ExaMath groups	**Exam in Mathematics** **Section:** G.S.	**Prepared by:** Ezzeddine Hnaini **Edited by:** H. Ahmad
Number of questions: 5	**Sample 01 - 2022** **Duration:** 180 min	**Name:** **N°:**

- *This exam includes five problems. It is inscribed on four pages, numbered from 1 to 4.*
- *The use of a non-programmable calculator is allowed.*

I - (1.5 points)
In the table below, only one among the proposed answers to each question is correct.
Choose, **with justification**, the correct answer.

N^o	Question	Proposed answers A	B	C
1.	Let z be a complex number such that: $z = 1 - e^{-i\frac{\pi}{3}}$. Then $\overline{z} =$	$e^{-i\frac{\pi}{3}}$	$e^{i\frac{\pi}{3}}$	$1 - \dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$
2.	For every real number x in the interval $]-1;0[$, we have: $e^{\lvert \ln(x+1)\rvert} =$	$-x-1$	$x+1$	$\dfrac{1}{x+1}$
3.	$\lim\limits_{x\to 0} x\ln\left(1+\dfrac{1}{x}\right) =$	0	e	$+\infty$

II - (3.5 points)
The plane is referred to an orthonormal system $(O\ ;\ \vec{u}\ ;\ \vec{v})$.

Consider the points A, B and C of respective affixes $z_A = i$, $z_B = 2$ and $z_C = -1-i$.

M and M' are two points of the plane with respective affixes z and z' such that $z' = \dfrac{iz-4+2i}{z-2}$ with $z \neq 2$.

1) Determine the trigonometric form of $\dfrac{z_A - z_B}{z_A - z_C}$. Deduce the nature of triangle ABC.

2) In the case where $z' = -3i$, find the exponential form of z.

3) a) Verify that $z' - i = \dfrac{-4+4i}{z-2}$.

   b) Show that $AM' \times BM = 4\sqrt{2}$ and $\left(\vec{u}\ ;\ \overrightarrow{AM'}\right) + \left(\vec{u}\ ;\ \overrightarrow{BM}\right) = \dfrac{3\pi}{4} + 2k\pi$ where $k \in \mathbb{Z}$.

   c) Determine the set of points M' when M moves on the circle of center B and radius 4.

   d) Determine the set of points M such that $\arg(z' - i) = \dfrac{\pi}{4}(2\pi)$.

III - (3.5 points)

Given a well balanced coin and two urns U and V such that:

- The urn U contains 4 red balls, 3 green balls and one yellow ball.
- The urn V contains 1 red ball, 3 green balls and 4 yellow balls.

1) In this part, the coin is tossed three times in a row.
Consider the following events:
A: « The three tosses show head »;
B: « At least one toss is tail ».
Verify that $p(A) = \dfrac{1}{8}$ and find $p(B)$.

2) In this part, a game is played as follows:
The coin is tossed three times in a row:

   - If the three tosses show head, then three balls are chosen randomly and simultaneously from the urn U.
   - Otherwise, three balls are drawn randomly and successively with replacement from the urn V.

   Consider the event D: « The three drawn balls are of three different colors ».

   a) Show that $p(D/A) = \dfrac{3}{14}$ and calculate $p(D \cap A)$.

   b) Calculate $p(D \cap B)$, then deduce that $p(D) = \dfrac{537}{3584}$.

   c) Knowing that the three drawn balls are of three different colors, calculate the probability that exactly two tosses are tail.

   d) Calculate the probability that the three drawn balls have the same color.

IV - (7 points)

Part A:

Consider the function g defined over $\mathbb{R}$ by $g(x) = xe^{x+1} + 2$.

1) Find $g'(x)$ and set up the table of variations of g. (It is not required to determine the limits of g at $+\infty$ and $-\infty$).

2) Deduce that $g(x) > 0$ for every $x \in \mathbb{R}$.

Part B:

Consider the function f defined over $\mathbb{R}$ by $f(x) = (x-1)e^{x+1} + 2x$.
Let (C) be its representative curve in an orthonormal system $\left(O\ ;\ \vec{i}\ ;\ \vec{j}\right)$.

