Solved Problems

N°1.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the mapping T of the plane ,that associates to every point $M(x;y)$ of the plane a point $M'(x';y')$, such that:

$$M\begin{cases}x\\y\end{cases}\longrightarrow M'\begin{cases}x'=x+y+1\\y'=-x+y-1\end{cases}.$$

1) Find the invariant point under T .
2) Is T involutive?
3) Determine the transformation T^{-1} inverse of T .
4) Let (d) be the straight line of equation $y=2x-1$.
   Determine the image of (d) by T .

N°2.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the mapping of the plane T , that to every point $M(x;y)$ distinct of O associates the point $M'(x';y')$, such that:

$$M\begin{cases}x\\y\end{cases}\longrightarrow M'\begin{cases}x'=\dfrac{x}{x^2+y^2}\\[2ex]y'=\dfrac{y}{x^2+y^2}\end{cases}$$

1) Determine the set of invariant points by T .
2) Show that T is involutive and determine its inverse transformation T^{-1}.
3) Let (C) be the circle of equation $x^2+y^2-2x-4y=0$.
   Find the image of (C) by T .
4) Denote by (C') the circle of equation $x^2+y^2-2x-4y+1=0$.
   Find the image of (C') by T .
5) Let (d) be the straight line of equation $y=x-1$, find the image of (d) by T .

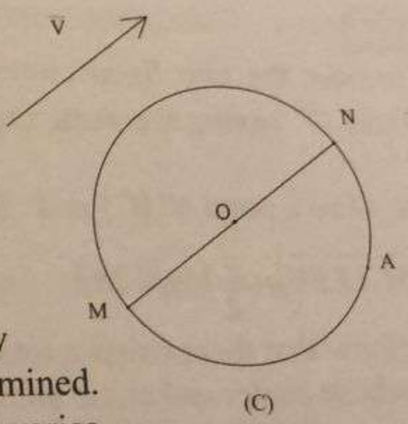
N° 3.

In the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$, consider the translation t of vector $\vec{v}(3;-2)$ and let $M'(x';y')$ be the image of the point $M(x;y)$ by t.

1) Calculate x and y in terms of x' and y'.
2) Consider the circle (C) of center $I(1;-1)$ and radius 2.
   Find the image of (C) by t.
3) Let r be the rotation of center O and angle $\frac{\pi}{2}$ and let $M'(x';y')$ be the image of point $M(x;y)$ by r.
   a- Express x and y in terms of x' and y'.
   b- Find the image of (C) by r.

N° 4.

(C) is a variable circle of center O, and of a constant radius R passing through a fixed point A.
$[MN]$ is a variable diameter such that $\overrightarrow{MN}=\vec{V}$ where $\vec{V}$ is a given vector.



1) Determine the set of points O as the circle varies.
2) Show that N is the image of O by a simple transformation to be determined.
   Deduce the set of points N as (C) varies.
3) Determine the set of points M as the circle (C) varies.

N° 5.

On a fixed axis $x'Ox$, consider a variable point A and construct an isosceles triangle of base $[OA]$ and vertex M.
Let w be the center of circle (C) circumscribed about triangle OAM.
Suppose that the radius of (C) is constant.

1) Determine the set of points w as A varies.
2) By which simple transformation is w mapped onto M?
3) Determine the set of points M.

N°6.

(C) and (C') are two fixed circles intersecting in two points A and B and of respective centers O and O'.
A variable secant (d) passing through A cuts (C) and (C') in I and J respectively.
The perpendicular through I to (d) cuts (C) in K.
The perpendicular through J to (d) cuts (C') in L.
The parallel through I to (OO') cuts (JL) in M.
The parallel through J to (OO') cuts (KI) in N.

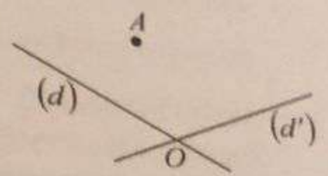
1) Prove that the points K, B and L, are collinear.
2) a- Show that $\overrightarrow{IM} = 2\overrightarrow{OO'}$.
   b- Deduce the set of points M as (d) varies.
   c- Find the set of points N as (d) varies.

N°7.

Consider the two fixed circles (C) and (C') of respective centers O and O' having the same radius R and tangent externally at a point J.
Let M be a point of (C) and M' a point of (C') such that $\left(\overrightarrow{OM},\overrightarrow{O'M'}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$.

