Extra-math

Mathematics Book
Higher Education

Acknowledgement

To the soul of my grandfather and the soul of my brother

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- *Book Title*

Integral| For Higher Education

- *From Series*

Extra-math

- *Author*

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Preface

"No problem can be solved for the same level of consciousness that created it"
Albert Einstein.

We believe that the time has come to add our touch in Mathematics world.
This book would not had been possible without the help and assistance of so many people whose names may not all be enumerated.
Their contributions are sincerely appreciated and gratefully acknowledged.

This book is addressed to university students (Mathematics major – Engineering major – Physics major) and prepares them to master all the integration concepts and techniques

Extra-math book covers the basic and advanced skills and competences needed to pass the integration university exams

Also, the university professor can benefit from this book and it will help him(her) to prepare their courses, sheets , examination papers etc …
We have respected the contents and the aims (for integration course) of the program for math – physics and engineering students

Extra-math book covers the following topics

- All Integral formulas
- Indefinite and definite integrals
- All integration techniques: direct integration – integration by substitution – integration by parts – evaluating improper integrals etc …
- Gamma function – incomplete gamma function – beta function – incomplete beta function - zeta function – Dirichlet eta function – digamma function – polygamma function etc …
- Special integrals: error function erf (x); exponential integral Ei(x); logarithmic integral li(x); polylogarithmic function $Li_s(z)$ etc …
- Elliptic integrals & Jacobi elliptic functions
- Integrals with series
- Supplementary & Miscellaneous integral exercises

I wish you a fruitful and successful work.
I thank you for your confidence.
All your suggestions and notes are sincerely welcomed.
Finally, I express a sense of love to my family and to all my friends for their support, strength, and help.

Special thanks to my friend Bassel Yassine for his support

The Author
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Hoping that this book will be an important reference for the teacher and the learner

Chapter 1: Introduction To Integration

1. Primitive of a Function:

Let f be continuous function over an interval I subset of $\mathbb{R}$

We call a primitive of f every continuous function F over I such that F ' $(x) = f(x)$, we write

$$\mathrm{F}(x) = \int f(x)\,dx$$

Remark:

If F and G are two primitives of a continuous function f, then:

- $\mathrm{F}(x) - \mathrm{G}(x) = c$, where c is a constant
- $\mathrm{F}'(x) = \mathrm{G}'(x)$

Example 1:

The functions $f(x) = 3x^2 - 1$ is defined and continuous over $\mathbb{R}$, moreover we have $(x^3 - x)' = 3x^2 - 1$, then we say that $x^3 - x$ is a primitive of $3x^2 - 1$ over $\mathbb{R}$ and we write

$$\int (3x^2 - 1)dx = x^3 - x + c$$

Example 2:

Consider the two functions G and H defined over $\mathbb{R}$ by G(x) = $\frac{x^2+1}{x^2+1}$ and H(x) = $\frac{2x^2+3}{x^2+2}$.

We will show, by two different methods that G and H are the antiderivatives of the same function over $\mathbb{R}$.

Solution:

First method

$\mathrm{G}'(x) = \frac{2x(x^2+2)-2x(x^2+1)}{(x^2+2)^2} = \frac{2x}{(x^2+2)^2}$ and $\mathrm{H}'(x) = \frac{4x(x^2+2)-2x(2x^2+3)}{(x^2+2)^2} = \frac{2x}{(x^2+2)^2}$.

G'(x) = H'(x), so, G and H are the antiderivatives of the same function over $\mathbb{R}$.

Second method

$\mathrm{H}(x) - \mathrm{G}(x) = \frac{2x^2+3}{x^2+2} - \frac{x^2+1}{x^2+1} = \frac{2x^2+3-x^2-1}{x^2+2} = \frac{x^2+2}{x^2+2} = 1.$

H(x) = G(x) + 1= G(x) + constant, so, G and H are the antiderivatives of the same function over $\mathbb{R}$.

Properties:

Let f and g be two continuous functions over an interval I, then the following holds:

- $\int kf(x)dx = k\int f(x)dx$, where k is a constant.
- $\int [f(x) \pm g(x)]\,dx = \int f(x)dx \pm \int g(x)dx$.
- $\int [af(x) \pm bg(x)]dx = a\int f(x)dx \pm b\int g(x)dx$, where a and b are two constants.

2. Some Basic Primitives:

Function f	Primitive F
0	$k = \text{constant}$
$k = \text{constant}$	$kx + c$
ax^n, where $n \neq -1$	$a\dfrac{x^{n+1}}{n+1} + c$
$u' \times u^n$, where $n \neq -1$	$\dfrac{u^{n+1}}{n+1} + c$
$u'(x) \pm v'(x)$	$u(x) \pm v(x) + c$
$\cos(ax+b)$	$\dfrac{1}{a}\sin(ax+b) + c$
$\sin(ax+b)$	$-\dfrac{1}{a}\cos(ax+b) + c$
$1 + \tan^2 x$	$\tan x + c$
e^{ax+b}	$\dfrac{1}{a}e^{ax+b} + c$
$u'(x)e^{u(x)}$	$e^{u(x)} + c$
$\dfrac{1}{x}$	$\ln\lvert x\rvert + c$
$\dfrac{u'(x)}{u(x)}$	$\ln\lvert u(x)\rvert + c$
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x + c$
$\dfrac{-1}{\sqrt{1-x^2}}$	$\arccos x + c$
$\dfrac{1}{1+x^2}$	$\arctan x + c$
$\dfrac{u'(x)}{\sqrt{a^2 - u'(x)}}$	$\arcsin u\left[\dfrac{u(x)}{a}\right] + c$
$\dfrac{-u'(x)}{\sqrt{a^2 - u'(x)}}$	$\arccos\left[\dfrac{u(x)}{a}\right] + c$
$\dfrac{u'(x)}{a^2 + [u(x)]^2}$	$\dfrac{1}{a}\arctan\left[\dfrac{u(x)}{a}\right] + c$

Function f	Primitive F
$\sec x$	$\begin{cases} \frac{1}{2}\ln\left\lvert\frac{1+\sin x}{1-\sin x}\right\rvert + c \\ \ln\lvert\sec x+\tan x\rvert + c \\ \ln\left\lvert\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right\rvert + c \end{cases}$
$\csc x$	$-\ln\lvert\csc x+\cot x\rvert + c$
$a^x;\ a > 0\ \&\ a \neq 1$	$\frac{1}{\ln a}a^x + c$
$\sec^2 u$	$\tan u + c$
$\csc^2 u$	$-\cot u + c$
$\sec u\tan u$	$\sec u + c$
$\csc u\cot u$	$-\csc u + c$
$\sinh u$	$\text{coh}u + c$
$\cosh u$	$\sinh u + c$
$\tanh u$	$\ln\cosh u + c$
$\coth u$	$\ln\sinh u + c$
$\text{sech}^2 u$	$\tanh u + c$
$\text{csch}^2 u$	$-\coth u + c$
$\frac{u'(x)}{u^2-a^2}$	$\frac{1}{2a}\ln\left(\frac{u-a}{u+a}\right) + c$
$\frac{u'(x)}{u\sqrt{u^2-a^2}}$	$\frac{1}{a}\text{arcsec}\frac{\lvert u\rvert}{a} + c$
$\frac{1}{\sqrt{x^2-a^2}};\ x > a$	$\cosh^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{a^2-x^2};\ \lvert x\rvert < a$	$\tanh^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{\sqrt{x^2+a^2}};\ x \in \mathbb{R}$	$\sinh^{-1}\left(\frac{x}{a}\right) + c$
$\frac{1}{a^2-x^2};\ \lvert x\rvert > a$	$\coth^{-1}\left(\frac{x}{a}\right) + c$

Function f	Primitive F
$\dfrac{-1}{x\sqrt{x^2-a^2}};\quad 0<x<a$	$\operatorname{sech}^{-1}\left(\dfrac{x}{a}\right)+c$
$\dfrac{1}{\lvert x\rvert\sqrt{x^2-a^2}};\quad x\neq 0$	$\operatorname{csch}^{-1}\left(\dfrac{x}{a}\right)+c$

3. <u>**Derivatives (Recall):**</u>

Function f	Derivative f'
$k=\text{constant}$	0
$u\pm v$	$u'\pm v'$
$u.v$	$u'v+v'u$
$\dfrac{u}{v}$	$\dfrac{u'v-v'u}{v^2}$
x^n	nx^{n-1}
$\sqrt{x}$	$\dfrac{1}{2\sqrt{x}}$
$\sqrt{u}$	$\dfrac{u'}{2\sqrt{u}}$
$\dfrac{k}{u}$	$-\dfrac{ku'}{u^2}$
$a^x;\ a>0\ \&\ a\neq 1$	$a^x\ln a$
$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x\tan x$
$\csc x$	$-\csc x\cot x$
$\ln u$	$\dfrac{u'}{u}$
e^u	$u'e^u$

Function f	Derivative f'
$\arctan x$	$\dfrac{1}{1+x^2}$
$\arcsin x$	$\dfrac{1}{\sqrt{1-x^2}}$
$\arccos x$	$\dfrac{-1}{\sqrt{1-x^2}}$
$\operatorname{arccot} x$	$\dfrac{-1}{1+x^2}$
$\operatorname{arcsec} x$	$\dfrac{1}{x\sqrt{x^2-1}}$
$\operatorname{arccsc} x$	$\dfrac{-1}{x\sqrt{x^2-1}}$
$\sinh x$	$\cosh x$
$\cosh x$	$\sinh x$
$\tanh x$	$1-\tanh^2 x$
$\coth x$	$1-\coth^2 x;\ \ x\neq 0$
$\operatorname{sech} x$	$-\tanh x \operatorname{sech} x$
$\operatorname{csch} x$	$-\coth x \operatorname{csch} x\ \ x\neq 0$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$
$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}};\ \ 1<x$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2};\ \ \lvert x\rvert<1$
$\coth^{-1} x$	$\dfrac{1}{1-x^2};\ \ 1<\lvert x\rvert$
$\operatorname{sech}^{-1} x$	$-\dfrac{1}{\lvert x\rvert\sqrt{1-x^2}};\ \ 0<x<1$
$\operatorname{csch}^{-1} x$	$-\dfrac{1}{\lvert x\rvert\sqrt{1+x^2}};\ \ x\neq 0$

4. Trigonometry (Recall):

Remarkable Properties

$\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$; $\cot\alpha = \dfrac{1}{\tan\alpha}$; $\cos^2\alpha = \dfrac{1}{1+\tan^2\alpha}$ and $\sin^2\alpha = \dfrac{\tan^2\alpha}{1+\tan^2\alpha}$

$\sec x = \dfrac{1}{\cos x}$; $\csc x = \dfrac{1}{\sin x}$; $\sec^2 x - \tan^2 x = 1$ and $\csc^2 x - \cot^2 x = 1$

Some Formulas

$\cos(-\alpha) = \cos\alpha$	$\cos(\pi-\alpha) = -\cos\alpha$	$\cos(\pi+\alpha) = -\cos\alpha$
$\sin(-\alpha) = -\sin\alpha$	$\sin(\pi-\alpha) = \sin\alpha$	$\sin(\pi+\alpha) = -\sin\alpha$
$\tan(-\alpha) = -\tan\alpha$	$\tan(\pi-\alpha) = -\tan\alpha$	$\tan(\pi+\alpha) = \tan\alpha$
$\cot(-\alpha) = -\cot\alpha$	$\cot(\pi-\alpha) = -\cot\alpha$	$\cot(\pi+\alpha) = \cot\alpha$

$\cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$	$\cos\left(\frac{\pi}{2}+\alpha\right) = -\sin\alpha$
$\sin\left(\frac{\pi}{2}-\alpha\right) = \cos\alpha$	$\sin\left(\frac{\pi}{2}+\alpha\right) = \cos\alpha$
$\tan\left(\frac{\pi}{2}-\alpha\right) = \cot\alpha$	$\tan\left(\frac{\pi}{2}+\alpha\right) = -\cot\alpha$
$\cot\left(\frac{\pi}{2}-\alpha\right) = \tan\alpha$	$\cot\left(\frac{\pi}{2}+\alpha\right) = -\tan\alpha$

Sum and Difference

$\cos(a+b) = \cos a\cos b - \sin a\sin b$	$\sin(a+b) = \sin a\cos b + \cos a\sin b$
$\cos(a-b) = \cos a\cos b + \sin a\sin b$	$\sin(a-b) = \sin a\cos b - \cos a\sin b$

$$\tan(a+b) = \frac{\tan a + \tan b}{1-\tan a\tan b} \quad\text{and}\quad \tan(a-b) = \frac{\tan a - \tan b}{1+\tan a\tan b}$$

Double and Half Angle Formulas

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1 = 1 - 2\sin^2\alpha$$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$ and

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}.$$

$\cos^2\alpha = \dfrac{1+\cos 2\alpha}{2}$ and

$$\sin^2\alpha = \frac{1-\cos 2\alpha}{2}.$$

$$\cos\alpha = \cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2} - 1 = 1 - 2\sin^2\frac{\alpha}{2}$$

$\sin\alpha = 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$ and

$$\tan\alpha = \frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}.$$

$\cos^2\frac{\alpha}{2} = \dfrac{1+\cos\alpha}{2}$ and

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}.$$

Triple Angle Formulas

$$\sin 3\alpha = 3\sin\alpha - 4\sin^3\alpha \quad \text{and} \quad \cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha.$$

Further Formulas

$$\cos a + \cos b = 2\cos\frac{a+b}{2}.\cos\frac{a-b}{2} \quad \text{and} \quad \cos a - \cos b = -2\sin\frac{a+b}{2}.\sin\frac{a-b}{2}$$

$$\sin a + \sin b = 2\sin\frac{a+b}{2}.\cos\frac{a-b}{2} \quad \text{and} \quad \sin a - \sin b = 2\cos\frac{a+b}{2}.\sin\frac{a-b}{2}$$

$$\tan a + \tan b = \frac{\sin(a+b)}{\cos a\cos b} \quad \text{and} \quad \tan a - \tan b = \frac{\sin(a-b)}{\cos a\cos b}$$

$$\cos a\cos b = \frac{1}{2}[\cos(a+b) + \cos(a-b)]$$

$$\sin a\sin b = \frac{1}{2}[\cos(a-b) - \cos(a-b)]$$

$$\sin a\cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$$

Remark

$$\cos a = \frac{1-\tan^2\frac{a}{2}}{1+\tan^2\frac{a}{2}}, \qquad \sin a = \frac{2\tan\frac{a}{2}}{1+\tan^2\frac{a}{2}} \qquad \text{and} \qquad \tan a = \frac{2\tan\frac{a}{2}}{1-\tan^2\frac{a}{2}}$$

5. Hyperbolic Functions (Recall):

Hyperbolic functions are analogues of the ordinary trigonometric functions, but defined using the hyperbola rather than the circle, they are defined as:

$$\sinh x = \frac{e^x - e^{-x}}{2}; \quad \cosh x = \frac{e^x + e^{-x}}{2}; \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}; \ \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}} \ \& \ \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x + e^{-x}}$$

Some Identities:

- $\cosh x + \sinh x = e^x$; $\cosh x - \sinh x = e^{-x}$ & $\cosh^2 x - \sinh^2 x = 1$
- $\operatorname{sech}^2 x = 1 - \tanh^2 x$ & $\operatorname{csch}^2 x = \coth^2 x - 1$

Sum and Difference of Arguments:

- $\sinh(x-y) = \sinh x\cosh y + \cosh x\sinh y$
- $\cosh(x-y) = \cosh x\cosh y + \sinh x\sinh y$
- $\tanh(x+y) = \frac{\tanh x + \tanh y}{1+\tanh x\tanh y}$
- $\sinh(x-y) = \sinh x\cosh y - \cosh x\sinh y$
- $\cosh(x-y) = \cosh x\cosh y - \sinh x\sinh y$
- $\tanh(x-y) = \frac{\tanh x - \tanh y}{1-\tanh x\tanh y}$

Double and Half of Arguments:

- $\cosh 2x = \cosh^2 x + \sinh^2 x = 2\sinh^2 x + 1 = 2\cosh^2 x - 1$
- $\sinh 2x = 2\sinh x\cosh x$
- $\tan 2x = \frac{2\tanh x}{1+\tanh^2 x}$
- $\sinh\frac{x}{2} = \frac{\sinh x}{\sqrt{2\cosh x + 2}}$; $\cosh\frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$ & $\tanh\frac{x}{2} = \frac{\sinh x}{\cosh x + 1}$

Inverse Hyperbolic Functions:

- $\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right)$
- $\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right); \quad x \geq 1$
- $\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right); \quad |x| < 1$
- $\coth^{-1} x = \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right); \quad |x| > 1$
- $\operatorname{sech}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} - 1}\right); \quad 0 < x \leq 1$
- $\operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \sqrt{\frac{1}{x^2} + 1}\right); \quad x \neq 0$

Formulas for Powers of Hyperbolic Functions:

- $\sinh^2 x = \frac{\cosh 2x - 1}{2}$ and $\cosh^2 x = \frac{\cosh 2x + 1}{2}$
- $\sinh^3 x = \frac{\sinh 3x - 3\sinh x}{2}$ and $\cosh^3 x = \frac{\cosh 3x + 3\cosh x}{2}$
- $\sinh^4 x = \frac{\cosh 4x - 4\cosh 2x + 3}{2}$ and $\cosh^4 x = \frac{\cosh 4x + 4\cosh 2x + 3}{2}$

Solved Exercises

Evaluate each of the following integrals:

1. $\int (3x^2 + 2x - 4)dx$
2. $\int 15\sqrt{x}(1 + x)dx$
3. $\int \frac{2x^3+x^2-1}{x^2} dx$
4. $\int x^2(1 + x)^2 dx$
5. $\int \sqrt{x\sqrt[4]{x^3}} dx$
6. $\int \frac{(x+1)^3}{x^5} dx$
7. $\int |2x - 2| dx$
8. $\int \frac{1}{x\sqrt{x}} dx$
9. $\int \frac{(1+\sqrt{x})^2}{\sqrt{x}} dx$
10. $\int (x^2 + 1)^3 dx$
11. $\int \frac{x\sqrt{x}}{x^{-2}} dx$
12. $\int \left(\frac{1-x}{x}\right)^2 dx$
13. $\int (1 - x)(1 - 2x)(1 - 3x)dx$
14. $\int \left(\frac{x^2-1}{x^2}\right)\sqrt{x\sqrt{x}} dx$
15. $\int x^3 \left(x + \frac{1}{x}\right) dx$
16. $\int \frac{x^3}{x+1} dx$
17. $\int \frac{x}{x^2-1} dx$
18. $\int \frac{2x-1}{(x-1)^2} dx$
19. $\int xe^{-x^2} dx$
20. $\int 4^x e^x dx$
21. $\int \cos x\, e^{\sin x} dx$
22. $\int \sec^2 x\, e^{\tan x} dx$
23. $\int [\ln x^2 - 2\ln(2x)]dx$
24. $\int \cos(2x - 1)\, dx$
25. $\int \sin(3x + 2)\, dx$
26. $\int \cos 3x \cos 2x\, dx$
27. $\int \sin 4x \sin 3x\, dx$
28. $\int \sin 3x \cos 2x\, dx$
29. $\int \sin x \sin(x + a)\, dx$
30. $\int \cos^2 x\, dx$
31. $\int \sin^2 2x\, dx$
32. $\int \tan^2 x\, dx$
33. $\int (\sin x + \cos x)^2\, dx$
34. $\int (2x - 3)^5 dx$
35. $\int \frac{2}{1+\cos 2x} dx$
36. $\int \frac{\arctan x}{1+x^2} dx$
37. $\int \arccos(\sin x) dx$
38. $\int \frac{dx}{(\arcsin x)^2\sqrt{1-x^2}}$
39. $\int \frac{dx}{(2x+1)^5}$

40. $\int \frac{2x-3}{(x^2-3x+4)^4} dx$

41. $\int \sqrt{3x+2} dx$

42. $\int x\sqrt{x^2+1} dx$

43. $\int \tanh x \, dx$

44. $\int x^2 \operatorname{csch} x^3 dx$

45. $\int \cosh^2 x \, dx$

46. $\int \cosh^3 x \, dx$

47. $\int \csc 3x \cot 3x \, dx$

48. $\int \sin^4 x \, dx$

49. $\int (\tan x - \cot x) \, dx$

50. $\int \frac{1+x}{1-x} dx$

51. $\int \frac{x dx}{(x^2+1)^4}$

52. $\int (x+1)(x^2+2x-3)^2 dx$

53. $\int 3(3x-4)^6 dx$

54. $\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3} dx$

55. $\int \frac{x^6-x^3}{(2x^3+1)^3} dx$

56. $\int (2^x+3^x)^2 dx$

57. $\int \frac{2^x}{1+4^x} dx$

58. $\int \left(\frac{2^{x+1}-5^{x-1}}{10^x}\right) dx$

59. $\int \frac{2^x \times 3^x}{9^x-4^x} dx$

60. $\int x^{\frac{x}{\ln x}} dx$

61. $\int \frac{x+1}{\sqrt{x^2+2x+4}} dx$

62. $\int \frac{1+\sin^2 x}{\cos^2 x} dx$

63. $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx$

64. $\int \frac{e^{\tan x}}{\cos^2 x} dx$

65. $\int \frac{x+\cos 6x}{3x^2+\sin 6x} dx$

66. $\int (\cos^4 x - \sin^4 x) \, dx$

67. $\int (\tan x + \tan^3 x) dx$

68. $\int \sin x \cos^4 x \, dx$

69. $\int \sin^2 x \cos^2 x \, dx$

70. $\int \frac{2x^2-3x+2}{x-1} dx$

71. $\int 4^{\ln x^5} dx$

72. $\int \frac{\tan x}{\cos x} dx$

73. $\int \frac{dx}{\sin x}$

74. $\int \frac{dx}{1+\sin 3x}$

75. $\int \frac{dx}{\sin^2(2x+\frac{\pi}{4})}$

76. $\int \frac{dx}{\sin^2 x \cos^2 x}$

77. $\int \frac{dx}{\cos^4 x}$

78. $\int \frac{\sin^2(\frac{x}{2})}{x-\sin x} dx$

79. $\int \frac{2-3\sin x}{1-\sin^2 x}dx$

80. $\int \sqrt{1+\sin x}\,dx$

81. $\int \sin x\sqrt{1+\tan^2 x}\,dx$

82. $\int x^2\sqrt[3]{1+x^3}dx$

83. $\int \sqrt{x^2+x^4}dx$

84. $\int \frac{e^{\sin x}}{\tan x\csc x}dx$

85. $\int \sin x\cos x\,dx$

86. $\int \sin x\sec x\,dx$

87. $\int (\cos x+\sin x)^2\,dx$

88. $\int \frac{\sin x}{(2+\cos x)^2}dx$

89. $\int \frac{1+\sin 2x+\cos 2x}{\sin x+\cos x}dx$

90. $\int \frac{x^2}{x^2+1}dx$

91. $\int \frac{x}{1+x^4}dx$

92. $\int \frac{dx}{1+3x^2}$

93. $\int (\sin^4 x+\cos^4 x)\,dx$

94. $\int \sec^6 x\,dx$

95. $\int (\tan^2 x+\tan^4 x)\,dx$

96. $\int (1+\tan x)\,e^x\sec x\,dx$

97. $\int \frac{dx}{3x^2-2x+2}$

98. $\int \frac{xdx}{\sqrt{1-x^4}}$

99. $\int \frac{dx}{\sqrt{x-x^2}}$

100. $\int \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx$

101. $\int \frac{\cosh x+\sinh x}{\cosh x-\sinh x}dx$

102. $\int \frac{x+1}{x^2+2x+2}dx$

103. $\int \frac{x^2}{x-1}dx$

104. $\int \frac{1}{x^2+(x-1)^2}dx$

105. $\int \frac{1}{x^4+x}dx$

106. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$

107. $\int \frac{\ln x}{x}dx$

108. $\int \frac{dx}{x\ln x}$

109. $\int \frac{e^x+1}{e^x+x}dx$

110. $\int \frac{e^{2x}-1}{e^{2x}+x}dx$

111. $\int \frac{e^{3x}+1}{e^x+1}dx$

112. $\int (x+1)e^{x^2+2x+3}\,dx$

113. $\int \frac{e^{4x}+e^{2x}+3}{e^{2x}}dx$

114. $\int e^x\sqrt{e^x+1}dx$

115. $\int \frac{2e^x}{e^x+2}dx$

116. $\int \frac{e^{\arctan x}}{1+x^2}dx$

117. $\int 2x^{2x}(1+\ln x)dx$

118. $\int \frac{dx}{2+9x^2}$

119. $\int \frac{x^n}{1+x^{2n+2}}dx$

120. $\int \frac{dx}{x^2-4x+7}$

121. $\int \frac{dx}{\sqrt{x+2}+\sqrt{x}}$

122. $\int \sqrt{\tan x}\sec^4 x\, dx$

123. $\int \frac{\sec x+1}{\sin x}dx$

124. $\int \frac{\sec x}{3-4\cos^2 x}dx$

125. $\int \frac{dx}{x^2+x+1}$

126. $\int \frac{dx}{2x^2-8x+10}$

127. $\int \frac{dx}{x+x^5}$

128. $\int \frac{e^{\log x}}{x}dx$

129. $\int \arctan\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx$

130. $\int \arctan\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)dx$

131. $\int \frac{1}{\sqrt{3x-x^2}}dx$

132. $\int \frac{2x-5}{x^2+2x+2}dx$

133. $\int \frac{x^4}{x^2+1}dx$

134. $\int \frac{1}{e^x+e^{-x}}dx$

135. $\int \frac{dx}{x^2-5x+6}$

136. $\int \frac{\cos 2x}{\cos x+\sin x}dx$

137. $\int \frac{1-\cos x}{1+\cos x}dx$

138. $\int \frac{\sqrt{1-\sin x}}{1+\cos x}dx$

139. $\int \frac{\sec 2x-1}{\sec 2x+1}dx$

140. $\int \frac{\sin 2x}{\sin x-\cos x-1}dx$

141. $\int \frac{ae^x+b}{ae^x-b}dx$

142. $\int \frac{dx}{\sqrt{1-\tan^2 x}}$

143. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx$

144. $\int \frac{1}{1+\cot x}dx$

145. $\int \frac{1}{1-\tan x}dx$

146. $\int \frac{dx}{\sin^2 x+2\cos^2 x}$

147. $\int \frac{\sin x}{\sin x-\cos x}dx$

148. $\int \frac{5\sin x}{\sin x-2\cos x}dx$

149. $\int \frac{\cos x}{\sqrt{2+\cos 2x}}dx$

150. $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx$

151. $\int \left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x\, dx$

152. $\int \frac{\cos 4x}{\cos x}dx$

153. $\int \frac{e^x}{e^{1-x}+e^{x-1}}\,dx$

154. $\int \frac{x^3}{3+x}\,dx$

155. $\int \frac{1}{1-\tan^2 x}\,dx$

156. $\int \frac{1}{x-x^{\frac{3}{5}}}\,dx$

157. $\int \frac{1}{1+\cos x}\,dx$

158. $\int \frac{1}{1+\sin x}\,dx$

159. $\int \frac{dx}{\sqrt{\sin x\cos^3 x}}$

160. $\int \frac{1}{1-x^2}\ln\left(\frac{1+x}{1-x}\right)dx$

161. $\int \cos x\,.\cos 2x\,.\cos 3x\,dx$

162. $\int \cot^4 x\,dx$

163. $\int \frac{\sin 2x}{(1-\sin x)^3}\,dx$

164. $\int \sqrt{\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}}\,dx$

165. $\int \sec^5 x\tan^3 x\,dx$

166. $\int \sqrt{\csc x-\sin x}\,dx$

167. $\int \frac{dx}{x\ln x\ln(\ln x)}$

168. $\int \frac{\cos(x+a)}{\sin(x+b)}\,dx$

169. $\int \frac{\sin x}{\sin(x-a)}\,dx$

170. $\int \frac{x-1}{x+x^2\ln x}\,dx$

171. $\int \frac{3e^{-2x}+5e^{3x}}{e^{-2x}+e^{3x}}\,dx$

172. $\int \frac{\tanh x}{1+e^x}\,dx$

173. $\int \frac{dx}{x^{\frac{41}{25}}+x^{\frac{9}{25}}}$

174. $\int \frac{(2+x)e^{-x}}{x^3}\,dx$

175. $\int \frac{2}{(\cos x-\sin x)^2}\,dx$

176. $\int \sqrt{\frac{x}{1-x^3}}\,dx$

177. $\int \sqrt{\frac{x}{1+x^3}}\,dx$

178. $\int \sqrt{1-\sin 2x}\,dx$

179. $\int \frac{(\ln x)^2-1}{x(\ln x)^2}\,dx$

180. $\int \sin x\sec x\tan x\,dx$

181. $\int \frac{1}{\csc^3 x}\,dx$

182. $\int \sqrt{\frac{1-\sin 2x}{1+\cos 2x}}\,dx$

183. $\int \frac{x^5-x^3+x^2-1}{x^4-x^3+x-1}\,dx$

184. $\int \frac{\tan(\arctan x-\arctan 2x)}{x}\,dx$

185. $\int \frac{1-\tan x}{1+\tan x}\,dx$

186. $\int \frac{1}{\sqrt{x}(1+x)}\,dx$

187. $\int \frac{1}{1-\cos nx}\,dx$

188. $\int \frac{x^6-1}{x^2+1}\,dx$

189. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$

190. $\int \frac{\cos 7x-\cos 8x}{1+2\cos 5x}dx$

191. $\int \frac{\cos x-\sin x}{(1+\sin 2x)^8}dx$

192. $\int \frac{x}{\sqrt{x^2+x+1}}dx$

193. $\int \frac{1}{1+a^x}dx$

194. $\int \left(\frac{x}{e}\right)^x \ln x\, dx$

195. $\int \frac{x^3+4x^2-2x-5}{x^2-4x+4}dx$

196. $\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx$

197. $\int \frac{1}{\sin x+\cos x}dx$

198. $\int (4-x^2)(2+x)^n dx$

199. $\int \left(\sqrt{\frac{4-x}{x}}-\sqrt{\frac{x}{4-x}}\right)dx$

200. $\int \frac{dx}{1+e^x+e^{2x}+e^{3x}}$

201. $\int \frac{\tan x+\sec x-1}{\tan x-\sec x+1}dx$

202. $\int \frac{1}{(x-1)(x-2)(x-3)(x-4)}dx$

203. $\int \sqrt{x\sqrt{x\sqrt{x\sqrt{x\ldots}}}}\,dx$

204. $\int \frac{1-\sin 2x}{1+\sin 2x}dx$

205. $\int \tan x \tan 2x\, dx$

206. $\int \frac{\csc x}{2\cos^2 x+\cos 2x}dx$

207. $\int \sin x \cos x \cos 2x \cos 4x\, dx$

208. $\int \frac{2000x^{2014}+14}{x^{2015}-x}dx$

209. $\int \frac{dx}{4\sin^2 x+5\cos^2 x}$

210. $\int \frac{\sin x\cos x}{\sqrt{a^2\sin^2 x+b^2\cos^2 x}}dx$

211. $\int \frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx$

212. $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx$

213. $\int \frac{x^2}{(1-x)^{100}}dx$

214. $\int \frac{(x^2+\sin^2 x)}{1+x^2}.\sec^2 x\, dx$

215. $\int \tan x \tan 2x \tan 3x\, dx$

216. $\int \frac{\cos x-\sin x}{\sqrt{1-\sin x}+\sqrt{1+\sin x}}dx$

217. $\int \frac{1}{\sqrt{2}+\sin x-\cos x}dx$

218. $\int \frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx$

219. $\int \frac{dx}{\sqrt{1+\arcsin x-x^2-x^2\arcsin x}}$

220. $\int \sin^2\left(\arctan\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx$

221. $\int \sin(\ln x)\, dx$

222. $\int \frac{1}{\tan x+\cot x+\sec x+\csc x}dx$

223. $\int \frac{\tan 2x}{\sqrt{\sin^4 x+4\cos^2 x}-\sqrt{\cos^4 x+4\sin^2 x}}dx$

224. $\int \sin^4 x \cos^4 x\,(\cos x+\sin x)(\cos x-\sin x)dx$

225. $\int \sqrt{(x+3)(x+4)(x+5)(x+6)+1}\,dx$

Solutions of Exercises

1. $\int(3x^2+2x-4)dx = 3\left(\frac{x^3}{3}\right)+2\left(\frac{x^2}{2}\right)-4x+c = x^3+x^2-4x+c$

2. $\int 15\sqrt{x}(1+x)dx = \int\left(15\sqrt{x}+15x\sqrt{x}\right)dx = 15\left(\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}\right)+15\left(\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1}\right)+c$

$= 15\left(\frac{2}{3}x^{\frac{3}{2}}\right)+15\left(\frac{2}{5}x^{\frac{5}{2}}\right)+c = 10x\sqrt{x}+6x^2\sqrt{x}+c$

3. $\int\frac{2x^3+x^2-1}{x^2}dx = \int\left(\frac{2x^3}{x^2}+\frac{x^2}{x^2}-\frac{1}{x^2}\right)dx = \int(2x+1-x^{-2})dx = 2\left(\frac{x^2}{2}\right)+x-\frac{x^{-2+1}}{-2+1}+c$

$= x^2+x+x^{-1}+c = x^2+x+\frac{1}{x}+c$

4. $\int x^2(1+x)^2dx = \int x^2(1+2x+x^2)dx = \int(x^2+2x^3+x^4)dx = \frac{x^3}{3}+2\left(\frac{x^4}{4}\right)+\frac{x^5}{5}+c$

$= \frac{x^3}{3}+\frac{x^4}{2}+\frac{x^5}{5}+c.$

5. $\int\sqrt{x\sqrt[4]{x^3}}dx = \int\sqrt{x.x^{\frac{3}{4}}}dx = \int\sqrt{x^{1+\frac{3}{4}}}\,dx = \int\sqrt{x^{\frac{7}{4}}}dx = \int x^{\frac{7}{8}}dx = \frac{x^{\frac{15}{8}}}{\frac{15}{8}}+c = \frac{8}{15}x^{\frac{15}{8}}+c$

6. $\int\frac{(x+1)^3}{x^5}dx = \int\frac{x^3+3x^2+3x+1}{x^5}dx = \int\left(\frac{x^3}{x^5}+\frac{3x^2}{x^5}+\frac{3x}{x^5}+\frac{1}{x^5}\right)dx$

$= \int(x^{-2}+3x^{-3}+3x^{-4}+x^{-5})dx = \frac{x^{-1}}{-1}+3\frac{x^{-2}}{-2}+3\frac{x^{-3}}{-3}++\frac{x^{-4}}{-4}+c$

$= -\frac{1}{x}-\frac{3}{2x^2}-\frac{1}{x^3}-\frac{1}{4x^4}+c.$

7. $\int|2x-2|dx$, we have $|2x-2| = \begin{cases}2x-2 & \text{if } x\geq 0\\ -2x+2 & \text{if } x<0\end{cases}$, so:

$\int|2x-2|dx = \begin{cases}x^2-2x & \text{if } x\geq 0\\ -x^2+2x & \text{if } x<0\end{cases}$

8. $\int\frac{1}{x\sqrt{x}}dx = \int x^{-\frac{3}{2}}dx = \frac{x^{-\frac{3}{2}+1}}{-\frac{3}{2}+1}+c = -2x^{-\frac{1}{2}}+c = -\frac{2}{\sqrt{x}}+c.$

9. $\int\frac{(1+\sqrt{x})^2}{\sqrt{x}}dx = \int\frac{1+2\sqrt{x}+x}{\sqrt{x}}dx = \int\left(\frac{1}{\sqrt{x}}+2+\frac{x}{\sqrt{x}}\right)dx = \int\left(x^{-\frac{1}{2}}+2+x^{\frac{1}{2}}\right)dx$

$= \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+2x+\frac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1}+c = 2\sqrt{x}+2x+\frac{2x\sqrt{x}}{3}+c.$

10. $\int(x^2+1)^3dx = \int(x^6+3x^4+3x^2+1)dx = \frac{x^7}{7}+\frac{3x^5}{5}+x^3+x+c.$

11. $\int\frac{x\sqrt{x}}{x^{-2}}dx = \int x^{\frac{3}{2}}x^2dx = \int x^{\frac{7}{2}}dx = \frac{x^{\frac{7}{2}+1}}{\frac{7}{2}+1}+c = \frac{x^{\frac{9}{2}}}{\frac{9}{2}}+c = \frac{2x^4\sqrt{x}}{9}+c$

12. $\int\left(\frac{1-x}{x}\right)^2dx = \int\left(\frac{1}{x}-1\right)^2dx = \int\left(\frac{1}{x^2}-\frac{2}{x}+1\right)dx = -\frac{1}{x}-2\ln|x|+x+c$

13. $\int(1-x)(1-2x)(1-3x)dx = \int(1-6x+11x^2-6x^3)\,dx = x-3x^2+\frac{11}{3}x^3-\frac{3}{2}x^4+c$

14. $\int\left(\frac{x^2-1}{x^2}\right)\sqrt{x\sqrt{x}}dx=\int\left(1-\frac{1}{x^2}\right)x^{\frac{1}{2}}x^{\frac{1}{4}}dx=\int\left(1-\frac{1}{x^2}\right)x^{\frac{3}{4}}dx=\int\left(x^{\frac{3}{4}}-x^{-\frac{5}{4}}\right)dx$

$=\frac{4}{7}x^{\frac{7}{4}}+4x^{-\frac{1}{4}}+c=\frac{4(x^2+7)}{7\sqrt[4]{x}}+c$

15. $\int x^3\left(x+\frac{1}{x}\right)dx=\int\left[x\left(x+\frac{1}{x}\right)\right]^3dx=\int(x^2+1)^3dx=\int(x^6+3x^4+3x^2+1)dx$

$=\frac{1}{7}x^7+\frac{3}{5}x^5+x^3+x+c$

16. $\int\frac{x^3}{x+1}dx=\int\frac{x^3+1-1}{x+1}dx=\int\left(\frac{x^3+1}{x+1}-\frac{1}{x+1}\right)dx=\int\left[\frac{(x+1)(x^2-x+1)}{x+1}-\frac{1}{x+1}\right]dx$

$=\int\left(x^2-x+1-\frac{1}{x+1}\right)dx=\frac{1}{3}x^3-\frac{1}{2}x^2+x-\ln|x+1|+c$

17. $\int\frac{x}{x^2-1}dx=\frac{1}{2}\int\frac{2x}{x^2-1}dx=\frac{1}{2}\int\frac{(x^2-1)'}{x^2-1}dx=\frac{1}{2}\ln|x^2-1|+c$

18. $\int\frac{2x-1}{(x-1)^2}dx=\int\frac{2x-1-1+1}{(x-1)^2}dx=\int\frac{2x-2+1}{(x-1)^2}dx=\int\frac{2(x-1)+1}{(x-1)^2}dx=\int\left[\frac{2}{x-1}+\frac{1}{(x-1)^2}\right]dx$

$=2\ln|x-1|-\frac{1}{x-1}+c$

19. $\int xe^{-x^2}dx=-\frac{1}{2}\int-2xe^{-x^2}dx=-\frac{1}{2}\int(-x^2)'e^{-x^2}dx=-\frac{1}{2}e^{-x^2}+c$

20. $\int 4^xe^xdx=\int(4e)^xdx=\frac{1}{\ln(4e)}(4e)^x+c$

21. $\int\cos x\,e^{\sin x}dx=\int(\sin x)'\,e^{\sin x}dx=e^{\sin x}+c$

22. $\int\sec^2 x\,e^{\tan x}dx=\int(\tan x)'e^{\tan x}dx=e^{\tan x}+c$

23. $\int[\ln x^2-2\ln(2x)]dx=\int[2\ln x-2(\ln 2+\ln x)]dx=\int(2\ln x-\ln 4-2\ln x)dx$

$=\int-\ln 4\,dx=-x\ln 4+c$

24. $\int\cos(2x-1)\,dx=\frac{1}{2}\sin(2x-1)+c$

25. $\int\sin(3x+2)\,dx=-\frac{1}{3}\cos(3x+2)+c$

26. $\int\cos 3x\cos 2x\,dx=\int\frac{1}{2}[\cos(3x+2x)+\cos(3x-2x)]dx=\frac{1}{2}\int(\cos 5x+\cos x)dx$

$=\frac{1}{2}\left(\frac{1}{5}\sin 5x+\sin x\right)+c$

27. $\int\sin 4x\sin 3x\,dx=\int-\frac{1}{2}[\cos(4x+3x)-\cos(4x-3x)]dx=-\frac{1}{2}\int(\cos 7x-\cos x)dx$

$=-\frac{1}{2}\left(\frac{1}{7}\sin 7x-\sin x\right)+c$

28. $\int\sin 3x\cos 2x\,dx=\int\frac{1}{2}[\sin(3x+2x)+\sin(3x-2x)]dx=\frac{1}{2}\int(\sin 5x+\sin x)dx$

$=\frac{1}{2}\left(-\frac{1}{5}\cos 5x-\cos x\right)+c$

29. $\int\sin x\sin(x+a)\,dx=\frac{1}{2}\int[\cos a-\cos(2x+a)]dx=\frac{1}{2}x\cos a-\frac{1}{4}\sin(2x+a)+c$

30. $\int\cos^2 x\,dx=\int\frac{1}{2}(1+\cos 2x)dx=\frac{1}{2}\left(x+\frac{1}{2}\sin 2x\right)+c$

31. $\int\sin^2 2x\,dx=\int\frac{1}{2}(1-\cos 4x)dx=\frac{1}{2}\left(x-\frac{1}{4}\sin 4x\right)+c$

32. $\int\tan^2 x\,dx=\int[-1+(1+\tan^2 x)]dx=-x+\tan x+c$

33. $\int (\sin x + \cos x)^2\, dx = \int (\sin^2 x + 2\sin x\cos x + \cos^2 x)\, dx = \int (1 + \sin 2x)\, dx$
$= x - \frac{1}{2}\cos 2x + c$

34. $\int (2x-3)^5 dx = \frac{1}{2}\left[\frac{(2x-3)^6}{6}\right] + c = \frac{(2x-3)^6}{12} + c$

35. $\int \frac{2}{1+\cos 2x} dx = \int \frac{2}{2\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx = \int \sec^2 x\, dx = \tan x + c$

36. $\int \frac{\arctan x}{1+x^2} dx = \int \frac{1}{1+x^2} \arctan x\, dx = \int \arctan x\, (\arctan x)' dx = \frac{1}{2}(\arctan x)^2 + c$

37. $\int \arccos(\sin x) dx = \int \arccos\left[\cos\left(\frac{\pi}{2} - x\right)\right] dx = \int \left(\frac{\pi}{2} - x\right) dx = \frac{\pi}{2}x - \frac{1}{2}x^2 + c$

38. $\int \frac{dx}{(\arcsin x)^2\sqrt{1-x^2}} = \int \frac{1}{(\arcsin x)^2}(\arcsin x\,')dx = -\frac{1}{\arcsin x} + c$

39. $\int \frac{dx}{(2x+1)^5} = \int (2x+1)^{-5} dx = \frac{1}{2}\left[\frac{(2x+1)^{-4}}{-4}\right] + c = -\frac{(2x+1)^{-4}}{8} + c$

40. $\int \frac{2x-3}{(x^2-3x+4)^4} dx = \int (2x-3)(x^2-3x+4)^{-4} dx = \int (x^2-3x+4)'(x^2-3x+4)^{-4} dx$
$= \frac{(x^2-3x++4)^{-3}}{-3} + c = -\frac{(x^2-3x++4)^{-3}}{3} + c$

41. $\int \sqrt{3x+2} dx = \frac{1}{3}\left[\frac{(\sqrt{3x+2})^{\frac{3}{2}}}{\frac{3}{2}}\right] + c = \frac{2}{9}(3x+2)\sqrt{3x+2} + c$

42. $\int x\sqrt{x^2+1} dx = \frac{1}{2}\int (x^2+1)'\sqrt{x^2+1} dx = \frac{1}{2}\left[\frac{\left(\sqrt{x^2+1}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right] + c = \frac{1}{3}(x^2+1)\sqrt{x^2+1} + c$

43. $\int \tanh x\, dx = \int \frac{\sinh x}{\cosh x} dx = \int \frac{(\cosh x)'}{\cosh x} dx = \ln(\cosh x) + c$

44. $\int x^2 \operatorname{csch} x^3 dx = \frac{1}{3}\int \operatorname{csch} x^3 (3x^2) dx = -\frac{1}{3}\coth x^3 + c$

45. $\int \cosh^2 x\, dx = \int \frac{1}{2}(\cosh 2x + 1) dx = \frac{1}{4}\sinh 2x + \frac{1}{2}x + c$

46. $\int \cosh^2 x \cosh x\, dx = \int (1 + \sinh^2 x)(\sinh x)' dx = \sinh x + \frac{1}{3}\sinh^3 x + c$

47. $\int \csc 3x \cot 3x\, dx = -\frac{1}{3}\int -3\csc 3x \cot 3x\, dx = -\frac{1}{3}\int (\csc 3x)' dx = -\frac{1}{3}\csc 3x + c$

48. $\int \sin^4 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx = \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x)$
$= \frac{1}{4}\int \left(1 - 2\cos 2x + \frac{1+\cos 4x}{2}\right) = \frac{1}{8}\int (3 - 4\cos 2x + \cos 4x)$
$= \frac{3}{8}x - \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + c$

49. $\int (\tan x - \cot x)\, dx = \int (\tan^2 x - 2\tan x \cot x + \cot^2 x)\, dx = \int (\tan^2 x - 2 + \cot^2 x)\, dx$
$= \int [(1 + \tan^2 x) + (1 + \cot^2 x) - 4]\, dx = \tan x - \cot x - 4x + c$

50. $\int \frac{1+x}{1-x} dx = \int \frac{1+x-2+2}{1-x} dx = \int \frac{-(1-x)+2}{1-x} dx = \int \left(-1 + \frac{2}{1-x}\right) dx = -x - 2\ln|1-x| + c$

51. $\int \frac{xdx}{(x^2+1)^4} = \int x(x^2+1)^{-4}dx = \frac{1}{2}\left[\frac{(x^2+1)^{-3}}{-3}\right] + c = -\frac{1}{6}(x^2+1)^{-3} + c$

52. $\int (x+1)(x^2+2x-3)^2 dx = \frac{1}{2}\left[\frac{(x^2+2x-3)^3}{3}\right] + c = \frac{(x^2+2x-3)^3}{6} + c$

53. $\int 3(3x-4)^6 dx = \frac{(3x-4)^7}{7} + c$

54. $\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3}dx = \int \frac{\sqrt{(x^2+x^{-2})^2}}{x^3}dx = \int \frac{x^2+\frac{1}{x^2}}{x^3}dx = \int \left(\frac{1}{x}+\frac{1}{x^5}\right)dx = \ln|x| - \frac{1}{4x^4} + c$

55. $\int \frac{x^6-x^3}{(2x^3+1)^3}dx = \int \frac{\frac{x^6-x^3}{x^6}}{\frac{(2x^3+1)^3}{x^6}}dx = \int \frac{1-\frac{1}{x^3}}{\left(\frac{2x^3+1}{x^2}\right)^3}dx = \int \frac{1-\frac{1}{x^3}}{\left(2x+\frac{1}{x^2}\right)^3}dx = \frac{1}{2}\int \frac{2-\frac{2}{x^3}}{\left(2x+\frac{1}{x^2}\right)^3}dx$

$= \frac{1}{2}\int \frac{\left(2x+\frac{1}{x^2}\right)'}{\left(2x+\frac{1}{x^2}\right)^3}dx = -\frac{1}{4\left(2x+\frac{1}{x^2}\right)^2} + c$

56. $\int (2^x+3^x)^2 dx = \int (4^x + 2\times 6^x + 9^x)dx = \frac{1}{\ln 4}4^x + \frac{2}{\ln 6}6^x + \frac{1}{\ln 9}9^x + c$

57. $\int \frac{2^x}{1+4^x}dx = \frac{1}{\ln 2}\int \frac{(\ln 2)2^x}{1+(2^x)^2}dx = \frac{1}{\ln 2}\int \frac{(2^x)'}{1+(2^x)^2}dx = \frac{1}{\ln 2}\arctan 2^x + c$

58. $\int \left(\frac{2^{x+1}-5^{x-1}}{10^x}\right)dx = \int \left(\frac{2^x\times 2-5^x\times\frac{1}{5}}{10^x}\right)dx = \int \left[2\left(\frac{1}{5}\right)^x - \frac{1}{5}\left(\frac{1}{2}\right)^x\right]dx$

$= -\frac{2}{\ln 5}\left(\frac{1}{5}\right)^x + \frac{1}{5\ln 2}\left(\frac{1}{2}\right)^2 + c$

59. $\int \frac{2^x\times 3^x}{9^x+4^x}dx = \int \frac{\frac{2^x\times 3^x}{4^x}}{\frac{9^x+4^x}{4^x}}dx = \int \frac{\left(\frac{3}{2}\right)^x}{\left[\left(\frac{3}{2}\right)^x\right]^2+1}dx = \frac{1}{\ln\left(\frac{3}{2}\right)}\int \frac{\ln\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)^x}{\left[\left(\frac{3}{2}\right)^x\right]^2+1}dx = \frac{1}{\ln\left(\frac{3}{2}\right)}\int \frac{\left[\left(\frac{3}{2}\right)^x\right]'}{\left[\left(\frac{3}{2}\right)^x\right]^2+1}dx$

$= \frac{1}{\ln 3 - \ln 2}\arctan\left(\frac{3}{2}\right)^x + c$

60. $\int x^{\frac{x}{\ln x}}dx = \int e^{\ln\left(x^{\frac{x}{\ln x}}\right)}dx = \int e^{\frac{x}{\ln x}\ln x}dx = \int e^x dx = e^x + c = x^{\frac{x}{\ln x}} + c$

61. $\int \frac{x+1}{\sqrt{x^2+2x+4}}dx = \int (x+1)(x^2+2x+4)^{-\frac{1}{2}}dx = \frac{1}{2}\left[\frac{(x^2+2x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right] + c$

$= \sqrt{x^2+2x+4} + c$

62. $\int \frac{1+\sin^2 x}{\cos^2 x}dx = \int \left(\frac{1}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\right)dx = \int (\sec^2 x + \tan^2 x)dx = \int (\sec^2 x + \sec^2 x - 1)dx$

$= \int (2\sec^2 x - 1)dx = 2\tan x - x + c$

63. $\int \frac{\cos 2x}{\sin^2 x \cos^2 x} dx = \int \frac{\cos^2 x - \sin^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{1}{\sin^2 x} - \frac{1}{\cos^2 x}\right) dx = \int (\csc^2 x - \sec^2 x) dx$
$= -\cot x - \tan x + c$

64. $\int \frac{e^{\tan x}}{\cos^2 x} dx = \int \sec^2 x\, e^{\tan x} dx = \int (\tan x)' e^{\tan x} dx = e^{\tan x} + c$

65. $\int \frac{x+\cos 6x}{3x^2+\sin 6x} dx = \frac{1}{6}\int \frac{6x+6\cos 6x}{3x^2+\sin 6x} dx = \frac{1}{6}\int \frac{(3x^2+\sin 6x)'}{3x^2+\sin 6x} dx = \frac{1}{6}\ln|3x^2 + \sin 6x| + c$

66. $\int (\cos^4 x - \sin^4 x)\, dx = \int (\cos^2 x + \sin^2 x)(\cos^2 x - \sin^2 x) dx = \int (\cos^2 x - \sin^2 x) dx$
$= \int \cos 2x\, dx = \frac{1}{2}\sin 2x + c$

67. $\int (\tan x + \tan^3 x) dx = \int (1 + \tan^2 x) \tan x\, dx = \int (\tan x)' \tan x\, dx = \frac{1}{2}\tan^2 x + c$

68. $\int \sin x \cos^4 x\, dx = -\int (-\cos x)' \cos^4 x\, dx = -\frac{1}{5}\cos^5 x + c$

69. $\int \sin^2 x \cos^2 x\, dx = \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx = \frac{1}{4}\int (1 - \cos^2 2x) dx = \frac{1}{4}\int \sin^2 2x$
$= \frac{1}{4}\int \left(\frac{1-\cos 4x}{2}\right) dx = \frac{1}{8}\int (1 - \cos 4x) dx = \frac{1}{2}\left(x - \frac{1}{4}\sin 4x\right) + c$

70. $\int \frac{2x^2-3x+2}{x-1} dx = \int \frac{2x^2-3x+1+1}{x-1} dx = \int \frac{(2x-1)(x-1)+1}{x-1} dx = \int \left(2x - 1 + \frac{1}{x-1}\right) dx$
$= x^2 - x + \ln|x-1| + c$

71. $\int 4^{\ln x^5} dx = \int \left(e^{\ln 4}\right)^{\ln x^5} dx = \int \left(e^{\ln x^5}\right)^{\ln 4} dx = \int x^{5\ln 4} dx = \int x^{\ln 4^5} dx = \int x^{\ln 1024} dx$
$= \frac{x^{\ln 024+1}}{\ln 1024+1} + c$

72. $\int \frac{\tan x}{\cos x} dx = \int \frac{\frac{\sin x}{\cos x}}{\cos x} dx = \int \frac{\sin x}{\cos^2 x} dx = \frac{(\cos x)^{-1}}{-1} + c = -\frac{1}{\cos x} + c$

73. $\int \frac{dx}{\sin x} = \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \int \frac{\frac{1}{\cos^2\frac{x}{2}}}{\frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}}} dx = \int \frac{\sec^2 x}{2\tan\frac{x}{2}} dx = \int \frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}} dx = \int \frac{(\tan\frac{x}{2})'}{\tan\frac{x}{2}} dx$
$= \ln\left|\tan\left(\frac{x}{2}\right)\right| + c$

74. $\int \frac{dx}{1+\sin 3x} = \int \frac{dx}{1+\cos(\frac{\pi}{2}-3x)} = \int \frac{dx}{2\cos^2\left[\frac{1}{2}\left(\frac{\pi}{4}-\frac{3x}{2}\right)\right]} = \int \frac{dx}{2\cos^2\left(\frac{\pi}{4}-\frac{3x}{2}\right)} = \frac{1}{2}\int \sec^2\left(\frac{\pi}{4} - \frac{3x}{2}\right) dx$
$= -\frac{2}{3} \times \frac{1}{2}\tan\left(\frac{\pi}{4} - \frac{3x}{2}\right) + c = -\frac{1}{3}\tan\left(\frac{\pi}{4} - \frac{3x}{2}\right) + c$

75. $\int \frac{dx}{\sin^2(2x+\frac{\pi}{4})} = \int \csc^2\left(2x + \frac{\pi}{4}\right) dx = -\frac{1}{2}\cot\left(2x + \frac{\pi}{4}\right) + c$

76. $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x}\right) dx = \int (\sec^2 x + \csc^2 x) dx$
$= \tan x - \cot x + c$

77. $\int \frac{dx}{\cos^4 x} = \int \frac{1}{\cos^2 x}\frac{1}{\cos^2 x} dx = \int \sec^2 x \sec^2 x\, dx = \int (1 + \tan^2 x)(\tan x)'\, dx$
$= \tan x + \frac{1}{3}\tan^3 x + c$

78. $\int \frac{\sin^2(\frac{x}{2})}{x-\sin x} dx = \int \frac{\frac{1-\cos x}{2}}{x-\sin x} dx = \frac{1}{2}\int \frac{1-\cos x}{x-\sin x} dx = \frac{1}{2}\int \frac{(x-\sin x)'}{x-\sin x} dx = \frac{1}{2}\ln|x - \sin x| + c$

79. $\int \frac{2-3\sin x}{1-\sin^2 x}dx = \int \frac{2-3\sin x}{\cos^2 x}dx = \int \left(\frac{2}{\cos^2 x} - \frac{3\sin x}{\cos x \cos x}\right)dx = \int (2\sec^2 x - 3\sec x \tan x)dx$
$= 2\tan x - 3\sec x + c$

80. $\int \sqrt{1+\sin x}\,dx = \int \sqrt{\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right) + 2\sin\frac{x}{2}\cos\frac{x}{2}}\,dx = \int \sqrt{\left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)^2}$
$= \int \left(\sin\frac{x}{2} + \cos\frac{x}{2}\right)dx = -2\cos\frac{x}{2} + 2\sin\frac{x}{2} + c$

81. $\int \sin x\sqrt{1+\tan^2 x}\,dx = \int \sin x\sqrt{\sec^2 x}\,dx = \int \sin x \sec x\,dx = \int \frac{\sin x}{\cos x}dx$
$= -\int \frac{(\cos x)'}{\cos x}dx = -\ln|\cos x| + c$

82. $\int x^2\sqrt[3]{1+x^3}dx = \frac{1}{3}\int 3x^2\sqrt[3]{1+x^3}dx = \frac{1}{3}\int \sqrt[3]{1+x^3}(1+x^3)'dx = \frac{1}{4}(1+x^3)^{\frac{4}{3}} + c$

83. $\int \sqrt{x^2+x^4}dx = \int \sqrt{x^2(1+x^2)}dx = \int x\sqrt{1+x^2}dx = \frac{1}{2}\int 2x\sqrt{1+x^2}dx$
$= \frac{1}{2}\int (1+x^2)'\sqrt{1+x^2}dx = \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)(1+x^2)^{\frac{3}{2}} + c = \frac{1}{3}(1+x^2)^{\frac{3}{2}} + c$

84. $\int \frac{e^{\sin x}}{\tan x \csc x}dx = \int \frac{e^{\sin x}}{\frac{\sin x}{\cos x}\frac{1}{\sin x}}dx = \int \cos x\, e^{\sin x}dx = \int (\sin x)'e^{\sin x}dx = e^{\sin x} + c$

85. $\int \sin x \cos x\,dx = \frac{1}{2}\int \sin 2x\,dx = \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + c = -\frac{\cos 2x}{4} + c$

86. $\int \sin x \sec x\,dx = \int \frac{\sin x}{\cos x}dx = -\int \frac{(\cos x)'}{\cos x}dx = -\ln|\cos x| + c$

87. $\int (\cos x + \sin x)^2\,dx = \int (\cos^2 x + 2\sin x\cos x + \sin^2 x)dx = \int (1+\sin 2x)dx$
$= x - \frac{1}{2}\cos 2x + c$

88. $\int \frac{\sin x}{(2+\cos x)^2}dx = -\int \frac{-(2+\cos x)'}{(2+\cos x)^2}dx = -\frac{1}{2+\cos x} + c$

89. $\int \frac{1+\sin 2x+\cos 2x}{\sin x+\cos x}dx = \int \frac{2\cos^2 x+2\sin x\cos x}{\sin x+\cos x}dx = \int \frac{2\cos x(\sin x+\cos x)}{\sin x+\cos x}dx = \int 2\cos x\,dx$
$= 2\sin x + c$

90. $\int \frac{x^2}{x^2+1}dx = \int \frac{x^2+1-1}{x^2+1}dx = \int \left(1 - \frac{1}{x^2+1}\right)dx = x - \arctan x + \text{c}$

91. $\int \frac{x}{1+x^4}dx = \int \frac{x}{1+(x^2)^2}dx = \frac{1}{2}\int \frac{(x^2)'dx}{1+(x^2)^2} = \frac{1}{2}\arctan x^2 + c.$

92. $\int \frac{dx}{1+3x^2} = \int \frac{dx}{1+(x\sqrt{3})^2} = \frac{1}{\sqrt{3}}\arctan x\sqrt{3} + \text{c}$

93. $\int (\sin^4 x + \cos^4 x)dx = \int (\sin^4 x + \cos^4 x + 2\sin^2 x\cos^2 x - 2\sin^2 x\cos^2 x)dx$
$= \int [(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x]dx = \int \left[1 - \frac{1}{2}(2\sin x\cos x)^2\right]dx$
$= \int \left(1 - \frac{1}{2}\sin^2 2x\right)dx = \int \left[1 - \frac{1}{2}\left(\frac{1-\cos 4x}{2}\right)\right]dx = \int \left(\frac{3}{4} + \frac{1}{4}\cos 4x\right)dx$
$= \frac{3}{4}x + \frac{1}{16}\sin 4x + c$

94. $\int \sec^6 x\,dx = \int \sec^4 x\sec^2 x\,dx = \int (\sec^2 x)^2\sec^2 x\,dx = \int (\tan^2 x + 1)^2\sec^2 x\,dx$
$= \int (\tan^4 x + 2\tan^2 x + 1)\sec^2 x\,dx$
$= \int \tan^4 x\sec^2 x\,dx + \int 2\tan^2 x\sec^2 x\,dx + \int \sec^2 x\,dx$

$= \int \tan^4 x\,(\tan x)'dx + \int 2\tan^2 x\,(\tan x)'dx + \int (\tan x)'dx$

$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + c$

95. $\int (\tan^2 x + \tan^4 x)\,dx = \int \tan^2 x\,(1+\tan^2 x)\,dx = \int \tan^2 x \sec^2 x\,dx$

$= \int \tan^2 x\,(\tan x)'\,dx = \frac{1}{3}\tan^3 x + c$

96. $\int (1+\tan x)\,e^x \sec x\,dx = \int (\sec x + \sec x \tan x)\,e^x dx = \int [\sec x + (\sec x)']\,e^x dx$

$= \int [f(x) + f'(x)]e^x dx = f(x)e^x + c = e^x \sec x + c$

97. $\int \frac{dx}{3x^2-2x+2} = \frac{1}{3}\int \frac{dx}{x^2-\frac{2}{3}x+\frac{2}{3}} = \frac{1}{3}\int \frac{dx}{\left(x^2-\frac{2}{3}x+\frac{1}{9}\right)+\frac{2}{3}-\frac{1}{9}} = \frac{1}{3}\int \frac{dx}{\left(x-\frac{1}{3}\right)^2+\frac{5}{9}} = \frac{2}{5}\int \frac{dx}{1+\left(\frac{3x-1}{\sqrt{5}}\right)^2}$

$= \frac{3}{5} \times \frac{\sqrt{5}}{3}\arctan\left(\frac{3x-1}{\sqrt{5}}\right) + \mathrm{c} = \frac{1}{\sqrt{5}}\arctan\left(\frac{3x-1}{\sqrt{5}}\right) + \mathrm{c}$

98. $\int \frac{xdx}{\sqrt{1-x^4}} = \int \frac{xdx}{\sqrt{1-(x^2)^2}} = \frac{1}{2}\int \frac{(x^2)'dx}{\sqrt{1-(x^2)^2}} = \frac{1}{2}\arcsin x^2 + \mathrm{c}.$

99. $\int \frac{dx}{\sqrt{x-x^2}} = \int \frac{dx}{\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}} = 2\int \frac{dx}{\sqrt{1-4\left(x-\frac{1}{2}\right)^2}} = 2\int \frac{dx}{\sqrt{1-(2x-1)^2}} = \frac{2}{2}\arcsin(2x-1) + \mathrm{c},$

therefore $\int \frac{dx}{\sqrt{x-x^2}} = \arcsin(2x-1) + \mathrm{c}.$

100. $\int \sqrt{1+\left(x-\frac{1}{4x}\right)^2}\,dx = \int \sqrt{1+x^2-\frac{1}{2}+\frac{1}{16x^2}}\,dx = \int \sqrt{x^2+\frac{1}{2}+\frac{1}{16x^2}}\,dx$

$= \int \sqrt{\left(x+\frac{1}{4x}\right)^2}\,dx = \int \left(x+\frac{1}{4x}\right)dx = \frac{1}{2}x^2 + \frac{1}{4}\ln|x| + c$

101. $\int \frac{\cosh x+\sinh x}{\cosh x-\sinh x}dx = \int \frac{\frac{e^x+e^{-x}}{2}+\frac{e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}}{2}-\frac{e^x-e^{-x}}{2}}dx = \int \frac{\frac{e^x+e^{-x}+e^x-e^{-x}}{2}}{\frac{e^x+e^{-x}-e^x+e^{-x}}{2}}dx = \int \frac{e^x}{e^{-x}}dx$

$= \int e^{2x}\,dx = \frac{1}{2}e^{2x} + c$

102. $\int \frac{x+1}{x^2+2x+2}dx = \frac{1}{2}\int \frac{2x+2}{x^2+2x+2}dx = \frac{1}{2}\int \frac{(x^2+2x+2)'}{x^2+2x+2}dx = \frac{1}{2}\ln|x^2+2x+2| + c$

103. $\int \frac{x^2}{x-1}dx = \int \frac{x^2-1+1}{x-1}dx = \int \left(1-\frac{1}{x-1}\right)dx = x - \ln|x-1| + c$

104. $\int \frac{1}{x^2+(x-1)^2}dx = \int \frac{1}{2x^2-2x+1}dx = \frac{1}{2}\int \frac{1}{x^2-x+\frac{1}{2}}dx = \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}dx$

$= \frac{1}{2}\int \frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}dx = \frac{1}{2} \times \frac{1}{\frac{1}{2}}\arctan\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right) + c = \arctan(2x-1) + c$

105. $\int \frac{1}{x^4+x}dx = \int \frac{1}{x^4(1+x^{-3})}dx = \int \frac{x^{-4}}{1+x^{-3}}dx = -\frac{1}{3}\int \frac{-3x^{-4}}{1+x^{-3}}dx = -\frac{1}{3}\int \frac{(1+x^{-3})'}{1+x^{-3}}dx$

$= -\frac{1}{3}\ln|1+x^{-3}| + c$

106. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{1}{a^2-b^2}\int \frac{a^2-b^2}{(x^2+a^2)(x^2+b^2)}dx = \frac{1}{a^2-b^2}\int \frac{(x^2+a^2)-(x^2+b^2)}{(x^2+a^2)(x^2+b^2)}dx$

$= \frac{1}{a^2-b^2}\int \left(\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right)dx = \frac{1}{a^2-b^2}\left[\frac{1}{b}\arctan\left(\frac{x}{b}\right) - \frac{1}{a}\arctan\left(\frac{x}{a}\right)\right] + c$

107. $\int \frac{\ln x}{x}dx = \int \frac{1}{x}\ln x\,dx = \int (\ln x)' \ln x\,dx = \frac{1}{2}(\ln x)^2 + c$

108. $\int \frac{dx}{x\ln x} = \int \frac{\frac{1}{x}}{\ln x}dx = \int \frac{(\ln x)'}{\ln x}dx = \ln|\ln x| + c$

109. $\int \frac{e^x+1}{e^x+x}dx = \int \frac{(e^x+x)'}{e^x+x}dx = \ln(e^x + x) + c$

110. $\int \frac{e^{2x}-1}{e^{2x}+1}dx = \int \frac{e^{-x}(e^{2x}-1)}{e^{-x}(e^{2x}+1)}dx = \int \frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \int \frac{(e^x+e^{-x})'}{e^x+e^{-x}}dx = \ln(e^x + e^{-x}) + c$

111. $\int \frac{e^{3x}+1}{e^x+1}dx = \int \frac{(e^x)^3+1^3}{e^x+1}dx = \int \frac{(e^x+1)(e^{2x}-e^x+1)}{e^x+1}dx = \int (e^{2x} - e^x + 1)dx$

$= \frac{1}{2}e^{2x} - e^x + x + c$

112. $\int (x + 1)e^{x^2+2x+3}\,dx = \frac{1}{2}\int (2x + 2)e^{x^2+2x+3}\,dx = \frac{1}{2}\int (x^2 + 2x + 3)' e^{x^2+2x+3}\,dx$

$= \frac{1}{2}e^{x^2+2x+3} + c$

113. $\int \frac{e^{4x}+e^{2x}+3}{e^{2x}}dx = \int \left(\frac{e^{4x}}{e^{2x}} + \frac{e^{2x}}{e^{2x}} + \frac{3}{e^{2x}}\right)dx = \int (e^{2x} + 1 + 3e^{-2x})dx$

$= \frac{1}{2}e^{2x} + x - \frac{3}{2}e^{-2x} + c$

114. $\int e^x\sqrt{e^x + 1}dx = \int (e^x + 1)'\sqrt{e^x + 1}dx = \frac{(e^x+1)^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{2}{3}(e^x + 1)\sqrt{e^x + 1} + c$

115. $\int \frac{2e^x}{e^x+2}dx = 2\int \frac{e^x}{e^x+2}dx = 2\int \frac{(e^x+2)'}{e^x+2}dx = 2\ln(e^x + 2) + c$

116. $\int \frac{e^{\arctan x}}{1+x^2}dx = \int (\arctan x)' e^{\arctan x}dx = e^{\arctan x} + c$

117. $\int 2x^{2x}(1 + \ln x)dx = \int e^{2x\ln x}(2 + 2\ln x)dx = \int e^{2x\ln x}(2x\ln x)'dx = e^{2x\ln x} + c$

$= x^{2x} + c$

118. $\int \frac{dx}{2+9x^2} = \int \frac{dx}{(\sqrt{2})^2+(3x)^2} = \frac{1}{3}\int \frac{3dx}{(\sqrt{2})^2+(3x)^2} = \frac{1}{3}\int \frac{(3x)'dx}{(\sqrt{2})^2+(3x)^2} = \frac{1}{3\sqrt{2}}\arctan\frac{3x}{\sqrt{2}} + c$

119. $\int \frac{x^n}{1+x^{2n+2}}dx = \frac{1}{n+1}\int \frac{(n+1)x^n}{1+x^{2(n+1)}}dx = \frac{1}{n+1}\int \frac{(n+1)x^n}{1+(x^{n+1})^2}dx = \frac{1}{n+1}\int \frac{(x^{n+1})'}{1+(x^{n+1})^2}dx$

$= \frac{1}{n+1}\arctan(x^{n+1}) + c$

120. $\int \frac{dx}{x^2-4x+7} = \int \frac{dx}{(x^2-4x+4)+7-4} = \int \frac{dx}{3+(x-2)^2} = \int \frac{dx}{(\sqrt{3})^2+(x-2)^2}$

$= \frac{1}{\sqrt{3}}\arctan\left(\frac{x-2}{\sqrt{3}}\right) + c$

121. $\int \frac{dx}{\sqrt{x+2}+\sqrt{x}} = \int \frac{1}{\sqrt{x+2}+\sqrt{x}} \times \frac{\sqrt{x+2}-\sqrt{x}}{\sqrt{x+2}-\sqrt{x}} dx = \int \frac{\sqrt{x+2}-\sqrt{x}}{x+2-x} dx = \frac{1}{2}\int\left(\sqrt{x+2}-\sqrt{x}\right)dx$

$= \frac{1}{2}\left[\frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right] + c = \frac{1}{3}(x+2)\sqrt{x+2} - \frac{1}{3}x\sqrt{x} + c$

122. $\int \sqrt{\tan x}\sec^4 x\, dx = \int \sqrt{\tan x}\sec^2 x \sec^2 x\, dx = \int \sqrt{\tan x}\,(1+\tan^2 x)\sec^2 x\, dx$

$= \int \tan^{\frac{1}{2}} \sec^2 x\, dx + \int \tan^{\frac{5}{2}}\sec^2 x\, dx = \int \tan^{\frac{1}{2}}(\tan x)'\, dx + \int \tan^{\frac{5}{2}}(\tan x)'\, dx$

$= \frac{2}{3}\tan^{\frac{3}{2}} x + \frac{2}{7}\tan^{\frac{7}{2}} x + c$

123. $\int \frac{\sec x+1}{\sin x} dx = \int \frac{\sec x}{\sin x} dx + \int \frac{1}{\sin x} dx = \int \frac{1}{\cos x \sin x} dx + \int \csc x\, dx$

$= 2\int \frac{1}{2\cos x \sin x} dx + \int \csc x\, dx = 2\int \frac{1}{\sin 2x} dx + \int \csc x\, dx = 2\int \csc 2x\, dx + \int \csc x\, dx$

$= -\ln|\csc 2x + \cot 2x| - \ln|\csc x + \cot x| + c$

124. $\int \frac{\sec x}{3-4\cos^2 x} dx = \int \frac{\frac{1}{\cos x}}{3-4\cos^2 x} dx = \int \frac{1}{3\cos x - 4\cos^3 x} dx = -\int \frac{1}{4\cos^3 x - 3\cos x} dx$

$= -\int \frac{1}{\cos 3x} dx = -\int \sec 3x\, dx = -\int \frac{\sec 3x(\sec 3x + \tan 3x)}{\sec 3x + \tan 3x} dx$

$= -\frac{1}{3}\int \frac{3\sec^2 3x + \sec 3x \tan 3x}{\sec 3x + \tan 3x} dx = -\frac{1}{3}\int \frac{(\sec 3x+\tan 3x)'}{\sec 3x+\tan 3x} dx = -\frac{1}{3}\ln|\sec 3x + \tan 3x| + c$

125. $\int \frac{dx}{x^2+x+1} = \int \frac{dx}{x^2+x+\frac{1}{4}+\frac{3}{4}} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}}\arctan\left[\frac{2}{\sqrt{3}}\left(x+\frac{1}{2}\right)\right] + c$

$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2x+1}{\sqrt{3}}\right) + c$

126. $\int \frac{dx}{2x^2-8x+10} = \frac{1}{2}\int \frac{dx}{x^2-4x+5} = \frac{1}{2}\int \frac{dx}{x^2-4x+4+1} = \frac{1}{2}\int \frac{dx}{1+(x-2)^2}$

$= \frac{1}{2}\arctan(x-2) + c$

127. $\int \frac{dx}{x+x^5} = \int \frac{dx}{x(1+x^4)} = \int \frac{x^3}{x^4(1+x^4)} dx = \int \frac{x^3+x^7-x^7}{x^4(1+x^4)} dx = \int \frac{x^3(1+x^4)-x^7}{x^4(1+x^4)} dx$

$= \int \frac{dx}{x} - \int \frac{x^3}{1+x^4} dx = \int \frac{dx}{x} - \frac{1}{4}\int \frac{4x^3}{1+x^4} dx = \int \frac{dx}{x} - \frac{1}{4}\int \frac{(x^4)'}{1+x^4} dx$

$= \ln|x| - \frac{1}{4}\ln(1+x^4) + c$

128. $\int \frac{e^{\log x}}{x} dx = \int \frac{e^{\frac{\ln x}{\ln 10}}}{x} dx = \int \frac{\left(e^{\ln x}\right)^{\frac{1}{\ln 10}}}{x} dx = \int \frac{x^{\frac{1}{\ln 10}}}{x} dx = \int x^{\frac{1}{\ln 10}-1}\, dx$

$= \ln 10\, x^{\frac{1}{\ln 10}} + c$

129. $\int \arctan\left(\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right) dx = \int \arctan\left(\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right) dx = \int \arctan\left(\sqrt{\tan^2 x}\right) dx$

$= \int \arctan(\tan x)\, dx = \int x dx = \frac{1}{2}x^2 + c$

130. $\int \arctan\left(\sqrt{\frac{1-\sin x}{1+\sin x}}\right)dx = \int \arctan\left(\sqrt{\frac{1-\cos(\frac{\pi}{2}-x)}{1+\cos(\frac{\pi}{2}-x)}}\right)dx$

$= \int \arctan\left(\sqrt{\frac{2\sin^2(\frac{\pi}{4}-\frac{x}{2})}{2\cos^2(\frac{\pi}{4}-\frac{x}{2})}}\right)dx = \int \arctan\left(\sqrt{\tan^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}\right)dx$

$= \int \arctan\left(\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)dx = \int\left(\frac{\pi}{4}-\frac{x}{2}\right)dx = \frac{\pi}{4}x - \frac{1}{4}x^2 + c$

131. $\int \frac{1}{\sqrt{3x-x^2}}dx = \int \frac{1}{\sqrt{-(x^2-3x)}}dx = \int \frac{1}{\sqrt{-\left(x^2-3x+\frac{9}{4}\right)+\frac{9}{4}}}dx = \int \frac{1}{\sqrt{\left(\frac{3}{2}\right)^2-\left(x-\frac{3}{2}\right)^2}}dx$

$= \arcsin\left(\frac{x-\frac{3}{4}}{\frac{3}{2}}\right) + c = \arcsin\left(\frac{2}{3}x - \frac{1}{2}\right) + c$

132. $\int \frac{2x-5}{x^2+2x+2}dx = \int \frac{2x+2-7}{x^2+2x+2}dx = \int \frac{2x+2}{x^2+2x+2}dx - \int \frac{7}{x^2+2x+2}dx$

$= \int \frac{(x^2+2x+2)'}{x^2+2x+2}dx - 7\int \frac{1}{(x+1)^2+1}dx = \ln|x^2+2x+2| - 7\arctan(x+1) + c$

133. $\int \frac{x^4}{x^2+1}dx = \int \frac{x^4}{x^2+1}.\frac{x^2-1}{x^2-1}dx = \int \frac{(x^4-1)+1}{x^4-1}.(x^2-1)dx$

$= \int\left(1+\frac{1}{x^4-1}\right)(x^2-1)\,dx = \int\left[(x^2-1)+\frac{x^2-1}{x^4-1}\right]dx = \int\left(x^2-1+\frac{1}{x^2+1}\right)dx$

$= \frac{1}{3}x^3 - x + \arctan x + c$

134. $\int \frac{1}{e^x+e^{-x}}dx = \int \frac{1}{e^x+\frac{1}{e^x}}dx = \int \frac{e^x}{e^{2x}+1}dx = \int \frac{(e^x)'}{(e^x)^2+1}dx = \arctan(e^x) + c$

135. $\int \frac{dx}{x^2-5x+6} = \int \frac{dx}{(x-2)(x-3)} = \int \frac{x-x+3-2}{(x-2)(x-3)}dx = \int \frac{(x-2)-(x-3)}{(x-2)(x-3)}dx$

$= \int\left[\frac{x-2}{(x-2)(x-3)} - \frac{x-3}{(x-2)(x-3)}\right]dx = \int\left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx = \ln|x-3| - \ln|x-2| + c$

$= \ln\left|\frac{x-3}{x-2}\right| + c$

136. $\int \frac{\cos 2x}{\cos x+\sin x}dx = \int \frac{\cos 2x}{\cos x+\sin x}\times\frac{\cos x-\sin x}{\cos x-\sin x}dx = \int \frac{(\cos^2 x-\sin^2 x)(\cos x-\sin x)}{\cos^2 x-\sin^2 x}dx$

$= \int(\cos x - \sin x)\,dx = \sin x + \cos x + c$

137. $\int \frac{1-\cos x}{1+\cos x}dx = \int \frac{2\sin^2(\frac{x}{2})}{2\cos^2(\frac{x}{2})}dx = \int\left(\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}\right)^2 dx = \int \tan^2\frac{x}{2}\,dx = \int\left(\sec^2\frac{x}{2} - 1\right)dx$

$= 2\tan\left(\frac{x}{2}\right) - x + c$

138. $\int \frac{\sqrt{1-\sin x}}{1+\cos x}dx = \int \frac{\sqrt{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}-2\sin\frac{x}{2}\cos\frac{x}{2}}}{2\cos^2\frac{x}{2}}dx = \int \frac{\sqrt{\left(\cos\frac{x}{2}-\sin\frac{x}{2}\right)^2}}{2\cos^2\frac{x}{2}}dx$

$= \int \frac{\cos\frac{x}{2}-\sin\frac{x}{2}}{2\cos^2\frac{x}{2}}dx = \int\left(\frac{1}{2}\sec\frac{x}{2} - \frac{\frac{1}{2}\sin\frac{x}{2}}{\cos^2\frac{x}{2}}\right)dx = \ln\left|\sec\frac{x}{2}+\tan\frac{x}{2}\right| - \sec\frac{x}{2} + c$

139. $\int \frac{\sec 2x-1}{\sec 2x+1}dx = \int \frac{\frac{1}{\cos 2x}-1}{\frac{1}{\cos 2x}+1}dx = \int \frac{\frac{1-\cos 2x}{\cos 2x}}{\frac{1+\cos 2x}{\cos 2x}}dx = \int \frac{1-\cos 2x}{1+\cos 2x}dx = \int \frac{2\sin^2 x}{2\cos^2 x}dx$

$= \int \tan^2 x\, dx = \int(\sec^2 x - 1)dx = \tan x - x + c$

140. $\int \frac{\sin 2x}{\sin x-\cos x-1}dx = \int \frac{2\sin x\cos x}{\sin x-\cos x-1}dx = \int \frac{(2\sin x\cos x)(\sin x-\cos x+1)}{(\sin x-\cos x-1)(\sin x-\cos x+1)}dx$

$= \int \frac{(2\sin x\cos x)(\sin x-\cos x+1)}{(\sin x-\cos x)^2-1}dx = \int \frac{(2\sin x\cos x)(\sin x-\cos x+1)}{\sin^2 x+\cos^2 x-1-2\sin x\cos x}dx$

$= \int \frac{(2\sin x\cos x)(\sin x-\cos x+1)}{-2\sin x\cos x}dx = \int(-\sin x + \cos x - 1)dx = \cos x + \sin x - x + c$

141. $\int \frac{ae^x+b}{ae^x-b}dx = \int \frac{2ae^x-ae^x+b}{ae^x-b}dx = \int \frac{2ae^x-(ae^x-b)}{ae^x-b}dx = \int \left(\frac{2ae^x}{ae^x-b} - 1\right)dx$

$= 2\int \frac{(ae^x-b)'}{ae^x-b}dx - \int dx = 2\ln|ae^x - b| - x + c$

142. $\int \frac{dx}{\sqrt{1-\tan^2 x}} = \int \frac{dx}{\sqrt{1-\frac{\sin^2 x}{\cos^2 x}}} = \int \frac{dx}{\sqrt{\frac{\cos^2 x-\sin^2 x}{\cos^2 x}}} = \int \frac{\cos x}{\sqrt{1-2\sin^2 x}}dx$

$= \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}\cos x}{\sqrt{1-(\sqrt{2}\sin x)}}dx = \frac{1}{\sqrt{2}}\int \frac{(\sqrt{2}\sin x)'}{\sqrt{1-(\sqrt{2}\sin x)}}dx = \frac{1}{\sqrt{2}}\arcsin(\sqrt{2}\sin x) + c$

143. $\int \frac{1}{\sqrt{x+a}+\sqrt{x+b}}dx = \int \frac{1}{\sqrt{x+a}+\sqrt{x+b}} \times \frac{\sqrt{x+a}-\sqrt{x+b}}{\sqrt{x+a}-\sqrt{x+b}}dx = \int \frac{\sqrt{x+a}-\sqrt{x+b}}{(x+a)-(x+b)}dx$

$= \int \frac{\sqrt{x+a}-\sqrt{x+b}}{a-b}dx = \frac{1}{a-b}\int\left(\sqrt{x+a} - \sqrt{x+b}\right)dx = \frac{1}{a-b}\left[\frac{(x+a)^{\frac{3}{2}}}{\frac{3}{2}} - \frac{(x+b)^{\frac{3}{2}}}{\frac{3}{2}}\right] + c$

$= \frac{2}{3(a-b)}\left[(x+a)^{\frac{3}{2}} - (x+b)^{\frac{3}{2}}\right] + c$

144. $\int \frac{1}{1+\cot x}dx = \int \frac{1}{1+\frac{\cos x}{\sin x}}dx = \int \frac{\sin x}{\sin x+\cos x}dx = \frac{1}{2}\int \frac{2\sin x}{\sin x+\cos x}dx$

$= \frac{1}{2}\int \frac{(\sin x+\cos x)+(\sin x-\cos x)}{\sin x+\cos x}dx = \frac{1}{2}\left[\int \frac{\sin x+\cos x}{\sin x+\cos x}dx + \int \frac{\sin x-\cos x}{\sin x+\cos x}dx\right]$

$= \frac{1}{2}\left[\int dx - \int \frac{\cos x-\sin x}{\sin x+\cos x}dx\right] = \frac{1}{2}\left[\int dx - \int \frac{(\sin x+\cos x)'}{\sin x+\cos x}dx\right] = \frac{1}{2}[x - \ln|\sin x + \cos x|] + c$

145. $\int \frac{1}{1-\tan x}dx = \int \frac{1}{1-\frac{\sin x}{\cos x}}dx = \int \frac{\cos x}{\cos x-\sin x}dx = \frac{1}{2}\int \frac{2\cos x}{\cos x-\sin x}dx$

$= \frac{1}{2}\int \frac{(\cos x-\sin x)+(\sin x+\cos x)}{\cos x-\sin x}dx = \frac{1}{2}\left[\int \frac{\cos x-\sin x}{\cos x-\sin x}dx + \int \frac{\sin x+\cos x}{\cos x-\sin x}dx\right]$

$= \frac{1}{2}\left[\int dx + \int \frac{\sin x+\cos x}{\cos x-\sin x}dx\right] = \frac{1}{2}\left[\int dx - \int \frac{(\cos x-\sin x)'}{\cos x-\sin x}dx\right] = \frac{1}{2}[x - \ln|\cos x - \sin x|] + c$

146. $\int \frac{dx}{\sin^2 x+2\cos^2 x} = \int \frac{\frac{1}{\cos^2 x}}{\frac{\sin^2 x+2\cos^2 x}{\cos^2 x}}dx = \int \frac{\sec^2 x}{\tan^2 x+2}dx = \int \frac{(\tan x)'}{\tan^2 x+(\sqrt{2})^2}dx$

$= \frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\tan x\right) + c$

147. $\int \frac{\sin x}{\sin x-\cos x}dx = \frac{1}{2}\int \frac{2\sin x}{\sin x-\cos x}dx = \frac{1}{2}\int \frac{(\sin x-\cos x)+(\sin x+\cos x)}{\sin x-\cos x}dx$

$= \frac{1}{2}\int \left(\frac{\sin x-\cos x}{\sin x-\cos x}+\frac{\sin x+\cos x}{\sin x-\cos x}\right)dx = \frac{1}{2}\int \left[1+\frac{(\sin x-\cos x)'}{\sin x-\cos x}\right]dx$

$= \frac{1}{2}x + \frac{1}{2}\ln|\sin x - \cos x| + c$

148. $\int \frac{5\sin x}{\sin x-2\cos x}dx = \int \frac{(\sin x-2\cos x)+(2\cos x+4\sin x)}{\sin x-2\cos x}dx$

$= \int \frac{\sin x-2\cos x}{\sin x-2\cos x}dx + 2\int \frac{\cos x+2\sin x}{\sin x-2\cos x}dx = \int dx + 2\int \frac{(\sin x-2\cos x)'}{\sin x-2\cos x}dx$

$= x + 2\ln|\sin x - 2\cos x| + c$

149. $\int \frac{\cos x}{\sqrt{2+\cos 2x}}dx = \int \frac{\cos x}{\sqrt{2+1-2\sin^2 x}}dx = \int \frac{\cos x}{\sqrt{3-2\sin^2 x}}dx = \int \frac{(\sin x)'}{\sqrt{3-2\sin^2 x}}dx$

$= \frac{1}{\sqrt{2}}\arcsin\left(\sqrt{\frac{2}{3}}\sin x\right) + c$

150. $\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}dx = \frac{1}{2}\int \frac{\sin 2x}{\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x}dx =$

$\frac{1}{2}\int \frac{\sin 2x}{1-\frac{1}{2}(2\sin x\cos x)^2}dx = \frac{1}{2}\int \frac{\sin 2x}{1-\frac{1}{2}\sin^2 2x}dx = \frac{1}{2}\int \frac{\sin 2x}{1-\frac{1}{2}\left(1-\cos^2 2x\right)}dx =$

$\frac{1}{4}\int \frac{2\sin 2x}{\frac{1+\cos^2 2x}{2}}dx = \frac{1}{2}\int \frac{(\cos 2x)'}{1+(\cos 2x)^2}dx$

$= \frac{1}{2}\arctan(\cos 2x) + c$

151. $\int \left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x\,dx = \int \left(\frac{2+2\sin x\cos x}{2\cos^2 x}\right)e^x\,dx = \int \left(\frac{1}{\cos^2 x}+\frac{\sin x\cos x}{\cos^2 x}\right)e^x\,dx$

$= \int(\sec^2 x + \tan x)e^x dx = \int[(\tan x)' + \tan x]e^x dx = e^x\tan x + c$

Remark: $[e^x f(x)]' = [f(x) + f'(x)]e^x$

152. $\int \frac{\cos 4x}{\cos x}dx = \int \frac{\cos^2 2x-\sin^2 2x}{\cos x}dx = \int \frac{\left(\cos^2 x-\sin^2 x\right)^2-(2\sin x\cos x)^2}{\cos x}dx$

$= \int \frac{\left(\cos^2 x+\sin^2 x\right)^2-4\sin^2 x\cos^2 x-4\sin^2 x\cos^2 x}{\cos x}dx = \int \frac{1-8\sin^2 x\cos^2 x}{\cos x}dx$

$= \int \sec x\,dx - 8\int \sin^2 x\cos x\,dx = \int \sec x\left(\frac{\tan x+\sec x}{\tan x+\sec x}\right)dx - 8\int \sin^2 x\cos x\,dx$

$= \int \frac{\sec x(\tan x+\sec x)}{\tan x+\sec x}dx - 8\int \sin^2 x\cos x\,dx = \int \frac{(\tan x+\sec x)'}{\tan x+\sec x}dx - 8\int \sin^2 x\,(\sin x)'dx$

$= \ln|\tan x + \sec x| - \frac{8}{3}\sin^3 x + c$

153. $\int \frac{e^x}{e^{1-x}+e^{x-1}}dx = \int \frac{e^x}{e^{x-1}\left(\frac{e^{1-x}}{e^{x-1}}+1\right)}dx = \int \frac{e^x.e^{1-x}}{1+e^{1-x}.e^{1-x}}dx = \int \frac{e}{1+e^{2(1-x)}}dx$

$= \int \frac{e}{1+e^{2(1-x)}}\times\frac{e^{2x-2}}{e^{2x-2}}dx = e\int \frac{e^{2x-2}}{1+e^{-2(1-x)}}dx = \frac{1}{2}e\int \frac{2e^{2x-2}}{1+e^{2x-2}}dx = \frac{1}{2}e\int \frac{\left(1+e^{2x-2}\right)'}{1+e^{2x-2}}dx$

$= \frac{1}{2}e\ln(1 + e^{2x-2}) + c$

154. $\int \frac{x^3}{3+x} dx = \int \frac{x^3+27-27}{x+3} dx = \int \left(\frac{x^3+27}{x+3} - \frac{27}{x+3}\right) dx$

$= \int \left[\frac{(x+3)(x^2-3x+9)}{x+3} - \frac{27}{x+3}\right] dx = \int \left(x^2 - 3x + 9 - \frac{27}{x+3}\right) dx$

$= \frac{1}{3}x^3 - \frac{3}{2}x^2 + 9x - 27\ln|x+3| + c$

155. $\int \frac{1}{1-\tan^2 x} dx = \int \frac{1}{1-\frac{\sin^2 x}{\cos^2 x}} dx = \int \frac{\cos^2 x}{\cos^2 x - \sin^2 x} dx = \frac{1}{2}\int \frac{2\cos^2 x - 1 + 1}{\cos^2 x - \sin^2 x} dx$

$= \frac{1}{2}\int \frac{\cos 2x + 1}{\cos 2x} dx = \frac{1}{2}\int \left(1 + \frac{1}{\cos 2x} d\right) x = \frac{1}{2}\int (1 + \sec 2x\, d)x$

$= \frac{1}{2}x + \frac{1}{4}\ln|\sec 2x + \tan 2x| + c$

156. $\int \frac{1}{x - x^{\frac{3}{5}}} dx = \int \frac{1}{x^{\frac{3}{5}}\left(x^{\frac{2}{5}} - 1\right)} dx = \frac{5}{2}\int \frac{\frac{2}{5}x^{-\frac{3}{5}}}{x^{\frac{2}{5}} - 1} dx = \frac{5}{2}\int \frac{\left(x^{\frac{2}{5}} - 1\right)'}{x^{\frac{2}{5}} - 1} dx = \frac{5}{2}\ln\left|x^{\frac{2}{5}} - 1\right| + c$

157. $\int \frac{1}{1+\cos x} dx = \int \frac{1}{1 + 2\cos^2\frac{x}{2} - 1} dx = \int \frac{1}{2\cos^2\frac{x}{2}} dx = \frac{1}{2}\int \sec^2\frac{x}{2} dx = \tan\frac{x}{2} + c$

158. $\int \frac{1}{1+\sin x} dx = \int \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx =$

$\int \frac{1-\sin x}{1-\sin^2 x} dx = \int \frac{1-\sin x}{\cos^2 x} dx = \int \left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx$

$= \int (\sec^2 x - \sec x \tan x) dx = \tan x - \sec x + c$

159. $\int \frac{dx}{\sqrt{\sin x \cos^3 x}} = \int \frac{dx}{\sqrt{\cos^2 x \cos x \sin x}} = \int \frac{dx}{\cos x\sqrt{\sin x \cos x}} = \int \frac{dx}{\cos x\sqrt{\cos^2 x \tan x}}$

$= \int \frac{dx}{\cos^2 x\sqrt{\tan x}} = \int \frac{\sec^2 x}{\sqrt{\tan x}} dx = 2\sqrt{\tan x} + c$

160. $\int \frac{1}{1-x^2}\ln\left(\frac{1+x}{1-x}\right) dx$, we have $\left[\ln\left(\frac{1+x}{1-x}\right)\right]' = \frac{\left(\frac{1+x}{1-x}\right)'}{\frac{1+x}{1-x}} = \frac{\frac{2}{(1-x)^2}}{\frac{1+x}{1-x}} = \frac{2}{(1-x)(1+x)} = \frac{2}{1-x^2}$,

so: $\int \frac{1}{1-x^2}\ln\left(\frac{1+x}{1-x}\right) dx = \frac{1}{2}\int \frac{2}{1-x^2}\ln\left(\frac{1+x}{1-x}\right) dx = \frac{1}{2}\int \left[\ln\left(\frac{1+x}{1-x}\right)\right]'\ln\left(\frac{1+x}{1-x}\right) dx$

$= \frac{1}{2}\left[\frac{1}{2}\ln^2\left(\frac{1+x}{1-x}\right)\right] + c = \frac{1}{4}\ln^2\left(\frac{1+x}{1-x}\right) + c$

161. $\int \cos x \,.\, \cos 2x \,.\, \cos 3x\, dx = \int \cos x \,.\left[\frac{1}{2}(\cos 5x + \cos x)\right] dx$

$= \int \frac{1}{2}[\cos x \cos 5x + \cos^2 x] dx = \int \frac{1}{2}\left[\frac{1}{2}(\cos 6x + \cos 4x) + \frac{1}{2} + \frac{1}{2}\cos 2x\right] dx$

$= \int \left(\frac{1}{4} + \frac{1}{4}\cos 6x + \frac{1}{4}\cos 4x + \frac{1}{4}\cos 2x\right) dx = \frac{1}{4}x + \frac{1}{24}\sin 6x + \frac{1}{16}\sin 4x + \frac{1}{8}\sin 2x + c$

162. $\int \cot^4 x\, dx = \int \cot^2 x \cot^2 x\, dx = \int \cot^2 x (\csc^2 x - 1) dx$

$= \int (\cot^2 x \csc^2 x - \cot^2 x) dx$

$= \int \cot^2 x \csc^2 x\, dx - \int (\csc^2 x - 1) dx = -\int \cot^2 x (\cot x)' dx - \int (\csc^2 x - 1) dx$

$= -\frac{1}{3}\cot^3 x + \cot x + x + c$

163. $\int \frac{\sin 2x}{(1-\sin x)^3}dx = \int \frac{2\sin x\cos x}{(1-\sin x)^3}dx = -2\int \frac{(1-\sin x-1)\cos x}{(1-\sin x)^3}dx$

$= -2\int\left[\frac{(1-\sin x)\cos x}{(1-\sin x)^3} - \frac{\cos x}{(1-\sin x)^3}\right]dx = 2\int\left[\frac{(-\sin x)'}{(1-\sin x)^2} - \frac{(-\sin x)'}{(1-\sin x)^3}\right]dx$

$= -\frac{2}{1-\sin x} + \frac{1}{(1-\sin x)^2} + c$

164. $\int \sqrt{\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}}dx = \int\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^{\frac{1}{2}}\frac{1}{\sqrt{1+x^2}}dx$

$= \int\left[\ln\left(x+\sqrt{1+x^2}\right)\right]^{\frac{1}{2}}\left[\ln\left(x+\sqrt{1+x^2}\right)\right]'dx = \frac{2}{3}\ln^{\frac{3}{2}}\left(x+\sqrt{1+x^2}\right)+c$

165. $\int \sec^5 x\tan^3 x\,dx = \int \sec^4 x\tan^2 x\,(\sec x\tan x)dx$

$= \int \sec^4 x\,(\sec^2 x - 1)\,(\sec x\tan x)dx$

$= \int(\sec^6 x - \sec^4 x)(\sec x)'\,dx = \frac{1}{7}\sec^7 x - \frac{1}{5}\sec^5 x + c$

166. $\int\sqrt{\csc x - \sin x}\,dx = \int\sqrt{\frac{1}{\sin x} - \sin x\,dx} = \int\sqrt{\frac{1-\sin^2 x}{\sin x}}dx = \int\sqrt{\frac{\cos^2 x}{\sin x}}dx$

$= \int\sqrt{\frac{1}{\sin x}}\cos x\,dx \int\frac{1}{\sqrt{\sin x}}(\sin x)'dx = 2\sqrt{\sin x}+c$

167. $\int\frac{dx}{x\ln x\ln(\ln x)} = \int\frac{1}{\ln(\ln x)}\times\frac{1}{x\ln x}dx = \int\frac{1}{\ln(\ln x)}[\ln(\ln x)]'\,dx = \ln|\ln(\ln x)|+c$

168. $\int\frac{\cos(x+a)}{\sin(x+b)}dx = \int\frac{\cos(x+b+a-b)}{\sin(x+b)}dx = \int\frac{\cos(x+b)\cos(a-b)-\sin(x+b)\sin(a-b)}{\sin(x+b)}dx$

$= \cos(a-b)\int\frac{\cos(x+b)}{\sin(x+b)}dx - \int\sin(a-b)\,dx = \cos(a-b)\int\frac{[\sin(x+b)]'}{\sin(x+b)}dx - \int\sin(a-b)\,dx$

$= \cos(a-b)\ln|\sin(x+b)| - x\sin(a-b)+c$

169. $\int\frac{\sin x}{\sin(x-a)}dx = \int\frac{\sin(x-a+a)}{\sin(x-a)}dx = \int\frac{\sin[(x-a)+a]}{\sin(x-a)}dx$

$= \int\frac{\sin(x-a)\cos a+\cos(x-a)\sin a}{\sin(x-a)}dx = \int\left[\frac{\sin(x-a)\cos a}{\sin(x-a)}+\frac{\cos(x-a)\sin a}{\sin(x-a)}\right]dx$

$= \int[\cos a + \sin a\cot(x-a)]\,dx = (\cos a)x + (\sin a)\ln|\sin(x-a)|+c$

170. $\int\frac{x-1}{x+x^2\ln x}dx = \int\frac{x-1}{x+x^2\ln x}dx = \int\frac{x^2\left(\frac{1}{x}-\frac{1}{x^2}\right)}{x^2\left(\frac{1}{x}+\ln x\right)}dx = \int\frac{-\frac{1}{x^2}+\frac{1}{x}}{\frac{1}{x}+\ln x}dx = \int\frac{\left(\frac{1}{x}+\ln x\right)'}{\frac{1}{x}+\ln x}dx$

$= \ln\left|\frac{1}{x}+\ln x\right|+c$

171. $\int\frac{3e^{-2x}+5e^{3x}}{e^{-2x}+e^{3x}}dx$, $3e^{-2x}+5e^{3x} = \mathrm{A}(e^{-2x}+e^{3x}) + \mathrm{B}(-2e^{-2x}+3e^{3x}) \Longrightarrow$

$3e^{-2x}+5e^{3x} = e^{-2x}(\mathrm{A}-2\mathrm{B}) + e^{3x}(\mathrm{A}+3\mathrm{B})$, so $\begin{cases}\mathrm{A}-2\mathrm{B}=3\\ \mathrm{A}+3\mathrm{B}=5\end{cases}$, then $\mathrm{A}=\frac{19}{5}$ and $\mathrm{B}=\frac{2}{5}$, so:

$\int\frac{3e^{-2x}+5e^{3x}}{e^{-2x}+e^{3x}}dx = \int\frac{\frac{19}{5}(e^{-2x}+e^{3x})+\frac{2}{5}(e^{-2x}+e^{3x})}{e^{-2x}+e^{3x}}dx = \frac{19}{5}\int dx + \frac{2}{5}\int\frac{-2e^{-2x}+3e^{3x}}{e^{-2x}+e^{3x}}dx$

$= \frac{19}{5}\int dx + \frac{2}{5}\int\frac{(e^{-2x}+e^{3x})'}{e^{-2x}+e^{3x}}dx = \frac{19}{5}x + \frac{2}{5}\ln|e^{-2x}+e^{3x}|$

172. $\int \frac{1}{1+e^x}\cdot\frac{e^x-e^{-x}}{e^x+e^{-x}}dx = \int \frac{e^x-1}{e^{2x}+1}dx = \int\left(\frac{e^x}{e^{2x}+1}-\frac{1}{e^{2x}+1}\right)dx$

$= \int\left(\frac{(e^x)'}{(e^x)^2+1}-\frac{1}{e^{2x}+1}\right)dx = \int\left(\frac{(e^x)'}{(e^x)^2+1}+\frac{1}{2}\frac{-2e^{-2x}}{e^{-2x}+1}\right)dx = \int\left(\frac{(e^x)'}{(e^x)^2+1}+\frac{1}{2}\frac{(e^{-2x}+1)'}{e^{-2x}+1}\right)dx$

$= \arctan e^x + \frac{1}{2}\ln(1+e^{-2x}) + c$

173. $\int \frac{dx}{x^{\frac{41}{25}}+x^{\frac{9}{25}}} = \int \frac{dx}{x^{\frac{9}{25}}\left(x^{\frac{32}{25}}+1\right)} = \int \frac{x^{-\frac{9}{25}}}{1+x^{\frac{32}{25}}}dx = \int \frac{x^{-\frac{9}{25}}}{1+\left(x^{\frac{16}{25}}\right)^2}dx = \frac{25}{16}\int \frac{\frac{16}{25}x^{-\frac{9}{25}}}{1+\left(x^{\frac{16}{25}}\right)^2}dx$

$= \frac{25}{16}\int \frac{\left(x^{\frac{16}{25}}\right)'}{1+\left(x^{\frac{16}{25}}\right)^2}dx = \frac{25}{16}\arctan x^{\frac{16}{25}} + c$

174. $\int \frac{(2+x)e^{-x}}{x^3}dx = \int \frac{x(2+x)e^{-x}}{x^4}dx = \int \frac{x(2+x)e^x}{x^4e^{2x}}dx = \int \frac{(x^2e^x)'}{(x^2e^x)^2}dx = -\frac{1}{x^2e^x} + c$

175. $\int \frac{2}{(\cos x-\sin x)^2}dx = \int \frac{2}{\left[\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos x-\frac{\sqrt{2}}{2}\sin x\right)\right]^2}dx = \int \frac{2}{\left[\sqrt{2}\left(\cos x\cos\frac{\pi}{4}-\sin x\sin\frac{\pi}{4}\right)\right]^2}dx$

$= \int \frac{2}{\left(\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)\right)^2}dx = \int \frac{1}{\cos^2\left(x+\frac{\pi}{4}\right)}dx = \int \sec^2\left(x+\frac{\pi}{4}\right)dx = \tan\left(x+\frac{\pi}{4}\right) + c$

176. $\int \sqrt{\frac{x}{1-x^3}}dx = \frac{2}{3}\int \frac{\frac{3}{2}\sqrt{x}}{\sqrt{1-x^3}}dx = \frac{2}{3}\int \frac{\left(x^{\frac{3}{2}}\right)'}{\sqrt{1-\left(x^{\frac{3}{2}}\right)^2}}dx = \frac{2}{3}\arcsin x^{\frac{3}{2}} + c$

177. $\int \sqrt{\frac{x}{1+x^3}}dx\,\frac{2}{3}\int \frac{\frac{3}{2}\sqrt{x}}{\sqrt{1+x^3}}dx = \frac{2}{3}\int \frac{\left(x^{\frac{3}{2}}\right)'}{\sqrt{1+\left(x^{\frac{3}{2}}\right)^2}}dx = \frac{2}{3}\operatorname{arcsinh} x^{\frac{3}{2}} + c$

178. $\int \sqrt{1-\sin 2x}\,dx = \int \sqrt{1-2\sin x\cos x}\,dx = \int \sqrt{\sin^2 x+\cos^2 x-2\sin x\cos x}\,dx$

$= \int \sqrt{(\sin x-\cos x)^2}\,dx = \int \pm(\sin x-\cos x)dx = \pm(\sin x-\cos x) + c$

179. $\int \frac{(\ln x)^2-1}{x(\ln x)^2}dx = \int\left(1-\frac{1}{(\ln x)^2}\right)\times\frac{1}{x}dx = \int\left(1-\frac{1}{(\ln x)^2}\right)(\ln x)'dx = \ln x + \frac{1}{\ln x} + c$

180. $\int \sin x\sec x\tan x\,dx = \int \sin x\times\frac{1}{\cos x}\times\frac{\sin x}{\cos x}dx = \int \frac{\sin^2 x}{\cos^2 x}dx = \int \frac{1-\cos^2 x}{\cos^2 x}dx$

$= \int\left(\frac{1}{\cos^2 x}-1\right)dx = \int(\sec^2 x-1)dx = \tan x - x + c$

181. $\int \frac{1}{\csc^3 x}dx = \int \sin^3 x\,dx = \int \sin x\sin^2 x\,dx = \int \sin x\,(1-\cos^2 x\,dx)$

$= \int \sin x\,dx - \int \cos^2 x\sin x\,dx = \int \sin x\,dx + \int \cos^2 x\,(\cos x)'dx = -\cos x + \frac{1}{3}\cos^3 x + c$

182. $\int \sqrt{\frac{1-\sin 2x}{1+\cos 2x}}\,dx = \int \sqrt{\frac{\cos^2 x+\sin^2 x-\sin 2x}{2\cos^2 x}}\,dx = \int \sqrt{\frac{(\cos x-\sin x)^2}{2\cos^2 x}}\,dx$

$\int \frac{\cos x-\sin x}{\sqrt{2}\cos x}dx = \frac{1}{\sqrt{2}}\int\left(1-\frac{\sin x}{\cos x}\right)dx = \frac{1}{\sqrt{2}}\int dx + \frac{1}{\sqrt{2}}\int\frac{(\cos x)'}{\cos x}dx = \frac{1}{\sqrt{2}}(x+\ln|\cos x|)+c$

183. $\int \frac{x^5-x^3+x^2-1}{x^4-x^3+x-1}dx = \int \frac{x^3(x^2-1)+x^2-1}{x^3(x-1)+x-1}dx = \int \frac{(x^2-1)(x^3-1)}{(x-1)(x^3-1)}dx = \int \frac{x^2-1}{x-1}dx$

$= \int \frac{(x-1)(x+1)}{x-1}dx = \int (x+1)dx = \frac{1}{2}x^2+x+c$

184. $\int \frac{\tan(\arctan x-\arctan 2x)}{x}dx$, $\tan(a-b) = \frac{\tan a-\tan b}{1+\tan a\tan b}$, so we get:

$\tan(\arctan x-\arctan 2x) = \frac{\tan\arctan x-\tan\arctan 2x}{1+\tan\arctan x.\tan\arctan 2x} = \frac{x-2x}{1+x\times 2x} = -\frac{1}{1+2x^2}$, so:

$\int \frac{\tan(\arctan x-\arctan 2x)}{x}dx = -\int\frac{1}{1+2x^2}dx = -\int\frac{1}{1+\left(\sqrt{2}x\right)^2}dx = -\frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}x\right)+c$

185. $\int \frac{1-\tan x}{1+\tan x}dx = \int \frac{1-\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}dx = \int \frac{\cos x-\sin x}{\cos x+\sin x}dx = \int \frac{(\sin x+\cos x)'}{\sin x+\cos x}dx$

$= \ln(\sin x+\cos x)+c$

186. $\int \frac{1}{\sqrt{x}(1+x)}dx = \int \frac{x^{-\frac{1}{2}}}{1+x}dx = 2\int\frac{\frac{1}{2}x^{-\frac{1}{2}}}{1+(\sqrt{x})^2}dx = 2\int\frac{(\sqrt{x})'}{1+(\sqrt{x})^2}dx = 2\arctan\sqrt{x}+c$

187. $\int \frac{1}{1-\cos nx}dx = \int \frac{1}{1-\cos nx}.\frac{1+\cos nx}{1+\cos nx}dx = \int\frac{1+\cos nx}{1-\cos^2 nx}dx = \int\frac{1+\cos nx}{\sin^2 nx}dx$

$= \int \csc^2 nx\,dx + \int\frac{\cos nx}{\sin^2 nx}dx = \int\csc^2 nx\,dx + \frac{1}{n}\int\frac{n\cos nx}{\sin^2 nx}dx$

$= \int\csc^2 nx\,dx + \frac{1}{n}\int\frac{(\sin nx)'}{\sin^2 nx}dx = -\frac{1}{n}\cot nx - \frac{1}{n\sin nx}+c$

188. $\int\frac{x^6-1}{x^2+1}dx = \int\frac{x^6+1-2}{x^2+1}dx = \int\frac{\left(x^2\right)^3+1}{x^2+1}dx - 2\int\frac{1}{x^2+1}dx$

$= \int\frac{(x^2+1)(x^4-x^2+1)}{x^2+1}dx - 2\int\frac{1}{x^2+1}dx = \int(x^4-x^2+1)\,dx - 2\int\frac{1}{x^2+1}dx$

$= \frac{1}{5}x^5-\frac{1}{3}x^3+x-2\arctan x+c$

189. $\int\frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx = \int\frac{x^2-1}{x^3\sqrt{x^4(2-2x^{-2}+x^{-4})}}dx = \int\frac{x^2-1}{x^5\sqrt{2-2x^{-2}+x^{-4}}}dx$

$= \int\frac{x^{-3}-x^{-5}}{\sqrt{x^{-4}-2x^{-2}+2}}dx = \frac{1}{4}\int\frac{(x^{-4}-2x^{-2}+2)'}{\sqrt{x^{-4}-2x^{-2}+2}}dx = \frac{1}{2}\sqrt{x^{-4}-2x^{-2}+2}+c$

190. $\int\frac{\cos 7x-\cos 8x}{1+2\cos 5x}dx = \int\frac{(\cos 7x+\cos 3x)-(\cos 8x+\cos 2x)-(\cos 3x-\cos 2x)}{1+2\cos 5x}dx$

$= \int\frac{2\cos 5x\cos 2x-2\cos 5x\cos 3x-(\cos 3x-\cos 2x)}{1+2\cos 5x}dx$

$= \int\frac{-2\cos 5x(\cos 3x-\cos 2x)-(\cos 3x-\cos 2x)}{1+2\cos 5x}dx = \int\frac{-(\cos 3x-\cos 2x)(1+2\cos 5x)}{1+2\cos 5x}dx$

$= \int (\cos 2x - \cos 3x)dx = \frac{1}{2}\sin 2x - \frac{1}{3}\sin 3x + c$

Remark: $\cos a + \cos b = 2\cos\frac{a+b}{2}.\cos\frac{a-b}{2}$

191. $\int \frac{\cos x - \sin x}{(1+\sin 2x)^8}dx = \int \frac{\cos x - \sin x}{\left(\sin^2 x + \cos^2 x + 2\sin x\cos x\right)^8}dx = \int \frac{\cos x - \sin x}{\left[(\sin x + \cos x)^2\right]^8}dx$

$= \int \frac{\cos x - \sin x}{(\sin x + \cos x)^{16}}dx = \int (\sin x + \cos x)^{-16}(\cos x - \sin x)\, dx$

$= \int (\sin x + \cos x)^{-16}(\sin x + \cos x)'\, dx = -\frac{1}{15}(\sin x + \cos x)^{-15} + c$

192. $\int \frac{x}{\sqrt{x^2+x+1}}dx = \frac{1}{2}\int \frac{2x+1-1}{\sqrt{x^2+x+1}}dx = \frac{1}{2}\int \frac{2x+1}{\sqrt{x^2+x+1}}dx - \frac{1}{2}\int \frac{1}{\sqrt{x^2+x+1}}dx$

$= \int \frac{1}{2}\int (x^2+x+1)'(x^2+x+1)^{-\frac{1}{2}}dx - \frac{1}{2}\int \frac{1}{\sqrt{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}}dx$

$= \sqrt{x^2+x+1} - \frac{1}{2}\ln\left|x + \frac{1}{2} + \sqrt{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}}\right| + c$

193. $\int \frac{1}{1+a^x}dx = \int \frac{1}{a^x(1+a^{-x})}dx = \int \frac{a^{-x}}{1+a^{-x}}dx = -\frac{1}{\ln a}\int \frac{-\ln a \times a^{-x}}{1+a^{-x}}dx$

$= -\frac{1}{\ln a}\int \frac{(1+a^{-x})'}{1+a^{-x}}dx = -\frac{1}{\ln a}\ln|1+a^{-x}| + c$

Remark: $(a^{-x})' = -a^{-x}\ln a$

194. $\int \left(\frac{x}{e}\right)^x \ln x\, dx = \int x^x e^{-x}\ln x\, dx = \int \left(e^{\ln x}\right)^x e^{-x}\ln x\, dx = \int e^{x\ln x}e^{-x}\ln x\, dx$

$= \int e^{x\ln x - x}\ln x\, dx = \int e^{x\ln x - x}(x\ln x - x)'dx$, with $(x\ln x - x)' = \ln x$

$\int \left(\frac{x}{e}\right)^x \ln x\, dx = \int (x\ln x - x)e^{x\ln x - x}\, dx = e^{x\ln x - x} + c = e^{x\ln x}e^{-x} + c = \left(e^{\ln x}\right)^x e^{-x} + c$

$= x^x e^{-x} + c = \left(\frac{x}{e}\right)^x + c$

195. $\int \frac{x^3+4x^2-2x-5}{x^2-4x+4}dx = \int \frac{(x+8)(x-2)^2+26(x-2)+15}{(x-2)^2}dx$

$= \int \left(x + 8 + \frac{26}{x-2} + \frac{15}{(x-2)^2}\right)dx = \frac{1}{2}x^2 + 8x + 26\ln|x-2| - \frac{3}{x-15} + c$

196. $\int \frac{\sqrt{x^2+1}-\sqrt{x^2-1}}{\sqrt{x^4-1}}dx = \int \left(\frac{\sqrt{x^2+1}}{\sqrt{x^4-1}} - \frac{\sqrt{x^2-1}}{\sqrt{x^4-1}}\right)dx = \int \left(\sqrt{\frac{x^2+1}{x^4-1}} - \sqrt{\frac{x^2-1}{x^4-1}}\right)dx$

$= \int \left(\sqrt{\frac{x^2+1}{(x^2-1)(x^2+1)}} - \sqrt{\frac{x^2-1}{(x^2-1)(x^2+1)}}\right)dx = \int \left(\sqrt{\frac{1}{x^2-1}} - \sqrt{\frac{1}{x^2+1}}\right)dx$

$= \int \frac{dx}{\sqrt{x^2-1}} - \int \frac{dx}{\sqrt{x^2+1}} = \cosh^{-1}x + \sinh^{-1}x + c$

197. $\int \frac{1}{\sin x + \cos x}dx = \frac{1}{\sqrt{2}}\int \frac{1}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x}dx = \frac{1}{\sqrt{2}}\int \frac{1}{\sin x\cos\left(\frac{\pi}{4}\right) + \cos x\sin\left(\frac{\pi}{4}\right)}dx$

$= \frac{1}{\sqrt{2}}\int \frac{1}{\sin\left(x+\frac{\pi}{4}\right)}dx = \frac{1}{\sqrt{2}}\int \csc\left(x+\frac{\pi}{4}\right)dx = -\frac{1}{\sqrt{2}}\ln\left|\csc\left(x+\frac{\pi}{4}\right) + \cot\left(x+\frac{\pi}{4}\right)\right| + c$

198. $\int(4-x^2)(2+x)^n dx=\int(2+x)(2-x)(2+x)^n dx=\int(2-x)(2+x)^{n+1}dx$

$=\int[4-(2+x)](2+x)^{n+1}dx=4\int(2+x)^{n+1}dx-\int(2+x)^{n+2}dx$

$=\frac{4}{n+2}(2+x)^{n+2}-\frac{1}{n+3}(2+x)^{n+3}+c$

199. $\int\left(\sqrt{\frac{4-x}{x}}-\sqrt{\frac{x}{4-x}}\right)dx=\int\left(\frac{\sqrt{4-x}}{\sqrt{x}}-\frac{\sqrt{x}}{\sqrt{4-x}}\right)dx=\int\left(\frac{\sqrt{4-x}\sqrt{4-x}}{\sqrt{x}\sqrt{4-x}}-\frac{\sqrt{x}\sqrt{x}}{\sqrt{x}\sqrt{4-x}}\right)dx$

$=\int\left(\frac{4-x}{\sqrt{x}\sqrt{4-x}}-\frac{x}{\sqrt{x}\sqrt{4-x}}\right)dx=\int\frac{4-2x}{\sqrt{x}\sqrt{4-x}}dx=\int\frac{4-2x}{\sqrt{4x-x^2}}dx=\int\frac{(4x-x^2)'}{\sqrt{4x-x^2}}dx$

$=\int(4x-x^2)'(4x-x^2)^{-\frac{1}{2}}dx=2\sqrt{4x-x^2}+c$

200. $\int\frac{dx}{1+e^x+e^{2x}+e^{3x}}=\frac{1}{2}\int\frac{2-e^{2x}+e^{2x}}{1+e^x+e^{2x}+e^{3x}}dx$

$=\frac{1}{2}\int\frac{1+e^{2x}}{(e^x+1)(e^{2x}+1)}dx-\frac{1}{2}\int\frac{e^{2x}-1}{(e^x+1)(e^{2x}+1)}dx=\frac{1}{2}\int\frac{1}{e^x+1}dx-\frac{1}{2}\int\frac{(e^x-1)(e^x+1)}{(e^x+1)(e^{2x}+1)}dx$

$=\frac{1}{2}\int\frac{e^{-x}}{e^{-x}(e^x+1)}dx-\frac{1}{2}\int\frac{e^x-1}{e^{2x}+1}dx=\frac{1}{2}\int\frac{e^{-x}}{e^{-x}+1}dx-\frac{1}{2}\int\frac{e^x}{e^{2x}+1}dx-\frac{1}{4}\int\frac{-2e^{-2x}}{e^{-2x}(e^{2x}+1)}dx$

$=-\frac{1}{2}\int\frac{(e^{-x}+1)'}{e^{-x}+1}dx-\frac{1}{2}\int\frac{(e^x)'}{(e^x)^2+1}dx-\frac{1}{4}\int\frac{(e^{-2x}+1)'}{e^{-2x}+1}dx$

$=-\frac{1}{2}\ln(1+e^{-x})-\frac{1}{2}\arctan(e^x)-\frac{1}{4}\ln(1+e^{-2x})+c$

201. $\int\frac{\tan x+\sec x-1}{\tan x-\sec x+1}dx=\int\frac{\frac{2\tan(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}+\frac{1+\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}-\frac{1-\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}}{\frac{2\tan(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}-\frac{1+\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}+\frac{1-\tan^2(\frac{x}{2})}{1-\tan^2(\frac{x}{2})}}dx=\int\frac{2\tan(\frac{x}{2})(1+\tan(\frac{x}{2}))}{2\tan(\frac{x}{2})(1-\tan(\frac{x}{2}))}dx$

$=\int\frac{\tan(\frac{\pi}{4})+\tan(\frac{x}{2})}{\tan(\frac{\pi}{4})-\tan(\frac{x}{2})}dx=\int\frac{\sin(\frac{\pi}{4}+\frac{x}{2})}{\sin(\frac{\pi}{4}-\frac{x}{2})}dx=\int\frac{\cos(\frac{\pi}{4}-\frac{x}{2})}{\sin(\frac{\pi}{4}-\frac{x}{2})}dx=-2\ln\left|\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+c$

202. $\int\frac{1}{(x-1)(x-2)(x-3)(x-4)}dx=\frac{1}{2}\int\frac{(x-2)(x-3)-(x-1)(x-4)}{(x-1)(x-2)(x-3)(x-4)}dx$

$=\frac{1}{2}\int\frac{(x-2)(x-3)}{(x-1)(x-2)(x-3)(x-4)}dx-\frac{1}{2}\int\frac{(x-1)(x-4)}{(x-1)(x-2)(x-3)(x-4)}dx$

$=\frac{1}{2}\int\frac{1}{(x-1)(x-4)}dx-\frac{1}{2}\int\frac{1}{(x-2)(x-3)}dx=\frac{1}{6}\int\frac{(x-1)-(x-4)}{(x-1)(x-4)}dx-\frac{1}{2}\int\frac{(x-2)-(x-3)}{(x-2)(x-3)}dx$

$=\frac{1}{6}\int\left(\frac{1}{x-4}-\frac{1}{x-1}\right)dx-\frac{1}{2}\int\left(\frac{1}{x-3}-\frac{1}{x-2}\right)dx=\frac{1}{6}\ln\left|\frac{x-4}{x-1}\right|+\frac{1}{2}\ln\left|\frac{x-2}{x-3}\right|+c$

203. $\int\sqrt{x\sqrt{x\sqrt{x\sqrt{x\,\dots}}}}\,dx=\int x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots}dx=\int x^{\frac{\frac{1}{2}}{1-\frac{1}{2}}}dx=\int x\,dx=\frac{1}{2}x^2+c$

204. $\int\frac{1-\sin 2x}{1+\sin 2x}dx=\int\frac{(\cos^2 x+\sin^2 x)-2\sin x\cos x}{(\cos^2 x+\sin^2 x)+2\sin x\cos x}dx=\int\frac{(\cos x-\sin x)^2}{(\cos x+\sin x)^2}dx$

$=\int\left(\frac{\cos x-\sin x}{\cos x+\sin x}\right)^2dx=\int\left(\frac{\frac{\cos x-\sin x}{\cos x}}{\frac{\cos x+\sin x}{\cos x}}\right)^2dx=\int\left(\frac{1-\tan x}{1+\tan x}\right)^2dx=\int\left(\frac{\tan\frac{\pi}{4}-\tan x}{1+\tan\frac{\pi}{4}\tan x}\right)^2dx$

$=\int\tan^2\left(\frac{\pi}{4}-x\right)dx=\int\left[\sec^2\left(\frac{\pi}{4}-x\right)-1\right]dx=-\tan\left(\frac{\pi}{4}-x\right)-x+c$

205. $\int \tan x \tan 2x \, dx = \int \tan x \frac{2\tan x}{1-\tan^2 x} dx = \int \frac{2\tan^2 x}{1-\tan^2 x} dx = \int \frac{(\tan^2 x+1)+(\tan^2 x-1)}{1-\tan^2 x} dx$

$= \int \frac{\tan^2 x+1}{1-\tan^2 x} dx + \int \frac{-(1-\tan^2 x)}{1-\tan^2 x} dx = \int \frac{\sec^2 x}{1-\tan^2 x} dx - \int dx = \int \frac{(\tan x)'}{1-(\tan x)^2} dx - \int dx$

$= \tanh^{-1}(\tan x) - x + c$

206. $\int \frac{\csc x}{2\cos^2 x+\cos 2x} dx = \int \frac{\csc x}{2(1-\sin^2 x)+(1-2\sin^2 x)} dx = \int \frac{\csc x}{3-4\sin^2 x} dx$

$= \int \frac{\csc x}{3-4\sin^2 x} \cdot \frac{\sin x}{\sin x} dx = \int \frac{1}{3\sin x - 4\sin^3 x} dx = \int \frac{1}{\sin 3x} dx = \int \csc x \, dx$

$= \frac{1}{3} \ln|\csc 3x + \cot 3x| + c$

207. $\int \sin x \cos x \cos 2x \cos 4x \, dx = \frac{1}{2} \int 2\sin x \cos x \cos 2x \cos 4x \, dx$

$= \frac{1}{2} \int \sin 2x \cos 2x \cos 4x \, dx = \frac{1}{4} \int 2\sin 2x \cos 2x \cos 4x \, dx = \frac{1}{4} \int \sin 4x \cos 4x \, dx$

$= \frac{1}{8} \int 2\sin 4x \cos 4x \, dx = \frac{1}{8} \int \sin 8x \, dx = -\frac{1}{64} \cos 8x + c$

208. $\int \frac{2000x^{2014}+14}{x^{2015}-x} dx = \int \frac{2015x^{2014}-1-(15x^{2014}-15)}{x^{2015}-x} dx$

$= \int \frac{2015x^{2014}-1}{x^{2015}-x} dx - 15 \int \frac{x^{2014}-1}{x(x^{2014}-1)} dx = \int \frac{(x^{2015}-x)'}{x^{2015}-x} dx - 15 \int \frac{dx}{x}$

$= \ln|x^{2015} - x| - 15 \ln x + c = \ln|x^{2015} - x| - \ln x^{15} + c = \ln\left|\frac{x^{2014}-1}{x^{14}}\right| + c$

209. $\int \frac{dx}{4\sin^2 x+5\cos^2 x} = \int \frac{dx}{4\sin^2 x+5\cos^2 x} \times \frac{\sec^2 x}{\sec^2 x} = \int \frac{\sec^2 x}{\tan^2 x+5} dx$

$= \frac{1}{4} \int \frac{\sec^2 x}{\tan^2 x+\left(\frac{\sqrt{5}}{2}\right)^2} dx = \frac{1}{4} \int \frac{(\tan x)'}{(\tan x)^2+\left(\frac{\sqrt{5}}{2}\right)^2} dx = \frac{1}{4} \times \frac{2}{\sqrt{5}} \arctan\left(\frac{2\tan x}{\sqrt{5}}\right) + c$

$= \frac{1}{2\sqrt{5}} \arctan\left(\frac{2\tan x}{\sqrt{5}}\right) + c$

210. $\int \frac{\sin x \cos x}{\sqrt{a^2\sin^2 x+b^2\cos^2 x}} dx = \frac{1}{2} \int \frac{2\sin x \cos x}{\sqrt{a^2\sin^2 x+b^2(1-\sin^2 x)}} dx$

$= \frac{1}{2} \int \frac{2\sin x \cos x}{\sqrt{a^2\sin^2 x+b^2-b^2\sin^2 x}} dx = \frac{1}{2} \int \frac{2\sin x \cos x}{\sqrt{(a^2-b^2)\sin^2 x+b^2}} dx = \frac{1}{2} \int \frac{(\sin^2 x)'}{\sqrt{(a^2-b^2)\sin^2 x+b^2}} dx$

$= \frac{1}{2(a^2-b^2)} \int \frac{(a^2-b^2)(\sin^2 x)'}{\sqrt{(a^2-b^2)\sin^2 x+b^2}} dx = \frac{1}{2(a^2-b^2)} \int \frac{[(a^2-b^2)\sin^2 x+b^2]'}{\sqrt{(a^2-b^2)\sin^2 x+b^2}} dx$

$= \frac{1}{2(a^2-b^2)} \int [(a^2 - b^2)\sin^2 x + b^2]^{-\frac{1}{2}} [(a^2 - b^2)\sin^2 x + b^2]' dx$

$= \frac{1}{2\left(a^2-b^2\right)}\ 2\sqrt{\int[(a^2-b^2)\sin^2 x + b^2]} + c = \frac{\sqrt{\left(a^2-b^2\right)\sin^2 x+b^2}}{a^2-b^2} + c$

$= \frac{\sqrt{a^2\sin^2 x+b^2\cos^2 x}}{a^2-b^2} + c$

211. $\int \frac{\cos 2x-\cos 2\alpha}{\cos x-\cos\alpha}dx = \int \frac{-2\sin\left(\frac{2x+2\alpha}{2}\right)\sin\left(\frac{2x-2\alpha}{2}\right)}{-2\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}dx = \int \frac{\sin(x+\alpha)\sin(x-\alpha)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}dx$

$= \int \frac{2\sin\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x+\alpha}{2}\right).2\sin\left(\frac{x-\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)}{\sin\left(\frac{x+\alpha}{2}\right)\sin\left(\frac{x-\alpha}{2}\right)}dx = 4\int\cos\left(\frac{x+\alpha}{2}\right)\cos\left(\frac{x-\alpha}{2}\right)dx$

$= 4\int\frac{1}{2}\cos\left(\frac{x+\alpha+x-\alpha}{2}\right)+\frac{1}{2}\cos\left(\frac{x+\alpha-x+\alpha}{2}\right)dx = 2\int(\cos x+\cos\alpha)dx$

$= 2\sin x + x\cos\alpha + c$

212. $\int \frac{\sin^8 x-\cos^8 x}{1-2\sin^2 x\cos^2 x}dx = \int \frac{\left(\sin^4 x-\cos^4 x\right)\left(\sin^4 x+\cos^4 x\right)}{\left(\sin^2 x+\cos^2 x\right)^2-2\sin^2 x\cos^2 x}dx$

$= \int \frac{\left(\sin^2 x-\cos^2 x\right)\left(\sin^2 x+\cos^2 x\right)\left(\sin^4 x+\cos^4 x\right)}{\sin^4 x+\cos^4 x+2\sin^2 x\cos^2 x-2\sin^2 x\cos^2 x}dx$

$= \int \frac{\left(\sin^2 x-\cos^2 x\right)\left(\sin^4 x+\cos^4 x\right)}{\sin^4 x+\cos^4 x}dx = \int(\sin^2 x-\cos^2 x)dx = -\int\cos 2x\,dx$

$= -\frac{1}{2}\sin 2x + c$

213. $\int \frac{x^2}{(1-x)^{100}}dx = \int \frac{(x-1+1)^2}{(1-x)^{100}}dx \int \frac{(x-1)^2+2(x-1)+1}{(1-x)^{100}}dx$

$= \int\left[\frac{(1-x)^2}{(1-x)^{100}} - \frac{2(1-x)}{(1-x)^{100}} + \frac{1}{(1-x)^{100}}\right]dx = \int[(1-x)^{-98} - 2(1-x)^{-99} + (1-x)^{-100}]dx$

$= \frac{1}{97}(1-x)^{-97} - \frac{1}{49}(1-x)^{-98} + \frac{1}{99}(1-x)^{-99} + c$

214. $\int \frac{\left(x^2+\sin^2 x\right)}{1+x^2}.\sec^2 x\,dx = \int \frac{\left(x^2+1-\cos^2 x\right)}{1+x^2}.\sec^2 x\,dx = \int\left(1-\frac{\cos^2 x}{1+x^2}\right).\sec^2 x\,dx$

$= \int\left(\sec^2 x - \frac{1}{1+x^2}\right)dx = \tan x - \arctan x + c$

215. $\int \tan x\tan 2x\tan 3x\,dx$

We know that $\tan(a+b) = \frac{\tan a+\tan b}{1-\tan a\tan b}$, for $a = x$ and $b = 2x$, we get

$\tan 3x = \frac{\tan x+\tan 2x}{1-\tan x\tan 2x}$, then $\tan 3x - \tan x\tan 2x\tan 3x = \tan x + \tan 2x$, so we get

$\tan x\tan 2x\tan 3x = \tan 3x - \tan 2x - \tan x$, so:

$\int \tan x\tan 2x\tan 3x\,dx = \int(\tan 3x - \tan 2x - \tan x)dx$

$= -\frac{1}{3}\ln|\cos 3x| + \frac{1}{2}\ln|\cos 2x| + \ln|\cos x| + c$

216. $\int \frac{\cos x-\sin x}{\sqrt{1-\sin x}+\sqrt{1+\sin x}}dx$, we have $1-\sin x = 2\sin^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$,

$1+\sin x = 2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)$ and $\cos x - \sin x = 2\cos^2\left(\frac{x}{2}\right) - 1$, then:

$$\int \frac{\cos x-\sin x}{\sqrt{1-\sin x}+\sqrt{1+\sin x}}dx = \int \frac{\cos x-\sin x}{\sqrt{1-\sin x}+\sqrt{1+\sin x}} \times \frac{\sqrt{1+\sin x}-\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}dx$$

$$= \int \frac{(\cos x-\sin x)\left(\sqrt{1+\sin x}-\sqrt{1-\sin x}\right)}{1+\sin x-1+\sin x}dx = \int \frac{\left(2\cos^2\left(\frac{x}{2}\right)-1\right)\left(\sqrt{2}\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)-\sqrt{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)}{2\sin x}dx$$

$$= \int \frac{\left(2\cos^2\left(\frac{x}{2}\right)-1\right)\left(\sqrt{2}.\sqrt{2}\cos\left(\frac{\pi}{4}+\frac{\pi}{4}-\frac{x}{2}\right)\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}dx = \int \frac{\left(2\cos^2\left(\frac{x}{2}\right)-1\right)\sin\left(\frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}dx$$

$$= \int \left[2\cos\left(\frac{x}{2}\right)-\sec\left(\frac{x}{2}\right)\right]dx = 4\sin\left(\frac{x}{2}\right)-2\ln\left|\sec\left(\frac{x}{2}\right)+\tan\left(\frac{x}{2}\right)\right|+c$$

217. $$\int \frac{1}{\sqrt{2}+\sin x-\cos x}dx = \int \frac{1}{\sqrt{2}-\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)}dx$$

$$= \int \frac{1}{\sqrt{2}-\sqrt{2}\left(\sin\left(\frac{\pi}{4}\right)\cos x-\cos\left(\frac{\pi}{4}\right)\sin x\right)}dx = \int \frac{1}{\sqrt{2}-\sqrt{2}\sin\left(\frac{\pi}{4}-x\right)}dx = \int \frac{1}{\sqrt{2}\left(1-\sin\left(\frac{\pi}{4}-x\right)\right)}dx$$

$$= \int \frac{1}{2\sqrt{2}\sin^2\left(\frac{\pi}{8}+\frac{x}{2}\right)}dx = \frac{1}{2\sqrt{2}}\int \csc^2\left(\frac{\pi}{8}+\frac{x}{2}\right)dx = -\frac{1}{\sqrt{2}}\cot\left(\frac{\pi}{8}+\frac{x}{2}\right)+c$$

218. $$\int \frac{\cos 5x+\cos 4x}{1-2\cos 3x}dx = \int \frac{2\cos\left(\frac{5x+4x}{2}\right)\cos\left(\frac{5x-4x}{2}\right)}{1-2\cos 3x}dx = \int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)}{1-2\cos 3x}dx$$

$$= \int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)\sin 3x}{\sin 3x-2\cos 3x\sin 3x}dx = \int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}{\sin 3x-\sin 6x}dx$$

$$= \int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}{2\cos\left(\frac{3x+6x}{2}\right)\sin\left(\frac{3x-6x}{2}\right)}dx = \int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)2\sin\left(\frac{3x}{2}\right)\cos\left(\frac{3x}{2}\right)}{2\cos\left(\frac{9x}{2}\right)\sin\left(\frac{-3x}{2}\right)}dx$$

$$= -\int \cos\left(\frac{x}{2}\right)\cos\left(\frac{3x}{2}\right)dx = -\int(\cos 2x+\cos(-x))dx = -\frac{1}{2}\sin 2x-\sin x+c$$

219. $$\int \frac{dx}{\sqrt{1+\arcsin x-x^2-x^2\arcsin x}} = \int \frac{dx}{\sqrt{(1+\arcsin x)-x^2(1+\arcsin x)}}$$

$$= \int \frac{dx}{\sqrt{(1+\arcsin x)(1-x^2)}} = \int \frac{dx}{\sqrt{1-x^2}}.\frac{1}{\sqrt{1+\arcsin x}} = \int(1+\arcsin x)'(1+\arcsin x)^{-\frac{1}{2}}dx$$

$$= 2\sqrt{1+\arcsin x}+c$$

220. $$\int \sin^2\left(\arctan\sqrt{\frac{1-\cos 2x}{1+\cos 2x}}\right)dx = \int \sin^2\left(\arctan\sqrt{\frac{2\sin^2 x}{2\cos^2 x}}\right)dx$$

$$= \int \sin^2\left(\arctan\sqrt{\tan^2 x}\right)dx = \int \sin^2(\arctan(\tan x))\,dx = \int \sin^2 x\,dx = \int\left(\frac{1-\cos 2x}{2}\right)dx$$

$$= \frac{1}{2}x-\frac{1}{4}\sin 2x+c$$

221. $$\int \sin(\ln x)\,dx = \frac{1}{2}\int 2\sin(\ln x)\,dx$$

$$= \frac{1}{2}\int\left[\sin\left(\ln x\right)+\cos\left(\ln x\right)+\sin\left(\ln x\right)-\cos\left(\ln x\right)\right]dx$$

$$= \frac{1}{2}\int[x\sin(\ln x)]'\,dx-\frac{1}{2}\int[x\cos(\ln x)]'\,dx = \frac{1}{2}x[\sin(\ln x)-\cos(\ln x)]+c$$

222. $$\int \frac{1}{\tan x+\cot x+\sec x+\csc x}dx = \int \frac{1}{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}+\frac{1}{\cos x}+\frac{1}{\sin x}}dx$$

$$= \int \frac{\sin x\cos x}{\sin^2 x+\cos^2 x+\sin x+\cos x}dx$$

$= \int \frac{\sin x \cos x}{1+\sin x+\cos x}dx = \int \frac{\sin x \cos x}{1+\sin x+\cos x} \times \frac{\sin x+\cos x-1}{\sin x+\cos x-1}dx = \int \frac{\sin x \cos x(\sin x+\cos x-1)}{(\sin x+\cos x)^2-1}dx$

$= \int \frac{\sin x \cos x(\sin x+\cos x-1)}{\sin^2 x+\cos^2 x+2\sin x \cos x-1}dx = \int \frac{\sin x \cos x(\sin x+\cos x-1)}{2\sin x \cos x}dx$

$= \frac{1}{2}\int(\sin x + \cos x - 1)dx = \frac{1}{2}(\sin x - \cos x - x) + c$

223. $\int \frac{\tan 2x}{\sqrt{\sin^4 x+4\cos^2 x}-\sqrt{\cos^4 x+4\sin^2 x}}dx$

$= \int \frac{\tan 2x}{\sqrt{\sin^4 x-4\sin^2 x+4}-\sqrt{\cos^4 x-4\cos^2 x+4}}dx$

$= \int \frac{\tan 2x}{\sqrt{\left(\sin^2 x-2\right)^2}-\sqrt{\left(\cos^2 x-2\right)^2}}dx=\int \frac{\tan 2x}{\left(\sin^2 x-2\right)-\left(\cos^2 x-2\right)}dx = \int \frac{\tan 2x}{\sin^2 x-\cos^2 x}dx$

$= -\int \frac{\tan 2x}{\cos 2x}dx = -\frac{1}{2}\int 2\sec x \tan x\, dx = -\frac{1}{2}\sec 2x + c$

224. $\int \sin^4 x \cos^4 x\,(\cos x + \sin x)(\cos x - \sin x)dx$

$= \int(\sin x \cos x)^4(\cos^2 x - \sin^2 x)dx = \int \left(\frac{1}{2}\sin 2x\right)^4 \cos 2x\, dx = \frac{1}{16}\int(\sin 2x)^4 \cos 2x\, dx$

$= \frac{1}{32}\int(\sin 2x)^4(2\cos 2x)dx = \frac{1}{32}\int(\sin 2x)^4(\sin 2x)'dx = \frac{1}{32}\left(\frac{\sin^5 2x}{5}\right) + \text{c} = \frac{1}{160}\sin^5 2x + c$

225. $\int \sqrt{(x+3)(x+4)(x+5)(x+6)+1}\,dx = \int \sqrt{(x+3)(x+6)(x+4)(x+5)+1}\,dx$

$= \int \sqrt{(x^2+9x+18)(x^2+9x+20)+1}\,dx$

Simplifying $(x^2+9x+18)(x^2+9x+20)+1$. Let's write $x^2+9x=\alpha$, then

$(x^2+9x+18)(x^2+9x+20)+1 = (\alpha+18)(\alpha+20)+1 = \alpha^2+20\alpha+18\alpha+360+1$

$= \alpha^2+38\alpha+361 = (\alpha+19)^2$, with $\alpha = x^2+9x$, so**:**

$(x^2+9x+18)(x^2+9x+20)+1 = (x^2+9x+19)^2$, then**:**

$\int \sqrt{(x+3)(x+4)(x+5)(x+6)+1}\,dx = \int \sqrt{(x^2+9x+19)^2}\,dx = \int(x^2+9x+19)dx$

$= \frac{1}{3}x^3 + \frac{9}{2}x^2 + 19x + c$

End of Chapter 1

Chapter 2: Methods Of Integration

1. Integration by Substitution:

In calculus, integration by substitution, also known as u-substitution or change of variables, is a method for evaluating integrals and anti-derivatives. It is the counterpart to the chain rule for differentiation (reverse chain rule).

The first and the most vital step is to be able to write our integral in the form**:**

$$\int \boldsymbol{f}(\mathbf{g}(\boldsymbol{x})) \times \mathbf{g}'(\boldsymbol{x})\boldsymbol{dx} = \int \boldsymbol{f}(\boldsymbol{u})\boldsymbol{du}\,;\ \ \textbf{where } \boldsymbol{u} = \mathbf{g}(\boldsymbol{x}) \textbf{ and } \mathbf{g}'(\boldsymbol{x}) = \frac{\boldsymbol{du}}{\boldsymbol{dx}}$$

Then we can integrate $f(u)$ and finish by putting $\mathrm{g}(x)$ back as u.

Example: $\int 2x\cos(x^2)\,dx = \int \cos u\,du$, where $u = x^2$ and $2x = \frac{du}{dx}$, so we get**:**

$\int 2x\cos(x^2)\,dx = \sin u + c = \sin(x^2) + c$

2. Integration Using Trigonometric Identities:

Trigonometric substitution is a technique for evaluating integrals for which we use trigonometric identities to simplify certain integrals (especially some forms containing the radical sign **…**)

a. Integrands containing $\boldsymbol{a^2 - x^2}$ **(or** $\sqrt{\boldsymbol{a^2 - x^2}}$**)**, use the change of variable $\boldsymbol{x = a\sin\theta}$

b. Integrands containing $\boldsymbol{a^2 + x^2}$ **(or** $\sqrt{\boldsymbol{a^2 + x^2}}$**)**,, use the change of variable $\boldsymbol{x = a\tan\theta}$

c. Integrands containing $\boldsymbol{x^2 - a^2}$ **(or** $\sqrt{\boldsymbol{x^2 - a^2}}$**)**, use the change of variable $\boldsymbol{x = a\sec\theta}$

d. Integrands containing $\sqrt{\boldsymbol{\frac{a-x}{a+x}}}$ **(or** $\sqrt{\boldsymbol{\frac{a+x}{a-x}}}$**)**, use the change of variable $\boldsymbol{x = a\cos 2\theta}$

e. Integrands containing $\sqrt{\boldsymbol{\frac{x-a}{b-x}}}$ **(or** $\sqrt{\boldsymbol{(x-a)(b-x)}}$**)**, use the change of variable $\boldsymbol{x = a\cos^2\theta + b\sin^2\theta}$

f. Integrands containing $\boldsymbol{2ax - x^2}$ **(or** $\sqrt{\boldsymbol{2ax - x^2}}$**)**, use the change of variable $\boldsymbol{x = a(1-\cos\theta)}$

g. Integrands containing $\sqrt{\boldsymbol{\frac{x}{a+x}}}$ **(or** $\sqrt{\boldsymbol{\frac{a+x}{x}}}$ **or** $\sqrt{\boldsymbol{x(a+x)}}$**)**, use the change of variable $\boldsymbol{x = a\tan^2\theta}$ or $\boldsymbol{x = a\cot^2\theta}$

h. Integrands containing $\sqrt{\boldsymbol{\frac{x}{a-x}}}$ **(or** $\sqrt{\boldsymbol{\frac{a-x}{x}}}$ **or** $\sqrt{\boldsymbol{x(a-x)}}$ **or** $\boldsymbol{\frac{1}{\sqrt{x(a-x)}}}$**)**, use the change of variable $\boldsymbol{x = a\sin^2\theta}$ or $\boldsymbol{x = a\cos^2\theta}$

i. Integrands containing $\sqrt{\boldsymbol{\frac{x}{x-a}}}$ **(or** $\sqrt{\boldsymbol{\frac{x-a}{x}}}$ **or** $\sqrt{\boldsymbol{x(x-a)}}$ **or** $\boldsymbol{\frac{1}{\sqrt{x(x-a)}}}$**)**, use the change of variable $\boldsymbol{x = a\sec^2\theta}$ or $\boldsymbol{x = a\csc^2\theta}$

Remark: For the forms $a^2 - b^2x^2$, $a^2 + b^2x^2$ and $b^2x^2 - a^2$, we may use $x = \frac{a}{b}\sin\theta$, $x = \frac{a}{b}\tan\theta$ and $x = \frac{a}{b}\sec\theta$ respectively

Example: $\int \frac{dx}{\sqrt{a^2-x^2}}$, according to the above information, we take the change of variable $x = a\sin\theta$, then $dx = a\cos\theta\, d\theta$, then:

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2-a^2\sin^2\theta}} = \int \frac{a\cos\theta d\theta}{\sqrt{a^2(1-\sin^2\theta)}} = \int \frac{a\cos\theta d\theta}{a\sqrt{\cos^2\theta}} = \int d\theta = \theta + c, \text{ but:}$$

$x = a\sin\theta$, so $\sin\theta = \frac{x}{a}$ and so $\theta = \arcsin\left(\frac{x}{a}\right)$, therefore, $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right)$

3. **Integration Using Hyperbolic Identities:**
   - a. Integrands containing $\boldsymbol{x^2 + a^2}$ **(or** $\sqrt{\boldsymbol{x^2+a^2}}$**)**, use the change of variable $\boldsymbol{x = a\sinh t}$
   - b. Integrands containing $\boldsymbol{x^2 - a^2}$ **(or** $\sqrt{\boldsymbol{x^2-a^2}}$**)**,, use the change of variable $\boldsymbol{x = a\cosh t}$

Example: $\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{dx}{\sqrt{x^2-2^2}}$, let $x = 2\cosh t$, then $dx = 2\sinh t\, dt$, so:

$$\int \frac{dx}{\sqrt{x^2-4}} = \int \frac{2\sinh t}{\sqrt{4\cosh^2 t-4}}dt = \int \frac{2\sinh t}{\sqrt{4(\cosh^2 t-1)}}dt = \int \frac{2\sinh t}{2\sqrt{\sinh^2 t}}dt = \int dt = t + c, \text{ but}$$

$x = 2\cosh t$, so $\cosh t = \frac{x}{2}$, then $t = \cosh^{-1}\left(\frac{x}{2}\right) = \ln\left(\frac{x}{2} + \sqrt{\frac{x^2}{4} - 1}\right) = \ln\left[\frac{1}{2}\left(x + \sqrt{x^2-4}\right)\right]$

therefore, $\int \frac{dx}{\sqrt{x^2-4}} = \ln\left[\frac{1}{2}\left(x+\sqrt{x^2-4}\right)\right] + c = \ln\left(x + \sqrt{x^2-4}\right) + k$

4. **Decomposition Into Partial Fractions:**
   Partial-fraction decomposition is the process of starting with the simplified answer and taking it back apart, of "decomposing" the final expression into its initial polynomial fractions.

Factor in the denominator	Term in partial fraction decomposition
$\boldsymbol{ax+b}$	$\frac{\mathrm{A}}{ax+b}$
$\boldsymbol{(ax+b)^k}$	$\frac{\mathrm{A_1}}{ax+b} + \frac{\mathrm{A_2}}{(ax+b)^2} + \cdots + \frac{\mathrm{A_k}}{(ax+b)^k}$; $k = 1,2,3,\ldots$
$\boldsymbol{ax^2+bx+c}$	$\frac{\mathrm{A}x+\mathrm{B}}{ax^2+bx+c}$
$\boldsymbol{(ax^2+bx+c)^k}$	$\frac{\mathrm{A_1}x+\mathrm{B_1}}{ax^2+bx+c} + \frac{\mathrm{A_2}x+\mathrm{B_2}}{(ax^2+bx+c)^2} + \cdots + \frac{\mathrm{A_k}x+\mathrm{B_k}}{(ax^2+bx+c)^k}$ $k = 1,2,3,\ldots$

To find the partial fraction decomposition (PFD) of a rational expression, follow these steps:

Step1: Factor the denominator

Step2: Find the form of the PFD

a. For each term $(x+a)^n$, the PFD has terms : $\frac{C_1}{x+a}+\frac{C_2}{(x+a)^2}+\cdots+\frac{C_n}{(x+a)^n}$

b. For each term $(x^2+ax+b)^n$, the PFD has terms**:**

$$\frac{C_1x+D_1}{x^2+ax+b}+\frac{C_2x+D_2}{(x^2+ax+b)^2}+\cdots+\frac{C_nx+D_n}{(x^2+ax+b)^n}$$

Step3: Solve for the coefficients $C_1, C_2, \ldots, C_n$

The best way is to clear all denominators and then subsutitute "intelligent" values of x (e.g., the roots of the linear factors).

Sometimes plugging in other values of x is necessary, in order to find the coefficients of the higher terms

One can also employ other more clever methods, such as taking derivatives

Step4: Evaluate the integral

a. Terms of the form : $\frac{C}{(x+a)^n}$ can be integrated directly using the power rule

b. Terms of the form : $\frac{Cx+D}{(x^2+ax+b)^n}$ should be separated further as

$$\frac{E(2x+a)}{(x^2+ax+b)^n}+\frac{F}{(x^2+ax+b)^n}$$

The first term can be integrated by substituting $u=x^2+ax+b$ and the second term can be integrated by completing the square as $\left(x+\frac{a}{2}\right)^2+\left(b-\frac{a^2}{4}\right)=d^2\left[\left(\frac{x+c}{d}\right)^2+1\right]$

Example 1: $\int\frac{1}{x^2-1}dx$, let's decompose : $\frac{1}{x^2-1}$ into partial fractions**:**

Then : $\frac{1}{x^2-1}=\frac{1}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$, then $1=A(x+1)+B(x-1)$, taking $x=1$, we get $1=2A$, so $A=\frac{1}{2}$ and taking $x=-1$, we get $1=-2B$, so $B=-\frac{1}{2}$, so we get

$$\int\frac{1}{x^2-1}dx=\frac{1}{2}\int\left(\frac{1}{x-1}-\frac{1}{x+1}\right)dx=\frac{1}{2}\ln|x-1|-\frac{1}{2}\ln|x+1|+c=\frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|+c$$

Example 2: $\int\frac{2x^2+x+3}{x^3+x^2+x+1}dx$, let's decompose : $\frac{2x^2+x+3}{x^3+x^2+x+1}$ into partial fractions**:**

Then : $\frac{2x^2+x+3}{x^3+x^2+x+1}=\frac{2x^2+x+3}{x^2(x+1)+(x+1)}=\frac{2x^2+x+3}{(x^2+1)(x+1)}=\frac{Ax+B}{x^2+1}+\frac{C}{x+1}$, then we get

$2x^2+x+3=(x+1)(Ax+B)+(x^2+1)C$, $2x^2+x+3=(A+C)x^2+(A+B)x+B+C$

By comparing the both sides of the equality we get the system $\begin{cases}A+C=2\\A+B=1\\B+C=3\end{cases}$, by solving this system we get**:** $A=0$, $B=1$ and $C=2$, so we get : $\frac{2x^2+x+3}{x^3+x^2+x+1}=\frac{1}{x^2+1}+\frac{2}{x+1}$, so we get**:**

$$\int\frac{2x^2+x+3}{x^3+x^2+x+1}dx=\int\left(\frac{1}{x^2+1}+\frac{2}{x+1}\right)dx=\arctan x+2\ln|x+1|+c$$

5. **Weierstrass Substitution:**

The **Weierstrass substitution**, named after German mathematician Karl Weierstrass (1815 – 1897), is used for converting rational expressions of trigonometric functions into algebraic rational functions, which may be easier to integrate.

This method is also called tangent **half-angle substitution** as it implies the following half-angle identities**:**

$\boldsymbol{t = \tan\frac{x}{2}}$	$\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2}$
$\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$	$\tan x = \frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{2t}{1-t^2}$
$\cot x = \frac{1-\tan^2\frac{x}{2}}{2\tan\frac{x}{2}} = \frac{1-t^2}{2t}$	$\sec x = \frac{1+\tan^2\frac{x}{2}}{1-\tan^2\frac{x}{2}} = \frac{1+t^2}{1-t^2}$
$\csc x = \frac{1+\tan^2\frac{x}{2}}{2\tan\frac{x}{2}} = \frac{1+t^2}{2t}$	$dx = d(2\arctan t) = \frac{2dt}{1+t^2}$

Example 1: $\int \frac{dx}{1+\sin x}$, let $t = \tan\frac{x}{2}$, then $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$, then**:**

$$\int \frac{dx}{1+\sin x} = \int \frac{\frac{2}{1+t^2}dt}{1+\frac{2t}{1+t^2}} = \int \frac{2dt}{1+t^2+2t} = \int \frac{2dt}{(t+1)^2} = -\frac{2}{t+1} + c = -\frac{2}{\tan\frac{x}{2}+1} + c$$

Example 2: $\int \frac{dx}{\sin x+\cos x+1}$, let $t = \tan\frac{x}{2}$, then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$,

then**:** $\int \frac{dx}{\sin x+\cos x+1} = \int \frac{\frac{2}{1+t^2}dt}{\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}+1} = \int \frac{\frac{2}{1+t^2}dt}{\frac{2t+1-t^2+1+t^2}{1+t^2}} = \int \frac{2}{2t+2}dt = \int \frac{1}{t+1}dt$

$= \ln|t+1| + c = \ln\left|\tan\frac{x}{2}+1\right| + c$

Example 3: $\int \csc x\, dx = \int \frac{dx}{\sin x}$, let $t = \tan\frac{x}{2}$, then $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2}{1+t^2}dt$, then**:**

$$\int \csc x\, dx \int \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}}dt = \int \left(\frac{1+t^2}{2t}\right)\left(\frac{2}{1+t^2}\right)dt = \int \frac{dt}{t} = \ln|t| + c = \ln\left|\tan\left(\frac{x}{2}\right)\right| + c$$

6. Bioche's Rules:

Are rules to evaluate indefinite integrals in which the integrand contains sines and cosines
Each of the following rules gives a change of variables that brings back the calculation of $\int f(x)dx$, where $f(x)$ is a rational expression in $\sin x$ and $\cos x$
Suppose that $w(x) = \mathrm{F}(\cos x, \sin x)dx$ be the differential element of the asked integral, then:

1. If $w(-x) = w(x)$, put $t = \cos x$
2. If $w(\pi - x) = w(x)$, put $t = \sin x$
3. If $w(\pi + x) = w(x)$, put $t = \tan x$
4. If two of the preceding relations both hold, a good change of variable is $t = \cos 2x$
5. In all other cases we can use the universal substitution $t = \tan\left(\frac{x}{2}\right)$

Remark:

Bioche's rules formulated by the French mathematician Charles Bioche (1859 – 1949)

Example 1: $\int \sin^5 x\, dx$

$\sin^5(-x)\, d(-x) = (-\sin^5 x)(-dx) = \sin^5 x\, dx$, then let $t = \cos x$ and $dt = -\sin x\, dx$, so:

$\int \sin^5 x\, dx = \int (\sin^2 x)^2 \sin x\, dx = \int (1 - \cos^2 x)^2 \sin x\, dx = \int (1 - t^2)^2\,(-dt)$

$= \int(-t + 2t^2 - t^4)dt = -t + \frac{2}{3}t^3 - \frac{1}{5}t^5 + c = -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$

Example 2: $\int \frac{1}{1+\tan x}dx$

We have : $\frac{d(\pi+x)}{1+\tan(\pi+x)} = \frac{dx}{1+\tan x}$, then let $t = \tan x$, then $dt = (1 + \tan^2 x)dx = (1 + t^2)dx$

So: $\int \frac{1}{1+\tan x}dx = \int \frac{1}{(1+t^2)(1+t)}dt = \int \left[\frac{1}{2(1+t)} + \frac{1-t}{2(1+t^2)}\right]dx$

$= \frac{1}{2}\int \frac{dt}{1+t} + \frac{1}{2}\int \frac{dt}{1+t^2} - \frac{1}{2}\int \frac{t}{1+t^2}dt = \frac{1}{2}\ln|1 + t| + \frac{1}{2}\arctan t - \frac{1}{4}\ln(1 + t^2) + c$

$= \frac{1}{2}\ln|1 + \tan x| + \frac{1}{2}\arctan(\tan x) - \frac{1}{4}\ln(\sec^2 x) + c$

Remark:

If we are dealing with integrands having the form $\mathrm{F}(\cosh x, \sinh x)$, where F is a rational function, then we can use each of the following rules:

1. If $\int \mathrm{F}(\cos x, \sin x)\, dx$ could be performed using the change of variable $t = \cos x$, then it will be able to set $t = \cosh x$ to perform $\int \mathrm{F}(\cosh x, \sinh x)\, dx$
2. If $\int \mathrm{F}(\cos x, \sin x)\, dx$ could be performed using the change of variable $t = \sin x$, then it will be able to set $t = \sinh x$ to perform $\int \mathrm{F}(\cosh x, \sinh x)\, dx$
3. If $\int \mathrm{F}(\cos x, \sin x)\, dx$ could be performed using the change of variable $t = \tan x$, then it will be able to set $t = \tanh x$ to perform $\int \mathrm{F}(\cosh x, \sinh x)\, dx$
4. If $\int \mathrm{F}(\cos x, \sin x)\, dx$ could be performed using the change of variable $t = \cos 2x$, then it will be able to set $t = \cosh 2x$ to perform $\int \mathrm{F}(\cosh x, \sinh x)\, dx$
5. In any other cases, we can use the change of variable $t = \tanh\left(\frac{x}{2}\right)$
6. In general we can perform hyperbolic integrals with the change of variable $t = e^x$

Note: To evaluate the integral $\int \sin^m x \cos^n x\, dx$, where $m, n \in \mathbb{N}$

1. Check if m is odd, then put $t = \cos x$
2. If n is odd then put $t = \sin x$
3. If both m and n are odd then put $t = \sin x$ if $m \geq n$ and $t = \cos x$ otherwise
4. If both m and n are even, then use the power reducing formulas
$$\sin^2 x = \frac{1 - \cos 2x}{2} \quad \text{and} \quad \cos^2 x = \frac{1 + \cos 2x}{2}$$
5. **Special case:** If $m + n$ is a negative even integer then put $t = \tan x$

Example: $\int \cos^5 x \sin^2 x\, dx$, $n = 5$ odd, then let $t = \sin x$, so $dt = \cos x\, dx$, then**:**
$\int \cos^5 x \sin^2 x\, dx = \int (\cos^2 x)^2 \sin^2 x \cos x\, dx = \int (1 - \sin^2 x)^2 \sin^2 x \cos x\, dx$
$= \int (1 - t^2)^2 t^2 dt = \int (t^2 - 2t^4 + t^6) dt = \frac{1}{3}t^3 - \frac{2}{5}t^5 + \frac{1}{7}t^7 + c$
Therefore**:** $\int \cos^5 x \sin^2 x\, dx = \frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + c$

7. **Euler's Substitution:**

Euler substitution is a method for evaluating integrals of the form $\int \mathrm{R}\left(x, \sqrt{ax^2 + bx + c}\right)$, where R is a rational function.

Euler's first substitution:

It is used when $a > 0$ and we substitute $\sqrt{ax^2 + bx + c} = \pm x\sqrt{a} + t$ and so we get
$$x = \frac{c - t^2}{\pm 2t\sqrt{a} - b}$$

Euler's second substitution:

It is used when $c > 0$ and we substitute $\sqrt{ax^2 + bx + c} = xt \pm \sqrt{c}$ and so we get
$$x = \frac{\pm 2t\sqrt{c} - b}{a - t^2}$$

Euler's third substitution:

If the polynomial $ax^2 + bx + c$ has real roots α and β, then we can substitute
$$(x - \alpha)t = \sqrt{ax^2 + bx + c} = \sqrt{a(x - \alpha)(x - \beta)} \quad \text{and so we have} \quad x = \frac{a\beta - \alpha t^2}{a - t^2}$$

Example 1: $\int \frac{dx}{\sqrt{x^2+2}}$, we can use **Euler's first substitution**, $x = \frac{t^2-2}{2t}$ and $dx = \frac{t^2+2}{2t^2}dt$, so**:**

$\int \frac{dx}{\sqrt{x^2+2}} = \int \frac{\frac{t^2+2}{2t^2}}{\frac{t^2+2}{2t}} dt$, since $\sqrt{x^2 + 2} = -x + t = -\frac{t^2-2}{2t} + t = \frac{-t^2+2+2t^2}{2t} = \frac{t^2+2}{2t}$, then**:**

$\int \frac{dx}{\sqrt{x^2+2}} = \int \frac{dt}{t} = \ln|t| + c = \ln\left|x + \sqrt{x^2 + 2}\right| + c$

Example 2: $\int \frac{dx}{x\sqrt{-x^2+x+2}}$, we can use **Euler's second substitution**

$x = \frac{1-2\sqrt{2}t}{t^2+1}$ and $dx = \frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{\left(t^2+1\right)^2} dt$, so**:**

$\int \frac{dx}{x\sqrt{-x^2+x+2}} = \int \frac{\frac{2\sqrt{2}t^2-2t-2\sqrt{2}}{\left(t^2+1\right)^2}}{\frac{1-2\sqrt{2}t}{t^2+1}\cdot\frac{-\sqrt{2}t^2+t+\sqrt{2}}{t^2+1}}dt$, since $\sqrt{-x^2+x+2} = xt+\sqrt{2} = \frac{-\sqrt{2}t^2+t+\sqrt{2}}{t^2+1}$ so:

$$\int \frac{dx}{x\sqrt{-x^2+x+2}} = \int \frac{-2}{-2\sqrt{2}t+1}dt = \frac{1}{\sqrt{2}}\int \frac{-2\sqrt{2}}{-2\sqrt{2}t+1}dt = \frac{1}{\sqrt{2}}\ln\left|2\sqrt{2}t-1\right| + c$$

$$= \frac{1}{\sqrt{2}}\ln\left|2\sqrt{2}\,\frac{\sqrt{-x^2+x+2}-\sqrt{2}}{x} - 1\right| + c$$

Example 3: $\int \frac{1}{\sqrt{-x^2+3x-2}}dx$, we can use **Euler's third substitution**

$\sqrt{-x^2+3x+2} = \sqrt{-(x-2)(x-1)} = (x-2)t$, then $x = \frac{-2t^2-1}{-t^2-1}$ and $dx = \frac{2t}{\left(-t^2-1\right)^2}dt$

then we evaluate the integral in terms of t and then we get the final answer in terms of x.

__Remark:__ To evaluate an integral of the form $\int \mathrm{F}\left(x, \sqrt{\frac{ax+b}{cx+d}}\right)$, use the change of variable $t = \sqrt{\frac{ax+b}{cx+d}}$, so that $x = \frac{b-dt^2}{ct^2-a}$ and $dx = 2\frac{ad-bc}{\left(ct^2-a\right)^2}dt$

8. **__Binomial Integral:__**

It is the integral of the form $\int x^m(a+bx^n)^p dx$, where a and b are real numbers while m, n and p are rational numbers.

If m, n and p are all integers then the integrand is a rational function integration

- If m and n are fractions and p is an integer, then the integral can be solved using the substitution $x = t^s$, where s is the least common denominator of m and n
- If p is a fraction and : $\frac{m+1}{n}$ is an integer, then the integral can be solved using the substitution $a+bx^n = t^s$, where s is the denominator of p
- Is p is a fraction and : $\frac{m+1}{n+p}$ is an integer,, then the integral can be solved using the substitution $ax^{-n}+b = t^s$, where s is the denominator of p

__Remark__

1. To evaluate the integral $\int \frac{dx}{(ax+b)\sqrt{px+q}}$ use the change of variable $px+q = u^2$
2. To evaluate the integral $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$ use the change of variable $px+q = u^2$
3. To evaluate the integral $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ use the change of variable $ax+b = \frac{1}{u}$
4. To evaluate the integral $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$ use the change of variable $x = \frac{1}{u}$

5. To evaluate the integral $\int \frac{dx}{(ax+b)^m\sqrt{ax^2+bx+c}}$ use the change of variable $ax+b=\frac{1}{u}$

6. To evaluate the integral $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$ use the change of variable $x=a\cos^2 t+b\sin^2 t$

Example 1: $\int \frac{dx}{(2x+3)\sqrt{x+1}}$

Let $u^2=x+1$, then $2udu=dx$, $2x+3=2(x+1)+1=2u^2+1$, so:

$$\int \frac{dx}{(2x+3)\sqrt{x+1}}=\int \frac{2udu}{(2u^2+1).u}=2\int \frac{1}{1+2u^2}du=\frac{2}{\sqrt{2}}\int \frac{\sqrt{2}}{1+(\sqrt{2}u)^2}du=\sqrt{2}\arctan(\sqrt{2}u)+c$$

$$=\sqrt{2}\arctan(\sqrt{2x+2})+c$$

Example 2: $\int \frac{dx}{(x-1)\sqrt{x^2-2}}$, let $x-1=\frac{1}{u}\left(x=\frac{1}{u}+1=\frac{u+1}{u}\right)$, $dx=-\frac{1}{u^2}du$, so:

$$\int \frac{dx}{(x-1)\sqrt{x^2+1}}=\int \frac{-\frac{1}{u^2}du}{\frac{1}{u}\sqrt{\left(\frac{u+1}{u}\right)^2-2}}=-\int \frac{du}{u\times\frac{1}{u}\sqrt{(u+1)^2-2u^2}}=-\int \frac{du}{\sqrt{1+2u-u^2}}=-\int \frac{du}{\sqrt{2-(u-1)^2}}$$

$$\int \frac{dx}{(x-1)\sqrt{x^2-2}}=-\int \frac{du}{\sqrt{(\sqrt{2})^2-(u-1)^2}}=-\arcsin\left(\frac{u-1}{\sqrt{2}}\right)+c=-\arcsin\left(\frac{\frac{1}{x-1}-1}{\sqrt{2}}\right)+c$$

$$\int \frac{dx}{(x-1)\sqrt{x^2-2}}=-\arcsin\left(\frac{2-x}{\sqrt{2}x-\sqrt{2}}\right)+c$$

9. **Integration by Parts:**

Let u and v be two differentiable functions and let u' and v' be their respective derivatives which are continuous, then:

$$\int u(x)\times v'(x)dx=(x)\times v(x)-\int u'(x)\times v(x)dx$$

Example: $\int xe^{2x}\,dx$, let $u=x$, then $u'=1$ and let $v'=e^{2x}$, then $v=\frac{1}{2}e^{2x}$, then:

$$\int xe^{2x}\,dx=\frac{1}{2}xe^{2x}-\int \frac{1}{2}e^{2x}dx=\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+c=\frac{1}{4}(2x-1)e^{2x}+c$$

Integration Techniques Summary

Substitution or Change of Independent Variable

Integral I $= \int f(x)dx$ is changed to $\int f(w(t)) \times w'(t)dt$, by a suitable substitution $x = w(t)$ provided the later integral is easier to integrate

Some Standard Substitution:

$$\int [f(x)]^n f'(x)dx \quad \text{or} \quad \int \frac{f'(x)}{[f(x)]^n}dx; \quad \text{put } f(x) = t \ \& \text{ proceed}$$

$$\int \frac{dx}{ax^2+bx+c}; \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}; \quad \int \sqrt{ax^2+bx+c}\,dx$$

Write ax^2+bx+c in canonical form (perfect square form) & then apply the standard results

$$\int \frac{px+q}{ax^2+bx+c}dx; \quad \int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$$

Express $px+q =$ A (differential coefficient of denominator) +B

$$\int e^x[f(x)+f'(x)]dx = e^x.f(x)+c \quad \& \quad \int [f(x)+xf'(x)]dx = xf(x)+c$$

$$\int \frac{dx}{x(x^n+1)} \quad n \in \mathbb{N}; \text{ take } x^n \text{ common } \& \ put\ 1 = x^{-n} = \mathrm{t}$$

$$\int \frac{dx}{x^2(x^n+1)^{\frac{n-1}{n}}} \quad n \in \mathbb{N}; \text{ take } x^n \text{ common } \& \ put\ 1 = x^{-n} = \mathrm{t}^n$$

$$\int \frac{dx}{x^n(1+x^n)^{\frac{1}{n}}}; \text{ take } x^n \text{ common } \& \ put\ 1 = x^{-n} = \mathrm{t}$$

$$\int \frac{dx}{a+b\sin^2 x} \quad \text{or} \quad \int \frac{dx}{a+b\cos^2 x} \quad \text{or} \quad \int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x}$$

Multiply Numerator & Denominator by $\sec^2 x$ & then put $\tan x = t$

$$\int \frac{dx}{a+b\sin x} \quad \text{or} \quad \int \frac{dx}{a+b\cos x} \quad \text{or} \quad \int \frac{dx}{a+b\sin x + c\cos x}$$

Convert sines & cosines into their respective tangents of half the angles and then put $\tan\frac{x}{2} = t$

$$\int \frac{a\cos x + b\sin x + c}{p\cos x + q\sin x + r}dx$$

Express Numerator (N$^{\mathrm{r}}$) $\equiv l(\mathrm{D^r}) + m\frac{d}{dx}(\mathrm{D^r}) + n$ and proceed

$$\int \frac{x^2+1}{x^4+kx^2+1}dx \quad \text{or} \quad \int \frac{x^2-1}{x^4+kx^2+1}dx; \quad k \text{ is a constant}$$

Divide N$^{\mathrm{r}}$ & D$^{\mathrm{r}}$ by x^2 & then put $x - \frac{1}{x} = t$ or $x + \frac{1}{x} = t$ respectively and proceed

$$\int \frac{dx}{(ax+b)\sqrt{px+q}} \quad \& \quad \int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}; \quad \text{put } px+q = t^2$$

$$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}} \left(\text{put } ax+b = \frac{1}{t}\right); \quad \int \frac{dx}{(ax^2+bx+c)\sqrt{px^2+qx+r}} \left(\text{put } x = \frac{1}{t}\right)$$

$$\int \sqrt{\frac{x-\alpha}{\beta-x}}dx \quad \text{or} \quad \int \sqrt{(x-\alpha)(\beta-x)}dx; \quad \text{put } x = \alpha\cos^2\theta + \beta\sin^2\theta$$

Integration Techniques Summary

$$\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx \text{ or } \int \sqrt{(x-\alpha)(x-\beta)}\,dx\,; \text{ put } x = \alpha\sec^2\theta - \beta\tan^2\theta$$

$$\int \frac{dx}{\sqrt{(x-\alpha)(x-\beta)}}; \text{ put } x-\alpha = t^2 \text{ or } x-\beta = t^2$$

To Integrate: $\int \sin^m x \cos^n x\,dx$

1. If m is odd positive integer then put $\cos x = t$
2. If n is odd positive integer put $\sin x = t$
3. If $m+n$ is negative even integer then put $\tan x = t$
4. If m and n both even positive integer the use the identities:

$$\sin^2 x = \frac{1-\cos 2x}{2} \quad \& \quad \cos^2 x = \frac{1+\cos 2x}{2}$$

Integration by Parts: $\int u.v'dx = uv - \int u'vdx$

Substitutions

$$\int \mathrm{F}(ax+b)\,dx = \frac{1}{a}\int \mathrm{F}(u)du \qquad \text{where} \quad \boldsymbol{u = ax+b}$$

$$\int \mathrm{F}(\sqrt{ax+b})\,dx = \frac{2}{a}\int u\mathrm{F}(u)du \qquad \text{where} \quad \boldsymbol{u = \sqrt{ax+b}}$$

$$\int \mathrm{F}(\sqrt[n]{ax+b})\,dx = \frac{n}{a}\int u^{n-1}\mathrm{F}(u)du \qquad \text{where} \quad \boldsymbol{u = \sqrt[n]{ax+b}}$$

$$\int \mathrm{F}\left(\sqrt{a^2-x^2}\right)dx = a\int \mathrm{F}(a\cos u)\cos u\,du \qquad \text{where} \quad \boldsymbol{x = a\sin u}$$

$$\int \mathrm{F}\left(\sqrt{x^2+a^2}\right)dx = a\int \mathrm{F}(a\sec u)\sec^2 u\,du \qquad \text{where} \quad \boldsymbol{x = a\tan u}$$

$$\int \mathrm{F}\left(\sqrt{x^2-a^2}\right)dx = a\int \mathrm{F}(a\tan u)\sec u\tan u\,du \qquad \text{where} \quad \boldsymbol{x = a\sec u}$$

$$\int \mathrm{F}(e^{ax})\,dx = \frac{1}{a}\int \frac{\mathrm{F}(u)}{u}du \qquad \text{where} \quad \boldsymbol{u = e^{ax}}$$

$$\int \mathrm{F}(\ln x)\,dx = \int \mathrm{F}(u)\,e^u du \qquad \text{where} \quad \boldsymbol{u = \ln x}$$

$$\int \mathrm{F}\left(\sin^{-1}\frac{x}{a}\right)dx = a\int \mathrm{F}(u)\cos u\,du \qquad \text{where} \quad \boldsymbol{u = \sin^{-1}\frac{x}{a}}$$

Similar results apply for other inverse trigonometric functions

$$\int \mathrm{F}(\sin x, \cos x)dx = 2\int \mathrm{F}\left(\frac{2u}{1+u^2}, \frac{1-u^2}{1+u^2}\right).\frac{du}{1+u^2} \qquad \text{where} \quad \boldsymbol{u = \tan\frac{x}{2}}$$

Solved Exercises

Evaluate each of the following integrals:

1. $\int (2x+1)^3 dx$
2. $\int \frac{1}{(3-2x)^4} dx$
3. $\int 2x\sqrt{x^2-1}dx$
4. $\int (x+2)\sin(x^2+4x-6)dx$
5. $\int 3x^3\sqrt{x^2+4}\, dx$
6. $\int \frac{\cot(\ln x)}{x} dx$
7. $\int x^2 e^{x^3} dx$
8. $\int e^{e^x+x} dx$
9. $\int \frac{e^{\frac{1}{x}}}{x^2} dx$
10. $\int \frac{1}{\sqrt{x}e^{\sqrt{x}}} dx$
11. $\int \frac{1}{\sqrt{x}}\cos\left(\sqrt{x}+3\right) dx$
12. $\int \frac{\tan\sqrt{x}}{\sqrt{x}} dx$
13. $\int 2^{-x}\tanh 2^{1-x}\, dx$
14. $\int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}} dx$
15. $\int e^x \sin(e^x)\, dx$
16. $\int (3x+2)e^x dx$
17. $\int xe^{-2x} dx$
18. $\int 2\sec^2 4x\, dx$
19. $\int (2x-1)\cos x\, dx$
20. $\int (1-x)\sin x\, dx$
21. $\int \frac{e^x-2x}{e^x-x^2} dx$
22. $\int \frac{x}{1+x^4} dx$
23. $\int \frac{x}{\sqrt{x+1}} dx$
24. $\int \frac{x dx}{\sqrt{1-x^4}}$
25. $\int \frac{\arcsin x}{\sqrt{1-x^2}} dx$
26. $\int \frac{x^3}{4+x^2} dx$
27. $\int \frac{e^{\frac{x}{2}}}{1+e^x} dx$
28. $\int \frac{\ln^4 x}{x} dx$
29. $\int \frac{\sqrt[3]{\ln^2 x}}{x} dx$
30. $\int \ln x\, dx$
31. $\int x\ln x\, dx$
32. $\int \frac{\ln x}{x^2} dx$
33. $\int \ln^2 x\, dx$
34. $\int x\tan x^2 \sec x^2\, dx$
35. $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right) dx$

36. $\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}}\,dx$

37. $\int \frac{\ln(2x)}{x\ln x}\,dx$

38. $\int \frac{1}{x-x\ln x}\,dx$

39. $\int \frac{\ln(\ln x)}{x\ln x}\,dx$

40. $\int \frac{\tan^3 x}{\cos^2 x}\,dx$

41. $\int \frac{\sec^2 x}{\tan^2 x}\,dx$

42. $\int \frac{\sin^6 x}{\cos^8 x}\,dx$

43. $\int \sec^4 x \tan x\,dx$

44. $\int \operatorname{sech} x\,dx$

45. $\int \tan x\,dx$

46. $\int \sinh x \cosh^2 x\,dx$

47. $\int \cos 3x \sin^5 3x\,dx$

48. $\int \sin x \sin 2x\,dx$

49. $\int \sec^6 x\,dx$

50. $\int \frac{\operatorname{sech}^2(\ln x)}{x}\,dx$

51. $\int \frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx$

52. $\int x^2 \operatorname{sech} x^3 \tanh x^3\,dx$

53. $\int x\sin(4-x^2)\,dx$

54. $\int \frac{\cos x}{2-3\sin x}\,dx$

55. $\int \frac{dx}{\sin^2 x\cos^2 x}$

56. $\int \frac{1}{(ax+b)^n}\,dx$

57. $\int \frac{x^2-2x}{x^3-3x^2+1}\,dx$

58. $\int \frac{x\ln(x^2+5)}{x^2+5}\,dx$

59. $\int \frac{1+\tan x}{x+\ln(\sec x)}\,dx$

60. $\int \frac{\operatorname{sech}^2 x}{3+2\tanh x}\,dx$

61. $\int \frac{e^{\sin x}}{\tan x\csc x}\,dx$

62. $\int \frac{e^x}{(1+e^x)^2}\,dx$

63. $\int \frac{e^x}{\sqrt{e^{2x}+1}}\,dx$

64. $\int \frac{dx}{\sqrt{e^{2x}-1}}$

65. $\int \cos(\arcsin x)\,dx$

66. $\int \frac{e^x\arctan(e^x)}{e^{2x}+1}\,dx$

67. $\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}\,dx$

68. $\int \frac{\sin(\arctan x)}{1+x^2}\,dx$

69. $\int \arctan x\,dx$

70. $\int \arcsin x\,dx$

71. $\int \sqrt{1-x^2}\,dx$

72. $\int \arcsin(\sqrt{x})\,dx$

73. $\int x(1-x)^{1991}\,dx$

74. $\int x\sqrt{x-4}dx$

75. $\int x\sqrt{2x-5}dx$

76. $\int \frac{dx}{3-\sqrt{x}}$

77. $\int \sin^5 x\, dx$

78. $\int \sec^5 x \tan^3 x\, dx$

79. $\int \frac{\ln(\tan x)}{\sin 2x}dx$

80. $\int \frac{\ln(\sin x)}{\tan x}dx$

81. $\int \frac{1}{x\sqrt{4x^2-64}}dx$

82. $\int \frac{1}{x^2\times\sqrt[4]{(x^4+1)^3}}dx$

83. $\int (x^6+x^3)\sqrt[3]{x^3+2}\,dx$

84. $\int \frac{dx}{\sqrt{(x+2)(3-x)}}$

85. $\int \frac{3}{x^2+4x-45}dx$

86. $\int \frac{1}{x^3+x^2}dx$

87. $\int \frac{1}{x^3-4x}dx$

88. $\int \frac{x+1}{\sqrt{4-x^2}}dx$

89. $\int \cot(\ln\sin x)\,dx$

90. $\int \frac{e^{-2x}}{e^{-x}+1}dx$

91. $\int \frac{x^5}{2+x^{12}}dx$

92. $\int x\log_2 x\,dx$

93. $\int \frac{1+\tan^2 x}{a^2+\tan^2 x}dx$

94. $\int \frac{\tan^2 x}{\sqrt[3]{\tan x - x}}dx$

95. $\int \frac{e^x}{e^x-e^{-x}}dx$

96. $\int \sqrt{\frac{e^x}{e^{-x}+1}}dx$

97. $\int \frac{\sin 2x}{1+\sin^4 x}dx$

98. $\int \frac{\sin 2x}{\sqrt{1+\sin x}}dx$

99. $\int \frac{x^2+1}{x^4-x^2+1}dx$

100. $\int \frac{dx}{x\sqrt{\sqrt{x}-1}}$

101. $\int \frac{x\cos x - \sin x}{x^2+\sin^2 x}dx$

102. $\int \frac{x}{\sqrt{x+1}}dx$

103. $\int \sin(2\arcsin x)\,dx$

104. $\int \frac{x}{1+\cot^2 x}dx$

105. $\int \ln\left(\sqrt{1+x^2}\right)dx$

106. $\int \sqrt{x}\sqrt{x\sqrt{x}+1}dx$

107. $\int x\sqrt{1-\sqrt{x}}dx$

108. $\int \frac{x^3}{\sqrt{1+x^2}}dx$

109. $\int \sqrt{\frac{x}{1-x^3}}dx$

110. $\int \frac{1}{\sqrt{1+\sqrt{x}}}dx$

111. $\int \frac{\sin x}{a+b\cos x}\,dx$

112. $\int \frac{dx}{1+\cos\frac{x}{2}}$

113. $\int \frac{dx}{1+\cos 2x}$

114. $\int x\ln(1-x^2)\,dx$

115. $\int \sec x\,dx$

116. $\int \sec^3 x\,dx$

117. $\int \frac{1}{\sqrt{1+x^2}}\,dx$

118. $\int \frac{dx}{x\sqrt{9x^2-4}}$

119. $\int \frac{1}{x\sqrt{x^n-1}}\,dx$

120. $\int \frac{dx}{x(x^5+1)}$

121. $\int x^2\sqrt{4-x^2}\,dx$

122. $\int x\sin x\sec x\tan x\,dx$

123. $\int \frac{1}{\cos^2 x(1+\sqrt{\tan x})^4}\,dx$

124. $\int \frac{\sqrt{\tan x}}{\sin x\cos x}\,dx$

125. $\int \frac{\sin 2x-4\cos x}{5-\cos^2 x-4\sin x}\,dx$

126. $\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}}\,dx$

127. $\int \frac{\ln(x+1)-\ln x}{x(x+1)}\,dx$

128. $\int \frac{\sin 4x}{\sin^4 x+\cos^4 x}\,dx$

129. $\int \frac{(x+1)(x+\ln x)^2}{x}\,dx$

130. $\int \frac{x}{1+\sin x}\,dx$

131. $\int \sqrt{\tanh x}\,dx$

132. $\int \sin x\ln(\sin x)\,dx$

133. $\int \frac{dx}{(x+1)\sqrt{x-2}}$

134. $\int \frac{x}{x^4+x^2+1}\,dx$

135. $\int \frac{dx}{(x+1)^{\frac{3}{4}}(x-2)^{\frac{5}{4}}}$

136. $\int \frac{6-x}{(x-3)(2x+5)}\,dx$

137. $\int \frac{2x-1}{(x-1)(x-2)(x-3)}\,dx$

138. $\int \frac{1}{x^2+x-6}\,dx$

139. $\int \frac{2x+4}{x^3-4x}\,dx$

140. $\int \frac{x^2+1}{(x^2+4)(x^2+25)}\,dx$

141. $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx$

142. $\int \frac{e^x}{\sqrt{e^{2x}+e^x+1}}\,dx$

143. $\int \frac{x^9}{(4x^2+1)^6}\,dx$

144. $\int \frac{dx}{\sqrt{x\sqrt{x}-x^2}}$

145. $\int \sqrt{x+\sqrt{x^2+1}}\,dx$

146. $\int \sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}\,dx$

147. $\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}}$

148. $\int \frac{\sqrt[4]{x^4-x}}{x^5}\,dx$

149. $\int \frac{dx}{x^n(1+x^n)^{\frac{1}{n}}}$

150. $\int \left(\frac{\sec x+\tan x-1}{\tan x-\sec x+1}\right)dx$

151. $\int \frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$

152. $\int \frac{1}{\sqrt{x}\left(\sqrt[4]{x}+1\right)^{10}}\,dx$

153. $\int \frac{dx}{1+\sin x+\cos x}$

154. $\int \frac{dx}{5+3\cos x}$

155. $\int \frac{dx}{\sqrt{2}+\sin x}$

156. $\int \frac{dx}{13\cosh x-5}$

157. $\int \frac{3+2\cos x}{(2+3\cos x)^2}\,dx$

158. $\int \frac{1}{(1-x\cot x)^2}\,dx$

159. $\int \frac{\sin x-1}{\cos^2 x\sqrt{\sec x+\tan x}}\,dx$

160. $\int \left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ln x\,dx$

161. $\int \frac{1-\tanh x}{\sqrt{\tanh x}}\,dx$

162. $\int \frac{1}{\sqrt{\sin^3 x\cos x}}\,dx$

163. $\int \frac{\sinh^3 x}{\cosh x\left(2+\sinh^2 x\right)}\,dx$

164. $\int \frac{1}{x\sqrt{1-x^3}}\,dx$

165. $\int \frac{\cos^2 x}{\sin^6 x}\,dx$

166. $\int \frac{\left[\ln\left(\tan\frac{x}{2}\right)\right]^3}{\sin x}\,dx$

167. $\int \frac{\ln x}{(1+\ln x)^2}\,dx$

168. $\int \sqrt{\csc x+1}\,dx$

169. $\int \frac{dx}{3\sin^2 x+4}$

170. $\int \frac{1}{\sqrt{(1-x^2)(1+\arcsin x)}}\,dx$

171. $\int \frac{\left(x+\sqrt{1+x^2}\right)^{20}}{\sqrt{1+x^2}}\,dx$

172. $\int \frac{dx}{\cos^3 x\sqrt{2\sin 2x}}$

173. $\int \frac{3x^2-3x+8}{x^3-3x^2+4x-12}\,dx$

174. $\int \frac{dx}{3+5\sin x+3\cos x}$

175. $\int \frac{\cosh x}{\cosh^2 x-2}\,dx$

176. $\int \frac{\cos^5 7x}{\sin^2 7x}\,dx$

177. $\int x\frac{\sqrt{1+x^2}}{\sqrt{1-x^2}}\,dx$

178. $\int \frac{\sin 2x}{(3+4\cos x)^2}\,dx$

179. $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}}dx$

180. $\int \frac{1}{\sqrt{x^2-x}}dx$

181. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$

182. $\int \frac{1}{a^x-1}dx$

183. $\int \frac{2^x}{\sqrt{4^x+9}}dx$

184. $\int \frac{1}{(25+x^2)^{\frac{3}{2}}}dx$

185. $\int \frac{1}{\left(a^2-b^2x^2\right)^{\frac{3}{2}}}dx$

186. $\int \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}dx$

187. $\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$

188. $\int \sqrt{1+\sec x}\,dx$

189. $\int \frac{\cos 4x-1}{\cot x-\tan x}dx$

190. $\int \frac{1}{x\left[1+\sin^2(\ln x)\right]}dx$

191. $\int \frac{\sqrt{x\ln x(1-\ln x)}}{x^3+\ln^3 x}dx$

192. $\int \frac{1}{\sqrt{x^8-x^2}}dx$

193. $\int \frac{\sqrt{x-\sqrt{a}}}{x}dx$

194. $\int \frac{\sqrt{a+x^2}}{x^6}dx$

195. $\int \frac{t^x}{\sqrt{1-t^{2x}}}dx$

196. $\int \frac{1}{\sqrt{x^7-x^2}}dx$

197. $\int \frac{dx}{(1+\sqrt{x})^8}$

198. $\int \sqrt{x^2-3}\,dx$

199. $\int \sqrt{16-25x^2}\,dx$

200. $\int \frac{1}{\sqrt{x^2-16}}dx$

201. $\int \frac{\sqrt{9-x^2}}{x^2}dx$

202. $\int \frac{1}{\sqrt{x^2+9}}dx$

203. $\int \frac{x^5}{\sqrt{x^2+2}}dx$

204. $\int \frac{1}{e^x+e^{-x}-1}dx$

205. $\int \frac{x}{1+e^{x^2}}dx$

206. $\int \frac{1-x\sin x}{x\left(1-x^2e^{2\cos x}\right)}dx$

207. $\int \frac{\ln x}{(x-3)^2}dx$

208. $\int \left(\frac{x^2-x+1}{x^2+1}\right)e^{\operatorname{arccot} x}dx$

209. $\int \frac{\tan\left(\sec^{-1}\left(\sqrt{e^x}\right)\right)}{\sqrt{1+\sinh x-\cosh x}}dx$

210. $\int \frac{dx}{e^{\frac{x}{2}}+e^x}$

211. $\int \sec 2x\,\sqrt{\sec 2x-1}\,dx$

212. $\int \frac{\sin x \cos x}{\cos^4 x+\sin^4 x} dx$

213. $\int \frac{1}{\sin^6 x+\cos^6 x} dx$

214. $\int \sqrt{e^x + 1} dx$

215. $\int \frac{3}{x^2+4x+29} dx$

216. $\int \frac{\cos x+x\sin x}{x(x+\cos x)} dx$

217. $\int \frac{1}{5\cosh x+3\sinh x+4} dx$

218. $\int \frac{x^2+1}{x^4+1} dx$

219. $\int \frac{1-x^2}{1+3x^2+x^4} dx$

220. $\int \frac{\sin 2x\cos 2x}{\sqrt{4-\sin^4 2x}} dx$

221. $\int \frac{\sqrt{\cos x-\cos^3 x}}{\sqrt{1-\cos^3 x}} dx$

222. $\int \frac{dx}{\sqrt{x}\left(4+\sqrt[3]{x}\right)}$

223. $\int \frac{(\ln x-1)^2}{\left(1+(\ln x)^2\right)^2} dx$

224. $\int \frac{4^x+1}{2^x+1} dx$

225. $\int e^{e^{e^x}+e^x+x} dx$

226. $\int e^x x^{e^x}\left(\ln x + \frac{1}{x}\right) dx$

227. $\int \frac{2^{x^3}x^2\left(1-\cos\left(2^{x^3}\right)\right)}{1+\cos\left(2^{x^3}\right)} dx$

228. $\int \frac{1}{\sin^4 x+\cos^4 x} dx$

229. $\int \frac{1}{\cos 2x+\cos^2 x} dx$

230. $\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5-\sqrt[6]{x^7}}} dx$

231. $\int \sqrt{x + \sqrt{x^2 + 2}}\, dx$

232. $\int \cos x \,.\, \cos(\sin x) \,.\, \cos(\sin(\sin x))$

233. $\int \frac{1}{3\sin^2 x-4\sin x\cos x-4\cos^2 x} dx$

234. $\int \frac{x+\sqrt[3]{x^2}+\sqrt[6]{x}}{x\left(1+\sqrt[3]{x}\right)} dx$

235. $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$

236. $\int \frac{\sec^4 x\tan x}{\sec^4 x+4} dx$

237. $\int \frac{\sin^3\sqrt{x}\cos^5\sqrt{x}}{\sqrt{x}} dx$

238. $\int \frac{\sec x\csc x}{\ln(\tan x)} dx$

239. $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}} dx$

240. $\int \frac{x^2+1}{x^4-5x^2+1} dx$

241. $\int \frac{x^{5m-1}+2x^{4m-1}}{\left(x^{2m}+x^m+1\right)^3} dx$

242. $\int \frac{1}{x-\sqrt{1-x^2}} dx$

243. $\int \frac{1}{x+\sqrt{x^2-1}} dx$

244. $\int \frac{1}{e^x+e^{-x}} dx$

245. $\int \tan x \ln(\cos x)\, dx$

246. $\int \frac{\cos 4x-1}{\cot x-\tan x} dx$

247. $\int x^3\sqrt{1-x^2} dx$

248. $\int x^2\sqrt{1+x^2} dx$

249. $\int \frac{dx}{1+\sin^2 x}$

250. $\int \frac{1}{e^x+2e^{-x}+2} dx$

251. $\int \frac{\sqrt{e^x-1}}{e^x} dx$

252. $\int \frac{1}{e^x\sqrt{e^{2x}-16}} dx$

253. $\int \frac{1}{\sqrt{e^x-1}} dx$

254. $\int \frac{1}{\sqrt{x+1}+\sqrt{(x+1)^3}} dx$

255. $\int \frac{8x-3}{\sqrt{-4x^2+12x-5}} dx$

256. $\int \frac{\sqrt{x^3-2}}{x} dx$

257. $\int \frac{1}{x^n-x} dx$

258. $\int \frac{1}{e^{2x}+e^{3x}} dx$

259. $\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}} dx$

260. $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1} dx$

261. $\int \frac{1}{(a^2+x^2)^2} dx$

262. $\int \sec x\sqrt{\sec x+\tan x}\, dx$

263. $\int \frac{\sin 2x}{\alpha\cos^2 x+\beta\sin^2 x} dx$

264. $\int \frac{dx}{4+5\cos^2 x}$

265. $\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x} dx$

266. $\int \frac{3+2\cos x}{(2+3\cos x)^2} dx$

267. $\int \frac{dx}{9+16\cos^2 x}$

268. $\int \frac{1}{a^2+\tan^2 x} dx$

269. $\int \frac{2\cos 2x}{\left(1-\sin^2 2x\right)\left(2-\sin^2 2x\right)} dx$

270. $\int \frac{\sqrt{x}}{1+x} dx$

271. $\int \frac{1-\sin x}{\sin x(1+\sin x)} dx$

272. $\int \frac{dx}{\sin^2 x+\tan^2 x}$

273. $\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{\sqrt{2}(\cos x+\sin x)} dx$

274. $\int \frac{1}{\cos(x+a)\cos(x+b)} dx$

275. $\int \frac{1}{\sqrt{\sin^3 x\sin(x+a)}} dx$

276. $\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$

277. $\int \frac{\sqrt{x^2+1}}{x^4} dx$

278. $\int \frac{\sqrt{x^2-1}}{x} dx$

279. $\int \frac{\sqrt{x^2+1}}{x}dx$

280. $\int \frac{1}{(x^2-2x+4)^{\frac{3}{2}}}dx$

281. $\int \sqrt{a+\sqrt{b+\sqrt{x}}}\,dx$

282. $\int \sqrt{\frac{1-x}{1+x}}dx$

283. $\int \frac{x^2-1}{x\sqrt{x^4+3x^2+1}}dx$

284. $\int \frac{1}{x^2\sqrt{x^2+4}}dx$

285. $\int \frac{x^3}{(4x^2+9)\sqrt{4x^2+9}}dx$

286. $\int \frac{dx}{(x^2-4)\sqrt{x}}$

287. $\int \sqrt{\frac{x}{(1-x)^3}}dx$

288. $\int \frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx$

289. $\int \frac{\cos(x+\ln a)}{\cos(x+\ln b)}dx$

290. $\int \frac{\sqrt{1+\ln x}}{x\ln x}dx$

291. $\int e^{4x}\sqrt{1+e^{2x}}dx$

292. $\int \frac{1+\ln x}{\sqrt{1+x^x}}dx$

293. $\int \frac{1}{e^x\sqrt{\sinh 2x}}dx$

294. $\int \frac{x^2-2}{(x^4+5x^2+4)\arctan\left(\frac{x^2+2}{x}\right)}dx$

295. $\int \frac{x^{-\frac{1}{2}}}{1+x^{\frac{1}{3}}}dx$

296. $\int \frac{8}{3\cos 2x+1}dx$

297. $\int \frac{1}{5+4\sqrt{x}+x}dx$

298. $\int \frac{e^{\tan x}}{1-\sin^2 x}dx$

299. $\int \frac{\sin x+\cos x}{9+16\sin 2x}dx$

300. $\int x^{\frac{x}{\ln x}}dx$

301. $\int 2^{\ln x}dx$

302. $\int a^x.a^{a^x}.a^{a^{a^x}}dx$

303. $\int \frac{1}{\sqrt[3]{(x^2-1)^2(x-1)^2}}dx$

304. $\int \frac{1}{x+\sqrt{x^2-1}}dx$

305. $\int \sqrt{x+\frac{1}{\sqrt{x}}}dx$

306. $\int \frac{x^2}{1-x^4}dx$

307. $\int \sqrt{\frac{x}{a^3-x^3}}dx$

308. $\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}}dx$

309. $\int \frac{1}{x^2-1}.\ln\left|\frac{x-1}{x+1}\right|dx$

310. $\int \frac{1}{(x^2+1)\sqrt{x^2-1}}dx$

311. $\int \frac{1}{\sqrt{1+\cos nx}}dx$

312. $\int \frac{\ln^3 x}{x\sqrt{\ln^2 x-4}}dx$

313. $\int \frac{x^2(1-\ln x)}{\ln^4 x-x^4}dx$

314. $\int \frac{\sin[\ln(\sin(\ln x))]\cos(\ln x)}{x\sin(\ln x)}dx$

315. $\int \frac{\tan^2 x}{1-\tan^2 x}dx$

316. $\int \frac{\sin^3 2x}{8+8\sin^2 x}dx$

317. $\int \sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}dx$

318. $\int \sqrt{\frac{1-\cos x}{\cos\alpha-\cos x}}dx$

319. $\int \frac{dx}{\cos^3 x-\sin^3 x}$

320. $\int \frac{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}{3+\sin 2x}dx$

321. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx$

322. $\int \frac{1}{2+2\sin x+\cos x}dx$

323. $\int \frac{dx}{\sqrt{1-\sin^4 x}}$

324. $\int \frac{dx}{x+3\sqrt{x}+2}$

325. $\int \frac{3+3x^2}{3x^4+2x^3+6x^2-2x+3}dx$

326. $\int \frac{e^{3x}-e^x}{e^{4x}+e^{2x}+1}dx$

327. $\int \frac{\sqrt{x^2+4}}{x^2}dx$

328. $\int \frac{x+1}{x(xe^x+1)^2}dx$

329. $\int \frac{\tan x+\cot x+1}{\ln(e^x\tan x)}dx$

330. $\int \frac{\operatorname{arcsec}\left(\frac{1+x^2}{1-x^2}\right)}{x\arctan x}dx$

331. $\int \frac{x-x\ln x+1}{x(x+1)^2+x\ln^2 x}dx$

332. $\int \frac{1}{5\cos^2 x+4\sin 2x+3}dx$

333. $\int \frac{\sin x}{\sin 4x}dx$

334. $\int \frac{1}{3(\sec x+\tan x)-\cos x}dx$

335. $\int \frac{\sqrt{\tan x}}{\sin x(\sin x+\cos x)}dx$

336. $\int \frac{\cos^4 x-\sin^4 x}{\left(1+\sqrt{e^x}+\sqrt[3]{e^x}+\sqrt[6]{e^x}\right)\cos 2x}dx$

337. $\int \frac{x^2-1}{x^2+1}\cdot\frac{1}{\sqrt{1+x^4}}dx$

338. $\int \frac{1+x^2}{1-x^2}\cdot\frac{dx}{\sqrt{1+3x^2+x^4}}$

339. $\int \ln\left(2x^2+1+2x\sqrt{1+x^2}\right)dx$

340. $\int \frac{x\arctan x}{(1+x^2)^3}dx$

341. $\int \ln\left(x+\sqrt{x}\right)dx$

342. $\int \frac{dx}{x^3+1}$

343. $\int \frac{dx}{x^4+4}$

344. $\int \frac{dx}{x^6+1}$

345. $\int \frac{a+bx^2-2bx^2}{x\sqrt{x^2-(bx^2-a)^2}}dx$

346. $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx$

347. $\int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx$

348. $\int \frac{\sqrt{(x^2-a^2)^3}}{x^3}dx$

349. $\int \frac{\sqrt{x}}{1+x+x^2+x^3}dx$

350. $\int \frac{2\cos x+\sin 2x}{(\sin x-1)\sqrt{\sin x+\sin^2 x+\sin^3 x}}dx$

351. $\int \frac{1}{x\sqrt{(\ln x+1)(\ln x+2)(\ln x+3)(\ln x+4)+1}}dx$

352. $\int x\sin(x^2)\tan^2[\cos(x^2)]\,dx$

353. $\int (\cos x)^{\cos x+1}\tan x\,(1+\ln\cos x)dx$

354. $\int \sin(\arccos x+2\arcsin x)\,dx$

355. $\int \sqrt{\sin x\cos^2 x\tan^3 x\cot^4 x\csc^5 x\sec^6 x}\,dx$

356. $\int (\sin x)^{e^x}.e^x.(\cot x+\ln\sin x)dx$

357. $\int \frac{\cos x}{\sin x+\tan x}dx$

358. $\int \frac{dx}{1+a^2-2a\cos x}$

359. $\int \frac{dx}{\sin^2 x+\tan^2 x}$

360. $\int \sqrt{2e^x-e^{2x}}dx$

361. $\int \sqrt{\frac{e^x+2}{e^x-2}}dx$

362. $\int \sqrt{\frac{a-x}{x-b}}dx$

363. $\int \sqrt{\frac{x}{x-a}}dx$

364. $\int \frac{x}{\sqrt{\sqrt{x}-\sqrt{a}}}dx$

365. $\int \sqrt{1+\frac{\pi}{x}}dx$

366. $\int \frac{\sqrt{\sqrt{\sqrt{x}}}}{\sqrt{\sqrt{x}}+1}dx$

367. $\int \frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}dx$

368. $\int \frac{dx}{x(x^n+1)}$

369. $\int \left(x+\sqrt{1+x^2}\right)^n dx$

370. $\int \frac{x}{1-x^2+\sqrt{1-x^2}}dx$

371. $\int \frac{1}{(4x^2+8x+13)^2}dx$

372. $\int \frac{dx}{(x+a)^2(x+b)^2}$

373. $\int \sqrt{2ax-x^2}dx$

374. $\int \frac{1}{\sqrt{x(x-a)}}dx$

375. $\int \sqrt{\frac{x^2-a}{x^2-b}}dx$

376. $\int \frac{e^x}{(e^{2x}+8e^x+7)^{\frac{3}{2}}}dx$

377. $\int \frac{1}{x^8-1}dx$

378. $\int \sqrt{\tan x}\, dx$

379. $\int \sqrt[3]{\tan x}\, dx$

380. $\int \sqrt{1+\tan x}\, dx$

381. $\int \frac{1}{x^5+1} dx$

382. $\int \frac{dx}{\sqrt{(1+\sin x)(2+\sin x)}}$

383. $\int \frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}\left(1+x\sqrt{1-x^2}\right)} dx$

384. $\int \left(\frac{1+\cos x}{\sin x}\right)^4 dx$

385. $\int \left(\frac{1-\tan x}{1+\tan x}\right)^4 dx$

386. $\int \frac{\sqrt{\sin x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^5} dx$

387. $\int \left(\sqrt{\tan x}+\sqrt{\cot x}\right) dx$

388. $\int \frac{1}{\sqrt[4]{1+x^4}} dx$

389. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx$

390. $\int \frac{dx}{x+\sqrt{x^2+2x+2}}$

391. $\int \frac{dx}{x-2+\sqrt{x^2-2x+2}}$

392. $\int \frac{dx}{(x+1)\sqrt{-x^2+x+1}}$

393. $\int \frac{1-\sin^4 x}{\left(1+\sin^4 x\right)\sqrt{1+\sin^2 x}} dx$

394. $\int \frac{dx}{\left(x^2-x-2\right)\sqrt{x^2+x+1}}$

395. $\int \frac{dx}{3-3x^2-(4x+5)\sqrt{1-x^2}}$

396. $\int \frac{x}{\sqrt{1+x^2+\sqrt{\left(1+x^2\right)^3}}} dx$

397. $\int \frac{1}{\sqrt{1+x}+\sqrt{1-x}+2} dx$

398. $\int \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}} dx$

399. $\int \frac{dx}{\tan^3 x+1}$

400. $\int \frac{\tan x}{(1-\sin x)^3} dx$

401. $\int \frac{\left(x^2-1\right)\sqrt{x^4+2x^3-x^2+2x+1}}{x^3} dx$

402. $\int \frac{\tan(\ln x)\tan\left[\ln\left(\frac{x}{2}\right)\right]}{x} dx$

403. $\int \frac{dx}{x\ln x\sqrt{\ln^2 x+2\ln x+4}}$

404. $\int \frac{\sqrt[6]{2x-4}}{\sqrt[4]{2x-4}+\sqrt{2x-4}} dx$

405. $\int \frac{1+x^2}{x^4+2x^2\cos\left(\frac{2\pi}{5}\right)+1} dx$

406. $\int \frac{dx}{x^{n-2}+C_n^1 x^{n-3}+C_n^2 x^{n-4}+\dots+x^{-2}}$

407. $\int \frac{dx}{x(x+1)(x+2)\dots(x+m)}$

408. $\int \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}}\, dx$

409. $\int \frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)} dx$

Solutions of Exercises

1. $\int (2x+1)^3 dx = \frac{1}{2}\int 2(2x+1)^3 dx$, let $u = 2x+1$, so $du = 2dx$, so

$\int (2x+1)^3 dx = \frac{1}{2}\int u^3 du = \frac{1}{2}\left(\frac{1}{4}u^4\right) + c = \frac{1}{8}u^4 + c = \frac{1}{8}(2x+1)^4 + c$

2. $\int \frac{1}{(3-2x)^4}dx = -\frac{1}{2}\int \frac{-2}{(3-2x)^4}dx$, let $u = 3-2x$, then $du = -2dx$, so:

$\int \frac{1}{(3-2x)^4}dx = -\frac{1}{2}\int \frac{du}{u^4} = -\frac{1}{2}\int u^{-4}du = -\frac{1}{2}\left(\frac{u^{-3}}{-3}\right) + c = \frac{1}{6u^3} + c = \frac{1}{6(3-2x)^3} + c$

3. $\int 2x\sqrt{x^2-1}dx$, let $u = x^2-1$, then $du = 2xdx$, so:

$\int 2x\sqrt{x^2-1}dx = \int \sqrt{u}du = \frac{2}{3}u^{\frac{3}{2}} + c = \frac{2}{3}u\sqrt{u} + c = \frac{2}{3}(x^2-1)\sqrt{x^2-1} + c$

4. $\int (x+2)\sin(x^2+4x-6)dx$, let $u = x^2+4x-6$, then $du = 2(x+2)dx$, so:

$\int (x+2)\sin(x^2+4x-6)dx = \frac{1}{2}\int \sin(x^2+4x-6)[2(x+2)]dx = \frac{1}{2}\int \sin u\, du$

$= -\frac{1}{2}\cos u + c = -\frac{1}{2}\cos(x^2+4x-6) + c$

5. $\int 3x^3\sqrt{x^2+4}\,dx$, let $u = x^2+4$, then $du = 2xdx$ and $x^2 = u-4$ so:

$\int 3x^3\sqrt{x^2+4}\,dx = \frac{3}{2}\int x^2\sqrt{x^2+4}\,2xdx = \frac{3}{2}\int (u-4)\sqrt{u}\,du = \frac{3}{2}\int \left(u^{\frac{3}{2}} - 4u^{\frac{1}{2}}\right)du$

$= \frac{3}{2}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{8}{3}u^{\frac{3}{2}}\right) + c = \frac{3}{5}(x^2+4)^{\frac{5}{2}} - 4(x^2+4)^{\frac{3}{2}} + c$

6. $\int \frac{\cot(\ln x)}{x}dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, so: $\int \frac{\cot(\ln x)}{x}dx = \int \cot u\, du = \int \frac{\cos u}{\sin u}du$

$= \int \frac{(\sin u)'}{\sin u}du = \ln|\sin u| + c = \ln|\sin(\ln x)| + c$

7. $\int x^2 e^{x^3}dx = \frac{1}{3}\int 3x^2 e^{x^3}dx$, let $u = x^3$, then $du = 3x^2dx$, so:

$\int x^2 e^{x^3}dx = \frac{1}{3}\int e^u du = \frac{1}{3}e^u + c = \frac{1}{3}e^{x^3} + c$

8. $\int e^{e^x+x}dx = \int e^{e^x}.e^x dx$, let $t = e^x$, then $dt = e^x dx$, so:

$\int e^{e^x+x}dx = \int e^t dt = e^t + c = e^{e^x} + c$

9. $\int \frac{e^{\frac{1}{x}}}{x^2}dx = -\int -\frac{1}{x^2}e^{\frac{1}{x}}dx$, let $u = \frac{1}{x}$, then $du = -\frac{1}{x^2}dx$, so:

$\int \frac{e^{\frac{1}{x}}}{x^2}dx = -\int e^u du = -e^u + c = -e^{\frac{1}{x}} + c$

10. $\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx = 2\int \frac{1}{e^{\sqrt{x}}} \times \frac{1}{2\sqrt{x}}dx$, let $t = \sqrt{x}$, then $dt = \frac{1}{2\sqrt{x}}dx$, then:

$\int \frac{1}{\sqrt{x}e^{\sqrt{x}}}dx = 2\int \frac{1}{e^t}dt = 2\int e^{-t}dt = -2e^{-t} + c = -2e^{-\sqrt{x}} + c$

11. $\int \frac{1}{\sqrt{x}}\cos(\sqrt{x}+3)\,dx = 2\int \frac{1}{2\sqrt{x}}\cos(\sqrt{x}+3)\,dx$, let $u = \sqrt{x}+3$, then $du = \frac{1}{2\sqrt{x}}dx$, so:

$\int \frac{1}{\sqrt{x}}\cos(\sqrt{x}+3)\,dx = 2\int \cos u\, du = 2\sin u + c = 2\sin(\sqrt{x}+3) + c$

12. $\int \frac{\tan\sqrt{x}}{\sqrt{x}}dx = 2\int \frac{1}{2\sqrt{x}}\tan\sqrt{x}\,dx$, let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, so:

$\int \frac{\tan\sqrt{x}}{\sqrt{x}}dx = 2\int \tan u\,du = -2\int \frac{-\sin u}{\cos u}du = -2\int \frac{(\cos u)'}{\cos u}du = -2\ln|\cos u| + c$

$= -2\ln\left|\cos\sqrt{x}\right| + c$

13. $\int 2^{-x}\tanh 2^{1-x}\,dx$, let $u = 2^{1-x}$, then $du = -2^{1-x}(\ln 2)dx$, then:

$\int 2^{-x}\tanh 2^{1-x}\,dx = -\frac{1}{2\ln 2}\int \tanh u\,du = -\frac{1}{2\ln 2}\int \frac{\sinh u}{\cosh u}du = -\frac{1}{2\ln 2}\int \frac{(\cosh u)'}{\cosh u}du$

$= -\frac{1}{2\ln 2}\ln\cosh u + c = -\frac{1}{2\ln 2}\cosh 2^{1-x} + c$

14. $\int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}}dx = \int \frac{1}{x^5}\left(1+\frac{1}{x^4}\right)^{-\frac{3}{4}}dx$, let $t = \frac{1}{x^4}$, then $dt = -\frac{4}{x^5}dx$, so:

$\int \frac{1}{x^2(x^4+1)^{\frac{3}{4}}}dx = -\frac{1}{4}\int(1+t)^{-\frac{3}{4}}dt = -\frac{1}{4}\left[\frac{(1+t)^{\frac{1}{4}}}{\frac{1}{4}}\right] + c = -(1+t)^{\frac{1}{4}} + c = -\left(1+\frac{1}{x^4}\right)^{\frac{1}{4}} + c$

15. $\int e^x\sin(e^x)\,dx$, let $u = e^x$, then $du = e^x dx$, so:

$\int e^x\sin(e^x)\,dx = \int \sin u\,du = -\cos x + c = -\cos(e^x) + c$

16. $\int(3x+2)e^x dx$, let $u = 3x+2 \Longrightarrow u' = 3$ and let $v' = e^x \Longrightarrow v = e^x$, so:

$= (3x+2)e^x - \int 3e^x = (3x+2)e^x - 3e^x + c = (3x-1)e^x + c$

17. $\int xe^{-2x}dx$, let $u = x \Longrightarrow u' = 1$ and let $v' = e^{-2x} \Longrightarrow v = -\frac{1}{2}e^{-2x}$, so:

$\int xe^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{2}e^{-2x}dx = -\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c = -\frac{1}{4}(2x+1)e^{-2x} + c$

18. $\int 2\sec^2 4x\,dx$, let $u = 4x$, then $du = 4dx$, so:

$\int 2\sec^2 4x\,dx = \frac{1}{2}\int \sec^2 u\,dx = \frac{1}{2}\tan u + c = \frac{1}{2}\tan(4x) + c$

19. $\int(2x-1)\cos x\,dx$, let $u = 2x-1 \Longrightarrow u' = 2$ and let $v' = \cos x \Longrightarrow v = \sin x$, so:

$\int(2x-1)\cos x\,dx = (2x-1)\sin x - \int 2\sin x\,dx = (2x-1)\sin x + 2\cos x + c$

20. $\int(1-x)\sin x\,dx$, let $u = 1-x \Longrightarrow u' = -1$ and let $v' = \sin x \Longrightarrow v = -\cos x$, so:

$\int(1-x)\sin x\,dx = (1-x)(-\cos x) - \int \cos x\,dx = (x-1)\cos x - \sin x + c$

21. $\int \frac{e^x-2x}{e^x-x^2}dx$. let $u = e^x - x^2$, so $du = (e^x-2x)dx$, then:

$\int \frac{e^x-2x}{e^x-x^2}dx = \int \frac{du}{u} = \ln|u| + c = \ln|e^x - x^2| + c$

22. $\int \frac{x}{1+x^4}dx = \int \frac{x}{1+(x^2)^2}dx$, let $u = x^2$, so $du = 2xdx$, $\frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\arctan u + c$.

Therefore $\int \frac{x}{1+x^4}dx = \frac{1}{2}\arctan x^2 + \mathrm{c}$

23. $\int \frac{x}{\sqrt{x+1}}dx$, let $u = x+1$, $x = u-1$ and $dx = du$, so:

$\int \frac{x}{\sqrt{x+1}}dx = \int \frac{u-1}{\sqrt{u}}du = \int\left(\frac{u}{\sqrt{u}} - \frac{1}{\sqrt{u}}\right)du = \int\left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)du = \frac{2}{3}u^{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{2}{3}u^{\frac{3}{2}} - u^{\frac{1}{2}} + c = \frac{2}{3}(x+1)^{\frac{3}{2}} - (x+1)^{\frac{1}{2}} + c = \frac{2}{3}(x+1)\sqrt{x+1} - \sqrt{x+1} + c$

24. $\int \frac{xdx}{\sqrt{1-x^4}} = \int \frac{xdx}{\sqrt{1-(x^2)^2}}$, let $u = x^2$, so $du = 2xdx$, $\frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\arcsin u + \text{c}$, therefore

$\int \frac{xdx}{\sqrt{1-x^4}} = \frac{1}{2}\arcsin x^2 + \text{c}.$

25. $\int \frac{\arcsin x}{\sqrt{1-x^2}} dx$, let $x = \sin u$, then $dx = \cos u\, du$ and $u = \arcsin x$, so:

$\int \frac{\arcsin x}{\sqrt{1-x^2}} dx = \int \frac{u}{\sqrt{1-\sin^2 u}} \cos u\, du = \int \frac{u}{\cos u} \cos u\, du = \int u du = \frac{1}{2}u^2 + c$

$= \frac{1}{2}(\arcsin x)^2 + c$

26. $\int \frac{x^3}{4+x^4} dx$, let $u = x^2$, then $du = 2xdx$, so:

$\int \frac{x^3}{4+x^2} dx = \frac{1}{2}\int \frac{x^2}{4+x^2}(2x)dx = \frac{1}{2}\int \frac{u}{u+4} du = \frac{1}{2}\int \frac{u+4-4}{u+4} du = \frac{1}{2}\int \left(1 - \frac{4}{u+4}\right) du$

$= \frac{1}{2}(u - 4\ln|u+4|) + c = \frac{1}{2}x^2 - 2\ln|x^2+4| + c$

27. $\int \frac{e^{\frac{x}{2}}}{1+e^x} dx = 2\int \frac{\frac{1}{2}e^{\frac{x}{2}}}{1+\left(e^{\frac{x}{2}}\right)^2} dx$, let $u = e^{\frac{x}{2}}$, then $du = \frac{1}{2}e^{\frac{x}{2}}$, so:

$\int \frac{e^{\frac{x}{2}}}{1+e^x} dx = 2\int \frac{du}{1+u^2} = 2\arctan u = 2\arctan\left(e^{\frac{x}{2}}\right) + c$

28. $\int \frac{\ln^4 x}{x} dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, so:

$\int \frac{\ln^4 x}{x} dx = \int u^4 du = \frac{1}{5}u^5 + c = \frac{1}{5}\ln^5 x + c$

29. $\int \frac{\sqrt[3]{\ln^2 x}}{x} dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, so:

$\int \frac{\sqrt[3]{\ln^2 x}}{x} dx = \int \sqrt[3]{u^2} du = \int u^{\frac{2}{3}} du = \frac{3}{5}u^{\frac{5}{3}} + c = \frac{3}{5}\ln^{\frac{5}{3}} x + c = \frac{3}{5}\sqrt[3]{\ln^5 x}\, dx + c$

30. $\int \ln x\, dx$, let $u = \ln x \Rightarrow u' = \frac{1}{x}$ and let $v' = 1 \Rightarrow v = x$, so:

$\int \ln x\, dx = x\ln x - \int \frac{1}{x} \times dx = x\ln x - \int dx = x\ln x - x + c$

31. $\int x\ln x\, dx$, let $u = \ln x \Rightarrow u' = \frac{1}{x}$ and let $v' = x \Rightarrow v = \frac{1}{2}x^2$, so:

$\int x\ln x\, dx = \frac{1}{2}x^2\ln x - \int \frac{1}{x} \times \frac{1}{2}x^2 dx = \frac{1}{2}x^2\ln x - \int \frac{1}{2}x dx = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$

32. $\int \frac{\ln x}{x^2} dx$, let $u = \ln x \Rightarrow u' = \frac{1}{x}$ and let $v' = \frac{1}{x^2} \Rightarrow v = -\frac{1}{x}$, so:

$\int \frac{\ln x}{x^2} dx = -\frac{\ln x}{x^2} - \int \left(\frac{1}{x}\right)\left(-\frac{1}{x}\right) dx = -\frac{\ln x}{x} + \int \frac{1}{x^2} dx = -\frac{\ln x}{x} - \frac{1}{x} + c = -\frac{\ln x+1}{x} + c$

33. $\int \ln^2 x\, dx$, let $u = \ln x \Rightarrow u' = \frac{1}{x}$ and let $v' = \ln x \Rightarrow v = x\ln x - x$, so:

$\int \ln^2 x\, dx = \ln x\,(x\ln x - x) - \int \frac{1}{x}x(\ln x - 1)\, dx = x\ln^2 x - x\ln x - \int (\ln x - 1)\, dx$

$= x\ln^2 x - x\ln x - (x\ln x - x - x) + c = x\ln^2 x - x\ln x - x\ln x + 2x + c$

$= x\ln^2 x - 2x\ln x + 2x + c$

34. $\int x\tan x^2 \sec x^2\, dx = \frac{1}{2}\int 2x\tan x^2\sec x^2\, dx$, let $u = x^2$, so $du = 2xdx$, then:

$\int x\tan x^2\sec x^2\, dx = \frac{1}{2}\int \tan t \sec t\, dt = \frac{1}{2}\sec t + c = \frac{1}{2}\sec x^2 + c$

35. $\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right) dx$, let $u = \cos\left(\frac{1}{x}\right)$, then $du = \frac{1}{x^2}\sin\left(\frac{1}{x}\right) dx$, so:

$\int \frac{1}{x^2}\sin\left(\frac{1}{x}\right)\cos\left(\frac{1}{x}\right) dx = \int u\, du = \frac{1}{2}u^2 + c = \frac{1}{2}\cos^2\left(\frac{1}{x}\right) + c$

36. $\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}} dx = \int \frac{\sin\sqrt{x}}{\sqrt{x}}\frac{1}{\sqrt{\cos^3\sqrt{x}}} dx = -2\int \frac{-\sin\sqrt{x}}{2\sqrt{x}}\frac{1}{\sqrt{\cos^3\sqrt{x}}} dx$

Let $t = \cos\sqrt{x}$, then $dt = -\frac{\sin\sqrt{x}}{2\sqrt{x}} dx$, so:

$\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}} dx = -2\int \frac{1}{\sqrt{t^3}} dt = -2\int t^{-\frac{3}{2}} dt = -2\frac{t^{-\frac{1}{2}}}{-\frac{1}{2}} + c = \frac{4}{\sqrt{t}} + c = \frac{4}{\sqrt{\cos\sqrt{x}}} + c$

37. $\int \frac{\ln(2x)}{x\ln x} dx = \int \frac{\ln 2 + \ln x}{\ln x}.\frac{dx}{x}$, let $u = \ln x$, so $du = \frac{1}{x}dx$, so:

$\int \frac{\ln(2x)}{x\ln x} dx = \int \left(\frac{\ln 2 + u}{u}\right) du = \int \left(\frac{\ln 2}{u} + 1\right) du = \ln 2\ln|u| + u + c = (\ln 2)\ln|\ln x| + \ln x + c$

38. $\int \frac{1}{x - x\ln x} dx = \int \frac{1}{x(1-\ln x)} dx = -\int \frac{-1}{x(1-\ln x)} dx$, let $u = 1 - \ln x$, then $du = -\frac{1}{x}dx$, so:

$\int \frac{1}{x - x\ln x} dx = -\int \frac{du}{u} = -\ln|u| + c = -\ln|1 - \ln x| + c$

39. $\int \frac{\ln(\ln x)}{x\ln x} dx$, let $u = \ln(\ln x)$, then $du = \frac{(\ln x)'}{\ln x} dx = \frac{\frac{1}{x}}{\ln x} dx = \frac{1}{x\ln x} dx$, so:

$\int \frac{\ln(\ln x)}{x\ln x} dx = \int u du = \frac{1}{2}u^2 + c = \frac{1}{2}\ln^2(\ln x) + c$

40. $\int \frac{\tan^3 x}{\cos^2 x} dx = \int \sec^2 x\tan^3 x\, dx$, let $u = \tan x$, then $du = \sec^2 x\, dx$, so:

$\int \frac{\tan^3 x}{\cos^2 x} dx = \int u^3 du = \frac{1}{4}u^4 + c = \frac{1}{4}\tan^4 x + c$

41. $\int \frac{\sec^2 x}{\tan^2 x} dx$, let $u = \tan x$, then $du = \sec^2 x\, dx$, so:

$\int \frac{\sec^2 x}{\tan^2 x} dx = \int \frac{du}{u^2} = -\frac{1}{u} + c = -\frac{1}{\tan x} + c = -\cot x + c$

42. $\int \frac{\sin^6 x}{\cos^8 x} dx = \int \frac{\sin^6 x}{\cos^6 x\cos^2 x} dx = \int \tan^6 x\sec^2 x$, let $t = \tan x$, so $dt = \sec^2 x\, dx$, then:

$\int \frac{\sin^6 x}{\cos^8 x} dx = \int t^6 dt = \frac{1}{7}t^7 + c = \frac{1}{7}\tan^7 x + c$

43. $\int \sec^4 x\tan x\, dx = \int \sec^3 x\,(\sec x\tan x)dx$, let $u = \sec x$, so $du = \sec x\tan x\, dx$, then:

$\int \sec^4 x\tan x\, dx = \int u^3 du = \frac{1}{4}u^4 + c = \frac{1}{4}\sec^4 x + c$

44. $\int \operatorname{sech} x\, dx = \int \frac{1}{\cosh x} dx = \int \frac{\cosh x}{\cosh^2 x} dx = \int \frac{\cosh x}{1+\sinh^2 x} dx$, let $u = \sinh x$, $di = \cosh x\, dx$,

$\int \operatorname{sech} x\, dx = \int \frac{du}{1+u^2} = \arctan u + c = \arctan(\sinh x) + c$

45. $\int \tan x\, dx = \int \frac{\sin x}{\cos x} dx$, let $u = \cos x$, then $du = -\sin x\, dx$, so:

$\int \tan x\, dx = -\int \frac{du}{u} = -\ln|u| + c = -\ln|\cos x| + c$

46. $\int \sinh x \cosh^2 x \, dx$, let $u = \cosh x$, then $du = \sinh x \, dx$, so:

$\int \sinh x \cosh^2 x \, dx = \int u^2 du = \frac{1}{3}u^3 + c = \frac{1}{3}\cosh^3 x + c$

47. $\int \cos 3x \sin^5 3x \, dx = \frac{1}{3}\int 3\cos 3x \sin^5 3x \, dx$, let $u = \sin 3x$, then $du = 3\cos 3x \, dx$, so:

$\int \cos 3x \sin^5 3x \, dx = \frac{1}{3}\int u^5 du = \frac{1}{3}\left(\frac{1}{6}u^6\right) + c = \frac{1}{18}\sin^6 3x + c$

48. $\int \sin x \sin 2x \, dx = \int \sin x \times 2\sin x \cos x \, dx = 2\int \sin^2 x \cos x \, dx$

Let $u = \sin x$, then $du = \cos x \, dx$, so:

$\int \sin x \sin 2x \, dx = 2\int u^2 du = \frac{2}{3}u^3 + c = \frac{2}{3}\sin^3 x + c$

49. $\int \sec^6 x \, dx = \int (\sec^2 x)^2 \sec^2 x \, dx = \int (1 + \tan^2 x)^2 \sec^2 x \, dx$

$= \int (\tan^4 x + 2\tan^2 x + 1)\sec^2 x \, dx$, let $u = \tan x$, then $du = \sec^2 x \, dx$, so:

$\int \sec^6 x \, dx = \int (u^4 + 2u^2 + 1)du = \frac{1}{5}u^5 + \frac{2}{3}u^3 + u + c = \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + c$

50. $\int \frac{\operatorname{sech}^2(\ln x)}{x} dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, so:

$\int \frac{\operatorname{sech}^2(\ln x)}{x} dx = \int \operatorname{sech}^2 u \, du = \tanh u + c = \tanh(\ln x) + c$

51. $\int \frac{\sinh\sqrt{x}}{\sqrt{x}} dx = 2\int \sinh\sqrt{x}\frac{dx}{2\sqrt{x}}$, let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, so:

$\int \frac{\sinh\sqrt{x}}{\sqrt{x}} dx = 2\int \sinh u \, du = 2\cosh u + c = 2\cosh\sqrt{x} + c$

52. $\int x^2 \operatorname{sech} x^3 \tanh x^3 \, dx = \frac{1}{3}\int 3x^2 \operatorname{sech} x^3 \tanh x^3 \, dx$, let $u = x^3$, then $du = 3x^2 dx$, so:

$\int x^2 \operatorname{sech} x^3 \tanh x^3 \, dx = \frac{1}{3}\int \operatorname{sech} u \tanh u \, du = -\frac{1}{3}\operatorname{sech} u + c = -\frac{1}{3}\operatorname{sech}(x^3) + c$

53. $\int x\sin(4 - x^2) \, dx = -\frac{1}{2}\int -2x\sin(4 - x^2) \, dx$, let $t = 4 - x^2$, so $dt = -2xdx$, then:

$\int x\sin(4 - x^2) \, dx = -\frac{1}{2}\int \sin t \, dt = -\frac{1}{2}\cos t + c = -\frac{1}{2}\cos(4 - x^2) + c$

54. $\int \frac{\cos x}{2-3\sin x} dx = -\frac{1}{3}\int \frac{-3\cos x}{2-3\sin x} dx$, let $u = 2 - 3\sin x$, then $du = -3\cos x \, dx$, so:

$\int \frac{\cos x}{2-3\sin x} = -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\ln|u| + c = -\frac{1}{3}\ln|2 - 3\sin x| + c$

55. $\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sec^2 x . \sec^2 x}{\tan^2 x} dx = \int \frac{1+\tan^2 x}{\tan^2 x}\sec^2 x \, dx$, let $u = \tan x$, then $du = \sec^2 x \, dx$,

$\int \frac{dx}{\sin^2 x \cos^2 x} = \int \left(\frac{1+t^2}{t^2}\right) dx = \int \left(1 + \frac{1}{t^2}\right) dx = t - \frac{1}{t} + c = \tan x - \cot x + c$

56. $\int \frac{1}{(ax+b)^n} dx = \frac{1}{a}\int \frac{a}{(ax+b)^n} dx$, let $u = ax + b$, then $du = adx$, so:

$\int \frac{1}{(ax+b)^n} dx = \frac{1}{a}\int \frac{du}{u^n} = \frac{1}{a}\int u^{-n} du = \frac{1}{a} . \frac{u^{-n+1}}{-n+1} + c = \frac{1}{a(1-n)}u^{1-n} + c$

$= \frac{1}{a(1-n)}(ax + b)^{1-n} + c$

57. $\int \frac{x^2-2x}{x^3-3x^2+1} dx = \frac{1}{3}\int \frac{3x^2-6x}{x^3-3x^2+1} dx$, let $u = x^3 - 3x^2 + 1$, so $du = (3x^2 - 6x)dx$, then:

$\int \frac{x^2-2x}{x^3-3x^2+1} dx = \frac{1}{3}\int \frac{du}{u} = \frac{1}{3}\ln|u| + c = \frac{1}{3}\ln|x^3 - 3x^2 + 1| + c$

58. $\int \frac{x\ln(x^2+5)}{x^2+5}dx = \frac{1}{2}\int \frac{2x}{x^2+5}\ln(x^2+5)\,dx$, let $u = \ln(x^2+5)$, then $du = \frac{2x}{x^2+5}dx$, so:

$\int \frac{x\ln(x^2+5)}{x^2+5}dx = \frac{1}{2}\int u du = \frac{1}{2}\left(\frac{u^2}{2}\right) + c = \frac{1}{4}\ln^2(x^2+5) + c$

59. $\int \frac{1+\tan x}{x+\ln(\sec x)}dx$, let $u = x + \ln(\sec x)$, then $du = \left[1 + \frac{(\sec x)'}{\sec x}\right]dx = \left(1 + \frac{\sec x \tan x}{\sec x}\right)dx$

so, $du = (1+\tan x)dx$, so $\int \frac{1+\tan x}{x+\ln(\sec x)}dx = \int \frac{du}{u} = \ln|u| + c = \ln|x + \ln(\sec x)| + c$

60. $\int \frac{\operatorname{sech}^2 x}{3+2\tanh x}dx = \frac{1}{2}\int \frac{2\operatorname{sech}^2 x}{3+2\tanh x}dx$, let $u = 3 + 2\tan h\,dx$, then $du = 2\operatorname{sech}^2 x\,dx$, so:

$\int \frac{\operatorname{sech}^2 x}{3+2\tan hx}dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|x| + c = \frac{1}{2}\ln|3 + 2\tanh x| + c$

61. $\int \frac{e^{\sin x}}{\tan x \csc x}dx = \int \frac{e^{\sin x}}{\frac{\sin x}{\cos x}\frac{1}{\sin x}}dx = \int e^{\sin x}\cos x\,dx$, let $u = \sin x$, then $du = \cos x\,dx$, so:

$\int \frac{e^{\sin x}}{\tan x \csc x}dx = \int e^u du = e^u + c = e^{\sin x} + c$

62. $\int \frac{e^x}{(1+e^x)^2}dx$, let $u = 1 + e^x$, then $du = e^x dx$, so:

$\int \frac{e^x}{(1+e^x)^2}dx = \int \frac{du}{u^2} = -\frac{1}{u} + c = -\frac{1}{1+e^x} + c$

63. $\int \frac{e^x}{\sqrt{e^{2x}+1}}dx$, let $u = e^x$, then $du = e^x du$, so:

$\int \frac{e^x}{\sqrt{e^{2x}+1}}dx = \int \frac{1}{\sqrt{u^2+1}}dx = \operatorname{arcsinh} u + c = \operatorname{arcsinh}(e^x) + c$

64. $\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{dx}{\sqrt{(e^x)^2-1}}$, let $u = e^x$, then $du = e^x dx = u dx$, so:

$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{du}{u\sqrt{u^2-1}} = \operatorname{arcsec}|u| + c = \operatorname{arcsec}(e^x) + c$

65. $\int \cos(\arcsin x)\,dx$, let $u = \arcsin x$, so $\sin u = x$ and $\cos u\,du = dx$, then:

$\int \cos(\arcsin x)\,dx = \int \cos u \cos u\,du = \int \cos^2 u\,du = \frac{1}{2}\int (1 + \cos 2u)du$

$= \frac{1}{2}\left(u + \frac{1}{2}\sin 2u\right) + c = \frac{1}{2}u + \frac{1}{2}\sin u \cos u + c = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + c$

66. $\int \frac{e^x \arctan(e^x)}{e^{2x}+1}dx$, let $u = \arctan(e^x)$, then $du = \frac{e^x}{1+e^{2x}}dx$, so:

$\int \frac{e^x \arctan(e^x)}{e^{2x}+1}dx = \int u du = \frac{1}{2}u^2 + c = \frac{1}{2}[\arctan(e^x)]^2 + c$

67. $\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}dx$, let $u = \arcsin x$, then $du = \frac{1}{\sqrt{1-x^2}}dx$, so:

$\int \frac{e^{\arcsin x}}{\sqrt{1-x^2}}dx = \int e^u du = e^u + c = e^{\arcsin x} + c$

68. $\int \frac{\sin(\arctan x)}{1+x^2}dx$, let $t = \arctan x$, then $dt = \frac{1}{1+x^2}dx$, so:

$\int \frac{\sin(\arctan x)}{1+x^2}dx = \int \sin t\,dt = -\cos t + c = -\cos(\arctan x) + c$

69. $\int \arctan x\,dx$, let $u = \arctan x$, then $u' = \frac{1}{1+x^2}$ and let $v' = 1$, so $v = x$, then**:**

$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}dx = x\arctan x - \frac{1}{2}\int \frac{2x}{1+x^2}dx$

$= x\arctan x - \frac{1}{2}\int \frac{(1+x^2)'}{1+x^2}dx = x\arctan x - \frac{1}{2}\ln(1+x^2) + c$

70. $\int \arcsin x\,dx$, let $u = \arcsin x$ then $u' = \frac{1}{\sqrt{1-x^2}}$ and let $v' = 1$ the $v = x$, so

$\int \arcsin x\,dx = x\arcsin x - \int \frac{x}{\sqrt{1-x^2}}dx = x\arcsin x - \sqrt{1-x^2} + \mathrm{c}$

Let's evaluate: $\int \frac{x}{\sqrt{1-x^2}}dx$, let $u = 1 - x^2$, so $du = -2xdx$, then

$\int \frac{x}{\sqrt{1-x^2}}dx - \int \frac{dt}{2\sqrt{t}} = -\sqrt{t} + \mathrm{c} = -\sqrt{1-x^2} + \mathrm{c}.$

Therefore, $\int \arcsin x\,dx = x\arcsin x - \sqrt{1-x^2} + \mathrm{c}$

71. $\int \sqrt{1-x^2}dx$, let $u = \sqrt{1-x^2}$ then $u' = \frac{-x}{\sqrt{1-x^2}}$ and let $v' = 1$ the $v = x$, so**:**

$\int \sqrt{1-x^2}dx = x\sqrt{1-x^2} - \int \frac{-x^2}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} - \int \frac{1-x^2-1}{\sqrt{1-x^2}}dx$

$= x\sqrt{1-x^2} - \int \frac{1-x^2}{\sqrt{1-x^2}}dx + \int \frac{1}{\sqrt{1-x^2}}dx = x\sqrt{1-x^2} - \int \frac{1-x^2}{\sqrt{1-x^2}}dx + \arcsin x$

$\int \sqrt{1-x^2}dx = x\sqrt{1-x^2} - \int \frac{\sqrt{1-x^2}\sqrt{1-x^2}}{\sqrt{1-x^2}}dx + \arcsin x$

$\int \sqrt{1-x^2}dx = x\sqrt{1-x^2} - \int \sqrt{1-x^2}dx + \arcsin x$, so

$2\int \sqrt{1-x^2}dx = x\sqrt{1-x^2} + \arcsin x + k$, then $\int \sqrt{1-x^2}dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin x + c$

72. $\int \arcsin(\sqrt{x})\,dx$, let $x = \sin^2 t$, then $dx = 2\sin t\cos t\,dt = \sin 2t\,dt$, then**:**

$\int \arcsin(\sqrt{x})\,dx = \int t\sin 2t\,dt$, let $u = t$ then $u' = 1$ and let $v' = \sin 2t$ the $v = -\frac{1}{2}\cos 2t$, so**:**

$\int \arcsin(\sqrt{x})\,dx = -\frac{1}{2}t\cos 2t + \frac{1}{2}\int \cos 2t dt = -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t + c$

$= -\frac{1}{2}t(1 - 2\sin^2 t) + \frac{1}{2}\sin t\sqrt{1-\sin^2 t} + c = -\frac{1}{2}(1-2x)\arcsin(\sqrt{x}) + \frac{1}{2}\sqrt{x}\sqrt{1-x} + c$

73. $\int x(1-x)^{1991}dx$, let $u = 1 - x$, then $du = -du$, so**:**

$\int x(1-x)^{1991}dx = -\int (1-u)u^{1991}du = \int (u^{1992} - u^{1991})dx = \frac{1}{1993}u^{1993} - \frac{1}{1992}u^{1992} + c$

$= \frac{1}{1993}(1-x)^{1993} - \frac{1}{1992}(1-x)^{1992} + c$

74. $\int x\sqrt{x-4}dx$, let $u = x - 4$ $(x = u + 4)$, then $du = dx$, so**:**

$\int x\sqrt{x-4}dx = \int (u+4)\sqrt{u}du = \int (u+4)u^{\frac{1}{2}}du = \int \left(u^{\frac{3}{2}} + 4u^{\frac{1}{2}}\right)du = \frac{2}{5}u^{\frac{5}{2}} + \frac{8}{3}u^{\frac{3}{2}} + c$

$= \frac{2}{5}(x-4)^{\frac{5}{2}} + \frac{8}{3}(x-4)^{\frac{3}{2}} + c$

75. $\int x\sqrt{2x-5}dx$, let $u=2x-5$, then $du=2dx$, $\left(x=\frac{u+5}{2}\right)$, so:

$\int x\sqrt{2x-5}dx=\int\frac{u+5}{2}.\sqrt{u}.\frac{1}{2}du=\frac{1}{4}\int(u+5)u^{\frac{1}{2}}du=\frac{1}{4}\int\left(u^{\frac{3}{2}}+5u^{\frac{1}{2}}\right)du$

$=\frac{1}{4}\left(\frac{2}{5}u^{\frac{5}{2}}+\frac{10}{3}u^{\frac{3}{2}}\right)+c=\frac{1}{10}u^{\frac{5}{2}}+\frac{5}{6}u^{\frac{3}{2}}+c=\frac{1}{10}(2x-5)^{\frac{5}{2}}+\frac{5}{6}(2x-5)^{\frac{3}{2}}+c$

76. $\int\frac{dx}{3-\sqrt{x}}$, let $u=3-\sqrt{x}$, then $x=(3-u)^2=(u-3)^2$ and so $dx=2(u-3)du$, so:

$\int\frac{dx}{3-\sqrt{x}}=\int\frac{1}{u}\times2(u-3)du=2\int\left(1-\frac{3}{u}\right)du=2(u-3\ln|u|)+c$

$=2\left(3-\sqrt{x}-3\ln\left|3-\sqrt{x}\right|\right)+c$

77. $\int\sin^5x\,dx=\int\sin x\,(\sin^2x)^2dx=\int\sin x\,(1-\cos^2x)dx$

Let $u=\cos x$, then $du=-\sin x\,dx$, so:

$\int\sin^5x\,dx=-\int(1-u^2)^2du=\int(-u^4+2u^2-1)du=-\frac{1}{5}u^5+\frac{2}{3}u^3-u+c$

$=-\frac{1}{5}\cos^5x+\frac{2}{3}\cos^3x-\cos x+c$

78. $\int\sec^5x\tan^3x\,dx=\int\sec^4x\tan^2x\sec x\tan x\,dx=\int\sec^4x\,(\sec^2x-1)\sec x\tan x\,dx$

Let $u=\sec x$, then $du=\sec x\tan x\,dx$, so:

$\int\sec^5x\tan^3x\,dx=\int u^4(u^2-1)du=\int(u^6-u^4)du=\frac{1}{7}u^7-\frac{1}{5}u^5+c$

$=\frac{1}{7}\sec^7x-\frac{1}{5}\sec^5x+c$

79. $\int\frac{\ln(\tan x)}{\sin2x}dx=\int\frac{\ln(\tan x)}{2\sin x\cos x}dx$

Let $u=\ln(\tan x)$, then $du=\frac{\sec^2x}{\tan x}dx=\frac{\frac{1}{\cos^2x}}{\frac{\sin x}{\cos x}}dx=\frac{1}{\sin x\cos x}dx$, so:

$\int\frac{\ln(\tan x)}{\sin2x}dx=\frac{1}{2}\int udu=\frac{1}{4}u^2+c=\frac{1}{4}\ln^2(\tan x)+c$

80. $\int\frac{\ln(\sin x)}{\tan x}dx=\int\frac{\ln(\sin x)}{\tan x}\times\frac{\cot x}{\cot x}dx=\int\cot x\ln(\sin x)\,dx$

Let $u=\ln(\sin x)$, then $du=\frac{(\sin x)'}{\sin x}dx=\frac{\cos x}{\sin x}dx=\cot x\,dx$, so:

$\int\frac{\ln(\sin x)}{\tan x}dx=\int udu=\frac{1}{2}u^2+c=\frac{1}{2}\ln^2(\sin x)+c$

81. $\int\frac{1}{x\sqrt{4x^2-64}}dx$, let $u=2x$, then $du=2dx$, so:

$\int\frac{1}{x\sqrt{4x^2-64}}dx=\int\frac{1}{u\sqrt{u^u-64}}du=\int\frac{1}{u\sqrt{u^u-8^2}}du=\frac{1}{8}\text{arcsec}\frac{|u|}{8}+c=\frac{1}{8}\text{arcsec}\frac{|x|}{4}+c$

82. $\int\frac{1}{x^2\times\sqrt[4]{(x^4+1)^3}}dx=\int\frac{1}{x^2\times\sqrt[4]{[x^2(1+x^{-4})]^3}}dx=\int\frac{1}{x^2x^3\times\sqrt[4]{(1+x^{-4})^3}}dx=\int\frac{x^{-5}}{\sqrt[4]{(x^{-4}+1)^3}}dx$

Let $u=1+x^{-4}$, then $du=-4x^{-5}$, so:

$\int\frac{1}{x^2\times\sqrt[4]{(x^4+1)^3}}dx=-\frac{1}{4}\int\frac{-4x^{-5}}{\sqrt[4]{(x^{-4}+1)^3}}dx=-\frac{1}{4}\int\frac{du}{\sqrt[4]{u^3}}=-\frac{1}{4}\int u^{-\frac{3}{4}}du=-\sqrt{u}+c$

$=-\sqrt{1+x^{-4}}+c$

83. $\int (x^6 + x^3)\sqrt[3]{x^3+2}\,dx = \int (x^5 + x^2) \times x \times \sqrt[3]{x^3+2}\,dx = \int (x^5 + x^2)\sqrt[3]{x^6+2x^3}\,dx$

$= \frac{1}{6}\int 6(x^5 + x^2)\sqrt[3]{x^6+2x^3}\,dx$

Let $u = x^6 + 2x^3$, then $du = (6x^5 + 6x^2)dx = 6(x^5 + x^2)dx$, so**:**

$\int (x^6 + x^3)\sqrt[3]{x^3+2}\,dx = \frac{1}{6}\int u^{\frac{1}{3}}du = \frac{1}{6} \times \frac{3}{4}u^{\frac{4}{3}} + c = \frac{1}{8}\sqrt[3]{(x^6+2x^3)^4} + c$

84. $\int \frac{dx}{\sqrt{(x+2)(3-x)}} = \int \frac{dx}{\sqrt{6+x-x^2}} = \int \frac{dx}{\sqrt{6-\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)}} = \int \frac{dx}{\sqrt{\frac{25}{4}-\left(x-\frac{1}{2}\right)^2}}$, let $x - \frac{1}{2} = u$, $dx = du$,

$\int \frac{dx}{\sqrt{(x+2)(3-x)}} = \int \frac{dx}{\sqrt{\left(\frac{5}{2}\right)^2 - u^2}} = \arcsin\left(\frac{u}{\frac{5}{2}}\right) + c = \arcsin\left(\frac{2x-1}{5}\right) + c$

85. $\int \frac{3}{x^2+4x-45}dx = \int \frac{3}{(x+9)(x-5)}dx = \frac{3}{14}\int\left(-\frac{1}{x+9} + \frac{1}{x-5}\right)dx$

$= \frac{3}{14}(\ln|x-5| - \ln|x+9|) + c = \frac{3}{14}\ln\left|\frac{x-5}{x+9}\right| + c$

Remark : $\frac{1}{(x+9)(x-5)} = \frac{\mathrm{A}}{x+9} + \frac{\mathrm{B}}{x-5}$ (method of partial fractions)

Then , $\frac{1}{(x+9)(x-5)} = \frac{\mathrm{A}(x-5)+\mathrm{B}(x+9)}{(x+9)(x-5)} = \frac{(\mathrm{A}+\mathrm{B})x+(-5\mathrm{A}+9\mathrm{B})}{(x+9)(x-5)}$, so we get $\begin{cases} \mathrm{A}+\mathrm{B} = 0 \\ -5\mathrm{A}+9\mathrm{B} = 1 \end{cases}$'

therefore, $\mathrm{A} = -\frac{1}{14}$ and $\mathrm{B} = \frac{1}{14}$.

86. $\int \frac{1}{x^3+x^2}dx = \int \frac{1}{x^2(x+1)}dx$

Remark : $\frac{1}{x^2(x+1)} = \frac{\mathrm{A}}{x} + \frac{\mathrm{B}}{x^2} + \frac{\mathrm{C}}{x+1}$ (method of partial fractions), then**:**

$1 = \mathrm{A}x(x+1) + \mathrm{B}(x+1) + \mathrm{C}x^2$, for $x = 0$, $\mathrm{B} = 1$, for $x = -1$, $\mathrm{C} = 1$ and for $x = 1$,

$1 = 2\mathrm{A} + 3$, so $\mathrm{A} = -1$, then**:**

$\int \frac{1}{x^3+x^2}dx = \int\left(\frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right)dx = -\ln|x| - \frac{1}{x} + +\ln|x+1| + c$

87. $\int \frac{1}{x^3-4x}dx = \int \frac{1}{x^2(x-4)}dx = \int\left(-\frac{1}{4x^2} - \frac{1}{16x} + \frac{1}{16(x-4)}\right)dx$

$= \frac{1}{4x} - \frac{1}{16}\ln|x| + \frac{1}{16}\ln|x-4| + c$

Remark : $\frac{1}{x^2(x-4)} = \frac{\mathrm{A}}{x^2} + \frac{\mathrm{B}}{x} + \frac{\mathrm{C}}{x-4}$ (method of partial fractions), then**:**

$\mathrm{A}(x-4) + \mathrm{B}x(x-4) + \mathrm{C}x^2 = 1$, then $(\mathrm{C}+\mathrm{B})x^2 + (\mathrm{A}-4\mathrm{B})x - 4\mathrm{A} = 1$, so $-4\mathrm{A} = 1$, then

$\mathrm{A} = -\frac{1}{4}$, $\mathrm{A} - 4\mathrm{B} = 0$, so $\mathrm{B} = \frac{1}{4}\mathrm{A} = -\frac{1}{16}$ and $\mathrm{C} + \mathrm{B} = 0$, so $\mathrm{C} = -\mathrm{B} = \frac{1}{16}$.