1) a) Determine $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x}$. Interpret your answer.

   b) Determine $\lim\limits_{x \to -\infty} f(x)$ and show that the straight line (d) of equation $y = 2x$ is an asymptote to (C).

   c) Study the relative positions of (C) and (d).

2) a) Show that for every $x \in \mathbb{R}$, $f'(x) = g(x)$ and set up the table of variations of f.

b) Show that the equation $f(x) = 0$ admits over $\mathbb{R}$ a unique solution α and verify that $\alpha \in]0.7\ ;\ 0.8[$.

c) Show that the curve (C) admits an inflection point I whose coordinates are to be determined.

3) Draw (d) and (C).

Part C:
Consider the function h defined as $h(x) = \ln[f(x) - 2x]$.

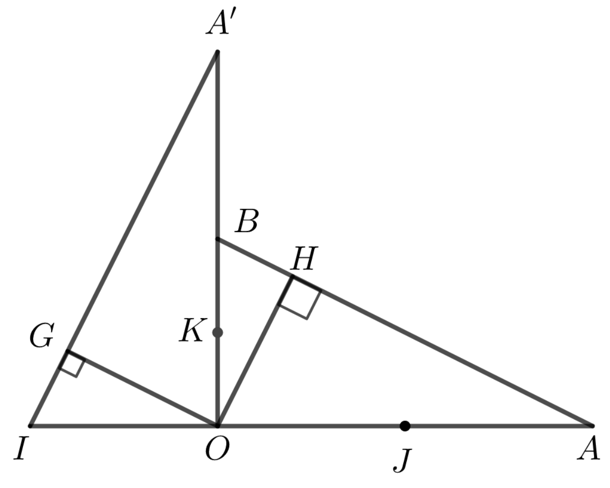
1) Determine the domain of definition of h.

2) Determine the limits of h at the open boundaries of its domain of definition.

3) Let (H) be the representative curve of h in the same system. Show that (H) has at $+\infty$ an asymptotic direction parallel to straight line (L) of equation $y = x$.

V - (4.5 points)
In the figure below:

- OAB is a triangle such that $\left(\overrightarrow{OA}\ ;\ \overrightarrow{OB}\right) = \frac{\pi}{2}(2\pi)$, $OA = 8$ and $OB = 4$.
- J and K are respectively the midpoints of segments $[OA]$ and $[OB]$.
- I is the symmetric of J with respect to O.
- A' is the symmetric of O with respect to B.
- H and G are respectively the orthogonal projections of O on (AB) and (IA').



1) Let r be the rotation of center O and angle $\frac{\pi}{2}$.

   a) Determine $r(A)$ and $r(B)$.

   b) Prove that $r(H) = G$.

2) Let S be the direct plane similitude such that $S(O) = A$ and $S(B) = O$.

   a) Determine the ratio of S and verify that $\frac{\pi}{2}$ is a mesure of its angle.

b) Justify that $S(K) = J$.

c) Determine the images of the lines (HO) and (AB) by S. Deduce that H is the center of S.

d) Justify that (HK) and (HJ) are perpendicular.

3) The perpendicular to (OA) at point A intersects (KH) at C.

a) Determine the images of (OA) and (HJ) by S.

b) Deduce $S(J)$.

c) Prove that $HC = OA = AC$.

4) Let L be the symmetric of O with respect to I. Let $h = S \circ r^{-1}$.

a) Determine $h\,(I)$.

b) Show that h is a dilation, determine its ratio, and show that L is its center.

5) The plane is referred to the orthonormal system $\left(O\ ;\ \overrightarrow{OJ}\ ;\ \overrightarrow{OB}\right)$.

a) Determine the complex form of S. Deduce the affix of point H.

b) Determine the complex form of r and the affix of point G.

