PLANES AND LINES IN SPACE
CHAPTER REVIEW

1. **Definition 1:**
Consider a point A and $\vec{u}$ a non- zero vector in the space: The set of points M of the space such that: $\overrightarrow{AM} = t\vec{u}$, t is a real number is a straight line (d) pass through A and of direction vector $\vec{u}$.
Definition 2:
Consider a point A and $\vec{n}$ a non - zero vector of the space. Then the set of points M of the space such that: $\overrightarrow{AM}.\vec{n} = 0$ is the plane (P) pass through A, and of a normal vector $\vec{n}$.

2. **Components of the vector product:**
Consider the two vectors $\vec{u}(x;y;z)$ and $\vec{v}(x';y';z')$.
The vector product: $\vec{u}\wedge\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ x' & y' & z' \end{vmatrix} = (yz'-y'z)\vec{i} - (xz'-x'z)\vec{j} + (xy'-x'y)\vec{k}$.

3. **Applications of calculation of the areas:**
The area of a parallelogram ABCD is $a_p = \left\|\overrightarrow{AB}\wedge\overrightarrow{AD}\right\|$.square units of area.
The area of triangle ABD is $a_t = \frac{1}{2}a_p = \frac{1}{2}\left\|\overrightarrow{AB}\wedge\overrightarrow{AD}\right\|$ square units of area.

4. **The triple scalar product of three vectors:**
Definition:
We called the triple scalar product of three vectors $\vec{u}$, $\vec{v}$ et $\vec{w}$ in that order the real: $\vec{u}.(\vec{v}\wedge\vec{w})$.
Property:
The points O; A; B and C are coplanar if and only if $\overrightarrow{OA}.(\overrightarrow{OB}\wedge\overrightarrow{OC}) = 0$.

5. **Analytic expression of the triple scalar product:**
Consider the three vectors $\vec{u}(x;y;z)$, $\vec{v}(x';y';z')$ and $\vec{w}(x'';y'';z'')$.
$$\vec{u}.(\vec{v}\wedge\vec{w}) = \begin{vmatrix} x & y & z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix} = x(y'z''-y''z') - y(x'z''-x''z') + z(x'y''-x''y').$$

6. **Applications of calculation of the volumes:**
The volume of the parallelepiped OABCDEFG is $V = \left|\overrightarrow{OA}.(\overrightarrow{OB}\wedge\overrightarrow{OC})\right|$ cubic units of volume.
The volume of the tetrahedron OABC is $v = \frac{1}{6}V = \frac{\left|\overrightarrow{OA}.(\overrightarrow{OB}\wedge\overrightarrow{OC})\right|}{6}$ cubic units of volume.

7. Cartesian equation of a plane:

A set of points is a plane if and only if it has an equation of the form $ux+vy+wz+r=0$ Where u, v, w are real's and all not equal to zero at the same time.

Example 1:

The equation of the plane pass through the point A (2; -2; 3) and of director vectors $\vec{u}(-1;2;5)$ *and* $\vec{v}(-1;3;3)$ is:

$$\begin{vmatrix} x-2 & y+2 & z-3 \\ -1 & 2 & 5 \\ -1 & 3 & 3 \end{vmatrix} = 0 \Leftrightarrow (x-2)(7x)-(y+2)(2)+(z-3)(1)=0 \Leftrightarrow -7x-2y+z+7=0.$$

Example 2

The equation of the plane passes through the points A (1; 3; 5), B (-1; 4; 8) and C (2; -3; 1) is:

$$\begin{vmatrix} x-1 & y-3 & z-5 \\ -2 & 1 & 3 \\ 1 & -6 & -4 \end{vmatrix} = 0 \Leftrightarrow (x-1)(14)-(y-3)(5)+(z-5)(11)=0 \Leftrightarrow 14x-5y+11z-54=0.$$

8. System of parametric equations of a straight line:

A set of points is a straight line if and only if it has a system of parametric equations of the form: $\begin{cases} x=at+x_0 \\ y=bt+y_0 \\ z=ct+z_0 \end{cases}$ where a, b, c are non – zero real numbers.

Example:

A system of parametric equations of the straight line passes through the points A (2; -3; 6) and B (-1; 4; 7) is given by: $\begin{cases} x=-3t-1 \\ y=7t+4 \\ z=t+7 \end{cases} \quad (t\in IR).$

9. Relative positions of two planes:

Let (P) and (P') be two planes of respective equations: u x + v y + w z + r = 0 and $u'x+v'y+w'z+r'=0$. $\vec{n}(u;v;w)$ is a normal vector to (P) and $\vec{n'}(u';v';w')$ is a normal vector to (P'). The possible cases are:

a- The two planes are confounded if we have: $u=tu';\ v=tv';\ w=tw'$ *and* $r=tr'$. Where t is a non – zero real number.

b- The two planes are parallels if we have: $u=tu';v=tv';\ w=tw'$ *and* $r\neq tr'$.

c- The two planes are intersecting along a straight line (d) if we have the vectors $\vec{n}$ *and* $\vec{n'}$ are not collinear. $\left(\frac{u}{u'}\neq\frac{v}{v'}\neq\frac{w}{w'}\right)$.

d- The two planes are perpendiculars if we have: $\vec{n}.\vec{n'}=uu'+vv'+ww'=0$.

10. Relative positions of a straight line and a plane:

Let (P) be a plane of equation: u x + v y + w z + r = 0 and (d) a line of parametric equations: x = a t + x $_0$; y = b t + y_0; z = c t + z_0.

$\vec{n}(u;v;w)$ is a normal vector to (P) and $\vec{m}(a;b;c)$ is a direction vector for (d).

The possible cases are the following:

a- (d) is lies in the plane (P) if and only if $\vec{n}.\vec{m}=au+bv+cw=0$, and if (P) and (d) have a common point.

b- (d) is parallel to the plane (P) if and only if $\vec{n}.\vec{m} = au + bv + cw = 0$, and (P) and (d) have no point in common.

c- (d) cuts (P) in a unique point if and only if $\vec{n}.\vec{m} = au + bv + cw \neq 0$.

d- (d) is perpendicular to the plane (P) $\Leftrightarrow \vec{n} = t\vec{m}$, ($\vec{n}$ and $\vec{m}$ are collinear).

11. <u>Relative position of two lines:</u>

Let (d) and (d') be two lines with respective parametric equations:

(d) : $x = a\,t + x_0$, $y = b\,t + y_0$; $z = c\,t + z_0$ and $(d'): x = a'n + x'_0;\ y = b'n + y'_0;\ z = c'n + z'_0$.

$\vec{v}(a;b;c)$ and $\vec{v'}(a';b';c')$ are direction vectors for (d) and (d') respectively.

The possible cases are that:

a- The two lines are either parallel or coinciding if and only if there exists a non - zero real number t such that: $a = ta';\ b = tb';\ c = tc'$.

b- The two lines are coplanar then they are intersecting in a point if and only if the vectors $\vec{v}$ and $\vec{v'}$ are not collinear.

c- (d) and (d') are not coplanar.

d- The two lines are orthogonal $\Leftrightarrow \vec{v} \perp \vec{v'}$.

12. <u>Distance from a point to a plane:</u>

Let (P) be a plane of equation: $u\,x + v\,y + w\,z + r = 0$ and A (x_0; y_0; z_0) be a point:

The distance from A to (P) is given by the formula: $d = \dfrac{|ux_0 + vy_0 + wz_0 + r|}{\sqrt{u^2 + v^2 + w^2}}$.

13. <u>Distance from a point to a line:</u>

Let (d_1) be a line with parametric equations: $x = a\,t + x_0$; $y = b\,t + y_0$; $z = c\,t + z_0$ and let A (x_1; y_1; z_1) be any point not belongs to(d_1).

The distance from A to (d_1) is given by the formula: $d = \dfrac{\left\|\vec{v} \wedge \overrightarrow{AB}\right\|}{\left\|\vec{v}\right\|}$. Where B is a point of (d_1).

<u>Example:</u>

Calculate the distance from the point A (1; -1; 2) to the line (d): $x = 2\,t + 1$; $y = -\,t + 2$; $z = t + 4$.

$d(A;(d)) = \dfrac{\left\|\vec{v} \wedge \overrightarrow{AB}\right\|}{\left\|\vec{v}\right\|}$, Where B (1; 2; 4) $\in (d)$ for t = 0.

$$\vec{v} \wedge \overrightarrow{AB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -1 & 1 \\ 0 & 3 & 2 \end{vmatrix} = \vec{i}(-5) - \vec{j}(4) + \vec{k}(6). \text{Thus} \quad d(A;(d)) = \frac{\sqrt{25+16+36}}{\sqrt{4+1+1}} = \frac{\sqrt{77}}{\sqrt{6}}\ u.$$

14. <u>System of parametric equations of the line of intersection of two planes:</u>

Find a system of parametric equations of the line (d) intersection of two planes (P) and (P') with respective equations: $x - 2\,y + z + 1 = 0$ and $2\,x + y - 3z + 2 = 0$.

Let $z = t \Rightarrow \begin{cases} x - 2y = -t - 1 \\ 2x + y = 3t - 2 \end{cases} \Leftrightarrow \begin{cases} x - 2y = -t - 1 \\ 4x + 2y = 6t - 4 \end{cases} \Leftrightarrow \begin{cases} 5x = 5t - 5 \\ y = t \end{cases} \Leftrightarrow \begin{cases} x = t - 1 \\ y = t \\ z = t \end{cases} \quad (t \in IR)$.

15. <u>Point of intersection of a plane and a line:</u>

Find the coordinates of point A the intersection of the line (d): $\{x = 2m - 3; y = m + 1; z = -m$ and the plane (P) of equation: $x + y - z - 2 = 0$.

We replaced x, y and z of the line (d) in the equation of the plane (P) then we have:

$2\,m - 3 + m + 1 + m - 2 = 0 \Leftrightarrow 4m = 4 \Leftrightarrow m = 1$, Thus A (-1, 2, -1).

LINES AND PLANES IN THE SPACE

Exercise 1

In the space referred to a direct orthonormal system $(O,\vec{i},\vec{j},\vec{k})$, consider the points $A(1;-1;1)$ and $B(0;1;-1)$, the planes $(P): 2x+3y+z=0$ and $(Q): x+y+2z+4=0$ and the straight lines

$$(d):\begin{cases} x=3t \\ y=-t+1 \\ z=2t-1 \end{cases} \text{ and } (d'):\begin{cases} x=2m+3 \\ y=m \\ z=3m+1 \end{cases}.$$

In the table given below, only one among the proposed answers to each question is correct. Write down the number of each question and give, with justification, the answer corresponding to it.

N°	Questions	Answers		
		A	B	C
1	The equation of the mediator plane of the segment $[AB]$ is $x-2y+(m+2)z-2n=0$ when:	$\begin{cases} m=0 \\ n=1/4 \end{cases}$	$\begin{cases} m=-3 \\ n=1 \end{cases}$	$\begin{cases} m=3 \\ n=-1 \end{cases}$
2	$\frac{5\sqrt{14}}{14}$ is the distance of the point A to……	(d)	(Q)	(P)
3	A direction vector of the line of intersection of the two planes (P) and (Q) is:	$\vec{V}(5;-3;-1)$	$\vec{V}(2;3;2)$	$\vec{V}(1;2;-1)$
4	A system of parametric equations of one of the bisectors of the angles formed by (d) and (d') is:	$\begin{cases} x=k+3 \\ y=0 \\ z=k+1 \end{cases}$	$\begin{cases} x=5k+1 \\ y=-1 \\ z=5k+1 \end{cases}$	$\begin{cases} x=-k+3 \\ y=-k \\ z=k+1 \end{cases}$

Exercise 2

In the space referred to a direct orthonormal system $(O,\vec{i},\vec{j},\vec{k})$, consider the points $A(-2;-1;3)$, $B(-1;1;0)$, $C(0;2;2)$ et $D(1;0;1)$ and the straight line (d) of system of parametric equations $\begin{cases} x=-t+1 \\ y=-t+2 \\ z=t \end{cases}$

1) a. Verify that a Cartesian equation of the plane (BCD) is: $x+y-z=0$
   b. Prove that the points A, B, C and D formed a tetrahedron then calculate its volume.
2) a. Verify that (d) passes through A and that (d) is perpendicular to the plane (BCD).
   b. Determine the coordinates of the point H, the orthogonal projection of A on the plane (BCD).
3) a. Find a normal vector of the plan (ACD).
   b. H' is the orthogonal projection of B on the plane (ACD). Show that (d) and (BH') are secants
   c. Calculate $\overrightarrow{BH}.\overrightarrow{CD}$ and prove that H is the orthocenter of the triangle BCD.
4) Let M be a variable point of (d). Determine t so that the volume of the tetrahedron MBCD is double of that the tetrahedron ABCD.
5) Find an equation of plane (Q) the symmetry of the plane (ABC) with respect to the plane (BCD).

Exercise 3

In the space referred to a direct orthonormal system $(O,\vec{i},\vec{j},\vec{k})$ consider the straight line

$$(D)\begin{cases} 2x-3y+z-1=0 \\ x-y+5=0 \end{cases}$$ and the point $A(1;-1;0)$.

1. a. Verify that a system of parametric equations of (D) is: $x=t-5; y=t$ and $z=t+11$

b. Find a system of parametric equations of the straight line (D') passes through A and parallel to (D').

c. Find the coordinates of point A' the symmetry of A with respect to (D)

2. Consider the plane $(P): 2x - y + z + 1 = 0$ and the point $B(-5;0;11) \in (D)$.

   a. Find the coordinates of point J the intersection of (D) and (P).

   b. Verify that $H(\frac{-17}{3};\frac{1}{3};\frac{32}{3})$ is the orthogonal projection of B on (P).

3. Find an equation of plane plan (Q) containing (D) and perpendicular to (P).

Exercise 4

In the space is referred to a direct orthonormal system $\left(O; \vec{i}, \vec{j}, \vec{k}\right)$, Consider the points A (2; -2; 0) and B (2; 0; 2) and the plane (P) of equation: 2 x – y + z – 6 = 0.

1) Verify that the points A and B belong to the plane (P), and that the triangle AOB is isosceles.
2) Find an equation of the plane (AOB) and prove that the planes (P) and (AOB) are perpendicular.
3) Calculate the distance from O to the plane (P).
   Deduce that the circle of center O, radius $\sqrt{6}$, and lying in the plane (AOB) is tangent to the straight line (AB), and calculate the coordinates of the point of tangency E.
4) Let (D) be the straight line of parametric equations: $\begin{cases} x = 3 \\ y = 2t + 2 \\ z = 2t + 2 \end{cases} \quad (t \in IR).$
   a- Show that (D) is parallel to (AB) and that (D) is lying in (P).
   b- F is any point on the straight line (D).
   Prove that the volume of the tetrahedron FOAB is constant.

Exercise 5

In the space is referred to a direct orthonormal system $\left(O; \vec{i}, \vec{j}, \vec{k}\right)$, consider the two straight lines (d) and (d') of respective parametric equations:

$$(d): \begin{cases} x = -t + 1 \\ y = 5t + 1 \\ z = 4t - 2 \end{cases} (t \in IR) \quad and \quad (d'): \begin{cases} x = 4m - 3 \\ y = m \\ z = 5m - 7 \end{cases} \quad (m \in IR).$$

1. a – Verify that (d) and (d') intersect at point A (1; 1; - 2).
   b- Find an equation of plane (P) determined by (d) and (d').
   c- Verify that the point B (2; 3; 1) belongs to (P).
2. Let (Q) be a plane of equation: x + 2 y + 3 z – 11 = 0.
   a- Show that (P) and (Q) are perpendicular.
   b- Designate by (D) the line of intersection of (P) and (Q).
   Without finding the system of parametric equations of (D), determine the coordinates of points E and F intersection of (D) with (d) and (d') respectively.
   c- Show that the triangle AEF is equilateral. Deduce that [AB) is the bisector of angle $E\hat{A}F$.

Exercise 6

In the space referred to a direct orthonormal system $\left(O;\vec{i}, \vec{j}, \vec{k}\right)$, consider the points A (1; -1; 2); B (-1; 1; - 2); C (3; -2; 1) and M (t – 1; t + 2; 1). (t is a real parameter).

1. Determine t so that the point M is on the plane (ABC).
2. Let E (0;1;1) be a point in the space.
   a- Calculate the area of the triangle ABE.

b- Calculate the volume of the tetrahedron ABEC, then deduce the length of the height from the point C to the plane (ABE).

3. Let N (x; y; z) be a variable point in the space.
   Find a relation between x, y and z in each of the following cases:
   a- N in on the plane (ABC).
   b- N is on the mediator plane of the segment [EC].
   c- N is on the straight line (AB).
4. Let R (α; β; γ) be a point in the space.
   Determine the coordinates of the points R so that the straight line (AR) is perpendicular to the plane (ABC).

Exercise 7

In the space is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, Consider two straight lines (d) and (d') with respective parametric equations:

$$(d):\begin{cases}x=2m+1\\y=-m+3\\z=m-4\end{cases}\quad(m\in IR)\ \text{and}\ (d'):\begin{cases}x=t+4\\y=2t+4\\z=-t-4\end{cases}\quad(t\in IR).$$

1. Show that the two straight lines (d) and (d') are intersects at the point A (3; 2; - 3)
2. Find the Cartesian equation of the plane (P) determined by (d) and (d').
3. Consider the plane (Q) with Cartesian equation: x – 2 y – 4 z – 6 = 0.

   Find a system of parametric equations of the straight line (D) the intersection of the two planes (P) and (Q).
4. Let I (1; 3; - 4) be a point on the straight line (d).

   a- Determine the coordinates of the point J on the straight line (d') such that AI = AJ.

   b- Deduce a system of parametric equations of the bisector of the angle $I\hat{A}J$.

Exercise 8

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, consider the points:

A (-1; 0; 6), B (4; 2; 5) and C (2; - 6; 9).

1. Verify that ABC is right triangle.
2. Let (C) be the circumscribed circle of the triangle ABC and (P) the plane of the circle.
   a- Precise the center I and the radius R of the circle (C).
   b- Verify that: y + 2 z – 12 = 0 is the equation of the plane (P).
3. Let (T) be the tangent to (C) at the point A in the plane (P).
   a- Verify that $\vec{W}(5;8;-4)$ is a director vector of the straight line (T).
   b- Determine a system of parametric equations of the straight line (T).
4. Let (Q) be the plane passing through B and perpendicular to the straight line (BC).
   In the plane (P) the tangent (T') to (C) at the point B cuts (T) in S.
   a- Determine an equation of the plan (Q).
   b- Deduce the coordinates of the point S.
   c- Determine a system of parametric equations of the straight line (T').

Exercise 9

The space is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$.

Given the points A (-1; 2; 1), B (1; -6; -1), C (2; 2; 2) and I (5; 1; -1).

1. Verify that A, B and C determine a plane (P) of equation: x + y – 3z + 2 = 0.
2. a- Determine the coordinates of the point H orthogonal projection of I on (P).
   b- Determine the coordinates of the point S the symmetry of the point I with respect to (P).

c- Deduce the area of the triangle IAS.

3. Let (d) be a straight line of equation: $x = m+2;\ y = 2m-5;\ z = m-4\ (m \in IR)$ parallel to (P).
   a- Determine an equation of the plane (Q) containing (d) and perpendicular to (P).
   b- Let E (4; -1; -2) be a point of (d). Find the coordinates of the point F orthogonal projection of E on (P).
   c- Deduce a system of parametric equations of the line (D) intersection of (P) and (Q).
4. Consider in the plane (P) the circle (C) with center J (1; -6; -1).
   Write a system of parametric equations of the straight line (d_1) tangent to (C) at the point T (5; -1; 2).

Exercise 10

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, given the plane (P): x +2y −z +1=0; the straight line (d): $x = t-2;\ y = -t+1;\ z = 2t+3,\ (t \in IR)$ and the point A (-1; 0; 2).

Let (C) be a circle in the plane (P) of center I (1; 1; 4) and of radius R=6.

1. a. Prove that the equation of plane (Q) that passes through A and contains (d) is: x + y +1=0.
   b. Determine the system of parametric equations of the straight line (D) intersection of two planes (P) and (Q).
   c. Prove that the distance d from the point I to the straight line (D) is: d= $3\sqrt{2}$.
   d. Deduce that (C) cuts (Q) in two points E and F.
   e. Calculate the area of triangle IEF.
2. Let (R) be the mediator plane of the segment [EF].
   a. Verify that (R) has an equation: x – y – z + 4 =0.
   b. Deduce the coordinates of point H midpoint of segment [EF].
3. Let (Δ) be a straight line passes through I and perpendicular to (P).
   a. Write a system of parametric equations of (Δ).
   b. J is a point of (Q) where I the orthogonal projection of J on (P).
      Calculate the coordinates of J.
   c. Calculate the volume of the tetrahedron JEIF.

Exercise 11

In the space referred to a direct orthonormal system $\left(O;\ \vec{i},\ \vec{j},\ \vec{k}\right)$, consider the points A (-1 ; 2 ; 1) and B (1; 1; 0) and the two straight lines (d) and (d') defined by their parametric equations:

$$(d):\begin{cases} x = t+3 \\ y = -t \\ z = 2t-1 \end{cases} \quad (t \in IR) \quad and \quad (d'):\begin{cases} x = m \\ y = -m+2 \\ z = -m+1 \end{cases} \quad (m \in IR).$$

1. Prove that (d) and (d') are orthogonal and not coplanar.
2. Let (P) be the plane determined by A and (d). Prove that the equation of the plane (P) is:
   3 x + 5 y + z – 8 = 0.
3. Prove that (d') cuts (P) in B.
4. Let C (2; 1; - 3) be a point of (d).
   a- Write an equation of the plane (Q) passing through A and perpendicular to (BC).
   b- Deduce a system of parametric equations of the height issued of A to [BC] in ABC.
   c- Let H be the point of intersection of the plane (Q) and the straight line (BC).
      What does the point H represent for the point A? Deduce the distance from the point A to the straight line (BC).
5. The circle (C) in the plane (P) with center A and of radius $2\sqrt{6}$ cuts (d) in two points.
   Verify that E (3; 0; - 1) is one of common points to (d) and (C) and find the other point F.
6. Find a system of parametric equations of the straight line (D) passing through the point

K (-4; -3; 0), parallel to (P) and orthogonal to (d).

7. Prove that: $x - y + 2z + 1 = 0$ is an equation of the mediator plane of the segment [EF].

Exercise 12

In the space referred to a direct orthonormal system (O, $\vec{i}, \vec{j}, \vec{k}$), consider the points A (–1; 2; 1), B(1 ; 0 ; 3) and the plane (P) of equation: $x - y - 2z - 1 = 0$.

1. Show that the straight line (AB) is parallel to (P).
2. a- Prove that A'(0 ; 1; –1) is the orthogonal projection of A on (P).

   Deduce the coordinates of the point B', the orthogonal projection of B on (P).

   b- Find a system of parametric equations of the straight line (d) the symmetry of the straight line (A'B) with respect to the plane (P).
3. Find an equation of the mediator plane (Q) of the segment [AB].
4. Deduce that the system of parametric equations of the perpendicular bisector (δ) of [A'B'] in the plane (P) is: $\begin{cases} x = m+1 \\ y = m \\ z = 0 \end{cases}$.

Exercise 13

In the space referred to a direct orthonormal system (O, $\vec{i}, \vec{j}, \vec{k}$), consider the two planes (P) and (Q) of respective equations:(P): $x + y + z = 0$ and (Q): $3x + 2y + z - 12 = 0$,

And the straight line (D) with parametric equations: $\begin{cases} x = m+6 \\ y = -2m \\ z = m-6 \end{cases}$.

Let (C) be a circle in the plane (P) with center O and of radius 9.

1. Verify that the planes (P) and (Q) are intersecting along the line (D).
2. a- Show that the distance of O to the straight line (D) is equal to $6\sqrt{2}$.

   b- Deduce that (C) cuts (D) in two points A and B.

   c- Calculate the area of the triangle OAB.
3. Let (d) be a straight line passing through O and perpendicular to (P).

   a- Write a system of parametric equations of (d).

   b- Calculate the coordinates of the point J of (Q) such that the orthogonal projection of J on the plane is the point O.

   c- Calculate the volume of the tetrahedron JAOB.

Exercise 14

In the space referred to a direct orthonormal system $\left(O; \vec{i}, \vec{j}, \vec{k}\right)$,

Consider the points A (+1; - 1; +2); B (-1; - 2; 2); C (2; 0; 3) and D (2m+1; m; 2m+2).

1. Determine m so that the points A, B, C and D are coplanar.
2. Find a normal vector of the plane (EBC), where E (1; 0; 2).
3. a - Calculate the area of the triangle ABC.

   b- Calculate the volume of the tetrahedron EABC.

   c- Deduce the length of the height EH relative to the plane (ABC).
4. Let M (x; y; z) be a variable point in the space.

   Find a relation between x, y and z in each of the following cases:

a- M is on the mediator plane of the segment [AB].

b- M is on the plane (EBC).

<u>Exercise 15</u>

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, consider the plane (P): y + z – 5 = 0 and the two straight lines: $(D_1):\begin{cases}x=t+4\\y=t+5\\z=t+6\end{cases}$; $(D_2):\begin{cases}x=m+2\\y=4m\\z=5\end{cases}$ $(t \text{ and } m \in IR)$.

1) Justify that (D 1) and (D 2) are intersecting at the point A (3 ;4 ;5) and that A $\notin$ (P).
2) Consider the three points B (1 ;2 ;3), C (2 ;0 ;5) and D (2 ;4 ;1).
   a- Verify that D is a point of (P) and that $\begin{cases}x=\alpha+4\\y=4\\z=4\alpha+9\end{cases}$ is the parametric equations of the straight line (AD).
   b- Prove that {B} = $(D_1)\cap(P)$ and $\{C\}=(D_2)\cap(P)$.
3) Let I be the midpoint of the segment [CD] and (Q) be the mediator plane of the segment [CD].
   a- Calculate the scalar products $\overrightarrow{BC}.\overrightarrow{CD}$ and $\overrightarrow{AI}.\overrightarrow{CD}$.
   b- Deduce that (Q) = (ABI) and that (P) and (Q) are perpendiculars.
4) Consider the point H (3 ;2 ;3).
   a- Prove that H is the orthogonal projection of A on (P).
   b- Prove that BCHD is a Rhombus. Deduce that H is a point of (Q).
5) In the plane (P), consider the circle (γ) with diameter [CD].
   a. The tangent (T) at C to (γ) cuts (DH) in E. Calculate the coordinates of E.
   b. Let M be a variable point on (γ) distinct of C and D, suppose $\widehat{CIM}=\alpha$.
   Calculate in terms of α, the volume V of the tetrahedron AIMC.
   Deduce the two positions of M on (γ) when V is maximum.

INVERSE FUNCTIONS
CHAPTER REVIEW

I CONTINUOUS FUNCTION:

Let f be a continuous function on an interval I.

1. If is strictly monotone; then
   a) f is a bijection of I onto f(I).
   b) f (I) is an interval of the same nature as I.
2. If I is a closed interval [a; b]; then
   a) f(I) is a closed interval f(I) = [m; M];
   b) f is bounded by m and M;
   c) f attends their bounds;
   d) theorem of the intermediate values:
      f attends all the values included between m and M.
3. If f (a) x f (b) ≤ 0 , then equation f(x) = 0 admits at least one root in the interval [a; b].

II COMPOSITE FUNCTIONS :

1. $g \circ f : x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x))$.
2. If f and g are continuous functions, then $g \circ f$ is continuous.
3. If f and g are differentiable then $g \circ f$ is differentiable and we have

$$(g \circ f)' = g'(f(x)) \times f'(x).$$

III INVERSE FUNCTIONS

1. If f : I $\to J$ is continuous and strictly monotone function, the nit has an inverse function $f^{-1} : J \to I$.
2. f^{-1} (When it exists) is characterized by x= $f^{-1}(y), with\ y \in f(I)$ if and only if y=f(x), with x$\in I$.
3. Determination of the inverse function of the function f:
   - Verified if the function f admits an inverse function:
   - Determination of the domain of definition J of f^{-1}, J = f (I);
   - Determination if possible, the expression of f^{-1}, x in terms of y
4. If f is strictly increasing, then f^{-1} is strictly increasing.
5. If f if strictly decreasing, then f^{-1} is strictly decreasing.
6. If f is differentiable at x_0, then f^{-1} is differentiable at y_0 = f (x_0) and

$$\left(f^{-1}\right)'(y_0) = \frac{1}{f'(x_0)}.$$

IV CURVE OF THE INVERSE FUNCTION

In an orthonormal system the curve of f^{-1} is the symmetric of the curve of the function f with respect to the first bisector (d) of equation y = x.

