IV. Part A: Consider the function $g(x) = x^2 - 2\ln x$ defined over $]0,+\infty[$

1) Find the limits of g at the endpoints of its domain.
2) Calculate g' and set up the table of variation of g.
3) Deduce the sign of $g(x)$.

Part B: f is a function defined over $]0,+\infty[$ by $f(x) = \dfrac{x}{2} + \dfrac{1+\ln x}{x}$.

1) Find the limit of f as x tends to 0. Interpret.
2) a) Find the limit of f as x tends to $+\infty$.

   b) Prove that the line $(d): y = \dfrac{x}{2}$ is an asymptote to (C).

   c) Study the relative position of (C) and (d).

3) Calculate f' and set up the table of variation of f.
4) Prove that there exists a point B belongs to (C) where the tangent (T) to (C) is parallel to (d).
5) Prove that the equation $f(x) = 0$ admits a unique root $\alpha \in]0.34, 0.35[$.
6) Draw (d), (T) and (C).

I A) $g(x) = x^2 - 2\ln x$, $]0, +\infty[$

1) $\lim_{0^+} g(x) = +\infty$ $\frac{1}{4}$ $\lim_{+\infty} g(x) = +\infty$ $\frac{1}{4}$

2) $g'(x) = 2x - \frac{2}{x} = \frac{2x^2 - 2}{x} = \frac{2(x-1)(x+1)}{x}$ $\frac{1}{2}$

x	0		1		$+\infty$
g'	‖	$-$	0	$+$	
g	‖ $+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

$\frac{1}{2}$

3) $g(x) > 0$ for every $x \in]0, +\infty[$ $\frac{1}{2}$

(B) $f(x) = \frac{x}{2} + \frac{1 + \ln x}{x}$ over $]0, +\infty[$

1) $\lim_{0^+} f(x) = -\infty$ $\frac{1}{2}$ $x = 0$ V.A 2) a) $\lim_{+\infty} f(x) = +\infty$ $\frac{1}{2}$

b) $\lim_{+\infty} \left(f(x) - \frac{x}{2}\right) = \lim_{+\infty} \frac{1 + \ln x}{x} = \lim_{+\infty} \frac{\frac{1}{x}}{1} = 0$, $y = \frac{x}{2}$ O.A. $\frac{1}{2}$

c) $f(x) - y_{(d)} = \frac{1 + \ln x}{x}$ $\quad 1 + \ln x \geq 0 \quad \ln x \geq -1, \quad x \geq e^{-1}$ $\frac{1}{2}$

for $x = e^{-1}$, (C) cuts (d), for $x > e^{-1}$; (C) above (d)

for $x < e^{-1}$, (C) is below (d)

3) $f'(x) = \frac{1}{2} + \frac{\left(\frac{1}{x} \times x - 1 - \ln x\right)}{x^2} = \frac{x^2 - 2\ln x}{2x^2} = \frac{g(x)}{2x^2} > 0$, $\frac{1}{2}$

x	0	$+\infty$
f'	‖	$+$
f	‖ $-\infty$	$\nearrow$ $+\infty$

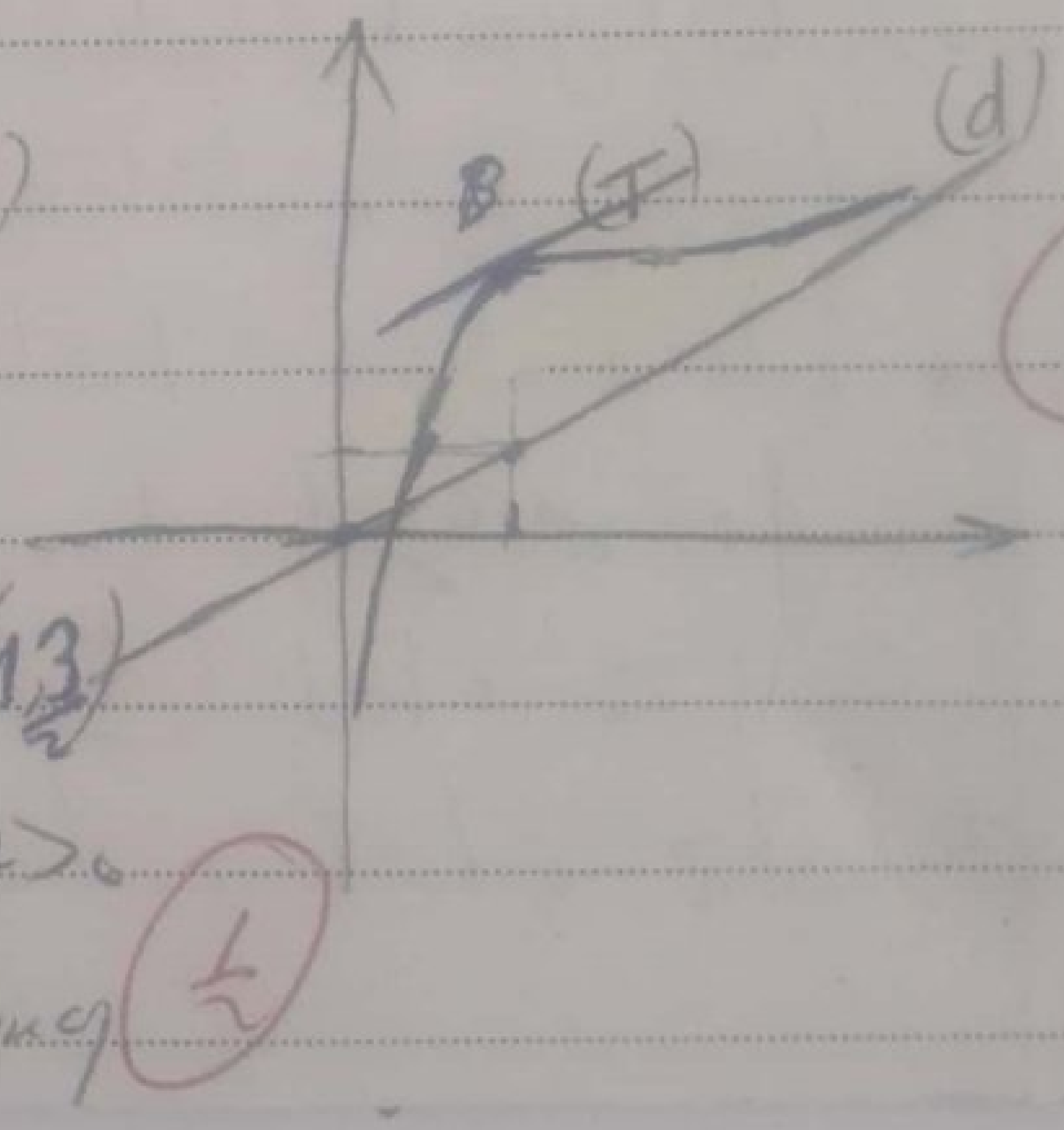
$\frac{1}{2}$

4) $(T) \parallel (d) \Rightarrow f'(x_0) = \text{Slope}(T) = \text{Slope}(D)$

$f'(x_0) = \frac{1}{2} \Rightarrow \frac{x_0^2 - 2\ln x_0}{2x_0^2} = \frac{1}{2}$

1 $x_0^2 - 2\ln x_0 = x_0^2$, $-2\ln x_0 = 0$, $x_0 = 1$

$x_0 = 1$, $y_0 = f(1) = \frac{1}{2} + \frac{1+0}{1} = 1$, $B(1, \frac{3}{2})$



5) $f(0.34) = -0.06 < 0 \qquad f(0.35) = 0.032 > 0$

and f is continuous and increasing $\frac{1}{2}$

IV. Part A: g is a function defined over $]0,+\infty[$ by
$$g(x) = 2x^3 - 1 + 2\ln x.$$

1) Find the Limits of g, Calculate $g'(x)$ and set up the table of variation of g.

2) Verify that the equation $g(x)=0$ admits a unique root $\alpha \in]0.8, 0.9[$. Deduce the sign of $g(x)$.

Part B: Consider the function $f(x) = 2x - \dfrac{\ln x}{x^2}$

a) calculate the limits of f at the endpoints of its domain.

b) Prove that the Line (d): $y = 2x$ is an oblique asymptote then study the relative position of (C_f) and (d)

c) verify that $f'(x)$ has the same sign as $g(x)$

d) set up the table of variation of f

e) prove that f admits an inverse function f^{-1} over $[\alpha, +\infty[$ and set up its table of variation.

f) write the equation of the tangent to $(C_{f^{-1}})$ at $a = 2$

g) Find the point of (C_f) where the tangent is parallel to (d).

h) Let $\alpha = 0.85$.

