- *This exam consists of three problems inscribed on two pages.*
- *The use of a non-programmable calculator is allowed.*

I – (4 ½ points)

A student in LS section answered a math quiz in **Logarithmic function**.
The quiz consists of five independent parts. He got 3 over 5 (1 mark for each answer).
The parts of the quiz and the student's answers are shown below.

Quiz:

Let f be the function defined by $f(x)=\dfrac{1-\ln x}{\ln x+1}$.

1) The domain of definition of f is:
2) $\lim\limits_{x\to+\infty} f(x)=$
3) The representative curve of f is above the line of equation $y=2$ for $x\in$
4) The value of the expression $A=f\left(\dfrac{1}{\sqrt{e}}\right)+\ln\left(5-2\sqrt{6}\right)+\ln\left(5+2\sqrt{6}\right)=$
5) $f\left(\left[1;e^2\right]\right)=$

Answers **3 / 5**

1) $]0;+\infty[$
2) $\lim\limits_{x\to+\infty} f(x)=-1$
3) $\left]\dfrac{1}{e};\dfrac{1}{e^{\frac{1}{3}}}\right[$
4) $A=0$
5) $\left[-\dfrac{1}{3};1\right]$

Indicate with justification, the parts answered correctly by the student and then correct the wrong answers.

II – (6 ½ points)

Consider two urns U and V:
U contains 7 balls: 4 red, 2 white and 1 black
V contains 8 balls: 3 red, 2 white and 3 black
Consider a perfect die whose 6 faces are numbered: 1 ; 1 ; 1 ; 2 ; 2 ; 3.

Part A

We roll this die.

- If we obtain a face numbered 1, we draw successively with replacement **three** balls from the urn U.
- If we obtain a face numbered 2, we draw simultaneously **three** balls from the urn V.
- If we obtain a face numbered 3, we select **one** ball from **each** urn.

Consider the following events:
A: « The obtained face of the die is numbered 1 ».
B: « The obtained face of the die is numbered 2 ».
C: « The obtained face of the die is numbered 3 ».
R: « Among the drawn balls we obtain exactly 2 red balls ».

1) Calculate $P(A)$, $P(B)$ and $P(C)$.

2) **a)** Show that $P(R\cap A)=\dfrac{72}{343}$ then calculate $P(R\cap B)$ and $P(R\cap C)$.

b) Deduce $P(R)$.

3) Consider the event M: « The drawn balls have the same color ».

a) Show that $P(M)=\dfrac{2879}{16464}$.

b) Calculate $P(M\cup B)$.

c) The drawn balls don't have the same color. Calculate the probability that the die shows an odd number.

Part B

In this part all balls of the two urns U and V are placed in a new urn W.
The player selects successively without replacement 3 balls from the urn W.
The player scores +1 point for each selected white ball, – 1 point for each selected black ball, and 0 point for each selected red ball.
Calculate the probability of obtaining three balls where the sum of scored points is equal to zero.

III – (9 points)

Part A

Consider the function g that is defined over $\mathbb{R}$ by: $g(x)=(2x-1)e^{-x+1}-1$.

1) Calculate $\lim\limits_{x\to-\infty} g(x)$ and $\lim\limits_{x\to+\infty} g(x)$.

2) Calculate $g'(x)$, then set up the table of variations of g .

3) Prove that the equation $g(x)=0$ admits on $\mathbb{R}$ exactly two roots one of them is 1 and the other is α such that $2.2<\alpha<2.4$.

4) Study the sign of the function g for all real numbers x .

Part B

Consider the function f that is defined over $\mathbb{R}$ by $f(x)=(4x+2)e^{-x+1}+2x-1$.

Let (C) be its representative curve in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.

1) Calculate $\lim\limits_{x\to-\infty} f(x)$ and $f(2)$.

2) **a)** Calculate $\lim\limits_{x\to+\infty} f(x)$,then prove that the line (d) of equation $y=2x-1$ is an asymptote to (C).

b) Study, according to the values of x , the relative position of (C) and (d).

