Name: _______________

NUMERICAL SEQUENCES

Exercise 1

Let (U_n) be a sequence defined by: $U_1 = \frac{1}{4}$ and $U_{n+1} = \left(\frac{n+1}{4n}\right)U_n$, n is a natural integer, $n \geq 1$.

1. a – Prove that $U_n \geq 0$, for all -natural integer $n \geq 1$.
   b – Prove that the sequence (U_n) is decreasing.
   c- Deduce that (U_n) is convergent. Precise its limit.
2. a – Prove that $U_n = \frac{n}{4^n}$, for all - natural integer $n \geq 1$.
   b- Calculate $\lim_{n\to+\infty} \ln(U_n)$. Deduce $\lim_{n\to+\infty} U_n$.

Exercise 2

Consider the sequence U that is defined over IN by: $U_0 = 0$ and $U_{n+1} = \sqrt{1+\frac{1}{4}(U_n)^2}$.

1. a) Prove by the mathematical induction that for all naturel number n: $0 \leq U_n \leq \frac{2\sqrt{3}}{3}$.
   b) Prove that the sequence U is increasing.
2. Consider the sequence V that is defined over IN by: $V_n = (U_n)^2 - \frac{4}{3}$.
   a) Prove that V is a geometric sequence whose common ratio and first term are to be determined.
   b) Express V_n then U_n in terms of n.
   c) Calculate in terms of n: $S_n = U_0^2 + U_1^2 + \ldots\ldots\ldots\ldots\ldots + U_n^2$.
3. Consider the sequence T that is defined on IN by: $T_0 = 1$ and $T_{n+1} = \frac{1}{4}T_n + V_n$. Assume that: $H_n = (4)^n T_n$.
   a) Prove that H is an arithmetic sequence of common difference $r = \frac{-16}{3}$.
   b) Calculate H_n then T_n in terms of n.

Exercise 3

Consider the sequence (U_n) defined over ℕ by $U_0 = 3$ and for all $n \in IN: U_{n+1} = \sqrt{\frac{1+U_n}{2}}$.

1 - Calculate: U_1; U_2 and U_3, then prove by mathematical induction that for all $n \in IN$: $U_n > 1$.

2- Show that the sequence (U_n) is decreasing, then calculate its limit.

Exercise 4

Consider the sequence (U_n) that is defined by: $\begin{cases} u_0 = 3 \\ u_{n+1} = 4\left(1 - \dfrac{1}{u_n}\right) \end{cases}$ For all n ∈ ℕ.

1.
   a- Show that by mathematical induction, that for all n ∈ ℕ, we have: $u_n > 2$.
   b- Verify that, for all n ∈ ℕ, we have: $u_{n+1} - u_n = \dfrac{-(-2+u_n)^2}{u_n}$.
   Deduce the sense of variations of the sequence (U_n).
   c- Deduce that the sequence (U_n) is convergent, and find its limit.
2. Suppose that: $v_n = \dfrac{1}{-2+u_n}$ for all n ∈ ℕ.
   a- Prove that (V_n) is an arithmetic sequence, and precise its common difference.
   b- Express v_n then u_n in terms of n.
   c- Find again $\lim\limits_{n\to+\infty} u_n$.

Exercise 5

Consider the sequence (U_n) defined by: $\begin{cases} U_0 = 2 \\ U_{n+1} = \dfrac{1+3U_n}{3+U_n} \end{cases}$ for all- natural number n.

1. Prove that for all -natural number n, we have: $U_n > 1$.
2. a – Determine the sense of variation of the sequence (U_n).
   b- Deduce that the sequence (U_n) is convergent, and determine $\lim\limits_{n\to+\infty} U_n$.
3. Consider the sequence (V_n) defined by: $V_n = \dfrac{-1+U_n}{1+U_n}$.
   a- Prove that (V_n) is a geometric sequence whose common ratio q and first term V_0 are to be determined.
   b- Express V_n then U_n in terms of n.
   c- Deduce $\lim\limits_{n\to+\infty} U_n$.
4. a – Prove that: $\dfrac{2}{1+U_n} = 1 - V_n$.
   b- Calculate in terms of n the sum: $S = \dfrac{4}{1+U_0} + \dfrac{4}{1+U_1} + \dfrac{4}{1+U_2} + \dots\dots\dots\dots + \dfrac{4}{1+U_n}$.

Exercise 6

Consider the sequence (U_n) defined for all naturel number by $u_0 > 0$ and $u_{n+1} = \dfrac{1}{2}\left(u_n + \dfrac{2}{u_n}\right)$.

1. Suppose that $u_0 = \sqrt{2}$. Prove that (U_n) is a constant sequence.
2. Consider the function f defined over $\left]\sqrt{2}; +\infty\right[$ by: $f(x) = \dfrac{1}{2}\left(x + \dfrac{2}{x}\right)$.
   Determine $f'(x)$ and set up the table of variations of f.
3. In all what follows we suppose that $u_0 > \sqrt{2}$.

a- Prove that for all naturel number $u_n \geq \sqrt{2}$.

b- Prove that for all $x \geq \sqrt{2}$, $f(x) \leq x$.

c- Deduce that the sequence (U_n) is decreasing.

d- Prove that (U_n) is convergent and determine its limit.

Exercise 7

Consider the sequence (U_n) defined by: $\begin{cases} u_0 = 2 \\ u_{n+1} = \frac{2}{5}u_n + 3. \end{cases}$

Part A :

1. Calculate u_1 and u_2. Is the sequence (U_n) geometric? Is (U_n) is an arithmetic sequence?
2. Prove that for all - natural number n we have: $2 \leq u_n \leq 5$.
3. Prove that the sequence (U_n) is increasing on IN.
4. Deduce that (U_n) is convergent and calculate its limit.

Part B:

Assume that for all -natural number n: $V_n = U_n - 5$.

1. Show that (V_n) is a geometric sequence.
2. Express V_n then U_n in terms of n.
3. Calculate $\lim\limits_{n\to+\infty} V_n$ *then* $\lim\limits_{n\to+\infty} U_n$.
4. Given the two expressions: $S_n = v_0 + v_1 + ...+ v_n$ *and* $S'_n = u_0 + u_1 + ...+ u_n$.
   a) Express S_n then S'_n in terms of n.
   b) Calculate $\lim\limits_{n\to+\infty} S_n$ *and* $\lim\limits_{n\to+\infty} S'_n$.

Exercise 8

Consider the sequence (I_n) that is defined as: $\begin{cases} I_0 = \int_0^{\frac{\pi}{2}} \sin x dx \\ I_n = \int_0^{\frac{\pi}{2}} x^n \sin x dx \quad \forall n \in IN^*. \end{cases}$

1. Calculate I_0 and I_1.
2. By using the integration by parts prove that for all $n \in IN^* : I_n = n\int_0^{\frac{\pi}{2}} x^{n-1} \cos x dx$.
3. a) By using the integration by parts for all $n \geq 2 : I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$.

   b) Deduce the values of I_2 and I_3.

   c) Then calculate: $J = \int_0^{\frac{\pi}{2}} \left(1 - 2x + x^2 - x^3\right)\sin x dx$.

Exercise 9

Consider the sequence (U_n) that is defined by: $\begin{cases} u_0 = 1 \\ u_{n+1} = \dfrac{u_n}{2+u_n} & \forall n \in IN^*. \end{cases}$

1. a) Calculate u_1 and u_2.
   b) Prove that the sequence (U_n) neither arithmetic nor geometric.
2. a) Prove that for all-natural number n we have: $U_n > 0$.
   b) Prove that the sequence (U_n) is decreasing.

   c) Deduce that the sequence (U_n) is convergent and calculate its limit.
3. Consider the sequence (V_n) defined by: $v_n = \dfrac{u_n}{1+u_n} \quad \forall n \in IN.$
   a) Calculate v_0 and prove that (V_n) is a geometric sequence.
   b) Determine the limit of the sequence (V_n).
   c) Prove that: $u_n = \dfrac{1}{2^{n+1}-1} \quad \forall n \in IN.$
   d) Find again the limit of the sequence (U_n).

Exercise 10

Consider the sequence (U_n) that is defined over IN by: $U_0 = 2 \quad and \quad U_{n+1} = \dfrac{U_n^{\,2} + U_n}{U_n^{\,2} + 1}; \quad n \in IN.$

1. Prove that for all - natural number n, $1 < U_n \le 2$.
2. a- Prove that the sequence (U_n) is decreasing.
   b- Prove that (U_n) is convergent and then calculate its limit.
3. a- Prove that for all - natural number n, $(U_{n+1} - 1) \le \dfrac{1}{2}(U_n - 1)$.

   b- Deduce that for all- natural number n, $U_n - 1 \le \left(\dfrac{1}{2}\right)^n$.

   c- Find again $\lim\limits_{n\to+\infty} U_n$.

Exercise 11

Consider the sequence (U_n) defined over IN by: $U_0 = -\dfrac{1}{2} \quad and \quad U_n = U^2_{\,n-1} + 2U_{n-1}; \quad n \ge 1.$

1. Prove that: $U_n > -1$ for all $n \ge 1$.
2. Let (V_n) be a sequence defined by $V_n = \ln(1+U_n)$.
   a- Prove that (V_n) is a geometric sequence.
   b- Express V_n then U_n in terms of n.

Exercise 12

Consider the sequence (I_n) that is defined as: $I_n = \int_0^1 \frac{x^n}{\sqrt{1+x^2}}\,dx$.

1. a- Prove that (I_n) is decreasing.

   b- Prove that: $0 \le I_n \le 1$ and that (I_n) is convergent.

   c- Prove that: $\frac{1}{\sqrt{2}(n+1)} \le I_n \le \frac{1}{n+1}$ and then $\lim_{n\to+\infty} I_n$.

2. Consider the function f that is defined by: $f(x) = \ln\left(x+\sqrt{1+x^2}\right)$.

   a- Calculate $f'(x)$ and then deduce the value of I_0.

   b- By using integration by parts prove that: $(n+2)I_{n+2} + (n+1)I_n = \sqrt{2}$.

   c- Calculate I_1 and then deduce I_2.

Exercise 13

Consider the function f that is defined over $]1,+\infty[$ by: $f(x) = \frac{x}{\ln(x)}$.

1. Study the sense of variations of f.
2. Consider the sequence (U_n) that is defined by: $U_0 = 5 \quad and \quad U_{n+1} = f(U_n)$.

   a- Prove that: $U_n \ge e$ for all n.

   b- Prove that (U_n) is decreasing.

COMPLEX NUMBERS

Exercise 1

The complex plane P is referred to a direct orthonormal system $(O;\vec{u};\vec{v})$

1. In the complex plane P, consider the points A and B with respective affixes

$$Z_A = 3 + i\sqrt{3}\ ;\ Z_B = 3 - i\sqrt{3} \text{ and } Z_M = 2$$

a- Determine the modulus and an argument of each of the complex numbers Z_A and Z_B.

b- Write Z_A and Z_B in its exponential form.

c- Prove that the points A, B and O are on a circle of center M whose radius is to be determined.

d- Plot the points A and B in the system $(O;\vec{u};\vec{v})$.

2. Consider the point C, the image of point O by the translation of the vector $\overrightarrow{AB}$.

a- Plot the point C in the system $(O;\vec{u};\vec{v})$

b- Determine the algebraic form of the affix z_C of the point C.

c- Prove that $OB = OC$. Give with justification, the nature of triangle OAB.

d- Deduce the nature of the quadrilateral OABC.

3. Consider the point D, the image of point A by the rotation of center O and of angle $\frac{\pi}{3}$.

a- Plot the point D in the system $(O;\vec{u};\vec{v})$

b- Determine the algebraic form of the affix Z_D of the point D.

c- Prove that the quadrilateral ABCD is a trapezoid whose opposite sides have same lengths.

d- Prove that the points A, C and M on a part and B, M and D on the other part are collinear.

Exercise 2

The complex plan is referred to a direct orthonormal system $(O;\vec{u};\vec{v})$ graphic units2 cm.

1. Solve in the set of the complex numbers the equation: $Z^2 + 2Z + 2 = 0$
2. Let A and C be two points of the complex plane with respective affixes:

$$Z_A = -1 + i \text{ and } Z_C = \left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right) + \left(\frac{\sqrt{3}}{2} + \frac{1}{2}\right)i.$$

Determine the modulus of Z_A and the modulus of Z_C then give an argument of Z_A.

3. a-Suppose that: $Z = \frac{Z_C}{Z_A}$, Prove that: $Z = \frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)i.$

b-Prove that: $Z = e^{\frac{i\pi}{3}}$

c-Deduce that the point C is the image of the point A by the rotation of center O and of angle $-\frac{\pi}{3}$.

4. Plot the point A then construct the point C by using the result of the preceding question. Write the steps of the construction.
5. Let B be the image of the point O by the translation of the vector $\overrightarrow{CA}$. Construct the point B and prove that OCAB is a Rhombus.

Exercise 3

1. In the plane referred to a direct orthonormal system $(O;\vec{u};\vec{v})$ (unit2 cm), consider the points A of affix $Z_A = 2$, B of affix $Z_B = -1 + i\sqrt{3}$ and C of affix $Z_C = -1 - i\sqrt{3}$.

a- Plot the points A, B and C

b- Determine the nature of triangle ABC. Justify your answer.

2. Consider the rotation R of center O and of angle $\frac{\pi}{6}$ and denote by A', B' and C' the respective images of A, B and C by R.

a- Determine the exponential forms of Z_A, Z_B, and Z_C then of $Z_{A'}$, $Z_{B'}$, and $Z_{C'}$.

b- Plot A', B' and C' on the preceding figure.

c- Verify that $Z_{A'}$, $Z_{B'}$, and $Z_{C'}$ are the solutions of the equation $Z^3 = 8i$.

Exercise 4

The plane is referred to a direct orthonormal system $(O;\vec{u};\vec{v})$, unit1 cm.

1. Solve in the set of complex numbers the following equations:

$(1):\ Z^2 - 10Z + 50 = 0$ $(2):\ Z + 2 = i\sqrt{3}\,Z - 6$.

2. a - Let A be a point of affix $Z_A = 5 - 5i$.Determine the modulus an argument and exponential form of ZA.

b- Let B be a point of affix Z_B, Z_B is the conjugate of Z_A.

Determine the exponential form of Z_B and that of $\frac{Z_B}{Z_A}$.

Deduce that B is the image of A by a rotation of center O whose angle is to be determined.
Construct the triangle OAB in the given system and indicate its nature.

3. Let C be a point of affix: $Z_C = -2 - 2i\sqrt{3}$

Prove that the image of C by the rotation of center O and of angle $-\frac{\pi}{2}$ is the point D of affix

$Z_D = -2\sqrt{3} + 2i$.

Calculate the distance OC and then construct the triangle OCD.

4. Let K be the midpoint of the segment [AC].

Calculate the affixes of the vectors $\overrightarrow{OK}$ and $\overrightarrow{DB}$ then prove that the straight lines (DB) and (OK) are perpendiculars

Exercise 5

The complex plan is referred to a direct orthonormal system $(O;\vec{u};\vec{v})$ graphic unit2 cm.

1. Solve in the set $\mathbb{C}$ the equation: $Z^2 - 2Z + 4 = 0$
2. Consider the points A and B of respective affixes: $Z_A = 1 + i\sqrt{3}$ and $Z_B = 1 - i\sqrt{3}$.

a- Determine the modulus and an argument of Z_A and Z_B.Find the exponential form of Z_A .

b-Plot the points A and B in the system $(O;\vec{u};\vec{v})$

3. Designate by R a transformation of the complex plane that is to every point M of affix z we associate the point M' of affix Z' so that: $Z' = e^{\frac{2i\pi}{3}} Z$

a- Indicate the nature of the transformation R and precise its characteristic elements.

b- Let C be the image of the point A by the transformation R.
Determine the exponential of the affix Z_C of the point C. Deduce its algebraic form.

c- Plot the point C.

d- Prove that the point B is the image of point C by the transformation R. What is the nature of Triangle ABC ?

Exercise 6

The complex plane is referred to a direct orthonormal system $(O;\vec{u};\vec{v})$, graphic unit4 cm.

1. a- Solve in the set $\mathbb{C}$ the equation: $Z^2 - 2\sqrt{3}\,Z + 4 = 0$

   b- Designate by Z_1 and Z_2 the solutions, Z_1 is the root with negative imaginary part.

   Write Z_1 and Z_2 in their exponential forms.

2. Let A be a point of affix Z_1 and B that of affix Z_2.

   Plot the points A and B then prove that the triangle OAB is equilateral.

3. Let E be a point of affix: $Z_3 = e^{-i\frac{\pi}{3}}$ and F of affix: $Z_4 = e^{i\frac{\pi}{4}}$

   F is the image of E by a transformation? Precise the nature of this transformation and its Characteristic elements. Prove that F is the midpoint of the segment $[OB]$.

4. a- Consider the application R of P onto P that is for every point M of affix Z associate the point M' of

   affix Z' so that: $Z' = e^{i\frac{\pi}{4}}Z$. Characterized geometrically the application R.

   b- Plot the point A' the image of point A under R.

   c- Calculate the trigonometric and the algebraic forms of the affix of the point A'.

   d- Deduce the exact values of $\cos\left(\frac{\pi}{12}\right)$ and $\sin\left(\frac{\pi}{12}\right)$.

5. Let D be the image of E by a translation of the vector $2\vec{v}$.

   a- Plot the points D, E and F.

   b- Determine the affix of the point D and prove that: $OD = DB$.

   c- What can you deduce for the straight line (AD)?

Exercise 7

The plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$ (graphic unit: 1 cm).

Part A: Let $P(Z) = Z^3 + 4Z^2\sqrt{3} + 24Z + 24\sqrt{3}$ where Z is a complex variable.

1. Verify that: $P(Z) = \left(Z + 2\sqrt{3}\right)\left(Z^2 + 2Z\sqrt{3} + 12\right)$
2. Solve the equation: $Z^2 + 2Z\sqrt{3} + 12 = 0$
3. Deduce the solutions of the equation $P(Z) = 0$.

Part B

1. Plot the points *A*, *B* and *C* with respective affixes $Z_A = -2\sqrt{3}$, $Z_B = -\sqrt{3} + 3i$, $Z_C = -\sqrt{3} - 3i$
2. a- Determine the modulus and an argument of Z_A, Z_B and Z_C.

   b- Find the exponential forms Z_A, Z_B and Z_C.
3. R is a rotation of center O and of angle $-\frac{\pi}{3}$.

   a- Find the complex form of the rotation R

   b-Prove that the image of the point *A* by R is the point *B*.

   c-Find the algebraic form of the affix of point *D*, image of point *B* by R.
4. Let (C) be a circle of diameter [*CD*].

   a- Justify that *O* is the center of (C).

   b- Prove that les points A and B are belonging to(C).

   c- Deduce the nature of the triangles *CAD* and *CBD*.

Exercise 8

1. a- Find the algebraic form of the following complex number $(2+3i)^2$.

   b- Solve in the set of complex numbers the following equation:

   $$Z^2-(2+7i)Z-10+4i=0.$$

2. Suppose that: $f(Z)=Z^3-(5+7i)Z^2+(-4+25i)Z+30-12i$.
   a- Calculate f (3).
   b- Deduce that: $f(Z)=(Z-3)(Z^2+bZ+c)$ where b and c are two complex numbers are to be determined.
   c- Solve the equation: f (Z) = 0.

3. The plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$. Consider the points A, B and C with respective affixes: $Z_A=2i,\ Z_B=3\ \ and\ \ Z_C=2+5i$.

   a- Prove that: $\dfrac{Z_C-Z_A}{Z_B-Z_A}=i$.

   b- Deduce that ABC is right triangle at A.

Exercise 9

1. Solve in the set of complex numbers the equation (E): $Z^2-\sqrt{2}(1+i)Z-1+i=0$.
2. Let α be a real number in the interval $]0;\pi[$.

   Solve the following equation $(E_\alpha): Z^2-2e^{i\alpha}Z+e^{2i\alpha}-1=0$.
3. The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

   Consider the points A and B of respective affixes: $Z_A=1+e^{i\alpha}\ \ and\ \ Z_B=-1+e^{i\alpha}$.

   a) Verify that: $Z_A=2\cos\left(\dfrac{\alpha}{2}\right)e^{i\left(\frac{\alpha}{2}\right)}\ \ and\ \ Z_B=2\sin\left(\dfrac{\alpha}{2}\right)e^{i\left(\frac{\alpha}{2}\right)}$.

   b) Deduce that OAB is right triangle at O.

   c) Determine the value of the realα so that: OAB is an isosceles triangle of vertex O.

Exercise 10

1. Prove that: $(2+2i)^2=8i$.
2. Solve the following equation: $Z^2-6Z+9-2i=0$.
3. Let $f(Z)=Z^3-(6+i)Z^2+(9+4i)Z-9i-2=0;\ Z\in C$.
   a) Verify that i is a root of the equation f (Z) = 0.
   b) Determine the complex numbers a and b so that: $f(Z)=(Z-i)(Z^2+aZ+b)$.
   c) Solve then the equation: f (Z) = 0.
4. The complex plan is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

   Consider the points A, B and C with respective affixes 2 – i, i and 4 + i.

   a) Plot the points A, B and C in the system $(O;\vec{u},\vec{v})$.

   b) Prove that ABC is right isosceles triangle.
5. Determine and construct the set des points M of affix Z so that: $|Z-i|=|Z-2+i|$.

Exercise 11

1. a) Verify that: $(3+i)^2 = 8+6i$.

   b) Solve the equation: $Z^2-(5+3i)Z+2+6i=0$.

2. Let $f(Z)=Z^3-(5+i)Z^2+4(2-i)Z-12+4i$.

   a) Calculate f (-2i).

   b) Determine the complex numbers b and c so that: $f(Z)=(Z+2i)(Z^2+bZ+c)$.

   c) Solve the equation f (Z) = 0.

3. In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points A, B and C with respective affixes: $Z_A=-2i$; $Z_B=1+i$; $Z_C=4+2i$.

   a) Plot the points A, B and C in the system $(O;\vec{u},\vec{v})$.

   b) Prove that ABC is an isosceles triangle.

   c) Determine the affix Z_D of point D so that ABCD is a Rhombus.

Exercise 12

Part A: Consider the equation: $(E): Z^2+(-5+i)Z+8-i=0$.

1. Verify that: $Z'=2+i$ is a solution of the equation (E).
2. Without solving the equation (E), find Z'' the second solution of (E)

Part B: The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

Consider the points A, B and C of respective affixes: $Z_A = 2 + i$, $Z_B = -1$ and $Z_C = 3 - 2i$.

1. Plot the points A, B and C.
2. Determine the affix of the point I midpoint of the segment [BC].
3. a) Calculate the distances AB, AC and BC.

   b) Deduce the nature of triangle ABC.
4. a) Determine the affix of point D the symmetric of point A with respect to I.

   b) What is the nature of the quadrilateral ABDC?

Exercise 13

Let P be a polynomial defined by: $P(Z)=Z^3+2(\sqrt{2}-1)Z^2+4(1-\sqrt{2})Z-8$.

1. Verify that P (2) = 0, then deduce the factorization of P (Z).

2. Solve in the set C of complex numbers the equation: P (Z) = 0.

3. Denote by Z_1 and Z_2 the solutions of P (Z) = 0 ,other than 2, where Z_1 is the root with positive imaginary part , verify that : $Z_1 + Z_2 = -2\sqrt{2}$.

   a. Plot in the plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$ the points A (2) , B (Z_1) and C(Z_2) .Designate by I the midpoint of [AB].

   b. Show that OAB is an isosceles triangle. Deduce $(\vec{u},\overrightarrow{OI})$.

   c. Find the algebraic form of Z_I the affix of I then calculate the modulus of Z_I.

   d. Write Z_I in its trigonometric form, then deduce the exact values of $\cos\left(\frac{3\pi}{8}\right)$ *and* $\sin\left(\frac{3\pi}{8}\right)$.

Exercise 14

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Designate by (C) the circle of center O and of radius 1 and by I and A the points of respective affixes $Z_I = 1$ and $Z_A = \sqrt{3}+i$.

1. a) Give the exponential form of Z_A.
   b) Locate the point A.
2. Let B be a point of affix $Z_B = \dfrac{Z_A - 1}{1-\overline{Z_A}}$.
   a) Verify that: $Z_B.\overline{Z_B} = 1$. Deduce that the point B belongs to the circle (C).
   b) Prove that: $\dfrac{Z_B - 1}{Z_A - 1}$ is real. Deduce that the points A, B and I are collinear.
   c) Locate the point B in the system $\left(O;\vec{u},\vec{v}\right)$.
3. Let θ be an argument of the complex number Z_B.
   Prove that: $\cos\theta = \dfrac{2\sqrt{3}-3}{5-2\sqrt{3}}$ *and* $\sin\theta = \dfrac{2-2\sqrt{3}}{5-2\sqrt{3}}$.

Exercise 15

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

To every point M of affix $Z \neq -i$, we associate the point M' of affix $Z' = \dfrac{Z+2i}{1-iZ}$ and let B and C be two points of respective affixes $Z_B = -i$ *and* $Z_C = -2i$.

1. a) Verify that for all $Z \neq -i$ we have: $-iZ' = \dfrac{Z+2i}{Z+i}$.
   b) Deduce the set of the points M so that Z' is real.
2. a) Prove that : $|Z'| = \dfrac{CM}{BM}$.
   b) Deduce the set of the points M in the case M' describes the circle C (O (0,0) ; R = 1).
3. Consider the complex number: $W = \dfrac{Z'-i}{Z-i}$, $(Z \neq -i \text{ and } Z \neq i)$.
   a) Verify that for all complex number Z we have: $(Z-i)(1-iZ) = -i\left(1+Z^2\right)$.
   b) Deduce that: $W = -\dfrac{1}{Z^2+1}$.
4. Suppose that: $Z = e^{i\theta}$; $\theta \in \left[0;\dfrac{\pi}{2}\right[$.
   Verify that: $W = \dfrac{-e^{-i\theta}}{e^{i\theta}+e^{-i\theta}}$. Deduce in terms of θ the modulus and an argument of W.

Exercise 16

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

To every point M of affix Z, we associate the point M' of affix Z' such that: $Z'=(1+i)Z+2-i.$

1. Determine Z in its algebraic and in its exponential form in the case: $Z'=\left(1-\sqrt{3}\right)+i\left(-2+\sqrt{3}\right).$
2. Determine Z' in the case $Z=2e^{\frac{i\pi}{4}}$.
3. Suppose that : $Z=x+iy$ *and* $y'=x'+iy'.$
   a) Prove that: $x'=x-y+2$ *et* $y'=x+y-1.$
   b) Find the set of points M' in the case M moves on a straight line (d) of equation: y = 2.
4. Given the points A, B and C of respective affixes $Z_A=2+i;\ Z_B=3+4i$ *and* $Z_C=8-i.$
   a) Calculate $\dfrac{Z_B-Z_A}{Z_C-Z_A}$, and then deduce the nature of triangle ABC.
   b) Find the affix of point D the fourth vertex of the square ABDC.
5. a) Prove that : $Z'=(1+i)\left(Z+\dfrac{1-3i}{2}\right).$

   b) Find the set of points M' such that M moves on a circle $C\left(I\left(-\frac{1}{2};\frac{3}{2}\right);\ R=3\sqrt{2}\right).$

Exercise 17

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$ graphic unit: 1 cm.

