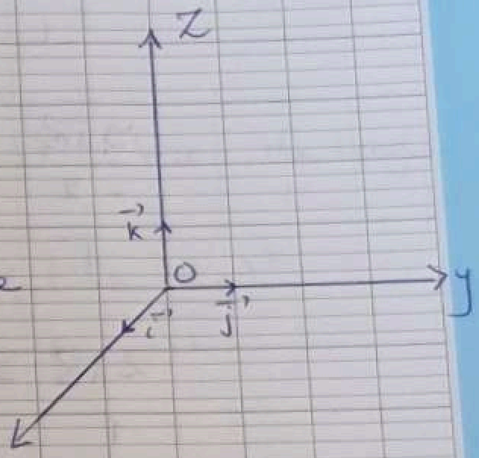
Space

Vectors in space:

* In this lesson the plane is referred to a direct orthonormal system $(O; \vec{i}, \vec{j}, \vec{k})$ ($\vec{i}$, $\vec{j}$ and $\vec{k}$ are unit vectors)
* Every vector $\vec{u}(x, y, z)$ can be written in the form $\vec{u} = x\vec{i} + y\vec{j} + z\vec{k}$



Obj 1: Find the components of a vector.

Let A and B be two points in space with $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$ then

$$\overrightarrow{AB}(x_B - x_A, y_B - y_A, z_B - z_A)$$

second point − first point

To write $\overrightarrow{AB}$ in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$:

$$\overrightarrow{AB} = (x_B - x_A)\vec{i} + (y_B - y_A)\vec{j} + (z_B - z_A)\vec{k}.$$

Examples:

1) $A(3, 4, 5)$ $B(2, 3, 4)$ Find the components of vector $\overrightarrow{AB}$.

Sol.
$$\left.\begin{aligned} x_{\overrightarrow{AB}} &= x_B - x_A = 2 - 3 = -1 \\ y_{\overrightarrow{AB}} &= y_B - y_A = 3 - 4 = -1 \\ z_{\overrightarrow{AB}} &= z_B - z_A = 4 - 5 = -1 \end{aligned}\right\} \quad \overrightarrow{AB}(-1, -1, -1)$$

$\overrightarrow{AB}$ in terms of $\vec{i}$, $\vec{j}$ and $\vec{k}$:

$$\overrightarrow{AB} = -\vec{i} - \vec{j} - \vec{k}$$

2) $A(-4, 3, -1)$ and $\vec{AB} = 3\vec{i} - 2\vec{j} + \vec{k}$.

Find the coordinates of point B.

Sol. $\vec{AB} = 3\vec{i} - 2\vec{j} + \vec{k}$

$\vec{AB}(3, -2, 1)$

$$\left.\begin{aligned}
x_{\vec{AB}} &= x_B - x_A \\
3 &= x_B - (-4) \\
-1 &= x_B \\
\\
y_{\vec{AB}} &= y_B - y_A \\
-2 &= y_B - 3 \\
1 &= y_B \\
\\
z_{\vec{AB}} &= z_B - z_A \\
1 &= z_B - (-1) \\
0 &= z_B
\end{aligned}\right\} \Rightarrow B(-1, 1, 0)$$

Note: Coordinates of the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$

$\vec{i}(1, 0, 0)$

$\vec{j}(0, 1, 0)$

$\vec{k}(0, 0, 1)$

Obj. 2 : Find the length of a vector.

If $\vec{u}\,(x, y, z)$

Length of $\vec{u}$ is $\|\vec{u}\| = \sqrt{x^2 + y^2 + z^2}$

Note: Length of the unit vectors $\vec{i}$, $\vec{j}$ and $\vec{k}$:

$\|\vec{i}\| = \|\vec{j}\| = \|\vec{k}\| = 1$.

Ex. If $A(3, -4, 7)$ and $B(0, 2, 1)$. Find the length of $\overrightarrow{AB}$.

Sol. $\|\overrightarrow{AB}\| = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$

$= \sqrt{(0-3)^2 + (2-(-4))^2 + (1-7)^2}$

$= \sqrt{(-3)^2 + (6)^2 + (-6)^2}$

$= \sqrt{9 + 36 + 36}$

$= \sqrt{81} = 9$

Dot product or Scalar Product:

Obj 3: Find the scalar product (dot product) of two vectors.

Rule 1: If $\vec{u}(x, y, z)$ and $\vec{v}(x', y', z')$
then $\vec{u} \cdot \vec{v} = xx' + yy' + zz'$.

Notes: * The dot product of two vectors is a real number and not a vector.

* The dot product of two vectors could be positive, negative or zero.

Rule 2: If $\vec{u}(x, y, z)$ and $\vec{v}(x', y', z')$
then $\vec{u} \cdot \vec{v} = \underbrace{\|\vec{u}\|}_{\text{length of vector } \vec{u}} \cdot \underbrace{\|\vec{v}\|}_{\text{length of vector } \vec{v}} \cdot \cos\underbrace{(\vec{u}, \vec{v})}_{\text{angle between } \vec{u} \text{ and } \vec{v}}$

* To find the angle between $\vec{u}$ and $\vec{v}$.
using rule 2:

$$\cos(\vec{u}, \vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \cdot \|\vec{v}\|}$$

$$= \frac{xx' + yy' + zz'}{\|\vec{u}\| \times \|\vec{v}\|}.$$

If $\cos(\vec{u}, \vec{v}) = 0$ then $\vec{u} \perp \vec{v}$

If $\cos(\vec{u}, \vec{v}) > 0$ then the angle is acute.

If $\cos(\vec{u}, \vec{v}) < 0$ then the angle is obtuse.

Exercises:

1- Given the vectors $\vec{u}(5, 2, -1)$ and $\vec{v}(-2, 1, 0)$
Find $\vec{u}\cdot\vec{v}$

2- Given the vectors $\vec{u}(2, 1, -1)$ and $\vec{v}(1, -2, 0)$
Are the two vectors $\vec{u}$ and $\vec{v}$ orthogonal? Justify your answer.
perpendicular

3- Given $\|\vec{u}\| = 2cm$, $\|\vec{v}\| = 3\ cm$ and $(\vec{u}, \vec{v}) = \frac{\pi}{3}$
Calculate $\vec{u}\cdot\vec{v}$

4- Given $\vec{u}(1, 1, 1)$ and $\vec{v}(2, 0, -1)$
Calculate $\cos(\vec{u}, \vec{v})$

5- Given $A(1, 5, 2)$, $B(3, -1, 0)$ and $C(1, 1, 1)$
Calculate $\cos(\vec{AB}, \vec{AC})$

Solution:

1- $\vec{u}\cdot\vec{v} = 5(-2) + 2(1) + (-1)(0) = -10 + 2 + 0 = -8$

2- $\vec{u}\cdot\vec{v} = 2(1) + 1(-2) + (-1)(0) = 2 - 2 + 0 = 0$
so $\vec{u}$ and $\vec{v}$ are orthogonal.

3- $\vec{u}\cdot\vec{v} = \|\vec{u}\|.\|\vec{v}\|.\cos(\vec{u}, \vec{v})$
$= 2 \times 3 \times \cos\left(\frac{\pi}{3}\right) = 2 \times 3 \times \frac{1}{2} = 3$

4- $$\cos(\vec{u}, \vec{v}) = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|.\|\vec{v}\|} = \frac{1(2) + 1(0) + 1(-1)}{\sqrt{1^2+1^2+1^2}\cdot\sqrt{2^2+0^2+(-1)^2}}$$
$$= \frac{2+0-1}{\sqrt{3}\cdot\sqrt{5}} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15} = 0.258$$

5- $A(1, 5, 2)$ $B(3, -1, 0)$ $C(1, 1, 1)$

$\overrightarrow{AB}\ (3-1\ ,\ -1-5\ ,\ 0-2)$

$\overrightarrow{AB}\ (2\ ,\ -6\ ,\ -2)$

$\overrightarrow{AC}\ (1-1\ ,\ 1-5\ ,\ 1-2)$

$\overrightarrow{AC}\ (0\ ,\ -4\ ,\ -1)$

$$\cos(\overrightarrow{AB}, \overrightarrow{AC}) = \frac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \cdot \|\overrightarrow{AC}\|}$$

$$= \frac{2(0) - 6(-4) - 2(-1)}{\sqrt{2^2 + (-6)^2 + (-2)^2} \times \sqrt{0^2 + (-4)^2 + (-1)^2}}$$

$$= \frac{0 + 24 + 2}{\sqrt{4 + 36 + 4} \times \sqrt{0 + 16 + 1}}$$

$$= \frac{26}{\sqrt{44} \times \sqrt{17}}$$

$$= \frac{26}{2\sqrt{11} \times \sqrt{17}} = \frac{26}{2\sqrt{187}}$$

$$= \frac{13}{\sqrt{187}} = 0.95$$

Note:

- If $\vec{u} \cdot \vec{v} > 0$ then the angle between $\vec{u}$ and $\vec{v}$ is acute.
- If $\vec{u} \cdot \vec{v} < 0$ then the angle between $\vec{u}$ and $\vec{v}$ is obtuse.
- If $\vec{u} \cdot \vec{v} = 0$ then $\vec{u}$ and $\vec{v}$ are orthogonal (perpendicular)

Vector product or Cross Product

Obj. 4 : Find the vector product (cross product) of two vectors.

Let $\vec{u}(x, y, z)$ and $\vec{v}(x', y', z')$
the vector product of $\vec{u}$ and $\vec{v}$ is a new vector $\vec{w} = \vec{u} \wedge \vec{v}$ where $\vec{w}$ is perpendicular to the plane formed by $\vec{u}$ and $\vec{v}$.
To find the components of $\vec{w}$:

$$\vec{w} = \vec{u} \wedge \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix}$$

$$= (yz' - y'z)\vec{i} - (xz' - x'z)\vec{j} + (xy' - x'y)\vec{k}$$

Ex. Given $\vec{u}(1, 2, 3)$ and $\vec{v}(3, 4, 1)$.
Determine $\vec{u} \wedge \vec{v}$.

Sol. $$\vec{u} \wedge \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 4 & 1 \end{vmatrix}$$

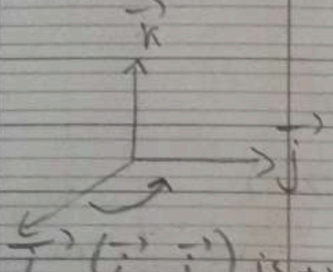
$$= (2 - 12)\vec{i} - (1 - 9)\vec{j} + (4 - 6)\vec{k}$$

$$= -10\vec{i} + 8\vec{j} - 2\vec{k}$$

so $\vec{u} \wedge \vec{v}(-10, 8, -2)$

Note: If $\vec{u} \wedge \vec{v} = \vec{0}$ then $\vec{u}$ and $\vec{v}$ have same direction (parallel or confounded (collinear))

Note: In the direct orthonormal system $(O, \vec{i}, \vec{j}, \vec{k})$ we have:



$(\vec{i}, \vec{j})$ is +ve angle

but

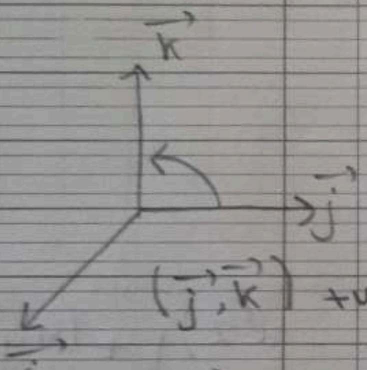
$(\vec{j}, \vec{i})$ is -ve angle

$$\vec{i} \wedge \vec{i} = \vec{j} \wedge \vec{j} = \vec{k} \wedge \vec{k} = \vec{0}$$

$$\vec{i} \wedge \vec{j} = \vec{k} \quad \text{and} \quad \vec{j} \wedge \vec{i} = -\vec{k}$$

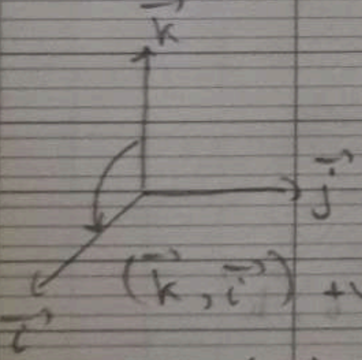
$$\vec{j} \wedge \vec{k} = \vec{i} \quad \text{and} \quad \vec{k} \wedge \vec{j} = -\vec{i}$$

$$\vec{k} \wedge \vec{i} = \vec{j} \quad \text{and} \quad \vec{i} \wedge \vec{k} = -\vec{j}$$



$(\vec{j}, \vec{k})$ +ve angle

but $(\vec{k}, \vec{j})$ -ve angle



$(\vec{k}, \vec{i})$ +ve angle

but $(\vec{i}, \vec{k})$ -ve angle

H.W. Determine $\vec{i} \wedge \vec{i}$, $\vec{j} \wedge \vec{j}$, $\vec{k} \wedge \vec{k}$, $\vec{i} \wedge \vec{j}$, $\vec{j} \wedge \vec{k}$, $\vec{k} \wedge \vec{i}$.