3) Prove that $f'(x)=-2g(x)$, then set up the table of variations of f .

4) Prove that (C) admits an inflection point I whose coordinates is to be determined.

5) Write an equation of the tangent (T) to (C) at point I.

6) Show that $f(\alpha)=\dfrac{4\alpha^2+3}{2\alpha-1}$

7) Draw (d), (T) and (C) (take $\alpha\approx 2.3$).

<table>
<tr><th>QI</th><th>Answers</th><th>4 ½ pts.</th></tr>
<tr><td>1)</td><td>f is defined for $\begin{cases} x>0 \\ \ln x \neq -1 \end{cases}$; $\begin{cases} x>0 \\ x \neq e^{-1} \end{cases}$; $\begin{cases} x>0 \\ x \neq \dfrac{1}{e} \end{cases}$.
Then the domain of definition of f is $\left]0;\dfrac{1}{e}\right[\cup \left]\dfrac{1}{e};+\infty\right[$.
This part is answered wrongly.</td><td>¾</td></tr>
<tr><td>2)</td><td>$\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty} \dfrac{1-\ln x}{\ln x + 1} \overset{\text{H.R}}{=} \lim\limits_{x \to +\infty} \dfrac{-\frac{1}{x}}{\frac{1}{x}} = -1$. This part is answered correctly.</td><td>¾</td></tr>
<tr><td>3)</td><td>$f(x) - 2 = \dfrac{1-\ln x}{\ln x + 1} - 2 = \dfrac{1-\ln x - 2\ln x - 2}{\ln x + 1} = \dfrac{-1-3\ln x}{\ln x + 1}$.
$$\begin{array}{c|ccccccc} x & 0 & & \frac{1}{e} & & \frac{1}{e^{\frac{1}{3}}} & & \\ \hline -1-3\ln x & \| & + & & + & 0 & - & \\ \hline \ln x + 1 & \| & - & 0 & + & & + & \\ \hline \dfrac{-1-3\ln x}{\ln x + 1} & \| & - & \| & + & 0 & - & \end{array}$$
The representative curve of f is above line of equation $y=2$ for $\dfrac{-1-3\ln x}{\ln x + 1} > 0$;
that is for $x \in \left]\dfrac{1}{e};\dfrac{1}{e^{\frac{1}{3}}}\right[$. This part is answered correctly.</td><td>1</td></tr>
<tr><td>4)</td><td>$A = f\left(\dfrac{1}{\sqrt{e}}\right) + \ln\left(5-2\sqrt{6}\right) + \ln\left(5+2\sqrt{6}\right) = \dfrac{1-\ln\left(\frac{1}{\sqrt{e}}\right)}{\ln\left(\frac{1}{\sqrt{e}}\right)+1} + \ln\left[\left(5-2\sqrt{6}\right)\times\left(5+2\sqrt{6}\right)\right]$
$= \dfrac{1+\ln\left(\sqrt{e}\right)}{-\ln\left(\sqrt{e}\right)+1} + \ln(25-24) = \dfrac{1+\frac{1}{2}}{-\frac{1}{2}+1} + \ln 1 = \dfrac{\frac{3}{2}}{\frac{1}{2}} + 0 = 3$. This part is answered wrongly.</td><td>1</td></tr>
<tr><td>5)</td><td>$f'(x) = \dfrac{-\frac{(\ln x + 1)}{x} - \frac{(1-\ln x)}{x}}{(\ln x + 1)^2} = \dfrac{\frac{-\ln x - 1 - 1 + \ln x}{x}}{(\ln x + 1)^2} = \dfrac{-2}{x(\ln x + 1)^2} < 0$ for every x belongs to its domain.
f is continuous and strictly decreasing over its domain, and in particular over $\left[1;e^2\right]$
$f(1) = \dfrac{1-\ln 1}{\ln 1 + 1} = \dfrac{1-0}{0+1} = 1$.
$f(e^2) = \dfrac{1-\ln\left(e^2\right)}{\ln\left(e^2\right)+1} = \dfrac{1-2\ln e}{2\ln e + 1} = \dfrac{1-2}{2+1} = -\dfrac{1}{3}$.
Thus $f\left(\left[1;e^2\right]\right) = \left[f(e^2);f(1)\right] = \left[-\dfrac{1}{3};1\right]$. This part is answered correctly.</td><td>1</td></tr>
</table>

