Examath Groups	Mathematics Exam Section :L.S.	Prepared by: Randa Chehade Edited by: Hassan Ahmad
Number of questions: 3	**Sample 03 – year 2022 Duration: 90 min**	**Name: Nº:**

- *This exam includes three problems inscribed on two pages.*
- *The use of a non-programmable calculator is allowed.*

I- (5 points)

Consider the function f defined over $]0\,;+\infty[$ by $f(x) = a(\ln x)^2 + b\ln x$ (a and b are real numbers).
Designate by (C) its representative curve in an orthonormal system $(O;\ \vec{\imath}\ ;\ \vec{\jmath}\ )$.
Given that:

- (C) passes through the point E $(e; -1)$;
- (C) admits at point E a tangent parallel to the x-axis.

We define over $]0\,;+\infty[$ the function g defined by $g(x) = e^{f(x)}$.
Designate by (G) its representative curve in the same orthonormal system.
Tell, **with justification**, whether each of the following statements is true or false:

1) The values of a and b are respectively 1 and -2.
2) The curve (C) cuts the x –axis in two distinct points.
3) The solution set of the inequality $f(x) > -1$ is $]e\,;+\infty[$.
4) The slope of the tangent to the curve (G) at point of abscissa 1 is -2.

II- (6 points)

Consider 2 urns U and V are such that:

- U contains 3 balls each carrying the number 0 and two balls each carrying the number 1.
- V contains 2 balls each carrying the number 0 and three balls each carrying the number 1.

Part A:
One ball is randomly selected from U and one ball is randomly selected from V.
1) Show that the probability of selecting two balls carrying the number 0 is 0.24.
2) Calculate the probability of selecting two balls carrying the same number.
3) Calculate the probability of selecting two balls carrying different numbers.

Part B:
In this part one ball is randomly selected from U:

- if the selected ball from urn U carries number 0, then two balls are randomly and simultaneously selected from urn V.
- if the selected ball from urn U carries number 1, then three balls are randomly and simultaneously selected from urn V.

Consider the following events:
E: "The selected ball from urn U carries number 0".
F: "The selected balls from urn V carry the same number".
1) Determine the probability P(E).
2) Show that P(F / E) = 0.4 , and deduce $P(E \cap F)$.
3) Calculate $P(\overline{E} \cap F)$, and deduce P(F).
4) Knowing that the selected balls from urn V carry the same number, what is the probability that the selected ball from urn U carries number 1?

Part C:

The ten balls from the two urns U and V are placed in one urn W.
Three balls are selected one after the other without replacement from the urn W.
Calculate the probability of the event H:
"The product of numbers on the three selected balls is equal to 0".

III- (9 points)

Part A:

Let g be the function defined over $\mathbb{R}$ by $g(x) = 1 + (1 - x)e^x$.

1) **a)** Calculate $\lim_{x \to +\infty} g(x)$ and $\lim_{x \to -\infty} g(x)$.

b) Calculate $g'(x)$, and set up the table of variations of g.

2) **a)** Show that the equation $g(x) = 0$ admits a unique solution α and verify that $1.27 < \alpha < 1.28$.

b) Discuss, according to the values of x in $\mathbb{R}$, the sign of $g(x)$.

Part B:

Let f be the function defined over $\mathbb{R}$ by: $f(x) = (2 - x)e^x + x - 2$.
Designate by (C) its representative curve in an orthonormal system $(O; \vec{\imath}; \vec{\jmath})$..

1) Determine $\lim_{x \to +\infty} f(x)$ and calculate $f(2.5)$ to the nearest 10^{-2} .

2) **a)** Determine $\lim_{x \to -\infty} f(x)$, and prove that the straight line (d) of equation: $y = x - 2$ is an oblique asymptote to (C).

b) Study the relative position of (C) and (d).

3) Verify that $f'(x) = g(x)$, then set up the table of variations of f.

4) Show that $f(\alpha) = \frac{(\alpha - 2)^2}{\alpha - 1}$.

5) Draw (d) and (C) (take $\alpha \approx 1.275$).

Q.I.	Answers	5 points
1.	$f(e) = -1\,;\ a(\ln e)^2 + b\ln e = -1\,;\ \ a + b = -1.$ $f'(x) = \dfrac{2a\ln x}{x} + \dfrac{b}{x}\ ;\ f'(e) = 0\,;\ \dfrac{2a\ln e}{e} + \dfrac{b}{e} = 0\,;\ 2a + b = 0.$ Then $a = 1\,; b = -2$, hence $f(x) = (\ln x)^2 - 2\ln x$ **(True).**	**1 ¼**
2.	$f(x) = 0\,; (\ln x)^2 - 2\ln x = 0\,;\ \ln x\,(\ln x - 2) = 0\,;\ \ln x = 0\ or\ \ \ln x = 2\,;$ $x = 1\ or\ x = e^2.$ Then (C) cuts the x –axis in 2 distinct points of respective abscissas 1 and e^2**(True).**	**1**
3.	$f(x) > -1\,;\ (\ln x)^2 - 2\ln x + 1 > 0\,; (\ln x - 1)^2 > 0$ which is true for all values of x such that $(\ln x - 1) \neq 0\ and\ x > 0\ \ ; that\ is\ x \neq e\ and\ x > 0.$ Then the solution set is $]0; e[\cup]e; +\infty[$ **(False).**	**1 ¼**
4.	$g'(x) = f'(x)e^{f(x)}$. The slope of tangent at point of abscissa 1 is $g'(1) = f'(1)e^{f(1)}$. $f'(1) = \dfrac{2(\ln 1 - 1)}{1} = -2\,;\ f(1) = (\ln 1)^2 - 2\ln 1 = 0$. Then $g'(1) = -2e^0 = -2$ **(True).**	**1 ½**

