Summary: Functions (LS & GS)

Definition of a function:

A function f is a **mapping** (relation) from I to J, that associates to every x in I exactly **one image** y in J.

$f : I \longmapsto J.$

$\quad x \longmapsto y = f(x).$

Domain of definition of a function:

- The domain of definition $\boldsymbol{D_f}$ of f is the set of **possible values of $\boldsymbol{x}$**. This domain is sometimes given.
- If $\boldsymbol{f(x)} = \frac{\boldsymbol{A}}{\boldsymbol{B}}$, then f is defined for $\boldsymbol{B \neq 0}$.
- If $\boldsymbol{f(x)} = \sqrt{\boldsymbol{A}}$, then f is defined for $\boldsymbol{A \geq 0}$.
- If $\boldsymbol{f(x)} = \mathbf{g}(\boldsymbol{x}) \pm \boldsymbol{h(x)}$ or $\boldsymbol{f(x)} = \mathbf{g}(\boldsymbol{x}).\boldsymbol{h(x)}$, then $\boldsymbol{D_f} = \boldsymbol{D_{\mathbf{g}}} \cap \boldsymbol{D_h}$.
- If $\boldsymbol{f(x)} = \frac{\mathbf{g}(\boldsymbol{x})}{\boldsymbol{h(x)}}$, then $\boldsymbol{D_f} = \boldsymbol{D_{\mathbf{g}}} \cap \boldsymbol{D_h}$ with $\boldsymbol{h(x) \neq 0}$.

Parity of a function (even function – odd function):

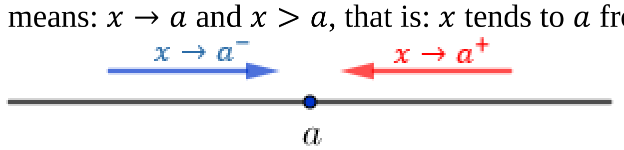
- f is said to be **even function** over I if: ***I* is centered at zero** and $\boldsymbol{f(-x) = f(x)}$.
  In this case (C) admits **y-axis** as an **axis of symmetry**.
  The **symmetric** of the **point** $(\boldsymbol{a}; \boldsymbol{b})$ with respect to **y-axis** is $(\boldsymbol{-a}; \boldsymbol{b})$.
- f is said to be **odd function** over I if: ***I* is centered at zero** and $\boldsymbol{f(-x) = -f(x)}$.
  In this case (C) admits the **origin $\boldsymbol{O}$** as a **center of symmetry**.
  The **symmetric** of the point $(\boldsymbol{a}; \boldsymbol{b})$ with respect to $\boldsymbol{O}$ is $(\boldsymbol{-a}; \boldsymbol{-b})$.
- I is centered at zero if for every $\boldsymbol{x \in I, -x \in I}$. For example: $\mathbb{R}$, $\mathbb{R}^*$, $[-a; a]$ …
- There exist functions neither even nor odd.

Axis of symmetry – Center of symmetry:

- The **line** of equation $\boldsymbol{x = a}$ is an **axis of symmetry** of (C) of a function f if: $\boldsymbol{f(2a - x) = f(x)}$.
- The **point** $\boldsymbol{I(a; b)}$ is a **center of symmetry** of (C) of a function f if: $\boldsymbol{f(2a - x) + f(x) = 2b}$.

Limits of a function:

- We **calculate** the **limit** of a function f at the **open boundaries** of its domain of definition $\boldsymbol{D_f}$.
- If $\boldsymbol{a \in D_f}$, then $\lim\limits_{x \to a} f(x) = f(a)$ (we calculate the **image** of a by f).
- The writing: $\boldsymbol{x \to a^-}$ means: $x \to a$ and $x < a$, that is: x tends to a from the **left**.
  The writing: $\boldsymbol{x \to a^+}$ means: $x \to a$ and $x > a$, that is: x tends to a from the **right**.



