Solved Problems

N°1.

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Check, with justification, if the following statements are true or false?

1) Let f be the mapping of the plane whose complex form is: $z' = (-2+2i)z + 1 - i$. Then, f is a direct plane similitude of ratio $2\sqrt{2}$ and angle $\frac{\pi}{4}$.

2) The composite of the rotation $r\left(O;\frac{\pi}{4}\right)$ and the dilation $h(O;-2)$ is a direct plane similitude of center O and angle $\frac{\pi}{4}$ and of ratio 2.

3) A direct plane similitude of ratio k multiplies the areas by k.

N°2.

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Let f be the mapping in the complex plane that associates to every point M of affix z the point M' of affix z' defined by $z' = \frac{1+i}{\sqrt{2}}z$.

Are the following statements true ?

Justify your answers :

1) f is the composite of a dilation of ratio $\frac{1}{\sqrt{2}}$ and a rotation of angle $\frac{\pi}{4}$.

2) The image of a circle of center O and radius R by f is a circle

of center O and radius $\frac{R}{\sqrt{2}}$.

3) The image of a circle (C) of center $A(1;-2)$ and radius 2 by f is a circle (C') of equation $x^2+y^2-3\sqrt{2}\,x+\sqrt{2}y+3=0$.

N°3.

Complete the following table :

Complex Form	Affix of the center	Ratio	angle
$z'=iz+1$			
	$1+i$	2	$\frac{\pi}{3}$

N°4.

The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

Let T be the mapping of the plane defined by :

$$M\begin{cases}x\\y\end{cases}\xrightarrow{T}M'\begin{cases}x'=x+y\sqrt{3}+4\\y'=-x\sqrt{3}+y-2\end{cases}.$$

Let $z'=x'+iy'$ and $z=x+iy$.

1) Express z' in terms of z.
2) Determine the nature and characteristic elements of T.

N°5.

The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

1) Let S be the transformation of the plane that associates to every point $M(z)$ the point $M'(z')$ such that $z'=(1+i)z+3i$. Determine the nature and the elements of S.
2) Consider the rotation r of center O and angle $-\frac{\pi}{2}$, let $f=r\circ S$. Determine the nature and the elements of f.

N°6.
The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

Given the two fixed points A and B of respective affixes 12 and $9i$.
Designate by S the transformation that to each point $M(z)$ associates the point $M'(z')$ such that $z' = -\frac{3}{4}iz + 9i$.

1) Determine the nature and the elements of S.
2) Determine the images of the points A and O by S.
3) Denote by Ω the center of S.
   a- Show that Ω is a common point to the circles (C_1) and (C_2) of respective diameters $[OA]$ and $[OB]$.
   b- Prove that Ω is the foot of the perpendicular drawn through O in triangle AOB.
   c- Using S, show that $\Omega A \times \Omega B = \Omega O^2$.

N°7.
The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

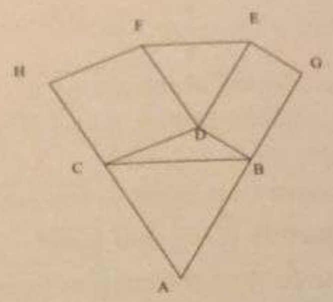
Consider the points A_0, A_1 and A_2 of respective affixes $z_0 = 5-4i$, $z_1 = -1-4i$ and $z_2 = -4-i$.

1) a- Determine the complex form of the direct plane similitude S that transforms A_0 onto A_1 and A_1 onto A_2.
   b- Deduce the affix ω of the point Ω center of S, as well as the ratio and an angle of S.
2) Let M be the point of affix z and $M'(z')$ the image of M by S. Verify that $\omega - z' = i(z - z')$, and deduce the nature of triangle $\Omega MM'$.

N°8.
The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

In the figure below, triangles ABC and DEF are two equilateral triangles such that $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{3}(\bmod 2\pi)$ and $\left(\overrightarrow{DE};\overrightarrow{DF}\right)=\frac{\pi}{3}(\bmod 2\pi)$.

Let G and H be two points such that $EDBG$ and $CDFH$ are two parallelograms.



