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Math - Grade 12 - GS

Ch.:15 Integral

Ch.:16 Properties of the Integral

Ch.:17 Integration Techniques

Ch:19 Applications to Integration

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a: $\int x^n dx = \frac{1}{n+1}x^{n+1} + c \quad (n \neq -1)$

b: $\int U'\,U^n dx = \frac{1}{n+1}U^{n+1} + c \quad (n \neq -1)$

c: $\int \frac{1}{x}dx = ln|x| + c$

d: $\int \frac{U'}{U}dx = ln|U| + c$

e: $\int e^x dx = e^x + c$

f: $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + c$

g: $\int U'e^U dx = e^U + c$

Eg1: $\int x^3 dx = \frac{x^4}{4} + c$

Eg2: $\int x^2 dx = \frac{x^3}{3} + c$

Eg3: $\int x dx = \frac{x^2}{2} + c$

Eg4: $\int dx = \int x^0\, dx = x + c$,

Eg5: $\int 5(5x-8)dx$ $\qquad u = 5x - 8 \rightarrow u' = 5$

$= \frac{(5x-8)^2}{2} + c$

Eg6: $\int (2x+1)(x^2+x+3)^5 dx$ $\qquad u = x^2 + x + 3 \rightarrow u' = 2x + 1$

$= \frac{(x^2+x+3)^6}{6} + c$

Eg7: $\int (x-1)(x^2-2x+3)^6 dx$ $\qquad u = x^2 - 2x + 3 \rightarrow u' = 2x - 2 = 2\,(x-1)$

$= \frac{1}{2}\int 2(x-1)(x^2-2x+3)^6 dx$

$= \frac{1}{2}\frac{(x^2-2x+3)^7}{7} + c = \frac{(x^2-2x+3)^7}{14} + c$

Eg8: $\int [4x - \frac{1}{x}]\, dx = 4\frac{x^2}{2} - ln|x| + c = 2x^2 - ln|x| + c$

Eg9: $\int \frac{x^2+3x-2}{x} dx = \int \left[x + 3 - \frac{2}{x}\right] dx = \frac{x^2}{2} + 3x - 2ln|x| + c$

Eg10: $\int \frac{2x-1}{x^2-x+3} dx = ln|x^2 - x + 3| + c$ $\qquad$ Let $U = x^2 - x + 3$ then $U' = 2x - 1$

Eg11: $\int \frac{x-1}{x^2-2x+3} dx =$ $\qquad$ Let U = $x^2 - 2x + 3$ then $U' = 2x - 2 = 2(x-1)$

$$=\frac{1}{2}\int\frac{2(x-1)}{x^2-2x+3}dx=\frac{1}{2}ln|x^2-2x+3|+c$$

Eg12: $\int[4\mathrm{x}-\mathrm{e}^{\mathrm{x}}]\mathrm{dx}=2\mathrm{x}^2-\mathrm{e}^{\mathrm{x}}+\mathrm{c}$

Eg13: $\int[6\mathrm{x}+3-5\mathrm{e}^{\mathrm{x}}]\mathrm{dx}=3\mathrm{x}^2+3\mathrm{x}-5\mathrm{e}^{\mathrm{x}}+\mathrm{c}$

Eg14: $\int \mathrm{e}^{4\mathrm{x}-1}\mathrm{dx}=\frac{1}{4}\mathrm{e}^{4\mathrm{x}-1}+\mathrm{c}$

Eg15: $\int(2\mathrm{x}+1)\mathrm{e}^{\mathrm{x}^2+\mathrm{x}}\mathrm{dx}=\mathrm{e}^{\mathrm{x}^2+\mathrm{x}}+\mathrm{c}$ $\quad \mathrm{U}=\mathrm{x}^2+\mathrm{x}$ $\quad \mathrm{U}'=2\mathrm{x}+1$

Eg16: $\int(\mathrm{x}+1)\mathrm{e}^{\mathrm{x}^2+2\mathrm{x}+1}\mathrm{dx}$ $\quad U=x^2+2x+1$ then $U'=2(x+1)$

$$=\frac{1}{2}\int 2(\mathrm{x}+1)\mathrm{e}^{\mathrm{x}^2+2\mathrm{x}+1}\mathrm{dx}=\frac{1}{2}\mathrm{e}^{\mathrm{x}^2+2\mathrm{x}+1}+\mathrm{c}$$

Definite integrals

1 - Definition Let f be a continuous function on an interval I.