88. $\int \frac{x+1}{\sqrt{4-x^2}}dx$, let $x = 2\sin\theta$, then $dx = 2\cos\theta\,d\theta$, so**:**

$\int \frac{x+1}{\sqrt{4-x^2}}dx = \int \frac{2\sin\theta+1}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,d\theta = \int \frac{2\sin\theta+1}{\sqrt{4\cos^2\theta}}2\cos\theta\,d\theta = \int(2\sin\theta+1)d\theta$

$= \theta - 2\cos\theta + c = \arcsin\left(\frac{x}{2}\right) - \sqrt{4-x^2} + c$

89. $\int \cot(\ln\sin x)\,dx$, let $u = \ln(\sin x)$, then $du = \frac{\cos x}{\sin x}dx = \cot x\,dx$, then**:**

$\int \cot(\ln\sin x)\,dx = \int u\,du = \frac{1}{2}u^2 + c = \frac{1}{2}\ln^2(\sin x) + c$

90. $\int \frac{e^{-2x}}{e^{-x}+1}dx$, let $u=e^{-x}+1$, then $du=-e^{-x}dx$, so:

$\int \frac{e^{-2x}}{e^{-x}+1}dx=-\int \frac{e^{-x}}{e^{-x}+1}(-e^{-x})dx=-\int \frac{u-1}{u}du=\int \frac{1-u}{u}dx=\int \left(\frac{1}{u}-1\right)du$

$=\ln|u|-u+c=\ln|e^{-x}+1|-e^{-x}-1+c$

91. $\int \frac{x^5}{2+x^{12}}dx=\frac{1}{6}\int \frac{6x^5}{\left(\sqrt{2}\right)^2+\left(x^6\right)^2}dx$, let $u=x^6$, then $du=6x^5dx$, so:

$\int \frac{x^5}{2+x^{12}}dx=\frac{1}{6}\int \frac{du}{\left(\sqrt{2}\right)^2+u^2}=\frac{1}{6\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)+c=\frac{1}{6\sqrt{2}}\arctan\left(\frac{x^6}{\sqrt{2}}\right)+c$

92. $\int x\log_2 x\,dx$, let $u=\log_2 x$, then $u'=\frac{1}{x\ln 2}$ and let $v'=x$, then $v=\frac{1}{2}x^2$, then:

$\int x\log_2 x\,dx=\frac{1}{2}x^2\log_2 x-\int \frac{1}{2}x^2\frac{1}{x\ln 2}dx+c=\frac{1}{2}x^2\log_2 x-\frac{1}{2\ln 2}\int x\,dx+c$

$=\frac{1}{2}x^2\log_2 x-\frac{1}{4\ln 2}x^2+c$

93. $\int \frac{1+\tan^2 x}{a^2+\tan^2 x}dx$, let $t=\tan x$, then $dt=(1+\tan^2 x)dx$, so:

$\int \frac{1+\tan^2 x}{a^2+\tan^2 x}dx=\int \frac{dt}{a^2+t^2}=\frac{1}{a}\arctan\left(\frac{t}{a}\right)+c=\frac{1}{a}\arctan\left(\frac{\tan x}{a}\right)+c$

94. $\int \frac{\tan^2 x}{\sqrt[3]{\tan x-x}}dx=\int \frac{\sec^2 x-1}{\sqrt[3]{\tan x-x}}dx$, let $u=\tan x-x$, then $du=(\sec^2 x-1)dx$, so:

$\int \frac{\tan^2 x}{\sqrt[3]{\tan x-x}}dx=\int \frac{du}{\sqrt[3]{u}}=\int u^{-\frac{1}{3}}du=\frac{1}{-\frac{1}{3}+1}u^{-\frac{1}{3}+1}+c=\frac{3}{2}u^{\frac{2}{3}}+c=\frac{3}{2}(\tan x-x)^{\frac{2}{3}}+c$

95. $\int \frac{e^x}{e^x-e^{-x}}dx=\int \frac{e^x}{e^x-e^{-x}}\times\frac{e^x}{e^x}dx=\int \frac{e^{2x}}{e^{2x}-1}dx=\frac{1}{2}\int \frac{2e^{2x}}{e^{2x}-1}dx$

Let $u=e^{2x}-1$, then $du=2e^{2x}dx$, so:

$\int \frac{e^x}{e^x-e^{-x}}dx=\frac{1}{2}\int \frac{du}{u}=\frac{1}{2}\ln|u|+c=\frac{1}{2}\ln|e^{2x}-1|+c$

96. $\int \sqrt{\frac{e^x}{e^{-x}+1}}dx=\int \sqrt{\frac{e^{2x}}{1+e^x}}dx=\int \frac{e^x}{\sqrt{1+e^x}}dx$, let $u=1+e^x$, then $du=e^xdx$, so:

$\int \sqrt{\frac{e^x}{e^{-x}+1}}dx=\int \frac{1}{\sqrt{u}}du=2\sqrt{u}+c=2\sqrt{1+e^x}+c$

97. $\int \frac{\sin 2x}{1+\sin^4 x}dx=\int \frac{2\sin x\cos x}{1+\left(\sin^2 x\right)^2}dx$, let $u=\sin^2 x$, then $du=2\sin x\cos x\,dx$, so:

$\int \frac{\sin 2x}{1+\sin^4 x}dx=\int \frac{du}{1+u^2}=\arctan u+c=\arctan(\sin^2 x)+c$

98. $\int \frac{\sin 2x}{\sqrt{1+\sin x}}dx=\int \frac{2\sin x\cos x}{\sqrt{1+\sin x}}dx$, let $u=1+\sin x$, then $du=\cos x\,dx$, so:

$\int \frac{\sin 2x}{\sqrt{1+\sin x}}dx=2\int \frac{u-1}{\sqrt{u}}dx=2\int \left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)du=\frac{4}{3}u^{\frac{3}{2}}-2\sqrt{u}+c$

$=\frac{4}{3}(1+\sin x)^{\frac{3}{2}}-2\sqrt{1+\sin x}+c$

99. $\int \frac{x^2+1}{x^4-x^2+1} dx = \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}} dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+1} dx$, let $u = x - \frac{1}{x}$, then $du = \left(1 + \frac{1}{x^2}\right) dx$, so:

$\int \frac{x^2+1}{x^4-x^2+1} dx = \int \frac{du}{1+u^2} = \arctan u + c = \arctan\left(x - \frac{1}{x}\right) + c$

100. $\int \frac{dx}{x\sqrt{\sqrt{x}-1}}$, let $u = \sqrt{\sqrt{x}} = x^{\frac{1}{4}}$, then $du = \frac{1}{4}x^{-\frac{3}{4}}dx = \frac{1}{4}\left(x^{\frac{1}{4}}\right)^{-3} dx = \frac{1}{4}u^{-3}dx$

then $dx = 4u^3 du$, so we get:

$\int \frac{dx}{x\sqrt{\sqrt{x}-1}} = \int \frac{4u^3 du}{u^4\sqrt{u^2-1}} = 4\int \frac{du}{u\sqrt{u^2-1}} = 4\sec^{-1} u + c = 4\sec^{-1}\left(x^{\frac{1}{4}}\right) + c =$

$4\sec^{-1}\left(\sqrt{\sqrt{x}}\right) + c$

101. $\int \frac{x\cos x - \sin x}{x^2+\sin^2 x} dx = \int \frac{\frac{\cos x}{x} - \frac{\sin x}{x^2}}{1+\left(\frac{\sin x}{x}\right)^2} dx = \int \frac{\frac{x\cos x-\sin x}{x^2}}{1+\left(\frac{\sin x}{x}\right)^2} dx$

Let $u = \frac{\sin x}{x}$, then $du = \left(\frac{x\cos x - \sin x}{x^2}\right) dx$, so:

$\int \frac{x\cos x-\sin x}{x^2+\sin^2 x} dx = \int \frac{du}{1+u^2} = \arctan u + c = \arctan\left(\frac{\sin x}{x}\right) + c$

102. **First Method: By parts** $\int \frac{x}{\sqrt{x+1}} dx = \int x\,(1+x)^{-\frac{1}{2}} dx$

Let $u = x$, then $u' = 1$ and let $v' = (1+x)^{-\frac{1}{2}}$, then $v = 2(x+1)^{\frac{1}{2}} = 2\sqrt{x+1}$, then:

$\int \frac{x}{\sqrt{x+1}} dx = 2x\sqrt{x+1} - 2\int (x+1)^{\frac{1}{2}} dx + c = 2x\sqrt{x+1} - \frac{4}{3}(x+1)^{\frac{3}{2}} + c$

Second Method: By substitution $\int \frac{x}{\sqrt{x+1}} dx$, let $u = x+1$, then $du = dx$, so:

$\int \frac{x}{\sqrt{x+1}} dx = \int \frac{u-1}{\sqrt{u}} du = \int (u-1)u^{-\frac{1}{2}} du = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du = \frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + c$

$= \frac{2}{3}(1+x)^{\frac{3}{2}} - 2\sqrt{1+x} + c$

103. $\int \sin(2\arcsin x)\, dx$, we know that $\sin 2\theta = 2\sin\theta\cos\theta$, with $\theta = \arcsin x$, so:

$\sin(2\arcsin x) = 2\sin(\arcsin x)\cos(\arcsin x) = 2x.\sqrt{1-\sin^2 \arcsin x} = 2x\sqrt{1-x^2}$, then:

$\int \sin(2\arcsin x)\, dx = \int 2x\sqrt{1-x^2} dx$, let $u = 1-x^2$, so $du = -2xdx$, then:

$\int \sin(2\arcsin x)\, dx = -\int \sqrt{u} du = -\frac{2}{3}u^{\frac{3}{2}} + c = -\frac{2}{3}\sqrt{(1-x^2)^3} + c$

104. $\int \frac{x}{1+\cot^2 x} dx = \int \frac{x}{\csc^2 x} dx = \int x\sin^2 x\, dx = \int x\left(\frac{1-\cos 2x}{2}\right) dx$

$= \frac{1}{2}\int (x - x\cos 2x)\, dx = \frac{1}{4}x^2 - \frac{1}{2}\int x\cos 2x\, dx$

Let $u = x$, then $u' = 1$ and let $v' = \cos 2x$, then $v' = \frac{1}{2}\sin 2x$, then we get:

$\int \frac{x}{1+\cot^2 x} dx = \frac{1}{4}x^2 - \frac{1}{2}\left(\frac{1}{2}x\sin 2x - \frac{1}{2}\int \sin 2x dx\right) + c = \frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$

105. $\int \ln\left(\sqrt{1+x^2}\right)dx = \frac{1}{2}\int \ln(1+x^2)dx$

Let $u = \ln(1+x^2)$, then $u' = \frac{2x}{1+x^2}$ and let $v' = 1$, so $v = x$, so:

$\int \ln\left(\sqrt{1+x^2}\right)dx = \frac{1}{2}x\ln(1+x^2) - \int \frac{x^2}{1+x^2}dx = \frac{1}{2}x\ln(1+x^2) - \int\left(1-\frac{1}{1+x^2}\right)dx$

$= \frac{1}{2}x\ln(1+x^2) - x + \arctan x + c$

106. $\int \sqrt{x}\sqrt{x\sqrt{x}+1}dx = \frac{2}{3}\int \frac{3}{2}\sqrt{x}\sqrt{x\sqrt{x}+1}dx$, let $t = x\sqrt{x}+1$, then $dt = \frac{3}{2}\sqrt{x}dx$, so:

$\int \sqrt{x}\sqrt{x\sqrt{x}+1}dx = \frac{2}{3}\int \sqrt{t}dt = \left(\frac{2}{3}\right)\left(\frac{2}{3}\right)t^{\frac{3}{2}} + c = \frac{4}{9}t\sqrt{t} + c$

107. $\int x\sqrt{1-\sqrt{x}}dx$, let $u = 1-\sqrt{x}$, so $x = (1-u)^2 = (u-1)^2$ and $dx = 2(u-1)du$, so:

$\int x\sqrt{1-\sqrt{x}}dx = \int (u-1)^2\sqrt{u}.2(u-1)du = \int 2(u-1)^3\sqrt{u}du$

$= \int(2u^3-6u^2+6u-2)\sqrt{u}du = \int\left(2u^{\frac{7}{2}} - 6u^{\frac{5}{2}} + 6u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)du$

$= \frac{4}{9}u^{\frac{9}{2}} - \frac{12}{7}u^{\frac{7}{2}} + \frac{12}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} + c$

$= \frac{4}{9}(1-\sqrt{x})^{\frac{9}{2}} - \frac{12}{7}(1-\sqrt{x})^{\frac{7}{2}} + \frac{12}{5}(1-\sqrt{x})^{\frac{5}{2}} - \frac{4}{3}(1-\sqrt{x})^{\frac{3}{2}} + c$

108. $\int \frac{x^3}{\sqrt{1+x^2}}dx = \int x^2 \times \frac{x}{\sqrt{1+x^2}}dx$, let $t = \sqrt{1+x^2}$, so $x^2 = t^2-1$ and $dt = \frac{x}{\sqrt{1+x^2}}dx$,

so: $\int \frac{x^3}{\sqrt{1+x^2}}dx = \int(t^2-1)dt = \frac{1}{3}t^3 - t + c = \frac{1}{3}\left(\sqrt{1+x^2}\right)^3 - \sqrt{1+x^2} + c$

109. $\int\sqrt{\frac{x}{1-x^3}}dx$, let $u = x^{\frac{3}{2}}$, then $du = \frac{3}{2}x^{\frac{1}{2}}dx$, so:

$\int\sqrt{\frac{x}{1-x^3}}dx = \frac{2}{3}\int\frac{\frac{3}{2}\sqrt{x}}{\sqrt{1-x^3}}dx = \frac{2}{3}\int\frac{1}{\sqrt{1-u^2}}du = \frac{2}{3}\arcsin u + c = \frac{2}{3}\arcsin\left(x^{\frac{3}{2}}\right) + c$

110. $\int\frac{1}{\sqrt{1+\sqrt{x}}}dx$, let $u = 1+\sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, $dx = 2\sqrt{x}du = 2(u-1)du$, so:

$\int\frac{1}{\sqrt{1+\sqrt{x}}}dx = 2\int\frac{u-1}{\sqrt{u}}du = 2\int\left(\sqrt{u}-\frac{1}{\sqrt{u}}\right)du = 2\left(\frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u}\right) + c$

$= \frac{4}{3}(1+\sqrt{x})^{\frac{3}{2}} - 4\sqrt{1+\sqrt{x}} + c = \frac{4}{3}\sqrt{(1+\sqrt{x})^3} - 4\sqrt{1+\sqrt{x}} + c$

111. $\int\frac{\sin x}{a+b\cos x}dx$, let $u = \cos x$, then $du = -\sin x\,dx$, so:

$\int\frac{\sin x}{a+b\cos x}dx = -\int\frac{1}{a+bu}dt = -\frac{1}{b}\int\frac{b}{a+bu}dt = -\frac{1}{b}\int\frac{(a+bu)'}{a+bu}dt = -\frac{1}{b}\ln|a+bu| + c$

$= -\frac{1}{b}\ln|a+b\cos x| + c$

112. $\int\frac{dx}{1+\cos\frac{x}{2}}$, let $t = \tan\frac{x}{4}$, $d\left(\frac{x}{2}\right) = \frac{2dt}{1+t^2}$ and $\cos\frac{x}{2} = \frac{1-t^2}{1+t^2}$, so:

$\int\frac{dx}{1+\cos\frac{x}{2}} = 2\int\frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} = 4\int\frac{dt}{1+t^2+1-t^2} = 2\int dt = 2t + c = 2\tan\frac{x}{4} + c$

113. $\int \frac{dx}{1+\cos 2x}$; let $t=\tan x$, $dx=\frac{dt}{1+t^2}$ and $\cos 2x=\frac{1-t^2}{1+t^2}$, so:

$$\int \frac{dx}{1+\cos 2x}=\int \frac{\frac{dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}=\int \frac{dt}{1+t^2+1-t^2}=\frac{1}{2}\int dt=\frac{1}{2}t+c=\frac{1}{2}\tan x+c$$

114. $\int x\ln(1-x^2)\,dx=-\frac{1}{2}\int -2xx\ln(1-x^2)\,dx$, let $y=1-x^2$, then $dy=-2xdx$, so:

$$\int x\ln(1-x^2)\,dx=-\frac{1}{2}\int \ln y\,dy$$

Let $u=\ln y$, then $u'=\frac{1}{y}$ and let $v'=1$, then $v=y$, then we get:

$$\int x\ln(1-x^2)\,dx=-\frac{1}{2}\int \ln y\,dy=-\frac{1}{2}\left[y\ln y-\int \frac{1}{y}ydy\right]=-\frac{1}{2}y\ln y+-\frac{1}{2}y+c$$

$$=-\frac{1}{2}(1-x^2)\ln(1-x^2)+\frac{1}{2}(1-x^2)+c$$

115. **First Method:** $\int \sec x\,dx=\int \frac{1}{\cos x}dx$, let $t=\tan\frac{x}{2}$, so $\cos x=\frac{1-t^2}{1+t^2}$ and $dx=\frac{2dt}{1+t^2}$,

$$\int \sec x\,dx=\int \frac{1+t^2}{1-t^2}\times\frac{2dt}{1+t^2}=\int \frac{2}{1-t^2}dt=\int \frac{(1-t)+(1+t)}{(1-t)(1+t)}dt=\int\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt$$

$$=\ln|1+t|-\ln|1-t|+c=\ln\left|\frac{1+t}{1-t}\right|+c=\ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|+c=\ln\left|\frac{\tan\frac{\pi}{4}+\tan\frac{x}{2}}{1-\tan\frac{\pi}{4}\tan\frac{x}{2}}\right|+c$$

$$=\ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|+c$$

Second Method: $\int \sec x\,dx=\int \frac{1}{\cos x}dx=\int \frac{\cos x}{\cos^2 x}dx=\int \frac{\cos x}{1-\sin^2 x}dx=\int \frac{\cos x}{(1-\sin x)(1+\sin x)}dx$

$$=\frac{1}{2}\int \frac{\cos x(1-\sin x+1+\sin x)}{(1-\sin x)(1+\sin x)}dx=\frac{1}{2}\int \frac{\cos x}{1+\sin x}dx+\frac{1}{2}\int \frac{\cos x}{1-\sin x}dx$$

$$=\frac{1}{2}\int \frac{(1+\sin x)'}{1+\sin x}dx-\frac{1}{2}\int \frac{(1-\sin x)'}{1-\sin x}dx=\frac{1}{2}\ln|1+\sin x|-\frac{1}{2}\ln|1-\sin x|+c$$

$$=\frac{1}{2}\ln\left|\frac{1+\sin x}{1-\sin x}\right|+c$$

Third Method: $\int \sec x\,dx=\int \sec x\times\frac{\sec x+\tan x}{\sec x+\tan x}dx=\int \frac{\sec x\tan x+\sec^2 x}{\sec x+\tan x}dx$

$$=\int \frac{(\sec x+\tan x)'}{\sec x+\tan x}dx=\ln|\sec x+\tan x|+c$$

116. $\int \sec^3 x\,dx=\int \sec x\sec^2 x\,dx$

Let $u=\sec x$, then $du=\sec x\tan x\,dx$ and let $v'=\sec^2 x$, then $v=\tan x$, so:

$\int \sec^3 x\,dx=\sec x\tan x-\int \sec x\tan^2 x\,dx=\sec x\tan x-\int \sec x\,(\sec^2 x-1)dx$, then:

$\int \sec^3 x\,dx=\sec x\tan x-\int \sec^3 x\,dx+\int \sec x\,dx$, so:

$2\int \sec^3 x\,dx=\sec x\tan x+\int \sec x\,dx=\sec x\tan x+\ln|\sec x+\tan x|+k$, therefore:

$$\int \sec^3 x\,dx=\frac{1}{2}(\sec x\tan x+\ln|\sec x+\tan x|)+c$$

117. $\int \frac{1}{\sqrt{1+x^2}}dx$, let $x=\tan\theta$, then $.\,dx=\sec^2\theta\,d\theta$, so:

$$\int \frac{1}{\sqrt{1+x^2}}dx=\int \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}d\theta=\int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta=\int \sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+c$$

$$=\ln\left|x+\sqrt{1+x^2}\right|+c$$

118. $\int \frac{dx}{x\sqrt{9x^2-4}} = \int \frac{dx}{x\sqrt{4\left(\frac{9x^2}{4}-1\right)}} = \int \frac{dx}{2x\sqrt{\left(\frac{3x}{2}\right)^2-1}}$, let $u = \frac{3}{2}x$, then $du = \frac{3}{2}dx$, so:

$\int \frac{dx}{x\sqrt{9x^2-4}} = \int \frac{1}{2\left(\frac{2}{3}u\right)\sqrt{u^2-1}}\frac{2}{3}du = \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}du = \frac{1}{2}\operatorname{arcsec} u + c$

$= \frac{1}{2}\operatorname{arcsec}\left(\frac{3}{2}x\right) + c$

119. $\int \frac{1}{x\sqrt{x^n-1}}dx = \int \frac{1}{x\sqrt{x^n-1}}.\frac{x^n}{x^n}dx = \int \frac{x^{n-1}}{x^n\sqrt{x^n-1}}dx = \frac{1}{n}\int \frac{nx^{n-1}}{x^n\sqrt{x^n-1}}dx$

Let $u = \sqrt{x^n-1}$, so $u^2 = x^n - 1$ and $u^2 + 1 = x^n$, then $2udu = nx^{n-1}dx$, so:

$\int \frac{1}{x\sqrt{x^n-1}}dx = \frac{2}{n}\int \frac{u}{u(u^2+1)}du = \frac{2}{n}\int \frac{1}{u^2+1}du = \frac{2}{n}\arctan u + c = \frac{2}{n}\arctan\left(\sqrt{x^n-1}\right) + c$

120. $\int \frac{dx}{x(x^5+1)} = \int \frac{dx}{x\left[x^5\left(\frac{x^5+1}{x^5}\right)\right]} = \int \frac{dx}{x^6\left(1+\frac{1}{x^5}\right)} = -\frac{1}{5}\int \frac{-5dx}{x^6\left(1+\frac{1}{x^5}\right)}$

Let $u = 1 + \frac{1}{x^5}$, then $du = -\frac{1}{x^6}dx$, so:

$\int \frac{dx}{x(x^5+1)} = -\frac{1}{5}\int \frac{dt}{t} = -\frac{1}{5}\ln|t| + c = -\frac{1}{5}\ln\left|1 + \frac{1}{x^5}\right| + c$

121. $\int x^2\sqrt{4-x^2}dx$, let $x = 2\sin t$ so $t = \arcsin\left(\frac{x}{2}\right)$, then $dx = 2\cos t\,dt$, so:

$\int x^2\sqrt{4-x^2}dx = \int 4\sin^2 t\sqrt{4-4\sin^2 t} \times 2\cos t\,dt = 4\int 2\cos t\sin^2 t\sqrt{4(1-\sin^2 t)}dt$

$= 4\int 4\cos^2 t\sin^2 t\,dt = 4\int(2\sin t\cos t)^2dt = 4\int \sin^2 2t\,dt = 4\int \frac{1-\cos 4t}{2}dt$

$= 2\int(1-\cos 4t)dt = 2\left(t - \frac{1}{4}\sin 4t\right) + c = 2\left[\arcsin\left(\frac{x}{2}\right) - \frac{1}{4}\sin\left(4\arcsin\left(\frac{x}{2}\right)\right)\right] + c$

122. $\int x\sin x\sec x\tan x\,dx = \int x\sin x\frac{1}{\cos x}\tan x\,dx = \int x\tan^2 x\,dx$

Let $u = x$, then $u' = 1$ and let $v' = \tan^2 x$, then $v = \int(\sec^2 x - 1)\,dx = \tan x - x$, so:

$\int x\sin x\sec x\tan x\,dx = x(\tan x - x) - \int(\tan x - x)dx + c$

$= x(\tan x - x) - \int \frac{\sin x}{\cos x}dx + \frac{1}{2}x^2 + c = x\tan x - x^2 + \int \frac{(\cos x);}{\cos x}dx + \frac{1}{2}x^2 + c$

$= x\tan x - \frac{1}{2}x^2 + \ln|\cos x| + c$

123. $\int \frac{1}{\cos^2 x(1+\sqrt{\tan x})^4}dx = \int \frac{\sec^2 x}{(1+\sqrt{\tan x})^4}dx$, let $t^2 = \tan x$, then $2tdt = \sec^2 x\,dx$, so:

$\int \frac{1}{\cos^2 x(1+\sqrt{\tan x})^4}dx = \int \frac{dt}{(1+t)^4} = -\frac{1}{3(t+1)^2} + c = -\frac{1}{3(1+\sqrt{\tan x})^2} + c$

124. $\int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int \frac{\sqrt{\tan x}\times\cos x}{\sin x\cos x\times\cos x}dx = \int \frac{\sqrt{\tan x}}{\tan x\cos^2 x}dx = \int \frac{\sec^2 x}{\sqrt{\tan x}}dx$

Let $u = \tan x$, $du = \sec^2 x\,dx$, $\int \frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int \frac{dt}{\sqrt{t}} = 2\sqrt{t} + c = 2\sqrt{\tan x}$

125. $\int \frac{\sin 2x - 4\cos x}{5-\cos^2 x - 4\sin x}dx = \int \frac{2\sin x\cos x - 4\cos x}{5-(1-\sin^2 x)-4\sin x}dx = \int \frac{2\cos x(\sin x - 2)}{\sin^2 x - 4\sin^2 x + 4}dx$

Let $t = \sin x$, then $dt = \cos x\,dx$, so:

$$\int \frac{\sin 2x-4\cos x}{5-\cos^2 x-4\sin x}dx = \int \frac{2(t-2)}{t^2-4t+4}dt = \int \frac{2(t-2)}{(t-2)^2}dt = 2\int \frac{1}{t-2}dt = 2\ln|t-2| + c$$

$= 2\ln|\sin x - 2| + c$

126. $\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}}dx = -2\int \frac{1}{\sqrt{\cos^3\sqrt{x}}}.\frac{(-\sin\sqrt{x})}{2\sqrt{x}}dx$, let $t = \cos\sqrt{x}$, so $dt = -\frac{1}{2\sqrt{x}}\sin\sqrt{x}\,dx$,

then: $\int \frac{\sin\sqrt{x}}{\sqrt{x\cos^3\sqrt{x}}}dx = -2\int \frac{1}{\sqrt{t^3}}dt = -2\int t^{-\frac{3}{2}}dt = \frac{4}{\sqrt{t}} + c = \frac{1}{\sqrt{\cos\sqrt{x}}} + c$

127. $\int \frac{\ln(x+1)-\ln x}{x(x+1)}dx$, let $u = \ln(x+1) - \ln x$, then $du = \left(\frac{1}{x+1} - \frac{1}{x}\right)dx = -\frac{1}{x(x+1)}dx$,

so: $\int \frac{\ln(x+1)-\ln x}{x(x+1)}dx = -\int u du = -\frac{1}{2}u^2 + c = -\frac{1}{2}[\ln(x+1) - \ln x]^2 + c$

128. $\int \frac{\sin 4x}{\sin^4 x+\cos^4 x}dx$, let $u = \sin^4 x + \cos^4 x$, then:

$du = 4\sin^3 x\cos x - 4\cos^3 x\sin x = -2(2\sin x\cos x)(\cos^2 x - \sin^2 x)dx$

$du = -2\sin 2x\cos 2x\,dx = -\sin 4x\,dx$, so:

$\int \frac{\sin 4x}{\sin^4 x+\cos^4 x}dx = -\int \frac{du}{u} = -\ln|u| + c = -\ln|\sin^4 x + \cos^4 x| + c$

129. $\int \frac{(x+1)(x+\ln x)^2}{x}dx = \int \left(\frac{x+1}{x}\right)(x+\ln x)^2 dx = \int \left(1 + \frac{1}{x}\right)(x+\ln x)^2 dx$

Let $t = x + \ln x$, then $dt = \left(1 + \frac{1}{x}\right)dx$, so:

$\int \frac{(x+1)(x+\ln x)^2}{x}dx = \int t^2 dt = \frac{1}{3}t^3 + c = \frac{1}{3}(x+\ln x)^3 + c$

130. $\int \frac{x}{1+\sin x}dx = \int \frac{x}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}dx = \int \frac{x-x\sin x}{1-\sin^2 x}dx = \int \frac{x-x\sin x}{\cos^2 x}dx$

$= \int (x\sec^2 x - x\sec x\tan x)\,dx = x\tan x - \int \tan x\,dx - x\sec x + \int \sec x\,dx$ **(by parts)**

$= x\tan x + \ln|\cos x| - x\sec x + \ln|\sec x + \tan x| + c$

131. $\int \sqrt{\tanh x}\,dx$, let $u = \sqrt{\tanh x}$, so $x = \tanh^{-1}(u^2)$ and $dx = \frac{2u}{1-(u^2)^2}du$, so:

$\int \sqrt{\tanh x}\,dx = \int u.\frac{2u}{1-u^4}du = \int \frac{2u^2}{1-u^4}du = \int \frac{(u^2-1)+(u^2+1)}{(1-u^2)(1+u^2)}du$

$= \int \left[-\frac{(1-u^2)}{(1-u^2)(1+u^2)} + \frac{(1+u^2)}{(1-u^2)(1+u^2)}\right]du = -\int \frac{1}{1+u^2}du + \int \frac{1}{1-u^2}du$

$= -\tan^{-1} u + \tanh^{-1} u + c = \tan^{-1}\left(\sqrt{\tanh x}\right) + \tanh^{-1}\left(\sqrt{\tanh x}\right) + c$

132. $\int \sin x\ln(\sin x)\,dx$, let $u = \ln(\sin x)$, then $u' = \frac{\cos x}{\sin x}$ and let $v' = \sin x$, so $v = -\cos x$

$\int \sin x\ln(\sin x)\,dx = -\cos x\ln(\sin x) + \int \frac{\cos^2 x}{\sin x}dx = -\cos x\ln(\sin x) + \int \frac{1-\sin^2 x}{\sin x}dx$

$= -\cos x\ln(\sin x) + \int \csc x\,dx - \int \sin x dx = -\cos x\ln(\sin x) - \ln|\csc x + \cot x| + \cos x + c$

133. $\int \frac{dx}{(x+1)\sqrt{x-2}}$, let $t=\sqrt{x-2}$, $t^2=x-2$, then $dx=2tdt$, so:

$$\int \frac{dx}{(x+1)\sqrt{x-2}}=\int \frac{2t}{(t^2+3)t}dt=\int \frac{2}{t^2+3}dt=2\int \frac{1}{t^2+(\sqrt{3})^2}dt=\frac{2}{\sqrt{3}}\arctan\left(\frac{t}{\sqrt{3}}\right)+c$$

$$=\frac{2}{\sqrt{3}}\arctan\left(\sqrt{\frac{x-2}{3}}\right)+c$$

134. $\int \frac{x}{x^4+x^2+1}dx=\frac{1}{2}\int \frac{2x}{(x^2)^2+x^2+1}dx$, let $t=x^2$, then $dt=2xdx$, so:

$$\int \frac{x}{x^4+x^2+1}dx=\frac{1}{2}\int \frac{1}{t^2+t+1}dt=\frac{1}{2}\int \frac{1}{t^2+t+\frac{1}{4}+\frac{3}{4}}dt=\frac{1}{2}\int \frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt$$

$$=\frac{1}{2}\times\frac{2}{\sqrt{3}}\arctan\left(\frac{t+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c=\frac{1}{\sqrt{3}}\arctan\left(\frac{2t+1}{\sqrt{3}}\right)+c=\frac{1}{\sqrt{3}}\arctan\left(\frac{2x^2+1}{\sqrt{3}}\right)+c$$

135. $\int \frac{dx}{(x+1)^{\frac{3}{4}}(x-2)^{\frac{5}{4}}}=\int \frac{dx}{(x+1)^{\frac{3}{4}}(x-2)^{\frac{5}{4}}\times\frac{(x-2)^{\frac{3}{4}}}{(x-2)^{\frac{3}{4}}}}=\int \frac{dx}{\left(\frac{x+1}{x-2}\right)^{\frac{3}{4}}(x-2)^2}$

Let $u=\frac{x+1}{x-2}$, $du=\frac{-2}{(x-2)^2}d$ so:

$$\int \frac{dx}{(x+1)^{\frac{3}{4}}(x-2)^{\frac{5}{4}}}=-\frac{1}{3}\int -\frac{1}{u^{\frac{3}{4}}}du=-\frac{1}{3}\frac{u^{\frac{1}{4}}}{\frac{1}{4}}+c=-\frac{4}{3}\left(\frac{x+1}{x-2}\right)^{\frac{1}{4}}+c$$

136. $\int \frac{6-x}{(x-3)(2x+5)}dx=\int\left[\frac{3}{11}\left(\frac{1}{x-3}\right)-\frac{17}{11}\left(\frac{1}{2x+5}\right)\right]dx=\frac{3}{11}\ln|x-3|-\frac{17}{22}\ln|2x+5|+c$

Remark : $\frac{6-x}{(x-3)(2x+5)}=\frac{\mathrm{A}}{x-3}+\frac{\mathrm{B}}{2x+5}$ (method of partial fractions), then**:**
$6-x=\mathrm{A}(2x+5)+\mathrm{B}(x-3)$, then $6-x=(5\mathrm{A}-\mathrm{B})+(2\mathrm{A}+\mathrm{B})x$, so $5\mathrm{A}-\mathrm{B}=6$ and $2\mathrm{A}+\mathrm{B}=-1$, so $\mathrm{A}=\frac{3}{11}$ and $\mathrm{B}=-\frac{17}{11}$.

137. $\int \frac{2x-1}{(x-1)(x-2)(x-3)}dx$

We have : $\frac{2x-1}{(x-1)(x-2)(x-3)}=\frac{\mathrm{A}}{x-1}+\frac{\mathrm{B}}{x-2}+\frac{\mathrm{C}}{x-3}$ (method of partial fractions)

$\mathrm{A}=\left[\frac{2x-1}{(x-2)(x-3)}\right]_{x=1}=\frac{1}{2}$, $\mathrm{B}=\left[\frac{2x-1}{(x-1)(x-3)}\right]_{x=2}=-3$ and $\mathrm{C}=\left[\frac{2x-1}{(x-1)(x-2)}\right]_{x=3}=\frac{5}{2}$, so**:**

$$\int \frac{2x-1}{(x-1)(x-2)(x-3)}dx=\frac{1}{2}\int \frac{1}{x-1}dx-3\int \frac{1}{x-2}dx+\frac{5}{2}\int \frac{1}{x-3}dx$$

$$=\frac{1}{2}\ln|x-1|-3\ln|x-2|+\frac{5}{2}\ln|x-3|+c$$

138. $\int \frac{1}{x^2+x-6}dx=\int \frac{1}{(x-2)(x+3)}dx$

Method of partial fractions : $\frac{1}{(x-2)(x+3)}=\frac{\mathrm{A}}{x-2}+\frac{\mathrm{B}}{x+3}$, so $1=(x+3)\mathrm{A}+(x-2)\mathrm{B}$

for $x=2$, $1=5\mathrm{A}$, so $\mathrm{A}=\frac{1}{5}$ and for $x=-3$, $1=-5\mathrm{B}$, so $\mathrm{B}=-\frac{1}{5}$, then we get**:**

$$\int \frac{1}{x^2+x-6}dx=\frac{1}{5}\int\left(\frac{1}{x-2}-\frac{1}{x+3}\right)dx=\frac{1}{5}(\ln|x-2|+\ln|x+3|)+c=\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|+c$$

139. $\int \frac{2x+4}{x^3-4x}dx = \int \frac{2(x+2)}{x(x^2-4)}dx = \int \frac{2(x+2)}{x(x-2)(x+2)}dx = \int \frac{2}{x(x-2)}dx$

Method of partial fractions : $\frac{2}{x(x-2)} = \frac{\mathrm{A}}{x} + \frac{\mathrm{B}}{x-2}$, so $2 = (x-2)\mathrm{A} + x\mathrm{B}$

for $x = 0$, $2 = -2\mathrm{A}$, so $\mathrm{A} = -1$ and for $x = 2$, $2 = 2\mathrm{B}$, so $\mathrm{B} = 1$, then we get:

$\int \frac{2x+4}{x^3-4x}dx = \int \left(-\frac{1}{x} + \frac{1}{x-2}\right)dx = -\ln|x| + \ln|x+2| + c = \ln\left|1 + \frac{2}{x}\right| + c$

140. $\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx$

Decomposition into partial fractions : $\frac{y+1}{(y+4)(y+25)} = \frac{\mathrm{A}}{y+4} + \frac{\mathrm{B}}{y+25}$, then

$y + 1 = \mathrm{A}(y+25) + \mathrm{B}(y+4)$, for $y = -4$, $-3 = 21\mathrm{A}$, so $\mathrm{A} = -\frac{1}{7}$, and for $y = -25$, we have

$-24 = -21\mathrm{B}$, so $\mathrm{B} = \frac{8}{7}$, then : $\frac{y+1}{(y+4)(y+25)} = -\frac{1}{7}\frac{1}{y+4} + \frac{8}{7}\frac{1}{y+25}$, set $y = x^2$, we get

$\int \frac{x^2+1}{(x^2+4)(x^2+25)}dx = \int \left(-\frac{1}{7}\frac{1}{x^2+4} + \frac{8}{7}\frac{1}{x^2+25}\right)dx = -\frac{1}{7}\int \frac{1}{x^2+2^2}dx + \frac{8}{7}\int \frac{1}{x^2+5^2}dx$

$= -\frac{1}{14}\arctan\left(\frac{x}{2}\right) + \frac{8}{35}\arctan\left(\frac{x}{5}\right) + c$

141. $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx$, let $t = e^x$, then $dt = e^x dx$, so:

$\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}dx = \int \frac{1}{\sqrt{5-4t-t^2}}dt = \int \frac{1}{\sqrt{5-(t^2+4t)}}dt = \int \frac{1}{\sqrt{5-(t^2+4t+4-4)}}dt$

$= \int \frac{1}{\sqrt{9-(t+2)^2}}dt = \int \frac{1}{\sqrt{3^2-(t+2)^2}}dt = \arcsin\left(\frac{t+2}{3}\right) + c = \arcsin\left(\frac{e^x+2}{3}\right) + c$

142. $\int \frac{e^x}{\sqrt{e^{2x}+e^x+1}}dx$, let $t = e^x$, then $dt = e^x dx$, so:

$\int \frac{e^x}{\sqrt{e^{2x}+e^x+1}}dx = \int \frac{dt}{\sqrt{t^2+t+1}} = \int \frac{dt}{\sqrt{t^2+t+\frac{1}{2}+\frac{3}{4}}} = \int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}} = \sinh^{-1}\left[\frac{2}{\sqrt{3}}\left(t+\frac{1}{2}\right)\right] + c$

$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}t + \frac{1}{\sqrt{3}}\right) + c$

143. $\int \frac{x^9}{(4x^2+1)^6}dx = \int \frac{x^9}{\left[x^2\left(4+\frac{1}{x^2}\right)\right]^6}dx = \int \frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)^6}dx = \int \frac{1}{x^3\left(4+\frac{1}{x^2}\right)^6}dx$

$= -\frac{1}{2}\int \frac{-2}{x^3\left(4+\frac{1}{x^2}\right)^6}dx$, let $u = 4 + \frac{1}{x^2}$, then $du = -\frac{2}{x^3}dx$, so:

$\int \frac{x^9}{(4x^2+1)^6}dx = -\frac{1}{2}\int \frac{du}{u^6} = \frac{1}{10t^5} + c = \frac{1}{10\left(4+\frac{1}{x^2}\right)^5} + c$

144. $\int \frac{dx}{\sqrt{x\sqrt{x}-x^2}} = \int \frac{dx}{\sqrt{x^{\frac{3}{2}}-x^2}} = \int \frac{dx}{\sqrt{x^{\frac{3}{2}}(1-\sqrt{x})}} = \int \frac{x^{-\frac{3}{4}}}{\sqrt{1-\left(x^{\frac{1}{4}}\right)^2}}dx$, let $u = x^{\frac{1}{4}}$, $du = \frac{1}{4}x^{-\frac{3}{4}}dx$,

then: $\int \frac{dx}{\sqrt{x\sqrt{x}-x^2}} = \int \frac{4du}{\sqrt{1-u^2}} = 4\arcsin u + c = 4\arcsin\left(x^{\frac{1}{4}}\right) + c$

145. $\int \sqrt{x+\sqrt{x^2+1}}\,dx$

Let $\operatorname{arcsinh} x = \ln\left(x+\sqrt{x^2+1}\right)$, then ,$\frac{1}{2}\operatorname{arcsinh} x = \frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)$

so: $e^{\frac{1}{2}\operatorname{arcsinh} x} = e^{\frac{1}{2}\ln\left(x+\sqrt{x^2+1}\right)} = e^{\ln\left(x+\sqrt{x^2+1}\right)^{\frac{1}{2}}} = \left(x+\sqrt{x^2+1}\right)^{\frac{1}{2}} = \sqrt{x+\sqrt{x^2+1}}$, so:

$\int \sqrt{x+\sqrt{x^2+1}}\,dx = \int e^{\frac{1}{2}\operatorname{arcsinh} x}\,dx$, let $u = \operatorname{arcsinh} x$, $du = \frac{1}{\sqrt{1+x^2}}dx$, then:

$dx = \sqrt{1+x^2}du = \sqrt{1+\sinh^2 u}\,du = \cosh u\,du$ ($u = \operatorname{arcsinh} x$ so $x = \sinh u$), so:

$\int \sqrt{x+\sqrt{x^2+1}}\,dx = \int e^{\frac{1}{2}u}\cosh u\,du = \int e^{\frac{1}{2}u}\left(\frac{e^u+e^{-u}}{2}\right)du = \frac{1}{2}\int\left(e^{\frac{3}{2}u}+e^{-\frac{1}{2}u}\right)du$

$= \frac{1}{2}\left(\frac{2}{3}e^{\frac{3}{2}u} - 2e^{-\frac{1}{2}u}\right) + c = \frac{1}{3}e^{\frac{3}{2}u} - e^{-\frac{1}{2}u} + c = \frac{1}{3}\left(x+\sqrt{x^2+1}\right)^{\frac{3}{2}} - \left(x+\sqrt{x^2+1}\right)^{-\frac{1}{2}} + c$

146. $\int \sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}}\,dx = \int \sqrt{\frac{1}{x}+\frac{1}{\sqrt{x}}}\,dx = \int \sqrt{\frac{x+\sqrt{x}}{x\sqrt{x}}}\,dx = \int \sqrt{\frac{1+\sqrt{x}}{x}}\,dx = \int \frac{\sqrt{1+\sqrt{x}}}{\sqrt{x}}dx$

$= 2\int \frac{\sqrt{1+\sqrt{x}}}{2\sqrt{x}}dx$. Let $u = 1+\sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, so:

$\int \sqrt{\frac{1}{x}+\sqrt{\frac{1}{x}}} = 2\int \sqrt{u}du = 2\left(\frac{2}{3}u^{\frac{3}{2}}\right) + c = \frac{4}{3}u\sqrt{u} + c = \frac{4}{3}\left(1+\sqrt{x}\right)\sqrt{1+\sqrt{x}} + c$

147. $\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}} = \int \frac{dx}{(x+1)^2\sqrt{(x+1)^2+1}}$, let $x+1 = \tan\theta$, so $dx = \sec^2\theta\,d\theta$, then:

$\int \frac{dx}{(x+1)^2\sqrt{x^2+2x+2}} = \int \frac{\sec^2\theta d\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}} = \int \frac{\sec^2\theta d\theta}{\tan^2\theta\sqrt{\sec^2\theta}} = \int \frac{\sec^2\theta d\theta}{\tan^2\theta\sec\theta} = \int \frac{\sec\theta d\theta}{\tan^2\theta}$

$= \int \frac{\cos\theta}{\sin^2\theta}d\theta = \int \frac{(\sin\theta)}{\sin^2\theta}d\theta = -\frac{1}{\sin\theta} + c = -\frac{\sqrt{1+\tan^2\theta}}{\tan\theta} + c = -\frac{\sqrt{x^2+2x+2}}{x+1} + c$

148. $\int \frac{\sqrt[4]{x^4-x}}{x^5}dx = \int \frac{\sqrt[4]{x^4\left(1-\frac{1}{x^3}\right)}}{x^5}dx = \int \frac{x\sqrt[4]{1-\frac{1}{x^3}}}{x^5}dx = \int \frac{\sqrt[4]{1-\frac{1}{x^3}}}{x^4}dx$

Let $t^4 = 1-\frac{1}{x^3}$, then $4t^3dt = \frac{3}{x^4}dx$, so:

$\int \frac{\sqrt[4]{x^4-x}}{x^5}dx = \frac{4}{3}\int t.t^3\,dt = \frac{4}{3}\int t^4\,dt = \frac{4}{3}\left(\frac{t^5}{5}\right) + c = \frac{4}{15}t^5 + c = \frac{4}{15}\left(1-\frac{1}{x^3}\right)^{\frac{5}{4}} + c$

149. $\int \frac{dx}{x^n(1+x^n)^{\frac{1}{n}}} = \int \frac{dx}{x^n\left[x^n\left(1+\frac{1}{x^n}\right)\right]^{\frac{1}{n}}} = \int \frac{dx}{x^n \times x\left(1+\frac{1}{x^n}\right)^{\frac{1}{n}}} = \int \frac{dx}{x^{n+1}\left(1+\frac{1}{x^n}\right)^{\frac{1}{n}}}$

$= -\frac{1}{n}\int \frac{ndx}{x^{n+1}\left(1+\frac{1}{x^n}\right)^{\frac{1}{n}}}$. Let $t = 1 + \frac{1}{x^n}$, then $dt = -\frac{n}{x^{n+1}}dx$, so:

$\int \frac{dx}{x^n(1+x^n)^{\frac{1}{n}}} = -\frac{1}{n}\int \frac{dt}{t^{\frac{1}{n}}} = -\frac{1}{n}\int t^{-\frac{1}{n}}dt = -\frac{1}{n}\cdot\frac{t^{1-\frac{1}{n}}}{1-\frac{1}{n}} + c = \frac{1}{1-n}\left(1+\frac{1}{x^n}\right)^{1-\frac{1}{n}} + c$

150. $\int \left(\frac{\sec x+\tan x-1}{\tan x-\sec x+1}\right)dx = \int \frac{\sec x+\tan x-\left(\sec^2 x-\tan^2 x\right)}{\tan x-\sec x+1}dx$

$= \int \frac{\sec x+\tan x-(\sec x-\tan x)(\sec x+\tan x)}{\tan x-\sec x+1}dx = \int \frac{(\sec x+\tan x)(\tan x-\sec x+1)}{\tan x-\sec x+1}dx$

$= \int(\sec x + \tan x)dx = \int \sec x\, dx - \int \frac{(\cos x)'}{\cos x}dx = \ln|\sec x + \tan x| - \ln|\cos x| + c$

$= \ln|\sec x + \tan x| + \ln|\sec x| + c$

151. $\int \frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx = \int \frac{\sin x+\cos x}{\sqrt{1-(1-\sin 2x)}}dx = \int \frac{\sin x+\cos x}{\sqrt{1-\left(\sin^2 x+\cos^2 x-\sin 2x\right)}}dx$

$= \int \frac{\sin x+\cos x}{\sqrt{1-\left(\sin^2 x-2\sin x\cos x+\cos^2 x\right)}}dx = \int \frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}dx$

Let $t = \sin x - \cos x$, then $dt = (\sin x + \cos x)dx$, so:

$\int \frac{\sin x+\cos x}{\sqrt{\sin 2x}}dx = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + c = \arcsin(\sin x - \cos x) + c$

152. $\int \frac{1}{\sqrt{x}\left(\sqrt[4]{x}+1\right)^{10}}dx = \int \frac{1}{x^{\frac{1}{2}}\left(x^{\frac{1}{4}}+1\right)^{10}}dx$

Let $u = x^{\frac{1}{4}} + 1$, then $du = \frac{1}{4}x^{-\frac{3}{4}}dx$, then $dx = 4x^{\frac{3}{4}}du \Longrightarrow \frac{4x^{\frac{3}{4}}}{x^{\frac{1}{2}}}du = 4x^{\frac{1}{4}}du = 4(u-1)du$, so:

$\int \frac{1}{\sqrt{x}\left(\sqrt[4]{x}+1\right)^{10}} = \int \frac{4(u-1)}{u^{10}}du = 4\int\left(\frac{1}{u^9} - \frac{1}{u^{10}}\right)du = 4\int(u^{-9} - u^{-10})du$

$= -\frac{1}{2u^8} + \frac{4}{9u^9} + c = -\frac{1}{2\left(\sqrt[4]{x}+1\right)^8} + \frac{4}{9\left(\sqrt[4]{x}+1\right)^9} + c$

153. $\int \frac{dx}{1+\sin x+\cos x}$, let $t = \tan\frac{x}{2}$, so $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$, then:

$\int \frac{dx}{1+\sin x+\cos x} = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} = \int \frac{2}{1+t^2+2t+1-t^2}dt = \int \frac{1}{1+t}dt = \ln|1+t| + c$

$= \ln\left|1 + \tan\frac{x}{2}\right| + c$

154. $\int \frac{dx}{5+3\cos x}$, let $t=\tan\frac{x}{2}$, so $\cos x=\frac{1-t^2}{1+t^2}$ and $dx=\frac{2dt}{1+t^2}$, then:

$$\int \frac{dx}{5+3\cos x}=\int \frac{\frac{2dt}{1+t^2}}{5+3\left(\frac{1-t^2}{1+t^2}\right)}=\int \frac{\frac{2dt}{1+t^2}}{\frac{8+2t^2}{1+t^2}}=\int \frac{dt}{4+t^2}=\frac{1}{2}\arctan\left(\frac{t}{2}\right)+c=\frac{1}{2}\arctan\left(\frac{1}{2}\tan\frac{x}{2}\right)+c$$

155. $\int \frac{dx}{\sqrt{2}+\sin x}$, let $t=\tan\frac{x}{2}$, then $\sin x=\frac{2t}{1+t^2}$ and $dx=\frac{2}{1+t^2}dt$, so:

$$\int \frac{dx}{\sqrt{2}+\sin x}=\int \frac{\frac{2}{1+t^2}dt}{\sqrt{2}+\frac{2t}{1+t^2}}=\int \frac{\sqrt{2}\times\sqrt{2}dt}{\sqrt{2}\left(1+t^2\right)+2t}=\sqrt{2}\int \frac{dt}{t^2+\sqrt{2}t+1}=\sqrt{2}\int \frac{dt}{t^2+\sqrt{2}t+\frac{1}{2}+\frac{1}{2}}$$

$$=\sqrt{2}\int \frac{dt}{\left(t+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}=\sqrt{2}\times\sqrt{2}\arctan\left(\sqrt{2}\left(t+\frac{1}{\sqrt{2}}\right)\right)+c=2\arctan\left(1+\sqrt{2}t\right)+c$$

$$=2\arctan\left(1+\sqrt{2}\tan\frac{x}{2}\right)+c$$

156. $\int \frac{dx}{13\cosh x-5}$, let $t=\tanh\left(\frac{x}{2}\right)$, then $\cosh x=\frac{1+t^2}{1-t^2}$ and $dx=\frac{2}{1-t^2}dt$, so:

$$\int \frac{dx}{13\cosh x-5}=\int \frac{1}{13\left(\frac{1+t^2}{1-t^2}\right)-5}\cdot\frac{2}{1-t^2}dt=\int \frac{2}{13+13t^2-5+5t^2}dt=\int \frac{2}{8+18t^2}dt$$

$$=\int \frac{dt}{4+9t^2}=\int \frac{dt}{2^2+(3t)^2}=\frac{1}{6}\arctan\left(\frac{3}{2}t\right)+c=\frac{1}{6}\arctan\left(\frac{3}{2}\tanh\frac{x}{2}\right)+c$$

157. $\int \frac{3+2\cos x}{(2+3\cos x)^2}dx=\int \frac{(3+2\cos x)\csc^2 x}{(2+3\cos x)^2\csc^2 x}dx=\int \frac{3\csc^2 x+2\cot x\csc x}{(2\csc x+3\cot x)^2}dx$

Let $t=2\csc x+3\cot x$, then $dt=-(3\csc^2 x+2\cot x\csc x)dx$, so:

$$\int \frac{3+2\cos x}{(2+3\cos x)^2}dx=-\int \frac{dt}{t^2}=\frac{1}{t}+c=\frac{1}{2\csc x+3\cot x}+c$$

158. $\int \frac{1}{(1-x\cot x)^2}dx=\int \frac{1}{(1-x\cot x)^2}\times\frac{\tan^2 x}{\tan^2 x}dx=\int \frac{\tan^2 x}{\left[\left(1-\frac{x}{\tan x}\right)\tan x\right]^2}dx$

$=\int \frac{\tan^2 x}{(\tan x-x)^2}dx$. Let $u=\tan x-x$, then $du=(1+\tan^2 x-1)dx=\tan^2 x\,dx$, so:

$$\int \frac{1}{(1-x\cot x)^2}dx=\int \frac{du}{u^2}=-\frac{1}{u}+c=-\frac{1}{\tan x-x}+c=\frac{1}{x-\tan x}+c$$

159. $\int \frac{\sin x-1}{\cos^2 x\sqrt{\sec x+\tan x}}dx=\int \frac{\sin x-1}{\cos^2 x\sqrt{\sec x+\tan x}}\times\frac{\sqrt{\sec x-\tan x}}{\sqrt{\sec x-\tan x}}dx$

$$=\int \frac{\sec x\tan x-\sec^2 x}{\sqrt{\sec^2 x-\tan^2 x}}\sqrt{\sec x-\tan x}\,dx=\int(\sec x\tan x-\sec^2 x)\sqrt{\sec x-\tan x}\,dx$$

Let $u=\sec x-\tan x$, then $du=(\sec x\tan x-\sec^2 x)dx$, so:

$$\int \frac{\sin x-1}{\cos^2 x\sqrt{\sec x+\tan x}}dx=\int \sqrt{u}du=\frac{2}{3}u^{\frac{3}{2}}+c=\frac{2}{3}\sqrt{(\sec x-\tan x)^3}+c$$

160. $\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ln x\,dx$, let $t=\left(\frac{x}{e}\right)^x$, then $\ln t=x\ln x-x$, so $,\frac{1}{t}dt=\ln x\,dx$, then:

$$\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ln x\,dx=\int\left(t+\frac{1}{t}\right)\frac{1}{t}dt=\int\left(1+\frac{1}{t^2}\right)dt=t-\frac{1}{t}+c=\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+c$$

161. $\int \frac{1-\tanh x}{\sqrt{\tanh x}}dx = \int \frac{1-\frac{e^x-e^{-x}}{e^x+e^{-x}}}{\sqrt{\frac{e^x-e^{-x}}{e^x+e^{-x}}}}dx = \int \frac{\frac{e^x+e^{-x}-e^x+e^{-x}}{e^x+e^{-x}}}{\sqrt{\frac{(e^x-e^{-x})(e^x+e^{-x})}{(e^x+e^{-x})^2}}}dx = \int \frac{\frac{2e^{-x}}{e^x+e^{-x}}}{\frac{\sqrt{e^{2x}-e^{-2x}}}{e^x+e^{-x}}}dx$

$= \int \frac{2e^{-x}}{\sqrt{e^{2x}-e^{-2x}}}dx = \int \frac{2e^{-x}}{\sqrt{e^{2x}(1-e^{-4x})}}dx = \int \frac{2e^{-x}}{e^x\sqrt{1-e^{-4x}}}dx = \int \frac{2e^{-2x}}{\sqrt{1-e^{-4x}}}dx$

$= -\int \frac{-2e^{-2x}}{\sqrt{1-(e^{-2x})^2}}dx$, let $u = e^{-2x}$, then $du = -2e^{-2x}dx = -2udu$, so:

$\int \frac{1-\tanh x}{\sqrt{\tanh x}}dx = -\int \frac{du}{\sqrt{1-u^2}} = -\arcsin u + c = -\arcsin(e^{-2x}) + c$

162. $\int \frac{1}{\sqrt{\sin^3 x \cos x}}dx = \int \frac{1}{\sqrt{\sin^3 x \cos x \times \frac{\cos^3 x}{\cos^3 x}}}dx = \int \frac{1}{\sqrt{\cos^4 x \times \frac{\sin^3 x}{\cos^3 x}}}dx$

$= \int \frac{1}{\cos^2 x\sqrt{\tan^3 x}}dx = \int \frac{\sec^2 x}{\sqrt{\tan^3 x}}dx$, let $t = \tan x$, then $dt = \sec^2 x\, dx$, so:

$\int \frac{1}{\sqrt{\cos^4 x \times \frac{\sin^3 x}{\cos^3 x}}}dx = \int \frac{1}{t^{\frac{3}{2}}}dt = -\frac{2}{\sqrt{t}} + c = -\frac{2}{\sqrt{\tan x}} + c$

163. $\int \frac{\sinh^3 x}{\cosh x(2+\sinh^2 x)}dx = \int \frac{\sinh^2 x}{\cosh x(2+\sinh^2 x)}\sinh x\, dx$

Let $t = \cosh x$, then $dt = \sinh x\, dx$, with $\sinh^2 x = t^2 - 1$, then:

$\int \frac{\sinh^3 x}{\cosh x(2+\sinh^2 x)}dx = \int \frac{t^2-1}{t(t^2+1)}dt = \int \frac{2t^2-(t^2+1)}{t(t^2+1)}dt = \int \left(\frac{2t}{t^2+1} - \frac{1}{t}\right)dt$

$= \int \left[\frac{(t^2+1)'}{t^2+1} - \frac{1}{t}\right]dt = \ln|t^2+1| - \ln|t| + c = \ln\left|\frac{t^2+1}{t}\right| + c = \ln\left|t + \frac{1}{t}\right| + c$

$= \ln\left|\cosh x + \frac{1}{\cosh x}\right| + c$

164. $\int \frac{1}{x\sqrt{1-x^3}}dx = -\frac{1}{3}\int \frac{-3x^2}{x^3\sqrt{1-x^3}}dx$, let $u^2 = 1 - x^3$, then $2udu = -3x^2dx$, so:

$\int \frac{1}{x\sqrt{1-x^3}}dx = -\frac{1}{3}\int \frac{2u}{u(1-u^2)}du = \frac{1}{3}\int \frac{2}{u^2-1}du = \frac{1}{3}\int \frac{u+1-(u-1)}{(u+1)(u-1)}du$

$= \frac{1}{3}\int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \frac{1}{3}(\ln|u-1| - \ln|u+1|) + c = \frac{1}{3}\ln\left|\frac{u-1}{u+1}\right| + c = \frac{1}{3}\ln\left|\frac{\sqrt{1-x^3}-1}{\sqrt{1-x^3}+1}\right| + c$

165. $\int \frac{\cos^2 x}{\sin^6 x}dx = \int \frac{\cos^2 x}{\sin^2 x \sin^4 x}dx = \int \frac{\cot^2 x}{\sin^4 x}dx = \int \cot^2 x \csc^4 x\, dx$

$= \int \cot^2 x \csc^2 x \csc^2 x\, dx = \int \cot^2 x\,(1 + \cot^2 x)\csc^2 x\, dx = \int (\cot^2 x + \cot^4 x)\csc^2 x\, dx$

Let $u = \cot x$, then $du = -\csc^2 x\, dx$, so:

$\int \frac{\cos^2 x}{\sin^6 x}dx = -\int (t^2 + t^4)dt = -\frac{1}{3}t^3 - \frac{1}{5}t^5 + c = -\frac{1}{3}\csc^3 x - \frac{1}{5}\csc^5 x + c$

166. $\int \frac{\left[\ln\left(\tan\frac{x}{2}\right)\right]^3}{\sin x}dx$, let $u = \frac{x}{2}$, then $x = 2u$ and $dx = 2du$, so:

$$\int \frac{\left[\ln\left(\tan\frac{x}{2}\right)\right]^3}{\sin x}dx = \int \frac{2[\ln(\tan u)]^3}{\sin 2u}du = \int \frac{2[\ln(\tan u)]^3}{2\sin u\cos u}du = \int \frac{[\ln(\tan u)]^3}{\sin u\cos u}du$$

Let $t = \ln(\tan u)$, then $dt = \frac{\sec^2 u}{\tan u}du = \frac{\frac{1}{\cos^2 u}}{\frac{\sin u}{\cos u}}du = \frac{1}{\sin u\cos u}du$, so:

$$\int \frac{\left[\ln\left(\tan\frac{x}{2}\right)\right]^3}{\sin x}dx = \int t^3 dt = \frac{1}{4}t^4 + c = \frac{1}{4}\ln^4(\tan u) + c = \frac{1}{4}\ln^4\left(\tan\frac{x}{2}\right) + c$$

167. $\int \frac{\ln x}{(1+\ln x)^2}dx$, let $t = \ln x$, then $x = e^t$, so $dx = e^t dt$, so:

$$\int \frac{\ln x}{(1+\ln x)^2}dx = \int \frac{t}{(1+t)^2}e^t dt = \int \frac{t+1-1}{(1+t)^2}e^t dt = \int \left[\frac{1}{t+1} - \frac{1}{(1+t)^2}\right]e^t dt$$

$$= \int \left[\frac{1}{t+1} + \left(\frac{1}{t+1}\right)'\right]e^t dt = \frac{e^t}{t+1} + c = \frac{x}{\ln x+1} + c$$

168. $\int \sqrt{\csc x + 1}dx = \int \sqrt{\frac{1}{\sin x} + 1}dx = \int \sqrt{\frac{1+\sin x}{\sin x}}dx = \int \sqrt{\frac{1+\sin x}{\sin x}} \times \frac{2\cos x}{2\cos x}dx$

$$= \int \sqrt{\frac{1+\sin x}{\sin x}} \times \frac{2\cos x}{2\sqrt{1-\sin^2 x}}dx = 2\int \frac{\sqrt{1+\sin x}}{\sqrt{1-\sin^2 x}}\left(\frac{\cos x}{2\sqrt{\sin x}}\right)dx = 2\int \frac{1}{\sqrt{1-\sin x}}\left(\frac{\cos x}{2\sqrt{\sin x}}\right)dx$$

$= 2\int \frac{1}{\sqrt{1-\left(\sqrt{\sin x}\right)^2}}\left(\frac{\cos x}{2\sqrt{\sin x}}\right)dx$, let $u = \sqrt{\sin x}$, then $du = \left(\frac{\cos x}{2\sqrt{\sin x}}\right)dx$, so:

$$\int \sqrt{\csc x + 1}dx = 2\int \frac{du}{\sqrt{1-u^2}} = 2\arcsin u + c = 2\arcsin\left(\sqrt{\sin x}\right) + c$$

169. $\int \frac{dx}{3\sin^2 x+4} = \int \frac{\frac{1}{\cos^2 x}}{\frac{3\sin^2 x+4}{\cos^2 x}}dx = \int \frac{\frac{1}{\cos^2 x}}{3\left(\frac{\sin x}{\cos x}\right)^2+\frac{4}{\cos^2 x}}dx = \int \frac{\sec^2 x}{3\tan^2 x+4\sec^2 x}dx$

$$= \int \frac{\sec^2 x}{3\tan^2 x+4\left(\tan^2 x+1\right)}dx = \int \frac{\sec^2 x}{7\tan^2 x+4}dx = \frac{1}{7}\int \frac{\sec^2 x}{\tan^2 x+\left(\frac{2}{\sqrt{7}}\right)^2}dx$$

Let $t = \tan x$, then $dt = \sec^2 x\, dx$, then:

$$\int \frac{dx}{3\sin^2 x+4} = \frac{1}{7}\int \frac{dt}{t^2+\left(\frac{2}{\sqrt{7}}\right)^2} = \frac{1}{2\sqrt{7}}\arctan\left(\frac{\sqrt{7}}{2}t\right) + c = \frac{1}{2\sqrt{7}}\arctan\left(\frac{\sqrt{7}}{2}\tan x\right) + c$$

170. $\int \frac{1}{\sqrt{(1-x^2)(1+\arcsin x)}}dx = \int \frac{1}{\sqrt{1-\arcsin x}} \cdot \frac{1}{\sqrt{1-x^2}}dx$

Let $u = 1 + \arcsin x$, $du = \frac{1}{\sqrt{1-x^2}}dx$ so:

$$\int \frac{1}{\sqrt{(1-x^2)(1+\arcsin x)}}dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} + c = 2\sqrt{1 + \arcsin x} + c$$

171. $\int \frac{\left(x+\sqrt{1+x^2}\right)^{20}}{\sqrt{1+x^2}}dx$, let $t = x+\sqrt{1+x^2}$, then $t = \frac{\left(x+\sqrt{1+x^2}\right)\left(x-\sqrt{1+x^2}\right)}{x-\sqrt{1-x^2}}$, so:

$t = \frac{x^2-1-x^2}{x-\sqrt{1-x^2}} = \frac{-1}{x-\sqrt{1-x^2}}$, so $-\frac{1}{t} = x-\sqrt{1+x^2}$ and so

$t-\frac{1}{t} = x+\sqrt{1+x^2}+x-\sqrt{1+x^2} = 2x$, so $dx = \frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$, then:

$$\int \frac{\left(x+\sqrt{1+x^2}\right)^{20}}{\sqrt{1+x^2}}dx = \int \frac{\frac{1}{2}\left(1+\frac{1}{t^2}\right).t^{20}}{\frac{1}{2}\left(t+\frac{1}{t}\right)}dt = \int \frac{\frac{1}{2}\left(t+\frac{t}{t^2}\right).t^{19}}{\frac{1}{2}\left(t+\frac{1}{t}\right)}dt = \int \frac{\frac{1}{2}\left(t+\frac{1}{t}\right).t^{19}}{\frac{1}{2}\left(t+\frac{1}{t}\right)}dt = \int t^{19}dt$$

$$= \frac{1}{20}t^{20}+c = \frac{1}{20}\left(x+\sqrt{1+x^2}\right)^{20}+c$$

172. $\int \frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = \int \frac{dx}{\cos^3 x\sqrt{4\cos x\sin x}} = \int \frac{dx}{2\cos^3 x\sqrt{\cos x\sin x\times\frac{\cos x}{\cos x}}} = \int \frac{dx}{2\cos^4 x\sqrt{\tan x}}$

$$= \int \frac{\sec^4 x}{2\sqrt{\tan x}}dx = \int \frac{(\sec^2 x)(\sec^2 x)}{2\sqrt{\tan x}}dx = \int \frac{(1+\tan^2 x)(\sec^2 x)}{2\sqrt{\tan x}}dx$$

Let $t^2 = \tan x$, so $2tdt = \sec^2 x\,dx$, then:

$$\int \frac{dx}{\cos^3 x\sqrt{2\sin 2x}} = \int \frac{(1+t^4)2tdt}{2t} = \int(1+t^4)dt = t+\frac{1}{5}t^5+c = \sqrt{\tan x}+\frac{1}{5}\tan^{\frac{5}{2}} x+c$$

173. $\int \frac{3x^2-3x+8}{x^3-3x^2+4x-12}dx = \int \frac{3x^2-3x+8}{x^2(x-3)+4(x-3)}dx = \int \frac{3x^2-3x+8}{(x-3)(x^2+4)}dx$

Partial fraction method : $\frac{3x^2-3x+8}{(x-3)(x^2+4)} = \frac{\mathrm{A}}{x-3}+\frac{\mathrm{B}x+\mathrm{C}}{x^2+4}$, $\mathrm{A} = \left[\frac{3x^2-3x+8}{x^2+4}\right]_{x=3} = 2$

$3x^2-3x+8 = 2(x^2+4)+(\mathrm{B}x+\mathrm{C})(x-3) = (2+\mathrm{B})x^2+(-3\mathrm{B}+\mathrm{C})x+8-3c$

$2+\mathrm{B} = 3$, so $\mathrm{B} = 1$ and $8-3\mathrm{C} = 8$, so $\mathrm{C} = 0$, so:

$$\int \frac{3x^2-3x+8}{x^3-3x^2+4x-12}dx = \int\left(\frac{2}{x-3}+\frac{x}{x^2+4}\right)dx = 2\int\frac{1}{x-3}dx+\frac{1}{2}\int\frac{2\mathrm{x}}{x^2+4}dx$$

$$= 2\ln|x-3|+\frac{1}{2}\ln(x^2+4)$$

174. $\int \frac{dx}{3+5\sin x+3\cos x}$, let $t = \tan\frac{x}{2}$, so $\cos x = \frac{1-t^2}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$, then:

$$\int \frac{dx}{3+5\sin x+3\cos x} = \int \frac{\frac{2dt}{1+t^2}}{3+5\left(\frac{2t}{1+t^2}\right)+3\left(\frac{1-t^2}{1+t^2}\right)} = \int \frac{2}{3(1+t^2)+10t+3(1-t^2)}dt$$

$$= \int \frac{2}{10t+6}dt = \int \frac{1}{5t+3}dt = \frac{1}{5}\int\frac{5}{5t+3}dt = \frac{1}{5}\int\frac{(5t+3)'}{5t+3}dt = \frac{1}{5}\ln|5t+3|+c$$

$$= \frac{1}{5}\ln\left|5\tan\left(\frac{x}{2}\right)+3\right|+c$$

175. $\int \frac{\cosh x}{\cosh^2 x-2}dx = \int \frac{\cosh x}{(\cosh^2 x-1)-1}dx = \int \frac{\cosh x}{\sinh^2 x-1}dx$

Let $u = \sinh x$, then $du = \cosh x\,dx$, so:

$\int \frac{\cosh x}{\cosh^2 x-2} dx = \int \frac{1}{u^2-1} du = \frac{1}{2}\int \frac{(u+1)-(u-1)}{(u+1)(u-1)} dx = \frac{1}{2}\int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)$

$= \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + c = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c = \frac{1}{2}\ln\left|\frac{\sinh x-1}{\sinh x+1}\right| + c$

176. $\int \frac{\cos^5 7x}{\sin^2 7x} dx = \int \cos^3 7x \frac{\cos^2 7x}{\sin^2 7x} dx = \int \cos^3 7x \cot^2 7x\, dx =$, let $u = 7x$, then $du = 7dx$, so: $\int \frac{\cos^5 7x}{\sin^2 7x} dx = \frac{1}{7}\int (1-\sin^2 u)(\csc^2 u - 1)\cos u$, let $v = \sin u$, then $dv = \cos u\, du$, so:

$\int \frac{\cos^5 7x}{\sin^2 7x} dx = \frac{1}{7}\int (1-v^2)\left(\frac{1}{v^2} - 1\right) dv = \frac{1}{7}\int \left(\frac{1}{v^2} + v^2 - 2\right) dv = \frac{1}{7}\left(-\frac{1}{v} + \frac{1}{3}v^3 - 2v\right) + c$

$= \frac{1}{7}\left(-\frac{1}{\sin u} + \frac{1}{3}\sin^3 u - 2\sin u\right) + c = \frac{1}{7}\left(-\csc 7x + \frac{1}{3}\sin^3 7x - 2\sin 7x\right) + c$

177. $\int x \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} dx = \frac{1}{2}\int 2x \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} dx$, let $u = x^2$, then $du = 2xdx$, so:

$\int x \frac{\sqrt{1+x^2}}{\sqrt{1-x^2}} dx = \frac{1}{2}\int \frac{\sqrt{1+u}}{\sqrt{1-u}} du = \frac{1}{2}\int \frac{\sqrt{1+u}}{\sqrt{1-u}} \times \frac{\sqrt{1+u}}{\sqrt{1+u}} du = \frac{1}{2}\int \frac{1+u}{\sqrt{1-u^2}} du$

$= \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}} du - \frac{1}{2}\int \frac{-2u}{2\sqrt{1-u^2}} du = \frac{1}{2}\int \frac{1}{\sqrt{1-u^2}} du - \frac{1}{2}\int \frac{(1-u^2)'}{2\sqrt{1-u^2}} du$

$= \frac{1}{2}\arcsin u - \frac{1}{2}\left(\frac{1}{2}\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right) + c = \frac{1}{2}\arcsin u - \frac{1}{2}\sqrt{u} + c = \frac{1}{2}\arcsin x^2 - \frac{1}{2}\sqrt{1-x^2} + c$

178. $\int \frac{\sin 2x}{(3+4\cos x)^2} dx = \int \frac{2\sin x\cos x}{(3+4\cos x)^2} dx = -\frac{1}{2}\int \frac{-4\sin x\cos x}{(3+4\cos x)^2} dx$

Let $t = 3+4\cos x$ so $\cos x = \frac{t-3}{4}$ and $dt = -4\sin x\, dx$, then:

$\int \frac{\sin 2x}{(3+4\cos x)^2} dx = -\frac{1}{2}\int \frac{\frac{t-3}{4}}{t^2} dt = -\frac{1}{8}\int \frac{t-3}{t^2} dt = -\frac{1}{8}\int \left(\frac{1}{t} - \frac{3}{t^2}\right) dt = -\frac{1}{8}\left(\ln|t| + \frac{3}{t}\right) + c$

$= -\frac{1}{8}\left(\ln|3+4\cos x| + \frac{3}{3+4\cos x}\right) + c$

179. $\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}} dx$, let $x = a\sin^{\frac{2}{3}}\theta$, then $dx = \frac{2}{3}a\sin^{-\frac{1}{3}}\theta\cos\theta\, d\theta$, so:

$\int \frac{\sqrt{a\sin^{\frac{2}{3}}\theta}}{\sqrt{a^3-a^3\sin^2\theta}} \frac{2}{3}a\sin^{-\frac{1}{3}}\theta\cos\theta\, d\theta = \frac{2}{3}\int \frac{a^{\frac{1}{2}}\sin^{\frac{1}{3}}\theta}{a^{\frac{3}{2}}\sqrt{\cos^2\theta}} a\sin^{-\frac{1}{3}}\theta\cos\theta\, d\theta$

$= \frac{2}{3}\int \frac{1}{\cos\theta}\cos\theta\, d\theta = \frac{2}{3}\int d\theta = \frac{2}{3}\theta + c$, but $x = a\sin^{\frac{2}{3}}\theta$, so $\sin^{\frac{2}{3}}\theta = \frac{x}{a}$, $\sin\theta = \left(\frac{x}{a}\right)^{\frac{3}{2}}$, so $\sin\theta = \frac{x}{a}\sqrt{\frac{x}{a}}$, so $\theta = \arcsin\left(\frac{x}{a}\sqrt{\frac{x}{a}}\right)$, therefore:

$\int \frac{\sqrt{x}}{\sqrt{a^3-x^3}} dx = \frac{2}{3}\arcsin\left(\frac{x}{a}\sqrt{\frac{x}{a}}\right) + c$

180. $\int \frac{1}{\sqrt{x^2-x}}dx = 2\int \frac{1}{2\sqrt{x}\sqrt{x-1}}dx$, let $u = \sqrt{x}$, thdxen $du = \frac{1}{2\sqrt{x}}dx$, so:

$\int \frac{1}{\sqrt{x^2-x}}dx = 2\int \frac{1}{\sqrt{u^2-1}}dx$, let $u = \sec\theta$, then $du = \tan\theta\sec\theta\, d\theta$, so:

$2\int \frac{1}{\sqrt{u^2-1}}dx = 2\int \frac{\tan\theta\sec\theta}{\sqrt{\sec^2\theta-1}}d\theta = 2\int \frac{\tan\theta\sec\theta}{\tan\theta}d\theta = 2\int \sec\theta\, d\theta = 2\ln|\sec\theta+\tan\theta| + c$

$= 2\ln\left|\sec\theta + \sqrt{\sec^2\theta-1}\right| + c = 2\ln\left|u+\sqrt{u^2-1}\right| + c = 2\ln\left|\sqrt{x}+\sqrt{x-1}\right| + c$

181. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx = \int \frac{x^2-1}{x^3\sqrt{x^4\left(2-\frac{2}{x^2}+\frac{1}{x^4}\right)}}dx = \int \frac{x^2-1}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx = \int \frac{\frac{1}{x^3}-\frac{1}{x^5}}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$

$= \frac{1}{4}\int \frac{\frac{4}{x^3}-\frac{4}{x^5}}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}dx$, let $t = 2-\frac{2}{x^2}+\frac{1}{x^4}$, then $dt = \left(\frac{4}{x^3}-\frac{4}{x^5}\right)dx$, so:

$\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{t}} = \frac{1}{4}\times 2\sqrt{t} + c = \frac{\sqrt{2x^4-2x^2+1}}{2x^2} + c$

182. $\int \frac{1}{a^x-1}dx = \int \frac{\frac{1}{a^x}}{\frac{a^x-1}{a^x}}dx = \int \frac{a^{-x}}{1-a^{-x}}dx = \frac{1}{\ln a}\int \frac{a^{-x}\ln a}{1-a^{-x}}dx$

Let $u = 1-a^{-x}$, then $du = a^{-x}\ln a\, dx$, so:

$\int \frac{1}{a^x-1}dx = \frac{1}{\ln a}\int \frac{du}{u} = \frac{1}{\ln a}\ln|u| + c = \frac{1}{\ln a}\ln|1-a^{-x}| + c$

183. $\int \frac{2^x}{\sqrt{4^x+9}}dx = \int \frac{2^x}{\sqrt{(2^x)^2+9}}dx = \frac{1}{\ln 2}\int \frac{2^x\ln 2}{\sqrt{(2^x)^2+9}}dx$, let $u = 2^x$, then $du = 2^x\ln 2\, dx$,

so: $\int \frac{2^x}{\sqrt{4^x+9}}dx = \frac{1}{\ln 2}\int \frac{du}{\sqrt{u^2+9}}$, let $u = 3\tan\theta$, then $du = 3\sec^2\theta\, d\theta$, then:

$\int \frac{2^x}{\sqrt{4^x+9}}dx = \frac{1}{\ln 2}\int \frac{3\sec^2\theta}{\sqrt{9\sec^2\theta}}d\theta = \frac{1}{\ln 2}\int \sec\theta\, d\theta = \frac{1}{\ln 2}\ln|\sec\theta+\tan\theta| + c$

$= \frac{1}{\ln 2}\ln\left(\frac{4+\sqrt{u^2+9}}{3}\right) + c = \frac{1}{\ln 2}\ln\left(\frac{2^x+\sqrt{4^x+9}}{3}\right) + c = \frac{1}{\ln 2}\ln\left(2^x+\sqrt{4^x+9}\right) + k$

184. $\int \frac{1}{(25+x^2)^{\frac{3}{2}}}dx$, let $x = 5\tan\theta$, then $dx = 5\sec^2\theta\, d\theta$, so:

$\int \frac{1}{(25+x^2)^{\frac{3}{2}}}dx = \int \frac{1}{(25+25\tan^2\theta)^{\frac{3}{2}}}.5\sec^2\theta\, d\theta = \int \frac{1}{\left(\sqrt{25(1+\tan^2\theta)}\right)^3}.5\sec^2\theta\, d\theta$

$= \int \frac{1}{\left(\sqrt{25\sec^2\theta}\right)^3}.5\sec^2\theta\, d\theta = \int \frac{5\sec^2\theta}{125\sec^3\theta}d\theta = \frac{1}{25}\int \frac{1}{\sec\theta}d\theta = \frac{1}{25}\int \cos\theta\, d\theta = \frac{1}{25}\sin\theta + c$

But $\tan\theta = \frac{x}{5}$, then $\sin\theta = \sqrt{\frac{\left(\frac{x}{5}\right)^2}{1+\left(\frac{x}{5}\right)^2}} = \sqrt{\frac{\frac{x^2}{25}}{1+\frac{x^2}{25}}} = \sqrt{\frac{x^2}{25+x^2}} = \frac{x}{\sqrt{25+x^2}}$, therefore:

$$\int \frac{1}{(25+x^2)^{\frac{3}{2}}}dx = \frac{x}{25\sqrt{25+x^2}} + c$$

185. $$\int \frac{1}{\left(a^2-b^2x^2\right)^{\frac{3}{2}}}dx = \int \frac{1}{\left[x^2\left(\frac{a^2}{x^2}-b^2\right)\right]^{\frac{3}{2}}}dx = \int \frac{1}{x^3\left(\frac{a^2}{x^2}-b^2\right)^{\frac{3}{2}}}dx = -\frac{1}{2a^2}\int \frac{-2a^2}{x^3\left(\frac{a^2}{x^2}-b^2\right)^{\frac{3}{2}}}dx$$

Let $t = \frac{a^2}{x^2} - b^2$, then $dt = -\frac{-2a^2}{x^3}dx$, so:

$$\int \frac{1}{\left(a^2-b^2x^2\right)^{\frac{3}{2}}}dx = -\frac{1}{2a^2}\int \frac{1}{t^{\frac{3}{2}}}dt = \frac{1}{a^2\sqrt{t}} + c = \frac{1}{a^2\sqrt{\left(\frac{a^2}{x^2}-b^2\right)}} + c$$

186. $\int \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}dx = \int \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}} \times \sqrt{\frac{1-\sqrt{x}}{1-\sqrt{x}}}dx = \int \frac{\sqrt{1-x}}{1-\sqrt{x}}dx$, let $x = \sin^2\theta$, $dx = 2\sin\theta\cos\theta\, d\theta$, so:

$$\int \sqrt{\frac{1+\sqrt{x}}{1-\sqrt{x}}}dx = \int \frac{\sqrt{1-\sin^2\theta}}{1-\sqrt{\sin^2\theta}} 2\sin\theta\cos\theta\, d\theta = \int \frac{\cos\theta}{1-\sin\theta} 2\sin\theta\cos\theta\, d\theta$$

$$= \int \frac{2\sin\theta\cos^2\theta}{1-\sin\theta}d\theta = \int \frac{2\sin\theta\cos^2\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta}d\theta = \int \frac{\left(2\sin\theta\cos^2\theta\right)1+\sin\theta}{1-\sin^2\theta}d\theta$$

$$= \int \frac{(2\sin\theta)\left(1-\sin^2\theta\right)(1+\sin\theta)}{1-\sin^2\theta}d\theta = 2\int \sin\theta\,(1+\sin\theta)d\theta = 2\int(\sin\theta + \sin^2\theta)d\theta$$

$$= 2\int \sin\theta + 2\int\left(\frac{1-\sin 2\theta}{2}\right)d\theta = 2\int \sin\theta\, d\theta + \int(1-\cos 2\theta)d\theta$$

$$= -2\cos\theta + \theta - \frac{1}{2}\sin 2\theta + c = -2\cos\theta + \theta - \sin\theta\cos\theta + c$$

$$= -2\sqrt{1-x} + \arcsin\sqrt{x} - \sqrt{x(1-x)} + c$$

187. $\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx$, let $x = \cos^2\theta$, then $dx = -2\cos\theta\sin\theta\, d\theta$, so:

$$\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx = \int \frac{-2}{\cos^2\theta}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\cos\theta\sin\theta\, d\theta = -2\int \frac{\sin\theta}{\cos\theta}\sqrt{\frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}}}d\theta$$

$$= -2\int \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos\theta}.\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}d\theta = -2\int \frac{2\sin^2\frac{\theta}{2}}{\cos\theta}d\theta = -2\int \frac{1-\cos\theta}{\cos\theta}d\theta = -2\int\left(\frac{1}{\cos\theta} - 1\right)d\theta$$

$$= -2\int(\sec\theta - 1)d\theta = -2[\ln|\sec\theta + \tan\theta| - \theta] + c$$

But $x = \cos^2\theta$, then $\theta = \arccos\sqrt{x}$, so:

$$\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}dx = -2\left[\ln\left|\sec\left(\arccos\sqrt{x}\right) + \tan\left(\arccos\sqrt{x}\right)\right| - \arccos\sqrt{x}\right] + c$$

188. $\int \sqrt{1+\sec x}\,dx = \int \sqrt{1+\frac{1}{\cos x}}\,dx = \int \sqrt{\frac{\cos x+1}{\cos x}}\,dx = \int \frac{\sqrt{\cos x+1}}{\sqrt{\cos x}}\,dx$

$= \int \frac{\sqrt{2\cos^2\frac{x}{2}}}{\sqrt{1-2\sin^2\frac{x}{2}}}\,dx = \int \frac{\sqrt{2}\cos\frac{x}{2}}{\sqrt{1-2\sin^2\frac{x}{2}}}\,dx$, let $u=\sin\frac{x}{2}$, then $du=\frac{1}{2}\cos\frac{x}{2}$, so:

$\int \sqrt{1+\sec x}\,dx = 2\sqrt{2}\int \frac{du}{\sqrt{1-2u^2}} = 2\sqrt{2}\int \frac{du}{\sqrt{1-\left(\sqrt{2}u\right)^2}} = 2\arcsin\left(\sqrt{2}u\right)+x$

$= 2\arcsin\left(\sqrt{2}\sin\frac{x}{2}\right)+c$

189. $\int \frac{\cos 4x-1}{\cot x-\tan x}\,dx = \int \frac{1-2\sin^2 2x-1}{\frac{\cos x}{\sin x}-\frac{\sin x}{\cos x}}\,dx = \int \frac{-2\sin^2 2x}{\frac{\cos^2 x-\sin^2 x}{\sin x\cos x}}\,dx = \int \frac{-2\sin^2 2x\sin x\cos x}{\cos 2x}\,dx$

$= -\int \frac{\left(1-\cos^2 2x\right)\sin 2x}{\cos 2x}\,dx$, let $t=\cos 2x$, then $dt=-2\sin 2x\,dx$, so:

$\int \frac{\cos 4x-1}{\cot x-\tan x}\,dx = \frac{1}{2}\int \frac{1-t^2}{t}\,dx = \frac{1}{2}\int\left(\frac{1}{t}-t\right)dx = \frac{1}{2}\left[\ln|t|-\frac{1}{2}t^2\right]+c$

$= \frac{1}{2}\left[\ln|\cos 2x|-\frac{1}{2}\cos^2 2x\right]+c$

190. $\int \frac{1}{x\left[1+\sin^2(\ln x)\right]}\,dx$, let $u=\ln x$, then $du=\frac{1}{x}dx$, so:

$\int \frac{1}{x\left[1+\sin^2(\ln x)\right]}\,dx = \int \frac{1}{1+\sin^2 u}\,du = \int \frac{\frac{1}{\cos^2 u}}{\frac{1+\sin^2 u}{\cos^2 u}}\,du = \int \frac{\sec^2 u}{\sec^2 u+\tan^2 u}\,du$

$= \int \frac{\sec^2 u}{1+\tan^2 u+\tan^2 u}\,du = \int \frac{\sec^2 u}{1+2\tan^2 u}\,du$, let $w=\tan u$, so $dw=\sec^2 u\,du$, so:

$\int \frac{1}{x\left[1+\sin^2(\ln x)\right]}\,dx = \int \frac{1}{1+2w^2}\,dw = \int \frac{1}{1+\left(\sqrt{2}w\right)^2}\,dw = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}w\right)+c$

$= \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan u\right)+c = \frac{1}{\sqrt{2}}\arctan\left[\sqrt{2}\tan(\ln x)\right]+c$

191. $\int \frac{\sqrt{x\ln x}(1-\ln x)}{x^3+\ln^3 x}\,dx = \int \frac{\frac{\sqrt{x\ln x}(1-\ln x)}{x^3}}{\frac{x^3+\ln^3 x}{x^3}}\,dx = \int \frac{\sqrt{\frac{\ln x}{x}}}{1+\left(\frac{\ln x}{x}\right)^3}\cdot\frac{1-\ln x}{x^2}\,dx$

$= \int \frac{\sqrt{\frac{\ln x}{x}}}{1+\left(\sqrt{\frac{\ln x}{x}}\right)^6}\cdot\frac{1-\ln x}{x^2}\,dx$, let $y=\sqrt{\frac{\ln x}{x}}$, then $y^2=\frac{\ln x}{x}$ and $2ydy=\frac{1-\ln x}{x^2}$, so:

$\int \frac{\sqrt{x\ln x}(1-\ln x)}{x^3+\ln^3 x}\,dx = \int \frac{y}{1+y^6}.2ydy = 2\int \frac{y^2}{1+y^6}\,dy = \frac{2}{3}\int \frac{3y^2}{1+y^6}\,dy = \frac{2}{3}\int \frac{\left(y^3\right)'}{1+\left(y^3\right)^2}\,dy$

$= \frac{2}{3}\arctan\left(y^3\right)+c = \frac{2}{3}\arctan\left(\sqrt{\frac{\ln^3 x}{x^3}}\right)+c$

192. $\int \frac{1}{\sqrt{x^8-x^2}}dx = \int \frac{1}{x\sqrt{x^6-1}}dx = \frac{1}{3}\int \frac{3x^2}{x^3\sqrt{(x^3)^2-1}}dx$, let $u = x^3$, then $du = 3x^2dx$, so:

$$\int \frac{1}{\sqrt{x^8-x^2}}dx = \frac{1}{3}\int \frac{du}{u\sqrt{u^2-1}} = \frac{1}{3}\operatorname{arcsec} u + c = \frac{1}{3}\operatorname{arcsec}(x^3) + c$$

193. $\int \frac{\sqrt{x-\sqrt{a}}}{x}dx$, let $u = \sqrt{x-\sqrt{a}}$, $u^2 = x-\sqrt{a}$, then $2udu = dx$, so:

$$\int \frac{\sqrt{x-\sqrt{a}}}{x}dx = \int \frac{2u^2du}{u^2+\sqrt{a}} = 2\int \frac{u^2du}{u^2+\sqrt{a}}x = 2\int \frac{u^2+\sqrt{a}-\sqrt{a}}{u^2+\sqrt{a}}du = 2\int\left(1-\frac{\sqrt{a}}{u^2+\sqrt{a}}\right)du$$

$$= 2\int du - 2\sqrt{a}\int \frac{1}{u^2+\sqrt{a}}du = 2\int du - 2\sqrt{a}\int \frac{1}{u^2+\left(\sqrt{\sqrt{a}}\right)^2}du = 2u - \frac{2\sqrt{a}}{\sqrt{\sqrt{a}}}\arctan\left(\frac{u}{\sqrt{\sqrt{a}}}\right) + c$$

$$= 2\sqrt{x-\sqrt{a}} - 2\sqrt{\sqrt{a}}\arctan\left(\frac{x-\sqrt{a}}{\sqrt{\sqrt{a}}}\right) + c$$

194. $\int \frac{\sqrt{a+x^2}}{x^6}dx = \int \frac{\sqrt{x^2\left(1+\frac{a}{x^2}\right)}}{x^6}dx = \int \frac{\sqrt{1+\frac{a}{x^2}}}{x^5}dx = \int \frac{1}{x^2\times x^3}\sqrt{1+\frac{a}{x^2}}dx$

Let $t = 1+\frac{a}{x^2}$, then $dt = \frac{-2a}{x^3}dx$, so:

$$\int \frac{\sqrt{a+x^2}}{x^6}dx = -\frac{1}{2a^2}\int (t-1)\sqrt{t}dt = -\frac{1}{2a^2}\int\left(t^{\frac{3}{2}} - t^{\frac{1}{2}}\right)dt = -\frac{1}{2a^2}\left(\frac{2}{5}t^{\frac{5}{2}} - \frac{2}{3}t^{\frac{3}{2}}\right) + c$$

$$= \frac{1}{3a^2}t^{\frac{3}{2}} - \frac{1}{5a^2}t^{\frac{5}{2}} + c = \frac{1}{3a^2}\left(1+\frac{a}{x^2}\right)^{\frac{3}{2}} - \frac{1}{5a^2}\left(1+\frac{a}{x^2}\right)^{\frac{5}{2}} + c$$

195. $\int \frac{t^x}{\sqrt{1-t^{2x}}}dx = \frac{1}{\ln t}\int \frac{t^x\ln t}{\sqrt{1-t^{2x}}}dx$, let $u = t^x$, then $du = t^x\ln t\, dx$, so:

$$\int \frac{t^x}{\sqrt{1-t^{2x}}}dx = \frac{1}{\ln t}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{\ln t}\arcsin u + c = \frac{1}{\ln t}\arcsin(t^x) + c$$

196. $\int \frac{1}{\sqrt{x^7-x^2}}dx = \int \frac{1}{x\sqrt{x^5-1}}dx = \int \frac{x^4}{x^5\sqrt{x^5-1}}dx$, let $u = x^5-1$, then $du = 5x^4dx$, so:

$$\int \frac{1}{\sqrt{x^7-x^2}}dx = \frac{1}{5}\int \frac{du}{(u+1)\sqrt{u}} = \frac{1}{5}\int \frac{u^{-\frac{1}{2}}}{1+\left(u^{\frac{1}{2}}\right)^2}du = \frac{2}{5}\int \frac{\left(\frac{1}{2}u^{-\frac{1}{2}}\right)}{1+\left(u^{\frac{1}{2}}\right)^2}du = \frac{2}{5}\int \frac{\left(u^{\frac{1}{2}}\right)'}{1+\left(u^{\frac{1}{2}}\right)^2}du$$

$$= \frac{2}{5}\arctan\left(u^{\frac{1}{2}}\right) + c = \frac{2}{5}\arctan\left(\sqrt{x^5-1}\right) + c$$

197. $\int \frac{dx}{(1+\sqrt{x})^8}$, let $t = \sqrt{x}$, then $t^2 = x$, so $2tdt = dx$, then:

$$\int \frac{dx}{(1+\sqrt{x})^8} = \int \frac{2t}{(1+t)^8}dt = 2\int \frac{1+t-1}{(1+t)^8}dt = 2\int\left[\frac{1+t}{(1+t)^8} - \frac{1}{(1+t)^8}\right]dt$$

$$= 2\int\left[\frac{1}{(1+t)^7} - \frac{1}{(1+t)^8}\right]dt = 2\left[-\frac{1}{6(1+t)^6} + \frac{1}{7(1+t)^7}\right] + c = 2\left[-\frac{1}{6(1+\sqrt{x})^6} + \frac{1}{7(1+\sqrt{x})^7}\right] + c$$

198. $\int \sqrt{x^2-3}\,dx = \int \sqrt{x^2-\left(\sqrt{3}\right)^2}\,dx$, let $x = \sqrt{3}\cosh\theta$, then $dx = \sinh\theta\, d\theta$, so:

$\int \sqrt{x^2-3}\,dx = \int \sqrt{3\cosh^2\theta - 3}\sinh\theta\, d\theta = \sqrt{3}\int \sqrt{\cosh^2\theta - 1}d\theta \sinh\theta\, d\theta$

$= \sqrt{3}\int \sqrt{\sinh^2\theta}\sinh\theta\, d\theta = \sqrt{3}\int \sinh^2\theta\, d\theta = \sqrt{3}\int \frac{\cosh 2\theta - 1}{2}d\theta = \frac{\sqrt{3}}{2}\int(\cosh 2\theta - 1)d\theta$

$= \frac{\sqrt{3}}{2}\left(\frac{1}{2}\sinh 2\theta - \theta\right) + c = \frac{\sqrt{3}}{2}\sinh\theta\cosh\theta - \frac{\sqrt{3}}{2}\theta + c$

$\sinh\theta\cosh\theta = \sqrt{\cosh^2\theta - 1}\cosh\theta = \sqrt{\frac{x^2}{3} - 1} \times \frac{x}{\sqrt{3}} = \frac{1}{3}x\sqrt{x^2-3}$, therefore:

$\int \sqrt{x^2-3}\,dx = \frac{\sqrt{3}}{6}x\sqrt{x^2-3} - \frac{\sqrt{3}}{2}\operatorname{arccosh}\left(\frac{x}{\sqrt{3}}\right) + c$

199. $\int \sqrt{16-25x^2}\,dx = \int \sqrt{25\left(\frac{16}{25} - x^2\right)}\,dx = 5\int \sqrt{\left(\frac{4}{5}\right)^2 - x^2}\,dx$

Let $x = \frac{4}{5}\sin\theta$, then $dx = \frac{4}{5}\cos\theta\, d\theta$, $\left(\theta = \arcsin\left(\frac{5}{4}x\right)\right)$, so:

$\int \sqrt{16-25x^2}\,dx = 5\int \sqrt{\frac{16}{25} - \frac{16}{25}\sin^2\theta} \times \frac{4}{5}\cos\theta\, d\theta = \frac{16}{5}\int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta$

$= \frac{16}{5}\int \sqrt{\cos^2\theta}\cos\theta\, d\theta = \frac{16}{5}\int \cos^2\theta\, d\theta = \frac{16}{5}\int \frac{1+\cos 2\theta}{2}d\theta = \frac{8}{5}\int(1+\cos 2\theta)d\theta$

$= \frac{8}{5}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c = \frac{8}{5}\left[\arcsin\left(\frac{5}{4}x\right) + \frac{5}{16}x\sqrt{16-25x^2}\right] + c$

Remark : $\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta = \sin\theta\sqrt{1-\sin^2\theta} = \frac{5}{4}x\sqrt{1-\left(\frac{5}{4}x\right)^2} = \frac{5}{16}x\sqrt{16-25x^2}$

200. $\int \frac{1}{\sqrt{x^2-16}}dx$, let $x = 4\sec\theta$, then $dx = 4\sec\theta\tan\theta\, d\theta$, so:

$\int \frac{1}{\sqrt{x^2-16}}dx = \int \frac{1}{\sqrt{16\sec^2\theta - 16}}4\sec\theta\tan\theta\, d\theta = \int \frac{1}{4\sqrt{\sec^2\theta - 1}}4\sec\theta\tan\theta\, d\theta$

$= \int \frac{1}{\sqrt{\tan^2\theta}}\sec\theta\tan\theta\, d\theta = \int \frac{1}{\tan\theta}\sec\theta\tan\theta\, d\theta = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + c$

But : $\frac{\sqrt{x^2-16}}{4} = \frac{4\sqrt{\sec^2\theta - 1}}{4} = \sqrt{\tan^2\theta} = \tan\theta$, then:

$\int \frac{1}{\sqrt{x^2-16}}dx = \ln\left|\frac{x}{4} + \frac{\sqrt{x^2-16}}{4}\right| + c = \ln\left|x + \sqrt{x^2-16}\right| + k$

201. $\int \frac{\sqrt{9-x^2}}{x^2}dx$, let $x = 3\sin\theta$, $\left(\theta = \arcsin\left(\frac{x}{3}\right)\right)$, then $dx = 3\cos\theta\, d\theta$, so:

$\int \frac{\sqrt{9-x^2}}{x^2}dx = \int \frac{\sqrt{9-9\sin^2\theta}}{9\sin^2\theta}(3\cos\theta)d\theta = \int \frac{3\sqrt{1-\sin^2\theta}}{3\sin^2\theta}\cos\theta\, d\theta = \int \frac{\sqrt{\cos^2\theta}}{\sin^2\theta}\cos\theta\, d\theta$

$= \int \frac{\cos^2\theta}{\sin^2\theta}d\theta = \int \cot^2\theta\, d\theta = \int(\csc^2\theta - 1)d\theta = -\cot\theta - \theta + c$

But $x = 3\sin\theta$, then : $\frac{\sqrt{9-x^2}}{x} = \frac{\sqrt{9-9\sin^2\theta}}{3\sin\theta} = \frac{3\sqrt{\cos^2\theta}}{3\sin\theta} = \cot\theta$, then $-\cot\theta = -\frac{\sqrt{9-x^2}}{x}$, so:

$\int \frac{\sqrt{9-x^2}}{x^2}dx = -\frac{\sqrt{9-x^2}}{x} - \arcsin\left(\frac{x}{3}\right) + c$

202. $\int \frac{1}{\sqrt{x^2+9}}dx$, let $x = 3\tan\theta$, then $dx = 3\sec^2\theta\, d\theta$, so:

$\int \frac{1}{\sqrt{x^2+9}}dx = \int \frac{3\sec^2\theta}{\sqrt{9\tan^2\theta+9}}d\theta = \int \frac{3\sec^2\theta}{3\sqrt{1+\tan^2\theta}}d\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}d\theta = \int \sec\theta\, d\theta$

$= \ln|\sec\theta + \tan\theta| + c$, but $x = 3\tan\theta$, so $x^2 + 9 = 9(1 + \tan^2 x) = 9\sec^2 x$, so:

$\int \frac{1}{\sqrt{x^2+9}}dx = \ln\left|\frac{\sqrt{x^2+9}}{3} + \frac{x}{3}\right| + c = \ln\left|x + \sqrt{x^2+9}\right| + k$

203. $\int \frac{x^5}{\sqrt{x^2+2}}dx$, let $x = \sqrt{2}\tan\theta$, then $dx = \sqrt{2}\sec^2\theta\, d\theta$ and

$\sqrt{x^2+2} = \sqrt{2\tan^2\theta + 2} = \sqrt{2}\sqrt{\tan^2\theta + 1} = \sqrt{2}\sqrt{\sec^2\theta} = \sqrt{2}\sec\theta$, then:

$\int \frac{x^5}{\sqrt{x^2+2}}dx = \int \frac{\left(\sqrt{2}\right)^5 \tan^5\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec^2\theta\, d\theta = \int \left(\sqrt{2}\right)^5 \tan^5\theta \sec\theta\, d\theta$

$= 2^{\frac{5}{2}} \int \tan^4\theta \tan\theta \sec\theta\, d\theta = 2^{\frac{5}{2}} \int (\sec^2\theta - 1)^2 \tan\theta \sec\theta\, d\theta$

Let $u = \sec\theta$, then $du = \tan\theta\sec\theta\, d\theta$, so

$\int \frac{x^5}{\sqrt{x^2+2}}dx = 2^{\frac{5}{2}} \int (u^2-1)^2 du = 2^{\frac{5}{2}} \int (u^4 - 2u^2 + 1)du = 2^{\frac{5}{2}}\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + c$

$= 2^{\frac{5}{2}}\left(\frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta\right) + c$

But $\tan\theta = \frac{x}{\sqrt{2}}$, then $\sec\theta = \sqrt{1+\tan^2\theta} = \sqrt{1 + \left(\frac{x}{\sqrt{2}}\right)^2} = \sqrt{\frac{2}{2} + \frac{x^2}{2}} = \frac{\sqrt{x^2+2}}{\sqrt{2}}$ so:

$\int \frac{x^5}{\sqrt{x^2+2}}dx = 2^{\frac{5}{2}}\left[\frac{1}{5}\left(\frac{\sqrt{x^2+2}}{\sqrt{2}}\right)^5 - \frac{2}{3}\left(\frac{\sqrt{x^2+2}}{\sqrt{2}}\right)^3 + \frac{\sqrt{x^2+2}}{\sqrt{2}}\right] + c$

$= 4\sqrt{x^2+2} - \frac{4}{3}(x^2+2)^{\frac{3}{2}} + \frac{1}{5}(x^2+2)^{\frac{5}{2}} + c$

204. $\int \frac{1}{e^x+e^{-x}-1}dx = \int \frac{e^x}{e^x(e^x+e^{-x}-1)}dx = \int \frac{e^x}{e^{2x}-e^x+1}dx$, let $t = e^x$, then $t = e^x dx$

so: $\int \frac{1}{e^x+e^{-x}-1}dx = \int \frac{dt}{t^2-t+1} = \int \frac{dt}{t^2-t+\frac{1}{4}+\frac{3}{4}} = \int \frac{dt}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}}\arctan\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$

$= \frac{2}{\sqrt{3}}\arctan\left(\frac{2t-1}{\sqrt{3}}\right) + c = \frac{2}{\sqrt{3}}\arctan\left(\frac{2e^x-1}{\sqrt{3}}\right) + c$

205. $\int \frac{x}{1+e^{x^2}}dx = \frac{1}{2}\int \frac{2x}{1+e^{x^2}}dx$, let $u = x^2$, then $du = 2xdx$, so:

$\int \frac{x}{1+e^{x^2}}dx = \frac{1}{2}\int \frac{1}{1+e^u}du$, let $t = e^u$, then $dt = e^u du = t du$, so:

$\int \frac{x}{1+e^{x^2}}dx = \frac{1}{2}\int \frac{1}{t(t+1)}dt = \frac{1}{2}\int \frac{(1+t)-t}{t(t+1)}dt = \frac{1}{2}\int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \frac{1}{2}(\ln|t| - \ln|t+1|) + c$

$= \frac{1}{2}\ln\left|\frac{t}{t+1}\right| + c = \frac{1}{2}\ln\left|\frac{e^u}{e^u+1}\right| + c = \frac{1}{2}\ln\left|\frac{e^{x^2}}{e^{x^2}+1}\right| + c$

206. $\int \frac{1-x\sin x}{x\left(1-x^2e^{2\cos x}\right)}dx = \int \frac{e^{\cos x}(1-x\sin x)}{xe^{\cos x}\left[1-\left(xe^{\cos x}\right)^2\right]}dx,$

let $t = xe^{\cos x}$, $dt = e^{\cos x}(1 - x\sin x)dx$,

$\int \frac{1-x\sin x}{x\left(1-x^2e^{2\cos x}\right)}dx = \int \frac{dt}{t\left(1-t^2\right)}$, let $t = \frac{1}{u}$, then $dt = -\frac{1}{u^2}du$, then:

$\int \frac{1-x\sin x}{x\left(1-x^2e^{2\cos x}\right)}dx = \int \frac{-\frac{1}{u^2}du}{\frac{1}{u}\left(1-\frac{1}{u^2}\right)} = -\int \frac{u^3}{u^2\left(u^2-1\right)}du = -\int \frac{u}{u^2-1}du = -\frac{1}{2}\int \frac{2u}{u^2-1}du$

$= -\frac{1}{2}\int \frac{\left(u^2-1\right)'}{u^2-1}du = -\frac{1}{2}\ln|u^2 - 1| + c = -\frac{1}{2}\ln\left|\frac{1-t^2}{t^2}\right| + c = -\frac{1}{2}\ln\left|\frac{1-x^2e^{2\cos x}}{x^2e^{2\cos x}}\right| + c$

207. $\int \frac{\ln x}{(x-3)^2}dx = \int (x-3)^{-2}\ln x\, dx$

Let $u = \ln x$, then $u' = \frac{1}{x}$ and let $v' = (x-3)^{-2}$, then $v = -(x-3)^{-1}$, so we get:

$\int \frac{\ln x}{(x-3)^2}dx = -(x-3)^{-1}\ln x + \int \frac{1}{x(x-3)}dx$

Decomposition into partial fractions : $\frac{1}{x(x-3)} = \frac{\mathrm{A}}{x} + \frac{\mathrm{B}}{x-3}$, so $1 = \mathrm{A}(x-3) + \mathrm{B}x$, for $x = 0$ we get $\mathrm{A} = -\frac{1}{3}$ and for $x = 3$, we get $\mathrm{B} = \frac{1}{3}$, so $\int \frac{1}{x(x-3)}dx = -\frac{1}{3}\int \frac{1}{x}dx + \frac{1}{3}\int \frac{1}{x-3}dx$

$= -\frac{1}{3}\ln|x| + \frac{1}{3}\ln|x-3| + c = \frac{1}{3}\ln\left|\frac{x-3}{x}\right| + c = \frac{1}{3}\ln\left|1 - \frac{3}{x}\right| + c$, therefore:

$\int \frac{\ln x}{(x-3)^2}dx = \frac{1}{3}\ln\left|1 - \frac{3}{x}\right| - \frac{\ln x}{x-3} + c$

208. $\int \left(\frac{x^2-x+1}{x^2+1}\right)e^{\operatorname{arccot} x}dx = \int (x^2 - x + 1)\frac{1}{x^2+1}e^{\operatorname{arccot} x}dx$, let $t = \operatorname{arccot} x$, then $dt = -\frac{1}{1+x^2}dx$, $\cot t - \csc^2 t = \cot \operatorname{arccot} x - \csc^2 \operatorname{arccot} x = x - \left(\sqrt{1 + \cot^2(\operatorname{arccot} x)}\right)^2$

$= x - 1 - x^2 = -(x^2 - x + 1)$, then:

$\int \left(\frac{x^2-x+1}{x^2+1}\right)e^{\operatorname{arccot} x}dx = \int e^t(\cot t - \csc^2 t\, dt) = \int e^t(\cot x + (\cot x)')dx$

$= \int e^t[f(t) + f'(t)]dt = e^t(t) + c = e^t.\cot t + c = e^{\operatorname{arccot} x}.\cot \operatorname{arccot} x + c$

$= xe^{\operatorname{arccot} x} + c$

209. $\int \frac{\tan\left(\sec^{-1}\left(\sqrt{e^x}\right)\right)}{\sqrt{1+\sinh x-\cosh x}}dx$, let $y = \sec^{-1}\left(\sqrt{e^x}\right)$, then $\sec y = \sqrt{e^x}$, so $\sec^2 y = e^x$

then $\tan^2 y = e^x - 1 = e^x(1 - e^{-x}) = e^x\left(1 + \frac{1}{2}e^x - \frac{1}{2}e^x - \frac{1}{2}e^{-x} - \frac{1}{2}e^{-x}\right)$

$\tan^2 y = e^x\left[1 + \frac{1}{2}(e^x - e^{-x}) - \frac{1}{2}(e^x + e^{-x})\right] = e^x\sqrt{1 + \sinh x + \cosh x}$, then

$\tan y = \sqrt{e^x}\sqrt{1 + \sinh x - \cosh x}$, with $y = \sec^{-1}\left(\sqrt{e^x}\right)$, then we get

$\tan\left(\sec^{-1}\left(\sqrt{e^x}\right)\right) = \sqrt{e^x}\sqrt{1+\sinh x - \cosh x}$, so:

$$\int \frac{\tan\left(\sec^{-1}\left(\sqrt{e^x}\right)\right)}{\sqrt{1+\sinh x-\cosh x}}dx = \int \frac{\sqrt{e^x}\sqrt{1+\sinh x-\cosh x}}{\sqrt{1+\sinh x-\cosh x}}dx = \int \sqrt{e^x}dx = 2\sqrt{e^x} + c$$

210. $\int \frac{dx}{e^{\frac{x}{2}}+e^x}$, let $t = e^{\frac{x}{2}}$, then $t^2 = e^x$, so $2tdt = e^x dx$, then $2tdt = t^2 dx$, $2dt = tdx$, then

$dx = \frac{2}{t}dt$, so: $\int \frac{dx}{e^{\frac{x}{2}}+e^x} = \int \frac{\frac{2}{t}dt}{t+t^2} = 2\int \frac{dt}{t^2(1+t)}$

Method of partial fractions : $\frac{1}{t^2(1+t)} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1+t}$, then $1 = t(t+1)A + (1+t)B + t^2C$

$1 = (A+C)t^2 + (A+B)t + B$, by comparing both sides with get $\begin{cases} A+C=0 \\ A+B=0 \\ B=1 \end{cases}$, so $A = -B = -1$

and $C = -A = 1$, so : $\frac{1}{t^2(1+t)} = -\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t+1}$, so:

$$\int \frac{dx}{e^{\frac{x}{2}}+e^x} = 2\int \left(-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t+1}\right)dt = -\frac{2}{t} - 2\ln t + 2\ln(1+t) + c$$

$$= -\frac{2}{e^{\frac{x}{2}}} - 2\ln e^{\frac{x}{2}} + 2\ln\left(1+e^{\frac{x}{2}}\right) + c = -2e^{-\frac{x}{2}} - x + 2\ln\left(1+e^{\frac{x}{2}}\right) + c$$

211. $\int \sec 2x\sqrt{\sec 2x - 1}\,dx$, we have $\sec 2x = \frac{1+\tan^2 x}{1-\tan^2 x}$, then: $\int \sec 2x\sqrt{\sec 2x - 1}\,dx =$

$$\int \left(\frac{1+\tan^2 x}{1-\tan^2 x}\right)\sqrt{\frac{1+\tan^2 x}{1-\tan^2 x} - 1}dx = \int \left(\frac{1+\tan^2 x}{1-\tan^2 x}\right)\sqrt{\frac{1+\tan^2 x - 1 + \tan^2 x}{1-\tan^2 x}}dx$$

$= \int \left(\frac{\sec^2 x}{1-\tan^2 x}\right)\sqrt{\frac{2\tan^2 x}{1-\tan^2 x}}dx = \int \frac{\sqrt{2}\tan x}{\left(1-\tan^2 x\right)^{\frac{3}{2}}}\sec^2 x\,dx$, let $t = \tan x$, so $dt = \sec^2 x\,dx$, so:

$$\int \sec 2x\sqrt{\sec 2x - 1}\,dx = \int \frac{\sqrt{2}t}{\left(1-t^2\right)^{\frac{3}{2}}}dt = -\frac{\sqrt{2}}{2}\int \frac{2t}{\left(1-t^2\right)^{\frac{3}{2}}}dt = -\frac{\sqrt{2}}{2}\int \frac{\left(1-t^2\right)'}{\left(1-t^2\right)^{\frac{3}{2}}}dt$$

$$= -\frac{\sqrt{2}}{2}\left[-2\left(1-t^2\right)^{-\frac{1}{2}}\right] + c = \frac{\sqrt{2}}{\sqrt{1-t^2}} + c = \frac{\sqrt{2}}{\sqrt{1-\tan^2 x}} + c$$

212. $\int \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx = \int \frac{\frac{\sin x\cos x}{\cos^4 x}}{\frac{\cos^4 x+\sin^4 x}{\cos^4 x}}dx = \int \frac{\left(\frac{\sin x}{\cos x}\right)\left(\frac{1}{\cos^2 x}\right)}{1+\frac{\sin^4 x}{\cos^4 x}}dx = \int \frac{\tan x\sec^2 x}{1+\tan^4 x}dx$

Let $u = \tan^2 x$, then $du = 2\tan x\sec^2 x\,dx$, then:

$$\int \frac{\sin x\cos x}{\cos^4 x+\sin^4 x}dx = \frac{1}{2}\int \frac{du}{1+u^2} = \frac{1}{2}\arctan u + c = \frac{1}{2}\arctan(\tan^2 x) + c$$

213. $\int \frac{1}{\sin^6 x+\cos^6 x}dx = \int \frac{1}{\left(\sin^2 x\right)^3+\left(\cos^2 x\right)^3}dx$

$$= \int \frac{1}{\left(\sin^2 x+\cos^2 x\right)\left(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x\right)}dx = \int \frac{1}{\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x}dx$$

$$= \int \frac{1}{\left(\sin^2 x\right)^2 - 2\sin^2 x\cos^2 x + \left(\cos^2 x\right)^2 + \sin^2 x\cos^2 x}\,dx$$

$$= \int \frac{1}{\left(\sin^2 x - \cos^2 x\right)^2 + (\sin x\cos x)^2}\,dx = \int \frac{1}{\cos^2 2x + \left(\frac{1}{2}\sin 2x\right)^2}\,dx = \int \frac{1}{\cos^2 2x + \frac{1}{4}\sin^2 2x}\,dx$$

$$= 4\int \frac{1}{4\cos^2 2x + \sin^2 2x}\,dx = 4\int \frac{\frac{1}{\cos^2 2x}}{\left(4\cos^2 2x + \sin^2 2x\right)\frac{1}{\cos^2 2x}}\,dx = 4\int \frac{\sec^2 2x}{4 + \tan^2 2x}\,dx$$

Let $u = \tan 2x$, then $du = 2\sec^2 2x\,dx$, so:

$$\int \frac{1}{\sin^6 x + \cos^6 x}\,dx = 2\int \frac{1}{4+u^2}\,du = 2\int \frac{1}{2^2+u^2}\,du = 2\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right] + c$$

$$= \arctan\left(\frac{\tan 2x}{2}\right) + c$$

214. $\int \sqrt{e^x + 1}dx$, let $u = \sqrt{e^x + 1} \Longrightarrow u^2 = e^x + 1 \Longrightarrow 2udu = e^x dx \Longrightarrow$ $2udu = (u^2 - 1)dx$, so:

$$\int \sqrt{e^x + 1}dx = \int \frac{2u^2}{u^2-1}\,du = 2\int \frac{u^2-1+1}{u^2-1}\,du = 2\int \left(1 + \frac{1}{u^2-1}\right)du$$

$$= 2\int du + 2\int \frac{1}{2}\left[\frac{-(u-1)+(u+1)}{(u-1)(u+1)}\right]du = 2\int du + \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$$

$$= 2u + \ln|u-1| - \ln|u+1| + c = 2\sqrt{e^x+1} + \ln\left|\frac{\sqrt{e^x+1}-1}{\sqrt{e^x+1}+1}\right| + c$$

215. $\int \frac{3}{x^2+4x+29}\,dx = \int \frac{3}{(x+2)^2+25}\,dx = \frac{3}{25}\int \frac{1}{\left(\frac{x+2}{5}\right)^2+1}\,dx$, let $u = \frac{x+2}{5}$, $du = \frac{1}{5}dx$, then:

$$\int \frac{3}{x^2+4x+29}\,dx = \frac{3}{25}\int \frac{5}{1+u^2}\,du = \frac{3}{5}\arctan u + c = \frac{3}{5}\arctan\left(\frac{x+2}{5}\right) + c$$

216. $\int \frac{\cos x + x\sin x}{x(x+\cos x)}\,dx = \int \frac{x+\cos x+x\sin x-x}{x(x+\cos x)}\,dx = \int \frac{(x+\cos x)+x(\sin x-1)}{x(x+\cos x)}\,dx$

$= \int \frac{1}{x}dx + \int \frac{\sin x-1}{x+\cos x}\,dx = \int \frac{1}{x}dx - \int \frac{1-\sin x}{x+\cos x}\,dx$, let $u = x + \cos x$, so $du = (1-\sin x)dx$,

so: $\int \frac{\cos x+x\sin x}{x(x+\cos x)}\,dx = \int \frac{1}{x}dx - \int \frac{du}{u} = \ln|x| - \ln|x+\cos x| + c = \ln\left|\frac{x}{x+\cos x}\right| + c$

217. $\int \frac{1}{5\cosh x + 3\sinh x + 4}\,dx = \int \frac{1}{5\left(\frac{e^x+e^{-x}}{2}\right)+3\left(\frac{e^x-e^{-x}}{2}\right)+4}\,dx$

$= \int \frac{2e^x}{5\left(e^{2x}+1\right)+3\left(e^{2x}-1\right)+8e^x}\,dx$, let $u = e^x$, then $du = e^x dx$, then we get:

$$\int \frac{1}{5\cosh x+3\sinh x+4}\,dx = \int \frac{2}{5\left(u^2+1\right)+3\left(u^2-1\right)+8u}\,du = \int \frac{2}{8u^2+8u+2}\,du$$

$$= \int \frac{1}{4u^2+4u+1}\,du = \int \frac{1}{(2u+1)^2}\,du = -\frac{1}{2(2u+1)} + c = -\frac{1}{2(2e^x+1)} + c$$

218. $\int \frac{x^2+1}{x^4+1}\,dx = \int \frac{x^2+1}{x^4+1}.\frac{x^{-2}}{x^{-2}}\,dx = \int \frac{x^{-2}+1}{x^2+x^{-2}}\,dx = \int \frac{x^{-2}+1}{x^2+x^{-2}-2+2}\,dx$

$= \int \frac{x^{-2}+1}{\left(x-x^{-1}\right)^2+2}\,dx$, let $u = x - x^{-1}$, then $du = (1 + x^{-2})dx$, so:

$$\int \frac{x^2+1}{x^4+1}\,dx = \int \frac{du}{u^2+2} = \int \frac{du}{u^2+\left(\sqrt{2}\right)^2} = \frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}u\right) + c = \frac{1}{\sqrt{2}}\arctan\left(\frac{x-x^{-1}}{\sqrt{2}}\right) + c$$

219. $\int \frac{1-x^2}{1+3x^2+x^4}dx = \int \frac{x^2\left(\frac{1}{x^2}-1\right)}{x^2\left(\frac{1}{x^2}+3+x^2\right)}dx = -\int \frac{1-\frac{1}{x^2}}{\frac{1}{x^2}+3+x^2}dx.$

Let $u = x + \frac{1}{x}$, then $du = \left(1 - \frac{1}{x^2}\right)dx$ and $x^2 + \frac{1}{x^2} = u^2 - 2$, then we get:

$\int \frac{1-x^2}{1+3x^2+x^4}dx = -\int \frac{1}{1+u^2}du = -\arctan u + c = -\arctan\left(x + \frac{1}{x}\right) + c$

220. $\int \frac{\sin 2x \cos 2x}{\sqrt{4-\sin^4 2x}}dx = \frac{1}{4}\int \frac{4\sin 2x \cos 2x}{\sqrt{4-\sin^4 2x}}dx$, let $t = \sin^2 2x$, so $dt = 4\sin 2x \cos 2x\, dx$

$\int \frac{\sin 2x \cos 2x}{\sqrt{4-\sin^4 2x}}dx = \frac{1}{4}\int \frac{dt}{\sqrt{2^2-t^2}} = \frac{1}{4}\arcsin\left(\frac{1}{2}t\right) + c = \frac{1}{4}\arcsin\left(\frac{1}{2}\sin^2 2x\right) + c$

221. $\int \frac{\sqrt{\cos x-\cos^3 x}}{\sqrt{1-\cos^3 x}}dx = \int \frac{\sqrt{\cos x(1-\cos^2 x)}}{\sqrt{1-\cos^3 x}}dx = \int \frac{\sqrt{\cos x \sin^2 x}}{\sqrt{1-\cos^3 x}}dx = \int \frac{\sin x\sqrt{\cos x}}{\sqrt{1-\cos^3 x}}dx$

Let $t = \cos x$, then $dt = -\sin x\, dx$, so:

$\int \frac{\sqrt{\cos x \sin^2 x}}{\sqrt{1-\cos^3 x}}dx = -\int \frac{\sqrt{t}}{\sqrt{1-t^3}}dt = -\int \frac{\sqrt{t}}{\sqrt{1-\left(t^{\frac{3}{2}}\right)^2}}dt$, let $u = t^{\frac{3}{2}}$, then $du = \frac{3}{2}\sqrt{t}dt$, so:

$\int \frac{\sqrt{\cos x-\cos^3 x}}{\sqrt{1-\cos^3 x}}dx = -\frac{2}{3}\int \frac{du}{\sqrt{1-u^2}} = -\frac{2}{3}\arcsin u + c = -\frac{2}{3}\arcsin t^{\frac{3}{2}} + c$

$= -\frac{2}{3}\arcsin\left(\cos^{\frac{3}{2}} x\right) + c$

222. $\int \frac{dx}{\sqrt{x}(4+\sqrt[3]{x})}$, let $u = \sqrt[6]{x}$, then $x = u^6$ and $dx = 6u^5du$, so:

$\int \frac{dx}{\sqrt{x}(4+\sqrt[3]{x})} = 6\int \frac{u^5du}{u^3(4+u^2)} = 6\int \frac{u^2du}{4+u^2} = 6\int \frac{u^2+4-4}{4+u^2}du = 6\int \left(1 - \frac{4}{4+u^2}\right)du$

$= 6\int \left(1 - \frac{4}{2^2+u^2}\right)du = 6\left[u - 4 \times \frac{1}{2}\arctan\left(\frac{u}{2}\right)\right] + c = 6u - 12\arctan\left(\frac{1}{2}u\right) + c$

$= 6\sqrt[6]{x} - 12\arctan\left(\frac{1}{2}\sqrt[6]{x}\right) + c$

223. $\int \frac{(\ln x-1)^2}{\left(1+(\ln x)^2\right)^2}dx = \int \frac{1+(\ln x)^2-2\ln x}{\left(1+(\ln x)^2\right)^2}dx = \int \left[\frac{1+(\ln x)^2}{\left(1+(\ln x)^2\right)^2} - \frac{2\ln x}{\left(1+(\ln x)^2\right)^2}\right]dx$

$= \int \left[\frac{1}{1+(\ln x)^2} - \frac{2\ln x}{\left(1+(\ln x)^2\right)^2}\right]dx$, let $t = \ln x$, $x = e^t$, then $dx = e^t dt$, so:

$\int \frac{(\ln x-1)^2}{\left(1+(\ln x)^2\right)^2}dx = \int e^t\left[\frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2}\right]dt = \int e^t[f(t) + f'(t)]dt = e^t f(x) + c$

$= e^t\left(\frac{1}{1+t^2}\right) + c = \frac{x}{1+(\ln x)^2} + c$

224. $\int \frac{4^x+1}{2^x+1}dx = \int \frac{e^{2x\ln 2}+1}{e^{x\ln 2}+1}dx$, let $u = e^{x\ln 2}$, then $du = \ln 2\, e^{x\ln 2}dx = u\ln 2\, dx$, then:

$\int \frac{4^x+1}{2^x+1}dx = \int \frac{u^2+1}{u+1}\frac{1}{u\ln 2}dx = \frac{1}{\ln 2}\int \frac{u^2+u+1-u}{u(u+1)}du = \frac{1}{\ln 2}\int \left[1+\frac{1-u}{u(u+1)}\right]du$

$= \frac{1}{\ln 2}\int \left(1+\frac{1}{u}-\frac{2}{u+1}\right)du = \frac{1}{\ln 2}(u+\ln u - 2\ln(u+1)) + c$

$= x + \frac{2^x-2\ln(2^x+1)}{\ln 2} + c$

Remark : $\frac{1-u}{u(u+1)} = \frac{A}{u}+\frac{B}{u+1}$ (method of partial fractions), so : $\frac{1-u}{u(u+1)} = \frac{A(u+1)+Bu}{u(u+1)}$,

then , $\frac{1-u}{u(u+1)} = \frac{(A+B)u+A}{u(u+1)}$, by comparing the two sides of the equality we get $A = 1$ and $A + B = -1$, so $B = -1-1 = -2$

225. $\int e^{e^{e^x}+e^x+x}dx = \int e^{e^{e^x}}e^{e^x}e^x dx$, let $u = e^{e^{e^x}}$, then $du = \left(e^{e^{e^x}}\right)'dx$,

$du = (e^{e^x})'e^{e^{e^x}}du = (e^x)'\, e^{e^{e^x}}e^{e^x}dx = e^{e^{e^x}}e^{e^x}e^x dx$, so:

$\int e^{e^{e^x}+e^x+x}dx = \int du = u + c = e^{e^{e^x}+e^x+x} + c$

226. $\int e^x x^{e^x}\left(\ln x+\frac{1}{x}\right)dx$, we have $x^{e^x} = e^{\ln x^{e^x}} = e^{e^x\ln x}$

And we have also , $\frac{d}{dx}\left(e^{e^x\ln x}\right) = e^{e^x\ln x}\frac{d}{dx}(e^x\ln x) = x^{e^x}\left(e^x\ln x + e^x\frac{1}{x}\right)$

$= e^x x^{e^x}\left(\ln x+\frac{1}{x}\right)$, so $\int e^x x^{e^x}\left(\ln x+\frac{1}{x}\right)dx = \int d\left(e^{e^x\ln x}\right) = x^{e^x} + c$

227. $\int \frac{2^{x^3}x^2\left(1-\cos\left(2^{x^3}\right)\right)}{1+\cos\left(2^{x^3}\right)}dx$, let $t = 2^{x^3}$, then $dx = \frac{1}{3x^2 2^{x^3}\ln 2}dt$, so:

$\int \frac{2^{x^3}x^2\left(1-\cos\left(2^{x^3}\right)\right)}{1+\cos\left(2^{x^3}\right)}dx = \frac{1}{3\ln 2}\int \frac{1-\cos t}{1+\cos t}dt = \frac{1}{3\ln 2}\int \frac{1-\cos t}{1+\cos t}\times\frac{1-\cos t}{1-\cos t}dt$

$= \frac{1}{3\ln 2}\int \frac{(1-\cos t)^2}{1-\cos^2 t}dt = \frac{1}{3\ln 2}\int \frac{(1-\cos t)^2}{\sin^2 t}dt = \frac{1}{3\ln 2}\left(\int \frac{1}{\sin^2 t}dt - 2\int \frac{\cos t}{\sin^2 t}dt + \int \frac{\cos^2 t}{\sin^2 t}dt\right)$

$= \frac{1}{3\ln 2}(-\cot t - 2(-\csc t) + (-\cot t - t)) + c$

$= \frac{1}{3\ln 2}\left(-2\cot(2^{x^3}) - 2\csc(2^{x^3}) - 2^{x^3}\right) + c$

228. $\int \frac{1}{\sin^4 x+\cos^4 x}dx = \int \frac{1}{\left(\sin^2 x+\cos^2 x\right)^2 - 2\sin^2 x\cos^2 x}dx$

$= \int \frac{1}{\left(\sin^2 x+\cos^2 x\right)^2 - 2\left(\frac{\sin 2x}{2}\right)^2}dx = \int \frac{1}{1-\frac{1}{2}\sin^2 2x}dx = \int \frac{2}{2-\sin^2 2x}dx$

$= \int \frac{2\sec^2 2x}{2\sec^2 2x - \tan^2 2x}dx = \int \frac{2\sec^2 2x}{\tan^2 2x+2}dx$

Let $u = \tan 2x$, then $du = 2\sec^2 2x$, so: $\int \frac{1}{\sin^4 x+\cos^4 x}dx = \int \frac{du}{u^2+2} = \frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right) + c$

$= \frac{1}{\sqrt{2}}\arctan\left(\frac{\tan 2x}{\sqrt{2}}\right) + c$

229. $\int \frac{1}{\cos 2x+\cos^2 x}dx = \int \frac{1}{\cos^2 x-\sin^2 x+\cos^2 x}dx = \int \frac{1}{2\cos^2 x-\sin^2 x}dx$

$= \int \frac{\sec^2 x}{2-\tan^2 x}dx$. Let $u = \tan x$, then $du = \sec^2 x\, dx$, so:

$\int \frac{1}{\cos 2x+\cos^2 x}dx = \int \frac{du}{2-u^2} = -\int \frac{du}{u^2-2} = -\frac{1}{2\sqrt{2}}\int \frac{(u+\sqrt{2})-(u-\sqrt{2})}{(u+\sqrt{2})(u-\sqrt{2})}du$

$= -\frac{1}{2\sqrt{2}}\int \left(\frac{1}{u-\sqrt{2}} - \frac{1}{u+\sqrt{2}}\right)du = \frac{1}{2\sqrt{2}}\left[\ln|u+\sqrt{2}| - \ln|u-\sqrt{2}|\right] + c$

$= \frac{1}{2\sqrt{2}}\ln\left|\frac{u+\sqrt{2}}{u-\sqrt{2}}\right| + c = \frac{1}{2\sqrt{2}}\ln\left|\frac{\tan x+\sqrt{2}}{\tan x-\sqrt{2}}\right|$

230. $\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5-\sqrt[6]{x^7}}}dx$, let $x = t^{12}$, then $dx = 12t^{11}dt$, then:

$\int \frac{\sqrt{x}+\sqrt[3]{x}}{\sqrt[4]{x^5-\sqrt[6]{x^7}}}dx = \int \frac{t^6+t^4}{t^{15}-t^{14}}12t^{11}dt = \int \frac{t^2+1}{t-1}12t dt = 12\int \frac{t^3+t}{t-1}dt = 12\int \frac{t^3-1+t+1}{t-1}dt$

$= 12\int \frac{t^3-1}{t-1}dt + 12\int \frac{t-1}{t-1} + 12\int \frac{2}{t-1}dt = 12\int (t^2+t+1)dt + 12t + 24\ln|t-1| + c$

$= 4t^3 + 6t^2 + 12t + 12t + 24\ln|t-1| + c = 4t^3 + 6t^2 + 24t + 24\ln|t-1| + c$

$= 4\left(x^{\frac{1}{12}}\right)^3 + 6\left(x^{\frac{1}{12}}\right)^2 + 24x^{\frac{1}{12}} + 24\ln\left|x^{\frac{1}{12}} - 1\right| + c$

$= 4\sqrt[4]{x} + 6\sqrt[6]{x} + 24\sqrt[12]{x} + 24\ln|\sqrt[12]{x} - 1| + c$

231. $\int \sqrt{x+\sqrt{x^2+2}}\,dx$, let $x+\sqrt{x^2+2} = t$, then $\sqrt{x^2+2} = t-x$, squaring both sides, we get $x^2+2 = t^2+x^2-2tx$, then $x = \frac{t^2-2}{2t} = \frac{t}{2} - \frac{1}{t}$ and $dx = \left(\frac{1}{2}+\frac{1}{t^2}\right)dt$, so we get:

$\int \sqrt{x+\sqrt{x^2+2}}\,dx = \int \sqrt{t}\left(\frac{1}{2}+\frac{1}{t^2}\right)dt = \frac{1}{2}\int \sqrt{t}dt + \int t^{-\frac{3}{2}}dt = \frac{1}{2}t\sqrt{t} - \frac{1}{2\sqrt{t}} + c$

$= \frac{1}{2}\left(x+\sqrt{x^2+2}\right)\sqrt{x+\sqrt{x^2+2}} - \frac{1}{2\sqrt{x+\sqrt{x^2+2}}} + c$

232. $\int \cos x \,.\, \cos(\sin x) \,.\, \cos(\sin(\sin x))\,dx$

Let $u = \sin(\sin x)$, then $du = \cos(\sin x) \,.\, \cos x\,dx$, so:

$\int \cos x \,.\, \cos(\sin x) \,.\, \cos(\sin(\sin x))\,dx = \int \cos u\,du = \sin u + c = \sin(\sin(\sin x)) + c$

233. $\int \frac{1}{3\sin^2 x-4\sin x\cos x-4\cos^2 x}dx = \int \frac{\sec^2 x}{3\tan^2 x-4\tan x-4}dx$

(we divide by $\cos^2 x$ up and down)

Let $u = \tan x$, then $du = \sec^2 x\,dx$, then:

$\int \frac{1}{3\sin^2 x-4\sin x\cos x-4\cos^2 x}dx = \int \frac{du}{3u^2-4u-4} = \int \frac{du}{(3u+2)(u-2)}$

$= \int \left(\frac{1}{8(u-2)} - \frac{3}{8(3u+2)}\right)du = \frac{1}{8}\ln|u-2| - \frac{1}{8}\ln|3u+2| + c$

$= \frac{1}{8}\ln|\tan x - 2| - \frac{1}{8}\ln|3\tan x + 2| + c = \frac{1}{8}\ln\left|\frac{\tan x-2}{3\tan x+2}\right| + c$

Remark : $\frac{1}{(3u+2)(u-2)} = \frac{A}{3u+2} + \frac{B}{u-2}$ (method of partial fractions), $1 = (u-2)A + (3u+2)B$

For $u = 2$, $1 = 8B$, so $B = \frac{1}{8}$ and for $u = -\frac{2}{3}$, $1 = \left(-\frac{2}{3} - 2\right)A$, so $A = -\frac{3}{8}$.

234. $\int \frac{x+\sqrt[3]{x}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx = \int \frac{x+x^{\frac{2}{3}}+x^{\frac{1}{6}}}{x\left(1+x^{\frac{1}{3}}\right)}dx$, let $u = x^{\frac{1}{6}}$, so $x^{\frac{1}{3}} = u^2$ and $x^{\frac{2}{3}} = u^4$,

moreover $x = u^6$ and $du = \frac{1}{6x^{\frac{5}{6}}}dx$, so:

$$\int \frac{x+\sqrt[3]{x}+\sqrt[6]{x}}{x(1+\sqrt[3]{x})}dx = 6\int \frac{u^5+u^3+1}{u^2+1}du = 6\int \left(\frac{u^3(u^2+1)}{u^2+1} + \frac{1}{u^2+1}\right)du$$

$$= 6\int \left(u^3 + \frac{1}{u^2+1}\right)du = 6 \times \frac{u^4}{4} + 6\arctan u + c = \frac{3}{2}x^{\frac{2}{3}} + 6\arctan\left(x^{\frac{1}{6}}\right) + c$$

235. $\int \frac{\sec^2 x}{(1+\tan x)^3}dx$, let $u = 1 + \tan x$, then $du = \sec^2 x\, dx$, so:

$$\int \frac{\sec^2 x}{(1+\tan x)^3}dx = \int \frac{du}{u^3} = -\frac{1}{2u^2} + c = -\frac{1}{2(1+\tan x)^2} + c$$

236. **First Method:** $\int \frac{\sec^4 x \tan x}{\sec^4 x+4}dx = \int \frac{\frac{\sec^4 x\tan x}{\sec^2 x}}{\frac{\sec^4 x+4}{\sec^2 x}}dx = \int \frac{\sec^2 x\tan x}{\sec^2 x+\frac{4}{\sec^2 x}}dx =$

$\int \frac{\sec^2 x\tan x}{\tan^2 x+1+\frac{4}{\tan^2 x+1}}dx$, let $u = \tan x$, then $du = \sec^2 x\, dx$, then:

$$\int \frac{\sec^4 x\tan x}{\sec^4 x+4}dx = \int \frac{u}{u^2+1+\frac{4}{u^2+1}}du = \int \frac{u^3+u}{u^4+2u^2+5}du = \frac{1}{4}\int \frac{4u^3+4u}{u^4+2u^2+5}du$$

$$= \frac{1}{4}\int \frac{(u^4+2u^2+5)'}{u^4+2u^2+5}du = \frac{1}{4}\ln(u^4+2u^2+5) + c = \frac{1}{4}\ln(\tan^4 x + 2\tan^2 x + 5) + c$$

Second Method: $\int \frac{\sec^4 x\tan x}{\sec^4 x+4}dx = \int \frac{\frac{\sec^4 x\tan x}{\sec^3 x}}{\frac{\sec^4 x+4}{\sec^3 x}}dx = \int \frac{\sec x\tan x}{\sec x+\frac{4}{\sec^3 x}}dx$

Let $u = \sec x$, then $du = \sec x\tan x\, dx$, then:

$$\int \frac{\sec^4 x\tan x}{\sec^4 x+4}du = \int \frac{1}{u+\frac{4}{u^3}}du = \int \frac{u^3}{u^4+4}du = \frac{1}{4}\int \frac{4u^3}{u^4+4}du = \frac{1}{4}\int \frac{(u^4+4)'}{u^4+4}du$$

$$= \frac{1}{4}\ln(u^4+4) + c = \frac{1}{4}\ln(\sec^4 x + 4) + c$$

237. $\int \frac{\sin^3\sqrt{x}\cos^5\sqrt{x}}{\sqrt{x}}dx = 2\int \frac{\sin^3\sqrt{x}\cos^5\sqrt{x}}{2\sqrt{x}}dx$, let $u = \sqrt{x}$, then $du = \frac{1}{2\sqrt{x}}dx$, then:

$$\int \frac{\sin^3\sqrt{x}\cos^5\sqrt{x}}{\sqrt{x}}dx = 2\int \sin^3 u\cos^5 u\, du = 2\int \sin^2 u\cos^5 u\sin u\, du$$

$$= 2\int (1-\cos^2 u)\cos^5 u\sin u\, du = 2\int (\cos^5 u - \cos^7 u)\sin u\, du$$

Let $y = \cos u$, then $dy = -\sin u\, du$, so:

$\int \frac{\sin^3\sqrt{x}\cos^5\sqrt{x}}{\sqrt{x}}dx = -2\int(y^5-y^7)dy = -2\left(\frac{1}{6}y^6 - \frac{1}{8}y^8\right) + c = \frac{1}{4}y^8 - \frac{1}{3}y^6 + c$

$= \frac{1}{4}\cos^8 u - \frac{1}{3}\cos^6 u + c = \frac{1}{4}\cos^8\sqrt{x} - \frac{1}{3}\cos^6\sqrt{x} + c$

238. $\int \frac{\sec x\csc x}{\ln(\tan x)}dx = \int \frac{\frac{\sqrt{1+\tan^2 x}}{\tan x}\times\sqrt{1+\tan^2 x}}{\ln(\tan x)}dx = \int \frac{\frac{1+\tan^2 x}{\tan x}}{\ln(\tan x)}dx$

Let $u = \tan x$, then $du = \sec^2 x\,dx = (1+\tan^2 x)dx = (1+u^2)du$, so:

$\int \frac{\sec x\csc x}{\ln(\tan x)}dx = \int \frac{\frac{1+u^2}{u}}{\ln(u)}\frac{1}{1+u^2}du = \int \frac{\frac{1}{u}}{\ln u}du = \int \frac{(\ln u)'}{\ln u}du = \ln|\ln u| + c = \ln|\ln\tan x| + c$

239. $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx$, let $x = t^6$, then $dx = 6t^5dt$, $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx = \int \frac{6t^5}{t^3+t^2}dt = 6\int \frac{t^3}{1+t}dt$, so:

$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}dx = 6\int \frac{t^3+1-1}{1+t}dx = 6\int\left(\frac{t^3+1}{1+t} - \frac{1}{1+t}\right)dt = 6\int\left(\frac{(t+1)(t^2-t+1)}{1+t} - \frac{1}{1+t}\right)dt$

$= 6\int\left(t^2 - t + 1 - \frac{1}{1+t}\right)dt = 6\left[\frac{t^3}{3} - \frac{t^2}{2} + t - \ln|t+1|\right] + \text{c} = 2t^3 - 3t^2 + 6t - 6\ln|t+1| + c$

$= 2\sqrt{x} - 3\sqrt[3]{x} + 6\sqrt[6]{x} - 6\ln\left(1+\sqrt[6]{x}\right) + c$

240. $\int \frac{x^2+1}{x^4-5x^2+1}dx = \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}-5}dx = \int \frac{1+\frac{1}{x^2}}{x^2-2+\frac{1}{x^2}-3}dx = \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2-3}dx$

Let $t = x - \frac{1}{x}$, $dt = \left(1+\frac{1}{x^2}\right)dx$, so: $\int \frac{x^2+1}{x^4-5x^2+1}dx = \int \frac{dt}{t^2-3} = \int \frac{dt}{(t-\sqrt{3})(t+\sqrt{3})}$

$= \frac{1}{2\sqrt{3}}\int\left(\frac{1}{t-\sqrt{3}} - \frac{1}{t+\sqrt{3}}\right)dt = \frac{1}{2\sqrt{3}}\left(\ln|t-\sqrt{3}| - \ln|t+\sqrt{3}|\right) + c = \frac{1}{2\sqrt{3}}\ln\left|\frac{t-\sqrt{3}}{t+\sqrt{3}}\right| + c$

$= \frac{1}{2\sqrt{3}}\ln\left|\frac{x-\frac{1}{x}-\sqrt{3}}{x-\frac{1}{x}+\sqrt{3}}\right| + c$

Remark : $\frac{1}{(t-\sqrt{3})(t+\sqrt{3})} = \frac{\text{A}}{t-\sqrt{3}} + \frac{\text{B}}{t+\sqrt{3}}$ (method of partial fractions), then:

$1 = (t+\sqrt{3})\text{A} + (t-\sqrt{3})\text{B}$, for $t = \sqrt{3}$, $\text{A} = \frac{1}{2\sqrt{3}}$ and for $= -\sqrt{3}$, $\text{B} = -\frac{1}{2\sqrt{3}}$

241. $\int \frac{x^{5m-1}+2x^{4m-1}}{\left(x^{2m}+x^m+1\right)^3}dx = \int \frac{x^{5m-1}+2x^{4m-1}}{\left[x^{2m}\left(1+x^{-m}+x^{-2m}\right)\right]^3}dx = \int \frac{x^{5m-1}+2x^{4m-1}}{x^{6m}\left(1+x^{-m}+x^{-2m}\right)^3}dx$

$= \int \frac{x^{-m-1}+2x^{-2m-1}}{\left(1+x^{-m}+x^{-2m}\right)^3}dx = -\frac{1}{m}\int \frac{-mx^{-m-1}-2mx^{-2m-1}}{\left(1+x^{-m}+x^{-2m}\right)^3}dx$

Let $t = 1 + x^{-m} + x^{-2m}$, then $dt = (-mx^{-m-1} - 2mx^{-2m-1})dx$, so:

$\int \frac{x^{5m-1}+2x^{4m-1}}{\left(x^{2m}+x^m+1\right)^3}dx = -\frac{1}{m}\int \frac{1}{t^3}dt = \frac{1}{2mt^2} + \text{c} = \frac{1}{2m\left(1+x^{-x}+x^{-2m}\right)^2} + \text{c}$

$= \frac{x^{4m}}{2m\left(x^{2m}+x^m+1\right)^2} + c$

242. $\int \frac{1}{x-\sqrt{1-x^2}}dx$, let $x=\cos\theta$, then $dx=-\sin\theta\, d\theta$, so:

$$\int \frac{1}{x-\sqrt{1-x^2}}dx=\int \frac{1}{x-\sqrt{1-x^2}}dx=-\int \frac{\sin\theta}{\cos\theta-\sqrt{1-\cos^2\theta}}d\theta=-\int \frac{\sin\theta}{\cos\theta-\sin\theta}d\theta$$

$$=-\int \frac{\sin\theta(\cos\theta+\sin\theta)}{(\cos\theta-\sin\theta)\,(\cos\theta+\sin\theta)}d\theta=-\int \frac{\sin\theta\cos\theta+\sin^2\theta}{\cos^2\theta-\sin^2\theta}d\theta$$

$$=-\int \frac{\sin\theta\cos\theta}{\cos 2\theta}d\theta-\int \frac{\sin^2\theta}{1-2\sin^2\theta}d\theta$$

$$=\frac{1}{4}\int \frac{-2\sin 2\theta}{\cos 2\theta}d\theta+\frac{1}{2}\int \frac{-2\sin^2\theta+1-1}{1-2\sin^2\theta}d\theta=\frac{1}{4}\int \frac{(\cos 2\theta)'}{\cos 2\theta}d\theta+\frac{1}{2}\int\left(1-\frac{1}{\cos 2\theta}\right)d\theta$$

$$=\frac{1}{4}\ln|\cos 2\theta|+\frac{1}{2}\theta-\frac{1}{2}\int \sec 2\theta\, d\theta$$

$$=\frac{1}{4}\ln|\cos 2\theta|+\frac{1}{2}\theta-\frac{1}{4}\ln|\sec 2\theta+\tan 2\theta|+c$$

$$=\frac{1}{4}\ln|2\cos^2\theta-1|+\frac{1}{2}\theta-\frac{1}{4}\ln\left|\frac{1+\sin 2\theta}{\cos 2\theta}\right|+c$$

$$=\frac{1}{4}\ln|2\cos^2\theta-1|+\frac{1}{2}\theta-\frac{1}{4}\ln\left|\frac{1+2\sin\theta\cos\theta}{2\cos^2\theta-1}\right|+c$$

$$=\frac{1}{4}\ln|2x^2-1|+\frac{1}{2}x-\frac{1}{4}\ln\left|\frac{1+2x\sqrt{1-x^2}}{2x^2-1}\right|+c$$

243. $\int \frac{1}{x+\sqrt{x^2-1}}dx$, let $x=\cosh t$, then $du=\sinh t\, dt$, so:

$$\int \frac{1}{x+\sqrt{x^2-1}}dx=\int \frac{1}{\cosh t+\sqrt{\cosh^2 t-1}}\sinh t\, dt=\int \frac{1}{\cosh t+\sqrt{\sinh^2 t}}\sinh t\, dt$$

$$=\int \frac{\sinh t}{\cosh t+\sinh t}dt=\int \frac{\frac{e^t-e^{-t}}{2}}{\frac{e^t+e^{-t}}{2}+\frac{e^t-e^{-t}}{2}}dt=\int \frac{e^t-e^{-t}}{2e^t}dt=\frac{1}{2}\int(1-e^{-2t})dt$$

$=\frac{1}{2}t+\frac{1}{4}e^{-2t}+c$, but $x=\cosh t$, then $t=\operatorname{arccosh} x=\ln\left(x+\sqrt{x^2-1}\right)$, so:

$$\int \frac{1}{x+\sqrt{x^2-1}}dx=\frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)+\frac{1}{4}e^{-2\ln\left(x+\sqrt{x^2-1}\right)}+c$$

$$=\frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)+\frac{1}{4}e^{\ln\left[\frac{1}{\left(x+\sqrt{x^2-1}\right)^2}\right]}+c$$

$$=\frac{1}{2}\ln\left(x+\sqrt{x^2-1}\right)+\frac{1}{4}\times\frac{1}{2x^2-1+2x\sqrt{x^2-1}}+c$$

244. $\int \frac{1}{e^x+e^{-x}}dx=\int \frac{e^x}{e^{2x}+1}dx$, let $u=e^x$, then $du=e^x dx$, so:

$$\int \frac{1}{e^x+e^{-x}}dx=\int \frac{1}{u^2+1}du=\arctan u+c=\arctan e^x+c$$

245. $\int \tan x\ln(\cos x)\,dx$, let $u=\ln\cos x$, $du=-\frac{\sin x}{\cos x}dx=-\tan x\, dx$, then:

$$\int \tan x\ln(\cos x)\,dx=\int -u\,du=-\frac{1}{2}u^2+c=-\frac{1}{2}\ln^2\cos x+c$$

246. $\int \frac{\cos 4x-1}{\cot x-\tan x}dx = \int \frac{1-2\sin^2 2x-1}{\frac{\cos x}{\sin x}-\frac{\sin x}{\cos x}}dx = \int \frac{-2\sin^2 2x}{\frac{\cos^2 x-\sin^2 x}{\sin x\cos x}}dx = \int \frac{-2\sin^2 2x}{\frac{\cos 2x}{\sin x\cos x}}dx$

$= -\int \frac{\sin^2 2x(2\sin x\cos x)}{\cos 2x}dx = -\int \frac{\sin 2x(1-\cos^2 2x)}{\cos 2x}dx$

Let $t = \cos 2x$, then $dt = -2\sin 2x\, dx$, so:

$\int \frac{\cos 4x-1}{\cot x-\tan x}dx = \frac{1}{2}\int \frac{1-t^2}{t}dt = \frac{1}{2}\int \left(\frac{1}{t}-t\right)dt = \frac{1}{2}\ln|t| - \frac{t^2}{4} + c = \frac{1}{2}\ln|\cos 2x| - \frac{\cos^2 2x}{4} + c$

247. $\int x^3\sqrt{1-x^2}dx$, let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$, so:

$\int x^3\sqrt{1-x^2}dx = \int \sin^3\theta\sqrt{1-\sin^2\theta}\cos\theta\, d\theta = \int \sin^3\theta\sqrt{\cos^2\theta}\cos\theta\, d\theta$

$= \int \sin^3\theta\cos^2\theta\, d\theta = \int \sin^3\theta(1-\sin^2\theta)d\theta = \int \sin^3\theta\, d\theta - \int \sin^5\theta\, d\theta$

Evaluating: $\int \sin^3\theta\, d\theta = \int \sin\theta\sin^2\theta\, d\theta = \int \sin\theta(1-\cos^2\theta)\, d\theta$

$= \int \sin\theta\, d\theta - \int \cos^2\theta\sin\theta\, d\theta = \int \sin\theta\, d\theta + \int \cos^2\theta(\cos\theta)'d\theta = -\cos\theta + \frac{1}{3}\cos^3\theta + c_1$

Evaluating: $\int \sin^5\theta\, d\theta = \int \sin\theta(\sin^2\theta)^2 d\theta = \int \sin\theta(1-\cos^2\theta)^2 d\theta$

$= \int \sin\theta(1-2\cos^2\theta+\cos^4\theta)d\theta = \int \sin\theta\, d\theta + 2\int \cos^2\theta(\cos\theta)'d\theta - \int \cos^4\theta(\cos\theta)'d\theta$

$= -\cos\theta + \frac{2}{3}\cos^3\theta + \frac{1}{5}\cos^5\theta + c_2$, so:

$\int x^3\sqrt{1-x^2}dx = -\cos\theta + \frac{1}{3}\cos^3\theta + \cos\theta - \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta$

$= -\frac{1}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + c$, but $\sin\theta = x$, then $\cos\theta = \sqrt{1-x^2}$, therefore:

$\int x^3\sqrt{1-x^2}dx = -\frac{1}{3}\left(\sqrt{1-x^2}\right)^3 - \frac{1}{5}\left(\sqrt{1-x^2}\right)^5 + c$

248. $\int x^2\sqrt{1+x^2}dx$, let $x = \sinh t$, then $dx = \cosh t\, dt$, so:

$\int x^2\sqrt{1+x^2}dx = \int \sinh^2 t\sqrt{1+\sinh^2 t}\cosh t\, dt = \int \sinh^2 t\sqrt{\cosh^2 t}\cosh t\, dt$

$= \int \sinh^2 t\cosh^2 t\, dt = \int(\sinh t\cosh t)^2 dt = \int \left(\frac{\sinh 2t}{2}\right)^2 dt = \frac{1}{4}\int \sin^2 2t\, dt$

$= \frac{1}{4}\int \left(\frac{\cosh 4t-1}{2}\right) = \frac{1}{8}\int(\cosh 4t-1)dt = \frac{1}{8}\left(\frac{1}{4}\sinh 4t - t\right) + c$

$= \frac{1}{8}\left(\frac{1}{2}\sinh 2t\cos 2t - t\right) + c = \frac{1}{8}[\sinh t\cos t.(1+2\sin^2 t) - t] + c$

$= \frac{1}{8}\left[x\sqrt{1+x^2}(1+2x^2) - \sinh^{-1}x\right] + c$

249. $\int \frac{dx}{1+\sin^2 x}$, let $t = \tan x$, then $\sin^2 x = \frac{\tan^2 x}{1+\tan^2 x} = \frac{t^2}{1+t^2}$ and $dt = (1+\tan^2 x)dx$

so, $dt = (1+t^2)dx$, then:

$\int \frac{dx}{1+\sin^2 x} = \int \frac{\frac{1}{1+t^2}}{1+\frac{t^2}{1+t^2}}dt = \int \frac{1}{1+2t^2}dt = \int \frac{1}{1+\left(\sqrt{2}t\right)^2}dt = \frac{1}{\sqrt{2}}\int \frac{\sqrt{2}}{1+\left(\sqrt{2}t\right)^2}dt$

$= \frac{1}{\sqrt{2}}\int \frac{\left(\sqrt{2}t\right)'}{1+\left(\sqrt{2}t\right)^2}dt = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}t\right) + c = \frac{1}{\sqrt{2}}\arctan\left(\sqrt{2}\tan x\right) + c$

250. $\int \frac{1}{e^x+2e^{-x}+2}dx = \int \frac{1}{e^x+2e^{-x}+2}\cdot\frac{e^x}{e^x}dx = \int \frac{e^x}{(e^x)^2+2e^x+2}dx$

Let $u = e^x$, then $du = e^x dx$, so:

$\int \frac{1}{e^x+2e^{-x}+2}dx = \int \frac{1}{u^2+2u+2}dx = \int \frac{1}{(u+1)^2+1}dx = \arctan(u+1)+c$

$= \arctan(e^x+1)+c$

251. $\int \frac{\sqrt{e^x-1}}{e^x}dx = \int \frac{\sqrt{e^x(1-e^{-x})}}{e^x}dx = \int \frac{e^{\frac{x}{2}}\sqrt{1-e^{-x}}}{e^x}dx = \int e^{-\frac{x}{2}}\sqrt{1-\left(e^{-\frac{x}{2}}\right)^2}\,dx$

Let $t = e^{-\frac{x}{2}}$, then $dt = -\frac{1}{2}e^{-\frac{x}{2}}$, so $\int \frac{\sqrt{e^x-1}}{e^x}dx = \int -2\sqrt{1-t^2}dt$

Let $t = \cos\theta$, $dt = -\sin\theta d\theta$, so $\int \frac{\sqrt{e^x-1}}{e^x}dx = \int 2\sin\theta\sqrt{1-\cos^2\theta}\,d\theta$

$= 2\int \sin\theta\sqrt{\sin^2\theta}\,d\theta = 2\int \sin^2\theta\,d\theta = \int(1-\cos 2\theta)d\theta = \theta - \frac{1}{2}\sin 2\theta + c$

$= \theta - \sin\theta\cos\theta + c = \arccos t - t\sqrt{1-t^2} + c = \arccos\left(e^{-\frac{x}{2}}\right) - \sqrt{e^{-x}-e^{-2x}} + c$

252. $\int \frac{1}{e^x\sqrt{e^{2x}-16}}dx = \int \frac{e^x}{e^{2x}\sqrt{e^{2x}-16}}dx$, let $u = e^x$, then $du = e^x dx$, so:

$\int \frac{1}{e^x\sqrt{e^{2x}-16}}dx = \int \frac{1}{u^2\sqrt{u^2-16}}du$, let $u = 4\sec\theta$, then $du = 4\sec\theta\tan\theta\,d\theta$, so:

$\int \frac{1}{e^x\sqrt{e^{2x}-16}}dx = \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta-16}}d\theta = \int \frac{4\sec\theta\tan\theta}{16\sec^2\theta\times 4\sqrt{\tan^2\theta}}d\theta$

$= \frac{1}{16}\int \frac{\tan\theta}{\sec\theta\tan\theta}d\theta = \frac{1}{16}\int \frac{1}{\sec\theta}d\theta = \frac{1}{16}\int \cos\theta\,d\theta = \frac{1}{16}\sin\theta + c = \frac{1}{16}\sin\left(\operatorname{arcsec}\left(\frac{u}{4}\right)\right) + c$

$= \frac{1}{16}\sin\left(\operatorname{arcsec}\left(\frac{e^x}{4}\right)\right) + c$

253. $\int \frac{1}{\sqrt{e^x-1}}dx$, let $u = \sqrt{e^x-1}$, then $u^2 = e^x - 1$, $e^x = u^2+1$, then $e^x dx = 2udu$ and so $dx = \frac{2u}{e^x}du = \frac{2u}{u^2+1}du$, then:

$\int \frac{1}{\sqrt{e^x-1}}dx = \int \frac{2u}{u^2+1}.\frac{1}{u}du = 2\int \frac{1}{u^2+1}du = 2\arctan u + c = 2\arctan\left(\sqrt{e^x-1}\right) + c$

254. $\int \frac{1}{\sqrt{x+1}+\sqrt{(x+1)^3}}dx$, let $u = \sqrt{x+1}$, then $u^2 = x+1$ and $2udu = dx$

With $\sqrt{(x+1)^3} = \sqrt{(u^2)^3} = u^3$, then:

$\int \frac{1}{\sqrt{x+1}+\sqrt{(x+1)^3}}dx = \int \frac{2u}{u+u^3}du = \int \frac{2u}{u(1+u^2)}du = 2\int \frac{1}{1+u^2}du = 2\arctan u + c$

$= 2\arctan\left(\sqrt{x+1}\right) + c$

255. $\int \frac{8x-3}{\sqrt{-4x^2+12x-5}}dx = -\int \frac{-8x+3}{\sqrt{-4x^2+12x-5-4+4}}dx = -\int \frac{-8x+12-9}{\sqrt{4-(2x-3)^2}}dx$

$= -\left[\int \frac{-8x+12}{\sqrt{4-(2x-3)^2}}dx - \int \frac{9}{\sqrt{4-(2x-3)^2}}dx\right] = -\left[\int \frac{-8x+12}{\sqrt{4-(2x-3)^2}}dx - \frac{9}{2}\int \frac{1}{\sqrt{1-\left(\frac{2x-3}{2}\right)^2}}dx\right]$

$= -\int \frac{-8x+12}{\sqrt{-4x^2+12x-5}}dx + \frac{9}{2}\int \frac{1}{\sqrt{1-\left(\frac{2x-3}{2}\right)^2}}dx$

Evaluating: $\int \frac{-8x+12}{\sqrt{-4x^2+12x-5}}dx$, let $u = -4x^2 + 12x - 5$, so $du = -8x + 12$, then**:**

$$\int \frac{-8x+12}{\sqrt{-4x^2+12x-5}}dx = \int \frac{du}{\sqrt{u}} = 2\sqrt{u} = 2\sqrt{-4x^2 + 12x - 5} + c_1$$

Evaluating: $\int \frac{1}{\sqrt{1-\left(\frac{2x-3}{2}\right)^2}}dx$, let $v = \frac{2x-3}{2}$, then $dv = dx$, so**:**

$$\int \frac{1}{\sqrt{1-\left(\frac{2x-3}{2}\right)^2}}dx = \int \frac{1}{\sqrt{1-v^2}}dv = \arcsin v + c_2 = \arcsin\left(\frac{2x-3}{2}\right) + c_2$$

Therefore, $\int \frac{8x-3}{\sqrt{-4x^2+12x-5}}dx = -2\sqrt{-4x^2 + 12x - 5} + \arcsin\left(\frac{2x-3}{2}\right) + c$

256. $\int \frac{\sqrt{x^3-2}}{x}dx$, let $x = \sqrt[3]{u}$, then $dx = \frac{1}{3}u^{-\frac{2}{3}}du$, then we get**:**

$$\int \frac{\sqrt{x^3-2}}{x}dx = \int \frac{\sqrt{u-2}}{\sqrt[3]{u}}\frac{1}{3}u^{-\frac{2}{3}}du = \frac{1}{3}\int \frac{\sqrt{u-2}}{u}du, \text{ let } t = \sqrt{u-2},\ dt = \frac{1}{2\sqrt{u-2}}du = \frac{1}{2t}du, \text{ so:}$$

$$\int \frac{\sqrt{x^3-2}}{x}dx = \frac{1}{3}\int \frac{t}{t^2+2}2tdt = \frac{2}{3}\int \frac{t^2}{t^2+2}dt = \frac{2}{3}\int \frac{t^2+2-2}{t^2+2}dt = \frac{2}{3}\int \left(1 - \frac{2}{t^2+2}\right)dt$$

$$= \frac{2}{3}\int \left(1 - \frac{2}{t^2+\left(\sqrt{2}\right)^2}\right)dt = \frac{2}{3}\left(t - \frac{2}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right) + c = \frac{2}{3}t - \frac{2\sqrt{2}}{3}\arctan\frac{t}{\sqrt{2}} + c$$

$$= \frac{2}{3}\sqrt{u-2} - \frac{2\sqrt{2}}{3}\arctan\frac{\sqrt{u-2}}{\sqrt{2}} + c = \frac{2}{3}\sqrt{x^3-2} - \frac{2\sqrt{2}}{3}\arctan\frac{\sqrt{x^3-2}}{\sqrt{2}} + c$$

257. $\int \frac{1}{x^n-x}dx = \int \frac{1}{x\left(x^{n-1}-1\right)}dx = \int \left(\frac{x^{n-2}}{x^{n-1}-1} - \frac{1}{x}\right)dx = \frac{1}{n-1}\int \frac{(n-1)x^{n-2}}{x^{n-1}-1}dx - \int \frac{1}{x}dx$

$$= \frac{1}{n-1}\int \frac{\left(x^{n-1}-1\right)'}{x^{n-1}-1}dx - \int \frac{1}{x}dx = \frac{1}{n-1}\ln|x^{n-1}| - \ln|x| + c$$

258. $\int \frac{1}{e^{2x}+e^{3x}}dx = \int \frac{1}{e^{2x}(1+e^x)}dx$, let $t = \frac{1+e^x}{e^x}$, so $e^x = \frac{1}{1-t}$ and $e^x dx = -\frac{1}{(t-1)^2}dt$,

so: $\boldsymbol{dx} = -\frac{1}{t-1}dt$, so $\int \frac{1}{e^{2x}+e^{3x}}dx = \int \frac{-\frac{1}{t-1}dt}{\frac{1}{(t-1)^2}\left[1+\frac{1}{t-1}\right]} = -\int \frac{(t-1)^2}{t}dt = \int \left(2 - t - \frac{1}{t}\right)dt$

$$= 2t - \frac{1}{2}t^2 - \ln t + c = 2(1 + e^{-x}) - \frac{1}{2}(1 + e^{-x})^2 - \ln(1 + e^{-x}) + c$$

259. $\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}dx$, let $x = \sin^2\theta$, then $dx = 2\sin\theta\cos\theta\, d\theta$, then we get**:**

$$\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}dx = \int \frac{1}{(1+\sqrt{x})\sqrt{x}\sqrt{1-x}}dx = \int \frac{1}{\left(1+\sqrt{\sin^2\theta}\right)\sqrt{\sin^2\theta}\sqrt{1-\sin^2\theta}}2\sin\theta\cos\theta\, d\theta$$

$$= \int \frac{1}{(1+\sin\theta)(\sin\theta)(\cos\theta)}2\sin\theta\cos\theta\, d\theta = \int \frac{2}{1+\sin\theta}d\theta = 2\int \frac{1-\sin\theta}{1-\sin^2\theta}d\theta$$

$= 2\int\left(\frac{1}{\cos^2\theta} - \frac{1}{\cos\theta} \times \frac{\sin\theta}{\cos\theta}\right)d\theta = 2\int(\sec^2\theta - \sec\theta\tan\theta)d\theta = 2\tan\theta - 2\sec\theta + c$

$= \frac{2(\sqrt{x}-1)}{\sqrt{1-x}} + c$

260. $\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx$, let $u = \sqrt{x+1}+1$, then $x+1 = (u-1)^2$, so $dx = 2(u-1)du$

also $\sqrt{x+1} - 1 = u - 1 - 1 = u - 2$, then:

$\int \frac{\sqrt{x+1}+1}{\sqrt{x+1}-1}dx = \int \frac{2(u-1)u}{u-2}du = 2\int \frac{u^2-u}{u-2}du = 2\int \frac{u^2-2u+u}{u-2}du = 2\int \frac{u(u-2)+u}{u-2}du$

$= 2\int\left(u + \frac{u}{u-2}\right)du = 2\int\left(u + \frac{u-2+2}{u-2}\right)du = 2\int\left(u + 1 + \frac{2}{u-2}\right)du$

$= u^2 + 2y + 4\ln|u-2| + c = \left(\sqrt{x+1}+1\right)^2 + 2\left(\sqrt{x+1}+1\right) + \ln\left|\sqrt{x+1}-1\right| + c$

$= x + 4 + 4\sqrt{x+1} + 4\ln\left|\sqrt{x+1}-1\right| + c$

261. $\int \frac{1}{\left(a^2+x^2\right)^2}dx$, let $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$, then:

$\int \frac{1}{\left(a^2+x^2\right)^2}dx = \int \frac{a\sec^2\theta}{\left[a^2\left(1+\tan^2\theta\right)\right]^2}d\,\theta = \frac{1}{a^3}\int \frac{\sec^2\theta}{\sec^4\theta}d\theta = \frac{1}{a^3}\int \cos^2\theta d\theta$

$= \frac{1}{2a^3}\int(1+\cos 2\theta)d\theta = \frac{1}{2a^3}\left(\sin\theta\cos\theta + \theta\right) + c$

$= \frac{1}{2a^3} + \frac{x}{\sqrt{a^2+x^2}} \cdot \frac{a}{\sqrt{a^2+x^2}} + \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + c = \frac{x}{2a^2\left(a^2+x^2\right)} + \frac{1}{2a^3}\arctan\left(\frac{x}{a}\right) + c$

262. $\int \sec x\sqrt{\sec x + \tan x}\,dx = \int \frac{\sec x(\tan x + \sec x)}{\sqrt{\sec x + \tan x}}dx$

Let $u = \sec x + \tan x$, then $du = \sec x\,(\tan x + \sec x)dx$, then we get:

$\int \sec x\sqrt{\sec x + \tan x}\,dx = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} + c = 2\sqrt{\sec x + \tan x} + c$

263. $\int \frac{\sin 2x}{\alpha\cos^2 x + \beta\sin^2 x}dx$, let $u = \alpha\cos^2 x + \beta\sin^2 x$, then:

$du = \{\alpha[2\cos x\,(-\sin x)] + \beta[2\sin x\,(\cos x)]\}dx$

$du = [-\alpha(2\sin x\cos x) + \beta(2\sin x\cos x)]dx$, with $\sin 2x = 2\sin x\cos x$

$du = (-\alpha\sin 2x + \beta\sin 2x)dx = (\beta - \alpha)\sin 2x\,dx$, so:

$\int \frac{\sin 2x}{\alpha\cos^2 x + \beta\sin^2 x}dx = \int \frac{1}{\beta-\alpha} \times \frac{1}{u}du = \frac{1}{\beta-\alpha}\ln|u| + c = \frac{1}{\beta-\alpha}\ln|\alpha\cos^2 x + \beta\sin^2 x| + c$

264. $\int \frac{dx}{4+5\cos^2 x} = \int \frac{1}{4+5\cos^2 x} \times \frac{\sec^2 x}{\sec^2 x}dx = \int \frac{\sec^2 x}{4\sec^2 x + 5}dx = \int \frac{\sec^2 x}{4\left(\tan^2 x + 1\right)+5}dx$

$= \int \frac{\sec^2 x}{4\tan^2 x + 9}dx = \frac{1}{2}\int \frac{2\sec^2 x}{(2\tan x)^2 + 3^2}dx$, let $t = 2\tan x$, then $dt = 2\sec^2 x\,dx$, so:

$\int \frac{dx}{4+5\cos^2 x} = \frac{1}{2}\int \frac{dt}{t^2+3^2} = \frac{1}{6}\arctan\left(\frac{1}{3}t\right) + c = \frac{1}{6}\arctan\left(\frac{2\tan x}{3}\right) + c$

265. $\int \frac{\cos^3 x + \cos^5 x}{\sin^2 x + \sin^4 x}dx = \int \frac{\left(\cos^2 x + \cos^4 x\right)\cos x}{\sin^2 x + \sin^4 x}dx = \int \frac{\left[1 - \sin^2 x + \left(1 - \sin^2 x\right)^2\right]\cos x}{\sin^2 x + \sin^4 x}dx$

Let $t = \sin x$, then $dt = \cos x\,dx$, so:

$$\int \frac{\cos^3 x+\cos^5 x}{\sin^2 x+\sin^4 x}dx = \int \frac{\left[1-t^2+\left(1-t^2\right)^2\right]}{t^2+t^4}dt = \int \frac{2-3t^2+t^4}{t^2+t^4}dt = \int \frac{\left(t^4+t^2\right)+\left(2-4t^2\right)}{t^2+t^4}dt$$

$$= \int dt + \int \frac{2-4t^2}{t^2+t^4}dt = \int dt + \int \left(\frac{2}{t^2} - \frac{6}{1+t^2}\right)dt = t - \frac{2}{t} - 6\arctan t + c$$

$$= \sin x - \frac{2}{\sin x} - 6\arctan(\sin x) + c$$

266. $$\int \frac{3+2\cos x}{(2+3\cos x)^2}dx = \int \frac{\frac{3+2\cos x}{\sin^2 x}}{\frac{(2+3\cos x)^2}{\sin^2 x}}dx = \int \frac{3\csc^2 x+2\csc x\cot x}{(2\csc x+3\cot x)^2}dx$$

Let $t = 2\csc x + 3\cot x$, then $dt = -(2\csc x\cot x - 3\csc^2 x)dx$, so:

$$\int \frac{3+2\cos x}{(2+3\cos x)^2}dx = -\int \frac{dt}{t^2} = \frac{1}{t} + \mathrm{c} = \frac{1}{2\csc x+3\cot x} + c = \frac{\sin x}{2+3\cos x} + c$$

267. $$\int \frac{dx}{9+16\cos^2 x} = \int \frac{\frac{1}{\cos^2 x}}{\frac{9+16\cos^2 x}{\cos^2 x}}dx = \int \frac{\sec^2 x}{\frac{9}{\cos^2 x}+16}dx = \int \frac{\sec^2 x}{9\left(1+\tan^2 x\right)+16}dx$$

$= \int \frac{\sec^2 x}{9\tan^2 x+25}dx$, let $t = \tan x$, then $dt = \sec^2 x\, dx$, so:

$$\int \frac{dx}{9+16\cos^2 x} = \int \frac{dt}{9t^2+25} = \int \frac{dt}{(3t)^2+5^2} = \frac{1}{5}.\frac{1}{3}\arctan\left(\frac{3}{5}t\right) + \mathrm{c} = \frac{1}{15}\arctan\left(\frac{3}{5}\tan x\right) + \mathrm{c}$$

268. $$\int \frac{1}{a^2+\tan^2 x}dx = \int \frac{1+\tan^2 x}{\left(a^2+\tan^2 x\right)\left(1+\tan^2 x\right)}dx = \int \frac{\sec^2 x}{\left(a^2+\tan^2 x\right)\left(1+\tan^2 x\right)}dx$$

Let $t = \tan x$, then $dt = \sec^2 x\, dx$, so:

$$\int \frac{1}{a^2+\tan^2 x}dx = \int \frac{dt}{\left(1+t^2\right)\left(a^2+t^2\right)}dx = \frac{1}{a^2-1}\int \frac{\left(a^2+t^2\right)-\left(1+t^2\right)}{\left(1+t^2\right)\left(a^2+t^2\right)}\mathrm{dt}$$

$$= \frac{1}{a^2-1}\int \frac{1}{1+t^2} - \frac{1}{a^2-1}\int \frac{1}{a^2+t^2} = \frac{1}{a^2-1}\left[\arctan t - \frac{1}{a}\arctan\left(\frac{t}{a}\right)\right] + \mathrm{c}$$

$$= \frac{1}{a^2-1}\left[\arctan(\tan x) - \frac{1}{a}\arctan\left(\frac{\tan x}{a}\right)\right] + c = \frac{1}{a^2-1}\left[x - \frac{1}{a}\arctan\left(\frac{\tan x}{a}\right)\right] + c$$

269. $\int \frac{2\cos 2x}{\left(1-\sin^2 2x\right)\left(2-\sin^2 2x\right)}dx$, let $u = \sin 2x$, then $du = 2\cos 2x\, dx$, so:

$$\int \frac{2\cos 2x}{\left(1-\sin^2 2x\right)\left(2-\sin^2 2x\right)}dx = \int \frac{du}{(1-u^2)(2-u^2)} = \int \frac{du}{(u^2-1)(u^2-2)} = \int \frac{(u^2-1)-(u^2-2)}{(u^2-1)(u^2-2)}du$$

$$= \int \frac{1}{u^2-2}du - \int \frac{1}{u^2-1}du = \frac{1}{2\sqrt{2}}\int \frac{(u+\sqrt{2})-(u-\sqrt{2})}{(u+\sqrt{2})(u-\sqrt{2})}du - \frac{1}{2}\int \frac{(u+1)-(u-1)}{(u+1)(u-1)}du$$

$$= \frac{1}{2\sqrt{2}}\int \left(\frac{1}{u-\sqrt{2}} - \frac{1}{u+\sqrt{2}}\right)du - \frac{1}{2}\int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du = \frac{1}{2\sqrt{2}}\ln\left|\frac{u-\sqrt{2}}{u+\sqrt{2}}\right| - \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + \mathrm{c}$$

$$= \frac{1}{2\sqrt{2}}\ln\left|\frac{\sin 2x-\sqrt{2}}{\sin 2x+\sqrt{2}}\right| - \frac{1}{2}\ln\left|\frac{\sin 2x-1}{\sin 2x+1}\right| + c$$

270. $\int \frac{\sqrt{x}}{1+x}dx$, let $u=\sqrt{x}$, then $du=\frac{1}{2\sqrt{x}}dx=\frac{1}{2u}dx$, so:

$$\int \frac{\sqrt{x}}{1+x}dx=\int \frac{u}{1+u^2}2udu=\int \frac{2u^2}{1+u^2}du=\int \frac{2(u^2+1)-2}{1+u^2}du=\int\left(2-\frac{2}{1+u^2}\right)du$$
$$=2u-2\arctan u+c=2\sqrt{x}-2\arctan(\sqrt{x})+c$$

271. $\int \frac{1-\sin x}{\sin x(1+\sin x)}dx=\int \frac{1}{\sin x(1+\sin x)}dx-\int \frac{\sin x}{\sin x(1+\sin x)}dx$

$$=\int \frac{1}{\sin x(1+\sin x)}dx-\int \frac{1}{1+\sin x}dx=\int \frac{1}{\sin x}dx-\int \frac{1}{1+\sin x}dx-\int \frac{1}{1+\sin x}dx$$
$$=\int \frac{1}{\sin x}dx-\int \frac{2}{1+\sin x}dx=\int \csc x\,dx-2\int \frac{1-\sin x}{(1+\sin x)(1-\sin x)}dx$$
$$=\int \csc x\,dx-2\int \frac{1-\sin x}{1-\sin^2 x}dx=\int \csc x\,dx-2\int \frac{1-\sin x}{\cos^2 x}dx$$
$$=\int \csc x\,dx-2\int \frac{1}{\cos^2 x}dx-2\int \frac{-\sin x}{\cos^2 x}dx=\int \csc x\,dx-2\int(\tan x)'dx-2\int \frac{(\cos x)'}{\cos^2 x}dx$$
$$=\ln|\csc x-\cot x|-2\tan x+\frac{2}{\cos x}+c=\ln|\csc x-\cot x|-2\tan x+2\sec x+c$$

Remark : $\frac{1}{\sin x(1+\sin x)}=\frac{1}{\sin x}-\frac{1}{1+\sin x}$ has been solved using the method of partial fractions, so that : $\frac{1}{u(u+1)}=\frac{a}{u}+\frac{b}{1+u}$, $a=\left[\frac{1}{1+u}\right]_{u=0}=1$ and $b=\left[\frac{1}{u}\right]_{u=-1}=-1$

272. $\int \frac{dx}{\sin^2 x+\tan^2 x}=\int \frac{dx}{\tan^2 x\left(\frac{\sin^2 x}{\tan^2 x}+1\right)}=\int \frac{dx}{\tan^2 x(\cos^2 x+1)}$

$$=\int \frac{\sec^2 x}{\tan^2 x(\cos^2 x+1)\sec^2 x}dx=\int \frac{\sec^2 x}{\tan^2 x(1+\sec^2 x)}dx=\int \frac{\sec^2 x}{\tan^2 x(1+1+\tan^2 x)}dx$$

$=\int \frac{\sec^2 x}{\tan^2 x(2+\tan^2 x)}dx$, let $u=\tan x$, then $du=\sec^2 x\,dx$, so:

$$\int \frac{dx}{\sin^2 x+\tan^2 x}=\int \frac{du}{u^2(u^2+2)}$$

Method of partial fractions : $\frac{1}{u^2(u^2+2)}=\frac{\mathrm{A}}{u}+\frac{\mathrm{B}}{u^2}+\frac{\mathrm{C}u+\mathrm{D}}{u^2+2}$, then

$1=(u^2+2)(\mathrm{A}u+\mathrm{B})+u^2(\mathrm{C}u+\mathrm{D})$, for $u=0$, $1=2\mathrm{B}$, then $\mathrm{B}=\frac{1}{2}$

Coefficient of u^3, $0=\mathrm{A}+\mathrm{C}$, so $\mathrm{C}=-\mathrm{A}$, coefficient of u^2, $0=\mathrm{B}+\mathrm{D}$, then $\mathrm{D}=-\frac{1}{2}$

Coefficient of u, $0=2\mathrm{A}$, so $\mathrm{A}=0$ and $\mathrm{C}=0$, then:

$$\int \frac{dx}{\sin^2 x+\tan^2 x}=\frac{1}{2}\int \frac{du}{u^2}-\frac{1}{2}\int \frac{1}{u^2+2}=-\frac{1}{2u}-\frac{1}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)+c$$
$$=-\frac{1}{2\tan x}-\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)+c$$

273. $\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{\sqrt{2}(\cos x+\sin x)}dx=\int \frac{\sqrt{\frac{\cos x.\sin x.\cos x}{\sin x}}-\sqrt{\frac{\sin x.\sin x.\cos x}{\cos x}}}{\sqrt{2\sin x\cos x}(\cos x+\sin x)}dx=\int \frac{\cos x-\sin x}{\sqrt{\sin 2x}(\cos x+\sin x)}dx$

$$=\int \frac{(\cos x-\sin x)(\cos x+\sin x)}{\sqrt{\sin 2x}(\cos x+\sin x)}dx=\int \frac{\cos^2 x-\sin^2 x}{\sqrt{\sin 2x}(\cos x+\sin x)^2}dx$$

$$= \int \frac{\cos^2 x - \sin^2 x}{\sqrt{\sin 2x}\left(\cos^2 x + \sin^2 x + 2\sin x\cos x\right)} dx = \int \frac{\cos 2x}{\sqrt{\sin 2x}(1+\sin 2x)} dx = \int \frac{\cos 2x}{\sqrt{\sin 2x}\left(1+\left(\sqrt{\sin 2x}\right)^2\right)} dx$$

Let $u = \sqrt{\sin 2x}$, then $du = \frac{\cos 2x}{\sqrt{\sin 2x}} dx$, then:

$$\int \frac{\sqrt{\cot x} - \sqrt{\tan x}}{\sqrt{2}(\cos x + \sin x)} dx = \int \frac{du}{1+u^2} = \arctan u + c = \arctan\left(\sqrt{\sin 2x}\right) + c$$

274. $$\int \frac{1}{\cos(x+a)\cos(x+b)} dx = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x+a)\cos(x+b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(a-b+x-x)}{\cos(x+a)\cos(x+b)} dx = \frac{1}{\sin(a-b)} \int \frac{\sin[(x+a)-(x+b)]}{\cos(x+a)\cos(x+b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \frac{\sin(x+a)\cos(x+b) - \cos(x+a)\sin(x+b)}{\cos(x+a)\cos(x+b)} dx$$

$$= \frac{1}{\sin(a-b)} \int \left[\frac{\sin(x+a)}{\cos(x+a)} - \frac{\sin(x+b)}{\cos(x+b)}\right] dx = \frac{1}{\sin(a-b)} \int [\tan(x+a) - \tan(x+b)]\, dx$$

$$= \frac{1}{\sin(a-b)} [\ln|\sec(x+a)| - \ln|\sec(x+b)|] + c = \frac{1}{\sin(a-b)} \ln\left|\frac{\sec(x+a)}{\sec(x+b)}\right| + c$$

$$= \frac{1}{\sin(a-b)} \ln\left|\frac{\cos(x+b)}{\cos(x+a)}\right| + c$$

275. $$\int \frac{1}{\sqrt{\sin^3 x \sin(x+a)}} dx = \int \frac{1}{\sqrt{\sin^3 x(\sin x\cos a + \cos x \sin a)}} dx$$

$$= \int \frac{1}{\sqrt{\sin^4 x\left(\frac{\sin x\cos a + \cos x\sin a}{\sin x}\right)}} dx = \int \frac{1}{\sqrt{\sin^4 x(\cos a + \cot x \sin a)}} dx$$

$$= \int \frac{1}{\sin^2 x\sqrt{(\cos a + \cot x\sin a)}} dx = \int \frac{\csc^2 x}{\sqrt{(\cos a + \cot x \sin a)}} dx$$

Let $t = \cos a + \sin a \cot x$, then $dt = -\sin a \csc^2 x\, dx$, $\csc^2 x\, dx = -\frac{1}{\sin a} dt$, then:

$$\int \frac{1}{\sqrt{\sin^3 x\sin(x+a)}} dx = -\frac{1}{\sin a}\int \frac{1}{\sqrt{t}} dt = -\frac{2\sqrt{t}}{\sin a} + c = -\frac{2\sqrt{\cos a + \cot x \sin a}}{\sin a} + c$$

$$= -\frac{2\sqrt{\cos a + \frac{\cos x}{\sin x}\sin a}}{\sin a} + c = -\frac{2}{\sin a}\sqrt{\frac{\sin x\cos a + \cos x \sin a}{\sin x}} + c = -\frac{2}{\sin a}\sqrt{\frac{\sin(x+a)}{\sin x}} + c$$

276. $$\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \int \frac{dx}{\sqrt[4]{\left(\frac{x-1}{x+2}\right)(x-1)^2(x+2)^6}}$$. Let $y = \sqrt[4]{\frac{x-1}{x+2}} = \left(\frac{x-1}{x+2}\right)^{\frac{1}{4}}$, then

$$dy = \frac{1}{4}\left(\frac{x-1}{x+2}\right)^{-\frac{3}{4}} \frac{3dx}{(x+2)^2} = \frac{\frac{3}{4}}{(x+2)^2\sqrt[4]{\left(\frac{x-1}{x+2}\right)^3}} dx \Longrightarrow dx = \frac{4}{3}\sqrt[4]{(x-1)^3(x+2)^5}\, dy$$, so:

$$\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \frac{4}{3}\int dy = \frac{4}{3}y + c = \frac{4}{3}\sqrt[4]{\frac{x-1}{x+2}} + c$$

277. $\int \frac{\sqrt{x^2+1}}{x^4}dx$, let $x = \tan t$, then $dx = \sec^2 t\,dt$, so:

$$\int \frac{\sqrt{x^2+1}}{x^4}dx = \int \frac{\sqrt{\tan^2 t+1}}{\tan^4 t}\sec^2 t\,dt = \int \frac{\sqrt{\sec^2 t}}{\tan^4 t}\sec^2 t\,dt = \int \frac{\sec^3 t}{\tan^4 t}dt = \int \frac{1}{\cos^3 t.\frac{\sin^4 t}{\cos^4 t}}dt$$

$= \int \frac{\cos t}{\sin^4 t}dt$, let $u = \sin t$, then $du = \cos t\,dt$, so:

$$\int \frac{\sqrt{x^2+1}}{x^4}dx = \int \frac{du}{u^4} = -\frac{1}{3u^3} + c = -\frac{1}{3\sin^3 t} + c = -\frac{1}{3\sin^3(\arctan x)} + c$$

278. $\int \frac{\sqrt{x^2-1}}{x}dx$, let $x = \sec\theta$, then $dx = \sec\theta\tan\theta\,d\theta$, so:

$$\int \frac{\sqrt{x^2-1}}{x}dx = \int \frac{\sqrt{\sec^2\theta-1}}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int \tan\theta\tan\theta\,d\theta = \int \tan^2\theta\,d\theta = \int(\sec^2\theta - 1)d\theta$$

$$= \tan\theta - \theta + c = \sqrt{\sec^2\theta - 1} - \theta + c = \sqrt{x^2-1} - \arctan\sqrt{x^2-1} + c$$

279. **First Method:** $\int \frac{\sqrt{x^2+1}}{x}dx$, let $u = \sqrt{x^2+1}$, then $u^2 = x^2 + 1$ and $2udu = 2xdx$, then $dx = \frac{u}{x}du = \frac{u}{\sqrt{u^2-1}}du$, so:

$$\int \frac{\sqrt{x^2+1}}{x}dx = \int \frac{u}{\sqrt{u^2-1}}\frac{u}{\sqrt{u^2-1}}du = \int \frac{u^2}{u^2-1}du = \int \frac{u^2-1+1}{u^2-1}du = \int \left[1 + \frac{1}{(u-1)(u+1)}\right]du$$

$$= \frac{1}{2}\int \left[2 + \frac{(u+1)-(u-1)}{(u-1)(u+1)}\right]du = \frac{1}{2}\int\left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du = u + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + c$$

$$= u + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + c = \sqrt{x^2+1} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + c$$

First Method: $\int \frac{\sqrt{x^2+1}}{x}dx$, let $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$, so:

$$\int \frac{\sqrt{x^2+1}}{x}dx = \int \frac{\sqrt{1+\tan^2\theta}}{\tan\theta}\sec^2\theta\,d\theta = \int\int \frac{\sqrt{\sec^2\theta}}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec^3\theta}{\tan\theta}d\theta$$

$$= \int \frac{1}{\cos^3\theta}\cdot\frac{\cos\theta}{\sin\theta}d\theta = \int \frac{1}{\cos^2\theta}\cdot\frac{1}{\sin\theta}d\theta = \int \sec^2\theta\csc\theta\,d\theta = \int(1 + \tan^2\theta)\csc\theta\,d\theta$$

$$= \int \csc\theta\,d\theta + \int \tan\theta\sec\theta\,d\theta = -\ln|\csc\theta + \cot\theta| + \sec\theta + c$$

$$= \frac{1}{2}\ln\left|\frac{1-\cos\theta}{1+\cos\theta}\right| + \sqrt{1+\tan^2\theta} + c = \frac{1}{2}\ln\left|\frac{1-\frac{1}{\sqrt{1+\tan^2\theta}}}{1+\frac{1}{\sqrt{1+\tan^2\theta}}}\right| + \sqrt{1+\tan^2\theta} + c$$

$$= \sqrt{x^2+1} + \frac{1}{2}\ln\left|\frac{\sqrt{x^2+1}-1}{\sqrt{x^2+1}+1}\right| + c$$

280. $\int \frac{1}{(x^2-2x+4)^{\frac{3}{2}}}dx = \int \frac{1}{(x^2-2x+1+3)^{\frac{3}{2}}}dx = \int \frac{1}{\left[(x-1)^2+3\right]^{\frac{3}{2}}}dx$

Let $x-1=\sqrt{3}\tan t$, then $dx=\sqrt{3}\sec^2 t\,dt$, so:

$\int \frac{1}{(x^2-2x+4)^{\frac{3}{2}}}dx = \int \frac{\sqrt{3}\sec^2 t}{(3\tan^2 t+3)^{\frac{3}{2}}}dt = \int \frac{\sqrt{3}\sec^2 t}{3^{\frac{3}{2}}\left(1\tan^2 t+1\right)^{\frac{3}{2}}}dt = \int \frac{\sqrt{3}\sec^2 t}{3\sqrt{3}(\sec^2 t)^{\frac{3}{2}}}dt$

$= \frac{1}{3}\int \frac{\sec^2 t}{\sec^3 t}dt = \frac{1}{3}\int \frac{1}{\sec t}dt = \frac{1}{3}\int \cos t\,dt = \frac{1}{3}\sin t + c$, with $\tan x = \frac{x-1}{\sqrt{3}}$, therefore:

$\int \frac{1}{(x^2-2x+4)^{\frac{3}{2}}}dx = \frac{1}{3}\sqrt{\frac{\left(\frac{x-1}{\sqrt{3}}\right)^2}{1+\left(\frac{x-1}{\sqrt{3}}\right)^2}} + c = \frac{1}{3}\frac{(x-1)}{\sqrt{x^2-2x+4}} + c$

281. $\int \sqrt{a+\sqrt{b+\sqrt{x}}}\,dx$, let $u=\sqrt{a+\sqrt{b+\sqrt{x}}}$, then $u^2 = a+\sqrt{b+\sqrt{x}}$ and $(u^2-a)^2 = b+\sqrt{x}$, then $\sqrt{x}=(u^2-a)^2-b$, so $x=[(u^2-a)^2-b]^2$, so:

$dx = 2[(u^2-a)^2-b].2(u^2-a)2udu = (8u^3-8au)\left(u^4-2au^2+(a^2-b)\right)du$, then:

$\int \sqrt{a+\sqrt{b+\sqrt{x}}}\,dx = \int (8u^4-8au^2)\left(u^4-2au^2+(a^2-b)\right)du$

$= \int [8u^8+8au^6(2a-3)+8(a^2-b)u^4-8a(a^2-b)u^2]du$

$= \frac{8}{9}u^9+\frac{8}{7}u^7a(2a-3)+\frac{8}{5}u^5(a^2-b)-\frac{8}{3}u^3a(a^2-b)+c$, where $u=\sqrt{a+\sqrt{b+\sqrt{x}}}$

282. $\int \sqrt{\frac{1-x}{1+x}}dx = \int \frac{1-x}{\sqrt{1-x^2}}dx$, let $x=\sin u$, then $dx=\cos u\,du$, so:

$\int \sqrt{\frac{1-x}{1+x}}dx = \int \frac{(1-\sin u)\cos u}{\sqrt{1-\sin^2 u}}du = \int \frac{(1-\sin u)\cos u}{\cos u}du = \int (1-\sin u)du$

$= u+\cos u + c = \arcsin x + \cos(\arcsin x) + c = \arcsin x + \sqrt{1-x^2}+c$

283. $\int \frac{x^2-1}{x\sqrt{x^4+3x^2+1}}dx = \int \frac{1-\frac{1}{x^2}}{\sqrt{x^2+\frac{1}{x^2}+2+1}}dx = \int \frac{1-\frac{1}{x^2}}{\sqrt{\left(x+\frac{1}{x}\right)^2+1}}dx,$

let $t=x+\frac{1}{x}$, then $dt=\left(1-\frac{1}{x^2}\right)dx$, so:

$\int \frac{x^2-1}{x\sqrt{x^4+3x^2+1}}dx \int \frac{dt}{\sqrt{t^2+1}}dx = \ln\left|t+\sqrt{t^2+1}\right|+c = \ln\left|x+\frac{1}{x}+\sqrt{\left(x+\frac{1}{x}\right)^2+1}\right|+c$

284. $\int \frac{1}{x^2\sqrt{x^2+4}}dx$, let $u=2\tan\theta$, then $du=2\sec^2\theta\,d\theta$, so:

$\int \frac{1}{x^2\sqrt{x^2+4}}dx = \int \frac{2\sec^2\theta}{4\tan^2\theta\sqrt{4\left(\tan^2\theta+1\right)}}d\theta = \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sec\theta}d\theta = \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}d\theta$

$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}d\theta = \frac{1}{4}\int \frac{(\sin\theta)'}{\sin^2\theta}d\theta = -\frac{1}{4}\left(\frac{1}{\sin\theta}\right)+c = -\frac{\sqrt{x^2+4}}{4x}+c$

285. $\int \frac{x^3}{(4x^2+9)\sqrt{4x^2+9}}dx = \int \frac{x^3}{(4x^2+9)^{\frac{3}{2}}}dx$, let $x = \frac{3}{2}\tan\theta$, $dx = \frac{3}{2}\sec^2\theta\, d\theta$, so:

$$\int \frac{x^3}{(4x^2+9)\sqrt{4x^2+9}}dx = \int \frac{\frac{27}{8}\tan^3\theta}{\left(9\tan^2\theta+9\right)^{\frac{3}{2}}}\frac{3}{2}\sec^2\theta\, d\theta = \int \frac{\frac{27}{8}\tan^3\theta}{27\sec^3\theta}\frac{3}{2}\sec^2\theta\, d\theta = \frac{3}{16}\int \frac{\tan^3\theta}{\sec\theta}d\theta$$

$= \frac{3}{16}\int \frac{\sin^3\theta}{\cos^2\theta}d\theta = \frac{3}{16}\int \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\, d\theta$, let $u = \cos\theta$, $du = -\sin\theta\, d\theta$, so:

$$\int \frac{x^3}{(4x^2+9)\sqrt{4x^2+9}}dx = -\frac{3}{16}\int \left(\frac{1-u^2}{u^2}\right)du = -\frac{3}{16}\int \left(\frac{1}{u^2}-1\right) = \frac{3}{16}\left(u+\frac{1}{u}\right)+c$$

$= \frac{3}{16}\left(\cos\theta+\frac{1}{\cos\theta}\right)+c$, but $x = \frac{3}{2}\tan\theta$, then $\tan\theta = \frac{2}{3}x$, $\cos\theta = \frac{1}{\sqrt{1+\left(\frac{2}{3}x\right)^2}} = \frac{3}{\sqrt{4x^2+9}}$

$$\int \frac{x^3}{(4x^2+9)\sqrt{4x^2+9}}dx = \frac{3}{16}\left(\frac{3}{\sqrt{4x^2+9}}+\frac{\sqrt{4x^2+9}}{3}\right)+c$$

286. $\int \frac{dx}{(x^2-4)\sqrt{x}}$, let $t = \sqrt{x}$ and $x = t^2$, so $dx = 2tdt$, then:

$$\int \frac{dx}{(x^2-4)\sqrt{x}} = 2\int \frac{t}{t(t^4-4)}dt = 2\int \frac{1}{t^4-4}dt = 2\int \frac{1}{(t^2-2)(t^2+2)}dt$$

Method of partial fractions : $\frac{1}{(z-2)(z+2)} = \frac{\text{A}}{z+2}+\frac{\text{B}}{z-2}$, $1 = (z-2)\text{A}+(z+2)\text{B}$

For $z = 2$, $1 = 4\text{B}$, so $\text{B} = \frac{1}{4}$ and $z = -2$, $1 = -4\text{A}$, so $\text{A} = -\frac{1}{4}$

Then : $\frac{1}{(z-2)(z+2)} = \frac{-\frac{1}{4}}{z+2}+\frac{\frac{1}{4}}{z-2}$, put $z = t^2$, we get : $\frac{1}{(t^2-2)(t^2+2)} = \frac{-\frac{1}{4}}{t^2+2}+\frac{\frac{1}{4}}{t^2-2}$, so:

$$\int \frac{dx}{(x^2-4)\sqrt{x}} = \frac{1}{2}\int \frac{1}{t^2-2}dt - \frac{1}{2}\int \frac{1}{t^2+2}dt = \frac{1}{4\sqrt{2}}\int \frac{(t+\sqrt{2})-(t-\sqrt{2})}{(t-\sqrt{2})(t+\sqrt{2})}dt - \frac{1}{2}\int \frac{1}{t^2+\left(\sqrt{2}\right)^2}dt$$

$$= \frac{1}{4\sqrt{2}}\int \left(\frac{1}{t-\sqrt{2}}-\frac{1}{t+\sqrt{2}}\right)dt - \frac{1}{2}\int \frac{1}{t^2+\left(\sqrt{2}\right)^2}dt = \frac{1}{4\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| - \frac{1}{2\sqrt{2}}\arctan\left(\frac{t}{\sqrt{2}}\right)+c$$

$$= \frac{1}{4\sqrt{2}}\ln\left|\frac{\sqrt{x}-\sqrt{2}}{\sqrt{x}+\sqrt{2}}\right| - \frac{1}{2\sqrt{2}}\arctan\left(\sqrt{\frac{x}{2}}\right)+c$$

287. $\int \sqrt{\frac{x}{(1-x)^3}}dx = \int \sqrt{\frac{x}{(1-x)^2(1-x)}}dx = \int \frac{1}{1-x}\sqrt{\frac{x}{1-x}}dx$

Let $t = \sqrt{\frac{x}{1-x}}$, then $t^2 = \frac{x}{1-x}$ and so $x = \frac{t^2}{1+t^2}$, then $dx = \frac{2t}{\left(1+t^2\right)^2}dt$, so:

$$\int \sqrt{\frac{x}{(1-x)^3}}dx = \int \frac{1}{1-\frac{t^2}{1+t^2}}\times t\times \frac{2t}{\left(1+t^2\right)^2}dt = \int \frac{2t^2}{1+t^2}dt = 2\int \frac{1+t^2-1}{1+t^2}dt$$

$$= 2\int \left(1-\frac{1}{1+t^2}\right)dt = 2(t-\arctan t)+c = 2\sqrt{\frac{x}{1-x}} - 2\arctan\left(\sqrt{\frac{x}{1-x}}\right)+c$$

288. $\int \frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx$, let $u=\sin x+\cos x$, then $du=(\cos x-\sin x)dx$

and $\sqrt{\sin 2x}=\sqrt{2\sin x\cos x}=\sqrt{(\cos x+\sin x)^2-1}=\sqrt{u^2-1}$, then:

$\int \frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx=\int \frac{-1}{\sqrt{u^2-1}}du$, let $u=\sec\theta$, then $du=\sec\theta\tan\theta\,d\theta$, then we get:

$\int \frac{\sin x-\cos x}{\sqrt{\sin 2x}}dx=-\int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta-1}}d\theta=-\int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta=-\int \sec\theta\,d\theta=$

$-\ln|\sec\theta+\tan\theta|+c=-\ln\left|u+\sqrt{u^2-1}\right|+c=-\ln\left|\sin x+\cos x+\sqrt{\sin 2x}\right|+c$

289. $\int \frac{\cos(x+\ln a)}{\cos(x+\ln b)}dx$, let $t=x+\ln b$, then $dx=dt$, so:

$\int \frac{\cos(x+\ln a)}{\cos(x+\ln b)}dx=\int \frac{\cos(t-\ln b+\ln a)}{\cos t}dt=\int \frac{\cos\left(t+\ln\left(\frac{a}{b}\right)\right)}{\cos t}dt$

$=\int \frac{\cos t\cos\left(\ln\left|\frac{a}{b}\right|\right)-\sin t\sin\left(\ln\left|\frac{a}{b}\right|\right)}{\cos t}dt=\int \left[\cos\left(\ln\left|\frac{a}{b}\right|\right)-\tan t\sin\left(\ln\left|\frac{a}{b}\right|\right)\right]dt$

$=\cos\left(\ln\left|\frac{a}{b}\right|\right)t+\sin\left(\ln\left|\frac{a}{b}\right|\right)\ln|\cos t|+c$

$=\cos\left(\ln\left|\frac{a}{b}\right|\right)(x+\ln b)+\sin\left(\ln\left|\frac{a}{b}\right|\right)\ln|\cos(x+\ln b)|+c$

290. $\int \frac{\sqrt{1+\ln x}}{x\ln x}dx$, let $\ln x=\tan^2 y$, so , $\frac{dx}{x}=2\tan y\sec^2 y\,dy$, then:

$\int \frac{\sqrt{1+\ln x}}{x\ln x}dx=\int \frac{\sqrt{1+\tan^2 y}}{\tan^2 y}.2\tan y\sec^2 y\,dy=\int \frac{\sec y}{\tan^2 y}.2\tan y\sec^2 y\,dy$

$=2\int \frac{\tan y\sec y\left(1+\tan^2 y\right)}{\tan^2 y}dy=2\int \frac{\tan y\sec y}{\tan^2 y}dy+2\int \frac{\tan y\sec y\tan^2 y}{\tan^2 y}dy$

$=2\int \frac{\sec y}{\tan y}dy+2\int \tan y\sec y\,dy=2\int \csc y\,dy+2\int \tan y\sec y\,dy$

$=-2\ln|\csc y+\cot y|+2\sec y+c=-2\ln\left|\frac{\sqrt{1+\ln x}+1}{\sqrt{\ln x}}\right|+2\sqrt{1+\ln x}+c$

291. $\int e^{4x}\sqrt{1+e^{2x}}dx=\int e^{3x}\sqrt{1+e^{2x}}e^x dx=\int (e^x)^3\sqrt{1+(e^x)^2}e^x dx$

Let $u=e^x$, then $du=e^x dx$, so:

$\int e^{4x}\sqrt{1+e^{2x}}dx=\int u^3\sqrt{1+u^2}dx$, let $u=\tan\theta$, then $du=\sec^2\theta\,d\theta$, so:

$\int e^{4x}\sqrt{1+e^{2x}}dx=\int \tan^3\theta\sqrt{1+\tan^2\theta}\sec^2\theta\,d\theta=\int \tan^3\theta\sqrt{\sec^2\theta}\sec^2\theta\,d\theta$

$=\int \tan^3\theta\sec^3\theta\,d\theta=\int \tan^2\theta\sec^2\theta\,(\sec\theta\tan\theta)d\theta=\int (\sec^2\theta-1)\sec^2\theta\,(\sec\theta\tan\theta)d\,\theta$

$=\int (\sec^4\theta-\sec^2\theta)(\sec\theta\tan\theta)d\theta=\int (\sec^4\theta-\sec^2\theta)(\sec\theta)'d\theta$

$=\frac{1}{5}\sec^5\theta-\frac{1}{3}\sec^3\theta+c$, but $u=\tan\theta$, so $\sec\theta=\sqrt{1+\tan^2\theta}=\sqrt{1+u^2}$, then:

$\int e^{4x}\sqrt{1+e^{2x}}dx=\frac{1}{5}(1+u^2)^{\frac{5}{2}}-\frac{1}{3}(1+u^2)^{\frac{3}{2}}+c=\frac{1}{5}(1+e^{2x})^{\frac{5}{2}}-\frac{1}{3}(1+e^{2x})^{\frac{3}{2}}+c$

292. $\int \frac{1+\ln x}{\sqrt{1+x^x}}dx$, let $u^2=1+x^x$, then $2udu=x^x(1+\ln x)dx$,

so $(1+\ln x)dx=\frac{2u}{u^2-1}du$, then:

$$\int \frac{1+\ln x}{\sqrt{1+x^x}}dx = \int \frac{\frac{2u}{u^2-1}}{u}du = \int \frac{2}{u^2-1}du = \int \left[\frac{(u+1)-(u-1)}{(u+1)(u-1)}\right]du = \int \left(\frac{1}{u-1} - \frac{1}{u+1}\right)du$$

$$= \ln|u-1| - \ln|u+1| + c = \ln\left|\frac{u-1}{u+1}\right| + c = \ln\left|\frac{\sqrt{x^x+1}-1}{\sqrt{x^x+1}+1}\right| + c$$

293. $$\int \frac{1}{e^x\sqrt{\sinh 2x}}dx = \int \frac{1}{e^x\sqrt{\frac{e^{2x}-e^{-2x}}{2}}}dx = \int \frac{\sqrt{2}}{e^x\sqrt{e^{2x}-e^{-2x}}}dx = \int \frac{\sqrt{2}}{e^x\sqrt{e^{2x}(1-e^{-4x})}}dx$$

$= \int \frac{\sqrt{2}}{e^{2x}\sqrt{1-(e^{-2x})^2}}dx = \int \frac{\sqrt{2}e^{-2x}}{\sqrt{1-(e^{-2x})^2}}dx$, let $u = e^{-2x}$, then $du = -e^{-2x}dx$, so:

$$\int \frac{1}{e^x\sqrt{\sinh 2x}}dx = -\sqrt{2}\int \frac{du}{\sqrt{1-u^2}}dx = -\sqrt{2}\arcsin u + c = -\sqrt{2}\arcsin(e^{-2x}) + c$$

294. $$\int \frac{x^2-2}{(x^4+5x^2+4)\arctan\left(\frac{x^2+2}{x}\right)}dx = \int \frac{x^2-2}{\left[(x^2+2)^2+x^2\right]\arctan\left(\frac{x^2+2}{x}\right)}dx$$

$$= \int \frac{x^2-2}{(x^2+2)^2+x^2}.\frac{1}{\arctan\left(\frac{x^2+2}{x}\right)}dx = \int \frac{1-\frac{2}{x^2}}{\left(\frac{x^2+2}{x}\right)^2+1}.\frac{1}{\arctan\left(\frac{x^2+2}{x}\right)}dx$$

Let $u = \arctan\left(\frac{x^2+2}{x}\right)$, then $du = \frac{1-\frac{2}{x^2}}{\left(\frac{x^2+2}{x}\right)^2+1}$, so $\int \frac{x^2-2}{(x^4+5x^2+4)\arctan\left(\frac{x^2+2}{x}\right)}dx = \int \frac{1}{u}du$

$$= \ln|u| + c = \ln\left|\arctan\left(\frac{x^2+2}{x}\right)\right| + c$$

295. $\int \frac{x^{-\frac{1}{2}}}{1+x^{\frac{1}{3}}}dx = \int \frac{x^{-\frac{3}{6}}}{1+x^{\frac{2}{6}}}dx$, let $u = x^{\frac{1}{6}}$, $du = \frac{1}{6}x^{-\frac{5}{6}}dx = \frac{1}{6}u^{-5}dx$, so $dx = 6u^5du$, so:

$$\int \frac{x^{-\frac{1}{2}}}{1+x^{\frac{1}{3}}}dx = \int \frac{u^{-3}.6u^5}{1+u^2}dx = 6\int \frac{u^2}{1+u^2}dx = 6\int \frac{u^2+1-1}{1+u^2}dx = 6\int \left(1 - \frac{1}{1+u^2}\right)dx$$

$$= 6u - 6\arctan u + c = 6x^{\frac{1}{6}} - 6\arctan\left(x^{\frac{1}{6}}\right) + c$$

296. $\int \frac{8}{3\cos 2x+1}dx$, let $t = \tan x$, so $dx = \frac{dt}{1+t^2}$ and $\cos 2x = \frac{1-t^2}{1+t^2}$, then we get

$$\int \frac{8}{3\cos 2x+1}dx = \int \frac{8(1+t^2)}{(1+t^2)(3-3t^2+1+t^2)}dt = \int \frac{8}{4-2t^2}dt = 4\int \frac{1}{2-t^2}dt$$

$$= 4\int \frac{1}{(\sqrt{2}-t)(\sqrt{2}+t)}dt = \sqrt{2}\left(\int \frac{1}{\sqrt{2}-t}dt + \int \frac{1}{\sqrt{2}+t}dt\right) = \sqrt{2}\left(-\ln|\sqrt{2}-t| + \ln|\sqrt{2}+t|\right) + c$$

$$= \sqrt{2}\ln\left|\frac{t+\sqrt{2}}{t-\sqrt{2}}\right| + c = \sqrt{2}\ln\left|\frac{\tan x+\sqrt{2}}{\tan x-\sqrt{2}}\right| + c$$

Remark : $\frac{1}{(\sqrt{2}-t)(\sqrt{2}+t)} = \frac{A}{\sqrt{2}-t} + \frac{B}{\sqrt{2}+t}$ (method of partial fractions) $A = \left[\frac{1}{\sqrt{2}+t}\right]_{t=\sqrt{2}} = \frac{1}{2\sqrt{2}}$ and $B = \left[\frac{1}{\sqrt{2}-t}\right]_{t=-\sqrt{2}} = \frac{1}{2\sqrt{2}}$.

297. $\int \frac{1}{5+4\sqrt{x}+x}dx = \int \frac{1}{(\sqrt{x}+2)^2+1}dx$, let $u = \sqrt{x}+2$, then $du = \frac{1}{2\sqrt{x}}dx$, so

$dx = 2\sqrt{x}du = 2(u-2)du$, then:

$\int \frac{1}{5+4\sqrt{x}+x}dx = \int \frac{2u-4}{u^2+1}du = \int \left(\frac{2u}{u^2+1} - \frac{4}{u^2+1}\right)du = \ln(u^2+1) - 4\arctan u + c$

$= \ln\left(5+4\sqrt{x}+x\right) - 4\arctan\left(\sqrt{x}+2\right) + c$

298. $\int \frac{e^{\tan x}}{1-\sin^2 x}dx = \int \frac{e^{\tan x}}{\cos^2 x}dx = \int \sec^2 x\, e^{\tan x}dx$, let $u = \tan x$, then $du = \sec^2 x\, dx$,

so: $\int \frac{e^{\tan x}}{1-\sin^2 x}dx = \int e^u du = e^u + c = e^{\tan x} + c$

299. $\int \frac{\sin x+\cos x}{9+16\sin 2x}dx$, let $t = \sin x - \cos x$, then $dt = (\sin x + \cos x)dx$

$(\sin x - \cos x)^2 = t^2$, $\sin^2 x + \cos^2 x - 2\sin x\cos x = t^2$, $1 - \sin 2x = t^2$, $\sin 2x = 1 - t^2$

$\int \frac{\sin x+\cos x}{9+16\sin 2x}dx = \int \frac{dt}{9+16\left(1-t^2\right)} = \int \frac{dt}{25-16t^2} = \frac{1}{16}\int \frac{dt}{\left(\frac{5}{4}\right)^2 - t^2} = \frac{1}{16}\int \frac{dt}{\left(\frac{5}{4}-t\right)\left(\frac{5}{4}+t\right)}$

$= \frac{1}{16}\times\frac{2}{5}\int\left(\frac{1}{\frac{5}{4}+t} + \frac{1}{\frac{5}{4}-t}\right)dt = \frac{1}{40}\left(\ln\left|\frac{5}{4}+t\right| - \ln\left|\frac{5}{4}-t\right|\right) + c = \frac{1}{40}\ln\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right| + c$

$= \frac{1}{40}\ln\left|\frac{\frac{5}{4}+t}{\frac{5}{4}-t}\right| + c = \frac{1}{40}\ln\left|\frac{5+4t}{5+4t}\right| + c = \frac{1}{40}\ln\left|\frac{5+4(\sin x-\cos x)}{5+4(\sin x-\cos x)}\right| + c$

Remark : $\frac{1}{\left(\frac{5}{4}-t\right)\left(\frac{5}{4}+t\right)} = \frac{\mathrm{A}}{\frac{5}{4}+t} + \frac{\mathrm{B}}{\frac{5}{4}-t}$ (method of partial fractions)

$1 = \mathrm{A}\left(\frac{5}{4}-\mathrm{t}\right) + \mathrm{B}\left(\frac{5}{4}+\mathrm{t}\right)$, for $t = -\frac{5}{4}$, $\mathrm{A} = \frac{2}{5}$ and for $t = \frac{5}{4}$, $\mathrm{B} = \frac{2}{5}$.

300. $\int x^{\frac{x}{\ln x}}dx$, let $u = \ln x$, $x = e^u$, so $dx = e^u du$, so:

$\int x^{\frac{x}{\ln x}}dx = \int (e^u)^{\frac{e^u}{u}} e^u du = \int e^{e^u} e^u du$

Let $v = e^u$, $dv = e^u du = v du$, so $\int x^{\frac{x}{\ln x}}dx = \int e^v dv = e^v + c = e^{e^u} + c = e^{e^{\ln x}} + c$

$\int x^{\frac{x}{\ln x}}dx = e^x + c$

301. $\int 2^{\ln x}dx$, let $u = \ln x$, so $x = e^u$, then $dx = e^u du$, so:

$\int 2^{\ln x}dx = \int 2^u e^u du = \int (2e)^u du = \frac{(2e)^u}{\ln(2e)} + c = \frac{(2e)^{\ln x}}{\ln 2+\ln e} + c = \frac{2^{\ln x}e^{\ln x}}{\ln 2+1} + c$

$\int 2^{\ln x}dx = \frac{x2^{\ln x}}{\ln 2+1} + c$

302. $\int a^x.a^{a^x}.a^{a^{a^x}}dx$, let $u = a^x$, then $du = a^x\ln a\, dx = u\ln a\, dx$, so:

$\int a^x.a^{a^x}.a^{a^{a^x}}dx = \int u.a^u.a^{a^u}\frac{du}{u\ln a} = \frac{1}{\ln a}\int a^u.a^{a^u}du$

Let $v = a^u$, then $dv = a^u\ln a\, du = v\ln a\, du$, so:

$\int a^x.a^{a^x}.a^{a^{a^x}}dx = \frac{1}{\ln a}\int v.a^v\frac{dv}{v\ln a} = \frac{1}{(\ln a)^2}\int a^v dv = \frac{1}{(\ln a)^3}a^v + c = \frac{1}{(\ln a)^3}a^{a^u} + c$

$= \frac{1}{(\ln a)^3}a^{a^{a^x}} + c$

303. $\int \frac{1}{\sqrt[3]{(x^2-1)^2(x-1)^2}}dx = \int \frac{1}{\sqrt[3]{(x-1)^2(x+1)^2(x-1)^2}}dx = \int \frac{1}{\sqrt[3]{(x+1)^2(x-1)^4}}dx$

Let $x = \cos 2t$, then $dx = -2\sin 2t\,dt$, then:

$\int \frac{1}{\sqrt[3]{(x^2-1)^2(x-1)^2}}dx = \int \frac{-2\sin 2t}{\sqrt[3]{(\cos 2t+1)^2(\cos 2t-1)^4}}dt = \int \frac{-2\sin 2t}{\sqrt[3]{(2\cos^2 t)^2(2\sin^2 t)^4}}dt$

$= \int \frac{-4\sin t\cos t}{4\sqrt[3]{\cos^4 t\sin^8 t}}dt = -\int \frac{\tan t\cos^2 t}{\sqrt[3]{\cos^{12} t\tan^8 t}}dt = -\int \frac{\tan t\cos^2 t}{\cos^4 t\sqrt[3]{\tan^8 t}}dt = -\int \frac{\tan t\sec^2 t}{\sqrt[3]{\tan^8 t}}dt$

$= -\int \tan t\tan^{-\frac{8}{3}}\sec^2 t\,dt = \int \tan^{-\frac{5}{3}}\sec^2 t\,dt = \int \tan^{-\frac{5}{3}}(\tan t)'dt = \frac{3}{2}\tan^{-\frac{2}{3}}t + c$

$= \frac{3}{2}\sqrt[3]{\cot^2 \mathrm{t}} + c$, but $\cot^2 \mathrm{t} = \frac{1}{\sin^2 t} - 1 = \frac{2}{1-\cos 2t} - 1$, $\int \frac{1}{\sqrt[3]{(x^2-1)^2(x-1)^2}}dx = \frac{3}{2}\sqrt[3]{\frac{2}{1-x} - 1}$

304. $\int \frac{1}{x+\sqrt{x^2-1}}dx$

let $x = \cosh t$, then $dx = \sinh t\,dt$ and $x^2 - 1 = \cosh^2 t - 1 = \sinh^2 t$, so:

$\int \frac{1}{x+\sqrt{x^2-1}}dx = \int \frac{\sinh t}{\cosh t+\sinh t}dt$, but $\sinh t = \frac{e^t-e^{-t}}{2}$ and $\sinh t = \frac{e^t+e^{-t}}{2}$, so:

$\int \frac{1}{x+\sqrt{x^2-1}}dx = \int \frac{e^t-e^{-t}}{e^t+e^{-t}+e^t-e^{-t}}dx = \int \frac{e^t-e^{-t}}{2e^t}dx = \frac{1}{2}\int(1-e^{-2t})dt = \frac{1}{2}t + \frac{1}{4}e^{-2t} + c$

But $t = \cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$, then:

$\int \frac{1}{x+\sqrt{x^2-1}}dx = \frac{1}{2}\ln\left(x + \sqrt{x^2-1}\right) + \frac{1}{4}e^{-2\ln\left(x+\sqrt{x^2-1}\right)} + c$

$= \frac{1}{2}\ln\left(x + \sqrt{x^2-1}\right) + \frac{1}{2}x^2 - \frac{1}{2}x\sqrt{x^2-1} + c$

305. $\int \sqrt{x + \frac{1}{\sqrt{x}}}dx = \int \sqrt{\frac{x\sqrt{x}+1}{\sqrt{x}}}dx = \int \sqrt{x^{\frac{3}{2}} + 1}\,.\,x^{-\frac{1}{4}}dx$

Let $x^{\frac{3}{4}} = \sinh u$, $1 + x^{\frac{3}{2}} = 1 + \sinh^2 u = \cosh^2 u$, $x^{-\frac{1}{4}}dx = \frac{4}{3}\cosh u\,du$, then we get:

$\int \sqrt{x + \frac{1}{\sqrt{x}}}dx = \frac{4}{3}\int \cosh u\cosh u\,du = \frac{4}{3}\int \cosh^2 u\,du = \frac{2}{3}\int(1 + \cosh 2u)du$

$= \frac{2}{3}u + \frac{1}{3}\sinh 2u + c$, but $u = \operatorname{arcsinh} x^{\frac{3}{4}} = \ln\left(x^{\frac{3}{4}} + \sqrt{x^{\frac{3}{2}} + 1}\right)$ and $\cosh b = \sqrt{x^{\frac{3}{2}} + 1}$, then:

$\int \sqrt{x + \frac{1}{\sqrt{x}}}dx = \frac{2}{3}\left[\ln\left(x^{\frac{3}{4}} + \sqrt{x^{\frac{3}{2}} + 1}\right) + x^{\frac{3}{4}}.\sqrt{x^{\frac{3}{2}} + 1}\right] + c$

306. $\int \frac{x^2}{1-x^4}dx = \int \frac{x^2}{(1-x^2)(1+x^2)}dx$, let $x = \tan\theta$, then $dx = \sec^2\theta\,d\theta$, then:

$\int \frac{x^2}{1-x^4}dx = \int \frac{\tan^2\theta}{\sec^2\theta\left(1-\tan^2\theta\right)}\sec^2\theta\,d\theta = \int \frac{\tan^2\theta}{1-\tan^2\theta}d\theta = \int\left(\frac{1}{1-\tan^2\theta} - 1\right)d\theta$

$$= \int \left(\frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta} - 1\right) d\theta = \int \frac{\cos^2\theta}{\cos^2\theta - \sin^2\theta} d\theta - \theta = \frac{1}{2}\int \frac{2\cos^2\theta - 1 + 1}{2\cos^2\theta - 1} d\theta - \theta$$

$$= \frac{1}{2}\int \left(1 + \frac{1}{\cos 2\theta}\right) d\theta - \theta = \frac{1}{2}\int (1 + \sec 2\theta) d\theta - \theta = \frac{1}{4}\ln|\sec 2\theta + \tan 2\theta| - \frac{1}{2}\theta + c$$

$$= \frac{1}{4}\ln\left|\frac{1+x}{1-x}\right| - \frac{1}{2}\arctan x + c$$

307. $\int \sqrt{\frac{x}{a^3 - x^3}} dx = \int \frac{\sqrt{x}}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - \left(x^{\frac{3}{2}}\right)^2}} dx$, let $x^{\frac{3}{2}} = a^{\frac{3}{2}}\sin\theta$, then , $\frac{3}{2}\sqrt{x}dx = a^{\frac{3}{2}}\cos\theta\, d\theta$, so:

$$\int \sqrt{\frac{x}{a^3 - x^3}} dx = \frac{2}{3}\int \frac{a^{\frac{3}{2}}\cos\theta}{\sqrt{\left(a^{\frac{3}{2}}\right)^2 - \left(a^{\frac{3}{2}}\sin\theta\right)^2}} d\theta = \frac{2}{3}\int \frac{a^{\frac{3}{2}}\cos\theta}{a^{\frac{3}{2}}\sqrt{1 - \sin^2\theta}} d\theta = \frac{2}{3}\int \frac{\cos\theta}{\sqrt{\cos^2\theta}} d\theta = \frac{2}{3}\int d\theta$$

$= \frac{2}{3}\theta + c$, but $x^{\frac{3}{2}} = a^{\frac{3}{2}}\sin\theta$, so $\sin\theta = \left(\frac{x}{a}\right)^{\frac{3}{2}}$, so $\theta = \arcsin\left(\frac{x}{a}\right)^{\frac{3}{2}}$, therefore:

$$\int \sqrt{\frac{x}{a^3 - x^3}} dx = \frac{2}{3}\arcsin\left(\frac{x}{a}\right)^{\frac{3}{2}} + c$$

308. $\int \frac{x^2 - 1}{(x^2+1)\sqrt{x^4+1}} dx = \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)\sqrt{x^2 + \frac{1}{x^2}}} dx = \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)\sqrt{\left(x + \frac{1}{x}\right)^2 - 2}} dx$

Let $u = x + \frac{1}{x}$, $du = \left(1 - \frac{1}{x^2}\right) dx$ so: $\int \frac{x^2 - 1}{(x^2+1)\sqrt{x^4+1}} dx = \int \frac{1}{u\sqrt{u^2 - 2}} du = \int \frac{1}{u\sqrt{u^2 - (\sqrt{2})^2}} du$

$$= \frac{1}{\sqrt{2}}\operatorname{arcsec}\left(\frac{u}{\sqrt{2}}\right) + c = \frac{1}{\sqrt{2}}\operatorname{arcsec}\left(\frac{x + \frac{1}{x}}{\sqrt{2}}\right) + c$$

309. $\int \frac{1}{x^2 - 1}.\ln\left|\frac{x-1}{x+1}\right| dx = \int \frac{1}{(x-1)(x+1)}.\ln\left|\frac{x-1}{x+1}\right| dx = \frac{1}{2}\int \left(\frac{1}{x-1} - \frac{1}{x+1}\right)\ln\left|\frac{x-1}{x+1}\right| dx$

Let $u = \ln|x-1| - \ln|x+1|$, then $du = \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$, so:

$$\int \frac{1}{x^2 - 1}.\ln\left|\frac{x-1}{x+1}\right| dx = \frac{1}{2}\int u\,du = \frac{1}{4}u^2 + c = \frac{1}{4}\left(\ln\left|\frac{x-1}{x+1}\right|\right)^2 + c$$

Remark : $\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}$ (method of partial fractions), then:

$1 = A(x+1) + B(x-1)$, for $x = 1$, $A = \frac{1}{2}$ and for $x = -1$, $B = -\frac{1}{2}$.

310. $\int \frac{1}{(x^2+1)\sqrt{x^2-1}} dx = \int \frac{\sqrt{x^2-1}}{(x^2+1)(x^2-1)} dx$, let $x = \sec\theta$, then $dx = \tan\theta\sec\theta\, d\theta$, so:

$$\int \frac{1}{(x^2+1)\sqrt{x^2-1}} dx = \int \frac{\tan^2\theta\sec\theta}{(\sec^2\theta + 1)\tan^2\theta} d\theta = \int \frac{\sec\theta}{\sec^2\theta + 1} d\theta = \int \frac{\cos\theta}{1 + \cos^2\theta} d\theta = \int \frac{\cos\theta}{2 - \sin^2\theta} d\theta$$

Let $u = \sin\theta$, then $du = \cos\theta\, d\theta$, so: $\int \frac{1}{(x^2+1)\sqrt{x^2-1}} dx = \int \frac{1}{2 - u^2} du$

$$= \frac{1}{2\sqrt{2}}\int \left(\frac{1}{\sqrt{2} - u} + \frac{1}{\sqrt{2} + u}\right) du = \frac{1}{2\sqrt{2}}\left(\ln|\sqrt{2} + u| - \ln|\sqrt{2} - u|\right) + c = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2} + u}{\sqrt{2} - u}\right| + c$$

$= \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+u}{\sqrt{2}-u}\right| + c = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin\theta}{\sqrt{2}-\sin\theta}\right| + c = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin\operatorname{arcsec}x}{\sqrt{2}-\sin\operatorname{arcsec}x}\right| + c$

$= \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}+\frac{1}{\sqrt{1-x^2}}}{\sqrt{2}-\frac{1}{\sqrt{1-x^2}}}\right| + c$

311. $\int \frac{1}{\sqrt{1+\cos nx}}dx = \int \frac{1}{\sqrt{2\cos^2\left(\frac{nx}{2}\right)}}dx = \int \frac{1}{\sqrt{2}\cos\left(\frac{nx}{2}\right)}dx = \frac{1}{\sqrt{2}}\int \sec\left(\frac{nx}{2}\right)dx$

Let $t = \frac{nx}{2}$, then $dt = \frac{n}{2}dx$, so $\int \frac{1}{\sqrt{1+\cos nx}}dx = \frac{1}{\sqrt{2}} \times \frac{2}{n}\int \sec t\, dt = \frac{\sqrt{2}}{n}\int \sec t\, dt$

$= \frac{\sqrt{2}}{n}\ln|\sec t + \tan t| + c = \frac{\sqrt{2}}{n}\ln\left|\sec\left(\frac{nx}{2}\right) + \tan\left(\frac{nx}{2}\right)\right|$

312. $\int \frac{\ln^3 x}{x\sqrt{\ln^2 x-4}}dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, so $\int \frac{\ln^3 x}{x\sqrt{\ln^2 x-4}}dx = \int \frac{u^3}{\sqrt{u^2-4}}du$

Let $u = 2\sec t$, then $du = 2\sec t\tan t\, dt$, so we getb:

$\int \frac{\ln^3 x}{x\sqrt{\ln^2 x-4}}dx = \int \frac{2\sec^3 t}{\sqrt{4(\sec^2 t-1)}}2\sec t\tan t\, dt = \int \frac{4\sec^4 t}{2\sqrt{\tan^2 t}}\tan t\, dt = 2\int \sec^4 t\, dt$

$= 2\int \sec^2 t\sec^2 t\, dt = 2\int(1+\tan^2 t)(\tan t)'dt = 2\tan t + \frac{2}{3}\tan^3 t + c$

$= 2\sqrt{\sec^2 t - 1} + \frac{2}{3}(\sec^2 t - 1)^{\frac{3}{2}} + c = 2\sqrt{\frac{u^2}{4} - 1} + \frac{2}{3}\left(\frac{u^2}{4} - 1\right)^{\frac{3}{2}} + c$

$= \sqrt{u^2 - 4} + \frac{1}{3\sqrt{2}}(u^2 - 4)^{\frac{3}{2}} + c = \sqrt{\ln^2 x - 4} + \frac{1}{3\sqrt{2}}(\ln^2 x - 4)^{\frac{3}{2}} + c$

313. $\int \frac{x^2(1-\ln x)}{\ln^4 x - x^4}dx = \int \frac{x^2(1-\ln x)}{x^4\left[\left(\frac{\ln x}{x}\right)^4 - 1\right]}dx = \int \frac{\frac{1-\ln x}{x^2}}{\left(\frac{\ln x}{x}\right)^4 - 1}dx$

Let $u = \frac{\ln x}{x}$, then $du = \left(\frac{1-\ln x}{x^2}\right)dx$, so:

$\int \frac{x^2(1-\ln x)}{\ln^4 x - x^4}dx = \int \frac{du}{u^4-1} = \frac{1}{2}\int \frac{(u^2+1)-(u^2-1)}{(u^2+1)(u^2-1)}du = \frac{1}{2}\int\left(\frac{1}{u^2-1} - \frac{1}{u^2+1}\right)du$

$= \frac{1}{4}\int \frac{(u+1)-(u-1)}{(u+1)(u-1)}du - \frac{1}{2}\int \frac{1}{u^2+1}du = \frac{1}{4}\int\left(\frac{1}{u-1} - \frac{1}{u+1}\right)du - \frac{1}{2}\int \frac{1}{u^2+1}du$

$= \frac{1}{4}\ln|u - 1| - \frac{1}{4}\ln|u + 1| - \frac{1}{2}\arctan u + c = \frac{1}{4}\ln\left|\frac{\ln x - x}{\ln x + x}\right| - \frac{1}{2}\arctan\left(\frac{\ln x}{x}\right) + c$

314. $\int \frac{\sin[\ln(\sin(\ln x))]\cos(\ln x)}{x\sin(\ln x)}dx$, let $u = \sin(\ln x)$, then $du = \frac{\cos(\ln x)}{x}dx$, then:

$\int \frac{\sin[\ln(\sin(\ln x))]\cos(\ln x)}{x\sin(\ln x)}dx = \int \frac{\sin(\ln u)}{u}du$, let $t = \ln u$, then $dt = \frac{1}{u}du$, so:

$\int \frac{\sin[\ln(\sin(\ln x))]\cos(\ln x)}{x\sin(\ln x)}dx = \int \sin t\, dt = -\cos t + c = -\cos\ln(u) + c =$

$-\cos[\ln(\sin(\ln x))] + c$

315. $\int \frac{\tan^2 x}{1-\tan^2 x}dx$, let $t = \tan x$, then $dt = \sec^2 x\, dx = (1+\tan^2 x)dx = (1+t^2)dx$, so:

$\int \frac{\tan^2 x}{1-\tan^2 x}dx = \int \frac{t^2}{(1-t^2)(1+t^2)}dt$

Remark : $\frac{z}{(1-z)(1+z)} = \frac{A}{1-z} + \frac{B}{1+z}$ **(method of partial fractions)**, $A = \left[\frac{z}{1+z}\right]_{z=1} = \frac{1}{2}$, and $B = \left[\frac{z}{1-z}\right]_{z=-1} = -\frac{1}{2}$, so : $\frac{z}{(1-z)(1+z)} = \frac{1}{2(1-z)} - \frac{1}{2(1+z)}$.

Set $z = t^2$, then we get : $\frac{t^2}{(1-t^2)(1+t^2)} = \frac{1}{2(1-t^2)} - \frac{1}{2(1+t^2)}$ and so we have:

$\int \frac{\tan^2 x}{1-\tan^2 x}dx = \int \left[\frac{1}{2(1-t^2)} - \frac{1}{2(1+t^2)}\right]dt = \int \left[\frac{1}{4(1+t)} + \frac{1}{4(1-t)} - \frac{1}{2(1+t^2)}\right]dt$

$= \frac{1}{4}\ln\left|\frac{1+t}{1-t}\right| - \frac{1}{2}\arctan t + c = \frac{1}{4}\ln\left|\frac{1+\tan x}{1-\tan x}\right| - \frac{1}{2}\arctan\tan t + c = \frac{1}{4}\ln\left|\frac{1+\tan x}{1-\tan x}\right| - \frac{1}{2}x + c$

316. $\int \frac{\sin^3 2x}{8+8\sin^2 x}dx = \int \frac{(2\sin x\cos x)^3}{8(1+\sin^2 x)}dx = \int \frac{8\sin^3 x\cos^3 x}{8(1+1-\cos^2 x)}dx = \int \frac{\sin^3 x\cos^3 x}{2-\cos^2 x}dx$

$= \int \frac{\sin^3 x}{2-\cos^2 x}.\cos^3 x\sin x\, dx = \int \frac{1-\cos^2 x}{2-\cos^2 x}.\cos^3 x\sin x\, dx$, let $t = \cos x$, so $dt = -\sin x\, dx$,

$\int \frac{\sin^3 2x}{8+8\sin^2 x}dx = \int \frac{1-t^2}{t^2-2}.t^3 dt = \int \frac{t^3-t^5}{t^2-2}dt$

By performing the long division of $t^3 - t^5$ by $t^2 - 2$, we get $t^3 + t + \frac{2t}{t^2-2}$, so we get:

$\int \frac{\sin^3 2x}{8+8\sin^2 x}dx = \int \left(t^3 + t + \frac{2t}{t^2-2}\right)dt = \frac{1}{4}t^4 + \frac{1}{2}t^2 + \ln|t^2-2| + c$

$= \frac{1}{4}\cos^4 x + \frac{1}{2}\cos^2 x + \ln|\cos^2 x - 2| + c$

317. $\int \sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}dx = \int \sqrt{\frac{\cos x(1-\cos^2 x)}{1-\cos^3 x}}dx = \int \sqrt{\frac{\cos x\sin^2 x}{1-\cos^3 x}}dx = \int \sin x\sqrt{\frac{\cos x}{1-\cos^3 x}}dx$

Let $t^2 = \cos^3 x$, $\cos x\sqrt{\cos x} = t$, so $\left(-\sin x\sqrt{\cos x} - \frac{1}{2}\sin x\sqrt{\cos x}\right)dx = dt$, then

$\sin x\sqrt{\cos x} = -\frac{2}{3}dt$, so $\int \sqrt{\frac{\cos x-\cos^3 x}{1-\cos^3 x}}dx = -\frac{2}{3}\int \frac{dt}{\sqrt{1-t^2}} = \frac{2}{3}\arccos t + c$

$= \frac{2}{3}\arccos\left(\cos x\sqrt{\cos x}\right) + c$

318. $\int \sqrt{\frac{1-\cos x}{\cos\alpha-\cos x}}dx = \int \sqrt{\frac{2\sin^2\frac{x}{2}}{1+\cos\alpha-(1+\cos x)}}dx = \int \frac{\sqrt{2}\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{\alpha}{2}-2\cos^2\frac{x}{2}}}dx$

$= \int \frac{\sqrt{2}\sin\frac{x}{2}}{\sqrt{2\cos^2\frac{\alpha}{2}\left[1-\frac{\cos^2\frac{x}{2}}{\cos^2\frac{\alpha}{2}}\right]}}dx = \int \frac{\sqrt{2}\sin\frac{x}{2}}{\sqrt{2}\cos\frac{\alpha}{2}\sqrt{1-\left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right)^2}}dx = 2\int -\frac{-\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{\alpha}{2}}}{\sqrt{1-\left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right)^2}}dx$

Let $t=\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}$, then $dt=-\frac{1}{2}\frac{\sin\frac{x}{2}}{\cos\frac{\alpha}{2}}dx$, so:

$$\int\sqrt{\frac{1-\cos x}{\cos\alpha-\cos x}}dx=2\int-\frac{dt}{\sqrt{1-t^2}}=2\arccos t+c=2\arccos\left(\frac{\cos\frac{x}{2}}{\cos\frac{\alpha}{2}}\right)+c$$

319. $$\int\frac{dx}{\cos^3 x-\sin^3 x}=\int\frac{dx}{(\cos x-\sin x)\left(\cos^2 x+\cos x\sin x+\sin^2 x\right)}$$

$$=\int\frac{dx}{(\cos x-\sin x)(1+\cos x\sin x)}=\int\frac{\cos^2 x+\sin^2 x-2\sin x\cos x+2\sin x\cos x}{(\cos x-\sin x)(1+\cos x\sin x)}dx$$

$$=\int\frac{(\cos x-\sin x)^2+2\sin x\cos x}{(\cos x-\sin x)(1+\cos x\sin x)}dx=\int\frac{\cos x-\sin x}{1+\cos x\sin x}dx+2\int\frac{\cos x\sin x}{(\cos x-\sin x)(1+\cos x\sin x)}dx$$

$$=\frac{2}{3}\int\frac{\cos x-\sin x}{1+(\sin x+\cos x)^2}dx+\frac{2}{3}\int\frac{\cos x-\sin x}{2-(\sin x+\cos x)^2}dx$$

Let $u=\sin x+\cos x$, then $du=(\cos x-\sin x)dx$, then we get:

$$\int\frac{dx}{\cos^3 x-\sin^3 x}=\frac{2}{3}\int\frac{du}{1+u^2}+\frac{2}{3}\int\frac{du}{2-u^2}=\frac{2}{3}\int\frac{du}{1+u^2}+\frac{2}{3}\int\frac{du}{\left(\sqrt{2}\right)^2-u^2}$$

$$=\frac{2}{3}\arctan u+\frac{1}{3\sqrt{2}}\ln\left|\frac{\sqrt{2}+u}{\sqrt{2}-u}\right|+c=\frac{2}{3}\arctan(\sin x+\cos x)+\frac{1}{3\sqrt{2}}\ln\left|\frac{\sqrt{2}+\sin x+\cos x}{\sqrt{2}-\sin x-\cos x}\right|+c$$

320. $$\int\frac{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}{3+\sin 2x}dx=\int\frac{\sqrt{2}\left[\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}\right]}{3+\sin 2x}dx=\int\frac{\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right]}{3+\sin 2x}dx$$

$$=\int\frac{\sin x+\cos x}{3+\sin 2x}dx=\int\frac{\sin x+\cos x}{4-(1-\sin 2x)}dx=\int\frac{\sin x+\cos x}{4-\left(\sin^2 x+\cos^2 x-\sin 2x\right)}dx$$

$=\int\frac{\sin x+\cos x}{4-(\sin x-\cos x)^2}dx$, let $t=\sin x-\cos x$, then $dt=(\sin x+\cos x)dx$, so:

$$\int\frac{\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)}{3+\sin 2x}dx=\int\frac{dt}{4-t^2}=\int\frac{dt}{(2-t)(2+t)}=\frac{1}{4}\int\frac{(2+t)+(2-t)}{(2-t)(2+t)}dt=\frac{1}{4}\int\left(\frac{1}{2-t}+\frac{1}{2+t}\right)dt$$

$$=\frac{1}{4}(-\ln|2-t|+\ln|2+t|)+c=\frac{1}{4}\ln\left|\frac{2+t}{2-t}\right|+c=\frac{1}{4}\ln\left|\frac{2+\sin x-\cos x}{2-\sin x+\cos x}\right|+c$$

321. $$\int\frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx=\int\frac{x^2-1}{(x+1)^2\sqrt{x^3+x^2+x}}dx=\int\frac{x^2\left(1-\frac{1}{x^2}\right)}{(x^2+2x+1)\sqrt{x^2(x+1+x^{-1})}}dx$$

$$=\int\frac{x^2\left(1-\frac{1}{x^2}\right)}{x^2(x+2+x^{-1})\sqrt{x+1+x^{-1}}}dx=\int\frac{1-\frac{1}{x^2}}{(x+2+x^{-1})\sqrt{x+1+x^{-1}}}dx$$

Let $t^2=x+1+x^{-1}$, then $\left(1-\frac{1}{x^2}\right)dx=2tdt$, so:

$$\int\frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx=\int\frac{2tdt}{(t^2+1).t}=2\int\frac{dt}{t^2+1}=2\arctan t+c=2\arctan(x+1+x^{-1})^{\frac{1}{2}}+c$$

322. $$\int\frac{1}{2+2\sin x+\cos x}dx$$

Let $t=\tan\left(\frac{x}{2}\right)\Rightarrow dx=\frac{2dt}{1+t^2}$; $\cos x=\frac{1-t^2}{1+t^2}$ and $\sin x=\frac{2t}{1+t^2}$; then:

$$\int \frac{1}{2+2\sin x+\cos x}dx=\int \frac{1}{2+2\left(\frac{2t}{1+t^2}\right)+\frac{1-t^2}{1+t^2}}\frac{2}{1+t^2}dt=2\int \frac{1}{2\left(1+t^2\right)+4t+1-t^2}dt$$

$$=2\int \frac{1}{t^2+4t+3}dt=2\int \frac{1}{(t+1)(t+3)}dt=\frac{2}{2}\int\left(\frac{1}{t+1}-\frac{1}{t+3}\right)\,dt=\ln|t+1|-\ln|t+3|+c$$

$$=\ln\left|\tan\left(\frac{x}{2}\right)+1\right|-\ln\left|\tan\left(\frac{x}{3}\right)+3\right|+c=\ln\left|\frac{\tan\left(\frac{x}{2}\right)+1}{\tan\left(\frac{x}{2}\right)+3}\right|+c$$

Remark : $\frac{1}{(t+1)(t+3)}=\frac{\mathrm{A}}{t+1}+\frac{\mathrm{B}}{t+3}$ (method of partial fractions), then**:**

$1=\mathrm{A}(t+3)+\mathrm{B}(t+1)$, for $t=-1$, $\mathrm{A}=\frac{1}{2}$, and for $x=-3$, $\mathrm{B}=-\frac{1}{2}$.