Consider in the set of the complex numbers, the equation (E) of unknown z:

$$Z^3+(-8+i)Z^2+(17-8i)Z+17i=0.$$

Part A :

1. Verify that -i is a solution of the equation (E).
2. Determine the real numbers a, b and c such that:
   $Z^3+(-8+i)Z^2+(17-8i)Z+17i=(Z+i)\left(aZ^2+bZ+c\right).$
3. Solve the equation (E) in the set of the complex numbers.

Part B :

Consider the points A, B and C of respective affixes $4+i;\ 4-i$ *and* $-i.$

1. Plot the points A, B and C.
2. To every point M of affix Z, we associate the point M' of affix Z' such that: $Z'=iZ-2i+2.$
   a. Determine the affix of point S associate of the point A.
   b. Prove that the points A, B, C and S belongs to the same circle (C_1) whose center and radius are to be determined. Draw (C_1).

Part C :

To every point N of affix Z, we associate the point N' of affix Z' such that: $Z'=\dfrac{iZ+10-2i}{Z-2}.$

1. Determine the affixes of point's A', B' *et* C' associates respectively to the points A, B and C.

2. Verify that the points A', B' and C' belong to the circle (C_2) of center P of affix i. determine its radius and then draw it.
3. For all complex number Z different than 2, prove that: $|Z'-i|\times|Z-2|=10$.
4. Let M be a point of (C_1) affix Z. Prove that $|Z'-i|=2\sqrt{5}$. Deduce the set of points M'.

Exercise 18

Consider the equation (E): $e^{-i\theta}Z^2-2Z+2i\sin(\theta)=0$, where θ is a real in $]0;\pi[$.

1. Prove that: $1-2i\sin(\theta)e^{-i\theta}=e^{-2i\theta}$. Then solve (E).
2. In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the points A, B and C of respective affixes: $Z_A=2e^{i\theta},\ Z_B=1+e^{i\theta}\ and\ Z_C=-1+e^{i\theta}$.
   a. Write Z_B and Z_C in its exponential form and verify that: $\frac{Z_C}{Z_B}=i\tan\left(\frac{\theta}{2}\right)$.
   b. Prove that the quadrilateral OBAC is a rectangle.
   c. Determine θ in the case OBAC is a square.

Exercise 19

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, 1unit = 3 cm.

Let Z_M = Z, where Z is a complex number such that $Z\neq i,\ Z_{M'}=Z'$ with $Z'=\frac{Z-2-2i}{Z-i}$.

Let $Z_A=i\ and\ Z_C=2+2i$.

Part A :

1. Write Z in the exponential form in the case $M=M'$.
2. Interpret geometrically: $|Z'|$ and $\arg(Z')$. Deduce the set (γ) of points M' such that: MA=MC.
3. Suppose that : $Z=x+iy\ and\ Z'=x'+iy'$.
   a. Verify that: $x'=\frac{x^2+y^2-2x-3y+2}{x^2+(y-1)^2}\ and\ y'=\frac{-x+2y-2}{x^2+(y-1)^2}$.
   b. Deduce the set of points M in the case M' describes on $(x'0x)$.

Part B:

Let Z_B = 1, $Z_D=2+i$ and (C) be the circle of center A and of radius $R=\sqrt{5}$.

1. Calculate $|Z_D|$. Draw (C).
2. Justify that: $(Z'-1)(Z-i)=-2-i$.
3. Deduce that when M moves on (C), the point M' moves on the circle (C') of center B and of radius $R'=1$. Draw (C') in the same system of (C).
4. Verify that (C) and (C') are intersects at two points E and F such that: $Z_E=2\ and\ Z_F=1-i$.
5. What does the point B represents for the le triangle AEF ? Justify.
6. The straight line (EF) cuts $(y'y)$ in J. Calculate the affix of point J

LINES AND PLANS IN THE SPACE

Exercise 1

In the space referred to a direct orthonormal system $\left(O;\vec{i},\ \vec{j},\ \vec{k}\right)$.

Consider the straight line (d) of equations: x = m – 1, y = 5m, z = 4m – 2 $\left(m \in IR\right)$, the plane (P) of equation: 2 x + y – z = 0.

1. Find the equation of plane (Q) contains the straight line (d) and perpendicular to the plane (P).
2. Write a system of parametric equations of the straight line (d') intersection of two planes (P) and (Q).
3. a) Let F (0; 5; 2) be a point of the straight line (d).
   Verify that the point $F'\left(-\frac{2}{7};\frac{23}{14};\frac{15}{14}\right)$ is the orthogonal projection of the point F on the straight line (d').
   b) Deduce the distance from the point F to the straight line (d').
4. Consider in the plane (Q) a circle (C) of center F and radius $\sqrt{14}$.
   a) Prove that the circle (C) cuts the straight line (d') at two points A and B.
   b) Calculate the area of the triangle FAB.
   c) Find an equation of the mediator plane (R) of the segment [AB].

Exercise 2

The space is referred to a direct orthonormal system (O; $\vec{i}$, $\vec{j}$, $\vec{k}$) .

Consider the two straight lines (d) and (d') of respective parametric equations:

(d):{x=t-1; y = - t + 2 and Z = 2 t – 1 ; (d') :{x = - m +2 ; y = m – 1 and Z = m + 2.(t and m ∈ℝ)

1. Prove that the two straight lines (d) and (d') are perpendiculars.
2. Find the coordinates of the point A intersection of the two straight lines (d) and (d').
3. Find an equation of the plane (P) determined by (d) and (d').
4. Write a system of parametric equations of (D) the interior bisector of the angle between (d) and (d').
5. Designate by (R) the plane passes through the point W (1; 4; 1) and parallel to the plane (Q).
   Consider in the plane (R) the circle (C) of center W and of radius 3.

   a- Write the equation of the plane (R)

   b- Prove that B (3; 2; 0) is a point of the circle (C).

   c- Write a system of parametric equations of the tangent (T) at B to (C).

Exercise 3

In the space referred to a direct orthonormal system $\left(O;\ \vec{i},\ \vec{j},\ \vec{k}\right)$.

Consider the points A (1; - 8; - 1), B (- 1; - 2; - 1), C (0; 1; 0) and the plane (P) of equation:
x – y + 2 z – 7 = 0.

1. Show that the equation of the plane (Q) determined by the points A, B and C is:
   3 x + y – 6 z – 1 = 0.
2. Designate by (d) the line of intersection of two planes (P) and (Q).

a) Verify that the line (D) of system parametric equations $\begin{cases} x=-t+2 \\ y=-3t-5 \\ z=-t \end{cases}$ $(t \in IR)$ is included to the plane (P).
b) Prove that the straight line (d) is parallel to the straight line (D).
c) Verify that the point A belongs to the plane (P).
d) Deduce a system of parametric equations of the straight line (d).

3. Let E (1; - 2; 0) be a point in the plane (Q).
Prove that the point $E'\left(\frac{7}{3};-\frac{10}{3};\frac{8}{3}\right)$ is symmetry of the point E with respect to the plane (P).
4. Consider in the plane (Q) a circle (C) of center E and of radius $\sqrt{6}$.
   a) Calculate the distance from the point E to the straight line (d).
   b) Deduce that the straight line (d) cuts the circle (C) at two points R and S.
   c) Find the equation of the mediator plane of the segment [RS].
5. Let D (3; - 2; 1) be a point on the straight line (d).
   a) Write a system of parametric equations of the straight line (DE).
   b) Calculate the cosine of the acute angle between (DE) and (P).
6. Let M (- t + 2; - 3 t – 5; - t) be a variable point on the straight line (D).
Determine the values of the parameter t so that the volume of the tetrahedron MCBE is equal to 3 units' cube of volume.
7. Consider the straight line (d') of system of parametric equations $\begin{cases} x=m \\ y=-m+3 \\ z=2m-1 \end{cases}$
   a) Prove that the two straight lines (D) and (d') are skew.
   b) Let H (2; - 5; 0) be a point on the straight line (D).
   Find the coordinates of the point H' the orthogonal projection of the point H on the straight line (d').

Exercise 4

In the space referred to a direct orthonormal system $\left(O; \vec{i}, \vec{j}, \vec{k}\right)$, given the points A (1;-1 ;0), B (-1 ; 1 ;- 1) and the straight line (d) of parametric equations : $\begin{cases} x=-2m+1 \\ y=m \\ z=4m+2 \end{cases}$ $(m \in IR)$.

Designate by (Q) the plane of equation: 2 x – y – 4 z – 1 = 0.

1. Prove that B belongs to the plane (Q) and that (d) is perpendicular to (Q).
2. a) Prove that the point $H\left(\frac{5}{3};-\frac{1}{3};\frac{2}{3}\right)$ is the orthogonal projection of the point B on the straight line (d).
   b) Find the coordinates of the point C, symmetric of B with respect to (d).
3. Designate by (P) the plane formed by the point A and the straight line (d).
   a) Prove that: x – 2 y + z – 3 = 0 is the equation of (P).
   b) Prove that the two planes (P) and (Q) are perpendiculars.
   c) Justify that (P) and (Q) are intersect along the straight line (D) of parametric equations:
   (D) : $x=3\lambda+\frac{5}{3};\ y=2\lambda-\frac{1}{3}\ \ and\ \ z=\lambda+\frac{2}{3}$.

4. Designate by (C) the circle of the plane (P) with center A and of radius r = $\sqrt{21}$.
   a) Prove that E (- 1; 0; 4) is a point of (C).
   b) Find a system of parametric equations of the tangent (T) to (C) at the point E.
5. a) Prove that (AB) and (D) are not coplanar straight lines.
   b) Find an equation of the plane (R) contains the straight line (AB) and parallel to (D).

Exercise 5

In the space referred to an orthonormal system $\left(O; \vec{i}, \vec{j}, \vec{k}\right)$, given the straight lines

(d): {x = t +1; y = 4 t – 1 ; z = t ; (d') : x = m +2 ; y = m ; z = 4 m +4. (t and m ∈ℝ)

1. a- Verify that (d) and (d') are intersecting at the point A (1; - 1; 0).
   b- Determine the equation of the plane (P) determined by (d) and (d').
   c- Verify that the point $B\left(2;\frac{3}{2};\frac{5}{2}\right)$ is a point of (P).
2. Let (Q) be a plane of equation: 2 x + 5 y + 5 z – 24 = 0.
   a- Prove that the plane (Q) is perpendicular to the plane (P).
   b- Let (Δ) be the line of intersection of (P) and (Q). Determine the coordinates of E and F the respective intersection points of (Δ) with the straight lines (d) and (d').
   c- Prove that AEF is an equilateral triangle.
   d- Deduce that the straight line (AB) is the bisector of the angle $E\hat{A}F$.
3. Let (C) be a circle in the plane (P) of center I (3; 0; 9) and of radius R = $\sqrt{2}$.
   a- Prove that (C) cuts (d') at two points T and S.
   b- Find the coordinates of points T and S.
   c- Find a system of parametric equations of the tangent (D) to (C) at the point T (3 ; 1 ; 8).

Exercise 6

In the space referred to a direct orthonormal system $\left(O; \vec{i}, \vec{j}, \vec{k}\right)$, consider the points

A (1 ;2 ;1), B (2; -2; 0), C (- 1; 4; 1) and the plane (Q) of equation: 2 x – y – 3 z – 3 = 0.

1. Show that an equation of plane (P) formed by the points A, B and C is: x + y – 3 z = 0.
2. Write a system of parametric equations of the straight line (d) intersection of the two planes (P) and (Q).
3. Let E (3 ;4; - 5) be a point of the space.
   a- Prove that A is the orthogonal projection of the point E on the plane (P).
   b- Calculate the area of the triangle ABC, then deduce the volume of the tetrahedron EABC.
   c- Show that the point F (-1; 0; 7) is the symmetry of the point E with respect to the plane (P), then deduce the area of the triangle EBF.
4. In the plane (P), we consider the circle (C) with center A and of radius R = $2\sqrt{2}$.
   a- Prove that (d) cuts (C) in two points I and J.
   b- Find an equation of the mediator plane (R) of the segment [IJ].
   c- Calculate the coordinates of the points I and J.
   d- Find a system of parametric equations of the tangent (T) to (C) at the point I (3; 0; 1).

Exercise 7

In the space referred to a direct orthonormal system $\left(O; \vec{i}, \vec{j}, \vec{k}\right)$, Given the points A (1, -2,1),

B (2 ;-1 ;3) ; C (1 ;1 ;4) and the line (d) defined by: x = t ; y = t; z = - t + 2. (t ∈ℝ).

1. Write an equation of the plane (P) determined by the points A, B and C.
2. a – Prove that the straight line (d) is perpendicular to the plan (P) at a point H whose coordinates are to be determined.

b- Determine the coordinates of point L on (d) such that $LH = \sqrt{3}$. ($x_L < 0$)

c- Deduce the coordinates of point L' such that (P) is the mediator plane of [LL'].

3. Let M be a variable point on (d) and E (2 ;2 ;0) is a point on (d).
4. Determine the points M so that the volume of the tetrahedron MABC is equal to twice of the volume of the tetrahedron EABC.

__Exercise 8__

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, consider the points A (-1; 2; 1) and B (1; 1; 0) and the two straight lines (d) and (d') defined by their parametric equations:

(d): { x = t +3; y = - t ; z = 2 t – 1; (d'):{x = m ; y = - m +2 ; z = - m +1. (t and m ∈ℝ)

1. Prove that (d) and (d') are orthogonal and not coplanar.
2. Let (P) be the plane determined by A and (d). Prove that the equation of the plane (P) is:
   $3x + 5y + z - 8 = 0$.
3. Prove that (d') cuts (P) in B.
4. Let C (2; 1; - 3) be a point of (d).
   a- Write an equation of the plane (Q) passing through A and perpendicular to (BC).
   b- Deduce a system of parametric equations of the height issued of A to [BC] in ABC.
   c- Let H be the point of intersection of the plane (Q) and the straight line (BC).
   What does the point H represent for the point A? Deduce the distance from the point A to the straight line (BC).
5. The circle (C) in the plane (P) with center A and of radius $2\sqrt{6}$ cuts (d) in two points.
   Verify that E (3; 0; - 1) is one of common points to (d) and (C) and find the other point F.
6. Find a system of parametric equations of the straight line (D) passing through the point K (-4 ; -3 ; 0), parallel to (P) and orthogonal to (d).
7. Prove that: $x - y + 2z + 1 = 0$ is an equation of the mediator plane of the segment [EF].

__Exercise 9__

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, consider the plane (P) of equation $x + z - 7 = 0$ and the points: F (4; - 4; 3), H (3 ;0 ;4) and G (1; - 4 ;0).

1. Write the equation of the plan (Q) determined by the points F, H and G.
2. Verify that (FH) is the line of intersection of the two planes (P) and (Q).
3. Prove that the system of parametric equations of the straight line (D) included in the plane (Q), parallel to (P) and passing by G is: (D) $\begin{cases} x = -k+1 \\ y = 4k-4 \\ z = k \end{cases} \quad (k \in IR)$.
4. Determine the coordinates of the point E, the orthogonal projection of H on (D).
5. Consider in the plane (Q), the parabola (Σ) of focus F (4; - 4; 3) and directrix the straight line (D).
   a) Verify that H is a point on (Σ).
   b) Determine the coordinates of H' the point of intersection of the parabola (Σ) with the plane (P).
   c) Verify that (GF) is the focal of (Σ) and deduce the coordinates of the vertex S of the parabola (Σ).
   d) Write a system of parametric equations of the tangent (T) to the parabola (Σ) at the point H.

Exercise 10

The space is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$. Consider the point A (-1; 1;0), the plane (P) of equation: x – 2 y + 2 z – 6 = 0, and the straight line (D) of equations:
x = m + 1; y = 2 m + 1; z = 3 m + 2 ($m \in \mathbb{R}$).

1. Prove that A and (D) determine a plane (Q) and determine the equation of (Q).
2. a) Prove that (P) cuts (Q) along a straight line (Δ) defined by:
   x = 2; y = t – 2; z = t ($t \in \mathbb{R}$).

   b) Prove that the coordinates of A', the orthogonal projection of A on (Δ), are $\left(2;-\frac{1}{2};\frac{3}{2}\right)$.
3. M is a variable point of (Δ). α is a measure of the angle formed by (AM) and (P)
   a) Find the coordinates of H, the orthogonal projection of A on (P).
   b) Calculate the distance from A to the plane (P), and prove that: AM $\times$ sin α = 3.
   c) Determine the position of M for which α is maximum. Calculate sin α in this case.
   d) What does the value of represents α for the two planes (P) and (Q)?
4. Consider the circle (C) of center A which is tangent to (Δ) and included in the plane (Q). The orthogonal projection of (C) on the plane (P) is an ellipse (E).
   a) Calculate the radius of (C).
   b) Determine the coordinates of the center of (E).
   c) Calculate the eccentricity of (E).
   d) Determine a system of parametric equations of the focal axis of (E).

Exercise 11

The space is referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.
Consider the straight line (d): x = - t + 1; y = t – 2 and z = t – 1 (t is a real parameter), and the plane (P) of equation: x + 2 y – z + 8 = 0, and the point E (- 1; 0; 1).

1. Prove that E belongs to the straight line (d) and then prove that (d) is parallel to the plane (P).
2. a) Write an equation of the plane (Q) that containing (d) and perpendicular to the plane (P).

   b) Deduce a system of parametric equations of the straight line (d') the orthogonal projection of the straight line (d) on the plane (P).
3. a) Let F (- 2; - 2; 2) be a point on (d').
   Prove that F is the orthogonal projection of the point E on the plane (P).

   b) Deduce the coordinates of the point E' the symmetric of the point E with respect to the plane (P).
4. Let (C) be a circle in the plane (P) with center F and of radius $\sqrt{3}$.
   a) The circle (C) cuts the straight line (d') at two points A and B.
   Determine the coordinates of the points A and B.
   b) Write a system of parametric equations of the tangent (T) to the circle (C) at the point A.

Exercise 12

$\left(O;\vec{i},\vec{j},\vec{k}\right)$ is an orthonormal system in the space. Consider the two straight lines D and D' dedined by:

$$D:\begin{cases} x=-1+2\alpha \\ y=1-\alpha \\ z=\alpha \end{cases} \quad (\alpha \in IR) \quad and \quad D':\begin{cases} x=1+\beta \\ y=1+2\beta \\ z=1-\beta \end{cases} \quad (\beta \in IR).$$

1. Prove that D and D' are not orthogonals and not coplanars.
2. a) Write an equation of plane (P) contains D and parallel to D'.

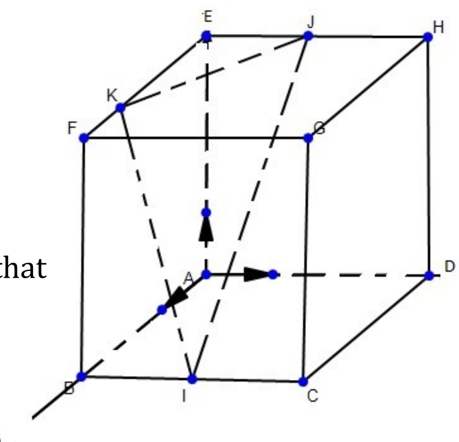
   b) Give a director vector of straight line Δ which is perperdicular to D and to D'.

3. a) Write an equation of plane (P') contains D' and perpendicular to (P).
   b) Determine a system of parametric equations of Δ.
4. Let A (1 ; 0 ; -1) be a point in the space. Calculate the distance from the point A to D.
5. Let (P_m) be a family of planes of equation : (2m – 1)x + (2 – m)y – (m +1)z + m + 1 = 0 where m is a real parameter.
   a) Verify that for all real m, (P_m) is a plane.
   b) Prove that all the planes (P_m) contains a fixed straight line is to be determined.
6. Let $M \in D$ *and* $M' \in D'$.
   a) What is the relation between α and β in the case straight line (MM') is parallel to the plane (P_1).
   b) In this case determine the set of points I midpoint of segment $[MM']$.

Exercise 13

The space is referred to a direct orthonormal system $\left(A;\vec{i},\vec{j},\vec{k}\right)$.

Considere the parallelopipede ABCDEFGH such that : $\overrightarrow{AB}=3\vec{i},\ \overrightarrow{AD}=4\vec{j}\ \ and\ \ \overrightarrow{AE}=3\vec{k}$.



1. Verify that: $\overrightarrow{AG}=3\vec{i}+4\vec{j}+3\vec{k}$.
2. Let I and J be the respective midpoints of edges [BC] and [EH].
   Let K be a point defined by: $\overrightarrow{EK}=\dfrac{2}{3}\overrightarrow{AB}$.
   a) Determine the coordinates of I and J and verify that the coordonnées of point K is (2 ; 0 ; 3).
   b) Determine the components of the vectors $\overrightarrow{KI}\ and\ \overrightarrow{KJ}$ and verify that $\overrightarrow{KI}\wedge\overrightarrow{KJ}=6\vec{i}+6\vec{j}+6\vec{k}$.
3. a) Calculate the area of triangle IJK.
   b) Calculate the volume of tetrahedron IJKG.
   c) Deduce the distance from the point G to the plane (IJK).
4. Write an equation of the plane (IJK).
5. Let G' be a point such that (IJK) is the mediator plane of segment $[GG']$ and N is the orthogonal projection of G on the plane (IJK).
   a) Prove that the coordinates of the point N are $\left(\dfrac{4}{3},\dfrac{7}{3},\dfrac{4}{3}\right)$.
   b) Deduce the coordinates of the point G'.

Exercise 14

The space is referred to a direct orthonormal system $\left(A;\vec{i},\vec{j},\vec{k}\right)$ let ABCDEFGH be a parallelopipede such that $\overrightarrow{AB}=2\vec{i},\ \overrightarrow{AD}=4\vec{j}\ \ and\ \ \overrightarrow{AE}=3\vec{k}$.

1. a) Verify that: $\overrightarrow{AG}=2\vec{i}+4\vec{j}+3\vec{k}$.
   b) Determine the components of each of the vectors $\overrightarrow{EB},\ \overrightarrow{EG}\ and\ \overrightarrow{EB}\wedge\overrightarrow{EG}$.
   c) Determine an equation of the plane (EBG).
2. Let α be a real number different from 1 and M be the point of coordinates (2α, 4α, 3α).
   a) Verify that M moves on the straight line (AG) deprived the point G.
   b) Prove that M not belongs to the plane (EBG).
3. Let V be the volume of tetrahedron MEBG.
   a) Express V in terms of α.
   b) Calculate the volume of tetrahedron AEBG.

c) For what values of α do we have V is equal to the volume of parallelopipede ABCDEFGH ?

Exercise 15

The space is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$.

Consider the points: A (2 ; 5 ; -1), B (2 ; 2 ; -1), C (0 ; 3 ; -3), D (0 ; 6 ; -3) and the straight line

$$(d):\begin{cases} x = t+3 \\ y = 5 \quad\quad , \quad t \in IR. \\ z = -t-2 \end{cases}$$

1. Justify that $\overrightarrow{AB} = \overrightarrow{DC}$ *and that* $\left\|\overrightarrow{BA}\right\| = \left\|\overrightarrow{BC}\right\|$. Deduce that ABCD is a Rhombus.
2. Designate by (P) the plane (ABCD) and S is a variable point on (d).
   a. Verify that: x – z – 3 =0 is an equation of the plane (P).
   b. Prove that (d) is perpendicular at A to (P).
   c. Prove that the volume of tetrahedron SABC is $V = \sqrt{2}SA$ units of volumes.
   d. Deduce the coordinates of S, in the case V = 2 units of volumes.
3. Let I be the center of the Rhombus ABCD. In the plane (P), consider the circle (γ) of center I and of radius R= $\sqrt{3}$.
   a. Justify that (γ) passes through A. Deduce that C is a point of (γ).
   b. The tangent (Δ) at A to (γ), cuts (CD) at E. Calculate the coordinates of E. (Page 6)
   c. Let M be a variable point on (d) and (T) $= (MAD) \cap (MBC)$. Determine a director vector of (T).

LOGQRITHEMIC FUNCTIONS

Exercise 1

Consider the function f that is defined over $]-2;+\infty[$ as: $f(x)=x-\dfrac{\ln^2(x+2)}{(x+2)}$.

(C) is the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Determine $\lim\limits_{x\to -2} f(x)$. Deduce an asymptote to (C).
2. Determine $\lim\limits_{x\to +\infty} f(x)$ and verify that the line (D) of equation y = x is an asymptote to (C).
3. Let ϕ be a function defined over $]-2;+\infty[$ by: $\phi(x)=(x+2)^2-2\ln(x+2)+\ln^2(x+2)$.
   The adjacent table is the table of variations of ϕ.

x	-2		-1		$+\infty$
$\phi'(x)$		$-$	0	$+$	
$\phi(x)$	$+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

   Prove that: $f'(x)=\dfrac{\phi(x)}{(x+2)^2}$ and then deduce the sense of variations of f.

4. a- Prove that the straight line (D) is tangent to (C) at the point A (-1 ;-1) and that (D) is above (C) for all $x\neq 1$.
   b- Verify that the tangent (T) to (C) at a point of abscissa e^2-2 is parallel to (D).
5. Prove that the equation f (x) = 0 has a unique root α and verify that: $0.3<\alpha<0.4$.
6. Set up the table of variations of f and then draw (D), (T) and (C).
7. a- Prove that f admits an inverse function f^{-1} over $]-2,+\infty[$.
   b- Precise the domain of definition of f^{-1}, then draw (C') the representative curve of the function f^{-1} in the same system as (C).
8. a- Calculate $\int_{-1}^{\alpha} -f(x)dx$ in terms of α.
   b- Deduce in terms of α the area of the domain bounded by (C), (C') and the lines x = 0 and y = 0.

Exercise 2

Let f be a function defined over $]-1;2[$ by: $f(x)=\ln(x+1)-\ln(2-x)$ and let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

A

1) Calculate $\lim\limits_{x\to -1} f(x)$, $\lim\limits_{x\to +2} f(x)$ and then deduce the asymptotes of (C).
2) Prove that f is strictly increasing and then set up its table of variations.
3) Prove that the curve (C) admits an inflection point of abscissa $\dfrac{1}{2}$.
4) The straight line (d) of equation y = x cuts (C) at a point of abscissa α.
   Verify that: $1.41<\alpha<1.42$.
5) Draw (C).
6) The function f has an inverse function g.
   a- Precise the domain of definition of g and study the sense of variations of g.

b- Draw (Γ) the representative curve of g in the same system as (C).

c- Prove that: $g(x)=\dfrac{2e^{x}-1}{e^{x}+1}$.

7) a- Verify that the function F defined over $]-1;2[$ by: $F(x)=(x+1)\ln(x+1)+(2-x)\ln(2-x)$ is a primitive of f.

b- Calculate in terms of α, the area of the domain bounded by (C), (Γ), the x-axis and the y – axis.

B Consider the sequence (U_n) defined by $U_0 = 0$ and for all naturel number n, $U_{n+1}=g(U_n)$.

