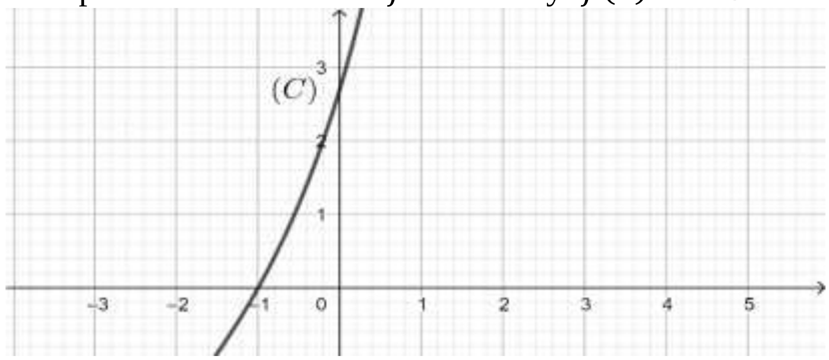
Grade 12 LS **Integrals (Extra sheet)** **2021-2022**

Exercise 1

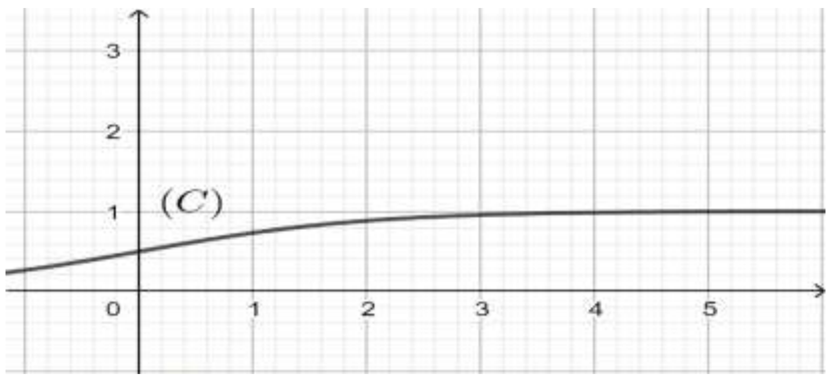
The curve (C) is the representative curve of f defined by $f(x) = x + e^{x+1}$



Calculate the area of the region bounded by (C) , the x-axis and the y-axis.

Exercise 2

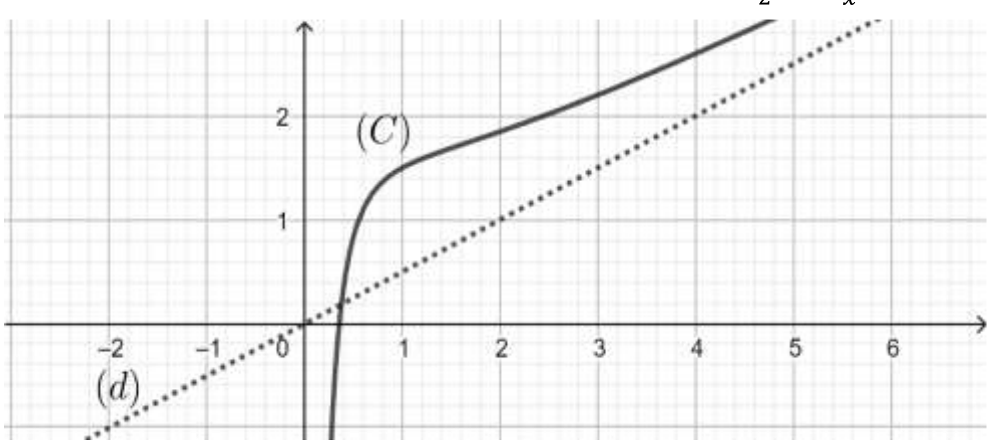
The curve (C) is the representative curve of f defined by $f(x) = \frac{e^x}{e^x+1}$



Calculate the area of the region bounded by (C) , the x-axis and the lines of equations $x = 1$ and $x = 3$.

Exercise 3

The curve (C) is the representative curve of f defined by $f(x) = \frac{x}{2} + \frac{1+\ln x}{x}$.

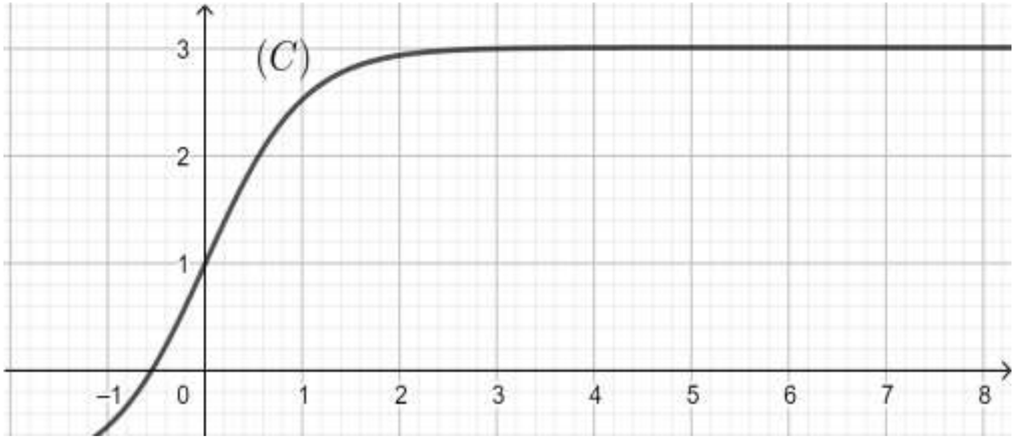


Calculate the area of the region bounded by (C) , line $(d): y = \frac{x}{2}$ and the two lines of equations

$x = 1$ and $x = e$.

