- *This exam consists of three problems inscribed on two pages.*
- *The use of a non-programmable calculator is allowed.*

I – (4 points)

In the table below, only one of the proposed answers to each question, is correct.
Write down the number of each question and give, **with justification**, the answer corresponding to it.

Nº	Questions	Answers a	Answers b	Answers c
1)	The equation $(x-1)\ln(x-1)=0$ has over $\mathbb{R}$	no solution	one solution	two solutions
2)	If $f(x)=\dfrac{x}{\ln^2 x}$, then $f'\left(e^{-1}\right)=$	-3	e^{-1}	3
3)	$\lim\limits_{x\to+\infty}\left[x-\ln\left(1+e^{x}\right)\right]=$	$-\infty$	0	$+\infty$
4)	Let A and B be two events of a sample space Ω and p be a probability such that $p(A\cup B)=0.8$, $p(A\cap B)=0.3$ and $p(A/B)=0.6$; then	$p(A)=0.5$	A and B are two independent events	$p(B)=0.6$
5)	A class contains 6 girls and 4 boys. A group of 3 students is chosen randomly. The number of possible groups containing at least one girl is	116	100	36

II – (6 points)

Consider a bag S and the two urns U and V such that:
- S contains 6 cards: 3 red, 2 green and 1 black.
- U contains 6 balls numbered: 1, 2, 2, 2, 4, 4.
- V contains 4 balls numbered: 1, 1, 1, 4.

A player selects at random one card from the bag S:
- If the selected card is red, then he draws randomly and simultaneously two balls from urn U.
- If the selected card is green, then he draws successively and with replacement two balls from urn V.
- If the selected card is black, then he draws randomly one ball from urn U and one ball from urn V.

Consider the following events:

R: ‹‹ The selected card from the bag is red ››.
G: ‹‹ The selected card from the bag is green ››.
B: ‹‹ The selected card from the bag is black ››.
A: ‹‹ The product of the numbers shown in the two drawn balls is equal to 4 ››.

1) Calculate the probabilities $p(R)$, $p(G)$ and $p(B)$.

2) **a)** Show that $p(A/R)=\dfrac{1}{3}$ and calculate $p(A\cap R)$.

b) Show that $p(A\cap G)=\dfrac{1}{8}$ and calculate $p(A\cap B)$.

c) Deduce that $p(A)=\dfrac{49}{144}$.

3) The product of the numbers shown in the two drawn balls is different from 4. Calculate the probability that these balls were drawn from urn V.

4) Calculate the probability that the sum of the numbers shown in the two drawn balls is equal to 4 knowing that their product is also equal to 4.

III – (10 points)

Part A

Consider the function g defined over $\mathbb{R}$ by $g(x)=x+e^{x}$.

1) Set up the table of variations of g.

2) Show that the equation $g(x)=0$ has a unique solution α and verify that $-0.7<\alpha<-0.5$.

3) Deduce the sign of $g(x)$ over $\mathbb{R}$.

Part B

Consider the function f defined over $\mathbb{R}$ by $f(x)=\dfrac{(x-1)e^{x}}{e^{x}+1}$

and let (C) be its representative curve in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.

1) **a)** Determine $\lim\limits_{x\to-\infty} f(x)$, deduce an asymptote to (C).

b) Determine $\lim\limits_{x\to+\infty} f(x)$ and show that the line (d) of equation $y=x-1$ is an asymptote to (C).

c) Discuss according to the values of x in $\mathbb{R}$ the relative positions of (C) and (d) .

2) **a)** Verify that $f(\alpha)=\alpha$.

b) Show that $f'(x)=\dfrac{g(x).e^{x}}{\left(e^{x}+1\right)^{2}}$ and set up the table of variations of the function f (take $\alpha\approx-0.6$).

3) Draw (d) and (C) **(1unit = 2cm).**

Part C

Let h be the function defined by $h(x)=\ln\left[-f(x)\right]$ and denote by (H) its representative curve in an orthonormal system $(O\,;\vec{i}\,;\vec{j})$.

1) Determine the domain of definition of h .

2) Calculate the slope of the tangent to (H) at point of abscissa 0.

3) Show that the equation $h(x)=0$ has no solution.