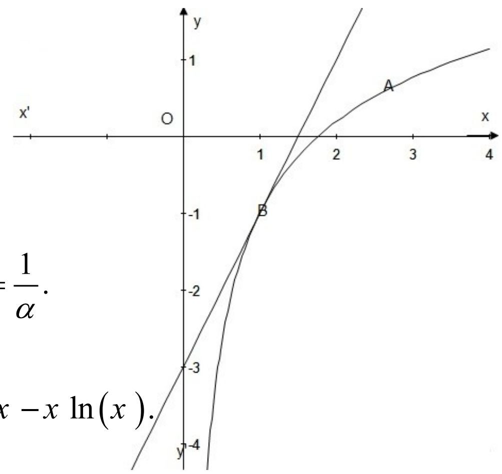
1) Prove by mathematical induction that for all n we have $U_n<\alpha$.
2) Prove that the sequence(U_n) is strictly increasing.
3) Deduce that the sequence (U_n) is convergent and determine its limit.

Exercise 3

The adjacent curve(C)is the representative curve of the function g that is defined over $]0;+\infty[$ by:

$$g(x)=a\ln(x)+\frac{b}{x}.$$

(C) passes through the point A $(e;1-e^{-1})$. (T) is the tangent to(C) at a point B (1;-1).



1. a) Calculate g (1), g (e) and $g'(1)$.

b) Determine an equation of (T).

c) Set up the table of variations of g over $]0;+\infty[$.

2. a) Calculate $g'(x)$.

b) Determine the real's a and b.

c) Prove that there exists a unique real α in $]1;e[$ such that $\ln(\alpha)=\dfrac{1}{\alpha}$.

d) Deduce the sign of the function g over $]0;+\infty[$.

3. Let f be a function that is defined over $]0;+\infty[$ by: $f(x)=\ln(x)+x-x\ln(x)$.

a- Calculate $\lim\limits_{x\to0^{+}} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$.

b- Calculate $f'(x)$ and then deduce $f'(x)=-g(x)$.

c- Set up the table of variations of f over $]0;+\infty[$.

d- Prove that: $f(\alpha)=\dfrac{1}{\alpha}+\alpha-1$. Take $\alpha=1.7$, Calculate f (1.7).

e - Draw (C_f) the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$

4. a- Prove that f has an inverse function f^{-1} over $]0\ ;\ \alpha[$. Precise $D_{f^{-1}}$ and draw $(C_{f^{-1}})$.

b- Calculate f (1).

c- Calculate $(f^{-1})'(1)$.

Exercise 4

Consider the function f that is defined over $]0;+\infty[$ by: $f(x)=\frac{x^2}{2}-4x+(4x-x^2)\ln(x)$, designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Calculate $\lim_{x\to 0^+} f(x)$ *and* $\lim_{x\to+\infty} f(x)$, then calculate f (1) and f(e).
2. Verify that: $f'(x)=(4-2x)\ln(x)$.
3. Set up the table of variations of f.
4. a. Set up the table of variations of $h(x)=f''(x)$ and deduce that (C) has an inflection point I of abscissa $\alpha\in]1.4\ ;\ 1.5[$.

   b. Verify that $\ln(\alpha)=\frac{2-\alpha}{\alpha}$ and deduce the expression of f(α).
5. Suppose that α=1.4. Plot I and draw (C).
6. a. Let F be a function defined over $]0;+\infty[$ by: $F(x)=\left(2x^2-\frac{x^3}{3}\right)\ln(x)-x^2+\frac{x^3}{9}$.

   Prove that F is a primitive of the function $g(x)=(4x-x^2)\ln(x)$ *over* $]0;+\infty[$.

   b. Let D be the domain bounded by the curve (C), the x-axis and the two straight lines x = λ and x = 1 where 0 < λ < 1. Calculate A (λ) the area of the domain D.

   c. Calculate $\lim_{\lambda\to 0} A(\lambda)$.

Exercise 5

Let f be a function that is defined on $]0,+\infty[$ by: $f(x)=1-\frac{1}{x}+\ln(x)$. Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$, Graphic unit: 2cm.

1. Calculate $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to+\infty} f(x)$. deduce an asymptote to (C).
2. Write an equation of the tangent (T) to (C) at a point of abscissa 1.
3. Consider the function g that is defined on IR* by: $g(x)=e^{\frac{2}{x}}-x$.
   a. Prove that the equation g (x) =0 admits a unique solution α so that: 2<α<3.
   b. Let M (x_0, y_0) be a point of (C) so that the tangent (D) to (C) at a point M passes through the origin O. Calculate x_0.
4. Calculate $f'(x)$, and then set up the table of variations of f.
5. Draw (T), (D) and (C).
6. a. Prove that f has an inverse function h. Precise the domain of definition of h.

   b. Set up the table of variations of h. Deduce the sign of h (x).

   c. Calculate h (0) and $h'(0)$.

   d. Draw the representative curve (C') of h in the same system as (C).

   e. Construct the tangent (T') to (C') at the point of intersection of (C') with the axis of ordinates Write an equation of (T') without using $h'(x)$.

f. Determine the tangent of the acute angle θ between (T) and (T').

g. (T) and (T') cuts respectively y' y and x' x at E and F. (T) and (T') are intersects at G. Calculate the area of the triangle EFG.

7. Calculate, in cm^2, the area of the domain defined by the points M (x; y) so that:

$$1 \le x \le e \ \ and \ \ 0 \le y \le f(x).$$

Exercise 6

Consider the function f that is defined as: $f(x) = 2x - 3 + \ln\left(\frac{x^2 - 2x + 2}{x^2}\right)$ and let (C) be its representative curve in an orthonormal system.

1. a) Verify that the domain of definition of is IR*.
   b) Calculate $\lim\limits_{x \to 0} f(x)$, interpret graphically.
   c) Calculate $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.
   d) Prove that (C) has two asymptotes one of them (D)of equation y = 2x -3.
   Study the relative position of (C) and (D).
2. a) Calculate $f'(x)$ then verify that: $f'(x) = \frac{2(x-1)}{x}\phi(x)$ where ϕ is a function strictly positive for all $x \neq 0$ is to be determined.
   b) Set up the table of variations of f.
   c) Prove that the equation f (x) = 0 has three different solutions $\alpha,\ \beta$ and γ.
   d) Draw (C).
3. a) Verify that for all real x we have: $\frac{2x-4}{x^2-2x+2} = \frac{2x-2}{x^2-2x+2} - \frac{2}{1+(x-1)^2}$.
   b) Calculate $A = \int_2^{1+\sqrt{3}} \frac{2x-2}{x^2-2x+2}dx.$
   c) Suppose that: x = 1 + tan (t) for all $t \in \left[0; \frac{\pi}{2}\right[$; calculate $B = \int_2^{1+\sqrt{3}} \frac{2}{1+(x-1)^2}dx.$
   d) Deduce from the preceding the value of the area S of the domain bounded by (C) and the three straight lines of equations: y = 2 x -3; x = 2 and x = 1 + $\sqrt{3}$.

Exercise 7

Consider the function f that is defined by: $f(x) = 2 + \ln(x^2 - 2x)$ and let (C) be its representative curve.

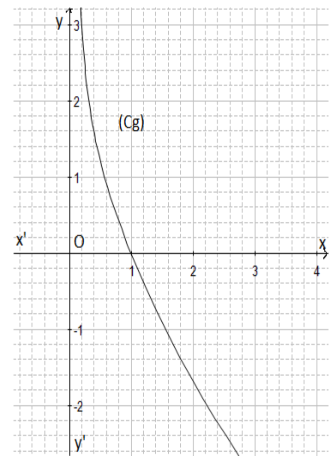
1. Verify that the domain of f is $I = \left]-\infty; 0\right[\cup \left]2; +\infty\right[$.
2. a. Calculate $\lim\limits_{x \to 0^-} f(x)$, $\lim\limits_{x \to 2^+} f(x)$, $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.
   b. Deduce the asymptotes of (C).
3. Calculate $f'(x)$ and set up the tableau of variations of f.
4. Solve the equation f(x)=0.
5. Draw (C).
6. a. Prove that f has over $\left]-\infty; 0\right[$ an inverse function g, and find D$_g$.
   b. Verify that: $g(x) = 1 - \sqrt{1 + e^{x-2}}$. Draw (G) the representative curve of g.

c. Calculate $g'(x)$ and then deduce the slope of the tangent to the curve (C) at a point A of abscissa $1-\sqrt{1+\frac{1}{e^2}}$.

<u>Exercise 8</u>

<u>Part A :</u>

Let g be a function defined over]0; +∞[by: $g(x)=\frac{1}{x}-[\ln(x)]^2-\ln(x)-1$. Designate by ($C_g$) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. (The adjacent figure)



1. Set up the table of variations of g.
2. Calculate g (1) then study, according to the values of x, the sign of g (x).

<u>Part B :</u>

Let f be a function defined over]0; +∞ [by: $f(x)=\frac{1+\ln(x)}{1+x\ln(x)}$ and designate by (C_f) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. (unit: 2 cm).

1. Determine $\lim\limits_{x\to 0+} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$. Deduce the asymptotes to (C_f).
2. Prove that $f'(x)=\frac{g(x)}{[1+x\ln(x)]^2}$ and set up the table of variations of f.
3. Verify that (T) of equation $y=\left(\frac{e^2}{e-1}\right)x-\frac{e}{e-1}$ is the equation of the tangent to (C_f) at its point of intersection with the x – axis.
4. Draw (C_f) and (T).
5. Prove that the curve (C_f) cuts the straight line (d): y = x in two points of respective abscissas α and 1, then verify that: 0.5 < α < 0.6.
6. The curve (C_f) admits over]0 ;1] an inverse function h.
   a- Give the domain of definition of h and draw (C_h).
   b- Find the domain of definition of the function t (x) = ln (α – h (x)).
7. a – Verify that $\int_{\alpha}^{1} f(x)dx=\ln\left(\frac{1}{1+\alpha\ln(\alpha)}\right)=\ln(\alpha)-\ln(1+\ln(\alpha))$.
   b- Calculate, in terms of α, the area of the region bounded by the two curves (C_f) and (C_h).

<u>Exercise 9</u>

<u>Part A:</u>

Consider the function g that is defined over $]0;+\infty[$ as: $g(x)=\ln\left(1+\frac{1}{x^2}\right)-\frac{2}{x^2+1}$.

Let (C_g) be its representative curve in the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ graphic unit 2 cm.

1. Calculate $\lim\limits_{x\to 0^+} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x)$, deduce the asymptotes to (C_g).

2. a- Prove that: $g'(x)=\dfrac{2(x^2-1)}{x(x^2+1)^2}$.

   b- Set up the table of variations of g.
3. Prove that the equation g (x) = 0 has a unique solution α such that: $0.5 < \alpha < 0.6$.
4. Draw the curve of the function g.
5. a- Prove that the function g has an inverse function g^{-1} over $]0;\ \alpha]$.

   b- Precise the domain of definition of g^{-1}.

   c- Draw the representative curve of the function g^{-1} in the same system as that of f.

<u>Part B :</u>

Consider the function f that is defined over $]0;\ +\infty[$ as: $f(x)=\begin{cases} x\ln\left(1+\dfrac{1}{x^2}\right) & \text{If } x>0 \\ 0 & \text{If } x=0. \end{cases}$

Let (C_f) be its representative curve in an orthonormal system $(O;\ \vec{i},\ \vec{j})$ graphic unit: 2 cm.

1. Prove that: $\lim\limits_{x\to 0^+} x\ln\left(1+\dfrac{1}{x^2}\right)=0$.
2. a- Calculate $\lim\limits_{x\to+\infty} x\,f(x)$; and then deduce $\lim\limits_{x\to+\infty} f(x)$.

   b- Deduce an asymptote to (C_f).
3. Study the differentiability f at 0. Precise the tangent to the curve (C_f) at x = 0.
4. a- Prove that for all x > 0 we have: $f'(x)=g(x)$.

   b- Deduce the sense of variations of f on $]0;+\infty[$.

   c- Set up the table of variations of f.
5. Prove that (C_f) admits an inflection point I. Determine the coordinates of point I.
6. Write an equation of the tangent (T) to (C_f) at the point I.
7. Prove that: $f(\alpha)=\dfrac{2\alpha}{\alpha^2+1}$. (Assume that: α = 0.54).
8. Draw the curve (C_f) and (T).

<u>Exercise 10</u>

<u>Part A:</u>

Let g be a function defined over $]0;+\infty[$ by: g (x) = x ln (x) – x + 1, and let (C) be its representative curve in an orthonormal system $(O;\ \vec{i},\ \vec{j})$.

1. a - Set up the table of variations of g.

   b- Deduce the sign of g (x) according to the values of x.
2. a- Study the relative positions of the curve (C) and the representative curve (C') of the function ln (x).

   b- Calculate the integral: $J=\int_1^e (x-1)\ln(x)\,dx$.

<u>Part B:</u> Consider the function f that is defined over $]1;+\infty[$ as: $f(x)=\left(\dfrac{1}{x-1}\right)\ln(x)$.

1. Calculate the limits of f at $+\infty$ and at 1.
2. Set up the table of variation of f.

3. Prove that the equation $f(x)=\frac{1}{2}$ admits a unique solution α so that: $3.5 \prec \alpha \prec 3.6$.
4. Construct the representative curve of f in the system $(O;\vec{i},\vec{j})$.

Part C: Consider the function h that is defined over $]1;+\infty[$ as: $h(x)=\ln(x)+\frac{1}{2}x+\frac{1}{2}$.

1. Prove that α is a solution of the equation h (x) = x.
2. Study the sense of variation of h.
3. Suppose that I = [3; 4]. Prove that for all x belongs to I, we have:

$$h(x)\in I \ \ and \ \ |h'(x)|\leq\frac{5}{6}.$$

4. Consider the sequence $(U_n)_{n\in IN}$ defined as: $U_0 = 3$ and $U_{n+1} = h(U_n)$ for all $n\geq 1$.
   a- Prove that for all n belongs to IN, U_n belongs to I,
   b- Prove that: $\forall n\in IN,\ |U_{n+1}-\alpha|\leq\frac{5}{6}|U_n-\alpha|$.
   c- Deduce that for all $n\in IN$, $|U_n-\alpha|\leq\left(\frac{5}{6}\right)^n$.

Exercise 11

Consider the function f that is defined over $]1;+\infty[$ as: f (x) = 3 + 2ln (x – 1) – ln (x + 1) and let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

Part A:

1. a - Calculate $\lim\limits_{x\to 1} f(x)$ *and* $\lim\limits_{x\to +\infty} f(x)$. Deduce an asymptote to (C).
   b- Calculate $f'(x)$ and set up the table of variations of f.
   a- Prove that the equation f (x) = 0 has on an interval $]1;+\infty[$ a unique solution α such that: 1.34 < α < 1.35.
   b- Verify that: f (3) = 3 and then draw (C).
2. a - Prove that if x belongs to the interval I = [3, 5], then f (x) belongs to I.
   b- Study the sense of variations of f' over I.
   c- Prove that for all x of I, we have: $\frac{1}{3}\leq f'(x)\leq\frac{3}{4}$.
   d- Deduce that for all x of I: $0\leq f(x)-3\leq\frac{3}{4}(x-3)$.

Part B:

Consider the sequence (U_n) that is defined as: $U_0 = 5$ and for all non – zero naturel number n, $U_{n+1} = f(U_n)$.

1. Prove that for all naturel number n we have:
   a- U_n belongs to I.
   b- $0\leq U_{n+1}-3\leq\frac{3}{4}(U_n-3)$.
   c- $0\leq U_n-3\leq 2\times\left(\frac{3}{4}\right)^n$.
2. Prove that the sequence (U_n) is convergent and calculate its limit L.

Exercise 12

1. Consider the function F that is defined over $]0;+\infty[$ by: $F(x)=\frac{1}{2}x^2(3-2\ln(x))+1$.

   Prove that F has an extension by continuity at 0 by the function f defined by:

   $$\begin{cases} f(x)=\frac{1}{2}x^2(3-2\ln(x))+1 & if \quad x>0 \\ f(0)=1 \end{cases}$$

   Denote by (C) the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$. G.U = 2 cm.

2. a- Calculate $\lim\limits_{x\to 0}\frac{f(x)-1}{x}$, What can you say about the curve (C) at the point A (0 ;1).

   b- Determine $\lim\limits_{x\to+\infty} f(x)$.

3. Verify that, for all x > 0, $f'(x)=2x[1-\ln(x)]$ and set up the table of variations of f.

4. a- Prove that the equation f (x) = 0 has a unique root α in the interval $[e;+\infty[$.

   b- Verify that: 4.6 < α < 4.7.

5. Write an equation of the tangent (T) to (C) at a point of abscissa 1.

6. Consider the function g defined over $]0;+\infty[$ by: $g(x)=f(x)-\left(2x+\frac{1}{2}\right)$.

   a- Calculate $g'(x)$ and Verify that: $g''(x)=-2\ln(x)$.

   b- Set up the table of variations of g ‘, (without calculating the limits) and deduce the sign of g ‘(x) over $]0;+\infty[$.

   c- Study the position of the tangent (T) with respect to the curve (C).

7. Draw (T) and (C).

8. a- By using the integration by parts, calculate $\int_1^e x^2\ln(x)\,dx$.

   b- Calculate, in cm 2, the area of the region bounded by the curve (C), and the two straight lines of equations x = 1and x = e.

Exercise 13

Consider the function f defined over the interval $]0;+\infty[$ by: f (x) = ln (x) – x ln (x) + x.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

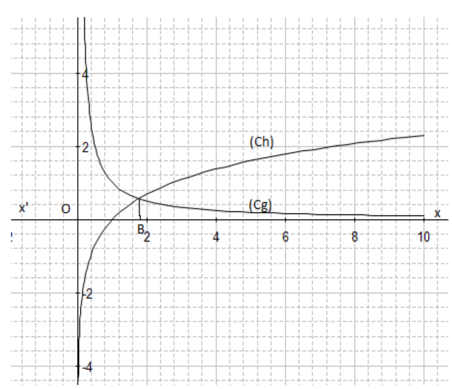
1. Determine $\lim\limits_{x\to 0^+} f(x)$; $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to+\infty}\frac{f(x)}{x}$.

2. a- Prove that for all x ∈ $]0;+\infty[$, $f'(x)=\frac{1}{x}-\ln(x)$.

   b- The curves (C_g) and (C_h) in the adjacent figure representing respectively the functions g and h defined over $]0;+\infty[$ by: $g(x)=\frac{1}{x}$ and $h(x)=\ln(x)$.

   (C_g) and (C_h) are intersect at a point of abscissa β, Such that: 1 < β < 2.

   By graphical reading give the sign of $f'(x)$.



   c- Set up the table of variations of f.

d- Prove that: $f(\beta)=\beta+\frac{1}{\beta}-1$ and verify that f (β) > 0.

3. Study the relative position of (C) and (C_h).
4. Prove that the straight line of equation y = x is tangent to (C) at a point of abscissa 1.
5. Prove that (C) cuts the x - axis at two points x_1 and x_2 such that:
   $0.4 < x_1 < 0.5$ and $3.8 < x_2 < 3.9$.
6. Suppose that β = 1.76, Draw the curve (C).
7. Prove that f admits over the interval $]0;\beta[$ an inverse function f^{-1}, determine the domain of f^{-1} and draw its curve (C') in the same system as that of (C).
8. For all real positive number t > β, we designate by A (t) the area of the domain bounded by the curves (C_g) and (C_h) and the line x = t. Prove that: $A(t)=f(\beta)-f(t)$.

Exercise 14

Part A: Consider the function g defined over]0; +∞[by: $g(x)=1-\frac{1}{x^2}+2\frac{\ln x}{x^2}$.

1. a. Calculate $\lim_{x\to 0^+} g(x)$ *and* $\lim_{x\to +\infty} g(x)$.

   b. Verify that: $g'(x)=\frac{4(1-\ln(x))}{x^3}$.

2. a. Set up the table of variations of g.

   b. Calculate g (1) and determine the sign of g (x).

Part B: We defined over]0; +∞[the functions h and f such that:

$$h(x)=\frac{1+2\ln(x)}{x} \quad and \quad f(x)=x-h(x).$$

Designate by (C) the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$.

1. a- Set up the table of variations of h.

   b- Write an equation of the tangent (d) to (C) at a point of abscissa $\sqrt{e}$.

2. a- Calculate $\lim_{x\to 0^+} f(x)$ *and* $\lim_{x\to +\infty} f(x)$.

   b- Verify that the straight line (D) of equation: y=x is an asymptote to (C), and determine the point of intersection of (C) and (D).

3. a- Prove that: $f'(x)=g(x)$, and set up the table of variations of f.

   b- Prove that the curve (C) admits an inflection point F to be determined.

   c- Draw (D), (d) and (C).

4. Calculate, in cm^2, the area of the region bounded by (C), (D) and the two straight lines of equations x=1 and x=e. (G.U = 2cm.)

5. a. Prove that f has over [1; +∞[an inverse function f^{-1}.

   b. The curve (C') of the function f^{-1} admits a tangent (d') parallel to (d).

Determine its contact point and draw (d') and (C') in the same system as that of (C).

Exercise 15

Part A :

In an orthonormal system $\left(O;\vec{i},\vec{j}\right)$ consider the function f, defined over $\left[0;+\infty\right[$, by:

$f\left(x\right)=x-\ln\left(1+x^2\right)$ and designate by (C) its representative curve.

1. a- Prove that for all x > 0, $f\left(x\right)=x-2\ln\left(x\right)-\ln\left(1+\frac{1}{x^2}\right)$.

   b- Calculate $\lim\limits_{x\to+\infty} f\left(x\right)$ *and* $\lim\limits_{x\to+\infty}\frac{f\left(x\right)}{x}$.

   c- Calculate $\lim\limits_{x\to+\infty}\left[f\left(x\right)-x\right]$ and interpret graphically the result.

2. Prove that: $f'\left(x\right)=\frac{\left(x-1\right)^2}{1+x^2}$ and set up the table of variations of f.
3. a- Write an equation of the tangent (d) to (C) at the point O.
   b- Study the relative position of the straight line (d) and the curve (C).
4. Prove that (C) admits an inflection point whose coordinates are to be determined.
5. Draw in the system $\left(O;\vec{i},\vec{j}\right)$ the straight line (d) and the curve (C).
6. a- Prove that f admits over the interval $\left[0;+\infty\right[$ an inverse function f^{-1} whose domain of definition is to be determined.
   b- Draw in the same system at that of (C') the representative curve of f^{-1}.
   c- M is a point on (C) and M' its symmetric with respect to the straight line of equation y = x. Calculate the abscissa of M such that: MM' = $\sqrt{2}$.

Part B :

Consider the function G that is defined over $\left[0;\frac{\pi}{2}\right[$ by: $G\left(x\right)=\int_0^{\tan x}\frac{1}{1+t^2}dt.$

1. a- Determine the derivative of the function G (x).

   b- Prove that for all x belongs to the interval $\left[0;\frac{\pi}{2}\right[$, G (x) = x.

   c- Deduce the value of $\int_0^1\frac{1}{1+t^2}dt.$

2. Designate by A the area of the region bounded by (C), (C') and the two lines of equations x = 1 and y = 1.

   a- Prove that: $\int_0^1\ln\left(1+x^2\right)dx=\ln 2-2+2\int_0^1\frac{1}{1+x^2}dx.$

   b- Deduce the value of A.

Part C :

We defined the sequence (U_n) by $U_0 = 1$ and $U_{n+1}=U_n-\ln\left(1+U_n^{\,2}\right)$ for all- natural number n.

1. Prove that for all -natural number n that: $0<U_n\le 1$.
2. Prove that (U_n) is decreasing.
3. Deduce that (U_n) is convergent and calculate its limit.

Exercise 16

Let f be a function that is defined over ℝ by: f (x) $= \begin{cases} 2x\ln(x)-2x & x>0 \\ 0 & x=0 \end{cases}$

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\ \vec{j})$.

1. a- Study the continuity and the differentiability of f at the point 0.

   b- Prove that: $f(x)=2x\ln\left(\frac{x}{e}\right)$. Determine the limit of f at + ∞.

   c- Find the point of intersection E of (C) with the x - axis and the equation of the tangent (T) at E to (C).

   d- Calculate $f'(x)$, set up the table of variations of f and draw (C) and (T).

   e- Let F be a function defined by: $F(x)=\begin{cases} x\ln(x^2)-2x & x\neq 0 \\ 0 & x=0 \end{cases}$

   Prove that F is an odd function and then deduce the drawn of its representative curve.

   f- Consider the equation (E): $x^2=e^{\frac{m}{x}+2}$.
   Discuss according to the values of m the existence of the roots of (E).

2. a- Prove that f admits an inverse function f^{-1} for all x > e.
   Draw the representative curve (C') of f^{-1} and determine the coordinates of the point of intersection L of (C) and (C').

   b- Designate by (d) the tangent at L to (C) and by (d') the tangent at L to (C').
   Calculate the trigonometric tangent of the angle ((d); (d')).

3. a – Prove that the function G defined by G (x) = $x^2\ln(x)-\frac{3x^2}{2}$ is a primitive of f over $]0;+\infty[$.

   b – Calculate the area of the region bounded by (C), the x – axis and the two straight lines of equations x = 1 and x = 2.

EXPONENTIAL FUNCTIONS

Exercise 1

Part A: The below table is the table of variations of a differentiable function f over ℝ:

x	$-\infty$		0		2		$+\infty$
$f'(x)$		-	0	+	0	-	
$f(x)$	$+\infty$	↘	0	↗	$4\ e^{-2}$	↘	0

1. Let F be a function defined over ℝ by: $F(x)=\int_2^x f(t)\,dt$. Determine F ‘(x).
2. Determine the sense of variations of the function F over ℝ.
3. Prove that: $0 \leq F(4) \leq 8e^{-2}$.

Part B: Consider the function f defined over ℝ by: $f(x)=x^2e^{-x}$.

Let (C) be the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$. U.G :2 cm.

1. Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote to (C).
2. Prove that: $f'(x)=(2x-x^2)e^{-x}$ and that the table of variations of f is given above in the part A.
3. Draw (C).
4. a- Prove that the equation f (x) = 1 admits a unique solution α such that: $-1 < \alpha < 0$.

   b- Prove that α verify the relation: $\alpha=-e^{\frac{\alpha}{2}}$.
5. a- Prove that f admits over the interval [0;2] an inverse function f^{-1} whose domain of definition is to be determined.

   b- Find an equation of the tangent to ($C_{f^{-1}}$) at a point of abscissa e^{-1}.

Exercise 2

Part A: Consider the differentiable equation (E): $y'+2y^2e^x-y=0.\ (y\neq 0)$

Suppose that: $z=\dfrac{1}{y}$, where z is a differentiable function over ℝ.

1. Find a differential equation (E’) satisfied by z. Solve (E’).
2. Deduce the particular solution of (E) whose representative curve passing through the point A (0; 0.5).

Part B: Consider the function f that is defined over ℝ by: $f(x)=\dfrac{1}{e^x+e^{-x}}$.