Exercise 4

Let f be the function defined on IR by $f(x) = 3 - \frac{4}{e^{2x}+1}$ and designate by (C) its representative curve in an orthonormal system $(O\ ;\ \vec{i}\ ,\ \vec{j}\ )$. Unit = 2 cm

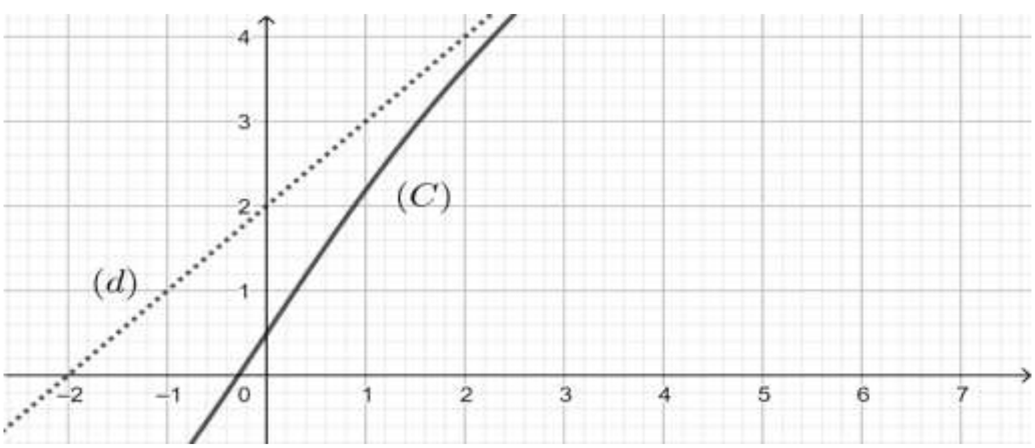


1) Verify that $f(x) = -1 + \frac{4e^{2x}}{e^{2x}+1}$ and deduce an antiderivative F of f.

2) Calculate , in terms of cm^2 ,the area of the region bounded by (C) , the $x - axis$, the $y - axis$ and the line with equation $x = \ln 2$.

Exercise 5

Let f be the function defined on IR by $f(x) = x + 2 - \frac{3}{1+e^{x}}$ and designate by (C) its representative curve in an orthonormal system $(O\ ;\ \vec{i}\ ,\ \vec{j}\ )$.

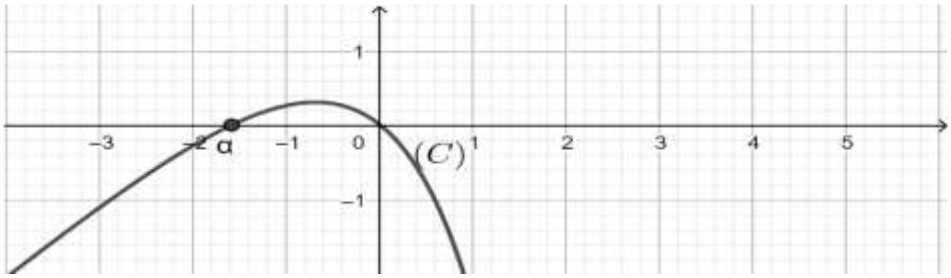


1) Verify that $f(x) = x + 2 - \frac{3e^{-x}}{1+e^{-x}}$.

2) Let A(λ) the area of the region bounded by (C) ,the line (d) :$y = x + 2$ and the lines with equations $x = 0$ and $x = \lambda$ $such\ that$ $\lambda > 0$ .Find $\lim\limits_{\lambda \to +\infty} A(\lambda)$.

Exercise 6

The curve (C) is the representative curve of f defined by $f(x) = x + 2 - 2\, e^{x}$.
The equation $f(x) = 0$ admits a root α between $]-\infty, 0[$

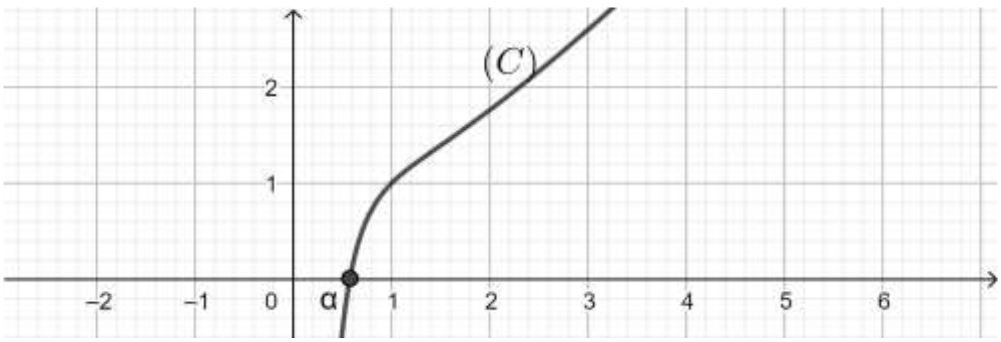


Denote by A(α) the area of the region bounded by (C) and the axis of abscissa.

Show that $A(\alpha)=-\frac{\alpha^2}{2} - \alpha$

Exercise 7

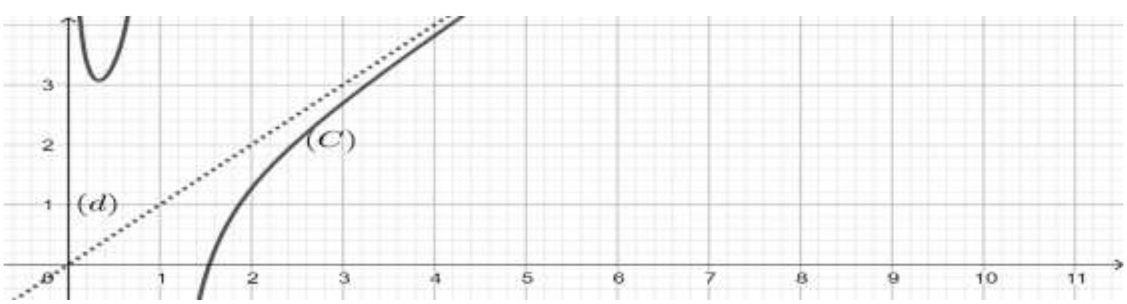
The curve (C) is the representative curve of f defined by $f(x) = x - \frac{(\ln x)^2}{x}$



1- Find the value of the area of the region bounded by (C) and the axis of abscissa. and the line of equation $x = 1$.