Rules of limit:

Let a be a real number:

- $+\infty + \infty = +\infty$; $-\infty - \infty = -\infty$; $\infty \pm \boldsymbol{a} = \infty$; $a \times \infty = \infty$; $\frac{\infty}{a} = \infty$.
- $\frac{a}{\infty} = 0$; $\frac{a}{0} = \infty$; $\frac{\infty}{0} = \infty$.
- $(\infty)^2 = +\infty$; $(-\infty)^3 = -\infty$; $\sqrt{+\infty} = +\infty$.

Indeterminate limits:

It is one of the four forms: $\frac{\mathbf{0}}{\mathbf{0}}$; $\frac{\infty}{\infty}$; $+\infty-\infty$ or $\mathbf{0}\times\infty$, that is the limit is not appeared.

To solve this problem, we must **change** the **form** of the function and calculate the limit again.

The limits at ∞ of polynomial and rational functions:

- To calculate the limit at ∞ of a **polynomial** function, we take the **monomial** of **highest exponent**.
- To calculate the limit at ∞ of a **rational** function, we take the **monomial of highest exponent** in the **numerator** to that in the **denominator**.

Derivative of a function at a point:

The **derivative** of f at a point a, denoted by $\boldsymbol{f'(a)}$, is given by: $\boldsymbol{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$.

Graphically: $\boldsymbol{f'(a)}=\boldsymbol{a_{(T)}}$, the **slope** of the **tangent** to (C) at a **point** $\boldsymbol{a}$.

Table of principle formulas of derivative function:

Function $f(x)$	Derivative function $f'(x)$
a (constant)	0
ax	a
x^n	nx^{n-1}
ax^n	$a.nx^{n-1}$
$\frac{1}{x}$ $(x\neq 0)$	$\frac{-1}{x^2}$
$\sqrt{x}$ $(x>0)$	$\frac{1}{2\sqrt{x}}$
u^n (u is a function of x)	$nu^{n-1}.u'$
$u\pm v$	$u'\pm v'$
$u.v$	$u'.v+v'.u$
$\frac{u}{v}$ $(v\neq 0)$	$\frac{u'.v-v'.u}{v^2}$
$\frac{1}{u}$ $(u\neq 0)$	$\frac{-u'}{u^2}$
$\sqrt{u}$ $(u>0)$	$\frac{u'}{2\sqrt{u}}$
$a.u$	$a.u'$

Equation of a tangent:

An **equation** of the **tangent** (T) to (C) of a function f at a point of abscissa $x=a$ is given by:
$\boldsymbol{(T): y-f(a)=f'(a)(x-a)}$.

Remarks:

- If (C) has a **horizontal tangent** (T) at a point of abscissa a, then $\boldsymbol{f'(a)=0}$.
- If (C) has an **extremum** at a point $\boldsymbol{A(a;b)}$, then $\boldsymbol{f(a)=b}$ and $\boldsymbol{f'(a)=0}$.
- If (C) has a **vertical tangent** (T) at a point of abscissa a, then $\boldsymbol{f'(a)}=\infty$.
- If (C) has an **oblique tangent** (T) at a point of abscissa a, then: $\boldsymbol{f'(a)}=\boldsymbol{\frac{y_2-y_1}{x_2-x_1}}$, where $\boldsymbol{(x_1;y_1)}$ and $\boldsymbol{(x_2;y_2)}$ are **two points on** $\boldsymbol{(T)}$.

Sense of variation of a function:

To study the variation of a function f, we must find the **derivative** function $f'(x)$, then determine the **roots** and make a table of **signs**:

- For $\boldsymbol{f'(x) > 0}$, f is **increasing**.
- For $\boldsymbol{f'(x) < 0}$, f is **decreasing**.
- For $\boldsymbol{f'(x) = 0}$ and **change its sign**, f admits an **extremum**.
- If $\boldsymbol{f'(x) > 0}$ **for every** $\boldsymbol{x \in I}$, then f is **strictly increasing over** $\boldsymbol{I}$.
- If $\boldsymbol{f'(x)} < 0$ for every $x \in I$, then f is **strictly decreasing over** $\boldsymbol{I}$.