1/ Draw (C_f) and $(C_{f^{-1}})$ on the same system

2) Find $(C_f) \cap (C_{f^{-1}})$

IV. $g(x) = 2x^3 - 1 + 2\ln x$

($1\frac{1}{2}$) 1) $\lim\limits_{0^+} g(x) = -\infty$, $\lim\limits_{+\infty} g(x) = +\infty$; $g'(x) = 6x^2 + \frac{2}{x} > 0$;

x	0		$+\infty$
g'		$+$	
g	$-\infty$	$\nearrow$	$+\infty$

2) $\alpha \in]0.8 ; 0.9[$?

($\frac{1}{2}$) g is continuous and strictly increasing over $]0.8, 0.9[$ and

($\frac{1}{2}$) $g(0.8) = -0.42$; $g(0.9) = 0.24$ then there exists $\alpha \in]0.8, 0.9[$ such that $g(\alpha) = 0$;

x	0		α		$+\infty$
$g(x)$		$-$	0	$+$	

Part B

$$f(x) = 2x - \frac{\ln x}{x^2}$$

($\frac{1}{2}$) 1) $\lim\limits_{0^+} f(x) = +\infty$, $\lim\limits_{+\infty} f(x) = +\infty$

($\frac{1}{2}$) 2) $\lim\limits_{x \to +\infty} f(x) - 2x = \lim\limits_{+\infty} \frac{-\ln x}{x^2} = 0$, (d): $y = 2x$ is an oblique Asymptote ; $x = 1$ (C) cuts (d)

($\frac{1}{2}$) * R.P.: $f(x) - 2x = \frac{-\ln x}{x^2}$: for $0 < x < 1$, (C) is above (d)
$x > 1$; (C) is below (d)

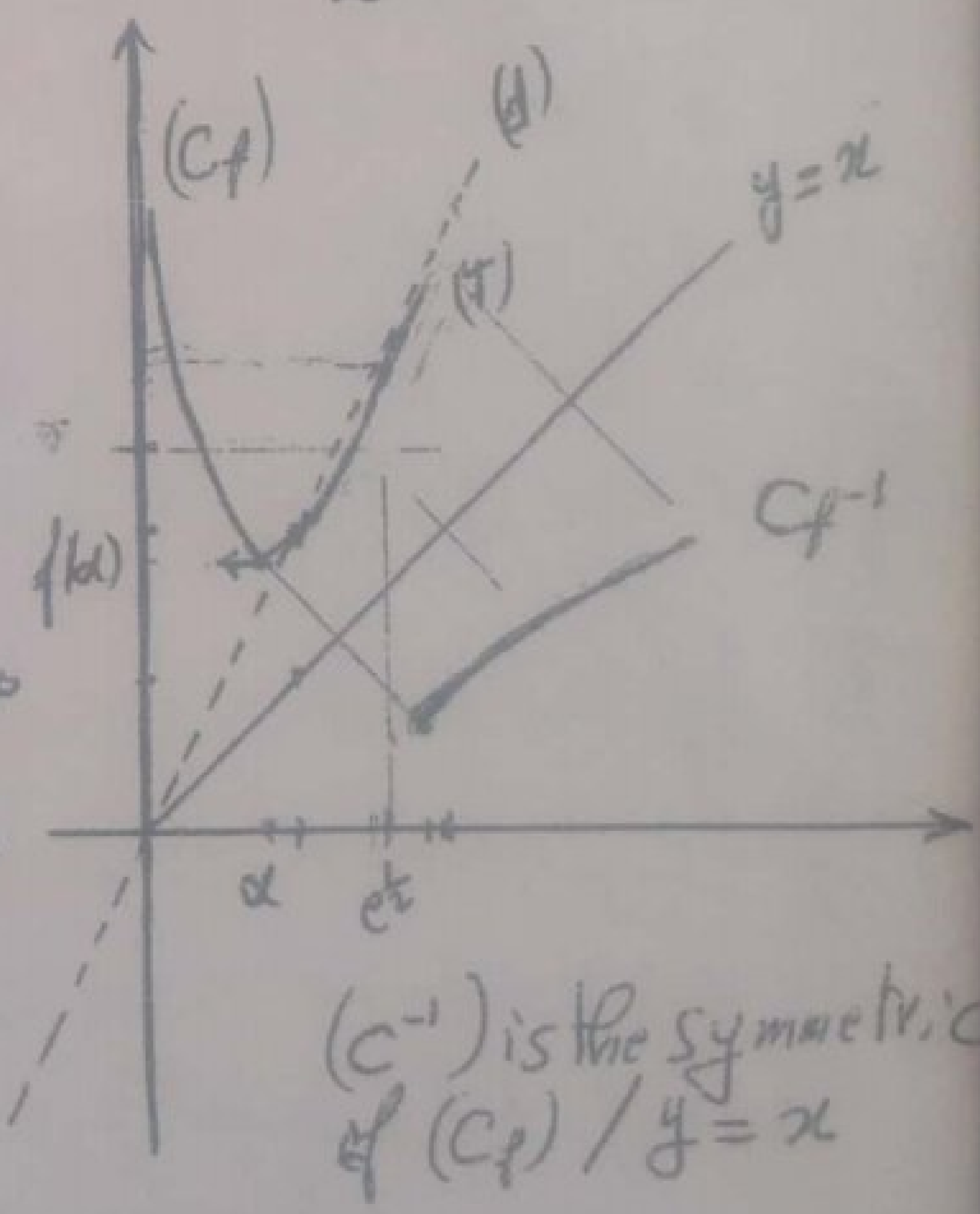
($\frac{1}{2}$) 3) $f'(x) = 2 - \left(\frac{\frac{1}{x} \times x^2 - 2x\ln x}{x^4}\right) = \frac{2x^3 - 1 + 2\ln x}{x^3} = \frac{g(x)}{x^3}$

$f'(x)$ has the same sign as $g(x)$

($\frac{1}{2}$) 4)

x	0		α		$+\infty$
f'		$-$	0	$+$	
f	$+\infty$	$\searrow$	$f(\alpha)$	$\nearrow$	$+\infty$

($1\frac{1}{2}$)



(C^{-1}) is the symmetric of (C_f) / $y = x$

($\frac{1}{2}$) 5) over $[\alpha, +\infty[$, f is continuous and strictly increasing, then it admits an inverse f^{-1}

x	$f(\alpha)$		$+\infty$
$(f^{-1})'$		$+$	
f^{-1}	α	$\nearrow$	$+\infty$

($\frac{3}{2}$) 6) $y = (f^{-1})'(2)(x-2) + f^{-1}(2)$

$y = \frac{1}{f'(1)}(x-2) + 1 = x - 1$

($\frac{1}{2}$) 7) tangent $\parallel$ (d) $\Rightarrow$ slope(t) $= 2$, $f'(x_0) = 2$, $\frac{2x_0^3 - 1 + 2\ln x_0}{x_0^3} = 2$

$-1 + 2\ln x_0 = 0$, $\ln x_0 = \frac{1}{2} = \ln e^{\frac{1}{2}}$, $x_0 = e^{\frac{1}{2}}$, $y_0 = f(e^{\frac{1}{2}}) = 2e^{\frac{1}{2}} - \frac{\frac{1}{2}}{e}$

($\frac{1}{2}$) 8) $\alpha = 0.85$, $f(\alpha) = 2(0.85) - \frac{\ln(0.85)}{(0.85)^2} = 1.924$

b) $(C_f) \cap (C_{f^{-1}}) = \emptyset$ because (C_f) does not cut the line $y = x$

$e^{\frac{1}{2}} = 1.64$, $f(e^{\frac{1}{2}}) = 3.17$

 2/12/2013

II. part A: g is a function defined over $]0,+\infty[$ by $g(x) = x^2 - 2\ln x$.

1) Determine $\lim\limits_{x \to 0} g(x)$ and $\lim\limits_{x \to +\infty} g(x)$.

2) Calculate $g'(x)$ and set up the table of variation of g.

3) Deduce the sign of $g(x)$.

Part B: Consider the function $f(x) = \frac{x}{2} + \frac{1+\ln x}{x}$ (C)

1) Calculate $\lim\limits_{x \to 0} f(x)$, deduce the equation of the asymptote

2) a) Determine $\lim\limits_{x \to +\infty}$ and show that the line $(\Delta): y = \frac{x}{2}$ is an oblique asymptote.

b) Study the relative position of (C) and (Δ).

3) Calculate $f'(x)$ and set up the table of var.

4) calculate the coordinates of the point B on (C), the tangent (T) to (C) is parallel to (Δ).

