Complex numbers

N°1) Find the algebraic form of each of the following complex numbers.

1) $z = 2i(2+3i)$ 2) $z=(1-i)(1+2i)$ 3) $z=(1-i)^3$ 4) $z=(2i-1)^2$

N°2) Determine the real and imaginary parts of each of the following complex numbers

1) $z = 2+i(1-\sqrt{3})$ 2) $z=(i+2)(-1-2i)$ 3) $z=1+i+i^2+i^3+i^4+i^5$ 4) $z=(1+i\sqrt{3})^3$

N°3) Determine the real number m so that the following complex number is a real number.

1) $z=(m-2)+(2m+3)i$ 2) $z=(1+2i)^2-m(3+i)$ 3) $z=(m+i)^2+(m-i)^2$

N°4) Determine the real number t so that the following complex number is a pure imaginary number.

1) $z=2t+(t+2)i$ 2) $z=t-i+(t+2i)^2$ 3) $z=i(t-i)-t(1+i)$.

N°5) $z_1=\frac{1}{2}+i\frac{\sqrt{3}}{2}$ and $z_2=\frac{\sqrt{3}}{2}+\frac{1}{2}i$ are two given complex numbers

1) Calculate z_1^2, z_2^2, z_1z_2 and $z_1\bar{z}_2+\bar{z}_1z_2$.
2) show that the complex number $\frac{z_1+z_2}{1+z_1z_2}$ is real
3) show that the complex number $\frac{z_1^2-1}{z_1}$ is pure imaginary.

N°6) Let $z=\frac{\sqrt{3}+i}{\sqrt{3}-i}$. Calculate $z+\bar{z}$, $z-\bar{z}$ and deduce the real and imaginary parts of z.

N°7) solve, in $\mathbb{C}$, each of the following equations.

1) $(3+2i)z+1-2i=2iz-2+i$ 2) $2z^2+3z+2=0$ 3) $z^2+1=0$
4) $2z^2-6z+5=0$ 5) $z^4-1=0$ 6) $2z^2-2(1+i)z+2+i=0$
7) $z^3+1=0$ 8) $z^3-8=0$ 9) $z^4-(2+3i)^4=0$

N°8) Determine, in $\mathbb{C}$, the square roots of the following numbers.

1) $z=-6+8i$ 2) $z=5-12i$ 3) $z=2i$

N°9) 1) place, in the plane, the points A, B and C of respective affixes $z_A=-1+i$, $z_B=2+i$ and $z_C=-\frac{1}{2}-\frac{1}{2}i$

2) Calculate the affixes of vectors $\vec{AB}$, $\vec{AC}$ and $\vec{BC}$
3) Calculate the distances AB, AC and BC. is the triangle ABC right at C

N°10) M is a point in the complex plane of affix $z=x+iy$.
Find the set of points $M(x,y)$ in each of the following cases.

1) $Re(z)=2$ 2) $Im(z)=-3$ 3) $Re(z)=Im(z)$ 4) $z\bar{z}=9$
5) $z-i$ is a real number 6) $z+1$ is a pure imaginary number.
7) $\frac{z+i}{z-i}$ is a real number 8) $\frac{5z-2}{z-1}$ is a pure imaginary number

N°11) The complex plane is referred to an orthonormal system $(O, \vec{u}, \vec{v})$.
Consider a point M of affix $z = x + iy$ and let M' be the point of affix $z' = x' + iy'$ such that $z' = (z-4)(\bar{z} + 2i)$

1) Determine the algebraic form of z' when $z = 1 - i$
3) Determine the set of points M in each of the following cases:
   a) z' is real b) z' is pure imaginary c) M' moves on the line (d): $x + y - 1 =$
2) Express x' and y' in terms of x and y

N°12) 1) Given $P(z) = z^3 - 2(\sqrt{3} + i)z^2 + 4(1 + i\sqrt{3})z - 8i$
   a) Calculate $P(2i)$
   b) Determine the real numbers a and b so that
   $$P(z) = (z - 2i)(z^2 + az + b)$$
   c) Solve, in $\mathbb{C}$, the equation $P(z) = 0$
2) Given in the complex plane $(O, \vec{u}, \vec{v})$ the point M of affix $z = x + iy$ and the point M' of affix z' such that $z' = \frac{3 + \bar{z}}{1 - iz}$.
   a) Write z in its algebraic form, when $z' = i$
   b) If $z'\bar{z}' = 1$ then show that $3x^2 - y^2 - 2y - 1 = 0$
   c) Determine the set of points M when z' is real.

