ExaMath Groups	Mathematics Exam Class: LS	Prepared by: Simon Semaan Edited by: Randa Chehade
Number of questions: 3	Sample 06 – year 2023 Duration: 1½ hours	Name: N°:

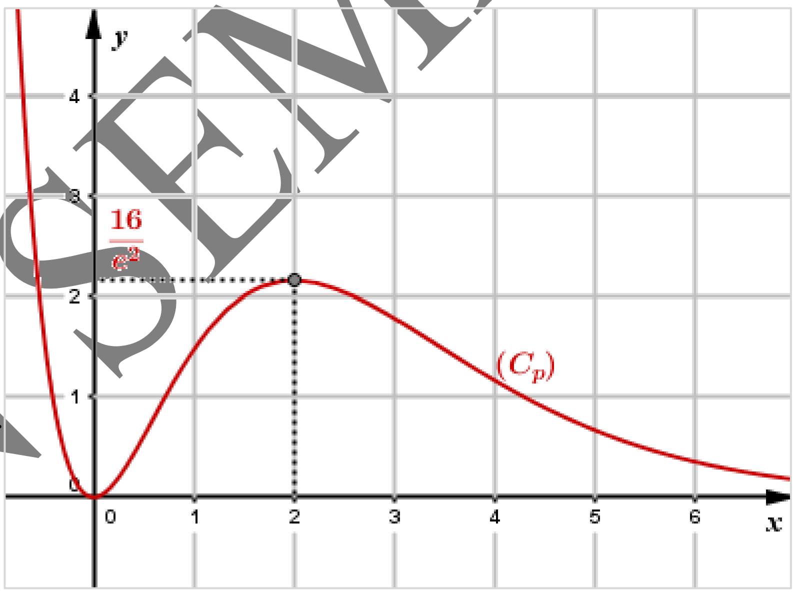
- *This exam consists of three problems inscribed on three pages.*
- *The use of a non-programmable calculator is allowed.*

I – (5 points)

Consider the function p defined over $\mathbb{R}$ by $p(x)=(ax^2+bx+c)e^{-x}$ (where a,b and c are real numbers).

Denote by (C_p) its representative curve in an orthonormal system $(O;\vec{i};\vec{j})$ as shown in the figure below.

The extrmums of the function p are reached at $x=0$ and $x=2$.



1) Using the graph:

a) Determine $p(0)$, $p'(0)$, $p(2)$ and $p'(2)$.

b) Set up the table of variations of p over $\mathbb{R}$.

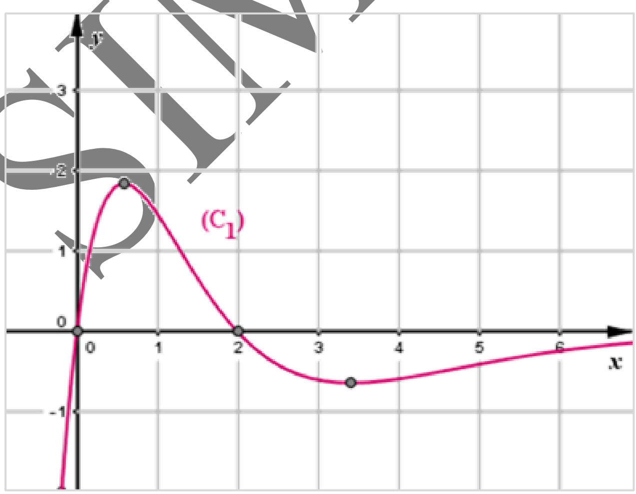
c) Consider the function r defined by $r(x)=\ln[p(x)]$.

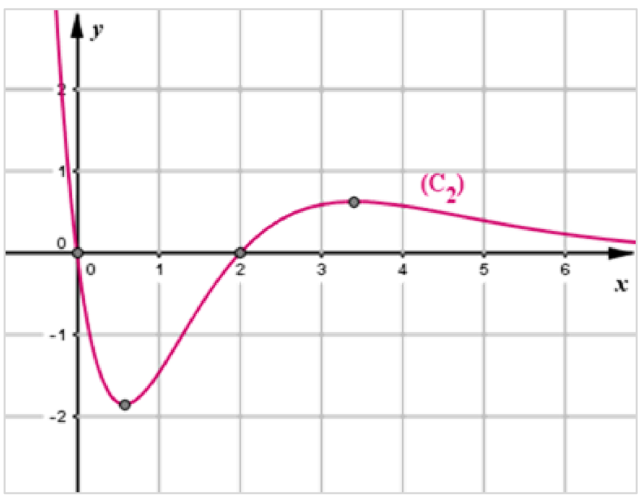
Determine the domain of definition of r.

