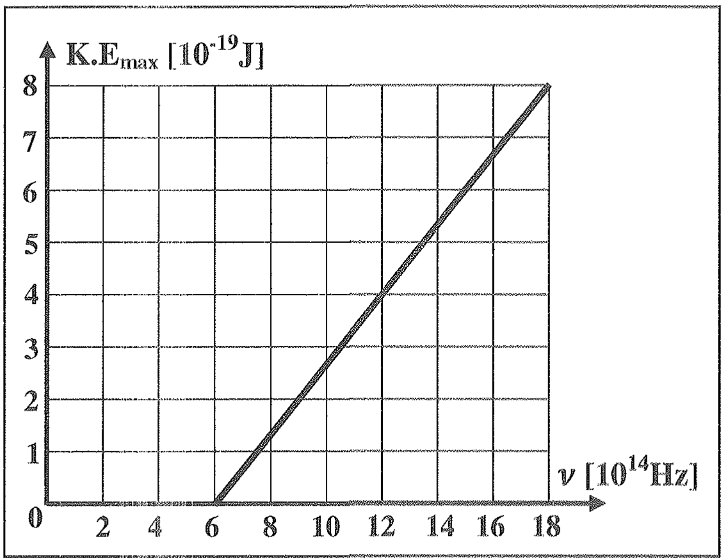
Metal	Cesium	Sodium	Potassium
W_0 [eV]	2.08	2.28	2.2

Table 2

1-
   1.1- Trace the curve representing the variations of KE_{max} in terms of frequency ν.
   1.2- The obtained graph confirms with Einstein's relation concerning the photoelectric effect. Justify.
   1.3- Name the physical constant that is represented by the slope of this graph.
2- Using the graph, determine the value of:
   2.1- This physical constant.
   2.2- The threshold frequency.
   2.3- Deduce the value of W_0.
3- Referring to table 2, indicate the nature of the used metal.
4- Sketch on the same graph the curve of KE_{max} in terms of frequency ν for a metal of larger threshold frequency. Justify your answer by commenting on the slope and the y-intercept.
5- The stopping potential of a sodium metal plate is 4.5eV. The plate is exposed to a suitable electromagnetic radiation of wavelength λ.
   5.1- Calculate the maximum kinetic energy of an ejected electron then deduce its maximum speed.
   5.2- Calculate λ.

Exercise 14*:

The graph below shows the variation of the maximum kinetic energy of photoelectrons emitted from the surface of sodium metal in vacuum as a function of the frequency of the illuminating light.



1- Using the graph:

  1.1- Determine the value of Planck's constant.

  1.2- Determine the value of the work function of sodium.

2- Plot the corresponding graph for zinc that has a threshold frequency of 10.4×10^{14} Hz.

Exercise 15:

In a photoelectric experiment, we illuminate the cathode of a photocell successively and separately each time with radiations of different frequencies and we determine the maximum kinetic energy (KE_{max}) of the emitted photoelectrons for each radiation of frequency ν and wavelength λ. We obtain the results shown in the table below.

$\nu\,[\times 10^{14} Hz]$	6	7	8	9	10
$K.E_{max}\,[eV]$	0.25	0.64	1	1.5	1.9
$\lambda\,[10^{-7} m]$					
$\frac{1}{\lambda}\,[10^{6} m^{-1}]$					

1- Complete the table by adding the values of λ and $\frac{1}{\lambda}$.

2- Write the expression of the KE_{max} in terms of λ.

3- Trace the curve representing the variations of KE_{max} in terms of $\frac{1}{\lambda}$.

  Scale: horizontal axis:1div $\rightarrow 0.5 \times 10^{-6} m^{-1}$; vertical axis: 1div $\rightarrow 0.5 eV$.

4- Using the graph, determine the value of:

  4.1- The Planck's constant.

  4.2- The threshold wavelength.

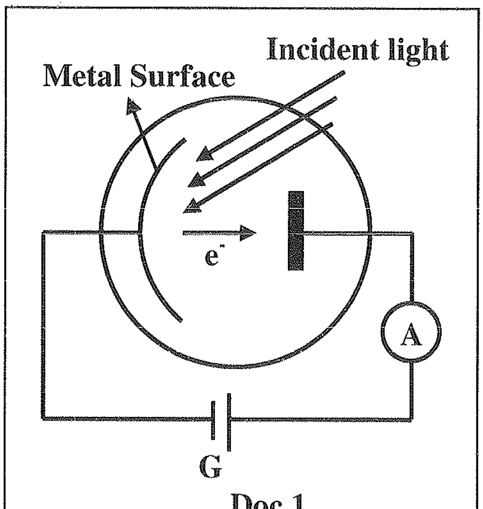
5- Deduce the work function W_0 of the metal.

6-

  6.1- Write the expression of the square of the maximum speed of an extracted electron in terms of ν.

  6.2- Calculate the maximum speed of the ejected electrons when the cathode is illuminated with radiations of frequency $\nu = 12 \times 10^{14} Hz$.

Exercise 16:



Doc.1

Document (1) is a schematic diagram of the circuit of a photocell which includes: a generator (G) of constant voltage, an ammeter (A), a cathode (a pure metal), and an anode.
We consider two different photo cells: the cathode of the first one is made of cesium and the other is made of magnesium. The work function of cesium is 2.1eV and of magnesium is 3.7eV.
We illuminate each one of the above cells by a radiation of wavelength $\lambda = 550$nm in vacuum.

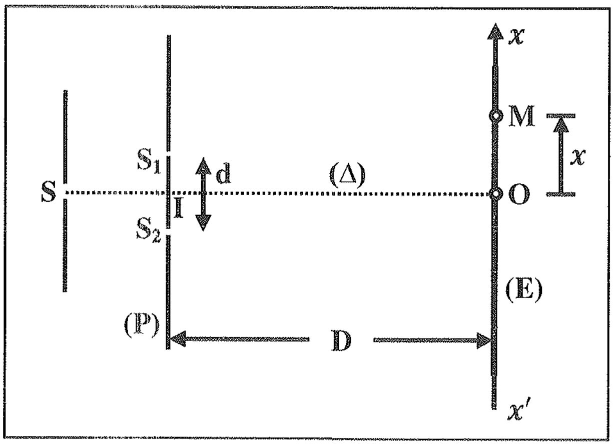
1- Specify the metal where photoelectric effect takes place.
2- Determine the maximum speed of the electron liberated from the surface of the metals .
3- The electrons liberated by the cathode are attracted by the anode, thus forming an average current of value $I = 30\mu A$.
   3.1- Calculate the number of electrons liberated from the surface of the cathode during one second.
   3.2- The quantum efficiency of the cell is 2%. Determine the power of the used radiation.
   3.3- In order to increase the kinetic energy of the liberated electrons, should we vary the power P of the source or the wavelength λ of the used light? Justify.

Exercise 17:

A laser source (S), of wavelength $\lambda_0 = 602$nm in vacuum, illuminates a photoelectric cell whose cathode is covered with cesium of work function $W_S = 1.9$eV. The power received by the cathode is $P = 2 \times 10^{-3}$W and the maximum speed of the emitted photoelectrons is 2.37×10^5 m/s.

1- Calculate, using two different methods, the cesium threshold wavelength λ_S.
2- The anode of the photoelectric cell captures the emitted electrons by the cathode with quantum efficiency r = 0.9%.
   2.1- Calculate the number N_0 of the incident photons received by the cathode C in one second.
   2.2- Determine the average current I in the circuit.
   2.3- Calculate the power of the source of light knowing that the cathode of the photoelectric cell has a surface area $s = 2\text{cm}^2$ placed at a distance D = 1.25m from the source.
3- The power is doubled. Layan suggests that:
   3.1- the number of photons is doubled
   3.2- number of emitted electrons is halved.
   3.3- λ_0 remains the same
   3.4- kinetic energy of the emitted electrons increases

   Choose the correct answer(s).



4- The source (S) illuminates two narrow and parallel slits S_1 and S_2 that are separated by a distance a = 0.8mm. A screen (E) is placed at a distance D = 1.6m from the plane of the slits.
   Consider on (E) a point M such that $x = \overline{OM}$.
   4.1- Draw a figure showing on it the zone and the field of interference on (E).
   4.2- Determine the expression of the abscissa x of M in terms of λ, D and a when M belongs to a dark fringe.
   4.3- Show that O is the center of the central fringe and deduce its nature.

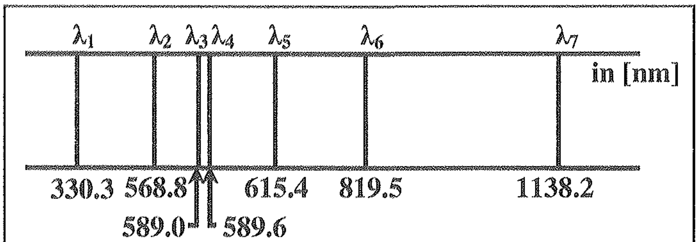
4.4- Determine the expression of the inter-fringe distance i in terms of λ, D and a. Calculate i.
4.5- Specify the nature of the fringe of abscissa $x = -4.2mm$.

5- We displace (S) slowly back and forth along the axis (Δ). We observe that the central fringe does not move and that the inter-fringe distance is not affected. Justify this observation.

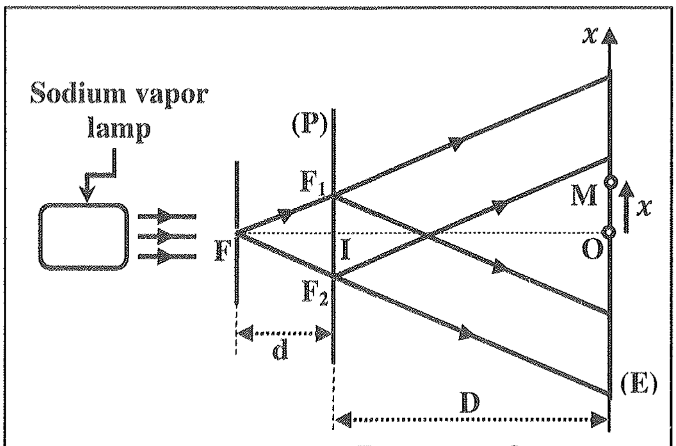
6- The source (S) is now on the axis (Δ) at a distance d = 8 mm from I. We displace (S) slowly and perpendicularly to (Δ) till the central fringe takes the place of the upper first bright fringe. Let z be the displacement of (S), and the path difference δ' is now given by: $\delta' = \frac{ax}{D} + \frac{az}{d}$.
6.1- In which direction is (S) moved? Justify.
6.2- Show that the inter-fringe distance is not affected and determine the value of z.

Exercise 18*:

A sodium vapor lamp emits mainly a yellow light called doublet of wavelengths 589 nm and 589.6 nm that are more intense than other radiations.
Document 1 represents some of the lines of the emission spectrum of sodium lamp.
The setup of Young's double-slit experiment is shown in document 2. F_1 and F_2 are separated by a = 0.5 mm and D = 1.6m.



Document 1



Document 2

1- Using document 1, explain the meaning of the term doublet.
2- The presence of the single-slit F, between the source of light and the double slits, is necessary to obtain stable interference pattern. Explain.
3- Describe the interference pattern.
4- Consider a point M of the screen (E), such that $x = \overline{OM}$.
4.1- Write the expression of the optical path difference at M, $\delta_M = F_2M - F_1M$, as function of a, x and D.
4.2- M is the center of the first dark fringe, where the path difference is $\delta_M = 0.2945\mu m$. Determine the wavelength λ and compare it with the emission spectrum of sodium.
5- The sodium vapor lamp illuminates the surface of a metal whose work function is $W_0 = 2.5eV$.
5.1- Calculate the value of the threshold wavelength λ_0.
5.2- Using document 1, specify the radiation(s) that can extract electrons from the surface of the metal.
5.3- Determine the maximum speed of an emitted electron.

CHAPTER 7 – CORPUSCULAR ASPECT OF LIGHT (PHOTOELECTRIC EFFECT)
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 3:

Part	Answer key
	The energy required to extract the least tightly bound electron is called the extraction energy W_0 of the metal. Electrons which are largely bound require more energy to be extracted. Hence in photoelectric emission, the least tightly bound electrons are emitted with maximum kinetic energy.

Exercise 5:

Part	Answer key
	The phenomenon of the photoelectric emission occurs within a delay of time extremely small less than $10^{-9}s$ after illumination, which makes the phenomenon almost instantaneous even at low intensity. When radiation strikes the target material, electrons are emitted almost instantaneously, even at very law intensities of incident radiations. Thus almost absence of lag time contradicts our understanding based on classical physics which predicts that for low energy radiation, it would take significant time before irradiated electrons could gains sufficient energy to leave the metal surface; however, such an energy build up is not observed.

Exercise 6:

Part	Answer key
	The threshold wavelength of zinc is: $\lambda_0 = \frac{hc}{W_0} = \frac{6.6\times10^{-34}\times3\times10^{8}}{4.31\times1.6\times10^{-19}} = 2.87\times10^{-7}m = 287nm$. $\lambda \leq \lambda_0 \Rightarrow \lambda \leq 287nm$.

Exercise 11:

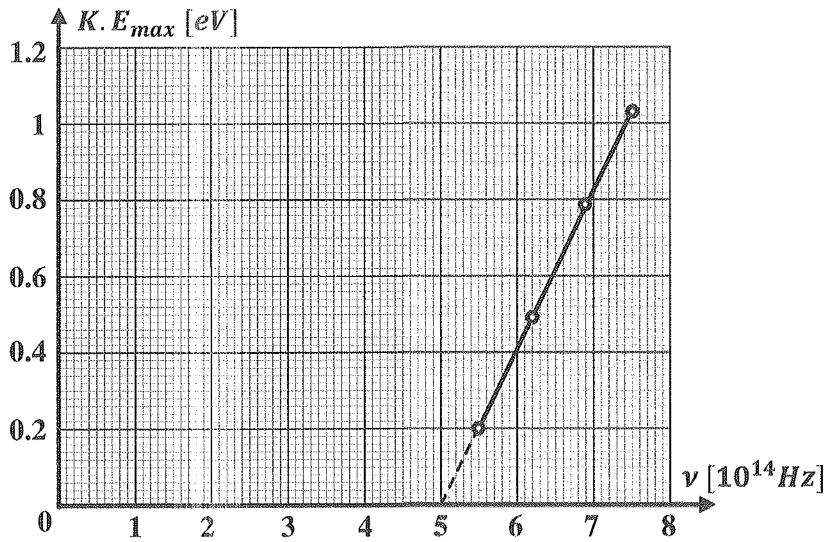
Part	Answer key
1	The speed of light in vacuum is $c = 3\times10^{8}m/s$.
2	The three properties of a photon are: - A photon is non-charged particle. - A photon has a zero mass. - A photon has the speed of light in vacuum $c = 3\times10^{8}m/s$.
3	The work function W_0 of a metal is the minimum energy needed to extract an electron from the surface of this metal.
4	$W_0 = 1.1\times1.6\times10^{-19} = 1.76\times10^{-19}J$. $E_{ph} = 2.7\times1.6\times10^{-19} = 4.32\times10^{-19}J$.
5	$E_{ph} = \frac{h.c}{\lambda} \Rightarrow \lambda = \frac{h.c}{E_{ph}} = \frac{6.62\times10^{-34}\times3\times10^{8}}{4.32\times10^{-19}} = 4.597\times10^{-7}m$. $W_0 = \frac{h.c}{\lambda_0} \Rightarrow \lambda_0 = \frac{h.c}{W_0} = \frac{6.62\times10^{-34}\times3\times10^{8}}{1.76\times10^{-19}} = 11.284\times10^{-7}m$. Thus $\lambda_0 > \lambda$, then the photon is absorbed and the electron is extracted from the surface of the metal with a kinetic energy (photoelectric emission takes place).
6.1	$E_{ph} = W_0 + K.E_{max} \Rightarrow K.E_{max} = E_{ph} - W_0 = 4.32\times10^{-19} - 1.76\times10^{-19} = 2.56\times10^{-19}J$ But $\Rightarrow K.E_{max} = \frac{1}{2}mV_{max}^2 \Rightarrow V_{max} = \sqrt{\frac{2K.E_{max}}{m}} = \sqrt{\frac{2\times2.56\times10^{-19}}{9.1\times10^{-31}}} = 750091.5695m/s$.
6.2	$K.E_{max} = E_{ph} - W_0 \Rightarrow \frac{1}{2}mV_{max}^2 = E_{ph} - W_0 = h\nu - W_0$. $V_{max} = \sqrt{\frac{2(h\nu - W_0)}{m}}$ or $V_{max} = \sqrt{\frac{2\left(\frac{hc}{\lambda} - W_0\right)}{m}}$.

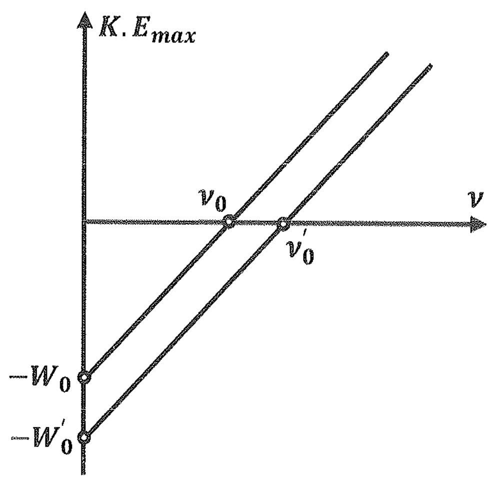
	This expression shows that V_{max} depends on the frequency ν or the wavelength λ of the incident radiation and not on the intensity of the incident radiation. Particular, to increase V_{max} , we must decreases the wavelength λ or increase the frequency ν and not increase (triple) the intensity of the incident radiation.
6.3	In order to change the maximum speed (V_{max}) of the extracted electron, we must modify the frequency γ or the wavelength λ of the incident radiation.
7	$P = \frac{E_{total}}{t} \Longrightarrow E_{total} = P \times \Delta t$ but the average power of the incident radiation is $P = N \times E_{Ph}$ where N is the number of proton per second. $E_{total} = N \times E_{Ph} \times \Delta t = 2 \times 10^{16} photons/s \times 4.3 \times 10^{-19} J \times 1 = 8.6 \times 10^{-3} J.$
8	$r = \frac{N}{N_{total}} \Longrightarrow N = r \times N_{total} = 0.01 \times 2 \times 10^{16} = 2 \times 10^{14}$ photoelectrons/s.

Exercise 12:

Part	Answer key
1	$W_0 = \frac{hc}{\lambda_0} \Longrightarrow \lambda_0 = \frac{hc}{W_0} = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{2.2 \times 1.6 \times 10^{-19}} = 5.64 \times 10^{-7} m.$
2.1	$P_S = 2W \longrightarrow 4\pi R^2$ $P_0 \rightarrow s = 2 \times 10^{-4} m^2$ Then $P_0 = \frac{P_s \times s}{4\pi R^2} = \frac{2 \times 2 \times 10^{-4}}{4\pi R^2} = \frac{4 \times 10^{-4}}{4\pi (1.25)^2} = 2.04 \times 10^{-5} W$
2.2	$E_{ph} = h\nu = \frac{hc}{\lambda} = \frac{6.62 \times 10^{-34} \times 3 \times 10^{8}}{486.13 \times 10^{-9}} = 4.085 \times 10^{-19} J$ $E_0 = P_0 \times t = 2.04 \times 10^{-5} \times 1 = 2.04 \times 10^{-5} J.$ But $E_0 = N_0 \times E_{Ph} \Longrightarrow N_0 = \frac{E_0}{E_{Ph}} = \frac{2.04 \times 10^{-5}}{4.085 \times 10^{-19}} = 5 \times 10^{13}$ photons/s.
2.3	$r = \frac{N_{eff}}{N_0} \Longrightarrow N_{eff} = r \times N_0 = 0.875 \times 5 \times 10^{13} = 4.375 \times 10^{13}$ photons/s. But $Q = N_{eff} \times e = I \times \Delta t \Longrightarrow I = \frac{N_{eff} \times e}{\Delta t} = \frac{4.375 \times 10^{13} \times 1.6 \times 10^{-19}}{1} = 7 \times 10^{-6} A = 0.007 mA.$

Exercise 13:

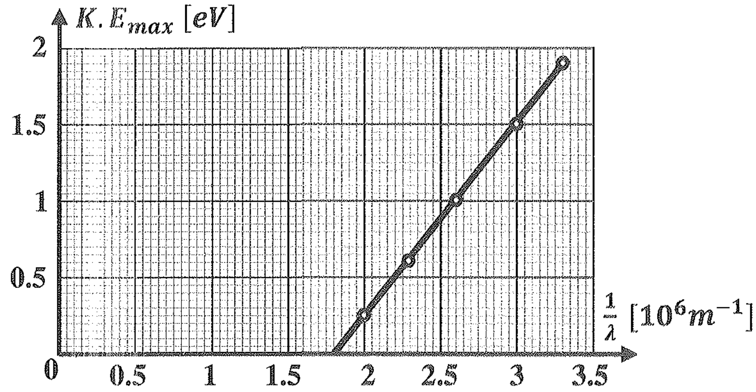
Part	Answer key
1.1	
1.2	The shape of the graph of $K.E_{max}$ versus ν is a straight line not passing through the origin of general equation: $K.E_{max} = a\nu + b$ ... (1) where a is the slope and b in the $K.E_{max}$ -intercept (intersection of the graph with $K.E_{max}$ axis).

	Einstein's relation is given by: $E_{ph} = W_0 + K.E_{max} \Rightarrow K.E_{max} = h\nu - W_0$ ... (2). The two equations (1) and (2) are compatible where $a = h$ and $b = -W_0$.
1.3	The physical constant that is represented by the slope of this graph is the Planck's constant h.
2.1	$h = a = \frac{\Delta K.E_{max}}{\Delta \nu} = \frac{(1.03-0.2)\times 1.6\times 10^{-19}}{(7.5-5.5)\times 10^{14}} = 6.64 \times 10^{-34} Js$.
2.2	The threshold frequency is the intersection between the graph and ν-axis. Graphically, $\nu_0 = 5 \times 10^{14}\ Hz$ for $K.E_{max} = 0eV$.
2.3	$K.E_{max} = 6.64 \times 10^{-34}\nu - W_0$. For $\nu = 5.5 \times 10^{14} Hz$; $K.E_{max} = 0.2eV = 0.2 \times 1.6 \times 10^{-19} = 0.32 \times 10^{-19} J$. $0.32 \times 10^{-19} = 6.64 \times 10^{-34} \times 5.5 \times 10^{14} - W_0 \Rightarrow W_0 = 3.332 \times 10^{-19} J = 2.08eV$.
3	The metal is the cesium.
4	 For a metal of work function $W_0' > W_0$ and threshold frequency $\nu_0' > \nu_0$. According to Einstein's relation the expression of the maximum kinetic energy of this metal is: $K.E_{max} = h\nu - W_0'$. The two straight lines have the same slope ($a = h$), then they are parallel. For $K.E_{max} = 0$; $\nu = \nu_0' \Rightarrow h\nu_0' = W_0 \Rightarrow \nu_0' = \frac{W_0'}{h}$. $W_0' > W_0 \Rightarrow \nu_0' > \nu_0$.
5.1	$K.E_{max} = eU_0$ where U_0 is the stopping potential. $K.E_{max} = 4.5eV = 4.5 \times 1.6 \times 10^{-19} = 7.2 \times 10^{-19} J$. But $K.E_{max} = \frac{1}{2} m_e V_{max}^2 \Rightarrow V_{max} = \sqrt{\frac{2K.E_{max}}{m_e}} = \sqrt{\frac{2\times 7.2\times 10^{-19}}{9.1\times 10^{-31}}} \Rightarrow V_{max} = 1257941.804 m/s$.
5.2	$E_{ph} = W_0 + K.E_{max} = 2.28 + 4.5 = 6.78eV$. But $E_{ph} = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E_{ph}} = \frac{6.64\times 10^{-34}\times 3\times 10^{8}}{6.78\times 1.6\times 10^{-19}} = 1.83628 \times 10^{-7} m = 183.628 nm$.

Exercise 15:

Part	Answer key
1	(table below)
2	$E_{ph} = W_0 + K.E_{max} \Rightarrow \frac{hc}{\lambda} = W_0 + K.E_{max} \Rightarrow K.E_{max} = hc\frac{1}{\lambda} - W_0$.

$\nu\ [\times 10^{14} Hz]$	6	7	8	9	10
$K.E_{max}\ [eV]$	0.25	0.64	1	1.5	1.9
$\lambda = \frac{c}{\nu}\ [10^{-7} m]$	5	4.28	3.75	3.33	3
$\frac{1}{\lambda}\ [10^{6} m^{-1}]$	2	2.3	2.6	3	3.3

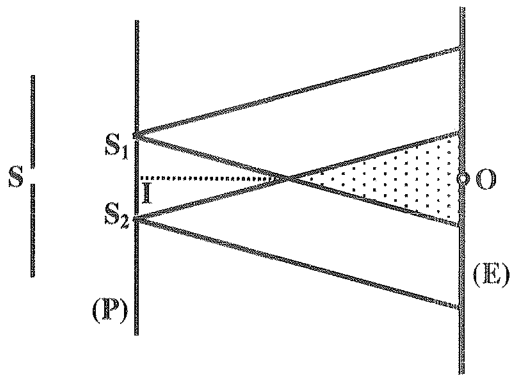
3	
4.1	The general equation of a straight line is: $K.E_{max} = a\frac{1}{\lambda} + b$ where a is the slope and b is the intersection between the straight line and the $K.E_{max}$ axis. $a = \frac{\Delta K.E_{max}}{\Delta\frac{1}{\lambda}} = \frac{(1.5-0.25)\times1.6\times10^{-19}}{(3-2)\times10^6} = 2\times10^{-25} Jm.$ $a = hc \Longrightarrow h = \frac{a}{c} = \frac{2\times10^{-25} Jm}{3\times10^8 m/s} = 6.67\times10^{-34} Js.$
4.2	Graphically, for $K.E_{max} = 0; \frac{1}{\lambda_0} = 1.8\times10^6 m^{-1} \Longrightarrow \lambda_0 = 5.556\times10^{-7} m = 555.6 nm$
5	$W_0 = \frac{hc}{\lambda_0} = \frac{6.67\times10^{-34}\times3\times10^8}{5.556\times10^{-7}\times1.6\times10^{-19}} = 2.25 eV.$
6.1	$\frac{1}{2}mV_{max}^2 = hv - W_0 \Longrightarrow V_{max}^2 = \frac{2(hv-W_0)}{m}.$
6.2	$V_{max} = \sqrt{\frac{2(hv-W_0)}{m}} = \sqrt{\frac{2(6.62\times10^{-34}\times12\times10^{14}-2.25\times1.6\times10^{-19})}{9.1\times10^{-31}}} = 1.12\times10^6 m/s.$

Exercise 16:

Part	Answer key
1	The work function of the cesium metal is: $W_{0_{Ce}} = \frac{hc}{\lambda_{0_{Ce}}} \Longrightarrow \lambda_{0_{ce}} = \frac{hc}{W_{0_{Ce}}} = \frac{6.62\times10^{-34}\times3\times10^8}{2.1\times1.6\times10^{-19}} = 5.91\times10^{-7} m = 591 nm.$ The work function of magnesium metal is: $W_{0_{Mg}} = \frac{hc}{\lambda_{0_{Mg}}} \Longrightarrow \lambda_{0_{Mg}} = \frac{hc}{W_{0_{Mg}}} = \frac{6.62\times10^{-34}\times3\times10^8}{3.7\times1.6\times10^{-19}} = 3.3547\times10^{-7} m = 335.47 nm.$ Then, $\lambda_{radiation} = 550\ nm < \lambda_{0_{Ce}} = 591\ nm$ then the photoelectric effect takes place for cesium cathode metal. In addition $\lambda_{radiation} = 550\ nm > \lambda_{0_{Mg}} = 335.47 nm$ then the photoelectric effect does not occur with magnesium cathode metal.
2	$E_{ph} = W_0 + K.E_{max} \Longrightarrow K.E_{max} = E_{ph} - W_0 \Longrightarrow \frac{1}{2}mV_{max}^2 = \frac{hc}{\lambda} - W_0.$ $V_{max} = \sqrt{\frac{2\left(\frac{hc}{\lambda}-W_0\right)}{m}} = \sqrt{\frac{2\left(\frac{6.62\times10^{-34}\times3\times10^8}{550\times10^{-9}}\right)-2.1\times1.6\times10^{-19}}{9.1\times10^{-31}}} = 234829.4172 m/s.$
3.1	$Q = It = Ne$ where N is the number of the photoelectrons $= N_{eff}$. $N = N_{eff} = \frac{It}{e} = \frac{30\times10^{-6}\times1}{1.6\times10^{-19}} = 1.875\times10^{14} electrons.$

3.2	$r = \frac{N_{eff}}{N_{\substack{total \\ or\ received}}} \Longrightarrow N_{total} = \frac{N_{eff}}{r} = \frac{1.875\times10^{14}}{0.02} = 9.375\times10^{15} photons/s.$ $P = \frac{E_{total}}{t} = \frac{N_{total}\times E_{ph}}{1} = N_{total}\times\frac{hc}{\lambda} = 9.375\times10^{15}\times\frac{6.62\times10^{-34}\times3\times10^{8}}{550\times10^{-9}} = 3.385\times10^{-3}W.$
3.3	Since $K.E_{max} = \frac{hc}{\lambda} - W_0$ where W_0 is constant related to the irradiated metal, then to increase $K.E_{max}$ we must decrease the wavelength λ of the incident radiation.

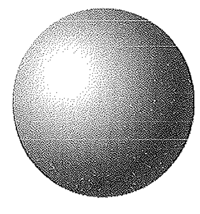
Exercise 17:

Part	Answer key
1	**First method:** $W_S = \frac{hc}{\lambda_S} \Longrightarrow \lambda_s = \frac{hc}{W_s} = \frac{6.62\times10^{-34}\times3\times10^{8}}{1.9\times1.6\times10^{-19}} = 653.3\times10^{-9}m = 653.3nm.$ **Second method:** $E_{ph} = W_S + K.E_{max} \Longrightarrow \frac{hc}{\lambda_0} = \frac{hc}{\lambda_S} + \frac{1}{2}mV_{max}^2 \Longrightarrow \frac{hc}{\lambda_S} = \frac{hc}{\lambda_0} - \frac{1}{2}mV_{max}^2.$ $\lambda_S = \frac{hc}{\frac{hc}{\lambda_0}-\frac{1}{2}mV_{max}^2} = \frac{6.62\times10^{-34}\times3\times10^{8}}{\frac{6.62\times10^{-34}\times3\times10^{8}}{602\times10^{-9}}-\frac{1}{2}\times9.1\times10^{-31}\times(2.37\times10^{5})^2} = 652.5\times10^{-9}m = 652.5nm.$ Therefore, $\lambda_S \simeq 653\ nm.$
2.1	$P_r = \frac{N_{total}\times E_{ph}}{\Delta t} = N_0\times\frac{hc}{\lambda_0}$ where $N_{total} = N_0 = N_{received}$. $N_0 = \frac{P_r\times\lambda_0}{hc} = \frac{2\times10^{-3}\times602\times10^{-9}}{6.62\times10^{-34}\times3\times10^{8}} = 6.062437059\times10^{15}\ photons/s.$
2.2	$r = \frac{N_{eff}}{N_0}$ but N_{eff} = number of emitted electrons N_{e-}. $N_{e-} = r\times N_0 = 0.009\times6.062437059\times10^{15} = 5.4561933353\times10^{13}\ electrons.$ $but\ Q = It = N_{e-}e \Longrightarrow I = \frac{N_{e-}e}{t} = \frac{5.4561933353\times10^{13}\times1.6\times10^{-19}}{1}.$ $I = 8.729909366\times10^{-6}\ A \Longrightarrow I \simeq 8.73\mu A.$
2.3	$P_s \to 4\pi D^2.$ $P_r \to S.$ Then, $P_s = \frac{P_r\times4\pi D^2}{S} = \frac{2\times10^{-3}\times4\times\pi\times(1.25)^2}{2\times10^{-4}} = 196.25W.$
3	The number of received photons is $N = \frac{P\times t}{h\times\nu} = \frac{P\times t\times\lambda_0}{hc}.$ So, if we double the power (P), $N = \frac{2\times P\times t}{h\times\nu} \Longrightarrow N$ is doubled.
4.1	
4.2	M belongs to a dark fringe then $\delta = (2k+1)\frac{\lambda}{2}$ where $k \in \mathbb{Z}$. But the path difference is $\delta = \frac{ax}{D}$. Thus $\delta = \frac{ax}{D} = (2k+1)\frac{\lambda}{2} \Longrightarrow x = (2k+1)\frac{\lambda D}{2a}.$
4.3	The path difference of 0 $is\ \delta_O = S_2O - S_1O$ but O belongs to the perpendicular bisector of

	$[S_1S_2] \Longrightarrow S_1O = S_2O$. $\delta_O = 0$ but $\delta_0 = \frac{ax_0}{D} \Longrightarrow \frac{ax_0}{D} = 0 \Longrightarrow x_0 = 0$. Since $a \neq 0$ *and* $D \neq 0$ then O is the central fringe. In addition $\delta_O = k\lambda = 0$ $(for\ k = 0) \Longrightarrow$ O is the center of bright fringe , so O is the central bright fringe.
4.4	Using two consecutive bright fringes of orders k and $k+1$. Then $x_k = \frac{k\lambda D}{a}$ and $x_{k+1} = (k+1)\frac{\lambda D}{a}$. Then, the inter fringe distance is: $i = x_{k+1} - x_k = \frac{(k+1)\lambda D}{a} - \frac{K\lambda D}{a} = \frac{\lambda D}{a}(k+1-k) = \frac{\lambda D}{a}$. $i = \frac{\lambda D}{a} = \frac{602\times10^{-9}\times1.6}{0.8\times10^{-3}} = 1.204 \times 10^{-3}m = 1.204mm$.
4.5	If the fringe is dark then $x = (2k+1)\frac{\lambda D}{2a} \Longrightarrow -4.2 \times 10^{-3} = (2k+1) \times \frac{602\times10^{9}\times1.6}{2\times0.8\times10^{-3}}$. $-4.2 \times 10^{-3} = (2k+1) \times 0.602 \times 10^{-3} \Longrightarrow k = -\frac{2.401}{0.602} = -3.988 \approx -4$. Thus the fringe is the 4th dark fringe on the negative side.

