First Edition

2006

Publishers

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# MASTERING MATHEMATICS

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## *Preface*

*Dear colleagues, Dear students,*

*Mastering Mathematics for the General Sciences (GS) cycle is a noteworthy scholastic endeavor in at least two respects, the problems and their solutions presented alongside their conceptual frameworks are exclusively limited to General Sciences students. Second, it furbishes learners with mathematical skills and competencies.*

*In short, this book seeks to remedy students' deficiencies in mathematics and strengthen their already acquired skills. It provides the students with the following competencies:*

- *A sharper comprehension of mathematical concepts and principles on a chapter by chapter basis .*
- *A clearer grasp of problems and their solutions, starting from basic to more general problems .*
- *A wider access to supplementary problems with indications .*
- *Sample Tests for a global preparation to examinations.*

*The simple reading of the solutions provided in the book is inadequate for attaining a rigorous understanding of mathematical concepts, problems and their solutions. Thus, we recommend students to solve the problems by themselves before referring to the solutions provided.*

*We hope that this work will be of great help to students interested in success and performance in the national examinations as well as in their respective majors in higher education*

*We welcome with interest all the suggestions addressed to us to enrich this work.*

***The Authors***

Dar AL Amal

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# CHAPTER 1

# Logic

**Chapter Review :**

- Definition:
  A statement is a sentence that can be either True or False.
  If $p$ is a statement, then $\overline{p}$ is its negation, also noted as $non(p)$ or $\neg p$ .
  If $\neg p$ is true then $p$ is false .
- Principle of contradiction:
  A statement cannot be true or false at the same time.
- Principle of the excluded middle:
  A statement is either true or false and nothing else.
- Logical Conjunction :
  If $p$ and $q$ are two statements, then « $p$ and $q$ » is a statement called the conjunction of $p$ and $q$, denoted by $p \wedge q$.
  $p \wedge q$ is true when the two statements $p$ and $q$ are true at the same time and false in the other cases.
- Truth Table for Logical Conjunction :

| $p$ | $Q$ | $p \wedge q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ |
| $F$ | $T$ | $F$ |
| $F$ | $F$ | $F$ |

- Logical Disjunction :
  If $p$ and $q$ are two statements, then « $p$ or $q$ » is a statement called the disjunction of $p$ and $q$, denoted by $p \vee q$.

$p \vee q$ is false when the two statements $p$ and $q$ are false at the same time, and true in the other cases.

- Truth Table for Logical disjunction :

| $p$ | $q$ | $p \vee q$ |
|---|---|---|
| $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ |
| $F$ | $F$ | $F$ |

- The conditional « If........Then » :
  $p$ and $q$ are two statements , the statement $\neg(p) \vee q$ is denoted by $p \Rightarrow q$ .
  $p \Rightarrow q$ is true when : $p$ is false or $q$ is true .

- The biconditional « If and only If » :
  $p$ and $q$ are two statements , the statement $(p \Rightarrow q \text{ and } q \Rightarrow p)$ is denoted by $p \Leftrightarrow q$ .
  $p \Leftrightarrow q$ is true when $p$ and $q$ are true at the same time or $p$ and $q$ are false at the same time .
  We say that $p$ and $q$ are equivalent.

- Law of Contradiction :
  To prove that a statement $p$ is true .
  We suppose that $p$ is false, it leads to prove that a proposition $q$ and its negation $\neg(q)$ are true at the same time then we conclude that $p$ is true .
  *Ex* : Let $(d')$ and $(d'')$ be two distinct straight lines in a plane such that :
  $(d') \perp (d)$ and $(d'') \perp (d)$, let's prove that $(d')$is parallel to $(d'')$.
  Suppose that $(d')$and $(d'')$are not parallel, then :
  $(d')$ and $(d'')$ intersect at a point $O$ so we will have through a point $O$ two distinct straight lines perpendicular to $(d)$, which is impossible . Hence, $(d')$ and $(d'')$ are parallel .

# Solved Problems

**N° 1.**

Give the truth value of each of the following statements:
1) $n$ is an even natural number and $n$ is an odd natural number .
2) All multiples of 6 are even integers .
3) If an integer is divisible by 4 and by 6 then it is divisible by 24 .
4) If the water is clear, then you can see the bottom of the pond or you are short-sighted .
5) If there are inhabitants in Mars then 5 divides 18 .
6) If 27 is a prime number then it is not even and if it is divisible by 3 then it is not a prime number.

**N° 2.**

Given the following three statements :

$p$ : Sami is a dentist .

$q$ : Hani is a surgeon .

$r$ : Jad is a pharmacist .

Write the following statements using the statements $p$ , $q$ and $r$ and the logical operators .
1) Sami is a dentist or Hani is not a surgeon.
2) If Jad is a pharmacist and Hani is a surgeon then Sami is a dentist.
3) Hani is a surgeon if and only if Sami is not a dentist.

**N° 3.**

By setting up their truth tables , verify that each of the following statements is a tautology:

1) $\neg(p \vee q) \Leftrightarrow (\neg p) \wedge (\neg q)$.
2) $non(p \Rightarrow q) \Leftrightarrow (p \wedge non\, q)$
3) $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$.
4) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$.

**N° 4.**

1) Without using the truth table, prove that the following statement is a tautology :
$[p \Rightarrow (q \wedge r)] \Leftrightarrow [(p \Rightarrow q) \wedge (p \Rightarrow r)]$.

2) Use the truth table to prove that the following statement is a tautology $[(p \vee q) \Rightarrow q] \Leftrightarrow [(\neg p) \vee q]$.

**N° 5.**

If $p$ and $q$ are two statements , denote by , $p \uparrow q$ the statement : $\neg(p \wedge q)$.

1) Prove that :
   a- $\neg p$ is equivalent to $p \uparrow q$ .
   b- $p \wedge q$ is equivalent to $(p \uparrow q) \uparrow (p \uparrow q)$.
2) a- Express $p \vee q$ using $p$ , $q$ and the connector $\uparrow$ only once.
   b- Express $p \Rightarrow q$ using $p$ , $q$ and the connector $\uparrow$ only once.

**N° 6.**

Salim, Walid and Wissam are having a ride on their bikes but each one is riding the bike of one of his other two friends and puts on the hat of the other friend.
We know that the one that puts on Wissam's hat is on Walid's bike.
Who is riding Salim's bike ?

**N° 7.**

Sahar lost her timetable for the next day.
She still remembers that the last period is "Physical Education", but she totally forgot the first five periods.
She called her friends who decided to tease her .
Tarek said : " The third period is Science and History is the first".
Rana said : " The second period is English and Math is the fourth".
Ziad said : " The fifth period is History and Science is the fourth".
Amal said : " The fifth period is French and English is the second".
Zeina said : " The third period is French and Math is the fourth".
Sahar is confused. But her friends admit that each of them said one true statement and another false one.
Help Sahar to figure out the order of her classes.

**N° 8.**

To choose a minister out of three candidates $A$ , $B$ and $C$ , an oriental king subjected them to an experiment:

He puts on the head of each candidate a ball that he can't see.
Each candidate can see the balls placed on the heads of the other two and knows that the balls are chosen from three black balls and two white. The first that knows the color of the ball on his head becomes the minister.
Candidate $A$ seeing a black ball on each of his friends' heads and noticing that they said nothing declares « Mine is black ».
Explain his reasoning .

**N°9.**

A prison cell has two doors one leads to freedom, another to death.
In front of each door, there stands a guard that answers, only by yes or no to each question asked.
One of the guards always lies , the other always says the truth.
The prisoner doesn't know which guard says the truth. He may ask one question only to one of the guards.
What should he ask to know the door that leads to freedom?

# Solution of Problems

N° 1.

1) If $p$ is the statement « $n$ is even » then $\overline{p}$ is the statement « $n$ is odd » but using the principle of contradiction, $p$ and $\overline{p}$ cannot be true and false at the same time.
   Hence, this statement is false .
2) All integers $n$ multiples of 6 can be written in the form : $n = 6k = 2 \times (3k) = 2k'$, then it is even , consequently this statement is true.
3) 12 is divisible by 4 and by 6 but it is not divisible by 24 , then this statement is false.
4) The statement « water is clear » is true and the logical disjunction « you can see the bottom of the pond or you are shortsighted » is true , then the logical implication is true .
5) The statement « There are inhabitants in Mars » is false , then the logical implication is true.
6) The statement « if 27 is prime then it is not even » is true since 27 is prime is a false statement .
   The statement « if it is divisible by 3 then it is not a prime number » is a true statement .
   Then, the logical conjunction of the statements is true .

N° 2.

1) Sami is a dentist or Hani is not a surgeon, is written symbolically as: $p \vee (\neg q)$.
2) If Jad is a pharmacist and Hani is a surgeon then Sami is a dentist, written symbolically as: $r \wedge q \Rightarrow p$.
3) Hani is a surgeon if and only if Sami is not a dentist is written symbolically as: $q \Leftrightarrow (\neg p)$.

N° 3.

1)

| $p$ | $q$ | $p \vee q$ | $\neg(p \vee q)$ | $\neg p$ | $\neg q$ | $\neg p \wedge \neg q$ | $\Leftrightarrow$ |
|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

2)

| $p$ | $q$ | $p \Rightarrow q$ | $\neg(p \Rightarrow q)$ | $\neg q$ | $p \wedge \neg q$ | $\Leftrightarrow$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $T$ | $F$ | $T$ |

3)

| $p$ | $q$ | $p \Rightarrow q$ | $\neg q$ | $\neg p$ | $\neg q \Rightarrow \neg p$ | $\Leftrightarrow$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

4) Denote by $s$ the statement $(p \vee q) \wedge (p \vee r)$

| $p$ | $q$ | $r$ | $q \wedge r$ | $p \vee (q \wedge r)$ | $p \vee q$ | $p \vee r$ | $s$ | $\Leftrightarrow$ |
|---|---|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $T$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $F$ | $T$ |

N° 4.

1) If $p \Rightarrow (q \wedge r)$ is false then the implication $[p \Rightarrow (q \wedge r)] \Rightarrow [(p \Rightarrow q) \wedge (p \Rightarrow r)]$ is true .

We suppose that $p \Rightarrow (q \wedge r)$ is true and we show that $(p \Rightarrow q) \wedge (p \Rightarrow r)$ is true .

We will face two possibilities:

- $p$ is false , which affirms that $p \Rightarrow q$ is true and $p \Rightarrow r$ is true , then $(p \Rightarrow q) \wedge (p \Rightarrow r)$ is true and consequently
- $p$ is true so $q \wedge r$ is true since $p \Rightarrow (q \wedge r)$ is true , therefore $q$ is true and $r$ is true, so $p \Rightarrow q$ and $p \Rightarrow r$ are true and consequently $(p \Rightarrow q) \wedge (p \Rightarrow r)$.

2)

| $p$ | $q$ | $p \vee q$ | $(p \vee q) \Rightarrow q$ | $\neg p$ | $(\neg p) \vee q$ | $\Leftrightarrow$ |
|---|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $T$ | $F$ | $T$ | $T$ |
| $T$ | $F$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $T$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ | $T$ |

Hence $[(p \vee q) \Rightarrow q] \Leftrightarrow [(\neg p) \vee q]$ is a tautology.

N° 5.

1)

| P | q | $p \wedge q$ | $p \uparrow q$ | $\neg p$ | $\Leftrightarrow$ |
|---|---|---|---|---|---|
| $T$ | $T$ | $T$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $T$ |
| $F$ | $T$ | $F$ | $T$ | $T$ | $T$ |
| $F$ | $F$ | $F$ | $T$ | $T$ | $T$ |

a- Hence $\neg p$ is equivalent to $p \uparrow q$.

b- $(p \uparrow q) \Leftrightarrow \neg(p \wedge q)$ then

$(p \uparrow q) \uparrow (p \uparrow q) \Leftrightarrow \neg(p \uparrow q \wedge p \uparrow q)$

$\neg(p \uparrow q \wedge p \uparrow q) \Leftrightarrow \neg(\neg(p \wedge q) \wedge \neg(p \wedge q))$

$\neg(\neg(p \wedge q) \wedge \neg(p \wedge q)) \Leftrightarrow \neg(\neg(p \wedge q)) \Leftrightarrow p \wedge q$

we deduce that $(p \uparrow q) \uparrow (p \uparrow q) \Leftrightarrow p \wedge q$.

2) a- $p \vee q \Leftrightarrow \neg p \uparrow \neg q$.

b- $[p \Rightarrow q] \Leftrightarrow [(\neg p) \vee q] \Leftrightarrow \neg(\neg p) \uparrow \neg q \Leftrightarrow p \uparrow \neg q$.

**N°6.**

The one who puts on Wissam's hat is on Walid's bike is third boy ,so he is Salim.
The one who is on Salim's bike puts on Walid's or Wissam's hat, but Salim puts on Wissam's hat, hence the one that is on Salim's bike puts on Walid's hat. Therefore, Wissam is on Salim's bike.

**N°7.**

We may use the following table:

| | First Period | 2nd Period | 3rd Period | 4th Period | 5th Period |
|---|---|---|---|---|---|
| Tarek | History | | Science | | |
| Rana | | English | | Math | |
| Ziad | | | | Science | History |
| Amal | | English | | | French |
| Zeina | | | French | Math | |

If History is the first period then Science is not third. So, Science is fourth. Consequently, Math is not fourth so English is second and French is third.
As a result, Math is the fifth period.

**N°8.**

$A$ thought in the following way :
If $B$ saw on my head a white ball then, $B$ seeing a black ball on $C$'s head and a white on my head , he should have known that the ball on his head is black , otherwise if the ball on his head is white then $C$ seeing two white balls on the heads of $A$ and $B$, he will know directly that the ball on his head is black. But, since $C$ said nothing , and $B$ said nothing then the ball on my head is black.

N°9.

Designating by $A$ and $B$ the two guards and by $p_A$ the door where $A$ stands and by $p_B$ the door where $B$ stands.

If the prisoner asks $A$ the following question :

If $I$ ask $B$ to tell me the door that leads me to freedom, what will his answer be?

The prisoner should choose the opposite door of that indicated in the answer.

Justification :

For example, if $A$ says that the answer of $B$ is $p_A$ then there are two possibilities :

- If $A$ is the guard that says the truth then $B$ is the liar then $B$ will guide to the door leading to death so $I$ choose door $p_B$.
- If $A$ is the guard that always lies, then $B$'s answer $(p_B)$ is true and since $B$ always says the truth then his door leads to freedom, so $I$ have to choose $p_B$.

# CHAPTER 2

# Inverse Trigonometric Functions

**Chapter Review :**

- Function: *arcsine :*

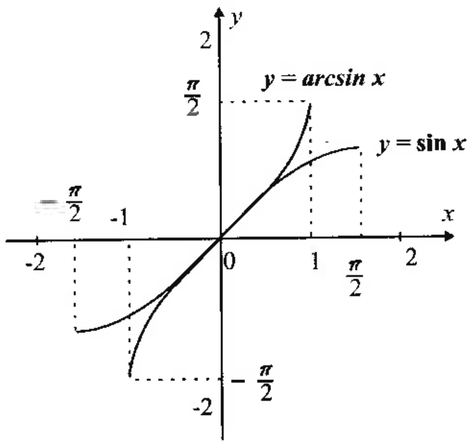
  * The function $f$ defined over $\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ by $f(x)=\sin x$ is continuous and strictly increasing, then it admits an inverse function $f^{-1}$ defined over $[-1;+1]$ and denoted by arcsin or $\sin^{-1}$.

$$\left[-\frac{\pi}{2};\frac{\pi}{2}\right] \xrightarrow{f} [-1;+1] \xrightarrow{f^{-1}} \left[-\frac{\pi}{2};\frac{\pi}{2}\right]$$

$$x \longrightarrow y=\sin x \longrightarrow x=\arcsin y$$

  * ***Ex :*** $\frac{1}{2}=\sin\frac{\pi}{6}$ is equivalent to $\frac{\pi}{6}=\arcsin\frac{1}{2}$.

  * The representative curve of the function arcsine is symmetric to the representative curve of the sine function with respect to the straight line of equation $y=x$ in an orthonormal system $(O;\vec{i},\vec{j})$.



* Derivative :

$$(\arcsin)'(x) = \frac{1}{\sqrt{1-x^2}} \quad ; \quad (\arcsin)'(u(x)) = \frac{u'(x)}{\sqrt{1-(u(x))^2}}$$

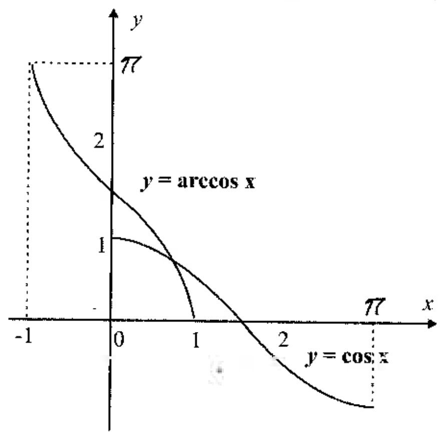
* Integral : $\int_a^b \frac{u'(x)}{\sqrt{1-(u(x))^2}}dx = \arcsin u(x)\Big|_a^b$

- Function *arccosin* :
  - * The function $f$ defined over $[0;\pi]$ by $f(x) = \cos x$ is continuous and strictly decreasing, then it admits an inverse function $f^{-1}$ defined over $[-1;+1]$ and denoted by arccos or $\cos^{-1}$.

$$\begin{cases} \alpha = \arccos\beta \\ -1 \le \beta \le 1 \\ 0 \le \alpha \le \pi \end{cases} \quad \text{is equivalent to} \quad \beta = \cos\alpha.$$

* ***Ex :*** $\frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$ is equivalent to $\frac{\pi}{4} = \arccos\frac{\sqrt{2}}{2}$.

* The representative curve of the function arccos is symmetric to that of the function cos with respect to the straight line of equation $y = x$ in an orthonormal system $(O;\vec{i},\vec{j})$.




* Derivative

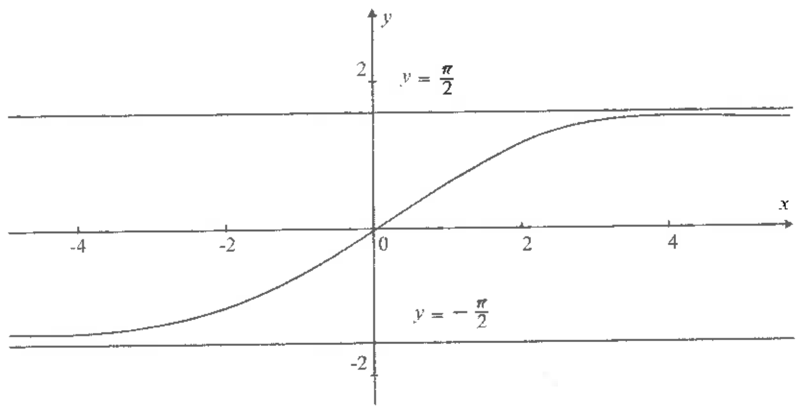
$$(\arccos)'(x) = \frac{-1}{\sqrt{1-x^2}} \quad ; \quad (\arccos)'(u(x)) = \frac{-u'(x)}{\sqrt{1-(u(x))^2}}.$$

- Function *arctan* :

* The function $f$ defined over $\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ by $f(x)=\tan x$ is continuous and strictly increasing , then it admits an inverse function $f^{-1}$ defined over $IR$ and denoted by arctan or $\tan^{-1}$.

$$\begin{cases}\alpha = \arctan\beta \\ \beta \in IR \\ -\frac{\pi}{2} < \alpha < \frac{\pi}{2}\end{cases} \quad \text{is equivalent to} \quad \beta = \tan\alpha .$$

* ***Ex :*** $\sqrt{3} = \tan\frac{\pi}{3}$ is equivalent to $\frac{\pi}{3} = \arctan\sqrt{3}$.
* The representative curve of the function arctan is symmetric to that of the function tan with respect to the straight line of equation $y = x$ in an orthonormal system $(O;\vec{i},\vec{j})$.




* Derivative:

$$(\arctan)'(x) = \frac{1}{1+x^2} \quad ; \quad (\arctan)'(u(x)) = \frac{u'(x)}{1+(u(x))^2}.$$

* Integral : $\int_a^b \frac{u'(x)}{1+u^2(x)}dx = \left[\arctan u(x)\right]_a^b$.

- $\cos(\arccos x) = x$ ; $\sin(\arcsin x) = x$ ; $\tan(\arctan x) = x$.

# Solved Problems

**N° 1.**

Calculate. Justify your work:

1) $\arccos\left[\cos\left(\frac{11\pi}{3}\right)\right]$

2) $\arccos\left[\sin\left(\frac{11\pi}{3}\right)\right]$

3) $\sin\left[\arccos\left(-\frac{1}{2}\right)\right]$

4) $\cos\left[\arcsin\left(-\frac{\sqrt{2}}{2}\right)\right]$

5) $\arccos\left[\cos\left(\frac{7\pi}{6}\right)\right]$

6) $\arctan\left[\tan\left(\frac{7\pi}{6}\right)\right]$

**N° 2.**

Find a relation between :

1) $\arccos(-x)$ and $\arccos(x)$
2) $\arcsin(-x)$ and $\arcsin(x)$
3) $\arctan(-x)$ and $\arctan(x)$

**N° 3.**

1) Given $\alpha = \arccos\frac{3}{5}$ and $\beta = \arcsin\frac{12}{13}$.
   Calculate $\cos(\alpha+\beta)$.
2) Calculate $\arctan 2 + \arctan 3$.

**N° 4.**

Simplify each of the following expressions :

1) $\tan(2\arctan x)$
2) $\cos(2\arccos x)$
3) $\cos(4\arctan x)$
4) $\sin(2\arcsin x)$
5) $\cos(2\arctan x)$
6) $\cos^2(2\arcsin x)$

**N° 5.**

Given the equation (E) : $\arcsin x + \arcsin\frac{x}{2} = \frac{\pi}{4}$

Does this equation admit solutions for $-1 < x < 0$ ?

**N° 6.**

Solve each of the following equations :

1) $\arctan 2x + \arctan 3x = \frac{\pi}{4}$.

2) $\arcsin 2x + \arcsin\frac{1}{2} = \frac{\pi}{2}$.

3) $\arctan x + \arctan 3 = \frac{3\pi}{4}$.

4) $\arctan 2x + \arccos x = \frac{\pi}{2}$.

5) $\arcsin(2x-1) + 2\arctan\sqrt{\frac{1-x}{x}} = \frac{\pi}{2}$.

6) $\arcsin x + \arccos\frac{1}{3} = \arcsin\frac{1}{2}$.

7) $\arctan 4x + \arctan\frac{12}{13} = \arctan 1$.

8) $\arcsin\sqrt{\frac{2x}{1+x}} = \frac{\pi}{2} - \arcsin\sqrt{x}$.

**N° 7.**

Prove each of the following equalities :

1) $\arctan\frac{1}{3} + \arctan\frac{1}{4} = \arctan\frac{7}{11}$

2) $2\arctan\frac{2}{3} = \arctan\frac{12}{5}$

3) $\arctan\frac{1}{2} + \arccos\frac{\sqrt{5}}{5} = \frac{\pi}{2}$

4) $2\arccos\frac{2}{3} = \pi - \arccos\frac{1}{9}$

## N°8.

Calculate each of the following integrals :

1) $\displaystyle\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}\,dx$

2) $\displaystyle\int_0^{1} \frac{1}{\sqrt{9-x^2}}\,dx$

3) $\displaystyle\int_{\frac{3}{2}}^{2} \frac{1}{\sqrt{-x^2+4x-3}}\,dx$

4) $\displaystyle\int_0^{\frac{1}{2}} \frac{\arcsin t}{\sqrt{1-t^2}}\,dt$

5) $\displaystyle\int_0^{1} \frac{\arctan t}{1+t^2}\,dt$

6) $\displaystyle\int_1^{2} \frac{1}{4x^2+4x+2}\,dx$

7) $\displaystyle\int_{-1}^{+1} \frac{dx}{x^2+2x+5}$

8) $\displaystyle\int_0^{1} \frac{dx}{2x^2-2x+1}$

9) $\displaystyle\int_{-1}^{0} \frac{x}{x^2+2x+2}\,dx$.

## N°9.

Consider the function $f$ defined over $IR$ by $f(x)=\sqrt{x^2+1}-x$.

1) Show that $f$ is strictly decreasing over $IR$.
2) Determine $\lim\limits_{x\to+\infty} f(x)$ and deduce the sign of $f(x)$ over $IR$.
3) Let $g$ be the function defined over $IR$ by $g(x)=\arctan f(x)$.
   a- Show that the equation $g(x)=\dfrac{\pi}{2}+\arccos x$ has no solutions in $IR$.
   b- Prove that $g(x)=\dfrac{\pi}{4}-\dfrac{1}{2}\arctan x$ and deduce $\tan\dfrac{\pi}{8}$.

## N° 10.

$f$ is the function defined over $\left[-\dfrac{1}{2};+\dfrac{1}{2}\right]$ by :

$f(x)=\arccos 2x+\arcsin 2x$.

1) Show that $f(x)=\dfrac{\pi}{2}$.
2) Solve the equation $\arcsin x+\arcsin 2x=\arccos x+\arccos 2x$.

**N° 11.**

Consider the function $f$ defined over $]-1;+1[$ by

$f(x)=\arctan\dfrac{2x}{1-x^2}$.

1) Calculate $f'(x)$ and deduce that $f(x)=2\arctan x$.
2) Solve the equation $f(x)=\dfrac{\pi}{2}-2\arctan\dfrac{1}{2}$.

**N° 12.**

Consider the function $f$ defined over $IR$ by $f(x)=\dfrac{1-x^2}{1+x^2}$.

1) Study the variations of $f$ and deduce that $-1<f(x)\leq 1$.
2) Determine the domain of definition of the function $g$ defined by

   $g(x)=\arccos\left(\dfrac{1-x^2}{1+x^2}\right)$.
3) Calculate $g'(x)$ and deduce that $g(x)=2\arctan x$ for $x\in]0;+\infty[$.

**N° 13.**

Consider the function $f$ defined over $[0;2]$ by $f(x)=2+\arcsin(x-1)$ and designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of $f$ and trace $(C)$.
2) Let $(C')$ be the curve representing a function $g$ defined over $[-1;1]$ by $g(x)=\arcsin x$.

   Show that $(C')$ can be deduced from $(C)$ by the translation of vectror $\vec{v}(-1;-2)$ then trace $(C')$.
3) Show that $f$ admits an inverse function $f^{-1}$.

   Determine the domain of definition of $f^{-1}$ and find $f^{-1}(x)$

   Trace the curve representative of $f^{-1}$.

**N° 14.**

Let $f$ be the function defined over $]0;+\infty[$ by $f(x)=\arctan\sqrt{x}$.

Designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $f(1)$ and $f(3)$.
2) Show that $\arctan\sqrt{x}+\arctan\frac{1}{\sqrt{x}}=\frac{\pi}{2}$.
3) Calculate $f'(x)$ and draw the table of variations of $f$.
4) Trace $(C)$.

**N° 15.**

Let $f$ be the function defined over $IR-\{0\}$ by $f(x)=\arctan\left(1+\frac{2}{x}\right)$, and designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to-\infty}f(x)$, $\lim\limits_{x\to+\infty}f(x)$, $\lim\limits_{\substack{x\to0\\x<0}}f(x)$ and $\lim\limits_{\substack{x\to0\\x>0}}f(x)$.
2) Calculate $f'(x)$ and draw the table of variations of $f$.
3) Find the coordinates of $A$, the point of intersection of $(C)$ with the axis $x'x$ and give an equation of the tangent $(T)$ at $A$ to $(C)$.
4) Trace $(C)$.

**N° 16.**

1) a- Prove that $arctan\,u+arc\,cot\,u=\frac{\pi}{2}$ for $u>0$.

   b- Deduce the derivative of $h(u)=arc\,cot\,u$.
2) Let $f$ be the function defined over $IR$ by :
   $f(x)=arc\cot(2x-1)-arc\cot(2x+1)$.

   a- Calculate $f'(x)$.

   b- Show that $f(x)=arc\cot(2x^2)$.
3) Deduce a simple expression of the sum:
   $S_n=arc\cot(2\times1^2)+arc\cot(2\times2^2)+..................+arc\cot(2\times n^2)$.
   where $n$ is a non-zero natural integer.
   Calculate $\lim\limits_{n\to+\infty}S_n$.

**N° 17.**

Prove that if $\arccos\alpha + \arccos\beta + \arccos\gamma = \pi$ then
$\alpha^2 + \beta^2 + \gamma^2 = 1 - 2\alpha\beta\gamma$.

**N° 18.**

Solve the equation:
$2\arctan x + \arctan 3x = \operatorname{arccot} x + 2\operatorname{arccot} 3x$.

**N° 19.**

Consider the function $f$ defined over $\left[\frac{\sqrt{2}}{2};1\right]$ by :

$f(x) = \arcsin\left(2x\sqrt{1-x^2}\right)$ and designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $f$ is differentiable over $\left]\frac{\sqrt{2}}{2};1\right[$ and calculate $f'(x)$.
2) Show that $f(x) = \pi - 2\arcsin x$.
3) Trace $(C)$.

**N° 20.**

$x$ is real number greater than or equal to 1.

1) Prove that $\arctan\frac{1}{2x-1} - \arctan\frac{1}{2x+1} = \arctan\frac{1}{2x^2}$.
2) Deduce a simple expression of the sum :
   $S_n = \arctan\frac{1}{2} + \arctan\frac{1}{8} + \arctan\frac{1}{18} + \cdots\cdots + \arctan\frac{1}{2n^2}$.
   Calculate $\lim\limits_{n\to+\infty} S_n$.

**N° 21.**

Let $f$ be the function defined by $f(x) = \arctan\sqrt{\frac{1-x}{1+x}} + \frac{1}{2}\arcsin x$.

1) Determine the domain of definition of $f$.
2) Show that $f(x)$ is a constant to be determined.

# Solution of Problems

**N°1.**

1) $arc\cos\left(\cos\frac{11\pi}{3}\right)=arc\cos\left[\cos\left(4\pi-\frac{\pi}{3}\right)\right]=arc\cos\left(\cos\frac{\pi}{3}\right)=\frac{\pi}{3}$.

2) $arc\,cos\left(sin\frac{11\pi}{3}\right)=arc\,cos\left[sin\left(4\pi-\frac{\pi}{3}\right)\right]=arc\,cos\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$.

3) $sin\left[arc\,cos\left(-\frac{1}{2}\right)\right]=sin\left(\frac{2\pi}{3}\right)=\frac{\sqrt{3}}{2}$ .

4) $cos\left[arc\,sin\left(\frac{-\sqrt{2}}{2}\right)\right]=cos\left(-\frac{\pi}{4}\right)=\frac{\sqrt{2}}{2}$.

5) $arc\cos\left(\cos\frac{7\pi}{6}\right)=arc\cos\left(\cos\left(\pi+\frac{\pi}{6}\right)\right)=arc\cos\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$.

6) $arc\,tan\left(tan\frac{7\pi}{6}\right)=arc\,tan\left(tan\frac{\pi}{6}\right)=\frac{\pi}{6}$.

**N°2.**

1) $arc\,cos(-x)=\pi-arc\,cos(x)$.
2) $arc\,sin(-x)=-arc\,sin(x)$.
3) $arc\,tan(-x)=-arc\,tan(x)$.

**N°3.**

1) $\alpha=arc\,cos\frac{3}{5}$ is equivalent to $cos\,\alpha=\frac{3}{5}$ where $\alpha\in\left]0;\frac{\pi}{2}\right[$.

$\beta=arc\,sin\frac{12}{13}$ is equivalent to $sin\,\beta=\frac{12}{13}$ where $\beta\in\left]0;\frac{\pi}{2}\right[$.

$$cos(\alpha+\beta)=cos\,\alpha\,cos\,\beta-sin\,\alpha\,sin\,\beta$$
$$=\frac{3}{5}\times\sqrt{1-\frac{144}{169}}-\sqrt{1-\frac{9}{25}}\times\frac{12}{13}$$
$$=\frac{3}{5}\cdot\frac{5}{13}-\frac{4}{5}\cdot\frac{12}{13}=\frac{15}{65}-\frac{48}{65}=-\frac{33}{65}.$$

2) $arc\tan 2=\alpha$ is equivalent to $tan\,\alpha=2$ where $\alpha\in\left]\frac{\pi}{4};\frac{\pi}{2}\right[$ .

$arc\tan 3=\beta$ is equivalent to $tan\,\beta=3$ where $\beta\in\left]\frac{\pi}{4};\frac{\pi}{2}\right[$ .

$$tan(\alpha+\beta)=\frac{tan\,\alpha+tan\,\beta}{1-tan\,\alpha\cdot tan\,\beta}=\frac{2+3}{1-6}=\frac{5}{-5}=-1.$$

But $\frac{\pi}{2}<\alpha+\beta<\pi$ therefore $\alpha+\beta=\frac{3\pi}{4}$ .

N°4.

1) $arc\tan x=\alpha$ is equivalent to $tan\,\alpha=x$ with $\alpha\in\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ .

$$tan(2\,arc\tan x)=tan\,2\alpha=\frac{2\tan\alpha}{1-tan^2\alpha}=\frac{2x}{1-x^2}.$$

2) $arc\cos x=\alpha$ is equivalent to $cos\,\alpha=x$ with $\alpha\in[0;\pi]$ .

$$cos(2arc\cos x)=cos\,2\alpha=2cos^2\alpha-1=2x^2-1.$$

3) $arc\tan x=\alpha$ is equivalent to $\tan\alpha=x$ with $\alpha\in\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ .

$$\cos(4\arctan x)=\cos(4\alpha)=2\cos^2 2\alpha-1=2\cdot\frac{1-\tan^2\alpha}{1+\tan^2\alpha}-1$$
$$=\frac{2(1-x^2)}{1+x^2}-1=\frac{2-2x^2-1-x^2}{1+x^2}=\frac{1-3x^2}{1+x^2}.$$

4) $arc\sin x=\alpha$ is equivalent to $sin\,\alpha=x$ where $\alpha\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right]$ .

$$sin(2arc\sin x)=sin\,2\alpha=2\sin\alpha\cos\alpha=2x\sqrt{1-x^2}.$$

5) $arc\tan x=\alpha$ is equivalent to $tan\,\alpha=x$ and $\alpha\in\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ .

$\cos(2arc\tan x) = \cos 2\alpha = 2\cos^2\alpha - 1$, but $\cos^2\alpha = \dfrac{1}{1+\tan^2\alpha}$

Then $\cos(2arc\tan x) = \dfrac{2}{1+\tan^2\alpha} - 1 = \dfrac{2}{1+x^2} - 1 = \dfrac{1-x^2}{1+x^2}$.

6) $arc\sin x = \alpha$ is equivalent to $\sin\alpha = x$ with $\alpha \in \left[-\dfrac{\pi}{2};\dfrac{\pi}{2}\right]$.

$\cos^2(2arc\sin x) = \cos^2 2\alpha$, but $\cos 2\alpha = 1 - 2\sin^2\alpha = 1 - 2x^2$.

Therefore, $\cos^2(2arc\sin x) = (1-2x^2)^2$.

**N° 5.**

If $-1 < x < 0$ then $-\dfrac{\pi}{2} < arc\sin x < 0$.

$-1 < x < 0$ gives $-\dfrac{1}{2} < \dfrac{x}{2} < 0$ then $-\dfrac{\pi}{2} < arc\sin\dfrac{x}{2} < 0$.

We get then $-\pi < arc\sin x + arc\sin\dfrac{x}{2} < 0$ .

Consequently, the equation $arc\sin x + arc\sin\dfrac{x}{2} = \dfrac{\pi}{4}$ has no solution for $-1 < x < 0$.

**N° 6.**

1) Since $arctan\, 2x + arctan\, 3x = \dfrac{\pi}{4}$, we get

$tan(arc\tan 2x + arc\tan 3x) = tan\dfrac{\pi}{4}$ , then

$\dfrac{tan(arc\tan 2x) + tan(arc\tan 3x)}{1 - tan(arc\tan 2x)\cdot tan(arc\tan 3x)} = 1$, which gives

$\dfrac{2x+3x}{1-6x^2} = 1$ and $5x = 1 - 6x^2$ , so $6x^2 + 5x - 1 = 0$,

this is a qudrartic equation whose roots are $x = -1$ and $x = \dfrac{1}{6}$.

For $x = -1$ we get $arc\tan 2x < 0$ and $arc\tan 3x < 0$ then $arc\tan 2x + arc\tan 3x < 0$ , so $x = -1$ is rejected and consequently, $x=\frac{1}{6}$ is the accepted solution .

2) This equation is defined when $-1\le 2x\le 1$ which gives $-\frac{1}{2}\le x\le \frac{1}{2}$.

We have: $arc\sin 2x+arc\sin\frac{1}{2}=\frac{\pi}{2}$ , then

$arc\sin 2x=\frac{\pi}{2}-arc\sin\frac{1}{2}$ which gives $arc\sin 2x=\frac{\pi}{2}-\frac{\pi}{6}=\frac{\pi}{3}$.

So $2x=\sin\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ , therefore $x=\frac{\sqrt{3}}{4}$ is accepted .

3) $arc\tan x+arc\tan 3=\frac{3\pi}{4}$ then $arc\tan x=\frac{3\pi}{4}-arc\tan 3$ which gives $\tan(arc\tan x)=\tan\left(\frac{3\pi}{4}-arc\tan 3\right)$.

Therefore: $x=\dfrac{\tan\frac{3\pi}{4}-3}{1+\tan\frac{3\pi}{4}\times 3}=\frac{-4}{1-3}=2$, which is accepted since $x\in IR$.

4) Let :

$\alpha=\arctan 2x$, then $\tan\alpha=2x$ with $\alpha\in\left]-\frac{\pi}{2};\frac{\pi}{2}\right[$ and $x\in IR$.

$\beta=\arccos x$, then $\cos\beta=x$ with $\beta\in[0;\pi]$ and $x\in[-1;1]$.

Therefore $\alpha+\beta=\frac{\pi}{2}$, which gives $\alpha=\frac{\pi}{2}-\beta$, hence

$\tan\alpha=\tan\left(\frac{\pi}{2}-\beta\right)=\cot\beta=\frac{\cos\beta}{\sin\beta}$ .

But $\sin\beta=\sqrt{1-\cos^2\beta}=\sqrt{1-x^2}$ , therefore $2x=\frac{x}{\sqrt{1-x^2}}$

$x=0$ is a solution of the equation.

Dividing by $x$, we get $2=\frac{1}{\sqrt{1-x^2}}$, then $2\sqrt{1-x^2}=1$

which gives $1-x^2=\frac{1}{4}$, which is equivalent to $x^2=\frac{3}{4}$.

Consequently, $x=\frac{\sqrt{3}}{2}$ or $x=-\frac{\sqrt{3}}{2}$, the three solutions are accepted

5) This equation is defined for $\frac{1-x}{x}\geq 0$ and $2x-1\in[-1,1]$ then for $x\in\left]0,1\right]$.

Let: $\begin{cases}\alpha=\arcsin(2x-1)\text{ then }\sin\alpha=2x-1;\alpha\in\left]-\frac{\pi}{2};\frac{\pi}{2}\right]\\ \beta=\arctan\left(\sqrt{\frac{1-x}{x}}\right)\text{ then }\tan\beta=\sqrt{\frac{1-x}{x}};\beta\in\left[0,\frac{\pi}{2}\right[\end{cases}$

But $\alpha+2\beta=\frac{\pi}{2}$ which is equivalent to $2\beta=\frac{\pi}{2}-\alpha$.

Therefore: $\cos 2\beta=\cos\left(\frac{\pi}{2}-\alpha\right)$ which is equivalent to

$\frac{1-\tan^2\beta}{1+\tan^2\beta}=\sin\alpha$ which gives $\frac{1-\frac{1-x}{x}}{1+\frac{1-x}{x}}=2x-1$ and consequently

$2x-1=2x-1$, which is true for all real numbers, hence the set of solutions of this equation is $S=]0,1]$.

6) This equation is defined for $-1\leq x\leq 1$.

We have $\arcsin x=\arcsin\frac{1}{2}-\arccos\frac{1}{3}=\frac{\pi}{6}-\arccos\frac{1}{3}$

$\arccos\frac{1}{3}=\alpha$ is equivalent to $\cos\alpha=\frac{1}{3}$ with $\alpha\in\left[0;\frac{\pi}{2}\right[$

Therefore, $\arcsin x=\frac{\pi}{6}-\alpha$, which gives

$\sin(\arcsin x)=\sin\left(\frac{\pi}{6}-\alpha\right)$ which gives

$x = sin\left(\frac{\pi}{6} - \alpha\right) = sin\frac{\pi}{6}cos\,\alpha - cos\frac{\pi}{6}sin\,\alpha$

$= \frac{1}{2}\times\frac{1}{3} - \frac{\sqrt{3}}{2}\times\sqrt{1-\frac{1}{9}} = \frac{1}{6} - \frac{\sqrt{3}}{2}\times\frac{2\sqrt{2}}{3} = \frac{1-2\sqrt{6}}{6}$ accepted.

7) $arc\tan 4x = \frac{\pi}{4} - arc\tan\frac{12}{13}$, then

$\tan(arc\tan 4x) = \tan\left(\frac{\pi}{4} - arc\tan\frac{12}{13}\right)$, which gives :

$4x = tan\left(\frac{\pi}{4} - arc\,tan\frac{12}{13}\right) = \frac{1-\frac{12}{13}}{1+\frac{12}{13}} = \frac{1}{25}$, then $x = \frac{1}{100}$ is

accepted since $x \in IR$.

8) This equation is defined for $\sqrt{\frac{2x}{1+x}} \in [0;1]$ and $\sqrt{x} \in [0;1]$

$\sqrt{\frac{2x}{1+x}} \in [0;1]$ gives $0 \le \frac{2x}{1+x} \le 1$ and $\sqrt{x} \in [0;1]$ gives $0 \le x \le 1$.

So $0 \le x \le 1$ and $\frac{2x}{1+x} - 1 \le 0$ which gives $\frac{x-1}{1+x} \le 0$.

Therefore $0 \le x \le 1$ and $-1 < x \le 1$, consequently $0 \le x \le 1$.

$\alpha = \arcsin\sqrt{\frac{2x}{1+x}}$ is equivalent to $\sin\alpha = \sqrt{\frac{2x}{1+x}}$ ; $\alpha \in \left[0;\frac{\pi}{2}\right]$

$\beta = \arcsin\sqrt{x}$ is equivalent to $\sin\beta = \sqrt{x}$ ; $\beta \in \left[0;\frac{\pi}{2}\right]$

Since $\alpha = \frac{\pi}{2} - \beta$ we get: $\sin\alpha = \sin\left(\frac{\pi}{2} - \beta\right)$ or $\sin\alpha = \cos\beta$.

Which gives $\sqrt{\frac{2x}{1+x}} = \sqrt{1-x}$ and consequently: $x^2 + 2x - 1 = 0$.

The roots of this equation are:

$x' = -1 + \sqrt{2} \in [0, 1]$. So, it is accepted.

$x'' = -1 - \sqrt{2} \notin [0, 1]$. So, it is rejected.

N°7.

1) Let:

$\alpha = \arctan\frac{1}{3}$, then $\tan\alpha = \frac{1}{3}$ with $0 < \alpha < \frac{\pi}{4}$.

$\beta = \arctan\frac{1}{4}$, then $\tan\beta = \frac{1}{3}$ with $0 < \beta < \frac{\pi}{4}$.

$\gamma = \arctan\frac{7}{11}$, then $\tan\gamma = \frac{7}{11}$ with $0 < \gamma < \frac{\pi}{4}$.

We need to prove that $\alpha + \beta = \gamma$.

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{\frac{1}{3}+\frac{1}{4}}{1-\frac{1}{12}} = \frac{\frac{7}{12}}{\frac{11}{12}} = \frac{7}{11} = \tan\gamma$$

Therefore, $\alpha + \beta = \gamma + k\pi$, which gives $\alpha + \beta - \gamma = k\pi$, $k \in Z$.

But $-\frac{\pi}{4} < \alpha + \beta - \gamma < \frac{\pi}{2}$, then $-\frac{\pi}{4} < k\pi < \frac{\pi}{2}$, so $-\frac{1}{4} < k < \frac{1}{2}$.

consequently, $k = 0$ and we get $\alpha + \beta = \gamma$.

2) Let:

$\arctan\frac{2}{3} = \alpha$ then $\tan\alpha = \frac{2}{3}$ ; $\alpha \in \left]0, \frac{\pi}{4}\right[$ since $0 < \frac{2}{3} < 1$

$\arctan\frac{12}{5} = \beta$ then $\tan\beta = \frac{12}{5}$ ; $\beta \in \left]\frac{\pi}{4}, \frac{\pi}{2}\right[$ since $1 < \frac{12}{5}$

Since we have: $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \dfrac{\frac{4}{3}}{1-\frac{4}{9}} = \dfrac{12}{5} = \tan\beta$.

we get: $2\alpha = \beta + k\pi$ which gives $2\alpha - \beta = k\pi$, $k \in Z$ but:

$$\begin{array}{rl} & 0 < 2\alpha < \frac{\pi}{2} \\ \oplus & -\frac{\pi}{2} < -\beta < -\frac{\pi}{4} \\ \hline & -\frac{\pi}{2} < 2\alpha - \beta < +\frac{\pi}{4} \end{array}$$

So, we get then: $-\frac{\pi}{2} < k\pi < \frac{\pi}{4}$ then $-\frac{1}{2} < k < \frac{1}{4}$.

$k \in Z$, hence $k = 0$ and we get $2\alpha = \beta$.

3) $arc\,tan\frac{1}{2} + arc\,cos\frac{\sqrt{5}}{5} = \frac{\pi}{2}$

$arc\,tan\frac{1}{2} = \alpha$ is equivalent to $tan\,\alpha = \frac{1}{2}$ with $0 < \alpha < \frac{\pi}{4}$

$arc\,cos\frac{\sqrt{5}}{5} = \beta$ is equivalent to $cos\,\beta = \frac{\sqrt{5}}{5}$ with $0 < \beta < \frac{\pi}{2}$

we need to prove that $\alpha + \beta = \frac{\pi}{2}$ or $\alpha = \frac{\pi}{2} - \beta$.

$$\tan\left(\frac{\pi}{2} - \beta\right) = \cot\beta = \frac{\cos\beta}{\sin\beta} = \frac{\frac{\sqrt{5}}{5}}{\sqrt{1-\frac{1}{5}}} = \frac{1}{2} = \tan\alpha$$ , then:

$\alpha = \frac{\pi}{2} - \beta + k\pi$ then $\alpha + \beta - \frac{\pi}{2} = k\pi$, $k \in Z$.

But $-\frac{\pi}{2} < \alpha + \beta - \frac{\pi}{2} < \frac{\pi}{4}$ so $-\frac{\pi}{2} < k\pi < \frac{\pi}{4}$, which gives

$-\frac{1}{2} < k < \frac{1}{4}$, consequently $k = 0$ as a result:

$arc\,tan\frac{1}{2} + arc\,cos\frac{\sqrt{5}}{5} = \frac{\pi}{2}$

4) Let $arc\,cos\frac{2}{3} = \alpha$ , then $cos\,\alpha = \frac{2}{3}$ with $0 < \alpha < \frac{\pi}{2}$.

$arc\,cos\frac{1}{9} = \beta$, then $cos\,\beta = \frac{1}{9}$ with $0 < \beta < \frac{\pi}{2}$

We need to prove that $2\alpha = \pi - \beta$.

$\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{8}{9} - 1 = \frac{-1}{9}$, $\cos(\pi - \beta) = -\cos\beta = -\frac{1}{9}$.

Therefore, $\cos 2\alpha = \cos(\pi - \beta)$.

$0 < \alpha < \dfrac{\pi}{2}$ , so $0 < 2\alpha < \pi$ and since $\cos 2\alpha < 0$ then

$\dfrac{\pi}{2} < 2\alpha < \pi$.

$0 < \beta < \dfrac{\pi}{2}$ , so $\dfrac{\pi}{2} < \pi - \beta < \pi$.

Therefore $2\alpha = \pi - \beta$.

$N°8.$

1) $\dfrac{1}{\sqrt{1-4x^2}} = \dfrac{1}{\sqrt{1-(2x)^2}} = \dfrac{1}{2} \times \dfrac{2}{\sqrt{1-(2x)^2}}$ so :

$$\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}} dx = \frac{1}{2} \arcsin 2x \Big]_0^{\frac{1}{4}} = \frac{1}{2}\left[\arcsin\frac{1}{2} - \arcsin 0\right] = \frac{\pi}{12}.$$

2) $$\int_0^1 \frac{1}{\sqrt{9-x^2}} dx = \int_0^1 \frac{1}{\sqrt{1-\left(\frac{x}{3}\right)^2}} \times \left(\frac{1}{3}\right) dx = \arcsin\frac{x}{3}\Big]_0^1 = \arcsin\frac{1}{3}.$$

3) $-x^2 + 4x - 3 = -\left(x^2 - 4x + 4 - 4\right) - 3 = -(x-2)^2 + 1$.

Hence

$$\int_{\frac{3}{2}}^{2} \frac{1}{\sqrt{-x^2+4x-3}} dx = \int_{\frac{3}{2}}^{1} \frac{1}{\sqrt{1-(x-2)^2}} dx = \arcsin(x-2)\Big]_{\frac{3}{2}}^{2} = \frac{\pi}{6}.$$

4) $$\int_0^{\frac{1}{2}} \frac{\arcsin t}{\sqrt{1-t^2}} dt = \int_0^{\frac{1}{2}} (\arcsin t)(\arcsin t)' \, dt$$

$$= \frac{(\arcsin t)^2}{2}\Big]_0^{\frac{1}{2}} = \frac{\frac{\pi^2}{36} - 0}{2} = \frac{\pi^2}{72}.$$

5) $$\int_0^1 \frac{\arctan t}{1+t^2} dt = \int_0^1 (\arctan t)(\arctan t)' \, dt$$

$$= \frac{(\arctan t)^2}{2}\Big]_0^1 = \frac{\left(\frac{\pi}{4}\right)^2 - 0}{2} = \frac{\pi^2}{32}.$$

6) $\displaystyle\int_1^2 \frac{1}{4x^2+4x+2}dx = \int_1^2 \frac{1}{(2x+1)^2+1}dx = \frac{1}{2}\int_1^2 \frac{2}{1+(2x+1)^2}dx$

$$= \frac{1}{2}\arctan(2x+1)\Big]_1^2 = \frac{1}{2}[\arctan 5 - \arctan 3].$$

7) $\displaystyle\frac{1}{x^2+2x+5} = \frac{1}{(x+1)^2+4} = \frac{1}{4}\times\frac{1}{1+\left(\frac{x+1}{2}\right)^2}$

If $u(x) = \dfrac{x+1}{2}$ then $u'(x) = \dfrac{1}{2}$, therefore $\dfrac{1}{x^2+2x+5} = \dfrac{2}{4}\times\dfrac{u'}{1+u^2}$

so $\displaystyle\int_{-1}^{+1} \frac{1}{x^2+2x+5}dx = \frac{1}{2}\arctan\frac{x+1}{2}\Big]_{-1}^{+1}$

$$= \frac{1}{2}\arctan 1 - \frac{1}{2}\arctan 0 = \frac{\pi}{8}.$$

8) $\displaystyle 2x^2-2x+1 = 2\left(x^2-x+\frac{1}{2}\right) = 2\left[\left(x-\frac{1}{2}\right)^2 - \frac{1}{4}+\frac{1}{2}\right]$

$$= 2\left[\left(x-\frac{1}{2}\right)^2+\frac{1}{4}\right] = \frac{2}{4}\left[(2x-1)^2+1\right]$$

so $\displaystyle\int_0^1 \frac{dx}{2x^2-2x+1} = \frac{1}{4}\int_0^1 \frac{2}{1+(2x-1)^2}dx = \frac{1}{4}\arctan(2x-1)\Big]_0^1$

$$= \frac{1}{4}[\arctan 1 - \arctan(-1)] = \frac{\pi}{8}.$$

9) $\displaystyle\frac{x}{x^2+2x+2} = \frac{1}{2}\times\frac{2x}{x^2+2x+2} = \frac{1}{2}\times\frac{2x+2-2}{x^2+2x+2}$

so $\displaystyle\frac{x}{x^2+2x+2} = \frac{1}{2}\times\frac{2x+2}{x^2+2x+2} - \frac{1}{(x+1)^2+1}.$

Therefore $\displaystyle\int_{-1}^{0} \frac{x}{x^2+2x+2}dx = \left[\frac{1}{2}\ln\left(x^2+2x+2\right) - \arctan(x+1)\right]_{-1}^{0}$

$$= \frac{\ln 2}{2} - \frac{\pi}{4}.$$

N°9.

1) $f'(x)=\dfrac{2x}{2\sqrt{x^2+1}}-1=\dfrac{x-\sqrt{x^2+1}}{\sqrt{x^2+1}}$.