N°13) 1) Write, in the algebraic form, the complex number $\left(\frac{i\sqrt{2}}{1-i}\right)^3$
2) a) Solve, in $\mathbb{C}$, the equation $z^4 - 1 = 0$
   b) Deduce the solutions of the equation $\left(\frac{2z_1 + 1}{z_1 - 1}\right)^4 = 1$
3) In the complex plane referred to a direct orthonormal system $(O, \vec{u}, \vec{v})$, consider the points M and M' of respective affixes $z = x + iy \neq 0$ and $z' = x' + iy'$ such that $z' = \frac{1}{2}\left(z + \frac{1}{z}\right)$
   a) Determine z when $z = z'$
   b) Show that $x' = \dfrac{x^3 + xy^2 + x}{2(x^2 + y^2)}$ and $y' = \dfrac{x^2y + y^3 - y}{2(x^2 + y^2)}$
   c) Determine the set of points M when z' is pure imaginary.
   d) Show that if $OM = 1$ then z' is real.

c- Show that if z is real, then M' moves on a straight line whose equation is to be determined.

Complex numbers

I) The complex plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$. For all points M of the plane with affix $z \neq 0$, we associate the point M' with affix z' such that $z' = \frac{z-5i}{z}$.

1) Write z in exponential form in the case where $z' = \frac{1}{2} - \frac{1}{2}i$.
2) Denote by E the point with affix $z_E = 1$.
   a- Verify that $z' - 1 = \frac{-5i}{z}$.
   b- Calculate EM' when $OM = 5$.
3) Suppose that $z = x + iy$ and $z' = x' + iy'$ with x, y, x' and y' being real numbers.
   a- Show that $x' = \frac{x^2 + y^2 - 5y}{x^2 + y^2}$ and $y' = \frac{-5x}{x^2 + y^2}$.
   b- Deduce that when M' moves on the line with equation $y = x$, M moves on a circle whose center and radius are to be determined.

II) In the complex plane referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$, consider the points $E(2i)$, $A(-i)$, $M(z)$ and $M'(z')$ where z and z' are two complex numbers such that: $z' = 2i - \frac{2}{z}$. $(z \neq 0)$.

1) a- Show that $z(z' - 2i) = -2$.
   b- Calculate $\arg(z) + \arg(z' - 2i)$.
2) a- Verify that: $z' = \frac{2i(z+i)}{z}$.
   b- Show that $OM' = \frac{2AM}{OM}$.
   c- As M moves on the perpendicular bisector of $[OA]$, prove that M' moves on a circle (C) whose center and radius are to be determined.
3) Suppose that $z = x + iy$ and $z' = x' + iy'$ where x, y, x' and y' are real numbers.
   a- Show that $x' = \frac{-2x}{x^2 + y^2}$ and $y' = 2 + \frac{2y}{x^2 + y^2}$.
   b- If $x = y$, show that the lines (OM) and (EM') are perpendicular.

III) In the complex plane referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$, consider the points A, B, M and M' of respective affixes 2, $-i$, z and z' where $z' = \frac{iz-1}{z-2}$. $(z \neq 2)$.

1) Find the coordinates of M when $z' = 1+2i$.
2) Give a geometric interpretation for $|z - 2|$ and for $|iz - 1|$ and determine the set of points M such that $|z - 2| = |iz - 1|$.
3) Let $z = x + iy$ and $z' = x' + iy'$ (x, y, x' and y' are real numbers).
   a- Calculate x' and y' in terms of x and y.
   b- Show that if z' is pure imaginary, then M moves on a straight line whose equation is to be determined.
   c- Show that if z is real, then M' moves on a straight line whose equation is to be determined.

IV) In the complex plane referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points A(1), M(z) and M'(z') so that: $z'=(1-i)z+i$ with $z\neq 1$.

1) a- Verify that $z'-1=(1-i)(z-1)$.

   b- Verify that $AM'=AM\sqrt{2}$. Deduce that if M moves on the circle with center A and radius $\sqrt{2}$, then M′ moves on a circle (C) whose center and radius should be determined.

   c- Prove that: $\left(\vec{u};\overrightarrow{AM'}\right)=-\frac{\pi}{4}+\left(\vec{u};\overrightarrow{AM}\right)+2k\pi$ with $k\in\mathbb{Z}$.

   d- Compare $|z'-z|$ and $|z-1|$, then prove that the triangle AMM' is right isosceles.

