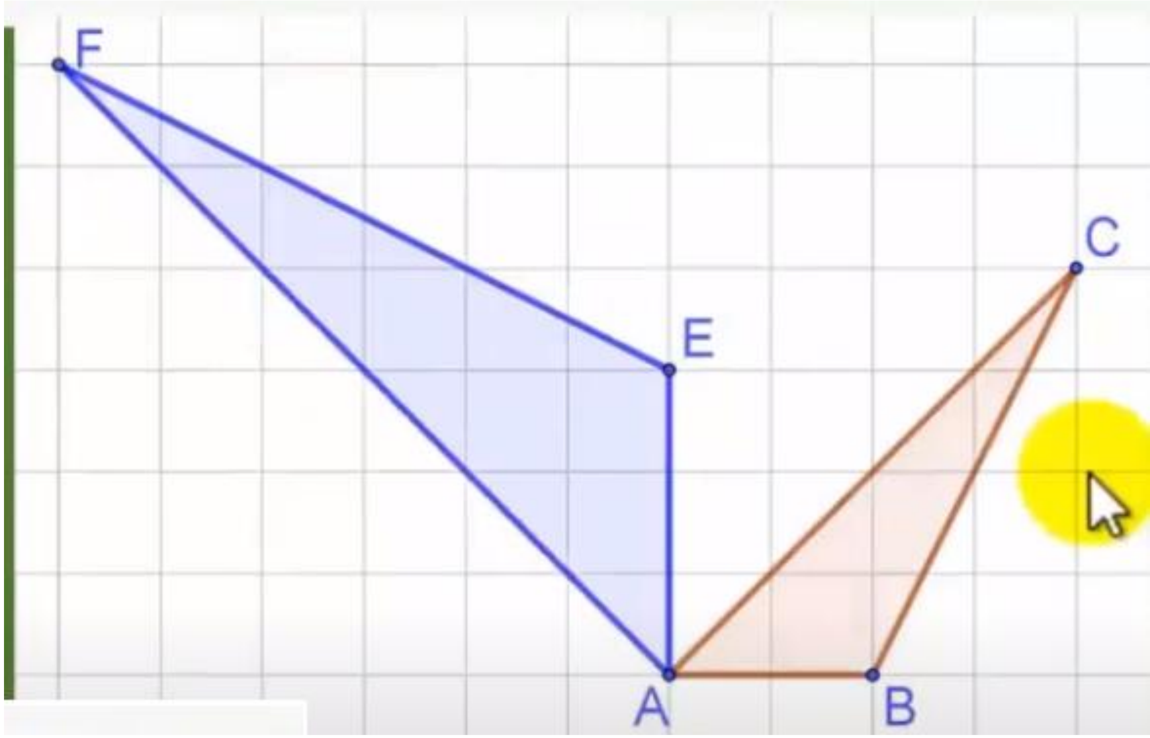
Direct plane - similitude



Introduction

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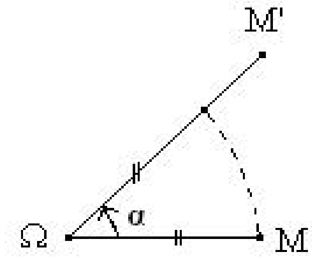
I-Definition



► Definition 1:

A direct plane similitude, noted S, is a transformation of the plane such that it is a translation, or a composite of a positive homothecy and a rotation ..

► Definition 2:

- Let Ω be a point on the plane P and k a strictly positive real and α a non-zero real. We call direct plane similitude of center Ω and of ratio k and angle α the transformation in P, denoted by S (Ω, k, α):

  which, at any point M, associates a single point

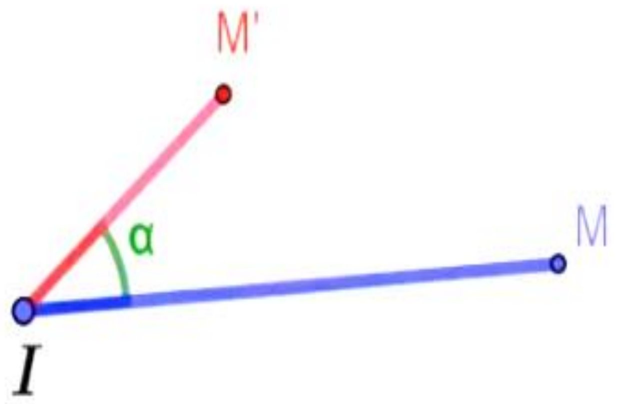
  M 'defined by:

$$S = S(\Omega , k , \alpha): P \rightarrow P$$

$$M \mapsto M' = S(M) \text{ with } \Omega M' = k\Omega M \text{ and } (\overrightarrow{\Omega M} , \overrightarrow{\Omega M'}) = \alpha\ (2\pi).$$