5) show that the equation $f(x) = 0$ has a unique soln $\alpha \in]0.34 ; 0.35[$.

6) Draw (Δ), (T) and (C)

7) Let h be the function defined over $]0, +\infty[$ by $h(x) = \dfrac{1 + \ln x}{x}$

a) Find an antiderivative of h

b) Calculate the area of the region bounded by (C), (Δ) and the lines with equations $x = 1$ and $x = e$.

8) prove that f admits an inverse function f^{-1}, specify its domain and Draw its curve on the same system.

9) write the equation of the tangent to (C^{-1}) at $A\left(\frac{3}{2}, 1\right)$

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z_{-1} $\overline{z_{-1}}$ real number.

II (A) $g(x) = x^2 - 2\ln x$

1) $\lim\limits_{0^+} g(x) = +\infty$, $\lim\limits_{+\infty} g(x) = \lim\limits_{+\infty} x\left(x - 2\frac{\ln x}{x}\right) = +\infty$ $\frac{1}{2}$

$\frac{1}{2}$ 2) $g'(x) = 2x - \frac{2}{x} = \frac{2x^2-2}{x} = \frac{2(x+1)(x-1)}{x}$ 1

x	0		1		$+\infty$
g'		$-$	0	$+$	
g	$+\infty$	$\searrow$	1	$\nearrow$	$+\infty$

$\frac{1}{2}$ 3) $g(x) > 0$ for every $x \in]0, +\infty[$.
(Min $= 1$)

(B) $f(x) = \frac{x}{2} + \frac{1+\ln x}{x}$

1) $\lim\limits_{0^+} f(x) = 0 + \frac{1-\infty}{0} = -\infty$, $x = 0$ V.A $\frac{1}{2}$

2) a) $\lim\limits_{+\infty} f(x) = \lim\limits_{+\infty}\left(\frac{x}{2} + \frac{1}{x} + \frac{\ln x}{x}\right) = +\infty$ $\frac{1}{4}$ $\frac{1}{2}$

$\lim\limits_{+\infty}\left(f(x) - \frac{x}{2}\right) = \lim\limits_{+\infty} \frac{1+\ln x}{x} = \lim\limits_{+\infty}\left(\frac{1}{x} + \frac{\ln x}{x}\right) = 0$, $y = \frac{x}{2}$ O.A.

b) $f(x) - \frac{x}{2} = \frac{1+\ln x}{x}$, $\ln x + 1 = 0$ when $x = e^{-1}$

$\frac{3}{4}$ for $x < e^{-1}$, (C) is below (Δ)
for $x > e^{-1}$, (C) is above (Δ)
for $x = e^{-1}$, (C) cuts (Δ) at $\left(e^{-1}, \frac{e^{-1}}{2}\right)$

3) $f'(x) = \frac{1}{2} + \frac{\left(\frac{1}{x} \times x - 1 - \ln x\right)}{x^2} = \frac{1}{2} + \frac{-\ln x}{x^2} = \frac{x^2 - 2\ln x}{2x^2}$

$\frac{1}{2}$ $f'(x) = \frac{g(x)}{2x^2}$, $f'(x)$ and $g(x)$ have the same sign

$\frac{1}{2}$

x	0		$+\infty$
f'		$+$	
f	$-\infty$	$\nearrow$	$+\infty$

4) $B? \in (C), (T) // (\Delta) \Rightarrow a_{(T)} = a_{(\Delta)} = \frac{1}{2}$

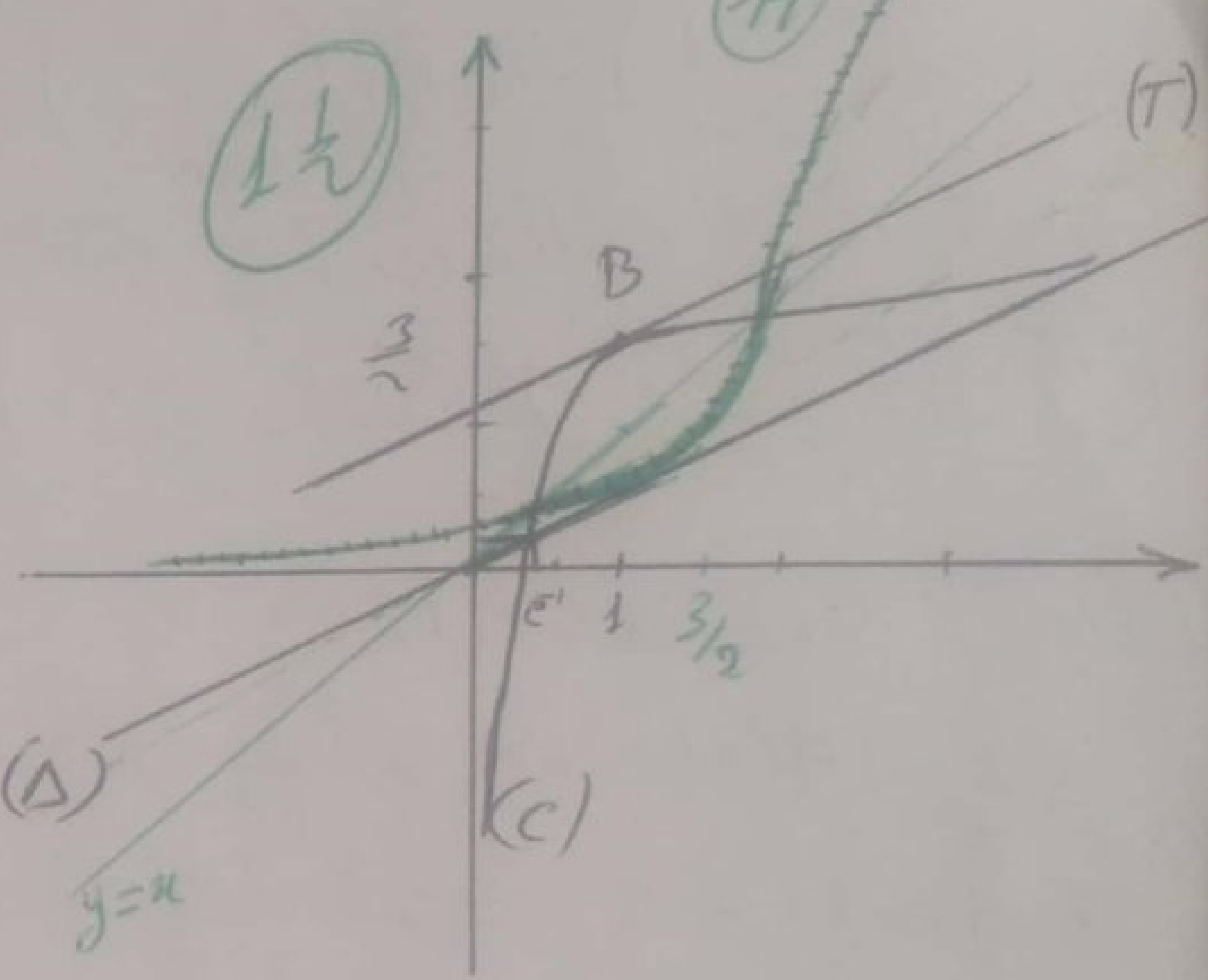
1 $f'(x_0) = \frac{1}{2}$, $\frac{x_0^2 - 2\ln x_0}{2x_0^2} = \frac{1}{2}$, $x_0^2 - 2\ln x_0 = x_0^2$, $\ln x_0 = 0$

$x_0 = 1$, $y_0 = 3/2$, $B(1, 3/2)$

5) $f(x)$ continuous and strictly increasing from $-\infty$ to $+\infty$

$\frac{1}{2}$ then there exists α such that $f(\alpha) = 0$

$f(0.34) = -0.06$ $f(0.35) = 0.03$ then $\alpha \in]0.34, 0.35[$



7) $h(x) = \frac{1 + \ln x}{x} = \frac{1}{x} + \frac{1}{x}\ln x$

3/4 a) $\int h(x)\,dx = \ln x + \frac{\ln^2 x}{2}$

b) $A_{(C)(\Delta)} = \int_1^e \left[f(x) - \frac{x}{2}\right] dx$

3/4 $= \int_1^e \frac{1 + \ln x}{x}\,dx$

$= \left[\ln x + \frac{\ln^2 x}{2}\right]_1^e = \frac{3}{2}\,u^2$

8) f is continuous and strictly increasing then it admits an inverse f^{-1} defined over $]-\infty, +\infty[$.