In the table of variations of f, we put in the **row of** $\boldsymbol{x}$: the **domain** and the **roots of** $\boldsymbol{f'(x)}$, in the **row of** $\boldsymbol{f'(x)}$: the **sign** and **zero**, and in the **row of** $\boldsymbol{f(x)}$: the **arrows** and **limits or images**.

Drawing the representative curve of a function:

To **draw** the **representative curve** of a function f, we follow these steps:

- Draw the **asymptotes** (vertical asymptote, horizontal asymptote, oblique asymptote).
- Locate the **extremums**.
- Locate some **particular points** (using calculator Mode 7).
- Draw the curve **as seen in the table of variation**.

Graphical reading of a function:

Mathematical writing	Graphical meaning
$f(a) = b$	Point $(a; b)$ belongs to (C)
$f(x) = 0$	(C) cuts x-axis
$f(x) > 0$	(C) is above x-axis
$f(x) < 0$	(C) is below x-axis
$f(x) \geq 0$	(C) above and cuts x-axis
$f(x) \leq 0$	(C) below and cuts x-axis
$f(x) = k$	(C) cuts the line $y = k$

Limits and asymptotes:

- If $\boldsymbol{\lim_{x \to a} f(x)} = \infty$, then the line of equation $\boldsymbol{x = a}$ is a **vertical asymptote** to (C).
- If $\boldsymbol{\lim_{x \to \infty} f(x) = a}$, then the line of equation $\boldsymbol{y = a}$ is a **horizontal asymptote** to (C) at ∞.
- To prove that the line (D) of equation: $y = ax + b$ is an **oblique asymptote** to (C), we must prove that: $\boldsymbol{\lim_{x \to \infty} \left[f(x) - y_{(D)}\right] = 0}$.

Asymptotic directions (only for G.S):

- If $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \infty$, then (C) admits at ∞ a **vertical asymptotic direction** (**fast function**).
- If $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \mathbf{0}$, then (C) admits at ∞ a **horizontal asymptotic direction** (**weak function**).
- If $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to\infty} \frac{f(x)}{x} = \boldsymbol{a} \in \mathbb{R}^*$, then we calculate $\lim\limits_{x\to+\infty} [f(x) - ax]$:
  - If $\lim\limits_{x\to\infty} [f(x) - ax] = \infty$, then (C) admits at ∞ an **oblique asymptotic direction** parallel to the line of equation: $\boldsymbol{y = ax}$.
  - If $\lim\limits_{x\to\infty} [f(x) - ax] = \boldsymbol{b} \in \mathbb{R}$, then (C) admits at ∞ an **oblique asymptote** of equation: $y = ax + b$.

Relative position of two curves:

To study the **relative position** of **two curves** $(\boldsymbol{C})$ **and** $(\boldsymbol{C'})$ with respective functions f and g, we must study the **sign** of the expression: $\boldsymbol{R(x) = f(x) - g(x)}$:

- For $\boldsymbol{R(x) > 0}$, (C) is **above** (C').
- For $\boldsymbol{R(x) < 0}$, (C) is **below** (C').
- For $\boldsymbol{R(x) = 0}$, (C) **cuts** (C').

Remarks:

- The **horizontal asymptote** and the **oblique asymptote may cut** the curve (C) of a function f.
- The **vertical asymptote don't cut** the curve (C) of a function f.
- To find the **point(s)** of **intersection** of $(\boldsymbol{C})$ **and x-axis**, we **solve** $\boldsymbol{f(x) = 0}$.
- To find the **point** of **intersection** of $(\boldsymbol{C})$ **and y-axis**, we **calculate** $\boldsymbol{f(0)}$.

