ExaMath groups	Exam in Mathematics Section: G.S.	Prepared by: Fredric Al Bayeh Edited by: H. Ahmad
Number of questions: 5	Sample 02 - 2022 Duration: 180 min	Name: N°:

- *This exam includes five problems. It is inscribed on four pages, numbered from 1 to 4.*
- *The use of a non-programmable calculator is allowed.*

I - (2 points)

In the table below, only one among the proposed answers to each question is correct.
Choose, **with justification**, the correct answer.

N°	Question	Proposed answers A	B	C
1.	The expression: $A=\ln\left(\dfrac{1}{e^2}\right)-e^{3\ln(2)}\times e^{-\ln(3)}$ can be written as:	$A=-\dfrac{14}{3}$	$A=-26$	$A=-\dfrac{2}{3}$
2.	The domain of definition of the function f defined by $f(x)=\ln\left[(\ln x)^2-2\ln x-3\right]$ is:	$]0\ ;\ +\infty[$	$]0\ ;\ e^{-1}[\ \cup\ ]e^3\ ;\ +\infty[$	$]0\ ;\ -1[\ \cup\ ]3\ ;\ +\infty[$
3.	In an orthonormal system, A, B and C are three points of the plane of respective affixes z_A, z_B and z_C such that : $\dfrac{z_A-z_B}{z_A-z_C}=i$. The triangle ABC is:	a right isosceles triangle at A	an equilateral triangle	a triangle right angled at A
4.	An argument of the complex number $z=-2i\left(\cos\dfrac{\pi}{3}-i\sin\dfrac{\pi}{3}\right)$ is:	$\dfrac{5\pi}{6}$	$-\dfrac{5\pi}{6}$	$\dfrac{\pi}{6}$

II - (4 points)

In the plane referred to a direct orthonormal system $(O\ ;\ \vec{u}\ ;\ \vec{v})$, consider the points A, B and D of respective affixes i, $-2i$ and 1.
Let f be the mapping that associates, to each point M of affix z such that $z\neq i$, the point M' of affix z' such that $z'=f(z)=\dfrac{2z-i}{iz+1}$.

1) a) Determine in algebraic form the affix $z_{D'}$ of the point D', image of the point D by f.
   b) For what values of the natural number n, is the complex number $(z_A+z_{D'})^n$ a real number?

2) a) Prove that, for every complex number z such that $z\neq i$, we have $(z'+2i)(z-i)=1$.
   b) Deduce that $BM'\times AM=1$ and $\left(\vec{u}\ ;\ \overrightarrow{BM'}\right)=-\left(\vec{u}\ ;\ \overrightarrow{AM}\right)+2k\pi$ for $k\in\mathbb{Z}$.

c) Prove that if the point M varies on the circle of center A and radius 2, then the point M' varies on a circle of center and radius are to be determined.

3) Let E be the point of affix $z_E = \dfrac{1+\sqrt{3}}{2}(1+i)$.

   a) Calculate $|z_A - z_D|$ and prove that $|z_A - z_E| = \sqrt{2}$.

   b) Deduce that the points D and E belong to the same circle of center and radius are to be determined.

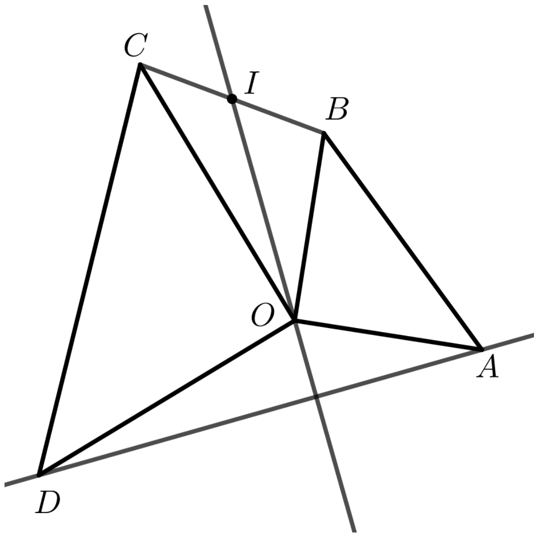
4) Let $z = x + iy$ and $z' = x' + iy'$ where x, y, x' and y' are real numbers.

   a) Prove that $x' = \dfrac{x}{x^2 + (1-y)^2}$ and $y' = \dfrac{-2x^2 - 2y^2 + 3y - 1}{x^2 + (1-y)^2}$.

   b) Deduce the set of points M of affix z when z' is pure imaginary.