Part A .

t_1 is the translation of vector $\overrightarrow{BD}$ and t_2 the translation of vector $\overrightarrow{DC}$.

r is the rotation of center D and angle $\frac{\pi}{3}$.

Let $f=t_2\circ r\circ t_1$.

1) a- Show that f is a rotation whose angle is to be determined.
   b- Determine $f(B)$ and deduce the center of f.
2) Determine the image of G by f and show that triangle AGH is equilateral.

Part B .

Designate by a, b, c, d, e, f, g and h the respective affixes of the points A, B, C, D, E, F, G and H.

1) a- Show that $c-a=e^{i\frac{\pi}{3}}(b-a)$.
   b- Express $f-d$ in terms of $e-d$.
2) a- Express g in terms of b, d and e.
   b- Express h in terms of c, d and f.
3) Show that $h-a=e^{i\frac{\pi}{3}}(g-a)$ and deduce the nature of triangle AGH.

N°9

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

Consider the points $M_0(1+5i)$, $M_1(1+i)$ and $M_2(-1-i)$.

1) Show that there exists a direct plane similitude S such that $S(M_0)=M_1$ and $S(M_1)=M_2$ and determine the elements of S.
2) Let $S^n = S \circ S \circ \cdots\cdots \circ S$, n times, where n is an integer greater than 1.
   a- Precise the nature and elements of S^n.
   b- For what values of n, is S^n a dilation ?
3) Let M be a point of affix z and $M_n = S^n(M)$.
   We define the sequence (u_n) by $u_0 = \left\|\overrightarrow{\Omega M_0}\right\|$ and for all natural numbers n, $u_n = \left\|\overrightarrow{\Omega M_n}\right\|$ where Ω is the center of S^n.
   a- Show that the sequence (u_n) is a geometric sequence whose ratio is to be determined.
   b- Express u_n in terms of n and calculate $\lim\limits_{n\to+\infty} u_n$.

N°10

$ABCD$ is a rectangle such that $AB=2$, $AD=4$ and $(\overrightarrow{AB};\overrightarrow{AD})=\frac{\pi}{2}\pmod{2\pi}$, E is a point of $[BC]$ such that $BE=1$.

Let S be the similitude that transforms A onto B and D onto A.

1) Determine the ratio k and the angle α of S.
2) a- Show that $S(B)=E$ and deduce that (AE) and (BD) are perpendicular.
   b- Let H be the point of intersection of (AE) and (BD), show that H is the center of S.
   c- Deduce that $HB^2 = HA \times HE$.
3) The plane is referred to the system $(A;\vec{u},\vec{v})$ such that $\overrightarrow{AB}=2\vec{u}$.
   a- Determine the complex form of S and deduce the affix of H.
   b- S^{-1} is the inverse of S, write the complex form of S^{-1}.

c- Let $h = S \circ S$, determine the nature of S.

d- Show that $\overrightarrow{HB} = -\frac{1}{4}\overrightarrow{HD}$.

e- Write the complex form of h.

4) (A_n) is the sequence of points defined by $A_0 = A$ and $A_{n+1} = S(A_n)$.

Let $\ell_n = A_n A_{n+1}$ where n is a natural integer.

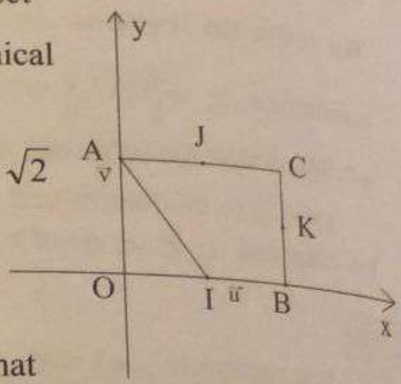
a- Show that the sequence (ℓ_n) is geometric.

b- Calculate $\lim\limits_{n \to +\infty} \ell_n$.

N°11.

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$ of graphical unit 5 cm, consider the points A, B and C of affixes: $z_A = i$, $z_B = \sqrt{2}$ and $z_C = \sqrt{2} + i$.

I, J and K are the respective midpoints of $[OB]$, $[AC]$ and $[BC]$.



Let S be the direct plane similitude that transforms A onto I and O onto B.