QII	Answers	6 ½ pts.
A.1)	$P(A)=\frac{3}{6}=\frac{1}{2}$. $P(B)=\frac{2}{6}=\frac{1}{3}$. $P(C)=\frac{1}{6}$.	¾
A.2)a)	$P(R\cap A)=P(R/A)\times P(A)=\frac{3!}{2!}P(RR\overline{R})\times\frac{1}{2}=\frac{3!}{2!}\times\frac{4}{7}\times\frac{4}{7}\times\frac{3}{7}\times\frac{1}{2}=\frac{72}{343}$. $P(R\cap B)=P(R/B)\times P(B)=P(2R\ \&\ 1\overline{R})\times\frac{1}{3}=\frac{C_3^2\times C_5^1}{C_8^3}\times\frac{1}{3}=\frac{5}{56}$. $P(R\cap C)=P(R/C)\times P(C)=\frac{4}{7}\times\frac{3}{8}\times\frac{1}{6}=\frac{1}{28}$.	¾ ¾ ½
A.2)b)	$P(R)=P(R\cap A)+P(R\cap B)+P(R\cap C)=\frac{72}{343}+\frac{5}{56}+\frac{1}{28}=\frac{919}{2744}$.	½
A.3)a)	$P(M)=P(M\cap A)+P(M\cap B)+P(M\cap C)=$ $P(M/A)\times P(A)+P(M/B)\times P(B)+P(M/C)\times P(C)=$ $\left[\left(\frac{4}{7}\right)^3+\left(\frac{2}{7}\right)^3+\left(\frac{1}{7}\right)^3\right]\times\frac{1}{2}+\frac{C_3^3+C_3^3}{C_8^3}\times\frac{1}{3}+\left[\left(\frac{4}{7}\times\frac{3}{8}\right)+\left(\frac{2}{7}\times\frac{2}{8}\right)+\left(\frac{1}{7}\times\frac{3}{8}\right)\right]\times\frac{1}{6}=\frac{2879}{16464}$.	1
A.3)b)	$P(M\cup B)=P(M)+P(B)-P(M\cap B)=\frac{2879}{16464}+\frac{1}{3}-\frac{C_3^3+C_3^3}{C_8^3}\times\frac{1}{3}=\frac{8171}{16464}$.	½
A.3)c)	$P(\overline{B}/\overline{M})=\frac{P(\overline{B}\cap\overline{M})}{P(\overline{M})}=\frac{P(\overline{B\cup M})}{P(\overline{M})}=\frac{1-P(B\cup M)}{1-P(M)}=\frac{8293}{13585}$.	¾
B.	To get zero sum, means to select 3 red balls each carrying 0 point or to select 1 white ball, 1 black ball and 1 red ball carrying +1; – 1 and 0 point respectively. Thus P (sum of scored points is equal to zero) = $P(RRR)+\frac{3!}{1!\times1!\times1!}P(WBR)=\frac{A_7^3}{A_{15}^3}+\frac{3!}{1!\times1!\times1!}\times\frac{A_4^1\times A_4^1\times A_7^1}{A_{15}^3}=\frac{21}{65}$.	1