INVERSE FUNCTIONS

Exercise 1

Consider the function f that is defined over $[2;+\infty[$ by: $f(x)=\dfrac{2}{\sqrt{2x-4}+2}$,

Let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Calculate $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote to (C).
2. Calculate f (2); f (4) and f (10).
3. Prove that: $f'(x)=\dfrac{-2}{\left(\sqrt{2x-4}\right)\left(\sqrt{2x-4}+2\right)^2}$.
4. Set up the table of variation of the function f.
5. Write an equation of the tangent line (T) to the curve (C) at the point A with abscissa 4.
6. Draw (T) and (C).
7.
   a- Prove that f has over the interval [2; +∞ [an inverse function f^{-1}.
   b- Determine the domain of definition of f^{-1}.
   c- Draw (C') the representative curve of f^{-1} in the same system as that of the curve (C).
   d- Calculate $\left(f^{-1}\right)'\left(\dfrac{1}{2}\right)$, then deduce the equation of the tangent (T') to the curve (C') at a point A' with abscissa $\dfrac{1}{2}$.
   e- Prove that: $f^{-1}(x)=\dfrac{4x^2-4x+2}{x^2}$.
8. Consider the function g that is defined over [1; +∞[by: $g(x)=\dfrac{x-1}{x+1}$.
   a- Prove that h = g o f exists.
   b- Precise the domain of definition of h.
   c- Calculate h '(4).

Exercise 2

Consider the function f defined over]1; +∞[by: $f(x)=\dfrac{2x}{(x-1)^2}$, Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Determine $\lim\limits_{x\to+1^+} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$. Deduce the asymptotes to (C).
2. Calculate $f'(x)$ and set up the table of variation of f over]1; +∞[.
3. Determine f ([2 ;5]).
4. Prove that the equation f (x) = 2 admits a unique solution unique α in the interval]2.6; 2.7[.
5. Let A be a point of (C) with abscissa 2.
   Find the equation of the tangent (T) to (C) at A.
6. Prove that f admits over]1; +∞ [an inverse function f^{-1} whose domain of definition is to be determined.
7. Designate by (C') the representative curve of f^{-1}.
   Without finding the expression of f^{-1}, calculate $\left(f^{-1}\right)'(4)$.
8. Let A' be a point of (C') with abscissa 4.
   Deduce the equation of the tangent (T') to (C') at A'.
9. Draw the curve (C) and the curve (C') in the same system. Determine $f^{-1}(x)$.

Exercise 3

Let f be a function that is defined over]0; +∞ [by: $f(x)=x+1-\frac{1}{x}$.

Designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1) Calculate $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to +\infty} f(x)$.
2) Show that the line (d) of equation y = x +1 is an asymptote to (C).
3) Calculate $f'(x)$, then set up the table of variations of f.
4) Draw (d) and (C).
5)
   a- Prove that f admits an inverse function f^{-1} over]0; +∞ [, and determine the domain of definition of f^{-1}.
   b- Find the coordinates of point of intersection of two curves (C) and (C') where (C') is the representative curve of the function f^{-1}.
   c- Write an equation of the tangent (T ') to (C') at a point of abscissa 1.
   d- Draw (C').
   e- Determine $f^{-1}(x)$.

Exercise 4

Consider the function f that is defined over [-1; + ∞[as: $f(x)=1+\sqrt{x+1}$. Designate by (C) its representative curve in an orthonormal system (O; $\vec{i}$, $\vec{j}$).

1. Study the sense of variations of f over [-1; + ∞[, and then draw (C).

2. a- Prove that f admits over [-1; + ∞[an inverse function f^{-1}.

   b- Precise the domain of definition of f^{-1}.
   c- Draw (C') the representative curve of the function f^{-1}.
   d- Find the expression of $f^{-1}(x)$.

3. Set up the table of variations of the function g defined on $[-1;+\infty[$ by: $g(x)=\frac{1}{f(x)}$.

Exercise 5

In the plane referred to a direct orthonormal system $(O;\vec{i},\vec{j})$ of unit 2 cm, designate by (C) the curve of the function f that is defined as: $f(x)=\frac{1}{\sqrt{x^2+4}}$

1. a- Justify that f is defined over IR.
   b- Calculate $\lim\limits_{x\to -\infty} f(x)$ *and* $\lim\limits_{x\to +\infty} f(x)$. Deduce an asymptote to the curve (C).

2. a- Verify that: $f'(x)=\frac{-x}{\left(x^2+4\right)\sqrt{x^2+4}}$

   b- Set up the table of variations of the function f. Draw (C).
3. Let f^{-1} be the inverse function of the function f on $[0;+\infty[$ and (C') its curve.
   a- Justify the existence of the function f^{-1}.
   b- Set up the table of variations of f^{-1}.
   c- Draw (C') in the same system as (C).

4. Let B be a point of (C) of abscissa 2 and B' the point of (C') of abscissa $\frac{\sqrt{2}}{4}$.

   (T) and (T') are the tangents at B and at B' to (C) and (C') respectively.
   a- Determine the slope of the tangent (T). Deduce the slope of the tangent (T').

b- Verify that (T) and (T') are intersects at a point I.

c- Find a relation between the coordinates of the point I.

Exercise 6

The plane is referred to an orthonormal system (O; $\vec{i}$; $\vec{j}$) Graphic unit: 2 cm.

Let (C) be the curve of the function f defined on IR – {0} by: $f(x) = x - 1 + \dfrac{4}{x}$

1. a- Determine: $\lim\limits_{x \to 0^-} f(x)$ and $\lim\limits_{x \to 0^+} f(x)$. Deduce an asymptote to (C).

   b- Determine: $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$

   c- Verify that the straight line (D) of equation: y = x – 1 is an asymptote to (C).
2. Let f' be the derived function of the function f.

   a- Prove that: $f'(x) = \dfrac{(x-2)(x+2)}{x^2}$

   b- Set up the table of variation of f. Draw (C).
3. Prove that the point W (0; -1) is the center of symmetry of (C).
4. a- Say why the function f^{-1} exists on the interval $[2;+\infty[$?

   b- Set up the tableau of variation of the function f^{-1}.

   c- (C') cuts (C) at L. calculate the coordinates of the point L.

   d- Write an equation of the tangent (T') at the point L to (C').

   e- Draw (C') in the same system as (C).

Exercise 7

The table below is the table variations of a function f.

x	0		1		$+\infty$
$f'(x)$		+	0	-	
$f(x)$	$-\infty$	↗	-2	↘	-3

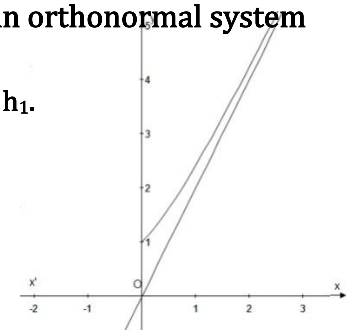
1) What is the domain of definition of the function f.
2) Determine $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$. Interpret graphically these results.
3) Prove that f has an inverse function over the interval] 0; 1].
4) a -Draw the curve (C) of the function f.

   b- Set up the table of variations of f^{-1} and draw (C') the representative curve of the function f^{-1} in the same system as that of (C).

Exercise 8

Consider the function f defined over I = [0; +∞[by: $f(x) = x + \sqrt{x^2+1}$, and the function g that is defined over J = [1; +∞[by: $g(x) = \dfrac{x+1}{2x-1}$.

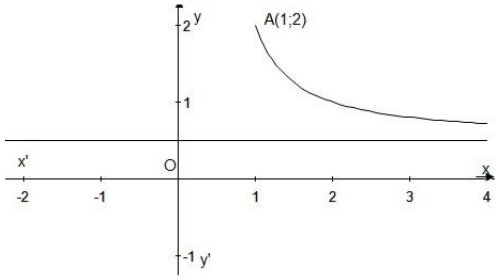
Designate by (C_1) and (C_2) the representative curves of f and g in the an orthonormal system (O; $\vec{i}$, $\vec{j}$).



1. Prove that: $h_1 = g \circ f$ exist, and then find the domain of definition of h_1.
2. Find the expression of $h_1(x)$.
3. Calculate $h_1'(1)$, without using the expression of $h_1(x)$.
4. Does the function $h_2 = f \circ g$ exist? Justify.
5. Prove that f has an inverse function f^{-1} over I.
6. Precise the domain of definition of f^{-1}.
7. Find the expression of the function $f^{-1}(x)$.

8. Calculate $\left(f \circ f^{-1}\right)(x)$.

9. Calculate f (1), and then deduce $\left(f^{-1}\right)'\left(1+\sqrt{2}\right)$.



10) The adjacent curve is the curve of the function f over I.
Draw (C'$_1$) the representative curve of the functionf^{-1}.

11)
a- Prove that g has an inverse function g^{-1}.
Precise the domain of definition of g^{-1}, and then find the expression of $g^{-1}(x)$
b- The below curve is the representative curve of the function (C$_2$)
In the interval J. Draw (C'$_2$) the representative curve of g^{-1}

Exercise 9

Consider the function f that is defined over [0; $+\infty$[by: f (x) = $\sqrt{x+1}-\sqrt{x}$, and let (C) be its representative curve in an orthonormal system (O ; $\vec{i}$, $\vec{j}$).

1. Calculate $f'(x)$, and prove that f is strictly deceasing over [0; $+\infty$[.
2. Prove that $\lim\limits_{x \to +\infty} f(x) = 0$.Deduce an asymptote to (C).
3. Set up the table of variations of the function f.
4. Prove that f has an inverse function $f^{-1}$and the find its domain of definition.
5. Prove that: $f^{-1}(x) = \dfrac{\left(1-x^2\right)^2}{4x^2}$.

Exercise 10

Consider the function f that is defined over I = [0; + ∞[as: $f(x) = 1+\sqrt{x^2+4}$.
Designate by (C) its representative curve in an orthonormal system (O; $\vec{i}$, $\vec{j}$)

1. Calculate f (0) and $\lim\limits_{x\to+\infty} f(x)$.
2. Prove that the straight line (d) of equation: y = x +1 is an asymptote to (C).
3. Calculate $f'(x)$, and then set up the table of variations of f.
4. Draw (d) and (C).
5. a- Prove that f has an inverse function f^{-1} over I.
b- Find the domain of definition of f^{-1}, and then find the expression of $f^{-1}(x)$.
c- Draw (C') the representative curve of the function f^{-1} in the same system as that of (C).
6. Prove that the equation $f(x) = f^{-1}(x)$ has no solution on I.

Exercise 11

Consider the functions f and g defined respectively on the intervals I = [0; 2] and J = [-4; 0] by:

$$f(x) = x^2 - 4x \quad ; \quad g(x) = 2x - \sqrt{x^2+9}.$$

1. Prove that the composite function h$_1$ of f and g exist.$\left(h_1 = g \circ f\right)$.
Precise the domain of definition of the function h$_1$. Then find the expression of the function h$_1$(x). Calculate $h_1'(x)$ without using the expression of the function h$_1$(x).
2. Does the function $h_2 = f \circ g$ the composite function of g and f exists? Justify your answer.
3. a) Prove that f has an inverse function f^{-1}on I.
b) Determine the domain of definition of f^{-1}.
c) Find the expression of the function $f^{-1}(x)$.
d) Draw (C) and (C') the respective curves of the functions f and f^{-1}in an orthonormal system,

NATUREL LOGARITHM
CHAPTER REVIEW

I-Napery's logarithmic function

The function f defined over the interval $]0;+\infty[$ by: $f(x)=\dfrac{1}{x}$ is differentiable and continuous. So it has a primitive over this interval.

1. Definition:

   The natural logarithm is the primitive over $]0;+\infty[$ of the function x $\rightarrow \dfrac{1}{x}$ that is vanishes for x = 1. For all real x strictly positive ln x is the natural logarithm of the real x.

2. Consequences:
   a) The function ln is defined over the interval $]0;+\infty[$.
   b) Ln 1 = 0.
   c) The function ln is differentiable over the interval $]0;+\infty[$ and for all real x > 0, $\left[\ln(x)\right]'=\dfrac{1}{x}$.

II Algebraic properties :

1. Fundamental properties:

   For all real a and b strictly positive: $\ln(a\times b)=\ln(a)+\ln(b)$.

   Proof:

   a is strictly positive real number, we consider the function f defined over $]0;+\infty[$ by:

   $f(x)=\ln(ax)-\ln(x)$. f is differentiable over $]0;+\infty[$ since the sum and the composite of the functions is differentiable.

   For all real x > 0, $f'(x)=a\times\dfrac{1}{ax}-\dfrac{1}{x}=\dfrac{1}{x}-\dfrac{1}{x}=0$.

   The derived function f is always null, so f is a constant function over $]0;+\infty[$,
   In particular f (1) = ln (a) – ln (1) = ln (a), So for all x > 0, f (x) = ln (a).
   Thus for all real x > 0, ln (a x) – ln (x)= ln (a).
   So finally for all real x > 0, ln (a x) = ln (a) + ln (x).

2. Other Rules of calculation

   For all real numbers a and b strictly positive and n is an integer:
   a) $\ln\left(\dfrac{1}{a}\right)=-\ln(a)$.
   b) $\ln\left(\dfrac{a}{b}\right)=\ln(a)-\ln(b)$.
   c) $\ln\left(a^n\right)=n\ln(a)$.
   d) $\ln\left(\sqrt{a}\right)=\dfrac{1}{2}\ln(a)$.

III Study of the function ln:

1. Variation

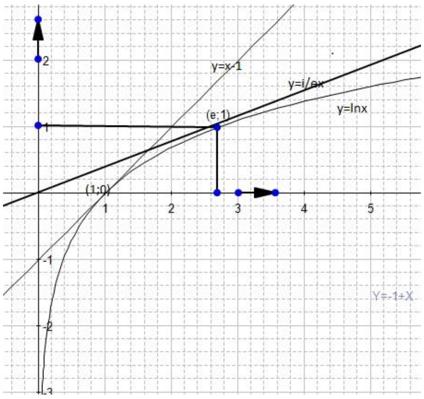
   The logarithmic function is continuous and strictly increasing over $]0;+\infty[$.

   Proprieties

   We deduce from this theorem the following properties:
   For all real a and b strictly positive:
   ln (a) = ln (b) if and only if a = b
   ln(a) > ln (b) if and only if, a > b.
   Since ln (1) = 0:
   For all real x strictly positive:

ln (x) = 0 if and only if x = 1, ln (x) >0 $\Leftrightarrow x>1, \ln(x)<0 \Leftrightarrow 0<x<1$.

2. Limits
   a) Study the limit at $+\infty$:
      The function ln a has a limit $+\infty$ *at* $+\infty$: $\lim\limits_{x\to+\infty} \ln(x) = +\infty$.
   b) Study the limit at 0:
      The function ln has a limit $-\infty$ at 0: $\lim\limits_{x\to 0} \ln(x) = -\infty$.
      The axis of ordinates is asymptote to the curve of the equation y = ln (x).
   c) The number e :
      The number e is the number so that: ln (e)=1.
      For all integer n, ln (e^n) = n ln (e) = n.
   d) Representative curve:
      Remarquable tangents:
      The tangent to the representative curve of the function ln at the point (1; 0) has an equation: y = x – 1
      The tangent to the representative curve of the function ln at the point (e; 1) has an equation: $y=\frac{1}{e}(x-e)+1 \Leftrightarrow y=\frac{1}{e}x$.



   e) Importants limits :
      $\lim\limits_{x\to+\infty} \frac{\ln(x)}{x} = 0$ *and* $\lim\limits_{x\to 0} x\ln(x) = 0$.
      For all real x > 0, ln (x) $\le$ x

IV Study of the function ln (u)

1. Definition:
   Let u be a function defined and strictly positive on an interval I.
   The composite of the function u and the function ln is denoted by ln (u).
2. Limits:
   For the study a limit of a function ln (u), we use the theorem over the limit of a composite function.
3. Derivative:
   Let u be a differentiable function and strictly positive over an interval I.
   The function ln (u) is differentiable over I and $(\ln(u))' = \frac{u'}{u}$.
4. Primitives:
   Let u be a differentiable function and strictly positive over an interval I.
   A primitive of $\frac{u'}{u}$ over I is ln (u).

NATURAL LOGARITHMES

Exercise 1

Simplify each of the following numbers:

A = ln (16) – 7 ln (2) + 4 ln (32) + 3 $\ln\left(\frac{1}{8}\right)$; B = 3 ln (125) – 2 ln (25) + 6 ln $\left(\frac{1}{5}\right)$.

C = $8\ln\left(\sqrt{3}\right)+5\ln(9)-\ln\left(9\sqrt{3}\right)$; D = $\dfrac{\frac{1}{3}\ln(9)-4\ln\sqrt{3}-\ln\left(\frac{1}{3}\right)}{\ln(3)}$.

E = $\dfrac{\ln\left(\sqrt{3}-1\right)+\ln\left(\sqrt{3}-1\right)}{2}$; F = $\dfrac{\ln(36)}{\ln(3)+\ln(2)}$.

G = $\ln\left(\sqrt{3}+\sqrt{2}\right)+\ln\left(\sqrt{3}-\sqrt{2}\right)$; H = $2\ln\sqrt{8}-3\ln\left(\frac{1}{4}\right)$.

I = $\dfrac{\ln(5)-\ln(10)}{2\ln\left(\sqrt{2}\right)}$; J = $\dfrac{\ln(12)-\ln(9)}{\ln\left(\frac{1}{27}\right)+\ln(64)}$.

Exercise 2

Solve the following equations:

1. ln (1- 2 x) = ln (x + 2) + ln (3).
2. ln (1 – x ^{2}) = ln (2 x – 1).
3. ln (x – 2) + ln (x + 3) = ln (3 x + 2).
4. $\ln\left(\sqrt{2x-2}\right)=\ln(4-x)-\frac{1}{2}\ln(x)$.
5. $\ln\left(\frac{x+2}{x}\right)=1$.
6. $2\left(\ln(x)\right)^2+3\ln(x)-2=0$.

Exercise 3

Solve the following in equations:

1. $\ln(x-2)\le\ln(2x+1)$.
2. $\ln(3x+2)\ge\ln\left(x^2+\frac{1}{4}\right)$.
3. $\ln\left(1+\frac{2}{x}\right)\le\ln(x)$.
4. $\ln\left(\frac{x}{x+1}\right)\le-1$.
5. $\ln(2x+1)-\ln(x-1)\le1$.
6. $\left(\ln(x)\right)^2-\ln(x)-6\le0$.

Exercise 4

Part A:

Consider the function g that is defined over] 0; + ∞ [as: $g(x)$ = x ^{2} – 1 + ln x.

1. Calculate $\lim\limits_{x\to0^+} g(x)$ and $\lim\limits_{x\to+\infty} g(x)$.
2. Calculate g '(x) and set up the table of variations of g.
3. Calculate g (1) then determine the sign of $g(x)$ over $]0;+\infty[$.

Part B:

Consider the function f that is defined over $]0;+\infty[$ by:

f(x) = $f(x)=-\frac{1}{2}x+1+\frac{\ln x}{2x}$, and designate by (C) its representative curve in an orthonormal system (O, $\vec{i},\vec{j}$).

1. Calculate $\lim\limits_{x\to0^+} f(x)$, $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote to (C).

2. Prove that the straight line (d) of equation $y = -\frac{1}{2}x+1$ is an asymptote to (C).

3. Study the relative position between (C) and (d).

4. Prove that : $f'(x) = \frac{-2g(x)}{4x^2}$.

5. Deduce the sign of $f'(x)$ and set up the table of variations of f.

6. Prove that the equation $f(x) = 0$ admits two solutions α and β such that:
   $\frac{1}{3} < \alpha < \frac{1}{2}$ *and* $2 < \beta < \frac{5}{2}$.

7. Draw (C) and (d). (graphic unit: 2cm).

8. Calculate, in cm ^{2}, the area of the region bounded by (C), (d) and the two straight lines of equations x = 1 and x = e.

9. Prove that f has over] 0; 1[an inverse function f^{-1} whose domain of definition is to be determined.

10. Draw (C') the representative curve of f^{-1}.

Exercise 5

Part A:

Let g be a function defined over]0; + ∞[by: $g(x) = 3 - \frac{2}{x} + \ln\left(\frac{x}{2}\right)$.

Designate by (γ) its representative curve in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$.

1. Calculate $\lim\limits_{x\to0+} g(x)$, $\lim\limits_{x\to+\infty} g(x)$, g (2) and g (4). Deduce an asymptote to (γ).
2. Calculate g '(x) and set up the table of variations of g.
3. Prove that the equation g (x) = 0 admits over]0; +∞[a unique root α. Verify that 0.9 < α < 0.91.
4. Write an equation of the tangent (T) to (γ) at the point A of abscissa 2.
5. Draw (T) and (γ).
6. Deduce the sign of g (x) for all real x > 0.
7. a – Let H be a function defined over]0 ; +∞[by H (x) = x ln $\left(\frac{x}{2}\right)$ - x.

   Prove that H is a primitive of the function h (x) = $\ln\left(\frac{x}{2}\right)$.

   b- Calculate the area A (α) of the region bounded by (γ), the x – axis and the straight line of equation x = 2. Prove that: A (α) = $\frac{(\alpha-2)^2}{\alpha}$.
8. Prove that g admits over]0; +∞ [an inverse function g $^{-1}$. Determine the domain of definition of g $^{-1}$ and draw its representative curve (γ') in the same system as that of (γ).

Part B:

Let f be a function defined over]0; +∞[by f (x) = $(x-2)\left(2+\ln\left(\frac{x}{2}\right)\right)$. Designate by (C) its representative curve in a new system.

1. Calculate $\lim\limits_{x\to0+} f(x)$, $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote to (C).

2. Determine the points of intersection of (C) and the x - axis.

3. Prove that: f (α) = $-\dfrac{(2-\alpha)^2}{\alpha}$.

4. Prove that: f '(x) = g (x) and set up the table of variations of f.

5. Draw (C). (Assume that: α = 0,91).

<u>Exercise 6</u>

<u>Part A:</u> Let g be a function defined over $]0;+\infty[$ by: $g(x)=\dfrac{1}{x^2}+1-4\ln(x)$. Designate by ($C_1$) the representative curve of g in an orthonormal system $(O;\vec{i},\vec{j})$ Graphic unit: 2 cm.

1. Calculate $\lim\limits_{x\to0^+} g(x)$ and $\lim\limits_{x\to+\infty} g(x)$. Deduce an asymptote to (C_1).
2. Calculate $g'(x)$, then set up the table of variations of g.
3. Prove that the equation g (x) = 0 admits a unique solution unique α such that: 1 < α < 2.
4. Draw (C_1), and then deduce the sign of g (x) for all x $\in]0;+\infty[$.
5. Calculate, in cm^2, the area of the region bounded by (C_1), the x – axis and the two straight lines of equations x = 2 and x = e.

<u>Part B:</u> Consider the function f defined over $]0;+\infty[$ by: $f(x)=\dfrac{\ln(x)}{(1+x^2)^2}$.

Let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ Graphic unit: 2 cm.

1. Calculate $\lim\limits_{x\to0^+} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$. Deduce the asymptotes to (C).
2. Prove that: $f'(x)=\dfrac{xg(x)}{(1+x^2)^3}$. Deduce the sense of variations of f.
3. Prove that: $f(\alpha)=\dfrac{1}{4\alpha^2(1+\alpha^2)}$. Solve the equation f (x) = 0.
4. Suppose that: α = 1.45 and $f(\alpha)=0.04$. Set up the table of variations of f.
5. Write an equation of the tangent (T) to (C) at the point A with abscissa 1.
6. Draw (T) and (C).

<u>Part C:</u>

1. Prove that f admits over $]0;\,1.45]$ an inverse function f^{-1} and precise its domain of definition.
2. Draw (C') the representative curve of f^{-1} in the same system as that of (C).

<u>Exercise 7</u> (G.S Class)

<u>Part A:</u>

Consider the function g that is defined over]0; +∞ [as: $g(x)=x^2+3-2\ln(x)$.

Designate by (G) the representative curve of g in an orthonormal system $(O;\vec{i},\vec{j})$, Graphic unit = 2 cm.

1. Calculate $\lim\limits_{x\to0^+} g(x)$. Deduce an asymptote to (G).
2. Calculate $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to+\infty} \dfrac{g(x)}{x}$. Interpret graphically the results.
3. Calculate $g'(x)$, then set up the table of variations of g.
4. Draw (G). Deduce the sign of g (x) for all x > 0.
5. a- Prove that the function H (x) = x ln (x) – x is a primitive of h (x) = ln(x).
   b- calculate the area of the region bounded by (G), the x – axis and the two straight lines x = 1 and x = e.

Part B:

Consider the function f that is defined over]0; +∞[as: $f(x)=x+1-\frac{1}{x}+2\frac{\ln(x)}{x}$.

Let (C) be its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$. G. U = 2 cm.

1. Calculate $\lim_{x\to 0^+} f(x)$. Deduce an asymptote to (C).
2. Calculate $\lim_{x\to +\infty} f(x)$, and then prove that the straight line (d) of equation y = x +1 is an asymptote to (C). Study the relative positions between the curve (C) and the straight line (d).
3. a- Prove that for all x ∈]0; +∞ [: $f'(x)=\frac{g(x)}{x^2}$.

   b- Set up the table of variation of f.

   c- Prove that the equation f (x) = 0 has a unique solution α such that: 0.8 < α < 0.9.
4. Determine the coordinates of point B on (C) where the tangent (T) to (C) at B is parallel to (d).
5. Draw (d), (T) and (C).
6. a- Prove that f has over]0; +∞ [an inverse function f^{-1}, and precise its domain of definition.

   b- Calculate f (1) and $\left(f^{-1}\right)'(1)$.

   c- Draw (C') in the same system as that of (C).

7) Calculate in terms of α the following integral: $\int_{\alpha}^{1} f(x)dx$. Deduce the area of the region bounded by (C), (C') and the two straight lines x = 0 and y = 0.

Exercise 8

Part A : Consider the function g that is defined on $]0;+\infty[$ by: $g(x)=x+(x-2)\ln(x)$.

1. a- Calculate $\lim_{x\to 0^+} g(x)$ *and* $\lim_{x\to +\infty} g(x)$.

   b- Prove that: $g'(x)=\frac{2x-2}{x}+\ln(x)$.

   c- Deduce that if x > 1 then $g'(x)>0$ and if 0 < x < 1 then $g'(x)<0$.
2. a- Study the variations of g and set up the table of variations of g.

   b- Deduce the sign of g(x) for all real x > 0.

Part B : Consider the function f that is defined on $]0;+\infty[$ by: $f(x)=1+x\ln(x)-\ln^2(x)$. Let (C) be its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$, $(G.U=2cm)$.

1. a- Calculate $\lim_{x\to 0^+} f(x)$. Deduce an asymptote to (C).

   b- Prove that $\lim_{x\to +\infty} f(x)=+\infty$, and then calculate f (2), f (e) and f (4).
2. a- Verify that: $f'(x)=\frac{g(x)}{x}$, and then study the sense of variations of f.

   b- Write an equation of the tangent (T) to (C) at a point A of abscissa 1.
3. Prove that the equation f (x) = 0 admits a unique solution α and verify that: 0.4 < α < 0.5.
4. Set up the table of variations of f and then draw (d), (T) and (C).
5. a- Prove that f admits an inverse function f^{-1} on the interval] 0;1].

   b- Determine the domain of definition of f^{-1}, and then draw (C') the representative curve of f^{-1} in the same system as (C).
6. a- Prove that the function H defined on $]0;+\infty[$ by: $H(x)=\frac{3x^2}{2}+x-\frac{1}{2}\left(x^2+4x\right)\ln(x)+x\ln^2(x)$ is a primitive of the function $h(x)=x-1-x\ln(x)+\ln^2(x)$.

b- Calculate, in cm², the area of the domain bounded by (d): y = x, (C) and the two straight lines of equations: x =1 and x=2.