III - (4 points)

In the below figure, OAB and OCD are two direct right isosceles triangles at O.
I is the midpoint of the segment $[BC]$.
Let r be the rotation of center O and angle $\dfrac{\pi}{2}$ and h be the dilation of center B and ratio 2.



1) Let $S = r \circ h$.

   a) Determine the image of the point I by S and prove that $S(O) = A$.

   b) Show that S is a similitude. Determine its ratio and a mesure of its angle.

   c) Construct geometrically the center H of S.

   d) Deduce that $AD = 2OI$ and that the lines (AD) and (OI) are perpendicular.

   e) Prove that the image of the line (OI) by S is the line (AD).

   f) Let J be the midpoint of the segment $[AB]$.
   Prove that $S(J) = B$ and construct the point F such that $F = S(A)$

2) a) Determine the nature and characteristic elements of the transformation $S \circ S$.

   b) Deduce that $\overrightarrow{HF} + 4\overrightarrow{HO} = \vec{0}$.

3) The plane is referred to a direct orthonormal system $(O\ ;\ \vec{u}\ ;\ \vec{v})$ such that $\overrightarrow{OA} = 3\vec{u}$. Write the complex form of S and determine the affix of the point H.

IV - (3 points)

Part A

Consider a perfect cubic die **A** having one green side, two black sides and three red sides.
A game consists of throwing the die twice in a row and denoting the obtained color after each throw.
For $i \in \{1\ ;\ 2\}$, consider the following events:
G_i: « The obtained side at the i^{th} throw is green »;
B_i: « The obtained side at the i^{th} throw is black »;
R_i: « The obtained side at the i^{th} throw is red ».

1) Show that $P\left(G_1 \cap G_2\right) = \dfrac{1}{36}$ then calculate $P(B_1 \cap B_2)$ and $P(R_1 \cap R_2)$.

2) Deduce that the probability of obtaining two sides of the same color by the end of the game is $\dfrac{7}{18}$.

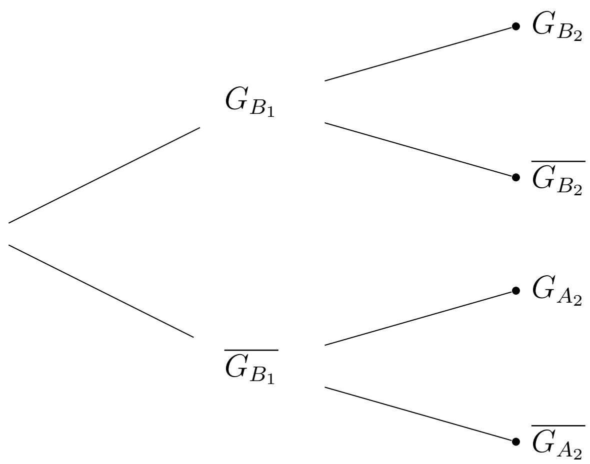
Part B

Consider now a second cubic die **B** having four green sides and two black sides. A second game consists of throwing the die **B**:

- If the obtained side is green, we throw again the die **B** and we denote the obtained color.
- If the obtained side is black, we throw the die **A** and we denote the obtained color.

For $i \in \{1\ ;\ 2\}$, consider the following events:
G_{Ai}: « The obtained side at the i^{th} throw of the die **A** is green »;
G_{Bi}: « The obtained side at the i^{th} throw of the die **B** is green ».

1) Complete the following probability tree representing the situation.



2) Verify that $P\left(G_{B_1} \cap G_{B_2}\right) = \dfrac{4}{9}$.