CHAPTER 8 – THE ATOM
COURSE

8.1- HISTORY OF THE ATOM

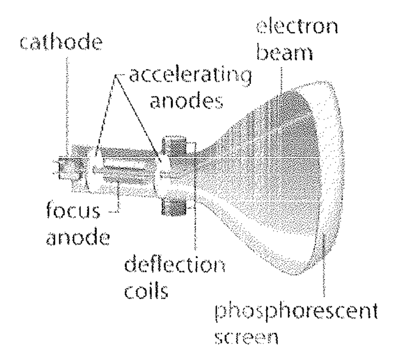


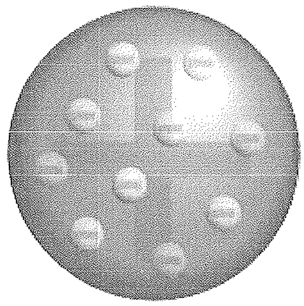
Democritus was a Greek philosopher who was the first person to use the term atom (from the Greek word *atomos* which means indivisible). He thought that if you take a piece of matter and divide it and continue to divide it you will eventually come to a point where you could not divide it any more. This fundamental or basic unit was what Democritus called an atom. This theory was adapted by Dalton. Later discoveries proved that it can be divided into a great number of smaller particles.

Thomson's atomic model (1897)

J. J. Thomson was a British scientist and a professor of physics at the University of Cambridge. He is credited for discovering the electron. He studied Cathode rays, which are negatively charged particles emitted from the cathode of a vacuum tube.

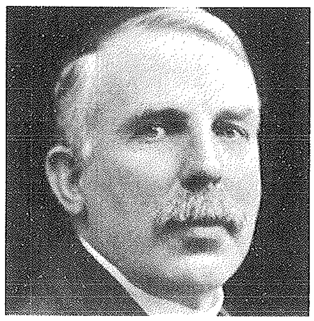
- He found that all kinds of matter contain identical negatively charged particles (same charge and same mass) called electrons. In addition, he measured its charge-mass ratio $\frac{e}{m}$.





- Because the ordinary atom is neutral in charge, then any atom should contain a positive charge equal and opposite to the charge of the electrons.
- Because the mass of the electrons is much smaller than the mass of the atom, Thomson proposed that an atom is constituted of a sphere of positive charge, which has almost all the mass of the atom, while electrons are embedded in it (plum pudding model).

Rutherford's model of the atom (1910's)



Ernest Rutherford was born in New Zealand and was working at the University of Manchester in England.

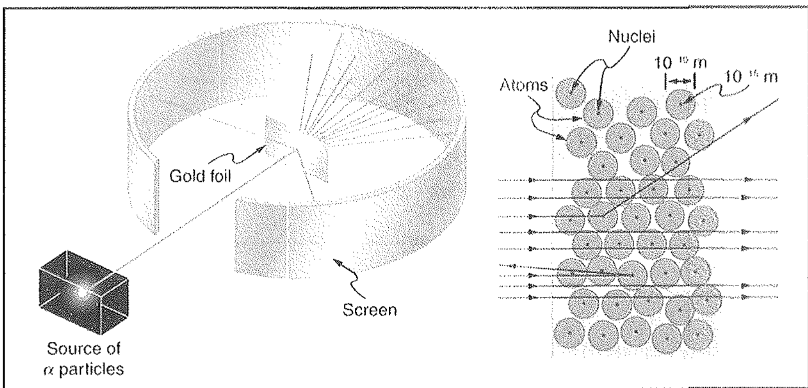
Ernest Rutherford was not convinced about the model of the atom proposed by Thomson. He thus set up his now famous gold foil experiment.

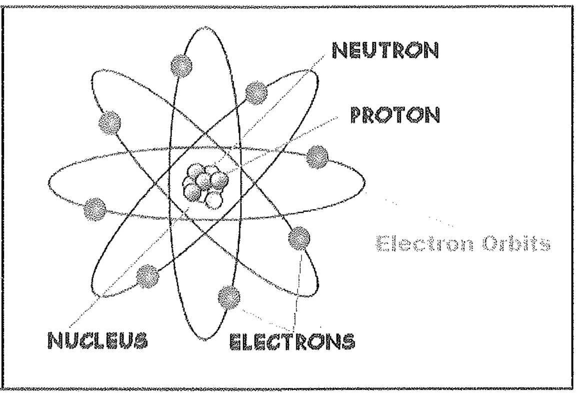
Rutherford studied α-particles, which are dense positively charged particles. He directed a stream of α-particles towards a very thin gold foil. He found that the majority of the α-particles cross the foil without deviation, and few of them either deviated from the original path or rebounded back. He proposed that the mass of the atom is concentrated in a small space at the center of the atom (nucleus), and that the electrons revolve around it.

Rutherford's atomic model (the planetary model)

- The atom consisted of a small dense core of positively charged particles at the center (nucleus).
- The nucleus is surrounded by a swirling ring of electrons.

- A problem raised: Why are the negatively charged particles not attracted by the positively charged nucleus (why electrons don't collapse into the nucleus)?
- Rutherford stated that the atom was like a mini solar system and that the electrons orbited the nucleus in a wide orbit. That is why it is known as the planetary model.

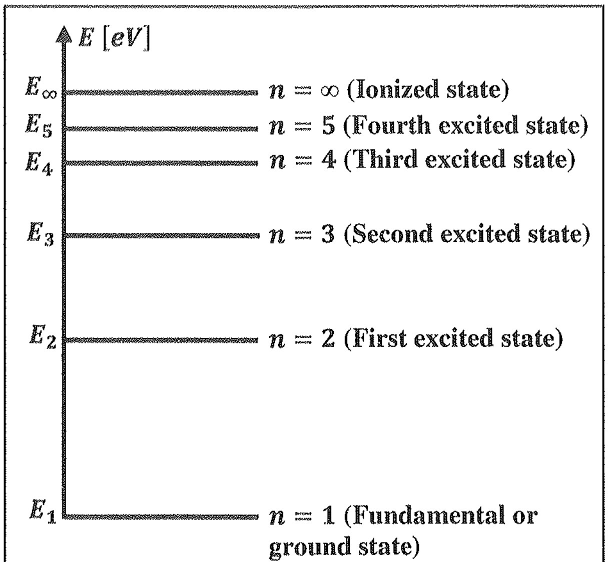




Bohr's model:

Niels Bohr agreed with the planetary model of the atom, but also knew that it had a few flaws. Using his knowledge of energy and quantum physics, he was able to perfect Rutherford's model. He was able to answer why the electrons did not collapse into the nucleus and explained the discrete line spectrum of the atom. According to Rutherford model, an electron revolves around the nucleus leading to centripetal acceleration. Accelerating charges radiate energy where electrons lose their energy continuously in the form of electromagnetic radiations, and is captured by the nucleus. This is not the case.

Niels Bohr was a Danish scientist that worked with Rutherford. "Bohr Theory of the Atom" was the closest to the modern atomic theory, and it reemphasized the idea of electrons circling the nucleus. Bohr suggested that electrons orbited around the nucleus in seven different quantum levels, or shells. The evidence that Bohr used to build this theory was the measurement of the line spectrum emitted by hydrogen gas. Bohr determined that different energy levels could be found by using mathematical formulas, which measured the wavelength of the different energy levels. Bohr went on to suggest that electrons would only occupy the lowest possible energy level on the respective level they were on. Furthermore, electrons would only move up a level (increasing energy) if the lower levels were full. Bohr's model was not entirely correct but it would lead to Schrodinger's idea of the modern atomic model.



Energy levels of an atom

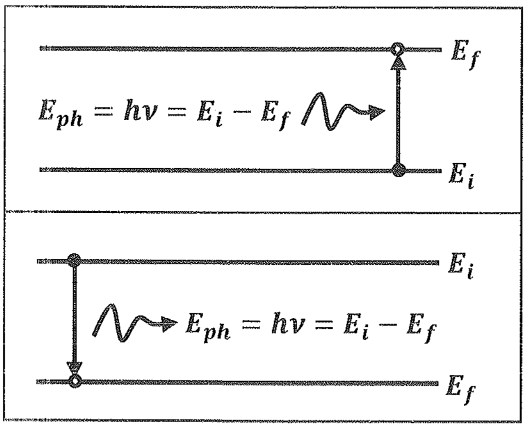
Bohr atomic model was developed after the investigation of the discrete emission spectrum of the hydrogen atom together with Planck's hypothesis about the quantization of electromagnetic energy.

Bohr's atomic model postulates

- Each atom has its own set of allowed energy levels.
- Each level has a quantum number n and energy E_n where $n = 1, 2, 3, ...$
  The level with $n = 1$ and energy E_n has the lowest energy and called the fundamental or ground state.

The level with $n = \infty$ and energy E_∞ has the highest energy and called the ionized state.
A level of quantum number n and energy E_n where $1 < n < \infty$ is called the $(n-1)^{th}$ excited state.

- The energy of the atom is **quantized** i.e. the atom can have energy equal to one of these levels and any intermediate energy between levels is strictly forbidden.
- All isolated atoms of the same element have identical sets of energy levels, while atoms of different elements have different sets of energy levels.
- **Upwards transition (atom excitation):** when an atom passes from one energy level E_i to a higher one E_f, it absorbs a photon of frequency depending on the equivalent energy "jump".
  The energy of the emitted photon is:
  $E_{ph} = h\nu = E_f - E_i$ where $E_f > E_i$
  In vacuum, $\nu = \frac{c}{\lambda} \Rightarrow E_{ph} = h\nu = \frac{hc}{\lambda} = E_f - E_i$.



- **Downwards transition (atom de-excitation):** when an atom passes from one energy level E_i to a lower one E_f, it emits photons whose frequency depends on the equivalent "jump".
  The energy of the emitted photon is:
  $E_{ph} = h\nu = E_i - E_f$ where $E_i > E_f$
  In vacuum, $\nu = \frac{c}{\lambda} \Rightarrow E_{ph} = h\nu = \frac{hc}{\lambda} = E_i - E_f$.

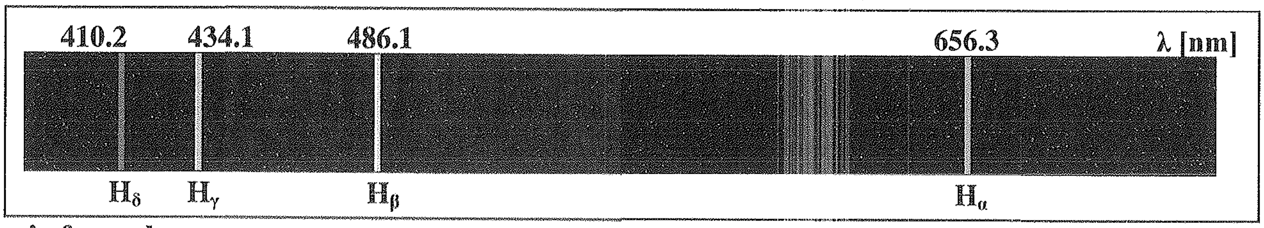
Bohr also suggested that, contrary to classical electromagnetic theory, no photons are emitted (i.e. no radiation of energy takes place) as long as the atom remains at the same level (i.e. as long as the electrons remain in their stable orbits).

Notes:

- The light is emitted or absorbed as a photon with energy: $E = E_{high} - E_{low}$.
- When an atom is excited, the atom doesn't remain in its excited state more than 10^{-8}s. The atom returns back to the ground state directly or by steps emitting thus photons.
- For an electron to cause a transition of an atom from an energy level E_P to a higher energy level E_n, its energy must be at least equal to the difference of the energies (E_n- E_P) of the atom.
- During one electron-atom collision, the atom absorbs, from the electron, an amount of energy enough to ensure a transition. The rest of the energy is carried by the electron as kinetic energy.

8.2- RYDBERG'S FORMULA AND ENERGY OF A HYDROGEN ATOM

The spectrum of hydrogen was studied extensively during the year 1913. Hydrogen gas in a discharge tube shows a spectrum of a series of lines as displaced in the document below. Starting from the visible line of longest wavelength (i.e. lowest frequency), the lines are called H_α, H_β, H_γ and H_δ.



Balmer's formula

At that time, Balmer proposed an empirical formula that calculated the four wavelengths.

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right)$$

λ is the wavelength of a specific line in the spectrum, n is a natural number which may take the values 3, 4,

5, ... and R = 1.097 × 10^{7} m^{-1} is a suitable constant (called Rydberg's constant) chosen to make the above relation fit the measured values of the wavelength.
For $n = 3$, the relation gives the wavelength of the H_α line: $\frac{1}{\lambda} = (1.097 \times 10^7)\left(\frac{1}{4} - \frac{1}{9}\right) \Longrightarrow \lambda = 656.3nm$.
For $n = 4$, we obtain the wavelength of the H_β line, and so on. For very large values of n ($n \to \infty$), we obtain the smallest wavelength $\lambda = 364.6nm$, which corresponds to H_∞.

Rydberg's formula
Johannes Rydberg generalized Balmer equation for all the radiations emitted by a hydrogen atom.
Using this formula, we can calculate the wavelength of the radiation emitted by an atom that undergoes a transition from energy level n to the energy level m.

$$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$$

m and n are the principal quantum numbers (whole numbers) of the levels where $n > m$.

If we now substitute the expression of λ given by Rydberg in that of the energy exchange, suggested by Bohr, we get:

$$E = \frac{hc}{\lambda} = hcR\left(\frac{1}{m^2} - \frac{1}{n^2}\right) = -\frac{hcR}{n^2} + \frac{hcR}{m^2}$$

Planck's constant h = 6.625 × 10^{-34}J.s; speed of light in vacuum c = 2.988 × 10^{8}m.s^{-1}.
This fits with $E = E_i - E_f$ if we choose: $E_i = -\frac{hcR}{n^2}$ and $E_f = -\frac{hcR}{m^2}$.
Note that the energies of the levels are negative because the potential energy of the electron-nucleus interaction is zero at infinity (i.e. just outside the atom).
The Balmer series, and others, suggest that the hydrogen atom has a series of discrete energy levels E_n defined by: $E_n = -\frac{hcR}{n^2}$
n is the principal quantum number ($n = 1, 2, 3, \ldots$).

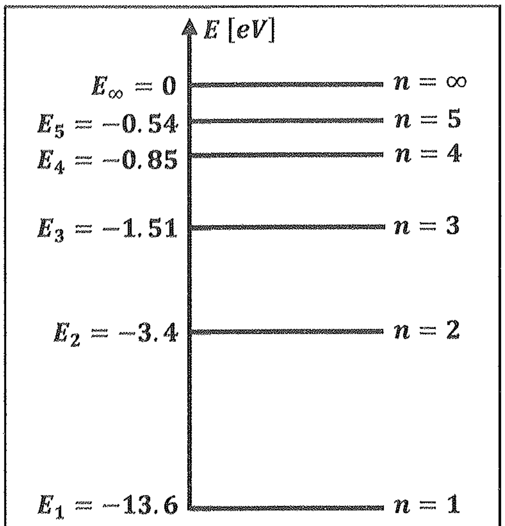
ATTENTION

For any hydrogen like element $E_n = -\frac{ZhcR}{n^2}$ where Z is the atomic number.

The numerical value of hcR is:
$hcR = (6.625 \times 10^{-34} J.s)(2.988 \times 10^8 m.s^{-1})(1.097 \times 10^7 m^{-1}) = 2.179 \times 10^{-18} J = -13.6eV$.
This energy is a very small quantity when expressed in joules. A much smaller unit of energy is the electron-volt [eV] which is the energy needed to accelerate an electron of charge $-e = -1.602 \times 10^{-19} C$ under a potential difference of 1 volt. Therefore, $1eV = 1.602 \times 10^{-19} J$.
This means that the energy of level n of the hydrogen atom is:

$$E_n = -\frac{hcR}{n^2} = -\frac{13.6eV}{n^2}$$

For $n = 1$; $E_1 = \frac{-13.6}{1^2} = -13.6eV$ (fundamental or ground state)
For $n = 2$; $E_2 = \frac{-13.6}{2^2} = -3.4eV$ (first excited state)
For $n = 3$; $E_3 = \frac{-13.6}{3^2} = -1.51eV$ (second excited state)
For $n \to \infty$; $E_\infty = 0$ (ionized state)



Energy levels of a hydrogen atom

Ionizing an atom means giving it energy to extract an electron.
The ionization energy is the energy needed for the atom to absorb for it to release its electron without speed.
The ionization energy W_i of an atom found in the ground state is the minimum energy absorbed by the atom in order to release an electron with zero speed.

$$W_i = E_\infty - E_1$$

If a photon is incident on the atom with energy $E_{ph} \geq W_i$, electron is liberated from the atom with kinetic energy: $K.E = E_{ph} - W_i$.

8.3- SPECTRAL SERIES OF THE HYDROGEN ATOM

Lyman series corresponds to a downwards transition from an energy level of n = 2 or higher, to the energy level n = 1.

$$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n^2}\right) \text{ with } n = 2, 3, 4 \ldots$$

The emitted radiations belong to the ultraviolet domain.

Balmer series corresponds to a downwards transition from an energy level of n = 3 or higher, to the energy level n = 2.

$$\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) \text{ with } n = 2, 3, 4 \ldots$$

The only four visible lines are:

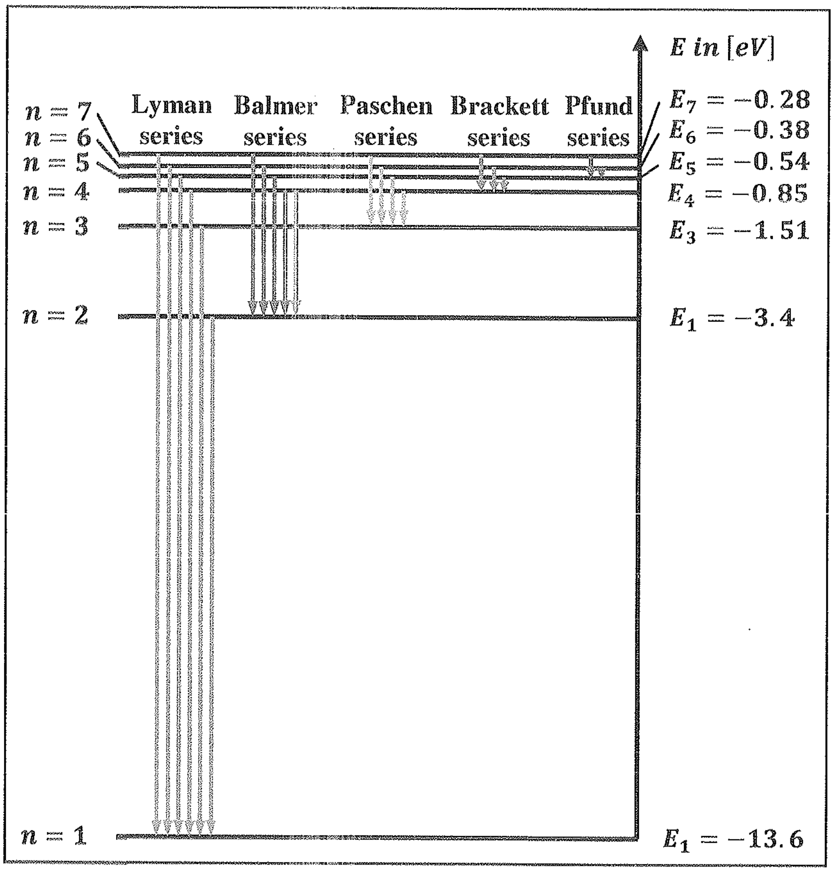
H_α of wave length $\lambda_\alpha = 656.3$nm corresponds to the transition from n = 3 to n = 2).

H_β of wavelength $\lambda_\beta = 486.1$nm corresponds to the transition from n = 4 to n = 2.

H_γ of wavelength $\lambda_\gamma = 434.1$nm corresponds to the transition from n = 5 to n = 2.

H_δ of wavelength $\lambda_\delta = 410.2$nm corresponds to the transition from n = 6 to n = 2.

The other emitted radiations belong to the ultraviolet domain.



Paschen series corresponds to a transition for energy levels corresponds to n = 4 or higher, to the energy level n = 3.

$$\frac{1}{\lambda} = R\left(\frac{1}{3^2} - \frac{1}{n^2}\right) \text{ with } n = 4, 5, 6 \ldots$$

The emitted radiations belong to the infrared domain.

Brackett series corresponds to a downwards transition from an energy level of n = 5 or higher, to the energy level n = 4.

$$\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n^2}\right) \text{ with } n = 5, 6, 7 \ldots$$

The emitted radiations belong to the infrared domain.

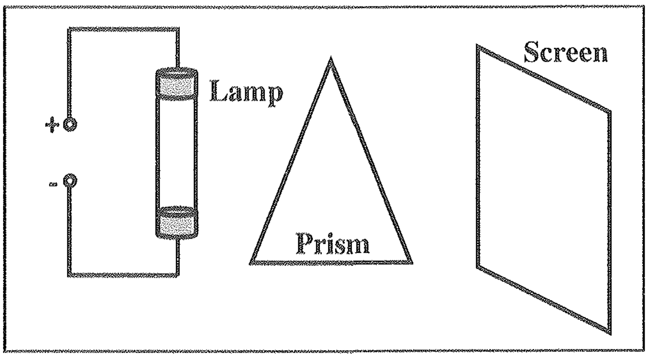
Pfund series corresponds to a downwards transition from an energy level of n = 6 or higher, to the energy level n = 5.

$$\frac{1}{\lambda} = R\left(\frac{1}{5^2} - \frac{1}{n^2}\right) \text{ with } n = 6, 7, 8 \ldots$$

The emitted radiations belong to the infrared domain.

8.4- EMISSION AND ABSORPTION SPECTRA

Discrete emission spectrum

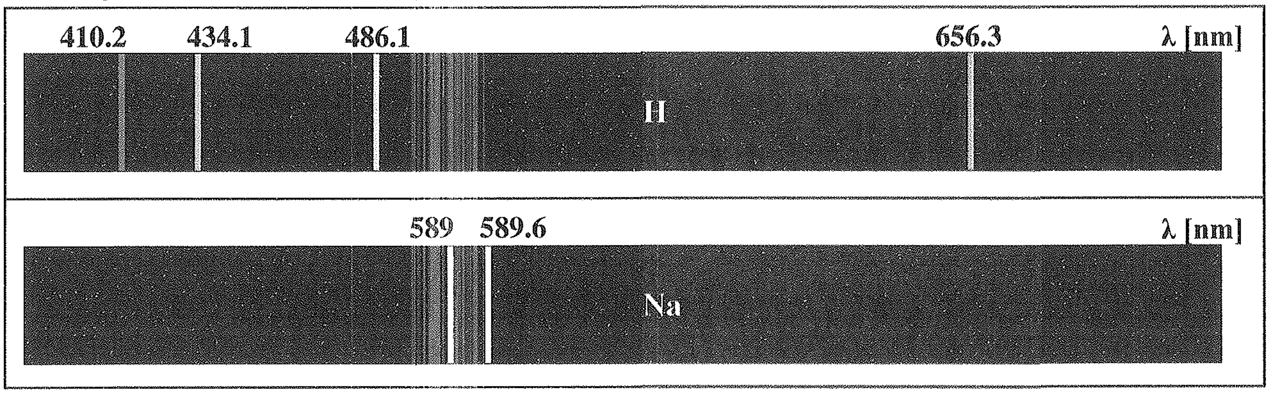


Equipment

- Hydrogen and sodium vapor lamps.
- Spectroscope (prism).
- High voltage DC source (few kV).
- Observation screen.

Procedure an observation

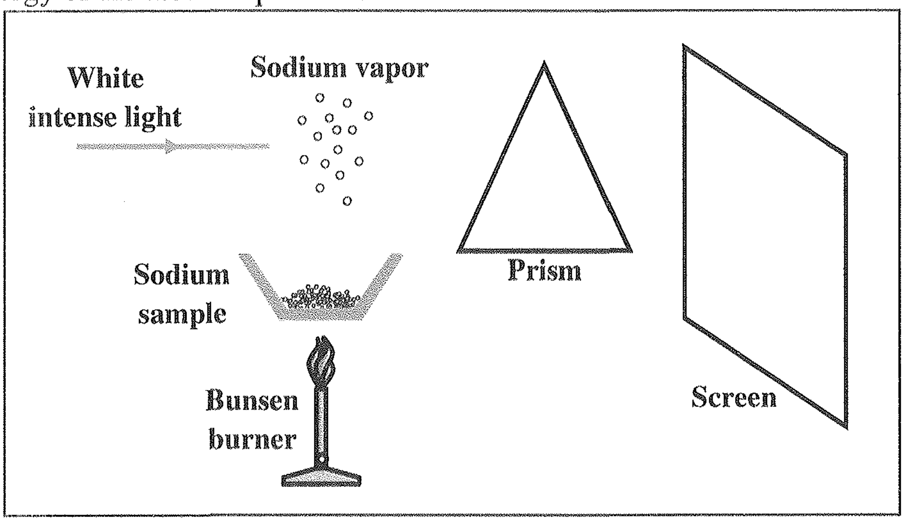
- Hydrogen gas is kept inside the lamp at low pressure.
- The lamp glows when a high voltage is applied to it.
- Direct the spectroscope towards the light issued by the lamp in a dark room.
- You observe the emission spectrum of hydrogen, which consists of four discrete lines.
- If the sodium lamp is used, the emission spectrum obtained consists of two very close yellow lines of wavelengths $\lambda_1 = 589nm$ and $\lambda_2 = 586.6nm$ known as the D-doublet of sodium.



Conclusion

- The spectra obtained by sodium and hydrogen lamps are called the emission spectra. The discrete emission spectrum is a set of discrete bright lines against a dark background.
- Each bright line in the emission spectrum corresponds to wavelength of a photon emitted when the atom is de-excited from a higher energy level to a lower one.
- The spectrum is discrete since the energy of the atom is quantized.

Discrete absorption spectrum



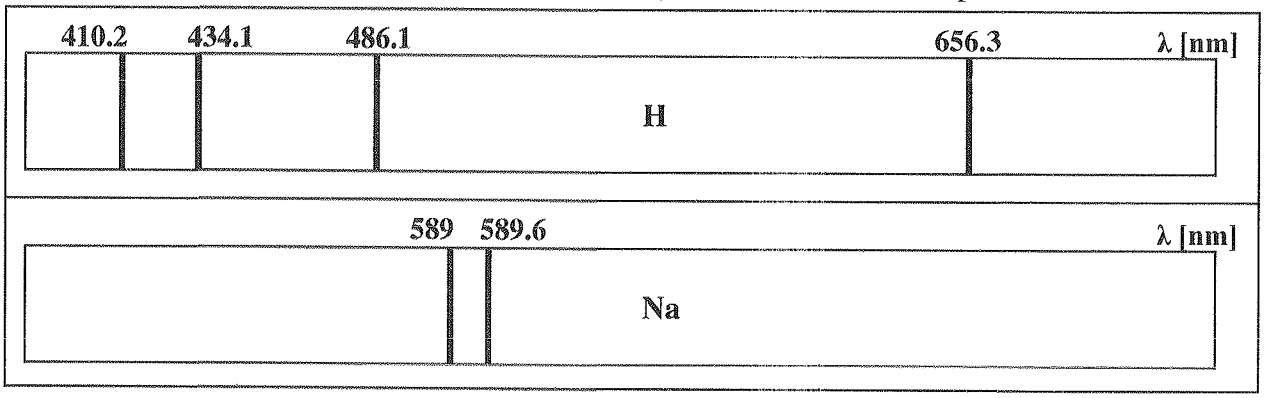
Equipment

- Crucible.
- Bunsen burner.
- Solid sodium sample.
- A spectroscope.
- An observation screen.

Procedure and observation

- Put the sodium sample in the crucible.
- Heat the sample using the Bunsen burner.
- Sodium vapor forms above the crucible.

- Direct intense white light towards the vapor.
- Direct the spectroscope towards the illuminated sodium vapor.
- You observe two very close black lines against a continuously illuminated background. These two lines are missing from the continuous spectrum of white light.
- The same experiment could be carried out with hydrogen gas contained in a chamber and illuminated with an intense light. We then observe four black lines missing from the continuous spectrum.



Conclusion

- The black lines (missing lines), observed in each case, are called the absorbed lines and they form the absorption spectrum of each element.
- The frequencies of the missing lines in the absorption spectrum of a given element are equal to the frequencies of the corresponding lines in the emission spectrum of the same element.
- The spectrum is a set of discrete dark lines on a bright band of colors.
- Each dark line corresponds to a wavelength of a photon absorbed by the gas atom as it is excited from a lower energy level to a higher one.

Application

The absorption spectrum of an element has many practical applications. The continuous spectrum of radiation emitted by the sun must pass through the coolor gases of the solar atmosphere, and throgh the Earth's atmosphere. The various absorption lines obserevd in the solar spectrum allow identifying elements in the solar atmosphere.
In the early studies of the solar spectrum, scientists found some lines which did not correspond to those of any element. This element was named helium (helios is a Greek word for the Sun).
Using this technique, scientists where also able to analyze the spectra of stars other than the Sun, and know their constituents.

CHAPTER 8 – THE ATOM
EXERCISES AND PROBLEMS

Exercise 1*:

Ernest Rutherford and Niles Bohr achieved major contributions in the study of the structure of the atom.

1- Outline Rutherford's work and state the failures of his model.
2- Explain Bohr's model.

Exercise 2*:

The electron in a hydrogen atom makes a transition from the energy level at -13.6eV to the level at -0.38eV when a photon is absorbed. Calculate the wavelength of the absorbed photon.

Take: $h = 6.62 \times 10^{-34}$ J.s; $c = 3 \times 10^{8}$ m/s; 1eV = 1.6×10^{-19} J.

Exercise 3:

The energy levels of the hydrogen atom are given by: $E_n = \frac{-13.6eV}{n^2}$; where n is a non-zero positive integer.

A hydrogen atom, initially in the energy level of n = 7, absorbs a photon of frequency 2×10^{14} Hz. The atom is ionized and the liberated electron has a kinetic energy of 0.551eV.

Given: speed of light in air is $c = 3 \times 10^{8}$ m/s; 1eV = 1.6×10^{-19} J.

1- Calculate the energy of the photon.
2- Calculate the value of Planck's constant.

Exercise 4:

The ionization energy of the hydrogen atom, in the ground state, is equal to 13.6eV

A hydrogen atom is found in an excited state of energy –1.51eV.

Calculate the wavelength of the radiation emitted by the de-excitation of the atom as it returns to the ground state. Take: $h = 6.62 \times 10^{-34}$ J.s; $c = 3 \times 10^{8}$ m/s; 1eV = 1.6×10^{-19} J.

Exercise 5*:

Hydrogen is used in various industrial applications; these include metalworking (primarily in metal alloying), flat glass production (hydrogen used as protective gas), the electronics industry (used as a protective and carrier gas, in deposition processes, for cleaning, in etching, in reduction ...

The energy levels of the hydrogen atom are given by: $E_n = \frac{-13.6eV}{n^2}$; where n is a non-zero positive integer.

Given: $h = 6.62 \times 10^{-34}$Js; $c = 3 \times 10^{8}$m/s; 1eV = 1.6×10^{-19}J.

1- For the Balmer series, calculate the longest and shortest wavelength possible.
2- Is any of the frequency of Lyman series in the visible region? Show your work.
3- A Hydrogen atom initially in its ground state absorbs a photon and ends up in n = 4 level.
   3.1- Calculate the frequency of the photon.
   3.2- The electron makes spontaneous emission and comes back to the ground state.
   Calculate the possible frequencies of the photons emitted during this process. Identify the part of the electromagnetic spectrum they belong to.