QIII	Answers	9 pts.
A.1)	$\lim\limits_{x\to-\infty}g(x)=\lim\limits_{x\to-\infty}(2x-1)e^{-x+1}-1=-\infty\times e^{+\infty}=-\infty$. $\lim\limits_{x\to+\infty}g(x)=\lim\limits_{x\to+\infty}(2x-1)e^{-x+1}-1=0-1=-1$; since $\lim\limits_{x\to+\infty}(2x-1)e^{-x+1}=\lim\limits_{x\to+\infty}\frac{(2x-1)}{e^{x-1}}\overset{\text{H.R}}{=}\lim\limits_{x\to+\infty}\frac{2}{e^{x-1}}=0$.	¼ ½
A.2)	$g'(x)=2e^{-x+1}-(2x-1)e^{-x+1}=(3-2x)e^{-x+1}$, same sign as $(3-2x)$ since $e^{-x+1}>0$ for every $x\in\mathbb{R}$. Table of variations of g : $\begin{array}{c\|ccccc} x & -\infty & & 1.5 & & +\infty \\ \hline g'(x) & & + & 0 & - & \\ \hline g(x) & -\infty & \nearrow & 2e^{-\frac{1}{2}}-1 & \searrow & -1 \end{array}$	

A.3)	• Over $]-\infty;1.5[$, g is continuous , strictly increasing and changes its sign from negative $(-\infty)$ to positive $(2e^{-\frac{1}{2}}-1)$ then the equation $g(x)=0$ admits a unique solution over $]-\infty;1.5[$. • Over $]1.5;+\infty[$, g is continuous , strictly decreasing and changes its sign from positive $(2e^{-\frac{1}{2}}-1)$ to negative (-1) then the equation $g(x)=0$ admits a unique solution α over $]1.5;+\infty[$ • In addition: $g(1)=0$, then 1 is a root. Then the equation $g(x)=0$ admits exactly two solutions 1 and α . • $g(2.2)\approx 0.024>0$ and $g(2.4)\approx -0.063<0$, therefore $2.2<\alpha<2.4$.	**1**
A.4)	• Over $]-\infty;1[$, g increases from negative $(-\infty)$ to positive (0) , thus $g(x)<0$ over $]-\infty;1[$. • Over $]1;\alpha[$, g increases from positive (0) to positive $(2e^{-\frac{1}{2}}-1)$ then decreases from positive $(2e^{-\frac{1}{2}}-1)$ to positive (0) , thus $g(x)>0$ over $]1;\alpha[$. • Over $]\alpha;+\infty[$, g decreases from positive (0) to negative (-1) , thus $g(x)<0$ over $]\alpha;+\infty[$. **<u>Conclusion:</u>** • $g(x)<0$ for $x\in]-\infty;1[\cup]\alpha;+\infty[$. • $g(x)=0$ for $x\in\{1;\alpha\}$. • $g(x)>0$ for $x\in]1;\alpha[$.	**1**
B.1)	$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}(4x+2)e^{-x+1}+2x-1=-\infty-\infty=-\infty.$ $f(2)\approx 6.68$.	**½**
B.2)a)	$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}(4x+2)e^{-x+1}+2x-1=0+\infty=+\infty$ since $\lim\limits_{x\to+\infty}(4x+2)e^{-x+1}=\lim\limits_{x\to+\infty}\frac{(4x+2)}{e^{x-1}}\overset{\text{H.R}}{=}\lim\limits_{x\to+\infty}\frac{4}{e^{x-1}}=0.$ $\lim\limits_{x\to+\infty}\left[f(x)-y_{(d)}\right]=\lim\limits_{x\to+\infty}\left[(4x+2)e^{-x+1}+2x-1-2x+1\right]=\lim\limits_{x\to+\infty}(4x+2)e^{-x+1}=0$, thus the line (d) of equation $y=2x-1$ is an oblique asymptote to (C).	**¾**
B.2)b)	$f(x)-y_{(d)}=(4x+2)e^{-x+1}$, same sign as $(4x+2)$ since $e^{-x+1}>0$ for every $x\in\mathbb{R}$. • $f(x)-y_{(d)}<0$ for $x<-\frac{1}{2}$; (C) is below (d) if $x\in\left]-\infty;-\frac{1}{2}\right[$. • $f(x)-y_{(d)}>0$ for $x>-\frac{1}{2}$; (C) is above (d) if $x\in\left]-\frac{1}{2};+\infty\right[$. • $f(x)-y_{(d)}=0$ for $x=-\frac{1}{2}$; (C) cuts (d) at point of coordinates $\left(-\frac{1}{2};-2\right)$.	**¾**
B.3)	$f'(x)=4e^{-x+1}-(4x+2)e^{-x+1}+2=-(4x-2)e^{-x+1}+2=-2\left[(2x-1)e^{-x+1}-1\right]$ $=-2g(x)$, verified. $f'(x)$ and $g(x)$ have opposite sign over $\mathbb{R}$.	**1 ¼**