3) Calculate the probability of the event L: « The obtained side during the 2^{nd} throw is green ».

V - (7 points)

Part A

Let h be the function defined over $]0\,;\,+\infty[$ by $h(x) = x^3 - 1 + 2\ln x$.

1) Calculate $h'(x)$ and set up the table of variations of h.

2) Calculate $h(1)$ and deduce the sign of h over $]0\,;\,+\infty[$.

Part B

Let f be the function defined over $]0\,;\,+\infty[$ by $f(x) = x - 2 - \dfrac{\ln x}{x^2}$.

Denote by (C) the representative curve of f in an orthonormal system $\left(O\,;\,\vec{i}\,;\,\vec{j}\right)$.

1) Calculate $\lim\limits_{x\to 0^+} f(x)$ and deduce the equation of an asymptote (d) to (C).

2) a) Calculate $\lim\limits_{x\to +\infty} f(x)$.

   b) Prove that the line (D) of equation $y = x - 2$ is an asymptote to (C).

   c) Study the relative position of (C) and (D).

3) Prove that for every $x \in]0\,;\,+\infty[$, $f'(x) = \dfrac{h(x)}{x^3}$ and set up the table of variations of f.

4) Show that the equation $f(x) = 0$ admits over $]0\,;\,+\infty[$ two solutions α and β such that $0.59 < \alpha < 0.61$ and $2.15 < \beta < 2.17$.

5) a) Prove that for every $x \in]0\,;\,+\infty[$, $f''(x) = \dfrac{5 - 6\ln x}{x^4}$.

   b) Deduce that the curve (C) admits an inflection point I where the coordinates are to be determined.

6) Draw (d), (D) and (C).

Part C

Consider the function g defined by $g(x) = e^{f(x)}$.

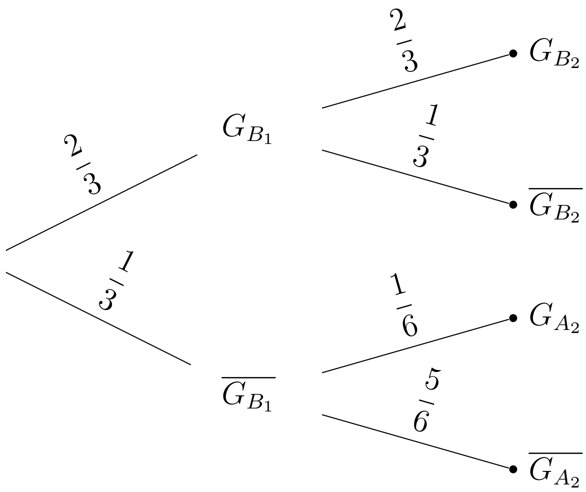
1) Determine the domain of definition of the function g.

2) Show that the function g is strictly increasing over the interval $]\beta\,;\,+\infty[$.

QI	Answers	Grade
1.	$A=-2-8+\frac{1}{3}=-\frac{14}{3}$.	**1/2**
2.	$x>0$ and $(\ln x)^2-2\ln x-3>0$; $x\in\left]0\,;\,e^{-1}\right[\cup\left]e^3\,;\,+\infty\right[$.	**1/2**
3.	$\frac{z_{\overrightarrow{BA}}}{z_{\overrightarrow{CA}}}=i$; $BA=CA$ and $\left(\overrightarrow{CA}\,;\,\overrightarrow{BA}\right)=\frac{\pi}{2}(2\pi)$; ABC is a right isosceles triangle at A.	**1/2**
4.	$z=2e^{-i\frac{\pi}{2}}\times e^{-i\frac{\pi}{3}}=2e^{-i\frac{5\pi}{6}}$; $\arg z=-\frac{5\pi}{6}(2\pi)$.	**1/2**