Exercise 6:

The sodium atom, being in the ground state of energy -5.14eV, is hit successively by the electrons (x) and (y) of respective kinetic energies 1.01eV and 3.5eV. The first, second and third excited states are of respective energies -3.03, -1.93 and -1.51eV.

1- Determine the electron that can interact with the sodium atom.
2- Specify the state of the sodium atom after each impact.
3- Deduce, after impact, the kinetic energy of the electron that interacts with the sodium atom.

Exercise 7*:

An electron in an atom can occupy four energy levels.

1- Draw a rough sketch to represent the energy levels. Show on the diagram all possible transitions between the energy levels.

2- Draw a rough diagram to represent the spectral emission lines (in terms of frequency) that correspond to the maximum number of transitions between these energy levels. Explain the position of the spectral lines.

Exercise 8:

The energy levels of the hydrogen atom are given by the relation:

$E_n = -\frac{E_0}{n^2}$, with $E_0 = 13.6$eV and n is whole non zero and positive number.

Given: h = 6.62×10^{-34}J.s; c = 3×10^8 m/s; 1eV = 1.6×10^{-19} J; 400 nm $\leq \lambda_{visible} \leq$ 800 nm.

1- The downward transition of a hydrogen atom from an excited level (n > 1) to the fundamental state gives the Lyman series. Calculate the longest wavelength of the radiations corresponding to this series.

2- The downward transition of a hydrogen atom from an excited level (n > 2) to the first excited level gives the Balmer series. Calculate the longest wavelength of the radiations corresponding to this series.

3- Deduce, with justification, the series containing visible radiations.

Exercise 9*:

The energy levels of the hydrogen atom are given by the relation: $E_n = -\frac{E_0}{n^2}$, with $E_0 = 13.6$eV and n is whole non zero and positive number.

Given: h = 6.62×10^{-34}J.s; c = 3×10^8 m/s; 1eV = 1.6×10^{-19} J; 400nm $\leq \lambda_{visible} \leq$ 800nm.

1-

1.1- Calculate the energy of the hydrogen atom when it is:

1.1.1- in the fundamental state;

1.1.2- in the first excited state;

1.1.3- in the ionized state.

1.2- The energy levels of the hydrogen atom are quantized. Justify

2- This atom, taken in a given energy level E_p, receives a photon of energy E and of wavelength λ in vacuum. Thus, the hydrogen atom passes to an energy level E_m such that m > p.

2.1- Write the relation among E, E_p and E_m.

2.2- Deduce the relation among E_0, p, m, h, c and λ.

Exercise 10*:

A lithium atom is an atom of the chemical element lithium. Lithium is composed of three electrons. Lithium atoms emit red light of wavelength 671 nm. The values the energies of the energy levels E_n and the quantum number "n" of the corresponding electronic states are tabulated below:

n	1	2	3	4
E_n [eV]	-5.4	-3.548	-2.017	-1.508

Take: h = 6.62×10^{-34} J.s; c = 3×10^8 m/s; 1eV = 1.6×10^{-19} J.

1- Explain the procedure that allows an atom to emit photons in the visible spectrum.

2- Calculate the energy of the emitted photons of red radiation.

3- Specify the difference between the energy levels of the lithium atom that are responsible for this emission.

4- Specify the value of the energy of the ground state. What is the name given to the other electronic states?

5- Calculate the energy transition between the energy levels of n = 3 and n = 1. Specify if the emitted photon belong to the visible spectrum.

6- The lithium atom is in the ground state. Calculate the ionization energy of the atom.

7- Give one application for emission of photons in the visible spectrum.

Exercise 11:

The object of this exercise is to determine the visible emission spectrum of a mercury vapor lamp.

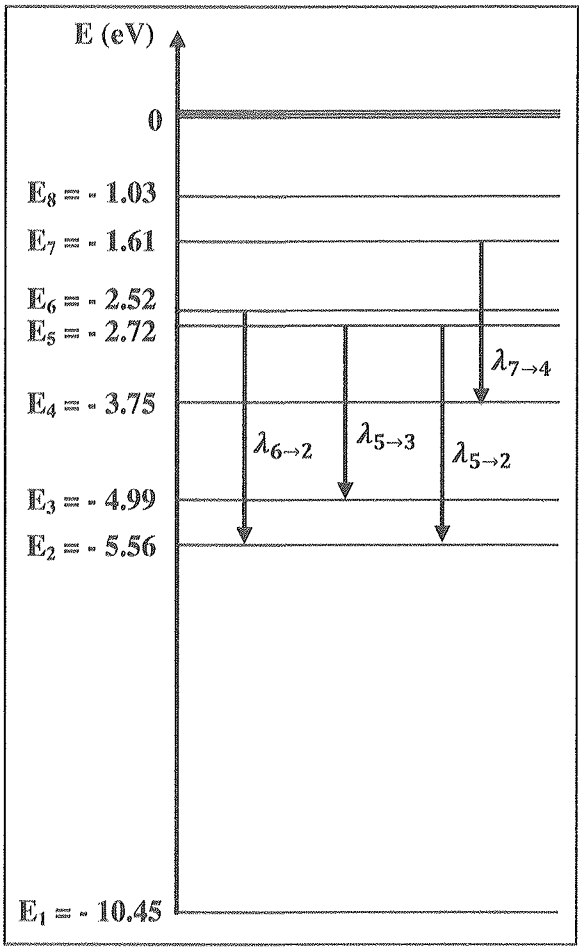
The adjacent diagram gives, in a simplified way, the energy level of the ground state, those of the excited states E_2, E_3, E_4, E_5, E_6, E_7, E_8 and the ionization energy level E = 0 of the mercury atom.

Given:

Planck's constant $h = 6.62 \times 10^{-34}$J.s;

Speed of light in vacuum: $c = 3 \times 10^8$m/s;

1eV= 1.6×10^{-19}J;



I- Quantization of the energy of the atom

1- The energy of the mercury atom is quantized. What is meant by "quantized energy"?

2.1- What is meant by « ionizing » an atom?

2.2- Calculate, in eV, the ionization energy of a mercury atom taken in the ground state.

2- Interaction photon-atom.

A photon cannot cause the transition of an atom from an energy level E_P to a higher energy level E_n unless its energy is exactly the same as the difference of the energies (E_n - E_P) of the atom.

The mercury atom is being in the ground state.

3.1- Determine the maximum wavelength of the wave associated to a photon capable of exciting this atom.

3.2- The mercury atom is hit with a photon of wavelength $\lambda_1 = 2.062 \times 10^{-7}$m.

3.2.1- Show that this photon cannot be absorbed.

3.2.2- What is then the state of this atom?

3.3- The atom receives now a photon of wavelength λ_2. The atom is thus ionized and the extracted electron is at rest. Calculate λ_2.

II- Emission by a mercury vapor lamp

For an electron to cause a transition of an atom from an energy level E_P to a higher energy level E_n, its energy must be at least equal to the difference of the energies (E_n- E_P) of the atom.

During one electron-atom collision, the atom absorbs, from the electron, an amount of energy enough to ensure a transition. The rest of the energy is carried by the electron as kinetic energy.

When the mercury vapor lamp is under a convenient voltage, an electric discharge takes place. Some electrons, each of kinetic energy 9eV, moving in the vapor of mercury between the electrodes of the lamp, hit the gaseous atoms giving them energy. For that lamp, the atoms are initially in the ground state.

1- Verify that an atom may not overpass the energy level E_7.

2- The visible emission spectrum due to the downward transition of the mercury atom, is formed of four rays of wavelengths: $\lambda_{7\to4}$; $\lambda_{6\to2}$; $\lambda_{5\to2}$; $\lambda_{5\to3}$ (refer to the diagram).

Determine the wavelengths of the limits of the visible spectrum of the mercury vapor lamp.

Exercise 12:

The energies of the different energy levels of the hydrogen atom are given by the relation:

$$E_n = -\frac{13.6}{n^2}\ (in\ eV) \qquad \text{where } n \text{ is a positive whole number.}$$

Given:

Planck's constant: $h = 6.63 \times 10^{-34}$ J.s; $1eV = 1.60 \times 10^{-19}$ J;
Speed of light in vacuum: $c = 3 \times 10^{8}$ m/s; $1nm = 10^{-9}$ m.

A- Energy of the hydrogen atom

1- The energies of the atom are quantized. Justify this using the expression of E_n.
2- Determine the energy of the hydrogen atom when it is:
2.1- in the fundamental state,
2.2- in the second excited state.
3- Give the name of the state for which the energy of the atom is zero.

B- Spectrum of the hydrogen atom

1- Emission spectrum

The Balmer's series of the hydrogen atom is the set of the radiations corresponding to the downward transitions to the level of n = 2.
The values of the wavelengths in vacuum of the visible radiations of this series are:
411nm; 435nm; 487nm; 658nm

1.1- Specify, with justification, the wavelength λ_1 of the visible radiation carrying the greatest energy.
1.2- Determine the initial level of the transition giving the radiation of wavelength λ_1.
1.3- Deduce the three initial levels corresponding to the emission of the other visible radiations.

2- Absorption spectrum

A beam of Sunlight crosses a gas formed mainly of hydrogen. The study of the absorption spectrum reveals the presence of dark spectral lines.
Give, with justification, the number of these lines and their corresponding wavelengths.

C- Interaction photon - hydrogen atom

1- We send on the hydrogen atom, being in the fundamental state, separately, two photons of respective energies 3.4eV and 10.2eV.
Specify, with justification, the photon that is absorbed.
2- A hydrogen atom found in its fundamental state absorbs a photon of energy 14.6eV. The electron is thus ejected.
2.1- Justify the ejection of the electron.
2.2- Calculate, in eV, the kinetic energy of the ejected electron.

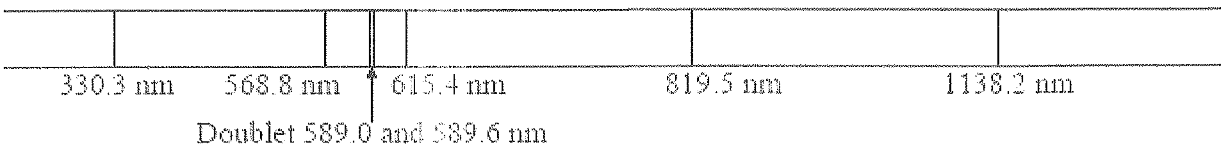
Exercise 13:

Sodium vapor lamps are used to illuminate roads. These lamps contain sodium vapor under very low pressure.
This vapor is excited by a beam of electrons that cross the tube containing the vapor. The electrons yield energy to the sodium atoms which give back this received energy during their downward transition towards the ground state in the form of electromagnetic radiations.

Given: $h = 6.62 \times 10^{-34}$ J.s; $c = 3 \times 10^{8}$ ms^{-1}; $e = 1.60 \times 10^{-19}$ C; 1 nm = 10^{-9} m.

1- What do each of the quantities h, c and e represent?

2- The analysis of the emission spectrum of a sodium vapor lamp shows the presence of lines of well-determined wavelengths λ. The figure below represents some of the lines of this spectrum.



2.1- The yellow doublet of wavelengths, in vacuum, $\lambda_1 = 589.0$nm and $\lambda_2 = 589.6$nm is more intense than the other lines.

2.1.1- To what range: visible, infrared or ultraviolet, does each of the other lines of the spectrum belong?

2.1.2- The sodium vapor lamps are characterized by the emission of yellow light. Why?

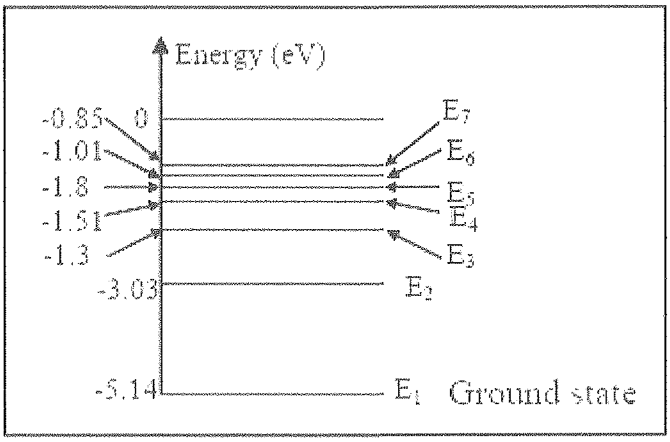
2.2- Is the visible light emitted by the sodium lamp monochromatic or polychromatic? Justify your answer.

3-

3.1- Referring to the diagram of the energy levels of the sodium atom in the adjacent figure:

3.1.1- Specify an indicator that justifies the discontinuity of the emission spectrum of the sodium vapor lamp.

3.1.2- Verify that the emission of the line of wavelength λ_1 corresponds to the downward transition from the energy level E_2 to the ground state.



3.2- In fact, the energy level E_2 is double, i.e., it is constituted of two energy levels that are very close to each other. Draw a diagram that shows the preceding downward transition as well as the downward transition corresponding to the emission of the radiation of wavelength λ_2.

4- The sodium atom, being in the ground state, is hit successively by the electrons (a) and (b) of respective kinetic energies 1.01eV and 3.03eV.

4.1- Determine the electron that can interact with the sodium atom.

4.2- Specify the state of the sodium atom after each impact.

4.3- Deduce, after impact, the kinetic energy of the electron that interacts with the sodium atom.

Exercise 14:

Rydberg found in 1885 an empirical formula that gives the wavelengths of the lines of Balmer series; other series are discovered after that date.

An atom in an excited state n, passes to a lower energy state m, emits electromagnetic rays of wavelength λ, such that: $\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$, λ in meter and $R = 1.097 \times 10^7 \text{m}^{-1}$.

Given: Speed of light in vacuum $c = 2.998 \times 10^8$m/s;

Planck's constant $h = 6.626 \times 10^{-34}$J.s;

$1\text{ev} = 1.60 \times 10^{-19}$J.

1- Show that the energy E_n of the hydrogen atom, corresponding to an energy level n, can be expressed as $E_n = -\frac{hcR}{n^2}$.

2- Deduce that the energy E_n, expressed in eV, may be written in the form $E_n = -\frac{13.6}{n^2}$.

3- Calculate the value of the:

3.1- maximum energy of the hydrogen atom;

3.2- minimum energy of the hydrogen atom;

3.3- energy of the hydrogen atom in the first excited state E_2;

3.4- energy of the atom in the second excited state E_3.

4- Deduce that the energy of the atom is quantized.

5- Give three characteristics of a photon.

6-

6.1- Define the ionization energy W_i of the hydrogen atom, found in the ground state.

6.2- Calculate the value of W_i.

6.3- Calculate the value of the wavelength of the radiation capable of producing this ionization.

7- The Lyman series corresponds to the lines emitted by the excited hydrogen atom in a downward transition to the fundamental state.

7.1- Determine the shortest and the longest wavelengths of this series.

7.2- To what domain (visible, infrared, ultraviolet) does it belong?

8-

8.1- Calculate the frequencies $\nu_{3\to1}, \nu_{2\to1}$ and $\nu_{3\to2}$ of the emitted photons corresponding respectively to the transitions $E_{3\to1}, E_{2\to1}$, and $E_{3\to2}$ of the hydrogen atom.

8.2- Verify Ritz relation: $\nu_{3\to1} = \nu_{3\to2} + \nu_{2\to1}$.

Exercise 15:

Document 1 represents some of the energy levels of the sodium atom.

Given: $h = 6.6 \times 10^{-34}$J.s; $c = 3 \times 10^8$m/s;
$1eV = 1.6 \times 10^{-19}$J; $1u = 931.5MeV/c^2$.

The aim of this exercise is to study the excitation and the de-excitation of the sodium atom.

1- Excitation of the sodium atom

Consider a sample of sodium atoms, initially in the ground state.

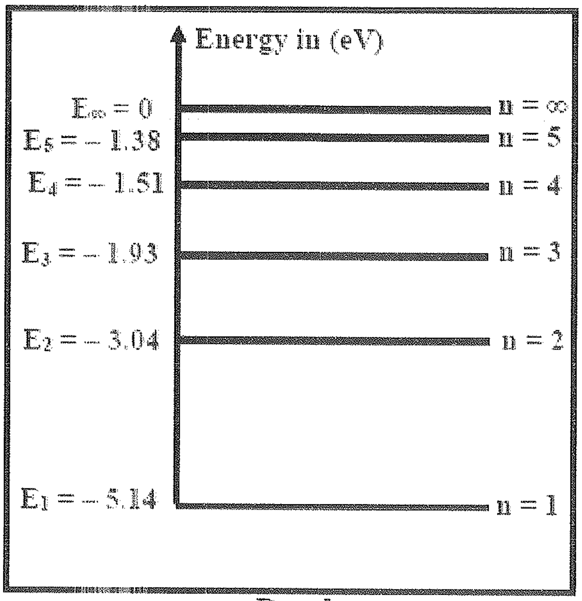
This sample is illuminated by white light that contains all the visible radiations: $0.4\mu m \le \lambda_{visible} \le 0.8\mu m$.

1.1- Using document 1, show that the energy of the sodium atom is quantized.

1.2- Determine, in eV, the maximum energy and the minimum energy of the photons in the white light.

1.3- Using document 1, show that white light is not capable to ionize the sodium atom.

1.4- Determine, in nm, the wavelength of the photon that excites the sodium atom to the first excited state.



Doc. 1

2- De-excitation of the sodium atom

The emission spectrum, obtained from the low-pressure sodium vapor lamp, contains two very close yellow lines of wavelengths $\lambda_1 = 589.0$nm and $\lambda_2 = 589.6$nm, called the D-doublet of sodium.

2.1- The sodium atom de-excites from the energy level E_n to the ground state and emits the photon of wavelength $\lambda_1 = 589.0$nm. Specify the value of E_n in eV.

2.2- The sodium atom undergoes a transition from the energy level E_3 to the energy level E_1. During this transition it loses energy $E_{3\to1}$ and its mass decreases by Δm.

2.2.1- Calculate, in MeV, the value of $E_{3\to1}$.

2.2.2- Deduce, in u, the value of Δm.

2.1- The power of the radiations of wavelengths λ_1 and λ_2 emitted by the sodium vapor lamp is $P = 6W$. The power P_1 of the radiation of wavelengths λ_1 is twice the power P_2 of the radiation of wavelengths λ_2.

2.3.1- Show that $P_1 = 4W$.

2.3.2- Determine the number of photons of the radiation of wavelength λ_1 emitted from the sodium vapor lamp in one second.

CHAPTER 8 – THE ATOM
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 3:

Part	Answer
1	$E_7 = \frac{-13.6}{7^2} = 0.277eV.$ $E_\infty = 0eV.$ $E_p = (E_\infty - E_7) + K.E = (0 + 0.277) + 0.551 = 0.828eV.$
2	$E_p = h\nu \Longrightarrow h = \frac{E_p}{\nu} = \frac{0.828\times1.6\times10^{-19}}{2\times10^{14}} = 6.624 \times 10^{-34}Js.$

Exercise 4:

Part	Answer
1	$E_{ion} = E_\infty - E_1 \Longrightarrow 13.6 = 0 - E_1 \Longrightarrow E_1 = -13.6eV.$ $E_{photon} = E_{high} - E_{low}.$ $\frac{hc}{\lambda} = E_{high} - E_{low} \Longrightarrow \lambda = \frac{hc}{E_{high}-E_{low}} = \frac{6.62\times10^{-34}\times3\times10^8}{(-1.51+13.6)\times1.6\times10^{-19}} = 1.026 \times 10^{-7}m.$

Exercise 6:

Part	Answer
1	$E = -5.14 + 1.01 = -1.13eV < -3.03eV$ (Electron x can't interact with the atom). $E = -5.14 + 3.5 = -1.64eV > -1.93eV$ (Electron y can interact with the atom).
2	After hitting the atom with electron x, it remains in the ground state. After hitting the atom with electron y, it may undergo transition to one of the levels -3.03eV or -1.93eV.
3	$K.E_1 = 3.5 - (-3.03 + 5.14) = 1.39eV.$ $K.E_2 = 3.5 - (-1.93 + 5.14) = 0.29eV.$

Exercise 8:

Part	Answer
1	Lyman series $\Longrightarrow n_f = 1 \Longrightarrow E_f = E_1 = -\frac{13.6}{1^2} = -13.6eV.$ $E_p = E_i - E_f \Longrightarrow \frac{hc}{\lambda} = E_i - E_1 \Longrightarrow \lambda = \frac{hc}{E_i-E_1}.$ $\lambda_{max} = \frac{hc}{(E_i-E_1)_{min}} = \frac{hc}{E_2-E_1}$ with $E_2 = -\frac{13.6}{2^2} = -3.4eV.$ $\lambda_{max} = \frac{6.62\times10^{-34}\times3\times10^8}{(-3.4+13.6)\times1.6\times10^{-19}} = 1.217 \times 10^{-7}m = 121.7nm.$
2	Balmer series $\Longrightarrow n_f = 2 \Longrightarrow E_f = E_2 = -3.4eV.$ $E_p = E_i - E_f \Longrightarrow \frac{hc}{\lambda} = E_i - E_2 \Longrightarrow \lambda = \frac{hc}{E_i-E_2}.$ $\lambda_{max} = \frac{hc}{(E_i-E_2)_{min}} = \frac{hc}{E_3-E_2}$ with $E_3 = -\frac{13.6}{3^2} = -1.51\ eV.$ $\lambda_{max} = \frac{6.62\times10^{-34}\times3\times10^8}{(-1.51+3.4)\times1.6\times10^{-19}} = 6.567 \times 10^{-7}m = 656.7nm.$
3	Balmer series, since $400nm \leq \lambda_{max} \leq 800nm$

Exercise 11:

Part	Answer key
I.1	Only specific values of energy are allowed.
I.2.1	Giving the atom energy to extract an electron.
I.2.2	E (ionization) = $E - E_1 = 0 - (-10.45) = 10.45$eV.

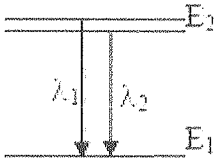
I.3.1	$\lambda = \frac{hc}{E_n - E_1}$. λ_{max} corresponds to $(E_n)_{min} \Leftrightarrow n = 2 \Rightarrow \lambda_{max} = 2.54 \times 10^{-7} m$.
I.3.2.1	For λ_1, the energy of the photon is: $E = \frac{hc}{\lambda} = 9.63 \times 10^{-19} J = 6.02 eV$; The energy level of the atom must be $E_1 + E = -4.43$eV; but this level does not exist in the energy diagram, so the photon is not absorbed.
I.3.2.2	The atom remains in the ground state.
I.3.3	$W = 10.45 eV \Rightarrow \lambda_2 = 1.188 \times 10^{-7} m$.
II.1	$E_1 + 9 = (-10.45) + 9 = -1.45 eV < E_8$.
II.2	$(\Delta E)_{6\to2} = 3.04 eV$; $(\Delta E)_{5\to3} = 2.27 eV$; $(\Delta E)_{7\to4} = 2.14 eV$. $(\Delta E)_{5\to2} = 2.84 eV$. $(\Delta E)_{max} = 3.04 eV$ and $(\Delta E)_{min} = 2.14 eV$. $\lambda = \frac{hc}{\Delta E}$. $\lambda_{min} \Rightarrow (\Delta E)_{max} = E_6 - E_2 \Rightarrow \lambda_{6\to2} = 408.3 nm$. $\lambda_{max} \Rightarrow (\Delta E)_{min} = E_7 - E_4 \Rightarrow \lambda_{7\to4} = 580.0 nm$.

Exercise 12:

Part	Answer key
A.1	$E_1 = -13.6$eV; $E_2 = -3.4$eV; $E_3 = -1.51$eV; $E_\infty = 0$. $\Rightarrow$ The values of energies are discontinuous.
A.2.1	$E_{fund.}$ corresponding to $n = 1 \Rightarrow E_{fund.} = -13.6 eV$.
A.2.2	Second excited state corresponding to $n = 3 \Rightarrow E_3 = -1.51 eV$.
A.3	Ionize state.
B.1.1	$E = \frac{hc}{\lambda}$ or E is inversely prop. to λ. $\Rightarrow \lambda_1 = 411 nm$.
B.1.2	$\frac{hc}{\lambda} = E_i - E_f \Rightarrow \frac{hc}{\lambda} = \left(-\frac{13.6}{n^2} + \frac{13.6}{4}\right) \times 1.6 \times 10^{-19} J$; For $\lambda = \lambda_1; n = 6$.
B.1.3	The other three levels are: n = 5; n = 4; n = 3 to n = 2.
B.2	The dark lines of the absorption spectrum corresponding to the bright lines of same wavelength of the emission spectrum. We have 4 bright lines $\Rightarrow$ we have 4 dark lines of wavelengths: 411nm; 487nm; 658nm
C.1	$-13.6 eV + 3.4 eV = -10.2 eV = -\frac{13.6}{n^2} \Rightarrow n = 1.15$; n is not a whole number $\Rightarrow$ not absorbed. $-13.6 eV + 10.2 eV = -3.4 = -\frac{13.6}{n^2} \Rightarrow n = 2$ (whole no) $\Rightarrow$ absorbed.
C.2.1	The energy of the photon is greater than the ionization energy.
C.2.2	$K.E = -13.6 eV + 14.6 eV = 1 eV$.

Exercise 13:

Part	Answer key
1	h: Planck's constant; c: speed of light in vacuum, e: elementary charge.
2.1.1	330.3nm ultraviolet domain; 568.8nm, 589nm and 615.4nm visible domain; 819.5nm and 1138.2nm infra-red domain.
2.1.2	Because this yellow light is much more intense than the others.
2.2	It is polychromatic because it is made of several radiations of different frequencies.

3.1.1	The discontinuity of the emission spectrum is justified by the discontinuous energy levels of the sodium atom.
3.1.2	$E = \frac{hc}{\lambda_1} = \frac{6.62\times10^{-34}\times3\times10^{8}}{589\times10^{-9}\times1.6\times10^{-19}} = 2.11eV.$ $E_1 + E = -5.14 + 2.11 = -3.03eV = E_2.$
3.2	
4.1	$1.01 + (-5.14) = -4.13eV$, this energy level does not exist $\Longrightarrow$ the electron (a) does not interact with the atom. $3.03 + (-5.14) = -2.11eV$, $-3.03eV < -2.11eV < -1.93eV$, $\Longrightarrow$ the electron (b) interacts with the atom.
4.2	In case of the electron (a) the atom remains in the ground state. In case of the electron (b) the atom attains level E_2.
4.3	For the electron (b), $K.E = 3.03 - (-3.03 + 5.14) = 0.92eV$.

Exercise 14:

Part	Answer key
1	$E_n - E_m = \frac{hc}{\lambda} = hcR\left(\frac{1}{m^2} - \frac{1}{n^2}\right) \Longrightarrow E_n = -\frac{hCR}{n^2}.$
2	$hcR = 6.626 \times 10^{-34} \times 3 \times 10^{8} \times 1.097 \times 10^{7} = 2.18 \times 10^{-18}J = 13.6eV.$ $\Longrightarrow E_n = -\frac{13.6}{n^2}.$
3.1	As $n \to \infty, E_{max} \to 0.$
3.2	As $n \to 1; E_{min} = -13.6eV.$
3.3	$E_2 = -\frac{13.6}{2^2} = -13.6eV.$
3.4	$E_3\ for\ n = 3 \Longrightarrow E_3 = -1.51eV.$
4	Only certain values of E_n (–13.6; –3.4; –1.51; –0.85 ……) are allowed.
5	The photon: no mass, no charge, speed in vacuum is c, of energy hv.
6.1	The ionization energy is the energy needed for the atom to absorb for it to release its electron without speed.
6.2	$W_i + (-13.6) = 0 \Longrightarrow W_i = 13.6eV.$
6.3	$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$ for $n \to \infty$ and $m = 1$, $\frac{1}{\lambda} = R = 1.097 \times 10^{7} \Longrightarrow \lambda = 0.911 \times 10^{-7}m.$
7.1	$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right)$; for $m = 1$ and $n = 2$, we obtain $\lambda_{max} = 0.121 \times 10^{-6}m.$ For $m = 1$ and $n \to \infty$, we obtain $\lambda_{min} = 0.091 \times 10^{-6}m.$
7.2	Ultra-violet.
8.1	$\frac{1}{\lambda} = R\left(\frac{1}{m^2} - \frac{1}{n^2}\right).$ For $m = 1$ and $n = 3$, $\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{3^2}\right) = \frac{8}{9}R = 0.975 \times 10^{7}.$ $v = \frac{C}{\lambda} \Longrightarrow v_{3\to1} = 2.92 \times 10^{15}Hz.$ For $m = 1$ and $n = 2$, $\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{2^2}\right) = \frac{3}{4}R = 0.82275 \times 10^{7}.$

	$\Rightarrow \nu_{2\to1} = 2.47 \times 10^{15} Hz$ For $m = 2$ and $n = 3$, $\frac{1}{\lambda} = R\left(\frac{1}{2^2} - \frac{1}{3^2}\right) = \frac{5}{36}R = 0.15236 \times 10^7$. $\Rightarrow \nu_{3\to2} = 0.46 \times 10^{15} Hz$
8.2	$\nu_{3\to1}, = \nu_{3\to2} + \nu_{2\to1}$ is verified.

Exercise 15:

Part	Answer key
1.1	Each energy level has a specific value; therefore the energy of the atom is quantized.
1.2	$E_{ph} = \frac{hc}{\lambda}$; E_{ph} max if λ is minimum; $E_{ph(max)} = \frac{6.6\times10^{-34}\times3\times10^8}{0.4\times10^{-6}} = 4.95 \times 10^{-19} J = 3.093 eV$. $E_{ph(min)} = \frac{6.6\times10^{-34}\times3\times10^8}{0.8\times10^{-6}} = 2.475 \times 10^{-19} J = 1.546 eV$.
1.3	$W_{ion} = E_\infty - E_1 = 0 — 5.14 = 5.14 eV$. $E_{ph(max)} = 3.093 eV < W_{ion} = 5.14 eV$. Therefore the white light cannot ionize the atom.
1.4	$E_{ph} = E_2 - E_1$, then $\frac{hc}{\lambda} = -3.04 + 5.14 = 2.1 eV = 3.36 \times 10^{-19} J$. $\lambda = \frac{6.6\times10^{-34}\times3\times10^8}{3.36\times10^{-19}} = 0.589 \times 10^{-6} m = 589 nm$.
2.1	$E_n = E_2 = -3.04 eV$ since this photon excites the atom from E_1 to E_2 so it is emitted when the atom de-excites from E_2 to E_1. OR: $E_n - E_1 = E_{photon}$; $E_{photon} = \frac{hc}{\lambda} = \frac{6.6\times10^{-34}\times3\times10^8}{589\times10^{-9}\times1.6\times10^{-19}} = 2.1 eV$. $E_n = E_{photon} + E_1 = 2.1 - 5.14 = -3.04 eV$.
2.2.1	$E_{3/1} = E_3 - E_1 = 3.21 eV = 3.21 \times 10^{-6} MeV$
2.2.2	$E_{3/1} = \Delta mc^2$. $\Delta m = \frac{3.21\times10^{-6}}{931.5} = 3.446 \times 10^{-9} u$.
2.3.1	$P = P_1 + P_2$ But $P_1 = 2P_2$, then $P = 3P_2$, thus $P_2 = 2W$ and $P_1 = 4W$.
2.3.2	$P_1 = \frac{nE_1}{t}$ then $n = \frac{t\times P_1}{E_1} = \frac{1\times4}{3.36\times10^{-19}} = 1.19 \times 10^{19} photons$.

CHAPTER 9 – ATOMIC NUCLEUS
COURSE

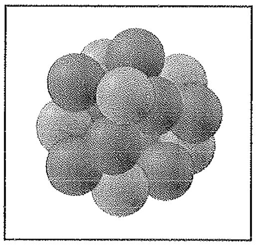
9.1- INTRODUCTION

The gold foil experiment gave a clear picture of the structure of an atom which consists of protons (nucleus) and same number of electrons outside the nucleus. But scientists soon realized that the atomic model offered by Rutherford is not complete. Various experiments showed that the mass of the nucleus is approximately twice than that of the protons. Rutherford postulated the existence of some neutral particles having mass similar to proton but there was no direct experimental evidence.

9.2- ATOMIC NUCLEUS

Composition of an atomic nucleus

The model of the atomic nucleus, elaborated in the early 1930's with the famous discovery of neutron by James Chadwick, considers the nucleus essentially constituted of two types of particles: the protons and the neutrons; both are called nucleons.



Representation of a nucleus

An atomic nucleus is denoted by ${}^{A}_{Z}X$ where X is the symbol of the chemical element, Z is charge number (number of protons) and A is mass number or nucleon number (total number of nucleons).