QI	Answers	4 pts.
1)	Condition of existence: $x-1>$; $x>1$. $(x-1)\ln(x-1)=0$; $x-1=0$; $x=1$ (rejected) or $\ln(x-1)=0=\ln 1$; $x-1=1$; $x=2$ (accepted). Thus the equation $(x-1)\ln(x-1)=0$ has in $\mathbb{R}$ one solution. Hence the correct answer is **b.**	¾
2)	$f'(x)=\dfrac{\ln^2 x-2\ln x}{\ln^4 x}$; $f'(e^{-1})=\dfrac{\ln^2 e^{-1}-2\ln e^{-1}}{\ln^4 e^{-1}}=\dfrac{(-1)^2+2}{(-1)^4}=\dfrac{1+2}{1}=\dfrac{3}{1}=3$. Hence the correct answer is **c.**	¾
3)	$\lim\limits_{x\to+\infty}\left[x-\ln\left(1+e^x\right)\right]=\lim\limits_{x\to+\infty}\left[\ln\left(e^x\right)-\ln\left(1+e^x\right)\right]=\lim\limits_{x\to+\infty}\ln\left(\dfrac{e^x}{1+e^x}\right)$, but $\lim\limits_{x\to+\infty}\dfrac{e^x}{1+e^x}\overset{\text{H.R}}{=}\lim\limits_{x\to+\infty}\dfrac{e^x}{e^x}=1$; so $\lim\limits_{x\to+\infty}\ln\left(\dfrac{e^x}{1+e^x}\right)=\ln 1=0$. Hence the correct answer is **b.**	1
4)	$p(A/B)=\dfrac{p(A\cap B)}{p(B)}$; so $p(B)=\dfrac{p(A\cap B)}{p(A/B)}=\dfrac{0.3}{0.6}=0.5$. $P(A\cup B)=P(A)+P(B)-P(A\cap B)$; so $0.8=P(A)+0.5-0.3$; $P(A)=0.8-0.5+0.3=0.6$. $p(A/B)=P(A)=0.6$ (**OR:** $P(A)\times P(B)=0.6\times0.5=0.3=P(A\cap B)$); thus A and B are two independent events. Hence the correct answer is **b.**	1
5)	The number of possible groups containing at least one girl = The number of all possibilities $-$ The number of all possible groups containing 3 boys $=C_{10}^3-C_4^3=120-4=116$. Hence the correct answer is **a.**	½

QII	Answers	6 pts.
1)	$p(R)=\dfrac{3}{6}=\dfrac{1}{2}$; $p(G)=\dfrac{2}{6}=\dfrac{1}{3}$; $p(B)=\dfrac{1}{6}$.	¾
2)a)	$p(A/R)=\dfrac{C_3^2+C_1^1C_2^1}{C_6^2}=\dfrac{5}{15}=\dfrac{1}{3}$, verified. $p(A\cap R)=p(A/R)\times p(R)=\dfrac{1}{3}\times\dfrac{1}{2}=\dfrac{1}{6}$.	½ ½
2)b)	$p(A\cap G)=p(A/G)\times p(G)=\left[\left(\dfrac{3}{4}\times\dfrac{1}{4}\right)\times 2!\right]\times\dfrac{1}{3}=\dfrac{3}{8}\times\dfrac{1}{3}=\dfrac{1}{8}$, verified. $p(A\cap B)=p(A/B)\times p(B)=\left(\dfrac{1}{6}\times\dfrac{1}{4}+\dfrac{2}{6}\times\dfrac{3}{4}\right)\times\dfrac{1}{6}=\dfrac{7}{24}\times\dfrac{1}{6}=\dfrac{7}{144}$.	1 1
2)c)	$p(A)=p(A\cap R)+p(A\cap G)+p(A\cap B)=\dfrac{1}{6}+\dfrac{1}{8}+\dfrac{7}{144}=\dfrac{49}{144}$, verified.	½

3)	$P\left(G/\overline{A}\right)=\dfrac{p\left(G\cap\overline{A}\right)}{p\left(\overline{A}\right)}=\dfrac{p\left(G\right)-p\left(G\cap A\right)}{1-P\left(A\right)}=\dfrac{\frac{1}{3}-\frac{1}{8}}{1-\frac{49}{144}}=\dfrac{6}{19}.$	¾
4)	$p\left(S=4/A\right)=\dfrac{p\left(\left(S=4\right)\cap A\right)}{p\left(A\right)}$; $p\left(\left(S=4\right)\cap A\right)=p\left(2,2\right)=\dfrac{C_3^2}{C_6^2}\times\dfrac{1}{2}=\dfrac{1}{10}.$ Thus $p\left(S=4/A\right)=\dfrac{\frac{1}{10}}{\frac{49}{144}}=\dfrac{72}{245}.$	**1**