If $x\leq 0$ then $f'(x)<0$ .

If $x>0$ then

$$f'(x)=\frac{\left(x-\sqrt{x^2+1}\right)\left(x+\sqrt{x^2+1}\right)}{\sqrt{x^2+1}\left(x+\sqrt{x^2+1}\right)}=\frac{-1}{\sqrt{x^2+1}\left(x+\sqrt{x^2+1}\right)}<0$$

hence $f'(x)<0$. So $f$ is strictly decreasing over $IR$ .

2) $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\dfrac{x^2+1-x^2}{\sqrt{x^2+1}+x}=\lim\limits_{x\to+\infty}\dfrac{1}{\sqrt{x^2+1}+x}=0$.

| $x$ | $-\infty$ | $+\infty$ |
|---|---|---|
| $f'(x)$ | $-$ | |
| $f(x)$ | $\searrow$ | $0$ |

From the table of variations of $f$, we notice that $f(x)>0$ over $IR$ .

3) a- $f(x)>0$ then $g(x)\in\left[0;\dfrac{\pi}{2}\right[$ and $\arccos x\in[0;\pi]$.

Therefore, $\dfrac{\pi}{2}\leq\dfrac{\pi}{2}+\arccos x<\dfrac{3\pi}{2}$ and $0<g(x)<\dfrac{\pi}{2}$.

Hence the equation $g(x)=\dfrac{\pi}{2}+\arccos x$ does not admit solutions in $IR$ .

b- $\alpha=\arctan\left(\sqrt{1+x^2}-x\right)$ then $\tan\alpha=\sqrt{1+x^2}-x$ , $\alpha\in\left]0,\dfrac{\pi}{2}\right[$

$\beta=\arctan x$ then $\tan\beta=x$ , $\beta\in\left]-\dfrac{\pi}{2},\dfrac{\pi}{2}\right[$

We have to prove that: $\alpha=\dfrac{\pi}{4}-\dfrac{1}{2}\beta$, so $2\alpha=\dfrac{\pi}{2}-\beta$.

As a result:

- $\tan(2\alpha)=\dfrac{2\tan\alpha}{1-\tan^2\alpha}=\dfrac{2\left(\sqrt{1+x^2}-x\right)}{1-\left[\sqrt{1+x^2}-x\right]^2}$

$=\dfrac{2\left(\sqrt{1+x^2}-x\right)}{1-\left(1+x^2+x^2-2x\sqrt{1+x^2}\right)}=\dfrac{2\left(\sqrt{1+x^2}-x\right)}{2x\left(\sqrt{1+x^2}-x\right)}=\dfrac{1}{x}$.

- $\tan\left(\dfrac{\pi}{2}-\beta\right)=\cot\beta=\dfrac{1}{\tan\beta}=\dfrac{1}{x}$.

Therefore, $\tan(2\alpha)=\tan\left(\dfrac{\pi}{2}-\beta\right)$ and since

$2\alpha\in\left]0,\pi\right[$ and $\dfrac{\pi}{2}-\beta\in\left]0,\pi\right[$, we obtain: $2\alpha=\dfrac{\pi}{2}-\beta$

since the tangent function is a bijection on this interval.

So we get that: $\arctan\left(\sqrt{1+x^2}-x\right)=\dfrac{\pi}{4}-\dfrac{1}{2}\arctan x$.

Since this equality is true for all $x$, it remains true for $x = 1$ therefore:

$\arctan\left(\sqrt{1+1}-1\right)=\dfrac{\pi}{4}-\dfrac{1}{2}\arctan 1$ which gives

$\arctan\left(\sqrt{2}-1\right)=\dfrac{\pi}{4}-\dfrac{\pi}{8}=\dfrac{\pi}{8}$ and consequently

$\tan\dfrac{\pi}{8}=\sqrt{2}-1$.

**N° 10.**

1) $f'(x)=\dfrac{-2}{\sqrt{1-4x^2}}+\dfrac{2}{\sqrt{1-4x^2}}=0$, then $f(x)=k$ for all

$x\in\left[-\dfrac{1}{2};\dfrac{1}{2}\right]$ and since $f(0)=\arccos 0+\arcsin 0=\dfrac{\pi}{2}$, we deduce

that $f(x)=\dfrac{\pi}{2}$.

2) The equation is equivalent to :

$arc\sin x+arc\sin 2x=\dfrac{\pi}{2}-arc\sin x+\dfrac{\pi}{2}-arc\sin 2x$.

So $arc\sin x + arc\sin 2x = \frac{\pi}{2}$, therefore $arc\sin 2x = \frac{\pi}{2} - arc\sin x$.

Which gives $\sin(arc\sin 2x) = \sin\left(\frac{\pi}{2} - arc\sin x\right) = \cos(arc\sin x)$.

Therefore, $2x = \sqrt{1-\sin^2(\arcsin x)} = \sqrt{1-x^2}$, consequently $4x^2 = 1 - x^2$.

Then, $5x^2 = 1$ giving $x = \pm\frac{1}{\sqrt{5}}$ both acceptable.

**N° 11.**

1) $f'(x) = 2\cdot\dfrac{\dfrac{1-x^2+2x^2}{(1-x^2)^2}}{1+\dfrac{4x^2}{(1-x^2)^2}} = \dfrac{2\cdot\dfrac{1+x^2}{(1-x^2)^2}}{\dfrac{(1+x^2)^2}{(1-x^2)^2}} = \dfrac{2}{1+x^2}$.

$f(x)$ and $2arc\tan x$ have the same derivative, then $f(x) = 2arc\tan x + k$ for all $x \in ]-1;1[$ and since $f(0) = 2arc\tan 0 + k$, we get $0 = 0 + k$, therefore $k = 0$ and consequently $f(x) = 2arc\tan x$.

2) $f(x) = \frac{\pi}{2} - 2arc\tan\frac{1}{2}$ gives $2arc\tan x = \frac{\pi}{2} - 2arc\tan\frac{1}{2}$.

Therefore $arc\tan x = \frac{\pi}{4} - arc\tan\frac{1}{2}$ which gives

$\tan(arc\tan x) = \tan\left(\frac{\pi}{4} - arc\tan\frac{1}{2}\right)$,

hence : $x = \dfrac{\tan\frac{\pi}{4} - \frac{1}{2}}{1 + \tan\frac{\pi}{4}\cdot\frac{1}{2}} = \dfrac{\frac{1}{2}}{\frac{3}{2}} = \frac{1}{3}$.

**N° 12.**

1) Noticing that $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty} \frac{-x^2}{x^2} = -1$.

$f'(x) = \dfrac{-2x(1+x^2) - 2x(1-x^2)}{(1+x^2)^2} = \dfrac{-4x}{(1+x^2)^2}$.

$f'(x) \geq 0$ for $x \leq 0$, then the table of variations of $f$ is :

| $x$ | $-\infty$ | | $0$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f'(x)$ | | $+$ | $0$ | $-$ | |
| $f(x)$ | $-1$ | $\nearrow$ | $1$ | $\searrow$ | $-1$ |

From the table of variations, we note that $-1 < f(x) \leq 1$.

2) $g$ is defined when $-1 \leq \dfrac{1-x^2}{1+x^2} \leq 1$ which is true when $-1 \leq f(x) \leq 1$ and since this double inequality is verified for all real numbers $x$ then the domain of definition of $g$ is $IR$ .

3) $$g'(x) = \frac{-f'(x)}{\sqrt{1-[f(x)]^2}} = \frac{\dfrac{4x}{(1+x^2)^2}}{\sqrt{1-\dfrac{(1-x^2)^2}{(1+x^2)^2}}} = \frac{\dfrac{4x}{(1+x^2)^2}}{\sqrt{\dfrac{4x^2}{(1+x^2)^2}}}$$

$$= \frac{\dfrac{4x}{(1+x^2)^2}}{\dfrac{2|x|}{1+x^2}} = \frac{\dfrac{4x}{(1+x^2)^2}}{\dfrac{2x}{1+x^2}} = \frac{2}{1+x^2} \quad \text{since } x > 0.$$

4) $g(x)$ has the same derivative as $2\arctan x$ hence $g(x) = 2\arctan x + k$.
But $x \in ]0;+\infty[$ and since $g(1) = 2\arctan 1 + k$ then $\arccos 0 = 2\arctan 1 + k$ , so $\dfrac{\pi}{2} = 2 \times \dfrac{\pi}{4} + k$, which gives $k = 0$, and consequently $g(x) = 2\arctan x$.

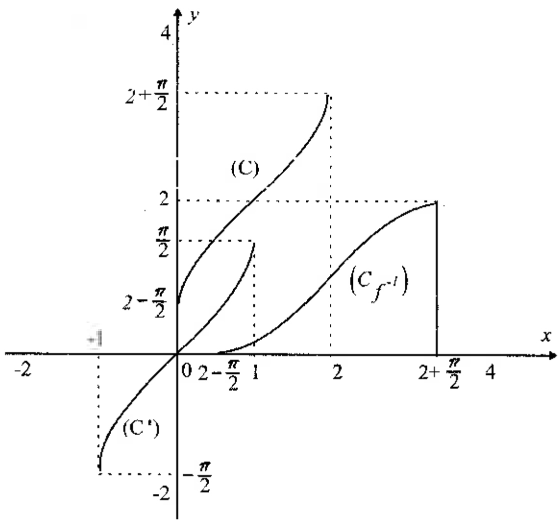
**N° 13.**

1) $f'(x) = \dfrac{1}{\sqrt{1-(x-1)^2}} > 0$ for $x \in ]0;2[$,

Remark that $\lim\limits_{x \to 0} f'(x) = +\infty$ and $\lim\limits_{x \to 2} f'(x) = +\infty$

Hence, at the points of abscissas $0$ and $2$ the tangents to $(C)$ are vertical .

| $x$ | $0$ | | $2$ |
|---|---|---|---|
| $f'(x)$ | ‖ | $+$ | ‖ |
| $f(x)$ | $2-\frac{\pi}{2}$ | $\nearrow$ | $2+\frac{\pi}{2}$ |




2) $(C')$ is the curve representative of the function $g$ defined by $g(x) = arc\sin x$ ,

The translation of vector $\vec{v}$ is defined analytically by $\begin{cases} x' = x - 1 \\ y' = y - 2 \end{cases}$

which gives $\begin{cases} x = x' + 1 \\ y = y' + 2 \end{cases}$.

Replacing $x$ and $y$ by their values in the equation of $(C)$ $y = 2 + arc\sin(x - 1)$, we get $y' + 2 = 2 + arc\sin(x' + 1 - 1)$ , therefore $y' = arc\sin x'$ that is the equation of $(C')$ , hence $(C')$ can be deduced from $(C)$ by $t_{\vec{v}}$ where $\vec{v}(-1;-2)$ .

3) $f$ is continuous and strictly increasing over $[0;2]$ then it admits an inverse function $f^{-1}$.

$D_{f^{-1}} = \left[2-\frac{\pi}{2};2+\frac{\pi}{2}\right]$.

$y = 2 + arc\sin(x-1)$ gives $y-2 = arc\sin(x-1)$, then

$x-1 = \sin(y-2)$ and consequently $x = 1+\sin(y-2)$, hence

$f^{-1}(x) = 1+\sin(x-2)$.

The representative curve of $f^{-1}$ is symmetric to that of $f$ with respect to the first bisector.

**N° 14.**

1) $f(1) = arc\tan(1) = \frac{\pi}{4}$ , $f(3) = arc\tan(3) = \frac{\pi}{3}$.

2) $arc\tan\sqrt{x} = \alpha$ is equivalent to $\tan\alpha = \sqrt{x}$ with $\alpha \in \left]0;\frac{\pi}{2}\right[$.

$arc\tan\frac{1}{\sqrt{x}} = \beta$ is equivalent to $\tan\beta = \frac{1}{\sqrt{x}}$ with $\beta \in \left]0;\frac{\pi}{2}\right[$.

We have to prove that $\alpha+\beta = \frac{\pi}{2}$ or $\alpha = \frac{\pi}{2} - \beta$.

But $\tan\left(\frac{\pi}{2}-\beta\right) = \cot\beta = \frac{1}{\tan\beta} = \sqrt{x} = \tan\alpha$.

Therefore, $\alpha = \frac{\pi}{2} - \beta + k\pi$, which can be written as

$\alpha + \beta - \frac{\pi}{2} = k\pi$, $k \in Z$.

$0 < \alpha < \frac{\pi}{2}$ and $0 < \beta < \frac{\pi}{2}$, then $-\frac{\pi}{2} < \alpha + \beta - \frac{\pi}{2} < \frac{\pi}{2}$,

so $-\frac{\pi}{2} < k\pi < \frac{\pi}{2}$, which gives $-\frac{1}{2} < k < \frac{1}{2}$ and consequently $k = 0$,

Hence, $arc\tan\sqrt{x} + arc\tan\frac{1}{\sqrt{x}} = \frac{\pi}{2}$.

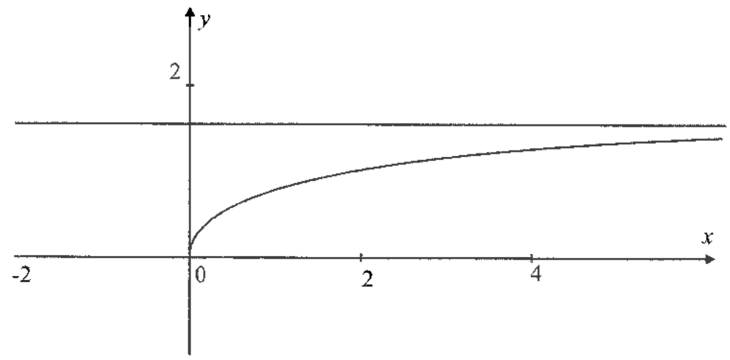
3) $f'(x)=\dfrac{\frac{1}{2\sqrt{x}}}{1+\left(\sqrt{x}\right)^2}=\dfrac{1}{2\sqrt{x}(1+x)}$ then $f'(x)>0$ over $]0;+\infty[$, the table of variations of $f$ is as follows:

| $x$ | $0$ | | $+\infty$ |
|---|---|---|---|
| $f'(x)$ | ‖ | $+$ | |
| $f(x)$ | $0$ | $\nearrow$ | $\frac{\pi}{2}$ |

$\lim\limits_{x\to+\infty} f(x)=\dfrac{\pi}{2}$ since $arc\tan\sqrt{x}\in\left[0;\dfrac{\pi}{2}\right[$ and $\lim\limits_{x\to 0} f'(x)=+\infty$

hence the tangent at the point $O(0,0)$ to the graph of $f$ is vertical.

4)




N° 15.

1) $\lim\limits_{x\to-\infty} f(x)=arc\tan 1=\dfrac{\pi}{4}$.

$\lim\limits_{x\to+\infty} f(x)=arc\tan 1=\dfrac{\pi}{4}$.

The straight line $(d)$ of equation $y=\dfrac{\pi}{4}$ is an asymptote to $(C)$.

$\lim\limits_{\substack{x\to0\\x<0}} f(x)=arc\tan(-\infty)=-\dfrac{\pi}{2}$.

$\lim\limits_{\substack{x \to 0 \\ x>0}} f(x) = arc\tan(+\infty) = \dfrac{\pi}{2}$ .

2) $f'(x) = \dfrac{\dfrac{-2}{x^2}}{1+\left(1+\dfrac{2}{x}\right)^2} < 0$ for all real numbers $x \in IR - \{0\}$,

therefore the table of variations of $f$ is :

| $x$ | $-\infty$ | $0$ | $+\infty$ |
|---|---|---|---|
| $f'(x)$ | $-$ | ‖ | $-$ |
| $f(x)$ | $\frac{\pi}{4} \searrow \frac{-\pi}{2}$ | ‖ | $\frac{\pi}{2} \searrow \frac{\pi}{4}$ |

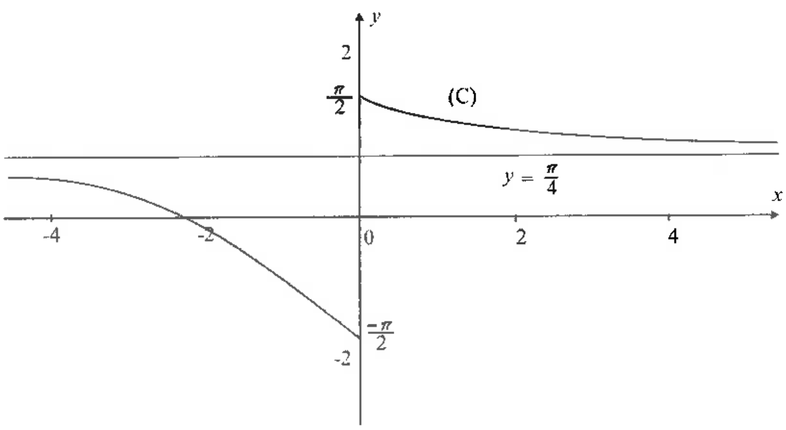
3) $(C)$ cuts $x'x$ when $f(x) = 0$, which gives $arctan\left(1+\dfrac{2}{x}\right) = 0$ ,

$1+\dfrac{2}{x} = 0$ , then $\dfrac{2}{x} = -1$ and consequently $x = -2$ hence $A(-2;0)$.

An equation of the tangent at $A$ to $(C)$ is :

$y - 0 = \dfrac{-1}{2}(x+2)$ so $y = -\dfrac{1}{2}x - 1$ .

4)



N° 16.

1) a- $arc\tan u=\alpha$ is equivalent to $\tan\alpha=u>0$ with $\alpha\in\left]0;\frac{\pi}{2}\right[$.

$\tan\alpha=u$ then $\cot\left(\frac{\pi}{2}-\alpha\right)=u$.

But $0<\alpha<\frac{\pi}{2}$ then $-\frac{\pi}{2}<-\alpha<0$ therefore $0<\frac{\pi}{2}-\alpha<\frac{\pi}{2}$.

Since $u>0$ then $\frac{\pi}{2}-\alpha=arc\cot u$ ,and we get

$\frac{\pi}{2}=arc\tan u+arc\cot u$.

b- $arc\cot u=\frac{\pi}{2}-arc\tan u$ , then

$(arc\cot u)'=\left(\frac{\pi}{2}-arc\tan u\right)'=-\frac{u'}{1+u^2}$.

2) $f(x)=arc\cot(2x-1)-arc\cot(2x+1)$

a- $f'(x)=\frac{-2}{1+(2x-1)^2}+\frac{2}{1+(2x+1)^2}$

$=\frac{-2}{4x^2+2-4x}+\frac{2}{4x^2+4x+2}=\frac{-4x}{1+4x^4}$ .

b- Since $(arc\cot)'(2x^2)=\frac{-4x}{1+4x^4}$ we deduce that $f(x)$ and $arc\cot(2x^2)$ have the same derivative .

Hence, $f(x)=arc\cot(2x^2)+k$.

$f(0)=arc\cot(0)+k$ , then

$arc\cot(-1)-arc\cot(1)=arc\cot 0+k$

$\frac{3\pi}{4}-\frac{\pi}{4}=\frac{\pi}{2}+k$ which gives $k=0$ and consequently

$f(x)=arc\cot(2x^2)$ .

3) $arc\,cot(2x-1)+arc\,cot(2x+1)=arc\,cot(2x^2)\ \forall x\in IR$

For $x=1$ , $arc\,cot\,1-arc\,cot\,3=arc\,cot(2\times 1^2)$

For $x=2$ , $arc\,cot\,3-arc\,cot\,5=arc\,cot(2\times 2^2)$

For $x=3$ , $arc\,cot\,5-arc\,cot\,7=arc\,cot(2\times 3^2)$

$\vdots$

$\vdots$

For $x=n$ $arc\,cot(2n-1)-arc\,cot(2n+1)=arc\,cot(2\times n^2)$

Adding side to side we get $S_n=arc\,cot\,1-arc\,cot(2n+1)$.

$S_n=\frac{\pi}{4}-arc\cot(2n+1)$ so

$\lim\limits_{n\to+\infty}S_n=\lim\limits_{n\to 0}\left[\frac{\pi}{4}-arc\cot(2n+1)\right]=\frac{\pi}{4}-0=\frac{\pi}{4}$ .

$\boxed{N^\circ 17.}$

$arc\cos\alpha=u$ is equivalent to $u\in[0;\pi]$ and $\cos u=\alpha$ .

$arc\cos\beta=v$ is equivalent to $v\in[0;\pi]$ and $\cos v=\beta$ .

$arc\cos\gamma=w$ is equivalent to $w\in[0;\pi]$ and $\cos w=\gamma$ .

If $u+v=\pi-w$ then :

$\cos(u+v)=\cos(\pi-w)$ ,

$\cos u\cos v-\sin u\sin v=-\cos w$ , which gives:

$\alpha\beta-\sqrt{1-\alpha^2}\cdot\sqrt{1-\beta^2}=-\gamma$ ,which is equivalent to :

$\alpha\beta+\gamma=\sqrt{1-\alpha^2}\cdot\sqrt{1-\beta^2}$ , squaring both sides we get :

$\alpha^2\beta^2+\gamma^2+2\alpha\beta\gamma=1-\beta^2-\alpha^2+\alpha^2\beta^2$, consequently :

$\alpha^2+\beta^2+\gamma^2=1-2\alpha\beta\gamma$ .

$\boxed{N^\circ 18.}$

$2arc\tan x+arc\tan 3x=arc\cot x+2arc\cot 3x$

We know that $\arctan\alpha+arc\cot\alpha=\frac{\pi}{2}$, then the given equation is equivalent to :

$2arc\tan x+arc\tan 3x=\frac{\pi}{2}-arc\tan x+2\left(\frac{\pi}{2}-arc\tan 3x\right)$.

Which gives $3\arctan 3x + 3\arctan x = \frac{3\pi}{2}$ , then

$\arctan x = \frac{\pi}{2} - \arctan 3x$ , so :

$\tan(\arctan x) = \tan\left(\frac{\pi}{2} - \arctan 3x\right) = \cot(\arctan 3x)$.

We get $x = \frac{1}{3x}$, consequently $3x^2 = 1$ therefore $x = \frac{\sqrt{3}}{3}$ or $x = -\frac{\sqrt{3}}{3}$.

N° 19.

1) The function $x \longrightarrow 2x\sqrt{1-x^2}$ is differentiable over ]-1 ; 1[ and the function $x \longrightarrow \arcsin x$ is differentiable over ]-1 ; 1[ .
   Therefore the function $x \longrightarrow \arcsin\left(2x\sqrt{1-x^2}\right)$ is differentiable for $x \in ]-1\,;\,1[$ and $2x\sqrt{1-x^2} \neq \pm 1$.

   Which is true for $x \neq \pm\frac{\sqrt{2}}{2}$.

$$f'(x) = \frac{2\sqrt{1-x^2} + \dfrac{-4x^2}{2\sqrt{1-x^2}}}{\sqrt{1-4x^2(1-x^2)}} = \frac{2(1-x^2)-2x^2}{\sqrt{1-x^2}\sqrt{1-4x^2(1-x^2)}}.$$

$$= \frac{2(1-2x^2)}{\sqrt{1-x^2}\sqrt{(1-2x^2)^2}} = \frac{2(1-2x^2)}{\sqrt{1-x^2}(-1+2x^2)} = \frac{-2}{\sqrt{1-x^2}}$$

***N.B***

$1-2x^2 < 0$ in the interval $\left[\frac{\sqrt{2}}{2};1\right]$ , that's why

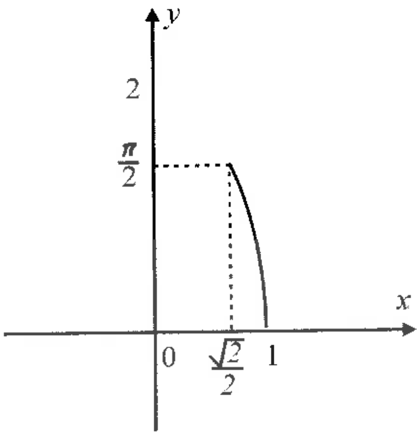
$\sqrt{(1-2x^2)^2} = -1+2x^2$.

2) Let $g(x) = -2\arcsin x$, we have :

   $g'(x) = -\frac{2}{\sqrt{1-x^2}} = f'(x)$, so $f(x) = g(x) + k$ , but $x \in \left[\frac{\sqrt{2}}{2};1\right]$,
   then we give $x$ the value 1 and we get $k = \pi$.
   hence $f(x) = \pi - 2\arcsin x$.

3) The table of variations of $f$ is :

| $x$ | $\frac{\sqrt{2}}{2}$ | | $1$ |
|---|---|---|---|
| $f'(x)$ | | $-$ | |
| $f(x)$ | $\frac{\pi}{2}$ | $\searrow$ | $0$ |




N°20.

1) $arc\tan\left(\dfrac{1}{2x-1}\right)=\alpha$ is equivalent to $\tan\alpha=\dfrac{1}{2x-1}$ with $\alpha\in\left]0;\dfrac{\pi}{2}\right[$.

$arc\tan\left(\dfrac{1}{2x+1}\right)=\beta$ is equivalent to $\tan\beta=\dfrac{1}{2x+1}$ with $\beta\in\left]0;\dfrac{\pi}{2}\right[$.

$z=\alpha-\beta$ , we have :

$$\tan z=\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\cdot\tan\beta}$$

$$\tan z=\frac{\dfrac{1}{2x-1}-\dfrac{1}{2x+1}}{1+\dfrac{1}{(2x-1)(2x+1)}}=\frac{\dfrac{2x+1-2x+1}{(2x-1)(2x+1)}}{\dfrac{4x^2-1+1}{(2x-1)(2x+1)}}$$

$$=\frac{2}{4x^2}=\frac{1}{2x^2}.$$

But $0<\dfrac{1}{2x+1}<\dfrac{1}{2x-1}$ then $0<\beta<\alpha$ and consequently $\alpha-\beta\in\left[0;\dfrac{\pi}{2}\right]$ which gives that $z\in\left[0;\dfrac{\pi}{2}\right]$, consequently $z=\alpha-\beta=\arctan\dfrac{1}{2x^2}$.

Therefore, $\arctan\left(\dfrac{1}{2x-1}\right)-\arctan\left(\dfrac{1}{2x+1}\right)=\arctan\dfrac{1}{2x^2}$ for $x\geq 1$.

2) For:

$x=1$ ; $\arctan 1-\arctan\dfrac{1}{3}=\arctan\dfrac{1}{2\times 1^2}$

$x=2$ ; $\arctan\dfrac{1}{3}-\arctan\dfrac{1}{5}=\arctan\dfrac{1}{2\times 2^2}$

$x=3$ ; $\arctan\dfrac{1}{5}-\arctan\dfrac{1}{7}=\arctan\dfrac{1}{2\times 3^2}$

$\vdots$

$x=n$ ; $\arctan\dfrac{1}{2n-1}-\arctan\dfrac{1}{2n+1}=\arctan\dfrac{1}{2n^2}$

Adding side to side , we get :

$S_n=\arctan 1-\arctan\dfrac{1}{2n+1}$ , so $S_n=\dfrac{\pi}{4}-\arctan\dfrac{1}{2n+1}$

$\lim\limits_{n\to+\infty}S_n=\dfrac{\pi}{4}-\arctan 0=\dfrac{\pi}{4}$.

N°21.

1) $f$ is defined if $\begin{cases}\dfrac{1-x}{1+x}\geq 0\\ -1\leq x\leq 1\end{cases}$ which gives $-1<x\leq 1$.

2) $$f'(x)=\frac{\dfrac{-1-x-1+x}{2(1+x)^2\sqrt{\dfrac{1-x}{1+x}}}}{\left[1+\dfrac{1-x}{1+x}\right]}+\frac{1}{2}\times\frac{1}{\sqrt{1-x^2}}$$

$$=\frac{-1}{(1+x)^2\sqrt{\dfrac{1-x}{1+x}}\cdot\dfrac{2}{1+x}}+\frac{1}{2\sqrt{1-x^2}}$$

$$=\frac{-1}{\sqrt{1-x^2}\cdot 2}+\frac{1}{2\sqrt{1-x^2}}=0\ .$$

Hence, $f(x)$ is a constant .

$f(0)=\arctan 1+\dfrac{1}{2}\arcsin 0=\dfrac{\pi}{4}$ then $f(x)=\dfrac{\pi}{4}$ .

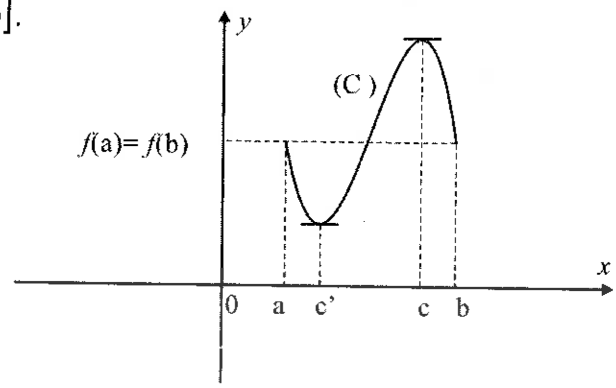
# CHAPTER 3

# Differential Calculus

**Chapter Review :**

- Rolle's Theorem.
  - $f$ is a function defined over an interval $[a;b]$ with $a<b$, if :
    $f$ is continuous over $[a;b]$.
    $f$ is differentiable over $]a;b[$.
    $f(a)=f(b)$.
    Then, there exists a real number $c \in ]a;b[$ such that $f'(c)=0$.

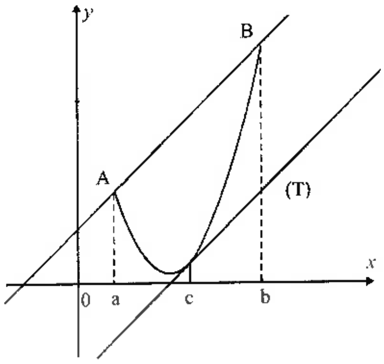
  - Graphical Interpretation:
    If a function satisfies the conditions of Rolle's theorem then its representative curve admits at least an extremum in the interval $[a;b]$.




- Mean Value Theorem.
  - $f$ is a function defined on the interval $[a;b]$ with $a<b$, if :
    $f$ is continuous over $[a;b]$.
    $f$ is differentiable over $]a;b[$.
    Then, there exists at least a value $c \in ]a;b[$ such that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$ or $f(b) - f(a) = (b - a)f'(c)$.

* Graphical Interpretation :
  If a function $f$ satisfies the conditions of the mean value Theorem then there exists at least one point , of abscissa $c$ belonging to the representative curve of $f$ such that the tangent to the curve at this point is parallel to the straight line joining the two points: $A(a; f(a))$ and $B(b; f(b))$.




* Inequality of the Mean Value Theorem.
  If $f$ is a function that satisfies the conditions of the mean value theorem over $[a;b]$ with $a < b$.
  If $m \le f'(x) \le M$ then: $m(b-a) \le f(b) - f(a) \le M(b-a)$.
  If $|f'(x)| \le k$ then $|f(b) - f(a)| \le k|b - a|$.

- Approximate Value of a Function .
  The inequality $|f(x) - f(a)| \le k|x - a|$ can be interpreted as follows:
  $f(a)$ is an approximate value of $f(x)$ with an error less than $k|x - a|$.

***Ex :*** $f(x) = \sqrt{x}$ , applying the mean value theorem to $f$ over [40000; 40001] , we get:

$\sqrt{40001} - \sqrt{40000} = \dfrac{1}{2\sqrt{c}}$, but $40000 < c < 40001$,

then:

$200 < \sqrt{c} < \sqrt{40001}$ gives $\dfrac{1}{\sqrt{40001}} < \dfrac{1}{\sqrt{c}} < \dfrac{1}{200}$.

Therefore, $\left|\sqrt{40001} - \sqrt{40000}\right| < \dfrac{1}{2 \times 200}$.

So $\left|\sqrt{40001} - 200\right| < \dfrac{1}{400}$.

Hence, 200 is an approximate value of $\sqrt{40001}$ with an error less than $\dfrac{1}{400} = 0.0025$.

- Extension by Continuity .
  Let $f$ be a function defined over $I$ except at a point $a$ of $I$.
  If $f$ is continuous elsewhere on $I$ and if $\lim\limits_{x \to a} f(x) = \ell$ where $\ell$ is a real number, then $f$ can be extended by continuity at $a$.
  The function that extends $f$ by continuity is defined as
  $$\varphi(x) = \begin{cases} f(x) & \text{for } x \neq a \\ \ell & \text{for } x = a \end{cases}$$

- L'Hopital's Rule :
  $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$, if $f(a) = g(a) = 0$ and $g'(a) \neq 0$ for all $x \neq a$.
  ***N.B*** :
  * We may apply L'Hopital's rule several times .
  * We may apply L'Hopital's rule in the following cases:
    $\lim\limits_{x \to a} f(x) = \infty$ and $\lim\limits_{x \to a} g(x) = \infty$ with $a$ finite or infinite .
    $\lim\limits_{x \to a} f(x) = 0$ and $\lim\limits_{x \to a} g(x) = 0$ with $a$ finite or infinite .

# Solved Problems

**N° 1.**

Find an extension by continuity for each of the following functions at the given point :

1) $f(x)=\dfrac{\sqrt{x+1}-2\sqrt{x-2}}{x-3}$ at the point $x=3$.

2) $f(x)=x^2\sin\dfrac{1}{x}$ at the point $x=0$.

3) $f(x)=\dfrac{\sqrt{x+4}-2}{x^2}$ at the point $x=0$.

**N° 2.**

Consider the function $f$ defined over $IR$ by $f(x)=x^3-x$.
Apply Rolle's theorem to $f$ over $[-1;+1]$ and calculate the value of $c$ then interpret the result graphically.

**N° 3.**

Let $f$ be the function defined over $[-\sqrt{3};2]$ by

$$f(x)=\begin{cases} x^2 & \text{if } -\sqrt{3}\le x\le 1 \\ 2x-1 & \text{if } 1<x\le 2 \end{cases}$$

Show that the conditions of Rolle's theorem are satisfied by $f$ over $[-\sqrt{3};2]$ and then calculate the constant $c$ of this formula .

**N° 4.**

Let $f$ be the function defined over $IR$ by $f(x)=x^3-|x|$.
Check whether the conditions of Rolle's theorem are satisfied by $f$ over $[-1;+1]$.

**N° 5.**

$f$ is the function defined over $IR$ by $f(x)=1-\sqrt[3]{x}$.
Check whether the conditions of Rolle's theorem are satisfied by

$f$ over $[-1;+1]$.

N°6.

1) Using the inequality of the Mean Value Theorem, show that $|\sin x| \leq |x|$.

2) Let $g$ be the function defined over $[1;+\infty[$ by $g(x)=\sqrt{x^2+8}$.

   a- Verify that $g'(x) \geq \dfrac{1}{3}$.

   b- Show that $\dfrac{1}{3}(x+8) \leq \sqrt{x^2+8}$.

N°7.

Let $f$ be the function defined over $[0;+\infty[$ by $f(x)=\arctan x$.

1) Show that: $0 < f'(x) \leq 1$.

2) Deduce that $0 \leq f(x) \leq x$.

N°8.

1) $h$ is the function defined over $[0;+\infty[$ by $h(x)=\dfrac{-1}{(x+1)^2}$.

   Study the variations of $h$ and show that $|h(x)| \leq 1$.

2) Using the Mean Value Theorem inequality, prove that :

   $1-x \leq \dfrac{1}{x+1} \leq 1+x$ for $x \in [0;+\infty[$.

N°9.

Let $f$ be the function defined over $IR$ by:

$f(x)=x(x-1)(x+1)+3$.

Without calculating $f'(x)$, show that the equation $f'(x)=0$ admits two distinct real roots .

N° 10.

Let $f$ be the function defined over $[-1;+1]$ by $f(x)=\arcsin x$.

Applying the Mean Value Theorem to $f$ over $[0;a]$ with $0<a<1$, prove that $\dfrac{\arcsin a}{a} < \dfrac{1}{\sqrt{1-a^2}}$.

**N° 11.**

Let $f$ be the function defined over $\left[0;\frac{1}{2}\right]$ by $f(x)=\sqrt{x+1}$.

1) Applying the Mean Value Theorem to $f$ over $[0;x]$,

   show that : $1+\frac{x}{\sqrt{6}} \leq f(x) \leq 1+\frac{x}{2}$.

2) Deduce a bounding of $\sqrt{1.1}$ to the nearest $10^{-2}$.

**N° 12.**

Let $f$ be the function defined over $]-1;+\infty[$ by $f(x)=\frac{x^2+2x+2}{x+1}$.

1) Determine $a$, $b$ and $c$ for which $f(x)$ can be written in the form:

   $f(x)=ax+b+\frac{c}{x+1}$.

2) Apply the Mean Value Theorem to $f$ over $[x;x+h]$ with $h>0$ and find the value of the constant $c$ of this formula.

3) Let $c=x+\varphi(h)$, determine $\lim\limits_{h\to 0}\frac{\varphi(h)}{h}$.

**N° 13.**

1) Applying the Mean Value Theorem over $[n;n+1]$, $n\in IN$ to the function $f$ defined over $IR$ by $f(x)=\arctan x$, prove that

   $\frac{1}{1+(n+1)^2} < \arctan(n+1)-\arctan n < \frac{1}{1+n^2}$.

2) Deduce that $\frac{\pi}{4}+\frac{1}{5} < \arctan 2 < \frac{\pi}{4}+\frac{1}{2}$.

3) Let $E(n)=(1+n^2)[\arctan(n+1)-\arctan(n)]$.
   Determine $\lim\limits_{n\to+\infty} E(n)$.

**N° 14.**

Let $f$ be the function defined over $[0;+\infty[$ by $f(x)=1-\frac{x^2}{2}-\cos x$.

1) Study the variations of $f'$ and draw its table.

2) Deduce that $1-\frac{x^2}{2} \leq \cos x$.
3) Let $g$ be the function defined over $[0;+\infty[$ by
   $g(x)=x-\frac{x^3}{6}-\sin x$.
   a- Study the variations of $g$ and show that $x-\frac{x^3}{6} \leq \sin x$.
   b- Prove that $\sin x \leq x-\frac{x^3}{3!}+\frac{x^5}{5!}$.
   c- Deduce that $x-\frac{x^3}{6}$ is an approximation of $\sin x$ with an error inferior to $\frac{x^5}{5!}$.
   d- Show that $\frac{\pi}{9}-\frac{1}{6}\left(\frac{\pi}{9}\right)^3$ is an approximate value of $\sin\frac{\pi}{9}$ with an error to be determined.
4) Prove that $\cos x \leq 1-\frac{x^2}{2!}+\frac{x^4}{4!}$ for $x \geq 0$.

### N° 15.

1) Given $f(x)=\sin(ax)$, determine $f^{(n)}(x)$.
2) Deduce the derivative of order $n$ of each of the following functions :
   $g(x)=\sin^2(2x)$.
   $h(x)=\cos^2 x$.
   $\varphi(x)=\sin(4x)\times\cos(3x)$.
   $k(x)=\sin(4x)\times\sin(3x)$.

**N° 16.**

1) Given $f(x)=\dfrac{1}{ax+b}$, show that $f^{(n)}(x)=\dfrac{(-1)^n \times a^n \times n!}{(ax+b)^{n+1}}$.

2) a- Determine the $n^{th}$ derivative of $y=\dfrac{1}{2x-1}$ and deduce the $n^{th}$ derivative of $z=\dfrac{1}{(2x-1)^2}$.

b- Let $g$ be the function defined over $IR-\{1;3\}$ by $g(x)=\dfrac{1}{x^2-4x+3}$.

Determine $m$ and $n$ so that we can write $g(x)$ in the form $g(x)=\dfrac{m}{x-1}+\dfrac{n}{x-3}$ and deduce $g^{(n)}(x)$.

**N° 17.**

Let $f$ be the function defined over $IR$ by $f(x)=\arctan x$.
Prove, by mathematical induction, that the $n^{th}$ derivative of $f$ is given by $f^{(n)}(x)=\dfrac{p_{n-1}(x)}{(x^2+1)^n}$ where $p_{n-1}(x)$ is a polynomial of degree $n-1$.

**N° 18.**

Let $f$ be the function defined over $IR$ by $f(x)=x\cos x$.
Show, by mathematical induction, that the $n^{th}$ derivative of $f$ is given by $f^{(n)}(x)=n\cos\left(x+(n-1)\dfrac{\pi}{2}\right)+x\cos\left(x+n\dfrac{\pi}{2}\right)$.

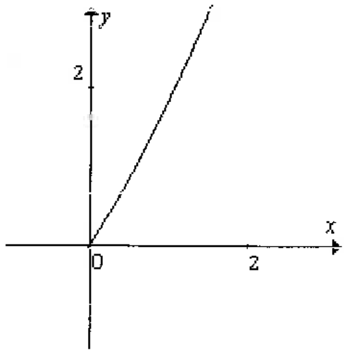
**N° 19.**

Let $f$ be the function defined over $[0;+\infty[$ by :

$f(x)=\sqrt{x+1}-1-\dfrac{x}{2}+\dfrac{x^2}{8}$.

***Part A:***

The curve below is the graphical representation of the function defined over $[0;+\infty[$ by $h(x)=(1+x)\sqrt{x+1}-1$.




1) Study the variations of the function $f'$, derivative of $f$ and draw its table of variations.
2) Deduce that :
   a- $f$ is strictly increasing over $[0;+\infty[$.

   b- $1+\frac{x}{2}-\frac{x^2}{8}\leq\sqrt{x+1}$.

***Part B:***

Let $g$ be the function defined over $[0;+\infty[$

by $g(x)=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\sqrt{x+1}$.

1) Determine the sign of $g^{(3)}(x)$, the derivative of order 3 of $g(x)$.
2) Study the variations of $g$ over $[0;+\infty[$.
3) Deduce that $\sqrt{x+1}\leq 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$.
4) Find a bounding of $\sqrt{x+1}$ and deduce an approximate value of $\sqrt{x+1}$ around $\frac{x^3}{16}$.

# Solution of Problems

N° 1.

1) $\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \dfrac{\sqrt{x+1}-2\sqrt{x-2}}{x-3} = \dfrac{0}{0}$, indeterminate form ,

applying L'Hopital's rule , we get:

$\lim\limits_{x\to 3} f(x) = \lim\limits_{x\to 3} \dfrac{\dfrac{1}{2\sqrt{x+1}} - \dfrac{1}{\sqrt{x-2}}}{1} = \dfrac{-3}{4}$, then $f$ can be extended by continuity at the point $x = 3$.

The function that extends $f$ by continuity is :

$$\psi(x) = \begin{cases} \dfrac{\sqrt{x+1}-2\sqrt{x-2}}{x-3} & \textit{for } x \neq 3 \\ -\dfrac{3}{4} & \textit{for } x = 3 \end{cases}$$

2) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0 \times \sin\dfrac{1}{0}$, indeterminate form.

We know that $-1 \leq \sin\dfrac{1}{x} \leq 1$, then $-x^2 \leq x^2 \sin\dfrac{1}{x} \leq x^2$ we deduce that $\lim\limits_{x\to 0} x^2 \sin\dfrac{1}{x} = 0$ $x^2$ since $\lim\limits_{x\to 0}(-x^2) = \lim\limits_{x\to 0} x^2 = 0$ then $f$ is extended by continuity at $x = 0$.

The function extension by continuity of $f$ is :

$$\psi(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \textit{for } x \neq 0 \\ 0 & \textit{for } x = 0 \end{cases}$$

3) $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \dfrac{\sqrt{x+4}-2}{x^2} = \dfrac{0}{0}$, indeterminate form.

Applying L'Hopital's rule, we get:

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\dfrac{1}{2\sqrt{x+4}}}{2x} = \lim_{x\to 0} \frac{1}{4x\sqrt{x+4}} = \pm\infty,$$

then $f$ cannot be extended by continuity at $x=0$.

N°2.

$f$ is continuous over $[-1;+1]$ and differentiable over $]-1;+1[$ since it is a polynomial function and $f(-1)=f(1)=0$ then we can apply Rolle's theorem to $f$, hence there exists at least a value $c \in ]-1;+1[$ such that $f'(c)=0$.

$f'(x)=3x^2-1$, then $f'(c)=3c^2-1=0$, which gives $c=\frac{\sqrt{3}}{3}$ or $c=-\frac{\sqrt{3}}{3}$.

So, at the points of abscissas $\frac{\sqrt{3}}{3}$ and $-\frac{\sqrt{3}}{3}$ of the curve representative of $f$, there is a tangent parallel to the axis $x'x$.

N°3.

The function $f$ is continuous over $[-\sqrt{3};1]$, since it is a polynomial function, similarly it is continuous over $]1;2]$.
Studying its continuity at the point of abscissa 1.
$f(1)=(1)^2=1$, then $f$ is defined at 1.
$\lim\limits_{\substack{x\to 1\\ x>1}} f(x)=\lim\limits_{\substack{x\to 1\\ x>1}}(2x-1)=1=f(1)$.
Hence $f$ is continuous to the right of 1 and consequently over $[-\sqrt{3};2]$.
Similarly, $f$ is differentiable over $[-\sqrt{3};1]$ and over $]1;2]$, studying its differentiability at 1, we get:

$$\lim_{\substack{x\to 1\\ x>1}}\frac{f(x)-f(1)}{x-1}=\lim_{\substack{x\to 1\\ x>1}}\frac{2x-1-1}{x-1}=\lim_{\substack{x\to 1\\ x>1}}\frac{2(x-1)}{x-1}=2=f_r'(1).$$

$$\lim_{\substack{x\to 1\\ x<1}}\frac{f(x)-f(1)}{x-1}=\lim_{\substack{x\to 1\\ x<1}}\frac{x^2-1}{x-1}=\lim_{\substack{x\to 1\\ x<1}}\frac{(x-1)(x+1)}{x-1}=\lim_{\substack{x\to 1\\ x<1}}(x+1)=2=f_l'(1)$$

Since $f_r'(1)=f_l'(1)$ then $f$ is differentiable at 1 then it is differentiable at $[-\sqrt{3};2]$.

$f(-\sqrt{3})=(-\sqrt{3})^2=3$, $f(2)=2\times 2-1=3$.

Hence, the conditions of Rolle's theorem are satisfied by $f$ over $[-\sqrt{3};2]$, consequently there exists at least a value $c\in]-\sqrt{3};+2[$ such that $f'(c)=0$ .

$f'(x)=2x$, therefore $f'(c)=2c=0$, which gives $c=0$.

N° 4.

$$f(x)=\begin{cases} x^3+x & \text{if } x\in[-1;0] \\ x^3-x & \text{if } x\in]0;1] \end{cases}.$$

Studying the differentiability of $f$ at 0 :

$$\lim_{\substack{x\to 0\\ x>0}}\frac{f(x)-f(0)}{x}=\lim_{\substack{x\to 0\\ x>0}}\frac{x^3-x}{x}=\lim_{\substack{x\to 0\\ x>0}}\frac{x(x^2-1)}{x}=\lim_{\substack{x\to 0\\ x>0}}(x^2-1)=-1=f_r'(0)$$

$$\lim_{\substack{x\to 0\\ x<0}}\frac{f(x)-f(0)}{x}=\lim_{\substack{x\to 0\\ x<0}}\frac{x^3+x}{x}=\lim_{\substack{x\to 0\\ x<0}}\frac{x(x^2+1)}{x}=\lim_{\substack{x\to 0\\ x<0}}(x^2+1)=1=f_r'(0)$$

Since $f_r'(0)\neq f_l'(0)$ then $f$ is not differentiable at 0 and consequently the conditions of Rolle's theorem are not satisfied by $f$ over $[-1;+1]$ .

N° 5.

The function $f$ is continuous over $[-1;+1]$ but it is not differentiable over $]-1;+1[$ ; more precisely it is not differentiable at the point of abscissa 0 , since : $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x}=\lim\limits_{x\to 0}\frac{1-\sqrt[3]{x}-1}{x}=\lim\limits_{x\to 0}\frac{-1}{\sqrt[3]{x^2}}=-\infty$.

Hence $f$ does not satisfy the conditions of the Mean Value Theorem over $[-1;+1]$ .

**N° 6.**

1) The function $f$ defined over $IR$ by $f(x)=\sin x$ is continuous and differentiable of derivative $f'(x)=\cos x$, moreover $|f'(x)|\leq 1$.
Hence, by applying the inequality of the Mean Value Theorem to $f$ over the interval $[0;x]$, we get:
$|\sin x-\sin 0|\leq 1\times|x-0|$ , which gives $|\sin x|\leq|x|$.

2) a- $g'(x)=\dfrac{2x}{2\sqrt{x^2+8}}=\dfrac{x}{\sqrt{x^2+8}}=\dfrac{1}{\sqrt{1+\dfrac{8}{x^2}}}$.

But $x\geq 1$ gives $x^2\geq 1$, therefore $\dfrac{1}{x^2}\leq 1$ consequently $\dfrac{8}{x^2}\leq 8$.

So, we get then $1+\dfrac{8}{x^2}\leq 9$ which gives $\sqrt{1+\dfrac{8}{x^2}}\leq\sqrt{9}$, finally we get $\dfrac{1}{\sqrt{1+\dfrac{8}{x^2}}}\geq\dfrac{1}{3}$ hence $g'(x)\geq\dfrac{1}{3}$.

b- Applying the Mean Value Theorem to $g$ over the interval $[1;x]$ we get :
$g(x)-g(1)=(x-1)g'(c)$, then $\sqrt{x^2+8}-3=(x-1)g'(c)$.
$x\geq 1$ gives $x-1\geq 0$ and since $g'(c)\geq\dfrac{1}{3}$ we get
$(x-1)g'(c)\geq\dfrac{1}{3}(x-1)$ and consequently
$\sqrt{x^2+8}-3\geq\dfrac{1}{3}(x-1)$ which gives $\sqrt{x^2+8}\geq\dfrac{1}{3}(x-1)+3$, so
$\sqrt{x^2+8}\geq\dfrac{x+8}{3}$. Therefore, $\dfrac{1}{3}(x+8)\leq\sqrt{x^2+8}$.

**N° 7.**

1) $f'(x)=\dfrac{1}{1+x^2}$, but $1+x^2\geq 1$ so $0<\dfrac{1}{1+x^2}\leq 1$ and consequently $0<f'(x)\leq 1$.

2) Applying the Mean Value Theorem to $f$ over the interval $[0;x]$, we get :

$|\arctan x - \arctan 0| \leq 1 \times |x - 0|$, but $x \geq 0$ so $\arctan x \geq 0$ ,therefore $|\arctan x| = \arctan x$ and $|x| = x$ hence the inequality $|\arctan x - \arctan 0| \leq 1 \times |x - 0|$ is equivalent to : $0 \leq \arctan x \leq x$ , so we get $0 \leq f(x) \leq x$.

N°8.

1) $h(x) = \dfrac{-1}{(x+1)^2} = -(x+1)^{-2}$, then $h'(x) = 2(x+1)^{-3} = \dfrac{2}{(x+1)^3}$.
But, $x \geq 0$, so $h'(x) > 0$ and $h$ is strictly increasing over $[0;+\infty[$, then: $x \geq 0$ gives $h(x) \geq h(0)$, so $h(x) \geq -1$ and since $\lim\limits_{x \to +\infty} h(x) = 0$ then $h(x) < 0$ over $[0;+\infty[$ therefore $-1 \leq h(x) < 0$ and $-1 \leq h(x) < 0 < 1$, consequently $|h(x)| \leq 1$.

2) Let $f$ be the function defined over $[0;+\infty[$ by $f(x) = \dfrac{1}{x+1}$,
$f'(x) = h(x)$, applying the inequality of the Mean Value Theorem to $f$ over $[0;x]$ we get :
$|f(x) - f(0)| \leq 1 \times |x - 0|$, which gives $-x \leq f(x) - 1 \leq x$.
Therefore, $1 - x \leq f(x) \leq 1 + x$ and consequently
$1 - x \leq \dfrac{1}{x+1} \leq 1 + x$.

N°9.

$f$ is continuous over $[-1;0]$ and differentiable over $]-1;0[$ since it is a polynomial function. Also, $f(-1) = f(0) = 3$ so we may apply Rolle's theorem to $f$ over $[-1;0]$, so there exists at least a value of the constant $c_1 \in ]-1;0[$ such that $f'(c_1) = 0$ .
Similarly, the function $f$ is continuous over $[0;1]$ and differentiable over $]0;1[$ since it is a polynomial function and since $f(0) = f(1) = 3$ then we may apply Rolle's theorem to $f$ over $[0;1]$, hence there exists at least one value of the constant $c_2 \in ]0;1[$ such that $f'(c_2) = 0$ .

Since $c_1$ and $c_2$ are distinct since they belong to two disjoint intervals $]-1;0[$ and $]0;1[$ but $f'(x)$ is a second degree polynomial therefore the equation $f'(x)=0$ admits two distinct real roots.