2) Let $z=x+iy$ and $z'=x'+iy'$ where x, y, x' and y' are real numbers.

   a- Express x' and y' in terms of x and y.

   b- Verify that if M′ moves on a line (D) with equation $y=x$, then M moves on a line (Δ) to be determined.

V) The complex plane is referred to the direct orthonormal system $(O;\ \vec{u},\vec{v})$.
Consider the points A, B and C with affixes $z_A=-2+2i$, $z_B=-2i$ and $z_C=4$.
For every point M with affix z, assign the point M′ with affix z′ such that $z'=\frac{2z+4i}{iz+2+2i}$ with $z\neq -2+2i$.

1) In the case where $z=0$, give the exponential form of z′.
2) Write $\frac{z_A-z_B}{z_C-z_B}$ in algebraic form. Deduce the nature of triangle ABC.
3) a- Verify that $z'=\frac{2(z-z_B)}{i(z-z_A)}$

   b- Deduce that $OM'=\frac{2BM}{AM}$

   c- Show that when M moves on the perpendicular bisector of [AB], the point M′ moves on a circle whose center and radius are to be determined.

VI) The complex plane is referred to a direct orthonormal system $\left(O;\ \vec{u},\vec{v}\right)$.

A and B are two points with respective affixes $z_A=-4$ and $z_B=2$.

M and M′ are two points with respective affixes z and z′ such that $z'=\frac{\overline{z}+4}{\overline{z}-2}$, where $z\neq -4$ and $z\neq 2$.

1) Determine the coordinates of points M in the case where M and M′ are confounded.

2) a- Express $|z'|$ in terms of MA and MB and verify that $\arg(z')=\arg\left(\frac{z-2}{z+4}\right)+2k\pi$, $(k\in\mathbb{Z})$.

   b- Show that if M′ varies on the circle (C) with center O and radius 1, then M varies on a straight line (Δ) to be determined.

   c- Determine the set of points M if z′ is a strictly negative real number.

   d- Given the complex number $u=e^{-i\frac{\pi}{9}}$.
   Determine the nature of triangle MBA when u is a cubic root of z′.

VII) The complex plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$.

Consider the points A, M and M ' with affixes $z_A = -i$, $z_M = z$ and $z_{M'} = z'$ such that

$z' = \frac{z+i}{i\bar{z}}$ with $z \neq 0$.

Suppose that $z = x + iy$ and $z' = x' + iy'$ where x, y, x' and y' are real numbers.

1) Write z' in exponential form in the case where $z = e^{i\frac{\pi}{2}}$.
2) Write z in algebraic form in the case where $z' = z$.
3) a- Show that $OM' = \frac{AM}{OM}$.

   b- Show that when M varies on the perpendicular bisector of [OA], the point M' varies on a circle (C) whose center and radius are to be determined.
4) In this part $x > 0$ and $y > 0$.

   a- Show that $\frac{z'+i}{z} = \frac{2y+1}{x^2+y^2}$ and deduce that (OM) and (M'A) are parallel.

   b- Show that $z' - z = \frac{i + z - iz\bar{z}}{i\bar{z}}$ and deduce that if M belongs to (C), then $MM' = OA$.

VIII) In the plane referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$, consider the points A, B and C with respective affixes $z_A = i$, $z_B = 3-2i$ and $z_C = 1$.

1) Prove that the points A, B and C are collinear.
2) Consider the complex number $w = z_C - z_A$.

   Write w in exponential form and deduce that w^{20} is a real negative number.
3) Let M be a point in the plane with affix z.

   a- Give a geometric interpretation to $|z-i|$ and $|z-1|$.

   b- Suppose that $|z-i| = |z-1|$; show that the point M moves on a line to be determined.

   c- Prove that if $(z-i)\times(\bar{z}+i) = 16$, then the point M moves a circle whose center and radius to be determined.

IX) The complex plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$.

Consider the points A, B, M and M' with affixes $z_A = i$, $z_B = 2i$, $z_M = z$ and $z_{M'} = z'$ such that

$z' = \frac{iz-1}{z}$ for all $z \neq 0$.