Let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Prove that f is an even function.
2. Determine $\lim\limits_{x\to+\infty} f(x)$ *and* $\lim\limits_{x\to-\infty} f(x)$. Deduce an asymptote to (C).
3. Calculate f ‘(x) and study its sign. Set up the table of variation of f.
4. a- Set up the table of variation of the function g defined over $[0;+\infty[$ by g (x) = f (x) – x.

   b- Deduce that the equation f (x) = x admits a unique solution α and that: $0.4 < \alpha < 0.5$.
5. Draw (C) (G.U: 4 cm).
6. a- Prove that f admits over $[0;+\infty[$ an inverse function h.

   b- Give the domain of definition of h and calculate h (x).

   c- Draw the curve (γ) of h in the same system.

Part C: Consider the sequence (U_n) that is defined over ℕ by: $U_n = \int_0^n f(x)\,dx.$

1. a- Prove that, for all $x \geq 0$, $f(x) < e^{-x}$.
   b- Deduce that, for all natural number n, $U_n < 1 - e^{-n}$.
2. a- Verify that: $U_{n+1} - U_n = \int_n^{n+1} f(x)\,dx.$
   b- Deduce that the sequence (U_n) is strictly increasing.
   c- Prove that the sequence (U_n) converge tends t a limit L such that: $0 \leq L \leq 1$.
3. Verify that: $f(x) = \dfrac{e^x}{1+e^{2x}}$. Calculate then U_n in terms of n and determine L.
4. Calculate, in cm^2, the area of the region bounded by (γ), y' y, x' x and the line y = 2.

Exercise 3

Part A: Consider the function g defined over ℝ by: $g(x) = (x-2)e^x + 2.$

Designate by (γ) its representative curve in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$.

1. Calculate $\lim\limits_{x\to-\infty} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x)$. Deduce an asymptote (d) to (γ).
2. Calculate $g'(x)$ and set up the table of variations of g.
3. Prove that the equation g (x) = 0 admits over ℝ two solutions 0 and α.
   Verify that: $1.5 < \alpha < 1.6$.
4. Draw (d) and (γ).
5. Calculate the area A (α) of the region bounded by (γ) and the x - axis.
6. The function g admits over $[1; +\infty[$ an inverse function g^{-1}. Designate by (γ') the representative curve of g^{-1} in the same system as that of (γ).
   a- Find the coordinates of the point of intersection of (γ) and (γ').
   b- Draw (γ').

Part B: Consider the differential equation (E): $y' - \left(1 - \dfrac{2}{x}\right)y = \dfrac{1}{x^2}$. Assume that: $z = x^2 y$.

1. Find a differential equation (E') satisfied by z.
2. Solve (E'). Deduce the general solution of (E).
3. Determine a particular solution of (E) whose representative curve passing through the point A (1; e – 1).

Part C: Let f be a function defined over $]-\infty;0[\cup]0;+\infty[$ by: $f(x) = \dfrac{e^x - 1}{x^2}$.

Designate by (C) its representative curve in a new orthonormal system $\left(O; \vec{i}, \vec{j}\right)$.

1.
   a- Calculate $\lim\limits_{x\to 0} f(x)$. Deduce an asymptote (D) to (C).
   b- Calculate $\lim\limits_{x\to-\infty} f(x)$, $\lim\limits_{x\to+\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x}$. Interpret.
2. Prove that: $f'(x) = \dfrac{g(x)}{x^3}$. Deduce that $f'(x) < 0$ over $]0; \alpha[$.
3. Prove that: $f(\alpha) = \dfrac{1}{\alpha(2-\alpha)}$. Set up the table of variations of f.
4. Draw (C).

Exercise 4

Part A: Consider the differential equation (E): $y''-4y'+4y=4x$. Assume that: z = y – x.

1. Write a differential equation (F) satisfied by z.
2. Solve the differential equation (F). Deduce the general solution of the equation (E).
3. Find the particular solution of (E) such that the tangent at A (0; 1) is parallel to the straight line of equation: y = 2x.

Part B: Consider the function f that is defined over ℝ by: f (x) = x + (1 – x) e ^{2x}.

Designate by (C) its representative curve in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$.

1.
   a- Calculate $\lim\limits_{x \to -\infty} f(x)$. Prove that the line (d) of equation y = x is an asymptote to (C).
   b- Study the relative position of (C) with respect to (d).
2. Prove that: $\lim\limits_{x \to +\infty} f(x) = -\infty$ and calculate $\lim\limits_{x \to +\infty} \frac{f(x)}{x}$. Interpret.
3.
   a- Calculate $f'(x)$, $\lim\limits_{x \to -\infty} f'(x)$ *and* $\lim\limits_{x \to +\infty} f'(x)$.
   b- Prove that: $f''(x) = -4xe^{2x}$. Set up the table of variations of f ‘.
   c- Prove that the equation f ‘(x) = 0 admits over ℝ a unique root α. Verify that: 0.63 < α < 0.65.
   d- Prove that: $f(\alpha) = \frac{2\alpha^2}{2\alpha - 1} - 1$. Set up the table of variations of f.
   e- Prove that f admits an inflection point E. Find an equation of the tangent (T) to (C) at E.
4. Suppose that: α = 0.64. Draw (d) and (C).
5.
   a- Find a primitive of $(1-x)e^{2x}$ over ℝ.
   b- Calculate the area of the region bounded by (C), (d) and the y - axis.
6. The function f admits over] - ∞; α] an inverse function h.
   a- Determine the domain of definition of h and draw its representative curve (C’) in the same system as that of (C).
   b- Find the coordinates of the point F of (C’) where the tangent is parallel to (d).
   c- Solve the in equation: ln [1 – h (x)] > 0.

Part C : Let n be a natural number such that : n ≥ 2.

1. Prove by mathematical induction that: $f^{(n)}(x) = -2^{n-1}\left[2x+n-2\right]e^{2x}$.
2. Consider the sequence (U_n) defined, for all n ≥ 2 by: $U_n = f^{(n)}(0)$.
   a- Prove that the sequence (U_n) is decreasing.
   b- Prove that (U_n) is not convergent.

Exercise 5

Part A: Consider the differential equation (E): $y''-3y'+2y=-4x+1$. Suppose that: y = z – 2 x - $\frac{5}{2}$.

1. Determine a differential equation (E’) satisfied by z.
2. Solve (E’) then deduce the general solution of (E).
3. Determine a particular solution of (E) whose representative curve in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$, passing through O (0;0) and admits at this point a tangent parallel to the x – axis.

Part B: Consider the function f, defined over $\mathbb{R}$ by: $f(x) = -\frac{1}{2}e^{2x} + 3e^{x} - 2x - \frac{5}{2}$. (C) its representative curve in an orthonormal system $(O; \vec{i}, \vec{j})$. (G.U: 2 cm).

1.
   a- Determine $\lim\limits_{x\to-\infty} f(x)$ and verify that the straight line (d) of equation y = $-2x - \frac{5}{2}$ is an asymptote to (C).
   b- Study, according to the values of x the relative position of (C) and (d).
   c- Prove that: $\lim\limits_{x\to+\infty} f(x) = -\infty$ *and* $\lim\limits_{x\to\infty} \frac{f(x)}{x} = -\infty$.
2. Verify that: $f'(x) = (1-e^{x})(e^{x}-2)$ and set up the table of variations of f.
3. Prove that the equation f (x) = 0 admits two solutions 0 et α such that: 0.9 < α < 1.
4. Prove that (C) admits an inflection point whose abscissa is to be determine.
5. Prove that there exist a unique point A of (C) where the tangent (T) at A is parallel to (d) whose coordinates are to be determined.
6. Justify that the equation of (T) is of the form y = - k x + k where k is a natural number is to be determined.
7. Draw (C), (T) and (d).
8. Prove that the area of the region bounded by (C) and the x - axis is equal to $\left(\frac{3}{2}e^{\alpha} - \alpha^{2} - \frac{3}{2}\alpha - \frac{3}{2}\right)$ unit square of area.

Exercise 6

Part A: Consider the differential equation (E): $y'' - 2y' + y = (4x+4)e^{-x}$. Assume that:

$$y = z + (x+2)e^{-x}.$$

1. Determine a differential equation (F) satisfied by z.
2. Solve (F) and deduce the general solution of (E).
3. Determine the particular solution of (E) whose representative curve is tangent at a point A (0 ;2) to the straight line (d) of equation: y = 3 x + 2.

Part B: Consider the function f that is defined over $\mathbb{R}$ as: $f(x) = 5 - (x-2)^{2} e^{x}$;

Designate by (C) the representative curve in an orthonormal system $(O; \vec{i}, \vec{j})$.

1.
   a- Prove that: $\lim\limits_{x\to-\infty} f(x) = 5$ and deduce an asymptote (d) to (C).
   b- Determine $\lim\limits_{x\to+\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} \frac{f(x)}{x}$, interpret graphically the result.
3. a- Prove that: $f'(x) = x(2-x)e^{x}$.
   b- Set up the table of variations of f.
   c- Prove that the equation f (x) = 0 admits a unique solution α such that: 2.6 < α < 2.7.
4. Draw (d) and (C).
5. Calculate the area of the region bounded by (C) and the coordinates axes.
6. Use the curve (C) to solve, according to the values of the parameter m, the equation $(x-2)^{2} e^{x} = m$.
7. We defined over $\mathbb{R}$ the function g by: $g(x) = e^{5-f(x)}$. Set up the table of variations of g.

Exercise VII

Part A: Consider the differential equation (E): $xy' - (2x+1)y = 8x^2$. assume that: y = x. z.

1. Find a differential equation (E') satisfied by z.
2. Solve (E'). Deduce the general solution of (E).
3. Find a particular solution g of (E) such that: g (ln (2)) = 0.

Part B: Consider the function h that is defined over $[0;+\infty[$ by: $h(x) = (2x+1)e^{2x} - 4$.

1. Calculate h '(x), then set up the table of variations of h.
2. Prove that the equation h (x) = 0 admits a unique solution α such that: 0.39 < α < 0.4.

Part C: Consider the function f that is defined over [0; +∞ [as: $f(x) = xe^{2x} - 4x$.

1. Determine $\lim\limits_{x\to+\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} \frac{f(x)}{x}$.
2. Study the variations of f and set up the table of variations of f.
3. Prove that: $f(\alpha) = \frac{-8\alpha^2}{2\alpha+1}$.
4. Draw the curve (C) of the function f in an orthonormal system $(O; \vec{i}, \vec{j})$. (Take α = 0.4).
5. a- Calculate $\int_0^{\ln 2} xe^{2x}dx$.

   b- Deduce the area of the region bounded by (C) and the x - axis.

Part D: For all n ∈ ℕ; suppose that: $I_n = \int_0^1 x^n e^{2x}dx$.

1. Calculate I_0.
2. Verify that for all n ∈ ℕ; we have: $0 \le I_n \le \frac{e^2-1}{2}$.
3. Prove that: $I_{n+1} = \frac{e^2}{2} - \left(\frac{n+1}{2}\right)I_n$.
4. Deduce the exact values of I_1 and I_2.
5. Prove that the sequence (I_n) is decreasing, and deduce that (I_n) is convergent.

Exercise VIII

Part A: Consider the differential equation (E): $y' + 2y = -4xe^{-2x}$. Suppose that: $z = y + 2x^2e^{-2x}$.

1. Write a differential equation (E') satisfied by z.
2. Solve (E') and then deduce the general solution of (E).
3. Determine a particular solution f of (E) such that: f (0) = 0.

Part B: consider the function f that is defined over ℝ by: $f(x) = -2x^2e^{-2x}$.

let (C) be its representative curve in an orthonormal system $(O; \vec{i}, \vec{j})$. G.U: 2 cm.

1. Calculate $\lim\limits_{x\to+\infty} f(x)$ and deduce an asymptote (d) to (C).
2. Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to-\infty} \frac{f(x)}{x}$.
3. Calculate $f'(x)$ and set up the table of variations of f.
4. Prove that the equation f (x) = - 1 admits a unique solution α and verify that: -0.5 < α < -0.4.
5. Draw (d) and (C).
6. a- Prove that f admits over]-∞; 0] an inverse function g and draw its curve (G) in the preceding

system.

b- Calculate, in terms of α, the area of the region bounded by (G), the x - axis and the two straight lines of equations x = - 1 and x = 0.

Part C: For all non - zero natural number, consider the function f_n defined over ℝ by:

$$f_n(x) = -2x^2e^{-2nx}.$$

Denote by (C_n) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

We defined, for all non – zero natural number n, $I_n = \int_{-1}^{0} f_n(x)dx.$

1. a – Justify that, for all non – zero natural number n and for all real x of [-1;0]:
$$f_{n+1}(x) = e^{-2x}f_n(x).$$
b- Deduce that for all non – zero natural numbers n and for all real number x in [-1;0]:
$$f_{n+1}(x) \le f_n(x).$$
c- Determine then the sense of variations of I_n.
2. a- Justify that, for all non – zero natural number n and for all x of [-1;0]:
$$-2e^{-2x} \le f_n(x) \le 0.$$
b- Deduce a boundary of I_n.
c- What can you deduce about the limit of (I_n)? Justify.

Exercise IX

Part A: Consider the differential equation (E): $y' + y = 1 + e^{-x}$. suppose that: $y = z + x + xe^{-x}$.

1. Prove that the differential equation (E') satisfied by z is: $z' + z = -x.$
2. Find a particular solution of (E') of the form z = a x + b.
3. Solve (E'). Deduce the general solution of (E).
4. Determine a particular solution of (E) such that: y (0) = 2.

Part B: Consider the function f that is defined over ℝ as: $f(x) = (x+1)e^{-x} + 1$

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1. Calculate $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote (d) to (C).
2. Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x}$.
3. Calculate $f'(x)$ and set up the table of variations of f.
4. Prove that (C) admits an inflection point I whose coordinates are to be determined.
5. Find an equation of the tangent (T) to (C) at I. Draw (d), (T) and (C).
6. Calculate the area of the region bounded by (C), (d) and the lines x = - 1 and x = 0.
7. Prove that f admits over [0; +∞[an inverse function f^{-1} and draw its curve (C') in the preceding system. Calculate $\left(f^{-1}\right)'\left(\dfrac{2}{e}+1\right)$.
8. Consider the function f_n defined by: $f_n(x) = (1+x)^n e^{-x} + 1$ and the sequence (U_n) defined by:
$$U_n = \int_0^1 \left[f_n(x) - 1\right]dx.$$
a- Prove that: $1 - \dfrac{1}{e} \le U_n \le \left(1 - \dfrac{1}{e}\right).2^n.$

b- Prove that the sequence (U_n) is increasing.

Exercise 7

Part A :

Consider the differential equation (E): $y''+3y'+2y=\left(\frac{x-1}{x^2}\right)e^{-x}$.

1. Prove that the function P defined on]0; +∞ [by: $P(x)=e^{-x}\ln x$ is a particular solution of the equation (E).
2. Determine the general solution of the equation: $y''+3y'+y=0$.
3. Deduce the general solution of (E) and the particular solution whose curve passes through the points: $A\left(1;\frac{3}{e}\right)$ *and* $B\left(2;\frac{3+\ln 2}{e^2}\right)$.

Part B :

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

Consider the function g defined over]0; + ∞ [by: $g(x)=-3-\ln x+\frac{1}{x}$ and designate by (C_g) its representative curve.

1. Study the variations of g and set up its table of variations.
2. Prove that the equation g (x) = 0 has a unique solution α and verify that: 0.45 < α <0,.46.
3. Deduce the sign of g (x) over]0 ; + ∞[.
4. Draw the curve (C_g).

Part C: Consider the function f defined over]0; + ∞[by: $f(x)=e^{-x}(3+\ln x)$ and designate by (C) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1. Calculate the limits of f at the boundaries of its domain of definition and deduce the equations of the asymptotes to (C).
2. a – Prove that for all x of]0; + ∞ [, $f'(x)=e^{-x}.g(x)$.

   b- Study, according to the values of x, the sign of $f'(x)$ and set up the tableau of variations of f.
3. Prove that $f(\alpha)=\frac{e^{-\alpha}}{\alpha}$. Give a boundary of f (α).
4. Calculate $f\left(e^{-3}\right)$ and draw (C).
5. Let A be the area of the region bounded by (C), the x – axis and the straight lines of equations $x=e^{-3}$ *and* $x=\alpha$. Prove that: $A\le e^{-\alpha}$.

Exercise 8

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$, graphic unit 2 cm.

Part A :

Consider the differential equation (E): $y''+4y'+4y=-4x$.

1. Determine m and n so that h (x) = m x + n is a particular solution of (E).
2. Consider the differential equation (E'): $y''+4y'+4y=0$.
   a- Determine the general solution of (E ').
   b- Deduce the general solution of (E).
3. Determine the particular solution of (E) such that: y (0) = 2 and y '(0) = - 2.

Part B :

Let (γ) be the curve of the function g defined on ℝ by: $g(x)=e^{2x}+2x+1$.

1. Calculate $\lim_{x\to-\infty} g(x)$; $\lim_{x\to+\infty} g(x)$, and g (0.5).
2. Calculate $g'(x)$, then set up the table of variations of g.
3. Prove that the equation g (x) = 0, has a unique solution α such that:
   - 0.7 < α < - 0. 6
4. Prove that the straight line (d) of equation: y = 2 x + 1 is asymptote to (γ) at - ∞.
5. Study the relative position of (γ) and (d).
6. Draw (γ) and (d). [Assume that α = - 0.65]. Deduce the sign of g (x) according to the values of x.
7. Calculate the area of the region bounded by (γ), (d), and the two straight lines of equations x = - 1 and x = 0.

Part C :

Let (C) be the curve of the function f defined on ℝ by: $f(x)=(x+1)e^{-2x}+1-x$.

1. Calculate $\lim_{x\to-\infty} f(x)$ *and* $f(-1)$.
2. Calculate $\lim_{x\to+\infty} f(x)$ *and* $f(0)$.
3. a- Prove that the straight line (D) of equation y = 1 – x is asymptote to (C) at +∞.
   b- Study the relative position of (C) and (D).
4. a- Prove that: $f'(x)=e^{-2x}\left[-g(x)\right]$.
   b- Set up the table of variations of f.
5. Write an equation of the tangent (T) to (C) at a point of abscissa 0.
6. Draw (C) and (D) in a new system.
7. Let m be a strictly positive real number.
   a- Calculate in cm ^{2}, the area A (m) of the region bounded by (C), (D) and the straight lines of equations x = 0 and x = m.
   b- Calculate $\lim_{m\to+\infty} A(m)$.

Exercise 9

Part A: Consider the differential equation (E): $y''+2y'+y=2e^{-x}$.

Assume that: $y=ze^{-x}$.

1. Find a differential equation (E’) satisfied by z.
2. Solve the differential equation (E’) thus obtained. Deduce the general solution of (E).
3. Determine a particular solution of (E) such that: $y(1)=y'(1)=\frac{1}{e}$.

Part B :

Consider the function f defined over ℝ by: $f(x)=x^2e^{-x}$. Let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. (Graphic unit 2 cm).

1. Calculate $\lim_{x\to+\infty} f(x)$, $\lim_{x\to-\infty} f(x)$ *and* $\lim_{x\to-\infty} \frac{f(x)}{x}$. Deduce an asymptote to (C).
2. Prove that: $f'(x)=(2x-x^2)e^{-x}$ and set up the table of variations of f.
3. Write an equation of the tangent (T) to (C) at a point of abscissa – 1.
4. Draw (C).
5. a) Prove that the equation f (x) = 1 admits a unique solution α such that: - 0.8 < α < - 0.6.

b) Prove that: $\alpha = -e^{\frac{\alpha}{2}}$.

6. a) Prove that f admits over the interval [2; + ∞[an inverse function f^{-1} whose domain of definition is to be determined. Draw $\left(C_{f^{-1}}\right)$.

   b) Calculate $\left(f^{-1}\right)'\left(9e^{-3}\right)$.

7. a) Determine the real numbers a, b and c so that the function h (x) = $\left(ax^2+bx+c\right)e^{-x}$ is a primitive of f.

   b) Calculate, in cm ^{2}, the area of the region bounded by (C), the x - axis and the two straight lines of equations x = 0 and x = 2.

8. Consider the function F defined over ℝ by: $F(x)=\int_2^x f(t)\,dt$.

   a) Determine the sense of variations of the function F over ℝ.

   b) Prove that: $0 \le F(4) \le 8e^{-2}$.

Exercise 10

Part A :

Consider the differential equation (E): $4y''-4y'+y=-x+3$, where y is a function of x.

Let Z be a function of x defined by: Z = y + x + 1.

1. Determine a differential equation (E') satisfied by Z.
2. Solve (E') then deduce the general solution of (E).
3. Determine the particular solution of (E) whose representative curve, in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$, is tangent at the point O (0; 0) to the straight line of equation y = $-\frac{1}{2}x$.

Part B :

Consider the function f, defined over ℝ, by: $f(x)=e^{\frac{x}{2}}-x-1$.

(C) is the representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1. Calculate $\lim\limits_{x\to+\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} \frac{f(x)}{x}$. Interpret graphically the results.
2. a- Calculate $\lim\limits_{x\to-\infty} f(x)$.

   b- Prove that the straight line (d) of equation y = - x – 1 is an asymptote to (C).

   a- Study the relative position of (C) and (d).
3. Calculate $f'(x)$ and set up the table of variations of f.
4. Prove that the equation f (x) = 0 admits two solutions one of them 0 and the other α such that: 2.5 < α < 2.6.
5. Draw (d) and (C).
6. Calculate the area of the region bounded by (C), (x' O x) and the two straight lines x = 0 and x= ln 4.
7. Prove that f has an inverse function f^{-1} on $\left]-\infty;\ln 4\right]$, find the domain of definition of f^{-1} then draw ($C_{f^{-1}}$).
8. Show that the equation f (x) = 1 admits two solutions a and b such that:

   $$-1.6 < a < -1.5 \quad and \quad 3.3 < b < 3.4.$$
9. Consider the function g (x) = $\sqrt{\ln\left(e^{\frac{x}{2}}-x-1\right)}$.

a- Determine the domain of definition of g.
b- Set up the table of variations of g.

10. Consider the function h, defined on $]-1;+\infty[$, by: $h(x)=2\ln(x+1)$.
    a- Set up the table of variations of h.
    b- Prove that: h (α) = α.
    c- Prove that $h(x)\geq 2$ for all x ≥ 2.

<u>Exercise 11</u>

<u>Part A:</u> Consider the function f defined on ℝ as: $f(x)=e^{-x}\ln(1+e^{x})$, (C) is the representative curve of f in an orthogonal system $(O;\vec{i},\vec{j})$ (Graphic unit: 1 cm on the x - axis and 2 cm on the y - axis)

1. a) Determine the limit of f at - ∞.
   b) Verify that $f(x)=\dfrac{x}{e^{x}}+e^{-x}\ln(1+e^{-x})$. Determine the limit of f at + ∞.
   c) Deduce that the curve (C) admits two asymptotes are to be determined.
2. Consider the function g defined on $]-1;+\infty[$ as: $g(x)=\dfrac{x}{1+x}-\ln(1+x)$.
   a) Prove that the function g is strictly decreasing on $]0;+\infty[$.
   b) Deduce the sign of g (x) when x > 0.
3. a) Express $f'(x)$ in terms of $g(e^{x})$.
   b) Set up the table of variations of f.
4. Draw (C).

<u>Part B :</u> Consider the function F defined on ℝ as: $F(x)=\int_{0}^{x}f(t)\,dt$.

1. Study the sense of variations of the function F.
2. a) Verify that: $\dfrac{1}{1+e^{t}}=1-\dfrac{e^{t}}{1+e^{t}}$ and determine $\int_{0}^{x}\dfrac{dt}{1+e^{t}}$.
   b) Deduce by using the integration by parts, F (x).
   c) Verify that: $F(x)=x-\ln(1+e^{x})-f(x)+2\ln 2$, and that: $F(x)=\ln\left(\dfrac{e^{x}}{1+e^{x}}\right)-f(x)+2\ln 2$.
3. Determine $\lim\limits_{x\to-\infty}F(x)$.
4. Determine $\lim\limits_{x\to-\infty}\left[F(x)-x\right]$. Give a geometric interpretation of this result.

<u>Part C:</u> Let (U_n) be a sequence defined on IN^{*} as: $U_n=f(1)+f(2)+\dots\dots\dots+f(n)$.

1. Shad on the graphical representation the domain whose area in square units is U_4.
2. Determine the sense of variation of the sequence (U_n).
3. Prove that: $f(K)\leq\int_{k-1}^{k}f(t)\,dt$, where $1\leq k\leq n$, then compare U_n and F (n).
4. Is The sequence (U_n) convergent? Justify.

<u>Exercise 12</u>

<u>Part A :</u>

Let g be a function defined over ℝ by: $g(x)=xe^{x}-e$.

1. Study the sense of variations of the function g.

2. Determine: $\lim_{x\to-\infty} g(x)$ and $\lim_{x\to+\infty} g(x)$, then calculate $g(1)$. Deduce the sign of $g(x)$.

3. Set up the table of variations of the function g.

Part B :

Consider the function f defined over $\mathbb{R}$ by: $f(x)=(x-1)e^x-ex+2$.

Let (C) be the representative curve of f in an orthonormal system $(O;\vec{i};\vec{j})$.

1. Determine: $\lim_{x\to-\infty} f(x)$, $\lim_{x\to+\infty} f(x)$ and $\lim_{x\to+\infty} \frac{f(x)}{x}$.

2. Calculate $f'(x)$, the derivative of $f(x)$ and set up the table of variations of f.

3. Prove that the equation $f(x)=0$ has two solutions α and β such that:

$0.4<\alpha<0.41$ and $1.45<\beta<1.46$.

4. Consider the function h defined over $\mathbb{R}-\{1\}$ by : $g(x)=\frac{e^x-2}{e^x-e}$, and let (Γ) be its representative curve in the system $(O;\vec{i};\vec{j})$.

Prove that (Γ) cuts the straight line of equation $y=x$ in two points A and B whose coordinates are to be determined in terms of α and β.

CONICS

Exercise 1

Part A: The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$. Consider the two straight lines (D) and (D') pf respective equations (D): x + y + 1 = 0, (D'): x – y +1 = 0 and the point L (- 2; 0).

1. Draw (D) and (D'). Solve graphically the inequation: $(x+1)^2 - y^2 > 0$.
2. For all point M (x; y) of the plane, we designate by d_1 and d_2 the respective distances from M to (D) and (D'). Let (H) be the set of points M (x; y) whose coordinates verifying the two conditions:

$$\begin{cases} d_1 \times d_2 = \dfrac{1}{2} \\ (x+1)^2 - y^2 > 0. \end{cases}$$

   a- Justify that $x^2 - y^2 + 2x = 0$ is the equation of (H).
   b- Prove that (H) is a hyperbola and that (x' x) it's an axis of symmetry.
   c- Prove that L and O are the vertices of (H).
   d- Deduce the center of (H). Draw (H).

Part B :

Let $S\left(I;\frac{1}{\sqrt{2}};-\frac{\pi}{4}\right)$ be a direct plane similitude of center I, ratio $K=\frac{1}{\sqrt{2}}$ and angle $\alpha=-\frac{\pi}{4}\,[2\pi]$.

Designate by (C) the curve of equation x y + y – 1 = 0. For all point M of affix Z = x + i y, we designate by M' of affix $Z'=x'+iy'$, the image of M by S.

