- *This exam consists of three problems inscribed on two pages.*
- *The use of a non-programmable calculator is allowed.*

I – (4 points)

In the table below, only one of the proposed answers to each question, is correct.

Write down the number of each question and give, **with justification**, the answer corresponding to it.

Nº	Questions	Answers a	Answers b	Answers c
1)	$\lim\limits_{x\to e^+}\dfrac{(\ln x)^3-1}{(\ln x)^2-1}$ is equal to	$\dfrac{1}{e}$	1	$\dfrac{3}{2}$
2)	The equation $(\ln x)^2+3\ln x=4$ has two roots x_1 and x_2. The product $x_1.x_2$ is equal to	-4	e^{-4}	e^{-3}
3)	The inequality $\ln x>\ln(2x-1)$ is verified in the interval	$]-\infty;1[$	$\left]\dfrac{1}{2};1\right[$	$\left]\dfrac{1}{2};1\right]$
4)	Let A and B be two events of a sample space Ω and p be a probability such that $p(A)=\dfrac{19}{24}$, $p(B)=\dfrac{1}{4}$ and $p(A/\overline{B})=\dfrac{1}{2}$, then $P(A\cap B)=$	$\dfrac{5}{12}$	$\dfrac{3}{8}$	$\dfrac{7}{12}$

II – (6 points)

Three balls are drawn simultaneously and randomly from an urn containing three white, three black, three green and three red balls. Suppose that the draws are equiprobable.

All results will be given as irreducible fractions.

1) Consider the events:

A: "The drawn balls contain exactly two white balls";
B: "The drawn balls have the same color";
C: "The drawn balls contain at least two white balls".

a) Calculate $P(A)$ and $P(B)$.

b) Show that $P(C)=\dfrac{7}{55}$.

2) A game is played such as:

- If a player isn't a cheater, he wins the game when he draws at least two white balls.
- If he's a cheater, he wins the game with a probability equals to $\dfrac{1}{2}$.
- A player out of 10 is a cheater.

Consider the events:
T : "The player is a cheater";
G : "The player wins the game".

a) Calculate the probability of the event "the player wins the game knowing that he is not a cheater".
Deduce that the probability of the event $G \cap \overline{T}$ is $\frac{63}{550}$.

b) Calculate $P(G \cap T)$.

c) Prove that the probability of the event G is $\frac{181}{1100}$.

d) Calculate the probability that a player is a cheater knowing that he has won the game.

III – (10 points)

Part A

Consider the function g defined over $\mathbb{R}$ by: $g(x) = 1 - (1+x)e^x$.

1) Calculate $\lim\limits_{x \to +\infty} g(x)$ and $\lim\limits_{x \to -\infty} g(x)$.

2) Calculate $g'(x)$, then set up the table of variations of g.

3) Calculate $g(0)$, then study according to the values of x in $\mathbb{R}$ the sign of $g(x)$.

Part B

Consider the function f defined over $\mathbb{R}$ by $f(x) = \dfrac{2+x}{1+e^x}$.

Denote by (C) the representative curve of function f in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.

1) Calculate $\lim\limits_{x \to +\infty} f(x)$. Interpret graphically.

2) a) Calculate $\lim\limits_{x \to -\infty} f(x)$, then prove that the line (d) of equation $y = x+2$ is an asymptote to (C).

b) Study, according to the values of x, the relative position of (C) and (d).

3) Prove that $f'(x) = \dfrac{g(x)}{(1+e^x)^2}$, then set up the table of variations of f.

4) Draw (d) and (C) **(1 unit = 2cm).**

Part C

Consider the function h defined by $h(x) = \ln[f(x)]$ and denote by (H) its representative curve in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.

1) Determine the domain of definition of h.

2) Calculate the slope of the tangent to (H) at point of abscissa -1.

3) Show that the equation $h(x) = 1$ has no solution.