QII	Answers	Grade
1.a.	$z_{D'}=\frac{1}{2}-\frac{3}{2}i$.	**1/2**
1.b.	$(z_A+z_{D'})^n=\left(c\frac{\sqrt{2}}{2}e^{-i\frac{\pi}{4}}\right)^n=\left(\frac{\sqrt{2}}{2}\right)^n e^{-i\frac{n\pi}{4}}$, is real if $-n\frac{\pi}{4}=k\pi$ where $k\in\mathbb{Z}$; As $n\in\mathbb{N}$ then n is a positive multiple of 4.	**1/2**
2.a.	$(z'+2i)(z-i)=\left(\frac{2z-i-2z+2i}{iz+1}\right)(z-i)=\frac{i}{i\,(z-i)}(z-i)=1$;	**1/2**
2.b.	$z_{\overrightarrow{BM'}}\times z_{\overrightarrow{AM}}=1$; $\left\|z_{\overrightarrow{BM'}}\times z_{\overrightarrow{AM}}\right\|=1$ then $BM'\times AM=1$; $\arg\left(z_{\overrightarrow{BM'}}\times z_{\overrightarrow{AM}}\right)=arg(1)$; $\left(\vec{u}\,;\,\overrightarrow{BM'}\right)+\left(\vec{u}\,;\,\overrightarrow{AM}\right)=0+2k\pi$ with $k\in\mathbb{Z}$.	**1/2**
2.c.	$BM'\times AM=1$ with $AM=2$ so $BM'=\frac{1}{2}$; M varies on the circle of center B and radius $\frac{1}{2}$.	**1/2**
3.a.	$\|z_A-z_D\|=\sqrt{2}$; $\|z_A-z_E\|=\sqrt{2}$; $DA=EA=\sqrt{2}$; D and E belong to the circle of center A and radius $\sqrt{2}$.	**1/2**
4.a.	$x'+iy'=\frac{2(x+iy)-i}{i(x+iy)+1}$; $x'=\frac{x}{x^2+(1-y)^2}$; $y'=\frac{-2x^2-2y^2+3y-1}{x^2+(1-y)^2}$.	**1/2**
4.b.	z' is pure imaginary if $x'=0$ and $y'\neq 0$; $x=0$ and $-2x^2-2y^2+3y-1\neq 0$; The set of points M is the line of equation $x=0$ deprived of points of coordinates $(0\,;\,1)$ and $\left(0\,;\,\frac{1}{2}\right)$.	**1/2**

QIII	Answers	Grade
1.a.	$S(I)=r[h(I)]=r(C)=D$; $S(O)=r[h(O)]=r(O')$ where O' is the symmetric of B with respect to O ; so $S(O)=A$ as $OO'=OA$ and $\left(\overrightarrow{OO'}\ ;\ \overrightarrow{OA}\right)=\dfrac{\pi}{2}(2\pi)$.	**1/2**
1.b.	$S=r\left(O\ ;\ \dfrac{\pi}{2}\right)\circ h(B\ ;\ 2)$; S is a similarity of ratio 2 and angle $\dfrac{\pi}{2}$.	**1/2**
1.c.	$S(I)=D$ so $\left(\overrightarrow{HI}\ ;\ \overrightarrow{HD}\right)=\dfrac{\pi}{2}(2\pi)$; $S(O)=A$ so $\left(\overrightarrow{HO}\ ;\ \overrightarrow{HA}\right)=\dfrac{\pi}{2}(2\pi)$; H is one of the points of intersection of two circles of diameters $[ID]$ and $[OA]$ such that $\left(\overrightarrow{HI}\ ;\ \overrightarrow{HD}\right)=\dfrac{\pi}{2}(2\pi)$.	**1/2**
1.d.	$S(I)=D$ and $S(O)=A$ so $DA=2OI$ and $\left(\overrightarrow{IO}\ ;\ \overrightarrow{DA}\right)=\dfrac{\pi}{2}(2\pi)$; Then $DA=2OI$ and (DA) and (IO) are perpendicular.	**1/2**
1.e.	$S((OI))$ is the line passing through A and perpendicular to (OI); So $S((OI))=(AD)$.	**1/2**
1.f	$S(J)=r[h(J)]=r(A)=B$; JOA is a direct right isosceles triangle at J; $S(J)=B$ and $S(O)=A$ then F is the 3$^{\text{rd}}$ vertex of the direct triangle BAF right isosceles at B.	**1/2**
2.a	$S\circ S=S\left(H\ ;\ 2\ ;\dfrac{\pi}{2}\right)\circ S\left(H\ ;\ 2\ ;\dfrac{\pi}{2}\right)=S(H\ ;\ 4\ ;\pi)$; $S\circ S=h(H\ ;\ -4)$.	**1/2**
2.b	$S\circ S(O)=S[S(O)]=S(A)=F$ and $S\circ S=h(H\ ;\ -4)$, so $\overrightarrow{HF}=-4\overrightarrow{HO}$.	**1/2**
2.c	$S:z'=az+b$; $a=2e^{i\frac{\pi}{2}}=2i$; $S(O)=A$ so $b=3$; $S:z'=2iz+3$; $z_H=\dfrac{b}{1-a}=\dfrac{3}{5}+\dfrac{6}{5}i$.	**1/2**

