CHAPTER REVIEW

1- Definition

The exponential function, denoted by exp, is the inverse function of the function ln.

$$\ln :]0 : +\infty[\longrightarrow \mathbb{R}$$

$$\exp : \mathbb{R} \longrightarrow]0 : +\infty[$$

$\ln(\exp(x)) = x$ for any real number x, and $\exp(\ln x) = x$ for any positive real number x.

Since $\ln(\exp(x)) = x$, then $\ln(\exp(x)) = x\ln e = \ln e^x$, thus $\boxed{\exp(x) = e^x}$

2-Properties

- $f(x) = e^x$, so $f'(x) = e^x > 0$, the exponential function is strictly increasing on $\mathbb{R}$.
- $y = e^x \Leftrightarrow x = \ln y$ where $y > 0$.
- For any real number x, $\ln(e^x) = x$.
- For any real number $x > 0$, $e^{\ln(x)} = x$.
- a and b are real numbers, we have :
  - $e^a < e^b \Leftrightarrow a < b$ ▸ $e^a = e^b \Leftrightarrow a = b$ ▸ $e^a > e^b \Leftrightarrow a > b$
  - $e^a < 1 \Leftrightarrow a < 0$ ▸ $e^a = 1 \Leftrightarrow a = 0$ ▸ $e^a > 1 \Leftrightarrow a > 0$
- a and b are real numbers, we have :
  - $e^{a+b} = e^a \times e^b$ ▸ $e^{a-b} = \dfrac{e^a}{e^b}$ ▸ $e^{-b} = \dfrac{1}{e^b}$ ▸ $(e^a)^b = e^{ab}$

3- Limits

▸ $\lim\limits_{x\to+\infty} e^x = +\infty$ ▸ $\lim\limits_{x\to-\infty} e^x = 0^+$ ▸ $\lim\limits_{x\to-\infty} xe^x = 0^-$;

▸ $\lim\limits_{x\to 0} \dfrac{e^x-1}{x} = 1$ ▸ $\lim\limits_{x\to+\infty} \dfrac{e^x}{x} = +\infty$

4- Derivative and antiderivative

- If $f(x) = e^x$, then $f'(x) = e^x$.
- a and b are real numbers, $(a \neq 0)$; we have $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$.

5- real powers of a positive number

For any real numbers a and b (with $a > 0$), the number "a power b", denoted by a^b, is the number defined as $a^b = e^{b\ln(a)}$.

If $f(x) = a^x$, then $f'(x) = a^x \ln a$.

If $g(x) = a^{u(x)}$ where u is a differentiable function, then $g'(x) = u'(x) \ln a\, g(x)$.

$$\int a^x dx = \frac{a^x}{\ln a} + C.$$

CHAPTER 7
EXPONENTIAL FUNCTIONS

TEST YOUR KNOWLEDGE

In this chapter, all systems of reference $(O\,;\,\vec{i}=\overrightarrow{OI}\,,\,\vec{j}=\overrightarrow{OJ})$ are supposed to be orthonormal, unless otherwise given.

I- MULTIPLE CHOICE QUESTIONS (M.C.Q)

For each question, exactly one answer is correct. Choose, with justification, the correct answers:

1-• $f(x) = xe^{x^2} + x^2e^{2x}$. The number derivative $f'(1)$ is equal to:

a. $e(3 + 4e)$ b. $e(4 + 3e)$ c. $11e$

2- The integral $\int_0^1 x^2e^x dx$ is equal to:

a. $2 - e$ b. $e - 2$ c. $5e - 2$

3- Let $L = \lim\limits_{x\to 0} \dfrac{x}{\sqrt{3 + e^x} - 2}$. L is equal to:

a. -4 b. 0 c. 4

4- Given $\varphi(x) = x^{\ln x}$ where $x > 0$. The number derivative $\varphi'(e)$ is equal to:

a. 1 b. 2 c. e

5-• g is the inverse function of the function f where $f(x) = \ln(1 + e^x)$.

The expression $g(x)$ is equal to:

a. $\ln(1 + x)$ b. $\ln(1 - e^x)$ c. $\ln(e^x - 1)$

6-• $H(x) = \ln(e^{2x} + e^x - 6)$.

a. The domain of definition of H is:

a. $]\ln 2\,;\,+\infty[$ b. $]-\infty\,;\,\ln 2[$ c. $]\ln 2\,;\,\ln 3[$

b. $\lim\limits_{x\to+\infty} (H(x) - 2x)$ is equal to:

a. $-\infty$ b. 0 c. $+\infty$

7- The equation $e^{\ln(x-1)^2} = \ln e^{x-1}$ is verified for:

a. $x = 2$ b. $x = 1$ c. $x = e$

8- The curve (C) with equation $y = f(x) = (x - 3)e^x + 2e$ has a point of inflection E with coordinates:

a. $(1\,;\,2e)$ b. $(1\,;\,1)$ c. $(1\,;\,0)$

9- $f(x) = \ln(1 + \ln x)$ and $g(x) = e^{x-1}$. The domain of definition of $f \circ g$ is:

a. $]0\,;\,+\infty[$ b. $]1\,;\,+\infty[$ c. $]\frac{1}{e}\,;\,+\infty[$

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EXPONENTIAL FUNCTIONS

II- TRUE OU FALSE (WITH JUSTIFICATION)

A) Consider the function f such that $f(x) = (3 - x)\, e^{-x}$ and let C be be its representative curve.

1- For every $x \geq 0$, we have $f(x) \geq -x + 3$.

2- The line with equation $y = 0$ is asymptote to the curve C.

3- The derivative of f is $f'(x) = (2 - x)e^{-x}$.

4- The function f has a unique extremum.

5- For every real number $m \neq -e^{-4}$, the equation $f(x) = m$ has either 0 or 2 solutions.

6- The function g such that $g(x) = (3 - x)\, e^{-|x|}$ is not differentiable at 0.

7- The function f is a solution of the differential equation $y' + y = -e^{-x}$.

8- The average of f between 0 and 1 is $2 - \frac{1}{e}$.

B) Let $T(x) = 2^x + 3^x - x\ln 6$.

1- The number derivative $T'(0) = 0$.

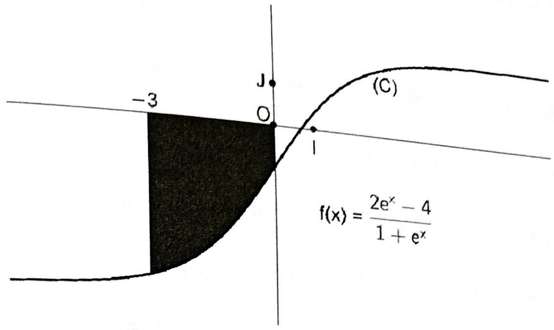
2-• $\int_0^1 T(x)dx = \frac{\ln 2 + \ln 6}{\ln 2 \times \ln 3} - \ln\sqrt{3}$.

CHAPTER 7
EXPONENTIAL FUNCTIONS

SOLVED EXERCISES

Exercise 1

Calculate the area of the shaded region.



Exercise 2

Define on $\mathbb{R}$ a function f by $f(x) = -x + e^x$.

Find an equation of the tangent (T) issued from the origin O to the representative curve C of f.

Exercise 3

Define on $\mathbb{R}$ a function T by $T(x) = \displaystyle\int_{x+e^{-x}}^{x+e^{x}} \ln(t^2 - t + 1)dt$.

Calculate T(0) and T′(0).

Exercise 4

Determine the real numbers a and b for which the function $F: x \longrightarrow e^x (a \sin x + b \cos x)$ is an anti-derivative of the function $f: x \longrightarrow e^x \sin x$ on $\mathbb{R}$.

Exercise 5

Let f and $\overset{h}{g}$ be two functions defined on $\mathbb{R}$ by $f(x) = e^{2x} + 2e^x$ and $h(x) = \ln(f(x))$.

1- Solve the equation $f(x) = 3$.

1- g is the inverse function of f.

a. Find the domain of definition of g.

b. Calculate $g'(3)$.

c. Calculate $g(x)$.

3- Calculate $\lim\limits_{x \to +\infty} (h(x) - 2x)$ and interpret the result.

Exercise 6

Define on $]0\,;+\infty[$ a function g by $g(x) = x - \ln(2x)$.

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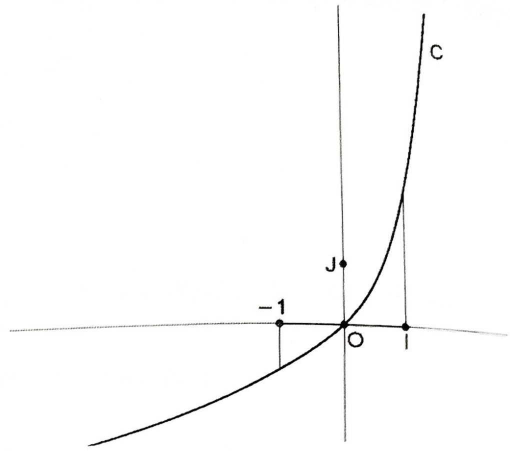
Let f be a function defined on $\mathbb{R}$ by $f(x) = x^2 - e^x$.

Show, for $x > 0$, that $g(x) > 0$ and deduce that f is strictly decreasing on $\mathbb{R}$.

Exercise 7



The curve C in the adjacent figure has equation

$y = f(x) = (e^x - 1)\sqrt{e^x - x}$.

1- Write an equation of the tangent (T) at O to C.

2- Calculate the area of the shaded region.

3- a. Let $T_k(x) = xe^{kx}$ where k is a real number different from zero.

Show that $T_k'(x) = e^{kx} + kT_k(x)$. Deduce a primitive of T_k on $\mathbb{R}$.

b. Calculate the volume generated by the rotation of the shaded region around the x-axis.

Exercise 8

f is a function described by $f(x) = x + 1 + \ln(e^{2x} - e^x + 1)$ and let (C) be its representative curve.

1- Verify that the domain of definition of f is $\mathbb{R}$.

2- Calculate $\lim\limits_{x \to -\infty} (f(x) - x)$ and $\lim\limits_{x \to +\infty} (f(x) - 3x)$. Write four immediate conclusions.

2- Show that f is strictly increasing on $\mathbb{R}$.

3- g is the inverse function of f. Calculate f(0) and g′(1).

✓ Exercise 9

Let f be the function defined on $\mathbb{R}$ as $f(x) = x + 1 - \dfrac{4e^x}{1 + e^x}$ whose representative curve is (C).

1- Show that the point I(0 ; −1) is a center of symmetry of (C).

2- Determine $\lim\limits_{x \to -\infty} [f(x) - (x+1)]$ and $\lim\limits_{x \to +\infty} [f(x) - (x - 3)]$. Interpret the results graphically.

3- Set up the table of variations of f.

4- Draw (C).

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Exercise 10

Let f be the function defined on $\mathbb{R}$ by $f(x) = 1 - (2x + 1)e^{2x}$ with representative curve (C).

1- a. Calculate $\lim\limits_{x \to -\infty} f(x)$, $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x}$. Interpret the results graphically.

b. Set up the table of variations of f.

c. Draw (C).

2- Let g be the function defined on $\mathbb{R}$ by $g(x) = x + 3 - xe^{2x}$. Designate by (Γ) the representative curve of g.

a. Show that the line (d) with equation $y = x + 3$ is asymptote to (Γ) ; then study , according to the values of x, the relative positions of (Γ) and (d).

b. Calculate $\lim\limits_{x \to +\infty} \dfrac{g(x)}{x}$ then write two immediate conclusions.

c. Prove that $g'(x) = f(x)$ and then set up the table of variations of g.

d. Draw (Γ) and (d) in the given system.

Exercise 11

Let f be the function defined on $\mathbb{R}$ by $f(x) = e^{-2x} - 4e^{-x} + 3$.

Designate by (C) its representative curve (Graphical unit: 2cm).

1- Calculate $\lim\limits_{x \to +\infty} f(x)$ and deduce an asymptote (d) to (C).

Determine the coordinates of the point of intersection of (C) with its asymptote (d).

2- Show that $\lim\limits_{x \to -\infty} f(x) = +\infty$, find $f'(x)$ and set up the table of variations of f.

3- Prove that (C) has the origin O as a point of inflection and write an equation of the tangent (T) at O to (C).

4- Draw (d), (T) and (C).

5-Calculate the area of the region bounded by (C) and the axis of abscissas.

6- Let φ be the restriction of f on $[-\ln 2 ; +\infty[$.

Show that φ has an inverse function g, set up the table of variations of g, draw the representative curve (G) of g and calculate g(x).