$\frac{1}{2}$ * (C^{-1}) is the symetric of (C) / $y = x$

9) $d: y = f'^{-1}(3/2)(x - 3/2) + f^{-1}(3/2)$ $f'^{-1}(3/2) = \frac{1}{f'(1)} = \frac{1}{\frac{1}{2}} =$

3/4 $y = 2(x - 3/2) + 1 = 2x - 2$

$y = 2x - 2$

AL MAHDI SCHOOL
3^{rd} year, LS

MATH EXAM

Question 1

In the complex plane referred to a direct orthonormal system$(O;\ \vec{u}, \vec{v})$, consider the points A, B, C, and D such that $z_A = 1 + 2i$, $z_B = 5$, $z_C = 4 + 3i$, $z_D = 3 - \sqrt{3} + i(1 - 2\sqrt{3})$. Let M and M′ be two points of respective affixes z and z′, where $z \neq 2$ and $z' = \dfrac{-1+2i}{z-2}$.

1) Find the exponential form of z′ when $z = \dfrac{1+i}{2}$.
2) a)Prove that $z_C - z_A = i(z_B - z_C)$.
b)Deduce the nature of the triangle ABC.
3) a)Show that $\dfrac{z_C - z_D}{z_A - z_B}$ is pure imaginary.
b)What can you say about the two straight-lines (AB) and (CD)?
4) Prove that: If M′ moves on the circle of center O and radius 1, then M moves on a circle whose radius and center are to be determined.
5) Let $z = x + iy$ and $z' = x' + iy'$. Find the set of the points M when z′ is a real number.

Question 2

The following table is the table of variations of the function f that is defined on J.

X	$-\infty$		-3		1		2		$+\infty$
f′(x)		$-$	-5	$-$	‖	$-$	0	$+$	
f(x)	1	↘	0	↘ $-\infty$	‖ $+\infty$	↘	4	↗	$+\infty$

Answer by TRUE or FALSE, and justify your answer.

1) $J = \mathbb{R}$
2) The image of J is $\mathbb{R}$
3) The equation $f(x) = 0$ has a unique root
4) f admits, on J, an inverse function
5) The curve of f has two extrema
6) $f(x) \geq 0$ for any x in J
7) f admits, on $]1, +\infty[$, an inverse function
8) f admits, on $[-3, 2]$, an inverse function
9) f admits, on $]-\infty, 0]$, an inverse function
10) When f^{-1} exists, $\left(f^{-1}\right)'(0) = -\dfrac{1}{5}$

Question 3

Let f be the function defined on $]0\,,+\infty[$. $\lim\limits_{x\to0^+} f(x) = -\infty$ and $\lim\limits_{x\to+\infty} f(x) = +\infty$. (C) is the representative curve of f in an orthonormal system $(O;\vec{i},\vec{j})$ and $e \approx 2.7$. Using the above given and the following table:

x	0		e		$e\sqrt{e}$		$+\infty$
$f''(x)$	$\|\|$	$-$		$-$	0	$+$	
$f'(x)$	$\|\|$	$+\infty \searrow$	1	$\searrow$	$1-e^{-3}$	$\nearrow$	1

1) Show that f is strictly increasing on its domain of definition.
2) Set up the table of variations of f.
3) Prove that the equation $f(x) = 0$ has a unique solution.
4) Prove that $f(x) = \lambda$ has a unique solution, where λ is a real number.
5) Let $G\left(e\,,\, e+\dfrac{2}{e}\right)$ be a point of (C). Write an equation of the line (D) that is tangent to (C) at the point G.
6) Prove that the curve (C) has an inflection point L whose abscissa is to be determined.
7) a)Prove that f admits an inverse function f^{-1} whose table of variations is to be constructed.

b)Calculate $\left(f^{-1}\right)'(a)$, where $a = e+\dfrac{2}{e}$.

Question 4

Part A: Consider the function g defined, on $]0\,,+\infty[$, by: $g(x) = x-1+\ln x$.

- a- Find the limits of g and setup its table of variation.
- b- Find $g(1)$ then study the sign of $g(x)$ on $]0\,,+\infty[$.

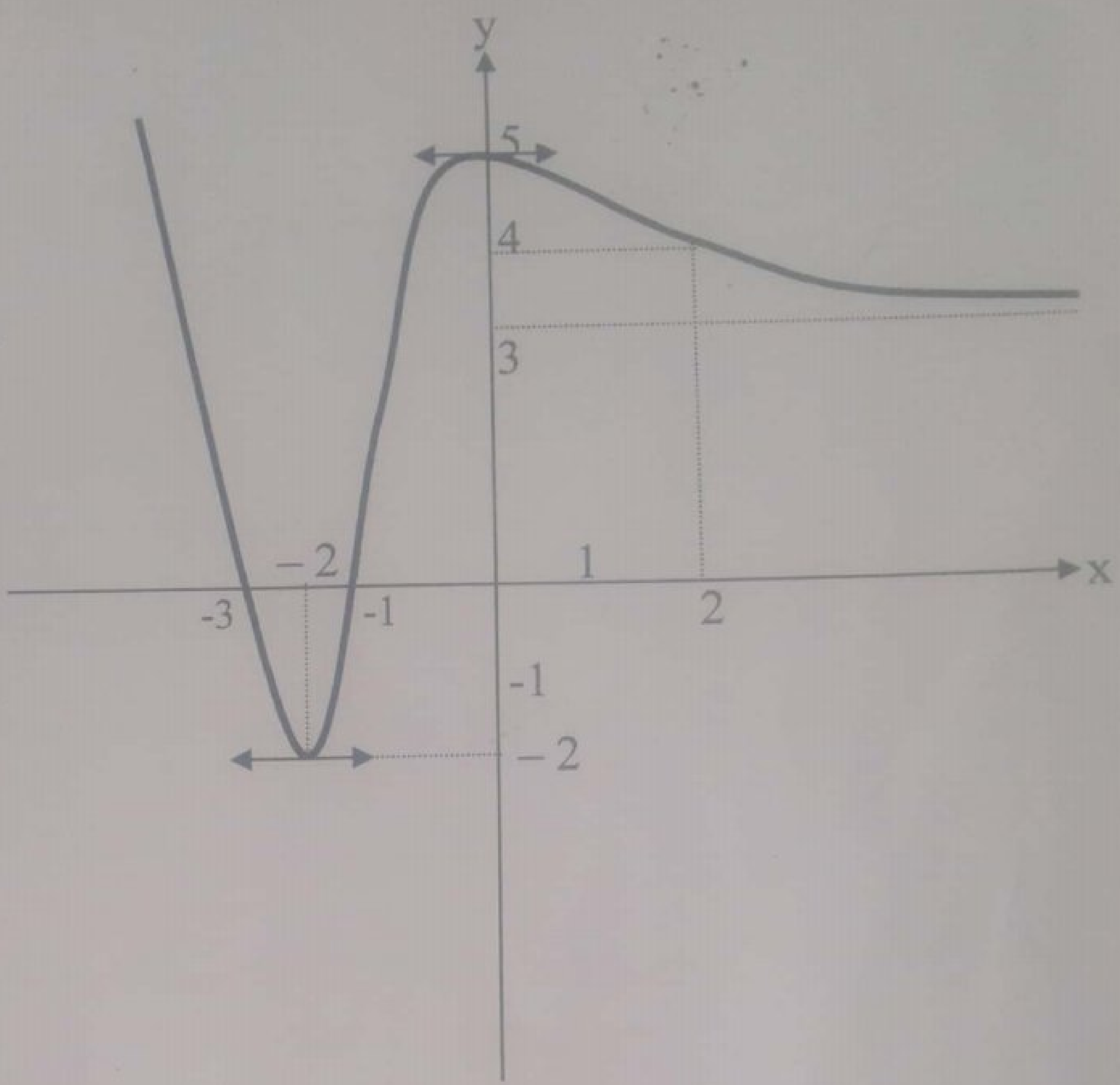
Part B :Consider the function f defined, on $]0\,,+\infty[$, by: $f(x) = \dfrac{x-1}{x}\ln x$. Let (C) its curve

1)Determine the limits of f. Deduce an equation of the asymptote.
2)Show that $f'(x)$ has the same sign as $g(x)$. Set up the table of variation of f.
3)Determine the equation of the tangent (T) to (C) at $x= e$.
4)Draw (T), then (C).
5)a)Prove that the function f admits, on $]0\,,1[$, an inverse function f^{-1}.
b)Construct the table of variations of f^{-1}.
c)Draw (G), the representative curve of f^{-1} in the same system of (C).

e 3 : . (C) is the curve of a function f.