QI	Answers	4 pts.
1)	$\lim\limits_{x\to e^+}\dfrac{(\ln x)^3-1}{(\ln x)^2-1}=\dfrac{0}{0}$ (I.F). $\lim\limits_{x\to e^+}\dfrac{(\ln x)^3-1}{(\ln x)^2-1}\overset{\text{H.R}}{=}\lim\limits_{x\to e^+}\dfrac{\frac{3(\ln x)^2}{x}}{\frac{2\ln x}{x}}=\lim\limits_{x\to e^+}\dfrac{3\ln x}{2}=\dfrac{3}{2}$. Thus, the correct answer is **c.**	1
2)	Condition: x > 0. Let $t=\ln x$, then $t^2+3t=4$; $t^2+3t-4=0$. $t_1=1$ and $t_2=-4$; $\ln x_1=1$ and $\ln x_2=-4$; $x_1=e$ and $x_2=e^{-4}$ (both are accepted). $x_1.x_2=e\times e^{-4}=e^{-3}$. Thus, the correct answer is **c.**	1
3)	Condition: $\begin{cases}x>0\\2x-1>0\end{cases}$; $\begin{cases}x>0\\x>\frac{1}{2}\end{cases}$; $x\in\left]\frac{1}{2},+\infty\right[$. $\ln x>\ln(2x-1)$; $x>2x-1$; $-x>-1$; $x<1$. Thus $x\in\left]\frac{1}{2},1\right[$. Thus, the correct answer is **b.**	1
4)	$P(A\cap B)=P(A)-P(A\cap\overline{B})=P(A)-p\left(A/\overline{B}\right)\times p\left(\overline{B}\right)=\dfrac{19}{24}-\dfrac{1}{2}\times\left(1-\dfrac{1}{4}\right)=\dfrac{5}{12}$. Thus, the correct answer is **a.**	1

QII	Answers	6 pts.
1) a)	$P(A)=\dfrac{C_3^2\times C_9^1}{C_{12}^3}=\dfrac{27}{220}$; $P(B)=\dfrac{C_3^3+C_3^3+C_3^3+C_3^3}{C_{12}^3}=\dfrac{1}{55}$.	1½
1) b)	$P(C)=P(2\text{ white of }3)+P(3\text{ white})=P(A)+\dfrac{C_3^3}{C_{12}^3}=\dfrac{27}{220}+\dfrac{1}{220}=\dfrac{28}{220}=\dfrac{7}{55}$.	¾
2) a)	$P\left(G/\overline{T}\right)=P(C)=\dfrac{7}{55}$; $P\left(G\cap\overline{T}\right)=P\left(G/\overline{T}\right)\times P\left(\overline{T}\right)=\dfrac{7}{55}\times\dfrac{9}{10}=\dfrac{63}{550}$.	¾
2) b)	$P(G\cap T)=P(G/T)\times P(T)=\dfrac{1}{2}\times\dfrac{1}{10}=\dfrac{1}{20}$.	1
2) c)	Using the total probabilities formula: $P(G)=P(G\cap T)+P\left(G\cap\overline{T}\right)=\dfrac{1}{20}+\dfrac{63}{550}=\dfrac{181}{1100}$.	1
2) d)	$P(T/G)=\dfrac{P(T\cap G)}{P(G)}=\dfrac{\frac{1}{20}}{\frac{181}{1100}}=\dfrac{55}{181}$.	1