Let F be the anti-derivative of f. a and b are two points in I.

The integral of f from a to b, denoted by $\int_a^b f(x)dx$, is

$$\boldsymbol{\int_a^b f(x)dx=[F(x)]_a^b=F(b)-F(a)}$$

Example: Calculate the following integrals

A = $\int_1^2(4x-1)dx$

B = $\int_1^3(x+1)(x^2+2x+3)^4dx$

C = $\int_1^e\frac{1}{x}dx$

D = $\int_0^1\frac{2x+1}{x^2+x+1}dx$

$A = \int_1^2 (4x-1)\,dx = \left[4\frac{x^2}{2} - x\right]_1^2$

$= \left[2x^2 - x\right]_1^2 = \left[2(2)^2 - (2)\right] - \left[2(1)^2 - (1)\right]$

$= 6 - 1 = 5$

$B = \int_1^3 (x+1)(x^2+2x+3)^4\,dx$

$u = x^2+2x+3$

$u' = 2x+2 = 2(x+1)$

$= \frac{1}{2}\int_1^3 2(x+1)(x^2+2x+3)^4\,dx = \left[\frac{1}{2}\,\frac{(x^2+2x+3)^5}{5}\right]_1^3$

$= \frac{1}{10}(3^2+2(3)+3)^5 - \frac{1}{10}(1^2+2(1)+3)^5 = 188179.2$

$C = \int_1^e \frac{1}{x}\,dx = [\ln x]_1^e$

$= \ln e - \cancel{\ln 1}^{0} = 1$

$D = \int_0^1 \frac{2x+1}{x^2+x+1}\,dx$

$u = x^2+x+1$

$u' = 2x+1$

$= \left[\ln(x^2+x+1)\right]_0^1$

$= \ln(1^2+1+1) - \ln(0^2+0+1)$

$= \ln 3$

2 - Properties

1: $\int_a^b dx = b - a$

2: $\int_a^a f(x)dx = 0$

3: $\int_a^b f(x)dx = -\int_b^a f(x)dx$

4: $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha \int_a^b f(x)dx + \beta \int_a^b g(x)dx$

5: $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$ where a, b and c are points in I.

6: If f $\geq$ 0 on [a, b] then $\int_a^b f(x)dx \geq 0$

7: If f(x) $\leq$ g(x) on [a, b] then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

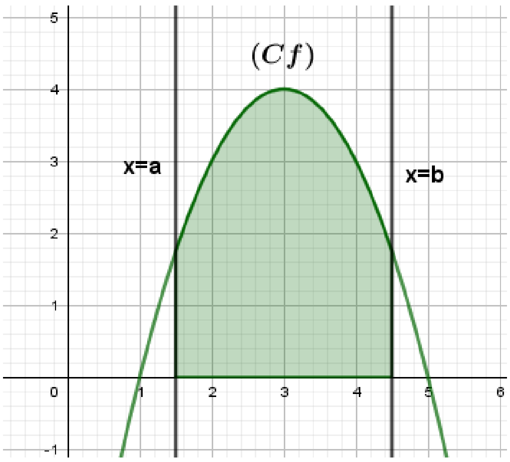
8: If f is even on [–a, a] then $\int_{-a}^{a} f(x)dx = 2\int_0^a f(x)dx$

9: If f is odd on [–a, a] then $\int_{-a}^{a} f(x)dx = 0$

****GS-Solve** $\frac{1-3-5}{150}$ $\frac{6(1-4)}{150}$ $\frac{2}{159}$ $\frac{4(1-2-5)}{159}$ $\frac{5(1-3-5)}{159}$ $\frac{6(1-5-7)}{159}$ $\frac{7(6)}{159}$ $\frac{8(4)}{159}$