**N° 10.**

$f$ is continuous over $[0;a]$ and differentiable over $]0;a[$, then we may apply the Mean Value Theorem, consequently there exists at least a value of the constant $c \in ]0;a[$ such that:

$\arcsin a - \arcsin 0 = (a-0)f'(c)$.

But $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$, then $f'(c)=\dfrac{1}{\sqrt{1-c^2}}$ and since $\arcsin 0 = 0$

we get $\arcsin a = \dfrac{a}{\sqrt{1-c^2}}$.

$0<c<a$ which gives $0<c^2<a^2$ et $-a^2<-c^2<0$, then

$1-a^2<1-c^2<1$ and consequently $\sqrt{1-a^2}<\sqrt{1-c^2}<1$, therefore

$1<\dfrac{1}{\sqrt{1-c^2}}<\dfrac{1}{\sqrt{1-a^2}}$ and since $a>0$, we get

$a<\dfrac{a}{\sqrt{1-c^2}}<\dfrac{a}{\sqrt{1-a^2}}$ and consequently $a<\arcsin a<\dfrac{a}{\sqrt{1-a^2}}$.

We finally get $\dfrac{\arcsin a}{a}<\dfrac{1}{\sqrt{1-a^2}}$.

**N° 11**

1) $f'(x)=\dfrac{1}{2\sqrt{x+1}}$, but $0\le x\le\dfrac{1}{2}$, then $1\le 1+x\le\dfrac{3}{2}$ which gives

$2\le 2\sqrt{1+x}\le 2\sqrt{1.5}$ and consequently $\dfrac{1}{2\sqrt{1.5}}\le\dfrac{1}{2\sqrt{1+x}}\le\dfrac{1}{2}$.

Then $\dfrac{\sqrt{6}}{6}\le f'(x)\le\dfrac{1}{2}$.

Applying the Mean Value Theorem to $f$ over $[0;x]$, there exists at least one value $c\in]0;x[$ such that : $f(x)-f(0)=xf'(c)$, but

$\dfrac{\sqrt{6}}{6}\le f'(c)\le\dfrac{1}{2}$, therefore $\dfrac{\sqrt{6}}{6}x\le xf'(c)\le\dfrac{1}{2}x$.

We deduce that $\frac{\sqrt{6}}{6}x \le f(x)-1 \le \frac{1}{2}x$ and consequently

$1+\frac{x}{\sqrt{6}} \le f(x) \le 1+\frac{x}{2}$.

2) For $x=0.1$, we get $1+\frac{0.1}{\sqrt{6}} \le f\,(0.1) \le 1+\frac{0.1}{2}$, then :

$1+\frac{0.1}{\sqrt{6}} \le \sqrt{1.1} \le 1+\frac{0.1}{2}$, so $1.04 \le \sqrt{1.1} \le 1.05$

N° 12

1) Dividing $x^2+2x+2$ by $x+1$, we get :

$\frac{x^2+2x+2}{x+1} = x+1+\frac{1}{x+1}$, so $a=1$, $b=1$ and $c=1$, which gives

$f(x) = x+1+\frac{1}{x+1}$.

2) $f$ is continuous over $[x;x+h]$ since $x>-1$ and it is differentiable over $]x;x+h[$ ; therefore:

$f(x+h)-f(x) = h\,f'(c)$ where $c \in \,]x;x+h[$.

But $f'(x) = 1-\frac{1}{(x+1)^2}$, which gives :

$$x+h+1+\frac{1}{x+h+1}-x-1-\frac{1}{x+1} = h-\frac{h}{(c+1)^2}$$

$$\frac{1}{x+h+1}-\frac{1}{x+1} = -\frac{h}{(c+1)^2}, \quad \frac{-h}{(x+h+1)(x+1)} = -\frac{h}{(c+1)^2}$$

$(c+1)^2 = (x+1)(x+h+1)$ but $-1<x<c<x+h$ then $c+1>0$ and consequently $c+1=\sqrt{(x+1)(x+h+1)}$ therefore

$c = -1+\sqrt{(x+1)(x+h+1)}$.

3) $c = x+\varphi(h)$, so $\varphi(h) = c-x = -1-x+\sqrt{(x+1)(x+h+1)}$.

$$\lim_{h\to 0}\frac{\varphi(h)}{h} = \lim_{h\to 0}\frac{-1-x+\sqrt{(x+1)(x+h+1)}}{h} = \frac{0}{0}.$$

Applying L'Hopital's rule we get:

$$\lim_{h\to 0}\frac{\varphi(h)}{h} = \lim_{h\to 0}\frac{(x+1)}{2\sqrt{(x+1)(x+h+1)}} = \frac{x+1}{2(x+1)} = \frac{1}{2}.$$

N° 13

1) $f$ is continuous and differentiable over $[n;n+1]$, the Mean Value Theorem can be applied, therefore:
$f(n+1)-f(n)=(n+1-n)f'(c)$ with $c\in ]n;n+1[$.
Then: $\arctan(n+1)-\arctan(n)=(n+1-n)f'(c)$
with $c\in ]n;n+1[$.

And since $f'(x)=\dfrac{1}{1+x^2}$ we get:

$\arctan(n+1)-\arctan(n)=\dfrac{1}{1+c^2}$, but $n<c<n+1$, which gives

$1+n^2<1+c^2<1+(n+1)^2$ and consequently:

$\dfrac{1}{1+(n+1)^2}<\dfrac{1}{1+c^2}<\dfrac{1}{1+n^2}$ , therefore :

$$\frac{1}{1+(n+1)^2}<\arctan(n+1)-\arctan(n)<\frac{1}{1+n^2} \quad (1)$$

2) For $n=1$ we get $\dfrac{1}{5}<\arctan(2)-\arctan(1)<\dfrac{1}{2}$, so

$\dfrac{1}{5}<\arctan(2)-\dfrac{\pi}{4}<\dfrac{1}{2}$, which gives $\dfrac{\pi}{4}+\dfrac{1}{5}<\arctan 2<\dfrac{\pi}{4}+\dfrac{1}{2}$.

3) Multiplying the terms of the inequality $(1)$ by $1+n^2$, we get

$\dfrac{1+n^2}{1+(n+1)^2}<E(n)<\dfrac{1+n^2}{1+n^2}$ and since:

$\lim\limits_{n\to+\infty}\dfrac{1+n^2}{1+(1+n)^2}=1=\lim\limits_{n\to+\infty}\dfrac{1+n^2}{1+n^2}$ we get $\lim\limits_{n\to+\infty}E(n)=1$.

N° 14

1) $f'(x)=-x+\sin x$ and $f''(x)=-1+\cos x$ .
But $\cos x\leq 1$ then $f''(x)\leq 0$, so $f'$ is strictly decreasing.
The table of variations of $f'$ is as follows:

| $x$ | $0$ | | $+\infty$ |
|---|---|---|---|
| $f''(x)$ | $0$ | $-$ | |
| $f'(x)$ | $0$ | $\searrow$ | |

2) $f'$ is strictly decreasing and since $x \geq 0$, then $f'(x) \leq f'(0)$ but since $f'(0) = 0$ then $f'(x) \leq 0$, hence $f$ is strictly decreasing but since $x \geq 0$, then $f(x) \leq f(0)$ and since $f(0) = 0$ we get $f(x) \leq 0$ consequently $1 - \dfrac{x^2}{2} - \cos x \leq 0$ and finally, $1 - \dfrac{x^2}{2} \leq \cos x$.

3) a- $g'(x) = 1 - \dfrac{x^2}{2} - \cos x = f(x)$, then $g'(x) \leq 0$ and consequently $g$ is strictly decreasing and since $x \geq 0$, we get

$g(x) \leq g(0)$ and since $g(0) = 0$ we get $x - \dfrac{x^3}{6} - \sin x \leq 0$.

Consequently, $x - \dfrac{x^3}{6} \leq \sin x$.

b- $h$ is the function defined for $x \geq 0$ by:

$$h(x) = \sin x - x + \frac{x^3}{3!} - \frac{x^5}{5!}.$$

$$h'(x) = \cos x - 1 + \frac{x^2}{2} - \frac{x^4}{24}$$

$$h''(x) = -\sin x + x - \frac{x^3}{6} = g(x)$$

Since $g(x) \leq 0$ then $h''(x) \leq 0$.
Hence $h'$ is strictly decreasing for $x \geq 0$.
Then $h'(x) \leq h'(0) = 0$.
Hence, $h$ is strictly decreasing and $x \geq 0$ gives $h(x) \leq h(0) = 0$.

Therefore, $\sin x \leq x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!}$.

c- For $x \geq 0$ we can write:

$$x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Then, $x-\frac{x^3}{3!}-\frac{x^5}{5!}\le x-\frac{x^3}{3!}\le \sin x\le x-\frac{x^3}{3!}+\frac{x^5}{5!}$.

Which gives $-\frac{x^5}{5!}\le \sin x-\left(x-\frac{x^3}{3!}\right)\le +\frac{x^5}{5!}$ and consequently

$\left|\sin x-\left(x-\frac{x^3}{3!}\right)\right|\le \frac{x^5}{5!}$.

Hence, $x-\frac{x^3}{6}$ is an approximation of $\sin x$ with an error inferior to $\frac{x^5}{5!}$.

d- For $x=\frac{\pi}{9}$ we get: $\sin\frac{\pi}{9}\approx\frac{\pi}{9}-\frac{1}{6}\left(\frac{\pi}{9}\right)^3$ with an error inferior to $\frac{1}{5!}\left(\frac{\pi}{9}\right)^5$.

4) Let $\varphi$ be the function defined for $x\ge 0$ by:

$\varphi(x)=\cos x-1+\frac{x^2}{2!}-\frac{x^4}{4!}$.

$\varphi'(x)=-\sin x+x-\frac{x^3}{6}$.

$\varphi''(x)=-\cos x+1-\frac{x^2}{2}$.

$\varphi'''(x)=\sin x-x\le 0$.

Hence, $\varphi''$ is strictly decreasing for $x\ge 0$, and since $x\ge 0$ we get: $\varphi''(x)\le\varphi''(0)=0$.

So, $\varphi'$ is strictly decreasing for $x\ge 0$.

Consequently, $\varphi'(x)\le\varphi'(0)=0$.

As a result $\varphi$ is strictly decreasing and $x\ge 0$ then $\varphi(x)\le\varphi(0)=0$.

Therefore, $\cos x\le 1-\frac{x^2}{2!}+\frac{x^4}{4!}$.

N° 15

1) $f'(x) = a\cos(ax) = a\sin\left(ax + \frac{\pi}{2}\right)$

$f''(x) = a^2\cos\left(ax + \frac{\pi}{2}\right) = a^2\sin\left(ax + 2\times\frac{\pi}{2}\right)$.

Suppose that $f^{(n)}(x) = a^n\sin\left(ax + n\frac{\pi}{2}\right)$ and we prove that:

$f^{(n+1)}(x) = a^{n+1}\sin\left(ax + (n+1)\frac{\pi}{2}\right)$.

$f^{(n+1)}(x) = \left(f^{(n)}(x)\right)' = a^n \times a\cos\left(ax + n\frac{\pi}{2}\right)$.

$f^{(n+1)}(x) = a^{n+1}\sin\left(ax + n\frac{\pi}{2} + \frac{\pi}{2}\right) = a^{n+1}\sin\left(ax + (n+1)\frac{\pi}{2}\right)$.

Hence, for all real numbers $x$ and for all non-zero natural numbers $n$ we get: $f^{(n)}(x) = a^n\sin\left(ax + n\frac{\pi}{2}\right)$.

2) **n^th Derivative of g:**

$g(x) = \sin^2 2x$ then $g'(x) = 4\sin 2x \times \cos 2x = 2\sin 4x$.

Hence, $g^{(n)}(x) = 2\sin^{(n-1)}(4x) = 2\times 4^{n-1}\sin\left(4x + (n-1)\frac{\pi}{2}\right)$.

**n^th Derivative of $h$:**

$h(x) = \cos^2 x$ then $h'(x) = -2\sin x\cos x = -\sin 2x$.

Therefore $h^{(n)}(x) = -\sin^{(n-1)}(2x) = -2^{n-1}\sin\left(2x + (n-1)\frac{\pi}{2}\right)$.

$h^{(n)}(x) = 2^{n-1}\cos\left(2x + n\frac{\pi}{2}\right)$.

**n^th Derivative of $\varphi$:**

We know that: $\sin a\cos b = \frac{1}{2}[\sin(a+b) + \sin(a-b)]$ then

$\sin(4x)\times\cos(3x) = \frac{1}{2}[\sin 7x + \sin x]$.

Therefore $\varphi^{(n)}(x) = \frac{1}{2}[\sin(7x)]^{(n)} + \frac{1}{2}[\sin x]^{(n)}$, so:

$$\varphi^{(n)}(x)=\frac{1}{2}\left[7^n\sin\left(7x+n\frac{\pi}{2}\right)\right]+\frac{1}{2}\left[\sin\left(x+n\frac{\pi}{2}\right)\right].$$

**n^th Derivative of $k$ :**

We know that: $\sin a\sin b=\frac{1}{2}[\cos(a-b)-\cos(a+b)]$ then

$$\sin(4x)\times\sin(3x)=\frac{1}{2}[\cos x-\cos 7x].$$

But $\cos x=(\sin x)'$ then

$$(\cos x)^{(n)}=(\sin x)^{(n+1)}=\sin\left(x+(n+1)\frac{\pi}{2}\right)=\cos\left(x+n\frac{\pi}{2}\right)$$

Therefore, $k^{(n)}(x)=\left[\frac{1}{2}\cos(x)\right]^{(n)}-\left[\frac{1}{2}\cos(7x)\right]^{(n)}.$

$$=\frac{1}{2}\left[\cos\left(x+n\frac{\pi}{2}\right)+7^n\cos\left(7x+n\frac{\pi}{2}\right)\right]$$

**N° 16**

1) $f(x)=\dfrac{1}{ax+b}=(ax+b)^{-1}$ , then :

$$f'(x)=-1\times a\times(ax+b)^{-2}=(-1)^1\times a^1\times 1!(ax+b)^{-2}$$

$$=\frac{(-1)^1\times a^1\times 1!}{(ax+b)^2}$$

Using mathematical induction over $n$, we suppose that:

$f^{(n)}(x)=\dfrac{(-1)^n\times a^n\times n!}{(ax+b)^{n+1}}$ then we prove that:

$$f^{(n+1)}(x)=\frac{(-1)^{n+1}\times a^{n+1}\times(n+1)!}{(ax+b)^{n+2}}.$$

$f^{(n+1)}(x)=\left(f^{(n)}(x)\right)'$, or $f^{(n)}(x)=(-1)^n\times a^n\times n!\times(ax+b)^{-n-1}$

Then, $f^{(n+1)}(x)=(-1)^n\times a^n\times n!\times(-n-1)\times a\times(ax+b)^{-n-2}$

$f^{(n+1)}(x)=(-1)^n\times(-1)\times(n+1)\times a^{n+1}\times n!\times(ax+b)^{-n-2}$ ,

So :

$$f^{(n+1)}(x)=(-1)^{n+1}(n+1)!\times a^{n+1}\times(ax+b)^{-n-2}=\frac{(-1)^{n+1}(n+1)!\times a^{n+1}}{(ax+b)^{n+2}}$$

Then for all $n \in IN^*$, $f^{(n)}(x) = \dfrac{(-1)^n \times a^n \times n!}{(ax+b)^{n+1}}$.

2) a- $y = \dfrac{1}{2x-1}$ gives $y^{(n)} = \dfrac{(-1)^n \times n! \times 2^n}{(2x-1)^{n+1}}$.

Remark that $z = \dfrac{1}{(2x-1)^2} = -\dfrac{1}{2}y'$

then $z^{(n)} = -\dfrac{1}{2}y^{(n+1)} = -\dfrac{1}{2}\dfrac{(-1)^{n+1} \times (n+1)! \times 2^{n+1}}{(2x+1)^{n+2}}$.

b- $\dfrac{m}{x-1} + \dfrac{n}{x-3} = \dfrac{(m+n)x - 3m - n}{(x-1)(x-3)}$, therefore :

$\dfrac{(m+n)x - 3m - n}{(x-1)(x-3)} = \dfrac{1}{x^2 - 4x + 3}$ which gives :

$(m+n)x - 3m - n = 1$

results in: $m + n = 0$ and $-3m - n = 1$, which gives $m = -\dfrac{1}{2}$ and $n = \dfrac{1}{2}$.

Hence, $g(x) = \dfrac{-\frac{1}{2}}{x-1} + \dfrac{\frac{1}{2}}{x-3}$ and consequently :

$$g^{(n)}(x) = -\frac{1}{2}\left(\frac{1}{x-1}\right)^{(n)} + \frac{1}{2}\left(\frac{1}{x-3}\right)^{(n)}.$$

$$g^{(n)}(x) = -\frac{1}{2} \times \frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{1}{2} \times \frac{(-1)^n n!}{(x-3)^{n+1}}.$$

**N° 17.**

Using mathematical induction over $n$.

$f(x) = \arctan x$.

$f'(x) = \dfrac{1}{(x^2+1)^1}$, since 1 is a polynomial of degree 0 so

$f^{(1)}(x) = \dfrac{p_0(x)}{(x^2+1)^1}$.

Suppose now that $f^{(n)}(x)=\dfrac{p_{n-1}(x)}{\left(x^2+1\right)^n}$ we then prove that

$$f^{(n+1)}(x)=\frac{p_n(x)}{\left(x^2+1\right)^{n+1}}.$$

$$f^{(n+1)}(x)=\left[\frac{p_{n-1}(x)}{\left(x^2+1\right)^n}\right]'=\frac{p'_{n-1}(x)\left(x^2+1\right)^n-n\left(x^2+1\right)^{n-1}\times 2x\times p_{n-1}(x)}{\left(x^2+1\right)^{2n}}$$

$$f^{(n+1)}(x)=\frac{\left(x^2+1\right)^{n-1}\left[\left(x^2+1\right)p'_{n-1}(x)-2nx\times p_{n-1}(x)\right]}{\left(x^2+1\right)^{2n}}$$

$$f^{(n+1)}(x)=\frac{\left(x^2+1\right)p'_{n-1}(x)-2nx\times p_{n-1}(x)}{\left(x^2+1\right)^{n+1}}$$

Let $q(x)=p'_{n-1}(x)\left(x^2+1\right)-2nx\times p_{n-1}(x)$.

Since $p_{n-1}(x)=ax^{n-1}+bx^{n-2}+\dots\dots\dots\dots$ is a polynomial of degree $n-1$ with $a\neq 0$, we get:

$p'_{n-1}(x)=(n-1)ax^{n-2}+(n-2)bx^{n-3}+\dots\dots\dots\dots$

which gives that: $q(x)=\left(x^2+1\right)\left((n-1)ax^{n-2}+\dots\right)-2nx\left(ax^n+\dots.\right)$

$q(x)=(na-a-2an)x^n+\dots\dots\dots\dots$

$q(x)=-(n+1)ax^n+\dots\dots\dots\dots$ and as $(n+1)a\neq 0$ then the polynomial $q(x)$ is of degree $n$, $q(x)=p_n(x)$

Hence, the property is true for the order $n+1$.

**N° 18.**

$f'(x)=\cos x-x\sin x=\cos x+x\cos\left(x+\dfrac{\pi}{2}\right)$, then

the formula is true for $n=1$.

Suppose that:

$$f^{(n-1)}(x)=(n-1)\cos\left(x+(n-2)\frac{\pi}{2}\right)+x\cos\left(x+(n-1)\frac{\pi}{2}\right)$$

Therefore, $f^{(n)}(x)=\left[(n-1)\cos\left(x+(n-2)\dfrac{\pi}{2}\right)+x\cos\left(x+(n-1)\dfrac{\pi}{2}\right)\right]'$

$$f^{(n)}(x) = -(n-1)\sin\left(x+(n-2)\frac{\pi}{2}\right) + \cos\left(x+(n-1)\frac{\pi}{2}\right) - x\sin\left(x+(n-1)\frac{\pi}{2}\right).$$

$$f^{(n)}(x) = (n-1)\cos\left(x+(n-1)\frac{\pi}{2}\right) + \cos\left(x+(n-1)\frac{\pi}{2}\right) + x\cos\left(x+n\frac{\pi}{2}\right)$$

$$f^{(n)}(x) = n\cos\left(x+(n-1)\frac{\pi}{2}\right) + x\cos\left(x+n\frac{\pi}{2}\right).$$

**N° 19.**

***Part A:***

1) $f'(x) = \dfrac{1}{2\sqrt{x+1}} - \dfrac{1}{2} + \dfrac{x}{4}$.

$$f''(x) = \frac{1}{2}\,\frac{-\frac{1}{2\sqrt{1+x}}}{x+1} + \frac{1}{4} = \frac{-1}{4(1+x)\sqrt{1+x}} + \frac{1}{4}, \text{ then:}$$

$$f''(x) = \frac{(1+x)\sqrt{1+x}-1}{4(1+x)\sqrt{1+x}} = \frac{h(x)}{4(1+x)\sqrt{1+x}}.$$

But $(1+x)\sqrt{1+x} > 0$ over $[0;+\infty[$ and $h(x) \geq 0$ from the graph, then $f''(x) \geq 0$ over $[0;+\infty[$, hence $f'$ is strictly increasing over $[0;+\infty[$.

Therefore the table of variations of $f'$ is as follows:

| $x$ | $0$ | | $+\infty$ |
|---|---|---|---|
| $f''(x)$ | | $+$ | |
| $f'(x)$ | $0$ | $\nearrow$ | |

2) a- From the table of variations, we notice that $f'(x) \geq 0$.

Then $f$ is strictly increasing over $[0;+\infty[$ and since $x \geq 0$ then $f(x) \geq f(0)$, but $f(0) = 0$ therefore $f(x) \geq 0$.

b- $f(x) \geq 0$ gives $\sqrt{x+1} - 1 - \dfrac{x}{2} + \dfrac{x^2}{8} \geq 0$, which gives

$$1+\frac{x}{2}-\frac{x^2}{8}\leq\sqrt{x+1}.$$

***Part B:***

1) $g'(x)=\frac{1}{2}-\frac{x}{4}+\frac{3x^2}{16}-\frac{1}{2}(x+1)^{-\frac{1}{2}}$

$g''(x)=-\frac{1}{4}+\frac{3x}{8}+\frac{1}{4}(x+1)^{-\frac{3}{2}}$

$g^{(3)}(x)=\frac{3}{8}-\frac{3}{8}(x+1)^{-\frac{5}{2}}=\frac{3}{8}\left[1-(x+1)^{-\frac{5}{2}}\right].$

$x\geq 0$ then $x+1\geq 1$, which gives $(x+1)^{\frac{5}{2}}\geq 1$ and consequently $(x+1)^{-\frac{5}{2}}\leq 1$, therefore $1-(x+1)^{-\frac{5}{2}}\geq 0$ hence $g^{(3)}(x)\geq 0$.

2) $g^{(3)}(x)\geq 0$ then $g''$ is strictly increasing over $[0;+\infty[$.
$x\geq 0$ gives $g''(x)\geq g''(0)$, but $g''(0)=0$, therefore $g''(x)\geq 0$.
Consequently $g'$ is strictly increasing over $[0;+\infty[$ .
$x\geq 0$ gives $g'(x)\geq g'(0)$, but $g'(0)=0$, so $g'(x)\geq 0$
consequently $g$ is strictly increasing over $[0;+\infty[$.

3) $x\geq 0$ gives $g(x)\geq g(0)$ but $g(0)=0$ so $g(x)\geq 0$ .

Therefore, $1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\sqrt{x+1}\geq 0$ and consequently

$\sqrt{x+1}\leq 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}.$

4) From part ***A*** and part ***B*** :

$1+\frac{x}{2}-\frac{x^2}{8}\leq\sqrt{x+1}\leq 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}$ which gives :

$1+\frac{x}{2}-\frac{x^2}{8}-\frac{x^3}{16}\leq 1+\frac{x}{2}-\frac{x^2}{8}\leq\sqrt{x+1}\leq 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}.$

therefore $-\frac{x^3}{16}\leq\sqrt{x+1}-\left(1+\frac{x}{2}-\frac{x^2}{8}\right)\leq+\frac{x^3}{16}$ hence

$1+\frac{x}{2}-\frac{x^2}{8}$ is an approximate value of $\sqrt{x+1}$ to the nearest $\frac{x^3}{16}$.

# CHAPTER 4

# Sequences

**Chapter review :**

- Definition:
  A sequence is a mapping from the set $IN$ or a part of $IN$ to the set $IR$.
- Arithmetic Sequences and Geometric Sequences :

| Arithmetic Sequences | Geometric Sequences |
|---|---|
| An arithmetic sequence is defined by its first term $u_0 = a$ and by the relation $u_{n+1} - u_n = d$ where $d$ is the common difference different from 0. | A geometric sequence is defined by its first term $u_0 = a$ and by the relation $\frac{u_{n+1}}{u_n} = r$, where $r$ is the common ratio different from 1. |
| Properties:<br>1. $u_n = u_k + (n-k)d$.<br>2. $u_{n+1} - u_n = u_n - u_{n-1} = d$.<br>3. $u_n = \frac{u_{n-1} + u_{n+1}}{2}$.<br>4. Sum of first $n$ terms:<br>$S_n = u_0 + ..... + u_{n-1} = \frac{n}{2}(u_0 + u_{n-1})$<br>$S_{n+1} = u_0 + ... + u_n = \frac{n+1}{2}(u_0 + u_n)$ | Properties:<br>1. $u_n = u_k \times r^{n-k}$<br>2. $\frac{u_{n+1}}{u_n} = \frac{u_n}{u_{n-1}} = r$<br>3. $u_n^2 = u_{n-1} \times u_{n+1}$<br>4. Sum of first n terms:<br>$S_n = u_0 + u_1 + ... + u_{n-1} = u_0 \frac{1-r^n}{1-r}$<br>$S_{n+1} = u_0 + ... + u_n = u_0 \frac{1-r^{n+1}}{1-r}$ |

- Determination of a Sequence:
  * A sequence can be determined by expressing its general term

$u_n$ in terms of $n$.

* ***Ex:*** $u_n = \dfrac{1}{2^n}$.
* A sequence can be determined by stating its first term or first two terms and a recurring relation between two or three consecutive terms. Ex: $u_0 = -1, u_{n+1} = \sqrt{u_n + 2}$ or $u_0 = 1$ ; $u_1 = 2$ and $u_{n+1} = u_n + u_{n-1}$

- Sense of Variation of a sequence:
  Given a sequence $(u_n)$ defined over $IN$.
  * $(u_n)$ is said to be increasing if for all natural numbers $n$, $u_{n+1} \geq u_n$.
  * $(u_n)$ is said to be strictly increasing if for all natural numbers $n$, $u_{n+1} > u_n$.
  * $(u_n)$ is said to be decreasing if for all natural numbers $n$, $u_{n+1} \leq u_n$.
  * $(u_n)$ is said to be strictly decreasing if for all natural numbers $n$, $u_{n+1} < u_n$.
- Bounded Sequences:
  * A sequence is bounded from above if for all $n \in IN$ there exists a real number $B$ such that $u_n \leq B$.
  * A sequence is said to be bounded below if for all $n \in IN$, there exists a real number $A$ such that $u_n \geq A$.
  * A sequence is said to be bounded if it is bounded from above and below.
- Convergent Sequences:
  * A sequence is said to be convergent to a limit $l$ if $\lim\limits_{n \to +\infty} u_n = l$.
  * All increasing sequences and bounded above are convergent.
  * All decreasing sequences bounded below are convergent.
  * If $u_{n+1} = f(u_n)$ and $(u_n)$ is convergent to a limit $\ell$ then $\ell$ is solution of the equation $\ell = f(\ell)$.
  * Let $(u_n)$ and $(v_n)$ two sequences, if $u_n \leq v_n$ and $\lim\limits_{n \to +\infty} u_n = +\infty$

then $\lim\limits_{n\to+\infty} v_n = +\infty$.

* If $u_n \le v_n$ and $\lim\limits_{n\to+\infty} v_n = -\infty$ then $\lim\limits_{n\to+\infty} u_n = -\infty$.
* Let $(u_n)$ , $(v_n)$ and $(w_n)$ be three sequences.
  If $w_n \le u_n \le v_n$ and $\lim\limits_{n\to+\infty} v_n = \ell = \lim\limits_{n\to+\infty} w_n$ then $\lim\limits_{n\to+\infty} u_n = \ell$.

- Remarks:
  * Let $(u_n)$ be an arithmetic sequence of common difference $d$.
    If $d > 0$ then $(u_n)$ is increasing and $\lim\limits_{n\to+\infty} u_n = +\infty$.
    If $d < 0$ then $(u_n)$ is decreasing and $\lim\limits_{n\to+\infty} u_n = -\infty$.
  * Let $(u_n)$ be a geometric sequence of common ratio $r$.
    If $|r| < 1$ then $(u_n)$ is convergent to 0.
    If $0 < r < 1$ then $(u_n)$ is decreasing.
  * The sequence of general term $u_n = an + b$ is an arithmetic sequence of common difference $a$.

# Solved Problems

**N° 1.**

Consider the sequence $(u_n)$ defined over $IN$ by:

$u_0 = 1$ and for all natural numbers $n$, $u_{n+1} = \dfrac{u_n}{\sqrt{1+u_n^2}}$.

Express the general term $u_n$ in terms of $n$ and deduce that it is convergent.

**N° 2.**

Consider the sequence $(u_n)$ defined over $IN$ by $u_n = \dfrac{3n+\cos n}{2n+1}$.

1) Show that $0 < \dfrac{3}{2} - u_n \le \dfrac{5}{4n}$.
2) Deduce the limit of $u_n$ as $n$ tends to $+\infty$.

**N° 3.**

Let $u_0$ be a real number and $(u_n)$ the sequence defined over $IN$ by its first term $u_0$ and the recurring relation $u_{n+1} = \dfrac{u_n}{2+u_n^2}$.

1) Prove that $|u_{n+1}| \le \dfrac{|u_n|}{2}$ for all natural numbers $n$.
2) Deduce that for all natural numbers $n$, $|u_n| \le \dfrac{|u_0|}{2^n}$.
3) What is the limit of the sequence $(u_n)$?

**N° 4.**

For all natural numbers $n > 0$ define the sequence $(u_n)$, by $u_n = \dfrac{n^2}{2^n}$.

1) For all natural numbers $n > 0$ let $v_n = \dfrac{u_{n+1}}{u_n}$.

a- Determine $\lim\limits_{n\to+\infty} v_n$.

b- Prove that, $v_n > \frac{1}{2}$ for all $n > 0$.

c- Find the smallest natural number $N$ such that if $n \geq N$ then $v_n < \frac{3}{4}$.

d- Deduce that if $n \geq N$ then $u_{n+1} < \frac{3}{4}u_n$.

2) Suppose that, for $n \geq 5$, $S_n = u_5 + u_6 + \dots\dots\dots + u_n$.

a- Prove that $u_{n+1} \leq \left(\frac{3}{4}\right)^{n-5} u_5$.

b- Show that for all natural numbers $n \geq 5$,

$$S_n \leq \left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots\dots\dots\dots + \left(\frac{3}{4}\right)^{n-5}\right] u_5.$$

c- Deduce that, $S_n \leq 4u_5$ for all natural numbers $n \geq 5$.

d- Show that the sequence $(S_n)$ is increasing then deduce that it converges.

**N° 5.**

Let $(u_n)$ be a sequence defined over $IN$ by the relations:

$$\begin{cases} u_0 = 2 \\ u_{n+1} = \sqrt{1 + u_n^{\ 2}} \end{cases}$$

1) Show that $(u_n)$ is a divergent sequence.

2) Let $(v_n)$ be a sequence defined over $IN$ by $v_n = u_n^2$.

a- Show that $(v_n)$ is an arithmetic sequence whose common difference is to be determined.

b- Calculate $v_n$ then $u_n$ in terms of $n$.

c- Determine $\lim\limits_{n\to+\infty} u_n$.

d- Determine $\lim\limits_{n\to+\infty} \left(\frac{u_n}{\sqrt{n}}\right)$.

N°6.

Given the sequence $(u_n)$, defined by:

$u_1 = \frac{1}{3}$ and for $n \geq 2$, $u_{n+1} = \frac{4-3u_n}{9(1-u_n)}$.

1) Let $f$ be the function defined over $]1;+\infty[$ by $f(x) = \frac{-3x+4}{9(-x+1)}$.

   Study the variations of $f$.

2) a- Show that the sequence $(u_n)$ is bounded above by $\frac{2}{3}$.

   b- Show that $(u_n)$ is increasing.

   c- Deduce that $(u_n)$ is convergent and calculate its limit.

3) Noting that $u_1 = \frac{1}{3}$, $u_2 = \frac{3}{6}$, $u_3 = \frac{5}{9}$ and $u_4 = \frac{7}{12}$,

   express $u_n$ in terms of $n$.

N°7.

Consider the sequence $(u_n)$ defined over $IN$ by:

$$\begin{cases} u_0 = 1 \\ u_{n+1} = \frac{1}{3}u_n + n - 1 \end{cases}.$$

Let $(v_n)$ be the sequence defined by $v_n = 4u_n - 6n + 15$ for all natural numbers $n$.

1) Show that $(v_n)$ is a geometric sequence whose first term and ratio are to be determined.

2) Express $v_n$ in terms of $n$ and deduce that $u_n = \frac{19}{4} \times \frac{1}{3^n} + \frac{6n-15}{4}$.

3) Show that the sequence $(u_n)$ can be written in the form $u_n = t_n + w_n$ where $(t_n)$ is a geometric sequence and $(w_n)$ is an an arithmetic sequence.

4) Calculate $T_n = t_0 + t_1 + .......... + t_n$ and $W_n = w_0 + w_1 + ........ + w_n$ then deduce $U_n = u_0 + u_1 + .......... + u_n$.

**N°8.**

Let $f$ be the function defined for $x > \frac{1}{2}$ by $f(x) = \frac{x^2}{2x-1}$.

Define the sequence $(u_n)$ by :

$$\begin{cases} u_0 = 2 \\ u_{n+1} = f(u_n) = \dfrac{{u_n}^2}{2u_n - 1} \end{cases}$$ for all natural numbers $n$.

1) a- Prove that for all $x > 1$, $f(x) > 1$.
   b- Deduce that $u_n > 1$ for all natural numbers $n$.
2) Consider the two sequences $(v_n)$ and $(w_n)$ such that

   $v_n = \dfrac{-1 + u_n}{u_n}$ and $w_n = \ln(v_n)$.

   a- Verify that $(v_n)$ and $(w_n)$ are defined for all natural numbers $n$.
   b- Prove that $(w_n)$ is a geometric sequence of common ratio $r = 2$ whose first term is to be determined.
   c- Express $w_n$ then $v_n$ in terms of $n$.
   d- Deduce the expression of $u_n$ and calculate $\lim\limits_{n \to +\infty} u_n$.

**N°9.**

Let $(u_n)$ be the sequence defined over $IN$ by its first term $u_0$ and for all natural numbers $n$, $u_{n+1} = \dfrac{2 + 4u_n}{3 + u_n}$.

1) Suppose that $u_0 = 2$, show that $(u_n)$ is a constant sequence.
2) Suppose that $u_0 = 1$.
   a- Prove that $0 < u_n < 2$ for all natural numbers $n$.
   b- Show that this sequence is increasing for all natural numbers $n$.
   c- Deduce the limit of this sequence .

N° 10.

Consider the sequence $(u_n)$ defined over $IN$ by $u_0 = 1$ and
$u_{n+1} = \dfrac{u_n + 8}{2u_n + 1}$.

1) Calculate $u_1, u_2$ and $u_3$.

2) Let $h$ be the function defined over $\left]-\dfrac{1}{2};+\infty\right[$ by $h(x) = \dfrac{x+8}{2x+1}$ and let $(H)$ be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

   a- Draw $(H)$ and the straight line (d) of equation $y = x$ in the given system.

   b- Construct the points of $(H)$ and (d) of respective abscissas $u_0$ , $u_1$ , $u_2$ and $u_3$ .

   c- What do you notice about the convergence of the sequence $(u_n)$?

3) Let $(v_n)$ be the sequence defined over $IN$ by $v_n = \dfrac{u_n - 2}{u_n + 2}$.

   a- Calculate: $v_0$ , $v_1$ and $v_2$ .

   b- Show that $(v_n)$ is a geometric sequence whose first term and common ratio are to be determined.

   c- Determine $\lim\limits_{n\to+\infty} v_n$.

   d- Express $u_n$ in terms of $v_n$ and deduce the limit of $(u_n)$.

N° 11.

Consider the sequence $(I_n)$ defined for all non-zero natural numbers $n$ by $I_n = \int_0^1 x^n e^{-x} dx$.

1) Show that this sequence is bounded below by 0.
2) Show that this sequence is decreasing.
3) a- Prove that $I_{n+1} = (n+1)I_n - \dfrac{1}{e}$.

b- Calculate $I_1$ and deduce the value of $I_2$

**N° 12.**

For all natural numbers $n$ of $IN^*$, consider the sequence $(I_n)$ defined by $I_n = \int_1^e (\ln x)^n \, dx$.

1) Show that the sequence $(I_n)$ is decreasing.
2) a- Calculate $I_1$.
   b- Prove, using integration by parts, that:
   $I_{n+1} = e - (n+1)I_n$.
3) a- Prove that for all natural numbers $n$ of $IN^*$, $(n+1)I_n \le e$.
   b- Deduce the limit of $I_n$.
   c- Determine the value of $nI_n + (I_n + I_{n+1})$ and deduce the limit of $nI_n$.

**N° 13.**

$(u_n)$ and $(v_n)$ are two sequences of real numbers such that $u_1 = 12$, $v_1 = 1$ and for all natural numbers $n \in IN^*$, $u_{n+1} = \dfrac{u_n + 2v_n}{3}$ and $v_{n+1} = \dfrac{u_n + 3v_n}{4}$.

1) For all natural numbers $n \in IN^*$, let $w_n = v_n - u_n$.
   a- Show that $(w_n)$ is a geometric sequence whose first term and common ratio are to be determined.
   b- Express $w_n$ in terms of $n$ and determine $\lim\limits_{n\to+\infty} w_n$.
2) Prove that the sequence $(u_n)$ is decreasing and that $(v_n)$ is increasing.
3) a- Prove that $u_n > v_n$ for all natural numbers $n \in IN^*$.
   b- Deduce that the two sequences are convergent.
4) For all natural numbers $n \in IN^*$, let $t_n = 3u_n + 8v_n$.
   a- Prove that the sequence $(t_n)$ is a constant sequence.

b- Deduce the expressions of $u_n$ and $v_n$ in terms of $n$.

c- Show that $(u_n)$ and $(v_n)$ are convergent to the same limit.

**N° 14.**

For all natural numbers $n$, consider the sequence $(u_n)$ defined by

$$u_n = \int_0^1 \frac{e^x}{e^{nx}(1+e^x)}dx.$$

1) Show that $u_0 = \ln\frac{1+e}{2}$.
2) Show that $u_0 + u_1 = 1$ then deduce $u_1$.
3) Prove that the sequence $(u_n)$ is bounded below by 0.
4) Prove that the sequence $(u_n)$ is decreasing for all natural numbers $n$.
5) a- Prove that $u_{n-1} + u_n = \frac{1-e^{1-n}}{n-1}$ for all natural numbers $n$.

   b- Calculate $u_2$.
6) Let $(v_n)$ be the sequence defined by $v_n = \frac{u_{n-1}+u_n}{2}$.

   a- Calculate $\lim\limits_{n\to+\infty} v_n$.

   b- Prove that $0 \le u_n \le v_n$ for all natural numbers $n$.

   c- Deduce $\lim\limits_{n\to+\infty} u_n$.

**N° 15.**

In the plane of an orthonormal system $(O;\vec{i},\vec{j})$, consider the function $f$ defined over $]0;+\infty[$ by $f(x) = \frac{\ln x + e}{x^2}$.

Consider the sequences defined by $I_n = \int_{e^n}^{e^{n+1}} \frac{\ln t}{t^2}dt$ and $A_n = \int_{e^n}^{e^{n+1}} f(t)dt$ where $n$ is a natural number.

1) Prove that $I_n = \frac{n+1}{e^n} - \frac{n+2}{e^{n+1}}$.

2) a- Show that $A_n = I_n + \frac{e-1}{e^n}$.

b- Calculate $I_0$ and $A_0$.

c- Give a graphical interpretation of $A_0$.

3) Show that the sequence $(A_n)$ converges to 0.

**N° 16.**

Consider the function $f$ defined over $[0;+\infty[$ by $f(x)=1-x^2e^{1-x^2}$, and designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

***Part A:***

1) Study the variations of $f$ and show that $f(x)\geq 0$.
2) Trace $(C)$.
3) Let $k$ be a given real number, study according to the values of $k$ the number of solutions of the equation $f(x)=k$ in the interval $[0;+\infty[$.
4) $n$ is a non-zero natural number, determine the values of $n$ for which the equation $f(x)=\frac{1}{n}$ admits two distinct solutions.

***Part B.***

1) Let $n$ be a natural number greater than or equal to 2.

   Show that the equation $f(x)=\frac{1}{n}$ admits two solutions $u_n$ and $v_n$ included in the intervals $[0;1]$ and $[1;+\infty[$ respectively.
2) Determine the sense of variations of $(u_n)$ and $(v_n)$.
4) Show that the sequence $(u_n)$ is convergent and determine its limit. Proceed in a similar way for the sequence $(v_n)$.

**N° 17.**

A student should answer questions in an exam successively.
We admit that if he answers the n$^{th}$ question right then the probability that he answers right to the question that follows that is to the $(n+1)^{th}$ question is 0.8 and that if he answers wrong to the n$^{th}$ question then the probability that he answers right to the question that follows is 0.6

Suppose that the probability that he answers the first question right is 0.7 .

Designate by $A_n$ the event:

« the student answered the n$^{th}$ question right » .

1) Verify that $p(A_2)=0.74$

3) Designate by $p_n$ the probability of the event $A_n$ and by $p_{n+1}$ the probability of the event $A_{n+1}$.

   a- Show that $p_{n+1}=0.2p_n+0.6$

   b- For $n\geq 1$ , let $u_n=p_n-0.75$, show that $(u_n)$ is a geometric sequence of common ratio 0.2 ,and deduce an expression of $p_n$ in terms of $n$ and determine $\lim\limits_{n\to+\infty} p_n$.

**N° 18**

***Part A :***

Let $f$ be the function defined over $]0;+\infty[$ by $f(x)=x+\ln\left(\dfrac{x}{2x+1}\right)$.

Designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine $\lim\limits_{x\to 0} f(x)$ and deduce an asymptote to $(C)$.

   b- Determine $\lim\limits_{x\to+\infty} f(x)$.

2) Study the variations of $f$ and draw its table of variations .

3) a- Show that the straight line $(d)$ of equation $y=x-\ln 2$ is an asymptote to $(C)$ and study the position of $(C)$ with respect to $(d)$.

   b- Trace $(C)$ .

4) Show that the equation $f(x)=0$ admits over $]0;+\infty[$ a unique solution $\alpha$ and such that $1<\alpha<\dfrac{5}{4}$.

***Part B:***

Let $g$ be the function defined over $[0;+\infty[$ by $g(x)=(2x+1)e^{-x}$.
and designate by $(\gamma)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim_{x \to +\infty} g(x)$ and deduce an asymptote to $(\gamma)$.

2) Study the variations of $g$ and draw its table of variations.

3) Trace $(\gamma)$.

***Part C:***

1) Show that $\alpha$ is a solution of the equation $g(x) = x$.

2) Prove that if $x \in \left[1;\frac{5}{4}\right]$, then $g(x) \in \left[1;\frac{5}{4}\right]$.

3) Study the variations of $g'$ and show that for all $x \in \left[1;\frac{5}{4}\right]$ then

   $|g'(x)| \leq \frac{1}{2}$.

4) Consider the sequence $(u_n)$, defined over $IN$ by $u_0 = 1$ and $u_{n+1} = g(u_n)$, show that:

   a- For all natural numbers $n$, $u_n \in \left[1;\frac{5}{4}\right]$.

   b- For all natural numbers $n$, $|u_{n+1} - \alpha| \leq \frac{1}{2}|u_n - \alpha|$.

   c- For all natural numbers $n$, $|u_n - \alpha| \leq \frac{1}{2^{n+2}}$.

   d- The sequence $(u_n)$ is convergent to $\alpha$.

**N° 19**

Consider the sequences $(u_n)$ and $(v_n)$ defined over $IN$ by

$$\begin{cases} u_0 = 1 \\ u_{n+1} = \left(1 + i\sqrt{3}\right)u_n + 3 \end{cases} \quad \text{and} \quad v_{n+1} = u_{n+1} - i\sqrt{3}.$$

1) Calculate $v_0$ and write it in exponential form.

2) Show that the sequence $(v_n)$ is a geometric sequence of common ratio $r = 1 + i\sqrt{3}$.

3) Express $v_n$ in terms of $n$.

# Supplementary Problems

N° 1.

Consider a sequence $(u_n)$, whose terms are all different from 0, defined over $IN$ and consider the sequence $(v_n)$ defined over $IN$ by $v_n = u_n + 2$.

Indicate whether each of the following statements is true or false and justify your answer:

1) If $(u_n)$ is convergent, then $(v_n)$ is convergent.
2) If $(u_n)$ is bounded below by 2, then $(v_n)$ is bounded below by $-1$.
3) If $(u_n)$ is decreasing, then $(v_n)$ is decreasing.
4) If $(u_n)$ is divergent, then $(v_n)$ converges to 0.

N° 2.

Consider the sequence $(u_n)$ defined by : $u_0 = e$ and for all natural numbers $n$, $u_{n+1} = \sqrt{u_n}$.

Suppose, for all natural numbers $n, v_n = \ln u_n$.

1) a- Prove that, for all natural numbers $n$, $(v_n)$ is a geometric sequence of common ratio $r = \dfrac{1}{2}$ and determine its first term.

   b- Express $v_n$ then $u_n$ in terms of $n$.

2) $S_n = v_0 + v_1 + v_2 + \dots\dots\dots + v_n$ and $P_n = u_0 \times u_1 \times u_2 \times \dots\dots\dots\dots \times u_n$.

   a- Prove that $P_n = e^{S_n}$.

   b- Express $S_n$ then $P_n$ in terms of $n$.

   c- Determine $\lim\limits_{n\to+\infty} S_n$ and $\lim\limits_{n\to+\infty} P_n$.

N° 2.

Given a sequence $(u_n)$ defined by the recurring relation:

$u_{n+1} = u_n^2 - 2u_n + 2$ ($u_0$ is a real number).

1) Express $u_{n+1}-1$ in terms of $u_n-1$.
2) Show, by mathematical induction, that for all $n\geq 0$ ,
   $u_n-1=(u_0-1)^{2^n}$.
3) What can be said about $(u_n)$ if $u_0=1, u_0=0$ or $u_0=2$ ?
4) a- Suppose that $u_0\in]0;1[\cup]1;2[$, determine the limit of $(u_n)$.
   b- Determine $\lim\limits_{n\to+\infty} u_n$ if $u_0\in]-\infty;0[\cup]2;+\infty[$.

**N°4.**

Consider the sequence $(u_n)$ defined by $\begin{cases} u_1=e^2 \\ u_n^2\cdot e=u_{n-1}\end{cases}$ for all $n\in IN-\{0;1\}$ and let $v_n=\dfrac{1+\ln u_n}{2}$.

1) Verify that $(v_n)$ is defined for all natural numbers $n\geq 1$.
2) Express $v_n$ in terms of $u_{n-1}$, then in terms of $v_{n-1}$.
3) Deduce that $(v_n)$ is a geometric sequence whose common ratio and first term are to be determined.
4) Express $v_n$ then $u_n$ in terms of $n$.
5) Determine $\lim\limits_{n\to+\infty} u_n$ and $\lim\limits_{n\to+\infty} v_n$.

**N°5.**

Let $I_n=\int_1^e x^2(\ln x)^n dx$ where $n$ is a non-zero natural number.

1) a- Calculate $I_1$.
   b- Show that for all non-zero natural numbers $n$, we have :
   $I_{n+1}=\dfrac{e^3}{3}-\dfrac{n+1}{3}I_n$ .
   c- Deduce $I_2$.
2) a- Prove that $I_n\geq 0$.
   b- Show that the sequence $(I_n)$ is decreasing and deduce that it is convergent.

N° 6.

$(a_n)$ and $(b_n)$ are two sequences defined by :

$a_0 = 1$, $b_0 = 7$ and for all natural numbers $n \in IN$ $a_{n+1} = \dfrac{2a_n + b_n}{3}$ and

$b_{n+1} = \dfrac{a_n + 2b_n}{3}$.

On an axis $x'Ox$ consider the points $A_n$ and $B_n$ of respective abscissas $a_n$ and $b_n$ where $n \in IN$.

1) For all natural numbers $n \in IN$, let $u_n = b_n - a_n$.
   a- Show that $(u_n)$ is a geometric sequence whose first term and common ratio are to be determined.
   b- Express $u_n$ in terms of $n$.
2) Prove that the sequence $(a_n)$ is increasing and that $(b_n)$ is decreasing.
3) a- Compare $a_n$ and $b_n$.
   b- Deduce that the two sequences converge to the same limit.
4) For all natural numbers $n \in IN^*$, let $v_n = a_n + b_n$.
   a- Prove that the sequence $(v_n)$ is a constant sequence.
   b- Deduce that the segments $[A_nB_n]$ all have the same midpoint $I$.

N° 7.

Consider the sequence defined by $u_0 = \dfrac{1}{2}$ and $u_{n+1} = \dfrac{1}{2}\left(u_n + \dfrac{1}{u_n}\right)$ for all natural numbers $n$.

1) Show that this sequence is bounded below by 0.
2) Let $v_n = \dfrac{u_n - 1}{u_n + 1}$ for all natural numbers $n$.
   Find a relation between $v_{n+1}$ and $v_n$ then deduce the expression of $v_n$ in terms of $n$.
3) Find $u_n$ in terms of $n$ and determine $\lim\limits_{n\to+\infty} u_n$.

**N°8.**

Let $(u_n)$ be the sequence defined by $u_n = \int_{n-1}^{n} e^{-\frac{1}{2}x}\,dx$ for $n \in IN^*$.

1) Calculate $u_n$ in terms of $n$.
2) Show that $(u_n)$ is a geometric sequence whose first term and common ratio are to be determined.
3) $S_n$ is the sum of the first $n$ terms of the sequence $(u_n)$.
   Show that $S_n = \int_{0}^{n} e^{-\frac{1}{2}x}\,dx$.
4) Determine $\lim\limits_{n\to+\infty} S_n$.

**N°9.**

Consider the sequence $(u_n)$of general term $u_n = \ln\dfrac{n}{n+1}$, $n \in IN^*$.

1) Determine the sense of variations of the sequence $(u_n)$.
2) Determine the sign of $u_n$ for $n \in IN^*$.
3) Show that the sequence $(u_n)$is convergent and determine its limit.
4) Let $S_n = u_1 + u_2 + \ldots\ldots\ldots\ldots + u_n$.
   Express $S_n$ in terms of $n$ and determine $\lim\limits_{n\to+\infty} S_n$.

**N°10.**

Consider the sequence defined by $I_n = \int_{0}^{1} \dfrac{x^{2n+1}}{1+x^2}\,dx$ where $n$ is a natural integer.

1) Prove that $I_n \geq 0$ for all natural numbers $n$.
2) a- Calculate $I_0$.
   b- Calculate $I_0 + I_1$ and deduce the value of $I_1$.
3) Prove that the sequence $(I_n)$ is decreasing for all $n$.
4) Show that $I_n \leq \dfrac{1}{2n+2}$ and deduce the limit of $(I_n)$.

**N°11.**

Let $(u_n)$ be the sequence defined by $u_{n+1} = \sqrt{2+u_n}$ and $u_0 = 4$.

1) Show that this sequence is bounded below by 2.
2) Verify that $u_{n+1} - u_n = \dfrac{(1+u_n)(2-u_n)}{u_{n+1}+u_n}$.
   Deduce that the sequence $(u_n)$ is decreasing.
3) Prove that $|u_{n+1} - 2| \leq \dfrac{1}{4}|u_n - 2|$.
   Deduce that $|u_n - 2| \leq \dfrac{2}{4^n}$.
4) Deduce that $(u_n)$is convergent and determine its limit.

**N° 12.**

Consider the sequences $(u_n)$ and $(v_n)$defined by $u_n = \left(\dfrac{1}{2}\right)^n$ and $v_n = n\dfrac{\pi}{3}$.

Designate by $M_n$ the point of the complex plane of affix $z_n = u_n e^{i v_n}$.

Let $(a_n)$ be the sequence defined by $a_n = \left\|\overrightarrow{M_n M_{n+1}}\right\|$.

1) Show that $(a_n)$is a geometric sequence whose first term and common ratio are to be determined.
2) Determine $\lim\limits_{n \to +\infty} a_n$.