<table>
<tr><th>QIII</th><th>Answers</th><th>10 pts.</th></tr>
<tr><td>A 1)</td><td>$\lim\limits_{x\to+\infty} g(x)=\lim\limits_{x\to+\infty}\left[1-(1+x)e^{x}\right]=1-\infty=-\infty.$
$\lim\limits_{x\to-\infty} g(x)=\lim\limits_{x\to-\infty}\left[1-(1+x)e^{x}\right]=\lim\limits_{x\to-\infty}\left[1-e^{x}-xe^{x}\right]=1-0-0=1,$
since $\lim\limits_{x\to-\infty} xe^{x}=0$ (rule).</td><td>¼
½</td></tr>
<tr><td>A 2)</td><td>$g'(x)=-e^{x}-(1+x)e^{x}=(-2-x)e^{x}$, same sign as $(-2-x)$ since $e^{x}>0$ for every x.
$g'(x)=0$ for $x=-2$. $g(-2)=1+e^{-2}$.
Table of variations of g :
<table>
<tr><td>x</td><td>$-\infty$</td><td></td><td>-2</td><td></td><td>0</td><td></td><td>$+\infty$</td></tr>
<tr><td>$g'(x)$</td><td></td><td>$+$</td><td>0</td><td>$-$</td><td>¦</td><td></td><td></td></tr>
<tr><td>$g(x)$</td><td>1</td><td>$\nearrow$</td><td>$1+e^{-2}$</td><td>$\searrow$</td><td>0</td><td>$\searrow$</td><td>$-\infty$</td></tr>
</table></td><td>1 ¼</td></tr>
<tr><td>A 3)</td><td>$g(0)=0$.
Over $]-\infty;0[$, g increases from positive (1) to positive $\left(1+e^{-2}\right)$ then decreases from positive$\left(1+e^{-2}\right)$ to 0, thus $g(x)>0$ over $]-\infty;0[$.
Over $]0;+\infty[$, g decreases from 0 to negative $(-\infty)$, thus $g(x)<0$ over $]0;+\infty[$.
<u>Conclusion:</u>
• $g(x)>0$ for $x\in]-\infty;0[$.
• $g(x)=0$ for $x=0$.
• $g(x)<0$ for $x\in]0;+\infty[$.</td><td>1</td></tr>
<tr><td>B 1)</td><td>$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\dfrac{2+x}{1+e^{x}}\overset{\text{H.R}}{=}\lim\limits_{x\to+\infty}\dfrac{1}{e^{x}}=0$.
Thus, the y – axis is a horizontal asymptote to (C) at $(+\infty)$.</td><td>½</td></tr>
<tr><td>B 2)a)</td><td>$\lim\limits_{x\to-\infty} f(x)=\lim\limits_{x\to-\infty}\dfrac{2+x}{1+e^{x}}=\dfrac{-\infty}{0}=-\infty$.
$\lim\limits_{x\to-\infty}\left[f(x)-y_{(d)}\right]=\lim\limits_{x\to-\infty}\left[\dfrac{2+x}{1+e^{x}}-x-2\right]=\lim\limits_{x\to-\infty}\left[\dfrac{2+x+\left(1+e^{x}\right)(-x-2)}{1+e^{x}}\right]=$
$\lim\limits_{x\to-\infty}\left[\dfrac{2+x-x-2-xe^{x}-2e^{x}}{1+e^{x}}\right]=\lim\limits_{x\to-\infty}\left[\dfrac{-xe^{x}-2e^{x}}{1+e^{x}}\right]=\dfrac{0-0}{1+0}=0$.
so, the line (d) of equation: $y=x+2$ is an oblique asymptote to (C) at $(-\infty)$.</td><td>1</td></tr>
<tr><td>B 2)b)</td><td>$f(x)-y_{(d)}=\dfrac{-xe^{x}-2e^{x}}{1+e^{x}}=\dfrac{(-x-2)e^{x}}{1+e^{x}}$ same sign as $(-x-2)$,
since $1+e^{x}>0$ and $e^{x}>0$ for every $x\in\mathbb{R}$.
• $f(x)-y_{(d)}<0$ for $x>-2$; thus (C) is below (d) if $x\in]-2;+\infty[$.
• $f(x)-y_{(d)}>0$ for $x<-2$; thus (C) is above (d) if $x\in]-\infty;-2[$.
• $f(x)-y_{(d)}=0$ for $x=-2$; thus (C) cuts (d) at point of coordinates $(-2;0)$.</td><td>1</td></tr>
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<tr><td>B 3)</td><td>

$f'(x)=\dfrac{1+e^{x}-e^{x}(2+x)}{(1+e^{x})^{2}}=\dfrac{1-(1+x)e^{x}}{(1+e^{x})^{2}}=\dfrac{g(x)}{(1+e^{x})^{2}}$; verified.