Continuity of a function at a point of the curve:

- f is said to be **continuous at** a **point** of abscissa $\boldsymbol{a}$ if: $\lim\limits_{x\to a^-} f(x) = \lim\limits_{x\to a^+} f(x) = f(a)$.

  In this case the **curve** (C) **has no gap** (no jump) at this point.
- f is said to be **discontinuous at** a **point** of abscissa $\boldsymbol{a}$ if $\lim\limits_{x\to a^-} f(x) \neq \lim\limits_{x\to a^+} f(x)$.

  In this case the **curve** (C) **has a gap** at this point.

Remarks:

- The function f is **continuous over** an **interval** I if it is **continuous at every point** of I.
- The **polynomial functions** are **continuous over** $\mathbb{R}$.
- The **rational functions** are **continuous over** its **domain** of definition D_f.

Differentiability (derivativity) of a function at a point:

- f is said to be **differentiable at** a **point** of abscissa $\boldsymbol{a}$ if $\boldsymbol{f'(a)}$ **exist**, that is: $\boldsymbol{f'(a^-) = f'(a^+)}$.
  In this case (C) **admits one tangent** $(\boldsymbol{T})$ at a, that is (C) has **no angular** (broken) **point** at a.

- f is said to be **not differentiable at** a **point** of abscissa $\boldsymbol{a}$ if $\boldsymbol{f'(a^-) \neq f'(a^+)}$.
  In this case (C) **admits two semi-tangents** $(\boldsymbol{T_1})$ **and** $(\boldsymbol{T_2})$ at a, one from the left and one from the right, that is (C) has an angular point at a. The equations of (T_1) and (T_2) are given by:
  $(\boldsymbol{T_1}): \boldsymbol{y - f(a) = f'(a^-)(x - a)}$; $(\boldsymbol{T_2}): \boldsymbol{y - f(a) = f'(a^+)(x - a)}$.

Remarks:

- If $f'(a) = \infty$, then f is not differentiable at a.
- The function f is **differentiable over an interval** I if f is **differentiable at every point** of I.
- If f is **discontinuous** at a, then f **isn't differentiable** at a.
- If f is **differentiable** at a, then f is **continuous** at a. The **converse isn't necessary true**.

Limits by comparison:

Let f, g and h be a three functions:

- If $\boldsymbol{f(x) \geq g(x)}$ and $\boldsymbol{\lim\limits_{x \to +\infty} g(x)} = +\infty$, then $\boldsymbol{\lim\limits_{x \to +\infty} f(x)} = +\infty$.
- If $\boldsymbol{\lim\limits_{x \to +\infty} f(x)} > -\infty$, then we **can't conclude any result** about the limit.
- If $\boldsymbol{f(x) \leq g(x)}$ and $\boldsymbol{\lim\limits_{x \to +\infty} g(x)} = -\infty$, then $\boldsymbol{\lim\limits_{x \to +\infty} f(x)} = -\infty$.
- If $\boldsymbol{\lim\limits_{x \to +\infty} f(x)} < +\infty$, then we **can't conclude any result** about the limit.
- If $\boldsymbol{g(x) \leq f(x) \leq h(x)}$ and $\boldsymbol{\lim\limits_{x \to +\infty} g(x) = \lim\limits_{x \to +\infty} h(x) = k}$, then $\boldsymbol{\lim\limits_{x \to +\infty} f(x) = k}$ (**sandwish rule**).

Symmetric and translation:

Let f and g be two functions of respective representative curves (C) and (C'):

- If $\boldsymbol{g(x) = -f(x)}$, then (C') is the image of (C) by **symmetric with respect to x-axis**.
  Note that, the symmetric of a point $(\boldsymbol{a}; \boldsymbol{b})$ with respect to x-axis is $(\boldsymbol{a}; -\boldsymbol{b})$.

