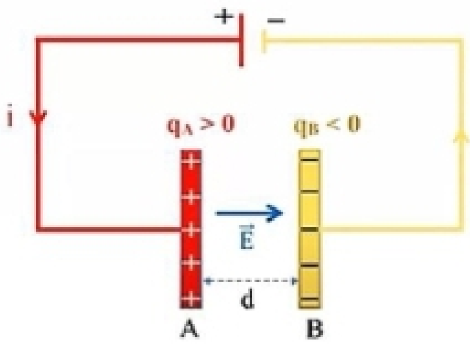
Chapter Outline:

- I- Introduction
- II- R-C series circuit under square voltage
- III- Charging process of a capacitor
- IV- Discharging process of a capacitor
- V- Energy stored in a capacitor

****

I- Introduction

Definition of a capacitor:



A capacitor is an electric component composed of two parallel metallic plates separated by an insulator.

A voltage applied across the plates creates a uniform electric field in the capacitor, which stores energy. The capacitor starts charging, and its armatures carry electric charges that are equal but opposite +q and -q. The net charge of a capacitor is zero.

The process of accumulating electric charges (or storing electric energy) by the plates of the capacitor is called: "**charging phase**", and the process of releasing electric charges or the stored electric energy is called: "**discharging phase**".

- For every capacitor, there is a quantity called: "**capacitance**", denoted by "C", which measures the ability of a capacitor to trap electric charges or electric energy.
  - This capacitance C is expressed in "**farad**" denoted by "**F**"; named after the famous physicist Michael Faraday.
  - The capacitance C depends only on the geometry of the armatures of each capacitor.
- Capacitors are used in almost all kinds of electronic circuit for storing, surge suppression (protection of circuits) and filtering (allowing some frequencies and blocking others).
  For example: there are thousands of them on modern computer motherboards, in camera flashes, and also they are essential components of the tuning circuits of radio and television transmitters and receivers.

Voltage between the terminals of a capacitor:

The voltage between the terminals of a capacitor of capacitance C is proportional to the charge q on its armatures: $U_C = \frac{q}{C}$

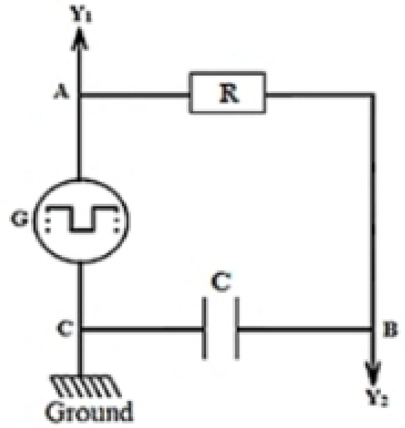
With:

- C: the capacitance of the capacitor in farads (F);
- u_C: the voltage across the capacitor in volts (V);
- q: the electric charge in coulombs (C).

Unit	Multiplier
Farad (F)	1
Milifarad (mF)	10^{-3}
Microfarad (µF)	10^{-6}
Nanofarad (nF)	10^{-9}
Picofarad (pF)	10^{-12}

II- R-C circuit under square voltage

Experimental study: Visualization of the growth and the decay of voltage across a capacitor



Material:

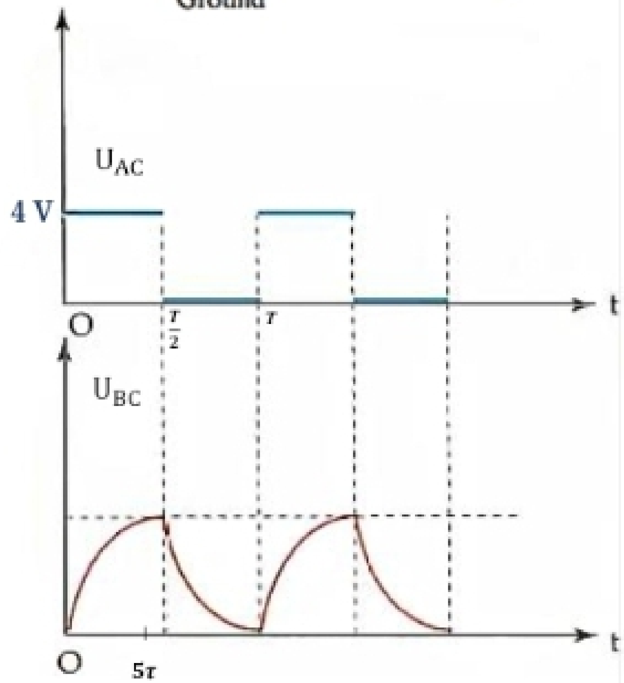
- A low frequency generator (L.F.G);
- An oscilloscope;
- A capacitor of capacitance C;
- A resistor of resistance R.