Exercise 9

PART A : Consider the function g that is defined over $]0;+\infty[$ as: $g(x)=1-\frac{1}{x}+\ln(x)$.

Let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. G.U = 2 cm.

1. Calculate $\lim\limits_{x\to 0^+}[g(x)]$ *and* $\lim\limits_{x\to+\infty}[g(x)]$. Deduce an asymptote to the curve (C).
2. Calculate g (1), g (2) and g (e).
3. Calculate $g'(x)$, then study the sense of variations of function g.
4. Set up the table of variations of the function g.
5. Write an equation of the tangent line (T) to (C) at a point A of abscissa 1.
6. Draw (T) and (C).
7. a- Prove that g has an inverse function g^{-1} over the interval $]0;+\infty[$.

   b- Determine the domain of definition of the function g^{-1}.

   c- Draw (C') the representative curve of the function g^{-1} in the same system as (C).

PART B: Consider the function f that is defined over $]0;+\infty[$ as: $f(x)=-1+(x-1)\ln(x)$.

The below table is the table of variations of the function f over $]0;+\infty[$:

x	0		1		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	-1	$\nearrow$	$+\infty$

1. Prove that the equation f (x) = 0 has exactly two roots α and β such that: 0.2 < α < 0.3 and 2.2 < B < 2.3.
2. Designate by (E) the region bounded by the curve (C) of the function g, the x-axis and the two straight lines x = α and x = β. Let A be the area of the region (E).

   a- Prove that for all x $\in\,]0;+\infty[$ we have: $f'(x)=g(x)$.

   b- Prove that: $A=\int_1^{\alpha} g(x)dx+\int_1^{\beta} g(x)\,dx$.

   c- Deduce the value of A in terms of α and β.

Exercise 10

Part A: Consider the function g that is defined over]-2 ; +∞[by : g(x)= x²+4x+3+ln(x+2).

1. Calculate $\lim\limits_{x\to-2} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x)$.
2. Calculate $g'(x)$, and then prove that g is strictly increasing over]-2; +∞[.
3. Set up the table of variations of g.
4. Calculate g (-1), and then deduce the sign of g(x) for all x> -2.

Part B : Consider the function f that is defined over]-2 ; +∞[by : $f(x)=x-1-\frac{\ln(x+2)}{(x+2)}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Calculate $\lim\limits_{x\to-2} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$, deduce an asymptote to (C).
2. Prove that the straight line (d) of equation y=x-1 is an asymptote to (C).

3. Montrer que $f'(x)=\dfrac{g(x)}{(x+2)^2}$, puis étudier le sens de variations de f.
4. Write an equation of the tangent line (T) to (C) at a point A of abscissa 0.
5. Draw (d) and (C).
6. Calculate the area of the region limited by (C), (d) and the two straight lines of equations x =-1 and x=0.
7. a. Prove that f admits an inverse function f^{-1} over [-1; $+\infty$[.
   b. Precise the domain of definition of f^{-1}.
   c. Draw (C') the representative curve of f^{-1} in the same system of (C).

Exercise 11

Consider the function f that is defined over $]-\infty;-1[\cup]+1;+\infty[$ by: $f(x)=2x+1-3\ln\left(\dfrac{x-1}{x+1}\right)$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$, $G.U=2cm$.

1. Calculate $\lim\limits_{x\to-\infty} f(x)$, $\lim\limits_{x\to-1^-} f(x)$, $\lim\limits_{x\to+1^+} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$. Deduce the asymptotes of (C).
2. Prove that the straight line (d) of equation y =2x+1 is an asymptote to (C).
3. Calculate $f'(x)$, and then study the sense of variations of f.
4. Prove that the point E (0; 1) is a center of symmetry of (C).
5. Set up the table of variations of f.
6. Draw (d) and (C).
7. a. Prove that the function G defined over $]-\infty;-1[\cup]+1;+\infty[$ by:

   $G(x)=-3x\ln\left(\dfrac{x-1}{x+1}\right)+3\ln\left(x^2-1\right)$ is a primitive of the function $g(x)=-3\ln\left(\dfrac{x-1}{x+1}\right)$.

   b. Deduce, in cm^2, the area of the domain bounded by (C), (d) and the two straight lines of equations: x=2 and x=e.
8. a. Prove that f admits an inverse function f^{-1} over $[2;+\infty[$.

   b. Determine the domain of definition of f^{-1}.

   c. Draw (d') the symmetry of the straight line (d) with respect to the first bisector y =x and then draw (C') the representative curve of the function f^{-1} in the same system of (C).

Exercise 12

Part A:

Consider the representative curve (C), of a function g defined over $]0;+\infty[$, in the plane referred to an orthonormal system with graphic unit = 2 cm. This curve is represented on the document below. The points of intersection of (C) and the x-axis have the respective coordinates (1; 0) and (3; 0).

1. Let *a* and *b* are two real numbers such that, for all real $x\in]0;+\infty[$, $g(x)=\dfrac{x^2+ax+b}{x}$.

   By using the coordinates of the points of intersection of the curve (C) with the x-axis, determine a and b
2. Prove that g(*x*) can be written in the form: $g(x)=x-4+\dfrac{3}{x}$.

Part B:

Let h be the function defined on $]0;+\infty[$ by: $h(x)=x^2+1-2\ln x$.

1. Study the sense of variations of h and set up the table of variations of h.
2. Calculate h (1). Deduce that h (x) is strictly positive for all real number x in $]0;+\infty[$.

Part C:

Let f be the function defined on $]0;+\infty[$ as: $f(x)=x-4+\frac{1+2\ln x}{x}$ and denote by (C') its representative Curve In the system $(O;\vec{i},\vec{j})$.

1. a- Calculate $\lim\limits_{x\to0^+} f(x)$.

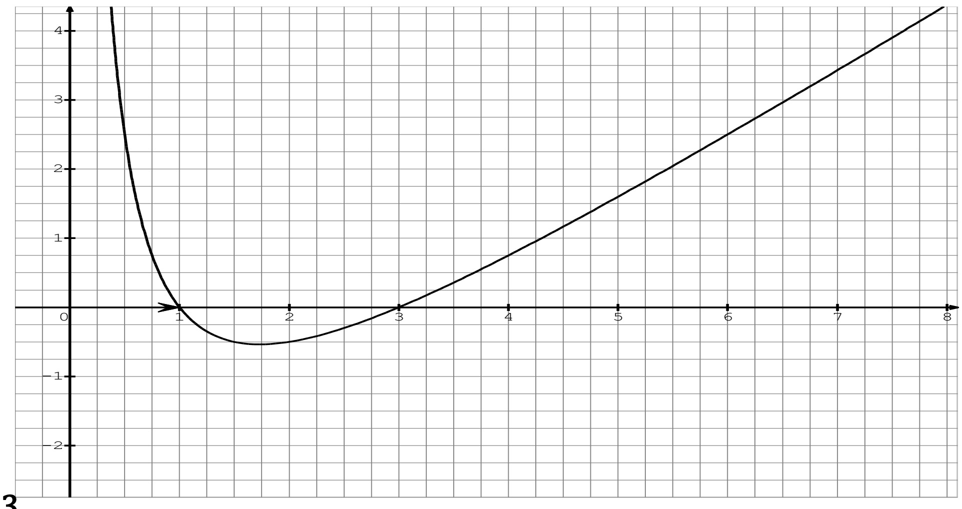
   b- Deduce that the curve (C') admits an asymptote is to be determined.
2. Calculate $\lim\limits_{x\to+\infty} f(x)$. Prove that the straight line (d) of equation y = x - 4 is an asymptote to (C').
3. For all x in the interval $]0;+\infty[$ prove that: $f'(x)=\frac{h(x)}{x^2}$, and then deduce the table of variations of f.
4. a- Calculate $\lim\limits_{x\to+\infty}\left[f(x)-g(x)\right]$, interpret graphically this result.

   b- Determine the coordinates of the point of intersection of the two curves (C') and (C).

   c- Study, according to the values of x, the relative positions of (C) and (C').
5. Draw (C').

Part D:

1. Prove that f (x) – g (x) has a primitive function K defined on $]0;+\infty[$ by: $K(x)=(\ln(x)-1)^2$.
2. Calculate the area of the region bounded by the two curves (C) and (C').



Exercise 13

Part A: Let g be a function defined over $]0;+\infty[$ as: $g(x)=x^3-1+2\ln(x)$.

1. Calculate $g'(x)$ and study its sign.
2. Set up the table of variations of the function g. Calculate g (1).
3. Deduce the sign of g (x) over the interval $]0;+\infty[$.

Part B: Consider the function f that is defined over $]0;+\infty[$ by: $f(x)=x-1-\frac{\ln x}{x^2}$.

Let (C) be its representative curve in an orthonormal system $(O;\vec{i};\vec{j})$.

1. a) Determine $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to 0} f(x)$.

   b) Prove that the straight line (D) of equation: $y = x-1$ is an asymptote to (C).
   Is there another asymptote to (C)? If yes, give its equation.

   c) Prove that: $f'(x) = \frac{g(x)}{x^3}$

   d) Determine the sign of $f'(x)$, and then set up the table of variations of the function f.

   e) Calculate the coordinates of point of intersection between the asymptote (D) and the curve (*C*). Study the position of the curve (*C*) with respect to the straight line (*D*).

   f) Draw in the system $(O;\vec{i};\vec{j})$ the curve (C) and the straight line (D).

2. a) Prove that the function H defined by : $H(x) = \frac{-1}{x}(1+\ln x)$ is a primitive of the function *h* defined

   b) Calculate the area of the region bounded by (*D*), (*C*) and the straight lines of equations $x = 1$ and $x = \sqrt{e}$.

<u>Exercise 14</u>

<u>Part A:</u>

Consider the function g that is defined over interval I =] 0; +∞[by: $g(x) = -4\ln x + x^2 + 6$.

1. a- Calculate $g'(x)$ for all $x \in]0;+\infty[$.

   b- Prove that $g'(x) = 0$ over I has a unique solution $x = \sqrt{2}$.

   c- Study the sign of $g'(x)$ over I.

   d- Prove that the table of variation of the function g is given by:

X	0		$\sqrt{2}$		$+\infty$
$g'(x)$		----	0	+	
$g(x)$		↘	$g(\sqrt{2})$	↗	

2. a- Calculate the exact value of $g(\sqrt{2})$.

   b- Prove that g is positive function positive over the interval I .

   c- Calculate $\lim_{x\to 0} g(x)$ *and* $\lim_{x\to+\infty} g(x)$, and then complete the table of variations of g.

<u>Part B:</u>

Consider the function f that is defined over] 0 ; +∞[by : $f(x) = \frac{x}{4} - \frac{1}{2x} + \frac{\ln x}{x}$

Denote by (C) the representative curve of f in an orthonormal system $(O;\vec{i};\vec{j})$.

1. Calculate $\lim_{x\to 0^+} f(x)$. Deduce an asymptote to the curve (C).
2. Calculate $\lim_{x\to+\infty} f(x)$.
3. Let (D) be a straight line of equation: $y = \frac{x}{4}$. Let h be a function defined on]0 ; +∞ [by :

$$h(x) = f(x) - \frac{x}{4}.$$

   a- Prove that (D) is an asymptote to the curve (C).

   b- Calculate the coordinates of point of intersection of (C) and (D).

   c- Study the relative position of (C) and (D) over]0 ; +∞[

4. a- Calculate $f'(x)$ for all $x \in]0;+\infty[$.

   b- Verify that for all x in] 0; + ∞[: $f'(x) = \frac{g(x)}{4x^2}$.

   c- Deduce from the part A the sense of variations of f over]0 ; +∞[.

5. Determine an equation of the tangent (T) to the curve (C) at a point A of abscissa 1.
6. Draw (C), (T) and the asymptotes to the curve (C) in the system $(O;\vec{i};\vec{j})$.
7. Prove that the equation f (x) = 0 has a unique solution α and verify that: 1 < α < 2.

Part C:

Let K be the function defined on the interval] 0; +∞ [by : $k(x)=(\ln x)^2$

1. Calculate $K'(x)$ for all real number x of the interval $]0;+\infty[$.
2. Deduce a primitive H of the function h over the interval] 0 ; +∞ [such that H (0) = 1.
3. Solve in IR the equation $u(u+1)=0$, and then deduce the solutions of $H(x)$ =0 in the interval I.

4- Consider the function h that is defined over]0 ; +∞ [by: $h(x)=-\frac{1}{2x}+\frac{\ln x}{x}$.

a- We noticed that $\frac{\ln x}{x}$ is of the form $u'(x).u(x)$, determine a primitive H of h.

b- Calculate in cm^2, the area of the domain bounded by the curve (C), the line *(D)* and the two straight lines of equations $x=\sqrt{e}$ *and* $x=e$.

Exercise 15

Part A: Consider the function f that is defined over $I=]-2;+\infty[$ *by* : $f(x)=\ln(x+2)+\frac{x}{x+2}$.

Let (C f) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Verify that for all x belongs to I ; $f'(x)=\frac{x+4}{(x+2)^2}$.
2. Set up the table of variations of f.
3. Show that f has an inverse function f^{-1} defined on an interval J to be determined.

Part B: Consider the function g that is defined over I by: g (x) = f (x) – x.

1. a - Study the sense of variations of g.
   b- Deduce the equation g (x) = 0 admits in I exactly two solutions -1 and α.
   c- Verify that: 1.8 < α < 2.
2. Study the relative position of (C f) and the straight line Δ of equation: y = x.
3. Find an equation of the tangent (T) to (C f) at a point of abscissa 0.
4. Draw (T), Δ, (C f) and (C'). (Take: α = 1.9).

Part C: Let F be a function defined by: F(x) = x ln (x +2).

1. Prove that F is a primitive of f over I.
2. Prove that: $F(\alpha)=\frac{\alpha^2(\alpha+1)}{\alpha+2}$.

COMPLEX NUMBERS
CHAPTER REVIEW

1. Definitions

a- Algebraic form:

The set ℂ of the complex numbers is the set of numbers of the form $x+iy$, i verify the equality: $i^2=-1$, x and y are real numbers.

Let $z=x+iy$. then x = Re (z) and y = Im (z).

If y = 0, then z is real. If a = 0, then z is pure imaginary.

- Equality : Two complex numbers $z=x+iy \text{ and } z'=x'+iy'$ are equal if and only if $x=x' \text{ and } y=y'$.

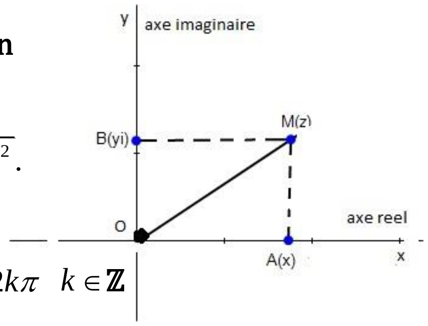
b- Geometric representation:

In the plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, every complex $z=x+iy$, we associate a point M (x; y) and conversely we associate a complex number.

M (x; y) is the image of $z=x+iy$ and z is the affix of M (x; y);

z is equal to the affix of the vector: $\overrightarrow{OM}=x\vec{u}+y\vec{v}$.

c- Trigonometric form



Let z = x + i y and M its image in the plane referred to an Orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

- The modulus of z is the positive real r = $|z|=\sqrt{x^2+y^2}$.

  Geometrically, $|z|=OM=\left\|\overrightarrow{OM}\right\|$.

- An argument of z $(z\neq 0)$ is the number θ defined to $2k\pi \quad k\in\mathbb{Z}$

  By: $\cos(\theta)=\dfrac{x}{r} \text{ and } \sin(\theta)=\dfrac{y}{r}$.

  Geometrically θ to the nearest $2k\pi$ the measure of the angle $\left(\vec{u};\overrightarrow{OM}\right)$.

  Then we have $z=r\left(\cos(\theta)+i\sin(\theta)\right)$: is the trigonometric form of z.

Examples:

Let $z=-3i$. then $|z|=3 \text{ and } \arg(z)=\dfrac{3\pi}{2}+2k\pi,\ z=3\sin\left(\dfrac{3\pi}{2}\right)i.$

Let z = 1 + i. then $|z|=\sqrt{2} \text{ and } \arg(z)=\dfrac{\pi}{4}+2k\pi.\ z=\sqrt{2}\left(\cos\left(\dfrac{\pi}{4}\right)+i\sin\left(\dfrac{\pi}{4}\right)\right).$

2. Operations

a- Conjugate complex number:

Definition:

Let z = x + i y.

The conjugate of the complex number z is $\bar{z}=x-iy$.

Geometrically, the image M' of $\bar{z}$ is the symmetry of the image M of z with respect to $(x'x)$.

We have: $|\bar{z}|=|z| \text{ and } \arg(\bar{z})=-\arg(z)+2k\pi$.

Immediate properties:

$\operatorname{Re}(z)=\dfrac{1}{2}(z+\bar{z});\ \operatorname{Im}(z)=\dfrac{1}{2}(z-\bar{z})\ ;\ z\bar{z}=|z|^2$.

b- Addition :

$$\begin{cases} z = x+iy \\ z' = x'+iy' \end{cases} \Rightarrow z+z' = (x+x')+(y+y')i.$$

Geometrically if M and M' are the images of z and z', then:

The point P image of $z+z'$ is the point such that: $\overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{OM'}$, and the point Q image of $z'-z$ is the point such that: $\overrightarrow{OQ} = \overrightarrow{OM'} - \overrightarrow{OM} = \overrightarrow{MM'}$.

$z'-z$ is then the affix of the vector $\overrightarrow{MM'}$ and $MM' = |z'-z|$.

c- <u>Product :</u>

$$\begin{cases} z = x+iy \\ z' = x'+iy' \end{cases} \Rightarrow zz' = (xx'-yy')+(xy'+x'y)i.$$

<u>Properties:</u>

$|zz'| = |z|\times|z'|,\quad \arg(zz') = \arg(z)+\arg(z')+2k\pi.$

d- <u>Inverse and quotient :</u>

By using the above formulas, we verify that:

<u>Properties:</u>

$\left|\frac{1}{z}\right| = \frac{1}{|z|}\ (z\neq 0) \qquad \arg\left(\frac{1}{z}\right) = -\arg(z)+2k\pi.$

$\left|\frac{z}{z'}\right| = \frac{|z|}{|z'|}\ (z'\neq 0) \qquad \arg\left(\frac{z}{z'}\right) = \arg(z)-\arg(z')+2k\pi.$

e- <u>Power :</u>

Let $z = r(\cos(\theta)+i\sin(\theta)),\ (z\neq 0)$. By using the properties of the product, we prove by mathematical induction that for all natural n, we have: $z^n = r^n(\cos(n\theta)+i\sin(n\theta))$.

Example:

$$1-i = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right] \Rightarrow (1-i)^{20} = \left(\sqrt{2}\right)^{20}\left[\cos(-5\pi)+i\sin(-5\pi)\right] = -1024.$$

3. <u>Exponential form:</u>

1. <u>De Moivre's Formula :</u>

For all complex numbers of modulus1, $z = \cos\theta + i\sin\theta$, we have for all natural number n, De Moivre's formula:

$\left[\cos(\theta)+i\sin(\theta)\right]^n = \cos(n\theta)+i\sin(n\theta).$

$e^{i\theta} = \cos(\theta)+i\sin(\theta).$

Examples: For $\theta = \pi,\ e^{i\pi} = -1$. For $\theta = \frac{\pi}{2},\quad e^{i\frac{\pi}{2}} = i.$

2. <u>Exponential notation of a complex number:</u>

<u>Property:</u> Every non – zero complex number of modulus r and of argument θ can be written as $re^{i\theta}$.

Example: Calculation of the modulus and an argument of $z = \frac{i}{1+i}$.

$$z = \frac{e^{i\frac{\pi}{2}}}{\sqrt{2}e^{i\frac{\pi}{4}}} = \frac{1}{\sqrt{2}}e^{i\left(\frac{\pi}{2}-\frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}};\ |z| = \frac{1}{\sqrt{2}}\ \ and\ \arg(z) = \frac{\pi}{4}+2k\pi.$$

3. <u>Euler Formula :</u>

$e^{i\theta} = \cos(\theta)+i\sin(\theta)$, so $e^{-i\theta} = \cos(-\theta)+i\sin(-\theta) = \cos(\theta)-\sin(\theta)$.

By addition we have: $2\cos(\theta)=e^{i\theta}+e^{-i\theta}$ and by subtraction we have $2i\sin(\theta)=e^{i\theta}-e^{-i\theta}$.

$$\cos(\theta)=\frac{e^{i\theta}+e^{-i\theta}}{2},\quad \sin(\theta)=\frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

<u>Application: linearization of sine and cosine:</u>

<u>Example:</u> Linearization of $\cos^3(\theta)$ *and of* $\sin^3(\theta)$

$$\cos^3(\theta)=\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)^3=\frac{1}{8}\left(e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}+e^{-3i\theta}\right)=\frac{1}{4}\left[\cos(\theta)+3\cos(\theta)\right].$$

$$\sin^3(\theta)=\left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^3=-\frac{1}{8i}\left(e^{3i\theta}+3e^{i\theta}+3e^{-i\theta}-e^{-3i\theta}\right)=\frac{1}{4}\left[-\sin(\theta)+3\sin(\theta)\right].$$

<u>COMPLEXE NUMBERS</u>

<u>Exercise 1</u>

1. Consider in the set of the complex numbers the following complex numbers:

$$Z_1=\sqrt{2}\left(\sqrt{3}+i\right);\quad Z_2=4(1-i)\quad and\quad Z=\frac{Z_1^{\ 2}}{Z_2}.$$

   a- Determine the modulus and an argument of Z_1 and Z_2.

   b- Write Z in its exponential form, and then write Z in its algebraic form.

   c- Deduce the exact values of $\cos\left(\frac{7\pi}{12}\right)$ and $\sin\left(\frac{7\pi}{12}\right)$.

2. Denote by E and F the points of respective affixes $Z_E=-i$ *and* $Z_F=3i.$

   To every point M of affix Z = x + i y$(Z\neq -i)$, we associate a point M' of affix $Z'=x'+iy'$ such that: $z'=\frac{iz+3}{z+i}$.

   a- Express x' *and* y' in terms of x and y.

   b- Determine the set (d) of the points M so that Z' is pure imaginary.

   c- Determine the set (C) of the points M so that Z' is real. Precise the elements of the set (C).

<u>Exercise 2</u>

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, given the points A, B, M and M' of respective affixes: $Z_A=1,\ Z_B=i\sqrt{2},\ Z_M=Z\ (Z\neq 1)$ *and* $Z_{M'}=Z'=\frac{Z-i\sqrt{2}}{Z-1}$.

1. a - Prove that: $|Z'|=\frac{BM}{AM}$.

   b- Deduce the set (d) of the points M so that: $|Z'|=1$.

2. Find the set of the points M so that: $\arg(Z')=\frac{\pi}{2}+2k\pi.$

3. a - Prove that for all $Z\neq 1$ we have: $|Z'-1|\times|Z-1|=\sqrt{3}.$

   b- Deduce that if M moves on a circle of center A and of radius $2\sqrt{3}$ then M' moves on a circle whose center and radius is to be determined.

4. Let D be a point of affix $Z_D=e^{i\theta}$ *with* $\theta\in\left]0;\pi\right[$.

   a) Verify that: $e^{i\theta}-1=2i\sin\left(\frac{\theta}{2}\right).e^{\frac{i\theta}{2}}.$

b) Deduce AD in terms of θ.

5. Determine the value of θ so that ABD is an isosceles triangle of vertex A

Exercise 3

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points A and B of respective affixes: $Z_A = 1-i$ and $Z_B = 2+\sqrt{3}+i$.

1. Write Z_A in its exponential form.
2. a - Write $\frac{Z_B}{Z_A}$ in its algebraic form.

   b- Prove that: $\frac{Z_B}{Z_A} = \left(1+\sqrt{3}\right)e^{i\frac{\pi}{3}}$.

   c- Deduce the exponential form of the complex number Z_B.

   d- Determine then $\cos\left(\frac{\pi}{12}\right)$.
3. Let C be the symmetric of the point B with respect to the axis $\left(O;\vec{u}\right)$.

   Prove that: $Z_C = \left(\sqrt{2}+\sqrt{6}\right)e^{-i\frac{\pi}{12}}$.

Exercise 4

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

To every point M of affix $Z \neq 1$, we associate the point M' of affix $Z' = \frac{1}{Z-1}$.

Let (C) be a circle of center A of affix 1 and of radius 2.

1. a - Calculate Z' in the case $Z = 4 + i\sqrt{3}$, write Z' in its exponential form.

   b- Solve in the complex set the equation: $Z' = -Z$, write the solutions in the exponential forms.
2. a - Express $|Z'|$ in terms of $|Z-1|$ and $\arg(Z')$ in terms of arg (Z – 1).

   b- Prove that if M describes the circle (C) then M' moves on a circle (C') whose center and radius is to be determined.
3. Suppose that: $Z = x+iy$ and $Z' = x'+iy'$ where x, y, x' and y' are real numbers.
   a- Express x' and y' in terms of x and y.
   b- On which line the point M moves if the point M' describes the axis $y'y$?
4. Consider the straight line (D) of equation: $x = \frac{1}{2}$.
   a- Prove that M belongs to (D) if and only if $|Z-1| = |Z|$.
   b- Prove that if M belongs to (D) then $|Z'+1| = 1$, deduce the locus of the point M'.

Exercise 5

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

To every point M of affix Z, we associate the point M' of affix Z' such that: $Z' = \frac{Z-1+i}{Z-2}$; $(Z \neq 2)$.

1. Write Z' in its exponential form in the case: $Z = \frac{9}{5} - \frac{3}{5}i$.
2. Write Z in its algebraic form in the case: $Z' = 2e^{-\frac{i\pi}{3}}$.

3. Suppose that : $Z = x + iy$ and $Z' = x' + iy'$.

   a) Prove that: $\begin{cases} x' = \dfrac{x^2 + y^2 - 3x + y + 2}{(x-2)^2 + y^2} \\ y' = \dfrac{x - y - 2}{(x-2)^2 + y^2} \end{cases}$.

   b) Prove that if M moves on the x-axis deprived the point (2; 0) then M' moves on a straight line (d) whose equation is to be determined.

   c) On which line the point M moves in the case Z' is real.

4. Consider the points A, B and C with respective affixes: $Z_A = 1 - i;\ Z_B = 2$ and $Z_C = 3 - i$.

   a) Interpret geometrically: $|Z - 1 + i|$, $|Z - 2|$ and $|Z'|$.

   b) Determine the set of points M' in the case M moves on the perpendicular bisector of segment [AB].

5. Calculate: $\dfrac{Z_A - Z_B}{Z_C - Z_B}$, and then deduce the nature of triangle ABC.

Exercise 6

The complex plane P is referred to an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$. Consider in the plane P the points A, B and C of respective affixes $Z_A = 1 + i\sqrt{3},\ Z_B = -1 - i$ and $Z_C = -\left(2 + \sqrt{3}\right) + i$.

1. a - Calculate the modulus and an argument of the complex number $W = \dfrac{Z_C - Z_B}{Z_A - Z_B}$.

   b- Deduce the nature of triangle ABC.

2. a - Write the complex number $\dfrac{Z_A}{Z_B}$ in its algebraic form.

   b- Write the numbers Z_A and Z_B in trigonometric forms. Deduce the trigonometric form of $\dfrac{Z_A}{Z_B}$.

   c- Deduce the exact values of $\cos\left(\dfrac{\pi}{12}\right)$ and $\sin\left(\dfrac{\pi}{12}\right)$.

