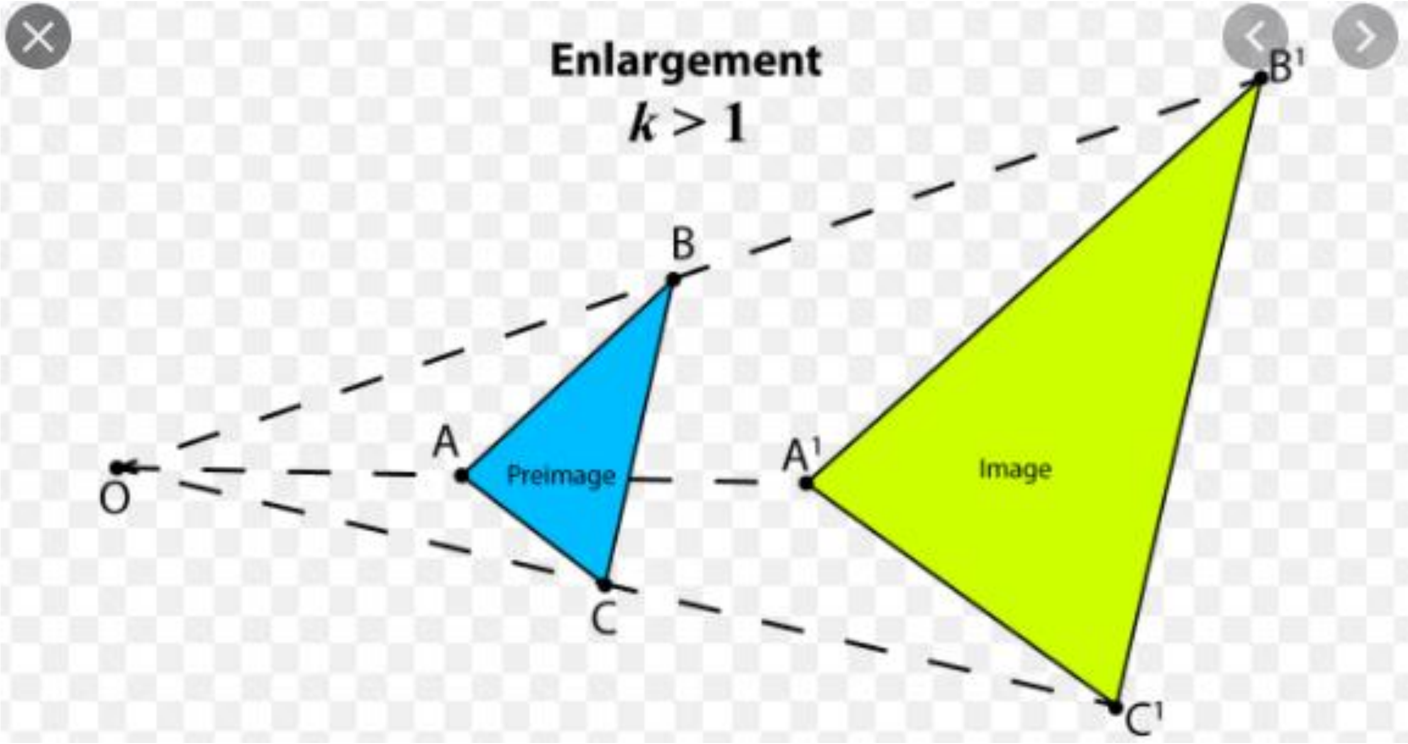
Transformation-Dilation(homothecy)



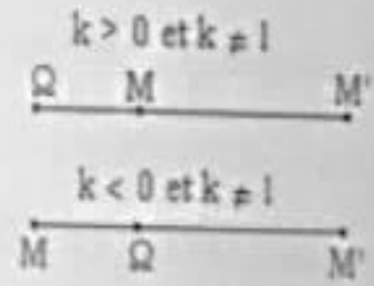
Transformations (Dilation)

***P* designate a set of points in a plane**

I) <u>Definition :</u> Let Ω be a point of the plane P and K is a non-zero number . We call dilation (homothecy) of center Ω and ratio K the transformation in P , denote by h (Ω,K) , which , to every point M , associates a unique point M' defined by : $\overrightarrow{\Omega M'} = k\overrightarrow{\Omega M}$

We write : $h = h(\Omega, k): P \to P$

$$M \mapsto M' = h(M) \text{ with } \overrightarrow{\Omega M'} = k\overrightarrow{\Omega M}.$$



- Every point M and its image M' by h are collinear with the center Ω
- If $k > 0$, the homothecy is said to be positive , and if $k < 0$ the homothecy is said to be negative
- The inverse of the dilation $h(\Omega, k)$ is the dilation $h(\Omega, \frac{1}{k})$.
- If $k = 1$,the homothecy $h(\Omega, 1)$ is an **identity mapping** in the plane P (Id_P)
- If $k = -1$, the homothecy $h(\Omega, -1)$is a **central symmetry** of center Ω
- If $k \neq 1$, $h(\Omega, k)$ **has only one double point** or **invariant** which is the center Ω