A direct plane similitude denoted by $S(I, k, \alpha)$ is a plane transformation that maps every point M of the plane onto a point M' so that $\left(\overrightarrow{IM}, \overrightarrow{IM'}\right) = \alpha + 2n\pi$ and $\frac{IM'}{IM} = k$. $(k > 0)$

- I is the center of S
- k is the ratio or (scale factor)
- α is an angle of S.
- I is invariant under S
- M' is the image of M under S
- M is the preimage of M'



► Application 1

1) A and B two given points and $S\left(A\ ;\ \frac{2}{3}\ ;\ -\frac{\pi}{6}\right)$ is the similitude S. Construct C = S (B).

► special cases of similitudes:

Translations, rotations, dilation are direct plane similitudes.

- S (Ω, 1, 0) is the identical application of the plane P (IdP).
- S (Ω, 1, α) where $\alpha \neq 0$ is the rotation r (Ω, α).
- S (Ω, 1, π) is the central symmetry with center Ω.
- S (Ω, k, 0) where k> 0 is the positive homothecy h (Ω, k).
- S (Ω, k, π) where k> 0 is the negative homothecy h (Ω, - k).
- .

II- Properties:

a) S (Ω, k, α) has a single double or invariant point which is the center Ω.

b) Characteristic property: Let S (Ω, k, α) be a direct plane similitude.If S (M) = M 'and S (N) = N' is equivalent to M'N '= k MN and $(\overrightarrow{MN} ; \overrightarrow{M'N'}) = \alpha\ (2\pi)$.

c) The image of a segment of length L by the direct plane similitude S (Ω, k, α) is a segment of length “kL”.

d) The image of a line (D) by the direct plane similitude S (Ω, k, α) is a line (D ') such that ((D); (D')) = α (2π).

e) The image of a circle C (O, R) by the direct plane similitude S (Ω, k, α) is a circle C '(O', kR) with O '= S (O).

f) Direct plane similitude preserves collinearity, parallelism, orthogonality, midpoint, oriented angles.

g) Direct plane similitude multiplies the lengths by k and the area by k^2.

h)The inverse of the direct plane similitude S (Ω, k, α) is the direct plane similitude $S(\Omega, \frac{1}{k}, -\alpha)$

III) Complex form of a direct plane similitude:

The plane P is referred to a direct orthonormal coordinate system $(O;u;v)$.
Consider the direct plane similitude $S(\Omega, k, \alpha)$.

If $M(z)$ and $M'(z')$ such that $S(M) = M'$ then $\mathbf{z' = az + b}$ where $\mathbf{a = ke^{i\alpha}}$ and $\mathbf{b = (1 - a)\, z_\Omega}$.

- The affix of Ω the center of S is $z_\Omega = \dfrac{b}{1-a}$

Complex form of a transformation f	Condition of a	Nature of f	Characteristic elements of f
$z' = az + b$	If $a = 1$	**translation**	➢ ***Vector*** $\vec{v}$ such that $z_{\vec{v}} = b$
	If $a = e^{i\alpha}$ $(\alpha \in \mathbb{R}^*)$	**rotation**	➢ ***center*** Ω such that $z_\Omega = \dfrac{b}{1-a}$ ➢ ***Angle*** α.
	If $a = k \in \mathbb{R} - \{0;1\}$	**Homothecy (dilation)**	➢ ***Centre*** Ω such that $z_\Omega = \dfrac{b}{1-k}$ ➢ ***Ratio*** k
	If $a = ke^{i\alpha}$ $(k \in \mathbb{R}^{+*}$ and $\alpha \in \mathbb{R})$	**Similitude**	➢ ***Center*** Ω such that $z_\Omega = \dfrac{b}{1-a}$ ➢ ***Ratio*** k ➢ ***Angle*** α

Application 2

The plane is referred to a direct orthonormal coordinate system (O;;).

- 1) Determine the nature and the characteristic elements of the transformation f defined by its complex form z '= (1 + i) z + 2 - 3i.
- 2) Write the complex form of the similitude $S\left(\Omega\ ;\ \sqrt{2}\ ;\ \frac{\pi}{4}\right)$ where Ω is the point of affix 1 - i.