EXPONENTIAL FUNCTIONS

Exercise 12

1-Let h be the function defined on $\mathbb{R}$ as $h(x) = (2 - x)e^x - 2$.

a. Calculate $\lim_{x\to-\infty} h(x)$ and $\lim_{x\to+\infty} \frac{h(x)}{x}$. Interpret the results.

b. Set up the table of variations of h.

c. Draw the curve (γ).

d. The equation $h(x) = 0$ has two roots 0 and α. Show that $1.5 < \alpha < 1.6$.

2- Let f be the function defined on $\mathbb{R}$ as $f(x) = \frac{e^x - 2}{e^x - 2x}$. Designate by (C) its representative curve.

a. Show that the lines with equations $y = 0$ and $y = 1$ are asymptotes to (C).

b. Prove that $f'(x)$ and $h(x)$ have the same sign and deduce the table of variations of f.

c. Prove that $f(\alpha) = \frac{1}{\alpha - 1}$ and deduce an approximate value to the nearest 10^{-1} of $f(\alpha)$ when $\alpha = 1.55$.

d. Determine the point of intersection of (C) with the line (d) with equation $y = 1$.

Draw (C), without using the approximate values of α and $f(\alpha)$.

e. Let A_n be the area of the region bounded by (C), the line (d) and the lines with equations $x = 1$ and $x = n$, where $n > 1$. Calculate A_n and find the limit of A_n as n tends to $+\infty$.

Exercise 13

Part A

Let f be the function defined on $\mathbb{R}$ as $f(x) = e^{0.5x} - e^x$.

1- Calculate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} \frac{f(x)}{x}$. Write three immediate conclusions.

2- Set up the table of variations of f.

3- Draw the representative curve (C) of f.

4-Calculate the area of the region bounded by (C), the line with equation $x = 1$ and the x-axis.

Part B

Let g be the function given by $g(x) = \ln(e^x - e^{0.5x})$; (C') is its representative curve.

1-Verify that the domain of definition of g is $]0 ; +\infty[$.

2- Show that $g(x) - x = \ln(1 - e^{-0.5x})$ and deduce that the line (d) with equation $y = x$ is asymptote to (C').

3- Show that (C') is below (d).

4- Calculate $\lim\limits_{\substack{x \to 0 \\ x>0}} g(x)$ and $\lim\limits_{x\to+\infty} g(x)$; calculate $g'(x)$ and set up the table of variations of g.

5- Solve the equation $g(x) = 0$ and draw (C').

6- Show that g has an inverse function h and calculate h(x).

Exercise 14

Let (C) be the representative curve of the function f defined on $\mathbb{R}$ as $f(x) = \dfrac{3e^x - 1}{e^x + 1}$.

1- a. Show that, for every real number x, we have: $f(-x) + f(x) = 2$.

Deduce that (C) has a center of symmetry to be designated by A.

b. Show that (C) has two asymptotes whose equations are to be determined.

c. Find $f'(x)$ and set up the table of variations of f.

2- a. Determine an equation of the tangent (T) to (C) at the point with abscissa 0.

b. Consider the function δ defined on $\mathbb{R}$ by $\delta(x) = f(x) - (x + 1)$.

Prove that $\delta'(x) = -\left(\dfrac{e^x - 1}{e^x + 1}\right)^2$. Deduce the sense of variation of δ then study its sign (Specify $\delta(0)$).

c. Study, according to the values of x, the relative positions of (C) and (T).

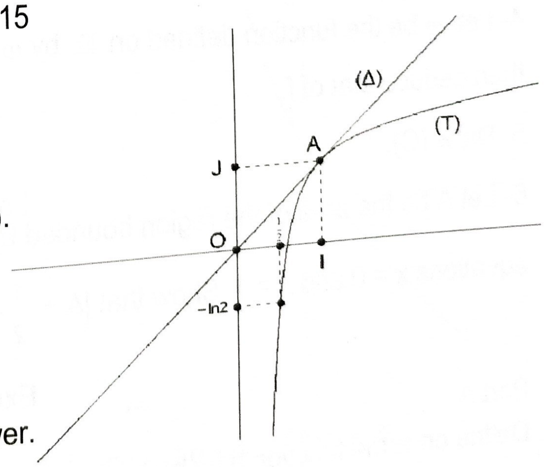
3- Draw (C) and (T).

4- Calculate the area of the region bounded by (C) and the axis of abscissas and the lines with equations $x = 0$ and $x = 1$.

5- f has an inverse function g. Calculate g(x) and draw the representative curve (G) of g.

Exercise 15

Part A



Given the representative curve (T) of a function g continuous and differentiable on $]0;+\infty[$.(unit:2 cm).

(Δ) is the tangent at A to the curve (T).

1- Study the variations of g on $]0;+\infty[$ and find $g(1)$, $g(\frac{1}{2})$. Determine $g'(1)$ and interpret the answer.

2- Suppose that, for every x in $]0;+\infty[$, $g(x) = \ln x + \dfrac{2}{x} - \dfrac{1}{x^2}$.

a. Verify all results obtained in preceding question.

b. Show that the equation $g(x) = 0$ has a unique real root α ; prove that $0.5 < \alpha < 0.6$ and study the sign of g on $]0 ; +\infty[$.

Part B

Let f be the function defined on $]0 ; +\infty[$ by $f(x) = e^x\left(\ln x + \frac{1}{x}\right)$.

1- a. Determine the limit of f at $+\infty$.

b. Verify that $f(x) = \frac{e^x}{x}(x\ln x + 1)$ and deduce the limit of f at 0.

2-a. Determine the derivative function f ' of f.

b. Set up the table of variations of f.

3- Denote by (C) the representative curve of f (graphical unit: 2cm).

a. Determine an equation of the tangent (L) to (C) at the point E with abscissa 1.

b. Draw (C) and (L) by taking $\alpha = 0.59$ and $f(\alpha) = 2.6$.

Exercise 16

Consider the function f defined on $\mathbb{R}$ by $f(x) = \ln(e^x + e^{-x})$; (C) is its representative curve.

Designate by (d) and (d′) the lines with equations $y = x$ and $y = -x$ respectively.

1- Show that f is even.

2- Show that (C) is below (d); deduce the relative position of (C) with respect to (d).

3- Calculate $\lim\limits_{x\to+\infty}[f(x)-x]$.Deduce four immediate conclusions.

4- Let φ be the function defined on $\mathbb{R}$ by $\varphi(x) = e^x + e^{-x}$; study the sense of variations of φ then deduce that of f.

5- Draw (C).

6- Let A be the area of the region bounded by (C), the axis of abscissas and the two lines with equations $x = 0$ and $x = 1$. Show that $|A + \frac{1}{2} - \ln(1 + e)| \le \frac{1}{2}$.

Exercise 17

Part A

Define on $\mathbb{R}$ the function h by $h(x) = x + 2 - e^x$.

1- Verify that $\lim\limits_{x\to+\infty} h(x) = -\infty$ and find $\lim\limits_{x\to-\infty} h(x)$.

2- Set up the table of variation of h.

3- a. Show that the equation $h(x) = 0$ has exactly two real roots $\mathbb{R}$.

Denote these roots by α and β, where $\alpha < \beta$.

Verify that $-1.85 < \alpha < -1.84$ and $1.14 < \beta < 1.15$.

b. Deduce the sign of $h(x)$ according to the values of x.

4- Noticing that, for every real number x, $h(-x) \leq 1$, show that $e^x - xe^x - 1 \leq 0$.

Deduce the sign of $xe^x + 1$.

Part B

Define on $\mathbb{R}$ the function f by $f(x) = \dfrac{e^x - 1}{xe^x + 1}$, C is its representative curve.

Take graphical unit: 2cm on the axis of abscissas and 5cm on the axis of ordinates.

1- Find $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.

2-Verify that $f(\alpha) = \dfrac{1}{\alpha + 1}$ and determine an interval, with amplitude 10^{-2} , and containing $f(\alpha)$.

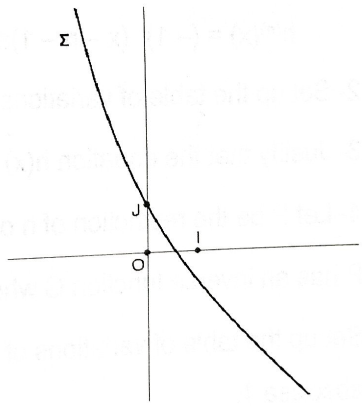
3- Prove that $f'(x) = \dfrac{e^x h(x)}{(xe^x+1)^2}$ and set up the table of variation of f.

4-Prove that $f(x) - x = \dfrac{(x+1)(e^x - xe^x - 1)}{xe^x + 1}$ then study the positions of C relative to the line (L) with equation $y = x$.

5- Draw the line (L) and the curve C.

6- Designate by $S(\alpha)$ the area, in cm^2, of the region bounded by (C), the axis of abscissas and the two lines with equations $x = \alpha$ and $x = 0$.

Exercise 18



Given a function f defined on $\mathbb{R}$ by $f(x) = e^{-x} - x$ whose representative curve is Σ.

1- a. Prove that f has on $\mathbb{R}$ an inverse function h and draw its representative curve Σ'.

b. Calculate $h'(1)$.

2- The equation $f(x) = 0$ has a unique solution α.

Find an interval with amplitude 0.1 and containing α. Determine the sign of f(x) on $\mathbb{R}$.

3- Calculate, in terms of α the area $A(\alpha)$ of the region bounded by the curve Σ and the two axes of coordinates. Write $A(\alpha)$ in the form of a polynomial in α.

4- Let D be the region bounded by Σ and the lines with equations $y = -x$, $x = -1$ and $x = 0$.

Calculate the volume generated by the rotation of D around the axis of abscissas.

5- Let $U(x) = f^2(x) - x^2$.

Show that the nth derivative of U is $U^{(n)}(x) = (-2)^n e^{-2x} + 2(-1)^{n+1}(x - n)e^{-x}$.

6- Let g be the function defined on $\mathbb{R} - \{\alpha\}$ by $g(x) = \dfrac{x}{f(x)}$.

Designate by Ω the representative curve of g.

a. Precise, according to the values of x, the sign of g(x).

b.Find the limits of g at the open ends of the domain and set up the table of variations of g.

7- Let $J =]-\infty\,;0]$.

a. Define on J the function V by $V(x) = f\circ g(x)$. Calculate $V'(0)$.

b. Define on J the function W by $W(x) = \int_{-x^2}^{0} g(t)\,dt$. Prove that W is strictly increasing.

✓ Exercise 19

Part A

Define on $\mathbb{R}$ a function h as $h(x) = (x - 1)e^{-x} + 2$.

1- a. Verify that h(x) is a solution of the differential equation: $y''(x) + 2y'(x) + y(x) = 2$.

b. Show, by mathematical induction, that the nth derivative of h is:

$h^{(n)}(x) = (-1)^n (x - n - 1)e^{-x}$.

2- Set up the table of variations of h.

3- Justify that the equation $h(x) = 0$ has a unique solution α. Show that $-0.4 < \alpha < -0.3$.

4- Let P be the restriction of h on $]-\infty\,;2[$.

P has an inverse function G whose representative curve is Σ.

Set up the table of variations of g and write an equation of the tangent to Σ at the point A with abscissa 1.

Part B

Define on $\mathbb{R}$ a function f by $f(x) = 2x + 1 - xe^{-x}$. Let C be its representative curve.

1- a. Study the relative positions of C and the line (d) with equation $y = 2x + 1$.

b. Show that $\lim\limits_{x\to+\infty}(f(x) - 2x - 1) = 0$. Write two immediate conclusions.

2- Calculate $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}$. Write two immediate conclusions.

3- a. Verify that $f'(x) = h(x)$ and $f(\alpha) = \dfrac{(\alpha+1)(2\alpha-1)}{\alpha-1}$ then deduce that $f(\alpha) > 0$.

b. Set up the table of variations of f.

4- Write an equation of the tangent (Δ) to C at the point with abscissa 0 and the equation of the tangent (Δ') to C at the point with abscissa 1.

5- Construct (d), (Δ), (Δ') and C.

6- Calculate the area of the region bounded by C, (d) and the line with equation $x = 1$.

Part C

Let $T = h \circ f$.

Calculate the number-derivative $T'(0)$ and find $\lim\limits_{x\to-\infty} T(x)$.

CHAPTER 7
EXPONENTIAL FUNCTIONS

SUPPLEMENTARY EXERCISES

Exercise 1

Let g be the function defined on $\mathbb{R}$ as $g(x) = (x^2 + x)\, e^{-x}$.

1- Verify that: $g(x) + 2\,g'(x) + g''(x) = 2\,e^{-x}$. Deduce a primitive of g on $\mathbb{R}$.

2- Calculate the area of the region bounded by (C), the axis of abscissas and the two lines with equations $x = -1$ and $x = 0$.

Exercise 2

Define on $\mathbb{R}$ a function g by $g(x) = -x + \dfrac{2}{1 + e^x}$.