Q.II.	Answers	6 points
A.1.	P (selecting 2 balls carrying the number 0) = $\frac{3}{5} \times \frac{2}{5} = \frac{6}{25}$ = 0.24 .	½
A.2.	P (selecting 2 balls carrying the same number) = $\frac{3}{5} \times \frac{2}{5} + \frac{2}{5} \times \frac{3}{5} = \frac{12}{25}$ = 0.48	½
A.3.	P (selecting 2 balls carrying different numbers) = 1– P(selecting 2 balls carrying the same number) = 1 – 0.48 = 0.52 .	½
B.1.	P(E) = $\frac{3}{5}$ = 0.6 .	½
B.2.	P (F / E) = $\frac{C_2^2 + C_3^2}{C_5^2} = 0.4$. P(E ∩ F) = P(F /E) × P(E) = $0.4 \times 0.6 = 0.24$.	**1**
B.3.	$P(\overline{E} \cap F) = P(F/\overline{E}) \times P(\overline{E}) = \frac{C_3^3}{C_5^3} \times 0.4 = 0.1 \times 0.4 = 0.04$. P (F) = P (E ∩ F) + $P(\overline{E} \cap F) = 0.24 + 0.04 = 0.28.$	**1**
B.4.	$P(\overline{E}/F) = \frac{P(\overline{E} \cap F)}{P(F)} = \frac{0.04}{0.28} = \frac{1}{7}$.	**1**
C.	P(H) = P(at least one ball carries number 0) = 1− P(no ball carries number 0) $= 1 - \frac{A_5^3}{A_{10}^3} = \frac{11}{12}.$	**1**

Q.III.	Answers	9 points
A.1.a.	$\lim_{x\to+\infty} g(x) = \lim_{x\to+\infty}[1+(1-x)e^x] = -\infty.$, $\lim_{x\to-\infty} g(x) = \lim_{x\to-\infty}[1+(1-x)e^x] = \lim_{x\to-\infty}[1+e^x-xe^x] = 1+0-0 = 1.$	¾
A.1.b.	$g'(x) = -e^x + (1-x)e^x = -xe^x$ same sign as $-x$ since $e^x > 0$ for every $x \in \mathbb{R}$ $\begin{array}{c\|ccccc} x & -\infty & & 0 & & +\infty \\ \hline g'(x) & & + & 0 & - & \\ \hline g(x) & & & 2 & & \\ & 1 & \nearrow & & \searrow & -\infty \end{array}$	¾
A.2.a.	Over $]-\infty;0]$: g is continuous and strictly increasing from positive (1) to positive (2), then $g(x) > 0$ over $]-\infty;0]$. Over $]0;+\infty[$: g is continuous, strictly decreasing and changes its sign from positive (2) to negative ($-\infty$) , then the equation $g(x) = 0$ admits one solution α over $]0;+\infty[$. Then the equation $g(x) = 0$ admits a unique solution α. In addition, $g(1.27) \approx 0.003 > 0$ and $g(1.28) \approx -0.007 < 0$, therefore: $1.27 < \alpha < 1.28$.	**1**
A.2.b.	$g(x) > 0 \ if \ x < 0$. Over $]0;+\infty[$: g is continuous, strictly decreasing from 2 to $-\infty$ and $g(\alpha) = 0$, so: $g(x) > 0 \ if \ 0 < x < \alpha$ and $g(x) < 0 \ if \ x > \alpha$. Therefore: • $g(x) < 0 \ if \ x > \alpha$. • $g(x) = 0 \ if \ x = \alpha$. • $g(x) > 0 \ if \ x < \alpha$.	¾
B.1.	$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}[(2-x)e^x + x - 2] = \lim_{x\to+\infty}(x-2)[-e^x+1] = -\infty.$ $f(2.5) = (2-2.5)e^{2.5} + 2.5 - 2 = -0.5e^{2.5} + 0.5 \approx -5.59.$	¾
B.2.a.	$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}[(2-x)e^x + x - 2] = \lim_{x\to-\infty}(x-2)[-e^x+1] = -\infty(0+1) = -\infty.$ $\lim_{x\to-\infty}[f(x) - y_d] = \lim_{x\to-\infty}(2-x)e^x = \lim_{x\to-\infty}(2e^x - xe^x) = 0$; then the line (d) of equation: $y = x - 2$ is an oblique asymptote to (C) at $-\infty$.	¾
B.2.b.	$f(x) - y_d = (2-x)e^x$; has the same sign as $(2-x)$ since $e^x > 0$ for every $x \in \mathbb{R}$ • $f(x) - y_d > 0 \ if \ x < 2$; (C) is above (d) if $x \in]-\infty;2[$. • $f(x) - y_d < 0 \ if \ x > 2$; (C) is below (d) if $x \in]2;+\infty[$. • $f(x) - y_d = 0 \ if \ x = 2$; (C) cuts (d) at point of coordinates (2; 0).	**1**