Exercise 8

The curve (C) is the representative curve of f defined by $f(x) = x - \frac{1}{x\, ln\, x}$

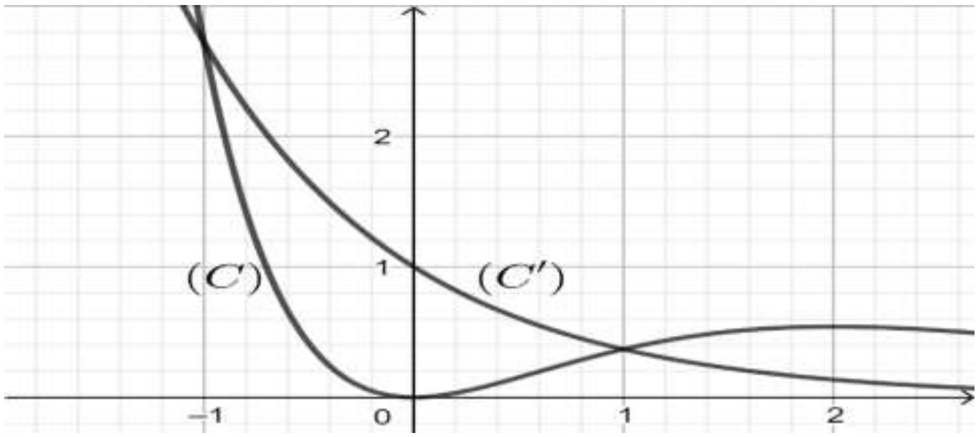


1- Find A(t) the area of the region bounded by (C) , (d) : $y = x$ and the two lines of equations $x = e \; and \; x = t$ such that $t > e$

2- Show that , $\forall\, t > e$, $A(t) < t$.

Exercise 9

The curves (C) and (C') are the representative curves of f and g defined by $f(x) = x^2\, e^{-x}$ and $g(x) = e^{-x}$ respectively.



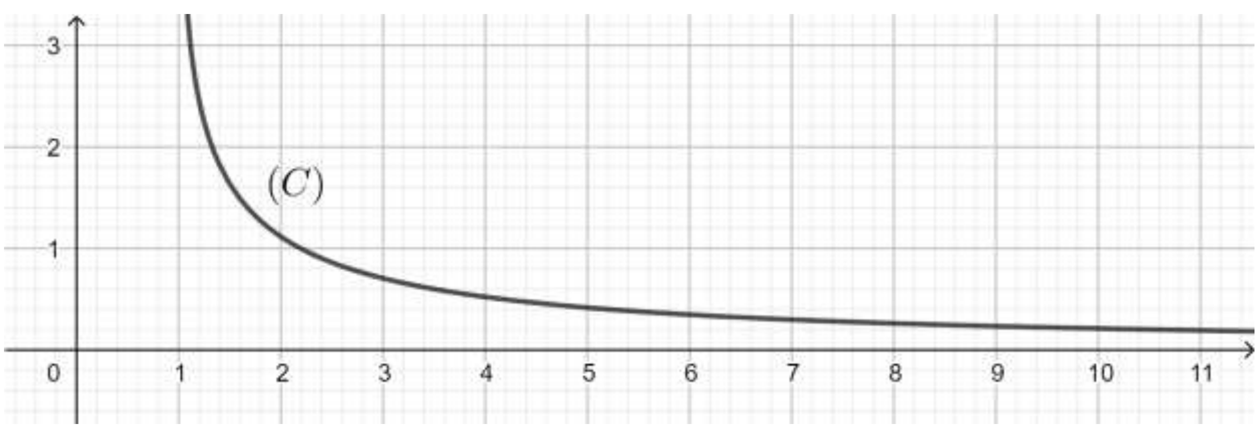
Unit = 2 cm

Let h be the function defined on ℝ by $h(x) = (x^2 + 2x + 2)e^{-x}$

1- Calculate h'(x).Deduce an antiderivative of f .
2- Calculate in cm^2 the area S of the region bounded by (C) ,(C') and the two lines of equations $x = -1$ and $x = 1$.
3- Let $A(\alpha)$ be the area of the region bounded by (C) ,(C') and the two lines of equations $x = 1$ and $x = \alpha$. Calculate , in cm^2, $A(\alpha)$ and prove that $\lim_{\alpha \to +\infty} A(\alpha) = S \quad (\alpha > 1)$

Exercise 10

The curve (C) *is* the representative curve of f defined by $f(x) = \ln(\frac{x+1}{x-1})$

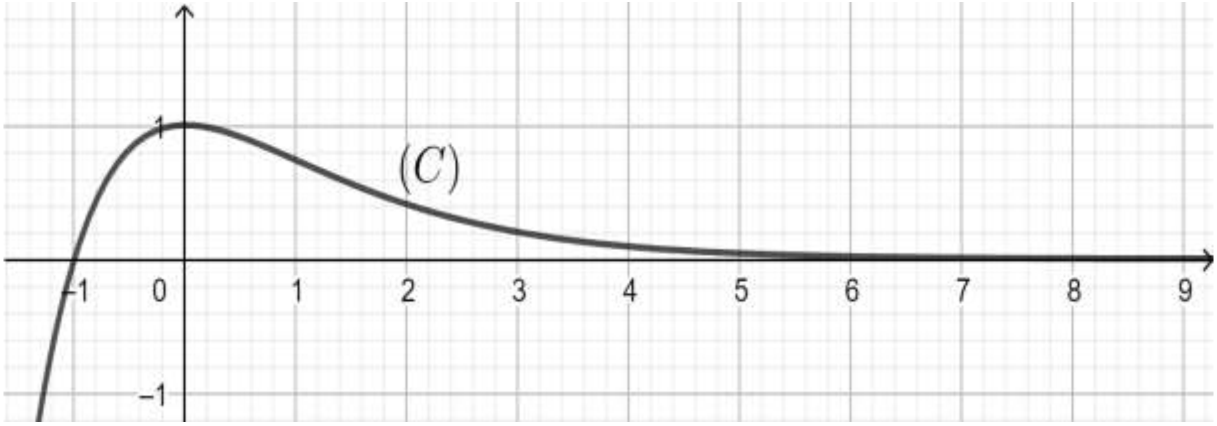


Let h be the function defined by $h(x) = xf(x)$

1- Verify that $f(x) = h'(x) + \frac{2x}{x^2-1}$ *and* Deduce an antiderivative of f .
2- Calculate the area of the region bounded by (C) ,the axis of abscissa and the line with equation $x = 2$ *and* $x = 3$.