323. $$\int \frac{dx}{\sqrt{1-\sin^4 x}}=\int \frac{1}{\sqrt{\left(1-\sin^2 x\right)\left(1+\sin^2 x\right)}}dx=\int \frac{1}{\sqrt{\cos^2 x\left(1+\sin^2 x\right)}}dx$$

$$=\int \frac{1}{\cos x\sqrt{1+\sin^2 x}}dx=\int \frac{\sec^2 x}{\sec^2 x\cos x\sqrt{1+\sin^2 x}}dx=\int \frac{\sec^2 x}{\sec x\sqrt{1+\sin^2 x}}dx$$

$$=\int \frac{\sec^2 x}{\sqrt{\left(1+\sin^2 x\right)\sec^2 x}}dx=\int \frac{\sec^2 x}{\sqrt{\sec^2 x+\tan^2 x}}dx=\int \frac{\sec^2 x}{\sqrt{1+\tan^2 x+\tan^2 x}}dx$$

$=\int \frac{\sec^2 x}{\sqrt{1+2\tan^2 x}}dx=\frac{1}{\sqrt{2}}\int \frac{\sec^2 x}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+\tan^2 x}}dx$, let $u=\tan x$, then $du=\sec^2 x\,dx$, so**:**

$$\int \frac{dx}{\sqrt{1-\sin^4 x}}=\frac{1}{\sqrt{2}}\int \frac{du}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2+u^2}}=\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{u}{\frac{1}{\sqrt{2}}}\right)+c=\frac{1}{\sqrt{2}}\sinh^{-1}\left(\sqrt{2}\tan x\right)+c$$

324. $\int \frac{dx}{x+3\sqrt{x}+2}=\int \frac{dx}{\sqrt{x}\left(\sqrt{x}+3+\frac{2}{\sqrt{x}}\right)}=2\int \frac{dx}{2\sqrt{x}\left(\sqrt{x}+3+\frac{2}{\sqrt{x}}\right)}$, let $u=\sqrt{x}$, then $du=\frac{1}{2\sqrt{x}}dx$

$$\int \frac{dx}{x+3\sqrt{x}+2}=2\int \frac{du}{u+3+\frac{2}{u}}=2\int \frac{u}{u^2+3u+2}du=\int \frac{2u}{(u+1)(u+2)}du=\int \frac{4u+4-2u-4}{(u+1)(u+2)}du$$

$$=\int \frac{4(u+1)-2(u+2)}{(u+1)(u+2)}du=\int\left(\frac{4}{u+2}-\frac{2}{u+1}\right)du=4\ln|u+2|-2\ln|u+1|+c$$

$$=4\ln\left|\sqrt{x}+2\right|-2\ln\left|\sqrt{x}+1\right|+c$$

325. $$\int \frac{3+3x^2}{3x^4+2x^3+6x^2-2x+3}dx=\int \frac{3\left(1+\frac{1}{x^2}\right)}{3\left(x^2+\frac{1}{x^2}\right)+2\left(x-\frac{1}{x}\right)+6}dx=\int \frac{1+\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}\right)+\frac{2}{3}\left(x-\frac{1}{x}\right)+2}dx$$

Let $t=x-\frac{1}{x}$, $\left(x-\frac{1}{x}\right)^2=t^2$, so $x^2-2+\frac{1}{x^2}=t^2$, so $x^2+\frac{1}{x^2}=t^2+2$ and $dt=\left(1+\frac{1}{x^2}\right)dx$

$$\int \frac{3+3x^2}{3x^4+2x^3+6x^2-2x+3}dx=\int \frac{dt}{t^2+2+\frac{2}{3}t+2}=\int \frac{dt}{t^2+\frac{2}{3}t+4}=\int \frac{dt}{\left(t+\frac{1}{3}\right)^2+\frac{35}{9}}=\frac{9}{35}\int \frac{dt}{\left(\frac{3t+1}{\sqrt{35}}\right)^2+1}$$

$$=\frac{9}{35}\times\frac{\sqrt{35}}{3}\arctan\left(\frac{3t+1}{\sqrt{35}}\right)+c=\frac{3}{\sqrt{35}}\arctan\left(\frac{3\left(x-\frac{1}{x}\right)+1}{\sqrt{35}}\right)+c$$

326. $\int \frac{e^{3x}-e^x}{e^{4x}+e^{2x}+1}dx = \int \frac{e^{2x}-1}{e^{4x}+e^{2x}+1}e^x dx$, let $u = e^x$, so $du = e^x dx$, then:

$$\int \frac{e^{3x}-e^x}{e^{4x}+e^{2x}+1}dx = \int \frac{u^2-1}{u^4+u^2+1}du = \int \frac{\frac{u^2-1}{u^2}}{\frac{u^4+u^2+1}{u^2}}du = \int \frac{1-\frac{1}{u^2}}{u^2+1+\frac{1}{u^2}}du = \int \frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-1}du$$

Let $t = u + \frac{1}{u}$, then $dt = \left(1 - \frac{1}{u^2}\right)du$, so:

$$\int \frac{e^{3x}-e^x}{e^{4x}+e^{2x}+1}dx = \int \frac{1}{t^2-1}dt = \frac{1}{2}\int \frac{(t+1)-(t-1)}{(t-1)(t+1)}dt = \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right)dt = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + c$$

$$= \frac{1}{2}\ln\left|\frac{u+\frac{1}{u}-1}{u+\frac{1}{u}+1}\right| + c = \frac{1}{2}\ln\left|\frac{u^2-u+1}{u^2+u+1}\right| + c = \frac{1}{2}\ln\left|\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right| + c$$

327. $\int \frac{\sqrt{x^2+4}}{x^2}dx$, let $x = 2\tan u$, then $dx = 2\sec^2 u\,du$, then:

$$\int \frac{\sqrt{x^2+4}}{x^2}dx = \int \frac{2\sqrt{1+\tan^2 u}}{4\tan^2 u}2\sec^2 u\,du = \int \frac{\sec u \sec^2 u}{\tan^2 u}du = \int \frac{\frac{1}{\cos^3 u}}{\frac{\sin^2 u}{\cos^2 u}}dx = \int \frac{1}{\sin^2 u \cos u}du$$

$$= \int \frac{1}{(1-\cos^2 u)\cos u}du = \int \left(\frac{1}{\cos u} + \frac{\cos u}{1-\cos^2 u}\right)du = \int \left(\sec u + \frac{\cos u}{\sin^2 u}\right)du$$

$$= \int \left(\sec u + \frac{(\sin u)'}{\sin^2 u}\right)du = \ln|\sec u + \tan u| - \frac{1}{\sin u} + c$$

$$= \ln\left|\sec\left(\arctan\left(\frac{x}{2}\right)\right) + \tan\left(\arctan\left(\frac{x}{2}\right)\right)\right| - \csc\left(\arctan\left(\frac{x}{2}\right)\right) + c$$

$$= \ln\left|\frac{x+\sqrt{x^2+4}}{2}\right| - \frac{\sqrt{x^2+4}}{x} + c$$

328. $\int \frac{x+1}{x(xe^x+1)^2}dx = \int \frac{(x+1)e^x}{xe^x(xe^x+1)^2}dx$, let $t = 1 + xe^x$, $dt = (e^x + xe^x)dx$, so:

$$\int \frac{x+1}{x(xe^x+1)^2}dx = \int \frac{dt}{(t-1)t^2} = \int \frac{t^{-2}dt}{t-1} = \int \frac{t^{-2}dt}{\left(t^{-1}\right)^{-1}-1}, \text{ let } u = t^{-1},\ du = -t^{-2}dt, \text{ so:}$$

$$\int \frac{x+1}{x(xe^x+1)^2}dx = -\int \frac{du}{u^{-1}-1} = -\int \frac{du}{\frac{1}{u}-1} = -\int \frac{du}{\frac{1-u}{u}} = -\int \frac{u}{1-u}du = \int \frac{-u+1-1}{1-u}du$$

$$= \int \left(1 - \frac{1}{1-u}\right)du = \int \left(1 + \frac{(1-u)'}{1-u}\right)du = u + \ln|1-u| + c = \frac{1}{t} + \ln\left|1 - \frac{1}{t}\right| + c$$

$$= \frac{1}{1+xe^x} + \ln|xe^x| - \ln|1 + xe^x| + c$$

329. $\int \frac{\tan x+\cot x+1}{\ln(e^x\tan x)}dx = \int \frac{\tan x+\cot x+1}{\ln e^x+\ln(\tan x)}dx = \int \frac{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}+1}{x+\ln\left(\frac{\sin x}{\cos x}\right)}dx$

Let $u = \ln\left(\frac{\sin x}{\cos x}\right) + x$, then $du = \left[\frac{\frac{1}{\cos^2 x}}{\frac{\sin x}{\cos x}} + 1\right]dx = \left(\frac{1}{\sin x\cos x} + 1\right)dx$

$du = \left(\frac{\cos^2 x+\sin^2 x}{\sin x \cos x} + 1\right) dx = \left(\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + 1\right) dx$, then:

$\int \frac{\tan x+\cot x+1}{\ln(e^x \tan x)} dx = \int \frac{du}{u} = \ln|u| + c = \ln|\ln(\tan x) + x|$

330. $\int \frac{\operatorname{arcsec}\left(\frac{1+x^2}{1-x^2}\right)}{x \arctan x} dx$, let $x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

Also , $\frac{1+x^2}{1-x^2} = \frac{1+\tan^2\theta}{1-\tan^2\theta} = \frac{1}{\cos 2\theta} = \sec 2\theta$

$\int \frac{\operatorname{arcsec}\left(\frac{1+x^2}{1-x^2}\right)}{x \arctan x} dx = \int \frac{\operatorname{arcsec}(\sec 2\theta)}{\tan\theta \arctan(\tan\theta)} \sec^2\theta\, d\theta = \int \frac{2\theta}{\theta \tan\theta} \sec^2\theta\, d\theta = 2\int \frac{d\theta}{\sin\theta\cos\theta}$

$= 4\int \frac{d\theta}{\sin 2\theta} = 4\int \csc 2\theta\, d\theta = 4\int \csc 2\theta \frac{\csc 2\theta - \cot 2\theta}{\csc 2\theta - \cot 2\theta} d\theta = 2\int \frac{(\csc 2\theta - \cot 2\theta)'}{\csc 2\theta - \cot 2\theta} d\theta$

$= 2\ln|\csc 2\theta - \cot 2\theta| + c$

331. $\int \frac{x - x\ln x + 1}{x(x+1)^2 + x\ln^2 x} dx = \int \frac{x - x\ln x+1}{x(x+1)^2\left[1+\frac{\ln^2 x}{(x+1)^2}\right]} dx$, let $t = \frac{\ln x}{x+1}$, $dt = \frac{x - x\ln x + 1}{x(x+1)^2} dx$, so:

$\int \frac{x - x\ln x + 1}{x(x+1)^2 + x\ln^2 x} dx = \int \frac{1}{1+t^2} dt = \arctan t + c = \arctan\left(\frac{\ln x}{x+1}\right) + c$

332. $\int \frac{1}{5\cos^2 x + 4\sin 2x + 3} dx = \int \frac{1}{5\cos^2 x + 8\sin x\cos x + 3} dx$

$= \int \frac{\frac{1}{\cos^2 x}}{\frac{5\cos^2 x + 8\sin x \cos x + 3}{\cos^2 x}} dx = \int \frac{\sec^2 x}{5 + 8\tan x + 3\sec^2 x} dx = \int \frac{\sec^2 x}{5+8\tan x + 3(\tan^2 x + 1)} dx$

$= \int \frac{\sec^2 x}{8 + 8\tan x + 3\tan^2 x} dx$, let $t = \tan x$, then $dt = \sec^2 x\, dx$, so:

$\int \frac{1}{5\cos^2 x + 4\sin 2x + 3} dx = \int \frac{1}{3t^2 + 8t + 8} dt$

$3t^2 + 8t + 8 = 3\left(t^2 + \frac{8}{3}t + \frac{8}{3}\right) = 3\left(t^2 + \frac{8}{3}t + \frac{16}{9} - \frac{16}{9} + \frac{8}{3}\right) = 3\left[\left(t+\frac{4}{3}\right)^2 + \frac{8}{9}\right]$

$= 3\left[\left(t+\frac{4}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2\right]$, then:

$\int \frac{1}{5\cos^2 x + 4\sin 2x + 3} dx = \frac{1}{3}\int \frac{1}{\left(t+\frac{4}{3}\right)^2 + \left(\frac{2\sqrt{2}}{3}\right)^2} dt = \frac{1}{3}\left[\frac{3}{2\sqrt{2}} \arctan\left(\frac{3}{2\sqrt{2}}\left(t + \frac{4}{3}\right)\right)\right] + c$

$= \frac{1}{2\sqrt{2}} \arctan\left(\frac{3}{2\sqrt{2}} t + \frac{2}{\sqrt{2}}\right) + c = \frac{1}{2\sqrt{2}} \arctan\left(\frac{3}{2\sqrt{2}} \tan x + \frac{2}{\sqrt{2}}\right) + c$

333. $\int \frac{\sin x}{\sin 4x} dx = \int \frac{\sin x}{2\cos 2x \sin 2x} dx = \int \frac{\sin x}{2\cos 2x(2\sin x\cos x)} dx = \int \frac{1}{4\cos x\cos 2x} dx$

$= \int \frac{\cos x}{4\cos^2 x\cos 2x} dx = \int \frac{\cos x}{4(1-\sin^2 x)(1-2\sin^2 x)} dx$, let $t = \sin x$, then $dt = \cos x\, dt$, so:

$\int \frac{\sin x}{\sin 4x} dx = \frac{1}{4}\int \frac{1}{(1-t^2)(1-2t^2)} dt = \frac{1}{4}\int \frac{1}{(t^2-1)(2t^2-1)} dt$

Method of partial fractions : $\frac{1}{(z-1)(2z-1)} = \frac{A}{z-1} + \frac{B}{2z-1}$, so $1 = (2z-1)A + (z-1)B$, For $z = 1$ we get A = 1 and for $z = \frac{1}{2}$ we get $1 = \frac{1}{2}$B, so B = 2

So : $\frac{1}{(z-1)(2z-1)} = \frac{1}{z-1} + \frac{2}{2z-1}$, put $z = t^2$ we get : $\frac{1}{(t^2-1)(2t^2-1)} = \frac{1}{t^2-1} + \frac{2}{2t^2-1}$, so:

$\int \frac{\sin x}{\sin 4x} dx = \frac{1}{4}\int \frac{1}{t^2-1} dt + \frac{1}{4}\int \frac{2}{2t^2-1} dt = \frac{1}{8}\int \frac{(t+1)-(t-1)}{(t+1)(t-1)} dt + \frac{1}{4}\int \frac{(\sqrt{2}t+1)-(\sqrt{2}t-1)}{(\sqrt{2}t+1)(\sqrt{2}t-1)} dt$

$= \frac{1}{8}\int \left(\frac{1}{t-1} - \frac{1}{t+1}\right) dt + \frac{1}{4}\int \left(\frac{1}{\sqrt{2}t-1} - \frac{1}{\sqrt{2}t+1}\right) dt = \frac{1}{8}\ln\left|\frac{t-1}{t+1}\right| + \frac{1}{4\sqrt{2}}\ln\left|\frac{\sqrt{2}t-1}{\sqrt{2}t+1}\right| + c$

$= \frac{1}{8}\ln\left|\frac{\sin x-1}{\sin x+1}\right| + \frac{1}{4\sqrt{2}}\ln\left|\frac{\sqrt{2}\sin x-1}{\sqrt{2}\sin x+1}\right| + c$

334. $\int \frac{1}{3(\sec x+\tan x)-\cos x} dx = \int \frac{1}{3(\sec x+\tan x)-\cos x} \times \frac{\cos x}{\cos x} dx$

$= \int \frac{\cos x}{3(1+\sin x)-\cos^2 x} dx = \int \frac{\cos x}{3+3\sin x-(1-\sin^2 x)} dx = \int \frac{\cos x}{\sin^2 x+3\sin x+2} dx$

$= \int \frac{\cos x}{(\sin x+1)(\sin x+2)} dx$, let $t = \sin x$, then $dt = \cos x\, dx$, so:

$\int \frac{1}{3(\sec x+\tan x)-\cos x} dx = \int \frac{1}{(t+1)(t+2)} dt$

Decomposition into partial fractions : $\frac{1}{(t+1)(t+2)} = \frac{A}{t+1} + \frac{B}{t+2}$, $A = \left[\frac{1}{t+2}\right]_{t=-1} = 1$ and $B = \left[\frac{1}{t+1}\right]_{t=-2} = -1$, so we get : $\frac{1}{(t+1)(t+2)} = \frac{1}{t+1} - \frac{1}{t+2}$ and so:

$\int \frac{1}{3(\sec x+\tan x)-\cos x} dx \int \left(\frac{1}{t+1} - \frac{1}{t+2}\right) dt = \ln|t + 1| - \ln|t + 2| + c = \ln\left|\frac{t+1}{t+2}\right| + c$

$= \ln\left|\frac{\sin x+1}{\sin x+2}\right| + c$

335. $\int \frac{\sqrt{\tan x}}{\sin x(\sin x+\cos x)} dx = \int \frac{\sqrt{\tan x}}{\sin^2 x+\sin x\cos x} dx = 2\int \frac{\sqrt{\tan x}}{2\sin^2 x+2\sin x\cos x} dx$

$= 2\int \frac{\sqrt{\tan x}}{1-\cos 2x+\sin 2x} dx$, let $t = \tan x$, then $\cos 2x = \frac{1-t^2}{1+t^2}$, $\sin 2x = \frac{2t}{1+t^2}$, and $dx = \frac{dt}{1+t^2}$,

$\int \frac{\sqrt{\tan x}}{\sin x(\sin x+\cos x)} dx = 2\int \frac{\sqrt{t}}{1-\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}} \frac{dt}{1+t^2} = 2\int \frac{\sqrt{t}}{1+t^2-1+t^2+2t} dt = \int \frac{\sqrt{t}}{t^2+t} dt$

$= 2\int \frac{1}{(\sqrt{t})^2+1} . \frac{1}{2\sqrt{t}} dt$, let $u = \sqrt{t}$, so $du = \frac{1}{2\sqrt{t}} dt$, then:

$\int \frac{\sqrt{\tan x}}{\sin x(\sin x+\cos x)} dx = 2\int \frac{1}{1+u^2} du = 2\arctan u + c = 2\arctan(\sqrt{t}) + c$

$= 2\arctan(\sqrt{\tan x}) + c$

336. $\int \frac{\cos^4 x-\sin^4 x}{(1+\sqrt{e^x}+\sqrt[3]{e^x}+\sqrt[6]{e^x})\cos 2x} dx = \int \frac{(\cos^2 x+\sin^2 x)(\cos^2 x-\sin^2 x)}{(1+\sqrt{e^x}+\sqrt[3]{e^x}+\sqrt[6]{e^x})(\cos^2 x-\sin^2 x)} dx$

$= \int \frac{1}{1+\sqrt{e^x}+\sqrt[3]{e^x}+\sqrt[6]{e^x}} dx$, let $u = \sqrt[6]{e^x}$, $u^6 = e^x$, so $x = 6\ln u$, then $dx = \frac{6}{u} du$, so:

$\int \frac{\cos^4 x-\sin^4 x}{(1+\sqrt{e^x}+\sqrt[3]{e^x}+\sqrt[6]{e^x})\cos 2x} dx = \int \frac{6}{(1+u^3+u^2+u)u} du = \int \frac{6}{(1+u)(1+u^2)u} du$

$= \int \left(\frac{6}{u} - \frac{3}{1+u} - \frac{3u+3}{1+u^2}\right) du = 6\int \frac{1}{u} du - 3\int \frac{1}{1+u} du - \frac{3}{2}\int \frac{(1+u^2)'}{1+u^2} du - 3\int \frac{1}{1+u^2} du$

$= 6\ln|u| - 3\ln|1+u| - \frac{3}{2}\ln|1+u^2| - 3\arctan u + c$

$= 6\ln\left|\sqrt[6]{e^x}\right| - 3\ln\left|1+\sqrt[6]{e^x}\right| - \frac{3}{2}\ln\left|1+\sqrt[3]{e^x}\right| - 3\arctan\left(\sqrt[6]{e^x}\right) + c$

Remark : $\frac{1}{(1+u)(1+u^2)u} = \frac{A}{u} + \frac{B}{1+u} + \frac{Cu+D}{1+u^2}$ (method of partial fractions)

$A = \left[\frac{1}{(1+u)(1+u^2)}\right]_{u=0} = 1,\ B = \left[\frac{1}{(1+u^2)u}\right]_{u=-1} = -\frac{1}{2}$, then

$$\frac{1}{(1+u)(1+u^2)u} = \frac{1}{u} - \frac{1}{2(1+u)} + \frac{Cu+D}{1+u^2}; \text{then}$$

$1 = (1+u)(1+u^2) - \frac{1}{2}u(1+u^2) + u(1+u)(Cu+D)$, taking $u=1$, $C+D=-1$ and taking $u=2$, $2C+D=-\frac{3}{2}$, so we get $\begin{cases} C+D=-1 \\ 2C+D=-\frac{3}{2}\end{cases}$, then $C=D=-\frac{1}{2}$.

337. $\int \frac{x^2-1}{x^2+1}.\frac{1}{\sqrt{1+x^4}}dx$, let $y = \frac{x^2-1}{x^2+1}$, then $x^2 = \frac{1+y}{1-y}$, so $2xdx = \frac{2}{(1-y)^2}dy$,

so $dx = \frac{1}{x}.\frac{1}{(1-y)^2}dy$, then $dx = \sqrt{\frac{1-y}{1+y}}.\frac{1}{(1-y)^2}dy$, so $\int \frac{x^2-1}{x^2+1}.\frac{1}{\sqrt{1+x^4}}dx =$

$\int y.\frac{1}{\sqrt{1+\left(\frac{1+y}{1-y}\right)^2}}.\sqrt{\frac{1-y}{1+y}}.\frac{1}{(1-y)^2}dy$

$= \int y.\frac{1-y}{\sqrt{(1-y)^2+(1-y)^2}}.\sqrt{\frac{1-y}{1+y}}.\frac{1}{(1-y)^2}dy = \frac{1}{\sqrt{2}}\int \frac{y}{\sqrt{1-y}\sqrt{1+y}}.\frac{1}{\sqrt{1+y^2}}dy$

$= \frac{1}{2\sqrt{2}}\int \frac{2y}{\sqrt{1-y^2}\sqrt{1+y^2}}dy = \frac{1}{2\sqrt{2}}\int \frac{2y}{\sqrt{1-y^4}}dy$, let $u = y^2$, so $dy = 2ydy$, then:

$\int \frac{x^2-1}{x^2+1}.\frac{1}{\sqrt{1+x^4}}dx = \frac{1}{2\sqrt{2}}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2\sqrt{2}}\arcsin u + c = \frac{1}{2\sqrt{2}}\arcsin y^2 + c$

$= \frac{1}{2\sqrt{2}}\arcsin\left[\left(\frac{x^2-1}{x^2+1}\right)^2\right] + c$

338. $\int \frac{1+x^2}{1-x^2}.\frac{dx}{\sqrt{1+3x^2+x^4}} = \int \frac{1+x^2}{(1-x^2)\sqrt{1+3x^2+x^4}}dx = \int \frac{\frac{1+x^2}{x^2}}{\frac{1}{x^2}(1-x^2)\sqrt{1+3x^2+x^4}}dx$

$= \int \frac{1+\frac{1}{x^2}}{\left(\frac{1-x^2}{x}\right)\sqrt{\frac{1+3x^2+x^4}{x^2}}}dx = \int \frac{1+\frac{1}{x^2}}{\left(\frac{1}{x}-x\right)\sqrt{x^2+3+\frac{1}{x^2}}}dx = \int \frac{1+\frac{1}{x^2}}{\left(\frac{1}{x}-x\right)\sqrt{x^2-2+\frac{1}{x^2}+5}}dx$

$= \int \frac{1+\frac{1}{x^2}}{\left(\frac{1}{x}-x\right)\sqrt{\left(x-\frac{1}{x}\right)^2+5}}dx$, let $t = x - \frac{1}{x}$, then $dt = \left(1+\frac{1}{x^2}\right)dx$, so:

$\int \frac{1+x^2}{1-x^2}.\frac{dx}{\sqrt{1+3x^2+x^4}} = -\int \frac{1}{t\sqrt{t^2+5}}dt$, let $z^2 = t^2+5$, then $2tdt = 2zdz$, so:

$$\int \frac{1+x^2}{1-x^2}\cdot\frac{dx}{\sqrt{1+3x^2+x^4}} = -\int \frac{1}{z^2-5}dt = -\frac{1}{2\sqrt{5}}\ln\left|\frac{z-\sqrt{5}}{z+\sqrt{5}}\right| + c$$

$$= -\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{t^2+5}-\sqrt{5}}{\sqrt{t^2+5}+\sqrt{5}}\right| + c = -\frac{1}{2\sqrt{5}}\ln\left|\frac{\sqrt{x^2+\frac{1}{x^2}+3}-\sqrt{5}}{\sqrt{x^2+\frac{1}{x^2}+3}+\sqrt{5}}\right| + c$$

339. $\int \ln\left(2x^2+1+2x\sqrt{1+x^2}\right)dx = \int \ln\left(x^2+2x\sqrt{1+x^2}+\left(\sqrt{x^2+1}\right)^2\right)dx$

$= \int \ln\left(x+\sqrt{1+x^2}\right)^2 dx = 2\int \ln\left(x+\sqrt{1+x^2}\right)dx$, let $x = \tan y$, so $dx = \sec^2 y\, dy$, so:

$\int \ln^2\left(x+\sqrt{1+x^2}\right)dx = 2\int \ln\left(\tan y + \sqrt{1+\tan^2 y}\right)\sec^2 y\, dy$

$= 2\int \ln(\tan y + \sec y)\sec^2 y\, dy$

Let $u = \ln(\tan y + \sec y)$, so $u' = \frac{\sec^2 y + \sec y \tan y}{\tan y + \sec y} = \frac{\sec y(\sec y + \tan y)}{\tan y + \sec y} = \sec y$ and

let $v' = \sec^2 y$, then $v = \tan y$, then:

$\int \ln\left(2x^2+1+2x\sqrt{1+x^2}\right)dx = 2\tan y \ln(\sec y + \tan y) - 2\int \tan y \sec y\, dy$

$= 2\tan y\ln(\sec y + \tan y) - 2\sec y + c$

$= 2\tan y\ln\left(\sqrt{1+\tan^2 y} + \tan y\right) - 2\sqrt{1+\tan^2 y} + c = 2x\ln\left(\sqrt{1+x^2}x\right) - 2\sqrt{1+x^2} + c$

340. $\int \frac{x\arctan x}{\left(1+x^2\right)^3}dx$

Let $u = \arctan x$, then $u' = \frac{1}{1+x^2}$ and let $v' = \frac{x}{\left(1+x^2\right)^3}$, then $v = -\frac{1}{4\left(1+x^2\right)^2}$, so:

$\int \frac{x\arctan x}{\left(1+x^2\right)^3}dx = -\frac{\arctan x}{4\left(1+x^2\right)^2} + \frac{1}{4}\int \frac{1}{\left(1+x^2\right)^3}dx$

Evaluating: $\int \frac{1}{\left(1+x^2\right)^3}dx$, let $x = \tan t$, then $dx = \sec^3 t\, dt$, then:

$\int \frac{1}{\left(1+x^2\right)^3}dx = \int \frac{1}{\left(1+\tan^2 t\right)^3}\sec^2 t\, dt = \int \frac{1}{\left(\sec^2 t\right)^3}\sec^2 t\, dt = \int \frac{1}{\sec^4 t}dt = \int \cos^4 t\, dt$

$= \int \left(\frac{1+\cos 2t}{2}\right)^2 dt = \frac{1}{4}\int(1+2\cos 2t + \cos^2 2t\, dt) = \frac{1}{4}\int\left(1+2\cos 2t + \frac{1+\cos 4t}{2}dt\right)$

$= \frac{3}{8}t + \frac{1}{4}\sin 2t + \frac{1}{32}\sin 4t + c$, then:

$\int \frac{x\arctan x}{\left(1+x^2\right)^3}dx = -\frac{\arctan x}{4\left(1+x^2\right)^2} + \frac{1}{32}\left(\frac{\sin 4t}{4} + 3t + 2\sin 2t\right) + c$

$$= -\frac{\arctan x}{4\left(1+x^2\right)^2} + \frac{1}{32}\left(\frac{x\left(1-x^2\right)}{\left(1+x^2\right)^2} + 3\arctan x + \frac{4x}{1+x^2}\right) + c$$

341. $\int \ln\left(x+\sqrt{x}\right)dx = \int \ln\sqrt{x}\, dx + \int \ln\left(\sqrt{x}+1\right)dx = \frac{1}{2}\int \ln x\, dx + \int \ln\left(\sqrt{x}+1\right)dx$

$= \frac{1}{2}x\ln x - \frac{1}{2}x + \int \ln\left(\sqrt{x}+1\right)dx$

Evaluating: $\int \ln\left(\sqrt{x}+1\right)dx$, let $y = \sqrt{x}+1$, so $x = (y-1)^2$ and $dx = 2(y-1)dy$, so:

$\int \ln\left(\sqrt{x}+1\right)dx = \int 2(y-1)\ln y\, dy$

Let $u = \ln y$, then $u' = \frac{1}{y}$ and let $v' = 2(y-1) = 2y-2$, so $v = y^2 - 2y$, then:

$\int \ln(\sqrt{x}+1)dx = (y^2-2y)\ln y - \int \frac{1}{y}(y^2-2y)dy + c$

$= (y^2-2y)\ln y - \int (y-2)dy + c = (y^2-2y)\ln y - \frac{1}{2}y^2 + 2y + c$

$= \left((\sqrt{x}+1)^2 - 2(\sqrt{x}+1)\right)\ln(\sqrt{x}+1) - \frac{1}{2}(\sqrt{x}+1)^2 + 2(\sqrt{x}+1) + c$

$= (x-1)\ln(\sqrt{x}+1) - \frac{1}{2}x - \sqrt{x} + \frac{3}{2} + c$, therefore:

$\int \ln(x+\sqrt{x})dx = \frac{1}{2}x\ln x - \frac{1}{2}x + (x-1)\ln(\sqrt{x}+1) - \frac{1}{2}x - \sqrt{x} + \frac{3}{2} + c$

$= (x-1)\ln(\sqrt{x}+1) + \frac{1}{2}x\ln x - x - \sqrt{x} + \frac{3}{2} + c$

342. $\int \frac{dx}{x^3+1} = \int \frac{dx}{(x+1)(x^2-x+1)} = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}dx$

$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}.\frac{1}{2}\int \frac{2x-4}{x^2-x+1}dx = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\int \frac{2x-1-3}{x^2-x+1}dx$

$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\left(\int \frac{2x-1}{x^2-x+1}dx - \int \frac{3}{x^2-x+1}dx\right)$

$= \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{6}\left(\int \frac{2x-1}{x^2-x+1}dx - \int \frac{3}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx\right)$

$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{2}.\frac{2}{\sqrt{3}}\arctan\left[\frac{x-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right] + c$

$= \frac{1}{3}\ln|x-1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\arctan\left(\frac{2x-1}{\sqrt{3}}\right) + c$

Remark: $\frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$ (method of partial fractions), then:

A $= \left[\frac{1}{x^2-x+1}\right]_{x=-1} = \frac{1}{3}$, now for $x = 0$ we get $1 = \frac{1}{3}$ + C, so C $= \frac{2}{3}$, now for $x = 1$, we get: $\frac{1}{2} = \frac{1}{6}$ + B + $\frac{2}{3}$; then B $= -\frac{1}{3}$

343. $\int \frac{dx}{x^4+4} = \int \frac{dx}{x^4+4x^2+4-4x^2} = \int \frac{dx}{(x^2+2)^2-(2x)^2} = \int \frac{dx}{(x^2+2x+2)(x^2-2x+2)}$

Method of partial fractions : $\frac{1}{(x^2+2x+2)(x^2-2x+2)} = \frac{\mathrm{A}x+\mathrm{B}}{x^2+2x+2} + \frac{\mathrm{C}x+\mathrm{D}}{x^2-2x+2}$, then:

$1 = (\mathrm{A}+\mathrm{C})x^3 + (\mathrm{B}-2\mathrm{A}+2\mathrm{C}+\mathrm{D})x^2 + (2\mathrm{A}-2\mathrm{B}+2\mathrm{C}+2\mathrm{D})x + 2\mathrm{B} + 2\mathrm{D}$, by comparison, we get $\mathrm{A}+\mathrm{C} = 0$, $\mathrm{B}-2\mathrm{A}+2\mathrm{C}+\mathrm{D} = 0$, $2\mathrm{A}-2\mathrm{B}+2\mathrm{C}+2\mathrm{D} = 0$ and $2\mathrm{B}+2\mathrm{D} = 1$ and so we get: $\mathrm{A} = \frac{1}{8}$, $\mathrm{B} = \frac{1}{4}$, $\mathrm{C} = -\frac{1}{8}$ and $\mathrm{C} = \frac{1}{4}$, thus:

$\int \frac{dx}{x^4+4} = \frac{1}{8}\int \frac{x+2}{x^2+2x+2}dx - \frac{1}{8}\int \frac{x-2}{x^2-2x+2}dx$

$= \frac{1}{8}\int \frac{x+1}{(x+1)^2+1}dx + \frac{1}{8}\int \frac{dx}{(x+1)^2+1} - \frac{1}{8}\int \frac{x-1}{(x-1)^2+1}dx + \frac{1}{8}\int \frac{dx}{(x-1)^2+1}$

$= \frac{1}{16}\int \frac{2x+2}{x^2+2x+2}dx + \frac{1}{8}\int \frac{dx}{(x+1)^2+1} - \frac{1}{16}\int \frac{2x-2}{x^2-2x+2}dx + \frac{1}{8}\int \frac{dx}{(x-1)^2+1}$

$$= \frac{1}{16}\int \frac{(x^2+2x+2)'}{x^2+2x+2}dx + \frac{1}{8}\int \frac{dx}{(x+1)^2+1} - \frac{1}{16}\int \frac{(x^2-2x+2)'}{x^2-2x+2}dx + \frac{1}{8}\int \frac{dx}{(x-1)^2+1}$$

$$= \frac{1}{16}\ln(x^2+2x+2) + \frac{1}{8}\arctan(x+1) - \frac{1}{16}\ln(x^2-2x+2) + \frac{1}{8}\arctan(x-1) + c$$

344. $$\int \frac{dx}{x^6+1} = \int \frac{1}{(x^2+1)(x^4-x^2+1)}dx = \int \frac{x^2+1-x^2}{(x^2+1)(x^4-x^2+1)}dx$$

$$= \int \frac{1}{x^4-x^2+1}dx - \int \frac{x^2}{x^6+1}dx = \int \frac{1}{x^4-x^2+1}dx - \int \frac{x^2}{(x^3)^2+1}dx$$

Let's evaluate: $\int \frac{1}{x^4-x^2+1}dx = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4-x^2+1}dx$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4-x^2+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4-x^2+1}dx = \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+1}dx - \frac{1}{2}\int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right)^2-(\sqrt{3})^2}dx = \frac{1}{2}\int \frac{\left(x-\frac{1}{x}\right)'}{\left(x-\frac{1}{x}\right)^2+1}dx - \frac{1}{2}\int \frac{\left(x+\frac{1}{x}\right)'}{\left(x+\frac{1}{x}\right)^2-(\sqrt{3})^2}dx$$

$$= \frac{1}{2}\arctan\left(x-\frac{1}{x}\right) - \frac{1}{2\sqrt{3}}\operatorname{arctanh}\left[\frac{1}{\sqrt{3}}\left(x+\frac{1}{x}\right)\right] + c$$

Then: $\int \frac{dx}{x^6+1} = \int \frac{1}{x^4-x^2+1}dx - \int \frac{x^2}{(x^3)^2+1}dx$

$$= \frac{1}{2}\arctan\left(x-\frac{1}{x}\right) - \frac{1}{2\sqrt{3}}\operatorname{arctanh}\left[\frac{1}{\sqrt{3}}\left(x+\frac{1}{x}\right)\right] - \frac{1}{3}\arctan(x^3) + c$$

345. $$\int \frac{a+bx^2-2bx^2}{x\sqrt{x^2-(bx^2-a)^2}}dx = \int \frac{a+bx^2}{x\sqrt{x^2\left[1-\frac{(bx^2-a)^2}{x^2}\right]}}dx - \int \frac{2bx^2}{x\sqrt{x^2-(bx^2-a)^2}}dx$$

$$= \int \frac{a+bx^2}{x^2\sqrt{1-\left(\frac{bx^2-a}{x}\right)^2}}dx - \int \frac{2bx}{\sqrt{-\left[b^2x^4-(2ab+1)x^2+a^2\right]}}dx$$

$$= \int \frac{\frac{a}{x^2}+b}{\sqrt{1-\left(\frac{bx^2-a}{x}\right)^2}}dx - \int \frac{2bx^2}{\sqrt{\frac{4ab+1}{4b^2}-\left(bx^2-\frac{2ab+1}{2b}\right)^2}}dx$$

Let $u = bx - \frac{a}{x}$, then $du = \left(b+\frac{a}{x^2}\right)dx$, so:

$$\int \frac{\frac{a}{x^2}+b}{\sqrt{1-\left(\frac{bx^2-a}{x}\right)^2}}dx = \int \frac{du}{\sqrt{1-u^2}} = \arcsin u + c = \arcsin\left(bx-\frac{a}{x}\right) + c$$

Let $t = bx^2 - \frac{2ab+1}{2b}$, then $dt = 2bxdx$, so:

$$\int \frac{2bx^2}{\sqrt{\frac{4ab+1}{4b^2}-\left(bx^2-\frac{2ab+1}{2b}\right)^2}}dx = \int \frac{dt}{\sqrt{\left(\sqrt{\frac{4ab+1}{4b^2}}\right)^2-t^2}} = \arcsin\left(\frac{t}{\sqrt{\frac{4ab+1}{4b^2}}}\right)+c$$

$\arcsin\left(\frac{bx^2-\frac{2ab+1}{2b}}{\sqrt{\frac{4ab+1}{4b^2}}}\right)+c$, therefore we get**:**

$$\int \frac{a+bx^2-2bx^2}{x\sqrt{x^2-(bx^2-a)^2}}dx = \arcsin\left(bx-\frac{a}{x}\right)-\arcsin\left(\frac{bx^2-\frac{2ab+1}{2b}}{\sqrt{\frac{4ab+1}{4b^2}}}\right)+c$$

346. $\int \frac{1}{\sqrt{(x-a)(x-b)}}dx$, let $x = a\sec^2\theta - b\tan^2\theta$,

then $dx = 2a\sec^2\theta\tan\theta - 2b\tan\theta\sec^2\theta$, $dx = [2(a-b)\sec^2\theta\tan\theta]d\theta$, then**:**

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}dx = \int \frac{2(a-b)\sec^2\theta\tan\theta d\theta}{\sqrt{\left(a\sec^2\theta-a-b\tan^2\theta\right)\left(a\sec^2\theta-b-b\tan^2\theta\right)}} =$$

$$2\int \frac{(a-b)\sec^2\theta\tan\theta d\theta}{\sqrt{\left(a\tan^2\theta-b\tan^2\theta\right)\left(a\sec^2\theta-b\sec^2\theta\right)}}d\theta = 2\int \frac{(a-b)\sec^2\theta\tan\theta d\theta}{\sqrt{(a-b)\tan^2\theta(a-b)\sec^2\theta}}d\theta$$

$$= 2\int \frac{(a-b)\sec^2\theta\tan\theta d\theta}{\sqrt{(a-b)^2\tan^2\theta\sec^2\theta}}d\theta = 2\int \frac{(a-b)\sec^2\theta\tan\theta d\theta}{(a-b)\tan\theta\sec\theta}d\theta = 2\int \sec\theta\, d\theta$$

$= 2\ln|\sec\theta+\tan\theta|+c$

But $x = a\sec^2\theta - b\tan^2\theta$, then $x = a\sec^2\theta - b(\sec^2\theta-1) = a\sec^2\theta - b\sec^2\theta + b$,

then $x-b = (a-b)\sec^2\theta$, so $\sec\theta = \sqrt{\frac{x-b}{a-b}}$

Also $x = a(1+\tan^2\theta) - b\tan^2\theta = a + a\tan^2\theta - b\tan^2\theta$, $x-a = (a-b)\tan^2\theta$, then

$\tan\theta = \sqrt{\frac{x-a}{a-b}}$. Then**:**

$$\int \frac{1}{\sqrt{(x-a)(x-b)}}dx = 2\ln\left|\sqrt{\frac{x-b}{a-b}}+\sqrt{\frac{x-a}{a-b}}\right|+c = 2\ln\left|\sqrt{x-a}+\sqrt{x-b}\right|+k$$

347. **First Method:** $\int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = \int \frac{x^2}{(x^2+a^2)^{\frac{3}{2}}}dx$, let $x = a\sinh t$, $dx = a\cosh t\, dt$

$$\int \frac{x^2}{\sqrt{(x^2+a^2)^3}}dx = \int \frac{a^2\sinh^2 t}{\left(a^2\sinh^2 t+a^2\right)^{\frac{3}{2}}}a\cosh t\, dt = \int \frac{a^3\sinh^2 t\cosh t}{a^3\left(\sinh^2 t+1\right)^{\frac{3}{2}}}dt = \int \frac{\sinh^2 t\cosh t}{\left(\cosh^2 t\right)^{\frac{3}{2}}}dt =$$

$$\int \frac{\sinh^2 t\cosh t}{\cosh^3 t}dx = \int \frac{\sinh^2 t}{\cosh^2 t}dt = \int \tanh^2 t\, dt = \int (1-\operatorname{sech}^2 t)dt = t - \tanh t + c$$

$$= t - \frac{\sinh t}{\cosh t} + c = t - \frac{\sinh t}{\sqrt{1+\sinh^2 t}} + c = \sinh^{-1}\left(\frac{x}{a}\right) - \frac{x}{a\sqrt{1+\left(\frac{x}{a}\right)^2}} + c$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2+x^2}} + c$$

Second Method: $\int \frac{x^2}{\sqrt{(x^2+a^2)^3}} dx$, $x = a\tan\theta$, then $dx = a\sec^2\theta\, d\theta$, then**:**

$$\int \frac{x^2}{\sqrt{(x^2+a^2)^3}} dx = \int \frac{a^2\tan^2\theta}{(a^2\tan^2\theta + a^2)^{\frac{3}{2}}} a\sec^2\theta\, d\theta = \int \frac{a^3\tan^2\theta\sec^2\theta}{a^3(\tan^2\theta+1)^{\frac{3}{2}}} d\theta = \int \frac{\tan^2\theta\sec^2\theta}{(\sec^2\theta)^{\frac{3}{2}}} d\theta$$

$$= \int \frac{\tan^2\theta\sec^2\theta}{\sec^3\theta} d\theta = \int \frac{\tan^2\theta}{\sec\theta} d\theta = \int \frac{\sec^2\theta - 1}{\sec\theta} d\theta = \int(\sec\theta - \cos\theta)\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| - \sin\theta + c = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{a^2+x^2}} + c$$

$$= \ln\left|\frac{a\sqrt{1+\left(\frac{x}{a}\right)^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{a^2+x^2}} + c = \sinh^{-1}\left(\frac{x}{a}\right) - \frac{x}{\sqrt{a^2+x^2}} + c$$

348. $\int \frac{\sqrt{(x^2-a^2)^3}}{x^3} dx$, let $x = a\sec y$, then $dx = a\sec y\tan y\, dy$, so**:**

$$\int \frac{\sqrt{(x^2-a^2)^3}}{x^3} dx = \int \frac{\sqrt{(a^2\sec^2 y - a^2)^3}}{a^3\sec^3 y} a\sec y\tan y\, dy = \int \frac{\sqrt{[a^2(\sec^2 y - 1)]^3}}{a^2\sec^2 y} \tan y\, dy$$

$$= \int \frac{\sqrt{a^6\tan^6 y}}{a^2\sec^2 y} \tan y\, dy = \int \frac{a^3\tan^3 y}{a^2\sec^2 y} \tan y\, dy = a\int \frac{\tan^2 y\tan^2 y}{\sec^2 y} dy$$

$$= a\int \frac{\tan^2 y(\sec^2 y - 1)}{\sec^2 y} dy = a\int \frac{\tan^2 y\sec^2 y - \tan^2 y}{\sec^2 y} dy = a\int \tan^2 y\, dy - a\int \frac{\tan^2 y}{\sec^2 y} dy$$

$$= a\int(\sec^2 y - 1)dy - a\int \sin^2 y\, dy = a\int(\sec^2 y - 1)dy - \frac{a}{2}\int(1 - \cos 2y)\, dy$$

$$= a\tan y - y - \frac{a}{2}y + \frac{a}{4}\sin 2y + c$$

But $x = a\sec y$, so $\cos y = \frac{a}{x}$, $\tan y = \sqrt{\sec^2 y - 1} = \sqrt{\frac{x^2}{a^2} - 1} = \frac{1}{a}\sqrt{x^2 - a^2}$

and $\sin 2y = 2\sin y\cos y = 2\frac{a}{x}\sqrt{1 - \frac{a^2}{x^2}} = \frac{2a}{x}\sqrt{\frac{x^2-a^2}{x^2}} = \frac{2a}{x^2}\sqrt{x^2 - a^2}$, so**:**

$$\int \frac{\sqrt{(x^2-a^2)^3}}{x^3} dx = \sqrt{x^2 - a^2} - \left(\frac{a+2}{2}\right)\sec^{-1}\left(\frac{x}{a}\right) + \frac{a^2}{2x^2}\sqrt{x^2 - a^2} + c$$

$$= \left(\frac{2x^2+a^2}{2x^2}\right)\sqrt{x^2 - a^2} - \left(\frac{a+2}{2}\right)\sec^{-1}\left(\frac{x}{a}\right) + c$$

349. $\int \frac{\sqrt{x}}{1+x+x^2+x^3}dx$, let $y=\sqrt{x}$, then $dy=\frac{1}{2\sqrt{x}}dx=\frac{1}{2y}dx$, so:

$$\int \frac{\sqrt{x}}{1+x+x^2+x^3}dx=\int \frac{y}{1+y^2+y^4+y^6}.2ydy=\int \frac{2y^2}{1+y^2+y^4+y^6}dy$$
$$=\int \frac{2y^2}{(1+y^2)+y^4(1+y^2)}dy=\int \frac{2y^2}{(1+y^2)(1+y^4)}dy$$

Method of partial fractions : $\frac{2z}{(1+z)(1+z^2)}=\frac{\mathrm{A}}{1+z}+\frac{\mathrm{B}z+\mathrm{C}}{1+z^2}$

So, $2z=(1+z^2)\mathrm{A}+(1+z)(\mathrm{B}z+\mathrm{C})$, for $z=-1$, we get $-2=2\mathrm{A}$, then $\mathrm{A}=-1$, for $z=0$, we get $0=-1+\mathrm{C}$, so $\mathrm{C}=1$ and for $z=1$, we get $2=-2+2(\mathrm{B}+1)$, so $\mathrm{B}=1$.

So, we get , $\frac{2z}{(1+z)(1+z^2)}=\frac{-1}{1+z}+\frac{z+1}{1+z^2}$, put $z=t^2$, so , $\frac{2y^2}{(1+y^2)(1+y^4)}=\frac{-1}{1+y^2}+\frac{y^2+1}{1+y^4}$, so:

$$\int \frac{\sqrt{x}}{1+x+x^2+x^3}dx=\int\left(-\frac{1}{1+y^2}+\frac{y^2+1}{1+y^4}\right)dy$$

Evaluating: $\int \frac{y^2+1}{1+y^4}dy=\int \frac{\frac{y^2+1}{y^2}}{\frac{1+y^4}{y^2}}dy=\int \frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy=\int \frac{1+\frac{1}{y^2}}{y^2-2+\frac{1}{y^2}+2}dy=\int \frac{1+\frac{1}{y^2}}{\left(y-\frac{1}{y}\right)^2+2}dy$

$$=\int \frac{1+\frac{1}{y^2}}{\left(y-\frac{1}{y}\right)^2+(\sqrt{2})^2}dy, \text{ let } w=y-\frac{1}{y}, \text{ then } dw=\left(1+\frac{1}{y^2}\right)dy, \text{ so:}$$

$$\int \frac{y^2+1}{1+y^4}dy=\int \frac{dw}{w^2+(\sqrt{2})^2}=\frac{1}{\sqrt{2}}\arctan\left(\frac{w}{\sqrt{2}}\right)+c_1=\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\left(y-\frac{1}{y}\right)\right)+c_1, \text{ therefore:}$$

$$\int \frac{\sqrt{x}}{1+x+x^2+x^3}dx=\int\left(-\frac{1}{1+y^2}+\frac{y^2+1}{1+y^4}\right)dy=\frac{1}{\sqrt{2}}\arctan\left(\frac{1}{\sqrt{2}}\left(y-\frac{1}{y}\right)\right)-\arctan y+c$$
$$=\frac{1}{\sqrt{2}}\arctan\left(\sqrt{\frac{x}{2}}-\frac{1}{\sqrt{2x}}\right)-\arctan\sqrt{x}+c$$

350. $\int \frac{2\cos x+\sin 2x}{(\sin x-1)\sqrt{\sin x+\sin^2 x+\sin^3 x}}dx=\int \frac{2\cos x+2\sin x\cos x}{(\sin x-1)\sqrt{\sin x+\sin^2 x+\sin^3 x}}dx$

$$=\int \frac{2\cos x(\sin x+1)}{(\sin x-1)\sqrt{\sin x+\sin^2 x+\sin^3 x}}dx=\int 2.\frac{\sin x+1}{\sin x-1}.\frac{\cos x}{\sqrt{\sin x+\sin^2 x+\sin^3 x}}dx$$

Let $u^2=\sin x$, so $2udu=\cos x\,dx$, then:

$$\int \frac{2\cos x+\sin 2x}{(\sin x-1)\sqrt{\sin x+\sin^2 x+\sin^3 x}}dx=\int 2.\frac{u^2+1}{u^2-1}.\frac{2u}{\sqrt{u^2+u^4+u^6}}du$$

$$=4\int \frac{u^2+1}{u^2-1}.\frac{u}{\sqrt{u^4\left(1+u^2+\frac{1}{u^2}\right)}}du=4\int \frac{u+\frac{1}{u}}{u^2-1}.\frac{du}{\sqrt{\left(u-\frac{1}{u}\right)^2-3}}=4\int \frac{1+\frac{1}{u^2}}{u-\frac{1}{u}}.\frac{du}{\sqrt{\left(u-\frac{1}{u}\right)^2+3}}$$

Let $t=u-\frac{1}{u}$, then $dt=\left(1+\frac{1}{u^2}\right)du$, then:

$$\int \frac{2\cos x+\sin 2x}{(\sin x-1)\sqrt{\sin x+\sin^2 x+\sin^3 x}}dx=4\int \frac{dt}{t\sqrt{t^2+3}}, \text{ let } w^2=t^2+3, \text{ then } 2wdw=2tdt, \text{ so,}$$

$wdw = \sqrt{w^2-3}dt$, then $\int \frac{2\cos x+\sin 2x}{(\sin x-1)\sqrt{\sin x+\sin^2 x+\sin^3 x}}dx = 4\int \frac{wdw}{w\sqrt{w^2-3}\sqrt{w^2-3}}$

$= 4\int \frac{dw}{w^2-3} = \frac{2}{\sqrt{3}}\int \frac{(w+\sqrt{3})-(w-\sqrt{3})}{(w+\sqrt{3})(w-\sqrt{3})}dw = \frac{2}{\sqrt{3}}\int \left(\frac{1}{w-\sqrt{3}} - \frac{1}{w+\sqrt{3}}\right)dw$

$= \frac{2}{\sqrt{3}}\left(\ln|w-\sqrt{3}| - \ln|w+\sqrt{3}|\right) + c = \frac{2}{\sqrt{3}}\ln\left|\frac{w-\sqrt{3}}{w+\sqrt{3}}\right| + c = \frac{2}{\sqrt{3}}\ln\left|\frac{\sqrt{t^2+3}-\sqrt{3}}{\sqrt{t^2+3}+\sqrt{3}}\right| + c$

$= \frac{2}{\sqrt{3}}\ln\left|\frac{\sqrt{\left(u-\frac{1}{u}\right)^2+3}-\sqrt{3}}{\sqrt{\left(u-\frac{1}{u}\right)^2+3}+\sqrt{3}}\right| + c = \frac{2}{\sqrt{3}}\ln\left|\frac{\sqrt{u^2+\frac{1}{u^2}+1}-\sqrt{3}}{\sqrt{u^2+\frac{1}{u}^2+1}+\sqrt{3}}\right| + c$ (with $u^2 = \sin x$)

$= \frac{2}{\sqrt{3}}\ln\left|\frac{\sqrt{\sin^4 x+\sin^2 x+1}-\sqrt{3}\sin x}{\sqrt{\sin^4 x+\sin^2 x+1}+\sqrt{3}\sin x}\right| + c$

351. $\int \frac{1}{x\sqrt{(\ln x+1)(\ln x+2)(\ln x+3)(\ln x+4)+1}}dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, so:

$\int \frac{1}{x\sqrt{(\ln x+1)(\ln x+2)(\ln x+3)(\ln x+4)+1}}dx = \int \frac{1}{\sqrt{(u+1)(u+2)(u+3)(u+4)+1}}dx$

$= \int \frac{1}{\sqrt{[(u+1)(u+4)][(u+2)(u+3)]+1}}dx = \int \frac{1}{\sqrt{(u^2+5u+4)(u^2+5u+6)+1}}dx$

Factoring: $(u^2+5u+4)(u^2+5u+6)+1$, let $t = u^2+5u+4$

then $u^2+5u+6 = t+2$, then $(u^2+5u+4)(u^2+5u+6)+1 = t(t+2)+1$

$= t^2+2t+1 = (t+1)^2 = (u^2+5u+5)^2$, so:

$\int \frac{1}{x\sqrt{(\ln x+1)(\ln x+2)(\ln x+3)(\ln x+4)+1}}dx = \int \frac{du}{\sqrt{(u^2+5u+5)^2}} = \int \frac{du}{u^2+5u+5}$

$= \int \frac{du}{\left(u+\frac{5}{4}\right)^2-\frac{5}{4}} = \frac{1}{\sqrt{5}}\int \frac{\left[\left(u+\frac{5}{4}\right)+\frac{\sqrt{5}}{2}\right]-\left[\left(u+\frac{5}{4}\right)-\frac{\sqrt{5}}{2}\right]}{\left[\left(u+\frac{5}{4}\right)+\frac{\sqrt{5}}{2}\right]\left[\left(u+\frac{5}{4}\right)-\frac{\sqrt{5}}{2}\right]}du = \frac{1}{\sqrt{5}}\int\left[\frac{1}{\left(u+\frac{5}{4}\right)-\frac{\sqrt{5}}{2}} - \frac{1}{\left(u+\frac{5}{4}\right)+\frac{\sqrt{5}}{2}}\right]du$

$= \frac{1}{\sqrt{5}}\left[\ln\left|\left(u+\frac{5}{4}\right)-\frac{\sqrt{5}}{2}\right| - \ln\left|\left(u+\frac{5}{4}\right)+\frac{\sqrt{5}}{2}\right|\right] + c = \frac{1}{\sqrt{5}}\ln\left|\frac{4\ln x-2\sqrt{5}+5}{4\ln x+2\sqrt{5}+5}\right| + c$

352. $\int x\sin(x^2)\tan^2[\cos(x^2)]\,dx = -\frac{1}{2}\int -2x\sin(x^2)\tan^2[\cos(x^2)]\,dx$

Let $u = \cos(x^2)$, then $du = -2x\sin(x^2)\,dx$, so:

$\int x\sin(x^2)\tan^2[\cos(x^2)]\,dx = -\frac{1}{2}\int \tan^2 u\,du = -\frac{1}{2}\int(\sec^2 u - 1)du$

$= -\frac{1}{2}\tan u + \frac{1}{2}u + c = -\frac{1}{2}\tan[\cos(x^2)] + \frac{1}{2}\cos(x^2) + c$

353. $\int(\cos x)^{\cos x+1}\tan x\,(1+\ln\cos x)dx$,

let $u = 1+\ln\cos x$, then $du = -\tan x\,dx$ and $\cos x = e^{u-1}$, so:

$\int(\cos x)^{\cos x+1}\tan x\,(1+\ln\cos x)dx = -\int(e^{u-1})^{e^{u-1}+1}u\,du = -\int(e^{u-1})^{e^{u-1}}ue^{u-1}du$

Let $y = e^{u-1}$, then $dy = e^{u-1}du$ and $u = \ln y + 1$

$\int(\cos x)^{\cos x+1}\tan x\,(1+\ln\cos x)dx = -\int y^y(\ln y+1)dy$

Let $v = y^y$, so $\ln v = y \ln y$, so $, \frac{1}{v} dv = (1 + \ln y)dy$, then:

$\int (\cos x)^{\cos x + 1} \tan x\,(1 + \ln\cos x)dx = -\int v \frac{1}{v} dv = -\int dv = -v + c = -y^y + c$

$= -(e^{u-1})^{e^{u-1}} + c = -(\cos x)^{\cos x} + c$

354. $\int \sin(\arccos x + 2\arcsin x)\, dx$

Simplifying: $\sin(\arccos x + 2\arcsin x)$, take $\arccos x = a$ and $\arcsin x = \frac{b}{2}$,

$\sin(\arccos x + 2\arcsin x) = \sin(a + b) = \sin a \cos b + \cos a \sin b$

$= \sin\arccos x\,.\cos(2\arcsin x) + \cos\arccos x + \sin(2\arcsin x)$

$= \sqrt{1 - x^2}(1 - 2x^2) + x\left(2x\sqrt{1 - x^2}\right) = \sqrt{1 - x^2} - 2x^2\sqrt{1 - x^2} + 2x^2\sqrt{1 - x^2} = \sqrt{1 - x^2}$

$\int \sin(\arccos x + 2\arcsin x)\, dx = \int \sqrt{1 - x^2}\, dx$, let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$, so:

$\int \sin(\arccos x + 2\arcsin x)\, dx = \int \sqrt{1 - \sin^2\theta} \cos\theta\, d\theta = \int \cos\theta \cos\theta\, d\theta = \int \cos^2\theta\, d\theta$

$= \frac{1}{2}\int (1 + \cos 2\theta)d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + c$

$= \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1 - x^2} + c$

355. $\int \sqrt{\sin x \cos^2 x \tan^3 x \cot^4 x \csc^5 x \sec^6 x}\, dx$

$= \int \sqrt{\sin x \times \cos^2 x \times \frac{\sin^3 x}{\cos^3 x} \times \frac{\cos^4 x}{\sin^4 x} x \times \frac{1}{\sin^5 x} \times \frac{1}{\cos^6 x} x\, dx} = \int \frac{\sin^{-\frac{5}{2}} x}{\cos^{\frac{3}{2}} x} dx$

Let $t = \tan x$, then $dt = \sec^2 x\, dx = (1 + \tan^2 x)dx = (1 + t^2)dx$, so:

$$\int \sqrt{\sin x \cos^2 x \tan^3 x \cot^4 x \csc^5 x \sec^6 x}\, dx = \int \frac{\left[\frac{t}{(1+t^2)^{\frac{1}{2}}}\right]^{-\frac{5}{2}}}{\left[\frac{1}{(1+t^2)^{\frac{1}{2}}}\right]^{\frac{3}{2}}} . \frac{dt}{1+t^2}$$

$$= \int \frac{(1+t^2)^{\frac{5}{4}}}{t^{\frac{5}{2}}} . \frac{(1+t^2)^{\frac{3}{4}}}{1+t^2} dt = \int \left(\frac{1+t^2}{t^{\frac{5}{2}}}\right) dt = \int \left(t^{-\frac{5}{2}} + t^{-\frac{1}{2}}\right) dt = -\frac{2}{3}t^{-\frac{3}{2}} + 2\sqrt{t} + c$$

$= -\frac{2}{3}(\tan x)^{-\frac{3}{2}} + 2\sqrt{\tan x} + c$

356. $\int (\sin x)^{e^x} . e^x . (\cot x + \ln\sin x)dx$, let $y = (\sin x)^{e^x}$, then $\ln y = e^x . \ln\sin x$,

then $, \frac{dy}{y} = \left(e^x . \ln\sin x + e^x . \frac{\cos x}{\sin x}\right) dx$, so $dy = y(e^x . \ln\sin x + e^x . \cot x)dx$, then

$dy = (\sin x)^{e^x} . e^x . (\cot x + \ln\sin x)dx$, so:

$\int (\sin x)^{e^x} . e^x . (\cot x + \ln\sin x)dx = \int dy = y + c = (\sin x)^{e^x} + c$

357. **First Method**

$$\int \frac{\cos x}{\sin x + \tan x} dx = \int \frac{\cos x}{\sin x + \frac{\sin x}{\cos x}} dx = \int \frac{\cos^2 x}{\sin x \cos x + \sin x} dx = \int \frac{\cos^2 x}{\sin x(1 + \cos x)} dx$$

$= \int \frac{\cos^2 x , \sin x}{\sin^2 x(1 + \cos x)} dx = \int \frac{\cos^2 x}{(1 - \cos^2 x)(1 + \cos x)} \sin x\, dx$, let $u = \cos x$, so $du = -\sin x\, dx$, so:

$\int \frac{\cos x}{\sin x+\tan x}dx = \int -\frac{u^2}{(1-u^2)(1+u)}du = -\int \frac{u^2}{(1-u)(1+u)^2}du = \int \frac{u^2}{(u-1)(u+1)^2}du$

But : $\frac{u^2}{(u-1)(u+1)^2} = \frac{A}{u-1} + \frac{B}{u+1} + \frac{C}{(u+1)^2}$, $A = \left[\frac{u^2}{(u+1)^2}\right]_{u=1} = \frac{1}{4}$, $C = \left[\frac{u^2}{u-1}\right]_{u=-1} = -\frac{1}{2}$ and

we can substitute $u = 0$ to get B, so $0 = -A + B + C$, then $B = A - C = \frac{3}{4}$, so:

$\int \frac{\cos x}{\sin x+\tan x}dx = \int \left(\frac{\frac{1}{4}}{u-1} + \frac{\frac{3}{4}}{u+1} - \frac{\frac{1}{2}}{(u+1)^2}\right)du = \frac{1}{4}\ln|u-1| + \frac{3}{4}\ln|u+1| + \frac{1}{2(u+1)} + c$

$= \frac{1}{4}\ln|\cos x - 1| + \frac{3}{4}\ln|\cos x + 1| + \frac{1}{2(\cos x+1)} + c$

Second Method $\int \frac{\cos x}{\sin x+\tan x}dx$

Let $t = \tan\left(\frac{x}{2}\right)$, so $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\tan x = \frac{2t}{1-t^2}$, then:

$\int \frac{\cos x}{\sin x+\tan x}dx = \int \frac{\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}+\frac{2t}{1-t^2}}\frac{2dt}{1+t^2} = \frac{1}{2}\int \frac{(1-u^2)^2}{u(1+u^2)}du = \frac{1}{2}\int \left(u + \frac{1}{u} - \frac{4u}{1+u^2}\right)du$

$= \frac{1}{4}u^2 + \frac{1}{2}\ln|u| - \ln|1+u^2| + c = \frac{1}{4}\tan^2\left(\frac{x}{2}\right) + \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}\right)\right| - \ln\left|1+\tan^2\left(\frac{x}{2}\right)\right| + c$

358. $\int \frac{dx}{1+a^2-2a\cos x} = \int \frac{dx}{1+a^2-2a\left(1-2\sin^2\frac{x}{2}\right)} = \int \frac{1}{1-2a+a^2+4a\sin^2\frac{x}{2}} \times \frac{\sec^2\frac{x}{2}}{\sec^2\frac{x}{2}}dx =$

$\int \frac{\sec^2\frac{x}{2}}{(1-a)^2\sec^2\frac{x}{2}+4a\tan^2\frac{x}{2}}dx = \int \frac{\sec^2\frac{x}{2}}{\left[(1-a)^2+4a\right]\tan^2\frac{x}{2}+(1-a)^2}dx =$

$\int \frac{\sec^2\frac{x}{2}}{(1+a)^2\tan^2\frac{x}{2}+(1-a)^2}dx = 2\int \frac{\frac{1}{2}\sec^2\frac{x}{2}}{(1+a)^2\tan^2\frac{x}{2}+(1-a)^2}dx$

Let $t = \tan\frac{x}{2}$, then $dt = \frac{1}{2}\sec^2\frac{x}{2}$, so:

$\int \frac{dx}{1+a^2-2a\cos x} = 2\int \frac{dt}{(1+a)^2t^2+(1-a)^2} = \frac{2}{(1+a)^2}\int \frac{dt}{t^2+\left(\frac{1-a}{1+a}\right)^2}$

$= \frac{2}{(1+a)^2} \times \frac{1+a}{1-a}\arctan\left(\frac{1+a}{1-a}t\right) + c = \frac{2}{1-a^2}\arctan\left(\frac{1+a}{1-a}\tan\frac{x}{2}\right) + c$

359. $\int \frac{dx}{\sin^2 x+\tan^2 x} = \int \frac{dx}{\tan^2 x(\cos^2 x+1)} = \int \frac{1}{\tan^2 x(\cos^2 x+1)}\cdot\frac{\sec^2 x}{\sec^2 x}dx$

$= \int \frac{\sec^2 x}{\tan^2 x(1+\sec^2 x)}dx = \int \frac{\sec^2 x}{\tan^2 x(1+1+\tan^2 x)}dx = \int \frac{\sec^2 x}{\tan^2 x(2+\tan^2 x)}dx$

Let $u = \tan x$, then $du = \sec^2 x\,dx$, so: $\int \frac{dx}{\sin^2 x+\tan^2 x} = \int \frac{du}{u^2(2+u^2)}$

Decomposing into partial fractions : $\frac{1}{y(2+y)} = \frac{A}{y} + \frac{B}{2+y}$, so $1 = (2+y)A + By$

For $y = 0$, we get $A = \frac{1}{2}$ and for $y = -2$, we get $B = -\frac{1}{2}$.

So : $\frac{1}{y(2+y)} = \frac{1}{2y} - \frac{1}{2(2+y)}$, put $y = u^2$, we get : $\frac{1}{u^2(2+u^2)} = \frac{1}{2u^2} - \frac{1}{2(2+u^2)}$, so:

$\int \frac{dx}{\sin^2 x+\tan^2 x} = \int \left[\frac{1}{2u^2} - \frac{1}{2(2+u^2)}\right] du = -\frac{1}{2u} - \frac{1}{2\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right) + c$

$= -\frac{1}{2\tan x} - \frac{1}{2\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right) + c$

360. $\int \sqrt{2e^x - e^{2x}}dx = \int \sqrt{1 - 1 - (e^{2x} - 2e^x)}dx = \int \sqrt{1 - (e^{2x} - 2e^x + 1)}dx$

$= \int \sqrt{1 - (1 - e^x)^2}\, dx$, let $u = 1 - e^x$, then $du = -e^x dx = -(1-u)dx$, so:

$\int \sqrt{2e^x - e^{2x}}dx = -\int \sqrt{1-u^2}\frac{1}{1-u}du = -\int \frac{\sqrt{1-u^2}}{1-u}du = -\int \frac{\sqrt{1-u}\sqrt{1+u}}{\sqrt{1-u}\sqrt{1-u}}du = -\int \frac{\sqrt{1+u}}{\sqrt{1-u}}du$

$= -\int \frac{\sqrt{1+u}\sqrt{1+u}}{\sqrt{1-u}\sqrt{1+u}}du = -\int \frac{1+u}{\sqrt{1-u^2}}du = -\int \frac{du}{\sqrt{1-u^2}} - \int \frac{u}{\sqrt{1-u^2}}du$

$= -\int \frac{du}{\sqrt{1-u^2}} + \frac{1}{2}\int \frac{-2u}{\sqrt{1-u^2}}du = -\int \frac{du}{\sqrt{1-u^2}} + \frac{1}{2}\int \frac{(1-u^2)'}{\sqrt{1-u^2}}du = -\arcsin u + \sqrt{1-u^2} + c$

$= -\arcsin(1-e^2) + \sqrt{2e^x - e^{2x}} + c$

361. $\int \sqrt{\frac{e^x+2}{e^x-2}}dx$, let $u = \sqrt{\frac{e^x+2}{e^x-2}}$, then $u^2 = \frac{e^x+2}{e^x-2}$ and so $e^x = \frac{2u^2+2}{u^2-1}$, then

$e^x dx = \frac{4u(u^2-1)-2u(2u^2+2)}{(u^2-1)^2}du = \frac{-8u}{(u^2-1)^2}du$, then $dx = e^{-x}\frac{-8u}{(u^2-1)^2}du$, so

$dx = \frac{u^2-1}{2u^2+2}.\frac{-8u}{(u^2-1)^2}du = \frac{-4u}{u^4-1}du$, so we get:

$\int \sqrt{\frac{e^x+2}{e^x-2}}dx = \int u.\frac{-4u}{u^4-1}du = -4\int \frac{u^2}{u^4-1}du = -2\int \frac{(u^2-1)+(u^2+1)}{(u^2+1)(u^2-1)}du$

$= -2\int \left(\frac{1}{u^2+1} + \frac{1}{u^2-1}\right)du = -2\int \frac{1}{u^2+1}du + 2\int \frac{1}{1-u^2}du = -2\tan^{-1}u + 2\tanh^{-1}u + c$

$= -2\tan^{-1}\left(\sqrt{\frac{e^x+2}{e^x-2}}\right) + 2\tanh^{-1}\left(\sqrt{\frac{e^x+2}{e^x-2}}\right) + c$

362. $\int \sqrt{\frac{a-x}{x-b}}dx$, let $x = a\sin^2\theta + b\cos^2\theta$, then $dx = (2a\sin\theta\cos\theta - 2b\sin\theta\cos\theta)d\theta$,

then $dx = 2\sin\theta\cos\theta\,(a-b)d\theta$, so:

$\int \sqrt{\frac{a-x}{x-b}}dx = \int \sqrt{\frac{a-a\sin^2\theta-b\cos^2\theta}{a\sin^2\theta+b\cos^2\theta-b}}\,2\sin\theta\cos\theta\,(a-b)d\theta$

$= \int \sqrt{\frac{a(1-\sin^2\theta)-b\cos^2\theta}{a\sin^2\theta-b(1-\cos^2\theta)}}\,2\sin\theta\cos\theta\,(a-b)d\theta$

$= \int \sqrt{\frac{a\cos^2\theta-b\cos^2\theta}{a\sin^2\theta-b\sin^2\theta}}\,2\sin\theta\cos\theta\,(a-b)d\theta = \int \sqrt{\frac{(a-b)\cos^2\theta}{(a-b)\sin^2\theta}}\,2\sin\theta\cos\theta\,(a-b)d\theta$

$= 2\int \cot\theta\,.\sin\theta\,.\cos\theta\,(a-b)d\theta = 2(a-b)\int \frac{\cos\theta}{\sin\theta}.\sin\theta\,.\cos\theta$

$= 2(a-b)\int \cos^2\theta\, d\theta = 2(a-b)\int \left(\frac{1+\cos 2\theta}{2}\right)d\theta = (a-b)\int(1+\cos 2\theta)d\,\theta$

$= (a-b)\left(\theta + \frac{1}{2}\sin 2\theta\right) + c = (a-b)(\theta + \sin\theta\cos\theta) + c$

But $x = a\sin^2\theta + b\cos^2\theta$, so $x - b = a\sin^2\theta - b(1-\cos^2\theta) = a\sin^2\theta - b\sin^2\theta$, **so:**

$x - b = (a-b)\sin^2\theta \Rightarrow \sin\theta = \sqrt{\frac{x-b}{a-b}} \Rightarrow \theta = \arcsin\left(\sqrt{\frac{x-b}{a-b}}\right)$

$x - a = -a(1-\sin^2\theta) + b\cos^2\theta = -a\cos^2\theta + b\cos^2\theta = (b-a)\cos^2\theta$

$\Rightarrow \cos\theta = \sqrt{\frac{x-a}{b-a}}$, therefor**e:**

$\int\sqrt{\frac{a-x}{x-b}}\,dx = (a-b)\left[\arcsin\left(\sqrt{\frac{x-b}{a-b}}\right) + \sqrt{\frac{x-b}{a-b}} \times \sqrt{\frac{x-a}{b-a}}\right] + c$

$= (a-b)\arcsin\left(\sqrt{\frac{x-b}{a-b}}\right) + \sqrt{(x-b)(a-x)} + c$

363. $\int\sqrt{\frac{x}{x-a}}\,dx$, let $x = a\sec^2\theta$, then $dx = (2a\sec^2\theta\tan\theta)d\theta$ and

$x - a = a\sec^2\theta - a = a(\sec^2\theta - 1) = a\tan^2\theta$, then**:**

$\int\sqrt{\frac{x}{x-a}}\,dx = \int\sqrt{\frac{a\sec^2\theta}{a\tan^2\theta}}\,2a\sec^2\theta\tan\theta\,d\theta = \int\sqrt{\frac{1}{\cos^2\theta}\times\frac{\cos^2\theta}{\sin^2\theta}}\,2a\sec^2\theta\tan\theta\,d\theta$

$= \int\sqrt{\frac{1}{\sin^2\theta}}\,2a\sec^2\theta\tan\theta\,d\theta = 2a\int\frac{1}{\sin\theta}\sec^2\theta\tan\theta\,d\theta = 2a\int\frac{1}{\sin\theta}\frac{1}{\cos^2\theta}\frac{\sin\theta}{\cos\theta}\,d\theta$

$= 2a\int\sec^3\theta\,d\theta = 2a\int\sec\theta\sec^2\theta\,d\theta = 2a\int\sqrt{1+\tan^2\theta}\,\sec^2\theta\,d\theta$

Let $u = \tan\theta$, then $du = \sec^2\theta\,d\theta$, so**:** $2a\int\sec^3\theta\,d\theta = 2a\int\sqrt{1+u^2}\,du$

$= 2a\left(\frac{1}{2}u\sqrt{u^2+1} + \frac{1}{2}\ln\left|z+\sqrt{z^2+1}\right|\right) + c$

$= 2a\left(\frac{1}{2}\tan\theta\sec\theta + \frac{1}{2}\ln|\tan\theta+\sec\theta|\right) + c = a\tan\theta\sec\theta + a\ln|\tan\theta+\sec\theta| + c$

But $x = a\sec^2\theta$ and $x - a = a\tan^2\theta$, so $\sec\theta = \sqrt{\frac{x}{a}}$ and $\tan\theta = \sqrt{\frac{x-a}{a}}$, then**:**

$\int\sqrt{\frac{x}{x-a}}\,dx = a\sqrt{\frac{x-a}{a}}\sqrt{\frac{x}{a}} + a\ln\left|\sqrt{\frac{x-a}{a}} + \sqrt{\frac{x}{a}}\right| + c = \sqrt{x(x-a)} + a\ln\left|\sqrt{x}+\sqrt{x-a}\right| + c$

364. $\int\frac{x}{\sqrt{\sqrt{x}-\sqrt{a}}}\,dx$, let $\sqrt{x} = \sqrt{a}\cosh^2\theta \Rightarrow x = a\cosh^4\theta \Rightarrow dx = 4a\cosh^3\theta\sinh\theta\,d\theta$

$\int\frac{x}{\sqrt{\sqrt{x}-\sqrt{a}}}\,dx = \int\frac{a\cosh^4\theta.4a\cosh^3\theta\sinh\theta}{\sqrt{\sqrt{a}\cosh^2\theta-\sqrt{a}}}\,d\theta = \int\frac{a\cosh^4\theta.4a\cosh^3\theta\sinh\theta}{\sqrt{\sqrt{a}\cosh^2\theta-\sqrt{a}}}\,d\theta$

$= \int\frac{4a^2\cosh^7\theta\sinh\theta}{\sqrt{\sqrt{a}\left(\cosh^2\theta-1\right)}}\,d\theta = \int\frac{4a^2\cosh^7\theta\sinh\theta}{\sqrt{\sqrt{a}}\sqrt{\sinh^2\theta}}\,d\theta = \int\frac{4a^2\cosh^7\theta\sinh\theta}{\sqrt{\sqrt{a}}\sinh\theta}\,d\theta$

$= \frac{4a^2}{\sqrt{\sqrt{a}}}\int\cosh^7\theta\,d\theta = \frac{4a^2}{\sqrt{\sqrt{a}}}\int\cosh^6\theta.\cosh\theta\,d\theta = \frac{4a^2}{\sqrt{\sqrt{a}}}\int(\cosh^2\theta)^2.\cosh\theta\,d\theta$

$= \frac{4a^2}{\sqrt{\sqrt{a}}}\int(1+\sinh^2\theta)^2.\cosh\theta\,d\theta = \frac{4a^2}{\sqrt{\sqrt{a}}}\int(1+3\sinh^2\theta+3\sinh^4\theta+\sinh^6\theta)^2.\cosh\theta\,d\theta$

$= \frac{4a^2}{\sqrt{\sqrt{a}}}\int(1+3\sinh^2\theta+3\sinh^4\theta+\sinh^6\theta)^2.(\sinh\theta)'d\theta$

$= \frac{4a^2}{\sqrt{\sqrt{a}}}\left(\sinh\theta+\sinh^3\theta+\frac{3}{5}\sinh^5\theta+\frac{1}{7}\sinh^7\theta\right) + c$

But $\cosh^2\theta = \sqrt{\frac{x}{a}} \Rightarrow 1+\sinh^2\theta = \sqrt{\frac{x}{a}} \Rightarrow \sinh^2\theta = \sqrt{\frac{x}{a}}-1 \Rightarrow \sinh\theta = \sqrt{\sqrt{\frac{x}{a}}-1}$, so:

$$\int \frac{x}{\sqrt{\sqrt{x}-\sqrt{a}}}dx = \frac{4a^2}{\sqrt{\sqrt{a}}}\left[\sqrt{\sqrt{\frac{x}{a}}-1}+\left(\sqrt{\frac{x}{a}}-1\right)^{\frac{3}{2}}+\frac{3}{5}\left(\sqrt{\frac{x}{a}}-1\right)^{\frac{5}{2}}+\frac{1}{7}\left(\sqrt{\frac{x}{a}}-1\right)^{\frac{3}{7}}\right]+c$$

365. $\int \sqrt{1+\frac{\pi}{x}}dx = \int \sqrt{\frac{x+\pi}{x}}dx = \int \frac{\sqrt{x+\pi}}{\sqrt{x}}dx$

Let $u = \sqrt{x+\pi}$, then $u^2 = x+\pi$, so $x = u^2-\pi$, then $dx = 2udu$, so:

$\int \sqrt{1+\frac{\pi}{x}}dx = \int \frac{u.2udu}{\sqrt{u^2-\pi}} = 2\int \frac{u^2}{\sqrt{u^2-\pi}}dx$, let $u = \sqrt{\pi}\cosh\theta$, then $u^2 = \pi\cosh^2\theta$ and $du = \sqrt{\pi}\sinh\theta\, d\theta$, then we get:

$$\int \sqrt{1+\frac{\pi}{x}}dx = 2\int \frac{\pi\cosh^2\theta.\sqrt{\pi}\sinh\theta}{\sqrt{\pi\cosh^2\theta-\pi}}d\theta = 2\pi\sqrt{\pi}\int \frac{\cosh^2\theta.\sinh\theta}{\sqrt{\pi}\sqrt{\cosh^2\theta-1}}d\theta = \frac{2\pi\sqrt{\pi}}{\sqrt{\pi}}\int \frac{\cosh^2\theta.\sinh\theta}{\sqrt{\sinh^2\theta}}d\theta$$

$$= 2\pi\int \frac{\cosh^2\theta.\sinh\theta}{\sinh\theta}d\theta = 2\pi\int \cosh^2\theta\, d\theta = 2\pi\int\left(\frac{\cosh 2\theta+1}{2}\right)d\theta = \pi\int(1+\cosh 2\theta)d\theta$$

$$= \pi\theta+\frac{\pi}{2}\sinh 2\theta + c = \pi\theta+\pi\sinh\theta\cosh\theta + c$$

But $\cosh\theta = \frac{u}{\sqrt{\pi}}$ with $\sinh\theta = \sqrt{\cosh^2\theta-1} = \sqrt{\frac{u^2}{\pi}-1} = \frac{\sqrt{u^2-\pi}}{\sqrt{\pi}}$, then we get:

$$\int \sqrt{1+\frac{\pi}{x}}dx = \pi\cosh^{-1}\left(\frac{u}{\sqrt{\pi}}\right)+\pi.\frac{u}{\sqrt{\pi}}.\frac{\sqrt{u^2-\pi}}{\sqrt{\pi}}+c = \pi\cosh^{-1}\left(\sqrt{1+\frac{\pi}{x}}\right)+\sqrt{x+\pi}\sqrt{x}+c$$

$$\int \sqrt{1+\frac{\pi}{x}}dx = \pi\cosh^{-1}\left(\sqrt{1+\frac{\pi}{x}}\right)+\sqrt{x^2+\pi x}+c$$

366. $\int \frac{\sqrt{\sqrt{\sqrt{x}}}}{\sqrt{\sqrt{x}}+1}dx = \int \frac{\sqrt{\sqrt{x^{\frac{1}{2}}}}}{\sqrt{x^{\frac{1}{2}}}+1}dx = \int \frac{\sqrt{x^{\frac{1}{4}}}}{x^{\frac{1}{4}}+1}dx = \int \frac{x^{\frac{1}{8}}}{x^{\frac{1}{4}}+1}dx$

Let $x^{\frac{1}{4}} = \tan^2\theta$, then $x = \tan^8\theta$, so $dx = 8\tan^7\theta\,.\sec^2\theta\, d\theta$, then we get:

$$\int \frac{\sqrt{\sqrt{\sqrt{x}}}}{\sqrt{\sqrt{x}}+1}dx = \int \frac{\tan\theta.8\tan^7\theta.\sec^2\theta d\theta}{\tan^2\theta+1}d\theta = 8\int \frac{\tan^8\theta.\sec^2\theta d\theta}{\sec^2\theta}d\theta = 8\int \tan^8\theta\, d\theta$$

$$= 8\int \tan^6\theta\tan^2\theta\, d\theta = 8\int \tan^6\theta\,(\sec^2\theta-1)d\theta = 8\int \tan^6\theta\sec^2\theta\, d\theta - 8\int \tan^6\theta\, d\theta$$

$$= \frac{8}{7}\tan^7\theta - 8\int \tan^4\theta\,(\sec^2\theta-1)\, d\theta = \frac{8}{7}\tan^7\theta-\frac{8}{5}\tan^5\theta+8\int \tan^4\theta\, d\theta$$

$$= \frac{8}{7}\tan^7\theta-\frac{8}{5}\tan^5\theta+8\int \tan^2\theta\,(\sec^2\theta-1)d\theta$$

$$= \frac{8}{7}\tan^7\theta-\frac{8}{5}\tan^5\theta+\frac{8}{3}\tan^3\theta-8\int(\sec^2\theta-1)d\theta$$

$$= \frac{8}{7}\tan^7\theta-\frac{8}{5}\tan^5\theta+\frac{8}{3}\tan^3\theta-8\tan\theta+8\theta+c$$

$$= \frac{8}{7}\left(x^{\frac{1}{8}}\right)^7 - \frac{8}{5}\left(x^{\frac{1}{8}}\right)^5 + \frac{8}{3}\left(x^{\frac{1}{8}}\right)^3 - 8x^{\frac{1}{8}} + 8\arctan\left(x^{\frac{1}{8}}\right) + c$$

$$= \frac{8}{7}x^{\frac{7}{8}} - \frac{8}{5}x^{\frac{5}{8}} + \frac{8}{3}x^{\frac{3}{8}} - 8x^{\frac{1}{8}} + 8\arctan\left(x^{\frac{1}{8}}\right) + c$$

367. $\int \frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}dx = \int \frac{\left(\frac{x+a}{x+b}\right)^{\frac{1}{n}}}{(x+a)(x+b)}dx = \int \frac{\left(\frac{x+a}{x+b}\right)^{\frac{1}{n}}}{(x+a)(x+b)} \times \left(\frac{x+b}{x+b}\right)dx$

$= \int \left(\frac{x+a}{x+b}\right)^{\frac{1}{n}} \times \frac{1}{(x+b)^2}\left(\frac{x+b}{x+a}\right)dx = \int \left(\frac{x+a}{x+b}\right)^{\frac{1}{n}} \times \frac{1}{(x+b)^2}\left(\frac{x+a}{x+b}\right)^{-1}dx = \int \left(\frac{x+a}{x+b}\right)^{\frac{1}{n}-1}\frac{1}{(x+b)^2}\,dx$

$= \frac{1}{b-a}\int \left(\frac{x+a}{x+b}\right)^{\frac{1}{n}-1}\frac{b-a}{(x+b)^2}\,dx$, let $t = \frac{x+a}{x+b}$, then $dt = \frac{x+b-x-a}{(x+b)^2}dx = \frac{b-a}{(x+b)^2}dx$, so:

$\int \frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}}dx = \frac{1}{b-a}\int t^{\frac{1}{n}-1}dt = \frac{1}{b-a} \times \frac{t^{\frac{1}{n}}}{\frac{1}{n}} + c = \frac{nt^{\frac{1}{n}}}{b-a} + c$

$= \frac{n}{b-a}\left(\frac{x+a}{x+b}\right)^{\frac{1}{n}} + c$

368. $\int \frac{dx}{x(x^n+1)} = \frac{1}{n}\int \frac{nx^{n-1}}{x^n(x^n+1)}dx$, let $t = x^n + 1$, then $dt = nx^{n-1}dx$, so:

$\int \frac{dx}{x(x^n+1)} = \frac{1}{n}\int \frac{dt}{t(t-1)} = \frac{1}{n}\int \frac{-t+1+t}{t(t-1)}dt = \frac{1}{n}\int \frac{-(t-1)+t}{t(t-1)}dt = \frac{1}{n}\int \left(-\frac{1}{t} + \frac{1}{t-1}\right)dt$

$= -\ln|t| + \ln|t-1| + c = \ln\left|\frac{t-1}{t}\right| + c = \ln\left|\frac{x^n}{x^n+1}\right| + c$

369. $\int \left(x + \sqrt{1+x^2}\right)^n dx$, let $t = x + \sqrt{1+x^2}$, then $dt = \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)dx$

$dt = \left(\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}\right)dx = \left(\frac{t}{\sqrt{1+x^2}}\right)dx$

But $x + \sqrt{1+x^2} = t$, then , $\frac{1}{t} = \frac{1}{\sqrt{1+x^2}+x} = \frac{\sqrt{1+x^2}-x}{1+x^2-x^2} = \sqrt{1+x^2} - x$

$x + \sqrt{1+x^2} + \sqrt{1+x^2} - x = 2\sqrt{1+x^2} = t + \frac{1}{t}$, so $\sqrt{1+x^2} = \frac{t^2+1}{2t}$ and so:

$dt = \left(\frac{t}{\frac{t^2+1}{2t}}\right)dx = \left(\frac{2t^2}{t^2+1}\right)dx$, then $dx = \left(\frac{t^2+1}{2t^2}\right)dt$, so:

$\int \left(x + \sqrt{1+x^2}\right)^n dx = \int t^n\left(\frac{t^2+1}{2t^2}\right)dt = \frac{1}{2}\int t^{n-2}(t^2+1)dt = \frac{1}{2}\int (t^n + t^{n-2})dt$

$= \frac{1}{2n+2}t^{n+1} + \frac{1}{2n-2}t^{n-1} + c = \frac{1}{2n+2}\left(x + \sqrt{1+x^2}\right)^{n+1} + \frac{1}{2n-2}\left(x + \sqrt{1+x^2}\right)^{n-1} + c$

370. $\int \frac{x}{1-x^2+\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{-2x}{1-x^2+\sqrt{1-x^2}}dx$, let $u = 1 - x^2$, then $du = -2xdx$, so:

$\int \frac{x}{1-x^2+\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{du}{u+\sqrt{u}}$, let $v = \sqrt{u}$, then $v^2 = u$, so $2vdv = du$, then:

$\int \frac{x}{1-x^2+\sqrt{1-x^2}}dx = -\frac{1}{2}\int \frac{2v}{v^2+v}dv = \int \frac{1}{v+1}dv = \ln|v+1| + c = \ln\left|\sqrt{u}+1\right| + c$

$= \ln\left|\sqrt{x^2-1}+1\right| + c$

371. $\int \frac{1}{\left(4x^2+8x+13\right)^2}dx = \int \frac{1}{\left[4\left(x^2+2x+1\right)+13-4\right]^2}dx = \int \frac{1}{\left[4(x+1)^2+9\right]^2}dx$

Let $u^2 = 4(x+1)^2$ and $u = 2(x+1)$, so $du = 2dx$ and so:

$\int \frac{1}{\left(4x^2+8x+13\right)^2}dx = \frac{1}{2}\int \frac{1}{\left(u^2+9\right)^2}du$, let $u = 3\tan\theta$, then $du = 3\sec^2\theta\, d\theta$, so:

$\int \frac{1}{\left(4x^2+8x+13\right)^2}dx = \frac{1}{2}\int \frac{3\sec^2\theta}{\left[9\left(1+\tan^2\theta\right)\right]^2}d\theta = \frac{1}{2}\int \frac{3\sec^2\theta}{81\left(\sec^2\theta\right)^2}d\theta = \frac{1}{54}\int \frac{\sec^2\theta}{\sec^4\theta}d\theta$

$= \frac{1}{54}\int \frac{1}{\sec^2\theta}d\theta = \frac{1}{54}\int \cos^2\theta\, d\theta = \frac{1}{54}\int \left(\frac{1+\cos 2\theta}{2}\right)d\theta = \frac{1}{108}\left(\theta + \frac{1}{2}\sin 2\theta\right) + c$

$= \frac{1}{108}\theta + \frac{1}{108}\sin\theta\cos\theta + c = \frac{1}{108}\left(\arctan\frac{u}{3} + \frac{u}{\sqrt{u^2+9}}.\frac{3}{\sqrt{u^2+9}}\right) + c$

$= \frac{1}{108}\left(\arctan\frac{u}{3} + \frac{3u}{u^2+9}\right) + c = \frac{1}{108}\left[\arctan\frac{2(x+1)}{3} + \frac{6(x+1)}{4(x+1)^2+9}\right] + c$

372. $\int \frac{dx}{(x+a)^2(x+b)^2}$, let $t = x+a$, then $dt = dx$, so

$\int \frac{dx}{(x+a)^2(x+b)^2} = \int \frac{dt}{t^2(t+b-a)^2} = \int \frac{dt}{t^2(t+c)^2}$ (setting $c = b-a$)

Decomposition into partial fractions : $\frac{1}{t^2(t+c)^2} = \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t+c} + \frac{D}{(t+c)^2}$, then:

$At(t+c)^2 + B(t+c)^2 + Ct^2(t+c) + Dt^2 = 1$

$At^3 + 2Act^2 + Ac^2t + Bt^2 + 2Bct + Bc^2 + Ct^3 + Cct^2 + Dt^2 = 1$, then:

$(A+C)t^3 + (2Ac+B+D+Cc)t^2 + (Ac^2+2Bc)t + Bc^2 = 1$, then we get the following

system $\begin{cases} A+C=0 \\ 2Ac+B+D+Cc=0 \\ Ac^2+2Bc=0 \\ Bc^2=1 \end{cases}$, by solving this system we get: $A = -\frac{2}{c^3}$, $B = \frac{1}{c^2}$, $C = \frac{2}{c^3}$ and

$D = \frac{1}{c^2}$, so $\int \frac{dx}{(x+a)^2(x+b)^2} = \int \left(\frac{-\frac{2}{c^3}}{t} + \frac{\frac{1}{c^2}}{t^2} + \frac{\frac{2}{c^3}}{t+c} + \frac{\frac{1}{c^2}}{(t+c)^2}\right)dt$

$= \frac{2}{c^3}\ln\left(\frac{t+1}{t}\right) - \frac{1}{c^2}\left(\frac{1}{t+c}\right) - \frac{1}{c^2}\left(\frac{1}{t}\right) + c = \frac{2}{(b-a)^3}\ln\left(\frac{x+b}{x+a}\right) - \frac{1}{(b-a)^2}\left(\frac{1}{x+b} + \frac{1}{x+a}\right) + c$

373. $\int \sqrt{2ax-x^2}dx$, let $x = a(1-\cos\theta)$, then $dx = a\sin\theta\, d\theta$ and

$\sqrt{2ax-x^2} = \sqrt{2a^2(1-\cos\theta) - a^2(1-\cos\theta)^2}$

$= \sqrt{a^2(2-2\cos\theta-1+2\cos\theta-\cos^2\theta)} = \sqrt{a^2(1-\cos^2\theta)} = \sqrt{a^2\sin^2\theta} = a\sin\theta$, so:

$\int \sqrt{2ax-x^2}dx = \int a\sin\theta \times a\sin\theta\, d\theta = a^2\int \sin^2\theta\, d\theta = \frac{1}{2}a^2\int (1-\cos 2\theta)d\theta$

$= \frac{1}{2}a^2\left(\theta - \frac{1}{2}\sin 2\theta\right) + c = \frac{1}{2}a^2\theta - \frac{1}{4}a^2\sin 2\theta + c$

But $x = a(1-\cos\theta)$, then $\cos\theta = \frac{a-x}{a}$ and so $\theta = \arccos\left(\frac{a-x}{a}\right)$

$\sin 2\theta = 2\cos\theta\sin\theta = 2\left(\frac{a-x}{a}\right)\sqrt{1-\left(\frac{a-x}{a}\right)^2} = 2\left(\frac{a-x}{a}\right)\sqrt{\frac{2ax-x^2}{a^2}} = 2\left(\frac{a-x}{a^2}\right)\sqrt{2ax-x^2}$

Then: $\sqrt{2ax-x^2} = \frac{1}{2}a^2\arccos\left(\frac{a-x}{a}\right) - \frac{1}{2}(a-x)\sqrt{2ax-x^2}+c$

374. $\int \frac{1}{\sqrt{x(x-a)}}dx$, let $x = a\sec^2\theta$, then $dx = 2a\sec^2\theta\tan\theta\, d\theta$ and

$\sqrt{x(x-a)} = \sqrt{a\sec^2\theta\,(a\sec^2\theta - a)} = a\sqrt{\sec^2\theta\,(\sec^2\theta-1)} = a\sqrt{\sec^2\theta\tan^2\theta}$

$\sqrt{x(x-a)} = a\sec\theta\tan\theta$, so:

$\int \frac{1}{\sqrt{x(x-a)}}dx = \int \frac{1}{a\sec\theta\tan\theta}\, 2a\sec^2\theta\tan\theta\, d\theta = 2\int\sec\theta\, d\theta = 2\ln|\sec\theta+\tan\theta|+c$

But $x = a\sec^2\theta$, then $\sec\theta = \sqrt{\frac{x}{a}}$ and $\tan\theta = \sqrt{\sec^2\theta-1} = \sqrt{\frac{x}{a}-1} = \sqrt{\frac{x-a}{a}}$, so:

$\int \frac{1}{\sqrt{x(x-a)}}dx = 2\ln\left|\sqrt{\frac{x}{a}}+\sqrt{\frac{x-a}{a}}\right| + c = 2\ln\left|\sqrt{x}+\sqrt{x-a}\right|+k$

375. $\int\sqrt{\frac{x^2-a}{x^2-b}}dx$, let $t^2 = \frac{x^2-a}{x^2-b}$, then $x^2 = \frac{bt^2-a}{t^2-1}$, so $xdx = (a-b)\frac{t}{\left(t^2-1\right)^2}dt$ and so

$dx = \frac{1}{x}(a-b)\frac{t}{\left(t^2-1\right)^2}dt = \frac{(a-b)t}{\sqrt{bt^2-a}\left(t^2-1\right)^{\frac{3}{2}}}dt$, so:

$\int\sqrt{\frac{x^2-a}{x^2-b}}dx = \int\frac{(a-b)t^2}{\sqrt{bt^2-a}\left(t^2-1\right)^{\frac{3}{2}}}dt$, let $u = t^2 \Longrightarrow t = \sqrt{u}$ and $dt = \frac{1}{2\sqrt{u}}du$, so:

$\int\sqrt{\frac{x^2-a}{x^2-b}}dx = \frac{a-b}{2}\int\frac{\sqrt{u}}{\sqrt{bu-a}(u-1)^{\frac{3}{2}}}du$, let $u = \sec^2\theta$, then $du = 2\sec^2\theta\tan\theta$, so:

$\int\sqrt{\frac{x^2-a}{x^2-b}}dx = \frac{a-b}{2}\int\frac{2\sec^2\theta\tan\theta d\theta}{\sqrt{b\sec^2\theta - a\times\sec^3\theta}} = (a-b)\int\frac{\cos\theta\tan\theta}{\sqrt{b-a\cos^2\theta}}d\theta$

$= (a-b)\int\frac{\sin\theta}{\sqrt{b-a\cos^2\theta}}d\theta$, let $w = \cos\theta$, $dw = -\sin\theta\, d\theta$, so:

$\int\sqrt{\frac{x^2-a}{x^2-b}}dx = (b-a)\int\frac{dw}{\sqrt{b-aw^2}} = \frac{b-a}{\sqrt{a}}\arcsin\left(\sqrt{\frac{a}{b}}w\right)+c = \frac{b-a}{\sqrt{a}}\arcsin\left(\sqrt{\frac{a}{b}}\cos\theta\right)+c$

$= \frac{b-a}{\sqrt{a}}\arcsin\left(\sqrt{\frac{a}{bu}}\right)+c = \frac{b-a}{\sqrt{a}}\arcsin\left(\frac{1}{t}\sqrt{\frac{a}{b}}\right)+c = \frac{b-a}{\sqrt{a}}\arcsin\left(\sqrt{\frac{a\left(x^2-b\right)}{b\left(x^2-a\right)}}\right)+c$

376. $\int \frac{e^x}{(e^{2x}+8e^x+7)^{\frac{3}{2}}}dx$, let $u=e^x$, then $du=e^x dx$, so:

$$\int \frac{e^x}{(e^{2x}+8e^x+7)^{\frac{3}{2}}}dx=\int \frac{1}{(u^2+8u+7)^{\frac{3}{2}}}dx=\int \frac{1}{(u^2+8u+16-9)^{\frac{3}{2}}}dx=\int \frac{1}{\left[(u+4)^2-9\right]^{\frac{3}{2}}}dx$$

Let $u+4=3\sec t$, then $du=3\sec t\tan t\,dt$, so:

$$\int \frac{e^x}{(e^{2x}+8e^x+7)^{\frac{3}{2}}}dx=\int \frac{3\sec t\tan t}{\left[3^2(\sec^3 t-1)\right]^{\frac{3}{2}}}dt=\int \frac{3\sec t\tan t}{27(\tan^2 t)^{\frac{3}{2}}}dt=\int \frac{\sec t\tan t}{9\tan^3 t}dt=\frac{1}{9}\int \frac{\sec t}{\tan^2 t}dt$$

$$=\frac{1}{9}\int \frac{\cos t}{\sin^2 t}dt=\frac{1}{9}\int \frac{(\sin t)'}{\sin^2 t}dt=-\frac{1}{9\sin t}+c=-\frac{\sec t}{9\tan t}+c=-\frac{\sec t}{9\sqrt{\sec^2 t-1}}+c$$

$$=-\frac{\frac{u+4}{3}}{9\sqrt{\left(\frac{u+4}{3}\right)^2-1}}+c=-\frac{\frac{u+4}{3}}{3\sqrt{u^2+8u+7}}+c=-\frac{u+4}{9\sqrt{u^2+8u+7}}+c=-\frac{e^x+4}{9\sqrt{e^{2x}+8e^x+7}}+c$$

377. $\int \frac{1}{x^8-1}dx=\frac{1}{2}\int \frac{(x^4+1)-(x^4-1)}{x^8-1}dx=\frac{1}{2}\int \frac{x^4+1}{x^8-1}dx-\frac{1}{2}\int \frac{x^4-1}{x^8-1}dx$

Let's evaluate: $\int \frac{x^4+1}{x^8-1}dx=\int \frac{x^4+1}{(x^4-1)(x^4+1)}dx=\int \frac{1}{x^4-1}dx=\frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4-1}dx$

$$=\frac{1}{2}\int \frac{x^2+1}{x^4-1}dx-\frac{1}{2}\int \frac{x^2-1}{x^4-1}dx=\frac{1}{2}\int \frac{x^2+1}{(x^2+1)(x^2-1)}dx-\frac{1}{2}\int \frac{x^2-1}{(x^2+1)(x^2-1)}dx$$

$$=\frac{1}{2}\int \frac{1}{x^2-1}dx-\frac{1}{2}\int \frac{1}{x^2+1}dx=\frac{1}{4}\int \frac{(x+1)-(x-1)}{x^2-1}dx-\frac{1}{2}\arctan x$$

$$=\frac{1}{4}\int \frac{(x+1)-(x-1)}{(x-1)(x+1)}dx-\frac{1}{2}\arctan x=\frac{1}{4}\left[\int \frac{1}{x-1}dx-\int \frac{1}{x+1}dx\right]-\frac{1}{2}\arctan x$$

$$=\frac{1}{4}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{2}\arctan x+c$$

Let's evaluate: $\int \frac{x^4-1}{x^8-1}dx=\int \frac{x^4-1}{(x^4+1)(x^4-1)}dx=\int \frac{1}{x^4+1}dx=\frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^4+1}dx$

$$=\frac{1}{2}\int \frac{x^2+1}{x^4+1}dx-\frac{1}{2}\int \frac{x^2-1}{x^4+1}dx=\frac{1}{2}\int \frac{1+x^{-2}}{x^2+x^{-2}}dx-\frac{1}{2}\int \frac{1-x^{-2}}{x^2+x^{-2}}dx$$

$$=\frac{1}{2}\int \frac{1+x^{-2}}{(x-x^{-1})^2+2}dx-\frac{1}{2}\int \frac{1-x^{-2}}{(x+x^{-1})^2-2}dx=\frac{1}{2}\int \frac{(x-x^{-1})'}{(x-x^{-1})^2+2}dx-\frac{1}{2}\int \frac{1-x^{-2}}{(x+x^{-1})^2-2}dx$$

$$=\frac{1}{2\sqrt{2}}\arctan\left(\frac{x-x^{-1}}{\sqrt{2}}\right)-\frac{1}{4\sqrt{2}}\ln\left|\frac{x+x^{-1}-\sqrt{2}}{x+x^{-1}+\sqrt{2}}\right|+c\text{, so:}$$

$$\int \frac{1}{x^8-1}dx=\frac{1}{8}\ln\left|\frac{x-1}{x+1}\right|-\frac{1}{4}\arctan x-\frac{1}{4\sqrt{2}}\arctan\left(\frac{x^2-1}{x\sqrt{2}}\right)+\frac{1}{8\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|+c$$

378. $\int \sqrt{\tan x}\,dx$, let $u=\sqrt{\tan x}$, then $du=\frac{1+\tan^2 x}{2\sqrt{\tan x}}dx=\frac{1+u^4}{2u}dx$, so:

$$\int \sqrt{\tan x}\,dx=\int \frac{2u^2}{1+u^4}du=\int \frac{\frac{2u^2}{u^2}}{\frac{1+u^4}{u^2}}du=\int \frac{2}{u^2+\frac{1}{u^2}}du=\int \frac{2}{\left(u+\frac{1}{u}\right)^2-2}du$$

$$= \int \left[\frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2} + \frac{1+\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2} \right] du = \int \frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2} du + \int \frac{1+\frac{1}{u^2}}{\left(u-\frac{1}{u}\right)^2+2} du = (1) + (2) \ldots$$

Evaluating 1: Let $t = u + \frac{1}{u}$, then $dt = \left(1 - \frac{1}{u^2}\right) du$, so:

$$\int \frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)^2-2} du = \int \frac{dt}{t^2-2} = \int \frac{dt}{(t-\sqrt{2})(t+\sqrt{2})} = \frac{1}{2\sqrt{2}} \int \frac{t+\sqrt{2}-(t-\sqrt{2})}{(t-\sqrt{2})(t+\sqrt{2})} dt$$

$$= \frac{1}{2\sqrt{2}} \int \left(\frac{1}{t-\sqrt{2}} - \frac{1}{t+\sqrt{2}} \right) dt = \frac{1}{2\sqrt{2}} \left(\ln|t-\sqrt{2}| - \ln|t+\sqrt{2}| \right) + c = \frac{1}{2\sqrt{2}} \ln \left| \frac{t-\sqrt{2}}{t+\sqrt{2}} \right| + c$$

$$= \frac{1}{2\sqrt{2}} \ln \left| \frac{u+\frac{1}{u}-\sqrt{2}}{u+\frac{1}{u}+\sqrt{2}} \right| + c$$

Evaluating 2: Let $w = u - \frac{1}{u}$, then $dw = \left(1 + \frac{1}{u^2}\right) du$, so:

$$\int \frac{1+\frac{1}{u^2}}{\left(u-\frac{1}{u}\right)^2+2} du = \int \frac{dw}{w^2+2} = \int \frac{dw}{w^2+\left(\sqrt{2}\right)^2} = \frac{1}{\sqrt{2}} \arctan\left(\frac{w}{\sqrt{2}}\right) + c = \frac{1}{\sqrt{2}} \arctan\left(\frac{u-\frac{1}{u}}{\sqrt{2}}\right) + c$$

So: $\int \sqrt{\tan x}\, dx = \frac{1}{2\sqrt{2}} \ln \left| \frac{u+\frac{1}{u}-\sqrt{2}}{u+\frac{1}{u}+\sqrt{2}} \right| + \frac{1}{\sqrt{2}} \arctan\left(\frac{u-\frac{1}{u}}{\sqrt{2}}\right) + k$

Therefore: $\int \sqrt{\tan x}\, dx = \frac{1}{2\sqrt{2}} \ln \left| \frac{\sqrt{\tan x}+\sqrt{\cot x}-\sqrt{2}}{\sqrt{\tan x}+\sqrt{\cot x}+\sqrt{2}} \right| + \frac{1}{\sqrt{2}} \arctan\left(\frac{\sqrt{\tan x}-\sqrt{\cot x}}{\sqrt{2}}\right) + k$

379. $\int \sqrt[3]{\tan x}\, dx$, let $u = \sqrt[3]{\tan x}$, $u^3 = \tan x$ and $3u^2 du = \sec^2 x\, dx$

$u^6 + 1 = \tan^2 x + 1 = \sec^2 x dx$, so $3u^2 du = (u^6+1) dx$, so:

$\int \sqrt[3]{\tan x}\, dx = \int u . \frac{3u^2}{u^6+1} du = \int \frac{3u^3}{u^6+1} du = \frac{3}{2} \int \frac{2u^2 . u}{\left(u^2\right)^3+1} du$, let $t = u^2$, so $dt = 2u du$, then:

$$\int \sqrt[3]{\tan x}\, dx = \frac{3}{2} \int \frac{t}{t^3+1} dt = \frac{3}{2} \int \frac{t}{(t+1)\left(t^2-t+1\right)} dt = \frac{3}{2} \times \frac{1}{3} \int \left(-\frac{1}{t+1} + \frac{t+1}{t^2-t+1} \right) dt$$

$$= \frac{1}{2} \int \left(-\frac{1}{t+1} + \frac{t+1}{t^2-t+1} \right) dt$$

Evaluating: $\int \frac{t+1}{t^2-t+1} dt = \int \frac{t+1}{\left(t-\frac{1}{2}\right)^2+\frac{3}{4}} dt$, let $w = t - \frac{1}{2}$ then $dw = dt$, so:

$$\int \frac{t+1}{t^2-t+1} dt = \int \frac{w+\frac{3}{2}}{w^2+\frac{3}{4}} dt = \int \frac{w}{w^2+\frac{3}{4}} dt + \frac{3}{2} \int \frac{1}{w^2+\left(\frac{\sqrt{3}}{2}\right)^2} dt = \frac{1}{2} \int \frac{2w}{w^2+\frac{3}{4}} dt + \frac{3}{2} \int \frac{1}{w^2+\left(\frac{\sqrt{3}}{2}\right)^2} dt$$

$$= \frac{1}{2} \ln\left| w^2 + \frac{3}{4} \right| + \frac{3}{2} \times \frac{2}{\sqrt{3}} \arctan\left(\frac{2}{\sqrt{3}} w\right) + c = \frac{1}{2} \ln\left| w^2 + \frac{3}{4} \right| + \sqrt{3} \arctan\left(\frac{2}{\sqrt{3}} w\right) + c$$

$= \frac{1}{2} \ln|t^2 - t + 1| + \sqrt{3} \arctan\left[\frac{2}{\sqrt{3}}\left(t - \frac{1}{2}\right)\right] + c$, so:

Remark : $\frac{t}{(t+1)\left(t^2-t+1\right)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$, $A = \left[\frac{t}{t^2-t+1}\right]_{t=-1} = -\frac{1}{3}$, for $t = 0$ we get $C = \frac{1}{3}$

and for $t = 1$, we get $B = \frac{1}{3}$.

$$\int \sqrt[3]{\tan x}\, dx = \frac{1}{2} \int -\frac{1}{t+1} dt + \frac{1}{2} \int \frac{t+1}{t^2-t+1} dt$$

$$= -\frac{1}{2}\ln|t+1| + \frac{1}{4}\ln|t^2-t+1| + \frac{\sqrt{3}}{2}\arctan\left(\frac{2t-1}{\sqrt{3}}\right) + c$$

$$= -\frac{1}{2}\ln|u^2+1| + \frac{1}{4}\ln|u^4-u^2+1| + \frac{\sqrt{3}}{2}\arctan\left(\frac{2u^2-1}{\sqrt{3}}\right) + c$$

$$= -\frac{1}{2}\ln\left|\tan^{\frac{2}{3}}x+1\right| + \frac{1}{4}\ln\left|\tan^{\frac{4}{3}}x-\tan^{\frac{2}{3}}x+1\right| + \frac{\sqrt{3}}{2}\arctan\left(\frac{2\tan^{\frac{2}{3}}x-1}{\sqrt{3}}\right) + c$$

380. $\int\sqrt{1+\tan x}\,dx$, let $u=\sqrt{1+\tan x}$, $u^2=1+\tan x$ then $u^2-1=\tan x$, so:

$x=\arctan(u^2-1)$, then $dx=\frac{2u}{1+(u^2-1)^2}du=\frac{2u}{u^4-2u^2+2}du$, then:

$$\int\sqrt{1+\tan x}\,dx=\int\frac{2u^2}{u^4-2u^2+2}du=\int\frac{2}{u^2-2+\frac{2}{u^2}}du=\int\frac{1+\frac{\sqrt{2}}{u^2}+1-\frac{\sqrt{2}}{u^2}}{u^2-2+\frac{2}{u^2}}du$$

$$=\int\frac{1+\frac{\sqrt{2}}{u^2}}{\left(u-\frac{\sqrt{2}}{u}\right)^2+2\sqrt{2}-2}du+\int\frac{1-\frac{\sqrt{2}}{u^2}}{\left(u+\frac{\sqrt{2}}{u}\right)^2-2\sqrt{2}-2}du \ = \mathbf{(1)+(2)\ldots}$$

Evaluating 1: Let $w=u-\frac{\sqrt{2}}{u}$, then $dw=\left(1+\frac{\sqrt{2}}{u^2}\right)du$, then:

$$\int\frac{1+\frac{\sqrt{2}}{u^2}}{\left(u-\frac{\sqrt{2}}{u}\right)^2+2\sqrt{2}-2}du=\int\frac{dw}{w^2+\left(\sqrt{2\sqrt{2}-2}\right)^2}=\frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left(\frac{w}{\sqrt{2\sqrt{2}-2}}\right)+c$$

$$=\frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left(\frac{u-\frac{\sqrt{2}}{u}}{\sqrt{2\sqrt{2}-2}}\right)+c=\frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left[\frac{1}{\sqrt{2\sqrt{2}-2}}\left(u-\frac{\sqrt{2}}{u}\right)\right]+c$$

Evaluating 2: Let $v=u+\frac{\sqrt{2}}{u}$, then $dv=\left(1-\frac{\sqrt{2}}{u^2}\right)du$, then:

$$\int\frac{1-\frac{\sqrt{2}}{u^2}}{\left(u+\frac{\sqrt{2}}{u}\right)^2-2\sqrt{2}-2}du=\int\frac{dv}{v^2-2\sqrt{2}-2}=\int\frac{dv}{v^2-\left(\sqrt{2\sqrt{2}+2}\right)^2}=-\frac{1}{\sqrt{2\sqrt{2}+2}}\operatorname{arccoth}\left(\frac{v}{\sqrt{2\sqrt{2}+2}}\right)+c'$$

$$=-\frac{1}{\sqrt{2\sqrt{2}+2}}\operatorname{arccoth}\left(\frac{u+\frac{\sqrt{2}}{u}}{\sqrt{2\sqrt{2}+2}}\right)+c'=-\frac{1}{\sqrt{2\sqrt{2}+2}}\operatorname{arccoth}\left[\frac{1}{\sqrt{2\sqrt{2}+2}}\left(u+\frac{\sqrt{2}}{u}\right)\right]+c'$$

Then $\int\sqrt{1+\tan x}\,dx=\int\frac{1+\frac{\sqrt{2}}{u^2}}{\left(u-\frac{\sqrt{2}}{u}\right)^2+2\sqrt{2}-2}du+\int\frac{1-\frac{\sqrt{2}}{u^2}}{\left(u+\frac{\sqrt{2}}{u}\right)^2-2\sqrt{2}-2}du$

$$=\frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left[\frac{1}{\sqrt{2\sqrt{2}-2}}\left(u-\frac{\sqrt{2}}{u}\right)\right]-\frac{1}{\sqrt{2\sqrt{2}+2}}\operatorname{arccoth}\left[\frac{1}{\sqrt{2\sqrt{2}+2}}\left(u+\frac{\sqrt{2}}{u}\right)\right]+k$$

$$=\frac{1}{\sqrt{2\sqrt{2}-2}}\arctan\left[\frac{1}{\sqrt{2\sqrt{2}-2}}\left(\sqrt{1+\tan x}-\frac{\sqrt{2}}{\sqrt{1+\tan x}}\right)\right]$$

$-\frac{1}{\sqrt{2\sqrt{2}+2}}\operatorname{arccoth}\left[\frac{1}{\sqrt{2\sqrt{2}+2}}\left(\sqrt{1+\tan x}+\frac{\sqrt{2}}{\sqrt{1+\tan x}}\right)\right]+k$

Remark: $u^2-2+\frac{2}{u^2}=\left(u-\frac{\sqrt{2}}{u}\right)^2+2\sqrt{2}-2=\left(u+\frac{\sqrt{2}}{u}\right)^2-2\sqrt{2}-2$

381. $\int\frac{1}{x^5+1}dx=\int\frac{1}{(x+1)(x^4-x^3+x^2-x+1)}dx$

Method of partial fractions : $\frac{1}{(x+1)(x^4-x^3+x^2-x+1)}=\frac{A}{x+1}+\frac{Bx^3+Cx^2+Dx+E}{x^4-x^3+x^2-x+1}$, then:

$1=(A+B)x^4+(-A+B+C)x^3+(A+C+D)x^2+(-A+D+E)x+(A+E)$, so:
$A+B=0$, so $A=-B$, $-A+B+C=0$, so $-2A+C=0$, then $C=2A$, $A+C+D=0$, then $3A+D=0$, so $D=-3A$, $-A+D+E=0$, $-4A=E$, so $E=4A$ and $A+E=1$, $5A=1$, then $A=\frac{1}{5}$, so $E=4A=\frac{4}{5}$, $D=-3A=-\frac{3}{5}$, $C=2A=\frac{2}{5}$ and $B=-A=-\frac{1}{5}$, then:

$\int\frac{1}{x^5+1}dx=\frac{1}{5}\int\frac{dx}{x+1}+\frac{1}{5}\int\frac{-x^3+2x^2-3x+4}{x^4-x^3+x^2-x+1}dx$

Let's evaluate: $\int\frac{-x^3+2x^2-3x+4}{x^4-x^3+x^2-x+1}dx=-\int\frac{x^3-2x^2+3-+4}{x^4-x^3+x^2-x+1}dx=-\frac{1}{4}\int\frac{4x^3-8x^2+12x-16}{x^4-x^3+x^2-x+1}dx$

$=-\frac{1}{4}\int\frac{4x^3-3x^2-5x^2+14x-2x-15-1}{x^4-x^3+x^2-x+1}dx$

$=-\frac{1}{4}\int\frac{4x^3-3x^2-2x-1}{x^4-x^3+x^2-x+1}dx-\frac{1}{4}\int\frac{-5x^2+14x-15}{x^4-x^3+x^2-x+1}dx$

$=-\frac{1}{4}\int\frac{(x^4-x^3+x^2-x+1)'}{x^4-x^3+x^2-x+1}dx-\frac{1}{4}\int\frac{-5x^2+14x-15}{x^4-x^3+x^2-x+1}dx$

$=-\frac{1}{4}\ln|x^4-x^3+x^2-x+1|-\frac{1}{4}\int\frac{-5x^2+14x-15}{x^4-x^3+x^2-x+1}dx$, then:

$\int\frac{1}{x^5+1}dx=-\frac{1}{20}\ln|x^4-x^3+x^2-x+1|-\frac{1}{20}\int\frac{-5x^2+14x-15}{x^4-x^3+x^2-x+1}dx$, then:

$\int\frac{1}{x^5+1}dx=-\frac{1}{20}\ln|x^4-x^3+x^2-x+1|+\frac{1}{20}\int\frac{5x^2-14x+15}{\left(x^2+\frac{\sqrt{5}-1}{2}x+1\right)\left(x^2-\frac{\sqrt{5}+1}{2}x+1\right)}dx$

Method of partial fractions : $\frac{5x^2-14x+15}{\left(x^2+\frac{\sqrt{5}-1}{2}x+1\right)\left(x^2-\frac{\sqrt{5}+1}{2}x+1\right)}=\frac{Px+Q}{x^2+\frac{\sqrt{5}-1}{2}x+1}+\frac{Rx+S}{x^2-\frac{\sqrt{5}+1}{2}x+1}$

$5x^2-14x+15=\left(x^2-\frac{\sqrt{5}+1}{2}x+1\right)(Px+Q)+\left(x^2+\frac{\sqrt{5}-1}{2}x+1\right)(Rx+S)$

After doing some calculations we get: $P=2\sqrt{5}$, $Q=\frac{13+15\sqrt{5}}{2\sqrt{5}}$, $R=2\sqrt{5}$ and $S=\frac{13-15\sqrt{5}}{2\sqrt{5}}$.

First : $\frac{1}{20}\int\frac{Px+Q}{x^2+\frac{\sqrt{5}-1}{2}x+1}dx=\frac{1}{20}\int\frac{2\sqrt{5}x+\frac{13+15\sqrt{5}}{2\sqrt{5}}}{x^2+\frac{\sqrt{5}-1}{2}x+1}dx=\frac{1}{4\sqrt{5}}\int\frac{2x+\frac{13+15\sqrt{5}}{10}}{x^2+\frac{\sqrt{5}-1}{2}x+1}dx$

$=\frac{1}{4\sqrt{5}}\int\frac{2x+\frac{\sqrt{5}-1}{2}}{x^2+\frac{\sqrt{5}-1}{2}x+1}dx=\frac{1}{40\sqrt{5}}\int\frac{10\sqrt{5}+18}{x^2+\frac{\sqrt{5}-1}{2}x+1}dx$

$$= \frac{1}{4\sqrt{5}} \int \frac{\left(x^2+\frac{\sqrt{5}-1}{2}x+1\right)'}{x^2+\frac{\sqrt{5}-1}{2}x+1} dx + \frac{10\sqrt{5}+18}{40\sqrt{5}} \int \frac{dx}{\left(x+\frac{\sqrt{5}-1}{4}\right)^2+\left(\sqrt{\frac{10+2\sqrt{5}}{16}}\right)^2}$$

$$= \frac{1}{4\sqrt{5}} \ln\left|x^2+\frac{\sqrt{5}-1}{2}x+1\right| + \frac{10\sqrt{5}+18}{40\sqrt{5}} \times \frac{4}{\sqrt{10+2\sqrt{5}}} \arctan\left(\frac{4x+\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}\right) + c$$

By the same approach we get : $\frac{1}{20}\int \frac{Rx+S}{x^2-\frac{\sqrt{5}+1}{2}x+1}$

$$= -\frac{1}{4\sqrt{5}} \ln\left|x^2-\frac{\sqrt{5}+1}{2}x+1\right| - \frac{18-10\sqrt{5}}{40\sqrt{5}} \times \frac{4}{\sqrt{10-2\sqrt{5}}} \arctan\left(\frac{4x-\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}}\right) + c'$$

Therefore, we get: $\int \frac{1}{x^5+1} dx$

$$= \frac{1}{5}\ln|x+1| - \frac{1}{20}\ln|x^4-x^3+x^2-x+1| + \frac{1}{4\sqrt{5}}\ln\left|x^2+\frac{\sqrt{5}-1}{2}x+1\right|$$

$$+ \frac{5\sqrt{5}+9}{2\sqrt{5}} \times \frac{1}{\sqrt{10+2\sqrt{5}}} \arctan\left(\frac{4x+\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}\right) - \frac{1}{4\sqrt{5}}\ln\left|x^2-\frac{\sqrt{5}+1}{2}x+1\right|$$

$$- \frac{9-5\sqrt{5}}{2\sqrt{5}} \times \frac{1}{\sqrt{10-2\sqrt{5}}} \arctan\left(\frac{4x-\sqrt{5}-1}{\sqrt{10-2\sqrt{5}}}\right) + k$$

382. $\int \frac{dx}{\sqrt{(1+\sin x)(2+\sin x)}}$, let $u = \sin x$, then $du = \cos x\, dx = \sqrt{1-\sin^2 x}\, dx$

$du = \sqrt{1-u^2}dx = \sqrt{(1-u)(1+u)}dx$, so**:**

$$\int \frac{dx}{\sqrt{(1+\sin x)(2+\sin x)}} = \int \frac{1}{\sqrt{(1+u)(2+u)}} \cdot \frac{1}{\sqrt{(1-u)(1+u)}} du = \int \frac{1}{(1+u)\sqrt{(1-u)(2+u)}} du$$

Let $t = \frac{1}{1+u}$, so $u = \frac{1}{t} - 1$ and $du = -\frac{1}{t^2}dt$, then**:**

$$\int \frac{dx}{\sqrt{(1+\sin x)(2+\sin x)}} = \int \frac{t\left(-\frac{1}{t^2}\right)dt}{\sqrt{\left(2-\frac{1}{t}\right)\left(1+\frac{1}{t}\right)}} = \int \frac{-\frac{dt}{t}}{\sqrt{\frac{1}{t^2}(2t-1)(t+1)}} = -\int \frac{dt}{\sqrt{(2t-1)(t+1)}}$$

$$= -\int \frac{dt}{\sqrt{2t^2+t-1}} = -\int \frac{\sqrt{8}}{\sqrt{8}\sqrt{2t^2+t-1}} dt = -\int \frac{\sqrt{8}}{\sqrt{8\left(2t^2+t-1\right)}} dt = -\int \frac{\sqrt{8}}{\sqrt{16t^2+8t-8}} dt$$

$$= -\int \frac{\sqrt{8}}{\sqrt{\left(16t^2+8t+1\right)-9}} dt = -\int \frac{\sqrt{8}}{\sqrt{(4t+1)^2-9}} dt$$

Let $4t+1 = 3\sec\theta$, then $4dt = 3\sec\theta\tan\theta\, d\theta$, se we get**:**

$$\int \frac{dx}{\sqrt{(1+\sin x)(2+\sin x)}} = -\sqrt{8}\int \frac{\frac{3}{4}\sec\theta\tan\theta}{\sqrt{9\left(\sec^2\theta-1\right)}} d\theta = -\frac{3\sqrt{2}}{2}\int \frac{\sec\theta\tan\theta}{3\sqrt{\tan^2\theta}} d\theta$$

$$= -\frac{\sqrt{2}}{2}\int \frac{\sec\theta\tan\theta}{\tan\theta} d\theta = -\frac{\sqrt{2}}{2}\int \sec\theta\, d\theta = -\frac{\sqrt{2}}{2}\ln|\sec\theta+\tan\theta| + c$$

But $4t+1=3\sec\theta$, so $\sec\theta=\frac{4t+1}{3}$ and $\tan^2\theta=\sec^2\theta-1$, so we get

$\tan\theta=\sqrt{\left(\frac{4t+1}{3}\right)^2-1}=\sqrt{\frac{16t^2+8t+1-9}{9}}=\frac{\sqrt{8(2t^2+t-1)}}{3}=\frac{2\sqrt{2}\sqrt{2t^2+t-1}}{3}$, then:

$$\int\frac{dx}{\sqrt{(1+\sin x)(2+\sin x)}}=-\frac{\sqrt{2}}{2}\ln\left|\frac{4t+1}{3}+\frac{2\sqrt{2}\sqrt{2t^2+t-1}}{3}\right|+c$$

$$=-\frac{\sqrt{2}}{2}\ln\left|(4t+1)+2\sqrt{2}\sqrt{2t^2+t-1}\right|+k=-\frac{\sqrt{2}}{2}\ln\left|\frac{5+u}{1+u}+\frac{2\sqrt{2}}{1+u}\sqrt{2-u-u^2}\right|+k$$

$$=-\frac{\sqrt{2}}{2}\ln\left|\frac{5+\sin x}{1+\sin x}+\frac{2\sqrt{2}}{1+\sin x}\sqrt{2-\sin x-\sin^2 x}\right|+k$$

383. $\int\frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}\left(1+x\sqrt{1-x^2}\right)}dx$, let $x=\sin y$, then $dx=\cos y\,dy$, so:

$$\int\frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}\left(1+x\sqrt{1-x^2}\right)}dx=\int\frac{\sqrt{1-\sin^2 y}-\sin y}{\sqrt{1-\sin^2 y}\left(1+\sin y\sqrt{1-\sin^2 y}\right)}\cos y\,dy$$

$$=\int\frac{\sqrt{\cos^2 y}-\sin y}{\sqrt{\cos^2 y}\left(1+\sin y\sqrt{\cos^2 y}\right)}\cos y\,dy=\int\frac{\cos y-\sin y}{\cos y(1+\sin y\cos y)}\cos y\,dy$$

$$=\int\frac{\cos y-\sin y}{1+\sin y\cos y}dy=2\int\frac{\cos y-\sin y}{2+2\sin y\cos y}dy=2\int\frac{\cos y-\sin y}{\cos^2 y+\sin^2 y+2\sin y\cos y+1}$$

$=2\int\frac{\cos y-\sin y}{(\cos y+\sin y)^2+1}$. Let $u=\sin y+\cos y$, then $du=(\cos y-\sin y)dy$, so:

$$\int\frac{\sqrt{1-x^2}-x}{\sqrt{1-x^2}\left(1+x\sqrt{1-x^2}\right)}dx=2\int\frac{du}{1+u^2}=2\arctan u+c=2\arctan(\sin y+\cos y)+c$$

$$=2\arctan[\sin(\arcsin x)+\cos(\arcsin x)]+c=2\arctan\left(x+\sqrt{1-x^2}\right)+c$$

384. $\int\left(\frac{1+\cos x}{\sin x}\right)^4dx=\int\left(\frac{2\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)^4dx=\int\left(\frac{\cos\frac{x}{2}}{\sin\frac{x}{2}}\right)^4dx=\int\left(\cot\frac{x}{2}\right)^4dx$

Let $t=\tan\frac{x}{2}$, then $dt=\frac{2dt}{1+t^2}$ and $\cot\frac{x}{2}=\frac{1}{t}$, then:

$$\int\left(\frac{1+\cos x}{\sin x}\right)^4dx=\int\frac{1}{t^4}.\frac{2dt}{1+t^2}=2\int\frac{1}{t^4(1+t^2)}dt$$

Method of partial fractions : $\frac{1}{z^2(1+z)}=\frac{\mathrm{A}}{z}+\frac{\mathrm{B}}{z^2}+\frac{\mathrm{C}}{1+z}$, so $1=z(z+1)\mathrm{A}+(\mathrm{z}+1)\mathrm{B}+z^2\mathrm{C}$

For $z=0$, then $\mathrm{B}=1$, for $z=-1$, $\mathrm{C}=1$ and for $z=1$, $1=2\mathrm{A}+2+10$, so $\mathrm{A}=-1$, then we get , $\frac{1}{z^2(1+z)}=\frac{-1}{z}+\frac{1}{z^2}+\frac{1}{1+z}$, put $z=t^2$, so , $\frac{1}{t^4(1+z^4)}=\frac{-1}{t^2}+\frac{1}{t^4}+\frac{1}{1+t^2}$, so:

$$\int\left(\frac{1+\cos x}{\sin x}\right)^4dx=2\int\left(\frac{-1}{t^2}+\frac{1}{t^4}+\frac{1}{1+t^2}\right)dt=2\left(\frac{1}{t}-\frac{1}{3t^3}+\arctan t\right)+c$$

$$=\frac{2}{\tan\frac{x}{2}}-\frac{2}{3\tan^3\frac{x}{2}}+2\arctan\left(\tan\frac{x}{2}\right)+c=\frac{2}{\tan\frac{x}{2}}-\frac{2}{3\tan^3\frac{x}{2}}+x+c$$

385. $\int\left(\frac{1-\tan x}{1+\tan x}\right)^4 dx=\int\left(\frac{\tan(\frac{\pi}{4})-\tan x}{1+\tan(\frac{\pi}{4})\tan x}\right)^4 dx=\int\tan^4\left(\frac{\pi}{4}-x\right)dx$

Let $u=\frac{\pi}{4}-x$, then $du=-dx$, so: $\int\left(\frac{1-\tan x}{1+\tan x}\right)^4 dx=-\int\tan^4 u\,du$

Let $t=\tan u$, so $dt=\sec^2 u\,du=(1+\tan^2 u)du=(1+t^2)du$, then:

$\int\left(\frac{1-\tan x}{1+\tan x}\right)^4 dx=-\int\frac{t^4}{1+t^2}dx=-\int\frac{t^4-1+1}{1+t^2}dx=-\int\frac{(t^2+1)(t^2-1)+1}{1+t^2}dx$

$=-\int\left(t^2-1+\frac{1}{1+t^2}\right)dt=-\frac{1}{3}t^3+t-\arctan t+c=-\frac{1}{3}\tan^3 u+\tan u-u+c$

$=-\frac{1}{3}\tan^3\left(\frac{\pi}{4}-x\right)+\tan\left(\frac{\pi}{4}-x\right)+x+k$

386. $\int\frac{\sqrt{\sin x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^5}dx=\int\frac{\frac{\sqrt{\sin x}}{\cos^{\frac{5}{2}}x}}{\frac{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^5}{\cos^{\frac{5}{2}}x}}dx=\int\frac{\frac{\sqrt{\sin x}}{\cos^2 x\sqrt{\cos x}}}{\frac{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^5}{(\sqrt{\cos x})^5}}dx$

$=\int\frac{\sqrt{\tan x}}{\cos^2 x\left(\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\cos x}}\right)^5}dx=\int\frac{\sqrt{\tan x}}{\cos^2 x(\sqrt{\tan x}+1)^5}dx=2\int\frac{(\sqrt{\tan x})^2}{2\sqrt{\tan x}\cos^2 x(\sqrt{\tan x}+1)^5}dx$

Let $t=\tan x$, then $dt=\frac{1}{2\sqrt{\tan x}}\times\frac{1}{\cos^2 x}dx$, so:

$\int\frac{\sqrt{\sin x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^5}dx=2\int\frac{t^2}{(t+1)^5}dt$, let $u=t+1$, then $du=dt$, so:

$\int\frac{\sqrt{\sin x}}{\left(\sqrt{\sin x}+\sqrt{\cos x}\right)^5}dx=2\int\frac{(u-1)^2}{u^5}du=2\int\frac{u^2-2u+1}{u^5}du=2\int\left(\frac{1}{u^3}-\frac{2}{u^4}+\frac{1}{u^5}\right)du$

$=2\left(-\frac{1}{2u^2}+\frac{2}{3u^3}-\frac{1}{4u^4}\right)+c=-\frac{1}{(t+1)^2}+\frac{4}{3(t+1)^3}-\frac{1}{2(t+1)^4}+c$

$=-\frac{1}{(\tan x+1)^2}+\frac{4}{3(\tan x+1)^3}-\frac{1}{2(\tan x+1)^4}+c$

387. **First Method:** $\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx$

$=\int\left(\frac{\sin x+\cos x}{\sqrt{\sin x\cos x}}\right)dx=\sqrt{2}\int\left(\frac{\sin x+\cos x}{\sqrt{2\sin x\cos x}}\right)dx=\sqrt{2}\int\left(\frac{\sin x+\cos x}{\sqrt{(\cos^2 x+\sin^2 x)-(\sin x-\cos x)^2}}\right)dx$

$=\sqrt{2}\int\left(\frac{\sin x+\cos x}{\sqrt{1-(\sin x-\cos x)^2}}\right)dx$, let b$t=\sin x-\cos x$, then $dt=(\cos x+\sin x)dx$, so:

$\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\sqrt{2}\int\frac{1}{\sqrt{1-t^2}}dt=\sqrt{2}\arcsin(\sin x-\cos x)+c$

Second Method: $\int(\sqrt{\tan x}+\sqrt{\cot x})dx=\int\frac{\sqrt{\tan x}}{\sqrt{\tan x}}\left(\sqrt{\tan x}+\frac{1}{\sqrt{\tan x}}\right)dx$

$= \int \sqrt{\tan x}\left(1+\frac{1}{\tan x}\right)dx$, let $t=\sqrt{\tan x}$, then $dt=\frac{1}{2\sqrt{\tan x}}\sec^2 x\,dx=\frac{1+\tan^2 x}{2\sqrt{\tan x}}dx$,

then: $dx=\frac{2t}{1+t^4}dt$, so:

$$\int(\sqrt{\tan x}+\sqrt{\cot x})dx=2\int t^2\left(1+\frac{1}{t^2}\right)\frac{t}{1+t^4}dt=2\int\frac{t^2+1}{t^4+1}dt=2\int\frac{\frac{t^2+1}{t^2}}{\frac{t^4+1}{t^2}}dt=2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}}dt$$

$= 2\int\frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}-2+2}dt=2\int\frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2}dt$, let $u=t-\frac{1}{t}$, so $du=\left(1-\frac{1}{t^2}\right)dt$, then:

$$\int(\sqrt{\tan x}+\sqrt{\cot x})dx=2\int\frac{1}{u^2+2}du=2\int\frac{1}{u^2+(\sqrt{2})^2}du=\frac{2}{\sqrt{2}}\arctan\left(\frac{u}{\sqrt{2}}\right)+c$$

$$=\sqrt{2}\arctan\left(\frac{t-\frac{1}{t}}{\sqrt{2}}\right)+c=\sqrt{2}\arctan\left(\frac{t^2-1}{\sqrt{2}t}\right)+c=\sqrt{2}\arctan\left(\frac{\tan x-1}{\sqrt{2\tan x}}\right)+c$$

388. $\int\frac{1}{\sqrt[4]{1+x^4}}dx$, let $t=x^4$, then $x=t^{\frac{1}{4}}$, so $dx=\frac{1}{4}t^{-\frac{3}{4}}dx$, then:

$$\int\frac{1}{\sqrt[4]{1+x^4}}dx=\frac{1}{4}\int\frac{t^{-\frac{3}{4}}}{\sqrt[4]{1+t}}dt=\frac{1}{4}\int\frac{1}{t^{\frac{3}{4}}\sqrt[4]{1+t}}dt=\frac{1}{4}\int\frac{1}{t\times t^{-\frac{1}{4}}\sqrt[4]{1+t}}dt=\frac{1}{4}\int\frac{1}{t\sqrt[4]{(1+t)t^{-1}}}dt$$

$=\frac{1}{4}\int\frac{1}{t\sqrt[4]{\frac{1+t}{t}}}dt$, let $u^4=\frac{1+t}{t}$, then $t=\frac{1}{u^4-1}$, so $dt=-\frac{4u^3}{(u^4-1)^2}du$, then:

$$\int\frac{1}{\sqrt[4]{1+x^4}}dx=-\int\frac{1}{u\times\frac{1}{u^4-1}}\times\frac{u^3}{(u^4-1)^2}du=-\int\frac{u^2}{u^4-1}du=\int\frac{u^2}{(u-1)(u+1)(u^2+1)}du$$

Method of partial fractions: $\frac{u^2}{(u-1)(u+1)(u^2+1)}=\frac{a}{u-1}+\frac{b}{u+1}+\frac{cu+d}{u^2+1}$

then $u^2=(u+1)(u^2+1)a+(u-1)(u^2+1)b+(u-1)(u+1)(cu+d)$

For $u=1$, $1=4a$, so $a=\frac{1}{4}$, for $u=-1$, $1=-4b$, so $b=-\frac{1}{4}$, for $u=0$, $0=\frac{1}{4}+\frac{1}{4}-d$, so $d=\frac{1}{2}$ and for $u=2$, $4=\frac{15}{4}-\frac{5}{4}+6c+\frac{3}{2}$, $4=6c+4$, so $c=4$, then:

$$\int\frac{1}{\sqrt[4]{1+x^4}}dx=\frac{1}{4}\int\frac{1}{u+1}du-\frac{1}{4}\int\frac{1}{u-1}du-\frac{1}{2}\int\frac{1}{u^2+1}du$$

$$=\frac{1}{4}\ln|u+1|-\frac{1}{4}\ln|u-1|-\frac{1}{2}\arctan u+c$$

$$=\frac{1}{4}\ln\left|\sqrt[4]{\frac{1+t}{t}}+1\right|-\frac{1}{4}\ln\left|\sqrt[4]{\frac{1+t}{t}}-1\right|-\frac{1}{2}\arctan\sqrt[4]{\frac{1+t}{t}}+c$$

$$=\frac{1}{4}\ln\left|\sqrt[4]{\frac{1+x^4}{x^4}}+1\right|-\frac{1}{4}\ln\left|\sqrt[4]{\frac{1+x^4}{x^4}}-1\right|-\frac{1}{2}\arctan\sqrt[4]{\frac{1+x^4}{x^4}}+c$$

$$=\frac{1}{4}\ln\left|\frac{\sqrt[4]{1+x^4}+x}{\sqrt[4]{1+x^4}-x}\right|-\frac{1}{2}\arctan\left(\frac{\sqrt[4]{1+x^4}}{x}\right)+c$$

389. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx = \int \frac{x-1}{(x+1)\sqrt{x(x^2+x+1)}}dx = 2\int \frac{x-1}{(x+1)\sqrt{x^2+x+1}}\cdot\frac{1}{2\sqrt{x}}dx$

Let $y=\sqrt{x}$, $(y^2=x)$ then $dy=\frac{1}{2\sqrt{x}}dx$, so:

$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx = 2\int \frac{y^2-1}{(y^2+1)\sqrt{y^4+y^2+1}}dy = 2\int \frac{y^2-1}{y\left(y+\frac{1}{y}\right)\sqrt{y^2\left(y^2+1+\frac{1}{y^2}\right)}}dy$

$= 2\int \frac{y^2-1}{y^2\left(y+\frac{1}{y}\right)\sqrt{y^2+2+\frac{1}{y^2}-1}}dy = 2\int \frac{1-\frac{1}{y^2}}{\left(y+\frac{1}{y}\right)\sqrt{\left(y+\frac{1}{y}\right)^2-1}}dy$, let $t=y+\frac{1}{y}$, so $dt=\left(1-\frac{1}{y^2}\right)dy$

$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx = 2\int \frac{1}{t\sqrt{t^2-1}}dt = 2\operatorname{arcsec} t + c = 2\operatorname{arcsec}\left(y+\frac{1}{y}\right)+c$

$= 2\operatorname{arcsec}\left(\frac{x+1}{\sqrt{x}}\right)+c$

390. $\int \frac{dx}{x+\sqrt{x^2+2x+2}}$, $a>0$, then we use the first Euler's substitution, then:

$\sqrt{x^2+2x+2}=x+t$ and then $x=\frac{t^2-2}{2(1-t)}$ and $dx=\frac{-t^2+2t-2}{2(1-t)^2}dt$, then we get:

$\int \frac{dx}{x+\sqrt{x^2+2x+2}} = \int \frac{\frac{-t^2+2t-2}{2(1-t)^2}}{\frac{t^2-2}{2(1-t)}+\frac{t^2-2}{2(1-t)}+t}dt = \int \frac{-t^2+2t-2}{(1-t)(2t-4)}dt = \frac{1}{2}\int \frac{(t-1)^2+1}{(t-1)(t-2)}dt$

$= \frac{1}{2}\int \frac{(t-1)^2}{(t-1)(t-2)}dt + \frac{1}{2}\int \frac{1}{(t-1)(t-2)}dt = \frac{1}{2}\mathrm{I}_1+\frac{1}{2}\mathrm{I}_2$

For I_1 $\int \frac{(t-1)^2}{(t-1)(t-2)}dt = \int \frac{t-1}{t-2}dt = \int \frac{t-1-1+1}{t-2}dt = \int \frac{(t-2)+1}{t-2}dt = \int\left(1+\frac{1}{t-2}\right)dt$

$= t+\ln|t-2|+c_1$

For I_2 $\int \frac{1}{(t-1)(t-2)}dt = \int \frac{(t-1)-(t-2)}{(t-1)(t-2)}dt = \int\left(\frac{1}{t-2}-\frac{1}{t-1}\right)dt = \ln|t-2|-\ln|t-1|+c_2$

$= \ln\left|\frac{t-2}{t-1}\right|+c_2$, then we get:

$\int \frac{dx}{x+\sqrt{x^2+2x+2}} = \frac{1}{2}\left[t+\ln|t-2|+c_1+\ln\left|\frac{t-2}{t-1}\right|+c_2\right]$

$= \frac{1}{2}t+\frac{1}{2}\ln|t-2|+\frac{1}{2}\ln\left|\frac{t-2}{t-1}\right|+c$, where $t=\sqrt{x^2+2x+2}-x$, therefore: $\int \frac{dx}{x+\sqrt{x^2+2x+2}} =$

$\frac{1}{2}\left(\sqrt{x^2+2x+2}-x\right)+\frac{1}{2}\ln\left|\sqrt{x^2+2x+2}-x-2\right|+\frac{1}{2}\ln\left|\frac{\sqrt{x^2+2x+2}-x-2}{\sqrt{x^2+2x+2}-1}\right|+c$

391. $\int \frac{dx}{x-2+\sqrt{x^2-2x+2}}$

Let $\sqrt{x^2-2x+2}=x+t$, then $x=\frac{2-t^2}{2(1+t)}$ and $dx=-\frac{t^2+2t+2}{2(1+t)^2}dt$, so:

$$\int \frac{dx}{x-2+\sqrt{x^2-2x+2}} = \int \frac{-\frac{t^2+2t+2}{2(1+t)^2}}{\frac{2-t^2}{2(1+t)}-2+\frac{2-t^2}{2(1+t)}+t}dt = \frac{1}{2}\int \frac{t^2+2t+2}{t(t+1)}dt$$

$$= \frac{1}{2}\int \frac{t^2+t+2t+2-t}{t(t+1)}dt = \frac{1}{2}\int \left[\frac{t(t+1)}{t(t+1)} + \frac{2(t+1)}{t(t+1)} - \frac{t}{t(t+1)}\right]dt$$

$= \frac{1}{2}\int \left(1+\frac{2}{t}-\frac{1}{t+1}\right)dt = \frac{1}{2}t + \ln|t| - \frac{1}{2}\ln|t+1| + c$, therefore: $\int \frac{dx}{x-2+\sqrt{x^2-2x+2}}$

$$= \frac{1}{2}\left(\sqrt{x^2-2x+2}-x\right) + \ln\left|\sqrt{x^2-2x+2}-x\right| - \frac{1}{2}\ln\left|\sqrt{x^2-2x+2}-x+1\right| + c$$

392. $\int \frac{dx}{(x+1)\sqrt{-x^2+x+1}}$, $c > 0$, then we use the third Euler's substitution, then:

$\sqrt{-x^2+x+1} = xt+1$ and so $x = \frac{1-2t}{t^2+1}$ and $dx = -2.\frac{-t^2+t+1}{\left(t^2+1\right)^2}dt$

$x+1 = \frac{1-2t}{t^2+1}+1 = \frac{t^2-2t+2}{t^2+1}$ and $\sqrt{-x^2+x+1} = \left(\frac{1-2t}{t^2+1}\right)t+1 = \frac{-t^2+t+1}{t^2+1}$, so:

$$\int \frac{dx}{(x+1)\sqrt{-x^2+x+1}} = \int \frac{-2.\frac{-t^2+t+1}{\left(t^2+1\right)^2}}{\frac{t^2-2t+2}{t^2+1}.\frac{-t^2+t+1}{t^2+1}}dt = -2\int \frac{dt}{t^2-2t+2} = -2\int \frac{dt}{(t-1)^2+1}$$

$= -2\arctan(t-1)+c$, where $t = \frac{\sqrt{-x^2+x+1}-1}{x}$, therefore:

$$\int \frac{dx}{(x+1)\sqrt{-x^2+x+1}} = -2\arctan\left(\frac{\sqrt{-x^2+x+1}-1}{x}-1\right)+c$$

393. $\int \frac{1-\sin^4 x}{\left(1+\sin^4 x\right)\sqrt{1+\sin^2 x}}dx = \int \frac{\left(1-\sin^2 x\right)\left(1+\sin^2 x\right)}{\left(1+\sin^4 x\right)\sqrt{1+\sin^2 x}}dx$

$$= \int \frac{\left(1-\sin^2 x\right)\left(\sqrt{1+\sin^2 x}\right)\left(\sqrt{1+\sin^2 x}\right)}{\left(1+\sin^4 x\right)\sqrt{1+\sin^2 x}}dx = \int \frac{\left(1-\sin^2 x\right)\left(\sqrt{1+\sin^2 x}\right)}{1+\sin^4 x}dx$$

$$= \int \frac{\cos^2 x\left(\sqrt{1+\sin^2 x}\right)}{1+\sin^4 x}dx = \int \frac{\cos x\left(\sqrt{1+\sin^2 x}\right)}{1+\sin^4 x}\cos x\,dx = \int \frac{\sqrt{1-\sin^2 x}\left(\sqrt{1+\sin^2 x}\right)}{1+\sin^4 x}\cos x\,dx$$

$$= \int \frac{\sqrt{1-\sin^4 x}}{1+\sin^4 x}\cos x\,dx = \int \frac{\sin x\frac{1}{\sin x}\sqrt{1-\sin^4 x}}{1+\sin^4 x}\cos x\,dx = \int \frac{\sin x\sqrt{\frac{1}{\sin^2 x}-\frac{\sin^4 x}{\sin^2 x}}}{1+\sin^4 x}\cos x\,dx$$

$$= \int \frac{\sin x\sqrt{\csc^2 x-\sin^2 x}}{1+\sin^4 x}\cos x\,dx = \int \frac{\sin^4 x\sqrt{\csc^2 x-\sin^2 x}}{\left(1+\sin^4 x\right)^2}.\frac{1+\sin^4 x}{\sin^3 x}.\cos x\,dx$$

$$= \int \frac{\sqrt{\csc^2 x - \sin^2 x}}{\left(\frac{1+\sin^4 x}{\sin^2 x}\right)^2}\left[\left(\frac{1}{\sin^2 x}\right)\left(\frac{\cos x}{\sin x}\right) + \sin x \cos x\right] dx$$

$$= \int \frac{\sqrt{\csc^2 x - \sin^2 x}}{\left(\csc^2 x + \sin^2 x\right)^2}[\csc^2 x \cot x + \sin x \cos x] dx$$

$$= \int \frac{\sqrt{\csc^2 x - \sin^2 x}}{\left(\csc^2 x - \sin^2 x\right)^2 + 4}[\csc^2 x \cot x + \sin x \cos x] dx$$

$$= \int \frac{\sqrt{\csc^2 x - \sin^2 x}\sqrt{\csc^2 x - \sin^2 x}}{\left(\csc^2 x - \sin^2 x\right)^2 + 4} \cdot \frac{2\csc^2 x \cot x + 2\sin x \cos x}{2\sqrt{\csc^2 x - \sin^2 x}} dx$$

$$= \int \frac{\csc^2 x - \sin^2 x}{\left(\csc^2 x - \sin^2 x\right)^2 + 4} \cdot \frac{2\csc^2 x \cot x + 2\sin x \cos x}{2\sqrt{\csc^2 x - \sin^2 x}} dx$$

Let $y = \sqrt{\csc^2 x - \sin^2 x}$, then $dy = -\frac{2\csc^2 x \cot x + 2\sin x \cos x}{2\sqrt{\csc^2 x - \sin^2 x}} dx$, so**:**

$$\int \frac{1 - \sin^4 x}{\left(1 + \sin^4 x\right)\sqrt{1 + \sin^2 x}} dx = -\int \frac{y^2}{y^4 + 4} dy = -\frac{1}{2}\int \frac{(y^2+2) + (y^2-2)}{y^4+4} dy$$

$$= -\frac{1}{2}\int \frac{1 + \frac{2}{y^2}}{y^2 + \frac{4}{y^2}} dy - \frac{1}{2}\int \frac{1 - \frac{2}{y^2}}{y^2 + \frac{4}{y^2}} dy = -\frac{1}{2}\int \frac{d\left(y - \frac{2}{y}\right)}{\left(y - \frac{2}{y}\right)^2 + 4} dy - \frac{1}{2}\int \frac{d\left(y + \frac{2}{y}\right)}{\left(y + \frac{2}{y}\right)^2 - 4} dy$$

$$= -\frac{1}{4}\tan^{-1}\left(\frac{y - \frac{2}{y}}{2}\right) + \frac{1}{4}\tanh^{-1}\left(\frac{y + \frac{2}{y}}{2}\right) + c = -\frac{1}{4}\tan^{-1}\left(\frac{y^2-2}{2y}\right) + \frac{1}{4}\tanh^{-1}\left(\frac{y^2+2}{2y}\right) + c$$

$$= -\frac{1}{4}\tan^{-1}\left(\frac{\csc^2 x - \sin^2 x - 2}{2\sqrt{\csc^2 x - \sin^2 x}}\right) + \frac{1}{4}\tanh^{-1}\left(\frac{\csc^2 x - \sin^2 x + 2}{2\sqrt{\csc^2 x - \sin^2 x}}\right) + c$$

394. $$\int \frac{dx}{(x^2 - x - 2)\sqrt{x^2 + x + 1}} = \int \frac{dx}{(x-2)(x+1)\sqrt{x^2 + x + 1}}$$

Method of partial fractions : $\frac{1}{(x-2)(x+1)} = \frac{\mathrm{A}}{x-2} + \frac{\mathrm{B}}{x+1}$, so $1 = \mathrm{A}(x+1) + \mathrm{B}(x-2)$

For $x = 2$, $1 = 3\mathrm{A}$, so $\mathrm{A} = \frac{1}{3}$ and for $x = -1$, $1 = -3\mathrm{B}$, so $\mathrm{B} = -\frac{1}{3}$, then**:**

$$\int \frac{dx}{(x^2 - x - 2)\sqrt{x^2 + x + 1}} = \frac{1}{3}\int \left(\frac{1}{x-2} - \frac{1}{x+1}\right)\frac{1}{\sqrt{x^2 + x + 1}} dx$$

$$= \frac{1}{3}\left[\int \frac{1}{(x-2)\sqrt{x^2 + x + 1}} dx - \int \frac{1}{(x+1)\sqrt{x^2 + x + 1}} dx\right]$$

Evaluating: $\int \frac{1}{(x-2)\sqrt{x^2 + x + 1}} dx$, let $x - 2 = \frac{1}{t}$, $x = 2 + \frac{1}{t}$, so $dx = -\frac{1}{t^2} dt$, then**:**

$$\int \frac{1}{(x-2)\sqrt{x^2+x+1}}dx = \int \frac{-\frac{1}{t^2}}{\frac{1}{t}\sqrt{\left(2+\frac{1}{t}\right)^2+\left(2+\frac{1}{t}\right)+1}}dt = -\int \frac{1}{t\sqrt{4+\frac{4}{t}+\frac{1}{t^2}+3+\frac{1}{t}}}dt$$

$$= -\int \frac{1}{\sqrt{t^2\left(\frac{1}{t^2}+\frac{5}{t}+7\right)}}dt = -\int \frac{1}{\sqrt{7t^2+5t+1}}dt = -\frac{1}{\sqrt{7}}\int \frac{1}{\sqrt{\left(t+\frac{5}{14}\right)^2+\left(\frac{\sqrt{3}}{14}\right)^2}}dt$$

$$= -\frac{1}{\sqrt{7}}\sinh^{-1}\left(\frac{t+\frac{5}{14}}{\frac{\sqrt{3}}{14}}\right)+c_1 = -\frac{1}{\sqrt{7}}\sinh^{-1}\left(\frac{14t+5}{\sqrt{3}}\right)+c_1 = -\frac{1}{\sqrt{7}}\sinh^{-1}\left(\frac{14\frac{1}{x-2}+5}{\sqrt{3}}\right)+c_1$$

$$= -\frac{1}{\sqrt{7}}\sinh^{-1}\left(\frac{5x+4}{\sqrt{3}x-2\sqrt{3}}\right)+c_1$$

Evaluating: $\int \frac{1}{(x+1)\sqrt{x^2+x+1}}dx$, let $x+1=\frac{1}{t}$, $x=-1+\frac{1}{t}$, so $dx=-\frac{1}{t^2}dt$, then**:**

$$\int \frac{1}{(x+1)\sqrt{x^2+x+1}}dx = \int \frac{-\frac{1}{t^2}}{\frac{1}{t}\sqrt{\left(-1+\frac{1}{t}\right)^2+\left(-1+\frac{1}{t}\right)+1}}dt = -\int \frac{1}{t\sqrt{1-\frac{2}{t}+\frac{1}{t^2}+\frac{1}{t}}}dt$$

$$= -\int \frac{1}{\sqrt{t^2\left(\frac{1}{t^2}-\frac{1}{t}+1\right)}}dt = -\int \frac{1}{\sqrt{t^2-t+1}}dt = -\frac{1}{\sqrt{7}}\int \frac{1}{\sqrt{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}}dt$$

$$= -\sinh^{-1}\left(\frac{t-\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c_2 = -\sinh^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)+c_2 = -\sinh^{-1}\left(\frac{2\frac{1}{x+1}-1}{\sqrt{3}}\right)+c_2$$

$= -\sinh^{-1}\left(\frac{1-x}{\sqrt{3}x+\sqrt{3}}\right)+c_2$. **Therefore:**

$$\int \frac{dx}{(x^2-x-2)\sqrt{x^2+x+1}} = \frac{1}{3}\left[\sinh^{-1}\left(\frac{1-x}{\sqrt{3}x+\sqrt{3}}\right)-\frac{1}{\sqrt{7}}\sinh^{-1}\left(\frac{5x+4}{\sqrt{3}x-2\sqrt{3}}\right)\right]+c$$

395. $\int \frac{dx}{3-3x^2-(4x+5)\sqrt{1-x^2}} = \int \frac{dx}{3(1-x^2)-(4x+5)\sqrt{1-x^2}}$, let $u=\sqrt{\frac{1-x}{1+x}}$, so $u^2=\frac{1-x}{1+x}$,

then $x=\frac{1-u^2}{1+u^2}$ and $dx=-\frac{4u}{(1+u^2)^2}du$, $\sqrt{1-x^2}=\sqrt{1-\left(\frac{1-u^2}{1+u^2}\right)^2}=\sqrt{\frac{4u^2}{(1+u^2)^2}}=\frac{2u}{1+u^2}$, so**:**

$$\int \frac{dx}{3-3x^2+(4x+5)\sqrt{1-x^2}} = -4\int \frac{1}{3\left(\frac{2u}{1+u^2}\right)^2+\left[5+4\left(\frac{1-u^2}{1+u^2}\right)\right]\left(\frac{2u}{1+u^2}\right)}\frac{u}{(1+u^2)^2}du$$

$$= -4\int \frac{1}{\frac{12u^2-10u(1+u^2)-8u+8u^3}{(1+u^2)^2}}\frac{u}{(1+u^2)^2}du = -4\int \frac{u}{12u^2-18u-2u^3}du$$

$$= 2\int \frac{1}{u^2-6u+9}du = 2\int \frac{1}{(u-3)^2}du = -\frac{2}{x-3}+c = -\frac{2}{\sqrt{\frac{1-x}{1+x}}-3}+c = -\frac{2\sqrt{1+x}}{\sqrt{1-x}-3\sqrt{1+x}}+c$$

396. $\int \frac{x}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}dx$, let $x=\tan t$, then $dx=\sec^2 t\,dt$, so:

$$\int \frac{x}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}dx=\int \frac{\tan t}{\sqrt{1+\tan^2 t+\sqrt{(1+\tan^2 t)^3}}}\sec^2 t\,dt=\int \frac{\tan t\sec^2 t}{\sqrt{\sec^2 t+\sqrt{(\sec^2 t)^3}}}dt$$

$$=\int \frac{\tan t\sec^2 t}{\sqrt{\sec^2 t+\sec^3 t}}dt=\int \frac{\tan t\sec^2 t}{\sqrt{\sec^2 t(1+\sec t)}}dt=\int \frac{\tan t\sec^2 t}{\sec t\sqrt{1+\sec t}}dt=\int \frac{\tan t\sec t}{\sqrt{1+\sec t}}dt$$

$$=\int \frac{\frac{\sin t}{\cos t}\times\frac{1}{\cos t}}{\sqrt{1+\frac{1}{\cos t}}}dt=\int \frac{\sin t}{\cos^2 t\sqrt{1+\frac{1}{\cos t}}}dt=\int \frac{\sin t}{\sqrt{\cos t\left(1+\frac{1}{\cos t}\right)}}dt=\int \frac{\sin t}{\sqrt{\cos t+1}}dt$$

Let $u=\cos t$, then $du=-\sin t\,dt$, so:

$$\int \frac{x}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}dx=-\int \frac{du}{\sqrt{u+1}}=-2\sqrt{u+1}+c=-2\sqrt{\cos t+1}+c$$

But $x=\tan t$, then $\cos t=\sqrt{\frac{1}{1+x^2}}$, therefore: $\int \frac{x}{\sqrt{1+x^2+\sqrt{(1+x^2)^3}}}dx=-2\sqrt{1+\sqrt{\frac{1}{1+x^2}}}+c$

397. $\int \frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}dx$, let $x=\sin 4t$, then $dx=4\cos 4t\,dt$, so:

$$\int \frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}dx=\int \frac{4\cos 4t}{\sqrt{1+\sin 4t}+\sqrt{1-\sin 4t}+2}dt$$

$$=\int \frac{4\cos 4t}{\sqrt{\cos^2 2t+\sin^2 2t+2\sin 2t\cos 2t}+\sqrt{\cos^2 2t+\sin^2 2t-2\sin 2t\cos 2t}+2}dt$$

$$=\int \frac{4\cos 4t}{\sqrt{(\cos 2t+\sin 2t)^2}+\sqrt{(\cos 2t-\sin 2t)^2}+2}dt=\int \frac{4\cos 4t}{\cos 2t+\sin 2t+\cos 2t-\sin 2t+2}dt$$

$$=\int \frac{4\cos 4t}{2\cos 2t+2}dt=\int \frac{4\cos 4t}{2(2\cos^2 t-1)+2}dt=\int \frac{4\cos 4t}{4\cos^2 t-2+2}dt=\int \frac{\cos 4t}{\cos^2 t}dt$$

$$=\int \frac{2\cos^2 2t-1}{\cos^2 t}dt=\int \frac{2\left(2\cos^2 t-1\right)^2-1}{\cos^2 t}dt=\int \frac{2\left(4\cos^4 t-4\cos^2 t+1\right)-1}{\cos^2 t}dt$$

$$=\int \frac{8\cos^4 t-8\cos^2 t+1}{\cos^2 t}dt=\int(8\cos^2 t-8+\sec^2 t)dt=\int\left(8\frac{1+\cos 2t}{2}-8+\sec^2 t\right)dt$$

$$=\int(4+4\cos 2t-8+\sec^2 t)\,dt=\int(4\cos 2t-4+\sec^2 t)\,dt=2\sin 2t-4t+\tan t+c$$

But $x=\sin 4t$, then $4t=\arcsin x$, so $t=\frac{1}{4}\arcsin x$, therefore:

$$\int \frac{1}{\sqrt{1+x}+\sqrt{1-x}+2}dx=2\sin\left(\frac{1}{2}\arcsin x\right)-4\arcsin x+\tan\left(\frac{1}{4}\arcsin x\right)+c$$

398. $\int \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}}dx=\int \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}}\times\frac{\sqrt{x}-\sqrt{6-x}}{\sqrt{x}-\sqrt{6-x}}dx=\int \frac{x-\sqrt{x(6-x)}}{2x-6}dx$

$$=\int \frac{x-3+3-\sqrt{x(6-x)}}{2x-6}dx=\int \frac{1}{2}dx+\frac{3}{2}\int \frac{dx}{x-3}+\frac{1}{2}\int \frac{\sqrt{x(6-x)}}{x-3}$$

$= \frac{1}{2}x + \frac{3}{2}\ln|x-3| + \frac{1}{2}\int \frac{\sqrt{9-(x-3)^2}}{x-3}dx = \frac{1}{2}x + \frac{3}{2}\ln|x-3| + \frac{1}{2}\int \sqrt{\frac{3^2-(x-3)^2}{(x-3)^2}}dx$

$= \frac{1}{2}x + \frac{3}{2}\ln|x-3| + \frac{1}{2}\int \sqrt{\left(\frac{3}{x-3}\right)^2 - 1}\, dx$

Evaluating: $\int \sqrt{\left(\frac{3}{x-3}\right)^2 - 1}\, dx$, let $\sec y = \frac{3}{x-3}$, then $\sec y \tan y\, dy = -\frac{3}{(x-3)^2}dx$, then:

$\int \sqrt{\left(\frac{3}{x-3}\right)^2 - 1}\, dx = \int \sqrt{\sec^2 y - 1}\sec y \tan y \left(-\frac{1}{3}\right)\left(\frac{9}{\sec^2 y}\right)dy = -3\int \tan^2 y \cos y\, dy$

$= -3\int (\sec^2 y - 1)\cos y\, dy = -3\int(\sec y - \cos y)dy = -3\ln|\sec y + \tan y| + 3\sin y + c$

$= -3\ln\left|\frac{3+\sqrt{6x-x^2}}{x-3}\right| + \sqrt{6x-x^2} + c$, therefore, we get:

$\int \frac{\sqrt{x}}{\sqrt{x}+\sqrt{6-x}}dx = \frac{1}{2}x + \frac{3}{2}\ln|x-3| - \frac{3}{2}\ln\left|\frac{3+\sqrt{6x-x^2}}{x-3}\right| + \frac{1}{2}\sqrt{6x-x^2} + c$

399. $\int \frac{dx}{\tan^3 x+1}$, let $u = \tan x$, then $du = \sec^2 x\, dx = (1+\tan^2 x)dx = (1+u^2)dx$, so:

$\int \frac{dx}{\tan^3 x+1} = \int \frac{du}{(u^3+1)(u^2+1)} = \int \frac{du}{(u+1)(u^2-u+1)(u^2+1)}$, let $v = u+1$, so $dv = du$, so:

$\int \frac{dx}{\tan^3 x+1} = \int \frac{dv}{(v-1+1)\left[(v-1)^2-(v-1)+1\right]\left[(v-1)^2+1\right]} = \int \frac{dv}{v(v^2-3v+3)(v^2-2v+2)}$

Decomposition into partial fractions : $\frac{1}{v(v^2-3v+3)(v^2-2v+2)} = \frac{A}{v} + \frac{Bv+C}{v^2-3v+3} + \frac{Dv+E}{v^2-2v+2}$,

$1 = A(v^2-3v+3)(v^2-2v+2) + (Bv+C)v(v^2-2v+2) + (Dv+E)v(v^2-3v+3)$, so:

$1 = (A+B+D)v^4 + (-5A-2B-3D+E+C)v^3 + (11A+2B+3D-2C-3E)v^2$

$+(-12A+2C+3E)v + 6A$, so we get the system $\begin{cases} A+B+D=0 \\ -5A-2B-3D+E+C=0 \\ 11A+2B+3D-2C-3E=0 \\ -12A+2C+3E=0 \\ 6A=1 \end{cases}$

So, $A = \frac{1}{6}$, $B = -\frac{2}{3}$, $C = 1$, $D = \frac{1}{2}$ and $E = 0$, then:

$\int \frac{dx}{\tan^3 x+1} = \frac{1}{6}\int \frac{dv}{v} + \int \frac{1-\frac{2}{3}v}{v^2-3v+3}dv + \int \frac{\frac{1}{6}}{v^2-2v+2}dv$

$= \frac{1}{6}\int \frac{dv}{v} + \int \frac{1-\frac{2}{3}v}{v^2-3v+3}dv + \frac{1}{2}\int \frac{v}{v^2-2v+2}dv = \frac{1}{6}\int \frac{dv}{v} - \frac{1}{3}\int \frac{2v-3}{v^2-3v+3}dv + \frac{1}{4}\int \frac{2v-2+2}{v^2-2v+2}dv$

$= \frac{1}{6}\int \frac{dv}{v} - \frac{1}{3}\int \frac{(v^2-3v+3)'}{v^2-3v+3}dv + \frac{1}{4}\int \frac{(v^2-2v+2)'}{v^2-2v+2}dv + \frac{1}{2}\int \frac{1}{(v-1)^2+1}dv$

$= \frac{1}{6}\ln|v| - \frac{1}{3}\ln|v^2-3v+3| + \frac{1}{4}\ln|v^2-2v+2| + \frac{1}{2}\arctan(v-1) + c$

$= \frac{1}{6}\ln|u+1| - \frac{1}{3}\ln|u^2-u+1| + \frac{1}{4}\ln|u^2+1| + \frac{1}{2}\arctan u + c$

$= \frac{1}{6}\ln|\tan x + 1| - \frac{1}{3}\ln|\tan^2 x - \tan x + 1| + \frac{1}{4}\ln|\tan^2 x + 1| + \frac{1}{2}\arctan(\tan x) + c$

$= \frac{1}{6}\ln|\tan x + 1| - \frac{1}{3}\ln|\sec^2 x - \tan x| + \frac{1}{4}\ln|\sec^2 x| + \frac{1}{2}x + c$

400. $\int \frac{\tan x}{(1-\sin x)^3}dx = \int \frac{\sin x}{\cos x(1-\sin x)^3}dx = \int \frac{\sin x}{\cos x(1-\sin x)^3} \times \frac{\cos x}{\cos x}dx$

$= \int \frac{\sin x}{\cos^2 x(1-\sin x)^3}\cos x\, dx = \int \frac{\sin x}{(1-\sin^2 x)(1-\sin x)^3}\cos x\, dx$

$= \int \frac{\sin x}{(1+\sin x)(1-\sin x)(1-\sin x)^3}\cos x\, dx = \int \frac{\sin x}{(1+\sin x)(1-\sin x)^4}\cos x\, dx$

Let $y = \sin x$, then $dy = \cos x\, dx$, so**:** $\int \frac{\tan x}{(1-\sin x)^3}dx = \int \frac{y}{(1+y)(1-y^4)}dy$

Partial fractions decomposition : $\frac{y}{(1+y)(1-y^4)} = \frac{A}{1+y} + \frac{B}{1-y} + \frac{C}{(1-y)^2} + \frac{D}{(1-y)^3} + \frac{E}{(1-y)^4}$,

$y = A(1-y)^4 + B(1+y)(1-y)^3 + C(1+y)(1-y)^2 + D(1+y)(1-y) + E(1+y)$

For $y = 1$, $1 = E(1+1)$, so $E = \frac{1}{2}$, for $y = -1$, $-1 = A(1+1)^4$, so $A = -\frac{1}{16}$.

$A(1 - 4y + 6y^2 - 4y^3 + y^4) + B(1 - 2y + 2y^3 - y^4) + C(1 - y - y^2 + y^3) + D(1 - y^2) +$
$E(1 + y) = y$, comparing the both sides we get $A - B = 0$, so $B = -\frac{1}{16}$, $-4A + 2B + C = 0$,
then $C = -\frac{2}{16} = -\frac{1}{8}$, $6A - C - D = 0$, then $D = -\frac{4}{16} = -\frac{1}{4}$, so we get**:**

$\int \frac{\tan x}{(1-\sin x)^3}dx = -\frac{1}{16}\int \frac{dy}{1+y} - \frac{1}{16}\int \frac{dy}{1-y} - \frac{1}{8}\int \frac{dy}{(1-y)^2} - \frac{1}{4}\int \frac{dy}{(1-y)^3} + \frac{1}{2}\int \frac{dy}{(1-y)^4}$

$= -\frac{1}{16}\ln|1+y| + \frac{1}{16}\ln|1-y| - \frac{1}{8(1-y)} - \frac{1}{8(1-y)^2} + \frac{1}{6(1-y)^4} + c$

$= \frac{1}{16}\ln\left|\frac{1-y}{1+y}\right| - \frac{1}{8(1-y)} - \frac{1}{8(1-y)^2} + \frac{1}{6(1-y)^4} + c$

$= \frac{1}{16}\ln\left|\frac{1-\sin x}{1+\sin x}\right| - \frac{1}{8(1-\sin x)} - \frac{1}{8(1-\sin x)^2} + \frac{1}{6(1-\sin x)^4} + c$

401. $\int \frac{(x^2-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^3}dx = \int \frac{(x^2-1)\sqrt{x^2\left(x^2+2x-1+\frac{2}{x}+\frac{1}{x^2}\right)}}{x^3}dx$

$= \int \frac{(x^2-1)x\sqrt{\left(x^2+2+\frac{1}{x^2}\right)+2\left(x+\frac{1}{x}\right)-3}}{x^3}dx = \int \sqrt{\left(x+\frac{1}{x}\right)^2 + 2\left(x+\frac{1}{x}\right) - 3}\left(1 - \frac{1}{x^2}\right)dx$

Let $u = x + \frac{1}{x}$, then $du = \left(1 - \frac{1}{x^2}\right)dx$, so**:**

$\int \frac{(x^2-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^3}dx = \int \sqrt{u^2 + 2u - 3}\,du = \int \sqrt{(u+1)^2 - 4}\,du$

Let $u + 1 = 2\cosh t$, then $\sqrt{(u+1)^2 - 4} = \sqrt{4(\cosh^2 t + 1)} = \sqrt{4\sinh^2 t} = 2\sinh t$
and $du = 2\sinh t\, dt$, so**:**

$\int \frac{(x^2-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^3}dx = \int 2\sinh t\, 2\sinh t\, dt = 4\int \sinh^2 t\, dt = 4\int \frac{\cosh 2t-1}{2}dt$

$= 2\int(\cosh 2t - 1)\,dt = \sinh 2t - 2t + k$

But $u+1=2\cosh t$, so $\cosh t=\frac{u+1}{2}$, so $\sinh 2t=2\sinh t\cosh t=\frac{1}{2}(u+1)\sqrt{(u+1)^2-4}$

and $t=\cosh^{-1}\left(\frac{u+1}{2}\right)=\ln\left|\frac{u+1}{2}+\sqrt{\left(\frac{u+1}{2}\right)^2-1}\right|$, **so:**

$\int\frac{(x^2-1)\sqrt{x^4+2x^3-x^2+2x+1}}{x^3}dx\ =\frac{1}{2}(u+1)\sqrt{(u+1)^2-4}-2\ln\left|\frac{u+1}{2}+\sqrt{\left(\frac{u+1}{2}\right)^2-1}\right|+k$

$=\frac{1}{2}(u+1)\sqrt{(u+1)^2-4}-2\ln\left|u+1+\sqrt{(u+1)^2-4}\right|\ +c$

$=\frac{1}{2}\left(x+\frac{1}{x}+1\right)\sqrt{\left(x+\frac{1}{x}+1\right)^2-4}-2\ln\left|x+\frac{1}{x}+1+\sqrt{\left(x+\frac{1}{x}+1\right)^2-4}\right|\ +c$

402. $\int\frac{\tan(\ln x)\tan\left[\ln\left(\frac{x}{2}\right)\right]}{x}dx=\int\frac{\tan(\ln x)\tan(\ln x-\ln 2)}{x}dx$, let $u=\ln x$, then $du=\frac{1}{x}dx$, so:

$\int\frac{\tan(\ln x)\tan\left[\ln\left(\frac{x}{2}\right)\right]}{x}dx=\int\tan u\tan(u-\ln 2)du=\int\tan u\times\frac{\tan u-\tan(\ln 2)}{1+\ln 2\tan u}du$, set $a=\ln 2$

$\int\frac{\tan(\ln x)\tan\left[\ln\left(\frac{x}{2}\right)\right]}{x}dx=\int\tan u\times\frac{\tan u-\tan a}{1+a\tan u}du$

Let $t=\tan u$, then $dt=\sec^2 u\,du=(1+\tan^2 u)du=(1+t^2)du$, **then:**

$\int\tan u\times\frac{\tan u-\tan(\ln 2)}{1+\ln 2\tan u}du=\int t\times\frac{t-a}{1+at}\times\frac{1}{1+t^2}dt=\int\frac{t^2-at}{(1+t^2)(1+at)}dt$

Decomposition into partial fractions : $\frac{t^2-at}{(1+t^2)(1+at)}=\frac{\mathrm{A}t+\mathrm{B}}{1+t^2}=\frac{\mathrm{C}}{1+at}$, then

$t^2-at=(\mathrm{A}t+\mathrm{B})(1+at)+\mathrm{C}(1+t^2)$, so $t^2-at=(a\mathrm{A}+\mathrm{C})t^2+(\mathrm{A}+a\mathrm{B})t+\mathrm{B}+\mathrm{C}$

Then we get the following system: $\begin{cases}a\mathrm{A}+\mathrm{C}=1\\ \mathrm{A}+a\mathrm{B}=-1\\ \mathrm{B}+\mathrm{C}=0\end{cases}$, by solving this system we get $\mathrm{A}=0$, $\mathrm{B}=-1$

and $\mathrm{C}=1$, and so we get : $\frac{t^2-at}{(1+t^2)(1+at)}=-\frac{1}{1+t^2}=\frac{1}{1+at}$, **then:**

$\int\frac{\tan(\ln x)\tan\left[\ln\left(\frac{x}{2}\right)\right]}{x}dx=-\int\frac{dt}{1+t^2}+\int\frac{dt}{1+at}=\arctan t+\frac{1}{a}\ln|1+at|+c$

$=-\arctan\tan u+\frac{1}{a}\ln|1+a\tan u|+c=-u+\frac{1}{a}\ln|1+a\tan u|+c$

$=-\ln x+\frac{1}{\ln 2}\ln|1+\ln 2\tan(\ln x)|+c$

403. $\int\frac{dx}{x\ln x\sqrt{\ln^2 x+2\ln x+4}}$, let $u=\ln x$, so $du=\frac{1}{x}dx$, **then:**

$\int\frac{dx}{x\ln x\sqrt{\ln^2 x+2\ln x+4}}=\int\frac{du}{u\sqrt{u^2+2u+4}}=\int\frac{du}{u\sqrt{(u^2+2u+1)+3}}=\int\frac{du}{u\sqrt{(u+1)^2+(\sqrt{3})^2}}$

Let $u+1=\sqrt{3}\tan v$, so $du=\sqrt{3}\sec^2 v\,dv$, **then:**

$\int\frac{dx}{x\ln x\sqrt{\ln^2 x+2\ln x+4}}=\int\frac{\sqrt{3}\sec^2 v}{(\sqrt{3}\tan v-1)\sqrt{3\tan^2 v+3}}dv=\int\frac{\sqrt{3}\sec^2 v}{(\sqrt{3}\tan v-1)\sqrt{3(1+\tan^2 v)}}dv$

$\int\frac{\sqrt{3}\sec^2 v}{(\sqrt{3}\tan v-1)\sqrt{3\sec^2 v}}dv=\int\frac{\sqrt{3}\sec^2 v}{(\sqrt{3}\tan v-1)\sqrt{3}\sec v}dv=\int\frac{\sec v}{(\sqrt{3}\tan v-1)}dv$

$$= \int \frac{1}{\cos v\left(\sqrt{3}\frac{\sin v}{\cos v}-1\right)} dv = \int \frac{1}{\sqrt{3}\sin v-\cos v} dv$$

Let $t = \tan\frac{v}{2}$, then $dv = \frac{2}{1+t^2}dt$, $\sin v = \frac{2t}{1+t^2}$ and $\cos v = \frac{1-t^2}{1+t^2}$, so:

$$\int \frac{dx}{x\ln x\sqrt{\ln^2 x+2\ln x+4}} = \int \frac{\frac{2}{1+t^2}}{\frac{2\sqrt{3}t}{1+t^2}-\frac{1-t^2}{1+t^2}} dt = \int \frac{2}{t^2+2\sqrt{3}t-1} dt = 2\int \frac{1}{\left(t+\sqrt{3}+2\right)\left(t+\sqrt{3}-2\right)} dt$$

Method of partial fractions : $\frac{1}{\left(t+\sqrt{3}+2\right)\left(t+\sqrt{3}-2\right)} = \frac{\mathrm{A}}{t+\sqrt{3}+2} + \frac{\mathrm{B}}{t+\sqrt{3}-2}$, then**:**

$1 = \mathrm{A}\left(t+\sqrt{3}-2\right) + \mathrm{B}\left(t+\sqrt{3}+2\right)$

For $t = -\sqrt{3}-2$, then $1 = \mathrm{A}\left(-\sqrt{3}-2+\sqrt{3}-2\right)$, so $\mathrm{A} = -\frac{1}{4}$

For $t = -\sqrt{3}+2$, then $1 = \mathrm{B}\left(-\sqrt{3}+2+\sqrt{3}+2\right)$, so $\mathrm{B} = \frac{1}{4}$, so**:**

$$\int \frac{dx}{x\ln x\sqrt{\ln^2 x+2\ln x+4}} = \frac{1}{2}\int \left(\frac{1}{t+\sqrt{3}-2} - \frac{1}{t+\sqrt{3}+2}\right) dt$$

$$= \frac{1}{2}\left(\ln\left|t+\sqrt{3}-2\right| - \ln\left|t+\sqrt{3}+2\right|\right) + c = \frac{1}{2}\ln\left|\frac{t+\sqrt{3}-2}{t+\sqrt{3}+2}\right| + c$$

$= \frac{1}{2}\ln\left|\frac{\tan\frac{v}{2}+\sqrt{3}-2}{\tan\frac{v}{2}+\sqrt{3}+2}\right| + c$, but $u+1 = \sqrt{3}\tan v$, then $\tan v = \frac{u+1}{\sqrt{3}}$, so $v = \tan^{-1}\left(\frac{u+1}{\sqrt{3}}\right)$, then**:**

$\int \frac{dx}{x\ln x\sqrt{\ln^2 x+2\ln x+4}} = \frac{1}{2}\ln\left|\frac{\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{u+1}{\sqrt{3}}\right)\right)+\sqrt{3}-2}{\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{u+1}{\sqrt{3}}\right)\right)+\sqrt{3}+2}\right| + c$, with $u = \ln x$, therefore**:**

$$\int \frac{dx}{x\ln x\sqrt{\ln^2 x+2\ln x+4}} = \frac{1}{2}\ln\left|\frac{\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{\ln x+1}{\sqrt{3}}\right)\right)+\sqrt{3}-2}{\tan\left(\frac{1}{2}\tan^{-1}\left(\frac{\ln x+1}{\sqrt{3}}\right)\right)+\sqrt{3}+2}\right| + c$$

404. $$\int \frac{\sqrt[6]{2x-4}}{\sqrt[4]{2x-4}+\sqrt{2x-4}} dx = \int \frac{\left(\sqrt[12]{2x-4}\right)^2}{\left(\sqrt[12]{2x-4}\right)^3+\left(\sqrt[12]{2x-4}\right)^6} dx$$

Let $y = \sqrt[12]{2x-4}$, so $y^{12} = 2x-4$, then $x = \frac{y^{12}+4}{2}$ and $dx = 6y^{11}dy$, so**:**

$$\int \frac{\sqrt[6]{2x-4}}{\sqrt[4]{2x-4}+\sqrt{2x-4}} dx = \int \frac{y^2}{y^3+y^6}.6y^{11}dy = \int \frac{6y^{13}}{y^3+y^6} dy = 6\int \frac{y^{10}}{1+y^3} dy$$

By performing Euclidean division of y^{10} by $1+y^3$, we get:

$$\frac{y^{10}}{1+y^3} = y^7 - y^4 + y - \frac{y}{1+y^3} = y^7 - y^4 + y - \frac{y}{(1+y)(1-y+y^2)}$$

Method of partial fractions : $\frac{y}{(1+y)\left(1-y+y^2\right)} = \frac{\mathrm{A}}{1+y} + \frac{\mathrm{B}y+\mathrm{C}}{1-y+y^2}$, then we get**:**

$y = \mathrm{A}(1-y+y^2) + (\mathrm{B}y+\mathrm{C})(y+1)$, for $y = -1$, we get $-1 = \mathrm{A}(1+1+1)$, so $\mathrm{A} = -\frac{1}{3}$,

For $y = 0$, we get $0 = -\frac{1}{3} + \mathrm{C}$, so $\mathrm{C} = \frac{1}{3}$, for $y = 1$, we get $1 = -\frac{1}{3} + 2\left(\mathrm{B}+\frac{1}{3}\right)$, so $\mathrm{B} = \frac{1}{3}$, so**:**

$$\frac{y^{10}}{1+y^3} = y^7 - y^4 + y - \frac{y}{1+y^3} = y^7 - y^4 + y + \frac{\frac{1}{3}}{1+y} - \frac{\frac{1}{3}y+\frac{1}{3}}{1-y+y^2} \quad \text{and so:}$$

$$\frac{6y^{10}}{1+y^3}=6y^7-6y^4+6y-\frac{6y}{1+y^3}=6y^7-6y^4+6y+\frac{2}{1+y}-\frac{2y+2}{1-y+y^2};\ \text{then:}$$

$$\int\frac{\sqrt[6]{2x-4}}{\sqrt[4]{2x-4}+\sqrt{2x-4}}=\frac{3}{4}y^8-\frac{5}{5}y^5+3y^2+2\ln|y+1|+c_1-\int\frac{2y+2}{y^2-y+1}dy$$

Evaluating: $\int\frac{2y+2}{y^2-y+1}dy=\int\frac{2y-1+3}{y^2-y+1}dy=\int\left[\frac{2y-1}{y^2-y+1}+\frac{3}{y^2-y+\frac{1}{4}+\frac{3}{4}}\right]dy$

$$=\int\left[\frac{(y^2-y+1)'}{y^2-y+1}+\frac{3}{\left(y-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]dy=\ln|y^2-y+1|+2\sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}y-\frac{1}{\sqrt{3}}\right)+c_2,$$

Therefore: $\int\frac{\sqrt[6]{2x-4}}{\sqrt[4]{2x-4}+\sqrt{2x-4}}$

$$=\frac{3}{4}(2x-4)^{\frac{2}{3}}-\frac{5}{5}(2x-4)^{\frac{5}{12}}+3(2x-4)^{\frac{1}{6}}+2\ln\left|\sqrt[12]{2x-4}+1\right|$$

$$-\ln\left|\sqrt[6]{2x-4}-\sqrt[12]{2x-4}+1\right|-2\sqrt{3}\arctan\left(\frac{2}{\sqrt{3}}\sqrt[12]{2x-4}-\frac{1}{\sqrt{3}}\right)+c$$

With $y=\sqrt[12]{2x-4}$

405. $\int\frac{1+x^2}{x^4+2x^2\cos\left(\frac{2\pi}{5}\right)+1}dx$

$$x^4+2x^2\cos\left(\frac{2\pi}{5}\right)+1=x^4+2x^2\left[1-2\sin^2\left(\frac{\pi}{5}\right)\right]+1=(x^4+2x^2+1)-4x^2\sin^2\left(\frac{\pi}{5}\right)$$

$=(x^2+1)^2-\left[2x\sin\left(\frac{\pi}{5}\right)\right]^2=\left(x^2+1+2x\sin\left(\frac{\pi}{5}\right)\right)\left(x^2+1-2x\sin\left(\frac{\pi}{5}\right)\right)$, then:

$$\int\frac{1+x^2}{x^4+2x^2\cos\left(\frac{2\pi}{5}\right)+1}dx=\int\frac{1+x^2}{\left(x^2+1+2x\sin\left(\frac{\pi}{5}\right)\right)\left(\left(x^2+1-2x\sin\left(\frac{\pi}{5}\right)\right)\right)}dx$$

$$=\frac{1}{2}\int\frac{\left(x^2+1+2x\sin\left(\frac{\pi}{5}\right)\right)+\left(x^2+1-2x\sin\left(\frac{\pi}{5}\right)\right)}{\left(x^2+1+2x\sin\left(\frac{\pi}{5}\right)\right)\left(\left(x^2+1-2x\sin\left(\frac{\pi}{5}\right)\right)\right)}dx$$

$$=\frac{1}{2}\int\frac{1}{x^2+1-2x\sin\left(\frac{\pi}{5}\right)}dx+\frac{1}{2}\int\frac{1}{x^2+1+2x\sin\left(\frac{\pi}{5}\right)}dx$$

$$=\frac{1}{2}\int\frac{1}{\left(x+\sin\left(\frac{\pi}{5}\right)\right)^2+1-\sin^2\left(\frac{\pi}{5}\right)}dx+\frac{1}{2}\int\frac{1}{\left(x-\sin\left(\frac{\pi}{5}\right)\right)^2+1-\sin^2\left(\frac{\pi}{5}\right)}dx$$

$$=\frac{1}{2}\int\frac{1}{\left(x+\sin\left(\frac{\pi}{5}\right)\right)^2+\left(\sqrt{1-\sin^2\left(\frac{\pi}{5}\right)}\right)^2}dx+\frac{1}{2}\int\frac{1}{\left(x-\sin\left(\frac{\pi}{5}\right)\right)^2+\left(\sqrt{1-\sin^2\left(\frac{\pi}{5}\right)}\right)^2}dx$$

$$=\frac{1}{2\sqrt{1-\sin^2\left(\frac{\pi}{5}\right)}}\arctan\left(\frac{x+\sin\left(\frac{\pi}{5}\right)}{\sqrt{1-\sin^2\left(\frac{\pi}{5}\right)}}\right)+\frac{1}{2\sqrt{1-\sin^2\left(\frac{\pi}{5}\right)}}\arctan\left(\frac{x-\sin\left(\frac{\pi}{5}\right)}{\sqrt{1-\sin^2\left(\frac{\pi}{5}\right)}}\right)+c$$

406. $\int\frac{dx}{x^{n-2}+C_n^1x^{n-3}+C_n^2x^{n-4}+\cdots+x^{-2}}=\int\frac{x^2}{x^2\left(x^{n-2}+C_n^1x^{n-3}+C_n^2x^{n-4}+\cdots+x^{-2}\right)}dx$

$=\int\frac{x^2}{x^n+C_n^1x^{n-1}+C_n^2x^{n-2}+\cdots+1}dx=\int\frac{x^2}{(1+x)^n}dx$, let $t=x+1$, then $dt=dx$, so:

$$\int\frac{dx}{x^{n-2}+C_n^1x^{n-3}+C_n^2x^{n-4}+\cdots+x^{-2}}=\int\frac{(t-1)^2}{t^n}dt=\int\frac{t^2-2t+1}{t^n}dt=\int(t^{2-n}-2t^{1-n}+t^{-n})\,dt$$

$= \frac{1}{3-n}t^{3-n} - \frac{2}{2-n}t^{2-n} + \frac{1}{1-n}t^{1-n} + c$

$= \frac{1}{3-n}(1+x)^{3-n} - \frac{2}{2-n}(1+x)^{2-n} + \frac{1}{1-n}(1+x)^{1-n} + c$

407. $\int \frac{dx}{x(x+1)(x+2)\ldots(x+m)}$

Partial fractions decomposition : $\frac{1}{x(x+1)(x+2)\ldots(x+m)} = \frac{a_0}{x} + \frac{a_1}{x+1} + \frac{a_2}{x+2} + \cdots + \frac{a_m}{x+m}$, so

$1 = a_0(x+1)(x+2)(x+3)\ldots(x+m) + a_1 x(x+2)(x+3)\ldots(x+m)$

$+ a_2 x(x+1)(x+3)\ldots(x+m) + \cdots + a_m x(x+1)(x+2)\ldots\big(x+(m-1)\big)$, then we have

$$a_0 = \frac{1}{(x+1)(x+2)(x+3)\ldots(x+m)}\bigg|_{x=0} = \frac{(-1)^0}{0!\,m!}$$

$$a_1 = \frac{1}{x(x+2)(x+3)\ldots(x+m)}\bigg|_{x=-1} = \frac{(-1)^1}{1!\,(m-1)!}$$

$$a_2 = \frac{1}{x(x+1)(x+3)\ldots(x+m)}\bigg|_{x=-2} = \frac{(-1)^2}{2!\,(m-2)!}$$

$$\ldots \quad a_k = \frac{(-1)^k}{k!\,(m-k)!}$$

$$\int \frac{dx}{x(x+1)(x+2)\ldots(x+m)} = \int \sum_{k=0}^{m} \frac{(-1)^k}{k!\,(m-k)!}\frac{1}{x+k}dx = \sum_{k=0}^{m} \frac{(-1)^k}{k!\,(m-k)!}\ln(x+k) + c$$

408. $\int \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}}\,dx$, let $t = \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}}$, then:

$t^2 = x + \sqrt{x+\sqrt{x+\sqrt{x+\cdots}}} = x + t$, so $t^2 - t = x$, then $t^2 - t + \frac{1}{4} = x + \frac{1}{4}$, so

$\left(t-\frac{1}{2}\right)^2 = x + \frac{1}{4}$, then $t - \frac{1}{2} = \pm\sqrt{x+\frac{1}{4}} = \pm\sqrt{\frac{1+4x}{4}} = \pm\frac{\sqrt{1+4x}}{2}$, then $t = \frac{1}{2} \pm \pm\frac{\sqrt{1+4x}}{2}$,

but $t \geq 0$ for all x, then: $t = \frac{1}{2}\big(1+\sqrt{1+4x}\big)$, then:

$$\int \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x+\cdots}}}}\,dx = \frac{1}{2}\int\big(1+\sqrt{1+4x}\big)dx = \frac{1}{2}x + \frac{1}{12}(1+4x)^{\frac{3}{2}} + c$$

409. **First Method:** $\int \frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}dx$

Let $x = \tan^4 y$ $\left(\sqrt{x} = \tan^2 y\right)$, then $dx = 4\tan^3 y \sec^2 y\, dy$, then we get:

$$\int \frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}dx = 4\int \frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\tan^2 y}{1+\tan^2 y}\right)\right)}\tan^3 y \sec^2 y\, dy$$

$= 4\int \frac{1-\tan\left(\frac{1}{2}\sin^{-1}(\cos 2y)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}(\cos 2y)\right)}\tan^3 y \sec^2 y\, dy$, let $f(y) = \arcsin(\cos 2y)$, then:

$f'(y) = \frac{-2\sin 2y}{\sqrt{1-\cos^2 2y}} = \frac{-2\sin 2y}{\sin 2y} = -2$, so $f(y) = -2y + k$, for $x = 0$, $f(0) = \arcsin 1 = k$

then $k=\frac{\pi}{2}$, so $f(y)=\frac{\pi}{2}-2y$, thus we get:

$$\int\frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}dx=4\int\frac{1-\tan\left(\frac{\pi}{4}-y\right)}{1+\tan\left(\frac{\pi}{4}-y\right)}\tan^3y\sec^2y\,dy$$

$$=4\int\frac{1-\frac{\tan\left(\frac{\pi}{4}\right)-\tan y}{1+\tan\left(\frac{\pi}{4}\right)\tan y}}{1+\frac{\tan\left(\frac{\pi}{4}\right)-\tan y}{1+\tan\left(\frac{\pi}{4}\right)\tan y}}\tan^3y\sec^2y\,dy=4\int\frac{1-\frac{1-\tan y}{1+\tan y}}{1+\frac{1-\tan y}{1+\tan y}}\tan^3y\sec^2y\,dy$$

$$=4\int\frac{\frac{1+\tan y-1+\tan y}{1+\tan y}}{\frac{1+\tan y+1-\tan y}{1+\tan y}}\tan^3y\sec^2y\,dy=4\int\frac{\frac{2\tan y}{1+\tan y}}{\frac{2}{1+\tan y}}\tan^3y\sec^2y\,dy=4\int\tan^4y\sec^2y\,dy$$

$$=4\int\tan^4y\,(\tan y)'\,dy=\frac{4}{5}\tan^5y+c=\frac{4}{5}(\tan^4y)^{\frac{5}{4}}+c=\frac{4}{5}x^{\frac{5}{4}}+c$$

Second Method: $\int\frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}dx$, let $\theta=\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)$, so $\sin\theta=\frac{1-\sqrt{x}}{1+\sqrt{x}}$, then

$\sqrt{x}=\frac{1-\sin\theta}{1+\sin\theta}$ and so $x=\left(\frac{1-\sin\theta}{1+\sin\theta}\right)^2=\left[\frac{\sin^2\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)-2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}{\sin^2\left(\frac{\theta}{2}\right)+\cos^2\left(\frac{\theta}{2}\right)+2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)}\right]^2$, then

$x=\left[\frac{\left(\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)\right)^2}{\left(\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)\right)^2}\right]^2=\left(\frac{\cos\left(\frac{\theta}{2}\right)-\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)+\sin\left(\frac{\theta}{2}\right)}\right)^4=\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)^4$, then:

$$\int\frac{1-\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}{1+\tan\left(\frac{1}{2}\sin^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right)\right)}dx=\int\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}.\,d\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)^4$$

$$=\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)^4-\int\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)^4.\,d\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)=\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)^5-\frac{1}{5}\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)^5+c$$

$$=\frac{4}{5}\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right)^5+c=\frac{4}{5}x^{\frac{5}{4}}+c$$

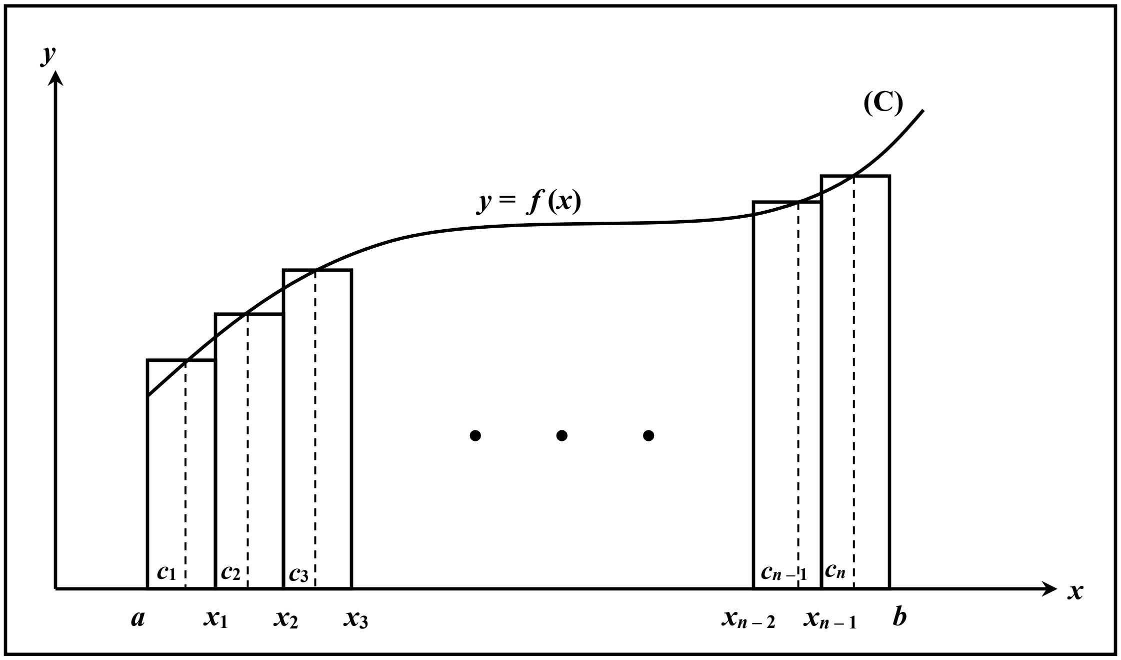
End of Chapter 2

Chapter 3: Definite Integrals

1. **Introduction:**

Let f be a continuous function over an interval $[a\ ;\ b]$ and let (C) be its representative curve in an orthonormal system $(O; \vec{\imath}, \vec{\jmath})$.

The concept of a definite integral is often motivated by consideration of the area bounded by the curve (C), the axis of abscissas and the two straight lines with equations $x = a$ and $x = b$.



In the above figure, we subdivide the interval $[a\ ;\ b]$ into n sub-intervals by means of the points $x_1, x_2 \dots x_n$ chosen arbitrary.

In each of the new intervals $]a, x_1[$, $]x_1, x_2[\ \dots\ ]x_{n-1}, b[$ we choose the points $c_1, c_2 \dots c_n$ arbitrary. Let's consider the sum:

$$\boldsymbol{f(c_1)(x_1 - a) + f(c_2)(x_2 - x_1) + \cdots + f(c_n)(b - x_{n-1})}$$

Then we write $a = x_0$, $b = x_n$ and $x_i - x_{i-1} = \Delta x_i$, then the sum becomes:

$$\boldsymbol{\sum_{i=1}^{n} f(x_i)\,\Delta x_i \quad \dots \quad (1)}$$

Geometrically the obtained sum represents the total area of all rectangles in the above figure.

We now let the number n of subdivisions increase in such a way that each of $\Delta x_i \to 0$

Then the sum (1) approaches a limit which does not depend on the mode of subdivisions, we denote this limit by:

$$\boldsymbol{\int_a^b f(x)dx}$$

And which is called the definite integral of $f(x)$ between a and b (a and b: limits of integration)

$f(x)dx$ is often called the integrand and $[a\ ;\ b]$ is called the range of integration

2. **Definition of Definite Integrals:**

Let f be a continuous function over an interval I subset of $\mathbb{R}$ and let F be an anti-derivative (primitive) of f over I.
Let *a* and *b* be two real numbers belongs to I.

We call the integral of the function f from a to b to be the real number F(b) – F(a), denoted by $\int_a^b f(x)dx$, so we have:

$$\int_a^b f(x)dx = \mathrm{F}(b) - \mathrm{F}(a)$$

This is called the **"Fundamental Theorem of Integral Calculus"**

Properties:

Let f and g be two continuous functions over the interval I = [*a;b*], then the following hold:

- $\int_a^a f(x)dx = 0$
- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$, where $a \leq c \leq b$
- $\int_a^b [\alpha f(x) \pm \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$
- $\left|\int_a^b f(x)\, dx\right| \leq \int_a^b |f(x)|dx$

Remarks:

- If $f(x) \geq 0$ over I, then $\int_a^b f(x)dx \geq 0$
- If $f(x) \leq 0$ over I, then $\int_a^b f(x)dx \leq 0$
- If $f(x) \leq g(x)$ over I, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
- If $f(x) \geq g(x)$ over I, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$
- If for $a \leq x \leq b$, $m \leq f(x) \leq \mathrm{M}$, where m and M are constants, then

  $m(b-a) \leq \int_a^b f(x)dx \leq \mathrm{M}(b-a)$

3. **Even and Odd Functions:**

Let f be a continuous function over the interval $[-a;a]$ where $a > 0$, then:

- $\int_{-a}^a f(x)\, dx = 2\int_0^a f(x)dx$, if f is an even function.
- $\int_{-a}^a f(x)dx = 0$, if f is an odd function.

Remarks

$$\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx & \text{if } f(-x) = f(x) \\ 0 & \text{if } f(-x) = -f(x) \end{cases}$$

$$\int_{0}^{2a} f(x)dx = \begin{cases} 2\int_{0}^{a} f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 & \text{if } f(2a-x) = -f(x) \end{cases}$$

4. **Mean Value Theorems for Integrals:**

First Mean Value Theorem: If $f(x)$ is continuous over $[a;b]$, then there exists $c \in]a;b[$ such that**:**

$$\boldsymbol{\int_{a}^{b} f(x)dx = (b-a)f(c)}$$

Generalized First Mean Value Theorem: If $f(x)$ and $g(x)$ are two continuous functions over $[a;b]$ and $g(x)$ does not change sign over the interval, then there exists $c \in]a;b[$ such that**:**

$$\boldsymbol{\int_{a}^{b} f(x)g(x)dx = f(c)\int_{a}^{b} g(x)dx}$$

Bonnet's Second Mean Value Theorem: If $f(x)$ and $g(x)$ are two continuous functions over $[a;b]$ and if $g(x)$ is a positive monotonic decreasing function, then there exists $c \in]a;b[$ such that**:**

$$\boldsymbol{\int_{a}^{b} f(x)g(x)dx = g(a)\int_{a}^{c} f(x)dx}$$

If $g(x)$ is a positive monotonic decreasing function, then there exists $c \in]a;b[$ such that**:**

$$\boldsymbol{\int_{a}^{b} f(x)g(x)dx = g(b)\int_{c}^{b} f(x)dx}$$

Generalized Bonnet's Second Mean Value Theorem: If $f(x)$ and $g(x)$ are two continuous functions over $[a;b]$ and if $g(x)$ is monotonic increasing or monotonic decreasing and not necessarily always positive, then there exists $c \in]a;b[$ such that**:**

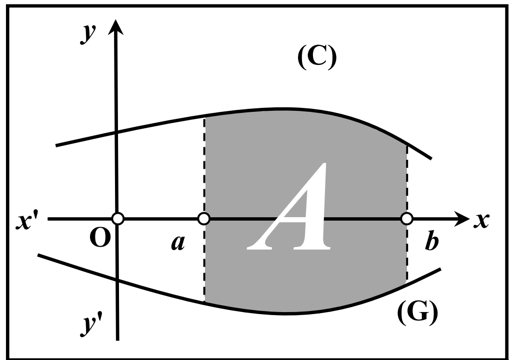
$$\boldsymbol{\int_{a}^{b} f(x)g(x)dx = g(a)\int_{a}^{c} f(x)dx + g(b)\int_{c}^{b} f(x)dx}$$

5. **Integrals and Areas:**

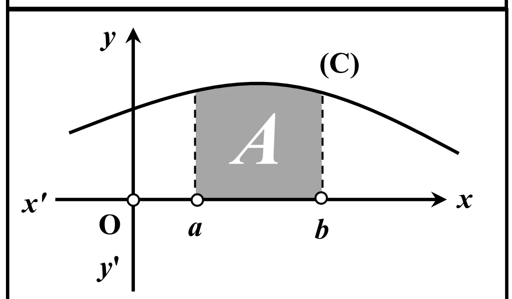
Let f and g be two continuous functions over the interval I = [a;b] such that $f(x) \geq g(x)$ over I.

The area $\mathcal{A}$ of the domain limited by (C) and (G) the respective curves of f and g is given by:

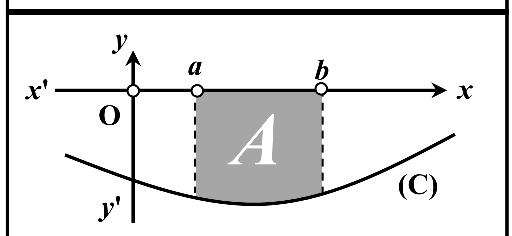
A = $\int_a^b [f(x) - g(x)]dx\ u^2$, where u^2 is the unit of area in the coordinate system **(Figure 1)**



- If $f(x) > 0$ over I then the area of the domain limited between (C) the representative curve of f, x-axis and the two straight lines of equations $x = a$ and $x = b$ is given by: A = $\int_a^b f(x)dx$. **(Figure 2)**



- If $f(x) < 0$ over I , then A = $-\int_a^b f(x)dx$. **(Figure 3)**



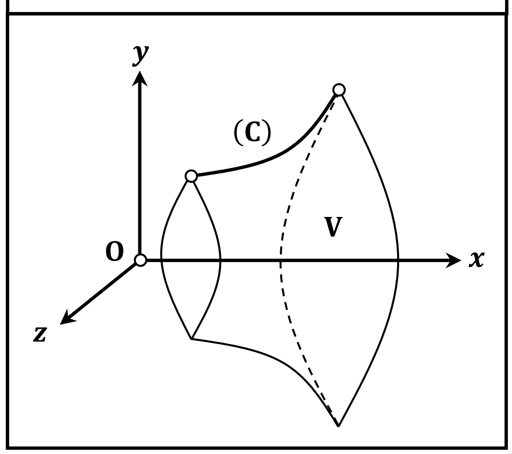
6. **Volumes and Integrals:**

Let f be a continuous function and designate by (C) its representative curve in an orthonormal system (O; $\vec{i}$, $\vec{j}$).

Let (D) be the domain limited by (C), the x-axis and the two straight lines of equations $x = a$ and $x = b$.

Let V be the volume of the solid obtained by rotating (D) about ($x'x$), then:

V = $\pi \int_a^b [f(x)]^2\, dx$, cubic units



7. **Improper Integrals:**

If the range of the integration $[a; b]$ is not finite or if the function $f(x)$ is not defined at one or more points of $[a; b]$, then the integral of $f(x)$ over this range is called an improper integral, i.e.

The integral $\int_a^b f(x)dx$ is called an improper integral if:

a. $a = -\infty$ or $b = +\infty$ or both (one or both integration limits is infinite)

b. $f(x)$ is unbounded at one or more points of $a \leq x \leq b$, such points are called singularities of $f(x)$

Integrals corresponding to (a) and (b) are called improper integrals of the first and second kind respectively

Integrals with both conditions (a) and (b) are called integrals of the second kind

Examples:

- $\int_0^{+\infty} \cos(x^2+1)\,dx$ is an improper integral of the first kind
- $\int_1^3 \frac{dx}{x-2}$ is an improper integral of the second kind
- $\int_0^{+\infty} \frac{e^x}{x}\,dx$ is an improper integral of the third kind

Remark

- $\int_a^{+\infty} f(x)\,dx = \lim\limits_{t\to+\infty} \int_a^t f(x)dx$ and $\int_{-\infty}^b f(x)\,dx = \lim\limits_{t\to-\infty} \int_t^b f(x)dx$
- $\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{+\infty} f(x)\,dx$

King Property of Definite Integrals

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$$

Proof:

Consider the substitution: $u = a+b-x \Longrightarrow du = -dx$ & $\begin{cases} x = a \\ x = b \end{cases} \Longrightarrow \begin{cases} u = b \\ u = a \end{cases}$; then we get:

$$\int_a^b f(x)dx = \int_b^a f(a+b-u)(-du) = \int_a^b f(a+b-u)du = \int_a^b f(a+b-x)dx$$

Two More Formulae:

$$\int_a^b f(x)\,dx = \int_a^{\frac{a+b}{2}} f(x)dx + \int_a^{\frac{a+b}{2}} f(a+b-x)dx \quad \&$$

$$\int_a^b f(x)\,dx = \int_{\frac{a+b}{2}}^{b} f(x)dx + \int_{\frac{a+b}{2}}^{b} f(a+b-x)dx$$

Worked Examples (1):

$$\mathbf{I} = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx$$

We can wite I as $\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(0+\frac{\pi}{4}-x\right)\right)dx = \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$; then :

$$\mathrm{I}+\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx + \int_0^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$$

$$2\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx + \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right) dx$$

$$= \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx + \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1-\tan x}{1+\tan x}\right) dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx + \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right) dx$$

Then: $2\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx + \int_0^{\frac{\pi}{4}} \ln 2\,dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\,dx = \int_0^{\frac{\pi}{4}} \ln 2\,dx$

$2\mathrm{I} = \frac{\pi}{4}\ln 2$; therefore; $\mathrm{I} = \frac{\pi}{8}\ln 2$

Worked Examples (2):

$$\mathbf{I} = \int_{-1}^{1} \tan^{-1}(e^x)\,dx$$

I can be written as $\mathrm{I} = \int_{-1}^{1} \tan^{-1}(e^{-1+1-x})\,dx = \int_{-1}^{1} \tan^{-1}(e^{-x})dx$; then we get:

$$\mathrm{I} + \mathrm{I} = 2\mathrm{I} = \int_{-1}^{1} \tan^{-1}(e^x)\,dx + \int_{-1}^{1} \tan^{-1}(e^{-x})dx = \int_{-1}^{1} [\tan^{-1}(e^x)dx + \tan^{-1}(e^{-x})]dx$$

$$2\mathrm{I} = \int_{-1}^{1} \frac{\pi}{2}dx = 2\int_0^1 \frac{\pi}{2}\mathrm{d}x = 2\left(\frac{\pi}{2}\right) \Longrightarrow \mathrm{I} = \frac{\pi}{2}$$

Worked Examples (3):

$$\mathbf{I} = \int_0^4 \frac{1}{4+2^x}\,dx$$

I can be written as $\mathrm{I} = \int_0^4 \frac{1}{4+2^{4+0-x}}\,dx = \int_0^4 \frac{1}{4+2^{4-x}}\,dx$; then we get:

$$\mathrm{I} + \mathrm{I} = 2\mathrm{I} = \int_0^4 \frac{1}{4+2^x}\,dx + \int_0^4 \frac{1}{4+2^{4-x}}\,dx = \int_0^4 \left(\frac{1}{4+2^x} + \frac{1}{4+2^{4-x}}\right) dx$$

$$2\mathrm{I} = \int_0^4 \left(\frac{1}{4+2^x} + \frac{2^x}{4\times 2^x + 16}\right) dx = \int_0^4 \left(\frac{4+2^x}{4\times 2^x+16}\right) dx = \frac{1}{4}\int_0^4 \left(\frac{4+2^x}{4+2^x}\right) dx;$$ then:

$$2\mathrm{I} = \frac{1}{4}\int_0^4 dx = \frac{1}{4}(4) = 1;\ \ 2\mathrm{I} = 1;\ \ \text{therefore;}\ \ \text{we get: } \mathrm{I} = \frac{1}{2}$$

Worked Examples (4):

$$\mathbf{I} = \int_0^{2\pi} \frac{\mathbf{1}}{\mathbf{1} + \boldsymbol{e}^{\sin x}}\,\boldsymbol{dx}$$

I can be written as: $\mathrm{I} = \int_0^{2\pi} \frac{1}{1 + e^{\sin(2\pi+0-x)}}\,dx = \int_0^{2\pi} \frac{1}{1 + e^{\sin(2\pi-x)}}\,dx = \int_0^{2\pi} \frac{1}{1 + e^{\sin(-x)}}\,dx$

Then we get: $\mathrm{I} = \int_0^{2\pi} \frac{1}{1 + e^{-\sin x}}\,dx$; adding the two forms we get:

$$2\mathrm{I} = \int_0^{2\pi} \frac{1}{1 + e^{\sin x}}\,dx + \int_0^{2\pi} \frac{1}{1 + e^{-\sin x}}\,dx = \int_0^{2\pi} \left(\frac{1}{1 + e^{\sin x}} + \frac{1}{1 + e^{-\sin x}}\right) dx$$

$$2\mathrm{I} = \int_0^{2\pi} \left(\frac{1}{1 + e^{\sin x}} + \frac{e^{\sin x}}{1 + e^{\sin x}}\right) dx = \int_0^{2\pi} \left(\frac{1 + e^{\sin x}}{1 + e^{\sin x}}\right) dx = \int_0^{2\pi} dx = 2\pi \Longrightarrow \mathrm{I} = \pi$$

Worked Examples (5):

$$\mathbf{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,\boldsymbol{dx}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^n\left(0 + \frac{\pi}{2} - x\right)}{\sin^n\left(0 + \frac{\pi}{2} - x\right) + \cos^n\left(0 + \frac{\pi}{2} - x\right)}\,dx = \int_0^{\frac{\pi}{2}} \frac{\cos^n\left(\frac{\pi}{2} - x\right)}{\sin^n\left(\frac{\pi}{2} - x\right) + \cos^n\left(\frac{\pi}{2} - x\right)}\,dx \text{ ; then:}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx;\ \ \text{so: } \mathrm{I} + \mathrm{I} = 2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^n x}{\sin^n x + \cos^n x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\cos^n x + \sin^n x}\,dx$$

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^n x + \sin^n x}{\cos^n x + \sin^n x}\,dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2} \Longrightarrow \mathrm{I} = \frac{\pi}{4}$$

Solved Exercises

Evaluate each of the following integrals:

1. $\int_0^1 (3x^2+2x+1)dx$
2. $\int_{-2}^{-1} \left(\frac{x^4+1}{x^2}\right) dx$
3. $\int_0^1 x\sqrt{x}dx$
4. $\int_0^1 x(1+x^2)^4 dx$
5. $\int_1^2 x\sqrt{x^2+1}dx$
6. $\int_0^1 \frac{x+1}{\sqrt{x^2+2x+4}} dx$
7. $\int_0^1 \frac{xdx}{(x^2+1)^4}$
8. $\int_1^2 (x+1)(x^2+2x-3)^2 dx$
9. $\int_0^1 |5x-3|\, dx$
10. $\int_0^2 |x^2-x| dx$
11. $\int_1^2 |x^2-3x+2| dx$
12. $\int_{-2}^2 x(x^8+x^4+1)^{10}\, dx$
13. $\int_{-1}^1 x\sqrt{x^4+1}\, dx$
14. $\int_1^e \frac{3}{2x-1} dx$
15. $\int_0^1 \frac{x-1}{x^2-2x+3} dx$
16. $\int_1^e \frac{\ln x}{x} dx$
17. $\int_e^{e^2} \frac{dx}{x\ln x}$
18. $\int_1^2 2e^{2x-1} dx$
19. $\int_0^{\ln 2} e^{-x}(e^{2x}+2e^x-2)dx$
20. $\int_0^1 \frac{e^x+1}{e^x+x} dx$
21. $\int_0^{\frac{1}{\sqrt{2}}} \frac{x\arcsin x^2}{\sqrt{1-x^4}} dx$
22. $\int_1^2 \frac{dx}{(x^2-2x+4)^{\frac{3}{2}}}$
23. $\int_{-1}^2 \frac{|x|}{x} dx$
24. $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}} dx$
25. $\int_0^{2\pi} e^x |\sin x|\, dx$
26. $\int_{-1}^1 |3^x-2^x|\, dx$
27. $\int_0^a \frac{f(x)}{f(x)+f(a-x)} dx$
28. $\int_a^b \frac{f(x)}{f(x)+f(a+b-x)} dx$
29. $\int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\sin x}{1+\cos x}\right) dx$
30. $\int_0^{\ln\sqrt{3}} \frac{e^{2x}}{(e^{2x}+1)\ln(e^{2x}+1)} dx$
31. $\int_{-2}^2 \sqrt{4-x^2}\, dx$
32. $\int_0^{\frac{\pi}{2}} \sin(2x)\cos(\cos x)\, dx$

33. $\int_0^1 x(1-x)^n\,dx$

34. $\int_{n\pi}^{(n+1)\pi} e^{-x}\sin x\,dx$

35. $\int_0^{\frac{1}{2}} \cos(5\arcsin x)\,dx$

36. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{\sin x}+1}\,dx$

37. $\int_{-1}^{1} \frac{x^2+\sin x}{x^2+1}\,dx$

38. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{\frac{1}{x}}}\,dx$

39. $\int_0^{\frac{\pi}{2}} \frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}\,dx$

40. $\int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}\,dx$

41. $\int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{1991} x}\,dx$

42. $\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$

43. $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

44. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin x\cos x}\,dx$

45. $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\alpha\cos^2 x+\beta\sin^2 x}\,dx$

46. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x\sin x}{\sqrt[x]{e}+1}\,dx$

47. $\int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}}\,dx$

48. $\int_0^{2\pi} \frac{x\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x}\,dx$

49. $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x}\,dx$

50. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\tan^2 x}{1+4^x}\,dx$

51. $\int_{-1}^{1} \frac{x^2}{1+2^{\sin x}}\,dx$

52. $\int_0^1 \ln\left(\frac{1}{x}-1\right)dx$

53. $\int_0^{+\infty} \frac{\ln(2x)}{1+x^2}\,dx$

54. $\int_0^{+\infty} \frac{\arctan x}{x\sqrt{x}+\sqrt{x}}\,dx$

55. $\int_0^{\frac{\pi}{2}} \sin 2x\ln(\tan x)\,dx$

56. $\int_0^{\frac{\pi}{4}} \ln(\cot x-\tan x)\,dx$

57. $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$

58. $\int_0^{\frac{\pi}{2}} \ln(\alpha\tan x)\,dx$

59. $\int_0^{\frac{\pi}{2}} \frac{a\tan x+b\cot x}{\tan x+\cot x}\,dx$

60. $\int_0^{\pi} \frac{x}{a^2\cos^2 x+b^2\sin^2 x}\,dx$

61. $\int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{1+\cos^2(2x)}\,dx$

62. $\int_{-2}^{4} \left[2\mathrm{E}\left(\frac{x}{2}\right)+1\right]dx$

63. $\int_0^{\frac{3}{2}} [x^2]\,dx$

64. $\int_0^9 \{\sqrt{x}\}\,dx$

65. $\int_0^{\pi}[2\sin x]dx$

66. $\int_0^{2} x[x]dx$

67. $\int_0^{n}\{\sqrt{x}\}\,dx$

68. $\int_0^{2n\pi}[\sin x+\cos x]\,dx$

69. $\int_{-2}^{1}\left[1+\cos\left(\frac{\pi x}{2}\right)\right]dx$

70. $\int_0^{3}\left([x]+\left[x+\frac{1}{2}\right]+\left[x+\frac{2}{3}\right]\right)dx$

71. $\int_0^{\pi}[\cot x]dx$

72. $\int_{e^{-e}}^{e^{-1}}\frac{\ln(\ln(-\ln x))}{x\ln x}dx$

73. $\int_0^{\frac{\pi}{2}}\frac{1}{a\cos^2 x+b\sin^2 x}dx$

74. $\int_{-\infty}^{+\infty}\frac{1}{e^{2x}+e^{-2x}}dx$

75. $\int_0^{+\infty}\frac{1}{(1+x^{1991})(1+x^2)}dx$

76. $\int_1^{+\infty}\frac{xe^x-1}{x\left[1+(e^x-\ln x)^2\right]}dx$

77. $\int_0^{+\infty}\frac{\sqrt[4]{e^{3x}}-\sqrt[4]{e^x}}{e^x-1}dx$

78. $\int_0^{+\infty}\frac{\tan(\arctan x-\arctan 2x)}{x}dx$

79. $\int_0^{+\infty}\frac{dx}{1+x+x^2+x^3}$

80. $\int_0^{1}\frac{\ln(x+1)}{x^2+1}dx$

81. $\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx$

82. $\int_0^{\pi} x\ln(\sin x)\,dx$

83. $\int_0^{+\infty}\ln\left(1+\frac{k^2}{x^2}\right)dx$

84. $\int_0^{+\infty}\frac{\ln x}{x^2+\alpha^2}dx$

85. $\int_0^{+\infty}\frac{\ln(x^2-x+1)}{(x^2+1)\ln x}dx$

86. $\int_0^{\frac{\pi}{2}}\frac{\sqrt{\tan x}}{\sin x(\sin x+\cos x)}dx$

87. $\int_0^{\pi}\frac{x}{1+\cos^2 x}dx$

88. $\int_0^{1}(1-x^2)^n\,dx$

89. $\int_0^{1}x^n\sqrt{1-x}\mathrm{d}x$

90. $\int_0^{\frac{\pi}{2}}\frac{1}{1+a^2\sin^2 x}dx$

91. $\int_0^{\pi}\frac{x}{1+\tan^4 x}dx$

92. $\int_0^{\frac{\pi}{2}}\frac{x^2\sin^2 x}{x^2+\left(\frac{\pi^2}{4}-\pi x\right)\cos^2 x}dx$

93. $\int_0^{1}\sqrt{\frac{1+x}{1-x}}dx$

94. $\int_0^{1}\frac{1}{\sqrt{x}\sqrt{1-x}}dx$

95. $\int_0^{\frac{\pi}{4}}(\sqrt{\tan x}+\sqrt{\cot x})dx$

96. $\int_0^{+\infty}\frac{\ln\left(x+\frac{1}{x}\right)}{1+x^2}dx$

97. $\int_{-\infty}^{+\infty}\frac{dx}{(x^2+1)^2}$

98. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x+\cos x} dx$

99. $\int_0^{\pi} \frac{(2x+1)\sin^3 x}{\cos^2 x+1} dx$

100. $\int_0^{\frac{\pi}{4}} \frac{1}{(\sec x+\tan x)^2} dx$

101. $\int_0^{\pi} \frac{\ln(x+\pi)}{x^2+\pi^2} dx$

102. $\int_e^{e^2} \left(\frac{1}{\ln x} + \ln(\ln x)\right) dx$

103. $\int_0^{\frac{1}{\sqrt{2}}} \frac{\arcsin x}{(1-x^2)^{\frac{3}{2}}} dx$

104. $\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$

105. $\int_1^e \left[\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x\right] \ln x \, dx$

106. $\int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\tan^{-1}(x^2)}{1+x^2} dx$

107. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\pi-4x)\tan x}{1-\tan x} dx$

108. $\int_1^{+\infty} \frac{1}{(ax+b)\sqrt{x-1}} dx$

109. $\int_0^{+\infty} \frac{1}{(1+x^a)(1+x^2)} dx$

110. $\int_0^{+\infty} \frac{x^{n-1}}{(1+x^n)(\pi^2+\ln^2 x)} dx$

111. $\int_{-2\pi}^{2\pi} \frac{\pi x^2+\cos^3 x}{1+e^{\pi x^3}} dx$

112. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{(\sin x+\cos x)^3} dx$

113. $\int_0^{\pi} \frac{x}{\tan^2 x-1} dx$

114. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec x+\cos x}{1+e^x} dx$

115. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin\left(x+\frac{\pi}{4}\right)} dx$

116. $\int_{-1}^{1} e^{\sin^{-1}(x)} dx$

117. $\int_{-\pi}^{\pi} \frac{\sin(nx)}{(1+2^x)\sin x} dx$

118. $\int_0^{\pi} \ln|1+\cos x| \, dx$

119. $\int_0^{\frac{\pi}{2}} \frac{1}{1+a\csc^2 x} dx$

120. $\int_0^{\pi} \frac{1}{(x^2+\pi^2)\sqrt{\pi^2-x^2}} dx$

121. $\int_0^{+\infty} \frac{1-\tanh x}{\sqrt[4]{\tanh x}} dx$

122. $\int_0^{+\infty} \frac{\ln\left(\frac{1+x^{11}}{1+x^3}\right)}{(1+x^2)\ln x} dx$

123. $\int_0^1 \cos^{-1}\left(\sqrt{1-\sqrt{x}}\right) dx$

124. $\int_0^{\pi} \frac{x^2 \sin 2x \sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi} dx$

125. $\int_0^{+\infty} \frac{1}{x^4+4x^2\cosh(2a)+4} dx$

126. $\int_0^{+\infty} \frac{\ln x}{(x+a)^2+1} dx$

127. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\cos 2x}{(1+\sin^2 x)(1+\cos^2 x)} dx$

128. $\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)\sqrt{4+3x^2}}$

129. $\int_{0}^{+\infty} \frac{x \arctan x}{x^4+x^2+1} dx$

130. $\int_{0}^{+\infty} \frac{\ln x}{(x+a)(x+b)} dx$

131. $\int_{-\infty}^{+\infty} \frac{x}{a^2 e^x - b^2 e^{-x}} dx$

132. $\int_{0}^{\ln 2} \frac{x}{e^x+2e^{-x}-2} dx$

133. $\int_{0}^{\pi} \frac{\pi - x}{\sin x \cos \theta + 1} dx$

134. $\int_{0}^{+\infty} \frac{dx}{(1+x)\left(\pi^2+\ln^4 x\right)}$

135. $\int_{0}^{+\infty} \frac{\sqrt[\varphi]{x} \arctan x}{(1+x^{\varphi})^2} dx$

136. $\int_{0}^{1} \ln^{2k} \left(\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)} \right) dx \; ; k \in \mathbb{Z}$

137. $\int_{0}^{2\pi} \frac{x}{\phi - \cos^2 x} dx$

Solutions of Exercises

1. $\int_0^1 (3x^2+2x+1)dx = [x^3+x^2+x]_0^1 = 3$

2. $\int_{-2}^{-1}\left(\frac{x^4+1}{x^2}\right)dx = \int_{-2}^{-1}(x^2+x^{-2})dx = \left[\frac{x^3}{3}-\frac{1}{x}\right]_{-2}^{-1} = \frac{17}{6}$

3. $\int_0^1 x\sqrt{x}dx = \int_0^1 x^{\frac{3}{2}}dx = \left[\frac{x^{\frac{5}{2}}}{\frac{5}{2}}\right]_0^1 = \frac{2}{5}$

4. $\int_0^1 x(1+x^2)^4dx = \frac{1}{2}\int_0^1 2x(1+x^2)^4dx = \frac{1}{2}\int_0^1 (1+x^2)'(1+x^2)^4dx$

$= \left[\frac{1}{2}\frac{(1+x^2)^5}{5}\right]_0^1 = \frac{31}{10}$

5. $\int_1^2 x\sqrt{x^2+1}dx = \frac{1}{2}\left[\frac{\left(\sqrt{x^2+1}\right)^{\frac{3}{2}}}{\frac{3}{2}}\right]_1^1 = \left[\frac{1}{3}(x^2+1)\sqrt{x^2+1}\right]_1^2 = \frac{1}{3}\left(5\sqrt{5}-2\sqrt{2}\right)$

6. $\int_0^1 \frac{x+1}{\sqrt{x^2+2x+4}}dx = \int_0^1 (x+1)(x^2+2x+4)^{-\frac{1}{2}}dx = \frac{1}{2}\left[\frac{(x^2+2x+4)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^1$

$= \left[\sqrt{x^2+2x+4}\right]_0^1 = \sqrt{7}-2$

7. $\int_0^1 \frac{xdx}{(x^2+1)^4} = \int_0^1 x(x^2+1)^{-4}d = \frac{1}{2}\left[\frac{(x^2+1)^{-3}}{-3}\right]_0^1 = \left[-\frac{1}{6}(x^2+1)^{-3}\right]_0^1 = \frac{7}{48}$

8. $\int_1^2 (x+1)(x^2+2x-3)^2dx = \frac{1}{2}\left[\frac{(x^2+2x-3)^3}{3}\right]_1^2 = \left[\frac{(x^2+2x-3)^3}{6}\right]_1^2 = \frac{125}{6}$

9. $\int_0^1 |5x-3|\,dx = \int_0^{\frac{3}{5}}|5x-3|\,dx + \int_{\frac{3}{5}}^1 |5x-3|\,dx = -\int_0^{\frac{3}{5}}(5x-3)\,dx + \int_{\frac{3}{5}}^1 (5x-3)\,dx$

$= \left[3x-\frac{5x^2}{2}\right]_0^{\frac{3}{5}} + \left[\frac{5x^2}{2}-3x\right]_{\frac{3}{5}}^1 = \left(\frac{9}{5}-\frac{9}{10}\right)+\left(-\frac{1}{2}+\frac{9}{10}\right) = \frac{13}{10}$

10. $\int_0^2 |x^2-x|dx = -\int_0^1 (x^2-x)dx + \int_1^2 (x^2-x)dx = \int_1^0 (x^2-x)dx + \int_1^2 (x^2-x)dx$

$= \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^0 + \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_1^2 = 1$

11. $\int_1^2 |x^2-3x+2|dx = -\int_1^2 (x^2-3x+2)dx = \left[-\frac{x^3}{3}+\frac{x^2}{2}-2x\right]_1^2 = -\frac{17}{6}$

12. $\int_{-2}^{2} x(x^8+x^4+1)^{10}\,dx = 0$ since $x(x^8+x^4+1)^{10}$ is an odd function

13. $\int_{-1}^{1} x\sqrt{x^4+1}\,dx = 0$ since $x\sqrt{x^4+1}$ is an odd function

14. $\int_1^e \frac{3}{2x-1}dx = \frac{3}{2}\int_1^e \frac{2}{2x-1}dx = \frac{3}{2}\int_1^e \frac{(2x-1)'}{2x-1}dx = \left[\frac{3}{2}\ln(2x-1)\right]_1^e = \frac{3}{2}\ln(2e-1)$

15. $\int_0^1 \frac{x-1}{x^2-2x+3}dx = \frac{1}{2}\int_1^e \frac{2x-2}{x^2-2x+3}dx = \frac{1}{2}\int_1^e \frac{(x^2-2x+3)'}{x^2-2x+3}dx = \left[\frac{1}{2}\ln(x^2-2x+3)\right]_0^1$
$= \frac{1}{2}\ln\left(\frac{2}{3}\right).$

16. $\int_1^e \frac{\ln x}{x}dx = \int_1^e (\ln x)'\ln x\,dx = \left[\frac{(\ln x)^2}{2}\right]_1^e = \frac{1}{2}$

17. $\int_e^{e^2} \frac{dx}{x\ln x}\int_e^{e^2} \frac{dx}{x\ln x} = \int_e^{e^2} \frac{(\ln x)'}{\ln x}dx = [\ln\ln x]_e^{e^2} = \ln 2$

18. $\int_1^2 2e^{2x-1}dx = [e^{2x-1}]_1^2 = e^3 - e$

19. $\int_0^{\ln 2} e^{-x}(e^{2x}+2e^x-2)dx = \int_0^{\ln 2}(e^x+2-2e^{-x})dx = [e^x+2x+2e^{-x}]_0^{\ln 2} = \ln 4$

20. $\int_0^1 \frac{e^x+1}{e^x+x}dx = \int_0^1 \frac{(e^x+x)'}{e^x+x}dx = [\ln(e^x+x)]_0^1 = \ln(1+e)$

21. $\int_0^{\frac{1}{\sqrt{2}}} \frac{x\arcsin x^2}{\sqrt{1-x^4}}dx = \frac{1}{2}\int_0^{\frac{1}{\sqrt{2}}} \frac{2x\arcsin x^2}{\sqrt{1-x^4}}dx$, let $u=\arcsin x^2$, $du = \frac{2x}{\sqrt{1-x^4}}dx$, so:
$\int_0^{\frac{1}{\sqrt{2}}} \frac{x\arcsin x^2}{\sqrt{1-x^4}}dx = \frac{1}{2}\int_{u_1}^{u_2} u\,dx = \frac{1}{2}\left[\frac{1}{2}u^2\right]_{u_1}^{u_2} = \frac{1}{4}[(\arcsin x^2)^2]_0^{\frac{1}{\sqrt{2}}} = \frac{1}{4}\left(\arcsin\frac{1}{2}\right)^2 = \frac{\pi^2}{144}$

22. $\int_1^2 \frac{dx}{(x^2-2x+4)^{\frac{3}{2}}} = \int_1^2 \frac{dx}{(x^2-2x+1+3)^{\frac{3}{2}}} = \int_1^2 \frac{dx}{\left[(x+1)^2+3\right]^{\frac{3}{2}}} = \int_1^2 \frac{dx}{\left[(x+1)^2+(\sqrt{3})^2\right]^{\frac{3}{2}}}$

Let $x-1=\sqrt{3}\tan u$, then $dx=\sqrt{3}\sec^2 u\,du$

For $x=2$, $u=\arctan 0 = 0$ and for $x=2$, $u=\arctan\left(\frac{1}{\sqrt{3}}\right)=\frac{\pi}{6}$, then:

$\int_1^2 \frac{dx}{(x^2-2x+4)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2 udu}{(3+3\tan^2 u)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2 udu}{(3\sec^2 u)^{\frac{3}{2}}} = \int_0^{\frac{\pi}{6}} \frac{\sqrt{3}\sec^2 udu}{3\sqrt{3}\sec^3 u} = \frac{1}{3}\int_0^{\frac{\pi}{6}} \frac{du}{\sec u}$

$= \frac{1}{3}\int_0^{\frac{\pi}{6}}\cos u\,du = \frac{1}{3}[\sin u]_0^{\frac{\pi}{6}} = \frac{1}{3}\left(\sin\frac{\pi}{6}-\sin 0\right) = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}$

23. $I = \int_{-1}^2 \frac{|x|}{x}dx$, we have : $\frac{|x|}{x} = \begin{cases} -1 & \text{when } -1<x<0 \\ 1 & \text{when } 0<x<2 \end{cases}$, then:
$I = \int_{-1}^0 \frac{|x|}{x}dx + \int_0^2 \frac{|x|}{x}dx = \int_{-1}^0(-1)dx + \int_0^2 dx = -1+2 = 1$

24. $\int_0^1 \frac{1}{\sqrt{1+x}+\sqrt{x}}dx = \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{(\sqrt{1+x}+\sqrt{x})(\sqrt{1+x}-\sqrt{x})}dx = \int_0^1 \frac{\sqrt{1+x}-\sqrt{x}}{1+x-x}dx$

$= \int_0^1(\sqrt{1+x}-\sqrt{x})dx = \left[\frac{2}{3}(1+x)^{\frac{3}{2}} - \frac{2}{3}x^{\frac{3}{2}}\right]_0^1 = \frac{2}{3}\left[2^{\frac{3}{2}}-1\right] - \frac{2}{3}[1-0] = \frac{4}{3}(\sqrt{2}-1)$

25. $\int_0^{2\pi} e^x|\sin x|\,dx = \int_0^{\pi} e^x \sin x\,dx - \int_{\pi}^{2\pi} e^x \sin x\,dx$

Performing integration by parts twice we get: $\int e^x \sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + c$

So, $\int_0^{2\pi} e^x|\sin x|\,dx = \left[\frac{1}{2}e^x(\sin x - \cos x)\right]_0^{\pi} - \left[\frac{1}{2}e^x(\sin x - \cos x)\right]_0^{2\pi} = \frac{1}{2}(e^{\pi}+1)^2$

26. $\int_{-1}^{1}|3^x - 2^x|\,dx = \int_{-1}^{0}(2^x - 3^x)\,dx = \int_0^1(3^x - 2^x)\,dx$

$= \left[\frac{2^x}{\ln 2} - \frac{3^x}{\ln 3}\right]_{-1}^{0} + \left[\frac{3^x}{\ln 3} - \frac{2^x}{\ln 2}\right]_0^1 = \frac{1}{\ln 2} - \frac{1}{\ln 3} - \frac{1}{2\ln 2} + \frac{1}{3\ln 3} + \frac{3}{\ln 3} - \frac{2}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 2}$

$= \frac{4}{3\ln 3} - \frac{1}{2\ln 2}$

27. $\mathrm{I} = \int_0^a \frac{f(x)}{f(x)+f(a-x)}dx$ **… (1)**, let $u = a - x$, then $du = -dx$

For $x = 0, u = a - 0 = a$ and for $x = a, u = a - a = 0$, then we get:

$\mathrm{I} = \int_a^0 \frac{f(a-u)}{f(a-u)+f(u)}(-du) = \int_0^a \frac{f(a-u)}{f(a-u)+f(u)}du = \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}dx$ **… (2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_0^a \frac{f(x)}{f(x)+f(a-x)}dx + \int_0^a \frac{f(a-x)}{f(a-x)+f(x)}dx$

$2\mathrm{I} = \int_0^a \frac{f(x)+f(a-x)}{f(x)+f(a-x)}dx = \int_0^a dx = [x]_0^a = a$, so $\mathrm{I} = \frac{a}{2}$

28. $\mathrm{I} = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx$ **… (1)**, let $u = a + b - x$, then $du = -dx$

For $x = a, u = a + b - a = b$ and for $x = b, u = a + b - b = a$, then:

$\mathrm{I} = \int_b^a \frac{f(a+b-u)}{f(a+b-u)+f(u)}(-du) = \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}dx$ **… (2)** Adding (1) and (2)

$2\mathrm{I} = \int_a^b \frac{f(x)}{f(x)+f(a+b-x)}dx + \int_a^b \frac{f(a+b-x)}{f(a+b-x)+f(x)}dx = \int_a^b \frac{f(x)+f(a+b-x)}{f(x)+f(a+b-x)}dx$

$2\mathrm{I} = \int_a^b dx = b - a$, therefore $\mathrm{I} = \frac{b-a}{2}$

29. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\sin x}{1+\cos x}\right)dx$, let $u = \frac{\pi}{2} - x$, then $du = -dx$

For $x = 0, u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}, u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then:

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \ln\left[\frac{1+\sin(\frac{\pi}{2}-u)}{1+\cos(\frac{\pi}{2}-u)}\right](-du) = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\cos u}{1+\sin u}\right)du = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\cos x}{1+\sin x}\right)dx$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{\frac{1+\sin x}{1+\cos x}}\right)dx = -\int_0^{\frac{\pi}{2}} \ln\left(\frac{1+\sin x}{1+\cos x}\right)dx = -\mathrm{I}$, $\mathrm{I} = -\mathrm{I}$, therefore $\mathrm{I} = 0$

30. $\mathrm{I} = \int_0^{\ln\sqrt{3}} \frac{e^{2x}}{(e^{2x}+1)\ln(e^{2x}+1)}dx$, let $u = e^{2x}$, then $du = 2e^{2x}dx$

For $x = 0, u = 1$ and for $x = \ln\sqrt{3} = \frac{1}{2}\ln 3, u = 3$, then:

$\mathrm{I} = \frac{1}{2}\int_1^3 \frac{1}{(u+1)\ln(u+1)}du = \frac{1}{2}\int_1^3 \frac{\frac{1}{u+1}}{\ln(u+1)}du = \frac{1}{2}\int_1^3 \frac{[\ln(u+1)]'}{\ln(u+1)}du = \frac{1}{2}[\ln(\ln(u+1))]_1^3$

$\mathrm{I} = \frac{1}{2}(\ln\ln 4 - \ln\ln 2) = \frac{1}{2}\ln\left(\frac{\ln 4}{\ln 2}\right) = \frac{1}{2}\ln\left(\frac{2\ln 2}{\ln 2}\right) = \frac{1}{2}\ln 2 = \ln\sqrt{2}$

31. First Method: $\mathrm{I} = \int_{-2}^{2} \sqrt{4-x^2}\, dx$, let $y = \sqrt{4-x^2}$, so $y^2 = 4 - x^2$, then $x^2 + y^2 = 4$ then $y = \sqrt{4-x^2}$ is an equation of the upper semi-circle of center O and radius 2

Then $\mathrm{I} = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi \times 2^2 = 2\pi$

Second Method: $\mathrm{I} = \int_{-2}^{2} \sqrt{4-x^2}\, dx$, let $x = 2\sin\theta$, then $dx = 2\cos\theta\, d\theta$

for $x = -2$, $\sin\theta = -1$ and so $\theta = -\frac{\pi}{2}$ and for $x = 2$, $\sin\theta = 1$ and so $\theta = \frac{\pi}{2}$, then:

$\mathrm{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{4(1-\sin^2\theta)}2\cos\theta\, d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos\theta\, 2\cos\theta\, d\theta = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta$

$\mathrm{I} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1+\cos 2\theta) d\theta = 2\left[\theta + \frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = 2\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi\right) - 2\left(-\frac{\pi}{2} + \frac{1}{2}\sin(-\pi)\right)$

$\mathrm{I} = \pi + \pi = 2\pi$

32. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \sin(2x)\cos(\cos x)\, dx = \int_0^{\frac{\pi}{2}} 2\sin x\cos x\cos(\cos x)\, dx$

Let $u = \cos x$, then $du = -\sin x\, dx$, for $x = 0$, $u = 1$ and for $x = \frac{\pi}{2}$, $u = 0$, so:

$\mathrm{I} = -2\int_1^0 u\cos u\, du = 2\int_0^1 x\cos x\, dx$

Let $u = x$, then $u' = 1$ and let $v' = \cos x$, so $v = \sin x$, then:

$\mathrm{I} = 2[x\sin x]_0^1 - 2\int_0^1 \sin x\, dx = 2\sin(1°) + 2[\cos x]_0^1 = 2[\sin(1°) + \cos(1°) - 1]$

33. $\mathrm{I} = \int_0^1 x(1-x)^n\, dx$, let $u = 1 - x$, then $du = -dx$

For $x = 0$, $u = 1 - 0 = 1$ and for $x = 1$, $u = 1 - 1 = 0$, so we get:

$\mathrm{I} = \int_1^0 (1-u)u^n\,(-du) = \int_0^1 (1-u)u^n du = \int_0^1 (u^n - u^{n+1}) du$

$= \left[\frac{1}{n+1}u^{n+1} - \frac{1}{n+2}u^{n+2}\right]_0^1 = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)}$

34. $\mathrm{I}_n = \int_{n\pi}^{(n+1)\pi} e^{-x}\sin x\,\mathrm{d}x$ let $u = \sin x$ then $u' = \cos x$ and let $v' = e^{-x}$ then $v = -e^{-x}$, so we get $\mathrm{I}_n = [-e^{-x}\sin x]_{n\pi}^{(n+1)\pi} + \int_{n\pi}^{(n+1)\pi} e^{-x}\cos x\,\mathrm{d}x = \int_{n\pi}^{(n+1)\pi} e^{-x}\cos x\,\mathrm{d}x$.

Let $u = \cos x$ then $u' = -\sin x$ and let $v' = e^{-x}$ then $v = -e^{-x}$, then we get

$\mathrm{I}_n = [-e^{-x}\cos x]_{n\pi}^{(n+1)\pi} - \int_{n\pi}^{(n+1)\pi} e^{-x}\sin x\,\mathrm{d}x$, so

$\mathrm{I}_n = -\mathrm{e}^{-(n+1)\pi}\cos[(n+1)\pi] + e^{-n\pi}\cos n\pi - \mathrm{I}_n$, so $2\mathrm{U}_n = \mathrm{e}^{-(n+1)\pi}(-1)^n + e^{-n\pi}(-1)^n$,

therefore $\mathrm{U}_n = (-1)^n e^{-n\pi} \times \frac{1+e^{-\pi}}{2}$

35. $\mathrm{I} = \int_0^{\frac{1}{2}} \cos(5\arcsin x)\, dx$, let $\theta = \arcsin x$, then $x = \sin\theta$ and $dx = \cos\theta\, d\theta$

For $x = 0$, $\theta = 0$ and for $x = \frac{1}{2}$, $x = \frac{\pi}{6}$, then we get:

$\mathrm{I} = \int_0^{\frac{\pi}{6}} \cos(5\theta)\cos\theta\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{6}} [\cos(5\theta + \theta) + \cos(5\theta - \theta)] d\theta$

$\mathrm{I} = \frac{1}{2}\int_0^{\frac{\pi}{6}} [\cos(6\theta) + \cos(4\theta)] d\theta = \left[\frac{1}{12}\sin(6\theta) + \frac{1}{8}\sin(4\theta)\right]_0^{\frac{\pi}{6}} = \frac{1}{12}\sin(\pi) + \frac{1}{8}\sin\left(\frac{2\pi}{3}\right)$

$\mathrm{I} = 0 + \frac{1}{8}\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{16}$

36. $\text{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{\sin x}+1} dx$ **… (1)**

Let $t = -x$, then $dt = -dx$, for $x = -\frac{\pi}{2}$, $t = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $t = -\frac{\pi}{2}$, then:

$$\text{I} = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{1}{e^{\sin(-t)}+1}(-dt) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{-\sin t}+1} dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{-\sin x}+1} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+1} dx \; \textbf{… (2)}$$

Adding (1) and (2) we get $2\text{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{e^{\sin x}+1} dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\sin x}}{e^{\sin x}+1} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\sin x}+1}{e^{\sin x}+1} dx$

$2\text{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dx = \left(\frac{\pi}{2} + \frac{\pi}{2}\right) = \pi$, therefore $\text{I} = \frac{\pi}{2}$

37. $\text{I} = \int_{-1}^{1} \frac{x^2+\sin x}{x^2+1} dx$ **… (1)**

Let $u = -x$, then $du = -dx$, for $x = -1$, $x = 1$ and for $x = 1$, $u = -1$, so:

$$\text{I} = \int_{1}^{-1} \frac{(-u)^2+\sin(-u)}{(-u)^2+1}(-du) = \int_{-1}^{1} \frac{u^2-\sin u}{u^2+1} du = \int_{-1}^{1} \frac{x^2-\sin x}{x^2+1} dx \; \textbf{… (2)}$$

Adding (1) and (2) we get $2\text{I} = \int_{-1}^{1} \frac{x^2+\sin x}{x^2+1} dx + \int_{-1}^{1} \frac{x^2-\sin x}{x^2+1} dx$

$2\text{I} = \int_{-1}^{1} \frac{x^2+\sin x+x^2-\sin x}{x^2+1} dx = \int_{-1}^{1} \frac{2x^2}{x^2+1} dx$, so $\text{I} = \int_{-1}^{1} \frac{x^2}{x^2+1} dx$, then:

$$\text{I} = \int_{-1}^{1} \frac{x^2+1-1}{x^2+1} dx = \int_{-1}^{1} dx - \int_{-1}^{1} \frac{1}{x^2+1} dx = [x - \arctan x]_{-1}^{1}$$

$$\text{I} = \left(1 - \frac{\pi}{4}\right) - \left(-1 + \frac{\pi}{4}\right) = 2 - \frac{\pi}{2}$$

38. $\text{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{\frac{1}{x}}} dx$ **… (1)**, let $u = -x$, then $du = -dx$, so:

$$\text{I} = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\cos(-u)}{1+e^{-\frac{1}{u}}}(-du) = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\cos u}{1+e^{-\frac{1}{u}}} du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{-\frac{1}{x}}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\frac{1}{x}}\cos x}{e^{\frac{1}{x}}\left(1+e^{-\frac{1}{x}}\right)} dx$$

$\text{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\frac{1}{x}}\cos x}{1+e^{\frac{1}{x}}} dx$ **… (2)** Adding (1) and (2) we get:

$$2\text{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^{\frac{1}{x}}} dx + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{\frac{1}{x}}\cos x}{1+e^{\frac{1}{x}}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x+e^{\frac{1}{x}}\cos x}{1+e^{\frac{1}{x}}} dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\left(1+e^{\frac{1}{x}}\right)\cos x}{1+e^{\frac{1}{x}}} dx$$

$2\text{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \, dx = [\sin x]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \sin\left(\frac{\pi}{2}\right) - \sin\left(-\frac{\pi}{2}\right) = 2$, therefore, I = 1

39. $\text{I} = \int_{0}^{\frac{\pi}{2}} \frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}} dx$ **… (1)**, let $u = \frac{\pi}{2} - x$, then $du = -dx$

For $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then:

$$\text{I} = \int_{\frac{\pi}{2}}^{0} \frac{2^{\sin\left(\frac{\pi}{2}-u\right)}}{2^{\sin\left(\frac{\pi}{2}-u\right)}+2^{\cos\left(\frac{\pi}{2}-u\right)}}(-du) = \int_{0}^{\frac{\pi}{2}} \frac{2^{\cos u}}{2^{\cos u}+2^{\sin u}} du = \int_{0}^{\frac{\pi}{2}} \frac{2^{\cos x}}{2^{\sin x}+2^{\cos x}} dx \; \textbf{… (2)}$$

Adding (1) and (2) we get $2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{2^{\sin x}}{2^{\sin x}+2^{\cos x}}dx + \int_0^{\frac{\pi}{2}} \frac{2^{\cos x}}{2^{\sin x}+2^{\cos x}}dx$

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{2^{\sin x}+2^{\cos x}}{2^{\sin x}+2^{\cos x}}dx = \int_0^{\frac{\pi}{2}} dx = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$, therefore $\mathrm{I} = \frac{\pi}{4}$

40. $\mathrm{I} = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx$ **… (1)** let $u = \pi - x$, then $du = -dx$

For $x = 0$, $u = \pi - 0 = \pi$ and for $x = \pi$, $u = \pi - \pi = 0$, then we get:

$\mathrm{I} = \int_{\pi}^{0} \frac{e^{\cos(\pi-u)}}{e^{\cos(\pi-u)}+e^{-\cos(\pi-u)}}(-du) = \int_0^{\pi} \frac{e^{-\cos u}}{e^{-\cos u}+e^{\cos u}}du = \int_0^{\pi} \frac{e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}dx$ **…(2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_0^{\pi} \frac{e^{\cos x}}{e^{\cos x}+e^{-\cos x}}dx + \int_0^{\pi} \frac{e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}dx$

$2\mathrm{I} = \int_0^{\pi} \frac{e^{-\cos x}+e^{-\cos x}}{e^{\cos x}+e^{-\cos x}}dx = \int_0^{\pi} dx = \pi$, therefore $\mathrm{I} = \frac{\pi}{2}$

41. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{1991} x}dx$ **… (1)**, let $u = \frac{\pi}{2} - x$, then $du = -dx$

For $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then:

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{1}{1+\tan^{1991}\left(\frac{\pi}{2}-u\right)}(-du) = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^{1991} u}du = \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^{1991} x}dx$

$= \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{\tan^{1991} x}}dx = \int_0^{\frac{\pi}{2}} \frac{\tan^{1991} x}{1+\tan^{1991} x}dx$ **… (2)** Adding (1) and (2) we get**:**

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^{1991} x}dx + \int_0^{\frac{\pi}{2}} \frac{\tan^{1991} x}{1+\tan^{1991} x}dx = \int_0^{\frac{\pi}{2}} \frac{1+\tan^{1991} x}{1+\tan^{1991} x}dx = \int_0^{\frac{\pi}{2}} dx$

$= [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$, therefore $\mathrm{I} = \frac{\pi}{4}$

42. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx$, let $u = \frac{\pi}{2} - x$, then $du = -dx$

For $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then:

$\int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_{\frac{\pi}{2}}^{0} \frac{\sqrt{\sin\left(\frac{\pi}{2}-u\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-u\right)}+\sqrt{\cos\left(\frac{\pi}{2}-u\right)}}(-du) = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos u}}{\sqrt{\cos u}+\sqrt{\sin u}}du$

$= \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx$, then**:**

$\mathrm{I} + \mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx + \int_0^{\frac{\pi}{2}} \frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}dx = \int_0^{\frac{\pi}{2}} dx$

$2\mathrm{I} = [x]_0^{\frac{\pi}{2}} = \frac{\pi}{2}$, therefore $\mathrm{I} = \frac{\pi}{4}$.

43. $\mathrm{I} = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx$, let $u = \pi - x$, then $du = -dx$

For $x = 0$, $u = \pi - 0 = \pi$ and for $x = \pi$, $u = \pi - \pi = 0$, then we get:

$\int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}dx = \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{1+\cos^2(\pi-u)}(-du) = \int_0^{\pi} \frac{(\pi-u)\sin u}{1+\cos^2 u}du = \int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

$\mathrm{I} = \pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx = -\pi \int_0^{\pi} \frac{(\cos x)'}{1+\cos^2 x} dx - \mathrm{I}$

$2\mathrm{I} = -\pi[-\arctan \cos x]_0^{\pi} = -\pi\left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{2}$, therefore $\mathrm{I} = \frac{\pi^2}{4}$.