Procedure and observation:

- Adjust the L.F.G to give a square signal of frequency f = 200 Hz.

$$T = 5 \text{ ms} \,;\, U_{AC} = U_G = \begin{cases} 4\text{ V} & \text{for } 0 \le t \le \frac{T}{2} \\ 0\text{ V} & \text{for } \frac{T}{2} \le t \le T \end{cases}$$

- Channel 1 of the oscilloscope is connected to measure the voltage $U_{AC} = U_G$ cross the generator;
- Channel 2 is connected to measure the voltage $U_{BC} = U_C$ across the capacitor.



Observation:

- When U_{AC} jumps from 0 V to 4 V, the voltage U_{BC} **increases gradually (charging phase).**
- When U_{AC} drops from 4 V to 0 V, the voltage U_{BC} **decreases gradually (discharging phase).**

Interpretation:

Charging process of the capacitor ($0 \le t \le \frac{T}{2}$)

- At $t_0 = 0$, the free electrons of the connecting wires and the armatures are attracted to the positive pole of (G), and repelled away from the negative pole. Electrons begin to flow forming an electric current in the opposite sense and charges starts accumulating on the capacitor.
- Armature B loses electrons and becomes positively charged, while the other, armature C, gains electrons and becomes negatively charged. Thus the capacitor is charging.
- As the capacitor starts charging, the potential difference across its plates slowly increases and the actual time it takes to reach 63% of its maximum possible voltage, is known as one **Time Constant τ** (tau).

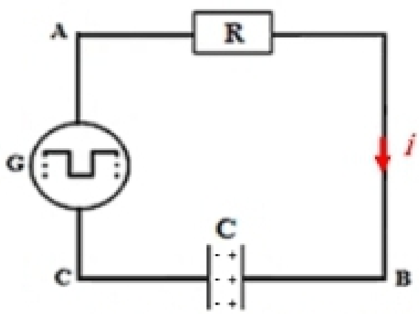


Figure a: RC circuit during charging phase

➔ τ: the time constant only specifies a rate of charge where R is in Ω and C in farads (F).

▪ The capacitor continues to charge and corresponding to it, the charging current also decreases. (transient phase)

▪ Now at $t = 5\tau$, the charging current finally diminishes to zero, the free electrons stop flowing. The capacitor now acts as an open switch and the voltage across it reaches the voltage U_G of the generator. the capacitor is said to be **fully charged**. (Steady-state phase)

➔ The capacitor remains fully charged as long as there is a constant supply applied to it.

➔ If the fully charged capacitor is disconnected from the generator, the stored energy during the charging process will stay indefinitely on its plates, keeping the voltage across its terminals constant.

Discharging process of the capacitor ($\frac{T}{2} \leq t \leq T$)

In this phase, the voltage of the generator decreases to zero. So, the capacitor starts discharging itself back through the resistor R.