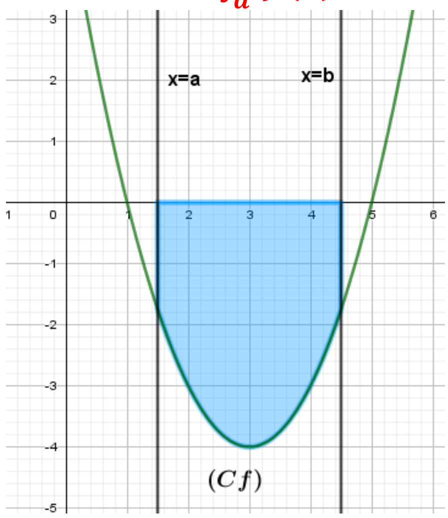
3- Graphical interpretation of the integral

Case1: Let f be a positive function over [a, b]. Area A of the region bounded between (C_f), x'ox & x = a & x = b is **A** = $\int_a^b f(x)dx$



Case 2: If f < 0 on [a,b]

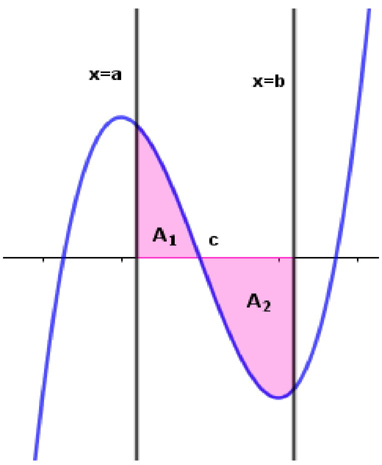
then **A** = $-\int_a^b f(x)dx$



Case3:

$A = A_1 + A_2$

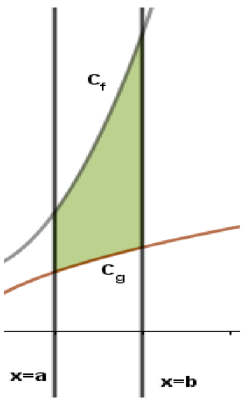
$= \int_a^c f(x)dx - \int_c^b f(x)dx$



Case4:

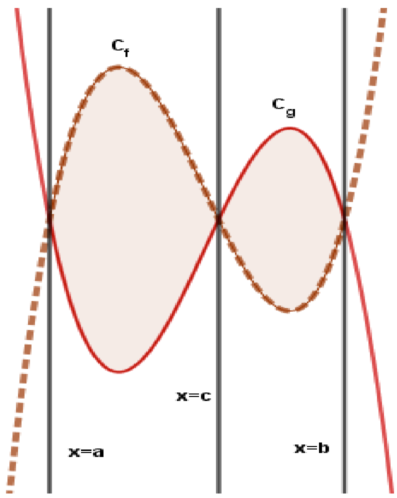
Area of the region bounded between (C_f), (C_g), x = a, and x = b, is

A=$\int_a^b [f(x) - g(x)]dx$

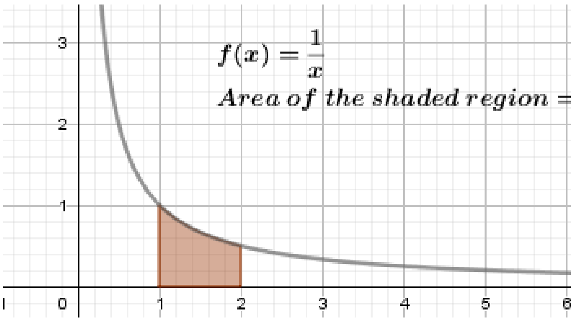


Case5:

A=$\int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx$



3Eg1: Calculate the shaded area.