Deduce $\vec{j} \wedge \vec{i}$, $\vec{k} \wedge \vec{j}$ and $\vec{i} \wedge \vec{k}$.

Solution:

$\vec{i}(1,0,0)$ $\quad \vec{j}(0,1,0)$ $\quad \vec{k}(0,0,1)$

* $\vec{i} \wedge \vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$
$= 0\vec{i} - 0\vec{j} + 0\vec{k}$

$\Rightarrow \vec{i} \wedge \vec{i} = \vec{j} \wedge \vec{j} = \vec{k} \wedge \vec{k} = \vec{0}$

* $\vec{i} \wedge \vec{j} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(1-0)$
$= 0\vec{i} - 0\vec{j} + \vec{k} = \vec{k}$
$(0,0,1)$

$\Rightarrow \vec{i} \wedge \vec{j} = \vec{k}$

and $\vec{j} \wedge \vec{i} = -\vec{k}$ $\quad (0,0,-1)$

* $\vec{j} \wedge \vec{k} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \vec{i}(1-0) - \vec{j}(0-0) + \vec{k}(0-0)$
$= \vec{i} - 0\vec{j} + 0\vec{k} = \vec{i}$
$(1,0,0)$

$\Rightarrow \vec{j} \wedge \vec{k} = \vec{i}$

and $\vec{k} \wedge \vec{j} = -\vec{i}$ $\quad (-1,0,0)$

* $\vec{k} \wedge \vec{i} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-1) + \vec{k}(0-0)$
$= 0\vec{i} + \vec{j} + 0\vec{k} = \vec{j}$
$(0,1,0)$

$\Rightarrow \vec{k} \wedge \vec{i} = \vec{j}$

and $\vec{i} \wedge \vec{k} = -\vec{j}$ $\quad (0,-1,0)$

Note:

The components of vectors can be added, subtracted, or multiplied by a constant.

example:

Given $\vec{u}(3, 2, 4)$ and $\vec{v}(-2, 1, 3)$

Find the coordinates of the following vectors.

1) $\vec{w} = \vec{u} + \vec{v}$

$$\left.\begin{aligned} x_{\vec{w}} &= x_{\vec{u}} + x_{\vec{v}} \\ &= 3 + (-2) = 1 \\ y_{\vec{w}} &= y_{\vec{u}} + y_{\vec{v}} \\ &= 2 + 1 = 3 \\ z_{\vec{w}} &= z_{\vec{u}} + z_{\vec{v}} \\ &= 4 + 3 = 7 \end{aligned}\right\} \vec{w}(1, 3, 7)$$

2) $\vec{l} = 2\vec{u} - 3\vec{v}$

$$\left.\begin{aligned} x_{\vec{l}} &= 2x_{\vec{u}} - 3x_{\vec{v}} \\ &= 2(3) - 3(-2) \\ &= 6 + 6 = 12 \\ y_{\vec{l}} &= 2y_{\vec{u}} - 3y_{\vec{v}} \\ &= 2(2) - 3(1) \\ &= 4 - 3 = 1 \\ z_{\vec{l}} &= 2z_{\vec{u}} - 3z_{\vec{v}} \\ &= 2(4) - 3(3) \\ &= 8 - 9 = -1 \end{aligned}\right\} \vec{l}(12, 1, -1)$$

3) $\vec{m} = 3\vec{v}$

$$\left.\begin{aligned} x_{\vec{m}} &= 3x_{\vec{v}} = 3(-2) = -6 \\ y_{\vec{m}} &= 3y_{\vec{v}} = 3(1) = 3 \\ z_{\vec{m}} &= 3z_{\vec{v}} = 3(3) = 9 \end{aligned}\right\} \vec{m}(-6, 3, 9)$$

H.W.

Consider the vectors $\vec{u}(2,-3,4)$, $\vec{v}(1,2,3)$ and $\vec{w}(2,0,1)$. Calculate the components of :

1) $\vec{u} \wedge \vec{v}$

2) $\vec{u} \wedge (\vec{v} + \vec{w})$

3) $\vec{w} \wedge (2\vec{u} - 3\vec{v})$

4) $\vec{u} \wedge (\vec{v} - 2\vec{u} + 2\vec{w})$

5) $\vec{u} \wedge (\vec{v} \wedge \vec{w})$

Solution:

1) $\vec{u} \wedge \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 1 & 2 & 3 \end{vmatrix}$

$= \vec{i}(-9-8) - \vec{j}(6-4) + \vec{k}(4+3)$

$= -17\vec{i} - 2\vec{j} + 7\vec{k}$

so $\vec{u} \wedge \vec{v}(-17, -2, 7)$

2) $\vec{v} + \vec{w}(3, 2, 4)$

$\vec{u} \wedge (\vec{v} + \vec{w}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 3 & 2 & 4 \end{vmatrix}$

$= \vec{i}(-12-8) - \vec{j}(8-12) + \vec{k}(4+9)$

$= -20\vec{i} + 4\vec{j} + 13\vec{k}$

so $\vec{u} \wedge (\vec{v} + \vec{w})(-20, 4, 13)$

3) $2\vec{u} - 3\vec{v}(1, -12, -1)$

$\vec{w} \wedge (2\vec{u} - 3\vec{v}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 1 & -12 & -1 \end{vmatrix}$

$= \vec{i}(0+12) - \vec{j}(-2-1) + \vec{k}(-24-0)$

$= 12\vec{i} + 3\vec{j} - 24\vec{k}$

$(12, 3, -24)$

4) $\vec{v} - 2\vec{u} + 2\vec{w}\ (1 , 8 , -3)$

$$\vec{u} \wedge (\vec{v} - 2\vec{u} + 2\vec{w}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 1 & 8 & -3 \end{vmatrix}$$

$$= \vec{i}\,(9 - 32) - \vec{j}\,(-6 - 4) + \vec{k}\,(16 + 3)$$

$$= -23\vec{i} + 10\vec{j} + 19\vec{k}$$

$(-23, 10, 19)$

5) $$\vec{v} \wedge \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \vec{i}\,(2 - 0) - \vec{j}\,(1 - 6) + \vec{k}\,(0 - 4)$$

$$= 2\vec{i} + 5\vec{j} - 4\vec{k}$$

$\vec{v} \wedge \vec{w}\ (2, 5, -4)$

$$\vec{u} \wedge (\vec{v} \wedge \vec{w}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 4 \\ 2 & 5 & -4 \end{vmatrix}$$

$$= \vec{i}\,(12 - 20) - \vec{j}\,(-8 - 8) + \vec{k}\,(10 + 6)$$

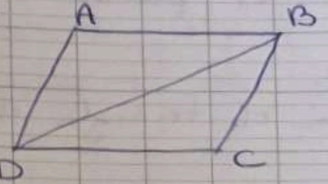
$$= -8\vec{i} + 16\vec{j} + 16\vec{k}$$

$(-8, 16, 16)$

Obj. 5: Calculate the area of a triangle and the area of a parallelogram using vector (cross) product.

Rule:

Consider the points A, B and D, the three vertices of a parallelogram ABCD.



- The area of parm ABCD is equal to:

$\| \overrightarrow{AB} \wedge \overrightarrow{AD} \|$ ⟶
- we find the coordinates of vectors $\overrightarrow{AB}$ & $\overrightarrow{AD}$
- we find $\overrightarrow{AB} \wedge \overrightarrow{AD}$
- we find the length (norm) of $\| \overrightarrow{AB} \wedge \overrightarrow{AB} \|$

- The area of triangle ABD is equal to: $\frac{1}{2} \| \overrightarrow{AB} \wedge \overrightarrow{AD} \|$

Example: Consider the points $A(2, 3, -1)$, $B(1, 3, -2)$ and $D(3, 0, 2)$ the three vertices of a parm ABCD.
Find the areas of parm ABCD and triangle ABD.

$\overrightarrow{AB} \left(x_B - x_A \ , \ y_B - y_A \ , \ z_B - z_A \right)$

$\left(1 - 2 \ , \ 3 - 3 \ , \ -2 - (-1) \right)$

$\left(-1 \ , \ 0 \ , \ -1 \right)$

$\overrightarrow{AD} \left(x_D - x_A \ , \ y_D - y_A \ , \ z_D - z_A \right)$

$\left(3 - 2 \ , \ 0 - 3 \ , \ 2 - (-1) \right)$

$\left(1 \ , \ -3 \ , \ 3 \right)$

$$\overrightarrow{AB} \wedge \overrightarrow{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & -1 \\ 1 & -3 & 3 \end{vmatrix}$$

$$= \vec{i}(0 - 3) - \vec{j}(-3 + 1) + \vec{k}(3 - 0)$$

$$= -3\vec{i} + 2\vec{j} + 3\vec{k}$$

$$\|\overrightarrow{AB} \wedge \overrightarrow{AD}\| = \sqrt{(-3)^2 + (2)^2 + (3)^2}$$
$$= \sqrt{9 + 4 + 9}$$
$$= \sqrt{22}.$$

area of parm ABCD $= \|\overrightarrow{AB} \wedge \overrightarrow{AD}\| = \sqrt{22}$ square units

area of triangle ABD $= \frac{1}{2}\|\overrightarrow{AB} \wedge \overrightarrow{AD}\| = \frac{1}{2}\sqrt{22}$ square units

Note: To find the area of a parm, we can use any 3 vertices of it.

$$\mathcal{A}_{ABCD} = \|\overrightarrow{AB} \wedge \overrightarrow{AD}\|$$
$$\text{or} = \|\overrightarrow{AB} \wedge \overrightarrow{AC}\|$$
$$\text{or} = \|\overrightarrow{BA} \wedge \overrightarrow{BC}\|$$
$$\text{or} = \|\overrightarrow{BA} \wedge \overrightarrow{BD}\|$$
$$\vdots$$

Find any 2 vectors and find the area of the parm.

H.W. Consider the points $A(1,2,3)$ and $B(3,2,1)$
Calculate the area of triangle OAB.
(Note that $O(0,0,0)$ is the origin of the orthonormal system $(O; \vec{i}, \vec{j}, \vec{k})$)

Solution:

$\overrightarrow{OA}(1,2,3)$ $\overrightarrow{OB}(3,2,1)$

$$\overrightarrow{OA} \wedge \overrightarrow{OB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} = \vec{i}(2-6) - \vec{j}(1-9) + \vec{k}(2-6)$$
$$= -4\vec{i} + 8\vec{j} - 4\vec{k}$$

$$\|\overrightarrow{OA} \wedge \overrightarrow{OB}\| = \sqrt{(-4)^2 + (8)^2 + (-4)^2} = \sqrt{16 + 64 + 16} = \sqrt{96}$$ square units

$$\mathcal{A}_{ABC} = \frac{1}{2}\|\overrightarrow{OA} \wedge \overrightarrow{OB}\| = \frac{1}{2}\sqrt{96} = 2\sqrt{6}$$ square units

Obj. 6 : Find the measure of the angle between two vectors.

Definition: If $\vec{u}$ and $\vec{v}$ are two vectors not having the same direction and $\vec{u} \wedge \vec{v} = \vec{w}$ then

- $\vec{w}$ is perpendicular to the plane formed by $\vec{u}$ and $\vec{v}$.
- $\|\vec{w}\| = \|\vec{u}\| \times \|\vec{v}\| \times \sin(\vec{u}, \vec{v})$

Note: We can find the angle α between two vectors using dot product (we can find $\cos\alpha$) or using the cross product (we can find $\sin\alpha$).