QIV	Answers	Grade
A.1.	$P\left(G_1\cap G_2\right)=\dfrac{1}{6}\times\dfrac{1}{6}=\dfrac{1}{36}$; $P\left(B_1\cap B_2\right)=\dfrac{2}{6}\times\dfrac{2}{6}=\dfrac{1}{9}$; $Pp\left(R_1\cap R_2\right)=\dfrac{3}{6}\times\dfrac{3}{6}=\dfrac{1}{4}$.	**1/2**
A.2.	$p\left(G_1\cap G_2\right)+p\left(B_1\cap B_2\right)+p\left(R_1\cap R_2\right)=\dfrac{7}{18}$.	**1/2**
B.1.	 	**1/2**
B.2.	$P\left(G_{B_1}\cap G_{B_2}\right)=P\left(G_{B_2}/G_{B_1}\right)\times P\left(G_{B_1}\right)=\dfrac{4}{9}$.	**1/2**
B.3.	$P(L)=P\left(G_{B_1}\cap G_{B_2}\right)+P\left(G_{A_2}\cap\overline{G_{B_1}}\right)=\dfrac{4}{9}+\dfrac{1}{18}=\dfrac{1}{2}$.	**1/2**

<table>
<tr><th>QV</th><th>Answers</th><th>Grade</th></tr>
<tr><td>A.1.</td><td>$h'(x)=3x^2+\dfrac{2}{x}>0$ for every $x\in]0\,;\,+\infty[$;
Table of variations of h :
<table>
<tr><td>x</td><td>0</td><td>$+\infty$</td></tr>
<tr><td>$h'(x)$</td><td>‖</td><td>$+$</td></tr>
<tr><td>$h(x)$</td><td>‖</td><td>$-\infty$ ↗ $+\infty$</td></tr>
</table></td><td>1/2</td></tr>
<tr><td>A.2.</td><td>$h(1)=0$;
• $h(x)>0$ if $x\in]1\,;\,+\infty[$;
• $h(x)<0$ if $x\in]0\,;\,1[$;
• $h(x)=0$ if $x=1$</td><td>1/2</td></tr>
<tr><td>B.1.</td><td>$\lim\limits_{x\to0^+}f(x)=+\infty$;
The line of equation $x=0$ is a vertical asymptote to (C).</td><td>1/2</td></tr>
</table>