1. Prove that: $Z'=\left(\frac{1}{2}-\frac{1}{2}i\right)Z-\frac{1}{2}-\frac{1}{2}i.$
2. Verify that: x = x '– y 'and y = x ' + y ' + 1.
3. Justify that (H) is the image of (C) by S.
4. Deduce that (C) is a rectangular hyperbola. Draw (C) in the same system as that of (H).
5. Let J (α ; β) be one of the points of intersection of (C) and (H).
   Designate by (T) the tangent at J to (H). Prove that (T) is normal to (C).

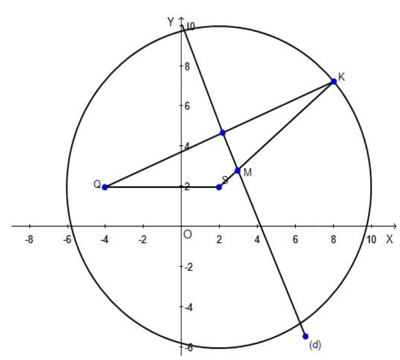
Exercise 2

In the plane referred to a direct orthonormal system $(O;\vec{i},\vec{j})$, given the points S (2;2) and Q (-4;2).
The adjacent figure to the right. (C) is a circle of center S and of radius 10, and K is a variable point on (C). The perpendicular bisector (d) of the segment [QK] cuts the straight line (SK) at M.

1. Prove that when K describes on the circle (C), then M moves on an ellipse (E) of foci S and Q and of length of the major axis is equal to 10.
2. a) Precise the center G of (E), and deduce that (E) has an equation:

$$\frac{(x+1)^2}{25}+\frac{(y-2)^2}{16}=1$$

   b) Calculate the value of the eccentricity " e " of the ellipse (E).
   c) Find the vertices and the directrices of (E). Draw (E).
   d) Calculate the area of the region bounded by (E) and its auxiliary circle.



Exercise 3

In the plane referred to an orthonormal system $\left(O;\ \vec{i},\ \vec{j}\right)$ consider the points

F (7; 0), F '(- 3; 0) and $L\left(-3;\frac{9}{4}\right)$. Let (H) be a hyperbola passing through L and of foci F and F '.

1. a – Calculate LF – LF ', then deduce the coordinates of the vertices on the focal axis of (H).

   b- Deduce that $9x^2 - 16y^2 - 36x - 108 = 0$ is an equation of (H) and determine the equations of the asymptotes of (H).
2. a – Write an equation of the tangent (T) to (H) at L.

   b- Let (d') be the straight line of equation: $x = -\frac{6}{5}$. What does (d') represent for (H)?

   Verify that (T) and (d') are intersecting at a point J on the x - axis.

   c- Draw (T), (d') and (H).

   d- Prove that $\tan\left(J\hat{L}F\right) = \frac{1}{e}$, where e is the eccentricity of (H).
3. Let (D) be the region bounded by (H), (T) and the straight line of equation x = - 2.
   Calculate the volume of the solid generated by revolving (D) around x ' x

Exercise 4

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\ \vec{j}\right)$.

Let (P) be a parabola with focus F (0; 2) and of directrix the x – axis.

1. a -Prove that $x^2 - 4y + 4 = 0$ is the equation of the parabola (P).

   b- Determine the vertex S and the focal axis (δ) of (P).

   c- Draw (P).
2. a- Calculate the area of the region bounded by (P), the x – axis, the y – axis and the straight line of equation x = 2.

   b- Deduce the area of the region D bounded by the parabola (P) and the straight line (d) of equation y = 2.

   c- Calculate the volume of the solid generated by revolving D around the y – axis.
3. Let M be any point on the parabola (P) with abscissa α. The tangent (T) and the normal (N) to (P) at M cuts the focal axis (δ) of (P) respectively at T and N.

   a- Find the equations of the tangent (T) and the normal (N).

   b- Calculate, in terms of α, the coordinates of T and N.

   c- Determine the values of α so that the triangle MNT is right isosceles.
4. Let (C) be a semi – circle with center F and of radius 2.

   (C') is a circle tangent to the straight line (d) and to the circle (C).

   Prove that the center I of the circle (C') belongs to the parabola (P).
5. Let (D) be a variable straight line of equation y = t (x – 3), t is a non- zero real number. In the case where the straight line (D) cuts the parabola (P) at two points M' and M" prove that the midpoints J of the segments [M'M"] moves on a fixed parabola (P') whose equation is to be determined.

Exercise 5

In an oriented plane consider a rectangle AOBC such that: AO = 15, OB = 4 and $\left(\overrightarrow{AO};\overrightarrow{AC}\right) = \frac{\pi}{2} + 2k\pi$.

F and D are two points of [OA] and [BC] respectively such that: BD = FO = 3.
Let (E) be an ellipse of foci O and F and of directrix the line (AC) associated to F and of eccentricity e.

A 1. Determine the center I of (E) and prove that: $e=\frac{1}{3}$.

2. a- Prove that the points B and D belong to (E).
   b- Determine the vertices A_1 and A_2 of the focal axis of (E).
   c- Draw (E).

B The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$ such that: $\overrightarrow{OF}=-3\vec{i}$ and $\overrightarrow{OB}=4\vec{j}$.

1. Verify that: $8x^2+9y^2+24x-144=0$ is an equation of (E).
2. Let (d) be a straight line passes through B and of slope $-\frac{1}{3}$. Prove that (d) is tangent to (E) at B.
3. Write an equation of the tangent (T) at D to (E) and verify that A belongs to (T).
4. The straight lines (d) and (T) are intersects at the point R and that (d) cuts the focal axis at J. Calculate the area of the region bounded by the triangle RJA and the semi – ellipse situated above the focal axis.

Exercise 6

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

Denote by (C) the circle with center I (0; 3) and radius 2, and by (d) the line with equation y = - 3.
Let L (α; β) be a variable point on (C). Denote by N the orthogonal projection of L on (d) and by M the midpoint of segment [LN].

1. Write an equation of (C).
2. a - Determine the coordinates of M in terms of α and β.

   b- As L moves on (C), prove that M moves on the ellipse (E) with equation: $\frac{x^2}{4}+y^2=1$

   c- Draw (E).
3. Let (P) be the parabola with vertex V (0 ; 1) and focus $F\left(0;\frac{3}{4}\right)$.

   a- Show that: $y = 1 - x^2$ is an equation of (P).
   b- Draw (P) in the same system as (E).
4. a - Calculate $\int_0^1\left(1-x^2\right)dx$.

   c- Deduce the area of the region that is above the x – axis and bounded by (E) and (P).
5. Let G (- 1; 0) be a point on (P) and (Δ) the tangent at G to (P). Denote by H the point of (P) where the tangent to (P) is perpendicular to (Δ). Prove that G, H and F are collinear.

Exercise 7

The plane is referred to an orthonormal system $\left(O;\vec{i},\vec{j}\right)$. Graphic unit 2 cm.

Let (P) be a parabola of focus F (2; 2) and directrix the straight line(d): x = - 2.

1. Prove that: $y^2-8x-4y+4=0$ is the equation of (P). Precise the vertex S of (P) and Draw (P).
2. Calculate, in cm^2, the area of the region bounded by (P), the straight line (D_1) of equation y = 2, the straight line (D_2) of equation y = 6 and the directrix (d).
3. Consider the two points A (8; 10) and $B\left(\frac{1}{2};0\right)$.

   a- Verify that the points A and B belong to (P) and that F, A and B are collinear.
   b- Find the equations of the tangents (T_A) and (T_B) at A and at B to (P).

c- Prove that (T_A) and (T_B) are perpendiculars and they are intersecting on the directrix (d) of the parabola (P).

4. Let I be the midpoint of [AB] and L its orthogonal projection on the focal axis of (P). Let (D) be a straight line with slope $-\frac{3}{4}$ and passes through par I, the straight line (D) cuts the focal axis of (P) in J. Prove that LJ is equal to the parameter of the parabola (P).

Exercise 8

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$. Graphic unit 2 cm.

Consider the les points $F\left(2\sqrt{2};0\right)$ *and* $F'\left(-2\sqrt{2};0\right)$.

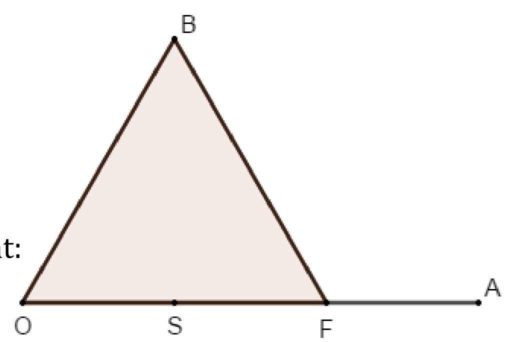
1. a – Let M (x; y) be a variable point variable in the plane.
   Prove that the set of points M such that: $MO^2 = MF \times MF'$ is a rectangular hyperbola (H) of equation: $x^2 - y^2 = 4$.
   b- Determine the directrices and the asymptotes of (H). Draw (H).
2. Let $P(\alpha;\beta)$ be a point of (H) with $\alpha \neq \pm 2$.
   Designate by (N) the normal at P to (H) and {L} = $(N)\cap(x'x)$.
   a- Verify that $y = -\frac{\beta}{\alpha}x + 2\beta$ is the equation of (N). Deduce the coordinates of the point L.
   b- The point P is given on (H). Construct the tangent at P to (H).
3. Consider the point $K\left(2\sqrt{2};\frac{2}{\sqrt{3}}\right)$.
   a- Calculate: KF + KF ‘. Deduce that K belongs to an ellipse (E): $x^2 + 3y^2 = 12$.
   b- Determine the coordinates of the common points to (E) and (H).
   Draw (E) in the same system as that of (H).
   c- Let R be one of the common point to (E) and (H).
   Prove that the tangents at R to (E) and (H) are perpendicular.
   d- Let (Δ) be the region bounded by (E) and (H) and containing the point F.
   Calculate, in cm^3, the volume of the solid generated by revolving (Δ) around (x’ x).

Exercise 9

The points O, S, F and A are collinear and OFB is a direct equilateral triangle such that:

OS = SF = FA =2.

Let (E) be an ellipse of foci O and F passes through A.



1. **a)** What does the point A represents for ellipse (E)? Justify
   b) Calculate the length of the major axis of (E).
   c) Prove that B is a vertex of the ellipse (E).
   d) The perpendicular bisector of [SA] cuts (E) in I and I′ such that: FI = FI′ = d. Calculate d.
   e) Plot I, I′ and the vertices of (E). Draw (E).
2. The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$ such that: $\overrightarrow{OA} = 6\vec{i}$.
   a) Prove that the equation of the ellipse (E) is: $\frac{(x-2)^2}{16} + \frac{y^2}{12} = 1$.

b) Determine the equation of the directrix (D) associated to F.

3. **a)** Let M (4; 3) and N (4 , – 3) be two points of (E).
   Determine the equations of the tangents (T) and (T') respectively at M and N to (E).
   b) (T) and (T′) are intersecting at the point R. Prove that R is the point of intersection of the straight lines (OF) and (D).
   c) (T) and (T′) cuts (SB) respectively at P and Q.
   Calculate the area of the region bounded by the triangle PQR and the ellipse (E).

<u>Exercise 10</u>

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

Consider a hyperbola (H) of equation: $16x^2-9y^2-32x-128=0$.

Designate by F the focus of (H) with positive abscissa and by (D) the directrix of (H) associated to F.

1. a) Write the reduced equation of (H) and then determine its center O'.
   b) Determine the coordinates of the vertices A and A', the foci F and F', the equations of the asymptotes (Δ) *and* (Δ'), the directress (D) and (D'), the eccentricity of (H).
   c) Draw (H).
2. The tangent (T) to (H) at the point $E\left(7;4\sqrt{3}\right)$ cuts the directrix (D): $x=\dfrac{14}{5}$ at a point K.
   a) Write the equation of the tangent line (T).
   b) Find the coordinates of the point K.

<u>Exercise 11</u>

In the plane referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$, consider the hyperbola (H) of equation:

$x^2-\dfrac{y^2}{4}=1$ and designed by M the point of coordinates $\left(\dfrac{1}{\cos(\theta)};2\tan(\theta)\right)$, θ is a real number $\in\left]0;\dfrac{\pi}{2}\right[$.

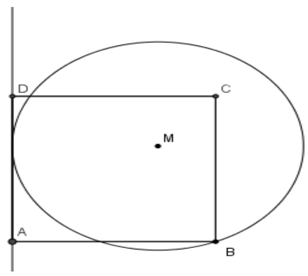
1. a - Determine the coordinates of the vertices and the foci of (H).
   b- Write the equations of the asymptotes (Δ_1) *and* (Δ_2).
   c- Draw (H).
   a- Verify that the point M belongs to (H).
2. Let (T_M) be the tangent to (H) at M.
   Show that the equation of (T_M) in the system $\left(O;\vec{i},\vec{j}\right)$ is $2x-y\sin(\theta)-2\cos(\theta)=0$
3. Let P_1 and P_2 be the respective intersection points of (T_M) with the asymptotes (Δ_1) *and* (Δ_2).
   a- Find the coordinates of the points P_1 and P_2.
   b- Prove that the area of triangle OP_1P_2 is independent of θ.

<u>Exercise 12</u>

Let (P) be a parabola of equation $y^2 = 2px$ where p is strictly positive real number. Let M (x_0; y_0) be a point of (P) and M 'the other intersection point of (P) with the straight line (FM). (F is the focus of the parabola (P)).

1. Determine the coordinates of the point N midpoint of segment [MM'].
2. Deduce the set of points N as M moves on (P).

<u>Exercise 13</u>



ABCD is a direct square of side 4 cm.
(C) is a variable circle of center M passing through B and tangent to (AD).

<u>Part A:</u>

1. Verify that the point M moves on a parabola
   (P) whose focus and directrix are to be determined.
3. Determine the focal axis, the vertex S and the

Parameter of (P).
4. What does the straight line (AC) represents for The parabola (P)?
5. Draw (P).

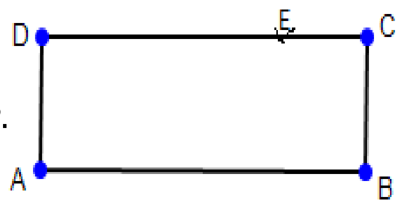
Part B:

The plan is referred to a direct orthonormal system $\left(A;\vec{i},\vec{j}\right)$, such that: $\vec{i}=\frac{\overrightarrow{AB}}{4}$.

1. Verify that the equation of (P) is $y^2 = 8x - 16$.
2. Let Δ be the region bounded by (P) and the straight line (BC).
   a- Calculate the area of the region Δ.
   b- Calculate the volume of the solid generated by rotating of Δ around the straight line (AB).
3. Let M (a; b) be a point of (P) distinct of S.
   a- Determine the equation of the tangent (t) and the equation of the normal (n) to (P) at M.
   b- The tangent (t) and the normal (n) at M to (P), cut the focal axis of (P) respectively at T and N.
   Prove that the vertex S is the midpoint of [HT], where H is the orthogonal projection of M on the focal axis of (P).
   c- Verify that the length NH is independent of the position of M on (P).

Exercise 14

In the adjacent figure we have, ABCD is a rectangle such that AB = 3 cm; BC = 1 cm and EC = 1 cm.
Let (P) be a parabola of focus C and of directrix (AB).



Part A:

1. Determine the focal axis of (P), its vertex S and its parameter P.
2. Verify that E belongs to the parabola (P).
3. Find another point on the parabola (P) with justification.

Part B:

The plane is referred to a direct orthonormal system $\left(A;\vec{i},\vec{j}\right)$, such that $\vec{i}=\frac{1}{3}\overrightarrow{AB}$.

1. Show that the equation of the parabola (P) is: $y=\frac{1}{2}x^2-3x+5$.
2. Draw (P).
3. The straight line of equation x = 5 cuts (P) at a point M.
   a- Verify that the straight line (CM) cuts again the parabola (P) at a point N whose coordinates are to be determined.
   b- Prove that the tangents (T_1) and (T_2) to (P) respectively at M and N are perpendicular, and that they are intersecting on (AB).
   c- Designate by Δ the region bounded by (P), the x – axis and the straight lines of equations x = 3 and x = m (m is a real parameter greater than 3). Calculate m for which the area of region Δ equal to $\frac{7}{3}$ cm².

Part C:

Consider the function f defined over]3; +∞ [by f (x) = ln (x – 3) + 1, and let (C) be its representative curve in the system $\left(A;\vec{i},\vec{j}\right)$,

1. Find the equation of the tangent (T) to (C) at its point G with abscissa 4.
2. Justify that (T) is tangent to (P).
3. Study the sense of variations of f and then draw (C) and (T).
4. Let λ be a real number in the interval]3;5].
   Calculate in terms of λ, the area $I(\lambda)$ of the region bounded by (C), (P) and the two straight lines of equations x = λ and x = 5. Then calculate $\lim_{\lambda\to3^+} I(\lambda)$.

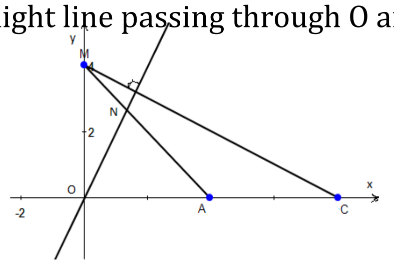
Exercise 15

Consider the equation: $4(x^2 + y^2) = (x - 6)^2$ where x and y are the coordinates of a variable point M on curve (C) situated in the plan referred to a direct orthonormal system $(O; \vec{i}, \vec{j})$.

1. By using the distance from the point M to the straight line (d) of equation: x = 6, prove that (C) is a conic whose nature and eccentricity are to be determined.
2. Deduce the coordinates of the following points of (C): The vertices on the focal axis of (C), the center, the vertices on the non- focal axis of (C), the other focus. Draw (C).
3. Calculate by using (C) the integral I = $\int_{-2}^{2} \frac{1}{2}\sqrt{-3x^2 - 12x + 36}\,dx$.
4. Determine the equations of the tangents to (C) at the points of intersection of (C) with the y – axis.
5. Prove that the orthogonal projections of the point O on these tangents are found on the principal circle of this conic.

Exercise 16

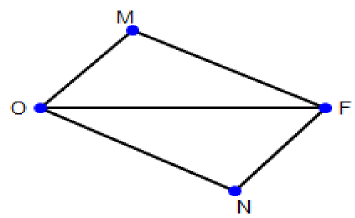
In an orthonormal system $(O; \vec{i}, \vec{j})$, given the points A (4; 0) and C (8; 0) and the variable point M on the y – axis of coordinates M (0; a) where a is a real number. The straight line passing through O and perpendicular to (CM) cuts (AM) at the point N.



1. Prove that the equations of straight lines (ON) and (AM) are respectively $y = \frac{8}{a}x$ and $y = -\frac{a}{4}x + a$.
2. Deduce that the locus of the point N is an ellipse (S) of equation: $2x^2 + y^2 - 8x = 0$.
3. Determine the center of (S) and verify that A is one of its vertices on the focal axis and determine the other vertex A'.
4. Determine the vertices B and B' on the non – focal axis of (E), $y_B > 0$.
6. Determine the foci and the directrices of (S). Draw (S).
7. M (p; q) is a point of (S) (p and q are real numbers) and G is the center of gravity of triangle MAB.
   a- Determine the coordinates of G in terms of p and q.
   b- Deduce as M varies on (S), the point G moves on an ellipse (S') whose equation is to be determined.
   c- Calculate the area of the region bounded by (S) and exterior to (S').

Exercise 17

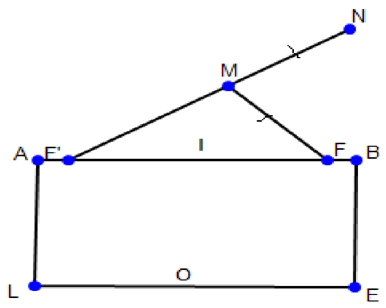
Consider in the plane a fixed segment [OF] of length 4 cm.
M and N are two variable points of the plane, such that the quadrilateral OMFN remains a parallelogram with perimeter 16 cm.



1. a- Prove that M varies on an ellipse (E) whose foci and the length of the major axis is to be determined.
   b- Draw (E).
2. The plane is referred to a direct orthonormal system $(O; \vec{i}, \vec{j})$ such that: $\vec{i} = \frac{1}{4}\overrightarrow{OF}$.
   a- Write the equation of the ellipse (E).
   b- Calculate the eccentricity of (E) and find the equation of the directrix (d) associated to O.
   c- Let Δ be the region bounded by (E) and its principal circle.
   Calculate the volume of the solid generated by revolving Δ around the focal axis of (E).
   d- Let M (x_0; y_0) be a point of (E). The tangent (T) and the normal (N) at M to (E) cuts respectively the x – axis in R and S.
   Prove that $\overline{OR} \times \overline{OS}$ is a constant number to be determined.

Exercise 18

In the below figure we have:



- LEBA is a rectangle such that: LE = $4\sqrt{3}$ *and* $BE = 2\sqrt{3}$.
- O is the midpoint of the segment [LE] and I that of [AB].
- F and F' are two points of [AB] such that: IF = IF' = 3.
- N is a variable point on the circle of center F' and radius AB.
- M is a point of [F'N] such that MF = MN.

Part A:

1. a- Prove that the point M moves on an ellipse (E) of foci F and F'.
   b- Determine the focal axis of (E) and the two corresponding vertices.
   c- Let J be the midpoint of [IO]. Prove that J is a vertex of (E), then determine the fourth vertex of (E).
2. S is a point such that: $\overrightarrow{FS} = \frac{1}{4}\overrightarrow{EB}$. Prove that S belongs to (E), then draw (E).
3. (P) is a parabola with focus O and of directrix (AB).
   a- Prove that J is the vertex of (P).
   b- Prove that the points L and E are two points of (P).
   c- Prove that the straight lines (LI) and (EI) are tangents to (P).
   d- Draw (P).

Part B:

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$ such that the coordinates of points E and B are given by $E\left(2\sqrt{3};0\right)$ *and* $B\left(0;2\sqrt{3}\right)$.

1. Find the equation of the parabola (P).
2. Find the equation of the ellipse (E).
3. Calculate the area of the region bounded by (P) and (LE).
4. (D) is the interior region to LEBA and situated between (P) and (E). Find the area of (D).
5. T is a point defined by: $\overrightarrow{IT} = \frac{4}{3}\overrightarrow{IF}$. Prove that (ST) is tangent to (E)

Exercise 19

In the plane referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$, Designate by (C) the representative curve of the function f defined by: $f(x) = -1 + \frac{3}{2}\sqrt{x^2 - 4x - 5}$.

1. Study the variations of f and draw (C).
2. Prove that (C) is a part of a hyperbola (H) whose reduced equation and its characteristic elements are to be determined.
3. Determine the equations of the tangent and the normal to (H) at its point M (-1; -1).

Exercise 20

Consider the curve (H) of equation: $y^2 = x^2 - 4x + 3$.

1. Prove that (H) is a hyperbola. Determine the coordinates of its center I and the vertices of (H), and the equations of the asymptotes of (H).
2. Let (d) be a variable straight line of equation y = m x where m is a real parameter. Discuss, according to the values of the parameter m the number of the points of intersection of (d) and (H).
3. In the case (d) intersect (H) at two points, Designate by M_1 and M_2 these two points. Determine the set (H') of points I midpoints of $[M_1M_2]$.

Exercise 21

In the plane referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$, Designate by (C) all the circle that passes through the points A (3;0) and B (-3; 0).

Let [MN] be the diameter of this circle which is parallel to the x – axis such that $x_M > 0$
Let H be the orthogonal projection of M on the x – axis.

1. a- Prove that (MH) tangent at M to (C).

   b- Prove that: $\hat{ABM} = \hat{AMH}$ and deduce that $HM^2 = HA \times HB$.

2. a- Prove that as (C) varies, the equation of the set (H) of points M and N is:

$$x^2 - y^2 = 9.$$

   b- Prove that (H) is a rectangular hyperbola, and then determine the foci, the vertices, and the equations of asymptotes. Draw (H).

   c- Let (D) be the domain bounded by (H) and the two straight lines x = 3 and x = 4. Calculate the volume of the solid generated by revolving (D) around the x – axis.

3. a- Let E be a point of (H) with ordinate 3 and with positive abscissa, F and F' are the foci, (Focus F has a positive abscissa). Calculate the distance EF'.

   b- Prove that y = $x\sqrt{2} - 3$ is the equation of the tangent to (H) at E.

   c- T is the common point of this tangent with the x – axis.

   Prove that: $\frac{TF}{TF'} = \frac{EF}{EF'}$.

TRANSFORMATIONS

Exercise 1

1. Consider the segment [AB] with I it's midpoint, and consider two dilations h = h (A, 2) and g = h (B, $\frac{3}{2}$), Let O be a point such that: $\overrightarrow{AO}=\frac{1}{4}\overrightarrow{AB}$.

   Plot the images of the points A, I and O under g ∘ h. Identify $g \circ h$.

2. By using the complex form of the rotation r with center O and an angle $\frac{\pi}{6}$, determine the equation of the straight line (d') the image of the straight line (d) of equation y = x under r.

Exercise 2

ABC is an isosceles triangle of vertex A such that : $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{6}\left[2\pi\right]$. M is a point of [BC]. The parallel through M to (AC) cuts [AB] in N and the parallel to (AB) through M cuts [AC] in P.

1. Prove that: AN = CP.
2. Deduce that there exists a rotation r that transforms A onto C and N onto P whose angle is to be determined.
3. Prove that r (B) = A and determine the center of the rotation r.

Exercise 3

ABC is right isosceles triangle and direct with vertex A. M is a variable point of [BC]. P and Q are the respective orthogonal projections of M on (AB) and (AC).
Let r be a rotation that transforms C onto A and A onto B.

1. Determine the angle α of the rotation r.
2. Determine $\left(r \circ r\right)\left(C\right)$. Deduce that the center I of r is the midpoint of the segment [BC].
3. Prove that: r (Q) = P. Deduce the nature of the triangle IPQ.

Exercise 4

Consider a right triangle ABC such that: $\left(\overrightarrow{AB};\ \overrightarrow{AC}\right)=\frac{\pi}{2}\left[2\pi\right]$ and AB = 2 AC.

Let J be the midpoint of [AB]. Let r be a rotation that transforms A onto J and C onto B.

1. Determine the angle of r and find a geometric construction for its center I.
2. Designate by (d) the perpendicular bisector of [AI]. Let O be a point of (d) and (C) is a circle with center O and passes through A. (C) cuts again (AC) at H and (AB) at K.
   a- Verify that [HK] is a diameter of (C).
   b- Determine the image of the line (HI). Deduce that IHK is right isosceles triangle.

Exercise 5

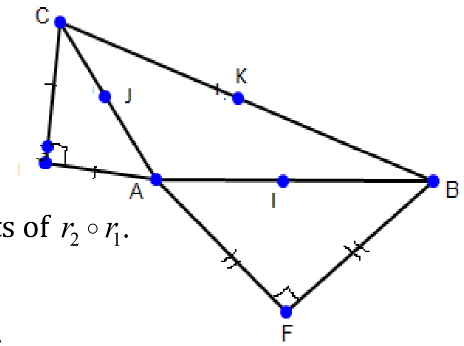
ABCD is a parallelogram with center J such that $\left(\overrightarrow{AB};\ \overrightarrow{AD}\right)=\frac{\pi}{3}\left[2\pi\right]$ and AOB is a direct and right isosceles triangle with vertex O. Let I be the symmetric of A with respect to O and r be a rotation of center O and of angle $\frac{\pi}{2}$.