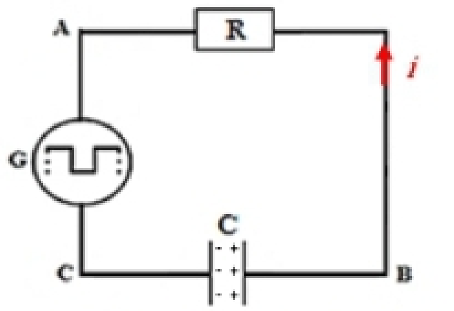


Figure b: RC circuit during discharging phase

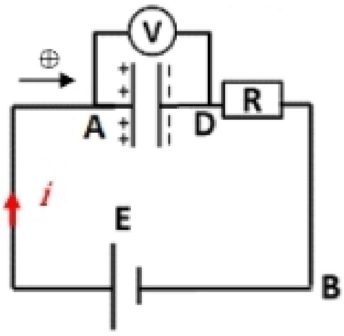
- At $t_0 = 0$, the free electrons of the connecting wires and the armatures are attracted to the positive armature B of the capacitor and repelled away from the negative armature C. Electrons begin to flow forming an electric current in the opposite sense. The sense of the discharging current is opposite to the sense of the charging current. (figure (b)).
- As the capacitor keeps on discharging, the voltage U_C across it decays gradually.
- The **time constant (τ)** always corresponds to the value of 63%. Thus during discharging, the voltage across the capacitor after one time constant τ, has dropped by 63% of its initial value; thus it is equal to: **1 – 0.63 = 0.37 or 37%** of its maximum value.
- As the discharging continues, the voltage U_C decreases, resulting in a less discharging current.
- Now at $t = 5\tau$, the capacitor is said to be **fully discharged**; the discharging current finally diminishes to zero and the capacitor now acts as a connecting wire and the voltage across it becomes zero.

III- Charging Process of a Capacitor

We connect a capacitor of capacitance C, and a resistor of resistance R in series with a DC generator of constant voltage E.

The capacitor is oriented positively from A to D. We will note the charge of the capacitor $\mathbf{q_A = q}$ and the voltage at its terminals $\mathbf{u_{AD} = u_c}$

Evolution of u_c as a function of time (theoretical study)



❖ Differential Equation

The law of addition of voltages gives: $u_{AB} = u_{AD} + u_{DB}$ => $\mathbf{E = u_c + R.i}$

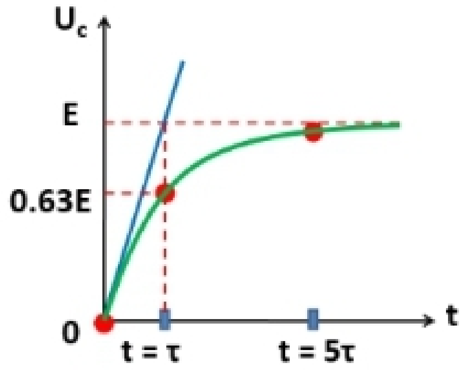
With: $\mathbf{i} = \frac{dq}{dt}$ (since the positive direction enters through the positive armature) and $\mathbf{q = C.u_c}$ thus, $\mathbf{i = C.}\frac{d\,u_c}{dt}$

then, $\mathbf{E = u_c + RC.}\frac{d\,u_c}{dt}$

This is the differential equation of first order which governs the variation of voltage u_c as a function of time during charge.

❖ Solution of the differential equation

The solution of this equation is: $\mathbf{u_c = E(1- e^{-\frac{t}{\tau}})}$



This expression represents an exponential growth of voltage u_C.

Particular instants:

- at $t = 0$, $u_c = 0$ V
- at $t = \tau$, $u_c = 0.63E$
- at $t = 5\tau$, $u_c = E$: the capacitor is fully charged.

The voltage of the capacitor is gradually established and reaches its maximum after $t = 5\tau$

Where: $\boldsymbol{\tau} = \mathbf{R.C}$

- $\boldsymbol{\tau}$ is in seconds (s);
- R in ohms (Ω);
- C in farads (F).

τ the *time constant* of the R-C circuit. It is the time after which the voltage of the capacitor reaches 63% of its maximum value during charging.

➔ The capacitance of a capacitor in an RC circuit, subjected to a constant voltage, does not charge instantly: the **charging** of a capacitor is **a transient phenomenon**.
➔ The duration of the charge of a capacitor increases when R or/and C increases and vice versa.
➔ The resistance of R represents that of the whole circuit. In case it is zero, the capacitor charges and discharges instantly. Rapid charging and discharging of a capacitor have negative effects on it.

Evolution of current i as a function of time

❖ Differential equation

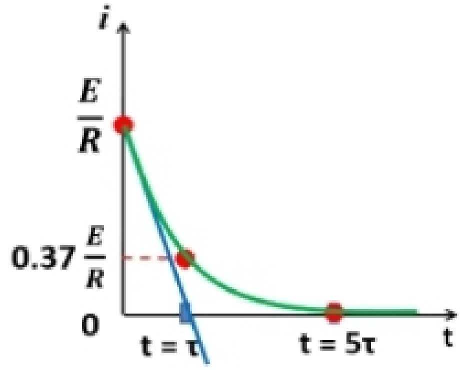
The law of addition of voltages gives: $u_{AB} = u_{AD} + u_{DB}$ => $\mathbf{E = U_c + R.i}$

With: $i = \frac{dq}{dt}$ and $q = C.u_c$ thus, $i = C.\frac{d\,u_c}{dt}$ => $\frac{d\,u_c}{dt} = \frac{i}{C}$

Derive the equation $E = u_c + R.i$ with respect to time: $0 = \frac{du_c}{dt} + R\frac{di}{dt}$ => $0 = \frac{i}{C} + R\frac{di}{dt}$ => $\frac{di}{dt} + \frac{1}{RC}i = 0$.