1) a- Determine the ratio k and the angle α of S.

b- Show that the complex form of S is $z' = \frac{\sqrt{2}}{2} i z + \sqrt{2}$.

c- Deduce the affix of the center Ω of S.

d- What is the image of rectangle $AOBC$ by S ?

2) Let $h = S \circ S$.

a- What are the images of the points O, B and A by $S \circ S$?

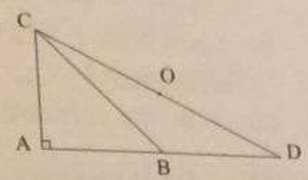
b- Show that h is a dilation whose center and ratio are to be determined.

c- Deduce that the straight lines (OC), (BJ) and (AK) are concurrent.

N° 12.

In the figure to the right, ABC is an isosceles triangle such that :

$AB = AC = \ell$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \frac{\pi}{2} \pmod{2\pi}$.



D is the symmetric of A with respect to B, and O is the midpoint of $[CD]$.

Let (C) be the circle of diameter $[CD]$.

S is the direct plane similitude that transforms D onto B and B onto C.

1) a- Determine the ratio k and angle α of S.

b- Let J be the center of S, show that

$\left(\overrightarrow{JD};\overrightarrow{JC}\right) = -\frac{\pi}{2} \pmod{2\pi}$ and that $JC = 2\,JD$.

c- Deduce that J belongs to (C) and that $JD = \ell$.

d- Show that (OB) is the perpendicular bisector of $[JC]$.

Determine the nature of quadrilateral $CADJ$ and place J.

2) The plane is referred to the direct orthonormal system $\left(A;\overrightarrow{AB},\overrightarrow{AC}\right)$.

a- Determine the complex form of S .

b- Deduce the affix of J .

c- Let M be a variable point of (C), what is the set of points M' image of M by S ?

d- S^{-1} is the inverse transformation of S .

Determine the nature, the elements and the complex form of S^{-1}.

3) a- Determine the image of circle (C) by the inversion $I(A;1)$.

b- Determine the image of the straight line (BC) by the inversion $I(A;1)$.

N° 13.

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Consider the points A , B , C and D of respective affixes

$z_A = 2+i$, $z_B = 1+2i$, $z_C = 6+3i$ and $z_D = -1+6i$.

1) a- Show that there exists a direct similitude f such that : $f(A) = B$ and $f(C) = D$.

b- Show that f is a rotation and precise its characteristic elements.

2) Let J be the point of affix $3+5i$.

Show that the rotation R of center J and angle $-\frac{\pi}{2}$ transforms A onto D and C onto B.

3) I is the point of affix $1+i$, M and N are the respective midpoints of segments $[AC]$ and $[BD]$.

Determine the nature of quadrilateral $IMJN$.

4) Consider the points P and Q such that quadrilaterals $IAPB$ and $ICQD$ are direct squares.

a- Calculate the affixes z_P et z_Q of points P and Q.

b- Determine $\frac{IP}{IA}$ and $\frac{IQ}{IC}$ as well ax a measure of the angles $\left(\overrightarrow{IA};\overrightarrow{IP}\right)$ and $\left(\overrightarrow{IC};\overrightarrow{IQ}\right)$.

c- Deduce the characteristic elements of the direct similitude g such that $g(A) = P$ and $g(C) = Q$.

d- Show that J is the image of M by g.

What can you deduce about the point J ?

N° 14.

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Consider the sequence of points A_n of affix z_n defined by :

$A_0 = O$ and $z_{n+1} = \frac{1}{1+i} z_n + i$ for all $n \in IN$.

1) Show that, for all $n \in IN$ the point A_{n+1} is the image of A_n by a direct similitude whose center Ω, ratio and angle are to be determined.

2) Prove that, for all $n \in IN$, the triangle $\Omega A_n A_{n+1}$ is right at A_{n+1}.

3) Consider the sequence (ℓ_n) defined by :

$\ell_0 = \Omega A_0$ and $\ell_n = \Omega A_n$.

a- Prove that the (ℓ_n) is a geometric sequence whose first term and ratio are to be determined.

b- What is the smallest value of n for which $\ell_n \leq 0.4$?