Exercise 1:

Consider the points $E(-1, 2, 0)$ and $F(2, 3, 0)$

a. Calculate $\vec{OE} \wedge \vec{OF}$

b. Find the coordinates of G so that OEGF is a parm.

c. Calculate the area of OEGF.

d. Calculate the distance from E to (OF).

Solution:

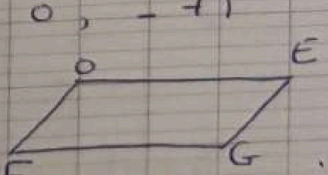
a. $\vec{OE}(x_E - x_O, y_E - y_O, z_E - z_O) \Rightarrow \vec{OE}(-1, 2, 0)$

$\vec{OF}(x_F - x_O, y_F - y_O, z_F - z_O) \Rightarrow \vec{OF}(2, 3, 0)$

$\vec{OE} \wedge \vec{OF} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix} = \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(-3-4)$

$= 0\vec{i} + 0\vec{j} - 7\vec{k}$

$(0, 0, -7)$



b. OEGF is a parm

$\vec{OE} = \vec{FG}$ (same sense)

$x_{\vec{FG}} = x_G - x_F = x_G - 2$

$x_{\vec{FG}} = x_{\vec{OE}}$

$x_G - 2 = -1 \Rightarrow x_G = 1$

$y_{\vec{FG}} = y_G - y_F = y_G - 3$

$y_{\vec{FG}} = y_{\vec{OE}}$

$y_G - 3 = 2 \Rightarrow y_G = 5$

$z_{\vec{FG}} = z_G - z_F = z_G - 0$

$z_{\vec{FG}} = z_{\vec{OE}}$

$z_G - 0 = 0 \Rightarrow z_G = 0$

$G(1, 5, 0)$

c. $\mathcal{A}_{OEGF} = \| \vec{OE} \wedge \vec{OF} \|$

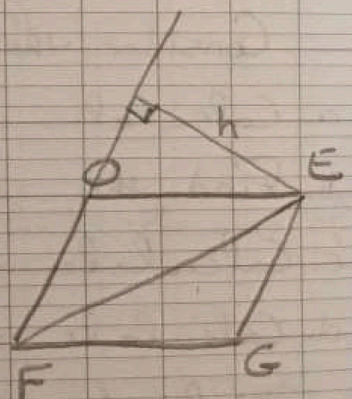
$= \sqrt{0^2 + 0^2 + (-7)^2}$

$= \sqrt{49} = 7$ square units.

d. distance from E to (OF) is the height issued from E to (OF) in ΔOEF.

$\mathcal{A}_{EOF} = \frac{1}{2} \| \vec{OE} \wedge \vec{OF} \|$

$\frac{1}{2} b \times h = \frac{1}{2} \| \vec{OE} \wedge \vec{OF} \|$

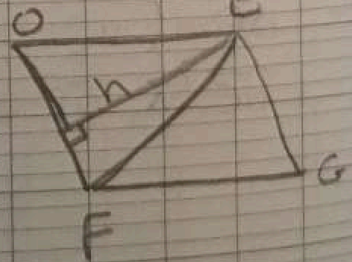


$b = \| \vec{OF} \| = \sqrt{(2)^2 + (3)^2 + (0)^2}$

$= \sqrt{4 + 9} = \sqrt{13}$.

$\cancel{\frac{1}{2}} \times \sqrt{13} \times h = \cancel{\frac{1}{2}} \times 7$

$h = \frac{7}{\sqrt{13}} \times \frac{\sqrt{13}}{\sqrt{13}} = \frac{7\sqrt{13}}{13}$.



Exercise 2:

Consider the points $A(1, 2, -3)$, $B(2, -1, 0)$ and $C(3, 6, 5)$

a. Show that the points O, A, B are not collinear.

b. Show that (OC) is perpendicular to the plane (OBA)

Solution:

a. $\left(\begin{array}{l} A, B \,\&\, O \text{ are collinear if } \overrightarrow{OA} \wedge \overrightarrow{OB} = \overrightarrow{0} \\ A, B \,\&\, O \text{ are not collinear if } \overrightarrow{OA} \wedge \overrightarrow{OB} \neq \overrightarrow{0} \\ (\text{we can use any two vectors formed by } A, B \,\&\, O) \end{array} \right)$

$\overrightarrow{OA}\,(x_A - x_0, y_A - y_0, z_A - z_0) \Rightarrow \overrightarrow{OA}\,(1, 2, -3)$

$\overrightarrow{OB}\,(x_B - x_0, y_B - y_0, z_B - z_0) \Rightarrow \overrightarrow{OB}\,(2, -1, 0)$

$$\overrightarrow{OA} \wedge \overrightarrow{OB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -3 \\ 2 & -1 & 0 \end{vmatrix} = \vec{i}(0-3) - \vec{j}(0+6) + \vec{k}(-1-4)$$

$$= -3\vec{i} - 6\vec{j} - 5\vec{k}$$

$$(-3, -6, -5)$$

$\overrightarrow{OA} \wedge \overrightarrow{OB} \neq \overrightarrow{0}$ so $A, B \,\&\, O$ are not collinear.

b. $\overrightarrow{OC}\,(x_C - x_0, y_C - y_0, z_C - z_0) \Rightarrow \overrightarrow{OC}\,(3, 6, 5)$

$\overrightarrow{OA} \wedge \overrightarrow{OB} = -3\vec{i} - 6\vec{j} - 5\vec{k}$ is a vector perpendicular to the plane (OBA).

Any vector whose coordinates are $k.(\overrightarrow{OA} \wedge \overrightarrow{OB})$ is also perpendicular to plane (OBA) where $k \in \mathbb{R}$.

$$\overrightarrow{OC} = 3\vec{i} + 6\vec{j} + 5\vec{k}$$

$$= -1 \times (-3\vec{i} - 6\vec{j} - 5\vec{k})$$

$$= -1 \times (\overrightarrow{OA} \wedge \overrightarrow{OB})$$

so (OC) is perpendicular to the plane (OBA).

Exercise 3:

Consider the points $A(1, 2, 3)$, $B(-1, 1, 2)$ and $C(3, 1, 0)$

a. What is the nature of triangle ABC ?

b. Calculate the area of triangle ABC.

c. Consider the vector $\vec{u}(m, 2n+1, 2)$. Determine m and n so that $\vec{u}$ is normal (perpendicular) to the plane (ABC).

Solution:

a. $\vec{AB}(x_B - x_A, y_B - y_A, z_B - z_A)$

$(-1-1, 1-2, 2-3)$

$(-2, -1, -1)$

$$\|\vec{AB}\| = \sqrt{(-2)^2 + (-1)^2 + (-1)^2}$$
$$= \sqrt{4+1+1} = \sqrt{6}.$$

$\vec{AC}(x_C - x_A, y_C - y_A, z_C - z_A)$

$(3-1, 1-2, 0-3)$

$(2, -1, -3)$

$$\|\vec{AC}\| = \sqrt{(2)^2 + (-1)^2 + (-3)^2}$$
$$= \sqrt{4+1+9} = \sqrt{14}$$

$\vec{BC}(x_C - x_B, y_C - y_B, z_C - z_B)$

$(3-(-1), 1-1, 0-2)$

$(4, 0, -2)$

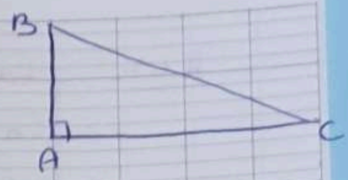
$$\|\vec{BC}\| = \sqrt{(4)^2 + 0^2 + (-2)^2}$$
$$= \sqrt{16+4} = \sqrt{20}$$

$BC^2 = AB^2 + AC^2$

$20 = 6 + 14$

so ΔABC is right at A.

b. $\mathcal{A}_{ABC} = \frac{b \times h}{2} = \frac{AB \times AC}{2}$



$= \frac{\sqrt{6} \times \sqrt{14}}{2} = \frac{2\sqrt{21}}{2}$

$= \sqrt{21}$ square units.

OR 2nd method:

$$\vec{AB} \wedge \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & -1 & -1 \\ 2 & -1 & -3 \end{vmatrix} = \vec{i}(3-1) - \vec{j}(6+2) + \vec{k}(2+2)$$
$$= 2\vec{i} - 8\vec{j} + 4\vec{k}$$

$$\|\vec{AB} \wedge \vec{AC}\| = \sqrt{(2)^2 + (-8)^2 + (4)^2}$$
$$= \sqrt{4 + 64 + 16} = \sqrt{84}$$
$$= 2\sqrt{21}$$

$$\mathcal{A}_{ABC} = \frac{1}{2}\|\vec{AB} \wedge \vec{AC}\|$$
$= \frac{1}{2} \times 2\sqrt{21} = \sqrt{21}$ square units.

c. $\vec{u} \perp (ABC)$

so $\vec{u} = k \times (\vec{AB} \wedge \vec{AC})$

$\vec{u} = k(2\vec{i} - 8\vec{j} + 4\vec{k})$

$z_{\vec{u}} = k \times 4$

$2 = k \times 4 \Rightarrow k = \frac{1}{2}$.

we start with $z_{\vec{u}}$ because it is given $= 2$ without unknown. (we use it to find the value of k)

$x_{\vec{u}} = k \times 2$

$m = \frac{1}{2} \times 2 = 1$

$y_{\vec{u}} = k \times (-8)$

$2n + 1 = \frac{1}{2}(-8)$

$2n = -4 - 1$

$n = \frac{-5}{2}$.

H.W. Exercise 4:

Consider a parallelogram ABCD with $A(3,-2,-1)$, $B(2,1,3)$ and $C(0,4,1)$

a. Find the coordinates of D.

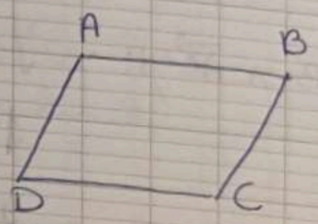
b. Calculate the area of this parallelogram.

c. Calculate the area of triangle ABD and that of ABC.

d. Calculate $\cos \widehat{BAC}$ and $\sin \widehat{BAC}$.

Solution:

a. $\overrightarrow{AB} = \overrightarrow{DC}$



$$
\left.
\begin{aligned}
x_{\overrightarrow{AB}} &= x_{\overrightarrow{DC}} \\
x_B - x_A &= x_C - x_D \\
2 - 3 &= 0 - x_D \\
-1 &= -x_D \Rightarrow x_D = 1 \\
\\
y_{\overrightarrow{AB}} &= y_{\overrightarrow{DC}} \\
y_B - y_A &= y_C - y_D \\
1 - (-2) &= 4 - y_D \\
3 &= 4 - y_D \\
-1 &= -y_D \Rightarrow y_D = 1 \\
\\
z_{\overrightarrow{AB}} &= z_{\overrightarrow{DC}} \\
z_B - z_A &= z_C - z_D \\
3 - (-1) &= 1 - z_D \\
4 &= 1 - z_D \\
3 &= -z_D \Rightarrow z_D = -3
\end{aligned}
\right\} \quad D(1,1,-3)
$$

b. $\overrightarrow{AB}\,(x_B - x_A \,,\, y_B - y_A \,,\, z_B - z_A)$

$(2 - 3 \,,\, 1-(-2) \,,\, 3-(-1))$

$(-1 \,,\, 3 \,,\, 4)$

$\vec{AC}\ (x_C - x_A\ ,\ y_C - y_A\ ,\ z_C - z_A)$
$(0 - 3\ ,\ 4 - (-2)\ ,\ 1 - (-1))$
$(-3\ ,\ 6\ ,\ 2)$

$$\vec{AB} \wedge \vec{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 4 \\ -3 & 6 & 2 \end{vmatrix} = \vec{i}(6-24) - \vec{j}(-2+12) + \vec{k}(-6+9)$$
$$= -18\vec{i} - 10\vec{j} + 3\vec{k}$$

$$\mathcal{A}_{ABCD} = \|\vec{AB} \wedge \vec{AC}\|$$
$$= \sqrt{(-18)^2 + (-10)^2 + (3)^2}$$
$$= \sqrt{324 + 100 + 9} = \sqrt{433} \text{ square uni}$$

c. $\mathcal{A}_{ABC} = \frac{1}{2}\|\vec{AB} \wedge \vec{AC}\| = \frac{1}{2}\sqrt{433}$ square unit