Characteristics of the nucleons

	Number	Charge	Mass
Proton	Z	$q_p = e = 1.6 \times 10^{-19}$C	$m_p = 1.6726 \times 10^{-27}$kg
Neutron	N = A - Z	$q_n = 0$	$m_n = 1.6749 \times 10^{-27}$kg

Mass of a nucleus

The mass m of a nucleus having Z protons and $N = A - Z$ neutrons is:

$$m = Zm_p + Nm_n = Zm_p + (A - Z)m_n$$

Nuclides and isotopes

A **nuclide** is a kind of nucleus characterized by a given charge number Z and a given mass number A or number of neutrons N.

Element	Nuclide	Number of protons	Number of neutrons
Carbon	${}^{12}_{6}C$	$n_p = Z = 6$	$N = A - Z = 6$
Uranium	${}^{238}_{92}U$	$n_p = Z = 92$	$N = A - Z = 146$

The **isotopes** of a chemical element are the set of nuclides having the same charge number but different mass numbers or number of neutrons.

Element		Isotopes		
	Name	Protium	Deuterium	Tritium (manmade)
Hydrogen	Symbol	${}^{1}_{1}H$	${}^{2}_{1}H$	${}^{3}_{1}H$
	Number of protons	$n_p = Z = 1$	$n_p = Z = 1$	$n_p = Z = 1$
	Number of neutrons	$N = A - Z = 0$	$N = A - Z = 1$	$N = A - Z = 2$
Carbon	Symbol	${}^{11}_{6}C$	${}^{12}_{6}C$	${}^{14}_{6}C$
	Number of protons	$n_p = Z = 6$	$n_p = Z = 6$	$n_p = Z = 6$
	Number of neutrons	$N = A - Z = 5$	$N = A - Z = 6$	$N = A - Z = 8$

9.3- VOLUME, MASS AND DENSITY OF THE ATOMIC NUCLEUS

Most of the atomic nuclei are represented by sphere. In 1911, Rutherford was the first to determine the approximate dimensions of the atomic nucleus. His experiments led to the empirical expression giving the radius of the atomic nucleus in terms of the mass number A.

Volume and radius of an atomic nucleus

The volume of a nucleon (proton or neutron), assumed to be a sphere of radius r_0, is $V_0 = \frac{4}{3}\pi r_0^3$.

The volume of the atomic nucleus, assumed to be a sphere of radius r, is $V = \frac{4}{3}\pi r^3$

$$V = AV_0 \Longrightarrow \frac{4}{3}\pi r^3 = A\frac{4}{3}\pi r_0^3 \Longrightarrow r = A^{\frac{1}{3}} \times r_0$$

A practical unit, the fermi (or femtometer) of symbol fm, is used to express such small values.

$$1fm = 10^{-15}m$$

Therefore, the radius of the atomic nucleus is: $r = r_0 A^{\frac{1}{3}}$ where $r_0 = 1.2fm = 1.2 \times 10^{-15}m$ is the radius of the hydrogen atom.

Average mass of a nucleus

Neglecting the mass of the electron with respect to that of the neutron or the proton, it is possible to consider the mass of the nucleus as that of the atom.

In nuclear physics, scientists use a unit of mass which is more convenient for tiny particles of small mass. It is the atomic mass unit represented by the letter u, and defined as:

The atomic mass unit u is equal to one-twelfth of the mass of the atom of the carbon isotope $^{12}_{6}C$.

$$1u = \frac{1}{12}m(^{12}_{6}C)$$

The mass of one mole of carbon $^{12}_{6}C$ which contains $N_A = 6.02217 \times 10^{23}$ atoms is 12g.

$$1u = \frac{1}{12}m(^{12}_{6}C) = \frac{1}{12} \times \frac{12\times10^{-3}}{6.02217\times10^{23}} = 1.66053 \times 10^{-27}kg$$

Relation between u and MeV/c^2.

$$E = mc^2 \Longrightarrow m = \frac{E}{c^2}$$

$$\frac{MeV}{c^2} = \frac{1.6022\times10^{-13}J}{(2.99792\times10^8 m.s^{-1})^2} = \frac{1.6022\times10^{-13}kg.m^2.s^{-2}}{(2.99792\times10^8)^2 m^2.s^{-2}} = 1.78269 \times 10^{-30}kg = \frac{1.78269\times10^{-30}}{1.66053\times10^{-27}}u$$

$$u = \frac{1.66053\times10^{-27}}{1.78296\times10^{-30}}MeV/c^2 \simeq 931.5MeV/c^2$$

The average mass of a nucleon is $m_0 = 1.7 \times 10^{-27}kg$.
The average mass of the atomic nucleus is $m = Am_0$.

$$\boxed{[J] = [N].[m] = \left[\frac{kgm}{s^2}\right].[m] = \left[\frac{kg.m^2}{s^2}\right]}$$

Density of the atomic nucleus

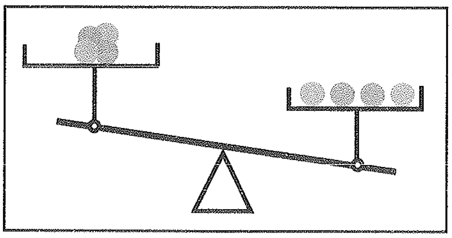
Since the volume of the nucleus is very small compared to that of the atom, all the matter that constitutes the atom is concentrated in the nucleus. The density of the latter is then enormous.

The density of the atomic nucleus is $\rho = \frac{m}{V} = \frac{Am_0}{A\frac{4}{3}\pi r_0^3} = \frac{3m_0}{4\pi r_0^3} = \frac{3\times1.7\times10^{-27}}{4\pi\times(1.2\times10^{-15})^3} = 2.3 \times 10^{17}kgm^{-3}$.

The density of an atomic nucleus is constant and independent of A.

9.4- MASS DEFECT AND BINDING ENERGY

Mass defect



Consider the neutral helium atom ${}^4_2He$ whose mass is $4.00150u$. If we neglect the mass of the two electrons constituting the electronic cloud of the atom, the mass of the atom is reduced to that of its atomic nucleus formed of two protons and two neutrons.
But the mass of the two protons is $2m_p = 2 \times 1.00728 = 2.01456u$ and that of the two neutrons is $2m_n = 2 \times 1.00866 = 2.01732u$.
$2m_p + 2m_n = 2.01456 + 2.01732 = 4.03188u$.
We observe that the mass of ${}^4_2He$ is smaller that $2m_p + 2m_n$.
The mass of an atomic nucleus is less than the sum o the masses of its different nucleons taken separately.
The difference Δm between the sum of the masses of the nucleons taken separately and the mass of the atomic nucleus is called the mass defect.

$$\Delta m = [Zm_p + (A - Z)m_n] - m_x$$

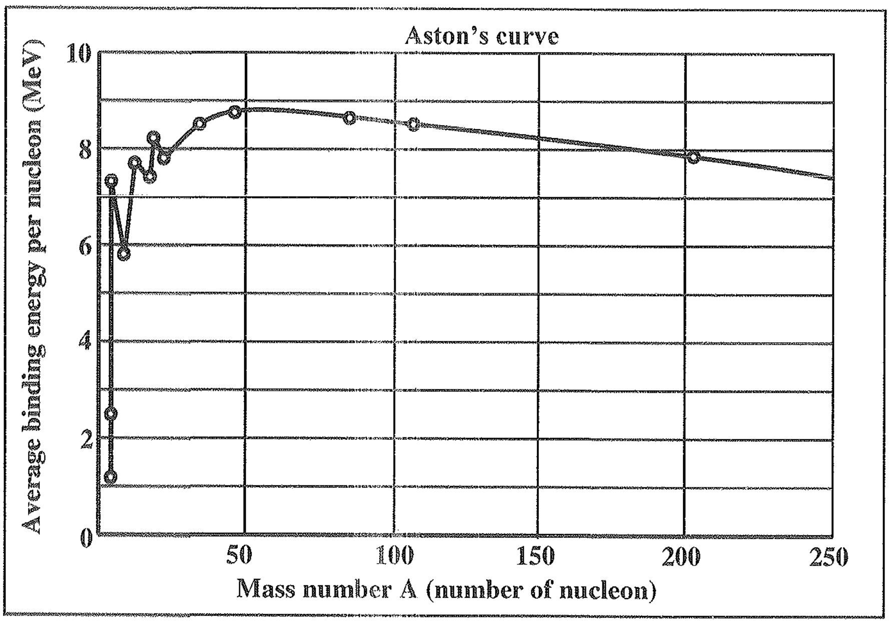
TIP

Binding energy is the minimum energy needed to break the nucleus into separate nucleons. The binding energy per nucleon is the minimum average energy needed to extract one nucleon from the nucleus.

Binding energy

The binding energy is the energy that should be given to the nucleus in order to break it up. Its expression is:

$$E_B = \Delta mc^2 = [Zm_p + (A - Z)m_n - m_x]c^2$$

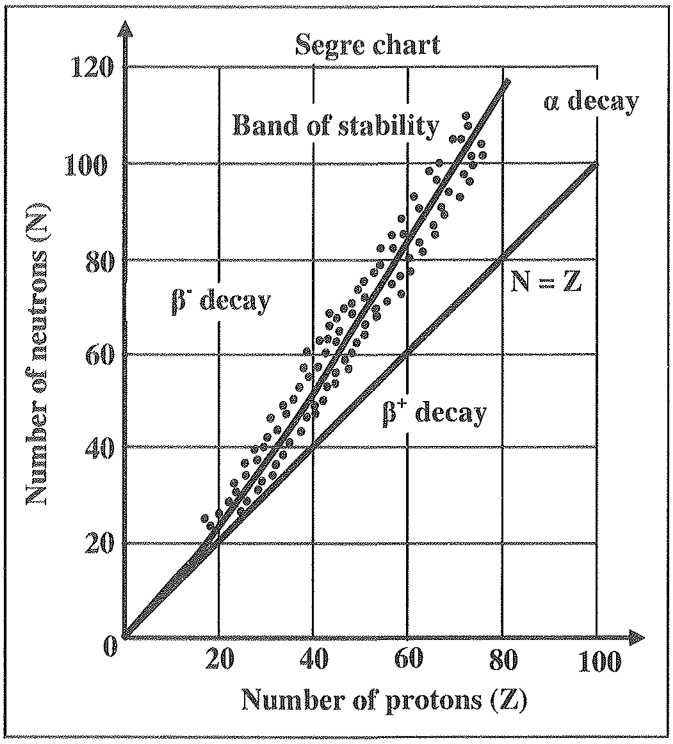


A nucleus is said to be stable if it is not subjected to any spontaneous evolution, which modifies its composition: it remains as one entity. Not all nuclei have the same binding energy. The binding energy per nucleon $\frac{E_b}{A}$ that represents the quotient of the binding energy to the number of nucleons in the nucleus is used to study the stability of a nucleus.
The adjacent graph, called Aston's graph, represents the vairations of the binding energy per nucleon $\frac{E_b}{A}$ as a function of the mass number A.
This graph increases, attains a level at about 8.8MeV, and then decreases slowly as the mass number A increases. It can be divided into three regions:

- For $1 < A < 20; \frac{E_b}{A} < 8MeV$.
- For $20 < A < 190; \frac{E_b}{A} > 8MeV$.
- For $A > 190; \frac{E_b}{A} < 8MeV$.

Since the binding energy is the energy needed to break up the nucleus, the most stable nuclei are those of large binding energy per nucleon, that is to say the nuclei of mass number A between 20 and 190.

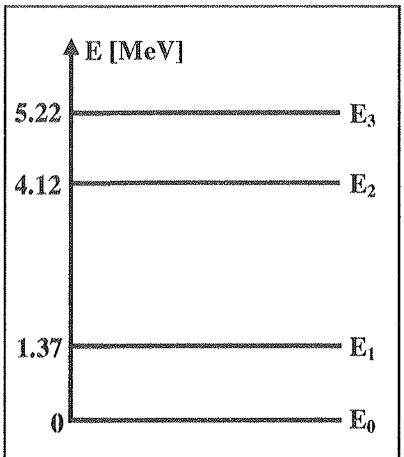
9.5- INTERACTION WITHIN THE NUCLEUS AND ITS STABILITY



Being all positively charged, protons of the nucleus mutually repel each other, which tend to break the nucleus. Neutrons, however, don't exert ant electrostatics interaction.
One then asks the question, why does the nucleus remain as one entity? Why does it not explode?
In fact, an interaction, whose origin is still not known, assures the stability of the nucleus and keeps the nucleons together. This interaction, called strong interaction, was discovered by Yukawa in 1935.
Although its magnitude us very large, this interaction between two nucleons, acts at short ranges only (within the nucleus). It becomes practically nil when the inter-nucleon distance exceeds the order of 10^{-15}m, contrary to gravitational and electrostatics interactions, which have a long range.
In fact, if the gravitational force is taken to be 1N, then the electrostatics interaction is 10^{37}N and the strong interaction is 10^{39}N.
It is important to note that the most stable nuclei having A > 20 contain more neutrons than protons. This is the cause of stability. This can be explained by the fact when the number Z of protons increases; we need a large number of neutrons to weaken the electrostatic repulsion. But if Z becomes very large, the electrostatic repulsion increases so the neutrons, whatever their number, cannot assure the stability of the nucleus. Practically, it is very difficult to find a stable nucleus for Z > 82.

9.6- ENERGY LEVELS OF THE ATOMIC NUCLEUS

In the previous chapter, we have seen that an atom can be found in quantized energy level or energy states depending on the nature of the atom. Similarly, a given nuclide can also be found in **quantized energy states**. The zero level of energy E_0 is attributed to the minimum energy called the **ground state**. The other levels of energies E_1, E_2, E_3 ... are called the **excited states**.



Energy levels of Mg-24 nuclide

We a magnesium-24 nuclide receives the required energy (5.22MeV for example), a transition takes place from the ground state to the third excited state of energy E_3. The nucleus cannot remain in this state for a long time; it tends to get rid of this excess energy in the form of electromagnetic radiation called γ rays, either partially when passing to another lower excited state (E_2 or E_1), or totally when passing to the ground state. We say that the nuclide is no longer in an excited state.

CHAPTER 9 – ATOMIC NUCLEUS
EXERCISES AND PROBLEMS

Given: Avogadro's number: $N_A = 6.022 \times 10^{23}$; $1u = 931.5 MeV/c^2 = 1.66 \times 10^{-27} kg$;
Plank's constant: $h = 6.62 \times 10^{-34} J.s$; $1 MeV = 1.6 \times 10^{-13} J$; speed of light in vacuum $c = 3 \times 10^8 m/s$;
Mass of Helium: $m_{He} = 4.0026u$; Mass of proton: $m_p = 1.00728u$; mass of a neutron: $m_n = 1.00866u$.
The radius of a proton or a neutron is $r_0 = 1.2 \times 10^{-15} m$

Exercise 1*:
Define the following terms:
1- Isotopes.
2- Nuclides.
3- Binding energy.

Exercise 2:
Consider the radius of the uranium nucleus ${}^{235}_{92}U$.
1- Calculate the radius of this nucleus.
2- The volume of the uranium atom is approximately $1.38 \times 10^{-29} m^3$. Determine the relation between the radius of the atom of uranium and the radius of its nucleus.

Exercise 3*:
Calculate the mass of a proton in u and MeV/c^2.

Exercise 4*:
Sketch the graph of the number of neutrons N as a function of the charge number Z of stable and unstable nuclei. Explain the obtained curve.

Exercise 5:
Calculate the binding energy per nucleon of the ${}^{235}_{92}U$ nucleus of mass 235.044 u. Is it a stable nucleus? Justify.

Exercise 6*:
A nucleus X, of mass charge number 5 and mass number 11, has a defect mass of 0.081798 u.
1- Calculate the precise mass of the nucleus.
2- Determine, in MeV and in joules, the binding energy per nucleon.

Exercise 7*:
Consider the (Z, A) couples of different elements: (8, 16), (6, 12), (8, 17), (5, 11), (8, 18). Specify the isotopes.

Exercise 8*:
An atomic nucleus is formed is formed of "A" nucleons.
Calculate the density of the corresponding atomic nucleus.

Exercise 9:
The mass of the ${}^{11}_{5}B$ isotope is 11.009 u, that of the ${}^{56}_{26}Fe$ isotope is 55.9349 u and that of the uranium ${}^{235}_{92}U$ isotope is 235.044 u.
1- Define binding energy per nucleon.
2- Determine, in MeV then in J, the binding energy per nucleon of each of these isotopes.
3- Specify which one is most stable.
4- Use Aston's curve to explain your result.

Exercise 10*:

Consider the copper isotope $^{63}_{29}Cu$.

1- Define isotopes of a chemical element.

2- Indicate the number of protons and that of neutrons in a $^{63}_{29}Cu$ nucleus.

3- Calculate the approximate value of the radius of a $^{63}_{29}Cu$ nucleus.

Exercise 11*:

Consider a nuclide $^{A}_{Z}X$. The average mass of a nucleon is $m_0 = 1.7 \times 10^{-27}\,kg$.

1- Write the expression of the:

  1.1- approximate volume of $^{A}_{Z}X$ in terms of A and r_0;

  1.2- approximate mass of $^{A}_{Z}X$ in terms of A and m_0.

2- Show that the density of $^{A}_{Z}X$ is constant. Find its value.

Exercise 12*:

Calculate the mass defect, binding energy and binding energy per nucleon of a $^{232}_{90}Th$ nucleus of mass 232.0380553u.

Exercise 13:

The object of this exercise is to compare the values of physical quantities characterizing the stability of different nuclei and to verify that, during nuclear reactions, certain nuclei are transformed into more stable nuclei with the liberation of energy.

Numerical data:

Mass of a neutron: $m_n = 1.0087u$; mass of a proton : $m_p = 1.0073u$; mass of an electron: $m_e = 0.00055u$;

Stability of atomic nuclei

Consider the table below that shows some physical quantities associated with certain nuclei.

Nucleus	$^{2}_{1}H$	$^{3}_{1}H$	$^{4}_{2}He$	$^{14}_{6}C$	$^{14}_{7}N$	$^{94}_{38}Sr$	$^{140}_{54}Xe$	$^{235}_{92}U$
Mass (u)	2.0136	3.0155	4.0015	14.0065	14.0031	93.8945	139.892	234.9935
Binding energy E_b (MeV)	2.23	8.57	28.41	99.54	101.44	810.50	1164.75	
Binding energy per nucleon $\frac{E_b}{A}$ (MeV/nucleon)	1.11		7.10		7.25	8.62		

1-

  1.1- Define the binding energy of a nucleus.

  1.2- Write the expression of the binding energy E_b of a nucleus $^{A}_{Z}X$ as a function of Z, A, m_p, m_n, m_X (the mass of the nucleus $^{A}_{Z}X$ and the speed of light in vacuum c.

  1.3- Calculate, in MeV, the binding energy of the uranium 235 nucleus.

  1.4- Complete the table by calculating the missing values of $\frac{E_b}{A}$.

  1.5- Give the name of the most stable nucleus in the above table. Justify your answer.

2- Each of the considered nuclei in the table belongs to one of the three groups given by: $A < 20$; $20 < A < 190$; $A > 190$.

Referring to the completed table, trace the shape of the curve representing the variation of $\frac{E_b}{A}$ as a function of A. Specify on the figure the three mentioned groups.

Exercise 14:

Consider the nuclei ${}^{11}_{5}B$; ${}^{56}_{26}Fe$; ${}^{40}_{20}Ca$.

1- Indicate the composition of the corresponding nuclei.
2- Define the binding energy of an atomic nucleus.
3- Calculate the binding energy per nucleon of the ${}^{11}_{5}B$ nucleus, given that its mass is 11.006562u.
4- Compare the value obtained to 8.7894MeV of ${}^{56}_{26}Fe$ and to 8.552MeV of ${}^{40}_{20}Ca$. Conclude.

Exercise 15:

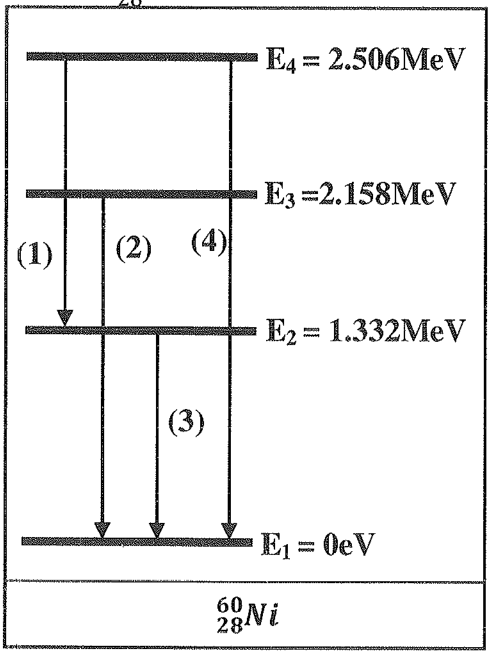
In the equation: $r = r_0 A^{\frac{1}{3}}$ with $r_0 = 1.2 \times 10^{-15} m$.

1- State the meaning of r, r_0 and A.
2- Sketch a graph of r as a function A.
3- Calculate the radius of the ${}^{56}_{26}Fe$ nucleus. Give your answer to 3 significant figures.
4- The density of the nucleus of ${}^{56}_{26}Fe$ is 2.3×10^{17} kg/m^3. Calculate the average mass of ${}^{56}_{26}Fe$ nucleus.
5- The ${}^{56}_{26}Fe$ nucleus is composed of protons and neutrons. Given: m_{Fe} = 59.9349u.
 5.1- State the number of protons and neutrons in the ${}^{56}_{26}Fe$ nucleus.
 5.2- Calculate the binding energy per nucleon of the ${}^{56}_{26}Fe$ nucleus.

Exercise 16:

The adjacent figure shows the first four energy levels of the nucleus of nickel ${}^{60}_{28}Ni$.

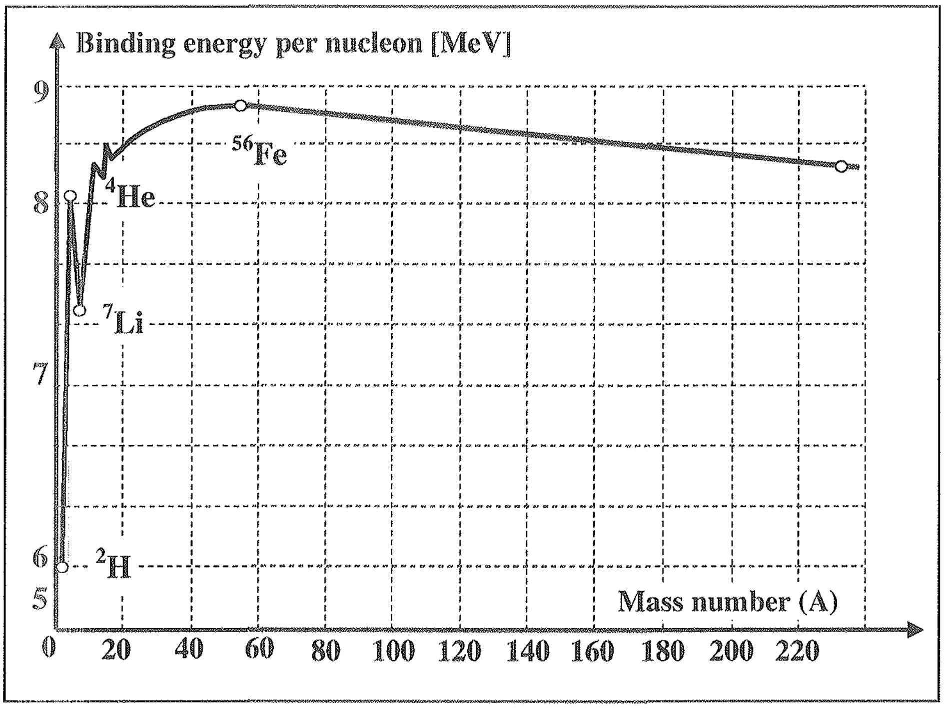
Type of Radiation	Frequency Range (Hz)	Wavelength Range	Type of Transition
Gamma rays	$10^{20} - 10^{24}$	< 1 pm	Nuclear
X-rays	$10^{17} - 10^{20}$	1nm – 1pm	Inner electron
Ultraviolet	$10^{15} - 10^{17}$	400nm – 1nm	Outer electron
Visible	$4 - 7.5\ (\times 10^{14})$	750nm – 400nm	Outer electron
Near-infrared	$1 - 4\ (\times 10^{14})$	2.5μm – 750nm	Outer electron molecular vibrations
Infrared	$10^{13} - 10^{14}$	25μm – 2.5μm	Molecular vibrations
Microwaves	$3 \times 10^{11} - 10^{13}$	1mm – 25μm	Molecular rotations, electron spin flips*
Radio waves	$< 3 \times 10^{11}$	> 1 mm	Nuclear spin flips*



1- The energy of the nickel nucleus is quantized. Explain.
2- Match between the energies of the energy levels of ${}^{60}_{28}Ni$ nucleus and their corresponding states.
3- State two differences between the energy levels of an atom and the energy levels of a nucleus.
4- Calculate the wavelengths of the emitted photons corresponding to the transitions labeled 1, 2, 3 and 4.
5- Use the table to indicate to which part of the spectrum do the emitted particles belong?
6- Compare the energy of the photon emitted in transition 4 to the energy of the photons emitted in transitions 1 and 3.

Exercise 17:

The adjacent diagram shows the "binding energy per nucleon" versus mass number.



1- Give the name of this curve.
2- What is meant by a nucleon?
3- Define Binding energy and binding energy per nucleon.
4- Specify the element that is the most "stable" in the graph. Give its approximate binding energy per nucleon.
5- The Binding energy per nucleon of ${}^{4}_{2}He$ is 7.07MeV.
   5.1- Calculate the mass defect.
   5.2- Deduce the mass of the Helium nucleus.
6- The curve is divided into three regions. Explain.

Exercise 18:

The nucleus ${}^{212}_{83}Bi$ is considered.

1- Define the following terms:
   1.1- Nuclides.
   1.2- Binding energy.
2- What do the numbers 83 and 212 represent for the Bismuth-212 nucleus?
3- What is the numbers of protons and neutrons in the nucleus ${}^{212}_{83}Bi$?
4- Knowing that the nucleus of Bismuth is spherical and that the average radius of a nucleon is $r_0 = 1.2 \times 10^{-15}$m. Show that the expression of the average radius of the Bismuth-212 nucleus is $r = r_0 \times A^{\frac{1}{3}}$. Calculate its value.
5- Deduce the volume of the Bismuth-212 nucleus.
6- Calculate the average mass of the Bismuth-212 nucleus. Deduce its density.
   Given: the average mass of a nucleon is $m_0 = 1.7 \times 10^{-27} kg$.
7- One of the Bismuth isotopes is ${}^{216}_{83}Bi$.
   7.1- Define isotopes.
   7.2- Is the density of ${}^{216}_{83}Bi$ is equal, smaller, or greater than that of ${}^{212}_{83}Bi$.
8- Calculate, in [MeV], the binding energy per nucleon of ${}^{212}_{83}Bi$.
   Mass of the ${}^{212}_{83}Bi$ nucleus: m_B = 211.991876 u.
9- The molar mass of ${}^{212}_{83}$Bi is 212 g/mole. Determine the approximate mass of 10^{12} nuclei of ${}^{212}_{83}Bi$.

Exercise 19:

The radius R of some isotopes and their mass number A are given in the adjacent table:

1- Copy and complete the adjacent table.
2- Draw a graph of R^3 as a function of A.
3- Determine the radius R_0 of the hydrogen nucleus.

Element	R / 10^{-15}m	R^3 / 10^{-45} m^3	A
Carbon	2.66		12
Silicon	3.43		28
Iron	4.35		56
Tin	5.49		120
Lead	6.66		208

CHAPTER 9 – ATOMIC NUCLEUS
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 2:

Part	Answer
1	$r = r_0 A^{\frac{1}{3}} = 1.2 \times 10^{-15} \times 235^{\frac{1}{3}} = 7.4 \times 10^{-15} m$
2	$V = \frac{4}{3}\pi r^3 \Longrightarrow r^3 = \frac{3V}{4\pi} \Longrightarrow r = \left(\frac{3V}{4\pi}\right)^{\frac{1}{3}} = 1.488 \times 10^{-10} m$ $\frac{r}{r_0} = \frac{1.488\times10^{-10}}{7.4\times10^{-15}} = 20108.1$

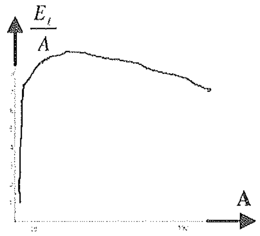
Exercise 5:

Part	Answer
	$\Delta m = Zm_P + (A-Z)m_n - m_u = 92 \times 1.00728 + 143 \times 1.00866 - 235.044$ $\Delta m = 1.86414u = 1736.44641 MeV/c^2$ $E_b = \Delta mc^2 = 1736.44641 MeV$ $\frac{E_b}{A} = 7.389 MeV < 8 MeV$ (The nucleus is not stable)

Exercise 9:

Part	Answer
1	The binding energy is the energy that should be given to the nucleus to break it up.
2	For $^{11}_{5}B$ $\Delta m = Zm_p + (A-Z)m_n - m_B = 5 \times 1.00728 + 6 \times 1.00866 - 11.009 = 0.07936u$ $\Delta m = 73.92384 MeV/c^2$ $E_b = \Delta mc^2 = 73.92384 MeV$ $\frac{E_b}{A} = 6.72 MeV = 1.0752 \times 10^{-12} J$ For $^{56}_{26}Fe$ $\Delta m = Zm_p + (A-Z)m_n - m_B = 26 \times 1.00728 + 30 \times 1.00866 - 55.9349 = 0.51418u$ $\Delta m = 478.95867 MeV/c^2$ $E_b = \Delta mc^2 = 478.95867 MeV$ $\frac{E_b}{A} = 8.55 MeV = 1.368 \times 10^{-12} J$ For $^{235}_{92}U$ $\Delta m = Zm_p + (A-Z)m_n - m_B = 92 \times 1.00728 + 143 \times 1.00866 - 235.044 = 1.86414u$ $\Delta m = 1736.44641 MeV/c^2$ $E_b = \Delta mc^2 = 1736.44641 MeV$ $\frac{E_b}{A} = 7.389 MeV = 1.18 \times 10^{-12} J$
3	$\left(\frac{E_b}{A}\right)_{Fe} > \left(\frac{E_B}{A}\right)_U > \left(\frac{E_b}{A}\right)_B$ Fe is the most stable nucleus.
4	For 1 < A < 20: the binding energy per nucleon is less than 8. For 20 < A < 190: the binding energy per nucleon is greater than 8. For A > 190: the binding energy per nucleon is less than 8.