QIII	Answers	10 pts.
A1)	$\lim\limits_{x\to-\infty}g\left(x\right)=\lim\limits_{x\to-\infty}\left(x+e^x\right)=-\infty+e^{-\infty}=-\infty.$ $\lim\limits_{x\to+\infty}g\left(x\right)=\lim\limits_{x\to+\infty}\left(x+e^x\right)=+\infty+e^{+\infty}=+\infty.$ $g'(x)=1+e^x>0$ for every x , hence g is strictly increasing over $\mathbb{R}$. Table of variations of g : $\begin{array}{c\|ccc} x & -\infty & \alpha & +\infty \\ \hline g'(x) & & + & \\ \hline g(x) & -\infty & \nearrow 0 \nearrow & +\infty \end{array}$	½
A2)	• g is continuous over $\mathbb{R}$, strictly increasing and changes its sign from negative $(-\infty)$ to positive ($+\infty$) one time only then the equation $g\left(x\right)=0$ admits a unique solution α over $\mathbb{R}$. • $g\left(-0.7\right)\approx-0.2<0$ and $g\left(-0.5\right)\approx0.1>0$, therefore $-0.7<\alpha<-0.5$.	**1**
A3)	According to the table of variations of g : • $g\left(x\right)<0$ if $x<\alpha$. • $g\left(x\right)=0$ if $x=\alpha$. • $g\left(x\right)>0$ for $x>\alpha$.	½
B1)a)	$\lim\limits_{x\to-\infty}f\left(x\right)=\lim\limits_{x\to-\infty}\dfrac{(x-1)e^x}{e^x+1}=\lim\limits_{x\to-\infty}\dfrac{xe^x-e^x}{e^x+1}=\dfrac{0-e^{-\infty}}{e^{-\infty}+1}=0$, since $\lim\limits_{x\to-\infty}xe^x=0$ (rule). Thus the line of equation $y=0$ is a H.A to (C) at $(-\infty)$.	½
B1)b)	$\lim\limits_{x\to+\infty}f\left(x\right)=\lim\limits_{x\to+\infty}\dfrac{(x-1)e^x}{e^x+1}=\dfrac{+\infty}{+\infty}$ (I.F), $\lim\limits_{x\to+\infty}\dfrac{(x-1)e^x}{e^x+1}\overset{\text{H.R}}{=}\lim\limits_{x\to+\infty}\dfrac{e^x+(x-1)e^x}{e^x}=\lim\limits_{x\to+\infty}\dfrac{xe^x}{e^x}=\lim\limits_{x\to+\infty}x=+\infty.$ $\lim\limits_{x\to+\infty}\left[f\left(x\right)-y_{(d)}\right]=\lim\limits_{x\to+\infty}\left[\dfrac{(x-1)e^x}{e^x+1}-(x-1)\right]=\lim\limits_{x\to+\infty}\left[\dfrac{(x-1)e^x-(x-1)\left(e^x+1\right)}{e^x+1}\right]=$	½ **1**

$$= \lim_{x\to+\infty}\left(\frac{xe^{x}-e^{x}-xe^{x}-x+e^{x}+1}{e^{x}+1}\right)=\lim_{x\to+\infty}\left(\frac{-x+1}{e^{x}+1}\right)=\frac{+\infty}{+\infty}\ \text{(I.F).}$$

but $\lim\limits_{x\to+\infty}\left(\dfrac{-x+1}{e^{x}+1}\right)\overset{\text{H.R}}{=}\lim\limits_{x\to+\infty}\left(\dfrac{-1}{e^{x}}\right)=\dfrac{-1}{+\infty}=0.$

so the line (d) of equation : $y=x-1$ is an oblique asymptote to (C) at $(+\infty)$.

B1)c) — 1

$f(x)-y_{(d)}=\dfrac{-x+1}{e^{x}+1}$, same sign as $(-x+1)$ since $e^{x}+1>0$ for every $x\in\mathbb{R}$.

- $f(x)-y_{(d)}<0$ for $x>1$; thus (C) is below (d) if $x\in\left]1;+\infty\right[$.
- $f(x)-y_{(d)}>0$ for $x<1$; thus (C) is above (d) if $x\in\left]-\infty;1\right[$.
- $f(x)-y_{(d)}=0$ for $x=1$; thus (C) cuts (d) at point of coordinates $(1;0)$.

B2)a) — ½

$g(\alpha)=0$; so $\alpha+e^{\alpha}=0$; $e^{\alpha}=-\alpha$.

$f(\alpha)=\dfrac{(\alpha-1)e^{\alpha}}{e^{\alpha}+1}=\dfrac{(\alpha-1)(-\alpha)}{-\alpha+1}=\dfrac{(\alpha-1)(-\alpha)}{-(\alpha-1)}=\alpha$; verified.