2) Show that: $a=4$, $b=0$ and $c=0$.

3) Determine the inflection points of the curve (C_p).

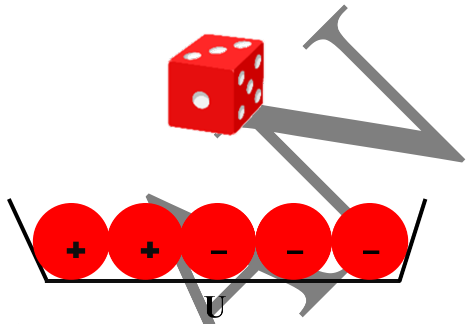
4) One of the curves (C_1) and (C_2) given below is the representative curve of the derivative function p' of the function p. Which of the below curves is that of function p'? Justify.





II – (5 points)

We have a perfect dice whose faces are numbered from 1 to 6 and an urn U containing:
two balls carrying strictly positive numbers and
three balls carrying strictly negative numbers.



The dice is rolled:
- If the number appearing is 1 or 2, **two** balls are drawn randomly and simultaneously from the urn U.
- If the number appearing is 3, 4, 5 or 6, **three** balls are drawn randomly and simultaneously from the urn U.

We consider the following events:

A : ‹‹ The number that appeared on the dice is 1 or 2 ››.
$\overline{A}$: ‹‹ The number that appeared on the dice is 3, 4, 5 or 6 ››.
B : ‹‹ The **product** of the numbers carried by the balls drawn from the urn U is **negative** ››.

1) Show that $P(A)=\dfrac{1}{3}$.

2) a) Show that $P(B/A)=0.6$ and deduce $P(B\cap A)$.

b) Show that $P(B/\overline{A})=0.4$ and deduce $P(B\cap \overline{A})$.

c) Calculate $P(B)$.

3) The product of the numbers carried by the balls drawn from the urn U is positive, what is the probability that the number appearing on the dice is 1 or 2?

III – (10 points)

The aim of this problem is to represent, graphically, two functions in the same coordinate system.

Consider the function f defined over $\mathbb{R}$ by $f(x)=e^{x}-x-1$,
and the function g defined over $]0;+\infty[$ by $g(x)=x-1-2\ln x$.
Denote by (C_f) and (C_g) the representative curves of the functions f and g respectively in an orthonormal system $(O;\vec{i};\vec{j})$.

<u>Part A:</u> Study of the function *f*

1) Determine $\lim\limits_{x\to+\infty} f(x)$ and calculate $f\left(\dfrac{3}{2}\right)$.

2) a) Determine $\lim\limits_{x\to-\infty} f(x)$.

b) Show that the line (d) of equation $y=-x-1$ is an asymptote to (C_f) at $-\infty$ and verify that (C_f) is above the line (d) for every $x\in\mathbb{R}$.

3) Calculate $f'(x)$ and set up the table of variations of f.

<u>Part B:</u> Study of the function *g*

1) Determine $\lim\limits_{x\to0^+} g(x)$ and deduce an asymptote to (C_g).

2) Determine $\lim\limits_{x\to+\infty} g(x)$.

3) Calculate $g'(x)$ and set up the table of variations of g.

4) Show that the equation $g(x)=0$ has on $]0;+\infty[$ two solutions α and β $(\alpha<\beta)$ and verify that $\alpha=1$ and that $3.5<\beta<3.6$.

<u>Part C:</u> Study of the relative position of (C_f) and (C_g)

The following table is the table of variations of the function h defined over $]0;+\infty[$ by:

$h(x)=f(x)-g(x)$.

x	0	$+\infty$
$h(x)$	$-\infty$ ↗	$+\infty$

1) It's clear that the equation $h(x)=0$ has over $]0;+\infty[$ a unique solution γ.
Verify that $0.7<\gamma<0.8$.

2) Study, according to the values of x over $]0;+\infty[$, the relative position of (C_f) and (C_g) (take $\gamma\approx 0.74$).