Exercise 7:

In the complex plane referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$, unit= 2cm, consider the points M and M' of respective affixes $Z = x + iy$ and $Z' = x' + iy'$ such that:

$$Z' = -2iZ - 3 + 4i.$$

1. Find Z' in the algebraic and the exponential forms in the case: $Z = 4 - 2\sqrt{3} + \dfrac{i}{2}$.
2. Find Z in the algebraic and the trigonometric forms in the case: $Z' = -4 + 3i$.
3. Find Z in the case M is confounded with M'.
4. a. Express x' and y' in terms of x and y.

   b. Prove that when M' moves on a circle (C') of center I (1, -1) and of radius 2, M moves on a circle (C) whose center and radius are to be determined.
5. Let A, B and C be three points of respective affixes:

   $$Z_A = 1 + i,\ Z_B = 1 \text{ and } Z_C = -2.$$

   a. Calculate $\dfrac{Z_A - Z_B}{Z_C - Z_B}$, and then deduce the nature of triangle ABC.

b. Calculate Z'_A, Z'_B and Z'_C, the respective affixes of points A', B' and C' the images of points A, B and C.

c- Prove that the triangle A'B'C' is right angled at B'.

Exercise 8

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$. Consider the points A, B and C with respective affixes: $Z_A=-2\sqrt{3},\ Z_B=-\sqrt{3}+3i$ and $Z_C=-\sqrt{3}-3i$.

1. Determine the modulus and an argument of Z_A, Z_B and Z_C.

2. Write Z_A, Z_B and Z_C in its trigonometric form.
3. Let $Z=\dfrac{2Z_B}{\sqrt{6}+i\sqrt{6}}$ be a complex number.
   a) Write the complex number Z in its algebraic form.
   b) Determine the modulus and an argument of the complex number Z.
   c) Write Z in its trigonometric form.
   d) Deduce the exact value of $\cos\left(\dfrac{11\pi}{12}\right)$.

Exercise 9

Consider the complex number $Z=-3\left(\dfrac{1+i\sqrt{3}}{i}\right)$.

1. Calculate the modulus of Z.
2. Find an argument of Z.
3. Deduce the trigonometric form of the complex number Z.
4. Determine the natural non zero number n so that Z^n is real.

Exercise 10

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, given the points A and B of respective affixes: $Z_A=-1$ and $Z_B=3i$.

To every point M of affixes Z, (M distinct from A), we associate the point M' of affix Z' such that:

$$Z'=\frac{iZ+3}{Z+1}.$$

1. Suppose that: $Z=x+iy$ and $Z'=x'+iy'$.
   a- Express x' and y' in terms of x and y.
   b- Deduce the set of point M so that Z' is pure imaginary.
2. Let C be a point of affix $Z_C=2-i$. Prove that there is a unique point D such that: f (D) = C.
3.
   a- Solve the equation f (Z) $=Z_B$ and then write its solution in the exponential form.
   b- Solve the equation f (Z) = i, and then deduce that the point M' is distinct from the point B.
4. Calculate $\dfrac{Z_B-Z_C}{Z_C-Z_A}$ and then deduce that the triangle ABC is right isosceles.
5. Let G be a point of the plane such that the triangle OCG is direct right and isosceles of vertex O.

a- Calculate $\frac{Z_G}{Z_C}$.

b- Deduce the affix of point G and prove that G belongs to the circle (C) of diameter [AB].

Exercise 11

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$ (G. U= 2 cm), consider the points A, B and C of respective affixes $Z_A = 2;\ Z_B = 1+i\sqrt{3}\ and\ Z_C = 1-i\sqrt{3}.$

Part A:

1. a. Give the exponential form of Z_B then of Z_C.
   b. Plot the points A, B and C.
2. Determine the nature of the quadrilateral OBAC.
3. Determine and construct the set (D) of points M of the plane such that $|Z| = |Z-2|$.
4. Consider the point D of affix $Z_D = 2e^{\frac{-i\pi}{6}}$.
   a. Prove that $Z_D = \sqrt{3} - i$.
   b. Deduce that $BOD = 90^\circ$.

Part B:

To every point M of affix Z such that $Z \neq Z_A$, we associate the point M' of affix Z' defined by:

$$Z' = \frac{-4}{Z-2}.$$

1. Prove that for all complex number Z distinct of 2, we have: $|Z'-2| = \frac{2|Z|}{|Z-2|}$.
2. Suppose in this question that M is any point of (D), where (D) is the set defined in the question 3, part A. Prove that the point M' associate to M moves on a circle (C) whose center and radius are to be determined.

Exercise 12

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points A, B, C and I of respective affixes: $Z_A = 2i\ ;\ Z_B = -4i;\ Z_C = 3-i\ and\ Z_I = -i.$

1. a- Locate the points A, B and C.
   b- Prove that ABC is right isosceles triangle.
   c- Determine the affix Z_D of the point D so that ACBD is a square.
2. To every point M of affix Z, (M is distinct of B), we associate the point M' of affix Z' such that:
$$Z' = \frac{Z-2i}{iZ-4}.$$
   a- Find Z' in the case Z = 2 – 3 i.
   b- Find Z in the case $Z' = 2-3i$.
3. a- Verify that for all $Z \neq -4i$; we have: $Z' = \frac{i(Z-2i)}{-Z-4i}$.
   b- Determine the set of points M so that: $|Z'| = 1$.
4. a- Prove that: $|Z+i| \times |Z+4i| = 6$.
   b- Deduce that if M belongs to the circle (C) of center B and of radius 2 then M' belongs to a circle (C') whose center and radius are to be determined.

Exercise 13

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, Consider the points A, B and C with respective affixes $Z_A=\sqrt{2}+i\sqrt{2}$, $Z_B=\sqrt{3}-i$ and $Z_C=Z_A+Z_B$.

1. a- Write Z_A and Z_B in its exponential form.

   b- Plot the points A, B and C.

2. Write $Z_A\times Z_B$ in its algebraic and trigonometric forms, then deduce $\cos\left(\frac{\pi}{12}\right)$ and $\sin\left(\frac{\pi}{12}\right)$.

3. a- Write $\frac{Z_A}{Z_B}$ and $1+\frac{Z_A}{Z_B}$ in their exponential forms.

   b- Deduce the exponential form of Z_A+Z_B.

Exercise 14

PART A:

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points A, B and C of respective affixes $Z_A=3-2i$, $Z_B=5+i$ and $Z_C=\frac{11}{2}-\frac{3}{2}i$.

1. Write the complex number $Z=\frac{Z_A-Z_C}{Z_B-Z_C}$ in their algebraic and exponential forms, and then deduce the nature of triangle ABC.
2. To every point M of affix Z, we associate a point M' of affix Z' such that: $Z'=iZ+3+i$.

   a- Find the affixes of the points A', B' and C' the respective images of the points A, B and C.

   b- Prove that $A'B'C'$ is right isosceles triangle of vertex C'.

PART B:

To every point M of affix ($Z\neq 5+i$) we associate a point M' of affix Z' such that:

$$Z'=\frac{Z-3+2i}{Z-5-i}.$$

1. Find the affix Z' of point M' in the case $Z=\frac{5}{2}+\frac{i}{2}$. Deduce the trigonometric and exponential forms of Z.
2. a- Give a geometric interpretation of $|Z-3+2i|$, $|Z-5-i|$ and $|Z'|$.

   b-Find the set of points M in the case M' moves on a circle (C) of center O(0;0) and of radius 1.
   Find the coordinates of the point M in the case Z'=-1 + i

Exercise 15

Given the complex number $Z=\frac{5+3i\sqrt{3}}{1-2i\sqrt{3}}$.

1. Write Z in its algebraic form.
2. Calculate Z^2, Z^3 and Z^{21}.
3. Prove that for all -natural number n we have: $Z^{3n+2}=-(2)^{3n+1}\left(1+i\sqrt{3}\right)$, and then deduce Z^{32}.

Exercise 16

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, every point M of affix Z we associate a point M' of affix Z' such that: $Z'=\frac{Z-i}{\overline{Z}+i}$ $\left(\overline{Z}\neq -i\right)$.

1. Calculate the modulus of Z.

2. Prove that the number $\frac{Z'+1}{Z-i}$ is real.

3. What is the set of the points M in the case Z ' = - i.?

Exercise 17

Given the two complex numbers $u=\sqrt{2}-i\sqrt{6}$ *and* $v=1+i$.

1. Determine the algebraic, the trigonometric and the exponential forms of u, v^3 and $Z=\frac{u}{v^3}$.
2. Write Z in its algebraic form.
3. Deduce the exact values of $\cos\left(\frac{11\pi}{12}\right)$ *and* $\sin\left(\frac{11\pi}{12}\right)$.

Exercise 18

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, given the point M of affix Z and the point M' of affix Z' such that: $Z'=\left(-\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)Z$.

1. A is a point of affix $Z_A=2$. Find the affix $Z'_{A'}$ of the point A' the image of the point A.
2. Calculate the affix of the point I the midpoint of the segment [AA']. Calculate $|Z_I|$.
3. By using the nature of triangle OAA', determine the measure of the oriented angle $\left(\vec{u},\overrightarrow{OI}\right)$.
4. Deduce the exact values of $\cos\left(\frac{3\pi}{8}\right)$ *and* $\sin\left(\frac{3\pi}{8}\right)$.

Exercise 19

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Designate by A and B the points of respective affixes: $Z_A = 2i$ and $Z_B = -i$.

To every point M different of A of affix Z we associate the point M' of affix Z' defined by: $Z'=\frac{Z}{iZ+2}$.

1. a - Determine the algebraic form of Z' *in the case*: $Z=2+i$.
   b- Determine the algebraic form of Z so that: $Z'=1-i$.
2. a- Prove that for all $Z\in C-\{2i\}$ we have: $(Z'+i)(Z-2i)=2$ and that: $BM'.AM=2$.
   b- Deduce that if M traces on the circle (C) of center A and of radius 1then M' traces on a circle (C') Whose center and radius are to be determined.
   c- Determine and construct the set Δ of points M (Z) such that: $|Z'|=1$.

3. Let Z be a complex number such that: Z = a + i where $a\in IR$. Designate by θ an argument of Z.
   a- Prove that: $Z'=\frac{Z}{i\overline{Z}}$. Deduce an argument of Z' in terms of θ.
   b- Prove that for all Z we have: $Z^2-2\sqrt{2}Z+4=\left(Z-\sqrt{2}\right)^2+2$.
   c- Solve then in the complex set numbers the equation (E): $Z^2-2\sqrt{2}Z+4=0$.
   Denote by: Z_1 and Z_2 the solutions of the equation (E) with Im $(Z_1) < 0$.
   d- Write Z_1 in its trigonometric form. Deduce the trigonometric form of Z_2.

Exercise 20

The complex plane is referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$.

Designate by (C) the circle of center O and of radius 1 and by I and A the points of respective affixes $Z_I = 1$ and $Z_A = \sqrt{3}+i$.

1. a - Give the exponential form of Z_A.
   b- Locate the point A.
2. Let B be a point of affix $Z_B = \dfrac{Z_A - 1}{1-\overline{Z_A}}$.
   a- Verify that: $Z_B.\overline{Z_B} = 1$. Deduce that the point B belongs to the circle (C).
   b- Prove that: $\dfrac{Z_B - 1}{Z_A - 1}$ is real. Deduce that the points A, B and I are collinear.
   c- Locate the point B in the system $\left(O; \vec{u}, \vec{v}\right)$.
3. Let θ be an argument of the complex number Z_B.
   Prove that: $\cos\theta = \dfrac{2\sqrt{3}-3}{5-2\sqrt{3}}$ *and* $\sin\theta = \dfrac{2-2\sqrt{3}}{5-2\sqrt{3}}$.

Exercise 21

The complex plane is referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$.

To every point M of affix $Z \neq -i$, we associate the point M' of affix $Z' = \dfrac{Z+2i}{1-iZ}$ and let B and C be two points of respective affixes $Z_B = -i$ *and* $Z_C = -2i$.

1. a - Verify that for all $Z \neq -i$ we have: $-iZ' = \dfrac{Z+2i}{Z+i}$.
   b- Deduce the set of the points M so that Z' is real.
2. a - Prove that : $|Z'| = \dfrac{CM}{BM}$.
   b- Deduce the set of the points M in the case M' describes the circle C (O (0,0); R = 1).
3. Consider the complex number: $W = \dfrac{Z'-i}{Z-i}$, $(Z \neq -i \text{ and } Z \neq i)$.
   a- Verify that for all complex number Z we have: $(Z-i)(1-iZ) = -i\left(1+Z^2\right)$.
   b- Deduce that: $W = -\dfrac{1}{Z^2+1}$.
4. Suppose that: $Z = e^{i\theta}$; $\theta \in \left[0; \dfrac{\pi}{2}\right[$.
   a) Verify that: $W = \dfrac{-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$.
   b) Deduce in terms of θ the modulus and an argument of W.

EXPONENTIAL FUNCTIONS
CHAPTER REVIEW

1. **definition**

   The logarithmic napery's function ln (x) is strictly increasing function defined on an interval $]0;+\infty[$, the domain of the definition of its inverse function is $]-\infty;+\infty[$. So the logarithmic function has an inverse function

   $y = f^{-1}(x) = e^x$: $y = e^x \Leftrightarrow x = \ln(y)$.

2. **Properties:**

   a) For all x, $e^x > 0$

   We have: $y = e^x$ implies that ln (y) = x so y > 0.

   b) $\ln(e^x) = x$ for all real x

   If $y = e^x$, ln (y) and $\ln(y) = \ln(e^x)$.

   c) $e^{\ln x} = x$ for all x > 0

   Because $y = e^x$ and x = ln (y) so $y = e^{\ln y}$.

   d) exp (a + b) = exp (a) + exp (b)

   We have ln (exp (a)exp (b)) = ln (exp(a)) + ln (exp (b)) = a + b and ln (exp(a + b)= a + b.

   So we have: ln (x) = ln (y) $\Rightarrow x = y$.

   e) exp(0) = 1 and exp (1) = e

   We have ln (1) = 0 so $e^0 = e^{\ln 1} = 1$, and ln (e) = 1 so $e^{\ln e} = e$.

   f) exp(n) = e^n for all naturel n

   We have ln (exp (n)) = n = n ln (e) = ln (e^n).

   a) Notation :

   - $e^x > 0$ for all $x \in \mathbb{R}$.
   - $e^{\ln(x)} = x$ For all real x strictly positive.
   - $\ln(e^x) = x$ For all real x strictly positive.
   - $e^1 = e,\ e^0 = 1 \text{ and } e^{a+b} = e^a e^b$ for all ordered pair (a ; b)

   b) Study of the function e^x

   a- The domain of definition :

   Exp (x) is the inverse function of ln (x), its domain of definition is $\mathbb{R}$ and its image $\mathbb{R}^+$.

   b- Derivative:

   $(e^x)' = e^x$, So the derivative is > 0 over $\mathbb{R}$.

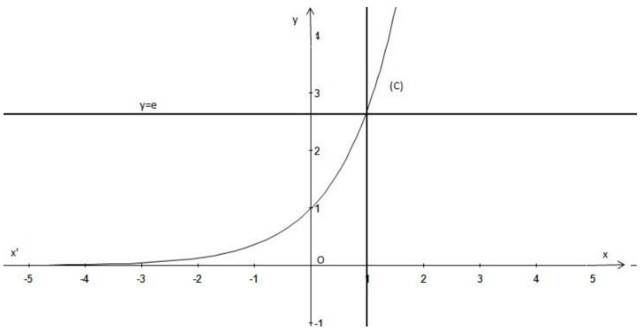
   c- Limits:

   $\lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to+\infty} e^x = +\infty.$

   d- Table of variation :

X	-∞	0	1	+∞
f '(x)		+		
f (x)	0	↗		+∞

   e- Curve



   f- Derivative and Primitives:

   $(e^u)' = u'e^u, \qquad \int u'e^u dx = e^u + c.$

EXPONENTIAL FUNCTIONS

Exercise 1

Part A: Consider the differential equation (E): $y''-2y'+y=2x-3$. Suppose that: y = Z +2x +1.

1. Formed a differential equation (E') satisfied by Z.
2. Solve (E'). Deduce the general solution of (E).
3. Find the particular solution of (E) such that: $y(0)=1$ *and* $y(1)=2$.

Part B: The plane is referred to a direct orthonormal system $\left(O;\vec{i},\ \vec{j}\right)$ of unit 2 cm. Let (C) is the graph of the function f defined on IR by: $f(x)=2x+1-xe^{x-1}$ and (Δ) the straight line of equation: y =2x +1.

1. Prove that: $\lim\limits_{x\to-\infty} f(x)=-\infty$ and calculate f (-1).
2. Prove that (Δ) is an asymptote to (C). study position of (C) with respect to (Δ).
3. Let f' the derivative function of f. The table below is the table of variations of the function f'.

x	$-\infty$		-2		1		$+\infty$
$f''(x)$		$+$	0	$-$			
$f'(x)$	$+2$	↗	$2+e^{-3}$	↘	0	↘	$-\infty$

   a - Prove that (C) admits a point of inflection W, whose coordinates are to be determined.
   b- Set up the table of variations of f. deduce that the equation f (x) =0 admits a unique root α such that: 1.9 < α < 2.

4. Let A is a point of (C) where the tangent (T) is parallel to (Δ).
   a. Write the equation of (T).
   b. Draw (Δ), (T) and (C).
5. Let D be the domain limited by (C), the x-axis and the two straight lines x =1 and x= α.
   a. Calculate in cm² and in terms of α, the area A (α) of the domain D.
   b- Verify that: $A(\alpha)=4(\alpha-1)\left(\alpha-\dfrac{1}{\alpha}\right)$.

Exercise 2

Part A: Consider the function g that is defined over IR by: $g(x)=(4-x)e^{-\frac{x}{2}}-1$.

1. Calculate $\lim\limits_{x\to-\infty} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x)$.
2. Calculate $g'(x)$, and then set up the table of variations of g.
3. Prove that the equation g (x) = 0 admits a unique solution α, and verify that: 1.6 < α < 1.8.
4. Deduce the sign of g (x) for all real x.

Part B:

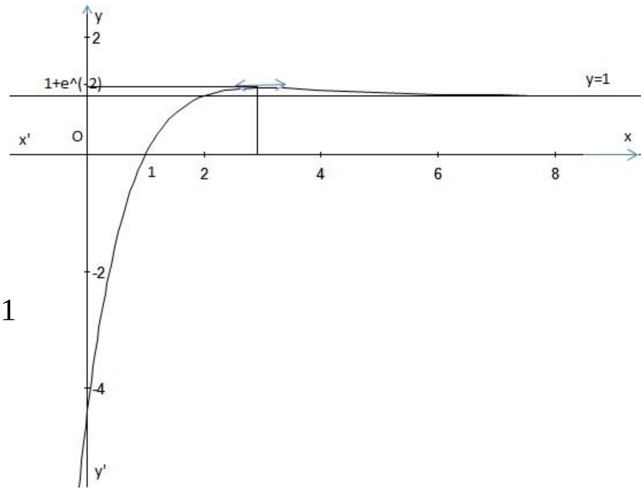
Consider the function f that is defined on IR by: $f(x)=(2x-4)e^{-\frac{x}{2}}+2-x$. Designate by (C) its representative curve in an orthonormal system $\left(O;\ \vec{i},\ \vec{j}\right)$. Graphic unit = 2 cm.

1. Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$.
2. Prove that the straight line (d) of equation y = 2 – x is an asymptote to (C) at + ∞.
3. Prove that: $f'(x)=g(x)$. study the sense of variations of f.

4. Prove that: $f(\alpha)=\dfrac{\alpha^2-4\alpha+4}{4-\alpha}$. Set up the tableau of variations of f.
5. a- Calculate the coordinates of the points of intersection of (C) and the axis of abscissas.
   b- Calculate the coordinates of E the intersection point of (C) and the axis of ordinates
   c- Write an equation of the tangent (T) to (C) at the point E.
6. Suppose that $\alpha = 1.7$. Draw (d), (T) and (C).
7. Calculate, in cm², the area of the region bounded by the curve (C), (d) and the straight lines of equations: x = 0 and x = 2.

Exercise 3

Part A : The adjacent figure to the right is the representative curve (G) of a function g defined on IR in an orthonormal system $\left(O;\vec{i},\ \vec{j}\right)$.



- (G) admits a local maximum (3; 1+e⁻²)
- The straight line (d) of equation y =1 is an asymptote to (G) at $+\infty$.
- The equation of the function g is in the form: $g(x)=(ax+b)e^{-x+1}+c$.

1. By graphical reading determine the values of the real numbers a, b and c.
2. Deduce the sign of the function $g(x)=(x-2)e^{-x+1}+1$ for all x in IR.

Part B:

Consider the differential equation (E): $y''+2y'+y=x$.

Suppose that: y = z+ x -2.

1. Find the differential equation (F) satisfied by z. Solve (F).
2. Deduce the general solution of (E).
3. Find a particular solution f of (E) such that: $y(0)=e-2;\ y(1)=-1$.

Part C:

Consider the function f that is defined on IR by: $f(x)=(-x+1)e^{-x+1}+x-2$.

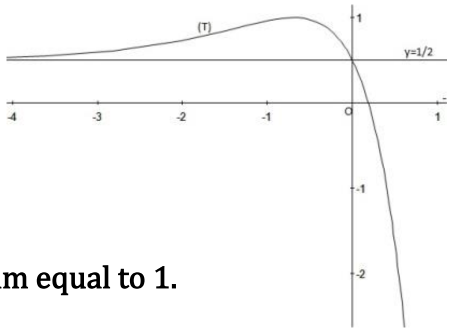
Let (C) be its representative curve in an orthonormal system $\left(O;\vec{i},\ \vec{j}\right)$. (G.U = 2 cm).

1. Calculate $\lim\limits_{x\to-\infty} f(x)$, $f(-1)$ and $f(-2)$.
2. Calculate $\lim\limits_{x\to+\infty} f(x)$, and then prove that the straight line (D) of equation: y = x -2 is an asymptote to (C).
3. Study according to the values of x the relative positions between (C) and (D).
4. Prove that $f'(x)=g(x)$, and then study the sense of variations of f.
5. Prove that f admits an inflection point I whose coordinates is to be determined.
6. Write an equation of the tangent line (T) to the curve (C) at a point of abscissa 0.
7. Set up the table of variations of f.
8. Draw (D) and (C).
9. Calculate, in cm², the area of the region bounded by (C) , (D) and the two straight lines of equations x = 1 and x = 2.

Exercise 4

Consider the differential equation (E): $y'' - 3y' + 2y = x - 1$.

1. Assume that: $z = y - \frac{1}{2}x - \frac{1}{4}$.
   a. Write a differential equation (F) satisfied by z and the solve (F).
   b. Deduce the general solution of (E).
2. Let f be a particular solution of (E). The below curve (T) is the representative curve of the function f' the derivative of the function f.



   - (T) cuts the x- axis at a point of abscissa $\ln\left(\frac{\sqrt{2}+1}{2}\right)$.
   - (T) cuts the y-axis at a point of ordinate $\frac{1}{2}$.
   - (T) admits at a point of abscissa – ln (2) a local maximum equal to 1.

   Prove that : $f(x) = 2e^{x} - e^{2x} + \frac{1}{2}x + \frac{1}{4}$.
3. Designate by (C) the representative curve of the function f in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$ graphical unit = 2cm.
   a. Calculate f (1) and $\lim\limits_{x \to +\infty} f(x)$.
   b. Calculate $\lim\limits_{x \to -\infty} f(x)$ and prove that the straight line (d) of equation $y = \frac{1}{2}x + \frac{1}{4}$ is an asymptote to (C).
   Determine, according to the values of x, the relative positions of (C) and (d).
   c. Verify that the point $I\left(-\ln(2); 1 - \frac{\ln(2)}{2}\right)$ is an inflection point of the curve (C).
   d. Prove that the equation f (x) = 0 admits two solutions α and β such that 0.8 < α < 0.9 and -1.4 < β <-1.3.
   e. Study the sense of variations of f and set up the table of variations of f.
   f. Draw (d) and (C).
4. Calculate, in cm², the area of the region bounded by (C), (d) and the two lines x = - ln(2) and x =0.

Exercise 5

Part A:

Consider the function f that is defined and differentiable over $\mathbb{R}$ by: $f(x) = e^{-x} + 2x - 3$

Denote by (C) the representative curve of the function f in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$.

1. a. Calculate $\lim\limits_{x \to +\infty} f(x)$.

   b. Prove that the straight line (D) of equation: $y = 2x - 3$ is an asymptote to the curve (C) at +∞.

2. a. Prove that for all real x we have the following equality: $f(x) = e^{-x}\left(1 + 2xe^{x} - 3e^{x}\right)$

   b. Deduce $\lim\limits_{x \to -\infty} f(x)$.

3. a. Prove that for all real x, $f'(x) = \frac{2e^{x} - 1}{e^{x}}$. Deduce the sign of $f'(x)$ over IR.

b. Set up the table of variations of f over $\mathbb{R}$.

4. Prove that the equation $f(x)=0$ admits a unique solution α over the interval $[1;\ 2]$.

5. Consider the point A of the curve (C) of abscissa $(-\ln 3)$.

a. Calculate the exact value of the ordinate of point A.

b. Denote by (T) the tangent to the curve (C) at a point A.
Write an equation of the tangent (T).

6. Construct the curve (C) and the straight line (*D*).

Part B:

1. Verify that the function F defined over IR by: $F(x)=-e^{-x}+x^2-3x$ is a primitive of f over $\mathbb{R}$.

2. Calculate the area of the domain bounded by (C), the x- axis and the straight lines of equations: $x=-1$ and $x=1$.

Exercise 6:

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$ Graphic unit = 2 cm.

Let (C) be the representative curve of the function f that is defined over IR by:

$$f(x)=(ax^2+bx+c)e^x,$$

Where a, b and c are three real numbers such that:

- The point A of coordinates (0; −1) belongs to the curve (C);
- The curve (C) admits at the point A a tangent line parallel to the x – axis.
- $f(1)=2e$.

Part A:

1. Prove that: $c=-1$.

2. Calculate $f'(x)$ in terms of a and of b, and then calculate a and b.

Part B:

We admits for all real *x*, $f(x)=\left(2x^2+x-1\right)e^x$.

1. a. Calculate $\lim\limits_{x\to+\infty} f(x)$..

b. Calculate $\lim\limits_{x\to-\infty} f(x)$. Interpret graphically this result.

2. a. Verify that, for all real *x*, $f'(x)=x(2x+5)e^x$.

b. Study the sign of $f'(x)$ according to the values of *x*.

c. Set up the table of variations of the function *f*.

3. Determine by the coordinates of the points of intersection of the curve (C) with the x - axis.

4. Draw the curve (C).

Part C:

Consider the functions F defined over $\mathbb{R}$ by: $F(x)=(2x^2-3x+2)e^x$.

1. Verify that the function F is a primitive of the function *f* over $\mathbb{R}$.

2. Calculate the area of the domain D bounded by (C), the x- axis, and the two straight lines of equations $x=-1$ and $x=1/2$.

Exercise 7:

Part A:

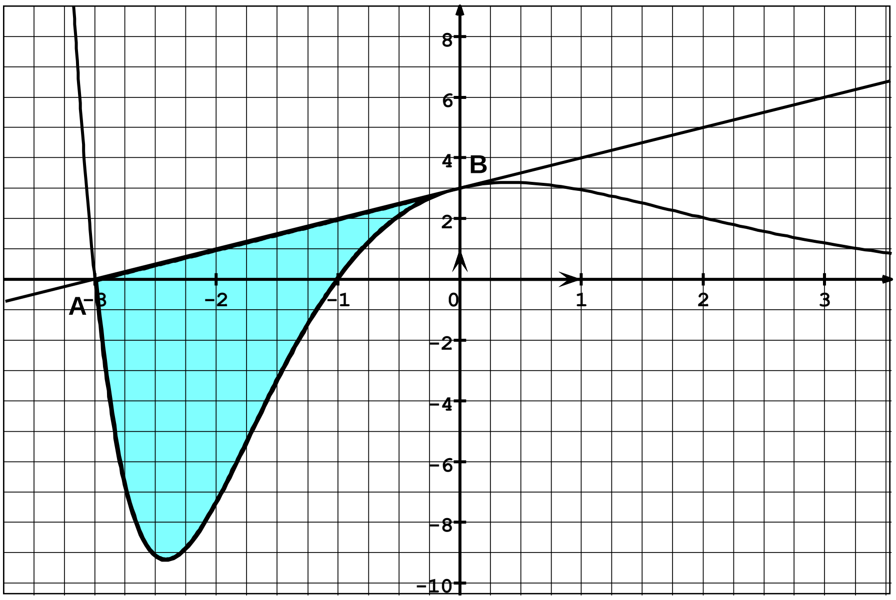
Consider the function that is defined and differentiable over R by: $f(x)=(ax^2+bx+c)e^{-x}$ where a, b and c are three real numbers are proposed to be determined. The below curve (C) is the representative curve of the function f in an orthonormal system $\left(O;\vec{i},\ \vec{j}\right)$.