**N° 13.**

Consider the sequence $(u_n)$defined by $u_0 = \dfrac{1}{2}$ and $u_{n+1} = u_n{}^2 + \dfrac{3}{16}$ for all natural numbers $n \geq 0$.

Show that :

1) For all natural numbers $n$, $u_n \in \left[\dfrac{1}{4};\dfrac{1}{2}\right]$.
2) For all natural numbers $n$, $\left|u_{n+1} - \dfrac{1}{4}\right| \leq \dfrac{3}{4}\left|u_n - \dfrac{1}{4}\right|$.
3) For all natural numbers $n$, $\left|u_n - \dfrac{1}{4}\right| \leq \dfrac{1}{2} \times \left(\dfrac{3}{4}\right)^n$.
4) The sequence $(u_n)$ is convergent.

N° 14.

**Part A:**

Consider the function $f_1$ defined over $[-1,+\infty[$ by

$f_1(x)=\sqrt{1+x}\ e^{-x}$ and designate by $(C_1)$ its representative graph in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Study the differentiability of $f_1$ to the right of $-1$.
And interpret, graphically the result .
b- Calculate $\lim\limits_{x\to+\infty} f_1(x)$ . Interpret the result graphically.
c- Study the variations of $f_1$ and draw its table of variations.
d- Write an equation of the tangent to $(C_1)$ at the point of abscissa 0.

2) Show that the equation $f_1(x)=x$ admits a unique solution $\alpha$ such that $0.5<\alpha<1$.

3) Draw the curve $(C_1)$ .

4) $\lambda$ is a real number greater than or equal to 1.
Let $S(\lambda)=\int_1^{\lambda} f_1(x)dx$ .
a- Give a graphical interpretation of $S(\lambda)$.
b- Show that for all $\lambda\ge1,\ 0\le S(\lambda)\le\frac{2}{\sqrt{e}}$.

**Part B:**

For all $n\in IN^*$, consider the function $f_n$ defined over $IR$ by :

$f_n(x)=\sqrt{1+x}\ e^{-\frac{x}{n}}$.

Suppose that $I=\int_{-1}^{1}\sqrt{1+x}dx$ and designate by $(A_n)$ the sequence

defined by $A_n=\int_{-1}^{1} f_n(x)dx$.

1) Calculate $I$ .

2) Show that for all $n\in IN^*$, we have $1-e^{\frac{1}{n}}\le e^{-\frac{1}{n}}-1$.

3) Deduce that for all $x\in[-1,1]$; $\left|e^{-\frac{x}{n}}-1\right|\le e^{\frac{1}{n}}-1$.

4) Prove that $\left|A_n - I\right| \leq \frac{4}{3}\sqrt{2}\left(e^{\frac{1}{n}} - 1\right)$.

Deduce that the sequence $(A_n)$ is convergent and find its limit.

N° 15.

Let $f$ be the function defined over $IR$ by $f(x) = e^{3x} - e^{x}$.
Designate by $(\Gamma)$ its representative curve in an orthonormal system $(O, \vec{i}, \vec{j})$.

1) Determine $\lim\limits_{x \to -\infty} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.

2) Study the variations of $f$ over $IR$.

3) Show that the equation $f(x) = 1$ admits a unique solution $\alpha$ over $[0;1]$ and that $\alpha = \frac{1}{2}\ln\left(1 + e^{-\alpha}\right)$.

4) Let $g$ be the function defined over $I = [0;+\infty[$ by $g(x) = \ln\left(1 + e^{-x}\right)$.
   a- Show that for all $x$ belonging to $I$, $g(x) \in I$.
   b- Show that for all $x$ belonging to $I$, $\left|g'(x)\right| \leq \frac{1}{4}$.

5) Define the sequence $(u_n)$ by : $\begin{cases} u_0 = 0 \\ u_{n+1} = g(u_n) \end{cases}$ for all natural numbers $n \geq 0$.
   a- Prove that for all natural numbers $n \geq 0$, $\left|u_{n+1} - \alpha\right| \leq \frac{1}{4}\left|u_n - \alpha\right|$.
   b- Deduce that $\left|u_n - \alpha\right| \leq \left(\frac{1}{4}\right)^n$.
   c- Determine the limit of the sequence $(u_n)$.

N° 16.

***Part A:***

Let $f$ be the function defined over $]0;+\infty[$ by $f(x) = 5\frac{\ln x}{\sqrt{x}}$.

Designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine the limits of $f$ at $+\infty$ and at $0$ then deduce the asymptotes of $(C)$.
2) Study the variations of $f$ and draw its table of variations.
3) Trace $(C)$.

***Part B:***

1) show that the equation $f(x)=-5$ admits over $]0;+\infty[$ a unique solution $\alpha$ and such that $0.4<\alpha<0.6$
2) a- Suppose that ,for $x>0$ ,$h(x)=e^{-\sqrt{x}}$, verify that $\alpha$ is a solution of the equation $h(x)=x$.

   b- Calculate $h'(x)$ then prove that for all real numbers $x\in[0.4;0.6]$, $h(x)\in[0.4;0.6]$ and $|h'(x)|\le 0.43$
3) Consider the sequence $(u_n)$, defined over $IN$ by $u_0=0.4$ and $u_{n+1}=h(u_n)$, show that :

   a- For all natural numbers $n$, $u_n\in[0.4;0.6]$.

   b- For all natural numbers $n$, $|u_{n+1}-\alpha|\le 0.43|u_n-\alpha|$.

   c- For all natural numbers $n$, $|u_n-\alpha|\le 0.2(0.43)^n$.

   d- The sequence $(u_n)$ is convergent to $\alpha$.

# Solution of Problems

**N° 1.**

$u_0 = 1$ then $u_1 = \dfrac{1}{\sqrt{1+1}}$ and $u_2 = \dfrac{1}{\sqrt{2+1}}$. Suppose that $u_n = \dfrac{1}{\sqrt{n+1}}$.

We show that the relation remains true for $(n+1)$:

$$u_{n+1} = \frac{u_n}{\sqrt{1+u_n^{\,2}}} = u_n \times \frac{1}{\sqrt{1+u_n^{\,2}}} = \frac{1}{\sqrt{n+1}} \times \frac{1}{\sqrt{1+\dfrac{1}{n+1}}}$$

$$u_{n+1} = \frac{1}{\sqrt{n+1}} \times \frac{1}{\sqrt{\dfrac{n+2}{n+1}}} = \frac{1}{\sqrt{n+2}}.$$

Hence for all natural numbers $n$ , $u_n = \dfrac{1}{\sqrt{n+1}}$.

$\lim\limits_{n\to+\infty} u_n = 0$, so the sequence $(u_n)$ is convergent to 0.

**N° 2.**

1) $\dfrac{3}{2} - u_n = \dfrac{3}{2} - \dfrac{3n+\cos n}{2n+1} = \dfrac{3-2\cos n}{2(2n+1)}$.

But $-1 \le \cos n \le 1$, then $1 \le 3-2\cos n \le 5$, hence $\dfrac{3-2\cos n}{2(2n+1)} > 0$.

$2(2n+1) = 4n+2$ gives $4n+2 > 4n$ so $\dfrac{1}{4n+2} < \dfrac{1}{4n}$ and since $1 \le 3-2\cos n \le 5$ then $\dfrac{3-2\cos n}{2(2n+1)} < \dfrac{5}{4n}$.

2) We have $0 < \dfrac{3}{2} - u_n \le \dfrac{5}{4n}$ and since $\lim\limits_{n\to+\infty} \dfrac{1}{4n} = 0$ , then $\lim\limits_{n\to+\infty}\left(\dfrac{3}{2} - u_n\right) = 0$ and consequently $\lim\limits_{n\to+\infty} u_n = \dfrac{3}{2}$.

N°3.

1) For all natural numbers $n$, $|u_{n+1}| = \dfrac{|u_n|}{2+u_n^{\ 2}}$.

Moreover, $2+u_n^{\ 2} \geq 2$, then $|u_{n+1}| \leq \dfrac{|u_n|}{2}$.

2) Proving by mathematical induction that $|u_n| \leq \dfrac{|u_0|}{2^n}$ :

$\dfrac{|u_0|}{2^0} = |u_0|$, then $|u_0| \leq \dfrac{|u_0|}{2^0}$ , hence the statement is true for $n = 0$.

Assuming that $|u_p| \leq \dfrac{|u_0|}{2^p}$.

$|u_{p+1}| \leq \dfrac{|u_p|}{2}$ , then $|u_{p+1}| \leq \dfrac{1}{2}\times\dfrac{|u_0|}{2^p}$, which gives $|u_{p+1}| \leq \dfrac{|u_0|}{2^{p+1}}$

then for all natural numbers $n$, we have $|u_n| \leq \dfrac{|u_0|}{2^n}$.

3) We know that $\lim\limits_{n\to+\infty} \dfrac{1}{2^n} = 0$, then $\lim\limits_{n\to+\infty} \dfrac{|u_0|}{2^n} = 0$ and consequently $\lim\limits_{n\to+\infty} u_n = 0$.

N°4.

1) a- $v_n = \dfrac{u_{n+1}}{u_n} = \dfrac{(n+1)^2}{2^{n+1}} \times \dfrac{2^n}{n^2} = \dfrac{1}{2}\left(\dfrac{n+1}{n}\right)^2$ ,then $\lim\limits_{n\to+\infty} v_n = \dfrac{1}{2}$.

b- $v_n - \dfrac{1}{2} = \dfrac{1}{2}\left(\dfrac{n+1}{n}\right)^2 - \dfrac{1}{2} = \dfrac{1}{2}\left[\left(\dfrac{n+1}{n}\right)^2 - 1\right]$.

$v_n - \dfrac{1}{2} = \dfrac{1}{2}\left[\left(\dfrac{n+1}{n} - 1\right)\left(\dfrac{n+1}{n} + 1\right)\right] = \dfrac{1}{2}\times\dfrac{1}{n}\left(\dfrac{n+1}{n}+1\right) > 0$.

c- $v_n < \dfrac{3}{4}$ gives $\dfrac{1}{2}\left(\dfrac{n+1}{n}\right)^2 < \dfrac{3}{4}$ then $\left(\dfrac{n+1}{n}\right)^2 < \dfrac{3}{2}$ and consequently $\dfrac{n+1}{n} < \sqrt{1.5}$, so $1+\dfrac{1}{n} < \sqrt{1.5}$, which gives

$\frac{1}{n} < \sqrt{1.5} - 1$ which also gives $n > \frac{1}{\sqrt{1.5} - 1}$ therefore

$n > 4.45$, hence $N = 5$.

d- For $n \geq N$ ; $v_n < \frac{3}{4}$ then $\frac{u_{n+1}}{u_n} < \frac{3}{4}$, which gives $u_{n+1} < \frac{3}{4} u_n$.

2) a- Proving this inequality by mathematical induction :

For $n = 5$, $\left(\frac{3}{4}\right)^{5-5} u_5 = u_5$, then $u_5 \leq \left(\frac{3}{4}\right)^{5-5} u_5$.

Assuming that $u_n \leq \left(\frac{3}{4}\right)^{n-5} u_5$ and proving that it is true for $n+1$.

$u_n \leq \left(\frac{3}{4}\right)^{n-5} u_5$ gives $\frac{3}{4} \times u_n \leq \frac{3}{4} \times \left(\frac{3}{4}\right)^{n-5} u_5$.

But $u_{n+1} \leq \frac{3}{4} u_n$, therefore $u_{n+1} \leq \frac{3}{4} \times u_n \leq \frac{3}{4} \times \left(\frac{3}{4}\right)^{n-5} u_5$.

So $u_{n+1} \leq \left(\frac{3}{4}\right)^{n-4} u_5$, hence the inequality is always true .

b- For $n = 5$ ; $u_5 \leq \left(\frac{3}{4}\right)^{5-5} u_5$

For $n = 6$ ; $u_6 \leq \left(\frac{3}{4}\right)^{6-5} u_5$

For $n = 7$ ; $u_7 \leq \left(\frac{3}{4}\right)^{7-5} u_5$

For $n$ , $u_n \leq \left(\frac{3}{4}\right)^{n-5} u_5$

Adding we get :

$$u_5 + u_6 + \dots\dots\dots + u_n \leq \left[\left(\frac{3}{4}\right)^0 + \left(\frac{3}{4}\right)^1 + \left(\frac{3}{4}\right)^2 + \dots\dots\left(\frac{3}{4}\right)^{n-5}\right] u_5$$

$$S_n \leq \left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots\dots\dots\dots + \left(\frac{3}{4}\right)^{n-5}\right] u_5.$$

c- The sum $1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+.............+\left(\frac{3}{4}\right)^{n-5}$ is the sum of four terms of a geometric sequence whose first term is 1 and common ratio $\frac{3}{4}$.

Then $1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+.............+\left(\frac{3}{4}\right)^{n-5}=\frac{1-\left(\frac{3}{4}\right)^{n-4}}{1-\frac{3}{4}}$

$$=4\left[1-\left(\frac{3}{4}\right)^{n-4}\right]\leq 4.$$

Consequently, $S_n \leq 4u_5$.

d- $S_{n+1}-S_n=(u_5+u_6+...+u_n+u_{n+1})-(u_5+u_6+...+u_n)$

$S_{n+1}-S_n=u_{n+1}>0$ hence the sequence $(S_n)$ is increasing .

The sequence $(S_n)$ is increasing and bounded above by $4u_5$ so it is convergent.

**N°5.**

1) If the sequence is convergent then its limit $\ell$ is a solution of the equation $f(\ell)=\ell$ where $f$ is the function defined by $u_{n+1}=f(u_n)$.

Therefore, $\ell=\sqrt{1+\ell^2}$ which gives $\ell^2=1+\ell^2$ which is impossible hence, the sequence $(u_n)$ is divergent.

2) a- $v_{n+1}-v_n=u_{n+1}^{\ 2}-u_n^{\ 2}=1+u_n^{\ 2}-u_n^{\ 2}=1$ , then $(v_n)$ is an arithmetic sequence of common difference 1 and first term $v_0=u_0^{\ 2}=4$.

b- $v_n=v_0+nd=4+n$ and $u_n=\sqrt{v_n}=\sqrt{4+n}$.

c- $\lim\limits_{n\to+\infty} u_n=+\infty$.

d- $\lim\limits_{n\to+\infty}\frac{u_n}{\sqrt{n}}=\lim\limits_{n\to+\infty}\frac{\sqrt{4+n}}{\sqrt{n}}=1$.

N° 6.

1) $f'(x)=\dfrac{1}{9(-x+1)^2}>0$ for all $x\in\left]1;+\infty\right[$ hence $f$ is an increasing function over $\left]1;+\infty\right[$.

2) a- $u_1=\dfrac{1}{3}<\dfrac{2}{3}$ , supposing that $u_n<\dfrac{2}{3}$ and proving that $u_{n+1}<\dfrac{2}{3}$.

$$u_{n+1}-\frac{2}{3}=\frac{3\left(u_n-\frac{2}{3}\right)}{9(1-u_n)}.$$

But, $u_n<\dfrac{2}{3}$ then $u_n-\dfrac{2}{3}<0$ and $1-u_n>0$ so $u_{n+1}-\dfrac{2}{3}<0$.

And consequently the sequence $(u_n)$ is bounded above by $\dfrac{2}{3}$.

b- Showing that $(u_n)$ is increasing .

$u_1=\dfrac{1}{3}$ and $u_2=\dfrac{1}{2}$ then $u_1<u_2$.

Assuming that $u_{n-1}<u_n$ and proving that $u_n<u_{n+1}$.

$u_{n-1}<u_n$ and $f$ is increasing hence $f(u_{n-1})<f(u_n)$.

So, $u_n<u_{n+1}$.

c- The sequence $(u_n)$ is increasing and bounded above then it is convergent to a limit $\ell$ which is the solution of the equation $f(\ell)=\ell$ which gives $\ell=\dfrac{4-3\ell}{9(1-\ell)}$, which is the quadratic equation : $-9\ell^2+12\ell-4=0$ that admits a double root $\ell'=\ell''=\dfrac{2}{3}$, hence this sequence is convergent to $\dfrac{2}{3}$.

3) Remark that $u_1=\dfrac{2\times1-1}{3\times1}$, $u_2=\dfrac{2\times2-1}{3\times2}$ and $u_3=\dfrac{2\times3-1}{3\times3}$

Suppose that $u_n=\dfrac{2\times n-1}{3\times n}$.

$$u_{n+1}=\frac{4-3u_n}{9(1-u_n)}=\frac{1}{9}\frac{4-3\left(\frac{2n-1}{3n}\right)}{1-\frac{2n-1}{3n}}=\frac{2n+1}{3(n+1)}=\frac{2(n+1)-1}{3(n+1)}.$$

Hence, for all non-zero natural numbers: $u_n=\frac{2n-1}{3n}$.

N°7.

1) $v_{n+1}=4u_{n+1}-6(n+1)+15=\frac{4}{3}u_n+4n-4-6n-6+15$.

$v_{n+1}=\frac{1}{3}(4u_n-6n+15)=\frac{1}{3}v_n$, then $(v_n)$ is a geometric sequence

of first term $v_0=4u_0+15=19$ and common ratio $r=\frac{1}{3}$.

2) $v_n=v_0\times r^n=19\times\left(\frac{1}{3}\right)^n$ and since $v_n=4u_n-6n+15$ we get

$4u_n-6n+15=19\times\left(\frac{1}{3}\right)^n$, which gives $u_n=\frac{19}{4}\times\frac{1}{3^n}+\frac{6n-15}{4}$.

3) For all natural numbers $n$, let $t_n=\frac{19}{4}\times\frac{1}{3^n}$ and $w_n=\frac{6n-15}{4}$

so $u_n=t_n+w_n$ where $(t_n)$ is a geometric sequence of common

ratio $r=\frac{1}{3}$ and of first term $t_0=\frac{19}{4}$ and $(w_n)$ is an arithmetic

sequence of common difference $d=\frac{3}{2}$ and first term $w_0=-\frac{15}{4}$

since $w_{n+1}-w_n=\frac{6(n+1)-15}{4}-\frac{6n-15}{4}=\frac{3}{2}$.

4) $T_n$ is the sum of $(n+1)$consecutive terms of a geometric sequence

then $T_n=\frac{19}{4}\times\frac{1-\left(\frac{1}{3}\right)^{n+1}}{1-\frac{1}{3}}$, so $T_n=\frac{3}{2}\times\frac{19}{4}\left(1-\frac{1}{3^{n+1}}\right)$.

$W_n$ is the sum of $(n+1)$consecutive terms of an arithmetic sequence hence:

$$W_n = \frac{n+1}{2}(w_0 + w_n) = \frac{n+1}{2}\left(-\frac{15}{4} + \frac{3}{2}n - \frac{15}{4}\right) = \frac{3(n-5)(n+1)}{4}$$

$u_n = t_n + w_n$ so $U_n = T_n + W_n = \dfrac{57}{8}\left(1 - \dfrac{1}{3^{n+1}}\right) + \dfrac{3(n-5)(n+1)}{4}$.

N°8.

1) a- $f(x) - 1 = \dfrac{x^2}{2x-1} - 1 = \dfrac{x^2 - 2x + 1}{2x-1} = \dfrac{(x-1)^2}{2x-1}$.

But $x > 1$ then $2x - 1 > 0$ and $(x-1)^2 > 0$ therefore $f(x) > 1$.

b- $u_0 = 2 > 1$, we assume that $u_n > 1$ and we prove that $u_{n+1} > 1$.
But $u_{n+1} = f(u_n) > 1$ ( from part a) .

2) a- $u_n > 1$ for all natural numbers $n$ then the sequence $(v_n)$ is defined since $u_n \neq 0$ .

$u_n > 1$ then $v_n > 0$ and consequently the sequence $(w_n)$ is defined for all natural numbers $n$.

b- $w_{n+1} = \ln(v_{n+1}) = \ln\left(\dfrac{u_{n+1} - 1}{u_{n+1}}\right)$, but

$$\frac{u_{n+1}-1}{u_{n+1}} = 1 - \frac{1}{u_{n+1}} = 1 - \frac{2u_n - 1}{u_n^{\,2}} = \frac{u_n^{\,2} - 2u_n + 1}{u_n^{\,2}} = \left(\frac{u_n - 1}{u_n}\right)^2.$$

Therefore, $w_{n+1} = \ln\left(\dfrac{u_n - 1}{u_n}\right)^2 = 2\ln\dfrac{u_n - 1}{u_n} = 2w_n$.

Hence, $(w_n)$ is a geometric sequence of common ratio $r = 2$ and first term $w_0 = \ln v_0 = \ln\dfrac{1}{2} = -\ln 2$.

c- $w_n = w_0 \times r^n = (-\ln 2) \times 2^n$.

$v_n = e^{w_n} = e^{(-\ln 2)\times 2^n}$.

d- $v_n = \dfrac{-1 + u_n}{u_n}$ gives $v_n u_n = -1 + u_n$ then $(v_n - 1)u_n = -1$ and consequently, $u_n = \dfrac{-1}{v_n - 1} = \dfrac{-1}{e^{(-\ln 2)\times 2^n} - 1} = \dfrac{-1}{\frac{1}{2}e^{2^n} - 1}$.

$\lim\limits_{n \to +\infty} u_n = 1$.

**N°9.**

1) $u_0 = 2$, then $u_1 = \dfrac{2 + 4u_0}{3 + u_0} = 2$.

For $u_n = 2$; $u_{n+1} = \dfrac{2 + 4u_n}{3 + u_n} = \dfrac{10}{5} = 2$.

Then $u_n = 2$ for all natural numbers $n$ ,consequently the sequence $(u_n)$ is constant .

2) a- $u_0 = 1$, then $u_0 > 0$, suppose that $u_n > 0$, we get $u_{n+1} > 0$.

$u_0 = 1$, then $u_0 < 2$, suppose that $u_n < 2$.

$u_{n+1} - 2 = \dfrac{2 + 4u_n}{3 + u_n} - 2 = \dfrac{2(u_n - 2)}{3 + u_n} < 0$ since $u_n < 2$.

Hence for all natural numbers $n$, $0 < u_n < 2$.

b- $u_{n+1} - u_n = \dfrac{2 + 4u_n}{3 + u_n} - u_n = \dfrac{-u_n^2 + u_n + 2}{3 + u_n}$.

The trinomial $-u_n^2 + u_n + 2$ admits two roots that are $-1$ and 2 therefore $u_{n+1} - u_n = \dfrac{-(u_n + 1)(u_n - 2)}{3 + u_n} > 0$ since $u_n + 1 > 0$ and $u_n - 2 < 0$.

Hence the sequence $(u_n)$ is increasing for all natural numbers $n$.

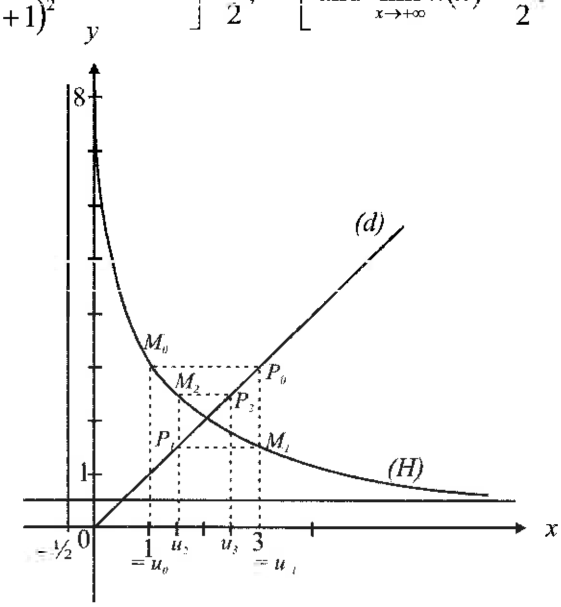
c- This sequence is increasing and bounded above by 2 for all natural numbers $n$ hence it is convergent to a limit $\ell$ solution of the equation $\ell = \dfrac{2 + 4\ell}{3 + \ell}$, which gives $\ell^2 - \ell - 2 = 0$ and that admits two roots of which $\ell = 2$ is accepted .

N° 10.

1) $u_0 = 1$, then $u_1 = \dfrac{u_0+8}{2u_0+1} = 3$, $u_2 = \dfrac{u_1+8}{2u_1+1} = \dfrac{11}{7}$ and

$u_3 = \dfrac{u_2+8}{2u_2+1} = \dfrac{67}{29}$.

2) a- $h'(x) = \dfrac{-15}{(2x+1)^2} < 0$ over $\left]-\dfrac{1}{2};+\infty\right[$ and $\lim\limits_{x\to+\infty} h(x) = \dfrac{1}{2}$.




b- We place on the axis $x'x$ the point of abscissa $u_0 = 1$ then over $(H)$ the point $M_0$ of abscissa $u_0$ of ordinate $u_1 = h(u_0)$.
The parallel through $M_0$ to the axis $x'x$ cuts the straight line $(d)$ at the point $P_0$ of abscissa $u_1$.
The point of the axis $x'x$ of abscissa $u_1$ is the orthogonal projection of $P_0$ on $x'x$.
The parallel to the axis $y'y$ through the point of abscissa $u_1$ of $x'x$ cuts $(H)$ at the point $M_1$ of abscissa $u_1$ and of ordinate $u_2$.
The parallel through $M_1$ to the axis $x'x$ cuts the straight line $(d)$ at the point $P_1$ of abscissa $u_2$.

The point of the axis $x'x$ of abscissa $u_2$ is the orthogonal projection of $P_1$ on $x'x$ .

c- The figure obtained permits us to conjecture that the sequence $(u_n)$ converges to a real number $\ell$ of abscissa the point of intersection of $(H)$ and $(d)$.

Therefore, $\ell = \dfrac{\ell+8}{2\ell+1}$, which gives $\ell = 2$.

3) a- $v_0 = \dfrac{u_0-2}{u_0+2} = -\dfrac{1}{3}$ , $v_1 = \dfrac{u_1-2}{u_1+2} = \dfrac{1}{5}$ , $v_2 = \dfrac{u_2-2}{u_2+2} = -\dfrac{3}{25}$

b- $v_{n+1} = \dfrac{u_{n+1}-2}{u_{n+1}+2} = \dfrac{\dfrac{u_n+8}{2u_n+1}-2}{\dfrac{u_n+8}{2u_n+1}+2} = \dfrac{-3u_n+6}{5u_n+10} = -\dfrac{3}{5}v_n$.

Hence $(v_n)$ is a geometric sequence of common ratio $r = -\dfrac{3}{5}$

and first term $v_0 = \dfrac{u_0-2}{u_0+2} = -\dfrac{1}{3}$.

c- $v_n = v_0 \times r^n = \dfrac{-1}{3} \times \left(-\dfrac{3}{5}\right)^n$, $\lim\limits_{n\to+\infty} v_n = 0$ since $-1 < r < 1$.

d- $v_n = \dfrac{u_n-2}{u_n+2}$ gives $v_n u_n + 2v_n = u_n - 2$ then

$u_n(v_n-1) = -2v_n - 2$ and consequently $u_n = \dfrac{-2v_n-2}{v_n-1}$ hence,

$\lim\limits_{n\to+\infty} u_n = 2$ since $\lim\limits_{n\to+\infty} v_n = 0$.

**N° 11.**

1) $0 \le x \le 1$, then $x^n \ge 0$ and since $e^{-x} > 0$ then $x^n e^{-x} > 0$ for all $0 \le x \le 1$ and consequently $I_n \ge 0$ and consequently this sequence is bounded below by 0 .

2) $I_{n+1} - I_n = \int_0^1 x^{n+1}e^{-x}dx - \int_0^1 x^n e^{-x}dx = \int_0^1 x^n e^{-x}(x-1)dx$

but $x-1\leq 0$ and $x^n e^{-x}>0$ then $I_{n+1}-I_n\leq 0$ and consequently, the sequence is decreasing.

3) a- $I_{n+1}=\int_0^1 x^{n+1}e^{-x}dx$, let $u=x^{n+1}$ and $v'=e^{-x}$, therefore

$u'=(n+1)x^n$ and $v=-e^{-x}$ so

$I_{n+1}=\left[-x^{n+1}e^{-x}\right]_0^1+\int_0^1 (n+1)x^n e^{-x}dx$, as a result

$I_{n+1}=-e^{-1}+(n+1)I_n$.

b- Taking $u=x$ and $v'=e^{-x}$, then $u'=1$ and $v=-e^{-x}$

which gives $\int_0^1 xe^{-x}dx=-xe^{-x}\Big|_0^1+\int_0^1 e^{-x}dx=-xe^{-x}\Big|_0^1-e^{-x}\Big|_0^1$.

Therefore, $I_1=1-2e^{-1}$.

For $n=1$, the relation $I_{n+1}=-e^{-1}+(n+1)I_n$ gives $I_2=-e^{-1}+2I_1$.

Therefore, $I_2=-e^{-1}+2-4e^{-1}=2-5e^{-1}$.

N° 12.

1) For all $n$ in $IN^*$, and $x$ in $[1;e]$,

$I_{n+1}-I_n=\int_1^e (\ln x)^{n+1}dx-\int_1^e (\ln x)^n dx=\int_1^e (\ln x)^n(\ln x-1)dx$.

$x\in[1;e]$ then $0\leq \ln x\leq 1$ therefore, $\ln^n x\geq 0$ and $\ln x-1\leq 0$ and consequently, $I_{n+1}-I_n\leq 0$, hence $I_{n+1}\leq I_n$.

So the sequence $(I_n)$ is decreasing.

2) a- $I_1=\int_1^e \ln x\,dx=[x\ln x-x]_1^e=1$.

b- $I_n=\int_1^e (\ln x)^n dx$ and $I_{n+1}=\int_1^e (\ln x)^{n+1}dx$.

Taking $\begin{cases}u=(\ln x)^{n+1}\\ v'=1\end{cases}$ which gives $\begin{cases}u'=(n+1)(\ln x)^n\times\dfrac{1}{x}\\ v=x\end{cases}$

Therefore, $I_{n+1} = \left[x(\ln x)^{n+1}\right]_1^e - \int_1^e (n+1)(\ln x)^n dx = e-(n+1)I_n$.

3) a- $\ln^n x \geq 0$ for $x \in [1;e]$ then $I_{n+1} \geq 0$ for all $n$ in $IN^*$.
Consequently $e-(n+1)I_n \geq 0$ which gives $(n+1)I_n \leq e$.

b- $I_n \geq 0$ and $(n+1)I_n \leq e$ which gives $0 \leq I_n \leq \dfrac{e}{n+1}$.

Since $\lim\limits_{n\to+\infty} \dfrac{e}{n+1} = 0$ hence $\lim\limits_{n\to+\infty} I_n = 0$.

c- $nI_n + (I_n + I_{n+1}) = nI_n + I_n + I_{n+1} = (n+1)I_n + e-(n+1)I_n = e$.
We deduce that $nI_n = e - I_n - I_{n+1}$, therefore $\lim\limits_{n\to+\infty} nI_n = e$.

**N° 13.**

1) a- $w_{n+1} = v_{n+1} - u_{n+1} = \dfrac{u_n+3v_n}{4} - \dfrac{u_n+2v_n}{3}$

$w_{n+1} = \dfrac{-u_n+v_n}{12} = \dfrac{1}{12}w_n$, hence $(w_n)$ is a geometric sequence of common ratio $r = \dfrac{1}{12}$ and first term $w_1 = v_1 - u_1 = -11$.

b- $w_n = w_1 \times r^{n-1} = -11 \times \left(\dfrac{1}{12}\right)^{n-1}$.

$\lim\limits_{n\to+\infty} w_n = 0$ since $0 < r < 1$.

2) $u_{n+1} - u_n = \dfrac{u_n+2v_n}{3} - u_n = \dfrac{2v_n - 2u_n}{3} = \dfrac{2}{3}w_n$ and since $w_n < 0$ then $u_{n+1} - u_n < 0$ and consequently the sequence $(u_n)$ is decreasing.

Similarly, $v_{n+1} - v_n = -\dfrac{1}{4}w_n$ hence $v_{n+1} > v_n$ and consequently the sequence $(v_n)$ is increasing .

3) a- $u_1 = 12$ and $v_1 = 1$ then $u_1 > v_1$.
Assume that $u_n > v_n$ and proving that $u_{n+1} > v_{n+1}$.

$u_{n+1} - v_{n+1} = \frac{1}{12}(u_n - v_n)$ then $u_{n+1} - v_{n+1} > 0$ so $u_{n+1} > v_{n+1}$.

b- The sequence $(u_n)$ is decreasing then $u_n < u_1$ but $v_n < u_n$ so $v_n < u_1$ for all natural numbers $n \in IN^*$ ,consequently $(v_n)$ is increasing and bounded above by $u_1$ hence it is convergent . Similarly the sequence $(u_n)$ being decreasing and bounded below by $v_1$ is convergent .

4) a- For all natural numbers $n \in IN^*$, we have:
$t_n = 3u_{n+1} + 8v_{n+1} = u_n + 2v_n + 2u_n + 6v_n = 3u_n + 8v_n = t_n$.
Hence $(t_n)$ is a constant sequence.

b- Since $(t_n)$ is a constant sequence, then $t_n = t_1 = 44$ therefore, $3u_n + 8v_n = 44$ and since $v_n - u_n = -11 \times \left(\frac{1}{12}\right)^{n-1}$ we get by solving the system that:

$$u_n = 4 + 8\left(\frac{1}{12}\right)^{n-1} \quad \text{and} \quad v_n = 4 - 3\left(\frac{1}{12}\right)^{n-1}.$$

c- $\lim\limits_{n\to+\infty} u_n = \lim\limits_{n\to+\infty}\left[4 + 8\left(\frac{1}{12}\right)^{n-1}\right] = 4 = \lim\limits_{n\to+\infty} v_n$.

**N° 14.**

1) $u_0 = \int_0^1 \frac{e^x}{1+e^x}dx = \ln(1+e^x)\Big]_0^1 = \ln(1+e) - \ln 2 = \ln\frac{1+e}{2}$.

2) $u_0 + u_1 = \int_0^1 \frac{e^x}{1+e^x}dx + \int_0^1 \frac{e^x}{e^x(1+e^x)}dx = \int_0^1 \frac{1+e^x}{1+e^x}dx = \int_0^1 dx = [x]_0^1 = 1$

$u_1 = 1 - u_0 = 1 - \ln\frac{1+e}{2}$.

3) $\frac{e^x}{e^{nx}(1+e^x)} > 0$ for all real numbers, then $\int_0^1 \frac{e^x}{e^{nx}(1+e^x)}dx > 0$.

Consequently, the sequence $(u_n)$ is bounded below by 0 .

4) Remark that $\dfrac{e^x}{e^{nx}\left(1+e^x\right)}=\dfrac{e^{(1-n)x}}{1+e^x}$.

$$u_{n+1}-u_n=\int_0^1\frac{e^{-nx}}{1+e^x}dx-\int_0^1\frac{e^{(1-n)x}}{1+e^x}dx=\int_0^1\frac{e^{-nx}\left(1-e^x\right)}{1+e^x}dx$$

$0\le x\le 1$ then $e^0\le e^x$, so $1-e^x\le 0$.

Consequently, $u_{n+1}-u_n\le 0$.

Hence, the sequence $(u_n)$ is decreasing for all natural numbers $n$.

5) a- $$u_{n-1}+u_n=\int_0^1\frac{e^{(2-n)x}}{1+e^x}dx+\int_0^1\frac{e^{(1-n)x}}{1+e^x}dx=\int_0^1\frac{e^{(1-n)x}\left(1+e^x\right)}{1+e^x}dx$$

$$u_{n-1}+u_n=\int_0^1 e^{(1-n)x}dx=\left[-\frac{1}{n-1}e^{(1-n)x}\right]_0^1=\frac{1}{n-1}\left(1-e^{1-n}\right).$$

b- From the previous relation, $u_1+u_2=\dfrac{1}{2-1}\left(1-e^{1-2}\right)=1-e^{-1}$

$$u_2=1-e^{-1}-1+\ln\frac{1+e}{2}=\ln\frac{1+e}{2}-e^{-1}.$$

6) a- $$\lim_{n\to+\infty}v_n=\lim_{n\to+\infty}\frac{u_{n-1}+u_n}{2}=\lim_{n\to+\infty}\frac{1}{2}\left[\frac{1-e^{-(n-1)}}{n-1}\right]=0.$$

b- $u_n\ge 0$ and $(u_n)$ is decreasing for $n\ge 2$

$u_n\le u_{n-1}$ therefore, $2u_n\le u_{n-1}+u_n$, so $u_n\le\dfrac{1}{2}\left[u_{n-1}+u_n\right]$

hence, $u_n\le v_n$.

c- Since $0\le u_n\le v_n$ and $\lim\limits_{n\to+\infty}v_n=0$ then $\lim\limits_{n\to+\infty}u_n=0$.

**N°15**

1) a- Using integration by parts :

Let $u=\ln t$ and $v'=\dfrac{1}{t^2}$ then $u'=\dfrac{1}{t}$ and $v=-\dfrac{1}{t}$.

$$I_n=-\frac{\ln t}{t}\bigg|_{e^n}^{e^{n+1}}+\int_{e^n}^{e^{n+1}}\frac{1}{t^2}dt=-\frac{\ln t}{t}\bigg|_{e^n}^{e^{n+1}}-\frac{1}{t}\bigg|_{e^n}^{e^{n+1}}$$

$$=-\frac{\ln e^{n+1}}{e^{n+1}}+\frac{\ln e^n}{e^n}-\frac{1}{e^{n+1}}+\frac{1}{e^n}=\frac{n+1}{e^n}-\frac{n+2}{e^{n+1}}.$$

2) a- $A_n = \int_{e^n}^{e^{n+1}} \left[\frac{\ln x}{x^2} + \frac{e}{x^2}\right]dt = I_n + e\left[-\frac{1}{x}\right]_{e^n}^{e^{n+1}}$

$A_n = I_n + e\left[-\frac{1}{e^{n+1}} + \frac{1}{e^n}\right] = I_n + \frac{e-1}{e^n}$.

b- $I_0 = 1 - \frac{2}{e}$ , $\quad A_0 = I_0 + e - 1 = 1 - \frac{2}{e} + e - 1 = e - \frac{2}{e}$.

c- $A_0$ represents the area of the domain limited by $(C)$ the axis $x'x$ and the two straight lines of equations $x = e^n$ and $x = e^{n+1}$.

3) $\lim_{n\to+\infty} A_n = 0$.

## N° 16.

**Part A:**

1) $f'(x) = -2xe^{1-x^2} + 2x^3e^{1-x^2} = 2xe^{1-x^2}(x^2-1)$.

$f'(x) \geq 0$ for $x^2 - 1 \geq 0$ which is true for $x \geq 1$ since $x \geq 0$.

$f'(x) < 0$ for $x < 1$.

Then the table of variations of $f$ is as follows:

| $x$ | 0 | | 1 | | $+\infty$ |
|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | 0 | $+$ | |
| $f(x)$ | | $\searrow$ | 0 | $\nearrow$ | |

$f(0) = 1$.

$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty}\left[1 - \frac{x^2}{e^{x^2-1}}\right] = \lim_{x\to+\infty}\left[1 - \frac{2x}{2xe^{x^2-1}}\right]$

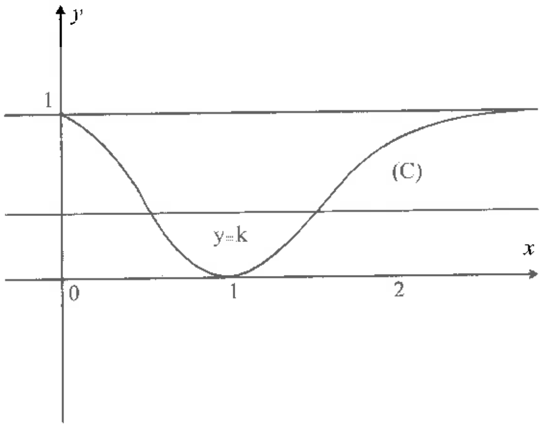
$= \lim_{x\to+\infty}\left[1 - \frac{1}{e^{x^2-1}}\right] = 1$ .

So, the straight line of equation $y = 1$ is an asymptote to $(C)$.

Then, $f$ admits at the point of abscissa 1 a minimum equal to $f(1) = 0$.

Then, $f(x) > 0$ over $[0;+\infty[$.

2)




3) The straight line of equation $y = k$ cuts $(C)$ in two distinct points for $0 < k < 1$ .
Hence, the equation $f(x) = k$ admits two distinct solutions; one in the interval $]0;1[$ and the other in the interval $]1;+\infty[$.
For $k = 0$ , there is a double root $x = 1$.
For $k = 1$ , there is one unique solution $x = 0$.
For $k > 1$ , there is no solution .
For $k < 0$ , there is no solution .

4) From the preceding question, the equation $f(x) = \frac{1}{n}$ admits two distinct roots for $\frac{1}{n} \in ]0;1[$ which is true when $n > 1$.

***Part B:***

1) $n \geq 2$ ; $\frac{1}{n}$ is then in the interval $]0;1[$.
Hence, the equation $f(x) = \frac{1}{n}$ admits two solutions $u_n$ and $v_n$ belonging to the intervals $[0;1]$ and $[1;+\infty[$ respectively.

2) $f(u_n) = \frac{1}{n}$ with $u_n \in [0;1]$ and $f(v_n) = \frac{1}{n}$ with $v_n \in [1;+\infty[$.

$n < n+1$ gives $\frac{1}{n} > \frac{1}{n+1}$ then $f(u_n) > f(u_{n+1})$.

But $f$ is decreasing over $[0;1]$ hence $u_n < u_{n+1}$.

As a result, the sequence $(u_n)$ is increasing.

Similarly, $n < n+1$ gives $\frac{1}{n} > \frac{1}{n+1}$ then $f(v_n) > f(v_{n+1})$.

But on $[1;+\infty[$ $f$ is increasing so $v_n > v_{n+1}$.

The sequence $(v_n)$ is hence decreasing.

4) The sequence $(u_n)$ is increasing and bounded above by 1 for all natural numbers $n \geq 2$ so it is convergent.

$\lim\limits_{n\to+\infty} \frac{1}{n} = 0$ then $\lim\limits_{n\to+\infty} f(u_n) = 0$ so from the table of variations of $f$, we conclude that $\lim\limits_{n\to+\infty} u_n = 1$.

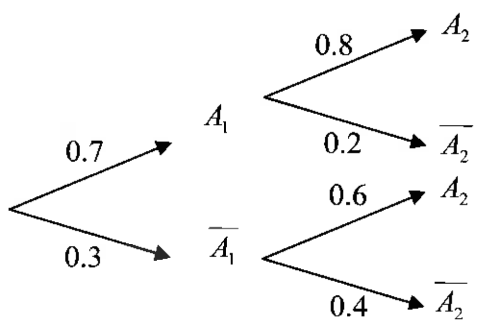
Similarly, we have $\lim\limits_{n\to+\infty} v_n = 1$.

**N° 17.**

1) $p(A_2) = p(A_2 \cap A_1) + p(A_2 \cap \overline{A_1})$

$p(A_2) = p(A_2 / A_1) \times p(A_1) + p(A_2 / \overline{A_1}) \times p(\overline{A_1})$

$p(A_2) = 0.8 \times 0.7 + 0.6 \times 0.3 = 0.74$




2) a- $p_{n+1} = p(A_{n+1}) = p(A_{n+1} \cap A_n) + p(A_{n+1} \cap \overline{A_n})$

$p_{n+1} = p(A_{n+1} / A_n) \times p(A_n) + p(A_{n+1} / \overline{A_n}) \times p(\overline{A_n})$

$p_{n+1} = 0.8 \times p_n + 0.6 \times (1 - p_n) = 0.2p_n + 0.6$

b- $u_{n+1} = p_{n+1} - 0.75 = 0.2p_n + 0.6 - 0.75 = 0.2p_n - 0.15$

$u_{n+1} = 0.2(p_n - 0.75) = 0.2u_n$

Then, $(u_n)$ is a geometric sequence of common ratio $0.2$ and of first term $u_1 = p_1 - 0.75 = 0.7 - 0.75 = -0.05$.

$u_n = u_1 \times r^{n-1} = -0.05 \times (0.2)^{n-1}$, and

$p_n = u_n + 0.75 = 0.75 - 0.05 \times (0.2)^{n-1}$.

$\lim\limits_{n \to +\infty} p_n = 0.75$

**N° 18.**

***Part A.***

1) a- $\lim\limits_{x \to 0} f(x) = -\infty$ then the axis $y'y$ is an asymptote to $(C)$.

b- $\lim\limits_{x \to +\infty} f(x) = \lim\limits_{x \to +\infty}\left(x + \ln\dfrac{x}{2x}\right) = +\infty$.

2) $f(x)$ can be written in the form $f(x) = x + \ln x - \ln(2x+1)$ then

$$f'(x) = 1 + \frac{1}{x} - \frac{2}{2x+1} = \frac{2x^2 + x + 1}{x(2x+1)} > 0 \text{ over } ]0;+\infty[$$

Hence, the table of variations of $f$ is as follows:

| $x$ | $0$ | | $+\infty$ |
|---|---|---|---|
| $f'(x)$ | $\Vert$ | $+$ | |
| $f(x)$ | $\Vert$ $-\infty$ | $\nearrow$ | $+\infty$ |

3) a- $\lim\limits_{x \to +\infty}(f(x) - y) = \lim\limits_{x \to +\infty}\left[\ln\left(\dfrac{x}{2x+1}\right) + \ln 2\right]$.

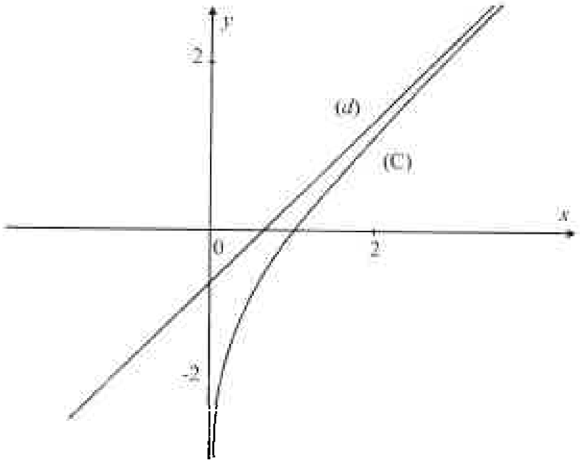
$$= \lim_{x \to +\infty} \ln\frac{2x}{2x+1} = \ln 1 = 0 \ .$$

Then the straight line $(d)$ of equation $y = x - \ln 2$ is an asymptote to $(C)$.

$f(x) - y = \ln\dfrac{2x}{2x+1}$ , but $\dfrac{2x}{2x+1} < 1$ hence $\ln\dfrac{2x}{2x+1} < 0$.

Consequently, $(C)$ is below $(d)$.

b-




4) $f$ is continuous and strictly increasing from $-\infty$ to $+\infty$, then $(C)$ cuts the axis $x'x$ at a unique point so the equation $f(x)=0$ admits over $]0;+\infty[$ a unique solution $\alpha$.

$f(1) = 1 - \ln 3 \approx -0.09 < 0$ and $f\left(\dfrac{5}{4}\right) \approx 0.22 > 0$ then $1 < \alpha < \dfrac{5}{4}$.

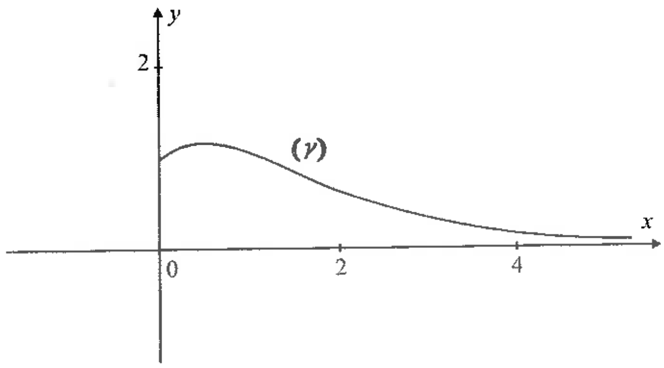
***Part B:***

1) $\lim\limits_{x\to+\infty} g(x) = \lim\limits_{x\to+\infty} \dfrac{2x+1}{e^x} = \lim\limits_{x\to+\infty} \dfrac{2}{e^x} = 0$ then $x'x$ is an asymptote to $(C)$.

2) $g'(x) = (-2x+1)e^{-x}$ , then the table of variations of $g$ is as follows :

| $x$ | $0$ | | $\frac{1}{2}$ | | $+\infty$ |
|---|---|---|---|---|---|
| $g'(x)$ | | $+$ | $0$ | $-$ | |
| $g(x)$ | $1$ | $\nearrow$ | $2e^{-\frac{1}{2}}$ | $\searrow$ | $0$ |

3)




***Part C:***

1) $\alpha$ verifies the equation $f(\alpha)=0$ therefore,

$\alpha+ln\left(\dfrac{\alpha}{2\alpha+1}\right)=0$, so $ln\left(\dfrac{\alpha}{2\alpha+1}\right)=-\alpha$ which gives

$\dfrac{\alpha}{2\alpha+1}=e^{-\alpha}$ hence $g(\alpha)=(2\alpha+1)e^{-\alpha}=\alpha$.

Consequently, $\alpha$ is a solution of the equation $g(x)=x$.

2) For $x\in\left[1;\dfrac{5}{4}\right]$, $g$ is decreasing hence, $g\left(\dfrac{5}{4}\right)\le g(x)\le g(1)$ which

gives: $3.5e^{-\frac{5}{4}}\le g(x)\le 3e^{-1}$, so $1.003\le g(x)\le 1.1$

hence $g(x)\in\left[1;\dfrac{5}{4}\right]$.

3) $g'(x) = (1-2x)e^{-x}$ and $g''(x) = (2x-3)e^{-x}$.

For $x \in \left[1;\frac{5}{4}\right]$, $2x-3<0$ hence $g''(x) \le 0$ therefore, $g'$ is decreasing over $\left[1;\frac{5}{4}\right]$ which gives $g'\left(\frac{5}{4}\right) \le g'(x) \le g'(1)$ which in turn gives: $-1{,}5e^{-\frac{5}{4}} \le g'(x) \le -e^{-1}$, so $-0.43 \le g'(x) \le -0.37$

Hence, for $x \in \left[1;\frac{5}{4}\right]$, $|g'(x)| \le \frac{1}{2}$.

4) a- $u_0 = 1$ then $1 \le u_0 \le \frac{5}{4}$, suppose that $1 \le u_n \le \frac{5}{4}$ we get $u_{n+1} = g(u_n) \in \left[1;\frac{5}{4}\right]$ hence for all natural numbers $n$, $u_n \in \left[1;\frac{5}{4}\right]$.

b- Applying the mean value theorem to $g$ over $[u_n;\alpha]$, we get:

$|g(u_n) - g(\alpha)| \le \frac{1}{2}|u_n - \alpha|$ but $g(\alpha) = \alpha$ therefore,

$|u_{n+1} - \alpha| \le \frac{1}{2}|u_n - \alpha|$.

c- For $n=0$ ; $|u_1 - \alpha| \le \frac{1}{2}|u_0 - \alpha|$

For $n=1$ ; $|u_2 - \alpha| \le \frac{1}{2}|u_1 - \alpha|$

For $n=2$ ; $|u_3 - \alpha| \le \frac{1}{2}|u_2 - \alpha|$

For the integer $n$, we have : $|u_n - \alpha| \le \frac{1}{2}|u_{n-1} - \alpha|$

Multiplying these positive terms by each other, we get :

$|u_n - \alpha| \le \left(\frac{1}{2}\right)^n |u_0 - \alpha|$.

But $u_0 = 1$ and $1 \le \alpha \le \frac{5}{4}$ hence, $|u_0 - \alpha| \le \frac{1}{4}$. Consequently,

$|u_n - \alpha| \le \left(\frac{1}{2}\right)^n \times \frac{1}{4}$.

So $|u_n - \alpha| \le \left(\frac{1}{2}\right)^{n+2}$ or $|u_n - \alpha| \le \frac{1}{2^{n+2}}$.

$\lim\limits_{n\to+\infty} \frac{1}{2^{n+2}} = 0$ so $\lim\limits_{n\to+\infty} |u_n - \alpha| = 0$ and consequently,

$\lim\limits_{n\to+\infty} u_n = \alpha$.

**N° 19.**

1) We have: $v_0 = u_0 - i\sqrt{3} = 1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}}$.
2) $v_{n+1} = u_{n+1} - i\sqrt{3} = (1 + i\sqrt{3})u_n + 3 - i\sqrt{3}$

   $v_{n+1} = (1 + i\sqrt{3})u_n - i\sqrt{3}(1 + i\sqrt{3}) = (1 + i\sqrt{3})(u_n - i\sqrt{3}) = (1 + i\sqrt{3})v_n$

   Then $(v_n)$ is a geometric sequence of common ratio $1 + i\sqrt{3}$ and of first term $v_0 = 1 + i\sqrt{3}$.
3) $v_n = v_0 \times q^n = 2e^{-i\frac{\pi}{3}} \times \left(2e^{i\frac{\pi}{3}}\right)^n = 2^{n+1} e^{i(n-1)\frac{\pi}{3}}$.