- If $\boldsymbol{g(x) = |f(x)|}$, then (C') is **confounded** with (C) when (C) **above** x-axis, and (C') is **symmetric** of (C) with respect to **x-axis** when (C) **below** x-axis.

- If $\boldsymbol{g(x) = f(x) + a}$, then (C') is the image of (C) by **translation** of a vector $\vec{\boldsymbol{v}} = \boldsymbol{a\vec{j}}$.
- If $\boldsymbol{g(x) = f(x + a)}$, then (C') is the image of (C) by **translation** of a vector $\vec{\boldsymbol{v}} = -\boldsymbol{a\vec{i}}$.
- If $\boldsymbol{g(x) = f(x + a) + b}$, then (C') is the image of (C) by **translation** of a vector $\vec{\boldsymbol{v}} = -\boldsymbol{a\vec{i}} + \boldsymbol{b\vec{j}}$.

Concavity of a function next to a point (only for G.S):

To study the **concavity** of a function f at a, we must **determine** $\boldsymbol{f''(a)}$ (second derivative):

- If $\boldsymbol{f''(a) > 0}$, then the **concavity** of (C) is **upward**, and $(\boldsymbol{C})$ is **above** the **tangent** to (C) at a.
- If $\boldsymbol{f''(a) > 0}$, then the **concavity** of (C) is **downward**, and $(\boldsymbol{C})$ is **below** the **tangent** to (C) at a.
- If $\boldsymbol{f''(a) = 0}$ **and change its sign**, then (C) admits an **inflection point** at a, and (C) **crosses** its **tangent** at a.

L' Hopital's rule:

- If $\boldsymbol{\lim\limits_{x \to a}} \frac{f(x)}{g(x)} = \boldsymbol{\frac{0}{0}}$ **or** $\frac{\infty}{\infty}$, then we can apply **L'hopital's rule:** $\boldsymbol{\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}}$.
- If, after applying the L'hopital's rule, we still obtain $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then we apply it **successively**.

Intermediate value theorem (I.V.T):

- If f is a **continuous** function over $[a; b]$ with $\boldsymbol{f(a) \times f(b) < 0}$, then the equation $\boldsymbol{f(x) = 0}$ admits **at least one root $\boldsymbol{\alpha}$** such that $\boldsymbol{a < \alpha < b}$, that is $\boldsymbol{f(\alpha) = 0}$.
  Note that: $f(x) = 0$ means graphically: (C) cuts the x-axis.

- If f is a **continuous** function and **strictly monotone** (strictly increasing or strictly decreasing) over $[a; b]$ such that $\boldsymbol{f(a) \times f(b) < 0}$, then the equation $\boldsymbol{f(x) = 0}$ admits a **unique** root α such that $\boldsymbol{a < \alpha < b}$.

Remark:

To prove that the equation $\boldsymbol{f(x) = E}$, where E is a function of x or constant, admits a solution α with $a < \alpha < b$, we take a **new function** $\boldsymbol{h(x) = f(x) - E}$ and verify that $\boldsymbol{h}$ is **continuous** over $[a; b]$ and $\boldsymbol{h(a) \times h(b) < 0}$.
Note that: $\boldsymbol{f(x) = E}$ means **graphically**: $(\boldsymbol{C})$ **cuts:** $\boldsymbol{y = E}$.

Image of an interval by a function:

- If f is a **continuous** function over a closed interval $\boldsymbol{I = [a; b]}$, then $\boldsymbol{f(I) = f([a; b]) = [m; M]}$, where m and M are respectively the **absolute minimum** and the **absolute maximum** of f, and for all $x \in [a; b]$, $m \leq f(x) \leq M$.

- If f is a **continuous** function and **strictly increasing** over $[a; b]$, then: $\boldsymbol{f([a; b]) = [f(a); f(b)]}$.

- If f is a **continuous** function and **strictly decreasing** over $[a; b]$, then $\boldsymbol{f([a; b]) = [f(b); f(a)]}$.