$\vec{AD}\ (x_D - x_A\ ,\ y_D - y_A\ ,\ z_D - z_A)$
$(1 - 3\ ,\ 1 - (-2)\ ,\ -3 - (-1))$
$(-2\ ,\ 3\ ,\ -2)$

$$\vec{AB} \wedge \vec{AD} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 3 & 4 \\ -2 & 3 & -2 \end{vmatrix} = \vec{i}(-6-12) - \vec{j}(2+8) + \vec{k}(-3+6)$$
$$= -18\vec{i} - 10\vec{j} + 3\vec{k}$$

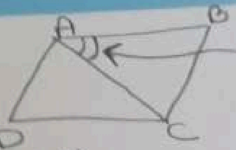
$$\|\vec{AB} \wedge \vec{AD}\| = \sqrt{(-18)^2 + (-10)^2 + (3)^2}$$
$$= \sqrt{324 + 100 + 9} = \sqrt{433} \text{ square uni}$$

$\mathcal{A}_{ABD} = \frac{1}{2}\|\vec{AB} \wedge \vec{AD}\| = \frac{1}{2}\sqrt{433}$ square units

OR 2nd method:

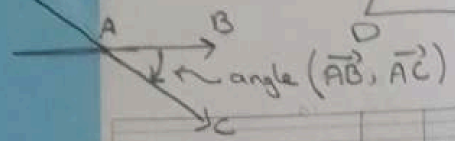
$$\mathcal{A}_{ABD} = \frac{1}{2}\mathcal{A}_{ABCD} = \frac{1}{2}\sqrt{433}$$

$$\mathcal{A}_{ABC} = \frac{1}{2}\mathcal{A}_{ABCD} = \frac{1}{2}\sqrt{433}$$

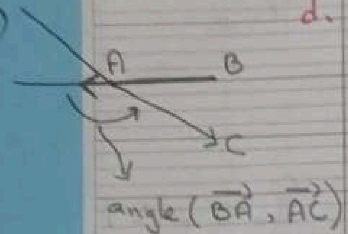


angle $\widehat{BAC}$ is included between vectors $\overrightarrow{AB}$ & $\overrightarrow{AC}$. (The vectors must start with letter A) $\left\{ \begin{array}{l} \text{we can't use } \overrightarrow{BA} \text{ or } \overrightarrow{CA} \\ \text{we will get wrong answers} \end{array} \right\}$

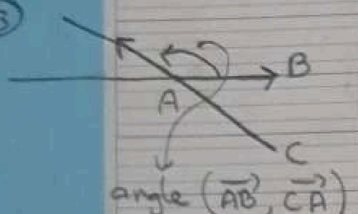
①



②

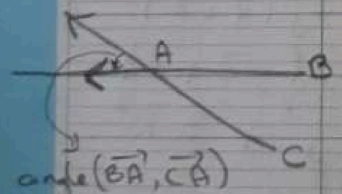


③



Notice that when we change the sense of one of the vectors, we get a different angle. However, we can change the sense of both vectors. In this case we will get the same angle.

④



$= \text{angle}(\overrightarrow{AB}, \overrightarrow{AC})$ (vertically opposite angles.)

d. $\cos \widehat{BAC} = \dfrac{\overrightarrow{AB} \cdot \overrightarrow{AC}}{\|\overrightarrow{AB}\| \times \|\overrightarrow{AC}\|}$

$\overrightarrow{AB}(-1, 3, 4)$

$\overrightarrow{AC}(-3, 6, 2)$

$$\begin{aligned} \overrightarrow{AB} \cdot \overrightarrow{AC} &= (-1)(-3) + (3)(6) + (4)(2) \\ &= 3 + 18 + 8 \\ &= 29 \end{aligned}$$

$$\begin{aligned} \|\overrightarrow{AB}\| &= \sqrt{(-1)^2 + (3)^2 + (4)^2} \\ &= \sqrt{1 + 9 + 16} = \sqrt{26} \end{aligned}$$

$$\begin{aligned} \|\overrightarrow{AC}\| &= \sqrt{(-3)^2 + (6)^2 + (2)^2} \\ &= \sqrt{9 + 36 + 4} = \sqrt{49} = 7 \end{aligned}$$

$$\cos \widehat{BAC} = \frac{29}{7\sqrt{26}}$$

$$\sin \widehat{BAC} = \frac{\|\overrightarrow{AB} \wedge \overrightarrow{AC}\|}{\|\overrightarrow{AB}\| \times \|\overrightarrow{AC}\|}$$

$$= \frac{\sqrt{433}}{7\sqrt{26}}$$

<u>OR</u> <u>second method</u>

$$\begin{aligned} \sin^2 \widehat{BAC} &= 1 - \cos^2 \widehat{BAC} \\ &= 1 - \left(\frac{29}{7\sqrt{26}}\right)^2 = 1 - \frac{841}{1274} \\ &= \frac{433}{1274} \end{aligned}$$

$$\sin \widehat{BAC} = \sqrt{\frac{433}{1274}} = \frac{\sqrt{433}}{7\sqrt{26}}$$

Obj. 7: Calculate the triple scalar product of three vectors.

Definition: The triple scalar product of three vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$ in this order is the real number: $\vec{u}\cdot(\vec{v}\wedge\vec{w})$

If $\vec{u}(x, y, z)$ $\vec{v}(x', y', z')$ & $\vec{w}(x'', y'', z'')$

then

$$\vec{u}\cdot(\vec{v}\wedge\vec{w}) = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} \begin{matrix} \to \text{components of } \vec{u} \\ \to \text{components of } \vec{v} \\ \to \text{components of } \vec{w} \end{matrix} \Bigg\} \text{we write them in order.}$$

$$= x(y'z'' - y''z') - y(x'z'' - x''z') + z(x'y'' - x''y')$$

Example 1: Consider the points $A(2, -1, 3)$, $B(3, 1, 2)$ and $C(0, -12, 9)$

Calculate $\vec{OA}\cdot(\vec{OB}\wedge\vec{OC})$

Solution: $\vec{OA}\,(x_A - x_O,\ y_A - y_O,\ z_A - z_O)$

$(2, -1, 3)$

$\vec{OB}\,(x_B - x_O,\ y_B - y_O,\ z_B - z_O)$

$(3, 1, 2)$

$\vec{OC}\,(x_C - x_O,\ y_C - y_O,\ z_C - z_O)$

$(0, -12, 9)$

$$\vec{OA}\cdot(\vec{OB}\wedge\vec{OC}) = \begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 2 \\ 0 & -12 & 9 \end{vmatrix}$$

$$= 2(9+24) - (-1)(27-0) + 3(-36-0)$$

$$= 66 + 27 - 108$$

$$= -15$$

Example 2: Consider the points $E(3,2,-1)$, $F(4,-3,2)$, $G(0,3,1)$ and $H(4,2,-2)$

Calculate $\overrightarrow{EF} \cdot (\overrightarrow{EG} \wedge \overrightarrow{EH})$

Solution: $\overrightarrow{EF}\,(x_F - x_E\,,\ y_F - y_E\,,\ z_F - z_E)$

$(4-3\,,\ -3-2\,,\ 2-(-1))$

$(1\,,\ -5\,,\ 3)$

$\overrightarrow{EG}\,(x_G - x_E\,,\ y_G - y_E\,,\ z_G - z_E)$

$(0-3\,,\ 3-2\,,\ 1-(-1))$

$(-3\,,\ 1\,,\ 2)$

$\overrightarrow{EH}\,(x_H - x_E\,,\ y_H - y_E\,,\ z_H - z_E)$

$(4-3\,,\ 2-2\,,\ -2-(-1))$

$(1\,,\ 0\,,\ -1)$

$$\overrightarrow{EF} \cdot (\overrightarrow{EG} \wedge \overrightarrow{EH}) = \begin{vmatrix} 1 & -5 & 3 \\ -3 & 1 & 2 \\ 1 & 0 & -1 \end{vmatrix}$$

$$= 1(-1-0) - (-5)(3-2) + 3(0-1)$$

$$= -1 + 5 - 3 = 1$$

H.W. 1- Consider the vectors $\vec{u}(2,3,1)$, $\vec{v}(-2,-1,4)$ and $\vec{w}(1,2,3)$. Calculate:

$\vec{u} \cdot (\vec{v} \wedge \vec{w})$ and $\vec{v} \cdot (\vec{w} \wedge \vec{v})$

2- Consider the points $A(2,1,3)$, $B(-2,2,1)$ and $C(0,3,-2)$. Calculate:

$\overrightarrow{OA} \cdot (\overrightarrow{OB} \wedge \overrightarrow{OC})$ and $\overrightarrow{OA} \cdot (\overrightarrow{AC} \wedge \overrightarrow{BC})$

Solution:

1- $\vec{u}\cdot(\vec{v}\wedge\vec{w}) = \begin{vmatrix} 2 & 3 & 1 \\ -2 & -1 & 4 \\ 1 & 2 & 3 \end{vmatrix}$

$= 2(-3-8) - 3(-6-4) + 1(-4+1)$

$= -22 + 30 - 3 = 5$

$\vec{v}\cdot(\vec{w}\wedge\vec{v}) = \begin{vmatrix} -2 & -1 & 4 \\ 1 & 2 & 3 \\ -2 & -1 & 4 \end{vmatrix}$

$= -2(8+3) - (-1)(4+6) + 4(-1+4)$

$= -22 + 10 + 12 = 0$

2- $\vec{OA}(x_A - x_O \ , \ y_A - y_O \ , \ z_A - z_O)$

$(2 \ , \ 1 \ , \ 3)$

$\vec{OB}(x_B - x_O \ , \ y_B - y_O \ , \ z_B - z_O)$

$(-2 \ , \ 2 \ , \ 1)$

$\vec{OC}(x_C - x_O \ , \ y_C - y_O \ , \ z_C - z_O)$

$(0 \ , \ 3 \ , \ -2)$

$\vec{OA}\cdot(\vec{OB}\wedge\vec{OC}) = \begin{vmatrix} 2 & 1 & 3 \\ -2 & 2 & 1 \\ 0 & 3 & -2 \end{vmatrix}$

$= 2(-4-3) - 1(4-0) + 3(-6-0)$

$= -14 - 4 - 18$

$= -36$

$\overrightarrow{OA}\,(2\ ,1\ ,3)$

$\overrightarrow{AC}\,(x_C - x_A\ ,\ y_C - y_A\ ,\ z_C - z_A)$

$(0 - 2\ ,\ 3 - 1\ ,\ -2 - 3)$

$(-2\ ,\ 2\ ,\ -5)$

$\overrightarrow{BC}\,(x_C - x_B\ ,\ y_C - y_B\ ,\ z_C - z_B)$

$(0 - (-2)\ ,\ 3 - 2\ ,\ -2 - 1)$

$(2\ ,\ 1\ ,\ -3)$

$$\overrightarrow{OA}\cdot(\overrightarrow{AC}\wedge\overrightarrow{BC}) = \begin{vmatrix} 2 & 1 & 3 \\ -2 & 2 & -5 \\ 2 & 1 & -3 \end{vmatrix}$$

$$= 2(-6+5) - 1(6+10) + 3(-2-4)$$

$$= -2 - 16 - 18$$

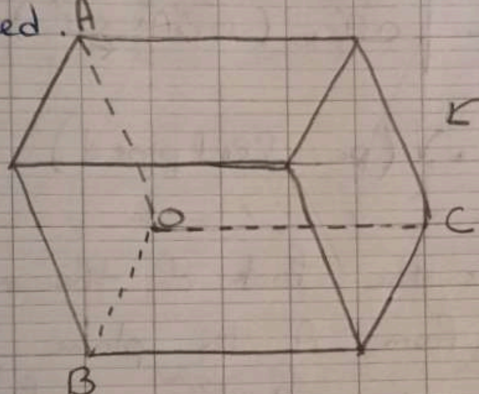
$$= -36$$

Property: Four points O, A, B and C are coplanar (form one plane only) if and only if $\overrightarrow{OA} \cdot (\overrightarrow{OB} \wedge \overrightarrow{OC}) = 0$

means that we can use it in both directions. ($\Longleftrightarrow$)

$$\begin{cases} \text{If } O, A, B \,\&\, C \text{ coplanar} \Longrightarrow \overrightarrow{OA} \cdot (\overrightarrow{OB} \wedge \overrightarrow{OC}) = 0 \\ \text{If } \overrightarrow{OA} \cdot (\overrightarrow{OB} \wedge \overrightarrow{OC}) = 0 \Longrightarrow O, A, B \,\&\, C \text{ coplanar.} \end{cases}$$

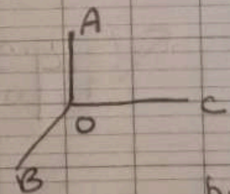
If $\overrightarrow{OA} \cdot (\overrightarrow{OB} \wedge \overrightarrow{OC}) \neq 0$ then the 4 points are vertices of a parallelpiped.