# Indications

N°2.

1) a- $v_{n+1} = \ln u_{n+1} = \ln\sqrt{u_n} = \frac{1}{2}\ln u_n = \frac{1}{2}v_n$.

2) a- $P_n = e^{v_0} \times e^{v_1} \times e^{v_2} \times \dots\dots\dots \times e^{v_n} = e^{v_0+v_1+\dots\dots+v_n} = e^{S_n}$.

b- $S_n = \ln e \times \dfrac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$.

N°3.

1) $u_{n+1} - 1 = (u_n - 1)^2$.

2) Suppose that $u_n - 1 = (u_0 - 1)^{2^n}$ then

$u_{n+1} - 1 = (u_n - 1)^2 = \left[(u_0 - 1)^{2^n}\right]^2 = (u_0 - 1)^{2^{n+1}}$.

N°4.

2) $v_n = \dfrac{1+\ln u_n}{2} = \dfrac{1+\ln\left(\sqrt{\dfrac{u_{n-1}}{e}}\right)}{2} = \dfrac{1}{2} \times \dfrac{1+\ln u_{n-1}}{2} = \dfrac{1}{2}v_{n-1}$.

N°5.

1) b- Let $u = (\ln x)^{n+1}$ and $v' = x^2$ then $u' = (n+1)(\ln x)^n \times \dfrac{1}{x}$ and $v = \dfrac{x^3}{3}$.

$$I_{n+1} = \left[\frac{x^3}{3}(\ln x)^{n+1}\right]_1^e - \frac{(n+1)}{3}I_n.$$

2) b- $I_{n+1} - I_n = \displaystyle\int_1^e x^2(\ln x)^n(\ln x - 1)dx$.

$1 \le x \le e$ then $(\ln x)^n \ge 0$ and $\ln x - 1 \le 0$ hence, $I_{n+1} - I_n \le 0$.

N°8.

1) $u_n = 2e^{-\frac{1}{2}n}\left(e^{\frac{1}{2}} - 1\right)$.

3) $S_n = \int_0^1 e^{-\frac{1}{2}x}dx + \int_1^2 e^{-\frac{1}{2}x}dx + \int_2^3 e^{-\frac{1}{2}x}dx + \ldots\ldots + \int_{n-1}^{n} e^{-\frac{1}{2}x}dx = \int_0^n e^{-\frac{1}{2}x}dx$.

N°9.

3) $\lim\limits_{n\to+\infty} u_n = 0$ 4) $S_n = \ln\frac{1}{2} + \ln\frac{2}{3} + \ln\frac{3}{4} + \ldots.. + \ln\frac{n}{n+1} = \ln\frac{1}{n+1}$

N°10.

2) b- $I_0 + I_1 = \int_0^1 \left(\frac{x}{1+x^2} + \frac{x^3}{1+x^2}\right)dx = \int_0^1 xdx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$.

4) $0 \le x^2 \le 1$ then $1 \le x^2 + 1 \le 2$ which gives $\frac{1}{2} \le \frac{1}{x^2+1} \le 1$

and $\frac{x^{2n+1}}{2} \le \frac{x^{2n+1}}{x^2+1} \le x^{2n+1}$ so $I_n \le \int_0^1 x^{2n+1}dx$.

N°11.

2) $u_{n+1} - u_n = \left(\sqrt{2+u_n} - u_n\right) \times \frac{\sqrt{2+u_n} + u_n}{\sqrt{2+u_n} + u_n} = \frac{-u_n^{\ 2} + u_n + 2}{\sqrt{2+u_n} + u_n}$.

3) $u_{n+1} - 2 = \left(\sqrt{2+u_n} - 2\right) \times \frac{\sqrt{2+u_n} + 2}{\sqrt{2+u_n} + 2} = \frac{u_n - 2}{\sqrt{2+u_n} + u_n}$

$u_n \ge 2$ then $\sqrt{2+u_n} + 2 \ge 4$ which gives $\frac{|u_n - 2|}{\sqrt{u_n + 2} + 2} \le \frac{1}{4}|u_n - 2|$.

N°12.

1) $z_n = u_n e^{i v_n} = \left(\frac{1}{2}\right)^n e^{n i \frac{\pi}{3}}$.

$a_n = \left\|\overrightarrow{M_n M_{n+1}}\right\| = |z_{n+1} - z_n| = \left|\left(\frac{1}{2}\right)^n e^{n i \frac{\pi}{3}}\left(-\frac{3}{4} + i\frac{\sqrt{3}}{4}\right)\right|$.

$a_n = \frac{\sqrt{3}}{2^{n+1}}$.

# CHAPTER 5

# Equations With Complex Coefficients

**Chapter Review :**

- $n^{th}$ roots of a complex number .
  - $w$ is an $n^{th}$ root of a complex number $z$ if $w^n = z$ .

    ***N.B.*** ***If*** $n \geq 3$ , it is preferable to find the $n^{th}$ roots in exponential form .

* ***Ex 1 :***

Find the square roots of the complex number $-3+4i$.

Let $z = x + iy$ be a square root of $-3+4i$ , then :

$z^2 = -3+4i$ and $|z^2| = |-3+4i|$.

Therefore $(x+iy)^2 = -3+4i$ and $x^2 + y^2 = 5$, as a result we get the following system:

$\begin{cases} x^2 + 2ixy - y^2 = -3+4i \\ x^2 + y^2 = 5 \end{cases}$ that is equivalent to the system

$$\begin{cases} x^2 - y^2 = -3 & (1) \\ 2xy = 4 & (2) \\ x^2 + y^2 = 5 & (3) \end{cases}$$

Adding equations $(1)$ and $(3)$, we get $2x^2 = 2$ ,

which gives $x = 1$ or $x = -1$.

Replacing $x = 1$ by its value in equation $(2)$, we get $y = 2$ ,

therefore $z_0 = 1 + 2i$ is a square root of $-3+4i$.

Replacing $x=-1$ by its value in equation $(2)$, we get
$y=-2$, therefore $z_1=-1-2i$ is a square root of $-3+4i$.

* *Ex 2:*
To find the cubic roots of the complex number $i$.
It is advisable to find the cubic roots in exponential form:
If $z=re^{i\theta}$ is a cubic root of $i$ then $z^3=i$, so $r^3e^{3i\theta}=e^{i\frac{\pi}{2}}$.
which gives $r=1$ and $3\theta=\frac{\pi}{2}+2k\pi$ therefore $\theta=\frac{\pi}{6}+\frac{2k\pi}{3}$.
The cubic roots are then:
$z_0=e^{i\frac{\pi}{6}}$, $z_1=e^{i\frac{5\pi}{6}}$ and $z_2=e^{i\frac{3\pi}{2}}$ obtained by giving $k$ the values $0;1$ and $2$.
* The sum of the $n^{th}$ roots of a complex number is 0.
* The images of the $n^{th}$ roots of a complex number are the vertices of a regular polygon of $n$ sides inscribed in a circle.

- Quadratic Equation :
All quadratic equations $az^2+bz+c=0$ with real or complex coefficients admit two solutions in the set of complex numbers.
$z'=\frac{-b-\omega}{2a}$ et $z''=\frac{-b+\omega}{2a}$ where $\omega^2=\Delta=b^2-4ac$.

*Ex :*
Consider the equation: $z^2+2iz+2-4i=0$.
We find $\Delta'=b'^2-ac=-3+4i$.
What we need now is to find $\Delta'$ as a perfect square .
We proceed as in the example *Ex1* , we find
$\Delta'=(1+2i)^2=(-1-2i)^2$, which gives :
$z'=-i-(1+2i)=-1-3i$ and $z''=-i+(1+2i)=1+i$.

# Solved Problems

**N° 1.**

1) Find the square roots of the number $p = 2i$.
2) Solve the equation $z^2 - 2iz - 1 - 2i = 0$.

**N° 2.**

1) Solve in the set of complex numbers the equation $(E): z^5 = 1$.
2) a- Prove that the sum of solutions of $(E)$ is equal to 0.

   b- Deduce that $\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} = -\frac{1}{2}$.

**N° 3.**

Consider the complex number $z = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$.

Let $S = z + z^2 + z^4$ and $T = z^3 + z^5 + z^6$.

1) Show that $S$ and $T$ are conjugates and that the imaginary part of $S$ is positive.
2) Calculate $S + T$ and $S \times T$ then deduce $S$ and $T$.
3) Deduce that :

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} = -\frac{1}{2}.$$

$$\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}.$$

**N° 4.**

Consider the equation $(E)$ : $z^3 - (1+2i)z^2 + 3(1+i)z - 10(1+i) = 0$.

1) Find the square roots of the complex number $p = 5 - 12i$.
2) Show that $(E)$ admits a solution that is pure imaginary to be determined.
3) Solve $(E)$.

N°5.

Given $\alpha = e^{2i\frac{\pi}{5}}$, $A = \alpha + \alpha^4$ and $B = \alpha^2 + \alpha^3$.

1) a- Show that $1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 = 0$.
   b- Deduce that $A$ and $B$ are solutions of the equation $(E):\ z^2 + z - 1 = 0$.
2) Express $A$ in terms of $\cos\frac{2\pi}{5}$.
3) Solve $(E)$ and deduce the value of $\cos\frac{2\pi}{5}$.

N°6.

1) Designate by $1$, $j$ and $j^2$ the cubic roots of unity.
   Show that $1 + j + j^2 = 0$.
2) Prove that if $\alpha$ is an $n^{th}$ root of the complex number $z$ and $\beta$ is an $n^{th}$ root of unity then, $\alpha \times \beta$ is an $n^{th}$ root of $z$.
3) Deduce the solutions of the equation $\left(\dfrac{z+1}{z}\right)^3 = -8$.

N°7.

1) Determine, in exponential form, the roots of the equation $(E):\ z^3 = 4\sqrt{2}(-1+i)$.
2) Using the cubic roots of unity write the roots of equation $(E)$ in algebraic form.
3) Deduce $\cos\frac{11\pi}{12}$ and $\sin\frac{11\pi}{12}$.
4) Calculate $(1+2i)^6$ then deduce the roots of the equation $z^6 = 117 + 44i$ in algebraic form.

N°8.

Given $p(z) = z^3 - (4+i)z^2 + (7+i)z - 4$.

1) Calculate $p(1)$ and determine $a$ and $b$ so that $p(z) = (z-1)(z^2 + az + b)$.

2) Solve the equation $p(z)=0$.
3) The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.
   Designate by $z_2$ and $z_3$ the roots of the equation $z^2+az+b=0$ and we suppose that $|z_2|>|z_3|$.
   Let $M_2$ and $M_3$ be the points of respective affixes $z_2$ and $z_3$.
   Prove that triangle $OM_2M_3$ is right at $O$.

N°9.

Given $p(z)=z^4+2\sqrt{3}z^3+8z^2+2\sqrt{3}z+7$.

1) Prove that if $\alpha$ is a root of the equation $p(z)=0$ then $\overline{\alpha}$ is also a root of the equation $p(z)=0$.
2) Calculate $p(i)$ and solve the equation $p(z)=0$.
3) The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.
   Consider the points $A$, $B$, $C$ and $D$ such that
   $z_A=i$ , $z_B=-i$ , $z_C=-\sqrt{3}+2i$ and $z_D=-\sqrt{3}-2i$.
   Prove that these points belong to the circle of diameter $[CD]$.

N° 10.

Consider the equation $(E)$ : $(z+i)^n=(z-i)^n$ with $n$ being a natural number such that $n\geq 2$.

1) Solve $(E)$ for $n=2$ and for $n=3$.
2) Prove that the solutions of $(E)$ are real.

N° 11.

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

1) Let $ABC$ be an equilateral triangle and designate by $a$, $b$ and $c$ the respective affixes of the points $A$, $B$ and $C$.
   Prove that $(a-b)^2+(b-c)^2+(c-a)^2=0$.
2) Deduce that the points of respective affixes $-2$ ; $1+i\sqrt{3}$ and $1-i\sqrt{3}$ are the vertices of an equilateral triangle.

**N° 12.**

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

Let $M$ be a point of affix $z = x + iy$.

1) Determine and construct the set $(E_1)$ of points $M$ such that $z^2 - (1+i)^2 = \bar{z}^2 - (1-i)^2$.
2) Determine and construct the set $(E_2)$ of points $M$ such that $(z-(1+i))(\bar{z}-(1-i)) = 8$.
3) Prove that $(E_1)$ and $(E_2)$ admit the same tangent at the point $A(-1;-1)$.

**N° 13.**

Given $S = 1 + \dfrac{\cos x}{\cos x} + \dfrac{\cos 2x}{\cos^2 x} + \ldots\ldots\ldots\ldots + \dfrac{\cos nx}{\cos^n x}$.

$S' = \dfrac{\sin x}{\cos x} + \dfrac{\sin 2x}{\cos^2 x} + \ldots\ldots\ldots\ldots + \dfrac{\sin nx}{\cos^n x}$.

1) Calculate $S + iS'$.
2) Deduce $S$ and $S'$.

**N° 14.**

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

Let $N$ be a point of affix $u$ and let $M'$ and $M''$ be the points of respective affixes $z'$ and $z''$ where $z'$ and $z''$ are the roots of the equation $(E)$ : $z^2 - 2(u+1)z + 2u^2 + 2u + 1 = 0$.

1) Solve $(E)$.
2) Find the set of points $N$ such that $M'M'' = 2$.

**N° 15.**

Given $p(z) = z^4 - 3z^3 + \dfrac{9}{2}z^2 - 3z + 1$ with $z$ being a complex number.

1) Show that if $u$ is a root of the equation $p(z) = 0$ then $\dfrac{1}{u}$ is also a root of the equation $p(z) = 0$.

2) a- Verify that $1+i$ is a root of the equation $p(z)=0$.

b- Write $p(z)$ as a product of two polynomials of degree 2 and with complex coefficients then solve the equation: $p(z)=0$.

**N° 16.**

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

Consider the equation $(E)$ : $z^2-2pz+q=0$, with $p=e^{i\alpha}$ and $q$ the complex number of modulus $2\sin\alpha$ and argument $\alpha+\dfrac{\pi}{2}$ with $\alpha\in[0;\pi]$.

$M_1$ and $M_2$ are the points of respective affixes $z_1$ and $z_2$ that are solutions of the equation $(E)$ and suppose that $\text{Re}(z_1)<0$.

Let $N$ be the midpoint of $[M_1M_2]$.

1) Without solving $(E)$, determine the set of points $N$.
2) a- Solve the equation $(E)$.

   b- Find the affix of the vector $\overrightarrow{M_1M_2}$.

   c- Deduce the set of points $M_1$ and $M_2$ as $\alpha$ varies.

**N° 17.**

In the complex plane referred to a direct orthonormal system $(O;\vec{u},\vec{v})$, consider the point $A$ of affix $z_A=1$ and the point $M$ of affix $z$.

1) Prove that the equation $(E):|z|=|1-z|$ has infinite number of solutions, and determine the locus of points $M$ as $z$ varies.
2) Let $z_0=re^{i\theta}$ be a solution of $(E)$.

   Calculate in terms of $r$ and $\theta$ the argument of $1-z_0$.
3) Consider the system $\begin{cases}|z|=|1-z|\\ z^n(1-z)=1\end{cases}$.

   Determine $n$ so that the system admits solutions then determine these solutions.

**N° 18.**

Consider the equation $(E)$ : $4z^3 - 6i\sqrt{3}z^2 - 3(3+i\sqrt{3})z - 4 = 0$.

1) Calculate the square roots of the complex number $6+6i\sqrt{3}$.
2) Show that $(E)$ admits a real root to be determined.
3) Solve $(E)$.
4) Write the roots of $(E)$ in exponential form.
5) Show that the roots of $(E)$ are three consecutive terms of a geometric sequence whose ratio is to be determined.

**N° 19.**

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

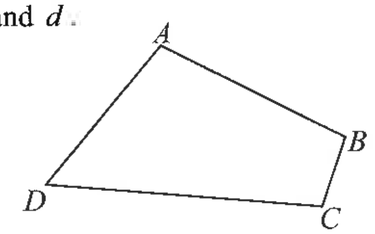
***Part A:***

Consider the points $E$, $F$ and $H$ of affixes $z_E = 3+i$, $z_F = 1+3i$ and $z_H = 3+3i$.

Calculate $\dfrac{z_E - z_H}{z_F - z_H}$ and deduce that triangle $EFH$ is right isosceles.

***Part B:***

$A$, $B$, $C$ and $D$ are four points of the plane of respective affixes $a$, $b$, $c$ and $d$.




On the exterior of quadrilateral $ABCD$, construct the direct right isosceles triangles $BIA$, $AJD$, $DKC$ and $CLB$ of right angles $B\hat{I}A$, $A\hat{J}D$, $D\hat{K}C$ and $C\hat{L}B$.

1) Designate by $z_I$, $z_L$, $z_K$ and $z_J$ the affixes of the points $I$, $L$, $K$ and $J$ respectively.

   Prove that $\left|\dfrac{b-z_I}{a-z_I}\right| = 1$ and that $\arg\left(\dfrac{b-z_I}{a-z_I}\right) = \dfrac{\pi}{2}(\text{mod}\, 2\pi)$ and then deduce that $z_I = \dfrac{ia-b}{i-1}$.

2) Using the result of part a) express :

$z_L$ in terms of b and c.

$z_K$ in terms of c and d.

$z_J$ in terms of a and d.

3) Prove that $z_L - z_J = i(z_K - z_I)$ and deduce that the straight lines $(JL)$ and $(IK)$ are perpendicular.

## *Supplementary Problems*

**N° 1.**

Determine the complex numbers $z \neq 0$, such that the points $M(z)$, $N(z^3)$ and $P(z^5)$ are collinear.

**N° 2.**

Let $u = -7 + 24i$.

1) Verify that $1 - 2i$ is a fourth root of $u$.
2) Deduce the solutions of the equation $z^4 = -7 + 24i$.

**N° 3.**

Consider the equation $(E)$ : $z^3 - (1+2i)z^2 + 3(1+i)z - 10(1+i) = 0$.

1) Calculate the square roots of the complex number $5 - 12i$.
2) Verify that $i$ is a root of the equation $(E)$.
3) Solve $(E)$.
4) Write the roots of $(E)$ in exponential form.
5) Show that the images of the roots of $(E)$ are the vertices of a right isosceles triangle.

**N° 4.**

Consider the complex number $z = 2e^{i\theta}$ with $0 < \theta < \dfrac{\pi}{2}$.

$M$, $N$ and $M'$ are the images of the complex numbers $z$, $z^2$ and $\bar{z}$.

1) Calculate $\theta$ in the case where the points $M'$, $O$ and N are collinear.
2) Calculate $\theta$ in the case where the triangle $M'MN$ is isosceles with vertex $M$.

**N° 5.**

1) Calculate the cubic roots of the complex number $-i$.
2) a- Calculate the square roots of the complex number $-5 + 12i$.
   b- Solve the equation $z^2 - (4+5i)z + 7i - 1 = 0$.

**N°6.**

Designate by $z_1$, $z_2$ and $z_3$ the respective affixes of the vertices $M_1$, $M_2$ and $M_3$ of a direct equilateral triangle $M_1M_2M_3$.

Prove that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3$.

**N°7.**

Given $p(z) = z^3 + (14 - i\sqrt{2})z^2 + (74 - 14i\sqrt{2})z - 74i\sqrt{2}$.

1) Show that the equation $p(z) = 0$ admits a pure imaginary solution to be determined .
2) Solve the equation $p(z) = 0$.
3) The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.
   - a- Place the $A$ , $B$ and $I$ in the system so that $z_A = -7 + 5i$, $z_B = -7 - 5i$ and $z_I = i\sqrt{2}$.
   - b- Place the point $C$ of affix $z_C = 1 + i$ and find the affix of the point $N$ so that $ABCN$ is a parallelogram .
   - c- Place the point $D$ of affix $z_D = 1 + 11i$.
   - d- Write in exponential form the complex number $Z = \dfrac{z_A - z_C}{z_D - z_B}$.
   - e- Prove that the straight lines $(AC)$ and $(BD)$ are perpendicular then deduce the nature of quadrilateral $ABCD$.

# Solution of Problems

**N° 1.**

1) Let $z = x + iy$ be a square root of $p = 2i$, so
$z^2 = 2i$ and $|z^2| = |2i|$, then $(x+iy)^2 = 2i$ and $x^2 + y^2 = 2$.
As a result we get the following system :
$\begin{cases} x^2 + 2ixy - y^2 = 2i \\ x^2 + y^2 = 2 \end{cases}$ that is equivalent to the system
$$\begin{cases} x^2 - y^2 = 0 & (1) \\ 2xy = 2 & (2) \\ x^2 + y^2 = 2 & (3) \end{cases}$$
Adding equations $(1)$ and $(3)$, we get $2x^2 = 2$, which gives $x = 1$ or $x = -1$.
Replacing $x = 1$ by its value in equation $(2)$, we get $y = 1$, therefore $z_0 = 1 + i$ is a square root of $p = 2i$.
Replacing $x = -1$ by its value in the equation $(2)$, we get $y = -1$, therefore $z_1 = -1 - i$ is a square root of $p = 2i$.
***N.B.***
If you note that $2i = (1+i)^2$, then we may replace $2i$ by $(1+i)^2$ which gives directly the square roots of $p = 2i$ as $z_0 = 1 + i$ and $z_1 = -1 - i$.
2) $\Delta'^2 = b'^2 - ac = i^2 + 1 + 2i = 2i = (1+i)^2$, the roots of the equation are then $z' = i + 1 + i = 1 + 2i$ and $z'' = i - 1 - i = -1$.

**N° 2.**

1) Taking $z = re^{i\theta}$, the equation $z^5 = 1$ will be equivalent to $r^5 e^{5i\theta} = e^{0i}$ which gives $r = 1$ and $5\theta = 2k\pi$ therefore $\theta = \dfrac{2k\pi}{5}$.
Hence the roots of this equation are:

$z_0 = 1$ , $z_1 = e^{i\frac{2\pi}{5}}$ , $z_2 = e^{i\frac{4\pi}{5}}$ , $z_3 = e^{i\frac{6\pi}{5}}$ and $z_4 = e^{i\frac{8\pi}{5}}$ obtained by giving $k$ the values $0;1;2;3$ and $4$ .

2) a- Let $S = z_0 + z_1 + z_2 + z_3 + z_4$, $S$ is the sum of the terms of a geometric sequence whose first term is $z_0 = 1$ and ratio $r = e^{i\frac{2\pi}{5}}$ , then $S = z_0 \times \dfrac{1-r^5}{1-r}$.

$$\text{So , } S = \frac{1-\left(e^{2i\frac{\pi}{5}}\right)^5}{1-e^{2i\frac{\pi}{5}}} = \frac{1-e^{2i\pi}}{1-e^{2i\frac{\pi}{5}}} = \frac{1-1}{1-e^{2i\frac{\pi}{5}}} = 0 .$$

b- Remark that $\dfrac{2\pi}{5} + \dfrac{8\pi}{5} = 2\pi$ so $z_1 = e^{i\frac{2\pi}{5}}$ and $z_4 = e^{i\frac{8\pi}{5}}$ are conjugates therefore

$z_1 + z_4 = e^{i\frac{2\pi}{5}} + e^{i\frac{8\pi}{5}} = 2\,\mathrm{Re}(z_1) = 2\cos\dfrac{2\pi}{5}$ , similarly

$z_2 + z_3 = e^{i\frac{4\pi}{5}} + e^{i\frac{6\pi}{5}} = 2\cos\dfrac{4\pi}{5}$ and since

$S = z_0 + z_1 + z_2 + z_3 + z_4 = 0$ then

$1 + 2\cos\dfrac{2\pi}{5} + 2\cos\dfrac{4\pi}{5} = 0$ which gives

$\cos\dfrac{2\pi}{5} + \cos\dfrac{4\pi}{5} = -\dfrac{1}{2}$.

**N°3.**

1) $z = \cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$ , using De Moivre's formula we get:

$z^2 = \cos\dfrac{4\pi}{7} + i\sin\dfrac{4\pi}{7}$ ; $z^3 = \cos\dfrac{6\pi}{7} + i\sin\dfrac{6\pi}{7}$

$z^4 = \cos\dfrac{8\pi}{7} + i\sin\dfrac{8\pi}{7}$ ; $z^5 = \cos\dfrac{10\pi}{7} + i\sin\dfrac{10\pi}{7}$

$z^6 = \cos\dfrac{12\pi}{7} + i\sin\dfrac{12\pi}{7}$ .

Remark that $\frac{4\pi}{7}+\frac{10\pi}{7}=2\pi$ then $z^2$ and $z^5$ are conjugates , so $\overline{z^2}=z^5$.

Similarly $\frac{8\pi}{7}+\frac{6\pi}{7}=2\pi$ then $z^4$ and $z^3$ are conjugates , so $\overline{z^4}=z^3$.

Similarly $\frac{2\pi}{7}+\frac{12\pi}{7}=2\pi$ then $z$ and $z^6$ are conjugates , so $\bar{z}=z^6$ , therefore,

$\overline{S}=\overline{z+z^2+z^4}=\bar{z}+\overline{z^2}+\overline{z^4}=z^6+z^5+z^3=T$.

The imaginary part of $S$ is $\text{Im}(S)=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}$.

$\sin\frac{8\pi}{7}=-\sin\frac{\pi}{7}$, then

$\text{Im}(S)=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}+\sin\frac{8\pi}{7}=\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{\pi}{7}$.

Since $\sin\frac{4\pi}{7}>\sin\frac{\pi}{7}$ then $\sin\frac{4\pi}{7}-\sin\frac{\pi}{7}>0$ and since $\sin\frac{2\pi}{7}>0$ then $\text{Im}(S)>0$.

2) $S+T=z+z^2+z^3+z^4+z^5+z^6$ is the sum of six consecutive terms of a geometric sequence whose first term is $z$ and common ratio $z$ , therefore, $S+T=z\times\frac{1-z^6}{1-z}=\frac{z-z^7}{1-z}=\frac{z-1}{1-z}=-1$

since $z^7=\left(e^{2i\frac{\pi}{7}}\right)^7=e^{2i\pi}=1$.

$S\times T=\left(z+z^2+z^4\right)\times\left(z^3+z^5+z^6\right)$

$=z^4+z^6+z^7+z^5+z^7+z^8+z^7+z^9+z^{10}$

and since $z^7=1$ we get :

$S\times T=z^4+z^6+1+z^5+1+z+1+z^2+z^3$

$S\times T=3+(z+z^2+z^3+z^4+z^5+z^6)=3-1=2$

$S$ and $T$ are the roots of the equation $X^2+X+2=0$.

Therefore, $X' = \dfrac{-1-\sqrt{7}i}{2}$ and $X'' = \dfrac{-1+\sqrt{7}i}{2}$.

Since $\operatorname{Im}(S) > 0$ then $S = \dfrac{-1+\sqrt{7}i}{2}$ and $S = \dfrac{-1+\sqrt{7}i}{2}$.

3) $S = z + z^2 + z^4 = \dfrac{-1+\sqrt{7}i}{2}$, so

$$S = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7} + \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7} + \cos\frac{8\pi}{7} + i\sin\frac{8\pi}{7}.$$

Therefore, $\cos\dfrac{2\pi}{7} + \cos\dfrac{4\pi}{7} + \cos\dfrac{8\pi}{7} = -\dfrac{1}{2}$

$\sin\dfrac{2\pi}{7} + \sin\dfrac{4\pi}{7} + \sin\dfrac{8\pi}{7} = \dfrac{\sqrt{7}}{2}$.

**N°4.**

1) Let $z = x + iy$ be a square root of $p = 5 - 12i$, then

$z^2 = 5 - 12i$ and $|z^2| = |5 - 12i|$, therefore, $(x+iy)^2 = 5 - 12i$ and

$x^2 + y^2 = 13$, we get then the system: :

$$\begin{cases} x^2 + 2ixy - y^2 = 5 - 12i \\ x^2 + y^2 = 13 \end{cases}$$ that is equivalent to the system

$$\begin{cases} x^2 - y^2 = 5 & (1) \\ xy = -6 & (2) \\ x^2 + y^2 = 13 & (3) \end{cases}$$

adding the equations $(1)$ and $(3)$ we get $2x^2 = 18$, which gives $x = 3$ or $x = -3$.

Replacing $x = 3$ by its value in the equation $(2)$, we get $y = -2$, then $z_0 = 3 - 2i$ is a square root of $p = 5 - 12i$.

The second root is $z_1 = -3 + 2i$.

2) Let $z = \lambda i$ be a pure imaginary solution of $(E)$, then

$(\lambda i)^3 - (1+2i)(\lambda i)^2 + 3(1+i)(\lambda i) - 10(1+i) = 0$, which gives

$\lambda^2 - 3\lambda - 10 + i(-\lambda^3 + 2\lambda^2 + 3\lambda - 10) = 0$, then

$\lambda^2 - 3\lambda - 10 = 0$ and $-\lambda^3 + 2\lambda^2 + 3\lambda - 10 = 0$.

The roots of the equation $\lambda^2 - 3\lambda - 10 = 0$ are $\lambda' = -2$ and $\lambda'' = 5$ .

Since $\lambda' = -2$ satisfies the second equation and $\lambda'' = 5$ does not satisfy the second equation then $z = -2i$ is a solution of $(E)$.

3) The polynomial $p(z) = z^3 - (1+2i)z^2 + 3(1+i)z - 10(1+i)$ is divisible by $z + 2i$ then $p(z) = (z+2i)(z^2 + az + b)$, so
$p(z) = z^3 + (a+2i)z^2 + (b+2ia)z + 2ib$ .
By comparison we get $a + 2i = -1 - 2i$ and $2ib = -10(1+i)$ which gives $a = -1 - 4i$ and $b = -5 + 5i$.
Consequently $p(z) = (z+2i)(z^2 - (1+4i)z - 5 + 5i)$.
$p(z) = 0$ gives $z = -2i$ or $z^2 - (1+4i)z - 5 + 5i = 0$.
Solving $z^2 - (1+4i)z - 5 + 5i = 0$ :
$\Delta = b^2 - 4ac = 5 - 12i = (3-2i)^2$, then :
$$z' = \frac{1+4i-(3-2i)}{2} = -1+3i \text{ and } z'' = \frac{1+4i+(3-2i)}{2} = 2+i.$$

$\boxed{N°5.}$

1) a- Let $S = 1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4$ $S$ is the sum of five terms of a geometric sequence whose first term is 1 and common ratio $r = e^{i\frac{2\pi}{5}}$ , therefore, $S = z_0 \times \dfrac{1-r^5}{1-r}$ , so
$$S = \frac{1-\left(e^{2i\frac{\pi}{5}}\right)^5}{1-e^{2i\frac{\pi}{5}}} = \frac{1-e^{2i\pi}}{1-e^{2i\frac{\pi}{5}}} = \frac{1-1}{1-e^{2i\frac{\pi}{5}}} = 0.$$

b- $A^2 + A - 1 = (\alpha + \alpha^4)^2 + \alpha + \alpha^4 - 1$
$= \alpha^2 + \alpha^8 + 2\alpha^5 + \alpha + \alpha^4 - 1$
$A^2 + A - 1 = \alpha^2 + \alpha^3 + 2 + \alpha + \alpha^4 - 1$
$= \alpha + \alpha^2 + \alpha^3 + \alpha^4 + 1 = 0$
Hence $A$ is a solution of the equation $z^2 + z - 1 = 0$.
Similarly we can prove that $B$ is a solution of the equation $z^2 + z - 1 = 0$.

2) $\frac{2\pi}{5}+\frac{8\pi}{5}=2\pi$ then $\alpha$ and $\alpha^4$ are conjugates , therefore

$$A=\alpha+\alpha^4=e^{2i\frac{\pi}{5}}+e^{8i\frac{\pi}{5}}=2R_e\left(e^{2i\frac{\pi}{5}}\right)=2\cos\frac{2\pi}{5}.$$

3) The roots of the equation $z^2+z-1=0$ are $z'=\frac{-1-\sqrt{5}}{2}$ and $z''=\frac{-1+\sqrt{5}}{2}$.

$A$ is a solution of the equation $z^2+z-1=0$.

Since $A=2\cos\frac{2\pi}{5}>0$ then $A=2\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{2}$ which gives $\cos\frac{2\pi}{5}=\frac{-1+\sqrt{5}}{4}$.

**N° 6.**

1) $1+j+j^2=\frac{1-j^3}{1-j}=\frac{1-1}{1-j}=0$ since $j^3=1$.

2) $\alpha$ is an $n^{th}$ of $z$ so $\alpha^n=z$ and $\beta$ is an $n^{th}$ root of unity so $\beta^n=1$ , therefore $(\alpha\times\beta)^n=z\times 1=z$ hence $\alpha\times\beta$ is an $n^{th}$ root of the complex number $z$.

3) Taking $Z=\frac{z+1}{z}$, the equation $Z^3=-8$ admits $Z_0=-2$ as a root.

The two other solutions are $Z_1=-2j=-2\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=1-i\sqrt{3}$

$$Z_2=-2j^2=-2\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=1+i\sqrt{3}.$$

The solutions of the equation $\left(\frac{z+1}{z}\right)^3=-8$ are :

$\dfrac{z+1}{z}=-2$ gives $z_0=-\dfrac{1}{3}$.

$\dfrac{z+1}{z}=1-i\sqrt{3}$ gives $z_1=\dfrac{i\sqrt{3}}{3}$.

$\dfrac{z+1}{z}=1+i\sqrt{3}$ gives $z_2=-\dfrac{i\sqrt{3}}{3}$.

**N°7.**

1) $|-1+i|=\sqrt{2}$ then $-1+i=\sqrt{2}\left(-\dfrac{1}{\sqrt{2}}+i\dfrac{1}{\sqrt{2}}\right)=\sqrt{2}e^{3i\frac{\pi}{4}}$

Therefore, $z^3=4\sqrt{2}(-1+i)=8\,e^{3i\frac{\pi}{4}}$.

Taking $z=re^{i\alpha}$, then $r^3e^{3i\alpha}=8e^{3i\frac{\pi}{4}}$, which gives $r=2$ and $3\alpha=\dfrac{3\pi}{4}+2k\pi$.

Therefore, $\alpha=\dfrac{\pi}{4}+\dfrac{2k\pi}{3}$.

The roots of this equation are then:

$z_0=2e^{i\frac{\pi}{4}}$ , $z_1=2e^{i\left(\frac{\pi}{4}+\frac{2\pi}{3}\right)}$ and $z_2=2e^{i\left(\frac{\pi}{4}+\frac{4\pi}{3}\right)}$.

2) $z_0=2e^{i\frac{\pi}{4}}=\sqrt{2}+i\sqrt{2}$, the other solutions are then

$$z_0\times j=\left(\sqrt{2}+i\sqrt{2}\right)\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)=\frac{-\sqrt{6}-\sqrt{2}}{2}+i\frac{\sqrt{6}-\sqrt{2}}{2}.$$

$$z_0\times j^2=\left(\sqrt{2}+i\sqrt{2}\right)\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)=\frac{\sqrt{6}-\sqrt{2}}{2}-i\frac{\sqrt{6}+\sqrt{2}}{2}.$$

3) $z_1=2e^{i\frac{11\pi}{12}}=\dfrac{-\sqrt{6}-\sqrt{2}}{2}+i\dfrac{\sqrt{6}-\sqrt{2}}{2}$, since $\cos\dfrac{11\pi}{12}<0$ and $\sin\dfrac{11\pi}{12}>0$, therefore, $2\cos\dfrac{11\pi}{12}=\dfrac{-\sqrt{6}-\sqrt{2}}{2}$ which gives $\cos\dfrac{11\pi}{12}=\dfrac{-\sqrt{6}-\sqrt{2}}{4}$ and $\sin\dfrac{11\pi}{12}=\dfrac{\sqrt{6}-\sqrt{2}}{4}$.

4) $(1+2i)^6 = \left[(1+2i)^2\right]^3 = (-3+4i)^3 = 117+44i$.

$z^6 = 117+44i$ is equivalent to $z^6 = (1+2i)^6$ therefore, $z_0 = 1+2i$.

The sixth roots of unity are solutions of the equation $u^6 = 1$ which gives $u^3 = 1$ that has as roots $1$ ; $j$ and $j^2$

$u^3 = -1$ that has as roots $-1$ ; $-j$ and $-j^2$.

Hence the roots of the equation $z^6 = 117+44i$ are :

$z_0 = 1+2i$ , $z_1 = z_0 \times -1 = -1-2i$,

$$z_2 = z_0 \times j = \left(-\frac{1}{2} - \sqrt{3}\right) + i\left(\frac{\sqrt{3}}{2} - 1\right)$$

$$z_3 = -z_0 \times j = \left(\frac{1}{2} + \sqrt{3}\right) + i\left(-\frac{\sqrt{3}}{2} + 1\right)$$

$$z_4 = z_0 \times j^2 = \left(-\frac{1}{2} + \sqrt{3}\right) - i\left(\frac{\sqrt{3}}{2} + 1\right).$$

$$z_5 = -z_0 \times j^2 = \left(\frac{1}{2} - \sqrt{3}\right) + i\left(\frac{\sqrt{3}}{2} + 1\right).$$

**N°8.**

1) $p(1) = 1-(4+i)+(7+i)-4 = 0$.

Then the polynomial $p(z)$ is divisible by $z-1$.

Hence, $p(z) = (z-1)(z^2+az+b)$, therefore

$p(z) = z^3 + (a-1)z^2 + (b-a)z - b$.

By comparison, we get $a = -3-i$ and $b = 4$.

Consequently, $p(z) = (z-1)(z^2-(3+i)z+4)$.

2) $p(z) = 0$ gives $z = 1$ or $z^2-(3+i)z+4 = 0$.

Solving $z^2-(3+i)z+4 = 0$:

$\Delta = b^2 - 4ac = -8+6i = (1+3i)^2$, therefore:

$z' = \dfrac{3+i-(1+3i)}{2} = 1-i$ and $z' = \dfrac{3+i+(1+3i)}{2} = 2+2i$.

3) Since $|2+2i| = 2\sqrt{2}$ and $|1-i| = \sqrt{2}$ then $z_2 = 2+2i$ and $z_3 = 1-i$.

$\dfrac{z_{\overrightarrow{OM_2}}}{z_{\overrightarrow{OM_3}}} = \dfrac{2+2i}{1-i} = \dfrac{2+2i}{1-i} \times \dfrac{1+i}{1+i} = 2i$, then

$\left(\overrightarrow{OM_3};\overrightarrow{OM_2}\right) = \dfrac{\pi}{2}(\bmod 2\pi)$, consequently triangle $OM_2M_3$ is right at $O$.

N°9.

1) $\alpha$ is a root of the equation $p(z)=0$ then $p(\alpha)=0$

$p(\overline{\alpha}) = (\overline{\alpha})^4 + 2\sqrt{3}(\overline{\alpha})^3 + 8(\overline{\alpha})^2 + 2\sqrt{3}(\overline{\alpha}) + 7$

$p(\overline{\alpha}) = (\overline{\alpha^4}) + 2\sqrt{3}(\overline{\alpha^3}) + 8(\overline{\alpha^2}) + 2\sqrt{3}(\overline{\alpha}) + 7$

$p(\overline{\alpha}) = \overline{\alpha^4 + 2\sqrt{3}\alpha^3 + 8\alpha^2 + 2\sqrt{3}\alpha + 7} = \overline{p(\alpha)} = 0$

Hence, $\overline{\alpha}$ is also a root of the equation $p(z)=0$.

2) $p(i) = i^4 + 2\sqrt{3}i^3 + 8i^2 + 2\sqrt{3}i + 7 = 0$

Then, $i$ is a root of the equation $p(z)=0$ so $\overline{i}=-i$ is also a root of the equation $p(z)=0$ consequently, $p(z)$ is divisible by $(z-i)(z+i) = z^2+1$ therefore,

$p(z) = (z^2+1)(z^2+bz+c) = z^4 + bz^3 + (c+1)z^2 + bz + c$.

By comparison, we get $b = 2\sqrt{3}$ and $c = 7$.

Consequently, $p(z) = (z^2+1)(z^2+2\sqrt{3}z+7)$.

The roots of the equation $z^2+2\sqrt{3}z+7=0$ are $z' = -\sqrt{3}+2i$ and $z'' = -\sqrt{3}-2i$.

3) The points $C$ and $D$ belong to the circle of diameter $[CD]$.

$\dfrac{z_D - z_A}{z_C - z_A} = \dfrac{-\sqrt{3}-2i-i}{-\sqrt{3}+2i-i} = \dfrac{-\sqrt{3}-3i}{-\sqrt{3}+i} \times \dfrac{-\sqrt{3}-i}{-\sqrt{3}-i} = \sqrt{3}i$

then the two straight lines $(AD)$ and $(AC)$ are perpendicular and consequently $A$ belongs to the circle of diameter $[CD]$. Similarly, we prove that $B$ belongs to the circle of diameter $[CD]$.

N° 10.

1) For $n=2$, the equation $(E)$ becomes $(z+i)^2=(z-i)^2$ so $z^2-1+2iz=z^2-1-2iz$ which gives $z=0$.
For $n=3$ the equation $(E)$ becomes $(z+i)^3=(z-i)^3$ so $(z+i)^3-(z-i)^3=0$ and by factorization, we get :
$(z+i-z+i)\left((z+i)^2+(z+i)(z-i)+(z-i)^2\right)=0$.
So, $2i\left(3z^2-1\right)=0$ which gives $z=\dfrac{\sqrt{3}}{3}$ or $z=-\dfrac{\sqrt{3}}{3}$.

2) Let $A$ be the point of affix $z_A=i$ and $B$ the point of affix $z_B=-i$ , $z+i=z_{\overrightarrow{BM}}$ and $z-i=z_{\overrightarrow{AM}}$.
$\left|(z+i)^n\right|=\left|(z-i)^n\right|$ then $|z+i|^n=|z-i|^n$ consequently, $|z+i|=|z-i|$.
Hence, $BM=AM$ and $M$ belongs to the perpendicular bisector of the segment $[AB]$ that is the axis $x'x$.
Hence, the solutions of $(E)$ are real.

N° 11.

1) $ABC$ is equilateral then: $\dfrac{z_{\overrightarrow{AC}}}{z_{\overrightarrow{AB}}}=e^{i\frac{\pi}{3}}=\dfrac{z_{\overrightarrow{BA}}}{z_{\overrightarrow{BC}}}$, so
$\dfrac{c-a}{b-a}=\dfrac{a-b}{c-b}$ which gives $(c-a)(c-b)+(a-b)^2=0$ which is equivalent to $a^2+b^2+c^2-ab-ac-bc=0$.
Expanding $(a-b)^2+(b-c)^2+(c-a)^2$, we get
$(a-b)^2+(b-c)^2+(c-a)^2=2\left(a^2+b^2+c^2-ab-ac-bc\right)=0$,
which proves that $(a-b)^2+(b-c)^2+(c-a)^2=0$.

2) Let $a=-2$ , $b=1+i\sqrt{3}$ , $c=1-i\sqrt{3}$ , we have:
$(a-b)^2+(b-c)^2+(c-a)^2=$
$\left(-2-1-i\sqrt{3}\right)^2+\left(1+i\sqrt{3}-1+i\sqrt{3}\right)^2+\left(1-i\sqrt{3}+2\right)^2=0$.
Hence, these points are the vertices of an equilateral triangle.

N° 12.

1) $z^2-(1+i)^2=\overline{z}^2-(1-i)^2$ is equivalent to

$z^2-\overline{z}^2=(1+i)^2-(1-i)^2$ then :

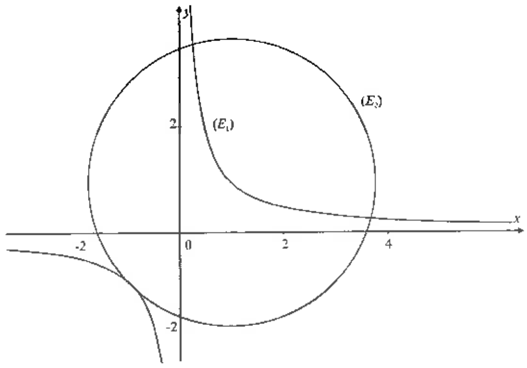
$(z-\overline{z})(z+\overline{z})=(1+i+1-i)(1+i-1+i)$

$(2iy)(2x)=4i$ and consequently the equation of $(E_1)$ is $xy=1$.

2) $(z-(1+i))(\overline{z}-(1-i))=8$ is equivalent to $(z-(1+i))(\overline{z-(1+i)})=8$.

Then $|z-(1+i)|^2=8$ and consequently the equation of

$(E_2)$ is $(x-1)^2+(y-1)^2=8$ which is a circle of center $(1;1)$ and radius $2\sqrt{2}$.




2) At point $A(-1;-1)$, the two curves have the same tangent.

Moreover, $y=\frac{1}{x}$ gives $y'=\frac{-1}{x^2}$ then the slope of the tangent at $A$ to $(E_1)$ is -1

Deriving the equation of the circle with respect to $x$, we get :

$2(x-1)+2(y-1)y'=0$ and at point $A(-1;-1)$ we get

$y'=-1$ hence the two curves have the same tangent at this point.

N° 13.

1) $S+iS'=$

$$1+\frac{\cos x+i\sin x}{\cos x}+\frac{\cos 2x+i\sin 2x}{\cos^2 x}+\ldots\ldots+\frac{\cos nx+i\sin nx}{\cos^n x}$$

$$=\left(\frac{e^{ix}}{\cos x}\right)^0+\left(\frac{e^{ix}}{\cos x}\right)^1+\left(\frac{e^{ix}}{\cos x}\right)^2+\ldots\ldots+\left(\frac{e^{ix}}{\cos x}\right)^n$$

Then $S+iS'$ is the sum of $n+1$ first terms of a geometric sequence of first term 1 and common ratio $\frac{e^{ix}}{\cos x}$, so :

$$S+iS'=\frac{1-\left(\frac{e^{ix}}{\cos x}\right)^{n+1}}{1-\frac{e^{ix}}{\cos x}}=\frac{1-\frac{\cos((n+1)x)+i\sin((n+1)x)}{\cos^{n+1}x}}{\frac{-i\sin x}{\cos x}}$$

$$S+iS'=\frac{1}{\sin x\cos^n x}\left[\sin((n+1)x)+i\left(\cos^{n+1}x-\cos((n+1)x)\right)\right].$$

2) $S=R_e(S+iS')=\dfrac{\sin((n+1)x)}{\sin x\cos^n x}$

$S'=I_m(S+iS')=\dfrac{\cos^{n+1}x-\cos((n+1)x)}{\sin x\cos^n x}$

**N° 14.**

1) $\Delta'=(u+1)^2-2u^2-2u-1=-u^2=u^2i^2$ then
$z'=u+1+ui=(1+i)u+1$ and $z'=u+1-ui=(1-i)u+1$

2) $M'M''=2$ is equivalent to $M'M''^2=4$ therefore, $|z''-z'|^2=4$, so $|2iu|=2$ which gives: $|i|\times|u|=1$ hence $|u|=1$ and consequently, the point $N$ varies on a circle of center O and radius 1.

**N° 15.**

1) $p(u)=u^4-3u^3+\frac{9}{2}u^2-3u+1$.

$$p\left(\frac{1}{u}\right)=\left(\frac{1}{u}\right)^4-3\left(\frac{1}{u}\right)^3+\frac{9}{2}\left(\frac{1}{u}\right)^2-3\left(\frac{1}{u}\right)+1.$$

$$p\left(\frac{1}{u}\right)=\frac{1}{u^4}-\frac{3}{u^3}+\frac{9}{2u^2}-\frac{3}{u}+1=\frac{2-6u+9u^2-6u^3+2u^4}{2u^4}.$$

$p\left(\frac{1}{u}\right)=\frac{2p(u)}{2u^4}=0$ then $\frac{1}{u}$ is also a root of the equation.

2) a- $p(1+i)=(1+i)^4-3(1+i)^3+\frac{9}{2}(1+i)^2-3(1+i)+1$.

$p(1+i)=-4-3(-2+2i)+9i-3-3i+1=0$.

Hence, $1+i$ is a root of the equation $p(z)=0$.

b- $1+i$ is a root of the equation $p(z)=0$ then $\frac{1}{1+i}=\frac{1}{2}-\frac{1}{2}i$ is a root of the equation $p(z)=0$.

Hence the polynomial $p(z)$ is divisible by

$(z-1-i)\left(z-\frac{1}{2}+\frac{1}{2}i\right)=z^2-\frac{1}{2}(3+i)z+1$.

Therefore, $p(z)=\left(z^2-\frac{1}{2}(3+i)z+1\right)\left(z^2+bz+c\right)$

expanding and comparing we get $c=1$ and $b=-\frac{3}{2}+\frac{1}{2}i$

then : $p(z)=\left(z^2-\frac{1}{2}(3+i)z+1\right)\left(z^2-\frac{1}{2}(3-i)z+1\right)$.

Solving $z^2-\frac{1}{2}(3-i)z+1=0$ or $2z^2-(3-i)z+2=0$ :

$\Delta=-8-6i=(1-3i)^2$ then:

$z'=\frac{3-i-1+3i}{4}=\frac{1}{2}+\frac{1}{2}i$ and $z''=\frac{3-i+1-3i}{4}=1-i$.

**N° 16.**

1) We know that $z_N=\frac{z'+z''}{2}=\frac{-b}{2a}=p=e^{i\alpha}$ then $|z_N|=1$ and consequently $N$ varies on the circle of center $O$ and radius 1.

2) a- $p=e^{i\alpha}$, $q=2\sin\alpha\, e^{i\left(\alpha+\frac{\pi}{2}\right)}$, then

$\Delta'=p^2-q$

$=\cos 2\alpha+i\sin 2\alpha-2\sin\alpha\left[\cos\left(\alpha+\frac{\pi}{2}\right)+i\sin\left(\alpha+\frac{\pi}{2}\right)\right]$

$\Delta'=\cos 2\alpha+i\sin 2\alpha-2\sin\alpha[-\sin\alpha+i\cos\alpha]$.

$\Delta' = \cos^2\alpha - \sin^2\alpha + 2i\sin\alpha\cos\alpha + 2\sin^2\alpha - 2i\sin\alpha\cos\alpha$.
$\Delta' = \cos^2\alpha + \sin^2\alpha + 2i\sin\alpha\cos\alpha - 2i\sin\alpha\cos\alpha = 1$.
Therefore, $z' = p-1$ and $z'' = p+1$.

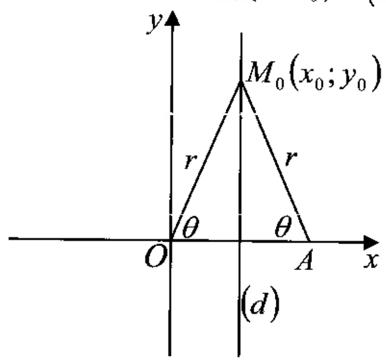
b- $p-1 = \cos\alpha - 1 + i\sin\alpha$ , since $\cos\alpha - 1 < 0$ then $z_1 = p-1$ and $z_2 = p+1$ therefore :
$z_{\overrightarrow{M_1M_2}} = p+1-p+1 = 2$ then $\overrightarrow{M_1M_2} = 2\vec{u}$.

c- $\overrightarrow{NM_2} = \vec{u}$ then $M_2$ is the image of $N$ by the translation of vector $\vec{u}$ , $N$ varies on the circle $(C)$ hence $M_2$ varies on the circle image of $(C)$ by this translation.
Similarly , $M_1$ varies on the circle image of $(C)$ by the translation of vector $-\vec{u}$.

## N°17.

1) Let $z = x + iy$ , $|z| = |1-z|$ gives $|x+iy|^2 = |1-x-iy|^2$ then $x^2+y^2 = (1-x)^2 + y^2$ , which gives $x = \frac{1}{2}$, hence, the equation admits an infinite number of solutions .
The set of points $M$ is the straight line $(d)$ of equation $x = \frac{1}{2}$.

2) $z_0 = re^{i\theta}$ is a solution of $(E)$, then the point $M_0(z_0)$ belongs to $(d)$ that is the perpendicular bisector of $[OA]$, hence triangle $OAM_0$ is isosceles at $M_0$ and consequently,
$|1-z_0| = MA = r$ and $\arg(1-z_0) = (\vec{u};\overrightarrow{M_0A}) = -\theta(\text{mod } 2\pi)$.



3) $|z| = |1-z|$ then the point $M(z)$ belongs to $(d)$then $z = re^{i\theta}$ and $1-z = re^{-i\theta}$.

The equation $z^n(1-z) = 1$ gives $r^{n+1}e^{in\theta}e^{-i\theta} = 1$ so $r^{n+1}e^{i(n-1)\theta} = 1$.

Therefore, $r = 1$ and $(n-1)\theta = 2k\pi$.

$r = 1$ gives that $M$ belongs to the circle $(C)$ of center $O$ and radius 1.