We admit that the straight line (D) passes through A and is tangent to the curve (C) at a point B.

1. a. By graphical reading, determine the coordinates of the points A and B.

Deduce the values of $f(-3)$ and $f(0)$.

b. Prove that the equation of the straight line (AB) is: $y = x + 3$. Deduce the value of $f'(0)$.



a) Prove that, for all x belongs to R, $f'(x) = \left(-ax^2 + (2a-b)x + b - c\right)e^{-x}$.

b- Deduce $f'(0)$ in terms of b and c.

3. By using the preceding questions, Prove that the real's a, b and c are solutions of the system

$$\begin{cases} 9a - 3b + c = 0 \\ b - c = 1 \\ c = 3 \end{cases}$$

Solve the system and then deduce the expression of $f(x)$ in terms of x.

Part B:

We suppose that f is defined over R by: $f(x) = (x^2 + 4x + 3)e^{-x}$.

1. a. Calculate $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote to the curve (C).

b. Calculate $\lim\limits_{x\to-\infty} f(x)$.

2. a. Verify that, for all x belongs to R, $f'(x) = (-x^2 - 2x + 1)e^{-x}$

b. Study the sense of variations of f, and set up the table of variations of the function f.

c. Calculate an approximate value to the nearest 10^{-1} the ordinate of each of the points of the curve (C) where the tangent at these points is parallel to the x- axis.

3. Prove that the equation: $f(x) = 2$ admits a unique solution α and then verify that: $-1 < \alpha < 0$.

Part C:

1. Let F be a function defined over R by: $F(x) = (-x^2 - 6x - 9)e^{-x}$.
   Prove that F is a primitive of f over R.
2. Deduce a primitive, G, of the function defined over R by: $g(x) = x + 3 - f(x)$.
3. Calculate the area of the domain bounded by the straight line (D), the curve (C) and the straight line of equation x = -3.

Exercise 8

Part A:

Consider the function g defined over $\mathbb{R}$ by: $g(x) = (x-1)e^x + 1$

1. Calculate $g'(x)$ and study the sign of g.

2. Set up the table of variations of the function *g*.
3. Deduce the sign of the function *g(x)*

Part B:

Consider the function f that is defined over $\mathbb{R}$ by: $f(x)=(x-2)e^{x}+x$

Let (C) be its representative curve in the plane referred to an orthonormal system $\left(O;\vec{i},\vec{j}\right)$,

Graphic unit = 2 cm.

1. Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$.
2. a. Verify that for all x in $\mathbb{R}$ we have: $f'x)=g(x)$.
   b. By using the part A, determine the table of variations of *f*.
3. a. Prove that the equation: $f(x)=0$ admits a unique solution noted .
   b. Calculate $f(1)$ and $f(2)$, and then deduce the bounded of .
4. a. Show that the straight line (d) of equation $y=x$ is an asymptote to the curve (C) at $-\infty$.
   b. Study the position of (C) with respect to (d).
   c. Prove that there exists a unique point A of the curve such that the tangent is parallel to .
   Determine the coordinates of point A.
5. Draw the curve (C) the straight line (d) and the tangent at A to the curve (C).
6. a. Consider the function H that is defined over $\mathbb{R}$ by: $H(x)=(x-3)e^{x}$, calculate $H'(x)$.
   b. Calculate, in cm² the area of the region bounded by (C), the straight line (d) and the two straight lines of equations: x = 0 and x = 2.

Exercise 9:

Part A:

Consider the function f that is defined over $\mathbb{R}$ by: $f(x)=\left(2x^{2}-5x+2\right)e^{x}$. Let (C) be the representative curve of the function f in an orthonormal system $(O;\vec{i},\vec{j})$, G .U = 2 cm

1. Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote to the curve (C)
2. a) Prove that for all real *x*, $f'(x)=\left(2x^{2}-x-3\right)e^{x}$.
   b) Study the sense of variations of f.
   c) Set up the table of variations of f.
3. Prove that the equation: $f(x)=2$ admits a unique solution α in the interval [2 ; 3].
4. Draw (C).

Partie B :

1. Designate by F the function defined over $\mathbb{R}$ by: $F(x)=\left(2x^{2}-9x+11\right)e^{x}$.
   Prove that the function F is a primitive of the function f over $\mathbb{R}$.
2. Calculate: $I=\int_{0,5}^{2} f(x)\,dx$.
3. Give a graphical interpretation of I.

Exercise 10:

Part A:

Consider the function f defined over the set of the real numbers IR by: $f(x)=\frac{1}{2}(2-x)e^{x}$. The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$ G.U = 2 cm. Let (C) be the representative curve of f.

1. Calculate $\lim\limits_{x\to+\infty} f(x)$..
2. Calculate $\lim\limits_{x\to-\infty} f(x)$. Deduce an asymptote to the curve (C).

3. a) Prove that : $f'(x)=\frac{1}{2}(1-x)e^{x}$.

b) Study the sense of variations of the function f.

c) Calculate $f(1)$.

d) Set up the table of variations of f on IR.

Part B:

1. Denote by (T) the tangent line to the curve (C) at a point of abscissa 2.

Prove that the equation of the straight line (T) is: $y=\frac{e^{2}}{2}(-x+2)$.

2. a) Verify that, for all real x, $\frac{e^{2}}{2}(-x+2)-f(x)=\frac{1}{2}(-x+2)\left(e^{2}-e^{x}\right)$

b) Study the sign of the expression $\frac{e^{2}}{2}(-x+2)-f(x)$.

c) Study the position of the curve (C) with respect to the straight line (T).

3. Draw (T) and (C) in the interval $[-4; 3]$.

Part C:

Consider the function g that is defined over $\mathbb{R}$ by: $g(x)=\frac{1}{2}(x-3)e^{x}+\frac{e^{2}}{2}\left(2x-\frac{x^{2}}{2}\right)$.

1. Calculate $g'(x)$, and then conclude.

2. Calculate the area of the region bounded by (C), (T) and the two straight lines x = 0 and x = 2.

Exercise 11:

Part A:

Consider the function g that is defined over $\mathbb{R}$ by: $g(x)=(2x-1)e^{x}+1$.

Given in the figure 1 a part of the table of variations of the function g over $\mathbb{R}$.

We know that the limit of the function g at $-\infty$ is 1 and that the limit of the function g at $+\infty$ is $+\infty$. Also we have: $g\left(-\frac{1}{2}\right)=\frac{-2}{\sqrt{e}}+1$

1. Complete the table of variations of the function g in the figure 1.

2. Calculate the exact value of $g(0)$.

3. Prove that the equation $g(x) = 0$ admits a unique solution α over the interval $\left[-2;-\frac{1}{2}\right]$.

4. Deduce the sign of $g(x)$ according to the values of x.

Part B:

Consider the function f that is defined over $\mathbb{R}$ by: $f(x)=(x-1)e^{2x}+e^{x}$.

In the figure 2, we find (C) the representative curve of the function f in the system $\left(O;\vec{i},\vec{j}\right)$

1. Calculate $\lim\limits_{x\to+\infty} f(x)$.

2. Calculate $\lim\limits_{x\to-\infty} f(x)$. Deduce an asymptote to the curve (C).

3. Calculate $f'(x)$ and prove that, for all real number x, $f'(x)=g(x)\times e^{x}$.

4. a) By using the result of the question 4 of the part A, determine according to the values of x the sign of $f'(x)$.

b) Set up the table of variations of the function f over $\mathbb{R}$.

c) Take $\alpha \approx -1,3$, determine an approximate value of $f(\alpha)$.

5. a) Consider the function F that is defined over $\mathbb{R}$ by: $F(x)=\left(\frac{x}{2}-\frac{3}{4}\right)e^{2x}+e^{x}$

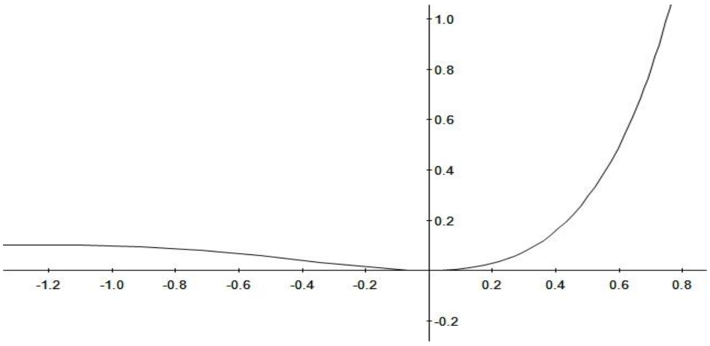
   Prove that the function F is a primitive of the function f over $\mathbb{R}$.

   b) Calculate the area of the domain bounded by (C), the x – axis and the two straight lines x = 0 and x = 1.

Figure 1

x	$-\infty$		$-1/2$		0		$+\infty$
$g'(x)$		$-$	0		$+$		
$g(x)$							

Figure 2



<u>Exercise 12</u>

Let f be a function defined over IR as: $f(x)=\frac{e^{-x}}{1+e^{x}}$. Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Calculate $\lim_{x\to-\infty} f(x)$ *and* $\lim_{x\to+\infty} f(x)$, then deduce an asymptote to (C).
2. a) Prove that for all real x, $f'(x)=\frac{-\left(2+e^{-x}\right)}{\left(1+e^{x}\right)^{2}}$.

   b) Set up the table of variations of f.

   c) Write an equation of the tangent (T) to (C) at a point of abscissa 0.

   d) Draw (T) and (C).
3. a) Prove that f admits an inverse function f^{-1} defined over $]0;+\infty[$.

   b) Draw the curve (C') *of* f^{-1} in the system $(O;\vec{i},\vec{j})$.
4. Let α be a strictly positive real number and A (α) the area of the part of the plane limited by the curve (C) the x - axis and the straight lines of equations: x = 0 and x = α.

   a) Prove that for all real x, $f(x)=e^{-x}-1+\frac{e^{x}}{1+e^{x}}$.

   a- Express A (α) in terms of α then calculate $\lim_{\alpha\to+\infty} A(\alpha)$.

Exercise 13 (G.S section)

Consider the function f that is defined over $]0;+\infty[$ by: $f(x)=\frac{1}{x^2}e^{\frac{1}{x}}$.

Let (C) be the representative curve of the function f in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Prove that: $f'(x)=-\frac{1}{x^4}e^{\frac{1}{x}}(2x+1)$.
2. Set up the table of variations of the function f.
3. Draw the curve (C).
4. For all naturel number n > 2 consider the integral I_n defined by : $I_n=\int_1^2 \frac{1}{x^n}e^{\frac{1}{x}}dx$.

   a. Calculate I_2.

   b. Prove by using integration by parts that for all naturel number n > 2 :

   $$I_{n+1}=e-\frac{\sqrt{e}}{2^{n+1}}+(1-n)I_n.$$

   c. Calculate I_3.

5. a. Establish for all real number x belongs to the interval [1 ; 2] we have: $0\le\frac{1}{x^n}e^{\frac{1}{x}}\le\frac{e}{x^n}$.

   b. Deduce the bounded of I_n then study the limit of I_n.

Exercise 14 (G.S section)

Consider the function f defined by: $\begin{cases} f(x)=\frac{1}{x^3}e^{\frac{-1}{x}} \text{ if } x>0. \\ f(0)=0. \end{cases}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. (G.U : 3 cm)

1. a. Prove that f is differentiable at 0 to the right and that: $f'(0^+)=0$.

   b. Prove that f is differentiable over $]0;+\infty[$ and for all $x\in]0;+\infty[: f'(x)=\frac{e^{\frac{-1}{x}}(1-3x)}{x^5}$.

   c. Set up the table of variations of f and construct (C).

2. Let α be a real number so that: α > 1.

   a. By using the integration by parts, calculate in cm², the area A (α) of the domain bounded by the curve (C) and the straight lines of equations: y = 0 ; x = 1 and x = α.

   b. Calculate $\lim_{\alpha\to+\infty} A(\alpha)$.

3. Let F be a function defined over $[1;+\infty[$ by : $F(x)=\int_1^x t^2 f(t)dt$.

   a. Prove that F differentiable over $[1;+\infty[$ and calculate $F'(x),\forall x\ge 1$.

b. Prove that for all real x ≥ 1; $\frac{1}{e}\ln(x) \leq F(x) \leq e^{\frac{1}{x}}\ln(x)$.

c. Set up the table of variations of F and then draw its curve (C_1). We precise the semi tangent to (C_1) at A (1; F (1)).

Exercise 15

Part A Consider the differential equation $(E): y'' + 2y' + y = 2x + 1$ and let $y = z + 2x - 3$.

1) a. Write a differential equation (E') satisfied by z then solve (E').

   b. Deduce the general solution of (E).

2) Let f be function so that f is a particular solution of (E). The below table is the table of variations of f 'the derivative of f.

x	$-\infty$		0		2		$+\infty$
$f''(x)$		$-$		$-$	0	$+$	
$f'(x)$	$+\infty$	$\searrow$	3	$\searrow$	$2-e^{-2}$	$\nearrow$	2

   Prove that: $f(x) = xe^{-x} + 2x - 3$

   Let (C) be the representative curve Of the function f in an orthonormal system $(O, \vec{i}, \vec{j})$ G.U = 2 cm.

3) a. Calculate $\lim\limits_{x \to -\infty} f(x)$

   b. Calculate $\lim\limits_{x \to +\infty} f(x)$ and prove that the straight line (D) *of equation*: $y = 2x - 3$ is an asymptote to the curve (C).

   c. Study according to the values of x the relative positions of (C) and (D).

   d. Verify that the point $W\,(2; 1 + \frac{2}{e^2})$ is an inflection point of (C).

4) a. Verify that f is strictly increasing over ℝ then set up its tableau of variations.

   b. Draw (D) and (C).

   c. Calculate, in cm^2, the area of the region bounded by (C), (D) and the two straights lines of equations $x = 0$ and x = 1.

5) a. Prove that f has over ℝ an inverse function g.

   b. Verify that $g(-3) = \frac{1}{3}$ then write an equation of the tangent to (G) the representative curve of g at a point with abscissa – 3.

Exercise 16

Part A: Consider the differential equation (E): y ' – 2 y = 4 e^x – 6. Suppose that z = y + 4 e^x – 3.

1. Determine the differential equation (E') satisfied by z.
2. Solve the equation (E'). Deduce the general solution of (E).
3. Determine the particular solution of (E) whose representative curve admits, at a point of abscissa 0 a tangent parallel to the straight line of equation: y = - 2 x.

Part B: Let f be a function defined over ℝ by : f (x) = e^{2x} – 4 e^x + 3. Designate by (C) the representative curve of the function f in an orthonormal system $(O; \vec{i}, \vec{j})$.

1. a- Calculate $\lim\limits_{x\to-\infty} f(x)$. Deduce an asymptote (d) to (C).

   b- Determine the coordinates of the point of intersection of (C) and (d).

c- Calculate $\lim\limits_{x\to+\infty} f(x)$ and calculate f (1).

2. Calculate f '(x) and set up the table of variations of f.
3. Prove that (C), has an inflection point I whose coordinates are to be determined.
4. Precise the points of intersection of the curve (C) with the x – axis.
5. Draw (d) and (C). (Graphic unit = 2 cm).
6. Calculate, in cm ^{2}, the area of the region bounded by (C), (d) and the two straight lines of equations x = 0 and x = ln (4).
7. Let g be the function defined by g (x) = ln [f (x)]. Designate by (C') its representative curve in an orthonormal system.
   a- Determine the domain of definition of the function g.
   b- Calculate $\lim\limits_{x\to+\infty} g(x)$. Prove that the straight line (Δ) of equation y = 2 x is an asymptote to (C'). Determine the other three asymptotes to (C').
   c- Set up the table of variations of g and draw (C') in a new system.
8. a – g, admits over]- ∞; 0[an inverse function g^{-1}. Precise the domain of definition of g^{-1} and draw its representative curve (γ) in the same system as that of (C').
   b- Determine the coordinates of the point of intersection of (C') and (γ).

Exercise 17

Let f be the function defined over ℝ as: $f(x)=2-\dfrac{4e^x}{1+e^x}$. Denote by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1. a- Determine $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$, then deduce that (C) has two asymptotes.
   b- Prove that f is an odd function and interpret graphically the result thus obtained.
2. a- Calculate f '(x) and set up the table of variations of f.
   b- Write an equation of (T), the tangent to (C) at O.
   c- Draw (T) and (C).
3. a- Prove that f has over ℝ an inverse function g.
   b- Determine the domain of definition of g, then express g (x) as function of x.
   c- Prove that the graph (C') of g is tangent at O to (C). Then draw (C') in the same system as that of (C).
4. Let (D) be the region bounded by (C'), (y' y) and the line with equation y = a where a > 0.
   a- Calculate in terms of a the area of (D).
   b- Calculate a so that this area is equal to 4 ln 2 unit of area.

Exercise 18

Let f be a function defined over $\mathbb{R}$ as $f(x)=(x^2+2x+2)e^{-x}$. Denote by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Graphic unit: 2 cm.

1. Calculate $\lim_{x\to-\infty} f(x)$ and $f(-1.5)$.
2. Calculate $\lim_{x\to+\infty} f(x)$ and deduce an asymptote (d) to (C).
3. Calculate f '(x) and set up the table of variations of f.
4. Verify that (C) has two inflection points one of them is W (0;2).
5. Draw (C).
6. Let F be a function defined over $\mathbb{R}$ as $F(x)=(ax^2+bx+c)e^{-x}$ where a, b and c are real numbers.
   a- Determine a, b and c so that F is a primitive of the function f.
   b- Let α be a positive real number. Designate by A (α) the area of the region bounded by (C), the x – axis and the two straight lines of equations x = 0 and x = α.
   Prove that: $A(\alpha)=24-4(\alpha^2+4\alpha+6)e^{-\alpha}cm^2$.
7. Let g be the inverse function of the function f over [0; +∞[and let (G) be its representative curve.
   a- Study the differentiability of g at x = 2.
   b- The equation g (x) = x admits a unique root β. Prove that 1.5 < β < 1.7.
   c- Draw (G) in the same system as that of (C).

Exercise 19

Part A:

Let g be a function defined over $\mathbb{R}$ as: g (x) = 1 + (1 – x) e ^{x}.

1. a- Determine $\lim_{x\to-\infty} g(x)$ and $\lim_{x\to+\infty} g(x)$.
   b- Calculate g '(x) and set up the table of variations of g.
2. a- Prove that the equation g (x) = 0 admits a unique root α, then verify that 1.27 < α < 1.28.
   b- Deduce the sign of g (x) for all real x.

DIFFERENTIAL EQUATIONS
CHAPTER REVIEW

1.General solution of the differential equation of first order:

- **Equations Of the type**: $y' = f(x)$

  Let f be a continuous function over an interval I. The solutions of the equation $y' = f(x)$ is given by: $y = \int f(x)dx + c$, where c is an arbitrary constant.

Example: The solutions of the equation: $y' = 3e^{3x} - 5x$ are

$$y = \int (3e^{3x} - 5x)dx = e^{3x} - \frac{5}{2}x^2 + c.$$

- **General solution of an equation with separable variables**:

  The solutions of the equation with separable variable $g(y)y' = f(x)$, are given by:

$$\int g(y)dy = \int f(x)dx + c, \text{ Where c is an arbitrary constant}$$

Example: Consider the differential equation: $y' - 2xy = 0$

$$\Rightarrow y' = 2xy \Rightarrow \frac{y'}{y} = 2x \Rightarrow \frac{dy}{ydx} = 2x \Rightarrow \frac{dy}{y} = 2xdx \Rightarrow \int \frac{dy}{y} = \int 2xdx$$ If y >0 then

$$\ln(y) = x^2 + c \Leftrightarrow y = e^{x^2} \times e^c.$$

- **General solution of a linear differential equation without second member:**

The equation $y' + ay = 0$ has a general solution $y = ce^{-ax}, x \in IR$, where c is an arbitrary constant.

Example: The general solution of the equation $y' - 4y = 0$ is $y = ce^{4x}$.

- **General solution of a linear differential equation with second member**:

If Y is a particular solution of the equation $y' + ay = f(x)$, then the general solution of this equation is: $y = ce^{-ax} + Y(x)$, where c is an arbitrary constant.

Example:

Find the particular solution of the equation: $y' - 2y = 4x + 2$ in the form y = a x + b.

After replacement, in this equation, we have a = b = -2. So the general solution is:

$$y(x) = ce^{2x} - 2(x+1).$$

- **Solution verifies an initial condition**.

There exists one and only one solution of the equation $y' + ay = f(x)$ satisfies the initial condition: $y(x_0) = y_0, x_0$ *and* y_0 are two real numbers.

Example:

The solution of the differential equation, of the preceding example such that:

y (0) = 1 is: 1 = c -2, So c =3. The solution: $y = 3e^{2x} - 2(x+1)$.

1. The general solution of the differential equation of 2nd order always contains two arbitraries constants.

- **Equations of the type** $y'' = f(x)$.

Consider the function f that is continuous over an interval I.

The general solution of the equation y = f(x) is obtained by two steps:

i) Calculating first: $y' = \int f(x)dx + c_1$,

ii) Second $y = \int y'(x)dx + c_2, c_1$ *and* c_2 are two arbitraries constants.

- **Equations Without second member:**

We call a characteristic equation of the equation $y'' + ay' + by = 0$, The equation of second degree: $r^2+ar+b=0$.

If r_1 and r_2 the roots of the characteristic equation of the differential equation $y''+ay'+by=0$.

1. If r_1 and r_2 are real and distinct, then the general solution of this equation is $y(x)=c_1e^{r_1x}+c_2e^{r_2x}$.
2. If $r_1=r_2=-\frac{a}{2}$, then the general solution of this equation is: $y(x)=(c_1+c_2x)e^{\frac{-ax}{2}}$.
3. If r_1 *and* r_2 are complexes of the form $r_1=\alpha+i\beta=\overline{r_2}$, then the general solution is $y(x)=e^{\alpha x}\left[c_1\cos(\beta x)+c_2\sin(\beta x)\right]$ c_1 and c_2 are two arbitraries constants.

Example: the differentials equations: $y''-3y'+2y=0,\ 4z''+4z'+z=0$ *and* $u''-4u'+5u=0$ have the respective general solutions: $y(x)=c_1e^x+c_2e^{2x}, z(x)=(c_1+c_2x)e^{\frac{-x}{2}}$ *and* $u(x)=e^{2x}\left[c_1\cos x+c_2\sin x\right]$.

• Equations of the type $y''+w^2y=k$.

The general of the differential equation: $y''+w^2y=k$, where w and k are two given real is $y(x)=c_1\cos(wx)+c_2\sin(wx)+\frac{k}{w^2}$.

DIFFERENTIAL EQUATIONS

Exercise 1

Part A:

Consider the differential equation (E): $y'+2y=x$, Assume that: $y=z+\frac{1}{2}x-\frac{1}{4}$.

1. Write a differential equation (F) satisfied by z.
2. Solve (F), and then find the general solution of (E).
3. Determine the particular solution of (E) such that: $y(0)=\frac{3}{4}$.

Part B:

Consider the function f that is defined over the set R by: $f(x)=\frac{1}{2}x-\frac{1}{4}+e^{-2x}$

Let (C) be its representative curve in an orthonormal system $(O;\vec{i};\vec{j})$.

1. a - Calculate $\lim\limits_{x\to+\infty} f(x)$.

   b- Justify that $f(x)=e^{-2x}\left(\frac{1}{2}xe^{2x}-\frac{1}{4}e^{2x}+1\right)$ and then deduce $\lim\limits_{x\to-\infty} f(x)$.

   c- Prove that the straight line (D) of equation: $y=\frac{1}{2}x-\frac{1}{4}$ is an asymptote to the (C) at $+\infty$, and precise the position of the curve (C) with respect to the straight line (D).
2. a- Calculate $f'(x)$.

   b- Solve the equation $e^{-2x}>\frac{1}{4}$ and then deduce the table of variations of the function f.

   c- Determine an equation of the tangent line (T) to the curve (C) at a point on (C) of abscissa 0.

   d- Prove that the equation $f(x)=\frac{1}{2}$ has a unique solution over the interval [1; 2].
3. Draw (D), (T) and (C).

Part C:

1. Let m be a real number such that m > ln2. Denote by A (m) the area of the domain bounded by (C), the straight line (D) and the two straight lines of equations x = ln2 and x = m.
2. Calculate the limit of A (m) as x tends to $+\infty$.

Exercise 2

The plane is referred to an orthonormal system $(O;\vec{i};\vec{j})$.

Part A:

Consider the differential equation (E): $y'+y=3e^{-x}+x+1$. Assume that : y = z + 3 x e $^{-x}$ +x.

1. Find a differential equation (F) satisfied by Z.
2. Solve (F), and then find the general solution of (E).
3. Determine the particular solution of (E) such that: y (0) = 2.

Part B:

Let g be a function defined over $\mathbb{R}$ as: $g(x)=e^{-x}(-3x+1)+1$.

1. Calculate $\lim\limits_{x\to-\infty} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x)$.
2. Calculate $g'(x)$.
3. Study the sense of variations of the function g, and then set up the table of variations of g.
4. Calculate $g\left(\frac{3}{4}\right)$ and deduce the sign of the function g on IR.

Part C:

Consider the function f that is defined over $\mathbb{R}$ by: $f(x)=e^{-x}(3x+2)+x$

Let (C) be its representative curve in an orthonormal system $(O;\vec{i};\vec{j})$.

1. a - Calculate $\lim\limits_{x\to-\infty} f(x)$.

   b- Calculate $\lim\limits_{x\to+\infty} f(x)$.
2. a- Prove that for all real number x $f'(x)=g(x)$.

   b- Deduce the table of variations of f.
3. Prove that the straight line (D) of equation y = x is an asymptote to the curve (C) at $+\infty$, and then precise the position of (C) with respect (D). Denote by A the point of intersection of (C) and (D).
4. Determine the abscissa of point B of the curve (C) where the tangent (T) is parallel to (D).
5. Draw (D), (T) and (C).

Part D :

1. Prove that the function F that is defined over IR by: $F(x)=-f(x)-3e^{-x}+\frac{x^2}{2}+x$ is a primitive of the function f over IR.
2. Deduce the area A of the domain bounded by the curve (C), the x – axis and the two straight lines x = 0 and x = 1.

Exercise 3

Part A:

1. Find the general solution of the following differential equation: $y'-2y=0$.
2. Determine the particular solution f of this differential equation verifies $f(0)=1$.

Part B:

Consider the function g that is defined for all real x by: $g(x)=3e^{x}+2x-4$.

Verify that g is the solution differential equation: $g'(x)-g(x)=6-2x$.

Part C:

1. Consider the function h that is defined for all real x by: $h(x)=e^{2x}-3e^{x}-2x+4$,

and denote by (C) its representative curve in an orthonormal system $(O;\vec{i};\vec{j})$.

2. Calculate $\lim_{x\to+\infty} h(x)$.
3. a - Determine $\lim_{x\to-\infty} h(x)$.

   b- Prove that the straight line (D) of equation y = -2x + 4is an asymptote to (C) at $-\infty$.

   c- Suppose that for all real x, $d(x)=h(x)+2x-4$

   • Verify that: $d(x)=e^{x}\left(e^{x}-3\right)$.

