Chapter 3	Class: Terminales GS - LS *Exponential function*	MATH 4 ALL

I- Definition and properties

Definition

Since the function " ln " is continuous and strictly increasing from $]0\,;\,+\infty[$ in $]-\infty\,;\,+\infty[$, so for every real number x in $]-\infty\,;\,+\infty[$, there exists a unique real number y in $]0\,;\,+\infty[$ such that $\ln y = x$. We then obtain $y = e^x$.

In this way, we define over $]-\infty\,;\,+\infty[$ a new function f having values in $]0\,;\,+\infty[$ which is the exponential function denoted $f(x) = \exp(x) = e^x$ (e is the real number such that $\ln e = 1$).

Properties

1) The domain of definition of the function $f : x \mapsto e^x$ is: $]-\infty;+\infty[$.
2) $e^0 = 1$ and $e^1 = e$ with $e \approx 2.71$.
3) For every x in $\mathbb{R}$, we have: $e^x > 0$.
4) For every real number x we have: $\ln e^x = x$, and for every real number $x > 0$ we have: $e^{\ln x} = x$.
5) For every real numbers x and y we have:
   - $e^{-x} = \dfrac{1}{e^x}$
   - $e^x \times e^y = e^{x+y}$
   - $\dfrac{e^x}{e^y} = e^{x-y}$
   - $\left(e^x\right)^y = e^{xy}$.
6) Differentiating both members of the equality $\ln y = x$ with respect to x, we obtain $\dfrac{y'}{y} = 1$

   $y' = y = e^x$. We conclude that the function $f = \exp$ is differentiable over $\mathbb{R}$ and that $f'(x) = \exp(x) = e^x$.

Application exercise 1

Simplify the following expressions:

1) $\left(e^x\right)^4 \times e^{-3x}$;
2) $\dfrac{e^{3x+4}}{e^{3x+2}}$
3) $\dfrac{e^{-x}+e^x}{e^{-x}}$
4) $e^{2x} + e^{-2x} - \left(e^x - e^{-x}\right)^2$
5) $\dfrac{e^{x^2} \times \left(e^x\right)^2}{e^{(x+1)^2}}$
6) $\left(e^x + e^{-x}\right)^2 - \left(e^x + e^{-x}\right)^2$.

Application exercise 2

Simplify the following expressions:

1) $\ln e^{-x} + 3\ln e^{x+1}$.
2) $e^{\ln x^2} - \ln e^{x^2+1}$.
3) $\ln\left[\left(e^{2x} + e^{-x}\right)^2 - \left(e^{2x} + e^{-x}\right)^2\right]$
4) $\ln\left(1 + e^{2x}\right) - \ln\left(1 + e^{-2x}\right)$
5) $e^{\ln 3}$
6) $e^{-\ln 2}$
7) $e^{-\frac{1}{2}\ln 4}$
8) $e^{\ln 2 - \ln 5}$
9) $e^{\ln 2 + \ln 3}$.

Application exercise 3

Simplify the following expressions:

1) $\ln e^{-x} + 3\ln e^{x+1}$.
2) $e^{\ln x^2} - \ln e^{x^2+1}$.
3) $\ln\left(1 + e^{2x}\right) - \ln\left(1 + e^{-2x}\right)$
4) $\ln\left[\left(e^{2x} + e^{-x}\right)^2 - \left(e^{2x} + e^{-x}\right)^2\right]$.

II- Equalities and inequalities

Property

Since the exponential function $f: x \mapsto e^x$ is differentiable over $\mathbb{R}$ and $f'(x) = e^x > 0$, then it is continuous and strictly increasing over $\mathbb{R}$, we deduce the following properties:
For every x and y in $\mathbb{R}$:

- $e^x > e^y \Leftrightarrow x > y$
- $e^x < e^y \Leftrightarrow x < y$
- $e^x = e^y \Leftrightarrow x = y$.

Application exercise 4

Solve in $\mathbb{R}$ each of the following equations:

1) $e^{-2x} = 1$ **2)** $e^{3x-8} = \dfrac{1}{e^2}$ **3)** $e^{x^2} = e^{-5}$

4) $e^{x^2+9} = e^{6x}$ **5)** $(e^x + 8)(e^x - e) = 0$ **6)** $e^{2x} - 3e^x + 2 = 0$

7) $e^{-2x} + e^{-x} - 2 = 0$ **8)** $e^{3x+1} - 5e^{2x+1} + 4e^{x+1} = 0$ **9)** $e^x - 4e^{-x} + 3 = 0$.

Application exercise 5

Solve in $\mathbb{R}$ each of the following inequalities:

1) $e^{3x+2} \le 2$ **2)** $3e^{2x+1} \le e^x$ **3)** $e^x > -2$

4) $(e^x - 1)(e^x + 3) < 0$ **5)** $\ln(e^x + 1) \le 2$ **6)** $e^{x^2-5} \le e^{-4x}$

7) $e^{x^2} \ge \dfrac{1}{e^{6x}}$ **8)** $e^{2x} - 3e^x > -2$ **9)** $e^x - 5e^{-x} + 4 \le 0$.