Exercise 11

The curve (C) is the representative curve of f defined over $]-\infty,+\infty[$ by $f(x)=(x+1)e^{-x}$



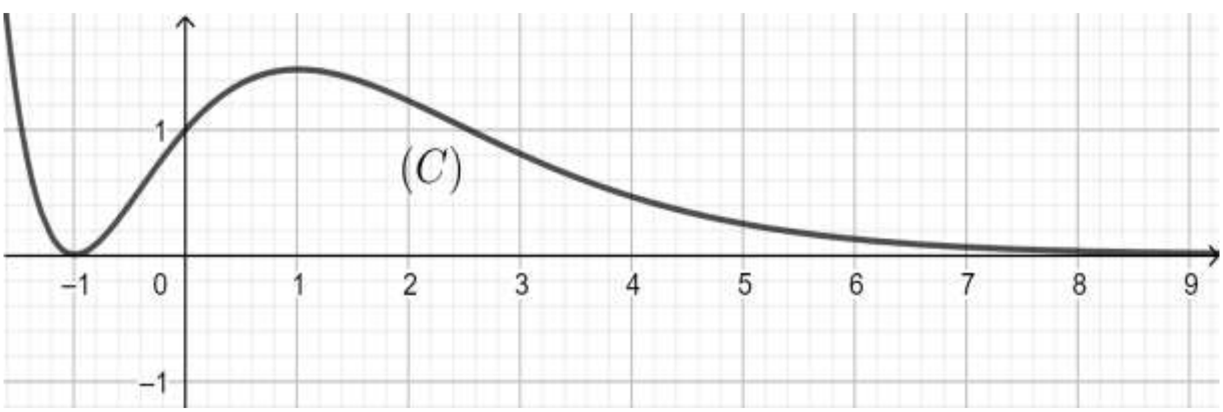
Let F be the function defined on $]-\infty,+\infty[$ by $F(x)=(ax+b)e^{-x}$

1- Determine a and b for which F is a antiderivative of f .

2- Calculate the area of the region bounded by (C) ,the axis of abscissa and the two lines with equations $x=0$ and $x=1$.

Exercise 12

The curve (C) is the representative curve of f defined over $]-\infty,+\infty[$ by $f(x)=(x+1)^2e^{-x}$



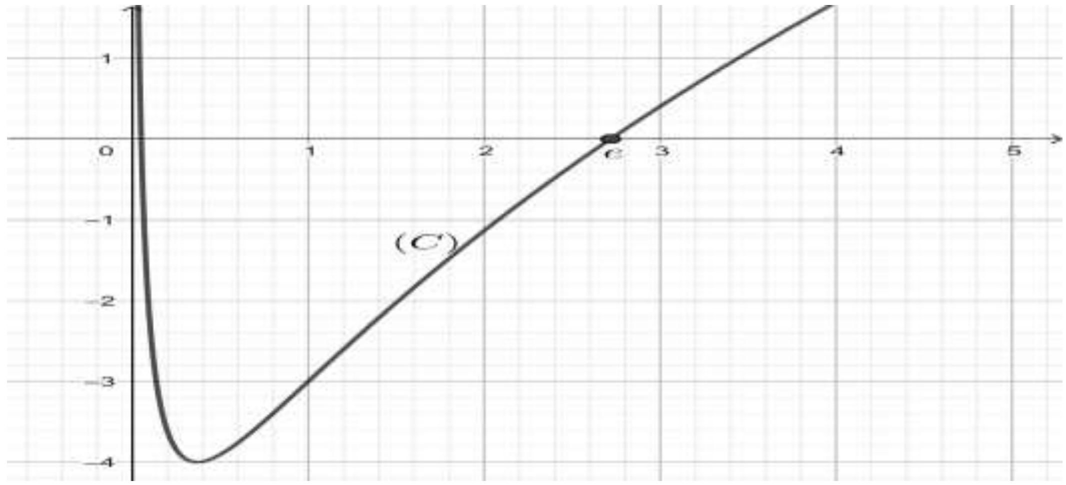
Let F be the function defined on $]-\infty,+\infty[$ by $F(x)=(px^2+qx+r)e^{-x}$

1- Determine p, q and r for which F is a primitive of f .

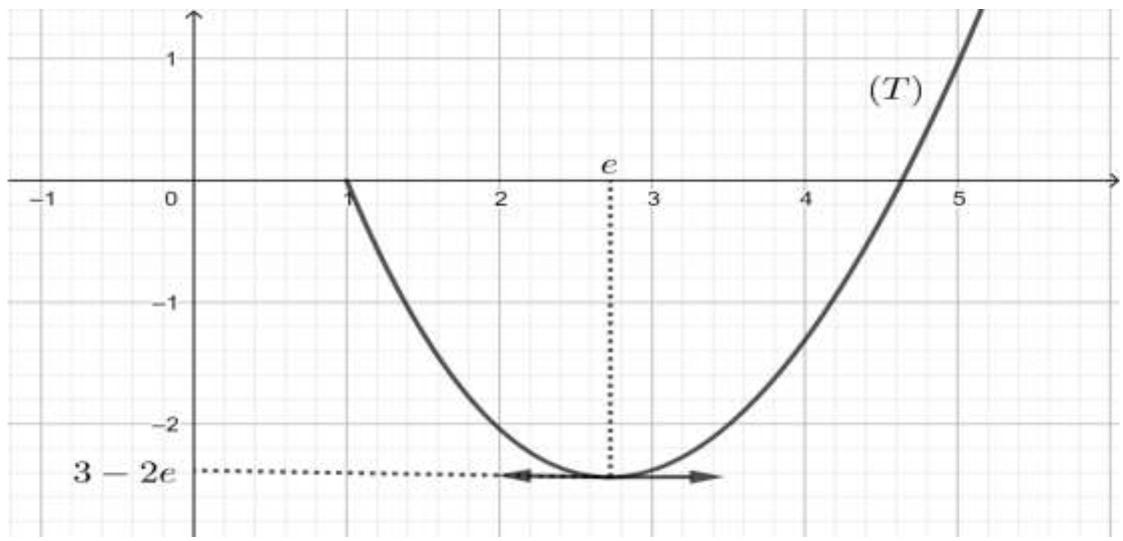
2- Calculate the area of the region bounded by (C) ,the axis of abscissa and the two lines with equations $x=0$ and $x=1$.