Each face of the parallelpiped is a parm.

O, A, B & C are four vertices of the parallelpiped. But they don't form a face of it.

we put them like an orthonormal system



because they are not coplanar, but not perpendicular

We can name the rest of the vertices using any letters.

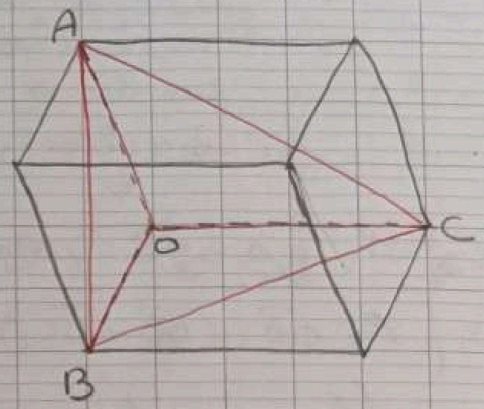
* Volume of parallelpiped:

$V = \left| \overrightarrow{OA} \cdot (\overrightarrow{OB} \wedge \overrightarrow{OC}) \right|$

we must put the absolute value because the volume can't be negative

* Four non coplanar points form a tetrahedron.

a tetrahedron (the red shape) has 4 faces. Each face is a triangle (OBC, OAB, OAC and ABC)



* Volume of tetrahedron:

$$V_{OABC} = \frac{1}{6} \left| \vec{OA} \cdot (\vec{OB} \wedge \vec{OC}) \right|$$

$$= \frac{1}{6} \cdot V(\text{parallelpiped})$$

If H is the foot of the normal (height) issued from A to plane (OBC) then

we use this to find the distance from A to plane (OBC).

$$V_{OABC} = \frac{1}{3} \underbrace{\mathcal{A}_{OBC}}_{\text{area of base}} \times \underbrace{AH}_{\text{height (distance from A to plane (OBC))}}$$

$$\frac{1}{6} \left| \vec{OA} \cdot (\vec{OB} \wedge \vec{OC}) \right| = \frac{1}{3} \left(\frac{1}{2} \| \vec{OB} \wedge \vec{OC} \| \times AH \right)$$

Exercise 1:

Consider the points $A(1,1,2)$, $B(0,2,0)$, $C(3,2,3)$ and $D(1,3,-1)$.

1) How many planes are determined by the points A, B, C and D?

2) The same question is asked for the points A, B, C and $E(3,0,5)$.

Solution:

1) $\overrightarrow{AB}\,(x_B - x_A\,,\ y_B - y_A\,,\ z_B - z_A)$
$(0-1\,,\ 2-1\,,\ 0-2)$
$(-1\,,\ 1\,,\ -2)$

$\overrightarrow{AC}\,(x_C - x_A\,,\ y_C - y_A\,,\ z_C - z_A)$
$(3-1\,,\ 2-1\,,\ 3-2)$
$(2\,,\ 1\,,\ 1)$

$\overrightarrow{AD}\,(x_D - x_A\,,\ y_D - y_A\,,\ z_D - z_A)$
$(1-1\,,\ 3-1\,,\ -1-2)$
$(0\,,\ 2\,,\ -3)$

$$\overrightarrow{AB}\cdot(\overrightarrow{AC}\wedge\overrightarrow{AD}) = \begin{vmatrix} -1 & 1 & -2 \\ 2 & 1 & 1 \\ 0 & 2 & -3 \end{vmatrix}$$

$$= -1(-3-2) - 1(-6-0) - 2(4-0)$$
$$= 5 + 6 - 8 \quad = +3$$

$\overrightarrow{AB}\cdot(\overrightarrow{AC}\wedge\overrightarrow{AD}) \neq 0$ so A, B, C & D are not coplanar. 4 planes are determined by A, B, C & D

{The points are not coplanar $\Longrightarrow$ they form a tetrahedron $\Longrightarrow$ They form 4 planes (because the tetrahedron has 4 faces.)}

2) $\vec{AB}(-1, 1, -2)$

$\vec{AC}(2, 1, 1)$

$\vec{AE}(x_E - x_A, y_E - y_A, z_E - z_A)$

$(3-1, 0-1, 5-2)$

$(2, -1, 3)$

$\vec{AB} \cdot (\vec{AC} \wedge \vec{AE}) = \begin{vmatrix} -1 & 1 & -2 \\ 2 & 1 & 1 \\ 2 & -1 & 3 \end{vmatrix}$

$= -1(3+1) - 1(6-2) - 2(-2-2)$

$= -4 - 4 + 8 = 0$

$\vec{AB} \cdot (\vec{AC} \wedge \vec{AE}) = 0 \Longrightarrow$ A, B, C & E are coplanar $\Longrightarrow$ They form only one plane.

Exercise 2:

Consider the points $A(3, -1, 2)$, $B(2, -2, 1)$, $C(0, 2, 3)$ and $M(\alpha, \beta, \gamma)$.

Find a relation between α, β & γ such that M is in the plane (ABC).

Solution:

M is in the plane (ABC) so A, B, C & M are coplanar so $\vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) = 0$

$\vec{AM}(x_M - x_A, y_M - y_A, z_M - z_A)$

$(\alpha - 3, \beta + 1, \gamma - 2)$

$\vec{AB}(x_B - x_A, y_B - y_A, z_B - z_A)$

$(2-3, -2-(-1), 1-2)$

$(-1, -1, -1)$

$\vec{AC}(x_C - x_A, y_C - y_A, z_C - z_A)$

$(0-3, 2-(-1), 3-2)$

$(-3, 3, 1)$

we can choose any 3 vectors formed by the points A, B, C & M.

It is easier to put the vector that contains the unknowns in the first row of the determinant.

$\overrightarrow{AM} \cdot (\overrightarrow{AB} \wedge \overrightarrow{AC}) = 0$

$$\begin{vmatrix} \alpha - 3 & \beta + 1 & \gamma - 2 \\ -1 & -1 & -1 \\ -3 & 3 & 1 \end{vmatrix} = 0$$

$(\alpha - 3)(-1-(-3)) - (\beta+1)(-1-3) + (\gamma - 2)(-3-3) = 0$

$(\alpha - 3)(2) - (\beta+1)(-4) + (\gamma - 2)(-6) = 0$

$2\alpha - 6 + 4\beta + 4 - 6\gamma + 12 = 0$

$2\alpha + 4\beta - 6\gamma + 10 = 0$

Exercise 3:

Consider the points $A(1,1,2)$, $B(2,3,4)$, $C(1,0,-3)$ and $D(2,2,2)$.

1) Are the points A, B, C & D coplanar?

2) For what values of m does the point $M(m, 1, 2m)$ belong to the plane (ABC)?

3) Calculate in terms of m the volume of the tetrahedron MABC when M does not belong to the plane (ABC).

Solution:

1) A, B, C & D are coplanar if $\overrightarrow{AB} \cdot (\overrightarrow{AC} \wedge \overrightarrow{AD}) = 0$.

$\overrightarrow{AB}\,(x_B - x_A, \; y_B - y_A, \; z_B - z_A)$

$(2 - 1, \; 3 - 1, \; 4 - 2)$

$(1, \; 2, \; 2)$

$\overrightarrow{AC}\,(x_C - x_A, \; y_C - y_A, \; z_C - z_A)$

$(1 - 1, \; 0 - 1, \; -3 - 2)$

$(0, \; -1, \; -5)$

$\overrightarrow{AD}\,(x_D - x_A, \; y_D - y_A, \; z_D - z_A)$

$(2 - 1, \; 2 - 1, \; 2 - 2)$

$(1, \; 1, \; 0)$

$$\vec{AB} \cdot (\vec{AC} \wedge \vec{AD}) = \begin{vmatrix} 1 & 2 & 2 \\ 0 & -1 & -5 \\ 1 & 1 & 0 \end{vmatrix}$$

$$= 1(0-(-5)) - 2(0-(-5)) + 2(0-(-1))$$

$$= 5 - 10 + 2 = -3 \neq 0$$

$\vec{AB} \cdot (\vec{AC} \wedge \vec{AD}) \neq 0$ so the points A, B, C & D are not coplanar.

2) M belongs to plane (ABC)

$\Rightarrow$ A, B, C & M are coplanar

$\Rightarrow \vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) = 0$

$\vec{AM}\,(x_M - x_A \,,\, y_M - y_A \,,\, z_M - z_A)$

$(m-1 \,,\, 1-1 \,,\, 2m-2)$

$(m-1 \,,\, 0 \,,\, 2m-2)$

$\vec{AB}\,(1 \,,\, 2 \,,\, 2)$

$\vec{AC}\,(0 \,,\, -1 \,,\, -5)$

$\vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) = 0$

$$\begin{vmatrix} m-1 & 0 & 2m-2 \\ 1 & 2 & 2 \\ 0 & -1 & -5 \end{vmatrix} = 0$$

$(m-1)(-10-(-2)) - 0(-5-0) + (2m-2)(-1-0) = 0$

$(m-1)(-8) - 0 + (2m-2)(-1) = 0$

$-8m + 8 - 2m + 2 = 0$

$-10m + 10 = 0$

$-10m = -10$

$m = \frac{-10}{-10} = 1.$

$M(1 \,,\, 1 \,,\, 2).$

3) M does not belong to plane (ABC)

$\vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) \neq 0$

$\vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) = -10m + 10$.

$V_{MABC} = \frac{1}{6} \left| \vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) \right|$

$= \frac{1}{6} \left| -10m + 10 \right|$ cubic units.

H.W. Exercise 4:

Consider the points $A(3, 2, -3)$, $B(2, 1, -1)$ and $C(4, 0, 1)$.

1) Show that the points O, A, B & C are not coplanar.
2) Calculate the volume of parallelpiped with sides [OA], [OB] and [OC].
3) Calculate the volume of the tetrahedron OABC.

Solution:

1) $\vec{OA}\,(x_A - x_O\,,\; y_A - y_O\,,\; z_A - z_O)$

$(3 - 0\,,\; 2 - 0\,,\; -3 - 0)$

$(3\,,\; 2\,,\; -3)$

$\vec{OB}\,(x_B - x_O\,,\; y_B - y_O\,,\; z_B - z_O)$

$(2 - 0\,,\; 1 - 0\,,\; -1 - 0)$

$(2\,,\; 1\,,\; -1)$

$\vec{OC}\,(x_C - x_O\,,\; y_C - y_O\,,\; z_C - z_O)$

$(4 - 0\,,\; 0 - 0\,,\; 1 - 0)$

$(4\,,\; 0\,,\; 1)$

$$\vec{OA} \cdot (\vec{OB} \wedge \vec{OC}) = \begin{vmatrix} 3 & 2 & -3 \\ 2 & 1 & -1 \\ 4 & 0 & 1 \end{vmatrix}$$

$= 3(1 - 0) - 2(2 - (-4)) - 3(0 - 4)$

$= 3 - 12 + 12 = 3$

$\vec{OA} \cdot (\vec{OB} \wedge \vec{OC}) \neq 0 \Rightarrow$ O, A, B & C are not coplanar.

2) $V_{p.p.} = |\overrightarrow{OA} \cdot (\overrightarrow{OB} \wedge \overrightarrow{OC})|$

$= |3| = 3$ cubic units.

3) $V_{OABC} = \frac{1}{6} |\overrightarrow{OA} \cdot (\overrightarrow{OB} \wedge \overrightarrow{OC})|$

$= \frac{1}{6} \cdot 3 = \frac{1}{2}$ cubic units.

Exercise 5:

Consider the points $A(3, 2, -1)$, $B(5, 3, 1)$ and $C(-1, 2, 3)$.

1) Find a vector $\vec{v}$ which is normal (orthogonal) to plane (ABC).