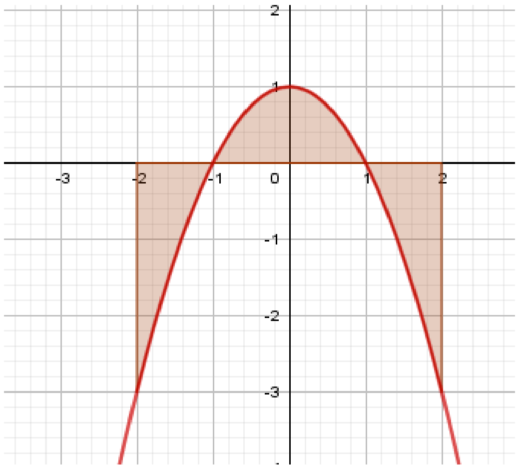
Solution:

$$A = \int_1^2 \frac{1}{x}\,dx$$

A = [ln$x$] = ln2 – ln1 = ln2 unit^2

3Eg2: Calculate the shaded area.

f(x) = 1 – x^2… (C)



Solution:

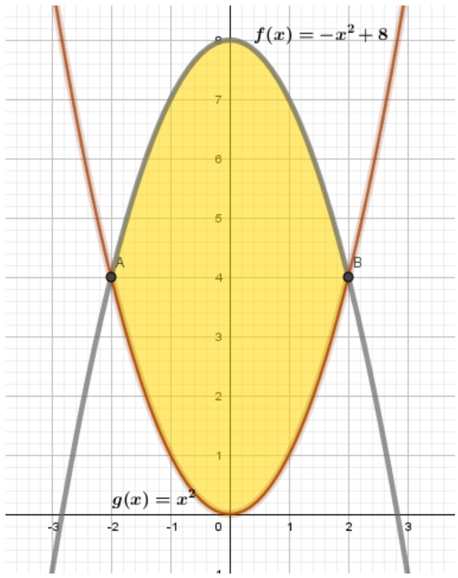
$$A = -\int_{-2}^{-1}(1-x^2)\,dx + \int_{-1}^{1}(1-x^2)\,dx - \int_{1}^{2}(1-x^2)\,dx$$

$$= -\left[x - \frac{x^3}{3}\right]_{-2}^{-1} + \left[x - \frac{x^3}{3}\right]_{-1}^{1} - \left[x - \frac{x^3}{3}\right]_{1}^{2}$$

$$= \frac{4}{3} + \frac{4}{3} + \frac{4}{3} = 4 \text{ unit}^2$$

3Eg3: Calculate the shaded area.

f(x) = – x^2 + 8 & g(x) = x^2



Solution:

Let's find the abscissas of the points of intersection of the two curves.

$-x^2 + 8 = x^2$ then $2x^2 = 8$, $x^2 = 4$, $x = \pm 2$

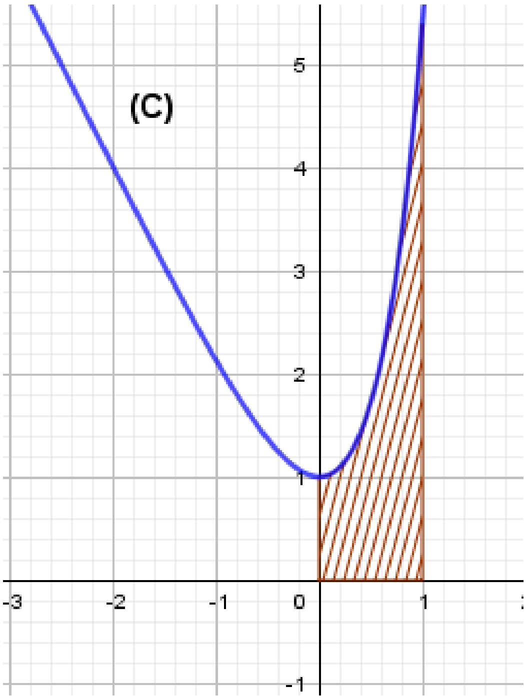
A=$\int_{-2}^{2}[f(x) - g(x)]dx = \int_{-2}^{2}[-x^2 + 8 - x^2]dx$

$=\int_{-2}^{2}[-2x^2 + 8]dx$

$= [-\frac{2x^3}{3} + 8x]_{-2}^{2} = \frac{64}{3}\ unit^2$

3Eg4: Calculate the shaded area.