Application 1:

Let Ω and M be two points :

a) Construct the point M' the image of M by the dilation $h(\Omega\,;\,\frac{1}{2})$.

b) Construct the point M' the image of M by the dilation $h(\Omega\,;-2)$

Application 2:

A , B , and C are three collinear points in this order , such that AB = 1 and BC = 4.

a) Characterize the dilation that transforms A to C

b) Characterize the dilation that transforms B to C

Solution of App2:

a) since $\overrightarrow{BC} = -4\overrightarrow{BA}$ thus the dilation that transforms

A to C is h(B;-4)

b) since $\overrightarrow{AC} = 5\overrightarrow{AB}$ thus the dilation that transforms

B to C is h(A;5)

2-Properties

a) **Characteristic property:** Let h = h(Ω, k) be a dilation. If h (M) = M 'and h (N) = N' is equivalent to $\overrightarrow{M'N'} = k\overrightarrow{MN}$.

b) The image of a segment of length L by the dilation h (Ω, k) is a parallel segment of length $|k| \times L$.

c) The image of a line by the dilation h (Ω, k) is a parallel line.

d) The image of a vector $\overline{AB}$ by the dilation h (Ω, k) is a vector such that $\overrightarrow{A'B'} = k\overrightarrow{AB}$.

e) The image of a circle C (O, R) by the dilation h (Ω, k) is a circle C '(O', | k | R) with O '= h (O).

f) The dilation preserves the collinearity, the parallelism, the orthogonality, the midpoint, the oriented angles.

g) The dilation multiplies the lengths by | k | and the area by k^2.

3) Complex form of a dilation:

The plane P is provided with a direct orthonormal coordinate system (O;u;v) .

- Let the dilation h (Ω, k) with real k different from 0 and 1. If M (z) and M '(z') such that h (M) = M 'then **$z' = k z + b$** and the affix of Ω the center of h is $z_\Omega = \frac{b}{1-k}$. Where k is a real number and b is a complex number.

$\overrightarrow{\Omega M'} = k\,\overrightarrow{\Omega M}$

$z_{M'} - z_\Omega = k(z_M - z_\Omega)$

$z' = k z_M - k z_\Omega + z_\Omega$

$z' = k z + z_\Omega (1 - k)$.

- Pay attention: If the center Ω of h is the point O origin of the orthonormal system, the complex form of h (O; k) is then z '= kz.

Application 4:

The plane P is provided with a direct orthonormal coordinate system $(O;\vec{u};\vec{v})$.

1) Let f be the transformation of complex form z '= 2z + 1 - i. Determine the nature and the characteristic elements of f.
2) a) Write the complex form of the dilation h with center Ω (0; 1) and ratio 0.5.
   b) Let A be the point with affix 2 + i. Determine the affix of point B image of A by h.
   c) Determine the complex form of the transformation h^{-1}.

Solution :

1) z'=2z+1-i

has the form z'=kz+b

so f is a dilation of center Ω such that

$$z_{\Omega} = \frac{b}{1-k} = \frac{1-i}{1-2} = -1+i$$

so $\Omega(-1;1)$ and ratio k=2.

2)a) z'=kz+b

k=0.5

b=$z_{\Omega}(1-k) = i(1-0.5) = 0.5i$

so z'=0.5z+0.5i

b) h(A)=B

$z'_B = 0.5z_A + 0.5i$

$= 0.5(2+i) + 0.5i$

$= 1+i$

$c) h^{-1}(\Omega;\frac{1}{k}) \Leftrightarrow h^{-1}(\Omega;2)$

thus

z'=kz+b

$= 2z + i(1-2)$

$= 2z - i$

$2nd$ method :

z'=0.5z+0.5i

$-0.5z = -z' + 0.5i$

$z = 2z' - i$

thus z'=2z-i

3-Composite of 2 dilations

a) Composite of two dilations of the same center:

Let h (Ω, k) and h '(Ω, k') be two dilations with the same center Ω in P.

- If kk '= 1(reciprocal ratios), the composite of these two dilations is the identical application of P (IdP).
- If kk '≠ 1, the composite of these two dilation is the dilation h''(Ω, kk'). where h '(Ω, k') ○ h (Ω, k) = h (Ω, k') ○ h' (Ω, k)= h '' (Ω, kk ').

b) Composite of two dilations with distinct centers:

Let h (Ω, k) and h '(Ω', k ') be two dilations with distinct centers Ω and Ω' in P.