Exercise 13:

<table>
<tr><th>Part</th><th>Answer</th></tr>
<tr><td>1.1</td><td>The binding energy of a nucleus is the minimum energy needed in order to break the nucleus into its nucleons.</td></tr>
<tr><td>1.2</td><td>$E_b = \left[Zm_p + (A - Z)m_n - m_X\right]c^2$.</td></tr>
<tr><td>1.3</td><td>$\Delta m = Zm_p + (A - Z)m_n - m_X = 92 \times 1.0073 + 143 \times 1.0087 - 234.9935 = 1.9222u$.
$E_b = 1.9222 \times 931.5 = 1790.53MeV$.</td></tr>
<tr><td>1.4</td><td><table>
<tr><td>$^{2}_{1}H$</td><td>$^{3}_{1}H$</td><td>$^{4}_{2}He$</td><td>$^{14}_{6}C$</td><td>$^{14}_{7}N$</td><td>$^{94}_{38}Sr$</td><td>$^{140}_{54}Xe$</td><td>$^{235}_{92}U$</td></tr>
<tr><td>1.11</td><td>2.86</td><td>7.10</td><td>7.11</td><td>7.25</td><td>8.62</td><td>8.32</td><td>7.62</td></tr>
</table></td></tr>
<tr><td>1.5</td><td>The nucleus that has greater binding energy per nucleon is more stable $\Rightarrow$ strontium is the most stable nucleus.</td></tr>
<tr><td>2</td><td>Shape of the curve.
</td></tr>
</table>

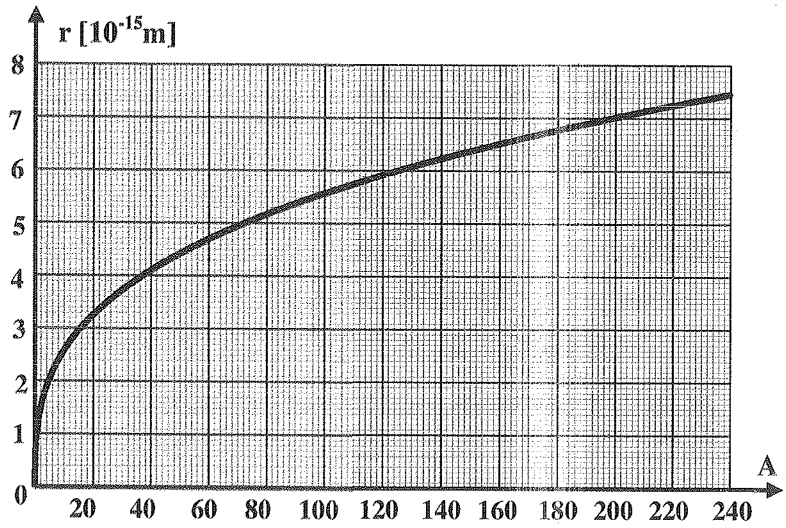
Exercise 14:

<table>
<tr><th>Part</th><th>Answer</th></tr>
<tr><td>1</td><td>For $^{11}_{5}B$
Number of protons = Z = 5
Number of neutrons = N = A – Z = 11 – 5 = 6

For $^{56}_{26}Fe$
Number of protons = Z = 26
Number of neutrons = N = A – Z = 56 – 26 = 30

For $^{40}_{20}Ca$
Number of protons = Z = 20
Number of neutrons = N = A – Z = 40 – 20 = 20</td></tr>
<tr><td>2</td><td>The binding energy is the energy that should be given to the nucleus to break it up.</td></tr>
<tr><td>3</td><td>$\Delta m = Zm_p + (A - Z)m_n - m_B = 5 \times 1.00728 + 6 \times 1.00866 - 11.006562 = 0.081798u$
$\Delta m = 76.194837MeV/c^2$
$E_b = \Delta mc^2 = 73.92384MeV$
$\frac{E_b}{A} = 6.927MeV$</td></tr>
<tr><td>4</td><td>$\left(\frac{E_b}{A}\right)_{Fe} > \left(\frac{E_b}{A}\right)_{Ca} > \left(\frac{E_b}{A}\right)_{B}$
Fe is more stable than Ca and B.</td></tr>
</table>

Exercise 15:

Part	Answer
1	r: radius of the nucleus. r_0: radius of a proton or a neutron. A: number of nucleons (protons and neutrons).
2	
3	$r = r_0A^{\frac{1}{3}} = 1.2 \times 10^{-15} \times 56^{\frac{1}{3}} = 4.59 \times 10^{-15}m$
4	$m = \rho \times V = \rho \times \frac{4}{3}\pi r^3 = 2.3 \times 10^{17} \times \frac{4\pi}{3} \times (4.59 \times 10^{-15})^3 = 9.3165 \times 10^{-26}kg = 56.12349398u$
5.1	Number of protons = Z = 26 Number of neutrons = 30
5.2	$\Delta m = Zm_p + (A - Z)m_n - m_B = 26 \times 1.00728 + 30 \times 1.00866 - 55.9349 = 0.51418u$ $\Delta m = 478.95867MeV/c^2$ $E_b = \Delta mc^2 = 478.95867MeV$ $\frac{E_b}{A} = 8.55MeV$

Exercise 16:

Part	Answer
1	Only specific values of energy are allowed (the spectrum is discontinuous).
2	E_1: Ground state. E_2: 1st excited state. E_3: 2nd excited state. E_4: 3rd excited state.
3	The energy levels of an atom are negative while that of a nucleus are positive. The energy levels of an atom are in the order of eV while that of a nucleus are in the order of MeV. For an atom, the energy of the ionized state is zero, while for a nucleus, the energy of the ground state is zero.
4	$E_p = E_h - E_l \Longrightarrow \frac{hc}{\lambda} = E_h - E_l \Longrightarrow \lambda = \frac{hc}{E_h - E_l}$. $\lambda_{4\to2} = \frac{hc}{E_4 - E_2} = \frac{6.62\times10^{-34}\times3\times10^{8}}{(2.506-1.332)\times1.6\times10^{-13}} = 1.057 \times 10^{-12}m = 1.057pm.$

	$\lambda_{3\to1} = \frac{hc}{E_3-E_1} = \frac{6.62\times10^{-34}\times3\times10^{8}}{(2.158-0)\times1.6\times10^{-13}} = 5.75\times10^{-13}m = 0.575pm.$ $\lambda_{2\to1} = \frac{hc}{E_2-E_1} = \frac{6.62\times10^{-34}\times3\times10^{8}}{(1.332-0)\times1.6\times10^{-13}} = 9.32\times10^{-13}m = 0.932pm.$ $\lambda_{4\to2} = \frac{hc}{E_4-E_1} = \frac{6.62\times10^{-34}\times3\times10^{8}}{(2.506-0)\times1.6\times10^{-13}} = 4.95\times10^{-13}m = 0.495pm.$
5	$\lambda_{4\to2}$: Gamma ray. $\lambda_{3\to1}, \lambda_{2\to1}$ and $\lambda_{4\to2}$: X-rays.
6	$E_{4\to1} = E_{4\to2} + E_{2\to1}$.

Exercise 17:

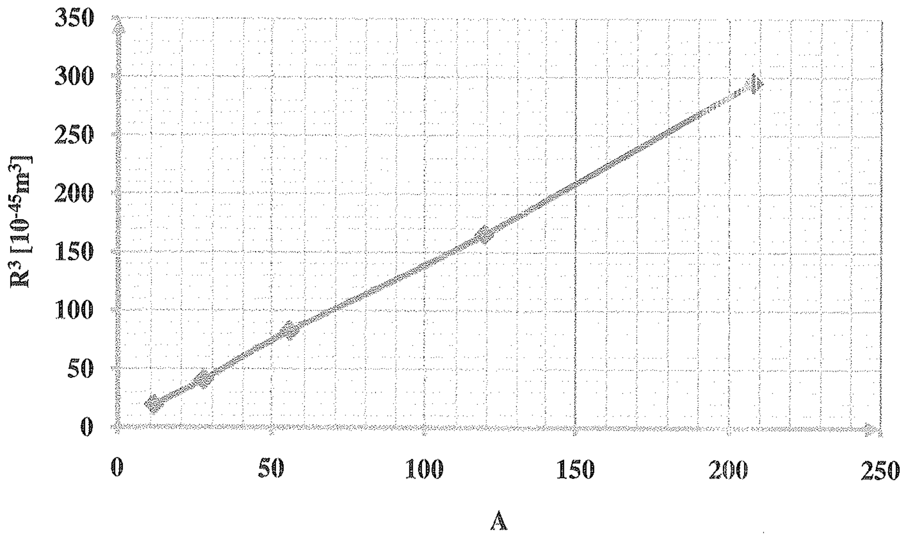
Part	Answer
1	The Aston curve.
2	A nucleon is a proton or neutron.
3	Binding energy: is the energy that should be given to the nucleus to break it up. Binding energy per nucleon: ratio of the binding energy to the mass number (number of nucleons).
4	The most stable nucleus is iron-56 (Fe-56) since it has the larger binding energy per nucleon. $\left(\frac{E_b}{A}\right)_{Fe} \simeq 8.8MeV.$
5.1	$\frac{E_b}{A} = 7.07 \Longrightarrow E_b = 7.07\times4 = 28.28MeV.$ $E_b = \Delta mc^2 \Longrightarrow \Delta m = \frac{E_b}{c^2} = 28.28MeV/c^2 = 0.030359635u.$
5.2	$\Delta m = Zm_p + (A-Z)m_n - m_{He}.$ $0.030359635 = 2\times1.00728 + 2\times1.00866 - m_{He}.$ $m_{He} = 4.001520365u.$
6	For 1 < A < 20: the binding energy per nucleon is less than 8. For 20 < A < 190: the binding energy per nucleon is greater than 8. For A > 190: the binding energy per nucleon is less than 8.

Exercise 18:

Part	Answer
1.1	A nuclide is a kind of nucleus characterized by a given atomic number Z and a given number of neutrons N.
1.2	The binding energy is the energy that should be given to the nucleus to break it up.
2	Z = 83 (charge number). A = 212 (mass number or number of nucleons).
3	Number of protons = Z = 83. Number of neutrons = N = A – Z = 212 – 83 = 129.
4	The volume of a nucleon is: $V_0 = \frac{4}{3}\pi r_0^3.$ The volume of the nucleus is: $V = \frac{4}{3}\pi r^3.$ $V = AV_0 \Longrightarrow \frac{4}{3}\pi r^3 = A\times\frac{4}{3}\pi r_0^3.$ $r^3 = r_0^3\times A \Longrightarrow r = r_0\times A^{\frac{1}{3}} = 1.2\times10^{-15}\times212^{\frac{1}{3}} = 7.155\times10^{-15}m.$
5	$V = \frac{4}{3}\pi r^3 = 1.534\times10^{-42}m^3.$
6	$m = Am_0 = 212\times1.7\times10^{-27} = 3.604\times10^{-25}kg.$ $\rho = \frac{m}{V} = \frac{3.604\times10^{-25}}{1.534\times10^{-42}} = 2.35\times10^{17}kg/m^3.$

7.1	Nuclei of the same element having the same charge number and different mass numbers.
7.2	The density of an atomic nucleus is constant, so they have the same density.
8	$\Delta m = Zm_p + (A-Z)m_n - m_{Bi} = 83 \times 1.00728 + 129 \times 1.00866 - 211.991876.$ $\Delta m = 1.728504u = 1611MeV/c^2.$ $E_b = \Delta mc^2 = 1611MeV.$
9	$n = \frac{m}{M} = \frac{N}{N_A} \Longrightarrow m = \frac{M \times N}{N_A} = \frac{212\times10^{12}}{6.022\times10^{23}} = 3.52 \times 10^{-10}g.$

Exercise 19:

<table>
<tr><th>Part</th><th>Answer</th></tr>
<tr><td>1</td><td>
<table>
<tr><th>Element</th><th>R / 10⁻¹⁵m</th><th>R³ / 10⁻⁴⁵ m³</th><th>A</th></tr>
<tr><td>Carbon</td><td>2.66</td><td>18.82</td><td>12</td></tr>
<tr><td>Silicon</td><td>3.43</td><td>40.35</td><td>28</td></tr>
<tr><td>Iron</td><td>4.35</td><td>82.31</td><td>56</td></tr>
<tr><td>Tin</td><td>5.49</td><td>165.47</td><td>120</td></tr>
<tr><td>Lead</td><td>6.66</td><td>295.4</td><td>208</td></tr>
</table>
</td></tr>
<tr><td>2</td><td>


</td></tr>
<tr><td>3</td><td>$R^3 = R_0^3 \times A.$
For: A = 12; $R^3 = 18.82 \times 10^{-45}m^3$.
$R_0 = \frac{R}{A^{\frac{1}{3}}} = \frac{2.66\times10^{-15}}{12^{\frac{1}{3}}} = 1.16 \times 10^{-15}m.$</td></tr>
</table>

CHAPTER 10 – RADIOACTIVITY
COURSE

10.1- RADIOACTIVITY

Radioactivity is the phenomenon resulting from the spontaneous disintegration of an unstable atomic nucleus. During this phenomenon, an unstable nucleus, called parent nucleus, is transformed into a more stable daughter nucleus with the emission of γ rays and particles α, β ... We say that the parent nucleus decays or that there is transmutation of the parent nucleus into a daughter nucleus.

TYPES OF RADIOACTIVITY

Natural radioactivity: the parent nucleus is a natural nucleus. Natural radioactivity is hazardous, spontaneous, non-controllable and unpredictable which takes place without any external intervention
Artificial radioactivity: the parent nucleus is artificially prepared.

General equation

$$\begin{matrix}\text{unstable nucleus} \\ \text{(parent nucleus)}\end{matrix} \rightarrow \begin{matrix}\text{more stable nucleus} \\ \text{(daughter nucleus)}\end{matrix} + \text{particle} + \text{radiations}$$

$$^{A}_{Z}X \rightarrow {}^{A'}_{Z'}Y + {}^{b}_{a}p + \gamma$$

10.2- LAWS OF CONSERVATION

Soddy's laws

Law of conservation of mass number: $\sum A_{before} = \sum A_{after} \Rightarrow A = A' + b$

Law of conservation of charge number: $\sum Z_{before} = \sum Z_{after} \Rightarrow Z = Z' + a$

Law of conservation of total energy

$$\sum E_{before} = \sum E_{after}$$

The total energy of a nucleus X is $E_X = E_{0X} + E_{kX} = m_X c^2 + E_{kX}$ where $E_{0X} = m_X c^2$ the rest energy of X and E_{kX} is the kinetic energy of X.

10.3- MASS DEFECT AND LIBERATED ENERGY

The mass defect, which accompanies a radioactive decay, can be written as: $\Delta m = m_{before} - m_{after}$.
In SI units, Δm is expressed in [kg].
The mass defect Δm is very small, so it is usually expressed in u or MeV/c^2 where:

$$1u = 1.6605 \times 10^{-27} kg = 931.5 MeV/c^2$$

The energy liberated, which accompanies a radioactive decay, can be written as: $E_{lib} = \Delta mc^2$.
In SI unit, E_{lib} is expressed in [J], Δm in [kg].
c = 3 × 10^8m/s is the speed of light in vacuum.
The energy liberated is very small, so it is usually expressed in [MeV] where $1MeV = 1.60219 \times 10^{-13} J$.

10.4- TYPES OF RADIOACTIVE DECAY

Not all radioactive nuclides emit the same particle and radiations, it depends whether it is heavy, rich in protons or rich in neutrons.

ATTENTION

The energies of a nucleus are quantized but have positive values.
The minimum energy is zero which corresponds to the ground state.

Gamma (γ) decay

Gamma radiations are emitted due to the de-excitation of the daughter nucleus when born in an excited state.

Characteristics of γ radiations

- They are neutral and massless. They do not deflect under the action of electric and magnetic fields.

- They propagate in vacuum at the speed of light.
- They have high penetrating power (many centimeters in lead).
- Their wavelengths are of the order of 10^{-11}m to 10^{-13}m.

General equation

$$_{Z'}^{A'}Y^* \rightarrow {}_{Z'}^{A'}Y + \gamma$$

Alpha (α) decay

Alpha decay is the emission of an α particle by a parent nucleus. This decay is a characteristic of heavy nuclei of A > 200. The emitted α particles are helium nuclei $_2^4He$, which have the following characteristics:
- They have a strong ionizing effect.
- Their speed is about 2×10^7m/s which is small compared to the speed of light in vacuum.
- They have low penetrating power (stopped by a sheet of paper).
- They are positively charged, so they are deflected under the action of electric and magnetic field.

General equation

$$_Z^AX \rightarrow {}_{Z-2}^{A-4}Y + {}_2^4He + \gamma$$

Liberated energy

$$\sum E_b = \sum E_a$$
$$E_{Kb} + m_bc^2 = E_{ka} + m_ac^2 + E_\gamma \text{ with } E_{kb} = 0$$
$$(m_b - m_a)c^2 = E_{ka} + E_\gamma \Rightarrow \Delta mc^2 = E_{ka} + E_\gamma$$
$$E_{lib} = E_{kY} + E_{k\alpha} + E_\gamma \text{ with } E_{kY} = 0 \text{ (heavy nucleus)}$$
$$E_{lib} = E_{k\alpha} + E_\gamma$$

Example

$$_{92}^{235}U \rightarrow {}_{90}^{231}Th + {}_2^4He + \gamma$$

Beta minus (β⁻) decay

This decay is a characteristic of nuclei rich in neutrons. It takes place with emission of β⁻ particle, which is nothing other than an electron $_{-1}^{0}e$ which has the following characteristics.
- It is less ionizing than α but 100 times more penetrating (about 7mm in aluminum).
- It propagates at a speed of 0.9c close to that of light in vacuum.
- They are negatively charged; they deflect under the action of electric and magnetic fields.

ATTENTION

The electron is emitted due to the decay of a neutron according to the following equation:
$$_0^1n \rightarrow {}_1^1p + {}_{-1}^{0}e + {}_0^0\bar{\nu}$$
The electron is formed inside the nucleus.

The study of the energy of the β⁻ decay has shown that the sum of the kinetic energy of the products and the energy of the γ ray is less than the energy liberated by the mass defect.
Apparently there is no conservation of total energy. To solve this enigma, Pauli has postulated, in 1930, the emission of a new particle, very difficult to detect. In 1934, Fermi called it antineutrino $_0^0\bar{\nu}$.
The conservation of total energy is then satisfied.

Characteristics of antineutrinos
- They have extremely small mass.
- They propagate at the speed of light.
- They are neutral and do not interact with matter (that is why they are hard to be detected).
- They have high penetrating power.

General equation

$$^A_ZX \to {}_{Z+1}^{\;\;A}Y + {}_{-1}^{\;0}e + {}_0^0\bar{\nu} + \gamma$$

Energy liberated

$$E_{lib} = \Delta mc^2 = E_{k\beta^-} + E_{\bar{\nu}} + E_{\gamma}$$

Example

$$^{210}_{83}Bi \to {}^{210}_{84}Po + {}_{-1}^{\;0}e + {}_0^0\bar{\nu} + \gamma$$

Beta plus (β^+) decay

The β+ decay was discovered in 1934 by Irene and Fredric Joliot-Curie. They bombarded a table boron nucleus ^8_5B with an α particle. The daughter nucleus is the unstable nitrogen nuclide $^{12}_7N$.

$$^8_5B + {}^4_2He \to {}^{12}_7N$$

This decay is a characteristic of nuclei rich in protons. It takes place with the emission of β^+ particle called a positron, positive electron or anti-particle of electron $_{+1}^{\;0}e$. Positron has the same mass of an electron but of opposite charge.

Using the same interpretation of β^- emission, positron emission is accompanied with the emission of a particle called neutrino $^0_0\nu$.

A neutrino is uncharged, together with extremely small mass, and moves at the speed of light with very large penetrating power.

ATTENTION

β^+ is emitted due to the decay of a proton according to the following equation:

$$^1_1p \to {}^1_0n + {}_{+1}^{\;0}e + {}^0_0\nu$$

General equation

$$^A_ZX \to {}_{Z-1}^{\;\;A}Y + {}_{+1}^{\;0}e + {}_0^0\nu + \gamma$$

Energy liberated

$$E_{lib} = \Delta mc^2 = E_{k\beta^+} + E_{\nu} + E_{\gamma}$$

Example

$$^{12}_7N \to {}^{12}_6C + {}_{+1}^{\;0}e + {}_0^0\bar{\nu} + \gamma$$

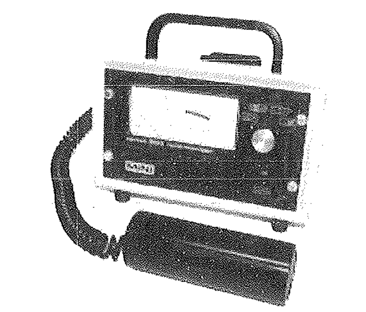
Types and Characteristics of Particles and Radiations

Radiation	α	β^-	β^+	γ
Name	Helium nucleus	Electron	Positron	Electromagnetic Radiation (photon)
Symbol	^4_2He	$_{-1}^{\;0}e$	$_{+1}^{\;0}e$	$^0_0\gamma$
Speed	20,000km/s	270,000 km/s	270,000 km/s	$c = 3 \times 10^8 m/s$
Charge	Positive	Negative	Positive	Neutral
Penetrating power	Low (stopped by a sheet of paper)	Good (can travel through several centimeters of aluminum)	Good (can travel through several centimeters of aluminum)	High (can pass through several centimeters of lead)
Mass	$4.0015u$	$5.5 \times 10^{-4}u$	$5.5 \times 10^{-4}u$	Massless
Reason behind emission	Usually emitted by heavy nuclides ($A > 200$)	Usually emitted by nuclides reach in neutrons Accompanied with anti-neutrino $^0_0\bar{\nu}$ emission	Usually emitted by nuclides reach in protons Accompanied with neutrino $^0_0\nu$ emission	Emitted due to the de-excitation of the daughter nucleus from an excited state to the ground state.

10.5- RADIOACTIVE DETECTORS AND COUNTERS

A radioactive detector is a device used to detect the presence of radioactive radiations.

Scintillation counter: certain compounds, like some plastics, zinc sulfide, sodium iodide, when crossed by radioactive ray produce lightening or scintillation of light. By using a device (photo-multiplier), scintillated light is transformed into electric impulses that can be amplified and recorded.



Geiger-Muller counter

Dosimeter: this device contains a film sensitive to radioactive rays. Once developed, the film permits the evaluation of the dose of rays received.

Geiger-Muller counter: is a device that detects radioactive radiations based on their ionizing properties.

10.5- LAWS OF RADIOACTIVE DECAY

Let N_0 be the initial number of nuclides present in a radioactive sample at the instant $t_0 = 0$ and N the number of nuclides present (not decayed or remained) at the instant t.

At the instant $t + dt$, the number of the non-decayed nuclides is $N + dN$.

The number of decayed nuclides during the time interval dt is:

$$N - (N + dN) = -dN.$$

The counting of the particles emitted shows that the number of decayed nuclides $(-dN)$ is directly proportional to N and to the time interval dt.

This can be written as:

$$-dN = \lambda N dt$$

The constant of proportionality λ represents the decay constant of the substance constituting the sample. In SI unit, it is expressed in [s^{-1}]. It varies according to the nuclides.

The preceding relation can be written as:

$$\frac{dN}{dt} + \lambda N = 0$$

The above equation is a first order differential equation and its solution has the form: $N = N_0 e^{-\lambda t}$

This is a decreasing exponential function of time. It expresses the decay law stated as follows:

The number of radioactive nuclides decreases exponentially with time.

ATTENTION

The radioactive decay law states that the probability per unit time that a nucleus will decay is a constant. This constant is called the decay constant and is denoted by λ.

TIP

$$\frac{dN}{N} = -\lambda dt$$

$$\int_0^{N_0} \frac{dN}{N} = -\lambda \int_0^t dt$$

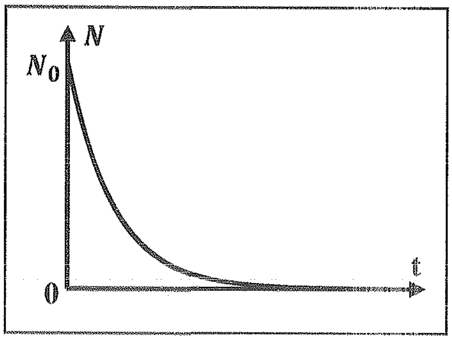
$$[\ln N]_{N_0}^{N} = -\lambda [t]_0^t$$

$$\ln N - \ln N_0 = -\lambda (t - 0)$$

$$\ln \frac{N}{N_0} = -\lambda t$$

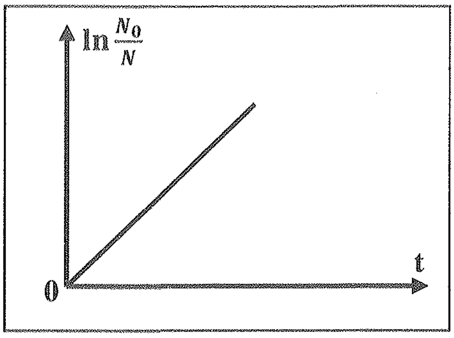
Graphical representation

Graph of N versus t



Graph of $\ln\frac{N_0}{N}$ versus t

$\ln\frac{N}{N_0} = -\lambda t \Longrightarrow \ln\frac{N_0}{N} = \lambda t$ which represents an equation of a straight line passing through origin with $slope = \lambda$



10.6- ACTIVITY AND PERIOD (HALF-LIFE)

The activity A of a sample represents the number of disintegrations per unit time. In SI units, the it is expressed in becquerel of symbol [Bq].
One becquerel represents one disintegration per second.

> **TIP**
> **For a small duration**
> $A = -\frac{\Delta N}{\Delta t}$

Law of radioactive decay

$A = -\frac{dN}{dt} = \lambda N \Longrightarrow \frac{dN}{dt} + \lambda N = 0$ (1st order differential equation in N)

Solution: $N = N_0 e^{-\lambda t}$ where $\begin{cases} N \text{ is the number of nuclei present (not decayed) at the instant } t \\ N_0 \text{ is the initial number of nuclei present at the instant } t_0 = 0s \\ \lambda \text{ is the decay constant in } [s^{-1}] \end{cases}$

The number of radioactive nuclei decreases exponentially with time.

$N = N_0 e^{-\lambda t}$ (multiply both sides by λ)

$\lambda N = \lambda N_0 e^{-\lambda t} \Longrightarrow A = A_0 e^{-\lambda t}$ where $\begin{cases} A \text{ is the activity at the instant } t \\ A_0 \text{ is the initial activity at the instant } t_0 = 0s \end{cases}$

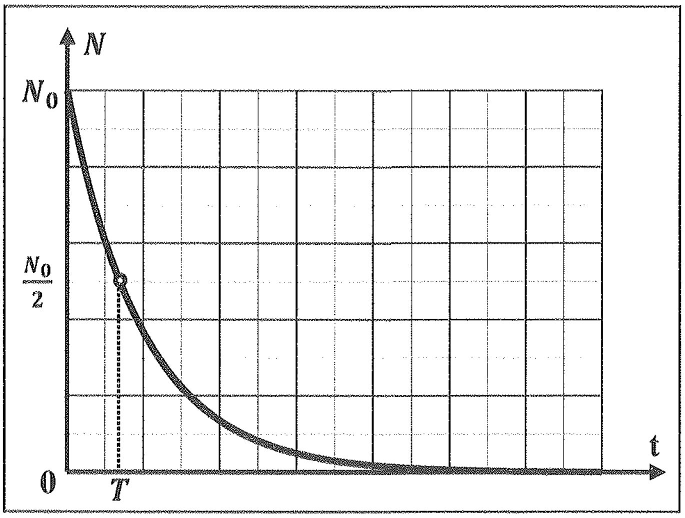
Similarly, $m = m_0 e^{-\lambda t}$ where $\begin{cases} m \text{ is the remaining mass at the instant } t \\ m_0 \text{ is the initial mass at the instant } t_0 = 0s \end{cases}$

> **TIP**
> **The half-life is a characteristic of the radioactive substance**

The period T (half-life) of a radioactive substance is the time interval after which the activity becomes equal to half its initial value.

At $t = nT$; $N = \frac{N_0}{2^n}$, $A = \frac{A_0}{2^n}$ and $m = \frac{m_0}{2^n}$

where n is the number of periods.



Relation between T and λ:

$N = N_0 e^{-\lambda t}$ with $N = \frac{N_0}{2}$ at $t = T$

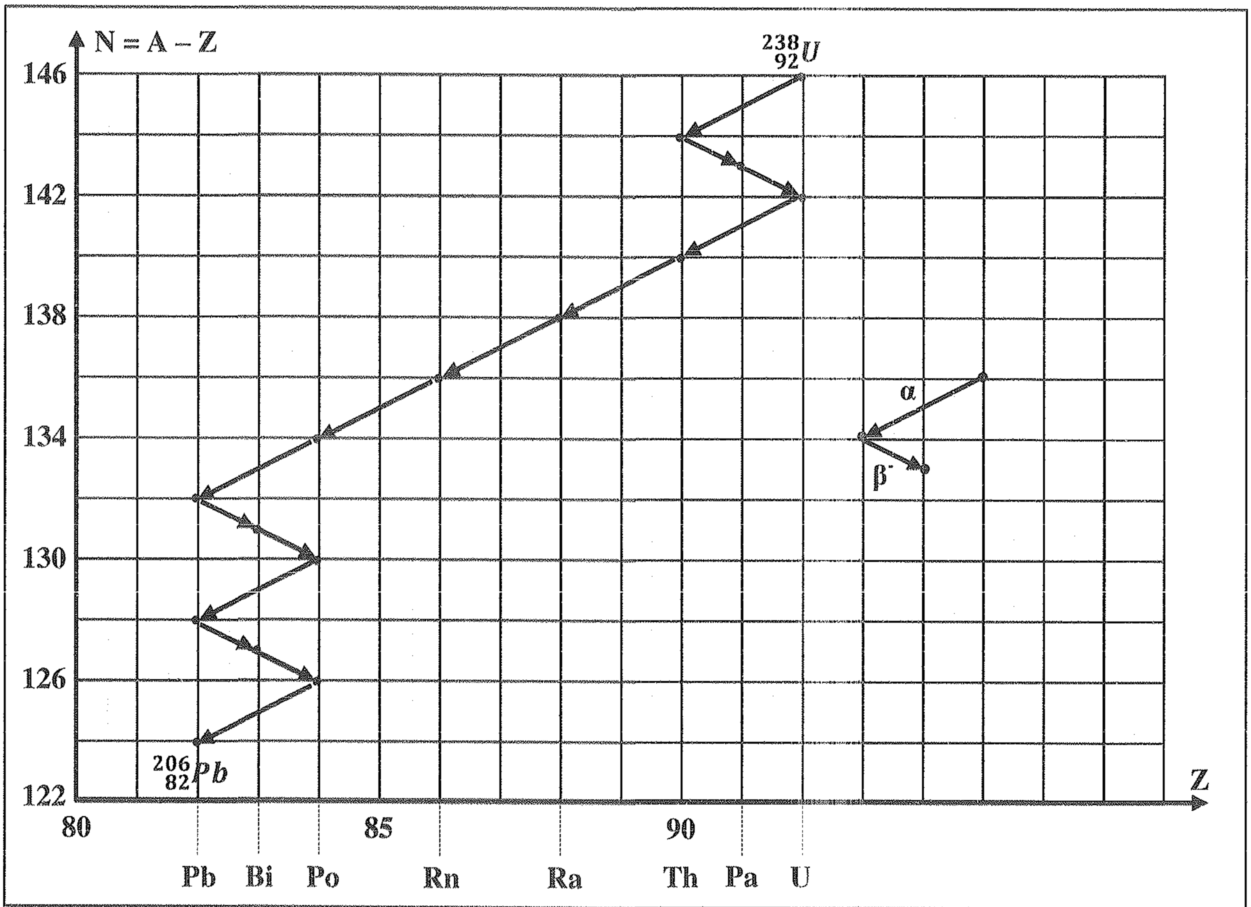
$\frac{N_0}{2} = N_0 e^{-\lambda t} \Longrightarrow e^{-\lambda t} = \frac{1}{2} \Longrightarrow \lambda = \frac{\ln 2}{T}$

Relation between power, activity and the liberated energy:

$P = \frac{E}{\Delta t} = \frac{NE_{lib}}{\Delta t} = A \times E_{lib}$.

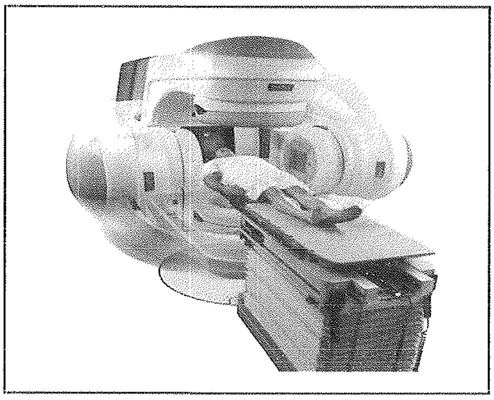
10.7- DECAY FAMILY

If, after disintegration, the daughter nucleus is itself radioactive, it disintegrates into another daughter nucleus, which can be also radioactive, and so on. These successive disintegrations will stop when the daughter nucleus obtained is stable. The set of these disintegrations constitute a decay family (or series). A radioactive family is given the name of the first element constituting it. The uranium ${}^{238}_{92}U$ family starts with the uranium ${}^{238}_{92}U$ nuclide and ends at the ${}^{206}_{82}Pb$ nuclueus after a series of α and β^- disintegrations. Other radioactive families are those of thorium and of unranium, which are natural. The only artifical radioactive family is that of neptunium.



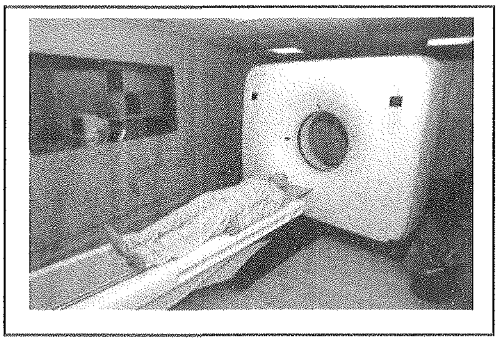
10.8- APPLICATIONS OF RADIOACTIVITY IN MEDICINE

Radiotherapy: is the destruction of the cancerous cells by irradiating the tumor by γ radioactive rays. This treatment has side effects like tiredness, loss of appetite, and vomiting.



Scintigraphy: a method to explore an organ by injecting a radioactive source which is used as a tracer. For example, the injection of the radioelement $^{131}_{53}I$ (Iodine), it becomes possible to visualize, localize, study and control the working of the thyroid gland, thanks to the detection of radioactive γ rays.