1- Show that g has an inverse function h and calculate $h'(1)$.

2- Calculate $g(x) + g(-x)$ and deduce that (C_h) has a point of inflection.

3- Calculate the area of the region bounded by (C_g) and the lines with equations $y = -x$, $x = 0$ and $x = \ln 2$.

Exercise 3

Define on $\mathbb{R}$ two functions f and U by $f(x) = 2x + 3e^x$ and $U(x) = x^2 + 3e^x + 1$.

1- Calculate $\lim\limits_{x \to -\infty} (f(x) - 2x)$ and $\lim\limits_{x \to +\infty} \dfrac{f(x)}{x}$. Give four immediate conclusions.

2- Show that the equation $f(x) = 0$ has a single real root k and find an approximate value of k to the nearest 0.1.

3- Draw (C_f). (Graphical unit: 2 cm)

4- Verify that $U(k) = (k - 1)^2$ and set up the table of variations of U.

5- Let S be the region bounded by (C_f), the axis of abscissas and the lines with equations $x = 0$ and $x = 1$. Calculate the volume generated by the rotation of S around the axis of abscissas.

Exercise 4

Consider the function f defined on $\mathbb{R}$ by $f(x) = e^{2x} + 2e^x - 3$.

1- Determine les limits of the function f at $+\infty$ and at $-\infty$.

2- Set up the table of variations of f.

3- Draw the representative curve C of the function f.

4- Let $g(x) = \ln(f(x))$.

a. Precise the domain of definition of g and find the limits of g at its domain.

b. Show that g has an inverse function h and calculate $h(x)$.

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Exercise 5

Consider the function g defined $\mathbb{R} - \{0\}$ by $g(x) = x - 1 + \dfrac{4e^x}{1 - e^x}$, and let (C_g) be its representative curve.

1- Calculate $\lim\limits_{\substack{x \to 0 \\ x<0}} [g(x)]$, $\lim\limits_{\substack{x \to 0 \\ x>0}} [g(x)]$ $\lim\limits_{x \to -\infty} [g(x) - x]$ and $\lim\limits_{x \to +\infty} [g(x) - x]$. What can be concluded?

2- Study the variations of g and draw (C_g).

3- Show that the expression $g(-x) + g(x)$ is independent of x. What do you deduce about (C_g).

4- Designate by D the region bounded by the curve (C_g), the line $y = x - 5$, the two lines with equations $x = 1$ and $x = m$, where m is a real number ; $m > 1$.

Calculate, in terms of m, the area A_m of the region D.

Deduce the limit of A_m as m tends to infinity.

Exercise 6

1-Let f be the function defined on $\mathbb{R}$ by $f(x) = -2xe^{-2x} + 2x - 3$.

Designate by (C) its representative curve.

a. Show that the line (d) with equation $y = 2x - 3$ is an asymptote to (C) at $+\infty$

b. Determine $\lim\limits_{x \to -\infty} f(x)$.

c. Study, according to the values of x, the relative position of (C) and (d).

2-The adjacent figure shows the table of variations of the function f ' , the derivative of f:

x	$-\infty$		0		1		$+\infty$
$f''(x)$		$+$		$+$	0	$-$	
$f'(x)$	$-\infty$	$\nearrow$	0	$\nearrow$	$2 + \frac{2}{e^2}$	$\searrow$	2

a. Set up the table of variations of f.

b. Show that (C) has a point of inflection.

c. Draw (C) and (d).

d. Calculate the area of the region bounded by (C), (d) and the lines with equations $x = 0$ and $x = \dfrac{3}{2}$.

Exercise 7

Part A

Let g be the function defined on $\mathbb{R}$ by $g(x) = 1 + xe^x$.

1- Calculate $\lim\limits_{x \to +\infty} g(x)$ and $\lim\limits_{x \to -\infty} g(x)$.

2- Study the variations of g. Deduce, according to the values of x, the signs of g(x).

Part B

Define on $\mathbb{R}$ a function f by $f(x) = x + (x - 1)e^x$ and let (C) be its representative curve.

1- Show that the line (D) with equation $y = x$ is asymptote to (C).

2- Calculate the coordinates of point A, the intersection of (C) and (D), then study the relative positions of (C) and (D).

3- $M(x\,;\,y)$ is any point on (C), and (T) be the tangent at M to (C).

Calculate the coordinates of M for which (T) is parallel to (D).

4- Study the variations of f and draw (C).

5- Calculate the area of the region D bounded by (C), the axis of abscissas and the two lines with equations $x = 1$ and $x = \frac{3}{2}$.

Exercise 8

Consider the function f defined on $\mathbb{R}$ by $y = f(x) = e^{-x^2 + 2x}$, (C) is its representative curve.

1- Show that the curve (C) has an axis of symmetry.

2- Study the variations of f and draw (C).

3- Deduce the graph (C') of the function g defined by $g(x) = e^{-x^2 + 2|x|}$.

(Draw (C') in a new system).

4-Let (S) be the curve with equation $y^2 + \ln x = 2e^{\ln|y|}$. Deduce how to draw (S).

Exercise 9

Let f be the function defined on $]\ln 2\,;+\infty[$ by: $y = f(x) = 3 - x - \dfrac{1}{e^x - 2}$, and designate by (C) its representative curve.

1-Calculate $\lim\limits_{\substack{x \to \ln 2 \\ x > \ln 2}} f(x)$ and $\lim\limits_{x \to +\infty} [f(x) - (-x + 3)]$; then deduce the asymptotes of (C).

2- Study, for all $x \in]\ln 2\,;+\infty[$, the position of (C) relative to the line (d) with equation: $y = -x+3$.

3- Noticing that: $e^{2x} - 5e^x + 4 = (e^x - 1)(e^x - 4)$, calculate $f'(x)$ and set up the table of variations of f.

4- Draw (C) and (d).

5- Verify, for $x \in]\ln 2\,;+\infty[$, that $\dfrac{1}{2}\left(\dfrac{e^x}{e^x - 2} - 1\right) = \dfrac{1}{e^x - 2}$.

6- Calculate the area A of the region D, bounded by (C), the line (d) and the two lines with equations $x = 2\ln 2$ and $x = 4\ln 2$.

CHAPTER 7
EXPONENTIAL FUNCTIONS

Exercise 10

Part A

Given the table of variation of a function f that is differentiable on $\mathbb{R}$.

x	$-\infty$		0		2		$+\infty$
$f(x)$	$+\infty$	$\searrow$	0	$\nearrow$	$\frac{4}{e^2}$	$\searrow$	0

Define the function F on $\mathbb{R}$ by $F(x) = \displaystyle\int_2^x f(t)dt$.

1-Study the variations of the function F.

2-Show that $0 < F(3) < 4e^{-2}$.

Part B

The function f considered in part A is the function defined on $\mathbb{R}$ as $f(x) = x^2e^{-x}$.

Designate by (C) the representative curve of f.

1-Show that the variations of the function f are as shown in part A.

2- Consider a real number k that is greater than or equal to 1.

Consider the region bounded by the curve (C), the axis of abscissas and the lines with equations $x = 1$ and $x = k$.

Determine the area $A(k)$ of this region . Determine the limit of $A(k)$ as k tends to $+\infty$.

CHAPTER 7
EXPONENTIAL FUNCTIONS

SOLUTION

TEST YOUR KNOWLEDGE

I- MULTIPLE CHOICE QUESTIONS (M.C.Q) — ANSWERS

1- $f(x) = xe^{x^2} + x^2e^{2x}$, which gives: $f'(x) = e^{x^2} + 2x^2e^{x^2} + 2xe^{2x} + 2x^2e^{2x}$;

$f'(1) = e + 2e + 2e^2 + 2e^2 = 3e + 4e^2 = e(3 + 4e)$. — **a**

2- $\int_0^1 x^2e^x dx = \left[(x^2 - 2x + 2)e^x\right]_0^1 = e - 2$. — **b**

x^2	$+$	e^x
$2x$	$-$	e^x
2	$+$	e^x
0		e^x

3- $L = \lim\limits_{x\to 0}\dfrac{x}{\sqrt{3+e^x}-2} = "\dfrac{0}{0}"$ Indeterminate Form

Applying L'Hôpital's rule ; $L = \lim\limits_{x\to 0}\dfrac{1}{\dfrac{e^x}{2\sqrt{3+e^x}}} = 4$. — **c**

4- $\varphi(x) = x^{\ln x}$; $\ln(\varphi(x)) = \ln x(\ln x) = (\ln x)^2$

$\dfrac{\varphi'(x)}{\varphi(x)} = \dfrac{2\ln x}{x}$; $\varphi'(x) = \dfrac{2\ln x}{x}\varphi(x)$; $\varphi'(e) = \dfrac{2\ln e}{e}\times\varphi(e) = 2$. — **b**

5- $z = g(x)$; $x = f(z) = \ln(1 + e^z)$; $1 + e^z = e^x$; $e^z = e^x - 1$, $z = \ln(e^x - 1)$. — **c**

6- $H(x) = \ln(e^{2x} + e^x - 6)$.

a. H is defined for $e^{2x} + e^x - 6 > 0$; $(e^x - 2)(e^x + 3) > 0$; $e^x - 2 > 0$; $x > \ln 2$. — **a**

b. $\lim\limits_{x\to+\infty}(H(x) - 2x) = \lim\limits_{x\to+\infty}\left[\ln(e^{2x} + e^x + 6) + \ln e^{-2x}\right] = \lim\limits_{x\to+\infty}\ln(1 + e^{-x} + 6e^{-2x}) = \ln 1 = 0$. — **b**

7- $e^{\ln(x-1)^2} = \ln e^{x-1}$ is equivalent to $(x-1)^2 = (x-1)$ with $x \neq 1$;

which gives: $x - 1 = 1$; $x = 2$. — **a**

8- $f'(x) = (x-3)e^x + e^x = (x-2)e^x$ and $f''(x) = (x-1)e^x$

$f''(x)$ vanishes while changing signs at $x = 1$; so $E(1 ; 0)$ is point of inflection. — **c**

9- The domain of definition of f o g is:

f is defined for $x > 0$ and $\ln x > -1$; which gives: $x > \dfrac{1}{e}$.

g is defined for every real number x.

f o g is defined for $x \in D_g$ and $g(x) \in D_f$; which gives: $e^{x-1} > \dfrac{1}{e}$, so: $x - 1 > -1$; $x > 0$. — **a**

CHAPTER 7
EXPONENTIAL FUNCTIONS

II- TRUE OR FALSE

A) Consider the function f such that $f(x) = (3 - x)e^{-x}$ and let (C) be its representative curve.

1- $f(x) + x - 3 = (3 - x)\,e^{-x} - (3 - x) = e^{-x}(x - 3)\,(e^{x} - 1) \geq 0$ for $x \leq 0$ or $x \geq 3$. (F)

2- $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\left(\frac{3}{e^x} - \frac{x}{e^x}\right) = 0$. The line with equation $y = 0$ is asymptote to C. (T)

3- $f'(x) = -e^{-x} - (3 - x)\,e^{-x} = (x - 4)e^{-x}$. (F)

4- $f'(x) = 0$ for $x = 4$; $f'(x) > 0$ for $x > 4$; $f'(x) < 0$ for $x < 4$.

The function f has a single extremum at the point $x = 4$. (T)

5- $\lim\limits_{x\to-\infty} f(x) = +\infty$ and $\lim\limits_{x\to-\infty} f'(x) = +\infty$.

It is enough to draw (C) for to confirm that, for $m > 0$, the equation $f(x) = m$ has

a single root. (F)

6- •If $x \geq 0$; $g(x) = g_1(x) = (3 - x)e^{-x}$

•If $x \leq 0$; $g(x) = g_2(x) = (3 - x)e^{x}$

$g_1(x) = g_2(x) \Leftrightarrow (3 - x)e^{-x} = (3 - x)e^{x} \Leftrightarrow (3 - x)(e^{-x} - e^{x}) = 0$

$x = 0$ is not a double root of the equation $g_1(x) = g_2(x)$. g is not differentiable at 0. (F)

7- $f'(x) + f(x) = \ldots = -e^{-x}$. (T)

8- The average (mean) value V_m of f between 0 and 1 is equal to $\frac{1}{1-0}\int_0^1 f(x)\,dx = \int_0^1 f(x)\,dx$

$V_m = \int_0^1 (-f'(x) - e^{-x})dx = \left[-f(x) + e^{-x}\right]_0^1 = \left[(x-2)e^{-x}\right]_0^1 = 2 - \frac{1}{e}$. (T)

B) $T(x) = 2^x + 3^x - x\ln 6$.