1. Determine r (B) and construct the point E the image of C under r.
2. Prove that: CI = BE and that the two straight lines (OJ) and (EB) are perpendiculars.
3. Prove that E is the image of D by a rotation r 'of center A and an angle $\frac{\pi}{2}$.

Exercise 6

ABC is a direct triangle. Designate by I, J and K are the respective midpoints of the segments [AB], [AC] and [BC].To the exterior of the triangle, we construct the triangles ABM and ACN, right and isosceles at M and N.

1. Prove that: IM = JK.
2. Find an angle of the rotation f that transforms I onto J and M onto K.
3. Prove that: f (K) = N.



4. Determine $(f \circ f)(M)$. Deduce the center O of f.
5. Consider two rotations r 1 and r 2 with same angles $\frac{\pi}{2}$ and with respective centers M and N.
6. Determine $(r_2 \circ r_1)(B)$. Deduce the nature and the elements of $r_2 \circ r_1$.

Exercise 7

Let ABCD be a square with center I such that: $(\overrightarrow{AB}; \overrightarrow{AD}) = \frac{\pi}{2}\ [2\pi]$.

Let r be a rotation of center A and an angle $\frac{\pi}{2}$.

1. a- Construct the point I' = r (I).
   b- Prove that the straight lines (BI) and (DI') are perpendiculars.
   c- Let M be a variable point on (BC). What is the set of points M'= r (M) when M moves on (BC)?
2. Consider the point I 1 such that: r (I 1) = I.
   a- Determine $r \circ r$.
   b- Precise $(r \circ r)(I_1)$, then Construct I 1.
3. Let r ' be a rotation that transforms I onto I ' and C onto A. Verify that the angle of r 'is $-\frac{\pi}{2}$ and determine the center of r '.
4. Let t = $r' \circ r$.
   a- Determine t (B) and t (I 1).
   b- Prove that t is a translation whose vector is to be determined.

Exercise 8

In the plane referred to an orthonormal system $(O; \vec{i}, \vec{j})$, consider the points A (- 2; 0) and B (2; 0) and the two dilations $h_1 = h\left(A; \frac{1}{3}\right)$ and $h_2 = h\left(B; \frac{3}{5}\right)$.

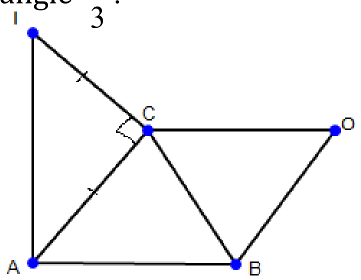
1. Study and determine the characteristic elements of h = $h_2 \circ h_1$.
2. Consider the circle (C 1) of equation : $x^2 + y^2 - 10x + 5 = 0$.
   What is the transformation (C 2) of (C 1) under h?

Exercise 9

ABC is a direct equilateral triangle, (C) is the circumscribed circle about the triangle ABC of center O.
Designate by I, J and K the respective midpoints of the segments [BC], [AC] and [AB].
The lines (OA), (OB) and (OC) cuts the circle (C) respectively at L, M and N.

1. a- What is the image of point A under the central symmetry of center O?
   b- What is the image of the point B by the translation of the vector $\overrightarrow{KJ}$?
2. Find the images of the points K and L by a rotation of center O and of angle $\frac{2\pi}{3}$.
3. Find the image of the point N by a rotation of center A and of angle $\frac{\pi}{3}$.

Exercise 10



ABOC is a Rhombus such that: $(\overrightarrow{AB}; \overrightarrow{AC}) = \frac{\pi}{4}\ [2\pi]$.

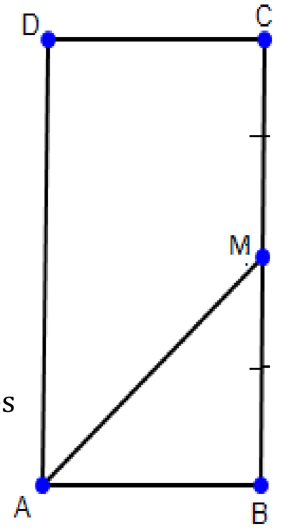
CAI is right isosceles triangle such that: $(\overrightarrow{CA}; \overrightarrow{CI}) = -\frac{\pi}{2}\ [2\pi]$.

1. R A is a rotation of center A that transforms B onto C.

R C is a rotation of center C and an angle $-\frac{\pi}{2}$. Suppose that : f = R C o R A.

a- Determine the images under f of the points A and B.
b- Prove that f is a rotation whose angle is to determined.
c- Prove that O is the center of f.

2. Let S be a direct plane similitude with center O that transforms A onto B.
   Let C'= S (C), H is the midpoint of [BC] and H' = S (H).
   a- Give the measure of the angle of S.
   b- Prove that C' belongs to the straight line (OA).
   c- Prove that H' is the midpoint of the segment [OB].
   d- Prove that (C'H') is perpendicular to (OB).
   e- Deduce that C' is the center of the circumscribed circle about the triangle OBC.
3. The plane is referred to a direct orthonormal system $\left(A; \vec{u}, \vec{v}\right)$ such that: $\overrightarrow{AB} = \sqrt{2}\vec{u}$.
   a- Prove that: $Z_C = 1 + i$ and $Z_O = 1 + \sqrt{2} + i$.
   b- Write the complex forms of R_A and R_C.
   c- Determine the complex form of S.



Exercise 11

In an oriented plane consider a rectangle ABCD such that:

AB = 3 cm, BC = 6 cm and $\left(\overrightarrow{AB}; \overrightarrow{AD}\right) = \frac{\pi}{2}\ [2\pi]$.

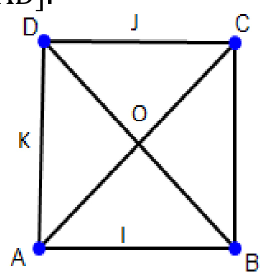
Let M be the midpoint of the segment [BC].

1. Let S be a direct plane similitude of center W, that transforms A onto M and B onto D.
   a- Determine the ratio and an angle of S.
   b- Determine and plot the images under S of the straight lines (AM) and (AD).
   c- Let S (M) = E. Construct E and then deduce the nature Of the quadrilateral AMDE.
   d- Determine T = S (C), deduce the area of the triangle MDT.
2. The complex plane is referred to a direct orthonormal system $\left(A; \vec{u}, \vec{v}\right)$ such that: $\vec{u} = \overrightarrow{AB}$.
   a- Determine the complex form of S and determine the affix of its center W.
   b- Determine the complex form of S o S.
3. Let S' be a direct plane similitude with center W, of ratio $\frac{1}{\sqrt{2}}$ and an angle $\frac{\pi}{4}$.
   a- Determine the nature and the characteristic elements of the transformation $S' \circ S$.
   b- Calculate the affix of the point B' the image of B by $S' \circ S$.

Exercise 12

The plane is oriented. In the below figure, ABCD is a square of center O such that: $\left(\overrightarrow{AB}, \overrightarrow{AD}\right) = \frac{\pi}{2}\ [2\pi]$.

Denote by I, J and K the respective midpoints of the segments [AB], [CD] and [AD].
Let S be a direct plane similitude that transforms C onto O and J onto A.



1. Determine the ratio K and an angle of S.
2. Determine S (D).
3. a- Determine the images of the straight lines (AO) and (OD) under S.
   b- Deduce the point E = S (O) and the point F = S (B). Plot E and F.
   c- Determine and construct the image of the square ABCD under S.
   d- Calculate the area of the triangle CJO if $A_{\Delta CJO} + A_{\Delta OAE} = 9u^2$.

4. Prove that the center Ω of S is the point of intersection of two circles one of them is the diameter [CE] and the other of diameter [JA'] where A' = S (A).
5. Let $S_n = S \circ S \circ S \circ \ldots\ldots\ldots \circ S$ n times where n is a natural number; $n \geq 2$.
   Find the values of n for that S_n is a dilation.
6. The complex plane is referred to a direct orthonormal system $\left(A;\vec{u},\vec{v}\right)$ with $\vec{u}=\frac{\overrightarrow{AB}}{4}$.
   a- Determine the complex forms of S, S_2 and S^{-1}.
   b- Let (E) be an ellipse of equation: $\frac{(x-4)^2}{16}+\frac{(y-4)^2}{4}=1$, and (E') its image by S.
   Determine an equation of the focal axis of (E'), its area and the coordinates of one of the two vertices.

__Exercise 13__

In an oriented plane, consider a circle (C) of center A and of radius 3 and a circle (C') of center B and of radius 1, such that: AB = 6.

1. Let S be a direct plane similitude of angle $\frac{\pi}{2}$ that transforms (C) onto (C').
   a- Determine the ratio of S and justify that its center I where we have: IA = 3 IB.
   b- Prove that: $IA=\frac{18}{\sqrt{10}}$ and construct I.
2. Let r be a rotation of center A and of angle $\frac{\pi}{2}$ and h be a dilation of center A and of ratio $\frac{2}{3}$.
   a- Construct the points D and E such that: D = r (B) and E = h (B).
   b- Prove that: S (D) = E.
   c- Prove that I belong to the circumscribed circle about the triangle ADE.
3. The plane is referred to a direct orthonormal system $\left(A;\vec{u},\vec{v}\right)$ with : $\vec{u}=\frac{1}{6}\overrightarrow{AB}$.
   a- Find the complex form of the similitude S. Deduce the affix of I.
   b- Find the complex forms of r and h.
   c- Determine the affix of each of the points D and E and verify that S (D) = E.

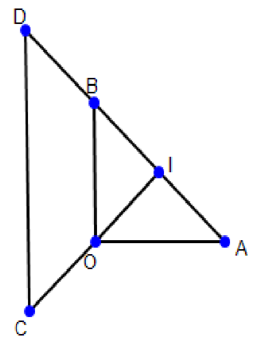
__Exercise 14__

In the below figure, OAB is right isosceles triangle such that: OA = OB and $\left(\overrightarrow{OA},\overrightarrow{OB}\right)=\frac{\pi}{2}\,[2\pi]$.

Designate by I the midpoint of the segment [AB] and by C and D the respective symmetry of the point I with respect to O and to B.

Let S be a direct plane similitude that transforms A onto D and O onto C.

1. Determine the ratio and a measure of the angle of S.
2. The straight lines (DO) and (AC) intersect at the point J.
   a- Prove that (DO) and (AC) are perpendiculars.
   b- Determine the images of the straight lines (OJ) and (AJ) under S.
   c- Deduce that J is the center of the similitude S.
3. The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$ such that:
   $\vec{u}=\overrightarrow{AB}$.
   a- Find the affixes of the points C and D.
   b- Find the complex form of S.
   c- Determine the affix of the center J of S.



__Exercise 15__

The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, (G.U : 5 cm).

Consider the points A of affix $\sqrt{2}$ and B of affix i. Let C be a point such that OACB is a rectangle.
Let I be the midpoint of the segment [OA]. J is the midpoint of [BC] and K that of [AI].

Plot these points in the plane.

1. Consider the transformation S in P such that for all point M of affix Z, we associate the point M' of affix Z ', such that: $Z' = -i\frac{\sqrt{2}}{2}Z + \frac{\sqrt{2}}{2} + i.$

   a- Prove that S is a direct plane similitude whose ratio K, angle θ and the affix of its center Ω are to be determined.

   b- Determine the images under S of the points O, A, C and B.

2. a- Calculate the measure of the angle $(\overrightarrow{\Omega B};\overrightarrow{\Omega A})$. Deduce that the points A, B and Ω are collinear.

   b- Prove that the points I, C and Ω are collinear.

   c- Deduce a construction of Ω. Plot Ω in the figure.

3. a- Prove that Ω belong to the circles Γ_1 *and* Γ_2 with respective diameters [BC] and [AI].

   b- Prove that $\overrightarrow{J\Omega}$ *and* $\overrightarrow{JK}$ are collinear.

   c- Prove that (ΩO) is common tangent to Γ_1 *and* Γ_2.

PROBABILITIES

Exercise 1

We disposed an urn U contains five dices:

- Three red dices, each one has four faces numbered 2 and two faces numbered 5.
- Two black dices, each one has two faces numbered 2 and four faces numbered 5.

A player selects randomly and simultaneously two dices from the urn U and then he rolled them once.
Consider the following events:

A : « The player Selects two red dices » B : « The player selects two black dices »
C : « The player selects one red die and one black die » E : « Obtaining exactly one face numbered 2 ».

1. Calculate the probabilities of the events A, B and C.
2. a- Verify that P (E /A) = $\frac{4}{9}$, and calculate $P(A \cap E)$.

   b- Calculate $P(E \cap B)$ *and* $P(E \cap C)$.

   c- Verify that P (E) = $\frac{23}{45}$.
3. Knowing that obtained exactly one face numbered 2, calculate the probability that the player selects two black dices.
4. In this part, the player selects randomly and simultaneously three dices from the urn U.
   Let X be a random variable that is equal to the number of red selected dices.
   a- Give the three possible values of X.
   b- Determine the probability distribution of X.

Exercise 2

We disposed two urns U and V.
Urn U contains 5 red balls and 4 green balls.
Urn V contains 2 red balls and 3 green balls.
A player selects randomly and simultaneously two balls from the urn U.

- If the two selected balls are red, then he put them in the urn V, and he selects randomly and simultaneously two balls from the urn V.
- If the two selected balls are green, then he put them in the urn V, and then he selects randomly one ball from the urn V.
- If the two selected balls have different colors, then the player selects randomly one ball from the urn V. Consider the following events:

  R: « the two selected balls from the urn U are red »
  V: « The two selected balls from the urn U are green »
  D: « The two selected balls from the urn U have different colors »
  E : « The selected ball (s) from the urn V are green »

1. Calculate P (R), P (V) and P (D).
2. Calculate $P(E/R)$, $P(E/V)$ *and* $P(E/D)$.
3. Prove that: P (E) = $\frac{41}{63}$.
4. The selected ball (s) from the urn V are red, What is the probability that the two drawn balls from the urn U are red.
5. Let X be a random variable that is equal the number of the red balls remaining in the urn V after the two drawns.
   a- Determine the four possible values of X.
   b- Prove that: $P(X = 2) = \frac{67}{126}$. Determine the probability distribution of X.

Exercise 3

A game consists a player selects randomly and simultaneously 4 balls from a bag containing one black ball and 9 white balls, then we throw a perfect die which has six faces numbered 1, 2, 3, 4, 5 and 6.

- If one of the selected balls is black, it should have obtained an even number with the die to win.

- If all the selected balls are not black, it should have obtained a six with the die to win.
  Consider the two events N: « One of the 4 selected balls is black » and
  G « The player wins the game ».

1.
   a- Determine the probability of the event N.

   b- Prove that the probability of the event G is equals to $\frac{3}{10}$.

   c- The player loses the game. What is the probability he selects one black ball?
2. To play this game, the player pays before 5 000 LL as a bet.
   If the player wins the game, he gets 20 000 LL.
   If he loses the game but he selects a black ball, the player is taking his bet.
   If he loses the game but didn't select a black ball, the player loses his bet.
   Let X be a random variable that is equal to the algebraic gain of the player.
   a- Determine the probability distribution of X.
   b- Calculate the excepted value of X.
3. The player repeats the game 5 times with the same conditions.
   Calculate the probability of winning the game at least one time.

Exercise 4

An urn U contains eight cards numbered: 2, 3, 4, 8, 9, 10, 14 and 15.
An urn V contains ten balls: 4 balls numbered 0, 2 balls numbered 1 and 4 balls numbered 2.

Part A :

A player selects at random one card from the urn U.

- If the selected card holds an even number, then the player selects randomly and simultaneously three balls from the urn V.
- If the selected card holds a number multiple of 3, then the player selects randomly and simultaneously two balls from the urn V.
  Consider the following events:
  B : « The selected card from the urn U holds an even number »
  S : « The sum of the numbers on the selected balls from the urn V is equals to 4 »

1. Calculate the probabilities P (B) and P(S/B).
2. Prove that: $P(B \cap S) = \frac{7}{48}$.
3. Calculate $P(\overline{B} \cap S)$. Deduce that: $P(S) = \frac{47}{240}$.
4. The sum of the numbers on the selected balls from the urn V different of 4, what is the probability that the three selected balls from the urn U?

Part B :

The player wins 5 000 L.L for each selected ball holding the number 0.
Designate by X a random that is equal the sum of money wins by the player.

1. Prove that: $P(X = 10000) = \frac{19}{80}$.
2. Determine the probability distribution of X.

Exercise 5

A game debuts a video game and makes several successive parts. We admits that:

- The probability wins the first part is 0.1.
- If he wins a part, the probability to wins the following part is 0.8.
- If he loses a part, the probability to wins the following part is 0.6.
  Denote for all non – zero natural number the following events:
  G_n « The player wins the n - part »
  P_n the probability of the event G_n. So we have: $P_1 = P(G_1) = 0.1$.

1. a- Calculate $P(G_1 \cap G_2)$ *and* $P(\overline{G_1} \cap G_2)$.

b- Deduce that: $P_2 = 0.62$.

2. The player wins the second part. Calculate the probability that he loses the first part.
3. Prove that for all non – zero natural number n, $P_{n+1} = \frac{1}{5}P_n + \frac{3}{5}$.
4. Prove that for all non – zero natural number n, $P_n = \frac{3}{4} - \frac{13}{4}\left(\frac{1}{5}\right)^n$.
5. Determine the limit of the sequence (P_n) as n tends to $+\infty$.

Exercise 6

We disposed two urns U and V.
The urn U contains five balls each one holds the number 0 and two balls each one holds the number 3.
The urn V contains one ball holds the number 3 and two balls each one holds the number 5.

Part A :

We select randomly and simultaneously two balls from the urn U and one ball from the urn V.
Let A be the event: « the sum of the numbers on the three drawn balls is greater than or equal to 8 », and B the event: « Obtained at least one ball holding the number 0 ».

1. Calculate P (A), P (B) and $P(A \cap B)$.
2. Are the two events A and B independents? Justify

Part B :

In this part we emptied all the balls of the two urns U and V in a new urn W.
A game consists to select randomly and successively without replacement two balls from the urn W.

- If the sum of the numbers on the two selected balls is greater than or equal to 8 then the game is end,
- If not, the player selects a third ball and then the game is end.

1. Calculate the probability that the game end in the second drawn.
2. To participate this game the player pays 200 000 LL ; he wins when the sum of the numbers on the drawn balls is greater than or equal to 8.

- If he wins two drawn, then he gets 500 000 LL.
- If he wins three drawn, then he gets 300 000 LL. If he loses then he gets nothing.
  Let X be a random variable that is equal to the algebraic gains by the player.

  a) Prove that: $P(X = 300000) = \frac{7}{45}$.

  b) Determine the probability distribution of X, and calculate E (X).

Exercise 7

We disposed a perfect cubic die whose faces are numbered from 1 to 6 and two urns U_1 and U_2.
U_1 contains 6 red balls and 4 black balls. U_2 contains 5 red balls and 5 black balls.
A player throws the die once.
If the player obtained the faces 1or 2, he selects randomly and simultaneously two balls from the urn U_1.
If not, then the player selects randomly and successively and with replacement two balls from the urn U_2. Consider the following events:

E: « The player obtained the faces 1 or 2 ».
R: « The selected balls are red ».

1. Calculate the probabilities P (R/E) and $P(R \cap E)$.
2. Prove that : P (R) = $\frac{5}{18}$.
3. The two selected balls are red. Calculate the probability that they are from the urn U_1.
4. Designate by X a random variable that is equal the number of the selected red balls.

   a- Verify that the probability $P(X = 1) = \frac{23}{45}$.

   b- Determine the probability distribution of X.

c- If the player repeats this game ten times with the same conditions, estimate the average number of the selected red balls.

Exercise 8

Three of them are green whose faces are numbered 1, 2, 3, 4, 5, 6 and one die is red whose faces are numbered 2,2,2,4, 5, 5.

A game consists as follows a player selects randomly one die, then he throws it two times one after the other. Consider the following events:

R : « The selected die is red »
V : « The selected die is green »
A : « Obtaining 2 times an even number »
B : « Obtaining one time an even number ».

1. Calculate the probability P (A/R) and prove that: $P(A \cap R) = \frac{1}{16}$.
2. Calculate P (A).
3. a- Verify that : $P(B) = \frac{3}{8}$.
   b- The two obtained faces are even. Calculate the probability that the selected die is red.
4. Let X be a random variable that is equal the number of times where the obtained face is an even number.
   a- Determine the probability distribution of X.
   b- Calculate the excepted value of X.

Exercise 9

We disposed two urns U_1 and U_2.

U_1 contains 4 red balls and 3 black balls. U_2 contains 3 red balls and 5 black balls.

A player selects randomly an urn, then he selects randomly and simultaneously three balls from the selected urn. Consider the following events:

A : « The player selects the urn U_1 » E : « The player selects exactly one red ball »
F : « The player selects exactly two red balls » G : « The player selects three red balls »

1. Calculate the probability of the event A.
2. Calculate P (E/A) and $P(E/\overline{A})$. Deduce that: $P(E) = \frac{123}{280}$.
3. Calculate P (F/A) and $P(F/\overline{A})$. Deduce P (F).
4. Calculate P (G).
5. Knowing that the player selects exactly two red balls, what is the probability that the player selects the urn U_2?
6. **In this part the player selects three balls from the urn U_2 randomly and simultaneously.** The player gets 5 000 LL for each selected red ball and he loses 3 000 LL for each selected black ball.
   Let X be a random variable that is equal to the algebraic gain by the player.
   a- Determine the probability distribution of X.
   b- Calculate the expected value E (x).

Exercise 10

We disposed two urns U and V.

Urn U contains 5 red balls and 4 green balls. Urn V contains 2 red balls and 3 green balls.

A player selects randomly and simultaneously two balls from the urn U.

- If the two selected balls are red, then he put them in the urn V, again he selects randomly and simultaneously two balls from the urn V.
- If the two selected balls are green, then he put them in the urn V, again he selects randomly one ball from the urn V.
- If the two selected balls have different colors, then he dosn't put them in the urn V and he selects randomly one ball from the urn V.

Consider the following events:

R: « The two selected balls from the urn U are red »

V: « The two selected balls from the urn U are green »

D: « the two selected balls from the urn U are of different colors »

E : « Th eselected ball' (s) from the urn V are green »

1. Calculate P (R), P (V) and P (D).
2. Calculate $P(E/R)$, $P(E/V)$ *and* $P(E/D)$.
3. Prove that: P (E) = $\frac{31}{63}$.
4. The selected ball' (s) from the urn V are red, calculate the probability that the selected balls from the urn U are red.
5. Let X be a random variable that is equals to the number of the red balls remaining in the urn V after the two drawns.
   a- Determine the four possible values of X.
   b- Prove that: $P(X=2)=\frac{67}{126}$.
   c- Determine the probability distribution of X.

Exercise 11

We disposed three urns U, V and W.
Urn U contains 4 red balls and 5 black balls.
Urn V contains 3 red balls and 4 black balls.
Urn W contains 4 red balls and 4 black balls.

Part A:

We select randomly one ball from the urn U.

- If the selected ball is red, then we put it in the urn V.
- If the selected ball is black, then we put them in the urn W.
- Then finaly we select two balls: one ball from the urn V and one ball from the urn W.

Consider the following events:

R: « The selected ball from the urn U is red »

C: « The selected ball from the urn V is red and the selected ball from the urn W is red ».

1. Calculate P (R), P (C/R) and verify that $P(C\cap R)=\frac{1}{9}$.
2. Prove that: $P(C)=\frac{41}{189}$, then calculate $P(\overline{R}/\overline{C})$.

Part B:

In this part we emptied all the balls in the urns U, V and W in a new unr T.
We select randomly and simultaneously three balls from the urn T.
Let X be a random variable that is equals to the number of the selected red balls.

1. Calculate P (X = 0) and P (X ≤ 1).
2. Calculate P (X ≤ 2 / X ≥ 1).

Exercise 12

We have two urns U and V. Urn U contains 5 white balls and 3 red balls and urn V contains 4 white balls and 4 red balls.
We consist the following game: A player rolled a perfect die numbered from 1 to 6.

- If the die shows up the face 6 then the player draws randomly and simultaneously two balls from the urn U.
- If the die shows up the face 1, 2 or 3 then the player draws randomly and simultaneously two balls from the urn V.
- If the die shows up the face 4 or 5 then he chooses at random one of the two urns then draws simultaneously and randomly two balls from the chosen urn.

Consider the following events:

A: « The die shows up the face 6 »;
B: « The die shows up the face 1, 2 or 3 »;
C: « The die shows up the face 4 or 5 »;
W: « The 2 drawn balls are white »; R: « The 2 drawn balls are red ».

1. Calculate the probability of each of the following events:
   P (W/A), P (W/B) and P (W/C), Deduce that: $P(W)=\frac{10}{21}$.
2. Calculate the probability of the event R.
3. Knowing that the two drawn balls are white, what is the probability that the die shows up the face 1, 2 or 3.
4. In this part we emptied all the balls of the two urns U and V in a new urn S.
   The player draws randomly and successively without replacement three balls from the urn S.
   Let X be a random variable equal to the number of red drawn balls.
   a) Determine the probability distribution of the random variable X.
   b) Calculate the excepted value E (X) of the random variable X.

Exercise 13

In a television game two boxes A and B are displayed.
Box A contains seven tokens out of which three carry the number 1, two carry the number 2 and two carry the number 3.
Box B contains fifteen keys out of which three only open the door of a room containing a car.
A player starts by selecting, simultaneously and randomly, three tokens from box A.

- If the sum of the numbers carried by the three tokens is 3, 4 or 5, the player must withdraw.
- If the sum of the numbers carried by the three tokens is 6 or 7, the player selects randomly one key from box B.
- If the sum of the numbers carried by the three tokens is 8, the player selects randomly from box B two keys one after another without replacement.
- If the player selects a key that opens the door of the room, then he wins the car.

1. a- Calculate the probability that the sum of the numbers carried by the three drawn tokens is equal to 3.
   b- Show that the probability that the player « must withdraw » is equal to $\frac{16}{35}$.
2. a- Calculate the probability that the player selects two keys.
   b- The player got a sum equal to 8. Prove that the probability that the player wins the car is $\frac{13}{35}$.
3. Calculate the probability that the player wins the car.

Exercise 14

Consider an urn containing 10 balls, n balls are green, m balls are red and others are white so that: $n \geq 2;\ m \geq 2\ \ and\ \ n+m \leq 8$.
A player pays 5 \$ and then he draws at random two balls from the urn.
The player wins 15 \$ for each drawn green ball, 5 \$ for each drawn red ball and loses 5 \$ for each drawn white ball.
Let X be the random variable equal to the algebraic gain of the player at the end of the game.

1. a- Determine the possible values of X.
   b- Calculate P (X = 25) and P (X = 15) in terms of n and m.
   a- Knowing that: P (X = 25) = $\frac{1}{15}$ $and\ \ P(X=15)=\frac{2}{15}$, determine n and m.
2. Suppose that in this part that the urn contains 3 green balls, 2 red balls and 5 white balls.
   a- Determine the probability distribution of X and calculate its expected value E (X).

b- Calculate the probability that the player draws 2 balls of the same color knowing that their algebraic gain is positive.

