Solved Problems

Construction

N°1.

$ABCD$ is a square of center O such that $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{2}(\text{mod }2\pi)$.

1) Construct the image of $ABCD$ by $S\left(A;\sqrt{2};\frac{\pi}{4}\right)$.

2) Construct the image of $ABCD$ by $S\left(O;\frac{\sqrt{2}}{2};\frac{\pi}{4}\right)$.

N°2.

ABD is a triangle right at B such that: $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{3}(\text{mod }2\pi)$.

1) a- Construct the point C image of B by $S\left(D;\frac{\sqrt{3}}{3};\frac{\pi}{2}\right)$.

b- Determine the nature of $ABCD$.

2) Construct the image of $ABCD$ by $S\left(B;\frac{1}{2};\frac{2\pi}{3}\right)$.

N°3.

$ABCD$ is a rectangle such that $AD=4$, $AB=2$ and $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{2}(\text{mod }2\pi)$.

Let I be the midpoint of $[AD]$ and J the symmetric of I with respect to (AC).

1) Let S be the direct plane similitude I, of ratio $\sqrt{2}$ and angle $\frac{3\pi}{4}$. Determine $S(A)$.

2) Let S' be the direct plane similitude of center J that transforms A onto C.

Precise the ratio k of S' and a measure of its angle.

N°4.

ABC is a triangle whose center of gravity is G.

M, N and P are the midpoints of the segments $[BC]$, $[AC]$ and $[AB]$.

1) Determine the image of triangle ABC by the direct plane similitude $S\left(G;\frac{1}{2};\frac{\pi}{3}\right)$.

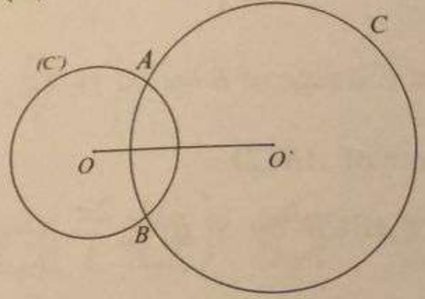
2) Construct the image of triangle ABM by the direct plane similitude $S\left(A;\frac{\sqrt{3}}{3};\frac{\pi}{6}\right)$.

N°5.

In an oriented plane, consider the two circles (C) and (C') of centers O and O' respectively, and of respective radii r and r'.

The two circles intersect in two points A and B.

1) Prove that there exists a direct plane similitude S of center A that transforms (C) onto (C') and precise its angle and ratio.



2) Let M be a point of (C) and M' its image by S.

a- Compare the angles $\left(\overrightarrow{OA};\overrightarrow{OM}\right)$ and $\left(\overrightarrow{O'A};\overrightarrow{O'M'}\right)$.

b- Prove that the points M, B and M' are collinear.

N°6.

In an oriented plane, consider a triangle ABC right and isosceles of vertex A such that : $AB = a$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$.

Denote by A' the symmetric of A with respect to C.

$\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{3}(\text{mod }2\pi)$.

1) Calculate AO and AC.
2) Let S be the direct similitude of center C, of angle $\frac{\pi}{6}$ and ratio $\frac{\sqrt{3}}{3}$.
   a- Prove that S transforms A onto B.
   b- Show that the image O' of point O is the midpoint of $[BC]$.
3) Denote by D' the image of D by S.
   a- Prove that D' belongs to the semi straight line $[CA)$.
   b- What is the measure of angle $\left(\overrightarrow{OD};\overrightarrow{O'D'}\right)$?
   c- Deduce a measure of angle $\left(\overrightarrow{BC};\overrightarrow{O'D'}\right)$.
4) Prove that D' is the center of the circle circumscribed about triangle BCD.

N°9.

Consider a circle (C) of diameter $[OB]$.
A is a point of the segment $[OB]$, distinct of O and of B, and I the midpoint of $[AB]$.
The perpendicular bisector of the segment $[AB]$ cuts the circle at M and M' such that : $\left(\overrightarrow{MO};\overrightarrow{MB}\right)=\frac{\pi}{2}(\text{mod }2\pi)$.

Designate by N the orthogonal projection of A on (OM).

1) a- Precise the nature of quadrilateral $AMBM'$
   b- Deduce that the straight line (AM') is perpendicular to (OM) and that the points N, A and M' are collinear.
   In what follows, let S be the direct plane similitude of center N such that: $S(M)=A$.
2) a- Precise the angle of S.
   b- Determine the images of the straight lines (MI) and (NA)

by S.

c- Deduce the image of M' by S.