2) Consider $M(x, y, z)$. Find a relation between x, y and z so that M is in the plane (ABC).

Solution:

1) $\vec{v} = \overrightarrow{AB} \wedge \overrightarrow{AC}$

$\overrightarrow{AB}\,(x_B - x_A, y_B - y_A, z_B - z_A)$

$(5 - 3, 3 - 2, 1 - (-1))$

$(2, 1, 2)$

$\overrightarrow{AC}\,(x_C - x_A, y_C - y_A, z_C - z_A)$

$(-1 - 3, 2 - 2, 3 - (-1))$

$(-4, 0, 4)$

$$\vec{v} = \overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & 2 \\ -4 & 0 & 4 \end{vmatrix}$$

$= \vec{i}(4 - 0) - \vec{j}(8 - (-8)) + \vec{k}(0 - (-4))$

$= 4\vec{i} - 16\vec{j} + 4\vec{k}$

$\vec{v}(4, -16, 4)$.

2) M is in plane (ABC)

so M, A, B + C are coplanar.

so $\overrightarrow{AM} \cdot (\overrightarrow{AB} \wedge \overrightarrow{AC}) = 0$

$\overrightarrow{AM}\ (x_M - x_A\ ,\ y_M - y_A\ ,\ z_M - z_A)$

$(x-3\ ,\ y-2\ ,\ z+1)$

$\overrightarrow{AB}\ (2\ ,\ 1\ ,\ 2)$

$\overrightarrow{AC}\ (-4\ ,\ 0\ ,\ 4)$

$$\overrightarrow{AM} \cdot (\overrightarrow{AB} \wedge \overrightarrow{AC}) = \begin{vmatrix} x-3 & y-2 & z+1 \\ 2 & 1 & 2 \\ -4 & 0 & 4 \end{vmatrix} = 0$$

$(x-3)(4-0) - (y-2)(8-(-8)) + (z+1)(0-($

$(x-3)(4) - (y-2)(16) + (z+1)(4) =$

$4x - 12 - 16y + 32 + 4z + 4 = 0$

$4x - 16y + 4z + 24 = 0$

<u>OR</u> <u>2nd method</u>:

$\overrightarrow{AM} \cdot (\overrightarrow{AB} \wedge \overrightarrow{AC}) = 0$

$\overrightarrow{AM} \cdot \vec{v} = 0$ $\quad \begin{cases} \overrightarrow{AM}\ (x-3\ ,\ y-2\ ,\ z+1) \\ \vec{v}\ (4\ ,\ -16\ ,\ 4) \end{cases}$

$xx' + yy' + zz' = 0$

$4(x-3) - 16(y-2) + 4(z+1) = 0$

$4x - 12 - 16y + 32 + 4z + 4 = 0$

$4x - 16y + 4z + 24 = 0$.

Exercise 6:

Consider the points $A(2, 1, -3)$, $B(1, -1, 2)$, $C(2, -2, 1)$ and $D(3, -2, -1)$

1) Calculate the components of $\overrightarrow{AB} \wedge \overrightarrow{BC}$.
2) Let H be the orthogonal projection of A on (BC). Show that $\overrightarrow{AB} \wedge \overrightarrow{BC} = \overrightarrow{AH} \wedge \overrightarrow{BC}$. and deduce the value of AH.
3) Calculate the area of triangle ABC.
4) Calculate the volume of the tetrahedron DABC and deduce the distance from D to plane (ABC).

Solution:

1) $\overrightarrow{AB}\,(x_B - x_A\,,\ y_B - y_A\,,\ z_B - z_A)$

$(1-2\,,\ -1-1\,,\ 2-(-3))$

$(-1\,,\ -2\,,\ 5)$

$\overrightarrow{BC}\,(x_C - x_B\,,\ y_C - y_B\,,\ z_C - z_B)$

$(2-1\,,\ -2-(-1)\,,\ 1-2)$

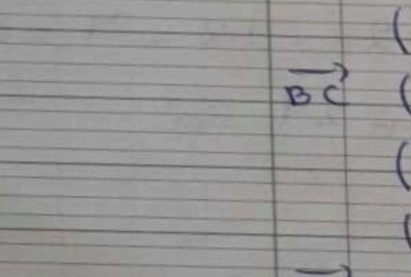
$(1\,,\ -1\,,\ -1)$

$$\overrightarrow{AB} \wedge \overrightarrow{BC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -2 & 5 \\ 1 & -1 & -1 \end{vmatrix}$$

$$= \vec{i}\,(2-(-5)) - \vec{j}\,(1-5) + \vec{k}\,(1-(-2))$$

$$= 7\vec{i} + 4\vec{j} + 3\vec{k}.$$

$\overrightarrow{AB} \wedge \overrightarrow{BC}\,(7, 4, 3)$.



2) $\overrightarrow{AH} \wedge \overrightarrow{BC} = (\overrightarrow{AB} + \overrightarrow{BH}) \wedge \overrightarrow{BC}$

$= \overrightarrow{AB} \wedge \overrightarrow{BC} + \overrightarrow{BH} \wedge \overrightarrow{BC}$

$= \overrightarrow{AB} \wedge \overrightarrow{BC} + \vec{0}$

$= \overrightarrow{AB} \wedge \overrightarrow{BC}$

B, H & C are collinear

so $\overrightarrow{BH}$ & $\overrightarrow{BC}$ have same direction

so $\overrightarrow{BH} \wedge \overrightarrow{BC} = \vec{0}$

value of AH :

$\mathcal{A}_{ABC} = \frac{1}{2} \| \overrightarrow{AB} \wedge \overrightarrow{BC} \|$

$\mathcal{A}_{ABC} = \frac{h \times b}{2} = \frac{AH \times BC}{2} = \frac{1}{2} (AH \times BC)$

so $\cancel{\frac{1}{2}} \| \overrightarrow{AB} \wedge \overrightarrow{BC} \| = \cancel{\frac{1}{2}} (AH \times BC)$

$\| \overrightarrow{AB} \wedge \overrightarrow{BC} \| = AH \times BC$.

$AH = \frac{\| \overrightarrow{AB} \wedge \overrightarrow{BC} \|}{BC}$

$= \frac{\sqrt{(7)^2 + (4)^2 + (3)^2}}{\sqrt{(1)^2 + (-1)^2 + (-1)^2}}$

$= \frac{\sqrt{49 + 16 + 9}}{\sqrt{1 + 1 + 1}} = \frac{\sqrt{74}}{\sqrt{3}}$ units of length.

3) $\mathcal{A}_{ABC} = \frac{1}{2} \| \overrightarrow{AB} \wedge \overrightarrow{BC} \| = \frac{1}{2} \sqrt{74}$ square units

or $\mathcal{A}_{ABC} = \frac{1}{2} AH \times BC = \frac{1}{2} \frac{\sqrt{74}}{\sqrt{3}} \times \sqrt{3} = \frac{1}{2} \sqrt{74}$ square units.

4) $V_{DABC} = \frac{1}{6} \left| \overrightarrow{DA} \cdot (\overrightarrow{AB} \wedge \overrightarrow{BC}) \right|$

$\overrightarrow{DA} \; (x_A - x_D \;,\; y_A - y_D \;,\; z_A - z_D)$

$(2 - 3 \;,\; 1 - (-2) \;,\; -3 - (-1))$

$(-1 \;,\; 3 \;,\; -2)$

$\overrightarrow{DA} \cdot (\overrightarrow{AB} \wedge \overrightarrow{BC}) = \begin{vmatrix} -1 & 3 & -2 \\ -1 & -2 & 5 \\ 1 & -1 & -1 \end{vmatrix}$

$= -1(2 - (-5)) - 3(1 - 5) - 2(1 - (-2))$

$= -7 + 12 - 6 = -1$

$V_{DABC} = \frac{1}{6} |-1| = \frac{1}{6}$ cubic units.

OR 2nd method:

$$V_{DABC} = \frac{1}{6} \left| \overrightarrow{DA} \cdot (\overrightarrow{AB} \wedge \overrightarrow{BC}) \right|$$

$\overrightarrow{DA} \, (-1, 3, -2)$

$\overrightarrow{AB} \wedge \overrightarrow{BC} \, (7, 4, 3)$

$$\begin{aligned}\overrightarrow{DA} \cdot (\overrightarrow{AB} \wedge \overrightarrow{BC}) &= xx' + yy' + zz' \\ &= -1(7) + 3(4) + (-2)(3) \\ &= -7 + 12 - 6 \\ &= -1\end{aligned}$$

$V_{DABC} = \frac{1}{6} \left| -1 \right| = \frac{1}{6}$ cubic units.

Distance from D to plane (ABC):

$$V_{DABC} = \frac{1}{3} \underbrace{\mathcal{A}_{ABC}}_{\text{area of base.}} \times \underbrace{DK}_{\text{height}}$$

height (distance from D to plane (ABC))

$$\frac{1}{6} = \frac{1}{3} \times \frac{1}{2} \sqrt{74} \times DK$$

$$\frac{\cancel{1}}{\cancel{6}} = \frac{\cancel{1}}{\cancel{6}} \sqrt{74} \times DK$$

$DK = \frac{1}{\sqrt{74}}$ units of length.

Distance from point D to plane (ABC) is the perpendicular distance (height).

so distance from D to plane (ABC) $= DK$

$= \frac{1}{\sqrt{74}}$ units of length.

Exercise 7:

Consider the points $A(-1, 2, 1)$, $B(-2, 1, 0)$, $C(-1, 1, -1)$ and $E(1, -3, 1)$

1) Show that the points A, B and C determine a plane (P).
2) Show that (EC) is perpendicular to (P).
3) Show that the planes (OEC) and (P) are perpendicular.
4) Calculate the area of triangle ABC.
5) Calculate the volume of tetrahedron EABC.

Solution:

1) $\overrightarrow{AB}\,(x_B - x_A \;,\; y_B - y_A \;,\; z_B - z_A)$

$(-2-(-1) \;,\; 1-2 \;,\; 0-1)$

$(-1 \;,\; -1 \;,\; -1)$

$\overrightarrow{AC}\,(x_C - x_A \;,\; y_C - y_A \;,\; z_C - z_A)$

$(-1-(-1) \;,\; 1-2 \;,\; -1-1)$

$(0 \;,\; -1 \;,\; -2)$

$$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= \vec{i}(2-1) - \vec{j}(2-0) + \vec{k}(1-0)$$

$$= \vec{i} - 2\vec{j} + \vec{k}$$

$\overrightarrow{AB} \wedge \overrightarrow{AC} \neq \vec{0} \Longrightarrow$ A, B & C are not collinear

$\Longrightarrow$ A, B & C determine a plane (P).

{ Note: Three non collinear points determine a plane. }

2) $\overrightarrow{EC}\,(x_C - x_E \;,\; y_C - y_E \;,\; z_C - z_E)$

$(-1-1 \;,\; 1-(-3) \;,\; -1-1)$

$(-2 \;,\; 4 \;,\; -2)$

$\vec{EC} = -2(\vec{AB} \wedge \vec{AC})$

$\vec{EC}$ and $\vec{AB} \wedge \vec{AC}$ have same direction

but $\vec{AB} \wedge \vec{AC}$ is perpendicular to (P)

so (EC) is perpendicular to (P).

3) $\vec{OE}\ (x_E - x_O \ , \ y_E - y_O \ , \ z_E - z_O)$

$(1 - 0 \ , \ -3 - 0 \ , \ 1 - 0)$

$(1 \ , \ -3 \ , \ 1)$

$\vec{OC}\ (x_C - x_O \ , \ y_C - y_O \ , \ z_C - z_O)$

$(-1 - 0 \ , \ 1 - 0 \ , \ -1 - 0)$

$(-1 \ , \ 1 \ , \ -1)$

$$\vec{OE} \wedge \vec{OC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{vmatrix}$$

$$= \vec{i}(3-1) - \vec{j}(-1-(-1)) + \vec{k}(1-3)$$

$$= 2\vec{i} - 2\vec{k}.$$

$\vec{OE} \wedge \vec{OC}\ (2, 0, -2)$ is normal vector to plane (OEC)

$\vec{AB} \wedge \vec{AC}\ (1, -2, 1)$ is normal vector to plane (P).

$$(\vec{OE} \wedge \vec{OC}) \cdot (\vec{AB} \wedge \vec{AC}) = xx' + yy' + zz'$$

$$= 2 - 0 - 2$$

$$= 0$$

so $\vec{OE} \wedge \vec{OC}$ and $\vec{AB} \wedge \vec{AC}$ are perpendicular

so planes (OEC) and (P) are perpendicular.