Positron Emission Tomography: is a nuclear medicine, functional imaging technique that is used to observe metabolic processes in the body. The system detects pairs of γ rays emitted indirectly by positron emitting radionuclide (tracer), which is introduced into the body on a biologically active molecule. Three dimensional images of tracer concentration within the body are then constructed by computer analysis,



CHAPTER 10 – RADIOACTIVITY
EXERCISES AND PROBLEMS

Exercise 1*:

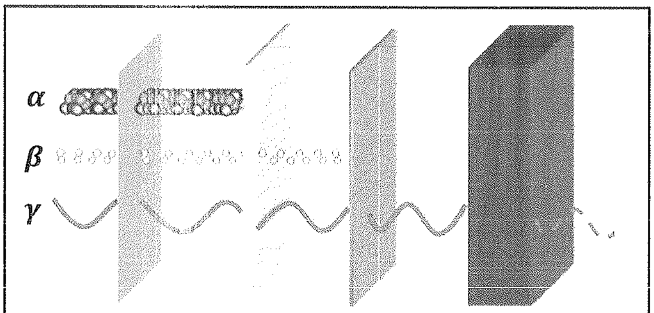
1- Define the following terms:
- **1.1-** Radioactivity.
- **1.2-** Isotopes.
- **1.3-** Nuclide.
- **1.4-** Activity.
- **1.5-** Half-life.

2- Determine the composition of the following nuclei:
- **2.1-** ${}^{14}_{6}C$.
- **2.2-** ${}^{18}_{8}O$.
- **2.3-** ${}^{235}_{92}U$.

3- Give the name, and the symbol of the following radioactive radiations:
- **3.1-** Alpha radiation.
- **3.2-** Beta minus radiation.
- **3.3-** Beta plus radiation.
- **3.4-** Gamma radiation.

4- List two properties for each of the radioactive radiations.

5- A radioactive source emits nuclear radiations. Match between the nuclear radiations and the substances that either stop or reduce their penetrating power.



6- Name a detector of radioactive radiation.

7- A 200g organ receives energy of $2 \times 10^{-2} J$. Calculate the absorbed dose by the organ.

8- An organ absorbs $50 \times 10^{-2} Gy$ of α rays. Calculate the physiological equivalence of the dose.
Given: $R.B.E(\alpha) = 20$.

9- The energy received by an organ per one kilogram is 8×10^{-2}J. Calculate the physiological equivalent of the dose in the following cases:
- **9.1-** The organ receives α radiations.
- **9.2-** The organ receives β radiations.

10- A 80kg man is exposed to alpha radiations. He receives energy of 8J.
Given: $R.B.E(\alpha) = 20$.
- **10.1-** Calculate, in [Gy], the absorbed dose by the man.
- **10.2-** Calculate, in [Sv], the physiological equivalent of a dose; then, deduce the effect of these radiations on the man.

P.E [Sv]	Effects
Higher than 10	death
5	50% death, 50% cancer and other risks
2	10% death, cancer and other risks
1	Troubles in digestion, sterility
0.05	Modification of the blood formula

11- A patient is believed to have a blood circulation problem. The doctor therefore introduces a medical tracer into the patient's blood stream to diagnose the problem.

11.1- What is a medical tracer?

11.2- What type of nuclear radiation should the tracer emit? Explain your answer briefly.

11.3- There are two radio-nuclides that emit the above nuclear radiation. One of the nuclides has a half-life much smaller than the other. Which nuclide should be used as the tracer? Explain your answer briefly.

Exercise 2*:

Indicate with justification the type of the radioactive decay in each of the following reactions.

1- $^{222}_{88}Ra \rightarrow {}^{4}_{2}He + {}^{218}_{86}Rn.$

2- $^{131}_{53}I \rightarrow {}^{131}_{54}Xe + {}^{0}_{-1}e.$

3- $^{22}_{11}Na \rightarrow {}^{0}_{+1}e + {}^{22}_{10}Ne.$

Exercise 3*:

Write a balanced nuclear equation for the decay of the following nuclides (The mode of decay is indicated in parentheses and denote the daughter nucleus by $^{A}_{Z}X$ where Z and A must be calculated).

1- $^{131}_{53}I$ (beta minus emission).

2- $^{235}_{92}U$ (alpha emission).

3- $^{50}_{25}Mn$ (beta plus emission).

Exercise 4:

Consider the two nuclear equations:

$$^{235}_{92}U \rightarrow {}^{231}_{90}Th^{*} + {}^{A_1}_{Z_1}X \ldots (1)$$
$$^{137}_{55}Cs \rightarrow {}^{137}_{56}Na + {}^{A_2}_{Z_2}Y \ldots (2)$$

1- For equation 1:

1.1- Determine A_1 and Z_1 by applying the convenient laws. Deduce the name and the symbol of the particle X.

1.2- Indicate the type of the radioactive decay.

1.3- Give two properties of the particle X.

1.4- The emission of X is accompanied by the emission of another radiation:

1.4.1- Write the de-excitation equation of the daughter nucleus Th.

1.4.2- Indicate the reason behind the emission of this radiation.

1.4.3- Indicate the nature of this radiation.

1.4.4- Give three of its properties.

2- For equation 2:

2.1- Calculate A_2 and Z_2 by applying the convenient laws.

2.2- Give the name and the symbol of the emitted particle Y.

2.3- Indicate the type of the radioactive decay.

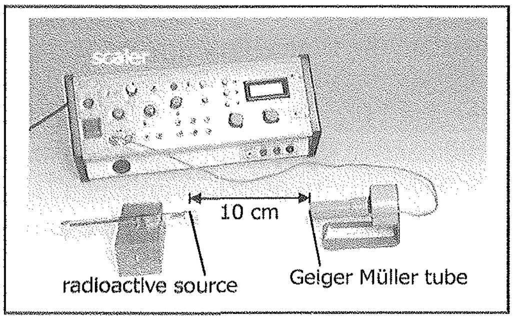
Exercise 5:

A student set up the following apparatus for detecting nuclear radiation.

1- Define a radioactive detector.

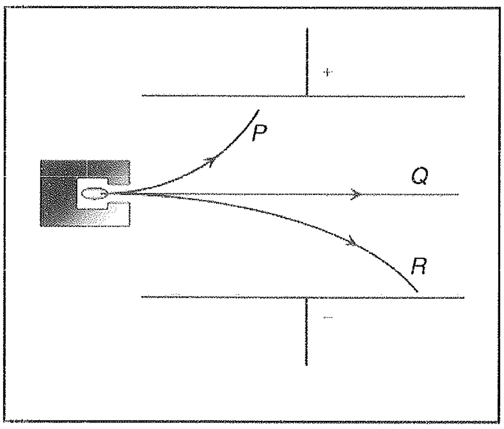
2- Which types of radiation can be detected using the above apparatus?

3- Give the names of two radioactive detectors other than the Geiger counter.



Exercise 6*:

A radioactive source emits α, β^- and γ radiations. The three types of radiation pass through a uniform electric field formed by two parallel charged plates as shown.



1- Indicate the type of charge of P, Q, and R.
2- What do P, Q and R represent?
3- Give:
  3.1- The name and the symbol of α and β^-.
  3.2- The nature of the γ radiation.
  3.3- Two properties (ionization and penetrating power) of each of P, Q, and R.
4- A β^+ particle is ejected between the two plates.
  4.1- Does this particle deflect? Justify.
  4.2- Indicate the direction of the deflection of this particle.

Exercise 7:

Which type of nuclear radiation:
1- is the most penetrating?
2- has the highest ionizing power?
3- is used to sterilize medical instruments?
4- is stopped by a piece of paper?
5- is stopped by lead block of thickness 25mm?
6- is stopped by aluminum sheet of thickness 5cm?

Exercise 8*:

The isotope carbon-14 ($^{14}_{6}C$) is radioactive and has a half-life of 5730 years. At t_0 = 0sec, a sample of $^{14}_{6}C$ contains 10^{10} nuclei.
Calculate the remaining number of nuclei of carbon-14 after t = 25000 years.

Exercise 9*:

At time t = 0s, a radioactive sample of $^{11}_{6}C$, of half life 20.4 min, contains 3.5×10^{-6}g.
The molar mass of $^{11}_{6}C$ is 11g/mole.
1- Determine the number N_0 of nuclei in the sample at t = 0 s
2- Define the radioactive constant and then calculate its value
3- Calculate the activity of the sample at t = 0s and at t = 8h.
4- How long does the carobn-11 sample take to lose $\frac{3}{4}$ of its initial number of nuclei?

Exercise 10:

A sample of the isotope I-131, which has a half-life of 8.04 days, has an activity of 5.0mCi at t = 0s.
Calculate the time elapsed when the activity of the sample is 2.1mCi.

Exercise 11*:

A radioactive sample, of decay constant λ and radioactive period T, contains 89×10^{18} nuclei of uranium-235. The activity of the sample is 2718Bq.
1- What does Bq stand for? Give another unit of activity.
2- Calculate λ.
3- Deduce T.

Exercise 12*:

A stationary radium nucleus ${}^{226}_{88}Ra$, undergoes an alpha decay where the daughter nucleus is radon ${}^{A}_{Z}Rn$ and is born in the ground state.

The masses are 226.025410u for ${}^{226}_{88}Ra$, 222.017578u for ${}^{A}_{Z}Rn$, and 4.002603u for ${}^{4}_{2}He$.

1- Write the equation of the nuclear decay.
2- Calculate Z and A by applying the convenient laws.
3- The above disintegration of radium-226 is not accompanied with the emission of gamma rays. Why?
4- Calculate the liberated energy by the nuclear decay of one nucleus of radium.
5- By applying the conservation laws of linear momentum and energy, determine the kinetic energy of the alpha particle after the decay.
6- Deduce the kinetic energy of the radon ${}^{A}_{Z}Rn$. Draw a conclusion.

Exercise 13*:

The disintegrations of ${}^{212}_{83}Bi$ gives rise to ${}^{208}_{81}Tl^{*}$ according to the following reaction:

$${}^{212}_{83}Bi \rightarrow {}^{208}_{81}Tl^{*} + {}^{4}_{2}He$$

Given: $m_{Bi} = 211.991876u$; $m_{Tl} = 207.982013u$ and $m_{He} = 4.0015u$.

$1u = 1.66 \times 10^{-27}kg$; $c = 3 \times 10^{8}$ m/s.

1- Write the de-excitation equation of the daughter nucleus Tl.
2- Give two properties of the radiation emitted by the daughter nucleus Tl.
3- Determine the energy liberated during the nuclear decay.
4- Deduce the energy liberated by 1kg of ${}^{212}_{83}Bi$.

Exercise 14:

Thorium 228 (${}^{228}_{90}Th$) decays to lead 212 (${}^{212}_{Z}Pb$) in a series of steps, through intermediate radioactive nuclides, via alpha decay. The net reaction can be written as follows:

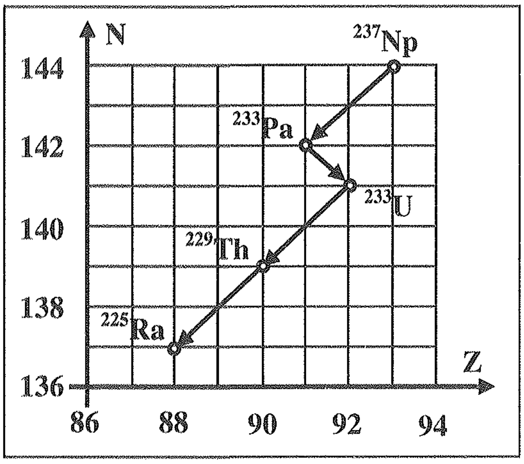
$${}^{228}_{90}Th \rightarrow {}^{212}_{Z}Pb + n\alpha$$

1- Is this reaction spontaneous or stimulated?
2- Give the name and the symbol of the α radiation.
3- State two of its properties.
4- Specify which nucleus thorium 228 or lead 212 is more stable.
5- Determine n and Z by applying the convenient laws.

Exercise 15:

The adjacent document represents part of the decay family of neptunium-237 nucleus. N represents the number of neutrons and Z is the charge number.

1- Define radioactivity.
2- Determine the composition of Np-237 nucleus.
3- Use the figure to write the complete decay equation of ${}^{237}_{93}Np \rightarrow {}^{225}_{88}Ra$
4- Neptunium-237 nucleus can decay to ${}^{209}_{83}Bi$ after series of α and β^{-} decays. Write the complete decay equation.



Exercise 16*:

Cobalt-60 is the longest lived isotope of cobalt, with a half-life of T = 5.3 years. Cobalt-60 is used in the inspection of materials to reveal internal structure, flaws, or foreign objects and in the sterilization of food. In medicine, it is used to treat cancer and to sterilize medical equipment.

Given:

$^{A}_{Z}X$	$^{60}_{27}Co$	$^{60}_{28}Ni$	$^{0}_{-1}e$
Mass (in u)	59.9190	59.9154	0.00055

- 1u = 931.5MeV/c^2; speed of light in vacuum: c = 3 × 10^8m.s^{-1}.
- Planck's constant: h = 6.63 × 10^{-34} J.s; Avogadro's constant: N_A = 6.02 × 10^{23}mol^{-1}.
- Molar mass of cobalt-60: M = 60g.mol^{-1}.

1- A sample of $^{60}_{27}Co$ has a mass m_0 = 1g at t = 0s. Determine the remaining number of $^{60}_{27}Co$ nuclei and the activity of this sample at the end of 10.6 years.

2- One of the disintegrations of $^{60}_{27}Co$ gives rise to the nickel isotope $^{60}_{28}Ni$.

2.1- Write, with justification, the equation of the disintegration of one cobalt nucleus $^{60}_{28}Ni$. Identify the emitted particle.

2.2- Calculate, in MeV, the energy liberated by this disintegration.

2.3- Determine the energy liberated by the disintegration of 1g of cobalt $^{60}_{27}Co$.

2.4- Knowing that the energy liberated from the complete combustion of 1g of coal is 30kJ, find the mass of coal that would liberate the same amount of energy calculated in part 2.3.

Exercise 17:

Cobalt $^{60}_{27}Co$ is a β^- radioactive. The daughter nucleus $^{A}_{Z}Ni$ undergoes a downward transition to the ground state. The energy due to this downward transition is $E(\gamma) = 2.5060MeV$.

The β^- particle is emitted with a kinetic energy $K.E(\beta^-) = 0.0010MeV$.

Numerical data:

Mass of the $^{60}_{27}Co$ nucleus: 59.91901u; mass of the $^{A}_{Z}Ni$ nucleus: 59.91544u;
mass of an electron: 5.486 × 10^{-4}u; 1u = 931.5MeV/c^2; 1MeV = 1.6 × 10^{-13}J.

A- Study of the disintegration

1- Determine A and Z.

2- Calculate, in u, the mass defect Δm during this disintegration.

3- Deduce, in MeV, the energy E liberated by this disintegration.

4- During this disintegration, the daughter nucleus is practically obtained at rest. In what form of energy does E appear?

5-

5.1- Deduce, from what preceded, that the electron, emitted by the considered disintegration, is accompanied by a certain particle.

5.2- Give the name of this particle.

5.3- Give the charge number and the mass number of this particle.

5.4- Deduce in MeV, the energy of this particle.

6- Write down the global equation of this disintegration.

B- The use of cobalt 60

In medicine, we use a source of radioactive cobalt $^{60}_{27}Co$ of activity A = 6 × 10^{19}Bq.

The emitted β^- particles are absorbed by the living organism.

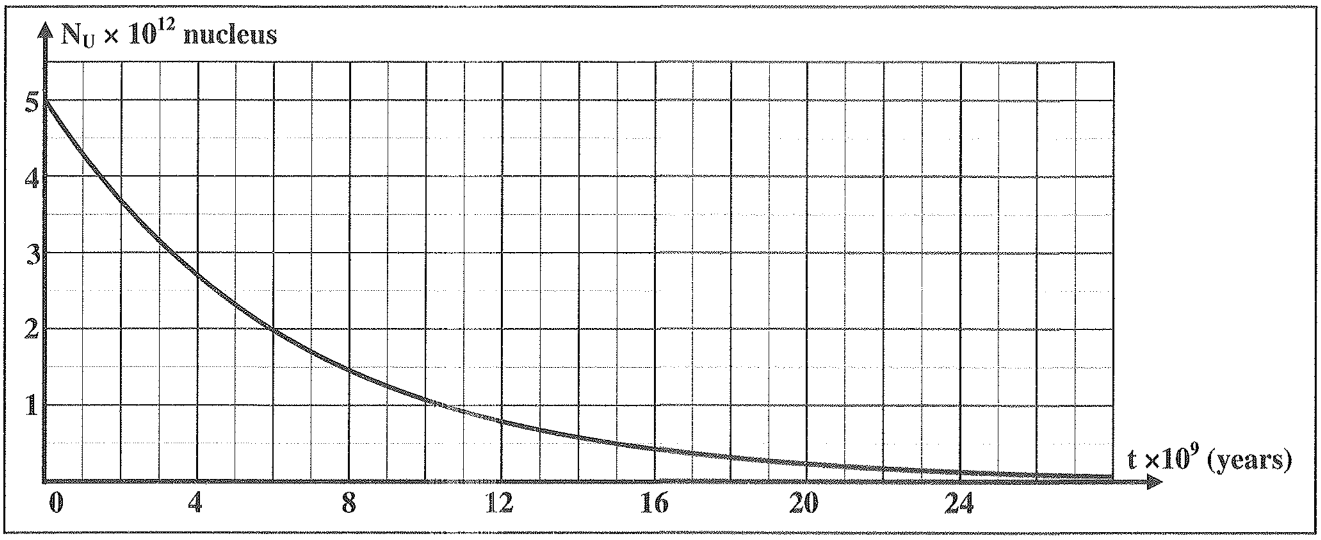
1- The energy of the particle mentioned in the question (A-5) is not absorbed by the living organism. Why?

2- Calculate, in watt, the power transferred to the organism.

3- This large power is used in radiotherapy. What is its effect?

Exercise 18:

The object of this exercise is to determine the age of the Earth using the disintegration of a uranium 238 nucleus ($^{238}_{92}U$) into a lead 206 nucleus ($^{206}_{82}Pb$).



When we determine the number of lead 206 nuclei in a sample taken out from a rock that did not contain lead when it was formed, we can then determine its age that is the same as that of the Earth.

The above figure represents the curve of the variation of the number N_U of uranium 238 nuclei as a function of time.

1 division on the axis of ordinates corresponds to 10^{12} nuclei.

1 division on the axis of abscissa corresponds to 10^9 years.

The equation of the disintegration of Uranium 238 into lead 206 is:

$$^{238}_{92}U \rightarrow {}^{206}_{82}Pb + x\beta^- + y\alpha$$

1- Determine, specifying the laws used, the values of x and y.

2- Referring to the curve, indicate the number N_{0U} of uranium 238 nuclei existing in the sample at the date of its birth $t_0 = 0$.

3- Referring to the curve, determine the period (half-life) of uranium 238. Deduce the value of the radioactive constant λ of uranium 238.

4-

4.1- Give, in terms of N_{0U}, λ and t, the expression of the number N_U of uranium 238 nuclei remaining in the sample at instant t.

4.2- Calculate the number of uranium 238 nuclei remaining in the sample at instant $t_1 = 2 \times 10^9$ years

4.3- Verify this result graphically:

5- The number of lead 206 nuclei existing in the sample at the instant of measurement (age of the Earth) is $N_{Pb} = 2.5 \times 10^{12}$ nuclei.

5.1- Give the relation among N_U, N_{0U} and N_{Pb}.

5.2- Calculate the number N_U of uranium nuclei remaining in the sample at the date of measurement.

5.3- Determine the age of the Earth.

Exercise 19:

Given: $1u = 1.66 \times 10^{-27}kg = 931.5MeV/c^2$; $1MeV = 1.6 \times 10^{-13}J$;

Mass of some nuclei: m (Po) = 209.9829u; m (Pb) = 205.9745u; m (α) = 4.0026u;

$h = 6.63 \times 10^{-34}J.s$; $c = 3 \times 10^8 m/s$.

A- Decay of polonium 210

The polonium $^{210}_{84}Po$ is an α emitter. The daughter nucleus produced by this decay is the lead $^{A}_{Z}Pb$.

1- Determine Z and A specifying the laws used.

2- Calculate, in MeV and in J, the energy liberated by this decay.

3- The nucleus $^{210}_{84}Po$ is initially at rest. We suppose that the daughter nucleus $^{A}_{Z}Pb$ is obtained at rest and in the fundamental state. Deduce the kinetic energy of the emitted α particle.

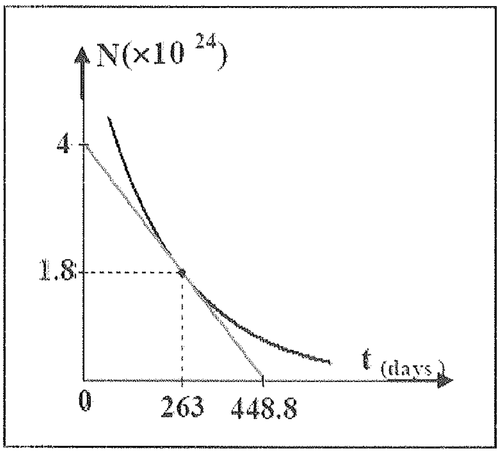
4- In general, the decay of $^{210}_{84}Po$ is accompanied by the emission of γ radiation.

4.1- Due to what is the emission of γ radiation?

4.2- The emitted γ radiation has the wavelength $\lambda = 1.35 \times 10^{-12}$m in vacuum. Using the conservation of total energy, determine the kinetic energy of the emitted α particle.

B- Radioactive period of polonium 210

The adjacent figure shows the curve representing the variations with time t of the number N of the nuclei present in the radioactive sample $^{210}_{84}Po$, this number being called N_0 at the instant $t_0 = 0$. The same figure shows also the tangent to that curve at the instant $t_1 = 263$ days.



1- Write down the expression of N as a function of t and specify what does each term represent.

2- The activity of the radioactive sample is given by: $A = -\frac{dN}{dt}$.

2.1- Define the activity A.

2.2- Using the given on the figure above, determine the activity A of the sample at the instant $t_1 = 263$ days.

3- Deduce the value of the radioactive constant and the value of the half-life (period) of polonium 210.

Exercise 20:

The aim of this exercise is to show evidence of some characteristics of iodine 131.

Iodine 131 ($^{131}_{53}I$) is radioactive and is a β^- emitter. Its radioactive period (half-life) is 8 days.

Given: Mass of an electron: $m_e = 5.5 \times 10^{-4}$u; 1MeV = 1.6×10^{-13}J; 1u = 931.5MeV/c^2.

Element	Iodine ($^{131}_{53}I$)	Cesium ($^{137}_{55}Cs$)	Xenon ($^{131}_{54}Xe$)
Mass of nucleus	130.8770 u	136.8773 u	130.8754 u

A- Disintegration of iodine 131

1- Write down the equation of the disintegration of iodine 131 and identify the daughter nucleus.

2- The disintegration of iodine 131 nucleus, is often, accompanied with the emission of γ rays. Due to what is this emission?

3- Calculate the radioactive constant λ of iodine 131 in day^{-1} and in s^{-1}.

4- Show that the energy liberated by the disintegration of one nucleus of iodine 131 is $E_{lib} = 1.56 \times 10^{-13}$J.

B- Application in medicine

During a medical examination of a thyroid gland of a patient, we inject this gland with a solution of iodine 131. The thyroid of this patient captures from this solution a number $N = 10^{11}$ of iodine nuclei.

1- Calculate, in Bq, the activity A corresponding to these N nuclei knowing that A = λN.

2- Calculate, in J, the energy liberated by the disintegration of these N nuclei.

3- Deduce, in J/kg, the value of the dose absorbed by the thyroid gland knowing that its mass is 25g.

C- Contamination

On the 26th of April 1986, an accident took place in the nuclear power plant of Chernobyl that provoked an explosion in one of the reactors. One of the many radioactive elements that were ejected to the atmosphere is the iodine 131. This element spread on the ground, absorbed by cows and contaminated their milk and then captured by the thyroid gland of consumers.

Every morning, a person drank a certain quantity of milk containing $N_0 = 2.6 \times 10^{16}$ nuclei of iodine 131. We suppose that all these nuclei were captured by the thyroid of that person, and that the person drank the first quantity at the instant $t_0 = 0$.

1- Determine, in terms of N_0 and λ (expressed in day^{-1}), the number of iodine 131 nuclei that remained in the thyroid, at the instant:
   1.1- $t_1 = 1$ day, (just after drinking the 2nd quantity of milk);
   1.2- $t_2 = 2$ days, (just after drinking the 3rd quantity of milk).
2- Deduce, at the instant $t_3 = 3$ days just after drinking the 4th quantity of milk that the number N_3 of the iodine 131 nuclei that remained in the thyroid is: $N_3 = N_0\,(1 + e^{-\lambda} + e^{-2\lambda} + e^{-3\lambda})$ where λ is expressed in day^{-1}.
3- Serious troubles in the thyroid gland will take place if the activity of the iodine 131 exceeds 75×10^9 Bq. Show that at the instant t_3, the person was in danger.

__Exercise 21:__

The radioactive carbon isotope $^{14}_{6}C$ is a β^- emitter. In the atmosphere, $^{14}_{6}C$ exists with the carbon 12 in a constant ratio.

When an organism is alive it absorbs carbon dioxide that comes indifferently from carbon 12 and carbon 14. Just after the death of an organism, this absorption stops and carbon 14, that it has, disintegrate with a half life T = 5700 years.

In living organisms, the ratio of the number of carbon 14 atoms to that of the number of carbon 12 atoms is:

$r_0 = \frac{\text{initial number of carbon 14 atoms}}{\text{number of carbon 12 atoms}} = \frac{N_0(^{14}_{6}C)}{N'(^{12}_{6}C)}$.

After the death of an organism by a time t, the ratio of the number of carbon 14 atoms to that of the number of carbon 12 atoms becomes: $r = \frac{\text{remaining number of carbon 14 atoms}}{\text{number of carbon 12 atoms}} = \frac{N(^{14}_{6}C)}{N'(^{12}_{6}C)}$.

1- The disintegration of $^{14}_{6}C$ is given by: $^{14}_{6}C \rightarrow {}^{A}_{Z}N + \beta^- + {}^{0}_{0}\bar{\nu}$.
   Calculate Z and A, specifying the laws used.
2- Calculate, in year^{-1}, the radioactive constant λ of carbon 14.
3- Using, the law of radioactive decay of carbon 14, $N(^{14}_{6}C) = N_0(^{14}_{6}C) \times e^{-\lambda t}$.
   Show that $r = r_0 e^{-\lambda t}$.
4- Measurements of $\frac{r}{r_0}$, for specimens a, b and c, are given in the following table:

ratio	specimen a	specimen b	specimen c
$\frac{r}{r_0}$	0.914	0.843	0.984

   4.1- Specimen b is the oldest. Why?
   4.2- Determine the age of specimen b.
5-
   5.1- Calculate the ratio $\frac{r}{r_0}$ for $t_0 = 0$, $t_1 = 2T$, $t_2 = 4T$ and $t_3 = 6T$.
   5.2- Trace then the curve $\frac{r}{r_0} = f(t)$ by taking the following scales:

- On the abscissa axis: 1cm → 2T
- On the ordinate axis: 1cm → $\frac{r}{r_0} = 0.2$

5.3- To determine the date of death of a living organism, it is just enough to measure $\frac{r}{r_0}$.
Explain why we cannot use the traced curve to determine the date of the death of an organism that died several millions years ago.

Exercise 22:

The radioactive isotope phosphorus 32 ($^{32}_{15}P$) is used in the diagnosing of cancer. Phosphorus 32, is injected into the human body, it decays and gives radiations. These radiations are detected by an appropriate device to create the image of the inside of the human body.
The aim of this exercise is to determine the dose of radiation absorbed by a tissue of a patient during 6 days.
Phosphorus 32 ($^{32}_{15}P$) is a β^- emitter; it disintegrates to give an isotope $^{A}_{Z}S$ of sulfur.
Given:
- mass of $^{32}_{15}P$: 31.965 678u;
- mass of $^{A}_{Z}S$: 31.963 293u;
- mass of electron: 5.486×10^{-4}u ;
- The radioactive period of $^{32}_{15}P$: 14.3days;
- $1\ u = 931.5 MeV/c^2$;
- $1MeV = 1.6 \times 10^{-13}J$.

1- Energy liberated by the decay of phosphorus 32

The disintegration of phosphorus 32 nucleus is given by the following reaction: $^{32}_{15}P \rightarrow {}^{A}_{Z}S + {}^{0}_{-1}e + {}^{0}_{0}\bar{\nu}$.

1.1- Determine A and Z.

1.2- Prove that the energy liberated by the above disintegration is $E_{lib} = 1.7106MeV$.

1.3- The sulfur nucleus is produced in the ground state. The emitted antineutrino carries energy of 1.011MeV.

1.3.1- The above disintegration of phosphorus 32 is not accompanied with the emission of gamma rays. Why?

1.3.2- Calculate the kinetic energy carried by the emitted electron knowing that phosphorus and sulfur are considered at rest.

2- Absorbed dose

A patient is injected by a pharmaceutical product containing phosphorus 32. The initial activity of phosphorus
32 in the pharmaceutical product at $t_0 = 0$, is $A_0 = 1.36 \times 10^6 Bq$.

2.1- Calculate, in s^{-1}, the radioactive constant of phosphorus 32.

2.2- Deduce the number N_0 of nuclei of phosphorus 32 present in the pharmaceutical product at $t_0 = 0$

2.3-

2.3.1- Determine the remaining number N of nuclei of phosphorus 32 at t = 6 days.

2.3.2- Deduce the disintegrated number N_d of nuclei of phosphorus 32 during the 6 days.

2.3.3- The number of the emitted electrons is $N_e = 6.12 \times 10^{11}$ electrons during the 6 days. Why?

2.1- The emitted radiation is absorbed by a tissue of mass M = 112g. The antineutrino does not interact with matter, and suppose that the energy of the emitted electrons is completely absorbed by the tissue.

2.4.1- Calculate the energy E_{abs} absorbed by the tissue during the 6 days.

2.4.2- The absorbed dose by the tissue is $D = \frac{E_{abs}}{M}$ during the 6 days. Deduce the value of D in J/kg.

Exercise 23:

The bones scintigraphy is a medical examination that permits to observe bones and articulations. The aim of this exercise is to study a radioactive sample used in this scintigraphy.

This medical examination uses technetium-99 produced due to the disintegration of molybdenum-99 according to the following nuclear reaction:

$$^{99}_{42}Mo \rightarrow ^{99}_{43}Tc + ^{A}_{Z}X + \gamma$$

The energy of the emitted gamma (γ) photon is 140keV.

Given: $c = 3\times10^{8}m.s^{-1}$; $1eV = 1.6 \times 10^{-19}J$; Planck's constant $h = 6.6 \times 10^{-34}J.s$.

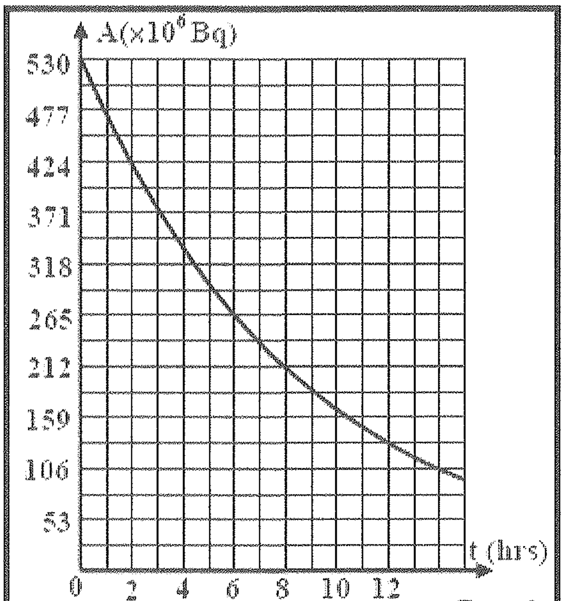
1- Identify the emitted particle $^{A}_{Z}X$, indicating the used laws.

2- The emitted particle $^{A}_{Z}X$ is always accompanied with the emission of another particle. Name this particle.

3- Indicate the cause of the emission of the gamma photon.

4- Calculate the wavelength of the emitted gamma photon.

5- Technetium-99 is a radioactive substance.
The graph of document 2 represents the activity of a sample of technetium-99 as a function of time.
Using document 2, show that the radioactive period (half-life) of technetium-99 is T = 6 hrs.



Doc. 2

6- In a session of scintigraphy examination, a patient is injected at $t_0 = 0$ by techenetium-99 of activity $A_0 = 530 \times 10^{6}Bq$. At the end of the examination session, the activity of technetium in the body of the patient is 63% of its initial value.

6.1- Write, at instant t, the expression of the activity A in terms of A_0, t and the decay constant λ.

6.2- Using the preceded expression, determine:

6.2.1- the duration of the examination session;

6.2.2- the ratio $\frac{A}{A_0}$ of technetium-99 at t = 40 hrs.

Exercise 24*:

In order to study the radioactivity of polonium $^{210}_{84}Po$ which is an α emitter, we take a sample of polonium 210 containing N_0 nuclei at the instant $t_0 = 0$.