Exercise 15

An urn U_1 contains 4 red balls and 6 black balls.
Another urn U_2 contains 1 red ball and 9 black balls.
A player has a perfect dice, he throws the dice, if it shows the number 1, he draws a ball from U_1 if not he draws a ball from U_2.
Consider the following events:

A: « the dice shows 1 »
R: « the drawn ball is red »

1. a- Calculate the probabilities P (A) and $P\left(R/A\right)$.

   b- Show that P (R) = 0.15.
2. The player repeats the game twice by replacing the ball obtained in the urn. After the two drawings, the player gets 3 points for each red ball and – 2 points for each black ball. Let X be the random variable equal to the sum of points after the drawings.
   a- Verify that the possible values of X are: 6 , + 1, and - 4.
   b- Determine the probability distribution of X and calculate E (X).

Exercise 18

An urn contains 4 red balls and 5 black balls, and a dice has a 6 faces numbered from 1 to 6. A player throw the dice one time.
If the face shows up 1 or 4, then he draws randomly and simultaneously 2 balls from the urn.
If the face shows up 2, 5 or 6, then he draws randomly and simultaneously 3 balls from the urn.
If the face shows up 3, then he draws randomly one ball form the urn.
Consider the following events: A: « The face shows up 1 or 4 ». B: « The face shows up 2, 5 or 6 ».
C : « The face shows up 3 »; E : « The drawn balls are red ».

1. Calculate P (A), P (B) and P (C),
2. a. Calculate P (E/A) and P ($E \cap A$).

   b. Calculate P (E/B) and P ($E \cap B$).

   c. Calculate P (E/C) and P ($E \cap C$). Then deduce P (E).
3. Knowing that the drawn balls are red, what is the probability that the face shows up 1 or 4.
4. In this part the player draws randomly and simultaneously three balls from the urn.
   Let X be a random variable equal to the number of the white drawn ball.
   a. Determine the probability distribution of X.
   b. Calculate the expected value E(x).

Exercise 19

The staff of a hospital is distributed into three categories: Doctors (D), Nurses (N) and Technicians (T).
20 % are doctors and 50 % are nurses.
75 % of the doctors are men and 80 % of the nurses are women.
We ask randomly one member of the staff.

1. Calculate the probability that this person is:
   a) A technician; b) a women knowing that she is a doctor; c) a man knowing that he is a nurse.
2. Calculate the probability that this person is:
   a) A women doctor; b) a women nurse.
3. Knowing that 51 % of the staff are women.
   a. Calculate the probability that the asked person is a women technician.
   b. Deduce the probability that the asked person is a women knowing that she is a technician.

Exercise 20

We disposed:

A cubic perfect die has: one red face, two green faces and three yellow faces.

An urn U contains: 3 black balls and 2 white balls.

An urn V contains: 1 black ball and 3 white balls.

Part A:

A game consists as follows: We throw the die one time.

If the obtained face is green, then we draw randomly and simultaneously two balls from the urn U.

If the obtained face is yellow, then we draw randomly and simultaneously 3 balls from the urn V.

If the obtained face is red, then we draw randomly one ball from the urn U and one ball from the urn V

Consider the following events:

G : « The die shows up the green face »;

Y: « The die shows up the yellow face »;

R: « The die shows up the red face »;

M: « The drawn balls have the same color »

1. Calculate P (G); P (Y) and P (R).
2. Prove that P (M / G) = $\frac{2}{5}$. and then calculate $P(M \cap G)$.
3. Calculate P (M).
4. Knowing that the drawn balls have the same color, calculate the probability that the drawn balls from the same urn.

Part B:

In this part, we emptied all the balls in a new urn W.

We draw at random one by one balls from the urn W and without replacement.

The selected is finish when we draw a black ball.

Calculate the probability that the game is ended in the third draw.

Exercise 21

We disposed an urn U contains seven dices:

- Three dices are red such that each die has Two faces numbered 3, three faces numbered 4 and one face numbered 6.
- Four dices are black such that each die has one face numbered 3, Four faces numbered 4 and one face numbered 6.

A player draws randomly and simultaneously two dices from the urn, then he throws them one time.

Consider the following events:

R : « The player selects two red dices » B : « The player selects two black dices » ;

D : « the player selects one red die and one black die»

E : « Obtaining one face numbered 3 and one face numbered 4 ».

1. Calculate P (R), P (B) and P (D).
2. Calculate P (E/R); P (E/B) and P (E/D).
3. Prove that: $P(E \cap R) = \frac{1}{21}$.
4. Calculate $P(E \cap B)$ *and* $P(E \cap D)$.
5. Prove that: P (E) = $\frac{29}{63}$.
6. Knowing that the player not obtaining neither face 3 nor face 4, Calculate the probability that he draws two red dices.

INVERSE TRIGONOMETRIC FUNCTIONS

Exercise 1

1. $2\arccos\left(\frac{8}{9}\right)=\arccos\left(\frac{47}{81}\right)=\frac{\pi}{2}-\arcsin\left(\frac{47}{81}\right).$
2. $2\arccos\left(\frac{8}{11}\right)=\arccos\left(\frac{7}{121}\right)=\frac{\pi}{2}-\arcsin\left(\frac{7}{121}\right).$
3. $2\arcsin\left(\frac{2}{9}\right)=\arcsin\left(\frac{4\sqrt{77}}{81}\right)=\frac{\pi}{2}-\arccos\left(\frac{4\sqrt{77}}{81}\right).$
4. $2\arcsin\left(\frac{4}{9}\right)=\arcsin\left(\frac{8\sqrt{65}}{81}\right)=\frac{\pi}{2}-\arccos\left(\frac{8\sqrt{65}}{81}\right).$
5. $2\arctan\left(\frac{4}{9}\right)=\arctan\left(\frac{72}{65}\right).$
6. $2\arctan\left(\frac{8}{9}\right)=\arctan\left(\frac{144}{17}\right).$
7. $2\arctan\left(\frac{4}{11}\right)=\arctan\left(\frac{88}{105}\right).$
8. $2\arctan\left(\frac{4}{13}\right)=\arctan\left(\frac{104}{153}\right).$
9. $2\arctan\left(\frac{4}{5}\right)=\arctan\left(\frac{40}{9}\right).$
10. $2\arctan\left(\frac{3}{5}\right)=\arctan\left(\frac{15}{8}\right).$
11. $\arccos x=\arccos\left(\frac{3}{7}\right)-\arccos\left(\frac{5}{7}\right).$
12. $\arccos x=\arccos\left(\frac{1}{7}\right)-\arccos\left(\frac{5}{7}\right).$
13. $\arccos\left(\frac{20+6\sqrt{22}}{49}\right)=\arccos\left(\frac{4}{7}\right)-\arccos\left(\frac{5}{7}\right).$
14. $\arccos\left(\frac{36+10\sqrt{42}}{121}\right)=\arccos\left(\frac{4}{11}\right)-\arccos\left(\frac{9}{11}\right).$
15. $\arccos\left(\frac{5+48\sqrt{5}}{121}\right)=\arccos\left(\frac{1}{11}\right)-\arccos\left(\frac{5}{11}\right).$
16. $\arccos\left(\frac{8+2\sqrt{34}}{27}\right)=\arccos\left(\frac{1}{3}\right)-\arccos\left(\frac{8}{9}\right).$
17. $\arccos\left(\frac{8+2\sqrt{114}}{33}\right)=\arccos\left(\frac{1}{3}\right)-\arccos\left(\frac{8}{11}\right).$
18. $\arccos\left(\frac{18+10\sqrt{2}}{33}\right)=\arccos\left(\frac{2}{3}\right)-\arccos\left(\frac{9}{11}\right).$
19. $\arctan\left(\frac{29}{31}\right)=\arctan\left(\frac{1}{4}\right)+\arctan\left(\frac{5}{9}\right).$
20. $\arctan\left(\frac{65}{11}\right)=\arctan\left(\frac{5}{4}\right)+\arctan\left(\frac{5}{9}\right).$
21. $\arctan\left(-\frac{65}{11}\right)=\arctan\left(\frac{4}{5}\right)+\arctan\left(\frac{9}{5}\right).$
22. $\arctan(2)=\arctan\left(\frac{2}{3}\right)+\arctan\left(\frac{4}{7}\right).$

Exercise 2

Solve in the set $\mathbb{R}$:

1. $\arctan x = \arctan 3 - \arctan\left(\frac{7}{9}\right)$.
2. $\arctan x = \arctan 3 - \arctan\left(\frac{1}{3}\right)$
3. $\arccos\frac{1}{3} + \arccos\frac{1}{4} = \arccos x$.
4. $\arctan 2x + \arctan 3x = -\frac{\pi}{4}$.

Exercise 3

1. Establish the following relations: $\arctan\frac{3}{4} + \arctan\frac{1}{7} = \frac{\pi}{4}$.

$$2\arctan\frac{1}{2} - \arctan\frac{1}{7} = \frac{\pi}{4}.$$

2. Deduce the value of the expression: $2\arctan\frac{1}{2} + \arctan\frac{3}{4}$.
3. Prove that: $2\arcsin\ \frac{4}{9}\ = \arcsin\ \frac{8\sqrt{65}}{81}$.
4. Solve in the set $\mathbb{R}$: $\arctan x = \arctan 3 - \arctan\ \frac{7}{9}$.

Exercise 4

Choose the right answer with justification:

1. Let x be a positive real number, $\arccos\left(\frac{1-x^2}{1+x^2}\right) = \ldots\ldots.$

   a) Arcsin (2x) ; b) 2 arctan (x) ; c) 2 arccos (x).

2. The set of the solutions of the equation: arctan (2x) = arcsin (x) is :

   a) {0} ; b) $\left\{0;\frac{\sqrt{3}}{2}\right\}$; c) $\left\{0;-\frac{\sqrt{3}}{2};\frac{\sqrt{3}}{2}\right\}$.

3. $\int\frac{dx}{x^2+9} =$ $a)\ \arctan(x+9)+c;$ $b)\arctan\left(\frac{x+1}{3}\right)+c;$ $c)\frac{1}{3}\arctan\left(\frac{x}{3}\right)+c.$

4. $\cos(2\arctan(x)) = \ldots..$ $a)\frac{1-x^2}{1+x^2};$ $b)\frac{1}{1+x^2};$ $c)2x^2-1.$

Exercise 5

Let $f(x) = \frac{2\arccos(x-2)-\pi}{\sqrt{x-2}}$.

1. Find the domain of definition of f.
2. Calculate $\lim\limits_{x\to 2^+} f(x)$.

SAMPLE TEST 1

Exercise 1

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1. Let (Γ) be the set of points M whose coordinates (x ; y) verifying : $y^2 - 4x + 4 = 0$.

   a- Prove that (Γ) is a parabola , and precise its vertex, focus and its directrix.

   b- Prove that the equation of the tangent (T) à (Γ) at the point A of abscissa 2 and positive ordinate is y = x.

   c- Draw (Γ) and (T).

2. Let (D) be the region bounded by (T) and (Γ) , the straight lines of equations x = 1 and x = 2. Calculate the volume of the generated solid by revolving (D) around the x – axis.

3. Let (Δ) be the straight line of equation x = $1+3\sqrt{2}$ and (Γ') the conic of focus $F'\left(1+\frac{3}{\sqrt{2}};0\right)$, of directrix associated (Δ) and of eccentricity $\frac{1}{\sqrt{2}}$.

   a- Indicate the nature de (Γ') then verify that the equation de (Γ') is: $(x-1)^2+2y^2=9$.

   b- Prove that the conics (Γ) and (Γ') are interesting at A and at another point B whose coordinates
   are to be determined on.

   c- Let H be the orthogonal projection of A on the ordinates axis, prove that H belongs to (Γ').
   Draw (Γ') in the same orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

Exercise 2

In the complex plane plan referred to a direct orthonormal system (O ; $\vec{u}$, $\vec{v}$) , Consider the points A and B with respective affixes : 2 and –2.

To every point M of affix z, we associate le point M' of affix z' such that z' = $\frac{\bar{z}(z-2)}{\bar{z}-2}$ with $z \neq 2$.

1. Give that algebraic form of z' such that z = 1 + i.
2. Determine the set of points M such that : z' = z.
3. a. Prove that : $|z'| = |z|$.

   b- Deduce that M and M' belong to the same circle with center O.
4. a. Prove that , for all complex number z, $(z-2)\left(\bar{z}-2\right)$ is real.

   b. Deduce that: $\frac{z'+2}{z-2}$ is real.

   c. Prove that the two straight lines (AM) and (BM') are parallel.

Exercise 3

In the space referred to a direct orthonormal system $\left(O;\vec{i},\vec{j},\vec{k}\right)$, Consider the lines ($d_1$) , ($d_2$) and ($d_3$) defined by : ($d_1$) : $\begin{cases} x=-\alpha+4 \\ y=3 \\ z=\alpha+2 \end{cases}$; (d_2) : $\begin{cases} x=-\beta+5 \\ y=\beta+1 \\ z=3 \end{cases}$ and (d_3) : $\begin{cases} x=-3 \\ y=\lambda+6 \\ z=-\lambda+6 \end{cases}$.

Where α, β and λ are real parameters.

1. a. Prove that (d_1) and (d_2) are coplanar straight lines.

   b. Determine the intersection point A of (d_1) and (d_2).

   c. Calculate the acute angle of (d_1) and (d_2).

d. Find an equation of the plane formed by (d_1) and (d_2).

2. Consider the plane (P) of equation: $x + y + z - 9 = 0$.

a- Prove that (d_3) belongs to the plane (P).

b- Let $(d_1)\cap(d_3) = \{B\}$ and $(d_2)\cap(d_3) = \{C\}$.

Find the coordinates of the points B and C, then prove that the triangle ABC is equilateral.

c- Write a system of parametric equations of one of the bisectors formed by (d_1) et (d_2).

3. Let E (1; –5; –5) and F (5; 11; 11). Prove that the tetrahedrons ABCE and ABCF have the same volume

4. Consider the points H (–1; 5; 5) and F' (–7; –1; –1).

a- Prove that H is the orthogonal projection of F on (P).

b- Prove that F' is the symmetry of F with respect to (P).

c- Find an equation of the plane (Q) symmetric of the plane (ABF) with respect to (P).

5. Consider the straight line (d_4) : $\begin{cases} x = 2t+1 \\ y = 3t+2 \\ z = -5t+3 \end{cases}$ Where t is a real parameter.

a. Prove that the straight line (d_4) is parallel to the plane (P).

b. Calculate the length of the common perpendicular to two straight lines (AB) and (d_4).

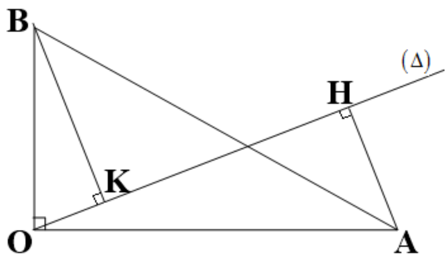
c. Let M be a variable point on the straight line (d_4).

Prove that the tetrahedron ABCM has a constant volume is to be determined.

<u>Exercise 4</u>

Consider, in an oriented plane, a right triangle AOB such that: $\left(\overrightarrow{OA};\overrightarrow{OB}\right) = \frac{\pi}{2} \quad (2\pi)$.

Let (Δ) be a variable straight line passing through O. The points H and K are the orthogonal projections of A and B on (Δ). Let S be a direct plane similitude that transforms O onto A and B onto O.



1. Determine the angle of S.
2. a. Prove that the center I of S belongs to two semis -circles of diameters [OA] and [OB].

b- Deduce that I is the orthogonal projection of O on [AB].

3. a. Determine the image by S each of the straight lines (BK) and (Δ).

b- Deduce that: S(K) = H.

c- Prove that as (Δ) varies, the circle (γ) of diameter [HK] passes through a fixed point to be determined.

4. Consider a dilation h of center B and of ratio 2. Let M be the midpoint of [OB], O' and B' the respective symmetric points of O and B with respect to I.

a- Prove that $S(B') = O'$ and determine $S\circ h(M)$ and $S\circ h(I)$.

b- Deduce that the median (IM) of the triangle IOB is a height of the triangle IAO'.

<u>Exercise 5</u>

<u>Part A :</u>

Consider the function K defined over $I = [0;+\infty[$ by : $k(x) = (2x+1)e^{2x} - x$.

1. Calculate $k(0)$, Then set up the table of variations of K.
2. Deduce for all $x \in I = [0;+\infty[$, the number of the solutions of the equation $k(x) = 0$.

<u>Part B :</u>

Let f be the general solution of the differential equation (E) : $y' - 2y = 2e^{2x}$.

1. Verify that : $g(x) = 2xe^{2x}$, is a particular solution of (E) .

2. Solve the differential equation $(E') : y' - 2y = 0$.

3. Prove that if f is a solution of (E), then $f - g$ is a solution of (E').

Deduce the general solution of (E), then the particular solution such that f (0) = 1.

Part C :

Consider the function f defined over $\mathbb{R}$ by $f(x) = (2x+1)e^{2x}$, and let (C) be its representative curve in an orthonormal system $(O;\vec{i};\vec{j})$.

1. Determine $\lim_{x\to-\infty} f(x)$, $\lim_{x\to+\infty} f(x)$ *and* $\lim_{x\to+\infty} \frac{f(x)}{x}$.. What can you deduce?
2. Calculate $f'(x)$, and set up the table of variations of f. Draw (C).
3. Prove that f admits over $[0;+\infty[$ an inverse function f^{-1}, Determine its domain of definition.

   Draw in the same system as that of (C), the curve (C') of the function f^{-1}.
4. Prove that (C) and (C') has no common point.
5. a- Write an equation of the tangent (T) to (C) at a point of abscissa $x = \frac{1}{2}$.

   b- Deduce the exact value of $(f^{-1})'(2e)$, then find an equation of the tangent (T') to the curve (C') at a point of abscissa $x = 2e$.
6. Calculate, **A** , the area of the region (D) bounded by : (C) , x' O x and the lines x = 0 and x = 1.
7. Calculate **V** , the volume of the solid generated by revolving (D) around x' O x.
8. Let h be the function defined over $\mathbb{R}$, by : $h(x) = (2|x|+1)e^{|2x|}$. Draw the curve of the function h.

SAMPLE TEST 2

Exercise 1

The space is referred to a direct orthonormal system $\left(O;\vec{i},\ \vec{j},\vec{k}\right)$, consider the straight line:

(D): x=-2m-2; y = 2m+1; z = m +2, $\left(m \in IR\right)$, and the three planes:

$\left(P\right): 2x+y+2z-1=0,\ \left(Q\right): x+2y-2z+4=0\ and\ \left(B\right): x-y+4z-5=0.$

1. Verify that (P) and (Q) are perpendicular and that (D) its line of intersection.
2. Let α and β be the acute angles formed respectively between (B) with (P) and (Q).
   a- Prove that: cos(α) = cos(β).
   b- Justify that (D) is included in the plane (B).
   c. Deduce that (B) is a bisector plane of the dihedral angle formed by (P) and (Q).
3. Consider the point A (-1; -2; 1) and let H, K and I be the respective orthogonal projections of A on (P), (Q) and (D). Let (R) be the plane passing through A, H and K.
   a- Verify that A is a point of (B). Deduce that AHIK is a square.
   b- Prove that: 2x -2y –z -1 =0 is the equation of (R). Deduce that I (0; -1; +1).
   a- Let (S) be a plane of equation: x + y + 2 =0. Prove that (S) is the mediator plane of [AI]. Deduce the parametric equations of (HK).
4. In the plane (R), consider the circle (C) with center A and passing through I. Let (T) be the tangent at I to (C).
   a- Write a system of parametric equations of (T).
   b- (T) cuts (AK) in L. Calculate the area of the triangle AIL.

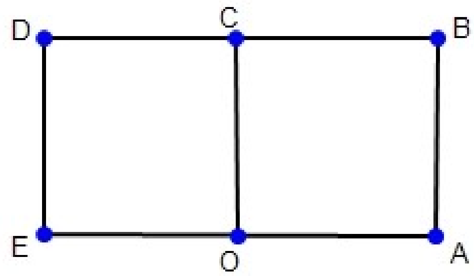
Exercise 2

On the adjacent figure, we have two squares OABC and OCDE such that:

$\left(\overrightarrow{OA};\overrightarrow{OC}\right)=\left(\overrightarrow{OC};\overrightarrow{OE}\right)=\frac{\pi}{2}\ [2\pi]$. Designate by I and J the respective midpoints of [CD] and [OC].

Assume that AB =a, (a > 0). Designate by S the direct plane similitude with angle $\alpha=\frac{\pi}{2}$, that transforms A onto I, and D onto E.

Part A :



1. Prove that the ratio of S is K $=\frac{1}{2}$.
2. Verify that S(B) =D. Deduce the image of C by S.
3. In this part we suppose that AB = 6 cm, and let A be the center of S.
   a- Draw the figure on its paper.
   b- Construct geometrically the point W. Plot W.
4. Let h = $S \circ S$.
   a- Determine the nature and the elements of h.
   b- Determine h (A) and h (B). Deduce another construction of the point W.

Part B :

The plane is referred to an orthonormal system $\left(O;\ \overrightarrow{OA},\ \overrightarrow{OC}\right)$.

1. Give the complex form of S, and determine the affix of W.
2. Let R be a rotation of center O and of angle $-\frac{\pi}{2}$. Suppose that: $f=S \circ R$.
   a- Determine f (D). Deduce the nature and the characteristic elements of f.
   b- Determine the complex form of f.

Exercise 3

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the conic (E) of equation:

$$\frac{x^2}{36}+\frac{y^2}{9}=1.$$

1. Determine the nature of (E) and precise its elements. Draw (E).
2. Determine the equations of hyperbolas with center O, whose vertices are the vertices of the conic (E) and the asymptotes are perpendiculars. Draw these hyperbolas.
3. Consider the function f defined over IR by: $f(x)=\int_0^{6\sin x}\sqrt{36-t^2}\,dt.$

   a- Prove that f is differentiable over IR and calculate $f'(x)$.

   b- Prove that the area of (E) is $A=2f\left(\frac{\pi}{2}\right)$.

   c- Deduce the value of A.

Exercise 4

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$ with unit 3 cm.

1. Consider the equation (E): $z^2-\sqrt{3}.z+1=0$.

   a- Verify that $Z_1=\frac{\sqrt{3}+i}{2}$ *and* $Z_2=\frac{\sqrt{3}-i}{2}$ are the roots of (E).

   b- Let $Z_{M_1}=Z_1$ *and* $Z_{M_2}=Z_2$. Write Z_1 and Z_2 in exponential form. Plot M_1 and M_2.

2. Consider the rotation R with center O and of angle $\frac{2\pi}{3}$.

   Let M_3=R(M_2). Calculate $Z_3=Z_{M_3}$. Plot M_3.

3. Let M_4 =T (M_2), where T is the translation of vector $\vec{w}$ with $Z_{\vec{w}}=-\frac{\sqrt{3}+i}{2}$.

   Calculate $Z_4=Z_{M_4}$ in exponential form. Plot M_4.

1. 4. Let $Z_{M_5}=Z_5=\frac{i}{2}\left(1+i\sqrt{3}\right)$ *and* $Z_{M_6}=Z_6=\frac{2}{i-\sqrt{3}}$.

   Write Z_5 and Z_6 in algebraic and exponential forms. Plot M_5 and M_6.

Exercise 5

Consider an urn which contains 9 balls: **6 black and 3 red.**

A game is consisting as follow: A player draws one ball from this urn.

- If the drawn ball is black ball, then he put it to the exterior of the urn.
- If the drawn ball is red, then he put it back in the urn and he adds one red ball in the urn.
- The game ends in the case the red balls equal to the number of black balls.

1. Prove that the game is end when the player draws exactly **3 balls.**
2. Knowing that the second drawn ball is black. Calculate the probability that the third ball is red.
3. The player wins 5000 L.L for each red drawn ball and loses 5000L.L for each drawn black ball. Let X be the random variable that is equal to the algebraic gains of the player.

   a- Verify that the possible values of X are: -15000; -5000; 5000 and 15000.
   Determine the probability distribution of X.

   b- Calculate the expected value of X.

Exercise 6

Part A: Consider the differential equation (E): $y''+3y'+2y=0$.

1. Find the general solution of (E).
2. Deduce the particular solution of (E) such that: $y(0)=3$ *and* $y'(0)=-2$.

Part B :

The plane is referred to an orthonormal system $(O;\vec{i},\ \vec{j})$ Graphic unit: 2 cm.

Let (C) be its representative curve of the function f defined over IR by: $f(x)=4e^{-x}-e^{-2x}$.

1. Prove that the x – axis is an asymptote to (C).
2. Verify that: $\lim\limits_{x\to-\infty} f(x)=-\infty$. Calculate $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}$.
3. Calculate $f'(x)$. Set up the table of variations of f.
4. Prove that f admits an inflection point I (0; 3).
5. Justify that (D): y =-2x+3 is the tangent at I to (C).
6. Solve the equation f (x) =0. Draw (D) and (C).
7. Let (Δ) be the region bounded by (C), (x' o x) and the lines: x = -ln (4) and x = -ln (2). Calculate in cm³, the volume of the solid generated by revolving (Δ) around (x' O x).

Part C :

Let (C') be the curve of the function h defined over]0; 4] by: $h(x)=\ln\left(\dfrac{2+\sqrt{4-x}}{x}\right)$.

1. Developed: $x\left(2+\sqrt{4-x}\right)^2$.
2. Justify that for all $x\in\left]0;4\right]$, *we have* : $(f\circ g)(x)=x$.
3. Deduce that h is the inverse function of f over $\left[-\ln 2;+\infty\right[$.

   Draw (C') in the same system as that of (C).
4. Let $J=\int_3^4 -h(x)\,dx$. Interpret J graphically. Deduce the value of J in cm²

SAMPLE TEST 3

Exercise 1

In the space referred to a direct orthonormal system $\left(O;\vec{i},\ \vec{j},\ \vec{k}\right)$, consider the straight lines (d_1) and (d_2) defined by: $(d_1):\begin{cases} x = m \\ y = m-1 \\ z = 1 \end{cases}$ and $(d_2):\begin{cases} x = -t+1 \\ y = t \\ z = -2t+4 \end{cases}$ (m and t are real numbers).

1. Prove that the straight lines (d_1) and (d_2) are orthogonal and not coplanar.
2. Verify that the vector $\vec{n}(-1;1;1)$ is orthogonal to (d_1) and to (d_2).
3. Prove that the equation of the plane (P) containing (d_1) and parallel to $\vec{n}$ is: x – y + 2 z – 3 = 0.
4. The straight line (d_2) cuts the plane (P) at B. Determine the coordinates of B.
5. Prove that the straight line (D) passes through B and has a direction vector $\vec{n}$ cuts the straight line (d_1) at the point A (1 ; 0 ; 1).
2. Let M be a variable point of (d_1) and M' be a variable point of (d_2).
   a- Prove that: $MM'^2 = MA^2 + AB^2 + M'B^2$.
   b- Deduce the minimum value of MM'.