44. $\mathrm{I} = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin x \cos x} dx$, let $t = \left(\frac{\pi}{3} + \frac{\pi}{6}\right) - x = \frac{\pi}{2} - x$, then $dt = -dx$

For $x = \frac{\pi}{6}$, $t = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$ and for $x = \frac{\pi}{3}$, $t = \frac{\pi}{2} - \frac{\pi}{3} = \frac{\pi}{6}$, then we get**:**

$\mathrm{I} = \int_{\frac{\pi}{3}}^{\frac{\pi}{6}} \frac{\frac{\pi}{2}-t}{\sin\left(\frac{\pi}{2}-t\right)\cos\left(\frac{\pi}{2}-t\right)}(-dt) = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\pi}{2}-t}{\cos t \sin t} dt = \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\frac{\pi}{2}-x}{\cos x \sin x} dx$

$\mathrm{I} = \frac{\pi}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos x \sin x} dx - \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{x}{\sin x \cos x} dx = \frac{\pi}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos x \sin x} dx - \mathrm{I}$, so

$2\mathrm{I} = \frac{\pi}{2} \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\frac{\sin 2x}{2}} dx = \pi \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \csc 2x \, dx = \frac{\pi}{2}[\ln(\csc 2x - \cot 2x)]_{\frac{\pi}{6}}^{\frac{\pi}{3}}$

$\mathrm{I} = \frac{\pi}{4}\left[\ln\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) - \ln\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right)\right]$, then $\mathrm{I} = \frac{\pi}{4} \ln 3$

45. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\alpha \cos^2 x + \beta \sin^2 x} dx$ Where $\alpha, \beta \in \mathbb{R}^{*+}$

Changing of variable: Let $= \alpha \cos^2 x + \beta \sin^2 x$, then

$du = \{\alpha[2 \cos x\,(-\sin x)] + \beta[2 \sin x\,(\cos x)]\}dx$

$du = [-\alpha(2 \sin x \cos x) + \beta(2 \sin x \cos x)]dx$, with $\sin 2x = 2 \sin x \cos x$

$du = (-\alpha \sin 2x + \beta \sin 2x)dx = (\beta - \alpha) \sin 2x \, dx$. So; $\sin 2x dx = \frac{1}{\beta-\alpha} du$

For $x = 0$ then $u = \alpha \cos^2 0 + \beta \sin^2 0 = \alpha$

For $x = \frac{\pi}{2}$ then $u = \alpha \cos^2 \frac{\pi}{2} + \beta \sin^2 \frac{\pi}{2} = \beta$. Then we get

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\alpha \cos^2 x + \beta \sin^2 x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin 2x dx}{\alpha \cos^2 x + \beta \sin^2 x} = \int_{\alpha}^{\beta} \frac{1}{\beta - \alpha} \times \frac{1}{u} = \left[\frac{1}{\beta - \alpha} \ln|u|\right]_{\alpha}^{\beta}$$

$$\mathrm{I} = \frac{1}{\beta - \alpha}(\ln \beta - \ln \alpha); \ \ \alpha, \beta > 0. \ \ \text{Therefore;} \ \ \mathrm{I} = \frac{\ln\left(\frac{\beta}{\alpha}\right)}{\beta - \alpha}$$

46. $\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x \sin x}{\sqrt[x]{e}+1} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x \sin x}{e^{\frac{1}{x}}+1} dx$**…(1)**, let $u = -x$, then $du = -dx$, so**:**

$\mathrm{I} = \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{-u \sin(-u)}{e^{-\frac{1}{u}}+1}(-du) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{u \sin u}{e^{-\frac{1}{u}}+1} du = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{u \sin u e^{\frac{1}{u}}}{\left(e^{-\frac{1}{u}}+1\right)e^{\frac{1}{u}}} du = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{u \sin u e^{\frac{1}{u}}}{e^{\frac{1}{u}}+1} du$

$\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x \sin x e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1} dx$ **… (2)** Adding (1) and (2) we get

$2\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x \sin x}{e^{\frac{1}{x}}+1} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x \sin x e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x \sin x + x \sin x e^{\frac{1}{x}}}{e^{\frac{1}{x}}+1} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{x \sin x\left(e^{\frac{1}{x}}+1\right)}{e^{\frac{1}{x}}+1} dx$

$2\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} x \sin x \, dx$, then $\mathrm{I} = \int_{0}^{\frac{\pi}{4}} x \sin x \, dx$

Let $u = x$, then $u' = 1$ and let $v' = \sin x$, then $v = -\cos x$, so:

$\mathrm{I} = [-x \cos x]_0^{\frac{\pi}{4}} + \int_0^{\frac{\pi}{4}} \cos x \, dx = -\frac{\pi}{4\sqrt{2}} + [\sin x]_0^{\frac{\pi}{4}} = \frac{1}{\sqrt{2}} - \frac{\pi}{4\sqrt{2}}$

47. $\mathrm{I} = \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}} dx$ **… (1)**, let $u = 6 - x$, then $du = -dx$

For $x = 2$, $u = 6 - 2 = 4$ and for $x = 4$, $u = 6 - 4 = 2$, then we get:

$\mathrm{I} = \int_4^2 \frac{\sqrt{\ln(3+6-u)}}{\sqrt{\ln(9-(6-u))}+\sqrt{\ln(3+6-u)}} (-du) = \int_2^4 \frac{\sqrt{\ln(9-u)}}{\sqrt{\ln(3+u)}+\sqrt{\ln(9-u)}} du$

$= \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(3+x)}+\sqrt{\ln(9-x)}} dx$**… (2)** Adding (1) and (2) we get:

$2\mathrm{I} = \int_2^4 \frac{\sqrt{\ln(3+x)}}{\sqrt{\ln(9-x)}+\sqrt{\ln(3+x)}} dx + \int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(3+x)}+\sqrt{\ln(9-x)}} dx$

$= \int_2^4 \frac{\sqrt{\ln(3+x)}+\sqrt{\ln(9-x)}}{\sqrt{\ln(3+x)}+\sqrt{\ln(9-x)}} dx = \int_2^4 dx = [x]_2^4 = 4 - 2 = 2$, therefore I = 1

48. $\mathrm{I} = \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$, let $u = 2\pi - x$, then $du = -dx$

For $x = 0$, $u = 2\pi - 0 = 2\pi$ and for $x = 2\pi$, $u = 2\pi - 2\pi = 0$, then:

$\mathrm{I} = \int_{2\pi}^{0} \frac{(2\pi-u)\sin^{2n}(2\pi-u)}{\sin^{2n}(2\pi-u)+\cos^{2n}(2\pi-u)} (-du) = \int_0^{2\pi} \frac{(2\pi-u)\sin^{2n}(-u)}{\sin^{2n}(-u)+\cos^{2n}(-u)} du$

$= \int_0^{2\pi} \frac{(2\pi-u)\sin^{2n} u}{\sin^{2n} u+\cos^{2n} u} du = \int_0^{2\pi} \frac{(2\pi-x)\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx$

$\mathrm{I} = \int_0^{2\pi} \frac{2\pi \sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx - \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx = \int_0^{2\pi} \frac{2\pi \sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx - \mathrm{I}$

So, $2\mathrm{I} = 2\pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx$, so:

$\mathrm{I} = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx = 4\pi \int_0^{\frac{\pi}{2}} \frac{\sin^{2n} x}{\sin^{2n} x+\cos^{2n} x} dx$ **… (1)**

Let $v = \frac{\pi}{2} - x$, so $dv = -dx$, for $x = 0$, $v = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $v = 0$, so:

$\mathrm{I} = 4\pi \int_{\frac{\pi}{2}}^{0} \frac{\sin^{2n}(\frac{\pi}{2}-v)}{\sin^{2n}(\frac{\pi}{2}-v)+\cos^{2n}(\frac{\pi}{2}-v)} (-dv) = 4\pi \int_0^{\frac{\pi}{2}} \frac{\cos^{2n} v}{\cos^{2n} v+\sin^{2n} v} dv$

$\mathrm{I} = 4\pi\int_0^{\frac{\pi}{2}}\frac{\cos^{2n}v}{\sin^{2n}v+\cos^{2n}v}dv = 4\pi\int_0^{\frac{\pi}{2}}\frac{\cos^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx$ **… (2)** Adding (1) and (2)**:**

$2\mathrm{I} = 4\pi\int_0^{\frac{\pi}{2}}\frac{\sin^{2n}x+\cos^{2n}x}{\sin^{2n}x+\cos^{2n}x}dx = 4\pi\int_0^{\frac{\pi}{2}}dx = 4\pi[x]_0^{\frac{\pi}{2}} = 4\pi\left(\frac{\pi}{2}\right)$, so $\mathrm{I} = 4\pi^2$

49. $\mathrm{I} = \int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx$ **… (1)**

Let $u = -x$, so $du = -dx$, for $x = -\pi$ then $u = \pi$ and for $x = \pi$, then $u = -\pi$, so**:**

$\mathrm{I} = \int_{\pi}^{-\pi}\frac{\cos^2(-u)}{1+a^{-u}}(-du) = \int_{-\pi}^{\pi}\frac{\cos^2 u}{1+a^{-u}}du = \int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^{-x}}dx = \int_{-\pi}^{\pi}\frac{a^x\cos^2 x}{a^x(1+a^{-x})}dx$

$\mathrm{I} = \int_{-\pi}^{\pi}\frac{a^x\cos^2 x}{1+a^x}dx$ **… (2)** Adding (1) and (2) we get**:**

$2\mathrm{I} = \int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx + \int_{-\pi}^{\pi}\frac{a^x\cos^2 x}{1+a^x}dx = \int_{-\pi}^{\pi}\frac{\cos^2 x+a^x\cos^2 x}{1+a^x}dx = \int_{-\pi}^{\pi}\frac{(1+a^x)\cos^2 x}{1+a^x}dx$

$\mathrm{I} = \int_{-\pi}^{\pi}\cos^2 x\,dx = \frac{1}{2}\int_{-\pi}^{\pi}(1+\cos 2x)dx = \frac{1}{2}\left[x+\frac{1}{2}\sin 2x\right]_{-\pi}^{\pi}$

$\mathrm{I} = \frac{1}{2}\left(\pi+\frac{1}{2}\sin 2\pi\right) - \frac{1}{2}\left(-\pi+\frac{1}{2}\sin(-2\pi)\right) = \frac{1}{2}\pi+\frac{1}{2}\pi = \pi$, therefore $\mathrm{I} = \frac{\pi}{2}$

50. $\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\tan^2 x}{1+4^x}dx$ **… (1)**

Let $u = -x$, so $du = -dx$, for $x = -\pi$ then $u = \pi$ and for $x = \pi$, then $u = -\pi$, so**:**

$\mathrm{I} = \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\frac{\tan^2(-u)}{1+4^{-u}}(-du) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{4^u\tan^2 u}{1+4^u}du = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{4^x\tan^2 x}{1+4^x}dx$ **… (2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\tan^2 x}{1+4^x}dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{4^x\tan^2 x}{1+4^x}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(1+4^x)\tan^2 x}{1+4^x}dx$

$2\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\tan^2 x\,dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(\sec^2 x-1)\,dx = 2\int_0^{\frac{\pi}{4}}(\sec^2 x-1)\,dx$

$\mathrm{I} = \int_0^{\frac{\pi}{4}}(\sec^2 x-1)\,dx = [\tan x - x]_0^{\frac{\pi}{4}} = 1-\frac{\pi}{4}$

51. $\mathrm{I} = \int_{-1}^{1}\frac{x^2}{1+2^{\sin x}}dx = \int_{-1}^{0}\frac{x^2}{1+2^{\sin x}}dx + \int_0^1\frac{x^2}{1+2^{\sin x}}dx = \mathrm{I}_1+\mathrm{I}_2$

$\mathrm{I}_1 = \int_{-1}^{0}\frac{x^2}{1+2^{\sin x}}dx$, let $u = -x$, then $du = -dx$, for $x = -1$, $u = 1$, $x = 0$, $u = 0$, so**:**

$\mathrm{I}_1 = \int_1^0\frac{(-u)^2}{1+2^{\sin(-u)}}(-du) = \int_0^1\frac{u^2}{1+2^{-\sin u}}du = \int_0^1\frac{2^{\sin u}u^2}{1+2^{\sin u}}du = \int_0^1\frac{2^{\sin x}x^2}{1+2^{\sin x}}dx$

Then $\mathrm{I} = \mathrm{I}_1+\mathrm{I}_2 = \int_0^1\frac{2^{\sin x}x^2}{1+2^{\sin x}}dx + \int_0^1\frac{x^2}{1+2^{\sin x}}dx = \int_0^1\frac{x^2+2^{\sin x}x^2}{1+2^{\sin x}}dx$

$\mathrm{I} = \int_0^1\frac{x^2\left(1+2^{\sin x}\right)}{1+2^{\sin x}}dx = \int_0^1 x^2dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

52. $\mathrm{I} = \int_0^1\ln\left(\frac{1}{x}-1\right)dx$ **… (1)**

Let $t = 1-x$, so $dt = -dx$, for $x = 0$, $t = 1$, for $x = 1$, $t = 0$, so

$\mathrm{I} = \int_1^0\ln\left(\frac{1}{1-t}-1\right)(-dt) = \int_0^1\ln\left(\frac{1}{1-t}-1\right)dt = \int_0^1\ln\left(\frac{1-1+t}{1-t}\right)dt = \int_0^1\ln\left(\frac{t}{1-t}\right)dt$

$\mathrm{I} = \int_0^1 \ln\left(\frac{x}{1-x}\right) dx$ **… (2)** Adding (1) and (2) we get

$2\mathrm{I} = \int_0^1 \ln\left(\frac{1}{x} - 1\right) dx + \ln\left(\frac{x}{1-x}\right) dx = \int_0^1 \ln\left(\frac{1-x}{x}\right) dx + \ln\left(\frac{x}{1-x}\right) dx$

$2\mathrm{I} = \int_0^1 \ln\left[\left(\frac{1-x}{x}\right)\left(\frac{x}{1-x}\right)\right] dx = \int_0^1 \ln 1\, dx = 0$, therefore I = 0

53. $\mathrm{I} = \int_0^{+\infty} \frac{\ln(2x)}{1+x^2} dx$ **… (1)**, let $u = \frac{1}{x}$, so $x = \frac{1}{u}$ and $dx = -\frac{1}{u^2} du$, then**:**

$\mathrm{I} = \int_{+\infty}^{0} \frac{\ln\left(\frac{2}{u}\right)}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2} du\right) = \int_0^{+\infty} \frac{\ln\left(\frac{2}{u}\right)}{1+u^2} du = \int_0^{+\infty} \frac{\ln\left(\frac{2}{x}\right)}{1+x^2} dx$ **… (2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_0^{+\infty} \frac{\ln(2x)}{1+x^2} dx + \int_0^{+\infty} \frac{\ln\left(\frac{2}{x}\right)}{1+x^2} dx = \int_0^{+\infty} \frac{\ln(2x)+\ln\left(\frac{2}{x}\right)}{1+x^2} dx$

Then $2\mathrm{I} = \int_0^{+\infty} \frac{\ln\left(2x\frac{2}{x}\right)}{1+x^2} dx = \int_0^{+\infty} \frac{\ln(4)}{1+x^2} dx = \ln 4 \int_0^{+\infty} \frac{1}{1+x^2} dx = \ln 4\, [\arctan x]_0^{+\infty}$

$\mathrm{I} = \frac{2\ln 2}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{2}\ln 2 = \pi \ln\sqrt{2}$

54. $\mathrm{I} = \int_0^{+\infty} \frac{\arctan x}{x\sqrt{x}+\sqrt{x}} dx = \int_0^{+\infty} \frac{\arctan x}{(x+1)\sqrt{x}} dx$, let $u = \frac{1}{x}$, so $x = \frac{1}{u}$ and $dx = -\frac{1}{u^2} du$, then**:**

$\mathrm{I} = \int_{+\infty}^{0} \frac{\arctan\left(\frac{1}{u}\right)}{\left(\frac{1}{u}+1\right)\sqrt{\frac{1}{u}}}\left(-\frac{1}{u^2} du\right) = \int_0^{+\infty} \frac{\arctan\left(\frac{1}{u}\right)}{(u+1)u\sqrt{\frac{1}{u}}} du = \int_0^{+\infty} \frac{\arctan\left(\frac{1}{u}\right)}{(u+1)\sqrt{\frac{u^2}{u}}} du$

$= \int_0^{+\infty} \frac{\arctan\left(\frac{1}{u}\right)}{(u+1)\sqrt{u}} du = \int_0^{+\infty} \frac{\arctan\left(\frac{1}{x}\right)}{(x+1)\sqrt{x}} dx = \int_0^{+\infty} \frac{\frac{\pi}{2}-\arctan x}{(x+1)\sqrt{x}} dx$

$= \int_0^{+\infty} \frac{\frac{\pi}{2}}{(x+1)\sqrt{x}} dx - \int_0^{+\infty} \frac{\arctan x}{(x+1)\sqrt{x}} dx = \int_0^{+\infty} \frac{\frac{\pi}{2}}{(x+1)\sqrt{x}} dx - \mathrm{I}$, so $2\mathrm{I} = \int_0^{+\infty} \frac{\frac{\pi}{2}}{(x+1)\sqrt{x}} dx$

$2\mathrm{I} = \pi \int_0^{+\infty} \frac{1}{\left(1+(\sqrt{x})^2\right)} \frac{dx}{2\sqrt{x}} = \pi \int_0^{+\infty} \frac{(\sqrt{x})'}{\left(1+(\sqrt{x})^2\right)} dx = \left[\pi \arctan(\sqrt{x})\right]_{\mathbf{0}}^{+\infty} = \pi\left(\frac{\pi}{2}\right) = \frac{1}{2}\pi^2.$

So, $\mathrm{I} = \frac{1}{4}\pi^2$

55. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \sin 2x \ln(\tan x)\, dx$ **… (1)**, let $u = \frac{\pi}{2} - x$, then $du = -dx$

For $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then**:**

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \sin 2\left(\frac{\pi}{2} - u\right) \ln\left(\tan\left(\frac{\pi}{2} - u\right)\right)(-du) = \int_0^{\frac{\pi}{2}} \sin(\pi - 2u) \ln(\cot u) du$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \sin 2u \ln(\cot u)\, du = \int_0^{\frac{\pi}{2}} \sin 2x \ln(\cot x)\, dx$ **… (2)** Adding (1) and (2) we get

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \sin 2x \ln(\tan x)\, dx + \int_0^{\frac{\pi}{2}} \sin 2x \ln(\cot x)\, dx$

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} [\sin 2x \ln(\tan x) + \sin 2x \ln(\cot x)] dx = \int_0^{\frac{\pi}{2}} \sin 2x\, [\ln(\tan x) + \ln(\cot x)] dx$

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \sin 2x \ln(\tan x \times \cot x)\, dx = \int_0^{\frac{\pi}{2}} \sin 2x \ln(1)\, dx = \int_0^{\frac{\pi}{2}} \sin 2x \times 0 dx = 0$

Therefore, I = 0

56. $\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln(\cot x - \tan x)\,dx$, let $u = \frac{\pi}{4} - x$, then $du = -dx$

For $x = 0,\ u = \frac{\pi}{4} - 0 = \frac{\pi}{4}$ and for $x = \frac{\pi}{4},\ u = \frac{\pi}{4} - \frac{\pi}{4} = 0$, then**:**

$\mathrm{I} = \int_{\frac{\pi}{4}}^{0} \ln\left[\cot\left(\frac{\pi}{4} - u\right) - \tan\left(\frac{\pi}{4} - u\right)\right](-dx) = \int_0^{\frac{\pi}{4}} \ln\left[\cot\left(\frac{\pi}{4} - u\right) - \tan\left(\frac{\pi}{4} - u\right)\right] du$

$= \int_0^{\frac{\pi}{4}} \ln\left[\cot\left(\frac{\pi}{4} - x\right) - \tan\left(\frac{\pi}{4} - x\right)\right] dx = \int_0^{\frac{\pi}{4}} \ln\left[\frac{1}{\tan\left(\frac{\pi}{4} - x\right)} - \tan\left(\frac{\pi}{4} - x\right)\right] dx$

$= \int_0^{\frac{\pi}{4}} \ln\left[\frac{1+\tan\left(\frac{\pi}{4}\right)\tan x}{\tan\left(\frac{\pi}{4}\right) - \tan x} - \frac{\tan\left(\frac{\pi}{4}\right) - \tan x}{1+\tan\left(\frac{\pi}{4}\right)\tan x}\right] dx = \int_0^{\frac{\pi}{4}} \ln\left[\frac{1+\tan x}{1-\tan x} - \frac{1-\tan x}{1+\tan x}\right] dx$

$= \int_0^{\frac{\pi}{4}} \ln\left[\frac{\cot x + 1}{\cot x - 1} - \frac{1-\tan x}{1+\tan x}\right] dx = \int_0^{\frac{\pi}{4}} \ln\left[\frac{(\cot x + 1 + 1 + \tan x) - (\cot x - 1 - 1 + \tan x)}{(\cot x - 1)(1 + \tan x)}\right] dx$

$= \int_0^{\frac{\pi}{4}} \ln\left(\frac{4}{\cot x + 1 - 1 - \tan x}\right) dx = \int_0^{\frac{\pi}{4}} [\ln 4 - \ln(\cot x - \tan x)\,dx]dx = \int_0^{\frac{\pi}{4}} \ln 4\,dx - \mathrm{I}$, so**:**

$2\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln 4\,dx = \ln 4\,[x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}(2\ln 2)$, therefore $\mathrm{I} = \frac{\pi}{4}\ln 2$

57. $\mathrm{I} = \int_0^{\pi} \frac{x}{1+\sin x}\,dx$ **… (1)** let $u = \pi - x$, then $du = -dx$

for $x = 0,\ u = \pi$ and for $x = \pi,\ u = 0$, then we get**:**

$\mathrm{I} = \int_{\pi}^{0} \frac{\pi - u}{1+\sin(\pi - u)}(-du) = \int_0^{\pi} \frac{\pi - u}{1+\sin u}\,du = \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx$ **…(2)** Adding (1) and (2) we get**:** $2\mathrm{I} = \int_0^{\pi} \frac{x}{1+\sin x}\,dx + \int_0^{\pi} \frac{\pi - x}{1+\sin x}\,dx = \int_0^{\pi} \frac{x + \pi - x}{1+\sin x}\,dx = \pi \int_0^{\pi} \frac{1}{1+\sin x}\,dx$

$2\mathrm{I} = \pi \int_0^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x}\,dx = \pi \int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x}\,dx = \pi \int_0^{\pi} \frac{1-\sin x}{\cos^2 x}\,dx$

$2\mathrm{I} = \pi \int_0^{\pi} \left(\sec^2 x - \frac{1}{\cos x} \times \frac{\sin x}{\cos x}\right) dx = \pi \int_0^{\pi} (\sec^2 x - \sec x \tan x)\,dx$

$2\mathrm{I} = \pi[\tan x - \sec x]_0^{\pi} = \pi[0 - (-1) - (0 - 1)] = 2\pi$

58. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \ln(\alpha \tan x)\,dx$; let $u = \frac{\pi}{2} - x;\ du = -dx$ for $x = 0;\ u = \frac{\pi}{2}$ and for $x = \frac{\pi}{2};\ u = 0$; so:

$$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \ln\left(\alpha \tan\left(\frac{\pi}{2} - u\right)\right)(-du) = \int_0^{\frac{\pi}{2}} \ln\left(\alpha \tan\left(\frac{\pi}{2} - u\right)\right) du = \int_0^{\frac{\pi}{2}} \ln(\alpha \cot u)\,du$$

$= \int_0^{\frac{\pi}{2}} \ln(\alpha \cot x)\,dx$; then we add the initial and the obtained form of I; then we get:

$$\mathrm{I} + \mathrm{I} = \int_0^{\frac{\pi}{2}} \ln(\alpha \tan x)\,dx + \int_0^{\frac{\pi}{2}} \ln(\alpha \cot x)\,dx = \int_0^{\frac{\pi}{2}} [\ln(\alpha \tan x) + \ln(\alpha \cot x)]dx$$

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \ln(\alpha \tan x \times \alpha \cot x)\,dx = \int_0^{\frac{\pi}{2}} \ln(\alpha^2)\,dx \quad (\tan x \times \cot x = 1)$$

$$2\mathrm{I} = \ln(\alpha^2)\int_0^{\frac{\pi}{2}} dx = \ln(\alpha^2)\left[\frac{\pi}{2} - 0\right] = 2\ln a \times \frac{\pi}{2} = \pi \ln a\text{. So; } \mathrm{I} = \frac{\pi}{2}\ln a$$

59. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{a\tan x + b\cot x}{\tan x + \cot x}dx$ **… (1)**

Let $u = \frac{\pi}{2} - x$, so $du = -dx$, for $x = 0$, $u = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = 0$, so:

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{a\tan\left(\frac{\pi}{2}-u\right) + b\cot\left(\frac{\pi}{2}-u\right)}{\tan\left(\frac{\pi}{2}-u\right) + \cot\left(\frac{\pi}{2}-u\right)}(-du) = \int_0^{\frac{\pi}{2}} \frac{a\cot u + b\tan u}{\cot u + \tan u}du = \int_0^{\frac{\pi}{2}} \frac{a\cot x + b\tan x}{\tan x + \cot x}dx$ **… (2)**

$\mathrm{I} + \mathrm{I} = 2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{a\tan x + b\cot x + a\cot x + b\tan x}{\tan x + \cot x}dx = \int_0^{\frac{\pi}{2}} \frac{(a+b)\tan x + (a+b)\cot x}{\tan x + \cot x}dx$

$= \int_0^{\frac{\pi}{2}} \frac{(a+b)(\tan x + \cot x)}{\tan x + \cot x}dx = (a+b)\int_0^{\frac{\pi}{2}} dx = (a+b)\frac{\pi}{2}$, therefore, $\mathrm{I} = \frac{\pi}{4}(a+b)$

60. $\mathrm{I} = \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx$

Let $u = \pi - x$, then $du = -dx$, for $x = 0$, $u = \pi$ and for $x = \pi$, $u = 0$, so:

$\mathrm{I} = \int_{\pi}^{0} \frac{\pi - u}{a^2\cos^2(\pi-u) + b^2\sin^2(\pi-u)}(-du) = \int_0^{\pi} \frac{\pi - u}{a^2\cos^2 u + b^2\sin^2 u}du$

$= \int_0^{\pi} \frac{\pi - x}{a^2\cos^2 x + b^2\sin^2 x}dx = \pi\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx - \int_0^{\pi} \frac{x}{a^2\cos^2 x + b^2\sin^2 x}dx$

$\mathrm{I} = \pi\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx - \mathrm{I}$, so $2\mathrm{I} = \pi\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx$

$\mathrm{I} = \frac{\pi}{2}\int_0^{\pi} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx = 2\frac{\pi}{2}\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2 x + b^2\sin^2 x}dx$

$= \pi\int_0^{\frac{\pi}{2}} \frac{1}{a^2\cos^2 x + b^2\sin^2 x} \times \frac{\sec^2 x}{\sec^2 x}dx = \pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{a^2 + b^2\tan^2 x}dx$

Let $t = b\tan x$, then $dt = b\sec^2 x\,dx$, for $x = 0$, $t = 0$ and for $x = \frac{\pi}{2}$, $t = +\infty$, then:

$\mathrm{I} = \frac{\pi}{b}\int_0^{+\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{b}\frac{1}{a}\left[\arctan\left(\frac{t}{a}\right)\right]_0^{+\infty} = \frac{\pi}{ab}\left[\frac{\pi}{2} - 0\right] = \frac{1}{2ab}\pi^2$

61. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x}{1 + \cos^2(2x)}dx$ **… (1)**

Let $u = \frac{\pi}{2} - x$, so $du = -dx$, for $x = 0$, $u = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = 0$, so:

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{\sin^4\left(\frac{\pi}{2}-u\right)}{1 + \cos^2\left[2\left(\frac{\pi}{2}-u\right)\right]}(-du) = \int_0^{\frac{\pi}{2}} \frac{\cos^4 u}{1 + \cos^2(\pi - 2u)}du = \int_0^{\frac{\pi}{2}} \frac{\cos^4 u}{1 + \cos^2(2u)}du$

$= \int_0^{\frac{\pi}{2}} \frac{\cos^4 x}{1 + \cos^2(2x)}dx$ **… (2)** Adding (1) and (2) we get:

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^4 x + \cos^4 x}{1 + \cos^2(2x)}dx = \int_0^{\frac{\pi}{2}} \frac{\left(\cos^2 x + \sin^2 x\right)^2 - 2\sin^2 x\cos^2 x}{1 + \cos^2(2x)}dx = \int_0^{\frac{\pi}{2}} \frac{1 - 2\sin^2 x\cos^2 x}{1 + \cos^2(2x)}dx$$

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1 - 2\left(\frac{1}{2}\sin 2x\right)^2}{1 + \cos^2(2x)}dx = \int_0^{\frac{\pi}{2}} \frac{1 - \frac{1}{2}\sin^2 2x}{1 + \cos^2(2x)}dx = \int_0^{\frac{\pi}{2}} \frac{1 - \frac{1}{2}\left(1 - \cos^2(2x)\right)}{1 + \cos^2(2x)}dx$$

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1-\frac{1}{2}+\frac{1}{2}\cos^2(2x)}{1+\cos^2(2x)} dx = \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}+\frac{1}{2}\cos^2(2x)}{1+\cos^2(2x)} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{1+\cos^2(2x)}{1+\cos^2(2x)} dx = \frac{1}{2}\int_0^{\frac{\pi}{2}} dx$

$2\mathrm{I} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$, therefore $\mathrm{I} = \frac{\pi}{8}$

62. $\mathrm{I} = \int_{-2}^{4} \left[2\mathrm{E}\left(\frac{x}{2}\right) + 1\right] dx$, the function $f(x) = 2\mathrm{E}\left(\frac{x}{2}\right) + 1$ is defined over $[-2, 4]$ by**:**

If $-2 \le x < 0$, then $-1 \le \frac{x}{2} < 0$, so $\mathrm{E}\left(\frac{x}{2}\right) = -1$, then $f(x) = -1$

If $0 \le x < 2$, then $0 \le \frac{x}{2} < 1$, so $\mathrm{E}\left(\frac{x}{2}\right) = 0$, then $f(x) = 1$

If $2 \le x < 4$, then $1 \le \frac{x}{2} < 2$, so $\mathrm{E}\left(\frac{x}{2}\right) = 1$, then $f(x) = 3$, hence

$\mathrm{I} = -\int_{-2}^{4} dx + \int_0^2 dx + \int_2^4 3dx = 6$

63. $\int_0^{\frac{3}{2}} [x^2]\, dx$, for $0 \le x < 1$, we have $0 \le x^2 < 1$, so $[x^2] = 0$

for $1 \le x < \sqrt{2}$, we have $1 \le x^2 < 2$, so $[x^2] = 1$

for $\sqrt{2} \le x < \frac{3}{2}$, we have $2 \le x^2 < \frac{9}{4}$, so $[x^2] = 2$, then we get**:**

$\int_0^{\frac{3}{2}} [x^2]\, dx = \int_0^1 0dx + \int_1^{\sqrt{2}} dx + \int_{\sqrt{2}}^{\frac{3}{2}} 2dx = 2 - \sqrt{2}$

64. $\int_0^9 \{\sqrt{x}\}\, dx$, where $\{x\}$ is the fractional part of $\sqrt{x}$, so we have $\{\sqrt{x}\} = \sqrt{x} - [\sqrt{x}]$, then**:**

$\{\sqrt{x}\} = \begin{cases} \sqrt{x} - 0 & \text{for } \ 0 \le x < 1 \\ \sqrt{x} - 1 & \text{for } \ 1 \le x < 4 \\ \sqrt{x} - 2 & \text{for } \ 4 \le x < 9 \end{cases}$, then we get**:**

$\int_0^9 \{\sqrt{x}\}\, dx = \int_0^1 \sqrt{x}\, dx + \int_1^4 (\sqrt{x} - 1) dx + \int_4^9 (\sqrt{x} - 2) dx$

$\int_0^9 \{\sqrt{x}\}\, dx = \frac{2}{3} + \frac{5}{3} + \frac{8}{3} = 5$

65. $\int_0^{\pi} [2\sin x] dx = \int_0^{\frac{\pi}{6}} [2\sin x] dx + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [2\sin x]\, dx + \int_{\frac{5\pi}{6}}^{\pi} [2\sin x] dx$

$\int_0^{\pi} 0dx = \int_0^{\frac{\pi}{6}} dx + \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [2\sin x]\, dx + \int_{\frac{5\pi}{6}}^{\pi} 0dx = \frac{5\pi}{6} - \frac{\pi}{6} = \frac{2\pi}{3}$

66. $\int_0^2 x[x] dx = \int_0^1 x[x] dx + \int_1^2 x[x] dx = \int_0^1 x \times 0dx + \int_1^2 x \times 1dx$

$\int_0^2 x[x] dx = \int_1^2 xdx = \left[\frac{x^2}{2}\right]_1^2 = 2 - \frac{1}{2} = \frac{3}{2}$

67. $\int_0^n \{\sqrt{x}\}\, dx = \int_0^n (x - [x])\, dx = \int_0^n xdx - \int_0^n [x] dx = \left[\frac{x^2}{2}\right]_0^n - \mathrm{J} = \frac{n^2}{2} - \mathrm{J}$

With $\mathrm{J} + \int_0^n [x] dx = \int_0^n [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx + \cdots + \int_{n-1}^n [x] dx$

$\mathrm{J} = \int_0^n 0dx + \int_1^2 1.dx + \int_2^3 2.dx + \cdots + \int_{n-1}^n (n-1) dx$

$\mathrm{J} = 0 + 1(2-1) + 2(3-2) + \cdots + (n-1)\big(n - (n-1)\big)$

$\mathrm{J} = 1 + 2 + 3 + \cdots + (n-1) = \frac{n(n-1)}{2} = \frac{n^2-n}{2}$, therefore:

$\mathrm{I} = \frac{n^2}{2} - \frac{n^2-n}{2} = \frac{n}{2}$

68. $\mathrm{I} = \int_0^{2n\pi}[\sin x + \cos x]\,dx$, it's a periodic function with period 2π, then:

$\mathrm{I} = \int_0^{2n\pi}[\sin x + \cos x]\,dx = n\int_0^{2\pi}[\sin x + \cos x]\,dx = n\int_0^{2\pi}\left[\sqrt{2}\sin\left(x+\frac{\pi}{4}\right)\right]dx$

Let $u = x + \frac{\pi}{4}$, then $du = dx$, for $x = 0$, $u = \frac{\pi}{4}$ and for $x = 2\pi$, $u = 2\pi + \frac{\pi}{4} = \frac{9\pi}{4}$, so:

$\mathrm{I} = n\int_{\frac{\pi}{4}}^{\frac{9\pi}{4}}[\sqrt{2}\sin u]du = n\int_{\frac{\pi}{4}}^{\frac{9\pi}{4}}[\sqrt{2}\sin x]dx$

$= n\left[\left(\frac{3\pi}{4}-\frac{\pi}{4}\right).1 + \left(\frac{5\pi}{4}-\pi\right)(-1) + \left(\frac{7\pi}{4}-\frac{5\pi}{4}\right)(-2) + \left(2\pi-\frac{3\pi}{4}\right)(-1)\right]$

$= n\left(\frac{2\pi}{4}-\frac{\pi}{4}-\frac{\pi}{4}-\frac{4\pi}{4}\right) = -n\pi$

69. $\int_{-2}^{1}\left[1+\cos\left(\frac{\pi x}{2}\right)\right]dx$

$=\int_{-2}^{-1}\left[1+\cos\left(\frac{\pi x}{2}\right)\right]dx + \int_{-1}^{0}\left[1+\cos\left(\frac{\pi x}{2}\right)\right]dx + \int_{0}^{1}\left[1+\cos\left(\frac{\pi x}{2}\right)\right]dx$

$= \int_{-2}^{-1}1.dx + \int_{-1}^{0}0.dx + \int_{0}^{1}1.dx = 1 + 0 + 1 = 2$

70. $\int_0^3\left([x] + \left[x+\frac{1}{2}\right] + \left[x+\frac{2}{3}\right]\right)dx$

$= \int_0^{\frac{1}{3}}0.dx + \int_{\frac{1}{3}}^{\frac{2}{3}}1.dx + \int_{\frac{2}{3}}^{1}2.dx + \int_1^{\frac{4}{3}}3.dx + \int_{\frac{4}{3}}^{\frac{5}{3}}4.dx + \int_{\frac{5}{3}}^{2}5.dx + \int_2^{\frac{7}{3}}6.dx + \int_{\frac{7}{3}}^{\frac{8}{3}}7.dx$

$+ \int_{\frac{8}{3}}^{3}8.dx = \frac{1}{3}(1+2+3+4+5+6+7+8) = \frac{1}{3}\times 8\times\frac{9}{2} = 12$

71. $\mathrm{I} = \int_0^{\pi}[\cot x]dx$ **… (1)**

Let $u = \pi - x$m then $du = -dx$, for $x = 0$, $u = \pi$ and for $x = \pi$, $u = 0$, then:

$\mathrm{I} = \int_{\pi}^{0}[\cot(\pi-u)](-du) = \int_0^{\pi}[-\cot u]du = \int_0^{\pi}[-\cot x]dx$ **… (2)**

Adding (1) and (2) we get: $2\mathrm{I} = \int_0^{\pi}[\cot x]dx + \int_0^{\pi}[-\cot x]dx$

$2\mathrm{I} = \int_0^{\pi}([\cot x] + [-\cot x])dx$

But $[x] + [-x] = -1$ if $x \notin \mathbb{Z}$ and $[x] + [-x] = 0$ if $x\ \mathbb{Z}$, so we get:

$2\mathrm{I} = \int_0^{\pi}(-1)dx = -\pi$, therefore $\mathrm{I} = -\frac{\pi}{2}$

72. $\mathrm{I} = \int_{e^{-e}}^{e^{-1}}\frac{\ln(\ln(-\ln x))}{x\ln x}dx$, let $t = \ln(-\ln x)$, so $-\ln x = e^t$ and $x = e^{-e^t}$, then

$dx = -e^t e^{-e^t}dt$, for $x = e^{-e}$, $t = \ln(-\ln e^{-e}) = \ln e = 1$ and for $x = e^{-1}$, $t = \ln(-\ln e^{-1})$

$t = \ln 1 = 0$, then we get:

$\mathrm{I} = \int_1^0\frac{\ln t}{-e^t e^{-e^t}}\left(-e^t e^{-e^t}dt\right) = -\int_0^1\ln t\,dt = -[t\ln t - t]_0^1 = 1$

73. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{1}{a\cos^2 x + b\sin^2 x}dx = \int_0^{\frac{\pi}{2}}\frac{1}{a\cos^2 x + b\sin^2 x}\times\frac{\sec^2 x}{\sec^2 x}dx = \int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{a + b\tan^2 x}dx$

Let $u = \tan x$, then $du = \sec^2 x\,dx$, for $x = 0$, $u = 0$ and for $x = \frac{\pi}{2}$, $u = +\infty$, so:

$\mathrm{I} = \int_0^{+\infty} \frac{1}{a+bu^2} du$, let $bu^2 = a\tan^2\theta$, so $u^2 = \frac{a}{b}\tan^2\theta$, then $u = \sqrt{\frac{a}{b}}\tan\theta$ and

$du = \sqrt{\frac{a}{b}}\sec^2\theta\, d\theta$, then $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{a}{b}}\sec^2\theta}{a+\frac{a}{b}b\tan^2\theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{a}{b}}\sec^2\theta}{a+a\tan^2\theta} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\frac{a}{b}}\sec^2\theta}{a\sec^2\theta} d\theta$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{ab}} d\theta = \frac{\pi}{2\sqrt{ab}}$

74. $\mathrm{I} = \int_{-\infty}^{+\infty} \frac{1}{e^{2x}+e^{-2x}} dx = \int_{-\infty}^{+\infty} \frac{1}{e^{2x}+\frac{1}{e^{2x}}} dx = \int_{-\infty}^{+\infty} \frac{1}{\frac{e^{4x}+11}{e^{2x}}} dx = \int_{-\infty}^{+\infty} \frac{e^{2x}}{e^{4x}+1} dx$

$\mathrm{I} = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{2e^{2x}}{e^{4x}+1} dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{(e^{2x})'}{(e^{2x})^2+1} dx = \frac{1}{2}[\arctan e^{2x}]_{-\infty}^{+\infty} = \frac{1}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{4}$

75. $\mathrm{I} = \int_0^{+\infty} \frac{1}{(1+x^{1991})(1+x^2)} dx$, let $u = \frac{1}{x}$, so $x = \frac{1}{u}$, so $dx = -\frac{1}{u^2} du$

For $x = 0$, $u = +\infty$ and for $x = +\infty$, $u = 0$, then:

$\mathrm{I} = \int_{+\infty}^{0} \frac{1}{\left(1+\frac{1}{u^{1991}}\right)\left(1+\frac{1}{u^2}\right)}\left(-\frac{1}{u^2}du\right) = \int_0^{+\infty} \frac{1}{u^2\left(1+\frac{1}{u^{1991}}\right)\left(1+\frac{1}{u^2}\right)} du$

$\mathrm{I} = \int_0^{+\infty} \frac{1}{u^2\left(\frac{1+u^{1991}}{u^{1991}}\right)\left(\frac{u^2+1}{u^2}\right)} du = \int_0^{+\infty} \frac{u^{1991}}{(1+u^{1991})(1+u^2)} du$

$\mathrm{I} = \int_0^{+\infty} \frac{x^{1991}}{(1+x^{1991})(1+x^2)} dx = \int_0^{+\infty} \frac{(x^{1991}+1)-1}{(1+x^{1991})(1+x^2)} dx$

$\mathrm{I} = \int_0^{+\infty} \frac{(x^{1991}+1)}{(1+x^{1991})(1+x^2)} dx - \int_0^{+\infty} \frac{1}{(1+x^{1991})(1+x^2)} dx = \int_0^{+\infty} \frac{1}{1+x^2} dx - \mathrm{I}$

$2\mathrm{I} = \int_0^{+\infty} \frac{1}{1+x^2} dx = [\arctan x]_0^{+\infty} = \frac{\pi}{2}$, therefore $\mathrm{I} = \frac{\pi}{4}$

76. $\mathrm{I} = \int_1^{+\infty} \frac{xe^x-1}{x\left[1+(e^x-\ln x)^2\right]} dx$, let $u = e^x - \ln x$, then $du = \left(e^x - \frac{1}{x}\right) dx = \left(\frac{xe^x-1}{2}\right) dx$

for $x = 1$, $u = e - \ln 1 = e$ and for $x = +\infty$, $u = \lim_{x\to+\infty}(e^x - \ln x) = +\infty$, then:

$\mathrm{I} = \int_e^{+\infty} \frac{du}{1+u^2} = [\arctan u]_e^{+\infty} = \arctan +\infty - \arctan e = \frac{\pi}{2} - \tan^{-1} e$

77. $\mathrm{I} = \int_0^{+\infty} \frac{\sqrt[4]{e^{3x}} - \sqrt[4]{e^x}}{e^x-1} dx = \int_0^{+\infty} \frac{e^{\frac{3x}{4}} - e^{\frac{x}{4}}}{e^x-1} dx = \int_0^{+\infty} \frac{e^{\frac{x}{4}}\left(e^{\frac{x}{2}}-1\right)}{\left(e^{\frac{x}{2}}+1\right)\left(e^{\frac{x}{2}}-1\right)} dx = \int_0^{+\infty} \frac{e^{\frac{x}{4}}}{e^{\frac{x}{2}}+1} dx$

$\mathrm{I} = \int_0^{+\infty} \frac{e^{\frac{x}{4}}}{\left(e^{\frac{x}{4}}\right)^2+1} dx = 4\int_0^{+\infty} \frac{\frac{1}{4}e^{\frac{x}{4}}}{\left(e^{\frac{x}{4}}\right)^2+1} dx = 4\int_0^{+\infty} \frac{\left(e^{\frac{x}{4}}\right)'}{\left(e^{\frac{x}{4}}\right)^2+1} dx = 4\left[\arctan e^{\frac{x}{4}}\right]_0^{+\infty}$

$\mathrm{I} = 4(\arctan +\infty - \arctan 1) = 4\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = 4\left(\frac{\pi}{4}\right) = \pi$

78. $I = \int_0^{+\infty} \frac{\tan(\arctan x - \arctan 2x)}{x} dx$, let $u = \arctan x$ and $v = \arctan 2x$

$\tan(\arctan x - \arctan 2x) = \tan(u - v) = \frac{\tan u - \tan v}{1 + \tan u \tan v}$

$= \frac{\tan(\arctan x) - \tan(\arctan 2x)}{1 + \tan(\arctan x)\tan(\arctan 2x)} = \frac{x - 2x}{1 + 2x^2} = -\frac{x}{1 + 2x^2}$, then:

$I = -\int_0^{+\infty} \frac{1}{1 + 2x^2} dx = -\frac{1}{2}\int_0^{+\infty} \frac{1}{\left(\frac{1}{\sqrt{2}}\right)^2 + x^2} dx = -\frac{1}{2}\left[\sqrt{2}\arctan(\sqrt{2}x)\right]_0^{+\infty} = -\frac{\pi}{2\sqrt{2}}$

79. $I = \int_0^{+\infty} \frac{dx}{1 + x + x^2 + x^3}$ **… (1)** Let $x = \frac{1}{u}$, then $dx = -\frac{1}{u^2}du$, then:

$I = \int_{+\infty}^{0} \frac{1}{1 + \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^3}} \frac{1}{u^2}(-du) = \int_0^{+\infty} \frac{du}{u^2 + u + 1 + \frac{1}{u}} = \int_0^{+\infty} \frac{u}{1 + u + u^2 + u^3} du$

$= \int_0^{+\infty} \frac{x}{1 + x + x^2 + x^3} du$ **… (2)** Adding (1) and (2) we get:

$2I = \int_0^{+\infty} \frac{x + 1}{1 + x + x^2 + x^3} dx = \int_0^{+\infty} \frac{x + 1}{(x + 1)(1 + x^2)} dx = \int_0^{+\infty} \frac{1}{1 + x^2} dx = [\arctan x]_0^{+\infty} = \frac{\pi}{2}$

$2I = \frac{\pi}{2}$, so $I = \frac{\pi}{4}$

80. First Method: $I = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx$, let $x = \tan t$ then $dx = \sec^2 t\, dt$, for $x = 0$, $t = 0$ and for $x = 1$, $t = \frac{\pi}{4}$, so $I = \int_0^{\frac{\pi}{4}} \frac{\ln(\tan t + 1)}{\tan^2 t + 1} \sec^2 t\, dt = \int_0^{\frac{\pi}{4}} \frac{\ln(\tan t + 1)}{\sec^2 t} \sec^2 t\, dt$

$I = \int_0^{\frac{\pi}{4}} \ln(1 + \tan t)\, dt$ **… (1)** Let $u = \frac{\pi}{4} - t$, so $du = -dt$, for $t = 0$, $u = \frac{\pi}{4}$ and for $t = \frac{\pi}{4}$, $u = 0$, so $I = \int_{\frac{\pi}{4}}^{0} \ln\left(1 + \tan\left(\frac{\pi}{4} - u\right)\right)(-du) = \int_0^{\frac{\pi}{4}} \ln\left(1 + \tan\left(\frac{\pi}{4} - u\right)\right) du$

$I = \int_0^{\frac{\pi}{4}} \ln\left(1 + \tan\left(\frac{\pi}{4} - t\right)\right) dt = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{\tan\left(\frac{\pi}{4}\right) - \tan t}{1 + \tan\left(\frac{\pi}{4}\right)\tan t}\right) dt = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan t}{1 + \tan t}\right) dt$

$I = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan t}\right) dt = \int_0^{\frac{\pi}{4}} [\ln 2 - \ln(1 + \tan t)]\, dt = \int_0^{\frac{\pi}{4}} \ln 2\, dx - \int_0^{\frac{\pi}{4}} \ln(1 + \tan t)\, dt$

$I = \ln 2 \int_0^{\frac{\pi}{4}} dx - I$, $2I = \ln 2\, [x]_0^{\frac{\pi}{4}} = \frac{\pi}{4}\ln 2$, therefore $I = \frac{\pi}{8}\ln 2$

Second Method: $I = \int_0^1 \frac{\ln(x+1)}{x^2+1} dx$, let $x = \frac{1-y}{1+y}$, so $dx = -\frac{2}{(1+y)^2}dy$

For $x = 0$, $y = 1$ and for $x = 1$, $y = 0$, so:

$I = -\int_1^0 \frac{\ln\left(\frac{1-y}{1+y} + 1\right)}{\left(\frac{1-y}{1+y}\right)^2 + 1} \frac{2}{(1+y)^2} dy = 2\int_0^1 \frac{\ln\left(\frac{2}{1+y}\right)}{(1-y)^2 + (1+y^2)} dy = \int_0^1 \frac{\ln 2 - \ln(y+1)}{y^2+1} dy$

$I = \int_0^1 \frac{\ln 2 - \ln(x+1)}{x^2+1} dx = \ln 2 \int_0^1 \frac{1}{x^2+1} dx - \int_0^1 \frac{\ln(x+1)}{x^2+1} dx = \ln 2\, [\arctan x]_0^1 - I$

$2I = \ln 2\left(\frac{\pi}{4}\right)$, therefore $I = \frac{\pi}{8}\ln 2$

81. $I = \int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx$ **… (1)**, let $u = \frac{\pi}{2} - x$, then $du = -dx$

For $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then:

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \ln\left(\sin\left(\frac{\pi}{2} - u\right)\right)(-du) = \int_0^{\frac{\pi}{2}} \ln(\cos u)du = \int_0^{\frac{\pi}{2}} \ln(\cos x)\, dx$ **… (2)**

Adding (1) and (2) we get**:** $2\mathrm{I} = \int_0^{\frac{\pi}{2}}[\ln(\sin x) + \ln(\cos x)]dx = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right) dx$

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx - \int_0^{\frac{\pi}{2}} \ln 2\, dx = \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx - \frac{\pi}{2}\ln 2 = \mathrm{J} - \frac{\pi}{2}\ln 2$

Evaluating: $\mathrm{J} = \int_0^{\frac{\pi}{2}} \ln(2\sin x)\, dx$, let $v = 2x$, so $dv = 2dx$, then**:**

$\mathrm{J} = \frac{1}{2}\int_0^{\pi} \ln(\sin v)\, dv = \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} \ln(\sin v)\, dv + \int_{\frac{\pi}{2}}^{\pi} \ln(\sin v)\, dv\right]$

Letting $v = \pi - w$ in the integral $\int_{\frac{\pi}{2}}^{\pi} \ln(\sin v)\, dv$ becomes $\int_0^{\frac{\pi}{2}} \ln(\sin w)\, dw$

So**:** $\mathrm{J} = \frac{1}{2}(\mathrm{I} + \mathrm{I}) = \mathrm{I}$, with $2\mathrm{I} = \mathrm{J} - \frac{\pi}{2}\ln 2$, so $2\mathrm{I} = \mathrm{I} - \frac{\pi}{2}\ln 2$, therefore $\mathrm{I} = -\frac{\pi}{2}\ln 2$

82. $\mathrm{I} = \int_0^{\pi} x\ln(\sin x)\, dx$, let $y = \pi - x$, then $dy = -dx$, for $x = 0$, $y = \pi$, for $x = \pi$, $y = 0$, so $\mathrm{I} = \int_{\pi}^{0}(\pi - y)\ln(\sin(\pi - y))\,(-dx) = \int_0^{\pi}(\pi - y)\ln(\sin y)\, dy$

$\mathrm{I} = \int_0^{\pi}(\pi - x)\ln(\sin x)\, dx = \pi\int_0^{\pi}\ln(\sin x)dx - \int_0^{\pi} x\ln(\sin x)dx$

$\mathrm{I} = \pi\int_0^{\pi}\ln(\sin x)dx - \mathrm{I}$, so $2\mathrm{I} = \pi\int_0^{\pi}\ln(\sin x)dx = 2\pi\left(\frac{1}{2}\int_0^{\pi}\ln(\sin x)dx\right)$

$= 2\pi\left(-\frac{\pi}{2}\ln 2\right)$ (using the results of the preceding part) $= -\pi^2\ln 2$

83. $\mathrm{I} = \int_0^{+\infty} \ln\left(1 + \frac{k^2}{x^2}\right) dx$

Let $u = \ln\left(1 + \frac{k^2}{x^2}\right)$, so $u' = -\frac{2k^2}{x(x^2+k^2)}$ and let $v' = 1$, so $v = x$, then**:**

$\mathrm{I} = \left[x\ln\left(1 + \frac{k^2}{x^2}\right)\right]_0^{+\infty} + \int_0^{+\infty}\frac{2k^2x}{x(x^2+k^2)}dx = 0 + 2k^2\int_0^{+\infty}\frac{1}{x^2+k^2}dx$

$\mathrm{I} = 2k^2\left[\frac{1}{k}\arctan\left(\frac{x}{k}\right)\right]_0^{+\infty} = 2k\left(\frac{\pi}{2}\right)$, therefore $\mathrm{I} = k\pi$

84. $\mathrm{I} = \int_0^{+\infty}\frac{\ln x}{x^2+\alpha^2}dx$, let $x = \frac{1}{t}$, then $dx = -\frac{1}{t^2}dt$, so**:**

$\mathrm{I} = \int_{+\infty}^{0}\frac{\ln\left(\frac{1}{t}\right)}{\left(\frac{1}{t}\right)^2+\alpha^2}\left(-\frac{1}{t^2}dt\right) = \int_0^{+\infty}\frac{-\ln t}{1+\alpha^2t^2}dt$, let $u = \alpha t$, so$du = \alpha dt$, then**:**

$\mathrm{I} = \frac{1}{\alpha}\int_0^{+\infty}\frac{-\ln\left(\frac{u}{\alpha}\right)}{1+u^2}du = \frac{1}{a}\int_0^{+\infty}\frac{\ln\alpha - \ln u}{1+u^2}du = \frac{1}{a}\int_0^{+\infty}\frac{\ln\alpha - \ln x}{1+x^2}dx$, then**:**

$\mathrm{I} = \frac{\ln\alpha}{a}\int_0^{+\infty}\frac{1}{1+x^2}dx - \frac{1}{a}\int_0^{+\infty}\frac{\ln x}{1+x^2}dx = \frac{\ln\alpha}{a}[\arctan x]_0^{+\infty} - \frac{1}{a}\int_0^{+\infty}\frac{\ln x}{1+x^2}dx$

$\mathrm{I} = \frac{\pi\ln\alpha}{2} - \frac{1}{a}\mathrm{J}$, where $\mathrm{J} = \int_0^{+\infty}\frac{\ln x}{1+x^2}dx$, let's evaluate J

$\mathrm{J} = \int_0^{1}\frac{\ln x}{1+x^2}dx + \int_1^{+\infty}\frac{\ln x}{1+x^2}dx = \mathrm{J}_1 + \mathrm{J}_2$, for J_2, let $x = \frac{1}{w}$, then $dx = -\frac{1}{w^2}dw$, so**:**

$$J_2 = \int_1^0 \frac{\ln\left(\frac{1}{w}\right)}{1+\left(\frac{1}{w}\right)^2}\left(-\frac{1}{w^2}dw\right) = \int_0^1 \frac{\ln\left(\frac{1}{w}\right)}{1+w^2}dw = \int_0^1 \frac{-\ln w}{1+w^2}dw = -\int_0^1 \frac{\ln x}{1+x^2}dx = -J_1$$

So, $J = J_1 - J_1 = 0$, therefore, $I = \int_0^{+\infty} \frac{\ln x}{x^2+\alpha^2} = \frac{\pi \ln \alpha}{2}$

85. $I = \int_0^{+\infty} \frac{\ln(x^2-x+1)}{(x^2+1)\ln x}dx$, let $t = \frac{1}{x}$, then $x = \frac{1}{t}$, and $dx = -\frac{1}{t^2}dt$, then we get

$$I = \int_{+\infty}^0 \frac{\ln\left(\frac{1}{t^2}-\frac{1}{t}+1\right)}{\left(\frac{1}{t^2}+1\right)\ln\left(\frac{1}{t}\right)}\left(-\frac{1}{t^2}\right)dt = \int_0^{+\infty} \frac{\ln\left(\frac{1}{t^2}(1-t+t^2)\right)}{(t^2+1)(-\ln t)}dt = \int_0^{+\infty} \frac{\ln(t^2-t+1)-\ln t^2}{(t^2+1)(-\ln t)}dt$$

$$I = -\int_0^{+\infty} \frac{\ln(x^2-x+1)-2\ln x}{(x^2+1)\ln x}d = -\int_0^{+\infty} \frac{\ln(x^2-x+1)}{(x^2+1)\ln x}dx + 2\int_0^{+\infty} \frac{1}{x^2+1}dx$$

$I = -I + 2[\arctan x]_0^{+\infty}$, so $2I = 2\left(\frac{\pi}{2}\right)$, therefore $I = \frac{\pi}{2}$

86. $I = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\sin x(\sin x+\cos x)}dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\sin^2 x+\sin x\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{\frac{1-\cos 2x}{2}+\frac{\sin 2x}{2}}dx$

$I = 2\int_0^{\frac{\pi}{2}} \frac{\sqrt{\tan x}}{1-\cos 2x+\sin 2x}dx$, let $u = \tan x$, then $du = \sec^2 x\, dx = (1+\tan^2 x)dx$

$du = (1+u^2)dx$, for $x = 0$, $u = 0$ and for $x = \frac{\pi}{2}$, $u = +\infty$, moreover $\cos 2x = \frac{1-u^2}{1+u^2}$ and

$\sin 2x = \frac{2u}{1+u^2}$, then $I = 2\int_0^{+\infty} \frac{\sqrt{u}}{1-\frac{1-u^2}{1+u^2}+\frac{2u}{1+u^2}}.\frac{1}{1+u^2}.du = \int_0^{+\infty} \frac{\sqrt{u}}{u^2+u}du = \int_0^{+\infty} \frac{1}{u\sqrt{u}+\sqrt{u}}du$

$$I = 2\int_0^{+\infty} \frac{1}{(\sqrt{u})^2+1}\frac{1}{2\sqrt{u}}du = 2\int_0^{+\infty} \frac{(\sqrt{u})'}{(\sqrt{u})^2+1}du = 2[\arctan u]_0^{+\infty} = 2\left(\frac{\pi}{2}\right) = \pi$$

87. $I = \int_0^{\pi} \frac{x}{1+\cos^2 x}dx$. Let $u = \pi - x$, then $du = -dx$, for $x = 0$, $u = \pi$ and for $x = \pi$,

$u = 0$, then: $I = \int_\pi^0 \frac{\pi-u}{1+\cos^2(\pi-u)}(-du) = \int_0^\pi \frac{\pi-u}{1+\cos^2 u}du = \int_0^\pi \frac{\pi-x}{1+\cos^2 x}dx$

$I = \pi\int_0^\pi \frac{1}{1+\cos^2 x}dx - \int_0^\pi \frac{x}{1+\cos^2 x}dx$, so $I = \pi\int_0^\pi \frac{1}{1+\cos^2 x}dx - I$, then

$$2I = 2\pi\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos^2 x}dx = 2\pi\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos^2 x}\times\frac{\sec^2 x}{\sec^2 x}dx = 2\pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\sec^2 x+1}dx$$

$$2I = 2\pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{1+\tan^2 x+1}dx = 2\pi\int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{2+\tan^2 x}dx = 2\pi\int_0^{\frac{\pi}{2}} \frac{(\tan x)'}{(\tan x)^2+\left(\sqrt{2}\right)^2}dx$$

$$I = \pi\left[\frac{1}{\sqrt{2}}\arctan\left(\frac{\tan x}{\sqrt{2}}\right)\right]_0^{\frac{\pi}{2}} = \pi\left(\frac{1}{\sqrt{2}}\frac{\pi}{2}\right) = \frac{1}{2\sqrt{2}}\pi^2$$

88. $I_n = \int_0^1 (1-x^2)^n\, dx$, $I_1 = \int_0^1 (1-x^2)^1\, dx = \left[x - \frac{x^3}{3}\right]_0^1 = \frac{2}{3}$

$I_n = \int_0^1 (1-x^2)^n\, dx$, let $u = (1-x^2)^n$, then $u' = (-2x)n(1-x^2)^{n-1}$ and let $v' = 1$ then

$v = x$, so $I_n = [x(1-x^2)^n]_0^1 - \int_0^1 (-2x^2)n(1-x^2)^{n-1}\, dx$, then

$I_n = 2n\int_0^1 x^2(1-x^2)^{n-1}\, dx = 2n\int_0^1 (x^2-1+1)(1-x^2)^{n-1}\, dx$, then

$\mathrm{I}_n = 2n\left[-\int_0^1 (1-x^2)^n\,dx + \int_0^1 (1-x^2)^{n-1}\,dx\right]$, then we get

$\mathrm{I}_n = -2n\mathrm{I}_n + 2n\mathrm{I}_{n-1}$, therefore, we get $\mathrm{I}_n = \frac{2n}{2n+1}\mathrm{I}_{n-1}$

$\mathrm{I}_n = \frac{2n}{2n+1}\mathrm{I}_{n-1}$, $\mathrm{I}_{n-1} = \frac{2n-2}{2n-1}\mathrm{I}_{n-1}$, $\mathrm{I}_{n-2} = \frac{2n-4}{2n-3}\mathrm{I}_{n-2}$ … $\mathrm{I}_1 = \frac{2}{3}$, multiplying we get**:**

$$\mathrm{I}_n = \frac{2}{3}\times\frac{4}{5}\times\frac{6}{7}\times\ldots\times\frac{2n}{2n+1}$$

89. $\mathrm{I}_n = \int_0^1 x^n\sqrt{1-x}\,\mathrm{d}x$, $\mathrm{I}_0 = \int_0^1 \sqrt{1-x}\,\mathrm{d}x = \left[-\frac{2}{3}\sqrt{(1-x)^3}\right]_0^1 = \frac{2}{3}$

Let $u = x^n$, then $du = nx^{n-1}dx$ and $dv = \sqrt{1-x}dx$, so $v = -\frac{2}{3}\sqrt{(1-x)^3}$, then,

$\mathrm{I}_n = -\frac{2}{3}\left[x^n\sqrt{(1-x)^3}\right]_0^1 - \frac{2}{3}n\int_0^1 x^{n-1}\sqrt{(1-x)^3}\mathrm{d}x = \frac{2}{3}n\int_0^1 x^{n-1}(1-x)\sqrt{1-x}\mathrm{d}x$, then

$\mathrm{I}_n = \frac{2}{3}n\int_0^1 x^{n-1}\sqrt{1-x}\mathrm{d}x - \frac{2}{3}n\int_0^1 x^n\sqrt{1-x}\mathrm{d}x$, so $\mathrm{I}_n = \frac{2}{3}n\mathrm{I}_{n-1} - \frac{2}{3}n\mathrm{I}_n$, so we get

$(3+2n)\mathrm{I}_n = 2n\mathrm{I}_{n-1}$, therefore we get $\mathrm{I}_n = \frac{2n}{2n+3}\mathrm{I}_{n-1}$.

We have $\mathrm{I}_n = \frac{2n}{2n+3}\mathrm{I}_{n-1}$, so $\mathrm{I}_{n-1} = \frac{2n}{2n+3}\mathrm{I}_{n-2}$, $\mathrm{I}_{n-2} = \frac{2n}{2n+3}\mathrm{I}_{n-3}$, … ,

$\mathrm{I}_2 = \frac{2\times2}{7}\mathrm{I}_1$, and $\mathrm{I}_1 = \frac{2\times1}{5}\mathrm{I}_0$, multiplying we get,

$$\mathrm{I}_n = \frac{2n\times2(n-1)\times\ldots\times(2\times2)\times(2\times1)}{(2n+3)(2n+1)\times\ldots\times7\times5}\times\frac{2}{3} = \frac{2^{n+1}\times n!}{(2n+3)(2n+1)\times\ldots\times7\times5\times3}$$

Therefore we get: $\mathrm{I}_n = \dfrac{2^{2n+2}\times n!\times(n+1)!}{(2n+3)!}$

90. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{1}{1+a^2\sin^2 x}dx = \int_0^{\frac{\pi}{2}}\frac{1}{\cos^2 x+\sin^2 x+a^2\sin^2 x}dx$

$\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{1}{\cos^2 x+(1+a^2)\sin^2 x}dx = \int_0^{\frac{\pi}{2}}\frac{1}{\cos^2 x+(1+a^2)\sin^2 x}\times\frac{\sec^2 x}{\sec^2 x}dx$

$\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{1+(1+a^2)\tan^2 x}dx$, let $u = \tan x$, then $du = \sec^2 x\,dx$, for $x = 0$, $u = 0$ and for

$x = \frac{\pi}{2}$, $u = +\infty$, then $\mathrm{I} = \int_0^{+\infty}\frac{1}{1+(1+a^2)u^2}du = \frac{1}{1+a^2}\int_0^{+\infty}\frac{1}{\left(\frac{1}{\sqrt{1+a^2}}\right)^2+u^2}du$

$\mathrm{I} = \frac{1}{1+a^2}\left(\sqrt{1+a^2}\right)[\arctan u]_0^{+\infty} = \frac{1}{\left(\sqrt{1+a^2}\right)^2}\left(\sqrt{1+a^2}\right)\frac{\pi}{2} = \frac{\pi}{2\sqrt{1+a^2}}$

91. $\mathrm{I} = \int_0^{\pi}\frac{x}{1+\tan^4 x}dx$, let $u = \pi - x$, then $du = -dx$, fot $x = 0$, $u = \pi$ and for $x = \pi$, $u =$ 0, then $\mathrm{I} = \int_{\pi}^{0}\frac{\pi-u}{1+\tan^4(\pi-u)}(-du) = \int_0^{\pi}\frac{\pi-u}{1+\tan^4 u}du = \int_0^{\pi}\frac{\pi-x}{1+\tan^4 x}dx$

$\mathrm{I} = \pi\int_0^{\pi}\frac{1}{1+\tan^4 x}dx - \int_0^{\pi}\frac{x}{1+\tan^4 x}dx = \pi\int_0^{\pi}\frac{1}{1+\tan^4 x}dx - \mathrm{I}$, then

$2\mathrm{I} = \pi\int_0^{\pi}\frac{1}{1+\tan^4 x}dx = 2\pi\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^4 x}dx$, so $\mathrm{I} = \pi\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^4 x}dx$ **… (1)**

Let $t = \frac{\pi}{2} - x$, $dt = -dx$, for $x = 0$, $t = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $t = 0$, so:

$$\mathrm{I} = \pi \int_{\frac{\pi}{2}}^{0} \frac{1}{1+\tan^4\left(\frac{\pi}{2}-t\right)}(-dt) = \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^4 t}\,dt = \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\cot^4 x}\,dx$$

$$\mathrm{I} = \pi \int_0^{\frac{\pi}{2}} \frac{1}{1+\frac{1}{\tan^4 x}}\,dt = \pi \int_0^{\frac{\pi}{2}} \frac{\tan^4 x}{1+\tan^4 x}\,dx \ \ldots \ \textbf{(2)}$$ Adding (1) and (2) we get:

$$2\mathrm{I} = \pi \int_0^{\frac{\pi}{2}} \frac{1+\tan^4 x}{1+\tan^4 x}\,dx = \pi \int_0^{\frac{\pi}{2}} dx = \pi\left(\frac{\pi}{2}\right) = \frac{\pi^2}{2}, \text{ so } \mathrm{I} = \frac{\pi^2}{4}$$

92. $$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{x^2\sin^2 x}{x^2+\left(\frac{\pi^2}{4}-\pi x\right)\cos^2 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{x^2\sin^2 x}{x^2+\left[\left(\frac{\pi}{2}-x\right)^2-x^2\right]\cos^2 x}\,dx$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{x^2\sin^2 x}{x^2-x^2\cos^2 x+\left(\frac{\pi}{2}-x\right)^2\cos^2 x}\,dx = \int_0^{\frac{\pi}{2}} \frac{x^2\sin^2 x}{x^2(1-\cos^2 x)+\left(\frac{\pi}{2}-x\right)^2\cos^2 x}\,dx$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{x^2\sin^2 x}{x^2\sin^2 x+\left(\frac{\pi}{2}-x\right)^2\cos^2 x}\,dx \ \ldots \ \textbf{(1)}$$

Let $u = \frac{\pi}{2} - x$, then $du = -dx$, for $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then:

$$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{\left(\frac{\pi}{2}-u\right)^2\sin^2\left(\frac{\pi}{2}-u\right)}{\left(\frac{\pi}{2}-u\right)^2\sin^2\left(\frac{\pi}{2}-u\right)+u^2\cos^2\left(\frac{\pi}{2}-u\right)}(-du) = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-u\right)^2\cos^2 u}{\left(\frac{\pi}{2}-u\right)^2\cos^2 u+u^2\sin^2 x}\,du$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)^2\cos^2 x}{\left(\frac{\pi}{2}-x\right)^2\cos^2 x+x^2\sin^2 x}\,dx \ \ldots \ \textbf{(2)}$$ Adding (1) and (2) we get:

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{x^2\sin^2 x}{x^2\sin^2 x+\left(\frac{\pi}{2}-x\right)^2\cos^2 x}\,dx + \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2}-x\right)^2\cos^2 x}{\left(\frac{\pi}{2}-x\right)^2\cos^2 x+x^2\sin^2 x}\,dx$$

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{x^2\sin^2 x+\left(\frac{\pi}{2}-x\right)^2\cos^2 x}{x^2\sin^2 x+\left(\frac{\pi}{2}-x\right)^2\cos^2 x}\,dx = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \text{ therefore } \mathrm{I} = \frac{\pi}{4}$$

93. $$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx = \int_0^1 \sqrt{\frac{1+x}{1-x}} \times \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int_0^1 \frac{1+x}{\sqrt{1-x^2}} = \int_0^1 \frac{1}{\sqrt{1-x^2}} + \int_0^1 \frac{x}{\sqrt{1-x^2}}$$

$$\int_0^1 \sqrt{\frac{1+x}{1-x}}\,dx = \left[\arcsin x - \sqrt{1-x^2}\right]_0^1 = \frac{\pi}{2} - (-1) = 1 + \frac{\pi}{2}$$

94. $\mathrm{I} = \int_0^1 \frac{1}{\sqrt{x}\sqrt{1-x}}\,dx$, let $x = \sin^2\theta$, then $dx = 2\sin\theta\cos\theta\,d\theta$

For $x = 0$ we get $\theta = 0$ and when $x = 1$, we get $\theta = \frac{\pi}{2}$, so:

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin^2\theta}\sqrt{1-\cos^2\theta}}\,2\sin\theta\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{\sin\theta\cos\theta}\,2\sin\theta\cos\theta\,d\theta = \int_0^{\frac{\pi}{2}} 2\,d\theta$$

$$\mathrm{I} = 2[\theta]_0^{\frac{\pi}{2}} = 2\left(\frac{\pi}{2}\right) = \pi$$

95. $I = \int_0^{\frac{\pi}{4}} (\sqrt{\tan x} + \sqrt{\cot x}) dx$, let $t^2 = \tan x$, then $2tdt = \sec^2 x\, dx = (1 + \tan^2 x)dx$

So, $2tdt = (1 + t^4)dx$, for $x = 0$, $t = 0$ and for $x = \frac{\pi}{4}$, $t = 1$, then**:**

$$I = \int_0^1 \left(t + \frac{1}{t}\right) \frac{2t}{1+t^4} dt = 2\int_0^1 \frac{t^2+1}{t^4+1} dt = 2\int_0^1 \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} dt = 2\int_0^1 \frac{1+\frac{1}{t^2}}{\left(t-\frac{1}{t}\right)^2+2} dt$$

$$I = 2\int_0^1 \frac{\left(t-\frac{1}{t}\right)'}{\left(t-\frac{1}{t}\right)^2+\left(\sqrt{2}\right)^2} dt = 2\left[\frac{1}{\sqrt{2}}\arctan\left[\frac{1}{\sqrt{2}}\left(t - \frac{1}{t}\right)\right]\right]_0^1 = \sqrt{2}[\tan^{-1} 0 - \tan^{-1}(-\infty)]$$

$$I = \sqrt{2}\left(0 + \frac{\pi}{2}\right) = \frac{\pi}{\sqrt{2}}$$

96. $I = \int_0^{+\infty} \frac{\ln\left(x+\frac{1}{x}\right)}{1+x^2} dx$, let $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$, then**:**

$$I = \int_0^{\frac{\pi}{2}} \frac{\ln\left(\tan\theta+\frac{1}{\tan\theta}\right)}{1+\tan^2\theta} \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \frac{\ln(\tan\theta+\cot\theta)}{\sec^2\theta} \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \ln(\tan\theta + \cot\theta)\, d\theta$$

$$I = \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\right) d\theta = \int_0^{\frac{\pi}{2}} \ln\left(\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}\right) d\theta = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{\sin\theta\cos\theta}\right) d\theta$$

$$I = -\int_0^{\frac{\pi}{2}} \ln(\sin\theta\cos\theta)\, d\theta = -\left[\int_0^{\frac{\pi}{2}} \ln(\sin\theta)\, d\theta + \int_0^{\frac{\pi}{2}} \ln(\cos\theta)\, d\theta\right]$$

But $\int_0^{\frac{\pi}{2}} \ln(\sin\theta)\, d\theta$ was solved previously in the exercises of this chapter and the result was $\int_0^{\frac{\pi}{2}} \ln(\sin\theta)\, d\theta = -\frac{\pi}{2}\ln 2$, moreover using the property $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ we get $\int_0^{\frac{\pi}{2}} \ln(\cos\theta)\, d\theta = -\frac{\pi}{2}\ln 2$

Therefore $I = -\left[-\frac{\pi}{2}\ln 2 - \frac{\pi}{2}\ln 2\right] = \pi\ln 2$

97. First Method: $I = \int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2} = 2\int_0^{+\infty} \frac{dx}{\left(x^2+1\right)^2} = 2J$, let $u = \frac{1}{x}$, so $x = \frac{1}{u}$ and

$dx = -\frac{1}{u^2} du$, then**:**

$$J = -\int_{+\infty}^0 \frac{1}{\left(\frac{1}{u^2}+1\right)^2} \frac{1}{u^2} du = \int_0^{+\infty} \frac{\frac{1}{u^2}}{\left(\frac{u^2+1}{u^2}\right)^2} du = \int_0^{+\infty} \frac{u^2}{\left(u^2+1\right)^2} du = \int_0^{+\infty} \frac{x^2}{\left(x^2+1\right)^2} dx$$

$$J + J = \int_0^{+\infty} \frac{dx}{\left(x^2+1\right)^2} + \int_0^{+\infty} \frac{x^2}{\left(x^2+1\right)^2} dx = \int_0^{+\infty} \frac{1+x^2}{\left(x^2+1\right)^2} dx = \int_0^{+\infty} \frac{1}{x^2+1} dx$$

$2J = [\arctan x]_0^{+\infty} = \frac{\pi}{2}$, so $J = \frac{\pi}{4}$, therefore $I = 2J = \frac{\pi}{2}$

Second Method: $I = \int_{-\infty}^{+\infty} \frac{dx}{\left(x^2+1\right)^2} = 2\int_0^{+\infty} \frac{dx}{\left(x^2+1\right)^2}$, let $x = \tan t$, then $dx = \sec^2 t dt$

For $x = 0$, $t = 0$ and for $x = +\infty$, $t = \frac{\pi}{2}$, so we get**:**

$\mathrm{I} = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 t}{\left(1+\tan^2 x\right)^2} dt = 2\int_0^{\frac{\pi}{2}} \frac{\sec^2 t}{\left(\sec^2 t\right)^2} dt = 2\int_0^{\frac{\pi}{2}} \frac{1}{\sec^2 t} dt = 2\int_0^{\frac{\pi}{2}} \cos^2 t\, dt$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} (1+\cos 2t) dt = \left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \left[\frac{\pi}{2} + \frac{1}{2}\sin\pi\right] - [0-0] = \frac{\pi}{2}$

98. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x + \cos x} dx$**… (1)**

Let $u = \frac{\pi}{2} - x$, then $du = -dx$, for $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then:

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{\sin^2\left(\frac{\pi}{2}-u\right)}{\sin\left(\frac{\pi}{2}-u\right)+\cos\left(\frac{\pi}{2}-u\right)} (-du) = \int_0^{\frac{\pi}{2}} \frac{\cos^2 u}{\cos u + \sin u} du = \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{\sin x + \cos x} dx$ **… (2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + \cos^2 x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x} dx$

Let $t = \tan\left(\frac{x}{2}\right)$, then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2}{1+t^2} dt$

For $x = 0$, $t = \tan 0 = 0$ and for $x = \frac{\pi}{2}$, $t = \tan\left(\frac{\pi}{4}\right) = 1$, then:

$2\mathrm{I} = \int_0^1 \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \frac{2}{1+t^2} dt = 2\int_0^1 \frac{1}{\left(\sqrt{2}\right)^2 - (t-1)^2} dt = \left[2 \times \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right|\right]_0^1$

$2\mathrm{I} = \frac{1}{\sqrt{2}}\ln|1| - \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right| = -\frac{1}{\sqrt{2}}\ln\left[\frac{1}{\left(\sqrt{2}+1\right)^2}\right] = \frac{2}{\sqrt{2}}\ln(\sqrt{2}+1)$, so $\mathrm{I} = \frac{\ln\left(1+\sqrt{2}\right)}{\sqrt{2}}$

99. $\mathrm{I} = \int_0^{\pi} \frac{(2x+1)\sin^3 x}{\cos^2 x + 1} dx$

Let $u = \pi - x$, then $du = -dx$, for $x = 0$, $u = \pi$ and for $x = \pi$, $u = 0$, so:

$\mathrm{I} = \int_{\pi}^{0} \frac{[2(\pi-u)+1]\sin^3(\pi-u)}{\cos^2(\pi-u)+1} (-du) = \int_0^{\pi} \frac{(2\pi-2u+1)\sin^3 u}{\cos^2 u + 1} du$

$= \int_0^{\pi} \frac{(2\pi+2-2u-1)\sin^3 u}{\cos^2 u + 1} du = 2(\pi+1)\int_0^{\pi} \frac{\sin^3 u}{\cos^2 u + 1} du - \int_0^{\pi} \frac{(2u+1)\sin^3 u}{\cos^2 u + 1} du$

$\mathrm{I} = 2(\pi+1)\int_0^{\pi} \frac{\sin^3 x}{\cos^2 x + 1} dx - \int_0^{\pi} \frac{(2x+1)\sin^3 x}{\cos^2 x + 1} dx$, so

$\mathrm{I} = 2(\pi+1)\int_0^{\pi} \frac{\sin^3 x}{\cos^2 x + 1} dx - \mathrm{I}$, then $2\mathrm{I} = 2(\pi+1)\int_0^{\pi} \frac{\sin^3 x}{\cos^2 x + 1} dx$

$\mathrm{I} = (\pi+1)\int_0^{\pi} \frac{\sin^3 x}{\cos^2 x + 1} dx = (\pi+1)\int_0^{\pi} \frac{\sin^2 x}{\cos^2 x + 1} \sin x\, dx$

$\mathrm{I} = (\pi+1)\int_0^{\pi} \frac{1-\cos^2 x}{\cos^2 x + 1} \sin x\, dx = (\pi+1)\int_0^{\pi} \frac{1-\cos^2 x + 1 - 1}{\cos^2 x + 1} \sin x\, dx$

$\mathrm{I} = (\pi+1)\int_0^{\pi} \frac{\left[-\left(\cos^2 x + 1\right)+2\right]}{\cos^2 x + 1} \sin x\, dx = (\pi+1)\int_0^{\pi} \left(\frac{2\sin x}{\cos^2 x + 1} - \sin x\right) dx$

$\mathrm{I} = (\pi+1)\int_0^{\pi} \left(-2\frac{(\cos x)'}{1+(\cos x)^2} - \sin x\right) dx = (\pi+1)[-2\arctan(\cos x) + \cos x]_0^{\pi}$

$\mathrm{I} = (\pi+1)[-2\arctan(-1) - 1 - (-2\arctan 1 + 1)] = (\pi+1)\left(2\frac{\pi}{4} - 1 + 2\frac{\pi}{4} - 1\right)$

$\mathrm{I} = (\pi+1)(\pi-2)$

100. $\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{1}{(\sec x + \tan x)^2} dx = \int_0^{\frac{\pi}{4}} \frac{1}{(\sec x + \tan x)^2} \times \frac{(\sec x - \tan x)^2}{(\sec x - \tan x)^2} dx$

$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{(\sec x - \tan x)^2}{(\sec x + \tan x)^2 (\sec x - \tan x)^2} dx = \int_0^{\frac{\pi}{4}} \frac{(\sec x - \tan x)^2}{[(\sec x + \tan x)(\sec x - \tan x)]^2} dx$

$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{(\sec x - \tan x)^2}{\sec^2 x - \tan^2 x} dx$, but $\sec^2 x - \tan^2 x = 1$, then we get**:**

$\mathrm{I} = \int_0^{\frac{\pi}{4}} (\sec x - \tan x)^2\, dx = \int_0^{\frac{\pi}{4}} (\sec^2 x - 2\tan x \sec x + \tan^2 x)\, dx$

$\mathrm{I} = \int_0^{\frac{\pi}{4}} (\sec^2 x - 2\tan x \sec x + \sec^2 x - 1)\, dx = \int_0^{\frac{\pi}{4}} (2\sec^2 x - 2\tan x \sec x - 1)\, dx$

$\mathrm{I} = [2\tan x - 2\sec x - x]_0^{\frac{\pi}{4}} = \left(2 - 2\sqrt{2} - \frac{\pi}{4}\right) - (0 - 2 - 0) = 4 - 2\sqrt{2} - \frac{\pi}{4}$

101. $\mathrm{I} = \int_0^{\pi} \frac{\ln(x+\pi)}{x^2+\pi^2} dx = \int_0^{\pi} \frac{\ln\left[\pi\left(\frac{x}{\pi}+1\right)\right]}{x^2+\pi^2} dx = \int_0^{\pi} \frac{\ln \pi + \ln\left(\frac{x}{\pi}+1\right)}{x^2+\pi^2} dx$

$\mathrm{I} = \ln \pi \int_0^{\pi} \frac{1}{x^2+\pi^2} dx + \int_0^{\pi} \frac{\ln\left(\frac{x}{\pi}+1\right)}{x^2+\pi^2} dx = \ln \pi \left[\frac{1}{\pi} \arctan\left(\frac{x}{\pi}\right)\right]_0^{\pi} + \mathrm{J} = \ln \pi \left(\frac{1}{\pi}\frac{\pi}{4}\right) + \mathrm{J}$

$\mathrm{I} = \frac{\ln \pi}{4} + \mathrm{J}$, with $\mathrm{J} = \int_0^{\pi} \frac{\ln\left(\frac{x}{\pi}+1\right)}{x^2+\pi^2} dx$, let $u = \frac{x}{\pi}$, then $dx = \pi du$, for $x = 0$, $u = 0$ and for $x = \pi$, $u = 1$, so $\mathrm{J} = \int_0^1 \frac{\ln(u+1)}{\pi^2 u^2 + \pi^2} \pi du = \int_0^1 \frac{\ln(u+1)}{\pi u^2 + \pi} du = \frac{1}{\pi} \int_0^1 \frac{\ln(u+1)}{u^2+1} du$

$\mathrm{J} = \frac{1}{\pi} \int_0^1 \frac{\ln(x+1)}{x^2+1} dx$, but $\int_0^1 \frac{\ln(x+1)}{x^2+1} dx$ was solved previously in this chapter such that $\int_0^1 \frac{\ln(x+1)}{x^2+1} dx = \frac{\pi}{8} \ln 2$, so $\mathrm{I} = \frac{\ln \pi}{4} + \frac{1}{\pi}\left(\frac{\pi}{8} \ln 2\right) = \frac{\ln \pi}{4} + \frac{\ln 2}{8}$

102. $\mathrm{I} = \int_e^{e^2} \left(\frac{1}{\ln x} + \ln(\ln x)\right) dx = \int_e^{e^2} \frac{1}{\ln x} dx + \int_e^{e^2} \ln(\ln x)\, dx = \int_e^{e^2} \frac{1}{\ln x} dx + \mathrm{J}$

$\mathrm{J} = \int_e^{e^2} \ln(\ln x)\, dx$, let $u = \ln(\ln x)$, then $u' = \frac{1}{x \ln x}$ and let $v' = 1$, so $v = x$, so**:**

$\mathrm{J} = [x \ln(\ln x)]_e^{e^2} - \int_e^{e^2} x \frac{1}{x \ln x} dx = e^2 \ln(\ln e^2) - e \ln(\ln e) - \int_e^{e^2} \frac{1}{\ln x} dx$

$\mathrm{J} = e^2 \ln 2 - e \ln 1 - \int_e^{e^2} \frac{1}{\ln x} dx = e^2 \ln 2 - \int_e^{e^2} \frac{1}{\ln x} dx$, therefore**:**

$\mathrm{I} = \int_e^{e^2} \frac{1}{\ln x} dx + e^2 \ln 2 - \int_e^{e^2} \frac{1}{\ln x} dx = e^2 \ln 2$

103. $\mathrm{I} = \int_0^{\frac{1}{\sqrt{2}}} \frac{\arcsin x}{(1-x^2)^{\frac{3}{2}}} dx$, let $x = \sin\theta$, then $dx = \cos\theta\, d\theta$, for $x = 0$, we get $\theta = 0$ and for $x = \frac{1}{\sqrt{2}}$ we get $\theta = \frac{\pi}{4}$, so $\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{\theta}{(1-\sin^2\theta)^{\frac{3}{2}}} \cos\theta\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\theta}{(\cos^2\theta)^{\frac{3}{2}}} \cos\theta\, d\theta$

$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{\theta}{\cos^3\theta} \cos\theta\, d\theta = \int_0^{\frac{\pi}{4}} \frac{\theta}{\cos^2\theta} d\theta = \int_0^{\frac{\pi}{4}} \theta \sec^2\theta\, d\theta$

Let $u = \theta$, then $u' = 1$ and let $v' = \sec^2\theta$, then $v = \tan\theta$, then we get**:**

$\mathrm{I} = [\theta \tan\theta]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} \tan\theta\, d\theta = \left(\frac{\pi}{4} - 0\right) - [\ln(\cos\theta)]_0^{\frac{\pi}{4}} = \frac{\pi}{4} - \left[\ln\left(\frac{1}{\sqrt{2}}\right) - \ln 1\right]$

$\mathrm{I} = \frac{\pi}{4} - \frac{1}{2}\ln 2$

104. $\mathrm{I} = \int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$

Let $u = \sin\left(\frac{\pi}{4} + \frac{x}{2}\right)$, then $u' = \frac{1}{2}\cos\left(\frac{\pi}{4} + \frac{x}{2}\right)$ and let $v' = e^x$ so $v = e^x$, then:

$\mathrm{I} = \left[e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]_0^{2\pi} - \frac{1}{2}\int_0^{2\pi} e^x \cos\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$

$\mathrm{I} = \left[e^{2\pi}\sin\left(\frac{5\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)\right] - \frac{1}{2}\mathrm{J} = \left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\mathrm{J} = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right) - \frac{1}{2}\mathrm{J}$

Integrating $\mathrm{J} = \int_0^{2\pi} e^x \cos\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$

Let $u = \cos\left(\frac{\pi}{4} + \frac{x}{2}\right)$, then $u' = -\frac{1}{2}\sin\left(\frac{\pi}{4} + \frac{x}{2}\right)$ and let $v' = e^x$ so $v = e^x$, then:

$\mathrm{J} = \left[e^x \cos\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx$, so:

$\mathrm{I} = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left[e^x \cos\left(\frac{\pi}{4} + \frac{x}{2}\right)\right]_0^{2\pi} + \frac{1}{2}\int_0^{2\pi} e^x \sin\left(\frac{\pi}{4} + \frac{x}{2}\right) dx\right]$

$\mathrm{I} = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right) - \frac{1}{2}\left[\left(-\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) + \frac{1}{2}\mathrm{I}\right]$, so $\mathrm{I} = -\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right) + \frac{1}{2}\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right) - \frac{1}{4}\mathrm{I}$, then:

$\mathrm{I} + \frac{1}{4}\mathrm{I} = -\frac{e^{2\pi}+1}{2\sqrt{2}}$, so $,\frac{5}{4}\mathrm{I} = -\frac{e^{2\pi}+1}{2\sqrt{2}}$, therefore $\mathrm{I} = -\frac{\sqrt{2}}{5}(e^{2\pi}+1)$

105. $\mathrm{I} = \int_1^e \left[\left(\frac{x}{e}\right)^{2x} - \left(\frac{e}{x}\right)^x\right] \ln x\, dx = \int_1^e \left(\frac{x}{e}\right)^{2x} \ln x\, dx + \int_1^e \left(\frac{e}{x}\right)^x \ln x\, dx = \mathrm{I}_1 + \mathrm{I}_2$

$\mathrm{I}_1 = \int_1^e \left(\frac{x}{e}\right)^{2x} \ln x\, dx$, let $t = \left(\frac{x}{e}\right)^{2x}$, so $\ln t = 2x\ln\left(\frac{x}{e}\right)$ and

$\left[2\ln\left(\frac{x}{e}\right) + 2x \times \frac{1}{\frac{x}{e}} \times \frac{1}{e}\right] = \frac{1}{t}dt$ after simplifying we get $dt = 2\left(\frac{x}{e}\right)^{2x}\ln x\, dx$, so:

$\mathrm{I}_1 = \frac{1}{2}\int_{\frac{1}{e^2}}^1 dt = \frac{1}{2}[t]_{\frac{1}{e^2}}^1 = \frac{1}{2}\left(1 - \frac{1}{e^2}\right)$

$\mathrm{I}_2 = \int_1^e \left(\frac{e}{x}\right)^x \ln x\, dx$, let $u = \left(\frac{e}{x}\right)^x$, then $\ln u = x\ln\left(\frac{e}{x}\right)$, $\left[\ln\left(\frac{e}{x}\right) + x \times \frac{1}{\frac{e}{x}} \times \frac{-e}{x^2}\right] dx = \frac{1}{u}du$

after simplifying we get $du = -\left(\frac{e}{x}\right)^x \ln x\, dx$, then we get:

$\mathrm{I}_2 = -\int_e^1 du = -(1-e) = e - 1$, so $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2 = \frac{1}{2}\left(1 - \frac{1}{e^2}\right) + e - 1$

106. $\mathrm{I} = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\tan^{-1}(x^2)}{1+x^2} dx$, let $u = \frac{1}{x}$, then $x = \frac{1}{u}$ and $dx = -\frac{1}{u^2}du$

For $x = \sqrt{3}$, $u = \frac{1}{\sqrt{3}}$ and for $x = \frac{1}{\sqrt{3}}$ then $u = \sqrt{3}$, then:

$\mathrm{I} = \int_{\sqrt{3}}^{\frac{1}{\sqrt{3}}} \frac{\tan^{-1}\left(\frac{1}{u^2}\right)}{1+\frac{1}{u^2}}\left(-\frac{1}{u^2}du\right) = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\tan^{-1}\left(\frac{1}{u^2}\right)}{1+u^2} du = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\tan^{-1}\left(\frac{1}{x^2}\right)}{1+x^2} dx$

$\mathrm{I} = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\frac{\pi}{2} - \tan^{-1}(x^2)}{1+x^2} dx = \frac{\pi}{2} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+x^2} dx - \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{\tan^{-1}(x^2)}{1+x^2} dx = \frac{\pi}{2} \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{1+x^2} dx - \mathrm{I}$

$2\mathrm{I} = \frac{\pi}{2} [\arctan x]_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} = \frac{\pi}{2}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi}{2}\left(\frac{\pi}{6}\right) = \frac{\pi^2}{12}$, therefore $\mathrm{I} = \frac{\pi^2}{24}$

107. $\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\pi - 4x)\tan x}{1 - \tan x} dx = 4\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\left(\frac{\pi}{4} - x\right)\tan x}{1 - \tan x} dx$

Let $u = \frac{\pi}{4} - x$, then $du = -dx$, for $x = -\frac{\pi}{4}$, $u = \frac{\pi}{2}$ and for $x = \frac{\pi}{4}$, $u = 0$, then:

$\mathrm{I} = 4\int_{\frac{\pi}{2}}^{0} \frac{u \tan\left(\frac{\pi}{4} - u\right)}{1 - \tan\left(\frac{\pi}{4} - u\right)} (-du) = 4\int_{0}^{\frac{\pi}{2}} \frac{u \frac{1-\tan u}{1+\tan u}}{1 - \frac{1-\tan u}{1+\tan u}} du = 4\int_{0}^{\frac{\pi}{2}} \frac{u(1-\tan u)}{2\tan u} du$

$\mathrm{I} = 2\int_{0}^{\frac{\pi}{2}} u \cot u \, du - 2\int_{0}^{\frac{\pi}{2}} u du = 2[u \ln \sin u]_{0}^{\frac{\pi}{2}} - 2\int_{0}^{\frac{\pi}{2}} \ln \sin u - \frac{\pi^2}{4}$

$\mathrm{I} = -2\int_{0}^{\frac{\pi}{2}} \ln \sin u - \frac{\pi^2}{4} = \pi \ln 2 - \frac{\pi^2}{4} = \pi\left(\ln 2 - \frac{\pi}{4}\right)$

Remark:

$\int_{0}^{\frac{\pi}{2}} \ln(\sin x)\, dx$ was solved previously in this chapter and the answer was $-\frac{\pi}{2}\ln 2$

108. $\mathrm{I} = \int_{1}^{+\infty} \frac{1}{(ax+b)\sqrt{x-1}} dx$, let $u = \sqrt{x-1}$, then $u^2 = x - 1$, so $2udu = dx$

For $x = 1$, $u = 0$ and for $x = +\infty$, $u = +\infty$, then:

$\mathrm{I} = \int_{0}^{+\infty} \frac{2u}{[a(u^2+1)+b]u} du = 2\int_{0}^{+\infty} \frac{1}{au^2 + (a+b)} du$

Let $au^2 = (a+b)\tan^2\theta$, then $u^2 = \left(\frac{a+b}{a}\right)\tan^2\theta$, so $u = \sqrt{\frac{a+b}{a}}\tan\theta$

then $du = \sqrt{\frac{a+b}{a}}\sec^2\theta\, d\theta$, for $u = 0$, $\theta = 0$ and for $u = +\infty$, $u = \frac{\pi}{2}$, so:

$\mathrm{I} = 2\int_{0}^{\frac{\pi}{2}} \frac{\sqrt{\frac{a+b}{a}}\sec^2\theta}{(a+b)\tan^2\theta + (a+b)} d\theta = 2\int_{0}^{\frac{\pi}{2}} \frac{\sqrt{\frac{a+b}{a}}\sec^2\theta}{(a+b)(\tan^2\theta + 1)} d\theta = 2\int_{0}^{\frac{\pi}{2}} \frac{\sqrt{\frac{a+b}{a}}\sec^2\theta}{(a+b)\sec^2\theta} d\theta$

$\mathrm{I} = 2\frac{1}{\sqrt{a(a+b)}}\frac{\pi}{2} = \frac{\pi}{\sqrt{a(a+b)}}$

109. $\mathrm{I} = \int_{0}^{+\infty} \frac{1}{(1+x^a)(1+x^2)} dx$ **… (1)**

Let $x = \frac{1}{t}$, then $dx = -\frac{1}{t^2}dt$, for $x = 0$, $t = +\infty$ and for $x = +\infty$, $t = 0$, then:

$\mathrm{I} = \int_{+\infty}^{0} \frac{1}{\left(1+\frac{1}{t^a}\right)\left(1+\frac{1}{t^2}\right)}\left(-\frac{1}{t^2}dt\right) = \int_{0}^{+\infty} \frac{t^2.t^a}{(1+t^a)(1+t^2)}\frac{dt}{t^2} = \int_{0}^{+\infty} \frac{t^2}{(1+t^a)(1+t^2)} dt$

$\mathrm{I} = \int_{0}^{+\infty} \frac{x^2}{(1+x^a)(1+x^2)} dx$ **… (2)** Adding (1) and (2) we get:

$$2\mathrm{I} = \int_0^{+\infty} \frac{1}{(1+x^a)(1+x^2)}dx + \int_0^{+\infty} \frac{x^2}{(1+x^a)(1+x^2)}dx = \int_0^{+\infty} \frac{1+x^2}{(1+x^a)(1+x^2)}dx$$

$2\mathrm{I} = \int_0^{+\infty} \frac{1}{1+x^2}dx = [\arctan x]_0^{+\infty} = \frac{\pi}{2}$, therefore $\mathrm{I} = \frac{\pi}{4}$

110. $\mathrm{I} = \int_0^{+\infty} \frac{x^{n-1}}{(1+x^n)\left(\pi^2+\ln^2 x\right)}dx = \int_0^{+\infty} \frac{\frac{1}{x}(x^n+1)-\frac{1}{x}}{(1+x^n)\left(\pi^2+\ln^2 x\right)}dx$

$$\mathrm{I} = \int_0^{+\infty} \frac{1}{x\left(\pi^2+\ln^2 x\right)}dx - \int_0^{+\infty} \frac{1}{x(1+x^n)\left(\pi^2+\ln^2 x\right)}dx = \int_0^{+\infty} \frac{1}{x\left(\pi^2+\ln^2 x\right)}dx - \mathrm{J}$$

For J, let $x = \frac{1}{t}$, then $dx = -\frac{1}{t^2}dt$, so $\mathrm{J} = \int_{+\infty}^{0} \frac{1}{\frac{1}{t}\left(1+\left(\frac{1}{t}\right)^n\right)\left(\pi^2+\ln^2\left(\frac{1}{t}\right)\right)}\left(-\frac{1}{t^2}dt\right)$

$$\mathrm{J} = \int_0^{+\infty} \frac{t^n}{t(1+t^n)\left(\pi^2+\ln^2 t\right)}dt = \int_0^{+\infty} \frac{t^{n-1}}{(1+t^n)\left(\pi^2+\ln^2 t\right)}dt = \int_0^{+\infty} \frac{x^{n-1}}{(1+x^n)\left(\pi^2+\ln^2 x\right)}dx$$

So, $\mathrm{J} = \mathrm{I}$, so $\mathrm{I} = \int_0^{+\infty} \frac{1}{x\left(\pi^2+\ln^2 x\right)}dx - \mathrm{I}$, then $2\mathrm{I} = \int_0^{+\infty} \frac{1}{x\left(\pi^2+\ln^2 x\right)}dx$

Let $u = \ln x$, then $du = \frac{1}{x}dx$, for $x = 0$, $u = -\infty$ and for $x = +\infty$, $u = +\infty$, then

$2\mathrm{I} = \int_{-\infty}^{+\infty} \frac{1}{\pi^2+u^2}du = 2\int_0^{+\infty} \frac{1}{\pi^2+u^2}du$, so $\mathrm{I} = \int_0^{+\infty} \frac{1}{\pi^2+u^2}du = \frac{1}{\pi}\left[\arctan\left(\frac{u}{\pi}\right)\right]_0^{+\infty}$

Therefore, $\mathrm{I} = \frac{1}{\pi}\left(\frac{\pi}{2}\right) = \frac{1}{2}$

111. $\mathrm{I} = \int_{-2\pi}^{2\pi} \frac{\pi x^2+\cos^3 x}{1+e^{\pi x^3}}dx = \int_{-2\pi}^{0} \frac{\pi x^2+\cos^3 x}{1+e^{\pi x^3}}dx + \int_0^{2\pi} \frac{\pi x^2+\cos^3 x}{1+e^{\pi x^3}}dx$

For $\int_{-2\pi}^{0} \frac{\pi x^2+\cos^3 x}{1+e^{\pi x^3}}dx$, let $u = -x$, so $du = -dx$, $x = -2\pi$, $u = 2\pi$ and $x = 0$, $u = 0$,

so $\int_{-2\pi}^{0} \frac{\pi x^2+\cos^3 x}{1+e^{\pi x^3}}dx = \int_{2\pi}^{0} \frac{\pi(-u)^2+\cos^3(-x)}{1+e^{\pi(-x)^3}}(-du) = \int_0^{2\pi} \frac{\pi x^2+\cos^3 x}{1+e^{-\pi x^3}}dx$, then

$$\mathrm{I}=\int_0^{2\pi} \frac{\pi x^2+\cos^3 x}{1+e^{-\pi x^3}}dx+\int_0^{2\pi} \frac{\pi x^2+\cos^3 x}{1+e^{\pi x^3}}dx=\int_0^{2\pi} \frac{\left(\pi x^2+\cos^3 x\right)e^{\pi x^3}}{1+e^{\pi x^3}}dx+\int_0^{2\pi} \frac{\pi x^2+\cos^3 x}{1+e^{\pi x^3}}dx$$

$$\mathrm{I} = \int_0^{2\pi} \frac{\left(\pi x^2+\cos^3 x\right)e^{\pi x^3}+x^2+\cos^3 x}{1+e^{\pi x^3}}dx = \int_0^{2\pi} \frac{\left(\pi x^2+\cos^3 x\right)\left(e^{\pi x^3}+1\right)}{1+e^{\pi x^3}}dx$$

$$\mathrm{I} = \int_0^{2\pi}(\pi x^2 + \cos^3 x)dx = \pi\int_0^{2\pi} x^2 dx + \int_{-\pi}^{\pi} \cos^3 x\, dx = \pi\left[\frac{x^3}{3}\right]_0^{2\pi} + 0 = \frac{8\pi^4}{3}$$

112. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin x}{(\sin x+\cos x)^3}dx$ **… (1)**

Let $u = \frac{\pi}{2} - x$, then $du = -dx$, for $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then:

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{\sin\left(\frac{\pi}{2}-u\right)}{\left(\sin\left(\frac{\pi}{2}-u\right)+\cos\left(\frac{\pi}{2}-u\right)\right)^3}(-du) = \int_0^{\frac{\pi}{2}} \frac{\cos u}{(\cos u+\sin u)^3}du = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(\sin x+\cos x)^3}dx$ **… (2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{(\sin x+\cos x)^3} dx = \int_0^{\frac{\pi}{2}} \frac{1}{(\sin x+\cos x)^2} dx$

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x+\cos^2 x+2\sin x\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\sin 2x} dx$

Let $t = \tan x$, then $\sin 2x = \frac{2t}{1+t^2}$ and $dx = \frac{1}{1+t^2} dt$, for $x = 0$, $t = 0$, for $x = \frac{\pi}{2}$, $t = +\infty$, then $2\mathrm{I} = \int_0^{+\infty} \frac{1}{1+\frac{2t}{1+t^2}} \frac{1}{1+t^2} dt = \int_0^{+\infty} \frac{1}{(t+1)^2} dt = \left[-\frac{1}{t+1}\right] = 1$, so $\mathrm{I} = \frac{1}{2}$

113. $\mathrm{I} = \int_0^{\pi} \frac{x}{\tan^2 x-1} dx$, let $t = \pi - x$, then $dt = -dx$, for $x = 0$, $t = \pi$, for $x = \pi$, $t = 0$, so: $\mathrm{I} = \int_{\pi}^{0} \frac{\pi-t}{\tan^2(\pi-t)-1}(-dt) = \int_0^{\pi} \frac{\pi-t}{\tan^2 t-1} dt = \int_0^{\pi} \frac{\pi-x}{\tan^2 x-1} dx$, then

$\mathrm{I} = \pi \int_0^{\pi} \frac{1}{\tan^2 t-1} dt - \int_0^{\pi} \frac{x}{\tan^2 x-1} dx = \pi \int_0^{\pi} \frac{1}{\tan^2 x-1} dx - \mathrm{I}$, so $2\mathrm{I} = \pi \int_0^{\pi} \frac{1}{\tan^2 x-1} dx$

Then $\mathrm{I} = \frac{\pi}{2} \int_0^{\pi} \frac{1}{\sec^2 x-1-1} dx = \frac{\pi}{2} \int_0^{\pi} \frac{1}{\sec^2 x-2} dx = -\frac{\pi}{2} \int_0^{\pi} \frac{\cos^2 x}{2\cos^2 x-1} dx$

$\mathrm{I} = -\frac{\pi}{2} \int_0^{\pi} \frac{\frac{1}{2}(2\cos^2 x-1)+\frac{1}{2}}{2\cos^2 x-1} dx = -\frac{\pi}{2} \int_0^{\pi} \frac{1}{2} dx - \frac{\pi}{4} \int_0^{\pi} \sec 2x\, dx = -\frac{\pi^2}{4} - \frac{\pi}{8} \int_0^{2\pi} \sec x\, dx$

$\mathrm{I} = -\frac{\pi^2}{4} - \frac{\pi}{8} [\ln|\sec u + \tan u|]_0^{2\pi} = -\frac{\pi^2}{4}$

114. $\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec x+\cos x}{1+e^x} dx$ **… (1)**

Let $u = -x$, so $du = -dx$, for $x = -\frac{\pi}{4}$, $u = \frac{\pi}{4}$ and for $x = \frac{\pi}{4}$, $u = -\frac{\pi}{4}$, then**:**

$\mathrm{I} = \int_{\frac{\pi}{4}}^{-\frac{\pi}{4}} \frac{\sec(-u)+\cos(-u)}{1+e^{-u}}(-du) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec u+\cos u}{1+\frac{1}{e^u}} du = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\sec u+\cos u)e^u}{1+e^u} du$

$\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\sec x+\cos x)e^x}{1+e^x} dx$ **… (2)** Adding (1) and (2) we get**:**

$2\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec x+\cos x}{1+e^x} dx + \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\sec x+\cos x)e^x}{1+e^x} dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{(\sec x+\cos x)(1+e^x)}{1+e^x} dx$

$2\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (\sec x + \cos x) dx = [\ln|\sec x + \tan x| + \sin x]_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$

$2\mathrm{I} = \left[\ln(\sqrt{2}+1) + \frac{1}{\sqrt{2}}\right] - \left[\ln(\sqrt{2}-1) - \frac{1}{\sqrt{2}}\right] = \ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \sqrt{2}$, then

$\mathrm{I} = \frac{1}{2}\left[\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right) + \sqrt{2}\right] = \ln\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}} + \frac{1}{\sqrt{2}}$

115. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin(x+\frac{\pi}{4})} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin x\cos(\frac{\pi}{4})+\cos x\sin(\frac{\pi}{4})} dx = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x} dx$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\sin^2 x}{\sin x+\cos x} dx$ **… (1)**

Let $u = \frac{\pi}{2} - x$, then $du = -dx$, for $x = 0$, $u = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = \frac{\pi}{2} - \frac{\pi}{2} = 0$, then**:**

$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{\sqrt{2}\sin^2\left(\frac{\pi}{2}-u\right)}{\sin\left(\frac{\pi}{2}-u\right)+\cos\left(\frac{\pi}{2}-u\right)}(-du) = \int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\cos^2 u}{\cos u+\sin u}du = \int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\cos^2 x}{\cos x+\sin x}dx$ **… (2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sqrt{2}\sin^2 x+\sqrt{2}\cos^2 x}{\sin x+\cos x}dx = \int_0^{\frac{\pi}{2}} \frac{\sqrt{2}}{\cos x+\sin x}dx$

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{\cos x\cos\left(\frac{\pi}{4}\right)+\sin x\sin\left(\frac{\pi}{4}\right)}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\cos\left(x-\frac{\pi}{4}\right)}dx$

Let $t = x - \frac{\pi}{4}$, then $dx = dt$, for $x = 0$, $t = -\frac{\pi}{4}$ and for $x = \frac{\pi}{2}$, $t = \frac{\pi}{4}$, then:

$2\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{\cos t}dt = 2\int_0^{\frac{\pi}{4}} \frac{1}{\cos t}dt = 2\int_0^{\frac{\pi}{4}} \sec t\, dt = 2[\ln(\tan t + \sec t)]_0^{\frac{\pi}{4}}$

Then $\mathrm{I} = \ln\left(\tan\frac{\pi}{4} + \sec\frac{\pi}{4}\right) - \ln(\tan 0 + \sec 0) = \ln\left(1+\sqrt{2}\right) - \ln 1 = \coth^{-1}\left(\sqrt{2}\right)$

116. $\mathrm{I} = \int_{-1}^{1} e^{\sin^{-1}(x)}dx$, let $u = \sin^{-1} x$, then $x = \sin u$ and $dx = \cos u\, du$

For $x = -1$, $u = -\frac{\pi}{2}$ and for $x = 1$, $u = \frac{\pi}{2}$, then:

$\mathrm{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^u \cos u\, du = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^x \cos x\, dx$

Let $\mathrm{J} = \int e^x \cos x$, integration by parts: $\mathrm{J} = e^x \cos x - \int e^x(-\sin x)dx$

$\mathrm{J} = e^x \cos x + \int e^x \sin x\, dx = e^x \sin x + [e^x \sin x - \int e^x \cos x\, dx]$

So, $\mathrm{J} = e^x \cos x + e^x \sin x - \mathrm{J} + k$, so $2\mathrm{J} = e^x(\cos x + \sin x) + k$, so

$\mathrm{J} = \frac{1}{2}e^x(\cos x + \sin x) + c$, then we get: $\mathrm{I} = \left[\frac{1}{2}e^x(\cos x + \sin x)\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$\mathrm{I} = \frac{1}{2}e^{\frac{\pi}{2}}(0+1) - \frac{1}{2}e^{-\frac{\pi}{2}}(0-1) = \frac{1}{2}\left(e^{\frac{\pi}{2}} + e^{-\frac{\pi}{2}}\right) = \cosh\left(\frac{\pi}{2}\right)$

117. $\mathrm{I}_n = \int_{-\pi}^{\pi} \frac{\sin(nx)}{\left(1+2^x\right)\sin x}dx = \int_0^{\pi} \frac{\sin(nx)}{\left(1+2^x\right)\sin x}dx + \int_{-\pi}^{0} \frac{\sin(nx)}{\left(1+2^x\right)\sin x}dx$, where $n \in \mathbb{N}$

For the second integral we make the change of variable $t = -x$, so $dt = -dx$. then we get $\mathrm{I}_n = \int_0^{\pi} \frac{\sin(nx)}{\left(1+2^x\right)\sin x}dx + \int_0^{\pi} \frac{\sin(nt)}{\left(1+2^{-t}\right)\sin t}dt$

$\mathrm{I}_n = \int_0^{\pi} \frac{\sin(nx)}{\left(1+2^x\right)\sin x}dx + \int_0^{\pi} \frac{\sin(nx)}{\left(1+2^{-x}\right)\sin x}dx = \int_0^{\pi} \frac{\sin(nx)}{\left(1+2^x\right)\sin x}dx + \int_0^{\pi} \frac{2^x\sin(nx)}{\left(1+2^x\right)\sin x}dx$

$\mathrm{I}_n = \int_0^{\pi} \frac{\sin(nx)+2^x\sin(nx)}{\left(1+2^x\right)\sin x}dx = \int_0^{\pi} \frac{\left(1+2^x\right)\sin(nx)}{\left(1+2^x\right)\sin x}dx = \int_0^{\pi} \frac{\sin(nx)}{\sin x}dx$, for $n \geq 2$, we have

$\mathrm{I}_n - \mathrm{I}_{n-2} = \int_0^{\pi} \frac{\sin(nx)-\sin((n-2)x)}{\sin x}dx\; 2\int_0^{\pi} \cos\left((n-1)x\right)dx$ with $\mathrm{I}_0 = 0$ and $\mathrm{I}_1 = \pi$

we conclude that $\mathrm{I}_n = \begin{cases} 0 & \text{if} \quad n \text{ is even} \\ \pi & \text{if} \quad n \text{ is odd} \end{cases}$

118. $\mathrm{I} = \int_0^{\pi} \ln|1+\cos x|\, dx$ **… (1)** let $u = \pi - x$, then $du = -dx$, then we get:

$\mathrm{I} = \int_{\pi}^{0} \ln|1+\cos(\pi-u)|\,(-du) = \int_0^{\pi} \ln|1-\cos u|du = \int_0^{\pi} \ln|1-\cos x|dx$ **… (2)**

Adding (1) and (2) we get: $2\mathrm{I} = \int_0^{\pi} \ln|1+\cos x|\, dx + \int_0^{\pi} \ln|1-\cos x|dx$

$2\mathrm{I} = \int_0^{\pi} \ln|(1-\cos x)(1+\cos x)|dx = \int_0^{\pi} \ln|1-\cos^2 x|dx = \int_0^{\pi} \ln\sin^2 x\, dx$

$2\mathrm{I} = \int_0^{\pi} \ln\left(\frac{1-\cos 2x}{2}\right) dx = \int_0^{\pi}[\ln(1-\cos 2x) - \ln 2]dx$

$2\mathrm{I} = \int_0^{\pi} \ln(1-\cos 2x)\,dx - \pi\ln 2$, $2\mathrm{I} = \mathrm{J} - \pi\ln 2$, with $\mathrm{J} = \int_0^{\pi}\ln(1-\cos 2x)\,dx$

Let $t = 2x$, then $dt = 2dx$, then $\mathrm{J} = \frac{1}{2}\int_0^{2\pi}\ln(1-\cos t)\,dt = \frac{1}{2}\int_0^{2\pi}\ln(1-\cos x)\,dx$

then $\mathrm{J} = \frac{1}{2}2\int_0^{\pi}\ln(1-\cos x)\,dx = \int_0^{\pi}\ln(1-\cos x)\,dx = \mathrm{I}$

So, $2\mathrm{I} = \mathrm{I} - \pi\ln 2$, therefore $\mathrm{I} = -\pi\ln 2$

119. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{1+a\csc^2 x}dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+a\frac{1}{\sin^2 x}}dx = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\sin^2 x + a}dx = \int_0^{\frac{\pi}{2}} \frac{\left(\sin^2 x + a\right)-a}{\sin^2 x + a}dx$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin^2 x + a}{\sin^2 x + a}dx - \int_0^{\frac{\pi}{2}} \frac{a}{\sin^2 x + a}dx = \int_0^{\frac{\pi}{2}} dx - a\int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x + a}dx = \frac{\pi}{2} - a\mathrm{J}$, where

$\mathrm{J} = \int_0^{\frac{\pi}{2}} \frac{1}{\sin^2 x + a}dx = \int_0^{\frac{\pi}{2}} \frac{1}{\frac{1}{\csc^2 x}+a}dx = \int_0^{\frac{\pi}{2}} \frac{\csc^2 x}{1+a\csc^2 x}dx = \int_0^{\frac{\pi}{2}} \frac{\csc^2 x}{1+a\left(1+\cot^2 x\right)}dx$

$\mathrm{J} = -\int_0^{\frac{\pi}{2}} \frac{-\csc^2 x}{a\cot^2 x + \left(\sqrt{a+1}\right)^2}dx$, let $u = \cot x$, then $du = -\csc^2 x\,dx$

For $x = 0$, $u = +\infty$ and for $x = \frac{\pi}{2}$, $u = 0$, then we get**:**

$\mathrm{J} = -\int_{+\infty}^{0} \frac{du}{au^2 + \left(\sqrt{a+1}\right)^2} = \int_0^{+\infty} \frac{du}{au^2 + \left(\sqrt{a+1}\right)^2} = \frac{1}{\sqrt{a}\sqrt{a+1}}\left[\arctan\sqrt{\frac{a}{a+1}}u\right]_0^{+\infty}$

$\mathrm{J} = \frac{1}{\sqrt{a}\sqrt{a+1}}\left(\frac{\pi}{2} - 0\right) = \frac{\pi}{2}.\frac{1}{\sqrt{a}\sqrt{a+1}}$, therefore

$\mathrm{I} = \frac{\pi}{2} - a\left(\frac{\pi}{2}.\frac{1}{\sqrt{a}\sqrt{a+1}}\right) = \frac{\pi}{2} - \frac{\pi}{2}\sqrt{\frac{a}{a+1}} = \left(1 - \sqrt{\frac{a}{a+1}}\right)\frac{\pi}{2}$

120. $\mathrm{I} = \int_0^{\pi} \frac{1}{\left(x^2+\pi^2\right)\sqrt{\pi^2 - x^2}}dx$, let $x = \frac{\pi}{u}$, then $dx = -\frac{\pi}{u^2}du$

For $x = 0$, $u = +\infty$ and for $x = \pi$, $u = 1$, then we get**:**

$\mathrm{I} = \int_{+\infty}^{1} \frac{1}{\left(\frac{\pi^2}{u^2}+\pi^2\right)\sqrt{\pi^2 - \frac{\pi^2}{u^2}}}\left(-\frac{\pi}{u^2}du\right) = \int_1^{+\infty} \frac{\pi}{\left(\frac{\pi^2}{u^2}+\pi^2\right)\sqrt{\pi^2 - \frac{\pi^2}{u^2}}}\frac{dy}{u^2}$

$\mathrm{I} = \int_1^{+\infty} \frac{\pi}{\frac{\pi^2}{u^2}\left(u^2+1\right)\sqrt{\frac{\pi^2}{u^2}\left(u^2-1\right)}}\frac{dy}{u^2} = \int_1^{+\infty} \frac{\pi}{\frac{\pi^2}{u^2}\left(u^2+1\right)\frac{\pi}{u}\sqrt{u^2-1}}\frac{dy}{u^2} = \int_1^{+\infty} \frac{u}{\pi^2\left(u^2+1\right)\sqrt{u^2-1}}du$

Let $t = \sqrt{u^2-1}$, so $t^2 + 1 = u^2$, then $2tdt - 2udu$, for $u = 1$, $t = 0$, for $u = +\infty$, $t = +\infty$, so $\mathrm{I} = \int_1^{+\infty} \frac{tdt}{\pi^2\left(t^2+2\right)t} = \int_1^{+\infty} \frac{dt}{\pi^2\left(t^2+2\right)} = \frac{1}{\pi^2}\left[\frac{1}{\sqrt{2}}\arctan\frac{t}{\sqrt{2}}\right]_1^{+\infty}$

$\mathrm{I} = \frac{1}{\pi^2\sqrt{2}}\left(\frac{\pi}{2}\right) = \frac{\pi}{2\sqrt{2}}$

121. $\mathrm{I}=\int_0^{+\infty}\frac{1-\tanh x}{\sqrt[4]{\tanh x}}dx$, let $y=\sqrt[4]{\tanh x}$, then $x=\tanh^{-1}(y^4)$, so:

$dx=\frac{4y^3}{1-y^8}dy$, for $x=+\infty$, $y=1$ and for $x=0$, $y=0$, so we get:

$$\mathrm{I}=\int_0^1\frac{1-y^4}{y}\cdot\frac{4y^3}{1-y^8}dy=4\int_0^1\frac{1-y^4}{y}\cdot\frac{y^3}{(1-y^4)(1+y^4)}dy=4\int_0^1\frac{y^2}{1+y^4}dy$$

$$\mathrm{I}=2\left(\int_0^1\frac{y^2+1}{1+y^4}dy+\int_0^1\frac{y^2-1}{1+y^4}dy\right)=2\left(\int_0^1\frac{1+\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy+\int_0^1\frac{1-\frac{1}{y^2}}{y^2+\frac{1}{y^2}}dy\right)$$

$$\mathrm{I}=2\left(\int_0^1\frac{1+\frac{1}{y^2}}{\left(y-\frac{1}{y}\right)^2+2}dy+\int_0^1\frac{1-\frac{1}{y^2}}{\left(y+\frac{1}{y}\right)^2-2}dy\right)=2\left(\int_0^1\frac{\left(y-\frac{1}{y}\right)'}{\left(y-\frac{1}{y}\right)^2+2}dy+\int_0^1\frac{\left(y+\frac{1}{y}\right)'}{\left(y+\frac{1}{y}\right)^2-2}dy\right)$$

$$\mathrm{I}=2\left[\frac{1}{\sqrt2}\arctan\left(\frac{1}{\sqrt2}\left(y-\frac{1}{y}\right)\right)-\frac{1}{\sqrt2}\tanh^{-1}\left(\frac{1}{\sqrt2}\left(y+\frac{1}{y}\right)\right)\right]_0^1=\frac{2}{\sqrt2}\left(\frac{\pi}{2}-\tanh^{-1}\sqrt2\right)$$

$$\mathrm{I}=\frac{\pi}{\sqrt2}-\frac{1}{\sqrt2}\ln\left(\frac{\sqrt2+1}{\sqrt2-1}\right)=\frac{\pi}{\sqrt2}+\frac{1}{\sqrt2}\ln\left(3-2\sqrt2\right)$$

122. $\mathrm{I}=\int_0^{+\infty}\frac{\ln\left(\frac{1+x^{11}}{1+x^3}\right)}{(1+x^2)\ln x}dx$, let $u=\frac{1}{x}$, then $x=\frac{1}{u}$, so $dx=-\frac{1}{u^2}du$

For $x=0$, $u=+\infty$ and for $x=+\infty$, $u=0$, then:

$$\mathrm{I}=\int_{+\infty}^0\frac{\ln\left(\frac{1+\frac{1}{u^{11}}}{1+\frac{1}{u^3}}\right)}{\left(1+\frac{1}{u^2}\right)\ln\left(\frac{1}{u}\right)}\left(-\frac{1}{u^2}du\right)=-\int_0^{+\infty}\frac{\ln\left[\frac{\frac{1}{u^{11}}(1+y^{11})}{\frac{1}{u^3}(1+u^3)}\right]}{(1+y^2)\ln y}dy=-\int_0^{+\infty}\frac{\ln\left(\frac{1}{u^8}\right)+\ln\left(\frac{1+y^{11}}{1+y^3}\right)}{(1+y^2)\ln y}dy$$

$$\mathrm{I}=-\int_0^{+\infty}\frac{-8\ln y}{(1+y^2)\ln y}dy-\int_0^{+\infty}\frac{\ln\left(\frac{1+y^{11}}{1+y^3}\right)}{(1+y^2)\ln y}dy=8\int_0^{+\infty}\frac{1}{1+x^2}dx-\int_0^{+\infty}\frac{\ln\left(\frac{1+x^{11}}{1+x^3}\right)}{(1+x^2)\ln x}dx$$

$\mathrm{I}=8[\arctan x]_0^{+\infty}-\mathrm{I}$, so $2\mathrm{I}=8\left(\frac{\pi}{2}\right)=4\pi$, therefore $\mathrm{I}=2\pi$

123. $\mathrm{I}=\int_0^1\cos^{-1}\left(\sqrt{1-\sqrt{x}}\right)dx$, let $u=\sqrt{1-\sqrt{x}}$, so $1-u^2=\sqrt{x}$, then

$x=(1-u^2)^2$, so $dx=-4u(1-u^2)du$, for $x=0$, $u=1$ and for $x=1$, $u=0$, so:

$$\mathrm{I}=\int_1^0\cos^{-1}(u)\,.\,(-4u(1-u^2)du)=4\int_0^1(u-u^3)\cos^{-1}u\,du$$

Let $t=\cos^{-1}u$, then $\cos t=u$ and $du=-\sin t\,dt$, for $u=0$, $t=\frac{\pi}{2}$ and for $u=1$, $t=0$, then

$$\mathrm{I}=4\int_{\frac{\pi}{2}}^0(\cos t-\cos^3 t)\,.\,t\,.\,(-\sin t)dt=4\int_0^{\frac{\pi}{2}}t(\cos t-\cos^3 t)\sin t\,dt$$

Let $u=t$, then $u'=1$ and let $v'=(\cos t-\cos^3 t)\sin t$, so $v=-\frac{1}{2}\cos^2 t+\frac{1}{4}\cos^4 t$, then:

$$\mathrm{I}=4\left[t\left(-\frac{1}{2}\cos^2 t+\frac{1}{4}\cos^4 t\right)\right]_0^{\frac{\pi}{2}}+4\int_0^{\frac{\pi}{2}}\left(\frac{1}{2}\cos^2 t-\frac{1}{4}\cos^4 t\right)dt$$

$$\mathrm{I}=0+2\int_0^{\frac{\pi}{2}}\cos^2 t\,dt-\int_0^{\frac{\pi}{2}}\cos^4 t\,dt=\int_0^{\frac{\pi}{2}}(1+\cos 2t)\,dt-\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2t}{2}\right)^2dt=\frac{5\pi}{16}$$

124. $I=\int_0^{\pi}\frac{x^2\sin 2x\sin(\cos x)}{2x-\pi}dx$, let $t=\pi-x$, then $dt=-dx$, then we get:

$$I=\int_{\pi}^{0}\frac{(\pi-t)^2\sin 2(\pi-t)\sin\left(\frac{\pi}{2}\cos(\pi-t)\right)}{2(\pi-t)-\pi}(-dt)=\int_0^{\pi}\frac{(\pi-x)^2\sin 2(\pi-x)\sin\left(\frac{\pi}{2}\cos(\pi-x)\right)}{2(\pi-x)-\pi}dx$$

$$I=\int_0^{\pi}\frac{\left(2\pi x-\pi^2-x^2\right)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

$$I=\int_0^{\pi}\frac{\pi(2x-\pi)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx-\int_0^{\pi}\frac{x^2\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx$$

$I=\int_0^{\pi}\frac{\pi(2x-\pi)\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)}{2x-\pi}dx-\mathrm{I}$, then we get

$2\mathrm{I}=\pi\int_0^{\pi}\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)dx$, then $\mathrm{I}=\frac{\pi}{2}\int_0^{\pi}\sin 2x\sin\left(\frac{\pi}{2}\cos x\right)$

$I=-\pi\int_0^{\pi}-\sin x\cos x\sin\left(\frac{\pi}{2}\cos x\right)dx$, let $u=\cos x$, then $du=-\sin x\,dx$, so:

$$I=-\pi\int_1^{-1}u\sin\left(\frac{\pi}{2}u\right)du=\pi\int_{-1}^{1}u\sin\left(\frac{\pi}{2}u\right)du$$

Let $w=u$, then $w'=1$ and let $y'=\sin\left(\frac{\pi}{2}u\right)$, then $y=-\frac{2}{\pi}\cos\left(\frac{\pi}{2}u\right)$, then:

$$I=\pi\left\{\left[-\frac{2u}{\pi}\cos\left(\frac{\pi}{2}u\right)\right]_{-1}^{1}+\int_{-1}^{1}\frac{2}{\pi}\cos\left(\frac{\pi}{2}u\right)du\right\}=\pi\left[\frac{2}{\pi}\frac{2}{\pi}\sin\left(\frac{\pi}{2}u\right)\right]_{-1}^{1}=\frac{4}{\pi}(1+1)=\frac{8}{\pi}$$

125. $I=\int_0^{+\infty}\frac{1}{x^4+4x^2\cosh(2a)+4}dx$

$$x^4+4x^2\cosh(2x)+4=x^4+4x^2\left(\frac{e^{2a}+e^{-2a}}{2}\right)+4=x^4+2x^2e^{2a}+2x^2e^{-2a}+4$$

$=x^2(x^2+2e^{2a})+2e^{-2a}(x^2+2e^{2a})=(x^2+2e^{-2a})(x^2+2e^{2a})$, then:

$$I=\int_0^{+\infty}\frac{1}{(x^2+2e^{-2a})(x^2+2e^{2a})}dx=\frac{1}{2(e^{2a}-e^{-2a})}\int_0^{+\infty}\frac{(x^2+2e^{2a})-(x^2+2e^{-2a})}{(x^2+2e^{-2a})(x^2+2e^{2a})}dx$$

$$I=\frac{1}{2(e^{2a}-e^{-2a})}\int_0^{+\infty}\frac{1}{x^2+2e^{-2a}}dx-\frac{1}{2(e^{2a}-e^{-2a})}\int_0^{+\infty}\frac{1}{x^2+2e^{2a}}dx$$

$$I=\frac{1}{2(e^{2a}-e^{-2a})}\left[\frac{1}{\sqrt{2}e^{-a}}\arctan\left(\frac{x}{\sqrt{2}e^{-a}}\right)\right]_0^{+\infty}-\frac{1}{2(e^{2a}-e^{-2a})}\left[\frac{1}{\sqrt{2}e^{a}}\arctan\left(\frac{x}{\sqrt{2}e^{a}}\right)\right]_0^{+\infty}$$

$$I=\frac{\pi}{4(e^{2a}-e^{-2a})}\left[\frac{1}{\sqrt{2}e^{-a}}-\frac{1}{\sqrt{2}e^{a}}\right]_0^{+\infty}=\frac{\pi}{4\sqrt{2}(e^{2a}-e^{-2a})}(e^a-e^{-a})$$

$$I=\frac{\pi}{4\sqrt{2}(e^a+e^{-a})(e^a-e^{-a})}(e^a-e^{-a})=\frac{\pi}{4\sqrt{2}(e^a+e^{-a})}=\frac{\pi}{8\sqrt{2}\left(\frac{e^a+e^{-a}}{2}\right)}=\frac{\pi}{8\sqrt{2}\cosh a}$$

126. $I=\int_0^{+\infty}\frac{\ln x}{(x+a)^2+1}dx=\int_0^{+\infty}\frac{\ln x}{x^2+2ax+1+a^2}dx$

Let $x=\frac{1+a^2}{u}$, then $dx=-\frac{1+a^2}{u^2}du$, then we get:

$$I=\int_{+\infty}^{0}\frac{\ln\left(\frac{1+a^2}{u}\right)}{\left(\frac{1+a^2}{u}\right)^2+2a\left(\frac{1+a^2}{u}\right)+1+a^2}\left(-\frac{1+a^2}{u^2}du\right)=\int_0^{+\infty}\frac{\ln(1+a^2)-\ln u}{(1+u^2)+2au+u^2}du$$

$I=\int_0^{+\infty}\frac{\ln(1+a^2)-\ln x}{(1+x^2)+2ax+x^2}dx=\ln(1+a^2)\int_0^{+\infty}\frac{dx}{(x+a)^2+1}dx-\int_0^{+\infty}\frac{\ln x}{(x+a)^2+1}dx$, so

$\mathrm{I} = \ln(1+a^2)\int_0^{+\infty}\frac{dx}{(x+a)^2+1}dx - \mathrm{I}$, then $2\mathrm{I} = \ln(1+a^2)\,[\arctan(x+a)]_0^{+\infty}$

$2\mathrm{I} = \ln(1+a^2)\left(\frac{\pi}{2} - \arctan a\right)$, therefore $\mathrm{I} = \frac{1}{2}\ln(1+a^2)\,\mathrm{arccot}\, a$

127. $\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos 2x}{(1+\sin^2 x)(1+\cos^2 x)}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{\cos^2 x-\sin^2 x}{(1+\sin^2 x)(1+\cos^2 x)}dx$

$\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1+\cos^2 x-\sin^2 x-1}{(1+\sin^2 x)(1+\cos^2 x)}dx = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{(1+\cos^2 x)-(1+\sin^2 x)}{(1+\sin^2 x)(1+\cos^2 x)}dx$

$\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{1+\sin^2 x}dx - \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1}{1+\cos^2 x}dx = 2\left[\int_0^{\frac{\pi}{4}}\frac{1}{1+\sin^2 x}dx - \int_0^{\frac{\pi}{4}}\frac{1}{1+\cos^2 x}dx\right]$

$\mathrm{I} = 2\,(\mathrm{J}-\mathrm{K})$, where $\mathrm{J} = \int_0^{\frac{\pi}{4}}\frac{1}{1+\sin^2 x}dx$ and $\mathrm{K} = \int_0^{\frac{\pi}{4}}\frac{1}{1+\cos^2 x}dx$

$\mathrm{J} = \int_0^{\frac{\pi}{4}}\frac{1}{1+\sin^2 x}dx = \int_0^{\frac{\pi}{4}}\frac{1}{1+\frac{\tan^2 x}{1+\tan^2 x}}dx = \int_0^{\frac{\pi}{4}}\frac{1}{\frac{1+2\tan^2 x}{1+\tan^2 x}}dx = \int_0^{\frac{\pi}{4}}\frac{1+\tan^2 x}{1+2\tan^2 x}dx$

$\mathrm{J} = \int_0^{\frac{\pi}{4}}\frac{(\tan x)'}{1+(\sqrt{2}\tan x)^2}dx = \frac{1}{\sqrt{2}}\left[\arctan(\sqrt{2}\tan x)\right]_0^{\frac{\pi}{4}}$

$\mathrm{K} = \int_0^{\frac{\pi}{4}}\frac{1}{1+\cos^2 x}dx = \int_0^{\frac{\pi}{4}}\frac{1}{1+\frac{1}{1+\tan^2 x}}dx = \int_0^{\frac{\pi}{4}}\frac{1}{\frac{2+\tan^2 x}{1+\tan^2 x}}dx = \int_0^{\frac{\pi}{4}}\frac{1+\tan^2 x}{2+\tan^2 x}dx$

$\mathrm{J} = \int_0^{\frac{\pi}{4}}\frac{(\tan x)'}{(\sqrt{2})^2+(\tan x)^2}dx = \frac{1}{\sqrt{2}}\left[\arctan\left(\frac{\tan x}{\sqrt{2}}\right)\right]_0^{\frac{\pi}{4}}$, then we get:

$\mathrm{I} = \frac{2}{\sqrt{2}}\left[\arctan(\sqrt{2}\tan x) - \arctan\left(\frac{\tan x}{\sqrt{2}}\right)\right]_0^{\frac{\pi}{4}} = \frac{2}{\sqrt{2}}\left[\arctan(\sqrt{2}) - \arctan\left(\frac{1}{\sqrt{2}}\right)\right]$

Using $\arctan x - \arctan y = \arctan\left(\frac{x-y}{1+xy}\right)$, we get:

$\arctan(\sqrt{2}) - \arctan\left(\frac{1}{\sqrt{2}}\right) = \arctan\left[\frac{\sqrt{2}-\frac{1}{\sqrt{2}}}{1+\sqrt{2}\left(\frac{1}{\sqrt{2}}\right)}\right] = \arctan\left(\frac{1}{2\sqrt{2}}\right) = \mathrm{arccot}(2\sqrt{2})$

Therefore we get: $\mathrm{I} = \frac{2}{\sqrt{2}}\mathrm{arccot}(2\sqrt{2}) = \sqrt{2}\,\mathrm{arccot}(2\sqrt{2})$

128. $\mathrm{I} = \int_{-\infty}^{+\infty}\frac{dx}{(1+x^2)\sqrt{4+3x^2}} = 2\int_0^{+\infty}\frac{dx}{(1+x^2)\sqrt{4+3x^2}}$

Let $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$; for $x = 0 \Rightarrow t = +\infty$ & for $x = +\infty \Rightarrow u = 0$; then:

$$\mathrm{I} = 2\int_0^{+\infty}\frac{dx}{(1+x^2)\sqrt{4+3x^2}} = 2\int_{+\infty}^{0}\frac{-\frac{1}{t^2}}{\left(1+\frac{1}{t^2}\right)\sqrt{4+\frac{3}{t^2}}}dt = 2\int_0^{+\infty}\frac{1}{(1+t^2)\sqrt{4+\frac{3}{t^2}}}dt$$

$$\mathrm{I} = 2\int_0^{+\infty} \frac{1}{(1+t^2)\sqrt{\frac{1}{t^2}(4t^2+3)}}dt = \int_0^{+\infty} \frac{2t}{(1+t^2)\sqrt{4t^2+3}}dt$$

Let $u^2 = 4t^2+3 \Longrightarrow 2udu = 8tdt \Longrightarrow 2tdt = \frac{1}{2}udu$; for $t = 0 \Longrightarrow u = \sqrt{3}$ & for $\mathrm{t} = +\infty \Longrightarrow u = +\infty$; then we get:

$$\mathrm{I} = \int_{\sqrt{3}}^{+\infty} \frac{\frac{1}{2}udu}{\left(1+\frac{u^2-3}{4}\right).u} = 2\int_{\sqrt{3}}^{+\infty} \frac{du}{1+u^2} = 2[\arctan u]_{\sqrt{3}}^{+\infty} = 2\left(\frac{\pi}{2}-\frac{\pi}{3}\right) = 2\left(\frac{\pi}{6}\right) = \frac{\pi}{3}$$

129. $\mathrm{I} = \int_0^{+\infty} \frac{x\arctan x}{x^4+x^2+1}dx$ **… (1)**

Let $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$, for $x = 0$, $t = +\infty$ and for $x = +\infty$, $t = 0$, then**:**

$$\mathrm{I} = -\int_{+\infty}^{0} \frac{\frac{1}{t}\arctan\left(\frac{1}{t}\right)}{\frac{1}{t^4}+\frac{1}{t^2}+1}\frac{1}{t^2}dt = \int_0^{+\infty} \frac{t\arctan\left(\frac{1}{t}\right)}{t^4+t^2+1}dt = \int_0^{+\infty} \frac{x\arctan\left(\frac{1}{x}\right)}{x^4+x^2+1}dx \;\ldots\; \textbf{(2)}$$

Adding (1) and (2) we get**:**

$$2\mathrm{I} = \int_0^{+\infty} \frac{x\arctan x}{x^4+x^2+1}dx + \int_0^{+\infty} \frac{x\arctan\left(\frac{1}{x}\right)}{x^4+x^2+1}dx = \int_0^{+\infty} \frac{x}{x^4+x^2+1}\left[\arctan x + \arctan\left(\frac{1}{x}\right)\right]dx$$

But $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$, then we get**:**

$\mathrm{I} = \frac{\pi}{2}\int_0^{+\infty} \frac{x}{x^4+x^2+1}dx = \frac{\pi}{4}\int_0^{+\infty} \frac{2x}{x^4+x^2+1}dx$, let $u = x^2$, so $du = 2xdx$, then**:**

$$2\mathrm{I} = \frac{\pi}{4}\int_0^{+\infty} \frac{1}{u^2+u+1}du = \frac{\pi}{4}\int_0^{+\infty} \frac{1}{\left(u+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}du = \frac{\pi}{4}.\frac{2}{\sqrt{3}}\left[\arctan\left(\frac{2}{\sqrt{3}}\left(u+\frac{1}{2}\right)\right)\right]_0^{+\infty}$$

$2\mathrm{I} = \frac{\pi}{2\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) = \frac{\pi}{2\sqrt{3}}\left(\frac{\pi}{3}\right) = \frac{\pi^2}{6\sqrt{3}}$, therefore $\mathrm{I} = \frac{\pi^2}{12\sqrt{3}}$

130. $\mathrm{I} = \int_0^{+\infty} \frac{\ln x}{(x+a)(x+b)}dx$

Let $u = \frac{ab}{x}$, then $x = \frac{ab}{u}$, so $dx = -\frac{ab}{u^2}du$, for $x = 0$, $u = +\infty$ and for $x = +\infty$, $u = 0$

So $\mathrm{I} = \int_{+\infty}^{0} \frac{\ln\left(\frac{ab}{u}\right)}{\left(\frac{ab}{u}+a\right)\left(\frac{ab}{u}+b\right)}\left(-\frac{ab}{u^2}du\right) = \int_0^{+\infty} \frac{\ln(ab)-\ln u}{(u+a)(u+b)\frac{ab}{u^2}}.\frac{ab}{u^2}du = \int_0^{+\infty} \frac{\ln(ab)-\ln u}{(u+a)(u+b)}du$

$\mathrm{I} = \int_0^{+\infty} \frac{\ln(ab)-\ln x}{(x+a)(x+b)}dx = \int_0^{+\infty} \frac{\ln(ab)}{(x+a)(x+b)}dx - \int_0^{+\infty} \frac{\ln(ab)-\ln x}{(x+a)(x+b)}dx$, then

$\mathrm{I} = \ln(ab)\int_0^{+\infty} \frac{dx}{(x+a)(x+b)} - \mathrm{I}$, then $2\mathrm{I} = \ln(ab)\int_0^{+\infty} \frac{dx}{(x+a)(x+b)}$

$$2\mathrm{I} = \frac{\ln(ab)}{a-b}\int_0^{+\infty} \frac{(x+a)-(x+b)}{(x+a)(x+b)}dx = \frac{\ln(ab)}{a-b}\int_0^{+\infty} \left(\frac{1}{x+b}-\frac{1}{x+a}\right)dx$$

$$2\mathrm{I}=\frac{\ln(ab)}{a-b}\left[\ln|x+b|-\ln|x+a|\right]_0^{+\infty}=\frac{\ln(ab)}{a-b}\left[\ln\left|\frac{x+b}{x+a}\right|\right]_0^{+\infty}=\frac{\ln(ab)}{a-b}\left[-\ln\left(\frac{b}{a}\right)\right]$$

$$2\mathrm{I}=\frac{\ln a+\ln b}{a-b}(\ln a-\ln a)=\frac{\ln^2 a-\ln^2 b}{a-b}, \text{ therefore } \mathrm{I}=\frac{\ln^2 a-\ln^2 b}{2(a-b)}$$

131. $\mathrm{I}=\int_{-\infty}^{+\infty}\frac{x}{a^2e^x-b^2e^{-x}}dx=\int_{-\infty}^{+\infty}\frac{xe^x}{a^2e^{2x}-b^2}dx$

Let $u=e^x\Longrightarrow x=\ln u\Longrightarrow dx=\frac{du}{u}$; for $x=-\infty; u=0$ & for $x=+\infty; u=+\infty$; so:

$$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{xe^x}{a^2e^{2x}-b^2}dx=\int_0^{+\infty}\frac{u\ln u}{a^2u^2-b^2}.\frac{1}{u}du=\int_0^{+\infty}\frac{\ln u}{a^2u^2-b^2}du;\ \text{ then we get:}$$

$$\mathrm{I}=\int_0^{+\infty}\frac{\ln u}{(au+b)(au-b)}du=\frac{1}{a^2}\int_0^{+\infty}\frac{\ln u}{\left(u+\frac{b}{a}\right)\left(u-\frac{b}{a}\right)}du$$

But $\displaystyle\int_0^{+\infty}\frac{\ln x}{(x+a)(x+b)}dx=\frac{\ln^2 a-\ln^2 b}{2(a-b)}$; was solved before; then we get:

$$\mathrm{I}=\frac{1}{a^2}\left[\frac{\ln^2\left(\frac{b}{a}\right)-\ln^2\left(-\frac{b}{a}\right)}{2\left(\frac{b}{a}-\frac{-b}{a}\right)}\right]=\frac{1}{a^2}.\frac{a}{4b}\left\{\ln^2\left(\frac{b}{a}\right)-\left[\ln\left(-1.\frac{b}{a}\right)\right]^2\right\}$$

$$\mathrm{I}=\frac{1}{4ab}.\left\{\ln^2\left(\frac{b}{a}\right)-\left[i\pi+\ln\left(\frac{b}{a}\right)\right]^2\right\}=\frac{1}{4ab}.\left\{\ln^2\left(\frac{b}{a}\right)-\left[-\pi^2+2\pi\ln\left(\frac{b}{a}\right)i+\ln^2\left(\frac{b}{a}\right)\right]\right\}$$

$$\mathrm{I}=\frac{1}{4ab}\left\{\ln^2\left(\frac{b}{a}\right)+\pi^2-\ln^2\left(\frac{b}{a}\right)-i2\pi\ln\left(\frac{b}{a}\right)\right\};\ \text{ considering the real part we get:}$$

$$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{x}{a^2e^x-b^2e^{-x}}dx=\frac{\pi^2}{4ab}$$

132. $\mathrm{I}=\int_0^{\ln 2}\frac{x}{e^x+2e^{-x}-2}dx=\int_0^{\ln 2}\frac{xe^x}{e^{2x}-2e^x+2}dx=\int_0^{\ln 2}\frac{xe^x}{(e^x-1)^2+1}dx$

Let $u=e^x-1\Longrightarrow x=\ln(1+u)\Longrightarrow dx=\frac{1}{1+u}du$; for $x=0\Longrightarrow u=0$ & for $x=\ln 2\Longrightarrow u=1$; then we get:

$$\mathrm{I}=\int_0^{\ln 2}\frac{xe^x}{(e^x-1)^2+1}dx=\int_0^1\frac{(1+u)\ln(1+u)}{u^2+1}.\frac{du}{1+u}=\int_0^1\frac{\ln(1+u)}{u^2+1}du$$

let $u=\tan\theta\Longrightarrow du=\sec^2\theta\,d\theta$; for $u=0\Longrightarrow\theta=0$ & for $u=1\Longrightarrow\theta=\frac{\pi}{4}$; then:

$$\mathrm{I}=\int_0^1\frac{\ln(1+u)}{u^2+1}du=\int_0^{\frac{\pi}{4}}\frac{\ln(1+\tan\theta)}{\tan^2\theta+1}.\sec^2\theta\,d\theta=\int_0^{\frac{\pi}{4}}\frac{\ln(1+\tan\theta)}{\sec^2\theta}.\sec^2\theta\,d\theta;\ \text{ then we}$$

get: $\displaystyle\mathrm{I}=\int_0^{\frac{\pi}{4}}\ln(1+\tan\theta)\,d\theta$

Let $y = \frac{\pi}{4} - \theta \Longrightarrow dy = -d\theta$; for $\theta = 0 \Longrightarrow y = \frac{\pi}{4}$ & for $x = \frac{\pi}{4} \Longrightarrow y = 0$; then we get:

$$I = \int_{\frac{\pi}{4}}^{0} \ln\left(1 + \tan\left(\frac{\pi}{4} - y\right)\right)(-dy) = \int_{0}^{\frac{\pi}{4}} \ln\left(1 + \tan\left(\frac{\pi}{4} - y\right)\right) dy\,;\ \text{ then we get:}$$

$$I = \int_{0}^{\frac{\pi}{4}} \ln\left(1 + \frac{\tan\frac{\pi}{4} - \tan y}{1 + \tan\frac{\pi}{4}\tan y}\right) dy = \int_{0}^{\frac{\pi}{4}} \ln\left(1 + \frac{1 - \tan y}{1 + \tan y}\right) dy$$

$$I = \int_{0}^{\frac{\pi}{4}} \ln\left(\frac{1 + \tan y + 1 - \tan y}{1 + \tan y}\right) dy = \int_{0}^{\frac{\pi}{4}} \ln\left(\frac{2}{1 + \tan y}\right) dy\,;\ \text{ then we get:}$$

$$I = \int_{0}^{\frac{\pi}{4}} \ln 2\, dy - \int_{0}^{\frac{\pi}{4}} \ln(1 + \tan y)\, dy = \frac{\pi}{4}\ln 2 - \int_{0}^{\frac{\pi}{4}} \ln(1 + \tan\theta)\, d\theta \Longrightarrow I = \frac{\pi}{4}\ln 2 - I$$

Then: $2I = \frac{\pi}{4}\ln 2$; therefore; we get: $I = \frac{\pi}{8}\ln 2$

133. $I = \int_0^\pi \frac{\pi - x}{\sin x \cos\theta + 1} dx$ **… (1)**

Let $t = \pi - x \Longrightarrow dt = -dx$, for $x = 0 \Longrightarrow t = \pi$ & for $x = \pi \Longrightarrow t = 0$, then we get**:**

$$I = \int_{0}^{\pi} \frac{\pi - x}{\sin x \cos\theta + 1} dx = \int_{\pi}^{0} \frac{t}{\sin(\pi - t)\cos\theta + 1}(-dt) = \int_{0}^{\pi} \frac{t}{\sin t \cos\theta + 1} dt \ \text{ and so we can}$$

write $I = \int_{0}^{\pi} \frac{x}{\sin x \cos\theta + 1} dx$ … (2); now adding (1) & (2) we get:

$$2I = \int_{0}^{\pi} \frac{\pi - x}{\sin x \cos\theta + 1} dx + \int_{0}^{\pi} \frac{x}{\sin x \cos\theta + 1} dx = \int_{0}^{\pi} \frac{\pi}{\sin x \cos\theta + 1} dx\,;\text{then we get:}$$

$$I = \frac{\pi}{2}\int_{0}^{\pi} \frac{1}{\sin x \cos\theta + 1} dx;\ \text{ using Weierstrass subtitution:}$$

Let $t = \tan\frac{x}{2} \Longrightarrow dx = \frac{2}{1 + t^2}dt$ & $\sin x = \frac{2t}{1 + t^2}$; new bounds are 0 & $+\infty$; *then*:

$$I = \frac{\pi}{2}\int_{0}^{+\infty} \frac{1}{\frac{2t}{1 + t^2}\cos\theta + 1} \cdot \frac{2}{1 + t^2} dt = \frac{\pi}{2}\int_{0}^{+\infty} \frac{2}{2t\cos\theta + 1 + t^2} dt;\ \text{ then we get:}$$

$$I = \frac{\pi}{2}\int_{0}^{+\infty} \frac{2}{(t^2 + 2t\cos\theta + \cos^2\theta) + \sin^2\theta} dt = \pi\int_{0}^{+\infty} \frac{1}{(t + \cos\theta)^2 + \sin^2\theta} dt$$

Let $u = t + \cos\theta \Longrightarrow du = dt$; for $t = 0; u = \cos\theta$ & for $t = +\infty$; $u = +\infty$; then:

$$I = \pi\int_{\cos\theta}^{+\infty} \frac{1}{u^2 + \sin^2\theta} dt = \pi.\frac{1}{\sin\theta}\left[\tan^{-1}\left(\frac{u}{\sin\theta}\right)\right]_{\cos\theta}^{+\infty} = \frac{\pi}{\sin\theta}\left[\frac{\pi}{2} - \tan^{-1}(\cot\theta)\right]$$

$$\mathrm{I} = \frac{\pi}{\sin\theta}[\tan^{-1}(\tan\theta)] = \frac{\pi}{\sin\theta}.\theta = \frac{\pi\theta}{\sin\theta}$$

134. $\mathrm{I} = \int_0^{+\infty} \frac{dx}{(1+x)\left(\pi^2+\ln^4 x\right)}$ **… (1)**

Let $x = \frac{1}{t} \Longrightarrow dx = -\frac{1}{t^2}dt$; for $x = 0 \Longrightarrow t = +\infty$ & for $x = +\infty \Longrightarrow t = 0$; then we get

$$\mathrm{I} = \int_0^{+\infty} \frac{dx}{(1+x)(\pi^2+\ln^4 x)} = \int_{+\infty}^{0} \frac{-\frac{1}{t^2}}{\left(1+\frac{1}{t}\right)\left(\pi^2+\ln^4\left(\frac{1}{t}\right)\right)}dt = \int_0^{+\infty} \frac{\frac{1}{t^2}}{\left(1+\frac{1}{t}\right)(\pi^2+\ln^4 t)}dt$$

$$\mathrm{I} = \int_0^{+\infty} \frac{1}{t^2\left(1+\frac{1}{t}\right)(\pi^2+\ln^4 t)}dt = \int_0^{+\infty} \frac{1}{t(1+t)(\pi^2+\ln^4 t)}dt;$$ then we get:

$$\mathrm{I} = \int_0^{+\infty} \frac{1}{x(1+x)(\pi^2+\ln^4 x)}dx \quad \ldots(2);$$ now adding (1) & (2) we get:

$$2\mathrm{I} = \int_0^{+\infty} \frac{dx}{(1+x)(\pi^2+\ln^4 x)} + \int_0^{+\infty} \frac{1}{x(1+x)(\pi^2+\ln^4 x)}dx = \int_0^{+\infty} \frac{1+x}{x(1+x)(\pi^2+\ln^4 x)}dx$$

Then; $2\mathrm{I} = \int_0^{+\infty} \frac{1}{x(\pi^2+\ln^4 x)}dx = \int_0^{+\infty} \frac{1}{x}.\frac{1}{\pi^2+\ln^4 x}dx$

Let $u = \ln x \Longrightarrow du = \frac{1}{x}dx$; for $x = 0 \Longrightarrow u = -\infty$ & for $x = +\infty \Longrightarrow u = +\infty$; then:

$$2\mathrm{I} = \int_{-\infty}^{+\infty} \frac{1}{\pi^2+u^4}du = 2\int_0^{+\infty} \frac{1}{\pi^2+u^4}du \Longrightarrow \mathrm{I} = \int_0^{+\infty} \frac{1}{\pi^2+u^4}du;$$ then we get:

$$\mathrm{I} = \frac{1}{2\pi}\int_0^{+\infty} \frac{(u^2+\pi)-(u^2-\pi)}{\pi^2+u^4}du = \frac{1}{2\pi}\int_0^{+\infty} \frac{u^2+\pi}{\pi^2+u^4}du - \frac{1}{2\pi}\int_0^{+\infty} \frac{u^2-\pi}{\pi^2+u^4}du$$

$$\mathrm{I} = \frac{1}{2\pi}\int_0^{+\infty} \frac{1+\pi u^{-2}}{u^2+\pi^2 u^{-2}}du - \frac{1}{2\pi}\int_0^{+\infty} \frac{1-\pi u^{-2}}{u^2+\pi^2 u^{-2}}du;$$ then we get:

$$\mathrm{I} = \frac{1}{2\pi}\int_0^{+\infty} \frac{1+\pi u^{-2}}{(u-\pi u^{-1})^2+2\pi}du - \frac{1}{2\pi}\int_0^{+\infty} \frac{1-\pi u^{-2}}{(u+\pi u^{-1})^2-2\pi}du$$

Let $p = u - \pi u^{-1} \Longrightarrow dp = (1+\pi u^{-2})du$ & let $q = u + \pi u^{-1} \Longrightarrow dq = (1-\pi u^{-2})du$

$$\mathrm{I} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dp}{p^2+2\pi} - \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dq}{q^2+2\pi};$$ with $\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dq}{q^2+2\pi} = 0$; then we get:

$$\mathrm{I} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{dp}{p^2+\left(\sqrt{2\pi}\right)^2} = \frac{1}{\pi}\int_0^{+\infty} \frac{dp}{p^2+\left(\sqrt{2\pi}\right)^2} = \frac{1}{\pi}.\frac{1}{\sqrt{2\pi}}\left[\tan^{-1}\left(\frac{p}{\sqrt{2\pi}}\right)\right]_0^{+\infty}$$

$$\mathrm{I} = \frac{1}{\pi}.\frac{1}{\sqrt{2\pi}}\left[\frac{\pi}{2}-0\right] = \frac{1}{2\sqrt{2\pi}} = \frac{1}{\sqrt{8\pi}}$$

135. $\mathrm{I}=\int_0^{+\infty}\frac{\sqrt[\varphi]{x}\arctan x}{(1+x^{\varphi})^2}dx$ **… (1)**

Let $x=\frac{1}{t}$, then $dx=-\frac{1}{t^2}dt$, for $x=0$, $t=+\infty$ and for $x=+\infty$, $t=0$, then**:**

$$\mathrm{I}=\int_{+\infty}^{0}\frac{\left(\frac{1}{t}\right)^{\frac{1}{\varphi}}\arctan\left(\frac{1}{t}\right)}{\left(1+\left(\frac{1}{t}\right)^{\varphi}\right)^2}\left(-\frac{1}{t^2}dt\right)=\int_0^{+\infty}\frac{t^{-\frac{1}{\varphi}}t^{-2}t^{2\varphi}\arctan\left(\frac{1}{t}\right)}{(1+t^{\varphi})^2}dt;\ \text{ then we get:}$$

$$\mathrm{I}=\int_0^{+\infty}\frac{t^{2\varphi-2-\frac{1}{\varphi}}\arctan\left(\frac{1}{\mathrm{t}}\right)}{(1+t^{\varphi})^2}dt=\int_0^{+\infty}\frac{x^{2\varphi-2-\frac{1}{\varphi}}\arctan\left(\frac{1}{x}\right)}{(1+x^{\varphi})^2}dx\ \dots(2)$$

Adding (1) and (2) we get**:**

$$2\mathrm{I}=\int_0^{+\infty}\frac{\sqrt[\varphi]{x}\arctan x}{(1+x^{\varphi})^2}dx+\int_0^{+\infty}\frac{x^{2\varphi-2-\frac{1}{\varphi}}\arctan\left(\frac{1}{x}\right)}{(1+x^{\varphi})^2}d$$

$$2\mathrm{I}=\int_0^{+\infty}\frac{\sqrt[\varphi]{x}\arctan x+x^{2\varphi-2-\frac{1}{\varphi}}\arctan\left(\frac{1}{x}\right)}{(1+x^{\varphi})^2}dx$$

$$2\mathrm{I}=\int_0^{+\infty}\frac{x^{\frac{1}{\varphi}}}{(1+x^{\varphi})^2}\left[\arctan x+\left(x^{2\varphi-2-\frac{2}{\varphi}}\right)\arctan\left(\frac{1}{x}\right)\right]dx$$

$$2\mathrm{I}=\int_0^{+\infty}\frac{x^{\frac{1}{\varphi}}}{(1+x^{\varphi})^2}\left[\arctan x+\left(x^{2\left(\varphi-1-\frac{1}{\varphi}\right)}\right)\arctan\left(\frac{1}{x}\right)\right]dx$$

But $\varphi-\frac{1}{\varphi}=1$ (φ golden ratio), so $x^{2\left(\varphi-1-\frac{1}{\varphi}\right)}=x^{2(1-1)}=x^0=1$, then we get**:**

$2\mathrm{I}=\int_0^{+\infty}\frac{x^{\frac{1}{\varphi}}}{(1+x^{\varphi})^2}\left[\arctan x+\arctan\left(\frac{1}{x}\right)\right]dx$, with $\arctan x+\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}$

So, we get**:** $2\mathrm{I}=\frac{\pi}{2}\int_0^{+\infty}\frac{x^{\frac{1}{\varphi}}}{(1+x^{\varphi})^2}dx$

Let $u=1+x^{\varphi}$, then $\varphi x^{\varphi-1}dx=du$, so $\varphi x^{\frac{1}{\varphi}}dx=du$, then we get**:**

$2\mathrm{I}=\frac{\pi}{2\varphi}\int_1^{+\infty}\frac{du}{u^2}du=\frac{\pi}{2\varphi}\left[-\frac{1}{u}\right]_1^{+\infty}=\frac{\pi}{2\varphi}$, therefore $\mathrm{I}=\frac{\pi}{4\varphi}$

136. $\mathrm{I}=\int_0^1\ln^{2k}\left(\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}\right)dx$

Let $x=\sin\theta$, then for $x=0,\theta=0$ and for $x=1,\theta=\frac{\pi}{2}$; $dx=\cos\theta\,d\theta$

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) = \ln\left(\frac{1-\sqrt{1-\sin\theta^2}}{\sin\theta}\right) = \ln\left(\frac{1-\sqrt{\cos\theta^2}}{\sin\theta}\right) \text{ with } 0<\theta<\frac{\pi}{2}; \text{then}$$

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) = \ln\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

From the double angle formulas we have**:**

$$\cos\theta = 1-2\sin^2\frac{\theta}{2}; \text{so } 1-\cos\theta = 2\sin^2\frac{\theta}{2} \text{ and } \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}; \text{ then we get}$$

$$\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right) = \ln\left(\frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right) = \ln\left(\frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) = \ln\left(\tan\frac{\theta}{2}\right)$$

Similarly it can be shown that: $\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right) = \ln\left(\cot\frac{\theta}{2}\right)$

So; we get: $$\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)} = \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\cot\frac{\theta}{2}\right)} = \frac{\ln\left(\tan\frac{\theta}{2}\right)}{\ln\left(\frac{1}{\tan\frac{\theta}{2}}\right)} = \frac{\ln\left(\tan\frac{\theta}{2}\right)}{-\ln\left(\tan\frac{\theta}{2}\right)} = -1$$

Consequently: $$\mathrm{I} = \int_0^1 \ln^{2k}\left(\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}\right)dx = \int_0^{\frac{\pi}{2}} \ln^{2k}(-1)\times\cos\theta\,d\theta$$

Now we will work on the complex domain: $\ln z = \ln|z| + i\arg z$, so $\ln(-1) = \ln|-1| + i\pi$, then

$$\ln^{2k}(-1) = (i\pi)^{2k} = (i)^{2l}\times\pi^{2k} = (i^2)^k\times\pi^{2k} = (-1)^k\pi^{2k}$$

$$\text{So } \mathrm{I} = \int_0^{\frac{\pi}{2}}(-1)^k\pi^{2k}\times\cos\theta\,d\theta = (-1)^k\pi^{2k}\int_0^{\frac{\pi}{2}}\cos\theta\,d\theta = (-1)^k\pi^{2k}[\sin\theta]_0^{\frac{\pi}{2}}$$

$$\mathrm{I} = (-1)^k\pi^{2k}\left[\sin\frac{\pi}{2}-\sin 0\right] = (-1)^k\pi^{2k}(1-0) = (-1)^k\pi^{2k}$$

$$\textbf{Therefore } \mathbf{I} = \int_0^1 \ln^{2k}\left(\frac{\ln\left(\frac{1-\sqrt{1-x^2}}{x}\right)}{\ln\left(\frac{1+\sqrt{1-x^2}}{x}\right)}\right)dx = (-1)^k\pi^{2k}$$

137. Let $u = x-\pi \Rightarrow du = dx$, for $x=0$, $u = 0-\pi = -\pi$ and for $x = 2\pi$, $u = 2\pi-\pi = \pi$, so**:**

$$\mathrm{I} = \int_{-\pi}^{\pi}\frac{\pi+u}{\phi-\cos^2(\pi+u)}du = \int_{-\pi}^{\pi}\frac{\pi+u}{\phi-(-\cos u)^2}du = \int_{-\pi}^{\pi}\frac{\pi+u}{\phi-\cos^2 u}du$$

$$\mathrm{I} = \int_{-\pi}^{\pi}\frac{\pi}{\phi-\cos^2 u}du + \int_{-\pi}^{\pi}\frac{u}{\phi-\cos^2 u}du\text{; but } \frac{u}{\phi-\cos^2 u} \text{ is an odd function so for } u\in[-\pi;\pi]$$

we have $\int_{-\pi}^{\pi} \frac{u}{\phi - \cos^2 u} du$; then I $= \int_{-\pi}^{\pi} \frac{\pi}{\phi - \cos^2 u} du + 0 = \int_{-\pi}^{\pi} \frac{\pi}{\phi - \cos^2 u} du$

$$= \int_{-\pi}^{\pi} \frac{\pi}{\phi - \cos^2 x} dx$$

But $\frac{\pi}{\phi - \cos^2 x}$ and $x \in [-\pi; \pi]$; so we get $I = 2\int_0^{\pi} \frac{\pi}{\phi - \cos^2 u} du$; let $v = \frac{\pi}{2} - x \Rightarrow dv = -dx$

For $x = 0$; $v = \frac{\pi}{2} - 0 = \frac{\pi}{2}$ and for $x = \pi$; $v = \frac{\pi}{2} - \pi = -\frac{\pi}{2}$ then we get

$$I = 2\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} \frac{\pi}{\phi - \cos^2\left(\frac{\pi}{2} - v\right)}(-dv) = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dv}{\phi - \sin^2 v} = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dv}{\phi - \sin^2 v} \times \frac{\frac{1}{\cos^2 v}}{\frac{1}{\cos^2 v}}$$

$$I = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \tan^2 v}{\frac{1}{\cos^2 v}\phi - \tan^2 v} dv = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \tan^2 x}{\frac{1}{\cos^2 x}\phi - \tan^2 x} dx = 2\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1 + \tan^2 x}{(1 + \tan^2 x)\phi - \tan^2 x} dx$$

Let $t = \tan x \Rightarrow dt = (1 + \tan^2 x)dx$; for $x = -\frac{\pi}{2}$; $t = -\infty$ and for $x = \frac{\pi}{2}$; $t = +\infty$; then

$$I = 2\pi \int_{-\infty}^{+\infty} \frac{dt}{(1 + t^2)\phi - t^2} = 4\pi \int_0^{+\infty} \frac{dt}{(1 + t^2)\phi - t^2} = \frac{4\pi}{\phi}\int_0^{+\infty} \frac{dt}{1 + t^2 - \frac{t^2}{\phi}} = \frac{4\pi}{\phi}\int_0^{+\infty} \frac{dt}{\left(1 - \frac{1}{\phi}\right)t^2 + 1}$$

$I = \frac{4\pi}{\phi}\int_0^{+\infty} \frac{dt}{\left(\frac{\phi - 1}{\phi}\right)t^2 + 1}$; but $\phi^2 = \phi + 1 \Rightarrow \phi(\phi - 1) = 1 \Rightarrow \frac{\phi(\phi - 1)}{\phi^2} = \frac{1}{\phi^2} \Rightarrow \frac{\phi - 1}{\phi} = \frac{1}{\phi^2}$; so

$I = \frac{4\pi}{\phi}\int_0^{+\infty} \frac{dt}{\left(\frac{1}{\phi^2}\right)t^2 + 1} = \frac{4\pi}{\phi}\int_0^{+\infty} \frac{dt}{\left(\frac{t}{\phi}\right)^2 + 1}$; let $w = \frac{t}{\phi} \Rightarrow t = \phi w \Rightarrow dt = \phi dw$; then we get

$$I = \frac{4\pi}{\phi}\int_0^{+\infty} \frac{\phi dw}{w^2 + 1} = \frac{4\pi\phi}{\phi}\int_0^{+\infty} \frac{dw}{1 + w^2} = 4\pi[\arctan w]_0^{\infty} = 4\pi(\arctan\infty - \arctan 0) = 4\pi\left(\frac{\pi}{2} - 0\right)$$

Therefore; $\mathbf{I = 2\pi^2}$

End of Chapter 3

Chapter 4: Special Integrals

1. Gamma Function:

Definition

Let $z \in \mathbb{C}^*$, if the real part of the complex number z is positive ($\operatorname{Re}(z) > 0$), then the integral:

$$\int_0^{+\infty} x^{z-1}e^{-x}dx \quad \text{coverges absolutely}$$

The gamma function is the function represented by the capital letter "Γ" from the Greek alphabet and defined by:

$$\boldsymbol{\Gamma(z) = \int_0^{+\infty} x^{z-1}e^{-x}dx \;\; ; \;\; \text{where } \operatorname{Re}(z) > 0}$$

Some Properties:

- $\boldsymbol{\Gamma(z+1) = z\Gamma(z)}$

Proof

$$\Gamma(z+1) = \int_0^{+\infty} x^z e^{-x}dx = [-x^z e^{-x}]_0^{+\infty} + \int_0^{+\infty} zx^{z-1}e^{-x}dx \; ; \text{(Integration by parts)}$$

$$= \lim_{x\to+\infty}(-x^z e^{-x}) + z\int_0^{+\infty} x^{z-1}e^{-x}dx = \lim_{x\to+\infty}(-x^z e^{-x}) + \mathrm{z}\Gamma(\mathrm{z}); \text{ but } \lim_{x\to+\infty}(-x^z e^{-x}) = 0$$

Therefore, $\Gamma(\mathrm{z}+1) = \mathrm{z}\Gamma(\mathrm{z})$

Remark

$$\Gamma(1) = \int_0^{+\infty} x^{1-1}e^{-x}dx = \int_0^{+\infty} e^{-x}dx = [-e^{-x}]_0^{+\infty} = 0 - 1$$
$$= 1$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^{+\infty} x^{\frac{1}{2}-1}e^{-x}dx = \int_0^{+\infty} x^{-\frac{1}{2}}e^{-x}dx \; ; \text{let } u = x^{\frac{1}{2}}; \text{then } du = \frac{1}{2}x^{-\frac{1}{2}}dt;$$

$$\text{so } t^{-\frac{1}{2}}dt = 2du \text{ and so we get : } \Gamma\left(\frac{1}{2}\right) = 2\int_0^{+\infty} e^{-u^2}du = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}$$

- $\boldsymbol{\Gamma(n) = (n-1)!}$ **, where** $\boldsymbol{n} \in \mathbb{N}^*$

Proof

We have $\Gamma(1) = 1$ and $\Gamma(n+1) = n\Gamma(n)$, so:

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)[(n-2)\Gamma(n-2)] = \cdots$$
$$= (n-1)(n-2) \ldots \times 2 \times 1 \times \Gamma(1)$$

Therefore, $\Gamma(n) = (n-1)(n-2) \ldots \times 3 \times 2 \times 1 = (n-1)!$

Remark: If x is a real number, then:

$$\boldsymbol{x! = \Gamma(x+1) = \int_0^{+\infty} t^x e^{-t}dt}$$

Additional Properties

- $\Gamma\left(\frac{n}{2}\right) = 2^{1-n} \times \frac{(n-1)!\sqrt{\pi}}{\left(\frac{n-1}{2}\right)!}$
- $\Gamma\left(\frac{1}{2} + n\right) = \frac{(2n)!}{4^n n!} \times \sqrt{\pi}$
- $\Gamma\left(\frac{1}{2} - n\right) = \frac{(-4)^n n!}{(2n)!} \times \sqrt{\pi}$
- $|\Gamma(ni)| = \sqrt{\frac{\pi}{n\sinh(n\pi)}}$
- $\Gamma(n)\Gamma\left(\frac{1}{2} + n\right) = 2^{1-2n}\sqrt{\pi}\,\Gamma(2n)$ (Duplication Formula)
- $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ (Euler's Reflection Formula)
- $\Gamma\left(\frac{1}{2} + z\right)\Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos(\pi z)}$
- $\Gamma(z)\Gamma\left(z + \frac{1}{n}\right) \dots \Gamma\left(z + \frac{n-1}{n}\right) = (2\pi)^{\frac{n-1}{2}} n^{\frac{1}{2} - nz}\,\Gamma(nz)$ (Gauss Multiplication Formula)

<u>Another Definitions of Gamma Function</u>

- **Euler's definition as infinite product**

We admit that $\lim\limits_{n\to+\infty} \frac{n!(n+1)^z}{(n+z)!} = 1$, multiplying the both sides by $z!$, we get,

$$z! = \lim_{n\to+\infty}\left[n!\frac{z!}{(n+z)!}(n+1)^z\right]$$

$$z! = \lim_{n\to+\infty}(1\times 2\times \dots \times n)\frac{1\times 2\times \dots \times z}{1\times 2\times \dots \times z\times(1+z)\times \dots \times(n+z)}\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\dots\left(1+\frac{1}{n}\right)\right]^z$$

$$z! = \prod_{n=1}^{\infty}\left[\frac{1}{1+\frac{z}{n}}\left(1+\frac{1}{n}\right)^z\right];\text{but } \Gamma(z) = (z-1)! = \frac{1}{z}z! = \frac{1}{z}\prod_{n=1}^{\infty}\left[\frac{1}{1+\frac{z}{n}}\left(1+\frac{1}{n}\right)^z\right]$$

$$\text{Therefore; } \Gamma(z) = \frac{1}{z}\prod_{n=1}^{\infty}\frac{\left(1+\frac{1}{n}\right)^z}{1+\frac{z}{n}}$$

The definition is valid for all complex numbers z except the non-positive integers

- **Weierstrass's definition**

$$\Gamma(z) = \frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1} e^{\frac{z}{n}};\text{where } \gamma \approx 0.577216 \text{ is the Euler} - \text{Mascheroni constant}$$

This definition is also valid for all complex numbers z except the non-positive integers

Remark: Leonhard Euler is the only mathematician to have two mathematical constants named after him. The first one is the **Euler's** number **"*e* "** which is an irrational number approximately equal to 2.71828182845904523536028747135 2………… The second number is called the **Euler - Mascheroni constant (γ-gamma),** it is approximately equal to 0.577216. It is not known whether γ is rational or irrational number.

$$\boldsymbol{\gamma = \lim_{n\to+\infty}\left[\left(1+\frac{1}{2}+\frac{1}{3}+ \dots\dots\dots\dots\dots\dots + \frac{1}{n}\right) - \ln n\right] = -\int_0^{+\infty} e^{-x}\ln x\,dx}$$

Derivative of Gamma Function

Weierstrass's definition:

$$\Gamma(z)=\frac{e^{-\gamma z}}{z}\prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)^{-1}e^{\frac{z}{n}};\ \text{taking the logarithm} -\ln\Gamma(z)=\ln z+\gamma z+\sum_{n=1}^{\infty}\left[\ln\left(1+\frac{z}{n}\right)-\frac{z}{n}\right]$$

Differentiating both sides with respect to z, we get:

$$-\frac{\Gamma'(z)}{\Gamma(z)}=\frac{1}{z}+\gamma+\sum_{n=1}^{\infty}\left[\frac{\frac{1}{n}}{1+\frac{z}{n}}-\frac{1}{n}\right]=\frac{1}{z}+\gamma+\sum_{n=1}^{\infty}\left[\frac{1}{n+z}-\frac{1}{n}\right];\text{then we get}$$

$$\Gamma'(z)=-\Gamma(z)\left\{\frac{1}{z}+\gamma+\sum_{n=1}^{\infty}\left[\frac{1}{n+z}-\frac{1}{n}\right]\right\}\quad\text{in integral form we }\Gamma'(z)=\int_{0}^{+\infty}t^{z-1}(\ln t)e^{-t}dt$$

Remarks

$$\Gamma'(1)=-\Gamma(1)\left\{1+\gamma+\left[\left(\frac{1}{2}-1\right)+\left(\frac{1}{3}-\frac{1}{2}\right)+\cdots+\left(\frac{1}{n+1}-\frac{1}{n}\right)+\cdots\right]\right\}=-(1+\gamma-1)=\gamma$$

$$\Gamma'(n)=-(n-1)!\left(\frac{1}{n}+\gamma-\sum_{k=1}^{n}\frac{1}{k}\right)$$

- For every positive real number x, we have: $\mathbf{\Gamma''(x)\,\Gamma(x)>\Gamma'(x)}$
- For every two positive real numbers x and y with $x<y$, we have

$$\left[\frac{\Gamma(y)}{\Gamma(x)}\right]^{\frac{1}{y-x}}>\exp\left[\frac{\Gamma'(x)}{\Gamma(x)}\right]$$

- The n^{th} derivative of the gamma function is given by:

$$\frac{d^n}{dz^n}\Gamma(z)=\int_{0}^{+\infty}x^{z-1}e^{-x}(\ln x)^n dx;\ \text{ where } \operatorname{Re}(z)>0$$

Incomplete gamma function

The upper and lower incomplete gamma functions are defined respectively as:

$$\Gamma(s,x)=\int_{x}^{+\infty}t^{s-1}e^{-t}dt\quad\text{and}\quad\gamma(s,x)=\int_{0}^{x}t^{s-1}e^{-t}dt$$

Using integration by parts we can find the following recurrence relations:

$\Gamma(s+1,x)=s\Gamma(s,x)+x^s e^{-x}$ and $\gamma(s+1,x)=s\gamma(s,x)-x^s e^{-x}$

Since the ordinary gamma function is defined as:

$$\Gamma(s)=\int_{0}^{+\infty}t^{s-1}e^{-t}dt;\ \text{ then }\Gamma(s)=\Gamma(s,0)=\lim_{x\to+\infty}\gamma(s,x)\ \&\ \gamma(s,x)+\Gamma(s,x)=\Gamma(s)$$

2. Pi Function:

Π(z) function is the gamma function when offset to coincide with the factorial

$$\Pi(z)=\Gamma(z+1)=z\Gamma(z)=\int_{0}^{+\infty}x^z e^{-x}dx;\ \text{ so that }\Pi(n)=n!\ \forall n\in\mathbb{N}^*$$

$$\Pi(z)\Pi(-z)=\frac{\pi z}{\sin(\pi z)}=\frac{1}{\operatorname{sinc} z};\ \text{ where }\operatorname{sinc} z\text{ is the normalized sine function}$$

3. The Euler's Beta Function:

The Euler beta function is given by:

$$\mathbf{B}(\boldsymbol{m},\boldsymbol{n}) = \int_0^1 \boldsymbol{x}^{\boldsymbol{m}-\mathbf{1}}(\mathbf{1}-\boldsymbol{x})^{\boldsymbol{n}-\mathbf{1}}\boldsymbol{dx}; \ \ \textbf{where Re}(\boldsymbol{m}) > 0 \text{ and } \mathrm{Re}(\boldsymbol{n}) > 0$$

Relation Between Euler Beta Function and Gamma Function

Consider the change of variable $x = \sin^2\theta$ then $dx = 2\sin\theta\cos\theta\, d\theta$, for $x = 0$, $\theta = 0$ and for $x = 1$, then $\theta = \frac{\pi}{2}$, then we get

$$\mathrm{B}(m,n) = \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)^{n-1}(\sin^2\theta)^{m-1} 2\sin\theta\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^{2n-2}\theta \sin^{2m-2}\theta\, 2\sin\theta\cos\theta\, d\theta$$

Therefore; we get $\mathrm{B}(m,n) = 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta\sin^{2m-1}\theta\, d\theta$

We have $\Gamma(m) = \int_0^{+\infty} t^{m-1}e^{-t}dt$; let $t = x^2$; then $dt = 2xdx$; $t = 0$; $x = 0$ and $t \to +\infty$; $x \to +\infty$; so

$$\Gamma(m) = \int_0^{+\infty} e^{-x^2}(x^2)^{m-1}2xdx = 2\int_0^{+\infty} e^{-x^2}x^{2m-1}dx \text{ ; similarly } \Gamma(n)$$

$$= 2\int_0^{+\infty} e^{-y^2}y^{2n-1}dy \text{ so we get}$$

$$\Gamma(m)\Gamma(n) = \left(2\int_0^{+\infty} e^{-x^2}x^{2m-1}dx\right)\left(2\int_0^{+\infty} e^{-y^2}y^{2n-1}dy\right)$$

$$= 4\int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}x^{2m-1}y^{2n-1}dxdy$$

Using polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$, then $r^2 = x^2 + y^2$ and $dxdy = |\mathrm{J}|drd\theta$, where $|\mathrm{J}|$ is the Jacobean, $\mathrm{J}\left(\frac{x,y}{r,\theta}\right) = \begin{vmatrix} \frac{\delta x}{\delta r} & \frac{\delta x}{\delta\theta} \\ \frac{\delta y}{\delta r} & \frac{\delta y}{\delta\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = \mathrm{r}\cos^2\theta + \mathrm{r}\sin^2\theta = \mathrm{r}$

Then we get; $\Gamma(m)\Gamma(n)$

$$= 4\int_0^{+\infty}\int_0^{+\infty} e^{-r^2}(r\cos\theta)^{2m-1}(r\sin\theta)^{2n-1}rdrd\theta \text{ ; then we get}$$

$$\Gamma(m)\Gamma(n) = 4\int_0^{+\infty}\int_0^{+\infty} e^{-r^2}r^{2m-1+2n-1+1}\cos^{2m-1}\theta\sin^{2n-1}\theta\, drd\theta \text{ ; then we get}$$

$$\Gamma(m)\Gamma(n) = 4\int_0^{+\infty}\int_0^{+\infty} e^{-r^2}r^{2(m+n)-1}\cos^{2m-1}\theta\sin^{2n-1}\theta\, drd\theta \text{ and so we get}$$

$$\Gamma(m)\Gamma(n) = 2\int_0^{+\infty} e^{-r^2} r^{2(m+n)-1} dr \times 2\int_0^{+\infty} \cos^{2m-1}\theta \sin^{2n-1} d\theta$$

But $2\int_0^{+\infty} e^{-r^2} r^{2(m+n)-1} dr = \Gamma(m+n)$ and $2\int_0^{\frac{\pi}{2}} \cos^{2m-1}\theta \sin^{2n-1}\theta\, d\theta = \mathrm{B}(n,m)$; so we get

$\Gamma(m)\Gamma(n) = \Gamma(m+n)\,\mathrm{B}(n,m)$; therefore we get; $\mathrm{B}(n,m) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Example: Evaluate the two integrals:

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \cos^5\theta \times \sin^7\theta\, d\theta \quad \text{and} \quad \mathrm{J}_n = \int_0^{\frac{\pi}{2}} \sin^{2n-1} 2\theta\, d\theta$$

For I, $n = 3$ and $m = 4$, so, $\mathrm{I} = \mathrm{B}(3;4) = \dfrac{\Gamma(3)\times\Gamma(4)}{\Gamma(7)} = \dfrac{3!\times 4!}{7!} = \dfrac{1}{35}$.

For $\mathbf{J}_n$; $\mathrm{J}_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^{2n-1} 2\theta\, d\theta = 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta \times \sin^{2n-1}\theta\, d\theta = \mathrm{B}(n;n) = \dfrac{\Gamma(n)\times\Gamma(n)}{\Gamma(n+n)}$

Therefore, $\mathrm{J}_n = \dfrac{n!\times n!}{(2n)!}$

Some Properties of Euler Beta Function

- $\mathrm{B}(x,y) = \mathrm{B}(y,x)$
- $\mathrm{B}(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \dfrac{(x-1)!(y-1)!}{(x+y-1)!} = \dfrac{x+y}{xy}\cdot\dfrac{1}{\binom{x+y}{x}}$
- $\mathrm{B}(x,y) = \mathrm{B}(x,y+1) + \mathrm{B}(x+1,y)$
- $\mathrm{B}(x+1,y) = \mathrm{B}(x,y) \times \dfrac{y}{x+y}$
- $\mathrm{B}(x,y+1) = \mathrm{B}(x,y) \times \dfrac{y}{x+y}$
- $\mathrm{B}(x,y).\,\mathrm{B}(x+y,1-y) = \dfrac{\pi}{x\sin(\pi y)}$
- $\mathrm{B}(x,1-x) = \dfrac{\pi}{\sin(\pi x)}\ x \notin \mathbb{Z}$
- $\mathrm{B}(1,x) = \dfrac{1}{x}$
- Other forms of Euler Beta function

$$\mathrm{B}(x,y) = \int_0^{+\infty} \frac{t^{x-1}}{(1+t)^{x+y}} dt \quad \text{and} \quad \mathrm{B}(x,y) = n\int_0^1 t^{nx-1}(1-t^n)^{y-1} dt$$

Series and Product forms of Euler Beta function:

$$\mathrm{B}(x,y) = \sum_{n=0}^{+\infty} \frac{\binom{n-y}{n}}{x+n} \quad \text{and} \quad \mathrm{B}(x;y) = \frac{x+y}{xy}\prod_{n=1}^{+\infty}\left[1 + \frac{xy}{n(x+n+y)}\right]^{-1}$$

Partial Derivatives of Euler Beta function:

$$\frac{\delta}{\delta x}\mathrm{B}(x,y)=\mathrm{B}(x,y)\left[\frac{\Gamma'(x)}{\Gamma(x)}-\frac{\Gamma'(x+y)}{\Gamma(x+y)}\right]=\mathrm{B}(x,y)[\Psi(x)-\Psi(x+y)]$$

Where $\Psi(x)$ is the digamma function such that $\Psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}$

***In the next paragraph we will discuss the digamma function**

<u>Sterling's approximation to the Beta Function:</u>

$$\mathrm{B}(x,y)\sim\sqrt{2\pi}\,\frac{x^{x-\frac{1}{2}}.y^{y-\frac{1}{2}}}{(x+y)^{x+y-\frac{1}{2}}}$$

The reciprocal beta function: is the function about the form $f(x,y)=\frac{1}{\mathrm{B}(x,y)}$

We can found the following integrals**:**

$$\int_0^{\pi}\sin^{x-1}\sin y\theta\,d\theta=\frac{\pi\sin\left(\frac{y\pi}{2}\right)}{2^{x-1}x\mathrm{B}\left(\frac{x+y+1}{2},\frac{x-y+1}{2}\right)}$$

$$\int_0^{\pi}\sin^{x-1}\cos y\theta\,d\theta=\frac{\pi\cos\left(\frac{y\pi}{2}\right)}{2^{x-1}x\mathrm{B}\left(\frac{x+y+1}{2},\frac{x-y+1}{2}\right)}$$

$$\int_0^{\pi}\cos^{x-1}\sin y\theta\,d\theta=\frac{\pi\cos\left(\frac{y\pi}{2}\right)}{2^{x-1}x\mathrm{B}\left(\frac{x+y+1}{2},\frac{x-y+1}{2}\right)}$$

$$\int_0^{\frac{\pi}{2}}\cos^{x-1}\cos y\theta\,d\theta=\frac{\pi}{2^{x}x\mathrm{B}\left(\frac{x+y+1}{2},\frac{x-y+1}{2}\right)}$$

Multivariate Beta function: The beta function can be extended to a function with more than two arguments as follows**:**

$$\mathrm{B}(x_1,x_2,\dots,x_n)=\frac{\Gamma(x_1)\Gamma(x_2)\dots\Gamma(x_n)}{\Gamma(x_1+x_2+\dots+x_n)}$$

<u>Incomplete Beta Function</u>

The incomplete beta function, a generalization of the beta function, is defined as:

$$\mathrm{B}(x;a,b)=\int_0^{x}t^{a-1}(1-t)^{b-1}dx$$

For $x=1$, the incomplete beta function coincides with the complete beta function

Remark: (Gaussian Integral)

$$\mathbf{I}=\int_{-\infty}^{+\infty}\mathbf{e}^{-x^2}\boldsymbol{dx}$$ is called the Gaussian Integral and it is equal to $\sqrt{\pi}$

Proof: $\mathrm{I}^2=\left(\int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}dx\right)^2=\left(\int_{-\infty}^{+\infty}\mathrm{e}^{-x^2}dx\right)\left(\int_{-\infty}^{+\infty}\mathrm{e}^{-y^2}dy\right)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}e^{-(x^2+y^2)}\,dxdy$

$\mathrm{I}^2=\int_0^{2\pi}\int_0^{+\infty}e^{-r^2}.rdrd\theta=2\pi\int_0^{+\infty}re^{-r^2}dr=2\pi\left(\frac{1}{2}\right)=\pi$ (by polar coordinates)

$\mathrm{I}^2=\pi$, therefore $\mathrm{I}=\sqrt{\pi}$

4. Di gamma & Polygamma Functions:

The digamma is defined as the logarithmic derivative of the gamma function, where:

$$\Psi(x)=\frac{d}{dx}\ln\Gamma(x)=\frac{\Gamma'(x)}{\Gamma(x)}\sim\ln x-\frac{1}{2x}$$

The digamma function is often denoted as $\Psi_0(x)$ or $\Psi^{(0)}(x)$

Relation to harmonic numbers:

We have $\Gamma(z+1)=z\Gamma(z)$, taking the derivative with respect to z we get:

$\Gamma'(z+1)=z\Gamma'(z)+\Gamma(z)$, now dividing by $\Gamma(z+1)$ or the equivalent $z\Gamma(z)$, we get:

$$\frac{\Gamma'(z+1)}{\Gamma(z+1)}=\frac{\Gamma'(z)}{\Gamma(z)}+\frac{1}{z}\quad\text{or}\quad\Psi(z+1)=\Psi(z)+\frac{1}{z}$$

But since the harmonic numbers are defined for positive integers n as:

$$H_n=\sum_{k=1}^{n}\frac{1}{k};\ \text{ then the digamma function is related to them by }\ \Psi(n)=H_{n-1}-\gamma$$

For the half-integer arguments the digamma function takes the values:

$$\Psi\left(n+\frac{1}{2}\right)=-\gamma-2\ln 2+\sum_{k=1}^{n}\frac{2}{2k-1}$$

Integral representation of the digamma function:

If the real part of z is positive then the digamma function has the following integral representation according to Gauss:

$$\Psi(z)=\int_0^{+\infty}\left(\frac{e^{-t}}{t}-\frac{e^{-zt}}{1-e^{-t}}\right)dt;\quad\text{and with }\ \gamma:\Psi(z+1)=-\gamma+\int_0^1\left(\frac{1-t^z}{1-t}\right)dt$$

An integral representation of digamma function according to Dirichlet is:

$$\Psi(z)=\int_0^{+\infty}\left(e^{-t}-\frac{1}{(1+t)^z}\right)\frac{dt}{t}$$

From the definition of Ψ and the integral representation of Gamma function we get:

$$\Psi(z)=\frac{1}{\Gamma(z)}\int_0^{+\infty}t^{z-1}\ln(t)\,e^{-t}dt$$

Infinite product representation of digamma function:

$$\frac{\Psi(z)}{\Gamma(z)}=-e^{2\gamma z}\prod_{k=0}^{\infty}\left(1-\frac{z}{x_k}\right)e^{\frac{z}{x_k}}$$

Here x_k is the kth zero of Ψ (x_k's roots of the digamma function), where:

$$x_n\approx -n+\frac{1}{\pi}\arctan\left(\frac{\pi}{\ln n}\right);n\geq 2$$

Series representation of digamma function:

$$\Psi(z+1)=-\gamma+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right)=-\gamma+\sum_{n=1}^{\infty}\left(\frac{z}{n(n+z)}\right);\ z\neq -1,-2,-3,\dots$$

$$\text{Equivalentlt: }\Psi(z)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+z}\right)=-\gamma+\sum_{n=0}^{\infty}\frac{z-1}{(n+1)(n+z)};\quad\text{where:}$$

$z \neq 0, -1, -2, -3, \ldots$

Taylor Series: The digamma function has a Taylor expansion at $z = 1$ according to zeta series **(we will discuss the zeta function in the next paragraph)**, such that**:**

$$\Psi(\mathrm{z}+1) = -\gamma - \sum_{k=1}^{+\infty} \zeta(k+1)(-z)^k \; ; \;\; \text{which converges for } |\mathrm{z}| < 1$$

Newton Series: the Newton series of the digamma function is given as**:**

$$\Psi(\mathrm{s}+1) = -\gamma - \sum_{k=1}^{+\infty} \frac{(-1)^k}{k}\binom{\mathrm{s}}{k}; \;\; \text{where } \binom{\mathrm{s}}{k} \;\text{ is the binomial coefficient}$$

Reflection formula: the reflection formula for digamma function (it satisfies a relation similar to that of Γ function) is given as:

$$\Psi(1-x) - \Psi(x) = \pi\cot(\pi x)$$

Legendre Duplication Formula:

$$2\Psi(2\mathrm{s}) = 2\ln 2 + \Psi(\mathrm{s}) + \Psi\left(\mathrm{s} + \frac{1}{2}\right)$$

Recurrence Formula: The digamma function satisfies the recurrence relation**:**

$$\Psi(x+1) = \Psi(x) + \frac{1}{x}$$

Remark:

$$\Psi(x+\mathrm{N}) = \Psi(x) + \sum_{k=0}^{\mathrm{N}-1} \frac{1}{x+k}$$

Special Values:

$$\Psi(1) = -\gamma \;\;\text{ and }\;\; \Psi\left(\frac{1}{2}\right) = -2\ln 2 - \gamma$$

Regularization:

The digamma function appears in the regularization of divergent integrals $\displaystyle\int_0^{+\infty} \frac{dx}{x+a}$ this integral can be approximated by a divergent general harmonic series, but the following value can be attached to the series $\displaystyle\sum_{n=0}^{+\infty} \frac{1}{n+a} = -\Psi(a)$

Derivative of digamma function:

$$\frac{d}{dz}\Psi(z) = \frac{d}{dz}\left\{-\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+z}\right)\right\} = \frac{1}{z^2} + \sum_{n=1}^{\infty}\frac{1}{(n+z)^2} = \sum_{n=0}^{\infty}\frac{1}{(n+z)^2}$$

Polygamma Function: **The polygamma function is defined as:**

$$\Psi^{(m)}(z) = \frac{d^m\Psi(z)}{dz^m} = \frac{d^{m+1}}{dz^{m+1}}\ln\Gamma(z)$$

The integral representation of polygamma function is given by:

$$\Psi^{(m)}(z) = (-1)^{m+1}\int_0^{+\infty}\frac{t^m e^{-zt}}{1-e^{-t}}dt = -\int_0^1\frac{t^{z-1}}{1-t}(\ln t)^m dt\,;\text{where } m>0 \text{ and } z>0$$

Remark: The polygamma function satisfies the following recurrence relation**:**

$$\Psi^{(m+1)}(z+1) = \Psi^{(m)}(z) + \frac{(-1)^m m!}{z^{m+1}}$$

5. Riemann Zeta Function:

The Riemann zeta function is the function represented by the letter "ζ" from the Greek alphabet and defined by:

$$\boldsymbol{\zeta(s) = \frac{1}{1^s}+\frac{1}{2^s}+\frac{1}{3^s}+\cdots = \sum_{n=1}^{+\infty}\frac{1}{n^s};\ \ \text{where } \mathrm{Re}(s)} > 1$$

Leonhard Euler discovered the formula $\zeta(s) = \frac{1}{1-2^{-s}}\cdot\frac{1}{1-3^{-s}}\cdot\frac{1}{1-5^{-s}}\dots = \prod_{p/\text{prime}}\frac{1}{1-p^{-s}}$

Relation Between Riemann Zeta Function and Gamma Function

We have $\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}dt$; let $t = nu$; then $dt = ndu$; we get $\Gamma(s) = \int_0^{+\infty} n^s u^{s-1}e^{-nu}du$

and so we get

$$\Gamma(s)\frac{1}{n^s} = \int_0^{+\infty} u^{s-1}e^{-nu}du \text{ and so } \Gamma(s)\sum_{n=1}^{+\infty}\frac{1}{n^s} = \sum_{n=1}^{+\infty}\int_0^{+\infty} u^{s-1}e^{-nu}du = \int_0^{+\infty} u^{s-1}\sum_{n=1}^{+\infty}e^{-nu}\,;$$

then weget

$$\Gamma(s)\zeta(s) = \int_0^{+\infty} u^{s-1}\left(\frac{1}{1-e^{-u}}-1\right)du \quad \left(\text{We have: } 1+\sum_{n=1}^{+\infty}e^{-nu} = \frac{1}{1-e^{-u}} \quad \text{geometric series}\right)$$

$$\text{Then; } \Gamma(s)\zeta(s) = \int_0^{+\infty} u^{s-1}\left(\frac{1}{1-e^{-u}}-\frac{1-e^{-u}}{1-e^{-u}}\right)du = \int_0^{+\infty} u^{s-1}\frac{e^{-u}}{1-e^{-u}}du\,;\text{therefore we get}$$

$$\boldsymbol{\Gamma(s)\zeta(s) = \int_0^{+\infty}\frac{u^{s-1}}{e^u-1}du}$$

Another relation between Riemann Zeta function and gamma function:

$$\boldsymbol{\left(1-\frac{1}{2^n}\right)\Gamma(n+1)\zeta(n+1) = \int_0^{+\infty}\frac{x^n}{e^x+1}dx}$$

Specific Values

For any positive even integer n, we have**:**

$$\zeta(2n) = \frac{(-1)^{n+1}B_{2n}(2\pi)^{2n}}{2(2n)};\ \ \text{where } B_{2n} \text{ is the } 2n-\text{th Bernoulli number}$$

For odd positive integers, no such simple expression is known. For non-positive integers we have

$$\zeta(-n) = (1)^n \frac{B_{n+1}}{n+1}; \ \text{ where } n \geq 0$$

We have $\zeta(-1) = -\frac{1}{12}, \zeta(0) = -\frac{1}{2}, \zeta(1) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty,$

$$\zeta(2) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6};\ \zeta(4) = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}\ \ldots\ \zeta(\infty) = 1$$

Note: Bernoulli numbers are defined by**:**

$$B_n = \sum_{m=0}^{n} \sum_{k=0}^{n} (-1)^k \binom{m}{k} \frac{k^n}{m+1} \ \text{ starting with } B_0 = 1$$

Noting that $B_n = 0$ for all odd n greater than 1, here are the first few Bernoulli numbers**:**

$$1;\ \frac{1}{2};\ \frac{1}{6};\ \ 0;\ -\frac{1}{30};\ \ 0;\ \frac{1}{42};\ \ 0;\ -\frac{1}{30};\ \ 0;\ \frac{5}{66};\ \ 0;\ -\frac{691}{2730};\ \ 0;\ \frac{7}{6};\ \ 0;\ -\frac{3617}{510}\ldots$$

Riemann's functional equation: The zeta function satisfies the following functional equation**:**

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

Remark: If $0 < \mathrm{Re}(s) < 1$, then we can write**:**

$$\zeta(s) = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s}$$

Note that: The Dirichlet **eta function** is given by**:**

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} = (1-2^{1-s})\zeta(s)$$

6. <u>Error Function:</u>

The **error function** also called the **"Gauss** error function**"** is denoted by **" erf "** and equals to**:**

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} dt\,; \ \text{ where } z \in \mathbb{C}$$

Complementary error function: $\operatorname{erfc} z = 1 - \operatorname{erf} z$

Imaginary error function: $\operatorname{erfi}(z) = -i \operatorname{erf}(iz)$

Scaled complementary error function is denoted by erfcx, such that:

$$\operatorname{erfc}(x) = e^{-x^2} \operatorname{erfcx}(x)$$

Here's some properties of the error function

- $\operatorname{erf}(-z) = -\operatorname{erf}(z)$
- $\operatorname{erf}(\overline{z}) = \overline{\operatorname{erf}(z)}$
- Taylor series of $\operatorname{erf} z$**:**

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!\,(2n+1)} = \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \frac{z^9}{216} - \cdots \right)$$

- Taylor series of $\operatorname{erfi}(z)$**:**

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{0}^{\infty} \frac{z^{2n+1}}{n!\,(2n+1)} = \frac{2}{\sqrt{\pi}} \left(z + \frac{z^3}{3} + \frac{z^5}{10} + \frac{z^7}{42} + \frac{z^9}{216} + \cdots \right)$$

- There is also exists a representation of erf(z) by an infinite sum containing double factorial such that**:**

$$\operatorname{erf}(z)=\frac{2}{\sqrt{\pi}}\sum_{0}^{\infty}\frac{(-2)^n(2n-1)!!}{(2n+1)!}z^{2n+1}$$

- Derivative of erf z, we have : $\frac{d}{dz}\operatorname{erf} z=\frac{2}{\sqrt{\pi}}e^{-z^2}$

  Derivative of erfi(z), we have : $\frac{d}{dz}\operatorname{erfi}(z)=\frac{2}{\sqrt{\pi}}e^{z^2}$
- Integral of error function with Gaussian density function:

$$\int_{-\infty}^{+\infty}\operatorname{erf}(ax+b)\frac{1}{\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}dx=\operatorname{erf}\left[\frac{a\mu+b}{\sqrt{1+2a^2\sigma^2}}\right];\ \text{ where } a,b,\mu,\sigma\in\mathbb{R}$$

- Error function in terms of the incomplete gamma function: $\operatorname{erf}(x)=1-\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2},x^2\right)$
- **Generalized Error Function:**

$$\mathrm{E_n}(x)=\frac{n!}{\sqrt{\pi}}\int_0^x e^{-t^n}dt=\frac{n!}{\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(nk+1)k!}x^{nk+1}$$

  $\mathrm{E_0}(x)$ is a straight line passing through the origin; where: $\mathrm{E_0}(x)=\frac{1}{e\sqrt{\pi}}x$

  $\mathrm{E_2}(x)$ is the error function, $\operatorname{erf}(x)$

  We can express $\mathrm{E_n}(x)$ in terms of the gamma & incomplete gamma functions; where:

$$\mathrm{E_n}(x)=\frac{1}{\sqrt{\pi}}\Gamma(n)\left[\Gamma\left(\frac{1}{n}\right)-\Gamma\left(\frac{1}{n},x^n\right)\right];\ \text{ where } x>0$$

7. Some Special Integrals:

Exponential Integral: Denoted by **Ei** is a special function and defined for every $x\neq 0$, by:

$$\mathrm{Ei}(x)=-\int_{-x}^{+\infty}\frac{e^{-t}}{t}dt=\int_{-\infty}^{x}\frac{e^{t}}{t}dt$$

Note that: Ei is not an elementary function

Define the function $\mathrm{E_1}(x)=-\mathrm{Ei}(-x)$, such that:

$$\mathrm{E_1}(x)=\int_1^{+\infty}\frac{e^{-tx}}{t}dt=\int_0^1\frac{e^{-\frac{x}{u}}}{u}du$$

Both Ei and E₁ can be written more simply using the entire function Ein and it is defined as:

$$\mathrm{Ein}(x)=\int_0^x(1-e^{-t})\frac{dt}{t}=\sum_{k=1}^{\infty}\frac{(-1)^{k+1}x^k}{kk!}$$

Ei and Ein are related by the following relation: $\mathrm{Ei}(x)=\gamma+\ln x-\mathrm{Ein}(-x),\ x>0$

E₁ and Ein are related by the following relation: $\mathrm{E_1}(z)=-\gamma-\ln x+\mathrm{Ein}(z),\ |\mathrm{Arg}(z)|<\pi$

The exponential integral may also generalized as:

$$\mathrm{E}_n(x)=\int_1^{+\infty}\frac{e^{-xt}}{t^n}dt$$

Logarithmic Integral: Denoted by **li** is a special function and defined for every $x \neq 1$, by:

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\ln t}$$

The offset logarithmic integral or Eulerian logarithmic integral is defined as:

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t} = \mathrm{li}(x) - \mathrm{li}(2)$$

Relation between exponential integral and logarithmic integral:

$$\mathrm{li}(x) = \mathrm{Ei}(\ln x) \quad \text{and} \quad \mathrm{li}(e^x) = \mathrm{Ei}(x)$$

Trigonometric integrals are a family of integrals involving trigonometric functions

Sine Integral:

The different sine integral definition are:

$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt \quad \text{and} \quad \mathrm{si}(x) = -\int_x^{+\infty} \frac{\sin t}{t} dt \quad \text{such that} \quad \mathrm{Si}(x) = \frac{\pi}{2} + \mathrm{si}(x)$$

Cosine Integral:

The different cosine integral definition are:

$$\mathrm{Cin}(x) = \int_0^x \frac{1-\cos t}{t} dt \quad \text{and} \quad \mathrm{Ci}(x) = -\int_x^{+\infty} \frac{\cos t}{t} dt \quad \text{such that} \quad \mathrm{Ci}(x) = \gamma + \ln x - \mathrm{Cin}(x)$$

Hyperbolic Sine and Hyperbolic Cosine Integrals:

$$\mathrm{Shi}(x) = \int_0^x \frac{\sinh t}{t} dt \quad \text{and} \quad \mathrm{Chi}(x) = \gamma + \ln x + \int_0^x \frac{1-\cosh t}{t} dt$$

8. Fresnel's Integrals:

The **Fresnel integrals** S(x) and C(x) are two transcendental functions named after Augustin-Jean Fresnel and they are closely related to the error function erf, such that:

$$\mathbf{S}(\boldsymbol{x}) = \int_0^x \sin(t^2)\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)!\,(4n+3)}$$

$$\mathbf{C}(\boldsymbol{x}) = \int_0^x \cos(t^2)\, dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+1}}{(2n)!\,(4n+1)}$$

Complex Fresnel Integrals:

$$\mathbf{S}(z) = \sqrt{\frac{\pi}{2}} \cdot \frac{1+i}{4}\left[\mathrm{erf}\left(\frac{1+i}{\sqrt{2}} z\right) - i\,\mathrm{erf}\left(\frac{1-i}{\sqrt{2}} z\right)\right]$$

$$\mathbf{C}(z) = \sqrt{\frac{\pi}{2}} \cdot \frac{1-i}{4}\left[\mathrm{erf}\left(\frac{1+i}{\sqrt{2}} z\right) + i\,\mathrm{erf}\left(\frac{1-i}{\sqrt{2}} z\right)\right]$$

Remark:

$$\int_0^{+\infty} \cos(x^2)\, dx = \int_0^{+\infty} \sin(x^2) dx = \sqrt{\frac{2\pi}{8}} = \sqrt{\frac{\pi}{8}} \approx 0.6267$$

$$\int_0^{+\infty} \frac{\cos(ax)}{\sqrt{x}}dx = \int_0^{+\infty} \frac{\sin(ax)}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2a}};\quad \text{where } a > 0$$

$$\int_0^{+\infty} e^{iax^2}dx = \sqrt{\frac{\pi}{8a}}(1+i)$$

Normalized Fresnel Integrals:

$$\mathrm{S}(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)dt \quad \text{and} \quad \mathrm{C}(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)dt$$

How to deal with integrals that contains these special functions in the integrand or in the answer:
Let us evaluate the following integrals:

I. $\int \ln(\ln x)\,dx$, we will use integration by parts, so:

Let $u = \ln(\ln x)$, then $u' = \frac{(\ln x)'}{\ln x} = \frac{1}{x\ln x}$ and let $v' = 1$, then $v = x$, so we get:

$$\int \ln(\ln x)\,dx = x\ln(\ln x) - \int x\frac{1}{x\ln x}dx = x\ln(\ln x) - \int \frac{1}{\ln x}dx = x\ln(\ln x) - \mathrm{li}(x) + c$$

II. $\int e^{e^x}dx$, let $u = e^x$, then $du = e^x dx = udx$, so $dx = \frac{1}{u}du$, then we get:

$$\int e^{e^x}dx = \int e^u.\frac{du}{u} = \int \frac{e^u}{u}du = \mathrm{Ei}(u) + c = \mathrm{Ei}(e^x) + \mathrm{c}$$

III. $\int \mathrm{Si}(x)dx$, let $u = \mathrm{Si}(x)$, then $u' = \frac{\sin x}{x}$ and let $v' = 1$, then $v = x$, then we get:

$$\int \mathrm{Si}(x)dx = x\mathrm{Si}(x) - \int x\left(\frac{\sin x}{x}\right)dx = x\mathrm{Si}(x) - \int \sin x\,dx = x\mathrm{Si}(x) + \cos x + c$$

IV. $\int \sin\left(\frac{1}{x}\right)dx$, let $u = \sin\left(\frac{1}{x}\right)$, then $u' = -\frac{1}{x^2}\cos\left(\frac{1}{x}\right)$ and let $v' = 1$, then $v = x$, so we get:

$$\int \sin\left(\frac{1}{x}\right)dx = x\sin\left(\frac{1}{x}\right) - \int x\left(-\frac{1}{x^2}\right)\cos\left(\frac{1}{x}\right)dx = x\sin\left(\frac{1}{x}\right) + \int \frac{1}{x}\cos\left(\frac{1}{x}\right)dx$$

Let $w = \frac{1}{x}$, then $x = \frac{1}{w}$ and $dx = -\frac{1}{w^2}dw$, then we get:

$$\int \sin\left(\frac{1}{x}\right)dx = x\sin\left(\frac{1}{x}\right) + \int w\cos w\left(-\frac{1}{w^2}\right)dw = x\sin\left(\frac{1}{x}\right) - \int \frac{\cos w}{w}dw\,;\ \text{so we get}$$

$$\int \sin\left(\frac{1}{x}\right)dx = x\sin\left(\frac{1}{x}\right) - \mathrm{Ci}(w) + c = x\sin\left(\frac{1}{x}\right) - \mathrm{Ci}\left(\frac{1}{x}\right) + c$$

V. $\int \frac{1}{\sqrt{1-\ln x}}dx$, let $u = \sqrt{1-\ln x}$, then $x = e^{1-u^2}$ and $dx = -2ue^{1-u^2}du$, then we get:

$$\int \frac{1}{\sqrt{1-\ln x}}dx = \int \frac{1}{u}\left(-2ue^{1-u^2}\right)du = -2e\int e^{-u^2}du = -2e\times\frac{\sqrt{\pi}}{2}\mathrm{erf}(u)\,;\text{therefore}$$

$$\int \frac{1}{\sqrt{1-\ln x}}dx = -e\sqrt{\pi}\,\mathrm{erf}\left(\sqrt{1-\ln x}\right) + c$$

VI. $\int \frac{1}{(\ln x)^2}dx = \int \frac{x}{x(\ln x)^2}dx$, let $u = x$, then $u' = 1$ and let $v' = \frac{1}{x(\ln x)^2}$, then $v = -\frac{1}{\ln x}$, so:

$$\int \frac{1}{(\ln x)^2}dx = -\frac{x}{\ln x} + \int \frac{1}{\ln x}dx = -\frac{x}{\ln x} + \mathrm{li}(x) + c$$

VII. $\int \frac{\sin x}{x^2}dx$, let $u = \sin x$, then $u' = \cos x$ and let $v' = \frac{1}{x^2}$, then $v = -\frac{1}{x}$, so:

$$\int \frac{\sin x}{x^2}dx = -\frac{\sin x}{x} + \int \frac{\cos x}{x}dx = -\frac{\sin x}{x} + \mathrm{Ci}(x) + c$$

VIII. $\int \frac{\sin(x^2)}{x^2}dx$, let $u = \sin(x^2)$, then $u' = 2x\cos(x^2)$ and let $v' = \frac{1}{x^2}$, then $v = -\frac{1}{x}$, so:

$$\int \frac{\sin(x^2)}{x^2}dx = -\frac{\sin(x^2)}{x} + \int 2x\cos(x^2).\frac{1}{x}dx = -\frac{\sin x}{x} + 2\int \cos(x^2)\,dx$$

$$\int \frac{\sin(x^2)}{x^2}dx = -\frac{\sin(x^2)}{x} + 2\mathrm{C}(x) + \mathrm{c}$$

IX. $\int xe^x \ln x\,dx$, let $u = \ln x$, then $u' = \frac{1}{x}$ and let $v' = xe^x$, then $v = (x-1)e^x$, then we get:

$$\int xe^x \ln x\,dx = (x-1)e^x \ln x - \int \frac{(x-1)e^x}{x}dx\text{; then we get:}$$

$$\int xe^x \ln x\,dx = (x-1)e^x \ln x - \int e^x dx + \int \frac{e^x}{x}dx\text{; therefore:}$$

$$\int xe^x \ln x\,dx = (x-1)e^x \ln x - e^x + \mathrm{Ei}(x) + \mathrm{c}$$

9. Gauss's Constant:

Gauss's constant denoted by **G**, is defined as the reciprocal of the arithmetic – geometric mean of 1 and $\sqrt{2}$ such that:

$$\mathrm{G} = \frac{1}{\mathrm{agm}(1,\sqrt{2})} = 0.8346268\ldots$$

It is named after the German Mathematician Carl Friedrich Gauss, who in 1799 discovered that:

$$\mathrm{G} = \frac{2}{\pi}\int_0^1 \frac{dx}{\sqrt{1-x^4}};\quad \text{so that}\quad \mathrm{G} = \frac{1}{2\pi}\mathrm{B}\left(\frac{1}{4},\frac{1}{2}\right)$$

Gauss's constant may be used to express gamma function at argument 0.25 such that:

$$\Gamma\left(\frac{1}{4}\right) = \sqrt{2\mathrm{G}\sqrt{2\pi^3}}$$

Since π and $\Gamma\left(\frac{1}{4}\right)$ are algebraically independent, then Gauss's constant is **transcendental**

Other formulas: $\frac{1}{\mathrm{G}} = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx$ and $\mathrm{G} = \int_0^{+\infty} \frac{dx}{\sqrt{\cosh(\pi x)}}$

10. The Polylogarithm Function:

The **polylogarithm** function is defined by a Dirichlet series in s by:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{+\infty} \frac{z^k}{k^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \cdots \quad \text{where } s \in \mathbb{C} \ \ \& \ \ |z| < 1$$

The special case $\boldsymbol{s = 1}$, involves the ordinary natural logarithm such that:

$$\mathrm{Li}_1(z) = -\ln(1-z)$$

The name of this function comes from the fact that it may also be defined as the repeated integral of itself such that:

$$\mathrm{Li}_{s+1}(z) = \int_0^z \frac{\mathrm{Li}_s(t)}{t} dt$$

Duplication formula: $\mathrm{Li}_s(-z) + \mathrm{Li}_s(z) = 2^{1-s}\mathrm{Li}_s(z^2)$

Some particular values of s:

$$\mathrm{Li}_1(z) = -\ln(1-z)\,; \quad \mathrm{Li}_0(z) = \frac{z}{z-1}; \quad \mathrm{Li}_{-1}(z) = \frac{z}{(1-z)^2}; \quad \mathrm{Li}_{-2}(z) = \frac{z(1+z)}{(1-z)^3} \dots$$

Remark: $\mathrm{Li}_1\left(\frac{1}{2}\right) = \ln 2$ **&** $\mathrm{Li}_2\left(\frac{1}{2}\right) = \frac{1}{12}\pi^2 - \frac{1}{2}(\ln 2)^2$

Relation to other functions:

For $z = 1$, the polylogarithm reduces to Riemann zeta function $\mathrm{Li}_s(1) = \zeta(s)$; $\mathrm{Re}(s) > 1$

The polylogarithm is related to Dirichlet eta function and the Dirichlet beta function such that: $\mathrm{Li}_s(-1) = -\eta(s)$ & $\mathrm{Li}_s(\pm i) = -2^{-s}\,\eta(s) \pm i\beta(s)$

Integral representation of polylogarithmic function:

The polylogarithmic function can be expressed in the integral form such that:

$$\mathrm{Li}_s(z) = \frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{\frac{e^t}{z} - 1} dt\,; \quad \text{where} \quad \mathrm{Re}(s) > 0$$

$$\text{Similarly we have: } \mathrm{Li}_s(-z) = -\frac{1}{\Gamma(s)} \int_0^{+\infty} \frac{t^{s-1}}{\frac{e^t}{z} + 1} dt$$

A complementary integral representation applied when $\mathrm{Re}(s) < 0$ such that:

$$\mathrm{Li}_s(z) = \int_0^{+\infty} \frac{t^{-s} \sin\left(s\frac{\pi}{2} - t\ln(-z)\right)}{\sinh(\pi t)} dt$$

$$\text{For } \ s \in \mathbb{N}; \ \text{ we can write : } \ \mathrm{Li}_{s+1}(z) = \frac{z.(-1)^s}{s!} \int_0^1 \frac{\ln^s t}{1 - tz} dt$$

11. Incomplete Elliptic Integral of the First Kind:

The **incomplete elliptic integral of the first kind** is defined by**:**

$$u = \mathrm{F}(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}; \quad \text{where} \quad 0 < k < 1$$

- ϕ is the amplitude of $\mathrm{F}(k,\phi)$ and it is denoted by $\phi = \mathrm{am}\, u$
- k id the modulus of $\mathrm{F}(k,\phi)$ and it is denoted by $k = \mathrm{mod}\, u$
- u is also called ***legendre's form*** for the elliptic integral of the **first** kind
- For $\phi = \frac{\pi}{2}$, the integral is called the ***complete elliptic integral of the first kind*** and it is denoted by $\mathrm{K}(k)$ or K

Incomplete Elliptic Integral of the Second Kind:

The **incomplete elliptic integral of the second kind** is defined by**:**

$$\mathrm{E}(k,\phi) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta; \quad \text{where} \quad 0 < k < 1$$

- E is also called ***legendre's form*** for the elliptic integral of the **second** kind
- For $\phi = \frac{\pi}{2}$, the integral is called the ***complete elliptic integral of the second kind*** and it is denoted by $\mathrm{E}(k)$ or E
- This integral arises in the determination of the length of arc of an ellipse and for this reason these integrals are called elliptic integrals

Incomplete Elliptic Integral of the Third Kind:

The **incomplete elliptic integral of the second kind** is defined by**:**

$$\Pi(k,n,\phi) = \int_0^{\phi} \frac{d\theta}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}; \quad \text{where} \quad 0 < k < 1 \;\; \& \;\; n \neq 0$$

- Π is also called ***legendre's form*** for the elliptic integral of the **third** kind
- For $\phi = \frac{\pi}{2}$, the integral is called the ***complete elliptic integral of the third kind***

Jacobi's Forms for the Elliptic Integrals:

We have $\mathrm{F}(k,\phi) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$; let's consider the change of variable $v = \sin\theta$

New lower limit of the integral $\sin 0 = 0$

New upper limit of the integral $v_{\text{upper}} = x = \sin\phi$

$$dv = \cos\theta\, d\theta = \sqrt{1-\sin^2\theta}\, d\theta = \sqrt{1-v^2}\,d\theta \Longrightarrow d\theta = \frac{1}{\sqrt{1-v^2}}dv; \quad \text{then } \mathrm{F}(k,\phi)$$

becomes: $\mathrm{F}_1(k,x) = \int_0^x \frac{dv}{\sqrt{(1-v^2)(1-k^2v^2)}}$

Similarly for $\mathrm{E}(k,\phi)$ we get $\mathrm{E}_1(k,x) = \int_0^x \sqrt{\frac{1-k^2v^2}{1-v^2}}\,dv$ & $\Pi(k,n,\phi)$ becomes:

$$\Pi_1(k,n,x) = \int_0^x \frac{dv}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}$$

Remark: Jacobi's Elliptic Functions

$$x = \sin\phi = \sin(\mathrm{am}\,u) = \mathrm{sn}\,u \quad \& \quad \sqrt{1-x^2} = \cos\phi = \cos(\mathrm{am}\,u) = \mathrm{cn}\,u$$

$$\sqrt{1-k^2x^2} = \sqrt{1-k^2\mathrm{sn}^2\,u} = \mathrm{dn}\,u \quad \& \quad \frac{x}{\sqrt{1-x^2}} = \frac{\mathrm{sn}\,u}{\mathrm{cn}\,u} = \mathrm{tn}\,u$$

$$\mathrm{sn}^2 + \mathrm{cn}^2 = 1$$

Minor Functions

$$\mathrm{ns}\,u = \frac{1}{\mathrm{sn}\,u};\quad \mathrm{nc}\,u = \frac{1}{\mathrm{cn}\,u};\quad \mathrm{sd}\,u = \frac{\mathrm{sn}\,u}{\mathrm{dn}\,u}$$

Addition Formulas

$$\mathrm{cn}(x+y) = \frac{\mathrm{cn}(x)\mathrm{cn}(y) - \mathrm{sn}(x)\mathrm{sn}(y)\mathrm{dn}(x)\mathrm{dn}(y)}{1-k^2\mathrm{sn}^2(x)\mathrm{sn}^2(y)}$$

$$\mathrm{sn}(x+y) = \frac{\mathrm{sn}(x)\mathrm{cn}(y)\mathrm{dn}(y) + \mathrm{sn}(y)\mathrm{cn}(x)\mathrm{dn}(x)}{1-k^2\mathrm{sn}^2(x)\mathrm{sn}^2(y)}$$

$$\mathrm{dn}(x+y) = \frac{\mathrm{dn}(x)\mathrm{dn}(y) - k^2\mathrm{sn}(x)\mathrm{sn}(y)\mathrm{cn}(x)\mathrm{cn}(y)}{1-k^2\mathrm{sn}^2(x)\mathrm{sn}^2(y)}$$

Derivatives

$$\frac{d}{dz}\mathrm{sn}(z) = \mathrm{cn}(z)\mathrm{dn}(z);\quad \frac{d}{dz}\mathrm{cn}(z) = -\mathrm{sn}(z)\mathrm{dn}(z) \quad \& \quad \frac{d}{dz}\mathrm{dn}(z) = -k^2\mathrm{sn}(z)\mathrm{cn}(z)$$

Inverse Functions

$$\mathrm{arcsn}(x,k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}};\quad \mathrm{arccn}(x,k) = \int_x^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2+k^2t^2)}}$$

$$\&\quad \mathrm{arcdn}(x,k) = \int_x^1 \frac{dt}{\sqrt{(1-t^2)(t^2+k^2-1)}}$$

Landen's Transformation:

Consider the transformation: $\tan\phi = \dfrac{\sin 2\phi_1}{k + \cos 2\phi_1}$ & $k\sin\phi = \sin(2\phi_1 - \phi)$

By using this transformation ***Landen's transformation***, we can show that**:**

$$\int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2\sin^2\phi}} = \frac{2}{1+k}\int_0^{\phi_1} \frac{d\phi_1}{\sqrt{1 - k^2\sin^2\phi_1}}; \text{ where } k_1 = \frac{2\sqrt{k}}{1+k}; \text{ so it can be}$$

written as $\mathrm{F}(k,\phi) = \dfrac{2}{1+k}\mathrm{F}(k_1,\phi_1)$ such that $k < k_1 < 1$

By using successive applications of ***Landen's transformation***, a sequence of moduli k_n, $n = 1, 2. 3, \dots$ is obtained such that $k < k_1 < k_2 < \cdots < 1$ and $\lim\limits_{n\to+\infty} k_n = 1$, then we have**:**

$$\mathrm{F}(k,\phi) = \sqrt{\frac{k_1k_2k_3\dots}{k}}\int_0^{\phi}\frac{d\theta}{\sqrt{1-\sin^2\theta}} = \sqrt{\frac{k_1k_2k_3\dots}{k}}\ln\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$$

Where**:**

$$k_1 = \frac{2\sqrt{k}}{1+k}; \quad k_2 = \frac{2\sqrt{k_1}}{1+k_1}; \quad k_3 = \frac{2\sqrt{k_2}}{1+k_2}\dots \quad \& \quad \phi = \lim_{n\to+\infty}\phi_n$$

Solved Exercises

Evaluate each of the following integrals:

1. $\int_0^{+\infty} x^3 e^{-x} dx$
2. $\int_0^{+\infty} x^6 e^{-2x} dx$
3. $\int_0^1 x^4(1-x)^3 dx$
4. $\int_0^2 \frac{x^2}{\sqrt{2-x}} dx$
5. $\int_0^{+\infty} \sqrt{x} e^{-x^3} dx$
6. $\int_0^{+\infty} 3^{-4x^2} dx$
7. $\int_0^1 x^4 \left(1-\sqrt{x}\right)^5 dx$
8. $\int_0^a x^4 \sqrt{a^2-x^2}\, dx$
9. $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta$
10. $\int_0^{\frac{\pi}{2}} \sin^4\theta \cos^5\theta\, d\theta$
11. $\int_0^1 \ln(-\ln x)\, dx$
12. $\int_0^{+\infty} x^2 e^{-x^2}\, dx$
13. $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$
14. $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin x}} dx$
15. $\int_0^{+\infty} \frac{x^a}{a^x} dx$
16. $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$
17. $\int_0^{\frac{\pi}{2}} \left(\frac{\tan x}{n}\right)^n dx$
18. $\int_0^{\frac{\pi}{2}} \sqrt{\cot x + \tan x}\, dx$
19. $\int \mathrm{Ei}(x) dx$
20. $\int \mathrm{erf}(x)\, dx$
21. $\int e^x \ln x\, dx$
22. $\int_0^{+\infty} \frac{dx}{(1+x^2)^\alpha}$
23. $\int_0^1 (x-x^2)^n dx$
24. $\int_0^1 \left(1-x^{\frac{1}{n}}\right)^n dx$
25. $\int e^{e^x} dx$
26. $\int \ln(\ln x)\, dx$
27. $\int_0^1 \frac{1}{\sqrt{1-x^3}} dx$
28. $\int_{-\infty}^{+\infty} e^{-e^{ax}} . e^{bx}\, dx$
29. $\int_0^{+\infty} \frac{x^{m+1}}{(1+x^2)^2} dx$
30. $\int_1^e \frac{dx}{1+\ln x}$
31. $\int_0^{+\infty} x^m e^{-ax^n} dx$
32. $\int_0^{+\infty} e^{-x^{\left(\frac{1}{m^2}+\frac{1}{n^2}\right)}} dx$
33. $\int_{-a}^{+\infty} \frac{e^{-px}}{\sqrt{x+a}} dx$
34. $\int_0^2 x\sqrt[3]{8-x^3} dx$
35. $\int_0^1 \sqrt{1-x^4} dx$

36. $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx$

37. $\int_0^1 (\ln x)^{\frac{1}{n}}\, dx$

38. $\int_{-\infty}^{+\infty} e^{-e^{-nx}}.e^{-nx}.x\, dx$

39. $\int_0^1 \frac{1}{\sqrt{1+x^4}}\, dx$

40. $\int_0^1 \frac{x^5}{\sqrt{1-x^4}}\, dx$

41. $\int_0^1 \sqrt{\sqrt{\ln x}}\, dx$

42. $\int_0^{+\infty} \frac{dx}{\cosh^n x}$

43. $\int_0^1 \frac{1}{\sqrt{x \ln\left(\frac{1}{x}\right)}}\, dx$

44. $\int_{-\infty}^{+\infty} e^{-(ax^2+bx+c)}\, dx$

45. $\int_0^{\frac{\pi}{2}} \sin^p \theta\, d\theta$

46. $\int_0^{\pi} x \sin^m x\, dx$

47. $\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^x}\, dx$

48. $\int_0^{+\infty} \frac{1}{1+x^{\alpha}}\, dx$

49. $\int_0^1 \ln^{n-1}\left(\frac{1}{x}\right) dx$

50. $\int_0^{+\infty} e^{-\sqrt[n]{x}}\, dx$

51. $\int_0^{+\infty} e^{-x^n}\, dx$

52. $\int_0^{+\infty} e^{-ax} \ln(bx)\, dx$

53. $\int_{-\infty}^{+\infty} \left(1+\frac{x^2}{n-1}\right)^{-\frac{n}{2}} dx$

54. $\int_{-\infty}^{+\infty} x e^x e^{-me^x}\, dx$

55. $\int_1^{+\infty} \left(\frac{\ln x}{x}\right)^n dx$

56. $\int_0^1 x^p (-\ln x)^{n-1}\, dx$

57. $\int_{-\infty}^{+\infty} e^{-ax^2-bx}\, dx$

58. $\int_0^1 \ln\left(\frac{x}{\ln x}\right) dx$

59. $\int_0^{+\infty} \frac{1}{1+x^{2e}}\, dx$

60. $\int \operatorname{li}(x) dx$

61. $\int_0^{+\infty} \frac{dx}{x^5\left(e^{\frac{1}{x}}-1\right)}$

62. $\int_0^1 x^n \ln\ln\left(\frac{1}{x}\right) dx$

63. $\int_0^{\frac{\pi}{2}} (\sec x - 1)^{\mu} \sin x\, dx$

64. $\int_1^e \frac{1}{x} \ln(\ln(1-\ln x))\, dx$

65. $\int_0^1 \frac{x^n}{\sqrt{\ln x}}\, dx$

66. $\int_0^1 \ln^n\left(1-\sqrt{x}\right) dx$

67. $\int_0^{e^{-1}} \ln(1+\ln x)\, dx$

68. $\int_1^2 (x-1)^5 \sqrt{\ln(x-1)}\, dx$

69. $\int_1^e \frac{\ln(\ln(\ln x))}{x}\, dx$

70. $\int_{e^{-n}}^0 \sqrt{\ln x + n}\, dx$

71. $\int_0^1 \frac{\sqrt{x}}{\sqrt{\sqrt{\ln x}}}\, dx$

72. $\int_0^1 \left[x - \frac{1}{(1-\ln x)^m}\right] \cdot \frac{dx}{x \ln x}$

73. $\int_0^1 \sqrt[4]{\frac{x}{1-x}} \cdot \frac{dx}{(1+x)^2}$

74. $\int_0^{\frac{1}{k}} \ln\left(-\frac{1}{n}\ln(kx)\right) dx$

75. $\int_0^{\frac{1}{e}} \frac{\sqrt[m]{x}}{x\sqrt{-(1+\ln x)}} dx$

76. $\int_{-a}^{a} (a+x)^{m-1}(a-x)^{n-1}\, dx$

77. $\int_0^{e} \frac{x}{\sqrt{1-\ln x}} dx$

78. $\int_0^{+\infty} \frac{x^{s-1}}{(e^x-1)^2} dx$

79. $\int e^{\tan x}\, dx$

80. $\int_0^1 \frac{\ln\ln(1-x)}{(1-x)^{\frac{n}{k}}} dx$

81. $\int_0^{\frac{\pi}{2}} \frac{\sin 2x}{\cos^4 x\left(e^{2\pi \tan x}-1\right)} dx$

82. $\int_0^1 \frac{\ln^n\left(\frac{1+x}{1-x}\right)}{\sqrt{1-x^2}} \frac{dx}{1+x}$

83. $\int_0^{\frac{1}{b}} \ln(a\ln(bx))\, dx$

84. $\int_0^{+\infty} \frac{\ln(\cosh x)}{\sinh x} dx$

85. $\int_0^1 \ln\left(\frac{e^{x^n}}{n^n \ln^n(1-x)}\right) dx$

86. $\int_0^1 \frac{x^a-x^b}{\sqrt{\ln x}} dx$

87. $\int_0^{+\infty} \frac{xe^{-\mu x^2}}{\sqrt{x^2+a^2}} dx$

88. $\int_0^1 \frac{dx}{a-\ln x}$

89. $\int_0^1 \frac{dx}{a^2-\ln^2 x}$

90. $\int_0^{+\infty} x^{-a\ln x+b-1}\, dx$

91. $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\tan\left(\frac{x}{2}\right)\sec\left(\frac{x}{2}\right) dx$

92. $\int_0^1 \cos^{-1}\left(\sqrt{1-\sqrt{x}}\right) dx$

93. $\int_0^{+\infty} \frac{x^{2(m+n)}}{(1+x^2)^{2m+1}.(1+x^{2n})} dx$

94. $\int_0^{e} x^{\frac{1}{m}}(1-\ln x)^{\frac{1}{n}}\, dx$

95. $\int_0^{+\infty} x^{-a\ln x}\ln x\, dx$

96. $\int_0^{\frac{\pi}{2}} \sin^3 x \ln(\ln\cos x)\, dx$

97. $\int_0^{\frac{\pi}{2}} \frac{1-\sin^4 x}{(1+\sin^4 x)\sqrt{1+\sin^2 x}} dx$

98. $\int_0^{+\infty} \frac{1}{1+x^{10}} dx$

99. $\int_0^{+\infty} \frac{x^{\varphi}}{(x+1)\sqrt{x^{4\varphi}+x^4}} dx$

100. $\int_0^{+\infty} \left(\frac{x^2}{x^4+2ax^2+1}\right)^m dx$

101. $\int [\Psi(x)-\Psi(1-x)]\, dx$

102. $\int_0^1 \ln\Gamma(x)\, dx$

103. $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin^5 x+\cos^5 x} dx$

104. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)^{\cos 2\alpha} dx$

105. $\int_{\frac{1}{2}}^{1} \tan^{-1}\left[\frac{\Gamma\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}+x\right)}{\Gamma(1-x)\Gamma(x)}\right] dx$

106. $\int \ln(\log_2 x + \log_3 x)\, dx$

107. $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}} dx$

108. $\int_0^{+\infty} \frac{\ln(1+x^n)}{x^n} dx$

109. $\int_0^{\frac{1}{2}} \frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)} dx$

110. $\int_0^1 \frac{\mathrm{li}(x)\ln(-\ln x)}{x} dx$

111. $\int_0^1 \Psi(x)\sin(\pi x)\cos(\pi x)\, dx$

112. $\int_0^1 \frac{\Psi\left(x+\frac{3}{2}\right)-\Psi\left(x-\frac{1}{2}\right)}{x} dx$

113. $\int_0^{+\infty} e^{-x}\ln\Gamma(1-e^{-x})\, dx$

114. $\int_0^{+\infty} \frac{1}{(1+x^{\varphi})^{\varphi}} dx$

115. $\int_0^1 \sqrt{\frac{\ln^m(1-x)}{\sqrt{\frac{\ln^m(1-x)}{\sqrt{\frac{\ln^m(1-x)}{\sqrt{\dots}}}}}}} dx$

116. $\int_0^{\frac{\pi}{2}} \left(\cos^{\frac{1}{n}} x + \sin^{\frac{1}{n}} x\right)^{-2n} dx$

117. $\int_0^{+\infty} \cos\left(2x^2 + \frac{1}{x^2}\right) dx$

118. $\int_0^{\frac{\pi}{2}} \frac{dx}{a\cos^4 x + b\sin^4 x}$

119. $\int_0^1 \ln\Gamma(x) . \cos^2(\pi x)\, dx$

120. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin^2 x + 2\cos^2 x}}$

121. $\int_0^t \frac{dx}{\sqrt{1-3\sin^2 x}}$

122. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{\sin x}}$

123. $\int_0^{\frac{\pi}{2}} \sqrt{1+4\sin^2 x}\, dx$

124. $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$

125. $\int_0^x \sqrt{1-4\sin^2 u}\, du$

126. $\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2-\cos x}}$

127. $\int_0^{\phi} \frac{d\phi}{\sqrt{1+k^2\sin^2\phi}}$

128. $\int_0^2 \frac{dx}{\sqrt{(4-x^2)(9-x^2)}}$

129. $\int_0^1 \frac{dx}{\sqrt{(1+x^2)(1+2x^2)}}$

130. $\int_4^6 \frac{dx}{\sqrt{(x-1)(x-2)(x-3)}}$

131. $\int_1^{+\infty} \frac{dx}{\sqrt{(x^2-1)(x^2+3)}}$

132. $\int_1^{+\infty} \frac{dx}{(3x^2+1)\sqrt{(x^2-1)(x^2+3)}}$

133. $\int_0^{\frac{\pi}{3}} \frac{\cos^2 x}{\sqrt{1+\cos^2 x}} dx$

134. $\int \mathrm{W}(x)\, dx$

135. $\int \ln\left(\frac{x}{\ln\left(\frac{x}{\ln\left(\frac{x}{\dots}\right)}\right)}\right) dx$

136. $\int_1^{+\infty} \frac{dx}{\sqrt{x^4-1}}$

137. $\int_{\frac{1}{2}}^{1} \frac{\Psi(x)}{1+\Gamma^2(x)}\, dx$

138. $\int_{-\infty}^{+\infty} \sin(x^2 + 2x + 2)\, dx$

139. $\int_{0}^{+\infty} e^{-ax^2} \sin(bx^2)\, dx$

140. $\int_{0}^{+\infty} e^{-ax^2} \cos(bx^2)\, dx$

141. $\int_{0}^{1} \frac{1}{\sqrt{x^p - x^2}}\, dx$ **;** $1 < p < 2$

Solutions of Exercises

1. $\int_0^{+\infty} x^3 e^{-x} dx = \int_0^{+\infty} x^{4-1} e^{-x} dx = \Gamma(4) = 3! = 6$

2. $\int_0^{+\infty} x^6 e^{-2x} dx$, let $y = 2x$, then $dy = 2dx$, then:

$\int_0^{+\infty} x^6 e^{-2x} dx = \frac{1}{2}\int_0^{+\infty} \left(\frac{y}{2}\right)^6 e^{-y} dy = \frac{1}{2^7}\int_0^{+\infty} y^6 e^{-y} dy = \frac{1}{2^7}\,\Gamma(7) = \frac{6!}{2^7} = \frac{45}{8}$

3. $\int_0^1 x^4 (1-x)^3 dx = \mathrm{B}(5;4) = \frac{\Gamma(5)\Gamma(4)}{\Gamma(5+4)} = \frac{\Gamma(5)\Gamma(4)}{\Gamma(9)} = \frac{4!3!}{8!} = \frac{1}{280}$

4. $\int_0^2 \frac{x^2}{\sqrt{2-x}} dx$, let $x = 2y$, then $dx = 2dy$, for $x = 0$, $y = 0$ and for $x = 2$, $y = 1$, then:

$\int_0^2 \frac{x^2}{\sqrt{2-x}} dx = \int_0^1 \frac{4y^2}{\sqrt{2-2y}} 2dy = \frac{8}{\sqrt{2}}\int_0^1 \frac{y^2}{\sqrt{1-y}} dy = 4\sqrt{2}\int_0^1 y^2 (1-y)^{-\frac{1}{2}} dy = 4\sqrt{2}\mathrm{B}\left(3,\frac{1}{2}\right)$

$\int_0^2 \frac{x^2}{\sqrt{2-x}} dx = \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{4\sqrt{2}\Gamma\left(3+\frac{1}{2}\right)} = \frac{\Gamma(3)\Gamma\left(\frac{1}{2}\right)}{4\sqrt{2}\Gamma\left(\frac{7}{2}\right)} = \frac{64\sqrt{2}}{15}$

5. $\int_0^{+\infty} \sqrt{x} e^{-x^3} dx$, let $y = x^3$, then $dy = 3x^2 dx = 3(x^3)^{\frac{2}{3}} dx = 3y^{\frac{2}{3}} dx$, so $dx = \frac{1}{3} y^{-\frac{2}{3}} dy$

So, $\int_0^{+\infty} \sqrt{x} e^{-x^3} dx = \frac{1}{3}\int_0^{+\infty} \sqrt{y^{\frac{1}{3}}} e^{-y} y^{-\frac{2}{3}} dy = \frac{1}{3}\int_0^{+\infty} y^{\frac{1}{6}} e^{-y} y^{-\frac{2}{3}} dy = \frac{1}{3}\int_0^{+\infty} y^{-\frac{1}{2}} e^{-y} dy$

$\int_0^{+\infty} \sqrt{x} e^{-x^3} dx = \frac{1}{3}\Gamma\left(\frac{1}{2}\right) = \frac{1}{3}\sqrt{\pi}$

6. $\int_0^{+\infty} 3^{-4x^2} dx = \int_0^{+\infty} \left(e^{\ln 3}\right)^{-4x^2} dx = \int_0^{+\infty} e^{-(4\ln 3)x^2} dx$, let $y = (4\ln 3)x^2$, then:

$dy = (4\ln 3).2x dx = 2(4\ln 3)\frac{y^{\frac{1}{2}}}{\sqrt{4\ln 3}} dx = 2\sqrt{4\ln 3}\, y^{\frac{1}{2}} dx$, so $dx = \frac{1}{2\sqrt{4\ln 3}} y^{-\frac{1}{2}} dy$, then:

$\int_0^{+\infty} 3^{-4x^2} dx = \int_0^{+\infty} 3^{-4x^2} dx = \frac{1}{2\sqrt{4\ln 3}}\int_0^{+\infty} y^{-\frac{1}{2}} e^{-y} dy = \frac{1}{2\sqrt{4\ln 3}}\,\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{4\sqrt{\ln 3}}$

7. $\int_0^1 x^4 \left(1-\sqrt{x}\right)^5 dx$, let $t = \sqrt{x}$, so $x = t^2$ and $dx = 2tdt$, then:

$\int_0^1 x^4 \left(1-\sqrt{x}\right)^5 dx = \int_0^1 (t^2)^4 (1-t)^5 (2tdt) = 2\int_0^1 t^9 (1-t)^5 dt = 2\int_0^1 t^{10-1} (1-t)^{6-1} dt$

$= 2\mathrm{B}(10;6) = 2\frac{\Gamma(10)\Gamma(6)}{\Gamma(16)} = 2\frac{9!5!}{15!} = \frac{2\times1\times2\times3\times4\times5}{15\times14\times13\times12\times11\times10} = \frac{1}{15015}$

8. $\int_0^a x^4\sqrt{a^2-x^2}\, dx$, let $x^2 = a^2 y$, then $x = a\sqrt{y}$ and $dx = \frac{a}{2\sqrt{y}} dy$

For $x = 0$, $y = 0$ and for $x = a$, $x = a$, then we get:

$\int_0^a x^4\sqrt{a^2-x^2}\, dx = \int_0^1 a^4 y^2 \sqrt{a^2-a^2 y}\frac{a}{2\sqrt{y}} dy = \frac{1}{2}a^6\int_0^1 y^{\frac{3}{2}}(1-y)^{\frac{1}{2}} dy = \frac{1}{2}a^6 \mathrm{B}\left(\frac{5}{2},\frac{3}{2}\right)$

9. $\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta$, we have $\mathrm{B}(m,n) = 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta \sin^{2m-1}\theta\, d\theta$, so we get:

$2m-1 = 6$, then $m = \frac{7}{2}$ and $2n-1 = 0$, then $n = \frac{1}{2}$, so we get:

$\int_0^{\frac{\pi}{2}} \sin^6\theta\, d\theta = \frac{1}{2}\mathrm{B}\left(\frac{7}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{7}{2}+\frac{1}{2}\right)} = \frac{\Gamma\left(\frac{7}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(4)} = \frac{5\pi}{32}$

10. $\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^5\theta\, d\theta$, we have $\mathrm{B}(m,n) = 2\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta \sin^{2m-1}\theta\, d\theta$, so we get:

$2m-1=4$ and $2n-1=5$, so $m=\frac{5}{2}$ and $n=3$, so we get:

$\int_0^{\frac{\pi}{2}} \sin^4\theta\cos^5\theta\, d\theta = \frac{1}{2}\mathrm{B}\left(\frac{7}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{2\Gamma\left(\frac{5}{2}+3\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{2\Gamma\left(\frac{11}{2}\right)} = \frac{8}{315}$

11. $\mathrm{I} = \int_0^1 \ln(-\ln x)\, dx$, let $t=-\ln x$, then $x=e^{-t}$ and $dx=-e^{-t}dt$, for $x=0$, $t=+\infty$ and for $x=1$, $t=0$, then we get:

$\mathrm{I} = \int_{+\infty}^0 \ln t\,(-e^{-t}dt) = \int_0^{+\infty} e^{-t}dt = -\gamma$

12. $\mathrm{I} = \int_0^{+\infty} x^2e^{-x^2}\,dx = \int_0^{+\infty} x\left(xe^{-x^2}\right)dx$, using IBP:

Let $u=x$, then $u'=1$ and let $v'=xe^{-x^2}$, then $v'=-\frac{1}{2}e^{-x^2}$, then we get:

$\mathrm{I} = \left[-\frac{1}{2}xe^{-x^2}\right]_0^{+\infty} + \frac{1}{2}\int_0^{+\infty} e^{-x^2}\,dx = 0+\frac{1}{2}\left(\frac{\sqrt{\pi}}{2}\right) = \frac{\sqrt{\pi}}{4}$

13. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx = \int_0^{\frac{\pi}{2}}(\sin x)^{\frac{1}{2}}\,dx = \int_0^{\frac{\pi}{2}}(\sin x)^{2\left(\frac{3}{4}\right)-1}(\cos x)^{2\left(\frac{1}{2}\right)-1}\,dx$, then

$\mathrm{I} = \frac{1}{2}\left(2\int_0^{\frac{\pi}{2}}(\sin x)^{2\left(\frac{3}{4}\right)-1}(\cos x)^{2\left(\frac{1}{2}\right)-1}\,dx\right) = \frac{1}{2}\mathrm{B}\left(\frac{3}{4},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{3}{4}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{3}{4}\right)}{2\Gamma\left(\frac{5}{4}\right)}$

14. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin x}}\,dx = \int_0^{\frac{\pi}{2}}(\sin x)^{-\frac{1}{2}}\,dx = \int_0^{\frac{\pi}{2}}(\sin x)^{2\left(\frac{1}{4}\right)-1}(\cos x)^{2\left(\frac{1}{2}\right)-1}\,dx$, then

$\mathrm{I} = \frac{1}{2}\left(2\int_0^{\frac{\pi}{2}}(\sin x)^{2\left(\frac{1}{4}\right)-1}(\cos x)^{2\left(\frac{1}{2}\right)-1}\,dx\right) = \frac{1}{2}\mathrm{B}\left(\frac{1}{4},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}$

15. $\mathrm{I} = \int_0^{+\infty} \frac{x^a}{a^x}\,dx$, let $e^t = a^x$, then $t = x\ln a$ and so $dt = dx\ln a$, so we get:

$\mathrm{I} = \int_0^{+\infty}\left(\frac{t}{\ln a}\right)^a e^{-t}\frac{dt}{\ln a} = \left(\frac{1}{\ln a}\right)^{a+1}\int_0^{+\infty} t^a e^{-t}dt = \left(\frac{1}{\ln a}\right)^{a+1}\int_0^{+\infty} t^{(a+1)-1}e^{-t}dt$

$= \left(\frac{1}{\ln a}\right)^{a+1}\Gamma(a+1)$

16. $\int_0^1 \frac{dx}{\sqrt{-\ln x}}$, let $y=-\ln x$, then $x=e^{-y}$ and $dx=-e^{-y}dy$, for $x=0$, $y=+\infty$ and for $x=1$, $y=0$, so $\int_0^1 \frac{dx}{\sqrt{-\ln x}} = \int_{+\infty}^0 \frac{dx}{\sqrt{y}}(-e^{-y}dy) = \int_0^{+\infty} y^{-\frac{1}{2}}e^{-y}dy = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

17. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\left(\frac{\tan x}{n}\right)^n dx = \frac{1}{n^n}\int_0^{\frac{\pi}{2}}\tan^n x\,dx = \frac{1}{n^n}\int_0^{\frac{\pi}{2}}\left(\frac{\sin x}{\cos x}\right)^n dx = \frac{1}{n^n}\int_0^{\frac{\pi}{2}}\sin^n x\cos^{-n}x\,dx$

$\mathrm{I} = \frac{1}{2n^n}\left(2\int_0^{\frac{\pi}{2}}\sin^{2\left(\frac{n+1}{2}\right)-1}x\cos^{2\left(\frac{1-n}{2}\right)-1}x\,dx\right) = \frac{1}{2n^n}\mathrm{B}\left(\frac{n+1}{2},\frac{1-n}{2}\right) = \frac{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2n^n\Gamma\left(\frac{n+1}{2}+\frac{1-n}{2}\right)}$

$\mathrm{I} = \frac{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2n^n\Gamma(1)} = \frac{1}{2n^n}\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right) = \frac{1}{2n^n}\frac{\pi}{\sin\left(\frac{\pi}{2}+n\frac{\pi}{2}\right)} = \frac{1}{2n^n}\frac{\pi}{\cos\left(n\frac{\pi}{2}\right)} = \frac{1}{2n^n}\sec\left(n\frac{\pi}{2}\right)$

18. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\sqrt{\cot x+\tan x}\,dx=\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}\,dx=\int_0^{\frac{\pi}{2}}\sqrt{\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}}\,dx=\int_0^{\frac{\pi}{2}}\sqrt{\frac{1}{\sin x\cos x}}\,dx$

$\mathrm{I}=\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}x\cos^{-\frac{1}{2}}x\,dx=\frac{1}{2}\left(2\int_0^{\frac{\pi}{2}}\sin^{2\left(\frac{1}{4}\right)-1}x\cos^{2\left(\frac{1}{4}\right)-1}x\,dx\right)$, then we get:

$\mathrm{I}=\frac{1}{2}\mathrm{B}\left(\frac{1}{4},\frac{1}{4}\right)=\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{1}{4}+\frac{1}{4}\right)}=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{1}{2}\right)}=\frac{\Gamma^2\left(\frac{1}{4}\right)}{2\sqrt{\pi}}$

19. $\int \mathrm{Ei}(x)dx$, let $u=\mathrm{Ei}(x)$, then $u'=\frac{e^x}{x}$ and let $v'=1$, so $v=x$, then we get:

$\int \mathrm{Ei}(x)dx=x\mathrm{Ei}(x)-\int\frac{e^x}{x}.xdx=\ x\mathrm{Ei}(x)-\int e^x dx=x\mathrm{Ei}(x)-e^x+c$

20. $\int \mathrm{erf}(x)\,dx$, let $u=\mathrm{erf}(x)$, then $u'=\frac{2}{\sqrt{\pi}}e^{-x^2}$ and let $v'=1$, then $v=x$, then we get:

$\int \mathrm{erf}(x)\,dx=x\,\mathrm{erf}(x)-\frac{2}{\sqrt{\pi}}\int xe^{-x^2}\,dx=x\,\mathrm{erf}(x)-\frac{1}{\sqrt{\pi}}\int 2xe^{-x^2}\,dx$

$\int \mathrm{erf}(x)\,dx=\ x\,\mathrm{erf}(x)+\frac{1}{\sqrt{\pi}}e^{-x^2}+c$

21. $\int e^x\ln x\,dx$, let $u=\ln x$, then $u'=\frac{1}{x}$ and let $v'=e^x$, then $v=e^x$, so we get:

$\int e^x\ln x\,dx=e^x\ln x-\int\frac{1}{x}.e^x dx=e^x\ln x-\int\frac{e^x}{x}dx=e^x\ln x-\mathrm{Ei}(x)+\mathrm{c}$

22. $\mathrm{I}=\int_0^{+\infty}\frac{dx}{\left(1+x^2\right)^\alpha}$, let $x=\tan\theta$, then $dx=\sec^2\theta\,d\theta$, for $x=0$, $\theta=0$ and for $x=+\infty$, $\theta=\frac{\pi}{2}$

$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta}{\left(1+\tan^2\theta\right)^\alpha}d\theta=\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta}{\left(\sec^2\theta\right)^\alpha}d\theta=\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta}{\left(1+\tan^2\theta\right)^\alpha}d\theta=\int_0^{\frac{\pi}{2}}\frac{\sec^2\theta}{\sec^{2\alpha}\theta}d\theta$

$\mathrm{I}=\int_0^{\frac{\pi}{2}}\sec^{2-2\alpha}\theta\,d\theta=\int_0^{\frac{\pi}{2}}\cos^{2\alpha-2}\theta\,d\theta=\int_0^{\frac{\pi}{2}}\cos^{2\left(\frac{2\alpha-1}{2}\right)-1}\theta\sin^{2\left(\frac{1}{2}\right)-1}\theta\,d\theta$

$\mathrm{I}=\frac{1}{2}\left(2\int_0^{\frac{\pi}{2}}\cos^{2\left(\frac{2\alpha-1}{2}\right)-1}\theta\sin^{2\left(\frac{1}{2}\right)-1}\theta\,d\theta\right)=\frac{1}{2}\mathrm{B}\left(\frac{2\alpha-1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{2\alpha-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2\alpha-1}{2}+\frac{1}{2}\right)}$

$\mathrm{I}=\frac{\Gamma\left(\frac{2\alpha-1}{2}\right)\sqrt{\pi}}{2\Gamma(\alpha)}=\frac{\sqrt{\pi}}{2}\frac{\Gamma\left(\frac{2\alpha-1}{2}\right)}{\Gamma(\alpha)}$

23. $\int_0^1(x-x^2)^n dx=\int_0^1\left(x(1-x)\right)^n dx=\int_0^1 x^n(1-x)^n dx=\int_0^1 x^{(n+1)-1}(1-x)^{(n+1-1)}dx$

$=\mathrm{B}(n+1,n+1)=\frac{\Gamma(n+1)\Gamma(n+1)}{\Gamma(n+1+n+1)}=\frac{[\Gamma(n+1)]^2}{\Gamma(2n+2)}=\frac{(n!)^2}{(2n+1)!}$

24. $\int_0^1\left(1-x^{\frac{1}{n}}\right)^n dx$, let $x^{\frac{1}{n}}=\sin^2\theta$, then $x=\sin^{2n}\theta$ and $dx=2n\sin^{2n-1}\theta\cos\theta\,d\theta$

For $x=0$, $\theta=0$ and for $x=1$, $\theta=\frac{\pi}{2}$, then we get:

$\int_0^1\left(1-x^{\frac{1}{n}}\right)^n dx=\int_0^1(1-\sin^2\theta)^n.2n\sin^{2n-1}\theta\cos\theta\,d\theta=n\left(2\int_0^1\cos^{2n+1}\theta\sin^{2n-1}\theta\,d\theta\right)$

$=n\mathrm{B}(n+1,n)=n\left(\frac{\Gamma(n+1)\Gamma(n)}{\Gamma(n+1+n)}\right)=n\left(\frac{\Gamma(n+1)\Gamma(n)}{\Gamma(2n+1)}\right)=n\left(\frac{n!(n-1)!}{(2n!)}\right)=\frac{n!(n-1)!n}{(2n!)}=\frac{n!n!}{(2n!)}$,

therefore $\int_0^1\left(1-x^{\frac{1}{n}}\right)^n dx=\frac{(n!)^2}{(2n!)}$

25. $\int e^{e^x} dx$, let $u = e^x$, then $du = e^x dx = udx$, so $dx = \frac{1}{u} du$, then we get:

$\int e^{e^x} dx = \int e^u . \frac{1}{u} du = \int \frac{e^u}{u} du = \mathrm{Ei}(u) + \mathrm{c} = \mathrm{Ei}(e^x) + \mathrm{c}$

26. $\int \ln(\ln x)\, dx$, let $u = \ln(\ln x)$, then $du = \frac{1}{x \ln x} dx$ and let $v' = 1$, then $v = x$, so we get:

$\int \ln(\ln x)\, dx = x \ln(\ln x) - \int x . \frac{1}{x \ln x} dx = x \ln(\ln x) - \int \frac{dx}{\ln x} = x \ln(\ln x) - \mathrm{li}(x) + \mathrm{c}$

27. $\mathrm{I} = \int_0^1 \frac{1}{\sqrt{1-x^3}} dx = \int_0^1 (1-x^3)^{-\frac{1}{2}} dx$, let $u = x^3$, then $x = u^{\frac{1}{3}}$, $dx = \frac{1}{3} u^{-\frac{2}{3}}$, then we get:

$\mathrm{I} = \frac{1}{3} \int_0^1 u^{-\frac{2}{3}} (1-u)^{-\frac{1}{2}} du = \frac{1}{3} \int_0^1 u^{\frac{1}{3}-1} (1-u)^{\frac{1}{2}-1} du = \frac{1}{3} \mathrm{B}\left(\frac{1}{3}, \frac{1}{2}\right) = \frac{1}{3} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{3}+\frac{1}{2}\right)} = \frac{1}{3} \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{6}\right)}$

$\mathrm{I} = \frac{\sqrt{\pi}}{3} \frac{\Gamma\left(\frac{1}{3}\right)}{\Gamma\left(\frac{5}{6}\right)} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{5}{6}\right)}$

28. $\mathrm{I} = \int_{-\infty}^{+\infty} e^{-e^{ax}} . e^{bx}\, dx$, let $u = e^{ax}$, then $u^{\frac{1}{a}} = e^x$ and $x = \frac{1}{a} \ln u$, then $dx = \frac{1}{au} du$

$e^{bx} = (e^x)^b = \left(u^{\frac{1}{a}}\right)^b = u^{\frac{b}{a}}$, for $x = -\infty$, $u = 0$ and for $x = +\infty$, $u = +\infty$, then:

$\mathrm{I} = \int_0^{+\infty} e^{-u} . u^{\frac{b}{a}} \frac{1}{au} du = \frac{1}{a} \int_0^{+\infty} e^{-u} . u^{\frac{b}{a}-1}\, du = \frac{1}{a} \Gamma\left(\frac{b}{a}\right)$

29. $\int_0^{+\infty} \frac{x^{m+1}}{(1+x^2)^2} dx$, let $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$, for $x = 0$, $\theta = 0$ and for $x = +\infty$, $\theta = \frac{\pi}{2}$,

$\int_0^{+\infty} \frac{x^{m+1}}{(1+x^2)^2} dx = \int_0^{\frac{\pi}{2}} \frac{\tan^{m+1}\theta}{\sec^4\theta} \sec^2\theta\, d\theta = \int_0^{\frac{\pi}{2}} \cos^2\theta . \frac{\sin^{m+1}\theta}{\cos^{m+1}\theta} d\theta$

$= \int_0^{\frac{\pi}{2}} \sin^{m+1}\theta \cos^{1-m} d\theta = \int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{m}{2}+1\right)-1}\theta \cos^{2\left(1-\frac{m}{2}\right)-1} d\theta = \frac{1}{2} \mathrm{B}\left(\frac{m}{2} + 1, 1 - \frac{m}{2}\right)$

$= \frac{1}{2} \frac{\Gamma\left(1+\frac{m}{2}\right)\Gamma\left(1-\frac{m}{2}\right)}{\Gamma\left(1+\frac{m}{2}+1-\frac{m}{2}\right)} = \frac{\Gamma\left(1+\frac{m}{2}\right)\Gamma\left(1-\frac{m}{2}\right)}{2}$

But $\Gamma(m)\Gamma(1-m) = \frac{\pi}{\sin(\pi m)}$, then $\Gamma\left(\frac{m}{2}\right)\Gamma\left(1-\frac{m}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi}{2}m\right)}$

so, $\frac{m}{2}\Gamma\left(\frac{m}{2}\right)\Gamma\left(1-\frac{m}{2}\right) = \frac{\pi m}{2\sin\left(\frac{\pi}{2}m\right)}$, then we get $\Gamma\left(1+\frac{m}{2}\right)\Gamma\left(1-\frac{m}{2}\right) = \frac{\pi m}{2\sin\left(\frac{\pi}{2}m\right)}$, therefore:

$\int_0^{+\infty} \frac{x^{m+1}}{(1+x^2)^2} dx = \frac{\pi m}{4\sin\left(\frac{\pi}{2}m\right)}$

30. $\mathrm{I} = \int_1^e \frac{dx}{1+\ln x}$, let $t = 1 + \ln x$, then $dt = \frac{1}{x} dx$, but $t - 1 = \ln x$, so $x = e^{t-1}$, then we get:

$dx = e^{t-1} dt$, for $x = 1$, $t = 1$ and for $x = e$, $t = 2$, then we get:

$\mathrm{I} = \int_1^2 \frac{e^{t-1}}{t} dt = \int_1^2 \frac{e^t e^{-1}}{t} dt = e^{-1} \int_1^2 \frac{e^t}{t} dt = \frac{1}{e} \int_1^2 \frac{e^t}{t} dt = \frac{1}{e} \int_{-\infty}^2 \frac{e^t}{t} dt - \frac{1}{e} \int_{-\infty}^1 \frac{e^t}{t} dt$

$\mathrm{I} = \frac{1}{e} \mathrm{Ei}(2) - \frac{1}{e} \mathrm{Ei}(1) = \frac{1}{e} [\mathrm{Ei}(2) - \mathrm{Ei}(1)]$

31. $\int_0^{+\infty} x^m e^{-ax^n} dx$, where m, n and a are constants

Let $y = ax^n$, then $x = \left(\frac{y}{a}\right)^{\frac{1}{n}} = \left(\frac{1}{a}\right)^{\frac{1}{n}} y^{\frac{1}{n}}$ and $dx = \frac{1}{n}\left(\frac{1}{a}\right)^{\frac{1}{n}} y^{\frac{1}{n}-1} dy$, so we get:

$\int_0^{+\infty} x^m e^{-ax^n} dx = \int_0^{+\infty} \left(\frac{y}{a}\right)^{\frac{m}{n}} e^{-y} \frac{1}{n}\left(\frac{1}{a}\right)^{\frac{1}{n}} y^{\frac{1}{n}-1} dy = \frac{1}{na^{\frac{m+1}{n}}} \int_0^{+\infty} y^{\frac{m+1}{n}-1} e^{-y} dy$

$\int_0^{+\infty} x^m e^{-ax^n} dx = \frac{1}{na^{\frac{m+1}{n}}} \Gamma\left(\frac{m+1}{n}\right)$

32. $\mathrm{I} = \int_0^{+\infty} e^{-x^{\left(\frac{1}{m^2}+\frac{1}{n^2}\right)}} dx$, let $u = x^{\frac{1}{m^2}+\frac{1}{n^2}} = x^{\frac{m^2+n^2}{(mn)^2}}$, then $x = u^{\frac{(mn)^2}{m^2+n^2}}$ and

$dx = \frac{(mn)^2}{m^2+n^2} . u^{\frac{(mn)^2}{m^2+n^2}-1}$, then we get:

$\mathrm{I} = \int_0^{+\infty} e^{-u}\, u^{\frac{(mn)^2}{m^2+n^2}-1} \frac{(mn)^2}{m^2+n^2} du = \frac{(mn)^2}{m^2+n^2} \Gamma\left(\frac{(mn)^2}{m^2+n^2}\right) = \Gamma\left(1 + \frac{(mn)^2}{m^2+n^2}\right)$, therefore

$\mathrm{I} = \Gamma\left(\frac{m^2+m^2n^2+n^2}{m^2+n^2}\right)$

33. $\mathrm{I} = \int_{-a}^{+\infty} \frac{e^{-px}}{\sqrt{x+a}} dx$, let, $t = \sqrt{x+a}$, then $t^2 = x + a$ and $dx = 2tdt$

For $x = -a$, then $t = 0$ and for $x = +\infty$, then $t = +\infty$, then we get:

$\mathrm{I} = \int_0^{+\infty} \frac{e^{-p\left(t^2-a\right)}}{t} . 2tdt = 2\int_0^{+\infty} e^{-pt^2+ap} dt = 2e^{ap} \int_0^{+\infty} e^{-pt^2} dt$, let $u = \sqrt{p}t$, $du = \sqrt{p}dt$,

so: $\mathrm{I} = 2e^{ap} \int_0^{+\infty} e^{-u^2} . \frac{du}{\sqrt{p}} = \frac{2}{\sqrt{p}} e^{ap} \int_0^{+\infty} e^{-u^2} du = \frac{1}{\sqrt{p}} e^{ap} \int_{-\infty}^{+\infty} e^{-u^2} du = \frac{1}{\sqrt{p}} e^{ap} \sqrt{\pi}$

Therefore, we get: $\mathrm{I} = \sqrt{\frac{\pi}{p}} . e^{ap}$

34. $\int_0^2 x\sqrt[3]{8-x^3} dx$, let $x^3 = 8y$, then $x = 2y^{\frac{1}{3}}$ and $dx = \frac{2}{3} y^{-\frac{2}{3}} dy$

For $x = 0$, $y = 0$ and for $x = 2$, $y = 1$, then we get:

$\int_0^2 x\sqrt[3]{8-x^3} dx = \int_0^1 2y^{\frac{1}{3}} . \sqrt[3]{8-8y} \frac{2}{3} y^{-\frac{2}{3}} dy = \frac{4}{3} \int_0^1 y^{-\frac{1}{3}} . \sqrt[3]{2^3(1-y)}\, dy$

$= \frac{8}{3} \int_0^1 y^{-\frac{1}{3}} . \sqrt[3]{1-y}\, dy = \frac{8}{3} \int_0^1 y^{-\frac{1}{3}} . (1-y)^{\frac{1}{3}} dy = \frac{8}{3} \mathrm{B}\left(\frac{2}{3}, \frac{4}{3}\right) = \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)} = \frac{8}{3} \frac{\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{4}{3}\right)}{\Gamma(2)}$

$\int_0^2 x\sqrt[3]{8-x^3} dx = \frac{8}{9} \Gamma\left(\frac{1}{3}\right) \Gamma\left(\frac{2}{3}\right) = \frac{8}{9} . \frac{\pi}{\sin\left(\frac{\pi}{3}\right)} = \frac{16\pi}{9\sqrt{3}}$

35. $\int_0^1 \sqrt{1-x^4} dx$, let $x^4 = y$, then $x = y^{\frac{1}{4}}$ and $dx = \frac{1}{4} y^{-\frac{3}{4}} dy$, then we get:

$\int_0^1 \sqrt{1-x^4} dx = \int_0^1 \sqrt{1-y} . \frac{1}{4} y^{-\frac{3}{4}} dy = \frac{1}{4} \int_0^1 y^{-\frac{3}{4}} (1-y)^{\frac{1}{2}} dy = \frac{1}{4} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{7}{4}\right)} = \frac{\sqrt{\pi}}{6} \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}$

With $\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$, therefore $\int_0^1 \sqrt{1-x^4} dx = \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{6\sqrt{2\pi}}$

36. $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{\frac{\sin x}{\cos x}} dx = \int_0^{\frac{\pi}{2}} \sin^{\frac{1}{2}} x \cos^{-\frac{1}{2}} x\, dx = \int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{3}{4}\right)-1} x \cos^{2\left(\frac{1}{4}\right)-1} x\, dx$

$= \frac{1}{2} \times 2\int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{3}{4}\right)-1} x \cos^{2\left(\frac{1}{4}\right)-1} x\, dx = \frac{1}{2} \mathrm{B}\left(\frac{3}{4}, \frac{1}{4}\right) = \frac{1}{2} \times \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)} = \frac{1}{2} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right)$

$= \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(1-\frac{3}{4}\right)$, but $\Gamma(a)\Gamma(1-a) = \frac{\pi}{\sin(\pi a)}$ **(Euler's Reflection Formula)**

For $a = \frac{3}{4}$, we get $\int_0^{\frac{\pi}{2}} \sqrt{\tan x}\, dx = \frac{1}{2}\frac{\pi}{\sin\left(\frac{3}{4}\pi\right)} = \frac{\sqrt{2}}{2}\pi$

37. $I = \int_0^1 (\ln x)^{\frac{1}{n}} dx$, let $-t = \ln x \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}dt$ and $\begin{cases} \text{for } x = 0 & t = +\infty \\ \text{for } x = 1 & t = 0 \end{cases}$, then:

$I = \int_0^{+\infty} (-t)^{\frac{1}{n}}.e^{-t}dt = \int_0^{+\infty} \left(e^{i\pi}\right)^{\frac{1}{n}}.t^{\frac{1}{n}}.e^{-t}dt = \int_0^{+\infty} e^{i\left(\frac{\pi}{n}\right)}.e^{-t}.t^{\frac{1}{n}+1-1}dt$

$I = e^{i\left(\frac{\pi}{n}\right)} \int_0^{+\infty} e^{-t}.t^{\frac{1}{n}+1-1}dt = e^{i\left(\frac{\pi}{n}\right)}\Gamma\left(1+\frac{1}{n}\right) = \left[\cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)\right]\Gamma\left(1+\frac{1}{n}\right)$

38. $I = \int_{-\infty}^{+\infty} e^{-e^{-nx}}.e^{-nx}.x\,dx$, let $x = -\frac{1}{n}\ln t$, $nx = -\ln t$ and $t = e^{-nx}$, then

$dt = -ne^{-nx}dx$, for $x = -\infty$, $t = +\infty$ and for $x = +\infty$, $t = 0$, then we get:

$I = \int_{+\infty}^{0} e^{-t}.t.\left(-\frac{1}{n}\ln t\right)\left(-\frac{1}{nt}dt\right) = -\frac{1}{n^2}\int_0^{+\infty} e^{-t}\ln t\, dt = \frac{\gamma}{n^2}$

39. $I = \int_0^1 \frac{1}{\sqrt{1+x^4}}dx$, let $x^2 = \tan u$, then $x = \sqrt{\tan u}$ and $dx = \frac{\sec^2 u}{\sqrt{\tan u}}du$

For $x = 0$, $u = 0$ and for $x = 1$, $u = \frac{\pi}{4}$, then we get:

$I = \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2 u}}\frac{\sec^2 u}{\sqrt{\tan u}}du = \int_0^{\frac{\pi}{4}} \frac{1}{\sec u}\frac{\sec^2 u}{\sqrt{\tan u}}du = \int_0^{\frac{\pi}{4}} \frac{\sec u}{\sqrt{\tan u}}du = \int_0^{\frac{\pi}{4}} \frac{\sec u}{\sqrt{\tan u}}du = \int_0^{\frac{\pi}{4}} \frac{du}{\sqrt{\sin u \cos u}}$

$I = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{du}{\sqrt{2\sin u\cos u}} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}} \frac{du}{\sqrt{\sin 2u}}$, let $t = 2u$, then $dt = 2du$, then we get:

$I = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{\sin t}} = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^{-\frac{1}{2}} t\, dt = \frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{1}{4}\right)-1} t \cos^{2\left(\frac{1}{2}\right)-1} t\, dt$

$I = \frac{1}{4\sqrt{2}}\left(2\int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{1}{4}\right)-1} t\cos^{2\left(\frac{1}{2}\right)-1} t\, dt\right) = \frac{1}{2\sqrt{2}}B\left(\frac{1}{4},\frac{1}{2}\right) = \frac{1}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)} = \frac{\sqrt{\pi}}{2\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}$

40. $\int_0^1 \frac{x^5}{\sqrt{1-x^4}}dx$, let $u = x^4$, then $du = 4x^3dx$, for $x = 0$, $u = 0$ and for $x = 1$, $u = 1$, so:

$\int_0^1 \frac{x^5}{\sqrt{1-x^4}}dx = \int_0^1 \frac{x^2.x^3}{\sqrt{1-x^4}}dx = \frac{1}{4}\int_0^1 \frac{u^{\frac{1}{2}}}{\sqrt{1-u}}dx = \frac{1}{4}\int_0^1 u^{\frac{1}{2}}(1-u)^{-\frac{1}{2}}du$

$= \frac{1}{4}\int_0^1 u^{\frac{3}{2}-1}(1-u)^{\frac{1}{2}-1}du = \frac{1}{4}B\left(\frac{3}{2},\frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{2}\right)} = \frac{1}{4}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{1}{8}\sqrt{\pi}\sqrt{\pi} = \frac{\sqrt{\pi}}{8}$

41. $I = \int_0^1 \sqrt{\sqrt{\ln x}}\, dx = \int_0^1 (\ln x)^{\frac{1}{4}}dx$, let $-t = \ln x$, then $x = e^{-t}$ and $dx = -e^{-t}dt$

For $x = 0$, then $t = +\infty$ and for $x = 1$, $t = 0$, then we get:

$I = \int_{+\infty}^{0} (-t)^{\frac{1}{4}}\left(-e^{-t}dt\right) = \int_0^{+\infty} (-1)^{\frac{1}{4}}t^{\frac{1}{4}}e^{-t}dt = (-1)^{\frac{1}{4}}\int_0^{+\infty} e^{-t}t^{\frac{1}{4}}dt = (-1)^{\frac{1}{4}}\int_0^{+\infty} e^{-t}t^{\frac{5}{4}-1}dt$

$I = (-1)^{\frac{1}{4}}\boldsymbol{\Gamma}\left(\frac{5}{4}\right) = \left(e^{i\pi}\right)^{\frac{1}{4}}\boldsymbol{\Gamma}\left(\frac{5}{4}\right) = e^{i\frac{\pi}{4}}\boldsymbol{\Gamma}\left(\frac{5}{4}\right) = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)\boldsymbol{\Gamma}\left(\frac{5}{4}\right)$

42. $\int_0^{+\infty}\frac{dx}{\cosh^n x}=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{2^n}{(e^x+e^{-x})^n}dx=2^{n-1}\int_{-\infty}^{+\infty}\frac{e^{nx}}{e^{nx}(e^x+e^{-x})^n}dx=2^{n-1}\int_{-\infty}^{+\infty}\frac{e^{nx}}{\left(e^{2x}+1\right)^n}dx$

$=2^{n-1}\int_{-\infty}^{+\infty}\frac{e^{nx}dx}{\left(e^{2x}+1\right)^n}.\frac{de^{2x}}{dx}.\frac{e^{-2x}}{2}=2^{n-2}\int_{-\infty}^{+\infty}\frac{\left(e^{2x}\right)^{\frac{n}{2}-1}de^{2x}}{\left(e^{2x}+1\right)^n}=2^{n-1}\mathrm{B}\left(\frac{n}{2},\frac{n}{2}\right)=\frac{\sqrt{\pi}\Gamma\left(\frac{n}{2}\right)}{2\Gamma\left(\frac{n+1}{2}\right)}$

43. $\mathrm{I}=\int_0^1\frac{1}{\sqrt{x\ln\left(\frac{1}{x}\right)}}dx$, let $u=\ln\left(\frac{1}{x}\right)$, so $u=-\ln x$ and $x=e^{-u}$, so $dx=-e^{-u}du$

for $x=0$, $u=+\infty$ and for $x=1$, $u=0$, then we get:

$\mathrm{I}=\int_{+\infty}^0\frac{1}{\sqrt{e^{-u}.u}}(-e^{-u}du)=\int_0^{+\infty}\frac{e^{-u}}{\sqrt{e^{-u}.u}}du=\int_0^{+\infty}e^{-\frac{1}{2}u}u^{-\frac{1}{2}}du$

Let $t=\frac{1}{2}u$, then $u=2t$ and $du=2dt$, then we get:

$\mathrm{I}=\int_0^{+\infty}e^{-t}2^{-\frac{1}{2}}t^{-\frac{1}{2}}.2dt=\sqrt{2}\int_0^{+\infty}e^{-t}.t^{\frac{1}{2}-1}dt=\sqrt{2}\Gamma\left(\frac{1}{2}\right)=\sqrt{2\pi}$

44. $\mathrm{I}=\int_{-\infty}^{+\infty}e^{-(ax^2+bx+c)}\,dx=e^{-c}\int_{-\infty}^{+\infty}e^{-(ax^2+bx)}\,dx=e^{-c}\int_{-\infty}^{+\infty}e^{-a\left[x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}\right]}dx$

$\mathrm{I}=e^{-c}\int_{-\infty}^{+\infty}e^{-a\left[\left(x+\frac{b}{2a}\right)^2-\frac{b^2}{4a^2}\right]}dx=e^{-c}\int_{-\infty}^{+\infty}e^{-a\left(x+\frac{b}{2a}\right)^2+\frac{b^2}{4a^2}}dx=e^{-c}e^{\frac{b^2}{4a^2}}\int_{-\infty}^{+\infty}e^{-a\left(x+\frac{b}{2a}\right)^2}dx$

$\mathrm{I}=e^{\left(-c+\frac{b^2}{4a^2}\right)}\int_{-\infty}^{+\infty}e^{-a\left(x+\frac{b}{2a}\right)^2}dx$, let $u=x+\frac{b}{2a}$, then $du=dx$, so we get:

$\mathrm{I}=e^{\left(-c+\frac{b^2}{4a^2}\right)}\int_{-\infty}^{+\infty}e^{-au^2}du=2e^{\left(-c+\frac{b^2}{4a^2}\right)}\int_0^{+\infty}e^{-au^2}du$, let $t=\sqrt{a}u$, then $dt=\sqrt{a}du$, so:

$\mathrm{I}=2e^{\left(-c+\frac{b^2}{4a^2}\right)}\int_0^{+\infty}e^{-t^2}\frac{dt}{\sqrt{a}}=\frac{2}{\sqrt{a}}e^{\left(-c+\frac{b^2}{4a^2}\right)}\int_0^{+\infty}e^{-t^2}dt=\frac{2}{\sqrt{a}}e^{\left(-c+\frac{b^2}{4a^2}\right)}\left(\frac{\sqrt{\pi}}{2}\right)$, therefore, we get: $\mathrm{I}=e^{\left(-c+\frac{b^2}{4a^2}\right)}\sqrt{\frac{\pi}{a}}$

45. $\int_0^{\frac{\pi}{2}}\sin^p\theta\,d\theta=\int_0^{\frac{\pi}{2}}\sin^p\theta\cos^0\theta\,d\theta$, so $2m-1=p$ and $2n-1=0$, then $m=\frac{1}{2}(p+1)$

and $n=\frac{1}{2}$, then $\int_0^{\frac{\pi}{2}}\sin^p\theta\,d\theta=\frac{1}{2}\mathrm{B}\left(\frac{1}{2}(p+1),\frac{1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}(p+1)\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}p+\frac{1}{2}+\frac{1}{2}\right)}=\frac{\Gamma\left(\frac{1}{2}(p+1)\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}(p+2)\right)}$

If $p=2r$, then the integral becomes:

$\int_0^{\frac{\pi}{2}}\sin^p\theta\,d\theta=\frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(r+1)}=\frac{\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\ldots\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)}{2r(r-1)\ldots1}=\frac{(2r-1)(2r-3)\ldots1}{2r(2r-2)\ldots2}\frac{\pi}{2}=\frac{1\times3\times5\times\ldots\times(2r-1)}{2\times4\times6\times\ldots\times2r}\frac{\pi}{2}$

If $p=2r+1$, then the integral becomes:

$\int_0^{\frac{\pi}{2}}\sin^p\theta\,d\theta=\frac{\Gamma(r+1)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(r+\frac{3}{2}\right)}=\frac{r(r-1)\ldots1.\sqrt{\pi}}{2\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)\ldots\frac{1}{2}.\sqrt{\pi}}=\frac{(2r-1)(2r-3)\ldots1}{2r(2r-2)\ldots2}\frac{\pi}{2}=\frac{2\times4\times6\times\ldots\times2r}{1\times3\times5\times\ldots\times(2r+1)}$

Remark: $\int_0^{\frac{\pi}{2}}\cos^p\theta\,d\theta=\int_0^{\frac{\pi}{2}}\sin^p\theta\,d\theta$ as seen letting $\theta=\frac{\pi}{2}-\varphi$

46. $\mathrm{I}=\int_0^{\pi} x\sin^m x\,dx$, let $u=\pi-x$, then $du=-dx$, for $x=0$, $u=\pi$, for $x=\pi$, $u=0$, then:

$\mathrm{I}=\int_{\pi}^{0}(\pi-u)\sin^m(\pi-u)(-du)=\int_0^{\pi}(\pi-u)\sin^m u\,du=\int_0^{\pi}(\pi-x)\sin^m x\,dx$

$\mathrm{I}=\pi\int_0^{\pi}\sin^m x\,dx-\int_0^{\pi}x\sin^m x\,dx=\pi\int_0^{\pi}\sin^m x\,dx-\mathrm{I}$, so $2\mathrm{I}=\pi\int_0^{\pi}\sin^m x\,dx$, then:

$2\mathrm{I}=2\pi\int_0^{\frac{\pi}{2}}\sin^m x\,dx$, then $\mathrm{I}=\pi\int_0^{\frac{\pi}{2}}\sin^m x\,dx=\pi\int_0^{\frac{\pi}{2}}\sin^m x\cos^0 x\,dx$, then:

$$\mathrm{I}=\pi\frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{m+2}{2}\right)}=\frac{\sqrt{\pi}\left(\frac{m-1}{2}\right)!}{\left(\frac{m}{2}\right)!}\frac{\pi}{2}$$

47. $\int_{-\infty}^{+\infty}\frac{e^{ax}}{1+e^x}dx$, let $u=e^{ax}$, then $du=ae^{ax}dx$, for $x=-\infty$, $u=0$ and for $x=+\infty$, $u=+\infty$, then we get: $\int_{-\infty}^{+\infty}\frac{e^{ax}}{1+e^x}dx=\int_{-\infty}^{+\infty}\frac{e^{ax}}{1+(e^{ax})^{\frac{1}{a}}}dx=\int_0^{+\infty}\frac{\frac{1}{a}}{1+u^{\frac{1}{a}}}du$

Let $u^{\frac{1}{a}}=\tan^2\theta$, then $u=\tan^{2a}\theta$ and $du=2a\tan^{2a-1}\theta\sec^2\theta\,d\theta$, for $u=0$, $\theta=0$ and for $u=+\infty$, $\theta=\frac{\pi}{2}$, then we get:

$$\int_{-\infty}^{+\infty}\frac{e^{ax}}{1+e^x}dx=\frac{1}{a}\int_0^{\frac{\pi}{2}}\frac{2a\tan^{2a-1}\theta\sec^2\theta}{\sec^2\theta}d\theta=2\int_0^{\frac{\pi}{2}}\tan^{2a-1}\theta\ d\theta$$

$$\int_{-\infty}^{+\infty}\frac{e^{ax}}{1+e^x}dx=2\int_0^{\frac{\pi}{2}}\sin^{2a-1}\theta\cos^{1-2a}\theta\ d\theta=2\int_0^{\frac{\pi}{2}}\sin^{2a-1}\theta\cos^{2(1-a)-1}\theta\ d\theta$$

$$=\mathrm{B}(a,1-a)=\frac{\Gamma(a)\times\Gamma(1-a)}{\Gamma(a+1-a)}=\frac{\Gamma(a)\times\Gamma(1-a)}{\Gamma(1)}=\Gamma(a)\Gamma(1-a)=\frac{\pi}{\sin(\pi a)}$$

48. $\mathrm{I}=\int_0^{+\infty}\frac{1}{1+x^{\alpha}}dx$, where $\alpha>1$

Let $u=\frac{1}{1+x^{\alpha}}$, then $x=\left(\frac{1}{u}-1\right)^{\frac{1}{\alpha}}$ and $dx=\frac{1}{\alpha}\left(\frac{1}{u}-1\right)^{\frac{1}{\alpha}-1}\left(-\frac{1}{u^2}\right)du$

For $x=0$, $u=1$ and for $x=+\infty$, then $u=0$, then we get:

$$\mathrm{I}=\int_1^0 u\frac{1}{\alpha}\left(\frac{1}{u}-1\right)^{\frac{1}{\alpha}-1}\left(-\frac{1}{u^2}\right)du=\frac{1}{\alpha}\int_0^1 u^{-1}\left(\frac{1}{u}-1\right)^{\frac{1}{\alpha}-1}du=\frac{1}{\alpha}\int_0^1 u^{-1}\left(\frac{1}{u}(1-u)\right)^{\frac{1}{\alpha}-1}du$$

$\mathrm{I}=\frac{1}{\alpha}\int_0^1 u^{-1}\left(\frac{1}{u}\right)^{\frac{1}{\alpha}-1}(1-u)^{\frac{1}{\alpha}-1}du=\frac{1}{\alpha}\int_0^1 u^{-\frac{1}{\alpha}}(1-u)^{\frac{1}{\alpha}-1}du$, then we get:

$$\mathrm{I}=\frac{1}{\alpha}\int_0^1 u^{\left(1-\frac{1}{\alpha}\right)-1}(1-u)^{\frac{1}{\alpha}-1}du=\frac{1}{\alpha}\mathrm{B}\left(1-\frac{1}{\alpha},\frac{1}{\alpha}\right)=\frac{1}{\alpha}\frac{\Gamma\left(1-\frac{1}{\alpha}\right)\times\Gamma\left(\frac{1}{\alpha}\right)}{\Gamma\left(1-\frac{1}{\alpha}+\frac{1}{\alpha}\right)}=\frac{\Gamma\left(1-\frac{1}{\alpha}\right)\times\Gamma\left(\frac{1}{\alpha}\right)}{\alpha\Gamma(1)}$$

$$\mathrm{I}=\frac{1}{\alpha}\Gamma\left(1-\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}\right)=\frac{\pi}{\alpha\sin\left(\frac{\pi}{\alpha}\right)}$$

49. $\int_0^1\ln^{n-1}\left(\frac{1}{x}\right)dx$, let $t=-\ln x$, then $x=e^{-t}$ and $dx=-e^{-t}dt$, for $x=0$, $t=+\infty$ and for $x=1$, $t=0$, then we get:

$$\int_0^1\ln^{n-1}\left(\frac{1}{x}\right)dx=\int_{+\infty}^0 t^{n-1}.(-e^{-t}dt)=\int_0^{+\infty}t^{n-1}e^{-t}dt=\Gamma(n)$$

50. $\int_0^{+\infty}e^{-\sqrt[n]{x}}dx=\int_0^{+\infty}e^{-x^{\frac{1}{n}}}dx$, let $u=x^{\frac{1}{n}}$, then $x=u^n$ and $dx=n.u^{n-1}du$

For $x=0$, $u=0$ and for $x=+\infty$, $u=+\infty$, then:

$$\int_0^{+\infty}e^{-\sqrt[n]{x}}dx=\int_0^{+\infty}n.u^{n-1}e^{-u}du=n\int_0^{+\infty}u^{n-1}e^{-u}du=n\,\Gamma(n)=n!$$

51. $\int_0^{+\infty} e^{-x^n}\,dx$, let $t = x^n$, so $x = t^{\frac{1}{n}}$, then $dx = \frac{1}{n}t^{\frac{1}{n}-1}$, $x = 0$, $t = 0$ and for $x = +\infty$, $t = +\infty$,

then: $\int_0^{+\infty} e^{-x^n}\,dx = \frac{1}{n}\int_0^{+\infty} t^{\frac{1}{n}-1}e^{-t}\,dt = \frac{1}{n}\,\Gamma\left(\frac{1}{n}\right)$

52. $\mathrm{I} = \int_0^{+\infty} e^{-ax}\ln(bx)\,dx$, let $u = ax$, then $du = adx$, then:

$\mathrm{I} = \int_0^{+\infty} e^{-u}\ln\left(\frac{b}{a}u\right).\frac{1}{a}du = \frac{1}{a}\int_0^{+\infty} e^{-u}\left[\ln\left(\frac{b}{a}\right) + \ln u\right]du$, then we get:

$\mathrm{I} = \frac{1}{a}\ln\left(\frac{b}{a}\right)\int_0^{+\infty} e^{-u}du + \frac{1}{a}\int_0^{+\infty} e^{-u}\ln u\,du = \frac{1}{a}\ln\left(\frac{b}{a}\right) + \frac{1}{a}(-\gamma)$, therefore: $\mathrm{I} = \frac{1}{a}\left[\ln\left(\frac{b}{a}\right) - \gamma\right]$

53. $\int_{-\infty}^{+\infty}\left(1 + \frac{x^2}{n-1}\right)^{-\frac{n}{2}}dx$, let $\tan^2\theta = \frac{x^2}{n-1}$, then $x = \sqrt{n-1}\tan\theta$ and $dx = \sqrt{n-1}\sec^2\theta\,d\theta$

For $x = -\infty$, $\theta = -\frac{\pi}{2}$, and for $x = +\infty$, $\theta = \frac{\pi}{2}$, then:

$\int_{-\infty}^{+\infty}\left(1 + \frac{x^2}{n-1}\right)^{-\frac{n}{2}}dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sec^{-n}\theta\sqrt{n-1}\sec^2\theta\,d\theta = \sqrt{n-1}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{n-2}\theta\,d\theta$

$= 2\sqrt{n-1}\int_0^{\frac{\pi}{2}}\cos^{n-2}\theta\,d\theta = 2\sqrt{n-1}\int_0^{\frac{\pi}{2}}\cos^{2\left(\frac{n-1}{2}\right)-1}\theta\sin^{2\left(\frac{1}{2}\right)-1}\theta\,d\theta = \sqrt{n-1}\mathrm{B}\left(\frac{n-1}{2},\frac{1}{2}\right)$

$= \sqrt{n-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{n-1}{2}\right)} = \sqrt{n-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} = \sqrt{\pi(n-1)}\frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$

54. $\mathrm{I} = \int_{-\infty}^{+\infty} xe^xe^{-me^x}dx$, let $u = e^x$, then $du = e^xdx$ and $\begin{cases}\text{for } x = -\infty & u = 0\\ \text{for } x = +\infty & u = +\infty\end{cases}$, then:

$\mathrm{I} = \int_0^{+\infty}\ln u\,.\,e^{-mx}dx$, let $t = mx$, then $dt = mdx$, so:

$\mathrm{I} = \frac{1}{m}\int_0^{+\infty}\ln\left(\frac{t}{m}\right)e^{-t}dt = \frac{1}{m}\int_0^{+\infty}e^{-t}(\ln t - \ln m)dt = \frac{1}{m}\int_0^{+\infty}\ln t\,e^{-t}dt - \frac{\ln m}{m}\int_0^{+\infty}e^{-t}\,dt$

$\mathrm{I} = -\frac{\gamma}{m} - \frac{\ln m}{m} = -\frac{1}{m}(\gamma + \ln m)$

55. Let $t = \ln x$, then $x = e^t$ and $dx = e^tdt$, for $x = 1$, $t = 0$ and as $x \to +\infty$, $t \to +\infty$, then

$$\int_1^{+\infty}\left(\frac{\ln x}{x}\right)^n dx = \int_0^{+\infty}\left(\frac{t}{e^t}\right)^n e^tdt = \int_0^{+\infty} t^ne^{(1-n)t}dt$$

Let $u = (n-1)t$, then $du = (n-1)dt$, for $t = 0$, $u = 0$ and as $t \to +\infty$, $u \to +\infty$, then, we get

$$\int_1^{+\infty}\left(\frac{\ln x}{x}\right)^n dx = \int_0^{+\infty} t^ne^{(1-n)t}dt = \int_0^{+\infty}\left(\frac{u}{n-1}\right)^n e^{-u}\frac{du}{n-1} = \frac{1}{(n-1)^{n+1}}\int_0^{+\infty}u^ne^{-u}du$$

But $\int_0^{+\infty}u^ne^{-u}du = \Gamma(n+1) = n!$; therefore we get; $\int_1^{+\infty}\left(\frac{\ln x}{x}\right)^n dx = \frac{n!}{(n-1)^{n+1}}$

56. Let $t = \ln x$, then $x = e^t$ and $dx = e^tdt$, for $x = 1$, $t = 0$ and as $x \to 0$, $t \to -\infty$, then

$$\int_0^1 x^p(-\ln x)^{n-1}dx = \int_{-\infty}^0 e^{pt}(-t)^{n-1}e^tdt = \int_{-\infty}^0 e^{(p+1)t}(-t)^{n-1}dt$$

Let $u = -(p+1)t$, then $du = -(p+1)dt$, for $t = 0$, $u = 0$ and as $u \to -\infty$, $u \to +\infty$, then:

$$\int_0^1 x^p(-\ln x)^{n-1}dx = \int_{-\infty}^0 e^{(p+1)t}(-t)^{n-1}dt = \int_{+\infty}^0 e^{-u}\left(\frac{u}{p+1}\right)^{n-1}\frac{(-dt)}{(p+1)}$$

$$= \frac{1}{(p+1)^n}\int_0^{+\infty} u^{n-1}e^{-u}du$$

With $\int_0^{+\infty} u^{n-1}e^{-u}du = \Gamma(n) = (n-1)!$; therefore; $\int_0^1 x^p(-\ln x)^{n-1}dx = \frac{(n-1)!}{(p+1)^n}$

57. $\mathrm{I} = \int_{-\infty}^{+\infty} e^{-ax^2-bx}\,dx$, we have:

$$ax^2 + bx = \left(\sqrt{a}x\right)^2 + bx + \left(\frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{4a} = \left(\sqrt{a}x + \frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{4a} \Longrightarrow$$

$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-\left[\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2 - \frac{b^2}{4a}\right]}dx = e^{\frac{b^2}{4a}}\int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}x+\frac{b}{2\sqrt{a}}\right)^2}dx$, let $u = \sqrt{a}x + \frac{b}{2\sqrt{a}}$, $du = \sqrt{a}dx$

$\mathrm{I} = \frac{1}{\sqrt{a}}e^{\frac{b^2}{4a}}\int_{-\infty}^{+\infty} e^{-u^2}\,du = \frac{2}{\sqrt{a}}e^{\frac{b^2}{4a}}\int_0^{+\infty} e^{-u^2}\,du$, let $u = \sqrt{t}$, then $du = \frac{1}{2\sqrt{t}}dt$, so:

$$\mathrm{I} = \frac{2}{\sqrt{a}}e^{\frac{b^2}{4a}}\int_0^{+\infty} e^{-t}\frac{1}{2\sqrt{t}}dt = \frac{1}{\sqrt{a}}e^{\frac{b^2}{4a}}\int_0^{+\infty} t^{-\frac{1}{2}}e^{-t}\,dt = \frac{1}{\sqrt{a}}e^{\frac{b^2}{4a}}\Gamma\left(-\frac{1}{2}+1\right) = \frac{1}{\sqrt{a}}e^{\frac{b^2}{4a}}\Gamma\left(\frac{1}{2}\right)$$

with $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, therefore, $\mathrm{I} = \sqrt{\frac{\pi}{a}}e^{\frac{b^2}{4a}}$

58. $\mathrm{I} = \int_0^1 \ln\left(\frac{x}{\ln x}\right)dx = \int_0^1[\ln x - \ln(\ln x)]dx = \int_0^1 \ln x\,dx - \int_0^1 \ln(\ln x)\,dx$

$\mathrm{I} = [x\ln x - x]_0^1 - \int_0^1 \ln(\ln x)\,dx = -1 - \int_0^1 \ln(\ln x)\,dx$

Let $-t = \ln x$, then $x = e^{-t}$, for $x = 0$, $t = +\infty$ and for $x = 1$, $t = 0$, then we get:

$\mathrm{I} = -1 - \int_0^{+\infty} e^{-t}\ln(-t)\,dt = -1 - \int_0^{+\infty}[e^{-t}\ln t + e^{-t}\ln(-1)]dt$

$\mathrm{I} = -1 - \int_0^{+\infty} e^{-t}\ln t\,dt - \int_0^{+\infty} e^{-t}\ln(-1)\,dt = -1 + \gamma - i\pi\int_0^{+\infty} e^{-t}dt$

$\mathrm{I} = -1 + \gamma - i\pi = (\gamma - 1) - i\pi$

59. $\mathrm{I} = \int_0^{+\infty}\frac{1}{1+x^{2e}}dx$, let $x^{2e} = u$, then $x = u^{\frac{1}{2e}}$ and $dx = \frac{1}{2e}u^{\frac{1}{2e}-1}$

For $x = 0$, $u = 0$ and for $x = +\infty$, $u = +\infty$, then we get:

$$\mathrm{I} = \frac{1}{2e}\int_0^{+\infty}\frac{u^{\frac{1}{2e}-1}}{1+u}du = \frac{1}{2e}\int_0^{+\infty}\frac{x^{\frac{1}{2e}-1}}{1+x}dx = \frac{1}{2e}\int_0^{+\infty}\frac{x^{\frac{1}{2e}-1}}{(1+x)^{\frac{1}{2e}+\left(1-\frac{1}{2e}\right)}}dx = \frac{1}{2e}\mathrm{B}\left(\frac{1}{2e}, 1-\frac{1}{2e}\right)$$

$$\mathrm{I} = \frac{1}{2e}\left(\frac{\pi}{\sin\left(\pi\frac{1}{2e}\right)}\right) = \frac{\pi}{2e}\csc\left(\frac{\pi}{2e}\right)$$

60. $\int \mathrm{li}(x)dx$, let $u = \mathrm{li}(x)$, then $u' = \frac{1}{\ln x}$ and let $v' = 1$, then $v = x$, so we get:

$\int \mathrm{li}(x)dx = x\mathrm{li}(x) - \int x.\frac{1}{\ln x}dx = x\mathrm{li}(x) - \int\frac{x}{\ln x}dx$

Now evaluating: $\int\frac{x}{\ln x}dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, so $x = e^u$ and $dx = e^u du$, so:

$\int\frac{x}{\ln x}dx = \int\frac{e^u}{u}e^u du = \int\frac{e^{2u}}{u}du$, let $w = 2u$, then $u = \frac{1}{2}w$ and $du = \frac{1}{2}dw$, then:

$\int \frac{x}{\ln x}dx = \int \frac{e^w}{\frac{1}{2}w}.\frac{1}{2}dw = \int \frac{e^w}{w}dw = \text{Ei}(w) = \text{Ei}(2u) = \text{Ei}(2\ln x)$, therefore:

$\int \text{li}(x)dx = x\text{li}(x) - \text{Ei}(2\ln x) + \text{c}$

61. $\text{I} = \int_0^{+\infty} \frac{dx}{x^5\left(e^{\frac{1}{x}}-1\right)}$, let $x = \frac{1}{t}$, then $dx = -\frac{1}{t^2}dt$, for $x = 0$, $t = +\infty$ and for $x = +\infty$, $t = 0$, so:

$\text{I} = \int_{+\infty}^0 \frac{1}{\left(\frac{1}{t}\right)^5(e^t-1)}\left(-\frac{1}{t^2}dt\right) = \int_0^{+\infty} \frac{t^5}{e^t-1}\frac{1}{t^2}dt = \int_0^{+\infty} \frac{t^3}{e^t-1}dt = \int_0^{+\infty} \frac{t^{4-1}}{e^t-1}dt = \Gamma(4)\zeta(4)$

Therefore, $\text{I} = 3! \times \frac{\pi^2}{90} = 6 \times \frac{\pi^2}{90} = \frac{\pi^2}{15}$

62. $\text{I} = \int_0^1 x^n \ln\ln\left(\frac{1}{x}\right)dx$, let $t = \ln\left(\frac{1}{x}\right)$, then $e^t = \frac{1}{x}$ and $x = e^{-t}$, $dx = -e^{-t}dt$

For $x = 0$, $t = +\infty$ and for $x = 1$, $t = 0$, then we get:

$\text{I} = \int_{+\infty}^0 -e^{-nt}.\ln t\,.e^{-t}dt = \int_0^{+\infty} e^{-t(n+1)}.\ln t\,dt$, let $u = t(n+1)$, then $du = (n+1)dt$

$\text{I} = \int_0^{+\infty} e^{-u}\ln\left(\frac{u}{n+1}\right).\frac{du}{n+1} = \frac{1}{n+1}\int_0^{+\infty} e^{-u}[\ln u - \ln(n+1)]\,du$

$\text{I} = \frac{1}{n+1}\int_0^{+\infty} e^{-u}\ln u\,du - \frac{\ln(n+1)}{n+1}\int_0^{+\infty} e^{-u}du = -\frac{\gamma}{n+1} - \frac{\ln n}{n+1} = -\frac{1}{n+1}[\gamma + \ln(n+1)]$

63. $\text{I} = \int_0^{\frac{\pi}{2}}(\sec x - 1)^\mu \sin x\,dx = \int_0^{\frac{\pi}{2}}\left(\frac{1}{\cos x} - 1\right)^\mu \sin x\,dx = \int_0^{\frac{\pi}{2}}\left(\frac{1-\cos x}{\cos x}\right)^\mu \sin x\,dx$

$\text{I} = \int_0^{\frac{\pi}{2}}\frac{(1-\cos x)^\mu}{\cos^\mu x}\sin x\,dx$, let $u = \cos x$, then $du = \sin x\,dx$, for $x = 0$, $u = 1$ and for $x = \frac{\pi}{2}$, $u = 0$, then we get:

$\text{I} = \int_1^0 \frac{(1-u)^\mu}{u^\mu}(-du) = \int_0^1 u^{-\mu}(1-u)^\mu du = \int_0^1 u^{(1-\mu)-1}(1-u)^{(\mu+1)-1}du$

$\text{I} = \text{B}(1-\mu, 1+\mu) = \frac{\Gamma(1-\mu)\Gamma(1+\mu)}{\Gamma(1-\mu+1+\mu)} = \frac{\Gamma(1-\mu)\mu\Gamma(\mu)}{\Gamma(2)} = \mu\Gamma(\mu)\Gamma(1-\mu) = \frac{\mu\pi}{\sin(\mu\pi)}$

64. $\text{I} = \int_1^e \frac{1}{x}\ln(\ln(1-\ln x))\,dx$, let $-t = \ln(1-\ln x) \Rightarrow 1 - \ln x = e^{-t} \Rightarrow \ln x = 1 - e^{-t} \Rightarrow$

$x = e^{1-e^{-t}}$ and $dx = e^{-t}.e^{1-e^{-t}}dt$ and $\begin{cases} \text{for } x = 1 & t = 0 \\ \text{for } x = e & t = +\infty \end{cases}$, then we get:

$\text{I} = \int_0^{+\infty} \frac{\ln(-t)e^{-t}.e^{1-e^{-t}}}{e^{1-e^{-t}}}dt = \int_0^{+\infty} e^{-t}\ln(-t)\,dt = \int_0^{+\infty} e^{-t}[\ln(-1) + \ln t]$

$\text{I} = \ln(-1)\int_0^{+\infty} e^{-t} + \int_0^{+\infty} e^{-t}\ln t\,dt = -\gamma + i\pi$

65. $\text{I} = \int_0^1 \frac{x^n}{\sqrt{\ln x}}dx$, let $u = \ln x$, then $x = e^u$ and $dx = e^u du$, for $x = 0$, $u = -\infty$ and for $x = 1$, $u = 0$, then we get:

$\text{I} = \int_{-\infty}^0 (e^u)^n.\frac{1}{\sqrt{u}}.e^u du = \int_{-\infty}^0 e^{u(n+1)}.u^{-\frac{1}{2}}du$, let $u(n+1) = -t$, then $du = -\frac{1}{n+1}dt$

for $u = -\infty$, $t = +\infty$ and for $u = 0$, $t = 0$, then we get:

$\text{I} = -\int_{+\infty}^0 e^{-t}\frac{1}{\sqrt{-\frac{t}{n+1}}}\left(-\frac{1}{n+1}dt\right) = -\frac{i}{\sqrt{n+1}}\int_0^{+\infty} e^{-t}t^{-\frac{1}{2}}dt = -\frac{i}{\sqrt{n+1}}\Gamma\left(\frac{1}{2}\right) = -i\sqrt{\frac{\pi}{n+1}}$

66. $\text{I} = \int_0^1 \ln^n(1-\sqrt{x})\,dx$, let $-t = \ln(1-\sqrt{x}) \Rightarrow 1 - \sqrt{x} = e^{-t} \Rightarrow \sqrt{x} = 1 - e^{-t} \Rightarrow$

$x = (1-e^{-t})^2$ and $dx = 2(1-e^{-t})e^{-t}dt$ and $\begin{cases} \text{for } x = 0 & t = 0 \\ \text{for } x = 1 & t = +\infty \end{cases}$, then we get:

$\mathrm{I} = \int_0^{+\infty} (-1)^n t^n . 2e^{-t}(1-e^{-t})dt = 2\int_0^{+\infty} (-1)^n . t^n . (e^{-t} - e^{-2t})$

$\mathrm{I} = 2(-1)^n \int_0^{+\infty} e^{-t} t^n dt - 2(-1)^n \int_0^{+\infty} e^{-2t} t^n dt = 2(-1)^n n! - 2(-1)^n \int_0^{+\infty} e^{-2t} t^n dt$

Let $u = 2t$ and $du = 2dt$, then $\mathrm{I} = 2(-1)^n n! - 2(-1)^n \int_0^{+\infty} e^{-u} \frac{u^n}{2^n} . \frac{1}{2} dt$, then:

$\mathrm{I} = 2(-1)^n n! - \frac{(-1)^n}{2^n} . n! = (-1)^n n! \left(\frac{2^{n+1}-1}{2^n}\right)$

67. $\mathrm{I} = \int_0^{e^{-1}} \ln(1+\ln x)\, dx$, let $u = 1 + \ln x$, then $u - 1 = \ln x$ and $x = e^{u-1}$, so $dx = e^{u-1} du$, for $x = 0$, $u = -\infty$ and for $x = e^{-1}$, $u = 1 - \ln e = 0$, then we get:

$\mathrm{I} = \int_{-\infty}^0 \ln u\, e^{u-1} du = \frac{1}{e} \int_{-\infty}^0 e^u \ln u$, let $u = -t$, then $du = -dt$, then:

$\mathrm{I} = \frac{1}{e} \int_{\infty}^0 e^{-t} \ln(-t)\,(-dt) = \frac{1}{e} \int_0^{+\infty} e^{-t} \ln(-t)\, dt$

$\mathrm{I} = \frac{1}{e} \int_0^{+\infty} e^{-t} \ln(-1)\, dt + \frac{1}{e} \int_0^{+\infty} e^{-t} \ln t\, dt = \frac{i\pi}{e} \int_0^{+\infty} e^{-t} dt + \frac{1}{e}(-\gamma) = \frac{1}{e}(-\gamma + i\pi)$

68. $\mathrm{I} = \int_1^2 (x-1)^5 \sqrt{\ln(x-1)}\, dx$, let $-t = \ln(x-1) \Rightarrow x - 1 = e^{-t} \Rightarrow x = e^{-t} + 1 \Rightarrow$ $dx = -e^{-t} dx$, for $x = 1$, $t = -\ln 0 = +\infty$ and for $x = 2$, $t = -\ln 1 = 0$, then:

$\mathrm{I} = \int_{+\infty}^0 e^{-5t} . \sqrt{-t} . (-e^{-t}) dt = \int_0^{+\infty} e^{-6t} . \sqrt{-1} . \sqrt{t} dt = i \int_0^{+\infty} e^{-6t} t^{\frac{1}{2}} dt$

Let $u = 6t$, then $du = 6dt$, then we get:

$\mathrm{I} = \frac{1}{6} i \int_0^{+\infty} e^{-u} . \frac{\sqrt{u}}{\sqrt{6}} du = \frac{1}{6\sqrt{6}} i \int_0^{+\infty} e^{-u} . u^{\frac{1}{2}} du = \frac{1}{6\sqrt{6}} i \Gamma\left(\frac{3}{2}\right) = \frac{1}{6\sqrt{6}} i \left(\frac{\sqrt{\pi}}{2}\right) = \frac{i}{12}\sqrt{\frac{\pi}{6}}$

69. $\mathrm{I} = \int_1^e \frac{\ln(\ln(\ln x))}{x} dx$, let $\ln(\ln x) = -t \Rightarrow \ln x = e^{-t} \Rightarrow x = e^{e^{-t}} \Rightarrow dx = -e^{-t} . e^{e^{-t}} dt$

for $x = 1$, $t = -\ln(\ln 1) = -\ln 0 = +\infty$ and for $x = e$, $t = -\ln(\ln e) = -\ln 1 = 0$, then:

$\mathrm{I} = \int_{+\infty}^0 \frac{\ln(-t) . \left(-e^{-t} \times e^{e^{-t}}\right)}{e^{e^{-t}}} dt = \int_0^{+\infty} e^{-t} \times \ln(-t)\, dt = \int_0^{+\infty} [e^{-t} \ln(-1) + e^{-t} \ln t] dt$

$\mathrm{I} = i\pi \int_0^{+\infty} e^{-t} dt + \int_0^{+\infty} e^{-t} \ln t\, dt = -\gamma + i\pi$

70. $\mathrm{I} = \int_{e^{-n}}^0 \sqrt{\ln x + n}\, dx$, let $-t = \ln x + n$, then $\ln x = -t - n$ and $x = e^{-n-t}$, so $dx = -e^{-n-t} dt$

For $x = e^{-n}$, $t = -\ln e^{-n} - n = n - n = 0$ and for $x = 0$, $t = +\infty$, then we get:

$\mathrm{I} = \int_0^{+\infty} \sqrt{-t}\,(-e^{-n-t} dt) = -e^{-n} \int_0^{+\infty} i\sqrt{t} e^{-t} dt = -ie^{-n} \int_0^{+\infty} e^{-t} t^{\frac{1}{2}} dt$, then we get:

$\mathrm{I} = -ie^{-n} \int_0^{+\infty} e^{-t} t^{\frac{3}{2}-1} dt = -ie^{-n} \Gamma\left(\frac{3}{2}\right) = -ie^{-n} \frac{\sqrt{\pi}}{2}$

71. $\mathrm{I} = \int_0^1 \frac{\sqrt{x}}{\sqrt{\sqrt{\ln x}}} dx = \int_0^1 \frac{x^{\frac{1}{2}}}{(\ln x)^{\frac{1}{4}}} dx$, let $\ln x = -t$, then $x = e^{-t}$ and $dx = -e^{-t} dt$

For $x = 0$, $t = +\infty$ and for $x = 1$, $t = 0$, then we get:

$\mathrm{I} = \int_{+\infty}^0 \frac{e^{-\frac{1}{2}t} . (-e^{-t})}{(-t)^{\frac{1}{4}}} dt = \int_0^{+\infty} \frac{e^{-\frac{3}{2}t}}{(-1)^{\frac{1}{4}} t^{\frac{1}{4}}} dt = \frac{1}{(-1)^{\frac{1}{4}}} \int_0^{+\infty} e^{-\frac{3}{2}t} . t^{\frac{1}{4}} dt = \frac{1}{(e^{i\pi})^{\frac{1}{4}}} \int_0^{+\infty} e^{-\frac{3}{2}t} . t^{\frac{1}{4}} dt$

$\mathrm{I} = e^{-i\frac{\pi}{4}} \int_0^{+\infty} e^{-\frac{3}{2}t} . t^{\frac{1}{4}} dt$, let $u = \frac{3}{2} t$, then $du = \frac{3}{2} dt$, so $dt = \frac{2}{3} du$, then we get:

$\mathrm{I} = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)\int_0^{+\infty} e^{-u}.\left(\frac{2}{3}u\right)^{\frac{1}{4}}.\frac{2}{3}du = \left(\frac{\sqrt{2}}{3} - i\frac{\sqrt{2}}{3}\right)\left(\frac{2}{3}\right)^{\frac{1}{4}}\int_0^{+\infty} e^{-u}.u^{\frac{1}{4}}du$, then we get:

$\mathrm{I} = \frac{2^{\frac{3}{4}}}{3^{\frac{5}{4}}}(1-i)\int_0^{+\infty} e^{-u}.u^{\frac{1}{4}}du = \frac{\sqrt[4]{8}}{\sqrt[4]{243}}(1-i)\int_0^{+\infty} e^{-u}.u^{\frac{5}{4}-1}du = (1-i)\sqrt[4]{\frac{8}{243}}\Gamma\left(\frac{5}{4}\right)$

72. $\mathrm{I} = \int_0^1 \left[x - \frac{1}{(1-\ln x)^m}\right].\frac{dx}{x\ln x}$, let $-t = \ln x$, then $x = e^{-t}$, then $dx = -e^{-t}dt$, for $x = 0$, $t = +\infty$ and for $x = 1$, then $t = 0$, so we get:

$\mathrm{I} = \int_{+\infty}^0 \left[e^{-t} - \frac{1}{(1+t)^m}\right].\frac{-e^{-t}dt}{e^{-t}(-t)} = -\int_0^{+\infty}\left[e^{-t} - \frac{1}{(1+t)^m}\right]\frac{dt}{t} = -\Psi(m)$

73. $\mathrm{I} = \int_0^1 \sqrt[4]{\frac{x}{1-x}}.\frac{dx}{(1+x)^2}$, let $u = \frac{2x}{1-x}$, then $x = \frac{u}{u+2}$ and $dx = \frac{2}{(u+2)^2}du$

For $x = 0$, $u = 0$ and for $x = 1$, $u = +\infty$, then we get:

$\mathrm{I} = \int_0^{+\infty}\sqrt[4]{\frac{\frac{u}{u+2}}{1-\frac{u}{u+2}}}.\frac{\frac{2}{(u+2)^2}}{\left(1+\frac{u}{u+2}\right)^2}du = \int_0^{+\infty}\sqrt[4]{\frac{\frac{u}{u+2}}{\frac{2}{u+2}}}.\frac{\frac{2}{(u+2)^2}}{\left(\frac{2u+2}{u+2}\right)^2}du = \int_0^{+\infty}\sqrt[4]{\frac{u}{2}}.\frac{\frac{2}{(u+2)^2}}{\frac{4(u+1)^2}{(u+2)^2}}du$

$\mathrm{I} = \frac{1}{2\sqrt[4]{2}}\int_0^{+\infty}\sqrt[4]{u}\frac{1}{(u+1)^2}du = \frac{1}{2\sqrt[4]{2}}\int_0^{+\infty}\frac{u^{\frac{1}{4}}}{(u+1)^2}du = \frac{1}{2\sqrt[4]{2}}\mathrm{B}\left(\frac{5}{4},\frac{3}{4}\right) = \frac{\frac{\pi}{4}\csc\left(\frac{\pi}{4}\right)}{2\sqrt[4]{2}\Gamma(2)} = \frac{\pi}{8}\sqrt[4]{2}$

74. $\mathrm{I} = \int_0^{\frac{1}{k}}\ln\left(-\frac{1}{n}\ln(kx)\right)dx$, let $t = -\frac{1}{n}\ln(kx)$, then $-nt = \ln(kx)$, so $kx = e^{-nt}$ and so $x = \frac{1}{k}e^{-nt}$ and $dx = -\frac{n}{k}e^{-nt}dt$, for $x = 0$, $t = +\infty$ and for $x = \frac{1}{k}$, $t = -\frac{1}{n}\ln 1 = 0$, then:

$\mathrm{I} = \int_{+\infty}^0 \ln(t)\left(-\frac{n}{k}e^{-nt}dt\right) = \frac{n}{k}\int_0^{+\infty}\ln t\, e^{-nt}dt$, let $u = nt$, then $du = ndt$, then:

$\mathrm{I} = \frac{n}{k}\int_0^{+\infty}\ln\left(\frac{u}{n}\right)e^{-u}\frac{1}{n}du = \frac{1}{k}\int_0^{+\infty}e^{-u}(\ln u - \ln n)du$

$\mathrm{I} = \frac{1}{k}\left[\int_0^{+\infty}e^{-u}\ln u\,du - \ln u\int_0^{+\infty}e^{-u}du\right] = -\frac{\gamma}{k} - \frac{1}{k}\ln n = -\frac{1}{k}(\gamma + \ln n)$

75. $\mathrm{I} = \int_0^{\frac{1}{e}}\frac{\sqrt[m]{x}}{x\sqrt{-(1+\ln x)}}dx$, let $u = -(1+\ln x)$, then $-u = 1 + \ln x$, so $\ln x = -u - 1$ and $x = e^{-u-1}$, for $x = 0$, $u = +\infty$ and for $x = \frac{1}{e}$, $u = 0$, then:

$\mathrm{I} = \int_{+\infty}^0 \frac{-e^{-\frac{1}{m}(u+1)}}{e^{-u-1}.u^{\frac{1}{2}}}.e^{-u-1}du = \int_0^{+\infty}u^{-\frac{1}{2}}.e^{-\frac{1}{m}}.e^{-\frac{u}{m}}du = e^{-\frac{1}{m}}\int_0^{+\infty}e^{-\frac{u}{m}}.u^{-\frac{1}{2}}du$

Let $t = \frac{u}{m}$, then $dt = \frac{1}{m}du$ and $du = mdt$, then:

$\mathrm{I} = e^{-\frac{1}{m}}\int_0^{+\infty}e^{-t}.m^{-\frac{1}{2}}.t^{-\frac{1}{2}}.mdt = \frac{\sqrt{m}}{\sqrt[m]{e}}\int_0^{+\infty}e^{-t}.t^{-\frac{1}{2}}dt = \frac{\sqrt{m}}{\sqrt[m]{e}}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{m\pi}}{\sqrt[m]{e}}$

76. $\mathrm{I} = \int_{-a}^{a}(a+x)^{m-1}(a-x)^{n-1}dx$

Let $x = a\cos 2\theta$, then $dx = -2a\sin 2\theta\, d\theta$, for $x = a$, $\cos 2\theta = 1$, $2\theta = 0$, $\theta = 0$ and for $x = -a$, $\cos 2\theta = 1$, $2\theta = \pi$, so $\theta = \frac{\pi}{2}$, then:

$\mathrm{I} = \int_{\frac{\pi}{2}}^0 (a + a\cos 2\theta)^{m-1}(a - a\cos 2\theta)^{n-1}(-2a\sin 2\theta\, d\theta)$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} a^{m-1}(1+\cos 2\theta)^{m-1}.\, a^{n-1}(1-\cos 2\theta)^{n-1}.\, 4a\sin\theta\cos\theta\, d\theta$$

$$\mathrm{I} = a^{m+n-1}\int_0^{\frac{\pi}{2}}(2\cos^2\theta)^{m-1}(2\sin^2\theta)^{n-1}.\, 4\sin\theta\cos\theta\, d\theta$$

$$\mathrm{I} = a^{m+n-1}\int_0^{\frac{\pi}{2}} 2^{m-1}\cos^{2m-2}\theta\, .\, 2^{n-1}\sin^{2n-2}\theta\, .\, 2^2\sin\theta\cos\theta\, d\theta$$

$$\mathrm{I} = 2^{m+n}a^{m+n-1}\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\, d\theta = 2^{m+n-1}a^{m+n-1}\left(2\int_0^{\frac{\pi}{2}}\cos^{2m-1}\theta\sin^{2n-1}\theta\, d\theta\right)$$

$$\mathrm{I} = (2a)^{m+n-1}\mathrm{B}(m,n)$$

77. $\mathrm{I} = \int_0^e \frac{x}{\sqrt{1-\ln x}}dx$, let $t = 1-\ln x \Rightarrow \ln x = 1-t \Rightarrow x = e^{1-t}$ and $dx = -e^{1-t}dt$

and $\begin{cases}\text{for } x = 0 & t = +\infty \\ \text{for } x = e & t = 0\end{cases}$, then we get:

$$\mathrm{I} = \int_{+\infty}^0 \frac{e^{1-t}(-e^{1-t})}{\sqrt{t}}dt = \int_0^{+\infty} e^{2-2t}.\, t^{-\frac{1}{2}}dt = e^2\int_0^{+\infty} e^{-2t}.\, t^{-\frac{1}{2}}dt$$

Now let $u = 2t$ then $dv = 2dt$ and then:

$$\mathrm{I} = e^2\int_0^{+\infty} e^{-u}.\left(\frac{u}{2}\right)^{-\frac{1}{2}}.\frac{1}{2}du = \frac{1}{2}e^2\int_0^{+\infty} e^{-u}u^{-\frac{1}{2}}.\sqrt{2}du = \frac{1}{\sqrt{2}}e^2\int_0^{+\infty} e^{-v}v^{-\frac{1}{2}}dv = \frac{1}{\sqrt{2}}e^2\Gamma\left(\frac{1}{2}\right)$$

Therefore $\mathrm{I} = e^2\sqrt{\frac{\pi}{2}}$

78. $\mathrm{I} = \int_0^{+\infty}\frac{x^{s-1}}{(e^x-1)^2}dx = \int_0^{+\infty}\frac{x^{s-1}(e^x-e^x+1)}{(e^x-1)^2}dx = \int_0^{+\infty}\frac{x^{s-1}\left(e^x-(e^x-1)\right)}{(e^x-1)^2}dx$

$$\mathrm{I} = \int_0^{+\infty}\frac{x^{s-1}e^x - x^{s-1}(e^x-1)}{(e^x-1)^2}dx = \int_0^{+\infty}\left[\frac{x^{s-1}e^x}{(e^x-1)^2} - \frac{x^{s-1}(e^x-1)}{(e^x-1)^2}\right]dx$$

$$\mathrm{I} = \int_0^{+\infty}\frac{x^{s-1}e^x}{(e^x-1)^2}dx - \int_0^{+\infty}\frac{x^{s-1}}{e^x-1}dx$$

But Relation between Zeta and Gamma functions: $\zeta(z)\Gamma(z) = \int_0^{+\infty}\frac{x^{s-1}}{e^x-1}dx$, then:

$\mathrm{I} = \int_0^{+\infty}\frac{x^{s-1}e^x}{(e^x-1)^2}dx - \zeta(s)\Gamma(s) = \mathrm{J} - \zeta(s)\Gamma(s)$, where $\mathrm{J} = \int_0^{+\infty}\frac{x^{s-1}e^x}{(e^x-1)^2}dx$, using IBP:

Let $u = x^{s-1}$, then $u' = (s-1)x^{s-2}$ and let $v' = \frac{e^x}{(e^x-1)^2}$, so $v = -\frac{1}{e^x-1}$, then we get:

$\mathrm{J} = \left[-\frac{x^{s-1}}{e^x-1}\right]_0^{+\infty} + (s-1)\int_0^{+\infty}\frac{x^{s-2}}{e^x-1}dx = 0 + (s-1)\,\zeta(s-1)\Gamma(s-1)$, then:

$\mathrm{J} = \zeta(s-1)[(s-1)\,\Gamma(s-1)] = \zeta(s-1)\,\Gamma(s)$, therefore:

$\mathrm{I} = \zeta(s-1)\,\Gamma(s) - \zeta(s)\Gamma(s) = \Gamma(s)[\zeta(s-1) - \zeta(s)]$

79. $\int e^{\tan x}dx$, let $u = \tan x$, then $x = \tan^{-1}u$ and $dx = \frac{1}{1+u^2}du$, then:

$$\int e^{\tan x}dx = \int e^u.\frac{1}{1+u^2}du = \int\frac{e^u}{(u-i)(u+i)}du$$

Method of partial fractions : $\frac{1}{(u-i)(u+i)} = \frac{\mathrm{A}}{u-i} + \frac{\mathrm{B}}{u+i}$, then $1 = \mathrm{A}(u+i) + \mathrm{B}(u-i)$

For $u = i$, then $1 = 2i\mathrm{A}$, then $\mathrm{A} = \frac{1}{2i}$ and for $u = -i$, $1 = \mathrm{B}(-2i)$, so $\mathrm{B} = -\frac{1}{2i}$, then:

$\int e^{\tan x}\,dx = \frac{1}{2i}\int\left(\frac{e^u}{u-i} - \frac{e^u}{u+i}\right)du = \frac{1}{2i}\int\left(\frac{e^i e^{-i} e^u}{u-i} - \frac{e^i e^{-i} e^u}{u+i}\right)du$

$\int e^{\tan x}\,dx = \frac{1}{2i}\int\left(\frac{e^i e^{u-i}}{u-i} - \frac{e^{-i} e^{u+i}}{u+i}\right)du = \frac{1}{2i}e^i\int\frac{e^{u-i}}{u-i}du - \frac{1}{2i}e^{-i}\int\frac{e^{u+i}}{u+i}du$

$\int e^{\tan x}\,dx = -\frac{1}{2}ie^i\mathrm{E_i}(u-i) + \frac{1}{2}ie^{-i}\mathrm{E_i}(u+i) + c$, therefore we get:

$\int e^{\tan x}\,dx = -\frac{1}{2}ie^i\mathrm{E_i}(\tan x - i) + \frac{1}{2}ie^{-i}\mathrm{E_i}(\tan x + i) + c$

80. $\mathrm{I} = \int_0^1 \frac{\ln\ln(1-x)}{(1-x)^{\frac{n}{k}}}dx$, let $-t = \ln(1-x)$, then $1-x = e^{-t}$ and $x = 1-e^{-t}$, then $dx = e^{-t}dt$

For $x = 0$, $t = 0$ and for $x = 1$, $t = +\infty$, then we get:

$\mathrm{I}\int_0^{+\infty}\frac{\ln(-t)}{e^{-\frac{nt}{k}}}e^{-t}dt = \int_0^{+\infty}\ln(-t)\,e^{-t\left(1-\frac{n}{k}\right)}dt$, let $u = t\left(1-\frac{n}{k}\right)$, then $dt = \frac{k}{(k-n)}du$, then:

$\mathrm{I} = \int_0^{+\infty}\ln\left(-\frac{k}{k-n}u\right).e^{-u}.\frac{k}{(k-n)}du$

$\mathrm{I} = \frac{k}{k-n}\int_0^{+\infty}\left[e^{-u}\ln(-1) + \frac{k}{k-n}\ln\left(\frac{k}{k-n}\right)e^{-u} + \ln u\,e^{-u}\right]du$

$\mathrm{I} = i\frac{k\pi}{k-n} + \frac{k}{k-n}\ln\left(\frac{k}{k-n}\right) - \frac{k\gamma}{k-n} = \frac{k}{k-n}\left[\ln\left(\frac{k}{k-n}\right) - \gamma\right] + i\frac{k\pi}{k-n}$

81. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{\sin 2x}{\cos^4 x\left(e^{2\pi\tan x}-1\right)}dx = \int_0^{\frac{\pi}{2}}\frac{2\sin x\cos x}{\cos x\cos x\left(e^{2\pi\tan x}-1\right)}dx = \int_0^{\frac{\pi}{2}}\frac{2\sin x}{\cos^3 x\left(e^{2\pi\tan x}-1\right)}dx$

$\mathrm{I} = 2\int_0^{\frac{\pi}{2}}\frac{\tan x}{\left(e^{2\pi\tan x}-1\right)\cos^2 x}dx = 2\int_0^{\frac{\pi}{2}}\frac{\tan x}{e^{2\pi\tan x}-1}.\sec^2 x\,dx$

Let $u = \tan x$, so $du = \sec^2 x\,dx$, for $x = 0$, $u = 0$ and for $x = \frac{\pi}{2}$, $u = +\infty$, then we get:

$\mathrm{I} = 2\int_0^{+\infty}\frac{u}{e^{2\pi u}-1}du$, let $t = 2\pi u$, then $u = \frac{t}{2\pi}$ then $du = \frac{1}{2\pi}dt$, then we get:

$\mathrm{I} = 2\int_0^{+\infty}\frac{\frac{t}{2\pi}}{e^t-1}.\frac{1}{2\pi}dt = \frac{1}{2\pi^2}\int_0^{+\infty}\frac{t}{e^t-1}dt = \frac{1}{2\pi^2}\int_0^{+\infty}\frac{x}{e^x-1}dx$

Relation between Zeta and Gamma functions: $\zeta(z)\Gamma(z) = \int_0^{+\infty}\frac{x^{s-1}}{e^x-1}dx$, so for $z = 2$

$\int_0^{+\infty}\frac{x^{2-1}}{e^x-1}dx = \int_0^{+\infty}\frac{x}{e^x-1}dx = \zeta(2)\Gamma(2) = \frac{\pi^2}{6}$, therefore $\mathrm{I} = \frac{1}{2\pi^2}\left(\frac{\pi^2}{6}\right) = \frac{1}{12}$

82. $\mathrm{I} = \int_0^1\frac{\ln^n\left(\frac{1+x}{1-x}\right)}{\sqrt{1-x^2}}\frac{dx}{1+x}$, where $n \geq 1$. Let $x = \cos 2\theta$, then $dx = -\sin 2\theta\,d\theta$

For $x = 0$, $2\theta = \frac{\pi}{2}$, so $\theta = \frac{\pi}{4}$ and for $x = 1$ we get $2\theta = 0$, so $\theta = 0$, then we get:

$\mathrm{I} = \int_{\frac{\pi}{4}}^0\frac{\ln^n\left(\frac{1+\cos 2\theta}{1-\cos 2\theta}\right)}{\sqrt{1-\cos^2 2\theta}}\frac{-2\sin 2\theta}{1+\cos 2\theta}d\theta = \int_0^{\frac{\pi}{4}}\frac{\ln^n\left(\frac{2\cos^2\theta}{2\sin^2\theta}\right)}{\sin 2\theta}\frac{2\sin 2\theta}{2\cos^2\theta}d\theta = \int_0^{\frac{\pi}{4}}\ln^n(\cot^2\theta)\sec^2\theta\,d\theta$

$\mathrm{I} = \int_0^{\frac{\pi}{4}}(2\ln\cot\theta)^n\sec^2\theta\,d\theta = 2^n\int_0^{\frac{\pi}{4}}\ln^n(\cot\theta)\sec^2\theta\,d\theta$

Let $y = \ln(\cot\theta)$, then $\cot\theta = e^y$ and $\tan\theta = e^{-y}$, with $\sec^2\theta\,d\theta = -e^{-y}dy$

For $\theta = 0$, $y = +\infty$ and for $\theta = \frac{\pi}{4}$, $y = 0$ then:

$\mathrm{I} = 2^n\int_{+\infty}^0 y^n.(-e^{-y})dy = 2^n\int_0^{+\infty}y^n e^{-y}dy = 2^n\Gamma(n+1)$

83. $\mathrm{I}=\int_0^{\frac{1}{b}}\ln(a\ln(bx))\,dx$, let $-t=a\ln(bx)\Longrightarrow\ln(bx)=-\frac{t}{a}\Longrightarrow bx=e^{-\frac{t}{a}}\Longrightarrow x=\frac{1}{b}e^{-\frac{t}{a}}$ and $dx=-\frac{1}{ab}e^{-\frac{t}{a}}dt$, for $x=0$, $t=-a\ln(0)=+\infty$ and for $x=\frac{1}{b}$, $t=-a\ln(1)=0$, so:

$\mathrm{I}=\int_{+\infty}^{0}\ln(-t).\left(-\frac{1}{ab}e^{-\frac{t}{a}}\right)dt=\frac{1}{ab}\int_0^{+\infty}e^{-\frac{t}{a}}\ln(-t)\,dt=\frac{1}{ab}\int_0^{+\infty}e^{-\frac{t}{a}}[\ln(-1)+\ln t]dt$

$\mathrm{I}=\frac{1}{ab}\int_0^{+\infty}e^{-\frac{t}{a}}\ln(-1)\,dt+\frac{1}{ab}\int_0^{+\infty}e^{-\frac{t}{a}}\ln t\,dt=\frac{i\pi}{ab}\int_0^{+\infty}e^{-\frac{t}{a}}dt+\frac{1}{ab}\int_0^{+\infty}e^{-\frac{t}{a}}\ln t\,dt$

$\mathrm{I}=\frac{i\pi}{ab}\left[-ae^{-\frac{t}{a}}\right]_0^{+\infty}+\frac{1}{ab}\int_0^{+\infty}e^{-\frac{t}{a}}\ln t\,dt=\frac{i\pi}{b}+\mathrm{J}$, with $\mathrm{J}=\frac{1}{ab}\int_0^{+\infty}e^{-\frac{t}{a}}\ln t\,dt$

Let $u=\frac{t}{a}$, then $t=ua$ and $dt=adu$, then:

$\mathrm{J}=\frac{1}{ab}\int_0^{+\infty}e^{-u}\ln(au)\,du=\frac{1}{ab}\int_0^{+\infty}e^{-u}[\ln u+\ln a]du$

$\mathrm{J}=\frac{1}{ab}\int_0^{+\infty}e^{-u}\ln u\,du+\frac{\ln a}{ab}\int_0^{+\infty}e^{-u}du=\frac{1}{ab}(\ln a-\gamma)$, therefore $\mathrm{I}=\frac{1}{ab}(\ln a-\gamma)+i\frac{\pi}{b}$

84. $\mathrm{I}=\int_0^{+\infty}\frac{\ln(\cosh x)}{\sinh x}dx=\int_0^{+\infty}\frac{\ln(\cosh x)}{\sinh^2 x}\sinh x\,dx=\int_0^{+\infty}\frac{\ln(\cosh x)}{\cosh^2 x-1}\sinh x\,dx$

Let $t=\cosh x$, then $dt=\sinh x\,dx$, for $x=0$, $t=1$ and for $x=+\infty$, $t=+\infty$, then we get:

$\mathrm{I}=\int_1^{+\infty}\frac{\ln t}{t^2-1}dt$, let $t=\frac{1}{s}$, then $dt=-\frac{1}{s^2}ds$, for $t=1$, $s=1$ and for $t=+\infty$, $s=0$, then:

$\mathrm{I}=\int_1^{0}\frac{\ln\left(\frac{1}{s}\right)}{\left(\frac{1}{s}\right)^2-1}\left(-\frac{1}{s^2}ds\right)=\int_0^{1}\frac{-\ln s}{\frac{1}{s^2}-1}\frac{1}{s^2}ds=-\int_0^{1}\frac{-\ln s}{1-s^2}ds=-\int\frac{1}{1-s^2}.\ln s\,ds$

Geometric sum $1+s^2+s^4+\cdots+s^{2n}=\frac{1-(s^2)^{n+1}}{1-s^2}$, with $0\le s\le 1$, so we get:

$1+s^2+s^4+\cdots=\frac{1}{1-s^2}$, then $\mathrm{I}=-\int(1+s^2+s^4+\cdots).\ln s\,ds$

Consider the integral $\mathrm{J}_n=\int_0^1 s^n\ln s\,ds$, using IBP:

Let $u=\ln s$, then $u'=\frac{1}{s}$ and let $v'=s^n$, then $v=\frac{1}{n+1}s^{n+1}$, so we get:

$\mathrm{J}_n=\left[\frac{1}{n+1}s^{n+1}\ln s\right]_0^1-\int_0^1\frac{1}{n+1}s^n ds=0-\frac{1}{(n+1)^2}=-\frac{1}{(n+1)^2}$, so we get:

$\mathrm{I}=-\int_0^1\ln s\,ds-\int_0^1 s^2\ln s\,ds-\int_0^1 s^4\ln s\,ds+\cdots$, then

$\mathrm{I}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots=\frac{3}{4}\zeta(2)=\frac{\pi^2}{8}$

85. $\mathrm{I}=\int_0^1\ln\left(\frac{e^{x^n}}{n^n\ln^n(1-x)}\right)dx=\int_0^1\left[\ln\left(e^{x^n}\right)-\ln(n^n)-\ln(\ln^n(1-x))\right]dx$

$\mathrm{I}=\int_0^1 x^n dx-\ln n^n\int_0^1 dx-\int_0^1\ln(\ln^n(1-x))dx=\frac{1}{n+1}-n\ln n-\int_0^1\ln(\ln^n(1-x))dx$

Let $-u=\ln(1-x)$, then $e^{-u}=1-x$, so $x=1-e^{-u}$ and $dx=e^{-u}du$

For $x=0$, $u=0$ and for $x=1$, $u=+\infty$, then we get:

$\mathrm{I}=\frac{1}{n+1}-n\ln n-\int_0^{+\infty}\ln[(-u)^n]e^{-u}du=\frac{1}{n+1}-n\ln n-\int_0^{+\infty}\ln[(-1)^n u^n]e^{-u}du$

$\mathrm{I}=\frac{1}{n+1}-n\ln n-\int_0^{+\infty}e^{-u}\ln[(-1)^n]\,du-\int_0^{1}e^{-u}\ln(u^n)\,du$

$\mathrm{I}=\frac{1}{n+1}-n\ln n-\ln[(-1)^n]\int_0^{+\infty}e^{-u}du-n\int_0^{1}e^{-u}\ln u\,du$

$\mathrm{I} = \frac{1}{n+1} - n\ln n - \ln\left[\left(e^{i\pi}\right)^n\right]\int_0^{+\infty} e^{-u}du + n\gamma$

$\mathrm{I} = \frac{1}{n+1} - n\ln n - \ln\left(e^{in\pi}\right) + n\gamma = \frac{1}{n+1} - n\ln n - in\pi + n\gamma$, therefore we get:

$\mathrm{I} = \left(\frac{1}{n+1} - n\ln n + n\gamma\right) - in\pi$

86. $\mathrm{I} = \int_0^1 \frac{x^a - x^b}{\sqrt{\ln x}}dx = \int_0^1 \frac{x^a}{\sqrt{\ln x}}dx - \int_0^1 \frac{x^b}{\sqrt{\ln x}}dx$

Let $\ln x = -t$, then $x = e^{-t}$ and $dx = -e^{-t}dt$, for $x = 0$, $t = +\infty$ and for $x = 1$, $t = 0$, then:

$\mathrm{I} = \int_{+\infty}^0 \frac{e^{-at}}{\sqrt{-t}}(-e^{-t}dt) - \int_{+\infty}^0 \frac{e^{-bt}}{\sqrt{-t}}(-e^{-t}dt)$

$\mathrm{I} = \int_0^{+\infty} \frac{e^{-t(a+1)}}{(-1)^{\frac{1}{2}}t^{\frac{1}{2}}}dt - \int_0^{+\infty} \frac{e^{-t(b+1)}}{(-1)^{\frac{1}{2}}t^{\frac{1}{2}}}dt = -i\int_0^{+\infty} e^{-t(a+1)}t^{-\frac{1}{2}}dt + i\int_0^{+\infty} e^{-t(b+1)}t^{-\frac{1}{2}}dt$

Let us evaluate: $\int_0^{+\infty} e^{-t(a+1)}t^{-\frac{1}{2}}dt$, let $u = t(a+1)$, then $du = (a+1)dt$, then:

$\int_0^{+\infty} e^{-t(a+1)}t^{-\frac{1}{2}}dt = \int_0^{+\infty} e^{-u}\left(\frac{u}{a+1}\right)^{-\frac{1}{2}}\frac{1}{a+1}du$, then:

$\mathrm{I} = -i\int_0^{+\infty} e^{-u}\left(\frac{u}{a+1}\right)^{-\frac{1}{2}}\frac{1}{a+1}du + i\int_0^{+\infty} e^{-u}\left(\frac{u}{b+1}\right)^{-\frac{1}{2}}\frac{1}{b+1}du$

$\mathrm{I} = -\frac{i}{\sqrt{a+1}}\int_0^{+\infty} e^{-u}.u^{-\frac{1}{2}}du + \frac{i}{\sqrt{b+1}}\int_0^{+\infty} e^{-u}.u^{-\frac{1}{2}}du$

$\mathrm{I} = -\frac{i}{\sqrt{a+1}}\int_0^{+\infty} e^{-u}.u^{\frac{1}{2}-1}du + \frac{i}{\sqrt{b+1}}\int_0^{+\infty} e^{-u}.u^{\frac{1}{2}-1}du$, then we get:

$\mathrm{I} = -\frac{i}{\sqrt{a+1}}\Gamma\left(\frac{1}{2}\right) + \frac{i}{\sqrt{b+1}}\Gamma\left(\frac{1}{2}\right) = -\frac{i}{\sqrt{a+1}}\sqrt{\pi} + \frac{i}{\sqrt{b+1}}\sqrt{\pi}$, therefore, we get:

$\mathrm{I} = i\left(\sqrt{\frac{\pi}{b+1}} - \sqrt{\frac{\pi}{a+1}}\right)$

87. $\mathrm{I} = \int_0^{+\infty} \frac{xe^{-\mu x^2}}{\sqrt{x^2+a^2}}dx$, let $t = \sqrt{x^2+a^2}$, then $t^2 = x^2 + a^2$ and $2tdt = 2xdx$, so $tdt = xdx$

For $x = 0$, $t = a$ and for $x = +\infty$, $t = +\infty$, then:

$\mathrm{I} = \int_a^{+\infty} \frac{t.e^{-\mu\left(t^2-a^2\right)}}{t}dt = \int_a^{+\infty} e^{-\mu(t^2-a^2)}dt = \int_a^{+\infty} e^{-\mu t^2}.e^{\mu a^2}dt = e^{\mu a^2}\int_a^{+\infty} e^{-\mu t^2}dt$

$\mathrm{I} = e^{\mu a^2}\left(\int_0^{+\infty} e^{-\mu t^2}dt - \int_0^a e^{-\mu t^2}dt\right)$

But **Error Function:** $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$, then $\operatorname{erf}(\sqrt{\mu}a) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{\mu}a} e^{-m^2}dm$, then

$\mathrm{I} = e^{\mu a^2}\left(\int_0^{+\infty} e^{-\mu t^2}dt - \frac{1}{\sqrt{\mu}}.\frac{\sqrt{\pi}}{2}\operatorname{erf}(\sqrt{\mu}a)\right) = e^{\mu a^2}\int_0^{+\infty} e^{-\mu t^2}dt - \frac{1}{2}\sqrt{\frac{\pi}{\mu}}.e^{\mu a^2}\operatorname{erf}(\sqrt{\mu}a)$

Let $y = \sqrt{\mu}t$, then $dy = \sqrt{\mu}dt$, then we get:

$\mathrm{I} = e^{\mu a^2}\int_0^{+\infty} e^{-y^2}\frac{dy}{\sqrt{\mu}} - \frac{1}{2}\sqrt{\frac{\pi}{\mu}}.e^{\mu a^2}\operatorname{erf}(\sqrt{\mu}a) = \frac{1}{\sqrt{\mu}}e^{\mu a^2}\int_0^{+\infty} e^{-v^2}dv - \frac{1}{2}\sqrt{\frac{\pi}{\mu}}.e^{\mu a^2}\operatorname{erf}(\sqrt{\mu}a)$

$\mathrm{I} = \frac{1}{\sqrt{\mu}}e^{\mu a^2}\frac{\sqrt{\pi}}{2} - \frac{1}{2}\sqrt{\frac{\pi}{\mu}}.e^{\mu a^2}\operatorname{erf}(\sqrt{\mu}a) = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{\mu a^2} - \frac{1}{2}\sqrt{\frac{\pi}{\mu}}.e^{\mu a^2}\operatorname{erf}(\sqrt{\mu}a)$, therefore:

$\mathrm{I} = \frac{1}{2}\sqrt{\frac{\pi}{\mu}}e^{\mu a^2}[1 - \operatorname{erf}(\sqrt{\mu}a)]$

88. $\mathrm{I} = \int_0^1 \frac{dx}{a-\ln x}$, let $-t = \ln x$, then $x = e^{-t}$ and $dx = -e^{-t}dt$, for $x = 0$, $t = +\infty$ and for $x = 1$, $t = 0$, then we get:

$\mathrm{I} = \int_{+\infty}^{0} \frac{-e^{-t}}{a+t} dt = \int_0^{+\infty} \frac{e^{-t}}{a+t} dt$, let $u = a + t$, then $du = dt$, then we get:

$\mathrm{I} = \int_a^{+\infty} \frac{e^{-(u-a)}}{u} du = \int_a^{+\infty} \frac{e^{-u}.e^{a}}{u} du = e^{a} \int_a^{+\infty} \frac{e^{-u}}{u} du$, let $u = -v$, then $du = -dv$, so:

$\mathrm{I} = e^{a} \int_{-a}^{-\infty} \frac{e^{v}}{-v}(-dv) = e^{a} \int_{-a}^{-\infty} \frac{e^{v}}{v} dv = -e^{a} \int_{-\infty}^{-a} \frac{e^{v}}{v} dv = -e^{a}\mathrm{Ei}(-a)$

89. $\mathrm{I} = \int_0^1 \frac{dx}{a^2-\ln^2 x} = \int_0^1 \frac{dx}{(a-\ln x)(a+\ln x)} = \frac{1}{2a}\int_0^1 \frac{(a-\ln x)+(a+\ln x)}{(a-\ln x)(a+\ln x)} dx$, then we get:

$\mathrm{I} = \frac{1}{2a}\int_0^1 \frac{dx}{a-\ln x} + \frac{1}{2a}\int_0^1 \frac{dx}{a+\ln x}$, but $\int_0^1 \frac{dx}{a-\ln x} = -e^{a}\mathrm{Ei}(-a)$ (proved before) and it can be shown that $\int_0^1 \frac{dx}{a+\ln x} = e^{-a}\mathrm{Ei}(a)$, therefore, $\mathrm{I} = \frac{1}{2a}[e^{-a}\mathrm{Ei}(a) - e^{a}\mathrm{Ei}(-a)]$

90. $\mathrm{I} = \int_0^{+\infty} x^{-a\ln x+b-1}\, dx = \int_0^{+\infty} e^{\ln(x^{-a\ln x+b-1})}\, dx = \int_0^{+\infty} e^{(-a\ln x+b-1)\ln x} dx$

$\mathrm{I} = \int_0^{+\infty} e^{-a\ln^2 x+b\ln x-\ln x} dx = \int_0^{+\infty} e^{-a\ln^2 x}.e^{b\ln x}.e^{-\ln x} dx$

Let $u = \ln x$, then $x = e^{u}$ and $dx = e^{u}du$, for $x = 0$, $u = -\infty$ and for $x = +\infty$, $u = +\infty$, then:

$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-au^2}.e^{bu}.e^{-u}.e^{u}du = \int_{-\infty}^{+\infty} e^{-au^2}.e^{bu}du = \int_{-\infty}^{+\infty} e^{-au^2+bu}du = \int_{-\infty}^{+\infty} e^{-(au^2-bu)}du$

$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-\left(au^2-bu+\frac{b^2}{4a}-\frac{b^2}{4a}\right)}du = \int_{-\infty}^{+\infty} e^{-\left(au^2-bu+\frac{b^2}{4a}\right)+\frac{b^2}{4a}}du = \int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}u-\frac{b}{2\sqrt{a}}\right)^2+\frac{b^2}{4a}}du$

$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}u-\frac{b}{2\sqrt{a}}\right)^2} e^{\frac{b^2}{4a}}\, du = e^{\frac{b^2}{4a}} \int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}u-\frac{b}{2\sqrt{a}}\right)^2} du$

Let $t = \sqrt{a}u - \frac{b}{2\sqrt{a}}$, then $u = \frac{1}{\sqrt{a}}\left(t + \frac{b}{2\sqrt{a}}\right)$ and $du = \frac{1}{\sqrt{a}}dt$, then we get:

$\mathrm{I} = e^{\frac{b^2}{4a}} \int_{-\infty}^{+\infty} e^{-t^2} \frac{1}{\sqrt{a}} dt = \frac{1}{\sqrt{a}} e^{\frac{b^2}{4a}} \int_{-\infty}^{+\infty} e^{-t^2}\, dt = \frac{1}{\sqrt{a}} e^{\frac{b^2}{4a}}(\sqrt{\pi}) = e^{\frac{b^2}{4a}}\sqrt{\frac{\pi}{a}}$

91. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \sqrt{\tan x}\tan\left(\frac{x}{2}\right)\sec\left(\frac{x}{2}\right) dx$, let $t = \frac{x}{2}$, so $x = 2t$, then $dx = 2dt$, for $x = 0$, $t = 0$ and for $x = \frac{\pi}{2}$, $t = \frac{\pi}{4}$, then we get:

$\mathrm{I} = 2\int_0^{\frac{\pi}{4}} \sqrt{\tan 2t}\tan t \sec t\, dt = 2\int_0^{\frac{\pi}{4}} \sqrt{\frac{2\tan t}{1-\tan^2 t}} \tan t \sqrt{1+\tan^2 t}\, dt$

Let $u = \tan t$, then $du = (1+\tan^2 t)dt = (1+u^2)dt$, so $dt = \frac{1}{1+u^2}du$, for $t = 0$, $u = 0$ and for $t = \frac{\pi}{4}$, $u = 1$, then we get:

$\mathrm{I} = 2\int_0^1 \sqrt{\frac{2u}{1-u^2}}.u.\sqrt{1+u^2}.\frac{1}{1+u^2}du = 2\sqrt{2}\int_0^1 \frac{u^{\frac{1}{2}}}{\sqrt{1-u^2}}.u.\frac{1}{\sqrt{1+u^2}}du$, then

$\mathrm{I} = 2\sqrt{2}\int_0^1 \frac{u^{\frac{3}{2}}}{\sqrt{(1-u^2)(1+u^2)}}du = 2\sqrt{2}\int_0^1 \frac{u^{\frac{3}{2}}}{\sqrt{1-u^4}}du = 2\sqrt{2}\int_0^1 u^{\frac{3}{2}}(1-u^4)^{-\frac{1}{2}}du$

Let $y = u^4$, then $u = y^{\frac{1}{4}}$, so $du = \frac{1}{4}y^{-\frac{3}{4}}$, then we get:

$\mathrm{I} = 2\sqrt{2} \times \frac{1}{4}\int_0^1 \left(y^{\frac{1}{4}}\right)^{\frac{3}{2}} (1-y)^{-\frac{1}{2}}.y^{-\frac{3}{4}}.dy = \frac{1}{\sqrt{2}}\int_0^1 y^{-\frac{3}{8}}(1-y)^{-\frac{1}{2}} dy$, then we get:

$$\mathrm{I}=\frac{1}{\sqrt{2}}\int_0^1 y^{1-\frac{5}{8}}(1-y)^{1-\frac{1}{2}}\,dy=\frac{1}{\sqrt{2}}\mathrm{B}\left(\frac{5}{8},\frac{1}{2}\right)=\frac{1}{\sqrt{2}}\mathrm{B}\left(\frac{5}{8},\frac{1}{2}\right)=\frac{1}{\sqrt{2}}\frac{\Gamma\left(\frac{5}{8}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{8}+\frac{1}{2}\right)}=\frac{1}{\sqrt{2}}\frac{\Gamma\left(\frac{5}{8}\right)\sqrt{\pi}}{\Gamma\left(\frac{9}{8}\right)}$$

Therefore, we get $\mathrm{I}=\sqrt{\frac{\pi}{2}}\frac{\Gamma\left(\frac{5}{8}\right)}{\Gamma\left(\frac{9}{8}\right)}$

92. $\mathrm{I}=\int_0^1\cos^{-1}\left(\sqrt{1-\sqrt{x}}\right)dx$, let $x=\sin^4 t$, then $dx=4\sin^3 t\cos t\,dt$, for $x=0$, $t=0$ and for $x=1$, $t=\frac{\pi}{2}$, then we get:

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\cos^{-1}\left(\sqrt{1-\sqrt{\sin^4 t}}\right)4\sin^3 t\cos t\,dt=\int_0^{\frac{\pi}{2}}\cos^{-1}\left(\sqrt{1-\sin^2 t}\right)4\sin^3 t\cos t\,dt$$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\cos^{-1}\left(\sqrt{\cos^2 t}\right)4\sin^3 t\cos t\,dt=\int_0^{\frac{\pi}{2}}\cos^{-1}(\cos t)\,4\sin^3 t\cos t\,dt$$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}t\,4\sin^3 t\cos t\,dt=4\int_0^{\frac{\pi}{2}}t\sin^3 t\cos t\,dt$$

Let $u=t$, then $u'=1$ and let $v'=4\sin^3 t\cos t$, then $v=\sin^4 t$, then we get:

$$\mathrm{I}=[t\sin^4 t]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin^4 dt=\frac{\pi}{2}-\int_0^{\frac{\pi}{2}}\sin^4 dt=\frac{\pi}{2}-\mathrm{J}$$

$$\mathrm{J}=\int_0^{\frac{\pi}{2}}\sin^4 dt=\int_0^{\frac{\pi}{2}}\sin^{2\left(\frac{5}{2}\right)-1}t\times\cos^{2\left(\frac{1}{2}\right)-1}t\,dt=\frac{1}{2}\left(2\int_0^{\frac{\pi}{2}}\sin^{2\left(\frac{5}{2}\right)-1}t\times\cos^{2\left(\frac{1}{2}\right)-1}t\,dt\right)$$

$$\mathrm{I}=\frac{1}{2}\mathrm{B}\left(\frac{5}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{2}+\frac{1}{2}\right)}=\frac{\Gamma\left(\frac{5}{2}\right)\sqrt{\pi}}{2\Gamma(3)}=\frac{\frac{3}{2}\Gamma\left(\frac{3}{2}\right)\sqrt{\pi}}{2\times2}=\frac{\frac{3}{4}\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}{4}=\frac{\frac{3}{4}\sqrt{\pi}\sqrt{\pi}}{4}=\frac{3}{16}\pi$$, therefore, we get:

$$\mathrm{I}=\frac{\pi}{2}-\frac{3}{16}\pi=\frac{5}{16}\pi$$

93. $\mathrm{I}=\int_0^{+\infty}\frac{x^{2(m+n)}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx=\int_0^{+\infty}\frac{x^{2m}x^{2n}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx$

$$\mathrm{I}=\int_0^{+\infty}\frac{x^{2m}[(x^{2n}+1)-1]}{(1+x^2)^{2m+1}.(1+x^{2n})}dx=\int_0^{+\infty}\frac{x^{2m}(x^{2n}+1)-x^{2m}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx$$

$$\mathrm{I}=\int_0^{+\infty}\frac{x^{2m}}{(1+x^2)^{2m+1}}dx-\int_0^{+\infty}\frac{x^{2m}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx$$

For $\int_0^{+\infty}\frac{x^{2m}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx$, let $x=\frac{1}{t}$, then $dx=-\frac{1}{t^2}dt$, for $x=0$, $t=+\infty$ and for $x=+\infty$, $t=0$, so $\int_0^{+\infty}\frac{x^{2m}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx=\int_{+\infty}^0\frac{t^{-3m}}{(1+t^{-2m})^{2m+1}.(1+t^{-2n})}-\frac{1}{t^2}dt$

$$\int_0^{+\infty}\frac{x^{2m}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx=\int_0^{+\infty}\frac{t^{-2m}.t^{4m+2}t^{2n}}{(1+t^2)^{2m+1}.(1+t^{2n})t^2}dt$$, then we get:

$$\int_0^{+\infty}\frac{x^{2m}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx=\int_0^{+\infty}\frac{t^{2m}.t^{2n}}{(1+t^2)^{2m+1}.(1+t^{2n})}dt=\int_0^{+\infty}\frac{x^{2(m+n)}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx,$$

so: $\int_0^{+\infty}\frac{x^{2m}}{(1+x^2)^{2m+1}.(1+x^{2n})}dx=\mathrm{I}$, then $\mathrm{I}=\int_0^{+\infty}\frac{x^{2m}}{(1+x^2)^{2m+1}}dx$, then we get:

$2\text{I} = \int_0^{+\infty} \frac{x^{2m}}{(1+x^2)^{2m+1}} dx$, so $\text{I} = \frac{1}{2}\int_0^{+\infty} \frac{x^{2m}}{(1+x^2)^{2m+1}} dx$

Let $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$, for $x = 0$, $t = 0$ and for $x = +\infty$, $t = \frac{\pi}{2}$, then we get**:**

$\text{I} = \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{\tan^{2m}\theta . \sec^2\theta}{\sec^{4m+2}\theta} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^{4m}\theta \frac{\sec^{2m}\theta}{\cos^{2m}\theta} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin^{2m}\theta . \cos^{2m}\theta\, d\theta$

So, $\text{I} = \frac{1}{4}\left(2\int_0^{\frac{\pi}{2}} \sin^{2m}\theta . \cos^{2m}\theta\, d\theta\right) = \frac{1}{4}\text{B}\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{1}{4}\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(m+\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}+m+\frac{1}{2}\right)}$

Therefore, we get $\text{I} = \frac{1}{4}\frac{\Gamma^2\left(m+\frac{1}{2}\right)}{\Gamma(2m+1)} = \frac{\Gamma^2\left(m+\frac{1}{2}\right)}{4\Gamma(2m+1)}$

94. $\text{I} = \int_0^e x^{\frac{1}{m}}(1-\ln x)^{\frac{1}{n}} dx$, let $t = 1 - \ln x$, then $\ln x = 1 - t$, so $x = e^{1-t}$ and $dx = -e^{1-t}dt$, for $x = 0$, $t = +\infty$ and for $x = e$, $t = 0$, then we get**:**

$\text{I} = \int_{+\infty}^0 (e^{1-t})^{\frac{1}{m}}.t^{\frac{1}{n}}(-e^{1-t}dt) = \int_0^{+\infty} e^{\frac{1}{m}}.e^{-\frac{t}{m}}.t^{\frac{1}{n}}.e.e^{-t}\,dt = \int_0^{+\infty} e^{\left(1+\frac{1}{m}\right)}.e^{-t\left(1+\frac{1}{m}\right)}.t^{\frac{1}{n}}dt$

$\text{I} = e^{\left(1+\frac{1}{m}\right)}\int_0^{+\infty} e^{-t\left(1+\frac{1}{m}\right)} t^{\frac{1}{n}}dt$, let $u = t\left(1+\frac{1}{m}\right)$, then $dt = \frac{m}{m+1}du$, then we get**:**

$\text{I} = e^{\left(1+\frac{1}{m}\right)}\int_0^{+\infty} e^{-u}\left(\frac{m}{m+1}u\right)^{\frac{1}{n}}.\frac{m}{m+1}du = \left(\frac{m}{m+1}\right)^{\frac{1}{n}} e^{\left(1+\frac{1}{m}\right)}\int_0^{+\infty} e^{-u}u^{\frac{1}{n}}du$, then we get**:**

$\text{I} = \left(\frac{m}{m+1}\right)^{\frac{1}{n}} e^{\left(1+\frac{1}{m}\right)}\int_0^{+\infty} e^{-u}u^{\left(\frac{1}{n}+1\right)-1}du = \left(\frac{m}{m+1}\right)^{\frac{1}{n}} e^{\left(1+\frac{1}{m}\right)}\Gamma\left(1+\frac{1}{n}\right)$

95. $\text{I} = \int_0^{+\infty} x^{-a\ln x}\ln x\, dx = \int_0^{+\infty} e^{\ln(x^{-a\ln x})}\ln x\, dx = \int_0^{+\infty} e^{-a\ln x\times\ln x}\ln x\, dx$

$\text{I} = \int_0^{+\infty} e^{-a\ln^2 x}\ln x\, dx$, let $u = \ln x$, then $x = e^u$ and $dx = e^u du$, for $x = 0$, $u = -\infty$ and for $x = +\infty$, $u = +\infty$, then we get**:**

$\text{I} = \int_{-\infty}^{+\infty} e^{-au^2}.u.e^u du = \int_{-\infty}^{+\infty} e^{-au^2+u}.u du = \int_{-\infty}^{+\infty} e^{-(au^2-u)}u du$

We have $au^2 - u = au^2 - u + \frac{1}{4a} - \frac{1}{4a} = \left(\sqrt{a}u - \frac{1}{2\sqrt{a}}\right) - \frac{1}{4a}$, then we get**:**

$\text{I} = \int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}u-\frac{1}{2\sqrt{a}}\right)^2+\frac{1}{4a}}u du = \int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}u-\frac{1}{2\sqrt{a}}\right)^2} e^{\frac{1}{4a}}u du = e^{\frac{1}{4a}}\int_{-\infty}^{+\infty} e^{-\left(\sqrt{a}u-\frac{1}{2\sqrt{a}}\right)^2} u du$

Let $t = \sqrt{a}u - \frac{1}{2\sqrt{a}}$, then $u = \frac{1}{\sqrt{a}}\left(t + \frac{1}{2\sqrt{a}}\right)$ and $du = \frac{1}{\sqrt{a}}dt$, then we get**:**

$\text{I} = e^{\frac{1}{4a}}\int_{-\infty}^{+\infty} e^{-t^2}.\frac{1}{\sqrt{a}}\left(t+\frac{1}{2\sqrt{a}}\right).\frac{1}{\sqrt{a}}dt = \frac{1}{a}e^{\frac{1}{4a}}\int_{-\infty}^{+\infty} e^{-t^2}.\left(t+\frac{1}{2\sqrt{a}}\right)dt$

$\text{I} = \frac{1}{a}e^{\frac{1}{4a}}\left(\int_{-\infty}^{+\infty} te^{-t^2}dt + \frac{1}{2\sqrt{a}}\int_{-\infty}^{+\infty} e^{-t^2}dt\right) = \frac{1}{a}e^{\frac{1}{4a}}\left(\left[-\frac{1}{2}e^{-t^2}\right]_{-\infty}^{+\infty} + \frac{1}{2\sqrt{a}}\int_{-\infty}^{+\infty} e^{-t^2}dt\right)$

$\text{I} = \frac{1}{a}e^{\frac{1}{4a}}\left(\frac{1}{2\sqrt{a}}\int_{-\infty}^{+\infty} e^{-t^2}dt\right) = \frac{1}{2a\sqrt{a}}e^{\frac{1}{4a}}\int_{-\infty}^{+\infty} e^{-t^2}dt = \frac{1}{2a\sqrt{a}}e^{\frac{1}{4a}}\sqrt{\pi}$

Therefore, we get**:** $\text{I} = \frac{1}{2a}e^{\frac{1}{4a}}\sqrt{\frac{\pi}{a}}$

96. $\text{I} = \int_0^{\frac{\pi}{2}} \sin^3 x \ln(\ln\cos x)\, dx = \int_0^{\frac{\pi}{2}} \sin^2 x \ln(\ln\cos x)\sin x\, dx$

$\text{I} = \int_0^{\frac{\pi}{2}} (1-\cos^2 x)\ln(\ln\cos x)\sin x\, dx$, let $u = \cos x$, then $du = -\sin x\, dx$, for $x = 0$, then $u = 1$ and for $x = \frac{\pi}{2}$ then $u = 0$, then we get**:**

$\mathrm{I}=\int_1^0(1-u^2)\ln\ln u\,(-du)=\int_0^1(1-u^2)\ln\ln u\,du=\int_0^1\ln\ln u\,du-\int_0^1 u^2\ln\ln u\,du$

Let $-t=\ln u$, then $u=e^{-t}$ and $du=-e^{-t}dt$, for $u=0$, then $t=+\infty$ and for $u=1$, $t=0$, so

$\mathrm{I}=\int_0^{+\infty}\ln(-t)\,.\,e^{-t}dt-\int_0^{+\infty}e^{-2t}.\ln(-t)\,.\,e^{-t}dt$

$\mathrm{I}=\int_0^{+\infty}[\ln(-1)+\ln t].\,e^{-t}dt-\int_0^{+\infty}e^{-3t}.\,[\ln(-1)+\ln t]dt$

$\mathrm{I}=\int_0^{+\infty}(i\pi+\ln t).\,e^{-t}dt-\int_0^{+\infty}e^{-3t}.\,(i\pi+\ln t)dt$

$\mathrm{I}=i\pi-\gamma-i\pi\int_0^{+\infty}e^{-3t}dt-\int_0^{+\infty}e^{-3t}\ln t\,dt=i\pi-\gamma-i\frac{\pi}{3}-\int_0^{+\infty}e^{-3t}\ln t\,dt$

Let $y=3t$, then $dy=3dt$, then we get:

$\mathrm{I}=\ i\pi-\gamma-i\frac{\pi}{3}-\int_0^{+\infty}e^{-y}\ln\left(\frac{y}{3}\right)\frac{1}{3}dy=\ i\pi-\gamma-i\frac{\pi}{3}-\frac{1}{3}\int_0^{+\infty}e^{-y}(\ln y-\ln 3)dy$

$\mathrm{I}=\ i\pi-\gamma-i\frac{\pi}{3}-\frac{1}{3}\int_0^{+\infty}e^{-y}\ln y\,dy+\frac{1}{3}\ln 3\int_0^{+\infty}e^{-y}dy$

$\mathrm{I}=\ i\pi-\gamma-i\frac{\pi}{3}+\frac{1}{3}\gamma+\frac{1}{3}\ln 3=\frac{1}{3}(\ln 3-2\gamma)+\frac{2\pi}{3}i$

97. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1-\sin^4 x}{(1+\sin^4 x)\sqrt{1+\sin^2 x}}dx$, let $y=\sin x$, then $x=\arcsin y$ and $dx=\frac{1}{\sqrt{1-y^2}}dy$

For $x=0$, $y=0$ and for $x=\frac{\pi}{2}$, $y=1$, then we get:

$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1-y^4}{(1+y^4)\sqrt{1+y^2}}.\frac{1}{\sqrt{1-y^2}}dx=\int_0^{\frac{\pi}{2}}\frac{1-y^4}{(1+y^4)\sqrt{(1+y^2)(1-y^2)}}dx=\int_0^{\frac{\pi}{2}}\frac{1-y^4}{(1+y^4)\sqrt{1-y^4}}dx$

$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-y^4}.\sqrt{1-y^4}}{(1+y^4)\sqrt{1-y^4}}dx=\int_0^{\frac{\pi}{2}}\frac{\sqrt{1-y^4}}{1+y^4}dx$

Let $y^2=\tan\theta$ and $y=\sqrt{\tan\theta}$, then $dy=\frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta$, for $x=0$, $\theta=0$ and for $x=1$, $\theta=\frac{\pi}{4}$

$\mathrm{I}=\int_0^{\frac{\pi}{4}}\frac{\sqrt{1-\tan^2\theta}}{1+\tan^2\theta}.\frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta=\int_0^{\frac{\pi}{4}}\frac{\sqrt{1-\tan^2\theta}}{\sec^2\theta}.\frac{\sec^2\theta}{2\sqrt{\tan\theta}}d\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sqrt{1-\tan^2\theta}}{\sqrt{\tan\theta}}d\theta$

$\mathrm{I}=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sqrt{1-\frac{\sin^2\theta}{\cos^2\theta}}}{\sqrt{\frac{\sin\theta}{\cos\theta}}}d\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sqrt{\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta}}}{\sqrt{\frac{\sin\theta}{\cos\theta}}}d\theta=\frac{1}{2}\int_0^{\frac{\pi}{4}}\frac{\sqrt{\cos^2\theta-\sin^2\theta}}{\sqrt{\sin\theta\cos\theta}}d\theta$

$\mathrm{I}=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{\sqrt{\cos^2\theta-\sin^2\theta}}{\sqrt{2\sin\theta\cos\theta}}d\theta=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{\sqrt{\cos 2\theta}}{\sqrt{\sin 2\theta}}d\theta$, let $\alpha=2\theta$, then $d\alpha=2d\theta$, then we get:

$\mathrm{I}=\frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{\sqrt{\cos\alpha}}{\sqrt{\sin\alpha}}d\alpha=\frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}\alpha\cos^{\frac{1}{2}}\alpha\,d\alpha=\frac{1}{2\sqrt{2}}\int_0^{\frac{\pi}{2}}\sin^{2\left(\frac{1}{4}\right)-1}\alpha\cos^{2\left(\frac{3}{4}\right)-1}\alpha\,d\alpha$

$\mathrm{I}=\frac{1}{4\sqrt{2}}\left(2\int_0^{\frac{\pi}{2}}\sin^{2\left(\frac{1}{4}\right)-1}\alpha\cos^{2\left(\frac{3}{4}\right)-1}\alpha\,d\alpha\right)=\frac{1}{4\sqrt{2}}\mathrm{B}\left(\frac{1}{4},\frac{3}{4}\right)=\frac{1}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}=\frac{1}{4\sqrt{2}}\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}$

$\mathrm{I}=\frac{1}{4\sqrt{2}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{1}{4\sqrt{2}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)=\frac{1}{4\sqrt{2}}\times\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}=\frac{1}{4\sqrt{2}}.\pi\sqrt{2}=\frac{\pi}{4}$

98. $\mathrm{I}=\int_0^{+\infty}\frac{1}{1+x^{10}}dx$, let $x=\sqrt[5]{\tan\theta}\Longrightarrow x^5=\tan\theta\Longrightarrow 5x^4dx=\sec^2\theta\,d\theta\Longrightarrow dx=\frac{\sec^2\theta}{5x^4}d\theta$

$\Longrightarrow dx=\frac{\sec^2\theta}{5\sqrt[5]{\tan^4\theta}}d\theta$, for $x=0$, $\theta=0$ and as $x\to+\infty$, $\theta\to\frac{\pi}{2}\Longrightarrow$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{1+\left(\sqrt[5]{\tan\theta}\right)^{10}} \cdot \frac{\sec^2\theta}{5\sqrt[5]{\tan^4\theta}} dx = \int_0^{\frac{\pi}{2}} \frac{1}{1+\tan^2\theta} \cdot \frac{1+\tan^2\theta}{5(\tan\theta)^{\frac{4}{5}}} dx = \frac{1}{5}\int_0^{\frac{\pi}{2}} \tan^{-\frac{4}{5}}\theta\, d\theta$$

$$= \frac{1}{5}\int_0^{\frac{\pi}{2}} \left(\frac{\sin\theta}{\cos\theta}\right)^{-\frac{4}{5}} \theta\, d\theta = \frac{1}{5}\int_0^{\frac{\pi}{2}} \sin^{-\frac{4}{5}}\theta \cos^{\frac{4}{5}}\theta\, d\theta = \frac{1}{5}\int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{1}{10}\right)-1}\theta \cos^{2\left(\frac{9}{10}\right)-1}\theta\, d\theta$$

Beta Function: $\mathrm{B}(n,m) = 2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta \times \cos^{2m-1}\theta\, d\theta$ and $\mathrm{B}(n,m) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)} \Longrightarrow$

$$\mathrm{I} = \frac{1}{5}\left[\frac{1}{2}\mathrm{B}\left(\frac{1}{10},\frac{9}{10}\right)\right] = \frac{1}{10} \times \frac{\Gamma\left(\frac{1}{10}\right)\times\Gamma\left(\frac{9}{10}\right)}{\Gamma\left(\frac{1}{10}+\frac{9}{10}\right)} = \frac{1}{10}\Gamma\left(\frac{1}{10}\right)\Gamma\left(\frac{9}{10}\right) = \frac{1}{10}\Gamma\left(\frac{1}{10}\right)\Gamma\left(1-\frac{1}{10}\right)$$

Euler's reflection formula: $\Gamma(\alpha)\times\Gamma(1-\alpha) = \frac{\pi}{\sin(\pi\alpha)}$, set $\alpha = \frac{1}{10}$, so $\mathrm{I} = \frac{\pi}{10}\left(\frac{1}{\sin\left(\frac{\pi}{10}\right)}\right)$

Let $\mathrm{A} = 18° = \frac{\pi}{10}\mathrm{rd} \Longrightarrow 5\mathrm{A} = 5\times 18° = 90° \Longrightarrow 2\mathrm{A} + 3\mathrm{A} = 90° \Longrightarrow 2\mathrm{A} = 90° - 3\mathrm{A}$

$\Longrightarrow \sin(2\mathrm{A}) = \sin(90° - 3\mathrm{A}) \Longrightarrow \sin(2\mathrm{A}) = \cos(3\mathrm{A}) \Longrightarrow 2\sin\mathrm{A}\cos\mathrm{A} = 4\cos^3\mathrm{A} - 3\cos\mathrm{A}$

$\Longrightarrow 2\sin\mathrm{A}\cos\mathrm{A} = \cos\mathrm{A}\,(4\cos^2\mathrm{A} - 4) \Longrightarrow 2\sin\mathrm{A} = 4\cos^2\mathrm{A} - 3 \Longrightarrow$

$2\sin\mathrm{A} = 4(\sin^2\mathrm{A} - 1) - 3 \Longrightarrow 4\sin^2\mathrm{A} + 2\sin\mathrm{A} - 1 = 0$ by solving a quadratic equation we get: $\sin\mathrm{A} = \frac{-1\pm\sqrt{5}}{4}$, with $0 < 18° < 90°$, so $\sin\mathrm{A} > 0$ then $\sin\mathrm{A} = \frac{\sqrt{5}-1}{4} \Longrightarrow$

$$\mathrm{I} = \frac{\pi}{10}\left(\frac{1}{\frac{\sqrt{5}-1}{4}}\right) = \left(\frac{1+\sqrt{5}}{2}\right)\left(\frac{\pi}{5}\right) = \frac{\pi}{5}\varphi$$

99. $\int_0^{+\infty} \frac{x^{\varphi}}{(x+1)\sqrt{x^{4\varphi}+x^4}} dx = \int_0^{+\infty} \frac{x^{\varphi}}{(x+1)\sqrt{x^4\left(x^{4\varphi-4}+1\right)}} dx = \int_0^{+\infty} \frac{x^{\varphi}}{(x+1)x^2\sqrt{x^{4\varphi-4}+1}} dx$

$= \int_0^{+\infty} \frac{x^{\varphi-2}}{(x+1)\sqrt{x^{4(\varphi-1)}+1}} dx$, with $\varphi = \frac{1}{\varphi} + 1$ **(Golden Ratio)**, so we get:

$\mathrm{I} = \int_0^{+\infty} \frac{x^{\frac{1}{\varphi}-1}}{(x+1)\sqrt{x^{\frac{4}{\varphi}}+1}} dx$, let $u = x^{\frac{1}{\varphi}}$, then $du = \frac{1}{\varphi}x^{\frac{1}{\varphi}-1}dx$, so $\varphi du = x^{\frac{1}{\varphi}-1}dx$, then:

$\mathrm{I} = \int_0^{+\infty} \frac{\varphi}{(u^{\varphi}+1)\sqrt{u^4+1}} du$ **… (1)** , let $t = \frac{1}{u}\left(u = \frac{1}{t}\right)$, then $du = -\frac{1}{t^2}dt$, then:

$\mathrm{I} = \int_{+\infty}^{0} \frac{\varphi}{\left(\frac{1}{t^{\varphi}}+1\right)t^2\sqrt{\frac{1}{t^4}+1}}(-dt) = \int_0^{+\infty} \frac{\varphi t^{\varphi}}{(1+t^{\varphi})\sqrt{1+t^4}} dt = \int_0^{+\infty} \frac{\varphi u^{\varphi}}{(1+u^{\varphi})\sqrt{1+u^4}} du$ **… (2)**

Adding (1) and (2) we get**:** $2\mathrm{I} = \int_0^{+\infty} \frac{\varphi}{(u^{\varphi}+1)\sqrt{u^4+1}} du + \int_0^{+\infty} \frac{\varphi u^{\varphi}}{(1+u^{\varphi})\sqrt{1+u^4}} du$

$2\mathrm{I} = \int_0^{+\infty} \frac{\varphi+\varphi u^{\varphi}}{(u^{\varphi}+1)\sqrt{u^4+1}} du = \int_0^{+\infty} \frac{\varphi(1+u^{\varphi})}{(u^{\varphi}+1)\sqrt{u^4+1}} du = \int_0^{+\infty} \frac{\varphi}{\sqrt{u^4+1}} du$, $\mathrm{I} = \int_0^{+\infty} \frac{\varphi}{2\sqrt{u^4+1}} du$

Let $u = \sqrt{\tan\theta}$, then $du = \frac{\sec^2\theta}{2\sqrt{\tan\theta}} d\theta$, so $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\varphi\sec^2\theta}{4\sqrt{1+\tan^2\theta}\times\sqrt{\tan\theta}} d\theta$, then**:**

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{\varphi\sec^2\theta}{4\sqrt{\sec^2\theta\times\sqrt{\tan\theta}}}d\theta=\int_0^{\frac{\pi}{2}}\frac{\varphi\sec^2\theta}{4\sec\theta\times\sqrt{\tan\theta}}d\theta=\int_0^{\frac{\pi}{2}}\frac{\varphi\sec\theta}{4\sqrt{\tan\theta}}d\theta=\frac{2}{8}\int_0^{\frac{\pi}{2}}\frac{\varphi}{\sqrt{\sin\theta}\sqrt{\cos\theta}}d\theta$$

$$=\frac{2\varphi}{8}\int_0^{\frac{\pi}{2}}\sin^{2\left(\frac{1}{4}\right)-1}\theta\cos^{2\left(\frac{1}{4}\right)-1}\theta\,d\theta=\frac{\varphi}{8}\mathrm{B}\left(\frac{1}{4},\frac{1}{4}\right)=\frac{\varphi}{8}\times\frac{\Gamma\left(\frac{1}{4}\right)\times\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{4}\right)}=\frac{\varphi}{8}\times\frac{\Gamma^2\left(\frac{1}{4}\right)}{\Gamma\left(\frac{1}{2}\right)}=\frac{\varphi\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt{\pi}}$$

100. $\mathrm{I}=\int_0^{+\infty}\left(\frac{x^2}{x^4+2ax^2+1}\right)^m dx=\int_0^{+\infty}\left(\frac{1}{x^2+2a+\frac{1}{x^2}}\right)^m dx=$

$$\int_0^{+\infty}\left(\frac{1}{x^2+\frac{1}{x^2}-2+2a+2}\right)^m dx=\int_0^{+\infty}\left[\frac{1}{\left(x-\frac{1}{x}\right)^2+2(a+1)}\right]^m dx$$

Remark: Consider the integral $\mathrm{J}=\int_0^{+\infty}f\left(\left(ax-\frac{b}{x}\right)^2\right)dx$ **… (1)**

Let $x=\frac{b}{at}$, then $dx=-\frac{b}{at^2}dt$, so $\mathrm{J}=\int_{+\infty}^{0}f\left(\left(\frac{b}{t}-at\right)^2\right)\left(-\frac{b}{at^2}dt\right)$

$\mathrm{J}=\int_0^{+\infty}f\left(\left(at-\frac{b}{t}\right)^2\right)\frac{b}{at^2}dt=\int_0^{+\infty}f\left(\left(ax-\frac{b}{x}\right)^2\right)\frac{b}{ax^2}dx$ **… (2)**

Adding (1) and (2) we get $2\mathrm{J}=\int_0^{+\infty}f\left(\left(ax-\frac{b}{x}\right)^2\right)dx+\int_0^{+\infty}f\left(\left(ax-\frac{b}{x}\right)^2\right)\frac{b}{ax^2}dx$

$2\mathrm{J}=\frac{1}{a}\int_0^{+\infty}\left(a+\frac{b}{x^2}\right)f\left(\left(ax-\frac{b}{x}\right)^2\right)dx$, let $y=ax-\frac{b}{x}$, then $dy=a+\frac{b}{x^2}$, for $x=0$, we get $y=-\infty$ and for $x=+\infty$, $y=+\infty$, then we get**:**

$2\mathrm{J}=\frac{1}{a}\int_{-\infty}^{+\infty}f(y^2)dy=\frac{2}{a}\int_0^{+\infty}f(y^2)dy=\frac{2}{a}\int_0^{+\infty}f(x^2)dx$, so $\mathrm{J}=\frac{1}{a}\int_0^{+\infty}f(x^2)dx$

Therefore, we get $\int_0^{+\infty}f\left(\left(ax-\frac{b}{x}\right)^2\right)dx=\frac{1}{a}\int_0^{+\infty}f(x^2)dx$

Go back to our integral, $\mathrm{I}=\int_0^{+\infty}\left[\frac{1}{\left(x-\frac{1}{x}\right)^2+2(a+1)}\right]^m dx=\int_0^{+\infty}\left[\frac{1}{x^2+2(a+1)}\right]^m dx$

Let $u=\frac{2(a+1)}{x^2+2(a+1)}$, then $du=-\frac{x(a+1)}{\left[x^2+2(a+1)\right]^{\mathbf{2}}}dx$ and $\begin{cases}\text{for } x=0; & u=1\\ \text{for } x=+\infty; & u=0\end{cases}$

We have also $x=\sqrt{2(a+1)}\sqrt{\frac{1}{u}-1}$, then we get**:**

$\mathrm{I}=\frac{1}{2}[2(a+1)]^{-m+\frac{1}{2}}\int_0^1 u^{m-\frac{3}{2}}(1-u)^{-\frac{1}{2}}du=\frac{1}{2}[2(a+1)]^{-m+\frac{1}{2}}\mathrm{B}\left(m-\frac{1}{2},\frac{1}{2}\right)$

101. $\mathrm{I}=\int[\Psi(x)-\Psi(1-x)]\,dx$

We have $\Gamma(x).\Gamma(1-x)=\frac{\pi}{\sin(\pi x)}$, then $\ln[\Gamma(x).\Gamma(1-x)]=\ln\left[\frac{\pi}{\sin(\pi x)}\right]$, then we get**:**

$\ln\Gamma(x)+\ln\Gamma(1-x)=\ln\pi-\ln\sin(\pi x)$, by deriving both sides with respect to x, we get**:**

$$\frac{\Gamma'(x)}{\Gamma(x)}-\frac{\Gamma'(1-x)}{\Gamma(1-x)}=-\pi\frac{\cos(\pi x)}{\sin(\pi x)}\Longrightarrow\frac{\Gamma'(x)}{\Gamma(x)}-\frac{\Gamma'(1-x)}{\Gamma(1-x)}=-\pi\cot(\pi x)$$

With $\frac{\Gamma'(x)}{\Gamma(x)}=\Psi(x)$ and $\frac{\Gamma'(1-x)}{\Gamma(1-x)}=\Psi(1-x)$; so $\Psi(x)-\Psi(1-x)=-\pi\cot(\pi x)$; so:

$\mathrm{I} = -\pi \int \cot(\pi x)\, dx = -\int \frac{\pi \cos(\pi x)}{\sin(\pi x)} dx = -\int \frac{[\sin(\pi x)]'}{\sin(\pi x)} dx = \ln|\sin(\pi x)| + c$

102. $\mathrm{I} = \int_0^1 \ln \Gamma(x)\, dx \ldots$ **(1)** , let $u = 1 - x$, then $du = -dx$ and $\begin{cases} \text{for} \quad x = 0; & u = 1 \\ \text{for} \quad x = 1; & u = 0 \end{cases}$, then we get: $\mathrm{I} = \int_1^0 \ln \Gamma(1-u)\,(-du) = \int_0^1 \ln \Gamma(1-u)\, du = \int_0^1 \ln \Gamma(1-x)\, dx \ldots$ **(2)**

Adding (1) and (2) we get: $2\mathrm{I} = \int_0^1 \ln \Gamma(x)\, dx + \int_0^1 \ln \Gamma(1-x)\, dx$

$2\mathrm{I} = \int_0^1 [\ln \Gamma(x) + \ln \Gamma(1-x)]\, dx = \int_0^1 \ln[\Gamma(x)\Gamma(1-x)]$

Euler's reflection formula: $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$, so we get: $2\mathrm{I} = \int_0^1 \ln\left(\frac{\pi}{\sin(\pi x)}\right) dx$

$2\mathrm{I} = \int_0^1 \ln(\pi)\, dx - \int_0^1 \ln \sin(\pi x)\, dx = \ln \pi - \int_0^1 \ln \sin(\pi x)\, dx$

Let $v = \pi x$, then $dx = \pi dx$, for $x = 0$, $v = 0$ and for $x = 1$, $v = \pi$, then we get:

$\mathrm{I} = \ln \pi - \frac{1}{\pi}\int_0^\pi \ln \sin v\, dv = \ln \pi - \frac{1}{\pi}\int_0^\pi \ln \sin x\, dx = \ln \pi - \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \ln \sin x\, dx$

Then $2\mathrm{I} = \ln \pi - \frac{2}{\pi}\mathrm{J}$, with $\mathrm{J} = \int_0^{\frac{\pi}{2}} \ln \sin x\, dx$

Let's find J: let $u = \frac{\pi}{2} - x$, then $du = -dx$, for $x = 0$, $u = \frac{\pi}{2}$ and for $x = \frac{\pi}{2}$, $u = 0$, then:

$\mathrm{J} = \int_{\frac{\pi}{2}}^0 \ln \sin\left(\frac{\pi}{2} - u\right)(-du) = \int_0^{\frac{\pi}{2}} \ln \cos u\, du = \int_0^{\frac{\pi}{2}} \ln \cos x\, dx$, so:

$\mathrm{J} + \mathrm{J} = \int_0^{\frac{\pi}{2}} \ln \sin x\, dx + \int_0^{\frac{\pi}{2}} \ln \cos x\, dx = \int_0^{\frac{\pi}{2}} [\ln \sin x + \ln \cos x] dx = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)\, dx$

$2\mathrm{J} = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right) dx = \int_0^{\frac{\pi}{2}} \ln \sin 2x\, dx - \int_0^{\frac{\pi}{2}} \ln 2\, dx = \int_0^{\frac{\pi}{2}} \ln \sin 2x\, dx - \frac{\pi}{2}\ln 2$

Let $w = 2x$, then $dw = 2dx$, for $x = 0$, $w = 0$ and for $x = \frac{\pi}{2}$, $w = \pi$, so we get:

$2\mathrm{J} = \frac{1}{2}\int_0^\pi \ln \sin w\, dw - \frac{\pi}{2}\ln 2 = \frac{1}{2}\left(2\int_0^\pi \ln \sin w\, dw\right) - \frac{\pi}{2}\ln 2 = \int_0^{\frac{\pi}{2}} \ln \sin w\, dw - \frac{\pi}{2}\ln 2$

so, $2\mathrm{J} = \int_0^{\frac{\pi}{2}} \ln \sin x\, dx - \frac{\pi}{2}\ln 2$, then $2\mathrm{J} = \mathrm{J} - \frac{\pi}{2}\ln 2$, therefore $\mathrm{J} = -\frac{\pi}{2}\ln 2 = -\pi \ln \sqrt{2}$

Back to I: we have $2\mathrm{I} = \ln \pi - \frac{2}{\pi}\mathrm{J}$, then $2\mathrm{I} = \ln \pi - \frac{2}{\pi}\left(-\frac{\pi}{2}\ln 2\right) = \ln \pi + \ln 2 = \ln 2\pi$

Therefore $\mathrm{I} = \frac{1}{2}\ln 2\pi = \ln \sqrt{2\pi}$

103. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin^5 x + \cos^5 x} dx$, let $u = \frac{\pi}{2} - x$, then $du = -dx$ and $\begin{cases} \text{for} \quad x = 0 & u = \frac{\pi}{2} \\ \text{for} \quad x = \frac{\pi}{2} & u = 0 \end{cases}$, then

we get: $\mathrm{I} = \int_{\frac{\pi}{2}}^0 \frac{\sin\left(\frac{\pi}{2} - u\right)}{\sin^5\left(\frac{\pi}{2} - u\right) + \cos^5\left(\frac{\pi}{2} - u\right)}(-du) = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos^5 u + \sin^5 u} du = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^5 x + \cos^5 x} dx$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin^5 x + \cos^5 x} \times \frac{\frac{1}{\cos^5 x}}{\frac{1}{\cos^5 x}} dx = \int_0^{\frac{\pi}{2}} \frac{\sec^4 x}{1 + \tan^5 x} dx = \int_0^{\frac{\pi}{2}} \frac{\left(+\tan^2 x\right)\sec^2 x}{1 + \tan^5 x} dx$

Let $t = \tan x$, then $dt = \sec^2 x\, dx$, for $x = 0$, $t = 0$ and for $x = \frac{\pi}{2}$, $t = +\infty$, then we get:

$\mathrm{I} = \int_0^{+\infty} \frac{1 + t^2}{1 + t^5} dt$, let $z = t^5$, then $t = z^{\frac{1}{5}}$ and $dt = \frac{1}{5} z^{-\frac{4}{5}} dz$, then we get:

$$\mathrm{I}=\frac{1}{5}\int_0^{+\infty}\left(\frac{1+z^{\frac{2}{5}}}{1+z}\right)z^{-\frac{4}{5}}dz=\frac{1}{5}\int_0^{+\infty}\frac{z^{-\frac{4}{5}}+z^{-\frac{2}{5}}}{1+z}dz=\frac{1}{5}\left(\int_0^{+\infty}\frac{z^{-\frac{4}{5}}}{1+z}dz+\int_0^{+\infty}\frac{z^{-\frac{2}{5}}}{1+z}dz\right)$$

Recall that: $\int_0^{+\infty}\frac{z^{-a}}{1+z}dz=\frac{\pi}{\sin(\pi a)}$, then we get**:**

$$\mathrm{I}=\frac{1}{5}\left(\frac{\pi}{\sin\left(\frac{4\pi}{5}\right)}+\frac{\pi}{\sin\left(\frac{2\pi}{5}\right)}\right)=\frac{\pi}{5}\left(\frac{1}{\sin\left(\frac{4\pi}{5}\right)}+\frac{1}{\sin\left(\frac{2\pi}{5}\right)}\right)=\frac{\pi}{5}\left(\frac{1}{\sin\left(\frac{\pi}{5}\right)}+\frac{1}{\sin\left(\frac{2\pi}{5}\right)}\right)=\frac{\pi}{5}\left(\frac{\sin\left(\frac{2\pi}{5}\right)+\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)}\right)$$

$$\mathrm{I}=\frac{\pi}{5}\left(\frac{2\sin\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{5}\right)+\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{\pi}{5}\right)\sin\left(\frac{2\pi}{5}\right)}\right)=\frac{\pi}{5}\left(\frac{2\cos\left(\frac{\pi}{5}\right)+1}{\sin\left(\frac{2\pi}{5}\right)}\right)=\frac{\pi}{5}\left(\frac{1+2\cos\left(\frac{\pi}{5}\right)}{2\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{5}\right)}\right)$$

But $\varphi=2\cos\left(\frac{\pi}{5}\right)$ = Golden Ratio $=\frac{1+\sqrt{5}}{2}$ and $\varphi^2-\varphi-1=0$, then we get**:**

$$\mathrm{I}=\frac{\pi}{5}\left(\frac{1+\varphi}{\varphi\sqrt{1-\left(\frac{\varphi}{2}\right)^2}}\right)=\frac{\pi}{5}\left(\frac{1+\varphi}{\frac{1}{2}\varphi\sqrt{4-\varphi^2}}\right)=\frac{2\pi}{5}\left(\frac{1+\varphi}{\varphi\sqrt{4-\varphi^2}}\right)=\frac{2\pi}{5}\sqrt{\frac{\varphi^2}{4-\varphi^2}}=\frac{2\pi}{5}\sqrt{\frac{1+\varphi}{3-\varphi}}$$

$\mathrm{I}=\frac{2\pi}{5}\sqrt{\frac{3+\sqrt{5}}{5-\sqrt{5}}}=\frac{2\pi}{5}\sqrt{1-\frac{2(1-\sqrt{5})}{5-\sqrt{5}}}=\frac{2\pi}{5}\sqrt{1+\frac{2\sqrt{5}}{5}}$, therefore we get**:**

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{\sin x}{\sin^5 x+\cos^5 x}dx=\frac{2\pi}{5}\sqrt{1+\frac{2\sqrt{5}}{5}}$$

104. $\mathrm{I}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{\cos x+\sin x}{\cos x-\sin x}\right)^{\cos 2\alpha}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{\frac{1}{\sqrt{2}}\cos x+\frac{1}{\sqrt{2}}\sin x}{\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x}\right)^{\cos 2\alpha}dx$

with $\sin\left(\frac{\pi}{4}\right)=\cos\left(\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$

Then $\mathrm{I}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{\sin\left(\frac{\pi}{4}\right)\cos x+\cos\left(\frac{\pi}{4}\right)\sin x}{\cos\left(\frac{\pi}{4}\right)\cos x-\sin\left(\frac{\pi}{4}\right)\sin x}\right)^{\cos 2\alpha}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\frac{\sin\left(\frac{\pi}{4}+x\right)}{\cos\left(\frac{\pi}{4}+x\right)}\right)^{\cos 2\alpha}dx$

$\mathrm{I}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left[\tan\left(\frac{\pi}{4}+x\right)\right]^{\cos 2\alpha}dx$, let $t=\frac{\pi}{4}+x$, then $dt=dx$, for $x=-\frac{\pi}{4}$, then $t=0$ and for $x=\frac{\pi}{4}$, then $t=\frac{\pi}{2}$, then we get**:** $\mathrm{I}=\int_0^{\frac{\pi}{2}}(\tan t)^{\cos 2\alpha}dt=\int_0^{\frac{\pi}{2}}\sin^{\cos 2\alpha}t\cos^{-\cos 2\alpha}t\,dt$

With $\mathrm{B}(m,n)=2\int_0^{\frac{\pi}{2}}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$ We have $2m-1=\cos 2\alpha$, so $m=\frac{1+\cos 2\alpha}{2}$ and $2n-1=-\cos 2\alpha$, so $n=\frac{1-\cos 2\alpha}{2}$, then we get**:**

$$\mathrm{I}=\frac{1}{2}\mathrm{B}\left(\frac{1+\cos 2\alpha}{2},\frac{1-\cos 2\alpha}{2}\right)=\frac{1}{2}.\frac{\Gamma\left(\frac{1+\cos 2\alpha}{2}\right)\Gamma\left(\frac{1-\cos 2\alpha}{2}\right)}{\Gamma\left(\frac{1+\cos 2\alpha}{2}+\frac{1-\cos 2\alpha}{2}\right)}=\frac{1}{2}.\frac{\Gamma\left(\cos^2\alpha\right)\Gamma\left(\sin^2\alpha\right)}{\Gamma(1)}$$

$$\mathrm{I}=\frac{1}{2}\Gamma(\cos^2\alpha)\,\Gamma(\sin^2\alpha)=\frac{1}{2}\Gamma(\cos^2\alpha)\,\Gamma(1-\cos^2\alpha)$$

But $\Gamma(p)\Gamma(1-p)=\frac{\pi}{\sin(\pi p)}$ (Euler's reflection formula), then we get**:**

$$\mathrm{I}=\frac{1}{2}\Gamma(\cos^2\alpha)\,\Gamma(1-\cos^2\alpha)=\frac{1}{2}\frac{\pi}{\sin\left(\pi\cos^2\alpha\right)}=\frac{\pi}{2}\csc(\pi\cos^2\alpha)$$

105. $\int_{\frac{1}{2}}^{1} \tan^{-1}\left[\frac{\Gamma\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}+x\right)}{\Gamma(1-x)\Gamma(x)}\right] dx$

We have $\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$, setting $x = y + \frac{1}{2}$, then we get:

$$\Gamma\left(y+\frac{1}{2}\right)\Gamma\left(1-y-\frac{1}{2}\right) = \Gamma\left(\frac{1}{2}+y\right)\Gamma\left(\frac{1}{2}-y\right) = \frac{\pi}{\sin\left(\pi\left(y+\frac{1}{2}\right)\right)} = \frac{\pi}{\sin\left(\pi y+\frac{\pi}{2}\right)} = \frac{\pi}{\cos(\pi x)},$$

so, we get $\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right) = \frac{\pi}{\cos(\pi x)}$, then we get : $\frac{\Gamma\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}+x\right)}{\Gamma(1-x)\Gamma(x)} = \frac{\frac{\pi}{\cos(\pi x)}}{\frac{\pi}{\sin(\pi x)}} = \tan(\pi x)$,

So, $\mathrm{I} = \int_{\frac{1}{2}}^{1} \tan^{-1}[\tan(\pi x)]\, dx = \int_{\frac{1}{2}}^{1} \pi x\, dx = \left[\pi\frac{x^2}{2}\right]_{\frac{1}{2}}^{1} = \frac{\pi}{2}\left(1-\frac{1}{4}\right) = \frac{3\pi}{8}$

106. $\int \ln(\log_2 x + \log_3 x)\, dx = \int \ln(\log_2 x + \log_2 x \log_3 2)\, dx$, then we get:

$$\int \ln(\log_2 x + \log_3 x)\, dx = \int \ln\big(\log_2 x\,(1+\log_3 2)\big)\, dx$$

$$\int \ln(\log_2 x + \log_3 x)\, dx = \int \ln(\log_2 x)\, dx + \int \ln(1+\log_3 2)\, dx$$

$$\int \ln(\log_2 x + \log_3 x)\, dx = \int \ln(\log_2 x)\, dx + x\ln(1+\log_3 2)$$

Now evaluating $\int \ln(\log_2 x)\, dx$ using integration by parts:

Let $u = \ln(\log_2 x) \Longrightarrow u' = \frac{\frac{1}{x\ln 2}}{\log_2 x} = \frac{1}{x\ln 2\log_2 x}$ & let $v' = 1 \Longrightarrow v = x$; then we get:

$$\int \ln(\log_2 x)\, dx = x\ln(\log_2 x) - \int \frac{1}{x\ln 2\log_2 x}.xdx = x\ln(\log_2 x) - \int \frac{1}{\ln 2\log_2 x}dx$$

$$\int \ln(\log_2 x)\, dx = x\ln(\log_2 x) - \int \frac{1}{\frac{\ln x}{\log_2 x}\log_2 x}dx = x\ln(\log_2 x) - \int \frac{1}{\ln x}dx$$

$$\int \ln(\log_2 x)\, dx = x\ln(\log_2 x) - \mathrm{li}(x) + c;$$ therefore we get:

$$\int \ln(\log_2 x + \log_3 x)\, dx = x\ln(1+\log_3 2) + x\ln(\log_2 x) - \mathrm{li}(x) + c$$

107. $\mathrm{I} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}dx$

Let $\frac{x}{a+x} = \frac{t}{a+1} \Longrightarrow (a+1)x = (a+x)t \Longrightarrow x = \frac{at}{a+1-t}$

$$dx = \frac{(a+1-t)a - at(-1)}{(a+1-t)^2}dt = \frac{(a^2+a-at+at)}{(a+1-t)^2}dt = \frac{a(a+1)}{(a+1-t)^2}dt$$

For $x = 0 \Longrightarrow t = 0$ & for $x = 1 \Longrightarrow t = 1$; then we get:

$$\mathrm{I} = \int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+x)^{m+n}}dx = \int_0^1 \frac{\left(\frac{at}{a+1-t}\right)^{m-1}\left(1-\frac{at}{a+1-t}\right)^{n-1}}{\left(a+\frac{at}{a+1-t}\right)^{m+n}}.\frac{a(a+1)}{(a+1-t)^2}dt$$

$$\mathrm{I} = \int_0^1 \frac{(at)^{m-1}(a+1-t-at)^{n-1}}{(a^2+a-at+at)^{m+n}}.a(a+1)dt$$

$$\mathrm{I} = \int_0^1 \frac{a^{m-1}.t^{m-1}(a+1)^{n-1}(1-t)^{n-1}}{a^{m+n}(a+1)^{m+n}}.a(a+1)dt$$

$$\mathrm{I} = \frac{1}{a^n(a+1)^m}\int_0^1 t^{m-1}(1-t)^{n-1}dt = \frac{1}{a^n(a+1)^m}\mathrm{B}(m,n)$$

108. $\mathrm{I} = \int_0^{+\infty}\frac{\ln(1+x^n)}{x^n}dx$, using integration by parts**:**

Let $u = \ln(1+x^n) \Rightarrow u' = \dfrac{nx^{n-1}}{1+x^n}$ and let $v' = \dfrac{1}{x^n} \Rightarrow v = \dfrac{1}{1-n}x^{1-n}$; then we get:

$$\mathrm{I} = \left[\frac{\ln(1+x^n)}{(1-n)x^{n-1}}\right]_0^{+\infty} - \int_0^{+\infty}\frac{nx^{n-1}}{1+x^n}.\frac{1}{1-n}x^{1-n}dx = \frac{n}{n-1}\int_0^{+\infty}\frac{1}{1+x^n}dx$$

Let $x^n = \tan^2\theta \Rightarrow x = \tan^{\frac{2}{n}}\theta \Rightarrow dx = \dfrac{2}{n}\tan^{\frac{2}{n}-1}\theta\,.\sec^2\theta\,d\theta$

For $x = 0 \Rightarrow \theta = 0$ & for $x = +\infty \Rightarrow \theta = \dfrac{\pi}{2}$; then we get:

$$\mathrm{I} = \frac{n}{n-1}\int_0^{+\infty}\frac{1}{1+x^n}dx = \frac{n}{n-1}\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^2\theta}.\frac{2}{n}\tan^{\frac{2}{n}-1}\theta\,.\sec^2\theta\,d\theta$$

$$\mathrm{I} = \frac{n}{n-1}\int_0^{\frac{\pi}{2}}\frac{1}{\sec^2\theta}.\frac{2}{n}\tan^{\frac{2}{n}-1}\theta\,.\sec^2\theta\,d\theta = \frac{n}{n-1}\int_0^{\frac{\pi}{2}}\frac{2}{n}\tan^{\frac{2}{n}-1}\theta\,d\theta\,;$$ then we get:

$$\mathrm{I} = \frac{2}{n-1}\int_0^{\frac{\pi}{2}}\sin^{\frac{2}{n}-1}\theta\cos^{\frac{2}{n}-1}\theta\,d\theta = \frac{1}{n-1}\left(2\int_0^{\frac{\pi}{2}}\sin^{\frac{2}{n}-1}\theta\cos^{\frac{2}{n}-1}\theta\,d\theta\right)$$

$$\mathrm{I} = \frac{1}{n-1}\mathrm{B}\left(\frac{1}{n};\frac{1}{n}\right) = \frac{1}{n-1}.\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right) = \frac{1}{n-1}.\frac{\pi}{\sin\left(\frac{\pi}{n}\right)} = \frac{\pi\csc\left(\frac{\pi}{n}\right)}{n-1}$$

109. $\mathrm{I} = \int_0^{\frac{1}{2}}\frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)}dx$

We have $\Gamma(x)\Gamma(1-x) = \dfrac{\pi}{\sin(\pi x)}$ and $\Gamma\left(\dfrac{1}{2}+x\right)\Gamma\left(\dfrac{1}{2}-x\right) = \dfrac{\pi}{\cos(\pi x)}$

then , $\dfrac{\Gamma(x)\Gamma(1-x)}{\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)} = \cot(\pi x)$ and so , $\dfrac{x\Gamma(x)\Gamma(1-x)}{\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)} = x\cot(\pi x)$, therefore we get**:**

$$\mathrm{I} = \int_0^{\frac{1}{2}}\frac{\Gamma(1+x)\Gamma(1-x)}{\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)}dx = \int_0^{\frac{1}{2}}\frac{x\Gamma(x)\Gamma(1-x)}{\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)}dx = \int_0^{\frac{1}{2}}x\cot(\pi x)\,dx$$ (using IBP)

$$\mathrm{I} = \left[\frac{x}{\pi}\ln\sin(\pi x)\right]_0^{\frac{1}{2}} - \frac{1}{\pi}\int_0^{\frac{1}{2}}\ln(\sin\pi x)\,dx = -\frac{1}{\pi}\int_0^{\frac{1}{2}}\ln(\sin\pi x)\,dx$$

Let $t = \pi x$, then $dt = \pi dx$, for $x = 0$, $t = 0$ and for $x = \dfrac{1}{2}$, $t = \dfrac{\pi}{2}$, then we get**:**

$\mathrm{I} = -\frac{1}{\pi^2}\left(\int_0^{\frac{\pi}{2}} \ln(\sin t)\, dt\right) = -\frac{1}{\pi^2}\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx$, but $\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx$ was solved previously in this book and we get $\int_0^{\frac{\pi}{2}} \ln(\sin x)\, dx = -\frac{\pi}{2}\ln 2$, so $\mathrm{I} = -\frac{1}{\pi^2}\left(-\frac{\pi}{2}\ln 2\right) = \frac{1}{2\pi}\ln 2 = \frac{\ln\sqrt{2}}{\pi}$

110. $\mathrm{I} = \int_0^1 \frac{\mathrm{li}(x)\ln(-\ln x)}{x}\, dx$ **;** using integration by parts**:**

Let $u = \mathrm{li}(x) \Rightarrow u' = \frac{1}{\ln x}$ and let $v' = \frac{\ln(-\ln x)}{x} \Rightarrow v = \ln x\,(\ln(-\ln x) - 1)$

Remark:

111. $\mathrm{I} = \int_0^1 \Psi(x)\sin(\pi x)\cos(\pi x)\, dx$ **… (1)**

Let $u = 1 - x \Rightarrow du = -dx$; for $x = 0 \Rightarrow u = 1$ & for $x = 1 \Rightarrow u = 0$; then we get:

$$\mathrm{I} = \int_0^1 \Psi(x)\sin(\pi x)\cos(\pi x)\, dx = \int_1^0 \Psi(1-u)\sin\big(\pi(1-u)\big)\cos\big(\pi(1-u)\big)\,(-du)$$

$$\mathrm{I} = \int_0^1 \Psi(1-u)\sin(\pi - \pi u)\cos(\pi - \pi u)\, du = -\int_0^1 \Psi(1-u)\sin(\pi u)\cos(\pi u)\, du$$

Then we get: $\mathrm{I} = -\int_0^1 \Psi(1-x)\sin(\pi x)\cos(\pi x)\, dx$ … (2); now adding (1) & (2) we get:

$$2\mathrm{I} = \int_0^1 \Psi(x)\sin(\pi x)\cos(\pi x)\, dx - \int_0^1 \Psi(1-x)\sin(\pi x)\cos(\pi x)\, dx$$

$$2\mathrm{I} = \int_0^1 \{\Psi(x) - \Psi(1-x)\}\sin(\pi x)\cos(\pi x)\, dx = -\pi\int_0^1 \sin(\pi x)\cos(\pi x)\cot(\pi x)$$

$2\mathrm{I} = -\pi\int_0^1 \cos^2(\pi x)\, dx$; let $u = \pi x \Rightarrow du = \pi dx$; then we get:

$2\mathrm{I} = -\int_0^\pi \cos^2 u\, du = -\frac{1}{2}\int_0^\pi (1 + \cos 2u) du = -\frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^\pi = -\frac{1}{2}\pi$; therefore;

$\mathrm{I} = -\frac{\pi}{4}$

112. $\mathrm{I} = \int_0^1 \frac{\Psi\left(x+\frac{3}{2}\right) - \Psi\left(x-\frac{1}{2}\right)}{x}\, dx$

We have $\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \Gamma\left(1 + \left(\frac{1}{2}+x\right)\right)\Gamma\left(1 + \left(\frac{1}{2}-x\right)\right)$

$\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}-x\right)$, then we get**:**

$\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)\left[\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}-x\right)\right]$

$\Gamma\left(\frac{3}{2}+x\right)\Gamma\left(\frac{3}{2}-x\right) = \left(\frac{1}{4}-x^2\right)\left[\frac{\pi}{\cos(\pi x)}\right] = \left(\frac{1}{4}-x^2\right)\pi\sec(\pi x)$, apply ln to both sides, we get**:**

$\ln\Gamma\left(\frac{3}{2}+x\right) + \ln\Gamma\left(\frac{3}{2}-x\right) = \ln\left(\frac{1}{4}-x^2\right) + \ln\pi + \ln\sec(\pi x)$, derive both sides w.r.t x, we get**:**

$$\frac{\Gamma'\left(\frac{3}{2}+x\right)}{\Gamma\left(\frac{3}{2}+x\right)}-\frac{\Gamma'\left(\frac{3}{2}-x\right)}{\Gamma\left(\frac{3}{2}-x\right)}=-\frac{2x}{\frac{1}{4}-x^2}+\frac{\pi\sec(\pi x)\tan(\pi x)}{\sec(\pi x)};\quad \text{then we get:}$$

$$\Psi\left(\frac{3}{2}+x\right)-\Psi\left(\frac{3}{2}-x\right)=\pi\tan(\pi\pi)-\frac{8x}{1-4x^2}\ \ldots(1)$$

We have $\ln\Gamma(x)+\ln\Gamma(1-x)=\ln\pi-\ln\sin(\pi x)$, by deriving both sides w.r.t x, we get:

$$\frac{\Gamma'(x)}{\Gamma(x)}-\frac{\Gamma'(1-x)}{\Gamma(1-x)}=-\pi\frac{\cos(\pi x)}{\sin(\pi x)}\Longrightarrow\frac{\Gamma'(x)}{\Gamma(x)}-\frac{\Gamma'(1-x)}{\Gamma(1-x)}=-\pi\cot(\pi x)$$

With $\frac{\Gamma'(x)}{\Gamma(x)}=\Psi(x)$ and $\frac{\Gamma'(1-x)}{\Gamma(1-x)}=\Psi(1-x)$; so $\Psi(x)-\Psi(1-x)=-\pi\cot(\pi x)$

So replacing x by $x-\frac{1}{2}$, we get:

$$\Psi\left(x-\frac{1}{2}\right)-\Psi\left(1-\left(x-\frac{1}{2}\right)\right)=\Psi\left(x-\frac{1}{2}\right)-\Psi\left(\frac{3}{2}-x\right)=-\pi\cot\left(\pi\left(x-\frac{1}{2}\right)\right);\ \text{then}$$

$$\Psi\left(\frac{3}{2}-x\right)-\Psi\left(x-\frac{1}{2}\right)=\pi\frac{\cos\left(\pi x-\frac{\pi}{2}\right)}{\sin\left(\pi x-\frac{\pi}{2}\right)}=\frac{\pi\sin(\pi x)}{-\pi\cos(\pi x)}=-\pi\tan(\pi x)\ \ldots(2)$$

then we get: $\Psi\left(x-\frac{1}{2}\right)-\Psi\left(\frac{3}{2}-x\right)=\pi\tan(\pi x)\ \ldots(2)$

Subtract equation (1) from equation (2) we get:

$$\Psi\left(x-\frac{1}{2}\right)-\Psi\left(\frac{3}{2}-x\right)-\Psi\left(\frac{3}{2}+x\right)+\Psi\left(\frac{3}{2}-x\right)=\pi\tan(\pi x)-\pi\tan(\pi\pi)+\frac{8x}{1-4x^2}$$

Therefore ; we get: $\Psi\left(x-\frac{1}{2}\right)-\Psi\left(x+\frac{3}{2}\right)=\frac{8x}{1-4x^2}$; so $\Psi\left(x+\frac{3}{2}\right)-\Psi\left(x-\frac{1}{2}\right)=\frac{8x}{4x^2-1}$

Then we get:

$$\mathrm{I}=\int_0^1\frac{1}{x}\cdot\frac{8x}{4x^2-1}dx=4\int_0^1\frac{2}{4x^2-1}dx=4\int_0^1\frac{2x+1-2x+1}{(2x+1)(2x-1)}dx\,;\text{then}$$

$$\mathrm{I}=4\int_0^1\frac{(2x+1)-(2x-1)}{(2x+1)(2x-1)}dx=4\int_0^1\left(\frac{1}{2x-1}-\frac{1}{2x+1}\right)dx=2\int_0^1\left(\frac{2}{2x-1}-\frac{2}{2x+1}\right)dx$$

$$\mathrm{I}=2\left[\ln\left|\frac{2x-1}{2x+1}\right|\right]_0^1=2\ln\frac{1}{3}-2\ln 1=\ln\left(\frac{1}{3}\right)^2=\ln\left(\frac{1}{9}\right)$$

113. $\mathrm{I}=\int_0^{+\infty}e^{-x}\ln\Gamma(1-e^{-x})\,dx$, let $u=e^{-x}$, then $du=-e^{-x}dx$, for $x=0$, $u=1$ and for $x=+\infty$, $u=0$, then we get:

$\mathrm{I}=\int_1^0\ln\Gamma(1-u)\,(-du)=\int_0^1\ln\Gamma(1-u)\,du=\int_0^1\ln\Gamma(1-x)\,dx$ **… (1)**

Let $v=1-u$, then $du=-dv$, for $u=0$, then $v=1$ and for $v=1$, $u=0$, then:

$\mathrm{I}=\int_1^0\ln\Gamma(v)\,(-dv)=\int_0^1\ln\Gamma(v)\,dv=\int_0^1\ln\Gamma(x)\,dx$ **… (2)** Adding (1) and (2) we get:

$2\mathrm{I}=\int_0^1\ln\Gamma(1-x)\,dx+\int_0^1\ln\Gamma(x)\,dx=\int_0^1[\ln\Gamma(1-x)+\ln\Gamma(x)]dx$, then:

$2\mathrm{I}=\int_0^1\ln[\Gamma(1-x)\Gamma(x)]\,dx$, but $\Gamma(1-x)\Gamma(x)=\frac{\pi}{\sin(\pi x)}$ (Euler's reflection formula), then:

$2\mathrm{I}=\int_0^1\ln\left[\frac{\pi}{\sin(\pi x)}\right]dx=\int_0^1\ln\pi\,dx-\int_0^1\ln\sin(\pi x)\,dx=\ln\pi-\int_0^1\ln\sin(\pi x)\,dx$, then

$2\mathrm{I}=\ln\pi-\mathrm{J}$, with $\mathrm{J}=\int_0^1\ln\sin(\pi x)\,dx$

Now let us evaluate J: Let $y=\pi x$, then $dy=\pi dx$, for $x=0$, $y=0$ and for $x=1$, $y=\pi$, so:

$\mathrm{J}=\frac{1}{\pi}\int_0^{\pi}\ln(\sin y)\,dy=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin y)\,dy$ **… (1)**, let $w=\frac{\pi}{2}-y$, then $dw=-dy$, for $y=0$, $w=\frac{\pi}{2}$ and for $y=\frac{\pi}{2}$, then $w=0$, then we get:

$\mathrm{J}=\frac{2}{\pi}\int_{\frac{\pi}{2}}^{0}\ln\left(\sin\left(\frac{\pi}{2}-w\right)\right)(-dw)=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\ln(\cos w)\,dw=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\ln(\cos y)\,dy$ **… (2)**

Adding (1) and (2) we get: $2\mathrm{J}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin y)\,dy+\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\ln(\cos y)\,dy$

$2\mathrm{J}=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}[\ln(\sin y)+\ln(\cos y)]dy$, so $\mathrm{J}=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin y\cos y)\,dy=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{2}\sin 2y\right)dy$

$\mathrm{J}=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}[\ln(\sin 2y)-\ln 2]dy=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin 2y)\,dy-\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln(2)\,dy$, then we get:

$\mathrm{J}=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin 2y)\,dy-\frac{1}{\pi}\frac{\pi}{2}\ln 2=\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\ln(\sin 2y)\,dy-\frac{1}{2}\ln 2$

Let $t=2y$, then $dt=2dy$, for $y=0$, $t=0$ and for $y=\frac{\pi}{2}$, $t=\pi$, then we get:

$\mathrm{J}=\frac{1}{\pi}\int_0^{\pi}\ln(\sin t)\frac{1}{2}dt-\frac{1}{2}\ln 2=\frac{1}{2\pi}\int_0^{\pi}\ln(\sin t)\,dt-\frac{1}{2}\ln 2=\frac{1}{2}\left(\frac{1}{\pi}\int_0^{\pi}\ln(\sin y)\,dy\right)-\frac{1}{2}\ln 2$

Then $\mathrm{J}=\frac{1}{2}\mathrm{J}-\frac{1}{2}\ln 2$, then $,\frac{1}{2}\mathrm{J}=-\frac{1}{2}\ln 2$, so $\mathrm{J}=-\ln 2$, then:

$2\mathrm{I}=\ln\pi-\int_0^1\ln\sin(\pi x)\,dx=\ln\pi-(-\ln 2)=\ln\pi+\ln 2=\ln(2\pi)$

Therefore $\mathrm{I}=\frac{1}{2}\ln(2\pi)=\ln(2\pi)^{\frac{1}{2}}=\ln\sqrt{2\pi}$

114. $\mathrm{I}=\int_0^{+\infty}\frac{1}{(1+x^{\varphi})^{\varphi}}dx$, φ: Golden Ratio $=\frac{1+\sqrt{5}}{2}$

First Method: Let $x^{\varphi}=\tan^2\theta\Longrightarrow x=\tan^{\frac{2}{\varphi}}\theta\Longrightarrow dx=\frac{2}{\varphi}\tan^{\frac{2}{\varphi}-1}\theta\sec^2\theta\,d\theta$, for $x=0$, $\theta=0$ and for $x=+\infty$, $\theta=\frac{\pi}{2}$, then we get:

$$\mathrm{I}=\int_0^{+\infty}\frac{1}{(1+x^{\varphi})^{\varphi}}dx=\int_0^{\frac{\pi}{2}}\frac{1}{(1+\tan^2\theta)^{\varphi}}.\frac{2}{\varphi}\tan^{\frac{2}{\varphi}-1}\theta\sec^2\theta\,d\theta$$

$$\mathrm{I}=\frac{2}{\varphi}\int_0^{\frac{\pi}{2}}\frac{1}{(\sec^2\theta)^{\varphi}}.\tan^{\frac{2}{\varphi}-1}\theta\sec^2\theta\,d\theta=\frac{2}{\varphi}\int_0^{\frac{\pi}{2}}\frac{1}{\sec^{2\varphi}\theta}.\tan^{\frac{2}{\varphi}-1}\theta\sec^2\theta\,d\theta$$

$$\mathrm{I}=\frac{2}{\varphi}\int_0^{\frac{\pi}{2}}\frac{1}{\sec^{2\varphi-2}\theta}.\tan^{\frac{2}{\varphi}-1}\theta\,d\theta=\frac{2}{\varphi}\int_0^{\frac{\pi}{2}}\frac{1}{\sec^{2\varphi-2}\theta}\frac{\sin^{\frac{2}{\varphi}-1}\theta}{\cos^{\frac{2}{\varphi}-1}\theta}d\theta=\frac{2}{\varphi}\int_0^{\frac{\pi}{2}}\cos^{2\varphi-2}\theta.\frac{\sin^{\frac{2}{\varphi}-1}\theta}{\cos^{\frac{2}{\varphi}-1}\theta}d\theta$$

Then we get: $\mathrm{I}=\frac{2}{\varphi}\int_0^{\frac{\pi}{2}}\sin^{\frac{2}{\varphi}-1}\theta\cos^{2\varphi-\frac{2}{\varphi}-1}\theta\,d\theta$

We have: $\mathrm{B}(m,n)=2\int_0^{\frac{\pi}{2}}\cos^{2n-1}\theta\sin^{2m-1}\theta\,d\theta$

$2m-1=\frac{2}{\varphi}-1$, then $m=\frac{1}{\varphi}$ and $2n-1=2\varphi-\frac{2}{\varphi}-1$, then $n=\varphi-\frac{1}{\varphi}$, then we get:

$$\mathrm{I}=\frac{1}{\varphi}\mathrm{B}\left(\frac{1}{\varphi},\varphi-\frac{1}{\varphi}\right)=\frac{1}{\varphi}.\frac{\Gamma\left(\frac{1}{\varphi}\right)\Gamma\left(\varphi-\frac{1}{\varphi}\right)}{\Gamma\left(\frac{1}{\varphi}+\varphi-\frac{1}{\varphi}\right)}=\frac{1}{\varphi}.\frac{\Gamma\left(\frac{1}{\varphi}\right)\Gamma\left(\varphi-\frac{1}{\varphi}\right)}{\Gamma(\varphi)}=\frac{\frac{1}{\varphi}.\Gamma\left(\frac{1}{\varphi}\right)\Gamma\left(\varphi-\frac{1}{\varphi}\right)}{\Gamma(\varphi)}$$

$$\mathrm{I}=\frac{\Gamma\left(1+\frac{1}{\varphi}\right)\Gamma\left(\varphi-\frac{1}{\varphi}\right)}{\Gamma(\varphi)}$$

But $\varphi^2=\varphi+1$, so $1+\frac{1}{\varphi}=\varphi$ and $\varphi-\frac{1}{\varphi}=1$, then: $\mathrm{I}=\frac{\Gamma(\varphi)\Gamma(1)}{\Gamma(\varphi)}=\Gamma(1)=1$

Second Method: Let $y=\frac{x^\varphi}{1+x^\varphi}=\frac{1+x^\varphi-1}{1+x^\varphi}=1-\frac{1}{1+x^\varphi}$ then $x=\left(\frac{y}{1-y}\right)^{\frac{1}{\varphi}}$, then

$dx=\frac{1}{\varphi}\left(\frac{y}{1-y}\right)^{\frac{1}{\varphi}-1}.\frac{1}{(1-y)^2}dy$, for $x=0$, $y=0$ and for $x=+\infty$, then $y=1$, so we get:

$$\mathrm{I}=\int_0^{+\infty}\left(\frac{1}{1+x^\varphi}\right)^\varphi dx=\int_0^1(1-y)^\varphi.\frac{1}{\varphi}\left(\frac{y}{1-y}\right)^{\frac{1}{\varphi}-1}.\frac{1}{(1-y)^2}dy, \text{then we get:}$$

$$\mathrm{I}=\frac{1}{\varphi}\int_0^1 y^{\frac{1}{\varphi}-1}.(1-y)^{\varphi-\frac{1}{\varphi}+1-2}dy=\frac{1}{\varphi}\int_0^1 y^{\frac{1}{\varphi}-1}.(1-y)^{\varphi-\frac{1}{\varphi}-1}dy=\frac{1}{\varphi}\mathrm{B}\left(\frac{1}{\varphi},\varphi-\frac{1}{\varphi}\right)$$

$$\mathrm{I}=\frac{1}{\varphi}.\frac{\Gamma\left(\frac{1}{\varphi}\right)\Gamma\left(\varphi-\frac{1}{\varphi}\right)}{\Gamma\left(\frac{1}{\varphi}+\varphi-\frac{1}{\varphi}\right)}=\frac{1}{\varphi}.\frac{\Gamma\left(\frac{1}{\varphi}\right)\Gamma\left(\varphi-\frac{1}{\varphi}\right)}{\Gamma(\varphi)}=\frac{\frac{1}{\varphi}.\Gamma\left(\frac{1}{\varphi}\right)\Gamma\left(\varphi-\frac{1}{\varphi}\right)}{\Gamma(\varphi)}=\frac{\Gamma\left(1+\frac{1}{\varphi}\right)\Gamma\left(\varphi-\frac{1}{\varphi}\right)}{\Gamma(\varphi)}$$

$$\mathrm{I}=\frac{\Gamma(\varphi)\Gamma(1)}{\Gamma(\varphi)}=\Gamma(1)=1$$

Third Method: Consider the function $f(x)=x(1+x^\varphi)^{1-\varphi}$, let us determine the derivative of f then:

$$\frac{df}{dx}=\frac{d}{dx}[x(1+x^\varphi)^{1-\varphi}]=(1+x^\varphi)^{1-\varphi}+x(1-\varphi)(1+x)^{-\varphi}.\varphi x^{\varphi-1}$$
$$=(1+x^\varphi)^{-\varphi}[1+x^\varphi+(\varphi-\varphi^2)x^2]$$

but $\varphi^2-\varphi=1$; so $\varphi-\varphi^2=-1$, then we get:

$f'(x)=(1+x^\varphi)^{-\varphi}[1+x^\varphi+(-1)x^\varphi]=(1+x^\varphi)^{-\varphi}=\left(\frac{1}{1+x^\varphi}\right)^\varphi=\frac{1}{(1+x^\varphi)^\varphi}$, so we get:

$$\mathrm{I}=\int_0^{+\infty}\frac{1}{(1+x^\varphi)^\varphi}dx=[x(1+x^\varphi)^{1-\varphi}]_0^{+\infty}=\lim_{x\to+\infty}\left[\frac{x}{(1+x^\varphi)^{\varphi-1}}\right]=1$$

115. $\mathrm{I}=\int_0^1\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dots}}}}}}}\,dx$, let $y=\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dots}}}}}}}$, then $y=\sqrt{\dfrac{\ln^m(1-x)}{y}}$, then by S.B.S we get

$y^2=\dfrac{\ln^m(1-x)}{y}$ and so $y^3=\ln^m(1-x)$, then $y=\ln^{\frac{m}{3}}(1-x)$, so $\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dfrac{\ln^m(1-x)}{\sqrt{\dots}}}}}}}=\ln^{\frac{m}{3}}(1-x)$

Then $\mathrm{I}=\int_0^1\ln^{\frac{m}{3}}(1-x)\,dx$, let $-t=\ln(1-x)$, then $1-x=e^{-t}$, so $x=1-e^{-t}$, then $dx=e^{-t}dt$, for $x=0$, $t=0$ and for $x=1$, $t=+\infty$, then we get**:**

$\mathrm{I}=\int_0^{+\infty}(-t)^{\frac{m}{3}}e^{-t}dt=\int_0^{+\infty}(-1)^{\frac{m}{3}}t^{\frac{m}{3}}e^{-t}dt=\left(e^{i\pi}\right)^{\frac{m}{3}}\int_0^{+\infty}e^{-t}.t^{\frac{m}{3}}dt$, then we get**:**

$\mathrm{I}=e^{i\frac{\pi m}{3}}\int_0^{+\infty}e^{-t}.t^{\left(\frac{m}{3}+1\right)-1}dt=e^{i\frac{\pi m}{3}}\Gamma\left(\frac{m}{3}+1\right)=\left(\frac{m}{3}\right)!\left[\cos\left(\frac{\pi m}{3}\right)+i\sin\left(\frac{\pi m}{3}\right)\right]$

116. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\left(\cos^{\frac{1}{n}}x+\sin^{\frac{1}{n}}x\right)^{-2n}dx$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\left(\cos^{\frac{1}{n}}x+\sin^{\frac{1}{n}}x\right)^{-2n}dx=\int_0^{\frac{\pi}{2}}\frac{1}{\left(\cos^{\frac{1}{n}}x+\sin^{\frac{1}{n}}x\right)^{2n}}dx=\int_0^{\frac{\pi}{2}}\frac{1}{\left[\cos^{\frac{1}{n}}x\left(1+\dfrac{\sin^{\frac{1}{n}}x}{\cos^{\frac{1}{n}}x}\right)\right]^{2n}}dx$$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1}{\left(\cos^{\frac{1}{n}}x\right)^{2n}\left(1+\tan^{\frac{1}{n}}x\right)^{2n}}dx=\int_0^{\frac{\pi}{2}}\frac{1}{\cos^2x\left(1+\tan^{\frac{1}{n}}x\right)^{2n}}dx=\int_0^{\frac{\pi}{2}}\frac{\frac{1}{\cos^2x}}{\left(1+\tan^{\frac{1}{n}}x\right)^{2n}}dx$$

Then we get: $\mathrm{I}=\displaystyle\int_0^{\frac{\pi}{2}}\frac{\sec^2x}{\left(1+\tan^{\frac{1}{n}}x\right)^{2n}}dx$

Let $u=\tan x\Longrightarrow du=\sec^2x\,dx$; $\begin{cases}x=0\\x=\frac{\pi}{2}\end{cases}\Longrightarrow\begin{cases}u=0\\u=+\infty\end{cases}$; then we get : $\mathrm{I}=\displaystyle\int_0^{+\infty}\frac{du}{\left(1+u^{\frac{1}{n}}\right)^{2n}}$

Let $u^{\frac{1}{n}}=\tan^2\theta\Longrightarrow u=\tan^{2n}\theta\Longrightarrow du=2n\tan^{2n-1}\theta\sec^2\theta\,d\theta$; $\begin{cases}u=0\\u=+\infty\end{cases}\Longrightarrow\begin{cases}\theta=0\\\theta=\frac{\pi}{2}\end{cases}$; so:

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{1}{(1+\tan^2\theta)^{2n}}.2n\tan^{2n-1}\theta\sec^2\theta\,d\theta=\int_0^{\frac{\pi}{2}}\frac{2n\tan^{2n-1}\theta\sec^2\theta}{(\sec^2\theta)^{2n}}d\theta$$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{2n\tan^{2n-1}\theta\sec^2\theta}{\sec^{4n}\theta}d\theta=2n\int_0^{\frac{\pi}{2}}\frac{\tan^{2n-1}\theta}{\sec^{4n-2}\theta}d\theta=2n\int_0^{\frac{\pi}{2}}\tan^{2n-1}\theta.\cos^{4n-2}\theta\;d\theta$$

$$I = 2n\int_0^{\frac{\pi}{2}} \frac{\sin^{2n-1}\theta}{\cos^{2n-1}\theta}.\cos^{4n-2}\theta\ d\theta = 2n\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta.\cos^{2n-1}\theta\ d\theta$$

$$I = n\left(2\int_0^{\frac{\pi}{2}} \sin^{2n-1}\theta.\cos^{2n-1}\theta\ d\theta\right) = n.\mathrm{B}(n,n) = n.\frac{\Gamma(n).\Gamma(n)}{\Gamma(n+n)} = n.\frac{[\Gamma(n)]^2}{\Gamma(2n)}$$

$$I = n.\frac{1}{\frac{n.n.2n.\Gamma(2n)}{2n.n\Gamma(n).n\Gamma(n)}} = n.\frac{1}{\frac{n^2}{2n}\binom{2n}{n}} = \frac{2}{\binom{2n}{n}}$$

117. $I = \int_0^{+\infty} \cos\left(2x^2 + \frac{1}{x^2}\right) dx$ **… (1)**

Let $u = \frac{1}{\sqrt{2}x} \Rightarrow x = \frac{1}{\sqrt{2}u} \Rightarrow dx = -\frac{1}{\sqrt{2}u^2}du$ & $\begin{cases}\text{for } x = 0 \Rightarrow u = +\infty \\ \text{for } x = +\infty \Rightarrow u = 0\end{cases}$; then we get:

$$I = \int_0^{+\infty} \cos\left(2x^2 + \frac{1}{x^2}\right) dx = \int_{+\infty}^{0} \cos\left(2\left(\frac{1}{\sqrt{2}u}\right)^2 + \frac{1}{\left(\frac{1}{\sqrt{2}u}\right)^2}\right)\left(-\frac{1}{\sqrt{2}u^2}du\right);\ \text{then:}$$

$$I = \int_0^{+\infty} \cos\left(\frac{1}{u^2} + 2u^2\right).\frac{1}{u^2\sqrt{2}}du = \int_0^{+\infty} \cos\left(2x^2 + \frac{1}{x^2}\right).\frac{1}{x^2\sqrt{2}}dx\ \ldots(\mathbf{2})$$

Now adding (1) & (2) we get:

$$I + I = \int_0^{+\infty} \cos\left(2x^2 + \frac{1}{x^2}\right) dx + \int_0^{+\infty} \cos\left(2x^2 + \frac{1}{x^2}\right).\frac{1}{x^2\sqrt{2}}dx;\ \text{then we get:}$$

$$2I = \int_0^{+\infty} \cos\left(2x^2 + \frac{1}{x^2}\right)\left(1 + \frac{1}{x^2\sqrt{2}}\right) dx = \int_0^{+\infty} \cos\left[\left(2x^2 + \frac{1}{x^2} - 2\sqrt{2}\right) + 2\sqrt{2}\right]\left(1 + \frac{1}{x^2\sqrt{2}}\right) dx$$

$$2I = \int_0^{+\infty} \cos\left[\left(x\sqrt{2} - \frac{1}{x}\right)^2 + 2\sqrt{2}\right].\left(1 + \frac{1}{x^2\sqrt{2}}\right) dx;\ \text{then we get:}$$

$$2I = \frac{1}{\sqrt{2}}\int_0^{+\infty} \cos\left[\left(x\sqrt{2} - \frac{1}{x}\right)^2 + 2\sqrt{2}\right].\left(\sqrt{2} + \frac{1}{x^2}\right) dx\ \ \&\ \text{so:}$$

$$I = \frac{1}{2\sqrt{2}}\int_0^{+\infty} \cos\left[\left(x\sqrt{2} - \frac{1}{x}\right)^2 + 2\sqrt{2}\right].\left(\sqrt{2} + \frac{1}{x^2}\right) dx$$

Let $t = x\sqrt{2} - \frac{1}{x} \Rightarrow dt = \left(\sqrt{2} + \frac{1}{x^2}\right) dx$ & $\begin{cases}\text{for } x = 0 \Rightarrow t = -\infty \\ \text{for } x = +\infty \Rightarrow u = +\infty\end{cases}$; then we get:

$$I = \frac{1}{2\sqrt{2}}\int_{-\infty}^{+\infty} \cos\left(t^2 + 2\sqrt{2}\right) dt = \frac{1}{2\sqrt{2}}\int_{-\infty}^{+\infty} \left(\cos t^2 \cos 2\sqrt{2} - \sin t^2 \sin 2\sqrt{2}\right) dt$$

Then we get: $I = \frac{\cos 2\sqrt{2}}{2\sqrt{2}}\int_{-\infty}^{+\infty} \cos t^2\, dt - \frac{\sin 2\sqrt{2}}{2\sqrt{2}}\int_{-\infty}^{+\infty} \sin t^2\, dt$

From Fresnel Integrals: $\displaystyle\int_{-\infty}^{+\infty}\cos t^2\,dt=\int_{-\infty}^{+\infty}\sin t^2\,dt=\sqrt{\frac{\pi}{2}}$; then we get:

$$I=\frac{\cos 2\sqrt{2}}{2\sqrt{2}}\sqrt{\frac{\pi}{2}}-\frac{\sin 2\sqrt{2}}{2\sqrt{2}}\sqrt{\frac{\pi}{2}}=\frac{\sqrt{\pi}}{4}\left(\cos 2\sqrt{2}-\sin 2\sqrt{2}\right)=\frac{\sqrt{\pi}}{4}.\frac{2}{\sqrt{2}}.\frac{\sqrt{2}}{2}\left(\cos 2\sqrt{2}-\sin 2\sqrt{2}\right)$$

$$I=\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{\sqrt{2}}{2}\cos 2\sqrt{2}-\frac{\sqrt{2}}{2}\sin 2\sqrt{2}\right)=\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\cos\frac{\pi}{4}\cos 2\sqrt{2}-\sin\frac{\pi}{4}\sin 2\sqrt{2}\right)$$

Therefore we get: $\displaystyle I=\int_{0}^{+\infty}\cos\left(2x^2+\frac{1}{x^2}\right)dx=\frac{1}{2}\sqrt{\frac{\pi}{2}}\cos\left(2\sqrt{2}+\frac{\pi}{4}\right)$

118. $\displaystyle I=\int_0^{\frac{\pi}{2}}\frac{dx}{a\cos^4 x+b\sin^4 x}$

$$I=\int_0^{\frac{\pi}{2}}\frac{dx}{a\cos^4 x+b\sin^4 x}=\int_0^{\frac{\pi}{2}}\frac{1}{a\cos^4 x+b\sin^4 x}\times\frac{\sec^4 x}{\sec^4 x}dx=\int_0^{\frac{\pi}{2}}\frac{\sec^4 x}{a+b\tan^4 x}dx;\ \text{ then:}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\sec^2 x}{a+b\tan^4 x}\times\sec^2 x\,dx=\int_0^{\frac{\pi}{2}}\frac{1+\tan^2 x}{a+b\tan^4 x}\times\sec^2 x\,dx$$

Let $t=\tan x\Longrightarrow dt=\sec^2 x\,dx$; $\begin{cases}x=0\\ x=\frac{\pi}{2}\end{cases}\Longrightarrow\begin{cases}t=0\\ t=+\infty\end{cases}$; then we get:

$$I=\int_0^{+\infty}\frac{1+t^2}{a+bt^4}dt;\ \text{ let } bt^4=ay\Longrightarrow t^4=\frac{a}{b}y\Longrightarrow t=\left(\frac{a}{b}\right)^{\frac{1}{4}}.y^{\frac{1}{4}}\ \&\ t^2=\sqrt{\frac{a}{b}}.\sqrt{y}\Longrightarrow$$

$$dt=\left(\frac{a}{b}\right)^{\frac{1}{4}}.\frac{1}{4}y^{-\frac{3}{4}}dy\quad\text{then we get:}$$

$$I=\int_0^{+\infty}\frac{1+\sqrt{\frac{a}{b}}.\sqrt{y}}{a+b\left(\frac{a}{b}y\right)}.\left(\frac{a}{b}\right)^{\frac{1}{4}}.\frac{1}{4}y^{-\frac{3}{4}}dy=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\frac{1+\sqrt{\frac{a}{b}}.\sqrt{y}}{1+y}.y^{-\frac{3}{4}}dy;\ \text{ then we get:}$$

$$I=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\frac{y^{-\frac{3}{4}}}{1+y}dy+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\sqrt{\frac{a}{b}}\int_0^{+\infty}\frac{\sqrt{y}.y^{-\frac{3}{4}}}{1+y}dy$$

$$I=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\frac{y^{-\frac{3}{4}}}{1+y}dy+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{3}{4}}\int_0^{+\infty}\frac{y^{-\frac{1}{4}}}{1+y}dy$$

Let $y=\tan^2 u\Longrightarrow dy=2\tan u.\sec^2 u\,du$; $\begin{cases}y=0\\ y=+\infty\end{cases}\Longrightarrow\begin{cases}u=0\\ u=\frac{\pi}{2}\end{cases}$; then we get:

$$I=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\frac{(\tan^2 u)^{-\frac{3}{4}}}{1+\tan^2 u}.2\tan u.\sec^2 u\,du+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{3}{4}}\int_0^{+\infty}\frac{(\tan^2 u)^{-\frac{1}{4}}}{1+\tan^2 u}.2\tan u.\sec^2 u\,du$$

$$I=\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\frac{\tan^{-\frac{3}{2}}u}{\sec^2 u}.\tan u.\sec^2 u\,du+\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{3}{4}}\int_0^{+\infty}\frac{\tan^{-\frac{1}{2}}u}{\sec^2 u}.\tan u.\sec^2 u\,du;\ \text{ then we get:}$$

$$\mathrm{I}=\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\tan^{-\frac{3}{2}}u\,.\tan u\,du+\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{3}{4}}\int_0^{+\infty}\tan^{-\frac{1}{2}}u\,.\tan u\,du$$

$$\mathrm{I}=\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\tan^{-\frac{1}{2}}u\,du+\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{3}{4}}\int_0^{+\infty}\tan^{\frac{1}{2}}u\,du$$

$$\mathrm{I}=\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\sin^{-\frac{1}{2}}u\,.\cos^{\frac{1}{2}}u\,du+\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{3}{4}}\int_0^{+\infty}\sin^{\frac{1}{2}}u\,.\cos^{-\frac{1}{2}}u\,du;\ \text{ then we get:}$$

$$\mathrm{I}=\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}\int_0^{+\infty}\sin^{2\left(\frac{1}{4}\right)-1}u\,.\cos^{2\left(\frac{3}{4}\right)-1}u\,du+\frac{1}{2}\left(\frac{a}{b}\right)^{\frac{3}{4}}\int_0^{+\infty}\sin^{2\left(\frac{3}{4}\right)-1}u\,.\cos^{2\left(\frac{1}{4}\right)-1}u\,du$$

$$\mathrm{I}=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\left[2\int_0^{+\infty}\sin^{2\left(\frac{1}{4}\right)-1}u\,.\cos^{2\left(\frac{3}{4}\right)-1}u\,du\right]+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{3}{4}}\left[2\int_0^{+\infty}\sin^{2\left(\frac{3}{4}\right)-1}u\,.\cos^{2\left(\frac{1}{4}\right)-1}u\,du\right]$$

$$\mathrm{I}=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\mathrm{B}\left(\frac{1}{4};\frac{3}{4}\right)+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{3}{4}}\mathrm{B}\left(\frac{3}{4};\frac{1}{4}\right)=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}.\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{3}{4}}.\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$$

$$\mathrm{I}=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}.\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)}{\Gamma(1)}+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{3}{4}}.\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{\Gamma(1)}=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}.\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{3}{4}}.\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

$$\mathrm{I}=\left[\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}+\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{3}{4}}\right]\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\left(1+\sqrt{\frac{a}{b}}\right)\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)$$

Euler's reflection formual $\Gamma(z)\Gamma(1-z)=\dfrac{\pi}{\sin\pi z}$; then $\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right)=\dfrac{\pi}{\sin\left(\frac{\pi}{4}\right)}$; then:

$$\mathrm{I}=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\left(1+\sqrt{\frac{a}{b}}\right)\frac{\pi}{\sin\left(\frac{\pi}{4}\right)}=\frac{1}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\left(1+\sqrt{\frac{a}{b}}\right)\frac{\pi}{\frac{1}{\sqrt{2}}}=\frac{\sqrt{2}\pi}{4}\left(\frac{a}{b}\right)^{\frac{1}{4}}\left(1+\sqrt{\frac{a}{b}}\right)$$

119. $\mathrm{I}=\int_0^1\ln\Gamma(x)\,.\cos^2(\pi x)\,dx\ \dots\ \mathbf{(1)}$

Let $u=1-x\Longrightarrow du=-dx$, $\begin{cases}x=0\\x=1\end{cases}\Longrightarrow\begin{cases}u=1\\u=0\end{cases}$, then we get**:**

$$\mathrm{I}=\int_0^1\ln\Gamma(x)\,.\cos^2(\pi x)\,dx=\int_1^0\ln\Gamma(1-u)\,.\cos^2\big(\pi(1-u)\big)\,(-du)$$

$$\mathrm{I}=\int_0^1\ln\Gamma(1-u)\,.\cos^2(\pi-\pi u)\,du=\int_0^1\ln\Gamma(1-u)\,.\cos^2(\pi u)\,du$$

$$\mathrm{I}=\int_0^1\ln\Gamma(1-x)\,.\cos^2(\pi x)\,dx\ \dots(\mathbf{2});\ \text{ now adding (1) \& (2) we get:}$$

$$\mathrm{I}+\mathrm{I}=2\mathrm{I}=\int_0^1\ln\Gamma(x)\,.\cos^2(\pi x)\,dx+\int_0^1\ln\Gamma(1-x)\,.\cos^2(\pi x)\,dx\,;\ \text{ then we get:}$$

$$2\mathrm{I} = \int_0^1 [\ln\Gamma(x) + \ln\Gamma(1-x)]\cos^2(\pi x)\,dx = \int_0^1 \ln\Gamma(x)\,\Gamma(1-x).\cos^2(\pi x)\,dx$$

Euler's reflection formula: $\Gamma(z)\Gamma(1-z) = \pi\csc(\pi z) = \dfrac{\pi}{\sin(\pi z)}$; then we get:

$$2\mathrm{I} = \int_0^1 \ln\left(\frac{\pi}{\sin(\pi z)}\right)\cos^2(\pi x)\,dx \Longrightarrow \mathrm{I} = \frac{1}{2}\int_0^1 \ln\left(\frac{\pi}{\sin(\pi z)}\right)\cos^2(\pi x)\,dx$$

Let $u = \pi x \Longrightarrow du = \pi dx$; $\begin{cases} x = 0 \\ x = 1 \end{cases} \Longrightarrow \begin{cases} u = 0 \\ u = \pi \end{cases}$; then we get:

$$\mathrm{I} = \frac{1}{2\pi}\int_0^\pi \cos^2 u \,.\ln\left(\frac{\pi}{\sin u}\right)du = \frac{1}{2\pi}\int_0^\pi \cos^2 u\,(\ln\pi - \ln\sin u)du\,;\quad \text{then we get:}$$

$$\mathrm{I} = \frac{\ln\pi}{2\pi}\int_0^\pi \cos^2 u\,du - \frac{1}{2\pi}\int_0^\pi \cos^2 u\ln\sin u\,du$$

$$\mathrm{I} = \frac{\ln\pi}{2\pi}\int_0^\pi \left(\frac{1+\cos 2u}{2}\right)du - \frac{1}{2\pi}\int_0^\pi \cos^2 u\ln\sin u\,du$$

$$\mathrm{I} = \frac{\ln\pi}{4\pi}\left[u + \frac{1}{2}\sin 2u\right]_0^\pi - \frac{1}{2\pi}\int_0^\pi \cos^2 u\ln\sin u\,du$$

$$\mathrm{I} = \frac{\ln\pi}{4\pi}(\pi) - \frac{1}{2\pi}\int_0^\pi \cos^2 u\ln\sin u\,du = \frac{\ln\pi}{4} - \frac{1}{2\pi}\int_0^\pi \cos^2 u\ln\sin u\,du = \frac{\ln\pi}{4} - \frac{1}{2\pi}\mathrm{I}_1$$

Where $\mathrm{I}_1 = \displaystyle\int_0^\pi \cos^2 u\ln\sin u\,du$

$$\mathrm{I}_1 = \frac{1}{2}\int_0^\pi (1+\cos 2u)\ln\sin u\,du = \frac{1}{2}\int_0^\pi \ln\sin u\,du + \frac{1}{2}\int_0^\pi \cos 2u\ln\sin u\,du$$

Notice that both $\ln\sin x$ & $\cos 2x$ are symmetric about $x = \dfrac{\pi}{2}$ because $\sin(\pi - x) = \sin x$ (and the same for cos) then we get:

$$\mathrm{I}_1 = \int_0^{\frac{\pi}{2}} \ln\sin u\,du + \int_0^{\frac{\pi}{2}} \cos 2u\ln\sin u\,du\,;\text{let } \mathrm{I}_2 = \int_0^{\frac{\pi}{2}} \ln\sin u\,du \ \ \&\ \mathrm{I}_3 = \int_0^{\frac{\pi}{2}} \cos 2u\ln\sin u\,du$$

Now for I_2: let $x = \dfrac{\pi}{2} - u \Longrightarrow dx = -du$; $\begin{cases} u = 0 \\ u = \frac{\pi}{2} \end{cases} \Longrightarrow \begin{cases} x = \frac{\pi}{2} \\ x = 0 \end{cases}$; then we get:

$$\mathrm{I}_2 = \int_{\frac{\pi}{2}}^{0} \ln\sin\left(\frac{\pi}{2} - x\right)(-dx) = \int_0^{\frac{\pi}{2}} \ln\sin\left(\frac{\pi}{2} - x\right)dx = \int_0^{\frac{\pi}{2}} \ln\cos x\,dx$$

$$\mathrm{I}_2 + \mathrm{I}_2 = 2\mathrm{I}_2 = \int_0^{\frac{\pi}{2}} \ln\sin x\,dx + \int_0^{\frac{\pi}{2}} \ln\cos x\,dx = \int_0^{\frac{\pi}{2}} (\ln\sin x + \ln\cos x)\,dx;\quad \text{then we get:}$$

$$2\mathrm{I}_2 = \int_0^{\frac{\pi}{2}} \ln(\sin x \cos x)\, dx = \int_0^{\frac{\pi}{2}} \ln\left(\frac{1}{2}\sin 2x\right) dx = \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx - \int_0^{\frac{\pi}{2}} \ln 2\, dx\,; \text{ then:}$$

$$2\mathrm{I}_2 = -\frac{\pi}{2}\ln 2 + \int_0^{\frac{\pi}{2}} \ln(\sin 2x)\, dx\,; \text{ let } u = 2x \Longrightarrow du = 2dx; \begin{cases} x = 0 \\ x = \frac{\pi}{2} \end{cases} \Longrightarrow \begin{cases} u = 0 \\ u = \pi \end{cases}; \text{ then:}$$

$$2\mathrm{I}_2 = -\frac{\pi \ln 2}{2} + \frac{1}{2}\int_0^{\pi} \ln \sin u\, du \Longrightarrow 2\mathrm{I}_2 = -\frac{\pi \ln 2}{2} + \mathrm{I}_2 \Longrightarrow \mathrm{I}_2 = -\frac{\pi \ln 2}{2}$$

Now for: $\mathrm{I}_3 = \int_0^{\frac{\pi}{2}} \cos 2x \ln \sin x\, dx$

Let $x = \arcsin u \Longrightarrow dx = \frac{du}{\sqrt{1-u^2}}$; $\begin{cases} x = 0 \\ x = \frac{\pi}{2} \end{cases} \Longrightarrow \begin{cases} u = 0 \\ u = 1 \end{cases}$; then we get:

$$\mathrm{I}_3 = \int_0^{\frac{\pi}{2}} \cos 2x \ln \sin x\, dx = \int_0^{\frac{\pi}{2}} (1 - 2\sin^2 x) \ln \sin x\, dx = \int_0^1 (1-2u^2) \ln u \,.\frac{du}{\sqrt{1-u^2}}$$

$$\mathrm{I}_3 = \int_0^1 \frac{(1-2u^2)\ln u}{\sqrt{1-u^2}} du = \int_0^1 \frac{\ln u}{\sqrt{1-u^2}} du - \int_0^1 \frac{2\ln u \,.\, u^2}{\sqrt{1-u^2}} du$$

$$\mathrm{I}_3 = \int_0^1 \frac{\ln x}{\sqrt{1-x^2}} dx - \int_0^1 \frac{2\ln x \,.\, x^2}{\sqrt{1-x^2}} dx$$

For first integral (by parts): $u = \ln x \Longrightarrow u' = \frac{1}{x}$ & $v' = \frac{1}{\sqrt{1-x^2}} \Longrightarrow v = \arcsin x$; then:

$$\int_0^1 \frac{\ln x}{\sqrt{1-x^2}} dx = [\ln x \,.\arcsin x]_0^1 - \int_0^1 \frac{\arcsin x}{x} dx$$

For first integral (by parts):

$u = 2\ln x \Longrightarrow u' = \frac{2}{x}$ & $v' = \frac{x^2}{\sqrt{1-x^2}} \Longrightarrow v = \frac{1}{2}\left(\arcsin x - x\sqrt{1-x^2}\right)$; then we get:

$$\int_0^1 \frac{2\ln x \,.\, x^2}{\sqrt{1-x^2}} dx = \left[2\ln x \times \frac{1}{2}\left(\arcsin x - x\sqrt{1-x^2}\right)\right]_0^1 - \int_0^1 \frac{\arcsin x - x\sqrt{1-x^2}}{x} dx\,; \text{ then:}$$

$$\mathrm{I}_3 = -\int_0^1 \frac{\arcsin x}{x} dx + \int_0^1 \frac{\arcsin x - x\sqrt{1-x^2}}{x} dx$$

$$\mathrm{I}_3 = -\int_0^1 \frac{\arcsin x}{x} dx + \int_0^1 \frac{\arcsin x}{x} dx - \int_0^1 \sqrt{1-x^2}\, dx = -\int_0^1 \sqrt{1-x^2}\, dx = -\frac{\pi}{4}$$

Therefore; $\mathrm{I} = \frac{\ln \pi}{4} - \frac{1}{2\pi}\left(-\frac{\pi \ln 2}{2} - \frac{\pi}{4}\right) = \frac{\ln \pi + \ln 2}{4} + \frac{1}{8} = \frac{1}{4}\ln 2\pi + \frac{1}{8}$

120. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin^2 x+2\cos^2 x}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin^2 x+2\left(1-\sin^2 x\right)}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin^2 x+2-2\sin^2 x}}$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{2-\sin^2 x}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{2\left(1-\frac{1}{2}\sin^2 x\right)}}=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-\frac{1}{2}\sin^2 x}}=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2 2\sin^2 x}}$$

$$\mathrm{I}=\frac{1}{\sqrt{2}}\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{2}\right)=\frac{1}{\sqrt{2}}\mathrm{K}\left(\frac{1}{\sqrt{2}}\right)$$

121. $\mathrm{I}=\int_0^{t}\frac{dx}{\sqrt{1-3\sin^2 x}}$, let $\sin\phi=\sqrt{3}\sin x\Longrightarrow\cos\phi\,d\phi=\sqrt{3}\cos x\,dx$, then we get:

$$dx=\frac{\cos\phi}{\sqrt{3}\cos x}d\phi=\frac{\cos\phi}{\sqrt{3}\sqrt{1-\sin^2 x}}d\phi=\frac{\cos\phi}{\sqrt{3}\sqrt{1-\left(\frac{\sin\phi}{\sqrt{3}}\right)^2}}d\phi=\frac{\cos\phi}{\frac{\sqrt{3}}{\sqrt{3}}\sqrt{3-\sin^2\phi}}d\phi$$

$dx=\dfrac{\cos\phi}{\sqrt{3-\sin^2\phi}}d\phi$; for $x=0\Longrightarrow\phi=0$ & $fot\ x=t\Longrightarrow\phi=\sin^{-1}\left(\sqrt{3}\sin\mathrm{t}\right)$

$$\mathrm{I}=\int_0^{t}\frac{dx}{\sqrt{1-3\sin^2 x}}=\int_0^{\phi}\frac{\frac{\cos\phi}{\sqrt{3-\sin^2\phi}}}{\sqrt{1-\sin^2\phi}}d\phi=\int_0^{\phi}\frac{\frac{\cos\phi}{\sqrt{3-\sin^2\phi}}}{\sqrt{\cos^2\phi}}d\phi=\int_0^{\phi}\frac{\frac{\cos\phi}{\sqrt{3-\sin^2\phi}}}{\cos\phi}d\phi$$

$$\mathrm{I}=\int_0^{\phi}\frac{d\phi}{\sqrt{3-\sin^2\phi}}=\int_0^{\phi}\frac{d\phi}{\sqrt{3\left(1-\frac{1}{3}\sin^2\phi\right)}}=\frac{1}{\sqrt{3}}\int_0^{\phi}\frac{d\phi}{\sqrt{1-\left(\frac{1}{\sqrt{3}}\right)\sin^2\phi}}=\frac{1}{\sqrt{3}}\mathrm{F}\left(\frac{1}{\sqrt{3}},\phi\right)$$

$$\mathrm{I}=\frac{1}{\sqrt{3}}\mathrm{F}\left(\frac{1}{\sqrt{3}},\sin^{-1}\left(\sqrt{3}\sin\mathrm{t}\right)\right)$$

122. **First Method:** $\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin x}}$, let $x=\frac{\pi}{2}-y\Longrightarrow dx=-dy$ & $\begin{cases}x=0\Longrightarrow y=\frac{\pi}{2}\\ x=\frac{\pi}{2}\Longrightarrow y=0\end{cases}$**;** then**:**

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin x}}=\int_{\frac{\pi}{2}}^{0}\frac{-dy}{\sqrt{\sin\left(\frac{\pi}{2}-y\right)}}=\int_0^{\frac{\pi}{2}}\frac{dy}{\sqrt{\cos y}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\cos x}}$$

Let $\cos x=\cos^2 u\Longrightarrow-\sin x\,dx=-2\sin u\cos u\,du\Longrightarrow dx=\dfrac{2\sin u\cos u}{\sin x}du$; then we get:

$dx=\dfrac{2\sin u\cos u}{\sqrt{1-\cos^2 x}}du=\dfrac{2\sin u\cos u}{\sqrt{1-(\cos^2 u)^2}}du=\dfrac{2\sin u\cos u}{\sqrt{1-\cos^4 u}}du$; then we have:

$dx=\dfrac{2\sin u\cos u}{\sqrt{(1-\cos^2 u)(1+\cos^2 u)}}du=\dfrac{2\sin u\cos u}{\sqrt{\sin^2 u\,(1+\cos^2 u)}}du=\dfrac{2\cos u}{\sqrt{1+\cos^2 u}}du$; so:

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\cos x}}=\int_0^{\frac{\pi}{2}}\frac{\frac{2\cos u}{\sqrt{1+\cos^2 u}}}{\sqrt{\cos^2 u}}du=2\int_0^{\frac{\pi}{2}}\frac{\frac{\cos u}{\sqrt{1+\cos^2 u}}}{\cos u}du=2\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{1+\cos^2 u}}\ \text{ and so we have}$$

$$\mathrm{I}=2\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{1+(1-\sin^2 u)}}=2\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{2-\sin^2 u}}=2\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{2\left(1-\frac{1}{2}\sin^2 u\right)}}=\frac{2}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{1-\frac{1}{2}\sin^2 u}}$$

$$\mathrm{I}=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{1-\frac{1}{2}\sin^2 u}}=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{du}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2 u}}=\sqrt{2}\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{2}\right)=\sqrt{2}\mathrm{K}\left(\frac{1}{\sqrt{2}}\right)$$

Second Method: $\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin x}}$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\sin x}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\cos x}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-\sin^2\frac{x}{2}-\sin^2\frac{x}{2}}}=\int_0^{\frac{\pi}{2}}\frac{dx}{\sqrt{1-2\sin^2\frac{x}{2}}}$$

Let $\sqrt{2}\sin\frac{x}{2}=\sin\phi\Longrightarrow\frac{\sqrt{2}}{2}\cos\frac{x}{2}dx=\cos\phi\,d\phi\Longrightarrow dx=\frac{\cos\phi}{\frac{\sqrt{2}}{2}\cos\frac{x}{2}}d\phi=\frac{\sqrt{2}\cos\phi}{\sqrt{1-\sin^2\frac{x}{2}}}d\phi$

$$dx=\frac{\sqrt{2}\cos\phi}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2\phi}}d\phi\Longrightarrow\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{\frac{\sqrt{2}\cos\phi}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2\phi}}}{\sqrt{1-\sin^2\phi}}d\phi=\int_0^{\frac{\pi}{2}}\frac{\frac{\sqrt{2}\cos\phi}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2\phi}}}{\cos\phi}d\phi$$

Therefore; we get: $\mathrm{I}=\sqrt{2}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2\phi}}=\sqrt{2}\mathrm{K}\left(\frac{1}{\sqrt{2}}\right)$

123. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\sqrt{1+4\sin^2 x}\,dx=\int_0^{\frac{\pi}{2}}\sqrt{1+4(1-\cos^2 x)}\,dx=\int_0^{\frac{\pi}{2}}\sqrt{1+4-4\cos^2 x}\,dx$

$\mathrm{I}=\int_0^{\frac{\pi}{2}}\sqrt{5-4\cos^2 x}\,dx$**;** let $x=\frac{\pi}{2}-y\Longrightarrow dx=-dy$ & $\begin{cases}x=0\Longrightarrow y=\frac{\pi}{2}\\ x=\frac{\pi}{2}\Longrightarrow y=0\end{cases}$**;** then we get**:**

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\sqrt{5-4\cos^2 x}\,dx=\int_{\frac{\pi}{2}}^{0}\sqrt{5-4\cos^2\left(\frac{\pi}{2}-y\right)}(-dy)=\int_0^{\frac{\pi}{2}}\sqrt{5-4\sin^2 y}\,dy;\ \text{ then we get:}$$

$$\mathrm{I} = \sqrt{5}\int_0^{\frac{\pi}{2}} \sqrt{1 - \frac{4}{5}\sin^2 y}\, dy = \sqrt{5}\int_0^{\frac{\pi}{2}} \sqrt{1 - \left(\frac{2}{\sqrt{5}}\right)^2 \sin^2 y}\, dy = \sqrt{5}\mathrm{E}\left(\frac{2}{\sqrt{5}}\right)$$

124. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$; let $\cos x = \cos^2 u \Longrightarrow -\sin x\, dx = -2\sin u \cos u\, du$, then we get:

$$dx = \frac{2\sin u\cos u}{\sin x}du = \frac{2\sin u\cos u}{\sqrt{1-\cos^2 x}}du = \frac{2\sin u\cos u}{\sqrt{1-\cos^4 u}}du = \frac{2\sin u\cos u}{\sqrt{1-\cos^2 u}\sqrt{1+\cos^2 u}}du$$

$$dx = \frac{2\sin u\cos u}{\sin u\sqrt{1+\cos^2 u}}du = \frac{2\cos u}{\sqrt{1+\cos^2 u}}du;\ \text{ then we get:}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \cos u \times \frac{2\cos u}{\sqrt{1+\cos^2 u}}du = 2\int_0^{\frac{\pi}{2}} \frac{\cos^2 u}{\sqrt{1+\cos^2 u}}du = 2\int_0^{\frac{\pi}{2}} \frac{(1+\cos^2 u)-1}{\sqrt{1+\cos^2 u}}du$$

$$\mathrm{I} = 2\int_0^{\frac{\pi}{2}} \frac{1+\cos^2 u}{\sqrt{1+\cos^2 u}}du - 2\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1+\cos^2 u}} = 2\int_0^{\frac{\pi}{2}} \frac{\left(\sqrt{1+\cos^2 u}\right)^2}{\sqrt{1+\cos^2 u}}du - 2\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1+\cos^2 u}}$$

$$\mathrm{I} = 2\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2 u}\, du - 2\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1+\cos^2 u}} = 2\int_0^{\frac{\pi}{2}} \sqrt{2-\sin^2 u}\, du - 2\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{2-\sin^2 u}}$$

$$\mathrm{I} = 2\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1-\frac{1}{2}\sin^2 u}\, du - \frac{2}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1-\frac{1}{2}\sin^2 u}}$$

$$\mathrm{I} = 2\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2 u}\, du - \sqrt{2}\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2 u}} = 2\sqrt{2}\mathrm{E}\left(\frac{1}{\sqrt{2}}\right) - \sqrt{2}\mathrm{K}\left(\frac{1}{\sqrt{2}}\right)$$

125. $\mathrm{I} = \int_0^{x} \sqrt{1-4\sin^2 u}\, du$; where $0 \le x \le \frac{\pi}{6}$

Let $\sqrt{4\sin^2 u} = 2\sin u = \sin\phi \Longrightarrow 2\cos u\, du = \cos\phi\, d\phi$; then we get:

$$du = \frac{\cos\phi}{2\cos u}d\phi = \frac{\cos\phi}{2\sqrt{1-\sin^2 u}}d\phi = \frac{\cos\phi}{2\sqrt{1-\left(\frac{1}{2}\sin^2\phi\right)}}d\phi = \frac{\cos\phi}{2\sqrt{1-\frac{1}{4}\sin^2\phi}}d\phi;\ \text{ then:}$$

$$\int_0^{x} \sqrt{1-4\sin^2 u}\, du = \int_0^{\phi} \sqrt{1-\sin^2\phi}\,.\frac{\cos\phi}{2\sqrt{1-\frac{1}{4}\sin^2\phi}}d\phi = \int_0^{\phi} \cos\phi\,.\frac{\cos\phi}{2\sqrt{1-\frac{1}{4}\sin^2\phi}}d\phi$$

$$\int_0^{x} \sqrt{1-4\sin^2 u}\, du = \frac{1}{2}\int_0^{\phi} \frac{\cos^2\phi}{\sqrt{1-\frac{1}{4}\sin^2\phi}}d\phi = \frac{1}{2}\int_0^{\phi} \frac{1-\sin^2\phi}{\sqrt{1-\frac{1}{4}\sin^2\phi}}d\phi;\ \text{ then we get:}$$

$$\int_0^{x} \sqrt{1-4\sin^2 u}\, du = \frac{1}{2}\int_0^{\phi} \frac{-3+4\left(1-\frac{1}{4}\sin^2\phi\right)}{\sqrt{1-\frac{1}{4}\sin^2\phi}}d\phi$$

$$\int_0^x \sqrt{1-4\sin^2 u}\,du = -\frac{3}{2}\int_0^{\phi} \frac{d\phi}{\sqrt{1-\frac{1}{4}\sin^2\phi}} + 2\int_0^{\phi}\sqrt{1-\frac{1}{4}\sin^2\phi}\;d\phi;\quad \text{therefore, we get:}$$

$$\int_0^x \sqrt{1-4\sin^2 u}\,du = -\frac{3}{2}\mathrm{F}\left(\frac{1}{2},\phi\right) + 2\mathrm{E}\left(\frac{1}{2},\phi\right);\quad \text{where } \phi = \sin^{-1}(2\sin x)$$

126. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2-\cos x}} = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2-\left(2\cos^2\frac{x}{2}-1\right)}} = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{3-2\cos^2\frac{x}{2}}} = \frac{1}{\sqrt{3}}\int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1-\frac{2}{3}\cos^2\frac{x}{2}}}$

Let $\frac{x}{2} = \frac{\pi}{2} - u \implies dx = -2du;\ \begin{cases} x = 0 \implies u = \frac{\pi}{2} \\ x = \frac{\pi}{2} \implies u = \frac{\pi}{4} \end{cases};$ then we get:

$$\mathrm{I} = \frac{1}{\sqrt{3}}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \frac{-2du}{\sqrt{1-\frac{2}{3}\cos^2\left(\frac{\pi}{2}-u\right)}} = \frac{2}{\sqrt{3}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{du}{\sqrt{1-\frac{2}{3}\sin^2 u}} = \frac{2}{\sqrt{3}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{du}{\sqrt{1-\left(\sqrt{\frac{2}{3}}\right)^2\sin^2 u}};\quad \text{then:}$$

$$\mathrm{I} = \frac{2}{\sqrt{3}}\left\{\int_0^{\frac{\pi}{2}} \frac{du}{\sqrt{1-\left(\sqrt{\frac{2}{3}}\right)^2\sin^2 u}} - \int_0^{\frac{\pi}{4}} \frac{du}{\sqrt{1-\left(\sqrt{\frac{2}{3}}\right)^2\sin^2 u}}\right\};\quad \text{therefore; we get:}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{2-\cos x}} = \frac{2}{\sqrt{3}}\left\{\mathrm{F}\left(\sqrt{\frac{2}{3}},\frac{\pi}{2}\right) - \mathrm{F}\left(\sqrt{\frac{2}{3}},\frac{\pi}{4}\right)\right\}$$

127. $\mathrm{I} = \int_0^{\phi} \frac{d\phi}{\sqrt{1+k^2\sin^2\phi}}$, let $k\sin\phi\, 2 = \tan u \implies k\cos\phi\, d\phi = \sec^2 u\, du$, then we get:

$$d\phi = \frac{\sec^2 u}{k\cos\phi}du = \frac{\sec^2 u}{k\sqrt{1-\sin^2\phi}}du = \frac{\sec^2 u}{k\sqrt{1-\frac{\tan^2 u}{k^2}}}du = \frac{\sec^2 u}{\frac{k}{k}\sqrt{k^2-\tan^2 u}}du \implies$$

$$d\phi = \frac{\sec^2 u}{\sqrt{k^2-\tan^2 u}}du \implies \mathrm{I} = \int_0^u \frac{\frac{\sec^2 u}{\sqrt{k^2-\tan^2 u}}}{\sqrt{1+\tan^2 u}}du = \int_0^u \frac{\frac{\sec^2 u}{\sqrt{k^2-\tan^2 u}}}{\sqrt{\sec^2 u}}du;\quad \text{then we get:}$$

$$\mathrm{I} = \int_0^u \frac{\frac{\sec^2 u}{\sqrt{k^2-\tan^2 u}}}{\sec u}du = \int_0^u \frac{\sec u}{\sqrt{k^2-\tan^2 u}}du = \int_0^u \frac{\frac{1}{\cos u}}{\sqrt{k^2-\frac{\sin^2 u}{\cos^2 u}}}du = \int_0^u \frac{du}{\sqrt{k^2\cos^2 u-\sin^2 u}}$$

$$\mathrm{I} = \int_0^u \frac{du}{\sqrt{k^2(1-\sin^2 u)-\sin^2 u}} = \int_0^u \frac{du}{\sqrt{k^2-(k^2+1)\sin^2 u}}$$

Let $\sqrt{k^2+1}\sin u = k\sin x \Longrightarrow \sqrt{k^2+1}\cos u\,du = k\cos x\,dx \Longrightarrow du = \dfrac{k\cos x}{\sqrt{k^2+1}\cos u}dx$

$$du = \frac{k\cos x}{\sqrt{k^2+1}\sqrt{1-\sin^2 u}}dx = \frac{k\cos x}{\sqrt{k^2+1}\sqrt{1-\left(\frac{k\sin x}{\sqrt{k^2+1}}\right)^2}}dx = \frac{k\cos x}{\sqrt{k^2+1}\sqrt{1-\frac{k^2\sin^2 x}{k^2+1}}}dx$$

$$du = \frac{k\cos x}{\frac{\sqrt{k^2+1}}{\sqrt{k^2+1}}\sqrt{k^2+1-k^2\sin^2 x}}dx = \frac{k\cos x}{\sqrt{k^2+1-k^2\sin^2 x}}dx;\ \text{ then:}$$

$$\mathrm{I} = \int_0^x \frac{\frac{k\cos x}{\sqrt{k^2+1-k^2\sin^2 x}}}{\sqrt{k^2-k^2\sin^2 x}}dx = \int_0^x \frac{\frac{k\cos x}{\sqrt{k^2+1-k^2\sin^2 x}}}{k\sqrt{1-\sin^2 x}}dx = \int_0^x \frac{\frac{k\cos x}{\sqrt{k^2+1-k^2\sin^2 x}}}{k\cos x}dx$$

$$\mathrm{I} = \int_0^x \frac{dx}{\sqrt{k^2+1-k^2\sin^2 x}} = \frac{1}{\sqrt{k^2+1}}\int_0^x \frac{dx}{\sqrt{1-\left(\frac{k}{\sqrt{k^2+1}}\right)^2\sin^2 x}};\ \text{ therefore; we get:}$$

$$\mathrm{I} = \frac{1}{\sqrt{k^2+1}}\mathrm{F}\left(\frac{k}{\sqrt{k^2+1}}, x\right);\quad \text{where}\quad x = \sin^{-1}\left(\frac{\sqrt{k^2+1}\sin\phi}{\sqrt{1+k^2\sin^2\phi}}\right)$$

128. $\mathrm{I} = \int_0^2 \frac{dx}{\sqrt{(4-x^2)(9-x^2)}}$, let $x = 2\sin\theta \Longrightarrow dx = 2\cos\theta\,d\theta$

For $x = 0 \Longrightarrow \sin\theta = 0 \Longrightarrow \theta = 0$ & for $x = 2 \Longrightarrow \sin\theta = 1 \Longrightarrow \theta = \frac{\pi}{2}$, then we get**:**

$$\mathrm{I} = \int_0^2 \frac{dx}{\sqrt{(4-x^2)(9-x^2)}} = \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\sqrt{(4-4\sin^2\theta)(9-4\sin^2\theta)}}d\theta;\ \text{ then we get:}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{2\cos\theta}{\sqrt{2\cos\theta\,(9-4\sin^2\theta)}}d\theta = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{9-4\sin^2\theta}} = \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{4}{9}\sin^2\theta}} = \frac{1}{3}\mathrm{F}\left(\frac{2}{3}, \frac{\pi}{2}\right)$$

129. $\mathrm{I} = \int_0^1 \frac{dx}{\sqrt{(1+x^2)(1+2x^2)}}$, let $x = \tan\theta \Longrightarrow dx = \sec^2\theta\,d\theta$

For $x = 0 \Longrightarrow \tan\theta = 0 \Longrightarrow \theta = 0$ & for $x = 1 \Longrightarrow \tan\theta = 1 \Longrightarrow \theta = \frac{\pi}{4}$, then we get**:**

$$\mathrm{I} = \int_0^1 \frac{dx}{\sqrt{(1+x^2)(1+2x^2)}} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{(1+\tan^2\theta)(1+2\tan^2\theta)}}d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\,d\theta}{\sqrt{\sec^2\theta\,(1+2\tan^2\theta)}}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sec\theta\sqrt{1+2\tan^2\theta}}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\sec\theta}{\sqrt{1+2\tan^2\theta}}\,d\theta = \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\cos\theta}}{\sqrt{1+2\frac{\sin^2\theta}{\cos^2\theta}}}\,d\theta;\ \text{ then we get:}$$

$$I=\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{\cos^2\theta+2\cos^2\theta\,.\frac{\sin^2\theta}{\cos^2\theta}}}=\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{\cos^2\theta+2\sin^2\theta}}=\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{\cos^2\theta+2(1-\cos^2\theta)}}$$

$$I=\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{\cos^2\theta+2-2\cos^2\theta}}=\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{2-\cos^2\theta}}=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{1-\frac{1}{2}\cos^2\theta}}$$

Let $\theta=\frac{\pi}{2}-\phi\Longrightarrow d\theta=-d\,\phi$ & $\begin{cases}\theta=0\Longrightarrow\phi=\frac{\pi}{2}\\ \theta=\frac{\pi}{4}\Longrightarrow\phi=\frac{\pi}{4}\end{cases}$; then we get:

$$I=\frac{1}{\sqrt{2}}\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\frac{-d\phi}{\sqrt{1-\frac{1}{2}\cos^2\left(\frac{\pi}{2}-\phi\right)}}=\frac{1}{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}};\text{ then we can write:}$$

$$I=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{4}}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}=\frac{1}{\sqrt{2}}\left\{F\left(\frac{1}{\sqrt{2}},\frac{\pi}{2}\right)-F\left(\frac{1}{\sqrt{2}},\frac{\pi}{4}\right)\right\}$$

130. $I=\int_4^6\frac{dx}{\sqrt{(x-1)(x-2)(x-3)}}$, let $\sqrt{x-3}=u$; $x-3=u^2$; $x=3+u^2\Longrightarrow dx=2udu$

for $x=4\Longrightarrow u=\sqrt{4-3}=1$ & for $x=6\Longrightarrow u=\sqrt{6-3}=\sqrt{3}$**;** then we get**:**

$$I=\int_4^6\frac{dx}{\sqrt{(x-1)(x-2)(x-3)}}=\int_1^{\sqrt{3}}\frac{2u}{\sqrt{(3+u^2-1)(3+u^2-2)u^2}}du$$

$$I=\int_1^{\sqrt{3}}\frac{2u}{u\sqrt{(u^2+2)(u^2+1)}}du=2\int_1^{\sqrt{3}}\frac{du}{\sqrt{(u^2+2)(u^2+1)}}$$

Let $u=\tan\theta\Longrightarrow du=\sec^2\theta\,d\theta$ & $\begin{cases}u=1\Longrightarrow\theta=\frac{\pi}{4}\\ u=\sqrt{3}\Longrightarrow\theta=\frac{\pi}{3}\end{cases}$; then we get:

$$I=2\int_1^{\sqrt{3}}\frac{du}{\sqrt{(u^2+2)(u^2+1)}}=2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2\theta\,d\theta}{\sqrt{(\tan^2\theta+2)(\tan^2\theta+1)}}=2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2\theta\,d\theta}{\sqrt{(\tan^2\theta+2)\sec^2\theta}}$$

$$I=2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec^2\theta}{\sec\theta\sqrt{\tan^2\theta+2}}d\theta=2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\sec\theta}{\sqrt{\tan^2\theta+2}}d\theta=2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{\frac{1}{\cos\theta}}{\sqrt{\frac{\sin^2\theta}{\cos^2\theta}+2}}d\theta$$

$$I=2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{d\theta}{\sqrt{2\cos^2\theta+\sin^2\theta}}=2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{d\theta}{\sqrt{2(1-\sin^2\theta)+\sin^2\theta}}=2\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{d\theta}{\sqrt{2-\sin^2\theta}};\text{ then:}$$

$$I=\frac{2}{\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}=\sqrt{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{d\theta}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2\theta}};\ \text{then we can write:}$$

$$I=\sqrt{2}\int_{0}^{\frac{\pi}{3}}\frac{d\theta}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2\theta}}-\sqrt{2}\int_{0}^{\frac{\pi}{4}}\frac{d\theta}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2\theta}}=\sqrt{2}\left\{F\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-F\left(\frac{1}{\sqrt{2}},\frac{\pi}{4}\right)\right\}$$

131. $I=\int_1^{+\infty}\frac{dx}{\sqrt{(x^2-1)(x^2+3)}}$, let $x=\sec\theta \Longrightarrow dx=\sec\theta\tan\theta\, d\theta$

for $x=1$, then $\cos\theta=1$ & $\theta=0$ & for $x=+\infty$, then $\cos\theta=0$, so $\theta=\frac{\pi}{2}$, then we get:

$$I=\int_{1}^{+\infty}\frac{dx}{\sqrt{(x^2-1)(x^2+3)}}=\int_{0}^{\frac{\pi}{2}}\frac{\sec\theta\tan\theta\, d\theta}{\sqrt{(\sec^2\theta-1)(\sec^2\theta+3)}}=\int_{0}^{\frac{\pi}{2}}\frac{\sec\theta\tan\theta\, d\theta}{\sqrt{\tan^2\theta\,(\sec^2\theta+3)}}$$

$$I=\int_{0}^{\frac{\pi}{2}}\frac{\sec\theta\tan\theta\, d\theta}{\tan\theta\sqrt{\sec^2\theta+3}}=\int_{0}^{\frac{\pi}{2}}\frac{\sec\theta\, d\theta}{\sqrt{\sec^2\theta+3}}=\int_{0}^{\frac{\pi}{2}}\frac{\frac{1}{\cos\theta}}{\sqrt{\frac{1}{\cos^2\theta}+3}}d\theta=\int_{0}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\frac{\cos^2\theta}{\cos^2\theta}+3\cos^2\theta}}$$

$$I=\int_{0}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+3\cos^2\theta}}=\int_{0}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+3(1-\sin^2\theta)}}=\int_{0}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{4-3\sin^2\theta}}=\frac{1}{2}\int_{0}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{3}{4}\sin^2\theta}}$$

$$I=\frac{1}{2}\int_{0}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2\sin^2\theta}}=\frac{1}{2}F\left(\frac{\sqrt{3}}{2},\frac{\pi}{2}\right)=\frac{1}{2}K\left(\frac{\sqrt{3}}{2}\right)$$

132. $I=\int_1^{+\infty}\frac{dx}{(3x^2+1)\sqrt{(x^2-1)(x^2+3)}}$, let $x=\sec\theta \Longrightarrow dx=\sec\theta\tan\theta\, d\theta$

for $x=1$, then $\cos\theta=1$ & $\theta=0$ & for $x=+\infty$, then $\cos\theta=0$, so $\theta=\frac{\pi}{2}$, then we get:

$$I=\int_{1}^{+\infty}\frac{dx}{(3x^2+1)\sqrt{(x^2-1)(x^2+3)}}=\int_{0}^{\frac{\pi}{2}}\frac{\sec\theta\tan\theta}{(3\sec^2\theta+1)\sqrt{(\sec^2\theta-1)(\sec^2\theta+3)}}\,d\theta;\ \text{then:}$$

$$I=\int_{0}^{\frac{\pi}{2}}\frac{\sec\theta\tan\theta}{(3\sec^2\theta+1)\sqrt{\tan^2\theta\,(\sec^2\theta+3)}}\,d\theta=\int_{0}^{\frac{\pi}{2}}\frac{\sec\theta\tan\theta}{(3\sec^2\theta+1)\tan\theta\sqrt{\sec^2\theta+3}}\,d\theta$$

$$I=\int_{0}^{\frac{\pi}{2}}\frac{\sec\theta}{(3\sec^2\theta+1)\sqrt{\sec^2\theta+3}}\,d\theta=\int_{0}^{\frac{\pi}{2}}\frac{\frac{1}{\cos\theta}}{\left(\frac{3}{\cos^2\theta}+1\right)\sqrt{\frac{1}{\cos^2\theta}+3}}\,d\theta;\ \text{then we get:}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{\frac{1}{\cos\theta}}{\frac{1}{\cos^2\theta}(3+\cos^2\theta).\frac{1}{\cos\theta}\sqrt{1+3\cos^2\theta}}\,d\theta=\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta}{(3+\cos^2\theta)\sqrt{1+3\cos^2\theta}}\,d\theta$$

$$I=\int_0^{\frac{\pi}{2}}\frac{(3+\cos^2\theta)-3}{(3+\cos^2\theta)\sqrt{1+3\cos^2\theta}}\,d\theta;\ \text{ then we can write:}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{3+\cos^2\theta}{(3+\cos^2\theta)\sqrt{1+3\cos^2\theta}}d\theta-3\int_0^{\frac{\pi}{2}}\frac{d\theta}{(3+\cos^2\theta)\sqrt{1+3\cos^2\theta}};\ \text{ then:}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+3\cos^2\theta}}-3\int_0^{\frac{\pi}{2}}\frac{d\theta}{(3+\cos^2\theta)\sqrt{1+3\cos^2\theta}}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+3(1-\sin^2\theta)}}-3\int_0^{\frac{\pi}{2}}\frac{d\theta}{(3+1-\sin^2\theta)\sqrt{1+3(1-\sin^2\theta)}}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{4-3\sin^2\theta}}-3\int_0^{\frac{\pi}{2}}\frac{d\theta}{(4-\sin^2\theta)\sqrt{4-3\sin^2\theta}}$$

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2\sin^2\theta}}-\frac{3}{8}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\left(1-\frac{1}{4}\sin^2\theta\right)\sqrt{1-\left(\frac{\sqrt{3}}{2}\right)^2\sin^2\theta}};\ \text{ therefore; we get:}$$

$$I=\frac{1}{2}\mathrm{F}\left(\frac{\sqrt{3}}{2},\frac{\pi}{2}\right)-\frac{3}{8}\Pi\left(\frac{\sqrt{3}}{2},-\frac{1}{4},\frac{\pi}{2}\right)$$

133. $I=\int_0^{\frac{\pi}{3}}\frac{\cos^2x}{\sqrt{1+\cos^2x}}dx=\int_0^{\frac{\pi}{3}}\frac{1-\sin^2x}{\sqrt{1+1-\sin^2x}}dx=\int_0^{\frac{\pi}{3}}\frac{1-\sin^2x}{\sqrt{2-\sin^2x}}dx$

$$I=\int_0^{\frac{\pi}{3}}\frac{dx}{\sqrt{2-\sin^2x}}-\int_0^{\frac{\pi}{3}}\frac{\sin^2x}{\sqrt{2-\sin^2x}}dx=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{3}}\frac{dx}{\sqrt{1-\frac{1}{2}\sin^2x}}-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{3}}\frac{\sin^2x}{\sqrt{1-\frac{1}{2}\sin^2x}}dx$$

$$I=\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{3}}\frac{dx}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2x}}-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{3}}\frac{\sin^2x\,dx}{\sqrt{1-\frac{1}{2}\sin^2x}}=\frac{1}{\sqrt{2}}\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-\frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{3}}\frac{\sin^2x}{\sqrt{1-\frac{1}{2}\sin^2x}}dx$$

$$\text{Let }J=\int_0^{\frac{\pi}{3}}\frac{\sin^2x}{\sqrt{1-\frac{1}{2}\sin^2x}}dx;\ \text{ then: }J=\int_0^{\frac{\pi}{3}}\frac{(1-1)+\frac{3}{2}\sin^2x-\frac{1}{2}\sin^2x}{\sqrt{1-\frac{1}{2}\sin^2x}}dx;\ \text{ then we get:}$$

$$\mathrm{J}=\int_0^{\frac{\pi}{3}}\frac{1-\frac{1}{2}\sin^2 x}{\sqrt{1-\frac{1}{2}\sin^2 x}}dx-\int_0^{\frac{\pi}{3}}\frac{dx}{\sqrt{1-\frac{1}{2}\sin^2 x}}+\frac{3}{2}\int_0^{\frac{\pi}{3}}\frac{\sin^2 x}{\sqrt{1-\frac{1}{2}\sin^2 x}}dx$$

$$\mathrm{J}=\int_0^{\frac{\pi}{3}}\frac{1-\frac{1}{2}\sin^2 x}{\sqrt{1-\frac{1}{2}\sin^2 x}}dx-\int_0^{\frac{\pi}{3}}\frac{dx}{\sqrt{1-\frac{1}{2}\sin^2 x}}+\frac{3}{2}\mathrm{J}$$

$$\mathrm{J}-\frac{3}{2}\mathrm{J}=\int_0^{\frac{\pi}{3}}\sqrt{1-\frac{1}{2}\sin^2 x}\,dx-\int_0^{\frac{\pi}{3}}\frac{dx}{\sqrt{1-\frac{1}{2}\sin^2 x}}=\mathrm{E}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)=-\frac{1}{2}\mathrm{J}$$

Then we get: $\mathrm{J}=-2\left\{\mathrm{E}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)\right\}=2\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-2\mathrm{E}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)$

Therefore: $\mathrm{I}=\frac{1}{\sqrt{2}}\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-\frac{1}{\sqrt{2}}\mathrm{J}=\frac{1}{\sqrt{2}}\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-\frac{1}{\sqrt{2}}\left\{2\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-2\mathrm{E}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)\right\}$

$$\mathrm{I}=\sqrt{2}\mathrm{E}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)-\frac{1}{\sqrt{2}}\mathrm{F}\left(\frac{1}{\sqrt{2}},\frac{\pi}{3}\right)$$

134. $\int \mathrm{W}(x)\,dx$, $\mathrm{W}(x)$**:** Lambert function

If $f(x)=xe^x \implies \mathrm{W}(x)=f^{-1}(x)\ \ \&\ \ \mathrm{W}(x)\mathrm{e}^{\mathrm{W}(x)}=x$

Let $u=\mathrm{W}(x)\implies x=ue^u\implies dx=(e^u+ue^u)du=e^u(1+u)du$, then we get**:**

$\int \mathrm{W}(x)\,dx=\int u.e^u(1+u)du=\int (u^2+u)e^u du$; now using integration by parts; we get:

$$\int \mathrm{W}(x)\,dx=(u^2+u)e^u-(2u+1)e^u+2e^u+c=e^u(u^2-u+1)+c$$

$\int \mathrm{W}(x)\,dx=u(ue^u)-ue^u+e^u+c$; therefore; we get:

$$\int \mathrm{W}(x)\,dx=x\mathrm{W}(x)-x+\mathrm{e}^{\mathrm{W}(x)}+\mathrm{c}=x(\mathrm{W}(x)-1)+\mathrm{e}^{\mathrm{W}(x)}+\mathrm{c}$$

Or we can write:

$\int \mathrm{W}(x)\,dx=x\mathrm{W}(x)-x+\mathrm{e}^{\mathrm{W}(x)}+\mathrm{c}=x\mathrm{W}(x)-x+\frac{x}{\mathrm{W}(x)}+\mathrm{c}$; because $\mathrm{W}(x)\mathrm{e}^{\mathrm{W}(x)}=x$

$$\implies \int \mathrm{W}(x)\,dx=x\left(\mathrm{W}(x)-1+\frac{1}{\mathrm{W}(x)}\right)+\mathrm{c}$$

135. $\int \ln\left(\frac{x}{\ln\left(\frac{x}{\ln\left(\frac{x}{\dots}\right)}\right)}\right)dx$

Let $y=\ln\left(\frac{x}{\ln\left(\frac{x}{\ln\left(\frac{x}{\dots}\right)}\right)}\right)$ infinitely nested $\implies y=\ln\left(\frac{x}{y}\right)\implies e^y=\frac{x}{y}\implies ye^y=x$

$ye^y=x\implies \mathrm{W}(ye^y)=\mathrm{W}(x)\implies y=\ \mathrm{W}(x)$; then we get:

$$\int \ln\left(\frac{x}{\ln\left(\frac{x}{\ln\left(\frac{x}{\ldots}\right)}\right)}\right)dx = \int y dx = \int \mathrm{W}(x)dx = x(\mathrm{W}(x)-1)+\mathrm{e}^{\mathrm{W}(x)}+\mathrm{c}$$

136. $\mathrm{I} = \int_{\mathbf{1}}^{+\infty} \frac{dx}{\sqrt{x^4-1}}$

Let $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt;\ \begin{cases} x = 1 \Rightarrow t = 1 \\ x = +\infty \Rightarrow t = 0 \end{cases};$ then we get:

$$\mathrm{I} = \int_{\mathbf{1}}^{+\infty} \frac{dx}{\sqrt{x^4-1}} = \int_1^0 \frac{-\frac{1}{t^2}dt}{\sqrt{\left(\frac{1}{t}\right)^4-1}} = \int_0^1 \frac{\frac{1}{t^2}}{\sqrt{\frac{1}{t^4}-1}}dt = \int_0^1 \frac{\frac{1}{t^2}}{\sqrt{\frac{1}{t^4}(1-t^4)}}dt = \int_0^1 \frac{\frac{1}{t^2}}{\frac{1}{t^2}\sqrt{1-t^4}}dt$$

Then we get: $\mathrm{I} = \int_0^1 \frac{dt}{\sqrt{1-t^4}}$

Let $t = \sin\phi \Rightarrow dt = \cos\phi\, d\phi;\ \begin{cases} t = 0 \Rightarrow \phi = 0 \\ t = 1 \Rightarrow \phi = \frac{\pi}{2} \end{cases};$ then we get:

$$\mathrm{I} = \int_0^1 \frac{dt}{\sqrt{1-t^4}} = \int_0^{\frac{\pi}{2}} \frac{\cos\phi}{\sqrt{1-\sin^4\phi}}d\phi = \int_0^{\frac{\pi}{2}} \frac{\cos\phi}{\sqrt{(1-\sin^2\phi)(1+\sin^2\phi)}}d\phi$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos\phi}{\sqrt{\cos^2\phi\,(1+\sin^2\phi)}}d\phi = \int_0^{\frac{\pi}{2}} \frac{\cos\phi}{\cos\phi\sqrt{1+\sin^2\phi}}d\phi = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1+\sin^2\phi}}d\phi;\ \text{then we get:}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1+(1-\cos^2\phi)}}d\phi = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2-\cos^2\phi}}d\phi$$

Let $\phi = \frac{\pi}{2} - \theta \Rightarrow d\phi = -d\theta;\ \begin{cases} \phi = 0 \Rightarrow \theta = \frac{\pi}{2} \\ \phi = \frac{\pi}{2} \Rightarrow \theta = 0 \end{cases};$ then we get:

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2-\cos^2\phi}}d\phi = \int_{\frac{\pi}{2}}^0 \frac{-d\theta}{\sqrt{2-\cos^2\left(\frac{\pi}{2}-\theta\right)}} = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{2-\sin^2\theta}} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$$

$$\mathrm{I} = \frac{1}{\sqrt{2}}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\left(\frac{1}{\sqrt{2}}\right)^2\sin^2\theta}} = \frac{1}{\sqrt{2}}\mathrm{K}\left(\frac{1}{\sqrt{2}}\right)$$

137. $\mathrm{I} = \int_{\frac{1}{2}}^{1} \frac{\Psi(x)}{1+\Gamma^2(x)} dx$

$$\mathrm{I} = \int_{\frac{1}{2}}^{1} \frac{\Psi(x)}{1+\Gamma^2(x)} dx = \int_{\frac{1}{2}}^{1} \frac{\frac{\Gamma'(x)}{\Gamma(x)}}{1+\Gamma^2(x)} dx = \int_{\frac{1}{2}}^{1} \frac{\Gamma'(x)}{\Gamma(x)[1+\Gamma^2(x)]} dx$$

Let $y = \Gamma(x) \Longrightarrow dy = \Gamma'(x)dx$; $\begin{cases} x = \frac{1}{2} \Longrightarrow y = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\ x = 1 \Longrightarrow y = \Gamma(1) = 1 \end{cases}$; then we get:

$$\mathrm{I} = \int_{\frac{1}{2}}^{1} \frac{\Gamma'(x)}{\Gamma(x)[1+\Gamma^2(x)]} dx = \int_{\sqrt{\pi}}^{1} \frac{dy}{y(1+y^2)} = \int_{\sqrt{\pi}}^{1} \frac{1+y^2-y^2}{y(1+y^2)} dy = \int_{\sqrt{\pi}}^{1} \left(\frac{1}{y} - \frac{y}{1+y^2}\right) dy$$

$$\mathrm{I} = \frac{1}{2}\int_{1}^{\sqrt{\pi}} \frac{2y}{1+y^2} dy - \int_{1}^{\sqrt{\pi}} \frac{1}{y} dy = \left[\frac{1}{2}\ln(1+y^2) - \ln y\right]_1^{\sqrt{\pi}} = \left[\ln\left(\frac{\sqrt{1+y^2}}{y}\right)\right]_1^{\sqrt{\pi}}$$

$$\mathrm{I} = \ln\left(\frac{\sqrt{1+\pi}}{\sqrt{\pi}}\right) - \ln\sqrt{2} = \ln\left(\frac{\sqrt{1+\pi}}{\sqrt{2\pi}}\right) = \ln\left(\sqrt{\frac{1+\pi}{2\pi}}\right) = \frac{1}{2}\ln\left(\frac{1+\pi}{2\pi}\right)$$

138. $\mathrm{I} = \int_{-\infty}^{+\infty} \sin(x^2+2x+2)\, dx$

$$\mathrm{I} = \int_{-\infty}^{+\infty} \sin(x^2+2x+2)\, dx = \int_{-\infty}^{+\infty} \sin[(x^2+2x+1)+1]\, dx = \int_{-\infty}^{+\infty} \sin[(x+1)^2+1]\, dx$$

Let $t = x+1 \Longrightarrow dt = dx$; then we get:

$$\mathrm{I} = \int_{-\infty}^{+\infty} \sin[(x+1)^2+1]\, dx = \int_{-\infty}^{+\infty} \sin(t^2+1)\, dt = \int_{-\infty}^{+\infty} (\cos 1 \sin t^2 + \sin 1 \cos t^2) dt$$

$$\mathrm{I} = \cos 1 \int_{-\infty}^{+\infty} \sin t^2\, dt + \sin 1 \int_{-\infty}^{+\infty} \cos t^2\, dt = 2\cos 1 \int_{0}^{+\infty} \sin t^2\, dt + 2\sin 1 \int_{0}^{+\infty} \cos t^2\, dt$$

But $\int_{0}^{+\infty} \sin t^2\, dt = \int_{0}^{+\infty} \cos t^2\, dt = \sqrt{\frac{\pi}{8}}$ (**Fresnel's Integrals**) $\Longrightarrow$

$$\mathrm{I} = 2\cos 1\sqrt{\frac{\pi}{8}} + 2\sin 1\sqrt{\frac{\pi}{8}} = \cos 1\sqrt{\frac{\pi}{2}} + \sin 1\sqrt{\frac{\pi}{2}} = (\cos 1 + \sin 1)\sqrt{\frac{\pi}{2}}$$

139. $\mathrm{I} = \int_{0}^{+\infty} e^{-ax^2} \sin(bx^2)\, dx$

$$\mathrm{I} = \int_{0}^{+\infty} e^{-ax^2} \sin(bx^2)\, dx = \int_{0}^{+\infty} e^{-ax^2}.Im\left(e^{ibx^2}\right) dx = \mathrm{Im}\left(\int_{0}^{+\infty} e^{-ax^2}.e^{ibx^2} dx\right) dx$$

$$\mathrm{I} = \mathrm{Im}\left(\int_{0}^{+\infty} e^{-ax^2+ibx^2}\right) dx = \mathrm{Im}\left(\int_{0}^{+\infty} e^{-x^2(a-ib)}\right) dx = \mathrm{Im}\left(\int_{0}^{+\infty} e^{-\left(x\sqrt{a-ib}\right)^2}\right) dx$$

Let $t = x\sqrt{a-ib} \Longrightarrow dt = \sqrt{a-ib}dx$; then we get:

$$I = \operatorname{Im}\left(\int_0^{+\infty} e^{-x^2(a-ib)}\right)dx = \operatorname{Im}\left\{\frac{1}{\sqrt{a-ib}}\int_0^{+\infty} e^{-t^2}dt\right\} = \operatorname{Im}\left\{\frac{1}{\sqrt{a-ib}}.\frac{\sqrt{\pi}}{2}\right\};\ \text{then:}$$

$$I = \frac{\sqrt{\pi}}{2}\operatorname{Im}\left\{\frac{1}{\sqrt{a^2+b^2}.e^{-i\tan^{-1}\left(\frac{b}{a}\right)}}\right\} = \frac{1}{2}.\frac{\sqrt{\pi}}{\sqrt{\sqrt{a^2+b^2}}}\operatorname{Im}\left\{e^{i\left[\frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right]}\right\};\ \text{then we get:}$$

$$I = \frac{1}{2}\sqrt{\frac{\pi}{\sqrt{a^2+b^2}}}\sin\left[\frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right]$$

140. $I = \int_0^{+\infty} e^{-ax^2}\cos(bx^2)\,dx$

$$I = \int_0^{+\infty} e^{-ax^2}\cos(bx^2)\,dx = \int_0^{+\infty} e^{-ax^2}.\operatorname{Re}\left(e^{ibx^2}\right)dx = \operatorname{Re}\left(\int_0^{+\infty} e^{-ax^2}.e^{ibx^2}dx\right)dx$$

$$I = \operatorname{Re}\left(\int_0^{+\infty} e^{-ax^2+ibx^2}\right)dx = \operatorname{Re}\left(\int_0^{+\infty} e^{-x^2(a-ib)}\right)dx = \operatorname{Re}\left(\int_0^{+\infty} e^{-\left(x\sqrt{a-ib}\right)^2}\right)dx$$

Let $t = x\sqrt{a-ib} \Rightarrow dt = \sqrt{a-ib}dx$; then we get:

$$I = \operatorname{Re}\left(\int_0^{+\infty} e^{-x^2(a-ib)}\right)dx = \operatorname{Re}\left\{\frac{1}{\sqrt{a-ib}}\int_0^{+\infty} e^{-t^2}dt\right\} = \operatorname{Re}\left\{\frac{1}{\sqrt{a-ib}}.\frac{\sqrt{\pi}}{2}\right\};\ \text{then:}$$

$$I = \frac{\sqrt{\pi}}{2}\operatorname{Re}\left\{\frac{1}{\sqrt{a^2+b^2}.e^{-i\tan^{-1}\left(\frac{b}{a}\right)}}\right\} = \frac{1}{2}.\frac{\sqrt{\pi}}{\sqrt{\sqrt{a^2+b^2}}}\operatorname{Re}\left\{e^{i\left[\frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right]}\right\};\ \text{then we get:}$$

$$I = \frac{1}{2}\sqrt{\frac{\pi}{\sqrt{a^2+b^2}}}\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{b}{a}\right)\right]$$

141. $I = \int_0^1 \frac{1}{\sqrt{x^p-x^2}}dx$; where $1 < p < 2$

$$I = \int_0^1 \frac{1}{\sqrt{x^p-x^2}}dx = \int_0^1 \frac{1}{\sqrt{x^p\left(1-\frac{x^2}{x^p}\right)}}dx = \int_0^1 \frac{1}{x^{\frac{1}{2}p}\sqrt{1-x^{2-p}}}dx$$

Let $y = x^{2-p} \Rightarrow x = y^{\frac{1}{2-p}} \Rightarrow dx = \frac{1}{2-p}y^{\frac{1}{2-p}-1}dy$; then we get:

$$I = \int_0^1 \frac{1}{\sqrt{x^p-x^2}}dx = \int_0^1 \frac{1}{x^{\frac{1}{2}p}\sqrt{1-x^{2-p}}}dx = \int_0^1 \frac{1}{\left(y^{\frac{1}{2-p}}\right)^{\frac{1}{2}p}\sqrt{1-y}}.\frac{1}{2-p}y^{\frac{1}{2-p}-1}dy$$

$$I = \frac{1}{2-p}\int_0^1 \frac{y^{\frac{1}{2-p}-1}}{y^{\frac{p}{2}\left(\frac{1}{2-p}\right)}\sqrt{1-y}}dy = \frac{1}{2-p}\int_0^1 y^{\frac{1}{2-p}-1-\frac{p}{2(2-p)}}.(1-y)^{-\frac{1}{2}}dy;\ \text{then we get:}$$

$$I = \frac{1}{2-p}\int_0^1 y^{\frac{2-4+2p-p}{2(2-p)}}.(1-y)^{-\frac{1}{2}}dy = \frac{1}{2-p}\int_0^1 y^{\frac{p-2}{2(2-p)}}.(1-y)^{-\frac{1}{2}}dy$$

$$\mathrm{I}=\frac{1}{2-p}\int_0^1 y^{\frac{-(2-p)}{2(2-p)}}.(1-y)^{-\frac{1}{2}}\,dy=\frac{1}{2-p}\int_0^1 y^{-\frac{1}{2}}.(1-y)^{-\frac{1}{2}}\,dy=\frac{1}{2-p}\mathrm{B}\left(\frac{1}{2},\frac{1}{2}\right)$$

$$\mathrm{I}=\frac{1}{2-\mathrm{p}}.\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)}=\frac{1}{2-p}.\frac{\left[\Gamma\left(\frac{1}{2}\right)\right]^2}{\Gamma(2)}=\frac{1}{2-p}\left(\sqrt{\pi}\right)^2=\frac{\pi}{2-p};\ \text{ therefore; we get:}$$

$$\int_0^1 \frac{1}{\sqrt{x^p-x^2}}\,dx=\frac{\pi}{2-p}$$

Chapter 5: Integrals With Series

1. **Maclaurin Series:**

A **Maclaurin Series** is a power series that allows us to calculate an approximation of a function $f(x)$ for input values close to zero, given that we know the values of the successive derivatives of the function at zero

A Maclaurin series can be used also to find the anti-derivative of a complicated function, or compute an otherwise un-computable sum

Partial sums of a Maclaurin series provide polynomial approximations of the function

A Macluarin series is a special case of a Taylor series, obtained by setting $x_0 = 0$

Taylor series:

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = f(a) + (x-a)f'(a) + (x-a)^2\frac{f"(a)}{2!} + \cdots + (x-a)^n\frac{f^{(n)}(a)}{n!} + \cdots$$

Maclaurin series:

$$\sum_{n=0}^{\infty} f^{(n)}(0)\frac{x^n}{n!} = f(0) + xf'(0) + x^2\frac{f"(0)}{2!} + x^2\frac{f^{(3)}(0)}{3!} + \cdots + x^n\frac{f^{(n)}(0)}{n!} + \cdots$$

Maclaurin Expansion of Some Elementary Functions

Exponential: $e^x = \sum_{n=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \quad x \in]-\infty; +\infty[$

Trigonometric: $\sin x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \ x \in]-\infty; +\infty[$

Trigonometric: $\cos x = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots x \in]-\infty; +\infty[$

Trigonometric: $\tan x = \sum_{n=0}^{\infty}\frac{\mathrm{B}_{2n}(-4)^n(1-4^n)}{(2n)!}x^{2n-1}$

$= x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots \quad \text{for } |x| < \frac{\pi}{2}$

Inv. Trig: $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad \text{for } |x| \le 1$

Inv. Trig: $\sin^{-1} x = \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1} = x + \frac{x^3}{6} + \frac{3x^5}{40} + \cdots \quad \text{for } |x| \le 1$

Inv. Trig: $\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2} - \sum_{n=0}^{\infty}\frac{(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1} \quad \text{for } |x| \le 1$

Geometric Series: $\dfrac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\cdots$ for $|x|<1$

Binomial Series: $(1+x)^k=\sum_{n=0}^{\infty}\dfrac{k(k-1)(k-2)\ldots\big(k-(n-1)\big)}{n!}x^n$

$=1+kx+\dfrac{k(k-1)}{2!}x^2+\dfrac{k(k-1)(k-2)}{3!}x^3+\cdots$ for $|x|<1$

Logarithmic: $\ln(1+x)=\sum_{n=0}^{\infty}(-1)^{n+1}\dfrac{x^n}{n}=x-\dfrac{x^2}{2}+\dfrac{x^3}{3}-\dfrac{x^4}{4}+\cdots$ for $|x|<1$

Logarithmic: $\ln(1-x)=-\sum_{n=0}^{\infty}\dfrac{x^n}{n}=-x-\dfrac{x^2}{2}-\dfrac{x^3}{3}-\dfrac{x^4}{4}-\cdots$ for $|x|<1$

Hyperbolic: $\sinh x=\sum_{n=0}^{\infty}\dfrac{x^{2n+1}}{(2n+1)!}=x+\dfrac{x^3}{3!}+\dfrac{x^5}{5!}+\dfrac{x^7}{7!}+\cdots\ x\in]-\infty;+\infty[$

Hyperbolic: $\cosh x=\sum_{n=1}^{\infty}\dfrac{x^{2n}}{(2n)!}=1+\dfrac{x^2}{2!}+\dfrac{x^4}{4!}+\dfrac{x^6}{6!}+\cdots x\in]-\infty;+\infty[$

Hyperbolic: $\tanh x=\sum_{n=0}^{\infty}\dfrac{\mathrm{B}_{2n}4^n(4^n-1)}{(2n)!}x^{2n-1}$

$=x-\dfrac{x^3}{3}+\dfrac{2x^5}{15}-\dfrac{17x^7}{315}+\cdots$ for $|x|<\dfrac{\pi}{2}$

Hyperbolic: $\sinh^{-1}x=\sum_{n=0}^{\infty}\dfrac{(-1)^n(2n)!}{4^n(n!)^2(2n+1)}x^{2n+1}=x-\dfrac{x^3}{6}+\dfrac{3x^5}{40}-\cdots$ for $|x|<1$

Hyperbolic: $\tanh^{-1}x=\sum_{n=0}^{\infty}\dfrac{x^{2n+1}}{2n+1}x^{2n+1}=x+\dfrac{x^3}{3}+\dfrac{x^5}{5}+\cdots$ for $|x|<1;\ x\neq\pm$

2. Dirichlet Beta Function:

The **Dirichlet Beta** function (denoted by $\beta(s)$) (also known as Catalan Beta function), is a special function, closely related to the Riemann zeta function and it is defined as**:**

$$\beta(s)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^s}\quad\text{or equivqlently: }\beta(s)=\frac{1}{\Gamma(s)}\int_0^{+\infty}\frac{x^{s-1}e^{-x}}{1+e^{-2x}}dx$$

Some special values

$$\beta(0)=\frac{1}{2};\quad\beta(1)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}=\arctan 1=\frac{\pi}{4};\quad\beta(2)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}=\mathrm{G}$$

$$\beta(3)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{\pi^3}{32}$$

Relation between Dirichlet Beta function and polygamma function

$$\beta(s)=\frac{1}{2^s}\sum_{n=0}^{+\infty}\frac{(-1)^n}{\left(n+\frac{1}{2}\right)^s}=\frac{1}{(-4)^s(s-1)!}\left[\Psi^{(s-1)}\left(\frac{1}{4}\right)-\Psi^{(s-1)}\left(\frac{3}{4}\right)\right]$$

Derivative of Dirichlet Beta function at 1

$$\beta'(1)=\sum_{n=1}^{+\infty}(-1)^{n+1}\frac{\ln(2n+1)}{2n+1}=\frac{\pi}{4}(\gamma-\ln\pi)+\pi\ln\Gamma\left(\frac{3}{4}\right)$$

Remark: Catalan's constant denoted by G and defined as:

$$\mathrm{G}=\beta(2)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}=\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots$$

Its numerical value is approximately 0.915 965 594 177 219 015 054 603 514 932 384 …

Some integral identities with Catalan's constant:

$$\mathrm{G}=\int_1^{+\infty}\frac{\ln t}{1+t^2}dt=-\int_0^1\frac{\ln t}{1+t^2}dt$$

$$\mathrm{G}=\int_0^{\frac{\pi}{4}}\frac{t}{\sin t\cos t}dt=\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{t}{\sin t}dt=\frac{1}{4}\int_0^{\frac{\pi^2}{4}}\csc\sqrt{t}dt$$

$$\mathrm{G}=\int_0^{\frac{\pi}{4}}\ln\cot t\,dt=-\int_0^{\frac{\pi}{4}}\ln\tan t\,dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\ln(\sec t+\tan t)dt$$

$$\mathrm{G}=\int_0^1\frac{\arccos t}{\sqrt{1+t^2}}dt=\int_0^1\frac{\operatorname{arcsinh} t}{\sqrt{1-t^2}}dt=\int_0^{+\infty}\operatorname{arccot} e^t\,dt=\frac{1}{2}\int_0^{+\infty}\frac{t}{\cosh t}dt$$

Relation with Gamma function:

$$\mathrm{G}=\frac{\pi}{4}\int_0^1\Gamma\left(1+\frac{x}{2}\right)\Gamma\left(1-\frac{x}{2}\right)dx=\frac{\pi}{2}\int_0^{\frac{1}{2}}\Gamma(1+y)\Gamma(1-y)dy$$

Some Useful Summations:

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots=\ln 2$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{2n-1}=1-2\sum_{n=1}^{\infty}\frac{1}{(4n-1)(4n+1)}=\frac{\pi}{4}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{k+1}}{n^2}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\cdots=\frac{\pi^2}{12}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots=\frac{\pi^2}{8}$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(2n-1)^3}=1-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\cdots=\frac{\pi^3}{32}$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^4}=1+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots=\frac{\pi^4}{96}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^p}=1+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\cdots=\zeta(p)$$

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^p}=1-\frac{1}{2^p}+\frac{1}{3^p}-\frac{1}{4^p}+\cdots=\left(1-2^{1-p}\right)\zeta(p)$$

$$\sum_{n=1}^{\infty}\frac{1}{n(2n+1)}=\frac{1}{1\times 3}+\frac{1}{2\times 5}+\frac{1}{3\times 7}+\frac{1}{4\times 9}=2-\ln 4$$

3. **Dirichlet Lambda Function:**

The **Dirichlet Lambda** function (denoted by $\lambda(s)$) is defined as**:**

$$\lambda(s)=\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^s}\quad \text{or equivqlently: } \lambda(s)=(1-2^{-s})\zeta(s);\ \ \mathrm{Re}(s)>1$$

Note that:

$$\frac{\boldsymbol{\zeta(s)}}{\boldsymbol{2^s}}=\frac{\boldsymbol{\lambda(s)}}{\boldsymbol{2^s-1}}=\frac{\boldsymbol{\eta(s)}}{\boldsymbol{2^s-2}}$$

$\boldsymbol{\zeta(s)}$**:** Riemann Zeta function

$\boldsymbol{\eta(s)}$**:** Dirichlet Eta function

Solved Exercises

1. $\int_0^1 \frac{\arctan x}{x}\,dx$
2. $\int_0^1 \frac{\ln(1-x)}{x}\,dx$
3. $\int_0^1 \frac{1-(1-x)^n}{x}\,dx$
4. $\int_0^1 \frac{\ln x}{1+x}\,dx$
5. $\int_0^{+\infty} x\ln(1-e^{-x})\,dx$
6. $\int_0^{+\infty} x\ln(1+e^{-x})\,dx$
7. $\int_0^1 \frac{\tanh^{-1}(\sqrt[4]{x})}{x}\,dx$
8. $\int_0^1 x^{a\ln x}\,dx$
9. $\int_0^{+\infty} \ln(1+e^{-x}+e^{-2x}+\cdots)\,dx$
10. $\int_0^{\frac{\pi}{2}} e^{\cos x}\,dx$
11. $\int_1^{+\infty} \frac{\ln^2 x}{x^2+x\ln x}\,dx$
12. $\int_0^1 \ln\left(\frac{1+x}{1-x}\right).\frac{dx}{x}$
13. $\int_0^{+\infty} x\tan^{-1}(e^{-\gamma x})\,dx$
14. $\int_0^1 \frac{\tan^{-1}(\ln x)}{\ln x}\,dx$
15. $\int_0^1 \frac{x^{a-1}\ln x}{1-x^{2a}}\,dx$
16. $\int_0^{+\infty} \ln\sqrt[n]{1-e^{-x}}\,dx$
17. $\int_0^1 \frac{1}{x}.\ln\left(\frac{1-x^m}{1+x^n}\right)dx$
18. $\int_0^1 \frac{\ln(1+\sqrt[n]{x})}{x}\,dx$
19. $\int_0^{+\infty} \frac{x}{\cosh x}\,dx$
20. $\int_0^{\frac{\pi}{2}} \sin 2x\ln(\sin x)\,dx$
21. $\int_0^{+\infty} xe^{-x}\coth\left(\frac{x}{2}\right)dx$
22. $\int_1^e \frac{\ln(1+\ln(1-\ln x))}{x}\,dx$
23. $\int_{-\infty}^{+\infty} \frac{e^{-ax}}{e^{e^{-x}-1}}\,dx$
24. $\int_0^{\frac{\pi}{4}} \ln(\tan x)\,dx$
25. $\int_0^1 \ln x\ln(1-x)\,dx$
26. $\int_0^1 x^{-nx}\,dx$
27. $\int_0^1 \frac{x}{1-x^2}.\ln^k x\,dx$
28. $\int_0^{+\infty} \frac{x^{n-1}}{e^x-1}\,dx$
29. $\int_0^{+\infty} \frac{\sqrt{x}}{\cosh(\sqrt{x})}\,dx$
30. $\int_0^a \tanh^{-1}\left(\frac{x}{a}\right).\frac{dx}{x}$
31. $\int_0^{+\infty} \frac{\ln x}{1+e^{ax}}\,dx$
32. $\int_2^5 \left(\sum_{n=1}^{+\infty}\frac{n+1}{x^n}\right)dx$
33. $\int_0^1 \left\{\frac{1}{x}\right\}dx$
34. $\int_0^{+\infty} \frac{x^2}{\sinh(\sqrt{x})}\,dx$
35. $\int_0^{\frac{\pi}{2}} \frac{\ln(\cos x)}{\sin x}\,dx$