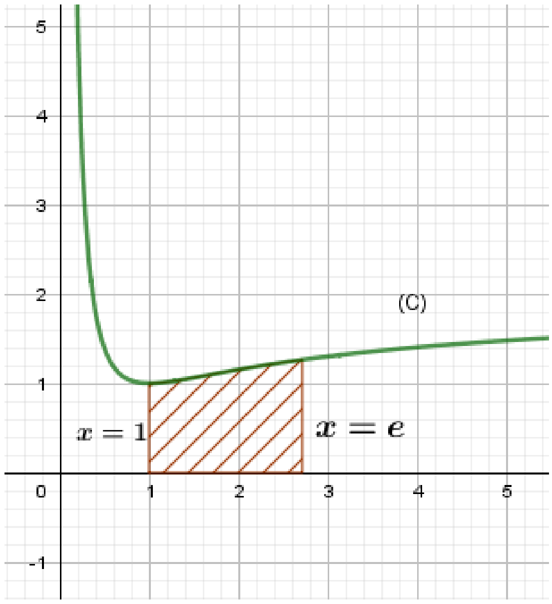
$f(x) = e^{2x} - 2x$……(C)



$$A=\int_0^1 (e^{2x} - 2x)$$
$$= \left[\frac{1}{2}e^{2x} - x^2\right]_0^1$$
$$=\left(\frac{1}{2}e^2 - 1\right) - \left(\frac{1}{2}e^0 - 0\right)$$
$$= \frac{1}{2}e^2 - \frac{3}{2}\ unit^2$$

3Eg5 : $f(x) = 2 - \frac{1}{x} - \frac{lnx}{x}$ ……(C)

Calculate the area of the shaded region.



$$A= \int_1^e f(x)dx=\int_1^e [2 - \frac{1}{x} - \frac{lnx}{x}]dx$$
$$= \left[2x - lnx - \frac{(lnx)^2}{2}\right]_1^e$$
$$=\left[2e - lne - \frac{(lne)^2}{2}\right] - \left[2 - ln1 - \frac{(ln1)^2}{2}\right]$$
$$=2e - 1 - \frac{1}{2} - 2$$
$$=2e - 3.5 \text{ unit}^2$$

4-Fundamental theorem of integral calculus

Rule: f: continuous on I. a is a constant. a∈I.

i: If F(x) = $\int_a^x f(t)dt$. Then F'(x) = f(x) and F(a)=0

ii: If F(x) = $\int_a^{u(x)} f(t)dt$. Then F'(x) = u'(x). f(u(x))

Eg1: Calculate $F'(x)$ in each of the following examples.

1: F(x) = $\int_a^x t^2dt$ → F'(x) = x^2.

2: F(x) = $\int_x^1 \sqrt{2t+6}dt$

F(x) = $-\int_1^x \sqrt{2t+6}dt$ → F'(x) = $-\sqrt{2x+6}$

3: F(x) = $\int_3^x \frac{e^{2t}}{1+te^t}dt$ → F'(x) = $\frac{e^{2x}}{1+xe^x}$

4: F(x) = $\int_1^{x^2} \frac{t^3+2}{1+t}dt \rightarrow F'(x) = \frac{(x^2)^3+2}{1+x^2}(x^2)' = \frac{(x^2)^3+2}{1+x^2}(2x) = 2x\frac{x^6+2}{1+x^2}$

Eg2: Given F(x) = $\int_1^x \frac{1}{1+t^2}dt$. Determine the sense of variation of F.

$\rightarrow F'(x) = \frac{1}{1+x^2} > 0$. Then F is increasing.