Application exercise 6

Determine the domain of definition of the function f in each case:

1) $f(x) = (x^2 - 4x + 5)e^x$ **2)** $f(x) = e^{\frac{1}{x}}$ **3)** $f(x) = \dfrac{e^x + x}{e^x - 1}$

4) $f(x) = \dfrac{e^x - 1}{e^x + 1}$ **5)** $f(x) = \dfrac{e^x}{x^2 - 1}$ **6)** $f(x) = \ln(e^x - 2)$

7) $f(x) = \dfrac{\ln(3 + e^x)}{e^x + 1}$ **8)** $f(x) = \dfrac{1}{x} + \dfrac{e^x}{e^x - 2}$ **9)** $f(x) = \dfrac{\ln(e^x - 1)}{x - 3}$.

III- Derivatives

Property

- The function $f: x \mapsto e^x$ is differentiable over $\mathbb{R}$ and $f'(x) = (e^x)' = e^x$.
- If the function U is differentiable over an interval I, then the function $x \mapsto e^U$ is differentiable over I and $(e^U)' = U'e^U$.

Application exercise 7

Calculate the derivative of the function f over the interval I in each of the following cases:

1) $f(x) = e^x - x - 4$; $I = \mathbb{R}$ **2)** $f(x) = e^{2x^2-3x+4}$; $I = \mathbb{R}$ **3)** $f(x) = e^{2x} - e^{-x}$; $I = \mathbb{R}$

4) $f(x) = \dfrac{1}{x}e^{x-1}$; $I = \mathbb{R}^*$ **5)** $f(x) = (x+1)e^{-x}$; $I = \mathbb{R}$ **6)** $f(x) = e^{\frac{1}{x}}$; $I = \mathbb{R}^*$

7) $f(x)=(x^2+2x)e^{1-x}$; $I=\mathbb{R}$

8) $f(x)=\dfrac{e^x-1}{2e^x+1}$; $I=\mathbb{R}$

9) $f(x)=\ln(e^x+1)$; $I=\mathbb{R}$

10) $f(x)=x+2-\dfrac{2e^x}{e^x+1}$; $I=\mathbb{R}$

11) $f(x)=(-x+2)^2 e^{-2x}$; $I=\mathbb{R}$

12) $f(x)=\dfrac{\ln(e^x+1)}{e^x}$; $I=\mathbb{R}$

IV- Important limits

Property

- $\lim\limits_{x\to+\infty} e^x=+\infty$; $\lim\limits_{x\to-\infty} e^x=0$
- $\lim\limits_{x\to+\infty} \dfrac{e^x}{x}=+\infty$; $\lim\limits_{x\to-\infty} xe^x=0$
- $\lim\limits_{x\to0^+} \dfrac{e^x-1}{x}=1$.

Properties

Limits and indeterminate forms

1) Indeterminate forms "$\dfrac{0}{0}$ and $\dfrac{\pm\infty}{\pm\infty}$":

To remove indeterminacy, we apply the Hôpital's rule.

Examples:

a) $\lim\limits_{x\to-\infty} \dfrac{e^{-x}+1}{x}$ is an indeterminate form "$\dfrac{+\infty}{-\infty}$". We apply the Hôpital's rule:

$$\lim_{x\to-\infty} \frac{e^{-x}+1}{x}=\lim_{x\to+\infty}\frac{\left[e^{-x}+1\right]'}{(x)'}=\lim_{x\to+\infty}\frac{-e^{-x}}{1}=\lim_{x\to+\infty}-e^{-x}=-\infty.$$

b) $\lim\limits_{x\to2}\dfrac{e^x-e^2}{x-2}$ is an indeterminate form "$\dfrac{0}{0}$". We apply the Hôpital's rule:

$$\lim_{x\to2}\frac{e^x-e^2}{x-2}=\lim_{x\to e}\frac{(e^x-e^2)'}{(x-2)'}=\lim_{x\to2}\frac{e^x}{1}=\lim_{x\to e}e^x=e^2.$$

Note: $\lim\limits_{x\to2}\dfrac{e^x-e^2}{x-2}=f'(2)$ where $f(x)=e^x$ (definition of the derivative of a function), so

$$\lim_{x\to2}\frac{e^x-e^2}{x-2}=f'(2)=e^2.$$

2) Indeterminate form "$\infty-\infty$":

To remove indeterminacy, we take a common factor which is often the exponential term:

Examples:

a) $\lim\limits_{x\to+\infty}(e^x-x)$ is an indeterminate form "$\infty-\infty$": we take the term "e^x" as a common factor:

$$\lim_{x\to+\infty}(e^x-x)=\lim_{x\to+\infty}e^x\left(1-\frac{x}{e^x}\right)=+\infty(1-0)=+\infty \text{ since } \lim_{x\to+\infty}\frac{x}{e^x}\overset{HR}{=}\lim_{x\to+\infty}\frac{1}{e^x}=0.$$

b) $\lim\limits_{x\to+\infty}(x^2-x-e^x)$ is an indeterminate form "$\infty-\infty$": we take the term "e^x" as a common factor: $\lim\limits_{x\to+\infty}(x^2-x-e^x)=\lim\limits_{x\to+\infty}e^x\left(\dfrac{x^2}{e^x}-\dfrac{x}{e^x}-1\right)=(+\infty)(0-0-1)=-\infty$ since

$$\lim_{x\to+\infty}\frac{x^2}{e^x}\overset{HR}{=}\lim_{x\to+\infty}\frac{2x}{e^x}\overset{HR}{=}\lim_{x\to+\infty}\frac{2}{e^x}=0 \text{ and } \lim_{x\to+\infty}\frac{x}{e^x}\overset{HR}{=}\lim_{x\to+\infty}\frac{1}{e^x}=0.$$

c) $\lim\limits_{x\to+\infty}\left(e^{2x}-e^{x}+1\right)$ is an indeterminate form "$\infty-\infty$": we take the term "e^{2x}" as a common factor: $\lim\limits_{x\to+\infty}\left(e^{2x}-e^{x}+1\right)=\lim\limits_{x\to+\infty}e^{2x}\left[1-\dfrac{1}{e^{x}}+\dfrac{1}{e^{2x}}\right]=(+\infty)(1-0+0)=+\infty$.

3) Indeterminate form "$0\times\infty$":
To remove indeterminacy, the expression is transformed into the form of a fraction in order to obtain one of the indeterminate forms "$\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$" and we apply the Hôpital's rule.

Example:

$\lim\limits_{x\to+\infty}xe^{-2x}$ is an indeterminate form "$+\infty\times 0$", we then write $\lim\limits_{x\to+\infty}xe^{-2x}=\lim\limits_{x\to+\infty}\dfrac{x}{e^{2x}}$ which is an indeterminate form "$\dfrac{+\infty}{+\infty}$" and we apply the Hôpital's rule:

$\lim\limits_{x\to+\infty}xe^{-2x}=\lim\limits_{x\to+\infty}\dfrac{x}{e^{2x}}\overset{HR}{=}\lim\limits_{x\to+\infty}\dfrac{1}{2e^{2x}}=0$.

Application exercise 8

Calculate the following limits:

1) $\lim\limits_{x\to+\infty}\left(e^{x}+e^{-x}\right)$ 2) $\lim\limits_{x\to-\infty}\left(e^{2x}-e^{x}\right)$ 3) $\lim\limits_{x\to+\infty}\left(2e^{3x}-4e^{2x}+3\right)$ 4) $\lim\limits_{x\to-\infty}\dfrac{e^{x}+1}{2x}$

5) $\lim\limits_{x\to+\infty}\dfrac{e^{x}-1}{3e^{x}-4}$ 6) $\lim\limits_{x\to-\infty}x\left(e^{x}-1\right)$ 7) $\lim\limits_{x\to+\infty}(3x-1)e^{-x}$ 8) $\lim\limits_{x\to+\infty}\left(3e^{x}-7x\right)$

9) $\lim\limits_{x\to+\infty}\left(x^{2}e^{-2x}\right)$ 10) $\lim\limits_{x\to-\infty}\left(xe^{\frac{1}{x}}-x\right)$ 11) $\lim\limits_{x\to0}\dfrac{e^{x}-e^{-x}}{x}$ 12) $\lim\limits_{x\to1}\dfrac{e^{x}-e}{x-1}$

13) $\lim\limits_{x\to0}\dfrac{e^{2x}-5e^{x}+4}{e^{2x}-1}$ 14) $\lim\limits_{x\to0}\dfrac{1}{x}e^{\frac{1}{x}}$ 15) $\lim\limits_{x\to0}xe^{\frac{1}{x}}$ 16) $\lim\limits_{x\to0}\dfrac{e^{2x}-1}{x}$

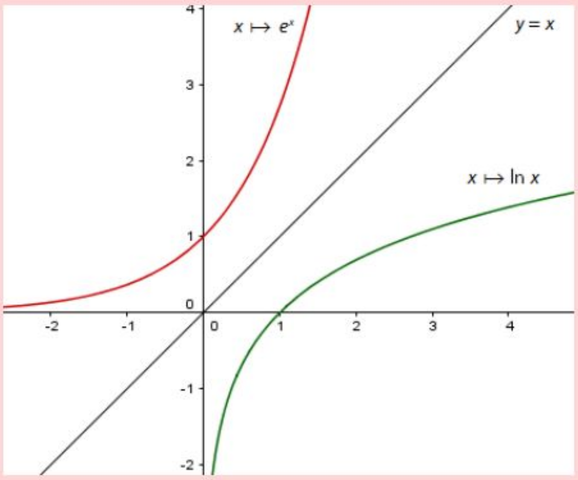
17) $\lim\limits_{x\to-\infty}\dfrac{\ln\left(e^{x}+1\right)}{e^{x}}$ 18) $\lim\limits_{x\to-\infty}\left[x-\ln\left(1+e^{x}\right)\right]$ 19) $\lim\limits_{x\to+\infty}\left[x-\ln\left(1+e^{x}\right)\right]$ 20) $\lim\limits_{x\to+\infty}\dfrac{e^{x}+1}{xe^{x}+2x}$.