1- $T'(x) = 2^x\ln 2 + 3^x\ln 3 - \ln 6$; $T'(0) = \ln 2 + \ln 3 - \ln 6 = \ln 6 - \ln 6 = 0$. (T)

2- $\int_0^1 T(x)dx = \left[\frac{2^x}{\ln 2} + \frac{3^x}{\ln 3} - \frac{\ln 6}{2}x^2\right]_0^1 = \frac{2}{\ln 2} + \frac{3}{\ln 3} - \frac{\ln 6}{2} - \frac{1}{\ln 2} - \frac{1}{\ln 3} = \frac{1}{\ln 2} + \frac{2}{\ln 3} - \frac{\ln 6}{2}$

$= \frac{\ln 2 + \ln 6}{\ln 2 \times \ln 3} - \ln\sqrt{6}$. (F)

Exercise 7

(C) : $y = f(x) = (e^x - 1)\sqrt{e^x - x}$.

1- O belongs to C ; an equation of the tangent (T) is $y = xf'(0)$.

$f'(x) = (e^x)\sqrt{e^x - x} + \dfrac{(e^x - 1)^2}{2\sqrt{e^x - x}}$; $f'(0) = 1 + 0 = 1$. So (T) : $y = x$.

2- A, to the nearest unit of area, is: $A = \displaystyle\int_0^{-1} f(x)dx + \int_0^1 f(x)dx$

$\displaystyle\int f(x)\,dx = \int (e^x - 1)\sqrt{e^x - x}\,dx$, let $u(x) = e^x - x$; $u'(x) = e^x - 1$

To calculate the area: $\displaystyle\int f(x)\,dx = \int u'(x)\sqrt{u(x)}dx = \frac{2}{3}u(x)\sqrt{u(x)} = \frac{2}{3}(e^x - x)\sqrt{e^x - x}$.

Which gives: $A = \left[\dfrac{2}{3}(e^x - x)\sqrt{e^x - x}\right]_0^{-1} + \left[\dfrac{2}{3}(e^x - x)\sqrt{e^x - x}\right]_0^1$

$$= \ldots = \frac{2}{3}\left[\frac{1+e}{e}\sqrt{\frac{1+e}{e}} + (e-1)\sqrt{e-1} - 2\right]$$

So $A = \dfrac{2}{3}\left[\dfrac{1+e}{e}\sqrt{\dfrac{1+e}{e}} + (e-1)\sqrt{e-1} - 2\right]u^2$.

3- a. $T_k'(x) = e^{kx} + kx\,e^{kx} = e^{kx} + kT_k(x)$; which gives: $T_k(x) = \dfrac{1}{k}(T_k'(x) - e^{kx})$.

To calculate the area: $\displaystyle\int T_k(x)\,dx = \frac{1}{k}(T_k(x) - \frac{1}{k}e^{kx}) = \frac{1}{k}e^{kx}(x - \frac{1}{k})$

b. $V = \pi V' u^3$; with $V' = \displaystyle\int_{-1}^{1} f^2(x)dx = \int_{-1}^{1}(e^x - 1)^2(e^x - x)dx = \int_{-1}^{1} e^x(e^x - 1)^2dx - \int_{-1}^{1} x(e^x - 1)^2dx$

$A = \displaystyle\int_{-1}^{1} e^x(e^x - 1)^2dx$ and $B = \displaystyle\int_{-1}^{1} x(e^x - 1)^2dx$

To calculate A:

Let $t(x) = e^x - 1$; $t'(x) = e^x$; thus $A = \dfrac{1}{3}\left[(e^x - 1)^3\right]_{-1}^{1} = \dfrac{1}{3}\left[(e-1)^3 + \dfrac{(e-1)^3}{e^3}\right] = \dfrac{(e^3 + 1)(e-1)^3}{3e^3}$.

To calculate B:

$B = \displaystyle\int_{-1}^{1}(xe^{2x} - 2xe^x + x)dx = \int_{-1}^{1}(T_2(x) - 2T_1(x) + x)dx = \left[\frac{1}{2}e^{2x}(x - \frac{1}{2}) - 2e^x(x-1) + \frac{1}{2}x^2\right]_{-1}^{1}$

$= \ldots = \dfrac{e^4 - 16e + 3}{4e^4}$.

So $V = \pi\left[\dfrac{(e^3 + 1)(e-1)^3}{3e^3} + \dfrac{e^4 - 16e + 3}{4e^4}\right] u^3$.

CHAPTER 7
EXPONENTIAL FUNCTIONS

Exercise 8

1-• $e^{2x} - e^x + 1 = (e^x - \frac{1}{2})^2 + \frac{3}{4} > 0$ for every real number x ; so $D_f = \mathbb{R}$.

2-• $\lim\limits_{x\to-\infty} (f(x) - x) = 1 + \lim\limits_{x\to-\infty} \ln(e^{2x} - e^x + 1) = 1.$

$\lim\limits_{x\to+\infty} (f(x) - 3x) = 1 + \lim\limits_{x\to+\infty} \left[\ln(e^{2x} - e^x + 1) + \ln(e^{-2x})\right] = 1 + \lim\limits_{x\to+\infty} \ln(1 - e^{-x} + e^{-2x}) = 1$

Conclusions :

- The line (d) with equation $y = x + 1$ is asymptote to (C) at $(-\infty)$;
- $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (x + 1) = -\infty$;
- The line (d′) with equation $y = 3x + 1$ is asymptote to (C) at $(+\infty)$;
- $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} (3x + 1) = +\infty$.

3-• $f'(x) = 1 + \dfrac{2e^{2x} - e^x}{e^{2x} - e^x + 1} = \dfrac{3e^{2x} - 2e^x + 1}{e^{2x} - e^x + 1} = \dfrac{\left(\sqrt{3}e^x - \frac{1}{\sqrt{3}}\right)^2 + \frac{2}{3}}{\underbrace{e^{2x} - e^x + 1}_{>0}} > 0$;

so f is strictly increasing on $\mathbb{R}$.

4-• $f(0) = 1$.

$f \circ g(x) = x$; which gives: $f'(g(x)) \times g'(x) = 1$ and $f'(g(1)) \times g'(1) = 1$;

$f'(g(1)) \times g'(1) = 1$ with $g(1) = 0$. So, $f'(0) \times g'(1) = 1$; thus $g'(1) = \frac{1}{2}$.

Exercise 9

1-• $f(x) = x + 1 - \dfrac{4e^x}{1 + e^x} = x + 1 - \dfrac{4(1 + e^x) - 4}{1 + e^x} = x - 3 + \dfrac{4}{1 + e^x}$

Starting with the formula $f(2a - x) = 2b - f(x)$ where $a = 0$ and $b = -1$,we get: $f(-x) + f(x) = -2$.

So: $f(-x) = -x + 1 - \dfrac{4e^{-x}}{1 + e^{-x}} = -x + 1 - \dfrac{4}{1 + e^x}$ and thus $f(-x) + f(x) = 2 - 4 = -2$.

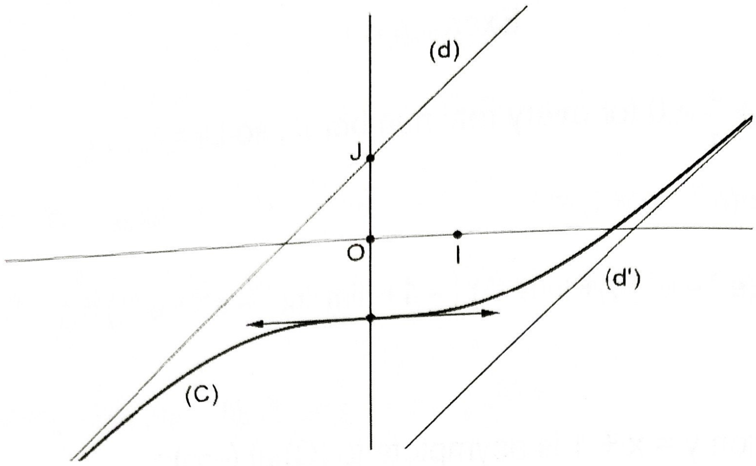
2-• $\lim\limits_{x\to-\infty} [f(x) - (x+1)] = \lim\limits_{x\to-\infty} \left[\dfrac{-4e^x}{1 + e^x}\right] = 0$ and $\lim\limits_{x\to+\infty} [f(x) - (x - 3)] = \lim\limits_{x\to+\infty} \left[\dfrac{4}{1 + e^x}\right] = 0$.

The two lines (d) and (d′) with equations $y = x + 1$ and $y = x - 3$ are the asymptotes to (C).

3-• $f'(x) = \dfrac{(e^x - 1)^2}{(1 + e^x)^2}$.

x	$-\infty$		0		$+\infty$
f '(x)		+	0	+	
f(x)	$-\infty$	↗	-1	↗	$+\infty$

4- Graph (C).



Exercise 10

1-• $f(x) = 1 - (2x + 1)\,e^{2x}$

a. $\lim\limits_{x\to-\infty} f(x) = 1$; $\lim\limits_{x\to+\infty} f(x) = -\infty$ and $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to+\infty}\left(\dfrac{1}{x} - (2+\dfrac{1}{x})e^{2x}\right) = +\infty$.

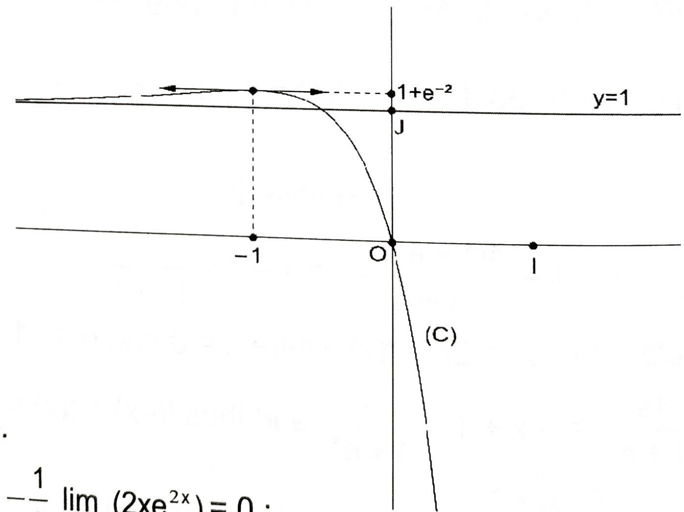
• The line with equation $y = 1$ is asymptote to (C) at $(-\infty)$.

• (C) has, at $(+\infty)$, a vertical asymptotic direction.

b. $f'(x) = -4(x + 1)\,e^{2x}$.

x	$-\infty$		-1		0		$+\infty$
f '(x)		+	0	—			
f(x)	1	↗	$1+e^{-2}$	↘	0	↘	$-\infty$

c. Curve (C).



2- $(\Gamma) : g(x) = x + 3 - xe^{2x}$.

a. $\lim\limits_{x\to-\infty}\left[g(x)-(x+3)\right] = -\dfrac{1}{2}\lim\limits_{x\to-\infty}(2xe^{2x}) = 0$;

so the line (d) with equation $y = x + 3$ is an asymptote to (Γ).

Let $d(x) = g(x) - (x + 3)$; $d(x) = -xe^{2x}$.

• If $d(x) > 0$ for $x < 0$; then: (Γ) is above (d).

• If $d(x) = 0$ for $x = 0$; then: (Γ) and (d) intersect at point A(0 ; 3).

• If $d(x) < 0$ for $x > 0$; then: (Γ) is below (d).

b. $\lim\limits_{x\to+\infty} \dfrac{g(x)}{x} = \lim\limits_{x\to+\infty} \left[1 + \dfrac{3}{x} - e^{2x}\right] = -\infty.$

Conclusion 1 : (Γ) has, at $(+\infty)$, a vertical asymptotic direction.

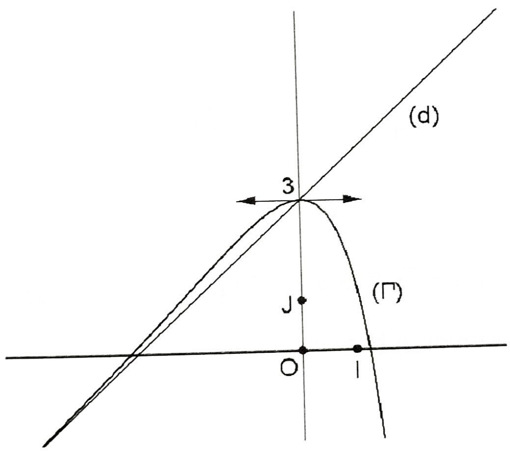
Conclusion 2 : $\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty} \left(\dfrac{g(x)}{x}\right)(x) =$ "$(-\infty)(+\infty)$" $= -\infty$.

c. $g'(x) = 1 - e^{2x} - 2xe^{2x} = 1 - (1+2x)e^{2x} = f(x)$.