Exercise 13

The curve $(C)\ is$ the representative curve of f defined by $f(x) = (\ln x)^2 + 2\ln x - 3$



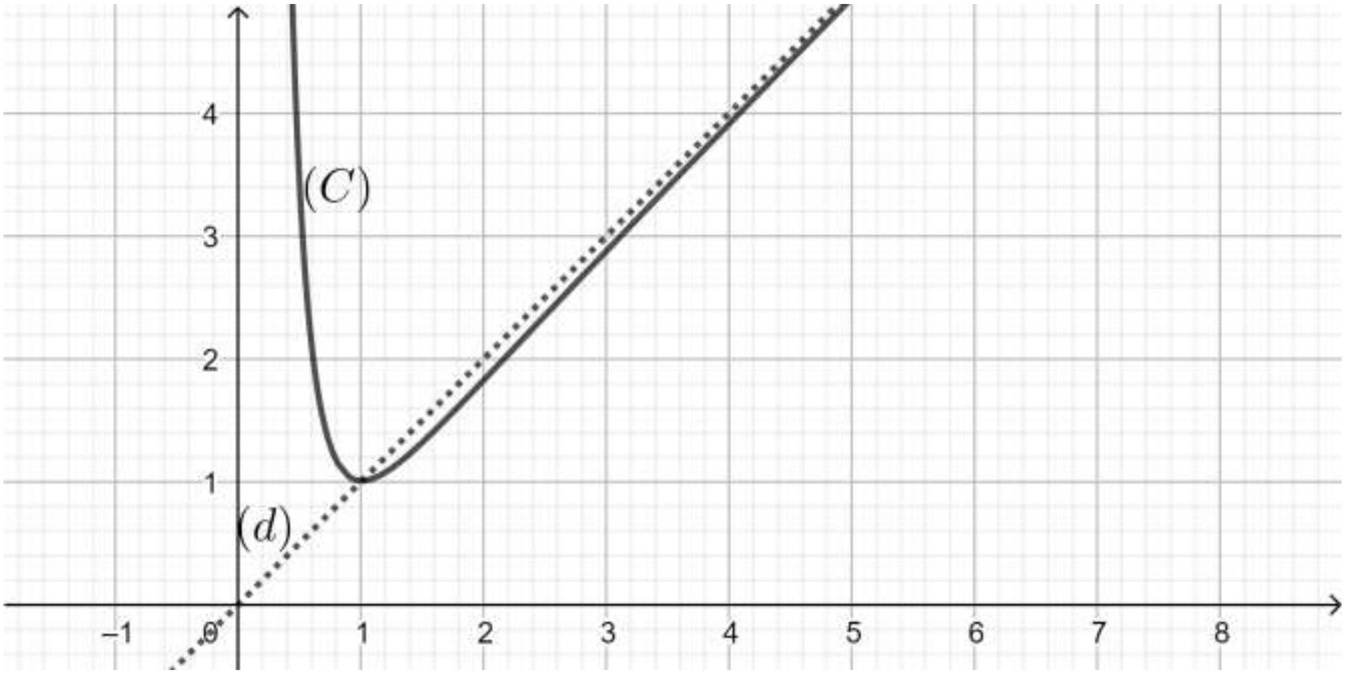
And the curve (T) shown below is the curve of the function F, where F is a primitive of f .



Calculate the area of the region bounded by (C) ,the axis of abscissa and the two lines with equations $x = 1\ and\ x = e.$

Exercise 14

The curve (C) is the representative curve of f defined by $f(x) = x - \frac{\ln x}{x^2}$.



Let α >1 , Designate by A(α) the area of the region bounded by (C) , line $(d) : y = x$ and the two lines with equations $x = 1$ *and* $x = \alpha$.

1- Verify that $\int \frac{\ln x}{x^2} dx = \frac{-1-\ln x}{x} + k$ where k is a real number.

2- Express A(α) as a function of α.

3- Show graphically that $A(\alpha) < \frac{(\alpha-1)^2}{2}$

Exercice 15

Let f be the function that is defined on IR by $f(x) = e^{2x} + x^2 - 2x$

The table below is that of f.

x	$-\infty$	0	$+\infty$
$f(x)$	↘	1	↗

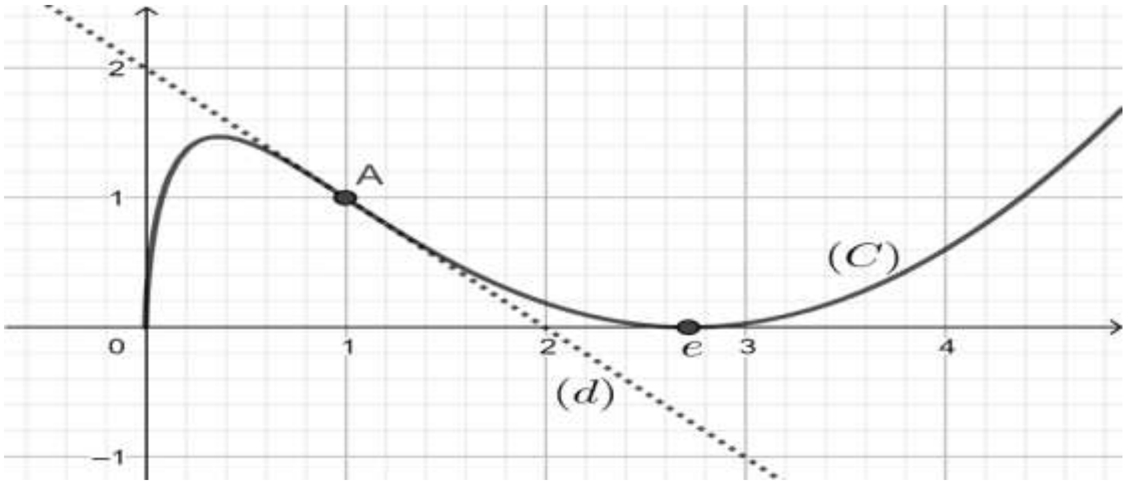
And Let F be the function that is defined on [0 ; + ∞ [by F(x) = $\int_0^x f(t)\, dt$

a- Determine the sense of variations of F.

b- What is the sign of F(x) ? Justify your answer.

Exercise 16

The curve (C) *is* the representative curve of f defined over $]0,+\infty[$ by $f(x) = x\,(\ln x - 1)^2$. (d): $y = -x + 2$ is tangent to (C) at A(1,1).