36. $\int_0^1 \frac{\ln(1-x^2)\ln^2 x}{x}\,dx$

37. $\int_{-\infty}^{+\infty} \frac{x^2 e^{-x}}{(1+e^{-x})^2}\,dx$

38. $\int_0^1 \ln(-\ln x)\,.\,(\ln x)^2\,dx$

39. $\int_0^{\frac{\pi}{4}} \ln^2(\cot x)\,dx$

40. $\int_0^1 \frac{\ln(1-x)}{1+x}\,dx$

41. $\int_0^1 \left(\frac{\ln x}{1-x}\right)^2 dx$

42. $\int_0^1 \ln x \tan^{-1} x\,dx$

43. $\int_0^1 \ln(1-x)\,.\,\frac{\ln^{n-1} x}{x}\,dx$

44. $\int_0^{\frac{\pi}{2}} \tan x \ln(\sin x)\ln(\cos x)\,dx$

45. $\int_0^1 \frac{\ln x \ln(1-x)}{x(1-x)}\,dx$

46. $\int_0^{\frac{\pi}{2}} \ln^n(\tan x)\,dx$

47. $\int_0^{+\infty} x^\alpha \tan^{-1}(e^{-x})\,dx$

48. $\int_0^1 \frac{x-\ln(1+x)}{x\ln x}\,dx$

49. $\int_0^{+\infty} \ln\left(\frac{e^x+1}{e^x-1}\right) dx$

50. $\int_0^1 \frac{\ln\left(1+x+x^2+\cdots+x^{n-1}\right)}{x}\,dx$

51. $\int_0^{+\infty} (2x-x^2)e^{-2x}\operatorname{sech} x\,dx$

52. $\int_0^{\frac{\pi}{2}} \frac{\sin x \ln(\sin x)\ln(\cos x)}{\cos^2 x}\,dx$

53. $\int_0^{\frac{\pi}{2}} \frac{x}{\sin x}\,dx$

54. $\int_0^1 \frac{\ln[1+\ln(1-x)]}{\ln(1-x)}\,dx$

55. $\int_0^{+\infty} \frac{x}{\sinh(ax)}\,dx$

56. $\int_0^1 \frac{\ln(1-x^2)\ln^2 x}{x}\,dx$

57. $\int_0^{\frac{\pi}{2}} \left[\frac{\ln(\tan x)}{\sin(\frac{\pi}{4}-x)}\right]^2 dx$

58. $\int_0^{+\infty} \ln\left(\frac{2^x+1}{2^x-1}\right) dx$

59. $\int_0^{+\infty} \frac{x^3 e^x}{\left(e^{2x}-1\right)^2}\,dx$

60. $\int_0^{+\infty} \frac{e^{-xt}\sin(a\sqrt{x})}{\pi x}\,dx$

61. $\int_0^{+\infty} \frac{\ln x}{e^x+e^{-x}}\,dx$

62. $\int_0^{\frac{\pi}{2}} \frac{\ln(\cos x)\sqrt{\ln(\sin x)}}{\tan x}\,dx$

63. $\int_0^1 x\left\{\frac{1}{x}\right\}\left\lfloor\frac{1}{x}\right\rfloor dx$

64. $\int_0^{2\pi} e^{e^{ix}}\,dx$

65. $\int_0^{+\infty} \ln\left(\frac{a+be^{-px}}{a+be^{-qx}}\right).\frac{dx}{x}$

66. $\int_0^{+\infty} \frac{\sin x}{e^x(e^x-1)}\,dx$

67. $\int_0^{+\infty} e^{-x}\frac{\sin(tx)}{\sinh x}\,dx$

68. $\int_0^{\frac{1}{n}} \frac{\cos(\ln nx)-1}{\ln nx}\,dx$

69. $\int_0^{\frac{\pi}{2}} \{\tan x\}\,dx$

70. $\int_1^{+\infty} \frac{\ln(\ln x)}{1+2x\cos\alpha+x^2}\,dx$

Solutions of Exercises

1. $\mathrm{I}=\int_0^1 \frac{\arctan x}{x}dx=\int_0^1 \frac{1}{x}\arctan x\,dx$, then:

$$\mathrm{I}=\int_0^1 \frac{1}{x}\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}x^{2n+1}dx=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^1 x^{2n}dx=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\left[\frac{1}{2n+1}x^{2n+1}\right]_0^1;\text{ then:}$$

$$\mathrm{I}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}.\frac{1}{2n+1}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}=\mathrm{G}$$

2. $\mathrm{I}=\int_0^1 \frac{\ln(1-x)}{x}dx=\int_0^1 \frac{1}{x}\ln(1-x)\,dx$, then:

$$\mathrm{I}=\int_0^1 \frac{1}{x}\left(-\sum_{n=1}^{+\infty}\frac{x^n}{n}\right)dx=-\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 x^{n-1}dx=-\sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{1}{n}\right)=-\sum_{n=1}^{+\infty}\frac{1}{n^2}=-\frac{\pi^2}{6}$$

3. $\mathrm{I}=\int_0^1 \frac{1-(1-x)^n}{x}dx$, let $t=1-x$. then $dt=-dx$, for $x=0$, $t=1$ and for $x=1$, $t=0$, so:

$\mathrm{I}=\int_1^0 \frac{1-t^n}{1-t}(-dt)=\int_0^1 \frac{1-t^n}{1-t}dt=\int_0^1 \frac{(1-t)\left(1+t+t^2+t^3+t^4+\cdots t^{n-1}\right)}{1-t}dt$, then we get:

$\mathrm{I}=\int_0^1 (1+t+t^2+t^3+\cdots t^{n-1})dt$, so we have:

$$\mathrm{I}=\int_0^1 \sum_{k=0}^{n-1}t^k dt=\int_0^1 \sum_{k=1}^{n}t^{k-1}dt=\sum_{k=1}^{n}\int_0^1 t^{k-1}dt=\sum_{k=1}^{n}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}=\mathrm{H}_n$$

4. $\mathrm{I}=\int_0^1 \frac{\ln x}{1+x}dx$ using integration by parts:

Let $u=\ln x$, then $u'=\frac{1}{x}$ and let $v'=\frac{1}{1+x}$, then $v=\ln(1+x)$, then we get:

$\mathrm{I}=[\ln x\ln(1+x)]_0^1-\int_0^1 \frac{\ln(1+x)}{x}dx=-\int_0^1 \frac{\ln(1+x)}{x}dx=-\int_0^1 \frac{1}{x}\ln(1+x)\,dx$, then we get:

$$\mathrm{I}=-\int_0^1 \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}x^{n-1}dx=-\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^{n-1}dx=-\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}.\frac{1}{n}$$

$$\text{Then } \mathrm{I}=-\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2}=-\frac{\pi^2}{12}$$

5. $\mathrm{I}=\int_0^{+\infty} x\ln(1-e^{-x})\,dx$

We have: $\ln(1-x)=-\sum_{n=1}^{+\infty}\frac{x^n}{n}$; then we have: $\ln(1-e^{-x})=-\sum_{n=1}^{+\infty}\frac{(e^{-x})^n}{n}=-\sum_{n=1}^{+\infty}\frac{e^{-nx}}{n}$

$$\text{Then we get: } \mathrm{I}=\int_0^{+\infty} x\left(-\sum_{n=1}^{+\infty}\frac{(e^{-x})^n}{n}\right)dx=-\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} xe^{-nx}dx$$

Now evaluating $\int_0^{+\infty} xe^{-nx}dx$ by parts:

Let $u=x$, then $u'=1$ and let $v'=e^{-nx}$, then $v=-\frac{1}{n}e^{-nx}$, then we get:

$\int_0^{+\infty} xe^{-nx}dx=\left[-\frac{x}{n}e^{-nx}\right]_0^{+\infty}+\int_0^{+\infty}\frac{1}{n}e^{-nx}dx=\frac{1}{n}\left[-\frac{1}{n}e^{-nx}\right]_0^{+\infty}=\frac{1}{n}\left(\frac{1}{n}\right)=\frac{1}{n^2}$, then we get:

$$\mathrm{I} = -\sum_{n=1}^{+\infty} \frac{1}{n} \cdot \frac{1}{n^2} = -\sum_{n=1}^{+\infty} \frac{1}{n^3} = -\zeta(3)$$

6. $\mathrm{I} = \int_0^{+\infty} x \ln(1 + e^{-x})\, dx$

We have: $\ln(1+x) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} x^n}{n}$; then: $\ln(1+e^{-x}) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} (e^{-x})^n}{n}$

and so $\ln(1+e^{-x}) = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} e^{-nx}}{n}$; then we get:

$$\mathrm{I} = \int_0^{+\infty} x \sum_{n=1}^{+\infty} \frac{(-1)^{n-1} e^{-nx}}{n} dx = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \int_0^{+\infty} x e^{-nx} dx = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n} \cdot \frac{1}{n^2}; \text{ then:}$$

$$\mathrm{I} = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^3} = \eta(3) = (1 - 2^{1-3})\zeta(3) = \left(1 - \frac{1}{4}\right)\zeta(3) = \frac{3}{4}\zeta(3)$$

7. $\mathrm{I} = \int_0^1 \frac{\tanh^{-1}(\sqrt[4]{x})}{x} dx = \int_0^1 \frac{1}{x} \tanh^{-1}(\sqrt[4]{x})\, dx$, we have:

$$\tanh^{-1} y = \sum_{n=1}^{+\infty} \frac{y^{2n-1}}{2n-1}; \text{ then } \mathrm{I} = \int_0^1 \frac{1}{x} \sum_{n=1}^{+\infty} \frac{\left(x^{\frac{1}{4}}\right)^{2n-1}}{2n-1} dx = \int_0^1 \frac{1}{x} \sum_{n=1}^{+\infty} \frac{x^{\frac{2n-1}{4}}}{2n-1} dx \text{; then we get:}$$

$$\mathrm{I} = \int_0^1 \sum_{n=1}^{+\infty} \frac{x^{\frac{2n-1}{4}-1}}{2n-1} dx = \int_0^1 \sum_{n=1}^{+\infty} \frac{x^{\frac{2n-5}{4}}}{2n-1} dx = \sum_{n=1}^{+\infty} \frac{1}{2n-1} \int_0^1 x^{\frac{2n-5}{4}} dx; \text{ then we get:}$$

$$\mathrm{I} = \sum_{n=1}^{+\infty} \frac{1}{2n-1} \left[\frac{4x^{2n-1}}{2n-1}\right]_0^1 = \sum_{n=1}^{+\infty} \frac{1}{2n-1} \cdot \frac{4}{2n-1} = 4 \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = 4\left(\frac{\pi^2}{8}\right) = \frac{\pi^2}{2}$$

8. $\mathrm{I} = \int_0^1 x^{a \ln x} dx = \int_0^1 e^{\ln(x^{a \ln x})} dx = \int_0^1 e^{a \ln x \ln x} dx = \int_0^1 e^{a(\ln x)^2} dx$, then we get:

$$\mathrm{I} = \int_0^1 \sum_{n=0}^{+\infty} \frac{(a \ln^2 x)^n}{n!} dx = \sum_{n=0}^{+\infty} \frac{a^n}{n!} \int_0^1 \ln^{2n} x\, dx$$

Let $t = -\ln x$, then $x = e^{-t}$ and $dx = -e^{-t} dt$, for $x = 0$, $t = +\infty$ and for $x = 1$, $t = 0$, then:

$$\mathrm{I} = \sum_{n=0}^{+\infty} \frac{a^n}{n!} \int_{+\infty}^{0} (-t)^{2n} (-e^{-t}) dt = \sum_{n=0}^{+\infty} \frac{a^n}{n!} \int_0^{+\infty} e^{-t} . t^{2n} dt = \sum_{n=0}^{+\infty} \frac{a^n}{n!} \Gamma(2n+1); \text{ then we get:}$$

$$\mathrm{I} = \sum_{n=0}^{+\infty} \frac{a^n}{n!} . (2n)! = 1 + 2a + 3 \times 4a^2 + 4 \times 5 \times 6a^3 + 5 \times 6 \times 7 \times 8a^4 + \cdots$$

9. $\mathrm{I} = \int_0^{+\infty} \ln(1 + e^{-x} + e^{-2x} + \cdots)\, dx$, then:

$$\mathrm{I} = \int_0^{+\infty} \ln\left(\sum_{k=0}^{+\infty} (e^{-x})^n\right) dx = \int_0^{+\infty} \ln\left(\frac{1}{1 - e^{-x}}\right) dx = -\int_0^{+\infty} \ln(1 - e^{-x}) dx; \text{ then we get:}$$

$$\mathrm{I} = -\int_0^{+\infty} \left(-\sum_{k=1}^{+\infty} \frac{(e^{-x})^k}{k}\right) dx = \int_0^{+\infty} \sum_{k=1}^{+\infty} \frac{e^{-kx}}{k} dx = \sum_{k=1}^{+\infty} \frac{1}{k} \int_0^{+\infty} e^{-kx} dx = \sum_{k=1}^{+\infty} \frac{1}{k} \left[-\frac{1}{k} e^{-kx}\right]_0^{+\infty}$$

$$\mathrm{I}=\sum_{k=1}^{+\infty}\frac{1}{k}\left(\frac{1}{k}\right)=\sum_{k=1}^{+\infty}\frac{1}{k^2}=\zeta(2)=\frac{\pi^2}{6}$$

10. $\mathrm{I}=\int_0^{\frac{\pi}{2}} e^{\cos x}\,dx$

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\sum_{n=0}^{+\infty}\frac{(\cos x)^n}{n!}dx=\sum_{n=0}^{+\infty}\frac{1}{n!}\int_0^{\frac{\pi}{2}}\cos^n x\,dx=\sum_{n=0}^{+\infty}\frac{1}{n!}\int_0^{\frac{\pi}{2}}\sin^{2\left(\frac{1}{2}\right)-1}x\cos^{2\left(\frac{n+1}{2}\right)-1}x\,dx\text{ ; then:}$$

$$\mathrm{I}=\sum_{n=0}^{+\infty}\frac{1}{n!}.\frac{1}{2}\mathrm{B}\left(\frac{1}{2},\frac{n+1}{2}\right)=\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{n+2}{2}\right)}=\frac{\sqrt{\pi}}{2}\sum_{n=0}^{+\infty}\frac{1}{n!}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

11. $\mathrm{I}=\int_1^{+\infty}\frac{\ln^2 x}{x^2+x\ln x}dx=\int_1^{+\infty}\left[\left(\frac{\ln x}{x}-1\right)+\frac{1}{1+\frac{\ln x}{x}}\right]dx$

$$\text{But }\frac{1}{1+\left(\frac{\ln x}{x}\right)}=\sum_{n=0}^{+\infty}(-1)^n x^n=1-\frac{\ln x}{x}+\left(\frac{\ln x}{x}\right)^2-\left(\frac{\ln x}{x}\right)^3+\cdots\text{ then we get:}$$

$$\mathrm{I}=\int_1^{+\infty}\left[\left(\frac{\ln x}{x}-1\right)+1-\frac{\ln x}{x}+\left(\frac{\ln x}{x}\right)^2-\left(\frac{\ln x}{x}\right)^3+\cdots\right]dx\text{ ; so we get:}$$

$$\mathrm{I}=\int_1^{+\infty}\left[\left(\frac{\ln x}{x}\right)^2-\left(\frac{\ln x}{x}\right)^3+\left(\frac{\ln x}{x}\right)^4-\left(\frac{\ln x}{x}\right)^5+\cdots\right]dx=\int_1^{+\infty}\sum_{n=2}^{+\infty}(-1)^n\left(\frac{\ln x}{x}\right)^n dx\text{ ; then:}$$

$$\mathrm{I}=\sum_{n=2}^{+\infty}(-1)^n\int_1^{+\infty}\left(\frac{\ln x}{x}\right)^n dx\text{ ; but }\int_1^{+\infty}\left(\frac{\ln x}{x}\right)^n dx=\frac{n!}{(n-1)^{n+1}}\text{ (Proved in chapter 4)}$$

$$\text{Therefore we get: }\mathrm{I}=\sum_{n=2}^{+\infty}(-1)^n\frac{n!}{(n-1)^{n+1}}=\frac{2!}{1^3}-\frac{3!}{2^4}+\frac{4!}{3^5}-\frac{5!}{4^6}+\frac{6!}{5^7}-\cdots$$

12. $\mathrm{I}=\int_0^1\ln\left(\frac{1+x}{1-x}\right).\frac{dx}{x}=\int_0^1[\ln(1+x)-\ln(1-x)].\frac{dx}{x}=\int_0^1\frac{\ln(1+x)}{x}dx-\int_0^1\frac{\ln(1-x)}{x}dx$

$$\int_0^1\frac{\ln(1+x)}{x}dx=\int_0^1\frac{1}{x}\left(\sum_{n=0}^{\infty}(-1)^{n+1}\frac{x^n}{n}\right)dx=\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\int_0^1 x^{n-1}dx=\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{1}{n}\right)$$

$$=\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2}=\frac{\pi^2}{12}$$

$$\int_0^1\frac{\ln(1-x)}{x}dx=\int_0^1\frac{1}{x}\left(-\sum_{n=1}^{+\infty}\frac{x^n}{n}\right)dx=-\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 x^{n-1}dx=-\sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{1}{n}\right)=-\sum_{n=1}^{+\infty}\frac{1}{n^2}=-\frac{\pi^2}{6}$$

$$\text{Therefore: }\mathrm{I}=\frac{\pi^2}{12}+\frac{\pi^2}{6}=\frac{\pi^2}{4}$$

13. $\mathrm{I}=\int_0^{+\infty}x\tan^{-1}(e^{-\gamma x})\,dx$

$$\text{We have }\arctan y=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}x^{2n+1}\text{ ; then we get}$$

$$\mathrm{I}=\int_0^{+\infty} x\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\left(e^{-\gamma x}\right)^{2n+1}dx=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty} x.e^{-\gamma(2n+1)x}\,dx$$

Let $t=\gamma(2n+1)x$, then $dt=\gamma(2n+1)dx$, so $dx=\frac{1}{\gamma(2n+1)}dt$, then we get**:**

$$\mathrm{I}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty}\frac{t}{\gamma(2n+1)}.e^{-t}.\frac{1}{\gamma(2n+1)}dx=\frac{1}{\gamma^2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\int_0^{+\infty} t.e^{-t}dt;\ \text{ then:}$$

$$\mathrm{I}=\frac{1}{\gamma^2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\int_0^{+\infty} t^{2-1}.e^{-t}dt=\frac{1}{\gamma^2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\Gamma(2)=\frac{1}{\gamma^2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}=\frac{1}{\gamma^2}\beta(3)$$

$$\text{Therefore, we get: }\ \mathrm{I}=\frac{1}{\gamma^2}\left(\frac{\pi^2}{32}\right)=\frac{\pi^2}{32\gamma^2}$$

14. $\mathrm{I}=\int_0^1\frac{\tan^{-1}(\ln x)}{\ln x}dx$, we have**:**

$$\tan^{-1}y=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}y^{2n+1};\ \text{ then we get }\tan^{-1}(\ln x)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}(\ln x)^{2n+1}$$

$$\text{and then we get: }\frac{\tan^{-1}(\ln x)}{\ln x}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}(\ln x)^{2n};\ \text{ so we get:}$$

$$\int_0^1\frac{\tan^{-1}(\ln x)}{\ln x}dx=\int_0^1\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}(\ln x)^{2n}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^1(\ln x)^{2n}\,dx$$

Let $u=-\ln x$, then $x=e^{-u}$ and $dx=-e^{-u}du$, for $x=0$, then $u=+\infty$ and for $x=1$, then $u=0$, so we get**:**

$$\mathrm{I}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_{+\infty}^{0}(-t)^{2n}\left(-e^{-t}\right)dt=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty} t^{2n}\,e^{-t}dt=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\Gamma(2n+1)$$

$$\text{Therefore, we get }\ \mathrm{I}=\sum_{n=0}^{+\infty}\frac{(-1)^n(2n)!}{2n+1}$$

15. $\mathrm{I}=\int_0^1\frac{x^{a-1}\ln x}{1-x^{2a}}dx$, let $\ln x=-t$, then $x=e^{-t}$ and $dx=-e^{-t}dt$, for $x=0$, then $t=+\infty$ and for $x=1$, then $t=0$, then we get**:** $\mathrm{I}=\int_{+\infty}^{0}\frac{-t.e^{-at+1}}{1-e^{-2at}}\left(-e^{-t}dt\right)=-\int_0^{+\infty} t.\frac{e^{-at}}{1-e^{-2at}}dt$, then**:**

$$\mathrm{I}=-\int_0^{+\infty} t.e^{-at}\frac{1}{1-e^{-2at}}dt=-\int_0^{+\infty} t.e^{-at}\sum_{n=0}^{\infty}\left(e^{-2at}\right)^n dt=-\int_0^{+\infty} t.e^{-at}\sum_{n=1}^{\infty}\left(e^{-2at}\right)^{n-1}dt$$

$$\mathrm{I}=-\int_0^{+\infty} t.e^{-at}\sum_{n=1}^{\infty}e^{-2atn+2at}\,dt=-\int_0^{+\infty} t.\sum_{n=1}^{\infty}e^{-2atn+2at-at}\,dt=-\int_0^{+\infty} t.\sum_{n=1}^{\infty}e^{-2atn+at}\,dt$$

$$\text{Then we get: }\ \mathrm{I}=-\int_0^{+\infty} t.\sum_{n=1}^{\infty}e^{-at(2n-1)}\,dt=-\sum_{n=1}^{\infty}\int_0^{+\infty} t.e^{-at(2n-1)}dt$$

Let $u=at(2n-1)$, then $du=a(2n-1)dt$, then we get**:**

$$\mathrm{I} = -\sum_{n=1}^{\infty}\int_{0}^{+\infty} \frac{u}{a(2n-1)}.e^{-u}.\frac{du}{a(2n-1)} = -\frac{1}{a^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\int_{0}^{+\infty} ue^{-u}du \;\;\&\;\; \int_{0}^{+\infty} ue^{-u}du = 1$$

Then we get $\mathrm{I} = -\frac{1}{a^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = -\frac{1}{a^2}\left(\frac{\pi^2}{8}\right) = -\frac{\pi^2}{8a^2}$ with $a > 0$

16. $\mathrm{I} = \int_0^{+\infty} \ln \sqrt[n]{1-e^{-x}}\, dx = \int_0^{+\infty} \ln(1-e^{-x})^{\frac{1}{n}}\, dx = \frac{1}{n}\int_0^{+\infty} \ln(1-e^{-x})\, dx$

Let $u = e^{-x}$, then $du = -e^{-x}dx = -udx$, for $x = 0$, then $u = 1$ and for $x = +\infty$, then $u = 0$, so we get**:**

$\mathrm{I} = \frac{1}{n}\int_1^0 \ln(1-u)\left(-\frac{1}{u}\right) du = \frac{1}{n}\int_0^1 \frac{\ln(1-u)}{u} du$, then we get**:**

$$\mathrm{I} = \frac{1}{n}\int_0^1 \frac{1}{u}\ln(1-u)\, du = \frac{1}{n}\int_0^1 \frac{1}{u}\left(-\sum_{k=1}^{+\infty}\frac{u^k}{k}\right) du = -\frac{1}{n}\sum_{k=1}^{+\infty}\frac{1}{k}\int_0^1 u^{k-1}du = -\frac{1}{n}\sum_{k=1}^{+\infty}\frac{1}{k}\left(\frac{1}{k}\right); \text{then}$$

$$\mathrm{I} = -\frac{1}{n}\sum_{k=1}^{+\infty}\frac{1}{k^2} = -\frac{1}{n}\left(\frac{\pi^2}{6}\right) = -\frac{\pi^2}{6n}$$

17. $\mathrm{I} = \int_0^1 \frac{1}{x}.\ln\left(\frac{1-x^m}{1+x^n}\right) dx = \int_0^1 \frac{1}{x}[\ln(1-x^m) - \ln(1+x^n)]\, dx$, then we get**:**

$\mathrm{I} = \int_0^1 \frac{1}{x}.\ln(1-x^m)\, dx - \int_0^1 \frac{1}{x}.\ln(1+x^n)\, dx$

$$\mathrm{I} = \int_0^1 \frac{1}{x}.\left[-\sum_{k=1}^{+\infty}\frac{1}{k}(x^m)^k\right] dx - \int_0^1 \frac{1}{x}.\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}(x^n)^k\, dx; \text{ then we get:}$$

$$\mathrm{I} = \int_0^1 \frac{1}{x}.\left[-\sum_{k=1}^{+\infty}\frac{1}{k}x^{mk}\right] dx - \int_0^1 \frac{1}{x}.\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}x^{nk}\, dx$$

$$\mathrm{I} = -\sum_{k=1}^{+\infty}\frac{1}{k}\int_0^1 x^{mk-1}dx - \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}\int_0^1 x^{nk-1}dx = -\sum_{k=1}^{+\infty}\frac{1}{k}\left[\frac{x^{mk}}{mk}\right]_0^1 - \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}\left[\frac{x^{nk}}{nk}\right]_0^1$$

$$\mathrm{I} = -\frac{1}{m}\sum_{k=1}^{+\infty}\frac{1}{k^2} - \frac{1}{n}\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k^2} = -\frac{1}{m}\left(\frac{\pi^2}{6}\right) - \frac{1}{n}\left(\frac{\pi^2}{12}\right) = -\frac{\pi^2}{12}\left(\frac{2}{m}+\frac{1}{n}\right)$$

18. $\mathrm{I} = \int_0^1 \frac{\ln(1+\sqrt[n]{x})}{x} dx = \int_0^1 \frac{\ln\left(1+x^{\frac{1}{n}}\right)}{x} dx$

We have: $\ln(1+t) = \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}t^k$; then $\ln\left(1+x^{\frac{1}{n}}\right) = \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}\left(x^{\frac{1}{n}}\right)^k$; then:

$\ln\left(1+x^{\frac{1}{n}}\right) = \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}x^{\frac{k}{n}}$; then we get: $\frac{\ln\left(1+x^{\frac{1}{n}}\right)}{x} = \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}x^{\frac{k}{n}}.\frac{1}{x}$; so we get:

$$\frac{\ln\left(1+x^{\frac{1}{n}}\right)}{x} = \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}x^{\frac{k}{n}-1}; \text{ so } \mathrm{I} = \int_0^1 \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}x^{\frac{k}{n}-1} = \sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}\int_0^1 x^{\frac{k}{n}-1}dx$$

So we get: $\mathrm{I}=\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}\left[\frac{1}{\frac{k}{n}}x^{\frac{k}{n}}\right]_0^1=\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k}.\frac{n}{k}=n\sum_{k=1}^{+\infty}\frac{(-1)^{k-1}}{k^2}=\frac{n\pi^2}{12}$

19. $\mathrm{I}=\int_0^{+\infty}\frac{x}{\cosh x}dx=\int_0^{+\infty}\frac{x}{\frac{1}{2}(e^x+e^{-x})}dx=2\int_0^{+\infty}\frac{x}{e^x+e^{-x}}dx=2\int_0^{+\infty}\frac{xe^{-x}}{1+e^{-2x}}dx$

Then we get $\mathrm{I}=2\int_0^{+\infty}\frac{xe^{-x}}{1-(-e^{-2x})}dx$, so

$$\mathrm{I}=2\int_0^{+\infty}xe^{-x}\left(\sum_{n=0}^{+\infty}(-e^{-2x})^n\right)dx=2\int_0^{+\infty}xe^{-x}\left(\sum_{n=0}^{+\infty}(-1)^n e^{-2nx}\right)dx;\ \text{ then we get:}$$

$$\mathrm{I}=2\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}xe^{-(2n+1)x}dx$$

Let $u=(2n+1)x$, then $du=(2n+1)dx$, then we get:

$$\mathrm{I}=2\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}\frac{u}{2n+1}e^{-u}.\frac{1}{2n+1}du=2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^{+\infty}u^{2-1}.e^{-u}du\,;\text{then}$$

$$\mathrm{I}=2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\Gamma(2)=2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}=2\mathrm{G}$$

20. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\sin 2x\ln(\sin x)\,dx=\int_0^{\frac{\pi}{2}}2\sin x\cos x\ln\sqrt{1-\cos^2 x}\,dx$

$\mathrm{I}=\int_0^{\frac{\pi}{2}}2\sin x\cos x\ln(1-\cos^2 x)^{\frac{1}{2}}dx=\int_0^{\frac{\pi}{2}}2\sin x\cos x.\frac{1}{2}\ln(1-\cos^2 x)\,dx$, then we get:

$\mathrm{I}=\int_0^{\frac{\pi}{2}}\sin x\cos x\ln(1-\cos^2 x)\,dx$, let $t=\cos^2 x$, then $dt=-2\sin x\cos x\,dx$, for $x=0$, then $t=1$ and for $x=\frac{\pi}{2}$, then $t=0$, so we get: $\mathrm{I}=\int_1^0-\frac{1}{2}\ln(1-t)\,dt=\frac{1}{2}\int_0^1\ln(1-t)\,dt$

$$\mathrm{I}=\frac{1}{2}\int_0^1\left(-\sum_{n=1}^{+\infty}\frac{t^n}{n}\right)dt=-\frac{1}{2}\int_0^1\sum_{n=1}^{+\infty}\frac{t^n}{n}dt=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 t^n dt=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}.\frac{1}{n+1}$$

Then we get: $\mathrm{I}=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n(n+1)}=-\frac{1}{2}\left(\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\cdots\right)$

Therefore we get: $\mathrm{I}=-\frac{1}{2}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots\right]=-\frac{1}{2}$

21. $\mathrm{I}=\int_0^{+\infty}xe^{-x}\coth\left(\frac{x}{2}\right)dx=\int_0^{+\infty}xe^{-x}\left(\frac{e^{\frac{x}{2}}+e^{-\frac{x}{2}}}{e^{\frac{x}{2}}-e^{-\frac{x}{2}}}\right)dx=\int_0^{+\infty}xe^{-x}\left(\frac{e^x+1}{e^x-1}\right)dx$

$\mathrm{I}=\int_0^{+\infty}xe^{-x}\left(\frac{1+e^{-x}}{1-e^{-x}}\right)dx=\int_0^{+\infty}\frac{xe^{-x}}{1-e^{-x}}dx+\int_0^{+\infty}\frac{xe^{-2x}}{1-e^{-x}}dx$, then we get:

$$\mathrm{I}=\int_0^{+\infty}xe^{-x}\sum_{n=0}^{+\infty}(e^{-x})^n\,dx+\int_0^{+\infty}xe^{-2x}\sum_{n=0}^{+\infty}(e^{-x})^n\,dx\,;\ \text{ then we get:}$$

$$\mathrm{I}=\int_0^{+\infty}xe^{-x}\sum_{n=0}^{+\infty}e^{-nx}\,dx+\int_0^{+\infty}xe^{-2x}\sum_{n=0}^{+\infty}e^{-nx}\,dx$$

$$\mathrm{I} = \int_0^{+\infty} xe^{-x}\sum_{n=1}^{+\infty} e^{-(n-1)x}\,dx + \int_0^{+\infty} xe^{-2x}\sum_{n=1}^{+\infty} e^{-(n-1)x}\,dx$$

$$\mathrm{I} = \int_0^{+\infty} x\sum_{n=1}^{+\infty} e^{-nx}\,dx + \int_0^{+\infty} x\sum_{n=1}^{+\infty} e^{-(n+1)x}\,dx = \sum_{n=1}^{+\infty}\int_0^{+\infty} xe^{-nx}dx + \sum_{n=1}^{+\infty}\int_0^{+\infty} xe^{-(n+1)x}dx$$

Using integration by parts we get: $\int_0^{+\infty} xe^{-nx}dx = \frac{1}{n^2}$ and $\int_0^{+\infty} xe^{-(n+1)x}dx = \frac{1}{(n+1)^2}$

$$\text{Then I} = \sum_{n=1}^{+\infty}\frac{1}{n^2} + \sum_{n=1}^{+\infty}\frac{1}{(n+1)^2} = \sum_{n=1}^{+\infty}\frac{1}{n^2} + \sum_{n=2}^{+\infty}\frac{1}{n^2} + \sum_{n=1}^{+\infty}\frac{1}{n^2} + \sum_{n=1}^{+\infty}\frac{1}{n^2} - \frac{1}{1^2} = 2\sum_{n=1}^{+\infty}\frac{1}{n^2} - 1$$

$$\text{Therefore we get: I} = 2\left(\frac{\pi^2}{6}\right) - 1 = \frac{\pi^2}{3} - 1$$

22. $\mathrm{I} = \int_1^e \frac{\ln(1+\ln(1-\ln x))}{x}dx$, let $\ln(1-\ln x) = -t$, then $1 - \ln x = e^{-t}$, so $\ln x = 1 - e^{-t}$ and $x = e^{1-e^{-t}}$, so $dx = e^{-t}.e^{1-e^{-t}}dt$, for $x = 1$, $t = -\ln(1-\ln 1) = -\ln 1 = 0$ and for $x = e$, $t = -\ln(1-\ln e) = -\ln(1-1) = -\ln 0 = +\infty$, then we get:

$\mathrm{I} = \int_0^{+\infty}\frac{\ln(1-t)}{e^{1-e^{-t}}}e^{-t}.e^{1-e^{-t}}dt = \int_0^{+\infty} e^{-t}\ln(1-t)$, then we get:

$$\mathrm{I} = \int_0^{+\infty} e^{-t}\left(-\sum_{n=1}^{+\infty}\frac{t^n}{n}\right)dt = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} e^{-t}t^n dt = -\sum_{n=1}^{+\infty}\frac{1}{n}.\Gamma(n+1) = -\sum_{n=1}^{+\infty}\frac{1}{n}.n! \quad \text{then}$$

$$\mathrm{I} = -\sum_{n=1}^{+\infty}\frac{1}{n}.n(n-1)! = -\sum_{n=1}^{+\infty}(n-1)! = -\{0! + 1! + 2! + 3! + \cdots\}$$

23. $\mathrm{I} = \int_{-\infty}^{+\infty}\frac{e^{-ax}}{e^{e^{-x}-1}}dx$, let $u = e^{-x}$, then $du = -e^{-x}dx = -udx$, for $x = -\infty$, then $u = +\infty$ and for $x = +\infty$, then $u = 0$, so we get:

$\mathrm{I} = \int_{+\infty}^{0}\frac{u^a}{e^{u-1}}\left(-\frac{du}{u}\right) = \int_0^{+\infty}\frac{u^{a-1}}{e^{u-1}}du$, then we get:

$$\mathrm{I} = \int_0^{+\infty} u^{a-1}.\frac{e^{-u}}{1-e^{-u}}du = \int_0^{+\infty} u^{a-1}\sum_{n=1}^{+\infty}(e^{-u})^n\,du = \int_0^{+\infty} u^{a-1}\sum_{n=1}^{+\infty} e^{-nu}\,du; \text{ then we get}$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\int_0^{+\infty} u^{a-1}.e^{-nu}du\,; \quad \text{let } t = nu \Longrightarrow dt = ndu \Longrightarrow du = \frac{1}{n}dt; \text{ then we get:}$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{t^{a-1}}{n^{a-1}}.e^{-t}.\frac{1}{n}dt = \sum_{n=1}^{+\infty}\frac{1}{n^a}\int_0^{+\infty} e^{-t}.t^{a-1}dt = \zeta(a)\Gamma(a)$$

24. $\mathrm{I} = \int_0^{\frac{\pi}{4}}\ln(\tan x)\,dx$, let $y = \tan x$, then $dy = (1+\tan^2 x)dx = (1+y^2)dx$

For $x = 0$, $y = 0$ and for $x = \frac{\pi}{4}$, $y = 1$, then we get:

$\mathrm{I} = \int_0^1 \ln y\,.\frac{1}{1+y^2}dy = \int_0^1\frac{\ln y}{1+y^2}dy$, let $\ln y = -z$, then $y = e^{-z}$ and $dy = -e^{-z}dz$

For $y = 0$, $z = +\infty$ and for $y = 1$, then ,$z = 0$, then:

$\mathrm{I} = \int_{+\infty}^{0}\frac{-z}{1+e^{-2z}}(-e^{-z}dz) = -\int_0^{+\infty}\frac{ze^{-z}}{1+e^{-2z}}dz = -\int_0^{+\infty} ze^{-z}.\frac{1}{1+e^{-2z}}dz$, then we get:

$$\mathrm{I} = -\int_0^{+\infty} ze^{-z}.\frac{1}{1-(-e^{-2z})}dz = -\int_0^{+\infty} ze^{-z}\sum_{n-0}^{+\infty}(-e^{-2z})^n dz = -\int_0^{+\infty} ze^{-z}\sum_{n=0}^{+\infty}(-1)^n e^{-2nz}dz$$

Then we get: $\mathrm{I} = -\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty} ze^{-(2n+1)z}dz$

Let $t = (2n+1)z$, then $dt = (2n+1)dz$ and $dz = \frac{1}{2n+1}dt$, so we get**:**

$$\mathrm{I} = -\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}\frac{t}{2n+1}.e^{-t}.\frac{1}{2n+1}dz = -\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^{+\infty} t.e^{-t}dt\,;\ \text{then we get:}$$

$$\mathrm{I} = -\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^{+\infty} t^{2-1}.e^{-t}dt = -\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\Gamma(2) = -\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2} = -\mathrm{G}$$

25. $\mathrm{I} = \int_0^1 \ln x\ln(1-x)\,dx$

$$\mathrm{I} = \int_0^1 \ln x\left(-\sum_{n=1}^{+\infty}\frac{x^n}{n}\right)dx = -\int_0^1 \ln x\sum_{n=1}^{+\infty}\frac{1}{n}x^n\,dx = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 x^n\ln x\,dx$$

Now let us evaluate $\int_0^1 x^n \ln x\,dx$, we will use integration by parts**:**

Let $u = \ln x$, then $u' = \frac{1}{x}$ and let $v' = x^n$, then $v = \frac{1}{n+1}x^{n+1}$, then we get**:**

$$\int_0^1 x^n\ln x\,dx = \left[\frac{1}{n+1}x^{n+1}\ln x\right]_0^1 - \int_0^1\frac{1}{n+1}x^{n+1}.\frac{1}{x}dx = -\frac{1}{n+1}\int_0^1 x^n dx\,;\ \text{then we get:}$$

$$\int_0^1 x^n\ln x\,dx = -\frac{1}{n+1}\left[\frac{1}{n+1}x^{n+1}\right]_0^1 = -\frac{1}{n+1}.\frac{1}{n+1} = -\frac{1}{(n+1)^2}\,;\text{then:}$$

$$\mathrm{I} = -\sum_{n=1}^{+\infty}\frac{1}{n}\left(-\frac{1}{(n+1)^2}\right) = \sum_{n=1}^{+\infty}\frac{1}{n(n+1)^2} = \sum_{n=1}^{+\infty}\frac{(n+1)-n}{n(n+1)^2} = \sum_{n=1}^{+\infty}\left[\frac{1}{n(n+1)} - \frac{1}{(n+1)^2}\right]$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\left[\frac{(n+1)-n}{n(n+1)} - \frac{1}{(n+1)^2}\right]\sum_{n=1}^{+\infty}\left[\frac{1}{n} - \frac{1}{n+1} - \frac{1}{(n+1)^2}\right]$$

$$\mathrm{I} = \left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots - \left[-1 + \sum_{n=1}^{+\infty}\frac{1}{n^2}\right] = 1 + 1 - \sum_{n=1}^{+\infty}\frac{1}{n^2} = 2 - \sum_{n=1}^{+\infty}\frac{1}{n^2}$$

Therefore, we get: $\mathrm{I} = 2 - \frac{\pi^2}{6}$

26. $\mathrm{I} = \int_0^1 x^{-nx}\,dx = \int_0^1 e^{\ln(x^{-nx})}dx = \int_0^1 e^{-nx\ln x}dx$

We have: $e^{-y} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{n!}y^n = 1 - \left(y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \cdots\right) = 1 - \sum_{m=1}^{+\infty}\frac{(-1)^{m+1}}{m!}y^m\,$; so:

$e^{-nx\ln x} = 1 - \sum_{m=1}^{+\infty}\frac{(-1)^{m+1}}{m!}(nx\ln x)^m\,$; then we get:

$$\mathrm{I}=\int_0^1\left[1-\sum_{m=1}^{+\infty}\frac{(-1)^{m+1}}{m!}(nx\ln x)^m\right]dx=1-\sum_{m=1}^{+\infty}\frac{(-1)^{m+1}}{m!}.n^m\int_0^1 x^m.\ln^m x\,dx$$

Now evaluating: $\int_0^1 x^m.\ln^m x\,dx$, let $\ln x=-t$, then $x=e^{-t}$ and $dx=-e^{-t}dt$, for $x=0$, $t=+\infty$ and for $x=1$, $t=0$, then**:**

$\int_0^1 x^m.\ln^m x\,dx=\int_{+\infty}^0 e^{-mt}(-t)^m(-e^{-t})dt=\int_0^{+\infty}(-1)^m e^{-t(m+1)}t^m dt$

Let $v=t(m+1)$, then $dv=(m+1)dt$, so**:**

$\int_0^1 x^m.\ln^m x\,dx=\int_0^{+\infty}(-1)^m e^{-v}\left(\frac{v}{m+1}\right)^m\frac{1}{m+1}dv=\frac{(-1)^m}{(m+1)^{m+1}}\int_0^{+\infty}e^{-w}.w^m dw$

Then $\int_0^1 x^m.\ln^m x\,dx=\frac{(-1)^m}{(m+1)^{m+1}}\Gamma(m+1)=\frac{(-1)^m}{(m+1)^{m+1}}m!$

$\mathrm{I}=1-\sum_{m=1}^{+\infty}\frac{(-1)^{m+1}}{m!}.n^m\int_0^1 x^m.\ln^m x\,dx=1-\sum_{m=1}^{+\infty}\frac{(-1)^{m+1}}{m!}.n^m.\frac{(-1)^m}{(m+1)^{m+1}}m!$; so we get

$\mathrm{I}=1-\sum_{m=1}^{+\infty}\frac{(-1)^{2m+1}}{(m+1)^{m+1}}.n^m=1+\sum_{m=1}^{+\infty}\frac{n^m}{(m+1)^{m+1}}$

27. $\mathrm{I}=\int_0^1\frac{x}{1-x^2}.\ln^k x\,dx$, let $\ln x=-t$, then $x=e^{-t}$ and $dx=-e^{-t}dt$, for $x=0$, $t=+\infty$ and for $x=1$, then $t=0$, so we get**:**

$\mathrm{I}=\int_{+\infty}^0\frac{e^{-t}}{1-e^{-2t}}.(-t)^k\,(-e^{-t}dt)=\int_0^{+\infty}(-1)^k t^k.\frac{e^{-2t}}{1-e^{-2t}}dt$, then we get**:**

$$\mathrm{I}=(-1)^k\int_0^{+\infty}t^k.\sum_{n=1}^{+\infty}(e^{-2t})^n\,dt=(-1)^k\int_0^{+\infty}t^k.\sum_{n=1}^{+\infty}e^{-2tn}\,dt=(-1)^k\sum_{n=1}^{+\infty}\int_0^{+\infty}e^{-2nt}.t^k dt$$

Let $u=2nt$, then $d=2ndt$, then we get**:**

$$\mathrm{I}=(-1)^k\sum_{n=1}^{+\infty}\int_0^{+\infty}e^{-u}.\left(\frac{u}{2n}\right)^k.\frac{1}{2n}du=(-1)^k\sum_{n=1}^{+\infty}\int_0^{+\infty}e^{-u}.u^k.\frac{1}{(2n)^{k+1}}du;\ \text{ then we get:}$$

$$\mathrm{I}=(-1)^k\sum_{n=1}^{+\infty}\frac{1}{(2n)^{k+1}}\int_0^{+\infty}e^{-u}.u^k.du=(-1)^k\sum_{n=1}^{+\infty}\frac{1}{(2n)^{k+1}}\int_0^{+\infty}e^{-u}.u^{(k+1)-1}.du;\ \text{ then}$$

$$\mathrm{I}=(-1)^k\sum_{n=1}^{+\infty}\frac{1}{(2n)^{k+1}}\Gamma(k+1)=(-1)^k\sum_{n=1}^{+\infty}\frac{1}{(2n)^{k+1}}k!=(-1)^k k!\sum_{n=1}^{+\infty}\frac{1}{2^{k+1}n^{k+1}}$$

$$\mathrm{I}=\frac{(-1)^k k!}{2^{k+1}}\sum_{n=1}^{+\infty}\frac{1}{n^{k+1}}=\frac{(-1)^k k!}{2^{k+1}}\zeta(k+1)$$

28. $\mathrm{I}=\int_0^{+\infty}\frac{x^{n-1}}{e^x-1}dx=\int_0^{+\infty}x^{n-1}.\frac{1}{e^x-1}dx=\int_0^{+\infty}x^{n-1}.\frac{e^{-1}}{1-e^{-x}}dx$, then we get**:**

$$\mathrm{I}=\int_0^{+\infty}x^{n-1}.\sum_{n=1}^{+\infty}(e^{-x})^k dx=\int_0^{+\infty}x^{n-1}.\sum_{k=1}^{+\infty}e^{-kx}dx=\sum_{k=1}^{+\infty}\int_0^{+\infty}x^{n-1}e^{-kx}dx$$

Let $t=kx$, then $dt=kdx$, $dx=\frac{1}{k}dt$, then we get**:**

$$\mathrm{I}=\sum_{k=1}^{+\infty}\int_{0}^{+\infty}\left(\frac{t}{k}\right)^{n-1}e^{-t}.\frac{1}{k}dt=\sum_{k=1}^{+\infty}\int_{0}^{+\infty}\frac{1}{k^{n-1}}.t^{n-1}e^{-t}.\frac{1}{k}dt=\sum_{k=1}^{+\infty}\frac{1}{k^{n}}\int_{0}^{+\infty}e^{-t}t^{n-1}dt$$

With $\displaystyle\sum_{k=1}^{+\infty}\frac{1}{k^{n}}=\zeta(n)$ and $\displaystyle\int_{0}^{+\infty}e^{-t}t^{n-1}dt\,d=\Gamma(n)$; therefore $\mathrm{I}=\zeta(n)\,\Gamma(n)$

29. $\mathrm{I}=\int_0^{+\infty}\frac{\sqrt{x}}{\cosh(\sqrt{x})}dx$, let $y=\sqrt{x}$, so $x=y^2$, then $dx=2ydy$, then we get**:**

$$\mathrm{I}=\int_0^{+\infty}\frac{y}{\cosh y}.2ydy=2\int_0^{+\infty}\frac{y^2}{\cosh y}dy=2\int_0^{+\infty}\frac{y^2}{\frac{1}{2}(e^y+e^{-y})}dy=4\int_0^{+\infty}\frac{y^2}{e^y+e^{-y}}dy$$

$\mathrm{I}=4\int_0^{+\infty}\frac{y^2}{e^y+e^{-y}}.\frac{e^{-y}}{e^{-y}}dy=4\int_0^{+\infty}\frac{y^2e^{-y}}{1+e^{-2y}}dy$, then we get**:**

$$\mathrm{I}=4\int_{0}^{+\infty}y^2e^{-y}\sum_{n=0}^{+\infty}(-1)^n(e^{-2y})^ndy=4\int_{0}^{+\infty}y^2e^{-y}\sum_{n=0}^{+\infty}(-1)^ne^{-2ny}dy$$

$$\mathrm{I}=4\sum_{n=0}^{+\infty}(-1)^n\int_{0}^{+\infty}y^2.e^{-(2n+1)y}dy$$

Let $t=(2n+1)y$, then $dt=(2n+1)dy$, so $dy=\frac{1}{2n+1}dt$, then we get**:**

$$\mathrm{I}=4\sum_{n=0}^{+\infty}(-1)^n\int_{0}^{+\infty}\left(\frac{t}{2n+1}\right)^2.e^{-t}.\frac{1}{2n+1}dt=4\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\int_{0}^{+\infty}t^2e^{-t}dt\,;\ \text{ then we get:}$$

$$\mathrm{I}=4\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\int_{0}^{+\infty}t^{3-1}e^{-t}dt=4\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\Gamma(3)=8\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}$$

Then we get $\mathrm{I}=8\left(\frac{\pi^3}{32}\right)=\frac{\pi^3}{4}$

30. $\mathrm{I}=\int_0^{a}\tanh^{-1}\left(\frac{x}{a}\right).\frac{dx}{x}$, let $u=\frac{x}{a}$, then $x=au$ and $dx=adu$, for $x=0$, $u=0$ and for $x=a$, then $u=1$, so we get**:**

$$\mathrm{I}=\int_0^{1}\tanh^{-1}u\frac{adu}{au}=\int_0^{1}\frac{\tanh^{-1}u}{u}du=\frac{1}{2}\int_0^1\ln\left(\frac{1+u}{1-u}\right)\frac{du}{u}=\frac{1}{2}\int_0^1[\ln(1+u)-\ln(1-u)]\frac{1}{u}du$$

$\mathrm{I}=\frac{1}{2}\int_0^1\frac{\ln(1+u)}{u}du-\frac{1}{2}\int_0^1\frac{\ln(1-u)}{u}du$, then we get**:**

$$\mathrm{I}=\frac{1}{2}\int_{0}^{1}\frac{1}{u}\times\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}u^n-\frac{1}{2}\int_{0}^{1}\frac{1}{u}\times\sum_{n=1}^{+\infty}\frac{-1}{n}u^n=\frac{1}{2}\int_{0}^{1}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}u^{n-1}+\frac{1}{2}\int_{0}^{1}\sum_{n=1}^{+\infty}\frac{1}{n}u^{n-1}$$

$$\mathrm{I}=\frac{1}{2}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\int_{0}^{1}u^{n-1}du+\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_{0}^{1}u^{n-1}du=\frac{1}{2}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2}+\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n^2}$$

$$\mathrm{I}=\frac{1}{2}\left(\frac{\pi^2}{8}-\frac{\pi^2}{24}\right)+\frac{1}{2}\left(\frac{\pi^2}{6}\right)=\frac{\pi^2}{8}$$

31. $\mathrm{I} = \int_0^{+\infty} \frac{\ln x}{1+e^{ax}} dx = \int_0^{+\infty} \ln x \,.\frac{1}{1+e^{ax}} dx = \int_0^{+\infty} \ln x \,.\frac{e^{-ax}}{1+e^{-ax}} dx = \int_0^{+\infty} \ln x \,.\frac{e^{-ax}}{1-(-e^{-ax})} dx$

$$\text{Then we get: I} = \int_0^{+\infty} \ln x \,.\sum_{n=1}^{+\infty}(-1)^{n+1}.\,e^{-akx} dx = \sum_{n=1}^{+\infty}(-1)^{n+1}\int_0^{+\infty} e^{-anx}.\ln x\, dx$$

Let $u = anx$, then $du = andx$, then we get**:**

$$\mathrm{I} = \sum_{n=1}^{+\infty}(-1)^{n+1}\int_0^{+\infty} e^{-u}.\ln\left(\frac{u}{an}\right).\frac{1}{an} du = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{an}\int_0^{+\infty} e^{-u}[\ln u - \ln(an)]du$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{an}\int_0^{+\infty} e^{-u}\ln u\, du - \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{an}\ln(an)\int_0^{+\infty} e^{-u} du\,;\ \text{ then we get:}$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{an}(-\gamma) - \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{an}(\ln a + \ln n)$$

$$\mathrm{I} = -\frac{\gamma}{a}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n} - \frac{\ln a}{a}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n} - \frac{1}{a}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\ln n$$

$$\text{But } \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n} = \ln 2\,;\text{ then we get I} = -\frac{\gamma}{a}\ln 2 - \frac{\ln a}{a}\ln 2 - \frac{1}{a}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\ln n$$

$$\text{But we have } \sum_{n=1}^{+\infty}\frac{(-1)^{n}\ln n}{n} = \gamma\ln 2 - \frac{1}{2}(\ln 2)^2;\ \text{ then we get}$$

$$\mathrm{I} = -\frac{\gamma}{2}\ln 2 - \frac{\ln a \ln 2}{a} - \frac{1}{a}\left[\frac{1}{2}(\ln 2)^2 - \gamma\ln 2\right] = -\frac{\ln a\ln 2}{a} - \frac{(\ln 2)^2}{2a} = -\frac{\ln 2}{2a}(2\ln a + \ln 2)$$

$$\text{Therefore; } \mathrm{I} = - -\frac{\ln 2\,(\ln a^2 + \ln 2)}{2a} = -\frac{\ln 2\times\ln(2a^2)}{2a} = \frac{1}{a}\ln\left(\frac{1}{\sqrt{2}}\right)\ln(2a^2)$$

32. $\mathrm{I} = \int_2^5 \left(\sum_{n=1}^{+\infty}\frac{n+1}{x^n}\right) dx$, we have**:**

$$\mathrm{S} = \sum_{n=1}^{+\infty}\frac{n+1}{x^n} = \frac{2}{x} + \frac{3}{x^2} + \frac{4}{x^3} + \cdots \Longrightarrow \frac{1}{x}\mathrm{S} = \frac{2}{x^2} + \frac{3}{x^3} + \frac{4}{x^4} + \cdots;\ \text{ then we get:}$$

$$\left(1-\frac{1}{x}\right)\mathrm{S} = \mathrm{S} - \frac{1}{x}\mathrm{S} = \left(\frac{2}{x} + \frac{3}{x^2} + \frac{4}{x^3} + \cdots\right) - \left(\frac{2}{x^2} + \frac{3}{x^3} + \frac{4}{x^4} + \cdots\right);\ \text{ then we get:}$$

$$\left(1-\frac{1}{x}\right)\mathrm{S} = \frac{2}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots = \frac{2}{x} + \left(\frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^4} + \cdots\right)$$

$$\left(1-\frac{1}{x}\right)\mathrm{S} = \frac{2}{x} + \frac{\frac{1}{x^2}}{1-\frac{1}{x}} = \frac{2}{x} + \frac{1}{x(x-1)} = \frac{2x-1}{x(x-1)};\ \forall \left|\frac{1}{x}\right| < 1 \Longrightarrow |x| > 1$$

$$\left(\frac{1-x}{x}\right)\mathrm{S} = \frac{2x-1}{x(x-1)} \Longrightarrow \mathrm{S} = \frac{x}{x-1}.\frac{2x-1}{x(x-1)} \Longrightarrow \mathrm{S} = \frac{2x-1}{(x-1)^2};\ \text{ then we get:}$$

$$\mathrm{I} = \int_2^5\left(\sum_{n=1}^{+\infty}\frac{n+1}{x^n}\right)dx = \int_2^5 \frac{2x-1}{(x-1)^2}dx$$

Let $y = x - 1$, then $x = y + 1$ and $dx = dy$, for $x = 2, y = 1$ and for $x = 5, y = 4$, then**:**

$$\mathrm{I}=\int_{1}^{4}\frac{2(y+1)-1}{y^2}dy=\int_{1}^{4}\left(\frac{2}{y}+\frac{1}{y^2}\right)dy=\left[2\ln y-\frac{1}{y}\right]_1^4=2\ln 4-\frac{1}{4}+1=\frac{3}{4}+4\ln 2$$

Therefore we get: $\mathrm{I}=\int_2^5\left(\sum_{n=1}^{+\infty}\frac{n+1}{x^n}\right)dx=\ln\left(16e^{\frac{3}{4}}\right)$

33. $\mathrm{I}=\int_0^1\left\{\frac{1}{x}\right\}dx=\int_0^1\left(\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor\right)dx$, then we get**:**

$$\mathrm{I}=\sum_{n=1}^{+\infty}\int_{\frac{1}{n+1}}^{\frac{1}{n}}\left(\frac{1}{x}-\left\lfloor\frac{1}{x}\right\rfloor\right)dx=\sum_{n=1}^{+\infty}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)-\int_{\frac{1}{n+1}}^{\frac{1}{n}}ndx\right]$$

Remark: $\frac{1}{n+1}\le x<\frac{1}{n}$; so $n\le\frac{1}{x}<n+1$; so $\left\lfloor\frac{1}{x}\right\rfloor=n$

$$\mathrm{I}=\sum_{n=1}^{+\infty}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)-n\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]=\sum_{n=1}^{+\infty}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)-n\left(\frac{1}{n(n+1)}\right)\right]$$

$$\mathrm{I}=\sum_{n=1}^{+\infty}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)-\frac{1}{n+1}\right]=\sum_{n=1}^{+\infty}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)\right]-\sum_{n=1}^{+\infty}\frac{1}{n+1}$$

$\mathrm{I}=\sum_{n=1}^{+\infty}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)\right]-\sum_{n=1}^{+\infty}\frac{1}{n+1}=\lim_{m\to+\infty}\sum_{n=1}^{m}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)\right]-\sum_{n=1}^{+\infty}\frac{1}{n+1}$; but

$$\sum_{n=1}^{m}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)\right]$$

$$=\ln 1-\ln\left(\frac{1}{2}\right)+\ln\left(\frac{1}{2}\right)-\ln\left(\frac{1}{3}\right)+\ln\left(\frac{1}{3}\right)+\cdots+\ln\left(\frac{1}{m}\right)-\ln\left(\frac{1}{m+1}\right)$$

So; $\sum_{n=1}^{m}\left[\ln\left(\frac{1}{n}\right)-\ln\left(\frac{1}{n+1}\right)\right]=-\ln\left(\frac{1}{m+1}\right)=\ln(m+1)$; so

$\mathrm{I}=\lim_{m\to+\infty}\sum_{n=1}^{m}\ln(m+1)-\sum_{n=1}^{+\infty}\frac{1}{n}=\lim_{m\to+\infty}\sum_{n=1}^{m}\ln(m+1)-\sum_{n=1}^{+\infty}\frac{1}{n}+1$; then we get

$$\mathrm{I}=\lim_{m\to+\infty}\sum_{n=1}^{m}\left[\ln m-\sum_{n=1}^{m}\frac{1}{n}\right]+1=-\lim_{m\to+\infty}\sum_{n=1}^{m}\left[\sum_{n=1}^{m}\frac{1}{n}-\ln m\right]+1=1-\gamma$$

34. $\mathrm{I}=\int_0^{+\infty}\frac{x^2}{\sinh(\sqrt{x})}dx$, let $y=\sqrt{x}$, so $x^2=y^4$ and $x=y^2$, then $dx=2ydy$, then we get**:**

$$\mathrm{I}=\int_0^{+\infty}\frac{y^4}{\sinh y}.2ydy=2\int_0^{+\infty}\frac{y^5}{\sinh y}dy=2\int_0^{+\infty}\frac{y^5}{\frac{1}{2}(e^y-e^{-y})}dy=4\int_0^{+\infty}\frac{y^5}{e^y-e^{-y}}dy$$

$\mathrm{I}=4\int_0^{+\infty}\frac{y^5}{e^y-e^{-y}}\cdot\frac{e^{-y}}{e^{-y}}dy=4\int_0^{+\infty}\frac{y^5e^{-y}}{1-e^{-2y}}dy$, then we get**:**

$$\mathrm{I}=4\int_0^{+\infty}y^5e^{-y}\sum_{n=0}^{+\infty}(e^{-2y})^n\,dy=4\int_0^{+\infty}y^5e^{-y}\sum_{n=0}^{+\infty}e^{-2yn}\,dy=4\int_0^{+\infty}y^5\sum_{n=0}^{+\infty}e^{-(2n+1)y}\,dy$$

Then we get: $\mathrm{I} = 4\sum_{n=0}^{+\infty}\int_{0}^{+\infty} y^5.e^{-(2n+1)y}dy$

Let $t = (2n+1)y$, then $dt = (2n+1)dy$, so $dy = \frac{1}{2n+1}dt$, then we get:

$$\mathrm{I} = 4\sum_{n=0}^{+\infty}\int_{0}^{+\infty}\left(\frac{t}{2n+1}\right)^5.e^{-t}.\frac{1}{2n+1}dt = 4\sum_{n=0}^{+\infty}\int_{0}^{+\infty}\frac{1}{(2n+1)^6}.t^5.e^{-t}d\,;\ \text{ then we get:}$$

$$\mathrm{I} = 4\sum_{\mathrm{n}=0}^{+\infty}\frac{1}{(2n+1)^6}\int_{0}^{+\infty} t^5.e^{-t}dt = 4\sum_{\mathrm{n}=0}^{+\infty}\frac{1}{(2n+1)^6}\int_{0}^{+\infty} t^{6-1}.e^{-t}dt = 4\sum_{\mathrm{n}=0}^{+\infty}\frac{1}{(2n+1)^6}\Gamma(6)$$

$$\mathrm{I} = 4\times 5!\sum_{\mathrm{n}=0}^{+\infty}\frac{1}{(2n+1)^6} = 480\sum_{\mathrm{n}=0}^{+\infty}\frac{1}{(2n+1)^6} = 480\left(\frac{\pi^6}{960}\right) = \frac{\pi^6}{2}$$

35. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{\ln(\cos x)}{\sin x}dx$, let $u = \cos x$, then $du = -\sin x\,dx = \sqrt{1-\cos^2 x}\,dx = \sqrt{1-u^2}dx$

For $x = 0$, $u = 1$ and for $x = \frac{\pi}{2}$, $u = 0$, then we get:

$\mathrm{I} = \int_1^0\frac{\ln u}{\sqrt{1-u^2}}.\frac{-du}{\sqrt{1-u^2}} = \int_0^1\frac{\ln u}{1-u^2}du$, then we get:

$$\mathrm{I} = \int_0^1 \ln u\left(\frac{1}{1-u^2}\right)du = \int_0^1 \ln u\sum_{n=0}^{+\infty}u^{2n}\,du = \sum_{n=0}^{+\infty}\int_0^1 u^{2n}\ln u\,du\,;\ \text{ now using IBP}$$

Let $\mathrm{J} = \int_0^1 u^{2n}\ln u\,du$, then $\mathrm{J} = \left[\ln u.\frac{u^{2n+1}}{2n+1}\right]_0^1 - \int_0^1\frac{u^{2n+1}}{2n+1}.\frac{1}{u}du = -\frac{1}{2n+1}\int_0^1 u^{2n}du$

Then $\mathrm{J} = -\frac{1}{2n+1}\left[\frac{u^{2n+1}}{2n+1}\right]_0^1 = -\frac{1}{(2n+1)^2}$, then we get:

$$\mathrm{I} = -\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2} = -\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots-\left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)\right];\ \text{ then we get}$$

$$\mathrm{I} = -\left[\sum_{n=1}^{+\infty}\frac{1}{n^2}-\sum_{n=1}^{+\infty}\frac{1}{(2n)^2}\right] = -\left(\sum_{n=1}^{+\infty}\frac{1}{n^2}-\frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2}\right) = -\frac{3}{4}\sum_{n=1}^{+\infty}\frac{1}{n^2} = -\frac{3}{4}\left(\frac{\pi^2}{6}\right) = -\frac{\pi^2}{8}$$

36. $\mathrm{I} = \int_0^1\frac{\ln(1-x^2)\ln^2 x}{x}dx = \int_0^1\frac{\ln^2 x}{x}\ln(1-x^2)\,dx$, then:

$$\mathrm{I} = \int_0^1\frac{\ln^2 x}{x}\left(-\sum_{n=1}^{+\infty}\frac{(x^2)^n}{n}\right)dx = -\int_0^1\frac{\ln^2 x}{x}\sum_{n=1}^{+\infty}\frac{x^{2n}}{n}dx = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1\ln^2 x\,.\,x^{2n-1}dx$$

Let $-t = \ln x$, then $x = e^{-t}$ and $dx = -e^{-t}dt$, for $x = 0$, thrn $t = +\infty$ and for $x = 1$, then $t = 0$, then we get:

$$\mathrm{I} = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_{+\infty}^{0}(e^{-t})^{2n-1}.(-t)^2.\left(-e^{-tdt}\right) = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} t^2.e^{-2nt}dt$$

Let $u = 2nt$, then $du = 2ndt$, then we get:

$$\mathrm{I} = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} e^{-u}.\left(\frac{u}{2n}\right)^2.\frac{1}{2n}du = -\sum_{n=1}^{+\infty}\frac{1}{n}.\frac{1}{8n^3}\int_0^{+\infty} u^2e^{-u}du = -\sum_{n=1}^{+\infty}\frac{1}{n}.\frac{1}{8n^3}\int_0^{+\infty} u^{3-1}e^{-u}du$$

$$\mathrm{I} = -\frac{1}{8}\sum_{n=1}^{+\infty}\frac{1}{n^4}\Gamma(3) = -\frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^4} = -\frac{1}{4}\left(\frac{\pi^4}{96}\right) = -\frac{\pi^4}{360}$$

37. $\mathrm{I} = \int_{-\infty}^{+\infty}\frac{x^2e^{-x}}{(1+e^{-x})^2}dx = \int_{-\infty}^{+\infty}\frac{x^2e^{-x}}{(1+e^{-x})^2}\times\frac{e^{2x}}{e^{2x}}dx = \int_{-\infty}^{+\infty}\frac{x^2e^{x}}{(1+e^{x})^2}dx = \int_0^{+\infty}\frac{2x^2e^{x}}{(1+e^{x})^2}dx$

Let $u = 2x^2$, then $u' = 4x$ and let $v' = \frac{e^x}{(1+e^x)^2}$, then $v' = -\frac{1}{1+e^x}$, so:

$\mathrm{I} = \left[-\frac{2x^2}{1+e^x}\right]_0^{+\infty} + \int_0^{+\infty}\frac{4x}{1+e^x}dx = 4\int_0^{+\infty}\frac{xe^{-x}}{1+e^{-x}}dx$, then we get:

$$\mathrm{I} = 4\int_0^{+\infty} x\sum_{n=1}^{+\infty}(-1)^{n+1}(e^{-x})^n dx = 4\int_0^{+\infty} x\sum_{n=1}^{+\infty}(-1)^{n+1}e^{-nx}\,dx = 4\sum_{n=1}^{+\infty}(-1)^{n+1}\int_0^{+\infty} xe^{-nx}dx$$

Let $u = nx$, then $du = ndx$, then we get:

$$\mathrm{I} = 4\sum_{n=1}^{+\infty}(-1)^{n+1}\int_0^{+\infty}\frac{u}{n}.e^{-u}.\frac{1}{n}dx = 4\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{n^2}\int_0^{+\infty}u.e^{-u}\,du$$

$$\mathrm{I} = 4\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{n^2}\int_0^{+\infty}u^{2-1}.e^{-u}\,du = 4\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{n^2}\Gamma(2) = 4\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{n^2};\ \text{then we get:}$$

$$\mathrm{I} = 4\left(\frac{\pi^2}{12}\right) = \frac{\pi^2}{3}$$

38. $\mathrm{I} = \int_0^1 \ln(-\ln x).(\ln x)^2\,dx$, let $u = -\ln x$, then $x = e^{-u}$ and $dx = -e^{-u}du$, for $x = 0$, then $u = +\infty$ and for $x = 1$, then $u = 0$, so we get:

$\mathrm{I} = \int_{+\infty}^0 \ln u.(-u)^2\,(-e^{-u})du = \int_0^{+\infty} e^{-u}.u^2\ln u\,du$

But we have $\Gamma'(n) = \int_0^{+\infty} e^{-x}x^{n-1}\ln x\,dx$, so $\int_0^{+\infty} e^{-u}.u^2\ln u\,du = \Gamma'(3)$, but the digamma function is equal to:

$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ so $\Gamma'(x) = \Gamma(x)\,\Psi(x) \Rightarrow \Gamma'(3) = \Gamma(3)\,\Psi(3) = 2!\,\Psi(3) = 2\,\Psi(3)$; but we have

$\Psi(x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{+\infty}\left(\frac{1}{k} - \frac{1}{k+x}\right)$; one of the series representations of digamma function

Then we get $\Psi(3) = -\gamma - \frac{1}{3} + \left(1-\frac{1}{4}\right) + \left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{3}-\frac{1}{6}\right) + \left(\frac{1}{4}-\frac{1}{7}\right) + \cdots$

Then $\Psi(3) = -\gamma - \frac{1}{3} + 1 + \frac{1}{2} + \frac{1}{3} = \frac{3}{2} - \gamma$, with $\mathrm{I} = 2\,\Psi(3) = 3 - 2\gamma$

39. $\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln^2(\cot x)\,dx$

Let $y = \ln(\cot x) \Longrightarrow \cot x = e^y \Longrightarrow x = \cot^{-1}(e^y) \Longrightarrow dx = -\frac{e^y}{1+e^{2y}}dy$

For $x = 0$, then $y = +\infty$ and for $x = \frac{\pi}{4}$, then $y = 0$, then we get:

$\mathrm{I} = \int_{+\infty}^0 y^2.\left(-\frac{e^y}{1+e^{2y}}dy\right) = \int_0^{+\infty}\frac{y^2e^y}{1+e^{2y}}dy = \int_0^{+\infty}\frac{y^2e^{-y}}{1+e^{-2y}}dy = \int_0^{+\infty}\frac{y^2e^{-y}}{1-(-e^{-2y})}dy$, then

$$\mathrm{I} = \int_0^{+\infty} y^2e^{-y}\sum_{n=0}^{+\infty}(-e^{-2y})^n\,dy = \int_0^{+\infty} y^2e^{-y}\sum_{n=0}^{+\infty}(-1)^n e^{-2ny}\,dy;\ \text{then we get:}$$

$$\text{I} = \sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} y^2 e^{-(2n+1)y} dy\,;$$ let $u = (2n+1)y$; then $du = (2n+1)dy$; then we get

$$\text{I} = \sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} \left(\frac{u}{2n+1}\right)^2 e^{-u}.\frac{1}{2n+1}du = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\int_0^{+\infty} u^2 e^{-u} du$$

$$\text{I} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3}\Gamma(3) = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^3} = 2\left(\frac{\pi^3}{32}\right) = \frac{\pi^3}{16}$$

40. $\text{I} = \int_0^1 \frac{\ln(1-x)}{1+x}dx$, let $t = \frac{1-x}{1+x}$, then $x = \frac{1-t}{1+t}$ and $dx = -\frac{2}{(1+t)^2}dt$, for $x = 0$, $t = 1$ and for $x = 1$, then $t = 0$, so we get:

$$\text{I} = \int_1^0 \frac{\ln\left(1-\frac{1-t}{1+t}\right)}{1+\left(\frac{1-t}{1+t}\right)}\left(-\frac{2}{(1+t)^2}\right)dt = 2\int_0^1 \frac{\ln\left(\frac{2t}{1+t}\right)}{\frac{2}{1+t}}\frac{1}{(1+t)^2}dt = \int_0^1 \frac{\ln(2t)-\ln(1+t)}{1+t}dt$$

$$\text{I} = \ln 2\int_0^1 \frac{1}{1+t}dt + \int_0^1 \frac{\ln t}{1+t}dt - \int_0^1 \frac{\ln(1+t)}{1+t}dt = \ln 2\ln 2 + \int_0^1 \frac{\ln t}{1+t} - \frac{1}{2}(\ln 2)^2,$$ then we get:

$\text{I} = \frac{1}{2}(\ln 2)^2 + \int_0^1 \frac{\ln t}{1+t}$, let $u = \ln t$, then $u' = \frac{1}{t}$ and let $v' = \frac{1}{1+t}$, then $v = \ln(1+t)$, so we get:

$$\text{I} = \frac{1}{2}(\ln 2)^2 + [\ln t\ln(1+t)]_0^1 - \int_0^1 \frac{\ln(1+t)}{t}dt = \frac{1}{2}(\ln 2)^2 - \int_0^1 \frac{\ln(1+t)}{t}dt,$$ then:

$$\text{I} = \frac{1}{2}(\ln 2)^2 - \int_0^1 \frac{1}{t}\ln(1+t)\,dt = \frac{1}{2}(\ln 2)^2 - \int_0^1 \frac{1}{t}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}t^n\,dt$$

$$\text{I} = \frac{1}{2}(\ln 2)^2 - \int_0^1 \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}t^{n-1}\,dt = \frac{1}{2}(\ln 2)^2 - \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\int_0^1 t^{n-1}\,dt;$$ then we get:

$$\text{I} = \frac{1}{2}(\ln 2)^2 - \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left[\frac{t^n}{n}\right]_0^1 = \frac{1}{2}(\ln 2)^2 - \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2} = \frac{1}{2}(\ln 2)^2 - \text{S}$$

Now Evaluating the Sum $\text{S} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2} = \text{S}_1(\text{for odd terms}) + \text{S}_2(\text{for even terms})$

For $n = 2k+1$; then $\text{S}_1 = \sum_{k=0}^{+\infty}\frac{(-1)^{2k+2}}{(2k+1)^2} = \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^2}$

Then we get: $\text{S}_1 = \sum_{k=0}^{+\infty}\frac{1}{(2k+1)^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots - \left(\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{6^2}+\frac{1}{8^2}+\cdots\right)$

$$\text{S}_1 = \sum_{i=1}^{+\infty}\frac{1}{i^2} - \sum_{i=1}^{+\infty}\frac{1}{(2i)^2} = \sum_{i=1}^{+\infty}\frac{1}{i^2} - \frac{1}{4}\sum_{i=1}^{+\infty}\frac{1}{i^2} = \frac{3}{4}\sum_{i=1}^{+\infty}\frac{1}{i^2} = \frac{3}{4}\zeta(2) = \frac{3}{4}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{8}$$

For $n = 2k$; then $\text{S}_2 = \sum_{k=1}^{+\infty}\frac{(-1)^{2k+1}}{(2k)^2} = -\sum_{k=1}^{+\infty}\frac{1}{(2k)^2} = -\frac{1}{4}\sum_{k=1}^{+\infty}\frac{1}{k^2} = -\frac{1}{4}\left(\frac{\pi^2}{6}\right) = -\frac{\pi^2}{24}$

Then we get $\text{I} = \frac{1}{2}(\ln 2)^2 - \left(\frac{\pi^2}{8} - \frac{\pi^2}{24}\right) = \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{12}$

41. $\mathrm{I} = \int_0^1 \left(\frac{\ln x}{1-x}\right)^2 dx = \int_0^1 \frac{1}{(1-x)^2} \ln^2 x\, dx$, but we have:

$$\frac{1}{1-x} = \sum_{n=0}^{+\infty} x^n \text{; derive both sides w.r.t } x\text{: } \frac{1}{(1-x)^2} = \sum_{n=1}^{+\infty} n x^{n-1} = \sum_{n=0}^{+\infty} (n+1)x^n \text{; then}$$

$$\mathrm{I} = \int_0^1 \sum_{n=0}^{+\infty} (n+1)x^n \ln^2 x\, dx = \sum_{n=0}^{+\infty} (n+1) \int_0^1 x^n \ln^2 x\, dx$$

Let $x = e^{-t}$ ($\ln x = -t$), then $dx = -e^{-t}dt$, for $x = 0$, then $t = +\infty$ and for $x = 1$, then $t = 0$, then we get:

$$\mathrm{I} = \sum_{n=0}^{+\infty} (n+1) \int_{+\infty}^{0} (e^{-t})^n (-t)^2 \,(-e^{-t}dt) = \sum_{n=0}^{+\infty} (n+1) \int_0^{+\infty} e^{-(n+1)t}.t^2 dt$$

Let $u = (n+1)t$, then $du = (n+1)dt$, then we get:

$$\mathrm{I} = \sum_{n=0}^{+\infty} (n+1) \int_0^{+\infty} e^{-u}.\left(\frac{u}{n+1}\right)^2 . \frac{1}{n+1} du = \sum_{n=0}^{+\infty} (n+1) . \frac{1}{(n+1)^3} \int_0^{+\infty} e^{-u}.u^2 du \text{; so we get}$$

$$\mathrm{I} = \sum_{n=0}^{+\infty} \frac{1}{(n+1)^2} \int_0^{+\infty} e^{-u}.u^{3-1} du = \sum_{n=0}^{+\infty} \frac{1}{(n+1)^2} \Gamma(3) = 2! \sum_{n=1}^{+\infty} \frac{1}{n^2} = 2\zeta(2) = 2\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{3}$$

42. $\mathrm{I} = \int_0^1 \ln x \tan^{-1} x\, dx$

$$\mathrm{I} = \int_0^1 \ln x . \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)} x^{2n-1} dx = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)} \int_0^1 x^{2n-1} \ln x\, dx$$

Let $\ln x = -t \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}dt$, for $x = 0$, then $t = +\infty$ and for $x = 1$, $t = 0$, so:

$$\mathrm{I} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)} \int_{+\infty}^{0} (e^{-t})^{2n-1}(-t)(-e^{-t})dt = -\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)} \int_0^{+\infty} e^{-2nt}.t dt$$

Let $u = 2nt \Rightarrow du = 2ndt$, then we get:

$$\mathrm{I} = -\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)} \int_0^{+\infty} e^{-u}.\frac{u}{2n}.\frac{1}{2n} dt = -\frac{1}{4}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)n^2} \int_0^{+\infty} u e^{-u} dt = -\frac{1}{4}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{(2n-1)n^2}$$

$$\mathrm{I} = -\frac{1}{4}\sum_{n=1}^{+\infty} (-1)^{n+1} \frac{[2n-(2n-1)]}{n^2(2n-1)} - \frac{1}{4}\sum_{n=1}^{+\infty} (-1)^{n+1} \left[\frac{2}{n(2n-1)} - \frac{1}{n^2}\right] \text{; then we get:}$$

$$\mathrm{I} = \frac{1}{4}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} - \frac{1}{2}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n(2n-1)} = \frac{1}{4}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} - \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}[2n-(2n-1)]}{2n(2n-1)} \text{; then:}$$

$$\mathrm{I} = \frac{1}{4}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} + \frac{1}{2}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} - \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{2n-1} = \frac{1}{4}\left(\frac{\pi^2}{8} - \frac{\pi^2}{24}\right) + \frac{1}{2}\ln 2 - \frac{\pi}{4}$$

Therefore we get: $\mathrm{I} = \frac{\pi^2}{48} - \frac{\pi}{4} + \ln\sqrt{2}$

43. $\mathrm{I} = \int_0^1 \ln(1-x) . \frac{\ln^{n-1} x}{x} dx$, let $-t = \ln x \Rightarrow x = e^{-t} \Rightarrow dx = -e^{-t}dt$, for $x = 0$, $t = +\infty$ and for $x = 1$, then $t = 0$, so we get:

$\mathrm{I} = \int_{+\infty}^{0} \ln(1-e^{-t}).\left[\frac{(-t)^{n-1}}{e^{-t}}\right](-e^{-t}dt) = \int_{0}^{+\infty} \ln(1-e^{-t}).(-t)^{n-1}dt$, then we get:

$\mathrm{I} = \int_{0}^{+\infty}(-1)^{n-1}t^{n-1}\ln(1-e^{-t})\,dt = (-1)^{n-1}\int_{0}^{+\infty} t^{n-1}\ln(1-e^{-t})\,dt$

$$\mathrm{I} = (-1)^{n-1}\int_{0}^{+\infty} t^{n-1}\left[-\sum_{k=1}^{+\infty}\frac{(e^{-t})^k}{k}\right]dt = (-1)^{n-1}\int_{0}^{+\infty} t^{n-1}\left(-\sum_{k=1}^{+\infty}\frac{e^{-kt}}{k}\right)dt;\ \text{ then we get:}$$

$$\mathrm{I} = (-1)^{n}\sum_{k=1}^{+\infty}\frac{1}{k}.\int_{0}^{+\infty} e^{-kt}.t^{n-1}dt$$

Now let $u = kt \Longrightarrow du = kdt \Longrightarrow dt = \frac{1}{k}du$, then we get:

$$\mathrm{I} = (-1)^{n}\sum_{k=1}^{+\infty}\frac{1}{k}.\int_{0}^{+\infty} e^{-u}.\left(\frac{u}{k}\right)^{n-1}.\frac{1}{k}dt = (-1)^{n}\sum_{k=1}^{+\infty}\frac{1}{k}\int_{0}^{+\infty} e^{-u}.u^{n-1}.\frac{1}{k^{n-1}}.\frac{1}{k}du\,;\text{ then we getL:}$$

$$\mathrm{I} = (-1)^{n}\sum_{k=1}^{+\infty}\frac{1}{k}.\frac{1}{k^n}\int_{0}^{+\infty} e^{-u}.u^{n-1}du = (-1)^{n}\sum_{k=1}^{+\infty}\frac{1}{k^{n+1}}.\Gamma(n) = (-1)^n\zeta(n+1)\,\Gamma(n)$$

Therefore we get: $\mathrm{I} = (-1)^n(n-1)!\,\zeta(n+1)$

44. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\tan x\ln(\sin x)\ln(\cos x)\,dx = \int_0^{\frac{\pi}{2}}\frac{\sin x}{\cos x}\ln(\sin x)\ln(\cos x)\,dx$

Let $u = \cos x$, then $du = -\sin x\,dx$, for $x = 0$, $u = 1$ and for $x = \frac{\pi}{2}$, then $u = 0$, then:

$\mathrm{I} = \int_1^0 \frac{\ln\sqrt{1-u^2}.\ln u}{u}(-du) = \frac{1}{2}\int_0^1 \ln u\,.\ln(1-u^2)\,.\frac{du}{u}$, then we get:

$$\mathrm{I} = \frac{1}{2}\int_0^1 \ln u\left(-\sum_{n=1}^{+\infty}\frac{(u^2)^n}{n}\right)\frac{du}{u} = -\frac{1}{2}\int_0^1 \ln u\sum_{n=1}^{+\infty}\frac{u^{2n}}{n}\frac{du}{u} = -\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 \ln u\,.u^{2n-1}\,du$$

Let $-t = \ln u$, then $u = e^{-t}$ and $du = -e^{-t}dt$, for $u = 0$, then $t = +\infty$ and for $u = 1$, then $t = 0$, then we get:

$$\mathrm{I} = -\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_{+\infty}^{0}(-t).(e^{-t})^{2n-1}(-e^{-t})dt = \frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 e^{-2nt}.t\,du$$

Let $w = 2nt$, then $dw = 2ndt$, then we get:

$$\mathrm{I} = \frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} e^{-w}.\frac{w}{2n}.\frac{1}{2n}dw = \frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{1}{4n^2}\right)\int_0^1 e^{-w}.wdw = \frac{1}{8}\sum_{n=1}^{+\infty}\frac{1}{n^3} = \frac{1}{8}\zeta(3)$$

45. $\mathrm{I} = \int_0^1 \frac{\ln x\ln(1-x)}{x(1-x)}dx = \int_0^1 \frac{[x+(1-x)]\ln x\ln(1-x)}{x(1-x)}dx$, then:

$\mathrm{I} = \int_0^1 \frac{x\ln x\ln(1-x)}{x(1-x)}dx + \int_0^1 \frac{(1-x)\ln x\ln(1-x)}{x(1-x)}dx = \int_0^1 \frac{\ln x\ln(1-x)}{1-x}dx + \int_0^1 \frac{\ln x\ln(1-x)}{x}dx$

Consider now: $\int_0^1 \frac{\ln x\ln(1-x)}{x}dx$, let $u = 1-x$, then $du = -dx$, for $x = 1$, $u = 0$ and for $x = 0$, $u = 1$, then we get:

$\int_0^1 \frac{\ln x\ln(1-x)}{1-x}dx = \int_1^0 \frac{\ln(1-u)\ln(u)}{u}(-du) = \int_0^1 \frac{\ln(1-u)\ln u}{u}du = \int_0^1 \frac{\ln x\ln(1-x)}{x}dx$

Then we get: $\mathrm{I} = \int_0^1 \frac{\ln x\ln(1-x)}{x}dx + \int_0^1 \frac{\ln x\ln(1-x)}{x}dx = 2\int_0^1 \frac{\ln x\ln(1-x)}{x}dx$

Let $x = e^{-t}$ $(t = -\ln x)$, then $dx = -e^{-t}dt$, for $x = 0$, $u = +\infty$ and for $x = 1$, $u = 0$, then:

$\mathrm{I} = 2\int_{+\infty}^{0} \frac{-t\ln(1-e^{-t})}{e^{-t}}(-e^{-t})dt = -2\int_0^{+\infty} t\ln(1-e^{-t})dt$, then we get:

$$\mathrm{I} = -2\int_0^{+\infty} t\left(-\sum_{n=1}^{+\infty}\frac{(e^{-t})^n}{n}\right)dt = 2\int_0^{+\infty} t\sum_{n=1}^{+\infty}\frac{e^{-nt}}{n}dx = 2\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} te^{-nt}dt$$

Now evaluating $\int_0^{+\infty} te^{-nt}dt$ by parts:

Let $u = t$, then $u' = 1$ and let $v' = e^{-nt}$, then $v = -\frac{1}{n}e^{-nt}$, then we get:

$\int_0^{+\infty} te^{-nt}dt = \left[-\frac{t}{n}e^{-nt}\right]_0^{+\infty} + \int_0^{+\infty}\frac{1}{n}e^{-nt}dt = \frac{1}{n}\left[-\frac{1}{n}e^{-nt}\right]_0^{+\infty} = \frac{1}{n}\left(\frac{1}{n}\right) = \frac{1}{n^2}$, then we get:

$$\mathrm{I} = 2\sum_{n=1}^{+\infty}\frac{1}{n}.\frac{1}{n^2} = 2\sum_{n=1}^{+\infty}\frac{1}{n^3} = 2\zeta(3)$$

46. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \ln^n(\tan x)\,dx$, let $u = \tan x$, then $du = (1+\tan^2 x)dx = (1+u^2)du$

For $x = 0$, $u = 0$ and for $x = \frac{\pi}{2}$, then $u = +\infty$, then we get:

$\mathrm{I} = \int_0^{+\infty} \ln^n u\,.\frac{1}{1+u^2}du = \int_0^{+\infty}\frac{\ln^n x}{1+u^2}du = \int_0^1 \frac{\ln^n x}{1+u^2}du + \int_1^{+\infty}\frac{\ln^n x}{1+u^2}du$

For $\int_1^{+\infty}\frac{\ln^n x}{1+u^2}du$, let $t = \frac{1}{u}$, then $u = \frac{1}{t}$ and $du = -\frac{1}{t^2}dt$, for $u = 1$, then $t = 1$ and for $u = +\infty$, then $t = 0$, then we get:

$$\int_1^{+\infty}\frac{\ln^n x}{1+u^2}du = \int_1^0 \frac{\ln^n\left(\frac{1}{t}\right)}{1+\left(\frac{1}{t}\right)^2}\left(-\frac{1}{t^2}\right)dt = \int_0^1\frac{\ln^n\left(\frac{1}{t}\right)}{1+t^2}dt = \int_0^1\frac{(-\ln t)^n}{1+t^2}dt = \int_0^1\frac{(-1)^n\ln^n t}{1+t^2}dt$$

Then we get: $\int_1^{+\infty}\frac{\ln^n x}{1+u^2}du = \int_0^1\frac{(-1)^n\ln^n u}{1+u^2}du$, so we get:

$\mathrm{I} = \int_0^1\frac{\ln^n x}{1+u^2}du + \int_0^1\frac{(-1)^n\ln^n u}{1+u^2}du = \int_0^1\frac{\ln^n x}{1+u^2}du + (-1)^n\int_0^1\frac{\ln^n x}{1+u^2}du$, so we get:

$\mathrm{I} = [1+(-1)^n]\int_0^1\frac{\ln^n x}{1+u^2}du$, let $\ln u = -t$, then $u = e^{-t}$ and $du = -e^{-t}dt$

For $u = 0$, then $t = +\infty$ and for $u = 1$, then $t = 0$, then we get:

$\mathrm{I} = [1+(-1)^n]\int_0^{+\infty}(-t)^n.\frac{e^{-t}}{1+e^{-2t}}dt = [1+(-1)^n](-1)^n\int_0^{+\infty}t^n.\frac{e^{-t}}{1+e^{-2t}}dt$

$\mathrm{I} = [(-1)^n+(-1)^{2n}]\int_0^{+\infty}t^n.\frac{e^{-t}}{1+e^{-2t}}dt = [(-1)^n+1]\int_0^{+\infty}t^n.\frac{e^{-t}}{1+e^{-2t}}dt$, then:

$$\mathrm{I} = [1+(-1)^n]\int_0^{+\infty} t^n\sum_{k=1}^{+\infty}(-1)^{k+1}e^{-(2k-1)t}dt\,;\ \text{let } v = (2k-1)t;\ \text{so } dv = (2k-1)dt;\ \text{so:}$$

$$\mathrm{I} = [1+(-1)^n]\sum_{k=1}^{+\infty}(-1)^{k+1}\int_0^{+\infty}\frac{v^n}{(2k-1)^{n+1}}e^{-v}dv\,;\ \text{then we get:}$$

$$\mathrm{I} = [1+(-1)^n]\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{(2k-1)^{n+1}}\int_0^{+\infty}v^ne^{-v}dv = [1+(-1)^n]\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{(2k-1)^{n+1}}.n!$$

$$\mathrm{I} = [1+(-1)^n].n!\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{(2k-1)^{n+1}} = [1+(-1)^n].n!\left(1-\frac{1}{3^{n+1}}+\frac{1}{5^{n+1}}-\frac{1}{7^{n+1}}+\cdots\right)$$

47. $\mathrm{I} = \int_0^{+\infty} x^\alpha \tan^{-1}(e^{-x})\,dx$, we have**:**

$$\tan^{-1} y = \sum_{n=0}^{\infty}(-1)^n\frac{y^{2n+1}}{2n+1} = y-\frac{y^3}{3}+\frac{y^5}{5}-\frac{y^7}{7}+\cdots \quad \text{for } |y|\le 1; \text{ then}$$

$$\tan^{-1}(e^{-x}) = \sum_{n=0}^{\infty}(-1)^n\frac{(e^{-x})^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}(-1)^n\frac{e^{-(2n+1)x}}{2n+1}; \text{ then we get:}$$

$$\mathrm{I} = \int_0^{+\infty} x^\alpha\sum_{n=0}^{\infty}(-1)^n\frac{e^{-(2n+1)x}}{2n+1}dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty} x^\alpha e^{-(2n+1)x}dx$$

Let $u = (2n+1)x$; then $du = (2n+1)dx$; then we get:

$$\mathrm{I} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty}\left(\frac{u}{2n+1}\right)^\alpha e^{-u}.\frac{1}{2n+1}du = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{\alpha+2}}\int_0^{+\infty} x^\alpha e^{-u}du; \text{ then}$$

$$\mathrm{I} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{\alpha+2}}\int_0^{+\infty} x^{(\alpha+1)-1}e^{-u}du\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^{\alpha+2}}\Gamma(\alpha+1) = \beta(\alpha+2)\,\Gamma(\alpha+1)$$

48. $\mathrm{I} = \int_0^1 \frac{x-\ln(1+x)}{x\ln x}dx$, let $x = e^{-t} \Rightarrow dx = -e^{-t}dt$, for $x = 0$, then $t = +\infty$ and for $x = 1$, then $t = 0$, then we get**:**

$\mathrm{I} = \int_{+\infty}^0 \frac{e^{-t}-\ln(1+e^{-t})}{-te^{-t}}(-e^{-t}dt) = -\int_0^{+\infty}\frac{e^{-t}-\ln(1+e^{-t})}{t}dt$ using IBP

Let $u = e^{-t}-\ln(1+e^{-t})$, then $u' = -e^{-t}+\frac{e^{-t}}{1+e^{-t}}$ and let $v' = \frac{1}{t}$, then $v = \ln t$, then**:**

$\mathrm{I} = -[\ln t\,(e^{-t}-\ln(1+e^{-t}))]_0^{+\infty} + \int_0^{+\infty}\ln t\left(-e^{-t}+\frac{e^{-t}}{1+e^{-t}}\right)dt$, then we get**:**

$\mathrm{I} = \int_0^{+\infty} e^{-t}\ln t\left(\frac{1}{1+e^{-t}}-1\right)dt = -\int_0^{+\infty} e^{-t}\ln t\,dt + \int_0^{+\infty}\ln t\frac{e^{-t}}{1+e^{-t}}dt$, then we get**:**

$\mathrm{I} = \gamma + \int_0^{+\infty}\ln t\frac{e^{-t}}{1+e^{-t}}dt$, so**:**

$$\mathrm{I} = \gamma + \int_0^{+\infty}\ln t.\sum_{k=1}^{+\infty}(-1)^{k+1}.e^{-kt}dt = \gamma + \sum_{k=1}^{+\infty}(-1)^{k+1}\int_0^{+\infty} e^{-kt}\ln t\,dt$$

Let $u = kt \Rightarrow du = kdt$, then we get**:**

$$\mathrm{I} = \gamma + \sum_{k=1}^{+\infty}(-1)^{k+1}\int_0^{+\infty} e^{-u}\ln\left(\frac{u}{k}\right).\frac{1}{k}.du = \gamma + \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}\int_0^{+\infty} e^{-u}\ln\left(\frac{u}{k}\right)du; \text{then we get:}$$

$$\mathrm{I} = \gamma + \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}\int_0^{+\infty} e^{-u}[\ln u-\ln k]du$$

$$\mathrm{I} = \gamma + \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}\int_0^{+\infty} e^{-u}\ln u\,du - \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}\ln k}{k}\int_0^{+\infty} e^{-u}du$$

$$\text{I} = \gamma - \gamma\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k} - \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}\ln k}{k} = \gamma - \gamma\ln 2 - \sum_{k=1}^{+\infty}\frac{(-1)^{k+1}\ln k}{k}$$

But we have $\displaystyle\sum_{k=1}^{+\infty}\frac{(-1)^{k}\ln k}{k} = \gamma\ln 2 - \frac{1}{2}(\ln 2)^2$; therefore we get:

$$\text{I} = \gamma - \gamma\ln 2 + \sum_{k=1}^{+\infty}\frac{(-1)^{k}\ln k}{k} = \gamma - \gamma\ln 2 + \gamma\ln 2 - \frac{1}{2}(\ln 2)^2 = \gamma - \frac{1}{2}(\ln 2)^2$$

49. $\text{I} = \int_0^{+\infty}\ln\left(\frac{e^x+1}{e^x-1}\right)dx = \int_0^{+\infty}\ln\left[\frac{e^{-x}(e^x+1)}{e^{-x}(e^x-1)}\right]dx = \int_0^{+\infty}\ln\left(\frac{1+e^{-x}}{1-e^{-x}}\right)dx$, then we get**:**

$\text{I} = \int_0^{+\infty}[\ln(1+e^{-x}) - \ln(1-e^{-x})]dx$, then**:**

$$\text{I} = \int_0^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}(e^{-x})^n\,dx - \int_0^{+\infty}\left[-\left(\sum_{n=1}^{+\infty}\frac{1}{n}(e^{-x})^n\right)\right]dx\,;\ \text{ then we get:}$$

$$\text{I} = \int_0^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}e^{-nx}\,dx + \int_0^{+\infty}\sum_{n=1}^{+\infty}\frac{1}{n}e^{-nx}\,dx \ \text{ and so we get:}$$

$$\text{I} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\int_0^{+\infty}e^{-nx}\,dx + \sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty}e^{-nx}\,dx$$

$$\text{I} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left[-\frac{1}{n}e^{-nx}\right]_0^{+\infty} + \sum_{n=1}^{+\infty}\frac{1}{n}\left[-\frac{1}{n}e^{-nx}\right]_0^{+\infty} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{1}{n}\right) + \sum_{n=1}^{+\infty}\frac{1}{n}\left(\frac{1}{n}\right)$$

Then we get: $\displaystyle\text{I} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n^2} + \sum_{n=1}^{+\infty}\frac{1}{n^2} = \sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{n^2} + \sum_{n=1}^{+\infty}\frac{1}{n^2}$; then we get:

$\text{I} = \eta(2) + \zeta(2) = (1-2^{1-2})\zeta(2) + \zeta(2) = (1 - 2^{-1} + 1)\zeta(2) = \left(2 - \frac{1}{2}\right)\zeta(2)$; so we get:

$$\text{I} = \frac{3}{2}\zeta(2) = \frac{3}{2}\left(\frac{\pi^2}{6}\right) = \frac{\pi^2}{4}$$

50. $\text{I} = \int_0^1\frac{\ln(1+x+x^2+\cdots+x^{n-1})}{x}dx$, where $n \geq 2$

$\text{I} = \int_0^1\frac{\ln(1+x+x^2+\cdots+x^{n-1})}{x}dx = \int_0^1\frac{\ln\left(\frac{1-x^n}{1-x}\right)}{x}dx = \int_0^1\frac{\ln(1-x^n)-\ln(1-x)}{x}dx$, then**:**

$$\text{I} = \int_0^1\frac{\ln(1-x^n)}{x}dx - \int_0^1\frac{\ln(1-x)}{x}dx$$

For $\int_0^1\frac{\ln(1-x^n)}{x}dx$, let $y = x^n$, then $x = y^{\frac{1}{n}}$ and $dx = \frac{1}{n}y^{\frac{1}{n}-1}dy$, then we get**:**

$$\int_0^1\frac{\ln(1-x^n)}{x}dx = \int_0^1\frac{\ln(1-y)}{y^{\frac{1}{n}}}.\frac{1}{n}y^{\frac{1}{n}-1}dy = \int_0^1\ln(1-y).\frac{1}{n}y^{-1}dy = \frac{1}{n}\int_0^1\frac{\ln(1-y)}{y}dx$$

So, $\int_0^1\frac{\ln(1-x^n)}{x}dx = \frac{1}{n}\int_0^1\frac{\ln(1-x)}{x}dx$, then we get**:**

$\text{I} = \frac{1}{n}\int_0^1\frac{\ln(1-x)}{x}dx - \int_0^1\frac{\ln(1-x)}{x}dx = \left(\frac{1}{n} - 1\right)\int_0^1\frac{\ln(1-x)}{x}dx$ and so, we get**:**

$$I = \left(\frac{1}{n} - 1\right)\int_0^1 \frac{1}{x}\left(-\sum_{k=1}^{+\infty}\frac{x^k}{k}\right)dx = \left(1 - \frac{1}{n}\right)\int_0^2 \sum_{k=1}^{+\infty}\frac{x^{k-1}}{k}dx = \left(1 - \frac{1}{n}\right)\sum_{k=1}^{+\infty}\frac{1}{k}\int_0^{+\infty} x^{k-1}dx\,;\ \text{then:}$$

$$I = \left(1 - \frac{1}{n}\right)\sum_{k=1}^{+\infty}\frac{1}{k}\left[\frac{1}{k}x^k\right]_0^1 = \left(1 - \frac{1}{n}\right)\sum_{k=1}^{+\infty}\frac{1}{k}.\frac{1}{k} = \left(1 - \frac{1}{n}\right)\sum_{k=1}^{+\infty}\frac{1}{k^2} = \left(\frac{n-1}{n}\right)\frac{\pi^2}{6}$$

51. $I = \int_0^{+\infty}(2x - x^2)e^{-2x}\operatorname{sech} x\,dx = \int_0^{+\infty}(2x - x^2)e^{-2x}.\frac{2}{e^x+e^{-x}}dx$

$I = \int_0^{+\infty}(4x - 2x^2).\frac{e^{-2x}}{e^x+e^{-x}}dx = \int_0^{+\infty}(4x - 2x^2).\frac{e^{-3x}}{1+e^{-2x}}dx$

Then we get**:** $I = \int_0^{+\infty}(4x - 2x^2).e^{-3x}\frac{1}{1+e^{-2x}}dx = \int_0^{+\infty}(4x - 2x^2).e^{-3x}\frac{1}{1-\left(-e^{-2x}\right)}dx$

$$I = \int_0^{+\infty}(4x - 2x^2)e^{-3x}\sum_{n=0}^{+\infty}(-e^{-2x})^n dx = \int_0^{+\infty}(4x - 2x^2)e^{-3x}\sum_{n=0}^{+\infty}(-1)^n e^{-2nx}dx$$

Then we get: $I = \sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}(4x - 2x^2)e^{-(2n+3)x}dx$

$$I = 4\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}x.e^{-(2n+3)x}dx - 2\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}x^2.e^{-(2n+3)x}dx$$

Let $u = (2n+3)x$, then $du = (2n+3)dx$, then we get**:**

$$I = 4\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}\frac{u}{2n+3}.e^{-u}.\frac{1}{2n+3}du - 2\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}\left(\frac{u}{2n+3}\right)^2.e^{-u}.\frac{1}{2n+3}du$$

$$I = 4\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+3)^2}\int_0^{+\infty}u.e^{-u}du - 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+3)^3}\int_0^{+\infty}u^2.e^{-u}du\,;\ \text{then we get:}$$

$$I = 4\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+3)^2}\int_0^{+\infty}u^{2-1}.e^{-u}du - 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+3)^3}\int_0^{+\infty}u^{3-1}.e^{-u}du$$

$$I = 4\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+3)^2}\Gamma(2) - 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+3)^3}\Gamma(3) = 4\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+3)^2} - 4\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+3)^3}$$

$$I = 4\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{(2n+1)^2} - 4\sum_{n=1}^{+\infty}\frac{(-1)^{n-1}}{(2n+1)^3} = -4\sum_{n=1}^{+\infty}\frac{(-1)^n}{(2n+1)^2} + 4\sum_{n=1}^{+\infty}\frac{(-1)^n}{(2n+1)^3}$$

Then: $I = -4G + 4\left(\frac{\pi^3}{32}\right) = \frac{\pi^3}{8} - 4G$

52. $I = \int_0^{\frac{\pi}{2}}\frac{\sin x\ln(\sin x)\ln(\cos x)}{\cos^2 x}dx$, let $u = \cos x$, then $du = -\sin x\,dx$, for $x = 0$, then $u = 1$ and for $x = \frac{\pi}{2}$, then $u = 0$, then we get**:**

$I = \int_1^0 \frac{\ln\sqrt{1-u^2}\ln u}{u^2}(-du) = \frac{1}{2}\int_0^1 \ln(1-u^2)\ln u.\frac{1}{u^2}du$

$$\mathrm{I}=\frac{1}{2}\int_0^1\left(-\sum_{n=1}^{+\infty}\frac{(x^2)^n}{n}\right)\ln u\,.\frac{1}{u^2}du=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 u^{2n-2}.\ln u\,du$$

Let $-t=\ln u$, then $u=e^{-t}$ and $du=-e^{-t}dt$, for $u=0$, then $t=+\infty$ and for $u=1$, then $t=0$, then we get**:**

$$\mathrm{I}=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_{+\infty}^{0}(e^{-t})^{2n-2}.(-t)(-e^{-t})dt=\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 t.e^{-t(2n-1)}dt$$

Let $y=t(2n-1)$, then $dy=(2n-1)dt$, then we get**:**

$$\mathrm{I}=\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1\frac{y}{2n-1}.e^{-y}.\frac{1}{2n-1}dy=\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n(2n-1)^2}\int_0^1 y.e^{-y}dy=\sum_{n=1}^{+\infty}\frac{1}{2n(2n-1)^2}$$

$$\mathrm{I}=\sum_{n=1}^{+\infty}\frac{2n-(2n-1)}{2n(2n-1)^2}=\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}-\sum_{n=1}^{+\infty}\frac{1}{2n(2n-1)};\ \text{ then we get:}$$

$$\mathrm{I}=\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}-\sum_{n=1}^{+\infty}\frac{2n-(2n-1)}{2n(2n-1)}=\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}-\sum_{n=1}^{+\infty}\frac{1}{2n-1}+\sum_{n=1}^{+\infty}\frac{1}{2n}$$

$$\mathrm{I}=\frac{\pi^2}{8}-\sum_{n=1}^{+\infty}\left(\frac{1}{2n-1}-\frac{1}{2n}\right)=\frac{\pi^2}{8}-\left\{\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\cdots\right\};\text{then we get:}$$

$$\mathrm{I}=\frac{\pi^2}{8}-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right)=\frac{\pi^2}{8}-\ln 2$$

$$\mathrm{I}=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n(2n-1)^2}\int_0^1 y^{2-1}.e^{-y}dy=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n(2n-1)^2}\Gamma(2)=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n(2n-1)^2}$$

53. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{x}{\sin x}dx$, using integration by parts**:**

Let $u=x$, then $u'=1$ and let $v'=\frac{1}{\sin x}=\csc x$, then $v=\ln|\csc x-\cot x|$, then**:**

$$\mathrm{I}=[x\ln|\csc x-\cot x|]_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\ln|\csc x-\cot x|\,dx=-\int_0^{\frac{\pi}{2}}\ln\left|\frac{1}{\sin x}-\frac{\cos x}{\sin x}\right|dx$$

$$\mathrm{I}=-\int_0^{\frac{\pi}{2}}\ln\left(\frac{1-\cos x}{\sin x}\right)dx=-\int_0^{\frac{\pi}{2}}\ln\left(\frac{2\sin^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right)dx=-\int_0^{\frac{\pi}{2}}\ln\left(\tan\left(\frac{x}{2}\right)\right)dx$$

Let $y=\frac{x}{2}$, then $x=2y$ and $dx=2dy$, for $x=0$, $y=0$ and for $x=\frac{\pi}{2}$, then $y=\frac{\pi}{4}$, so we get**:**

$$\mathrm{I}=-2\int_0^{\frac{\pi}{4}}\ln(\tan y)\,dy$$

Let $z=\tan y$, then $y=\tan^{-1}z$, so $dy=\frac{1}{1+z^2}dz$, for $y=0$, $z=0$, for $y=\frac{\pi}{4}$, $z=1$, then**:**

$$\mathrm{I}=-2\int_0^1\ln z\,.\frac{1}{1+z^2}dy=2\int_0^1\frac{-\ln z}{1+z^2}dy$$

Let $u=-\ln z$, then $z=e^{-u}$ and $dz=-e^{-u}du$, for $z=0$, $u=+\infty$ and for $z=1$, $u=0$, so**:**

$$\mathrm{I}=2\int_{+\infty}^0\frac{u}{1+e^{-2u}}(-e^{-u}du)=2\int_0^{+\infty}\frac{ue^{-u}}{1+e^{-2u}}du=2\int_0^{+\infty}\frac{ue^{-u}}{1-(-e^{-2u})}du$$

$$\mathrm{I} = 2\int_0^{+\infty} ue^{-u}\sum_{n=0}^{+\infty}(-e^{-2u})^n du = 2\int_0^{+\infty} ue^{-u}\sum_{n=0}^{+\infty}(-)^n e^{-2nu}du\,;\ \text{ then we get}$$

$$\mathrm{I} = 2\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty} u.e^{-(2n+1)u}du$$

Let $t = (2n+1)u$, then $dt = (2n+1)du$, then we get**:**

$$\mathrm{I} = 2\sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}\frac{t}{2n+1}.e^{-t}.\frac{1}{2n+1}dt = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^{+\infty} t.e^{-t}du$$

$$\text{Then }\ \mathrm{I} = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\Gamma(2) = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2} = 2\mathrm{G}$$

54. $\mathrm{I} = \int_0^1\frac{\ln[1+\ln(1-x)]}{\ln(1-x)}dx$, let $-t = \ln(1-x)$, then $1-x = e^{-t}$ and $x = 1-e^{-t}$

$dx = e^{-t}dt$, for $x = 0$, $t = 0$ and for $x = 1$, then $t = +\infty$, then we get**:**

$$\mathrm{I} = \int_0^{+\infty}\frac{\ln(1-t)}{-t}e^{-t}dt = \int_0^{+\infty} e^{-t}.\frac{1}{t}(-\ln(1-t))dt = \int_0^{+\infty} e^{-t}.\frac{1}{t}\sum_{n=1}^{+\infty}\frac{t^n}{n}dt = \int_0^{+\infty} e^{-t}\sum_{n=1}^{+\infty}\frac{t^{n-1}}{n}dt$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} e^{-t}.t^{n-1}dt = \sum_{n=1}^{+\infty}\frac{1}{n}[\Gamma(n)] = \sum_{n=1}^{+\infty}\frac{\Gamma(n)}{n} = 1+\frac{1!}{2}+\frac{2!}{3}+\frac{3!}{4}+\cdots$$

55. $\mathrm{I} = \int_0^{+\infty}\frac{x}{\sinh(ax)}dx = \int_0^{+\infty}\frac{x}{\frac{1}{2}(e^{ax}-e^{-ax})}dx = \int_0^{+\infty}\frac{2x}{e^{ax}-e^{-ax}}dx = \int_0^{+\infty}\frac{2xe^{ax}}{e^{2ax}-1}dx$

$\mathrm{I} = \int_0^{+\infty} 2x.\frac{e^{-ax}}{1-e^{-ax}}dx$, then we get**:**

$$\mathrm{I} = \int_0^{+\infty} 2x.\sum_{n=1}^{+\infty} e^{-ax(2n-1)}\,dx = 2\sum_{n=1}^{+\infty}\int_0^{+\infty} x.e^{-ax(2n-1)}dx$$

Let $u = ax(2n-1)$, then $du = a(2n-1)dx$, then we get**:**

$$\mathrm{I} = 2\sum_{n=1}^{+\infty}\int_0^{+\infty}\frac{u}{a(2n-1)}.e^{-u}.\frac{1}{a(2n-1)}du = \sum_{n=1}^{+\infty}\frac{2}{a^2(2n-1)^2}\int_0^{+\infty} u.e^{-u}du$$

$$\mathrm{I} = \frac{2}{a^2}\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2}\Gamma(2) = \frac{2}{a^2}\sum_{n=1}^{+\infty}\frac{1}{(2n-1)^2} = \frac{2}{a^2}\left(\frac{\pi^2}{8}\right) = \frac{\pi^2}{4a^2}$$

56. $\mathrm{I} = \int_0^1\frac{\ln(1-x^2)\ln^2 x}{x}dx = \int_0^1\frac{\ln^2 x}{x}\ln(1-x^2)\,dx$, then we get**:**

$$\mathrm{I} = \int_0^1\frac{\ln^2 x}{x}\left(-\sum_{n=1}^{+\infty}\frac{(x^2)^n}{n}\right)dx = -\int_0^1\frac{\ln^2 x}{x}\left(\sum_{n=1}^{+\infty}\frac{x^{2n}}{n}\right)dx = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^1 x^{2n-1}.(\ln x)^2\,dx$$

Let $-t = \ln x$, then $x = e^{-t}$ and $dx = -e^{-t}dt$, for $x = 0$, then $t = +\infty$ and for $x = 1$, then $x = 0$, then we get**:**

$$\mathrm{I} = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_{+\infty}^{0}(e^{-t})^{2n-1}.(t^2).(-e^{-t}dt) = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} e^{-2nt}.t^2dt$$

Let $u = 2nt$, then $du = 2ndt$, then we get**:**

$$\mathrm{I} = -\sum_{n=1}^{+\infty}\frac{1}{n}\int_0^{+\infty} e^{-u}.\left(\frac{u}{2n}\right)^2.\frac{1}{2n}du = -\sum_{n=1}^{+\infty}\frac{1}{8n^4}\int_0^{+\infty} u^2.e^{-u}du = -\sum_{n=1}^{+\infty}\frac{1}{8n^4}.\Gamma(3) = -\frac{1}{4}\sum_{n=1}^{+\infty}\frac{1}{n^4}$$

$$\mathrm{I} = -\frac{1}{4}\left(\frac{\pi^4}{90}\right) = -\frac{\pi^4}{360}$$

57. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\left[\frac{\ln(\tan x)}{\sin\left(\frac{\pi}{4}-x\right)}\right]^2 dx = \int_0^{\frac{\pi}{2}}\frac{\ln^2(\tan x)}{\left[\sin\left(\frac{\pi}{4}\right)\cos x-\cos\left(\frac{\pi}{4}\right)\sin x\right]^2}dx = \int_0^{\frac{\pi}{2}}\frac{\ln^2(\tan x)}{\left(\frac{1}{\sqrt{2}}\cos x-\frac{1}{\sqrt{2}}\sin x\right)^2}dx$

$$\mathrm{I} = 2\int_0^{\frac{\pi}{2}}\frac{\ln^2(\tan x)}{(\cos x-\sin x)^2}dx = 2\int_0^{\frac{\pi}{2}}\frac{\ln^2(\tan x)}{\cos^2 x+\sin^2 x-2\sin x\cos x}dx = 2\int_0^{\frac{\pi}{2}}\frac{\ln^2(\tan x)}{1-\sin 2x}dx$$

$$\mathrm{I} = 2\int_0^{\frac{\pi}{2}}\frac{\ln^2(\tan x)}{\cos^2 x\left(1-2\tan x+\tan^2 x\right)}dx = 2\int_0^{\frac{\pi}{2}}\frac{\ln^2(\tan x).\sec^2 x}{(1-\tan x)^2}dx$$

Let $u = \tan x$, then $du = \sec^2 x\, dx$, for $x = 0$, $u = 0$ and for $x = \frac{\pi}{2}$, then $u = +\infty$, then:

$$\mathrm{I} = 2\int_0^{+\infty}\frac{\ln^2 u}{(1-u)^2}du = 2\left[\int_0^1\frac{\ln^2 u}{(1-u)^2}du + \int_1^{+\infty}\frac{\ln^2 u}{(1-u)^2}du\right]$$

For $\int_1^{+\infty}\frac{\ln^2 u}{(1-u)^2}du$, let $t = \frac{1}{u}$, then $u = \frac{1}{t}$ and $du = -\frac{1}{t^2}dt$, for $u = 1$, then $t = 1$ and for $u = +\infty$ then $t = 0$, then we get:

$$\int_1^{+\infty}\frac{\ln^2 u}{(1-u)^2}du = \int_1^0\frac{\ln^2\left(\frac{1}{t}\right)}{\left(1-\frac{1}{t}\right)^2}\left(-\frac{1}{t^2}\right)dt = \int_0^1\frac{\ln^2 t}{(1-t)^2}dt = \int_0^1\frac{\ln^2 u}{(1-u)^2}du, \text{ then we get:}$$

$$\mathrm{I} = 2\left[\int_0^1\frac{\ln^2 u}{(1-u)^2}du + \int_0^1\frac{\ln^2 u}{(1-u)^2}du\right] = 4\int_0^1\frac{\ln^2 u}{(1-u)^2}du = 4\int_0^1(\ln u)^2(1-u)^{-2}du$$

$$\mathrm{I} = 4\int_0^1(\ln u)^2\sum_{n=1}^{+\infty}n.u^{n-1}\,du = 4\sum_{n=1}^{+\infty}n\int_0^1 u^{n-1}.(\ln u)^2du$$

Let $-t = \ln u$, then $u = e^{-t}$ and $du = -e^{-t}dt$, for $u = 0$, then $t = +\infty$ and for $u = 1$ then $t = 0$, then we get:

$$\mathrm{I} = 4\sum_{n=1}^{+\infty}n\int_{+\infty}^0(e^{-t})^{n-1}.(-t)^2(-e^{-t})du = 4\sum_{n=1}^{+\infty}n\int_0^{+\infty} t^2.e^{-nt}du$$

Let $w = nt$, then $dw = ndt$, then we get:

$$\mathrm{I} = 4\sum_{n=1}^{+\infty}n\int_0^{+\infty}\left(\frac{w}{n}\right)^2.e^{-w}.\frac{1}{n}dw = 4\sum_{n=1}^{+\infty}n.\frac{1}{n^3}\int_0^{+\infty} w^2.e^{-w}\,dw = 4\sum_{n=1}^{+\infty}\frac{1}{n^2}\int_0^{+\infty} w^{3-1}.e^{-w}\,dw$$

$$\mathrm{I} = 4\sum_{n=1}^{+\infty}\frac{1}{n^2}\Gamma(3) = 8\sum_{n=1}^{+\infty}\frac{1}{n^2} = 8\left(\frac{\pi^2}{6}\right) = \frac{4}{3}\pi^2$$

58. $\mathrm{I} = \int_0^{+\infty}\ln\left(\frac{2^x+1}{2^x-1}\right)dx$, using integration by parts:

Let $u = \ln\left(\frac{2^x+1}{2^x-1}\right)$, then $u' = -\frac{2^x\ln 4}{2^{2x}-1}$ and let $v' = 1$, so $v = x$, then we get:

$\mathrm{I}=\left[x\ln\left(\frac{2^x+1}{2^x-1}\right)\right]_0^{+\infty}+\int_0^{+\infty}\frac{2^x\ln 4}{2^{2x}-1}xdx=\int_0^{+\infty}\frac{2^{-x}\ln 4}{1-2^{-2x}}dx=\ln 4\int_0^{+\infty}\frac{x.2^{-x}}{1-2^{-2x}}dx$

Then we get: $\mathrm{I}=2\ln 2\int_0^{+\infty}\frac{x.e^{-x\ln 2}}{1-e^{-2x\ln 2}}dx=2\int_0^{+\infty}\frac{x.e^{-x\ln 2}}{1-e^{-2x\ln 2}}\ln 2\,dx$

Let $u=x\ln 2$, then $du=\ln 2\,dx$, so we get:

$\mathrm{I}=\frac{2}{\ln 2}\int_0^{+\infty}\frac{u.e^{-u}}{1-e^{-2u}}du=\frac{2}{\ln 2}\int_0^{+\infty}ue^{-u}.\frac{1}{1-e^{-2u}}du$, then we get:

$$\mathrm{I}=\frac{2}{\ln 2}\int_0^{+\infty}ue^{-u}\sum_{n=0}^{+\infty}(e^{-2u})^n\,du=\frac{2}{\ln 2}\int_0^{+\infty}ue^{-u}\sum_{n=0}^{+\infty}e^{-2nu}\,du=\frac{2}{\ln 2}\sum_{n=0}^{+\infty}\int_0^{+\infty}u.e^{-(2n+1)u}du$$

Let $t=(2n+1)u$, then $dt=(2n+1)du$, then we get:

$$\mathrm{I}=\frac{2}{\ln 2}\sum_{n=0}^{+\infty}\int_0^{+\infty}\frac{t}{2n+1}.e^{-u}.\frac{1}{2n+1}dt=\frac{2}{\ln 2}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}\int_0^{+\infty}t.e^{-u}dt\,;\ \text{ then we get}$$

$$\mathrm{I}=\frac{2}{\ln 2}\left(\frac{\pi^2}{8}\right)\Gamma(2)=\frac{\pi^2}{4\ln 2}$$

59. $\mathrm{I}=\int_0^{+\infty}\frac{x^3e^x}{\left(e^{2x}-1\right)^2}dx=\frac{1}{2}\int_0^{+\infty}x^3e^{-x}\frac{2e^{2x}}{\left(e^{2x}-1\right)^2}dx$, using integration by parts:

$\mathrm{I}=\frac{1}{2}\left[\frac{x^3e^{-x}}{e^{2x}-1}\right]_0^{+\infty}+\frac{1}{2}\int_0^{+\infty}\frac{(3x^2-x^3)}{e^{2x}-1}dx=\frac{1}{2}\int_0^{+\infty}\frac{(3x^2-x^3)e^{-3x}}{1-e^{-2x}}dx$, then we get:

$$\mathrm{I}=\frac{1}{2}\int_0^{+\infty}(3x^2-x^3)e^{-3x}\sum_{n=0}^{+\infty}(e^{-2x})^n\,dx=\frac{1}{2}\int_0^{+\infty}(3x^2-x^3)e^{-3x}\sum_{n=0}^{+\infty}e^{-2nx}\,dx\,;\text{then}$$

$$\mathrm{I}=\frac{1}{2}\sum_{n=0}^{+\infty}\int_0^{+\infty}(3x^2-x^3)e^{-(2n+3)x}\,dx$$

Let $u=(2n+3)x$, then $du=(2n+3)dx$, then we get:

$$\mathrm{I}=\frac{1}{2}\sum_{n=0}^{+\infty}\int_0^{+\infty}\left[3\left(\frac{u}{2n+3}\right)^2-\left(\frac{u}{2n+3}\right)^3\right]e^{-u}.\frac{1}{2n+3}du\,;\ \text{ then we get:}$$

$$\mathrm{I}=\frac{1}{2}\sum_{n=0}^{+\infty}\left[\frac{3}{(2n+3)^3}\int_0^{+\infty}u^2e^{-u}du-\frac{1}{(2n+3)^4}\int_0^{+\infty}u^3e^{-u}du\right]$$

$$\mathrm{I}=\frac{1}{2}\sum_{n=0}^{+\infty}\left[\frac{3}{(2n+3)^3}\Gamma(3)-\frac{1}{(2n+3)^4}\Gamma(4)\right]=\frac{1}{2}\sum_{n=0}^{+\infty}\left[\frac{3}{(2n+3)^3}\times 2-\frac{1}{(2n+3)^4}\times 6\right];\ \text{ then}$$

$$\mathrm{I}=3\sum_{n=0}^{+\infty}\frac{1}{(2n+3)^3}-3\sum_{n=0}^{+\infty}\frac{1}{(2n+3)^4}=\frac{3}{8}\sum_{n=0}^{+\infty}\frac{1}{\left(n+\frac{3}{2}\right)^3}-\frac{3}{16}\sum_{n=0}^{+\infty}\frac{1}{\left(n+\frac{3}{2}\right)^3}$$

$$\mathrm{I}=-\frac{3}{16}\Psi^{(2)}\left(\frac{3}{2}\right)-\frac{1}{32}\Psi^{(3)}\left(\frac{3}{2}\right)=-\frac{3}{16}\left(16-14\zeta(4)\right)-\frac{1}{32}\left(\pi^4-96\right)$$

60. $\mathrm{I} = \int_0^{+\infty} \frac{e^{-xt}\sin(a\sqrt{x})}{\pi x}dx = \frac{1}{\pi}\int_0^{+\infty} \frac{e^{-xt}\sin(a\sqrt{x})}{x}dx = \frac{1}{\pi}\int_0^{+\infty} \frac{e^{-xt}}{x}\sin(a\sqrt{x})\,dx$

We have $\sin z = \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} z^{2k+1}$; then we get:

$$\mathrm{I} = \frac{1}{\pi}\int_0^{+\infty} \frac{e^{-xt}}{x}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}\left(a\sqrt{x}\right)^{2k+1}dx = \frac{1}{\pi}\int_0^{+\infty} \frac{e^{-xt}}{x}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}a^{2k+1}\left(x^{\frac{1}{2}}\right)^{2k+1}dx;\text{ then}$$

$$\mathrm{I} = \frac{1}{\pi}\int_0^{+\infty} \frac{e^{-xt}}{x}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}a^{2k+1}x^{k+\frac{1}{2}}dx = \frac{1}{\pi}\int_0^{+\infty} e^{-xt}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}a^{2k+1}x^{k+\frac{1}{2}-1}dx \text{ so we get}$$

$$\mathrm{I} = \frac{1}{\pi}\int_0^{+\infty} e^{-xt}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}a^{2k+1}x^{k-\frac{1}{2}}dx = \frac{1}{\pi}\sum_{k=0}^{+\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!}\int_0^{+\infty} x^{k-\frac{1}{2}}e^{-xt}dx$$

Let $y = xt$, then $x = \frac{y}{t}$, so $dx = \frac{1}{t}dy$, then we get:

$$\mathrm{I} = \frac{1}{\pi}\sum_{k=0}^{+\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!}\int_0^{+\infty}\left(\frac{y}{t}\right)^{k-\frac{1}{2}}e^{-xt}.\frac{1}{t}dy = \frac{1}{\pi}\sum_{k=0}^{+\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!}\int_0^{+\infty}\frac{1}{t}.\frac{1}{t^{k-\frac{1}{2}}}y^{k-\frac{1}{2}}e^{-y}dy$$

$$\mathrm{I} = \frac{1}{\pi t}\sum_{k=0}^{+\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!\,t^{k-\frac{1}{2}}}\int_0^{+\infty} y^{k-\frac{1}{2}}e^{-y}dy = \frac{1}{\pi t}\sum_{k=0}^{+\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!\,t^{k-\frac{1}{2}}}\int_0^{+\infty} e^{-y}.y^{\left(k+\frac{1}{2}\right)-1}dy$$

$$\mathrm{I} = \frac{1}{\pi t}\sum_{k=0}^{+\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!\,t^{k-\frac{1}{2}}}\Gamma\left(k+\frac{1}{2}\right)$$

Using Duplication formula $\Gamma(n)\Gamma\left(n+\frac{1}{2}\right) = 2^{1-2n}\sqrt{\pi}\Gamma(2n)$; then we get:

$\Gamma\left(n+\frac{1}{2}\right) = \frac{2^{1-2n}\sqrt{\pi}\Gamma(2n)}{\Gamma(n)}$ and so $\Gamma\left(k+\frac{1}{2}\right) = \frac{(2k)!}{2^{2k}k!}$; then we get:

$$\mathrm{I} = \frac{1}{\pi t}\sum_{k=0}^{+\infty}\frac{(-1)^k a^{2k+1}}{(2k+1)!\,t^{k-\frac{1}{2}}}\frac{(2k)!}{2^{2k}k!} = \frac{2}{\sqrt{\pi}}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)k!}\left(\frac{a}{2\sqrt{t}}\right)^{2k+1} = \operatorname{erf}\left(\frac{a}{2\sqrt{t}}\right)$$

61. $\mathrm{I} = \int_0^{+\infty}\frac{\ln x}{e^x+e^{-x}}dx = \int_0^{+\infty}\frac{\ln x}{e^x+e^{-x}}\times\frac{e^{-x}}{e^{-x}}dx = \int_0^{+\infty}\frac{e^{-x}\ln x}{1+e^{-2x}}dx = \int_0^{+\infty}\frac{1}{1+e^{-2x}}.e^{-x}\ln x\,dx$

$$\mathrm{I} = \int_0^{+\infty}\frac{1}{1-(-e^{-2x})}.e^{-x}\ln x\,dx = \int_0^{+\infty}\ln x\,.e^{-x}\sum_{n=0}^{+\infty}(-e^{-2x})^n\,dx;\ \text{ then we get:}$$

$$\mathrm{I} = \int_0^{+\infty}\ln x\,.e^{-x}\sum_{n=0}^{+\infty}(-1)^n e^{-2nx}\,dx = \int_0^{+\infty}\ln x\,.\sum_{n=0}^{+\infty}(-1)^n e^{-(2n+1)x}\,dx$$

Then $\mathrm{I} = \sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}e^{-(2n+1)x}\ln x\,dx$ let $t = (2n+1)x$; so $dt = (2n+1)dx$; so

$$\mathrm{I} = \sum_{n=0}^{+\infty}(-1)^n\int_0^{+\infty}e^{-t}\ln\left(\frac{t}{2n+1}\right).\frac{1}{2n+1}dt = \sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty}e^{-t}[\ln t-\ln(2n+1)]\,dt$$

$$\mathrm{I}=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty}e^{-t}.\ln t\,dt-\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}.\ln(2n+1)\int_0^{+\infty}e^{-t}dt\,;\ \text{ then we get:}$$

$$\mathrm{I}=-\gamma\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}-\sum_{n=0}^{+\infty}\frac{(-1)^n\ln(2n+1)}{2n+1}=$$

Dirichlet beta function $\beta(s)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^s}$; with $\beta(1)=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}=\frac{\pi}{4}$

$$\beta'(s)=-\sum_{n=0}^{+\infty}\frac{(-1)^n\ln(2n+1)}{(2n+1)^s};\ \text{ then we get:}$$

$$\mathrm{I}=-\gamma\frac{\pi}{4}+\beta'(1)=-\gamma\frac{\pi}{4}+\frac{\pi}{4}(\gamma-\ln\pi)+\pi\ln\Gamma\left(\frac{3}{4}\right)=\pi\left[\ln\Gamma\left(\frac{3}{4}\right)-\frac{1}{4}\ln\pi\right]$$

62. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{\ln(\cos x)\sqrt{\ln(\sin x)}}{\tan x}dx$, let $t=\ln(\sin x)$ $(\sin x=e^t)$, then $dt=\frac{\cos x}{\sin x}dx=\frac{1}{\tan x}dx$

$\cos x=\sqrt{1-\sin^2 x}=\sqrt{1-e^{2t}}$, then we get**:**

For $x=0$, we get $t=-\infty$ and for $x=\frac{\pi}{2}$, then $t=0$, so we get**:**

$$\mathrm{I}=\int_{-\infty}^0\ln\left(\sqrt{1-e^{2t}}\right)\sqrt{t}dt=\int_{-\infty}^0\ln(1-e^{2t})^{\frac{1}{2}}\sqrt{t}dt=\frac{1}{2}\int_{-\infty}^0\ln(1-e^{2t})\sqrt{t}dt$$

The Maclaurin series for $\ln(1-x)$ is: $-\sum_{n=1}^{+\infty}\frac{x^n}{n}$

For $x\in[-1,1[$, substitute x by e^{2t} to get: $\ln(1-e^{2t})=-\sum_{n=1}^{+\infty}\frac{(e^{2t})^n}{n}=-\sum_{n=1}^{+\infty}\frac{e^{2nt}}{n}$

So for $t\in]-\infty,0[$ we get: $\mathrm{I}=-\frac{1}{2}\int_{-\infty}^{0}\sum_{n=1}^{+\infty}\frac{e^{2nt}}{n}\sqrt{t}\,dt=-\frac{1}{2}\sum_{n=1}^{+\infty}\frac{1}{n}\int_{-\infty}^{0}e^{2nt}.\sqrt{t}dt$

Let $2nt=-u$, then $2ndt=-du$, then we get**:**

$$\int_{-\infty}^{0}e^{2nt}.\sqrt{t}dt=\frac{1}{2n}\int_0^{+\infty}e^{-u}.\sqrt{-\frac{u}{2n}}du=\frac{i}{2n\sqrt{2n}}\int_0^{+\infty}e^{-u}\sqrt{u}du=\frac{i}{2n\sqrt{2n}}\int_0^{+\infty}e^{-u}u^{\frac{1}{2}}du$$

Then: $\int_{-\infty}^{0}e^{2nt}.\sqrt{t}dt=\frac{i}{2n\sqrt{2n}}\int_0^{+\infty}e^{-u}u^{\frac{3}{2}-1}du=\frac{i}{2n\sqrt{2n}}\Gamma\left(\frac{3}{2}\right)=\frac{i}{2n\sqrt{2n}}\Gamma\left(\frac{1}{2}+1\right)$

$$\int_{-\infty}^{0}e^{2nt}.\sqrt{t}dt=\frac{i}{2n\sqrt{2n}}\left[\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\right]=\frac{i}{2n\sqrt{2n}}\left(\frac{\sqrt{\pi}}{2}\right)=\frac{i\sqrt{\pi}}{4n\sqrt{2n}};\ \text{ then we get:}$$

$$\mathrm{I}=-\frac{i\sqrt{\pi}}{8\sqrt{2}}\sum_{n=1}^{+\infty}\frac{1}{n^2\sqrt{n}}=-\frac{i\sqrt{\pi}}{8\sqrt{2}}\sum_{n=1}^{+\infty}\frac{1}{n^{\frac{5}{2}}}=-\frac{i\sqrt{\pi}}{8\sqrt{2}}\zeta\left(\frac{5}{2}\right)=\frac{i\sqrt{2\pi}}{16}\zeta\left(\frac{5}{2}\right)$$

63. $\mathrm{I}=\int_0^1 x\left\{\frac{1}{x}\right\}\left\lfloor\frac{1}{x}\right\rfloor dx$, where $\{.\}$ denotes the fractional part function and $\lfloor.\rfloor$ denotes the floor function, let $u=\frac{1}{x}$, then $x=\frac{1}{u}$, so $dx=-\frac{1}{u^2}du$, for $x=0$, then $u=+\infty$ and for $x=1$, then $u=1$, then we get**:**

$\mathrm{I} = \int_{+\infty}^{1} \frac{1}{u}\{u\}\lfloor u\rfloor\left(-\frac{1}{u^2}\right)du = \int_{1}^{+\infty}\frac{\{u\}\lfloor u\rfloor}{u^3}du$

By definition, for $u \geq 0$, we have $\{u\} = u - \lfloor u\rfloor$, then we get**:**

$\mathrm{I} = \int_{1}^{+\infty}\frac{\{u\}\lfloor u\rfloor}{u^3}du = \int_{1}^{+\infty}\frac{(u-\lfloor u\rfloor)\lfloor u\rfloor}{u^3}du = \int_{1}^{+\infty}\left(\frac{\lfloor u\rfloor}{u^2} - \frac{\lfloor u\rfloor^2}{u^3}\right)du$

For $k \in \mathbb{Z}$, we have $\lfloor u\rfloor = k$ for $u \in [k, k+1[$, so we can transform $\int_{1}^{+\infty}\left(\frac{\lfloor u\rfloor}{u^2} - \frac{\lfloor u\rfloor^2}{u^3}\right)du$ into a sum of integrals over the domain $[k, k+1[$ as follows**:**

$$\mathrm{I} = \sum_{k=1}^{+\infty}\int_{k}^{k+1}\left(\frac{k}{u^2} - \frac{k^2}{u^3}\right)du = \sum_{k=1}^{+\infty}\left(\left[-\frac{k}{u}\right]_k^{k+1} - \left[-\frac{k^2}{2u^2}\right]_k^{k+1}\right) = \sum_{k=1}^{+\infty}\left(\frac{1}{k+1} + \frac{k^2}{2(k+1)^2} - \frac{1}{2}\right)$$

Note that: $\dfrac{k^2}{2(k+1)^2} = \dfrac{(k+1-1)^2}{2(k+1)^2} = \dfrac{(k+1)^2 - 2(k+1) + 1}{2(k+1)^2}$; then we get:

$\dfrac{k^2}{2(k+1)^2} = \dfrac{1}{2} - \dfrac{1}{k+1} + \dfrac{1}{2(k+1)^2}$; so

$$\mathrm{I} = \sum_{k=1}^{+\infty}\left(\frac{1}{k+1} + \frac{1}{2} - \frac{1}{k+1} + \frac{1}{2(k+1)^2} - \frac{1}{2}\right) = \sum_{k=1}^{+\infty}\frac{1}{2(k+1)^2} = \frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{(k+1)^2}$$

By re-indexing the sum we get: $\displaystyle\sum_{k=1}^{+\infty}\frac{1}{(k+1)^2} = \sum_{k=2}^{+\infty}\frac{1}{k^2} = \left(\sum_{k=1}^{+\infty}\frac{1}{k^2}\right) - 1$; so we get:

$$\mathrm{I} = \left(\frac{1}{2}\sum_{k=1}^{+\infty}\frac{1}{k^2}\right) - \frac{1}{2} = \frac{1}{2}\zeta(2) - \frac{1}{2} = \frac{1}{2}\left(\frac{\pi^2}{6}\right) - \frac{1}{2} = \frac{\pi^2}{12} - \frac{1}{2}$$

64. $\mathrm{I} = \int_0^{2\pi} e^{e^{ix}}dx$, we have**:**

$$\mathrm{I} = \int_0^{2\pi}\sum_{n=0}^{+\infty}\frac{\left(e^{ix}\right)^n}{n!}dx = \int_0^{2\pi}\left(1 + \sum_{n=1}^{+\infty}\frac{e^{inx}}{n!}\right)dx = \int_0^{2\pi}dx + \sum_{n=1}^{+\infty}\frac{1}{n!}\int_0^{2\pi}e^{inx}\,dx;\ \text{ then we get:}$$

$$\mathrm{I} = 2\pi + \sum_{n=1}^{+\infty}\frac{1}{n!}\left[\frac{1}{in}e^{inx}\right]_0^{2\pi} = 2\pi + \sum_{n=1}^{+\infty}\frac{1}{n!}\left[\frac{1}{in}\left(e^{i2\pi n}\right) - e^0\right] = 2\pi + \sum_{n=1}^{+\infty}\frac{1}{n!}[1-1] = 2\pi$$

65. $\mathrm{I} = \int_0^{+\infty}\ln\left(\frac{a+be^{-px}}{a+be^{-qx}}\right).\frac{dx}{x} = \int_0^{+\infty}\ln\left(\frac{1+\frac{b}{a}e^{-px}}{1+\frac{b}{a}e^{-qx}}\right).\frac{dx}{x} = \int_0^{+\infty}\left[\frac{\ln\left(1+\frac{b}{a}e^{-px}\right)}{x} - \frac{\ln\left(1+\frac{b}{a}e^{-qx}\right)}{x}\right]dx$

$$\mathrm{I} = \int_0^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{b}{a}e^{-px}\right)^n.\frac{1}{x}dx - \int_0^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{b}{a}e^{-qx}\right)^n.\frac{1}{x}dx;\ \text{ then we get:}$$

$$\mathrm{I} = \int_0^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{b}{a}\right)^n\frac{e^{-npx}}{x}dx - \int_0^{+\infty}\sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{b}{a}\right)^n\frac{e^{-nqx}}{x}dx$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{b}{a}\right)^n\int_0^{+\infty}\frac{e^{-npx}}{x}dx - \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{b}{a}\right)^n\int_0^{+\infty}\frac{e^{-nqx}}{x}dx;\ \text{ then we get:}$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\frac{(-1)^{n+1}}{n}\left(\frac{b}{a}\right)^n\int_0^{+\infty}\frac{e^{-npx} - e^{-nqx}}{x}dx$$

Remark: Frullani integrals are a specific type of improper integral named after the Italian mathematician Giulliano Frullani such that:

$$\int_0^{+\infty} \frac{f(ax)-f(bx)}{x}\,dx = \big(f(\infty)-f(0)\big)\ln\left(\frac{a}{b}\right)$$

Where f is a function defined for all non-negative real numbers that has a limit at ∞, then we get:

$$\mathrm{I} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\left(\frac{b}{a}\right)^n \int_0^{+\infty} \frac{e^{-npx}-e^{-nqx}}{x}\,dx = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\left(\frac{b}{a}\right)^n.\ln\left(\frac{nq}{np}\right);\ \text{ then we get:}$$

$$\mathrm{I} = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\left(\frac{b}{a}\right)^n.\ln\left(\frac{q}{p}\right) = \ln\left(1+\frac{b}{a}\right).\ln\left(\frac{q}{p}\right) = \ln\left(\frac{a}{a+b}\right).\ln\left(\frac{q}{p}\right)$$

66. $\mathrm{I} = \int_0^{+\infty} \frac{\sin x}{e^x(e^x-1)}dx = \int_0^{+\infty} \frac{[e^x-(e^x-1)]}{e^x(e^x-1)}\sin x\,dx = \int_0^{+\infty} \frac{\sin x}{e^x-1}dx - \int_0^{+\infty} \frac{\sin x}{e^x}dx$

$\mathrm{I} = \int_0^{+\infty} \frac{\sin x}{e^x-1}dx - \int_0^{+\infty} e^{-x}\sin x\,dx = \int_0^{+\infty} \sin x\,\frac{e^{-x}}{1-e^{-x}}dx - \int_0^{+\infty} e^{-x}\sin x\,dx$

$$\mathrm{I} = \int_0^{+\infty} \sin x\,.\sum_{n=1}^{+\infty}(e^{-x})^n\,dx - \int_0^{+\infty} e^{-x}\sin x\,dx = \int_0^{+\infty} \sin x\,.\sum_{n=1}^{+\infty} e^{-nx}\,dx - \int_0^{+\infty} e^{-x}\sin x\,dx$$

$$\mathrm{I} = \sum_{n=1}^{+\infty}\int_0^{+\infty} e^{-nx}\sin x\,dx - \int_0^{+\infty} e^{-x}\sin x\,dx$$

Recall: $\int_0^{+\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$; so for $a=b=1$; we get $\int_0^{+\infty} e^{-x}\sin x\,dx = \frac{1}{2}$; so

$$\mathrm{I} = \sum_{n=1}^{+\infty}\frac{1}{1+n^2} - \frac{1}{2};\ \text{ but we have } \sum_{n=}^{+\infty}\frac{1}{a^2+n^2} = \frac{\pi a-1}{2a^2} + \frac{\pi}{a(e^{2\pi a}-1)};\ \text{ then we get:}$$

$$\mathrm{I} = \frac{\pi-1}{2} + \frac{\pi}{e^{2\pi}-1} - \frac{1}{2} = \frac{\pi-2}{2} + \frac{\pi}{e^{2\pi}-1}$$

67. $\mathrm{I} = \int_0^{+\infty} e^{-x}\frac{\sin(tx)}{\sinh x}dx = \int_0^{+\infty}\frac{1}{e^x}.\frac{\sin(tx)}{\frac{1}{2}(e^x-e^{-x})}dx = \int_0^{+\infty}\frac{1}{e^x}.\frac{2\sin(tx)}{e^x-e^{-x}}dx$

$\mathrm{I} = 2\int_0^{+\infty}\sin(tx).\frac{1}{e^{2x}-1}dx = 2\int_0^{+\infty}\sin(tx).\frac{e^{-2x}}{1-e^{-2x}}dx$

$$\mathrm{I} = 2\int_0^{+\infty}\sin(tx).\sum_{n=1}^{+\infty}e^{-2nx}dx = 2\sum_{n=1}^{+\infty}\int_0^{+\infty}e^{-2nx}.\sin(tx)\,dx$$

But since $\int_0^{+\infty} e^{-ax}\sin(bx)\,dx = \frac{b}{a^2+b^2}$; then we get: $\mathrm{I} = 2\sum_{n=1}^{+\infty}\frac{t}{(2n)^2+t^2}$; then we get

$$\mathrm{I} = 2\sum_{n=1}^{+\infty}\frac{t}{4n^2+t^2} = 2t\sum_{n=1}^{+\infty}\frac{1}{(2n+it)(2n-it)}$$

From $\sum_{n=1}^{+\infty}\frac{1}{(n+a)(n+b)} = \frac{\Psi(b+1)-\Psi(a+1)}{b-a}$; where $b\neq a$; then we get:

$$\mathrm{I}=\frac{1}{2}t\left[\frac{\Psi\left(1+i\frac{t}{2}\right)-\Psi\left(1-i\frac{t}{2}\right)}{it}\right]=-\frac{1}{2}i\left[\Psi\left(1+i\frac{t}{2}\right)-\Psi\left(1-i\frac{t}{2}\right)\right]$$

$$\mathrm{I}=-\frac{1}{2}i\left[\Psi\left(\frac{it}{2}\right)+\frac{2}{it}-\Psi\left(1-i\frac{t}{2}\right)\right]=-\frac{1}{t}-\frac{1}{2}i\left[\Psi\left(\frac{it}{2}\right)-\Psi\left(1-i\frac{t}{2}\right)\right]$$

$$\mathrm{I}=-\frac{1}{t}-\frac{1}{2}i\left[-\pi\cot\left(\pi.\frac{it}{2}\right)\right]=-\frac{1}{t}+\frac{1}{2}\pi i\left[\cot\left(i\frac{\pi t}{2}\right)\right]$$

Where $\cot\left(i\frac{\pi t}{2}\right)=\frac{\cos\left(i\frac{\pi t}{2}\right)}{\sin\left(i\frac{\pi t}{2}\right)}=\frac{\frac{e^{i\left(i\frac{\pi t}{2}\right)}+e^{-i\left(i\frac{\pi t}{2}\right)}}{2}}{\frac{e^{i\left(i\frac{\pi t}{2}\right)}-e^{-i\left(i\frac{\pi t}{2}\right)}}{2i}}=-i\frac{\cosh\left(\frac{\pi t}{2}\right)}{\sinh\left(\frac{\pi t}{2}\right)}=-i\coth\left(\frac{\pi t}{2}\right)$; then:

$$\mathrm{I}=-\frac{1}{t}+i\frac{\pi}{2}\left[-i\coth\left(\frac{\pi t}{2}\right)\right]=\frac{\pi}{2}\coth\left(\frac{\pi t}{2}\right)-\frac{1}{t}$$

68. $\mathrm{I}=\int_0^{\frac{1}{n}}\frac{\cos(\ln nx)-1}{\ln nx}dx=\int_0^{\frac{1}{n}}\frac{1}{\ln nx}[\cos(\ln nx)-1]dx$

$$\mathrm{I}=\int_0^{\frac{1}{n}}\frac{1}{\ln nx}\left[1-\frac{(\ln nx)^2}{2!}+\frac{(\ln nx)^2}{4!}-\cdots-1\right]dx=-\int_0^{\frac{1}{n}}\frac{1}{\ln nx}\left[\frac{(\ln nx)^2}{2!}-\frac{(\ln nx)^2}{4!}+\cdots\right]dx$$

$$\mathrm{I}=-\int_0^{\frac{1}{n}}\left[\frac{\ln nx}{2!}-\frac{(\ln nx)^3}{4!}+\frac{(\ln nx)^5}{6!}-\frac{(\ln nx)^7}{8!}+\cdots\right]dx\text{; then we get:}$$

$$\mathrm{I}=-\int_0^{\frac{1}{n}}\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}.(\ln nx)^{2k-1}}{(2k)!}=-\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{(2k)!}\int_0^{\frac{1}{n}}(\ln nx)^{2k-1}dx$$

Let $-t=\ln nx$, then $nx=e^{-t}$ and $x=\frac{1}{n}e^{-t}$, so $dx=-\frac{1}{n}e^{-t}dt$

For $x=0$, $t=+\infty$ and for $x=\frac{1}{n}$, then $t=0$, then we get**:**

$$\mathrm{I}=-\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{(2k)!}\int_{+\infty}^{0}(-1)^{2k-1}.t^{2k-1}\left(-\frac{1}{n}e^{-t}\right)dt=-\frac{1}{n}\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}\int_0^{+\infty}e^{-t}.t^{2k-1}dt$$

$$\mathrm{I}=-\frac{1}{n}\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}\Gamma(2k)=-\frac{1}{n}\sum_{k=1}^{+\infty}\frac{(-1)^k}{(2k)!}(2k-1)!=-\frac{1}{2n}\sum_{k=1}^{+\infty}\frac{(-1)^k}{k}$$

$$\mathrm{I}=\frac{1}{2n}\sum_{k=1}^{+\infty}\frac{(-1)^{k+1}}{k}=\frac{1}{2n}\ln 2=\frac{\ln 2}{2n}$$

69. $\mathrm{I}=\int_0^{\frac{\pi}{2}}\{\tan x\}\,dx$, let $u=\tan x$, then $x=\arctan u$ and $dx=\frac{du}{1+u^2}du$

For $x=0$, $u=0$ and for $x=\frac{\pi}{2}$, then $u=+\infty$, then we get**:**

$\mathrm{I}=\int_0^{+\infty}\frac{\{u\}}{1+u^2}du$, but we have $\{u\}=u-\lfloor u\rfloor$, then $\mathrm{I}=\int_0^{+\infty}\frac{u-\lfloor u\rfloor}{1+u^2}du$ and so we can write**:**

$$\mathrm{I}=\lim_{\mathrm{N}\to+\infty}\int_0^{\mathrm{N}}\frac{u-\lfloor u\rfloor}{u^2+1}du=\lim_{\mathrm{N}\to+\infty}\int_0^{\mathrm{N}}\frac{u}{u^2+1}du-\lim_{\mathrm{N}\to+\infty}\int_0^{\mathrm{N}}\frac{\lfloor u\rfloor}{u^2+1}du\text{; then we get:}$$

$$\mathrm{I}=\lim_{\mathrm{N}\to+\infty}\frac{1}{2}\int_0^{\mathrm{N}}\frac{2u}{u^2+1}du-\lim_{\mathrm{N}\to+\infty}\int_0^{\mathrm{N}}\frac{\lfloor u\rfloor}{u^2+1}du=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\lim_{\mathrm{N}\to+\infty}\int_0^{\mathrm{N}}\frac{\lfloor u\rfloor}{u^2+1}du$$

Since $\lfloor u\rfloor=k\quad\forall\,u\in[k,k+1[$, where $k\in\mathbb{N}$, then we get:

$$\mathrm{I}=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\sum_{k=0}^{\mathrm{N}-1}\int_k^{k+1}\frac{k}{u^2+1}du=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\sum_{k=0}^{\mathrm{N}-1}k[\arctan u]_k^{k+1}$$

$$\mathrm{I}=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\sum_{k=0}^{\mathrm{N}-1}[k\arctan(k+1)-k\arctan k]$$

$$\mathrm{I}=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\sum_{k=0}^{\mathrm{N}-1}[k\arctan(k+1)-k\arctan k]$$

$$\mathrm{I}=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\sum_{k=0}^{\mathrm{N}-1}[(k+1)\arctan(k+1)-k\arctan k-\arctan(k+1)]$$

$\sum_{k=0}^{\mathrm{N}-1}[(k+1)\arctan(k+1)-k\arctan k]$ is telescoping and its equal to $\mathrm{N}\arctan(\mathrm{N}+1)$ then

$$\mathrm{I}=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\mathrm{N}\arctan(\mathrm{N}+1)+\sum_{k=1}^{\mathrm{N}}\arctan k$$

$$\mathrm{I}=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\mathrm{N}\arctan(\mathrm{N}+1)+\sum_{k=1}^{\mathrm{N}}\left[\frac{\pi}{2}-\arctan\left(\frac{1}{k}\right)\right]$$

$$\mathrm{I}=1+\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\sum_{k=1}^{\mathrm{N}}\arctan\left(\frac{1}{k}\right)$$

Since $\frac{1}{k}\leq 1\ \forall k$ we can use the series expansion of $\arctan\left(\frac{1}{k}\right)$ and substitute it into the last result; then we get:

$$\mathrm{I}=1+\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\sum_{k=1}^{\mathrm{N}}\sum_{n=0}^{+\infty}\frac{(-1)^k k^{-(2n+1)}}{2n+1};\ \text{ then we get:}$$

$$\mathrm{I}=1+\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\sum_{k=1}^{\mathrm{N}}\frac{1}{k}-\sum_{k=1}^{\mathrm{N}}\sum_{n=1}^{+\infty}\frac{(-1)^k k^{-(2n+1)}}{2n+1}$$

$$\mathrm{I}=1+\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\mathrm{H_N}-\sum_{k=1}^{\mathrm{N}}\sum_{n=1}^{+\infty}\frac{(-1)^k k^{-(2n+1)}}{2n+1}$$

Let's take a closer look at $\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\mathrm{H_N}$; where $\mathrm{H_N}\sim\ln\mathrm{N}+\gamma$; then we get:

$$\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)}{2}-\mathrm{H_N}=\lim_{\mathrm{N}\to+\infty}\frac{\ln(1+\mathrm{N}^2)-\ln(\mathrm{N}^2)}{2}-\gamma=-\gamma;\ \text{so}$$

$$\mathrm{I}=1-\gamma-\sum_{k=1}^{\mathrm{N}}\sum_{n=1}^{+\infty}\frac{(-1)^k k^{-(2n+1)}}{2n+1};$$ but both sums converge uniformly and we can interchange the order of summation to get:

$$\mathrm{I} = 1 - \gamma - \sum_{n=1}^{+\infty} \frac{(-1)^n}{2n+1} \sum_{k=1}^{+\infty} \frac{1}{k^{2n+1}} = 1 - \gamma - \sum_{n=1}^{+\infty} \frac{(-1)^n \zeta(2n+1)}{2n+1}$$

But we have $\Psi(1-x) = -\gamma - \sum_{n=1}^{+\infty} \zeta(2n+1)x^n$

$$\Psi(1-x) = -\gamma - \sum_{n=1}^{+\infty} \zeta(2n+1)x^{2n-1} - \sum_{n=1}^{+\infty} \zeta(2n+1)x^{2n}$$

And $\Psi(1+x) = -\gamma - \sum_{n=1}^{+\infty} \zeta(2n+1)(-1)^n x^n$

$\Psi(1+x) = -\gamma - \sum_{n=1}^{+\infty} \zeta(2n+1)x^{2n} + \sum_{n=1}^{+\infty} \zeta(2n+1)x^{2n-1}$; therefore we get:

$$\frac{\Psi(1+x) + \Psi(1-x)}{2} = -\gamma - \sum_{n=1}^{+\infty} \zeta(2n+1)x^{2n}$$

To get a form similar to our series we will intwgrate both sides w.r.t x to get:

$\frac{\ln[\Gamma(1+x)] - \ln[\Gamma(1-x)]}{2} + \gamma x = -\sum_{n=1}^{+\infty} \frac{\zeta(2n+1)x^{2n+1}}{2n+1}$; now let $x = i$; we get:

$\frac{\ln\left(\frac{\Gamma(1+i)}{\Gamma(1-i)}\right)}{2} + \gamma i = -i \sum_{n=1}^{+\infty} \frac{\zeta(2n+1)i^{2n}}{2n+1}$; with $i^{2n} = (i^2)^n = (-1)^n$; then we get:

$$\sum_{n=1}^{+\infty} \frac{(-1)^n \zeta(2n+1)}{2n+1} = -\frac{\ln\left(\frac{\Gamma(1+i)}{\Gamma(1-i)}\right)}{2i} - \gamma$$

Therefore; we get: $\mathrm{I} = 1 - \gamma - \left[-\frac{\ln\left(\frac{\Gamma(1+i)}{\Gamma(1-i)}\right)}{2i} - \gamma\right] 1 = 1 + \frac{\ln\left(\frac{\Gamma(1+i)}{\Gamma(1-i)}\right)}{2i}$

70. $I = \int_1^{+\infty} \frac{\ln(\ln x)}{1+2x\cos\alpha+x^2}dx$ **Malmsten Integral**

<u>This might be one of the coolest integrals in calculus, it shows the beauty of mathematics</u>

$$\mathbf{I} = \int_1^{+\infty} \frac{\ln(\ln x)}{1+2x\cos\alpha+x^2}dx; \quad \text{where } -\pi < \alpha < \pi$$

<u>Solution</u>

Let $u = \frac{1}{x}$, then $x = \frac{1}{u} \Longrightarrow dx = -\frac{1}{u^2}du$, for $x = 1 \Longrightarrow u = 1$ and for $x = +\infty \Longrightarrow u = 0$, then we get:

$$I = \int_1^0 \frac{\ln\left(\ln\left(\frac{1}{u}\right)\right)}{1+\frac{2}{u}\cos\alpha+\frac{1}{u^2}}\left(-\frac{1}{u^2}\right)du = \int_0^1 \frac{\ln\left(\ln\left(\frac{1}{u}\right)\right)}{u^2+2u\cos\alpha+1}du = \int_0^1 \frac{\ln\left(\ln\left(\frac{1}{x}\right)\right)}{1+2x\cos\alpha+x^2}dx$$

But $\cos\alpha = \frac{e^{i\alpha}+e^{-i\alpha}}{2} \Longrightarrow e^{i\alpha}+e^{-i\alpha} = 2\cos\alpha$, then we get:

$1+2x\cos\alpha+x^2 = 1+x\left(e^{i\alpha}+e^{-i\alpha}\right)+x^2 = \left(1+xe^{i\alpha}\right)\left(1+xe^{-i\alpha}\right)$, then we get:

$$\frac{1}{1+2x\cos\alpha+x^2} = \frac{1}{(1+xe^{i\alpha})(1+xe^{-i\alpha})} = \left(\frac{1}{1+xe^{i\alpha}}\right)\left(\frac{1}{1+xe^{-i\alpha}}\right); \text{ then we get}$$

$$\frac{1}{1+2x\cos\alpha+x^2} = \left[\frac{1}{1-(-xe^{i\alpha})}\right]\left[\frac{1}{1-(-xe^{-i\alpha})}\right]; \text{ with } \left|xe^{i\alpha}\right| = |x| < 1 \text{ and } \frac{1}{1-\mathrm{r}} = \sum_{n=0}^{\infty}\mathrm{r}^n; \text{ so}$$

$$\frac{1}{1+2x\cos\alpha+x^2} = \left[\sum_{n=0}^{\infty}\left(-xe^{i\alpha}\right)^n\right]\left[\sum_{n=0}^{\infty}\left(-xe^{-i\alpha}\right)^n\right] = \left[\sum_{n=0}^{\infty}(-1)^n\left(xe^{i\alpha}\right)^n\right]\left[\sum_{n=0}^{\infty}(-1)^n\left(xe^{-i\alpha}\right)^n\right]$$

Then $\frac{1}{1+2x\cos\alpha+x^2} = \left(1-xe^{i\alpha}+x^2e^{2i\alpha}-x^3e^{3i\alpha}+\cdots\right)\left(1-xe^{-i\alpha}+x^2e^{-2i\alpha}-x^3e^{-3i\alpha}+\cdots\right)$

$= 1-x\left(e^{i\alpha}+e^{-i\alpha}\right)+x^2\left(e^{2i\alpha}+1+e^{-2i\alpha}\right)-x^3\left(e^{3i\alpha}+e^{i\alpha}+e^{-i\alpha}+e^{-3i\alpha}\right)+\cdots$

Then $\frac{1}{1+2x\cos\alpha+x^2} = a_0-a_1x+a_2x^2-a_3x^3+\cdots = \sum_{n=0}^{\infty}(-1)^n a_n x^n$

$$a_0 = 1 = e^0 = e^{0i\alpha-2(0)i\alpha}; \; a_1 = e^{i\alpha}+e^{-i\alpha} = e^{i\alpha-2(0)i\alpha}+e^{i\alpha-2i\alpha} = \sum_{k=0}^{1}e^{i\alpha-2ki\alpha}$$

$$a_2 = e^{2i\alpha}+1+e^{-2i\alpha} = e^{2i\alpha}+e^{2i\alpha-2i\alpha}+e^{2i\alpha-4i\alpha} = \sum_{k=0}^{2}e^{2i\alpha-2ki\alpha} \text{ ... then we get}$$

$$a_n = \sum_{k=0}^{n}e^{in\alpha-2ki\alpha} = e^{in\alpha}\sum_{k=0}^{n}e^{-2ki\alpha} = e^{in\alpha}\sum_{k=0}^{n}\left(e^{-2i\alpha}\right)^k = e^{in\alpha}\left[\frac{1-\left(e^{-2i\alpha}\right)^{n+1}}{1-e^{-2i\alpha}}\right]$$

$$a_n = e^{in\alpha}\times\frac{e^{i\alpha}}{e^{i\alpha}}\left[\frac{1-e^{-2(n+1)i\alpha}}{1-e^{-2i\alpha}}\right] = e^{i(n+1)\alpha}\times\frac{1-e^{-2(n+1)i\alpha}}{e^{i\alpha}-e^{-i\alpha}} = \frac{e^{(n+1)i\alpha}-e^{-(n+1)i\alpha}}{e^{i\alpha}-e^{-i\alpha}}; \text{then we get:}$$

$$a_n = \frac{\frac{e^{(n+1)i\alpha}-e^{-(n+1)i\alpha}}{2}}{\frac{e^{i\alpha}-e^{-i\alpha}}{2}} = \frac{\sin\left((n+1)\alpha\right)}{\sin\alpha}; \text{ with } \frac{1}{1+2x\cos\alpha+x^2} = \sum_{k=0}^{1}e^{i\alpha-2ki\alpha} \Longrightarrow$$

$$\frac{1}{1+2x\cos\alpha+x^2}=\sum_{n=0}^{\infty}(-1)^n\frac{\sin\big((n+1)\alpha\big)}{\sin\alpha}x^n\ ;\ \text{ with } \ \mathrm{I}=\int_0^1\frac{\ln\left(\ln\left(\frac{1}{x}\right)\right)}{1+2x\cos\alpha+x^2}dx\ ;\ \text{ then we get}$$

$$\mathrm{I}=\int_0^1\ln\left(\ln\left(\frac{1}{x}\right)\right)\left(\frac{1}{1+2x\cos\alpha+x^2}\right)dx=\int_0^1\ln\left(\ln\left(\frac{1}{x}\right)\right)\left(\sum_{n=0}^{\infty}(-1)^n\frac{\sin\big((n+1)\alpha\big)}{\sin\alpha}x^n\right)dx\Rightarrow$$

$$\mathrm{I}=\frac{1}{\sin\alpha}\sum_{n=0}^{\infty}(-1)^n\sin\big((n+1)\alpha\big)\int_0^1\ln\left(\ln\left(\frac{1}{x}\right)\right)x^n dx=\frac{1}{\sin\alpha}\sum_{n=0}^{\infty}(-1)^n\sin\big((n+1)\alpha\big)\,\mathrm{J};\ \text{ where}$$

$$\mathrm{J}=\int_0^1\ln\left(\ln\left(\frac{1}{x}\right)\right)x^n dx\ ;\ \text{ let } y=\ln\left(\frac{1}{x}\right)=-\ln x\Rightarrow dy=-\frac{1}{x}dx\Rightarrow dx=-xdy=-e^{-y}dy$$

For $x=0\Rightarrow y=+\infty$ and for $x=1\Rightarrow y=0\Rightarrow$

$$\mathrm{J}=\int_{+\infty}^{0}\ln y\,(e^{-y})^n(-e^{-y})dy=\int_0^{+\infty}\ln y\,e^{-(n+1)y}dy\ \text{ let } v=(n+1)y\Rightarrow y=\frac{1}{n+1}v\Rightarrow dy=\frac{1}{n+1}dv$$

$$\Rightarrow\mathrm{J}=\int_0^{+\infty}\ln\left(\frac{v}{n+1}\right)e^{-v}.\frac{1}{n+1}dv=\frac{1}{n+1}\int_0^{+\infty}[\ln v-\ln(n+1)]e^{-v}dx\ ;\ \text{ then we get:}$$

$$\mathrm{J}=\frac{1}{n+1}\left[\int_0^{+\infty}e^{-v}\ln v\,dv-\ln(n+1)\int_0^{+\infty}e^{-v}dx\right]\ \text{ with }\int_0^{+\infty}e^{-v}\ln v\,dv=-\gamma\ \text{ and }\int_0^{+\infty}e^{-v}dx=1$$

$$\Rightarrow\mathrm{J}=-\frac{1}{n+1}[\gamma+\ln(n+1)];\ \text{ with }\ \mathrm{I}=\frac{1}{\sin\alpha}\sum_{n=0}^{\infty}(-1)^n\sin\big((n+1)\alpha\big)\,\mathrm{J};\ \text{ then we get:}$$

$$\mathrm{I}=\frac{1}{\sin\alpha}\sum_{n=0}^{\infty}(-1)^n\sin\big((n+1)\alpha\big)\left[-\frac{1}{n+1}[\gamma+\ln(n+1)]\right];\ \text{ then we get:}$$

$$\mathrm{I}=-\frac{1}{\sin\alpha}\left[\gamma\sum_{n=0}^{\infty}(-1)^n\frac{\sin\big((n+1)\alpha\big)}{n+1}+\sum_{n=0}^{\infty}(-1)^n\frac{\ln(n+1)}{n+1}\sin\big((n+1)\alpha\big)\right]$$

Then by shifting the sums by 1 we get: $\mathrm{I}=-\frac{1}{\sin\alpha}\left[\gamma\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin(k\alpha)}{k}+\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\ln k}{k}\sin(k\alpha)\right]$

But $(-1)^{k-1}\frac{\ln k}{k}\sin(k\alpha)=0$ for $k=1$; so $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\ln k}{k}\sin(k\alpha)=\sum_{k=2}^{\infty}(-1)^{k-1}\frac{\ln k}{k}\sin(k\alpha)$

$$\Rightarrow\ \mathrm{I}=-\frac{1}{\sin\alpha}\left[\gamma\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin(k\alpha)}{k}+\sum_{k=2}^{\infty}(-1)^{k-1}\frac{\ln k}{k}\sin(k\alpha)\right]=-\frac{1}{\sin\alpha}[\gamma\mathrm{A}+\mathrm{B}]$$

Where $\mathrm{A}=\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin(k\alpha)}{k}$ and $\mathrm{B}=\sum_{k=2}^{\infty}(-1)^{k-1}\frac{\ln k}{k}\sin(k\alpha)$

Calculating A: Consider the function $f(x)=x$ defined over the interval $]-\pi;\pi[$

The Fourier series of f is given by: $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi x}{\mathrm{L}}\right)+b_n\sin\left(\frac{n\pi x}{\mathrm{L}}\right)\right]$

Knowing that f is an odd function , so $a_0=0$ and $a_n=0$, then we get**:**

$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\mathrm{L}}\right)$ or $f(x) = \sum_{k=1}^{\infty} b_n \sin(kx)$ with $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(kx)$; then we get:

$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(kx)$; sing I. BP; let $u = x$; then $u' = 1$ and let $v' = \sin(kx)$; then $v = -\frac{1}{k}\cos(kx)$

Then $b_n = \frac{2}{\pi}\left(\left[-\frac{x}{k}\cos(kx)\right]_0^{\pi} + \int_0^{\pi}\frac{1}{k}\cos(kx)\,dx\right) = \frac{2}{\pi}\left(-\pi\cos\frac{(k\pi)}{k} + \frac{1}{k^2}[\sin kx]_0^{\pi}\right)$; then we get

$b_n = \frac{2}{\pi}\times\pi\times(-1)^1\times\frac{(-1)^k}{k} = \frac{2(-1)^{k+1}}{k} = \frac{2(-1)^{k-1}}{k}$; so $f(x) = \sum_{k=1}^{\infty}\frac{2}{k}(-1)^{k-1}\sin(kx)$

Now substitute $x = \alpha$; then we get $\alpha = \sum_{k=1}^{\infty}\frac{2}{k}(-1)^{k-1}\sin(k\alpha)$; then $\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\sin(k\alpha)}{k} = \frac{\alpha}{2}$

Therefore we get $\mathrm{A} = \frac{\alpha}{2}$

Now Calculating B: We will use the Fourier series expansion of the logarithm of the gamma function

$$\mathrm{B} = \sum_{k=2}^{\infty}(-1)^{k-1}\frac{\ln k}{k}\sin(k\alpha)$$

We have from Kummer's series:

$$\ln(\Gamma(x)) = \left(\frac{1}{2} - x\right)(\gamma + \ln 2) + (1 - x)\ln\pi - \frac{1}{2}\ln(\sin(\pi x)) + \frac{1}{\pi}\sum_{k=2}^{\infty}\frac{\ln k\sin(2\pi kx)}{k}$$

Now we will choose x such that: $2\pi kx = \pi k - \alpha k \Longrightarrow 2\pi x = \pi - \alpha \Longrightarrow x = \frac{\pi-\alpha}{2\pi} \Longrightarrow x = \frac{1}{2} - \frac{\alpha}{2\pi}$

Then we have , $\frac{1}{2} - x = \frac{\alpha}{2\pi}$ and $1 - x = \frac{1}{2} + \frac{\alpha}{2\pi}$ also we have: $\sin(\pi x) = \sin\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \cos\left(\frac{\alpha}{2}\right)$ and $\sin(2\pi kx) = \sin(\pi k - \alpha k) = \sin(\pi k)\cos(\alpha k) - \cos(\pi k)\sin(\alpha k) = 0 - \cos(\pi k)\sin(\alpha k)$

Then we get: $\sin(2\pi kx) = (-1)\times(-1)^k\sin(\alpha k) = (-1)^{k-1}\sin(\alpha k)$

Now we will substitute all the simplified values in Kummer's series to get:

$$\ln\left(\Gamma\left(\frac{1}{2} - \frac{\alpha}{2\pi}\right)\right) = \frac{\alpha}{2\pi}(\gamma + \ln 2) + \left(\frac{1}{2} + \frac{\alpha}{2\pi}\right)\ln\pi - \frac{1}{2}\cos\left(\frac{\alpha}{2}\right) + \frac{1}{\pi}\sum_{k=2}^{\infty}(-1)^{k-1}\frac{\sin(\alpha k)\ln k}{k}$$

With $\mathrm{B} = \sum_{k=2}^{\infty}(-1)^{k-1}\frac{\ln k}{k}\sin(k\alpha)$; then we get:

$$\mathrm{B} = \pi\ln\left(\Gamma\left(\frac{1}{2} - \frac{\alpha}{2\pi}\right)\right) - \frac{\alpha}{2}\gamma - \frac{\alpha}{2}\ln 2 - \frac{\pi}{2}\ln\pi - \frac{\alpha}{2}\ln\pi + \frac{\pi}{2}\ln\left(\cos\left(\frac{\alpha}{2}\right)\right)$$

$$\mathrm{B} = \pi\ln\left(\Gamma\left(\frac{1}{2} - \frac{\alpha}{2\pi}\right)\right) - \frac{\alpha}{2}\gamma - \frac{\alpha}{2}\ln(2\pi) - \frac{\pi}{2}\ln\pi + \frac{\pi}{2}\ln\left(\cos\left(\frac{\alpha}{2}\right)\right)$$

But we have $\mathrm{I} = -\frac{1}{\sin\alpha}[\gamma\mathrm{A} + \mathrm{B}]$; so substituting the results of A and B in I we get:

$\mathrm{I} = -\frac{1}{\sin\alpha}\left[\gamma\frac{\alpha}{2} + \pi\ln\left(\Gamma\left(\frac{1}{2} - \frac{\alpha}{2\pi}\right)\right) - \frac{\alpha}{2}\gamma - \frac{\alpha}{2}\ln(2\pi) - \frac{\pi}{2}\ln\pi + \frac{\pi}{2}\ln\left(\cos\left(\frac{\alpha}{2}\right)\right)\right]$; then we get:

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\left[-\ln\left(\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)\right)+\frac{\alpha}{2\pi}\ln(2\pi)+\frac{1}{2}\ln\pi-\frac{1}{2}\ln\left(\cos\left(\frac{\alpha}{2}\right)\right)\right]$$

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\left[-\ln\left(\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)\right)+\ln(2\pi)^{\frac{\alpha}{2\pi}}+\ln\sqrt{\pi}-\ln\left(\sqrt{\cos\left(\frac{\alpha}{2}\right)}\right)\right];\text{ then we get:}$$

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\left\{\ln(2\pi)^{\frac{\alpha}{2\pi}}+\ln\sqrt{\pi}-\left[\ln\left(\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)\right)+\ln\left(\sqrt{\cos\left(\frac{\alpha}{2}\right)}\right)\right]\right\}$$

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\left\{\ln\left[(2\pi)^{\frac{\alpha}{2\pi}}\sqrt{\pi}\right]-\ln\left[\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)\sqrt{\cos\left(\frac{\alpha}{2}\right)}\right]\right\};\text{ therefore we get:}$$

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\ln\left[\frac{(2\pi)^{\frac{\alpha}{2\pi}}\sqrt{\pi}}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)\sqrt{\cos\left(\frac{\alpha}{2}\right)}}\right]$$

Now we will use Euler's reflection formula: $\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin(\pi z)}$, for $z=\frac{1}{2}-\frac{\alpha}{2\pi}$, then

$1-z=1-\frac{1}{2}=\frac{\alpha}{2\pi}=\frac{1}{2}+\frac{\alpha}{2\pi}$, then we get:

$$\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)=\frac{\pi}{\sin\left[\pi\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)\right]}=\frac{\pi}{\sin\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)}=\frac{\pi}{\cos\left(\frac{\alpha}{2}\right)};\quad\text{then we get:}$$

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\ln\left[\frac{(2\pi)^{\frac{\alpha}{2\pi}}}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}\sqrt{\frac{\pi}{\cos\left(\frac{\alpha}{2}\right)}}\right]=\frac{\pi}{\sin\alpha}\ln\left[\frac{(2\pi)^{\frac{\alpha}{2\pi}}}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}.\sqrt{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)}\right];\text{then}$$

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\ln\left[\frac{(2\pi)^{\frac{\alpha}{2\pi}}}{\sqrt{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}.\sqrt{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}}.\sqrt{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}.\sqrt{\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)}\right];\quad\text{then we get:}$$

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\ln\left[\frac{(2\pi)^{\frac{\alpha}{2\pi}}}{\sqrt{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}}\sqrt{\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)}\right]=\frac{\pi}{\sin\alpha}\ln\left[(2\pi)^{\frac{\alpha}{2\pi}}\sqrt{\frac{\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}}\right];\quad\text{then we get:}$$

$$\mathrm{I}=\frac{\pi}{\sin\alpha}\ln\left[\left((2\pi)^{\frac{\alpha}{\pi}}\right)^{\frac{1}{2}}\left(\frac{\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}\right)^{\frac{1}{2}}\right]=\frac{\pi}{\sin\alpha}\ln\left[(2\pi)^{\frac{\alpha}{\pi}}\frac{\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}\right]^{\frac{1}{2}}=\frac{\pi}{2\sin\alpha}\left((2\pi)^{\frac{\alpha}{\pi}}\frac{\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}\right)$$

Therefore; we get: $$\boldsymbol{\int_1^{+\infty}\frac{\ln(\ln x)}{1+2x\cos\alpha+x^2}dx=\frac{\pi}{2\sin\alpha}\left((2\pi)^{\frac{\alpha}{\pi}}\frac{\Gamma\left(\frac{1}{2}+\frac{\alpha}{2\pi}\right)}{\Gamma\left(\frac{1}{2}-\frac{\alpha}{2\pi}\right)}\right)}$$

End of Chapter 5

Chapter 6: Miscellaneous Integrals

Evaluate each of the following integrals:

1. $\int_1^e \pi^{\ln x} dx$
2. $\int x^{-2} 5^{\frac{1}{x}} dx$
3. $\int_1^{\sqrt{e}} x.\left(\sqrt{e}\right)^{-\log_{\sqrt{e}}(x)} dx$
4. $\int_0^1 x^{70}(1-x)^{30} dx$
5. $\int_{-1}^1 \cos x \cos^{-1} x \, dx$
6. $\int \frac{x+\sin x}{1+\cos x} dx$
7. $\int \frac{\sec x}{\sqrt{\sin 2x}} dx$
8. $\int \frac{x^n}{x+x^{2n+1}} dx$
9. $\int \frac{\cos x}{\sin x \ln(\sin x)} dx$
10. $\int \frac{7^x\left(7+7^x\right)^{-1}}{e^{-\ln(\ln 7)}} dx$
11. $\int \frac{ax}{\sqrt{a^2-x^4}} dx$
12. $\int \frac{x}{\sqrt{a^4-x^4}} dx$
13. $\int \frac{dx}{2x^2+x-1}$
14. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sin 2x} dx$
15. $\int \frac{1+\tan x+\cot x}{\ln(e^x \tan x)} dx$
16. $\int \frac{1}{x\sqrt{x^6-16}} dx$
17. $\int \left(\frac{1-\sqrt{x}}{\sqrt{x}}\right) \cos\left(\sqrt{x}-\frac{1}{2}x\right) dx$
18. $\int \frac{2\ln x+1}{x\left(\ln^2 x+\ln x\right)} dx$
19. $\int \frac{dx}{\sqrt{7-6x-x^2}}$
20. $\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1+x^2}}$
21. $\int e^{\sin x+\cos x} \cos 2x \, dx$
22. $\int \frac{e^x(1+x\ln x)}{x} dx$
23. $\int \left(1+x-\frac{1}{x}\right) e^{\left(x+\frac{1}{x}\right)} dx$
24. $\int \frac{x}{x^4+2x^2+5} dx$
25. $\int e^x \sin(e^x) \cos(e^x) \, dx$
26. $\int_{-\infty}^{+\infty} e^{xt-e^t} \, dt$
27. $\int_{-1}^1 \frac{x^2}{1+|x|^x} dx$
28. $\int \frac{\arctan(\ln x)}{x+x\ln^2 x} dx$
29. $\int \frac{\sin x+\cos x}{\tan x+\cot x} dx$
30. $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$
31. $\int \frac{1}{x\sqrt{1+\ln x}} dx$
32. $\int e^{e^{-3x}} e^{-6x} dx$

33. $\int x^{\ln x}\left(\frac{2\ln x}{x}\right)dx$

34. $\int_0^{2\pi} e^{\frac{x}{2}}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$

35. $\int \frac{\cos 2x}{(\sin x+\cos x)^n}dx$

36. $\int \frac{x}{1+\sin x}dx$

37. $\int \frac{dx}{(1-\sin x)^2}$

38. $\int_1^e \sqrt{x}\ln x\,dx$

39. $\int_1^{+\infty} \frac{1}{2^x-\ln 2}dx$

40. $\int_1^{+\infty} \frac{1}{\varphi^x-1}dx$

41. $\int \sqrt{\frac{1}{\sin x}-1}\,dx$

42. $\int \frac{x+1}{x(1+xe^x)}dx$

43. $\int_0^{\pi} \frac{1}{1+(\sin x)^{\cos x}}dx$

44. $\int \frac{x}{1+x\tan x}dx$

45. $\int \frac{\tan^m x}{\sin x\cos x}dx$

46. $\int_{\frac{1}{e}}^{e} \frac{\arctan x}{x}dx$

47. $\int \frac{\sin 6x}{\sin 4x}dx$

48. $\int \frac{f'(f(\sqrt{x}))f'(\sqrt{x})}{\sqrt{x}}dx$

49. $\int_1^{+\infty} e^{-\lfloor x\rfloor}dx$

50. $\int \sqrt{\tan^2 x+\cot^2 x+2}\,dx$

51. $\int \sec^8 x\,dx$

52. $\int \frac{1+\cos 4x}{\cot x-\tan x}dx$

53. $\int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\cot x}{\cot x}\right)dx$

54. $\int_1^2 \frac{\ln x}{x^2-2x+2}dx$

55. $\int_{-n\pi}^{n\pi} \frac{\cos^5 x+1}{e^x+1}dx$

56. $\int \frac{(x+1)^3}{x^2-x}dx$

57. $\int \frac{dx}{x-2\sqrt{x}-3}$

58. $\int_0^1 x^m\ln^n x\,dx$

59. $\int \frac{1}{\sec x-1}dx$

60. $\int_0^1 \sqrt{1-x^{\pi}}\,dx$

61. $\int \cos x\ln(1+\cos x)\,dx$

62. $\int_0^{+\infty} \frac{dx}{(x^2+1)^2}$

63. $\int \frac{e^x}{(1+e^x)(2+e^x)}dx$

64. $\int_0^1 (1+xe^x)\,e^{e^x}dx$

65. $\int_{-\infty}^{+\infty} \frac{xe^x}{(a+e^x)^2}dx$

66. $\int \frac{1+\sin x}{1+\sin x+\cos x}dx$

67. $\int \frac{x^9}{(4x^2+1)^6}dx$

68. $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx$

69. $\int \frac{6^x}{9^x-4^x}dx$

70. $\int \frac{dx}{\sqrt[5]{(x^2-1)(x-1)^8}}$

71. $\int_0^{\frac{\pi}{2}} (\sin^2(\sin x) + \cos^2(\cos x))dx$

72. $\int \frac{x^2}{(a+bx)^2} dx$

73. $\int_0^1 (-1)^{\left[\frac{1}{x}\right]} dx$

74. $\int_0^{+\infty} e^{-\sqrt[\alpha]{\alpha x}} dx$

75. $\int \pi^{\pi^x} dx$

76. $\int \frac{1}{(\arcsin x)^2} dx$

77. $\int \frac{\sec x}{\sqrt{\sin(2x+\theta)+\sin\theta}} dx$

78. $\int \frac{dx}{\sqrt{9-\sqrt{9-x}}}$

79. $\int_0^1 x^2 \ln\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right) dx$

80. $\int \frac{1}{1+\sqrt{1-x^2}} dx$

81. $\int_0^1 \ln^n(1-x)\, dx$

82. $\int_0^{+\infty} \frac{\ln x}{x^2+9} dx$

83. $\int \sqrt{x^2+4x+13}\, dx$

84. $\int \frac{e^x}{\sqrt{e^{2x}+4e^x+1}} dx$

85. $\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}} dx$

86. $\int_0^1 \ln\left(\frac{x^{n+1}}{\ln x}\right) dx$

87. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx$

88. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sqrt{1-\cos 4x}} dx$

89. $\int x^2\sqrt{8-x^6} dx$

90. $\int_0^{\frac{e}{n}} \frac{x^n}{\sqrt{1-\ln(nx)}} dx$

91. $\int_0^{\pi} \frac{x\sin x}{\sqrt{1+\tan^2\alpha\sin^2 x}} dx$

92. $\int \frac{1}{\sqrt[3]{x^2+\sqrt[3]{x}+1}} dx$

93. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}} dx$

94. $\int_0^{\frac{\pi}{2}} \frac{4\sin x+2\pi}{\sin x+\cos x+\pi} dx$

95. $\int_0^{\frac{\pi}{2}} \cos 2x \tan\left(\frac{x}{2}\right) dx$

96. $\int \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}} dx$

97. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x\cos x} dx$

98. $\int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\sqrt{\sin 2x})^2} dx$

99. $\int \frac{1+e^{-x}}{\sqrt{1-e^{-x}}} dx$

100. $\int \frac{1}{\cos^3 x} dx$

101. $\int \frac{2}{x^2\sqrt{x^2-1}} dx$

102. $\int_a^b \frac{\ln(abx)}{x^2+ab}\,dx$

103. $\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}-2}$

104. $\int_1^n x.[x].\{x\}dx$

105. $\int_0^1 \tan^{-1}\left(\frac{4x-x^3}{x^4-6x^2+1}\right)dx$

106. $\int_0^{+\infty} \frac{\sinh x}{\sinh 3x}\,dx$

107. $\int \sqrt{\sin x + \cos x + \sqrt{2}}\,dx$

108. $\int_0^1 \frac{x\ln x}{\sqrt{1-x^4}}\,dx$

109. $\int_0^{+\infty} \frac{dx}{(1+x)(\pi^2+\ln^2 x)}$

110. $\int \frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}\,dx$

111. $\int \sqrt{e^{2x}+4e^x-1}\,dx$

112. $\int_3^7 \frac{\ln(x+2)}{\ln(24+10x-x^2)}\,dx$

113. $\int \frac{1}{1+\sqrt{x^2+2x+2}}\,dx$

114. $\int \frac{1}{1-(n+1)\sin^2 3x}\,dx$

115. $\int \frac{e^{\sqrt{\frac{1-x}{1+x}}}}{(1+x)\sqrt{1-x^2}}\,dx$

116. $\int \frac{1}{x\left(1+\sin^2(\ln x)\right)}\,dx$

117. $\int_0^{+\infty} \left(\frac{1}{x}\right)^{\ln x} dx$

118. $\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}}\,dx$

119. $\int_0^1 \ln(a+\ln x)\,dx$

120. $\int_0^{\pi} \frac{x^2\cos x}{(1+\sin x)^2}\,dx$

121. $\int_{-1}^1 \frac{\sin x\cos x}{1+x^2}\,dx$

122. $\int_0^1 \frac{(1-x)\arcsin x}{\sqrt{1-x^2}}\,dx$

123. $\int_0^{+\infty} \frac{e^{ax}-e^{bx}}{(1+e^{ax})(1+e^{bx})}\,dx$

124. $\int x\cos(\ln x)\,dx$

125. $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

126. $\int_0^{\frac{\pi}{2}} \frac{\tan x}{\cos^m x+\sec^m x}\,dx$

127. $\int \frac{\sqrt[3]{\sin x}}{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7}\,dx$

128. $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$

129. $\int_0^{+\infty} \frac{\ln x}{(x^2+4)(x^2+9)}\,dx$

130. $\int \sqrt{\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}}\,dx$

131. $\int \frac{1}{1+ab-a\cos x-b\sec x}\,dx$

132. $\int \frac{1}{(a^2-b^2x^2)\sqrt{a^2-b^2x^2}}\,dx$

133. $\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}\,dx$

134. $\int \frac{1+\cos^2 x}{\sqrt{\cos x}\left(1-\cos^2 x+2\cos x\right)} dx$

135. $\int_{-\infty}^{+\infty} e^{-x^2} \cosh x \, dx$

136. $\int_0^{\frac{1}{2}} \ln^2\left(\frac{1}{x}-1\right) dx$

137. $\int_a^b \frac{dx}{x\sqrt{\ln\left(\frac{x}{a}\right).\ln\left(\frac{b}{x}\right)}}$

138. $\int_0^{\frac{\pi}{2}} \frac{x\cos x-\sin x}{x^2+\sin^2 x} dx$

139. $\int_0^{+\infty} \frac{1}{\left(1+x^k\right)\left(1+x^2\right)} dx$

140. $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{dx}{\left(1+e^{\sin x}\right)(2-\cos 2x)}$

141. $\int_0^1 \frac{x}{\cos ax\cos(a-ax)} dx$

142. $\int_0^{\pi} \arctan\left(3^{\cos x}\right) dx$

143. $\int_0^1 \frac{\arctan x}{1+x} dx$

144. $\int_{-\infty}^{+\infty} \frac{p+qx}{x^2+2rx\cos\alpha+r^2} dx$

145. $\int \ln\left(\sqrt{x}+\sqrt{\pi}\right) dx$

146. $\int_0^1 \ln\left(\sqrt[3]{\ln\sqrt{1-x}}\right) dx$

147. $\int \frac{\sqrt{x+\sqrt{1+x^2}}}{1+x^2} dx$

148. $\int e^{-x}\ln(1+e^x)\, dx$

149. $\int_0^{+\infty} \frac{x\sqrt{x}}{\left(1+x^2\right)\left(1+a^2x^2\right)} dx$

150. $\int_0^{\pi} \frac{x}{1-\cos\alpha\sin x} dx$

151. $\int_{-1}^{1} \frac{\sin e}{1-2x\cos e+x^2} dx$

152. $\int_{-a}^{a} \frac{dx}{\sqrt[3]{(a-x)\left(a^2-x^2\right)}}$

153. $\int \frac{1}{\sqrt[6]{x^6+1}} dx$

154. $\int_0^{\pi} \frac{\sin^2 x}{1+2a\cos x+a^2} dx$

155. $\int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sqrt{1-\tan x}\, dx$

156. $\int_0^1 \ln^x\left(\frac{1}{a}\right).\ln^{1-x}\left(\frac{1}{b}\right) dx$

157. $\int_a^b \ln\left(\frac{\left(1+\frac{x}{a}\right)^{x-1}e^{\frac{b}{x}}}{\left(1+\frac{b}{x}\right)^{x-1}e^{\frac{x}{a}}}\right) dx$

158. $\int \frac{x^3}{\left(9-x^2\right)\sqrt{9-x^2}} dx$

159. $\int \frac{x^5}{\sqrt{1+x^2}} dx$

160. $\int \ln\left(\sqrt{1+x}-\sqrt{1-x}\right) dx$

161. $\int_0^a \frac{\ln(1+ax)}{1+x^2} dx$

162. $\int x^{-2}\tan\left(\frac{1}{2x}\right)\tan\left(\frac{1}{3x}\right)\tan\left(\frac{1}{6x}\right)$

163. $\int_{\frac{1}{e}}^{1} \frac{\ln\left(1+\ln^2 x\right)}{x\ln x} dx$

164. $\int_0^2 \frac{\ln(1+x)}{x^2-x+1} dx$

165. $\int_{-1}^{1} \frac{\left(\sin^{-1} x\right)^2}{1+2^x} dx$

166. $\int_0^{\alpha} \frac{dx}{x+\sqrt{\alpha^2-x^2}}$

167. $\int_0^{\frac{\pi}{4}} (\ln\tan x)^{\alpha}\, dx$

168. $\int \frac{1}{\alpha\sin x+\beta\cos x}\, dx$

169. $\int_1^{+\infty} \frac{1}{(x+a)\sqrt{x-1}}\, dx$

170. $\int_0^{a} \ln\left(\sqrt{a+x}+\sqrt{a-x}\right) dx$

171. $\int \frac{x+\sin\left(\cos^{-1}x\right)}{\cos\left(\sin^{-1}\left(x^2\right)\right)}\, dx$

172. $\int \frac{1}{(1+2x)\sqrt{(1+3x)(1+x)}}\, dx$

173. $\int \frac{1}{3\sin^4 x+3\cos^4 x-1}\, dx$

174. $\int_0^{+\infty} \frac{x-1}{\ln\left(2^x-1\right)\sqrt{2^x-1}}\, dx$

175. $\int_{-\infty}^{+\infty} \frac{\tan^{-1}\left(x+\sqrt{1+x^2}\right)}{4+x^2}\, dx$

176. $\int_0^{+\infty} \frac{e^{3x}}{e^{4x}+1}\, dx$

177. $\int_0^{\frac{1}{2}} \frac{\ln(1-x)}{2x^2-2x+1}\, dx$

178. $\int_0^{\frac{e}{\pi}} \frac{\tan^{-1}\left(\frac{\pi x}{e}\right)}{\pi x+e}\, dx$

179. $\int_0^{\pi} \left(\ln\tan\frac{x}{2}\right)^2 \sin x\, dx$

180. $\int \frac{1}{\sin x\sqrt[3]{\sin^3 x+\cos^3 x}}\, dx$

181. $\int_0^{\pi} \frac{\ln(x+\pi)}{x^2+\pi^2}\, dx$

182. $\int_0^{+\infty} \frac{\ln x}{\sqrt{\left(1+x^2\right)\left(2+x^2\right)}}\, dx$

183. $\int_0^{+\infty} \frac{\tan^{-1}x}{x^{\ln x+1}}\, dx$

184. $\int_{-\infty}^{+\infty} e^{x-\sinh^2 x}\, dx$

185. $\int_{-\infty}^{+\infty} \frac{dx}{\cosh\pi x-a}$

186. $\int_0^{+\infty} \tanh x\,.\,e^{-ax}\, dx$

187. $\int \frac{1}{1+\sqrt{x}}\cdot\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}\, dx$

188. $\int_0^{+\infty} \frac{dx}{\sqrt{x}\left(x^4+x^2+1\right)^{\frac{3}{4}}}$

189. $\int_0^{+\infty} \frac{1}{\sqrt{n+e^{\frac{\pi}{\sqrt{x}}}}}\cdot\frac{dx}{\sqrt{x^3}}$

190. $\int_0^{\frac{\pi}{2}} \ln|\ln(\tan x)|\, dx$

191. $\int_0^{\frac{\pi}{4}} \frac{\ln(\cot x)\sin^{\pi^e}(2x)}{\left(\sin^{\pi^e+1}(x)+\cos^{\pi^e+1}(x)\right)^2}\, dx$

192. $\int \sqrt[6]{\tan x}\, dx$

193. $\int_0^{\frac{\pi}{2}} \frac{1}{\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\ldots}\, dx$

194. $\int_0^{+\infty} \frac{x^2}{\left(x^2+a^2\right)\left(x^2+b^2\right)\left(x^2+c^2\right)}\, dx$

195. $\int_0^{+\infty} e^{-\sqrt{x}}\ln\left(1+\frac{1}{\sqrt{x}}\right) dx$

196. $\int \frac{x^n}{1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}\, dx$

197. $\int_0^{+\infty} \frac{x^2+1}{x^4+x^2+1}\cdot\frac{\ln\left(1-x+x^2-x^3+\cdots+x^{2m}\right)}{\ln x}\, dx$

198. $\int_{\frac{\pi}{6}}^{\frac{5\pi}{18}} \frac{2-\sin 3x}{\cos^3\left(\frac{3x}{2}\right)+\sin^3\left(\frac{3x}{2}\right)}\, dx$

199. $\int_0^1 (-1)^{\ln x} dx$

200. $\int \frac{dx}{\sqrt{\csc x - \cot x}}$

201. $\int \frac{\sqrt{1+x}}{1+\sqrt{x}} dx$

202. $\int_0^{+\infty} \frac{nx^{n-1}}{(x^2+1)^{\frac{n+2}{2}}} dx$; where $n > 0$

203. $\int_0^{+\infty} \frac{e^{-x^2}-e^{-x}}{x} dx$

204. $\int_0^{+\infty} \frac{x^{a-1}}{\sinh(bx)} dx$

205. $\int \frac{\sin x . \cos x}{\sin x + \cos x} dx$

206. $\int_0^1 \ln(-\ln x) \frac{x^{\alpha-1}}{\sqrt{-\ln x}} dx$

207. $\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right) dx$

208. $\int_0^1 x^b \ln\left(\frac{1}{x}\right) \ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{a-1} dx$

209. $\int \frac{\Delta}{\sin x} dx$; where $\Delta = \sqrt{1-k^2\sin^2 x}$

210. $\int \left(\frac{8}{27}\right)^x . \ln\left|\frac{4^x+9^x}{6^x+9^x}\right| dx$

211. $\int_0^{+\infty} \frac{1}{x^5+x^4+x^3+x^2+x+1} dx$

212. $\int_0^{\frac{\pi}{6}} \frac{\sin x \sin\left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x + \cos 3x} dx$

213. $\int_{-\infty}^{+\infty} e^{-x^2} \cos(2x^2)\, dx$

214. $\int_0^{+\infty} \sin x\,(\operatorname{erf}(x) - 1) dx$

215. $\int_0^{\frac{\pi}{2}} \ln(\alpha^2 \sin^2 x + \beta^2 \cos^2 x)\, dx$

216. $\int_0^1 \left(\left\lfloor\frac{\alpha}{x}\right\rfloor - \alpha\left\lfloor\frac{1}{x}\right\rfloor\right) dx$

217. $\int \frac{1}{x^7-4x^5+6x^3-4x} dx$

218. $\int_0^{+\infty} \frac{dx}{(1+x)\left(\varphi^2+\ln^2 x\right)}$

219. $\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}} dx$

220. $\int_0^{+\infty} xe^{-x} \ln(\cosh x)\, dx$

221. $\int_0^{+\infty} x^2 e^{-x^2} \cos x\, dx$

222. $\int_0^{+\infty} \frac{x^2 \cosh(ex)}{\sinh^2(ex)} dx$

223. $\int_0^{+\infty} \frac{\sqrt{x}\ln x}{x^2+1} dx$

224. $\int_0^{+\infty} \frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}} dx$

225. $\int_0^{\pi} \tan^{-1}(e^{\cos x})\, dx$

226. $\int_0^{\frac{\pi}{4}} \frac{x}{(\sin x + \cos x)\cos x} dx$

227. $\int_0^{+\infty} \frac{dx}{\left(1+x^{\delta_S}\right)^{\delta_S}}$; silver ratio

228. $\int_{-\infty}^{+\infty} \frac{x}{(a+e^x)(1-e^{-x})} dx$

229. $\int_{-\infty}^{+\infty} xe^{x-\mu e^{ax}} dx$

230. $\int_0^{+\infty} \frac{1}{2\sqrt{x}} \sin\left(\pi^2 x + \frac{1}{x}\right) dx$

231. $\int_0^{+\infty} \frac{e^{-\pi x}}{(\sinh(x\pi)+\varphi)\left((\cosh(x\pi)+\varphi)\right)} dx$

Solutions of Exercises

1. $\int_1^e \pi^{\ln x} dx = \int_1^e e^{\ln(\pi^{\ln x})} dx = \int_1^e e^{\ln x \ln \pi} dx = \int_1^e \left(e^{\ln x}\right)^{\ln \pi} dx = \int_1^e x^{\ln \pi} dx$

$= \left[\frac{1}{\ln \pi + 1} x^{\ln \pi + 1}\right]_1^e = \frac{e^{\ln \pi + \ln e}}{\ln \pi + 1} - \frac{1}{\ln \pi + 1} = \frac{e^{\ln(\pi e)} - 1}{\ln \pi + \ln e} = \frac{\pi e - 1}{\ln(\pi e)}$

2. $\int x^{-2} 5^{\frac{1}{x}} dx = \int \frac{1}{x^2} 5^{\frac{1}{x}} dx$, let $t = \frac{1}{x}$, then $dt = -\frac{1}{x^2} dx$, so we get:

$\int x^{-2} 5^{\frac{1}{x}} dx = -\int 5^t dt = -\frac{5^t}{\ln 5} + c = -\frac{5^{\frac{1}{x}}}{\ln 5} + c$

3. $\mathrm{I} = \int_1^{\sqrt{e}} x.\left(\sqrt{e}\right)^{-\log_{\sqrt{e}}(x)} dx = \int_1^{\sqrt{e}} x.\left(\sqrt{e}\right)^{\log_{\sqrt{e}}\left(\frac{1}{x}\right)} dx$, but we have $a^{\log_a(x)} = x$, then we get:

$\mathrm{I} = \int_1^{\sqrt{e}} x\left(\frac{1}{x}\right) dx = \int_1^{\sqrt{e}} dx = \sqrt{e} - 1$

4. $\mathrm{I} = \int_0^1 x^{70}(1-x)^{30} dx = \mathrm{B}(71; 31) = \frac{\Gamma(71)\Gamma(31)}{\Gamma(71+31)} = \frac{\Gamma(71)\Gamma(31)}{\Gamma(102)} = \frac{70!30!}{101!}$

5. $\mathrm{I} = \int_{-1}^1 \cos x \cos^{-1} x \, dx$, we have $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, then we get:

$$\mathrm{I} = \int_{-1}^{1} \left(\frac{\pi}{2} - \sin^{-1} x\right) \cos x \, dx = \frac{\pi}{2} \int_{-1}^{1} \cos x \, dx - \int_{-1}^{1} \sin^{-1} x \cos x \, dx$$

But we have: $\int_{-1}^{1} \sin^{-1} x \cos x \, dx = 0$; integral of odd function = odd × even; then we get:

$$\mathrm{I} = \frac{\pi}{2} \int_{-1}^{1} \cos x \, dx = \frac{\pi}{2} \times 2 \int_{0}^{1} \cos x \, dx = \pi[\sin x]_0^1 = \pi \sin(1)$$

6. $\int \frac{x + \sin x}{1 + \cos x} dx = \int \frac{x + 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}{2\cos^2\left(\frac{x}{2}\right)} dx = \int x.\frac{1}{2}.\sec^2\left(\frac{x}{2}\right) dx + \int \frac{\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} dx$ (use IBP)

$\int \frac{x + \sin x}{1 + \cos x} dx = x \tan\left(\frac{x}{2}\right) - \int \tan\left(\frac{x}{2}\right) dx + \int \tan\left(\frac{x}{2}\right) dx = x \tan\left(\frac{x}{2}\right) + c$

7. $\int \frac{\sec x}{\sqrt{\sin 2x}} dx = \int \frac{\frac{\sec x}{\cos x}}{\sqrt{\frac{2 \sin x \cos x}{\cos^2 x}}} dx = \frac{1}{\sqrt{2}} \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$, let $u = \tan x$, then $du = \sec^2 x \, dx$, so

$\int \frac{\sec x}{\sqrt{\sin 2x}} dx = \frac{1}{\sqrt{2}} \int u^{-\frac{1}{2}} du = \frac{1}{\sqrt{2}}\left(2\sqrt{u}\right) + c = \sqrt{2u} + c = \sqrt{2 \tan x} + c$

8. $\int \frac{x^n}{x + x^{2n+1}} dx = \int \frac{x^n}{x\left(1 + x^{2n}\right)} dx = \int \frac{x^{n-1}}{1 + x^{2n}} dx = \frac{1}{n} \int \frac{n x^{n-1}}{1 + (x^n)^2} dx = \frac{1}{n} \int \frac{(x^n)'}{1 + (x^n)^2} dx$

$\int \frac{x^n}{x + x^{2n+1}} dx = \frac{1}{n} \arctan(x^n) + c$

9. $\int \frac{\cos x}{\sin x \ln(\sin x)} dx = \int \frac{\cos x}{\sin x} . \frac{1}{\ln(\sin x)} dx = \int \cot x . \frac{1}{\ln(\sin x)} dx$

Let $y = \ln(\sin x)$, then $dy = \frac{(\sin x)'}{\sin x} dx = \frac{\cos x}{\sin x} dx = \cot x \, dx$, then we get:

$\int \frac{\cos x}{\sin x \ln(\sin x)} dx = \int \frac{dy}{y} = \ln|y| + c = \ln|\ln(\sin x)| + c$

10. $\int \frac{7^x(7+7^x)^{-1}}{e^{-\ln(\ln 7)}}dx = \int \frac{7^x e^{\ln(\ln 7)}}{7+7^x}dx = \int \frac{7^x \ln 7}{7+7^x}dx$, let $u = 7 + 7^x$, then $du = 7^x \ln 7\, dx$, so:

$\int \frac{7^x(7+7^x)^{-1}}{e^{-\ln(\ln 7)}}dx = \int \frac{du}{u} = \ln|u| + c = \ln|7 + 7^x| + c$

11. $\int \frac{ax}{\sqrt{a^2-x^4}}dx = a\int \frac{xdx}{\sqrt{a^2-(x^2)^2}} = \frac{a}{2}\int \frac{2xdx}{\sqrt{a^2-(x^2)^2}}$, let $y = x^2$, then $dy = 2xdx$, then:

$\int \frac{ax}{\sqrt{a^2-x^4}}dx = \frac{a}{2}\int \frac{dy}{\sqrt{a^2-y^2}} = \frac{a}{2}\sin^{-1}\left(\frac{y}{2}\right) + c = \frac{a}{2}\sin^{-1}\left(\frac{x^2}{2}\right) + c$

12. $\int \frac{x}{\sqrt{a^4-x^4}}dx = \int \frac{x}{\sqrt{(a^2)^2-x^4}}dx = \frac{1}{2}\int \frac{2x}{\sqrt{(a^2)^2-(x^2)^2}}dx = \frac{1}{2}\sin^{-1}\left(\frac{x^2}{a^2}\right) + c$

13. $\int \frac{dx}{2x^2+x-1} = \int \frac{dx}{2\left(x^2+\frac{1}{2}x-\frac{1}{2}\right)} = \frac{1}{2}\int \frac{dx}{x^2+\frac{1}{2}x-\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{x^2+\frac{1}{2}x+\frac{1}{16}-\frac{1}{16}-\frac{1}{2}} = \frac{1}{2}\int \frac{dx}{\left(x^2+\frac{1}{2}x+\frac{1}{16}\right)-\frac{9}{16}}$

$$\int \frac{dx}{2x^2+x-1} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2-\frac{9}{16}} = \frac{1}{2}\int \frac{dx}{\left(x+\frac{1}{4}\right)^2-\left(\frac{3}{4}\right)^2} = \frac{1}{2}\left(-\frac{1}{\frac{3}{4}}\right)\coth^{-1}\left(\frac{x+\frac{1}{4}}{\frac{3}{4}}\right) + c$$

$$\int \frac{dx}{2x^2+x-1} = -\frac{2}{3}\coth^{-1}\left(\frac{4x+1}{3}\right) + c$$

14. $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\sin 2x}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sqrt{\tan x}}{\frac{2\tan x}{1+\tan^2 x}}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2 x}{2\sqrt{\tan x}}dx = \left[\sqrt{\tan x}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}} = \sqrt{\sqrt{3}} - 1 = 3^{\frac{1}{4}} - 1$

15. $\int \frac{1+\tan x+\cot x}{\ln(e^x \tan x)}dx$, let $u = \ln(e^x \tan x) = \ln e^x + \ln(\tan x) = x + \ln(\tan x)$, then

$$du = \left(1 + \frac{\sec^2 x}{\tan x}\right)dx = \left(1 + \frac{\tan^2 x + 1}{\tan x}\right)dx = (1 + \tan x + \cot x)dx; \text{ then:}$$

$\int \frac{1+\tan x+\cot x}{\ln(e^x \tan x)}dx = \int \frac{1}{u}du = \ln|u| + c = \ln|\ln(e^x \tan x)| + c$

16. $\int \frac{1}{x\sqrt{x^6-16}}dx = \int \frac{x^2}{x^3\sqrt{(x^3)^2-16}}dx = \frac{1}{3}\int \frac{3x^2}{x^3\sqrt{(x^3)^2-16}}dx$, let $u = x^3$, then $du = 3x^2dx$, so:

$\int \frac{1}{x\sqrt{x^6-16}}dx = \frac{1}{3}\int \frac{du}{u\sqrt{u^2-4^2}} = \frac{1}{3}\times\frac{1}{4}\sec^{-1}\left(\frac{u}{4}\right) + c = \frac{1}{12}\sec^{-1}\left(\frac{x^3}{4}\right) + c$

17. $\int \left(\frac{1-\sqrt{x}}{\sqrt{x}}\right)\cos\left(\sqrt{x} - \frac{1}{2}x\right)dx = 2\int \frac{1}{2}\left(\frac{1}{\sqrt{x}} - 1\right)\cos\left(\sqrt{x} - \frac{1}{2}x\right)dx$

Let $u = \sqrt{x} - \frac{1}{2}x$, then $du = \left(\frac{1}{2\sqrt{x}} - \frac{1}{2}\right)dx = \frac{1}{2}\left(\frac{1}{\sqrt{x}} - 1\right)dx$, then we get:

$\int \left(\frac{1-\sqrt{x}}{\sqrt{x}}\right)\cos\left(\sqrt{x} - \frac{1}{2}x\right)dx = 2\int \cos u\, du = 2\sin u + c = 2\sin\left(\sqrt{x} - \frac{1}{2}x\right) + c$

18. $\int \frac{2\ln x+1}{x\left(\ln^2 x+\ln x\right)}dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, then we get:

$\int \frac{2\ln x+1}{x\left(\ln^2 x+\ln x\right)}dx = \int \frac{2u+1}{u^2+u}du = \int \frac{(u+1)+u}{u(u+1)}du = \int \left(\frac{1}{u} + \frac{1}{u+1}\right)du = \ln|u| + \ln|u + 1| + c$

$= \ln|\ln x| + \ln|\ln x + 1| + c = \ln|\ln x\,(\ln x + 1)| + c$

19. $\int \frac{dx}{\sqrt{7-6x-x^2}} = \int \frac{dx}{\sqrt{7-6x-x^2-9+9}} = \int \frac{dx}{\sqrt{7-(x^2+6x+9)+9}} = \int \frac{dx}{\sqrt{16-(x+3)^2}} = \int \frac{dx}{\sqrt{4^2-(x+3)^2}}$
$= \sin^{-1}\left(\frac{x+3}{4}\right) + c$

20. $\mathrm{I} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1+x^2}}$, let $x = \tan\theta$, then $dx = \sec^2\theta\, d\theta$, for $x = 0$, $\theta = 0$ and for $x = \frac{1}{2}$ then $\theta = \arctan\left(\frac{1}{2}\right)$, then we get:

$\mathrm{I} = \int_0^{\arctan\left(\frac{1}{2}\right)} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}} d\theta = \int_0^{\arctan\left(\frac{1}{2}\right)} \frac{\sec^2\theta}{\sqrt{\sec^2\theta}} d\theta = \int_0^{\arctan\left(\frac{1}{2}\right)} \sec\theta\, d\theta$, then we get:

$\mathrm{I} = [\ln|\sec\theta + \tan\theta|]_0^{\arctan\left(\frac{1}{2}\right)} = \ln\left|\sec\left(\arctan\left(\frac{1}{2}\right)\right) + \tan\left(\arctan\left(\frac{1}{2}\right)\right)\right| - \ln 1$

$\mathrm{I} = \ln\left[\sec\left(\arctan\left(\frac{1}{2}\right)\right) + \frac{1}{2}\right] = \ln\left[\sqrt{1 + \left(\tan\left(\arctan\left(\frac{1}{2}\right)\right)\right)^2} + \frac{1}{2}\right]$

$\mathrm{I} = \ln\left(\sqrt{1 + \left(\frac{1}{2}\right)^2} + \frac{1}{2}\right) = \ln\left(\sqrt{1 + \frac{1}{4}} + \frac{1}{2}\right) = \ln\left(\sqrt{\frac{5}{4}} + \frac{1}{2}\right) = \ln\left(\frac{\sqrt{5}}{2} + \frac{1}{2}\right) = \ln\left(\frac{1+\sqrt{5}}{2}\right)$,

therefore we get: $\mathrm{I} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1+x^2}} = \ln\varphi$, where $\varphi = \frac{1+\sqrt{5}}{2}$ golden ratio

21. $\int e^{\sin x+\cos x}\cos 2x\, dx = \int e^{\sin x+\cos x}(\cos^2 x - \sin^2 x)dx$
$\int e^{\sin x+\cos x}\cos 2x\, dx = \int e^{\sin x+\cos x}(\cos x + \sin x)(\cos x - \sin x)dx$
Let $u = \sin x + \cos x$, then $du = (\cos x - \sin x)dx$, then we get:
$\int e^{\sin x+\cos x}\cos 2x\, dx = \int ue^u du = (u-1)e^u + c = (\sin x + \cos x - 1)e^{\sin x+\cos x} + c$

22. $\int \frac{e^x(1+x\ln x)}{x} dx = \int \frac{e^x+e^x x\ln x}{x} dx = \int \left(\frac{e^x}{x} + e^x\ln x\right)$ (we will use IBP for the first one)
$= e^x\ln x - \int e^x\ln x\, dx + \int e^x\ln x\, dx = e^x\ln x + c$

23. $\int \left(1 + x - \frac{1}{x}\right) e^{\left(x+\frac{1}{x}\right)} dx = \int \left[1 + x\left(1 - \frac{1}{x^2}\right)\right] e^{\left(x+\frac{1}{x}\right)} dx$

$= \int e^{\left(x+\frac{1}{x}\right)} dx + \int x\left(1 - \frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)} dx$, now let us evaluate $\int x\left(1 - \frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)} dx$ using IBP

Let $u = x$, then $u' = 1$ and let $v' = \left(1 - \frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)}$, so $v = e^{\left(x+\frac{1}{x}\right)}$, then:

$\int x\left(1 - \frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)} dx = xe^{\left(x+\frac{1}{x}\right)} - \int e^{\left(x+\frac{1}{x}\right)} dx$, then we get:

$\int \left(1 + x - \frac{1}{x}\right) e^{\left(x+\frac{1}{x}\right)} dx = \int e^{\left(x+\frac{1}{x}\right)} dx + xe^{\left(x+\frac{1}{x}\right)} - \int e^{\left(x+\frac{1}{x}\right)} dx = xe^{\left(x+\frac{1}{x}\right)} + c$

24. $\int \frac{x}{x^4+2x^2+5} dx = \frac{1}{2}\int \frac{2xdx}{\left(x^2\right)^2+2x^2+5}$, let $y = x^2$, then $dy = 2xdx$, then we get:

$\int \frac{x}{x^4+2x^2+5} dx = \frac{1}{2}\int \frac{dy}{y^2+2y+5} = \frac{1}{2}\int \frac{dy}{y^2+2y+1+4} = \frac{1}{2}\int \frac{dy}{(y+1)^2+4} = \frac{1}{2}\int \frac{dy}{(y+1)^2+2^2}$

$= \frac{1}{2}\left(\frac{1}{2}\tan^{-1}\left(\frac{y+1}{2}\right)\right) + c = \frac{1}{4}\tan^{-1}\left(\frac{x^2+1}{2}\right) + c$

25. $\int e^x \sin(e^x)\cos(e^x)\,dx$, let $u = e^x$, then $du = e^x dx$, then we get:

$\int e^x \sin(e^x)\cos(e^x)\,dx = \int \sin u \cos u\,du = \frac{1}{2}\int 2\sin u\cos u\,du = \frac{1}{2}\int \sin 2u\,du$

$\int e^x \sin(e^x)\cos(e^x)\,dx = \frac{1}{2}\left(-\frac{1}{2}\cos 2u\right) + c = -\frac{1}{2}\cos 2u + c = -\frac{1}{2}\cos(2e^x) + c$

26. $\mathrm{I} = \int_{-\infty}^{+\infty} e^{xt-e^t}\,dt = \int_{-\infty}^{+\infty} e^{xt}.e^{-e^t}\,dt = \int_{-\infty}^{+\infty}(e^t)^x.e^{-e^t}\,dt$

Let $u = e^t \Rightarrow du = e^t dt \Rightarrow du = u dt \Rightarrow dt = \frac{du}{u}$, for $t = -\infty$, $u = e^{-\infty} = 0$ and for $t = +\infty$, $u = e^{+\infty} = +\infty$, then we get:

$$\mathrm{I} = \int_{-\infty}^{+\infty} e^{xt-e^t}\,dt = \int_0^{+\infty} u^x.e^{-u}.\frac{du}{u} = \int_0^{+\infty} u^{x-1}.e^{-u}\,du = \Gamma(x)$$

27. $\int \frac{\sin 2x}{\sin 5x \sin 3x}dx = \int \frac{\sin(5x-3x)}{\sin 5x\sin 3x}dx = \int \frac{\sin 5x\cos 3x - \cos 5x \sin 3x}{\sin 5x \sin 3x}dx$

$= \int\left(\frac{\cos 3x}{\sin 3x} - \frac{\cos 5x}{\sin 5x}\right)dx = \frac{1}{3}\int\frac{3\cos 3x}{\sin 3x}dx - \frac{1}{5}\int\frac{5\cos 5x}{\sin 5x}dx = \frac{1}{3}\ln|\sin 3x| - \frac{1}{5}\ln|\sin 5x| + c$

28. $\int \frac{\arctan(\ln x)}{x + x\ln^2 x}dx = \int \frac{\arctan(\ln x)}{x(1+\ln^2 x)}dx = \int\frac{\arctan(\ln x)}{1+\ln^2 x}.\frac{1}{x}dx$, let $u = \ln x$, so $du = \frac{1}{x}dx$, then:

$\int \frac{\arctan(\ln x)}{x+x\ln^2 x}dx = \int\frac{\arctan u}{1+u^2}du = \int(\arctan u)'\arctan u\,du = \frac{1}{2}(\arctan u)^2 + c$, then we get:

$\int \frac{\arctan(\ln x)}{x+x\ln^2 x}dx = \frac{1}{2}(\arctan(\ln x))^2 + c$

29. $\int\frac{\sin x + \cos x}{\tan x + \cot x}dx = \int\frac{\sin x+\cos x}{\frac{\sin x}{\cos x}+\frac{\cos x}{\sin x}}dx = \int\frac{\sin x+\cos x}{\frac{\sin^2 x+\cos^2 x}{\sin x\cos x}}dx = \int\frac{\sin x+\cos x}{\frac{1}{\sin x\cos x}}dx$

$\int\frac{\sin x+\cos x}{\tan x+\cot x}dx = \int(\sin x+\cos x)\sin x\cos x\,dx = \int(\sin^2 x\cos x + \cos^2 x\sin x)dx$

$\int\frac{\sin x+\cos x}{\tan x+\cot x}dx = \int(\sin^2 x\,(\sin x)' - \cos^2 x\,(\cos x)')dx = \frac{1}{3}(\sin^3 x - \cos^3 x) + c$

30. $\mathrm{I} = \int_{-1}^{1}\frac{x^2}{1+|x|^x}dx$ **… (1)** let $u = -x$, then $du = -dx$, then we get:

$\mathrm{I} = \int_1^{-1}\frac{(-u)^2}{1+|-u|^{-u}}(-du) = \int_{-1}^{1}\frac{u^2}{1+|u|^{-u}}du = \int_{-1}^{1}\frac{x^2}{1+|x|^{-x}}dx = \int_{-1}^{1}\frac{x^2|x|^x}{|x|^x+1}du$ **… (2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_{-1}^{1}\frac{x^2}{1+|x|^x}dx + \int_{-1}^{1}\frac{x^2|x|^x}{|x|^x+1}du = \int_{-1}^{1}\frac{x^2+x^2|x|^x}{1+|x|^x}dx$

$2\mathrm{I} = \int_{-1}^{1}\frac{x^2(1+|x|^x)}{1+|x|^x}dx = \int_{-1}^{1}x^2dx = \left[\frac{1}{3}x^3\right]_{-1}^{1} = \frac{2}{3}$, therefore $\mathrm{I} = \frac{1}{3}$

31. $\int\frac{1}{x\sqrt{1+\ln x}}dx = \int\frac{1}{\sqrt{1+\ln x}}.\frac{1}{x}dx$, let $u = \ln x$, then $du = \frac{1}{x}dx$, so we get:

$\int\frac{1}{x\sqrt{1+\ln x}}dx = \int\frac{1}{\sqrt{1+u}}du$, let $v = 1+u$, so $dv = du$, then we get:

$\int\frac{1}{x\sqrt{1+\ln x}}dx = \int\frac{1}{\sqrt{v}}dv = \int v^{-\frac{1}{2}}dv = 2v^{\frac{1}{2}} + c = 2\sqrt{u+1} + c = 2\sqrt{\ln x + 1} + c$

32. $\int e^{e^{-3x}}e^{-6x}dx = \int e^{e^{-3x}}e^{-3x}e^{-3x}dx$, let $u = e^{-3x}$, then $du = -3e^{-3x}dx$, then we get:

$\int e^{e^{-3x}}e^{-6x}dx = -\frac{1}{3}\int e^u u\,du = -\frac{1}{3}(ue^u - e^u) + c = -\frac{1}{3}ue^u + \frac{1}{3}e^u + c$, so:

$\int e^{e^{-3x}}e^{-6x}dx = -\frac{1}{3}e^{-3x}e^{e^{-3x}} + \frac{1}{3}e^{e^{-3x}} + c$

33. $\int x^{\ln x}\left(\frac{2\ln x}{x}\right)dx=\int e^{\ln(x^{\ln x})}\left(\frac{2\ln x}{x}\right)dx=\int e^{\ln x\ln x}\left(\frac{2\ln x}{x}\right)dx=\int e^{(\ln x)^2}\left(\frac{2\ln x}{x}\right)dx$

Let $u=(\ln x)^2$, then $du=\left(\frac{2\ln x}{x}\right)dx$, then we get**:**

$\int x^{\ln x}\left(\frac{2\ln x}{x}\right)dx=\int e^u du=e^u+c=e^{(\ln x)^2}+c=x^{\ln x}+c$

34. $\int_0^{2\pi} e^{\frac{x}{2}}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx$, let $t=\frac{x}{2}$, so $x=2t$ and $dt=2dx$, $\begin{cases}\text{for } & x=0; & t=0\\ \text{for } & x=2\pi; & t=\pi\end{cases}$, then we get

$\int_0^{2\pi} e^{\frac{x}{2}}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx=2\int_0^{\pi} e^t\sin\left(t+\frac{\pi}{4}\right)dt=\sqrt{2}\int_0^{\pi} e^t(\sin t+\cos t)dt$

$=\sqrt{2}\int_0^{\pi} e^t[\sin t+(\sin t)']dt=\sqrt{2}\int_0^{\pi}(\sin t\, e^t)'\,dt=\left[\sqrt{2}\sin t\, e^t\right]_0^{\pi}=0$

35. $\int\frac{\cos 2x}{(\sin x+\cos x)^n}dx=\int\frac{\cos 2x}{(\sin x+\cos x)^n}dx=\int\frac{\cos 2x}{\left[(\sin x+\cos x)^2\right]^{\frac{n}{2}}}dx$

$=\int\frac{\cos 2x}{\left(\cos^2 x+\sin^2 x+2\sin x\cos x\right)^{\frac{n}{2}}}dx=\int\frac{\cos 2x}{(1+\sin 2x)^{\frac{n}{2}}}dx=\frac{1}{2}\int\frac{2\cos 2x}{(1+\sin 2x)^{\frac{n}{2}}}dx$

Let $u=1+\sin 2x$, then $du=2\cos 2x$, then we get**:**

$\int\frac{\cos 2x}{(\sin x+\cos x)^n}dx=\frac{1}{2}\int\frac{du}{u^{\frac{n}{2}}}=\frac{1}{2}\int u^{-\frac{n}{2}}du=\frac{1}{2}\left(\frac{u^{-\frac{n}{2}+1}}{-\frac{n}{2}+1}\right)+c=\frac{1}{2}\left(\frac{u^{\frac{2-n}{2}}}{\frac{2-n}{2}}\right)+c$

$\int\frac{\cos 2x}{(\sin x+\cos x)^n}dx=\frac{1}{2-n}u^{\frac{2-n}{2}}+c=\frac{1}{2-n}(1+\sin 2x)^{\frac{2-n}{2}}+c$

36. $\int\frac{x}{1+\sin x}dx=\int\frac{x}{1+\cos\left(\frac{\pi}{2}-x\right)}dx=\int\frac{x}{2\cos^2\left(\frac{\pi}{4}-\frac{x}{2}\right)}dx=\frac{1}{2}\int x\sec^2\left(\frac{\pi}{4}-\frac{x}{2}\right)dx$ (now using IBP)

$\int\frac{x}{1+\sin x}dx=-x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+\int\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)dx=-x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+2\int\frac{\frac{1}{2}\sin\left(\frac{\pi}{4}-\frac{x}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)}dx$

$=-x\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)+2\ln\left|\cos\left(\frac{\pi}{4}-\frac{x}{2}\right)\right|+c$

37. $\int\frac{dx}{(1-\sin x)^2}=\int\frac{\csc^2 x}{(\csc x-1)^2}dx=\int\frac{\csc^2 x(\csc x+1)^2}{(\csc x-1)^2(\csc x+1)^2}dx=\int\frac{\csc^2 x(\csc x+1)^2}{[(\csc x-1)(\csc x+1)]^2}dx$

$=\int\frac{\csc^2 x(\csc x+1)^2}{\left(\csc^2 x-1\right)^2}dx=\int\frac{\csc^2 x(\csc x+1)^2}{\cot^4 x}dx=\int\frac{\csc^4 x}{\cot^4 x}dx+2\int\frac{\csc^3 x}{\cot^4 x}dx+\int\frac{\csc^2 x}{\cot^4 x}dx$

$=\int(\tan^2 x+1)\sec^2 x\,dx+2\int\frac{\sin x}{\cos^4 x}dx+\int\frac{\csc^2 x}{\cot^4 x}dx$

$=\frac{1}{3}\tan^3 x+\tan x+\frac{2}{3}\sec^3 x+\frac{1}{3}\tan^3 x+c=\frac{2}{3}\tan^3 x+\tan x+\frac{2}{3}\sec^3 x+c$

38. $\int_1^e\sqrt{x}\ln x\,dx$, let us first evaluate $\int\sqrt{x}\ln x\,dx$ using IBP

Let $u=\ln x$, then $u'=\frac{1}{x}$ and let $v'=\sqrt{x}$, then $v=\frac{2}{3}x\sqrt{x}$, then we get**:**

$\int\sqrt{x}\ln x\,dx=\frac{2}{3}x\sqrt{x}\ln x-\int\frac{2}{3}x\sqrt{x}.\frac{1}{x}dx=\frac{2}{3}x\sqrt{x}\ln x-\frac{2}{3}\int\sqrt{x}dx$

$=\frac{2}{3}x\sqrt{x}\ln x-\frac{2}{3}\times\frac{2}{3}x\sqrt{x}+c=\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x\sqrt{x}+c$, then we get**:**

$\int_1^e\sqrt{x}\ln x\,dx=\left[\frac{2}{3}x\sqrt{x}\ln x-\frac{4}{9}x\sqrt{x}\right]_1^e=\frac{2}{3}e\sqrt{e}-\frac{4}{9}e\sqrt{e}+\frac{4}{9}=\frac{2}{3}e\sqrt{e}+\frac{4}{9}$

39. $\int_1^{+\infty}\frac{1}{2^x-\ln 2}dx=\int_1^{+\infty}\frac{1}{2^x-\ln 2}\times\frac{2^{-x}}{2^{-x}}dx=\int_1^{+\infty}\frac{2^{-x}}{1-2^{-x}\ln 2}dx=\frac{1}{\ln^2 2}\int_1^{+\infty}\frac{\ln^2 2\times 2^{-x}}{1-2^{-x}\ln 2}dx$

But $(-2^{-x})'=2^{-x}\ln 2$, then $(1-2^{-x}\ln 2)'=2^{-x}\ln^2 2$, then:

$\int_1^{+\infty}\frac{1}{2^x-\ln 2}dx=\frac{1}{\ln^2 2}\int_1^{+\infty}\frac{(1-2^{-x}\ln 2)'}{1-2^{-x}\ln 2}dx=\frac{1}{\ln^2 2}[\ln(1-2^{-x}\ln 2)]_1^{+\infty}$

$\int_1^{+\infty}\frac{1}{2^x-\ln 2}dx=\frac{1}{\ln^2 2}(0-\ln(1-2^{-1}\ln 2))=\frac{1}{\ln^2 2}\left(-\ln\left(1-\frac{\ln 2}{2}\right)\right)=\frac{\ln 2-\ln(2-\ln 2)}{\ln^2 2}$

40. $\int_1^{+\infty}\frac{1}{\varphi^x-1}dx=\int_1^{+\infty}\frac{\varphi^{-x}}{1-\varphi^{-x}}dx=\frac{1}{\ln\varphi}\int_1^{+\infty}\frac{\varphi^{-x}\ln\varphi}{1-\varphi^{-x}}dx=\frac{1}{\ln\varphi}[\ln(1-\varphi^{-x})]_1^{+\infty}$

$\int_1^{+\infty}\frac{1}{\varphi^x-1}dx=\frac{1}{\ln\varphi}(0-\ln(1-\varphi^{-1}))=\frac{1}{\ln\varphi}\left(-\ln\left(\frac{\varphi-1}{\varphi}\right)\right)=\frac{\ln\varphi-\ln(\varphi-1)}{\ln\varphi}$

$\int_1^{+\infty}\frac{1}{\varphi^x-1}dx=1-\frac{\ln(\varphi-1)}{\ln\varphi}$, but $\varphi=\frac{1+\sqrt5}{2}=$ Golden Ratio and $\varphi^2=\varphi+1$, so $\varphi=1+\frac{1}{\varphi}$

$\int_1^{+\infty}\frac{1}{\varphi^x-1}dx=1-\frac{\ln\left(1+\frac{1}{\varphi}-1\right)}{\ln\varphi}=1-\frac{\ln\left(\frac{1}{\varphi}\right)}{\ln\varphi}=1-\frac{-\ln\varphi}{\ln\varphi}=1+\frac{\ln\varphi}{\ln\varphi}=1+1=2$

41. $\int\sqrt{\frac{1}{\sin x}-1}\,dx=\int\sqrt{\frac{1-\sin x}{\sin x}}dx$, let $u=\sin x$, then $du=\cos x\,dx=\sqrt{1-\sin^2 x}\,dx$

then $du=\sqrt{1-u^2}dx$, so $dx=\frac{1}{\sqrt{1-u^2}}$, then we get:

$$\int\sqrt{\frac{1}{\sin x}-1}\,dx=\int\sqrt{\frac{1-u}{u}}.\frac{1}{\sqrt{1-u^2}}du=\int\frac{\sqrt{1-u}}{\sqrt u\sqrt{1-u}\sqrt{1+u}}du=\int\frac{1}{\sqrt u\times\sqrt{1+u}}du$$

Let $y=\sqrt u$, then $u=y^2$ and $du=2ydy$, then we get:

$$\int\sqrt{\frac{1}{\sin x}-1}\,dx=\int\frac{2ydy}{y\sqrt{1+y^2}}=2\int\frac{dy}{\sqrt{1+y^2}}=2\sinh^{-1}y+c=2\sinh^{-1}\sqrt u+c$$

With $u=\sin x$; therefore; $\int\sqrt{\frac{1}{\sin x}-1}\,dx=2\sinh^{-1}\left(\sqrt{\sin x}\right)+c$

42. $\int\frac{x+1}{x(1+xe^x)}dx=\int\frac{e^x(x+1)}{e^x.x(1+xe^x)}dx=\int\frac{xe^x+e^x}{xe^x+(xe^x)^2}dx$, let $u=xe^x$, so $du=(xe^x+e^x)dx$, so

$\int\frac{x+1}{x(1+xe^x)}dx=\int\frac{du}{u+u^2}=\int\left[\frac{(u+1)-1}{u(u+1)}\right]du=\int\left(\frac1u-\frac{1}{u+1}\right)du=\ln|u|-\ln|u+1|+c$

$=\ln\left|\frac{u}{u+1}\right|+c=\ln\left|\frac{xe^x}{1+xe^x}\right|+c$

43. $\mathrm{I}=\int_0^\pi\frac{1}{1+(\sin x)^{\cos x}}dx$ **… (1)** Let $u=\pi-x$, then $du=-dx$, for $x=0$, $u=\pi$ and for $x=\pi$,

$u=0$, then $\mathrm{I}=\int_\pi^0\frac{1}{1+(\sin(\pi-u))^{\cos(\pi-u)}}(-du)=\int_0^\pi\frac{1}{1+(\sin u)^{-\cos u}}du$

$\mathrm{I}=\int_0^\pi\frac{(\sin u)^{\cos u}}{1+(\sin u)^{\cos u}}du=\int_0^\pi\frac{(\sin x)^{\cos x}}{1+(\sin x)^{\cos x}}dx$ **… (2)** Adding (1) and (2) we get:

$2\mathrm{I}=\int_0^\pi\frac{1}{1+(\sin x)^{\cos x}}dx+\int_0^\pi\frac{(\sin x)^{\cos x}}{1+(\sin x)^{\cos x}}dx=\int_0^\pi\frac{1+(\sin x)^{\cos x}}{1+(\sin x)^{\cos x}}dx=\int_0^\pi dx=\pi$

$2\mathrm{I}=\pi$, therefore $\mathrm{I}=\frac{\pi}{2}$

44. $\int \frac{x}{1+x\tan x}dx = \int \frac{x}{1+x\left(\frac{\sin x}{\cos x}\right)}dx = \int \frac{x\cos x}{\cos x+x\sin x}dx$

Let $u = \cos x + x\sin x \Longrightarrow du = (-\sin x + \sin x + x\cos x)dx$, then we get:
$du = \ x\cos x\, dx$, so:

$$\int \frac{x}{1+x\tan x}dx = \int \frac{du}{u} = \ln|u| + c = \ln|\cos x + x\sin x| + c$$

45. $\int \frac{\tan^m x}{\sin x\cos x}dx = \int \frac{\left(\frac{\sin x}{\cos x}\right)^m}{\sin x\cos x}dx = \int \frac{\frac{\sin^m x}{\cos^m x}}{\sin x\cos x}dx = \int \frac{\sin^m x}{\cos^m x\sin x\cos x}dx$, then we get:

$$\int \frac{\tan^m x}{\sin x\cos x}dx = \int \frac{\sin^{m-1} x}{\cos^{m+1} x}dx = \int \frac{\sin^{m-1} x}{\cos^{m-1} x\,.\cos^2 x}dx = \int \left(\frac{\sin x}{\cos x}\right)^{m-1}.\sec^2 x\, dx;\ \text{so:}$$

$$\int \frac{\tan^m x}{\sin x\cos x}dx = \int \tan^{m-1} x\,.\sec^2 x\, dx = \frac{1}{m}\tan^m x + c$$

46. $\mathrm{I} = \int_{\frac{1}{e}}^{e} \frac{\arctan x}{x}dx$, let $x = \frac{1}{u}$, then $dx = -\frac{1}{u^2}du$, for $x = \frac{1}{e}$, $u = e$ and for $x = e$, $u = \frac{1}{e}$, then:

$\mathrm{I} = \int_e^{\frac{1}{e}} \frac{\arctan\left(\frac{1}{u}\right)}{\frac{1}{u}}\left(-\frac{1}{u^2}du\right) = \int_{\frac{1}{e}}^{e} \frac{\arctan\left(\frac{1}{u}\right)}{u}du = \int_{\frac{1}{e}}^{e} \frac{\arctan\left(\frac{1}{x}\right)}{x}dx = \int_{\frac{1}{e}}^{e} \frac{\frac{\pi}{2}-\arctan u}{x}dx$, then

$\mathrm{I} = \frac{\pi}{2}\int_{\frac{1}{e}}^{e}\frac{1}{x}dx - \int_{\frac{1}{e}}^{e}\frac{\arctan x}{x}dx = \frac{\pi}{2}\int_{\frac{1}{e}}^{e}\frac{1}{x}dx - \mathrm{I}$, so $2\mathrm{I} = \frac{\pi}{2}\int_{\frac{1}{e}}^{e}\frac{1}{x}dx = \frac{\pi}{2}[\ln|x|]_{\frac{1}{e}}^{e}$, then

$2\mathrm{I} = \frac{\pi}{2}\left(\ln e - \ln\left(\frac{1}{e}\right)\right) = \frac{\pi}{2}(1+1) = \pi$, therefore $\mathrm{I} = \frac{\pi}{2}$

47. $\int \frac{\sin 6x}{\sin 4x}dx = \int \frac{\sin(4x+2x)}{\sin 4x}dx = \int \frac{\sin 4x\cos 2x+\cos 4x\sin 2x}{\sin 4x}dx$

$= \int \cos 2x\, dx + \int \frac{\cos 4x\sin 2x}{\sin 4x}dx = \int \cos 2x\, dx + \int \frac{\left(2\cos^2 2x-1\right)\sin 2x}{2\sin 2x\cos 2x}dx$

$= \int \cos 2x\, dx + + \int \cos 2x\, dx - \frac{1}{2}\int \sec 2x\, dx = \sin 2x - \frac{1}{4}\ln|\sec 2x + \tan 2x| + c$

48. $\int \frac{f'\left(f(\sqrt{x})\right)f'(\sqrt{x})}{\sqrt{x}}dx = 2\int \frac{f'\left(f(\sqrt{x})\right)f'(\sqrt{x})}{2\sqrt{x}}dx = 2\int f'\left(f\left(\sqrt{x}\right)\right)f'\left(\sqrt{x}\right)d\left(\sqrt{x}\right)$

$= 2\int f'\left(f\left(\sqrt{x}\right)\right)d\left(f\left(\sqrt{x}\right)\right) = 2f\left(f\left(\sqrt{x}\right)\right) + c$

49. $\mathrm{I} = \int_1^{+\infty} e^{-\lfloor x\rfloor}dx$, for any positive integer n, $\lfloor x\rfloor = n$ if $n \le x < n+1$

$$\mathrm{I} = \int_1^{+\infty} e^{-\lfloor x\rfloor}dx = \int_1^{2} e^{-\lfloor x\rfloor}dx + \int_2^{3} e^{-\lfloor x\rfloor}dx + \int_3^{4} e^{-\lfloor x\rfloor}dx + \cdots;\ \text{then we get:}$$

$$\mathrm{I} = \int_1^{2} e^{-1}dx + \int_2^{3} e^{-2}dx + \int_3^{4} e^{-3}dx + \cdots = e^{-1} + e^{-2} + e^{-3} + \cdots = \frac{1}{e}\left(1 + \frac{1}{e} + \frac{1}{e^2} + \cdots\right)$$

$$\mathrm{I} = \frac{1}{e}.\frac{1}{1-\frac{1}{e}} = \frac{1}{e}.\frac{e}{e-1} = \frac{1}{e-1}$$

50. $\int \sqrt{\tan^2 x + \cot^2 x + 2}\, dx = \int \sqrt{\tan^2 x + \frac{1}{\tan^2 x} + 2}\, dx = \int \sqrt{\frac{\tan^4 x+1+2\tan^2 x}{\tan^2 x}}dx$

$= \int \sqrt{\frac{\left(\tan^2 x\right)^2+1+2\tan^2 x}{\tan^2 x}}dx = \int \sqrt{\frac{\left(\sec^2 x-1\right)^2+1+2\left(\sec^2 x-1\right)}{\tan^2 x}}dx$

$= \int \sqrt{\frac{\sec^4 x - 2\sec^2 x + 1 + 1 + 2\sec^2 x - 2}{\tan^2 x}}\, dx = \int \sqrt{\frac{\sec^4 x}{\tan^2 x}}\, dx = \int \frac{\sec^2 x}{\tan x}\, dx = \int \frac{(\tan x)'}{\tan x}\, dx$

Therefore $\int \sqrt{\tan^2 x + \cot^2 x + 2}\, dx = \ln|\tan x| + c$

51. $\int \sec^8 x\, dx = \int \sec^6 x \sec^2 x\, dx = \int (\sec^2 x)^3 \sec^2 x\, dx = \int (\tan^2 x + 1)^3 \sec^2 x\, dx$

$\int \sec^8 x\, dx = \int (\tan^6 x + 3\tan^4 x + 3\tan^2 x + 1)\sec^2 x\, dx$

$\int \sec^8 x\, dx = \int (\tan^6 x + 3\tan^4 x + 3\tan^2 x + 1)(\tan x)' dx$

$\int \sec^8 x\, dx = \int (\tan^6 x + 3\tan^4 x + 3\tan^2 x + 1)\, d(\tan x)$

$\int \sec^8 x\, dx = \frac{1}{7}\tan^7 x + \frac{3}{5}\tan^5 x + \tan^3 x + \tan x + c$

52. $\int \frac{1+\cos 4x}{\cot x - \tan x}\, dx = \int \frac{1+\cos 4x}{\frac{\cos x}{\sin x} - \frac{\sin x}{\cos x}}\, dx = \int \frac{1+\cos 4x}{\frac{\cos^2 x - \sin^2 x}{\sin x \cos x}}\, dx = \int \frac{(1+\cos 4x)\sin x \cos x}{\cos^2 x - \sin^2 x}\, dx$

$\int \frac{1+\cos 4x}{\cot x - \tan x}\, dx = \int \frac{[1+(2\cos^2 2x - 1)]\left(\frac{1}{2}\sin 2x\right)}{\cos^2 x - \sin^2 x}\, dx = \int \frac{2\cos^2 2x\left(\frac{1}{2}\sin 2x\right)}{\cos 2x}\, dx$

$\int \frac{1+\cos 4x}{\cot x - \tan x}\, dx = \int \sin 2x \cos 2x\, dx = \frac{1}{2}\int \sin 4x\, dx = -\frac{1}{8}\cos 4x + c$

53. $\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln\left(\frac{1+\cot x}{\cot x}\right) dx = \int_0^{\frac{\pi}{4}} \ln\left(\frac{1}{\cot x} + 1\right) dx = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\, dx$

Let $u = \frac{\pi}{4} - x$, then $du = -dx$, for $x = 0$, $u = \frac{\pi}{4}$ and for $x = \frac{\pi}{4}$, $u = 0$, then we get**:**

$\mathrm{I} = \int_{\frac{\pi}{4}}^{0} \ln\left(1 + \tan\left(\frac{\pi}{4} - u\right)\right)(-du) = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{\tan\left(\frac{\pi}{4}\right) - \tan u}{1 + \tan\left(\frac{\pi}{4}\right)\tan u}\right) du = \int_0^{\frac{\pi}{4}} \ln\left(1 + \frac{1-\tan u}{1+\tan u}\right) du$

$\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan u}\right) du = \int_0^{\frac{\pi}{4}} [\ln 2 - \ln(1+\tan u)]\, du = \int_0^{\frac{\pi}{4}} \ln 2\, dx - \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\, dx$, so

$\mathrm{I} = \frac{\pi}{4}\ln 2 - \mathrm{I}$, then $2\mathrm{I} = \frac{\pi}{4}\ln 2$, therefore $\mathrm{I} = \frac{\pi}{8}\ln 2$

54. $\mathrm{I} = \int_1^2 \frac{\ln x}{x^2 - 2x + 2}\, dx = \int_1^2 \frac{\ln x}{(x^2+2x+1)+1}\, dx = \int_1^2 \frac{\ln x}{(x-1)^2+1}\, dx$

Let $x - 1 = \tan t$, so $x = 1 + \tan t$ and $dx = (1+\tan^2 t)dt = (1+(x-1)^2)dt$, then we get : $\frac{dx}{(x-1)^2+1} = dt$, for $x = 1$, $\tan t = 0$, so $t = 0$ and for $x = 2$, $\tan t = 1$, so $t = \frac{\pi}{4}$, so we get**:** $\mathrm{I} = \int_0^{\frac{\pi}{4}} \ln(1+\tan t)\, dt = \int_0^{\frac{\pi}{4}} \ln(1+\tan x)\, dx$, but this integral was solved in the previous part, therefore $\mathrm{I} = \int_1^2 \frac{\ln x}{x^2-2x+2}\, dx = \frac{\pi}{8}\ln 2$

55. $\mathrm{I} = \int_{-n\pi}^{n\pi} \frac{\cos^5 x + 1}{e^x + 1}\, dx$ **… (1)** let $u = -x$, then $du = -dx$, so we get**:**

$\mathrm{I} = \int_{n\pi}^{-n\pi} \frac{\cos^5(-u)+1}{e^{-u}+1}(-du) = \int_{-n\pi}^{n\pi} \frac{\cos^5 u + 1}{e^{-u}+1}\, du = \int_{-n\pi}^{n\pi} \frac{\cos^5 x + 1}{e^{-x}+1}\, dx$ **… (2)**

Adding (1) and (2) we get $2\mathrm{I} = \int_{-n\pi}^{n\pi} \frac{\cos^5 x + 1}{e^x+1}\, dx + \int_{-n\pi}^{n\pi} \frac{\cos^5 x + 1}{e^{-x}+1}\, dx$, so**:**

$2\mathrm{I} = \int_{-n\pi}^{n\pi} (\cos^5 x + 1)\left(\frac{1}{e^x+1} + \frac{1}{e^{-x}+1}\right) dx = \int_{-n\pi}^{n\pi} (\cos^5 x + 1)\left(\frac{1}{e^x+1} + \frac{e^x}{1+e^x}\right) dx$

$2\mathrm{I} = \int_{-n\pi}^{n\pi} (\cos^5 x + 1)\left(\frac{e^x+1}{e^x+1}\right) dx = \int_{-n\pi}^{n\pi} (\cos^5 x + 1)\, dx = 2n\pi + \int_{-n\pi}^{n\pi} \cos^5 x\, dx$

$\int_{-n\pi}^{n\pi} \cos^5 x\, dx = \int_{-n\pi}^{n\pi} \cos^5 x\, dx = \int_{-n\pi}^{n\pi} \cos x\,(1-\sin^2 x)^2\, dx$

$2\mathrm{I} = \int_{-n\pi}^{n\pi}(1 - 2\sin^2 x + \sin^4 x)\cos x\, dx = \left[\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x\right]_{-n\pi}^{n\pi} = 0$

So, $2\mathrm{I} = 2n\pi$, therefore $\mathrm{I} = n\pi\ \pi$

56. $\int \frac{(x+1)^3}{x^2-x}dx = \int \frac{(x+1)^3-8+8}{x^2-x}dx = \int \frac{[(x+1)-2]\left[(x+1)^2+2(x+1)+4\right]+8}{x(x-1)}dx$

$= \int \frac{(x-1)\left(x^2+4x+7\right)}{x(x-1)}dx + 8\int \frac{1}{x^2-x}dx = \int \left(x + 4 + \frac{7}{x}\right)dx + 8\int \frac{x^{-2}}{1-x^{-1}}dx$

$\int \frac{(x+1)^3}{x^2-x}dx = \frac{1}{2}x^2 + 4x + 7\ln|x| + 8\ln|1 - x^{-1}| + c$

$\int \frac{(x+1)^3}{x^2-x}dx = \frac{1}{2}x^2 + 4x + 7\ln|x| + 8\ln\left|\frac{x-1}{x}\right| + c$

57. $\int \frac{dx}{x-2\sqrt{x}-3} = \int \frac{dx}{(\sqrt{x}-3)(\sqrt{x}+1)} = \frac{1}{4}\int \frac{4\sqrt{x}+3-3}{(\sqrt{x}-3)(\sqrt{x}+1)\sqrt{x}}dx = \frac{1}{4}\int \frac{3\sqrt{x}+3+\sqrt{x}-3}{(\sqrt{x}-3)(\sqrt{x}+1)\sqrt{x}}dx$

$= \frac{1}{4}\int \frac{3(\sqrt{x}+1)+(\sqrt{x}-3)}{(\sqrt{x}-3)(\sqrt{x}+1)\sqrt{x}}dx = \frac{1}{4}\int \left(\frac{3}{(\sqrt{x}-3)\sqrt{x}} + \frac{1}{(\sqrt{x}+1)\sqrt{x}}\right)dx = \frac{3}{4}\int \frac{\frac{1}{\sqrt{x}}}{\sqrt{x}-3}dx + \frac{1}{4}\int \frac{\frac{1}{\sqrt{x}}}{\sqrt{x}+1}dx$

$\int \frac{dx}{x-2\sqrt{x}-3} = \frac{3}{2}\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}-3}dx + \frac{1}{2}\int \frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}+1}dx = \frac{3}{2}\int \frac{(\sqrt{x}-3)'}{\sqrt{x}-3}dx + \frac{1}{2}\int \frac{(\sqrt{x}+1)'}{\sqrt{x}+1}dx$, therefore:

$\int \frac{dx}{x-2\sqrt{x}-3} = \frac{3}{2}\ln\left|\sqrt{x} - 3\right| + \frac{1}{2}\ln\left(\sqrt{x} + 1\right) + c$

58. $\mathrm{I} = \int_0^1 x^m \ln^n x\, dx$, let $u = \ln x$, then $x = e^u$ and $dx = e^u du$ and $\begin{cases}\text{for } x = 0; & u = -\infty \\ \text{for } x = 1; & u = 0\end{cases}$, so:

$\mathrm{I} = \int_{-\infty}^{0}(e^u)^m u^n du = \int_{-\infty}^{0} e^{um}u^n du$, let $um = -y$, then $du = -\frac{1}{m}dy$ for $u = -\infty$, $y = +\infty$ and for $u = 0$, $y = 0$, then we get:

$\mathrm{I} = \int_{+\infty}^{0} e^{-y}\left(-\frac{y}{m}\right)^n\left(-\frac{1}{m}dy\right) = \frac{1}{m^{n+1}}\int_0^{+\infty} e^{-y}(-1)^n y^n dy = \frac{(-1)^n}{m^{n+1}}\int_0^{+\infty} e^{-y}y^n dy$

$\mathrm{I} = \frac{(-1)^n}{m^{n+1}}\int_0^{+\infty} e^{-y}y^{(n+1)-1}dy = \frac{(-1)^n}{m^{n+1}}\Gamma(n + 1) = \frac{(-1)^n n!}{m^{n+1}}$

59. $\int \frac{1}{\sec x-1}dx = \int \frac{\cos x}{1-\cos x}dx = \int \frac{-1+\cos x+1}{1-\cos x}dx = \int \frac{-(1-\cos x)+1}{1-\cos x}dx = \int \left(-1 + \frac{1}{1-\cos x}\right)dx$

$\int \frac{1}{\sec x-1}dx = -x + \int \frac{1}{1-\cos x}dx$, for the remaining integral, we will use the direct substitution $t = \tan\left(\frac{x}{2}\right)$, so $dt = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)dx$, then $dx = \frac{2dt}{1+t^2}$ and $\cos x = \frac{1-t^2}{1+t^2}$, then we get:

$\int \frac{1}{\sec x-1}dx = -x + \int \frac{1}{1-\cos x}dx = -x + \int \frac{1}{1-\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2} = -x + \int \frac{1}{t^2}dt = -x - \frac{1}{t} + c$

$\int \frac{1}{\sec x-1}dx = -x - \cot\left(\frac{x}{2}\right) + c$

60. $\mathrm{I} = \int_0^1 \sqrt{1 - x^\pi}\, dx$

Change of variable: Let $x^\pi = t \Rightarrow dt = \pi x^{\pi-1}dx \Rightarrow dx = \frac{dt}{\pi t^{\frac{\pi-1}{\pi}}} = \frac{1}{\pi}t^{\frac{1-\pi}{\pi}}dt$; then:

$$\mathrm{I} = \int_0^1 \sqrt{1 - x^\pi}\, dx = \frac{1}{\pi}\int_0^1 \sqrt{1 - t}\,.\,t^{\frac{1-\pi}{\pi}}dt = \frac{1}{\pi}\int_0^1 (1 - t)^{\frac{1}{2}}.\,t^{\frac{1-\pi}{\pi}}dt$$

$$I = \frac{1}{\pi}\int_0^1 (1-t)^{\frac{3}{2}-1} . t^{\frac{1}{\pi}-1} dt = \frac{1}{\pi}\mathrm{B}\left(\frac{3}{2};\frac{1}{\pi}\right) = \frac{1}{\pi}.\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{\pi}\right)}{\Gamma\left(\frac{3}{2}+\frac{1}{\pi}\right)}$$

61. $\int \cos x \ln(1+\cos x)\, dx$

Let $u = \ln(1+\cos x)$, then $u' = -\frac{\sin x}{1+\cos x}$ and $v' = \cos x$, so $v = \sin x$, then:

$\int \cos x \ln(1+\cos x)\, dx = \sin x \ln(1+\cos x) + \int \frac{\sin^2 x}{1+\cos x} dx$