V- Study of the exponential function

- The function $f:x\mapsto e^{x}$ is continuous over $\mathbb{R}$.
- $\lim\limits_{x\to-\infty}f(x)=0$ then the x-axis is a horizontal asymptote to the curve (C_f) of f at $-\infty$.
- $\lim\limits_{x\to+\infty}f(x)=+\infty$ and $\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}=+\infty$, so the curve (C_f) admits an asymptotic direction parallel to the y-axis at $+\infty$.
- The function $f:x\mapsto e^{x}$ is differentiable over $\mathbb{R}$ and $f'(x)=e^{x}>0$ for every $x\in\mathbb{R}$, so the function f is strictly increasing over $\mathbb{R}$.
- ***Table of variations of the exponential function*:**

x	$-\infty$		$+\infty$
$f'(x)$		$+$	
$f(x)=e^{x}$	0	$\nearrow$	$+\infty$

- ***Representative curve of*** $f : x \mapsto e^x$



Application exercise 9

Study the variations of each function f and draw its representative curve (C):

1) $f(x) = (x+1)e^{-x}$ **2)** $f(x) = \dfrac{e^x}{x}$ **3)** $f(x) = \dfrac{e^x}{e^x - 1}$.

Problem 1

Let f be the function defined over $\mathbb{R}$ by $f(x)=\dfrac{3e^x-1}{e^x+1}$.

Denote by (C_f) the representative curve of f in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$ of graphical unit 2 cm.

1) a) Calculate $\dfrac{f(x)+f(-x)}{2}$. What can be deduced about the point $I(0\,;1)$ with respect to (C_f).

b) Solve the equation $f(x)=0$.

2) Calculate $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$. Deduce the asymptotes to (C_f).

3) a) Justify that $f'(x)=\dfrac{4e^x}{(e^x+1)^2}$ and deduce the sense of variations of the function f.

b) Set up the table of variations of f.

4) Write the equation of the tangent (T) to (C_f) at the point I.

5) Draw (T) and (C_f).

6) Let t be a real number. Determine the values of t for which the curve (C_f) intersects the line of equation $y=t$ at a single point.

7) Let h be the function defined by $h(x)=f(-x)$. Explain how we can construct the representative curve (C_h) of h using (C_f) and plot (C_h) in the same coordinate system.

Problem 2

Part A

Let f be the function defined over $\mathbb{R}$ by $f(x)=(ax+b)e^{-x}+1$ where a and b are two real numbers.

Denote by (C_f) the representative curve of f in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$ of unit 1 cm.

Determine the values of a and b so that the point $A(-1\,;1)$ belongs to (C_f) and the slope of the tangent at A to (C_f) is $-e$.

Part B

Let g be the function defined over $\mathbb{R}$ by $g(x)=(-x-1)e^{-x}+1$ and let (C_g) be its representative curve in the same coordinate system.

1) Calculate $\lim\limits_{x\to-\infty} g(x)$.

2) Show that $\lim\limits_{x\to+\infty} g(x)=1$ and interpret graphically the result.

3) Calculate $g'(x)$ for every $x\in\mathbb{R}$ and set up the table of variations of g.

4) Show that the curve (C_g) admits an inflection point I whose coordinates will be determined.

5) Write the equation of the tangent at I to (C_g).

6) Trace (C_g).

7) Determine graphically according to the values of the real parameter m the number of solutions of the equation $g(x)=m$.

Part C

Let h be the function defined by $h(x)=\ln[f(x)-1]$.

1) Determine the domain of definition D_h of the function h.

2) Prove that for every x in D_h, $h'(x)=\dfrac{-x}{x+1}$ and determine the sense of variations of the function h.

Problem 3

Let f be the function defined over $\mathbb{R}$ by $f(x)=x+1-\dfrac{2e^x}{e^x+1}$.

Denote by (C_f) the representative curve of f in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.

1) a) Show that for every x in $\mathbb{R}$, $f(x)=x-1+\dfrac{2}{e^x+1}$.
   b) Deduce the limit of f at $+\infty$.
   c) Show that the line (D) of equation $y=x-1$ is an oblique asymptote to (C) at $+\infty$.
   d) Study the relative position of (C) and (D).