This is the first order differential equation that governs the variation of current as a function of time in an RC circuit during the phase of charge.

- **Expression of the current**



The solution of this equation is: $i = I_m.e^{-\frac{t}{\tau}}$

with: $\tau = R.C$ and $I_m = \frac{E}{R}$

- at t = 0, $i = \frac{E}{R}$
- at $t = \tau$, $i = 0.37\frac{E}{R}$
- at $t = 5\tau$, i = 0: when the capacitor is fully charged.

➔ The current i is zero before closing the switch and equal to $\frac{E}{R}$ just after closing. This intensity decreases gradually.

➔ At the end of the charging process, the current is zero. The capacitor, thus, acts as an open switch once charged completely under a certain voltage.

Evolution of charge q as a function of time

- **Differential equation**

The law of addition of voltages gives: $u_{AB} = u_{AD} + u_{DB}$ => $E = U_c + R.i$

With: $i = \frac{dq}{dt}$ and $q = C.u_c$ => $u_c = \frac{q}{C}$

$E = \frac{q}{C} + R\frac{dq}{dt}$ => $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$:

This is the first order differential equation that governs the variation of charge q as a function of time in an RC circuit during the phase of charge.

- **Expression of the charge**

The solution of this equation is: $q = Q_m(1- e^{-\frac{t}{\tau}})$

Determination of Q_m and τ:

$q = Q_m(1- e^{-\frac{t}{\tau}})$

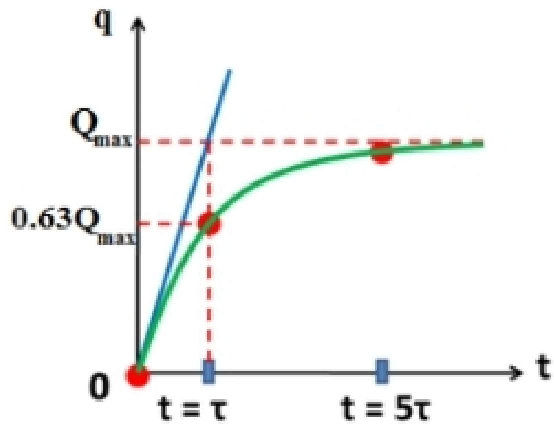
deriving with respect to time: $\frac{dq}{dt} = \frac{Q_m}{\tau}e^{-\frac{t}{\tau}}$

substitute q and $\frac{dq}{dt}$ in the differential equation:

=> $\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$ => $\frac{Q_m}{\tau}e^{-\frac{t}{\tau}} + \frac{1}{RC}Q_m\left(1 - e^{-\frac{t}{\tau}}\right) = \frac{E}{R}$

=> $\frac{Q_m}{\tau}e^{-\frac{t}{\tau}} + \frac{1}{RC}Q_m - \frac{1}{RC}Q_m e^{-\frac{t}{\tau}} = \frac{E}{R}$ => $Q_m e^{-\frac{t}{\tau}}\left(\frac{1}{\tau} - \frac{1}{RC}\right) + \frac{1}{RC}Q_m = \frac{E}{R}$

Then: $\frac{1}{\tau} - \frac{1}{RC}$ => $\tau = RC$ and $Q_m = C.E.$



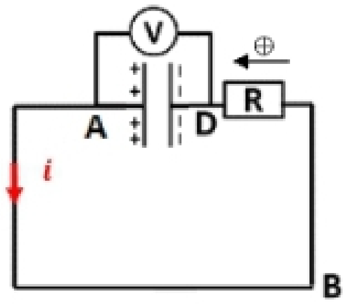
Particular values of q:

- at t = 0, q = 0 (the capacitor is initially neutral);

- at $t = \tau$, $q = 0.63\ Q_m$;
- at $t = 5\tau$, $q = 0.99\ Q_m \approx Q_m$: the capacitor is fully charged.

IV- Discharge of a capacitor through a resistor of resistance R

The generator is disconnected form the circuit. The capacitor starts discharging through the resistor. Electric current flows from armature A (+) to armature D (-) through the resistor (in the same sense as the chosen positive direction, and it is opposite to the direction of current during the charging phase).