1- Show that the function F defined on $]0,+\infty[$ as $F(x) = \frac{x^2}{2}[(lnx)^2 - 3lnx + \frac{5}{2}]$ is an antiderivative of f.

2- Deduce the area of the region bounded by (C), $(x'x)$ *and* (d) .

Grade 12 LS **Integrals (Solution of Extra sheet)** **2021-2022**

Exercise 1

$$A = \int_{-1}^{0} f(x)dx = \int_{-1}^{0} (x + e^{x+1})dx = \int_{-1}^{0} xdx + \int_{-1}^{0} e^{x+1}dx = \left(\frac{x^2}{2} + e^{x+1}\right)]_{-1}^{0}$$

$$= \frac{0^2}{2} + e^{0+1} - \frac{(-1)^2}{2} - e^{-1+1} = e - \frac{1}{2} - 1 = e - \frac{3}{2}\ unit^2$$

Exercise 2

$A = \int_1^3 f(x)dx = \int_1^3 \frac{e^x}{e^x+1}dx = \int_1^3 \frac{u'}{u}dx = (\ln u)]_1^3 = \ln(e^x+1)]_1^3$ $(\ u = e^x + 1\ \ then\ u' = e^x)$

$$= ln\ (e^3 + 1) - ln(e^1 + 1) = \ln\ (\frac{e^3 + 1}{e + 1})\ unit^2$$

Exercise 3

$A = \int_1^e \left[f(x) - \frac{x}{2}\right] dx = \int_1^e \frac{1+\ln x}{x} dx = \int_1^e (1 + \ln x)\frac{1}{x}dx = \int_1^e u\, u' dx$ $(\ u = 1 + lnx\ \ then\ u' = \frac{1}{x})$

$$= \frac{u^2}{2}]_1^e = \frac{(1+\ln x)^2}{2}]_1^e = \frac{(1+\ln e)^2}{2} - \frac{(1+\ln 1)^2}{2} = \frac{3}{2}\ unit^2$$

Exercise 4

1) $f(x) = 3 - \frac{4}{e^{2x}+1} = \frac{3e^{2x}+3-4}{e^{2x}+1} = \frac{3e^{2x}-1}{e^{2x}+1}$

$$-1 + \frac{4e^{2x}}{e^{2x}+1} = \frac{-e^{2x} - 1 + 4e^{2x}}{e^{2x}+1} = \frac{3e^{2x} - 1}{e^{2x}+1}$$

$$\int f(x)dx = \int(-1 + \frac{4e^{2x}}{e^{2x}+1})dx = \int -1\, dx + \int \frac{4e^{2x}}{e^{2x}+1}dx = \int -1\, dx + 2\int \frac{2e^{2x}}{e^{2x}+1}dx = -x + 2\int \frac{u'}{u}dx + c$$

$$(\ u = e^{2x} + 1\ then\ u' = 2e^{2x})$$

$$= -x + 2\ln u + c = -x + 2\ln(e^{2x} + 1) + c$$

2) $A = \int_0^{ln2} f(x)dx = -x + 2\ln(e^{2x} + 1)]_0^{ln2} = -ln2 + 2\ln(e^{2ln2} + 1) + o - 2\ln\ (e^{2(0)} + 1)$

$$= -\ln 2 + 2\ln(e^{\ln 4} + 1) - 2\ln\ 2 = -\ln 2 + 2\ln 5 - 2\ln 2 = \ln 25 - 3\ln 2 = \ln 25 - \ln 8$$

$$= \ln\frac{25}{8}\ unit^2 = \ln\frac{25}{8}\ (2\ cm)^2 = 4\ln\frac{25}{8}\ cm^2$$

Exercise 5

1) $x+2-\frac{3e^{-x}}{1+e^{-x}} = x+2-\frac{\frac{3}{e^x}}{1+\frac{1}{e^x}} = x+2-\frac{\frac{3}{e^x}}{\frac{e^x+1}{e^x}} = x+2-\frac{3}{e^x+1} = f(x)$.

2) $\mathrm{A}(\lambda) = \int_0^\lambda [x+2-f(x)]dx = \int_0^\lambda \frac{3e^{-x}}{1+e^{-x}}\,dx = \int_0^\lambda \frac{3e^{-x}}{1+e^{-x}}\,dx = -3\int_0^\lambda \frac{-e^{-x}}{1+e^{-x}}\,dx$

$$(\,u = 1+e^{-x}\ then\ u' = -e^{-x})$$

$$= -3\int_0^\lambda \frac{u'}{u}\,dx = -3\ln u]_0^\lambda = -3\ln(1+e^{-x}\,)]_0^\lambda = -3\ln\left(1+e^{-\lambda}\right) + 3\ln(1+e^{-0}\,)$$

$$= -3\ln\left(1+e^{-\lambda}\right) + 3\ln 2 \quad unit^2$$

$$\lim_{\lambda\to+\infty} A(\lambda) = \lim_{\lambda\to+\infty} -3\ln\left(1+e^{-\lambda}\right) + 3\ln 2 = -3\ln(1+e^{-\infty}) + 3\ln 2 = 3\ln 2 = \ln 8.$$

Exercise 6

$f(\alpha) = 0$

$\alpha + 2 - 2\,e^\alpha = 0\ \ then\ \ 2e^\alpha = \alpha + 2\ \ then\ \ e^\alpha = \dfrac{\alpha+2}{2}$

$$A(\alpha) = \int_\alpha^0 f(x)\,dx = \int_\alpha^0 (x+2-2e^x)dx = \left(\frac{x^2}{2} + 2x - 2e^x\right)]_\alpha^0$$

$$= \frac{0^2}{2} + 2(0) - 2e^0 - \frac{\alpha^2}{2} - 2\alpha + 2e^\alpha = -2 - \frac{\alpha^2}{2} - 2\alpha + 2e^\alpha$$

$$= -2 - \frac{\alpha^2}{2} - 2\alpha + 2\left(\frac{\alpha+2}{2}\right) == -2 - \frac{\alpha^2}{2} - 2\alpha + \alpha + 2 = -\frac{\alpha^2}{2} - \alpha$$