3) Draw (d), (C_f) and (C_g) in the same orthonormal system. (Unit: 2cm)

QI	Answers	5 pts
1)a)	$p(0)=0$; $p'(0)=0$; $p(2)=\frac{16}{e^2}$ and $p'(2)=0$.	½
1)b)	**Table of variations of p :** $\begin{array}{c\|ccccccc} x & -\infty & & 0 & & 2 & & +\infty \\ \hline p(x) & +\infty & \searrow & 0 & \nearrow & \frac{16}{e^2} & \searrow & 0 \end{array}$	1
1)c)	$r(x)=\ln(p(x))$ is defined for $p(x)>0$, that is if $x\in\left]-\infty;0\right[\cup\left]0;+\infty\right[$ (using the graph or the table of variations).	½
2)	• $p(0)=0$; $ce^0=0$; $\boldsymbol{c}=0$. • $p(2)=\frac{16}{e^2}$; $(4a+2b)e^{-2}=\frac{16}{e^2}=16e^{-2}$; $4\boldsymbol{a}+2\boldsymbol{b}=16$. $p'(x)=(2ax+b)e^{-x}+(-e^{-x})(ax^2+bx)=\left[-ax^2+(2a-b)x+b\right]e^{-x}$. • $p'(0)=0$; $be^0=0$; $\boldsymbol{b}=0$. Thus $4a+2\times0=16$; $a=\frac{16}{4}=4$.	1
3)	$p'(x)=(-4x^2+8x)e^{-x}$. $p''(x)=(4x^2-16x+8)e^{-x}$, has same sign as $(4x^2-16x+8)$ since $e^{-x}>0$ for every $x\in\mathbb{R}$. $p''(x)=0$ if $4x^2-16x+8=0$, if $x=2-\sqrt{2}$ or $x=2+\sqrt{2}$. $p''(x)$ vanishes and change the sign two times at $x=2-\sqrt{2}$ and $x=2+\sqrt{2}$, so (C_p) has two inflection points $(2-\sqrt{2};P(2-\sqrt{2}))$ and $(2+\sqrt{2};p(2+\sqrt{2}))$.	1
4)	If $x\in\left]-\infty;0\right[$; p is decreasing so $p'(x)<0$. If $x\in\left]0;2\right[$; p is increasing so $p'(x)>0$. If $x\in\left]0;+\infty\right[$; p is decreasing so $p'(x)<0$. So the curve (C_1) is the graphical representation of the function p'.	1

QII	Answers	5 pts
1)	$P(A)=\frac{2}{6}=\frac{1}{3}$.	½
2)a)	$P(B/A)=\frac{C_3^1\times C_2^1}{C_5^2}=\frac{6}{10}=0.6$. $P(B\cap A)=P(B/A)\times P(A)=0.6\times\frac{1}{3}=\frac{1}{5}$.	1
2)b)	$P(B/\overline{A})=\frac{C_3^1\times C_2^2+C_3^3}{C_5^2}=\frac{4}{10}=0.4$. $P(B\cap\overline{A})=(B/\overline{A})\times P(\overline{A})=0.4\times\frac{2}{3}=\frac{4}{15}$.	1

2)c)	$P(B)=P\left(B\cap\bar{A}\right)+P\left(B\cap\bar{A}\right)=\dfrac{7}{15}$.	**1**
3)	$P\left(A/\bar{B}\right)=\dfrac{P\left(A\cap\bar{B}\right)}{P\left(\bar{B}\right)}=\dfrac{P(A)-P(A\cap B)}{1-P(B)}=\dfrac{1}{4}=0.25$.	**1 ½**