1) Show that there exists a rotation that maps M onto M', whose center and angle are to be determined.
2) Show that the perpendicular bisector of $[MM']$ passes through a fixed point I.
3) Let I' be the symmetric of I with respect to the straight line (OO') and M'' the image of M by the rotation r' of center I' and angle $-\frac{\pi}{2}$.
   a- Determine $r'(O)$.
   b- Show that the points M' and M'' are diametrically opposite in (C').

N°8.
(d) and (d') are two straight lines intersecting at O.
Let A be a point not on (d) and not on (d').
Construct a right isosceles triangle ABC of vertex A and such that $B \in (d)$,
$C \in (d')$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}(\bmod 2\pi)$.



N°9.
ABC is a right isosceles triangle of vertex B and such that $\left(\overrightarrow{BA};\overrightarrow{BC}\right)=\frac{\pi}{2}(\bmod 2\pi)$.

Let (d) be the straight line passing through B and parallel to (AC).
Denote by $r_1 = S_{(AB)} \circ S_{(AC)}$, $r_2 = S_{(AB)} \circ S_{(d)}$ and $r_3 = S_{(AC)} \circ S_{(d)}$.

1) Determine the nature of each of the transformations r_1, r_2 and r_3.
2) Show that $r_2 \circ r_1(A) = C$.
3) Determine the nature and characteristic elements of the transformation $r_2 \circ r_1$.

N°10.
In the oriented plane , consider a triangle OAB right isosceles at O and such that $\left(\overrightarrow{OA};\overrightarrow{OB}\right)=\frac{\pi}{2}\bmod(2\pi)$.

Designate by R_A and R_B the rotations of centers A and B respectively and of the same angle $\frac{\pi}{2}$ and by S_O the symmetry of center O.
C is a point not on straight line (AB), we draw the direct squares $BEDC$ and $ACFG$.
We have then $\left(\overrightarrow{BE};\overrightarrow{BC}\right)=\frac{\pi}{2}\bmod(2\pi)$ and $\left(\overrightarrow{AC};\overrightarrow{AG}\right)=\frac{\pi}{2}\bmod(2\pi)$.

1) a- Determine $S_{(AO)} \circ S_{(AB)}$ the composite of the two axial symmetries of respective axes (AB) and (AO).
   b- Writing R_B in the form of the composite of two reflections ,

prove that $R_A \circ R_B = S_O$.

2) a- Determine the image of E by $R_A \circ R_B$.

b- Deduce that O is the midpoint of $[EG]$.

c- Denote by R_F and R_D the rotations of centers F and D respectively and having the same angle $\frac{\pi}{2}$.

Study the image of C by $R_F \circ S_O \circ R_D$.

Determine the transformation $R_F \circ S_O \circ R_D$.

d- Let H be the symmetric of D with respect to O.

Prove that $R_F(H) = D$.

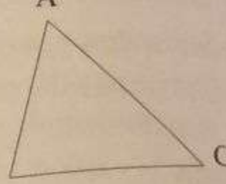
Prove that FOD is right isosceles at O.

N°11.

Consider a direct triangle ABC.

I, J and K are the midpoints of the segments $[BC]$, $[CA]$ and $[AB]$.

O and Ω are the centers of the squares constructed on the sides $[AB]$ and $[CA]$ respectively exterior to triangle ABC.



1) Let r be the rotation that transforms K onto J and O onto I, determine the angle of r.
2) Show that $r(I) = \Omega$.
3) Deduce that (IO) and $(I\Omega)$ are perpendicular and that $IO = I\Omega$.

N°12.

In the oriented plane, consider the figure below.

Triangles ABC and ACD are two direct equilateral triangles such that $\left(\overrightarrow{BC};\overrightarrow{BA}\right)=\frac{\pi}{3}\operatorname{mod}(2\pi)$ and $\left(\overrightarrow{DA};\overrightarrow{DC}\right)=\frac{\pi}{3}\operatorname{mod}(2\pi)$.

The points O and I are the midpoints of the segments $[CA]$ and $[AB]$ respectively. L and E are given points in the plane such that $\overrightarrow{OC}=\overrightarrow{CL}=\overrightarrow{LE}$.