$= \sin x \ln(1+\cos x) + \int \frac{1-\cos^2 x}{1+\cos x} dx = \sin x \ln(1+\cos x) + \int \frac{(1-\cos x)(1+\cos x)}{1+\cos x} dx$

$= \sin x \ln(1+\cos x) + \int (1-\cos x) dx = \sin x \ln(1+\cos x) + x - \sin x + c$

62. $I = \int_0^{+\infty} \frac{dx}{(x^2+1)^2}$, let $x = \tan t$, then $dx = \sec^2 t\, dt$ and $\begin{cases} \text{for } x = 0; \ \ t = 0 \\ \text{for } x = +\infty; \ \ t = \frac{\pi}{2} \end{cases}$, then we get:

$I = \int_0^{\frac{\pi}{2}} \frac{\sec^2 t}{(\tan^2 t+1)^2} dt = \int_0^{\frac{\pi}{2}} \frac{\sec^2 t}{\sec^4 t} dt = \int_0^{\frac{\pi}{2}} \frac{1}{\sec^2 t} dt = \int_0^{\frac{\pi}{2}} \cos^2 t\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2t)\, dt$

$I = \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}$

63. $\int \frac{e^x}{(1+e^x)(2+e^x)} dx = \int \frac{e^x}{e^{2x}+3e^x+2} dx = \int \frac{e^x}{\left(e^x+\frac{3}{2}\right)^2-\frac{1}{4}} dx = \frac{1}{4}\int \frac{e^x}{(2e^x+3)^2-1} dx$

$= \frac{1}{8}\coth^{-1}(2e^x+3) + c$

64. $I = \int_0^1 (1+xe^x)\, e^{e^x} dx = \int_0^1 e^{-x}(1+xe^x). e^x. e^{e^x} dx = \int_0^1 (e^{-x}+e^x)\, e^x e^{e^x} dx$

Let $u = e^{-x} + x$, then $u' = 1 - e^{-x}$ and let $v' = e^x e^{e^x}$, then $v = e^{e^x}$, so we get:

$\int (1+xe^x) e^{e^x} dx = e^{e^x}(e^{-x}+x) - \int (1-e^{-x}) e^{e^x} dx$

Now evaluating $\int (1-e^{-x}) e^{e^x} dx = \int e^x (1-e^{-x}) e^{-x} e^{e^x} dx = \int (e^x - 1) e^{-x} e^{e^x} dx$

Let $z = e^x - x$, then $dz = (e^x-1)dx$, so $\int (1-e^{-x}) e^{e^x} dx = \int e^z dz = e^z = e^{e^x - x} + k$

So, $\int (1+xe^x) e^{e^x} dx = e^{e^x}(e^{-x}+x) - e^{e^x} e^{-x} + c$, then we get:

$I = \int_0^1 (1+xe^x)\, e^{e^x} = \left[e^{e^x}(e^{-x}+x) - e^{e^x} e^{-x}\right]_0^1 = e^e$

65. $I = \int_{-\infty}^{+\infty} \frac{xe^x}{(a+e^x)^2} dx = \int_{-\infty}^{+\infty} \frac{xe^{-x}}{(1+ae^{-x})^2} dx$, let $ae^{-x} = \tan^2\theta$ and $\begin{cases} \text{for } x = -\infty; \ \ t = \frac{\pi}{2} \\ \text{for } x = +\infty; \ \ t = 0 \end{cases}$

$-ae^{-x} dx = 2\tan\theta \sec^2\theta\, d\theta$, with $x = -\ln\left(\frac{\tan^2\theta}{a}\right)$, then we get:

$I = \int_{\frac{\pi}{2}}^0 \frac{-\ln\left(\frac{\tan^2\theta}{a}\right)\left(-\frac{2}{a}\tan\theta\sec^2\theta\right)}{(1+\tan^2\theta)^2} d\theta = \int_{\frac{\pi}{2}}^0 \frac{(\ln a - 2\ln(\tan\theta))\left(-\frac{2}{a}\tan\theta\sec^2\theta\right)}{(1+\tan^2\theta)^2} d\theta$

$I = \int_{\frac{\pi}{2}}^0 \frac{(\ln a - 2\ln(\tan\theta))\left(-\frac{2}{a}\tan\theta\sec^2\theta\right)}{\sec^4\theta} d\theta = \frac{1}{a}\int_0^{\frac{\pi}{2}} 2\sin\theta\cos\theta\,(\ln a - 2\ln(\tan\theta))d\theta$

$I = \frac{\ln a}{a}\int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta - \frac{2}{a}\int_0^{\frac{\pi}{2}} \sin 2\theta \ln(\tan\theta)\, d\theta = \frac{\ln a}{a}\left[-\frac{\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}} - \frac{2}{a}\mathrm{J} = \frac{\ln a}{a} - \frac{2}{a}(0) = \frac{\ln a}{a}$

66. $\int \frac{1+\sin x}{1+\sin x+\cos x}dx$, let $t=\tan\left(\frac{x}{2}\right)$, then $dx=\frac{2dt}{1+t^2}$, $\sin x=\frac{2t}{1+t^2}$ and $\cos x=\frac{1-t^2}{1+t^2}$, so:

$$\int \frac{1+\sin x}{1+\sin x+\cos x}dx=\int \frac{1+\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}}\cdot\frac{2dt}{1+t^2}=\int \frac{1+t}{1+t^2}dt=\int \frac{1}{1+t^2}dt+\int \frac{t}{1+t^2}dt$$

$=\int \frac{1}{1+t^2}dt+\frac{1}{2}\int \frac{2t}{1+t^2}dt=\tan^{-1}t+\frac{1}{2}\ln(1+t^2)+c$, therefore we get:

$$\int \frac{1+\sin x}{1+\sin x+\cos x}dx=\frac{x}{2}+\frac{1}{2}\ln\left(1+\tan^2\left(\frac{x}{2}\right)\right)+c$$

67. $\int \frac{x^9}{\left(4x^2+1\right)^6}dx=\int \frac{x^9}{\left[x^2\left(4+\frac{1}{x^2}\right)\right]^6}dx=\int \frac{x^9}{x^{12}\left(4+\frac{1}{x^2}\right)^6}dx=\int \frac{1}{x^3\left(4+\frac{1}{x^2}\right)^6}dx$

Let $4+\frac{1}{x^2}=t$; then $-\frac{2}{x^3}dx=dt$ and so $\frac{1}{x^3}dx=-\frac{1}{2}dt$; then we get:

$$\int \frac{x^9}{(4x^2+1)^6}dx=-\frac{1}{2}\int \frac{1}{t^6}dt=-\frac{1}{2}\left(-\frac{1}{5}t^{-5}\right)+c=\frac{1}{10t^5}+c;\text{ therefore we get:}$$

$$\int \frac{x^9}{(4x^2+1)^6}dx=\frac{1}{10\left(4+\frac{1}{x^2}\right)^5}+c$$

68. $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx=\int \frac{x^4-1}{x^2\sqrt{x^2\left(x^2+1+\frac{1}{x^2}\right)}}dx=\int \frac{x^4-1}{x^3\sqrt{x^2+1+\frac{1}{x^2}}}dx$, then we get:

$$\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx=\int \frac{\frac{x^4-1}{x^3}}{\sqrt{x^2+1+\frac{1}{x^2}}}dx=\int \frac{x-\frac{1}{x^3}}{\sqrt{x^2+1+\frac{1}{x^2}}}dx$$

Let $t=x^2+1+\frac{1}{x^2}$; then $dt=\left(2x-\frac{2}{x^3}\right)dx$ so; $\left(x-\frac{1}{x^3}\right)dx=\frac{1}{2}dt$; then we get:

$$\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx=\frac{1}{2}\int \frac{1}{\sqrt{t}}dt=\frac{1}{2}\left(2\sqrt{t}\right)+c=\sqrt{t}+c;\text{ therefore we get:}$$

$$\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}dx=\sqrt{x^2+1+\frac{1}{x^2}}+c$$

69. $\int \frac{6^x}{9^x-4^x}dx=\int \frac{\frac{6^x}{4^x}}{\frac{9^x-4^x}{4^x}}dx=\int \frac{\left(\frac{3}{2}\right)^x}{\left(\frac{9}{4}\right)^x-1}dx=\int \frac{\left(\frac{3}{2}\right)^x}{\left(\frac{3}{2}\right)^{2x}-1}dx=\frac{1}{\ln\left(\frac{3}{2}\right)}\int \frac{\ln\left(\frac{3}{2}\right).\left(\frac{3}{2}\right)^x}{\left[\left(\frac{3}{2}\right)^x\right]^2-1}dx$

Let $t=\left(\frac{3}{2}\right)^x$, then $dt=\left(\frac{3}{2}\right)^x\ln\left(\frac{3}{2}\right)dx$, then we get:

$$\int \frac{6^x}{9^x-4^x}dx=\frac{1}{\ln\left(\frac{3}{2}\right)}\int \frac{dt}{t^2-1}=\frac{1}{\ln\left(\frac{3}{2}\right)}\int \frac{dt}{(t-1)(t+1)}=\frac{1}{\ln\left(\frac{3}{2}\right)}\frac{1}{2}\int \frac{(t+1)-(t-1)}{(t-1)(t+1)}dt$$

$$\int \frac{6^x}{9^x-4^x}dx = \frac{1}{2\ln\left(\frac{3}{2}\right)}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt = \frac{1}{2\ln\left(\frac{3}{2}\right)}(\ln|u-1|+\ln|u+1|)+c$$

$$\int \frac{6^x}{9^x-4^x}dx = \frac{1}{2\ln\left(\frac{3}{2}\right)}\ln\left|\frac{u-1}{u+1}\right|+c = \frac{1}{2\ln\left(\frac{3}{2}\right)}\ln\left|\frac{\left(\frac{3}{2}\right)^x-1}{\left(\frac{3}{2}\right)^x+1}\right|+c = \frac{1}{2\ln\left(\frac{3}{2}\right)}\ln\left|\frac{3^x-2^x}{3^x+2^x}\right|+c$$

70. $\int \frac{dx}{\sqrt[5]{(x^2-1)(x-1)^8}} = \int \frac{dx}{\sqrt[5]{(x+1)(x-1)(x-1)^8}} = \int \frac{dx}{\sqrt[5]{(x+1)(x-1)^9}} = \int \frac{dx}{(x+1)^{\frac{1}{5}}(x-1)^{\frac{9}{5}}}$

$= \int \frac{dx}{(x+1)^{\frac{1}{5}}(x-1)^{2-\frac{1}{5}}} = \int \frac{dx}{\left(\frac{x+1}{x-1}\right)^{\frac{1}{5}}(x-1)^2}$, let $t=\frac{x+1}{x-1}$, then , $\frac{dx}{(x-1)^2}=-\frac{1}{2}dt$, so:

$$\int \frac{dx}{\sqrt[5]{(x^2-1)(x-1)^8}} = -\frac{1}{2}\int\frac{1}{t^{\frac{1}{5}}}dt = -\frac{5}{8}t^{\frac{4}{5}}+c = -\frac{5}{8}\left(\frac{x+1}{x-1}\right)^{\frac{4}{5}}+c = -\frac{5}{8}\sqrt[5]{\left(\frac{x+1}{x-1}\right)^4}+c$$

71. $I = \int_0^{\frac{\pi}{2}}(\sin^2(\sin x)+\cos^2(\cos x))dx$ **… (1)**

Let $y=\frac{\pi}{2}-x$, then $dy=-dx$, for $x=0$, $y=\frac{\pi}{2}$ and for $x=\frac{\pi}{2}$, $y=0$, then we get:

$I = \int_{\frac{\pi}{2}}^{0}\left(\sin^2\left(\sin\left(\frac{\pi}{2}-y\right)\right)+\cos^2\left(\cos\left(\frac{\pi}{2}-y\right)\right)\right)(-dy)$, then we get:

$I = \int_0^{\frac{\pi}{2}}(\sin^2(\cos y)+\cos^2(\sin y))dy = \int_0^{\frac{\pi}{2}}(\sin^2(\cos x)+\cos^2(\sin x))dx$ **… (2)**

Adding (1) we get:

$2I = \int_0^{\frac{\pi}{2}}(\sin^2(\sin x)+\cos^2(\cos x))dx + \int_0^{\frac{\pi}{2}}(\sin^2(\cos x)+\cos^2(\sin x))dx$

$2I = \int_0^{\frac{\pi}{2}}[(\sin^2(\sin x)+\cos^2(\sin x))+(\sin^2(\cos x)+\cos^2(\cos x))]dx$

$2I = \int_0^{\frac{\pi}{2}}(1+1)dx = 2\int_0^{\frac{\pi}{2}}dx$, so $I=\int_0^{\frac{\pi}{2}}dx$, therefore $I=\frac{\pi}{2}$

72. $\int \frac{x^2}{(a+bx)^2}dx$, let $t=a+bx$, then $x=\frac{t-a}{b}$ and $dx=\frac{1}{b}dt$, then we get:

$$\int \frac{x^2}{(a+bx)^2}dx = \frac{1}{b}\int\frac{\left(\frac{t-a}{b}\right)^2}{t^2}dt = \frac{1}{b}\int\frac{\frac{1}{b^2}(t-a)^2}{t^2}dt = \frac{1}{b^3}\int\frac{(t-a)^2}{t^2}dt;\quad \text{then we get:}$$

$$\int \frac{x^2}{(a+bx)^2}dx = \frac{1}{b^3}\int\frac{t^2-2ta+a^2}{t^2}dt = \frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt$$

$$\int \frac{x^2}{(a+bx)^2}dx = \frac{1}{b^3}\left[t-2a\ln|t|-\frac{2a^2}{t}\right]+c;\quad \text{therefore we get:}$$

$$\int \frac{x^2}{(a+bx)^2}dx = \frac{a+bx}{b^3}-\frac{2a}{b^3}\ln|a+bx|-\frac{2a^2}{b^3(a+bx)}+c$$

73. $I=\int_0^1(-1)^{\left[\frac{1}{x}\right]}dx$, let $u=\frac{1}{x}$, then $x=\frac{1}{u}$, so $du=-\frac{1}{u^2}du$, for $x=0$, $u=+\infty$ and for $x=1$, $u=1$, then we get:

$$I = \int_{+\infty}^{1}(-1)^{[u]}\left(-\frac{1}{u^2}\right)du = \int_1^{+\infty}\frac{(-1)^{[u]}}{u^2}du = \sum_{k=1}^{+\infty}\int_k^{k+1}\frac{(-1)^{[u]}}{u^2}du \quad \text{and for } k\le u<k+1$$

we get $[u]=k$, then we get:

$$\mathrm{I}=\sum_{k=1}^{+\infty}\int_{k}^{k+1}\frac{(-1)^k}{u^2}du=\sum_{k=1}^{+\infty}(-1)^k\int_{k}^{k+1}\frac{1}{u^2}du=\sum_{k=1}^{+\infty}(-1)^k\left[-\frac{1}{u}\right]_k^{k+1}=\sum_{k=1}^{+\infty}(-1)^k\left(\frac{1}{k}-\frac{1}{k+1}\right)$$

$$\mathrm{I}=\sum_{k=1}^{+\infty}(-1)^k\left\{\int_0^1 t^{k-1}dt-\int_0^1 t^k dt\right\}=\sum_{k=1}^{+\infty}(-1)^k\int_0^1 t^{k-1}(1-t)dt\,;\ \text{ then we get:}$$

$$\mathrm{I}=\sum_{k=1}^{+\infty}(-1)^k\int_0^1 t^k.\frac{1}{t}(1-t)dt=\int_0^1\frac{1-t}{t}dt\sum_{k=2}^{+\infty}(-1)^k t^k=\int_0^1\frac{1-t}{t}.\frac{-t}{1+t}dt$$

$$\mathrm{I}=\int_0^1\frac{t-1}{t+1}dt=\int_0^1\left(1-\frac{2}{t+1}\right)dt=[t-2\ln(t+1)]_0^1=1-2\ln 2=1-\ln 4$$

74. $\mathrm{I}=\int_0^{+\infty}e^{-\sqrt[\alpha]{\alpha x}}dx$

Change of variable: let $\alpha x=u^\alpha\Longrightarrow\alpha dx=\alpha u^{\alpha-1}du\Longrightarrow dx=u^{\alpha-1}du$, then we get:

$$\mathrm{I}=\int_0^{+\infty}e^{-\sqrt[\alpha]{\alpha x}}dx=\int_0^{+\infty}e^{-(\alpha x)^{\frac{1}{\alpha}}}dx=\int_0^{+\infty}e^{-(u^\alpha)^{\frac{1}{\alpha}}}u^{\alpha-1}du=\int_0^{+\infty}e^{-u}u^{\alpha-1}du=\Gamma(\alpha)$$

75. $\int\pi^{\pi^x}dx$

Let $u=\pi^x\Longrightarrow du=\pi^x.\ln\pi\,dx\Longrightarrow dx=\dfrac{du}{\pi^x.\ln\pi}=\dfrac{du}{u.\ln\pi}$; then we get:

$$\int\pi^{\pi^x}dx=\int\pi^u.\frac{du}{u.\ln\pi}=\frac{1}{\ln\pi}\int\frac{\pi^u}{u}du=\frac{1}{\ln\pi}\mathrm{Ei}(u.\ln\pi)+c=\frac{1}{\ln\pi}\mathrm{Ei}(\pi^x.\ln\pi)+c$$

Remark: $\displaystyle\int\frac{a^x}{x}dx=\mathrm{Ei}(x\ln a)+\mathrm{c}$

76. $\int\dfrac{1}{(\arcsin x)^2}dx$, let $u=\arcsin x$, then $x=\sin u$ and so $dx=\cos u\,du$, then we get:

$$\int\frac{1}{(\arcsin x)^2}=\int\frac{1}{u^2}\cos u\,du\,;\ \text{ now we will use integration by parts:}$$

Then: $\displaystyle\int\frac{1}{(\arcsin x)^2}=-\frac{\cos u}{u}-\int\frac{\sin u}{u}du=-\frac{\cos u}{u}-\mathrm{Si}(u)+c$

$$\int\frac{1}{(\arcsin x)^2}=-\frac{\cos(\arcsin x)}{\arcsin x}-\mathrm{Si}(\arcsin x)+c=-\frac{\sqrt{1-x^2}}{\sin^{-1}x}-\mathrm{Si}(\sin^{-1}x)+\mathrm{c}$$

77. $\displaystyle\int\frac{\sec x}{\sqrt{\sin(2x+\theta)+\sin\theta}}dx=\int\frac{\sec x}{\sqrt{\sin 2x\cos\theta+\cos 2x\sin\theta+\sin\theta}}dx$

$$\int\frac{\sec x}{\sqrt{\sin(2x+\theta)+\sin\theta}}dx=\int\frac{\sec x}{\sqrt{2\sin x\cos x\cos\theta+\left(2\cos^2 x-1\right)\sin\theta+\sin\theta}}dx$$

$$\int\frac{\sec x}{\sqrt{\sin(2x+\theta)+\sin\theta}}dx=\int\frac{\sec x}{\sqrt{2\sin x\cos x\cos\theta+2\cos^2 x\sin\theta}}dx=\frac{1}{\sqrt{2}}\int\frac{\sec^2 x}{\sqrt{\tan x\cos\theta+\sin\theta}}dx$$

Let $u=\tan x\cos\theta+\sin\theta\Longrightarrow du=\cos\theta\sec^2 x\,dx$, then we get:

$$\int\frac{\sec x}{\sqrt{\sin(2x+\theta)+\sin\theta}}dx=\frac{1}{\sqrt{2}}\int\frac{du}{\cos\theta\sqrt{u}}=\frac{1}{\cos\theta\sqrt{2}}.2\sqrt{u}+c=\frac{\sqrt{2}}{\cos\theta}\sqrt{\tan x\cos\theta+\sin\theta}+c$$

78. $\int \frac{dx}{\sqrt{9-\sqrt{9-x}}}$, let $\sqrt{9-x}=t$, then $9-x=t^2$ and $-dx=2tdt$, then we get:

$$\int \frac{dx}{\sqrt{9-\sqrt{9-x}}}=\int \frac{-2tdt}{\sqrt{9-t}}=-\int \frac{2tdt}{\sqrt{9-t}};\text{ let } y=\sqrt{9-t}\Rightarrow 9-t=y^2\Rightarrow -dt=2ydy;\text{ so}$$

$$\int \frac{dx}{\sqrt{9-\sqrt{9-x}}}=-\int \frac{2(9-y^2)(-2ydy)}{y}=\int 4(9-y^2)dy=\int (36-4y^2)dy$$

$$\int \frac{dx}{\sqrt{9-\sqrt{9-x}}}=36y-\frac{4}{3}y^3+c=36\sqrt{9-t}-\frac{4}{3}\left(\sqrt{9-t}\right)^3+c;\text{ therefore; we get:}$$

$$\int \frac{dx}{\sqrt{9-\sqrt{9-x}}}=36\sqrt{9-\sqrt{9-x}}-\frac{4}{3}\left(\sqrt{9-\sqrt{9-x}}\right)^3+c$$

79. $\mathrm{I}=\int_0^1 x^2\ln\left(\frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\right)dx$, let $x=\sin\theta$, then $dx=\cos\theta\, d\theta$ and $\begin{cases}\text{for } x=0; & \theta=0\\ \text{for } x=1; & \theta=\frac{\pi}{2}\end{cases}$, then

$$\mathrm{I}=\int_0^{\frac{\pi}{2}} x^2\ln\left(\frac{1+\sqrt{1-\sin^2\theta}}{1-\sqrt{1-\sin^2\theta}}\right)\cos\theta\, d\theta=\int_0^{\frac{\pi}{2}}\sin^2\theta\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\cos\theta\, d\theta$$

Let $u=\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)$, then $u'=-\frac{2}{\sin\theta}$ and let $v'=\sin^2\theta\cos\theta$, then $v=\frac{1}{3}\sin^3\theta$, then:

$$\mathrm{I}=\left[\frac{1}{3}\sin^3\theta\ln\left(\frac{1+\cos\theta}{1-\cos\theta}\right)\right]_0^{\frac{\pi}{2}}+\frac{2}{3}\int_0^{\frac{\pi}{2}}\sin^2\theta\, d\theta=\frac{1}{3}(0-0)+\frac{2}{3}\int_0^{\frac{\pi}{2}}\frac{1-\cos 2\theta}{2}d\theta$$

$$\mathrm{I}=\frac{2}{3}\left[\frac{1}{2}\theta-\frac{1}{4}\sin 2\theta\right]_0^{\frac{\pi}{2}}=\frac{2}{3}\left(\frac{\pi}{4}\right)=\frac{\pi}{6}$$

80. $\int \frac{1}{1+\sqrt{1-x^2}}dx$, let $x=\sin u$, then $dx=\cos u\, du$, then we get:

$$\int \frac{1}{1+\sqrt{1-x^2}}dx=\int \frac{1}{1+\sqrt{1-\sin^2 u}}\cos u\, du=\int \frac{\cos u}{1+\cos u}du.\text{ Let } t=\tan\left(\frac{u}{2}\right)\text{, then } du=\frac{2dt}{1+t^2}$$

and $\cos u=\frac{1-t^2}{1+t^2}$, then we get:

$$\int \frac{1}{1+\sqrt{1-x^2}}dx=\int \frac{\frac{1-t^2}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2}=\int \frac{1-t^2}{2}\frac{2dt}{1+t^2}=\int \frac{1-t^2}{1+t^2}dt=\int \frac{-(1+t^2)+2}{1+t^2}dt$$

$$=\int\left(-1+\frac{2}{1+t^2}\right)dt=-t+2\arctan t+c=-\tan\left(\frac{u}{2}\right)+2\arctan\left(\tan\left(\frac{u}{2}\right)\right)+c$$

$$=-\tan\left(\frac{u}{2}\right)+2\left(\frac{u}{2}\right)+c=u-\tan\left(\frac{u}{2}\right)+c=\arcsin x-\tan\left(\frac{\arcsin x}{2}\right)+c$$

81. $\mathrm{I}=\int_0^1\ln^n(1-x)\, dx$, let $-t=\ln(1-x)$, then $1-x=e^{-t}$ so $x=1-e^{-t}$ and $dx=e^{-t}dx$

For $x=0$, then $t=0$ and for $x=1$, $t=+\infty$, then we get:

$\mathrm{I}=\int_0^{+\infty}(-t)^n e^{-t}dt=(-1)^n\int_0^{+\infty}e^{-t}.t^{(n+1)-1}dt=(-1)^n\Gamma(n+1)=(-1)^n n!$

82. $\mathrm{I}=\int_0^{+\infty}\frac{\ln x}{x^2+9}dx$

Let $x=\frac{9}{u}$, then $dx=-\frac{9}{u^2}du$, for $x=0$, $u=+\infty$ and for $x=+\infty$, $u=0$, then we get:

$\mathrm{I} = \int_{+\infty}^{0} \frac{\ln\left(\frac{9}{u}\right)}{\left(\frac{9}{u}\right)^2 + 9}\left(-\frac{9}{u^2}du\right) = \int_0^{+\infty} \frac{\ln 9 - \ln u}{u^2+9}du = \int_0^{+\infty} \frac{\ln 9 - \ln x}{x^2+9}dx$, then we get:

$\mathrm{I} = \ln 9 \int_0^{+\infty} \frac{1}{x^2+9}dx - \int_0^{+\infty} \frac{\ln x}{x^2+9}dx = \ln 9 \int_0^{+\infty} \frac{1}{x^2+9}dx - \mathrm{I}$, so $2\mathrm{I} = \ln 9 \int_0^{+\infty} \frac{1}{x^2+9}dx$

$2\mathrm{I} = \ln 9 \left[\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right]_0^{+\infty} = \ln 9\left(\frac{\pi}{6}\right) = 2\ln 3\left(\frac{\pi}{6}\right)$, therefore $\mathrm{I} = \frac{\pi \ln 3}{6}$

83. $\int \sqrt{x^2+4x+13}\,dx = \int \sqrt{(x^2+4x+4)+9}\,dx = \int \sqrt{(x+2)^2+3^2}\,dx$

Let $x+2 = 3\tan\theta$, then $dx = 3\sec^2\theta\, d\theta$, then we get:

$\int \sqrt{x^2+4x+13}\,dx = \int \sqrt{3^2(\tan^2\theta+1)}\sec^2\theta\, d\theta = 3\int \sqrt{\sec^2\theta}\,.\,3\sec^2\theta\, d\theta$

$\int \sqrt{x^2+4x+13}\,dx = 9\int \sec^3\theta\, d\theta$, using integration by parts:

$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\,(\sec^2\theta - 1)d\theta$

$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta\, d\theta + \int \sec\theta\, d\theta$, then we get:

$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \int \sec\theta\, d\theta$

$\int \sec^3\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta| + k$, then we get:

$\int \sqrt{x^2+4x+13}\,dx = \frac{9}{2}\sec\theta\tan\theta + \frac{9}{2}\ln|\sec\theta + \tan\theta| + c$

$x+2 = 3\tan\theta$, so $\tan\theta = \frac{x+2}{3}$ and $\sec\theta = \sqrt{1+\left(\frac{x+2}{3}\right)^2} = \frac{\sqrt{(x+2)^2+9}}{3} = \frac{\sqrt{x^2+4x+13}}{3}$

Then we get:

$$\int \sqrt{x^2+4x+13}\,dx = \frac{9}{2}\left(\frac{\sqrt{x^2+4x+13}}{3}\right)\left(\frac{x+2}{3}\right) + \frac{9}{2}\ln\left|\frac{\sqrt{x^2+4x+13}}{3} + \frac{x+2}{3}\right| + c$$

$$\int \sqrt{x^2+4x+13}\,dx = \frac{1}{2}(x+2)\sqrt{x^2+4x+13} + \frac{9}{2}\ln\left|(x+2)+\sqrt{x^2+4x+13}\right| + c'$$

84. $\int \frac{e^x}{\sqrt{e^{2x}+4e^x+1}}dx$, let $y = e^x$, then $dy = e^x dx$, so:

$$\int \frac{e^x}{\sqrt{e^{2x}+4e^x+1}}dx = \int \frac{1}{\sqrt{y^2+4y+1}}dy = \int \frac{1}{\sqrt{y^2+4y+4-3}}dy = \int \frac{1}{\sqrt{(y-2)^2-3}}dx$$

Let $y+2 = \sqrt{3}\sec\theta$, then $\sec\theta = \frac{y+2}{\sqrt{3}}$ and $dy = \sqrt{3}\sec\theta\tan\theta\, d\theta$, then:

$$\int \frac{e^x}{\sqrt{e^{2x}+4e^x+1}}d\theta = \int \frac{\sqrt{3}\sec\theta\tan\theta}{\sqrt{3\sec^2\theta - 3}}d\theta = \int \frac{\sqrt{3}\sec\theta\tan\theta}{\sqrt{3}\tan\theta}d\theta = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c = \ln\left|\frac{y+2}{\sqrt{3}} + \frac{\sqrt{(y+2)^2-3}}{\sqrt{3}}\right| + c = \ln\left|\frac{e^x+2+\sqrt{e^{2x}+4e^x+1}}{\sqrt{3}}\right| + c$$

85. $\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}dx = 2\int \frac{e^x-1}{\sqrt{3e^{2x}-6e^x-1}}e^x dx + \int \frac{1}{\sqrt{3e^{2x}-6e^x-1}}e^x dx$

Let $u = e^x$, then $du = e^x dx$, then we get:

$$\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}dx = 2\int \frac{u-1}{\sqrt{3u^2-6u-1}}du + \int \frac{1}{\sqrt{3u^2-6u-1}}du$$

Now evaluating: $2\int \frac{u-1}{\sqrt{3u^2-6u-1}}du$, let $w = 3u^2 - 6u - 1$, then $dw = 6(u-1)du$, then:

$$2\int \frac{u-1}{\sqrt{3u^2-6u-1}}du = 2\int \frac{1}{6\sqrt{w}}dw = \frac{1}{3}\left(2\sqrt{w}\right) + k = \frac{2\sqrt{3u^2-6u-1}}{3} + k$$

Now evaluating: $\int \frac{1}{\sqrt{3u^2-6u-1}}du = \int \frac{1}{\sqrt{\left(\sqrt{3}u-\sqrt{3}\right)^2-4}}du$, let $v = \frac{\sqrt{3}(u-1)}{2}$, so $du = \frac{2}{\sqrt{3}}dv$

$$\int \frac{1}{\sqrt{3u^2-6u-1}}du = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{v^2-1}}du = \frac{\ln\left(\sqrt{v^2-1}+v\right)}{\sqrt{3}} + k'$$, then we get:

$$\int \frac{1}{\sqrt{3u^2-6u-1}}du = \frac{\ln\left(\sqrt{\left(\frac{\sqrt{3}(u-1)}{2}\right)^2-1}+\frac{\sqrt{3}(u-1)}{2}\right)}{\sqrt{3}} + k'$$, so

$$\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}dx = \frac{2\sqrt{3u^2-6u-1}}{3} + \frac{\ln\left(\sqrt{\left(\frac{\sqrt{3}(u-1)}{2}\right)^2-1}+\frac{\sqrt{3}(u-1)}{2}\right)}{\sqrt{3}} + c$$

$$\int \frac{2e^{2x}-e^x}{\sqrt{3e^{2x}-6e^x-1}}dx = \frac{2\sqrt{3e^{2x}-6e^x-1}}{3} + \frac{\ln\left|\sqrt{3(e^x-1)^2-4}+\sqrt{3}(e^x-1)\right|}{\sqrt{3}} + c$$

86. $\mathrm{I} = \int_0^1 \ln\left(\frac{x^{n+1}}{\ln x}\right)dx = \int_0^1[\ln x^{n+1} - \ln(\ln x)]dx = (n+1)\int_0^1 \ln x\, dx - \int_0^1 \ln(\ln x)dx$

$\mathrm{I} = (n+1)[x\ln x - x]_0^1 - \int_0^1 \ln(\ln x)dx = -(n+1) - \int_0^1 \ln(\ln x)dx$

let $-u = \ln x$, then $x = e^{-u}$ and $dx = -e^{-u}du$, for $x = 0$, $u = +\infty$ and for $x = 1$, $u = 0$, so:

$\mathrm{I} = -(n+1) - \int_{+\infty}^0 \ln(-u)d(-e^{-u})du = -(n+1) - \int_0^{+\infty} e^{-u}\ln(-u)\, du$

$\mathrm{I} = -(n+1) - \int_0^{+\infty}[e^{-u}\ln(-1) + e^{-u}\ln u]du$

$\mathrm{I} = -(n+1) - i\pi\int_0^{+\infty} e^{-u}du - \int_0^{+\infty} e^{-u}\ln u\, du = -n - 1 - i\pi + \gamma$, therefore:

$\mathrm{I} = (\gamma - n - 1) - i\pi$

87. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx = \int \frac{x-1}{(x+1)\sqrt{x^2\left(x+1+\frac{1}{x}\right)}}dx = \int \frac{x-1}{x(x+1)\sqrt{x+1+\frac{1}{x}}}dx$

$= \int \frac{x-1}{x(x+1)\sqrt{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2-1}}dx$, let $u = \sqrt{x}$, then $x = u^2$, so $dx = 2udu$, then we get:

$$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx = \int \frac{(u^2-1).2u}{u^2(u^2+1)\sqrt{\left(u+\frac{1}{u}\right)^2-1}}du = 2\int \frac{1-\frac{1}{u^2}}{\left(u+\frac{1}{u}\right)\sqrt{\left(u+\frac{1}{u}\right)^2-1}}du$$

Let $u + \frac{1}{u} = z$, then $\left(1 - \frac{1}{u^2}\right)du = dz$, then we get:

$$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx = 2\int \frac{1}{z\sqrt{z^2-1}}dz = 2\int \frac{z}{z^2\sqrt{z^2-1}}dz$$

Let $t = \sqrt{z^2-1}$, then $t^2 = z^2 - 1$ and $z^2 = t^2 + 1$, with $2zdz = 2tdt$, then we get:

$$\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}}dx = 2\int \frac{t}{t(t^2+1)}dt = 2\int \frac{1}{t^2+1}dt = 2\arctan t + c$$

$$= 2\arctan\left(\sqrt{z^2-1}\right) + \text{c} = 2\arctan\left(\sqrt{\left(u+\frac{1}{u}\right)^2 - 1}\right) + c = 2\arctan\left(\sqrt{\left(\sqrt{x}+\frac{1}{\sqrt{x}}\right)^2 - 1}\right) + c$$

$$= 2\arctan\left(\sqrt{\frac{x^2+x+1}{x}}\right) + c$$

88. $\text{I} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sqrt{1-\cos 4x}}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sqrt{2\sin^2(2x)}}dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sqrt{2}\sin(2x)}dx = \frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\sin x\cos x}dx$

$$\text{I} = \frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\frac{\sin x\cos^2 x}{\cos x}}dx = \frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\tan x\cos^2 x}dx = \frac{1}{2\sqrt{2}}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\ln(\tan x)}{\tan x}\sec^2 x\, dx$$

Let $t = \tan x$, then $\sec^2 x\, dx = dt$, for $x = \frac{\pi}{4}$, $t = \tan\left(\frac{\pi}{4}\right) = 1$ and $x = \frac{\pi}{3}$, $t = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

Then $\text{I} = \frac{1}{2\sqrt{2}}\int_1^{\sqrt{3}} \frac{\ln(t)}{t}dt = \frac{1}{2\sqrt{2}}\int_1^{\sqrt{3}} \ln t\,(\ln t)'dt = \frac{1}{2\sqrt{2}}\left[\frac{1}{2}\ln^2 t\right]_1^{\sqrt{3}} = \frac{\sqrt{2}}{32}\ln^2 3$

89. $\int x^2\sqrt{8-x^6}dx$, let $x^3 = \sqrt{8}\sin y = 2\sqrt{2}\sin y$, then $3x^2dx = 2\sqrt{2}\cos y\, dy$, then we get:

$$\int x^2\sqrt{8-x^6}dx = \frac{1}{3}\int 3x^2\sqrt{8-(x^3)^2}dx = \frac{1}{3}\int \sqrt{8-\left(\sqrt{8}\sin y\right)^2}.2\sqrt{2}\cos y\, dy$$

$$\int x^2\sqrt{8-x^6}dx = \frac{1}{3}\int \sqrt{8(1-\sin^2 y)}.2\sqrt{2}\cos y\, dy = \frac{1}{3}\int 2\sqrt{2}\sqrt{\cos^2 y}.2\sqrt{2}\cos y\, dy$$

$$\int x^2\sqrt{8-x^6}dx = \frac{8}{3}\int \cos^2 y\, dy = \frac{8}{3}\int \left(\frac{1+\cos 2y}{2}\right)dy = \frac{4}{3}\int (1+\cos 2y)dy$$

$$\int x^2\sqrt{8-x^6}dx = \frac{4}{3}y + \frac{2}{3}\sin(2y) + c = \frac{4}{3}y + \frac{4}{3}\sin y\cos y + c$$

$$\int x^2\sqrt{8-x^6}dx = \frac{4}{3}\arcsin\left(\frac{x^3}{2\sqrt{2}}\right) + \frac{4}{3}\left(\frac{x^3}{2\sqrt{2}}\right)\sqrt{1-\left(\frac{x^3}{2\sqrt{2}}\right)^2} + c$$

$\int x^2\sqrt{8-x^6}dx = \frac{4}{3}\arcsin\left(\frac{x^3}{2\sqrt{2}}\right) + \frac{4}{3}\left(\frac{x^3}{2\sqrt{2}}\right)\sqrt{\frac{8-x^6}{8}} + c$, therefore we get:

$$\int x^2\sqrt{8-x^6}dx = \frac{4}{3}\arcsin\left(\frac{x^3}{2\sqrt{2}}\right) + \frac{1}{6}x^3\sqrt{8-x^6} + c$$

90. $\text{I} = \int_0^{\frac{e}{n}} \frac{x^n}{\sqrt{1-\ln(nx)}}dx$, let $t = 1 - \ln(nx)$, then $\ln(nx) = 1 - t$, so $nx = e^{1-t}$, $x = \frac{1}{n}e^{1-t}$ and $dx = -\frac{1}{n}e^{1-t}dt$ for $x = \frac{e}{n}$, $t = 1 - \ln e = 0$ and for $x = 0$, $t = -\ln 0 = +\infty$, then we get:

$$\text{I} = \int_{+\infty}^0 \frac{\frac{1}{n^n}e^{n-nt}}{\sqrt{t}}\left(-\frac{1}{n}e^{1-t}dt\right) = \frac{1}{n^{n+1}}\int_0^{+\infty} \frac{e^{n+1}.e^{-t(n+1)}}{\sqrt{t}}dt = \frac{e^{n+1}}{n^{n+1}}\int_0^{+\infty} e^{-t(n+1)}t^{-\frac{1}{2}}dt$$

Let $u = (n+1)t$, then $dn = (n+1)dt$, then we get:

$$\text{I} = \frac{e^{n+1}}{n^{n+1}}\int_0^{+\infty} e^{-u}\left(\frac{u}{n+1}\right)^{-\frac{1}{2}}t^{-\frac{1}{2}}\frac{du}{n+1} = \frac{e^{n+1}}{n^{n+1}\sqrt{n+1}}\int_0^{+\infty} e^{-u}t^{-\frac{1}{2}}du = \frac{e^{n+1}}{n^{n+1}\sqrt{n+1}}\Gamma\left(\frac{1}{2}\right)$$

Therefore, we get $\text{I} = \left(\frac{e}{n}\right)^{n+1}\sqrt{\frac{\pi}{n+1}}$

91. $\mathrm{I} = \int_0^{\pi} \frac{x\sin x}{\sqrt{1+\tan^2\alpha\sin^2 x}}dx$

Let $u = \pi - x$, then $du = -dx$, for $x = 0$, $u = \pi$ and for $x = \pi$, $u = 0$, then we get:

$$\mathrm{I} = \int_{\pi}^{0} \frac{(\pi-u)\sin(\pi-u)}{\sqrt{1+\tan^2\alpha\sin^2(\pi-u)}}(-du) = \int_0^{\pi} \frac{(\pi-u)\sin u}{\sqrt{1+\tan^2\alpha\sin^2 u}}du = \int_0^{\pi} \frac{(\pi-x)\sin x}{\sqrt{1+\tan^2\alpha\sin^2 x}}dx$$

$\mathrm{I} = \pi\int_0^{\pi} \frac{\sin x}{\sqrt{1+\tan^2\alpha\sin^2 x}}dx - \int_0^{\pi} \frac{x\sin x}{\sqrt{1+\tan^2\alpha\sin^2 x}}dx = \pi\int_0^{\pi} \frac{\sin x}{\sqrt{1+\tan^2\alpha\sin^2 x}}dx - \mathrm{I}$, then

$2\mathrm{I} = \pi\int_0^{\pi} \frac{\sin x}{\sqrt{1+\tan^2\alpha\sin^2 x}}dx = \pi\int_0^{\pi} \frac{\sin x}{\sqrt{1+\tan^2\alpha(1-\cos^2 x)}}dx$, then we get:

$$\mathrm{I} = \frac{\pi}{2}\int_0^{\pi} \frac{\sin x}{\sqrt{1+\tan^2\alpha-\tan^2\alpha\cos^2 x}}dx = \frac{\pi\cos\alpha}{2}\int_0^{\pi} \frac{\sin x}{\sqrt{\cos^2\alpha\left(1+\tan^2\alpha-\tan^2\alpha\cos^2 x\right)}}dx$$

$\mathrm{I} = \frac{\pi\cos\alpha}{2}\int_0^{\pi} \frac{\sin x}{\sqrt{1-\sin^2\alpha\cos^2 x}}dx$, let $y = \sin\alpha\cos x$, then $dy = -\sin\alpha\sin x\,dx$ and for $x = 0$, $y = \sin\alpha\cos 0 = \sin\alpha$ and for $x = \pi$, $y = \sin\alpha\cos\pi = -\sin\alpha$, then we get:

$$\mathrm{I} = \frac{\pi\cos\alpha}{2\sin\alpha}\int_{\sin\alpha}^{-\sin\alpha} -\frac{dy}{\sqrt{1-y^2}} = \frac{\pi\cos\alpha}{2\sin\alpha}\int_{-\sin\alpha}^{\sin\alpha} \frac{dy}{\sqrt{1-y^2}} = \frac{\pi\cos\alpha}{2\sin\alpha}[\arcsin(y)]_{-\sin\alpha}^{\sin\alpha}$$

$$\mathrm{I} = \frac{\pi\cos\alpha}{2\sin\alpha}(\arcsin(\sin\alpha) - \arcsin(-\sin\alpha)) = \frac{\pi\cos\alpha}{2\sin\alpha}(\alpha+\alpha) = \frac{\pi\alpha}{\tan\alpha}$$

92. $\int \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}dx = \int \frac{1}{x^{\frac{2}{3}}+x^{\frac{1}{3}}+\frac{1}{4}+\frac{3}{4}}dx = \int \frac{1}{\left(x^{\frac{1}{3}}+\frac{1}{2}\right)^2+\frac{3}{4}}dx$

Let $u = x^{\frac{1}{3}} + \frac{1}{2}$, then $du = \frac{1}{3}x^{-\frac{2}{3}}$, $dx = 3x^{\frac{2}{3}}$ and we have $x^{\frac{2}{3}} = \left(u - \frac{1}{2}\right)^2$, then we get:

$\int \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}dx = 3\int \frac{\left(u-\frac{1}{2}\right)^2}{u^2+\frac{3}{4}}du = 3\int \frac{u^2+\frac{3}{4}-u-\frac{2}{4}}{u^2+\frac{3}{4}}du$, then we get:

$$\int \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}dx = 3\int \frac{u^2+\frac{3}{4}}{u^2+\frac{3}{4}}du - \frac{3}{2}\int \frac{2u}{u^2+\frac{3}{4}}du - \sqrt{3}\int \frac{\frac{\sqrt{3}}{2}}{u^2+\frac{3}{4}}du$$

$$\int \frac{1}{\sqrt[3]{x^2}+\sqrt[3]{x}+1}dx = 3u - \frac{3}{2}\ln\left(u^2+\frac{3}{4}\right) - \sqrt{3}\arctan\left(\frac{2u}{\sqrt{3}}\right) + c$$

$$= 3\left(x^{\frac{1}{3}}+\frac{1}{2}\right) - \frac{3}{2}\ln\left(\left(x^{\frac{1}{3}}+\frac{1}{2}\right)^2+\frac{3}{4}\right) - \sqrt{3}\arctan\left(\frac{2\left(x^{\frac{1}{3}}+\frac{1}{2}\right)}{\sqrt{3}}\right) + c$$

93. $\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx$, let $t = \sqrt[4]{x}$, then $x = t^4$ and $dt = 4t^3dt$, then:

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx = \int \frac{\sqrt[3]{1+\sqrt[4]{t^4}}}{\sqrt{t^4}}4t^3dt = \int \frac{\sqrt[3]{1+t}}{t^2}4t^3dt = 4\int t\times\sqrt[3]{1+t}dt$$

Let $u = 1 + t$ with $t = u - 1$ and $dt = du$, then:

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx = 4\int (u-1)\sqrt[3]{u}du = 4\int (u-1)u^{\frac{1}{3}}du = 4\int \left(u^{\frac{4}{3}} - u^{\frac{1}{3}}\right)du$$

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx = 4\left(\frac{3}{7}u^{\frac{7}{3}} - \frac{3}{4}u^{\frac{4}{3}}\right) + c = \frac{12}{7}u^{\frac{7}{3}} - 3u^{\frac{4}{3}} + c = \frac{12}{7}(1+t)^{\frac{7}{3}} - 3(1+t)^{\frac{4}{3}} + c$$

$$\int \frac{\sqrt[3]{1+\sqrt[4]{x}}}{\sqrt{x}}dx = \frac{12}{7}(1+\sqrt[4]{x})^{\frac{7}{3}} - 3(1+\sqrt[4]{x})^{\frac{4}{3}} + c$$

94. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{4\sin x + 2\pi}{\sin x + \cos x + \pi}dx = \int_0^{\frac{\pi}{4}} \frac{4\sin x + 2\pi}{\sin x + \cos x + \pi}dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4\sin x + 2\pi}{\sin x + \cos x + \pi}dx$

For $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4\sin x + 2\pi}{\sin x + \cos x + \pi}dx$, let $u = \frac{\pi}{2} - x$, then $du = -dx$ and $\begin{cases} \text{for } \ x = 0 \quad u = \frac{\pi}{2} \\ \text{for } \ x = \frac{\pi}{2} \quad u = 0 \end{cases}$, then:

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4\sin x + 2\pi}{\sin x + \cos x + \pi}dx = \int_{-\frac{\pi}{4}}^{0} \frac{4\sin(\frac{\pi}{2} - u) + 2\pi}{\sin(\frac{\pi}{2} - u) + \cos(\frac{\pi}{2} - u) + \pi}(-du) = \int_0^{\frac{\pi}{4}} \frac{4\cos u + 2\pi}{\cos u + \sin u + \pi}du$$

Then $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{4\sin x + 2\pi}{\sin x + \cos x + \pi}dx = \int_0^{\frac{\pi}{4}} \frac{4\cos x + 2\pi}{\sin x + \cos x + \pi}dx$, then we get:

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{4\sin x + 2\pi}{\sin x + \cos x + \pi}dx + \int_0^{\frac{\pi}{4}} \frac{4\cos x + 2\pi}{\sin x + \cos x + \pi}dx = \int_0^{\frac{\pi}{4}} \frac{4\cos x + 4\sin x + 4\pi}{\sin x + \cos x + \pi}dx$$

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{4(\sin x + \cos x + \pi)}{\sin x + \cos x + \pi}dx = 4\int_0^{\frac{\pi}{4}} dx = 4\left(\frac{\pi}{4} - 0\right) = \pi$$

95. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \cos 2x \tan\left(\frac{x}{2}\right)dx = \int_0^{\frac{\pi}{2}} (1 - 2\sin^2 x)\tan\left(\frac{x}{2}\right)dx$

Let $y = \tan\left(\frac{x}{2}\right)$, then $dx = \frac{2}{1+y^2}dy$ and $\sin x = \frac{2y}{1+y^2}$, for $x = 0$, $y = 0$ and for $x = \frac{\pi}{4}$, $y = 1$

$$\mathrm{I} = \int_0^1 \left(1 - 2\left(\frac{2y}{1+y^2}\right)^2\right).y.\frac{2}{1+y^2}dy = \int_0^1 \frac{y^4 - 6y^2 + 1}{\left(1+y^2\right)^3}.2ydy$$

Let $z = 1 + y^2$, then $dz = 2ydy$, for $y = 0$, $z = 1$ and for $y = 1$, $z = 2$, so we get:

$$\mathrm{I} = \int_0^1 \frac{(z-1)^4 - 6(z-1)^2 + 1}{z^3}dz = \int_1^2 \left(\frac{1}{z} - \frac{8}{z^2} + \frac{8}{z^3}\right)dz = \left[\ln|z| + \frac{9}{z} - \frac{4}{z^2}\right]_1^2$$

$$\mathrm{I} = \ln 2 + 4 - 1 - 8 + 4 = \ln 2 - 1 = \ln 2 - \ln e = \ln\left(\frac{2}{e}\right)$$

96. $\int \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}}dx$, let $x = \tan\theta$, then $\arctan x = \theta$ and $dx = \sec^2\theta\, d\theta$, $1 + x^2 = \sec^2\theta$, then:

$$\int \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}}dx = \int \frac{\tan^2\theta e^\theta}{\sec\theta}\sec^2\theta\, d\theta = \int \sec\theta \tan^2\theta\, e^\theta d\theta$$

$$\int \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}}dx = \int \sec\theta(\sec^2\theta - 1)e^\theta d\theta = \int \sec^3\theta\, e^\theta d\theta - \int \sec\theta\, e^\theta d\theta$$

Let us evaluate $\int \sec^3\theta\, e^\theta d\theta$ using integration by parts:

Let $u = \sec\theta\, e^\theta$, then $u' = \sec\theta(\tan\theta + 1)e^\theta$ and let $v' = \sec^2\theta$, then $v = \tan\theta$, then:

$\int \sec^3\theta\, e^\theta d\theta = \sec\theta\tan\theta\, e^\theta - \int \sec\theta(\tan\theta + 1)\tan\theta\, e^\theta\, d\theta$, then:

$\int \sec\theta\tan^2\theta\, e^\theta d\theta = \sec\theta\tan\theta\, e^\theta - \int \sec\theta(\tan\theta + 1)\tan\theta\, e^\theta\, d\theta$

$\int \sec\theta\tan^2\theta\, e^\theta d\theta = \sec\theta\tan\theta\, e^\theta - \int \sec\theta\tan^2\theta\, e^\theta\, d\theta - \int \sec\theta\tan\theta\, e^\theta\, d\theta$, so:

$2\int \sec\theta\tan^2\theta\, e^{\theta} d\theta = \sec\theta\tan\theta\, e^{\theta} - \sec\theta\, e^{\theta} + k$, then we get

$\int \sec\theta\tan^2\theta\, e^{\theta} d\theta = \frac{1}{2}\left[\sec\theta\, e^{\theta}(\tan\theta - 1)\right] + c$, so:

$$\int \frac{x^2 e^{\arctan x}}{\sqrt{1+x^2}} dx = \frac{1}{2}\sec\theta\, e^{\theta}(\tan\theta - 1) + c = \frac{1}{2}e^{\arctan x}\sqrt{1+x^2}(x-1) + c$$

97. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x\cos x} dx$ **… (1)**, let $u = \frac{\pi}{2} - x$, then $du = -dx$ and $\begin{cases} \text{for } \ x = 0 & u = \frac{\pi}{2} \\ \text{for } \ x = \frac{\pi}{2} & u = 0 \end{cases}$, then**:**

$$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{\cos(\frac{\pi}{2}-u)}{1+\sin(\frac{\pi}{2}-u)\cos(\frac{\pi}{2}-u)}(-du) = \int_0^{\frac{\pi}{2}} \frac{\sin u}{1+\cos u\sin u} du = \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x\cos x} dx \ \textbf{… (2)}$$

Adding (1) and (2) we get**:** $2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x}{1+\sin x\cos x} dx + \int_0^{\frac{\pi}{2}} \frac{\sin x}{1+\sin x\cos x} dx$

$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{1+\sin x\cos x} dx$, so $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{2+2\sin x\cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{3-(1-2\sin x\cos x)} dx$, then**:**

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{3-\left(\sin^2 x+\cos^2 x-2\sin x\cos x\right)} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{3-(\sin x-\cos x)^2} dx$$

Let $y = \sin x - \cos x$, then $dy = (\cos x + \sin x)dx$, for $x = 0$, $y = \sin 0 - \cos 0 = -1$ and for $x = \frac{\pi}{2}$, $y = \sin\frac{\pi}{2} - \cos\frac{\pi}{2} = -1$, then we get**:**

$$\mathrm{I} = \int_{-1}^{1} \frac{1}{3-y^2} dy = \int_{-1}^{1} \frac{1}{\left(\sqrt{3}\right)^2 - y^2} dy = \frac{1}{2\sqrt{3}}\int_{-1}^{1} \frac{\sqrt{3}-y+\sqrt{3}+y}{\left(\sqrt{3}-y\right)\left(\sqrt{3}+y\right)} dy = \frac{1}{2\sqrt{3}}\int \left(\frac{1}{\sqrt{3}+v} + \frac{1}{\sqrt{3}-v}\right) dv$$

$$\mathrm{I} = \frac{1}{2\sqrt{3}}\left[\ln\left|\sqrt{3}+v\right| - \ln\left|\sqrt{3}-v\right|\right]_{-1}^{1} = \frac{1}{2\sqrt{3}}\left[\ln\left|\frac{\sqrt{3}+v}{\sqrt{3}-v}\right|\right]_{-1}^{1} = \frac{1}{2\sqrt{3}}\left[\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) - \ln\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)\right]$$

$$\mathrm{I} = \frac{1}{2\sqrt{3}}\left[\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) + \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)\right] = \frac{2}{2\sqrt{3}}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right) = \frac{1}{\sqrt{3}}\ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$$

Or we can directly write $\mathrm{I} = \int_{-1}^{1} \frac{1}{\left(\sqrt{3}\right)^2 - y^2} dy = \frac{2}{\sqrt{3}}\tanh^{-1}\left(\frac{1}{\sqrt{3}}\right)$

98. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\left(1+\sqrt{\sin 2x}\right)^2} dx$ **…(1)**, let $u = \frac{\pi}{2} - x$, then $du = -dx$ and $\begin{cases} \text{for } \ x = 0 & u = \frac{\pi}{2} \\ \text{for } \ x = \frac{\pi}{2} & u = 0 \end{cases}$, then**:**

$$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{\cos(\frac{\pi}{2}-u)}{\left(1+\sqrt{\sin 2(\frac{\pi}{2}-u)}\right)^2}(-du) = \int_0^{\frac{\pi}{2}} \frac{\sin u}{\left(1+\sqrt{\sin(\pi-2u)}\right)^2} du = \int_0^{\frac{\pi}{2}} \frac{\sin u}{\left(1+\sqrt{\sin 2u}\right)^2} du$$

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\sin x}{\left(1+\sqrt{\sin 2x}\right)^2} dx$ **… (2)** Adding (1) and (2) we get**:**

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{\left(1+\sqrt{\sin 2x}\right)^2} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{\left(1+\sqrt{2\sin x\cos x}\right)^2} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{\left(1+\sqrt{1-(1-2\sin x\cos x)}\right)^2} dx$$

$$2\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{\left(1+\sqrt{1-\left(\sin^2 x+\cos^2 x-2\sin x\cos x\right)}\right)^2} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x+\sin x}{\left(1+\sqrt{1-(\sin x-\cos x)^2}\right)^2} dx$$

Let $u = \sin x - \cos x$, then $du = (\cos x + \sin x)dx$ and $\begin{cases} \text{for} \quad x = 0 \quad u = -1 \\ \text{for} \quad x = \frac{\pi}{2} \quad u = 1 \end{cases}$, then we get:

$$2\mathrm{I} = \int_{-1}^{1} \frac{du}{\left(1+\sqrt{1-u^2}\right)^2} = 2\int_0^1 \frac{du}{\left(1+\sqrt{1-u^2}\right)^2}, \text{ so } \mathrm{I} = \int_0^1 \frac{du}{\left(1+\sqrt{1-u^2}\right)^2}$$

Let $u = \sin t$, then $du = \cos t\, dt$ and $\begin{cases} \text{for} \quad u = 0 \quad t = 0 \\ \text{for} \quad u = 1 \quad t = \frac{\pi}{2} \end{cases}$, then we get:

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos t}{\left(1+\sqrt{1-\sin^2 t}\right)^2} dt = \int_0^{\frac{\pi}{2}} \frac{\cos t}{(1+\cos t)^2} dt = \int_0^{\frac{\pi}{2}} \frac{\cos x}{(1+\cos x)^2} dx$$

Let $t = \tan\left(\frac{x}{2}\right)$, then $du = \frac{2dt}{1+t^2}$ and $\cos u = \frac{1-t^2}{1+t^2}$, $\begin{cases} \text{for} \quad x = 0 \quad u = 0 \\ \text{for} \quad x = \frac{\pi}{2} \quad t = 1 \end{cases}$, then:

$$\mathrm{I} = \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{\left(1+\frac{1-t^2}{1+t^2}\right)^2} \frac{2dt}{1+t^2} = \frac{1}{2}\int_0^1 (1-t^2)dt = \frac{1}{2}\left[t - \frac{1}{3}t^3\right]_0^1 = \frac{1}{3}$$

99. $\int \frac{1+e^{-x}}{\sqrt{1-e^{-x}}} dx$, let $e^{-x} = \sin^2\theta$, then $-e^{-x}dx = 2\sin\theta\cos\theta\, d\theta$, then

$-\sin^2\theta\, dx = 2\sin\theta\cos\theta\ d\theta$ so, we get $dx = -\frac{2\cos\theta}{\sin\theta}\ d\theta$, then:

$$\int \frac{1+e^{-x}}{\sqrt{1-e^{-x}}} dx = \int \frac{1+\sin^2\theta}{\sqrt{1-\sin^2\theta}} . -\frac{2\cos\theta}{\sin\theta}\ d\theta = \int \frac{1+\sin^2\theta}{\sqrt{\cos^2\theta}} . -\frac{2\cos\theta}{\sin\theta}\ d\theta$$

$$\int \frac{1+e^{-x}}{\sqrt{1-e^{-x}}} dx = \int \frac{1+\sin^2\theta}{\cos\theta} . -\frac{2\cos\theta}{\sin\theta}\ d\theta = -\int (1+\sin^2\theta) . \frac{2}{\sin\theta}\ d\theta$$

$$\int \frac{1+e^{-x}}{\sqrt{1-e^{-x}}} dx = -2\int (\csc\theta + \sin\theta)\, d\theta = -2\ln|\csc\theta - \cot\theta| + 2\cos\theta + c$$

$$\int \frac{1+e^{-x}}{\sqrt{1-e^{-x}}} dx = -2\ln\left|e^{-\frac{1}{2}x} - \sqrt{e^x - 1}\right| + 2\sqrt{1-e^{-x}} + c$$

100. $\int \frac{1}{\cos^3 x} dx = \int \frac{\cos x}{\cos^4 x} dx = \int \frac{\cos x}{\left(1-\sin^2 x\right)^2} dx =$, let $u = \sin x$, then $du = \cos x\, dx$,

then: $\int \frac{1}{\cos^3 x} dx = \int \frac{1}{\left(1-u^2\right)^2} du = \int \frac{1}{[(1-u)(1+u)]^2} du = \int \frac{1}{(u-1)^2(u+1)^2} du$

Using Partial Fractions Decomposition, we have:

$$\frac{1}{(u-1)^2(u+1)^2} = \frac{\mathrm{A}}{u-1} + \frac{\mathrm{B}}{(u-1)^2} + \frac{\mathrm{C}}{u+1} + \frac{\mathrm{D}}{(u+1)^2}; \ \text{ then}$$

$\mathrm{A}(u^2-1)(u+1) + \mathrm{B}(u+1)^2 + \mathrm{C}(u-1)(u^2-1) + \mathrm{D}(u-1)^2 = 1$, then:

$(\mathrm{A}+\mathrm{C})u^3 + (\mathrm{A}+\mathrm{B}-\mathrm{C}+\mathrm{D})u^2 + (-\mathrm{A}+2\mathrm{B}-\mathrm{c}-2\mathrm{D})u + (-\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D}) = 1$

then we get the following system $\begin{cases} \mathrm{A}+\mathrm{C} = 0 \\ \mathrm{A}+\mathrm{B}-\mathrm{C}+\mathrm{D} = 0 \\ -\mathrm{A}+2\mathrm{B}-\mathrm{c}-2\mathrm{D} = 0 \\ -\mathrm{A}+\mathrm{B}+\mathrm{C}+\mathrm{D} = 1 \end{cases}$, by solving this system we get:

$\mathrm{A} = -\frac{1}{4}$, $\mathrm{C} = \frac{1}{4}$ and $\mathrm{B} = \mathrm{D} = \frac{1}{4}$, then we get:

$\int \frac{1}{\cos^3 x} dx = \int \left[\frac{1}{4}\left(\frac{1}{u+1} - \frac{1}{u-1}\right) + \frac{1}{4}\left(\frac{1}{(u+1)^2} + \frac{1}{(u-1)^2}\right)\right] du$

$\int \frac{1}{\cos^3 x} dx = \frac{1}{4}\ln|u+1| - \frac{1}{4}\ln|u-1| - \frac{1}{4}\left(\frac{1}{u+1} + \frac{1}{u-1}\right) + c$

$\int \frac{1}{\cos^3 x} dx = \frac{1}{4}\ln\left|\frac{u+1}{u-1}\right| - \frac{1}{4}\left(\frac{1}{u+1} + \frac{1}{u-1}\right) + c$, then we get**:**

$\int \frac{1}{\cos^3 x} dx = \frac{1}{4}\ln\left|\frac{\sin x+1}{\sin x-1}\right| - \frac{1}{4}\left(\frac{1}{\sin x+1} + \frac{1}{\sin x-1}\right) + c = \frac{1}{4}\ln\left|\frac{\sin x+1}{\sin x-1}\right| - \frac{\sin x}{2\cos^2 x} + c$

101. **First Method:** $\int \frac{2}{x^2\sqrt{x^2-1}} dx$

Let $y = \sqrt{1 - \frac{1}{x^2}} \Rightarrow y^2 - 1 = -\frac{1}{x^2} \Rightarrow \frac{2}{x^3} dx = 2y dy$; then we get:

$$\int \frac{2}{x^2\sqrt{x^2-1}} dx = \int \frac{1}{\sqrt{1-\frac{1}{x^2}}} . \frac{2}{x^3} dx = \int \frac{1}{y} . 2y dy = \int 2dy = 2y + c; \text{ therefore:}$$

$$\int \frac{2}{x^2\sqrt{x^2-1}} dx = 2\sqrt{1-\frac{1}{x^2}} + c = \frac{2\sqrt{x^2-1}}{x} + c$$

Second Method: $\int \frac{2}{x^2\sqrt{x^2-1}} dx$

Let $x = \sec\theta \Rightarrow dx = \sec\theta . \tan\theta \, d\theta$; then we get:

$$\int \frac{2}{x^2\sqrt{x^2-1}} dx = \int \frac{2}{\sec^2\theta\sqrt{\sec^2\theta - 1}} . \sec\theta \tan\theta \, d\theta$$

$$\int \frac{2}{x^2\sqrt{x^2-1}} dx = \int \frac{2}{\sec^2\theta \tan\theta} . \sec\theta\tan\theta \, d\theta = \int \frac{2}{\sec\theta} d\theta = 2\int \cos\theta \, d\theta = 2\sin\theta + c$$

$$\int \frac{2}{x^2\sqrt{x^2-1}} dx = 2\sqrt{1-\cos^2\theta} + c = 2\sqrt{1-\frac{1}{\sec^2\theta}} + c = 2\sqrt{1-\frac{1}{x^2}} + c; \text{ therefore:}$$

$$\int \frac{2}{x^2\sqrt{x^2-1}} dx = \frac{2\sqrt{x^2-1}}{x} + c$$

102. $\mathrm{I} = \int_a^b \frac{\ln(abx)}{x^2+ab} dx$, let $y = \frac{1}{\sqrt{ab}} x$, then $x = \sqrt{ab} y$, so $dx = \sqrt{ab} dy$

For $x = a$, $y = \frac{1}{\sqrt{ab}} a = \frac{1}{\sqrt{ab}}\sqrt{a}\sqrt{a} = \sqrt{\frac{a}{b}}$ and for $x = b$, we get $y = \sqrt{\frac{b}{a}}$, then we get**:**

$$\mathrm{I} = \int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{\ln(ab\sqrt{ab}y)}{(\sqrt{ab}y)^2 + ab}\sqrt{ab} dy = \sqrt{ab}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{\ln\left((ab)^{\frac{3}{2}}y\right)}{aby^2+ab} dy = \frac{\sqrt{ab}}{ab}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{\frac{3}{2}\ln(ab) + \ln y}{y^2+1} dy$$

$$\mathrm{I} = \frac{3\sqrt{ab}\ln(ab)}{2ab}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{dy}{y^2+1} + \frac{\sqrt{ab}}{ab}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{\ln y}{y^2+1} dy = \frac{3\ln(ab)}{2\sqrt{ab}}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{dx}{x^2+1} + \frac{1}{\sqrt{ab}}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{\ln x}{x^2+1} dx$$

Let $\mathrm{J} = \int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{\ln x}{x^2+1} dx$, let $x = \frac{1}{y}$, then $dx = -\frac{1}{y^2} dy$, for $x = \sqrt{\frac{a}{b}}$, $y = \sqrt{\frac{b}{a}}$ and for $x = \sqrt{\frac{b}{a}}$, $y =$

$\sqrt{\frac{a}{b}}$, then J $= \int_{\sqrt{\frac{b}{a}}}^{\sqrt{\frac{a}{b}}} \frac{\ln\left(\frac{1}{y}\right)}{\left(\frac{1}{y}\right)^2+1}\left(-\frac{1}{y^2}dy\right) = \int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{-\ln y}{1+y^2}dy = -\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{\ln x}{x^2+1} = -$ J

So J + J = 0, 2J = 0, therefore J = 0, then we get:

$$\mathrm{I} = \frac{3\ln(ab)}{2\sqrt{ab}}\int_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} \frac{dx}{x^2+1} = \frac{3\ln(ab)}{2\sqrt{ab}}[\arctan x]_{\sqrt{\frac{a}{b}}}^{\sqrt{\frac{b}{a}}} = \frac{3\ln(ab)}{2\sqrt{ab}}\left[\arctan\left(\sqrt{\frac{b}{a}}\right) - \arctan\left(\sqrt{\frac{a}{b}}\right)\right]$$

$$\mathrm{I} = \frac{3\ln(ab)}{2\sqrt{ab}}\arctan\left(\frac{\sqrt{\frac{b}{a}}-\sqrt{\frac{a}{b}}}{1+\sqrt{\frac{b}{a}}\sqrt{\frac{a}{b}}}\right) = \frac{3\ln(ab)}{2\sqrt{ab}}\arctan\left(\frac{b-a}{2\sqrt{ab}}\right)$$

103. $\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}-2}$, let $x = y^4$, then $dx = 4y^3dy$, with $\sqrt[4]{x} = y$ and $\sqrt{x} = y^2$, then we get:

$$\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}-2} = \int \frac{4y^3}{y^2-y-2}dy = 4\int \frac{y^3-8+8}{y^2-y-2}dy = 4\int \frac{(y-2)(y^2+2y+4)+8}{(y+1)(y-2)}dy$$

$$\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}-2} = 4\int\left[\frac{y^2+2y+4}{y+1} + \frac{8}{(y+1)(y-2)}\right]dy = 4\int\left[\frac{(y^2+2y+1)+3}{y+1} + \frac{8}{(y+1)(y-2)}\right]dy$$

$$\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}-2} = 4\int\left[\frac{(y+1)^2+3}{y+1} + \frac{8}{(y+1)(y-2)}\right]dy = 4\int\left[y + 1 + \frac{3}{y+1} + \frac{8}{(y+1)(y-2)}\right]dy$$

Decomposition into partial fractions : $\frac{8}{(y+1)(y-2)} = \frac{\mathrm{A}}{y+1} + \frac{\mathrm{B}}{y-2}$, $8 = \mathrm{A}(y-2) + \mathrm{B}(y+1)$

For $y = -1$, then $8 = \mathrm{A}(-3)$, so $\mathrm{A} = -\frac{8}{3}$ and for $y = 2$, $8 = \mathrm{B}(3)$, so $\mathrm{B} = \frac{8}{3}$, then we get:

$$\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}-2} = 4\int\left[y + 1 + \frac{3}{y+1} - \frac{8}{3(y+1)} + \frac{8}{3(y-2)}\right]dy = 4\int\left[y + 1 + \frac{1}{y+1} + \frac{8}{3(y-2)}\right]dy$$

$\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}-2} = 2y^2 + 4y + \frac{4}{3}\ln|y+1| + \frac{32}{3}\ln|y-2| + c$, therefore we get:

$$\int \frac{dx}{\sqrt{x}-\sqrt[4]{x}-2} = 2\sqrt{x} + 4\sqrt[4]{x} + \frac{4}{3}\ln\left|\sqrt[4]{x}+1\right| + \frac{32}{3}\ln\left|\sqrt[4]{x}-2\right| + c$$

104. $\mathrm{I} = \int_1^n x.[x].\{x\}dx$

$$\mathrm{I} = \int_1^2 x.1.(x-1)dx + \int_2^3 x.2.(x-2)dx + \cdots + \int_{n-1}^n x.(n-1).(x-n+1)\,dx$$

For $\int_1^2 x.1.(x-1)dx$ let $u = x - 1 \Longrightarrow du = dx$; for $x = 1; u = 0$ & for $x = 2; u = 1$

Then we get: $\int_1^2 x.1.(x-1)dx = \int_0^1 u(u+1)du = \int_0^1 x(x+1)dx$

For $\int_2^3 x.2.(x-2)dx$ let $v = x - 2 \Longrightarrow dv = dx$; for $x = 2; u = 0$ & for $x = 3; u = 1$

Then we get: $\int_2^3 x.2.(x-2)dx = \int_0^1 2v(v+2)du = \int_0^1 2x(x+2)dx$

…

For $\int_{n-1}^{n} x.(n-1).(x-n+1)\,dx$

Let $w = x-n+1 \Longrightarrow dw = dx$; for $x = n-1; w = 0$ & for $x = n; w = 1$; then:

$$\int_{n-1}^{n} x.(n-1).(x-n+1)\,dx = \int_{0}^{1} (n-1)w(w+n-1)dw = \int_{0}^{1} (n-1)x(x+n-1)dx$$

Then we get: $\mathrm{I} = \int_0^1 x(x+1)dx + \int_0^1 2x(x+2)dx + \int_0^1 (n-1)x(x+n-1)dx$

$$\mathrm{I} = \sum_{k=0}^{n-1}\int_0^1 kx(x+k)\,dx = \sum_{k=0}^{n-1}\int_0^1 (kx^2+xk^2)\,dx = \sum_{k=0}^{n-1}\left[\frac{kx^3}{3}+\frac{k^2x^2}{2}\right]_0^1;$$ then we get:

$$\mathrm{I} = \sum_{k=0}^{n-1}\left(\frac{k}{3}+\frac{k^2}{2}\right) = \frac{1}{3}\sum_{k=0}^{n-1} k + \frac{1}{2}\sum_{k=0}^{n-1} k^2 = \frac{1}{3}\left[\frac{n(n-1)}{n}\right] + \frac{1}{2}\left[\frac{n(n-1)(2n-1)}{6}\right]$$

$$\int_1^n x.[x].\{x\}dx = \frac{n(n-1)}{12}(2n-1+2) = \frac{n(n-1)(2n+1)}{12}$$

105. $\mathrm{I} = \int_0^1 \tan^{-1}\left(\frac{4x-x^3}{x^4-6x^2+1}\right)dx$, let $x = \tan\theta$ then $dx = \sec^2\theta\, d\theta$ for $x = 0, \theta = 0$ and for $x = 1$, then $\theta = \frac{\pi}{4}$ then we get:

$\mathrm{I} = \int_0^{\frac{\pi}{4}} \tan^{-1}\left(\frac{4\tan\theta-\tan^3\theta}{\tan^4\theta-6\tan^2\theta+1}\right).\sec^2\theta\, d\theta$, with $\tan 4\theta = \frac{4\tan\theta-\tan^3\theta}{\tan^4\theta-6\tan^2\theta+1}$, then:

$$\mathrm{I} = \int_0^{\frac{\pi}{8}} 4\theta.\sec^2\theta\, d\theta + \int_{\frac{\pi}{8}}^{\frac{\pi}{4}}(4\theta-\pi).\sec^2\theta\, d\theta = 4\int_0^{\frac{\pi}{4}} 4\theta.\sec^2\theta\, d\theta - \pi\int_{\frac{\pi}{8}}^{\frac{\pi}{4}}\sec^2\theta\, d\theta$$

$\mathrm{I} = 4[\theta\tan\theta - \ln(\sec\theta)]_0^{\frac{\pi}{4}} - \pi[\tan\theta]_{\frac{\pi}{8}}^{\frac{\pi}{4}} = 4\left(\frac{\pi}{4}-\ln\sqrt{2}\right) - \pi\left(1-\left(\sqrt{2}-1\right)\right)$, therefore

$\mathrm{I} = \left(\sqrt{2}-1\right)\pi - \ln 4$

106. $\mathrm{I} = \int_0^{+\infty}\frac{\sinh x}{\sinh 3x}dx = \int_0^{+\infty}\frac{\frac{e^x-e^{-x}}{2}}{\frac{e^{3x}-e^{-3x}}{2}}dx = \int_0^{+\infty}\frac{e^x-e^{-x}}{e^{3x}-e^{-3x}}dx$

$$\mathrm{I} = \int_0^{+\infty}\frac{e^x-e^{-x}}{(e^x-e^{-x})\left(e^{2x}+1+e^{-2x}\right)}dx$$

$\mathrm{I} = \int_0^{+\infty}\frac{1}{e^{2x}+1+e^{-2x}}dx$, let $e^{2x} = t$, then $2e^x dx = dt$, so $2tdx = dt$, for $x = 0, t = 1$ and for $x = +\infty, t = +\infty$, then we get:

$$\mathrm{I} = \int_1^{+\infty}\frac{1}{t+1+\frac{1}{t}}.\frac{1}{2t}dt = \int_1^{+\infty}\frac{1}{2\left(t^2+t+1\right)}dt = \frac{1}{2}\int_0^{+\infty}\frac{1}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dt$$

Let $t+\frac{1}{2} = \frac{\sqrt{3}}{2}\tan\theta$, then $\mathrm{I} = \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\frac{1}{\left(\frac{\sqrt{3}}{2}\right)^2\tan^2\theta+\left(\frac{\sqrt{3}}{2}\right)^2}.\frac{\sqrt{3}d\theta}{2\cos^2\theta} = \frac{1}{\sqrt{3}}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}d\theta = \frac{\pi}{6\sqrt{3}}$

107. $\int \sqrt{\sin x+\cos x+\sqrt{2}}\,dx=\int \sqrt{\sqrt{2}\left(\frac{\sqrt{2}}{2}\sin x+\frac{\sqrt{2}}{2}\cos x\right)+\sqrt{2}}\,dx$

$\int \sqrt{\sin x+\cos x+\sqrt{2}}\,dx=\int \sqrt{\sqrt{2}\left(\sin\frac{\pi}{4}\sin x+\cos\frac{\pi}{4}\cos x\right)+\sqrt{2}}\,dx$; then we get:

$\int \sqrt{\sin x+\cos x+\sqrt{2}}\,dx=\int \sqrt{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)+\sqrt{2}}\,dx=\int \sqrt{\sqrt{2}\left(\cos\left(x-\frac{\pi}{4}\right)+1\right)}\,dx$

$\int \sqrt{\sin x+\cos x+\sqrt{2}}\,dx=\sqrt{\sqrt{2}}\int \sqrt{\cos\left(x-\frac{\pi}{4}\right)+1}\,dx=\sqrt[4]{2}\int \sqrt{\cos\left(x-\frac{\pi}{4}\right)+1}\,dx$

Let $u=x-\frac{\pi}{4}\Rightarrow du=dx$; then we get:

$\int \sqrt{\sin x+\cos x+\sqrt{2}}\,dx=\sqrt[4]{2}\int \sqrt{\cos u+1}\,du=\sqrt[4]{2}\int \sqrt{2\cos^2\frac{u}{2}-1+1}\,du$

$\int \sqrt{\sin x+\cos x+\sqrt{2}}\,dx=\sqrt[4]{2}\int \sqrt{2\cos^2\frac{u}{2}}\,du=\sqrt{2}\sqrt[4]{2}\int \cos\frac{u}{2}\,du=2\sqrt{2}\sqrt[4]{2}\sin\frac{u}{2}+c$

$\int \sqrt{\sin x+\cos x+\sqrt{2}}\,dx=\sqrt[4]{128}\sin\left(\frac{x-\frac{\pi}{4}}{2}\right)+c=\sqrt[4]{128}\sin\left(\frac{x}{2}-\frac{\pi}{8}\right)+c$

108. $\mathrm{K}=\int_0^1\frac{x\ln x}{\sqrt{1-x^4}}dx=\frac{1}{2}\int_0^1\frac{2x\ln x}{\sqrt{1-x^4}}dx$, let $u=x^2$, then $du=2xdx$, then**:**

$\mathrm{K}=\frac{1}{2}\int_0^1\frac{\ln\sqrt{u}}{\sqrt{1-u^2}}du=\frac{1}{2}\int_0^1\frac{\ln\left(u^{\frac{1}{2}}\right)}{\sqrt{1-u^2}}du=\frac{1}{2}\int_0^1\frac{\frac{1}{2}\ln u}{\sqrt{1-u^2}}du=\frac{1}{4}\int_0^1\frac{\ln u}{\sqrt{1-u^2}}dx$

Let $u=\sin\theta$, then $du=\cos\theta\,d\theta$, for $x=0,\theta=0$ and for $x=1,\theta=\frac{\pi}{2}$, then we get**:**

$\mathrm{K}=\frac{1}{4}\int_0^1\frac{\ln\sin\theta}{\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta=\frac{1}{4}\int_0^1\frac{\ln\sin\theta}{\sqrt{\cos^2\theta}}\cos\theta\,d\theta=\frac{1}{4}\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx=\frac{1}{4}\mathrm{I}$

Evaluating: $\mathrm{I}=\int_0^{\frac{\pi}{2}}\ln(\sin x)\,dx$ **… (1)**, let $u=\frac{\pi}{2}-x$, then $du=-dx$

For $x=0$, $u=\frac{\pi}{2}-0=\frac{\pi}{2}$ and for $x=\frac{\pi}{2}$, $u=\frac{\pi}{2}-\frac{\pi}{2}=0$, then**:**

$\mathrm{I}=\int_{\frac{\pi}{2}}^0\ln\left(\sin\left(\frac{\pi}{2}-u\right)\right)(-du)=\int_0^{\frac{\pi}{2}}\ln(\cos u)du=\int_0^{\frac{\pi}{2}}\ln(\cos x)\,dx$ **… (2)**

Adding (1) and (2) we get**:** $2\mathrm{I}=\int_0^{\frac{\pi}{2}}[\ln(\sin x)+\ln(\cos x)]dx=\int_0^{\frac{\pi}{2}}\ln\left(\frac{1}{2}\sin 2x\right)dx$

$2\mathrm{I}=\int_0^{\frac{\pi}{2}}\ln(\sin 2x)\,dx-\int_0^{\frac{\pi}{2}}\ln 2\,dx=\int_0^{\frac{\pi}{2}}\ln(\sin 2x)\,dx-\frac{\pi}{2}\ln 2=\mathrm{J}-\frac{\pi}{2}\ln 2$

Evaluating: $\mathrm{J}=\int_0^{\frac{\pi}{2}}\ln(2\sin x)\,dx$, let $v=2x$, so $dv=2dx$, then**:**

$\mathrm{J}=\frac{1}{2}\int_0^{\pi}\ln(\sin v)\,dv=\frac{1}{2}\left[\int_0^{\frac{\pi}{2}}\ln(\sin v)\,dv+\int_{\frac{\pi}{2}}^{\pi}\ln(\sin v)\,dv\right]$

Letting $v=\pi-w$ in the integral $\int_{\frac{\pi}{2}}^{\pi}\ln(\sin v)\,dv$ becomes $\int_0^{\frac{\pi}{2}}\ln(\sin w)\,dw$

So: $\mathrm{J}=\frac{1}{2}(\mathrm{I}+\mathrm{I})=\mathrm{I}$, with $2\mathrm{I}=\mathrm{J}-\frac{\pi}{2}\ln 2$, so $2\mathrm{I}=\ \mathrm{I}-\frac{\pi}{2}\ln 2$, therefore $\mathrm{I}=-\frac{\pi}{2}\ln 2$

Therefore, $\mathrm{K}=\frac{1}{4}\mathrm{I}=-\frac{\pi}{8}\ln 2$

109. $\mathrm{I}=\int_0^{+\infty}\frac{dx}{(1+x)\left(\pi^2+\ln^2 x\right)}$ **… (1)**

Let $y=\frac{1}{x}$, then $x=\frac{1}{y}$, so $dx=-\frac{1}{y^2}dy$, for $x=0$, $y=+\infty$ and for $x=+\infty$, $y=0$, then:

$$\mathrm{I}=\int_{+\infty}^{0}\frac{-\frac{1}{y^2}}{\left(1+\frac{1}{y}\right)\left(\pi^2+\ln^2\left(\frac{1}{y}\right)\right)}dy=\int_0^{+\infty}\frac{1}{y^2\left(1+\frac{1}{y}\right)\left(\pi^2+(-\ln y)^2\right)}dy=\int_0^{+\infty}\frac{1}{y(y+1)\left(\pi^2+\ln^2 y\right)}dy$$

then $\mathrm{I}=\int_0^{+\infty}\frac{1}{x(1+x)\left(\pi^2+\ln^2 x\right)}dx$ **… (2)** Adding (1) and (2) we get:

$$2\mathrm{I}=\int_0^{+\infty}\frac{dx}{(1+x)\left(\pi^2+\ln^2 x\right)}+\int_0^{+\infty}\frac{1}{x(1+x)\left(\pi^2+\ln^2 x\right)}dx=\int_0^{+\infty}\frac{x+1}{x(1+x)\left(\pi^2+\ln^2 x\right)}dx$$

$2\mathrm{I}=\int_0^{+\infty}\frac{1}{x\left(\pi^2+\ln^2 x\right)}dx$, let $t=\ln x$, then $dt=\frac{1}{x}dx$, for $x=0$, $t=-\infty$ and for $x=+\infty$, $t=+\infty$, then $2\mathrm{I}=\int_{-\infty}^{+\infty}\frac{dt}{\pi^2+t^2}=\left[\frac{1}{\pi}\arctan\left(\frac{t}{\pi}\right)\right]_{-\infty}^{+\infty}=\frac{1}{\pi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=1$, therefore $\mathrm{I}=\frac{1}{2}$

110. $\int\frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx$

$$\int\frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx=\int\frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}\times\frac{\frac{1}{x^2}}{\frac{1}{x^2}}dx;\ \text{ then we get:}$$

$$\int\frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx=\int\frac{1+\frac{1}{x^2}}{\left(1-\frac{1}{x^2}\right)\sqrt{x^4+x^2+1}}dx=\int\frac{1+\frac{1}{x^2}}{\left(1-\frac{1}{x^2}\right)\sqrt{x^2\left(x^2+\frac{1}{x^2}+1\right)}}dx$$

$$\int\frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx=\int\frac{1+\frac{1}{x^2}}{\left(1-\frac{1}{x^2}\right)x\sqrt{x^2+\frac{1}{x^2}+1}}dx=\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}+1}}dx$$

$$\int\frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx=\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)\sqrt{\left(x^2-2+\frac{1}{x^2}\right)+3}}dx;\ \text{ then we get:}$$

$$\int\frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx=\int\frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)\sqrt{\left(x-\frac{1}{x}\right)^2+3}}dx=\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)\sqrt{\left(x-\frac{1}{x}\right)^2+3}}$$

Let $x-\frac{1}{x}=\sqrt{3}\tan\theta\Longrightarrow\left(1+\frac{1}{x^2}\right)dx=\sqrt{3}\sec^2\theta\,d\theta$; then we get:

$$\int\frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx=\int\frac{\sqrt{3}\sec^2\theta}{\sqrt{3}\tan\theta\sqrt{3\tan^2\theta+3}}d\theta=\int\frac{\sqrt{3}\sec^2\theta}{\sqrt{3}\tan\theta\sqrt{3}\sqrt{\sec^2\theta}}d\theta$$

$$\int \frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx = \int \frac{\sec^2\theta}{\sqrt{3}\tan\theta\sec\theta}d\theta = \frac{1}{\sqrt{3}}\int \frac{\sec\theta}{\tan\theta}d\theta = \frac{1}{\sqrt{3}}\int \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta}}d\theta$$

$$\int \frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sin\theta}d\theta = \frac{1}{\sqrt{3}}\int \csc\theta\, d\theta = -\frac{1}{\sqrt{3}}\ln|\csc\theta+\cot\theta| + c$$

But Let $x-\frac{1}{x} = \sqrt{3}\tan\theta \Longrightarrow \theta = \tan^{-1}\left(\frac{x-\frac{1}{x}}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{x-x^{-1}}{\sqrt{3}}\right)$; therefore we get:

$$\int \frac{x^2+1}{(1-x^2)\sqrt{x^4+x^2+1}}dx = -\frac{1}{\sqrt{3}}\ln\left|\csc\left(\tan^{-1}\left(\frac{x-x^{-1}}{\sqrt{3}}\right)\right) + \cot\left(\tan^{-1}\left(\frac{x-x^{-1}}{\sqrt{3}}\right)\right)\right| + c$$

111. $\mathrm{I} = \int \sqrt{e^{2x}+4e^x-1}\,dx$

$$\int \sqrt{e^{2x}+4e^x-1} = \int \sqrt{e^{2x}+4e^x-1} \times \frac{\sqrt{e^{2x}+4e^x-1}}{\sqrt{e^{2x}+4e^x-1}}dx$$

Then we get: $\int \sqrt{e^{2x}+4e^x-1}\,dx = \int \frac{e^{2x}+4e^x-1}{\sqrt{e^{2x}+4e^x-1}}dx$

$$\int \sqrt{e^{2x}+4e^x-1}\,dx = \int \frac{e^{2x}+2e^x}{\sqrt{e^{2x}+4e^x-1}}dx + \int \frac{2e^x}{\sqrt{e^{2x}+4e^x-1}}dx - \int \frac{1}{\sqrt{e^{2x}+4e^x-1}}dx$$

$$\int \sqrt{e^{2x}+4e^x-1}\,dx = \sqrt{e^{2x}+4e^x-1} + \int \frac{2e^x}{\sqrt{(e^x+2)^2-5}}dx - \int \frac{1}{e^x\sqrt{1+4e^{-x}-e^{-2x}}}dx$$

$$\int \sqrt{e^{2x}+4e^x-1}\,dx = \sqrt{e^{2x}+4e^x-1} + 2\cosh^{-1}\left(\frac{e^x+2}{\sqrt{5}}\right) - \int \frac{e^{-x}}{\sqrt{5-(4-4e^{-x}-e^{-2x})}}dx$$

$$\int \sqrt{e^{2x}+4e^x-1}\,dx = \sqrt{e^{2x}+4e^x-1} + 2\cosh^{-1}\left(\frac{e^x+2}{\sqrt{5}}\right) - \int \frac{e^{-x}}{\sqrt{5-(2-e^{-x})^2}}dx$$

$$\int \sqrt{e^{2x}+4e^x-1}\,dx = \sqrt{e^{2x}+4e^x-1} + 2\cosh^{-1}\left(\frac{e^x+2}{\sqrt{5}}\right) - \sin^{-1}\left(\frac{2-e^{-x}}{\sqrt{5}}\right) + c$$

112. $\mathrm{I} = \int_3^7 \frac{\ln(x+2)}{\ln(24+10x-x^2)}dx = \int_3^7 \frac{\ln(x+2)}{\ln[(x+2)(12-x)]}dx = \int_3^7 \frac{\ln(x+2)}{\ln(x+2)+\ln(12-x)}dx$ **… (1)**

Let $y = 100-x \Longrightarrow x = 10-y$ & $dx = -dy$, for $x=3$, $y=7$ and for $x=7$, $y=3$, then we get**:**

$$\mathrm{I} = \int_7^3 \frac{\ln(10-y+2)}{\ln(10-y+2)+\ln(12-10+y)}(-dy) = \int_3^7 \frac{\ln(12-y)}{\ln(12-y)+\ln(2+y)}dy$$ we can write:

$$\mathrm{I} = \int_3^7 \frac{\ln(12-x)}{\ln(x+2)+\ln(12-x)}dx \quad \ldots(\mathbf{2})$$

Adding (1) & (2) we get**:**

$$2\mathrm{I} = \int_3^7 \frac{\ln(x+2)}{\ln(x+2)+\ln(12-x)}dx + \int_3^7 \frac{\ln(12-x)}{\ln(x+2)+\ln(12-x)}dx$$

$$2\mathrm{I} = \int_3^7 \frac{\ln(x+2)+\ln(12-x)}{\ln(x+2)+\ln(12-x)}dx = \int_3^7 dx = [x]_3^7 = 7-3 = 4 \Longrightarrow \mathrm{I} = \frac{4}{2} = 2$$

113. $\int \frac{1}{1+\sqrt{x^2+2x+2}}dx = \int \frac{1}{1+\sqrt{(x^2+2x+1)+1}}dx = \int \frac{1}{1+\sqrt{(x+1)^2+1}}dx$

Change of variable: let $y = x+1 \Longrightarrow dy = dx$, then we get**:**

$$\int \frac{1}{1+\sqrt{x^2+2x+2}}dx = \int \frac{1}{1+\sqrt{y^2+1}}dy = \int \frac{1}{1+\sqrt{y^2+1}} \times \frac{1-\sqrt{y^2+1}}{1-\sqrt{y^2+1}}dy$$

$$\int \frac{1}{1+\sqrt{x^2+2x+2}}dx = \int \frac{1-\sqrt{y^2+1}}{1-y^2-1}dy = -\int \frac{1-\sqrt{y^2+1}}{y^2}dy\text{; then we get:}$$

$$\int \frac{1}{1+\sqrt{x^2+2x+2}}dx = -\left(\int \frac{1}{y^2}dy - \int \frac{\sqrt{y^2+1}}{y^2}dy\right) = \int \frac{\sqrt{y^2+1}}{y^2}dy + \frac{1}{y} + k$$

For: $\int \frac{\sqrt{y^2+1}}{y^2}dy$; we will use integration by parts:

Let $u = \sqrt{y^2+1}$; then $u' = \frac{2y}{\sqrt{y^2+1}}$ and let $v' = \frac{1}{y^2}$; then $v = -\frac{1}{y}$; then we get:

$$\int \frac{\sqrt{y^2+1}}{y^2}dy = -\frac{\sqrt{y^2+1}}{y} + 2\int \frac{1}{\sqrt{y^2+1}}dy + k'$$

$$\int \frac{\sqrt{y^2+1}}{y^2}dy = -\frac{\sqrt{y^2+1}}{y} + 2\ln\left|y+\sqrt{y^2+1}\right| + k'\text{; then we get:}$$

$$\int \frac{1}{1+\sqrt{x^2+2x+2}}dx = \frac{1}{y} - \frac{\sqrt{y^2+1}}{y} + 2\ln\left|y+\sqrt{y^2+1}\right| + c\text{; with } y = x+1\text{; then:}$$

$$\int \frac{1}{1+\sqrt{x^2+2x+2}}dx = \frac{1-\sqrt{x^2+2x+2}}{x+1} + 2\ln\left|x+1+\sqrt{x^2+2x+2}\right| + c$$

114. $\int \frac{1}{1-(n+1)\sin^2 3x}dx = \int \frac{1}{\sin^2 3x+\cos^2 3x-(n+1)\sin^2 3x}dx$, then we get**:**

$$\int \frac{1}{1-(n+1)\sin^2 3x}dx = \int \frac{1}{\sin^2 3x + \cos^2 3x - n\sin^2 3x - \sin^2 3x}dx$$

$$\int \frac{1}{1-(n+1)\sin^2 3x}dx = \int \frac{1}{\cos^2 3x - n\sin^2 3x}dx = \int \frac{1}{\cos^2 3x - n\sin^2 3x} \times \frac{\sec^2 3x}{\sec^2 3x}dx$$

$$\int \frac{1}{1-(n+1)\sin^2 3x}dx = \int \frac{\sec^2 3x}{1-n\tan^2 3x}dx$$

Change of variable: let $y = \tan 3x \Longrightarrow dy = 3\sec^2 3x\, dx$, then we get**:**

$$\int \frac{1}{1-(n+1)\sin^2 3x}dx = \int \frac{1}{1-ny^2}.\frac{1}{3}dy = \frac{1}{3\sqrt{n}}\tanh^{-1}\left(\sqrt{n}y\right) + c\text{; therefore we get:}$$

$$\int \frac{1}{1-(n+1)\sin^2 3x}dx = \frac{1}{3\sqrt{n}}\tanh^{-1}\left(\sqrt{n}\tan 3x\right) + c$$

115. $\int \frac{e^{\sqrt{\frac{1-x}{1+x}}}}{(1+x)\sqrt{1-x^2}}dx$

Change of variable: let $u = \sqrt{\frac{1-x}{1+x}} \Longrightarrow x = \frac{1-u^2}{1+u^2} \Longrightarrow dx = -\frac{4u}{\left(1+u^2\right)^2}du$, then we get**:**

$$\int \frac{e^{\sqrt{\frac{1-x}{1+x}}}}{(1+x)\sqrt{1-x^2}}dx = \int \frac{e^u}{\left(1+\frac{1-u^2}{1+u^2}\right)\sqrt{1-\left(\frac{1-u^2}{1+u^2}\right)^2}} \cdot \frac{-4u}{(1+u^2)^2}du\,; \text{ then we get:}$$

$$\int \frac{e^{\sqrt{\frac{1-x}{1+x}}}}{(1+x)\sqrt{1-x^2}}dx = \int \frac{e^u}{\frac{2}{1+u^2}\sqrt{\frac{(1+u^2)^2-(1-u^2)^2}{(1+u^2)^2}}} \cdot \frac{-4u}{(1+u^2)^2}du$$

$$\int \frac{e^{\sqrt{\frac{1-x}{1+x}}}}{(1+x)\sqrt{1-x^2}}dx = \int \frac{e^u}{\sqrt{\frac{4u^2}{(1+u^2)^2}}} \cdot \frac{-2u}{1+u^2}du = \int \frac{e^u}{\frac{2u}{1+u^2}} \cdot \frac{-2u}{1+u^2}du = -\int e^u du$$

$$\int \frac{e^{\sqrt{\frac{1-x}{1+x}}}}{(1+x)\sqrt{1-x^2}}dx = -e^u + c = -e^{\sqrt{\frac{1-x}{1+x}}} + c$$

116. $\int \frac{1}{x\left(1+\sin^2(\ln x)\right)}dx = \int \frac{1}{1+\sin^2(\ln x)} \cdot \frac{1}{x}dx$

Change of variable : let $y = \ln x$, then $dy = \frac{1}{x}dx$, then we get**:**

$$\int \frac{1}{x(1+\sin^2(\ln x))}dx = \int \frac{1}{1+\sin^2 y}dy = \int \frac{1}{\cos^2 y + \sin^2 y + \sin^2 y}dy$$

$$\int \frac{1}{x(1+\sin^2(\ln x))}dx = \int \frac{1}{\cos^2 y + 2\sin^2 y}dy = \int \frac{1}{\cos^2 y + 2\sin^2 y} \times \frac{\sec^2 y}{\sec^2 y}dy; \text{ then:}$$

$$\int \frac{1}{x(1+\sin^2(\ln x))}dx = \int \frac{\sec^2 y}{\frac{\cos^2 y}{\cos^2 y} + 2\frac{\sin^2 y}{\cos^2 y}}dy = \int \frac{\sec^2 y}{1+2\tan^2 y}dy$$

Change of variable: let $z = \tan y$, then $dz = \sec^2 y\, dy$, then we get**:**

$$\int \frac{1}{x(1+\sin^2(\ln x))}dx = \int \frac{1}{1+2z^2}dz = \int \frac{1}{1+\left(\sqrt{2}z\right)^2}dz = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}z\right) + c; \text{ so we get:}$$

$$\int \frac{1}{x(1+\sin^2(\ln x))}dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan y\right) + c = \frac{1}{\sqrt{2}}\tan^{-1}\left(\sqrt{2}\tan(\ln x)\right) + c$$

117. $\mathrm{I} = \int_0^{+\infty} \left(\frac{1}{x}\right)^{\ln x} dx$

Change of variable: let $u = \ln x$ $(x = e^u)$, then $du = \frac{1}{x}dx \Longrightarrow dx = x du \Longrightarrow dx = e^u du$

New bounds: For $x = 0$, then $u = \ln 0 = -\infty$ and for $x = +\infty$, then $u = +\infty$, then we get**:**

$$\mathrm{I} = \int_0^{+\infty} \left(\frac{1}{x}\right)^{\ln x} dx = \int_{-\infty}^{+\infty} \left(\frac{1}{e^u}\right)^u . e^u du = \int_{-\infty}^{+\infty} (e^{-u})^u . e^u du = \int_{-\infty}^{+\infty} e^{-u^2} . e^u du = \int_{-\infty}^{+\infty} e^{-u^2+u} du$$

$$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-(u^2-u)} du = \int_{-\infty}^{+\infty} e^{-\left(u^2-u+\frac{1}{4}-\frac{1}{4}\right)} du = \int_{-\infty}^{+\infty} e^{-\left(u^2-u+\frac{1}{4}\right)+\frac{1}{4}} du = e^{\frac{1}{4}} \int_{-\infty}^{+\infty} e^{-\left(u^2-u+\frac{1}{4}\right)} du$$

$$\mathrm{I} = e^{\frac{1}{4}} \int_{-\infty}^{+\infty} e^{-\left(u-\frac{1}{2}\right)^2} du$$

Change of variable: let $y = u - \frac{1}{2}$, then $dy = du$, then we get**:**

$$\mathrm{I} = e^{\frac{1}{4}} \int_{-\infty}^{+\infty} e^{-y^2} dy \quad \text{with} \quad \int_{-\infty}^{+\infty} e^{-y^2} dy = \sqrt{\pi} \text{ (Gauss's Integral); therefore; we get:}$$

$$\mathrm{I} = \int_{0}^{+\infty} \left(\frac{1}{x}\right)^{\ln x} dx = e^{\frac{1}{4}}\sqrt{\pi} = \sqrt{\pi\sqrt{e}}$$

118. $\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \int \frac{\sec(x-a)}{\sqrt{\cos^2(x-b)}} dx = \int \frac{\frac{1}{\cos(x-a)}}{\sqrt{\cos^2(x-b)}} dx$, then we get**:**

$$\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \int \frac{1}{\cos(x-a)\cos(x-b)} dx = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b)}{\cos(x-a)\cos(x-b)} dx$$

$$\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \frac{1}{\sin(a-b)} \int \frac{\sin(a-b-x+x)}{\cos(x-a)\cos(x-b)} dx$$

$$\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \frac{1}{\sin(a-b)} \int \frac{\sin\big((x-b)-(x-a)\big)}{\cos(x-a)\cos(x-b)} dx$$

$$\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \cos(x-b)\sin(x-a)}{\cos(x-a)\cos(x-b)} dx$$

$$\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \frac{1}{\sin(a-b)} \int \left(\frac{\sin(x-b)}{\cos(x-b)} - \frac{\sin(x-a)}{\cos(x-a)}\right) dx$$

$$\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \frac{1}{\sin(a-b)} \int (\tan(x-b) - \tan(x-a)) dx\,; \quad \text{then we get:}$$

$$\int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \frac{1}{\sin(a-b)} (\ln|\sec(x-b)| - \ln|\sec(x-a)|) + c$$

$$\text{Therefore; we get: } \int \frac{\sec(x-a)}{\sqrt{1-\sin^2(x-b)}} dx = \frac{1}{\sin(a-b)} \ln\left|\frac{\sec(x-b)}{\sec(x-a)}\right| + c$$

119. $\mathrm{I} = \int_0^1 \ln(a + \ln x)\, dx$, using integration by parts**:**

Let $u = \ln(a + \ln x)$, then $u' = \frac{\frac{1}{x}}{a+\ln x} = \frac{1}{x}.\frac{1}{a+\ln x}$ and let $v' = 1$, so $v = x$, then we get**:**

$\mathrm{I} = [x\ln(a+\ln x)]_0^1 - \int_0^1 \frac{x}{a+\ln x}.\frac{1}{x} dx = \ln a - \int_0^1 \frac{1}{a+\ln x} dx = \ln a - \mathrm{J}$

$\int_0^1 \frac{dx}{a+\ln x}$, let $-t = \ln x$, then $x = e^{-t}$ and $dx = -e^{-t}dt$, for $x = 0$, $t = +\infty$ and for $x = 1$, $t = 0$,

then we get**:** $\mathrm{J} = \int_{+\infty}^0 \frac{-e^{-t}}{a-t} dt = \int_0^{+\infty} \frac{e^{-t}}{a-t} dt = e^{-a}\mathrm{Ei}(a)$

Therefore, we get $\mathrm{I} = \ln a - e^{-a}\mathrm{Ei}(a)$

120. $\mathrm{I} = \int_0^\pi \frac{x^2\cos x}{(1+\sin x)^2} dx$ using integration by parts**:**

Let $u = x^2$, then $u' = 2x$ and let $v' = \frac{\cos x}{(1+\sin x)^2}$, then $v = -\frac{1}{1+\sin x}$, so**:**

$\mathrm{I} = \left[-\frac{x^2}{1+\sin x}\right]_0^{\pi} + \int_0^{\pi} \frac{2x}{1+\sin x} dx = -\pi^2 + 2\int_0^{\pi} \frac{x}{1+\sin x} dx = -\pi^2 + \mathrm{J},$

$\mathrm{J} = \int_0^{\pi} \frac{x}{1+\sin x} dx$, let $t = \pi - x$, then $dt = -dx$, for $x = \pi$, $t = 0$ and for $x = 0$, $t = \pi$, then**:**

$\mathrm{J} = \int_{\pi}^{0} \frac{\pi - t}{1+\sin(\pi - t)}(-dt) = \int_0^{\pi} \frac{\pi - t}{1+\sin t} dt = \pi\int_0^{\pi} \frac{1}{1+\sin x} dx - \int_0^{\pi} \frac{x}{1+\sin x} dx$

So, $\mathrm{J} = \pi\int_0^{\pi} \frac{1}{1+\sin x} dx - \mathrm{J}$, then $2\mathrm{J} = \pi\int_0^{\pi} \frac{1}{1+\sin x} dx$, so $\mathrm{J} = \frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\sin x} dx$, then we get**:**

$\mathrm{J} = \frac{\pi}{2}\int_0^{\pi} \frac{1}{1+\sin x} \times \frac{1-\sin x}{1-\sin x} dx = \frac{\pi}{2}\int_0^{\pi} \frac{1-\sin x}{1-\sin^2 x} dx = \frac{\pi}{2}\int_0^{\pi} \frac{1-\sin x}{\cos^2 x} dx = \frac{\pi}{2}\int_0^{\pi}\left(\frac{1}{\cos^2 x} - \frac{\sin x}{\cos^2 x}\right) dx$

$\mathrm{J} = \frac{\pi}{2}\int_0^{\pi}(\sec^2 x - \sec x \tan x)dx = \frac{\pi}{2}[\tan x - \sec x]_0^{\pi} = \frac{\pi}{2}(1+1) = \pi$, therefore we get**:**

$\mathrm{I} = -\pi^2 + 2\mathrm{J} = -\pi^2 + 2\pi = \pi(2 - \pi)$

121. $\mathrm{I} = \int_{-1}^{1} \frac{\sin x \cos x}{1+x^2} dx$, but $\sin 2x = 2\sin x \cos x$, so $\mathrm{I} = \int_{-1}^{1} \frac{\frac{1}{2}\sin 2x}{1+x^2} dx$

$$\mathrm{I} = \int_{-1}^{1} \frac{\frac{1}{2}\sin 2x}{1+x^2} dx = \int_{-1}^{0} \frac{\frac{1}{2}\sin 2x}{1+x^2} dx + \int_{0}^{1} \frac{\frac{1}{2}\sin 2x}{1+x^2} dx = \mathrm{I}_1 + \mathrm{I}_2$$

For $\mathrm{I}_1 = \int_{-1}^{0} \frac{\frac{1}{2}\sin 2x}{1+x^2} dx$; let $x = -y$; then $dx = -dy$ and $\begin{cases} \text{for } x = -1; \ y = 1 \\ \text{for } x = 0; \ y = 0 \end{cases}$; then:

$$\mathrm{I}_1 = \int_{1}^{0} \frac{\frac{1}{2}\sin(-2y)}{1+(-y)^2}(-dy) = \int_{0}^{1} \frac{\frac{1}{2}\sin(-2y)}{1+y^2} dy = -\int_{0}^{1} \frac{\frac{1}{2}\sin 2y}{1+y^2} dy = -\int_{0}^{1} \frac{\frac{1}{2}\sin 2x}{1+x^2} dx = -\mathrm{I}_2$$

With: $\mathrm{I} = \mathrm{I}_1 + \mathrm{I}_2$; so $\mathrm{I} = -\mathrm{I}_2 + \mathrm{I}_2 = 0$

122. $\mathrm{I} = \int_0^1 \frac{(1-x)\arcsin x}{\sqrt{1-x^2}} dx$, let $y = \arcsin x$, then $x = \sin y$ and $dy = \frac{1}{\sqrt{1-x^2}} dx$

For $x = 0$, $y = 0$ and for $x = 1$, $y = \frac{\pi}{2}$, then we get**:**

$$\mathrm{I} = \int_0^1 \frac{(1-x)\arcsin x}{\sqrt{1-x^2}} dx = \int_0^{\frac{\pi}{2}} y(1 - \sin y)dy; \quad \text{using integration by parts:}$$

Let $u = y$; then $u' = 1$ and let $v' = 1 - \sin y$; then $v = y + \cos y$; then we get:

$$\mathrm{I} = [y(y + \cos y)]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}(y + \cos y)dy = \frac{\pi^2}{4} - \left[\frac{y^2}{2} + \sin y\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} - \frac{\pi^2}{8} - 1 = \frac{\pi^2}{8} - 1$$

123. $\mathrm{I} = \int_0^{+\infty} \frac{e^{ax} - e^{bx}}{(1+e^{ax})(1+e^{bx})} dx = \int_0^{+\infty} \frac{1+e^{ax}-1-e^{bx}}{(1+e^{ax})(1+e^{bx})} dx = \int_0^{+\infty} \frac{(1+e^{ax})-(1+e^{bx})}{(1+e^{ax})(1+e^{bx})} dx$

$\mathrm{I} = \int_0^{+\infty}\left(\frac{1}{1+e^{bx}} - \frac{1}{1+e^{bx}}\right) dx = \int_0^{+\infty} \frac{1}{1+e^{bx}} dx - \int_0^{+\infty} \frac{1}{1+e^{ax}} dx$

$\mathrm{I} = \int_0^{+\infty} \frac{e^{-bx}}{1+e^{-bx}} dx - \int_0^{+\infty} \frac{e^{-ax}}{1+e^{-ax}} dx = \int_0^{+\infty} \frac{e^{-bx}}{1+e^{-bx}} dx - \int_0^{+\infty} \frac{e^{-ax}}{1+e^{-ax}} dx$

$\mathrm{I} = -\frac{1}{b}\int_0^{+\infty} \frac{-be^{-bx}}{1+e^{-bx}} dx + \frac{1}{a}\int_0^{+\infty} \frac{-ae^{-ax}}{1+e^{-ax}} dx = -\frac{1}{b}\int_0^{+\infty} \frac{(1+e^{-bx})'}{1+e^{-bx}} dx + \frac{1}{a}\int_0^{+\infty} \frac{(1+e^{-ax})'}{1+e^{-ax}} dx$

$\mathrm{I} = -\frac{1}{b}\left[\ln(1 + e^{-bx})\right]_0^{+\infty} + \frac{1}{a}[\ln(1 + e^{-ax})]_0^{+\infty} = -\frac{1}{b}\ln 2 + \frac{1}{a}\ln 2 = \left(\frac{a-b}{ab}\right)\ln 2$

124. $\mathrm{I} = \int x\cos(\ln x)\,dx$, using integration by parts:

Let $u = \cos(\ln x)$, then $u' = -\sin(\ln x)\,.\frac{1}{x}$ and let $v' = x$, then $v = \frac{1}{2}x^2$, then we get:

$$\mathrm{I} = \int x\cos(\ln x)\,dx = \frac{1}{2}x^2\cos(\ln x) + \frac{1}{2}\int x\sin(\ln x)\,dx$$

For $\int x\sin(\ln x)\,dx$, integration by parts:

let $u = \sin(\ln x)$, then $u' = \cos(\ln x)\,.\frac{1}{x}$ and let $v' = x$, then $v = \frac{1}{2}x^2$, then we get:

$$\int x\sin(\ln x)\,dx = \frac{1}{2}x^2\sin\ln x - \frac{1}{2}\int x\cos(\ln x)\,dx\,; \quad \text{then we get:}$$

$$\mathrm{I} = \int x\cos(\ln x)\,dx = \frac{1}{2}x^2\cos(\ln x) + \frac{1}{2}\left[\frac{1}{2}x^2\sin\ln x - \frac{1}{2}\int x\cos(\ln x)\,dx\right]$$

$$\mathrm{I} = \frac{1}{2}x^2\cos(\ln x) + \frac{1}{2}\left[\frac{1}{2}x^2\sin\ln x - \frac{1}{2}\mathrm{I}\right] + \mathrm{c}$$

$$\mathrm{I} = \frac{1}{2}x^2\cos(\ln x) + \frac{1}{4}x^2\sin\ln x - \frac{1}{4}\mathrm{I} + \mathrm{c} \Longrightarrow \mathrm{I} + \frac{1}{4}\mathrm{I} = \frac{1}{2}x^2\cos(\ln x) + \frac{1}{4}x^2\sin\ln x + \mathrm{c}$$

$$\frac{5}{4}\mathrm{I} = \frac{1}{2}x^2\cos(\ln x) + \frac{1}{4}x^2\sin\ln x + \mathrm{c} \Longrightarrow \mathrm{I} = \frac{1}{5}x^2(2\cos(\ln x) + \sin(\ln x)) + \mathrm{c}$$

125. $\int \frac{1}{x^3\sqrt{x^2-1}}\,dx$

First Method: let $x = \sec\theta \Longrightarrow dx = \sec\theta\tan\theta\,d\theta$, then we get:

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{1}{\sec^3\theta\sqrt{\sec^2\theta - 1}}\sec\theta\tan\theta\,d\theta = \int \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta$$

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{1}{\sec^2\theta}\,d\theta = \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + c$$

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + c = \frac{1}{2}\sec^{-1}x + \frac{1}{2}\frac{\sqrt{x^2-1}}{x}.\frac{1}{x} + c; \quad \text{therefore:}$$

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \frac{1}{2}\sec^{-1}x + \frac{\sqrt{x^2-1}}{2x^2} + c$$

Second Method:

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int \frac{x^2-(x^2-1)}{x^3\sqrt{x^2-1}}\,dx = \int \frac{1}{x\sqrt{x^2-1}}\,dx - \int \frac{\sqrt{x^2-1}}{x^3}\,dx$$

With $\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + k$; then we write:

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \sec^{-1}x - \int \sqrt{1-\frac{1}{x^2}}.\frac{1}{x^2}\,dx\,; \quad \text{let } y = \frac{1}{x} \Longrightarrow dy = -\frac{1}{x^2}dx; \quad \text{then we get:}$$

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \sec^{-1}x - \int \sqrt{1-y^2}(-dy) = \sec^{-1}x + \int \sqrt{1-y^2}\,dy$$

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \sec^{-1}x + \frac{1}{2}y\sqrt{1-y^2} + \frac{1}{2}\sin^{-1}y + c$$

$$\int \frac{1}{x^3\sqrt{x^2-1}}\,dx = \sec^{-1}x + \frac{1}{2}.\frac{1}{x}\sqrt{1-\frac{1}{x^2}} + \frac{1}{2}\sin^{-1}\left(\frac{1}{x}\right) + c; \quad \text{therefore we get:}$$

$$\int \frac{1}{x^3\sqrt{x^2-1}}dx = \sec^{-1}x + \frac{1}{x} + \frac{\sqrt{x^2-1}}{2x^2}\sin^{-1}\left(\frac{1}{x}\right) + c$$

126. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\tan x}{\cos^m x+\sec^m x}dx = \int_0^{\frac{\pi}{2}} \frac{\tan x}{\cos^m x+\frac{1}{\cos^m x}}dx = \int_0^{\frac{\pi}{2}} \frac{\tan x}{\cos^m x+\frac{1}{\cos^m x}} \times \frac{\cos^m x}{\cos^m x}dx$, then:

$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\cos^m x \tan x}{1+\cos^{2m} x}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^m x \times \frac{\sin x}{\cos x}}{1+\cos^{2m} x}dx = \int_0^{\frac{\pi}{2}} \frac{\cos^{m-1} x}{1+\cos^{2m} x}.\sin x\,dx$

Let $u = \cos x$, then $du = -\sin x\,dx$, for $x = 0$, $u = \cos 0 = 1$ and for $x = \frac{\pi}{2}$, $u = \cos\frac{\pi}{2} = 0$,

so: $\mathrm{I} = \int_1^0 \frac{u^{m-1}}{1+u^{2m}}(-du) = \int_0^1 \frac{u^{m-1}}{1+u^{2m}}du$, let $t = u^m$, then $dt = mu^{n-1}du$, so we get:

$\mathrm{I} = \frac{1}{m}\int_0^1 \frac{1}{1+t^2}dt = \frac{1}{m}[\arctan t]_0^1 = \frac{1}{m}\left(\frac{\pi}{4}-0\right) = \frac{\pi}{4m}$

127. $\int \frac{\sqrt[3]{\sin x}}{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7}dx = \int \frac{\frac{\sqrt[3]{\sin x}}{\sqrt[3]{\cos x}}}{\frac{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7}{\sqrt[3]{\cos x}}}dx = \int \frac{\sqrt[3]{\tan x}}{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7 \sqrt[3]{\sec x}}dx$

$\int \frac{\sqrt[3]{\sin x}}{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7}dx = \int \frac{\sqrt[3]{\tan x}}{\left[\sqrt[3]{\cos x}\left(\sqrt[3]{\tan x}+1\right)\right]^7 \sqrt[3]{\sec x}}dx = \int \frac{\sqrt[3]{\tan x}}{\cos^{\frac{7}{3}} x\left(\sqrt[3]{\tan x}+1\right)^7 \sqrt[3]{\sec x}}dx$

$\int \frac{\sqrt[3]{\sin x}}{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7}dx = \int \frac{\sqrt[3]{\tan x}}{\sec^{-\frac{7}{3}+\frac{1}{3}} x\left(\sqrt[3]{\tan x}+1\right)^7}dx = \int \frac{\sqrt[3]{\tan x}}{\sec^{-2} x\left(\sqrt[3]{\tan x}+1\right)^7}dx$

$\int \frac{\sqrt[3]{\sin x}}{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7}dx = \int \frac{\sqrt[3]{\tan x}}{\left(\sqrt[3]{\tan x}+1\right)^7}\sec^2 x\,dx$

Let $u = \sqrt[3]{\tan x} + 1$, then $\tan x = (u-1)^3$, so $\sec^2 x\,dx = 3(u-1)^2 du$, then we get:

$\int \frac{\sqrt[3]{\sin x}}{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7}dx = \int \frac{u-1}{u^7}.3(u-1)^2 du = 3\int \frac{(u-1)^3}{u^7}du = 3\int \frac{u^3-3u^2+3u-1}{u^7}du$

$= 3\int (u^{-4} - 3u^{-5} + 3u^{-6} - u^{-7})du = -u^{-3} + \frac{9}{4}u^{-4} - \frac{9}{5}u^{-5} + \frac{1}{2}u^{-6} + c$, therefore:

$\int \frac{\sqrt[3]{\sin x}}{\left(\sqrt[3]{\sin x}+\sqrt[3]{\cos x}\right)^7}dx = \frac{1}{2\left(\sqrt[3]{\tan x}+1\right)^6} - \frac{9}{5\left(\sqrt[3]{\tan x}+1\right)^5} + \frac{9}{4\left(\sqrt[3]{\tan x}+1\right)^4} - \frac{1}{\left(\sqrt[3]{\tan x}+1\right)^3} + c$

128. $\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx$, let $x = \tan t$, then $dx = \sec^2 t\,dt$, then we get:

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right)\sec^2 t\,dt = \int \sin^{-1}\left(\frac{2\tan t}{1+\tan^2 t}\right)\sec^2 t\,dt$$

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int \sin^{-1}\left(\frac{2\frac{\sin t}{\cos t}}{1+\frac{\sin^2 t}{\cos^2 t}}\right)\sec^2 t\,dt = \int \sin^{-1}\left(\frac{2\frac{\sin t}{\cos t}}{\frac{\cos^2 t+\sin^2 t}{\cos^2 t}}\right)\sec^2 t\,dt$$

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int \sin^{-1}\left(\frac{2\frac{\sin t}{\cos t}}{\frac{1}{\cos^2 t}}\right)\sec^2 t\,dt = \int \sin^{-1}(2\sin t\cos t)\sec^2 t\,dt$$

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = \int \sin^{-1}(\sin 2t)\sec^2 t\,dt = \int 2t\sec^2 t\,dt$$

Using integration by parts**:** Let $u = 2t$, then $u' = 2$ and let $v' = \sec^2 t$, then $v = \tan t$, so**:**

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2t\tan t - \int 2\tan t = 2t\tan t - 2\ln|\sec t| + c;\ \text{ then we get:}$$

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2\tan^{-1}x \times x - 2\ln|\sec(\tan^{-1}x)| + c;\ \text{ therefore:}$$

$$\int \sin^{-1}\left(\frac{2x}{1+x^2}\right)dx = 2x\tan^{-1}x - 2\ln|\sec(\tan^{-1}x)| + c$$

129. $\mathrm{I} = \int_0^{+\infty} \frac{\ln x}{(x^2+4)(x^2+9)}dx$

Take: $\mathrm{J} = \int\limits_0^{+\infty} \frac{\ln x}{x^2+a^2}dx$; let $x = a\tan\theta \Longrightarrow dx = a\sec^2\theta\,d\theta$; for $x = 0$; $\theta = 0$ and for $x = +\infty$; $\theta = \frac{\pi}{2}$; then we get:

$$\mathrm{J} = \int\limits_0^{\frac{\pi}{2}} \frac{\ln(a\tan\theta)}{(a\tan\theta)^2 + a^2}a\sec^2\theta\,d\theta = \int\limits_0^{\frac{\pi}{2}} \frac{\ln(a\tan\theta)}{a^2\tan^2\theta + a^2}a\sec^2\theta\,d\theta$$

$$\mathrm{J} = \int\limits_0^{\frac{\pi}{2}} \frac{\ln(a\tan\theta)}{a^2(1+\tan^2\theta)}a\sec^2\theta\,d\theta = \int\limits_0^{\frac{\pi}{2}} \frac{\ln(a\tan\theta)}{a\sec^2\theta}\sec^2\theta\,d\theta = \frac{1}{a}\int\limits_0^{\frac{\pi}{2}} \ln(a\tan\theta)\,d\theta$$

$$\mathrm{J} = \frac{1}{a}\int\limits_0^{\frac{\pi}{2}} [\ln(a) + \ln(\tan\theta)]d\theta = \frac{1}{a}\int\limits_0^{\frac{\pi}{2}} \ln a\,d\theta + \frac{1}{a}\int\limits_0^{\frac{\pi}{2}} \ln(\tan\theta)\,d\theta = \frac{\pi\ln a}{2a} + \frac{1}{a}\int\limits_0^{\frac{\pi}{2}} \ln(\tan\theta)\,d\theta$$

$$\mathrm{J} = \frac{\pi\ln a}{2a} + \frac{1}{a}\int\limits_0^{\frac{\pi}{2}} \ln\left(\frac{\sin\theta}{\cos\theta}\right)d\theta = \frac{\pi\ln a}{2a} + \frac{1}{a}\left[\int\limits_0^{\frac{\pi}{2}} \ln(\sin\theta)\,d\theta - \int\limits_0^{\frac{\pi}{2}} \ln(\cos\theta)\,d\theta\right]$$

For $\int\limits_0^{\frac{\pi}{2}} \ln(\cos\theta)\,d\theta$; let $u = \frac{\pi}{2} - \theta \Longrightarrow du = d\theta$; for $\theta = 0$; $u = \frac{\pi}{2}$ and for $\theta = \frac{\pi}{2}$; $u = 0$

$$\int\limits_0^{\frac{\pi}{2}} \ln(\cos\theta)\,d\theta = \int\limits_{\frac{\pi}{2}}^{0} \ln\left(\cos\left(\frac{\pi}{2} - u\right)\right)(-du) = \int\limits_0^{\frac{\pi}{2}} \ln(\sin u)\,du = \int\limits_0^{\frac{\pi}{2}} \ln(\sin\theta)\,d\theta;\ \text{ then:}$$

$$\mathrm{J} = \frac{\pi\ln a}{2a} + \frac{1}{a}\left[\int\limits_0^{\frac{\pi}{2}} \ln(\sin\theta)\,d\theta - \int\limits_0^{\frac{\pi}{2}} \ln(\sin\theta)\,d\theta\right] = \frac{\pi\ln a}{2a} + \frac{1}{a}(0) = \frac{\pi\ln a}{2a}$$

Then: $\int\limits_0^{+\infty} \frac{\ln x}{x^2+a^2}dx = \frac{\pi\ln a}{2a}$; noe back to $\mathrm{I} = \int\limits_0^{+\infty} \frac{\ln x}{(x^2+4)(x^2+9)}dx$; we can write:

$$\frac{\ln x}{(x^2+4)(x^2+9)} = \frac{1}{5} \times \frac{5\ln x}{(x^2+4)(x^2+9)} = \frac{1}{5} \times \frac{9\ln x - 4\ln x}{(x^2+4)(x^2+9)}$$

$$\frac{\ln x}{(x^2+4)(x^2+9)} = \frac{1}{5} \times \frac{9\ln x - 4\ln x + x^2\ln x - x^2\ln x}{(x^2+4)(x^2+9)};\ \text{then we get:}$$

$$\frac{\ln x}{(x^2+4)(x^2+9)} = \frac{1}{5} \times \frac{\ln x\,(x^2+9) - \ln x\,(x^2+4)}{(x^2+4)(x^2+9)} = \frac{1}{5}.\frac{\ln x}{x^2+4} - \frac{1}{5}.\frac{\ln x}{x^2+9};\ \text{then}$$

$$\int_0^{+\infty} \frac{\ln x}{(x^2+4)(x^2+9)}dx = \frac{1}{5}\int_0^{+\infty} \frac{\ln x}{x^2+4}dx - \frac{1}{5}\int_0^{+\infty} \frac{\ln x}{x^2+9}dx = \frac{1}{5}\left(\frac{\pi\ln 2}{4} - \frac{\pi\ln 3}{6}\right);\ \text{therefore:}$$

$$\int_0^{+\infty} \frac{\ln x}{(x^2+4)(x^2+9)}dx = \frac{\pi}{60}\ln\left(\frac{8}{9}\right)$$

130. $\int \sqrt{\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}}dx$, let $x = \sinh u$, then $dx = \cosh u\,du$, then we get**:**

$$\int \sqrt{\frac{\ln\left(x+\sqrt{1+x^2}\right)}{1+x^2}}du = \int \sqrt{\frac{\ln\left(\sinh u + \sqrt{1+\sinh^2 u}\right)}{1+\sinh^2 u}}\cosh u\,du$$

$$= \int \sqrt{\frac{\ln\left(\sinh u + \sqrt{\cosh^2 u}\right)}{\cosh^2 u}}\cosh u\,du = \int \frac{\sqrt{\ln(\sinh u + \cosh u)}}{\cosh u}\cosh u\,du$$

$$= \int \sqrt{\ln(\sinh u + \cosh u)}\,du = \int \sqrt{\ln\left(\frac{e^u - e^{-u}}{2} + \frac{e^u + e^{-u}}{2}\right)}du = \int \sqrt{\ln\left(\frac{2e^u}{2}\right)}du$$

$$= \int \sqrt{\ln(e^u)}\,du = \int \sqrt{u}\,du = \frac{2}{3}u\sqrt{u} + c = \frac{2}{3}\sinh^{-1}x\sqrt{\sinh^{-1}x} + c$$

131. $\int \frac{1}{1+ab-a\cos x-b\sec x}dx = \int \frac{\cos x}{\cos x+ab\cos x-a\cos^2 x-b}dx$, then we get**:**

$$\int \frac{1}{1+ab-a\cos x-b\sec x}dx = \int \frac{\cos x}{a\cos x(b-\cos x)-(b-\cos x)}dx = \int \frac{\cos x}{(a\cos x-1)(b-\cos x)}dx$$

$$\int \frac{1}{1+ab-a\cos x-b\sec x}dx = \frac{1}{ab-1}\int \frac{1}{a\cos x-1}dx + \frac{b}{ab-1}\int \frac{1}{b-\cos x}dx$$

Let $t = \tan\left(\frac{x}{2}\right)$, then $dx = \frac{2}{1+t^2}dt$ and $\cos x = \frac{1-t^2}{1+t^2}$, then we get**:**

$$\int \frac{1}{1+ab-a\cos x-b\sec x}dx = \frac{1}{ab-1}\int \frac{1}{a\left(\frac{1-t^2}{1+t^2}\right)-1}\frac{2}{1+t^2}dt + \frac{b}{ab-1}\int \frac{1}{b-\left(\frac{1-t^2}{1+t^2}\right)}\frac{2}{1+t^2}dt$$

$$\int \frac{1}{1+ab-a\cos x-b\sec x}dx = \frac{2}{1-ab}\int \frac{dt}{t^2(a+1)-(a-1)} + \frac{2b}{ab-1}\int \frac{dt}{t^2(b+1)+(b-1)}$$

$$= \frac{2}{(1-ab)\sqrt{a^2-1}}\ln\left|\frac{t\sqrt{a+1}-\sqrt{a-1}}{t\sqrt{a+1}+\sqrt{a-1}}\right| + \frac{2b}{(ab-1)\sqrt{b^2-1}}\tan^{-1}\left(\sqrt{\frac{b+1}{b-1}}t\right) + c$$

$$= \frac{2}{(1-ab)\sqrt{a^2-1}}\ln\left|\frac{\tan\left(\frac{x}{2}\right)\sqrt{a+1}-\sqrt{a-1}}{\tan\left(\frac{x}{2}\right)\sqrt{a+1}+\sqrt{a-1}}\right| + \frac{2b}{(ab-1)\sqrt{b^2-1}}\tan^{-1}\left(\sqrt{\frac{b+1}{b-1}}\tan\left(\frac{x}{2}\right)\right) + c$$

132. $\int \frac{1}{(a^2-b^2x^2)\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{(a^2-b^2x^2)^{\frac{3}{2}}}dx = \int \frac{1}{\left[x^2\left(\frac{a^2}{x^2}-b^2\right)\right]^{\frac{3}{2}}}dx$, then we get**:**

$$\int \frac{1}{(a^2-b^2x^2)\sqrt{a^2-b^2x^2}}dx = \int \frac{1}{(x^2)^{\frac{3}{2}}\left(\frac{a^2}{x^2}-b^2\right)^{\frac{3}{2}}}dx = \int \frac{1}{x^3\left(\frac{a^2}{x^2}-b^2\right)^{\frac{3}{2}}}dx$$

Let $\frac{a^2}{x^2}-b^2=t$; then $-\frac{2a^2}{x^3}dx=dt$ and so $\frac{1}{x^3}dx=-\frac{1}{2a^2}dt$; then we get:

$$\int \frac{1}{(a^2-b^2x^2)\sqrt{a^2-b^2x^2}}dx = -\frac{1}{2a^2}\int \frac{1}{t^{\frac{3}{2}}}dt = -\frac{1}{2a^2}\int t^{-\frac{3}{2}}dt = -\frac{1}{2a^2}\left(-\frac{1}{\frac{1}{2}}t^{-\frac{1}{2}}\right)+c$$

$$\int \frac{1}{(a^2-b^2x^2)\sqrt{a^2-b^2x^2}}dx = \frac{1}{a^2\sqrt{t}}+c = \frac{1}{a^2\sqrt{\frac{a^2}{x^2}-b^2}}+c$$

133. $\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}dx$, let $u=\sqrt{\tan x}$, then $\tan x = u^2$ and $x=\arctan u^2$, then

$dx = \frac{2u}{1+u^4}du$ and $\sin 2x = 2\sin x\cos x = \frac{2\sin x\cos x}{\cos^2 x+\sin^2 x} = \frac{2\tan x}{1+\tan^2 x} = \frac{2u^2}{1+u^4}$, then we get**:**

$$\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}dx = \int \frac{\frac{1}{u}-u}{1+3\left(\frac{2u^2}{1+u^4}\right)}.\frac{2u}{1+u^4}du = 2\int \frac{1-u^2}{u^4+6u^2+1}du$$

$$\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}dx = 2\int \frac{1}{u^2+6+\frac{1}{u^2}}\left(\frac{1}{u^2}-1\right)du = 2\int \frac{1}{\left(u+\frac{1}{u}\right)^2+4}(-1)d\left(u+\frac{1}{u}\right)$$

$$\int \frac{\sqrt{\cot x}-\sqrt{\tan x}}{1+3\sin 2x}dx = \cot^{-1}\left(\frac{1}{2}\left(u+\frac{1}{u}\right)\right)+c = \cot^{-1}\left(\frac{\sqrt{\tan x}+\sqrt{\cot x}}{2}\right)+c$$

134. $\int \frac{1+\cos^2 x}{\sqrt{\cos x}(1-\cos^2 x+2\cos x)}dx$

Let $y=\sqrt{\cos x} \Longrightarrow x=\cos^{-1}(y^2) \Longrightarrow dx = -\frac{2y}{\sqrt{1-y^4}}dy$; then we get:

$$\int \frac{1+\cos^2 x}{\sqrt{\cos x}\,(1-\cos^2 x+2\cos x)}dx = \int \frac{1+y^4}{y(1-y^4+2y^2)}\left(-\frac{2y}{\sqrt{1-y^4}}dy\right);\ \text{ so}$$

$$\int \frac{1+\cos^2 x}{\sqrt{\cos x}\,(1-\cos^2 x+2\cos x)}dx = 2\int \frac{1+y^4}{(1+2y^2-y^4)\sqrt{1-y^4}}dy;\ \text{ so we can write:}$$

$$\int \frac{1+\cos^2 x}{\sqrt{\cos x}\,(1-\cos^2 x+2\cos x)}dx = 2\int \frac{1}{\left(\frac{1}{y^2}+2-y^2\right)\sqrt{\frac{1}{y^2}-y^2}}.\left(y+\frac{1}{y^3}\right)dy$$

Let $u=\sqrt{\frac{1}{y^2}-y^2} \Longrightarrow u^2=\frac{1}{y^2}-y^2 \Longrightarrow 2udu=-2\left(\frac{1}{y^3}+y\right)$; then we get:

$$\int \frac{1+\cos^2 x}{\sqrt{\cos x}\,(1-\cos^2 x+2\cos x)}dx = 2\int \frac{1}{(u^2+2).u}(-udu) = -2\int \frac{1}{u^2+2}du$$

$$\int \frac{1+\cos^2 x}{\sqrt{\cos x}\,(1-\cos^2 x+2\cos x)}dx = -2\int \frac{1}{u^2+\left(\sqrt{2}\right)^2}du = -\frac{2}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$$

$$\int \frac{1+\cos^2 x}{\sqrt{\cos x}\,(1-\cos^2 x+2\cos x)}dx = -\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\frac{1}{y^2}-y^2}}{\sqrt{2}}\right)+c;\ \text{ with } y=\sqrt{\cos x}\,;\text{then}$$

$$\int \frac{1+\cos^2 x}{\sqrt{\cos x}\,(1-\cos^2 x+2\cos x)}dx = -\sqrt{2}\tan^{-1}\left(\frac{\sqrt{\frac{1}{\cos x}-\cos x}}{\sqrt{2}}\right)+c;\ \text{ therefore we get:}$$

$$\int \frac{1+\cos^2 x}{\sqrt{\cos x}\,(1-\cos^2 x+2\cos x)}dx = -\sqrt{2}\tan^{-1}\left(\sqrt{\frac{1}{2\cos x}-\frac{\cos x}{2}}\right)+c$$

135. $\mathrm{I}=\int_{-\infty}^{+\infty} e^{-x^2}\cosh x\,dx$

We have $\cosh x+\sinh x = e^x$, then $\cosh x = e^x-\sinh x$, then:

$\mathrm{I}=\int_{-\infty}^{+\infty} e^{-x^2}(e^x-\sinh x)\,dx = \int_{-\infty}^{+\infty} e^{-x^2}.e^x\,dx - \int_{-\infty}^{+\infty} e^{-x^2}\sinh x\,dx$

But $\int_{-\infty}^{+\infty} e^{-x^2}\sinh x\,dx = 0$ (odd function), then we get:

$\mathrm{I}=\int_{-\infty}^{+\infty} e^{-x^2}.e^x\,dx = \int_{-\infty}^{+\infty} e^{-x^2+x}\,dx = \int_{-\infty}^{+\infty} e^{-\left(x^2-x+\frac{1}{4}\right)+\frac{1}{4}}\,dx = \int_{-\infty}^{+\infty} e^{-\left(x-\frac{1}{2}\right)^2+\frac{1}{4}}\,dx$

$\mathrm{I}=\int_{-\infty}^{+\infty} e^{-\left(x-\frac{1}{2}\right)^2}.e^{\frac{1}{4}}\,dx = e^{\frac{1}{4}}\int_{-\infty}^{+\infty} e^{-\left(x-\frac{1}{2}\right)^2}\,dx = \sqrt[4]{e}\int_{-\infty}^{+\infty} e^{-\left(x-\frac{1}{2}\right)^2}\,dx$

Let $u=x-\frac{1}{2}$, then $du=dx$, so we get:

$\mathrm{I}=\sqrt[4]{e}\int_{-\infty}^{+\infty} e^{-u^2}\,dx = \sqrt[4]{e}.\sqrt{\pi}$

136. $\mathrm{I}=\int_0^{\frac{1}{2}} \ln^2\left(\frac{1}{x}-1\right)dx$

Change of variable: let $t=\frac{1}{x}-1$, then $x=\frac{1}{t+1}$ and $dx=-\frac{1}{(t+1)^2}dt$

New bounds: for $x=\frac{1}{2}$, $t=2-1=1$ and for $x=0$, then $t=+\infty$, so we get:

$$\mathrm{I}=\int_0^{\frac{1}{2}} \ln^2\left(\frac{1}{x}-1\right)dx = \int_{+\infty}^{1} \ln^2 t.\left(-\frac{1}{(1+t)^2}dt\right) = \int_1^{+\infty}\frac{\ln^2 t}{(1+t)^2}dt\,;\quad \text{now using IBP:}$$

Let $u=\ln^2 t \Longrightarrow u'=\frac{2\ln t}{t}dt$ & $let\ v'=\frac{1}{(1+t)^2}\Longrightarrow v=-\frac{1}{1+t}$; then we get:

$$\mathrm{I}=\left[-\frac{\ln^2 t}{1+t}\right]_1^{+\infty}+2\int_1^{+\infty}\frac{\ln t}{t(1+t)}dt = 2\int_1^{+\infty}\frac{\ln t}{t(1+t)}dt$$

Let $\boldsymbol{y=\frac{1}{t}}\Longrightarrow t=\frac{1}{y}\Longrightarrow dt=-\frac{1}{y^2}dy$; for $t=1; y=1$ & $for\ t=+\infty; y=0$; then we get:

$$\mathrm{I}=2\int_1^0 \frac{\ln\left(\frac{1}{y}\right)}{\frac{1}{y}\left(1+\frac{1}{y}\right)}\left(-\frac{1}{y^2}dy\right) = 2\int_0^1\frac{-\ln y}{y+1}dy = -2\int_0^1\frac{\ln y}{1+y}dy = -2\int_0^1\left(\frac{1}{1+y}\right)\ln y\,dy$$

$$\mathrm{I} = -2\int_0^1 \sum_{n=0}^{+\infty}(-1)^n y^n \ln y\, dy = -2\sum_{n=0}^{+\infty}(-1)^n \int_0^1 y^n \ln y\, dy$$

Let $\boldsymbol{z = -\ln y}$ $\Rightarrow y = e^{-z} \Rightarrow dy = -e^{-z}dz$; for $y = 0; z = +\infty$ & for $y = 1; z = 0$; then:

$$\mathrm{I} = -2\sum_{n=0}^{+\infty}(-1)^n \int_{+\infty}^{0}(e^{-z})^n(-z)(-e^{-z}dz) = 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} e^{-(n+1)z}.zdz$$

Let $\boldsymbol{w = (n+1)z}$ $\Rightarrow z = \dfrac{1}{n+1}w \Rightarrow dz = \dfrac{1}{n+1}dw$; then we get:

$$\mathrm{I} = 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} e^{-w}.\frac{w}{n+1}.\frac{1}{n+1}dw = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n+1)^2}\int_0^{+\infty} \mathrm{w}e^{-w}dw$$

$$\mathrm{I} = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n+1)^2}\int_0^{+\infty} \mathrm{w}^{2-1}e^{-w}dw = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n+1)^2}\Gamma(2) = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(n+1)^2} = 2\eta(2) = \frac{\pi^2}{6}$$

137. $\mathrm{I} = \int_a^b \frac{dx}{x\sqrt{\ln\left(\frac{x}{a}\right).\ln\left(\frac{b}{x}\right)}} = \int_a^b \frac{dx}{x\sqrt{(\ln x - \ln a)(\ln b - \ln x)}}$, let $u = \ln x$, then $du = \frac{1}{x}dx$, then we get**:**

$\mathrm{I} = \int_{\ln a}^{\ln b}\frac{du}{\sqrt{(u-\ln a)(\ln b - u)}}$. Let $u = \ln a . \cos^2\theta + \ln b . \sin^2\theta$, then**:**

$du = (-2\ln a . \cos\theta\sin\theta + 2\ln b . \cos\theta\sin\theta)d\theta = (-\sin 2\theta . \ln a + \sin 2\theta . \ln b)d\theta$

$du = \sin 2\theta\,(\ln b - \ln a)d\theta = \ln\left(\frac{b}{a}\right).\sin 2\theta\, d\theta$ and $u = \ln a + \sin^2\theta\ln\left(\frac{b}{a}\right)$

For $x = \ln a$, then $\ln a = \ln a + \sin^2\theta\ln\left(\frac{b}{a}\right)$, so $\sin^2\theta = 0$, so $\theta = 0$ and for $x = \ln b$, then $\ln b = \ln a + (\ln b - \ln a)\sin^2\theta$, then $(\ln b - \ln a) = (\ln b - \ln a)\sin^2\theta$, so $\sin^2\theta = 1$, then we get $\theta = \frac{\pi}{2}$, then**:**

$$\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{\sin 2\theta\ln\left(\frac{b}{a}\right)}{\sqrt{\left(\sin^2\theta\ln\left(\frac{b}{a}\right)\right)\left(\cos^2\theta\ln\left(\frac{b}{a}\right)\right)}}d\theta = \int_0^{\frac{\pi}{2}}\frac{\sin 2\theta\ln\left(\frac{b}{a}\right)}{\sin\theta\cos\theta\sqrt{\ln\left(\frac{b}{a}\right).\ln\left(\frac{b}{a}\right)}}d\theta = \int_0^{\frac{\pi}{2}}\frac{\sin 2\theta\ln\left(\frac{b}{a}\right)}{\sin\theta\cos\theta\sqrt{\ln^2\left(\frac{b}{a}\right)}}d\theta$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{\sin 2\theta\ln\left(\frac{b}{a}\right)}{\sin\theta\cos\theta\ln\left(\frac{b}{a}\right)}d\theta = \int_0^{\frac{\pi}{2}}\frac{2\sin\theta\cos\theta\ln\left(\frac{b}{a}\right)}{\sin\theta\cos\theta\ln\left(\frac{b}{a}\right)}d\theta = \int_0^{\frac{\pi}{2}}2\,d\theta = 2\left(\frac{\pi}{2}\right) = \pi$$

138. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{x\cos x - \sin x}{x^2+\sin^2 x}dx = \int_0^{\frac{\pi}{2}}\frac{x\cos x - \sin x}{x^2+\sin^2 x}\times\frac{\frac{1}{\sin^2 x}}{\frac{1}{\sin^2 x}}dx = \int_0^{\frac{\pi}{2}}\frac{x\frac{\cos x}{\sin^2 x} - \frac{1}{\sin x}}{\left(\frac{x}{\sin x}\right)^2+1}dx$

$\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{\frac{x\cos x - \sin x}{\sin^2 x}}{\left(\frac{x}{\sin x}\right)^2+1}dx$, let $u = \frac{x}{\sin x}$, then $du = -\left(\frac{x\cos x - \sin x}{\sin^2 x}\right)dx$, for $x = 0$, then

$u = \lim\limits_{x\to 0}\left(\frac{x}{\sin x}\right) = 1$ and for $x = \frac{\pi}{2}$, then $u = \frac{\frac{\pi}{2}}{\sin\left(\frac{\pi}{2}\right)} = \frac{\pi}{2}$, then we get**:**

$$\mathrm{I} = -\int_1^{\frac{\pi}{2}}\frac{du}{1+u^2} = -[\arctan u]_1^{\frac{\pi}{2}} = \arctan(1) - \arctan\left(\frac{\pi}{2}\right) = \frac{\pi}{4} - \arctan\left(\frac{\pi}{2}\right)$$

139. **First Method:** $I=\int_0^{+\infty}\frac{1}{(1+x^k)(1+x^2)}dx$

Let $x=\tan t$, then $dx=(1+\tan^2 t)dt=(1+x^2)dt$, then $dt=\frac{dx}{1+x^2}$, for $x=0$, $t=0$ and for $x=+\infty$, $t=\frac{\pi}{2}$, then we get**:**

$I=\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^k x}dt$ **… (1)**, let $u=\frac{\pi}{2}-t$, then $du=-dt$, for $t=0$, $u=\frac{\pi}{2}$ and for $t=\frac{\pi}{2}$, $u=0$,

then**:** $I=\int_{\frac{\pi}{2}}^{0}\frac{1}{1+\tan^k(\frac{\pi}{2}-u)}(-du)=\int_0^{\frac{\pi}{2}}\frac{1}{1+\cot^k u}du=\int_0^{\frac{\pi}{2}}\frac{1}{1+\cot^k x}dx=\int_0^{\frac{\pi}{2}}\frac{1}{1+\frac{1}{\tan^k x}}dx$

$I=\int_0^{\frac{\pi}{2}}\frac{\tan^k x}{1+\tan^k x}dx$ **… (2)** Adding (1) and (2) we get**:'**

$2I=\int_0^{\frac{\pi}{2}}\frac{1}{1+\tan^k x}dt+\int_0^{\frac{\pi}{2}}\frac{\tan^k x}{1+\tan^k x}dx=\int_0^{\frac{\pi}{2}}\frac{1+\tan^k x}{1+\tan^k x}dx=\int_0^{\frac{\pi}{2}}dx=\frac{\pi}{2}$, therefore $I=\frac{\pi}{4}$

Second Method: $I=\int_0^{+\infty}\frac{1}{(1+x^k)(1+x^2)}dx$ **… (1)**

Let $x=\frac{1}{u}$, then $dx=-\frac{1}{u^2}du$, for $x=0$, $u=+\infty$ and for $x=+\infty$, $u=0$, then we get**:**

$I=\int_{+\infty}^{0}\frac{1}{\left(1+\left(\frac{1}{u}\right)^k\right)\left(1+\left(\frac{1}{u}\right)^2\right)}\left(-\frac{1}{u^2}du\right)=\int_0^{+\infty}\frac{1}{\left(1+\frac{1}{u^k}\right)(1+u^2)}du=\int_0^{+\infty}\frac{1}{\frac{1}{u^k}(1+u^k)(1+u^2)}du$

$I=\int_0^{+\infty}\frac{u^k}{(1+u^k)(1+u^2)}du=\int_0^{+\infty}\frac{x^k}{(1+x^k)(1+x^2)}dx$ **… (3)**

Adding (1) and (2) we get $2I=\int_0^{+\infty}\frac{1}{(1+x^k)(1+x^2)}dx+\int_0^{+\infty}\frac{x^k}{(1+x^k)(1+x^2)}dx$, then**:**

$2I=\int_0^{+\infty}\frac{1+x^k}{(1+x^k)(1+x^2)}dx=\int_0^{+\infty}\frac{1}{1+x^2}dx=[\arctan x]_0^{+\infty}=(\arctan\infty-\arctan 0)=\frac{\pi}{2}$,

$2I=\frac{\pi}{2}$, therefore $I=\frac{\pi}{4}$

140. $I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ **… (1)**, let $u=-x$, then $du=-dx$, then we get**:**

$I=\int_{\frac{\pi}{4}}^{-\frac{\pi}{4}}\frac{-du}{(1+e^{\sin(-u)})(2-\cos(-2x))}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{du}{(1+e^{-\sin u})(2-\cos 2u)}=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{(1+e^{-\sin x})(2-\cos 2x)}$

$I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{(1+e^{-\sin x})(2-\cos 2x)}$ **… (2)** Adding (1) and (2) we get**:**

$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{(1+e^{\sin x})(2-\cos 2x)}+\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{dx}{(1+e^{-\sin x})(2-\cos 2x)}$

$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{1+e^{-\sin x}+1+e^{\sin x}}{(1+e^{\sin x})(1+e^{-\sin x})(2-\cos 2x)}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{2+e^{-\sin x}+e^{\sin x}}{(1+e^{\sin x})(1+e^{-\sin x})(2-\cos 2x)}dx$

$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{2+e^{-\sin x}+e^{\sin x}}{(1+e^{-\sin x}+e^{\sin x}+e^{\sin x}e^{-\sin x})(2-\cos 2x)}dx$

$2I=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{2+e^{-\sin x}+e^{\sin x}}{(1+e^{-\sin x}+e^{\sin x}+1)(2-\cos 2x)}dx=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\frac{2+e^{-\sin x}+e^{\sin x}}{(2+e^{-\sin x}+e^{\sin x})(2-\cos 2x)}dx$

Then we get: $2\mathrm{I} = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1}{2-\cos 2x} dx$, let $u = \tan x$, then $\cos 2x = \frac{1-u^2}{1+u^2}$ and $dx = \frac{1}{1+u^2} du$

For $x = -\frac{\pi}{4}$, then $u = -1$ and for $x = \frac{\pi}{4}$, then $u = 1$, so we get:

$$2\mathrm{I} = \int_{-1}^{1} \frac{1}{2-\frac{1-u^2}{1+u^2}} \cdot \frac{1}{1+u^2} du = \int_{-1}^{1} \frac{1}{2(1+u^2)-(1-u^2)} du = \int_{-1}^{1} \frac{1}{2+2u^2-1+u^2} du$$

$$2\mathrm{I} = \int_{-1}^{1} \frac{1}{1+3u^2} du = \frac{1}{3}\int_{-1}^{1} \frac{1}{\frac{1}{3}+u^2} du = \frac{1}{3}\int_{-1}^{1} \frac{1}{\left(\frac{1}{\sqrt{3}}\right)^2+u^2} du = \frac{1}{3}\left[\sqrt{3}\arctan(\sqrt{3}u)\right]_{-1}^{1}$$

$$2\mathrm{I} = \frac{\sqrt{3}}{3}\left[\arctan(\sqrt{3}) - \arctan(-\sqrt{3})\right] = \frac{\sqrt{3}}{3}\left(\frac{\pi}{3}+\frac{\pi}{3}\right) = \frac{\sqrt{3}}{3}\left(\frac{2\pi}{3}\right), \text{ so } \mathrm{I} = \frac{\sqrt{3}}{3}\left(\frac{\pi}{3}\right) = \frac{\pi\sqrt{3}}{9}$$

141. $\mathrm{I} = \int_0^1 \frac{x}{\cos ax \cos(a-ax)} dx = \int_0^1 \frac{x}{\cos ax \cos(a(1-x))} dx$, let $u = 1-x$, then $du = -dx$, for $x = 0$, $u = 1$ and for $x = 1$, $u = 0$, then we get:

$$\mathrm{I} = \int_0^1 \frac{1-u}{\cos(a(1-u))\cos(au)}(-du) = \int_0^1 \frac{1-u}{\cos au \cos(a-au)} du = \int_0^1 \frac{1-x}{\cos ax \cos(a-ax)} dx, \text{ then:}$$

$$\mathrm{I} = \int_0^1 \frac{1}{\cos ax \cos(a-ax)} dx - \int_0^1 \frac{x}{\cos ax \cos(a-ax)} dx = \int_0^1 \frac{1}{\cos ax \cos(a-ax)} dx - \mathrm{I}, \text{ then:}$$

$$2\mathrm{I} = \int_0^1 \frac{1}{\cos ax \cos(a-ax)} dx, \text{ so: } \mathrm{I} = \frac{1}{2}\int_0^1 \frac{1}{\cos ax \cos(a-ax)} dx, \text{ then we get:}$$

$$\mathrm{I} = \frac{1}{2}\int_0^1 \frac{1}{\frac{1}{2}\cos(ax+a-ax)+\frac{1}{2}\cos(ax-a+ax)} dx = \int_0^1 \frac{1}{\cos(a)+\cos(2ax-a)} dx$$

$\mathrm{I} = \int_0^1 \frac{1}{\cos(a(2x-1))+\cos(a)} dx$, let $t = a(2x-1)$, then $dt = 2a dx$, for $x = 0$, $t = -a$ and for $x = 1$, $t = a$, then we get:

$$\mathrm{I} = \int_{-a}^{a} \frac{1}{\cos t+\cos a} \cdot \frac{1}{2a} dt = 2\int_0^a \frac{1}{\cos t+\cos a} \cdot \frac{1}{2a} dt = \frac{1}{a}\int_0^a \frac{1}{\cos t+\cos a} dt$$

Let $u = \tan\left(\frac{t}{2}\right)$, then $dt = \frac{2}{1+u^2} du$ and $\cos t = \frac{1-u^2}{1+u^2}$, then we get:

$$\mathrm{I} = \frac{1}{a}\int_0^{\tan\left(\frac{a}{2}\right)} \frac{\frac{2}{1+u^2}}{\frac{1-u^2}{1+u^2}+\cos a} du, \text{ putting } k = \cos a, \text{ we get: } \mathrm{I} = \frac{2}{a}\int_0^{\tan\left(\frac{a}{2}\right)} \frac{du}{(1+k)-u^2(1-k)}$$

$$\mathrm{I} = \frac{2}{a} \cdot \frac{1}{2\sqrt{1+k}} \int_0^{\tan\left(\frac{a}{2}\right)} \left[\frac{1}{\sqrt{1+k}-u\sqrt{1-k}} + \frac{1}{\sqrt{1+k}+u\sqrt{1-k}}\right] du$$

$$\mathrm{I} = \frac{1}{a\sqrt{1-k^2}}\left[\ln\left|\frac{\sqrt{1+k}+u\sqrt{1-k}}{\sqrt{1+k}-u\sqrt{1-k}}\right|\right]_0^{\tan\left(\frac{a}{2}\right)} = \frac{1}{a\sin a}\ln\left|\frac{1+\tan\left(\frac{a}{2}\right)\sqrt{\frac{1-coa\,2a}{1+\cos 2a}}}{1-\tan\left(\frac{a}{2}\right)\sqrt{\frac{1-coa\,2a}{1+\cos 2a}}}\right|, \text{ then we get:}$$

$$\mathrm{I} = \frac{1}{a\sin a}\ln\left|\frac{1+\tan^2\left(\frac{a}{2}\right)}{1-\tan^2\left(\frac{a}{2}\right)}\right| = \frac{1}{a}\csc a \ln(\sec a)$$

142. $\mathrm{I} = \int_0^{\pi} \arctan(3^{\cos x})\, dx$, let $u = \cos x$, then $x = \arccos u$, so $dx = -\frac{1}{\sqrt{1-u^2}} du$, for $x = 0$, $u = 1$ and for $x = \pi$, $u = -1$, then we get:

$$\mathrm{I} = \int_1^{-1} \arctan(3^u) \cdot \left(-\frac{1}{\sqrt{1-u^2}} du\right) = \int_{-1}^{1} \frac{\arctan(3^u)}{\sqrt{1-u^2}} du, \text{ then we can write:}$$

$$\mathrm{I}=\int_{-1}^{0}\frac{\arctan(3^{u})}{\sqrt{1-u^{2}}}du+\int_{0}^{1}\frac{\arctan(3^{u})}{\sqrt{1-u^{2}}}du$$

For $\int_{-1}^{0}\frac{\arctan(3^{u})}{\sqrt{1-u^{2}}}du$, let $v=-u$, then $dv=-du$, then we get:

$$\int_{-1}^{0}\frac{\arctan(3^{u})}{\sqrt{1-u^{2}}}du=\int_{1}^{0}\frac{\arctan(3^{-v})}{\sqrt{1-(-v)^{2}}}(-dv)=\int_{0}^{1}\frac{\arctan(3^{-v})}{\sqrt{1-v^{2}}}dv=\int_{0}^{1}\frac{\arctan(3^{-u})}{\sqrt{1-u^{2}}}du$$

So, we get: $\mathrm{I}=\int_{0}^{1}\frac{\arctan(3^{-u})}{\sqrt{1-u^{2}}}du+\int_{0}^{1}\frac{\arctan(3^{u})}{\sqrt{1-u^{2}}}du=\int_{0}^{1}\frac{\arctan(3^{-u})+\arctan(3^{u})}{\sqrt{1-u^{2}}}du$

But we know that $\arctan(\alpha)+\arctan\left(\frac{1}{\alpha}\right)=\frac{\pi}{2}$, then we can write:

$$\mathrm{I}=\frac{\pi}{2}\int_{0}^{1}\frac{1}{\sqrt{1-u^{2}}}du=\frac{\pi}{2}[\arcsin u]_{0}^{1}=\frac{\pi}{2}\left(\frac{\pi}{2}-0\right)=\frac{\pi^{2}}{4}$$

143. $\mathrm{I}=\int_{0}^{1}\frac{\arctan x}{1+x}dx\ldots$ **(1)**

Let $x=\frac{1-u}{1+u}\Rightarrow dx=-\frac{2}{(1+u)^{2}}du$; for $x=0\Rightarrow u=1$ & for $x=1\Rightarrow u=0$; so

$$\mathrm{I}=\int_{0}^{1}\frac{\arctan x}{1+x}dx=\int_{1}^{0}\frac{\arctan\left(\frac{1-u}{1+u}\right)}{1+\frac{1-u}{1+u}}\left\{-\frac{2}{(1+u)^{2}}du\right\}=2\int_{0}^{1}\frac{\arctan\left(\frac{1-u}{1+u}\right)}{(1+u)^{2}+(1-u)(1+u)}du$$

$$\mathrm{I}=2\int_{0}^{1}\frac{\arctan\left(\frac{1-u}{1+u}\right)}{1+2u+u^{2}+1-u^{2}}du=2\int_{0}^{1}\frac{\arctan\left(\frac{1-u}{1+u}\right)}{2+2u}du=\int_{0}^{1}\frac{\arctan\left(\frac{1-u}{1+u}\right)}{1+u}du$$

But $\tan\left(\frac{\pi}{4}-\arctan u\right)=\frac{\tan\left(\frac{\pi}{4}\right)-\tan(\arctan u)}{1+\tan\left(\frac{\pi}{4}\right)\tan(\arctan u)}=\frac{1-u}{1+u}$; then we get:

$$\mathrm{I}=\int_{0}^{1}\frac{\arctan\left(\frac{1-u}{1+u}\right)}{1+u}du=\int_{0}^{1}\frac{\arctan\left[\tan\left(\frac{\pi}{4}-\arctan u\right)\right]}{1+u}du=\int_{0}^{1}\frac{\frac{\pi}{4}-\arctan u}{1+u}du;\ \text{then:}$$

$$\mathrm{I}=\frac{\pi}{4}\int_{0}^{1}\frac{du}{1+u}-\int_{0}^{1}\frac{\arctan u}{1+u}du=\frac{\pi}{4}\int_{0}^{1}\frac{du}{1+u}-\int_{0}^{1}\frac{\arctan x}{1+x}dx=\frac{\pi}{4}\int_{0}^{1}\frac{du}{1+u}-\mathrm{I}$$

Then $2\mathrm{I}=\frac{\pi}{4}\int_{0}^{1}\frac{du}{1+u}=\frac{\pi}{4}[\ln(1+u)]_{0}^{1}=\frac{\pi}{4}\ln 2\Rightarrow \mathrm{I}=\frac{\pi}{8}\ln 2$

144. $\mathrm{I}=\int_{-\infty}^{+\infty}\frac{p+qx}{x^{2}+2rx\cos\alpha+r^{2}}dx=\int_{-\infty}^{+\infty}\frac{p+qx}{x^{2}+2rx\cos\alpha+r^{2}+r^{2}\cos^{2}\alpha-r^{2}\cos^{2}\alpha}dx$

$$=\int_{-\infty}^{+\infty}\frac{p+qx}{(x^{2}+2rx\cos\alpha+r^{2}\cos^{2}\alpha)+r^{2}-r^{2}\cos^{2}\alpha}dx=\int_{-\infty}^{+\infty}\frac{p+qx}{(x+r\cos\alpha)+r^{2}-r^{2}\cos^{2}\alpha}dx$$

$$=\int_{-\infty}^{+\infty}\frac{p+qx}{(x+r\cos\alpha)^{2}+r^{2}\left(1-\cos^{2}\alpha\right)}dx=\int_{-\infty}^{+\infty}\frac{p+qx}{(x+r\cos\alpha)^{2}+r^{2}\sin^{2}\alpha}dx$$

Let $y=x+r\cos\alpha$, then $dy=dx$, so we get:

$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{p+q(y-r\cos\alpha)}{y^{2}+r^{2}\sin^{2}\alpha}dy=\int_{-\infty}^{+\infty}\frac{p+qy-rq\cos\alpha}{y^{2}+r^{2}\sin^{2}\alpha}dy$, then we get:

$\mathrm{I} = (p - qr\cos\alpha)\int_{-\infty}^{+\infty}\frac{1}{y^2+r^2\sin^2\alpha}dy + q\int_{-\infty}^{+\infty}\frac{y}{y^2+r^2\sin^2\alpha}dy$, but : $\frac{y}{y^2+r^2\sin^2\alpha}$ is an odd function, so $\int_{-\infty}^{+\infty}\frac{y}{y^2+r^2\sin^2\alpha}dy = 0$, then we get:

$$\mathrm{I} = (p - qr\cos\alpha)\int_{-\infty}^{+\infty}\frac{1}{y^2+r^2\sin^2\alpha}dy = \frac{p-qr\cos\alpha}{r\sin\alpha}\left[\arctan\left(\frac{y}{r\sin\alpha}\right)\right]_{-\infty}^{+\infty}$$

$$\mathrm{I} = \frac{p-qr\cos\alpha}{r\sin\alpha}\left(\frac{\pi}{2}-\frac{\pi}{2}\right) = \frac{\pi}{r\sin\alpha}(p - qr\cos\alpha)$$

145. $\int \ln(\sqrt{x}+\sqrt{\pi})\,dx$, using integration by parts:

Let $u = \ln(\sqrt{x}+\sqrt{\pi})$, then $u' = \frac{1}{2\sqrt{x}(\sqrt{x}+\sqrt{\pi})} =$ and let $v' = 1$, so $v = x$, so we get:

$$\int \ln(\sqrt{x}+\sqrt{\pi})\,dx = x\ln(\sqrt{x}+\sqrt{\pi}) - \int x.\frac{1}{2\sqrt{x}(\sqrt{x}+\sqrt{\pi})}dx$$

$$\int \ln(\sqrt{x}+\sqrt{\pi})\,dx = x\ln(\sqrt{x}+\sqrt{\pi}) - \frac{1}{2}\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx$$

Let's now evaluate: $\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx$, let $x = \pi\tan^4\theta$, then $\sqrt{x} = \sqrt{\pi}\tan^2\theta$, then we have: $dx = 4\pi\tan^3\theta\sec^2\theta\,d\theta$, then we get:

$$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = \int\frac{\sqrt{\pi}\tan^2\theta}{\sqrt{\pi}\tan^2\theta+\sqrt{\pi}}.4\pi\tan^3\theta\sec^2\theta\,d\theta = 4\pi\int\frac{\sqrt{\pi}\tan^5\theta}{\sqrt{\pi}(1+\tan^2\theta)}\sec^2\theta\,d\theta$$

$$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = 4\pi\int\frac{\tan^5\theta}{\sec^2\theta}\sec^2\theta\,d\theta = 4\pi\int\tan^5\theta\,d\theta = 4\pi\int\tan^3\theta\tan^2\theta\,d\theta$$

$$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = 4\pi\int\tan^3\theta(\sec^2\theta-1)d\theta = 4\pi\int\tan^3\theta\sec^2\theta\,d\theta - 4\pi\int\tan^3\theta\,d\theta$$

$$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = 4\pi\int\tan^3\theta(\tan\theta)'d\theta - 4\pi\int\tan^3\theta\,d\theta = 4\pi\left(\frac{\tan^4\theta}{4}\right) - 4\pi\int\tan^3\theta\,d\theta$$

$$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = \pi\tan^4\theta - 4\pi\int\tan\theta\tan^2\theta\,d\theta = \pi\tan^4\theta - 4\pi\int\tan\theta(\sec^2\theta-1)d\theta$$

$$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = \pi\tan^4\theta - 4\pi\int\tan\theta\sec^2\theta\,d\theta - 4\pi\int\tan\theta\,d\theta$$

$$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = \pi\tan^4\theta - 4\pi\int\tan\theta(\tan\theta)'d\theta - 4\pi\int\frac{\sin\theta}{\cos\theta}d\theta$$

$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = \pi\tan^4\theta - 2\pi\tan^2\theta + 4\pi\int\frac{(\cos\theta)'}{\cos\theta}d\theta$, then we get:

$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = \pi\tan^4\theta - 2\pi\tan^2\theta + 4\pi\ln|\cos\theta| + c$ then we get:

$$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = x - 2\pi\left(\frac{\sqrt{x}}{\sqrt{\pi}}\right) + \pi\ln\left|\frac{1}{\sqrt{1+\left(\frac{\sqrt{x}}{\sqrt{\pi}}\right)^2}}\right| + c = x - 2\sqrt{\pi x} - \pi\ln\left(\frac{\pi+x}{\pi}\right) + c$$

$\int\frac{\sqrt{x}}{\sqrt{x}+\sqrt{\pi}}dx = x - 2\sqrt{\pi x} - \pi\ln(\pi+x) + k$, therefore, we get:

$$\int \ln(\sqrt{x}+\sqrt{\pi})\,dx = x\ln(\sqrt{x}+\sqrt{\pi}) - \frac{1}{2}\left(x - 2\sqrt{\pi x} - \pi\ln(\pi+x) + k\right)$$

146. $\mathrm{I} = \int_0^1 \ln\left(\sqrt[3]{\ln\sqrt{1-x}}\right)dx = \int_0^1 \ln\left[\ln(1-x)^{\frac{1}{2}}\right]^{\frac{1}{3}}dx = \frac{1}{3}\int_0^1 \ln\left[\frac{1}{2}\ln(1-x)\right]dx$

$$\mathrm{I} = \frac{1}{3}\int_0^1 \ln\left(\frac{1}{2}\right)dx + \frac{1}{3}\int_0^1 \ln(\ln(1-x))\,dx = -\frac{\ln 2}{3} + \frac{1}{3}\int_0^1 \ln(\ln(1-x))\,dx$$

Let $-t = \ln(1-x)$, then $x = 1 - e^{-t}$ and $dx = e^{-t}dt$, for $x = 0$, then $t = 0$ and for $x = 1$, then $x = +\infty$, then we get**:**

$$\mathrm{I} = -\frac{\ln 2}{3} + \frac{1}{3}\int_0^1 \ln(-t)\, e^{-t}dt = -\frac{\ln 2}{3} + \frac{1}{3}\int_0^1 [\ln(-1) + \ln t]e^{-t}dt$$

$$\mathrm{I} = -\frac{\ln 2}{3} + \frac{1}{3}\int_0^1 [i\pi + \ln t]e^{-t}dt = -\frac{\ln 2}{3} + \frac{i\pi}{3} - \frac{\gamma}{3} = -\frac{1}{3}(\gamma + \ln 2) + i\frac{\pi}{3}$$

147. $\int \frac{\sqrt{x+\sqrt{1+x^2}}}{1+x^2}dx$, let $y = \sqrt{x+\sqrt{1+x^2}} \Rightarrow x = \frac{y^4-1}{2y^2} \Rightarrow dx = \frac{y^4+1}{y^3}dx$, then we g**et**:

$$\int \frac{\sqrt{x+\sqrt{1+x^2}}}{1+x^2}dx = \int \frac{y}{1+\left(\frac{y^4-1}{2y^2}\right)^2} \cdot \frac{y^4+1}{y^3}dy = \int \frac{y(y^4+1)}{y^3\left(\frac{y^8+2y^4+1}{4y^4}\right)}dy = \int \frac{y^4+1}{y^2\left[\frac{(y^4+1)^2}{4y^4}\right]}dx$$

$$= \int \frac{y^4+1}{\frac{(y^4+1)^2}{4y^2}}dy = \int \frac{4y^2}{y^4+1}dy = \int \frac{4y^2}{y^4+1+2y^2-2y^2}dy = \int \frac{4y^2}{y^4+2y^2+1-(\sqrt{2}y)^2}dy$$

$$= \int \frac{4y^2}{(y^2+1)^2-(\sqrt{2}y)^2}dy = \int \frac{4y^2}{(y^2-\sqrt{2}y+1)(y^2+\sqrt{2}y+1)}dy$$

Partial Fractions : $\frac{4y^2}{(y^2-\sqrt{2}y+1)(y^2+\sqrt{2}y+1)} = \frac{Ay+B}{y^2-\sqrt{2}y+\mathbf{1}} + \frac{Cy+D}{y^2+\sqrt{2}y+1}$, then

$4y^2 = (Ay+B)(y^2+\sqrt{2}y+1) + (Cy+D)(y^2-\sqrt{2}y+1)$, then we g**et**:

$$4y^2 = (\mathrm{A}+C)y^3 + (\sqrt{2}\mathrm{A} + \mathrm{B} - \sqrt{2}\mathrm{C} + D)y^2 + (\mathrm{A} + \sqrt{2}\mathrm{B} + \mathrm{C} - \sqrt{2}D)y + \mathrm{B} + \mathrm{D}$$

By comparing the both sides we get the following system: $\begin{cases} \mathrm{A}+\mathrm{C} = 0 \\ \sqrt{2}\mathrm{A} + \mathrm{B} - \sqrt{2}\mathrm{C} + \mathrm{D} = 4 \\ \mathrm{A} + \sqrt{2}\mathrm{B} + \mathrm{C} - \sqrt{2}\mathrm{D} = 0 \\ \mathrm{B}+\mathrm{D} = 0 \end{cases}$

From (1) and (4) we have $\mathrm{C} = -\mathrm{A}$ and $\mathrm{D} = -\mathrm{B}$, substitute in (2), $2\sqrt{2}\mathrm{A} = 4$, so $\mathrm{A} = \sqrt{2}$ then $\mathrm{C} = -\sqrt{2}$, substitute C and A in (3), to get $\sqrt{2} + \sqrt{2}\mathrm{B} - \sqrt{2} + \sqrt{2}\mathrm{B} = 0$, $2\sqrt{2}\mathrm{B} = 0$, then $\mathrm{D} = 0$, therefore **:** $\frac{4y^2}{(y^2-\sqrt{2}y+1)(y^2+\sqrt{2}y+1)} = \frac{\sqrt{2}y}{y^2-\sqrt{2}y+\mathbf{1}} - \frac{\sqrt{2}y}{y^2+\sqrt{2}y+1}$, then**:**

$$\int \frac{\sqrt{x+\sqrt{1+x^2}}}{1+x^2}dx = \int \frac{\sqrt{2}y}{y^2-\sqrt{2}y+1}dy - \int \frac{\sqrt{2}y}{y^2+\sqrt{2}y+1}dy$$

$$= \frac{1}{\sqrt{2}}\left(\int \frac{2y-\sqrt{2}}{y^2-\sqrt{2}y+1}dy + \int \frac{\sqrt{2}}{y^2-\sqrt{2}y+1}dy - \int \frac{2y+\sqrt{2}}{y^2+\sqrt{2}y+1}dy + \int \frac{\sqrt{2}}{y^2+\sqrt{2}y+1}dy\right)$$

$$= \frac{1}{\sqrt{2}}\left[\int \frac{(y^2-\sqrt{2}y+1)'}{y^2-\sqrt{2}y+1}dy + \int \frac{\sqrt{2}}{\left(y-\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}dy - \int \frac{(y^2+\sqrt{2}y+1)'}{y^2+\sqrt{2}y+1}dy + \int \frac{\sqrt{2}}{\left(y+\frac{1}{\sqrt{2}}\right)^2+\left(\frac{1}{\sqrt{2}}\right)^2}d\boldsymbol{y}\right]$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{y^2-\sqrt{2}y+1}{y^2+\sqrt{2}y+1}\right| + \sqrt{2}\tan^{-1}\left(\sqrt{2}\left(y - \frac{1}{\sqrt{2}}\right)\right) + \sqrt{2}\tan^{-1}\left(\sqrt{2}\left(y + \frac{1}{\sqrt{2}}\right)\right) + c$$

$$= \frac{1}{\sqrt{2}} \ln \left| \frac{x+\sqrt{1+x^2}+1-\sqrt{2}\sqrt{x+\sqrt{1+x^2}}}{x+\sqrt{1+x^2}+1+\sqrt{2}\sqrt{x+\sqrt{1+x^2}}} \right| + \sqrt{2}\tan^{-1}\left(\sqrt{2}\sqrt{x+\sqrt{1+x^2}}-1\right)$$

$$+\sqrt{2}\tan^{-1}\left(\sqrt{2}\sqrt{x+\sqrt{1+x^2}}+1\right) + c$$

148. $\int e^{-x}\ln(1+e^x)\,dx$ using integration by parts**:**

Let $u = \ln(1+e^x)$, then $u' = \frac{(1+e^x)'}{1+e^x} = \frac{e^x}{1+e^x}$ and let $v' = e^{-x}$, then $v = -e^{-x}$, then**:**

$$\int e^{-x}\ln(1+e^x)\,dx = -e^{-x}\ln(1+e^x) - \int \frac{e^x}{1+e^x}(-e^{-x})dx\,; \text{ then we get:}$$

$$\int e^{-x}\ln(1+e^x)\,dx = -e^{-x}\ln(1+e^x) - \int \frac{1}{1+e^x}dx$$

$$\int e^{-x}\ln(1+e^x)\,dx = -e^{-x}\ln(1+e^x) - \int \frac{1}{1+e^x}\times\frac{e^{-x}}{e^{-x}}dx$$

$$\int e^{-x}\ln(1+e^x)\,dx = -e^{-x}\ln(1+e^x) - \int \frac{e^{-x}}{1+e^{-x}}dx = -e^{-x}\ln(1+e^x) + \int \frac{-e^{-x}}{1+e^{-x}}dx$$

$$\int e^{-x}\ln(1+e^x)\,dx = -e^{-x}\ln(1+e^x) + \int \frac{(1+e^{-x})'}{1+e^{-x}}dx$$

$$\int e^{-x}\ln(1+e^x)\,dx = -e^{-x}\ln(1+e^x) + \ln(1+e^{-x}) + c = (1-e^{-x})\ln(1+e^{-x}) + c$$

149. $\mathrm{I} = \int_0^{+\infty} \frac{x\sqrt{x}}{(1+x^2)(1+a^2x^2)}dx$, let $y = \frac{1}{x}$ then $x = \frac{1}{y}$ and $dx = -\frac{1}{y^2}dy$, for $x = 0$, $y = +\infty$ and for $x = +\infty$, $y = 0$, then we get**:**

$$\mathrm{I} = \int_{+\infty}^{0} \frac{\frac{1}{y}\sqrt{\frac{1}{y}}}{\left(1+\left(\frac{1}{y}\right)^2\right)\left(1+a^2\left(\frac{1}{y}\right)^2\right)}\left(-\frac{1}{y^2}dy\right) = \int_0^{+\infty} \frac{\frac{1}{y}\sqrt{\frac{1}{y}}}{\left(1+\frac{1}{y^2}\right)\left(1+\frac{a^2}{y^2}\right)}.\frac{1}{y^2}dy, \text{ then we get:}$$

$$\mathrm{I} = \int_0^{+\infty} \frac{\sqrt{y}}{(1+y^2)(a^2+y^2)}dy = \frac{1}{a^2-1}\int_0^{+\infty} \frac{\sqrt{y}(a^2-1)}{(1+y^2)(a^2+y^2)}dy, \text{ then we get:}$$

$$\mathrm{I} = \frac{1}{a^2-1}\int_0^{+\infty} \frac{\sqrt{y}(a^2-1+y^2-y^2)}{(1+y^2)(a^2+y^2)}dy = \frac{1}{a^2-1}\int_0^{+\infty} \frac{\sqrt{y}[(1+y^2)-(a^2+y^2)]}{(1+y^2)(a^2+y^2)}dy, \text{ then we get:}$$

$$\mathrm{I} = \frac{1}{a^2-1}\int_0^{+\infty} \frac{\sqrt{y}(1+y^2)-\sqrt{y}(a^2+y^2)}{(1+y^2)(a^2+y^2)}dy = \frac{1}{a^2-1}\int_0^{+\infty}\left(\frac{\sqrt{y}}{1+y^2}-\frac{\sqrt{y}}{a^2+y^2}\right)dy$$

Evaluating $\mathbf{J} = \int_0^{+\infty} \frac{\sqrt{y}}{a^2+y^2}dy$**:**

$$\mathrm{J} = \int_0^{+\infty} \frac{\sqrt{y}}{a^2+y^2}dy = \frac{1}{a^2}\int_0^{+\infty} \frac{\sqrt{y}}{1+\left(\frac{y}{a}\right)^2}dy, \text{ let } y = a\sqrt{z}, \text{ then } dy = a\frac{1}{2\sqrt{z}}dz, \text{ then we get:}$$

$$\mathrm{J} = \frac{1}{a^2}\int_0^{+\infty} \frac{\sqrt{a}z^{\frac{1}{4}}}{1+z}.a\frac{1}{2\sqrt{z}}dz = \frac{1}{2\sqrt{a}}\int_0^{+\infty} \frac{z^{\frac{3}{4}-1}}{1+z}dz = \frac{1}{2\sqrt{a}}.\frac{\pi}{\sin\left(\frac{3\pi}{4}\right)} = \frac{\pi}{\sqrt{2a}}$$

So, we get J $= \int_0^{+\infty} \frac{\sqrt{y}}{a^2+y^2} dy = \frac{\pi}{\sqrt{2a}}$ and for $a = 1$, $\int_0^{+\infty} \frac{\sqrt{y}}{1+y^2} dy = \frac{\pi}{\sqrt{2}}$, therefore:

$$\mathrm{I} = \frac{1}{a^2-1}\left(\frac{\pi}{\sqrt{2a}} - \frac{\pi}{\sqrt{2}}\right) = \frac{1}{a^2-1}\cdot\frac{\pi}{\sqrt{2}}\left(1 - \frac{1}{\sqrt{a}}\right) = \frac{\pi}{(1+\sqrt{a})(1+a)\sqrt{2a}}$$

150. $\mathrm{I} = \int_0^{\pi} \frac{x}{1-\cos\alpha\sin x} dx$, let $u = \pi - x$, then $du = -dx$, for $x = 0$, $u = \pi$ and for $x = \pi$, $u = 0$, then we get:

$$\mathrm{I} = \int_{\pi}^{0} \frac{\pi-u}{1-\cos\alpha\sin(\pi-u)}(-du) = \int_0^{\pi} \frac{\pi-u}{1-\cos\alpha\sin u} du = \int_0^{\pi} \frac{\pi-x}{1-\cos\alpha\sin x} dx$$

$\mathrm{I} = \pi\int_0^{\pi} \frac{1}{1-\cos\alpha\sin x} dx - \int_0^{\pi} \frac{x}{1-\cos\alpha\sin x} dx = \pi\int_0^{\pi} \frac{1}{1-\cos\alpha\sin x} dx - \mathrm{I}$, then we get:

$2\mathrm{I} = \pi\int_0^{\pi} \frac{1}{1-\cos\alpha\sin x} dx$, but $\sin x = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$, then:

$$2\mathrm{I} = \pi\int_0^{\pi} \frac{1}{1-\cos\alpha\left(\frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right)} dx = \pi\int_0^{\pi} \frac{1+\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)-2\cos\alpha\tan\left(\frac{x}{2}\right)} dx$$

$2\mathrm{I} = \pi\int_0^{\pi} \frac{\sec^2\left(\frac{x}{2}\right)}{\tan^2\left(\frac{x}{2}\right)-2\cos\alpha\tan\left(\frac{x}{2}\right)+1} dx$, let $y = \tan\left(\frac{x}{2}\right)$, then $dy = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) dx$, for $x = 0$, $y = 0$ and for $x = \pi$, then $y = +\infty$, so we get:

$$\mathrm{I} = \frac{\pi}{2}\int_0^{+\infty} \frac{2}{y^2-2y\cos\alpha+1} dy = \pi\int_0^{+\infty} \frac{2}{y^2-2y\cos\alpha+\cos^2\alpha+(1-\cos^2\alpha)} dy$$

$$\mathrm{I} = \pi\int_0^{+\infty} \frac{2}{(y-\cos\alpha)^2+\sin^2\alpha} dy = \frac{\pi}{\sin\alpha}\left[\frac{y-\cos\alpha}{\sin\alpha}\right]_0^{+\infty} = \frac{\pi}{\sin\alpha}\left[\arctan(+\infty) - \arctan\left(-\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

Then: $\mathrm{I} = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} - \arctan(-\cot\alpha)\right]$, but $\arctan(-\beta) = -\arctan\beta$, so:

$$\mathrm{I} = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \arctan(\cot\alpha)\right] = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2} + \arctan\left(\tan\left(\frac{\pi}{2} - \alpha\right)\right)\right] = \frac{\pi}{\sin\alpha}\left(\frac{\pi}{2} + \frac{\pi}{2} - \alpha\right)$$

Therefore we get: $\mathrm{I} = \frac{\pi}{\sin\alpha}(\pi - \alpha)$

151. $\mathrm{I} = \int_{-1}^{1} \frac{\sin e}{1-2x\cos e+x^2} dx$

Let $t = \frac{x-\cos e}{\sin e}$, then $x = \cos e + t\sin e$ and $dx = \sin e\, dt$, for $x = 1$, $t = \frac{1-\cos e}{\sin e} = t_2$ and for $x = -1$, $t = \frac{-1-\cos e}{\sin e} = t_2$, then we get:

$$\mathrm{I} = \int_{t_1}^{t_2} \frac{\sin^2 e}{1-2(\cos e+t\sin e)\cos e+(\cos e+t\sin e)^2} dt$$

$\mathrm{I} = \int_{t_1}^{t_2} \frac{\sin^2 e}{1-2\cos^2 e-2t\sin e\cos e+\cos^2 e+2t\sin e\cos e+t^2\sin^2 e} dt$, then we get:

$$\mathrm{I} = \int_{t_1}^{t_2} \frac{\sin^2 e}{1-2\cos^2 e+\cos^2 e+t^2\sin^2 e} dt = \int_{t_1}^{t_2} \frac{\sin^2 e}{1-\cos^2 e+t^2\sin^2 e} dt = \int_{t_1}^{t_2} \frac{\sin^2 e}{\sin^2 e+t^2\sin^2 e} dt$$

$\mathrm{I} = \int_{t_1}^{t_2} \frac{\sin^2 e}{\sin^2 e(1+t^2)} dt = \int_{t_1}^{t_2} \frac{1}{1+t^2} dt = \arctan t_2 - \arctan t_1$, then we get:

$$\mathrm{I} = \arctan\left(\frac{1-\cos e}{\sin e}\right) - \arctan\left(\frac{-1-\cos e}{\sin e}\right) = \arctan\left(\frac{1-\cos e}{\sin e}\right) + \arctan\left(\frac{1+\cos e}{\sin e}\right)$$

Applying the double angle formula we get:

$$\mathrm{I} = \arctan\left[\frac{\frac{1-\cos e}{\sin e}+\frac{1+\cos e}{\sin e}}{1-\left(\frac{1-\cos e}{\sin e}\right)\left(\frac{1+\cos e}{\sin e}\right)}\right] = \arctan\left[\frac{\frac{2}{\sin e}}{1-\frac{1-\cos^2 e}{\sin^2 e}}\right] = \arctan\left[\frac{\frac{2}{\sin e}}{1-\frac{\sin^2 e}{\sin^2 e}}\right]$$

$$\mathrm{I} = \arctan +\infty = \frac{\pi}{2}$$

152. $$\mathrm{I} = \int_{-a}^{a} \frac{dx}{\sqrt[3]{(a-x)(a^2-x^2)}}$$

$$\mathrm{I} = \int_{-a}^{a} \frac{dx}{\sqrt[3]{(a-x)(a^2-x^2)}} = \int_{-a}^{a} \frac{dx}{\sqrt[3]{(a-x)(a-x)(a+x)}} = \int_{-a}^{a} \frac{dx}{\sqrt[3]{(a-x)^2(a+x)}}$$

$$\mathrm{I} = \int_{-a}^{a} \frac{dx}{(a-x)^{\frac{2}{3}}(a+x)^{\frac{1}{3}}}$$

Let $x = a\cos 2y \Longrightarrow dx = -2a\sin 2y dy$; for $x = -a \Longrightarrow \cos 2y = -1 \Longrightarrow 2y = \pi \Longrightarrow y = \frac{\pi}{2}$

& for $x = a \Longrightarrow \cos 2y = 1 \Longrightarrow 2y = 0 \Longrightarrow y = 0$; then we get:

$$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{-2a\sin 2y}{(a-a\cos 2y)^{\frac{2}{3}}(a+a\cos 2y)^{\frac{1}{3}}}dy = \int_{0}^{\frac{\pi}{2}} \frac{2a\sin 2y}{a^{\frac{2}{3}}(1-\cos 2y)^{\frac{2}{3}}a^{\frac{1}{3}}(1+\cos 2y)^{\frac{1}{3}}}dy$$

$$\mathrm{I} = \int_{0}^{\frac{\pi}{2}} \frac{2a(2\sin y\cos y)}{a(2\sin^2 y)^{\frac{2}{3}}(2\cos^2 y)^{\frac{1}{3}}}dy = 4\int_{0}^{\frac{\pi}{2}} \frac{\sin y\cos y}{2^{\frac{2}{3}}\sin^{\frac{4}{3}} y\, 2^{\frac{1}{3}}\cos^{\frac{2}{3}} y}dy = 2\int_{0}^{\frac{\pi}{2}} \frac{\sin y\cos y}{\sin^{\frac{4}{3}} y\cos^{\frac{2}{3}} y}dy$$

$$\mathrm{I} = 2\int_{0}^{\frac{\pi}{2}} \sin^{-\frac{1}{3}} y\cos^{\frac{1}{3}} y\,dy = 2\int_{0}^{\frac{\pi}{2}} \sin^{2\left(\frac{1}{3}\right)-1} y\cos^{2\left(\frac{2}{3}\right)-1} y\,dy = \mathrm{B}\left(\frac{1}{3};\frac{2}{3}\right);\ \text{then we get:}$$

$$\mathrm{I} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)} = \Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right) = \frac{\pi}{\sin\left(\frac{1}{3}\pi\right)} = \frac{2\pi}{\sqrt{3}}$$

153. $$\int \frac{1}{\sqrt[6]{x^6+1}}dx = \int \frac{1}{(x^6+1)^{\frac{1}{6}}}dx = \int \frac{1}{\left[x^6(1+x^{-6})\right]^{\frac{1}{6}}}dx = \int \frac{1}{x(1+x^{-6})^{\frac{1}{6}}}dx$$

$= \int \frac{1}{x^{-6}(1+x^{-6})^{\frac{1}{6}}}x^{-7}dx$, let $y = (1+x^{-6})^{\frac{1}{6}} \Longrightarrow x^{-6} = y^6-1 \Longrightarrow -x^{-7}dx = y^5 dy$, then:

$$\int \frac{1}{\sqrt[6]{x^6+1}}dx = \int \frac{1}{(y^6-1)y}(-y^5)dy = \int \frac{y^4}{1-y^6}dy = \int \frac{y^4}{(1-y^3)(1+y^3)}dy$$

$$= \frac{1}{2}\int \frac{2y^4}{(1-y^3)(1+y^3)}dy = \frac{1}{2}\int \frac{y+y^4-y+y^4}{(1-y^3)(1+y^3)}dy = \frac{1}{2}\int \frac{y(1+y^3)-y(1-y^3)}{(1-y^3)(1+y^3)}dy$$

$$= \frac{1}{2}\int \left(\frac{y}{1-y^3}-\frac{y}{1+y^3}\right)dy = \frac{1}{2}\int \left[\frac{y}{(1-y)(1+y+y^2)}-\frac{y}{(1+y)(1-y+y^2)}\right]dy$$

But : $\frac{y}{(1-y)(1+y+y^2)} = \frac{\mathrm{A}}{1-y}+\frac{\mathrm{B}y+\mathrm{C}}{1+y+y^2}$, so $y = \mathrm{A}(1+y+y^2)+(\mathrm{B}y+\mathrm{C})(1-y)$

For $y=1$, we get $1=\mathrm{A}(1+1+1)$, then $\mathrm{A}=\frac{1}{3}$, for $y=0$, we get $0=\mathrm{C}+\frac{1}{3}$, so $\mathrm{C}=-\frac{1}{3}$ and for $x=-1$, $-1=\frac{1}{3}+2\left(-\mathrm{B}-\frac{1}{3}\right)$, then $\mathrm{B}=\frac{1}{3}$, so we get:

$$\frac{y}{(1-y)(1+y+y^2)}=\frac{\frac{1}{3}}{1-y}+\frac{\frac{1}{3}y-\frac{1}{3}}{1+y+y^2}=\frac{1}{3}\left(\frac{1}{1-y}+\frac{y-1}{1+y+y^2}\right)$$

Similarly using the same method we can get:

$$\frac{y}{(1+y)(1-y+y^2)}=\frac{\frac{1}{3}}{1+y}+\frac{-\frac{1}{3}y-\frac{1}{3}}{1-y+y^2}=\frac{1}{3}\left(\frac{1}{1+y}-\frac{y+1}{1-y+y^2}\right)$$

Then $\int\frac{1}{\sqrt[6]{x^6+1}}dx=\frac{1}{6}\left[\int\left(\frac{1}{1-y}+\frac{y-1}{y^2+y+1}\right)dx\right]+\frac{1}{6}\left[\int\left(\frac{1}{1+y}-\frac{y+1}{y^2-y+1}\right)dx\right]$

$$=\frac{1}{6}\left[-\ln|1-y|+\frac{1}{2}\int\frac{2y+1}{y^2+y+1}dy-\frac{3}{2}\int\frac{1}{\left(y+\frac{1}{2}\right)^2+\frac{2}{4}}dy\right]$$

$$+\frac{1}{6}\left[\ln|1+y|-\frac{1}{2}\int\frac{2y-1}{y^2-y+1}dy-\frac{3}{2}\int\frac{1}{\left(y-\frac{1}{2}\right)^2+\frac{3}{4}}dy\right]$$

$$=-\frac{1}{6}\ln|1-y|+\frac{1}{12}\ln|y^2+y+1|-\frac{1}{4}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(y+\frac{1}{2}\right)\right)+\frac{1}{6}\ln|1+y|$$

$$-\frac{1}{12}\ln|y^2-y+1|-\frac{1}{4}\cdot\frac{2}{\sqrt{3}}\arctan\left(\frac{2}{\sqrt{3}}\left(y-\frac{1}{2}\right)\right)+c$$

$$=\frac{1}{6}\ln\left|\frac{1+y}{1-y}\right|+\frac{1}{12}\ln\left|\frac{y^2+y+1}{y^2-y+1}\right|-\frac{1}{2\sqrt{3}}\arctan\left(\frac{2y+1}{\sqrt{3}}\right)-\frac{1}{2\sqrt{3}}\arctan\left(\frac{2y-1}{\sqrt{3}}\right)+c$$

$$=\frac{1}{6}\ln\left|\frac{x+(x^6+1)^{\frac{1}{6}}}{x-(x^6+1)^{\frac{1}{6}}}\right|+\frac{1}{12}\ln\left|\frac{(x^6+1)^{\frac{1}{3}}+x(x^6+1)^{\frac{1}{6}}+x^2}{(x^6+1)^{\frac{1}{3}}-x(x^6+1)^{\frac{1}{6}}+x^2}\right|-\frac{1}{2\sqrt{3}}\arctan\left(\frac{2(x^6+1)^{\frac{1}{6}}+x}{x\sqrt{3}}\right)$$

$$-\frac{1}{2\sqrt{3}}\arctan\left(\frac{2(x^6+1)^{\frac{1}{6}}-x}{x\sqrt{3}}\right)+c$$

154. $\mathrm{I}=\int_0^{\pi}\frac{\sin^2x}{1+2a\cos x+a^2}dx$, where $|a|\le 1$

$$\mathrm{I}=\int_0^{\pi}\frac{\sin^2x}{1+2a\cos x+a^2}dx=\int_0^{\pi}\frac{1-\cos^2x}{1+2a\cos x+a^2}dx=\frac{1}{4a^2}\int_0^{\pi}\frac{4a^2(1-\cos^2x)}{1+2a\cos x+a^2}dx$$

$$=\frac{1}{4a^2}\int_0^{\pi}\frac{4a^2(1-\cos^2x)}{(1+a^2)+2a\cos x}dx=\frac{1}{4a^2}\int_0^{\pi}\frac{[(1+a^2)^2-4a^2\cos^2x]+[4a^2-(1+a^2)^2]}{(1+a^2)+2a\cos x}dx$$

$$\mathrm{I}=\frac{1}{4a^2}\int_0^{\pi}\frac{\big((1+a^2)+2a\cos x\big)\big((1+a^2)-2a\cos x\big)-(1-a^2)^2}{(1+a^2)+2a\cos x}dx$$

$$\mathrm{I}=\frac{1}{4a^2}\int_0^{\pi}[(1+a^2)-2a\cos x]dx-\frac{(1-a^2)^2}{4a^2}\int_0^{\pi}\frac{1}{(1+a^2)+2a\cos x}dx$$

$$\mathrm{I} = \frac{1}{4a^2}[(1+a^2)x - 2a\sin x]_0^{\pi} - \frac{(1-a^2)^2}{4a^2}\int_0^{\pi} \frac{1}{(1+a^2)+2a\cos x}dx$$

$$\mathrm{I} = \frac{(1+a^2)\pi}{4a^2} - \frac{(1-a^2)^2}{4a^2}\int_0^{\pi} \frac{1}{(1+a^2)+2a\cos x}dx$$

For $\int \frac{1}{(1+a^2)+2a\cos x}dx$; let $t = \tan\left(\frac{x}{2}\right) \Rightarrow \cos x = \frac{1-t^2}{1+t^2}$ & $dx = \frac{2dt}{1+t^2}$ then:

$$\int \frac{1}{(1+a^2)+2a\cos x}dx = \int \frac{1}{(1+a^2)+2a\left(\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}dt$$

$$\int \frac{1}{(1+a^2)+2a\cos x}dx = \int \frac{2}{(1+a^2)(1+t^2)+2a(1-t^2)}dt$$

$$\int \frac{1}{(1+a^2)+2a\cos x}dx = \int \frac{2}{1+t^2+a^2+a^2t^2+2a-2at^2}dt$$

$$\int \frac{1}{(1+a^2)+2a\cos x}dx = \int \frac{2}{a^2+2a+1+t^2(1-2a+a^2)}dt$$

$$\int \frac{1}{(1+a^2)+2a\cos x}dx = \int \frac{2}{(a+1)^2+(a-1)^2t^2}dt = \frac{2}{(a-1)^2}\int \frac{1}{\left(\frac{a+1}{a-1}\right)^2+t^2}dt$$

$$\int \frac{1}{(1+a^2)+2a\cos x}dx = \frac{2}{(a-1)^2} \times \frac{1}{\frac{a+1}{a-1}}\tan^{-1}\left(\frac{t}{\frac{a+1}{a-1}}\right) + c$$

$$\int \frac{1}{(1+a^2)+2a\cos x}dx = \frac{2}{a^2-1}\tan^{-1}\left(\frac{a-1}{a+1}t\right) + c = \frac{2}{a^2-1}\tan^{-1}\left(\frac{a-1}{a+1}\tan\frac{x}{2}\right) + c$$

$$\int_0^{\pi} \frac{1}{(1+a^2)+2a\cos x}dx = \left[\frac{2}{a^2-1}\tan^{-1}\left(\frac{a-1}{a+1}\tan\frac{x}{2}\right)\right]_0^{\pi};\ \text{then we get:}$$

$$\mathrm{I} = \frac{(1+a^2)\pi}{4a^2} - \frac{(1-a^2)^2}{4a^2}\left[\frac{2}{a^2-1}\tan^{-1}\left(\frac{a-1}{a+1}\tan\frac{x}{2}\right)\right]_0^{\pi} = \frac{(1+a^2)\pi}{4a^2} - \frac{(a^2-1)\pi}{4a^2} = \frac{\pi}{2a^2}$$

155. $\mathrm{I} = \int_0^{\frac{\pi}{4}} \sqrt{\tan x}\sqrt{1-\tan x}\,dx$, let $t = \tan x$, then $dt = (1+\tan^2 x)dx = (1+t^2)dx$

for $x = 0$, $t = 0$ and for $x = \frac{\pi}{4}$, $t = 1$, then we get**:**

$\mathrm{I} = \int_0^1 \sqrt{t}\sqrt{1-t}\frac{dt}{1+t^2} = \int_0^1 \frac{\sqrt{t-t^2}}{1+t^2}dt$, now we will use Euler's substitution of the third kind, then

let $\sqrt{t-t^2} = tz$ $\left(z = \frac{\sqrt{t-t^2}}{t}\right)$, then $t = \frac{1}{1+z^2}$ and $dt = -\frac{2z}{\left(1+z^2\right)^2}dz$ and

$\begin{cases} \text{for } t=0 & z=+\infty \\ \text{for } t=1 & z=0 \end{cases}$, then we get**:**

$$\mathrm{I} = \int_{+\infty}^{0} \frac{\frac{z}{1+z^2}}{1+\left(\frac{1}{1+z^2}\right)^2}\left(-\frac{2z}{\left(1+z^2\right)^2}\right)dz = 2\int_0^{+\infty} \frac{z^2}{(z^4+2z^2+2)(z^2+1)}dz$$

$$\mathrm{I} = 2\int_0^{+\infty}\left(\frac{z^2+2}{z^4+2z^2+2} - \frac{1}{1+z^2}\right)dz = 2\int_0^{+\infty}\frac{z^2+2}{z^4+2z^2+2}dz - 2[\arctan z]_0^{+\infty}$$

$\mathrm{I} = 2\int_0^{+\infty} \frac{z^2+2}{z^4+2z^2+2}dz - 2\left(\frac{\pi}{2}\right) = 2\int_0^{+\infty} \frac{z^2+2}{z^4+2z^2+2}dz - \pi = 2\mathrm{J} - \pi$

Let $\mathrm{J} = \int_0^{+\infty} \frac{z^2+2}{z^4+2z^2+2}dz$

First: Let us take $z = \sqrt[4]{2}u$, then $dz = \sqrt[4]{2}du$, then we get:

$\mathrm{J} = \int_0^{+\infty} \frac{\left(\sqrt[4]{u}\right)^2+2}{\left(\sqrt[4]{u}\right)^4+2\left(\sqrt[4]{u}\right)^2+2}dz = \sqrt[4]{2}\int_0^{+\infty} \frac{\sqrt{2}u^2+2}{2u^4+2\sqrt{2}u^2+2}du$ **… (1)**

Second: Let us take $z = \frac{\sqrt[4]{u}}{u}$, then $dz = -\frac{\sqrt[4]{u}}{u^2}du$, then we get:

$\mathrm{J} = \int_{+\infty}^{0} \frac{\left(\frac{\sqrt[4]{u}}{u}\right)^2+2}{\left(\frac{\sqrt[4]{u}}{u}\right)^4+2\left(\frac{\sqrt[4]{u}}{u}\right)^2+2}\left(-\frac{\sqrt[4]{u}}{u^2}du\right) = \sqrt[4]{2}\int_0^{+\infty} \frac{\frac{\sqrt{2}}{u^2}+2}{\frac{2}{u^2}+2\sqrt{2}+2u^2}du$, then we get:

$\mathrm{J} = \sqrt[4]{2}\int_0^{+\infty} \frac{\sqrt{2}+2u^2}{2u^4+2\sqrt{2}u^2+2}du$ **… (2)** then by adding (1) and (2) we get:

$2\mathrm{J} = \sqrt[4]{2}\int_0^{+\infty} \frac{\sqrt{2}u^2+2}{2u^4+2\sqrt{2}u^2+2}du + \sqrt[4]{2}\int_0^{+\infty} \frac{\sqrt{2}+2u^2}{2u^4+2\sqrt{2}u^2+2}du = \sqrt[4]{2}\int_0^{+\infty} \frac{u^2(2+\sqrt{2})+(2+\sqrt{2})}{2u^4+2\sqrt{2}u^2+2}du$

$2\mathrm{J} = \sqrt[4]{2}(2+\sqrt{2})\int_0^{+\infty} \frac{u^2+1}{2u^4+2\sqrt{2}u^2+2}du = \frac{2+\sqrt{2}}{2}\sqrt[4]{2}\int_0^{+\infty} \frac{u^2+1}{u^4+\sqrt{2}u^2+1}du$, then we get:

$\mathrm{J} = \frac{2+\sqrt{2}}{4}\sqrt[4]{2}\int_0^{+\infty} \frac{u^2+1}{u^4+\sqrt{2}u^2+1}du = \frac{2+\sqrt{2}}{4\sqrt[4]{2}}\sqrt{2}\int_0^{+\infty} \frac{u^2+1}{u^4+\sqrt{2}u^2+1}du$

$\mathrm{J} = \frac{1+\sqrt{2}}{2\sqrt[4]{2}}\int_0^{+\infty} \frac{u^2+1}{u^4+\sqrt{2}u^2+1}du = \frac{1+\sqrt{2}}{2\sqrt[4]{2}}\int_0^{+\infty} \frac{1+\frac{1}{u^2}}{u^2+\sqrt{2}+\frac{1}{u^2}}du = \frac{1+\sqrt{2}}{2\sqrt[4]{2}}\int_0^{+\infty} \frac{1+\frac{1}{u^2}}{u^2-2+\frac{1}{u^2}+2+\sqrt{2}}du$

$\mathrm{J} = \frac{1+\sqrt{2}}{2\sqrt[4]{2}}\int_0^{+\infty} \frac{1+\frac{1}{u^2}}{\left(u-\frac{1}{u}\right)^2+2+\sqrt{2}}du$, let $v = u - \frac{1}{u}$, then $dv = \left(1+\frac{1}{u^2}\right)du$, so

$\mathrm{J} = \frac{1+\sqrt{2}}{2\sqrt[4]{2}}\int_{-\infty}^{+\infty} \frac{dv}{v^2+\left(\sqrt{2+\sqrt{2}}\right)^2} = \frac{1+\sqrt{2}}{2\sqrt[4]{2}}\left[\frac{1}{\sqrt{2+\sqrt{2}}}\arctan\left(\frac{v}{\sqrt{2+\sqrt{2}}}\right)\right]_0^{+\infty} = \frac{(1+\sqrt{2})\pi}{2\sqrt[4]{2}\sqrt{2+\sqrt{2}}}$

Then $\mathrm{J} = \frac{(1+\sqrt{2})\pi}{2\sqrt{\sqrt{2}(2+\sqrt{2})}} = \frac{(1+\sqrt{2})\pi}{2\sqrt{2(1+\sqrt{2})}}$, but we have $\mathrm{I} = 2\mathrm{J} - \pi$, then we get:

$\mathrm{I} = 2\left(\frac{(1+\sqrt{2})\pi}{2\sqrt{2(1+\sqrt{2})}}\right) - \pi = \frac{(1+\sqrt{2})\pi}{\sqrt{2(1+\sqrt{2})}} - \pi = \pi\left(\frac{(1+\sqrt{2})\pi}{\sqrt{2(1+\sqrt{2})}} - 1\right) = \pi\left(\sqrt{\frac{1+\sqrt{2}}{2}} - 1\right)$

156. $\mathrm{I} = \int_0^1 \ln^x\left(\frac{1}{a}\right).\ln^{1-x}\left(\frac{1}{b}\right)dx$

We know that $\int_0^1 u^x\,dx = \left[\frac{1}{\ln u}u^x\right]_0^1 = \frac{u-1}{\ln u}$, then put $u = \frac{a}{b}$, we get:

$\int_0^1 \left(\frac{a}{b}\right)^x dx = \frac{\frac{a}{b}-1}{\ln\left(\frac{a}{b}\right)} = \frac{\frac{a-b}{b}}{\ln\left(\frac{a}{b}\right)}$, so we get: $\int_0^1 \left(\frac{a}{b}\right)^x dx = \frac{a-b}{b\ln\left(\frac{a}{b}\right)}$, multiplying by b, we get:

$b\int_0^1 \left(\frac{a}{b}\right)^x dx = \frac{a-b}{\ln\left(\frac{a}{b}\right)} = \frac{a-b}{\ln a - \ln b}$. So, now we can write:

$$\ln\left(\frac{1}{b}\right)\int_0^1 \left(\frac{\ln\left(\frac{1}{a}\right)}{\ln\left(\frac{1}{b}\right)}\right)^x dx = \frac{\ln\left(\frac{1}{a}\right) - \ln\left(\frac{1}{b}\right)}{\ln\ln\left(\frac{1}{a}\right) - \ln\ln\left(\frac{1}{b}\right)}$$

$$\ln\left(\frac{1}{b}\right)\int_0^1 \left(\frac{\ln\left(\frac{1}{a}\right)}{\ln\left(\frac{1}{b}\right)}\right)^x dx = \ln\left(\frac{1}{a}\right)\int_0^1 \frac{\ln^x\left(\frac{1}{a}\right)}{\ln^x\left(\frac{1}{b}\right)}dx = \int_0^1 \frac{\ln^x\left(\frac{1}{a}\right)}{\ln^{x-1}\left(\frac{1}{b}\right)}dx = \int_0^1 \ln^x\left(\frac{1}{a}\right).\ln^{1-x}\left(\frac{1}{b}\right)dx$$

Therefore; we get $\int_0^1 \ln^x\left(\frac{1}{a}\right).\ln^{1-x}\left(\frac{1}{b}\right)dx = \frac{\ln\left(\frac{1}{a}\right) - \ln\left(\frac{1}{b}\right)}{\ln\ln\left(\frac{1}{a}\right) - \ln\ln\left(\frac{1}{b}\right)}$

157. $\mathrm{I} = \int_a^b \ln\left(\frac{\left(1+\frac{x}{a}\right)^{x^{-1}e^{\frac{b}{x}}}}{\left(1+\frac{b}{x}\right)^{x^{-1}e^{\frac{x}{a}}}}\right)dx = \int_a^b \left[\ln\left(\left(1+\frac{x}{a}\right)^{x^{-1}e^{\frac{b}{x}}}\right) - \ln\left(\left(1+\frac{b}{x}\right)^{x^{-1}e^{\frac{x}{a}}}\right)\right]dx$

$\mathrm{I} = \int_a^b x^{-1}e^{\frac{b}{x}}\ln\left(1+\frac{x}{a}\right)dx - \int_a^b x^{-1}e^{\frac{x}{a}}\ln\left(1+\frac{b}{x}\right)dx$, then we get:

$\mathrm{I} = \int_a^b \frac{e^{\frac{b}{x}}}{x}\ln\left(1+\frac{x}{a}\right)dx - \int_a^b \frac{e^{\frac{x}{a}}}{x}e^{\frac{x}{a}}\ln\left(1+\frac{b}{x}\right)dx = \mathrm{I}_1 - \mathrm{I}_2$, now let us work on I_1:

$\mathrm{I}_1 = \int_a^b \frac{e^{\frac{b}{x}}}{x}\ln\left(1+\frac{x}{a}\right)dx$, let $y = \frac{ab}{x}$, then $x = \frac{ab}{y}$ and $dx = -\frac{ab}{y^2}dy$, for $x = a$, $y = \frac{ab}{a} = b$ and for $x = b$, $y = \frac{ab}{b} = a$, then we get:

$\mathrm{I}_1 = \int_b^a \frac{e^{\frac{b}{\frac{ab}{y}}}}{\frac{ab}{y}}\ln\left(1+\frac{\frac{ab}{y}}{a}\right)\left(-\frac{ab}{y^2}dy\right) = \int_a^b \frac{y}{ab}e^{\frac{y}{a}}\ln\left(1+\frac{b}{y}\right)\frac{ab}{y^2}dy = \int_a^b \frac{1}{y}e^{\frac{y}{a}}\ln\left(1+\frac{b}{y}\right)dy$

$\mathrm{I}_1 = \int_a^b \frac{e^{\frac{y}{a}}}{y}e^{\frac{y}{a}}\ln\left(1+\frac{b}{y}\right)dy = \int_a^b \frac{e^{\frac{x}{a}}}{x}e^{\frac{x}{a}}\ln\left(1+\frac{b}{x}\right)dx = \mathrm{I}_2$.

Therefore $\mathrm{I}_1 = \mathrm{I}_2$ and so $\mathrm{I}_1 - \mathrm{I}_2 = 0$, hence $\mathrm{I} = \int_a^b \ln\left(\frac{\left(1+\frac{x}{a}\right)^{x^{-1}e^{\frac{b}{x}}}}{\left(1+\frac{b}{x}\right)^{x^{-1}e^{\frac{x}{a}}}}\right)dx = 0$

158. **First Method:** $\int \frac{x^3}{(9-x^2)\sqrt{9-x^2}}dx$. Let $x = 3\sin\theta$, then $dx = 3\cos\theta\, d\theta$, then we get:

$\int \frac{x^3}{(9-x^2)\sqrt{9-x^2}}dx = \int \frac{(3\sin\theta)^3}{\left(9-(3\sin\theta)^2\right)\sqrt{9-(3\sin\theta)^2}}3\cos\theta\, d\theta$, then we get:

$$\int \frac{x^3}{(9-x^2)\sqrt{9-x^2}}dx = \int \frac{27\sin^3\theta}{9(1-\sin^2\theta)\sqrt{9(1-\sin^2\theta)}}3\cos\theta\, d\theta$$

$$= \int \frac{27\sin^3\theta}{9\cos^2\theta\sqrt{9\cos^2\theta}}3\cos\theta\, d\theta = \int \frac{3\sin^3\theta}{\cos^2\theta(3\cos\theta)}3\cos\theta\, d\theta = 3\int \frac{\sin^3\theta}{\cos^2\theta}d\theta$$

$$= 3\int \frac{\sin^2\theta}{\cos^2\theta}\sin\theta\, d\theta = 3\int \frac{1-\cos^2\theta}{\cos^2\theta}\sin\theta\, d\theta = 3\int \frac{\cos^2\theta-1}{\cos^2\theta}(-\sin\theta)d\theta$$

Let $y = \cos\theta$, then $dy = -\sin\theta\, d\theta$, then we get:

$$\int \frac{x^3}{(9-x^2)\sqrt{9-x^2}}dx = 3\int \frac{y^2-1}{y^2}dy = 3\int \left(1-\frac{1}{y^2}\right)dy = 3\left(y+\frac{1}{y}\right)+c$$

$$\int \frac{x^3}{(9-x^2)\sqrt{9-x^2}}dx = 3\left(\cos\theta+\frac{1}{\cos\theta}\right)+c = 3\left(\frac{\sqrt{9-x^2}}{3}+\frac{3}{\sqrt{9-x^2}}\right)+c, \text{ therefore, we get:}$$

$$\int \frac{x^3}{(9-x^2)\sqrt{9-x^2}}dx = \frac{18-x^2}{\sqrt{9-x^2}}+c$$

Second Method: $\int \frac{x^3}{(9-x^2)\sqrt{9-x^2}}dx = \int \frac{x}{(9-x^2)^{\frac{3}{2}}}x^2dx$, using integration by parts:

Let $u = x^2$, then $u' = 2x$ and let $v' = \frac{x}{(9-x^2)^{\frac{3}{2}}}$, then $v = \frac{1}{\sqrt{9-x^2}}$, then we get:

$$\int \frac{x^3}{(9-x^2)\sqrt{9-x^2}}dx = \frac{x^2}{\sqrt{9-x^2}} - \int \frac{2x}{\sqrt{9-x^2}}dx = \frac{x^2}{\sqrt{9-x^2}} + 2\sqrt{9-x^2}+c = \frac{18-x^2}{\sqrt{9-x^2}}+c$$

159. **First Method:** $\int \frac{x^5}{\sqrt{1+x^2}}dx$. Let $x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$, then we get:

$$\int \frac{x^5}{\sqrt{1+x^2}}dx = \int \frac{\tan^5\theta}{\sqrt{1+\tan^2\theta}}\sec^2\theta\, d\theta = \int \frac{\tan^5\theta}{\sqrt{\sec^2\theta}}\sec^2\theta\, d\theta = \int \frac{\tan^5\theta}{\sec\theta}\sec^2\theta\, d\theta$$

$$= \int \tan^5\theta\sec\theta\, d\theta = \int (\tan^2\theta)^2.\tan\theta.\sec\theta\, d\theta = \int (\sec^2\theta-1)^2\tan\theta\sec\theta\, d\theta$$

$$= \int (\sec^4\theta - 2\sec^2\theta+1)\tan\theta\sec\theta\, d\theta = \int (\sec^4\theta-2\sec^2\theta+1)(\sec\theta)'d\theta$$

$= \frac{1}{5}\sec^5\theta - \frac{2}{3}\sec^3\theta + \sec\theta + c$, but $\tan\theta = x$, then $\sec^2\theta - 1 = x$, then $\sec\theta = \sqrt{1+x^2}$,

so: $\int \frac{x^5}{\sqrt{1+x^2}}dx = \frac{1}{5}\left(\sqrt{1+x^2}\right)^5 - \frac{2}{3}\left(\sqrt{1+x^2}\right)^3 + \sqrt{1+x^2}+c$

$$\int \frac{x^5}{\sqrt{1+x^2}}dx = \left(\frac{\sqrt{1+x^2}}{15}\right)\left[3\left(\sqrt{1+x^2}\right)^4 - 10\left(\sqrt{1+x^2}\right)^2+15\right]+c, \text{ then:}$$

$$\int \frac{x^5}{\sqrt{1+x^2}}dx = \left(\frac{\sqrt{1+x^2}}{15}\right)[3(1+x^2)^2 - 10(1+x^2)+15]+c$$

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \left(\frac{\sqrt{1+x^2}}{15}\right)[3 + 6x^2 + 3x^4 - 10 - 10x^2 + 15] + c$, therefore we get:

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \left(\frac{\sqrt{1+x^2}}{15}\right)(3x^4 - 4x^2 + 8) + c$

Second Method: $\int \frac{x^5}{\sqrt{1+x^2}} dx$. Let $x = \sinh t \Rightarrow dx = \cosh t\, dt$, then we get:

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \int \frac{\sinh^5 t}{\sqrt{1+\sinh^2 t}} \cosh t\, dt = \int \frac{\sinh^5 t}{\sqrt{\cosh^2 t}} \cosh t\, dt = \int \frac{\sinh^5 t}{\cosh t} \cosh t\, dt$

$= \int \sinh^5 t\, dt = \int \sinh^4 t \,.\sinh t\, dt = \int (\sinh^2 t)^2 . \sinh t\, dt = \int (\cosh^2 t - 1)^2 . \sinh t\, dt$

$= \int (\cosh^4 t - 2\cosh^2 t + 1) \sinh t\, dt = \int (\cosh^4 t - 2\cosh^2 t + 1)(\cosh t)' dt$

$= \frac{1}{5}\cosh^5 t - \frac{2}{3}\cosh^3 t + \cosh t + c = \frac{1}{15}\cosh t\, [3(\cosh^2 t)^2 - 10\cosh^2 t + 15] + c$

With $x = \sinh t$, then $\cosh t = \sqrt{1 + \sinh^2 t} = \sqrt{1 + x^2}$, then:

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \frac{1}{15}\cosh t\, [3(1 + x^2)^2 - 10(1 + x^2) + 15] + c$

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \left(\frac{\sqrt{1+x^2}}{15}\right)(3 + 6x^2 + 3x^4 - 10 - 10x^2 + 15) + c$, , therefore we get:

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \left(\frac{\sqrt{1+x^2}}{15}\right)(3x^4 - 4x^2 + 8) + c$

Third Method (Integration By Parts) : $\int \frac{x^5}{\sqrt{1+x^2}} dx = \int x^4 . \frac{x}{\sqrt{1+x^2}} dx$

Let $u = x^4$, then $u' = 4x^3$ and let $v' = \frac{x}{\sqrt{1+x^2}}$ then $v = \sqrt{1 + x^2}$, then we get:

$\int \frac{x^5}{\sqrt{1+x^2}} dx = x^4\sqrt{1 + x^2} - 4\int x^3\sqrt{1 + x^2} dx = x^4\sqrt{1 + x^2} - 4\int x(x^2 + 1 - 1)\sqrt{1 + x^2} dx$

$\int \frac{x^5}{\sqrt{1+x^2}} dx = x^4\sqrt{1 + x^2} - 2\int \left[2x(x^2 + 1)^{\frac{3}{2}} - 2x(x^2 + 1)^{\frac{1}{2}}\right] dx$

$\int \frac{x^5}{\sqrt{1+x^2}} dx = x^4\sqrt{1 + x^2} - 2\int \left[(x^2 + 1)' . (x^2 + 1)^{\frac{3}{2}} - (x^2 + 1)' . (x^2 + 1)^{\frac{1}{2}}\right] dx$

$\int \frac{x^5}{\sqrt{1+x^2}} dx = x^4\sqrt{1 + x^2} - 2\left[\frac{2}{5}(x^2 + 1)^{\frac{5}{2}} - \frac{2}{3}(x^2 + 1)^{\frac{3}{2}}\right] + c$

$\int \frac{x^5}{\sqrt{1+x^2}} dx = x^4\sqrt{1 + x^2} - \frac{4}{5}(x^2 + 1)^{\frac{5}{2}} + \frac{4}{3}(x^2 + 1)^{\frac{3}{2}} + c$

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \frac{1}{15}\sqrt{1 + x^2}[15x^4 - 12(x^2 + 1)^2 + 20(x^2 + 1)] + c$

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \frac{1}{15}\sqrt{1+x^2}[15x^4 - 12x^4 - 24x^2 - 12 + 20x^2 + 20] + c$, therefore we get:

$\int \frac{x^5}{\sqrt{1+x^2}} dx = \left(\frac{\sqrt{1+x^2}}{15}\right)(3x^4 - 4x^2 + 8) + c$

160. $\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}\int \ln\left[\left(\sqrt{1+x} - \sqrt{1-x}\right)\right]^2 dx$

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}\int \ln\left(1 + x - 2\sqrt{1-x}\sqrt{1+x} + 1 - x\right) dx$

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}\int \ln\left(2 - 2\sqrt{1-x^2}\right) dx$

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}\int \ln 2\, dx + \frac{1}{2}\int \ln\left(1 - \sqrt{1-x^2}\right)dx$, then we get:

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}x\ln 2 + \frac{1}{2}\int \ln\left(1 - \sqrt{1-x^2}\right)dx$, using integration by parts:

let $u = \ln\left(1 - \sqrt{1-x^2}\right)$, then $u' = \frac{x}{\left(1-\sqrt{1-x^2}\right)\sqrt{1-x^2}}$ and let $v' = 1$, so $v = x$, then we get:

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}x\ln 2 + \frac{1}{2}x\ln\left(1 - \sqrt{1-x^2}\right) - \frac{1}{2}\int \frac{x^2}{\left(1-\sqrt{1-x^2}\right)\sqrt{1-x^2}} dx$

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}x\ln 2 + \frac{1}{2}x\ln\left(1 - \sqrt{1-x^2}\right) - \frac{1}{2}\int \frac{x^2\left(1+\sqrt{1-x^2}\right)}{x^2\sqrt{1-x^2}} dx$

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}x\ln 2 + \frac{1}{2}x\ln\left(1 - \sqrt{1-x^2}\right) - \frac{1}{2}\int \frac{1+\sqrt{1-x^2}}{\sqrt{1-x^2}} dx$

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}x\ln 2 + \frac{1}{2}x\ln\left(1 - \sqrt{1-x^2}\right) - \frac{1}{2}\int \left(\frac{1}{\sqrt{1-x^2}} + 1\right) dx$

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}x\ln 2 + \frac{1}{2}x\ln\left(1 - \sqrt{1-x^2}\right) - \frac{1}{2}(\arcsin x + x) + c$

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = \frac{1}{2}x\ln\left(2 - 2\sqrt{1-x^2}\right) - \frac{1}{2}(\arcsin x + x) + c$, therefore, we get:

$\int \ln\left(\sqrt{1+x} - \sqrt{1-x}\right) dx = x\ln\left(\sqrt{1+x} - \sqrt{1-x}\right) - \frac{1}{2}\arcsin x - \frac{1}{2}x + c$

161. $I = \int_0^a \frac{\ln(1+ax)}{1+x^2} dx$

Let $t = \frac{a-x}{1+ax}$, then $x = \frac{a-t}{1+at}$ and so $dx = -\frac{1+a^2}{(1+at)^2}dt$, for $x = 0$, we get $t = a$ and for $x = a$, we get $t = 0$, so we get:

$I = \int_a^0 \frac{\ln\left(1+a\left(\frac{a-t}{1+at}\right)\right)}{1+\left(\frac{a-t}{1+at}\right)^2}\left(-\frac{1+a^2}{(1+at)^2}dt\right) = \int_0^a \frac{\ln\left(\frac{1+at}{1+at}+\frac{a^2-at}{1+at}\right)}{\left[1+\left(\frac{a-t}{1+at}\right)^2\right]\frac{(1+at)^2}{1+a^2}} dt$, then we get:

$I = \int_0^a \frac{\ln\left(\frac{1+at+a^2-at}{1+at}\right)}{\frac{(1+at)^2}{1+a^2}+\frac{(a-t)^2}{1+a^2}} dt = \int_0^a \frac{\ln\left(\frac{1+a^2}{1+at}\right)}{\frac{(1+at)^2+(a-t)^2}{1+a^2}} dt = \int_0^a \frac{\ln\left(\frac{1+a^2}{1+at}\right)}{\frac{1+2at+a^2t^2+a^2-2at+t^2}{1+a^2}} dt$

$\text{I} = \int_0^a \frac{\ln\left(\frac{1+a^2}{1+at}\right)}{\frac{1+a^2t^2+a^2+t^2}{1+a^2}} dt = \int_0^a \frac{\ln\left(\frac{1+a^2}{1+at}\right)}{\frac{(1+a^2)+\left(a^2t^2+t^2\right)}{1+a^2}} dt = \int_0^a \frac{\ln\left(\frac{1+a^2}{1+at}\right)}{\frac{(1+a^2)+t^2(1+a^2)}{1+a^2}} dt$, then we get:

$\text{I} = \int_0^a \frac{\ln\left(\frac{1+a^2}{1+at}\right)}{\frac{(1+a^2)(1+t^2)}{1+a^2}} dt = \int_0^a \frac{\ln\left(\frac{1+a^2}{1+at}\right)}{1+t^2} dt = \int_0^a \frac{\ln\left(\frac{1+a^2}{1+ax}\right)}{1+x^2} dx$, then we get:

$\text{I} = \int_0^a \frac{\ln(1+a^2)-\ln(1+ax)}{1+x^2} dx = \int_0^a \frac{\ln(1+a^2)}{1+x^2} dx - \int_0^a \frac{\ln(1+ax)}{1+x^2} dx$, then:

$\text{I} = \ln(1+a^2) \int_0^a \frac{1}{1+x^2} dx - \text{I}$, so we get $2\text{I} = \ln(1+a^2) \int_0^a \frac{1}{1+x^2} dx$

$\text{I} = \frac{1}{2}\ln(1+a^2)\,[\arctan x]_0^a$, therefore, we get: $\text{I} = \frac{1}{2}\ln(1+a^2)\arctan a$

162. $\int x^{-2} \tan\left(\frac{1}{2x}\right) \tan\left(\frac{1}{3x}\right) \tan\left(\frac{1}{6x}\right) dx$

Let $t = \frac{1}{6x}$, then $2t = 2\left(\frac{1}{6x}\right) = \frac{1}{3x}$ and $3t = 3\left(\frac{1}{6x}\right) = \frac{1}{2x}$ and $dt = -\frac{1}{6x^2}dx = -\frac{1}{6}x^{-2}dx$

So, $x^{-2}dx = -6dt$, then we get:

$\int x^{-2} \tan\left(\frac{1}{2x}\right) \tan\left(\frac{1}{3x}\right) \tan\left(\frac{1}{6x}\right) dx = -6\int \tan(3t)\tan(2t)\tan(t)\, dt$

We have: $\tan(3t) = \tan(2t+t) = \frac{\tan 2t + \tan t}{1 - \tan 2t \tan t}$; then we get:

$\tan 3t \tan 2t \tan t = \tan 3t - \tan 2t - \tan t$; so

$\text{I} = -6\int(\tan 3t - \tan 2t - \tan t)dt = -2\ln|\sec 3t| + 3\ln|\sec 2t| + 6\ln|\sec t| + c$

$\text{I} = -2\ln\left|\sec\left(\frac{1}{2x}\right)\right| + 3\ln\left|\sec\left(\frac{1}{3x}\right)\right| + 6\ln\left|\sec\left(\frac{1}{6x}\right)\right| + c$, therefore we get:

$$\text{I} = \ln\left|\frac{\sec^3\left(\frac{1}{3x}\right)\sec^6\left(\frac{1}{6x}\right)}{\sec^2\left(\frac{1}{2x}\right)}\right| + c$$

163. $\text{I} = \int_{\frac{1}{e}}^1 \frac{\ln\left(1+\ln^2 x\right)}{x\ln x} dx$, let $-t = \ln x$, then $x = e^{-t}$ and $dx = -e^{-t}dt$, for $x = \frac{1}{e}$, $t = 1$ and for $x = 1$, $t = 0$, then we get:

$$\text{I} = \int_1^0 \frac{\ln(1+t^2)}{e^{-t}(-t)}(-e^{-t})dt = -\int_0^1 \frac{\ln(1+t^2)}{t} dt = -\int_0^1 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}(t^2)^n}{n.t};\ \text{ then we get:}$$

$$\text{I} = -\int_0^1 \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}t^{2n}}{n.t} = -\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\int_0^1 t^{2n-1}dt = -\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\left[\frac{1}{2n}t^{2n}\right]_0^1$$

$$= -\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n}\left(\frac{1}{2n}\right) = -\frac{1}{2}\sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n^2} = -\frac{1}{2}\left(\frac{\pi^2}{12}\right) = -\frac{\pi^2}{24}$$

164. $\text{I} = \int_0^2 \frac{\ln(1+x)}{x^2-x+1} dx$, let $y = 1+x$, then $dy = dx$, for $x = 0$, so $y = 1$ and for $x = 2$, then $y = 3$ so we get:

$\text{I} = \int_1^3 \frac{\ln y}{(y-1)^2-(y-1)+1} dy = \int_1^3 \frac{\ln y}{y^2-2y+1-y+1+1} dy = \int_1^3 \frac{\ln y}{y^2-3y+3} dy$ **… (1)**

Let $z = \frac{3}{y}$, then $y = \frac{3}{z}$ and $dy = -\frac{3}{z^2}dz$, for $y = 1$, $z = 3$ and for $y = 3$, $z = 1$, so we get:

$\mathrm{I} = \int_3^1 \frac{\ln\left(\frac{3}{z}\right)}{\left(\frac{3}{z}\right)^2 - 3\left(\frac{3}{z}\right)+3}\left(-\frac{3}{z^2}dz\right) = \int_1^3 \frac{\ln\left(\frac{3}{z}\right)}{z^2-3z+3}dz = \int_1^3 \frac{\ln 3 - \ln z}{z^2-3z+3}dz$, then we get:

$\mathrm{I} = \ln 3\int_1^3 \frac{dz}{z^2-3z+3} - \int_1^3 \frac{\ln z}{z^2-3z+3}dz$, using **(1)**, $\mathrm{I} = \int_1^3 \frac{\ln y}{y^2-3y+3}dy = \int_1^3 \frac{\ln z}{z^2-3z+3}dz$

Then we get: $\mathrm{I} = \ln 3\int_1^3 \frac{dz}{z^2-3z+3} - \mathrm{I}$, so $2\mathrm{I} = \ln 3\int_1^3 \frac{dz}{z^2-3z+3}$, so we get:

$\mathrm{I} = \frac{\ln 3}{2}\int_1^3 \frac{dz}{z^2-3z+3} = \frac{\ln 3}{2}\int_1^3 \frac{dz}{\left(z-\frac{3}{2}\right)^2+\frac{3}{4}} = \frac{\ln 3}{2}\cdot\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2}{\sqrt{3}}\left(z-\frac{3}{2}\right)\right)\right]_1^3$; then we get:

$\mathrm{I} = \frac{\ln 3}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}) - \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right] = \frac{\ln 3}{\sqrt{3}}\left(\frac{\pi}{3}+\frac{\pi}{6}\right) = \frac{\pi\ln 3}{2\sqrt{3}}$

165. $\mathrm{I} = \int_{-1}^1 \frac{\left(\sin^{-1}x\right)^2}{1+2^x}dx$ **… (1)**

Let $u = -x$, then $du = -dx$, for $x = -1$, $u = 1$ and for $x = 1$, $u = -1$, then:

$\mathrm{I} = \int_1^{-1} \frac{\left(\sin^{-1}(-u)\right)^2}{1+2^{-u}}(-du) = \int_{-1}^1 \frac{\left(\sin^{-1}u\right)^2}{1+2^{-u}}du = \int_{-1}^1 \frac{\left(\sin^{-1}x\right)^2}{1+2^{-x}}dx$ **… (2)**

Adding (1) and (2) we get: $\mathrm{I}+\mathrm{I} = \int_{-1}^1 \frac{\left(\sin^{-1}x\right)^2}{1+2^x}dx + \int_{-1}^1 \frac{\left(\sin^{-1}x\right)^2}{1+2^{-x}}$, then we get:

$2\mathrm{I} = \int_{-1}^1 (\sin^{-1}x)^2\left(\frac{1}{1+2^x}+\frac{1}{1+2^{-x}}\right)dx = \int_{-1}^1 (\sin^{-1}x)^2\left(\frac{1}{1+2^x}+\frac{2^x}{1+2^x}\right)dx$

$2\mathrm{I} = \int_{-1}^1 (\sin^{-1}x)^2\left(\frac{1+2^x}{1+2^x}\right)dx = \int_{-1}^1 (\sin^{-1}x)^2dx$

Let $t = \sin^{-1}x$, then $x = \sin t$ and $dx = \cos t\,dt$, for $x = -1$, then $t = -\frac{\pi}{2}$, and for $x = 1$, then $t = \frac{\pi}{2}$, then we get:

$2\mathrm{I} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} t^2\cos t\,dt = 2\int_0^{\frac{\pi}{2}} t^2\cos t\,dt$, by performing IBP twice we get:

$2\mathrm{I} = 2[t^2\cos t + 2t\cos t - 2\sin t]_0^{\frac{\pi}{2}} = 2\left(\frac{\pi^2}{4}-2\right) = \frac{\pi^2}{2} - 4$, therefore, we get:

$\mathrm{I} = \frac{1}{2}\left(\frac{\pi^2}{2}-4\right) = \frac{\pi^2}{4} - 2$

166. $\mathrm{I} = \int_0^\alpha \frac{dx}{x+\sqrt{\alpha^2-x^2}}$, where $\alpha > 0$

Let $x = \alpha\sin u$, then $dx = \alpha\cos u\,du$, for $x = 0$, $u = 0$ and for $x = \alpha$, then $u = \frac{\pi}{2}$, so:

$$\mathrm{I} = \int_0^\alpha \frac{dx}{x+\sqrt{\alpha^2-x^2}} = \int_0^{\frac{\pi}{2}} \frac{\alpha\cos u}{\alpha\sin u + \sqrt{\alpha^2-\alpha^2\sin^2 u}}du = \int_0^{\frac{\pi}{2}} \frac{\alpha\cos u}{\alpha\sin u + \alpha\sqrt{1-\sin^2 u}}du\text{; then:}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{\alpha\cos u}{\alpha\sin u + \alpha\sqrt{\cos^2 u}}du = \int_0^{\frac{\pi}{2}} \frac{\alpha\cos u}{\alpha\sin u + \alpha\cos u}du = \int_0^{\frac{\pi}{2}} \frac{\cos u}{\sin u + \cos u}du \ldots (1)$$

Let $t = \frac{\pi}{2} - u$, then $dt = -du$, for $u = 0$, then $t = \frac{\pi}{2}$ and for $u = \frac{\pi}{2}$ then $u = 0$, then:

$$\mathrm{I} = \int_{\frac{\pi}{2}}^{0} \frac{\cos\left(\frac{\pi}{2} - t\right)}{\sin\left(\frac{\pi}{2} - t\right) + \cos\left(\frac{\pi}{2} - t\right)}(-dt) = \int_{0}^{\frac{\pi}{2}} \frac{\sin t}{\cos t + \sin t} dt = \int_{0}^{\frac{\pi}{2}} \frac{\sin u}{\cos u + \sin u} du \ldots (2)$$

Adding (1) & (2) we get:

$$2\mathrm{I} = \int_{0}^{\frac{\pi}{2}} \frac{\cos u}{\sin u + \cos u} du + \int_{0}^{\frac{\pi}{2}} \frac{\sin u}{\cos u + \sin u} du = \int_{0}^{\frac{\pi}{2}} \frac{\cos u + \sin u}{\cos u + \sin u} du = \int_{0}^{\frac{\pi}{2}} du = \frac{\pi}{2}$$

Therefore; we get: $\mathrm{I} = \frac{1}{2}\left(\frac{\pi}{2}\right) = \frac{\pi}{4}$

167. $\mathrm{I} = \int_0^{\frac{\pi}{4}} (\ln \tan x)^{\alpha}\, dx$

Change of variable: let $y = \tan x \Longrightarrow x = \tan^{-1} y \Longrightarrow dx = \frac{1}{1+y^2} dy$, then we get:

New bounds: for $x = 0$, $y = 0$ and for $x = \frac{\pi}{4}$, $y = 1$, then we get:

$$\mathrm{I} = \int_{0}^{\frac{\pi}{4}} (\ln \tan x)^{\alpha}\, dx = \int_{0}^{1} \frac{\ln^{\alpha} y}{1 + y^2} dy = \int_{0}^{1} \frac{1}{1 + y^2} . \ln^{\alpha} y\, dy = \int_{0}^{1} \sum_{n=0}^{+\infty} (-1)^n y^{2n} \ln^{\alpha} y\, dy$$

$$\mathrm{I} = \sum_{n=0}^{+\infty} (-1)^n \int_{0}^{1} y^{2n} \ln^{\alpha} y\, dy$$

Change of variable: let $z = -\ln y \Longrightarrow y = e^{-z} \Longrightarrow dy = -e^{-z} dz$

New bounds: for $y = 0$; $z = +\infty$ & for $y = 1$; $z = 0$; then we get:

$$\mathrm{I} = \sum_{n=0}^{+\infty} (-1)^n \int_{+\infty}^{0} (e^{-z})^{2n} (-z)^{\alpha} (-e^{-z} dz) = \sum_{n=0}^{+\infty} (-1)^n (-1)^{\alpha} \int_{0}^{+\infty} e^{-(2n+1)z} z^{\alpha} dz$$

Change of variable: let $u = (2n + 1)z \Longrightarrow z = \frac{1}{2n+1} u \Longrightarrow dz = \frac{1}{2n+1} du$; then:

$$\mathrm{I} = \sum_{n=0}^{+\infty} (-1)^n (-1)^{\alpha} \int_{0}^{+\infty} e^{-u} \left(\frac{1}{2n+1} u\right)^{\alpha} \frac{1}{2n+1} du$$

$$\mathrm{I} = \sum_{n=0}^{+\infty} (-1)^n (-1)^{\alpha} \int_{0}^{+\infty} e^{-u} \frac{1}{(2n+1)^{\alpha+1}} u^{\alpha} du; \text{ then we can wite:}$$

$$\mathrm{I} = (-1)^{\alpha} \sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^{\alpha+1}} \int_{0}^{+\infty} e^{-u} u^{\alpha} du$$

But $\sum_{n=0}^{+\infty} \frac{(-1)^n}{(2n+1)^{\alpha+1}} = \beta(\alpha + 1)$ & $\int_{0}^{+\infty} e^{-u} u^{\alpha} du = \int_{0}^{+\infty} e^{-u} u^{(\alpha+1)-1} du = \Gamma(\alpha + 1)$

Therefore; we get: $\mathrm{I} = \int_{0}^{\frac{\pi}{4}} (\ln \tan x)^{\alpha}\, dx = (-1)^{\alpha}\, \Gamma(\alpha + 1)\, \beta(\alpha + 1)$

168. $\int \frac{1}{\alpha \sin x+\beta \cos x} dx$, first let us make a substitutions which will make the shape of the integrand much easier, put $\alpha = r\cos t$ and $\beta = r\sin t$, then:

$$\int \frac{1}{\alpha \sin x + \beta \cos x} dx = \int \frac{1}{r\cos t \sin x + r\sin t \cos x} dx = \frac{1}{r}\int \frac{1}{\cos t \sin x + \sin t \cos x} dx$$

$$\int \frac{1}{\alpha \sin x + \beta \cos x} dx = \frac{1}{r}\int \frac{1}{\cos(x+t)} dx$$

Let $u = x + t$, then $du = dx$, then we get:

$$\int \frac{1}{\alpha \sin x + \beta \cos x} dx = \frac{1}{r}\int \frac{1}{\cos u} du = \frac{1}{r}\int \csc u\, du = \frac{1}{r}\ln\left|\tan\left(\frac{u}{2}\right)\right| + c; \text{ then we get:}$$

$$\int \frac{1}{\alpha \sin x + \beta \cos x} dx = \frac{1}{r}\ln\left|\tan\left(\frac{x+t}{2}\right)\right| + c$$

Now we must represent r and t in terms of α and β, first we have $\alpha = r\cos t$ and $\beta = r\sin t$, then: $\alpha^2 + \beta^2 = r^2\cos^2 t + r^2 \sin^2 t = r^2(\cos^2 t + \sin^2 t) = r^2 \Rightarrow r = \sqrt{\alpha^2+\beta^2}$

Moreover; $\frac{\beta}{\alpha} = \frac{r\sin t}{r\cos t} = \frac{\sin t}{\cos t} = \tan t \Rightarrow t = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$; therefore; we get:

$$\int \frac{1}{\alpha \sin x + \beta \cos x} dx = \frac{1}{r}\ln\left|\tan\left(\frac{x+t}{2}\right)\right| + c = \frac{1}{\sqrt{\alpha^2+\beta^2}}\ln\left|\tan\left(\frac{x+\tan^{-1}\left(\frac{\beta}{\alpha}\right)}{2}\right)\right| + c$$

169. $\mathrm{I} = \int_1^{+\infty} \frac{1}{(x+a)\sqrt{x-1}} dx$, where $a > 1$

Let $u = \sqrt{x-1}$, then $x = u^2 + 1$ and $dx = 2udu$, for $x = 1$, then $u = 0$ and for $x = +\infty$, then $u = +\infty$, then we get:

$$\mathrm{I} = \int_0^{+\infty} \frac{1}{(u^2+1+a)u}.2udu = \int_0^{+\infty}\frac{2}{u^2+1+a}du = 2\int_0^{+\infty}\frac{1}{u^2+\left(\sqrt{a+1}\right)^2}du, \text{ then we get:}$$

$$\mathrm{I} = \frac{2}{\sqrt{a+1}}\left[\tan\left(\frac{u}{\sqrt{a+1}}\right)\right]_0^{+\infty} = \frac{2}{\sqrt{a+1}}.\frac{\pi}{2} = \frac{\pi}{\sqrt{a+1}}$$

170. $\mathrm{I} = \int_0^a \ln\left(\sqrt{a+x}+\sqrt{a-x}\right) dx = \frac{1}{2}\int_0^a 2\ln\left(\sqrt{a+x}+\sqrt{a-x}\right) dx$, then we get:

$$\mathrm{I} = \int_0^a \ln\left(\sqrt{a+x}+\sqrt{a-x}\right) dx = \frac{1}{2}\int_0^a \ln\left[\left(\sqrt{a+x}+\sqrt{a-x}\right)^2\right] dx$$

$$\int_0^a \ln\left(\sqrt{a+x}+\sqrt{a-x}\right) dx = \frac{1}{2}\int_0^a \ln\left(a + x + 2\sqrt{a+x}\times\sqrt{a-x} + a - x\right) dx$$

$$\mathrm{I} = \frac{1}{2}\int_0^a \ln\left(2a + 2\sqrt{a^2-x^2}\right) dx = \frac{1}{2}\int_0^a \left[\ln 2 + \ln\left(a+\sqrt{a^2-x^2}\right)\right] dx; \text{ then we get:}$$

$$\mathrm{I} = \frac{1}{2}a\ln 2 + \frac{1}{2}\int_0^a \ln\left(a+\sqrt{a^2-x^2}\right) dx; \text{ now using integration by parts:}$$

Let $u = \ln\left(a+\sqrt{a^2-x^2}\right) \Rightarrow u' = -\frac{x}{\sqrt{a^2-x^2}\left(a+\sqrt{a^2-x^2}\right)}$ & *let* $v' = 1$ then $v = x$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}\left\{\left[x\ln\left(a+\sqrt{a^2-x^2}\right)\right]_0^a+\int_0^a\frac{x^2}{\sqrt{a^2-x^2}\left(a+\sqrt{a^2-x^2}\right)}dx\right\}$$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}\int_0^a\frac{x^2}{\sqrt{a^2-x^2}\left(a+\sqrt{a^2-x^2}\right)}dx$$

Let $x=a\sin\theta\Rightarrow dx=a\cos\theta\,d\theta$; then:

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}\int_0^a\frac{(a\sin\theta)^2}{\sqrt{a^2-(a\sin\theta)^2}\left(a+\sqrt{a^2-(a\sin\theta)^2}\right)}.\,a\cos\theta\,d\theta$$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}\int_0^a\frac{a^2\sin^2\theta}{\sqrt{a^2-a^2\sin^2\theta}\left(a+\sqrt{a^2-a^2\sin^2\theta}\right)}.\,a\cos\theta\,d\theta$$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}\int_0^a\frac{a^2\sin^2\theta}{\sqrt{a^2(1-\sin^2\theta)}\left(a+\sqrt{a^2(1-\sin^2\theta)}\right)}.\,a\cos\theta\,d\theta$$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}\int_0^a\frac{a^2\sin^2\theta}{\sqrt{a^2\cos^2\theta}\left(a+\sqrt{a^2\cos^2\theta}\right)}.\,a\cos\theta\,d\theta$$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}\int_0^a\frac{a^2\sin^2\theta}{a\cos\theta\,(a+a\cos\theta)}.\,a\cos\theta\,d\theta$$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}a\int_0^a\frac{\sin^2\theta}{1+\cos\theta}d\theta=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}a\int_0^a\frac{1-\cos^2\theta}{1+\cos\theta}d\theta$$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}a\int_0^a\frac{(1-\cos\theta)(1+\cos\theta)}{1+\cos\theta}d\theta$$

$$\mathrm{I}=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}a\int_0^a(1+\cos\theta)d\theta=\frac{1}{2}a\ln 2+\frac{1}{2}a\ln a+\frac{1}{2}a[\theta-\sin\theta]_0^a$$

$$\mathrm{I}=\frac{1}{2}a\ln(2a)+\frac{1}{2}a\left(\frac{\pi}{2}-1\right)\quad\therefore\int_0^a\ln\left(\sqrt{a+x}+\sqrt{a-x}\right)dx=\frac{1}{2}a\left(\ln 2a+\frac{\pi}{2}-1\right)$$

171. $\int\frac{x+\sin\left(\cos^{-1}x\right)}{\cos\left(\sin^{-1}\left(x^2\right)\right)}dx$, let $\alpha=\cos^{-1}x$, then $x=\cos\alpha$, then

$\sin(\cos^{-1}x)=\sin\alpha=\sqrt{1-\cos^2\alpha}=\sqrt{1-x^2}$

Let $\beta=\sin^{-1}(x^2)$, then $x^2=\sin\beta$, then $\cos(\sin^{-1}(x^2))=\cos\beta=\sqrt{1-\sin^2\beta}=\sqrt{1-x^4}$

Then we get**:**

$$\int\frac{x+\sin\left(\cos^{-1}x\right)}{\cos\left(\sin^{-1}\left(x^2\right)\right)}dx=\int\frac{x+\sqrt{1-x^2}}{\sqrt{1-x^4}}dx=\int\frac{x}{\sqrt{1-x^4}}dx+\int\frac{\sqrt{1-x^2}}{\sqrt{1-x^4}}dx$$

$\int\frac{x+\sin\left(\cos^{-1}x\right)}{\cos\left(\sin^{-1}\left(x^2\right)\right)}dx=\int\frac{x}{\sqrt{1-\left(x^2\right)^2}}dx+\int\frac{\sqrt{1-x^2}}{\sqrt{1-x^2}.\sqrt{1+x^2}}dx$, then we get**:**

$\int \frac{x+\sin(\cos^{-1}x)}{\cos(\sin^{-1}(x^2))}dx = \int \frac{x}{\sqrt{1-(x^2)^2}}dx + \int \frac{1}{\sqrt{1+x^2}}dx$, therefore, we get:

$\int \frac{x+\sin(\cos^{-1}x)}{\cos(\sin^{-1}(x^2))}dx = \frac{1}{2}\sin^{-1}(x^2) + \ln\left(x+\sqrt{1+x^2}\right) + c$

172. $\int \frac{1}{(1+2x)\sqrt{(1+3x)(1+x)}}dx = \int \frac{1}{(1+2x)\sqrt{3x^2+4x+1}}dx$

Let $y = \frac{1}{x}$, then $x = \frac{1}{y}$, so $dx = -\frac{1}{y^2}dy$, then we get:

$\int \frac{1}{(1+2x)\sqrt{(1+3x)(1+x)}}dx = \int \frac{1}{\left(1+\frac{2}{y}\right)\sqrt{\frac{3}{y^2}+\frac{4}{y}+1}}\left(-\frac{1}{y^2}dy\right) = -\int \frac{1}{(y+2)\sqrt{y^2+4y+3}}dy$, then:

$\int \frac{1}{(1+2x)\sqrt{(1+3x)(1+x)}}dx = -\int \frac{1}{(y+2)\sqrt{(y+2)^2-1}}dy$. Let $z = y+2$, so $dz = dy$, then:

$\int \frac{1}{(1+2x)\sqrt{(1+3x)(1+x)}}dx = -\int \frac{dz}{z\sqrt{z^2-1}} = -\sec^{-1}z + c = -\sec^{-1}(y+2) + c$, therefore,

$\int \frac{1}{(1+2x)\sqrt{(1+3x)(1+x)}}dx = -\sec^{-1}\left(\frac{1}{x}+2\right) + c = -\sec^{-1}\left(\frac{1+2x}{x}\right) + c$

173. $\int \frac{1}{3\sin^4 x+3\cos^4 x-1}dx = \int \frac{1}{3(\sin^4 x+\cos^4 x)-1}dx$

But we have $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x$

$\sin^4 x + \cos^4 x = 1 - 2\sin^2 x\cos^2 x = 1 - 2(\sin x\cos x)^2 = 1 - 2\left(\frac{1}{2}\sin 2x\right)^2$, then

$\sin^4 x + \cos^4 x = 1 - \frac{1}{2}\sin^2 2x$, then:

$$\int \frac{1}{3\sin^4 x + 3\cos^4 x - 1}dx = \int \frac{1}{3\left(1-\frac{1}{2}\sin^2 2x\right)-1}dx = \int \frac{1}{2-\frac{3}{2}\sin^2 2x}dx\text{; then:}$$

$$\int \frac{1}{3\sin^4 x + 3\cos^4 x - 1}dx = 2\int \frac{1}{4-3\sin^2 2x}dx = 2\int \frac{1}{4-3\left(\frac{1-\cos 4x}{2}\right)}dx$$

$$\int \frac{1}{3\sin^4 x + 3\cos^4 x - 1}dx = 4\int \frac{1}{5+3\cos 4x}dx$$

Change of variable: let $t = \tan 2x$; then $dx = \frac{dt}{2(1+t^2)}$ and $\cos 4x = \frac{1-t^2}{1+t^2}$; then:

$$\int \frac{1}{3\sin^4 x + 3\cos^4 x - 1}dx = 4\int \frac{1}{5+3\left(\frac{1-t^2}{1+t^2}\right)}\cdot\frac{dt}{2(1+t^2)} = 2\int \frac{1}{5+5t^2+3-3t^2}dt$$

$$\int \frac{1}{3\sin^4 x + 3\cos^4 x - 1}dx = 2\int \frac{1}{8+2t^2}dt = \int \frac{1}{4+t^2}dt = \int \frac{1}{2^2+t^2}dt\text{; then we get:}$$

$$\int \frac{1}{3\sin^4 x + 3\cos^4 x - 1}dx = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + c = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\tan 2x\right) + c$$

174. $\mathrm{I}=\int_0^{+\infty}\frac{x-1}{\ln(2^x-1)\sqrt{2^x-1}}dx$

Let $2^x-1=\tan^2\theta \Longrightarrow x=\log_2(1+\tan^2\theta)\Longrightarrow x=\frac{\ln(1+\tan^2\theta)}{\ln 2}$ & $dx=\frac{2\tan\theta}{\ln 2}d\theta$

For $x=0\Longrightarrow\theta=0$ & for $x=+\infty\Longrightarrow\theta=\frac{\pi}{2}$; then we get:

$$\mathrm{I}=\int_0^{\frac{\pi}{2}}\frac{x-1}{\ln(2^x-1)\sqrt{2^x-1}}dx=\int_0^{\frac{\pi}{2}}\frac{\frac{\ln(1+\tan^2\theta)}{\ln 2}-1}{\ln(\tan^2\theta)\sqrt{\tan^2\theta}}.\frac{2\tan\theta}{\ln 2}d\theta$$

$$\mathrm{I}=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{\ln(1+\tan^2\theta)-\ln 2}{2\ln(\tan\theta).\tan\theta}.2\tan\theta\,d\theta=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{\ln(\sec^2\theta)-\ln 2}{\ln(\tan\theta)}d\theta;\ \text{ then we get:}$$

$$\mathrm{I}=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{2\ln(\sec\theta)-\ln 2}{\ln(\tan\theta)}d\theta\,\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{2\ln\left(\frac{1}{\cos\theta}\right)-\ln 2}{\ln(\tan\theta)}d\theta$$

$$\mathrm{I}=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{-2\ln(\cos\theta)-\ln 2}{\ln(\tan\theta)}d\theta\quad\ldots(\mathbf{1})$$