1) Dress the table of variations of f.
2) Determine : $f([-2\,;0])$ and $f(]-2\,;2[)$.
3) a) Prove that f admits an inverse function $f^{-1}(x)$ over $[-2;0[$ then determine its domain and draw (C^{-1})

   b) Determine the point of intersection of (C) and (C^{-1}).
4) Solve $f(x) > 0$.



Exercise 4 : f is a function defined over $]0;+\infty[$ *par* $f(x)=1-\frac{1}{x}-\frac{\ln x}{x}$. (C) its curve.

1) Determine the limits of f at the endpoints of D_f.
2) Study the variations and set up the table of f.
3) Verify that the point $A(\frac{1}{e};1)$ belongs to (C).
4) Let (D) be the line of equation $y = 1$. Solve the equation $f(x) = 1$ and the inequation $f(x) > 1$, Then deduce the relative position of (C) and (D).
5) Write the equation of the tangent (T) to (C) at A.
6)) Prove that (C) has a point of inflection I, to be determined.
7) Draw (D), (T) and (C).
8) a) Verify that f admits an inverse function f^{-1} over $]1\,;+\infty[$ and find $D_{f^{-1}}$.

   b) Draw (C') the curve of f^{-1} on the same system.

   c) Write the equation of the tangent to (C') at $a = 1$.

1) The plane refers to an orthonormal system $(O; \vec{i}, \vec{j})$. One unit = 4cm. In all the problem we have: $x > 0$.

a) Consider the function

$$g(x) = -x^2 + 1 - \ln x.$$

1. Calculate $\lim\limits_{x \to 0^+} g(x)$ and $\lim\limits_{x \to +\infty} g(x)$.
2. Establish the table of variation of g.
3. Calculate $g(1)$. Deduce the sign of $g(x)$.

b) Consider the function

$$f : x \mapsto -\frac{1}{2}x + 1 + \frac{\ln x}{2x}.$$

1. Calculate the derivative $f'(x)$ and express $f'(x)$ in terms of $g(x)$.
2. Calculate $\lim\limits_{x \to 0^+} f(x)$; $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to +\infty} [f(x) + \frac{1}{2}x - 1]$
   Deduce the asymptotes to the graph (C) of f.
3. Establish the table of variation of f.

c) 1. Discuss the sign of $f(x) + \frac{x}{2} - 1$.
   Deduce the position of (C) with respect to (d):

$$y = -\frac{x}{2} + 1.$$

2. Draw (d) and (C).

2) The plane refers to an orthonormal system $(O; \vec{i}, \vec{j})$, one unit = 10 cm. In all the problem $x \in]0, 1[$.

Consider the function

$$f : x \mapsto x(\ln x)^2.$$

a) Prove that:

$$f'(x) = (2 + \ln x)\ln x.$$

b) Solve over $]0, 1[$ the equation $f'(x) = 0$. Deduce the sign of $f'(x)$.

c) Prove that for $x > 0$ we have:

$$x(\ln x)^2 = 4(\sqrt{x}\ln\sqrt{x})^2.$$

Deduce $\lim\limits_{x \to 0^+} f(x)$.

d) The following function g is the extention by continuity of f.

$$\begin{cases} g(x) = x(\ln x)^2 & \text{for } 0 < x \leq 1 \\ g(0) = 0. \end{cases}$$

The curve of g is obtained from that of f by adding the point $O(0, 0)$.

Calculate $\lim\limits_{x \to 0^+} \frac{g(x)}{x}$.

What can we say about the tangent to the graph (C) of f at O? Draw (C).

3) The plane refers to an orthonormal system $(O; \vec{i}, \vec{j})$. Consider the function f defined by

$$f(x) = (x - 1)^2 - \ln(x^2 - 2x + 2).$$

a) 1. Discuss the sign of $x^2 - 2x + 2$.
   Deduce the domain of definition of f.
2. Calculate $f(1 - x)$ and $f(1 + x)$.
   What can you say about the line (d): $x = 1$?

3. Prove that $f'(x) = \dfrac{2(x-1)^3}{x^2-2x+2}$.

4. Calculate $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.

5. Establish the table of variation of f.

b) A and B are two points on the graph (C) of f of abscissas $1-a$ and $1+a$ respectively. Calculate a for the tangents to (C) at A and B to be perpendicular.

4 The function f is defined by

$$f(x) = x - (\ln x)^2,\quad x \in]0, +\infty[.$$

a) Calculate $\lim\limits_{x\to0^+} f(x)$.

For $x > 0$ put $t = \sqrt{x}$. Prove that $f(t^2) = t^2 - 4\ln^2 t$.

Deduce $\lim\limits_{x\to+\infty} f(x)$.

b) Prove that $f''(x) = -2\left(\dfrac{1-\ln x}{x^2}\right)$.

c) Complete the following table

x	0	e	$+\infty$
$f''(x)$			
$f'(x)$		$1-\frac{2}{e}$	

Don't calculate the limits.

d) Create the table of variation of f.

e) The plane is equipped with an orthonormal system (one unit is 1 cm) (C) is the graph of f. What can you say about the point $A(e, e-1)$?

Draw (C).

5 $(O;\vec{i},\vec{j})$ is an orthonormal system in the plane. One unit is 1cm. f is the function defined by:

$$f: x \mapsto x - 1 + \frac{\ln x}{x^2},\quad x > 0.$$

a) 1. Consider the function g defined by $g(x) = x^3 + 1 - x$, $x > 0$. Calculate $g\left(\frac{1}{\sqrt{3}}\right)$ and create the table of variation of g.

2. Show that for $x > 0$, $x^3 + 1 - x > 0$.

3. Deduce that, for $x > 0$,

$$x^3 + 1 - 2\ln x > x - 2\ln x.$$

4. Create the table of variation of the function

$$h: x \mapsto x - 2\ln x,\quad x > 0.$$

5. Deduce the sign of $x^3 + 1 - 2\ln x$, for $x > 0$.

b) 1. Show that $f'(x) = \dfrac{x^3 + 1 - 2\ln x}{x^3}$

Deduce the sign of $f'(x)$

2. Calculate

$$\lim_{x\to0^+} f(x) \text{ and } \lim_{x\to+\infty} f(x).$$

3. Create the table of variation of f.

c) 1. Calculate $\lim\limits_{x\to+\infty} [f(x) - (x-1)]$.

2. Study the sign of $f(x) - (x-1)$ and the position of (C), graph of f, with respect to the line (d) of equation $y = x - 1$.

3. Draw (d) and (C).

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Pb2 $x \in]0,1[$, $f(x) = x(\ln x)^2$

a) $f'(x) = (\ln x)^2 + x\left(2\ln x \times \frac{1}{x}\right) = \ln x(\ln x + 2)$

b) $f'(x) = 0$, $\ln x = 0$ $\quad \ln x + 2 = 0$
$\quad x = 1 \quad x = e^{-2}$

x	$\cdot \quad e^{-2} \quad 1$
f'	$+ \quad 0 \quad - \quad 0 \quad +$

c) $x(\ln x)^2 = 4(\sqrt{x}\ln\sqrt{x})^2$??
$= 4(\sqrt{x}\ln\sqrt{x})(\sqrt{x}\ln\sqrt{x}) = 4\left(\sqrt{x} \times \frac{1}{2}\ln x\right)\left(\sqrt{x}\frac{1}{2}\ln x\right) = x\ln^2 x$

$\lim_{x \to 0^+} x(\ln x)^2 = \lim_{x \to 0^+} 4(\sqrt{x}\ln\sqrt{x})^2 = 4(0)^2 = 0$ $\quad (\lim_{X \to 0} X\ln X = 0)$

d) $g(x) = f(x)$ for $0 < x \leq 1$
$g(0) = 0$.

