# Summary: Logarithm

## Introduction:

Every **continuous** function should **have** an **antiderivative** function.

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$, where $n \neq -1$ ; What about $\int x^{-1}\,dx = \int \frac{1}{x}\,dx$ ?

## Logarithm function:

The **logarithm** function, denoted by $\ln x$, is an **antiderivative** of $\frac{1}{x}$ over $]0; +\infty[$.

## Domain of definition of Logarithm:

- If $f(x) = \ln x$, then $f$ is **defined** for $x > 0$, therefore $D_f = ]0; +\infty[$.
- If $f(x) = \ln u$, then $f$ is defined for $u > 0$.

## Particular points for Logarithm:

$\ln 1 = 0$ and $\ln e = 1$, where $e \simeq 2.71$ ... (**irrational number** called **exponential number**).

## Limits of Logarithm:

$\lim\limits_{x\to 0^+} \ln x = -\infty$ ; $\lim\limits_{x\to +\infty} \ln x = +\infty$ ; $\lim\limits_{x\to +\infty} \frac{\ln x}{x^\alpha} = 0^+$ $(\alpha > 0)$ ; $\lim\limits_{x\to +\infty} \frac{x^\alpha}{\ln x} = +\infty$ $(\alpha > 0)$.

Note that **$\ln x$ is weaker than $x$** (polynomial).

$\lim\limits_{x\to 0^+} x^\alpha . \ln x = 0^-$ $(\alpha > 0)$.

## Derivative of Logarithm:

- If $f(x) = \ln x$ ; $f'(x) = (\ln x)' = \frac{1}{x} > 0$ $\forall x > 0$, then $f$ is **strictly increasing** over $]0; +\infty[$.

| $x$ | $0$ | $1$ | $+\infty$ |
|---|---|---|---|
| $f'(x)$ | ‖ | $+$ | |
| $f(x)$ | ‖ $-\infty$ | $0$ | $+\infty$ |

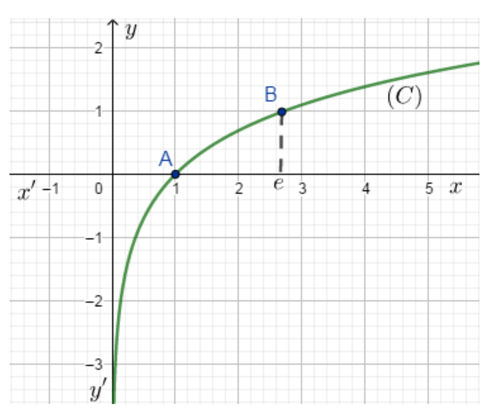
- If $f(x) = \ln u$, then $f'(x) = (\ln u)' = \frac{u'}{u}$.

## The representative curve of $\text{Ln}(x)$:

$\lim\limits_{x\to 0^+} \ln x = -\infty$, then **V.A: $x = 0$ (y-axis)**.

$\ln 1 = 0$, then the **point $A(1; 0)$ belongs to $(C)$**.

$\ln e = 1$, then the **point $B(e; 1)$ belongs to $(C)$**.




## The sign of $\text{Ln}(x)$:

- For $0 < x < 1$ ; $\ln x < 0$.
- For $x > 1$ ; $\ln x > 0$.
- For $x = 1$ ; $\ln x = 0$.

## Properties of Logarithm:

Let $a$ and $b$ be two real numbers such that $\boldsymbol{a > 0}$ and $\boldsymbol{b > 0}$:

- $\ln(ab) = \ln a + \ln b$.
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.

  In particular: $\ln\left(\frac{1}{a}\right) = -\ln a$ and $\ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)$.
- $\ln(a^n) = n\ln a$.

  In particular: $\ln e^n = n$.

## Equation and inequation with Logarithm:

- $\ln x = \ln y$ is equivalent to $x = y$ for $x > 0$ and $y > 0$.
- $\ln x > \ln y$ is equivalent to $x > y$ for $x > 0$ and $y > 0$.
- $\ln x < \ln y$ is equivalent to $x < y$ for $x > 0$ and $y > 0$.
- $\ln x = a$ is equivalent to $x = e^a$ for every real number $a$.

  Proof: $\ln x = a = a.1 = a\ln e = \ln e^a$, then $x = e^a$.
- $\ln x > a$ is equivalent to $x > e^a$ for every real number $a$.
- $\ln x < a$ is equivalent to $x < e^a$ for every real number $a$.

## Remark:

The function $f(x) = \ln|x| = \begin{cases} \ln x \ \text{ for } x > 0 \\ \ln(-x) \ \text{ for } x < 0 \end{cases}$ is defined over $\mathbb{R}^*$ and $f'(x) = \frac{1}{x}$.

## Antiderivative using logarithm function:

- $\int \frac{dx}{x} = \ln|x| + C \ \ (x \neq 0)$.
- $\int \frac{dx}{x+a} = \ln|x+a| + C \ \ (x \neq -a)$.
- $\int \frac{u'dx}{u} = \ln|u| + C$ (**change of variable**, $u \neq 0$).

## Remark:

In an **antiderivative using change of variable**, **any integral containing** $\ln x$, take $u = \ln x$.

## Integration by parts:

Let $u$ and $v$ be two functions of $x$. We have: $\int uv'\,dx = uv - \int u'v\,dx$.

Proof:

$(uv)' = u'v + v'u$ ; $uv' = (uv)' - u'v$, then: $\int uv'\,dx = \int (uv)'\,dx - \int u'v\,dx = uv - \int u'v\,dx$.

## Remarks:

- We **use** an **integration by parts** when we have **two functions** of **different types**.
- In the integral $\int P(x).\ln x\,dx$, take $u = \ln x$ and $v' = P(x)$, where $P$ is a **polynomial**.
- In the integral $\int P(x).e^x\,dx$, take $u = P(x)$ and $v' = e^x$.

## Log function (not required, just for entrance exam in university):

The function $\log_a x$ of base $a > 0$ and $a \neq 1$ is defined as $\log_a x = \frac{\ln x}{\ln a}$.