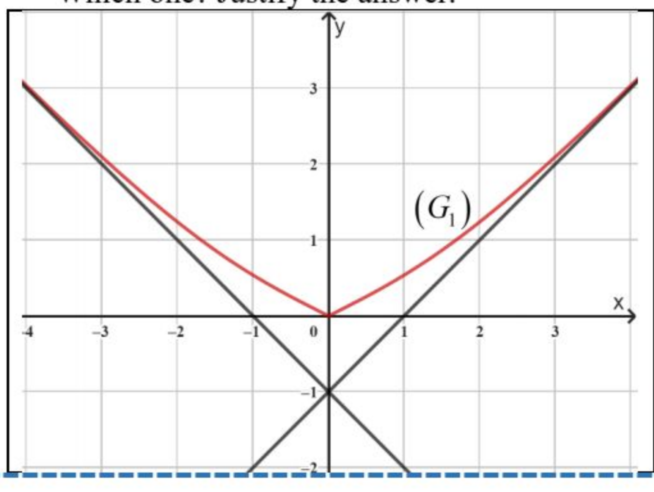
2) a) Show that for every x in $\mathbb{R}$, $f(x)=x+1-\dfrac{2}{e^{-x}+1}$.
   b) Deduce the limit of f at $-\infty$.
   c) Show that the line (D') of equation $y=x+1$ is an oblique asymptote to (C) at $-\infty$.
   d) Study the relative position of (C) and (D').

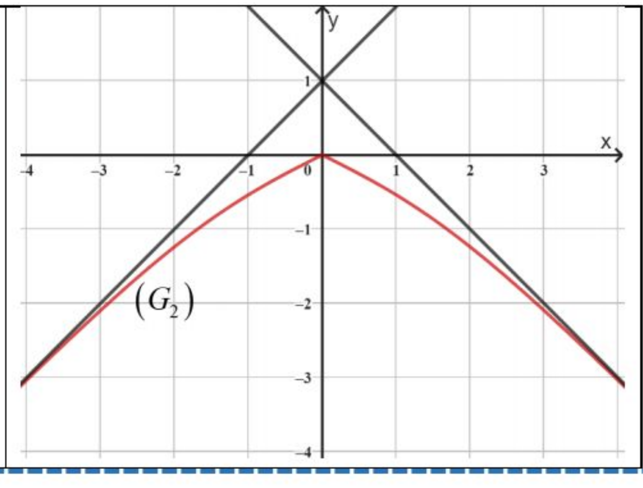
3) Show that for every x in $\mathbb{R}$, $f'(x)=\dfrac{e^{2x}+1}{(e^x+1)^2}$ and set up the table of variations of f.

4) Write an equation of the tangent (d) to (C) at the point A of abscissa 0.

5) Trace (d), (D), (D') and (C).

6) Let g be the function defined over $\mathbb{R}$ by $g(x)=f(|x|)$.
   a) Show that the function g is even.
   b) One of the curves (G_1) or (G_2) is the representative curve of g in an orthonormal system. Which one? Justify the answer.





Problem 4

Part A

Let g be the function defined over $\mathbb{R}$ by $g(x)=e^{3x}+3x+2$.

The table below is the table of variations of the function g.

x	$-\infty$		$+\infty$
$g'(x)$		$+$	
$g(x)$	$-\infty$	$\nearrow$	$+\infty$

1) Use the table to show that the equation $g(x)=0$ admits a unique solution α and justify that $-0.71<\alpha<-0.7$.

2) Deduce the sign of $g(x)$ according to the values of x in $\mathbb{R}$.

Part B

Let f be the function defined over $\mathbb{R}$ by $f(x)=-x+(x+1)e^{-3x}$.

Designate by (C_f) the representative curve of f in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$ of unit 2 cm.

1) **a)** Justify that $\lim\limits_{x\to-\infty} f(x)=-\infty$.

b) Calculate $\lim\limits_{x\to+\infty} f(x)$ and show that the line (Δ) of equation $y=-x$ is an oblique asymptote to (C_f) at $+\infty$.

c) Study the relative position of (C_f) and (Δ).

2) Show that $f(\alpha)=\dfrac{-3\alpha^2-3\alpha-1}{3\alpha+2}$.

3) Show that for every real number x, $f'(x)=\dfrac{-g(x)}{e^{3x}}$ and set up the table of variations of the function f.

4) Show that the curve (C_f) has a tangent (T) parallel to (Δ) whose equation to be determined.

5) Show that the curve (C_f) intersects the x-axis at two points of abscissas x_0 and x_1 $(x_0<x_1)$. Justify that $-1.1<x_0<-1$ and that $0.4<x_1<0.5$.

6) Trace (T), (Δ) and (C_f) (take $f(\alpha)\approx 3.15$).

7) Determine graphically according to the values of the real number m the number of solutions of the equation $f(x)=-x+m$.

Part C

Let h be the function defined by $h(x)=\ln[f(x)]$ and let (H) be its representative curve in the same system.

1) Show that the domain of definition of h is $]x_0\,;x_1[$.