Which gives the following table of variations of g:

x	$-\infty$		0		$+\infty$
g '(x)		+	0	—	
g(x)	$-\infty$	↗	3	↘	$-\infty$

d. Graph (Γ).



Exercise 11

1-• $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} (e^{-2x} - 4e^{-x} + 3) = 3$, which means: the line (d) with equation $y = 3$ is asymptote to (C) at $(+\infty)$.

• $f(x) = 3$ for $e^{-2x} - 4e^{-x} = 0$; $e^{-x}(e^{-x} - 4) = 0$; $e^{-x} = 4$; $x = -2\ln 2$.

Thus (d) cuts (C) at point $H(-2\ln 2 ; 3)$.

2-• $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} (e^{-2x} - 4e^{-x} + 3) = 3 + \lim\limits_{x\to-\infty} e^{-x}(e^{-x} - 4) = +\infty$.

• $f'(x) = -2e^{-2x} + 4e^{-x} = 2e^{-2x}(2e^{x} - 1)$

• Table of variations of f :

x	$-\infty$		$-\ln 2$		$+\infty$
f '(x)		—	0	+	
f(x)	$+\infty$	↘	-1	↗	3

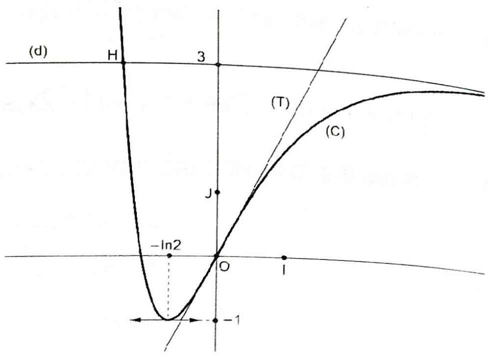
3- • $f''(x) = 4e^{-2x} - 4e^{-x} = 4e^{-2x}(1 - e^{x})$

So the origin O is the point of inflection of (C).

x	$-\infty$		0		$+\infty$
$f''(x)$		$+$	0	$-$	
(C)			0		

• (T) has equation $y = xf'(0)$ with $f'(0) = 2$; $y = 2x$.

4- Graph (C) :



$\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to-\infty} f'(x) = \lim\limits_{x\to-\infty} 2e^{-2x}(2e^{x} - 1) = +\infty$.

(C) has , at $(-\infty)$, a vertical asymptotic direction.

5- $f(x) = 0$ for $e^{-2x} - 4e^{-x} + 3 = 0$;

$(e^{-x} - 3)(e^{-x} - 1) = 0$; $e^{-x} = 3$ or $e^{-x} = 1$;

which gives: $x = -\ln 3$ or $x = 0$.

$A = A'\,u^2$ with $u^2 = OI \times OJ = 4\text{cm}^2$.

$$A' = \int_0^{-\ln 3} (e^{-2x} - 4e^{-x} + 3)\,dx = \left[-\frac{1}{2}e^{-2x} + 4e^{-x} + 3x\right]_0^{-\ln 3} = \ldots = 4 - 3\ln 3.$$

So $A = 4(4 - 3\ln 3)$ cm².

6-• φ is continuous and strictly monotone (↗), it is a bijection from $[-\ln 2\,;+\infty[$ onto $[-1\,;3[$, so it has an inverse bijection from $[-1\,;3[$ onto $[-\ln 2\,;+\infty[$.

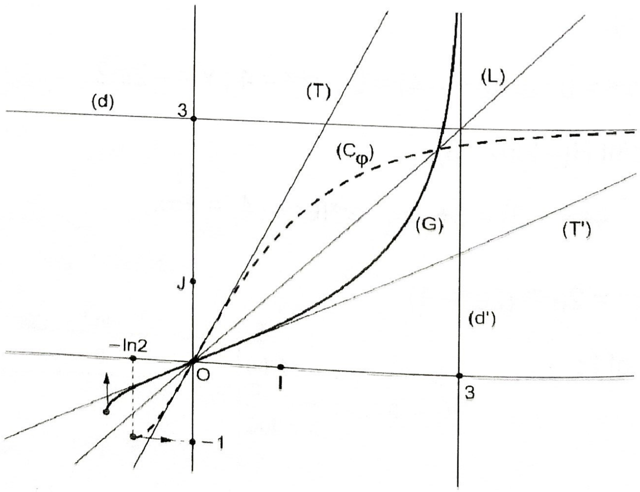
x	$-\ln 2$		$+\infty$
$\varphi'(x)$	0	$+$	
$\varphi(x)$	-1	↗	3

• Table of variations of g (adjacent figure):

x	-1		3
$g'(x)$	$+\infty$	$+$	
$g(x)$	$-\ln 2$	↗	$+\infty$

• Graph (G) :

(G) and ($C\varphi$) are symmetric with respect to the line (L) with equation $y = x$.



• Calculating an expression for g :

$z = g(x) \leftrightarrows x = \varphi(z) \leftrightarrows x = e^{-2z} - 4e^{-z} + 3$ with $-1 \leq x < 3$ and $z \geq -\ln 2$

Which gives: $x = (e^{-z} - 2)^2 - 1$; $(e^{-z} - 2)^2 = x + 1$; with $x + 1 \geq 0$ and $e^{-z} - 2 \leq 0$

Thus: $e^{-z} - 2 = -\sqrt{x+1}$; $e^{-z} = 2 - \sqrt{x+1}$; $z = -\ln(2 - \sqrt{x+1})$.

So $g(x) = -\ln(2 - \sqrt{x+1})$ with $D_g = [-1 ; 3[$.

Exercise 12

1- a. $\lim\limits_{x\to-\infty} h(x) = \lim\limits_{x\to-\infty} (2e^x - xe^x - 2) = -2$ et $\lim\limits_{x\to+\infty} \dfrac{h(x)}{x} = \lim\limits_{x\to+\infty} \left[(2-x)\dfrac{e^x}{x} - \dfrac{2}{x}\right] = -\infty$.

Conclusions:

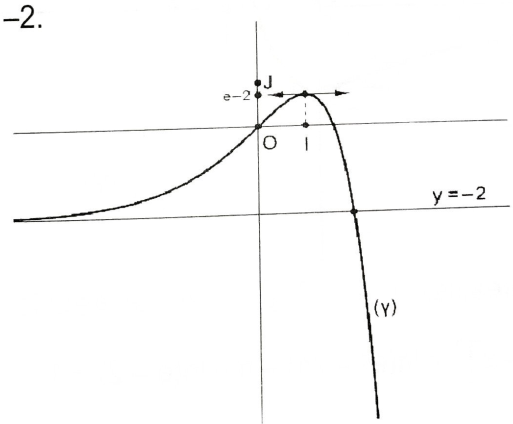
- (C) has, at $(-\infty)$, the line with equation $y = -2$ as horizontal asymptote.
- (C) has, at $(+\infty)$, a vertical asymptotic direction.
- $\lim\limits_{x\to+\infty} h(x) = \lim\limits_{x\to+\infty} \left(\dfrac{h(x)}{x}\right)(x) = " (-\infty)(+\infty) " = -\infty$.

b. $h'(x) = -e^x + (2 - x)e^x = (1 - x)e^x$

which gives the adjacent table of variations :

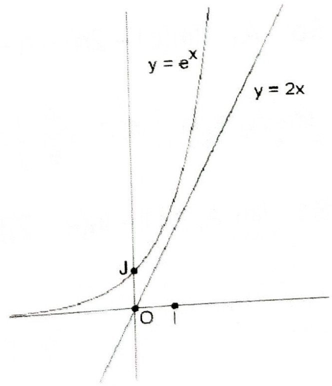
x	$-\infty$		1		$+\infty$
h'(x)		+	0	−	
h(x)	-2	↗	$e-2$	↘	$-\infty$

c. $h(0) = 0$ and $h(2) = -2$.



d. $h(1.5) \approx 0.24 > 0$ and $h(1.6) = -0.01 < 0$; so $1.5 < \alpha < 1.6$.

2- $f(x) = \dfrac{e^x - 2}{e^x - 2x}$.



Remark: From the adjacent figure, we deduce that: $e^x - 2x \neq 0$.

f is defined on $\mathbb{R}$.

a. $\lim\limits_{x\to-\infty} f(x) = 0$; so the line with equation $y = 0$ is asymptote to (C).

$$\lim_{x\to+\infty}[f(x)-1]=\lim_{x\to+\infty}\frac{2x-2}{e^x-2x}=\lim_{x\to+\infty}\frac{x(2-\frac{2}{x})}{x(\frac{e^x}{x}-2)}=0.$$

So, the line with equation y = 1 is asymptote to (C).

b. $f'(x)=\dfrac{e^x(e^x-2x)-(e^x-2)^2}{(e^x-2x)^2}=\ldots=\dfrac{2h(x)}{(e^x-2x)^2}$;

which gives the adjacent table of variations of f.

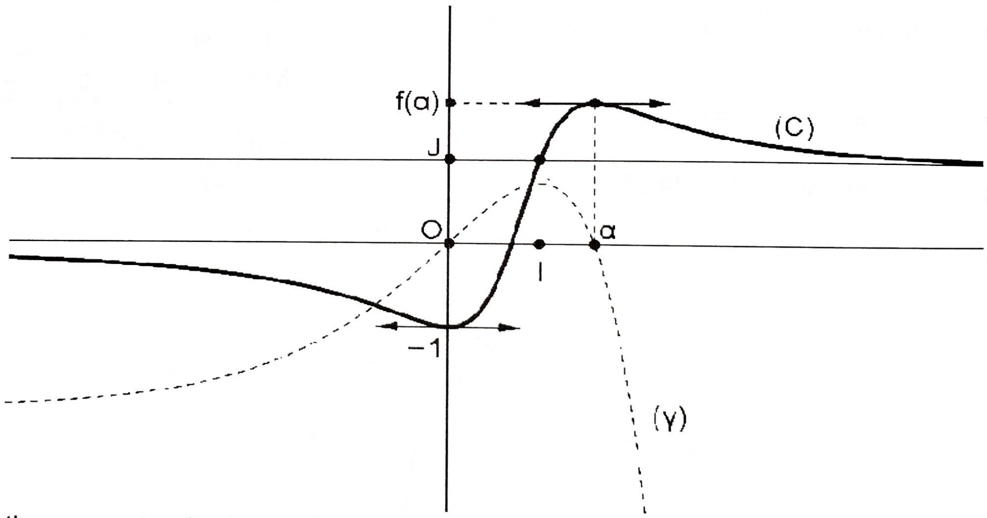
x	$-\infty$		0		α		$+\infty$
f '(x)		−	0	+	0	−	
f(x)	0	↘	−1	↗	f(α)	↘	1

c. $h(\alpha)=0$ which gives: $(2-\alpha)e^\alpha=2$;

$$f(\alpha)=\frac{(2-\alpha)e^\alpha-2(2-\alpha)}{(2-\alpha)e^\alpha-2(2-\alpha)\alpha}=\frac{2-2(2-\alpha)}{2-2(2-\alpha)\alpha}=\ldots=\frac{1}{\alpha-1}$$

For $\alpha\approx1.55$ we get $f(\alpha)\approx1.8$.

d. $f(x)=1$ for $e^x-2=e^x-2x$; so $x=1$ and thus (C) cuts the line (d) at point E(1 ; 1).



e. Area, to the nearest unit of area, is:

$$A_n=\int_1^n\left(\frac{e^x-2}{e^x-2x}-1\right)dx=\left[\ln(e^x-2x)-x\right]_1^n=\ln(e^n-2n)-n-\ln(e-2)+1.$$

So: $A_n=[\ln(e^n-2n)-n-\ln(e-2)+1]\,u^2$.

$$\lim_{n\to+\infty}A_n=\lim_{n\to+\infty}\left[\ln\left(1-\frac{2n}{e^n}\right)+1-\ln(e-2)\right]=1-\ln(e-2).$$

So: $\lim_{n\to+\infty}A_n=[1-\ln(e-2)]\,u^2$.