<table>
<tr><th>QV</th><th>Answers</th><th>Grade</th></tr>
<tr><td>B.2.a.</td><td>$\lim\limits_{x\to+\infty} f(x) = +\infty$.</td><td>1/2</td></tr>
<tr><td>B.2.b</td><td>$\lim\limits_{x\to+\infty} [f(x) - (x-2)] = \lim\limits_{x\to+\infty} \left(-\dfrac{\ln x}{x^2}\right) = 0$;
The line (D) of equation $y = x - 2$ is an oblique asymptote to (C) at $+\infty$.</td><td>1/2</td></tr>
<tr><td>B.2.c</td><td>$f(x) - (x-2) = -\dfrac{\ln x}{x^2}$ has the same sign as $-\ln x$ over $]0\,;\,+\infty[$;
• (C) is above (D) if $x \in]0\,;\,1[$;
• (C) is below (D) if $x \in]1\,;\,+\infty[$;
• (C) cuts (D) at the point of coordinates $(1\,;\,-1)$.</td><td>1/2</td></tr>
<tr><td>B.3.</td><td>$f'(x) = \dfrac{x^3 - 1 + 2\ln x}{x^3} = \dfrac{h(x)}{x^3}$;
Table of variations of f :
<table>
<tr><td>x</td><td>0</td><td></td><td>1</td><td></td><td>$+\infty$</td></tr>
<tr><td>$f'(x)$</td><td>$\|$</td><td>$-$</td><td>0</td><td>$+$</td><td></td></tr>
<tr><td>$f(x)$</td><td>$\|$ $+\infty$</td><td>$\searrow$</td><td>-1</td><td>$\nearrow$</td><td>$+\infty$</td></tr>
</table></td><td>1/2</td></tr>
<tr><td>B.4.</td><td>• over $]0\,;\,1[$, f is continuous and strictly decreasing and changes the sign, so the equation $f(x) = 0$ admits a unique solution α;
In addition $f(0.59) \approx 0.11 > 0$ and $f(0.61) \approx -0.06 < 0$ so:
$0.59 < \alpha < 0.61$;
• Over $]1\,;\,+\infty[$, f is continuous and strictly increasing and changes the sign, so the equation $f(x) = 0$ admits a unique solution β;
In addition $f(2.15) \approx -0.02 < 0$ and $f(2.17) \approx 0.01 > 0$ so:
$2.15 < \beta < 2.17$.</td><td>1/2</td></tr>
<tr><td>B.5.a.</td><td>$f''(x) = \dfrac{5 - 6\ln x}{x^4}$.</td><td>1/2</td></tr>
<tr><td>B.5.b.</td><td>$f''(x) = 0$ if $x = e^{\frac{5}{6}}$ and changes the sign, so the curve (C) admits an inflection point I of coordinates $\left(e^{\frac{5}{6}}\,;\,e^{\frac{5}{6}} - 2 - \dfrac{5}{6e^{\frac{5}{3}}}\right)$.</td><td>1/2</td></tr>
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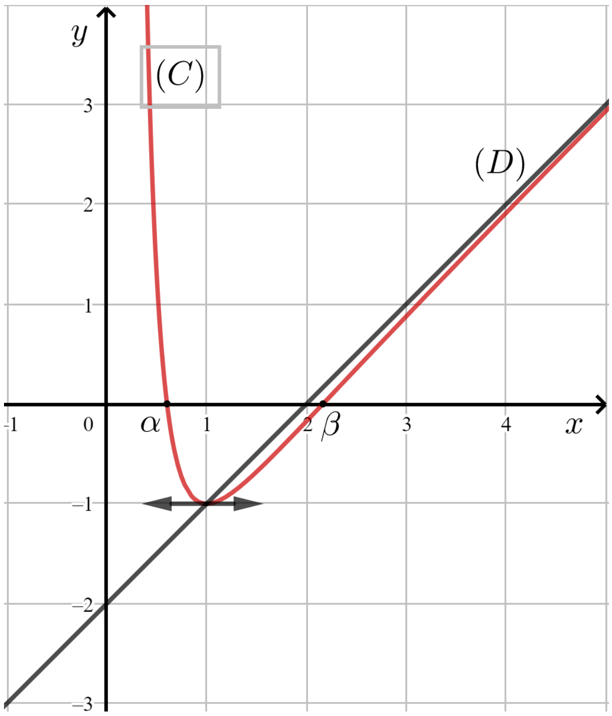
QV	Answers	Grade
B.6.		**1/2**
C.1.	The domain of g is $]0\,;\,+\infty[$.	**1/2**
C.2.	$g'(x) = f'(x)e^{f(x)} > 0$ for every $x \in\,]\beta\,;\,+\infty[$, then g is strictly increasing over $]\beta\,;\,+\infty[$.	**1/2**