Eg3: Determine the following limits

a: $\lim\limits_{x\to 0} \frac{\int_0^x ln(t+1)dt}{e^x-1} = \frac{\int_0^0 ln(t+1)dt}{e^0-1} = \frac{0}{0}$ I.F

$\lim\limits_{x\to 0} \frac{\int_0^x ln(t+1)dt}{e^x-1} = \lim\limits_{x\to 0} \frac{[\int_0^x ln(t+1)dt]'}{(e^x-1)'} = \lim\limits_{x\to 0} \frac{ln(x+1)}{e^x} = \frac{ln1}{1} = 0$

b: $\lim\limits_{x\to 1} \frac{\int_1^x \sqrt[3]{t^2-1}dt}{x-1} = \frac{\int_1^1 \sqrt[3]{t^2-1}dt}{1-1} = \frac{0}{0}$ I.F

$\lim\limits_{x\to 1} \frac{\int_1^x \sqrt[3]{t^2-1}dt}{x-1} = \lim\limits_{x\to 1} \frac{[\int_1^x \sqrt[3]{t^2-1}dt]'}{1} = \lim\limits_{x\to 1} \sqrt[3]{x^2-1} = 0$

Eg4: Given T(x) = $\int_1^{2x}(\sqrt{1+3\,ln^2 t})dt$ $x > 0$. Calculate $T'(0.5e)$.

$T'(x) = (2x)'\sqrt{1+3\,ln^2 2x} = 2\sqrt{1+3\,ln^2 2x}$

$T'(0.5e) = 2\sqrt{1+3\,ln^2 2(0.5e)} = 4$

****GS-Solve** $\frac{8}{150}$ $\frac{9}{151}$

5-Comparison of integrals

Recall:

If f ≥ 0 on [a, b] then $\int_a^b f(x)dx \geq 0$

If f(x) ≤ g(x) on [a, b] then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$

Example: Given $x \in [1,2]$

a: Show that $\frac{1}{17} \leq \frac{1}{1+x^4} \leq \frac{1}{2}$

b: Deduce The bounding of the integral $\int_1^2 \frac{1}{1+x^4}dx$

a: $1 \leq x \leq 2,\quad 1 \leq x^4 \leq 16,\quad 2 \leq 1 + x^4 \leq 17\ ,\quad \frac{1}{17} \leq \frac{1}{1+x^4} \leq \frac{1}{2}$

b: $\int_1^2 \frac{1}{17}dx \leq \int_1^2 \frac{1}{1+x^4}dx \leq \int_1^2 \frac{1}{2}dx$

$$\frac{1}{17}[x]_1^2 \leq \int_1^2 \frac{1}{1+x^4}dx \leq \frac{1}{2}[x]_1^2$$

$$\frac{1}{17} \leq \int_1^2 \frac{1}{1+x^4}dx \leq \frac{1}{2}$$

****GS-Solve** $\frac{9}{151}$

6- Integral of expression of the form $\frac{1}{a+e^x}$ where a is a constant.

Activity:

Calculate $\int_0^1 \frac{1}{3+e^x}dx$

$$\int_0^1 \frac{1}{3+e^x}dx = \frac{1}{3}\int_0^1 \frac{3}{3+e^x}dx = \frac{1}{3}\int_0^1 \frac{3+e^x-e^x}{3+e^x}dx = \frac{1}{3}\int_0^1 [1-\frac{e^x}{3+e^x}]dx =$$

$$= \frac{1}{3}[x - \ln(3+e^x)]_0^1 = \frac{1}{3}[1 - \ln(3+e) + ln4]$$

Calculate $\int_0^5 \frac{1}{1+e^x}dx$

$$\int_0^5 \frac{1}{1+e^x}dx = \int_0^5 \frac{1+e^x-e^x}{1+e^x}dx = \int_0^5 [1-\frac{e^x}{1+e^x}]dx =$$

$$= [x - \ln(1+e^x)]_0^5 = [5 - \ln(1+e^5) + ln2]$$

7-Integration by parts

$$\int U'V dx = UV - \int V'U dx$$

$$\int_a^b U'V dx = [UV]_a^b - \int_a^b V'\, U dx$$

Example:

$\int xlnxdx$: Let $U' = x$ and $V = lnx$

Then $U = \frac{x^2}{2}$ and $V' = \frac{1}{x}$

$$\int xlnxdx = \frac{x^2}{2}lnx - \int \frac{1}{x}\frac{x^2}{2}dx = \frac{x^2}{2}lnx - \int \frac{x}{2}dx = \frac{x^2}{2}lnx - \frac{x^2}{4} + c$$

Example:

$\int_1^e x^2 lnxdx$: Let $U' = x^2$ and $V = lnx$

Then $U = \frac{x^3}{3}$ and $V' = \frac{1}{x}$

$$\int_1^e x^2 lnxdx = \left[\frac{x^3}{3}lnx\right]_1^e - \int_1^e \frac{1}{x}\frac{x^3}{3}dx = \frac{e^3}{3}lne - \frac{1^3}{3}ln1 - \int_1^e \frac{x^2}{3}dx$$

$$= \frac{e^3}{3} - \left[\frac{x^3}{9}\right]_1^e = \frac{e^3}{3} - \left(\frac{e^3}{9} - \frac{1}{9}\right) = \frac{2e^3}{9} + \frac{1}{9}$$

Example:

$\int lnxdx$: Let $U' = 1$ and $V = lnx$

Then $U = x$ and $V' = \frac{1}{x}$

$$\int lnxdx = xlnx - \int \frac{1}{x}xdx = xlnx - \int dx = xlnx - x + c$$

$$\boxed{\int lnxdx = xlnx - x + c}$$

Example:

$\int xe^{2x}dx$: Let $U' = e^{2x}$ $\quad V = x$

$U = \frac{1}{2}e^{2x}$ $\quad V' = 1$

$$\int xe^{2x}dx = \frac{1}{2}e^{2x}x - \int \frac{1}{2}e^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

Second way: Tabular integration by parts

	Monomial		Exponential	
Derive until zero	x	+	e^{2x}	antiderivative
	1	-	$\frac{1}{2}e^{2x}$	
	0		$\frac{1}{4}e^{2x}$	

Then $\int xe^{2x}dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$

Example:

$$\int x^4 e^{-2x} dx$$

Monomial		Exponential
x^4	$+$	$e^{-2x}dx$
$4x^3$	$-$	$-\frac{1}{2}e^{-2x}$
$12x^2$	$+$	$\frac{1}{4}e^{-2x}$
$24x$	$-$	$-\frac{1}{8}e^{-2x}$
24	$+$	$\frac{1}{16}e^{-2x}$
0		$-\frac{1}{32}e^{-2x}$

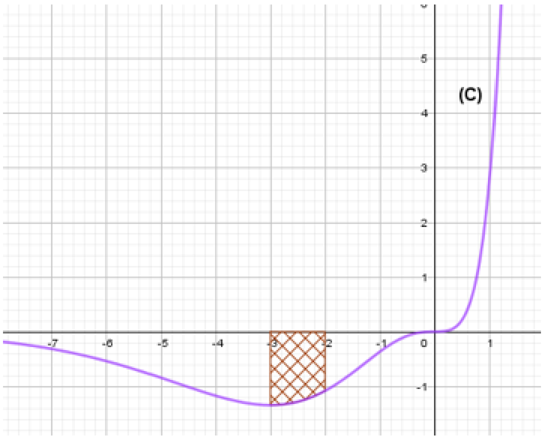
Then

$$\int x^4e^{-2x}dx = x^4 \cdot -\frac{1}{2}e^{-2x} - 4x^3 \cdot \frac{1}{4}e^{-2x} + 12x^2 \cdot -\frac{1}{8}e^{-2x} - 24x \cdot \frac{1}{16}e^{-2x} + 24 \cdot -\frac{1}{32}e^{-2x} + C$$

$$= -\frac{1}{2}x^4e^{-2x} - x^3e^{-2x} - \frac{3}{2}x^2e^{-2x} - \frac{3}{2}xe^{-2x} + -\frac{3}{4}e^{-2x} + C$$

$$= -\frac{1}{4}e^{-2x}\left(2x^4 + 4x^3 + 6x^2 + 6x + 3\right) + C$$

Example:



(C) Is the representative curve of the function f where $f(x) = x^3e^x$

Calculate the area of the shaded region

Area= $-\int_{-3}^{-2} f(x)dx \ unit^2$

x^3	$+$	e^x
$3x^2$	$-$	e^x
$6x$	$+$	e^x
6	$-$	e^x
0		e^x

$$\int x^3e^x\,dx = x^3e^x - 3x^2e^x + 6xe^x - 6e^x + c$$

$$= e^x(x^3 - 3x^2 + 6x - 6) + c$$

Then Area= $-[e^x(x^3 - 3x^2 + 6x - 6)]_{-3}^{-2}$

$= -[e^{-2}(-38) - e^{-3}(-78)]$
$= 38e^{-2} - 78e^{-3}unit^2$

****GS-Solve** $\frac{11(1-2-4-6-7-8)}{160}$ $\frac{12(2-3)}{160}$

8- Integration of rational functions $f(x) = \frac{P(x)}{Q(x)}$

1st case: If deg P(x) < deg Q(x):

i: Of the form $\int \frac{u'}{u}dx$

Eg $\int \frac{x+1}{x^2+2x-5}dx$ u = $x^2 + 2x - 5 \to$ u'=$2x + 2 = 2(x + 1)$

$= \frac{1}{2}\int \frac{2\,(x+1)}{x^2+2x-5}dx = \frac{1}{2}\ln|x^2 + 2x - 5| + c$

ii: If the denominator is quadratic with $\Delta = 0$ $\to$ Eg: $\int \frac{1}{x^2+6x+9}dx$

$$\int \frac{1}{x^2+6x+9}dx = \int \frac{1}{(x+3)^2}dx = \int (x+3)^{-2}dx = \frac{(x+3)^{-1}}{-1} + c = \frac{-1}{x+3} + c$$

iii: If the denominator is quadratic with $\Delta > 0$

Expand f(x) into a sum of partial fractions & then integrate.

→ Eg $\int \frac{4}{x^2-2x-3}dx$

$$\frac{4}{x^2-2x-3} = \frac{4}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{A(x-3)+B\,(x+1)}{(x+1)\,(x-3)} = \frac{Ax-3A+Bx+B}{(x+1)\,(x-3)}$$

$= \frac{x(A+B)-3A+B}{(x+1)\,(x-3)}$. Then:

$$\left.\begin{aligned} A+B&=0 \\ -3A+B&=4 \end{aligned}\right\} \quad \begin{aligned} A&=-1 \\ B&=1 \end{aligned}$$

Then $\int \frac{4}{x^2-2x-3}dx = \int\left(\frac{-1}{x+1} + \frac{1}{x-3}\right)dx = -\ln|x+1| + \ln|x-3| + c$

2nd case: If deg P(x) ≥ deg Q(x): **Use Euclidian division & then use the method of partial fraction then integrate.**

Eg: $\int \frac{2x^2-3x+5}{x^2-1}dx$ Using long division … $\frac{2x^2-3x+5}{x^2-1} = 2 + \frac{-3x+7}{x^2-1}$

$$\frac{-3x+7}{x^2-1} = \frac{-3x+7}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} = \frac{A(x+1)+B\,(x-1)}{(x-1)\,(x+1)} = \frac{Ax+A+Bx-B}{(x-1)\,(x+1)} = \frac{x(A+B)+A-B}{(x-1)\,(x+1)}$$

Then

$$\left.\begin{aligned} A+B&=-3 \\ A-B&=7 \end{aligned}\right\} \quad \begin{aligned} A&=2 \\ B&=-5 \end{aligned}$$

$$\int \frac{2x^2-3x+5}{x^2-1}dx = \int\left(2 + \frac{2}{x-1} + \frac{-5}{x+1}\right)dx = 2x + 2\ln|x-1| - 5\ln|x+1| + c$$

****GS-Solve** $\frac{ex10(1-3-4)}{172}$ $\frac{ex\ 11(1-2)}{172}$ $\frac{ex\ 12}{172}$.

THANK YOU

Hnaini