CHAPTER 7
EXPONENTIAL FUNCTIONS

Exercise 13

Part A

1-• $\lim\limits_{x\to-\infty} f(x) = 0$

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} e^{x}\left(e^{-0,5x}-1\right) = \text{"}(+\infty)(-1)\text{"} = -\infty$

$\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to+\infty} \dfrac{e^{0.5x}}{x}\left(1-e^{0.5x}\right) = \lim\limits_{x\to+\infty}(0,5)\dfrac{e^{0.5x}}{0.5x}\left(1-e^{0.5x}\right) = \text{"}(+\infty)(-\infty)\text{"} = -\infty$

Interpretations:

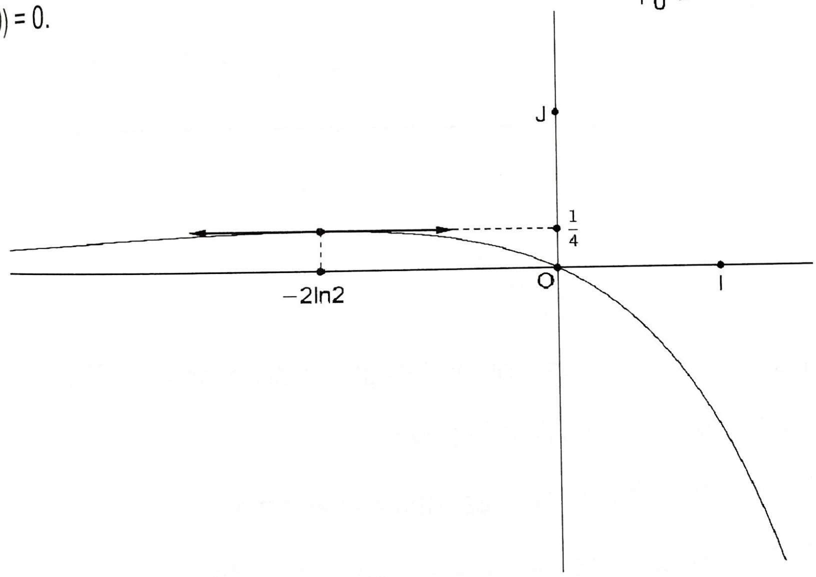
• (C) has, at $(-\infty)$, an asymptote with equation $y = 0$.

• (C) has, at $(+\infty)$, a vertical asymptotic direction.

2-$f'(x) = \dfrac{1}{2}e^{0,5x} - e^{x} = e^{0,5x}\left(\dfrac{1}{2} - e^{0,5x}\right)$.

x	$-\infty$		$-2\ln 2$		$+\infty$
$f'(x)$		$+$	0	$-$	
$f(x)$	0	$\nearrow$	$\dfrac{1}{4}$	$\searrow$	$-\infty$

3-• $f(0) = 0$.



4-A, to the nearest unit of area, is: $A = -\displaystyle\int_0^1 \left(e^{0.5x} - e^{x}\right)dx = \left[-0.5e^{0.5x} + e^{x}\right]_0^1 = e - \dfrac{1}{2}(\sqrt{e} + 1)$.

So $A = e - \dfrac{1}{2}(\sqrt{e} + 1)\ u^2$.

Part B

1-• g is defined for $e^{x} - e^{0,5x} > 0$; which gives: $e^{0,5x}(e^{0,5x} - 1) > 0$; which leads to $0,5x > 0$; $x > 0$.

So $D_g =]0\,;+\infty[$.

2-• $g(x) - x = \ln(e^{x} - e^{0.5x}) - \ln(e^{x}) = \ln[e^{-x}(e^{x} - e^{0.5x})] = \ln(1 - e^{-0.5x})$.

$\lim\limits_{x\to+\infty}[g(x) - x] = \lim\limits_{x\to+\infty} \ln(1-e^{-0,5x}) = \ln 1 = 0$; so the line (d) is asymptote to (C).

3-• $g(x) - x = \ln(1 - e^{-0.5x}) < 0$ car $1 - e^{-0.5x} < 1$ for every real number x. So (C') is below (d).

4- $\lim\limits_{\substack{x\to 0\\ x>0}} g(x) = -\infty$; so the line with equation $x = 0$ is a vertical asymptote to (C').

$\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty} (x) = +\infty$.

$g'(x) = \dfrac{e^x - 0.5e^{0.5x}}{e^x - e^{0.5x}} = \dfrac{e^{0.x} - 0.5}{e^{0.5x} - 1} > 0$, since $x > 0$.

Which gives adjacent table of variations of g :

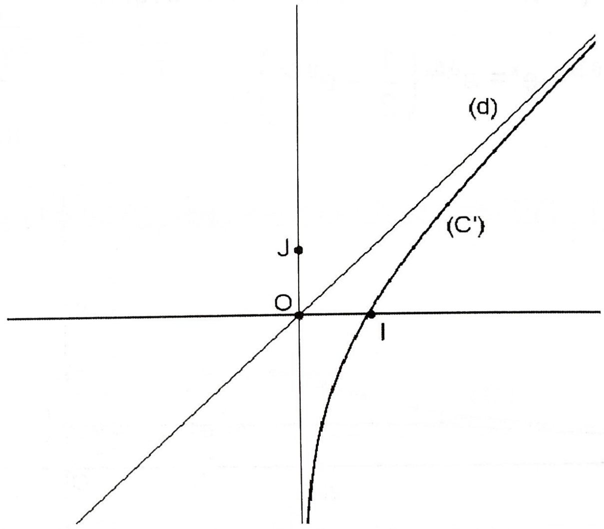
x	0		$+\infty$
g'(x)		+	
g(x)		$-\infty$ ↗	$+\infty$

5- $g(x) = 0$ for $e^x - e^{0.5x} = 1$; let $t = e^{0.5x} > 0$;

which gives : $t^2 - t - 1 = 0$; $4t^2 - 4t - 4 = 0$; $(2t - 1)^2 - 5 = 0$;

consequently $2t - 1 = \sqrt{5}$; $t = \dfrac{1+\sqrt{5}}{2}$.

$e^{0,5x} = \dfrac{1+\sqrt{5}}{2}$; so $x = 2\ln\dfrac{1+\sqrt{5}}{2} \approx 0.97$.



6- On $]0 ; +\infty[$, g is continuous and strictly monotone (↗) ,it is a bijection from $]0 ; +\infty[$ on $\mathbb{R}$, it has an inverse bijection ,denoted by h, from $\mathbb{R}$ to $]0 ; +\infty[$.

$z = h(x) \leftrightarrows x = g(z) = \ln(e^z - e^{0,5z})$, with x : any real number and $z > 0$.

$e^z - e^{0.5z} = e^x$; let $u = e^{0.5z} > 1$; which gives: $u^2 - u = e^x$; $4u^2 - 4u = 4e^x$;

Thus: $(2u - 1)^2 = 4e^x + 1$; $2u - 1 = \sqrt{4e^x + 1}$; $u = \dfrac{1+\sqrt{4e^x+1}}{2}$;

consequently: $\ln u = \ln\dfrac{1+\sqrt{4e^x+1}}{2}$; $0.5z = \ln\dfrac{1+\sqrt{4e^x+1}}{2}$; $z = 2\ln\dfrac{1+\sqrt{4e^x+1}}{2}$.

So: $h(x) = \ln\dfrac{1+\sqrt{4e^x+1}}{2}$; with $D_h = \mathbb{R}$.

Exercise 14

1- a. $f(-x) + f(x) = \dfrac{3e^{-x} - 1}{e^{-x} + 1} + \dfrac{3e^x - 1}{e^x + 1} = \dfrac{3 - e^x}{e^x + 1} + \dfrac{3e^x - 1}{e^x + 1} = \dfrac{2(e^x + 1)}{e^x + 1} = 2$.

Since $D_f = \mathbb{R}$ is centered at zero ; so the point A(0 ; 1) is a center of symmetry of (C).

Remark:

A(0 ; 1) is a center of symmetry of (C) and since $A \in (C)$, then it is point of inflection de (C).

b • $\lim\limits_{x\to-\infty} f(x) = -1$; so the line (d_1) with equation $y = -1$ is an asymptote to (C) at $(-\infty)$.

• $f(x) = \dfrac{3(e^x+1)-4}{e^x+1} = 3 - \dfrac{4}{e^x+1}$.

$\lim\limits_{x\to+\infty} f(x) = 3$; so the line (d_1) with equation $y = 3$ is an asymptote to (C) at $(+\infty)$.

c. $f'(x) = \dfrac{4e^x}{(e^x+1)^2} > 0$; which gives the adjacent table of variations of f :

x	$-\infty$		$+\infty$
$f'(x)$		+	
$f(x)$	-1	↗	3

2-a. An equation of (T) is: $y = (x-0)f'(0) + f(0)$; $y = x + 1$.

b. $\delta'(x) = \dfrac{4e^x}{(e^x+1)^2} - 1 = \dfrac{4e^x - e^{2x} - 2e^x - 1}{(e^x+1)^2} = -\dfrac{(e^x-1)^2}{(e^x+1)^2}$.

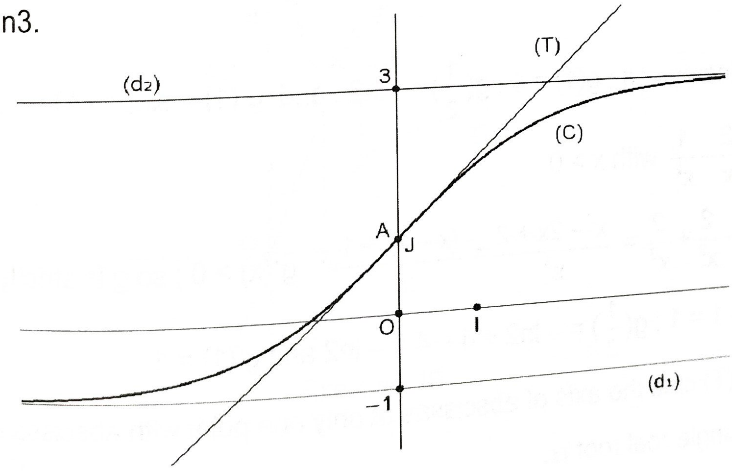
x	$-\infty$		0		$+\infty$
$\delta'(x)$			—		
$\delta(x)$		+ ↘	0	− ↘	

c. For $x < 0$; $\delta(x) > 0$; so (C) is above (T) ;

for $x = 0$; $\delta(x) = 0$; so (C) cuts (T) at the point A ;

for $x > 0$; $\delta(x) < 0$; so (C) is below (T) .

3- $f(x) = 0$ for $x = -\ln 3$.



4- A, to the nearest unit of area, is:

$A = \int_0^1 f(x)dx = \int_0^1 (3 - \dfrac{4}{1+e^x})dx = \int_0^1 (3 - \dfrac{4e^{-x}}{1+e^{-x}})dx = \left[3x + 4\ln(1+e^{-x})\right]_0^1$

$= 3 + 4\ln(\dfrac{1+e}{2e}) = -1 + 4\ln(\dfrac{1+e}{2})$.

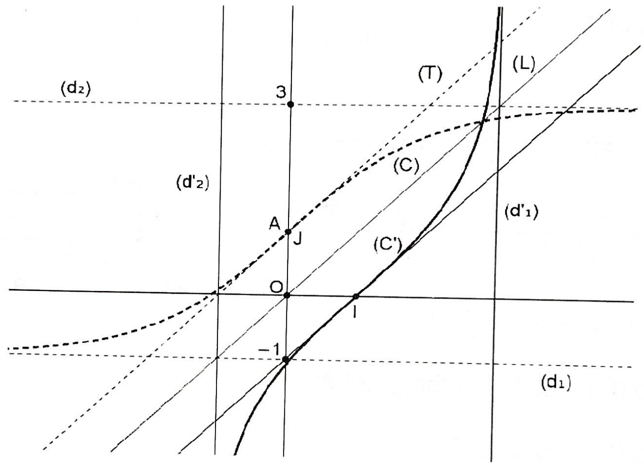
So: $A = -1 + 4\ln(\frac{1+e}{2}).u^2$.

5- •$z = g(x) \Leftrightarrow x = f(z) = \frac{3e^z - 1}{e^z + 1} \Leftrightarrow xe^z + x = 3e^z - 1 \Leftrightarrow (3 - x)e^z = x + 1$

$$\Leftrightarrow e^z = \frac{x+1}{3-x} > 0 \text{ and since } -1 < x < 3 \text{ ; we get: } z = \ln\left(\frac{x+1}{3-x}\right).$$

So $g(x) = \ln\left(\frac{x+1}{3-x}\right)$ with $D_g =]-1 ; 3[$.

• (C') is the symmetric of (C) with respect to the line with equation $y = x$.



Exercise 15

Part A

1-•g is strictly increasing ; $g(1) = 1$, $g(\frac{1}{2}) = -\ln 2$, and $g'(1) = \text{slope } (\Delta) = 1$.

2-•$g(x) = \ln x + \frac{2}{x} - \frac{1}{x^2}$ with $x > 0$.

a.•$g'(x) = \frac{1}{x} - \frac{2}{x^2} + \frac{2}{x^3} = \frac{x^2 - 2x + 2}{x^3} = \frac{(x-1)^2 + 1}{x^3}$; $g'(x) > 0$; so g is strictly increasing.

$g(1) = \ln 1 + 2 - 1 = 1$; $g(\frac{1}{2}) = -\ln 2 + 4 - 4 = -\ln 2$ and $g'(1) = 1$.

b. The curve (T) cuts the axis of abscissas at only one point with abscissa α ; so the equation $g(x) = 0$ has a single real root α.