Let r be the rotation of center A and whose angle has a measure $\frac{\pi}{3}$

and t the translation of vector $\overrightarrow{OA}$.

Denote by $r' = r \circ t$.

1) a- Determine $r'(O)$.

b- Determine a measure of angle $(\overrightarrow{IO};\overrightarrow{IA})$.

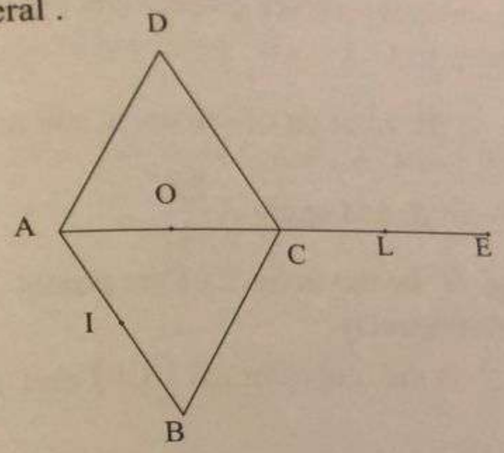
c- Determine the nature and characteristic elements of r'.

2) M is any point in the plane , denote by $N = r(M)$.

J is the midpoint of $[EM]$ and K the midpoint of $[ND]$.

a- Let P be the pre-image of M by t, what is the midpoint of $[LP]$?

b- Using $r'(L)$ and $r'(P)$, show that the two triangles ILD and IJK are equilateral .



N°13.

In the oriented plane , consider the two fixed points A and B . Designate by r_A and r_B the rotations of centers A and B respectively and having the same angle $\frac{\pi}{2}$.

For all points M of the plane, denote by M_1 and M_2 the images of M by r_A and r_B respectively.

1) Consider the transformation $T = r_B \circ r_A^{-1}$.

a- Construct the point C image of A by T.

b- Determine the nature and characteristic elements of T.

c- Deduce the nature of quadrilateral M_1M_2CA.

2) Suppose that M describes the circle (Γ) of diameter $[AB]$.
   a- Determine the set (Γ_2) described by M_2 when M describes the circle (Γ) and precise the center of (Γ_2).
   b- Let w and w' be the midpoints of segments $[AB]$ and $[BC]$ respectively.
   Determine the set of point I, midpoint of $[M_1M_2]$ as M describes circle (Γ).

N°14.

In the oriented plane, consider an equilateral triangle ABC such that $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{3}\bmod(2\pi)$.

Let I be the midpoint of $[BC]$ and J the point such that B is the midpoint of $[JC]$.

Designate by r_1 the rotation of center A and angle $\frac{\pi}{3}$ and by r_2 the rotation of center B and angle $-\frac{2\pi}{3}$.

1) Let A' and B' be the images of the points A and B by $r_2 \circ r_1$ respectively.
   Prove that I is the midpoint of $[AA']$ and B is the midpoint of $[AB']$.
2) Determine the nature of $r_2 \circ r_1^{-1}$ then prove that for all points M of the plane, the point I is the midpoint of $[M_1M_2]$ where $M_1 = r_1(M)$ and $M_2 = r_2(M)$.
3) Prove that $r_2 \circ r_1$ is a rotation whose center and angle are to be determined.

N°15.

Consider a triangle OAB right isosceles such that $OA = OB$ and $\left(\overrightarrow{OA};\overrightarrow{OB}\right)=\frac{\pi}{2}(\bmod 2\pi)$.

I, J and K are the midpoints of the segments $[AB]$, $[OB]$ and

$[OA]$ respectively.

Let r be the rotation of center I and angle $\frac{\pi}{2}$ and by t the translation of vector $\frac{1}{2}\overrightarrow{AB}$, let $f = r \circ t$ and $g = t \circ r$.

1) a- Determine $f(K)$, $f(I)$ and $f(A)$.
   b- Precise the nature of f and determine its characteristic elements.
2) a- Determine $g(J)$ and $g(O)$.
   b- Precise the nature of g and determine its characteristic elements.
3) Let $h = g \circ f^{-1}$.
   a- Determine $h(O)$ and find the nature of h.
   b- M being any point in the plane , let $M_1 = f(M)$ and $M_2 = g(M)$.
   Show that the vector $\overrightarrow{M_1M_2}$ is equal to a fixed vector .