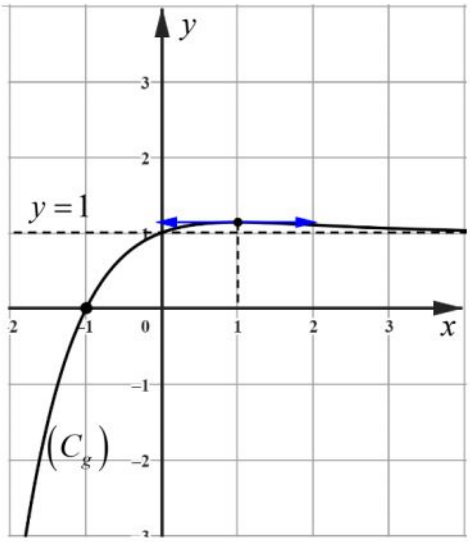
2) Determine the point A on the curve (H) where the tangent is parallel to the x-axis.

Problem 5

Part A

Let g be the function defined by $g(x) = 1 + xe^{-x-1}$.

The curve (C_g) in the opposite figure is the representative curve of g in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.



By graphical reading, determine:

1) The domain of definition D_g of g.
2) $\lim\limits_{x\to-\infty} g(x)$ and $\lim\limits_{x\to+\infty} g(x)$.
3) $g(-1)$.
4) The solution of the equation $g'(x) = 0$.
5) The sign of $g(x)$ according to the values of x in D_g.

Part B

Let f be the function defined over $\mathbb{R}$ by $f(x) = x - (x+1)e^{-x-1}$.

Denote by (C_f) the representative curve of f in the system $(O\,;\vec{i}\,;\vec{j})$ of unit 1 cm.

1) Calculate $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.
2) Show that for every x in $\mathbb{R}$, $f'(x) = g(x)$ and draw the table of variations of f.
3) a) Calculate $\lim\limits_{x\to+\infty}\left[f(x) - x\right]$ and interpret the result graphically.
   b) Study the relative position of (C_f) and the line of equation (Δ) $y = x$.
   c) Show that (C_f) admits a tangent (T) parallel to (Δ) whose equation to be determined.
4) Use the curve (C_g) in **Part A** to prove that the curve (C_f) admits an inflection point I whose coordinates to be determined.
5) Show that (C_f) intersects the x-axis at two points of abscissas α and β such that: $0.3 < \alpha < 0.4$ and. $-1.9 < \beta < -1.8$
6) Draw the lines (Δ) and (T) and the curve (C_f).

Part C

Let h be the function defined over $\mathbb{R}$ by $h(x) = e^{[f(x)]^2}$.

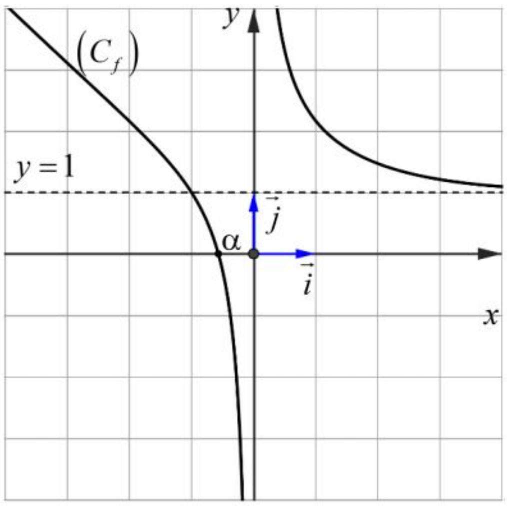
1) Justify that $h'(x) = 2f(x)f'(x)e^{[f(x)]^2}$.
2) Draw the table of variations of h.

Problem 6

The curve (C_f) in the opposite figure is the representative curve of a function f in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.

The y-axis and the line of equation $y=1$ are two asymptotes to (C_f).

(C_f) cuts the x-axis at a point of abscissa α.



Part A

1) Determine graphically the domain of definition of the function f.
2) Justify graphically that $f'(x)<0$ over its domain and set up the table of variations of the function f.
3) Study graphically the sign of $f(x)$ over its domain of definition.

Part B

Suppose, in what follows, that f is the function defined over $\mathbb{R}^*$ by $f(x)=\dfrac{e^x+x}{e^x-1}$.

1) Justify that $-0.6<\alpha<-0.5$.
2) Calculate $\lim\limits_{x\to-\infty}\left[f(x)+x\right]$ and interpret graphically the result.
3) Study the relative position of the curve (C_f) and the line (d) of equation $y=-x$.
4) Solve in $\mathbb{R}$ the inequality $\dfrac{e^x+x}{e^x-1}>1$.

Part C

Let g be the function defined by $g(x)=\ln\left[f(x)-1\right]$. Designate by (C_g) the representative curve of a function g in an orthonormal system.

1) Justify that the domain of definition of g is $D_g=\left]-\infty\,;\,-1\right[\cup\left]0\,;\,+\infty\right[$.
2) Calculate $\lim\limits_{x\to(-1)^-} g(x)$ and $\lim\limits_{x\to0^+} g(x)$. Deduce two asymptotes to the curve (C_g).
3) *For the G.S. section only*
   a) Show that (C_g) admits at $-\infty$ an asymptotic direction parallel to the x-axis.
   b) Show that (C_g) admits at $+\infty$ an asymptotic direction parallel to the line (d).
4) Show that g is strictly decreasing over $\left]-\infty\,;\,-1\right[\cup\left]0\,;\,+\infty\right[$ and set up its table of variations.
5) The equation $f(x)=2$ admits two solutions x_0 and x_1 such that $x_0<x_1$.
   a) Justify that $-1.8<x_0<-1.9$ and that $1.1<x_1<1.2$.
   b) Deduce that the curve (C_g) cuts the x-axis at two points whose coordinates to be determined.
6) Trace (C_g).