$* \lim_{x \to 0^+} \frac{g(x)}{x} = \lim_{x \to 0^+} \frac{x(\ln x)^2}{x} = \lim_{0^+} (\ln x)^2 = (-\infty)^2 = +\infty$

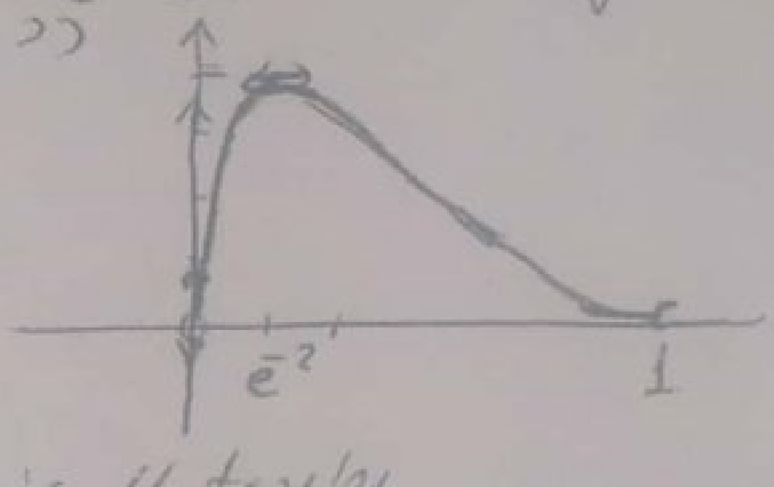
the tangent to (C) at 0 is parallel to y'y

$\text{“} \lim_{x \to 0^+} \frac{g(x)}{x} = \lim_{x \to 0^+} \frac{g(x) - g(0)}{x - 0} = g'(0) \text{”}$

x	$0 \quad e^{-2} \quad 1$
f'	$+ \quad 0 \quad -$
f	$0 \nearrow 4e^{-2} \searrow 0$

$\lim_{x \to 1} f'(x) = 0$

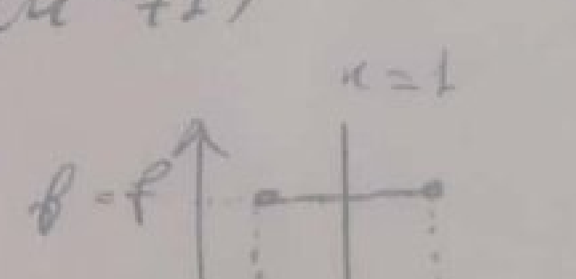
the tangent at 1 is // to x'x



Pb3 $f(x) = (x-1)^2 - \ln(x^2 - 2x + 2)$

a) 1) $x^2 - 2x + 2$; $\Delta' = 1 - 2 = -1 < 0$, $x^2 - 2x + 2 > 0$ for every $x \in \mathbb{R}$
$\Rightarrow D_f = \mathbb{R}$

2) $f(1-x) = (1-x-1)^2 - \ln((1-x)^2 - 2(1-x) + 2) = x^2 - \ln(x^2 + 1)$
$f(1+x) = (1+x-1)^2 - \ln((1+x)^2 - 2(1+x) + 2) = x^2 - \ln(x^2 + 1)$
$f(1-x) = f(1+x)$, $x = 1$ is an axis of symmetry
$\big(f(a+h) = f(a-h)$, $x = a$ is an axis of symmetry$\big)$
$\big(f(1+x) = f(1-x)$, $x = 1$ is an axis of symmetry



3) $f'(x) = 2(x-1)(1) - \frac{2x-2}{x^2-2x+2} = \text{-----} = \frac{2(x-1)^3}{x^2-2x+3}$

$f'(x) = 0$, $x - 1 = 0$, $\ominus x = 1 \oplus$

4) $\lim_{-\infty} f(x) = \lim_{-\infty} (x-1)^2\left[1 - \frac{\ln(x^2-2x+2)}{(x-1)^2}\right]$
$= \lim_{-\infty} (x-1)^2\left[1 - \frac{\ln(x^2-2x+2)}{(x-1)^2}\right]$
$= (-\infty - 1)^2(1 - 0) = +\infty$

$\lim_{-\infty} \frac{\ln x^2 - 2x + 2}{x^2 - 2x + 1} \overset{H}{=}$
$\lim_{-\infty} \frac{2x-2}{(2x-2)(x^2-2x+2)} =$
$\lim_{-\infty} \frac{1}{x^2\left(1 - \frac{2}{x} + \frac{2}{x^2}\right)} =$
$\frac{1}{(+\infty)(1 - 0 + 0)} = 0$

$\lim_{+\infty} f(x) = \lim_{+\infty} (x-1)^2\left[1 - \frac{\ln(x^2-2x+2)}{(x-1)^2}\right] = +\infty$

5)

x	$-\infty \qquad 1 \qquad +\infty$
f'	$- \quad 0 \quad +$
f	$+\infty \searrow 0 \nearrow +\infty$

b) $A(1-a, f(1-a))$
$B(1+a, f(1+a))$
tangents at A and B are perpendicular

$f'(x_A) \times f'(x_B) = -1$ $\quad \begin{pmatrix} \text{slope} \times \text{slope} = -1 \\ (T_1) \quad (T_2) \end{pmatrix}$

$f'(1-a) \times f'(1+a) = -1$

$\frac{2(1-a-1)^3}{(1-a)^2 - 2(1-a) + 3} \times \frac{2(1+a-1)^3}{(1+a)^2 - 2(1+a) + 3} = -1$

$\Rightarrow 4a^6 - a^4 - 2a^2 - 1 = 0$

Let $p(a)=4a^6-a^4-2a^2-1$

$p(1)=0 \Rightarrow (a-1)$ is a factor

$p(-1)=0 \Rightarrow (a+1)$ is a factor

$p(a)=(a-1)(a+1)Q(a)$

$p(a)=(a^2-1)Q(a)$

$$4a^6-a^4-2a^2-1 \;\Big|\; \frac{a^2-1}{4a^4+3a^2+1}$$

$p(a)=(a-1)(a+1)\underbrace{(4a^4+3a^2+1)}_{>0}$

$p(a)=0$ for $a=1$ and $a=-1$

Pb4 / 76

$f(x)=x-(\ln x)^2$, $x\in]0,+\infty[$

a) $\lim_{x\to0^+} f(x)=-\infty$, for $x>0$, let $t=\sqrt{x}$, $f(t^2)=t^2-(\ln t^2)^2$
$=t^2-(2\ln t)^2=t^2-4\ln^2 t$

$\lim_{x\to+\infty} f(x)=\lim_{t^2\to+\infty} t^2-4\ln^2 t=\lim_{t^2\to+\infty} t^2\left(1-\frac{4\ln^2 t}{t^2}\right)=+\infty(1-4(0))=+\infty$

$\left\{\lim_{X\to\infty}\frac{\ln X}{X}=0\right\}$

b) $f'(x)=1-2\ln x\times\frac{1}{x}=1-\frac{2\ln x}{x}$

$f''(x)=-2\left(\frac{\frac{1}{x}\times x-\ln x}{x^2}\right)=-2\frac{(1-\ln x)}{x^2}=\frac{2(\ln x-1)}{x^2}$, $\ln x-1=0$, ⊖ $x=e$ ⊕

c)

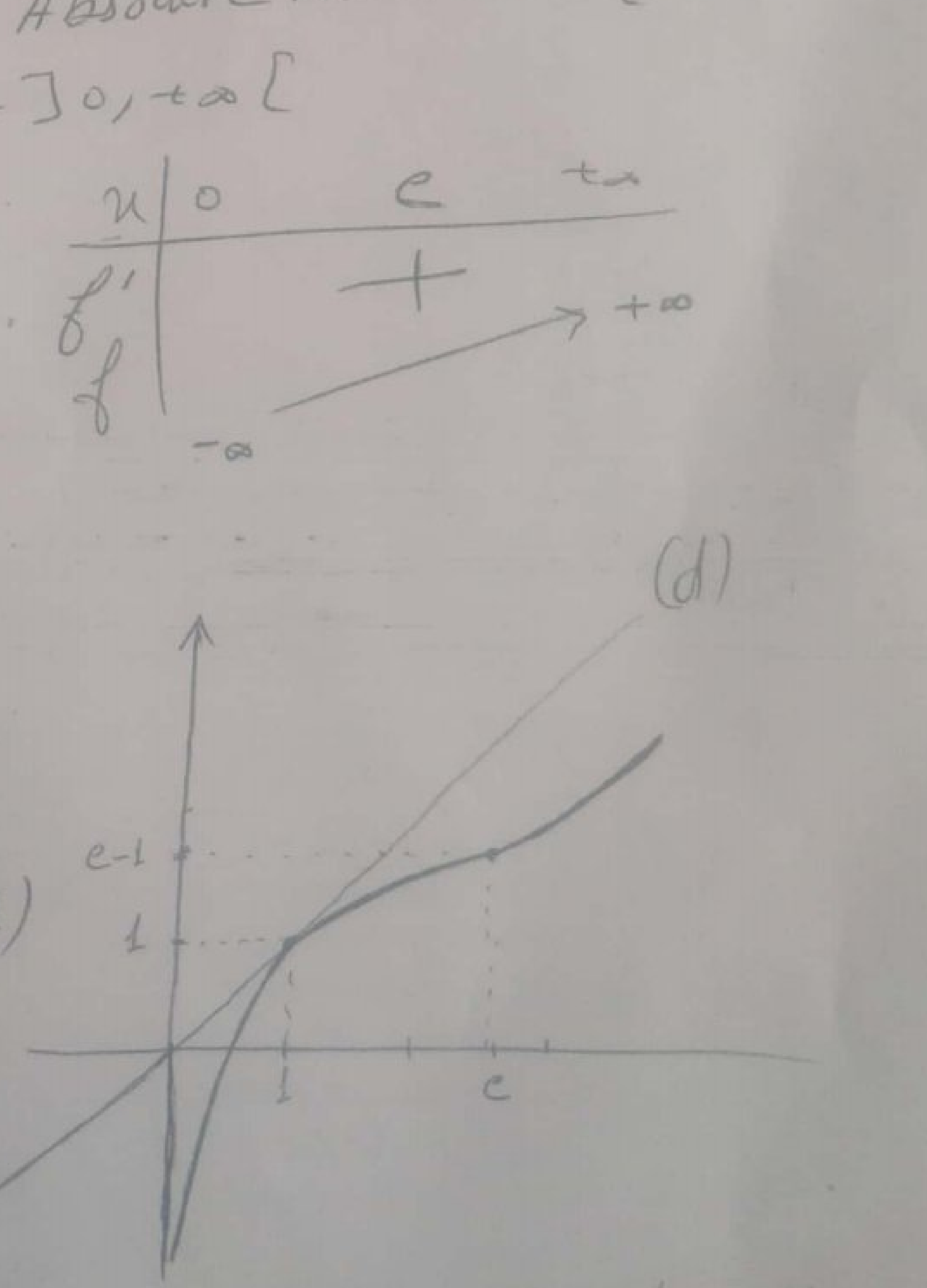
x	0		e		$+\infty$
f''		$-$	0	$+$	
f'	$+\infty$	$\searrow$	$1-\frac{2}{e}$	$\nearrow$	1