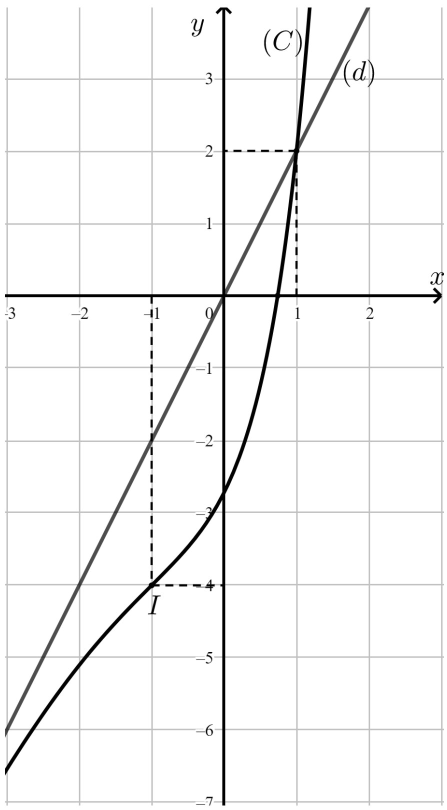
c) Prove that G is the midpoint of $[LH]$.

QI	Answers	Note
1.	$\overline{z} = \overline{1 - e^{i\frac{\pi}{3}}} = 1 - \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-i\frac{\pi}{3}}$; The correct answer is then C.	**1/2**
2.	$x \in]-1;0[$ so: $0 < x+1 < 1$; $\ln(x+1) < 0$ and $\lvert\ln(x+1)\rvert = -\ln(x+1)$; Then $e^{\lvert\ln(x+1)\rvert} = e^{-\ln(x+1)} = \frac{1}{x+1}$; The correct answer is then C.	**1/2**
3.	Let: $X = \frac{1}{x}$ then: $\lim\limits_{x\to 0^+} x\ln\left(1+\frac{1}{x}\right) = \lim\limits_{X\to+\infty} \frac{\ln(1+X)}{X} = 0$; **OR** $\lim\limits_{x\to 0^+} x\ln\left(1+\frac{1}{x}\right) = \lim\limits_{x\to 0^+} \frac{\ln\left(1+\frac{1}{x}\right)}{\frac{1}{x}} = \lim\limits_{x\to 0^+} \frac{\frac{-\frac{1}{x^2}}{1+\frac{1}{x}}}{-\frac{1}{x^2}} = \lim\limits_{x\to 0^+} \frac{1}{1+\frac{1}{x}} =$ 0; The correct answer is then A.	**1/2**

QII	Answers	Note
1.	$\frac{z_A - z_B}{z_A - z_C} = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$; • $\left\lvert\frac{z_A - z_B}{z_A - z_C}\right\rvert = 1$ then $BA = CA$; • $\arg\left(\frac{z_A - z_B}{z_A - z_C}\right) = \frac{\pi}{2}(2\pi)$ then $\left(\overrightarrow{CA}\ ;\ \overrightarrow{BA}\right) = \frac{\pi}{2}(2\pi)$; Then the triangle ABC is right isosceles at A.	**3/4**
2.	$z' = -3i$; $\frac{iz - 4 = 2i}{z-2} = -3i$ so $z = 1 - i$ so $z = \sqrt{2}e^{-i\frac{pi}{4}}$.	**1/2**
3.a.	$z' - i = \frac{iz - 4 + 2i}{z - 2} - i = \frac{-4+4i}{z-2}$.	**1/2**
3.b.	$z' - i = \frac{-4+4i}{z-2}$ gives $z_{\overrightarrow{AM'}} = \frac{-4+4i}{z_{\overrightarrow{BM}}}$; • $\lvert z_{\overrightarrow{AM'}}\rvert = \left\lvert\frac{-4+4i}{z_{\overrightarrow{BM}}}\right\rvert$; $AM' = \frac{4\sqrt{2}}{BM}$; $AM' \times BM = 4\sqrt{2}$; • $\arg\left(z_{\overrightarrow{AM'}}\right) = \arg(-4+4i) - \arg\left(z_{\overrightarrow{BM}}\right)$; $\left(\vec{u}\ ;\ \overrightarrow{AM'}\right) = \frac{3\pi}{4} - \left(\vec{u}\ ;\ \overrightarrow{BM}\right)$; $\left(\vec{u}\ ;\ \overrightarrow{AM'}\right) + \left(\vec{u}\ ;\ \overrightarrow{BM}\right) = \frac{3\pi}{4} + 2k\pi$ with $k \in \mathbb{Z}$.	**3/4**
3.c.	M moves on the circle of center B and radius 4 so $BM = 4$. Then $AM' = \sqrt{2}$; The set of points M' is the circle of center A and radius $\sqrt{2}$.	**1/2**
3.d.	$\arg(z' - i) = \frac{\pi}{4}(2\pi)$ so $\left(\vec{u}\ ;\ \overrightarrow{AM'}\right) = \frac{\pi}{4}(2\pi)$; $\left(\vec{u}\ ;\ \overrightarrow{BM}\right) = \frac{\pi}{2}(2\pi)$. Then the set of points M is the semi-line $]Bt)$ passing through B and orthogonal to $\vec{u}$ located above the x-axis.	**1/2**