3) a- I is the midpoint of $[MM']$, determine the position of point $I' = S(I)$.

b- Deduce that the straight line (NI) is tangent at N to the circle (C') of diameter $[OA]$.

N° 10.

ABC is a direct triangle, A', B' and C' are the points situated on the exterior of triangle such that $A'BC$, $B'CA$ and $C'AB$ are equilateral triangles. J, K and L are the centers of gravity of triangles $A'BC$, $B'CA$ and $C'AB$ respectively.

We need to prove that triangle JKL is equilateral.

Designate by S_A the direct plane similitude of center A that transforms K onto C and by S_B that of center B that transforms C onto J.

1) Determine the ratio and angle of S_A.
2) Determine the ratio and angle of S_B.
3) Show that $S_B \circ S_A$ is a rotation whose angle is to be determined.
   Prove that L is the center of $S_B \circ S_A$.
4) Deduce from part 3) that triangle JKL is direct equilateral.

N° 11.

$ABCD$ is a square such that $\left(\overrightarrow{AB};\overrightarrow{AD}\right) = \frac{\pi}{2} (\mathrm{mod}\, 2\pi)$.

Let r be the rotation of center A and angle $\frac{\pi}{2}$, t is the translation of vector $\overrightarrow{AB}$ and $h = h\left(C;\sqrt{3}\right)$.

1) a- Show that $t \circ r$ is a rotation whose angle is to be determined.
   Designate by $r' = t \circ r$.

   b- Determine the images of A and B by r'.
   Deduce the center of r'.

2) Let $f = r' \circ h$.

   a- Find the nature of f and precise its angle and ratio.

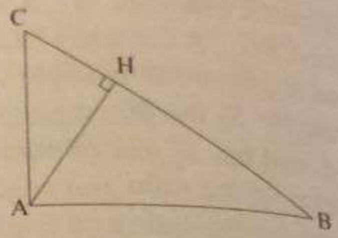
   b- Let I be the center of f.
   Determine the image of C by f, and show that

$\left(\overrightarrow{IC};\overrightarrow{ID}\right)=\frac{\pi}{2}(\text{mod }2\pi)$ and that $ID=\sqrt{3}\ IC$.

c- Find a measure of the angle $\left(\overrightarrow{CD};\overrightarrow{CI}\right)$ then construct I .

N°12.

ABC is a right triangle such that $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}(\text{mod }2\pi)$.

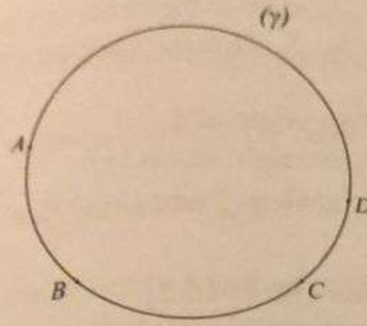


H is the foot of the perpendicular issued from A .
Let D be the point so that ACD is a direct right isosceles triangle of vertex A .
O is the foot of the perpendicular issued from D in triangle DBC and by K the foot of the perpendicular issued from A in triangle DAO .

1) a- Show that the rotation r of center A and angle $\frac{\pi}{2}$ transforms the straight line (CB) onto the straight line (DO) .
b- Determine the image of triangle AHC by r .
c- Deduce that $AHOK$ is a square .
2) Designate by Ω the point of intersection of (AB) and (KH).
Prove that there exists a dilation h that transforms triangle AKD onto triangle BHA .
3) Consider the transformation $S=h\circ r$.
a- Determine the image of the points H , C and A by S .
b- Determine the nature of S and precise its elements.

N°13.

A , B , C and D are four distinct points belonging to the circle (γ).
1) Consider the direct plane similitude S of center A that transforms C onto D . Designate by E the image of point B by S .
a- Prove that $\left(\overrightarrow{CB};\overrightarrow{DE}\right)=\left(\overrightarrow{AC};\overrightarrow{AD}\right)(\text{mod }2\pi)$.



b- Prove that E belongs to the straight line (BD).

c- Prove that : $AD \times BC = DE \times AC$.

2) a- Prove that : $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \left(\overrightarrow{AE};\overrightarrow{AD}\right) \pmod{2\pi}$

and that $\dfrac{AC}{AB} = \dfrac{AD}{AE}$.

b- Let S' be the similitude of center A that transforms B onto C .
Prove that $S'(E) = D$.

c- Prove that $AB \times CD = AC \times BE$.

3) Prove that $AC \times BD = AB \times CD + AD \times BC$.

$\boxed{N^{\circ}14.}$

In the oriented plane , consider an isosceles triangle ABC such that $AB = AC$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \dfrac{\pi}{4} \pmod{2\pi}$.