$\left\{\begin{array}{l}\lim_{x\to0} f'(x)=1-\frac{2(-\infty)}{0}=+\infty\\ \lim_{x\to+\infty} f'(x)=\lim_{+\infty}\left(1-2\frac{\ln x}{x}\right)=1-0=1.\end{array}\right\}$

according to this table f' has an Absolute Minim. $1-\frac{2}{e}>0$
so $f'(x)>0$ for every $x\in]0,+\infty[$

x	0		e		$+\infty$
f'			$+$		
f	$-\infty$		$\nearrow$		$+\infty$

e) for $x=e$, $f''(x)=0$
and changes its sign
from <0 to >0 then the
curve of f admits a point
of inflection $A(e,e-1)$



* consider the line (d): $y=x$
$f(x)-y_{(d)}=-(\ln x)^2\le 0$
for $x=1$, (C) cuts (d) at $(1,1)$
$x\neq1$, (C) is below (d)

* $\lim_{+\infty}\frac{f(x)}{x}=\lim_{+\infty}\left(1-\frac{(\ln x)^2}{x}\right)$
$=\lim_{+\infty}(1-0)=1$, (C) has an asymptotic direction $\parallel y=x$
$\left(a=\lim_{+\infty}\frac{f(x)}{x}=1\right)$

" $\lim_{+\infty}\frac{\ln^2 x}{x}\stackrel{H}{=}\lim_{+\infty}\frac{2\ln x}{x}=0$ "

Pb 1 Page 75 SG

a) $g(x) = -x^2 + 1 - \ln x$

1. $\lim\limits_{0^+} g(x) = +\infty \quad \lim\limits_{+\infty} g(x) = -\infty$

2) $g'(x) = -2x - \frac{1}{x} = -\left(2x + \frac{1}{x}\right) < 0$

x	0		$+\infty$
g'	‖	$-$	
g	‖	$+\infty \searrow$	$-\infty$

3) $g(1) = 0 \Rightarrow 0 < x < 1 : g(x) > 0, \quad x > 1 \; g(x) < 0$

b) $f(x) = -\frac{x}{2} + 1 + \frac{\ln x}{2x}$

$f'(x) = \frac{g(x)}{2x^2}$

2) $\lim\limits_{0^+} f(x) = -\infty \quad \lim\limits_{+\infty} f(x) = -\infty$

$\lim\limits_{+\infty} \left[f(x) + \frac{x}{2} - 1\right] = 0$

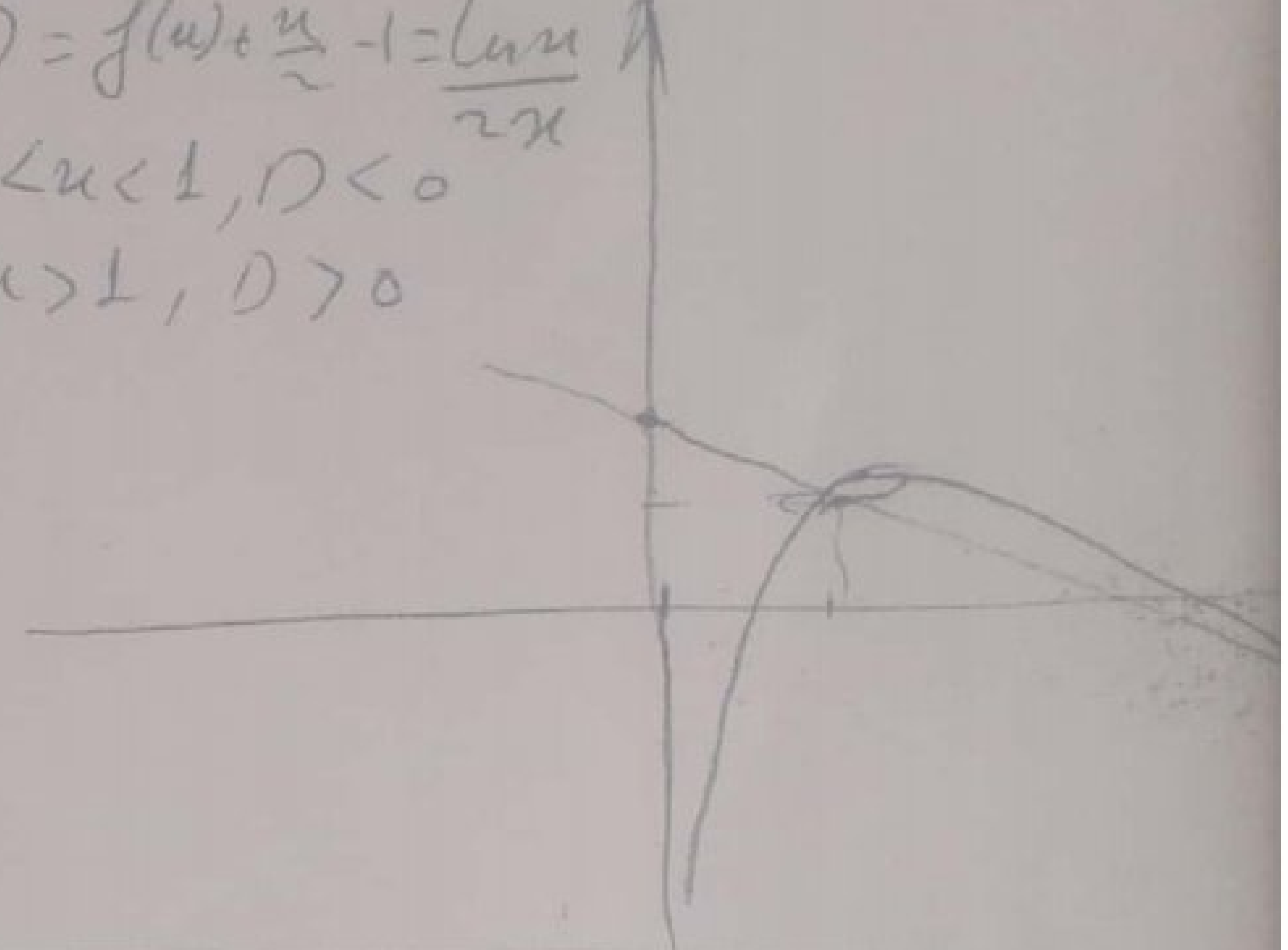
3)

x	0		1		$+\infty$
f'	‖	$+$	0	$-$	
f	‖ $-\infty$	$\nearrow$	$\frac{1}{2}$	$\searrow$	$-\infty$

c) $D = f(x) + \frac{x}{2} - 1 = \frac{\ln x}{2x}$

$0 < x < 1, D < 0$

$x > 1, D > 0$



Pb 2 Page 75

$f(x) = x(\ln x)^2$ sur $]0, 1[$.