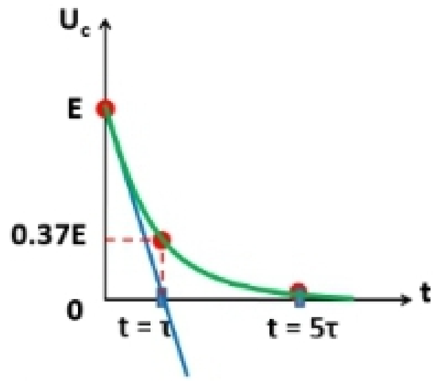
Evolution of u_c as a function of time (theoretical study)

- **Differential Equation**

The law of addition of voltages gives:

$u_{AD} = u_{AB} + u_{BD}$

$u_C = 0 + R.i \Rightarrow u_C - Ri = 0$



with: $i = -\frac{dq}{dt}$ (since the positive direction enters through the negative armature) and $q = C.u_c$

then : $i = -C.\frac{d\,u_c}{dt}$

so : $u_c + R.\ C.\frac{d\,u_c}{dt} = 0$

It is the differential equation which governs u_c in an RC circuit during discharge.

- **Solution of the differential equation**

The solution of this equation is: $u_c = E.e^{-\frac{t}{\tau}}$

This expression represents an exponential decay of voltage u_C.

Particular instants:

- at $t = 0$, $u_c = E$
- at $t = \tau$, $u_c = 0.37E$
- at $t = 5\tau$, $u_c = 0$: the capacitor is fully discharged.

The voltage of the capacitor decreases gradually and reaches zero after $t = 5\tau$

$\tau = R.C$: the *time constant* of the R-C circuit. It is the time after which the voltage of the capacitor reaches 37% of its maximum value during discharging.

➔ The capacitance of the capacitor does not discharge instantly: the **discharging** of a capacitor is **a transient phenomenon.**

➔ The duration of the discharge of a capacitor in an RC circuit increases when R or/and C increases and vice versa.

Evolution of current i as a function of time

Differential equation

The law of addition of voltages gives: $\mathbf{u_{AB} = u_{AD} + u_{DB} \Rightarrow 0 = u_c - R.i}$

With: $i = -\frac{dq}{dt}$ (since the positive direction enters through the negative armature)

and $q = C.u_c$ thus, $i = -C.\frac{d\,u_c}{dt} \Rightarrow \frac{d\,u_c}{dt} = -\frac{i}{C}$

Derive the equation $0 = u_c - R.i$ with respect to time: $0 = \frac{du_c}{dt} - R\frac{di}{dt} \Rightarrow 0 = -\frac{i}{C} - R\frac{di}{dt} \Rightarrow \boldsymbol{\frac{di}{dt} + \frac{1}{RC}i = 0}$.

This is the first order differential equation that governs the variation of current as a function of time in an RC circuit during the phase of discharge.

Expression of the current

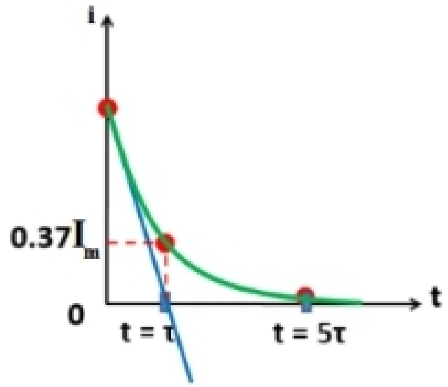
We can get the expression of electric current as a function of time from the expression we found of the voltage uc during discharge.

$u_c = E.e^{-\frac{t}{\tau}}$ and $i = -C.\frac{d\,u_c}{dt}$, then $i = -C\,(-\frac{E}{\tau}\,e^{-\frac{t}{\tau}})$

$\Rightarrow i = -\frac{E}{R}\,e^{-\frac{t}{\tau}} \Rightarrow i = I_m\,e^{-\frac{t}{\tau}}$

With $\mathbf{I_m} = \frac{E}{R}$: the maximum value (amplitude) of current

⇨ The current decreases exponentially with time.



V- Energy Stored in a Capacitor

During charging, a capacitor stores **electrical energy** which it releases during discharge.

