Math

Sequence

1. Definition

A sequence is a set of images of a mapping from the set $\mathbb{N}$ (or subset of $\mathbb{N}$) onto $\mathbb{R}$: $n \longmapsto U_n$

- $U_0, U_1, U_2 \dots$: are terms of the sequence (U_n).
- U_n is the general term of the sequence (U_n).
- n is called the index.

Remark:

- If U_n is expressed in terms of n, this sequence is said to be expressed explicitly.
- If a sequence is defined by its first term (or firsts terms) and a relation between two general terms (or several general terms), this sequence is said to be expressed recursively.

2. Sense of Variations

Let (U_n) be a given sequence defined for all $n \in E$, where $E \subset \mathbb{N}$.

(U_n) is said to be strictly increasing if and only if:

1. $U_{n+1} > U_n$ (or $U_{n+1} - U_n > 0$)
2. $n < m$ then $U_n < U_m$
3. $\frac{U_{n+1}}{U_n} > 1$ (if $U_n > 0$)

(U_n) is said to be strictly decreasing if and only if:

1. $U_{n+1} < U_n$ (or $U_{n+1} - U_n < 0$)
2. $n < m$ then $U_n > U_m$
3. $\frac{U_{n+1}}{U_n} < 1$ (if $U_n > 0$)

(U_n) is said to be constant when $U_{n+1} = U_n$ (or $U_{n+1} - U_n = 0$) for all n.

Remark: To study the variations of a sequence, we can also study:

- For a sequence defined explicitly, of the type $U_n = f(n)$:
 the sense of variation of f, so if f is increasing then (U_n) is increasing, and if f is decreasing then (U_n) is decreasing.

- For a sequence defined implicitly, of the type $U_{n+1} = f(U_n)$, U_0 being given:
 the sense of variation of f, compare the first two terms and by induction two consecutive general terms.

3. Arithmetic sequence

3. A. Definition

A sequence (U_n) is said to be arithmetic if $U_{n+1} - U_n = d$, where d is a constant.

d is called the common difference of (U_n).

Remark:

In order to prove that a sequence (U_n) is arithmetic, it is not enough to show that $U_2 - U_1 = U_3 - U_2 = U_4 - U_3$.

In fact, **we must show** that $U_n - U_{n-1} = d$ or $U_{n+1} - U_n = d$.

3. B. Property

For all natural numbers n and p, let (U_n) be an arithmetic sequence.

The general term of (U_n) is:

$$U_n = U_p + (n-p)d$$

3. C. Sum of terms of an Arithmetic Sequence:

let (U_n) be an arithmetic sequence. The sum of the terms of (U_n)

$$\sum_{i=m}^{n} u_i = u_m + \cdots + u_n = (n-m+1)\frac{u_m + u_n}{2}$$

That is to say: sum of consecutive terms of an arithmetic sequence

$$S = (\text{number of terms})\ \frac{(\textbf{first term} + \textbf{last term})}{2}$$

Remark:

What is the funny Σ symbol? It is called Sigma notation.

Σ (called sigma) means "sum up", and below and above it are shown the starting and ending values.

Start at this value

Go to this value

What to sum

$$\sum_{n=1}^{4} n = 1 + 2 + 3 + 4 = 10$$

It says "Sum up" n where n goes from 1 to 4. Answer = 10.

4. Geometric sequence

4.A. Definition:

A sequence (U_n) is said to be geometric if $U_{n+1} = U_n \times r$ that is $\frac{U_{n+1}}{U_n} = r$, where r is a constant.

r is called the common ratio of (U_n).

Remark:

In order to prove that a sequence (U_n) is **geometric**, it is not enough to show that $\frac{U_1}{U_0} = \frac{U_2}{U_1} = \frac{U_3}{U_2}$.

In fact, **we must show** that $\frac{U_{n+1}}{U_n} = r$ or $\frac{U_n}{U_{n-1}} = r$.

4.B. Property

For all natural numbers n and p, let (U_n) be a geometric sequence.