QIII	Answers	Note
1.	$p\left(A\right)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$; $p\left(B\right)=1-p\left(\text{no head}\right)=1-p\left(3\text{ tails}\right)=1-p\left(A\right)=\frac{7}{8}$.	**1/2**
2.a	$p\left(D/A\right)=\frac{C_4^1\times C_3^1\times C_1^1}{C_8^3}=\frac{3}{14}$; $p\left(D\cap A\right)=p\left(D/A\right)\times p\left(A\right)=\frac{3}{14}\times\frac{1}{8}=\frac{3}{112}$;	**3/4**
2.b	$p\left(D/B\right)=\frac{1}{8}\times\frac{3}{8}\times\frac{4}{8}\times 3!=\frac{9}{64}$; $p\left(D\cap B\right)=p\left(D/B\right)\times p\left(B\right)=\frac{9}{64}\times\frac{7}{8}=\frac{63}{512}$; $p\left(D\right)=p\left(D\cap A\right)+p\left(D\cap B\right)=\frac{3}{112}+\frac{63}{512}=\frac{537}{3584}$.	**3/4**
2.c.	Consider the event E : « Obtain exactly two tails among the three throws of the coin ». $p\left(E\right)=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{3!}{2!\times 1!}=\frac{3}{8}$; $p(E/D)=\frac{p(E\cap D)}{p(D)}=\frac{p(D/E)\times p(E)}{p(D)}=\frac{\frac{9}{64}\times\frac{3}{8}}{\frac{537}{3584}}=\frac{63}{179}$.	**3/4**
2.d.	Consider the event F : « the three drawn balls have the same color »; $p\left(F\right)=p\left(F\cap A\right)+p\left(F\cap B\right)=p\left(F/A\right)\times p\left(A\right)+p\left(F/B\right)\times p\left(B\right)$; $p\left(F\right)=\left(\frac{C_4^3+C_3^3}{C_8^3}\right)\times\frac{1}{8}+\left(\frac{1^3}{8^3}+\frac{3^3}{8^3}+\frac{4^3}{8^3}\right)\times\frac{7}{8}=\frac{1207}{7186}$	**3/4**

QIV	Answers	Note
A.1.	$g'(x)=(x+1)e^{x+1}$ has the same sign as $x+1$ over $\mathbb{R}$; $\begin{array}{c\|ccccc} x & -\infty & & -1 & & +\infty \\ \hline g'(x) & & - & 0 & + & \\ \hline g(x) & & \searrow & 1 & \nearrow & \end{array}$	1/2
A.2.	For every $x\in\mathbb{R}$ we have: $g(x)\geq 1$ then for every $x\in\mathbb{R}$ we have: $g(x)>0$.	**1/2**
B.1.a.	$\lim\limits_{x\to+\infty}f(x)=+\infty\times(+\infty)+\infty=+\infty$; $\lim\limits_{x\to+\infty}\frac{f(x)}{x}=\lim\limits_{x\to+\infty}\left[\left(1-\frac{1}{x}\right)e^{x+1}+2\right]=(1-0)(+\infty)+2=+\infty$; The (C) curve admits an asymptotic direction parallel to the y-axis at $+\infty$.	3/4
B.1.b.	$\lim\limits_{x\to-\infty}f\left(x\right)=\lim\limits_{x\to-\infty}\left(xe^x\times e-e^x\times e+2x\right)=0-\infty=-\infty$; $\lim\limits_{x\to-\infty}\left[f\left(x\right)-2x\right]=\lim\limits_{x\to-\infty}\left(xe^x\times e-e^x\times e\right)=0$; the line (d) of equation $y=2x$ is an oblique asymptote to (C) at $-\infty$.	**1/2**