4) Designate by a_k the area of triangle $\Omega A_k A_{k+1}$ and consider the sequence (a_k), $k \in IN$.

a- Prove that the sequence (a_k) is a geometric sequence whose first term and ratio are to be determined.

b- Let $S_n = a_0 + a_1 + a_2 + \cdots\cdots a_n$.

Express S_n in terms of n and determine $\lim\limits_{n \to +\infty} S_n$.

N° 15.

A and C are two distinct points of the plane, designate by (Γ) the circle of diameter $[AC]$ and center O, B is a point of (Γ) distinct of the points A and C.

The point D is such that triangle BCD is equilateral with $(\overrightarrow{BC}, \overrightarrow{BD}) = \frac{\pi}{3} (\text{mod } 2\pi)$.

The point G is the centroid of triangle BCD, the straight lines (AB) and (CG) intersect at M.

Part A.

1) Prove that the points O, D and G belong to the perpendicular bisector of $[BC]$ and that the point G is the midpoint of $[CM]$.

2) Determine the ratio k and angle α of the direct similitude S of center C that transforms B onto M.

Part B.

The plane is referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$ in such a way that the points A and C have affixes -1 and 1 respectively.

Let E be the point such that ACE is equilateral with

$$\left(\overrightarrow{AC}; \overrightarrow{AE}\right) = \frac{\pi}{3} (\text{mod } 2\pi).$$

1) Calculate the affix of E.

2) σ is the direct plane similitude of complex form:

$z' = \frac{3+i\sqrt{3}}{4}z + \frac{1-i\sqrt{3}}{4}$.

Determine the characteristic elements of σ, and deduce that σ is the inverse of S.

3) Find the affix of point E' image of E by σ.
4) Denote by (C) the set of points M as B traces (Γ) deprived of the points A and C.
   a- Show that E belongs to (C).
   b- Let O' be the image of O by S.
   Prove that O' is the center of gravity of triangle ACE.

N°16

The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

A, A', B and B' are the points of respective affixes $z_A = 1-2i$, $z_{A'} = -2+4i$, $z_B = 3-i$ and $z_{B'} = 5i$.

1) Place the points A, A', B and B' in the plane and prove that $ABB'A'$ is a rectangle.
2) Let S be the reflection such that $S(A) = A'$ and $S(B) = B'$ and denote by (Δ) its axis.
   Find an equation of (Δ).
3) Let z' be the affix of point M' image of point M of affix z by S.
   Knowing that the complex form of S is $z' = a\bar{z} + b$, show that
   $z' = \left(\frac{3}{5} + \frac{4}{5}i\right)\bar{z} + 2i - 1$.
4) Let g be the mapping of the plane that to each point M of affix z associates the point P of affix z' defined by:
   $z' = \left(-\frac{6}{5} - \frac{8}{5}i\right)\bar{z} + 5 - i$.
   a- Designate by C and D the images of A and B by g respectively.
   Determine the affixes of the points C and D.
   b- Ω is the point of affix $1+i$ and let h be the dilation of center Ω and ratio -2, show that C and D are the respective images of A' and B' by h.

c- M_1 is the point of affix z_1 image of M, of affix z by h.
Find the characteristic elements of h^{-1} and express z in terms of z_1.

5) Let $f = h^{-1} \circ g$.
a- Determine the complex form of f.
b- Identify f.

N° 17

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

T is the mapping of the plane defined by :

$$M\begin{cases}x\\y\end{cases} \xrightarrow{T} M'\begin{cases}x' = -3y+2\\y' = -3x+6\end{cases}.$$

1) Show that T admits only one invariant point Ω.
2) Let $z' = x' + iy'$ and $z = x + iy$.
Show that $z' = a\bar{z} + b$ where a and b are two complex numbers to be determined.
3) Show that T is the composite of a reflection of axis $x'x$ and a similitude to be determined.
4) Prove that T is the composite of a dilation $h(\Omega;-3)$ and of a reflection of axis (Δ) passing through the point Ω and of slope 1.

N° 18

The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Let f be the mapping of the plane that to each point M of affix z associates the point M' of affix z' defined by $z' = \dfrac{1}{z}$.

1) Show that f is the composite of an inversion and of a reflection to be determined.
2) (C) is the circle of equation $x^2 + y^2 - 4x - 2y = 0$.
Construct the image of (C) by f geometrically.