The general term of (U_n) is :

$$U_n = U_p \times r^{n-p}$$

4.C. Sum of terms of a Geometric Sequence

Let (U_n) be a geometric sequence. The sum of the terms of (U_n) is:

$$\sum_{i=m}^{n} u_i = u_m + \cdots + u_n = u_m \frac{1-r^{n-m+1}}{1-r}$$

That is to say: sum of consecutive terms of a geometric sequence:

$$S = \frac{\text{first term}\,(1-r^{\text{nb of terms}})}{1-r}$$

5. Limit of a Numerical sequence

5.A. Definition

Let (U_n) be a given sequence with general term U_n defined explicitly for all $n \in \mathbb{N}$.

We say that the limit of (U_n) exists when $\lim\limits_{n\to+\infty} U_n$ leads to a unique real number L.

5.B. Property

Let a be a real number.

$$\lim_{n\to+\infty} (a)^n = \begin{cases} +\infty & \text{if } a > 1 \\ 1 & \text{if } a = 1 \\ 0 & \text{if } -1 < a < 1 \end{cases}$$

6. Convergent sequence VS Divergent sequence

A sequence (U_n) is said to be **convergent** if and only if $\lim\limits_{n\to+\infty} U_n$ **exists** (unique $\in \mathbb{R}$).

A sequence that is not convergent is called a divergent sequence.

7. Sequences and Inequalities

- If (U_n) and (V_n) have limits and $U_n \le V_n$ for every n, then $\lim\limits_{n\to+\infty} U_n \le \lim\limits_{n\to+\infty} V_n$.

  If $\lim\limits_{n\to+\infty} U_n = +\infty$, then $\lim\limits_{n\to+\infty} V_n = +\infty$

  If $\lim\limits_{n\to+\infty} V_n = -\infty$, then $\lim\limits_{n\to+\infty} U_n = -\infty$

- If (U_n), (V_n) and (W_n) have limits and $U_n \le V_n \le W_n$ for every n,

  then $\lim\limits_{n\to+\infty} U_n \le \lim\limits_{n\to+\infty} V_n \le \lim\limits_{n\to+\infty} W_n$.

  If $\lim\limits_{n\to+\infty} U_n = \lim\limits_{n\to+\infty} W_n$, then $\lim\limits_{n\to+\infty} U_n = \lim\limits_{n\to+\infty} V_n = \lim\limits_{n\to+\infty} W_n$ (Sandwich Theorem).

8. Bounded Sequences

- A sequence (U_n) is said to be **bounded above** if there exists a real number k such that $U_n \le k$ for every $n \in \mathbb{N}$.
- A sequence (U_n) is said to be **bounded below** if there exists a real number m such that $m \le U_n$ for every $n \in \mathbb{N}$.
- A sequence (U_n) is said to be **bounded** if it is bounded above and bounded below; i.e. $m \le U_n \le k$ for every $n \in \mathbb{N}$.

9. Mathematical Induction

Definition: Mathematical Induction is a mathematical technique which is used to prove a statement is true for every natural number.

Steps:

If P(n) is a statement that depends on a natural number n. To show that P(n) is true, for any natural number $n \ge n_0$, it suffices to:

Step 1: verify if a statement is true for n = initial value, i.e. $P(n_0)$ is true.

Step 2: assume the statement is true for any value of n = k, i.e. P(k) is true.

Step 3: prove the statement is true for n = k+1, i.e. P(k + 1) is true.

10. Theorems of Monotonic Sequences

Property 1:

- An increasing sequence that is bounded from above is convergent.
- A decreasing sequence that is bounded from below is convergent.

Property 2:

Let (U_n) be a convergent sequence that is defined recursively, for every $n \in E$, with $E \subset \mathbb{N}$, and by the given of U_0 and the relation $U_{n+1} = f(U_n)$.

If $x \to f(x)$ is continuous, then the limit $L = \lim\limits_{n\to+\infty} U_n$ is a **root** of the equation $L = f(L)$