Exercise 7

1- $\int_\alpha^1 f(x)dx\ = \int_\alpha^1 \left(x - \frac{(\ln x)^2}{x}\right)dx = \int_\alpha^1 x\,dx - \int_\alpha^1 \frac{(\ln x)^2}{x}dx = \int_\alpha^1 x\,dx - \int_\alpha^1 (\ln x)^2\frac{1}{x}dx$

$$(u = \ln x\ \ then\ \ u' = \frac{1}{x})$$

$$= \int_\alpha^1 x\,dx - \int_\alpha^1 u^2u'dx = \left(\frac{x^2}{2} - \frac{u^3}{3}\right)]_\alpha^1 = \left(\frac{x^2}{2} - \frac{(\ln x)^3}{3}\right)]_\alpha^1 =$$

$$\frac{1^2}{2} - \frac{(\ln 1)^3}{3} - \frac{\alpha^2}{2} + \frac{(\ln\alpha)^3}{3} = \frac{1}{2} - \frac{\alpha^2}{2} + \frac{(\ln\alpha)^3}{3}\quad unit^2$$

Exercise 8

1- $A(t) = \int_e^t [x - f(x)]dx = \int_e^t \frac{1}{x\, lnx} dx = \int_e^t \frac{1}{x} \frac{1}{\ln x} dx = \int_e^t \frac{u'}{u} dx = \ln u]_e^t$

$$(u = \ln x \quad then\ u' = \frac{1}{x})$$

$= \ln(lnx)]_e^t = \ln(lnt) - \ln(lne) = \ln(\ln t)\ unit^2$

2- *The graph of* $\ln t$ *is below that of* e^t *then* $lnt < e^t$ *then* $\ln(lnt) < lne^t$ *then* $A(t) < t.$

Exercise 9

1- $h(x) = (x^2 + 2x + 2)e^{-x}$

$h'(x) = (2x + 2)e^{-x} + (x^2 + 2x + 2)(-e^{-x}) = (2x + 2 - x^2 - 2x - 2)e^{-x} = -x^2e^{-x}$

$h'(x) = -f(x)$

$$\int h'(x)\, dx = -\int f(x)\, dx$$

$$\int f(x)\, dx = -h(x) + c = -(x^2 + 2x + 2)e^{-x} + c.$$

2- $A = \int_{-1}^{1} [g(x) - f(x)]dx = \int_{-1}^{1} e^{-x}dx - \int_{-1}^{1} f(x)\, dx = (-e^{-x} + (x^2 + 2x + 2)e^{-x})\,]_{-1}^{1}$

$= (x^2 + 2x + 2 - 1)e^{-x}]_{-1}^{1} = (x^2 + 2x + 1)e^{-x}]_{-1}^{1} = (x^2 + 1)^2e^{-x}]_{-1}^{1}$

$= (1 + 1)^2e^{-1} - \left((-1) + 1\right)^2 e^1 = \frac{4}{e}\ unit^2 = \frac{4}{e}\ (2cm^2) = \frac{16}{e}\ cm^2$

3- $A(\alpha) = \int_1^\alpha [f(x) - g(x)]dx = \int_1^\alpha f(x)dx - \int_1^\alpha e^{-x}\, dx = (-(x^2 + 2x + 2)e^{-x} + e^{-x})\,]_1^\alpha$

$= (-x^2 - 2x - 2 + 1)e^{-x}]_1^\alpha = (-x^2 - 2x - 1)e^{-x}]_1^\alpha = -(x + 1)^2e^{-x}]_1^\alpha$

$= -(\alpha + 1)^2e^{-\alpha} + (1 + 1)^2e^{-1} = -(\alpha + 1)^2e^{-\alpha} + \frac{4}{e}\ unit^2 = -4(\alpha + 1)^2e^{-\alpha} + \frac{16}{e}\ cm^2$

$$\lim_{\alpha\to+\infty} A(\alpha) = \lim_{\alpha\to+\infty} -4(\alpha + 1)^2e^{-\alpha} + \frac{16}{e} = 0 + \frac{16}{e} = \frac{16}{e} = S$$

Since $\lim_{\alpha\to+\infty} -4(\alpha + 1)^2e^{-\alpha} = -4(+\infty + 1)e^{-\infty} = -\infty(0)\quad I.F.$

Using l'hopital 2 times we get $\lim_{\alpha\to+\infty} \frac{-4(\alpha+1)^2}{e^\alpha} = \lim_{\alpha\to+\infty} \frac{-8(\alpha+1)}{e^\alpha} = \lim_{\alpha\to+\infty} \frac{-8}{e^\alpha} = \frac{-8}{+\infty} = 0$

Exercise 10

1- $f'(x) = [\ln(x+1)]' - [\ln(x-1)]' = \frac{1}{x+1} - \frac{1}{x-1} = \frac{x-1-x-1}{x^2-1} = \frac{-2}{x^2-1}$

$h'(x) + \frac{2x}{x^2-1} = f(x) + xf'(x) + \frac{2x}{x^2-1} = f(x) - \frac{2x}{x^2-1} + \frac{2x}{x^2-1} = f(x).$

$$f(x) = h'(x) + \frac{2x}{x^2-1}$$

Then $\int f(x)\ dx = \int h'(x)\,dx + \int \frac{2x}{x^2-1}\ dx$

$\int f(x)\ dx = h(x) + \int \frac{u'}{u}\ dx$ $\qquad$ $(u = x^2 - 1 \quad then\ u' = 2x)$

$\int f(x)\ dx = h(x) + \ln u + c$

$\int f(x)\ dx = h(x) + \ln(x^2-1) + c$

2- $A = \int_2^3 f(x)dx = (x ln(\frac{x+1}{x-1}) + \ln(x^2-1))]_2^3$

$= 3\ln\left(\frac{3+1}{3-1}\right) + \ln(3^2-1) - 2\ln\left(\frac{2+1}{2-1}\right) - \ln(2^2-1) = 3ln2 + ln8 - 2ln3 - ln3 =$

$= ln2^3 + ln8 - 3ln3 = ln8 + ln8 - ln3^3 = 2ln8 - ln27 = ln8^2 - ln27$

$= ln64 - ln27 = \ln\left(\frac{64}{27}\right)\ unit^2$

Exercise 11

1- $F(x) = \int f(x)dx$

$(F(x))' = (\int f(x)dx)'$

$F'(x) = f(x)$

$ae^{-x} + (ax+b)(-e^{-x}) = (x+1)e^{-x}$

$(a - ax - b)e^{-x} = (x+1)e^{-x}$

$(-ax + a - b)e^{-x} = (x+1)e^{-x}$

Then $-a = 1\ \ so\ a = -1$

And $\ a - b = 1\ \ so - 1 - b = 1 \quad so\ b = -2$

Therefore $F(x) = (-x-2)e^{-x}$

2- $A = \int_0^1 f(x)\ dx = F(x)]_0^1 = (-x-2)e^{-x}\ ]_0^1 = (-1-2)e^{-1} - (-0-2)e^{-0} = -\frac{3}{e} + 2\ unit^2$