IV-composite of rotation and dilation

a) Composite of a rotation and a dilation with the same center:

- If $k > 0$, $k \neq 1$ and $\alpha \neq 0$ then
  $\mathbf{h(\Omega, k) \circ r(\Omega, \alpha) = r(\Omega, \alpha) \circ h(\Omega, k) = S(\Omega, k, \alpha)}$.
- If $k < 0$, $k \neq -1$ and $\alpha \neq 0$ then
  $\mathbf{h(\Omega, k) \circ r(\Omega, \alpha) = r(\Omega, \alpha) \circ h(\Omega, k) = S(\Omega, |k|, \alpha + \pi)}$.

b) composite of a rotation and a dilation of distinct centers:

- If $k > 0$, $k \neq 1$ and $\alpha \neq 0$ then $h(\Omega, k) \circ r(\Omega', \alpha) =$ $\mathbf{S(I, k, \alpha)}$
  (I distinct from Ω and Ω').
- If $k > 0$, $k \neq 1$ and $\alpha \neq 0$ then $r(\Omega', \alpha) \circ h(\Omega, k) =$ $\mathbf{S(I', k, \alpha)}$
  (I 'distinct from Ω and Ω').
- If $k < 0$ and $\alpha \neq 0$ then $h(\Omega, k) \circ r(\Omega', \alpha) =$ $\mathbf{S(I, |k|, \alpha + \pi)}$
  (I distinct from Ω and Ω').
- If $k < 0$ and $\alpha \neq 0$ then $r(\Omega', \alpha) \circ h(\Omega, k) =$ $\mathbf{S(I', |k|, \alpha + \pi)}$
  (I 'distinct from Ω and Ω').

IV-composite of two similitudes

c) composite of two similitudes with the same center:

Let S (Ω, k, α) and S '(Ω, k', α') be two direct plane similitudes with the same center Ω.

- **If $\alpha + \alpha' \neq 0$ and $kk' \neq 1$, then S '○ S is the similitude S $(\Omega, kk', \alpha + \alpha')$.**
- If $\alpha + \alpha' = 0$ and $kk' = 1$, then S '○ S is the identical application of the plane P (IdP).
- If $\alpha + \alpha' \neq 0$ and $kk' = 1$, then S '○ S is the rotation r $(\Omega, \alpha + \alpha')$.
- If $\alpha + \alpha' = 0$ and $kk' \neq 1$, then S '○ S is the dilation h (Ω, kk').

d) composite of two similitudes of distinct centers:

Let S (Ω, k, α) and S '(Ω', k', α') be two direct plane similitudes of distinct centers Ω and Ω'.

- **If $\alpha + \alpha' \neq 0$ and $kk' \neq 1$, then S '○ S is the similitude S $(I, kk', \alpha + \alpha')$ (I distinct from Ω and Ω').** (main formula)
- If $\alpha + \alpha' = 0$ and $kk' = 1$, then S '○ S is a translation.
- If $\alpha + \alpha' \neq 0$ and $kk' = 1$, then S '○ S is the rotation r $(I, \alpha + \alpha')$ (I distinct from Ω and Ω').
- If $\alpha + \alpha' = 0$ and $kk' \neq 1$, then S '○ S is the dilation h (I, kk') (I distinct from Ω and Ω').

Note : the new center I can be proved by showing that it is invariant by the transformation.

► Application 2

A is a given point, $r = r\left(A\,;\,\frac{\pi}{2}\right)$, $h = h(A\,;\,2)$, $S = S\left(A\,;\,\frac{1}{2}\,;\,-\frac{\pi}{2}\right)$ and $S' = S\left(A\,;\,3\,;\,\frac{\pi}{6}\right)$

Determine the nature and the caracteristic elements of : $r \circ h$, $h \circ r$, $r \circ S$, $S \circ h$ et $S' \circ S$.