Problem 7

Let f be the function defined over $\mathbb{R}$ by $f(x)=(x+2)(e^x-1)$.

Denote by (C_f) the representative curve of f in an orthonormal system $(O;\vec{i};\vec{j})$ of unit 2 cm.

1) Calculate $\lim_{x\to+\infty} f(x)$ and give a value of $f(1)$ to the nearest 10^{-1}.

2) Determine the coordinates of the points of intersection of (C_f) with the x-axis.

3) **a)** Calculate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to-\infty}[f(x)+x]$. Deduce that the curve (C_f) admits an oblique asymptote (Δ) whose equation will be determined.

b) Study the relative position of (C_f) and the line (Δ).

4) **a)** Calculate $f'(x)$, $f''(x)$ and set up the table of variations of the function f'.

b) Show that the equation $f'(x)=0$ admits over $\mathbb{R}$ a unique solution α and justify that $-0.8<\alpha<-0.7$.

c) Deduce the sign of $f'(x)$ over $\mathbb{R}$ and then set up the table of variations of the function f.

5) Write an equation of the tangent (T) to (C_f) which is parallel to (Δ).

6) Plot (Δ), (T) and (C_f) on the interval $]-\infty;1]$ (take $f(\alpha)\approx-0.7$).

7) Let g be the function defined by $g(x)=\dfrac{1}{f(x)}$.

a) Determine D_g the definition domain of g.

b) Show that, for every $x\in D_g$, $g'(x)=-\dfrac{f'(x)}{f^2(x)}$.

c) Set up the table of variation of g.

Problem 8

Let f be the function defined over $\mathbb{R}$ by $f(x)=2\ln(e^x+1)-x+1$.

Denote by (C_f) the representative curve of f in an orthonormal system $(O;\vec{i};\vec{j})$.

1) Calculate $\lim_{x\to-\infty} f(x)$ and show that the line (d_1) of equation $y=-x+1$ is an oblique asymptote to (C_f) at $-\infty$.

2) Show that the function f is even.

3) Deduce $\lim_{x\to+\infty} f(x)$ and that the curve (C_f) admits at $+\infty$ another oblique asymptote (d_2) whose equation will be determined.

4) Show that for every x in $\mathbb{R}$, $f'(x)=\dfrac{e^x-1}{e^x+1}$ and set up the table of variations of the function f.

5) Trace (d_1), (d_2) and (C_f).

6) Determine graphically according to the values of the real number m the number of solutions of the equation $2\ln(e^x+1)=x+m$.

7) Let g be the function defined by $g(x)=[f(x)]^2$.

Set up the table of variation of g.

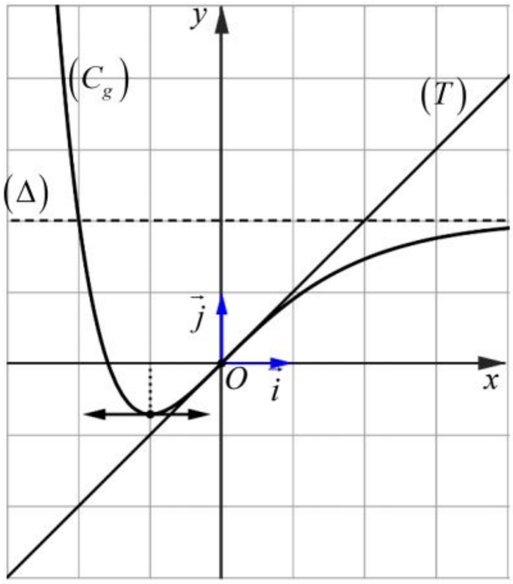
Problem 9

Part A

Let g be the function defined over $\mathbb{R}$ by $g(x)=(ax+b)e^{-x}+c$ where a, b and c are real numbers.

In the opposite figure:

- (C_g) is the representative curve of g in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.
- (T) is the tangent to (C_g) at O.
- (Δ) is an asymptote to (C_g) at $+\infty$.



1) By graphical reading:

a) Determine $\lim\limits_{x\to+\infty} g(x)$, $\lim\limits_{x\to-\infty} g(x)$, $g(0)$, $g'(0)$ and $g'(-1)$.

b) Set up the table of variations of g.

2) Using the previous question, determine the values of a, b and c.

3) Let $g(x)=(-x-2)e^{-x}+2$.

a) Show that the equation $g(x)=0$ admits two solutions, one of which is zero and the other is α such that $-1.75<\alpha<-1.5$.

b) Deduce the sign of $g(x)$ over $\mathbb{R}$.