Exercise 2

Consider two urns U and V.
The urn U contains eight balls: four balls holding the number 1, three balls holding the number 2 and one ball holding the number 4.
The urn V contains eight balls: three balls holding the number 1 and five balls holding the number 2.

1. We draw randomly and simultaneously, two balls from the urn U.
   Consider the following events:
   A: « The two drawn balls hold the same number ».
   B : « The product of the numbers on the two drawn balls is equal to 4 ».
   Calculate the probability P (A) of the event A and prove that $P(B) = \frac{1}{4}$.
2. We select at random one of the two urns U and V then we draw randomly and simultaneously two balls from the selected urn. Consider the following events:
   E: « The selected urn is V »
   F: « The product of the numbers on the two drawn balls is equal to 4 ».
   a- Verify that $P(F \cap E) = \frac{5}{28}$ and calculate $P\left(F \cap \overline{E}\right)$.
   b- Deduce P (E).
3. We draw randomly one ball from the urn U, and two balls randomly and simultaneously from the urn V.
   Calculate the probability of the event H: « the product of the 3 numbers on the three drawn balls is equal to 8 ».

Exercise 3

The plane is referred to an orthonormal system. (Graphic unit: 4 cm).

1. Consider the conic (H) of equation: $x^2 - y^2 + x + 1 = 0$.
   Prove that (H) is a rectangular hyperbola whose reduced equation, the center, the vertices and the asymptotes are to be determined.
2. Let (P) be a parabola of equation $y^2 = -x$.
   Determine the vertex S, the focus F and the directrix (d) of (P).
3. Prove that (H) and (P) have two points of intersection A and B (A has a positive ordinate), and that these two conics have the same tangents at these points.
4. Draw (H) and (P) in the same system.

5. Let (D) be the region bounded by (H), (P) and the straight lines of equations x = - 1 and x = 0. Calculate the volume of the solid generated by revolving (D), around (x' O x).

Exercise 4

In the complex plane referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$, consider the points A, B and C with respective affixes 1, i and $i\sqrt{3}$.

Let T be a transformation that to each point M of affix z we associate a point M' with affix z' such that:

$$z' = z^3 - 3z^2 + 3z.$$

1. Calculate the affixes of the images of O, B and C. Plot the points B and C and their images B' and C' in the plane. Is T preserves the collinearity?
2. Determine the double points by T.
3. a – Prove that, for all complex number z, the following equality: $z' - 1 = \left(z - 1\right)^3$.

   b- Deduce a relation between the distances AM' and AM and a relation between the angles $\left(\vec{u}, \overrightarrow{AM'}\right)$ and $\left(\vec{u}; \overrightarrow{AM}\right)$.

   c- Find the set of points M' such that M varies on the circle of center A and of radius $\sqrt{2}$.

   b- find the set of points M' such that M varies on the semi line]At) passing through B.

Exercise 5

Consider a direct square ABCD of side 10 and of center I.

Let J, K and L be the respective midpoints of segments [AB], [CD] and [DA]. Γ_1 the circle of diameter [AI] and Γ_2 the circle of diameter [BK].

Part A :

1. Determine the ratio and the angle of the similitude S such that S (A) = I and S (B) = K.
2. Prove that the circles Γ_1 *and* Γ_2 are interesting at two distinct points: the point J and the center Ω of the similitude S.
3. a- Determine the images by S the straight lines (AC) and (BC). Deduce the image of point C by S.

   b- Determine the image E of I by S.
4. Prove that the points A, Ω and E are collinear.

Part B :

Consider an orthonormal system $\left(A; \frac{1}{10}\overrightarrow{AB}, \frac{1}{10}\overrightarrow{AD}\right)$.

1. Determine the affixes of the points A, B, C and D.
2. Determine the complex form of S.
3. Calculate the affix W of the center Ω of S.
4. Calculate the affix of the point E and prove that again the points A, Ω and E are collinear.
5. Prove that the straight lines (AE), (CL) and (DJ) are concurrent at the point Ω.

Exercise 6

1. Consider the function g defined over ℝ by: g (x) = e^x – x – 1.

   a- Study the sense of variation of g. Calculate g (0).

   b- Deduce that the expression $\frac{e^x}{e^x - x}$ is defined for all real x.

2. Let f be a function defined over ℝ by: $f\left(x\right) = \frac{e^x}{e^x - x}$. Denote by (C) its representative curve in an Orthonormal system.

   a- Verify that for all real x, f (x) > 0.

   b- Determine $\lim\limits_{x \to -\infty} f\left(x\right)$ and prove that $\lim\limits_{x \to +\infty} f\left(x\right) = 1$. Deduce the asymptotes to (C).

   c- Calculate f '(x) and set up the table of variations of f. Draw (C).

3. Let h be a function defined over [0 ; +∞[by : $h(x)=\frac{1}{e^x-x}$.

Denote by (C’) its representative curve in an orthonormal system.

a- Calculate $\lim\limits_{x\to+\infty} h(x)$. Deduce an asymptote to (C’).

b- Calculate h ‘(x) and set up the table of variations of h. Draw (C’) in the same system.

c- Calculate the area of the region bounded by (C), (C’) and the straight lines of equations x = 1 and x = 2.

4. Let, for all- natural number n, $U_n=\int_0^n f(x)dx$.

a- Prove that the sequence (U_n) is increasing.

b- Prove that $f(x)=1+\frac{x}{e^x-x}$ and that, for all -natural number n, $U_n=n+\int_0^n \frac{x}{e^x-x}dx$.

c- Deduce the limit of the sequence (U_n) as n tends to + ∞.

5. Let, for all -natural number n, $V_n = U_n - n = \int_0^n \frac{x}{e^x-x}dx$.

a- Prove that the sequence (V_n) is increasing.

b- Prove that, for all x ≥ 0, $e^x-x\geq\frac{e^x}{2}$. Deduce that $V_n\leq\int_0^n 2xe^{-x}dx$.

c- Express $\int_0^n 2xe^{-x}dx$ in terms of n. Deduce that: $V_n \leq 2$.

SAMPLE TEST 4

Exercise 1

In the table given below, only one among the proposed answers to each question is correct. Write down the number of each question and give, **with justification**, the answer corresponding to it.

N^0	Questions	Answers a	b	c
1	Let $z \in \mathbb{C}$ verifying $\lvert z \rvert + \bar{z} = 3 + i$. The algebraic form z is:	$\frac{4}{3} + i$	$-\frac{4}{3} + i$	$\frac{4}{3} - i$
2	The curve of the function $F(x) = \int_0^{\arctan x} \sqrt{1 + \tan^2 t}\, dt$ admits:	Two inflection points	One inflection point	No inflection point
3	A primitive F of the function f defined over $\mathbb{R}$ by : $f(x) = \frac{1}{e^x + e^{-x}}$ is F(x) =	$\ln(e^x + e^{-x})$	$\arctan(e^x)$	$\arctan(1 + e^x)$

Exercise 2

In the space referred to a direct orthonormal system $(O; \vec{i}, \vec{j}, \vec{k})$.

Consider the point B(2 ; 3 ; 1) and the two straight lines (d) et (d') defined by :

$$(d): \begin{cases} x = m \\ y = -5m + 6 \\ z = -4m + 2 \end{cases} \quad \text{and} \quad (d'): \begin{cases} x = 4\lambda + 1 \\ y = \lambda + 1 \\ z = 5\lambda - 2 \end{cases}.$$

1. Calculate the acute angle α of the two straight lines (d) and (d').
2. Prove that (d) and (d') are intersect at the point A(1 ; 1 ; – 2)
3. Find an equation of the plane (P) formed by (d) and (d').
4. Prove that B is the center of the circle (C), of the plane (P), tangent to (d) and (d').
5. Find the coordinates of the point of tangency E of the circle (C) and the straight line (d).
6. Write a system of parametric equations of the straight line (AB).
7. Determine the coordinates of the points of intersection of (AB) with the circle (C).

Exercise 3

In the complex plane referred to direct orthonormal system $(O ; \vec{u}, \vec{v})$, Consider the points A and B with respective affixes : 2i and – i.

To every point M different from A with affix z, we associate a point M' with affix z' such that:

$$z' = \frac{z}{iz + 2}.$$

1. **a)** Give the algebraic form of z' in the case z = 2 + i .
   b) Give the algebraic form of z in the case z' = 1 – i .
2. **a)** Prove that, for all complex number $z \in \mathbb{C} - \{2i\}$, we have :
   $(z' + i)(z - 2i) = 2$ and BM'.AM = 2.
   b) Deduce that if M varies on the circle (C) of center A and radius 1 then M' varies on a circle (C') whose center and radius are be determined.
   c) Determine and draw the set (Δ) of points M such that : $\lvert z' \rvert = 1$.
3. Consider the complex number z such that: z = a + i where $a \in \mathbb{R}$.
   Designate by θ an argument of z.

a) Prove that: $z'=\frac{z}{iz}$. Deduce an argument of z' in terms of θ.

b) Prove that, for all complex number z, we have: $z^2-2\sqrt{2}z+4=\left(z-\sqrt{2}\right)^2+2$.

c) Then find the set of complex numbers which are verifies the equation (E) : $z^2-2\sqrt{2}z+4=0$.

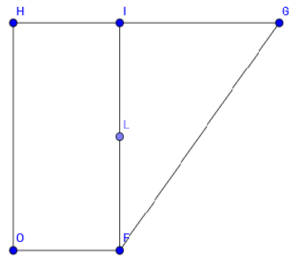
Denote: z_1 and z_2 the solutions of the equation (E) such that Im $(z_1)<0$.

d) Write z_1 in its trigonometric form. Deduce the trigonometric form of z_2.

<u>**Exercise 4**</u>

In the adjacent figure we have, OFGH is a rectangle trapezoid such that: OF = 2 ; OH = 4 and HG = 5.
Let I be the orthogonal projection of F on (HG) and L is the midpoint of [FI].
Let (P) be a parabola of focus F and of directrix (OH).

1. **a)** Prove that L is a point of (P) and that (OL) is tangent to (P).
   b) Prove that G belongs to (P).
   c) Determine the vertex S of (P). Draw (P).



2. The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$ such that:

$\overrightarrow{OF}=2\vec{i}$ and $\overrightarrow{OH}=4\vec{j}$.

a- Prove that the equation of (P) is: $y^2=4\left(x-1\right)$.

b- The straight line (FG) cuts (P) in E.

Prove that the coordinates of E are: $\left(\frac{5}{4};-1\right)$.

c- Prove that the tangents to (P) at G and E are perpendiculars and they are intersect at a point belongs to the straight line (OH).

<u>**Exercise 5**</u> We disposed an urn contains 5 red balls, 3 green balls and n white balls $\left(n\geq 2\right)$.

We draw at random and simultaneously two balls from the urn.

1. Calculate the probability of the drawn two white balls.
2. Denote by P(n) the probability of drawn two balls having the same color.

a- Prove that: $P(n)=\frac{n^2-n+26}{\left(n+7\right)\left(n+8\right)}$.

b- Calculate $\lim\limits_{n\to+\infty} P(n)$ and interpret the result.

3. Assume that n = 4.

A drawn consist to select simultaneously and randomly two balls from the urn.

A player makes two independents drawn, he put back the two drawn balls in the urn before selecting two balls from the urn in the second drawn.

. At the beginning he pays the sum of 30 000 L.L.

For each drawn :

- If the two drawn balls are of the same color, he wins 40 000 L.L.
- If not he wins 5 000 L.L.

Let X be the random variable that is equal to the algebraic gains after the two drawn.

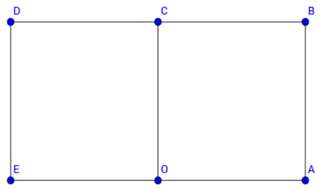
a) Determine the possible values of X.

b) Prove that: $P\left(X=-20000\right)=\left(\frac{47}{66}\right)^2$.

c) Determine the probability distribution of X.

<u>**Exercise 6**</u> In the adjacent figure, consider two squares OABC and OCDE such that:

$\left(\overrightarrow{OA};\overrightarrow{OC}\right)=\left(\overrightarrow{OC};\overrightarrow{OE}\right)=\frac{\pi}{2}\quad[2\pi]$. Let I and J be the respective midpoints of [CD] and [OC] and let AB = a , Designate by S the direct plane similitude of angle $\alpha = \frac{\pi}{2}$ that transforms: A onto I and D onto E.



Part A Let α be an angle of S and K its ratio.

1. Prove that the ratio of S is $K = \frac{1}{2}$.
2. Verify that S(B) = D. Deduce the image of C by S.
3. Suppose in this part that AB = 6 cm, and designate by W the center of S.
   a- Draw the figure on its paper.
   b- Construct geometrically the point W. Plot Won the figure.
4. Let h = S∘S.
   a- Determine the nature and the elements of h.
   b- Determine h(A) and h(B). Deduce another construction of the point W.

Part B: The plane is referred to an orthonormal system $\left(O;\overrightarrow{OA},\overrightarrow{OC}\right)$.

1. Give the complex form of S, and determine the affix of W.
2. Let R be a rotation of center O and of angle $-\frac{\pi}{2}$. Suppose that f = S∘R.
   a- Determine f(D). Deduce the nature and the elements of f.
   b- Determine the complex form of f.

Exercise 7

Consider the function f defined over $[0\,;+\infty[$ by: $f(x) = \frac{e^x - 1}{e^x - x}$. Designate by (C) the representative curve of f in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$ (unit: 5cm).

Part A 1.

Consider the function g defined over $[0\,;+\infty[$ by: $g(x) = e^x - x - 1$.

a- Prove that, for all x > 0, we have $g'(x) > 0$. Deduce the sense of variations of g over $[0\,;+\infty[$.

b- Calculate g (0). Deduce the sign of g(x) over $[0\,;+\infty[$.

2. Consider the function h defined over $[0\,;+\infty[$ by: $h(x) = (2-x)e^x - 1$.

a- Study the sense of variations of h and set up its table of variations.
b- Prove that the equation h(x) = 0 admits a unique solution α and verify that $1.84 < \alpha < 1.85$.
c- Precise, according to the values of the real numbers $x \geq 0$, the sign of h(x).

Part B **1. a -** Justify that f is defined over $[0\,;+\infty[$.

b- Calculate $\lim\limits_{x \to +\infty} f(x)$.

2. a- Prove that, for all $x \geq 0$, $f'(x) = \frac{h(x)}{\left(e^x - x\right)^2}$. Prove that: $f(\alpha) = \frac{1}{\alpha - 1}$.

   b- Study the sense of variations of f and set up its table of variations.

3. a- Prove that, for all $x \geq 0$, $f(x) - x = \frac{(1-x).g(x)}{e^x - x}$.

   b- Deduce, according to the values of the real number $x \geq 0$, the position of the curve (C) with respect to the straight line (D) of equation $y = x$.

4. a- Precise the tangent to (C) at a point of abscissa 0.
   b- Draw (C).

5. Calculate the area of the region bounded by (C), (D) and the lines of equations $x = 0$ and $x = 1$.

SAMPLE TEST 5

Exercise 1

In the table given below, only one among the proposed answers to each question is correct. Write down the number of each question and give, **with justification**, the answer corresponding to it.

N^0	Questions	Answers a	Answers b	Answers C
1	Let $z \in \mathbb{C}$ such that arg (Z) = θ+2kπ. Then an argument of $\frac{1-i}{(\overline{Z})^2}$ is:	$2\theta-\frac{\pi}{4}(2\pi)$	$-\frac{\pi}{4}-2\theta\ (2\pi)$	$\frac{3\pi}{4}+2\theta(2\pi)$
2	Consider the two points A and B of respective affixes: $Z_A=2$ *and* $Z_B=1+5i$. The complex form of the rotation of angle $\frac{\pi}{2}$ that transforms A onto B is:	$Z'=iZ-1+2i$	$Z'=\frac{\pi}{2}Z$	$Z'=iZ+1+3i$
3	M and M 'are two points of respective affixes Z and Z 'such that: $Z'=Z\overline{Z}+2Z-3$. If Z 'is a pure imaginary number then point M varies on:	The x – axis	The circle of center I (-1;0) and radius 2	The circle of center O and radius √3.
4	Knowing that the n^{th} derivative of the function $f(x)=\frac{1}{1-x}$ *is* $f^{(n)}(x)=\frac{n!}{(1-x)^{n+1}}$ Then the n th derivative of the function $g(x)=\ln(1-x)$ is:	$g^{(n)}(x)=-\frac{(n-1)!}{(1-x)^{n+1}}$	$g^{(n)}(x)=-\frac{n!}{(1-x)^{n}}$	$g^{(n)}(x)=-\frac{(n-1)!}{(1-x)^{n}}$

Exercise 2

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$. We consider the points A, B and C of respective affixes: $Z_A=2-3i$; $Z_B=i$ *and* $Z_C=6-i$.

1. **Calculate $\frac{Z_B-Z_A}{Z_C-Z_A}$. Deduce the nature of triangle ABC.**
2. **We define the function which to any point M of affix Z, distinct of B, associates the point M 'of affix Z' such that: $Z'=\frac{i(Z-2+3i)}{Z-i}$.**
   - a- **Determine the exponential form of Z 'when Z = 1 – i.**
   - b- **Determine Z when Z '= 2i.**
   - c- **Show that if M varies on the perpendicular bisector of [AB] then M 'moves on a circle whose center and radius are to be determined.**

Exercise 3

Two kinds of gift are displayed in a store: necklaces and bracelets. They can be either Gold or Silver.
40 % of the gifts are necklaces.
25 % of the necklaces are Gold.
1/3 of the bracelets are Silver.

Part A:

A customer chooses randomly a gift in this store. Consider the following events:
N: « The gift is a necklace »; B: « The gift is a bracelet »; S: « The gift is silver ».

1. What is the probability that the customer chooses a Gold necklace?
2. Show that $P(S \cap B) = \frac{1}{5}$.
3. What is the probability that the gift is Silver?
4. Knowing that the gift is Silver, Calculate the probability that it is a necklace.

Part B:

In this part, we suppose that there is 40 gifts in total.
We put all these gifts in a box and we draw one gift from this box.
If it is a necklace then we draw another gift. (We will have then 2 gifts)
Otherwise, we draw simultaneously 2 gifts from the box. (We will have then 3 gifts).

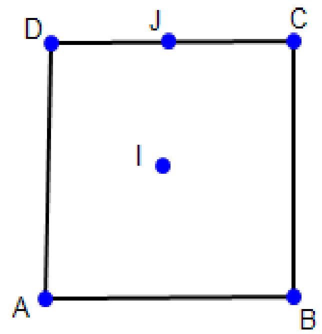
1. Calculate the probability that all gifts are of the same kind.
2. Calculate the probability that there is only one Gold necklace.

Exercise 4

ABCD is a square of side 4 such that $(\overrightarrow{AB};\overrightarrow{AD}) = \frac{\pi}{2} + 2k\pi$.

Let I be the center pf the square and J the midpoint of [CD].
Consider the direct plane similitude S that transforms A onto I and B onto J.

Part A:



1. Determine the ration and the angle of S.
2. Determine the image of line (BC) by S. Verify that S (C) = D.
3. Determine K image of I by S. Plot K.
4. Let h = S o S.
   a- Determine h (A).
   b- (AK) cuts the circle of diameter [AI] at A and M.
      Verify that M is the center of the similitude S.

Part B:

The complex plane is referred to a direct orthonormal system $(A;\overrightarrow{AB},\overrightarrow{AD})$.

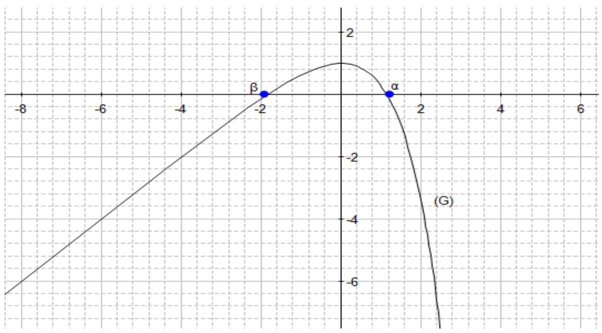
1. Write the complex form of S and that of h.
2. Determine the affix of M.
3. Calculate the affix of point E such that S (E) = A.

Exercise 5

Part A:

In the given figure, (G) is the representative curve of the function defined over ℝ by:
g (x) = x + 2 – e ^{x}.
The curve (G) cuts the x – axis at 2 points of respective abscissa **α** and **β**.

1. Show that the equation g (x) = 0 admits two roots **α** and **β**.
2. Verify that 1.1 < **α** < 1.2 and that -1.9 < **β** < -1.8.
3. Deduce the sign of g (x).



Part B:

Consider the function f (x) = $\frac{e^x - 1}{xe^x + 1}$ of representative Curve (C) in an orthonormal system $(O;\vec{i},\vec{j})$ (G.U:2 cm).

1. Consider the function h defined over ℝ by h (x) = x e ^{x} +1.
   a- Calculate h '(x) then set up the table of variations of h.
      Deduce that h (x) > 0 for all real numbers.
   b- Deduce that the function f is defined over ℝ.

2. a – Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$. Deduce the asymptotes to (C).

   b- Study the position of (C) and the straight line (d) of equation y = -1.

3. Show that $f'(x)=\dfrac{e^{x}g(x)}{\left(xe^{x}+1\right)^{2}}$ then set up the table of variations of f.

4. Verify that $f(\alpha)=\dfrac{1}{\alpha+1}$.

5. Consider that **α** = 1.14 and **β** = -1.85. Draw (C) and its asymptotes.

6. a – Show that $\dfrac{e^{x}-1}{xe^{x}+1}+1=\dfrac{(x+1)e^{x}}{xe^{x}+1}$.

   b- Calculate $\int_{-1}^{0}\left(\dfrac{e^{x}-1}{xe^{x}+1}+1\right)dx$ then deduce the area of the domain limited by the curve (C), the x – axis, and the lines x = -1 and x = 0.

<u>Exercise 6</u>

<u>Part A:</u>

Consider the function g that is defined over ℝ as: $g(x)=-1+(2-2x)e^{-2x+3}$.

1. Determine $\lim\limits_{x\to-\infty} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x)$.
2. a- Show that for all real x we have: $g'(x)=(4x-6)e^{-2x+3}$.

   b- Set up the table of variations of the function g.

   c- Justify that: $g\left(\dfrac{3}{2}\right)=-2$.
3. Show that the equation g (x) = 0 has a unique solution **α** such that: 0.86 < **α** < 0.87.

<u>Partie B:</u>

Consider the function f that is defined over ℝ as: f (x) = $-x+\left(x-\dfrac{1}{2}\right)e^{-2x+3}$.

Let (C) be its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1. Calculate $\lim\limits_{x\to-\infty} f(x)$ *and* $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x}$. Interpret graphically the results.
2. Calculate $\lim\limits_{x\to+\infty} f(x)$, then prove that the straight line (d) of equation y = - x is an asymptote to the curve (C) at +∞.
3. Study, according to the values of the real number x, the relative position of (C) and (d).
4. Prove that: f '(x) = g (x), then set up the table of variations of f.
5. Draw (C) and (d). (Assume that: **α** = 0.865 and f (**α**) = 0.4).
6. Let t be a real number such that t > 1.5.

   Let A (t) be the area of the region bounded by the curve (C), the straight line (d) and the two straight lines of equations x = 1.5 and x = t.

   a- Prove that: $\int_{1.5}^{t}\left(x-\dfrac{1}{2}\right)e^{-2x+3}dx=\dfrac{3}{4}-\dfrac{t}{2}e^{-2x+3}$.

   b- Deduce A (t).

   c- Calculate $\lim\limits_{t\to+\infty} A(t)$.

SAMPLE TEST 6

Exercise I

U_1 and U_2 are two given urns such that: U_1 contains 10 balls: 6 red and 4 yellow.
U_2 contains 10 balls: 5 red, 4 black and 1 green.
A fake coin is given such that the probability of having a head is three times more than that of having a tail. The coin is tossed:

- If we get a tail, we select, randomly and simultaneously, two balls from urn U_1.
- If we get a head, we select two balls from U_2 one after the other with replacement. Consider the following events:
  U_1: « The selected urn is U_1» and R: « The selected balls are red »

1. Show that $P(U_1) = \frac{1}{4}$.
2. Calculate $P(R/U_1)$ and $P(R \cap U_1)$. Deduce that: $P(R) = \frac{13}{48}$.
3. The two selected balls are red. Calculate the probability that they come from U_1.

Exercise II

Consider the function f defined on $\mathbb{R}$ as: $f(x) = \frac{2e^x}{e^x + 1} - x$, and denote by (C) its curve in an orthonormal system of axes $(O; \vec{i}, \vec{j})$.

1. a- Determine the limit of f (x) as x tends to $-\infty$, and show that the line (d): $y = -x$ is an asymptote to (C).
   b- Determine the limit of f (x) as x tends to $+\infty$, and show that the line (d'): $y = -x + 2$ is an asymptote to (C).
   c- Show that (C) is included between (d) and (d').
2. Show that point W (0;1) is the center of symmetry of curve (C).
3. a- For all real numbers x, prove that: $-1 < f'(x) < 0$.
   Set up the table of variations of the function f.
   b- Show that the equation $f(x) = 0$ has a unique root α, such that: $1.6 < \alpha < 1.7$.
   c- For all $x \in [0; \alpha[$, prove that: $0 \le f(x) \le \alpha - x$.
4. Draw (d), (d') and (C).

Exercise III

Consider the function f defined over $\mathbb{R}$ by: $f(x) = \ln(e^{2x} - e^x + 1) - 1$.

Let (C) be its representative curve in an orthonormal system $(O; \vec{i}, \vec{j})$.

1. Determine the limit of f at $-\infty$ and deduce an asymptote to (C).
2. a- Show that the line (D) with equation $y = 2x - 1$ is an asymptote to (C).
   b- Discuss according to the values of x, the relative position of (C) and (D).
3. Calculate $f'(x)$ and set up the table of variations of f.
4. Determine the coordinates of A, where the tangent to (C) is parallel to (D).
5. Draw (D) and (C).

Exercise IV

- Triangle ABC is direct and right at A.
- AB = 2, AC = 4.
- [AE] is an altitude in the triangle ABC.
- Let S be the similitude that transforms A onto B and E onto C.

Part A:

1. Determine an angle of S and show that the ratio of S is K = 5/2.
2. Let S (B) = F and S (C) = L.
   a- Construct F.
   b- Show that L is the meeting point of (CF) and (AB).
3. a- Construct (d) the image of the line (AF) under S, then determine S ((d)).
   b- Deduce that the center I of S is the meeting point of (d) and (AF).
4. Let h be the dilation that transforms F onto A and B onto C.
   a- Determine the center J of h, then verify that the ration of h is -4/5.
   b- Construct the point G the image of the point L under h.
5. a- Determine the nature of S o h.
   b- Show that C is the center of S o h.
   c- Deduce that E is the center of h o S.

Part B:

The complex plane is referred to the system $\left(A;\vec{u},\vec{v}\right)$ with $\vec{u}=\frac{1}{2}\overrightarrow{AB}$ and $\vec{v}=\frac{1}{4}\overrightarrow{AC}$.

1. a- Write the complex form of h o S.
   b- Deduce Z_E.
2. Determine h o S (C), then find Z_G.
3. Determine the nature of the quadrilateral LAGC.