Solution :

- $r \circ h$ is the composite of a dilation of ratio 2 and rotation of angle $\frac{\pi}{2}$ of same center so $r \circ h =$ $S\left(A\,;\,2\,;\,\frac{\pi}{2}\right)$.
- $h \circ r$: also same center so $h \circ r$ is A then :

  $$h \circ r = r \circ h = S\left(A\,;\,2\,;\,\frac{\pi}{2}\right)$$
- $r \circ S = S\left(A\,;\,1\,;\,\frac{\pi}{2}\right) \circ S\left(A\,;\,\frac{1}{2}\,;\,-\frac{\pi}{2}\right) = S\left(A\,;\,1\times\frac{1}{2}\,;\,\frac{\pi}{2}-\frac{\pi}{2}\right) = S\left(A\,;\,\frac{1}{2}\,;\,0\right) = h\left(A\,;\,\frac{1}{2}\right)$.
- $S \circ h = S\left(A\,;\,\frac{1}{2}\,;\,-\frac{\pi}{2}\right) \circ S(A\,;\,2\,;\,0) = S\left(A\,;\,\frac{1}{2}\times 2\,;\,-\frac{\pi}{2}+0\right) = S\left(A\,;\,1\,;\,-\frac{\pi}{2}\right)$.
- $S' \circ S = S\left(A\,;\,3\,;\,\frac{\pi}{6}\right) \circ S\left(A\,;\,\frac{1}{2}\,;\,-\frac{\pi}{2}\right) = S\left(A\,;\,3\times\frac{1}{2}\,;\,\frac{\pi}{6}-\frac{\pi}{2}\right) = S\left(A\,;\,\frac{3}{2}\,;\,-\frac{\pi}{3}\right)$.

Exercises

Entertainment Exercises :

I. ABC is a direct equilateral triangle of center of gravity G .

1) Precise the image of points A , B , and C by the similitude $S\left(G\ ;\ \frac{1}{2}\ ;\ \frac{\pi}{3}\right)$

2) Place the images of A , B , and C by similitude $S\left(G\ ;\ 2\ ;\ \frac{\pi}{3}\right)$

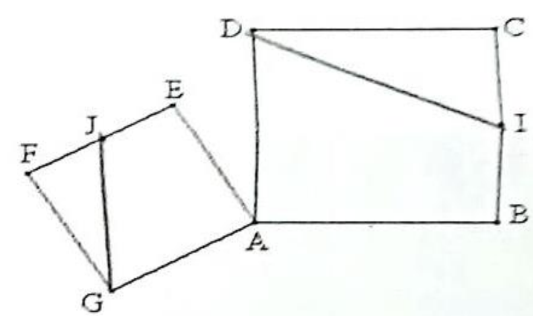
3) Let M the mi point of [AC] . Determine the image of triangle ABM by the direct similitude $S\left(A\ ;\ \frac{\sqrt{3}}{3}\ ;\ \frac{\pi}{6}\right)$.

II. ABCD is a direct square of center O

1) Precise the image of B and O by the direct similitude $S\left(A\ ;\ \sqrt{2}\ ;\ \frac{\pi}{4}\right)$.

2) Determine the image of square by the similitude $S\left(A\ ;\ \frac{\sqrt{2}}{2}\ ;\ \frac{\pi}{4}\right)$

III. The adjacent square are direct AB = 36, AG = 21 and $(\overrightarrow{AD}\ ;\ \overrightarrow{AE}) = \frac{\pi}{6}(2\pi)$.



1) Determine a direct similitude that transforms ABCD to AEFG

2) Give the measure of angle $(\overrightarrow{ID}\ ;\ \overrightarrow{JG})$, where I and J are the midpoints of [BC] and [EF] respectively

IV. ABC is a triangle such that $(\overrightarrow{AB}\ ;\ \overrightarrow{AC}) = \frac{\pi}{3}(2\pi)$. Consider the dilation $h = h\left(A\ ;\ \frac{1}{2}\right)$ and $h' = h\left(A\ ;\ -\frac{1}{2}\right)$ and the rotation $r = r\left(A\ ;\ \frac{\pi}{3}\right)$.

1) Place the points $D = h \circ r\,(B)$ and $E = h \circ r\,(C)$. Which is the direct similitude that transforms the triangle ABC to ADE ?

2) Place the points $D' = h' \circ r\,(B)$ and $E' = h' \circ r\,(C)$ .Which is the direct similitude that transforms the triangle ABC to AD'E'?

<u>Elements of a direct Similitude :</u>

V. ABC is a direct equilateral triangle of center G consider the dilation $h = h\left(C\ ;\ \frac{1}{2}\right)$ and the rotation $r = r\left(B\ ;\ \frac{\pi}{3}\right)$

1) Prices the image of segment [BC] by $h \circ r$

2) Determine $h \circ r\,(G)$.