   • Study the sign of $d(x)$ for all real number *x.*

   • Deduce the relative position of the curve (C) and the straight line (D).
4. a- Verify that for all real x we have: $h'(x)=\left(e^{x}-2\right)\left(2e^{x}+1\right)$.

   b- Study the sign of $h'(x)$ for all real x, deduce the sense of variations of h and set up the table of variations of h.

   c- Draw (D) and the curve (C).

Part C:

Calculate the area of the domain bounded by (C) the straight line (D) and the two straight lines of equations x = 0 and x = ln3.

Exercise 4:

Part A: Consider the differential equation (E): $y'+y=x+1$. Assume that: z = y –x.

1. Write a differential equation (F) satisfied by z.
2. Solve (F), and then find the general solution of (E).
3. Find the particular solution of (E) such that: y (0) = 1.

Part B:

Consider the function f that is defined over $\mathbb{R}$ by: $f(x)=x+e^{-x}$

Denote by (C) the representative curve of the function f in an orthonormal system $(O;\vec{i};\vec{j})$

(Graphic unit : 2 cm).

1. Calculate $\lim_{x\to+\infty} f(x)$.
2. Verify that for all real number x, we have: $f(x)=e^{-x}\left(xe^{x}+1\right)$, deduce $\lim_{x\to-\infty} f(x)$.
3. Calculate $f'(x)$, study the sense of variations of f and then set up the table of variations of f.
4. a - Prove that the straight line Δ of equation: y = x is an asymptote to the curve (C),

   b- Determine the relative position of the curve (C) and the straight line Δ.

   c- Draw Δ and (C).
5. a- Determine a primitive F of the function f over IR,

   b- Calculate in cm^2 the area of the region bounded by (C), the x – axis and the two straight lines of equations x = 0 and x = 2.

Exercise 5

Part A:

Consider the differential equation (E): $y''-4y'+4y=12$. Suppose that: y = z +3.

1. Write a differential equation (F) satisfied by z.
2. Solve (F), and then deduce the general solution of (E).
3. Find the particular solution of (E) such that: y (1) = -2 e +2 and $y'(1)=0$.

Part B:

Let g be a function defined over IR as: $g(x)=(2x-3)e^{2x-1}+2$.

1. Calculate $\lim_{x\to-\infty} g(x)$ *and* $\lim_{x\to+\infty} g(x)$.
2. Calculate $g'(x)$, and then set up the table of variations of g.

3. Prove that the equation g (x) = 0 admits two solutions one of them is $\frac{1}{2}$ and the other α such that: $1.2 < \alpha < 1.4$.
4. Deduce the sign of g (x) for all x belongs to IR.

Part C:

Consider the function f that is defined over IR by: $f(x)=\left(\frac{x}{2}-1\right)e^{2x-1}+x-\frac{1}{2}$. Let (C) be its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$. Graphic unit = 2 cm.

1. Calculate $\lim_{x\to+\infty} f(x)$, $f(1)$ *and* $f(2)$.
2. Calculate $\lim_{x\to-\infty} f(x)$ and prove that the straight line (d) of equation $y=x-\frac{1}{2}$ is an asymptote to (C). Study the relative positions of (C) and (d).
3. Prove that: $f'(x)=\frac{1}{2}g(x)$, and then study the sense of variations of f.
4. Set up the table of variations of f.
5. Prove that: $f(\alpha)=\frac{4\alpha^2-4\alpha-5}{4(2\alpha-5)}$.
6. Take: α = 1.3. Draw (d) and (C).
7. Calculate, in cm², the area of the domain bounded by (C), (d) and the two straight lines of equations x = 2 and x = 3.

Exercise 6

Part A:

Consider the differential equation (E): $y'+2y=\frac{1}{2}e^{-2x+1}+1$. Suppose that: $y=z+\frac{x}{2}e^{-2x+1}+\frac{1}{2}$.

1. Write a differential equation (F) satisfied by z.
2. Solve (F), and then deduce the general solution of (E).
3. Find the particular solution of (E) such that: $y\left(\frac{3}{2}\right)=\frac{1}{2}$.

Part B:

Let g be a function defined over IR as: $g(x)=\left(\frac{x}{2}-\frac{3}{4}\right)e^{-2x+1}+\frac{1}{2}$.

1. Calculate $\lim_{x\to-\infty} g(x)$ *and* $\lim_{x\to+\infty} g(x)$.
2. Calculate $g'(x)$, and then set up the table of variations of g.
3. Calculate $g\left(\frac{1}{2}\right)$, and then deduce the sign of g (x) for all x belongs to IR.

Part C:

Let f be a function defined over IR by: $f(x)=\left(-\frac{x}{4}+\frac{1}{4}\right)e^{-2x+1}+\frac{1}{2}x+1$. Let (C) be its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$. G.U: 2 cm.

1. Calculate $\lim_{x\to-\infty} f(x)$, $f(-1)$ *and* $f(2)$.
2. Calculate $\lim_{x\to+\infty} f(x)$ and prove that the straight line (d) of equation $y=\frac{1}{2}x+1$ is an asymptote to (C). Study the relative positions between (C) and (d).
3. Prove that: $f'(x)=g(x)$. Study the sense of variations of f.

4. Prove that f admits an inflection point $I\left(2;-\frac{1}{4}e^{-3}+2\right)$.
5. Set up the table of variations of f.
6. Find the coordinates of point A such that the tangent (T) at A to (C) is parallel to (d).
7. Draw (d), (T) and (C).
8. Calculate, in cm^2, the area of the domain bounded by (C), (d) and the straight lines x = 0.5 and x = 1.

<u>Exercise 7</u>

<u>Part A:</u>

Consider the differential equation (E) : $y''-4y'+4y=4$. Assume that: y = z +1.

1. Write a differential equation (F) satisfied by z.
2. Solve (F), and then deduce the general solution of (E).
3. Find a particular solution of (E) so that: y (0) = 2 and $y'(0)=0$.

<u>Part B:</u>

Let g be a function defined over IR as: $g(x)=1+(-2x+1)e^{2x}$.

1. Calculate $\lim\limits_{x\to-\infty} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x)$.
2. Study the sense of variations of g.
3. Prove that the equation g (x) = 0 has a unique solution α in the interval [0; 1].
4. Determine the sign of g (x) according to the values of x.

<u>Part C:</u>

Consider the function f that is defined over IR by: $f(x)=\frac{1}{2}(1-x)e^{2x}+\frac{1}{2}x$. Let (C) be its representative curve of f in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$. (G.U: 2 cm).

1. a - Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$.

   b- Prove that the straight line (d) of equation: y = $\frac{1}{2}x$ is an asymptote to (C).

   Study the relative positions between (C) and (d).

   c- Prove that: $f'(x)=\frac{1}{2}g(x)$. Study the sense of variations of f over IR.
2. Set up the table of variations of f.
3. Draw (d) and (C).
4. Calculate the area, in cm^2, of the domain bounded by the curve (C), the straight line (d) and the two straight lines of equations: x = 0 and x = 1.

<u>Exercise 8</u>

<u>Part A:</u>

Consider the differential equation (E) : $y''+2y'+y=2e^{-x}+1$. Assume that: $y=z+x^2e^{-x}+1$.

1. Write a differential equation (F) satisfied by z.
2. Solve (F), and then deduce the general solution of (E).
3. Find a particular solution of (E) so that: y (0) = 0 and y (1) = 1 + 2 e^{-1}.

<u>Part B:</u>

Consider the function g that is defined over IR by: $g(x)=\left(x^2+2x-1\right)e^{-x}+1$.

1. Calculate $\lim\limits_{x\to-\infty} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x)$.
2. Calculate $g'(x)$ and prove that $g'(x)$ *and* $\left(3-x^2\right)$ have the same sign.
3. Prove that the equation g (x) = 0 admits two roots one of them is 0 and the other is α such that: -2.4 <α < -2.3.
4. Deduce the sign of g (x) according to the values of x.

Part C:

Consider the function f that is defined over IR by: $f(x)=x-(x^2+4x+3)e^{-x}$. Let (C) be the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$; G.U: 2 cm.

1. Calculate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$.
2. a. Prove that, for all real x, we have : $f'(x)=g(x)$.

   b. Set up the table of variations of the function f.
3. Prove that the straight line (D) of equation y = x, is an asymptote to the curve (C).
4. a. Prove that the straight line (D) and the curve (C) are intersecting at two points A and B whose coordinates are to be determined.

   b. Study the relative positions between the straight line (D) and the curve (C).
5. Draw the curve (C) and the straight line (D).

Part D:

1. Let H be a function that is defined over IR as: $H(x)=(ax^2+bx+c)e^{-x}$.

   Determine the real's a, b and c such that the function H is a primitive of the function h defined by $:h(x)=(x^2+4x+3)e^{-x}$.
2. Determine the area, in cm², of the region bounded by the curve (C) and the straight line (D).
3. Let m un be a real number greater than -1. Consider the domain (D_m) bounded by the curve (C), the straight line (D) and the two straight lines of equations: x = -1 and x = m.
   a. Calculate the area (A_m) of the domain (D_m).
   b. Determine the limit of (A_m) as m tends to $+\infty$.

Exercise 9

Part A:

Consider the differential equation (E): $y'+y=-x-1$.

1. a - Solve the differential equation: $y'+y=0$.

   b- Determine the solution *h* of the differential equation: $y'+y=0$ so that h (1) = $1/e$.
2. Determine the real number *a* so that the function *u* defined over $\mathbb{R}$ by $u(x)=e^{-x}+ax$ is the solution of the differential equation (E).

Part B:

Consider the function f that is defined over IR by: $f(x)=e^{-x}-x$.

1. Determine the limits of the function *f* at $+\infty$ and at $-\infty$.
2. Calculate $f'(x)$, and then set up the table of variations of the function f.
3. Prove that the equation: $f(x)=0$ admits a unique solution α in the interval [0; l].
4. Precise the sign of f (x) over the interval [0; l].
5. Draw the curve (C) of the curve f in an orthonormal system.
6. Calculate in terms of α the area of the region bounded by the curve (C), the x – axis and the y – axis.

Part D:

The function g is defined over $]-\infty;\alpha[$ by: $g(x)=\dfrac{x}{e^{-x}-x}$ (where α is the real number that found in the part B) and denote by (C_g) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. a - Verify that, for all $x\in]-\infty;\alpha[$ $g(x)=\dfrac{xe^x}{1-xe^x}$

   b- Deduce the limit of the function g at $-\infty$ and interpret graphically this limit.
2. By using the results that founds in the part B question 4, determine the limit of the function g at α.

   Interpret graphically this limit.

3. a- Prove that for all $x \in]-\infty ; \alpha[$: $g'(x)=\dfrac{e^{-x}(x+1)}{(e^{-x}-x)^2}$.

b- Deduce the sense of variations of the function g over $]-\infty; \alpha[$ and set up the table of variations of the function g.

4. Draw the curve (C_g).

<u>Exercise 10</u>

<u>Part A:</u>

Consider the differential equation $(E): y'+5y=5x^3+3x^2+5$.

1. Solve the differential equation $(E_0): y'+5y=0$.
2. Determine the two real numbers a and b so that the function u, defined over $\mathbb{R}$ by $u(x)=a\,x^3+b$, is the particular solution of the differential equation (E).
3. Let h be a function defined over $\mathbb{R}$ by $h(x)=k\,e^{-5x}+x^3+1$ where k is a real number.

a- Verify that h is a solution of the equation (E).

b- Determine the real k such that: $h(0)=2$.

<u>Part B:</u>

Consider the function f that is defined over $\mathbb{R}$ by: $f(x)=-3e^{-5x}+x^3+1$.

1. a - Determine the limit of $f(x)$ as x tends to $-\infty$.

b- Determine the limit of $f(x)$ as x tends to $+\infty$.

2. a - Calculate $f'(x)$ pour tout real x.

b- Deduce the sense of variations of the function f over $\mathbb{R}$ and set up the table of variations of f.

3. a - Calculate $f(0)$ and $f(1)$.

b- Solve that the equation $f(x)=0$ admits a unique solution α in the interval [0; 1].

c- Determine according to the value of x the sign of f (x).

<u>Part C:</u>

Denote by (C) the representative curve of function f in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Let u be a function defined over $\mathbb{R}$ by: $u(x)=x^3+1$. The representative curve Γ of the function u, in the system $(O;\vec{i},\vec{j})$.

a- Suppose that, for all x: $d(x)=f(x)-u(x)$. Study the sign of d (x).

b- Study the relative positions of the curve (C) with respect to the curve Γ.

2. Copy and complete the below table.

x	$-0,2$	0	0,2	0,4	0,6	0,8	1	1,2
$f(x)$								

3. Draw the curve (C).
4. Calculate the area of the domain bounded by the curve (C), the x – axis and the two straight lines of equations: x = 0.5 and x = 1.

<u>Exercise 11</u>

<u>Part A:</u>

Consider the differential equation (E): $2y'+y=0$.

1. Solve the differential equation (E).
2. Let f be a particular solution of the differential equation (E) so that: $f(2)=e$.

Prove that, for all real x, $f(x)=e^{\frac{2-x}{2}}$.

3. For all real x, suppose that: $g(x)=(2x+1)[f(x)]^2-9$.

Prove that: $g(x)=(2x+1)e^{4-x}-9$.

Part B:

Designate by (C) the representative curve of the function g in an orthonormal system $(O;\vec{i},\vec{j})$..

1. Determine the limit of $g(x)$ as x tends to $-\infty$.
2. a- Prove that for all real number x, $g(x)=2e^{4}xe^{-x}+e^{4}e^{-x}-9$.
   b- Use this expression for determine the limit of $g(x)$ as x tends to $+\infty$.
   c- Deduce that the curve (C) admits an asymptote Δ whose equation is to be determined.
3. a- Prove that for all real x: $g'(x)=(1-2x)e^{4-x}$
   b- Determine the sense of variations of g and set up the table of variations of g over $\mathbb{R}$.
4. a- Calculate the exact value of the numbers $g(-1)$ and $g(0)$.
   b- Prove that the equation : $g(x) = 0$ admits a unique solution α in the interval $[-1; 0]$.
5. Determine an equation of the straight line (D) tangent to the curve (C) at a point of abscissa 4.
6. Draw Δ, the straight line (D) and then the curve (C).

Part C:

1. Prove that the function G that is defined over $\mathbb{R}$ by $G(x)=(-2x-3)e^{4-x}-9x$ is a primitive of g.
2. Calculate the area of the region bounded by the curve (C), the x- axis and the two straight lines of equations x = 0 and x = 4.

Exercise 12

Part A:

Consider the differential equation (E): $y''+4y'+4y=-4x$.

Suppose that : y = z – x +1.

1. Write a differential equation (F) satisfied by z. Solve (F).
2. Deduce the general solution of (E).
3. Find a particular solution of (E) such that y (0) =1 and $y'(0)=0$.

Part B :

Consider the function f that is defined on IR by: $f(x)=x.e^{-2x}-x+1$. Let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$, $G.U=2cm$.

1. Calculate $\lim\limits_{x\to-\infty} f(x)$ and $f(-1)$.
2. Calculate $\lim\limits_{x\to+\infty} f(x)$, and then prove that the straight line (d) of equation y = - x +1 is an asymptote to (C).
3. Study the relative positions between (d) and (C) according to the values of x.
4. The table below is the table of variations of f' the derivative of f.

X	$-\infty$	0	1	$+\infty$
$f''(x)$			0	
$f'(x)$	$+\infty$	0	$-1-e^{-2}$	-1

   a. Study the sense of variations of the function f over IR.
   b. Set up the table of variations of f.
   c. Prove that f has an inflection point I whose coordinates is to be determined.
   d. Write an equation of the tangent line (T) to the curve (C) at a point A of abscissa 1.
   e. Draw (d) and (C).

5. Calculate, in cm², the area of the region bounded by (C), (d) and the two straight lines of equations x = 0 and x = 1.

Exercise 13

Consider the differential equation (E): $4y''+4y'+y=-x-2$. Assume that: y = z + 2 – x.

Part A:

1. Write a differential equation (F) satisfied by z.
2. Solve (F), and then deduce the general solution of (E).
3. Find the particular solution of (E) so that: $y(0)=-2$ *and* $y'(0)=3$.

Part B:

Consider the function g that is defined over IR by: $g(x)=(4-x)e^{-\frac{x}{2}}-1$.

1. Calculate $\lim_{x\to-\infty} g(x)$ *and* $\lim_{x\to+\infty} g(x)$.
2. Calculate $g'(x)$, and then set up the table of variations of g.
3. Prove that the equation g (x) = 0 admits a unique solution α, and verify that: 1.6 <α< 1.8.
4. Deduce the sign of g (x) for all real x.

Part C:

Consider the function f that is defined on IR by: $f(x)=(2x-4)e^{-\frac{x}{2}}+2-x$. Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Graphic unit = 2 cm.

1. Calculate $\lim_{x\to-\infty} f(x)$ *and* $\lim_{x\to+\infty} f(x)$.
3. Prove that the straight line (d) of equation y = 2 – x is an asymptote to (C) at + ∞.
4. Prove that: $f'(x)=g(x)$. study the sense of variations of f.
5. Prove that: $f(\alpha)=\dfrac{\alpha^2-4\alpha+4}{4-\alpha}$. Set up the tableau of variations of f.
6. a- Calculate the coordinates of the points of intersection of (C) and the axis of abscissas.
   b- Calculate the coordinates of E the intersection point of (C) and the axis of ordinates
   b- Write an equation of the tangent (T) to (C) at the point E.
7. Suppose that α = 1.7. Draw (d), (T) and (C).
8. Calculate, in cm², the area of the region bounded by the curve (C), (d) and the straight lines x = 0 and x = ln 4.

Exercise 14

Part A : Consider the differential equation (E) : $y''+4y'+4y=4x-16$. *Suppose that* : $y=z+x-5$.

1. Find the differential equation (F) satisfied by z.
2. Solve (F), then deduce the general solution of (E).
3. Find the particular solution f of (E) so that: $f(0)=-2$ *and* $f'(0)=-3$.

Part B : Consider the function f defined over IR by: $f(x)=(2x+3)e^{-2x}+x-5$.

Let (C) be the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Calculate $\lim_{x\to-\infty} f(x)$, $\lim_{x\to+\infty} f(x)$ *and* $\lim_{x\to+\infty}\left[f(x)-(x-5)\right]$. Deduce an asymptote (d) to (C) at $+\infty$.
2. The table below is the table of variations of f' the derivative of f.

X	$-\infty$	-1/2	$+\infty$
$f''(x)$	−	0	+
$f'(x)$	1 ↘	1-2e	↗ $+\infty$

a- **Prove that (C) has an inflection point I whose coordinates are to be determined.**

b- **Prove that the equation $f'(x)=0$ has two roots α and β so that:** $-1<\alpha<0 \quad and \quad 1<\beta<1.1.$

c- **Deduce the sign of $f'(x)$ on IR.**

3. **Set up the table of variations of f. (Assume that: α= -1.2 and β= 1.05).**
4. **Draw (d) and (C).**

<u>Part C:</u> Let H be a primitive of the function h defined on IR by: $h(x)=(2x+3)e^{-2x}$.

1. **Prove that :** $H(x)=(-x-2)e^{-2x}+1.$
2. **Calculate the area A of the region bounded by (C), (d) and the two straight lines of equations** $x=-\dfrac{3}{2}$ and $x=0.$

INTEGRALS
CHAPTER REVIEW

Definition and notations:

1. Let f be a continuous function over an interval I, F is a primitive of f over I, a and b two points of I. We say that the integral of f from a to b, the real number F (b)-F (a). This number is denoted by:

$\int_a^b f(x)dx = F(b) - F(a).$ 2. $\int_a^a f(x)dx = 0.$

3. $\int_b^a f(x)dx = -\int_a^b f(x)dx.$ 4. $\int_a^b 1dx = b - a.$

5. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

6. Let f be a continuous function and positive over [a; b], $a \leq b$.
The area of the region bounded by the representative curve of f, (orthonormal system), the x – axis and the two straight lines of equations: x =a and x =b, expressed in unit square of areas is calculated by: $\int_a^b f(x)dx.$

When the function f is negative then we consider– f, and in this case, $\int_a^b -f(x)dx$ is the given Area.

7. Let f and g be two continuous functions over an interval I. a and b two real's of I.
for all real's α and β we have: $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx.$

8. Let f be a continuous function over an interval I and a is a real of I.
The defined function over I by: $x \to \int_a^x f(t)dt$ is a primitive of f over I, that vanishes at a.

9. Let f be a continuous function over an interval I and let a and b two real's of I.
If $a \leq b$ and if f≥ 0 over$[a,b]$ then: $\int_a^b f(x)dx \geq 0.$

10. Let f and g be two continuous functions over an interval I and let a< b be two real's of I.
If $f(x) \leq g(x)$ over$[a,b]$,then $\int_a^b f(t)dt \leq \int_a^b g(t)dt.$

11. If f is a continuous function over an interval I of center O, and a is any real of I.
a) If f is an even function then: $\int_{-a}^{+a} f(t)dt = 2\int_0^{+a} f(t)dt.$
b) If f is an odd function then: $\int_{-a}^{+a} f(t)dt = 0.$

12. Let f be a differentiable function over IR and a is a any real number. if f is periodic function of period T, then : $\int_a^{a+T} f(t)dt = \int_0^T f(t)dt.$

13. Integration by change of variable:
Let f be a continuous function on an interval I, and ϕ a function whose derivative is

continuous on an interval: $J=[\alpha,\beta]$ such that: $\phi(J)\subset I$.

We have: $\int_{\alpha}^{\beta} f\,(\phi(t))\phi'(t)\,dt=\int_{a}^{b} f\,(x)\,dx$ where $a=\phi(\alpha)$ et $b=\phi(\beta)$.

Example: $I=\int_{0}^{2} t.e^{t^2}dt$, suppose that: u= $\phi(t)=t^2 \Rightarrow u'=2t.dt=\phi'(t)$. thus

$$I=\frac{1}{2}\int_{0}^{4} e^{u}du=\frac{1}{2}\left[e^{u}\right]_{0}^{4}=\frac{1}{2}\left(e^{4}-1\right).$$

14. Primitives :

- $\int u^{\alpha}(x)u'(x)\,dx=\frac{1}{\alpha+1}u^{\alpha+1}(x)+c.\quad \alpha\neq -1.$
- $\int e^{u(x)}u'(x)\,dx=e^{u(x)}+c.$
- $\int \cos(u(x))u'(x)\,dx=\sin(u(x))+c.$
- $\int \sin(u(x))u'(x)\,dx=-\cos(u(x))+c.$
- $\int \frac{u'(x)}{\cos^{2}(u(x))}dx=\tan(u(x))+c.$
- $\int \frac{u'(x)}{\sin^{2}(u(x))}dx=-\cot(u(x))+c.$
- $\int \frac{u'(x)}{a^{2}+u^{2}(x)}dx=\frac{1}{a}\arctan\left(\frac{u(x)}{a}\right)+c.$
- $\int \frac{u'(x)}{u(x)}dx=\ln|u(x)|+c.$

15. Integration by parts:

Let u and v be two differentiable functions on an interval I such that u' and v' are Continuous functions on I. We have, for all real numbers a and b in I:

$$\int_{a}^{b} u'(x)v(x)\,dx=\left[u(x)v(x)\right]_{a}^{b}-\int_{a}^{b} u(x)v'(x)\,dx.$$

Example:

To calculate $I=\int_{0}^{1} te^{t}dt$, suppose that: $u'(t)=e^{t}$ *and* $v(t)=t$. Then $v'(t)=1$ and we

Choose $u(t)=e^{t}$. So $I=\int_{0}^{1} te^{t}dt=\left[te^{t}\right]_{0}^{1}-\int_{0}^{1} e^{t}dt=1.$

16. Linearization of trigonometric polynomials:

a) To calculate integrals such as:

$\int_{a}^{b} \sin(px)\sin(qx)\,dx$, $\int_{a}^{b} \sin(px)\cos(qx)\,dx$ *and* $\int_{a}^{b} \cos(px)\cos(qx)\,dx$ where p and q are positive integers, we use the following identities:

$$\sin(px)\sin(qx)=\frac{1}{2}\left[\cos(p-q)x-\cos(p+q)x\right];$$

$$\sin(px)\cos(qx)=\frac{1}{2}\left[\sin(p-q)x+\sin(p+q)x\right];$$

$$\cos(px)\cos(qx)=\frac{1}{2}\left[\cos(p-q)x+\cos(p+q)x\right].$$

b) To calculate integrals of the form $\int_{a}^{b} \sin^{p}(x)\cos^{q}(x)\,dx$.

We distinguish two cases:

i) If at least one of the two numbers p and q is an odd positive integer:

If, in the integral $I=\int_a^b \sin^p x \cos^q x dx$, $p=2k+1$, where k is positive integer, we write é

$\sin^p x=\sin^{2k} x \sin x=\left(1-\cos^2 x\right)^k .\sin x$ and then we perform the change of variable t = cos(x).

If q is an odd positive integer, we interchange the roles of sin(x) and cos(x).

ii) If, in the integral $\int_a^b \sin^p x \cos^q x dx$, p and q are even positive integers, we use the following identities:

$$\sin^2(x)=\frac{1-\cos(2x)}{2} \quad and \quad \cos^2(x)=\frac{1+\cos(2x)}{2}.$$

Example:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^4(x)dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(1-\cos(2x)\right)^2 dx=\frac{1}{2}\int_0^{\frac{\pi}{4}}\left[1+\cos^2(2x)-2\cos(2x)\right]dx$$

$$=\frac{\pi}{8}-\frac{1}{2}+\frac{1}{4}\int_0^{\frac{\pi}{4}}\left(1+\cos(4x)\right)dx=\frac{3\pi}{16}-\frac{1}{2}.$$

17. Area between two curves:

If f and g are two continuous functions on an interval [a, b], If $f \le g$ over $[a,b]$, the Area A of the region, bounded by the graphs of f and g and the two-vertical lines x = a and x = b, is given by: $A=\int_a^b\left[g(x)-f(x)\right]dx$.

18. Volume of a solid of revolution:

$$V=\int_a^b \pi\left[f(x)\right]^2 dx.$$

INTERGRALS

Exercise 1

Calculate the following integrals:

1) a- $\int_1^2 x^4 dx.$ **b-** $\int_0^1 \sqrt[3]{2x+1}dx$ **c-** $\int_1^2 \left(x^3+x-2\right)dx.$

2) a- $\int_1^3 \frac{1}{\sqrt[3]{x^2}}dx$; **b-** $\int_1^4 \frac{x}{\sqrt{2x}}dx$; **c-** $\int_1^3 \frac{1}{x^2}dx.$

3) a- $\int_0^1 x^2\left(x^3+2\right)^{11}dx$; **b-** $\int_0^1 (3x+2)\sqrt{3x^2+4x+5}dx$; **c-** $\int_1^2 (x+1)\sqrt{2x+2}dx.$

4) a- $\int_0^1 \frac{(x+2)}{\sqrt{x^2+4x+10}}dx$; **b-** $\int_0^2 \sin(-4x+5)dx$; **c-** $\int_0^3 \cos\left(\frac{1}{3}x+6\right)dx$.