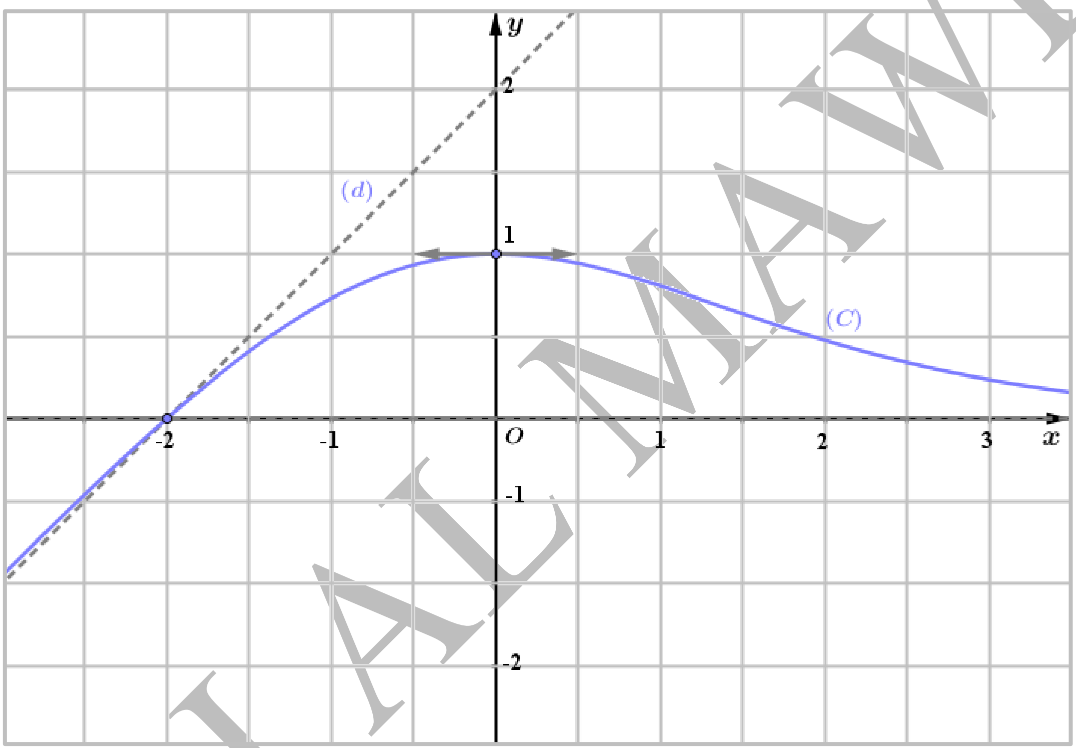
$f'(x)$ and $g(x)$ have same sign over $\mathbb{R}$ since $(1+e^{x})^{2}>0$ for every $x\in\mathbb{R}$.

$f'(x)=0$ for $g(x)=0$, that is for $x=0$. $f(0)=1$.

Table of variations of f :

<table>
<tr><td>x</td><td>$-\infty$ 0 $+\infty$</td></tr>
<tr><td>$f'(x)$</td><td>$+$ 0 $-$</td></tr>
<tr><td>$f(x)$</td><td>$-\infty \nearrow 1 \searrow 0$</td></tr>
</table>

</td><td>1 ½</td></tr>
<tr><td>B 4)</td><td>



</td><td>1 ½</td></tr>
<tr><td>C) 1)</td><td>

h is defined for $f(x)>0$.

Graphically, (C) is above $(x'Ox)$ if $x>-2$.

So $D_h=]-2;+\infty[$.

</td><td>½</td></tr>
<tr><td>C) 2)</td><td>

Slope $=h'(-1)=\dfrac{f'(-1)}{f(-1)}=\dfrac{\dfrac{1}{(1+e^{-1})^{2}}}{\dfrac{1}{1+e^{-1}}}=\dfrac{1}{1+e^{-1}}$.

Thus, the slope of the tangent to (H) at point of abscissa -1 is $\dfrac{1}{1+e^{-1}}$.

</td><td>½</td></tr>
<tr><td>C) 3)</td><td>

$h(x)=1$; $\ln[f(x)]=1=\ln e$; $f(x)=e$.

But graphically, the curve (C) and the line of equation $y=e$ don't intersect, thus, the equation $h(x)=1$ has no solution.

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