<table>
<tr><td>B.3.</td><td>$f'(x) = -e^x + (2-x)e^x + 1 = e^x - xe^x + 1 = (1-x)e^x + 1 = g(x)$.
$f'(x)$ and $g(x)$ have the same sign over $\mathbb{R}$.</td><td>½</td></tr>
<tr><td></td><td>
<table>
<tr><td>x</td><td>$-\infty$</td><td></td><td>α</td><td></td><td>$+\infty$</td></tr>
<tr><td>$f'(x)$</td><td></td><td>$+$</td><td>0</td><td>$-$</td><td></td></tr>
<tr><td>$f(x)$</td><td>$-\infty$</td><td>$\nearrow$</td><td>$f(\alpha)$</td><td>$\searrow$</td><td>$-\infty$</td></tr>
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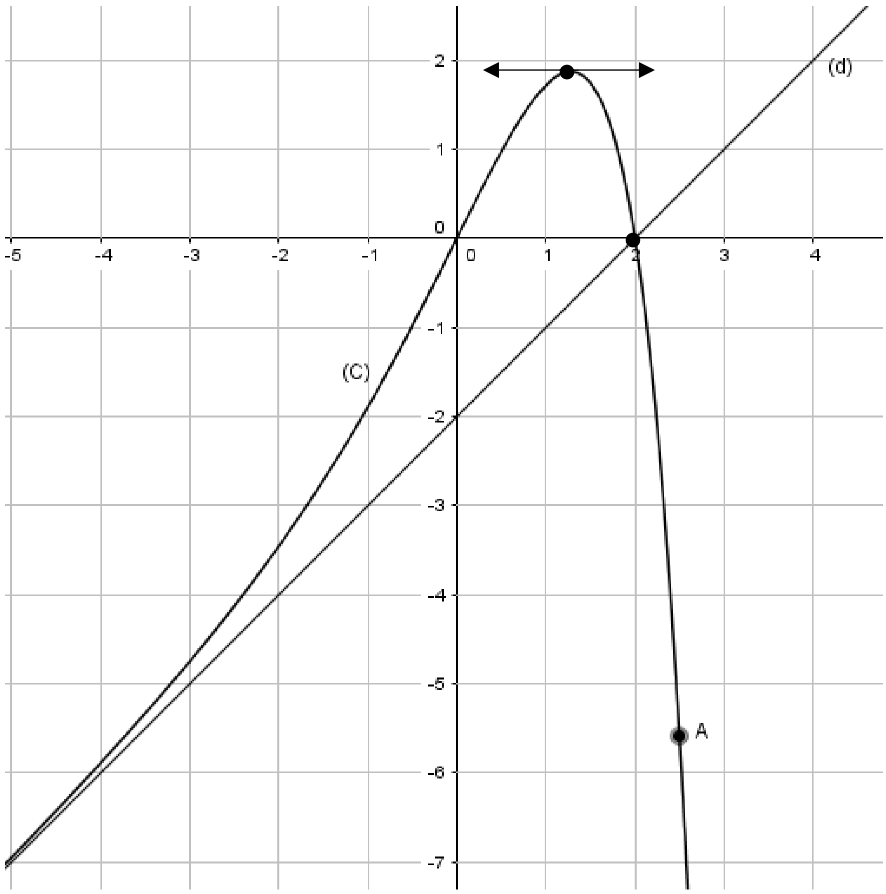
</td><td>¾</td></tr>
<tr><td>B.4.</td><td>$g(\alpha) = 0$; $e^{\alpha} = \frac{1}{\alpha-1}$;
$f(\alpha) = (2-\alpha)e^{\alpha} + \alpha - 2 = (2-\alpha)\frac{1}{\alpha-1} + \alpha - 2 = \frac{2-\alpha+(\alpha-2)(\alpha-1)}{\alpha-1}$;
$f(\alpha) = \frac{\alpha^2-4\alpha+4}{\alpha-1} = \frac{(\alpha-2)^2}{\alpha-1}$.</td><td>½</td></tr>
<tr><td>B.5.</td><td>
</td><td>1 ½</td></tr>
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