5) a- $\int_0^1 x^3\sin\left(x^4+2\right)dx$; **b-** $\int_{\sqrt{\pi}}^{\sqrt{2\pi}} x\cos\left(x^2\right)dx$; **c-** $\int_0^{\pi} \frac{\cos(x)}{\sqrt{4+\sin(x)}}dx$;

6) a- $\int_{\frac{\pi}{2}}^{\pi} \frac{\sin x}{\sqrt{6-4\cos x}}dx$; **b-** $\int_0^{\frac{\pi}{2}} \sin^2(2x)dx$; **c-** $\int_0^{\frac{\pi}{3}} \sin^2(3x)dx$;

7) a- $\int_0^{\frac{\pi}{6}} \cos^2(6x)dx$; **b-** $\int_0^{\pi} \cos^2\left(\frac{1}{2}x\right)dx$; **c-** $\int_0^{\frac{\pi}{3}} \frac{dx}{\cos^2(3x)}$;

8) a- $\int_0^{5\pi} \frac{dx}{\sin^2\left(\frac{x}{5}\right)}$; **b-** $\int_0^{\pi} \sin(2x)\cos^2(2x)dx$; **c-** $\int_0^{\pi} \cos(x)\sin^5(x)dx$;

9) a- $\int_0^{\pi} \sin(x)\cos^5\left(\frac{1}{2}x\right)dx$; **b-** $\int_0^{\frac{\pi}{3}} \sin(6x)\sin^7(3x)dx$; **c-** $\int_0^{\pi} \sin^2(2x)\cos^2(2x)dx$;

10) a- $\int_0^{\frac{\pi}{5}} \sin(5x)\cos(5x)dx$; **b-** $\int_0^{\pi} \sin(x)\sin(3x)dx$; **c-** $\int_0^{\pi} \cos(x)\cos(5x)dx$;

11) a- $\int_0^{\frac{\pi}{7}} \sin(x)\cos(7x)dx$; **b-** $\int_0^{\pi} \tan^2(x)dx$; **c-** $\int_0^{\pi} \tan^4(x)dx$;

12) a- $\int_0^4 |x-3|dx$ **b-** $\int_{-3}^2 |2x+5|dx$; **c-** $\int_{-1}^1 \left|x^2-1\right|dx$;

13) a- $\int_2^3 |x-1|dx$; **b-** $\int_{-1}^3 \left|x^2-4\right|dx$;

Exercise 2

Calculate the following integrals by using integration by parts:

1) a- $\int_0^{\frac{\pi}{2}} x.\sin(x)dx$; b- $\int_0^{\pi} x.\cos(2x)dx$; c- $\int_1^3 x.\sqrt{x+1}dx$;

2) a- $\int_0^{\frac{\pi}{3}} x^2\cos(x)dx$; b- $\int_0^{\pi} x^2\sin(x)dx$;

Exercise 3

Calculate the following integrals:

1) a- $\int_{-1}^{1}\left(t^3+t\right)dt$; b- $\int_{-2}^{2}\left(t^4+1\right)dt$; c- $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\left(t^5+\tan^3(t)\right)dt$;

2) a- $\int_{-5}^{5}\frac{t^3}{1+t^8}dt$; b- $\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}}\sin^3(\theta)d\theta$;

Exercise 4

Find the derivative of the function F (x) in each of the following cases:

1) a- $F(x)=\int_1^x\frac{1}{1+t^4}dt$; b- $F(x)=\int_0^x\frac{t}{\sqrt{1+t^4}}dt$;

2) a- $F(x)=\int_1^x t\sqrt{1+t^4}dt$; b- $F(x)=\int_0^x\sin^3(2t)dt$;

3) a- $F(x)=\int_{\sin x}^{\cos x}\frac{t}{1+t}dt$; b- $F(x)=\int_{x^2}^{x}\tan(t)dt$;

4) a- $F(x)=\int_0^x\sin\sqrt{t}dt$; b- $\int_{x^2}^{x^2+1}\cos^3(t)dt$;

Exercise 5

Given that: $\int_1^2 f(t)dt=4$ *and* $\int_1^5 f(t)dt=10$. **Calculate the following integrals:**

1) a- $\int_2^5 f(t)dt$; b- $\int_2^5 3f(t)dt$;

2) a- $\int_2^1\left[1+3.f(x)\right]dx$; b- $\int_5^2\left[x+3.f(x)\right]dx$;

Exercise 6

Consider the function f that is defined over]0; +∞[by: f (x) = x – (ln x) ^{2}, and let (C) be its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$. Graphic unit: 2 cm.

1) a- Determine $\lim\limits_{x\to+\infty} f(x)$ and calculate f (1), f (e) and f (3).

b- Prove that the y – axis is an asymptote to (C).

2) The below table is the table of variations of the function g defined over]0; + ∞ [by $g(x)=1-2\frac{\ln x}{x}$.

x	0		e		$+\infty$
$g'(x)$	‖	-	0	+	
$g(x)$	‖ $+\infty$	↘	$1-2e^{-1}$	↗	1

Show that $f'(x)=g(x)$. Deduce that f is strictly increasing.

3) Prove that the equation f (x) = 0 admits a unique solution α such that:

$$0.48 < \alpha < 0.52.$$

4) Prove that the curve (C) has an inflection point L whose coordinates are to be determined.

5) Write an equation of the tangent (T) at the point L to the curve (C).

6) Study the relative position between the curve (C) and the straight line (d): y = x.

7) Draw (C) , (d) and (T).

8) Calculate, in cm ^{2}, the area of the region bounded by (d), (C) and the two straight lines of equations x = 1 and x = e.

9) Prove that f has over]0;+∞[an inverse function h whose domain of definition is to be determined.

10) Draw (C') the representative curve of h in the same system as that of (C).

Exercise 7

Part A:

Let g be a function that is defined over]0;+∞ [as: g (x) = 1 + x ln (x).

1) Calculate $\lim_{x\to 0^+} g(x)$ *and* $\lim_{x\to +\infty} g(x)$.

2) Set up the table of variations of g.

3) Deduce that g (x) > 0 for all x ∈]0;+∞[.

Part B:

Consider the function f that is defined over]0; + ∞[by: f (x) = (x+1) ln (x) – x.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim_{x\to 0^+} f(x)$ *and* $\lim_{x\to +\infty} f(x)$. Deduce an asymptote to (C).

2) Show that: $f'(x)=\frac{g(x)}{x}$ and set up the table of variations of f.

3) Prove that (C) has an inflection point I whose coordinates are to be determined.

4) Show that the equation f (x) = 0 admits a unique root α such that: 1.9 < α < 2.

5) Write an equation of the tangent (T) to (C) at the point I.

6) The curve (C) cuts the straight line y = x at a point with abscissa β, Verify that:

$$5.4 < \beta < 5.5.$$

7) Draw (C) and (T).

8) Calculate the area of the region bounded by (C), the two axis x' o x , y ' o y and the straight line y = 1.

9) a- Prove that f admits over]0; + ∞[an inverse function f^{-1}.

b- Draw (C') the representative curve of f^{-1} in the same system as that of (C).

Exercise 8

Part A:

Let g be a function defined over I =]0; + ∞[by: g (x) = x^2 + ln (x) – 2.

1) Calculate $\lim\limits_{x \to 0^+} g(x)$ *and* $\lim\limits_{x \to +\infty} g(x)$.

2) Calculate g ' (x) then set up the table of variations of g.

3) Prove that the equation g (x) = 0 has a unique solution α such that:

$$1.3 < \alpha < 1.35.$$

4) Deduce the sign of g (x).

Part B:

Consider the function f that is defined over I =]0; + ∞[by: $f(x) = \frac{1}{x}\left(x^2 + 1 - \ln(x)\right)$.

Let (C) be its representative curve in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$. G.U = 2 cm.

1) Calculate $\lim\limits_{x \to 0^+} f(x)$ *and* $\lim\limits_{x \to +\infty} f(x)$.

2) Prove that $f'(x) = \frac{g(x)}{x^2}$. Set up the table of variations of f.

3) Prove that: $f(\alpha) = \frac{2\alpha^2 - 1}{\alpha}$.

4) Prove that the straight line (d) of equation y = x is an asymptote to (C).

5) Study the relative positions between (C) and (d).

6) Draw (C) and (d).

7. Calculate, in cm ^{2}, the area of the region bounded by (d), (C) and the two straight lines of equations: x = 1 and x = e.

PROBABILTIES
CHAPTER REVIEW

1. <u>Combinations</u>

   Let E be a set consisting of n elements, and let p be a natural number less than or equal to n.

   • A combination of p elements of E is a subset of E that is consists of p elements of E.

   Such a combination is also called a combination of p elements from n elements.

   The number of these combinations is denoted by C_n^p.

   <u>Example</u>

   $C_n^0 = 1$, Since E has a single subset that consists of zero elements (empty set).

   $C_n^1 = n$, Since E has n subset that consists of one element.

   $C_n^n = 1$, Since E has a single subset that consists of n elements.

   • Let n and p be two non-zero natural numbers $(p \leq n)$.

   The number of combinations of p elements out of n elements is given by the formula:

   $$C_n^p = \frac{n!}{p!(n-p)!} = \frac{n(n-1)(n-2)..........(n-p+1)}{p(p-1)(p-2)............2\times 1}.$$

   • For all natural numbers n and p $(p \leq n)$, we have:

   $$C_n^p = C_{n-1}^p + C_{n-1}^{p-1}.$$

2. <u>The Binomial Formula :</u>

   If a and b are any two numbers and n is a non – zero naturel number, then:

   $(a+b)^n = a^n + C_n^1 a^{n-1} b + ... + C_n^{n-1} a b^{n-1} + b^n$.

   Or $(a+b)^n = \sum_{p=0}^{p=n} C_n^p a^{n-p} b^p$.

3. <u>Conditional Probability:</u>

   p is a probability defined on a sample space Ω.

   • A and B are two events such that $p(A) \neq 0$. the probability that the event B will occur, given that event A has occurred is called conditional probability. It is denoted by $p(B/A)$, which is read « probability of B given A ».

   • A and B are two events such that: $p(A) \neq 0$. We have: $p(B/A) = \frac{p(A\cap B)}{p(A)}$.

   • A and B are two events such that: $p(A)p(B) \neq 0$, then

   $p(A\cap B) = p(A)p(B/A) = p(B)p(A/B)$.

4. <u>Independent events :</u>

   • We say that the two events are independent if the probability of one of them is

   not changes when the other event occurs or not which mean that:

   $p(B/A) = p(B)$ *or* $p(A/B) = p(A)$.

   • If A and B are two independent events then: $p(A\cap B) = p(A)p(B)$.

5. <u>Formula of the total Probabilities:</u>

If $A_1, A_2, \ldots\ldots\ldots\ldots\ldots, A_n$, n events that are formed a partition of the sample space Ω.

For all event B we have:

$p(B) = p(A_1)p(B/A_1) + p(A_2)p(B/A_2) + \ldots\ldots + p(A_n)p(B/A_n)$.

6. Random variable and the probability distribution:

• Let Ω be the sample space associated to a random experience. We say that the random variable defined on Ω all application of Ω onto IR. When Ω is defined, the random variable is discrete.

• We say that the probability distribution of a random variable every application p defined on $X(\Omega) = \{x_1, x_2, \ldots\ldots\ldots, x_n\}$ to the values in [0; 1] such that:

$p(X = x_1) + p(X = x_2) + \ldots\ldots + p(X = x_n) = 1$.

7. Distribution Function :

We denote the repartition function of a random variable X the function

$$F: IR \to [0,1]$$
$$x \to F(x) = p(X \le x).$$

8. Mean, Variance and the standard deviation of a random variable:

• Let $x_1, x_2, \ldots\ldots\ldots, x_n$ be the possible values of a random variable X and $p_1, \ldots, p_n$ be their corresponding probabilities.

The expected value of X is the real number E(X) that is defined by:

$$E(X) = p_1x_1 + p_2x_2 + \ldots\ldots\ldots\ldots + p_nx_n = \sum_{i=1}^{n} p_i x_i = \overline{X}.$$

• The variance of X, the mean value m, is the positive real number Var (X) defined by:

$$Var(X) = p_1x_1^2 + \ldots\ldots\ldots + p_nx_n^2 - m^2 = \sum_{i=1}^{n} x^2_i p_i - m^2.$$

• The standard deviation of X is the square root of its variance. $\sigma_x = \sqrt{\operatorname{var}(x)}$.

PROBABILITIES

Exercise 1

In an urn, there are:

- 3 black balls carry the numbers: 0; 1; 2.
- 4 red balls carry the numbers: 0; 1; 1; 2.
- 1 green ball carry the number: 3.

1. We draw successively and randomly two balls from the urn and without replacement.
   Consider the following events:
   E: « Getting two balls carries odd number and one ball carry an even number »
   F: « The sum of the numbers carries on the two drawn balls is even »
   a- Prove that: $P(E) = \frac{4}{7}$.
   b- Calculate P (F).
2. In this part we draw successively and randomly 3 balls from the urn and without replacement.
   Consider the following events:
   A: « The three drawing balls have the same color».
   B: « The carries numbers on the three drawn balls are consecutive and in an increasing order»

a - Calculate P (A), P (B) and $P(A\cap B)$.

b- Deduce the probability of the event: $P(A\cup B)$.

Exercise 2

Consider three urns U_1, U_2 and U_3 such that:

Urn U_1 contains 4 red balls and 2 black balls.

Urn U_2 contains 1 red ball and 5 black balls.

Urn U_3 contains 5 red balls and 1 black ball.

A game consists as follows:

A player selects one ball from each urn.

Consider the following events:

R: « The player selects one red ball from the urn U_2 »

B: « The player selects one black ball from the urn U_2 »

E: « The player selects two red balls and one black ball ».

1. Prove that: $P(E/R)=\frac{7}{18}$.
2. Calculate $P(E/B)$ and then deduce P (E).
3. Knowing that the player selects two red balls and one black ball, calculate the probability that he selects one red ball from the urn U_2.
4. Let X be a random variable that is equal to the number of selected red balls.

a- Prove that $P(X=1)=\frac{1}{3}$.

b- Determine the probability distribution of the random variable X.

Exercise 3

Part A:

1. An urn U contains 4 white balls and 5 black balls.
A player draws randomly and simultaneously two balls from the urn U.
Consider the following events:
A: « The two drawn balls are white »;
B: « The two drawn balls are black »
C: « The two drawn balls have different colors».

Calculate P (A), P (B) and P (C).

2. A game consists as follows:
The player draws randomly and simultaneously two balls from the urn U.
    - If the two drawn balls are white, then the player kept it outside the urn after which he draws two balls from the urn.
    - Otherwise the player replaced back the two drawn balls in the urn after which he draws two balls from the urn.

Consider the event E: «The paler wins the game if all the drawing balls in the second drawn are black ».

a) Calculate: $P\left(E/A\right)$, $P\left(E/B\right)$ and $P\left(E/C\right)$.

b) Calculate $P(A\cap E)$, $P(B\cap E)$ and $P(C\cap E)$.

c) Deduce P (E), and then calculate $P\left(B\cap\overline{E}\right)$.

d) Knowing that the two drawn balls in the second drawn are not black, calculate the probability that the two drawn balls in the first drawn are black.

Part B:

Consider another urn V which contains two white balls and three black balls.

We draw randomly and simultaneously two balls from the urn U and one ball from the urn V.

Let X be a random variable equal the number of white drawn balls.

1. Find the four possible values of X.
2. Prove that P (X = 1) = $\frac{4}{9}$.
3. Determine the probability distribution of X.

<u>Exercise 4</u>

An urn U contains 5 red balls and three black balls.
We proposed the following game we throw a coin:

- If the coin shows tail, then the player draws at random and simultaneously 3 balls from the urn.
- If the coin shows face, then the player draws at random and simultaneously 2 balls from the urn.

The player gains the game if all the drawn balls from the urn are red.
Consider the following events:

P: « the coin shows tail».
F: « the coin shows face».
G: « the player gains the game».

1. Calculate: $P\left(G/_P\right)$, $P\left(G/_F\right)$ and prove that: $P(G)=\frac{15}{56}$. (you may use the tree diagram).
2. The player gains the game. What is the probability that the coin shows tail?
3. In this part we have another urn V contains 3 red balls and two black balls.
   The game now is consisting to draw one ball from each of the two urns U and V.
   The player gains the game if the two drawn balls are red.
   a. Prove that the probability of the player gains the game is $\frac{3}{8}$.
   b. After noting the colors of the two drawn balls, they are replaced in the urns; the operation is then repeated to reach n times.
      i. Calculate the probability « p» for that the player gains the game at least one time.
      ii. Determine the smallest value of n so that: P > 0.94.

<u>Exercise 5</u>

<u>Part A:</u>

An urn U contains four black balls and two white balls. A player draws simultaneously and randomly two balls from the urn.

1. What is the probability that obtained two black balls?
2. What is the probability that obtained two white balls?
3. What is the probability that obtained two balls are of different colors?

<u>Part B:</u>

A game consists in the following manner: the player draws at random and simultaneously two balls from the urn U that contains four black balls and two white balls.

- If the two drawn balls are black, the player gains a thousand L.L $(a \geq 0)$ and the game end.
- If the two drawn balls are white, the player loses 6a thousand L.L and the game end.
- If the two drawn balls are of different colors, then he isn't back it in the urn and he draws again two balls randomly and simultaneously from the urn:
  * If the two drawn balls are black, he gains b thousand L.L (suppose that $b \geq 0,\ b \neq a$) and the game end.
  * If not, he loses 3 000 L.L and the game end.

Let X be a random variable equal to the values of the algebraic gains (positive or negative) to the player.

1. Find all the possible values of X.
2. What is the probability of the event A of drawing two black balls in the second drawing, knowing that he is draws two balls are of different colors in the first drawing?
3. Determine the probability distribution of X.
4. Express in terms of b the expected values E (X) of the random variable X. For what value of b the game is fair?

<u>Exercise 6</u>

An urn contains 9 balls: 6 black and 3 red.
A game is consisting as follow: A player draws one ball from the urn.

- If the drawing ball is black, he put it outside the urn.
- If the drawing ball is red, he back it in the urn and he add a red ball in the urn.
- The game ends in the case the number of the red balls is equal to the number of black balls.

1. Prove that the game end when the player draws exactly 3 balls.
2. Knowing that the second drawing ball is black. Calculate the probability that the third drawing ball is red.
3. The player gains 5000 L.L for each drawing red ball and he lose 5000 L.L for each drawing black ball.
   Let X be the random variable that is equal to the algebraic gain in the game.
   a. Prove that the possible values of X are: -15000; -5000; 5000 and 15000.
      Find the probability distribution of X.
   b. Calculate the expected value E (X) of X.

<u>Exercise 7</u>

An urn contains 6 balls 4 are reds and 2 are blacks.

1. We draw randomly and simultaneously two balls from this urn. Consider the following events:
   A_0: « the two drawing balls are red».
   A_1: « the two drawing balls are of two different colors».
   A_2: « the two drawing balls are black».
   Calculate the probability of each of the events A_0, A_1 and A_2.
2. After the first drawing, the urn contains 4 balls.
   We draw again randomly and simultaneously two balls from this urn. Consider the three events:
   B_0: « the two drawing balls are red».
   B_1: « the two drawing balls are of two different colors».
   B_2: « the two drawing balls are black».
   a. Calculate $P(B_0/A_0)$, $P(B_0/A_1)$ *and* $P(B_0/A_2)$. Deduce that: P (B_0) = 0.4.
   b. Calculate P (B_1) and P (B_2).
   c. Only one black ball obtained in the second drawing, calculate the probability one black ball obtained in the first drawing.
3. Calculate the probabilities of the event after the two drawings, the balls which are remains in the urn are red.

<u>Exercise 8</u>

The service of a production of a certain company is formed of 8 technicians (5 men and 3 women), of 6 workers (2 men and 4 women) and of 2 engineer (one man and one women).
This company decides to choose a group of three persons to represent the company in a forum.

1. Calculate the probability of each of the following events:
   A: « The chosen group is formed of three women ».
   B: « The chosen group is formed of three workers».

2. Calculate the probability that the 3 chosen persons are women knowing that they are workers.
3. The company decides to give to each chosen person 10 000 \$ if he is an engineer and 8 000 \$ if he is not.
   Designate by X the random variable equal to the sum of money given to the three chosen persons.
   Determine the probability distribution of X and calculate the expected value E (X).

Exercise 9

To encourage the clients, the director of a supermarket places in an urn 8 bills of 5 000 L.L, 4 bills of 10 000 L.L and 2 bills of 20 000 L.L.
The client who buys more than 100 000 L.L, draws randomly and simultaneously, three bills from the urn.

- If the three drawn bills are of three different values, the client gains these three bills.
- If two of the three drawn bills are of the same value and the third bill is of a different value, the client gains the bill which is not repeated.
- If the three drawn bills are of the same value, the client gains nothing.

1. Determine the probability of each of the following events:
   A: « the client draws three bills of 5 000 L.L »;
   B: « the client draws three bills of 10 000 L.L»;
   C: « the client draws three bills of three different values».
2. Designate by X the random variable that is equal to the sum of gain by one client who is participating in the drawing.
   a. Justify that the 5 possible values of X are: 0; 5 000; 10 000; 20 000 and 35 000.
   b. Prove that the probability $P(X = 10000) = \frac{29}{91}$.
   c. Determine the probability distribution of X.
   d- Estimate the sum paid by the director during one week if in each day 100 clients participate.

Exercise 10

Given two urns U_1 and U_2.
U_1 contains ten balls and U_2 contains eight balls.
Each of these balls holds a number and they are distributed according to the following table:

	Holding the number 1	Holding the number 2	Holding the number 5
Urn U_1	5	3	2
Urn U_2	4	4	0

1. We draw randomly and simultaneously two balls from the urn U_1.
   Calculate the probability of each of the following events:
   A: « the two drawn balls hold the same number».
   B: « the sum of the numbers held on the two drawn balls is equal to 3».
2. We choose randomly one of the urns U_1 and U_2 and then we draw randomly and simultaneously two balls from the chosen urn.
   Consider the following events:
   E: «the chosen urn is U_1».
   F: « the sum of the numbers held on the two drawn balls is 3».
   Calculate the probabilities $P(F \cap E)$ and $P(F \cap \overline{E})$. Deduce P (F).
3. We draw randomly one ball from the urn U_1 and one ball from the urn U_2.
   Let X be a random variable that is equal to the sum of the numbers held on the two drawn balls.

a. Determine the probability distribution of X.
b. Calculate the expected value E (X).

Exercise 11

A sports club offers its members three types of activities: Volley-ball, basket-ball and tennis.
Each member joins only one of these three activities.
* 30 % of the members join the volley-ball activity;
* 20 % of the members join the basket-ball activity;
* The remaining members join the tennis activity.

The club proposes an annual gathering day to all members.
- 20 % of members who join the volley-ball activity attend this gathering day;
- 25 % of members who join the basket-ball activity attend this gathering day;
- 70 % of members who join the tennis activity attend this gathering day.

A member of this club is randomly chosen. Consider the following events:
V: « The chosen member joins the volley-ball activity» B: « The chosen member joins the basket-ball activity» T: « The chosen member joins the tennis activity» R: « The chosen member attends the gathering day»

1. Verify that the probability P $(T \cap R)$ is equal to 0.35 and calculate $P(B \cap R)$ *and* $P(V \cap R)$.
2. The president of this club affirms that more than half of the members do not attend the gathering day. Justify his statement by calculation.
3. The annual membership fees for this year in the club are as follows:
   - 100 000 LL for the tennis activity,
   - 60 000 LL for either volley-ball or basket-ball.

   Moreover, an additional amount of 15 000 LL is required from each member who wants to attend the annual gathering day.
   Let X be the random variable equal to the total sum paid by a member in this club.
   a. Find the 4 possible values of X and prove that P (X= 75 000) is equal to 0.11.
   b. Determine the probability distribution of X.
   c. Calculate the expected value E(X).
   d. 200 members joined the club. Estimate the revenue of this club for this year.

Exercise 12

In order to prevent a certain disease, we vaccinated 40 % persons of a population.
Then we noticed that 85 % of the vaccinated persons were not affected by the disease and that 75 % of the persons who were not vaccinated are affected by the disease.
A person is chosen randomly from this population.
Consider the following events:
D: « the chosen person is affected by the disease»
V: « the chosen person is vaccinated».

1. a. Verify that the probability of the event $D \cap V$ is equal to $\frac{6}{100}$.
   b. What is the probability that the chosen person is affected by the disease and is not vaccinated?
   c. Deduce the probability P (D).
2. The chosen person is not affected by the disease. Calculate the probability that he/ she is vaccinated.
3. In this part, suppose that this population is formed of 300 persons.
   We choose randomly 3 persons from this population.
   What is the probability that at least one, among the 3 chosen persons, is vaccinated?

Exercise 13

A bookshop sells two series of books A and B.
A survey conducted on the customers of this bookshop revealed that:

- 20 % of the customers buy the series A.
- Among those who buy series A, 70 % buy also the series B.
- Among those who do not buy series A, 10 % buy the series B.

A customer is chosen randomly, consider the following events:

A: « the customer buys the series A»
B: « the customer buys the series B».

1. Construct a tree diagram that illustrates the given data.
2. Prove that the probability $P(A \cap B) = 0.14$.
3. Calculate P (B).
4. Calculate the probability that the client buys at least one of these two series.
5. Knowing that the client did not buy the series B, calculate the probability that he buys the series A.
6. The series A is sold for 125 000LL and the series B for 45 000LL. Let X be the random variable that is equal to the sum paid by the client.
   a. Justify that the four possible values of X are, 0, 45 000, 125 000 and 170 000.
   b. Determine the probability distribution of X.
   c. Calculate the expected value E (X) and give an interpretation for the obtained value.
7. The bookshop announces a discount of 20 % on the price of the two series.
   After this discount, a new survey revealed the following information:
   P (B) =0.3, $P(A \cap B) = 0.18$ *and* $P\left(\overline{A}/\overline{B}\right) = 0.75$.
   a. Prove that P (A) = 0.355.
   b. Calculate the new expected value.

Exercise 14

The staff of a hospital is distributed into three categories: Doctors (D), Nurses (N) and Technicians (T).
20 % are doctors and 50 % are nurses.
75 % of the doctors are men and 80 % of the nurses are women.
We ask randomly one member of the staff.

1. Calculate the probability that this person is:
   a) A technician; b) a woman knowing that she is a doctor; c) a man knowing that he is a nurse.
2. Calculate the probability that this person is:
   a) A women doctor; b) a women nurse.
3. Knowing that 51 % of the staff are women.
   a. Calculate the probability that the asked person is a women technician.
   b. Deduce the probability that the asked person is a woman knowing that she is a technician.

Exercise 15

In a television game two boxes A and B are displayed.
Box A contains seven tokens out of which three carry the number 1, two carry the number 2 and two carry the number 3.
Box B contains fifteen keys out of which three only open the door of a room containing a car.
A player starts by selecting, simultaneously and randomly, three tokens from box A.

- If the sum of the numbers carried by the three tokens is 3, 4 or 5, the player must withdraw.
- If the sum of the numbers carried by the three tokens is 6 or 7, the player selects randomly one key from box B.

- If the sum of the numbers carried by the three tokens is 8, the player selects randomly from box B two keys one after another without replacement.
- If the player selects a key that opens the door of the room, then he wins the car.

1. a- Calculate the probability that the sum of the numbers carried by the three drawn tokens is equal to 3.

   b- Show that the probability that the player « must withdraw » is equal to $\frac{16}{35}$.

2. a- Calculate the probability that the player selects two keys.

   b- The player got a sum equal to 8. Prove that the probability that the player wins the car is $\frac{13}{35}$.

3. Calculate the probability that the player wins the car.

Exercise 16

Consider an urn containing 10 balls, n balls are green, m balls are red and others are white so that: $n \geq 2;\ m \geq 2\ \ and\ \ n+m \leq 8.$

A player pays 5 $ and then he draws at random two balls from the urn.
The player wins 15 $ for each drawn green ball, 5 $ for each drawn red ball and loses 5 $ for each drawn white ball.
Let X be the random variable equal to the algebraic gain of the player at the end of the game.

1. a- Determine the possible values of X.

   b- Calculate P (X = 25) and P (X = 15) in terms of n and m.

   a- Knowing that: P (X = 25) = $\frac{1}{15}$ $and\ \ P(X=15)=\frac{2}{15}$, determine n and m.

2. Suppose that in this part that the urn contains 3 green balls, 2 red balls and 5 white balls.
   a- Determine the probability distribution of X and calculate its expected value E (X).
   b- Calculate the probability that the player draws 2 balls of the same color knowing that their algebraic gain is positive.