Let $y=\frac{\pi}{2}-\theta\Longrightarrow dy=-d\theta$; for $\theta=0\Longrightarrow y=\frac{\pi}{2}$ & for $\theta=\frac{\pi}{2}\Longrightarrow y=0$; then we get:

$$\mathrm{I}=\frac{1}{\ln^2 2}\int_{\frac{\pi}{2}}^{0}\frac{-2\ln\left(\cos\left(\frac{\pi}{2}-y\right)\right)-\ln 2}{\ln\left(\tan\left(\frac{\pi}{2}-y\right)\right)}(-dy)=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{-2\ln(\sin y)-\ln 2}{\ln(\cot y)}dy$$

$$\mathrm{I}=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{-2\ln(\sin y)-\ln 2}{\ln\left(\frac{1}{\tan y}\right)}dy=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{-2\ln(\sin y)-\ln 2}{-\ln(\tan y)}dy;\ \text{ then we get:}$$

$$\mathrm{I}=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{2\ln(\sin y)+\ln 2}{\ln(\tan y)}dy=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{2\ln(\sin\theta)+\ln 2}{\ln(\tan\theta)}d\theta\quad\ldots(\mathbf{2})\ \text{ Adding (1) \& (2):}$$

$$\mathrm{I}+\mathrm{I}=2\mathrm{I}=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{-2\ln(\cos\theta)-\ln 2}{\ln(\tan\theta)}d\theta+\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{2\ln(\sin\theta)+\ln 2}{\ln(\tan\theta)}d\theta;\ \text{ then we get:}$$

$$2\mathrm{I}=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{-2\ln(\cos\theta)-\ln 2+2\ln(\sin\theta)+\ln 2}{\ln(\tan\theta)}d\theta$$

$$2\mathrm{I}=\frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{-2\ln(\cos\theta)+2\ln(\sin\theta)}{\ln(\tan\theta)}d\theta=\frac{2}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{\ln\left(\frac{\sin\theta}{\cos\theta}\right)}{\ln(\tan\theta)}d\theta=\frac{2}{\ln^2 2}\int_0^{\frac{\pi}{2}}\frac{\ln(\tan\theta)}{\ln(\tan\theta)}d\theta$$

$$\mathrm{I} = \frac{1}{\ln^2 2}\int_0^{\frac{\pi}{2}} d\theta = \frac{1}{\ln^2 2} \times \frac{\pi}{2} = \frac{\pi}{2\ln^2 2}$$

175. $\mathrm{I} = \int_{-\infty}^{+\infty} \frac{\tan^{-1}\left(x+\sqrt{1+x^2}\right)}{4+x^2} dx$

$\mathrm{I} = \int_0^{+\infty} \frac{\tan^{-1}\left(x+\sqrt{1+x^2}\right)}{4+x^2} dx + \int_{-\infty}^{0} \frac{\tan^{-1}\left(x+\sqrt{1+x^2}\right)}{4+x^2} dx$. For the second integral take $y = -x$, so $dy = -dx$, for $x = -\infty$, then $y = +\infty$ and for $x = 0$, then $y = 0$, so, we get:

$\int_{-\infty}^{0} \frac{\tan^{-1}\left(x+\sqrt{1+x^2}\right)}{4+x^2} dx = \int_{+\infty}^{0} \frac{\tan^{-1}\left(-y+\sqrt{1+y^2}\right)}{4+y^2}(-dy)$, then we get:

$\int_{-\infty}^{0} \frac{\tan^{-1}\left(x+\sqrt{1+x^2}\right)}{4+x^2} dx = \int_0^{+\infty} \frac{\tan^{-1}\left(-y+\sqrt{1+y^2}\right)}{4+y^2} dy = \int_0^{+\infty} \frac{\tan^{-1}\left(\sqrt{1+x^2}-x\right)}{4+x^2} dx$, so:

$\mathrm{I} = \int_0^{+\infty} \frac{\tan^{-1}\left(x+\sqrt{1+x^2}\right)}{4+x^2} dx + \int_0^{+\infty} \frac{\tan^{-1}\left(\sqrt{1+x^2}-x\right)}{4+x^2} dx$

$\mathrm{I} = \int_0^{+\infty} \frac{\tan^{-1}\left[\left(x+\sqrt{1+x^2}\right)\times\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}-x}\right]}{4+x^2} dx + \int_0^{+\infty} \frac{\tan^{-1}\left(\sqrt{1+x^2}-x\right)}{4+x^2} dx$, then:

$\mathrm{I} = \int_0^{+\infty} \frac{\tan^{-1}\left(\frac{1+x^2-x^2}{\sqrt{1+x^2}-x}\right)}{4+x^2} dx + \int_0^{+\infty} \frac{\tan^{-1}\left(\sqrt{1+x^2}-x\right)}{4+x^2} dx$, then we get:

$\mathrm{I} = \int_0^{+\infty} \frac{\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}-x}\right)}{4+x^2} dx + \int_0^{+\infty} \frac{\tan^{-1}\left(\sqrt{1+x^2}-x\right)}{4+x^2} dx$, then we can write:

$\mathrm{I} = \int_0^{+\infty} \frac{\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}-x}\right)+\tan^{-1}\left(\sqrt{1+x^2}-x\right)}{4+x^2} dx$, but we have:

$\tan^{-1}\left(\frac{1}{\sqrt{1+x^2}-x}\right) + \tan^{-1}\left(\sqrt{1+x^2}-x\right) = \frac{\pi}{2}$, then $\mathrm{I} = \frac{\pi}{2}\int_0^{+\infty} \frac{1}{4+x^2} dx$, therefore:

$\mathrm{I} = \frac{\pi}{2}\left[\frac{1}{2}\arctan\left(\frac{x}{2}\right)\right]_0^{+\infty} = \frac{\pi}{2}\cdot\frac{\pi}{4} = \frac{\pi^2}{8}$

176. $\mathrm{I} = \int_0^{+\infty} \frac{e^{3x}}{e^{4x}+1} dx = \int_0^{+\infty} \frac{e^{2x}}{e^{4x}+1} . e^x dx$, let $y = e^x$, then $dy = e^x dx$, for $x = 0$, then $y = 1$ and for $x = +\infty$, then $y = +\infty$, so we get:

$\mathrm{I} = \int_1^{+\infty} \frac{y^2}{y^4+1} dy = \frac{1}{2}\int_1^{+\infty} \frac{(y^2+1)+(y^2-1)}{y^4+1} dy = \frac{1}{2}\int_1^{+\infty} \frac{y^2+1}{y^4+1} dy + \frac{1}{2}\int_1^{+\infty} \frac{y^2-1}{y^4+1} dy$, so:

$$\mathrm{I} = \frac{1}{2}\int_1^{+\infty} \frac{1}{\left(y-\frac{1}{y}\right)^2+2}.\left(1+\frac{1}{y^2}\right) dy + \frac{1}{2}\int_1^{+\infty} \frac{1}{\left(y+\frac{1}{y}\right)^2-2}.\left(1-\frac{1}{y^2}\right) dy$$

For the first integral we take $u = y - \frac{1}{y}$ and for the second one we take $v = y + \frac{1}{y}$, so we get:

$$I = \frac{1}{2}\int_0^{+\infty} \frac{1}{u^2+2}du + \frac{1}{2}\int_0^{+\infty} \frac{1}{v^2-2}dv = \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)\right]_0^{+\infty} + \frac{1}{2\sqrt{2}}\left[\ln\left(\frac{v-\sqrt{2}}{v+\sqrt{2}}\right)\right]_0^{+\infty}$$

$$I = \frac{\pi}{4\sqrt{2}} - \frac{1}{4\sqrt{2}}\ln\left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)$$

177. $I = \int_0^{\frac{1}{2}} \frac{\ln(1-x)}{2x^2-2x+1}dx$

Let $x = \frac{t}{1+t}$ $\left(t = \frac{x}{1-x}\right) \Rightarrow dx = \frac{1}{(1+t)^2}dt$; for thr bounds:

For $x = 0 \Rightarrow t = 0$ & for $x = \frac{1}{2} \Rightarrow t = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$; then we get:

$$I = \int_0^{\frac{1}{2}} \frac{\ln(1-x)}{2x^2-2x+1}dx = \int_0^1 \frac{\ln\left(1-\frac{t}{1+t}\right)}{2\left(\frac{t}{1+t}\right)^2 - 2\left(\frac{t}{1+t}\right)+1}.\frac{1}{(1+t)^2}dt = -\int_0^1 \frac{\ln(1+t)}{t^2+1}dt$$

Let $y = \frac{1-t}{1+t}$ $\left(t = \frac{1-y}{1+y}\right) \Rightarrow dt = -\frac{2}{(1+y)^2}dy$; for $t = 0 \Rightarrow y = 1$ & for $t = 1 \Rightarrow$ $y = 0$; then we get:

$$I = -\int_0^1 \frac{\ln(1+t)}{t^2+1}dt = -\int_1^0 \frac{\ln\left(1+\frac{1-y}{1+y}\right)}{\left(\frac{1-y}{1+y}\right)^2+1}\left(-\frac{2}{(1+y)^2}dy\right) = -\int_0^1 \frac{\ln\left(\frac{2}{1+y}\right)}{(1-y)^2+(1+y^2)}$$

$$I = -\int_0^1 \frac{\ln 2 - \ln(1+y)}{1-2y+y^2+1+2y+y^2}dy = -\frac{1}{2}\int_0^1 \frac{\ln 2 - \ln(1+y)}{1+y^2}dy\,;\ \text{then we get:}$$

$$I = -\frac{\ln 2}{2}\int_0^1 \frac{1}{1+y^2}dy + \frac{1}{2}\int_0^1 \frac{\ln(1+y)}{1+y^2}dy\,;\ \text{but } I = -\int_0^1 \frac{\ln(1+t)}{t^2+1}dt = -\int_0^1 \frac{\ln(1+y)}{y^2+1}dt$$

So; $I = -\frac{\ln 2}{2}[\tan^{-1} y]_0^1 - \frac{1}{2}I \Rightarrow \frac{3}{2}I = -\frac{\ln 2}{2}\left(\frac{\pi}{4}\right) \Rightarrow I = -\frac{2}{3}\left(\frac{\ln 2}{2}\right)\left(\frac{\pi}{4}\right) = -\frac{\pi \ln 2}{12}$

178. $I = \int_0^{\frac{e}{\pi}} \frac{\tan^{-1}\left(\frac{\pi x}{e}\right)}{\pi x+e}dx$

Change of variable: Let $t = \tan^{-1}\left(\frac{\pi x}{e}\right) \Rightarrow x = \frac{e}{\pi}\tan t \Rightarrow dx = \frac{e}{\pi}\sec^2 t\, dt$

For $x = 0 \Rightarrow t = 0$ & for $x = \frac{e}{\pi} \Rightarrow t = \tan^{-1}\left(\frac{\pi}{e}\times\frac{e}{\pi}\right) = \tan 1 = \frac{\pi}{4}$; then we get:

$$I = \int_0^{\frac{e}{\pi}} \frac{\tan^{-1}\left(\frac{\pi x}{e}\right)}{\pi x+e}dx = \int_0^{\frac{\pi}{4}} \frac{t}{\pi\left(\frac{e}{\pi}\tan t\right)+e}.\frac{e}{\pi}\sec^2 t\, dt = \frac{e}{\pi}\int_0^{\frac{\pi}{4}} \frac{t\sec^2 t}{e\tan t+e}dt\,;\text{then:}$$

$$\mathrm{I}=\frac{e}{\pi}\left(\frac{1}{e}\int_0^{\frac{\pi}{4}}\frac{t\sec^2 t}{\tan t+1}dt\right)=\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\frac{t\sec^2 t}{\tan t+1}dt\,;\ \text{now using integration by parts:}$$

Let $u=t\Rightarrow u'=1$ & let $v'=\dfrac{\sec^2 t}{\tan t+1}\Rightarrow v=\ln(1+\tan t)$; then we get:

$$\mathrm{I}=\frac{1}{\pi}[t\ln(1+\tan t)]_0^{\frac{\pi}{4}}-\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\ln(1+\tan t)\,dx=\frac{1}{\pi}\left(\frac{\pi}{4}\right)\ln\left(1+\tan\frac{\pi}{4}\right)-\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\ln(1+\tan t)\,dt$$

$$\mathrm{I}=\frac{1}{4}\ln 2-\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\ln(1+\tan t)\,dt=\frac{1}{4}\ln 2-\frac{1}{\pi}\mathrm{J};\ \text{where } \mathrm{J}=\int_0^{\frac{\pi}{4}}\ln(1+\tan t)\,dt$$

Let $y=\dfrac{\pi}{4}-t\Rightarrow dy=-dt$; for $t=0\Rightarrow y=\dfrac{\pi}{4}$ & for $t=\dfrac{\pi}{4}\Rightarrow y=0$; then we get:

$$\mathrm{J}=\int_{\frac{\pi}{4}}^{0}\ln\left(1+\tan\left(\frac{\pi}{4}-y\right)\right)(-dy)=\int_0^{\frac{\pi}{4}}\ln\left(1+\tan\left(\frac{\pi}{4}-y\right)\right)dy\,;\ \text{then we get:}$$

$$\mathrm{J}=\int_0^{\frac{\pi}{4}}\ln\left(1+\frac{\tan\frac{\pi}{4}-\tan y}{\tan\frac{\pi}{4}+\tan y}\right)dy=\int_0^{\frac{\pi}{4}}\ln\left(1+\frac{1-\tan y}{1+\tan y}\right)dy=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan y}\right)dy$$

$$\mathrm{J}=\int_0^{\frac{\pi}{4}}[\ln 2-\ln(1+\tan y)]dy=\int_0^{\frac{\pi}{4}}\ln 2\,dt-\int_0^{\frac{\pi}{4}}\ln(1+\tan y)\,dy=\frac{\pi}{4}\ln 2-\int_0^{\frac{\pi}{4}}\ln(1+\tan t)\,dt$$

So; $\mathrm{J}=\dfrac{\pi}{4}\ln 2-\mathrm{J}$; then $2\mathrm{J}=\dfrac{\pi}{4}\ln 2\ \Rightarrow \mathrm{J}=\dfrac{\pi}{8}\ln 2$

179. $\mathrm{I}=\int_0^{\pi}\left(\ln\tan\frac{x}{2}\right)^2\sin x\,dx$

Change of variable: let $x=2t\Longrightarrow dx=2dt$; for $x=0$; $t=0$ & $for\ x=\pi$; $t=\dfrac{\pi}{2}$, then we get

$$\mathrm{I}=\int_0^{\pi}\left(\ln\tan\frac{x}{2}\right)^2\sin x\,dx=\int_0^{\frac{\pi}{2}}(\ln\tan t)^2\sin 2t\,dt=4\int_0^{\frac{\pi}{2}}(\ln\tan t)^2\sin t\cos t\,dt\,;\text{then:}$$

$$\mathrm{I}=4\int_0^{\frac{\pi}{2}}(\ln\tan t)^2\sin t\cos t\times\frac{\cos t}{\cos t}dt=4\int_0^{\frac{\pi}{2}}(\ln\tan t)^2\left(\frac{\sin t}{\cos t}\right)\times\cos^2 t\,dt$$

$$\mathrm{I}=4\int_0^{\frac{\pi}{2}}(\ln\tan t)^2\tan t\times\cos^4 t\times\sec^2 t\,dt=4\int_0^{\frac{\pi}{2}}(\ln\tan t)^2\tan t\times(\cos^2 t)^2\times\sec^2 t\,dt$$

$$\mathrm{I}=4\int_0^{\frac{\pi}{2}}(\ln\tan t)^2\tan t\times\left(\frac{1}{1+\tan^2 t}\right)^2\times\sec^2 t\,dt$$

Change of variable: let $y = \tan t \Longrightarrow dy = \sec^2 t\, dt$, for $t = \frac{\pi}{2}$, $y = +\infty$ & for $x = 0$, $y = 0$, so

$$\mathrm{I} = 4\int_0^{+\infty} (\ln y)^2 \times y \times \left(\frac{1}{1+y^2}\right)^2 dy = 4\int_0^{+\infty} \frac{y\ln^2 y}{(1+y^2)^2} dy$$

Change of variable: let $y^2 = u \Longrightarrow y = \sqrt{u}$ & $2ydy = du \Longrightarrow ydy = \frac{1}{2}du$, then we get**:**

$$\mathrm{I} = 4\int_0^{+\infty} \frac{\ln^2\sqrt{u}}{(1+u)^2}.\frac{1}{2}du = 2\int_0^{+\infty} \frac{\left(\frac{1}{2}\ln u\right)^2}{(1+u)^2} du = 2\int_0^{+\infty} \frac{\frac{1}{4}(\ln u)^2}{(1+u)^2} du = \frac{1}{2}\int_0^{+\infty} \frac{(\ln u)^2}{(1+u)^2} du$$

$$\mathrm{I} = \frac{1}{2}\left\{\int_0^1 \frac{(\ln u)^2}{(1+u)^2} du + \int_1^{+\infty} \frac{(\ln u)^2}{(1+u)^2} du\right\}$$

For $\int_1^{+\infty} \frac{(\ln u)^2}{(1+u)^2} du$; let $v = \frac{1}{u} \Longrightarrow u = \frac{1}{v}$ & $du = -\frac{1}{v^2}dv$; for $u = 1$; $v = 1$ & for $u = +\infty$; then $v = 0$; so we get:

$$\int_1^{+\infty} \frac{(\ln u)^2}{(1+u)^2} du = \int_1^0 \frac{\left(\ln\frac{1}{v}\right)^2}{\left(1+\frac{1}{v}\right)^2}\left(-\frac{1}{v^2}dv\right) = \int_0^1 \frac{(-\ln v)^2}{\left(\frac{1+v}{v}\right)^2}.\frac{1}{v^2}dv = \int_0^1 \frac{(\ln v)^2}{\frac{1}{v^2}(1+v)^2}.\frac{1}{v^2}dv$$

So; $\int_1^{+\infty} \frac{(\ln u)^2}{(1+u)^2} du = \int_0^1 \frac{(\ln v)^2}{(1+v)^2} dv = \int_0^1 \frac{(\ln u)^2}{(1+u)^2} du$; then we get:

$$\mathrm{I} = \frac{1}{2}\left\{\int_0^1 \frac{(\ln u)^2}{(1+u)^2} du + \int_0^1 \frac{(\ln u)^2}{(1+u)^2} du\right\} = \int_0^1 \frac{(\ln u)^2}{(1+u)^2} du = \int_0^1 (\ln u)^2.\frac{1}{(1+u)^2} du$$

$$\mathrm{I} = \int_0^1 (\ln u)^2.\sum_{n=1}^{+\infty} (-1)^{n-1} n u^{n-1}\, du = \sum_{n=1}^{+\infty} (-1)^{n-1} n \int_0^1 u^{n-1}(\ln u)^2\, du$$

Change of variable: let $z = -\ln u \Longrightarrow u = e^{-z}$ & $du = -e^{-z}dz$, for $u = 0$, then $z = +\infty$ & for $u = 1$, then $z = 0$, then we get**:**

$$\mathrm{I} = \sum_{n=1}^{+\infty} (-1)^{n-1} n \int_{+\infty}^0 (e^{-z})^{n-1}(-z)^2\,(-e^{-z}dz) = \sum_{n=1}^{+\infty} (-1)^{n-1} n \int_0^{+\infty} z^2 e^{-nz}\, dz$$

Change of variable: let $w = nz \Longrightarrow z = \frac{1}{n}w$ & $dz = \frac{1}{n}dw$, then we get**:**

$$\mathrm{I} = \sum_{n=1}^{+\infty} (-1)^{n-1} n \int_0^{+\infty} \left(\frac{1}{n}w\right)^2 e^{-w}.\frac{1}{n}dw = \sum_{n=1}^{+\infty} (-1)^{n-1} n \int_0^{+\infty} \frac{1}{n^3}w^2 e^{-w}\, dw; \text{ then we get:}$$

$$\mathrm{I} = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2}\int_0^{+\infty} w^2 e^{-w}\, dw = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2}\int_0^{+\infty} w^{3-1} e^{-w}\, dw = \sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2}\Gamma(3)$$

$$\mathrm{I} = 2\sum_{n=1}^{+\infty} \frac{(-1)^{n-1}}{n^2} = 2\eta(2) = \frac{\pi^2}{6}$$

180. $\int \frac{1}{\sin x\sqrt[3]{\sin^3 x+\cos^3 x}}dx$

$$\int \frac{1}{\sin x\sqrt[3]{\sin^3 x+\cos^3 x}}dx=\int \frac{1}{\sin x\sqrt[3]{\sin^3 x\left(1+\frac{\cos^3 x}{\sin^3 x}\right)}}dx;\ \text{ then we get:}$$

$$\int \frac{1}{\sin x\sqrt[3]{\sin^3 x+\cos^3 x}}dx=\int \frac{1}{\sin^2 x\sqrt[3]{1+\cot^3 x}}dx=\int \frac{1}{\sqrt[3]{1+\cot^3 x}}.\csc^2 x\,dx;\ \text{ then}$$

let $y=\cot x$, so $dy=-\csc^2 x\,dx$, so we get:

$$\int \frac{1}{\sin x\sqrt[3]{\sin^3 x+\cos^3 x}}dx=-\int \frac{1}{\sqrt[3]{1+y^3}}dy=-\int \frac{1}{y^{-3}\sqrt[3]{1+y^{-3}}}.y^{-4}\,dy$$

Let $z=\sqrt[3]{1+y^{-3}}$, so $z^3=1+y^{-3}$, $y^{-3}=z^3-1$ and $-3y^{-4}dy=3z^2dz$, then we get:

$$\int \frac{1}{\sin x\sqrt[3]{\sin^3 x+\cos^3 x}}dx=\int \frac{z^2}{(z^3-1)z}dz=\int \frac{z}{z^3-1}dz=\frac{1}{3}\int\left(\frac{1}{z-1}-\frac{z-1}{z^2+z+1}\right)dz$$

$$\int \frac{1}{\sin x\sqrt[3]{\sin^3 x+\cos^3 x}}dx=\frac{1}{3}\ln|z-1|-\frac{1}{6}\ln|z^2+z+1|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)+c$$

$$=\frac{1}{3}\ln\left|\sqrt[3]{1+y^{-3}}-1\right|-\frac{1}{6}\ln\left|\left(\sqrt[3]{1+y^{-3}}\right)^2+\sqrt[3]{1+y^{-3}}+1\right|+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt[3]{1+y^{-3}}+1}{\sqrt{3}}\right)$$

$+c$; then we get $\int \frac{1}{\sin x\sqrt[3]{\sin^3 x+\cos^3 x}}dx=$

$$\frac{1}{3}\ln\left|\sqrt[3]{1+\tan^3 x}-1\right|-\frac{1}{6}\ln\left|\left(\sqrt[3]{1+\tan^3 x}\right)^2+\sqrt[3]{1+\tan^3 x}+1\right|$$

$$+\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2\sqrt[3]{1+\tan^3 x}+1}{\sqrt{3}}\right)+c$$

181. $\mathrm{I}=\int_0^{\pi}\frac{\ln(x+\pi)}{x^2+\pi^2}dx$

Let $x=\pi\tan u\Longrightarrow dx=\pi\sec^2 u\,du$; for $x=0\Longrightarrow u=0$ & for $x=\pi\Longrightarrow u=\frac{\pi}{4}$; then:

$$\mathrm{I}=\int_0^{\pi}\frac{\ln(x+\pi)}{x^2+\pi^2}dx=\int_0^{\frac{\pi}{4}}\frac{\ln(\pi\tan u+\pi)}{(\pi\tan u)^2+\pi^2}.\pi\sec^2 u\,du;\ \text{ then we get:}$$

$$\mathrm{I}=\int_0^{\frac{\pi}{4}}\frac{\ln(\pi\tan u+\pi)}{\pi^2(1+\tan^2 u)}.\pi\sec^2 u\,du=\int_0^{\frac{\pi}{4}}\frac{\ln(\pi\tan u+\pi)}{\pi\sec^2 u}.\sec^2 u\,du=\int_0^{\frac{\pi}{4}}\frac{\ln(\pi\tan u+\pi)}{\pi}du$$

$$\mathrm{I}=\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\ln\big(\pi(1+\tan u)\big)\,du=\frac{1}{\pi}\int_0^{\frac{\pi}{4}}[\ln\pi+\ln(1+\tan u)]du=\frac{\ln\pi}{4}+\frac{1}{\pi}\int_0^{\frac{\pi}{4}}\ln(1+\tan u)\,du$$

Now calculating $\mathrm{J}=\int_0^{\frac{\pi}{4}}\ln(1+\tan u)\,du$ ... (1)

Let $y=\frac{\pi}{4}-u\Longrightarrow dy=-du$; for $u=0\Longrightarrow y=\frac{\pi}{4}$ & for $u=\frac{\pi}{4}\Longrightarrow u=0$; then we get:

$$\mathrm{J} = \int_{\frac{\pi}{4}}^{0} \ln\left(1+\tan\left(\frac{\pi}{4}-y\right)\right)(-dy) = \int_{0}^{\frac{\pi}{4}} \ln\left(1+\tan\left(\frac{\pi}{4}-y\right)\right) dy; \text{ then we get:}$$

$$\mathrm{J} = \int_{0}^{\frac{\pi}{4}} \ln\left(1+\frac{\tan\frac{\pi}{4}-\tan y}{1+\tan\frac{\pi}{4}\tan y}\right) dy = \int_{0}^{\frac{\pi}{4}} \ln\left(1+\frac{1-\tan y}{1+\tan y}\right) dy = \int_{0}^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan y}\right) dy$$

$$\mathrm{J} = \int_{0}^{\frac{\pi}{4}} \ln(2)\, dy - \int_{0}^{\frac{\pi}{4}} \ln(1+\tan y)\, dy = \int_{0}^{\frac{\pi}{4}} \ln(2)\, dy - \int_{0}^{\frac{\pi}{4}} \ln(1+\tan u)\, du = \frac{\pi}{4}\ln 2 - \mathrm{J}$$

$$2\mathrm{J} = \frac{\pi}{4}\ln 2 \Longrightarrow \mathrm{J} = \frac{\pi}{8}\ln 2\,; \text{ then we get:}$$

$$\mathrm{I} = \frac{\ln\pi}{4} + \frac{1}{\pi}\left(\frac{\pi}{8}\ln 2\right) = \frac{\ln\pi}{4} + \frac{\ln 2}{8} = \frac{\ln\pi}{4} + \frac{\ln\sqrt{2}}{4} = \frac{\ln\left(\pi\sqrt{2}\right)}{4}$$

182. $\mathrm{I} = \int_0^{+\infty} \frac{\ln x}{\sqrt{(1+x^2)(2+x^2)}} dx$

Let $y = \frac{\sqrt{2}}{x}$ $\left(\& \ x = \frac{\sqrt{2}}{y}\right) \Longrightarrow dx = -\frac{\sqrt{2}}{y^2}dy$; for $x = 0 \Longrightarrow y = +\infty$ & for $x = +\infty \Longrightarrow y = 0$; then we get:

$$\mathrm{I} = \int_{0}^{+\infty} \frac{\ln x}{\sqrt{(1+x^2)(2+x^2)}} dx = \int_{0}^{+\infty} \frac{\ln\left(\frac{\sqrt{2}}{y}\right)}{\sqrt{\left[1+\left(\frac{\sqrt{2}}{y}\right)^2\right]\left[2+\left(\frac{\sqrt{2}}{y}\right)^2\right]}}\left(-\frac{\sqrt{2}}{y^2}dy\right); \text{ then we get:}$$

$$\mathrm{I} = \sqrt{2}\int_{0}^{+\infty} \frac{\ln\sqrt{2}-\ln y}{y^2\sqrt{\left(\frac{y^2+2}{y^2}\right)\left(\frac{2y^2+2}{y^2}\right)}} dy = \sqrt{2}\int_{0}^{+\infty} \frac{\ln\sqrt{2}-\ln y}{y^2 \times \frac{1}{y} \times \frac{\sqrt{2}}{y}\sqrt{(1+y^2)(2+y^2)}} dy$$

$$\mathrm{I} = \int_{0}^{+\infty} \frac{\ln\sqrt{2}-\ln y}{\sqrt{(1+y^2)(2+y^2)}} dy = \int_{0}^{+\infty} \frac{\ln\sqrt{2}}{\sqrt{(1+y^2)(2+y^2)}} dy - \int_{0}^{+\infty} \frac{\ln y}{\sqrt{(1+y^2)(2+y^2)}} dy$$

Then we get: $\mathrm{I} = \int_{0}^{+\infty} \frac{\ln\sqrt{2}}{\sqrt{(1+y^2)(2+y^2)}} dy - \int_{0}^{+\infty} \frac{\ln x}{\sqrt{(1+x^2)(2+x^2)}} dy$; so:

$$\mathrm{I} = \int_{0}^{+\infty} \frac{\ln\sqrt{2}}{\sqrt{(1+y^2)(2+y^2)}} dy - \mathrm{I} \Longrightarrow \mathrm{I} = \frac{1}{2}\int_{0}^{+\infty} \frac{\ln\sqrt{2}}{\sqrt{(1+y^2)(2+y^2)}} dy$$

Let $y = \sqrt{\frac{2\sin u}{1-\sin u}} \Longrightarrow dy = \frac{\cos u}{(1-\sin u)^2}\sqrt{\frac{1-\sin u}{2\sin u}}\,du$; for $y = 0 \Longrightarrow u = 0$ & for $y = +\infty \Longrightarrow 1-\sin u = 0 \Longrightarrow \sin u = 1 \Longrightarrow u = \frac{\pi}{2}$; then we get:

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\ln\sqrt{2}}{\sqrt{\left(1+\frac{2\sin u}{1-\sin u}\right)\left(2+\frac{2\sin u}{1-\sin u}\right)}}\times\frac{\cos u}{(1-\sin u)^2}\sqrt{\frac{1-\sin u}{2\sin u}}du;\ \text{ then:}$$

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\ln\sqrt{2}}{\sqrt{\left(\frac{1-\sin u+2\sin u}{1-\sin u}\right)\left(\frac{2-2\sin u+2\sin u}{1-\sin u}\right)}}\times\frac{\cos u}{(1-\sin u)^2}\sqrt{\frac{1-\sin u}{2\sin u}}du$$

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\ln\sqrt{2}}{\sqrt{\left(\frac{1+\sin u}{1-\sin u}\right)\left(\frac{2}{1-\sin u}\right)}}\times\frac{\cos u}{(1-\sin u)^2}\sqrt{\frac{1-\sin u}{2\sin u}}du$$

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\ln\sqrt{2}}{\sqrt{2\left[\frac{1+\sin u}{(1-\sin u)^2}\right]}}\times\frac{\cos u}{(1-\sin u)^2}\sqrt{\frac{1-\sin u}{2\sin u}}du;\ \text{ then we get:}$$

$$I=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\frac{1}{2}\ln 2}{\frac{\sqrt{2}}{(1-\sin u)}\sqrt{1+\sin u}}\times\frac{\cos u}{(1-\sin u)^2}\sqrt{\frac{1-\sin u}{2\sin u}}du$$

$$I=\frac{\ln 2}{8}\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{1+\sin u}}\times\frac{\cos u}{\sqrt{1-\sin u}}\sqrt{\frac{1}{\sin u}}du=\frac{\ln 2}{8}\int_0^{\frac{\pi}{2}}\frac{\cos u}{\sqrt{1-\sin^2 u}}\sqrt{\frac{1}{\sin u}}du;\text{then we get:}$$

$$I=\frac{\ln 2}{8}\int_0^{\frac{\pi}{2}}\frac{\cos u}{\sqrt{\cos^2 u}}\sqrt{\frac{1}{\sin u}}du=\frac{\ln 2}{8}\int_0^{\frac{\pi}{2}}\sqrt{\frac{1}{\sin u}}du=\frac{\ln 2}{8}\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}u\,du$$

$$\text{Beta function: } \mathrm{B}(m;n)=2\int_0^{\frac{\pi}{2}}(\sin x)^{2m-1}(\cos x)^{2n-1}dx$$

$$I=\frac{\ln 2}{8}\int_0^{\frac{\pi}{2}}\sin^{-\frac{1}{2}}u\,du=\frac{\ln 2}{8}\int_0^{\frac{\pi}{2}}(\sin u)^{2\left(\frac{1}{4}\right)-1}(\cos u)^{2\left(\frac{1}{2}\right)-1}du=\frac{\ln 2}{8}\left(\frac{1}{2}\mathrm{B}\left(\frac{1}{4};\frac{1}{2}\right)\right);\text{then:}$$

$$I=\frac{\ln 2}{8}\times\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{2}+\frac{3}{4}\right)}=\frac{\ln 2}{8}\times\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{2\Gamma\left(\frac{1}{2}+\frac{1}{4}\right)}=\frac{\ln 2}{16}.\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{2\Gamma\left(\frac{3}{4}\right)}$$

183. **First Method:** $I=\int_0^{+\infty}\frac{\tan^{-1}x}{x^{\ln x+1}}dx$

Change of variable: let $t=\ln x$ or $x=e^t$, then $dx=e^t dt$

New bounds: for $x=0$, $t=\ln 0=-\infty$ and for $x=+\infty$, then $t=\ln+\infty=+\infty$, then we get**:**

$$I=\int_0^{+\infty}\frac{\tan^{-1}x}{x^{\ln x+1}}dx=\int_{-\infty}^{+\infty}\frac{\tan^{-1}(e^t)}{(e^t)^{t+1}}.e^t dt=\int_{-\infty}^{+\infty}\frac{\tan^{-1}(e^t)}{e^{t^2}.e^t}.e^t dt=\int_{-\infty}^{+\infty}\frac{\tan^{-1}(e^t)}{e^{t^2}}dt;\ \text{ then:}$$

$$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-t^2} \tan^{-1}(e^t)dt \ \ldots\ (1)$$

Change of variable: let $u = -t$, then $t = -u$ and $dt = -du$

New bounds: for $t = -\infty$, then $u = +\infty$ and for $t = +\infty$, then $u = -\infty$, then we get**:**

$$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-t^2} \tan^{-1}(e^t)dt = \int_{+\infty}^{-\infty} e^{-(-u)^2} \tan^{-1}(e^{-u})(-du) = \int_{-\infty}^{+\infty} e^{-u^2} \tan^{-1}(e^{-u})du$$

$$\text{Then: } \mathrm{I} = \int_{-\infty}^{+\infty} e^{-t^2} \tan^{-1}(e^{-t})dt \ \ldots\ (2)$$

Adding (1) & (2) we get:

$$\mathrm{I} + \mathrm{I} = 2\mathrm{I} = \int_{-\infty}^{+\infty} e^{-t^2} \tan^{-1}(e^t)dt + \int_{-\infty}^{+\infty} e^{-t^2} \tan^{-1}(e^{-t})dt\,; \quad \text{then we get:}$$

$$2\mathrm{I} = \int_{-\infty}^{+\infty} e^{-t^2} [\tan^{-1}(e^t) + \tan^{-1}(e^{-t})]dt$$

$$\text{With: } \tan^{-1}(e^t) + \tan^{-1}(e^{-t}) = \frac{\pi}{2}; \text{ then we get: } 2\mathrm{I} = \int_{-\infty}^{+\infty} e^{-t^2} . \frac{\pi}{2} dt; \text{ then}$$

$$2\mathrm{I} = \frac{\pi}{2} \int_{-\infty}^{+\infty} e^{-t^2}\, dt \Longrightarrow \mathrm{I} = \frac{\pi}{4} \int_{-\infty}^{+\infty} e^{-t^2}\, dt; \text{ with } \int_{-\infty}^{+\infty} e^{-t^2}\, dt = \sqrt{\pi} \text{ (Guassian Integral)}$$

$$\text{Therefore: } \mathrm{I} = \frac{\pi}{4}\sqrt{\pi} = \frac{\pi\sqrt{\pi}}{4} = \frac{1}{4}\pi^{\frac{3}{2}}$$

Second Method: $\mathrm{I} = \int_0^{+\infty} \frac{\tan^{-1} x}{x^{\ln x + 1}} dx$

$$\mathrm{I} = \int_0^{+\infty} \frac{\tan^{-1} x}{x^{\ln x+1}} dx = \int_0^{+\infty} \frac{\tan^{-1} x}{x^{\ln x} . x} dx = \int_0^{+\infty} \frac{\tan^{-1} x}{(e^{\ln x})^{\ln x} . x} dx = \int_0^{+\infty} \frac{\tan^{-1} x}{x e^{\ln^2 x}} dx\,; \text{then we can write}$$

$$\mathrm{I} = \int_0^{1} \frac{\tan^{-1} x}{x e^{\ln^2 x}} dx + \int_1^{+\infty} \frac{\tan^{-1} x}{x e^{\ln^2 x}} dx = \mathrm{J}_1 + \mathrm{J}_2$$

$$\text{For } \mathrm{J}_2 = \int_1^{+\infty} \frac{\tan^{-1} x}{x e^{\ln^2 x}} dx\,; \text{let } y = \frac{1}{x} \Longrightarrow x = \frac{1}{y} \ \&\ dx = -\frac{1}{y^2} dy$$

For $x = 1$; then $y = 1$ and for $x = +\infty$; then $y = 0$; then we get:

$$\mathrm{J}_2 = \int_1^{+\infty} \frac{\tan^{-1} x}{x e^{\ln^2 x}} dx = \int_1^{0} \frac{\tan^{-1}\left(\frac{1}{y}\right)}{\frac{1}{y} e^{\ln^2\left(\frac{1}{y}\right)}} \left(-\frac{1}{y^2} dy\right) = \int_0^{1} \frac{\tan^{-1}\left(\frac{1}{y}\right)}{y e^{(-\ln y)^2}} dy = \int_0^{1} \frac{\tan^{-1}\left(\frac{1}{y}\right)}{y e^{\ln^2 y}} dy$$

$$\mathrm{J}_2 = \int_0^{1} \frac{\frac{\pi}{2} - \tan^{-1} y}{y e^{\ln^2 y}} dy = \int_0^{1} \frac{\frac{\pi}{2} - \tan^{-1} x}{x e^{\ln^2 x}} dx\,; \text{ then we get:}$$

$$\mathrm{I}=\int_0^1\frac{\tan^{-1}x}{xe^{\ln^2 x}}dx+\int_0^1\frac{\frac{\pi}{2}-\tan^{-1}x}{xe^{\ln^2 x}}dx=\int_0^1\frac{\tan^{-1}x}{xe^{\ln^2 x}}dx+\frac{\pi}{2}\int_0^1\frac{1}{xe^{\ln^2 x}}dx-\int_0^1\frac{\tan^{-1}x}{xe^{\ln^2 x}}dx$$

$$\mathrm{I}=\frac{\pi}{2}\int_0^1\frac{1}{xe^{\ln^2 x}}dx=\frac{\pi}{2}\int_0^1 e^{-\ln^2 x}.\frac{1}{x}dx$$

Let $t=\ln^2 x$ & for $0<x\le 1$; $\ln x=-\sqrt{t}\Longrightarrow \frac{1}{x}dx=-\frac{1}{2\sqrt{t}}dt$

For $x=0\Longrightarrow t=+\infty$ & for $x=1\Longrightarrow t=1$; then we get:

$$\mathrm{I}=\frac{\pi}{2}\int_{+\infty}^{0}e^{-t}\left(-\frac{1}{2\sqrt{t}}dt\right)=\frac{\pi}{4}\int_0^{+\infty}t^{-\frac{1}{2}}.e^{-t}dt=\frac{\pi}{4}\int_0^{+\infty}t^{\frac{1}{2}-1}.e^{-t}dt=\frac{\pi}{4}\Gamma\left(\frac{1}{2}\right)=\frac{\pi}{4}\sqrt{\pi}=\frac{\pi^{\frac{3}{2}}}{4}$$

184. $\mathrm{I}=\int_{-\infty}^{+\infty}e^{x-\sinh^2 x}dx$, we know that $\sinh x=\frac{e^x-e^{-x}}{2}$, then we get:

$$\mathrm{I}=\int_{-\infty}^{+\infty}e^{x-\left(\frac{e^x-e^{-x}}{2}\right)^2}dx=\int_{-\infty}^{+\infty}e^x.e^{-\left(\frac{e^x-e^{-x}}{2}\right)^2}dx$$

Let $u=e^x$, then $du=e^xdx$, for $x=-\infty$, $u=0$ and for $x=+\infty$, $u=+\infty$, then we get:

$$\mathrm{I}=\int_0^{+\infty}e^{-\left(\frac{u-u^{-1}}{2}\right)^2}du=\int_0^{+\infty}e^{-\left(\frac{1}{2}u-\frac{1}{2u}\right)^2}du\ \ldots(1)$$

Let $t=\frac{1}{u}$, then $u=\frac{1}{t}$ and $du=-\frac{1}{t^2}dt$ for $u=0$, $t=+\infty$ and for $u=+\infty$, $t=0$, then:

$$\mathrm{I}=\int_{+\infty}^{0}e^{-\left(\frac{1}{2t}-\frac{1}{2}t\right)^2}\left(-\frac{1}{t^2}\right)dt=\int_0^{+\infty}e^{-\left[-\left(\frac{1}{2}t-\frac{1}{2t}\right)\right]^2}.\frac{1}{t^2}dt=\int_0^{+\infty}\frac{1}{t^2}e^{-\left(\frac{1}{2}t-\frac{1}{2t}\right)^2}dt$$

Then we can write $\mathrm{I}=\int_0^{+\infty}\frac{1}{u^2}e^{-\left(\frac{1}{2}u-\frac{1}{2u}\right)^2}du\ \ldots(2)$

Now adding (1) & (2) we get:

$$2\mathrm{I}=\int_0^{+\infty}e^{-\left(\frac{1}{2}u-\frac{1}{2u}\right)^2}du+\int_0^{+\infty}\frac{1}{u^2}e^{-\left(\frac{1}{2}u-\frac{1}{2u}\right)^2}du=\int_0^{+\infty}\left(1+\frac{1}{u^2}\right)e^{-\left(\frac{1}{2}u-\frac{1}{2u}\right)^2}du$$

Let $y=\frac{1}{2}u-\frac{1}{2u}=\frac{1}{2}\left(u-\frac{1}{u}\right)$, then $dy=\frac{1}{2}\left(1+\frac{1}{u^2}\right)du\Rightarrow\left(1+\frac{1}{u^2}\right)du=2dy$

For $u=0$, $y=-\infty$ and for $u=+\infty$, then $y=+\infty$ so we get:

$$2\mathrm{I}=\int_{-\infty}^{+\infty}e^{-y^2}.2dy=2\int_{-\infty}^{+\infty}e^{-y^2}dy;\ \text{so}\ \mathrm{I}=\int_{-\infty}^{+\infty}e^{-y^2}dy=\sqrt{\pi}$$

185. $\mathrm{I}=\int_{-\infty}^{+\infty}\frac{dx}{\cosh\pi x-a}$, where $|a|<1$, we have:

$\cosh u=\frac{e^u+e^{-u}}{2}$; then $\cosh\pi x=\frac{e^{\pi x}+e^{-\pi x}}{2}$; then we get:

$$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{dx}{\frac{e^{\pi x}+e^{-\pi x}}{2}-a}=\int_{-\infty}^{+\infty}\frac{2}{e^{\pi x}+e^{-\pi x}-2a}dx=\int_{-\infty}^{+\infty}\frac{2}{e^{\pi x}+e^{-\pi x}-2a}\times\frac{e^{\pi x}}{e^{\pi x}}dx;\ \text{then}$$

$$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{2e^{\pi x}}{e^{2\pi x}-2ae^{\pi x}+1}dx=\int_{-\infty}^{+\infty}\frac{2e^{\pi x}}{e^{2\pi x}-2ae^{\pi x}+a^2+1-a^2}dx;\ \text{ then we get:}$$

$$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{2e^{\pi x}}{(e^{2\pi x}-2ae^{\pi x}+a^2)+\left(\sqrt{1-a^2}\right)^2}dx=\int_{-\infty}^{+\infty}\frac{2e^{\pi x}}{(e^{\pi x}-a)^2+\left(\sqrt{1-a^2}\right)^2}dx$$

Let $u=e^{\pi x}$, then $du=\pi e^{\pi x}dx$, then $e^{\pi x}dx=\frac{1}{\pi}du$, for $x=-\infty$, $u=0$ and for $x=+\infty$, $u=+\infty$, then we get**:**

$$\mathrm{I}=\int_0^{+\infty}\frac{2}{(u-a)^2+\left(\sqrt{1-a^2}\right)^2}\times\frac{1}{\pi}du=\frac{2}{\pi}\int_0^{+\infty}\frac{du}{(u-a)^2+\left(\sqrt{1-a^2}\right)^2};\ \text{ then we get:}$$

$$\mathrm{I}=\frac{2}{\pi}\left[\frac{1}{\sqrt{1-a^2}}\tan^{-1}\left(\frac{u-a}{\sqrt{1-a^2}}\right)\right]_0^{+\infty}=\frac{2}{\pi\sqrt{1-a^2}}\left[\tan^{-1}+\infty-\tan^{-1}\left(-\frac{a}{\sqrt{1-a^2}}\right)\right];\text{therefore}$$

$$\mathrm{I}=\frac{2}{\pi\sqrt{1-a^2}}\left[\frac{\pi}{2}-\tan^{-1}\left(-\frac{a}{\sqrt{1-a^2}}\right)\right]$$

186. $\mathrm{I}=\int_0^{+\infty}\tanh x\,.\,e^{-ax}dx=\int_0^{+\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}.\,e^{-ax}dx=\int_0^{+\infty}\frac{e^{2x}-1}{e^{2x}+1}.\,e^{-ax}dx$

Let $x=-\ln\sqrt{t}\ \ (t=e^{-2x})$, then $dx=-\dfrac{\frac{1}{2\sqrt{t}}}{\sqrt{t}}dt=-\dfrac{1}{2t}dt$, for $x=0$, $t=1$ and for $x=+\infty$, then $t=0$, then we get**:**

$$\mathrm{I}=\int_1^0\frac{\frac{1}{t}-1}{\frac{1}{t}+1}e^{-a(-\ln\sqrt{t})}.\left(-\frac{1}{2t}dt\right)=\frac{1}{2}\int_0^1\frac{1-t}{1+t}e^{a\ln\sqrt{t}}.\frac{1}{t}dt=\frac{1}{2}\int_0^1\frac{1-t}{1+t}e^{a\ln t^{\frac{1}{2}}}.\frac{1}{t}dt;\ \text{ then:}$$

$$\mathrm{I}=\frac{1}{2}\int_0^1\frac{1-t}{1+t}e^{\ln\left(t^{\frac{1}{2}}\right)^a}.\frac{1}{t}dt=\frac{1}{2}\int_0^1\frac{1-t}{1+t}e^{\ln\left(t^{\frac{a}{2}}\right)}.\frac{1}{t}dt=\frac{1}{2}\int_0^1\frac{1-t}{1+t}.t^{\frac{a}{2}}.\frac{1}{t}dt=\frac{1}{2}\int_0^1\frac{1-t}{1+t}.t^{\frac{a}{2}-1}dt$$

$$\mathrm{I}=\frac{1}{2}\left(\int_0^1\frac{t^{\frac{a}{2}-1}}{1+t}dt-\int_0^1\frac{t.t^{\frac{a}{2}-1}}{1+t}dt\right)=\frac{1}{2}\left(\int_0^1\frac{t^{\frac{a}{2}-1}}{1+t}dt-\int_0^1\frac{t^{\frac{a}{2}}}{1+t}dt\right)$$

We have $\displaystyle\int_0^1\frac{x^{s-1}}{1+x}dx=\frac{1}{2}\left(\Psi\left(\frac{1+s}{2}\right)-\Psi\left(\frac{s}{2}\right)\right)$

$$=\Psi(s)-\Psi\left(\frac{s}{2}\right)-\ln 2\,;\Psi\text{: digamma function}$$

Then we get : $\mathrm{I}=\dfrac{1}{2}\left[\Psi\left(\dfrac{a+2}{4}\right)-\dfrac{1}{2}\Psi\left(\dfrac{a}{4}\right)-\dfrac{1}{2}\Psi\left(1+\dfrac{a}{4}\right)\right]$

But we have: $\Psi(s+1)=\Psi(s)+\dfrac{1}{s}$; therefore; we get:

$$\mathrm{I}=\int_0^{+\infty}\tanh x\,.\,e^{-ax}dx=\frac{1}{2}\left(\Psi\left(\frac{a+2}{4}\right)-\Psi\left(\frac{a}{4}\right)-\frac{2}{s}\right)$$

187. $\mathrm{I}=\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx$

Change of variable: let $z^3=\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\Longrightarrow x=\left(\dfrac{z^3+1}{z^3-1}\right)^2\Longrightarrow\dfrac{1}{1+\sqrt{x}}=\dfrac{z^3-1}{2z^3}$ and

$$dx=\frac{-12z^2(z^3+1)}{(z^3-1)^2}dz;\quad \text{then we get:}$$

$$\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=\int\frac{z^3-1}{2z^3}.z.\frac{-12z^2(z^3+1)}{(z^3-1)^2}dz=-6\int\frac{z^3+1}{z^3-1}dz;\quad \text{then we get:}$$

$$\mathrm{I}=-6\int\frac{z^3+1-2+2}{z^3-1}dz=-6\int\left(\frac{z^3-1}{z^3-1}+2.\frac{1}{z^3-1}\right)dz=-6z-6\int\frac{2}{z^3-1}dz$$

$$\mathrm{I}=-6z-6\int\frac{2}{(z-1)(z^2+z+1)}dz$$

By using the method of decomposition into partial fractions we get**:**

$$\frac{2}{(z-1)(z^2+z+1)}=\frac{2}{3}.\frac{1}{z-1}-\frac{2}{3}.\frac{z+2}{z^2+z+1};\quad \text{then we get:}$$

$$\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=-6z-6\times\frac{2}{3}\int\left(\frac{1}{z-1}-\frac{z+2}{z^2+z+1}\right)dz;\quad \text{then:}$$

$$\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=-6z-4\ln|z-1|+4\int\frac{1}{2}.\frac{2z+4}{z^2+z+1}dz$$

$$\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=-6z-4\ln|z-1|+2\int\left(\frac{2z+1}{z^2+z+1}+\frac{3}{z^2+z+1}\right)dz$$

$$\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=-6z-4\ln|z-1|+2\ln|z^2+z+1|+2\int\frac{3}{z^2+z+1}dz$$

$$\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=-6z-4\ln|z-1|+2\ln|z^2+z+1|+6\int\frac{1}{\left(z+\frac{1}{2}\right)^2+\frac{3}{4}}dz$$

$$\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=-6z-4\ln|z-1|+2\ln|z^2+z+1|+6\int\frac{1}{\left(z+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dz$$

$$\mathrm{I}=-6z-4\ln|z-1|+2\ln|z^2+z+1|+6\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)+c$$

$$\mathrm{I}=-6z-4\ln|z-1|+2\ln|z^2+z+1|+4\sqrt{3}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)+c$$

Therefore; we get: $\displaystyle\int\frac{1}{1+\sqrt{x}}.\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}}dx=$

$$-6z - 4\ln\left|\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}} - 1\right| + 2\ln\left|\left(\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)^{\frac{2}{3}} + \sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}} + 1\right| + 4\sqrt{3}\tan^{-1}\left(\frac{2\sqrt[3]{\frac{\sqrt{x}-1}{\sqrt{x}+1}} + 1}{\sqrt{3}}\right)$$
$$+ c$$

188. $\mathrm{I} = \int_0^{+\infty} \frac{dx}{\sqrt{x}(x^4+x^2+1)^{\frac{3}{4}}}$

$$\mathrm{I} = \int_0^{+\infty} \frac{dx}{\sqrt{x}(x^4+x^2+1)^{\frac{3}{4}}} = \int_0^{+\infty} \frac{dx}{\sqrt{x}\left[x^2\left(x^2+1+\frac{1}{x^2}\right)\right]^{\frac{3}{4}}} = \int_0^{+\infty} \frac{dx}{\sqrt{x}\times(x^2)^{\frac{3}{4}}\left(x^2+1+\frac{1}{x^2}\right)^{\frac{3}{4}}}$$

$$\mathrm{I} = \int_0^{+\infty} \frac{dx}{x^{\frac{1}{2}}\times x^{\frac{3}{2}}\left(x^2+1+\frac{1}{x^2}\right)^{\frac{3}{4}}} = \int_0^{+\infty} \frac{dx}{x^2\left(x^2+\frac{1}{x^2}+1\right)^{\frac{3}{4}}} \ldots(\mathbf{1})$$

Let $t = \frac{1}{x}$ $\left(x = \frac{1}{t}\right) \Rightarrow dx = -\frac{1}{t^2}dt$; for $x = 0 \Rightarrow t = +\infty$ & for $x = +\infty \Rightarrow t = 0$; then we get:

$$\mathrm{I} = \int_0^{+\infty} \frac{dx}{x^2\left(x^2+\frac{1}{x^2}+1\right)^{\frac{3}{4}}} = \int_{+\infty}^{0} \frac{-\frac{1}{t^2}}{\left(\frac{1}{t}\right)^2\left(\frac{1}{t^2}+\frac{1}{\frac{1}{t^2}}+1\right)^{\frac{3}{4}}}dt = \int_0^{+\infty} \frac{dt}{\left(t^2+\frac{1}{t^2}+1\right)^{\frac{3}{4}}}$$

Then we can write: $\mathrm{I} = \int_0^{+\infty} \frac{dx}{\left(x^2+\frac{1}{x^2}+1\right)^{\frac{3}{4}}}$ ... $(\mathbf{2})$; now adding (1) & (2) we get:

$$\mathrm{I}+\mathrm{I} = 2\mathrm{I} = \int_0^{+\infty} \frac{dx}{x^2\left(x^2+\frac{1}{x^2}+1\right)^{\frac{3}{4}}} + \int_0^{+\infty} \frac{dx}{\left(x^2+\frac{1}{x^2}+1\right)^{\frac{3}{4}}} = \int_0^{+\infty} \frac{1+\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}+1\right)^{\frac{3}{4}}}dx; \text{ then we get:}$$

$$\mathrm{I} = \frac{1}{2}\int_0^{+\infty} \frac{1+\frac{1}{x^2}}{\left(x^2+\frac{1}{x^2}+1\right)^{\frac{3}{4}}}dx = \frac{1}{2}\int_0^{+\infty} \frac{1+\frac{1}{x^2}}{\left[\left(x-\frac{1}{x}\right)^2+3\right]^{\frac{3}{4}}}dx$$

Let $y = x - \frac{1}{x} \Rightarrow dy = \left(1+\frac{1}{x^2}\right)dx$; for $x = 0 \Rightarrow y = -\infty$ & for $x = +\infty \Rightarrow y = +\infty$; then:

$$\mathrm{I} = \frac{1}{2}\int_0^{+\infty} \frac{1+\frac{1}{x^2}}{\left[\left(x-\frac{1}{x}\right)^2+3\right]^{\frac{3}{4}}}dx = \frac{1}{2}\int_{-\infty}^{+\infty} \frac{1}{(y^2+3)^{\frac{3}{4}}}dy$$

Let $y = \sqrt{3}\tan\theta \Rightarrow dy = \sqrt{3}\sec^2\theta\, d\theta$; for $x = -\infty$; $\theta = -\frac{\pi}{2}$ & for $x = +\infty$; $\theta = \frac{\pi}{2}$; then:

$$\mathrm{I}=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sqrt{3}\sec^2\theta}{(3\tan^2\theta+3)^{\frac{3}{4}}}d\theta=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sqrt{3}\sec^2\theta}{(3\sec^2\theta)^{\frac{3}{4}}}d\theta=\frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sqrt{3}\sec^2\theta}{3^{\frac{3}{4}}\sec^{\frac{3}{2}}\theta}d\theta=\frac{1}{2\sqrt[4]{3}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sec^{\frac{1}{2}}\theta\, d\theta;\text{ so}$$

$$\mathrm{I}=\frac{1}{2\sqrt[4]{3}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^{-\frac{1}{2}}\theta\, d\theta=\frac{1}{\sqrt[4]{3}}\int_{0}^{\frac{\pi}{2}}\cos^{-\frac{1}{2}}\theta\, d\theta=\frac{1}{\sqrt[4]{3}}\int_{0}^{\frac{\pi}{2}}\cos^{2\left(\frac{1}{4}\right)-1}\theta\sin^{2\left(\frac{1}{2}\right)-1}\theta\, d\theta=\frac{1}{\sqrt[4]{3}}\left\{\frac{1}{2}\mathrm{B}\left(\frac{1}{4};\frac{1}{2}\right)\right\}$$

$$\mathrm{I}=\frac{1}{2\sqrt[4]{3}}\times\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}+\frac{1}{2}\right)}=\frac{1}{2\sqrt[4]{3}}\times\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}=\frac{\Gamma\left(\frac{1}{4}\right)\sqrt{\pi}}{2\sqrt[4]{3}\Gamma\left(\frac{3}{4}\right)}$$

189. $\mathrm{I}=\int_0^{+\infty}\frac{1}{\sqrt{n+e^{\frac{\pi}{\sqrt{x}}}}}.\frac{dx}{\sqrt{x^3}}$

Change of variable: let $u=\frac{\pi}{\sqrt{x}}$, then $du=-\frac{\pi}{2\sqrt{x}(\sqrt{x})^2}dx=-\frac{\pi}{2\sqrt{x^3}}dx\Rightarrow -du\frac{2}{\pi}=\frac{dx}{\sqrt{x^3}}$

New bounds: for $x=0$, $u=+\infty$ and for $x=+\infty$, then $u=0$, then we get:

$$\mathrm{I}=\int_0^{+\infty}\frac{1}{\sqrt{n+e^{\frac{\pi}{\sqrt{x}}}}}.\frac{dx}{\sqrt{x^3}}=\int_{+\infty}^{0}\frac{1}{\sqrt{n+e^u}}\left(-\frac{2}{\pi}du\right)=\frac{2}{\pi}\int_0^{+\infty}\frac{1}{\sqrt{n+e^u}}du$$

Change of variable: let $z=\sqrt{n+e^u}\Rightarrow z^2=n+e^u\Rightarrow 2zdz=e^u du$, then we get:

$$dz=\frac{e^u}{2z}du=\frac{z^2-n}{2z}du\Rightarrow du=\frac{2z}{z^2-n}dz$$

New bounds: for $u=0$, then $z=\sqrt{n+1}$ and for $u=+\infty$, then $z=+\infty$, then we get:

$$\mathrm{I}=\frac{2}{\pi}\int_0^{+\infty}\frac{1}{\sqrt{n+e^u}}du=\frac{2}{\pi}\int_{\sqrt{n+1}}^{+\infty}\frac{1}{z}.\frac{2z}{z^2-n}dz=\frac{2}{\pi}\int_{\sqrt{n+1}}^{+\infty}\frac{2}{z^2-n}dz=\frac{2}{\pi}\int_{\sqrt{n+1}}^{+\infty}\frac{2}{(z-\sqrt{n})(z+\sqrt{n})}dz$$

$$\mathrm{I}=\frac{2}{\pi}\times\frac{1}{\sqrt{n}}\int_{\sqrt{n+1}}^{+\infty}\frac{(\sqrt{n}+\sqrt{n})+(z-z)}{(z-\sqrt{n})(z+\sqrt{n})}dz=\frac{2}{\pi\sqrt{n}}\int_{\sqrt{n+1}}^{+\infty}\frac{(z+\sqrt{n})-(z-\sqrt{n})}{(z-\sqrt{n})(z+\sqrt{n})}dz\text{; then we get:}$$

$$\mathrm{I}=\frac{2}{\pi\sqrt{n}}\left(\int_{\sqrt{n+1}}^{+\infty}\frac{1}{z-\sqrt{n}}dz-\int_{\sqrt{n+1}}^{+\infty}\frac{1}{z+\sqrt{n}}dz\right)=\frac{2}{\pi\sqrt{n}}\left[\ln(z-\sqrt{n})-\ln(z+\sqrt{n})\right]_{\sqrt{n+1}}^{+\infty};\quad\text{then:}$$

$$\mathrm{I}=\frac{2}{\pi\sqrt{n}}\left[\ln\left(\frac{z-\sqrt{n}}{z+\sqrt{n}}\right)\right]_{\sqrt{n+1}}^{+\infty}=\frac{2}{\pi\sqrt{n}}\left[\lim_{a\to+\infty}\ln\left(\frac{z-\sqrt{n}}{z+\sqrt{n}}\right)\right]_{\sqrt{n+1}}^{a}$$

$$\mathrm{I}=\frac{2}{\pi\sqrt{n}}\left[\lim_{a\to+\infty}\ln\left(\frac{a-\sqrt{n}}{a+\sqrt{n}}\right)-\ln\left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\right]=\frac{2}{\pi\sqrt{n}}\left[\ln\left(\frac{a}{a}\right)-\ln\left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\right]$$

$$\mathrm{I}=\frac{2}{\pi\sqrt{n}}\left[\ln(1)-\ln\left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\right]=\frac{2}{\pi\sqrt{n}}\left[-\ln\left(\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n+1}+\sqrt{n}}\right)\right]=\frac{2}{\pi\sqrt{n}}\ln\left(\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\right)$$

$$\mathrm{I}=\frac{2}{\pi\sqrt{n}}\ln\left[\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\times\frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}-\sqrt{n}}\right]=\frac{2}{\pi\sqrt{n}}\ln\left[\frac{(\sqrt{n+1}+\sqrt{n})^2}{n+1-n}\right];\quad\text{then we get:}$$

$$\mathrm{I} = \frac{2}{\pi\sqrt{n}}\ln\left[\left(\sqrt{n+1}+\sqrt{n}\right)^2\right] = \frac{2}{\pi\sqrt{n}}.2\ln\left(\sqrt{n+1}+\sqrt{n}\right) = \frac{4}{\pi\sqrt{n}}\ln\left(\sqrt{n+1}+\sqrt{n}\right)$$

But we have: $\sinh^{-1} x = \ln\left(x+\sqrt{x^2+1}\right)$; so $\sinh^{-1}(\sqrt{n}) = \ln\left(\sqrt{n}+\sqrt{n+1}\right)$

Therefore: $\mathrm{I} = \frac{4}{\pi}.\frac{\sinh^{-1}(\sqrt{n})}{\sqrt{n}}$

190. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \ln|\ln(\tan x)|\,dx = \int_0^{\frac{\pi}{4}} \ln|\ln(\tan x)|\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln|\ln(\tan x)|\,dx$

For $\int_0^{\frac{\pi}{4}} \ln|\ln(\tan x)|\,dx$; let $u = \frac{\pi}{2} - x \Rightarrow du = -dx$; for $x = 0; u = \frac{\pi}{2}$ & for $x = \frac{\pi}{4}; u = \frac{\pi}{4}$ so

$$\int_0^{\frac{\pi}{4}} \ln|\ln(\tan x)|\,dx = \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} \ln\left|\ln\left(\tan\left(\frac{\pi}{2}-u\right)\right)\right|(-du) = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left|\ln\left(\tan\left(\frac{\pi}{2}-u\right)\right)\right|du\text{; then:}$$

$$\int_0^{\frac{\pi}{4}} \ln|\ln(\tan x)|\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln|\ln(\cot u)|\,du = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln\left|\ln\left(\frac{1}{\tan u}\right)\right|du = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln|-\ln(\tan u)|\,du$$

So; $\int_0^{\frac{\pi}{4}} \ln|\ln(\tan x)|\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln|\ln(\tan u)|\,du = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln|\ln(\tan x)|\,dx$; then we get:

$$\mathrm{I} = \int_0^{\frac{\pi}{2}} \ln|\ln(\tan x)|\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln|\ln(\tan x)|\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln|\ln(\tan x)|\,dx = 2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \ln|\ln(\tan x)|\,dx$$

Change of variable: Let $x = \tan^{-1}(e^t)$ & $e^t = \tan x$, so $t = \ln(\tan x)$, then we get**:**

$$dx = \frac{e^t}{1+(e^t)^2}dt = \frac{e^t}{1+e^{2t}}dt$$

For $x = \frac{\pi}{4}$, then $t = \ln 1 = 0$ and for $x = \frac{\pi}{2}$, then $t = \ln(+\infty) = +\infty$, then we get**:**

$$\mathrm{I} = 2\int_0^{+\infty} \ln t\,.\frac{e^t}{1+e^{2t}}dt = 2\int_0^{+\infty} \ln t\,.\frac{e^t}{1+e^{2t}} \times \frac{e^{-2t}}{e^{-2t}}dt = 2\int_0^{+\infty} \frac{e^{-t}\ln t}{1+e^{-2t}}dt\text{; then we can write:}$$

$$\mathrm{I} = 2\int_0^{+\infty} e^{-t}\ln t \sum_{n=0}^{+\infty}(-1)^n(e^{-2t})^n\,dt = 2\int_0^{+\infty} e^{-t}\ln t \sum_{n=0}^{+\infty}(-1)^n e^{-2nt}\,dt$$

$$\mathrm{I} = 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} e^{-(2n+1)t}\ln t\,dt$$

Let $y = (2n+1)t \Rightarrow t = \frac{1}{2n+1}y \Rightarrow dt = \frac{1}{2n+1}dy$; then we get:

$$\mathrm{I} = 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} e^{-y}\ln\left(\frac{y}{2n+1}\right).\frac{1}{2n+1}dy = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty} e^{-y}\ln\left(\frac{y}{2n+1}\right)dy\,;\ \text{ then:}$$

$$\mathrm{I} = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty}[e^{-y}\ln y - e^{-y}\ln(2n+1)]dy\,;\ \text{ then we get:}$$

$$\mathrm{I} = 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty} e^{-y}\ln y\,dy - 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{2n+1}\int_0^{+\infty} e^{-y}\ln(2n+1)\,dy$$

$$\mathrm{I} = -2\times\frac{\pi}{4}\times\gamma - 2\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}\ln(2n+1)}{2n+1} = -\frac{\pi\gamma}{2} + 2\left[\frac{\pi\gamma}{4} + \frac{\pi}{2}\ln\left\{\frac{\sqrt{2\pi}\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right\}\right]\text{; therefore:}$$

$$\mathrm{I} = \pi\ln\left[\frac{\sqrt{2\pi}\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{3}{4}\right)}\right]$$

191. $\mathrm{I} = \int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)\sin^{\pi^e}(2x)}{\left(\sin^{\pi^e+1}(x)+\cos^{\pi^e+1}(x)\right)^2}dx$, let us first consider the integral I_n such that**:**

$$\mathrm{I}_n = \int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)\sin^{n-1}(2x)}{(\sin^n(x)+\cos^n(x))^2}dx\ \&\ \mathrm{I} = \int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)\sin^{\pi^e}(2x)}{(\sin^{\pi^e+1}(x)+\cos^{\pi^e+1}(x))^2}dx\ (\text{for } n = \pi^e + 1)$$

$$\mathrm{I}_n = \int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)\sin^{n-1}(2x)}{\left[\sin^n(x)\left(1+\frac{\cos^n(x)}{\sin^n(x)}\right)\right]^2}dx = \int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)(2\sin x\cos x)^{n-1}}{[\sin^n(x)\,(1+\cot^n x)]^2}dx$$

$$\mathrm{I}_n = \int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)2^{n-1}\sin^{n-1}x\cos^{n-1}x}{\sin^{n-1}(x)\sin^{n+1}(x)\,(1+\cot^n x)^2}dx = 2^{n-1}\int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)\cos^{n-1}x}{\sin^{n+1}(x)\,(1+\cot^n x)^2}dx$$

$$\mathrm{I}_n = 2^{n-1}\int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)\left(\frac{\cos^{n-1}x}{\sin^{n-1}x}\right)}{\sin^2 x\,(1+\cot^n x)^2}dx = 2^{n-1}\int_0^{\frac{\pi}{4}}\frac{\ln(\cot x)\cot^{n-1}x}{\sin^2 x\,(1+\cot^n x)^2}dx$$

Change of variable: Let $u = 1+\cot^n x$; then $du = -n\cot^{n-1}x\left(\frac{1}{\sin^2 x}\right)dx$

New bounds: for $x = 0$; $u = +\infty$ & for $x = \frac{\pi}{4}$; $u = 1+1 = 2$; then we get:

$$\mathrm{I}_n = \frac{2^{n-1}}{n^2}\int_{+\infty}^{2} -\frac{\ln(u-1)}{u^2}du = \frac{2^{n-1}}{n^2}\int_2^{+\infty}\frac{\ln(u-1)}{u^2}du$$

Change of variable: let $t = u-1$; then $du = dt$

New bounds: for $u = 2$; $t = 1$ & for $u = +\infty$; $t = +\infty$; then we get:

$$\mathrm{I}_n = \frac{2^{n-1}}{n^2}\int_1^{+\infty}\frac{\ln t}{(t+1)^2}dt\,;\quad \text{now we will apply integration by parts:}$$

Let $u = \ln t \Longrightarrow u' = \dfrac{1}{t}$ & let $v' = \dfrac{1}{(t+1)^2} \Longrightarrow v = -\dfrac{1}{t+1}$; then we get:

$$\mathrm{I}_n = \frac{2^{n-1}}{n^2}\left\{\left[-\frac{\ln t}{t+1}\right]_1^{+\infty} + \int_1^{+\infty} \frac{1}{t(t+1)}dt\right\} = \frac{2^{n-1}}{n^2}\int_1^{+\infty} \frac{1}{t(t+1)}dt = \frac{2^{n-1}}{n^2}\int_1^{+\infty} \frac{t+1-t}{t(t+1)}dt$$

$$\mathrm{I}_n = \frac{2^{n-1}}{n^2}\int_1^{+\infty}\left(\frac{1}{t} - \frac{1}{t+1}\right)dt = \frac{2^{n-1}}{n^2}[\ln t - \ln(t+1)]_1^{+\infty} = \frac{2^{n-1}}{n^2}\left[\ln\left(\frac{t}{t+1}\right)\right]_1^{+\infty}$$; then we get:

$$\mathrm{I}_n = \frac{2^{n-1}}{n^2}\left[\lim_{t\to+\infty} \ln\left(\frac{t}{t+1}\right) - \ln\left(\frac{1}{2}\right)\right] = \frac{2^{n-1}}{n^2}(\ln 1 + \ln 2) = \frac{2^{n-1}\ln 2}{n^2}$$

Now putting $n = \pi^{e} + 1$; we get: $\mathrm{I} = \displaystyle\int_0^{\frac{\pi}{4}} \frac{\ln(\cot x)\sin^{\pi^{e}}(2x)}{(\sin^{\pi^{e}+1}(x) + \cos^{\pi^{e}+1}(x))^2}dx = \frac{2^{\pi^e}\ln 2}{(\pi^e+1)^2}$

192. $\int \sqrt[6]{\tan x}\,dx$

Let $z = \sqrt[6]{\tan x} \Longrightarrow z^6 = \tan x \Longrightarrow 6z^5dz = \dfrac{1}{1+\tan^2 x}dx \tan x \Longrightarrow 6z^5dz = \dfrac{1}{1+z^{12}}dx$

then $dx = \dfrac{6z^5}{1+z^{12}}dz$; then we get:

$$\int \sqrt[6]{\tan x}\,dx = \int z.\frac{6z^5}{1+z^{12}}dz = 6\int \frac{z^6}{1+z^{12}}dz = 6\int \frac{1}{z^{-6}(1+z^{12})}dz = 6\int \frac{1}{z^6 + \frac{1}{z^6}}dz$$

$$\int \sqrt[6]{\tan x}\,dx = 6\int \frac{1}{(z^2)^3 + \left(\frac{1}{z^2}\right)^3}dz$$

But $(a+b)^3 = a^3 + b^3 + 3ab(a+b) \Longrightarrow a^3 + b^3 = (a+b)^3 - 3ab(a+b) \Longrightarrow$

$(z^2)^3 + \left(\dfrac{1}{z^2}\right)^3 = \left(z^2 + \dfrac{1}{z^2}\right)^3 - 3\left(z^2 + \dfrac{1}{z^2}\right)\left(z^2 \times \dfrac{1}{z^2}\right) = \left(z^2 + \dfrac{1}{z^2}\right)^3 - 3\left(z^2 + \dfrac{1}{z^2}\right)$; then:

$$\int \sqrt[6]{\tan x}\,dx = 6\int \frac{1}{\left(z^2+\frac{1}{z^2}\right)^3 - 3\left(z^2+\frac{1}{z^2}\right)}dz = \frac{6}{2}\int \frac{\left(1-\frac{1}{z^2}\right)+\left(1+\frac{1}{z^2}\right)}{\left(z^2+\frac{1}{z^2}\right)^3 - 3\left(z^2+\frac{1}{z^2}\right)}dz$$

$$\int \sqrt[6]{\tan x}\,dx = 3\int \frac{1-\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3 - 3\left(z^2+\frac{1}{z^2}\right)}dz + 3\int \frac{1+\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3 - 3\left(z^2+\frac{1}{z^2}\right)}dz$$

For: $\displaystyle\int \frac{\mathbf{1}-\frac{\mathbf{1}}{\boldsymbol{z}^{\mathbf{2}}}}{\left(\boldsymbol{z}^{\mathbf{2}}+\frac{\mathbf{1}}{\boldsymbol{z}^{\mathbf{2}}}\right)^{\mathbf{3}} - \mathbf{3}\left(\boldsymbol{z}^{\mathbf{2}}+\frac{\mathbf{1}}{\boldsymbol{z}^{\mathbf{2}}}\right)}\boldsymbol{dz} = \int \frac{1-\frac{1}{z^2}}{\left[\left(z+\frac{1}{z}\right)^2 - 2\right]^3 - 3\left[\left(z+\frac{1}{z}\right)^2 - 2\right]}dz$

Let $t = z + \dfrac{1}{z} \Longrightarrow dt = \left(1 - \dfrac{1}{z^2}\right)dz$; then we get:

$$\int \frac{1-\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3 - 3\left(z^2+\frac{1}{z^2}\right)}dz = \int \frac{1}{(t^2-2)^2 - 3(t^2-2)}dt = \int \frac{1}{(t^2-2)[(t^2-2)^2-3]}dt$$

$$\int \frac{1-\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz = \int \frac{1}{(t^2-2)\left[(t^2-2)^2-\left(\sqrt{3}\right)^2\right]}dt\,; \text{ then we get:}$$

$$\int \frac{1-\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz = \int \frac{1}{(t^2-2)(t^2-2+\sqrt{3})(t^2-2-\sqrt{3})}dt$$

Using the method of decomposition into partial fractions ; we get:

$$\frac{1}{(t^2-2)(t^2-2+\sqrt{3})(t^2-2-\sqrt{3})} = \frac{\frac{1}{6}}{t^2-2+\sqrt{3}} + \frac{\frac{1}{6}}{t^2-2-\sqrt{3}} - \frac{\frac{1}{3}}{t^2-2}; \text{ then we get:}$$

$$\int \frac{1-\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz = \frac{1}{6}\int \frac{dt}{t^2-2+\sqrt{3}} + \frac{1}{6}\int \frac{dt}{t^2-2-\sqrt{3}} - \frac{1}{3}\int \frac{dt}{t^2-2}$$

$$= \frac{1}{6}\int \frac{dt}{t^2-\left(\sqrt{2-\sqrt{3}}\right)^2} + \frac{1}{6}\int \frac{dt}{t^2-\left(\sqrt{2+\sqrt{3}}\right)^2} - \frac{1}{3}\int \frac{dt}{t^2-(\sqrt{2})^2} =$$

$$\frac{1}{6}\times\frac{1}{2\sqrt{2-\sqrt{3}}}\ln\left|\frac{t-\sqrt{2-\sqrt{3}}}{t+\sqrt{2-\sqrt{3}}}\right| + \frac{1}{6}\times\frac{1}{2\sqrt{2+\sqrt{3}}}\ln\left|\frac{t-\sqrt{2+\sqrt{3}}}{t+\sqrt{2+\sqrt{3}}}\right| - \frac{1}{3}\times\frac{1}{2\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c$$

$$= \frac{1}{12\sqrt{2-\sqrt{3}}}\ln\left|\frac{t-\sqrt{2-\sqrt{3}}}{t+\sqrt{2-\sqrt{3}}}\right| + \frac{1}{12\sqrt{2+\sqrt{3}}}\ln\left|\frac{t-\sqrt{2+\sqrt{3}}}{t+\sqrt{2+\sqrt{3}}}\right| - \frac{1}{6\sqrt{2}}\ln\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right| + c$$

With $t = z+\frac{1}{z}$; then we get: $\int \frac{1-\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz$

$$= \frac{1}{12\sqrt{2-\sqrt{3}}}\ln\left|\frac{z+\frac{1}{z}-\sqrt{2-\sqrt{3}}}{z+\frac{1}{z}+\sqrt{2-\sqrt{3}}}\right| + \frac{1}{12\sqrt{2+\sqrt{3}}}\ln\left|\frac{z+\frac{1}{z}-\sqrt{2+\sqrt{3}}}{z+\frac{1}{z}+\sqrt{2+\sqrt{3}}}\right| - \frac{1}{6\sqrt{2}}\ln\left|\frac{z+\frac{1}{z}-\sqrt{2}}{z+\frac{1}{z}+\sqrt{2}}\right| + c$$

For: $\displaystyle\int \frac{\mathbf{1}+\frac{\mathbf{1}}{\boldsymbol{z^2}}}{\left(\boldsymbol{z^2}+\frac{\mathbf{1}}{\boldsymbol{z^2}}\right)^3-\mathbf{3}\left(\boldsymbol{z^2}+\frac{\mathbf{1}}{\boldsymbol{z^2}}\right)}\boldsymbol{dz} = \int \frac{1+\frac{1}{z^2}}{\left[\left(z-\frac{1}{z}\right)^2+2\right]^3-3\left[\left(z-\frac{1}{z}\right)^2+2\right]}dz$

Let $u = z-\frac{1}{z} \Longrightarrow dt = \left(1+\frac{1}{z^2}\right)dz$; then we get:

$$\int \frac{1+\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz = \int \frac{1}{(u^2+2)^2-3(u^2+2)}dt = \int \frac{dt}{(u^2+2)[(u^2+2)^2-3]}$$

$$\int \frac{1+\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz = \int \frac{1}{(u^2+2)\left[(u^2+2)^2-\left(\sqrt{3}\right)^2\right]}dt\,; \text{ then we get:}$$

$$\int \frac{1+\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz=\int \frac{1}{(u^2+2)(u^2+2+\sqrt{3})(u^2+2-\sqrt{3})}dt$$

Using the method of decomposition into partial fractions ; we get:

$$\frac{1}{(u^2+2)(u^2+2+\sqrt{3})(u^2+2-\sqrt{3})}=\frac{\frac{1}{6}}{u^2+2+\sqrt{3}}+\frac{\frac{1}{6}}{u^2+2-\sqrt{3}}-\frac{\frac{1}{3}}{u^2+2}$$; then we get:

$$\int \frac{1+\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz=\frac{1}{6}\int\frac{dt}{u^2+2+\sqrt{3}}+\frac{1}{6}\int\frac{dt}{u^2+2-\sqrt{3}}-\frac{1}{3}\int\frac{dt}{u^2+2}$$

$$=\frac{1}{6}\int\frac{dt}{u^2+\left(\sqrt{2+\sqrt{3}}\right)^2}+\frac{1}{6}\int\frac{dt}{u^2+\left(\sqrt{2-\sqrt{3}}\right)^2}-\frac{1}{3}\int\frac{dt}{u^2+(\sqrt{2})^2}$$

$$=\frac{1}{6}\times\frac{1}{\sqrt{2+\sqrt{3}}}\tan^{-1}\left(\frac{u}{\sqrt{2+\sqrt{3}}}\right)+\frac{1}{6}\times\frac{1}{\sqrt{2-\sqrt{3}}}\tan^{-1}\left(\frac{u}{\sqrt{2-\sqrt{3}}}\right)-\frac{1}{3}\times\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$$

$$=\frac{1}{6\sqrt{2+\sqrt{3}}}\tan^{-1}\left(\frac{u}{\sqrt{2+\sqrt{3}}}\right)+\frac{1}{6\sqrt{2-\sqrt{3}}}\tan^{-1}\left(\frac{u}{\sqrt{2-\sqrt{3}}}\right)-\frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right)+c$$

With $u = z-\frac{1}{z}$; then we get: $\int \frac{1+\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz$

$$=\frac{1}{6\sqrt{2+\sqrt{3}}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2+\sqrt{3}}}\right)+\frac{1}{6\sqrt{2-\sqrt{3}}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2-\sqrt{3}}}\right)-\frac{1}{3\sqrt{2}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2}}\right)+c$$

With $\int \sqrt[6]{\tan x}\,dx=3\int\frac{1-\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz+3\int\frac{1+\frac{1}{z^2}}{\left(z^2+\frac{1}{z^2}\right)^3-3\left(z^2+\frac{1}{z^2}\right)}dz$

$$\int \sqrt[6]{\tan x}\,dx=\frac{1}{4\sqrt{2-\sqrt{3}}}\ln\left|\frac{z+\frac{1}{z}-\sqrt{2-\sqrt{3}}}{z+\frac{1}{z}+\sqrt{2-\sqrt{3}}}\right|+\frac{1}{4\sqrt{2+\sqrt{3}}}\ln\left|\frac{z+\frac{1}{z}-\sqrt{2+\sqrt{3}}}{z+\frac{1}{z}+\sqrt{2+\sqrt{3}}}\right|$$

$$-\frac{1}{2\sqrt{2}}\ln\left|\frac{z+\frac{1}{z}-\sqrt{2}}{z+\frac{1}{z}+\sqrt{2}}\right|+\frac{1}{2\sqrt{2+\sqrt{3}}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2+\sqrt{3}}}\right)$$

$$+\frac{1}{2\sqrt{2-\sqrt{3}}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2-\sqrt{3}}}\right)-\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{z-\frac{1}{z}}{\sqrt{2}}\right)+k;$$

With $z=\sqrt[6]{\tan x}$; then we can write:

$$\int \sqrt[6]{\tan x}\,dx = \frac{1}{4\sqrt{2-\sqrt{3}}}\ln\left|\frac{\sqrt[6]{\tan x}+\frac{1}{\sqrt[6]{\tan x}}-\sqrt{2-\sqrt{3}}}{\sqrt[6]{\tan x}+\frac{1}{\sqrt[6]{\tan x}}+\sqrt{2-\sqrt{3}}}\right| + \frac{1}{4\sqrt{2+\sqrt{3}}}\ln\left|\frac{\sqrt[6]{\tan x}+\frac{1}{\sqrt[6]{\tan x}}-\sqrt{2+\sqrt{3}}}{\sqrt[6]{\tan x}+\frac{1}{\sqrt[6]{\tan x}}+\sqrt{2+\sqrt{3}}}\right|$$

$$-\frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt[6]{\tan x}+\frac{1}{\sqrt[6]{\tan x}}-\sqrt{2}}{\sqrt[6]{\tan x}+\frac{1}{\sqrt[6]{\tan x}}+\sqrt{2}}\right| + \frac{1}{2\sqrt{2+\sqrt{3}}}\tan^{-1}\left(\frac{\sqrt[6]{\tan x}-\frac{1}{\sqrt[6]{\tan x}}}{\sqrt{2+\sqrt{3}}}\right)$$

$$+\frac{1}{2\sqrt{2-\sqrt{3}}}\tan^{-1}\left(\frac{\sqrt[6]{\tan x}-\frac{1}{\sqrt[6]{\tan x}}}{\sqrt{2-\sqrt{3}}}\right) - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt[6]{\tan x}-\frac{1}{\sqrt[6]{\tan x}}}{\sqrt{2}}\right) + k$$

193. $\mathrm{I} = \int_0^{\frac{\pi}{2}} \frac{1}{\cos(\frac{x}{2})\cos(\frac{x}{4})\cos(\frac{x}{8})\cos(\frac{x}{16})\ldots}dx$

$\sin x = \sin\left(2\times\frac{x}{2}\right) = 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) = 2\cos\left(\frac{x}{2}\right)\sin\left(2\times\frac{x}{4}\right)$; then:

$\sin x = 2^2\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\sin\left(\frac{x}{4}\right) = 2^3\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\sin\left(\frac{x}{8}\right)$

$\sin x = 2^4\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\sin\left(\frac{x}{16}\right)$ then we get ... ...

$\sin x = 2^n\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\ldots\cos\left(\frac{x}{2^n}\right)\sin\left(\frac{x}{2^n}\right)$ and so we get:

$\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\ldots\cos\left(\frac{x}{2^n}\right) = \frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$; then:

$$\lim_{n\to+\infty}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\ldots\cos\left(\frac{x}{2^n}\right) = \lim_{n\to+\infty}\frac{\sin x}{2^n\sin\left(\frac{x}{2^n}\right)}$$

$$\lim_{n\to+\infty}\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\ldots\cos\left(\frac{x}{2^n}\right) = \frac{\sin x}{x}.\lim_{n\to+\infty}\frac{\frac{x}{2^n}}{\sin\left(\frac{x}{2^n}\right)} = \frac{\sin x}{x}\times 1$$

Therefore: $\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\ldots = \frac{\sin x}{x}$; then we get:

$$\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{1}{\cos\left(\frac{x}{2}\right)\cos\left(\frac{x}{4}\right)\cos\left(\frac{x}{8}\right)\cos\left(\frac{x}{16}\right)\ldots}dx = \int_0^{\frac{\pi}{2}}\frac{1}{\frac{\sin x}{x}}dx = \int_0^{\frac{\pi}{2}}\frac{x}{\sin x}dx$$

Using integration by parts: let $u = x \Longrightarrow u' = 1$ and let $v' = \frac{1}{\sin x} \Longrightarrow v = \ln\left(\tan\frac{x}{2}\right)$, then**:**

$$\mathrm{I} = \int_0^{\frac{\pi}{2}}\frac{x}{\sin x}dx = \left[x\ln\left(\tan\frac{x}{2}\right)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}}\ln\left(\tan\frac{x}{2}\right)dx = 0 - \int_0^{\frac{\pi}{2}}\ln\left(\tan\frac{x}{2}\right)dx$$

Let $\theta = \frac{x}{2} \Longrightarrow x = 2\theta \Longrightarrow dx = 2d\theta$; for $x = 0$; $\theta = 0$; and for $x = \frac{\pi}{2}$; $\theta = \frac{\pi}{4}$; then we get:

$$\mathrm{I} = -2\int_0^{\frac{\pi}{4}} \ln(\tan\theta)\,d\theta = 2\left(-\int_0^{\frac{\pi}{4}} \ln(\tan\theta)\,d\theta\right) = 2\mathrm{G}\ \ (\mathrm{G: Catalan's\ \ constant})$$

194. $\mathrm{I} = \int_0^{+\infty} \frac{x^2}{(x^2+a^2)(x^2+b^2)(x^2+c^2)}dx$, we can write I under the form**:**

$$\mathrm{I} = \frac{1}{a^2b^2c^2}\int_0^{+\infty} \frac{x^2}{\left(\frac{x^2}{a^2}+1\right)\left(\frac{x^2}{b^2}+1\right)\left(\frac{x^2}{c^2}+1\right)}dx$$

Let $\tan\theta = \frac{x}{a}$, then $x = a\tan\theta$ and $dx = a\sec^2\theta\,d\theta$, for $x = 0$, $\theta = 0$ and for $x = +\infty$, $\theta = \frac{\pi}{2}$, then we get**:**

$$\mathrm{I} = \frac{1}{a^2b^2c^2}\int_0^{\frac{\pi}{2}} \frac{a^2\tan^2\theta\, a\sec^2\theta}{(\tan^2\theta+1)\left(\frac{a^2}{b^2}\tan^2\theta+1\right)\left(\frac{a^2}{c^2}\tan^2\theta+1\right)}d\theta$$

with $\tan^2\theta + 1 = \sec^2\theta$, so we get**:**

$$\mathrm{I} = \frac{a}{b^2c^2}\int_0^{\frac{\pi}{2}} \frac{\tan^2\theta}{\left(\frac{a^2}{b^2}\tan^2\theta+1\right)\left(\frac{a^2}{c^2}\tan^2\theta+1\right)}d\theta$$

Let $t = \frac{\pi}{2} - \theta$, then $dt = -d\theta$, for $\theta = 0$, $t = \frac{\pi}{2}$ and for $\theta = \frac{\pi}{2}$, $t = 0$, then we get**:**

$$\mathrm{I} = \frac{a}{b^2c^2}\int_{\frac{\pi}{2}}^{0} \frac{\tan^2\left(\frac{\pi}{2}-t\right)}{\left(\frac{a^2}{b^2}\tan^2\left(\frac{\pi}{2}-t\right)+1\right)\left(\frac{a^2}{c^2}\tan^2\left(\frac{\pi}{2}-t\right)+1\right)}(-dt)\quad \text{then we get:}$$

$$\mathrm{I} = \frac{a}{b^2c^2}\int_0^{\frac{\pi}{2}} \frac{\cot^2 t}{\left(\frac{a^2}{b^2}\cot^2 t+1\right)\left(\frac{a^2}{c^2}\cot^2 t+1\right)}dt\quad \text{and so:}$$

$$\mathrm{I} = \frac{a}{b^2c^2}\int_0^{\frac{\pi}{2}} \frac{\cot^2 t}{\left(\frac{a^2}{b^2}\cot^2 t+1\right)\left(\frac{a^2}{c^2}\cot^2 t+1\right)}dt = \frac{a}{b^2c^2}\int_0^{\frac{\pi}{2}} \frac{\tan^2 t}{\left(\frac{a^2}{b^2}+\tan^2 t\right)\left(\frac{a^2}{c^2}+\tan^2 t\right)}dt$$

Setting $m = \frac{a^2}{b^2}$ and $n = \frac{a^2}{c^2}$, we get**:**

$$\mathrm{I} = \frac{a}{b^2c^2}\int_0^{\frac{\pi}{2}} \frac{\tan^2 t}{(m+\tan^2 t)(n+\tan^2 t)}dt$$

Decomposition in partial fractions: Let $y = \tan^2 t$, then**:**

$$\frac{y}{(y+m)(y+n)} = \frac{\mathrm{A}}{y+m} + \frac{\mathrm{B}}{y+n};\text{then}\ \ y = (y+n)\mathrm{A} + (y+m)\,\mathrm{B}$$

For $y = -m$, then $-m = (-m+n)\mathrm{A}$, $m = \mathrm{A}(m-n)$, so

$$\mathrm{A} = \frac{m}{m-n} = \frac{\frac{a^2}{b^2}}{\frac{a^2}{b^2} - \frac{a^2}{c^2}} = \frac{c^2}{c^2 - b^2}$$

and for $y = -n$, then $-n = (-n+m)\mathrm{B}$, then $n = (m-n)\mathrm{B}$, so

$$\mathrm{B} = -\frac{n}{m-n} = -\frac{\frac{a^2}{c^2}}{\frac{a^2}{b^2} - \frac{a^2}{c^2}} = -\frac{b^2}{c^2 - b^2}$$

Then we get:

$$\mathrm{I} = \frac{a}{b^2c^2} \times \frac{c^2}{c^2-b^2}\int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 t + \frac{a^2}{b^2}}dt - \frac{a}{b^2c^2} \times \frac{b^2}{c^2-b^2}\int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 t + \frac{a^2}{c^2}}dt; \text{ then we get:}$$

$$\mathrm{I} = \frac{a}{b^2(c^2-b^2)}\int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 t + \frac{a^2}{b^2}}dt - \frac{a}{c^2(c^2-b^2)}\int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 t + \frac{a^2}{c^2}}dt$$

The two obtained integrals are in the form: $\mathrm{I}_1 = \int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 x + \alpha^2}dx$, let us evaluate I_1:

$$\mathrm{I}_1 = \int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 x + \alpha^2}dx = \int_0^{\frac{\pi}{2}} \frac{(\sec^2 x - \tan^2 x) - \alpha^2 + \alpha^2}{\tan^2 x + \alpha^2}dx$$

$$\mathrm{I}_1 = \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{\tan^2 x + \alpha^2}dx - \int_0^{\frac{\pi}{2}} \frac{\tan^2 x + \alpha^2}{\tan^2 x + \alpha^2}dx + \alpha^2\int_0^{\frac{\pi}{2}} \frac{1}{\tan^2 x + \alpha^2}dx, \text{then we get:}$$

$\mathrm{I}_1 = \int_0^{\frac{\pi}{2}} \frac{(\tan x)'}{(\tan x)^2+\alpha^2}dx - \frac{\pi}{2} + \alpha^2\,\mathrm{I}_1$, then $(1-\alpha^2)\mathrm{I}_1 = \left[\frac{1}{\alpha}\arctan\left(\frac{\tan x}{\alpha}\right)\right]_0^{\frac{\pi}{2}} - \frac{\pi}{2}$, then we get:

$(1-\alpha^2)\mathrm{I}_1 = \frac{1}{\alpha}(\arctan(+\infty) - \arctan 0) = \frac{\pi}{2\alpha} - \frac{\pi}{2} = \frac{(1-\alpha)\pi}{2\alpha}$, therefore we have:

$\mathrm{I}_1 = \frac{1}{1-\alpha^2} \times \frac{(1-\alpha)\pi}{2\alpha} = \frac{(1-\alpha)\pi}{2\alpha(1-\alpha)(1+\alpha)} = \frac{\pi}{2\alpha(1+\alpha)}$, using this result, we can write:

$$\mathrm{I} = \frac{a}{b^2(c^2-b^2)} \times \frac{\pi}{2 \times \frac{a}{b}\left(1 + \frac{a}{b}\right)} - \frac{a}{c^2(c^2-b^2)} \times \frac{\pi}{2 \times \frac{a}{c}\left(1 + \frac{a}{c}\right)}; \text{ therefore we get:}$$

$$\mathrm{I} = \frac{\pi}{2(a+b)(a+c)(b+c)}$$

195. $\mathrm{I} = \int_0^{+\infty} e^{-\sqrt{x}} \ln\left(1 + \frac{1}{\sqrt{x}}\right)dx$, let $t = \sqrt{x}$, then $dt = \frac{1}{2\sqrt{x}}dx = \frac{1}{2t}dx$, so $dx = 2t dt$,

then: $\mathrm{I} = 2\int_0^{+\infty} te^{-t}\ln\left(1 + \frac{1}{t}\right)dt = 2\int_0^{+\infty} te^{-t}\ln\left(\frac{t+1}{t}\right)dt$, then we get:

$\mathrm{I} = 2\int_0^{+\infty} te^{-t}\ln(1+t)\,dt - 2\int_0^{+\infty} te^{-t}\ln t\,dt = 2\mathrm{J} - 2\int_0^{+\infty} te^{-t}\ln t\,dt$

Now evaluating: $\mathrm{J} = \int_0^{+\infty} te^{-t}\ln(1+t)\,dt$, using integration by parts:

Let $u = t\ln(1+t)$, then $u' = \ln(1+t) + \frac{t}{1+t}$ and let $v' = e^{-t}$, then $v = -e^{-t}$, then we get:

$\mathrm{J} = [-te^{-t}\ln(1+t)]_0^{+\infty} + \int_0^{+\infty}\left[\ln(1+t) + \frac{t}{1+t}\right]e^{-t}dt$

$J = \int_0^{+\infty} e^{-t}\ln(1+t)\,dt + \int_0^{+\infty} \frac{te^{-t}}{1+t}dt$

Note that: $\int_0^{+\infty} \frac{te^{-t}}{1+t}dt = \int_0^{+\infty} \frac{(t+1-1)e^{-t}}{1+t}dt = \int_0^{+\infty} e^{-t}\,dt - \int_0^{+\infty} \frac{e^{-t}}{1+t}dt = 1 - \int_0^{+\infty} \frac{e^{-t}}{1+t}dt$

Now integrating $\int_0^{+\infty} \frac{e^{-t}}{1+t}dt$ **by parts:** Let $u = e^{-t}$, then $u' = -e^{-t}$ and let $v' = \frac{1}{1+t}$, then $v = \ln(1+t)$, then we get: $\int_0^{+\infty} \frac{e^{-t}}{1+t}dt = [e^{-t}\ln(1+t)]_0^{+\infty} + \int_0^{+\infty} e^{-t}\ln(1+t)\,dt$

Then: $\int_0^{+\infty} \frac{e^{-t}}{1+t}dt = \int_0^{+\infty} e^{-t}\ln(1+t)\,dt$, then:

$J = \int_0^{+\infty} e^{-t}\ln(1+t)\,dt + \int_0^{+\infty} \frac{te^{-t}}{1+t}dt = \int_0^{+\infty} e^{-t}\ln(1+t)\,dt + 1 - \int_0^{+\infty} \frac{e^{-t}}{1+t}dt$

$J = \int_0^{+\infty} e^{-t}\ln(1+t)\,dt + 1 - \int_0^{+\infty} e^{-t}\ln(1+t)\,dt = 1$

So we get: $I = 2 - 2\int_0^{+\infty} te^{-t}\ln t\,dt = 2\left(1 - \int_0^{+\infty} te^{-t}\ln t\,dt\right)$

Now let's evaluate the integral: $\int_0^{+\infty} te^{-t}\ln t\,dt$

Definition of gamma function: $\Gamma(s) = \int_0^{+\infty} t^{s-1}e^{-t}dt$ and $\Gamma'(s) = \int_0^{+\infty} t^{s-1}e^{-t}\ln t\,dt$

Definition of digamma function: $\Psi(s) = \frac{s}{ds}\ln\big(\Gamma(s)\big) = \frac{\Gamma'(s)}{\Gamma(s)}$, so $\Gamma'(s) = \Gamma(s)\,\Psi(s)$, then we can write: $\Gamma(s)\,\Psi(s) = \int_0^{+\infty} t^{s-1}e^{-t}\ln t\,dt$, substituting $s = 2$, we get $\Gamma(2)\,\Psi(2) = \int_0^{+\infty} te^{-t}\ln t\,dt$

We have $\Gamma(2) = 1! = 1$ and $\Psi(s+1) = \Psi(s) + \frac{1}{2}$, so for $s = 1$ $\Psi(2) = \Psi(1) + 1$, but we have $\Psi(1) = -\gamma$, then we get: $\Psi(2) = 1 - \gamma$, then $\int_0^{+\infty} te^{-t}\ln t\,dt = \Gamma(2)\,\Psi(2) = 1 - \gamma$

Therefore: $I = 2[1 - (1-\gamma)] = 2\gamma$

196. $\int \frac{x^n}{1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}dx$

Let $f(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} \Longrightarrow f'(x) = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{n-1}}{(n-1)!} \Longrightarrow$

$f(x) - f'(x) = \frac{x^n}{n!} \Longrightarrow x^n = n!\,\big(f(x) - f'(x)\big)$; then we get:

$$\int \frac{x^n}{1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}dx = \int \frac{n!\,\big(f(x)-f'(x)\big)}{f(x)}dx = n!\int\left(1 - \frac{f'(x)}{f(x)}\right)dx \Longrightarrow$$

$$\int \frac{x^n}{1+x+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}}dx = n!\,(x - \ln|f(x)|) + c = n!\left[x - \ln\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}\right)\right] + c$$

197. $I = \int_0^{+\infty} \frac{x^2+1}{x^4+x^2+1}\cdot\frac{\ln\left(1-x+x^2-x^3+\cdots+x^{2m}\right)}{\ln x}dx$

Let $t = \frac{1}{x}$, then $x = \frac{1}{t}$, then $dx = -\frac{1}{t^2}dt$, for $x = 0$, then $t = +\infty$ and for $x = +\infty$, then $t = 0$, so we get:

$$I = \int_{+\infty}^{0} \frac{\frac{1}{t^2}+1}{\frac{1}{t^4}+\frac{1}{t^2}+1}\cdot\frac{\ln\left(1-\frac{1}{t}+\frac{1}{t^2}-\frac{1}{t^3}+\cdots+\frac{1}{t^{2m}}\right)}{\ln\left(\frac{1}{t}\right)}\left(-\frac{1}{t^2}dt\right);\ \text{then}$$

$$\mathrm{I}=\int_0^{+\infty}\frac{1+t^2}{t^4+t^2+1}\cdot\frac{\ln\left[\frac{1}{t^{2m}}(1-t+t^2-t^3+\cdots+t^{2m})\right]}{-\ln t}dt;\ \text{ then we get}$$

$$\mathrm{I}=-\int_0^{+\infty}\frac{1+t^2}{t^4+t^2+1}\cdot\frac{\ln(1-t+t^2-t^3+\cdots+t^{2m})-\ln(t^{2m})}{\ln t}dt$$

$$\mathrm{I}=-\int_0^{+\infty}\frac{1+t^2}{t^4+t^2+1}\cdot\frac{\ln(1-t+t^2-t^3+\cdots+t^{2m})-2m\ln t}{\ln t}dt$$

$$\mathrm{I}=-\int_0^{+\infty}\frac{t^2+1}{t^4+t^2+1}\cdot\frac{\ln(1-t+t^2-t^3+\cdots+t^{2m})}{\ln t}dt+2m\int_0^{+\infty}\frac{t^2+1}{t^4+t^2+1}dx;\ \text{ then:}$$

$$\mathrm{I}=-\int_0^{+\infty}\frac{x^2+1}{x^4+x^2+1}\cdot\frac{\ln(1-x+x^2-x^3+\cdots+x^{2m})}{\ln x}dt+2m\int_0^{+\infty}\frac{x^2+1}{x^4+x^2+1}dx$$

So, $\mathrm{I}=-\mathrm{I}+2m\int_0^{+\infty}\frac{x^2+1}{x^4+x^2+1}dx$, then: $2\mathrm{I}=2m\int_0^{+\infty}\frac{x^2+1}{x^4+x^2+1}dx$, $\mathrm{I}=m\int_0^{+\infty}\frac{x^2+1}{x^4+x^2+1}dx$

$\mathrm{I}=m\int_0^{+\infty}\frac{1-x^{-2}}{x^2+x^{-2}+1}dx=m\int_0^{+\infty}\frac{1-x^{-2}}{(x^2-2+x^{-2})+3}dx=m\int_0^{+\infty}\frac{1-x^{-2}}{(x-x^{-1})^2+3}dx$

Let $u=x-x^{-1}$, then $du=(1+x^{-2})dx$, for $x=0$, then $u=-\infty$ and for $x=+\infty$, then $u=+\infty$, then we get:

$$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{du}{u^2+3}=\int_{-\infty}^{+\infty}\frac{du}{u^2+(\sqrt{3})^2}=\frac{m}{\sqrt{3}}\left[\arctan\left(\frac{u}{\sqrt{3}}\right)\right]_{-\infty}^{+\infty}=\frac{m}{\sqrt{3}}\left(\frac{\pi}{2}+\frac{\pi}{2}\right)=\frac{\pi m}{\sqrt{3}}$$

$$\text{Therefore we get; } \mathrm{I}=\int_0^{+\infty}\frac{x^2+1}{{}^4+x^2+1}\cdot\frac{\ln(1-x+x^2-x^3+\cdots+x^{2m})}{\ln x}dx=\frac{\pi m}{\sqrt{3}}$$

198. $\mathrm{I}=\int_{\frac{\pi}{6}}^{\frac{5\pi}{18}}\frac{2-\sin 3x}{\cos^3\left(\frac{3x}{2}\right)+\sin^3\left(\frac{3x}{2}\right)}dx$

Let $\theta=\dfrac{3x}{2}\Longrightarrow x=\dfrac{2}{3}x$ & $dx=\dfrac{2}{3}dx$; $\begin{cases}x=\dfrac{\pi}{6}\\ x=\dfrac{5\pi}{18}\end{cases}\Longrightarrow\begin{cases}x=\dfrac{\pi}{4}\\ x=\dfrac{5\pi}{12}\end{cases}$; then we get:

$$\mathrm{I}=\int_{\frac{\pi}{6}}^{\frac{5\pi}{18}}\frac{2-\sin 3x}{\cos^3\left(\frac{3x}{2}\right)+\sin^3\left(\frac{3x}{2}\right)}dx=\frac{2}{3}\int_{\frac{\pi}{4}}^{\frac{5\pi}{12}}\frac{2-\sin 2\theta}{\cos^3\theta+\sin^3\theta}d\theta=\frac{2}{3}\int_{\frac{\pi}{4}}^{\frac{5\pi}{12}}\frac{2-2\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}d\theta$$

$$\mathrm{I}=\frac{4}{3}\int_{\frac{\pi}{4}}^{\frac{5\pi}{12}}\frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}d\theta;\quad \text{now let's evaluate}\quad \int\frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}d\theta$$

$$\text{Let } t=\tan\left(\frac{\theta}{2}\right)\Longrightarrow\sin\theta=\frac{2t}{1+t^2};\ \cos\theta=\frac{1-t^2}{1+t^2}\ \&\ d\theta=\frac{2}{1+t^2}dt;\ \text{ then we get:}$$

$$\int \frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}\,d\theta = \int \frac{1-\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)}{\left(\frac{2t}{1+t^2}\right)^3+\left(\frac{1-t^2}{1+t^2}\right)^3}\cdot\frac{2}{1+t^2}dt;\ \text{ then we get:}$$

$$\int \frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}\,d\theta = \int \frac{1-\frac{2t(1-t^2)}{(1+t^2)^2}}{\frac{(1-t^2)^3+8t^2}{(1+t^2)^3}}\cdot\frac{2}{1+t^2}dt\ \text{ \& after simplification we get:}$$

$$\int \frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}\,d\theta = -2\int \frac{t^4+2t^3+2t^2-2t+1}{t^6-3t^4-8t^3-3t^2-1}dt$$

$$\int \frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}\,d\theta = -2\int \frac{t^4+2t^3+2t^2-2t+1}{(t^4+2t^3+2t^2-2t+1)(t^2-2t-1)}dt;\ \text{ then we get:}$$

$$\int \frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}\,d\theta = -2\int \frac{1}{t^2-2t-1}dt = -2\int\frac{dt}{(t-1)^2-2} = -2\int\frac{dt}{(t-1)^2-\left(\sqrt{2}\right)^2}$$

$$\int \frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}\,d\theta = -2\left\{\frac{1}{2\sqrt{2}}\ln\left|\frac{t-1-\sqrt{2}}{t-1+\sqrt{2}}\right|\right\}+c = \frac{1}{\sqrt{2}}\ln\left|\frac{t-1+\sqrt{2}}{t-1-\sqrt{2}}\right|+c$$

$$\int \frac{1-\sin\theta\cos\theta}{\cos^3\theta+\sin^3\theta}\,d\theta = \frac{1}{\sqrt{2}}\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{\theta}{2}\right)-1-\sqrt{2}}\right|+c;\ \text{ then we get:}$$

$$\mathrm{I} = \frac{4}{3\sqrt{2}}\left[\ln\left|\frac{\tan\left(\frac{\theta}{2}\right)-1+\sqrt{2}}{\tan\left(\frac{\theta}{2}\right)-1-\sqrt{2}}\right|\right]_{\frac{\pi}{4}}^{\frac{5\pi}{12}}\quad \text{and so we get:}$$

$$\mathrm{I} = \frac{2\sqrt{2}}{3}\ln\left|\frac{\tan\left(\frac{5\pi}{24}\right)-1+\sqrt{2}}{\tan\left(\frac{5\pi}{24}\right)-1-\sqrt{2}}\right| - \frac{2\sqrt{2}}{3}\ln\left|\frac{\tan\left(\frac{\pi}{8}\right)-1+\sqrt{2}}{\tan\left(\frac{\pi}{8}\right)-1-\sqrt{2}}\right|$$

$$\text{If } -\frac{\pi}{2}<\theta<\frac{\pi}{2} \Longrightarrow \tan\left(\frac{\theta}{2}\right) = \frac{-1+\sqrt{1+\tan^2\theta}}{\tan\theta} \Longrightarrow \tan\left(\frac{\pi}{8}\right) = \frac{-1+\sqrt{1+\tan^2\left(\frac{\pi}{4}\right)}}{\tan\left(\frac{\pi}{4}\right)}$$

$$\text{Then we get: }\ \tan\left(\frac{\pi}{8}\right) = \frac{-1+\sqrt{2}}{2} \Longrightarrow \tan\left(\frac{\theta}{2}\right) = \sqrt{2}-1$$

$$\tan\left(\frac{5\pi}{24}\right) = \frac{-1+\sqrt{1+\tan^2\left(\frac{5\pi}{12}\right)}}{\tan\left(\frac{5\pi}{12}\right)} = \frac{-1+\sqrt{8+4\sqrt{3}}}{2+\sqrt{3}} = \left(\sqrt{\left(\sqrt{6}+\sqrt{2}\right)^2}-1\right)\left(2-\sqrt{3}\right)$$

$$\tan\left(\frac{5\pi}{24}\right) = \left(\sqrt{6}+\sqrt{2}-1\right)\left(2-\sqrt{3}\right) \Longrightarrow \tan\left(\frac{5\pi}{24}\right) = \sqrt{6}+\sqrt{3}-\sqrt{2}-2$$

$$\mathrm{I} = \frac{2\sqrt{2}}{3}\ln\left|\frac{\sqrt{6}+\sqrt{3}-3}{\sqrt{6}+\sqrt{3}-2\sqrt{2}-3}\right| - \frac{2\sqrt{2}}{3}\ln\left|1-\sqrt{2}\right|$$

$$\mathrm{I} = \frac{2\sqrt{2}}{3}\ln\left|\frac{\sqrt{6}+\sqrt{3}-3}{\left(\sqrt{6}+\sqrt{3}-2\sqrt{2}-3\right)\left(1-\sqrt{2}\right)}\right| = \frac{2\sqrt{2}}{3}\ln\left|\frac{\sqrt{6}+\sqrt{3}-3}{\sqrt{2}+1-\sqrt{3}}\right|;\ \text{ then we get:}$$

$$\mathrm{I} = \frac{2\sqrt{2}}{3}\ln\left|\frac{\sqrt{3}(\sqrt{2}+1-\sqrt{3})}{\sqrt{2}+1-\sqrt{3}}\right| = \frac{2\sqrt{2}}{3}\ln\sqrt{3} = \frac{2\sqrt{2}}{6}\ln 3$$

Therefore; $\displaystyle\int_{\frac{\pi}{6}}^{\frac{5\pi}{18}} \frac{2-\sin 3x}{\cos^3\left(\frac{3x}{2}\right)+\sin^3\left(\frac{3x}{2}\right)}dx = \frac{2\sqrt{2}}{6}\ln 3$

199. $\mathrm{I} = \int_0^1 (-1)^{\ln x}dx$, we have $e^{i\pi(2k+1)} = -1$, then we get**:**

$$\mathrm{I} = \int_0^1 (-1)^{\ln x}dx = \int_0^1 \left[e^{i\pi(2k+1)}\right]^{\ln x}dx = \int_0^1 e^{i(2k+1)\pi\ln x}dx$$

$$\mathrm{I} = \int_0^1 \{\cos[(2k+1)\pi\ln x] + i\sin[(2k+1)\pi\ln x]\}dx\,; \quad \text{then we get:}$$

$$\mathrm{I} = \int_0^1 \cos[(2k+1)\pi\ln x]\,dx + i\int_0^1 \sin[(2k+1)\pi\ln x]\,dx$$

Let $-t = \ln x \Longrightarrow x = e^{-t} \Longrightarrow dx = -e^{-t}dt$; for the bounds: $\begin{cases} x = 0 \\ x = 1 \end{cases} \Longrightarrow \begin{cases} t = +\infty \\ t = 0 \end{cases}$; then:

$$\mathrm{I} = \int_{+\infty}^{0} \cos[(2k+1)\pi(-t)]\,(-e^{-t}dt) + i\int_{+\infty}^{0} \sin[(2k+1)\pi(-t)]\,(-e^{-t}dt)$$

$$\mathrm{I} = \int_0^{+\infty} \cos[-(2k+1)\pi t]\,e^{-t}dt + i\int_0^{+\infty} \sin[-(2k+1)\pi t]\,e^{-t}dt$$

$$\mathrm{I} = \int_0^{+\infty} \cos[(2k+1)\pi t]\,e^{-t}dt - i\int_0^{+\infty} \sin[(2k+1)\pi t]\,e^{-t}dt$$

Recall that: $\displaystyle\int_0^{+\infty} e^{-ax}\cos bx\,dx = \frac{a}{a^2+b^2}$ & $\displaystyle\int_0^{+\infty} e^{-ax}\sin bx\,dx = \frac{b}{a^2+b^2}$

And so we can write: $\displaystyle\int_0^{+\infty} e^{-t}\cos[(2k+1)\pi t]\,dt = \frac{1}{1+(2k+1)^2\pi^2}$ &

$\displaystyle\int_0^{+\infty} e^{-t}\sin[(2k+1)\pi t]\,dt = \frac{(2k+1)\pi}{1+(2k+1)^2\pi^2}$; then we get:

$$\mathrm{I} = \int_0^1 (-1)^{\ln x}dx = \frac{1}{1+(2k+1)^2\pi^2} - i\frac{(2k+1)\pi}{1+(2k+1)^2\pi^2}; \quad \text{therefore; we have:}$$

$$\mathrm{Re}\left(\int_0^1 (-1)^{\ln x}dx\right) = \frac{1}{1+(2k+1)^2\pi^2} \quad \& \quad \mathrm{Im}\left(\int_0^1 (-1)^{\ln x}dx\right) = -\frac{(2k+1)\pi}{1+(2k+1)^2\pi^2}$$

200. $\int \frac{dx}{\sqrt{\csc x - \cot x}} = \int \frac{dx}{\sqrt{\frac{1}{\sin x} - \frac{\cos x}{\sin x}}} = \int \frac{dx}{\sqrt{\frac{1-\cos x}{\sin x}}} = \int \sqrt{\frac{\sin x}{1-\cos x}}\, dx$

Let $t = \tan\frac{x}{2} \Longrightarrow \sin x = \frac{2t}{1+t^2};\ \cos x = \frac{1-t^2}{1+t^2}\ \ \&\ \ dx = \frac{2dt}{1+t^2};$ then we get:

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = \int \sqrt{\frac{\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}}\cdot\frac{2dt}{1+t^2} = \int \sqrt{\frac{\frac{2t}{1+t^2}}{\frac{1+t^2-1+t^2}{1+t^2}}}\cdot\frac{2dt}{1+t^2}$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = \int \sqrt{\frac{\frac{2t}{1+t^2}}{\frac{2t^2}{1+t^2}}}\cdot\frac{2dt}{1+t^2} = \int \sqrt{\frac{2t}{2t^2}}\cdot\frac{2dt}{1+t^2} = 2\int \sqrt{\frac{1}{t}}\cdot\frac{dt}{1+t^2}$$

Let $\frac{1}{t} = y^2 \Longrightarrow t = \frac{1}{y^2}\ \left(t^2 = \frac{1}{y^4}\right)\ \ \&\ \ dt = -\frac{2}{y^3}dy;$ then we get:

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = 2\int y\,.\frac{1}{1+\frac{1}{y^4}}\left(-\frac{2}{y^3}dy\right) = -4\int \frac{y^2}{1+y^4}dy = 2\int \frac{-2y^2}{1+y^4}dy$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = 2\int \frac{1-y^2-1-y^2}{1+y^4}dy = 2\int \frac{1-y^2}{1+y^4}dy - 2\int \frac{1+y^2}{1+y^4}dy$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = 2\int \frac{\frac{1-y^2}{y^2}}{\frac{1+y^4}{y^2}}dy - 2\int \frac{\frac{1+y^2}{y^2}}{\frac{1+y^4}{y^2}}dy = 2\int \frac{\frac{1}{y^2}-1}{\frac{1}{y^2}+y^2}dy - 2\int \frac{\frac{1}{y^2}+1}{\frac{1}{y^2}+y^2}dy$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = 2\int \frac{\frac{1}{y^2}-1}{\left(\frac{1}{y}+y\right)^2-2}dy - 2\int \frac{\frac{1}{y^2}+1}{\left(\frac{1}{y}-y\right)^2+2}dy$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = 2\int \frac{\frac{1}{y^2}-1}{\left(\frac{1}{y}+y\right)^2-\left(\sqrt{2}\right)^2}dy - 2\int \frac{\frac{1}{y^2}+1}{\left(\frac{1}{y}-y\right)^2+\left(\sqrt{2}\right)^2}dy$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = -2\int \frac{d\left(\frac{1}{y}+y\right)}{\left(\frac{1}{y}+y\right)^2-\left(\sqrt{2}\right)^2}dy + 2\int \frac{d\left(\frac{1}{y}-y\right)}{\left(\frac{1}{y}-y\right)^2+\left(\sqrt{2}\right)^2}dy$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = \sqrt{2}\sin^{-1}\left(\frac{\frac{1}{y}+y}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{\frac{1}{y}-y}{\sqrt{2}}\right) + c;\ \text{ then we get:}$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = \sqrt{2}\sin^{-1}\left(\frac{\sqrt{t}+\sqrt{\frac{1}{t}}}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{\sqrt{t}-\sqrt{\frac{1}{t}}}{\sqrt{2}}\right) + c\ \text{ and so}$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = \sqrt{2}\sin^{-1}\left(\frac{\sqrt{\tan\frac{x}{2}} + \sqrt{\frac{1}{\tan\frac{x}{2}}}}{\sqrt{2}}\right) + \tan^{-1}\left(\frac{\sqrt{\tan\frac{x}{2}} - \sqrt{\frac{1}{\tan\frac{x}{2}}}}{\sqrt{2}}\right) + c$$

$$\int \frac{dx}{\sqrt{\csc x - \cot x}} = \sqrt{2}\sin^{-1}\left(\frac{\tan\frac{x}{2} + 1}{\sqrt{2\tan\frac{x}{2}}}\right) + \tan^{-1}\left(\frac{\tan\frac{x}{2} - 1}{\sqrt{2\tan\frac{x}{2}}}\right) + c$$

201. $\mathrm{I} = \int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx$, let $y = 1 + \sqrt{x} \Rightarrow x = (y-1)^2 \Rightarrow dx = 2(y-1)dy$; then we get:

$$\mathrm{I} = \int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = \int \frac{\sqrt{1+(y-1)^2}}{1+\sqrt{(y-1)^2}}.2(y-1)dy = 2\int \frac{\sqrt{1+(y-1)^2}}{1+y-1}(y-1)dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \frac{\sqrt{1+(y-1)^2}}{y}(y-1)dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{\sqrt{1+(y-1)^2}}{y}dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{\sqrt{1+(y-1)^2}}{y} \times \frac{\sqrt{1+(y-1)^2}}{\sqrt{1+(y-1)^2}}dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{y^2-2y+2}{y\sqrt{1+(y-1)^2}}dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{y(y-2)+2}{y\sqrt{1+(y-1)^2}}dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{y(y-2)}{y\sqrt{1+(y-1)^2}}dy - 2\int \frac{2}{y\sqrt{1+(y-1)^2}}dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{y-2}{\sqrt{1+(y-1)^2}}dy - 4\int \frac{1}{y\sqrt{1+(y-1)^2}}dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{y-2}{\sqrt{1+(y-1)^2}}dy - 4\int \frac{\frac{1}{y}}{\sqrt{y^2-2y+2}}dy$$

$$\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx = 2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{y-2}{\sqrt{1+(y-1)^2}}dy - 4\int \frac{\frac{1}{y^2}}{\sqrt{1-\frac{2}{y}+\frac{2}{y^2}}}dy$$

For the first two integrals: let $y = 1 + \tan\theta \Rightarrow dy = \sec^2\theta\, d\theta$; then we get:

$$2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{y-2}{\sqrt{1+(y-1)^2}}dy$$

$$= 2\int \sqrt{1+(1+\tan\theta-1)^2}\sec^2\theta\, d\theta - 2\int \frac{1+\tan\theta-2}{\sqrt{1+(1+\tan\theta-1)^2}}\sec^2\theta\, d\theta$$

$$= 2\int \sqrt{1+\tan^2\theta}\sec^2\theta\, d\theta - 2\int \frac{\tan\theta-1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\, d\theta$$

$$= 2\int \sqrt{\sec^2\theta}\,\sec^2\theta\,d\theta - 2\int \frac{\tan\theta - 1}{\sqrt{\sec^2\theta}}\sec^2\theta\,d\theta$$

$$= 2\int \sec^3\theta\,d\theta - 2\int (\tan\theta - 1)\sec\theta\,d\theta;\ \text{ then we get:}$$

$$2\int \sqrt{1+(y-1)^2}\,dy - 2\int \frac{y-2}{\sqrt{1+(y-1)^2}}dy = 2\int \sec^3\theta\,d\theta - 2\int (\tan\theta - 1)\sec\theta\,d\theta$$

$$= 2\int \sec^3\theta\,d\theta - 2\int (\tan\theta\sec\theta - \sec\theta)\,d\theta$$

$$= \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - 2\sec\theta + 2\ln|\sec\theta\tan\theta| + k$$

$$= \sec\theta\tan\theta + 3\ln|\sec\theta + \tan\theta| - 2\sec\theta + k$$

For the third integral: let $z = \dfrac{1}{y} \Longrightarrow dz = -\dfrac{1}{y^2}dy$; then we get:

$$-4\int \frac{\frac{1}{y^2}}{\sqrt{1-\frac{2}{y}+\frac{2}{y^2}}}dy = 4\int \frac{dz}{\sqrt{2z^2-2z+1}} = 4\sqrt{2}\int \frac{dz}{\sqrt{4z^2-4z+2}}$$

$$-4\int \frac{\frac{1}{y^2}}{\sqrt{1-\frac{2}{y}+\frac{2}{y^2}}}dy = 4\sqrt{2}\int \frac{dz}{\sqrt{(2z-1)^2+1}} = 2\sqrt{2}\sinh^{-1}(2z-1) + k'$$

Therefore; we get: $\displaystyle\int \frac{\sqrt{1+x}}{1+\sqrt{x}}dx$

$$= \sec\theta\tan\theta + 3\ln|\sec\theta + \tan\theta| - 2\sec\theta + 2\sqrt{2}\sinh^{-1}(2z-1) + c$$

$$= (y-1)\sqrt{1+(y-1)^2} - 2\sqrt{1+(y-1)^2} + 3\ln\left|y-1+\sqrt{1+(y-1)^2}\right|$$

$$+ 2\sqrt{2}\sinh^{-1}\left(\frac{2-y}{y}\right) + c$$

$$= (\sqrt{x}-2)\sqrt{1+x} + 3\ln\left|\sqrt{x}+\sqrt{1+x}\right| + 2\sqrt{2}\sinh^{-1}\left(\frac{1-\sqrt{x}}{1+\sqrt{x}}\right) + c$$

202. $\mathrm{I} = \int_0^{+\infty} \dfrac{nx^{n-1}}{(x^2+1)^{\frac{n+2}{2}}}dx$; where $n > 0$

$$\mathrm{I} = \int_0^{+\infty} \frac{nx^{n-1}}{(x^2+1)^{\frac{n+2}{2}}}dx$$

Let $x = \tan\theta \Longrightarrow dx = \sec^2\theta\,d\theta$ & for the bounds $\begin{cases} x = 0 \\ x = +\infty \end{cases} \Longrightarrow \begin{cases} \theta = 0 \\ \theta = \frac{\pi}{2} \end{cases}$; then we get:

$$\mathrm{I} = \int_0^{+\infty} \frac{nx^{n-1}}{(x^2+1)^{\frac{n+2}{2}}}dx = \int_0^{\frac{\pi}{2}} \frac{n\tan^{n-1}\theta}{(\tan^2\theta+1)^{\frac{n+2}{2}}}.\sec^2\theta\,d\theta = n\int_0^{\frac{\pi}{2}} \frac{\tan^{n-1}\theta}{(\sec^2\theta)^{\frac{n+2}{2}}}.\sec^2\theta\,d\theta$$

$$\mathrm{I} = n\int_0^{\frac{\pi}{2}} \frac{\tan^{n-1}\theta}{\sec^{n+2}\theta}.\sec^2\theta\,d\theta = n\int_0^{\frac{\pi}{2}} \frac{\tan^{n-1}\theta}{\sec^n\theta\,.\sec^2\theta}.\sec^2\theta\,d\theta = n\int_0^{\frac{\pi}{2}} \frac{\tan^{n-1}\theta}{\sec^n\theta}d\theta;\ \text{ then:}$$

$$\mathrm{I} = n\int_0^{\frac{\pi}{2}} \frac{\frac{\sin^{n-1}\theta}{\cos^{n-1}\theta}}{\frac{1}{\cos^n\theta}}d\theta = n\int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\,.\cos\theta\, d\theta; \text{ then we can write:}$$

$$\mathrm{I} = \frac{1}{2}n\left\{2\int_0^{\frac{\pi}{2}} \sin^{n-1}\theta\,.\cos\theta\, d\theta\right\} = \frac{1}{2}n\left\{2\int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{n}{2}\right)-1}\theta\,.\cos^{2(1)-1}\theta\, d\theta\right\}$$

With $\mathrm{B}(a,b) = 2\int_0^{\frac{\pi}{2}} \sin^{2a-1}\theta\,.\cos^{2b-1}\theta\, d\theta$; then we get:

$$\mathrm{I} = \frac{n}{2}\mathrm{B}\left(\frac{n}{2},1\right) = \frac{n}{2}.\frac{\Gamma\left(\frac{n}{2}\right)\Gamma(1)}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{n}{2}.\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)} = \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n}{2}+1\right)} = 1$$

Therefore; we get: $\displaystyle\int_0^{+\infty} \frac{nx^{n-1}}{(x^2+1)^{\frac{n+2}{2}}}dx = 1$

203. $\mathrm{I} = \int_0^{+\infty} \frac{e^{-x^2}-e^{-x}}{x}dx$; using integration by parts:

Let $u = e^{-x^2} - e^{-x} \Longrightarrow u' = -2xe^{-x^2} + e^{-x}$ & let $v' = \frac{1}{x} \Longrightarrow v = \ln x$; then we get:

$$\mathrm{I} = \int_0^{+\infty} \frac{e^{-x^2}-e^{-x}}{x}dx = \left[\left(e^{-x^2}-e^{-x}\right)\ln x\right]_0^{+\infty} - \int_0^{+\infty}\left(-2xe^{-x^2}+e^{-x}\right)\ln x\, dx$$

$$\mathrm{I} = \int_0^{+\infty}\left(2xe^{-x^2}-e^{-x}\right)\ln x\, dx = \int_0^{+\infty} 2xe^{-x^2}.\ln x\, dx - \int_0^{+\infty} e^{-x}.\ln x\, dx$$

For $\displaystyle\int_0^{+\infty} 2xe^{-x^2}.\ln x\, dx$; let $t = x^2 \Longrightarrow dt = 2xdx$; then we get:

$$\int_0^{+\infty} 2xe^{-x^2}.\ln x\, dx = \int_0^{+\infty} e^{-t}.\ln\sqrt{t}\, dt = \int_0^{+\infty} e^{-t}.\frac{1}{2}\ln t\, dt = \frac{1}{2}\int_0^{+\infty} e^{-t}.\ln t\, dt; \text{ then we get:}$$

$$\int_0^{+\infty} 2xe^{-x^2}.\ln x\, dx = \frac{1}{2}\int_0^{+\infty} e^{-t}.\ln t\, dt = \frac{1}{2}\int_0^{+\infty} e^{-x}.\ln x\, dx \quad \text{and so we can write:}$$

$$\mathrm{I} = \frac{1}{2}\int_0^{+\infty} e^{-x}.\ln x\, dx - \int_0^{+\infty} e^{-x}.\ln x\, dx = -\frac{1}{2}\int_0^{+\infty} e^{-x}.\ln x\, dx = -\frac{1}{2}(-\gamma) = \frac{1}{2}\gamma$$

204. $\mathrm{I} = \int_0^{+\infty} \frac{x^{a-1}}{\sinh(bx)}dx$; we have $\sinh bx = \frac{e^{bx}-e^{-bx}}{2}$, then:

$$\mathrm{I} = \int_0^{+\infty} \frac{x^{a-1}}{\sinh(bx)} = \int_0^{+\infty} \frac{x^{a-1}}{\frac{e^{bx}-e^{-bx}}{2}}dx = 2\int_0^{+\infty} \frac{x^{a-1}}{e^{bx}-e^{-bx}}dx = 2\int_0^{+\infty} \frac{x^{a-1}e^{bx}}{e^{2bx}-1}dx; \text{ then}$$

$$\mathrm{I} = 2\int_0^{+\infty} x^{a-1}.\frac{e^{-bx}}{1-e^{-2bx}}dx = 2\int_0^{+\infty} x^{a-1}.\sum_{n=1}^{\infty} e^{-(2n-1)bx}\,dx$$

$$\mathrm{I} = 2\sum_{n=1}^{\infty}\int_0^{+\infty} x^{a-1}.e^{-(2n-1)bx}dx\,;\quad \text{let } u = (2n-1)bx \Longrightarrow du = (2n-1)bdx;\ \text{ then:}$$

$$\mathrm{I} = 2\sum_{n=1}^{\infty}\int_0^{+\infty}\left[\frac{u}{(2n-1)b}\right]^{a-1}.e^{-u}.\frac{du}{(2n-1)b}du = \frac{2}{b^a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^a}\int_0^{+\infty} u^{a-1}.e^{-u}.du$$

$$\mathrm{I} = \frac{2}{b^a}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^a}\Gamma(a) = \frac{2}{b^a}\Gamma(a)\sum_{n=1}^{\infty}\frac{1}{(2n-1)^a}$$

We have $\zeta(a) = \frac{1}{1^a}+\frac{1}{2^a}+\frac{1}{3^a}+\frac{1}{4^a}+\cdots$

$$\zeta(a) = \frac{1}{1^a}+\frac{1}{3^a}+\frac{1}{5^a}+\frac{1}{7^a}+\cdots+\frac{1}{2^a}\left(\frac{1}{1^a}+\frac{1}{2^a}+\frac{1}{3^a}+\frac{1}{4^a}+\cdots\right);\ \text{ then we can write:}$$

$$\zeta(a) = \frac{1}{1^a}+\frac{1}{3^a}+\frac{1}{5^a}+\frac{1}{7^a}+\cdots+\frac{1}{2^a}.\zeta(a) \Longrightarrow \zeta(a)\left(\frac{2^a-1}{2^a}\right) = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^a};\ \text{ then:}$$

$$\mathrm{I} = 2^{-a}\Gamma(a).\zeta(a)(2^a-1)\times 2\times\frac{1}{b^a} = \frac{2^a-1}{2^{a-1}}.\frac{\Gamma(a)\zeta(a)}{b^a}$$

205. $\int\frac{\sin x.\cos x}{\sin x+\cos x}dx = \int\frac{\frac{1}{2}\sin 2x}{\sin x+\cos x}dx = \frac{1}{2}\int\frac{\sin 2x}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)}dx$

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = \frac{1}{2}\int\frac{\sin 2x}{\sqrt{2}\left(\cos\frac{\pi}{4}\sin x+\sin\frac{\pi}{4}\cos x\right)}dx = \frac{1}{2\sqrt{2}}\int\frac{\sin 2x}{\sin\left(\frac{\pi}{4}+x\right)}dx$$

Let $t = x+\frac{\pi}{4} \Longrightarrow x = t-\frac{\pi}{4} \Longrightarrow 2x = 2t-\frac{\pi}{2} \Longrightarrow dt = dx$; then we get:

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = \frac{1}{2\sqrt{2}}\int\frac{\sin\left(2t-\frac{\pi}{2}\right)}{\sin t}dt = -\frac{1}{2\sqrt{2}}\int\frac{\cos 2t}{\sin t}dt;\ \text{ then we get:}$$

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = -\frac{1}{2\sqrt{2}}\int\frac{1-2\sin^2 t}{\sin t}dt = -\frac{1}{2\sqrt{2}}\int\frac{1}{\sin t}dt+\frac{1}{\sqrt{2}}\int\frac{\sin^2 t}{\sin t}dt$$

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = \frac{1}{\sqrt{2}}\int\sin t\,dt-\frac{1}{2\sqrt{2}}\int\csc t\,dt;\ \text{ then we get:}$$

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = -\frac{1}{\sqrt{2}}\cos t-\frac{1}{2\sqrt{2}}\ln|\csc t-\cot t|+c$$

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = -\frac{1}{\sqrt{2}}\cos t-\frac{1}{2\sqrt{2}}\ln\left|\frac{1}{\sin t}-\frac{\cos t}{\sin t}\right|+c$$

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = -\frac{1}{\sqrt{2}}\cos t-\frac{1}{2\sqrt{2}}\ln\left|\frac{1-\cos t}{\sin t}\right|+c$$

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = -\frac{1}{\sqrt{2}}\cos\left(x+\frac{\pi}{4}\right)-\frac{1}{2\sqrt{2}}\ln\left|\frac{1-\cos\left(x+\frac{\pi}{4}\right)}{\sin\left(x+\frac{\pi}{4}\right)}\right|+c$$

$$\int\frac{\sin x\,.\cos x}{\sin x+\cos x}dx = -\frac{1}{\sqrt{2}}\cos\left(x+\frac{\pi}{4}\right)-\frac{1}{2\sqrt{2}}\ln\left|\frac{\sin x-\cos x+\sqrt{2}}{\sin x+\cos x}\right|+c$$

$$\int \frac{\sin x \,.\cos x}{\sin x + \cos x} dx = \frac{1}{2\sqrt{2}} \ln\left|\frac{\sin x + \cos x}{\sin x - \cos x + \sqrt{2}}\right| + \frac{1}{\sqrt{2}}(\sin x - \cos x) + c$$

206. $\quad \mathrm{I} = \int_0^1 \ln(-\ln x) \dfrac{x^{\alpha-1}}{\sqrt{-\ln x}} dx$

Let $u = -\ln x \Longrightarrow x = e^{-u}$ & $dx = -e^{-u}du$; $\begin{cases} x = 0 \\ x = 1 \end{cases} \Longrightarrow \begin{cases} u = +\infty \\ u = 0 \end{cases}$; then we get:

$$\mathrm{I} = \int_0^1 \ln(-\ln x)\frac{x^{\alpha-1}}{\sqrt{-\ln x}} dx = \int_{+\infty}^0 \ln u \frac{(e^{-u})^{\alpha-1}}{\sqrt{u}}(-e^{-u}du)\text{; then:}$$

$$\mathrm{I} = \int_0^{+\infty} \ln u \frac{e^{-u\alpha}.e^{u}}{\sqrt{u}} e^{-u}du = \int_0^{+\infty} \ln u \frac{e^{-u\alpha}}{\sqrt{u}} du$$

Let $y = u\alpha \Longrightarrow dy = \alpha du$; then we get:

$$\mathrm{I} = \int_0^{+\infty} \ln u \frac{e^{-u\alpha}}{\sqrt{u}} du = \int_0^{+\infty} \ln\left(\frac{y}{\alpha}\right)\frac{e^{-y}}{\sqrt{\frac{y}{\alpha}}}.\frac{dy}{\alpha} = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} \ln\left(\frac{y}{\alpha}\right)\frac{e^{-y}}{\sqrt{y}}.dy$$

$$\mathrm{I} = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} (\ln y - \ln\alpha)\frac{e^{-y}}{\sqrt{y}}.dy = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} \frac{e^{-y}\ln y}{\sqrt{y}}.dy - \frac{\ln\alpha}{\sqrt{\alpha}}\int_0^{+\infty} \frac{e^{-y}}{\sqrt{y}}.dy$$

$$\mathrm{I} = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} \frac{e^{-y}\ln y}{\sqrt{y}}.dy - \frac{\ln\alpha}{\sqrt{\alpha}}\int_0^{+\infty} y^{-\frac{1}{2}}e^{-y}.dy = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} \frac{e^{-y}\ln y}{\sqrt{y}}.dy - \frac{\ln\alpha}{\sqrt{\alpha}}\int_0^{+\infty} y^{\frac{1}{2}-1}e^{-y}.dy$$

$$\mathrm{I} = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} \frac{e^{-y}\ln y}{\sqrt{y}}.dy - \frac{\ln\alpha}{\sqrt{\alpha}}\Gamma\left(\frac{1}{2}\right) = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} \frac{e^{-y}\ln y}{\sqrt{y}}.dy - \frac{\ln\alpha}{\sqrt{\alpha}}\sqrt{\pi}\text{; then we can write:}$$

$$\mathrm{I} = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} \frac{e^{-y}\ln y}{\sqrt{y}}.dy - \ln\alpha\sqrt{\frac{\pi}{\alpha}} = \frac{1}{\sqrt{\alpha}}\int_0^{+\infty} y^{-\frac{1}{2}}e^{-y}\ln y\, dy - \ln\alpha\sqrt{\frac{\pi}{\alpha}}$$

We know: $\Gamma(n) = \displaystyle\int_0^{+\infty} x^{n-1}e^{-x}dx \Longrightarrow \Gamma'(n) = \int_0^{+\infty} x^{n-1}e^{-x}\ln x\, dx$

Now putting $n = \dfrac{1}{2} \Longrightarrow \Gamma'\left(\dfrac{1}{2}\right) = \displaystyle\int_0^{+\infty} x^{-\frac{1}{2}}e^{-x}\ln x\, dx$; then: $\mathrm{I} = \dfrac{1}{\sqrt{\alpha}}\Gamma'\left(\dfrac{1}{2}\right) - \ln\alpha\sqrt{\dfrac{\pi}{\alpha}}$

With $\Gamma'(n) = \Psi(n)\Gamma(n) \Longrightarrow \Gamma'\left(\dfrac{1}{2}\right) = \Psi\left(\dfrac{1}{2}\right)\Gamma\left(\dfrac{1}{2}\right) = (-\gamma - 2\ln 2)\sqrt{\pi}$; then we get:

$$\mathrm{I} = \frac{1}{\sqrt{\alpha}}(-\gamma - 2\ln 2)\sqrt{\pi} - \ln\alpha\sqrt{\frac{\pi}{\alpha}} = -\sqrt{\frac{\pi}{\alpha}}(\gamma + \ln 4\alpha)$$

207. $\quad \displaystyle\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(4x^2+8x+4)+9}}\right)dx\text{; then we get:}$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx = \int \sin^{-1}\left(\frac{2x+2}{\sqrt{(2x+2)^2+3^2}}\right)dx$$

Let $2x+2=3\tan\theta \Rightarrow 2dx=3\sec^2\theta\,d\theta \Rightarrow dx=\frac{3}{2}\sec^2\theta\,d\theta$; then we get:

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\int \sin^{-1}\left(\frac{3\tan\theta}{\sqrt{9\tan^2\theta+9}}\right)\times\sec^2\theta\,d\theta$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\int \sin^{-1}\left(\frac{3\tan\theta}{3\sqrt{\tan^2\theta+1}}\right)\times\sec^2\theta\,d\theta$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\int \sin^{-1}\left(\frac{\tan\theta}{\sqrt{\sec^2\theta}}\right)\times\sec^2\theta\,d\theta$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\int \sin^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)\times\sec^2\theta\,d\theta$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\int \sin^{-1}\left(\frac{\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}}\right)\times\sec^2\theta\,d\theta;\ \text{ then we get:}$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\int \sin^{-1}(\sin\theta)\times\sec^2\theta\,d\theta;\ \text{ and soassd we get:}$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\int \theta\times\sec^2\theta\,d\theta$$

Now using integration by parts:

Let $u=\theta \Rightarrow u'=1$ and let $v'=\sec^2\theta \Rightarrow v=\tan\theta$; then we get:

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\left[\theta\tan\theta-\int\tan\theta\,d\theta\right]$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}[\theta\tan\theta-\ln(\sec\theta)]+c$$

$$2x+2=3\tan\theta \Rightarrow \tan\theta=\frac{2x+2}{3};\ \ \theta=\tan^{-1}\left(\frac{2x+2}{3}\right)\ \ \&\ \sec\theta=\sqrt{1+\left(\frac{2x+2}{3}\right)^2}$$

$$\int \sin^{-1}\left(\frac{2x+2}{\sqrt{4x^2+8x+13}}\right)dx=\frac{3}{2}\left[\left(\frac{2x+2}{3}\right)\tan^{-1}\left(\frac{2x+2}{3}\right)-\ln\sqrt{1+\left(\frac{2x+2}{3}\right)^2}\right]+c$$

208. $\mathrm{I}=\int_0^1 x^b\ln\left(\frac{1}{x}\right)\ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{a-1}dx$

Let $t=-\ln x \Rightarrow x=e^{-t} \Rightarrow dx=-e^{-t}dx$; $\begin{cases}x=0\\x=1\end{cases} \Rightarrow \begin{cases}t=+\infty\\t=0\end{cases}$; then we get:

$$\mathrm{I}=\int_0^1 x^b\ln\left(\frac{1}{x}\right)\ln\left(\ln\frac{1}{x}\right)\left(\ln\frac{1}{x}\right)^{a-1}dx=\int_0^1 x^b(-\ln x)\ln(-\ln x)\,(-\ln x)^{a-1}dx$$

$$\mathrm{I}=\int_{+\infty}^{0}(e^{-t})^b.t\ln t.t^{a-1}(-e^{-t}dt)=\int_0^{+\infty} e^{-tb}.t\ln t.t^{a-1}e^{-t}dt$$

$$\mathrm{I}=\int_0^{+\infty} e^{-t(1+b)}.t^a.\ln t\,dt;\ \text{ let } u=t(1+b)\Rightarrow du=(1+b)dt;\ \text{ then we get:}$$

$$\mathrm{I}=\int_0^{+\infty} e^{-u}.\left(\frac{u}{1+b}\right)^a.\ln\left(\frac{u}{1+b}\right).\frac{du}{1+b}$$

$$\mathrm{I}=\int_0^{+\infty} e^{-u}.\frac{u^a}{(1+b)^a}.[\ln u-\ln(1+b)].\frac{1}{1+b}.du$$

$$\mathrm{I}=\frac{1}{(1+b)^{a+1}}\int_0^{+\infty} e^{-u}.u^a.[\ln u-\ln(1+b)].du\,;\quad \text{then we get:}$$

$$\mathrm{I}=\frac{1}{(1+b)^{a+1}}\int_0^{+\infty} e^{-u}.u^a\ln u\,du-\frac{\ln(1+\mathrm{b})}{(1+b)^{a+1}}\int_0^{+\infty} e^{-u}.u^a du$$

$$\mathrm{I}=\frac{1}{(1+b)^{a+1}}\Gamma'(a+1)-\frac{\ln(1+\mathrm{b})}{(1+b)^{a+1}}\Gamma(a+1)$$

$$\mathrm{I}=\frac{1}{(1+b)^{a+1}}\Gamma(a+1)\Psi(a+1)-\frac{\ln(1+\mathrm{b})}{(1+b)^{a+1}}\Gamma(a+1)$$

Therefore; we get: $\mathrm{I}=\dfrac{\Gamma(a+1)}{(1+b)^{a+1}}[\Psi(a+1)-\ln(1+\mathrm{b})]$

209. $\mathrm{I}=\int\dfrac{\Delta}{\sin x}dx$ **;** where $\Delta=\sqrt{1-k^2\sin^2 x}$

$$\mathrm{I}=\int\frac{\Delta}{\sin x}dx=\int\frac{\sqrt{1-k^2\sin^2 x}}{\sin x}dx=\int\frac{\sin x\sqrt{1-k^2\sin^2 x}}{\sin^2 x}dx=\int\frac{\sin x\sqrt{1-k^2\sin^2 x}}{1-\cos^2 x}dx$$

Let $u=\cos x\Longrightarrow du=-\sin x\,dx$; then we get:

$$\mathrm{I}=\int\frac{\sin x\sqrt{1-k^2\sin^2 x}}{1-\cos^2 x}dx=\int\frac{\sin x\sqrt{1-k^2(1-\cos^2 x)}}{1-\cos^2 x}dx$$

$$\mathrm{I}=\int\frac{\sin x\sqrt{k^2\cos^2 x+(1-k^2)}}{1-\cos^2 x}dx=-\int\frac{\sqrt{k^2u^2+(1-k^2)}}{1-u^2}du=-\int\frac{k\sqrt{u^2+\frac{1-k^2}{k^2}}}{1-u^2}du$$

Let $a^2=\dfrac{1-k^2}{k^2}$; then we can write: $\mathrm{I}=-k\displaystyle\int\frac{\sqrt{u^2+a^2}}{1-u^2}du$

Let $u^2=a^2\tan^2\theta\Longrightarrow u=a\tan\theta\Longrightarrow du=a\sec^2\theta\,d\theta$; then we get:

$$\mathrm{I}=-k\int\frac{\sqrt{a^2\tan^2\theta+a^2}}{1-a^2\tan^2\theta}.a\sec^2\theta\,d\theta=-ka^2\int\frac{\sqrt{\tan^2\theta+1}}{1-a^2\tan^2\theta}\sec^2\theta\,d\theta$$

$$\mathrm{I}=-ka^2\int\frac{\sqrt{\sec^2\theta}}{1-a^2\tan^2\theta}\sec^2\theta\,d\theta=-ka^2\int\frac{\sec^3\theta}{1-a^2\tan^2\theta}d\theta=-ka^2\int\frac{\frac{1}{\cos^3\theta}}{1-a^2\frac{\sin^2\theta}{\cos^2\theta}}d\theta$$

$$\mathrm{I}=-ka^2\int\frac{1}{\cos^3\theta-a^2\sin^2\theta\cos\theta}d\theta=-ka^2\int\frac{1}{\cos\theta\,(\cos^2\theta-a^2\sin^2\theta)}d\theta$$

$$\mathrm{I}=-ka^2\int\frac{\cos\theta}{\cos^2\theta\,(\cos^2\theta-a^2\sin^2\theta)}d\theta=-ka^2\int\frac{\cos\theta}{(1-\sin^2\theta)(1-\sin^2\theta-a^2\sin^2\theta)}d\theta$$

$$\mathrm{I}=-ka^2\int\frac{\cos\theta}{(1-\sin^2\theta)[1-(a^2+1)\sin^2\theta]}d\theta$$

Let $y=\sin\theta\Longrightarrow dy=\cos\theta\,d\theta$; then we get: $\mathrm{I}=-ka^2\displaystyle\int\frac{dy}{(1-y^2)[1-(a^2+1)y^2]}$

$$\mathrm{I}=\frac{-a^2k}{-a^2}\int\frac{[1-(a^2+1)y^2]-(a^2+1)(1-y^2)}{(1-y^2)[1-(a^2+1)y^2]}dy;\quad\text{then we get:}$$

$$I = k\int\left(\frac{1}{1-y^2} - \frac{a^2+1}{1-(a^2+1)y^2}\right)dy;\quad \text{then we can write:}$$

$$I = \frac{k}{2}\int\frac{(1+y)+(1-y)}{(1+y)(1-y)}dy + k(1+a^2)\int\frac{dy}{(a^2+1)y^2-1}$$

$$I = \frac{k}{2}\int\left(\frac{1}{1-y}+\frac{1}{1+y}\right)dy + k(1+a^2)\int\frac{dy}{(a^2+1)y^2-1}$$

$$I = \frac{k}{2}\ln\left|\frac{1+y}{1-y}\right| + k(1+a^2)\int\frac{dy}{\left(\sqrt{a^2+1}y+1\right)\left(\sqrt{a^2+1}y-1\right)}$$

$$I = \frac{k}{2}\ln\left|\frac{1+y}{1-y}\right| + \frac{k}{2}(1+a^2)\int\frac{\left(\sqrt{a^2+1}y+1\right)-\left(\sqrt{a^2+1}y-1\right)}{\left(\sqrt{a^2+1}y+1\right)\left(\sqrt{a^2+1}y-1\right)}dy$$

$$I = \frac{k}{2}\ln\left|\frac{1+y}{1-y}\right| + \frac{k}{2}(1+a^2)\int\left(\frac{1}{\sqrt{a^2+1}y-1} - \frac{1}{\sqrt{a^2+1}y+1}\right)dy$$

$$I = \frac{k}{2}\ln\left|\frac{1+y}{1-y}\right| + \frac{k}{2}\sqrt{a^2+1}\ln\left|\frac{\sqrt{a^2+1}y-1}{\sqrt{a^2+1}y+1}\right| + c;\quad \text{then we get:}$$

$$I = \frac{k}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + \frac{k}{2}\sqrt{a^2+1}\ln\left|\frac{\sqrt{a^2+1}\sin\theta-1}{\sqrt{a^2+1}\sin\theta+1}\right| + c$$

Where $a^2 = \dfrac{1-k^2}{k^2} \Longrightarrow a^2k^2 + k^2 = 1 \Longrightarrow k^2 = \dfrac{1}{1+a^2}$; then we get:

$$I = \frac{k}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right| + \frac{1}{2}\ln\left|\frac{\sin\theta-k}{\sin\theta+k}\right| + c;\quad \text{with } \tan\theta = \frac{u}{a};\quad \text{then we can write: } \theta$$

$$I = \frac{k}{2}\ln\left|\frac{\sqrt{u^2+a^2}+u}{\sqrt{u^2+a^2}-u}\right| + \frac{1}{2}\ln\left|\frac{u-k\sqrt{u^2+a^2}}{u+k\sqrt{u^2+a^2}}\right| + c$$

$$I = \frac{k}{2}\ln\left|\frac{\sqrt{\cos^2x+a^2}+\cos x}{\sqrt{\cos^2x+a^2}-\cos x}\right| + \frac{1}{2}\ln\left|\frac{\cos x-k\sqrt{\cos^2x+a^2}}{\cos x+k\sqrt{\cos^2x+a^2}}\right| + c;\quad \text{from}\quad a^2 = \frac{1-k^2}{k^2}$$

$$I = \frac{k}{2}\ln\left|\frac{\sqrt{k^2\cos^2x+1-k^2}+k\cos x}{\sqrt{k^2\cos^2x+1-k^2}-k\cos x}\right| - \frac{1}{2}\ln\left|\frac{\cos x-\sqrt{k^2\cos^2x+1-k^2}}{\cos x+\sqrt{k^2\cos^2x+1-k^2}}\right| + c$$

$$I = \frac{k}{2}\ln\left|\frac{\Delta+k\cos x}{\Delta-k\cos x}\right| - \frac{1}{2}\ln\left|\frac{\cos x-\Delta}{\cos x+\Delta}\right| + c = k\ln\sqrt{\frac{\Delta+k\cos x}{\Delta-k\cos x}} - \ln\sqrt{\frac{\cos x-\Delta}{\cos x+\Delta}} + c$$

210. $I = \int\left(\frac{8}{27}\right)^x .\ln\left|\frac{4^x+9^x}{6^x+9^x}\right|dx$

Let $t = \left(\frac{2}{3}\right)^x \Longrightarrow dt = \left(\frac{2}{3}\right)^x \ln\left(\frac{2}{3}\right)dx$; then we get:

$$I = \int\left(\frac{8}{27}\right)^x .\ln\left|\frac{4^x+9^x}{6^x+9^x}\right|dx = \int\left(\frac{2^3}{3^3}\right)^x .\ln\left|\frac{4^x+9^x}{6^x+9^x}\times\frac{\frac{1}{9^x}}{\frac{1}{9^x}}\right|dx = \int\left(\frac{2}{3}\right)^{3x} .\ln\left|\frac{\frac{4^x}{9^x}+1}{\frac{6^x}{9^x}+1}\right|dx$$

$$I = \int\left(\frac{2}{3}\right)^{3x} .\ln\left|\frac{\left[\left(\frac{2}{3}\right)^2\right]^x+1}{\left(\frac{2}{3}\right)^x+1}\right|dx = \int\left(\frac{2}{3}\right)^{3x} .\ln\left|\frac{\left[\left(\frac{2}{3}\right)^x\right]^2+1}{\left(\frac{2}{3}\right)^x+1}\right|dx;\quad \text{then we get:}$$

$$\mathrm{I} = \int t^3 . \ln\left|\frac{t^2+1}{t+1}\right| . \frac{dt}{t \ln\left(\frac{2}{3}\right)} = \frac{1}{\ln\left(\frac{2}{3}\right)} \int t^2 \ln\left(\frac{t^2+1}{t+1}\right) dt$$

I. B. P : Let $u = \ln\left(\frac{t^2+1}{t+1}\right) \Longrightarrow u' = \frac{t^2+2t-1}{(t+1)(t^2+1)}$ & let $v' = t^2 \Longrightarrow v = \frac{1}{3}t^3$; so we get:

$$\mathrm{I} = \frac{1}{\ln\left(\frac{2}{3}\right)}\left\{\frac{1}{3}t^3 \ln\left(\frac{t^2+1}{t+1}\right) - \int \frac{1}{3}t^3 . \frac{t^2+2t-1}{(t+1)(t^2+1)} dt\right\}$$

Decomposition into partial fractions: $\frac{t^2+2t-1}{(t+1)(t^2+1)} = \frac{a}{t+1} + \frac{bt+c}{t^2+1}$

$$t^2+2t-1 = (t^2+1)a + (t+1)(bt+c) \Longrightarrow t^2+2t-1 = at^2+a+bt^2+ct+bt+c$$

$$t^2+2t-1 = (a+b)t^2+(c+b)t+a+c \Longrightarrow \begin{cases} a+b=1 \\ b+c=2 \\ a+c=-1 \end{cases} \Longrightarrow \begin{cases} a=-1 \\ b=2 \\ c=0 \end{cases} ; \text{ then we get:}$$

$$\mathrm{I} = \frac{1}{\ln\left(\frac{2}{3}\right)}\left\{\frac{1}{3}t^3 \ln\left(\frac{t^2+1}{t+1}\right) - \int \frac{1}{3}t^3\left(\frac{2t}{t^2+1} - \frac{1}{1+t}\right) dt\right\}$$

$$\mathrm{I} = \frac{1}{3\ln\left(\frac{2}{3}\right)}\left\{t^3 \ln\left(\frac{t^2+1}{t+1}\right) - \int \left(\frac{2t^4}{t^2+1} - \frac{t^3}{1+t}\right) dt\right\}$$

$$\mathrm{I} = \frac{1}{3\ln\left(\frac{2}{3}\right)} t^3 \ln\left(\frac{t^2+1}{t+1}\right) - \frac{2}{3\ln\left(\frac{2}{3}\right)}\int \frac{t^4}{t^2+1} dt + \frac{1}{3\ln\left(\frac{2}{3}\right)}\int \frac{t^3}{t+1} dt$$

Now Evaluating: $\int \frac{t^4}{t^2+1} dt = \int \frac{t^2(1+t^2)-t^2}{t^2+1} dt = \int \left(t^2 - \frac{t^2}{1+t^2}\right) dt$; then we get:

$$\int \frac{t^4}{t^2+1} dt = \int \left(t^2 - \frac{t^2+1-1}{1+t^2}\right) dt = \int \left(t^2 - \frac{t^2+1}{1+t^2} + \frac{1}{1+t^2}\right) dt$$

Then we get: $\int \frac{t^4}{t^2+1} dt = \int \left(t^2 - 1 + \frac{1}{1+t^2}\right) dt$

Now Evaluating: $\int \frac{t^3}{t+1} dt = \int \frac{t^2(1+t)-t^2}{t+1} dt = \int \left(t^2 - \frac{t^2}{1+t}\right)$; then we get:

$$\int \frac{t^3}{t+1} dt = \int \left(t^2 - \frac{t^2-1+1}{1+t}\right) dt = \int \left(t^2 - \frac{t^2-1}{1+t} - \frac{1}{1+t}\right) dt$$

$$\int \frac{t^3}{t+1} dt = \int \left(t^2 - \frac{(t+1)(t-1)}{1+t} - \frac{1}{1+t}\right) dt = \int \left(t^2 - t + 1 - \frac{1}{1+t}\right) dt; \text{ so:}$$

$$\mathrm{I} = \frac{1}{3\ln\left(\frac{2}{3}\right)} t^3 \ln\left(\frac{t^2+1}{t+1}\right) - \frac{2}{3\ln\left(\frac{2}{3}\right)}\int \left(t^2 - 1 + \frac{1}{1+t^2}\right) dt + \frac{1}{3\ln\left(\frac{2}{3}\right)}\int \left(t^2 - t + 1 - \frac{1}{1+t}\right) dt$$

$$\mathrm{I} = \frac{1}{3\ln\left(\frac{2}{3}\right)} t^3 \ln\left(\frac{t^2+1}{t+1}\right) - \frac{2}{3\ln\left(\frac{2}{3}\right)}\left(\frac{1}{3}t^3 - t + \tan^{-1} t\right)$$

$$+\frac{1}{3\ln\left(\frac{2}{3}\right)}\left(\frac{1}{3}t^3-\frac{1}{2}t^2+t-\ln|1+t|\right)+c;\ \text{ with } t=\left(\frac{2}{3}\right)^x;\ \text{ therefore; we get:}$$

$$\mathrm{I}=\frac{1}{3\ln\left(\frac{2}{3}\right)}\left(\frac{2}{3}\right)^{3x}\ln\left(\frac{\left(\frac{2}{3}\right)^{2x}+1}{\left(\frac{2}{3}\right)^{3x}+1}\right)-\frac{2}{3\ln\left(\frac{2}{3}\right)}\left(\frac{1}{3}\left(\frac{2}{3}\right)^{3x}-\left(\frac{2}{3}\right)^{x}+\tan^{-1}\left(\frac{2}{3}\right)^{3x}\right)$$

$$+\frac{1}{3\ln\left(\frac{2}{3}\right)}\left(\frac{1}{3}\left(\frac{2}{3}\right)^{3x}-\frac{1}{2}\left(\frac{2}{3}\right)^{2x}+\left(\frac{2}{3}\right)^{x}-\ln\left|1+\left(\frac{2}{3}\right)^{3x}\right|\right)+c$$

211. $\mathrm{I}=\int_0^{+\infty}\frac{1}{x^5+x^4+x^3+x^2+x+1}dx$ **… (1)**

Let $y=\frac{1}{x}\Longrightarrow x=\frac{1}{y}\Longrightarrow dx=-\frac{1}{y^2}dy$; for the bounds: $\begin{cases}x=0\\x=+\infty\end{cases}\Longrightarrow\begin{cases}y=+\infty\\y=0\end{cases}$; then:

$$\mathrm{I}=\int_0^{+\infty}\frac{1}{x^5+x^4+x^3+x^2+x+1}dx=\int_{+\infty}^{0}\frac{1}{\left(\frac{1}{y}\right)^5+\left(\frac{1}{y}\right)^4+\left(\frac{1}{y}\right)^3+\left(\frac{1}{y}\right)^2+\frac{1}{y}+1}\left(-\frac{1}{y^2}dy\right)$$

$$\mathrm{I}=\int_0^{+\infty}\frac{\frac{1}{y^2}}{\frac{1}{y^5}+\frac{1}{y^4}+\frac{1}{y^3}+\frac{1}{y^2}+\frac{1}{y}+1}dy=\int_0^{+\infty}\frac{1}{\frac{1}{y^3}+\frac{1}{y^2}+\frac{1}{y}+1+y+y^2}dy$$

$$\mathrm{I}=\int_0^{+\infty}\frac{y^3}{1+y+y^2+y^3+y^4+y^5}dy=\int_0^{+\infty}\frac{x^3}{1+x+x^2+x^3+x^4+x^5}dx\ \ldots(\mathbf{2})$$

Now adding (1) & (2) we get:

$$\mathrm{I}+\mathrm{I}=2\mathrm{I}=\int_0^{+\infty}\frac{1}{x^5+x^4+x^3+x^2+x+1}dx+\int_0^{+\infty}\frac{x^3}{1+x+x^2+x^3+x^4+x^5}dx$$

$$2\mathrm{I}=\int_0^{+\infty}\frac{1+x^3}{1+x+x^2+x^3+x^4+x^5}dx=\int_0^{+\infty}\frac{1+x^3}{(1+x+x^2)+x^3(1+x+x^2)}dx$$

$$2\mathrm{I}=\int_0^{+\infty}\frac{1+x^3}{(1+x+x^2)(1+x^3)}dx=\int_0^{+\infty}\frac{1}{1+x+x^2}dx;\ \text{ then we can write:}$$

$$2\mathrm{I}=\int_0^{+\infty}\frac{1}{\left(x^2+x+\frac{1}{4}\right)+\frac{3}{4}}dx=\int_0^{+\infty}\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

Let $t=x+\frac{1}{2}\Longrightarrow dt=dx$; for the bounds: $\begin{cases}x=0\\x=+\infty\end{cases}\Longrightarrow\begin{cases}t=\frac{1}{2}\\t=+\infty\end{cases}$; then we get:

$$2\mathrm{I}=\int_{\frac{1}{2}}^{+\infty}\frac{1}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx=\left[\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)\right]_{\frac{1}{2}}^{+\infty}\Longrightarrow\mathrm{I}=\frac{1}{\sqrt{3}}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)=\frac{\pi}{3\sqrt{3}}$$

Therefore; we get: $\mathrm{I}=\int\limits_{0}^{+\infty}\frac{1}{x^5+x^4+x^3+x^2+x+1}dx=\frac{\pi}{3\sqrt{3}}$

212. $\mathrm{I}=\int_0^{\frac{\pi}{6}}\frac{\sin x\sin\left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx$

$$\sin x\sin\left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{2\pi}{3}\right)=\sin x\left\{\sin\left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{2\pi}{3}\right)\right\}$$

$$=\sin x\left\{\left(\sin x\cos\frac{\pi}{3}+\cos x\sin\frac{\pi}{3}\right)\left(\sin x\cos\frac{2\pi}{3}+\cos x\sin\frac{2\pi}{3}\right)\right\}$$

$$=\sin x\left(\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right)\left(-\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right)$$

$$=\sin x\left[\left(\frac{\sqrt{3}}{2}\cos x\right)^2-\left(\frac{1}{2}\sin x\right)^2\right]=\sin x\left(\frac{3}{4}\cos^2 x-\frac{1}{4}\sin^2 x\right)$$

$=\frac{1}{4}\sin x\,[3(1-\sin^2 x)-\sin^2 x]=\frac{1}{4}\sin x\,(3-3\sin^2 x-\sin^2 x);$ then we get:

$=\frac{1}{4}\sin x\,(3-4\sin^2 x)=\frac{1}{4}(3\sin x-4\sin^3 x)=\frac{1}{4}\sin 3x\,;$ then we can write:

$$\mathrm{I}=\int\limits_{0}^{\frac{\pi}{6}}\frac{\sin x\sin\left(x+\frac{\pi}{3}\right)\sin\left(x+\frac{2\pi}{3}\right)}{\sin 3x+\cos 3x}dx=\int\limits_{0}^{\frac{\pi}{6}}\frac{\frac{1}{4}\sin 3x}{\sin 3x+\cos 3x}dx=\frac{1}{4}\int\limits_{0}^{\frac{\pi}{6}}\frac{\sin 3x}{\sin 3x+\cos 3x}dx$$

Let $y=3x\Longrightarrow dy=3dx$; for $x=0\Longrightarrow y=0$ & for $x=\frac{\pi}{6}\Longrightarrow y=\frac{\pi}{2}$; then we get:

$$\mathrm{I}=\frac{1}{4}\int\limits_{0}^{\frac{\pi}{2}}\frac{\sin y}{\sin y+\cos y}.\frac{1}{3}dy=\frac{1}{12}\int\limits_{0}^{\frac{\pi}{2}}\frac{\sin y}{\sin y+\cos y}dy$$

Let $t=\frac{\pi}{2}-y\Longrightarrow dt=-dy$; for $y=0\Longrightarrow t=\frac{\pi}{2}$ & for $y=\frac{\pi}{2}\Longrightarrow t=0$; then we get:

$$\mathrm{I}=\frac{1}{12}\int\limits_{\frac{\pi}{2}}^{0}\frac{\sin\left(\frac{\pi}{2}-t\right)}{\sin\left(\frac{\pi}{2}-t\right)+\cos\left(\frac{\pi}{2}-t\right)}(-dt)=\frac{1}{12}\int\limits_{0}^{\frac{\pi}{2}}\frac{\cos t}{\cos t+\sin t}dt=\frac{1}{12}\int\limits_{0}^{\frac{\pi}{2}}\frac{\cos y}{\cos y+\sin y}dy$$

$$\mathrm{I}+\mathrm{I}=\frac{1}{12}\int\limits_{0}^{\frac{\pi}{2}}\frac{\sin y}{\sin y+\cos y}dy+\frac{1}{12}\int\limits_{0}^{\frac{\pi}{2}}\frac{\cos y}{\sin y+\cos y}dy=\frac{1}{12}\int\limits_{0}^{\frac{\pi}{2}}\frac{\sin y+\cos y}{\sin y+\cos y}dy$$

$$2\mathrm{I}=\frac{1}{12}\int\limits_{0}^{\frac{\pi}{2}}dy=\frac{1}{12}\left(\frac{\pi}{2}\right)=\frac{\pi}{24}\Longrightarrow\mathrm{I}=\frac{1}{2}\left(\frac{\pi}{24}\right)=\frac{\pi}{48}$$

213. $\mathrm{I}=\int_{-\infty}^{+\infty}e^{-x^2}\cos(2x^2)\,dx$, we have $\cos(2x^2)=\mathrm{Re}\left(e^{2ix^2}\right)$, then we can write**:**

$$\mathrm{I}=\int\limits_{-\infty}^{+\infty}e^{-x^2}\cos(2x^2)\,dx=\int\limits_{-\infty}^{+\infty}e^{-x^2}.\mathrm{Re}\left(e^{2ix^2}\right)dx=\mathrm{Re}\left(\int\limits_{-\infty}^{+\infty}e^{-x^2}e^{2ix^2}dx\right)$$

$$\mathrm{I} = \mathrm{Re}\left(\int_{-\infty}^{+\infty} e^{-x^2(1-2i)}dx\right)$$

Let $t = x\sqrt{1-2i} \Rightarrow dt = \sqrt{1-2i}dx \Rightarrow dx = \dfrac{1}{\sqrt{1-2i}}dt$; then we get:

$$\int_{-\infty}^{+\infty} e^{-x^2(1-2i)}dx = \int_{-\infty}^{+\infty} e^{-\left(x\sqrt{1-2i}\right)^2}dx = \int_{-\infty}^{+\infty} e^{-t^2}.\frac{1}{\sqrt{1-2i}}dt = \frac{1}{\sqrt{1-2i}}\int_{-\infty}^{+\infty} e^{-t^2}dt$$

$$\int_{-\infty}^{+\infty} e^{-x^2(1-2i)}dx = \frac{1}{\sqrt{1-2i}} \times \sqrt{\pi} = \frac{\sqrt{\pi}}{\sqrt{1-2i}};\text{ then we have:}$$

$$\mathrm{I} = \mathrm{Re}\left(\int_{-\infty}^{+\infty} e^{-x^2(1-2i)}dx\right) = \mathrm{Re}\left(\frac{\sqrt{\pi}}{\sqrt{1-2i}}\right) = \mathrm{Re}\left(\frac{\sqrt{\pi}}{\sqrt{1-2i}} \times \frac{\sqrt{1+2i}}{\sqrt{1+2i}}\right);\text{ then we get:}$$

$$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-x^2}\cos(2x^2)\,dx = \mathrm{Re}\left(\frac{\sqrt{\pi}}{\sqrt{5}}\sqrt{1+2i}\right) = \sqrt{\frac{\pi}{5}}\,\mathrm{Re}\left(\sqrt{1+2i}\right)$$

$\mathrm{Re}\left(\sqrt{1+2i}\right) = ?$ Let $\sqrt{1+2i} = a + ib \Rightarrow 1 + 2i = (a+ib)^2$

$$\Rightarrow 1 + 2i = a^2 - b^2 + 2iab \Rightarrow \begin{cases} 2ab = 2 \\ a^2 - b^2 = 1 \end{cases} \Rightarrow \begin{cases} a = \dfrac{1}{b} \\ a^2 - b^2 = 1 \end{cases} \Rightarrow \left(\frac{1}{b}\right)^2 - b^2 = 1$$

$$\Rightarrow \frac{1-b^4}{b^2} = 1 \Rightarrow b^4 + b^2 - 1 = 0;\text{ let } z = b^2 \Rightarrow z^2 = b^4;\text{ then the equation becomes:}$$

$$z^2 + z - 1 = 0 \Rightarrow z = b^2 = \frac{-1+\sqrt{5}}{2}$$

$$\text{Now } a = \frac{1}{b} = \frac{1}{\sqrt{\frac{-1+\sqrt{5}}{2}}} = \sqrt{\frac{2}{-1+\sqrt{5}}} = \sqrt{\frac{2}{-1+\sqrt{5}} \times \frac{-1-\sqrt{5}}{-1-\sqrt{5}}} = \sqrt{\frac{-2\left(1+\sqrt{5}\right)}{-4}}$$

$$a = \sqrt{\frac{1+\sqrt{5}}{2}} = \sqrt{\varphi};\text{ so } \mathrm{Re}\left(\sqrt{1+2i}\right) = a = \sqrt{\varphi};\text{ therefore; we get:}$$

$$\mathrm{I} = \int_{-\infty}^{+\infty} e^{-x^2}\cos(2x^2)\,dx = \sqrt{\frac{\pi}{5}} \times \sqrt{\varphi} = \sqrt{\frac{\pi\varphi}{5}}$$

214. $\mathrm{I} = \int_0^{+\infty} \sin x\,(\mathrm{erf}(x) - 1)dx$, using integration by parts**:**

Let $u = \mathrm{erf}(x) - 1 \Rightarrow u' = \dfrac{2}{\sqrt{\pi}}e^{-x^2}$ & let $v' = \sin x \Rightarrow v = -\cos x$; then we get:

$$\mathrm{I} = \int_0^{+\infty} \sin x\,(\mathrm{erf}(x) - 1)dx = [-\cos x\,(\mathrm{erf}(x) - 1)]_0^{+\infty} + \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-x^2}\cos x\,dx$$

$$\mathrm{I} = -1 + \frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-x^2}\cos x\,dx$$

Now Evaluating $\int_0^{+\infty} e^{-x^2}\cos x\,dx$: we have $\cos x = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}x^{2n}$; then we get:

$$\int_0^{+\infty} e^{-x^2}\cos x\,dx = \int_0^{+\infty} e^{-x^2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}x^{2n}\,dx = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} e^{-x^2}x^{2n}dx$$

Let $y = x^2 \Longrightarrow dy = 2xdx = 2\sqrt{y}dx \Longrightarrow dx = \frac{dy}{2\sqrt{y}}$; then we get:

$$\int_0^{+\infty} e^{-x^2}\cos x\,dx = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}.\frac{1}{2}\int_0^{+\infty} e^{-y}y^n.\frac{1}{\sqrt{y}}dy = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} e^{-y}y^{n-\frac{1}{2}}dy$$

$$= \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} e^{-y}y^{\left(n+\frac{1}{2}\right)-1}dy = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\Gamma\left(n+\frac{1}{2}\right) = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)}{(2n)!}$$

$$\int_0^{+\infty} e^{-x^2}\cos x\,dx = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n+1)} = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)}{2n\Gamma(2n)};\ \text{ then we get:}$$

$$\int_0^{+\infty} e^{-x^2}\cos x\,dx = \frac{1}{4}\sum_{n=0}^{+\infty}\frac{(-1)^n\Gamma\left(n+\frac{1}{2}\right)}{n\Gamma(2n)};\ \text{ but }\ \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(2n)} = \frac{\sqrt{\pi}}{2^{2n-1}\Gamma(n)};\ \text{ then:}$$

$$\int_0^{+\infty} e^{-x^2}\cos x\,dx = \frac{1}{4}\sum_{n=0}^{+\infty}\frac{(-1)^n\sqrt{\pi}}{n.2^{2n-1}\Gamma(n)} = \frac{2\sqrt{\pi}}{4}\sum_{n=0}^{+\infty}\frac{(-1)^n}{n.2^{2n}\Gamma(n)} = \frac{\sqrt{\pi}}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{1}{4}\right)^n}{n\Gamma(n)}$$

$$\int_0^{+\infty} e^{-x^2}\cos x\,dx = \frac{\sqrt{\pi}}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{1}{4}\right)^n}{\Gamma(n+1)} = \frac{\sqrt{\pi}}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{1}{4}\right)^n}{n!} = \frac{\sqrt{\pi}}{2}\sum_{n=0}^{+\infty}\frac{\left(-\frac{1}{4}\right)^n}{n!} = \frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}}$$

Therefore; $\int_0^{+\infty}\sin x\,(\operatorname{erf}(x)-1)dx = -1+\frac{2}{\sqrt{\pi}}\int_0^{+\infty} e^{-x^2}\cos x\,dx = -1+\frac{2}{\sqrt{\pi}}\left(\frac{\sqrt{\pi}}{2}e^{-\frac{1}{4}}\right)$

$$\int_0^{+\infty}\sin x\,(\operatorname{erf}(x)-1)dx = -1+e^{-\frac{1}{4}} = e^{-\frac{1}{4}}-1$$

215. $\mathrm{I} = \int_0^{\frac{\pi}{2}}\ln(\alpha^2\sin^2 x+\beta^2\cos^2 x)\,dx$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}}\ln(\alpha^2\sin^2 x+\beta^2\cos^2 x)\,dx = \int_0^{\frac{\pi}{2}}\ln(\alpha^2(1-\cos^2 x)+\beta^2\cos^2 x)\,dx;\ \text{ then we get:}$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}}\ln(\alpha^2-\alpha^2\cos^2 x+\beta^2\cos^2 x)\,dx = \int_0^{\frac{\pi}{2}}\ln(\alpha^2+(\beta^2-\alpha^2)\cos^2 x)\,dx$$

$$\mathrm{I} = \int_0^{\frac{\pi}{2}}\ln\left(\alpha^2\left(1+\left(\frac{\beta^2-\alpha^2}{\alpha^2}\right)\cos^2 x\right)\right)dx = \int_0^{\frac{\pi}{2}}\ln(\alpha^2)\,dx+\int_0^{\frac{\pi}{2}}\ln\left(1+\left(\frac{\beta^2-\alpha^2}{\alpha^2}\right)\cos^2 x\right)dx$$

Now, to simplify notation for the rest of the integral, allow $k = \dfrac{\beta^2 - \alpha^2}{\alpha^2}$; then:

$$I = 2\ln\alpha \int_0^{\frac{\pi}{2}} dx + \int_0^{\frac{\pi}{2}} \ln(1 + k\cos^2 x)\,dx = \pi\ln\alpha + \int_0^{\frac{\pi}{2}} \ln(1 + k\cos^2 x)\,dx$$

But $\ln(1+y) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n} y^n$; then we have $\ln(1 + k\cos^2 x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{n}(k\cos^2 x)^n$

So; $\ln(1 + k\cos^2 x) = \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}k^n}{n}\cos^{2n} x$; then we get:

$$I = \pi\ln\alpha + \int_0^{\frac{\pi}{2}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}k^n}{n}\cos^{2n} x\,dx = \pi\ln\alpha + \sum_{n=1}^{\infty} \int_0^{\frac{\pi}{2}} \frac{(-1)^{n+1}k^n}{n}\cos^{2n} x\,dx$$

$$I = \pi\ln\alpha + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}k^n}{n} \int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx$$

But beta function: $B(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1} x \cos^{2n-1} x\,dx$ and so we can write:

$$\int_0^{\frac{\pi}{2}} \cos^{2n} x\,dx = \int_0^{\frac{\pi}{2}} \sin^{2\left(\frac{1}{2}\right)-1} x \cos^{2\left(n+\frac{1}{2}\right)-1} x\,dx = \frac{1}{2}B\left(\frac{1}{2}, n+\frac{1}{2}\right);\ \text{then we get:}$$

$$I = \pi\ln\alpha + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}k^n}{n} B\left(\frac{1}{2}, n+\frac{1}{2}\right) = \pi\ln\alpha + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}k^n}{n}.\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+n+\frac{1}{2}\right)}$$

$$I = \pi\ln\alpha + \frac{1}{2}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}k^n}{n}.\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+1)}$$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$; $\Gamma(n+1) = n!$ & $\Gamma\left(n+\frac{1}{2}\right) = \sqrt{\pi}.\dfrac{1\times3\times5\times\ldots\times(2n-1)}{2^n}$; then we get:

$$I = \pi\ln\alpha - \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{(-k)^n}{n.n!}.\frac{1\times3\times5\times\ldots\times(2n-1)}{2^n}$$

$$I = \pi\ln\alpha - \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{(-k)^n}{n.n!}.\frac{1\times3\times5\times\ldots\times(2n-1)}{2^n}.\frac{2\times4\times6\times\ldots\times2n}{2\times4\times6\times\ldots\times2n}$$

$$I = \pi\ln\alpha - \frac{\pi}{2}\sum_{n=1}^{\infty} \frac{(-k)^n}{n.4^n}.\frac{(2n)!}{n!.n!} = \pi\ln\alpha - \frac{\pi}{2}\sum_{n=1}^{\infty} \binom{2n}{n}.\frac{\left(-\frac{k}{4}\right)^n}{n}$$

We have: $\sum_{k=0}^{\infty} \binom{2k}{k} x^k = \dfrac{1}{\sqrt{1-4x}}$; dividing both sides by x: $\sum_{k=0}^{\infty} \binom{2k}{k} x^{k-1} = \dfrac{1}{x\sqrt{1-4x}}$

Now integrating both sides with respect to x: $\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{k} = -2\ln\left(1+\sqrt{1-4x}\right)+c$

We can set $x = 0$ to solve for the constant of the integration, and we get:

$\sum_{k=0}^{\infty}\binom{2k}{k}\frac{x^k}{k} = 2\ln 2 - 2\ln\left(1+\sqrt{1-4x}\right)$; then we get:

$$\mathrm{I} = \pi\ln\alpha - \frac{\pi}{2}\left[2\ln 2 - 2\ln\left(1+\sqrt{k+1}\right)\right] = \pi\ln\alpha - \pi\ln 2 + \pi\ln\left(1+\sqrt{k+1}\right)$$

$$\mathrm{I} = \pi\ln\alpha - \pi\ln 2 + \pi\ln\left(1+\sqrt{\frac{\beta^2-\alpha^2}{\alpha^2}+1}\right) = \pi\ln\alpha - \pi\ln 2 + \pi\ln\left(1+\sqrt{\frac{\beta^2}{\alpha^2}}\right)$$

$$\mathrm{I} = \pi\ln\alpha - \pi\ln 2 + \pi\ln\left(1+\frac{\beta}{\alpha}\right) = \pi\ln\alpha - \pi\ln 2 + \pi\ln\left(\frac{\alpha+\beta}{\alpha}\right)$$

$$\mathrm{I} = \pi\ln\alpha - \pi\ln 2 + \pi\ln(\alpha+\beta) - \pi\ln\alpha = \pi\ln(\alpha+\beta) - \pi\ln 2 = \pi\ln\left(\frac{\alpha+\beta}{2}\right)$$

Therefore we get: $\displaystyle\int_0^{\frac{\pi}{2}} \ln(\alpha^2\sin^2 x + \beta^2\cos^2 x)\,dx = \pi\ln\left(\frac{\alpha+\beta}{2}\right)$

216. $\mathrm{I} = \int_0^1\left(\left\lfloor\frac{\alpha}{x}\right\rfloor - \alpha\left\lfloor\frac{1}{x}\right\rfloor\right)dx$, where $\alpha \in [0;1]$

Remember that: $x \to +\infty \Longrightarrow \mathrm{H}_x \sim \gamma + \ln x + \frac{1}{2x} \Longrightarrow \mathrm{H}_x - \mathrm{H}_y \sim \ln\left(\frac{x}{y}\right)$; $x, y \to +\infty$

$$\mathrm{I} = \int_0^1\left(\left\lfloor\frac{\alpha}{x}\right\rfloor - \alpha\left\lfloor\frac{1}{x}\right\rfloor\right)dx = \lim_{n\to+\infty}\int_{\frac{\alpha}{n}}^1\left(\left\lfloor\frac{\alpha}{x}\right\rfloor - \alpha\left\lfloor\frac{1}{x}\right\rfloor\right)dx = \lim_{n\to+\infty}\left\{\int_{\frac{\alpha}{n}}^1\left\lfloor\frac{\alpha}{x}\right\rfloor dx - \alpha\int_{\frac{\alpha}{n}}^1\left\lfloor\frac{1}{x}\right\rfloor dx\right\}$$

$$\mathrm{I} = \lim_{n\to+\infty}(\mathrm{I}_1 - \alpha\mathrm{I}_2);\ \text{ where } \mathrm{I}_1 = \int_{\frac{\alpha}{n}}^1\left\lfloor\frac{\alpha}{x}\right\rfloor dx \ \ \&\ \ \mathrm{I}_2 = \int_{\frac{\alpha}{n}}^1\left\lfloor\frac{1}{x}\right\rfloor dx$$

Let's evaluate $\boldsymbol{\mathrm{I}_1 = \int_{\frac{\alpha}{n}}^1\left\lfloor\frac{\alpha}{x}\right\rfloor dx}$

Let $u = \frac{\alpha}{x} \Longrightarrow x = \frac{\alpha}{u} \Longrightarrow dx = -\frac{\alpha}{u^2}du$; $\begin{cases} x = \frac{\alpha}{n} \\ x = 1\end{cases} \Longrightarrow \begin{cases} u = n \\ u = \alpha\end{cases}$; then we get:

$$\mathrm{I}_1 = \int_n^{\alpha}\lfloor u\rfloor\left(-\frac{\alpha}{u^2}du\right) = \alpha\int_{\alpha}^{n}\frac{\lfloor u\rfloor}{u^2}du = \alpha\left\{\int_{\alpha}^{\lceil\alpha\rceil}\frac{\lfloor\alpha\rfloor}{u^2}du + \sum_{k=\lceil\alpha\rceil}^{\lfloor n-1\rfloor}\int_k^{k+1}\frac{k}{u^2}du + \int_{\lfloor n\rfloor}^{n}\frac{\lfloor n\rfloor}{u^2}du\right\}$$

$$\mathrm{I}_1 = \alpha\left\{\lfloor\alpha\rfloor\left[-\frac{1}{u}\right]_{\alpha}^{\lceil\alpha\rceil} + \sum_{k=\lceil\alpha\rceil}^{\lfloor n-1\rfloor}k\left[-\frac{1}{u}\right]_k^{k+1} + \lfloor n\rfloor\left[-\frac{1}{u}\right]_{\lfloor n\rfloor}^{n}\right\}$$

$$\mathrm{I}_1 = \alpha\left\{\lfloor\alpha\rfloor\left(\frac{1}{\alpha} - \frac{1}{\lceil\alpha\rceil}\right) + \sum_{k=\lceil\alpha\rceil}^{\lfloor n-1\rfloor}k\left(\frac{1}{k} - \frac{1}{k+1}\right) + \lfloor n\rfloor\left(\frac{1}{\lfloor n\rfloor} - \frac{1}{n}\right)\right\};\ \text{ then we get:}$$

$$\mathrm{I}_1 = \alpha\left\{\lfloor\alpha\rfloor\left(\frac{1}{\alpha}-\frac{1}{\lceil\alpha\rceil}\right)+\sum_{k=\lceil\alpha\rceil}^{\lfloor n-1\rfloor}\frac{1}{k+1}+\lfloor n\rfloor\left(\frac{1}{\lfloor n\rfloor}-\frac{1}{n}\right)\right\}$$

$$\mathrm{I}_1 = \alpha\left\{\frac{\lfloor\alpha\rfloor}{\alpha}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}+\sum_{k=\lceil\alpha\rceil+1}^{\lfloor n\rfloor}\frac{1}{k}+\frac{\lfloor n\rfloor}{\lfloor n\rfloor}-\frac{\lfloor n\rfloor}{n}\right\} = \alpha\left\{\frac{\lfloor\alpha\rfloor}{\alpha}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}+\sum_{k=1}^{\lfloor n\rfloor}\frac{1}{k}-\sum_{k=1}^{\lceil\alpha\rceil}\frac{1}{k}+1-\frac{\lfloor n\rfloor}{n}\right\}$$

$$\mathrm{I}_1 = \alpha\left\{\frac{\lfloor\alpha\rfloor}{\alpha}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}+\mathrm{H}_{\lfloor n\rfloor}-\mathrm{H}_{\lceil\alpha\rceil}+1-\frac{\lfloor n\rfloor}{n}\right\}$$

Now evaluating $\mathbf{I_2} = \displaystyle\int_{\frac{\alpha}{n}}^{1}\left\lfloor\frac{1}{x}\right\rfloor dx$

Let $u=\dfrac{1}{x}\Longrightarrow x=\dfrac{1}{u}\Longrightarrow dx=-\dfrac{1}{u^2}du$; $\begin{cases}x=\dfrac{\alpha}{n}\\ x=1\end{cases}\Longrightarrow\begin{cases}u=\dfrac{n}{\alpha}\\ u=1\end{cases}$; then we get:

$$\mathrm{I}_2 = \int_{\frac{n}{\alpha}}^{1}\lfloor u\rfloor\left(-\frac{1}{u^2}du\right) = \int_{1}^{\frac{n}{\alpha}}\frac{\lfloor u\rfloor}{u^2}du$$

Same steps as I_1 but put 1 instead of α and $\dfrac{n}{\alpha}$ instead of n; then we can write:

$$\mathrm{I}_2 = \mathrm{H}_{\left\lfloor\frac{n}{\alpha}\right\rfloor}-\frac{\alpha}{n}\left\lfloor\frac{n}{\alpha}\right\rfloor$$

$$\mathrm{I} = \lim_{n\to+\infty}(\mathrm{I}_1-\alpha\mathrm{I}_2) = \mathrm{I} = \lim_{n\to+\infty}\left\{\alpha\left[\frac{\lfloor\alpha\rfloor}{\alpha}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}+\mathrm{H}_{\lfloor n\rfloor}-\mathrm{H}_{\lceil\alpha\rceil}+1-\frac{\lfloor n\rfloor}{n}\right]-\alpha\left[\mathrm{H}_{\left\lfloor\frac{n}{\alpha}\right\rfloor}-\frac{\alpha}{n}\left\lfloor\frac{n}{\alpha}\right\rfloor\right]\right\}$$

$$\mathrm{I} = \lfloor\alpha\rfloor+\alpha\left(1-\mathrm{H}_{\lceil\alpha\rceil}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}\right)+\alpha\lim_{n\to+\infty}\left(\mathrm{H}_{\lfloor n\rfloor}-\frac{\lfloor n\rfloor}{n}-\mathrm{H}_{\left\lfloor\frac{n}{\alpha}\right\rfloor}+\frac{\alpha}{n}\left\lfloor\frac{n}{\alpha}\right\rfloor\right)$$

$$\mathrm{I} = \lfloor\alpha\rfloor+\alpha\left(1-\mathrm{H}_{\lceil\alpha\rceil}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}\right)+\alpha\lim_{n\to+\infty}\left(\mathrm{H}_{\lfloor n\rfloor}-1-\mathrm{H}_{\left\lfloor\frac{n}{\alpha}\right\rfloor}+1\right)$$

$$\mathrm{I} = \lfloor\alpha\rfloor+\alpha\left(1-\mathrm{H}_{\lceil\alpha\rceil}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}\right)+\alpha\lim_{n\to+\infty}\left(\mathrm{H}_{\lfloor n\rfloor}-\mathrm{H}_{\left\lfloor\frac{n}{\alpha}\right\rfloor}\right)$$

$$\mathrm{I} = \lfloor\alpha\rfloor+\alpha\left(1-\mathrm{H}_{\lceil\alpha\rceil}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}\right)+\alpha\lim_{n\to+\infty}\ln\left(\frac{\lfloor n\rfloor}{\left\lfloor\frac{n}{\alpha}\right\rfloor}\right) = \lfloor\alpha\rfloor+\alpha\left(1-\mathrm{H}_{\lceil\alpha\rceil}-\frac{\lfloor\alpha\rfloor}{\lceil\alpha\rceil}\right)+\alpha\ln\alpha$$

$\alpha\in[0;1]$; then $\lfloor\alpha\rfloor=0$ & $\lceil\alpha\rceil=1\Longrightarrow \mathrm{I}=0+\alpha(1-\mathrm{H}_1-0)+\alpha\ln\alpha$

$$\mathrm{I} = \alpha(1-1-0)+\alpha\ln\alpha = \alpha\ln\alpha \Longrightarrow \int_0^1\left(\left\lfloor\frac{\alpha}{x}\right\rfloor-\alpha\left\lfloor\frac{1}{x}\right\rfloor\right)dx = \alpha\ln\alpha$$

217. $\displaystyle\int\frac{1}{x^7-4x^5+6x^3-4x}dx = \int\frac{1}{x^7-4x^5+6x^3-4x}\times\frac{x}{x}dx$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx = \int\frac{x}{x^8-4x^6+6x^4-4x^2}dx$$

Let $y=x^2\Longrightarrow dy=2xdx\Longrightarrow xdx=\dfrac{1}{2}dy$; then we get:

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{2}\int\frac{1}{y^4-4y^3+6y^2-4y}dy$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{2}\int\frac{1}{y^4-4y^3+6y^2-4y+1-1}dy=\frac{1}{2}\int\frac{1}{(y-1)^4-1}dy$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{2}\int\frac{1}{[(y-1)^2]^2-1^2}dy;\ \text{ then we get:}$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{2}\int\frac{1}{((y-1)^2-1)((y-1)^2+1)}dy$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{2}\times\frac{1}{2}\int\frac{(y-1)^2+1-((y-1)^2-1)}{((y-1)^2-1)((y-1)^2+1)}dy$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{4}\int\left[\frac{1}{(y-1)^2-1}-\frac{1}{(y-1)^2+1}\right]dy$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{4}\int\left[\frac{1}{y^2-2y+1-1}-\frac{1}{(y-1)^2+1}\right]dy$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{4}\int\left[\frac{1}{y(y-2)}-\frac{1}{(y-1)^2+1}\right]dy$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{4}\int\frac{1}{y(y-2)}dy-\frac{1}{4}\int\frac{1}{(y-1)^2+1}dy$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{4}\times\frac{1}{2}\int\frac{y-(y-2)}{y(y-2)}dy-\frac{1}{4}\int\frac{1}{(y-1)^2+1};\ \text{ then we get:}$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{8}\int\left(\frac{1}{y-2}-\frac{1}{y}\right)dy-\frac{1}{4}\int\frac{1}{(y-1)^2+1}$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{8}\ln|y-2|-\frac{1}{8}\ln|y|-\frac{1}{4}\tan^{-1}(y-1)+c$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{8}\ln\left|\frac{y-2}{y}\right|-\frac{1}{4}\tan^{-1}(y-1)+c;\ \text{ therefore; we get:}$$

$$\int\frac{1}{x^7-4x^5+6x^3-4x}dx=\frac{1}{8}\ln\left|\frac{x^2-2}{x^2}\right|-\frac{1}{4}\tan^{-1}(x^2-1)+c$$

218. $\mathrm{I}=\int_0^{+\infty}\frac{dx}{(1+x)\left(\varphi^2+\ln^2 x\right)}$ **… (1)**

Let $t=\frac{1}{x}\Longrightarrow x=\frac{1}{t}\Longrightarrow dx=-\frac{1}{t^2}dt$; $\begin{cases}x=0\Longrightarrow t=+\infty\\x=+\infty\Longrightarrow t=0\end{cases}$; then we get:

$$\mathrm{I}=\int_0^{+\infty}\frac{dx}{(1+x)(\varphi^2+\ln^2 x)}=\int_0^{+\infty}\frac{-\frac{1}{t^2}}{\left(1+\frac{1}{t}\right)\left(\varphi^2+\ln^2\frac{1}{t}\right)}dt=\int_0^{+\infty}\frac{\frac{1}{t^2}}{\left(1+\frac{1}{t}\right)[\varphi^2+(-\ln t^2)]}dt$$

$$\mathrm{I}=\int_0^{+\infty}\frac{dt}{t^2\left(1+\frac{1}{t}\right)(\varphi^2+\ln^2 t)}=\int_0^{+\infty}\frac{dt}{t(1+t)(\varphi^2+\ln^2 t)}=\int_0^{+\infty}\frac{dx}{x(1+x)(\varphi^2+\ln^2 x)}\ \dots\mathbf{(2)}$$

Now adding (1) & (2) we get: $I + I = \int_0^{+\infty} \frac{dx}{(1+x)(\varphi^2+\ln^2 x)} + \int_0^{+\infty} \frac{dx}{x(1+x)(\varphi^2+\ln^2 x)}$

$$2I = \int_0^{+\infty} \frac{dx}{(1+x)(\varphi^2+\ln^2 x)} + \int_0^{+\infty} \frac{\frac{1}{x}dx}{(1+x)(\varphi^2+\ln^2 x)} = \int_0^{+\infty} \frac{\left(1+\frac{1}{x}\right)}{(1+x)(\varphi^2+\ln^2 x)}dx$$

$$2I = \int_0^{+\infty} \frac{\left(\frac{1+x}{x}\right)}{(1+x)(\varphi^2+\ln^2 x)}dx = \int_0^{+\infty} \frac{\frac{1}{x}}{\varphi^2+\ln^2 x}dx = \int_0^{+\infty} \frac{1}{x(\varphi^2+\ln^2 x)}dx$$

Let $y = \ln x \Longrightarrow dy = \frac{1}{x}dx$; $\begin{cases} t = 0 \Longrightarrow y = -\infty \\ t = +\infty \Longrightarrow y = +\infty \end{cases}$; then we get:

$$2I = \int_{-\infty}^{+\infty} \frac{dy}{\varphi^2+y^2} = \frac{1}{\varphi}[\tan^{-1} y]_{-\infty}^{+\infty} = \frac{1}{\varphi}(\tan^{-1}+\infty - \tan^{-1}-\infty) = \frac{1}{\varphi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$2I = \frac{1}{\varphi}\left(\frac{\pi}{2}+\frac{\pi}{2}\right) = \frac{\pi}{\varphi} \Longrightarrow I = \int_0^{+\infty} \frac{dx}{(1+x)(\varphi^2+\ln^2 x)} = \frac{\pi}{2\varphi}$$

219. $\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}}dx$

Let $y = \sqrt{-1+\sqrt{x+2}} \Longrightarrow y^2 = -1+\sqrt{x+2} \Longrightarrow y^2+1 = \sqrt{x+2} \Longrightarrow x = (y^2+1)^2 - 2$

$dx = 2(y^2+1).2ydy = 4y(y^2+1)dy$; then we get:

$$\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}}dx = \int \frac{1}{(1+y^2+1)\sqrt{-1+y^2+1}}.4y(y^2+1)dy$$

$$\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}}dx = \int \frac{1}{(y^2+2)\sqrt{y^2}}.4y(y^2+1)dy;\ \text{then we get:}$$

$$\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}}dx = \int \frac{4y(y^2+1)}{y(y^2+2)}dy = 4\int \frac{y^2+1}{y^2+2}dy$$

$$\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}}dx = 4\int \frac{y^2+2-1}{y^2+2}dy = 4\int \left(\frac{y^2+2}{y^2+2} - \frac{1}{y^2+2}\right)dy$$

$$\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}}dx = 4\int \left(1 - \frac{1}{y^2+2}\right)dy = 4\int \left(1 - \frac{1}{y^2+\left(\sqrt{2}\right)^2}\right)dy$$

$$\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}}dx = 4y - \frac{4}{\sqrt{2}}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c = 4y - 2\sqrt{2}\tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c$$

Therefore; finally we get $\int \frac{1}{(1+\sqrt{x+2})\sqrt{-1+\sqrt{x+2}}}dx =$

$$4\sqrt{-1+\sqrt{x+2}}-2\sqrt{2}\tan^{-1}\left(\frac{\sqrt{-1+\sqrt{x+2}}\Rightarrow y^2}{\sqrt{2}}\right)+c$$

220. $\mathrm{I}=\int_0^{+\infty} xe^{-x}\ln(\cosh x)\,dx$, using integration by parts**:**

Let $u=\ln(\cosh x)\Rightarrow u'=\dfrac{\sinh x}{\cosh x}=\tanh x$ & let $v'=xe^{-x}\Rightarrow v=-e^{-x}(1+x)$; so:

$$\mathrm{I}=\int_0^{+\infty} xe^{-x}\ln(\cosh x)\,dx=[-e^{-x}(1+x)\ln(\cosh x)]_0^{+\infty}+\int_0^{+\infty} e^{-x}(1+x)\tanh x\,dx$$

$$\mathrm{I}=\int_0^{+\infty} xe^{-x}\ln(\cosh x)\,dx=\int_0^{+\infty} e^{-x}(1+x)\tanh x\,dx;\ \text{ then we get:}$$

$$\mathrm{I}=\int_0^{+\infty} xe^{-x}\ln(\cosh x)\,dx=\int_0^{+\infty} e^{-x}\tanh x\,dx+\int_0^{+\infty} xe^{-x}\tanh x\,dx=\mathrm{A}+\mathrm{B}$$

Where $\mathrm{A}=\int_0^{+\infty} e^{-x}\tanh x\,dx$ & $\mathrm{B}=\int_0^{+\infty} xe^{-x}\tanh x\,dx$

$$\mathrm{A}=\int_0^{+\infty} e^{-x}\tanh x\,dx=\int_0^{+\infty} e^{-x}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)dx;\ \text{ Let } t=e^{-x}\Rightarrow dt=-e^{-x}dx$$

For $x=0\Rightarrow t=e^{-0}=1$ & for $x=+\infty\Rightarrow t=e^{-\infty}=0$; then we get:

$$\mathrm{A}=\int_0^{+\infty} e^{-x}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)dx=\int_1^0\frac{1-t^2}{1+t^2}(-dt)=\int_0^1\frac{1-t^2}{1+t^2}dt=\int_0^1\frac{2-(1+t^2)}{1+t^2}dt$$

$$\mathrm{A}=\int_0^1\left(\frac{2}{1+t^2}-1\right)dt=[2\tan^{-1}t-t]_0^1=2\tan^{-1}1-1=2\left(\frac{\pi}{4}\right)-1=\frac{\pi}{2}-1$$

$$\mathrm{B}=\int_0^{+\infty} xe^{-x}\tanh x\,dx=\int_0^{+\infty} xe^{-x}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)dx;\ \text{ Let } t=e^{-x}\Rightarrow dt=-e^{-x}dx$$

For $x=0\Rightarrow t=e^{-0}=1$ & for $x=+\infty\Rightarrow t=e^{-\infty}=0$; then we get:

$$\mathrm{B}=\int_0^{+\infty} xe^{-x}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)dx=\int_0^1\frac{(t^2-1)\ln t}{1+t^2}dx=\int_0^1\ln t\,dt-2\int_0^1\frac{\ln t}{1+t^2}dt$$

$$\mathrm{B}=\int_0^{+\infty} xe^{-x}\left(\frac{1-e^{-2x}}{1+e^{-2x}}\right)dx=[t\ln t-t]_0^1-2\int_0^1\left(\frac{1}{1+t^2}\right)\ln t\,dt$$

$$\mathrm{B}=-1-2\int_0^1\sum_{n=0}^{+\infty}(-1)^n t^{2n}.\ln t\,dt=-1-2\sum_{n=0}^{+\infty}(-1)^n\int_0^1 t^{2n}\ln t\,dt$$

Let $t = e^{-u} \Rightarrow u = -\ln t$ & $dt = -e^{-u}du$; $\begin{cases} t = 0 \Rightarrow u = +\infty \\ t = 1 \Rightarrow u = 0 \end{cases}$; then we get:

$$B = -1 - 2\sum_{n=0}^{+\infty}(-1)^n \int_0^1 t^{2n} \ln t\, dt = -1 - 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} (e^{-u})^{2n}(-u)(e^{-u}du)$$

$$B = -1 + 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} e^{-2nu}e^{-u}.udu = -1 + 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} u.e^{-(2n+1)u}du$$

Let $y = (2n+1)u \Rightarrow dy = (2n+1)du \Rightarrow du = \dfrac{1}{2n+1}dy$; then:

$$B = -1 + 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} u.e^{-(2n+1)u}du = -1 + 2\sum_{n=0}^{+\infty}(-1)^n \int_0^{+\infty} \frac{y}{2n+1}.e^{-y}.\frac{1}{2n+1}dy$$

$$B = -1 + 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^{+\infty} y.e^{-y}dy = -1 + 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\int_0^{+\infty} y^{2-1}.e^{-y}dy$$

$$B = -1 + 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2}\Gamma(2) = -1 + 2\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n+1)^2} = -1 + 2\mathrm{G}$$

Therefore; we get: $\mathrm{I} = \displaystyle\int_0^{+\infty} xe^{-x}\ln(\cosh x)\, dx = \mathrm{A} + \mathrm{B} = \frac{\pi}{2} - 1 - 1 + 2\mathrm{G} = \frac{\pi}{2} + 2\mathrm{G} - 2$

221. $\mathrm{I} = \int_0^{+\infty} x^2 e^{-x^2}\cos x\, dx$

$$\mathrm{I} = \int_0^{+\infty} x^2e^{-x^2}\cos x\, dx = \int_0^{+\infty} x^2e^{-x^2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}x^{2n}\, dx = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} x^2e^{-x^2}.x^{2n}dx$$

$\mathrm{I} = \displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} x^{2n+2}e^{-x^2}dx$; let $y = x^2 \Rightarrow x = \sqrt{y}$ & $dx = \dfrac{1}{2\sqrt{y}}dy$; then we get:

$$\mathrm{I} = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} x^{2n+2}e^{-x^2}dx = \sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty}\left(y^{\frac{1}{2}}\right)^{2n+2} e^{-y}.\frac{1}{2\sqrt{y}}dy$$

$$\mathrm{I} = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} y^{n+1}e^{-y}.y^{-\frac{1}{2}}dy = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} y^{n+\frac{1}{2}}.e^{-y}\, dy$$

$\mathrm{I} = \dfrac{1}{2}\displaystyle\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\int_0^{+\infty} y^{\left(n+\frac{3}{2}\right)-1}.e^{-y}\, dy = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}\Gamma\left(n+\frac{3}{2}\right)$; then we get:

$$\mathrm{I} = \frac{1}{2}\sum_{n=0}^{+\infty}\frac{(-1)^n}{(2n)!}.\frac{(2n+2)!}{4^{n+1}(n+1)!}\sqrt{\pi} = \frac{1}{8}\sqrt{\pi}\sum_{n=0}^{+\infty}\left(-\frac{1}{4}\right)^n\left(\frac{(2n+2)(2n+1)}{(n+1)!}\right) = \frac{\sqrt{\pi}}{8\sqrt[4]{e}}$$

222. $\mathrm{I}=\int_0^{+\infty}\frac{x^2\cosh(ex)}{\sinh^2(ex)}dx$, let $t=ex\Longrightarrow dt=edx$, then we get**:**

$$\mathrm{I}=\int_0^{+\infty}\frac{x^2\cosh(ex)}{\sinh^2(ex)}dx=\int_0^{+\infty}\frac{\left(\frac{t}{e}\right)^2\cosh t}{\sinh^2 t}.\frac{1}{e}dt=\frac{1}{e^3}\int_0^{+\infty}\frac{t^2\cosh t}{\sinh^2 t}dt;\ \text{using I.B.P}$$

Let $u=t^2\Longrightarrow u'=2t$ & let $v'=\frac{\cosh t}{\sinh^2 t}\Longrightarrow v=-\frac{1}{\sinh t}$; then we can write:

$$\mathrm{I}=\frac{1}{e^3}\left[-\frac{t^2}{\sinh t}\right]_0^{+\infty}+\frac{1}{e^3}\int_0^{+\infty}\frac{2t}{\sinh t}dt=\frac{2}{e^3}\int_0^{+\infty}\frac{t}{\frac{e^t-e^{-t}}{2}}dt=\frac{4}{e^3}\int_0^{+\infty}\frac{t}{e^t-e^{-t}}dt$$

$$\mathrm{I}=\frac{4}{e^3}\int_0^{+\infty}\frac{t}{e^t-e^{-t}}\times\frac{e^{-t}}{e^{-t}}dt=\frac{4}{e^3}\int_0^{+\infty}\frac{te^{-t}}{1-e^{-2t}}dt=\frac{4}{e^3}\int_0^{+\infty}te^{-t}.\frac{1}{1-e^{-2t}}dt$$

$$\mathrm{I}=\frac{4}{e^3}\int_0^{+\infty}t\sum_{n=0}^{+\infty}e^{-(2n-1)t}\,dt=\frac{4}{e^3}\sum_{n=0}^{+\infty}\int_0^{+\infty}t.e^{-(2n-1)t}dt$$

Let $y=(2n-1)t\Longrightarrow t=\frac{1}{2n-1}y\Longrightarrow dt=\frac{1}{2n-1}dy$; then we get:

$$\mathrm{I}=\frac{4}{e^3}\sum_{n=0}^{+\infty}\int_0^{+\infty}t.e^{-(2n-1)t}dt=\frac{4}{e^3}\sum_{n=0}^{+\infty}\int_0^{+\infty}\frac{1}{2n-1}y.e^{-y}.\frac{1}{2n-1}dy$$

$$\mathrm{I}=\frac{4}{e^3}\sum_{n=0}^{+\infty}\frac{1}{(2n-1)^2}\int_0^{+\infty}y.e^{-y}dy=\frac{4}{e^3}\sum_{n=0}^{+\infty}\frac{1}{(2n-1)^2}\Gamma(2)=\frac{4}{e^3}\sum_{n=0}^{+\infty}\frac{1}{(2n+1)^2}$$

$$\mathrm{I}=\frac{4}{e^3}\sum_{n=0}^{+\infty}\frac{1}{4\left(n+\frac{1}{2}\right)^2}=\frac{1}{e^3}\sum_{n=0}^{+\infty}\frac{1}{\left(n+\frac{1}{2}\right)^2}=\frac{1}{e^3}\Psi^{(1)}\left(\frac{1}{2}\right)=\frac{\pi^2}{2e^3}$$

223. $\mathrm{I}=\int_0^{+\infty}\frac{\sqrt{x}\ln x}{x^2+1}dx=\int_0^1\frac{\sqrt{x}\ln x}{x^2+1}dx+\int_1^{+\infty}\frac{\sqrt{x}\ln x}{x^2+1}dx$

For the second integral: let $y=\frac{1}{x}\Longrightarrow x=\frac{1}{y}\Longrightarrow dx=-\frac{1}{y^2}dy$; $\begin{cases}x=1\Longrightarrow y=1\\x=+\infty\Longrightarrow y=0\end{cases}$; so:

$$\mathrm{I}=\int_0^1\frac{\sqrt{x}\ln x}{x^2+1}dx+\int_1^{+\infty}\frac{\sqrt{x}\ln x}{x^2+1}dx=\int_0^1\frac{\sqrt{x}\ln x}{x^2+1}dx+\int_1^0\frac{\sqrt{\frac{1}{y}}\ln\left(\frac{1}{y}\right)}{\left(\frac{1}{y}\right)^2+1}\left(-\frac{1}{y^2}dy\right)$$

$$\mathrm{I}=\int_0^1\frac{\sqrt{x}\ln x}{x^2+1}dx+\int_0^1\frac{\sqrt{\frac{1}{y}}\ln\left(\frac{1}{y}\right)}{\left(\frac{1}{y}\right)^2+1}.\frac{1}{y^2}dy=\int_0^1\frac{\sqrt{x}\ln x}{x^2+1}dx-\int_0^1\frac{\sqrt{\frac{1}{y}}\ln y}{1+y^2}dy$$

$$\mathrm{I}=\int_0^1 \frac{\sqrt{x}\ln x}{x^2+1}dx-\int_0^1 \frac{\ln y}{\sqrt{y}(1+y^2)}dy=\int_0^1 x^{\frac{1}{2}}\ln x\,.\frac{1}{x^2+1}dx-\int_0^1 \frac{\ln y}{y^{\frac{1}{2}}}.\frac{1}{(1+y^2)}dy$$

$$\mathrm{I}=\int_0^1 x^{\frac{1}{2}}\ln x\,.\sum_{n=0}^{+\infty}(-1)^n x^{2n}\,dx-\int_0^1 \frac{\ln y}{y^{\frac{1}{2}}}.\sum_{n=0}^{+\infty}(-1)^n y^{2n}\,dy$$

$$\mathrm{I}=\sum_{n=0}^{+\infty}(-1)^n\int_0^1 x^{\frac{1}{2}}\ln x\,.\,x^{2n}\,dx-\sum_{n=0}^{+\infty}(-1)^n\int_0^1 \frac{\ln y}{y^{\frac{1}{2}}}.\,y^{2n}dy;\ \text{ then we get:}$$

$$\mathrm{I}=\sum_{n=0}^{+\infty}(-1)^n\int_0^1 x^{2n+\frac{1}{2}}\ln x\,dx-\sum_{n=0}^{+\infty}(-1)^n\int_0^1 y^{2n-\frac{1}{2}}\ln y\,dy$$

By taking the change of variables $t=-\ln x$ & $u=-\ln y$ we will obtain the result:

$$\mathrm{I}=\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{\left(2n+\frac{3}{2}\right)^2}-\sum_{n=0}^{+\infty}\frac{(-1)^{n+1}}{\left(2n+\frac{1}{2}\right)^2}$$

$$\mathrm{I}=\sum_{n=0}^{+\infty}\left[\frac{1}{\left(4n+\frac{7}{2}\right)^2}-\frac{1}{\left(4n+\frac{3}{2}\right)^2}\right]-\sum_{n=0}^{+\infty}\left[\frac{1}{\left(4n+\frac{5}{2}\right)^2}-\frac{1}{\left(4n+\frac{1}{2}\right)^2}\right]$$

$$\mathrm{I}=\frac{1}{16}\left\{\sum_{n=0}^{+\infty}\left[\frac{1}{\left(n+\frac{7}{8}\right)^2}-\frac{1}{\left(n+\frac{3}{8}\right)^2}\right]-\sum_{n=0}^{+\infty}\left[\frac{1}{\left(n+\frac{5}{8}\right)^2}-\frac{1}{\left(n+\frac{1}{8}\right)^2}\right]\right\};\ \text{ then we can write:}$$

$$\mathrm{I}=\frac{1}{16}\left[\Psi^{(1)}\left(\frac{7}{8}\right)-\Psi^{(1)}\left(\frac{3}{8}\right)-\Psi^{(1)}\left(\frac{5}{8}\right)+\Psi^{(1)}\left(\frac{1}{8}\right)\right];\ \text{ but we know that:}$$

$$\Psi^{(1)}(1-\mathrm{z})\Psi^{(1)}(\mathrm{z})=\frac{\pi^2}{\sin^2(\pi z)}$$

$$\Rightarrow\begin{cases}\Psi^{(1)}\left(\frac{7}{8}\right)+\Psi^{(1)}\left(\frac{1}{8}\right)=\Psi^{(1)}\left(1-\frac{1}{8}\right)\Psi^{(1)}\left(\frac{1}{8}\right)=\dfrac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)}\\[2ex] \Psi^{(1)}\left(\frac{5}{8}\right)+\Psi^{(1)}\left(\frac{3}{8}\right)=\Psi^{(1)}\left(1-\frac{3}{8}\right)\Psi^{(1)}\left(\frac{3}{8}\right)=\dfrac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)}\end{cases}$$

$$\mathrm{I}=\frac{1}{16}\left[\frac{\pi^2}{\sin^2\left(\frac{\pi}{8}\right)}-\frac{\pi^2}{\sin^2\left(\frac{3\pi}{8}\right)}\right]=\frac{1}{16}\left[\frac{\pi^2}{\frac{1-\cos\left(\frac{\pi}{4}\right)}{2}}-\frac{2\pi^2}{\frac{1-\cos\left(\frac{3\pi}{4}\right)}{2}}\right];\ \text{ then we get:}$$

$$\mathrm{I}=\frac{\pi^2}{8}\left[\frac{1}{1-\cos\left(\frac{\pi}{4}\right)}-\frac{1}{1-\cos\left(\pi-\frac{\pi}{4}\right)}\right]=\frac{\pi^2}{8}\left[\frac{1}{1-\cos\left(\frac{\pi}{4}\right)}-\frac{1}{1+\cos\left(\frac{\pi}{4}\right)}\right]=\frac{\sqrt{2}\pi^2}{4}$$

224. $\mathrm{I}=\int_0^{+\infty}\frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}}dx$; let $t=e^{-(p+q)x}\Longrightarrow dt=-(p+q)e^{-(p+q)x}dx$

$\Longrightarrow dt=-(p+q)tdx\Longrightarrow dx=-\frac{1}{p+q}\left(\frac{dt}{t}\right)$; $\begin{cases} x=0\Longrightarrow t=1 \\ x=+\infty\Longrightarrow t=0\end{cases}$; then we get:

$$\mathrm{I}=\int_0^{+\infty}\frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}}dx=\frac{1}{p+q}\int_0^1\frac{t^{-\frac{p}{p+q}+}+t^{-\frac{q}{p+q}}}{1+t}dt$$

We can easily prove that $\int_0^1\frac{t^n}{1+t}dt=\frac{1}{2}\left[\Psi^{(0)}\left(\frac{n+2}{2}\right)-\Psi^{(0)}\left(\frac{n+1}{2}\right)\right]$; then we get:

$$\mathrm{I}=\frac{1}{p+q}\left(\int_0^1\frac{t^{-\frac{p}{p+q}+}}{1+t}dt+\int_0^1\frac{t^{-\frac{q}{p+q}}}{1+t}dt\right)$$

$$\mathrm{I}=\frac{1}{2(p+q)}\left[\Psi^{(0)}\left(\frac{p+2q}{2p+2q}\right)-\Psi^{(0)}\left(\frac{q}{2p+2q}\right)+\Psi^{(0)}\left(\frac{q+2p}{2p+2q}\right)-\Psi^{(0)}\left(\frac{p}{2p+2q}\right)\right]$$

$$\mathrm{I}=\frac{1}{2(p+q)}\left[\Psi^{(0)}\left(\frac{p+2q}{2p+2q}\right)-\Psi^{(0)}\left(\frac{p}{2p+2q}\right)+\Psi^{(0)}\left(\frac{q+2p}{2p+2q}\right)-\Psi^{(0)}\left(\frac{q}{2p+2q}\right)\right]$$

$$=\frac{1}{2(p+q)}\left[\Psi^{(0)}\left(1-\frac{p}{2p+2q}\right)-\Psi^{(0)}\left(\frac{p}{2p+2q}\right)+\Psi^{(0)}\left(1-\frac{q}{2p+2q}\right)-\Psi^{(0)}\left(\frac{q}{2p+2q}\right)\right]$$

Reflection Formula: $\Psi^{(0)}(1-\mathrm{z})-\Psi^{(0)}(\mathrm{z})=\pi\cot(\pi\mathrm{z})$; then we can write:

$$\mathrm{I}=\frac{\pi}{2(p+q)}\left[\cot\left(\frac{\pi p}{2p+2q}\right)+\cot\left(\frac{\pi q}{2p+2q}\right)\right]$$

But we know that: $\cot a+\cot b=\frac{2\sin(a+b)}{\cos(a-b)-\cos(a+b)}$; then we get:

$$\mathrm{I}=\frac{\pi}{2(p+q)}\times\frac{2\sin\left(\frac{\pi p}{2p+2q}+\frac{\pi q}{2p+2q}\right)}{\cos\left(\frac{\pi p}{2p+2q}-\frac{\pi q}{2p+2q}\right)-\cos\left(\frac{\pi p}{2p+2q}+\frac{\pi q}{2p+2q}\right)}$$

$$\mathrm{I}=\frac{\pi}{2(p+q)}\times\frac{2\sin\left(\frac{\pi(p+q)}{2(p+q)}\right)}{\cos\left(\frac{\pi p-\pi q}{2p+2q}\right)-\cos\left(\frac{\pi(p+q)}{2(p+q)}\right)}=\frac{\pi}{2(p+q)}\times\frac{2\sin\left(\frac{\pi}{2}\right)}{\cos\left(\frac{\pi p-\pi q}{2p+2q}\right)-\cos\left(\frac{\pi}{2}\right)}$$

$$\mathrm{I}=\frac{\pi}{p+q}\left[\frac{1}{\cos\left(\frac{\pi}{2}-\frac{\pi p}{p+q}\right)}\right]=\frac{\pi}{p+q}\left[\frac{1}{\sin\left(\frac{\pi p}{p+q}\right)}\right]=\frac{\pi}{p+q}\csc\left(\frac{\pi p}{p+q}\right)$$

Therefore; we get: $\int_0^{+\infty}\frac{e^{-px}+e^{-qx}}{1+e^{-(p+q)x}}dx=\frac{\pi}{p+q}\csc\left(\frac{\pi p}{p+q}\right)$

225. $\mathrm{I} = \int_0^{\pi} \tan^{-1}(e^{\cos x})\,dx$

Let $y = \cos x \Rightarrow x = \cos^{-1} y \Rightarrow dx = -\dfrac{1}{\sqrt{1-y^2}}dy$; $\begin{cases} x = 0 \Rightarrow y = 1 \\ x = \pi \Rightarrow y = -1 \end{cases}$; then we get:

$$\mathrm{I} = \int_1^{-1} \tan^{-1}(e^y).\left(-\frac{1}{\sqrt{1-y^2}}dy\right) = \int_{-1}^{1} \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy = \int_{-1}^{0} \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy + \int_0^1 \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy$$

For $\displaystyle\int_{-1}^{0} \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy$; let $z = -y \Rightarrow dz = -dy$; $\begin{cases} y = 0 \Rightarrow z = 0 \\ y = -1 \Rightarrow z = 1 \end{cases}$; then we get:

$$\int_{-1}^{0} \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy = \int_1^0 \frac{\tan^{-1}(e^{-z})}{\sqrt{1-(-z)^2}}(-dz) = \int_0^1 \frac{\tan^{-1}(e^{-z})}{\sqrt{1-z^2}}dz = \int_0^1 \frac{\tan^{-1}(e^{-y})}{\sqrt{1-y^2}}dy$$

Then we have $\displaystyle\mathrm{I} = \int_{-1}^{0} \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy + \int_0^1 \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy = \int_0^1 \frac{\tan^{-1}(e^{-y})}{\sqrt{1-y^2}}dy + \int_0^1 \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy$

$$\mathrm{I} = \int_0^1 \frac{\tan^{-1}\left(\frac{1}{e^y}\right)}{\sqrt{1-y^2}}dy + \int_0^1 \frac{\tan^{-1}(e^y)}{\sqrt{1-y^2}}dy = \int_0^1 \frac{\tan^{-1}\left(\frac{1}{e^y}\right) + \tan^{-1}(e^y)}{\sqrt{1-y^2}}dy$$

With $e^y, \dfrac{1}{e^y} > 0$ & $\tan^{-1}\left(\dfrac{1}{e^y}\right) + \tan^{-1}(e^y) = \dfrac{\pi}{2}$; then we can write:

$$\mathrm{I} = \int_0^1 \frac{\frac{\pi}{2}}{\sqrt{1-y^2}}dy = \frac{\pi}{2}\int_0^1 \frac{dy}{\sqrt{1-y^2}} = \frac{\pi}{2}[\sin^{-1} y]_0^1 = \frac{\pi}{2}\left(\frac{\pi}{2} - 0\right) = \frac{\pi^2}{4}$$

226. $\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{x}{(\sin x + \cos x)\cos x}dx$

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{x}{(\sin x + \cos x)\cos x}dx = \int_0^{\frac{\pi}{4}} \frac{x}{\left(\frac{\sin x + \cos x}{\cos x}\right)\cos^2 x}dx = \int_0^{\frac{\pi}{4}} \frac{x}{(\tan x + 1)\cos^2 x}dx$$

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{x}{\tan x + 1}.\frac{dx}{\cos^2 x} = \int_0^{\frac{\pi}{4}} \frac{x}{\tan x + 1}.\sec^2 x\,dx$$

Let $t = \tan x \Rightarrow x = \tan^{-1} t$ & $dt = \sec^2 x\,dx$; $\begin{cases} x = 0 \Rightarrow t = 0 \\ x = \frac{\pi}{4} \Rightarrow t = 1 \end{cases}$ then we get:

$$\mathrm{I} = \int_0^{\frac{\pi}{4}} \frac{x}{\tan x + 1}.\sec^2 x\,dx = \int_0^1 \frac{\tan^{-1} t}{t+1}dt$$

Let $t = \dfrac{1-y}{1+y} \Rightarrow dt = -\dfrac{2}{(1+y)^2}dy$; for $t = 0$; $y = 1$ & for $t = 1$; $y = 0$; then:

$$I=\int_0^1\frac{\tan^{-1}t}{t+1}dt=\int_1^0\frac{\tan^{-1}\left(\frac{1-y}{1+y}\right)}{\frac{1-y}{1+y}+1}\left[-\frac{2}{(1+y)^2}dy\right]=\int_0^1\frac{\tan^{-1}\left(\frac{1-y}{1+y}\right)}{\frac{2}{1+y}}.\frac{2}{(1+y)^2}dy$$

$$I=\int_0^1\frac{\tan^{-1}\left(\frac{1-y}{1+y}\right)}{1+y}dy;\ \text{ but we have: }\tan^{-1}u-\tan^{-1}v=\tan^{-1}\left(\frac{u-v}{1+uv}\right);\ \text{ so we have:}$$

$$I=\int_0^1\frac{\tan^{-1}1-\tan^{-1}y}{1+y}dy=\int_0^1\frac{\frac{\pi}{4}-\tan^{-1}y}{1+y}dy=\frac{\pi}{4}\int_0^1\frac{dy}{1+y}-\int_0^1\frac{\tan^{-1}y}{1+y}dy$$

$$I=\frac{\pi}{4}\int_0^1\frac{dy}{1+y}-\int_0^1\frac{\tan^{-1}y}{1+y}dy;\ \text{ with }I=\int_0^1\frac{\tan^{-1}t}{t+1}dt=\int_0^1\frac{\tan^{-1}y}{1+y}dy;\ \text{ then we get:}$$

$$I=\frac{\pi}{4}\int_0^1\frac{dy}{1+y}-I\tan x\Longrightarrow 2I=\frac{\pi}{4}\int_0^1\frac{dy}{1+y}\Longrightarrow I=\frac{\pi}{8}\int_0^1\frac{dy}{1+y}$$

$$I=\frac{\pi}{8}\int_0^1\frac{dy}{1+y}=I=\frac{\pi}{8}[\ln|1+y|]_0^1=I=\frac{\pi}{8}\ln 2\Longrightarrow\int_0^{\frac{\pi}{4}}\frac{x}{(\sin x+\cos x)\cos x}dx=\frac{\pi}{8}\ln 2$$

227. $I=\int_0^{+\infty}\frac{dx}{\left(1+x^{\delta_s}\right)^{\delta_s}}$ **;** where δ_s is the silver ratio

Remark: Two quantities a and b are in silver ratio if**:**

$$\frac{2a+b}{a}=\frac{a}{b}=\delta_s\Longrightarrow 2+\frac{b}{a}=\frac{a}{b}=\delta_s\Longrightarrow 2+\frac{1}{\delta_s}=\delta_s\Longrightarrow\delta_s^2-2\delta_s-1=0\Longrightarrow\delta_s=1+\sqrt{2}$$

$$I=\int_0^{+\infty}\frac{dx}{(1+x^{\delta_s})^{\delta_s}}=\int_0^{+\infty}\frac{dx}{\left(1+x^{1+\sqrt{2}}\right)^{1+\sqrt{2}}}$$

$$\text{Let }x=u^{\sqrt{2}-1}\Longrightarrow dx=\left(\sqrt{2}-1\right)u^{\sqrt{2}-2}du\ \ \&\ \ x^{1+\sqrt{2}}=\left(u^{\sqrt{2}-1}\right)^{1+\sqrt{2}}=u^{(\sqrt{2}-1)(1+\sqrt{2})}=u$$

$$I=\int_0^{+\infty}\frac{dx}{\left(1+x^{1+\sqrt{2}}\right)^{1+\sqrt{2}}}=\int_0^{+\infty}\frac{\left(\sqrt{2}-1\right)u^{\sqrt{2}-2}du}{(1+u)^{1+\sqrt{2}}}=\left(\sqrt{2}-1\right)\int_0^{+\infty}\frac{u^{\sqrt{2}-2}}{(1+u)^{1+\sqrt{2}}}du$$

$$I=\left(\sqrt{2}-1\right)\int_0^{+\infty}\frac{u^{(\sqrt{2}-1)-1}}{(1+u)^{(\sqrt{2}-1)+2}}du;\ \text{ but }B(x,y)=\int_0^{+\infty}\frac{t^{x-1}}{(1+t)^{x+y}};\ \text{ then we get:}$$

$$I=\left(\sqrt{2}-1\right)B\left(\sqrt{2}-1,2\right)=\left(\sqrt{2}-1\right).\frac{\Gamma\left(\sqrt{2}-1\right)\Gamma(2)}{\Gamma\left(\sqrt{2}-1+2\right)}=\frac{\left(\sqrt{2}-1\right)\Gamma\left(\sqrt{2}-1\right)}{\Gamma\left(\sqrt{2}+1\right)}$$

$$I=\frac{\Gamma\left(\sqrt{2}\right)}{\sqrt{2}\Gamma\left(\sqrt{2}\right)}=\frac{1}{\sqrt{2}};\ \text{ therefore; }\int_0^{+\infty}\frac{dx}{(1+x^{\delta_s})^{\delta_s}}=\frac{1}{\sqrt{2}}$$

228. $\mathrm{I}=\int_{-\infty}^{+\infty}\frac{x}{(a+e^x)(1-e^{-x})}dx=\int_{-\infty}^{+\infty}\frac{x}{(a+e^x)(1-e^{-x})}\times\frac{e^x}{e^x}dx=\int_{-\infty}^{+\infty}\frac{x}{(a+e^x)(e^x-1)}e^x dx$

Let $u=e^x\ \ (x=\ln u)\Rightarrow du=e^x dx;\ \begin{cases}x=-\infty\Rightarrow u=0\\ x=+\infty\Rightarrow u=+\infty\end{cases};$ then we get:

$$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{x}{(a+e^x)(e^x-1)}e^x dx=\int_0^{+\infty}\frac{\ln u}{(a+u)(u-1)}du=\int_0^{+\infty}\frac{\ln u}{(a+u)(u+e^{i\pi})}du$$

Let $u=\frac{ae^{i\pi}}{y}\Rightarrow du=-\,ae^{i\pi}.\frac{dy}{y^2};\ \begin{cases}u=0\Rightarrow y=+\infty\\ u=+\infty\Rightarrow y=0\end{cases};$ then we get:

$$\mathrm{I}=\int_0^{+\infty}\frac{\ln u}{(a+u)(u+e^{i\pi})}du=\int_{+\infty}^{0}\frac{\ln\left(\frac{ae^{i\pi}}{y}\right)}{\left(a+\frac{ae^{i\pi}}{y}\right)\left(\frac{ae^{i\pi}}{y}+e^{i\pi}\right)}\left(-\,ae^{i\pi}.\frac{dy}{y^2}\right)$$

$$\mathrm{I}=\int_0^{+\infty}\frac{[\ln(ae^{\mathrm{i}\pi})-\ln y]\,ae^{i\pi}}{a\left(1+\frac{e^{i\pi}}{y}\right).e^{i\pi}\left(\frac{a}{y}+1\right)}.\frac{dy}{y^2}=\int_0^{+\infty}\frac{\ln(ae^{\mathrm{i}\pi})-\ln y}{(y+e^{i\pi})(a+y)}dy;\ \text{then we can write:}$$

$$\mathrm{I}=\int_0^{+\infty}\frac{\ln(ae^{\mathrm{i}\pi})}{(y+e^{i\pi})(a+y)}dy-\int_0^{+\infty}\frac{\ln y}{(y+e^{i\pi})(a+y)}=\int_0^{+\infty}\frac{\ln(ae^{\mathrm{i}\pi})}{(y+e^{i\pi})(a+y)}dy-\mathrm{I}$$

$$\text{So;}\ \ 2\mathrm{I}=\int_0^{+\infty}\frac{\ln(ae^{\mathrm{i}\pi})}{(y+e^{i\pi})(a+y)}dy=\ln(ae^{\mathrm{i}\pi})\int_0^{+\infty}\frac{1}{(y+e^{i\pi})(a+y)}dy$$

$$2\mathrm{I}=\frac{\ln(ae^{\mathrm{i}\pi})}{a-e^{i\pi}}\int_0^{+\infty}\frac{(a+y)-(y+e^{i\pi})}{(y+e^{i\pi})(a+y)}dy=\frac{\ln(ae^{\mathrm{i}\pi})}{a-e^{i\pi}}\int_0^{+\infty}\left(\frac{1}{y+e^{i\pi}}-\frac{1}{a+y}\right)dy;\ \text{then:}$$

$$2\mathrm{I}=\frac{\ln(ae^{\mathrm{i}\pi})}{a-e^{i\pi}}\left[\ln\left|\frac{y+e^{i\pi}}{y+a}\right|_0^{+\infty}\right]=\frac{\ln(ae^{\mathrm{i}\pi})}{a-e^{i\pi}}\left(\lim_{y\to+\infty}\left|\frac{1+\frac{e^{i\pi}}{y}}{1+\frac{a}{y}}\right|-\ln\left|\frac{e^{i\pi}}{a}\right|\right);\ \text{then we get:}$$

$$2\mathrm{I}=-\frac{\ln(ae^{\mathrm{i}\pi})}{a-e^{i\pi}}.\ln\left|\frac{e^{i\pi}}{a}\right|=-\frac{\ln(ae^{\mathrm{i}\pi})}{a-e^{i\pi}}\{\ln e^{i\pi}-\ln a\}=-\frac{\ln(ae^{\mathrm{i}\pi})}{a-e^{i\pi}}\{i\pi-\ln a\}$$

$$2\mathrm{I}=-\frac{\ln(-a)}{a+1}\{i\pi-\ln a\}=\frac{1}{a+1}\{\ln a+i\pi\}\{\ln a-i\pi\}=\frac{(\ln a)^2+\pi^2}{a+1};\ \text{therefore we}$$

$$\mathrm{I}=\int_{-\infty}^{+\infty}\frac{x}{(a+e^x)(1-e^{-x})}dx=\frac{(\ln a)^2+\pi^2}{a+1}$$

229. $\mathrm{I}=\int_{-\infty}^{+\infty}xe^{x-\mu e^{ax}}dx$

Let $u=\mu e^{ax}\Rightarrow\left(\frac{u}{\mu}\right)^{\frac{1}{a}}=e^x\Rightarrow du=\mu ae^{ax}dx\Rightarrow dx=\frac{du}{a.u};\ \begin{cases}x=-\infty\Rightarrow u=0\\ x=+\infty\Rightarrow u=+\infty\end{cases}\Rightarrow$

$$\mathrm{I} = \int_{-\infty}^{+\infty} xe^{x-\mu e^{ax}}\,dx = \int_{0}^{+\infty} \frac{1}{a}\ln\left(\frac{u}{\mu}\right).\left(\frac{u}{\mu}\right)^{\frac{1}{a}}.e^{-u}.\frac{du}{au} = \frac{1}{a^2}.\frac{1}{\mu^{\frac{1}{a}}}\int_{0}^{+\infty} e^{-u}.u^{\frac{1}{a}-1}\ln\left(\frac{u}{\mu}\right)du\,;\ \text{then:}$$

$$\mathrm{I} = \frac{1}{a^2}.\frac{1}{\mu^{\frac{1}{a}}}\int_{0}^{+\infty} e^{-u}.u^{\frac{1}{a}-1}(\ln u - \ln \mu)du\,;\ \text{then we can write:}$$

$$\mathrm{I} = \frac{1}{a^2}.\frac{1}{\mu^{\frac{1}{a}}}\int_{0}^{+\infty} e^{-u}.u^{\frac{1}{a}-1}\ln u\,du - \frac{1}{a^2}.\frac{1}{\mu^{\frac{1}{a}}}\int_{0}^{+\infty} e^{-u}.u^{\frac{1}{a}-1}\ln \mu\,du\,;\ \text{but}\int_{0}^{+\infty} e^{-u}.u^{\frac{1}{a}-1}du = \Gamma\left(\frac{1}{a}\right)$$

$$\text{Then }\ \mathrm{I} = \frac{1}{a^2}.\frac{1}{\mu^{\frac{1}{a}}}\int_{0}^{+\infty} e^{-u}.u^{\frac{1}{a}-1}\ln u\,du - \frac{\ln\mu\,.\Gamma\left(\frac{1}{a}\right)}{a^2\mu^{\frac{1}{a}}}$$

$$\int_{0}^{+\infty} e^{-kx}.x^{n-1}dx = \frac{\Gamma(n)}{k^n} \Longrightarrow \int_{0}^{+\infty} e^{-kx}.x^{n-1}.\ln x\,dx = \frac{k^n\Gamma'(n) - \Gamma(n).k^n.\ln k}{k^{2n}}$$

$$\text{For }\ k = 1;\ \int_{0}^{+\infty} e^{-u}.x^{n-1}\ln u\,du = \Gamma'(n) \Longrightarrow \int_{0}^{+\infty} e^{-u}.u^{\frac{1}{a}-1}\ln u\,du = \Gamma'\left(\frac{1}{a}\right);\ \text{then we get:}$$

$$\mathrm{I} = \frac{\Gamma'\left(\frac{1}{a}\right)}{a^2\mu^{\frac{1}{a}}} - \frac{\ln\mu\,.\Gamma\left(\frac{1}{a}\right)}{a^2\mu^{\frac{1}{a}}} = \frac{\Gamma\left(\frac{1}{a}\right)\Psi\left(\frac{1}{a}\right)}{a^2\mu^{\frac{1}{a}}} - \frac{\ln\mu\,.\Gamma\left(\frac{1}{a}\right)}{a^2\mu^{\frac{1}{a}}} = \frac{\Gamma\left(\frac{1}{a}\right)}{a^2\mu^{\frac{1}{a}}}\left[\Psi\left(\frac{1}{a}\right) - \ln\mu\right];\ \text{therefore:}$$

$$\mathrm{I} = \int_{-\infty}^{+\infty} xe^{x-\mu e^{ax}}\,dx = \frac{\Gamma\left(\frac{1}{a}\right)}{a^2\mu^{\frac{1}{a}}}\left[\Psi\left(\frac{1}{a}\right) - \ln\mu\right]$$

230. $\mathrm{I} = \int_0^{+\infty} \frac{1}{2\sqrt{x}}\sin\left(\pi^2 x + \frac{1}{x}\right)dx$; let $x = t^2 \Longrightarrow dx = 2tdt$, then we get**:**

$$\mathrm{I} = \int_{0}^{+\infty} \frac{1}{2\sqrt{x}}\sin\left(\pi^2 x + \frac{1}{x}\right)dx = \int_{0}^{+\infty} \frac{1}{2t}\sin\left(\pi^2 t^2 + \frac{1}{t^2}\right).2tdt = \int_{0}^{+\infty} \sin\left(\pi^2 t^2 + \frac{1}{t^2}\right)dt\ \ (1)$$

$$\text{Let }\ t = \frac{1}{\pi y} \Longrightarrow dt = -\frac{1}{\pi y^2}dy;\ \begin{cases} t = 0 \Longrightarrow y = +\infty \\ t = +\infty \Longrightarrow y = 0 \end{cases};\ \text{then we get:}$$

$$\mathrm{I} = \int_{0}^{+\infty} \sin\left(\pi^2 t^2 + \frac{1}{t^2}\right)dt = \int_{+\infty}^{0} \sin\left(\pi^2\left(\frac{1}{\pi y}\right)^2 + \frac{1}{\left(\frac{1}{\pi y}\right)^2}\right)\left(-\frac{1}{\pi y^2}d\right)$$

$$\mathrm{I} = \int_{0}^{+\infty} \sin\left(\frac{1}{y^2} + \pi^2 y^2\right)\frac{1}{\pi y^2}dy = \frac{1}{\pi}\int_{0}^{+\infty} \sin\left(\frac{1}{y^2} + \pi^2 y^2\right).\frac{1}{y^2}dy \ldots (2)$$

$$\text{Adding (1) \& (2) we get: } \mathrm{I} + \mathrm{I} = \int_{0}^{+\infty} \sin\left(\pi^2 t^2 + \frac{1}{t^2}\right)dt + \frac{1}{\pi}\int_{0}^{+\infty} \sin\left(\frac{1}{y^2} + \pi^2 y^2\right).\frac{1}{y^2}dy$$

$$\mathrm{I}+\mathrm{I}=\int_0^{+\infty}\sin\left(\pi^2t^2+\frac{1}{t^2}\right)dt+\frac{1}{\pi}\int_0^{+\infty}\sin\left(\frac{1}{t^2}+\pi^2t^2\right).\frac{1}{t^2}dy;\ \text{ then we can write:}$$

$$2\mathrm{I}=\frac{1}{\pi}\int_0^{+\infty}\sin\left(\pi^2t^2+\frac{1}{t^2}\right)\left(\pi+\frac{1}{t^2}\right)dt=\frac{1}{\pi}\int_0^{+\infty}\sin\left(\pi^2t^2+\frac{1}{t^2}-2\pi+2\pi\right)\left(\pi+\frac{1}{t^2}\right)dt$$

$$2\mathrm{I}=\frac{1}{\pi}\int_0^{+\infty}\sin\left[\left(\pi t-\frac{1}{t}\right)^2+2\pi\right]\left(\pi+\frac{1}{t^2}\right)dt$$

Let $u=\ \pi t-\frac{1}{t}\Longrightarrow du=\left(\pi+\frac{1}{t^2}\right)dt;\ \begin{cases}t=0\Longrightarrow u=-\infty\\ t=+\infty\Longrightarrow u=+\infty\end{cases};$ then we get:

$$2\mathrm{I}=\frac{1}{\pi}\int_{-\infty}^{+\infty}\sin(u^2+2\pi)\,du=\frac{1}{\pi}\int_{-\infty}^{+\infty}\sin u^2\,du\Longrightarrow\mathrm{I}=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\sin u^2\,du$$

But $\displaystyle\int_{-\infty}^{+\infty}\sin u^2\,du=\sqrt{\frac{\pi}{2}}\Longrightarrow\mathrm{I}=\int_0^{+\infty}\frac{1}{2\sqrt{x}}\sin\left(\pi^2x+\frac{1}{x}\right)dx=\frac{1}{2\pi}\sqrt{\frac{\pi}{2}}=\frac{1}{\sqrt{8\pi}}$

231. $\mathrm{I}=\int_0^{+\infty}\frac{e^{-\pi x}}{(\sinh(x\pi)+\varphi)((\cosh(x\pi)+\varphi))}dx$

$$\mathrm{I}=\int_0^{+\infty}\frac{e^{-\pi x}}{(\sinh(x\pi)+\varphi)((\cosh(x\pi)+\varphi))}dx=\int_0^{+\infty}\frac{e^{-\pi x}}{\left(\frac{e^{x\pi}-e^{-x\pi}}{2}+\varphi\right)\left(\frac{e^{x\pi}+e^{-x\pi}}{2}+\varphi\right)}dx$$

$$\mathrm{I}=\int_0^{+\infty}\frac{4e^{-\pi x}}{(e^{x\pi}+2\varphi-e^{-x\pi})(e^{x\pi}+2\varphi+e^{-x\pi})}dx;\ \text{ by decompose into partial fractions}\Longrightarrow$$

$$\mathrm{I}=\int_0^{+\infty}\left(\frac{2}{e^{x\pi}+2\varphi-e^{-x\pi}}-\frac{2}{e^{x\pi}+2\varphi+e^{-x\pi}}\right)dx;\ \text{ by multiplying by } e^{x\pi} \text{ above } \&\ below\Longrightarrow$$

$$\mathrm{I}=2\int_0^{+\infty}\left(\frac{e^{x\pi}}{e^{2x\pi}+2\varphi e^{x\pi}-1}-\frac{e^{x\pi}}{e^{2x\pi}+2\varphi e^{x\pi}+1}\right)dx;\quad\text{then we can write:}$$

$$\mathrm{I}=2\int_0^{+\infty}\frac{e^{x\pi}}{e^{2x\pi}+2\varphi e^{x\pi}-1}dx-2\int_0^{+\infty}\frac{e^{x\pi}}{e^{2x\pi}+2\varphi e^{x\pi}+1}dx$$

$$\mathrm{I}=\frac{2}{\pi}\int_0^{+\infty}\frac{\pi e^{x\pi}}{(e^{x\pi}+\varphi)^2-(\varphi^2+1)}dx-\frac{2}{\pi}\int_0^{+\infty}\frac{\pi e^{x\pi}}{(e^{x\pi}+\varphi)^2-(\varphi^2-1)}dx$$

Let $u=e^{x\pi}+\varphi\Longrightarrow du=\ \pi e^{x\pi}dx;\ \begin{cases}x=0\Longrightarrow u=1+\varphi\\ x=+\infty\Longrightarrow+\infty\end{cases};$ with $\varphi^2=1+\varphi;$ so we get:

$$\mathrm{I}=\frac{2}{\pi}\int_{1+\varphi}^{+\infty}\frac{1}{u^2-\left(\sqrt{\varphi^2+1}\right)^2}du-\frac{2}{\pi}\int_{1+\varphi}^{+\infty}\frac{1}{u^2-\left(\sqrt{\varphi^2-1}\right)^2}du$$

$$\mathrm{I}=\frac{2}{\pi}\int_{\varphi^2}^{+\infty}\frac{1}{u^2-\left(\sqrt{\varphi^2+1}\right)^2}du-\frac{2}{\pi}\int_{\varphi^2}^{+\infty}\frac{1}{u^2-\left(\sqrt{\varphi}\right)^2}du\,;\ \text{ with }\int\frac{du}{y^2-a^2}=\frac{1}{2a}\ln\left|\frac{y-a}{y+a}\right|+c$$

$$\mathrm{I}=\frac{2}{\pi}\left[\frac{1}{2\sqrt{\varphi^2+1}}\ln\left(\frac{u-\sqrt{\varphi^2+1}}{u+\sqrt{\varphi^2+1}}\right)\right]_{\varphi^2}^{+\infty}-\frac{2}{\pi}\left[\frac{1}{2\sqrt{\varphi}}\ln\left(\frac{u-\sqrt{\varphi}}{u+\sqrt{\varphi}}\right)\right]_{\varphi^2}^{+\infty}$$

$$\mathrm{I}=-\frac{1}{\pi\sqrt{\varphi^2+1}}\ln\left(\frac{\varphi^2-\sqrt{\varphi^2+1}}{\varphi^2+\sqrt{\varphi^2+1}}\right)+\frac{1}{\pi\sqrt{\varphi}}\ln\left(\frac{\varphi^2-\sqrt{\varphi}}{\varphi^2+\sqrt{\varphi}}\right)$$

End of Chapter 6

Chapter 7: Integral Formulas

Indefinite Integral Formulas

$\int x^n dx = \frac{x^{n+1}}{n+1} + c$; where $n \neq -1$	$\int k du = ku + c$
$\int [f(u) \pm g(u)] du = \int f(u) du \pm \int g(u) du$	$\int c f(u) du = c \int f(u) du$
Integration by parts: $\int u dv = uv - \int v du$	$\int a^{bx+c} dx = \frac{1}{b} \cdot \frac{a^{bx+c}}{\ln a} + c$
$\int \cos x \, dx = \sin x + c$	$\int \sin x \, dx = -\cos x + c$
$\int \cos(ax+b) \, dx = \frac{1}{a} \sin(ax+b) + c$	$\int \sin(ax+b) \, dx = -\frac{1}{a} \cos(ax+b) + c$
$\int \tan x \, dx = \ln\lvert\sec x\rvert + c$	$\int \cot x \, dx = \ln\lvert\sin x\rvert + c$
$\int \sec^2 x \, dx = \tan x + c$	$\int \csc^2 x \, dx = -\cot x + c$
$\int \sec x \tan x \, dx = \sec x + c$	$\int \csc x \cot x \, dx = -\csc x + c$
$\int \sec x \, dx = \ln\lvert\sec x + \tan x\rvert + c$	$\int \csc x \, dx = \ln\lvert\csc x - \cot x\rvert + c$
$\int e^x dx = e^x + c$	$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b}$
$\int u'(x) e^{u(x)} \, dx = e^{u(x)} + c$	$\int [f(x) + f'(x)] \, e^x dx = e^x f(x) + c$
$\int a^x dx = \frac{1}{\ln a} a^x + c$	$\int \frac{1}{x} dx = \ln\lvert x\rvert + c$
$\int \frac{u'(x)}{u(x)} dx = \ln\lvert u(x)\rvert + c$	$\int \ln x \, dx = x \ln x - x + c$
$\int \log_a x \, dx = \frac{1}{\ln a}(x \ln x - x) + c = x \log_a x - \frac{x}{\ln a} + c$	

Indefinite Integral Formulas

$\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + c$	$\int \frac{dx}{\sqrt{1-x^2}} = -\cos^{-1} x + c$
$\int \frac{dx}{1+x^2} = \tan^{-1} x + c$	$\int \frac{dx}{1+x^2} = -\cot^{-1} x + c$
$\int \frac{dx}{x\sqrt{x^2-1}} = \sec^{-1} x + c$	$\int \frac{dx}{x\sqrt{x^2-1}} = -\csc^{-1} x + c$
$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$	$\int \frac{dx}{\sqrt{a^2-x^2}} = -\cos^{-1}\left(\frac{x}{a}\right) + c$
$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left\|\frac{x}{a}\right\| + c$	$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$
$\int \cos^{-1} x\, dx = x\cos^{-1} x - \sqrt{1-x^2} + c$	$\int \sin^{-1} x\, dx = x\sin^{-1} x + \sqrt{1-x^2} + c$
$\int \tan^{-1} x\, dx = x\tan^{-1} x - \frac{1}{2}\ln(1+x^2) + c$	
$\int \csc^{-1} x\, dx = x\csc^{-1} x + \ln\left(x + \sqrt{x^2-1}\right) + c$	
$\int \sec^{-1} x\, dx = x\sec^{-1} x - \ln\left(x + \sqrt{x^2-1}\right) + c$	
$\int \cot^{-1} x\, dx = x\cot^{-1} x + \frac{1}{2}\ln(1+x^2) + c$	
$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right) + c$	$\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2-a^2}\right) + c$
$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left\|\frac{a+x}{a-x}\right\| + c$	$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left\|\frac{x-a}{x+a}\right\| + c$
$\int \frac{dx}{a+bx} = \frac{1}{b}\ln\|a+bx\| + c$	$\int (a+bx)^n dx = \frac{(a+bx)^{n+1}}{b(n+1)} + c;\quad n \neq -1$
$\int \frac{a+x}{b+x} dx = x + (a-b)\ln\|b+x\| + c$	$\int \frac{dx}{(a+x)(b+x)} = \frac{1}{a-b}\ln\left\|\frac{b+x}{a+x}\right\| + c$

Indefinite Integral Formulas

$\int \cosh x\, dx = \sinh x + c$	$\int \sinh x\, dx = \cosh x + c$
$\int \operatorname{sech}^2 x\, dx = \tanh x + c$	$\int \operatorname{csch}^2 x\, dx = -\coth x + c$
$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x + c$	$\int \operatorname{csch} x \coth x\, dx = -\operatorname{csch} x + c$
$\int \tanh x\, dx = \ln(\cosh x) + c$	$\int \coth x\, dx = \ln\lvert\sinh x\rvert + c$
$\int \operatorname{sech} x\, dx = \tan^{-1}(\sinh x) + c$	$\int \operatorname{csch} x\, dx = \ln\left\lvert\tanh \frac{x}{2}\right\rvert + c$
$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + c$	$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + c$
$\int \frac{dx}{a^2-x^2} = \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + c$	$\int \frac{dx}{x\sqrt{a^2-x^2}} = \frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + c$
$\int \sqrt{a^2-x^2}dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right) + c$	
$\int \sqrt{x^2+a^2}dx = \frac{1}{2}x\sqrt{x^2+a^2} + \frac{1}{2}a^2\ln\left(x+\sqrt{x^2+a^2}\right) + c$	
$\int \sqrt{x^2-a^2}dx = \frac{1}{2}x\sqrt{x^2-a^2} - \frac{1}{2}a^2\ln\left(x+\sqrt{x^2-a^2}\right) + c$	
$\int e^{ax}\sin bx\, dx = \frac{1}{a^2+b^2}e^{ax}(a\sin bx - b\cos bx) + c$	
$\int e^{ax}\cos bx\, dx = \frac{1}{a^2+b^2}e^{ax}(a\cos bx + b\sin bx) + c$	
$\int \frac{dx}{(a^2+x^2)^{n+1}} = \frac{x}{2na^2(a^2+x^2)^n} + \frac{2n-1}{2na^2}\int \frac{dx}{(a^2+x^2)^n};\ \ n = 1,2\ldots$	
$\int \frac{dx}{(a^2-x^2)^{n+1}} = \frac{x}{2na^2(a^2-x^2)^n} + \frac{2n-1}{2na^2}\int \frac{dx}{(a^2-x^2)^n} + c;\ \ n = 1,2\ldots$	

Definite Integral Formulas

$\int_a^a f(x)dx = 0$	$\int_a^b f(x)dx = \int_a^b f(t)dt$
$\int_a^b f(x)dx = -\int_b^a f(x)dx$	$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$
$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$; where $a < c < b$	
$\int_0^a f(x)dx = \int_0^a f(a-x)dx$	$\int_a^b kf(x)dx = k\int_a^b f(x)dx$
$\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$	
$\int_{-a}^a f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(-x) = f(x) \\ 0 \ \text{ if } f(-x) = -f(x) \end{cases}$	
$\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx & \text{if } f(2a-x) = f(x) \\ 0 \ \text{ if } f(2a-x) = -f(x) \end{cases}$	

End of Chapter 7

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