Then there are two positions for $M$, that are the points of intersection of $(C)$ and $(d)$, therefore :

$\theta = \dfrac{\pi}{3}$ then $(n-1)\dfrac{\pi}{3} = 2k_1\pi$, which gives $n = 6k_1 + 1$, $k_1 \geq 0$.

$\theta = -\dfrac{\pi}{3}$ then $-(n-1)\dfrac{\pi}{3} = 2k_2\pi$, which gives $n = -6k_2 + 1, k_2 \leq 0$.

The two cases give that $n = 6k + 1$ with $k \geq 0$.

**N° 18.**

1) Let $z = x + iy$ be a square root of $6 + 6i\sqrt{3}$, $z^2 = 6 + 6i\sqrt{3}$ and $|z^2| = |6 + 6i\sqrt{3}|$.

Therefore, $(x+iy)^2 = 6 + 6i\sqrt{3}$ and $x^2 + y^2 = 12$, we get as a result the following system :

$\begin{cases} x^2 - y^2 = 6 \\ 2xy = 6\sqrt{3} \\ x^2 + y^2 = 12 \end{cases}$ that admits as a solution $z_0 = 3 + \sqrt{3}\,i$ and $z_1 = -3 - \sqrt{3}\,i$.

2) If $z_0 = r$ is a real root of the equation $(E)$ then $4r^3 - 6i\sqrt{3}r^2 - 3(3 + i\sqrt{3})r - 4 = 0$ which gives the system:

$\begin{cases} 2r^2 + r = 0 \\ 4r^3 - 9r - 4 = 0 \end{cases}$

that admits as a solution $r = -\dfrac{1}{2}$ therefore, $z_0 = -\dfrac{1}{2}$.

3) The polynomial $p(z)=4z^3-6i\sqrt{3}z^2-3\left(3+i\sqrt{3}\right)z-4$ is divisible by $z+\frac{1}{2}$.

Therefore, $p(z)=\left(z+\frac{1}{2}\right)\left(4z^2+bz+c\right)$, expanding and comparing the two identical polynomials , we get $b=-2-6i\sqrt{3}$ and $c=-8$, and consequently, $p(z)=\left(z+\frac{1}{2}\right)\left(4z^2-2\left(1+3i\sqrt{3}\right)z-8\right)$.

Solving the equation: $4z^2-2\left(1+3i\sqrt{3}\right)z-8=0$.

$\Delta'=\left(1+3i\sqrt{3}\right)^2+32=6+6i\sqrt{3}=\left(3+\sqrt{3}\,i\right)^2$, therefore,

$z'=\dfrac{1+3i\sqrt{3}-3-\sqrt{3}i}{4}=-\dfrac{1}{2}+i\dfrac{\sqrt{3}}{2}$ and

$z''=\dfrac{1+3i\sqrt{3}+3+\sqrt{3}i}{4}=1+i\sqrt{3}$.

4) $z_0=\frac{1}{2}e^{i\pi}$, $z'=e^{i\frac{2\pi}{3}}$ and $z''=2e^{i\frac{\pi}{3}}$.

5) Remark that $z'^2=z_0\times z''=e^{i\left(\frac{4\pi}{3}\right)}$, then the roots of $(E)$ are three consecutive terms of a geometric sequence of ratio $r=2e^{-i\frac{\pi}{3}}$.

N° 19.

***Part A:***

$$\frac{z_E-z_H}{z_F-z_H}=\frac{3+i-3-3i}{1+3i-3-3i}=i.$$

Then, $\dfrac{z_{\overrightarrow{HE}}}{z_{\overrightarrow{HF}}}=i$, which gives $\dfrac{HE}{HF}=1$ and $\left(\overrightarrow{HF};\overrightarrow{HE}\right)=\dfrac{\pi}{2}(\mathrm{mod}\,2\pi)$ and consequently triangle $EFH$ is right isosceles at $H$.

***Part B*.**

1) $\left|\dfrac{b-z_I}{a-z_I}\right| = \dfrac{|z_B - z_I|}{|z_A - z_I|} = \dfrac{\left|z_{\overrightarrow{IB}}\right|}{\left|z_{\overrightarrow{IA}}\right|} = \dfrac{IB}{IA} = 1$ and

$\arg\left(\dfrac{b-z_I}{a-z_I}\right) = \arg\left(\dfrac{z_{\overrightarrow{IB}}}{z_{\overrightarrow{IA}}}\right) = \left(\overrightarrow{IA};\overrightarrow{IB}\right) = \dfrac{\pi}{2}(\bmod 2\pi)$.

Therefore, $\dfrac{b-z_I}{a-z_I} = i$, which gives $b - z_I = ia - iz_I$ and

consequently, $z_I = \dfrac{ia-b}{i-1}$.

2) By analogy, we get : $z_J = \dfrac{id-a}{i-1}$ , $z_K = \dfrac{ic-d}{i-1}$ and $z_L = \dfrac{ib-c}{i-1}$.

3) $z_L - z_J = \dfrac{ib-c}{i-1} - \dfrac{id-a}{i-1} = \dfrac{ib-c-id+a}{i-1}$,

$z_K - z_I = \dfrac{ic-d}{i-1} - \dfrac{ia-b}{i-1} = \dfrac{ic-d-ia+b}{i-1}$,

therefore: $i(z_K - z_I) = \dfrac{ib-c-id+a}{i-1}$.

$\dfrac{z_L - z_J}{z_K - z_I} = i$ , then $\dfrac{z_{\overrightarrow{JL}}}{z_{\overrightarrow{IK}}} = i$ which gives $\left(\overrightarrow{IK};\overrightarrow{JL}\right) = \dfrac{\pi}{2}(\bmod 2\pi)$

and consequently, the two straight lines $(JL)$ and $(IK)$ are perpendicular.

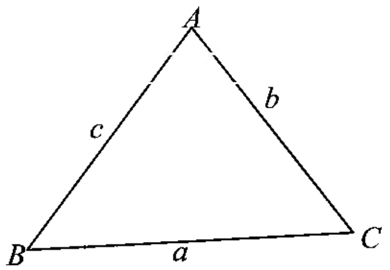
# CHAPTER 6

# Metric Relations

**Chapter Review :**

- First system of fundamental relations :

$$(I)\begin{cases} a^2 = b^2 + c^2 - 2\,bc\cos(\hat{A}) \\ b^2 = a^2 + c^2 - 2\,ac\cos(\hat{B}) \\ c^2 = a^2 + b^2 - 2\,ab\cos(\hat{C}) \end{cases}$$




- Second system of fundamental relations

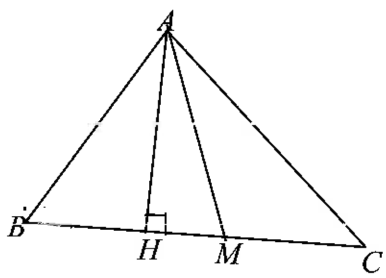
$$\begin{cases} \hat{A} + \hat{B} + \hat{C} = \pi \; rad \\ \dfrac{a}{\sin \hat{A}} = \dfrac{b}{\sin \hat{B}} = \dfrac{c}{\sin \hat{C}} = 2R \end{cases}$$

$R$ is the radius of the circle circumscribed about triangle $ABC$.

- Area of a triangle :

$$S = \frac{1}{2} b\,c \sin \hat{A} = \frac{1}{2} a\,b \sin \hat{C} = \frac{1}{2} a\,c \sin \hat{B}.$$

$$S = \frac{abc}{4R}.$$



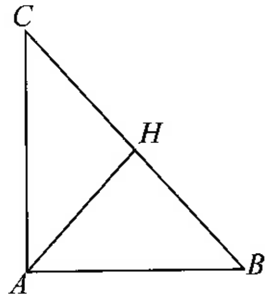

- $M$ is the mid point of $[BC]$ and $[AH]$ is a height in triangle $ABC$ :

$$AB^2 + AC^2 = 2AM^2 + \frac{BC^2}{2}.$$

$$AB^2 - AC^2 = 2\overline{BC} \times \overline{MH}.$$

- If $G$ is the center of gravity of triangle $ABC$ then :
  $\overrightarrow{GA} = -2\overrightarrow{GM}$ .

- In the triangle $ABC$ right at $A$, we have:
  $BC^2 = AB^2 + AC^2$.
  $AH^2 = HB \times HC$ .
  $AB^2 = BH \times BC$.
  $\dfrac{1}{AH^2} = \dfrac{1}{AB^2} + \dfrac{1}{AC^2}$ .



# Solved Problems

**N° 1.**

$ABC$ is a triangle such that : $\sin\hat{C} = 2\sin\hat{B}\cos\hat{A}$ .
Show that $ABC$ is isosceles.

**N° 2.**

$ABC$ is an isosceles triangle with vertex $A$ .

1) Calculate the area $S$ of this triangle in terms of $AB$ , $AC$ and the angle $\frac{\hat{A}}{2}$ .
2) Deduce that $\sin\hat{A} = 2\sin\frac{\hat{A}}{2}\cos\frac{\hat{A}}{2}$ .

**N° 3.**

$ABC$ is an isosceles triangle with vertex $A$ such that $AB = 2$ and $BC = \sqrt{3}$ .

1) Show that $\hat{A}$ is an acute angle .
2) a- Calculate the length of the median $BI$ .
   b- Let $H$ be the orthogonal projection of $B$ over $(AC)$.
   Calculate $IH$ and deduce that $BH = \frac{\sqrt{39}}{4}$ .
   c- Calculate $\sin\hat{A}$ , $\sin\hat{C}$ , the radius of the circle circumscribed about $ABC$ , as well as the area of triangle $ABC$ .

**N° 4.**

$ABC$ is a given triangle .
Designate by $a$ , $b$ and $c$ the measures of the sides $[BC]$ $[AC]$ and $[AB]$ respectively .

1) Prove that if $a^2 = b \times c$ then $\sin^2\hat{A} = \sin\hat{B} \times \sin\hat{C}$ .
2) Prove that if $\sin\hat{A} + \sin\hat{B} = 2\sin\hat{C}$ then $a + b = 2c$ .

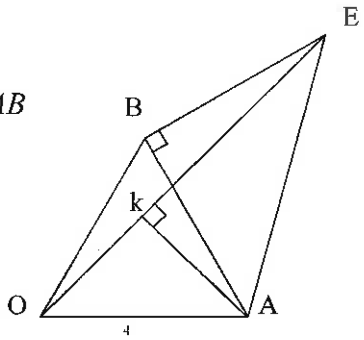
3) Prove that if $b^2+c^2=2a^2$ then $\cos\hat{A}=\dfrac{a^2}{2bc}$.

**N°5.**

Given an equilateral triangle $OAB$ such that $OA=4$ et $\left(\overrightarrow{OA};\overrightarrow{OB}\right)=\dfrac{\pi}{3}(\text{mod }2\pi)$.

Let $E$ be a point outside triangle $OAB$ and such that $BA=BE$ and $\left(\overrightarrow{BA};\overrightarrow{BE}\right)=\dfrac{\pi}{2}(\text{mod }2\pi)$.

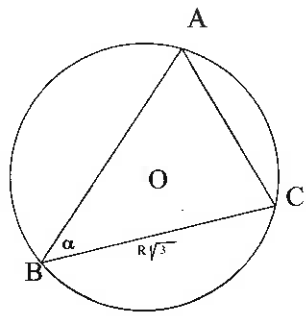
Designate by $K$ the orthogonal projection of $A$ over $(OE)$.




1) a- Determine the measures of the sides $OK$ and $OE$ .
   b- Deduce $\sin 105^\circ$ .
2) The plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$ such that $\overrightarrow{OA}=4\vec{u}$ .
   Determine the affixes of the points $A$ , $B$ , $E$ and $K$ .

**N°6.**

Consider a circle $(C)$ of center $O$ and radius $R$ .
$ABC$ is a triangle inscribed in the circle such that $BC=R\sqrt{3}$ .
Taking $\left(\overrightarrow{BC};\overrightarrow{BA}\right)=\alpha\,(\text{mod}\,2\pi)$ with $0<\alpha<\dfrac{\pi}{2}$ and $\hat{A}$ is an acute angle .




1) Calculate $AB$ and $AC$ in terms of $R$ and of $\alpha$ .
2) Let $y=AB-AC$ .
   a- Calculate $AB$ and $AC$ in terms of $R$ in the case where $y=R\sqrt{2}$ .
   b- Deduce the area of triangle $ABC$ in terms of $R$ .

**N° 7.**

$ABC$ is a given triangle such that $AB = c$, $AC = b$ and $BC = a$.
Let $O$ be a point interior to triangle $ABC$, designate by $M$, $N$ and $P$ be the orthogonal projections of $O$ over the sides $[BC]$, $[AC]$ and $[AB]$. Denote by $S$ the area of triangle $ABC$.

1) Prove that $2S = a\times OM + b\times ON + c\times OP$.
2) Deduce that $S = p\times r$ where $p$ is half the perimeter of the triangle $ABC$ and $r$ the radius of the circle inscribed in the triangle $ABC$.

**N° 8.**

$ABC$ is a given triangle such that $AB = c$, $AC = b$ and $BC = a$.
Establish the following relations :

1) $a\times\cos\hat{B} - b\times\cos\hat{A} = \dfrac{(a-b)(a+b)}{c}$.
2) $a = b\times\cos\hat{C} + c\times\cos\hat{B}$.
3) $\dfrac{\cos\hat{A}}{a} + \dfrac{\cos\hat{B}}{b} + \dfrac{\cos\hat{C}}{c} = \dfrac{a^2+b^2+c^2}{2abc}$.

   Deduce that if triangle $ABC$ is right at $A$, then
   $a = c\cos\hat{B} + b\cos\hat{C}$.

**N° 9.**

Given a triangle $ABC$ such that $AB = 1+\sqrt{2}$,
$AC = \sqrt{2}$ and $BC = \sqrt{3}$.

1) Calculate the angle $\hat{A}$ and the area of triangle $ABC$.
2) $[AI]$ and $[AH]$ representing respectively the median and the height $[AH]$ relative to $[BC]$, calculate $AI$ and $AH$.
3) Let $M$ be a variable point of the plane such that
   $MB^2 + MC^2 = 2\sqrt{2}$.
   a- Determine the set $(T)$ of points $M$.
   b- Determine the position of the point $A$ with respect to $(T)$.

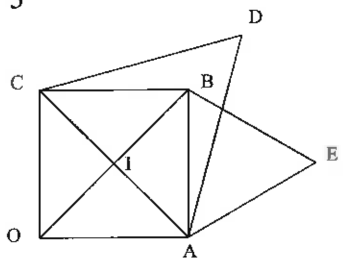
**N° 10.**

Given a square $OABC$ of center $I$, of side $a$ and such that

$\left(\overrightarrow{OA};\overrightarrow{OC}\right)=\frac{\pi}{2}(\bmod 2\pi)$.

$CAD$ and $BAE$ are two equilateral triangles such that

$\left(\overrightarrow{EA};\overrightarrow{EB}\right)=-\frac{\pi}{3}(\bmod 2\pi)$ and $\left(\overrightarrow{DA};\overrightarrow{DC}\right)=-\frac{\pi}{3}(\bmod 2\pi)$.




1) a- Calculate $OD$, $IE$ and $OE$ in terms of $a$.
   b- Deduce $\cos 15^\circ$ and $\sin 15^\circ$.
2) The plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$ with $\overrightarrow{OA}=2\vec{u}$.
   a- Determine the affixes of the points $A$, $B$, $C$, $I$, $D$ and $E$.
   b- Prove that triangle $AED$ is right isosceles.

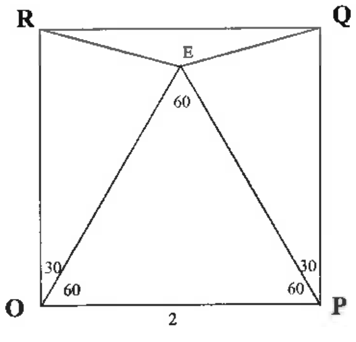
**N° 11.**

Given a square $OPQR$ of side 2 and such that

$\left(\overrightarrow{OP};\overrightarrow{OR}\right)=\frac{\pi}{2}(\bmod 2\pi)$.

$OEP$ is an equilateral triangle .

Designate by $F$ the orthogonal projection of $O$ on the straight line $(EP)$.



1) a- Find the measures of the angles of triangle $ORE$ and those of triangle $ERQ$.

   b- Deduce $\cos 15°$ and $\sin 15°$.

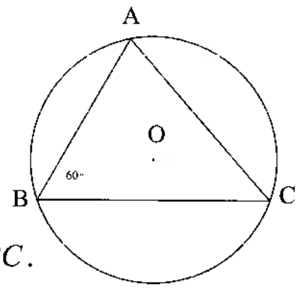
2) Calculate the area of triangle $ERQ$ and deduce the area of triangle $ORE$.

**N° 12.**

Consider the circle $(C)$, of center $O$ and of radius $R = 3$.

$ABC$ is a triangle inscribed in $(C)$ such that $BC = 5$ and $A\hat{B}C = 60°$.

Calculate $AB$ and the area of triangle $ABC$.




**N° 13.**

In a triangle $ABC$ right at $A$, let $AB = c$

$AC = b$ and $BC = 5$ and suppose that $\sin\hat{B} = 2\sin\hat{C}$.

1) Calculate $b$ and $c$.
2) Calculate $\sin\hat{B}$ and deduce the area of triangle $ABC$.

**N° 14.**

$ABC$ is a given triangle such that $AB = 1+\sqrt{3}$, $AC = \sqrt{6}$ and $\hat{A} = 45°$.

1) Calculate $BC$ and the angles $\hat{B}$ and $\hat{C}$.
2) Calculate the radius of the circle circumscribed about triangle $ABC$ as well as the area of this triangle.
3) Calculate the measure of the height $[AH]$ relative to $[BC]$.

**N° 15.**

$ABC$ is a given triangle such that $AB = c$, $AC = b$ and $BC = a$.

1) Determine the nature of triangle $ABC$ knowing that $a^2b^2 + c^4 = b^4 + a^2c^2$.
2) Prove that $b^2 - c^2 = a(b\cos\hat{C} - c\cos\hat{B})$.
3) Suppose that : $a = 1$ ; $b = 2$ and $c = \sqrt{3}$.

a- Calculate the angle $\hat{A}$ .

b- Using question 2) prove that:

$2\cos\hat{C}-\sqrt{3}\cos\hat{B}=1$ and $\sqrt{3}\cos\hat{B}-2\sqrt{3}\cos\hat{A}=-3$ .

c- Deduce $\cos\hat{B}$ and $\cos\hat{C}$ .

# Solution of Problems

N° 1.

$a^2 = b^2 + c^2 - 2bc\cos\hat{A}$ , then $\cos\hat{A} = \dfrac{b^2 + c^2 - a^2}{2bc}$.

Using the second system of fundamental relations, we get :

$\dfrac{a}{\sin\hat{A}} = \dfrac{b}{\sin\hat{B}} = \dfrac{c}{\sin\hat{C}} = 2R$ therefore $\sin\hat{B} = \dfrac{b}{2R}$ and $\sin\hat{C} = \dfrac{c}{2R}$.

Since $\sin\hat{C} = 2\sin\hat{B}\cos\hat{A}$ we get : $\dfrac{c}{2R} = 2\times\dfrac{b}{2R}\times\dfrac{b^2 + c^2 - a^2}{2bc}$,

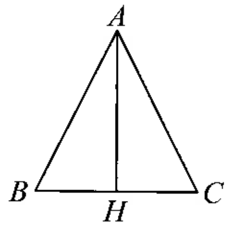
which gives after simplification $c^2 = b^2 + c^2 - a^2$ and consequently $a = b$ , hence triangle $ABC$ is isosceles at $C$.

N° 2.

1) Area of triangle $ABC$ is double that of triangle $ABH$, therefore

$S = 2\times\dfrac{1}{2}\times AB\times AH\times\sin\dfrac{\hat{A}}{2}$, but $AH = AC\cos\dfrac{\hat{A}}{2}$ then

$S = AB\times AC\times\cos\dfrac{\hat{A}}{2}\times\sin\dfrac{\hat{A}}{2}$.




2) We know that the area of triangle $ABC$ is :

$S = \dfrac{1}{2}AB\cdot AC\cdot\sin\hat{A}$ .

which gives the equality :

$\dfrac{1}{2}AB\times AC\times\sin\hat{A} = AB\times AC\times\cos\dfrac{\hat{A}}{2}\sin\dfrac{\hat{A}}{2}$.

So, we deduce that: $\sin\hat{A} = 2\sin\dfrac{\hat{A}}{2}\cos\dfrac{\hat{A}}{2}$.

N° 3.

1) $a^2 = b^2 + c^2 - 2bc\cos\hat{A}$, which gives

$cos\hat{A} = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{5}{8}$ , hence $cos\hat{A} > 0$ and consequently $\hat{A}$ is an acute angle .

2) a- We know that :

$BI^2 = AB^2 + AI^2 - 2\times AB\times AI\times\cos\hat{A} = 4+1-2\times2\times\dfrac{5}{8}$ ,

then : $BI^2 = \dfrac{5}{2}$ and consequently $BI = \sqrt{\dfrac{5}{2}} = \dfrac{\sqrt{10}}{2}$ .

b- In triangle $ABH$ , we have $\cos\hat{A} = \dfrac{AH}{AB}$ , therefore

$AH = 2\times\dfrac{5}{8} = \dfrac{5}{4}$ and consequently

$IH = AH - AI = \dfrac{5}{4} - 1 = \dfrac{1}{4}$.

In triangle $ABH$ we have $BH^2 = BI^2 - IH^2 = \dfrac{5}{2} - \dfrac{1}{16} = \dfrac{39}{16}$ .

Then, $BH = \dfrac{\sqrt{39}}{4}$.

c- In the right triangle $ABH$ , we have $sin\hat{A} = \dfrac{BH}{AB} = \dfrac{\sqrt{39}}{8}$ .

In the right triangle $BHC$ , we have

$\sin\hat{C} = \dfrac{BH}{BC} = \dfrac{\sqrt{39}}{4\sqrt{3}} = \dfrac{\sqrt{13}}{4}$.

- $\dfrac{a}{sin\,A} = 2R$ gives $2R = \dfrac{\sqrt{3}}{\dfrac{\sqrt{39}}{8}} = \dfrac{8\sqrt{3}}{\sqrt{39}} = \dfrac{8}{\sqrt{13}}$ then

$R = \dfrac{4}{\sqrt{13}} = \dfrac{4\sqrt{13}}{13}$ .

- Area $S$ of triangle $ABC$ is given by :

$S = \dfrac{1}{2}BH\times AC = \dfrac{1}{2}\times\dfrac{\sqrt{39}}{4}\times2 = \dfrac{\sqrt{39}}{4}$ square units .

**N°4.**

1) The second system of fundamental relations gives :

$\dfrac{a}{\sin\hat{A}} = \dfrac{b}{\sin\hat{B}} = \dfrac{c}{\sin\hat{C}} = 2R$, then :

$a = 2R\sin\hat{A}$ ; $b = 2R\sin\hat{B}$ and $c = 2R\sin\hat{C}$ .

$a^2 = bc$ gives $4R^2\sin^2\hat{A} = 2R\sin\hat{B}\times 2R\sin\hat{C}$ then :

$\sin^2\hat{A} = \sin\hat{B}\times\sin\hat{C}$ .

2) $\sin\hat{A} + \sin\hat{B} = 2\sin\hat{C}$ gives $\dfrac{a}{2R} + \dfrac{b}{2R} = \dfrac{2c}{2R}$ so $a + b = 2c$.

3) We know that $a^2 = b^2 + c^2 - 2bc\cos\hat{A}$ which gives :

$a^2 + 2bc\cos\hat{A} = b^2 + c^2$ and since $b^2 + c^2 = 2a^2$ we get

$a^2 + 2bc\cos\hat{A} = 2a^2$ therefore $\cos\hat{A} = \dfrac{a^2}{2bc}$ .

**N° 5.**

1) a- Triangle $OBE$ is isosceles since $BO = BE = 4$ .

$O\hat{B}E = 60° + 90° = 150°$ .

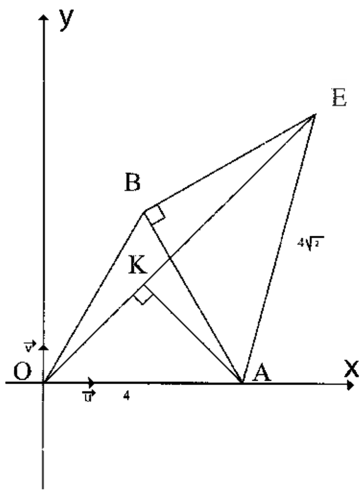
Then $B\hat{O}E = 15° = B\hat{E}O$ which gives

$A\hat{O}K = 60° - 15° = 45°$ .

In the right triangle $OKA$ , we have $\cos 45° = \dfrac{OK}{OA}$, then :

$OK = OA\cos 45°$ .

therefore , $OK = \dfrac{4\sqrt{2}}{2} = 2\sqrt{2}$ .




$A\hat{E}K = A\hat{E}B - K\hat{E}B = 45° - 15° = 30°$ .

In the right triangle $AKE$ , we have $\cos 30° = \dfrac{EK}{AE}$ .

But $AE = 4\sqrt{2}$ since $ABE$ is a right isosceles triangle, therefore: $EK = 4\sqrt{2} \times \frac{\sqrt{3}}{2} = 2\sqrt{6}$.

$OE = OK + KE = 2\sqrt{2} + 2\sqrt{6} = 2\sqrt{2}\left(1+\sqrt{3}\right)$.

b- The area $S$ of triangle $OAE$ is given by :

$S = \frac{1}{2} \times AO \times AE \times \sin 105 = \frac{1}{2} \times AK \times OE$.

But, $AK = OK = 2\sqrt{2}$ and $O\hat{A}E = 105^\circ$, therefore :

$\sin 105^\circ = \frac{AK \times OE}{AO \times AE} = \frac{2\sqrt{2} \times 2\sqrt{2}\left(1+\sqrt{3}\right)}{4 \times 4\sqrt{2}}$.

Consequently, $\sin 105^\circ = \frac{2\sqrt{2} \times 2\sqrt{2}\left(1+\sqrt{3}\right)}{4 \times 4\sqrt{2}} = \frac{\sqrt{2}+\sqrt{6}}{4}$.

2) $z_A = 4$.

- $x_B = OB\cos 60^\circ = 4 \times \frac{1}{2} = 2$

  $y_B = OB\sin 60^\circ = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3}$ , then $z_B = 2 + 2\sqrt{3}i$.

- $x_E = OE\cos 45^\circ = 2\sqrt{2}\left(1+\sqrt{3}\right) \times \frac{\sqrt{2}}{2} = 2\left(1+\sqrt{3}\right)$

  $y_E = OE \times \sin 45^\circ = 2\left(1+\sqrt{3}\right)$ , then :

  $z_E = 2\left(1+\sqrt{3}\right) + 2\left(1+\sqrt{3}\right)i$.

- $x_K = OK \times \cos 45^\circ = 2\sqrt{2} \times \frac{\sqrt{2}}{2} = 2$ ;

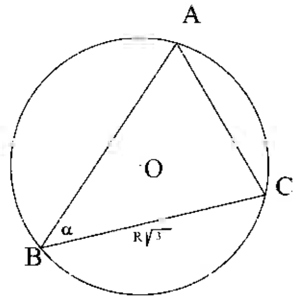
  $y_K = OK \times \sin 45^\circ = 2$. Therefore, $z_K = 2 + 2i$.

**N°6.**

1) Using the second system of fundamental relations :

$\frac{BC}{\sin \hat{A}} = \frac{AB}{\sin \hat{C}} = \frac{AC}{\sin \hat{B}} = 2R$ then $\sin \hat{A} = \frac{BC}{2R} = \frac{R\sqrt{3}}{2R} = \frac{\sqrt{3}}{2}$ which gives $\hat{A} = 60^\circ$ since $\hat{A}$ is an acute angle.

$\frac{AC}{\sin\alpha} = 2R$ implies $AC = 2R\sin\alpha$.




$\dfrac{AB}{\sin\hat{C}} = 2R$ implies :

$$AB = 2R\sin\hat{C} = 2R\sin\left[\pi - \left(\alpha + \frac{\pi}{3}\right)\right]$$

$$= 2R\sin\left(\alpha + \frac{\pi}{3}\right).$$

2) a- $y = AB - AC = R\sqrt{2}$ gives :

$2R\sin\left(\alpha + \frac{\pi}{3}\right) - 2R\sin\alpha = R\sqrt{2}$, therefore

$$\sin\left(\alpha + \frac{\pi}{3}\right) - \sin\alpha = \frac{\sqrt{2}}{2}.$$

Since $\sin p - \sin q = 2\sin\frac{p-q}{2}\cos\frac{p+q}{2}$, we get :

$$2\sin\left(\frac{\alpha + \frac{\pi}{3} - \alpha}{2}\right)\cos\left(\frac{\alpha + \frac{\pi}{3} + \alpha}{2}\right) = \frac{\sqrt{2}}{2}$$ , so :

$2\times\frac{1}{2}\cos\left(\alpha + \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2}$ and consequently

$\cos\left(\alpha + \frac{\pi}{6}\right) = \frac{\sqrt{2}}{2} = \cos\frac{\pi}{4}$ ,which gives $\alpha + \frac{\pi}{6} = \frac{\pi}{4}$ since $\alpha$ is an acute angle, then $\alpha = 15°$ .

Hence, $AC = 2R\sin 15°$ and $AB = 2R\sin 75°$ .

b- The area $S$ of triangle $ABC$ is equal to

$$S = \frac{1}{2}AB \times AC \times \sin\hat{A}.$$

$$S = \frac{1}{2}\times 2R\sin 75°\times 2R\sin 15°\times\frac{\sqrt{3}}{2} = R^2\sqrt{3}\times\sin 15\times\sin 75 .$$

But $\sin 75° = \sin(90 - 15) = \cos 15°$ , therefore :

$$S = R^2\sqrt{3}\times\sin 15°\times\cos 15° = R^2\sqrt{3}\times\frac{1}{2}\sin 30°.$$

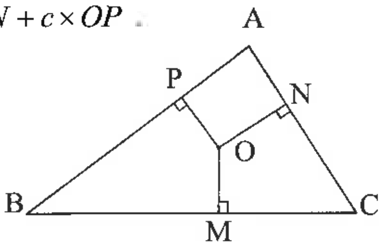
Finally , $S = \dfrac{R^2\sqrt{3}}{4}$

N°7.

1) $S$ = Area of triangle $BOC$ + Area of $COA$ + Area of $AOB$.

$S = \dfrac{1}{2}a \times OM + \dfrac{1}{2}b \times ON + \dfrac{1}{2}c \times OP$ .

Then $2S = a \times OM + b \times ON + c \times OP$ .




2) The center $O$ of the circle inscribed in triangle $ABC$ is the point of intersection of the three bisectors, in this case we have $OM = ON = OP = r$.

Therefore,

$2S = ar + br + cr = r(a+b+c)$ , then $2S = r \times 2p$ with

$p$ being half the perimeter of triangle $ABC$, $p = \dfrac{a+b+c}{2}$,

consequently, $S = p\,r$.

N°8.

1) $\cos\hat{A} = \dfrac{b^2+c^2-a^2}{2bc}$ and $\cos\hat{B} = \dfrac{a^2+c^2-b^2}{2ac}$, then:

$$a\cos\hat{B} - b\cos\hat{A} = a\left(\frac{a^2+c^2-b^2}{2ac}\right) - b\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

$$= \frac{a^2+c^2-b^2}{2c} - \frac{b^2+c^2-a^2}{2c} = \frac{2a^2-2b^2}{2c}$$

$$= \frac{(a-b)(a+b)}{c}.$$

2) $b\cos\hat{C} + c\cos\hat{B} = b\left(\dfrac{a^2+b^2-c^2}{2ab}\right) + c\left(\dfrac{a^2+c^2-b^2}{2ac}\right)$

$$= \frac{a^2+b^2-c^2+a^2+c^2-b^2}{2a} = a.$$

3) $\dfrac{\cos\hat{A}}{a} + \dfrac{\cos\hat{B}}{b} + \dfrac{\cos\hat{C}}{c} =$

$$\frac{b^2+c^2-a^2}{2abc}+\frac{a^2+c^2-b^2}{2abc}+\frac{a^2+b^2-c^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}.$$

For $\hat{A}=90°$ , $\cos\hat{A}=0$ and $a^2=b^2+c^2$ , the relation becomes:

$\frac{\cos\hat{B}}{b}+\frac{\cos\hat{C}}{c}=\frac{2a^2}{2abc}$ which gives $\frac{c\cos\hat{B}+b\cos\hat{C}}{bc}=\frac{a}{bc}$.

So, we get $a=c\cos\hat{B}+b\cos\hat{C}$ .

**N°9.**

1) In triangle $ABC$ and using the first system of fundamental relations we have:

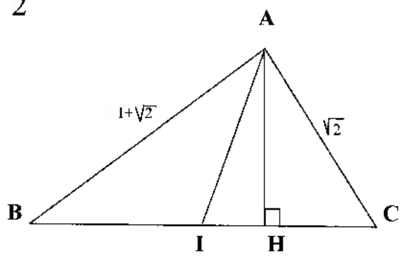
$$\cos\hat{A}=\frac{b^2+c^2-a^2}{2bc}=\frac{2+\left(1+2+2\sqrt{2}\right)-3}{2\sqrt{2}\left(1+\sqrt{2}\right)}$$

$$\cos\hat{A}=\frac{2+2\sqrt{2}}{2\sqrt{2}\left(1+\sqrt{2}\right)}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}=\cos\frac{\pi}{4}$$ then $\hat{A}=\frac{\pi}{4}$ since $0<A<\pi$.

The area $S$ of triangle $ABC$ is given by : $S=\frac{1}{2}AB\times AC\times\sin\frac{\pi}{4}$.

$S=\frac{1}{2}\left(1+\sqrt{2}\right)\times\sqrt{2}\times\frac{\sqrt{2}}{2}=\frac{1+\sqrt{2}}{2}$ square units .

2) In triangle $ABC$, we have :




$$AB^2+AC^2=2AI^2+\frac{BC^2}{2}$$

Then:

$$2AI^2=AB^2+AC^2-\frac{BC^2}{2}$$

$2AI^2=\left(1+\sqrt{2}\right)^2+\left(\sqrt{2}\right)^2-\frac{3}{2}=\frac{7+4\sqrt{2}}{2}$ ,consequently

$$AI=\frac{\sqrt{7+4\sqrt{2}}}{2}.$$

*Another method:*

In triangle $AIC$, we have :

$AB^2=AC^2+BC^2-2AC\times BC\times\cos\hat{C}$, so $\cos\hat{C}=\frac{b^2+a^2-c^2}{2ab}$.

Then $AI^2 = b^2 + \frac{a^2}{4} - 2b \times \frac{a}{2} \times \frac{b^2 + a^2 - c^2}{2ab}$, so

$AI^2 = \frac{b^2}{2} - \frac{a^2}{4} + \frac{c^2}{2} = \frac{7+4\sqrt{2}}{4}$, then $AI = \frac{\sqrt{7+4\sqrt{2}}}{2}$.

In triangle $ABC$, we have :

In triangle $ABC$, we know that $AB^2 - AC^2 = 2BC \times IH$.

Therefore, $1+2+2\sqrt{2}-2 = 2\sqrt{3} \times IH$ which gives

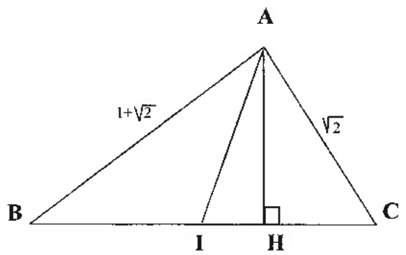
$IH = \frac{1+2\sqrt{2}}{2\sqrt{3}}$.

In the right triangle $AIH$, we have: $AH^2 = AI^2 - IH^2$ then:

$$AH^2 = \frac{7+4\sqrt{2}}{4} - \frac{1+8+4\sqrt{2}}{12} = \frac{21+12\sqrt{2}-9-4\sqrt{2}}{12}$$

$= \frac{12+8\sqrt{2}}{12}$, so $AH^2 = \frac{3+2\sqrt{2}}{3} = \frac{\left(1+\sqrt{2}\right)^2}{3}$

and consequently $AH = \frac{1+\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{3}+\sqrt{6}}{3}$.




3) $MB^2 + MC^2 = 2MI^2 + \frac{BC^2}{2}$ then $2\sqrt{2} = 2MI^2 + \frac{3}{2}$

Hence, $2MI^2 = 2\sqrt{2} - \frac{3}{2} = \frac{4\sqrt{2}-3}{2}$ so $MI^2 = \frac{4\sqrt{2}-3}{4}$ and

consequently $MI = \frac{\sqrt{4\sqrt{2}-3}}{2}$.

The set $(T)$ of points $M$ is the circle of center $I$ and radius

$$R = \frac{\sqrt{4\sqrt{2}-3}}{2}.$$

We have : $IA = \dfrac{\sqrt{4\sqrt{2}+7}}{2}$, then $IA > R$ and consequently $A$ is exterior to circle $(T)$.

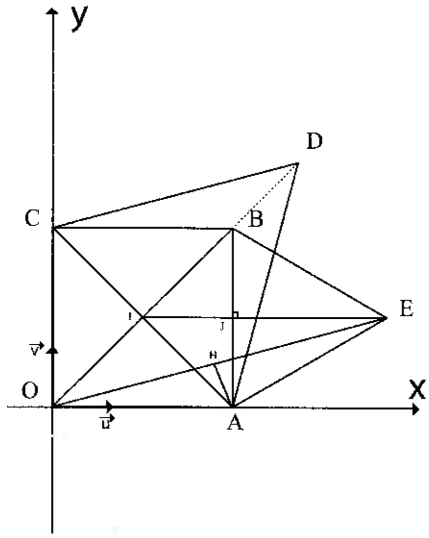
**N° 10.**

1) a- We have: $CA = a\sqrt{2}$. The points $O$ , $I$ , $B$ and $D$ are collinear since they belong to the perpendicular bisector of segment $[AC]$.

In the equilateral triangle $ACD$ the median $DI = AC \times \dfrac{\sqrt{3}}{2}$ ,

then $DI = a\sqrt{2} \times \dfrac{\sqrt{3}}{2} = \dfrac{a\sqrt{6}}{2}$ , moreover $OI = \dfrac{OB}{2} = \dfrac{a\sqrt{2}}{2}$ so :

$$OD = OI + ID = \frac{a\sqrt{2}}{2} + \frac{a\sqrt{6}}{2} = \frac{a\sqrt{2}}{2}\left(1+\sqrt{3}\right).$$




**Calculating** $IE$ :

Let $J$ be the midpoint of $[AB]$, the three points $I$ , $J$ and $E$ are collinear since they belong to the perpendicular bisector of $[AB]$.

But $IJ = \frac{CB}{2} = \frac{a}{2}$ and $JE = AE \times \frac{\sqrt{3}}{2} = \frac{a\sqrt{3}}{2}$, therefore:

$IE = IJ + JE = \frac{a}{2} + \frac{a\sqrt{3}}{2} = \frac{a}{2}(1+\sqrt{3})$.

**Calculating** $OE$ :

In triangle $OAE$ ,we have :

$$OE^2 = AO^2 + AE^2 - 2AO \times AE \times \cos(90+60)$$

$$= a^2 + a^2 - 2a \times a \times (-\sin 60) = 2a^2 + 2a^2 \frac{\sqrt{3}}{2}$$

$$= 2a^2\left(1 + \frac{\sqrt{3}}{2}\right) = a^2(2+\sqrt{3})\text{, therefore:}$$

$OE = a\sqrt{2+\sqrt{3}}$ .

b- Triangle $OAE$ is isosceles since $OA = AE = a$ and we have:

$\widehat{OAE} = 150°$. Let $H$ be the midpoint of $[OE]$

$$\cos 15° = \frac{OH}{OA} = \frac{OE}{2OA} = \frac{a\sqrt{2+\sqrt{3}}}{2a} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

$$\sin 15° = \sqrt{1-\cos^2 15°} = \sqrt{1 - \frac{2+\sqrt{3}}{4}} = \sqrt{\frac{2-\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

2) a- $z_A = 2$ since $\overrightarrow{OA} = 2\vec{u}$.

$z_B = 2+2i$ , $z_C = 2i$ and $z_I = \frac{z_O + z_B}{2} = 1+i$.

$z_D = OD\, e^{i\frac{\pi}{4}}$ with $OD = \sqrt{2}(1+\sqrt{3})$, therefore :

$$z_D = \sqrt{2}(1+\sqrt{3}) \times \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

$$= (1+\sqrt{3}) + i(1+\sqrt{3}).$$

$z_E = OE\, e^{i\frac{\pi}{15}}$ with $OE = 2\sqrt{2+\sqrt{3}}$ therefore:

$$z_E = 2\sqrt{2+\sqrt{3}} \times \left(\frac{\sqrt{2+\sqrt{3}}}{2} + i\frac{\sqrt{2-\sqrt{3}}}{2}\right) = (2+\sqrt{3}) + i.$$

b- We need to find $\dfrac{Z_{\overrightarrow{EA}}}{Z_{\overrightarrow{ED}}}$ ,

$Z_{\overrightarrow{EA}} = z_A - z_E = 2 - 2 - \sqrt{3} - i = -\sqrt{3} - i$

$Z_{\overrightarrow{ED}} = z_D - z_E = \left(1+\sqrt{3}\right) + i\left(1+\sqrt{3}\right) - \left(2+\sqrt{3}\right) - i = -1 + i\sqrt{3}$.

Therefore, $\dfrac{Z_{\overrightarrow{EA}}}{Z_{\overrightarrow{ED}}} = \dfrac{-\sqrt{3}-i}{-1+i\sqrt{3}} = i = e^{i\frac{\pi}{2}}$ , consequently, $\dfrac{EA}{ED} = 1$

and $\left(\overrightarrow{ED}, \overrightarrow{EA}\right) = \dfrac{\pi}{2}$ $(mod\, 2\pi)$ so triangle $AED$ is right isosceles .

N° 11.

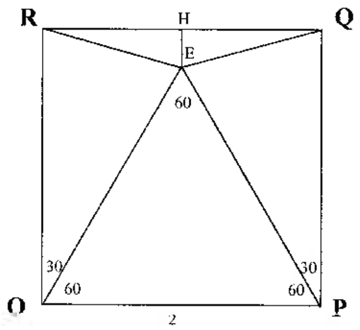
1) a- Triangle $ORE$ is isosceles since $OE = OR$ and since $E\hat{O}P = 60^\circ$ then $E\hat{O}R = 30^\circ$, therefore $O\hat{R}E = O\hat{E}R = 75^\circ$.

Similarly we have $P\hat{E}Q = P\hat{Q}E = 75^\circ$.

In the isosceles triangle $ERQ$ we have :

$E\hat{R}Q = E\hat{Q}R = 90^\circ - 75^\circ = 15^\circ$,

consequently $R\hat{E}Q = 180^\circ - 30^\circ = 150^\circ$.




b- In the isosceles triangle $ORE$ we have:

$OR = OE = 2$ and $R\hat{O}E = 30^\circ$, then :

$RE^2 = OR^2 + OE^2 - 2OR \times OE \times \cos 30^\circ$

So $RE^2 = 4 + 4 - 2 \times 2 \times 2 \times \dfrac{\sqrt{3}}{2} = 8 - 4\sqrt{3} = 4\left(2 - \sqrt{3}\right)$.

Therefore, $RE = EQ = 2\sqrt{2-\sqrt{3}}$.

Let $H$ be the midpoint of $[RQ]$ in triangle $REQ$.

$$\cos 15^\circ = \frac{RH}{RE} = \frac{1}{2\sqrt{2-\sqrt{3}}} = \frac{\sqrt{2+\sqrt{3}}}{2}.$$

$$\sin 15^\circ = \sqrt{1-\cos^2 15^\circ} = \sqrt{1-\frac{2+\sqrt{3}}{4}} = \frac{\sqrt{2-\sqrt{3}}}{2}.$$

2) Area of triangle $REQ$ is given by :

$$\frac{1}{2}RE \times RQ \times \sin 15^\circ = \frac{1}{2} \times 2\sqrt{2-\sqrt{3}} \times 2 \times \frac{\sqrt{2-\sqrt{3}}}{2}$$

$$= 2-\sqrt{3} \text{ square units.}$$

Let $S$ be the area of triangle $ORE$.

Area of triangle $PEQ$ is equal to $S$.

If $\mathcal{A}$ is the area of the square then :

$\mathcal{A} = S + S +$ area of triangle $OPE$ + area of triangle $REQ$.

Area of triangle $OEP$ =

$$\frac{1}{2}OE \times OP \times \sin 60^\circ = \frac{1}{2} \times 2 \times 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \text{ square units.}$$

Therefore, $\mathcal{A} = 4 = 2S + \sqrt{3} + 2 - \sqrt{3}$ and consequently, $S = 1$ square units.

**N° 12.**

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}} = 2R, \text{ then :}$$

$$\frac{5}{\sin \hat{A}} = \frac{b}{\frac{\sqrt{3}}{2}} = \frac{c}{\sin \hat{C}} = 6, \text{ which gives :}$$

$$\sin \hat{A} = \frac{5}{6} \text{ and } AC = b = \frac{6\sqrt{3}}{2} = 3\sqrt{3}.$$

In triangle $ABC$ , on a $b^2 = a^2 + c^2 - 2ac\cos \hat{B}$,

then $27 = 25 + c^2 - 10 \times c \times \frac{1}{2}$, which gives the quadratic equation:

$c^2-5c-2=0$ that has as roots $c=\dfrac{5-\sqrt{33}}{2}$ that is rejected and

$c=\dfrac{5+\sqrt{33}}{2}$ that is accepted.

Area $S$ of triangle $ABC$ is given by :

$$S=\frac{1}{2}a\times c\times\sin 60^\circ=\frac{1}{2}\times 5\times\frac{5+\sqrt{33}}{2}\times\frac{\sqrt{3}}{2}$$

$$=\frac{5\sqrt{3}\left(5+\sqrt{33}\right)}{8}\text{ square units.}$$

N° 13.

1) $\dfrac{a}{\sin\hat{A}}=\dfrac{b}{\sin\hat{B}}=\dfrac{c}{\sin\hat{C}}=2R$, then :

$\sin\hat{B}=\dfrac{b}{2R}$ and $\sin\hat{C}=\dfrac{c}{2R}$.

$\sin\hat{B}=2\sin\hat{C}$ gives $\dfrac{b}{2R}=2\times\dfrac{c}{2R}$ then $b=2c$.

Moreover, $a^2=b^2+c^2$ then $25=4c^2+c^2$.

Which gives $c^2=5$ and consequently $c=\sqrt{5}$ and $b=2\sqrt{5}$.

2) $\sin\hat{B}=\dfrac{b}{a}=\dfrac{2\sqrt{5}}{5}$.

Area of triangle $ABC$ is :

$$\frac{1}{2}BA\times BC\times\sin\hat{B}=\frac{1}{2}\times\sqrt{5}\times 5\times\frac{2\sqrt{5}}{5}=5\text{ square units}$$

N° 14.

1) • Calculating BC :

$$BC^2=AB^2+AC^2-2AB\times AC\times\cos 45^\circ$$

$$=\left(1+3+2\sqrt{3}\right)+6-2\left(1+\sqrt{3}\right)\times\sqrt{6}\times\frac{\sqrt{2}}{2}=4.$$

Then $BC=2$.

• Calculating $\hat{B}$ and $\hat{C}$ :

Using the first system of fundamental relations:

$$\frac{a}{\sin\hat{A}} = \frac{b}{\sin\hat{B}} = \frac{c}{\sin\hat{C}} = 2R$$

then $\dfrac{2}{\frac{\sqrt{2}}{2}} = \dfrac{\sqrt{6}}{\sin\hat{B}} = \dfrac{1+\sqrt{3}}{\sin\hat{C}} = 2R$.

Which gives $2\sin\hat{B} = \sqrt{3}$ so $\sin\hat{B} = \dfrac{\sqrt{3}}{2}$ and since

$\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{1}{2}$, we deduce that $\hat{B} = 60°$.

$\hat{C} = 180° - (60° + 45°)$ therefore $\hat{C} = 75°$.

2) $\dfrac{a}{\sin\hat{A}} = 2R$ gives $2R = \dfrac{4}{\sqrt{2}}$ therefore $R = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

The area $S$ of triangle $ABC$ is :

$$S = \frac{1}{2}AB \times AC \times \sin 45° = \frac{1}{2} \times \sqrt{6} \times \left(1+\sqrt{3}\right) \times \frac{\sqrt{2}}{2}$$

$$= \frac{\sqrt{3}\left(1+\sqrt{3}\right)}{2} = \frac{3+\sqrt{3}}{2} \text{ square units.}$$

3) The area $S$ of triangle $ABC$ is given by :

$S = \dfrac{1}{2}AH \times BC = \dfrac{3+\sqrt{3}}{2}$ which gives $AH = \dfrac{3+\sqrt{3}}{2}$.

## N° 15.

1) $a^2b^2 + c^4 = b^4 + a^2c^2$.

This relation can be written in the form :

$\left(a^2b^2 - a^2c^2\right) + \left(c^4 - b^4\right) = 0$

$a^2\left(b^2 - c^2\right) + \left(c^2 - b^2\right)\left(c^2 + b^2\right) = 0$

$\left(b^2 - c^2\right)\left[a^2 - c^2 - b^2\right] = 0$ which gives :

$b^2 - c^2 = 0$ or $a^2 = c^2 + b^2$, then :

$b = c$ or $a^2 = b^2 + c^2$ therefore triangle $ABC$ is isosceles at $A$ or right at $A$.

2) $a\left(b\cos\hat{C} - c\cos\hat{B}\right) = a\left[\dfrac{b\left(a^2+b^2-c^2\right)}{2ab} - \dfrac{c\left(a^2+c^2-b^2\right)}{2ac}\right]$

$$=\frac{a^2+b^2-c^2}{2}-\frac{a^2+c^2-b^2}{2}=\frac{2\left(b^2-c^2\right)}{2}=b^2-c^2.$$

3) a- $\cos\hat{A}=\dfrac{b^2+c^2-a^2}{2bc}=\dfrac{4+3-1}{4\sqrt{3}}=\dfrac{6}{4\sqrt{3}}=\dfrac{6\sqrt{3}}{12}=\dfrac{\sqrt{3}}{2}$

then $\hat{A}=30^0$ since $0<A<\pi$.

b- $b^2-c^2=a\left(b\cos\hat{C}-c\cos\hat{B}\right)$, then :

$4-3=1\left(2\cos\hat{C}-\sqrt{3}\cos\hat{B}\right)$, which gives

$2\cos\hat{C}-\sqrt{3}\cos\hat{B}=1$.

An analogous relation to that in the question can be written as:

$a^2-b^2=c\left(a\cdot\cos\hat{B}-b\cdot\cos\hat{A}\right)$ which gives :

$1-4=\sqrt{3}\left(\cos\hat{B}-2\cos\hat{A}\right)$ therefore the 2nd relation is

$\sqrt{3}\cos\hat{B}-2\sqrt{3}\cos\hat{A}=-3$

c- We have $\sqrt{3}\cos\hat{B}-2\sqrt{3}\cos\hat{A}=-3$ and since $\cos\hat{A}=\dfrac{\sqrt{3}}{2}$

we get $\sqrt{3}\cos\hat{B}-2\sqrt{3}\times\dfrac{\sqrt{3}}{2}=-3$ which gives

$\sqrt{3}\cos\hat{B}=-3+3=0$ then $\cos\hat{B}=0$ and $\cos\hat{C}=\dfrac{1}{2}$.

# CHAPTER 7

# Rotation and Translation

involutive

$f \circ f = Id.$

**Chapter Review :**

**I- *Generalities on Transformations :***

- A transformation is a mapping $f$ of the plane to itself that to every point $M$ associates a point $M'$, denoted by $M' = f(M)$.
- A point $M$ is said to be invariant if $M = f(M)$. غير متغيرة .
- A transformation $f$ is said to be involutive if $f \circ f = I_d$ or $f = f^{-1}$.

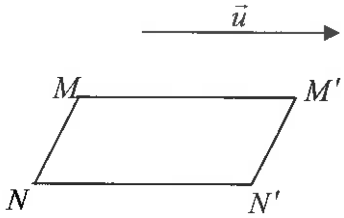
**II-*Translation :***

- Definition .

  $\vec{u}$ is a given non-zero vector, the translation of vector $\vec{u}$, denoted by $t_{\vec{u}}$ is the mapping of the plane that to every point $M$ associates a point $M'$ such that $\overrightarrow{MM'} = \vec{u}$.

  $t^{-1} = t_{-\vec{u}}$?

- $t^{-1} = t_{-\vec{u}}$.