<table>
<tr><th>QIV</th><th>Answers</th><th>Note</th></tr>
<tr><td>B.1.c.</td><td>$f\,(x)-2x=(x-1)\,e^{x+1}$ has the same sign as $x+1$ since $e^{x+1}>0$ for every $x\in\mathbb{R}$;
• $f\,(x)-2x=$0 if $x=$1; (C) cuts (d) at the point of coordinate $(1\ ;\ 2)$;
• $f\,(x)-2x>0$ if $x>1$; (C) is above (d) if $x\in\,]1\ ;\ +\infty[$;
• $f\,(x)-2x<0$ if $x<1$; (C) is below (d) if $x\in\,]-\infty\ ;\ 1[$;</td><td>1/2</td></tr>
<tr><td>B.2.a.</td><td>$f'(x)=xe^{x+1}+2=g(x)$ and $g(x)>0$ for every $x\in\mathbb{R}$ (part A.2.);
<u>Table of variations of f:</u>
<table>
<tr><td>x</td><td>$-\infty$ $+\infty$</td></tr>
<tr><td>$f'(x)$</td><td>$+$</td></tr>
<tr><td>$f(x)$</td><td>$-\infty$ $\nearrow$ $+\infty$</td></tr>
</table></td><td>3/4</td></tr>
<tr><td>B.2.b.</td><td>f is continuous over $\mathbb{R}$, strictly increasing and changes the sign, so the equation $f(x)=0$ admits a unique solution α over $\mathbb{R}$.
In addition: $f(0.7)\approx-0.24<0$ and $f(0.8)\approx0.39>0$, then $\alpha\in]0.7\ ;\ 0.8[$</td><td>1/2</td></tr>
<tr><td>B.2.c.</td><td>$f''(x)=(x+1)e^{(}x+1)$ has the same sign as $x+1$ over $\mathbb{R}$; $f''(x)$ is equal to zero if $x=-1$ and changes the sign so the curve (C) admits an inflection point I of coordinates $(-1\ ;\ f(-1))$, so $I\,(-1\,;-4)$.</td><td>1/2</td></tr>
<tr><td>B.3.</td><td>
</td><td>1</td></tr>
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QIV	Answers	Note
C.1.	$h(x) = \ln\left[(x-1)\,e^{x+1}\right] = \ln(x-1) + x + 1$; The domain of definition h is $]1\,;\,+\infty[$.	**1/4**
C.2.	• $\lim\limits_{x\to 1^+} h(x) = -\infty$; • $\lim\limits_{x\to +\infty} h(x) = +\infty$.	**1/2**
C.3.	• $\lim\limits_{x\to +\infty} h(x) = +\infty$; • $\lim\limits_{x\to +\infty} \dfrac{h(x)}{x} = 0 + 1 + 0 = 1$; • $\lim\limits_{x\to +\infty} [h(x) - x] = \lim\limits_{x\to +\infty} [\ln(x-1) + 1] = +\infty$; The curve (H) admits at $+\infty$ an asymptotic direction parallel to the line L of equation $y = x$.	**3/4**