The energy stored by a capacitor is: $\mathbf{W = \frac{1}{2}C.u_c^2 = \frac{1}{2}.\frac{q^2}{C} = \frac{1}{2}Q.u_C}$

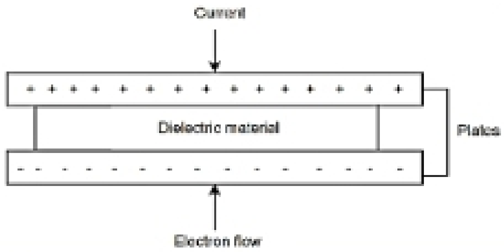
With:

- W: the electric energy in joules (J);
- C: the capacitance in farads (F);
- u_c: the voltage across the capacitor in volts (V);
- q: the electric charge on the armatures in coulombs (C);

Summary

Definition

A capacitor is an electric component composed of two parallel metallic plates separated by an insulator.
A capacitor stores electric energy in its electric field.



Characteristics of a capacitor

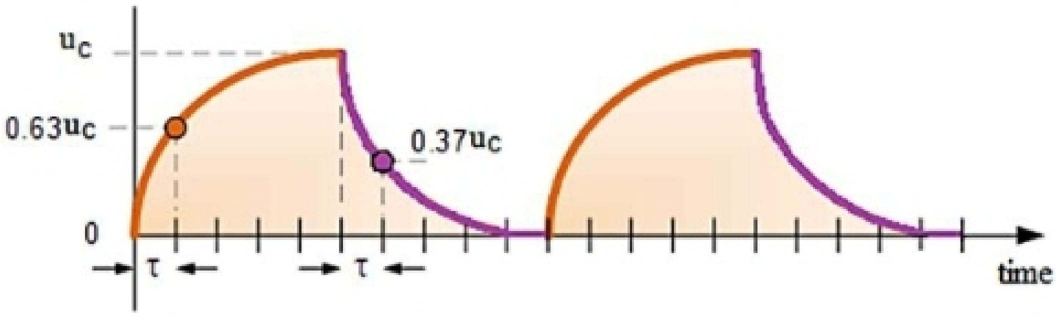
A capacitor is characterized by its **capacitance (C).**	The voltage across a capacitor is proportional to the charge on its armatures: $\mathbf{U = \frac{Q}{C}}$	**Electric energy** stored inside a capacitor: $\mathbf{W = \frac{1}{2}C.u_C^2 = \frac{1}{2}.\frac{q^2}{C} = \frac{1}{2}Q.u_C}$

Charging and discharging phases of a capacitor

	Phase of charge	**Phase of discharge**
Differential equation in u_C	$\frac{du_C}{dt} + \frac{1}{RC}u_C = \frac{E}{RC}$	$\frac{du_C}{dt} + \frac{1}{RC}u_C = 0$
Solution $u_C(t)$	$u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$	$u_C = E.e^{-t/\tau}$
Differential equation in q	$\frac{dq}{dt} + \frac{1}{RC}q = \frac{E}{R}$	$\frac{dq}{dt} + \frac{1}{RC}q = 0$
Solution q(t)	$q = Q_m\left(1 - e^{-\frac{t}{\tau}}\right)$	$q = Q_m.e^{-t/\tau}$
Differential equation in i	$\frac{di}{dt} + \frac{1}{RC}i = 0$	$\frac{di}{dt} + \frac{1}{RC}i = 0$
Solution i(t)	$i = I_m.e^{-t/\tau}$	$i = I_m.e^{-t/\tau}$
Time constant $\tau = RC$	The time interval after which the **voltage reaches 63% of its maximum value.**	The time after which the **voltage decreases to 37% of its maximum value.**

The sign of $\pm\frac{dq}{dt}$ is determined according to the positive sense chosen in the circuit. (usually taken in the same sense as the current)	$i = +\frac{dq}{dt}$ if the positive chosen direction enters through the positive armature.
	$i = -\frac{dq}{dt}$ if the positive chosen direction enters through the negative armature

Graphical study of the variation voltage across a capacitor



Charging phase	Discharging phase
$u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$ • at t = 0, $u_C = 0$ • at $t = \tau$: $u_C = 63\%$ **E** ➔ **63**% of the total charge is completed. • At the end of this phase($t = 5\tau$): $u_C = E$	$u_C = E.e^{-t/\tau}$ • at t = 0, u_C = **E** • at $t = \tau$: $u_C = 37\%$ E ➔ 63% of the total discharge is completed. • At the end of this phase ($t = 5\tau$): $u_C = 0$