Exercise 12

1- $F(x) = \int f(x)dx$

$(F(x))' = (\int f(x)dx)'$

$F'(x) = f(x)$

$(2px+q)e^{-x} + (px^2+qx+r)(-e^{-x}) = (x+1)^2e^{-x}$

$(2px+q-px^2-qx-r)e^{-x} = (x^2+2x+1)e^{-x}$

$(-px^2+(2p-q)x+q-r)e^{-x} = (x^2+2x+1)e^{-x}$

Then $-p = 1 \quad so\ p = -1$

And $2p - q = 2 \quad so -2-q = 2 \quad so\ q = -4$

And $q - r = 1 \quad so -4 - r = 1 \quad so\ r = -5$

Therefore $F(x) = (-x^2-4x-5)e^{-x}$

2- $A = \int_0^1 f(x)\,dx = F(x)]_0^1 = (-x^2-4x-5)e^{-x}\,]_0^1$

$= (-1^2-4(1)-5)e^{-1} - (-0^2-4(0)-5)e^{-0} = -\dfrac{10}{e} + 5 \quad unit^2$

Exercise 13

$$A = \int_1^e -f(x)\,dx = -F(x)]_1^e = -F(e) + F(1) = -(3-2e) + 0 = 2e - 3 \quad unit^2$$

Exercise 14

1- $\int \frac{\ln x}{x^2}dx = \frac{-1-\ln x}{x} + k$

$$(\int \frac{\ln x}{x^2}dx)' = (\frac{-1-\ln x}{x} + k)'$$

$$\frac{\ln x}{x^2} = \frac{-\frac{1}{x}(x) - (-1-\ln x)}{x^2} + 0$$

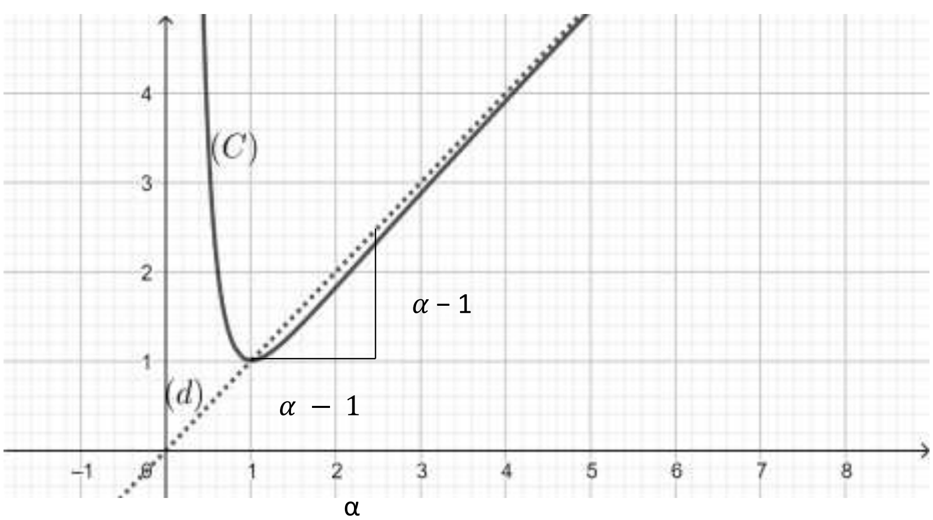
$$\frac{\ln x}{x^2} = \frac{-1+1+\ln x}{x^2}$$

$$\frac{\ln x}{x^2} = \frac{\ln x}{x^2}$$

2- $A(\alpha) = \int_1^\alpha [x - f(x)]dx = \int_1^\alpha \frac{\ln x}{x^2}dx = \frac{-1-\ln x}{x}]_1^\alpha$

$$= \frac{-1-\ln \alpha}{\alpha} - \frac{-1-\ln 1}{1} = \frac{-1-\ln \alpha}{\alpha} + 1 = \frac{-1-\ln \alpha + \alpha}{\alpha} \quad unit^2$$

3-



For any α $A(\alpha) < area\ this\ right\ isosceles\ triangle\ \frac{(\alpha-1)^2}{2}$

$$A(\alpha) < \frac{(\alpha-1)^2}{2}$$

Exercice 15

a- $F(x) = \int_0^x f(t)\, dt$

$F'(x) = (\int_0^x f(t)\, dt)'$

$F'(x) = f(x)d(x) = f(x)$

According to the table of f $min\ f(x) > 0\ so\ f(x) > 0$.

Therefore $F'(x) > 0$ and F is an increasing function.

b- $f(t) > 0\ and\ x \geq 0\ then\ \int_0^x f(t)dt \geq 0\ ,so\ \ F(x) \geq 0.$

OR : F is increasing and F(0) = 0 , then F(x) ≥ 0.

Exercise 16

1- $F'(x) = x\left[(lnx)^2 - 3lnx + \frac{5}{2}\right] - \frac{x^2}{2}\left(\frac{2lnx}{x} - \frac{3}{x}\right) = x\left[(lnx)^2 - 3lnx + \frac{5}{2}\right] - \frac{x}{2}(2lnx - 3)$

$= x\left(\ln^2 x - 3lnx + \frac{5}{2} - lnx + \frac{3}{2}\right) = x(\ln^2 x - 4lnx + 4) = x(lnx - 2)^2 = f(x)$

$2 - A = \int_1^e f(x)dx - area\ triangle\ ABC\ [B(1,0)\ and\ C(2,0)]$

$= F(e) - F(1) - \frac{1}{2} = \frac{e^2}{2}\left(\ln^2 e - 3lne + \frac{5}{2}\right) - \frac{1}{2}\left(\ln^2 1 - 3ln1 + \frac{5}{2}\right) - \frac{1}{2} = \frac{e^2}{4} - \frac{5}{4} - \frac{1}{2} = \frac{e^2-7}{4}\ \ u^2$