Table of variations of f :

x	$-\infty$		1		α		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	
$f(x)$	$-\infty$	$\nearrow$	7	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

B.4)

$f''(x)=-2g'(x)=-2(3-2x)e^{-x+1}=2(2x-3)e^{-x+1}$, same sign as $(2x-3)$ since $e^{-x+1}>0$ for every $x\in\mathbb{R}$.

$f''\left(\frac{3}{2}\right)=0$; $f\left(\frac{3}{2}\right)=8e^{-\frac{1}{2}}+2$.

f'' vanishes at $x=\frac{3}{2}$ and changes its sign, thus (C) admits an inflection point I whose coordinates are $\left(\frac{3}{2};8e^{-\frac{1}{2}}+2\right)$.

½

B.5)

$(T): y=f'\left(\frac{3}{2}\right)\left(x-\frac{3}{2}\right)+f\left(\frac{3}{2}\right)$; $f\left(\frac{3}{2}\right)=8e^{-\frac{1}{2}}+2$ and $f'\left(\frac{3}{2}\right)=-4e^{-\frac{1}{2}}+2$.

Thus $(T): y=\left(-4e^{-\frac{1}{2}}+2\right)\left(x-\frac{3}{2}\right)+8e^{-\frac{1}{2}}+2$; $(T): y=\left(-4e^{-\frac{1}{2}}+2\right)x+14e^{-\frac{1}{2}}-1$.

½

B.6)

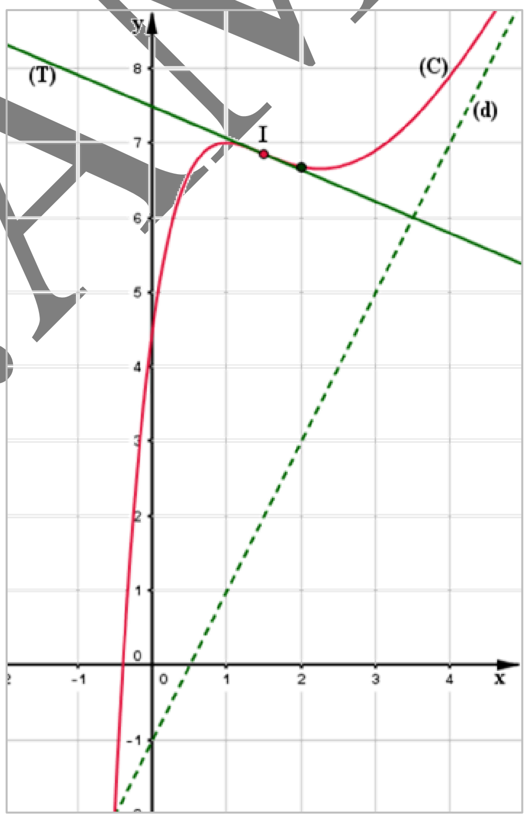
$g(\alpha)=0$; $(2\alpha-1)e^{-\alpha+1}-1=0$; $e^{-\alpha+1}=\frac{1}{2\alpha-1}$.

$f(\alpha)=(4\alpha+2)e^{-\alpha+1}+2\alpha-1=\frac{4\alpha+2}{2\alpha-1}+2\alpha-1=\frac{4\alpha+2+(2\alpha-1)^2}{2\alpha-1}=$

$\frac{4\alpha+2+4\alpha^2-4\alpha+1}{2\alpha-1}=\frac{4\alpha^2+3}{2\alpha-1}$.

½

B.7)



1 ½