Exercise 17

An urn U_1 contains 4 red balls and 6 black balls.
Another urn U_2 contains 1 red ball and 9 black balls.
A player has a perfect dice, he throws the dice, if it shows the number 1, he draws a ball from U_1 if not he draws a ball from U_2.
Consider the following events:

A: « the dice shows 1 »
R: « the drawn ball is red »

1. a- Calculate the probabilities P (A) and $P\left(R/A\right)$.

   b- Show that P (R) = 0.15.

2. The player repeats the game twice by replacing the ball obtained in the urn. After the two drawings, the player gets 3 points for each red ball and – 2 points for each black ball. Let X be the random variable equal to the sum of points after the drawings.
   a- Verify that the possible values of X are: 6 , + 1, and - 4.
   b- Determine the probability distribution of X and calculate E (X).

Exercise 18

An urn contains 4 red balls and 5 black balls, and a dice has a 6 faces numbered from 1 to 6. A player throw the dice one time.
If the face shows up 1 or 4, then he draws randomly and simultaneously 2 balls from the urn.
If the face shows up 2, 5 or 6, then he draws randomly and simultaneously 3 balls from the urn.
If the face shows up 3, then he draws randomly one ball form the urn.

Consider the following events:

A : « The face shows up 1 or 4 ».
B : « The face shows up 2, 5 or 6 ».
C : « The face shows up 3 ».
E : « The drawn balls are red ».

1. Calculate P (A), P (B) and P (C),
2. a. Calculate P (E/A) and P ($E \cap A$).

   b. Calculate P (E/B) and P ($E \cap B$).

   c. Calculate P (E/C) and P ($E \cap C$).

   d. Calculate P (E).
3. Knowing that the drawn balls are red, what is the probability that the face shows up 1 or 4.
4. In this part the player draws randomly and simultaneously three balls from the urn.
   Let X be a random variable equal to the number of the white drawn ball.
   a. Determine the probability distribution of X.
   b. Calculate the expected value E(x).

<u>Exercise 19</u>

We disposed two identic urns U and V.
Urn U contains four red balls and two white balls.
Urn V contains four red balls, three white balls and one black ball.

<u>Part A:</u>

We choose an urn and then we select randomly and simultaneously three balls.

1. Consider the following events:
   E: « Three selected balls from the urn U are red ».
   F: « Three selected balls are red ».

   a- Prove that the probability of the event E is equal to $\frac{1}{10}$.

   b- Calculate the probability of the event F.
2. a- What is the probability of obtaining one black ball out of the three selected balls?
   b- What is the probability of obtaining three balls having three different colors?

<u>Part B:</u>

We select randomly and simultaneously two balls from U and one ball from V.

1. Calculate the probability of obtaining one white ball.
2. Let X be a random variable that is equal to the number of white balls selected.
   a- Determine the probability distribution of X.
   b- Calculate the expected value of E (X).

SAMPLE TEST 1

Exercise 1

In the complex plane referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$, consider the points A (1+i), B (2) and C (1).

Let M be a point of affix z ≠2 and M' be a point of affix z' such that: $z' = \dfrac{z-(1+i)}{z-2}$.

1. In the case where $z = 2e^{i\frac{\pi}{3}}$, prove that $(z')^{12}$ is real.
2. Determine the set of points M' when M moves on the perpendicular bisector of [AB].
3. Assume that z = x + i y, determine the set of points M in the case z' is real.
4. a- Prove that: $\left(\overline{z'}-1\right)\left(\overline{z}-2\right)=1+i.$

   b- Let (C) be a circle with center B and of radius $\sqrt{2}$.
   Find the set of points M' when M moves on (C).

Exercise 2

We disposed two identic urns U and V.
Urn U contains four red balls and two white balls.
Urn V contains four red balls, three white balls and one black ball.

Part A :

We choose an urn and then we select randomly and simultaneously three balls.

1. Consider the following events:
   E : « Three selected balls from the urn U are red ».
   F : « Three selected balls are red ».

   c- Prove that the probability of the event E is equal to $\dfrac{1}{10}$.

   d- Calculate the probability of the event F.
2. a- What is the probability of obtaining one black ball out of the three selected balls?
   b- What is the probability of obtaining three balls having three different colors?

Part B :

We select randomly and simultaneously two balls from U and one ball from V.

1. Calculate the probability of obtaining one white ball.
2. Let X be a random variable that is equal to the number of white balls selected.
   a- Determine the probability distribution of X.
   b- Calculate the expected value of E (X).

Exercise 3

In the space referred to a direct orthonormal system $\left(O; \vec{i}, \vec{j}, \vec{k}\right)$, consider the point A (-1 ; 0 ; -1) and the straight line (d) defined by : $(d):\begin{cases} x = m+1 \\ y = -m \\ z = -m+1 \end{cases}$ Where m is a real number.

1. Write an equation of the plane (P) determined by A and (d).
2. Calculate the coordinates of the point H the orthogonal projection of A on (P).
3. Let M be a variable point on the perpendicular at A to the plane (P).
   Show that (d) is perpendicular to (MH).
4. Let (C) be a circle in the plane (P), of center A and of radius $\sqrt{11}$.
   (C) cuts (d) at two points E (0 ;1 ;2) and F.
   a- Calculate the coordinates of the point F.
   b- Write a system of parametric equations of the tangent at E to (C).

Exercise 4

Part A:

Consider the differential equation (E): $y''-3y'+2y=e^{x}$. Assume that: $y=z-xe^{x}$.

1. Find a differential equation (E') satisfied by z.
2. Solve (E') and deduce the particular solution g of (E) such that: y (0) = 0 and y'(0) = 1.

Part B:

Let g be a function that is defined over ℝ as: $g(x)=2e^{2x}-(x+2)e^{x}$.

1. a- Prove that g (x) = 0 admits two roots one of them is 0 and the other is α such that: $-1.6<\alpha<-1.5$.

   b- Determine the sign of g (x).

2. Consider the function f that is defined over ℝ by: f (x) = $e^{2x}-(x+1)e^{x}$.

   Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

   a- Determine $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$, deduce an asymptote to (C).

   b- Prove that: f '(x) = g (x), and that f '(x) < 0 for all $\alpha<x<0$.

   c- Set up the table of variations of f.

   d- Determine the equation of the tangent (d) to (C) at a point of abscissa x = 1.

   e- Assume that α = -1.55, Draw (C).

3. a- f admits over]0; +∞[an inverse function f^{-1}.

   Draw its representative curve $(C_{f^{-1}})$ in the same system.

   b- Prove that the equation f (x) = f^{-1} (x) has a unique root β such that: 0.7 < β < 0.8.

   c- Calculate, in terms of β, the area of the region bounded by (C) and $(C_{f^{-1}})$.

SAMPLE TEST 2

Exercise 1

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points A, B and F with respective affixes 2i, 1 and i and the points M of affix z (z ≠1), M' of affix z' such that: $z'=\dfrac{iz+2}{z-1}$.

Choose the right answer with justification:

N⁰	Questions	Answers a	B	c
1	Suposse that $z'=-1+3i$, then the exponential form of z is:	$\sqrt{2}.e^{i\frac{\pi}{4}}$	$\sqrt{2}.e^{-i\frac{\pi}{4}}$	$\sqrt{2}.e^{i\frac{5\pi}{4}}$
2	Consider the two real numbers $U=\lvert z-1\rvert$ and $U'=\lvert z'-1\rvert$ then $U\times U'=$	$\sqrt{5}$	$\sqrt{3}$	5
3	Suppose that z = x + i y and $z'=x'+iy'$ then:	$\begin{cases} x'=\dfrac{2x+y+2}{(x-1)^2+y^2} \\ y'=\dfrac{x^2+y^2-x-2y}{(x-1)^2+y^2}\end{cases}$	$\begin{cases} x'=\dfrac{2x+y-2}{(x-1)^2+y^2} \\ y'=\dfrac{x^2+y^2-x-2y}{(x-1)^2+y^2}\end{cases}$	$\begin{cases} x'=\dfrac{2x+y+2}{(x-1)^2+y^2} \\ y'=\dfrac{x^2+y^2+x+2y}{(x-1)^2+y^2}\end{cases}$

4	When z' is real then M moves on:	A straight line of equation: 2 x +y – 2 =0	A circle of equation $(x-1)^2+y^2=0$	A circle of center $I\left(\frac{1}{2};1\right)$ and of radius $R=\frac{\sqrt{5}}{2}$.

Exercise 2

In the space referred to a direct orthonormal system $(O;\vec{i},\ \vec{j},\ \vec{k})$ consider the two straight lines:

$(d):\begin{cases} x=m-1 \\ y=m \\ z=m+1 \end{cases}$ and $(d'):\begin{cases} x=2t \\ y=t \\ z=t+2 \end{cases}$ m and t are two real numbers and the point B (0 ;1 ;2).

1. Prove that (d) and (d') are skew straight lines.
2. Let (P) be a plane containing (d) and cuts (d') at the point E (0 ;0 ;2). Verify that the equation of (P) is: x – z + 2 = 0.
3. a- Write an equation of the plane (Q) perpendicular at E to (d').
   b- Write a system of parametric equations of the straight line (D) intersection of (P) and (Q) and verify that (d') and (D) are perpendiculars.
4. Let R (-1; 0; 1) be a point of (d).
   a- Find the points F on(d) so that the area of the triangle ERF should be equal to 2 u ^{2}.
   b- Verify that B is a point of (P) then write a system of parametric equations of the height issued of E in the triangle ERB.

Exercise 3

The 1000 students of a language class follow the courses of arabas, English and et French.
60 % of these students follow arabas among 70% of them are boys.
30 % of these students follow English among 25 % of them are boys.
57.5 % of these students are boys.

1. We select one student from this class and we consider the following events:

   G : « The selected student is a boy ». F : « The selected student follow the course of French ».

   a- Calculate $P(G\cap F)$ and deduce $P\left(\overline{G}\cup\overline{F}\right)$.

   b- Calculate $P\left(\overline{G}\cap\overline{F}\right)$.
2. The director of this class select successively and at random without replacement two students from this class. Let X be a random variable that is equal to the number of students whose follow the Arabic course. Determine the probability distribution of X.
3. The director chooses one course then he chooses simultaneously and randomly two students whose follow the chosen course. Consider the following events:
   A : « The chosen students follow the Arabic course »
   E : « The chosen students follow the English course »
   F : « The chosen students follow the French course »
   B : « The chosen students are boys ».
   a- Calculer $P(B/E),\ P(B/F),\ P(E\cap B)\ et\ P(F\cap C)$.
   b- Déduire P (B).
   c- The chosen students are not boys. Calculate the probability that they are follow the French course.

Exercise 4

Part A :

Consider the differential equation (E): $y''-4y'+4y=-8x$. Assume that: y = z – 2 x – 2.

1. Find a differential equation (E') satisfied by z.
2. Solve the equation (E'). Deduce the general solution of (E).

3. Determine a particular solution f of (E) such that the curve of f admits at E (0; -1) a tangent line Of equation y = x – 1.

Part B :

Consider the function g defined over ℝ by $g(x)=(2x+3)e^{2x}-2$.

1. Calculate $\lim_{x\to-\infty} g(x)$ and $\lim_{x\to+\infty} g(x)$.
2. Calculate g '(x) and set up the table of variations of g.
3. a- Show that the equation g (x) = 0 admits a unique root α such that: -0.2 < α < -0.1.
   b- Deduce the sign of g over ℝ.

Part C :

[Assume that α = -0.15]

Let f be a function defined over]- ∞; + ∞[by : $f(x)=(x+1)e^{2x}-2x-2$, Designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1. Calculate $\lim_{x\to+\infty} f(x)$ then calculate f (0) and f (0.5).
2. a- Calculate $\lim_{x\to-\infty} f(x)$.
   b- Prove that the straight line (Δ) of equation y = - 2 x – 2 is an asymptote to (C).
   c- Study the relative position of (C) and (Δ).
3. Verify that f '(x) = g (x), then set up the table of variations of f.
4. Draw (Δ) and (C).
5. Prove that the equation f (x) = 0 admits two roots one of them is – 1 and the other is β such that: 0< β< 0.5.
6. Assume that β = 0.25.
   Let h (x) = ln [f (x)].
   a- Find the domain of definition of h.
   b- Calculate h'(x) and set up the table of variations of h.

SAMPLE TEST 3

Exercise 1

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$.

Consider the points A (- 1; 2; 1), B (1; - 6; - 1), C (2 ;2 ;2) and I (5;1; - 1).

1. Verify that A, B and C determine a plane (P) of equation: x + y – 3 z + 2 = 0.
2. a- Determine the coordinates of the point H the orthogonal projection of I on (P).
   b- Determine the coordinates of the point S, the symmetry of the point I with respect to (P).
   c- Deduce the area of the triangle IAS.
3. Consider the straight line (d) of equation: $\begin{cases} x=m+2 \\ y=2m-5 \\ z=m-4 \end{cases}$ (m ∈ℝ) with (d) is parallel to (P).
   a- Determine the equation of the plane (Q) containing (d) and perpendicular to (P).
   b- Let E (4; - 1; - 2) be a point of (d). Determine the coordinates of point F, orthogonal projection of E on the plane (P).
   c- Deduce a system of parametric equations of (D) the intersection of (P) and (Q).
4. Let (C) be a circle in the plane (P) with center J (3 ;1 ;2).
   Find a system of parametric equations of the straight line (d_1) tangent to (C) at T (5; -1 ;2).

Exercise 2

We disposed two urns U_1 and U_2 each one of the two urns contains 2 red balls and 3 white balls.

Part A :

A player selects one ball from the urn U_1 and then he puts it in the urn U_2 and then he selects one ball from the urn U_2. Consider the following events:

E_1 : « The selected ball from the urn U_1 is red ». E_2 : « The selected ball from the urn U_2 is red».

1. a- Calculate la probability P (E_1) then P (E_2/E_1) and $P\left(E_2 / \overline{E_1}\right)$.

   P(E_1) = 2/5; P (E_2/E_1) = 3/6= ½; P ($E_2/\overline{E_1}$) = 2/6= 1/3.

   b- Deduce that $P\left(E_2\right)=\frac{2}{5}$.

2. Let X be a random variable that is equal to the number of the selected red balls at the end of the game.
   a- Determine the probability distribution of X.
   b- If the player repeats the game ten times with the same conditions, estimate the average number of the selected red balls.

Part B :

In this part, all the balls of the two urns U_1 and U_2 are emptied in another urn

We select randomly, successively and without replacement, two balls from the urn V.

1. Calculate the probability of the event: « the two selected balls are of different colors »
2. Calculate the probability of the event: « the second selected ball is white ».

Exercise 3

In the complex plane referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$, consider a point M with affix z, and we associate the point M' with affix z' such that: $z'=\frac{z-3i}{z+1} \quad (z \neq -1)$.

1. Prove that if z = $\sqrt{2}e^{-i\frac{\pi}{4}}$ then $z'=\frac{6}{5}-\frac{7}{5}i$.
2. Assume that: z = x + i y and z' = x' + i y '.

   a- Prove that: $x'=\frac{x^2+y^2+x-3y}{(x+1)^2+y^2}$ and $y'=\frac{-3x+y-3}{(x+1)^2+y^2}$.

   b- Deduce the set of points M in the case z' is real.
3. Consider the points A, B and C with respective affixes: $Z_A=-1,\ Z_B=3i$ and $Z_C=2-i$.

   Calculate $\frac{Z_B-Z_A}{Z_C-Z_A}$ and deduce that ABC is right isosceles triangle.
4. Find the set (E) of points M when M' moves on a circle with center O and of radius 1.
5. Find the set (F) of points M such that z' is pure imaginary.

Exercise 4

Part A :

Consider the differential equation (E): $xy'+y=1+\frac{1}{x}$. Assume that : $z=x.y$.

1. Find a differential equation (E') satisfied by z.
2. Solve (E') and deduce the general solution of (E).
3. Determine a particular solution of (E) whose representative curve passes through A(1 ;1)

Part B:

Consider the function f defined over]0;+∞[by: $f(x)=1+\frac{\ln x}{x}$.

Let (C) be the representative curve of f in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$ Graphic unit: 2 cm and let (d) be a straight line of equation y = 1.

1. a- Calculate $\lim_{x\to 0^+} f(x)$. Deduce an asymptote to (C).

   b- Show that (d) is an asymptote of (C) and study, according to the values of x, the relative position
   of (C) and (d).

2. Calculate f ' (x) and set up the table of variations of f.
3. Write an equation of the tangent (T) to (C) at a point A of (C) of abscissa 1.
4. Draw (d), (T) and (C).
5. Calculate, in cm ^{2}, the area of the region bounded by (C), (d) and the two straight lines of equations : x = 1 and x = e.
6. a- Prove that the function f, admits over]0 ; e], an inverse function g and then determine the domain of definition of g.
   b- Let (C') be the representative curve of g in the same system as that of (C). Prove that (C) and (C') are tangents between them at the point A. Draw (C').

SAMPLE TEST 4

Exercise 1

The space is referred to a direct orthonormal system $\left(O;\vec{i},\ \vec{j},\ \vec{k}\right)$.

Consider the points A (2; 3 ; 1), B (1 ; 2 ; 3), C (1 ;0 ;2) and D (0 ; -3 ; 0).

1. Show that the equation of the plane (P) determined by the points B, C and D is: x + y – 2 z + 3 = 0.
2. Prove that the straight line (AB) is perpendicular to the plane (P).
3. a- Find a system of parametric equations of the straight line (d) passing through the point A and parallel to (CD).
   b- Let M be a variable point on (d).
   Prove that the volume of the tetrahedron MBCD is constant.
4. a- Prove that the point A' (0; 1; 5) is the symmetry of A with respect to the plane (P).
   b- Find a system of parametric equations of the straight line (d') the symmetry of the straight line (d) with respect to the plane (P).
5. Prove that: $\frac{CB}{DB}=\frac{\left\|\overrightarrow{CA}\wedge\overrightarrow{CA'}\right\|}{\left\|\overrightarrow{DA}\wedge\overrightarrow{DA'}\right\|}$.

Exercise 2

We dispose a bag R contains 25 dices such that 10 of them are perfect such that the probability to appear face 6 is equal to $\frac{1}{2}$.

We select two dices from the bag simultaneously then we tossed them. Consider the following events:

I ; « The two selected dices are perfect ».
N : « The two selected dices are not perfect ».
D : « One of the two selected dices is perfect and the other one is not perfect ».
S : « The sum of the numbers on the two selected dices is equal to 12 ».

1. Calculate the probabilities of I and of N.
2. a- One of the selected dice is perfect and the other dice is not perfect, calculate the probability for the sum of the numbers on the two selected dices is 12.
   b- Verify that the probability $P\left(I\cap S\right)=\frac{3}{80}$.
   c- Prove that : P (S) = $\frac{4}{45}$.
3. Knowing that the sum of the numbers on the two selected dices is smaller than12, calculate the probability that the two selected dices are not perfect.
4. In this part, 40 % of the non-perfect dices are putted outside of the bag and then we add 5 perfect dices to the bag.
   Let X be a random variable that is equal the number of the perfect selected dices.
   a- Justify that the probability $P\left(X=1\right)=\frac{45}{92}$.
   b- Determine the probability distribution of X.

Exercise 3

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider a point M of affix z, and we associate the point M' of affix z' such that: $z'=\dfrac{z-2+i}{z+2i}\quad(z\neq -2i)$.

A- Assume that : $z = x + iy$ and $z' = x' + iy'$.
   1. Express x ' and y ' in terms of x and of y.
   2. Determine the set (F) of points M such that z' is non – zero pure imaginary number.

B- Let B be a point of affix $Z_B = -2i$.
   1. Calculate $|z'-1|\times|z+2i|$.
   2. Deduce that the points M', when the point M varies on the circle of center B and of radius $\sqrt{5}$, belongs to a circle (C').

Exercise 4

Part A :

Consider the differential equation (E): $y'+y=2(x+1)e^{-x}$.

1. Suppose that $g(x)=\left(x^2+2x\right)e^{-x}$. Verify that g is a solution of the equation (E).
2. Assume that : $z = y - g(x)$.
   a- Write a differential equation (E') satisfied by z and then solve (E').
   b- Deduce the general solution of (E).
   c- Let h be a particular solution of (E) and (C') its representative curve in an orthonormal system such that the tangent to (C') at the point with abscissa 0 is parallel to the x - axis. Determine h (x).

Part B :

Consider the function f defined over $\mathbb{R}$ by: $f(x)=\left(x^2+2x+2\right)e^{-x}$ and let (C) be its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$, Graphic unit: 2 cm.

1. Calculate $\lim\limits_{x\to-\infty} f(x)$ and f (-1.5).
2. Calculate $\lim\limits_{x\to+\infty} f(x)$ and deduce an asymptote (d) to (C)
3. Calculate f '(x) and set up the table of variations of f.
4. Verify that (C) admits two inflection points one of them is W (0 ;2).
5. Draw (C).
6. Let F be a function defined over $\mathbb{R}$ by $F(x)=\left(ax^2+bx+c\right)e^{-x}$ Where a, b and c are real.
   a- Determine a, b and c so that F is a primitive of the function f.
   b- Let α be a positive real number. Designate by A (α) the area of the region bounded by the curve (C), the x – axis and the two straight lines of equations x = 0 and x = α.
   Prove that: $A(\alpha)=24-4\left(\alpha^2+4\alpha+6\right)e^{-\alpha}$ cm².
7. Let g be the inverse function of f over [0 ;+∞[and (G) its representative curve.
   a- Study the differentiability of g at 2.
   b- The equation g (x) = x admits a unique root β. Prove that: 1.5 < β < 1.7.
   c- Draw (G) in the same preceding system.

SAMPLE TEST 5

Exercise 1

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points A, B, M and M' with respective affixes: 1 – 2 i ; 1 + i ; Z and Z ' such that:

$$Z' = \frac{Z-1+2i}{Z-1-i} \quad \text{Where } Z \neq 1+i \text{ and } Z \neq 1-2i.$$

1) Only in this question, we suppose that: Z = - 4 + i.
   a- Write the complex number 1 + i in the exponential form.
   b- Deduce that $(Z')^{42}$ is pure imaginary.
2) Suppose that Z = x + i y and Z ' = x ' + i y ' with x, y, x 'and y ' are real numbers.
   a- Express x 'and y 'in terms of x and y.
   b- Express the scalar product $\overrightarrow{AM}.\overrightarrow{BM}$ in terms of x and y.
   c- Deduce that if Z 'is pure imaginary then the two lines (AM) and (BM) are perpendicular.
3) a- Verify that $(Z'-1)(Z-1-i)=3i.$
   b- Deduce that if M varies on the circle (C) of center B and of radius 6, then M' varies on a circle (C') whose center and radius are to be determined.

Exercise 2

An urn contains 4 red balls and 5 black balls, a player rolled a perfect dice one time formed of six faces numbered from 1 to 6.

- If the dice shows up the faces 1 or 4, then he selects randomly and simultaneously 2 balls from the urn.
- If the dice shows up the faces 2, 5 or 6, then he selects randomly and simultaneously 3 balls from the urn.
- If the dice shows the face 3, then the player selects one ball randomly from the urn.

Consider the following events:

A: « The dice shows up the faces 1 or 4 ». B: « The dice shows up the faces 2, 5 or 6 ».
C: « The face shows up the face 3 ». E: « The selected balls are red ».

1. Calculate P (A), P (B) and P (C),
2. a- Calculate P (E/A), P ($E \cap A$), P (E/B), P ($E \cap B$), P (E/C) and P ($E \cap C$).
   b- Prove that P (E)=$\frac{29}{189}$.
3) Knowing that the selected balls are red, what is the probability that the dice shows up the faces 1or 4.
4) Let X be a random variable that is equal the number of red selected balls. Determine the probability distribution of X.

Exercise 3

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, consider the points A (1 ;2 ;3), B (0 ;2 ;0), C (1 ;3 ;0) and D (2 ;5 ;3).

1) Prove that ABCD is a tetrahedron and calculate its volume.
2) Calculate the area of the triangle BCD and deduce the distance of A to the plane (BCD).
3) Write an equation of the plane (BCD).
4) a- Prove that A on the axis (Δ) of the circle (C) circumscribed about the triangle BCD.
   b- Write a system of parametric equations of (Δ).
   a- Find the coordinates of the center W of (C) and calculate its radius.
   b- Find a system of parametric equations of the tangent (T) to (C) at the point D.

Exercise 4

Part A :

Consider the differential equation (E): $y''-2y'+y=-x+1$, where y is a function defined over IR.

1) Solve the differential equation (F): $y''-2y'+y=0$.
2) Find the two real numbers m and p such that u (x)= mx +p is a particular solution of the differential equation (E).
3) Deduce the general solution of (E).
4) Determine the particular solution of (E) such that y (0) = -1 and $y'(0)=0$.

Part B:

Consider the function f defined over IR by: $f(x)=xe^{x}-x-1$ and let (C) be its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Calculate the limits of f at $+\infty$ and at $-\infty$.
   b- Prove that the straight line (d) of equation: y= - x - 1 is an asymptote to (C).
2) Consider the functions h and g defined over IR by: $h(x)=xe^{x}$ and $g(x)=e^{x}-1$.
   a- Set up the table of variations of g.
   b- Study the signs of h and g.
   c- Deduce that $h(x)+g(x)\geq 0$ for x≥ 0.
3) a- Verify that, for all real x, $f'(x)=h(x)+g(x)$.
   b- Deduce the sense of variations of f.
4) a- Prove that the equation f (x)=0 admits two solutions α and β in ℝ.
   Let α be the negative root.
   b) Prove that : $-1.35<\alpha<-1.34$ *and that* $0.8<\beta<0.81$.
5) Draw (d) and (C).
6) Prove that f admits over $[1,+\infty[$ an inverse function f^{-1} and draw its representative curve in the same system as that of (C).

SAMPLE TEST 6

Exercise I

The following questions are independent:

1. a- Calculate the real numbers a, b and C so that: $\dfrac{2x^{2}+1}{x(2x-1)}=a+\dfrac{b}{x}+\dfrac{c}{2x-1}$.

   b- Deduce the exact value of the integral: I = $\displaystyle\int_{1}^{2}\frac{2x^{2}+1}{x(2x-1)}dx$.

2. Consider the function f that is defined over ℝ as: $f(x)=\ln(x^{2}-2x+2)$.

   Prove that the straight line (d) of equation x = 1 is an axis of symmetry of the representative curve of the function f in an orthonormal system $(O;\vec{i},\vec{j})$.

3. Prove that the equation x + ln x = 0 admits over]0; +∞[a unique root **α** such that:

   0.5 < **α** < 0.6.

Exercise 2

Consider an urn U contains five red balls and five green balls.

We select, at random and simultaneously three balls from the urn U.

Consider the following events:

E: « The three selected balls are red ».

F: « Among the three selected balls there are exactly two red balls ».

G: « Among the three selected balls there is at most one red ball ».

1. Calculate P (E) and P (F).
2. Prove that P (G) = ½.
3. In this question a game consists in the following manner:

   A player draws randomly and simultaneously three balls from the urn U.

   If the event G realized, then he gains nothing and the game is ending.

   If one of the two events E or F is realized, then he draws again a new ball from the seven remaining balls in the urn U.

   If the drawn ball is green, then he gains ten points.

   Otherwise, he gains two points.

   Consider the event D: « The player gains ten points ».

   a- Calculate P (D/E) and P (D/F). Prove that $P\left(D \cap E\right)=\dfrac{5}{84}$.

   b- Prove that P (D) = $\dfrac{25}{84}$.

   c- The player wins 10 points. Calculate the probability that he draws 3 red balls.

<u>Exercise III</u>

<u>Part A:</u>

Consider the function f that is defined over ℝ by: $f\left(x\right)=e^{2x}-2e^{x}$.

Designate by (C) its representative curve in an orthonormal system.

1. Calculate $\lim\limits_{x \to -\infty} f\left(x\right)$. Deduce an asymptote to (C).
2. Prove that: $\lim\limits_{x \to +\infty} f\left(x\right)=+\infty$.
3. Show that: $f'\left(x\right)=2e^{x}\left(e^{x}-1\right)$ and set up the table of variations of f.
4. Prove that the curve (C) cuts the x – axis at a point A (ln (2); 0).