Let I be the point such that CAI is right isosceles with $\left(\overrightarrow{CA};\overrightarrow{CI}\right) = -\dfrac{\pi}{2} \pmod{2\pi}$.

1) We denote by r_A the rotation of center A that transforms B onto C and r_C the rotation of center C and angle $-\dfrac{\pi}{2}$.

Let $f = r_C \circ r_A$.

a- Determine the images of A and of B by f.

b- Determine the nature of f and construct its center O .

c- What is the nature of quadrilateral $ABOC$?

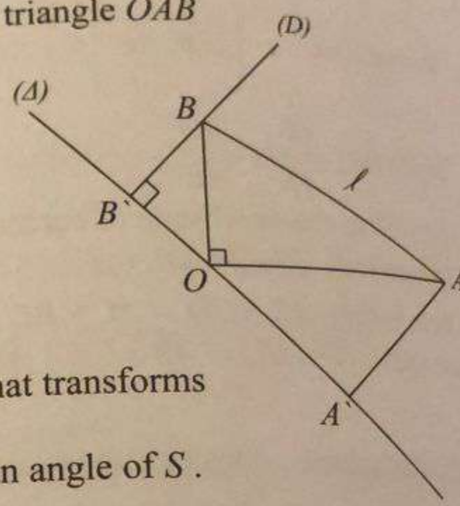
2) Let S be the direct similitude of center O that transforms A onto B. Denote by C' the image of C by S, H is the midpoint of $[BC]$ and H' its image by S.
   a- Determine the measure of the angle of S.
   Show that C' belongs to the straight line (OA).
   b- Find the image of segment $[OA]$ by S and show that H' is the midpoint of $[OB]$.
   c- Show that $(C'H')$ is perpendicular to (OB).
   Deduce that C' is the center of the circle circumscribed about triangle OBC.

N° 15.

In an oriented plane, consider a triangle OAB right angled at O such that $AB = \ell$ and :

$$\left(\overrightarrow{OA};\overrightarrow{OB}\right) = \frac{\pi}{2} \pmod{2\pi}$$

$$\left(\overrightarrow{AB};\overrightarrow{AO}\right) = \frac{\pi}{6} \pmod{2\pi}$$



Let S be the direct similitude that transforms B onto O and O onto A.

1) Determine the ratio k and an angle of S.
2) Let I be the center of S.
   a- Construct geometrically I.
   b- Prove that I is the foot of the perpendicular issued from O in triangle OAB.
3) (Δ) is a variable straight line passing through O and (D) is a straight line passing through B and perpendicular to (Δ), designate by A' and B' the orthogonal projections of A and B respectively on (Δ).
   a- Determine the image by S of (D) and of (Δ).
   b- Deduce the image of B' by S.
   c- Prove that the circle of diameter $[A'B']$ passes through a fixed point when (Δ) varies.

N°16.

OIJ is a right isosceles triangle such that $\left(\overrightarrow{OI};\overrightarrow{OJ}\right)=\frac{\pi}{2}(\text{mod }2\pi)$.

Let M be a variable point of (IJ) and N the point such that

$\left(\overrightarrow{NO};\overrightarrow{NM}\right)=\frac{\pi}{2}(\text{mod }2\pi)$ and $NO=NM$. I' is the midpoint of $[IJ]$.

1) Determine the ratio k and an angle α of the direct plane similitude S of center O that transforms M onto N.
2) a- Determine $S(I)$ and $S\circ S(I)$.
   b- Determine the image (Δ) of (IJ) by S.
3) Let S' be the direct plane similitude of center I', ratio $\sqrt{2}$ and angle $-\frac{\pi}{4}$.

   Precise the nature of the transformations $t_1=S'\circ S$ and $t_2=S\circ S'$.

N°17.

In an oriented plane, consider a triangle ABC such that :

$AB=2$, $AC=1+\sqrt{5}$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}(\text{mod }2\pi)$.

1) Let S be the direct plane similitude that transforms B onto A and A onto C.
   Determine the ratio k and a measure α of the angle of S.
2) Designate by Ω the center of S, construct Ω geometrically.
3) Let D be the image of C by S.
   a- Prove that the points A, Ω and D are collinear, and that the straight lines (CD) and (AB) are parallel.
   b- Construct the point D.
   c- Show that $CD=3+\sqrt{5}$.
4) Let E be the orthogonal projection of B on (CD).
   a- Explain the construction of the point F image of point E by S and place F on the figure.
   b- What is the nature of quadrilateral $BFDE$?