4) $\mathcal{A}_{ABC} = \frac{1}{2} \|\vec{AB} \wedge \vec{AC}\|$

$$= \frac{1}{2} \sqrt{(1)^2 + (-2)^2 + (1)^2}$$

$$= \frac{1}{2} \sqrt{6} \text{ square units.}$$

6) $V_{EABC} = \frac{1}{6}\left|\vec{EC} \cdot (\vec{AB} \wedge \vec{AC})\right|$

$\vec{EC}(-2, 4, -2)$

$\vec{AB}(-1, -1, -1)$

$\vec{AC}(0, -1, -2)$

$$\vec{EC} \cdot (\vec{AB} \wedge \vec{AC}) = \begin{vmatrix} -2 & 4 & -2 \\ -1 & -1 & -1 \\ 0 & -1 & -2 \end{vmatrix}$$

$$= -2(2-1) - 4(2-0) - 2(1-0)$$

$$= -2 - 8 - 2 = -12$$

$V_{EABC} = \frac{1}{6}\left|-12\right| = \frac{12}{6} = 2$ cubic units.

<u>OR</u> <u>2nd method</u>

$\vec{EC}(-2, 4, -2)$

$\vec{AB} \wedge \vec{AC}(1, -2, 1)$

$$\vec{EC} \cdot (\vec{AB} \wedge \vec{AC}) = xx' + yy' + zz'$$

$$= -2(1) + 4(-2) + (-2)(1)$$

$$= -2 - 8 - 2 = -12$$

$V_{EABC} = \frac{1}{6}\left|\vec{EC} \cdot (\vec{AB} \wedge \vec{AC})\right|$

$= \frac{1}{6}\left|-12\right| = 2$ cubic units.

Exercise 8:

Consider the points $A(2,1,-2)$, $B(3,2,0)$, $C(-1,2,-3)$ and $M(x,y,z)$.

1) Determine a normal vector to plane (ABC).
2) Find a relation between x, y and z in each of the following cases:
   a. M is on the plane (ABC)
   b. M is on the plane which is the perpendicular bisector of [AC].
   c. M is on the sphere with diameter [BC].
3) Find the relations between x, y and z if M is on the line (AB).

Solution:

1) $\overrightarrow{AB} \wedge \overrightarrow{AC}$ is a normal vector to plane (ABC).

$\overrightarrow{AB}\,(x_B - x_A \,,\, y_B - y_A \,,\, z_B - z_A)$

$(3-2 \,,\, 2-1 \,,\, 0-(-2))$

$(1 \,,\, 1 \,,\, 2)$

$\overrightarrow{AC}\,(x_C - x_A \,,\, y_C - y_A \,,\, z_C - z_A)$

$(-1-2 \,,\, 2-1 \,,\, -3-(-2))$

$(-3 \,,\, 1 \,,\, -1)$

$$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 2 \\ -3 & 1 & -1 \end{vmatrix}$$

$$= \vec{i}\,(-1-2) - \vec{j}\,(-1-(-6)) + \vec{k}\,(1-(-3))$$

$$= -3\,\vec{i} - 5\,\vec{j} + 4\,\vec{k}.$$

$\overrightarrow{AB} \wedge \overrightarrow{AC}\,(-3, -5, 4)$ is a normal vector to plane (ABC)

2)a. M is on plane (ABC)

$\Rightarrow$ M, A, B & C are coplanar.

$\Rightarrow \vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) = 0$

$\vec{AM}\ (x_M - x_A \ , \ y_M - y_A \ , \ z_M - z_A)$

$\quad\ (x - 2 \ , \ y - 1 \ , \ z + 2)$

$\vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) = 0$

$$\begin{vmatrix} x-2 & y-1 & z+2 \\ 1 & 1 & 2 \\ -3 & 1 & -1 \end{vmatrix} = 0$$

$(x-2)(-1-2) - (y-1)(-1-(-6)) + (z+2)(1-(-3)) = 0$

$(x-2)(-3) - (y-1)(5) + (z+2)(4) = 0$

$-3x + 6 - 5y + 5 + 4z + 8 = 0$

$-3x - 5y + 4z + 19 = 0$

OR second method:

$\vec{AM} \cdot (\vec{AB} \wedge \vec{AC}) = 0$

$\begin{cases} \vec{AM}(x-2 \ , \ y-1 \ , \ z+2) \\ \vec{AB} \wedge \vec{AC}(-3 \ , \ -5 \ , \ 4) \end{cases}$

$xx' + yy' + zz' = 0$

$-3(x-2) - 5(y-1) + 4(z+2) = 0$

$-3x + 6 - 5y + 5 + 4z + 8 = 0$

$-3x - 5y + 4z + 19 = 0$.

b. M is on the plane which is perpendicular bisector of [AC].

$\Rightarrow$ M is equidistant from A and C.

$\Rightarrow$ MA = MC. or $\|\vec{MA}\| = \|\vec{MC}\|$

$\vec{MA}\ (x_A - x_M \ , \ y_A - y_M \ , \ z_A - z_M)$

$\quad\ (2 - x \ , \ 1 - y \ , \ -2 - z)$

$\vec{MC}\ (x_C - x_M \ , \ y_C - y_M \ , \ z_C - z_M)$

$\quad\ (-1 - x \ , \ 2 - y \ , \ -3 - z)$

$\|\overrightarrow{MA}\| = \|\overrightarrow{MC}\|$

$\sqrt{(2-x)^2+(1-y)^2+(-2-z)^2} = \sqrt{(-1-x)^2+(2-y)^2+(-3-z)^2}$

(Square both sides)

$(2-x)^2+(1-y)^2+(-2-z)^2 = (-1-x)^2+(2-y)^2+(-3-z)^2$

(Expand all brackets)

Note

$(a+b)^2 = (-a-b)^2$

$= a^2 + b^2 + 2ab$

$(a-b)^2 = a^2 + b^2 - 2ab$

$4+x^2-4x+1+y^2-2y+4+z^2+4z$

$= 1+x^2+2x+4+y^2-4y+9+z^2+6z$

(x^2, y^2 and z^2 will be cancelled from both sides.)

$4-4x+1-2y+4+4z = 1+2x+4-4y+9+6z$

$9-4x-2y+4z = 14+2x-4y+6z$

(move all terms to one side.)

$9 \underline{-4x} \underline{-2y} + \underline{\underline{4z}} - 14 \underline{-2x} + \underline{4y} \underline{\underline{-6z}} = 0$

$-6x+2y-2z-5=0$

c. M is on the sphere with diameter [BC].

$\Rightarrow \widehat{BMC}$ is an inscribed angle standing on diameter

$\Rightarrow \widehat{BMC} = 90°$

$\Rightarrow \triangle BMC$ is right at M.

By Pythagora's theorem:

$BC^2 = BM^2 + CM^2$.

$\overrightarrow{BC}\,(x_C - x_B\,,\; y_C - y_B\,,\; z_C - z_B)$

$(-1-3\,,\; 2-2\,,\; -3-0)$

$(-4\,,\; 0\,,\; -3)$

$\overrightarrow{BM}\,(x_M - x_B\,,\; y_M - y_B\,,\; z_M - z_B)$

$(x-3\,,\; y-2\,,\; z-0)$

$\overrightarrow{CM}\,(x_M - x_C\,,\; y_M - y_C\,,\; z_M - z_C)$

$(x+1\,,\; y-2\,,\; z+3)$

$BC^2 = (-4)^2 + (0)^2 + (-3)^2 = 16 + 9 = 25$

$BH^2 = (x-3)^2 + (y-2)^2 + z^2$

$= x^2 + 9 - 6x + y^2 + 4 - 4y + z^2$

$= x^2 + y^2 + z^2 - 6x - 4y + 13$

$CM^2 = (x+1)^2 + (y-2)^2 + (z+3)^2$

$= x^2 + 1 + 2x + y^2 + 4 - 4y + z^2 + 9 + 6z$

$= x^2 + y^2 + z^2 + 2x - 4y + 6z + 14.$

$BC^2 = BH^2 + CM^2$

$25 = 2x^2 + 2y^2 + 2z^2 - 4x - 8y + 6z + 27$

$2x^2 + 2y^2 + 2z^2 - 4x - 8y + 6z + 2 = 0$

(we can divide by 2 to simplify the expression)

$x^2 + y^2 + z^2 - 2x - 4y + 3z + 1 = 0$

<u>OR</u> 2nd method:

$B\hat{M}C = 90° \Longrightarrow \overrightarrow{BM} \perp \overrightarrow{CM}$

$\Longrightarrow \overrightarrow{BM} \cdot \overrightarrow{CM} = 0$

$xx' + yy' + zz' = 0$

$(x-3)(x+1) + (y-2)(y-2) + (z)(z+3) = 0$

$x^2 + x - 3x - 3 + y^2 - 2y - 2y + 4 + z^2 + 3z = 0$

$x^2 + y^2 + z^2 - 2x - 4y + 3z + 1 = 0.$

3) M is on the line (AB)

$\Longrightarrow \overrightarrow{AM}$ and $\overrightarrow{AB}$ have same direction.

$\Longrightarrow \overrightarrow{AM} = k \times \overrightarrow{AB}$. where $k \in \mathbb{R}$. $\Longrightarrow \dfrac{x_{\overrightarrow{AM}}}{x_{\overrightarrow{AB}}} = \dfrac{y_{\overrightarrow{AM}}}{y_{\overrightarrow{AB}}} = \dfrac{z_{\overrightarrow{AM}}}{z_{\overrightarrow{AB}}}$

$\overrightarrow{AM}(x-2, y-1, z+2)$

$\overrightarrow{AB}(1, 1, 2)$

$$\frac{x-2}{1} = \frac{y-1}{1} = \frac{z+2}{2}$$

$$x - 2 = y - 1 = \frac{z+2}{2}.$$

Exercise 9:

Consider the points $A(2,3,1)$, $B(-1,0,-2)$, $C(-2,1,0)$ and $M(x,y,z)$.

1) Show that triangle ABC is a right triangle.

2) Calculate the area of triangle ABC.

3) Find a relation between x, y and z so that the vectors $\overrightarrow{AM}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are coplanar.

4) Determine a point M so that (AM) is perpendicular to plane (ABC). Calculate the volume of the tetrahedron MABC in this case.

Solution:

1) $\overrightarrow{AB}(x_B - x_A, y_B - y_A, z_B - z_A)$

$(-1-2, 0-3, -2-1)$

$(-3, -3, -3)$

$AB = \sqrt{(-3)^2 + (-3)^2 + (-3)^2}$

$= \sqrt{9+9+9} = \sqrt{27}$

$\overrightarrow{AC}(x_C - x_A, y_C - y_A, z_C - z_A)$

$(-2-2, 1-3, 0-1)$

$(-4, -2, -1)$

$AC = \sqrt{(-4)^2 + (-2)^2 + (-1)^2}$

$= \sqrt{16+4+1} = \sqrt{21}$

$\overrightarrow{BC}(x_C - x_B, y_C - y_B, z_C - z_B)$

$(-2-(-1), 1-0, 0-(-2))$

$(-1, 1, 2)$

$BC = \sqrt{(-1)^2 + (1)^2 + (2)^2}$

$= \sqrt{1+1+4} = \sqrt{6}$

$\left.\begin{array}{l} AB^2 = AC^2 + BC^2 \\ 27 = 21 + 6 \end{array}\right\} \Rightarrow \Delta ABC$ is right at C.

OR 2nd method:

$\overrightarrow{AC} \cdot \overrightarrow{BC} = xx' + yy' + zz'$

$= -4(-1) + (-2)(1) + (-1)(2)$

$= 4 - 2 - 2 = 0$

so $(AC) \perp (BC) \Rightarrow \Delta ABC$ is right at C.