a) $f'(x) = (2 + \ln x)\ln x$

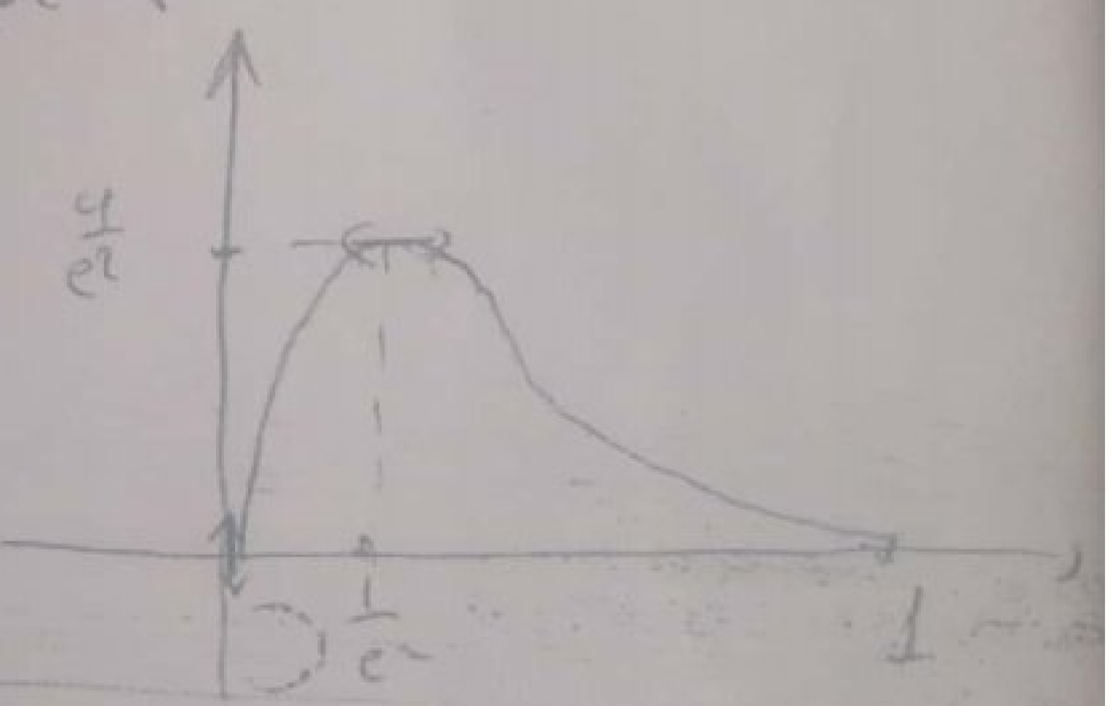
b) $f'(x) = 0 \Rightarrow \ln x < 0, \; \ln x + 2 = 0 \Rightarrow x = e^{-2} = \frac{1}{e^2} = 0.13$

$0 < x < \frac{1}{e^2}, f' > 0, \quad x = \frac{1}{e^2}, f' = 0, \quad x > \frac{1}{e^2} \; f' < 0.$

c) $x(\ln x)^2 = 4(\sqrt{x}\ln\sqrt{x})^2 \Rightarrow \lim\limits_{0^+} f(x) = 0$

d) $\begin{cases} g(x) = x(\ln x)^2 & 0 < x < 1. \\ g(0) = 0 \end{cases}$

$\lim\limits_{0^+} \frac{g(x)}{x} = +\infty \Rightarrow$ tang $// y'y$



Pb 5 Page 76

$f(x) = x - 1 + \frac{\ln x}{x^2}, \; x > 0$

a) 1) $g(x) = x^3 + 1 - x, \; x > 0$

$g\left(\frac{1}{\sqrt{3}}\right) = \frac{9 - 2\sqrt{3}}{9} > 0$

$g'(x) = 3x^2 - 1 \quad g'\left(\frac{1}{\sqrt{3}}\right) = 0$

	0		$\frac{1}{\sqrt{3}}$		
g'	‖	$-$	0	$+$	
g	‖	$\searrow$	$\frac{9-2\sqrt{3}}{9}$	$\nearrow$	$+\infty$

2) $\Rightarrow x^3 + 1 - x > 0 \quad \forall x > 0$

3) $x^3 + 1 > x$

$x^3 + 1 - 2\ln x > x - 2\ln x$

4) $h(x) = x - 2\ln x$

$h'(x) = 1 - \frac{2}{x} = \frac{x - 2}{x}$

x	0		2		$+\infty$
h'	‖	$-$	0	$+$	
h	‖	$\searrow$	$2 - 2\ln 2 = 0.61$	$\nearrow$	$+\infty$

$* \; h(x) > 0 \Rightarrow x - 2\ln x > 0$

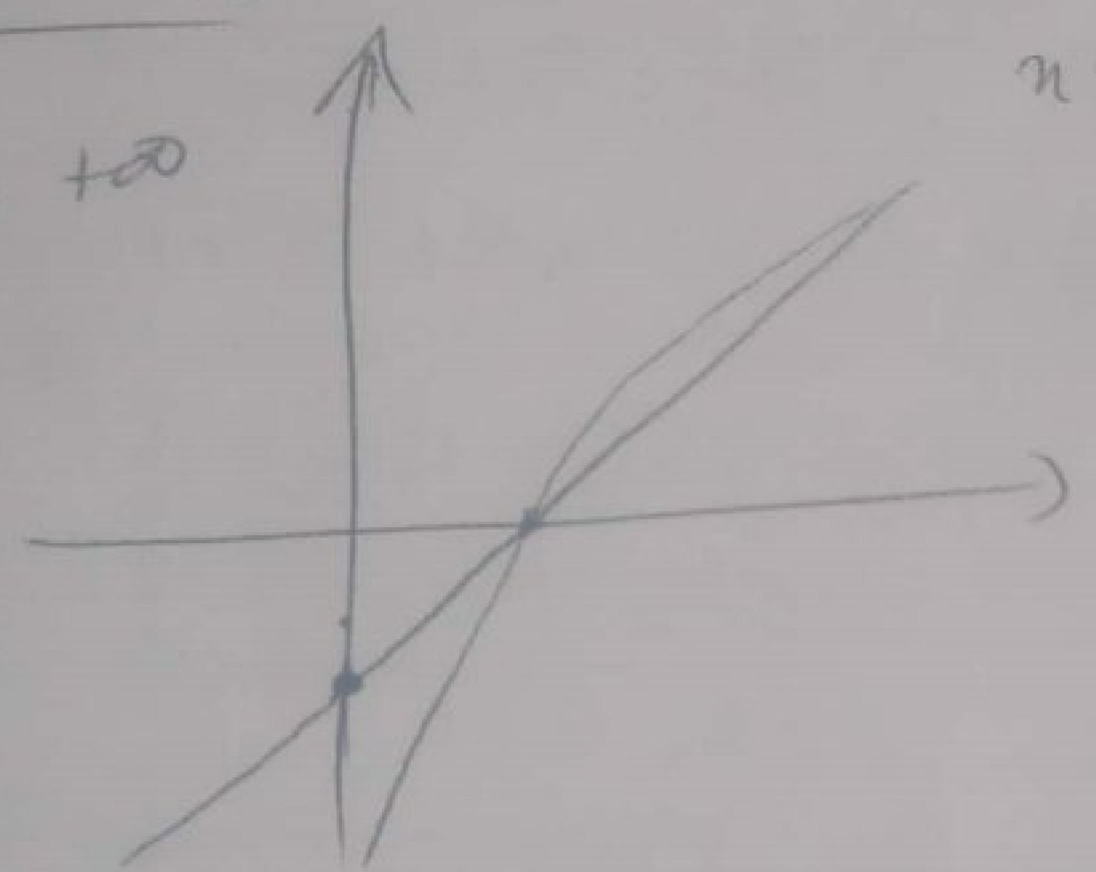
$\Rightarrow \boxed{x^3 + 1 - 2\ln x > 0}$

b) 1/ $f'(x) = \dfrac{x^3 + 1 - 2\ln x}{x^3} > 0$

2/ $\lim\limits_{x \to 0^+} f(x) = -\infty$ $\quad \lim\limits_{x \to +\infty} f(x) = +\infty$

3/

x	0		1		$+\infty$
f'			$+$		
f		$-\infty$	0	$+\infty$	



c) 1/ $\lim\limits_{+\infty} [f(x) - (x-1)] = \lim\limits_{+\infty} \dfrac{\ln x}{x^2} = 0$

2/ $D = f(x) - (x-1) = \dfrac{\ln x}{x^2}$

$0 < x < 1$ $\quad D < 0$ $\quad f$ au dessous

$x > 1$ $\quad D > 0$ $\quad f$ au dessus