3) Give the elements of the direct similitude $h \circ r$

ABC is a triangle such that $(\overrightarrow{AB}\ ;\ \overrightarrow{AC}) = \frac{\pi}{2}(2\pi)$ and $(\overrightarrow{BC}\ ;\ \overrightarrow{BA}) = \frac{\pi}{3}(2\pi)$. [AH] is a height . A describe a fixed straight line (d)

1) We suppose that B is fixed . Determine the geometrical locus of points H and C

2) We suppose that H is fixed . Determine the geometrical locus of point B and C

Problems :

XII. ABC is a triangle , I , J , and K are the midpoints of [BC] , [AC] , and [AB] respectively . Let $\alpha = (\overrightarrow{AB}\ ;\ \overrightarrow{AC})\ (2\pi)$, $\beta = (\overrightarrow{BC}\ ;\ \overrightarrow{BA})\ (2\pi)$, $\gamma = (\overrightarrow{CA}\ ;\ \overrightarrow{CB})\ (2\pi)$, $k = \frac{AC}{AB}$, $l = \frac{BA}{BC}$ and $m = \frac{CB}{CA}$.

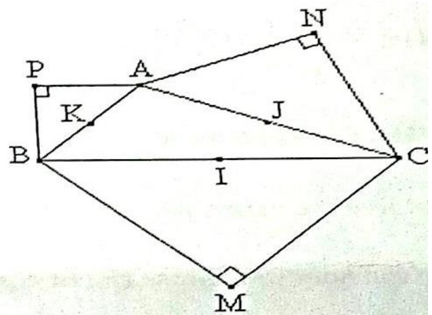
Let the similitude $S_A = S(A\ ;\ k\ ;\ \alpha)$, $S_B = S(B\ ;\ l\ ;\ \beta)$ and $S_C = S(C\ ;\ m\ ;\ \gamma)$.

1) Prove that $S_B \circ S_A \circ S_C$ is a centeral symmetry of center A. Determine the similitudes $S_C \circ S_B \circ S_A$ et $S_A \circ S_C \circ S_B$.

2) Determine $S_C \circ S_A \circ S_B$, $S_A \circ S_B \circ S_C$ and $S_B \circ S_C \circ S_A$

XIII. ABC is a triangle I , J , and K are the midpoints of [BC] , [CA] and [AB] respectively . The triangles APB , BMC and CNA are direct right isosceles .

Consider the direct similitude : $S = S\left(B\ ;\ \frac{1}{\sqrt{2}}\ ;\ \frac{\pi}{4}\right)$ and $S' = S\left(C\ ;\ \sqrt{2}\ ;\ \frac{\pi}{4}\right)$.



1) Determine S(M) and S(A) . Compare IP and MA and give the measure of angle $(\overrightarrow{MA}\ ;\ \overrightarrow{IP})$.

2) Determine S'(N) and S'(I) and prove that IN = IP and $(\overrightarrow{IN}\ ;\ \overrightarrow{IP}) = \frac{\pi}{2}(2\pi)$.

3) Prove that [CP] is the image of [MN] by a rotation of elements to be determined . Deduce the orthogonality of straight lines (CP) and (NM)

4) Prove that the straight lines (AM), (BN) , (CP) are concurrent

Hints to answer some questions in similitude

I) Image of a point by similitude S :

1) if you have all the characteristic elements of S such that S(Ω , k , α) then S(A)= A' can be found by verifying that

2) If the ratio k and the angle α are only given and moreover you have the image of another point B such that S(B)=B' , then we S(A)=A' can be found by verifying:

3) If the characteristic elements are not given or it is difficult to determine the image A' using the previous methods then we can use according to the given :

-The conservation of midpoint.

-The conservation of nature of triangle (equilateral , semi-equilateral, right isosceles)

-The conservation of special quadrilateral (parm , rectangle , rhombus ,square)

4) If A belong to a straight line (d) or circle (C) then S(A) belong S((d)) or S((C)).

II) Center of similitude S:

1) If S(Ω)=Ω (double point) then A is the center of similitude S

2)If S((d))= (d') and S((d'))= (d) then the center of Ω is the point of intersection of (d) and (d') since S(Ω) belong to (d) and (d') at the same tim.

3)If the angle of similitude ,and S(A)=A' and S(B)=B' then so the center Ω belong to both circles of respective diameters [AA'] and [BB] since thus it is the intersection between the 2 circles.

III-Image of straight lines , segment , vector:

1) If S(A)=A' and S(B) = B' then , the reverse is not necessarily correct.

2)If then S((d)) is a straigtht line perpendicular to (d) and passing through an image of a point of (d).