- If kk '= 1 (reciprocal), the composite of these two dilations is a translation. where h '(Ω', k ') ○ h (Ω, k) = $t_{\vec{v}}$ with $\vec{v} = \overrightarrow{\Omega\Omega''}$ and $\Omega'' = h'(\Omega)$.
- If kk '≠ 1, the composite of these two dilation is the dilation h''(I, kk') (I is distinct from Ω and from Ω 'and we can show that $\overrightarrow{\Omega I} = \frac{1-k'}{1-kk'}\overrightarrow{\Omega\Omega'}$). where h '(Ω', k ') ○ h (Ω, k) = h''(I, kk').

c) Composite of a dilation and a translation:

consider the dilation h (Ω, k) and the translation $t_{\vec{V}}$ be such that $k \neq 1$ and $\vec{V} \neq \vec{0}$ So:

- $t_{\vec{V}} \circ h(\Omega, k) = h(I, k)$ and I is distinct from Ω such that $\overrightarrow{\Omega I} = \frac{1}{1-k}\vec{V}$.
- $h(\Omega, k) \circ t_{\vec{V}} = h(I'; k)$ and I' is distinct from Ω such that $\overrightarrow{\Omega I'} = \frac{k}{1-k}\vec{V}$.

Application 5:

1) [AB] is a segment, given the dilations h = h (A; 2), and $h' = h(B; \frac{1}{2})$ $h'' = h(B; \frac{1}{4})$. Determine the nature and the elements of h '○ h and h ○ h', h '' ○ h and h ○ h ''.

2)ABC is a triangle. Let the dilation h = h (A; 2) and t be the translation $t = t_{\overrightarrow{BC}}$. Determine the nature and the elements of h ○ t and t ○ h.

*Application*5 :

1)$h' \circ h$, is a composite of 2 dilations with distinct centers

and kk'=1 , so it is a translation of vector $\vec{V} = \overrightarrow{AI}$

where I=h'(A) $\Leftrightarrow$ $\overrightarrow{BI} = \frac{1}{2}\overrightarrow{BA}$ thus I is the midpoint of [AB].

2) $h \circ h'$, is a composite of 2 dilations with distinct centers

and kk'=1 , so it is a translation of vector $\vec{V}_1 = \overrightarrow{BI'}$ where

I'=h(B) $\Leftrightarrow \overrightarrow{AI'} = 2\overrightarrow{AB}$ thus B is the mipoint of [AI'].

3)h"$\circ h$, is a composite of 2 dilations with distinct centers

and kk' $\neq$1 so it is a dilation of center I" distinct of A and B

such that : $\overrightarrow{AI''} = \frac{1-k'}{1-kk'}\overrightarrow{AB} = \frac{1-\frac{1}{4}}{1-\frac{1}{4}\times 2}\overrightarrow{AB} = \frac{3}{2}\overrightarrow{AB}.$

and ratio kk'=$\frac{1}{2}$

4)h$\circ h''$, is a composite of 2 dilations with distinct centers

and kk' $\neq$1 so it is a dilation of center I_1 distinct of A and B

such that : $\overrightarrow{BI_1} = \frac{1-k'}{1-kk'}\overrightarrow{BA} = \frac{1-2}{1-\frac{1}{4}\times 2}\overrightarrow{BA} = -2\overrightarrow{BA}.$

and ratio kk'=$\frac{1}{2}$.

*part*2)$h \circ t$, is a composite of dilation and translation so it is

a dilation of center I' distinct from Asuch that $\overrightarrow{AI'} = \frac{2}{1-2}\overrightarrow{BC}$ =-2$\overrightarrow{BC}$

and ratio = k =2

ExI –

Determine the nature and the elements of the transformation f defined by complex form in each of the following cases :

1) $z' = z - 1 + i$ 2) $z' = \sqrt{2}z + i - 1$ 3) $z' = e^{2i\frac{\pi}{3}}z + 3 + i$ 4) $z' = -\frac{5}{2}z + 3 + 2i$

*Ex*2 –

Let f be a dilation defined by a complex form $z'=u^2 z + u - 1$

Determine the set of complex number of u for which f is a dilation of ratio-2.

Ex 3-

Consider the point A and B of respective affixes $z_A = 1$ and $z_B = 2 + i$. let the *dilation* h=h(A;3) and h'=h(B;$\frac{1}{3}$)

1) *Write* the complex form of h,h' and of h' o h.

2) Deduce the nature and the charateristic elements of h' o h .

Ex 4 ;

ABCDis a trapezoid such that $\overrightarrow{AB} = 3\overrightarrow{DC}$, O is the intersection of its diagonals.

we denote by h the dilation of cneter O that transform A to C.

1)a-Determine h((AB)). Precise h(B).

b- Prove that the ratio of h is -$\frac{1}{3}$.

2) The parallel (d) drawn from C to (AD) cuts (DB) at I

a- Prove that h((AD))=(d).

b- Deduce that h(D)=I

3)The parallel (Δ) *drawn* from D to (BC) cuts (AC) at J.

a- Using the reason of the preceding question prove that h(C)=J.

b- Deduce that $\overrightarrow{IJ} = \frac{1}{3}\overrightarrow{CD}$.

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