$g(0.5) = -\ln 2 < 0$ and $g(0.6) = 0.045 > 0$; so $0.5 < \alpha < 0.6$.

For $0 < x < \alpha$: $g(x) < 0$;

For $x = \alpha$: $g(x) = 0$;

for $x > \alpha$; $g(x) > 0$.

Part B

$f(x) = e^x\left(\ln x + \dfrac{1}{x}\right)$ with $x > 0$.

1-a. $\lim\limits_{x\to+\infty} f(x) =$ " $+\infty(+\infty + 0)$ " $= +\infty$

b. $f(x) = e^x\left(\ln x + \dfrac{1}{x}\right) = e^x\left(\dfrac{x\ln x + 1}{x}\right) = \dfrac{e^x}{x}(x\ln x + 1)$; $\lim\limits_{\substack{x\to 0\\ x>0}} f(x) =$ " $+\infty(0+1)$ " $= +\infty$.

2- a. $f'(x) = e^x(\ln x + \dfrac{1}{x}) + e^x\left(\dfrac{1}{x} - \dfrac{1}{x^2}\right) = e^x\left(\ln x + \dfrac{2}{x} - \dfrac{1}{x^2}\right) = e^x\,(g(x))$.

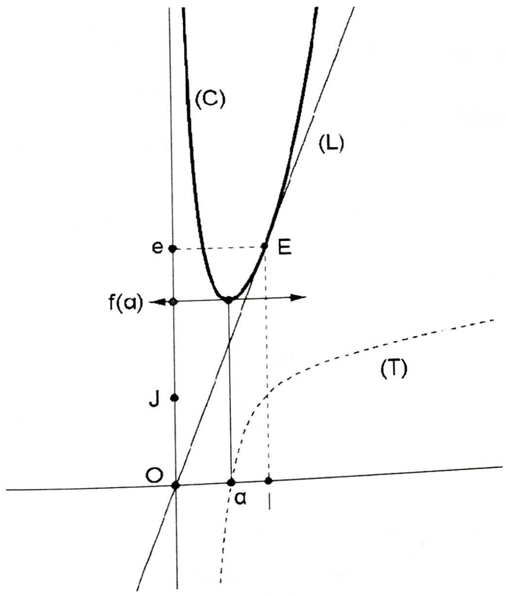
b. $f'(x)$ has the same sign as $g(x)$.

x	0		α		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

3-a. An equation of (L) is :

$y - f(1) = (x - 1)\,f'(1)$; so $y = ex$.

b. $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x} = \lim\limits_{x\to+\infty} \dfrac{e^x}{x}(\ln x + \dfrac{1}{x}) =$ " $+\infty(+\infty + 0)$ " $= +\infty$; the curve (C) has vertical asymptotic direction at $(+\infty)$.



Exercise 16

1-• $f(x) = \ln(e^x + e^{-x})$.

- $D_f = \mathbb{R}$ is centered at zero ;
- $f(-x) = \ln(e^{-x} + e^x) = f(x)$

So f is even.

2- We have : $f(x) - y_{(d)} = f(x) - x = \ln(e^x + e^{-x}) + \ln(e^{-x}) = \ln(1 + e^{-2x}) > 0$ since: $1 + e^{-2x} > 1$.

So (C) is above (d).

$f(x) - y_{(d')} = f(x) + x$; let $x = -t$

$f(x) - y_{(d')} = f(-t) - t = f(t) - t > 0$; so (C) is above (d′).

3-• $\lim\limits_{x\to+\infty}[f(x)-x] = \lim\limits_{x\to+\infty}\left[\ln(1+e^{-2x})\right] = \ln(1+0) = 0.$

Conclusions :

- (C) has, at $(+\infty)$, the line (d) with equation $y = x$ as oblique asymptote.
- $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}(x) = +\infty$
- $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{t\to+\infty} f(-t)$ où $x = -t$; so $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{t\to+\infty} f(t) = +\infty$
- $\lim\limits_{x\to+\infty}[f(x)-x] = \lim\limits_{t\to-\infty}[f(-t)+t] = \lim\limits_{t\to-\infty}[f(t)+t]$; which gives: $\lim\limits_{t\to-\infty}[f(t)+t] = 0.$

(C) has, at $(-\infty)$, the line (d′) with equation $y = -x$ as oblique asymptote.

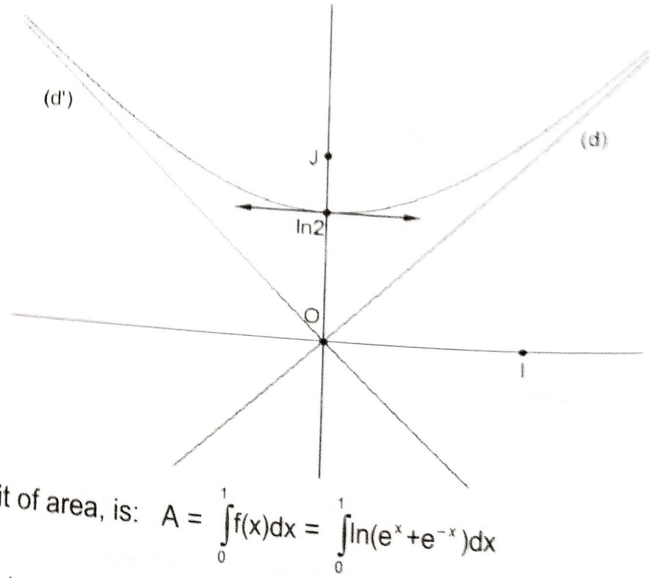
4- $\varphi'(x) = e^x - e^{-x} = e^{-x}(e^{2x}-1) = e^{-x}(e^x+1)(e^x-1)$

$f = \ln \circ \varphi$ and since ln is strictly increasing ;

so f and φ have the same sense of variations.

x	$-\infty$		0		$+\infty$
$\varphi'(x)$		$-$	0	$+$	
$\varphi(x)$	$+\infty$	$\searrow$	2	$\nearrow$	$+\infty$
$f(x)$	$+\infty$	$\searrow$	$\ln 2$	$\nearrow$	$+\infty$

5- Graph (C).



6- A, to the nearest unit of area, is: $A = \int_0^1 f(x)dx = \int_0^1 \ln(e^x+e^{-x})dx$

We have : $0 \le x \le 1$, which gives: $e^0 \le e^x \le e^1$ since $\exp_e$ function is strictly increasing ;

$1 \leq e^x \leq e$; $-1 \leq -x \leq 0$; $e^{-1} \leq e^{-x} \leq 1$.

Which gives: $1 + \frac{1}{e} \leq e^x + e^{-x} \leq e + 1$; $\ln(e + 1) - 1 \leq f(x) \leq \ln(e+1)$ since ln is strictly increasing.

So $\ln(e + 1) - 1 \leq A \leq \ln(e + 1)$; $-1 \leq A - \ln(e + 1) \leq 0$; $-1 + \frac{1}{2} \leq A + \frac{1}{2} - \ln(e + 1) \leq \frac{1}{2}$

$-\frac{1}{2} \leq A + \frac{1}{2} - \ln(e + 1) \leq \frac{1}{2}$; consequently $\left|A + \frac{1}{2} - \ln(1 + e)\right| \leq \frac{1}{2}$.

Exercise 17

Part A

1-• $\lim\limits_{x \to +\infty} h(x) = \lim\limits_{x \to +\infty} x\left(1 + \frac{2}{x} - \frac{e^x}{x}\right) = "+\infty(1 + 0 - \infty)" = -\infty$.

$\lim\limits_{x \to -\infty} h(x) = "-\infty + 2 - 0" = -\infty$.

2-• $h'(x) = 1 - e^x$

x	$-\infty$		α		0		β		$+\infty$
$h'(x)$			$+$		0		$-$		
$h(x)$	$-\infty$	$-$	0 ↗	$+$	1	$+$	↘ 0	$-$	$-\infty$

3- a. On $]-\infty ; 0]$, h is continuous and changes signs, which means that the the equation $h(x) = 0$ has at least one real root α. h is strictly increasing (↗) ; α is unique.

$h(-1,85) = -7,23 \times 10^{-3} < 0$ and $h(-1,84) = 1,18 \times 10^{-3} > 0$; so $-1,85 < \alpha < -1,84$.

On $[0 ; +\infty[$, h is continuous and changes signs, which means that the equation $h(x) = 0$ has at least one real root β. Since h is strictly decreasing , then β is unique.

$h(1.14) = 0.013 > 0$ and $h(1.15) = -8.19 \times 10^{-3} < 0$; so $1.14 < \beta < 1.15$.

b. $h(x) = 0$ for $x = \alpha$ or $x = \beta$.

$h(x) > 0$ for $\alpha < x < \beta$.

$h(x) < 0$ for $x < \alpha$ or $x > \beta$.

4- Le maximum de h is 1 for every real number x ; which gives: $h(-x) \leq 1$,

Thus: $-x + 2 - e^{-x} \leq 1$; to get $-x + 1 - e^{-x} \leq 0$;

Multiplying this last inequality by the positive number e^x ; we get : $e^x - xe^x - 1 \leq 0$.

On the other hand, $e^x \leq xe^x + 1$; so $xe^x + 1 > 0$ for every real number x.

Part B

1- $\lim\limits_{x \to -\infty} f(x) = \dfrac{0-1}{0+1} = -1$ and $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{1-e^{-x}}{x+e^{-x}} = " \dfrac{1-0}{+\infty + 0} " = 0.$

2-• $f(\alpha) = \dfrac{e^{\alpha}-1}{\alpha e^{\alpha}+1}$ with $h(\alpha) = 0$; which gives: $\alpha + 2 = e^{\alpha}$; so $f(\alpha) = \ldots = \dfrac{\alpha+1}{(\alpha+1)^2} = \dfrac{1}{\alpha+1}$.

$-1.85 < \alpha < -1.84$; $-0.85 < \alpha + 1 < -0.84$; $-1.19 < f(\alpha) < -1.17$.

3-• $f'(x) = \dfrac{e^x(xe^x+1)-(xe^x+e^x)(e^x-1)}{(xe^x+1)^2} = \ldots = \dfrac{e^x h(x)}{(xe^x+1)^2}$.

x	$-\infty$		α		β		$+\infty$
f '(x)		–	0	+	0	–	
f(x)	-1	↘	f(α)	↗	f(β)	↘	0

f '(x) has the same sign as h(x).

4-• $f(x) - x = \dfrac{e^x-1}{xe^x+1} - x = \dfrac{e^x - 1 - x^2e^x - x}{xe^x+1} = \dfrac{e^x(1-x^2)-(1+x)}{xe^x+1}$

$$= \frac{e^x(1-x)(1+x)-(1+x)}{xe^x+1} = \frac{(x+1)(e^x - xe^x - 1)}{xe^x+1}.$$

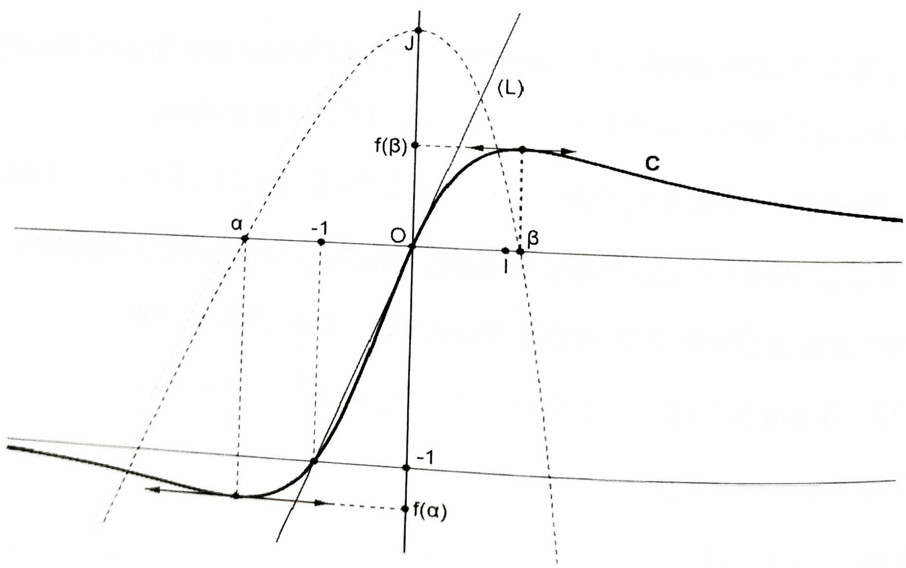
$e^x - xe^x - 1 = 0$ for $x = 0$ and $e^x - xe^x - 1 < 0$ for $x \neq 0$.