QV	Answers	Note
1.a.	$r(A) = A'$ since $OA = OA' = 8$ and $\left(\overrightarrow{OA}\,;\,\overrightarrow{AA'}\right) = \dfrac{\pi}{2}(2\pi)$; $r(B) = I$ since $OB = OI = 4$ and $\left(\overrightarrow{OB}\,;\,\overrightarrow{OI}\right) = \dfrac{\pi}{2}(2\pi)$.	**1/2**
1.b.	H is the orthogonal projection of O over (AB), then $r(H)$ is the orthogonal projection of $r(O)$ over $r((AB))$ since the rotation preserves the oriented angles; But $r(O) = O$ and $r((AB)) = (A'I)$, so $r(H)$ is the orthogonal projection of O over $(A'I)$ which is G; Finally $r(H) = B$.	**1/4**
2.a.	Let k be the ratio of S and α a measure of its angle; $S(O) = A$ and $S(B) = O$ then $k = \dfrac{AO}{OB} = 2$ and $\alpha = \left(\overrightarrow{OB}\,;\,\overrightarrow{AO}\right) = \dfrac{\pi}{2}(2\pi)$.	**1/2**
2.b.	K is the midpoint of $[OB]$ then $S(K)$ is the midpoint of $S([OB])$ because similitude preserves the midpoints; so $S(K)$ is the midpoint of $[AO]$ which is J; finally $S(K) = J$.	**1/4**
2.c.	$S(O) = A$ and the angle of S is $\alpha = \dfrac{\pi}{2}$, then $S((OH))$ is the line passing through A and perpendicular to (OH) which is (AB); so $S((OH)) = (AB)$. $S((AB))$ is the line passing through O and perpendicular to (AB) which is (OH); so $S((AB)) = (OH)$. $H \in (OH)$ then $S(H) \in S((OH))$ then $S(H) \in (AB)$; $H \in (AB)$ then $S(H) \in S(((AB))$ then $S(H) \in (OH)$; So $S(H)$ is the intersection of (AB) and (OH) which is H so $S(H) = H$ and H is the center of S.	**1/2**
2.d.	$S(K) = J$ and the center of S is H and its angle is $\alpha = \dfrac{\pi}{2}$ so $\left(\overrightarrow{HK}\,;\,\overrightarrow{HJ}\right) = \dfrac{\pi}{2}(2\pi)$, so (HK) and (HJ) are perpendicular.	**1/4**
3.a.	$S((OA)) = (AC)$; $S((HJ)) = (HC)$.	**1/4**
3.b.	J is the intersection of (OA) and (HJ) then $S(J)$ is the intersection of (AC) and (HC), then $S(J) = C$.	**1/4**

QV	Answers	Note
3.c.	$S(J) = C$ then $HC = 2HJ$ and $HJ = \frac{1}{2}OA$ then $HC = OA$; $S(O) = A$ and $S(J) = C$ then $AC = 2OJ = OA$; So $HC = OA = AC$.	**1/4**
4.a.	$h(I) = S\left[r^{-1}(I)\right] = S(B) = O$.	**1/4**
4.b.	$h = S\left(H\ ;\ 2\ ;\ \frac{\pi}{2}\right) \circ S\left(O\ ;\ 1\ ;\ -\frac{\pi}{2}\right) = S\,(?\ ; 2\ ;\ 0)$, so h is a dilation of ratio 2; h is a dilation of ratio 2 and $h(I) = O$; In addition we have: $\overrightarrow{LO} = 2\overrightarrow{LI}$, so L is the center of h.	**1/2**
5.a.	The complex form of S is: $z' = az + b$ with $a = 2e^{i\frac{\pi}{2}} = 2i$; and as $S(O) = A$ then $z_A = 2iz_O + b$ so $b = 2$; so $S : z' = 2iz + 2$; The affix of H is $z_H = \frac{b}{1-a} = \frac{2}{1-2i} = \frac{2}{5} + \frac{4}{5}i$.	**1/4**
5.b.	The complex form of r is: $z' = e^{i\frac{\pi}{2}}z = iz$; $r(H) = G$ then $z_G = iz_H = \frac{z_G}{i} = -\frac{4}{5} + \frac{2}{5}i$.	**1/4**
5.c.	$\frac{z_L + z_H}{2} = \frac{-4}{5} + \frac{2}{5}i = z_G$, then G is the midpoint of $[LH]$.	**1/4**