A- Determination of the half-life (period)

We measure, at successive instants, the number N of the remaining nuclei. We calculate the ratio N/N_0 and the result are tabulated as in the following table:

t (in days)	0	50	100	150	200	250	300
N/N_0	1	0.78	0.61	0.47	0.37	0.29	0.22
$-\ln(N/N_0)$	0	0.25				1.24	

1- Draw again the above table and complete it by calculating at each instant $-\ln(N/N_0)$.

2- Trace the curve representing the variation of $f(t) = -\ln(N/N_0)$, as a function of time, using the scale: 1cm on the abscissa represents 25 days; 1cm on the ordinate represents 0.1.

3-

3.1- Knowing that $\ln(N/N_0) = -\lambda t$, determine graphically the value of the radioactive constant λ of polonium 210.

3.2- Deduce the half-life of polonium 210.

B- Activity of polonium 210

1- Define the activity of a radioactive sample.

2- Give the expression of the activity A_0 of the sample at the instant $t_0 = 0$, in terms of λ and N_0. Calculate its value for $N_0 = 5 \times 10^{18}$.

3- Give the expression, in terms of t, of the activity A of the sample.

4- Calculate the activity A:

4.1- at the instant t = 90 days.

4.2- When t increases indefinitely.

C- Energy liberated by the disintegration of polonium 210

1- The disintegration of a nucleus of polonium produces a daughter nucleus which is an isotope of lead $^{A}_{Z}Pb$. Determine A and Z.

2- Calculate, in MeV, the energy liberated by the disintegration of one nucleus of polonium 210.

3- The disintegration of a polonium nucleus may take place with or without the emission of a photon. The energy of an emitted photon is 2.20MeV. Knowing that the daughter nucleus has a negligible velocity, determine in each case the kinetic energy of the emitted α particle.

4- The sample is put in an aluminum container. Thus, the α particles are stopped by the container whereas the photons are not.
Knowing that half of the disintegrations are accompanied by a γ emission, determine the power transferred to the aluminum container at the instant t = 90 days.

Numerical data:
Mass of a polonium 210 nucleus: 209.9828u.
Mass of lead (Pb) nucleus: 205.9745u.
Mass of an α particle: 4.0015u.
$1u = 1.66 \times 10^{-27}kg = 931.5MeV/c^2$.

CHAPTER 10 – RADIOACTIVITY
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 4:

Part	Answer
1.1	By applying the law of conservation of mass number: $235 = 231 + A_1 \Rightarrow A_1 = 4$. By applying the law of conservation of charge number: $92 = 90 + Z_1 \Rightarrow Z_1 = 2$. Name: Helium. Symbol: ${}^{4}_{2}He$.
1.2	Alpha decay (α decay).
1.3	Positively charged – low penetrating power.
1.4.1	${}^{231}_{90}Th^* \to {}^{231}_{90}Th + {}^{0}_{0}\gamma$.
1.4.2	Due to the de-excitation of the daughter nucleus (the daughter nucleus drops from an excited state to the ground state).
1.4.3	Electromagnetic.
1.4.4	Neutral – massless – high penetrating power.
2.1	By applying the law of conservation of mass number: $137 = 137 + A_2 \Rightarrow A_2 = 0$. By applying the law of conservation of charge number: $55 = 56 + Z_2 \Rightarrow Z_2 = -1$.
2.2	Name: electron. Symbol: ${}^{0}_{-1}e$.
2.3	Beta minus decay (β^- decay).

Exercise 5:

Part	Answer
1	A device used to detect and to study radioactive rays.
2	β radiation and γ radiation.
3	Scintillation counter-dosimeter.

Exercise 7:

Part	Answer		
1	Gamma radiation.	4	Alpha radiation.
2	Alpha radiation.	5	Gamma radiation.
3	Gamma radiation.	6	Beta minus radiation.

Exercise 10:

Part	Answer
	$A = \frac{A_0}{2^n} \Longrightarrow 2^n = \frac{A_0}{A} \Longrightarrow \ln 2^n = \frac{A_0}{A} \Longrightarrow n \ln 2 = \ln\frac{A_0}{A}$. $n = \ln\frac{A_0}{A} \times \frac{1}{\ln 2} = \ln\frac{5}{2.1} \times \frac{1}{\ln 2} = 1.25$. $t = nT = 1.25 \times 8.04 = 10.05 days$.

Exercise 14:

Part	Answer
1	Spontaneous.
2	Name: Helium. Symbol: ${}^{4}_{2}He$.
3	Positively charged – low penetrating power.
4	Lead 212 is more stable since the daughter nucleus is more stable than the parent nucleus.
5	By applying the law of conservation of mass number: $228 = 212 + 4n \Rightarrow n = 4$. By applying the law of conservation of charge number: $90 = Z + 4 \times 2 \Rightarrow Z = 82$.

Exercise 15:

Part	Answer
1	Radioactivity is the phenomenon resulting from the spontaneous disintegration of an unstable

	atomic nucleus. During this phenomenon, an unstable nucleus called the parent nucleus is transformed into more stable daughter nucleus with the emission of a radiations and particles.
2	Number of protons = Z = 93. Number of neutrons = N = A – Z = 237 – 93 = 144.
3	${}^{237}_{93}Np \rightarrow {}^{233}_{91}Pa + {}^{A_1}_{Z_1}X.$ By applying Soddy's laws: Law of conservation of mass number: $237 = 233 + A_1 \Rightarrow A_1 = 4.$ Law of conservation of charge number: $93 = 91 + Z_1 \Rightarrow Z_1 = 2.$ ${}^{A_1}_{Z_1}X = {}^{4}_{2}He$. ${}^{233}_{91}Pa \rightarrow {}^{233}_{92}U + {}^{A_2}_{Z_2}Y.$ By applying Soddy's laws: Law of conservation of mass number: $233 = 233 + A_2 \Rightarrow A_2 = 0.$ Law of conservation of charge number: $91 = 92 + Z_2 \Rightarrow Z_2 = -1.$ ${}^{A_2}_{Z_2}Y = {}^{0}_{-1}e.$ ${}^{233}_{92}U \rightarrow {}^{229}_{90}Th + {}^{A_3}_{Z_3}Z.$ By applying Soddy's laws: Law of conservation of mass number: $233 = 229 + A_3 \Rightarrow A_2 = 4.$ Law of conservation of charge number: $92 = 90 + Z_3 \Rightarrow Z_3 = 2.$ ${}^{A_3}_{Z_3}Z = {}^{4}_{2}He.$ ${}^{229}_{90}Th \rightarrow {}^{225}_{88}Ra + {}^{A_4}_{Z_4}L.$ By applying Soddy's laws: Law of conservation of mass number: $229 = 225 + A_4 \Rightarrow A_2 = 4.$ Law of conservation of charge number: $90 = 88 + Z_4 \Rightarrow Z_4 = 2.$ ${}^{A_4}_{Z_4}L = {}^{4}_{2}He$. Therefore, ${}^{237}_{93}Np \rightarrow {}^{225}_{88}Ra + 3{}^{4}_{2}He + {}^{0}_{-1}e.$
4	${}^{237}_{93}Np \rightarrow {}^{209}_{83}Bi + x{}^{4}_{2}He + y{}^{0}_{-1}e.$ By applying Soddy's laws: Law of conservation of mass number: $237 = 209 + 4x + 0 \Rightarrow x = 7.$ Law of conservation of charge number: $93 = 83 + 14 - y \Rightarrow y = 4.$ ${}^{237}_{93}Np \rightarrow {}^{209}_{83}Bi + 7{}^{4}_{2}He + 4{}^{0}_{-1}e$.

Exercise 17:

Part	Answer
A.1	The conservation of charge number and of mass number gives: Z = 28 and A = 60.
A.2	$\Delta m = m_{before} - m_{after} = 59.91901 - (59.91544 + 0.0005486) = 0.0030214u.$
A.3	$\Delta m = 0.0030214 \times 931.5MeV/c^2 = 2.8144MeV/c^2.$ $E = \Delta m \times c^2 = 2.8144MeV.$
A.4	E appears in the form of the kinetic energy of the obtained particles and of radiant energy of the γ photon.
A.5.1	$K.E(\beta^-) + E(\gamma) = 0.0010 + 2.5060 = 2.507MeV.$ Is smaller than E = 2.8144MeV; therefore there is no conservation of energy, thus there is a necessity to admit the emission of another particle other tan electron.
A.5.2	Antineutrino.
A.5.3	Z = 0 and A = 0.

A.5.4	E = 2.8144 – 2.507 = 0.3074MeV.
A.6	${}^{60}_{27}Co \rightarrow {}^{60}_{28}Ni + {}^{0}_{-1}e + {}^{0}_{0}\bar{\nu} + \gamma$.
B.1	Because the antineutrino does not interact with matter.
B.2	The activity corresponds to 6×10^{19} disintegrations per second, this means that 6×10^{19} electrons emitted per second $\Longrightarrow P = 6 \times 10^{19} \times 0.001 \times 1.6 \times 10^{-13} W = 9.6kW$.
B.3	The destruction of cells.

Exercise 18:

Part	Answer
1	${}^{238}_{92}U \rightarrow {}^{206}_{82}Pb + x{}^{0}_{-1}e + y{}^{4}_{2}He$. Law of conservation of mass number: $238 = 206 + 4y \Longrightarrow y = 8$. Law of conservation of charge number: $92 = 82 - x + 2y \Longrightarrow x = 6$.
2	$N_{0U} = 5 \times 10^{12} nuclei$
3	At t = T; $N_U = \frac{N_{0U}}{2} = 2.5 \times 10^{12} nuclei$. Graphically, $T = 4.5 \times 10^9 years$. $\lambda = \frac{\ln 2}{T} = \frac{0.693}{4.5 \times 10^9} = 1.54 \times 10^{-10} years$.
4.1	$N_U = N_{0U} e^{-\lambda t}$.
4.2	$N_U = 5 \times 10^{12} \times e^{-1.54 \times 10^{-10} \times 2 \times 10^9} = 3.675 \times 10^{12} nuclei$.
4.3	On the graph: 2×10^9 years corresponds 3.7×10^{12} nuclei.
5.1	$N_{0U} = N_U + N_{Pb}$.
5.2	$N_U = N_{0U} - N_{Pb} = 5 \times 10^{12} - 2.5 \times 10^{12} = 2.5 \times 10^{12} nuclei$.
5.3	$N_U = \frac{N_{0U}}{2}$; the age of the Earth is equal to the half-life of uranium 238. This age is 4.5×10^9 years.

Exercise 19:

Part	Answer
A.1	${}^{210}_{84}Po \rightarrow {}^{A}_{Z}Pb + {}^{4}_{2}He$. Law of conservation of mass number: $210 = A + 4 \Longrightarrow A = 206$. Law of conservation of charge number: $84 = Z + 2 \Longrightarrow Z = 84$.
A.2	$\Delta m = m_{before} - m_{after} = m_{Po} - (m_{Pb} + m_{He}) = 209.9829 - (205.9745 + 4.0026)$ $\Delta m = 5.8 \times 10^{-3} u = 5.8 \times 10^{-3} \times 931.5 = 5.4 MeV/c^2$. $E = \Delta m c^2 = 5.4027 MeV = 5.4027 \times 1.6 \times 10^{-13} = 8.64 \times 10^{-13} J$.
A.3	Law of conservation of energy: $E_{PO} = E_{Pb} + E_{He} + E_\gamma$. $m_{PO}c^2 + K.E(Po) = m_{Pb}c^2 + K.E(Pb) + m_{He}c^2 + K.E(He) + E_\gamma$. $K.E(Po) = K.E(Pb) = E_\gamma = 0$. $K.E(He) = E = 5.4MeV = 8.64 \times 10^{-13} J$.
A.4.1	If the obtained daughter nucleus is in an excited state and when drops to the ground state it emits γ rays.
A.4.2	$E_\gamma = \frac{hc}{\lambda} = \frac{6.63 \times 10^{-34} \times 3 \times 10^8}{1.35 \times 10^{-12} \times 1.6 \times 10^{-13}} = 0.92 MeV$. Law of conservation of energy: $E_{PO} = E_{Pb} + E_{He} + E_\gamma$. $m_{PO}c^2 + K.E(Po) = m_{Pb}c^2 + K.E(Pb) + m_{He}c^2 + K.E(He) + E_\gamma$. $E = K.E(He) + E_\gamma \Longrightarrow K.E(He) = 5.4 - 0.92 = 4.48 MeV$.
B.1	$N = N_0 e^{-\lambda t}$, N_0 being respectively the number of nuclei present at $t_0 = 0$ and at t, λ is the radioactive constant and t is the time.

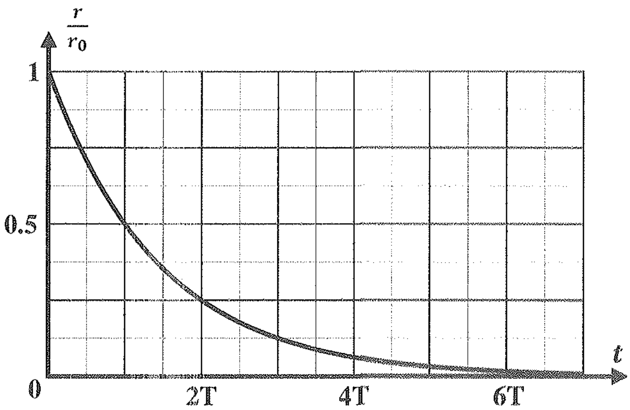
B.2.1	Activity is the number of decayed nuclei per unit time.
B.2.2	$A = -(slope\ of\ the\ curve) = \frac{4\times10^{12}}{448.8} = 8.91\times10^{21} decays/day.$
B.3	$A = \lambda N \Rightarrow \lambda = \frac{A}{N} = 0.00495 day^{-1}.$ $T = \frac{\ln 2}{\lambda} = \frac{0.69}{0.00495} = 140 days.$

Exercise 20:

Part	Answer
A.1	${}^{131}_{53}I \to {}^{A}_{Z}X + {}^{0}_{-1}e + {}^{0}_{0}\bar{\nu}.$ Law of conservation of mass number: $131 = A + 0 \Rightarrow A = 131.$ Law of conservation of charge number: $53 = Z - 1 \Rightarrow Z = 54.$ The daughter nucleus is Xenon (${}^{131}_{54}Xe$).
A.2	The daughter nucleus ${}^{131}_{54}Xe$ is in excited state and when it drops to the ground state (lower state) emits the γ ray (photon).
A.3	$\lambda = \frac{\ln 2}{T} = \frac{0.693}{8} = 0.087 day^{-1}.$ $\lambda = \frac{\ln 2}{T} = \frac{0.693}{8\times24\times3600} = 10^{-6}s^{-1}.$
A.4	$\Delta m = m_{before} - m_{after} = m_I - (m_{Xe} + m_e) = 130.8770 - (130.8754 + 5.5\times10^{-4})$ $\Delta m = 1.05\times10^{-3}u = 1.05\times10^{-3}\times931.5 = 0.978 MeV/c^2.$ $E_{lib} = \Delta mc^2 = 0.978 MeV = 0.978\times1.6\times10^{-13} = 1.56\times10^{-13}J.$
B.1	$A = \lambda N = 10^{-6}\times10^{11} = 10^5 Bq.$
B.2	The liberated energy is: $E = NE_{lib} = 10^{11}\times1.56\times10^{-13} = 1.56\times10^{-2}J.$
B.3	The absorbed dose is: $D = \frac{E}{m} = \frac{1.56\times10^{-2}}{0.025} = 0.624 J/kg.$
C.1.1	After the duration t_1= 1 day, according to the law of radioactive decay the number of remaining nuclei is $N_0e^{-\lambda t} = N_0e^{-\lambda}$ (λ in day^{-1}). An additional number N_0 is taken when drinks the second quantity next morning. Thus the number of non-decay nuclei is then: $N_1 = N_0 + N_0e^{-\lambda} = N_0(1 + e^{-\lambda})$.
C.1.2	On the 2nd day, an additional N_0 from the third quantity and the number remaining from the previous milk is $N_1e^{-\lambda}$: $N_2 = N_1e^{-\lambda} + N_0 = N_0(1 + e^{-\lambda})e^{-\lambda} + N_0 = N_0(1 + e^{-\lambda} + e^{-2\lambda}).$
C.2	On the 3rd day, an additional N_0 from the fourth quantity and the number remaining from the previous milk is $N_2e^{-\lambda}$. $N_3 = N_2e^{-\lambda} + N_0 = N_0(1 + e^{-\lambda} + e^{-2\lambda})e^{-\lambda} + N_0 = N_0(1 + e^{-\lambda} + e^{-2\lambda} + e^{-3\lambda}).$
C.3	At t =3 days, the number N_3 of nuclei is: $N_3 = N_0(1 + e^{-0.087} + e^{-2\times0.087} + e^{-3\times0.087}) = 9.17\times10^{16} nuclei$ The corresponding activity becomes: $A_3 = \lambda N_3 = 10^{-6}\times9.17\times10^{16} = 9.17\times10^{10}Bq = 91.7GBq > 75GBq$ At the instant t_3, the person is thus in danger.

Exercise 21:

Part	Answer
1	${}^{14}_{6}C \to {}^{A}_{Z}N + {}^{0}_{-1}e + {}^{0}_{0}\bar{\nu}.$ Law of conservation of mass number: $14 = A + 0 + 0 \Rightarrow A = 14.$ Law of conservation of charge number: $6 = Z - 1 + 0 \Rightarrow Z = 7.$

2	$\lambda = \frac{\ln 2}{T} = \frac{\ln 2}{5700} = 1.216 \times 10^{-4} year^{-1}$.
3	$r = \frac{N(^{14}_{6}C)}{N'(^{12}_{6}C)} = \frac{N_0(^{14}_{6}C)\times e^{-\lambda t}}{N'(^{12}_{6}C)}$ with $r_0 = \frac{N_0(^{14}_{6}C)}{N'(^{12}_{6}C)}$, we can write $r = r_0 e^{-\lambda t}$.
4.1	$\frac{r}{r_0} = e^{-\lambda t}$ as t increases then $e^{-\lambda t}$ decreases then $\frac{r}{r_0}$ decreases. Since specimen b has the lowest ratio then it is the oldest.
4.2	$\frac{r}{r_0} = e^{-\lambda t} = 0.843$ then $\ln 0.843 = -\lambda t$ thus the age of the specimen is: $t = \frac{-0.171}{-1.216\times 10^{-4}} = 1406.25 years$.
5.1	The ratio $\frac{r}{r_0} = e^{-\lambda t}$ for $t_0 = 0$ is $\frac{r}{r_0} = 1$; for t = 2T then $\frac{r}{r_0} = 0.25$; for t = 4T then $\frac{r}{r_0} = 0.0625$; for t = 6T then $\frac{r}{r_0} = 0.015625$.
5.2	
5.3	Since after millions of years the ratio $\frac{r}{r_0}$ becomes zero so we cannot determine the age of such organism.

Exercise 22:

Part	Answer
1.1	Law of conservation of mass number: $32 = A + 0 + 0 \Longrightarrow A = 32$. Law of conservation of charge number: $15 = Z - 1 + 0 \Longrightarrow Z = 16$.
1.2	$\Delta m = m_{before} - m_{after} = m_P - (m_S + m_e)$ $\Delta m = 31.965678 - (31.963293 + 5.486 \times 10^{-4}) = 1.8364 \times 10^{-3} u = 1.7106 MeV/c^2$. $E_{lib} = \Delta m c^2 = 1.7106 MeV$.
1.3.1	Gamma rays are not emitted in the above decay since the daughter nucleus (sulfur) is produced in the ground state.
1.3.2	$E_{lib} = K.E_{\beta^-} + K.E_{\bar{\nu}}$, so $1.7106 = K.E_{\beta^-} + 1.011 \Longrightarrow K.E_{\beta^-} = 0.6996 MeV$.
2.1	$\lambda = \frac{\ln 2}{T} = \frac{\ln 2}{14.3\times 24\times 3600} = 5.61 \times 10^{-7} s^{-1}$.
2.2	$A_0 = \lambda N_0 \Longrightarrow N_0 = \frac{A_0}{\lambda} = \frac{1.36\times 10^6}{5.61\times 10^{-7}} = 2.424 \times 10^{12} nuclei$.
2.3.1	$t = nT \Longrightarrow n = \frac{t}{T} = \frac{6}{14.3} = 0.4195$. Then, $N = \frac{N_0}{2^n} = \frac{2.424\times 10^{12}}{2^{0.4195}} = 1.812 \times 10^{12} nuclei$.
2.3.2	$N_d = N - N_0 = 6.12 \times 10^{11} nuclei$.
2.3.3	One electron is emitted in one decay of phosphorous-32, so $N_e = N_d = 6.12 \times 10^{11} nuclei$
2.4.1	$E_{abs} = N_e \times K.E_e = 6.12 \times 10^{11} \times 0.6996 \times 1.6 \times 10^{-13} = 6.8504 \times 10^{-2} J$.
2.4.2	$D = \frac{E_{abs}}{m} = \frac{6.8504\times 10^{-2}}{0.112} = 0.611 Gy = 0.611 J/kg$.

Exercise 23:

Part	Answer
1	Law of conservation of mass number: $99 = 99 + A \Longrightarrow A = 0$. Law of conservation of charge number: $42 = 43 + Z \Longrightarrow Z = -1$. The particle ${}^{A}_{Z}X$ is an electron (${}^{0}_{-1}e$).
2	Antineutrino.
3	The de-excitation of the daughter nucleus "Technetium".
4	$E = \frac{hc}{\lambda} \Longrightarrow \lambda = \frac{hc}{E} = \frac{6.6\times10^{-34}\times3\times10^{8}}{140\times1.6\times10^{-16}} = 8.839 \times 10^{-12}m$.
5	At t = 0; $A_0 = 530 \times 10^6 Bq$. At t = T; $A = \frac{A_0}{2} = \frac{530\times10^6}{2} = 265 \times 10^6 Bq$. Graphically, t = T = 6 hrs.
6.1	$A = A_0 e^{-\lambda t}$.
6.2.1	$A = 0.63A_0$. $0.63A_0 = A_0 e^{-\lambda t} \Longrightarrow \ln 0.63 = -\lambda t \Longrightarrow \ln 0.63 = -\frac{\ln 2}{T}t$. $t = -\frac{\ln 0.63}{\ln 2} \times T = -\frac{\ln 0.63}{\ln 2} \times 6 = 4hrs$.
6.2.2	$\frac{A}{A_0} = e^{-\lambda t} = e^{-\frac{\ln 2}{T}t} = e^{-\frac{\ln 2}{6}\times40} = 0.01 = 1\%$.

CHAPTER 11 – NUCLEAR REACTIONS
COURSE

11.1- STIMULATED NUCLEAR REACTIONS

In the previous chapter we defined radioactivity as the spontaneous transformation from a non-stable nucleus into a more stable nucleus with the emission of particles and radiations

In this chapter, we will study stimulated, provoked or triggered nuclear reactions which require an external intervention to be activated.

Stimulated nuclear reaction

Nuclear fission | **Nuclear fusion**

Notes:

- The laws of conservation of mass number and charge number (Soddy's laws) are applied in stimulated nuclear reactions.

$$\sum A_{before} = \sum A_{after}$$
$$\sum Z_{before} = \sum Z_{after}$$

- A stimulated nuclear reaction is accompanied with a mass defect, which is transformed into energy.

$$\Delta m = m_{before} - m_{after}$$
$$E_{lib} = \Delta mc^2$$

- During a stimulated nuclear reaction, the total energy is conserved.

$$(mc^2 + E_k)_{before} = (m'c^2 + E_k)_{after} + E\ of\ \gamma\ rays$$

11.2- NUCLEAR FISSION

Fission is a nuclear reaction in which a non-stable heavy nucleus is divided into two lighter and more stable nuclei under the impact of a neutron.

Nuclear fission reaction takes place each time a neutron having small kinetic energy, called a slow neutron (called also thermal neutron), enters into collision with the target nucleus and becomes embedded in it, thus forming a new unstable nucleus. In the case of uranium $^{235}_{92}U$, the kinetic energy of the thermal neutron should be about 0.1eV.

ATTENTION

Fissionable or fissile nucleus is the one that undergoes fission when bombarded by a thermal neutron such as U-235 and Pu-239.
A fertile nucleus is the one that gives rise to a fissionable nucleus by a nuclear reaction such as U-238

Example

$$^{235}_{92}U + ^{1}_{0}n \rightarrow ^{94}_{38}Sr + ^{140}_{54}Xe + 2^{1}_{0}n$$
$$^{235}_{92}U + ^{1}_{0}n \rightarrow ^{95}_{42}Mo + ^{139}_{57}La + 2^{1}_{0}n + 7^{\ 0}_{-1}e + 7^{0}_{0}\bar{\nu}$$

Law of conservation of total energy

$$E_{kb} + m_b c^2 = E_{ka} + m_a c^2$$
$$E_{kb} + \Delta mc^2 = E_{ka} \Longrightarrow E_{kb} + E_{lib} = E_{ka}$$
$$\text{For } E_{kb} \ll E_{lib} \text{ then } E_{lib} = E_{ka}.$$

The liberated energy during the fission reaction appears as kinetic energy of the fission fragments, the emitted neutrons, electrons, antineutrinos and γ-rays.
The most probable fission of one uranium nucleus produces energy around 200MeV.

Chain reaction
The fission of one nucleus of uranium 235 liberates 2 or 3 neutrons having high speeds. These neutrons have to be slowed down (in moderators) so they can be used to bombard another uranium nuclei. This helps to maintain a reaction called chain reaction, which increases rapidly in an explosive manner, as when an A-bomb explodes.

11.3- NUCLEAR FUSION

Fusion is a stimulated nuclear reaction in which two light nuclei are merged (combined) to form a heavier one and ejecting protons, neutrons and alpha particles.
Conditions of the reaction
It is necessary for the two positively charged merged nuclei to have high kinetic energy to overcome the electrostatic force of repulsion between them. The kinetic energy, about 0.1MeV, can be attained by thermal agitation at a very high temperature (about 10^8K). This is why such a reaction is called thermonuclear fusion or simply fusion.

> **ATTENTION**
> **Fusion reaction in the core of a star is the origin of its energy.**

Examples

$$^1_1H + ^1_1H \rightarrow ^2_1H + ^0_1e + ^0_0\nu \ (0.42MeV)$$
$$^1_1H + ^2_1H \rightarrow ^3_2He + \gamma \ (5.49MeV)$$
$$^3_2He + ^3_2He \rightarrow ^4_2He + 2^1_1H \ (12.86MeV)$$

The most frequent reaction is the fusion reaction between deuterium and tritium:

$$^2_1H + ^3_1H \rightarrow ^4_2He + ^1_0n$$

Nuclear fusion is more promising than nuclear fission for many reasons:
- Hydrogen is more abundant in nature than uranium.
- Contrary to fission, fusion does not produce any radioactive nuclei. The final products are stable, and so there is no problem of elimination of radioactive waste.
- For the same mass, the liberated energy by fusion is about 7 times more than that liberated by fission.

11.4- NUCLEAR WASTE

Nuclear waste is formed of solid or liquid radioactive substances coming essentially from uranium used in nuclear power plants (products of nuclear fission reactions).

Types of nuclear waste
Waste of short radioactive period (T < 30 years) that contains very active β^- and γ emitters such as cesium-137, strontium-90 and cobalt-60. Short-period waste is placed in metallic barrel and containers put in concrete then stocked underground or in oceans
Waste of long radioactive period (T > 30 years) that contains less radioactive α emitters such as neptunium-237, plutonium-239 and americium-243. Long-period wastes is stocked on the surface of the Earth, in order to assure their cooling, then they are enclosed in stainless steel tanks and buried underground at depths of few hundred meters.

11.5- EFFECT OF RADIATIONS ON LIVING THINGS

The exposure to radioactive radiations is very dangerous. Their effects depend on the absorbed does and on the type of radiations.

The absorbed dose D is defined as the energy received E per unit mass m:

$$D = \frac{E}{m}$$

In SI units, D is expressed in [J/kg] or gray [Gy], E in [J] and m in [kg].

D is also expressed in rad where 1rad = 10^{-2}Gy.

The equivalent dose ED is defined as the product of the quality factor QF of a given radiation and the absorbed dose D:

$$ED = D \times QF$$

In SI unit, ED is expressed in sievert [Sv].

Practically, we use also the rem (rad equivalent for men) where 1rem = 10^{-2}Sv.

Values of QF for some radiations and particle

Radiation or particle	QF
Gamma (γ)	1
Beta (β)	1
Neutron (n)	10
Alpha (α)	20

Effects of radiations based on the equivalent dose

ED [Sv]	Effects
Higher than 10	Death.
5	50% death, 50% cancer and other risks.
2	10% death, cancer and other risks.
1	Troubles in digestion, sterility.
0.05	Modification of the blood formula.

CHAPTER 11 – NUCLEAR REACTIONS
EXERCISES AND PROBLEMS

Exercise 1*:

1- Give the definition of:
  1.1- Fission reaction.
  1.2- Fusion reaction.

2- What are the conditions that must be satisfied to make a fusion reaction?

3- The fused nuclei must have a high kinetic energy. Explain why?

4- Give the properties of the projectile used in fission reactions.

5- In a fission reaction, what is the difference between the bombarded neutron and the emitted neutrons by this reaction?

6- What is the difference between a fissile and a fertile nucleus?

7- A nuclear fission reaction emits a certain number of neutrons. How can we obtain a chain reaction from them?

8- Give the order of energy liberated during one fission reaction.

9- Radioactive wastes:
  9.1- Define radioactive wastes.
  9.2- What are the two types of radioactive wastes? How can we distinguish between them?
  9.3- How can we get rid of these wastes?

10- How are people protected from the radioactive radiations?

11- List the advantages and disadvantages of fission and fusion.

Exercise 2*:

Indicate the type of the following nuclear reactions:

1- ${}^{232}_{90}Th \rightarrow {}^{228}_{88}Ra + {}^{4}_{2}He.$
2- ${}^{239}_{92}Pu + {}^{1}_{0}n \rightarrow {}^{96}_{38}Sr + {}^{141}_{56}Ba + 3{}^{1}_{0}n.$
3- ${}^{2}_{1}H + {}^{3}_{1}H \rightarrow {}^{4}_{2}He + {}^{1}_{0}n.$
4- ${}^{210}_{83}Bi \rightarrow {}^{210}_{84}Po + {}^{0}_{-1}e.$

Exercise 3:

The following fusion reaction is given: ${}^{2}_{1}H + {}^{2}_{1}H \rightarrow {}^{4}_{2}He + Q$.
The binding energy per nucleon of ${}^{2}_{1}H$ is 1.1MeV;
The binding energy per nucleon of ${}^{4}_{2}He$ is 7MeV.
Calculate the liberated energy Q.

Exercise 4*:

Consider the following nuclear reaction:

$${}^{235}_{92}U + {}^{1}_{0}n \rightarrow {}^{94}_{38}Sr + {}^{A}_{Z}Xe + 2{}^{1}_{0}n$$

Given that:

Particle	${}^{235}_{92}U$	${}^{1}_{0}n$	${}^{94}_{38}Sr$	${}^{A}_{Z}Xe$
Mass [u]	235.044	1.009	93.915	139.922

$$1u = 1.66 \times 10^{-27}kg;\ c = 3 \times 10^{8}m/s;\ 1MeV = 1.6 \times 10^{-13}J$$

1- Indicate the type of this reaction. Justify.
2- Determine Z and A specifying the used laws
3- Calculate, in [u] then in [kg], the mass defect of this reaction.
4- Calculate the energy liberated during the fission of one uranium nucleus.
5- Deduce the energy liberated by the fission of 1g of uranium.
6- The electrical power output of a large nuclear reactor is 900 MW. It has a 35% efficiency in converting nuclear power to electrical power. Calculate:
  6.1- the thermal nuclear power output,
  6.2- the number of ^{235}U nuclei that undergo fission each second,
  6.3- the mass of ^{235}U nuclei that undergo fission in one year of full-power operation.

Exercise 5*:

Consider the following nuclear reaction: $_1^2H + _1^2H \rightarrow _1^3H + _Z^AX$

Given:

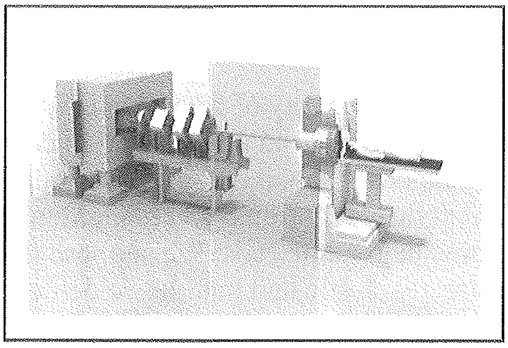
Nucleus or Particle	$_1^2H$	$_1^3H$	$_Z^AX$
Mass [u]	2.013	3.015	1.0072

$$1u = 1.66 \times 10^{-27} kg; c = 3 \times 10^8 m/s;\ 1MeV = 1.6 \times 10^{-13} J$$

1- This reaction is a nuclear fusion reaction. Why?
2- Give the name of a natural source of energy within which nuclear fusion occurs.
3- Each of the nuclei $_1^2H$ requires a great speed in order to approach each other enough to undergo fusion. Give the order of magnitude of the kinetic energy of each of the two nuclei.
4-
  4.1- Determine A and Z specifying the laws used.
  4.2- Give then the name and the symbol of the particle X.
5-
  5.1- Calculate, in kg, the mass defect in the above reaction.
  5.2- Deduce the energy liberated by this reaction.