1) a) Show that $z' - i = -\frac{1}{z}$.

   b) Write $z' - i$ in exponential form in the case where $z = e^{i\frac{\pi}{4}}$.
2) a) Show that $OM \times AM' = 1$ and that $(\vec{u}; \overrightarrow{OM}) + (\vec{u}; \overrightarrow{AM'}) = \pi \ (2\pi)$.

   b) Deduce that if M moves on the segment [OA] deprived of O and A, then M' moves on the semi-straight line [By) deprived of B.
3) Let $z = x + iy$ and $z' = x' + iy'$ where x, y, x' and y' are real numbers.

   a) Show that $x' = \frac{-x}{x^2+y^2}$ and $y' = \frac{x^2+y^2+y}{x^2+y^2}$.

   b) Show that if M' moves on the line (d) with equation $y = 2x$, then M moves on the circle with center I of affix $-1 - \frac{1}{2}i$ and radius $\frac{\sqrt{5}}{2}$.

The complex plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$.

Consider the points A, B, M and M′ with affixes $z_A = i$, $z_B = -2i$, $z_M = z$ and $z_{M'} = z'$ such that $z' = \frac{2z - i}{iz + 1}$ with $z \neq i$.

1) Write z' in exponential form in the case where $z = \frac{1}{2} + i$.

2) Determine the affix of point M in the case where the point M′ is the midpoint of segment [AB].

3) a) Verify that $(z' + 2i)(z - i) = 1$ for all $z \neq i$.

   b) Deduce that $BM' = \frac{1}{AM}$ and that $(\vec{u}; \overrightarrow{BM'}) = -(\vec{u}; \overrightarrow{AM}) + 2k\pi$ where k is an integer.

   c) When M varies on the circle (C) with center A and radius 2, show that M′ varies on a circle whose center and radius are to be determined.

   d) Show that if M varies on the semi line [Ay) deprived of point A, then O is a point on the segment [MM′].

The complex plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$.

Consider the points A, B, E, M and M′ with affixes $z_A = 1$, $z_B = \frac{3}{2} + i\frac{\sqrt{3}}{2}$, $z_E = -1$, $z_M = z$ and $z_{M'} = z'$ such that $z' = \frac{\bar{z} + 2}{\bar{z}}$ with $z \neq 0$.

1) a) Determine the exponential form of $(z_B - z_A)$.

   b) Deduce a measure of the angle $(\vec{u}; \overrightarrow{AB})$.

2) a) Calculate $\bar{z}(z' - 1)$.

   b) Deduce that $(\overrightarrow{OM}; \overrightarrow{AM'}) = 0 \ [2\pi]$.

   c) Show that if M moves on the y-axis deprived of O, then M′ moves on a line to be determined.

3) Let $z = x + iy$ and $z' = x' + iy'$ where x, y, x′ and y′ are real numbers.

   a) Show that $x' = 1 + \frac{2x}{x^2 + y^2}$ and $y' = \frac{2y}{x^2 + y^2}$.

   b) Show that if M moves on the line with equation $y = 2x$ then M′ moves on a line whose equation is to be determined.

4) a) Show that $\frac{z' + 1}{z + 1} = \frac{2z(\bar{z} + 1)}{z\bar{z}(z + 1)}$ where $z \neq -1$.

   b) Deduce that if M moves on the circle (C) with center O and radius 1, then $\overrightarrow{EM'} = 2\overrightarrow{EM}$.

The complex plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$. A, B and C are three points with respective affixes $z_A = i$, $z_B = -2i$, and $z_C = \frac{i}{2}$. For every point M with affix z, with $z \neq 0$ and $z \neq i$, we associate the point M′ with affix z′ such that $z' = \frac{2z - i}{iz + 1}$. (C) is a circle of center A and radius 1 unit.

1) Let $z = x + iy$ and $z' = x' + iy'$, where x, y, x′, and y′ are real numbers.

   a- Show that $x' = \frac{x}{x^2 + (y-1)^2}$ and $y' = \frac{-2x^2 - 2y^2 + 3y - 1}{x^2 + (y-1)^2}$

   b- Find the set of points M when z′ is pure imaginary.

2) Find the set of points M when $|z'| = 2$.

3) a- Verify that $(z' + 2i)(z - i) = 1$.

   b- Show that: if the point M moves on the circle (C), then the point M′ moves on a circle (C′) whose center and radius are to be determined.

4) Let D be a point with affix $z_D = a + (1 + a)i$, where a is a strictly positive real number.

   a- Give a measure of the angle $(\vec{u}, \overrightarrow{AD})$, and deduce the set of points D.

   b- Show that D is a point on a circle (C″) with center A and radius $a\sqrt{2}$ units.

   c- Deduce a geometric construction of the point D

Bekaa Youth Education Center
Math test
General sciences

Date: 14/11/2023
Duration: 60 minutes
Name:

I) (8 points)

In the complex plane referred to an orthonormal system $(0,\vec{u},\vec{v})$ consider the points A, B, M and M' of affixes $-i$, i, z and z' respectively such that $z' = \frac{1-\bar{z}}{1-i\bar{z}}$ with $z \neq -i$.