Part B

Let f be the function defined over $\mathbb{R}$ by $f(x)=(x+3)e^{-x}+2x$ and let (C_f) be its representative curve in the system $(O\,;\vec{i}\,;\vec{j})$.

1) Calculate $\lim\limits_{x\to-\infty} f(x)$ and $f(-3)$.

2) a) Calculate $\lim\limits_{x\to+\infty} f(x)$ and show that the line (D) with equation $y=2x$ is an asymptote oblique to (C_f) at $+\infty$.

b) Study the relative position of (C_f) and the line (D).

3) Show that for every x in $\mathbb{R}$, $f'(x)=g(x)$ and draw the table of variations of f.

4) Determine without calculation the limit $\lim\limits_{x\to\alpha}\dfrac{f(x)-f(\alpha)}{x-\alpha}$ and interpret graphically the result.

5) Show that $f(\alpha)=2\alpha+2+\dfrac{2}{\alpha+2}$ and determine a framing of $f(\alpha)$.

6) Show that the curve (C_f) admits a unique tangent (T') that is perpendicular to the line of equation $y=-\dfrac{1}{2}x$ at a point whose coordinates will be determined, and write an equation of (T').

7) Show that the curve (C_f) admits an inflection point I whose coordinates will be determined.

8) Show that (C_f) intersects the x-axis at a single point of abscissa β such that: $-2.7<\beta<-2.6$.

9) Trace (D), (T') and (C_f).

Problem 10

Let f be the function defined over $\mathbb{R}$ by $f(x)=x-e+\ln\left[1+2e^{-(x-e)}\right]$ and let (C_f) be its representative curve in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.

1) a) Show that for every real number x, $f(x)=-x+e+\ln\left[2+e^{2(x-e)}\right]$.

b) Show that the curve (C_f) admits two oblique asymptotes (D) and (D') of equations $y=x-e$ and $y=-x+\ln 2+e$ at $+\infty$ and $-\infty$ respectively.

c) Study the relative position of (C_f) with respect to the two lines (D) and (D').

d) Show that the line (Δ) of equation $x=\frac{1}{2}\ln 2+e$ is an axis of symmetry of the curve (C_f).

2) Study the sense of variations of the function f and set up its table of variations.

3) Trace (Δ), (D), (D') and (C_f).

4) Let (D_m) be the line of equation $y=mx-m\left(e+\frac{\ln 2}{2}\right)+\frac{\ln 2}{2}$ where m is a real parameter.

a) Justify that all lines (D_m) pass through the fixed point $A\left(\frac{\ln 2}{2}+e\,;\frac{\ln 2}{2}\right)$.

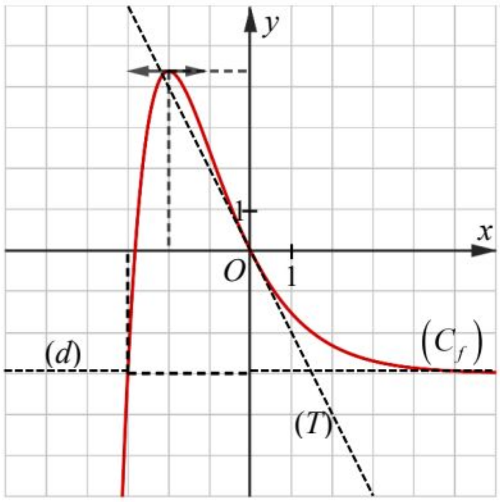
b) Determine according to the values of the real parameter m the number of points of intersection of the line (D_m) and the curve (C_f).

Problem 11

Let f be the function defined by $f(x)=(x+a)e^{-x}+b$ where a and b are two real numbers.

In the opposite figure:

- The curve (C_f) is the representative curve of f in an orthonormal system.
- The line (d) is an asymptote to (C_f) at $+\infty$.
- The line (T) is a tangent to (C_f) at O.



1) By graphical reading:

a) Determine the domain of definition of f.

b) Determine $f(-3)$, $f(0)$, $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.

c) Determine $f'(-2)$ and $f'(0)$.

d) Determine the sign of $f'(x)$ according to the values of x in $\mathbb{R}$.

2) Show that, for every x in $\mathbb{R}$, $f(x)=(x+3)e^{-x}-3$.

3) Set up the table of variations of f.

4) Show, by calculation, that the equation $f(x)=0$ admits in $\mathbb{R}$ exactly two solutions, one of which is 0 and the other is α such that $-2.9<\alpha<-2.7$.

5) Deduce the sign of $f(x)$ according to the values of x in $\mathbb{R}$.

6) Let g be the function defined by $g(x)=\ln\left[f(x)\right]$.

Designate by (C_g) the representative curve of g in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$

a) Justify that the domain of definition of g is $]\alpha\,;0[$.

b) Show that $\left(C_g\right)$ admits two vertical asymptotes.

c) Set up the table of variations of g.

d) Trace $\left(C_g\right)$.