Exercise 6:

Boron neutron capture therapy BNCT could be effective at treating tumors because it provides a way to deliver a very high radiation dose selectively to cancerous cells, even if they are spread out. In BNCT, boron captures neutron and transforms according to the following nuclear equation:



$$_5^{10}B + _0^1n \rightarrow _3^7Li^* + _Z^AX$$

1- What do 5 and 10 represent for a Boron nucleus?
2- Calculate the number of neutron presented in a Boron nucleus.
3-
  3.1- Calculate Z and A using the convenient laws.
  3.2- Deduce the name and the symbol of the particle X.
  3.3- Give two properties of the particle X.
4- Calculate the binding energy per nucleon of boron. Specify if it is a stable nucleus.
5- Calculate the radius of the boron nucleus. $r_0 = 1.2$ fm.
6- Calculate the density of the boron nucleus
7- Calculate in [u] then in [kg] the mass defect during the boron neutron capture.

8- Deduce in [J] then in [Mev] the energy liberated during the boron neutron capture.
9- Thc abovc cquation is accompanied by the emission of a dangerous radiation.
   9.1- Specify the nature of the emitted radiation.
   9.2- Write the de-excitation equation of the daughter nucleus ${}^{7}_{3}Li$.

Given: $1u = 1.66 \times 10^{-27}kg; c = 3 \times 10^{8}m/s; and\ 1MeV = 1.6 \times 10^{-13}J$

Nucleus or Particle	B	n	p	Li	X
Mass [u]	10.01294	1.0087	1.0073	7.016004	4.002

Exercise 7:
In the reactor of a nuclear power plant, the following fission reaction occurs:

$${}^{1}_{0}n + {}^{235}_{92}U \rightarrow {}^{138}_{56}Ba + {}^{95}_{36}Kr + 3{}^{1}_{0}n + \gamma$$

We suppose that all the neutrons emitted by the fission reactions have the same kinetic energy. The sum of the kinetic energy of the fission fragments is equal to 176MeV. The nucleus of a uranium-235 is at rest and the energy of the emitted γ photon is $E_{\gamma} = 18MeV$.
The nuclear power plant provides electrical power of 800MW and consumes 3kg of uranium-235 per day.

Given:
Masses of nuclei: ${}^{235}_{92}U$ = 234.9934 u; ${}^{138}_{x}Ba$ = 137.8742 u; ${}^{y}_{36}Kr$ = 94.8871u;
Mass of a neutron ${}^{1}_{0}$n = 1.0087u; molar mass of ${}^{235}_{92}U$ = 235 g/mole; Avogadro's number = $6.02 \times 10^{23}mol^{-1}$;
1 u = 931.5MeV/c^2; 1MeV = 10^6eV = 1.6×10^{-13}J.

1- Calculate in MeV, the energy liberated by the fission of a nucleus of uranium 235 nucleus.
2- Deduce the energy liberated by fission of 3 kg of uranium 235.
3- Determine the efficiency of the power plants.
4- Calculate the kinetic energy of a neutron emitted by this fission (the kinetic energy of the thermal neutron is 0.04eV).

Exercise 8*:
The kinetic energy of a neutron emitted by the fission of a ${}^{235}_{92}U$ nucleus is 2MeV. However, the kinetic energy of a neutron that may produce the fission of uranium 235 nucleus should be of the order of 0.04eV.
We suppose that all the neutrons emitted by the fission reactions have the same kinetic energy.
To produce a fission by an emitted neutron, it is necessary to slow it down by collisions with carbon 12 atoms in graphite rods. We suppose that each collision between a neutron and one carbon 12 atom is perfectly elastic and that the velocities before and after collision are collinear.
Take: m (${}^{1}_{0}n$) = 1 u and m (^{12}C) = 12 u.
Let $\vec{V}_0$ be the velocity of one emitted neutron just before collision and $\vec{V}_1$ its velocity just after its first collision with a carbon 12 atom supposed initially at rest.
1- Explain the role of a moderator.
2- What is meant by a chain reaction?
3- Determine the ratio $k = \left|\frac{V_1}{V_0}\right|$.
4- Determine the number of collisions needed for an emitted neutron with carbon 12 atoms, to slow down so that its kinetic energy is reduced to 0.04eV.

Exercise 9*:

The strength of explosives and bombs are often compared with that of TNT explosives. One ton of TNT releases about 4.2×10^9 J of energy.

1- The atomic bomb exploded over Hiroshima in 1945 released about the same amount of energy as 15 000 ton of TNT explosives by the fission of uranium-235.
   - **1.1-** Define nuclear fission.
   - **1.2-** Calculate the energy released by the atomic bomb.
   - **1.3-** The fission of one uranium-235 atom releases 3.2×10^{-11} J of energy. One uranium-235 atom has a mass of 3.9×10^{-25} kg. Determine the mass of uranium-235 that underwent fission in atomic bomb exploded over Hiroshima.

2- On the other hand, the hydrogen bomb, detonated by the Soviet Union in 1961, released about the same amount of energy as 5×10^7 tons of TNT explosives. It contained a fission bomb that was used to trigger nuclear fusion.
   - **2.1-** Define nuclear fusion.
   - **2.2-** State one reason why a fission bomb is needed to trigger nuclear fusion.
   - **2.3-** Calculate the energy released by the hydrogen bomb.
   - **2.4-** The fusion of hydrogen-2 and hydrogen-3 nuclei can be represented by the following equation:

$$ {}_1^2H + {}_1^3H \rightarrow {}_2^4He + {}_0^1n + \boldsymbol{energy}\ (\mathbf{2.8 \times 10^{-12}}J) $$

The ratio of the energy released in the fusion of 1 kg of hydrogen atoms to the energy released in the fission of 1 kg of uranium-235 atoms respectively is:

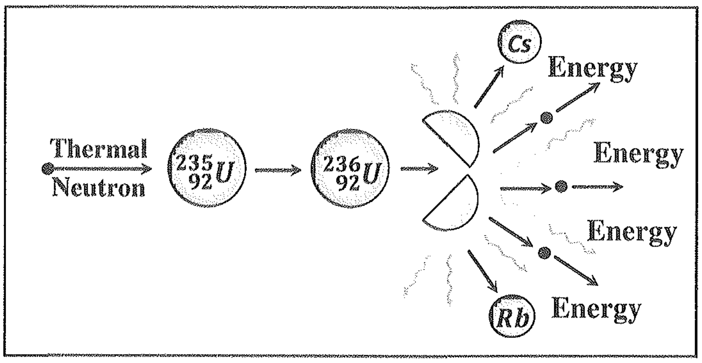
$$ \frac{E_H}{E_U} \approx 4.09 $$

Compare the energy released in the fission of 1 kg of uranium-235 atoms and the fusion of 1 kg of hydrogen atoms.

3- List the advantages of fusion over fission.

Exercise 10*:

Consider the following diagram:



Given that:

$m\,({}^{235}_{92}U) = 235.044u$

$m({}^{1}_{0}n) = 1.009u$

$m({}^{A}_{Z}Cs) = 139.909u$

$m({}^{93}_{37}Rb) = 92.922u$

$1\text{u} = 1.66 \times 10^{-27}\text{kg};\ \text{c} = 3 \times 10^8\text{m/s}.$

$1\text{MeV} = 1.6 \times 10^{-13}\text{J}$

1- Use the diagram to write the provoked nuclear reaction.
2- Specify the type of this reaction.
3- What is the role of the emitted neutrons?
4- Explain the term "thermal neutron".
5- Determine Z and A specifying the name of the used laws
6- Calculate in u then in kg the mass defect of this reaction.
7- Calculate the energy liberated during the fission of one uranium nucleus.
8- A small atomic bomb releases energy equivalent to the detonation of 20,000 tons of TNT; a ton of TNT releases 4×10^9J of energy when exploded.
   - **8.1-** Calculate in [J], then in [MeV] the energy released by this atomic bomb.
   - **8.2-** Find the number of reactions accomplished in order to achieve this reaction.

Exercise 11:

Given:

- Atomic mass unit 1u = 931.5MeV/c^2 = 1.66 × 10^{-27}kg;
- 1MeV = 1.6 × 10^{-13}J;
- Mass of particles (in u): antineutrino$_0^0\bar{\nu} \simeq 0$; electron $_{-1}^{0}e$: 5.5 × 10^{-4}; neutron$_0^1n$: 1.0087.

Element Uranium Neptunium Plutonium

Element	Molybdenum		Technetium		Tellurium
Nuclide	$^{101}_{42}Mo$	$^{102}_{42}Mo$	$^{101}_{43}Te$	$^{102}_{43}Te$	$^{135}_{52}Te$
Mass (in u)	100.9073	101.9103	100.9073	101.9092	134.9167

Element	Uranium		Neptunium	Plutonium
Nuclide	$^{235}_{92}U$	$^{238}_{92}Mo$	$^{239}_{93}Np$	$^{239}_{94}Pu$
Mass (in u)	235.0439	238.0508	239.0533	239.0530

Read carefully the following text about fast neutrons, and answer the questions that follow.

"...the basic substance used to obtain nuclear energy is the natural uranium which is mainly formed of the two isotopes: uranium 235 and uranium 238...

... The fast-neutron nuclear reactors (breeder reactors), use uranium 235 or plutonium 239 (or the two at the same time) as fuel. In each reactor we put around the core which is constituted of uranium 235($^{235}_{92}U$), a cover made essentially of fertile uranium 238 ($^{238}_{92}U$). This cover can trap fast neutrons issued from the fission reactions of uranium 235.

These reactors transform more uranium 238 atoms into plutonium 239.

Finally, in the very well-studied fast neutrons reactors, the quantity of fissionable matter that is created, exceeds notably the consumed quantity. For this reason, these reactors are called breeder reactors..."

Questions

1-

1.1- What is meant by isotopes of an element?

1.2- Give the composition of each of uranium 235 and uranium 238 nuclei.

2- In the reactor, the uranium 238 reacts with the fast neutrons according to the reaction:

$$^{238}_{92}U + {}^1_0n \rightarrow X \text{ (Reaction 1)}$$

The obtained nucleus X is radioactive and after two successive β^-emissions, it is transformed into plutonium:

$$X \rightarrow {}_{-1}^{0}e + {}_0^0\bar{\nu} + Y \text{ (Reaction 2)}$$

$$Y \rightarrow {}_{-1}^{0}e + {}_0^0\bar{\nu} + {}_{94}Pu \text{ (Reaction 3)}$$

2.1- Identify X and Y.

2.2- Deduce the nuclear reaction that occurs between a $^{238}_{92}U$ nucleus and a fast neutron that leads to the formation of a plutonium 239 (reaction 4).

3- The plutonium 239 ($^{239}_{94}Pu$) is fissile and can react with neutrons according to the reaction:

$$^{239}_{94}Pu + {}^1_0n \rightarrow B + {}^{135}_{52}Te + 3{}^1_0n \text{ (Reaction 5)}$$

3.1- Identify B.

3.2- Calculate, in MeV/c^2, the mass defect Δmin reaction 5.

3.3- Deduce, in Mev, the energy E liberated during the fission of a plutonium nucleus.

3.4- Find, in joules, the energy liberated by the fission of one kilogram of plutonium.

3.5- From reactions 4 and 5, justify the definition of a breeder reactor given in the text.

Exercise 12:

The object of this exercise is to compare the energy liberated per nucleon in a nuclear fission with that liberated in a nuclear fusion.

Given:

Symbol	${}_0^1n$	${}_1^2H$	${}_1^3H$	${}_2^4He$	${}_{92}^{235}U$	${}_Z^{94}Sr$	${}_{54}^{A}Xe$
Mass in u	1.00866	2.01355	3.01550	4.0015	234.9942	93.8945	138.8892

1u = 931.5MeV/c^2

A- Nuclear fission

The fission of uranium 235 is used to produce energy.

1- The fission of one uranium 235 nucleus takes place by bombarding this nucleus by a slow (thermal) neutron of kinetic energy around 0.025eV .The equation of this reaction is written as:

$${}_{92}^{235}U + {}_0^1n \rightarrow {}_Z^{94}Sr + {}_{54}^{A}Xe + 3{}_0^1n$$

1.1- Calculate A and Z specifying the laws used.

1.2- Show that the energy E liberated by the fission of one uranium nucleus is 179.947MeV.

1.3-

1.3.1- The number of nucleons participating in this reaction is 236. Why?

1.3.2- Calculate then E$_1$, the energy liberated per nucleon participating in this fission reaction.

2- Each of the obtained neutrons has an average kinetic energy $E_0 = \frac{E}{100}$.

2.1- In this case, the obtained neutrons do not, in general, provoke fission. Why?

2.2- What then should be done in order to obtain a fission reaction?

B- Nuclear fusion

Nowadays, many researches are performed in order to produce energy by nuclear fusion. The most accessible is the reaction between a deuterium nucleus ${}_1^2H$and a tritium nucleus${}_1^3H$.

1- The deuterium and the tritium are two isotopes of hydrogen. Write down the symbol of the third isotope of hydrogen.

2- Write down the fusion reaction of a deuterium nucleus with a tritium nucleus knowing that this reaction liberates a neutron and a nucleus ${}_Z^AX$. Calculate Z and A and give the name of thenucleus${}_Z^AX$.

3- Show that the energy liberated by this reaction is E' =17.596MeV.

4- Calculate $E_1'$the energy liberated per nucleon participating in this reaction.

C- Conclusion

Compare E$_1$ and E_1' and conclude.

Exercise 13:

Given: mass of a proton: m$_p$ = 1.0073u; mass of a neutron: m$_n$ = 1.0087u;
mass of ${}_{92}^{235}U$ nucleus = 235.0439u; mass of ${}_{36}^{90}Kr$ nucleus = 89.9197u;
mass of ${}_Z^{142}Ba$ nucleus =141.9164 u; molar mass of ${}_{92}^{235}U$ = 235g/mole;
Avogadro's number: N$_A$ = 6.022 × 10^{23}mol^{-1}; 1u = 931.5MeV/c^2 = 1.66 × 10^{-27} kg; 1MeV = 1.6 × 10^{-13}J.

A- Provoked nuclear reaction

As a result of collision with a thermal neutron, a uranium 235 nucleus undergoes the following reaction:

$${}_0^1n + {}_{92}^{235}U \rightarrow {}_{36}^{90}Kr + {}_Z^{142}Ba + y{}_0^1n$$

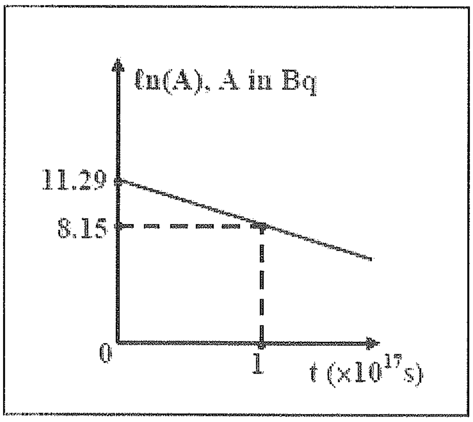
1-
 1.1- Determine y and z.
 1.2- Indicate the type of this provoked nuclear reaction.
2- Calculate, in MeV, the energy liberated by this reaction.
3- In fact, 7% of this energy appears as a kinetic energy of all the produced neutrons.
 3.1- Determine the speed of each neutron knowing that they have equal kinetic energy.
 3.2- A thermal neutron, that can provoke nuclear fission, must have a speed of few km/s; indicate then the role of the "moderator" in a nuclear reactor.
4- In a nuclear reactor with uranium 235, the average energy liberated by the fission of one nucleus is 170MeV.
 4.1- Determine, in joules, the average energy liberated by the fission of one kg of uranium $^{235}_{92}U$.
 4.2- The nuclear power of such reactor is 100MW. Calculate the time Δt needed so that the reactor consumes one kg of uranium $^{235}_{92}U$.

B- Spontaneous nuclear reaction

1- The nucleus krypton $^{90}_{36}Kr$ obtained is radioactive. It disintegrates into zirconium $^{90}_{40}Zr$, by a series of β^-disintegrations.
 1.1- Determine the number of β^-disintegrations.
 1.2- Specify, without calculation, which one of the two nuclides $^{90}_{36}Kr$ and $^{90}_{40}Zr$ is more stable.
2- Uranium $^{235}_{92}U$ is an αemitter.
 2.1- Write down the equation of disintegration of uranium $^{235}_{92}U$ and identify the nucleus produced.
 Given:

Actinium $_{89}$Ac	Thorium $_{90}$Th	Protactinium $_{91}$Pa

 2.2- The remaining number of nuclei of $^{235}_{92}U$ as a function of time is given by: $N = N_0 e^{-\lambda t}$ where N_0 is the number of the nuclei of $^{235}_{92}U$ at $t_0 = 0$ and λ is the decay constant of $^{235}_{92}U$.
  2.2.1- Define the activity A of a radioactive sample.
  2.2.2- Write the expression of A in terms of λ, N_0 and time t.
 2.3- Derive the expression of $\ell n(A)$ in terms of the initial activity A_0, λ and t.
 2.4- The adjacent figure represents the variation of $\ell n(A)$ of a sample of $^{235}_{92}U$as a function of time.



  2.4.1- Show that the shape of the graph, in the adjacent figure, agrees with the expression of $\ell n(A)$.
  2.4.2- Using the adjacent figure determine, in s^{-1}, the value of the radioactive constant λ.
  2.4.3- Deduce the value of the radioactive period T of $^{235}_{92}U$.

Exercise 14:

The nuclear chain fission reaction, conveniently controlled in a nuclear power plant, can be a source of a huge amount of energy able to generate electric power.

Given:

Masses of nuclei: ${}^{235}_{92}U$ = 234.9934u; ${}^{138}_{x}Ba$ = 137.8742u; ${}^{y}_{36}Kr$ = 94.8871u;
molar mass of ^{235}U = 235g.mol-1; Avogadro's number $N_A = 6.02 \times 10^{23}mol^{-1}$;
m(${}^{1}_{0}n$) = 1.0087u; 1u = 931.5MeV/c^2; 1eV = 1.6×10^{-19}J.

A – Efficiency of a nuclear power plant

In the reactor of a nuclear power plant, we use natural uranium enriched with uranium 235. The nucleus of a uranium 235 captures a thermal neutron and is transformed into a nucleus of uranium 236 in an excited state. The decay of this nucleus is accompanied by the emission of a photon γ of energy equal to 20MeV.

1-

- **1.1-** Complete the following reaction: ${}^{236}_{92}U^* \to \; .... + \gamma$
- **1.2-** Indicate the value of the excess of energy possessed by a uranium 236 nucleus in the excited state.

2- The obtained uranium nucleus, breaks instantaneously, producing two nuclides called fission fragments with the emission of some neutrons and γ photon, so the overall equation is:

$${}^{1}_{0}n + {}^{235}_{92}U \to {}^{138}_{x}Ba + {}^{y}_{36}Kr + 3{}^{1}_{0}n + \gamma$$

Determine:

- **2.1-** x and y;
- **2.2-** in MeV, the energy liberated by the fission of a uranium 235 nucleus;
- **2.3-** the energy liberated by the fission of 1 g of uranium 235;
- **2.4-** the efficiency of the nuclear power plant, knowing that it provides an electric power of 800MW and consumes 2.8 kg of uranium 235 per day.

B – Chain Reaction

The kinetic energy of a neutron that may produce the fission of uranium 235 nucleus should be of the order of 0.04eV.

We suppose that all the neutrons emitted by the fission reactions have the same kinetic energy.

1- The sum of the kinetic energies of the two fragments (Kr and Ba) is equal to 174MeV and the energy of the emitted γ photon is E_γ = 20MeV.

- **1.1-** Show, using the conservation of the total energy, that the kinetic energy of a neutron emitted by this fission is 2MeV.
- **1.2-** Deduce that the emitted neutrons cannot produce fission reactions of uranium 235.

2- To produce a fission by an emitted neutron, it is necessary to slow it down by collisions with carbon 12 atoms in graphite rods. We suppose that each collision between a neutron and one carbon 12 atom is perfectly elastic and that the velocities before and after collision are collinear.
Take: m (${}^{1}_{0}n$) = 1u and m (^{12}C) = 12u.

- **2.1-** Let $\vec{V}_0$ be the velocity of one emitted neutron just before collision and $\vec{V}_1$ its velocity just after its first collision with a carbon 12 atom supposed initially at rest. Show that: $\left|\frac{V_1}{V_0}\right| = k = \frac{11}{13}$.
- **2.2-**
  - **2.2.1-** Show that the ratio of the kinetic energies just after and just before the first collision of the emitted neutron is: $\frac{K.E_1}{K.E_0} = k^2$.
  - **2.2.2-** Determine the number of collisions needed for an emitted neutron with carbon 12 atoms, to slow down so that its kinetic energy is reduced to 0.04eV.

CHAPTER 11 – NUCLEAR REACTIONS
SOLUTION OF EXERCISES AND PROBLEMS

Exercise 3:

Part	Answer
	$E_B(^2_1H) = 1.1 \times 2 = 2.2MeV$. $E_B(^4_2He) = 7 \times 4 = 28MeV$. $Q = E_{B_{after}} - E_{B_{before}} = E_B(^4_2He) - 2E_B(^2_1H) = 28 - 2.2 \times 2 = 23.6MeV$.

Exercise 6:

Part	Answer
1	5: charge number 10: mass number
2	N = A – Z = 5
3.1	By applying law of conservation of mass number: 10 + 1 = 7 + A $\Rightarrow$ A = 4. By applying law of conservation of charge number: 5 + 0 = 3 + Z $\Rightarrow$ Z = 2.
3.2	Name: helium nucleus. Symbol: $^4_2$He
3.3	Positively charged particle – Low penetrating power.
4	$\Delta m = Zm_p + (A - Z)m_n - m_B = 5 \times 1.0073 + 5 \times 1.0087 - 10.01294 = 0.06706u$. $\Delta m = 62.46639MeV/c^2$. $E_b = \Delta mc^2 = 62.46639MeV$. $\frac{E_b}{A} = 6.246639MeV < 8MeV$ (not stable).
5	$r = r_0 A^{\frac{1}{3}} = 1.2 \times 10^{\frac{1}{3}} = 2.585fm$.
6	Average mass of boron-10 nucleus: $m = Am_0$. Volume of a boron-10 nucleus: $V = AV_0 = A \times \frac{4}{3}\pi r_0^2$. Density of a boron-10 nucleus: $\rho = \frac{m}{V} = \frac{Am_0}{A\times\frac{4}{3}\pi r_0^2} = \frac{m_0}{\frac{4}{3}\pi r_0^2} = \frac{1.7\times10^{-27}}{\frac{4}{3}\times\pi\times(1.2\times10^{-15})^3} = 2.3 \times 10^{17} kg/m^3$.
7	$\Delta m = m_b - m_a = [m_B + m_n] - [m_{Li} + m_{He}]$ $\Delta m = [10.01294u + 1.008u] - [7.016004u + 4.002u]$ $\Delta m = 11.02094u - 11.018004u = 0.002936u = 4.90312 \times 10^{-30} kg$.
8	$E = \Delta mc^2 = 4.412808 \times 10^{-13} J = 2.758005MeV$.
9.1	Electromagnetic radiation.
9.2	$^7_3Li^* \rightarrow {}^7_3Li + {}^0_0\gamma$.

Exercise 7:

Part	Answer
1	$\Delta m = m_{before} - m_{after} = (m_n + m_U) - (m_{Ba} + m_{Kr} + 3m_n)$. $\Delta m = (234.9934 + 1.0087) - (137.8742 + 94.8871 + 1.0087) = 0.2147u = 200MeV/c^2$. $E_l = \Delta mc^2 = 200MeV = 3.2 \times 10^{-11} J$.
2	$\frac{m}{M} = \frac{N}{N_A} \Rightarrow N = \frac{m\times N_A}{M} = \frac{3\times10^3\times6.02\times10^{23}}{235} = 7.685 \times 10^{24} nuclei$. $E_{nuclear} = NE_l = 2.4592 \times 10^{14} J$.
3	$P_{nuclear} = \frac{E_{nuclear}}{\Delta t} = \frac{2.4592\times10^{14}}{24\times3600} = 2846.3MeV$. $\rho = \frac{P_{electric}}{P_{nuclear}} = \frac{800}{2846.3} = 0.28 = 28\%$.
4	Law of conservation of total energy:

	$E_U + E_{n(thermal)} = E_{Ba} + E_{kr} + 3E_{n(kinetic)} + E_{\gamma}$. $m_U c^2 + K.E_U + K.E_{n(thermal)} = m_{Ba}c^2 + K.E_{Ba} + m_{Kr}c^2 + K.E_{Kr} + 3K.E_{n(kinetic)} + E_{\gamma}$. $K.E_{n(kinetic)} = \frac{E_l + K.E_{n(thermal)} - (K.E_{Ba} + K.E_{Kr}) - E_{\gamma}}{3} = \frac{200 + 0.04\times10^{-6} - 176 - 18}{3} = 2MeV$.

Exercise 11:

Part	Answer key
1.1	We call isotopes the nuclei that have the same atomic number Z and of different mass numbers A.
1.2	Uranium nuclei have 92 protons and 143 neutrons for $^{235}_{92}U$, 146 neutrons for $^{238}_{92}U$.
2.1	$^{238}_{92}U + ^{1}_{0}n \to ^{239}_{92}X$ (Reaction 1); $^{239}_{92}X$ is an isotope of uranium. $^{239}_{92}U \to ^{0}_{-1}e + ^{0}_{0}\overline{\nu} + ^{239}_{93}Y$ (Reaction 2); $^{239}_{93}Y$ is am isotope of neptunium $^{239}_{93}Np$. $^{239}_{93}Np \to ^{0}_{-1}e + ^{0}_{0}\overline{\nu} + ^{239}_{94}Pu$ (Reaction 3).
2.2	The addition of (1) + (2) + (3) gives the nuclear reaction leading to the formation of plutonium: $^{238}_{92}U + ^{1}_{0}n \to 2^{0}_{-1}e + 2^{0}_{0}\overline{\nu} + ^{239}_{94}Pu$ (Reaction 4).
3.1	$^{239}_{94}Pu + ^{1}_{0}n \to ^{A}_{Z}B + ^{135}_{52}Te + 3^{1}_{0}n$ (Reaction 5). Law of conservation of mass number: $239 + 1 = A + 135 + 3 \Longrightarrow A = 102$. Law of conservation of charge number: $94 + 0 = Z + 52 + 0 \Longrightarrow Z = 42$. $^{102}_{42}B$ is a nucleus of molybdenum $^{102}_{42}Mo$.
3.2	$\Delta m = m_b - m_a = [m_{Pu} + m_n] - [m_{Mo} + m_{Te} + 3m_n]$ $\Delta m = [239.0530 + 1.0087] - [101.9103 + 134.9167 + 3 \times 1.0087] = 0.2086u$. $\Delta m = 194.3MeV/c^2$.
3.3	$E = \Delta mc^2 = 0.2086 \times 931.5\frac{MeV}{c^2}.c^2 = 194.3MeV$.
3.4	The number of plutonium nuclei in 1 kg is: $N = \frac{1}{239.0530\times1.66\times10^{-27}} = 2.52 \times 10^{24} nuclei$. $E' = 2.52 \times 10^{24} \times 194.3 \times 1.6 \times 10^{-13} = 7.83 \times 10^{13}J$.
3.5	Under the action of an incident neutron, a plutonium nucleus reacts according to the equation (5) and liberates three neutrons during its fission. For these three neutrons: ❖ one is used to sustain the fission reaction of plutonium; ❖ the two others are available to react with the uranium 238 to create two new plutonium atoms. For one fissile plutonium nucleus consumed, two fissile plutonium nuclei are created. This justifies the appellation of breeder reactor giving to such reactor.

Exercise 12:

Part	Answer key
A.1.1	Conservation of nucleons number: 235 + 1 = 94 + A + 3 then A = 139. Conservation of charge number: 92 = Z + 54 then Z = 38.
A.1.2	$E = \Delta mc^2 = (234.9942 + 1.00866 - 93.8945 - 138.8892 - 3 \times 1.00866) \times 931.5$ $\Longrightarrow$ Energy = 179.947MeV.
A.1.3.1	We have 235 + 1 = 236 nucleons.
A.1.3.2	$E_1 = \frac{179.947}{236} = 0.76MeV/nucleon$.
A.2.1	$E_0 = \frac{179.947}{100} = 1.79947MeV$; which is much greater than 0.025eV.
A.2.2	They should be slowed down,
B.1	$^{1}_{1}H$.
B.2	$^{2}_{1}H + ^{3}_{1}H \to ^{4}_{Z}X + ^{1}_{0}n$.

	2 + 3 = A + 1 then A = 4. 1 + 1 = Z then Z = 2. The helium nucleus $^{4}_{2}He$.
B.3	E' = Δmc^2 = (2.01355 + 3.0155 – 4.0015 – 1.00866) × 931.5 = 17.596MeV.
B.4	We have 2 + 3 = 5 nucleons $\Rightarrow E_1' = \frac{17.596}{5} = 3.5912 MeV/nucleon.$
C	E_1' is greater than E_1; fusion is more efficient.

Exercise 13:

Part	Answer
A.1.2	Conservation of charge number: 92 + 0 = 36 + Z + 0 thus Z = 56. Conservation of mass number: 235 + 1 = 90 + 142 + y thus y = 4.
A.1.2	$\Delta m = [m_U + m_n] - [m_{Kr} + m_{Ba} + 4m_n]$ = 235.0439 – [89.9197 + 141.9164 + 3×1.0087] = 0.1817 u. E = Δmc^2 = [0.1817 × 931.5 Mev/c^2] c^2 = 169.253 MeV.
A.2	K.E of each neutron = $\frac{169.253\times\frac{7}{100}}{4}$ = 2.96 MeV = 2.96 × 1.6 × 10^{-13} K.E = 4.739 × 10^{-13} J. K.E = ½ mV^2 then V = $\sqrt{\frac{2K.E}{m}} = \sqrt{\frac{2\times4.739\times10^{-13}}{1.087\times1.66\times10^{-27}}}$ = 2.379 × 10^7 m/s = 23790 km/s.
A.3.1	A moderator will help in reducing their speed so as to provoke more such reactions.
A.3.2	$N = \frac{mass}{molar\ mass} \times N_A = \frac{100}{235} \times 6.022 \times 10^{23} = 2.5617 \times 10^{24} nuclei.$ $E = 170 \times 1.6 \times 10^{-13} \times 2.5617 \times 10^{24} = 6.97 \times 10^{13} J.$
A.4.1	$E = P \times \Delta t \Rightarrow \Delta t = \frac{6.97\times10^{13}}{10^8} = 6.97 \times 10^5 s = 8 days.$
A.4.2	$^{90}_{36}Kr \rightarrow ^{90}_{40}Zr + a\ ^{0}_{-1}\beta$ $a = 4$
B.1.1	A non-stable nucleus decays into a more stable one thus $^{90}_{40}Zr$ is more stable.
B.1.2	$^{235}_{92}U \rightarrow ^{4}_{2}He + ^{A}_{Z}X.$ $A = 231$ and $Z = 90 \Rightarrow$ X is thorium.
B.2.2.1	The activity is the number of decays per unit time.
B.2.2.2	$A = \lambda N = \lambda N_0 e^{-\lambda t}.$
B.2.3	$\ln(A) = -\lambda t + \ln(A_0).$
B.2.4.1	$\ln(A) = -\lambda t + \ln(A_0).$ Is a straight line of negative slope $\Rightarrow$ compatible with the graph.
B.2.4.2	$\lambda = -slope\ of\ curve = 3.14 \times 10^{-17} s^{-1}.$
B.2.4.3	$\lambda = \frac{\ln 2}{T} \Rightarrow T = 22.0747 \times 10^{15} s = 7 \times 10^8 years.$

Exercise 14:

Part	Answer
A.1.1	$^{236}_{92}U^* \rightarrow ^{236}_{92}U + \gamma.$
A.1.2	The excess of energy is 20MeV.
A.2.1	Conservation of mass number: 1 + 235 = 138 + y + 3 $\Rightarrow$ y =95. Conservation of charge number: 92 = x + 36 $\Rightarrow$ x = 56.
A.2.2	Δm = 1.0087 + 234.9934 - 137.8742 - 94.8871 - 3 × 1.0087 = 0.2147u. E = Δmc^2. Then E = 0.2147 × 931.5MeV/c^2.c^2 = 199.99 ≈ 200MeV.