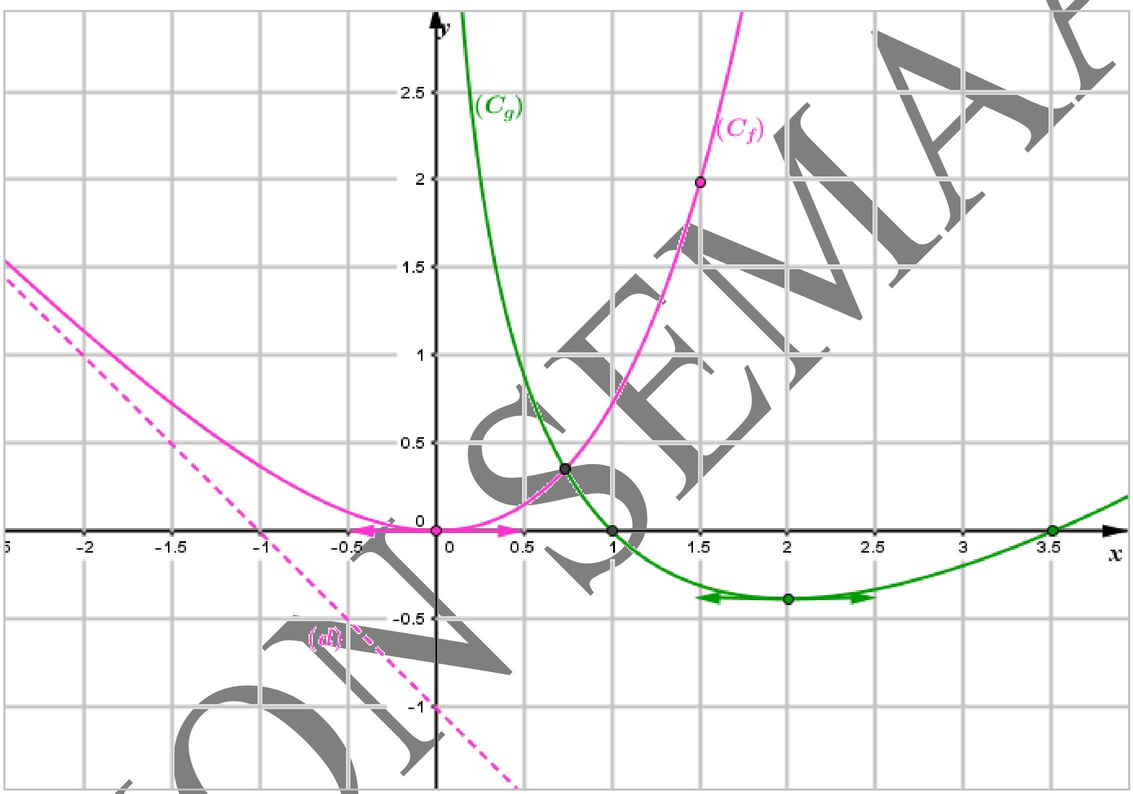
QIII	**Answers**	**10 pts**
A1)	$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty} x\left(\dfrac{e^x}{x}-1-\dfrac{1}{x}\right)=+\infty$. $f\left(\dfrac{3}{2}\right)=e^{\frac{3}{2}}-\dfrac{5}{2}$.	¾
A2)a)	$\lim\limits_{x\to-\infty} f(x)=+\infty$.	¼
A2)b)	$\lim\limits_{x\to-\infty}\left(f(x)-y_d\right)=\lim\limits_{x\to-\infty} e^x=0$ so (d) is an asymptote to (C_f) at $-\infty$. $f(x)-y_d=e^x>0$ for all $x\in\mathbb{R}$ so (C_f) is above (d) for all $x\in\mathbb{R}$.	½
A3)	$f'(x)=e^x-1$. If $e^x-1\geq 0$; $e^x\geq 1$; $x\geq 0$. $f(0)=0$. **Table of variations of f :**	**1 ½**

x	$-\infty$		0		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$+\infty$

B1)	$\lim\limits_{x\to0^+} g(x)=+\infty$ so $(y'Oy): x=0$ is an asymptote to (C_g).	½
B2)	$\lim\limits_{x\to+\infty} g(x)=\lim\limits_{x\to+\infty} x\left(1-\dfrac{1}{x}-2\dfrac{\ln x}{x}\right)=+\infty.$	½
B3)	$g'(x)=\dfrac{x-2}{x}$ $x>0$ so $g'(x)$ has the same sign of $(x-2)$. $x-2\geq 0$ if $x\geq 2$. $f(2)=1-2\ln 2$. **Table of variations of g :**	**1 ½**

x	0		2		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+\infty$	$\searrow$	$1-2\ln 2$	$\nearrow$	$+\infty$

B4)	• Over $]0;2[$: g is continuous , strictly decreasing and changes its sign from positive $(+\infty)$ to negative $(1-2\ln 2)$ so the equation $g(x)=0$ has a unique root α in this interval . Moreover $g(1)=0$ so $\alpha=1$. • Over $]2;+\infty[$: g is continuous , strictly increasing and changes its sign from negative $(1-2\ln 2)$ to positive $(+\infty)$ so the equation $g(x)=0$ has a unique root β in this interval. In addition : $g(3.5)=-0.006<0$ and $g(3.6)=0.04>0$ so $3.5<\beta<3.6$.	**1**

C1)	f & g are 2 continuous functions over $]0;+\infty[$; thus h is continuous function over $]0;+\infty[$. $h(0.7) = f(0.7) - g(0.7) = -0.1 < 0$ $h(0.8) = f(0.8) - g(0.8) = 0.18 > 0$ so $0.7 < \gamma < 0.8$	½
C2)	If $x \in]0;\gamma[$, then $h(x) < 0$ so $f(x) - g(x) < 0$ so (C_f) is below (C_g). If $x \in]\gamma;+\infty[$, then $h(x) > 0$ so $f(x) - g(x) > 0$ so (C_f) is above (C_g). If $x = 0.74$, then $h(x) = 0$ so $f(x) - g(x) = 0$ so (C_f) cuts (C_g) at point $(0.74\ ;\ h(0.74))$.	**1**
C3)		**2**

[**YouTube Channel: Simon Semaan**](https://www.youtube.com/@SimonSemaan)

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