- Characteristic Properties:
  A mapping is a translation if and only if for all points $M$ , $N$ of respective images $M'$ and $N'$ we have $\overrightarrow{MN} = \overrightarrow{M'N'}$.
- The image of a straight line $(d)$ by a translation $t_{\vec{u}}$ is a straight line $(d')$ that is parallel to it.
  It is sufficient to find the image $A'$ of a point $A$ of $(d)$ and to draw the straight line $(d')$ parallel to $(d)$ and passing through $A'$.

- The image of a circle $C(O;R)$ is a circle $C'(O';R)$ where $O'$ is the image of $O$ by $t_{\vec{u}}$.

- $t_{\vec{v_1}} \circ t_{\vec{v_2}} = t_{\vec{v_1}+\vec{v_2}} = t_{\vec{v_2}} \circ t_{\vec{v_1}}$.

***III- Rotation :***

- Definition :

$\alpha$ is a real number and $O$ a given point in the plane .
We define the rotation of center $O$ and angle $\alpha$ , denoted by $r(O;\alpha)$ the mapping of the plane that associates to every point $M$ of the plane a point $M'$ such that $OM = OM'$ and $(\overrightarrow{OM};\overrightarrow{OM'}) = \alpha \pmod{2\pi}$ .

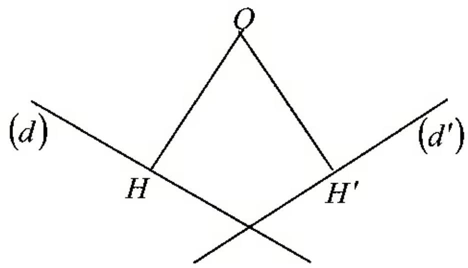
$r(O,\alpha)$

$r^{-1}(O,\alpha)$

- $O$ is invariant under $r$ .
- $r^{-1}(O;\alpha) = r(O;-\alpha)$.
- If $\alpha = 0 \pmod{2\pi}$ then $r = I_d$ .

$r(I,\alpha=0)$

- Characteristic Properties :
  A mapping is a rotation of angle $\alpha$ if and only if for all points $M$ , $N$ of respective images $M'$ and $N'$ we have :
  $MN = M'N'$ and $(\overrightarrow{MN};\overrightarrow{M'N'}) = \alpha \pmod{2\pi}$.
- The image of a straight line $(d)$ by a rotation is a straight line $(d')$.




- The image of a circle $C(w;R)$ is a circle $C'(w';R)$ where $w'$ is the image of $w$ by $r$.
- Composite of two rotations $r(O';\beta) \circ r(O;\alpha)$ :

  ***First Case :*** $\alpha + \beta = 0 \pmod{2\pi}$:

  * $r(O;-\alpha) \circ r(O;\alpha) = I_d$ .

  * $r(O';-\alpha) \circ r(O;\alpha) = t_{\vec{v}}$ .

  The vector $\vec{v} = \overrightarrow{OO''}$ where $O''$ is the image of $O$ by $r(O';-\alpha)$.

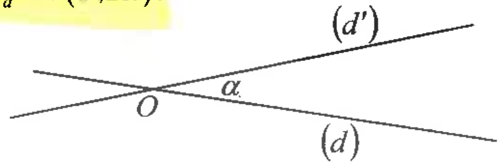
***Second Case:*** $\alpha+\beta \neq 0 (\text{mod}\, 2\pi)$ :

* $r(O;\beta)\circ r(O;\alpha) = r(O;\alpha+\beta)$.
* $r(O';\beta)\circ r(O;\alpha) = r(w;\alpha+\beta)$.

- Composite of a rotation and a translation :
  $r(O;\alpha)\circ t_{\vec{v}} = r(w;\alpha)$ .

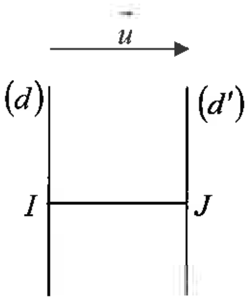
***IV- Symmetry :***

- Let $(d)$ and $(d')$ be two straight lines intersecting at point $O$ and let $\alpha$ be one of the angles determined by $(d)$ and $(d')$ .
  Designate by $S_d$ and $S_{d'}$ the two axial symmetries of respective axes $(d)$ and $(d')$. Intersecting
  $S_{d'}\circ S_d = r(O;2\alpha)$.




- Let $(d)$ and $(d')$ be two parallel straight lines and $[IJ]$ their common perpendicular , then:

$S_{d'}\circ S_d = t_{2\vec{IJ}}$ .



# Solved Problems

**N°1.**

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the mapping $T$ of the plane ,that associates to every point $M(x;y)$ of the plane a point $M'(x';y')$ , such that:

$$M\begin{cases}x\\y\end{cases}\longrightarrow M'\begin{cases}x'=x+y+1\\y'=-x+y-1\end{cases}$$

1) Find the invariant point under $T$.
2) Is $T$ involutive?
3) Determine the transformation $T^{-1}$ inverse of $T$.
4) Let $(d)$ be the straight line of equation $y=2x-1$.
   Determine the image of $(d)$ by $T$.

**N°2.**

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the mapping of the plane $T$, that to every point $M(x;y)$ distinct of $O$ associates the point $M'(x';y')$ , such that:

$$M\begin{cases}x\\y\end{cases}\longrightarrow M'\begin{cases}x'=\dfrac{x}{x^2+y^2}\\[2mm]y'=\dfrac{y}{x^2+y^2}\end{cases}$$

1) Determine the set of invariant points by $T$.
2) Show that $T$ is involutive and determine its inverse transformation $T^{-1}$.
3) Let $(C)$ be the circle of equation $x^2+y^2-2x-4y=0$.
   Find the image of $(C)$ by $T$.
4) Denote by $(C')$ the circle of equation $x^2+y^2-2x-4y+1=0$.
   Find the image of $(C')$ by $T$.
5) Let $(d)$ be the straight line of equation $y=x-1$, find the image of $(d)$ by $T$.

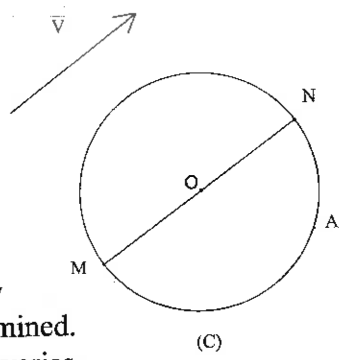
N°3.

In the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$, consider the translation $t$ of vector $\vec{v}(3;-2)$ and let $M'(x';y')$ be the image of the point $M(x;y)$ by $t$.

1) Calculate $x$ and $y$ in terms of $x'$ and $y'$.
2) Consider the circle $(C)$ of center $I(1;-1)$ and radius 2.
   Find the image of $(C)$ by $t$.
3) Let $r$ be the rotation of center $O$ and angle $\frac{\pi}{2}$ and let $M'(x';y')$ be the image of point $M(x;y)$ by $r$.
   a- Express $x$ and $y$ in terms of $x'$ and $y'$.
   b- Find the image of $(C)$ by $r$.

N°4.

$(C)$ is a variable circle of center $O$, and of a constant radius $R$ passing through a fixed point $A$.
$[MN]$ is a variable diameter such that $\overrightarrow{MN}=\vec{V}$ where $\vec{V}$ is a given vector.




1) Determine the set of points $O$ as the circle varies.
2) Show that $N$ is the image of $O$ by a simple transformation to be determined.
   Deduce the set of points $N$ as $(C)$ varies.
3) Determine the set of points $M$ as the circle $(C)$ varies.

N°5.

On a fixed axis $x'Ox$, consider a variable point $A$ and construct an isosceles triangle of base $[OA]$ and vertex $M$.
Let $w$ be the center of circle $(C)$ circumscribed about triangle $OAM$.
Suppose that the radius of $(C)$ is constant.

1) Determine the set of points $w$ as $A$ varies.
2) By which simple transformation is $w$ mapped onto $M$ ?
3) Determine the set of points $M$.

N° 6.

$(C)$ and $(C')$ are two fixed circles intersecting in two points $A$ and $B$ and of respective centers $O$ and $O'$.

A variable secant $(d)$ passing through $A$ cuts $(C)$ and $(C')$ in $I$ and $J$ respectively.

The perpendicular through $I$ to $(d)$ cuts $(C)$ in $K$.

The perpendicular through $J$ to $(d)$ cuts $(C')$ in $L$.

The parallel through $I$ to $(OO')$ cuts $(JL)$ in $M$.

The parallel through $J$ to $(OO')$ cuts $(KI)$ in $N$.

1) Prove that the points $K$, $B$ and $L$, are collinear.
2) a- Show that $\overrightarrow{IM} = 2\overrightarrow{OO'}$.
   b- Deduce the set of points $M$ as $(d)$ varies.
   c- Find the set of points $N$ as $(d)$ varies.

N° 7.

Consider the two fixed circles $(C)$ and $(C')$ of respective centers $O$ and $O'$ having the same radius $R$ and tangent externally at a point $J$.

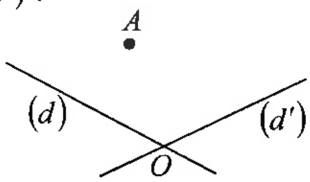
Let $M$ be a point of $(C)$ and $M'$ a point of $(C')$ such that

$\left(\overrightarrow{OM}, \overrightarrow{O'M'}\right) = \frac{\pi}{2} (\text{mod } 2\pi)$.

1) Show that there exists a rotation that maps $M$ onto $M'$, whose center and angle are to be determined.
2) Show that the perpendicular bisector of $[MM']$ passes through a fixed point $I$.
3) Let $I'$ be the symmetric of $I$ with respect to the straight line $(OO')$ and $M''$ the image of $M$ by the rotation $r'$ of center $I'$ and angle $-\frac{\pi}{2}$.
   a- Determine $r'(O)$.
   b- Show that the points $M'$ and $M''$ are diametrically opposite in $(C')$.

N°8.

$(d)$ and $(d')$ are two straight lines intersecting at $O$.
Let $A$ be a point not on $(d)$ and not on $(d')$.
Construct a right isosceles triangle $ABC$ of vertex $A$ and such that $B \in (d)$, $C \in (d')$ and $(\overrightarrow{AB};\overrightarrow{AC}) = \frac{\pi}{2}(\bmod 2\pi)$.




N°9.

$ABC$ is a right isosceles triangle of vertex $B$ and such that $(\overrightarrow{BA};\overrightarrow{BC}) = \frac{\pi}{2}(\bmod 2\pi)$.

Let $(d)$ be the straight line passing through $B$ and parallel to $(AC)$.
Denote by $r_1 = S_{(AB)} \circ S_{(AC)}$ , $r_2 = S_{(AB)} \circ S_{(d)}$ and $r_3 = S_{(AC)} \circ S_{(d)}$.

1) Determine the nature of each of the transformations $r_1$, $r_2$ and $r_3$.
2) Show that $r_2 \circ r_1(A) = C$.
3) Determine the nature and characteristic elements of the transformation $r_2 \circ r_1$.

N° 10.

In the oriented plane , consider a triangle $OAB$ right isosceles at $O$ and such that $(\overrightarrow{OA};\overrightarrow{OB}) = \frac{\pi}{2}\bmod(2\pi)$.

Designate by $R_A$ and $R_B$ the rotations of centers $A$ and $B$ respectively and of the same angle $\frac{\pi}{2}$ and by $S_O$ the symmetry of center $O$.

$C$ is a point not on straight line $(AB)$, we draw the direct squares $BEDC$ and $ACFG$.

We have then $(\overrightarrow{BE};\overrightarrow{BC}) = \frac{\pi}{2}\bmod(2\pi)$ and $(\overrightarrow{AC};\overrightarrow{AG}) = \frac{\pi}{2}\bmod(2\pi)$.

1) a- Determine $S_{(AO)} \circ S_{(AB)}$ the composite of the two axial symmetries of respective axes $(AB)$ and $(AO)$.

b- Writing $R_B$ in the form of the composite of two reflections , prove that $R_A \circ R_B = S_O$.

2) a- Determine the image of $E$ by $R_A \circ R_B$.

b- Deduce that $O$ is the midpoint of $[EG]$.

c- Denote by $R_F$ and $R_D$ the rotations of centers $F$ and $D$ respectively and having the same angle $\frac{\pi}{2}$.

Study the image of $C$ by $R_F \circ S_O \circ R_D$.

Determine the transformation $R_F \circ S_O \circ R_D$.

d- Let $H$ be the symmetric of $D$ with respect to $O$.

Prove that $R_F(H) = D$.

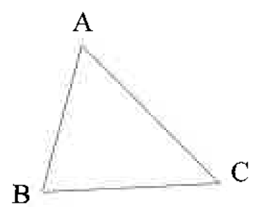
Prove that $FOD$ is right isosceles at $O$.

**N° 11.**

Consider a direct triangle $ABC$.

$I$, $J$ and $K$ are the midpoints of the segments $[BC]$, $[CA]$ and $[AB]$.

$O$ and $\Omega$ are the centers of the squares constructed on the sides $[AB]$ and $[CA]$ respectively exterior to triangle $ABC$.




1) Let $r$ be the rotation that transforms $K$ onto $J$ and $O$ onto $I$, determine the angle of $r$.

2) Show that $r(I) = \Omega$.

3) Deduce that $(IO)$ and $(I\Omega)$ are perpendicular and that $IO = I\Omega$.

**N° 12.**

In the oriented plane, consider the figure below.

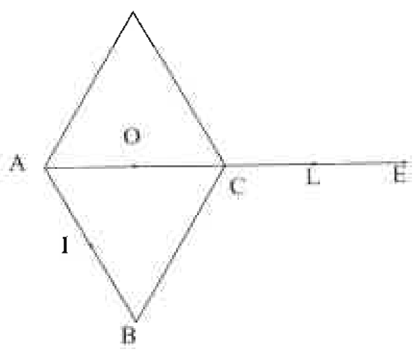
Triangles $ABC$ and $ACD$ are two direct equilateral triangles such that $(\overrightarrow{BC};\overrightarrow{BA}) = \frac{\pi}{3} \bmod(2\pi)$ and $(\overrightarrow{DA};\overrightarrow{DC}) = \frac{\pi}{3} \bmod(2\pi)$.

The points $O$ and $I$ are the midpoints of the segments $[CA]$ and $[AB]$ respectively. $L$ and $E$ are given points in the plane such that $\overrightarrow{OC} = \overrightarrow{CL} = \overrightarrow{LE}$.

Let $r$ be the rotation of center $A$ and whose angle has a measure $\frac{\pi}{3}$ and $t$ the translation of vector $\overrightarrow{OA}$.

Denote by $r' = r \circ t$.

1) a- Determine $r'(O)$.
   b- Determine a measure of angle $(\overrightarrow{IO};\overrightarrow{IA})$.
   c- Determine the nature and characteristic elements of $r'$.
2) $M$ is any point in the plane , denote by $N = r(M)$.
   $J$ is the midpoint of $[EM]$ and $K$ the midpoint of $[ND]$.
   a- Let $P$ be the pre-image of $M$ by $t$, what is the midpoint of $[LP]$ ?
   b- Using $r'(L)$ and $r'(P)$, show that the two triangles $ILD$ and $IJK$ are equilateral .




$\boxed{N°13.}$

In the oriented plane , consider the two fixed points $A$ and $B$. Designate by $r_A$ and $r_B$ the rotations of centers $A$ and $B$ respectively and having the same angle $\frac{\pi}{2}$.

For all points $M$ of the plane, denote by $M_1$ and $M_2$ the images of $M$ by $r_A$ and $r_B$ respectively.

1) Consider the transformation $T = r_B \circ r_A^{-1}$.
   a- Construct the point $C$ image of $A$ by $T$.
   b- Determine the nature and characteristic elements of $T$.
   c- Deduce the nature of quadrilateral $M_1M_2CA$.

2) Suppose that $M$ describes the circle $(\Gamma)$ of diameter $[AB]$.
   a- Determine the set $(\Gamma_2)$ described by $M_2$ when $M$ describes the circle $(\Gamma)$ and precise the center of $(\Gamma_2)$.
   b- Let $w$ and $w'$ be the midpoints of segments $[AB]$ and $[BC]$ respectively.
   Determine the set of point $I$, midpoint of $[M_1M_2]$ as $M$ describes circle $(\Gamma)$.

**N° 14.**

In the oriented plane , consider an equilateral triangle $ABC$ such that $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{3}\text{mod}(2\pi)$.

Let $I$ be the midpoint of $[BC]$ and $J$ the point such that $B$ is the midpoint of $[JC]$.

Designate by $r_1$ the rotation of center $A$ and angle $\frac{\pi}{3}$ and by $r_2$ the rotation of center $B$ and angle $-\frac{2\pi}{3}$.

1) Let $A'$ and $B'$ be the images of the points $A$ and $B$ by $r_2 \circ r_1$ respectively.
   Prove that $I$ is the midpoint of $[AA']$ and $B$ is the midpoint of $[AB']$.
2) Determine the nature of $r_2 \circ r_1^{-1}$ then prove that for all points $M$ of the plane, the point $I$ is the midpoint of $[M_1M_2]$ where $M_1 = r_1(M)$ and $M_2 = r_2(M)$.
3) Prove that $r_2 \circ r_1$ is a rotation whose center and angle are to be determined.

**N° 15.**

Consider a triangle $OAB$ right isosceles such that $OA = OB$ and $\left(\overrightarrow{OA};\overrightarrow{OB}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$.

$I$ , $J$ and $K$ are the midpoints of the segments $[AB]$, $[OB]$ and

$[OA]$ respectively.

Let $r$ be the rotation of center $I$ and angle $\frac{\pi}{2}$ and by $t$ the translation of vector $\frac{1}{2}\overrightarrow{AB}$ , let $f = r \circ t$ and $g = t \circ r$.

1) a- Determine $f(K)$ , $f(I)$ and $f(A)$.
   b- Precise the nature of $f$ and determine its characteristic elements.
2) a- Determine $g(J)$ and $g(O)$.
   b- Precise the nature of $g$ and determine its characteristic elements.
3) Let $h = g \circ f^{-1}$.
   a- Determine $h(O)$ and find the nature of $h$.
   b- $M$ being any point in the plane , let $M_1 = f(M)$ and $M_2 = g(M)$.
   Show that the vector $\overrightarrow{M_1M_2}$ is equal to a fixed vector .

# Supplementary Problems

**N° 1.**

$OABC$, $OMNP$ and $OIJK$ are three squares whose vertices $A$ $M$ and $I$ are on the same straight line $(d)$.

1) Show that the points $C$, $P$ and $K$ are collinear on a straight line $(d')$.
2) Prove that $(d')$ is perpendicular to $(d)$.

**N° 2.**

$OO'A$ is a right isosceles triangle such that: $\left(\overrightarrow{AO};\overrightarrow{AO'}\right)=\frac{\pi}{2}(\bmod 2\pi)$.

The circles $(C)$ and $(C')$ passing through $A$ and of centers $O$ and $O'$ respectively intersect in $B$.

Let $M$ be a point of $(C)$ and $M'$ a point of $(C')$ such that:

$\left(\overrightarrow{OM};\overrightarrow{O'M'}\right)=-\frac{\pi}{2}(\bmod 2\pi)$.

1) Show that there exists a rotation $r$ to be determined transforming $M$ onto $M'$.
2) $M$ being distinct from $B$, the straight lines $(BM)$ and $(BM')$ intersect again respectively $(C')$ in $N'$ and $(C)$ in $N$.
   Prove that $N'$ is the image of $N$ by $r$.

**N° 3.**

Given a parallelogram $ABCD$, consider the rotations:

$r_1=r\left(A;\left(\overrightarrow{AD};\overrightarrow{AB}\right)\right)$ ; $r_2=r\left(B;\left(\overrightarrow{BA};\overrightarrow{BC}\right)\right)$ ; $r_3=r\left(C;\left(\overrightarrow{CB};\overrightarrow{CD}\right)\right)$

and $r_4=r\left(D;\left(\overrightarrow{DC};\overrightarrow{DA}\right)\right)$.

1) a- Determine the image of $A$ by $r_2 \circ r_1$.
   b- Determine the nature of $r_2 \circ r_1$.
2) a- Determine $r_4 \circ r_3 \circ r_2 \circ r_1$.
   b- Under what condition $r_4 \circ r_3 \circ r_2 \circ r_1$ is the identity transformation?
   Precise, in this case, the nature of quadrilateral $ABCD$.

**N°4.**

Given a triangle $ABC$ such that $(\overrightarrow{AB};\overrightarrow{AC}) = \alpha (\text{mod } 2\pi)$ where $\alpha$ belongs to $]0;\pi[$.

Construct, externally, on this triangle the squares $ACDS$ , $BAMN$ and the parallelogram $MASD$ that $(\overrightarrow{AM};\overrightarrow{AB}) = \frac{\pi}{2}(\text{mod } 2\pi)$ and $(\overrightarrow{AC};\overrightarrow{AS}) = \frac{\pi}{2}(\text{mod } 2\pi)$ and designate by $I$ the center of this parallelogram. Let $r$ be the rotation of center $A$ and angle $\frac{\pi}{2}$.

1) What are the images of points $M$ and $C$ by $r$?
2) Denote by $S'$ the image of $S$ by $r$.
   Show that $A$ is the midpoint of $[CS']$.
3) Denote by $I'$ the image of $I$ by $r$.
   Show that $I'$ is the midpoint of $[BS']$.
4) Deduce that $(AD)$ is perpendicular to $(BC)$ and that $AD = BC$.

**N°5.**

$ABCD$ is a square of center $O$ such that $(\overrightarrow{OA};\overrightarrow{OB}) = \frac{\pi}{2}(\text{mod } 2\pi)$.

The points $M$, $N$, $P$ and $Q$ are the midpoints of the segments $[AB]$, $[BC]$, $[CD]$ and $[DA]$ respectively.

1) Let $r$ be the rotation of center $O$ and angle $-\frac{\pi}{2}$.
   a- Determine the image of $N$ by $r$ then the image of segment $[AN]$ by $r$.
   b- Determine the image of $P$ by $r$ then the image of segment $[BP]$ by $r$.
   c- Determine the point $G = r(F)$ and the nature of triangle $FOG$.
   d- Deduce that $EFGH$ is a square.
2) a- Justify that $AE = EH = DH$ and that $AE = 2\,QH$.
   b- Let $K$ be the image of $H$ by the symmetry $S$ of center $Q$.
   Prove that $AEHK$ is a square and compare its area to that of triangle $AED$.

c- Deduce the ratio of the two areas of squares $ABCD$ and $EFGH$.

N° 6.

Given a direct quadrilateral $ABCD$.
Construct the points $I$, $J$, $K$ and $L$ such that triangles $AIB$, $BCJ$, $CKD$ and $DAL$ are direct equilateral.

Denote by $r_1$ the rotation of center $A$ and angle $\frac{\pi}{3}$ and $r_2$ the rotation of center $C$ and angle $-\frac{\pi}{3}$.

1) Find $r_2 \circ r_1(I)$.
2) Determine $r_2 \circ r_1$.
3) Find $r_2 \circ r_1(L)$.
4) Deduce that $IJKL$ is a parallelogram.

N° 7.

In the oriented plane, given two distinct points $O$ and $I$.
Let $r$ be the rotation of center $O$ and angle $\frac{\pi}{2}$ and $S$ the symmetry of center $I$.

1) a- $OJO'G$ is a square of center $I$.
Determine the nature and elements of $S \circ r$.
b- Deduce that $J$ is the only point of the plane verifying $r(J) = S(J)$.
2) For all points $M$ and $N$ of the plane, denote by $r(M) = A$, $S(M) = B$, $r(N) = C$ and $S(N) = D$.
For $M \neq J$ with $J$ being the midpoint of $[MN]$, show that $ABCD$ is a square of center $G$.

N° 8.

Let $ABC$ be a triangle such that $(\overrightarrow{AB};\overrightarrow{AC}) = \alpha (\mathrm{mod}\, 2\pi)$ where $\alpha$ belongs to $]0;\pi[$.

Construct, externally, on this triangle the squares $ACDE$ , $AFGB$ such that $(\overrightarrow{AC};\overrightarrow{AE})=\frac{\pi}{2}(\text{mod}\,2\pi)$ and $(\overrightarrow{AF};\overrightarrow{AB})=\frac{\pi}{2}(\text{mod}\,2\pi)$ and designate by $O$ the center of $ACDE$ and by $O'$ the center of $AFGB$.
$I$ is the midpoint of $[BC]$ and $J$ is the midpoint of $[EF]$.

1) Using the rotation of center $A$ and angle $\frac{\pi}{2}$ , prove that

   $FC = BE$ and $(\overrightarrow{FC};\overrightarrow{BE})=\frac{\pi}{2}(\text{mod}\,2\pi)$.
2) Deduce that triangle $OIO'$ is right isosceles at $I$.
3) Prove that $JO'IO$ is a square .

**N°9.**

In the oriented plane, consider the two right isosceles triangles $IAB$ and $JAC$ at $I$ and $J$ and such that $(\overrightarrow{IB};\overrightarrow{IA})=\frac{\pi}{2}(\text{mod}\,2\pi)$ and $(\overrightarrow{JA};\overrightarrow{JC})=\frac{\pi}{2}(\text{mod}\,2\pi)$ , let $K$ be the midpoint of $[BC]$.

1) Let $r$ be the rotation of center $I$ and angle $\frac{\pi}{2}$ and $r'$ the rotation of center $J$ and angle $\frac{\pi}{2}$ and let $S = r' \circ r$.
   a- Determine $S(B)$ .
   b- Precise the nature and elements of $S$ .
2) Let $L = r'(I)$.
   a- What does $K$ represent for segment $[IL]$ ?
   b- Prove that triangle $IJK$ is right isosceles .

# Solution of Problems

N°1.

1) A point $M(x;y)$ is invariant under $T$ when $M = T(M)$.
Then $x = x'$ and $y = y'$ , so $x = x + y + 1$ and $y = -x + y - 1$, which gives $x = -1$ and $y = -1$ therefore $(-1;-1)$ is the invariant point.

2) $M\begin{cases} x \\ y \end{cases} \xrightarrow{T} M'\begin{cases} x' = x + y + 1 \\ y' = -x + y - 1 \end{cases} \xrightarrow{T} M''\begin{cases} x'' = x' + y' + 1 \\ y'' = -x' + y' - 1 \end{cases}$
Therefore $x'' = x + y + 1 - x + y - 1 + 1 = 2y + 1$ and $y'' = -2x - 3$.
Since $T \circ T(M) \neq M$ then $T$ is not reciprocal (involutive).

3) $M'\begin{cases} x' \\ y' \end{cases} \xrightarrow{T^{-1}} M\begin{cases} x \\ y \end{cases}$, is used to find $x$ and $y$ in terms of $x'$ and $y'$.

The system $\begin{cases} x' = x + y + 1 \\ y' = -x + y - 1 \end{cases}$ admits as a solution $\begin{cases} x = \dfrac{x' - y' - 2}{2} \\ y = \dfrac{x' + y'}{2} \end{cases}$

Therefore $M\begin{cases} x \\ y \end{cases} \xrightarrow{T^{-1}} M'\begin{cases} x' = \dfrac{x - y - 2}{2} \\ y' = \dfrac{x + y}{2} \end{cases}$

4) Replacing $x$ and $y$ by their values in the equation $y = 2x - 1$, we get $\dfrac{x' + y'}{2} = x' - y' - 2 - 1$, which gives:
$x' - 3y' - 6 = 0$ so the image of $(d)$ by $T$ is the straight line of equation $x - 3y - 6 = 0$.

N°2.

1) $M(x;y)$ is invariant under $T$ when $M = T(M)$.

Then $x=\dfrac{x}{x^2+y^2}$ and $y=\dfrac{y}{x^2+y^2}$, which gives

$x\left(x^2+y^2-1\right)=0$ and $y\left(x^2+y^2-1\right)=0$ therefore:

- $x=0$ and $y=0$ (rejected ).
- $x=0$ and $x^2+y^2-1=0$ which gives the two points $(0;1)$ and $(0;-1)$.
- $y=0$ and $x^2+y^2-1=0$ which gives the two points $(1;0)$ and $(-1;0)$.
- $x^2+y^2-1=0$ that is the circle of center $O$ and radius 1. The set of invariant points is the circle of center $O$ and radius 1.

2) $$M\begin{cases}x\\y\end{cases}\xrightarrow{T}M'\begin{cases}x'=\dfrac{x}{x^2+y^2}\\[2ex] y'=\dfrac{y}{x^2+y^2}\end{cases}\xrightarrow{T}M''\begin{cases}x''=\dfrac{x'}{x'^2+y'^2}\\[2ex] y''=\dfrac{y'}{x'^2+y'^2}\end{cases}$$

therefore $$x''=\frac{\dfrac{x}{x^2+y^2}}{\left(\dfrac{x}{x^2+y^2}\right)^2+\left(\dfrac{y}{x^2+y^2}\right)^2}=\frac{\dfrac{x}{x^2+y^2}}{\dfrac{x^2+y^2}{\left(x^2+y^2\right)^2}}=x.$$

Similarly, we get $y''=y$ hence $T$ is an involution .

Since $T$ is involutive then $T^{-1}=T$, so:

$$M'\begin{cases}x'\\y'\end{cases}\xrightarrow{T^{-1}}M\begin{cases}x=\dfrac{x'}{x'^2+y'^2}\\[2ex] y=\dfrac{y'}{x'^2+y'^2}\end{cases}$$

3) Taking $x'^2+y'^2=k$, we get $x=\dfrac{x'}{k}$ and $y=\dfrac{y'}{k}$.

Replacing $x$ and $y$ by their values in the equation

$x^2+y^2-2x-4y=0$ we get $\dfrac{x'^2}{k^2}+\dfrac{y'^2}{k^2}-\dfrac{2x'}{k}-\dfrac{4y'}{k}=0$

which gives $\dfrac{k}{k^2}-\dfrac{2x'}{k}-\dfrac{4y'}{k}=0$ therefore $1-2x'-4y'=0$.

The image of circle $(C)$ is the straight line of equation
$2x+4y-1=0$ .

4) Replacing $x$ and $y$ by their values in the equation:

$x^2+y^2-2x-4y+1=0$ we get : $\frac{x'^2}{k^2}+\frac{y'^2}{k^2}-\frac{2x'}{k}-\frac{4y'}{k}+1=0$

which gives

$\frac{1}{k}-\frac{2x'}{k}-\frac{4y'}{k}+1=0$ then $1-2x'-4y'+x'^2+y'^2=0$.

Consequently, the image of circle $(C')$ is the circle of equation
$x^2+y^2-2x-4y+1=0$.

5) Replacing $x$ and $y$ by their values in the equation $y=x-1$
we get $\frac{y'}{k}=\frac{x'}{k}-1$, which gives $x'^2+y'^2-x'+y'=0$.
Consequently, the image of the straight line $(d)$ is the circle of equation $x^2+y^2-x+y=0$.

**N°3.**

1) $t_{\vec{V}}: M\begin{cases}x\\y\end{cases} \longrightarrow M'\begin{cases}x'\\y'\end{cases}$ such that $\overrightarrow{MM'}=\vec{V}$, then:
$x'-x=3$ and $y'-y=-2$ which gives $x=x'-3$ and $y=y'+2$.

2) The image of circle $(C)$ by this translation is a circle of the same radius and of center the point $I'$ image of $I$ by this translation.
Since $x_{I'}=x_I+3=4$ and $y_{I'}=y_I-2=-3$, an equation of $(C')$ is $(x-4)^2+(y+3)^2=4$.

3) a- Let $z=x+iy$ be the affix of $M$ and $z'=x'+iy'$ the affix of $M'$.

We have: $\frac{z_{\overrightarrow{OM'}}}{z_{\overrightarrow{OM}}}=\frac{z'}{z}=\frac{OM'}{OM}e^{i(\overrightarrow{OM};\overrightarrow{OM'})}=e^{i\frac{\pi}{2}}=i$, then $z'=iz$.

Consequently, $x'+iy'=i(x+iy)=-y+ix$, which gives:
$x'=-y$ and $y'=x$. Therefore, $x=y'$ and $y=-x'$.

b- The image of circle $(C)$ by this rotation is the circle of the same radius and of center the point $I'$ image of $I$ by this rotation.

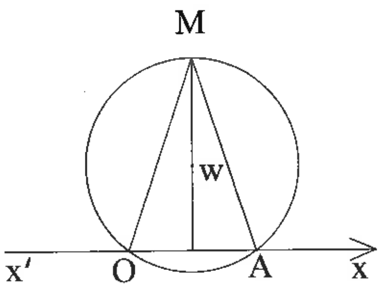
Since $I'(1;1)$ then an equation of the circle image is:
$(x-1)^2+(y-1)^2=4$.

**N°4.**

1) $AO = R$ with $A$ being fixed and $R$ a constant, then the point $O$ varies on the circle $(\gamma)$ of center $A$ and radius $R$.
2) $\overrightarrow{ON} = \frac{1}{2}\vec{V}$, then $N$ is the image of $O$ by the translation of vector $\frac{1}{2}\vec{V}$, and since $O$ varies on $(\gamma)$ then $N$ varies on the circle $(\gamma')$ image of $(\gamma)$ by this translation, the center of $(\gamma')$ is the image of $A$ by the translation and the radius of $(\gamma')$ is $R$, same as that of $(\gamma)$.
3) $\overrightarrow{OM} = -\frac{1}{2}\vec{V}$, then $M$ is the image of $O$ by the translation of vector $-\frac{1}{2}\vec{V}$, and since $O$ varies on $(\gamma)$ then $M$ varies on the circle $(\gamma'')$ image of $(\gamma)$ by this translation, the center of $(\gamma'')$ is the image of $A$ by the translation and the radius of $(\gamma'')$ is $R$, same as that of $(\gamma)$.

**N°5.**

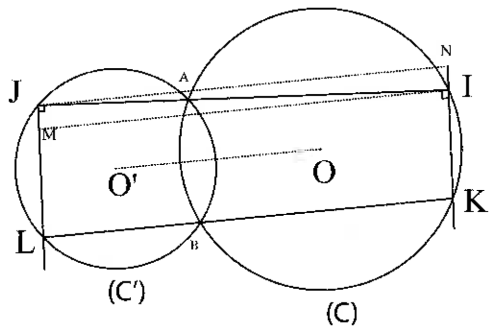
1) $Ow = R$ with $O$ being fixed and $R$ constant, then the point $w$ varies on the circle $(\gamma)$ of center $O$ and radius R.



2) $wO = wA$ and $MO = MA$ , then the straight line $(wM)$ is the perpendicular bisector of $[OA]$, so $(wM)$ is perpendicular to the axis $x'x$ and since $wM = R$ then the vector $\overrightarrow{wM}$ is equal to a fixed vector $\vec{V}$ .
$\overrightarrow{wM} = \vec{V}$ affirms that $M$ is the image of $w$ by the translation of vector $\vec{V}$ .
3) Since $w$ varies on the circle $(\gamma)$ of center $O$ and radius $R$ , then $M$ varies on the circle $(\gamma')$ image of $(\gamma)$ by this translation , the center $O'$ of $(\gamma')$ is the image of $O$ by the translation and the radius of $(\gamma')$ is $R$ .

**N°6.**

1) $A\hat{I}K = 90°$ then $[AK]$ is a diameter of $(C)$,similarly $[AL]$ is a diameter of $(C')$, $A\hat{B}K = 90°$ and $A\hat{B}L = 90°$ , angles inscribed in a semi-circle, then the three points $L$ , $B$ and $K$ are collinear .



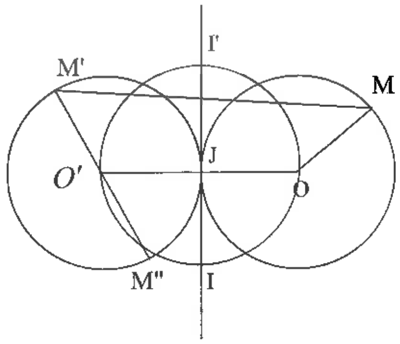

2) a- The straight line $(OO')$ joins the midpoints of the two sides in triangle $ALK$ then $\overrightarrow{KL} = 2\overrightarrow{OO'}$ .
Quadrilateral $IMLK$ is a parallelogram since $(IM)$ and $(LK)$ are parallel as well as $(ML)$ and $(IK)$, therefore:
$\overrightarrow{IM} = \overrightarrow{KL} = 2\overrightarrow{OO'}$ .
b- $\overrightarrow{IM} = 2\overrightarrow{OO'}$ then $M$ is the image of $I$ by the translation of vector $2\overrightarrow{OO'}$ and since $I$ varies on $(C)$, of center $O$ and radius $R$ then $M$ varies on the circle $(\gamma)$ image of $(C)$by this

translation . The radius of $(\gamma)$ is $R$ and its center is the point $w$ image of $O$ by the translation .

c- $\overrightarrow{JN}=2\overrightarrow{O'O}$ then $N$ is the image of $J$ by the translation of vector $2\overrightarrow{O'O}$ and since $J$ varies on $(C')$, then $N$ varies on the circle $(\gamma')$ image of $(C)$ by this translation.

$N°7.$

1) $OM = O'M'$ and $\left(\overrightarrow{OM},\overrightarrow{O'M'}\right)=\frac{\pi}{2}(\mathrm{mod}\,2\pi)$ then there exists a rotation $r\left(I;\frac{\pi}{2}\right)$ that transforms $O$ onto $O'$ and $M$ onto $M'$ .




$I$ belongs to the perpendicular bisector of $[OO']$ and to the semi-circle of diameter $[OO']$ verifying $\left(\overrightarrow{IO};\overrightarrow{IO'}\right)=\frac{\pi}{2}(\mathrm{mod}\,2\pi)$.

2) $M' = r(M)$ then $IM = IM'$ hence the perpendicular bisector of $[MM']$ passes through the fixed center of rotation $I$ .

3) a- $I'$ is the symmetric of $I$ with respect to the straight line $(OO')$ then $I'O = I'O'$ and $\left(\overrightarrow{I'O};\overrightarrow{I'O'}\right)=-\frac{\pi}{2}(\mathrm{mod}\,2\pi)$ hence $O' = r'(O)$.

b- $r:\begin{cases} O \longrightarrow O' \\ M \longrightarrow M' \end{cases}$ and $r':\begin{cases} O \longrightarrow O' \\ M \longrightarrow M'' \end{cases}$ then:

$OM = O'M'$ and $\left(\overrightarrow{OM};\overrightarrow{O'M'}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$

$OM = O'M''$ and $\left(\overrightarrow{OM};\overrightarrow{O'M''}\right)=-\frac{\pi}{2}(\text{mod}\,2\pi)$

$\left(\overrightarrow{O'M'},\overrightarrow{O'M''}\right)=\left(\overrightarrow{O'M'};\overrightarrow{OM}\right)+\left(\overrightarrow{OM};\overrightarrow{O'M''}\right)$

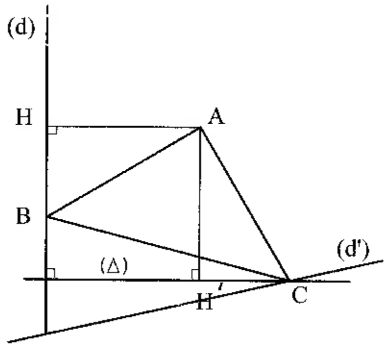
$=-\frac{\pi}{2}-\frac{\pi}{2}=-\pi(mod\,2\pi)$ .

Hence the points $M'$, $O'$ and $M''$ are collinear and since $M''$ belongs to $(C')$ then the points $M'$ and $M''$ are diametrically opposite.

**N°8.**

$AB = AC$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$

then $C$ is the image of $B$ by the rotation of center $A$ and angle $\frac{\pi}{2}$.

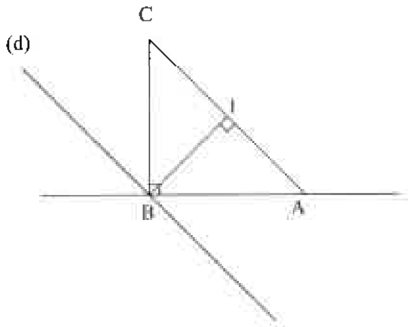



- $B \in (d)$,then its image $C$ belongs to the straight line $(\Delta)$image of $(d)$ by $r$. Since $C$ belongs to $(d')$ then $C$ is the point of intersection of the two straight lines $(d')$ and $(\Delta)$ and the point $B$ is the image of $C$ by the rotation $r^{-1}$ and the problem admits a unique solution .

- If the two straight lines are perpendicular and $A$ not belonging to the bisector of the angle formed by $(d)$ and $(d')$ then $(\Delta)$ is parallel to $(d')$ so triangle does not exist.

- If the two straight lines are perpendicular and $A$ belonging to the bisector of the angle formed by $(d)$ and $(d')$ then $(\Delta)$ is confounded with $(d')$ hence there are infinite number of solutions

**N°9.**




1) $S_{(AB)} \circ S_{(AC)}$ is a rotation of center $A$ and angle $2((AC);(AB)) = \frac{\pi}{2}$,

then $r_1 = r\left(A;\frac{\pi}{2}\right)$, similarly

$r_2 = S_{(AB)} \circ S_{(d)} = r\left(B;\frac{\pi}{2}\right)$.

$r_3 = S_{(AC)} \circ S_{(d)} = t_{2\overrightarrow{BI}}$ where $I$ is the midpoint of $[AC]$ since $(BI)$ is the common perpendicular of the two straight lines $(AC)$ et $(d)$.

2) $r_2 \circ r_1(A) = r_2(A) = C$ since $BA = BC$ and $\left(\overrightarrow{BA};\overrightarrow{BC}\right) = \frac{\pi}{2}(\text{mod}\, 2\pi)$.

3) $r_2 \circ r_1 = r\left(B;\frac{\pi}{2}\right) \circ r\left(A;\frac{\pi}{2}\right) = r(w;\pi)$, since $r_2 \circ r_1(A) = C$ then $w$ is the midpoint of $[AC]$ and $r_2 \circ r_1$ is the central symmetry of center $w$.

**N°10.**

1) a- $S_{(AO)} \circ S_{(AB)} = r(A;2((AB);(AO))) = r\left(A;\frac{\pi}{2}\right)$.

b- $R_A \circ R_B = S_{(AO)} \circ S_{(AB)} \circ S_{(AB)} \circ S_{(BO)} = S_{(AO)} \circ S_{(BO)} = r(O;\pi)$.

therefore $R_A \circ R_B = S_O$ .

2) a- $R_A \circ R_B(E) = R_A(C) = G$.

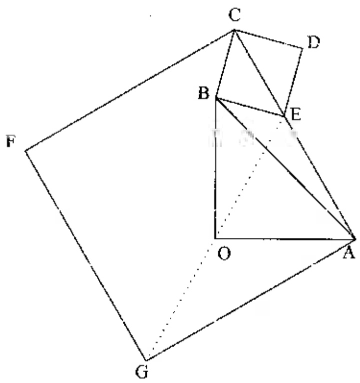
b- $R_A \circ R_B(E) = G$ and since $R_A \circ R_B = S_O$ then $S_O(E) = G$ and consequently $O$ is the midpoint of $[EG]$.

c- $R_F \circ S_O \circ R_D(C) = R_F \circ S_O(E) = R_F(G) = C$.

$R_F \circ S_O \circ R_D = R\left(F;\frac{\pi}{2}\right) \circ R(O;\pi) \circ R\left(D;\frac{\pi}{2}\right)_O = R(w;0) = I_d$.

d- $R_F \circ S_O \circ R_D(D) = R_F \circ S_O(D) = R_F(H)$

but $R_F \circ S_O \circ R_D(D) = I_d(D) = D$ , hence $R_F(H) = D$.




$R_F(H) = D$ then $FH = FD$ and $(\overrightarrow{FH};\overrightarrow{FD}) = \frac{\pi}{2}(\text{mod}\, 2\pi)$.

Triangle $FHD$ is right isosceles at $F$ and $O$ is the midpoint of $[HD]$ hence $(FO)$ is perpendicular to $(HD)$ and

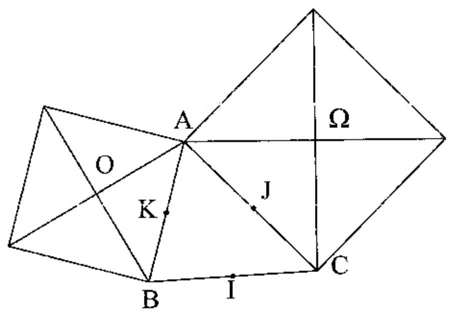
$FO = \frac{1}{2}HD = OD$.

Consequently, triangle $FOD$ is right isosceles at $O$.

## $N°11.$

1) $KO = \frac{1}{2}AB$ and $IJ = \frac{1}{2}AB$ then $KO = IJ$ and since the two straight lines $(KO)$ and $(IJ)$ are not parallel then there exists a rotation that transforms $K$ onto $J$ and $O$ onto $I$.

$(\overrightarrow{KO};\overrightarrow{JI}) = (\overrightarrow{KO};\overrightarrow{AB}) = \frac{\pi}{2}(\text{mod}\, 2\pi)$, then an angle of $r$ is $\frac{\pi}{2}$.



2) $KI = \frac{1}{2}AC = J\Omega$ and $\left(\overrightarrow{KI};\overrightarrow{J\Omega}\right)=\left(\overrightarrow{AC};\overrightarrow{J\Omega}\right)=\frac{\pi}{2}\bmod 2\pi$,
and since $r(K)=J$ then $r(I)=\Omega$.

3) $r:\begin{cases} I \longrightarrow \Omega \\ O \longrightarrow I \end{cases}$ then $\left(\overrightarrow{IO};\overrightarrow{\Omega I}\right)=\frac{\pi}{2}(\bmod 2\pi)$, consequently the two straight lines $(IO)$ and $(I\Omega)$ are perpendicular and $IO = I\Omega$.

**N° 12.**

1) a- $t(O)=A$ and $r(A)=A$ , then $r'(O)=A$.

b- $\left(\overrightarrow{IO};\overrightarrow{IA}\right)=\left(\overrightarrow{BC};\overrightarrow{BA}\right)=\frac{\pi}{3}(\bmod 2\pi)$.

c- The composite of a translation and a rotation $r$ is a rotation $r'$ of the same angle as $r$ hence $r' = r\left(I';\frac{\pi}{3}\right)$ .

Since $r'(O)=A$ then $I'O = I'A$ and $\left(\overrightarrow{I'O};\overrightarrow{I'A}\right)=\frac{\pi}{3}(\bmod 2\pi)$.

Hence triangle $I'OA$ is direct equilateral and since $IOA$ is direct equilateral then $I' = I$.

2) a- $M = t(P)$ then $\overrightarrow{PM}=\overrightarrow{OA}=\overrightarrow{EL}$, hence quadrilateral $EPML$ is a parallelogram hence the diagonals $[LP]$ and $[EM]$ bisect each other at $J$.

b- $r'(L)=r\circ t(L)=r(C)=D$ and consequently triangle $ILD$ is equilateral .

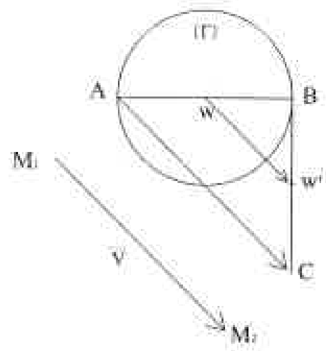
$P \xrightarrow{t} M \xrightarrow{r} N$ then $P \xrightarrow{r'} N$ and since $L \xrightarrow{r'} D$ then the midpoint $J$ of $[PL]$ has as an image the point $K$ midpoint of $[ND]$

$K = r'(J)$ gives $IJ = IK$ and $\left(\overrightarrow{IJ};\overrightarrow{IK}\right)=\frac{\pi}{3}(\bmod 2\pi)$ and consequently, triangle $IJK$ is equilateral .

**N° 13.**

1) a- $A \xrightarrow{r_A^{-1}} A \xrightarrow{r_B} C$, then :

$BA = BC$ and $\left(\overrightarrow{BA};\overrightarrow{BC}\right)=\frac{\pi}{2}(\bmod 2\pi)$.




b- $r_B \circ r_A^{-1} = r\left(B;\frac{\pi}{2}\right)\circ r\left(A;-\frac{\pi}{2}\right)=t_{\vec{V}}$ and since $r_B \circ r_A^{-1}(A)=C$

then $\vec{V}=\overrightarrow{AC}$ hence $T$ is the translation of vector $\overrightarrow{AC}$.

c- $M_1 \xrightarrow{r_A^{-1}} M \xrightarrow{r_B} M_2$ then:

$M_2 = r_B \circ r_A^{-1}(M_1)$ so

$M_2 = t_{\overrightarrow{AC}}(M_1)$ and consequently $\overrightarrow{M_1M_2}=\overrightarrow{AC}$ hence

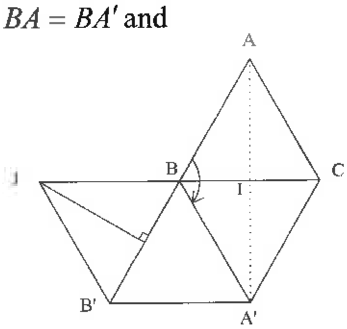
$M_1M_2CA$ is a parallelogram.

2) a- $M_2 = r_B(M)$ and since $M$ describes the circle $(\Gamma)$ of diameter $[AB]$ then the point $M_2$ describes the circle $(\Gamma')$ image of $(\Gamma)$ by the rotation $r_B$ . The center of $(\Gamma)$ is $w$ midpoint of $[AB]$ then the center of $(\Gamma')$ is the point $w' = r_B(w)$ which is the midpoint of $[BC]$ and the radius of $(\Gamma')$ is itself that of $(\Gamma)$.

b- $\overrightarrow{M_1M_2}=\overrightarrow{AC}$ and $\overrightarrow{M_2I}=\frac{1}{2}\overrightarrow{CA}$, then $I = t(M_2)$ by the translation of vector $\frac{1}{2}\overrightarrow{CA}$ and since $M_2$ describes the circle $(\Gamma')$ then the point $I$ describes the circle $(\Gamma'')$ image of $(\Gamma')$ by the translation of vector $\frac{1}{2}\overrightarrow{CA}$ .Center of $(\Gamma')$ is $w'$ then center of $(\Gamma'')$ is the image of $w'$ by the translation of vector $\frac{1}{2}\overrightarrow{CA}$, it is then the point $w$.

**N° 14.**

1) $r_1(A)=A$ and $r_2(A)=A'$ such that $BA=BA'$ and $(\overrightarrow{BA};\overrightarrow{BA'})=-\frac{2\pi}{3}(\text{mod}\,2\pi)$.




$ABC$ is equilateral then $A\hat{B}C=60°$ and since $A\hat{B}A'=120°$ then $C\hat{B}A'=60°$, triangle $ABA'$ is isosceles and since $[BI)$ is the bisector of angle $A\hat{B}A'$, then $[BI]$ is a median and consequently $I$ is the midpoint of $[AA']$.

$r_1(B)=C$ and $r_2(C)=B'$ such that $BC=BB'$ and $(\overrightarrow{BC};\overrightarrow{BB'})=-\frac{2\pi}{3}(\text{mod}\,2\pi)$.

$A\hat{B}B'=60°+120°=180°$ and $BA=BC=BB'$ then $B$ is the midpoint of $[AB']$.

2) $r_2\circ r_1^{-1}=r\left(B;-\frac{2\pi}{3}\right)\circ r\left(A;-\frac{\pi}{3}\right)=r(w;-\pi)=S_w$, central symmetry of center $w$.

But $A\xrightarrow{r_1^{-1}}A\xrightarrow{r_2}A'$ then $r_2\circ r_1^{-1}(A)=A'$ therefore $S_w(A)=A'$ and consequently $w$ is the midpoint of $[AA']$, then it is the point $I$.

Hence $r_2\circ r_1^{-1}=S_I$.

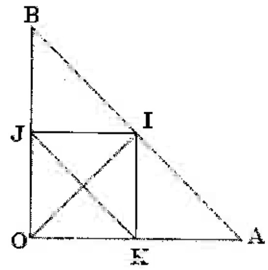
$r_1(M)=M_1$ then $r_1^{-1}(M_1)=M$ therefore :

$r_2\circ r_1^{-1}(M_1)=r_2(M)=M_2$ and since $r_2\circ r_1^{-1}=S_I$ then $S_I(M_1)=M_2$ hence $I$ is the midpoint of $[M_1M_2]$.

3) $r_2\circ r_1=r\left(B;-\frac{2\pi}{3}\right)\circ r\left(A;\frac{\pi}{3}\right)=r\left(w';-\frac{\pi}{3}\right)$

$r\left(w';-\frac{\pi}{3}\right):\begin{cases} B \longrightarrow B' \\ A \longrightarrow A' \end{cases}$, then $w'$ belongs to the perpendicular bisector of $[AA']$ and to the perpendicular bisector of $[BB']$ so it is the point $J$.