B2)b) — 1 ½

$$f'(x)=\frac{(xe^{x})(e^{x}+1)-(e^{x})(x-1)(e^{x})}{(e^{x}+1)^{2}}=\frac{(e^{x})(xe^{x}+x-xe^{x}+e^{x})}{(e^{x}+1)^{2}}=\frac{e^{x}g(x)}{(e^{x}+1)^{2}}.$$

$f'(x)$ and $g(x)$ have same sign over $\mathbb{R}$ $(e^{x}>0\,;e^{x}+1>0)$.

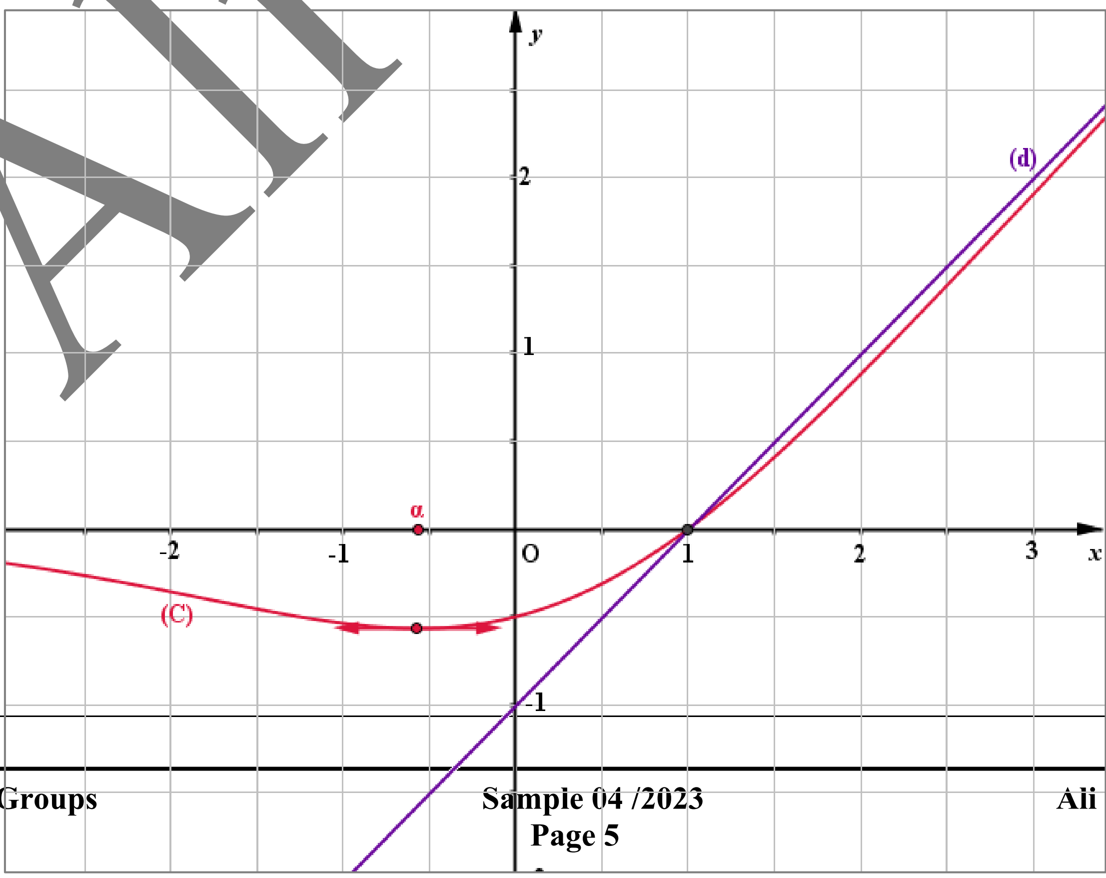
$g(x)=0$ at $x=\alpha\approx-0.6$

Table of variations of f :

x	$-\infty$		-0.6		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	0	$\searrow$	-0.6	$\nearrow$	$+\infty$

3) — 1 ½

$f(0)=-\dfrac{1}{2}$



C1)	h is defined for $-f(x)>0$, so $f(x)<0$. Graphically, (C) is below $(x'ox)$ if $x<1$. So $D_h = \left]-\infty;1\right[$.	½
C2)	Slope $=h'(0)=\dfrac{-f'(0)}{-f(0)}=\dfrac{f'(0)}{f(0)}=\dfrac{\frac{1}{4}}{-\frac{1}{2}}=-\dfrac{1}{2}$. Thus, the slope of the tangent to (H) at point of abscissa 0 is $-\dfrac{1}{2}$.	½
C3)	$h(x)=0$; $\ln\left[-f(x)\right]=0=\ln 1$; $-f(x)=1$; $f(x)=-1$. But graphically, the curve (C) and the line of equation $y=-1$ don't intersect, thus, the equation $h(x)=0$ has no solution.	½