2) $\mathcal{A}_{ABC} = \dfrac{h \times b}{2} = \dfrac{AC \times BC}{2} = \dfrac{\sqrt{21} \times \sqrt{6}}{2} = \dfrac{\sqrt{126}}{2}$ square units

OR 2nd method:

$\mathcal{A}_{ABC} = \dfrac{1}{2} \| \overrightarrow{AB} \wedge \overrightarrow{AC} \| \longrightarrow$ (we can use any 2 vectors of the ΔABC)

$$\overrightarrow{AB} \wedge \overrightarrow{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & -3 & -3 \\ -4 & -2 & -1 \end{vmatrix}$$

$= \vec{i}(3-6) - \vec{j}(3-12) + \vec{k}(6-12)$

$= -3\vec{i} + 9\vec{j} - 6\vec{k}$.

$\| \overrightarrow{AB} \wedge \overrightarrow{AC} \| = \sqrt{(-3)^2 + (9)^2 + (-6)^2}$

$= \sqrt{9 + 81 + 36} = \sqrt{126}$

$\mathcal{A}_{ABC} = \dfrac{1}{2} \| \overrightarrow{AB} \wedge \overrightarrow{AC} \| = \dfrac{1}{2}\sqrt{126}$ square units.

3) $\overrightarrow{AM}$, $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are coplanar $\Rightarrow \overrightarrow{AM} \cdot (\overrightarrow{AB} \wedge \overrightarrow{AC}) = 0$

$\overrightarrow{AM}\,(x_M - x_A \,,\; y_M - y_A \,,\; z_M - z_A)$

$(x - 2 \,,\; y - 3 \,,\; z - 1)$

$\overrightarrow{AM} \cdot (\overrightarrow{AB} \wedge \overrightarrow{AC}) = 0$

$$\begin{vmatrix} x-2 & y-3 & z-1 \\ -3 & -3 & -3 \\ -4 & -2 & -1 \end{vmatrix} = 0$$

$(x-2)(3-6) - (y-3)(3-12) + (z-1)(6-12) = 0$

$(x-2)(-3) - (y-3)(-9) + (z-1)(-6) = 0$

$-3x + 6 + 9y - 27 - 6z + 6 = 0$

$-3x + 9y - 6z - 15 = 0$ divide by (-3)

$x - 3y + 2z + 5 = 0$.

4) (AM) perpendicular to plane (ABC)

$\Longrightarrow$ $\overrightarrow{AM}$ and $\overrightarrow{AB} \wedge \overrightarrow{AC}$ have same direction.

$\overrightarrow{AM} = k.(\overrightarrow{AB} \wedge \overrightarrow{AC})$ where $k \in \mathbb{R}$.

$\overrightarrow{AM}(x-2\ ,\ y-3\ ,\ z-1)$

$\overrightarrow{AB} \wedge \overrightarrow{AC}(-3\ ,\ 9\ ,\ -6)$

$$\frac{x-2}{-3} = \frac{y-3}{9} = \frac{z-1}{-6} = k$$

For $k=1$ $\leadsto$ (we can take any value for k)

$\frac{x-2}{-3} = 1 \Longrightarrow x-2 = -3 \Longrightarrow x = -1$

$\frac{y-3}{9} = 1 \Longrightarrow y-3 = 9 \Longrightarrow y = 12$

$\frac{z-1}{-6} = 1 \Longrightarrow z-1 = -6 \Longrightarrow z = -5$

$M(-1\ ,12\ ,-5)$.

$V_{MABC} = \frac{1}{6} \left| \overrightarrow{AM}.(\overrightarrow{AB} \wedge \overrightarrow{AC}) \right|$

$$\begin{aligned}\overrightarrow{AM}.(\overrightarrow{AB} \wedge \overrightarrow{AC}) &= x - 3y + 2z + 5 \\ &= -1 - 3(12) + 2(-5) + 5 \\ &= -1 - 36 - 10 + 5 \\ &= -42\end{aligned}$$

$V_{MABC} = \frac{1}{6}\left|-42\right| = \frac{42}{6} = 7$ cubic units.

Exercise 10:

Consider the points $A(-2, 1, 3)$, $B(1, 0, 4)$, $C(3, -1, 2)$ and $D(1, 2, 4)$.

1) Show that the points A, B, C and D are not coplanar.
2) Determine a normal vector $\vec{v}$ to plane (ABC).
3) Let (δ) be the line passing through D perpendicular to plane (ABC), and let $M(\alpha, \beta, \gamma)$ be a point on (δ). Determine the relations between α, β and γ.
4) Determine α, β and γ such that M is also in the plane (ABC).
5) Deduce the distance from D to plane (ABC).

Solution:

1) A, B, C & D are not coplanar

$\Rightarrow \overrightarrow{AB} \cdot (\overrightarrow{AC} \wedge \overrightarrow{AD}) \neq 0$ → (we can use any 3 vectors formed by the points A, B, C & D.)

$\overrightarrow{AB}(x_B - x_A, y_B - y_A, z_B - z_A)$

$(1-(-2), 0-1, 4-3)$

$(3, -1, 1)$

$\overrightarrow{AC}(x_C - x_A, y_C - y_A, z_C - z_A)$

$(3-(-2), -1-1, 2-3)$

$(5, -2, -1)$

$\overrightarrow{AD}(x_D - x_A, y_D - y_A, z_D - z_A)$

$(1-(-2), 2-1, 4-3)$

$(3, 1, 1)$

$$\overrightarrow{AB} \cdot (\overrightarrow{AC} \wedge \overrightarrow{AD}) = \begin{vmatrix} 3 & -1 & 1 \\ 5 & -2 & -1 \\ 3 & 1 & 1 \end{vmatrix}$$

$$= 3(-2-(-1)) + 1(5-(-3)) + 1(5-(-6))$$

$$= -3 + 8 + 11$$

$$= 16 \neq 0$$

So A, B, C & D are not coplanar.

2) $\vec{v} = \vec{AB} \wedge \vec{AC}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -1 & 1 \\ 5 & -2 & -1 \end{vmatrix}$$

$= \vec{i}(1-(-2)) - \vec{j}(-3-5) + \vec{k}(-6-(-5))$

$= 3\vec{i} + 8\vec{j} - \vec{k}$.

$\vec{v}(3, 8, -1)$.

3) $(\delta) \perp$ plane (ABC)

$\Rightarrow \vec{DM}$ and $\vec{v}$ have same direction.

$\vec{DM}(x_M - x_D, y_M - y_D, z_M - z_D)$

$(\alpha - 1, \beta - 2, \gamma - 4)$

$\vec{DM} = k.\vec{v}$ where $k \in \mathbb{R}$.

$$\frac{x_{\vec{DM}}}{x_{\vec{v}}} = \frac{y_{\vec{DM}}}{y_{\vec{v}}} = \frac{z_{\vec{DM}}}{z_{\vec{v}}}$$

$$\frac{\alpha - 1}{3} = \frac{\beta - 2}{8} = \frac{\gamma - 4}{-1}$$

4) M is in the plane (ABC).

$\Rightarrow A, B, C$ & M are coplanar.

$\Rightarrow \vec{AM}.(\vec{AB} \wedge \vec{AC}) = 0$

$\vec{AM}(x_M - x_A, y_M - y_A, z_M - z_A)$

$(\alpha + 2, \beta - 1, \gamma - 3)$

$\vec{AB}(3, -1, 1)$

$\vec{AC}(5, -2, -1)$

$\vec{AM}.(\vec{AB} \wedge \vec{AC}) = 0$

$$\begin{vmatrix} \alpha + 2 & \beta - 1 & \gamma - 3 \\ 3 & -1 & 1 \\ 5 & -2 & -1 \end{vmatrix} = 0$$

$(\alpha+2)(1-(-2)) - (\beta-1)(-3-5) + (\gamma-3)(-6-(-5)) = 0$

$(\alpha+2)(3) - (\beta-1)(-8) + (\gamma-3)(-1) = 0$

$3\alpha + 6 + 8\beta - 8 - \gamma + 3 = 0$

$3\alpha + 8\beta - \gamma + 1 = 0$.

OR 2^{nd} method:

$$\overrightarrow{AM} \cdot (\overrightarrow{AB} \wedge \overrightarrow{AC}) = 0$$

$$\overrightarrow{AM} \cdot \vec{v} = 0$$

$$\begin{cases} \overrightarrow{AM}(\alpha+2, \beta-1, \gamma-3) \\ \vec{v}(3, 8, -1) \end{cases}$$

$$xx' + yy' + zz' = 0$$

$$3(\alpha+2) + 8(\beta-1) - 1(\gamma-3) = 0$$

$$3\alpha + 6 + 8\beta - 8 - \gamma + 3 = 0$$

$$3\alpha + 8\beta - \gamma + 1 = 0$$

(To find α, β & γ, we use the relations from part (3) and part (4))

$$\frac{\alpha-1}{3} = \frac{\beta-2}{8} = \frac{\gamma-4}{-1} = k$$

we find α, β & γ in terms of k, and substitute them in the 2^{nd} relation.

$$\frac{\alpha-1}{3} = k \Rightarrow \alpha - 1 = 3k$$

$$\Rightarrow \alpha = 3k + 1$$

$$\frac{\beta-2}{8} = k \Rightarrow \beta - 2 = 8k$$

$$\Rightarrow \beta = 8k + 2.$$

$$\frac{\gamma-4}{-1} = k \Rightarrow \gamma - 4 = -k$$

$$\Rightarrow \gamma = -k + 4.$$

$$3(3k+1) + 8(8k+2) - (-k+4) + 1 = 0$$

$$9k + 3 + 64k + 16 + k - 4 + 1 = 0$$

$$74k + 16 = 0$$

$$74k = -16 \Rightarrow k = \frac{-16}{74} = \frac{-8}{37}.$$

$$\alpha = 3k + 1 = 3\left(\frac{-8}{37}\right) + 1 = \frac{-24}{37} + 1 = \frac{13}{37}$$

$$\beta = 8k + 2 = 8\left(\frac{-8}{37}\right) + 2 = \frac{-64}{37} + 2 = \frac{10}{37}$$

$$\gamma = -k + 4 = -\left(\frac{-8}{37}\right) + 4 = \frac{8}{37} + 4 = \frac{156}{37}$$

$$\Rightarrow M\left(\frac{13}{37}, \frac{10}{37}, \frac{156}{37}\right)$$

5) (DM) is perpendicular to plane (ABC) so distance from D to plane (ABC) is equal to the length of DH.

$$DH = \sqrt{(x_H - x_D)^2 + (y_H - y_D)^2 + (z_H - z_D)^2}$$

$$= \sqrt{\left(\frac{13}{37} - 1\right)^2 + \left(\frac{10}{37} - 2\right)^2 + \left(\frac{156}{37} - 4\right)^2}$$

$$= \sqrt{\left(\frac{24}{37}\right)^2 + \left(\frac{-64}{37}\right)^2 + \left(\frac{8}{37}\right)^2}$$

$$= \sqrt{\frac{576 + 4096 + 64}{(37)^2}}$$

$$= \sqrt{\frac{4736}{(37)^2}} = \frac{8\sqrt{74}}{37} \text{ units of length}$$

<u>OR</u> 2nd method.

$$V_{DABC} = \frac{1}{3} \mathcal{A}_{ABC} \times \underbrace{DH}_{\text{distance from D to plane (ABC)}}$$

$$V_{DABC} = \frac{1}{6} \left| \vec{AB} \cdot (\vec{AC} \wedge \vec{AD}) \right|$$

$$= \frac{1}{6} |16| = \frac{16}{6} = \frac{8}{3} \text{ cubic units.}$$

$$\mathcal{A}_{ABC} = \frac{1}{2} \| \vec{AB} \wedge \vec{AC} \|$$

$$= \frac{1}{2} \| \vec{v} \| = \frac{1}{2} \sqrt{(3)^2 + (8)^2 + (-1)^2}$$

$$= \frac{1}{2} \sqrt{9 + 64 + 1} = \frac{\sqrt{74}}{2} \text{ square units.}$$

$$\frac{8}{3} = \frac{1}{3} \times \frac{\sqrt{74}}{2} \times DH$$

$$\frac{8}{3} = \frac{\sqrt{74}}{6} \times DH$$

$$DH = \frac{\frac{8}{3}}{\frac{\sqrt{74}}{6}} = \frac{8 \times \cancel{6}^{2}}{\cancel{3}_{1} \sqrt{74}} = \frac{16}{\sqrt{74}} \times \frac{\sqrt{74}}{\sqrt{74}}$$

$$= \frac{16\sqrt{74}}{74} = \frac{8\sqrt{74}}{37} \text{ units of length.}$$