$f(x) = x$ for $x = -1$ or $x = 0$; (L) cuts C at deux points $E(-1 ; -1)$ and the origin O.

For $x < -1$; $f(x) < x$; C is above (L).

For $x > -1$ (with $x \neq 0$) ; $f(x) > x$; C is below (L).

5-Graph C :



6- A, to the nearest unit of area, is: $A(\alpha) = \int_0^\alpha f(x)dx$;

$A(\alpha) = \int_0^\alpha \frac{1-e^{-x}}{x+e^{-x}}dx = \left[\ln\left|x+e^{-x}\right|\right]_0^\alpha = \ln|\alpha + e^{-\alpha}|$.

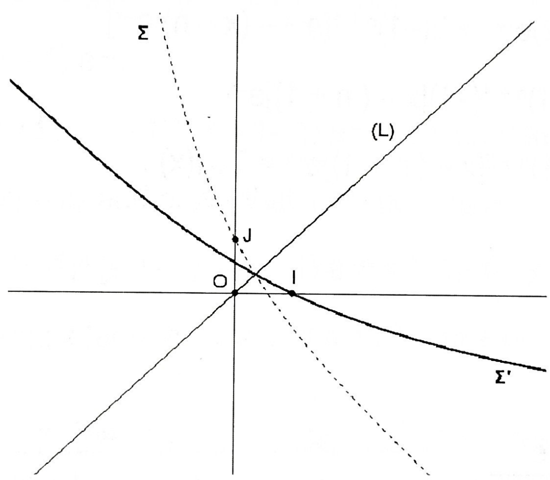
So $A(\alpha) = 10\ \ln|\alpha + e^{-\alpha}|\ cm^2$.

Remark: $e^\alpha = \alpha + 2$; $e^{-\alpha} = \frac{1}{\alpha+2}$; $\alpha + e^{-\alpha} = \frac{(\alpha+1)^2}{\alpha+2}$.

Exercise 18

1-a. f is continuous and strictly decreasing, it has an inverse function h.

Σ and Σ' are symmetric with respect to the line with equation $y = x$.



b. $f \circ h(x) = x$ which gives: $f'(h(x)) \times h'(x) = 1$; Thus: $f'(h(1)) \times h'(1) = 0$;

$f'(0) \times h'(1) = 1$; so $h'(1) = -\frac{1}{2}$.

2- It is easy to prove that: $0.5 < \alpha < 0.7$.

$f(x) = 0$ for $x = \alpha$; $f(x) > 0$ for $x < \alpha$ and $f(x) < 0$ for $x > \alpha$ since f is decreasing .

3-• $A(\alpha) = \int_0^\alpha f(x)dx = \left[-e^{-x} - \frac{x^2}{2}\right]_0^\alpha$; $A(\alpha) = \left(-e^{-\alpha} - \frac{\alpha^2}{2} + 1\right)\ u^2$

But $e^{-\alpha} = \alpha$, which gives: $A(\alpha) = \left(-\alpha - \frac{\alpha^2}{2} + 1\right)\ u^2$.

4-• $V = \pi V'\ u^3$.

$V' = \int_{-1}^0 (f^2(x) - x^2)dx = \int_{-1}^0 (e^{-2x} - 2xe^{-x})\ dx = \left[-\frac{1}{2}e^{-2x} + 2xe^{-x} + 2e^{-x}\right]_{-1}^0 = \frac{3+e^2}{2}$.

So $V = \frac{\pi(3+e^2)}{2}\ u^3$.

5- • $U(x) = f^2(x) - x^2 = e^{-2x} - 2xe^{-x}$

Let $T_n(x) = (-2)^n e^{-2x} + 2(-1)^{n+1}(x - n)e^{-x}$

Step 1:

$U'(x) = -2e^{-2x} - 2e^{-x} + 2xe^{-x}$.

$T_1(x) = (-2)^1 e^{-2x} + 2[(-1)^{1+1}](x-1)e^{-x} = -2e^{-2x} + 2(x-1)e^{-x}$

So $U'(x) = T_1(x)$.

Step 2:

Suppose that $U^{(n)}(x) = T_n(x)$, and prove that $U^{(n+1)}(x) = T_{n+1}(x)$.

$U^{(n+1)}(x) = [u^{(n)}]'(x) = (-2)^n(-2)e^{-2x} + 2(-1)^{n+1}[e^{-x} - (x-n)e^{-x}]$

$U^{(n+1)}(x) = (-2)^{n+1} e^{-2x} + 2(-1)^{n+1}(-1)[x - (n+1)]e^{-x}$

$= (-2)^{n+1} e^{-2x} + 2(-1)^{n+2}[x - (n+1)]e^{-x} = T_{n+1}(x)$.

B) 1-For $x < \alpha$; $g(x) > 0$

For $x = \alpha$; $g(x) = 0$

For $x > \alpha$; $g(x) < 0$

2- • $g(x) = \dfrac{x}{f(x)} = \dfrac{x}{e^{-x} - x} = \dfrac{xe^x}{1 - xe^x}$.

$\lim\limits_{x\to-\infty} g(x) = \text{"}\dfrac{-\infty}{+\infty}\text{"}$ Indeterminate Form

Applying L'Hôpital's rule: $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to-\infty} \dfrac{1}{f'(x)} = \lim\limits_{x\to-\infty} \dfrac{1}{-e^{-x}-1} = 0$.

$\lim\limits_{x\to+\infty} g(x) = \text{"}\dfrac{+\infty}{-\infty}\text{"}$ I.F.

Applying L'Hôpital's rule ; $\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty} \dfrac{1}{f'(x)} = \lim\limits_{x\to+\infty} \dfrac{1}{-e^{-x}-1} = -1$.

$\lim\limits_{\substack{x\to\alpha \\ x<\alpha}} g(x) = \text{"}\dfrac{\alpha}{0^+}\text{"} = +\infty$. $\lim\limits_{\substack{x\to\alpha \\ x>\alpha}} g(x) = \text{"}\dfrac{\alpha}{0^-}\text{"} = -\infty$

$g'(x) = \dfrac{f(x) - xf'(x)}{f^2(x)} = \dots = \dfrac{(x+1)e^{-x}}{f^2(x)}$.

x	0		-1		α		+∞
g'(x)		−	0	+	‖	+	
g(x)	0	↘	$\frac{-1}{1+e}$	↗	+∞ ‖ −∞	↗	-1

D) 1- • $V(x) = f \circ g(x)$; $V'(x) = f'(g(x)).g'(x)$

$V'(0) = f'(g(0)).g'(0) = f'(0).g'(0) = (-2)(1) = -2$.

2- $W(x) = \int_{-x^2}^{0} g(t)\,dt$.

$W'(x) = (2x)g(-x^2)$ with $x \le 0$ and $g(-x^2) \le 0$.

Which gives: $W'(x) \ge 0$ and $W'(x) = 0$ for $x = 0$ (but without changing signs)

So W is strictly increasing.

Exercise 19

Part A

1- a. $h(x) = (x-1)e^{-x}$; $h'(x) = (2-x)e^{-x}$ and $h''(x) = (x-3)e^{-x}$

By replacing these answers we get: $h''(x) + 2h'(x) + h(x) = \ldots\ldots = 1$.

b. Take $a_n = (-1)^n\,(x - n - 1)\,e^{-x}$.

Step 1: $h'(x) = (2-x)e^{-x}$ and $a_1 = (-1)^1\,(x - 1 - 1)\,e^{-x}$; so $h'(x) = a_1$.

Step 2: Suppose that $h^{(n)}(x) = a_n$ and prove that $h^{(n+1)}(x) = a_{n+1}$.

In fact: $h^{(n+1)}(x) = [h^{(n)}]'(x) = (-1)^n\,[e^{-x} - (x-n-1)\,e^{-x}] = (-1)^n\,(-1)(x-n-2)\,e^{-x} = a_{n+1}$.

Conclusion: For every natural integer n such that $n > 1: h^{(n)}(x) = (-1)^n\,(x-n-1)e^{-x}$.

2-

x	$-\infty$		2		$+\infty$
$h'(x)$		$+$	0	$-$	
$h(x)$	$-\infty$	$\nearrow$	$2+\frac{1}{e^2}$	$\searrow$	2

3- On $[2\,;+\infty[$: $h(x) > 0$.

On $]-\infty\,;2[$, h is continuous and changes signs, so for sure the equation $h(x) = 0$ has at least one real α; and since h is strictly increasing ; so α is unique.

$h(-0.4) \times h(-0.3) < 0$.

x	$-\infty$		2
$P'(x)$		$+$	0
$P(x)$		$\nearrow$	$2+\frac{1}{e^2}$

4-• $a = G(1) \leftrightarrows 1 = P(a) \leftrightarrows a = 0$; which gives: A (1 ; 0)

$G'(1) \times P'(0) = 1 \leftrightarrows 2 \times G'(1) = 1$

$G'(1) = \frac{1}{2}$ and (T) has an equation $y = \frac{1}{2}x - \frac{1}{2}$.

$2+\frac{1}{e^2}$

x	$-\infty$		$+\infty$
$G'(x)$		$+$	
$G(x)$	$-\infty$	$\nearrow$	2

Part B

1- a. Let $d(x) = f(x) - (2x + 1)$; $d(x) = -xe^{-x}$.

- $d(x) > 0$ for $x < 0$: (C) is above (d).
- $d(x) = 0$ for $x = 0$: (C) cuts (d) at the point $E(0\,;1)$.
- $d(x) < 0$ for $x > 0$: (C) is below (d).

b. $\lim\limits_{x\to+\infty}(f(x) - 2x - 1) = \lim\limits_{x\to+\infty}(-x)e^{-x} = 0$

Two conclusions:

- (d) is an asymptote to (C) at $(+\infty)$
- $\lim\limits_{x\to+\infty} f(x) = +\infty$.

2-• $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x} = \lim\limits_{x\to-\infty}\left[2 + \dfrac{1}{x} - e^{-x}\right] = -\infty$.

Two conclusions:

- (C) has vertical asymptotic direction at $(-\infty)$.
- $\lim\limits_{x\to-\infty} f(x) = +\infty$.

3- $f'(x) = \ldots\ldots = h(x)$ and $f(\alpha) = 2\alpha + 1 - \alpha e^{-\alpha}$ with $e^{-\alpha} = \dfrac{-2}{\alpha - 1}$;

which gives: $f(\alpha) = \ldots. = \dfrac{(\alpha + 1)(2\alpha - 1)}{\alpha - 1}$

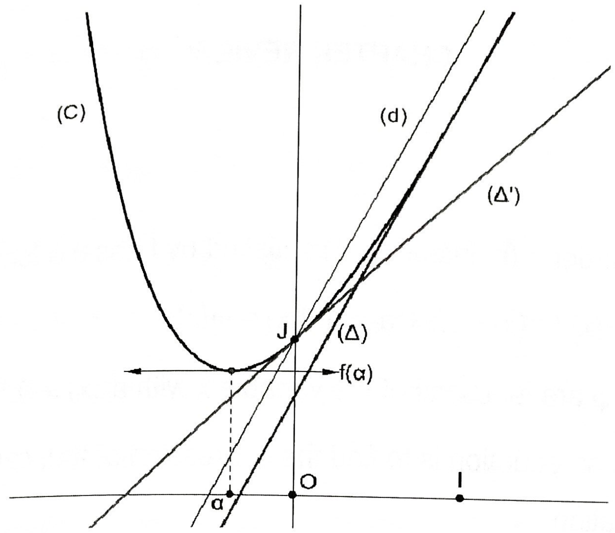
$-0.4 < \alpha < -0.3$; $\alpha + 1 > 0$; $2\alpha - 1 < 0$ and $\alpha - 1 < 0$; which gives: $f(\alpha) > 0$.

b. Table of variations of f:

x	$-\infty$		α		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

4- (Δ) has equation: $y - f(0) = x f'(0)$; $y = x + 1$

(Δ') has equation $y - f(1) = (x - 1)f'(1)$; $y = 2x + 1 - \dfrac{1}{e}$.



6-• $A = A'u^2$ with $A' = \int_0^1 xe^{-x}dx$.

Let $L(x) = xe^{-x}$, then: $L'(x) = e^{-x} - xe^{-x}$ and consequently $xe^{-x} = e^{-x} - L'(x)$;

So, $A' = \int_0^1 xe^{-x}dx = \left[(x+1)e^{-x}\right]_1^0 = 1 - \frac{2}{e}$.

Finally: $A = \frac{e-2}{e}u^2$.

Part C

$T'(x) = h'(f(x)) \times f'(x)$; $T'(0) = h'(1) \times f'(0) = \frac{1}{e} \times 1 = \frac{1}{e}$.

$\lim_{x\to-\infty} T(x) = \lim_{x\to-\infty} h(f(x)) = 2$.