1) Write z in exponential form in the case where $z' = 2+i$.
2) Determine the coordinates of points M in the case where $z' = 1 - z$.
3)a) Verify that $(z'+i)(\bar{z}+i) = -1+i$

b) Deduce that $AM' \times BM = \sqrt{2}$ and that $\left(\vec{u},\overrightarrow{AM'}\right) = \frac{3\pi}{4} + \left(\vec{u},\overrightarrow{BM}\right) + 2k\pi$.

c) Show that if M moves on the circle with center B and radius $\sqrt{2}$ then M' moves on the circle whose center and radius are to be determined.

4) Let $z = x+iy$ and $z' = x' + iy'$ whare x, y, x' and y' are real numbers.

a) Show that $x' = \frac{-x-y+1}{x^2+(y-1)^2}$ and $y' = \frac{-x^2-y^2+x+y}{x^2+(y-1)^2}$.

b) Show that if M' moves on a line of equation $y = -x-1$ then M moves on a line whose equation is to be determined.

II) (6 points)

The complex plane is referred to a direct orthonormal system $(0,\vec{u},\vec{v})$.
Let $z = re^{i\alpha}$ where r is a positive real number such that $r \neq 1$.
Consider the points A, B, C and D of respective affixes $z_A = z$, $z_B = \frac{1}{z}$, $z_C = \frac{\bar{z}}{z^2}$ and $z_D = -\bar{z}$.

1) Determine the exponential form of $\frac{z_A}{z_C}$. Deduce the set of values of α such that 0 belongs to the segment $]AC[$.

2) Suppose in this part that $\alpha = \frac{\pi}{4}$

a) Prove that $z_C - z_D = \overline{z_A - z_B}$

b) Calculate $z_A - z_D$ and $z_B - z_C$ in terms of r and prove that these numbers are two distinct positive real numbers.

c) Deduce that ABCD is an isosceles trapezoid whose diagonals intersect at O.

Good Work!!!

III) The complex plane is referred to an orthonormal system $(O; \vec{u}; \vec{v})$.

Consider the points A and B with respective affixes $z_A = i$ and $z_B = 1$.

For every point M of affix z we associate the point M' of affix z' such that: $z' = \frac{i\bar{z}-1}{\bar{z}-1}$ with $z \neq 1$.

1) In case where $z' = -1$ prove that z^{12} is a negative real number.
2) a) Show that for every point M distinct from B we have: $|z'| = \frac{AM}{BM}$.
   b) Deduct the set of points M' when M describes the perpendicular bisector of $[AB]$.
3) a) Show that $\arg(z') = \frac{\pi}{2} + (\overrightarrow{AM}; \overrightarrow{BM})[2\pi]$.
   b) Determine the set of points M when z' is a strictly negative real number.
4) In this part suppose that $z = 1 + \sqrt{2}e^{i\theta}$ where θ is a real number.
   a) Show that M describes the circle (φ) of center B and radius $\sqrt{2}$.
   b) Calculate $(z' - i)(\bar{z} - 1)$.
   c) Deduct the set of points M' when M describes the circle (φ).

IV) The complex plane is referred to a direct orthonormal system $(O; \vec{u}, \vec{v})$.

A- Let A be the point of affix 10 and (γ) the circle of diameter $[OA]$.

1- Prove that the points B and C of respective affixes $b=1+3i$ and $c=8-4i$ belong to (γ).

2- Let D be the point of affix $d=2+2i$.

- Calculate $\frac{b-d}{b-c}$ and $\frac{d}{b-c}$. Deduce that D is the orthogonal projection of O on (BC).
- Draw (γ) and plot the points A, B, C and D.

B- To each point M of plane with affix z, distinct from O, we associate the point M' of affix z' such that $z' = \frac{20}{\bar{z}}$.

1- Prove that the points O, M and M' are collinear.

2- Suppose in this part that M belongs to the straight line (Δ) of equation $x=2$.

a) Verify that $z+\bar{z}=4$ and prove that $5(z'+\bar{z'}) = z'\bar{z'}$. Deduce that M' belongs to (γ).

b) Take a point M on (Δ) and plot the associated point M'.