**N° 15.**



1) a- $f(K) = r \circ t(K) = r(J) = K$.
$f(I) = r \circ t(I) = r(B) = O$.
$f(A) = r \circ t(A) = r(I) = I$.

b- The composite of a rotation and a translation is a rotation then $f = r \circ t$ is a rotation of angle $\frac{\pi}{2}$ and of center the invariant point $K$, hence $f = r\left(K;\frac{\pi}{2}\right)$.

2) a- $g(J) = t \circ r(J) = t(K) = J$.
$g(O) = t \circ r(O) = t(A) = I$.

b- $g$ is a rotation of angle $\frac{\pi}{2}$ and of center the invariant point $J$, hence $g = r\left(J;\frac{\pi}{2}\right)$.

3) $h = g \circ f^{-1} = r\left(J;\frac{\pi}{2}\right) \circ r\left(K;-\frac{\pi}{2}\right)$

a- $h(O) = g \circ f^{-1}(O) = g(I) = B$.
$h = g \circ f^{-1} = r\left(J;\frac{\pi}{2}\right) \circ r\left(K;-\frac{\pi}{2}\right) = t_{\vec{V}}$ and since $h(O) = B$ then $h$ is the translation of vector $\overrightarrow{OB}$.

b- $r\left(K;\frac{\pi}{2}\right): M \longrightarrow M_1$ , $r\left(J;\frac{\pi}{2}\right): M \longrightarrow M_2$

therefore: $M_1 \xrightarrow{f^{-1}} M \xrightarrow{g} M_2$ hence $M_1 \xrightarrow{g \circ f^{-1}} M_2$ and consequently $M_2$ is the image of $M_1$ by the translation of vector $\overrightarrow{OB}$, therefore $\overrightarrow{M_1M_2} = \overrightarrow{OB}$.

# Indications

**N° 1.**

Prove that by the rotation $r\left(O;\frac{\pi}{2}\right)$ : $r(A)=C$ , $r(M)=P$ and $r(I)=K$ .

$A$ , $M$ and $I$ are collinear then their images $C$ , $P$ and $K$ are collinear.

**N° 2.**

1) $r\left(B;-\frac{\pi}{2}\right)$.

2) $\left(\overrightarrow{BN};\overrightarrow{BN'}\right)=\left(\overrightarrow{BM'};\overrightarrow{MB}\right)=\left(\overrightarrow{BM};\overrightarrow{BM'}\right)=-\frac{\pi}{2}(\bmod 2\pi)$.

The image of $(C)$ is $(C')$ and the image of $(BM')$ is $(BM)$ .
$N$ is the intersection of $(C)$ and $(BM')$ then its image is the point intersection of $(C')$ and of $(BM)$ .

**N° 3.**

1) a- $r_2(A)$ is the point $A'$ of $[BC]$ such that $BA'=BA$ .
$r_2 \circ r_1(A)=A'$ .

b- $r_2 \circ r_1$ is the central symmetry of center $w$ midpoint of $[AA']$.

2) a- $r_4 \circ r_3 \circ r_2 \circ r_1$ is a translation of vector $2\overrightarrow{ww'}$ where $w'$ is the midpoint of segment $[CC']$ with $C'=r_4(C)$.

**N° 5.**

1) a- $ON=OP$ and $\left(\overrightarrow{ON};\overrightarrow{OP}\right)=-\frac{\pi}{2}(\bmod 2\pi)$.

b- $r(P)=Q$ and $r(B)=C$ .

c- $[AN]$ and $[BP]$ intersect at $F$ , then their respective images $[BP]$ and $[CQ]$ intersect at $r(F)$ that is at $G$.

d- $r(F) = G$ and $r(H) = E$ then $OF = OG = OH = OE$ and $(OF)$ is perpendicular to $(OG)$ also $(OG)$ and $(OH)$, $(OH)$ and $(OE)$ .

Quadrilateral $EFGH$ has its diagonals the same length , and bisecting each other and perpendicular then it is a square.

**N°6.**

1) $r_1(I) = B$ since $AI = AB$ and $(\overrightarrow{AI}; \overrightarrow{AB}) = \frac{\pi}{3} (\mathrm{mod}\, 2\pi)$, $r_2(B) = J$ .

2) $r_2 \circ r_1 = t_{\overrightarrow{IJ}}$ .

3) $r_2 \circ r_1(L) = K$ .

**N°7.**

1) a- Remark that $r(J) = G$ and $S(G) = J$, $S \circ r = r\left(J; \frac{3\pi}{2}\right)$.

b- $M$ is such that $r(M) = S(M)$, we have then:

$S \circ r(M) = S \circ S(M) = M$ .

Hence, $M$ is invariant under $S \circ r$ and consequently $M = J$ .

2) $r(M) = A$ , $S(M) = B$, $r(N) = C$ and $S(N) = D$ give $MN = AC$ and $MN = BD$ hence $AC = BD$ .

$r(M) = A$ and $r(N) = C$ give $(\overrightarrow{MN}; \overrightarrow{AC}) = \frac{\pi}{2} (\mathrm{mod}\, 2\pi)$.

Hence $(MN)$ and $(AC)$ are perpendicular , $S$ transforms $(MN)$ onto $(BD)$ then $(MN)$ and $(BD)$ are parallel .

# CHAPTER 8

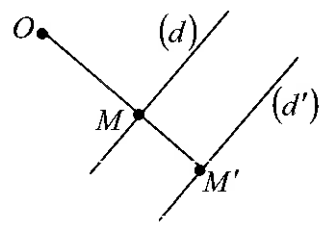
# Dilations (Homotheties)

**Chapter Review :**

- Definition .
  Given a non-zero real number $k$ and $O$ a given point of the plane. We define the dilation of center $O$ and ratio $k$, denoted by $h(O;k)$ the mapping of the plane that to every point $M$ of the plane associates the point $M'$ such that $\overrightarrow{OM'} = k\overrightarrow{OM}$.
  * $O$ is an invariant point under $h$.
  * If $k > 0$, the dilation is said to be positive.
  * If $k < 0$, the dilation is said to be negative .
  * If $k = 1$ then $h = I_d$.
  * $h^{-1}(O;k) = h\left(O;\frac{1}{k}\right)$.
- Characteristic properties:
  A mapping of the plane is a dilation of ratio $k$ if and only if for all points $M$, $N$ of respective images $M'$ and $N'$ we have : $\overrightarrow{M'N'} = k\overrightarrow{MN}$.

- Center of the dilation:
  * If $h(A) = A'$ and $h(B) = B'$ then the center of $h$ is the point of intersection of the two straight lines $(AA')$ and $(BB')$.
  * Given a dilation $h(w;k)$,if $h(A) = A'$ then $\overrightarrow{wA'} = k\overrightarrow{wA}$ which gives $\overrightarrow{wA} = \frac{1}{k-1}\overrightarrow{AA'}$ , $k \neq 1$.
- The image of a straight line $(d)$by a dilation is a straight line $(d')$.
  * If $(d)$ passes through the center of the dilation then $(d)$ is invariant as a whole $(d) = (d')$.

* If $(d)$ does not pass through the center of the dilation $O$ then $(d')$ is parallel to $(d)$.
  It is sufficient to find the image $A'$ of a point $A$ of $(d)$ and draw the straight line $(d')$ parallel to $(d)$ and passing through $A'$.




- The image of a circle $C(w;R)$ by $h(O;k)$ is a circle $C'(w';R')$ where $w'$ is the image of $w$ by $h$ and $R' = |k| \times R$.
- Dilations exchanging two circles:
  * $h\left(w;\dfrac{R'}{R}\right) : C(O;R) \longrightarrow C'(O';R')$ is the positive dilation .
    $w$ is the intersection of a common exterior tangent with the line of centers.
  * $h\left(w';-\dfrac{R'}{R}\right) : C(O;R) \longrightarrow C'(O';R')$ is the negative dilation.
    $w'$ is the intersection of a common interior tangent with the line of centers.
- Composite of two dilations: $h(O';k') \circ h(O;k)$ :
  ***First Case :*** $k \times k' = 1$ :
  * $h\left(O;\dfrac{1}{k}\right) \circ h(O;k) = I_d$.
  * $h\left(O';\dfrac{1}{k}\right) \circ h(O;k) = t_{\vec{v}}$.
    The vector $\vec{v} = \overrightarrow{OO''}$ with $O''$ being the image of $O$ by $h\left(O';\dfrac{1}{k}\right)$.

  ***Second case :*** $k \times k' \neq 1$ :
  * $h(O;k') \circ h(O;k) = h(O;k \times k')$.
  * $h(O';k') \circ h(O;k) = h(w;k \times k')$.

  Composite of a dilation and a translation:
  $h(O;k) \circ t_{\vec{v}} = h(w;k)$ .

# Solved Problems

**N°1**

In an oriented plane, consider a right triangle $ABC$.

1) Let $I$ be a point of $[AC]$ such that $\overrightarrow{AI} = 3\overrightarrow{CI}$.
   Determine the ratio of the dilation of center $A$ that transforms $I$ onto $C$.
2) Let $h$ be the dilation of ratio $\frac{3}{4}$ that transforms $A$ onto $B$.
   Determine the center of $h$.

**N°2**

Given a parallelogram $ABCD$ ; Designate by $h_A$ the dilation of center $A$ and ratio $\frac{1}{4}$ ; $h_B$ the dilation of center $B$ and ratio $\frac{1}{4}$ and by $T$ the translation of vector $\overrightarrow{CB}+\frac{1}{4}\overrightarrow{BA}$.

Let $f = t \circ h_B \circ h_A$.

1) Calculate the image of $A$ by $f$.
2) Show that $f$ is a dilation whose center and ratio are to be determined.

**N°3**

In an oriented plane, given a segment $[CD]$.

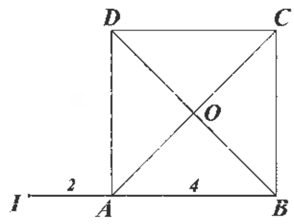
Consider the two dilations $h_1 = h(C;-2)$ and $h_2 = h\left(D;\frac{1}{3}\right)$.

1) Determine the nature of $h_2 \circ h_1$.
2) Determine $h_2 \circ h_1(C)$ and construct the center of $h_2 \circ h_1$.

**N°4**

In the figure to the right, given a square $ABCD$ of side $4$, of center $O$ and such that

$$\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{2}\ (\text{mod } 2\pi).$$




Let $r$ be the rotation of center $O$ and angle $\frac{\pi}{2}$, $h$ the dilation of center $I$ that transforms $A$ onto $B$ and $t$ the translation of vector $\overrightarrow{AD}$.

1) Determine $t\circ r(C)$ then identify $t\circ r$.
2) Determine a rotation $r'$ such that $r'\circ t=t\circ r$.
3) Find the nature and characteristic elements of $t\circ h$ and construct its center $w$.

**N°5**

The plane is referred to a direct orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

Let $f$ be the mapping of the plane defined by :

$$M\begin{cases}x\\ y\end{cases}\xrightarrow{f} M'\begin{cases}x'=3x+4\\ y'=3y-2\end{cases}$$

1) Determine the invariant point under $f$ then find the nature of $f$.
2) Let $h$ be the dilation of center $J(-1;2)$ and ratio $k=2$.
   a- Define $h$ analytically.
   b- Determine the image $(P')$ of the curve $(P)$ of equation $y=x^2$ by $h$
   c- Calculate the area of the domain limited by $(P)$, the axis $x'x$ and the two straight lines of equations $x=0$ and $x=1$.
   d- Deduce the area of the domain limited by $(P')$, the straight line $(d)$ of equation $y=-2$ and the two straight lines of equations $x=1$ and $x=3$.

**N°6**

$ABC$ is a right triangle such that $AB=4$, $AC=3$ and

$\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}\ (\text{mod } 2\pi)$.

$D$ is the midpoint of $[AB]$ and $E$ is the midpoint of $[BC]$.
Let $h$ be the dilation of center $A$ that transforms $D$ onto $B$.

$t$ is the translation of vector $\overrightarrow{DB}$.

1) a- Identify $t \circ h$ and locate its center $I$ in the figure.
   b- Identify $h \circ t$ and locate its center $J$ in the figure.

2) The plane is referred to a direct orthonormal system $\left(A;\vec{i},\vec{j}\right)$

   such that $\vec{i}=\frac{1}{2}\overrightarrow{AD}$.

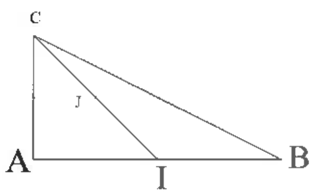
   Define, analytically, $h$, $t$ and $t \circ h$.

**N°7.**

$ABC$ is a triangle right at $A$ such that $AB = 4$, $AC = 2$ and

$\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}\ (\text{mod } 2\pi)$.

Let $I$ be the midpoint of $[AB]$
and $J$ that of $[CI]$.




1) a- Prove that $\overrightarrow{JA}+\overrightarrow{JB}+2\overrightarrow{JC}=\vec{0}$.
   b- Let $f$ be the mapping of the plane that to all points $M$ of the plane associates the point $M'$ defined by

   $\overrightarrow{M'M}=\overrightarrow{AM}+\overrightarrow{BM}+2\overrightarrow{CM}$.
   Show that $f$ is a dilation of center $J$.
   c- Let $(\gamma)$ be the circle circumscribed about triangle $ABC$.
   Determine the image $(\gamma')$ of $(\gamma)$ by $f$.

2) The plane is referred to a direct orthonormal system $\left(A;\vec{i},\vec{j}\right)$

   such that $\vec{i}=\frac{1}{4}\overrightarrow{AB}$.

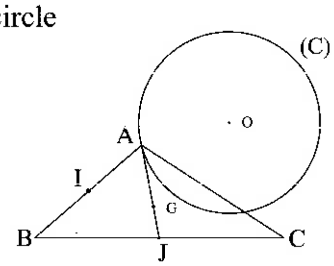
   a- Find the analytic expression of $f$.

b- Write an equation of $(\gamma')$ .

**N° 8.**

In the figure to the right. $B$ and $C$ are fixed and $A$ is a point that varies on a fixed circle $(C)$, of center $O$ and radius $R$ .

Let $I$ be the midpoint of $[AB]$ and $G$ the centroid of triangle $ABC$ .




1) Determine the set of points $I$ as $A$ varies .
2) Determine the set of points $G$ as $A$ varies .

**N° 9.**

In an oriented plane, consider the points $A$ and $B$ such that $AB = 16$ and the point $E$ such that $\overrightarrow{AE} = \frac{3}{4}\overrightarrow{AB}$ .

Let $C$ be a point distinct of $A$ such that $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \frac{\pi}{4}$ (mod $2\pi$).

The straight line parallel to $(BC)$ through $E$ cuts the straight line $(AC)$ at $F$ .

Let $I$ be the midpoint of $[BC]$ and $J$ the midpoint of $[EF]$ and $D$ the point of intersection of the straight lines $(EC)$ and $(BF)$ .

Designate by $h_A$ the dilation of center $A$ that transforms $B$ onto $E$ and by $h_D$ the dilation of center $D$ that transforms $E$ onto $C$ .

1) Determine $h_A(C)$ and $h_D(F)$ .
2) Deduce the nature and characteristic elements of $h_D \circ h_A$ Then of $h_A \circ h_D$ .
3) Let $E'$ be the image of $E$ by $h_A$ and $E''$ the image of $E'$ by $h_D$ . Represent $E'$ then construct $E''$ .
4) Determine the nature and characteristic elements of $h_D \circ h_A \circ h_A \circ h_D$ .
5) Determine the nature of the quadrilateral $BECE''$ .

**N° 10.**

In a plane $(P)$, consider a triangle $ABC$ .

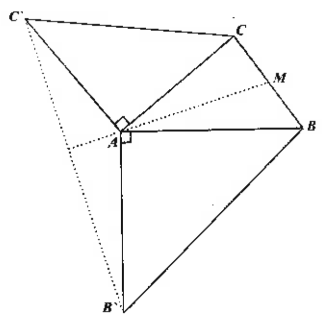
$I$ , $J$ and $K$ are the respective midpoints of $[BC]$ , $[AC]$ and $[AB]$ and let $G$ be the center of gravity of triangle $ABC$ .

For all points $M$ of the plane, designate by $P$ , $Q$ and $R$ the symmetrics of $M$ with respect $I$ , $J$ and $K$ respectively.

1) Prove that there exists a dilation $h_1$ that transforms $A$ , $B$ and $C$ onto $I$ , $J$ and $K$ respectively and determine this dilation .
2) Determine the dilation $h_2$ that transforms $I$ , $J$ and $K$ onto $P$ , $Q$ and $R$ respectively .
3) a- Precise the nature of $f = h_2 \circ h_1$ .
   b- Prove that the segments $[AP]$ , $[BQ]$ and $[CR]$ have the same midpoint $O$ .
   c- Prove that the points $O$ , $G$ and $M$ are collinear .

**N° 11.**

$ABC$ is a given triangle, $M$ is the midpoint of $[BC]$ , the triangles $BAB'$ and $CAC'$ are right isosceles at $A$ .




1) $h$ is the dilation of center $B$ and ratio $2$ .
   a- Determine the images of the points $A$ and $M$ by $h$ .
   b- Find a rotation $r$ knowing that $r \circ h$ transforms $A$ onto $B'$ and $M$ onto $C'$ .
2) Deduce that the straight lines $(AM)$ and $(B'C')$ are perpendicular and that $B'C' = 2AM$ .

N° 12.

Consider a circle $(\gamma)$ of center $O$ and diameter $[AB]$.
Let $C$ be a fixed point on $]O,A[$ and $(\Delta)$ a variable straight line through $C$. $(\Delta)$ cuts $(\gamma)$ at $M$ and $N$.
Let $I$ be the midpoint of $[MN]$.

1) Determine the set of points $I$ as $(\Delta)$ varies.
2) Let $h$ be the dilation of center $B$ and ratio $2$ and $H$ the image of $I$ by $h$.
   a- Show that $BMHN$ is a parallelogram.
   b- Prove that $H$ is the orthocenter of triangle $AMN$.
   c- Determine the locus of points $H$ as $(\Delta)$ varies.

N° 13.

$[AB]$ is a fixed segment such that $AB = 12$ and let $I$ be a point of $[AB]$ such that $AI = 4$.
Consider the circles $(C)$ and $(C')$ of respective diameters $[AI]$ and $[IB]$.
Let $M$ be a point of $(C)$ and $N$ a point of $(C')$ such that $(MN)$ is a common exterior tangent to $(C)$ and $(C')$.

1) The two straight lines $(AM)$ and $(BN)$ intersect in $K$.
   a- Define the negative dilation $h_1$ that transforms $(C)$ onto $(C')$.
   b- Determine $h_1(A)$ and $h_1(M)$.
   c- Prove that $\widehat{AKB} = 90°$ and deduce the set of points $K$.
2) Use the positive dilation $h'$ that transforms $(C)$ onto $(C')$ and prove that $\widehat{AKB} = 90°$.
3) Prove that the two straight lines $(IK)$ and $(AB)$ are perpendicular.

N° 14.

$ABCD$ is a rectangle such that $\left(\overrightarrow{AB};\overrightarrow{AD}\right) = \frac{\pi}{2} \pmod{2\pi}$.

On the exterior of this rectangle we construct the squares $AEFB$ and $ADGH$ and designate by $I$ the point of intersection of the straight

lines $(EG)$ and $(FH)$.
Let $h$ be the dilation of center $I$ that transforms $G$ onto $E$ and $h'$ the dilation of center $I$ that transforms $F$ onto $H$.

1) Determine the image of the straight line $(CG)$ by $h$, then the image of $(CG)$ by $h' \circ h$.
2) Determine the image of the straight line $(CF)$ by $h \circ h'$.
3) Justify that $h \circ h' = h' \circ h$ and deduce that the straight line $(AC)$ passes through $I$.

**N° 15.**

Given a quadrilateral $ABCD$.
The straight lines $(AB)$ and $(DC)$ intersect in $F$.
The straight lines $(AD)$ and $(BC)$ intersect in $E$.
Let $I$, $J$ and $K$ be the midpoints of the segments $[BD]$, $[AC]$ and $[EF]$ respectively.
$I'$ and $J'$ are the points such that $AFCI'$ and $BFDJ'$ are parallelograms.

1) Draw a clear figure.
2) Designate by $h_1$ the dilation of center $E$ that transforms $B$ onto $C$ and $h_2$ the dilation of center $E$ that transforms $D$ onto A.
   a- Determine the image of the straight line $(BJ')$ by $h_2 \circ h_1$.
   b- Determine the image of the straight line $(DJ')$ by $h_1 \circ h_2$.
   c- Deduce that the points $E$, $I'$ and $J'$ are collinear.
3) Prove that the points $I$, $J$ and $K$ are collinear.

**N° 16.**

$(C)$ and $(C')$ are two circles of different radii, of respective centers $O$ and $O'$ tangent externally at a point $A$.
A straight line $(d)$ passing through $A$ cuts $(C)$ again in $M$ and $(C')$ in $M'$.
A straight line $(d')$ distinct of $(d)$ passing through $A$ intersects $(C)$ in $N$ and $(C')$ in $N'$.

1) Designate by $h$ the dilation of center $A$ such that $h(O) = O'$.
   a- Show that $h(M) = M'$ and $h(N) = N'$.

b- Deduce that the straight lines $(MN)$ and $(M'N')$ are parallel.

2) Suppose that $[MN]$ is a diameter of $(C)$ .

a- Show that $[M'N']$ is a diameter of $(C')$ .

b- Show that $(MN')$ and $(M'N)$ intersect at a fixed point .

# Supplementary Problems

**N° 1.**

$(C)$ and $(C')$ are two circles of respective centers $O$ and $O'$ tangent externally at point $A$.
Let $M$ be a variable point of $(C)$ and $N$ a point of $(C')$ such that $(AN)$ and $(AM)$ are perpendicular.
Designate by $P$ the orthogonal projection of $A$ on $(MN)$ and $K$ the midpoint of $[MN]$.

1) a- Prove that $(OM)$ and $(O'N)$ are parallel .
   b- Deduce that the straight line $(MN)$ passes through a fixed point $I$ .
2) a- Determine the locus of $K$ as $M$ varies on $(C)$.
   b- Determine the locus of $P$.

**N° 2.**

$(C)$ is a circle of center $O$ and of diameter $AB = 12$.
Let $I$ be a point of $[OA]$ such that $OI = 2$ and let $[MM']$ be a variable diameter of $(C)$.
$(MI)$ and $(AM')$ intersect at a point $P$.
From $P$, we draw a line parallel to $(MM')$ that cuts $(AB)$ in $w$.

1) a- Prove that $\dfrac{\overline{Iw}}{\overline{IO}} = -\dfrac{\overline{Aw}}{\overline{AO}}$.
   b- Deduce that $w$ is fixed and calculate $Aw$.
2) a- Prove that there exists a dilation $h_1$ that transforms $M$ onto $P$ to be determined .
   b- Prove that there exists a dilation $h_2$ that transforms $M'$ onto $P$ to be determined .
   c- What is the locus of $P$ as $[MM']$ varies ?

**N° 3.**

$(C)$ is a fixed circle of center $O$ and radius $R = 2$.
Let $A$ be a fixed point such that $OA = 6$ and $M$ a variable

point of $(C)$.

The internal bisector of angle $M\hat{O}A$ cuts $(AM)$ in $I$ and the parallel through $I$ to $(OM)$ cuts $(OA)$ in $J$.

1) Show that triangle $OIJ$ is isosceles.
2) a- Show that $\frac{IM}{IA}=\frac{1}{3}$.

   b- Deduce the locus of $I$ as $M$ varies on $(C)$.

**N°4.**

$(C)$ and $(C')$ are two concentric circles of center $O$ such that $(C')$ is interior to $(C)$.

Consider the fixed point $A$ of $(C')$ and a variable point $M$ of $(C')$ distinct of $A$.

The perpendicular at $A$ to $(AM)$ cuts the circle $(C)$ in $P$ and $Q$ and the circle $(C')$ in another point $B$.

1) $I$ being the midpoint of $[AB]$, prove that $I$ is also the midpoint of $[PQ]$.

   Deduce that triangles $AMB$ and $MPQ$ have the same center of gravity.
2) Deduce the locus of points $I$ as $M$ varies on $(C')$.
3) Determine the locus of points $G$ center of gravity of triangle $AMB$ as $M$ varies on $(C')$.

**N°5.**

$M$ is a fixed point of segment $[AB]$ distinct of $A$ and $B$.

On the same side of $(AB)$, construct two equilateral triangles $APM$ and $MQB$, $(AP)$ and $(BQ)$ intersect at $C$.

1) Prove that $PMQC$ is a parallelogram.
2) Find the locus of points $O$ midpoint of $[PQ]$ as $M$ describes the segment $[AB]$.
3) Show that the perpendicular bisector of $[PQ]$ passes through a fixed point.
4) $I$ being the center of the circle circumscribed about triangle $CPQ$.

   a- Show that $I$ is the image of $O$ by a dilation to be determined.

b- Deduce the locus of points $I$ as $M$ varies on $[AB]$.

**N°6.**

$ABC$ is an isosceles triangle of vertex $A$, the points $D$ , $E$ , $F$ and $L$ are the respective midpoints of segments $[BC]$ , $[AC]$, $[AB]$ and $[DE]$, $G$ and $Q$ are the symmetrics of $D$ with respect to $C$ and to $B$.

The point $H$ is defined by $\overrightarrow{AH} = \frac{3}{2}\overrightarrow{AC}$.

The straight line $(EG)$ cuts $(AB)$ in $K$ and the straight line $(AD)$ cuts $(BE)$ in $M$.

Let $I$ be the point defined by $\overrightarrow{KI} = \frac{1}{2}\overrightarrow{AB}$.

1) Prove that the points $I$, $D$ and $H$ are collinear.( Use a translation).
2) Prove that the points $M$, $L$, $F$ and $C$ are collinear.( Use a dilation).
3) The parallel through $K$ to $(BC)$ cuts $(AC)$ in $P$.
   Prove that the points $P$, $F$ and $Q$ are collinear .( Use a dilation).
4) Let $J$ be the point of intersection of $(KG)$ and $(FP)$.
   Prove that the points $J$,$M$ and $D$ are collinear.
   ( You may use reflection).

**N°7.**

$(C)$ is a circle of center $O$,radius $R$ and diameter $[AB]$.

1) For all variable points $M$ of $(C)$ distinct of $A$ and $B$ we construct the point $Q$ such that $MABQ$ is a parallelogram..
   a- Determine the locus of the point $I$ midpoint of $[MQ]$.
   b- Determine the locus of the point $G$ center of gravity of triangle $BMQ$.
2) Let $N$ be the symmetric of $A$ with respect to $M$ and $P$ the point of intersection of the straight lines $(ON)$ and $(BM)$.
   Determine the locus of point $P$ as $M$ varies on $(C)$.

**N°8.**

$ABCD$ is a trapezoid such that $AB = 3\ell$ and $DC = \ell$ .

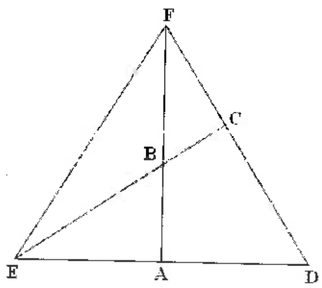
1) Determine the center $O$ and the ratio of the positive dilation $h$ that transforms $D$ onto $A$ and $C$ onto $B$.
2) Let $I$ be the midpoint of $[AB]$ and $J$ that of $[CD]$.
   Show that the points $O$ , $I$ and $J$ are collinear.

3) Let $O'$ be the point of intersection of the diagonals $[DB]$ and $[AC]$.
Show that the points $O$ , $I$ , $J$ and $O'$ are collinear .
4) Calculate the ratio of areas of the triangles $OAB$ and $ODC$ .

N° 9.

Four straight lines intersect two by two in six points $A$ , $B$ , $C$ , $D$, $E$ , $F$ . Let $\alpha$ , $\beta$ and $\gamma$ be the respective midpoints of the segments $[BD]$ , $[AC]$ and $[EF]$.

1) Consider the dilation of center $B$ and ratio 2 , construct the images $\beta'$ and $\gamma'$ of $\beta$ and $\gamma$ .




2) $(C\beta')$ intersect $(AD)$ in $G$ , $(A\beta')$ cuts $(CD)$ in $H$ .
   a- Show that the dilation $h_1$ that transforms $H$ onto $C$ , transforms $(AH)$ onto $(EC)$.
   b- Show that the dilation $h_2$ that transforms $G$ onto $A$ , transforms $(GC)$ onto $(AF)$.
3) Prove that $h_1 \circ h_2 = h_2 \circ h_1$ .
4) a- Show that $h_1 \circ h_2$ transforms $(HG)$ onto $(FE)$.
   b- Deduce that $h_1 \circ h_2$ transforms $\beta'$ onto $\gamma'$ .
5) Deduce that the points $\alpha$ , $\beta$ and $\gamma$ are collinear.

# Solution of Problems

**N° 1.**

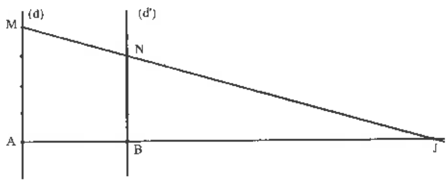
1) If $h(A;k): I \longrightarrow C$ then $\overrightarrow{AI} = 3\overrightarrow{CI}$ ,so $\overrightarrow{AI} = 3(\overrightarrow{CA} + \overrightarrow{AI})$ which gives $\overrightarrow{AC} = \frac{2}{3}\overrightarrow{AI}$ .Therefore $k = \frac{2}{3}$ .

2) If $J$ is the center of the dilation, then it should satisfy the relation:

$\overrightarrow{JB} = \frac{3}{4}\overrightarrow{JA}$, $4\overrightarrow{JB} = 3\overrightarrow{JA}$ , $4\overrightarrow{JB} = 3\overrightarrow{JB} + 3\overrightarrow{BA}$ which gives :

$\overrightarrow{JB} = 3\overrightarrow{BA}$ which can be written as $\overrightarrow{BJ} = 3\overrightarrow{AB}$ .

**Geometrical Method**: To construct the center $J$:
The ratio of $h$ is positive , then the point $J$ is exterior to segment $[AB]$.
We trace through the points $A$ and $B$ two parallel straight lines $(d)$ and $(d')$.




$4\overrightarrow{JB} = 3\overrightarrow{JA}$, for this we take on $(d)$ a point $M$ such that $AM = 4$ and on $(d')$ a point $N$ to the same side as $M$ with respect to $(AB)$ such that $BN = 3$.
$J$ is the intersection of the straight lines $(AB)$ and $(MN)$.

**N° 2.**

1) $f(A) = t \circ h_B \circ h_A(A)$, but $h_A(A) = A$ then $f(A) = t \circ h_B(A)$.

If $A' = h_B(A)$ then $\overrightarrow{BA'} = \frac{1}{4}\overrightarrow{BA}$, consequently $f(A) = t(A')$.

If $A'' = t(A')$ then $\overrightarrow{A'A''} = \overrightarrow{CB} + \frac{1}{4}\overrightarrow{BA} = \overrightarrow{CB} + \overrightarrow{BA'} = \overrightarrow{CA'}$,

so $\overrightarrow{A'A''} = \overrightarrow{CA'}$ which implies that $A''$ is the symmetric of $C$ with respect to $A'$ .

2) $h_B \circ h_A$ is a dilation of ratio $\frac{1}{4}\times\frac{1}{4}=\frac{1}{16}$ since the composite of two dilations is a dilation .

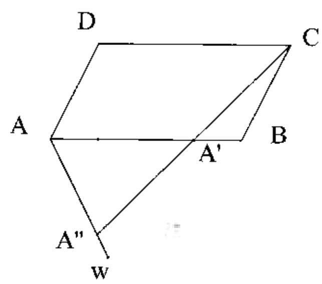
$f = t\circ(h_B \circ h_A)$ is a dilation of ratio $\frac{1}{16}$ , since the composite of a dilation and a translation is a dilation of the same ratio.
If $w$ is the center of $f$ , then it should satisfy the relation :

$\overrightarrow{wA''}=\frac{1}{16}\overrightarrow{wA}$ which is equivalent to $16\overrightarrow{wA''}=\overrightarrow{wA}$ .

Then $16\overrightarrow{wA''}=\overrightarrow{wA''}+\overrightarrow{A''A}$ which implies $\overrightarrow{A''w}=\frac{1}{15}\overrightarrow{AA''}$

**$N°3.$**

1) $h_2 \circ h_1$ being the composite of two dilations is a dilation of ratio $k=-2\times\frac{1}{3}=\frac{-2}{3}$ .




2) $h_2 \circ h_1(C)=h_2(C)=C'$

such that $\overrightarrow{DC'}=\frac{1}{3}\overrightarrow{DC}$,

which gives $3\overrightarrow{DC'}=\overrightarrow{DC'}+\overrightarrow{C'C}$

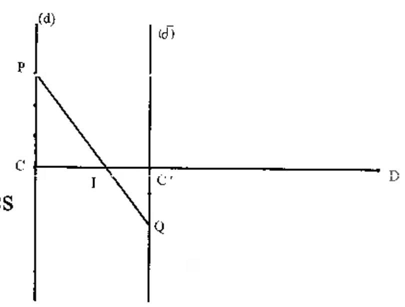
then $2\overrightarrow{DC'}=\overrightarrow{C'C}$

and $2\overrightarrow{DC'}=\overrightarrow{C'D}+\overrightarrow{DC}$, therefore $\overrightarrow{DC'}=\frac{1}{3}\overrightarrow{DC}$.

If $I$ is the center of $h_2\circ h_1$ , it should satisfy the relation :

$\overrightarrow{IC'}=\frac{-2}{3}\overrightarrow{IC}$, then $3\overrightarrow{IC'}+2\overrightarrow{IC}=\vec{0}$.

The ratio of $h$ is negative ,
then the point $J$ is interior to segment $[CC']$.
We trace through the points $C$ and $C'$ two parallel straight lines $(d)$ and $(\delta)$ respectively .



We place on $(d)$ a point $P$ such that $CP = 3$ and on $(\delta)$ a point $Q$ on the other side of $P$ with respect to $(CC')$ such that $C'Q = 2$.

The point $I$ of intersection of the straight lines $(CC')$ and $(PQ)$.

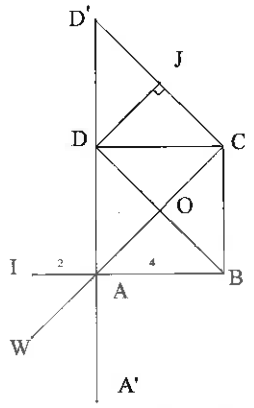
**N°4.**

1) We have $OC = OD$ and $(\overrightarrow{OC};\overrightarrow{OD}) = \frac{\pi}{2}(\text{mod } 2\pi)$, then $r(C) = D$.

$t \circ r(C) = t(D) = D'$ such that $\overrightarrow{DD'} = \overrightarrow{AD}$ then $D'$ is the symmetric of $A$ with respect to $D$.

$t \circ r$ is a rotation of the same angle as r.

Let $w'$ is the center of $t \circ r$, since $tor(C) = D'$, then $w'C = w'D'$ and $\left(\overrightarrow{w'C}, \overrightarrow{w'D'}\right) = \frac{\pi}{2}(\text{mod } 2\pi)$.




Hence $w'$ belongs to the perpendicular bisector of $[CD']$ and to the semi circle of diameter $[CD']$ verifying:

$\left(\overrightarrow{w'C}, \overrightarrow{w'D'}\right) = \frac{\pi}{2}(\text{mod } 2\pi)$ then it is the point $D$.

2) $r' \circ t = t \circ r$ implies that $(r' \circ t) \circ t^{-1} = (t \circ r) \circ t^{-1}$, so $r' \circ (t \circ t^{-1}) = (t \circ r) \circ t^{-1}$ implies that $r' = (t \circ r) \circ t^{-1}$ since $t \circ t^{-1} = I_d$.

But $t^{-1}$ is the translation of vector $\overrightarrow{DA}$ and $t \circ r$ is a rotation then $r'$ is a rotation of angle $\frac{\pi}{2}$ and of center $J$ to be determined.

Since $t^{-1}$ is the translation of vector $\overrightarrow{DA}$ it is available to find the image of $D$ by $(t \circ r) \circ t^{-1}$.

We have $t^{-1}(D) = A$ and $t \circ r(A) = C$, then $r'(D) = C$.

$r'(D)=C$ implies that $J$ belongs to the perpendicular bisector of $[CD]$ and to the semi circle of diameter $[CD]$ verifying $\left(\overrightarrow{JD};\overrightarrow{JC}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$, then it is the midpoint of $[CD']$.

3) $h$ transforms $A$ onto $B$ then it's ratio is $k=\frac{IB}{IA}=\frac{6}{2}=3$.

$t\circ h$ is a dilation of ratio $k=3$.
We have $toh(A)=t(B)=C$.

If $w$ is the center of $t\circ h$, we get : $\overrightarrow{wC}=3\overrightarrow{wA}$, $\overrightarrow{wA}+\overrightarrow{AC}=3\overrightarrow{wA}$

Then $\overrightarrow{wA}=\frac{1}{2}\overrightarrow{AC}$, consequently $w$ is the symmetric of $O$ with respect to $A$.

**N° 5.**

1) The invariant point is defined by $x'=x$ and $y'=y$.

$\begin{cases}x=3x+4\\ y=3y-2\end{cases}$ gives : $x=-2$ and $y=1$.

The center is $I(-2;1)$.

$$\overrightarrow{IM'}=(x'+2)\vec{i}+(y'-1)\vec{j}=(3x+6)\vec{i}+(3y-3)\vec{j}$$
$$=3\left[(x+2)\vec{i}+(y-1)\vec{j}\right]=3\overrightarrow{IM}.$$

Hence $f$ is a dilation of center $I(-2;1)$ and ratio $3$.

2) a- If $M'(x';y')$ is the image of $M(x;y)$ under $h$ we get $\overrightarrow{JM'}=2\overrightarrow{JM}$ which gives $\begin{cases}x'+1=2(x+1)\\ y'-2=2(y-2)\end{cases}$ then

$\begin{cases}x'=2x+1\\ y'=2y-2\end{cases}$

b- $h^{-1}$ is defined by $\begin{cases}x=\dfrac{x'-1}{2}\\ y=\dfrac{y'+2}{2}\end{cases}$

Replacing $x$ and $y$ by their values in the equation of $(P)$ we get : $\frac{y'+2}{2}=\left(\frac{x'-1}{2}\right)^2$ which gives $y'=\frac{1}{2}x'^2-x'-\frac{3}{2}$

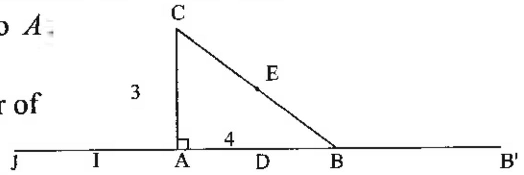
consequently an equation of $(P')$ is : $y=\frac{1}{2}x^2-x-\frac{3}{2}$ .

c- $S=\int_0^1 x^2dx=\left.\frac{x^3}{3}\right]_0^1=\frac{1}{3}$ square units .

2) The domain limited by $(P')$ , the straight line $(d)$ of equation $y=-2$ and the two straight lines of equations $x=1$ and $x=3$ is the image of the previous domain since $x'x$ of equation $y=0$ has as an image $(d)$ and the straight line of equation $x=0$ has as an image the straight line of equation $x=1$ .
The straight line of equation $x=1$ has as an image the straight line of equation has as an image the straight line of equation $x=3$ .
Therefore: $S'=k^2S=4\left(\frac{1}{3}\right)=\frac{4}{3}$ square units since the dilation of ratio $k$ multiplies the area by $k^2$ .

N° 6.

1) a- $h$ is the dilation of ratio $k=\frac{AB}{AD}=\frac{4}{2}=2$ .
$toh$ is a dilation, of ratio $2$ and of center $I$ to be determined.
We have $h(A)=A$.
If $A'=t(A)$ the :
$\overrightarrow{AA'}=\overrightarrow{DB}$ which gives $A'=D$.
$t\circ h(A)=D$ gives $h'(A)=D$, so $\overrightarrow{ID}=2\overrightarrow{IA}$, which gives that $\overrightarrow{IA}+\overrightarrow{AD}=2\overrightarrow{IA}$, therefore $\overrightarrow{AD}=\overrightarrow{IA}$, then $I$ is the symmetric of $D$ with respect to $A$.




b- $h\circ t=h(J;2)$, to determine the center of this dilation it is available to find the image of $D$ by $h\circ t$.

We have $t(D)=B$ , if $B'=h(B)$ then $\overrightarrow{AB'}=2\overrightarrow{AB}$.
hence $h'(D)=B'$ which gives $\overrightarrow{JB'}=2\overrightarrow{JD}$ and $\overrightarrow{JD}+\overrightarrow{DB'}=2\overrightarrow{JD}$
therefore $\overrightarrow{JD}=\overrightarrow{DB'}$, consequently $J$ is the symmetric of $B'$ with respect to $D$.

2) • $h$ : $M(x,y)\longrightarrow M'(x',y')$ such that :

$\overrightarrow{AM'}=2\overrightarrow{AM}$ then $\begin{cases}x'=2x\\ y'=2y\end{cases}$

• We have $\overrightarrow{DB}(2;0)$, so $t$ : $M(x,y)\longrightarrow M'(x',y')$ such that :

$\overrightarrow{MM'}=\overrightarrow{DB}$ which gives $\begin{cases}x'=2+x\\ y'=y\end{cases}$

• $t\circ h=h(I;2)$, so $\overrightarrow{IM'}=2\overrightarrow{IM}$ and since $I(-2;0)$ we get :

$\begin{cases}x'+2=2(x+2)\\ y'=2y\end{cases}$ therefore $\begin{cases}x'=2x+2\\ y'=2y\end{cases}$

$\boxed{N°7.}$

1) a- $\overrightarrow{JA}+\overrightarrow{JB}+2\overrightarrow{JC}=2\overrightarrow{JI}+2\overrightarrow{JC}=2\left(\overrightarrow{JI}+\overrightarrow{JC}\right)=\vec{0}$ since $J$ is the midpoint of $[CI]$.

b- $\overrightarrow{M'M}=\overrightarrow{AM}+\overrightarrow{BM}+2\overrightarrow{CM}$ implies that :
$\overrightarrow{JM}-\overrightarrow{JM'}=\overrightarrow{JM}-\overrightarrow{JA}+\overrightarrow{JM}-\overrightarrow{JB}+2\left(\overrightarrow{JM}-\overrightarrow{JC}\right)$.

We get : $\overrightarrow{JM'}=-3\overrightarrow{JM}$ since $\overrightarrow{JA}+\overrightarrow{JB}+2\overrightarrow{JC}=\vec{0}$ .
Then, $f$ is a dilation of center $J$ and ratio $-3$ .

c- The midpoint $w$ of $[BC]$ is the center of $(\gamma)$, the image of $(\gamma)$ by $f$ is the circle $(\gamma')$ of center $w'=h(w)$ such that $\overrightarrow{Jw'}=-3\overrightarrow{Jw}$ and of radius $R'=3R$, but $BC^2=16+4=20$ then $R=\sqrt{5}$ and $R'=3\sqrt{5}$.

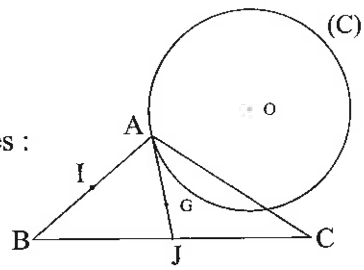
2) a- $f: M(x,y)\longrightarrow M'(x',y')$ gives $\overrightarrow{JM'}=-3\overrightarrow{JM}$.
We have $I(2,0)$ and $C(0,2)$ then $J(1,1)$ which gives

$x'-1=-3(x-1)$ and $y'-1=-3(y-1)$ so $\begin{cases} x'=-3x+4 \\ y'=-3y+4 \end{cases}$

b- $w(2;1)$ so $w'(-2;1)$ then an equation of $(\gamma')$ is
$(x+2)^2+(y-1)^2=45$.

N°8.

**Set of points I**




$\overrightarrow{BI}=\dfrac{1}{2}\overrightarrow{BA}$ since $I$ is the midpoint of $[AB]$ which gives : $I$ is the image $A$ by a dilation $h$ of center $B$ and ratio $\dfrac{1}{2}$.

As $A$ describes the circle $(C)$, its image $I$ describes the circle $(C')$ image of $(C)$ by $h$, the radius of $(C')$ is $R'=\dfrac{R}{2}$ and its center is $O'$ image of $O$.

$O'$ is defined by $\overrightarrow{BO'}=\dfrac{1}{2}\overrightarrow{BO}$.

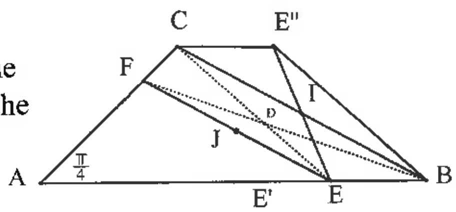
**Set of points G**

Let $J$ be the midpoint of $[BC]$.

We know that $\overrightarrow{JG}=\dfrac{1}{3}\overrightarrow{JA}$, center of gravity property, then $G$ is the image of $A$ by the dilation $h'$ of center $J$ and ratio $\dfrac{1}{3}$.

As $A$ describes $(C)$ its image $G$ describes the circle $(C'')$ of center $O''$ image of $O$ by $h'$ and radius $R''=\dfrac{R}{3}$.

N°9.




1) Noticing that the image of the straight line $(BC)$ by $h_A$ is the straight line passing through $E$ and parallel to $(BC)$,

it's the straight line $(EF)$.

Let $C' = h_A(C)$, since $C \in (BC)$then $C' \in (EF)$.

On the other hand $C' \in (AC)$ , then $C'$ is the point of intersection of $(EF)$and$(AC)$, then it is the point $F$ , $h_A(C) = F$ .

Similarly the image of the straight line $(EF)$ by $h_D$ is the straight line passing through $C$ and parallel to $(EF)$ , then it's the of the straight line $(BC)$.

Let $F' = h_D(F)$, since $F \in (EF)$ then $F' \in (BC)$ .

In addition $F' \in (DF)$ , then $F'$ is the point of intersection of $(DF)$ and $(BC)$, then it's the point $B$ , $h_D(F) = B$ .

$(CF')//(EF)$ moreover $F' \in (FD)$ then $F'$ the intersection of $(CB)$ and $(FD)$ is the point $B$ .

2) The ratio of $h_A$ is: $k = \frac{AE}{AB} = \frac{3}{4}$ .

The ratio of $h_D$ is: $k' = -\frac{DC}{DE}$ since $k' < 0$ .

But $\frac{DC}{DE} = \frac{DB}{DF} = \frac{CB}{FE}$ since $(EF)//(CB)$ .

Similarly, $\frac{AB}{AE} = \frac{BC}{EF} = \frac{AC}{AF}$ therefore: $\frac{DC}{DE} = \frac{AB}{AE} = \frac{4}{3}$ .

The ratio of $h_D \circ h_A$ is $\frac{3}{4} \times \frac{-4}{3} = -1$ then $h_D \circ h_A$ is a central symmetry , since $h_D \circ h_A(B) = h_D(E) = C$ then the center is the midpoint $I$ of $[BC]$ .

$h_A \circ h_D$ is a central symmetry since its ratio is $-1$ .

Since $h_A \circ h_D(E) = h_A(C) = F$ then the center $h_A \circ h_D$ is the point $J$ .

3) $E' = h_A(E)$ gives $\overrightarrow{AE'} = \frac{3}{4}\overrightarrow{AE} = \frac{3}{4} \times \frac{3}{4}\overrightarrow{AB}$ , so $AE' = 9$ .

$E'' = h_D(E')$ gives $h_D o h_A(E) = E''$ and since $h_D o h_A = S_I$ then $I$ Is the midpoint of $[EE'']$ .

4) $h_D \circ h_A \circ h_A \circ h_D = (h_D \circ h_A) \circ (h_A \circ h_D) = S_I \circ S_J$ .

But the composite of two central symmetries is a translation of vector $2\overrightarrow{JI}$ , then $h_D \circ h_A \circ h_A \circ h_D = t_{2\overrightarrow{JI}}$ .

5) $BECE''$ is a parallelogram since its diagonals bisect each other.

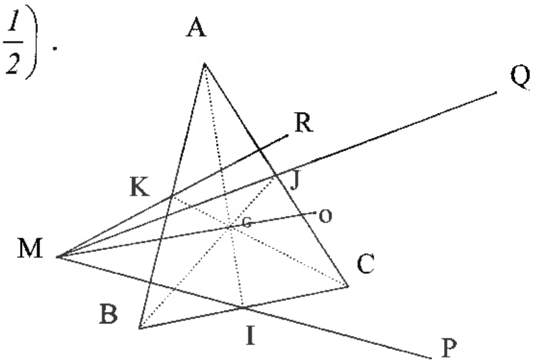
$N^\circ 10.$

1) $(IJ)$ and $(AB)$ are parallel , then there exists a negative dilation $h_1$ such that : $h_1 \begin{cases} A \longrightarrow I \\ B \longrightarrow J \end{cases}$

If such a dilation exists $h_1$ then $k_1 = -\frac{IJ}{AB} = -\frac{1}{2}$.

Its center belongs to $(AI)$ and $(BJ)$ then it is the point $G$ .

Hence, $h_1\left(G;-\frac{1}{2}\right)$ .




2) If there exists a dilation $h_2$ that transforms $I$ onto $P$, $J$ onto $Q$ and $K$ onto $R$, then its center is the point $M$ common to $(IP)$, $(JQ)$ and $(KR)$ .

Since $\overrightarrow{MP} = 2\overrightarrow{MI}$ , $\overrightarrow{MR} = 2\overrightarrow{MK}$ and $\overrightarrow{MQ} = 2\overrightarrow{MJ}$ then $h_2$ is the dilation of center $M$ and ratio 2.

3) a- $f = h_2 \circ h_1$ is a central symmetry since $k_1 \cdot k_2 = -1$ .

b- Let $O$ be the center of $h_2 \circ h_1$ .

$h_2 \circ h_1(A) = h_2(I) = P$ , then $O$ is the midpoint of $[AP]$ .

$h_2 \circ h_1(B) = h_2(J) = Q$ ,then $O$ is the midpoint of $[BQ]$ .

$h_2 \circ h_1(C) = h_2(K) = R$ ,then $O$ is the midpoint of $[CR]$ .

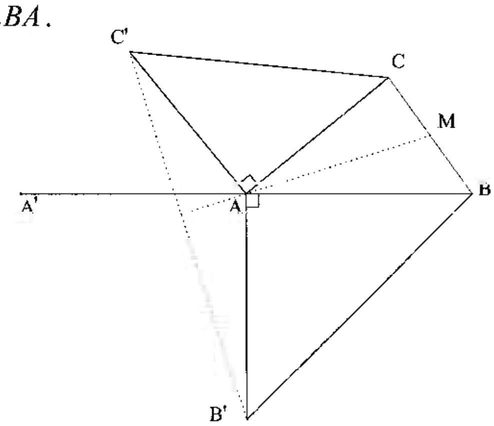
c- $h_2 \circ h_1(G) = h_2(G)$, let $h_2(G) = G'$ .

Since $G$ is the center of gravity of triangle $IJK$ then $G'$ is the center of gravity of triangle $PQR$ witch is the image of $PQR$ by $h_2$ .

$h_2 \circ h_1 = S_O$ and $h_2 \circ h_1(G) = G'$ then $O$ is the midpoint of $[GG']$ and since $h_2(G) = G'$ then $\overrightarrow{MG'} = 2\overrightarrow{MG}$ therefore the points $O$, $G$ and $M$ are collinear.

N° 11.

1) a- $M$ being the midpoint of $[BC]$, $\overrightarrow{BC} = 2\overrightarrow{BM}$ and consequently $h(M) = C$.
$h(A)$ is the point $A'$ symmetric of $B$ with respect to $A$ since $\overrightarrow{BA'} = 2\overrightarrow{BA}$.




b- $r \circ h(A) = B'$ is equivalent to $r(A') = B'$.
$r \circ h(M) = C'$ is equivalent to $r(C) = C'$.

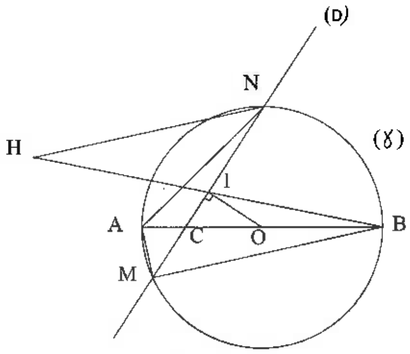
Then $r$ is the rotation of center $A$ and angle $\frac{\pi}{2}$ since this rotation transforms $A'$ onto $B'$ and $C$ onto $C'$ since triangles $AA'B'$ and $ACC'$ are right isosceles.

2) From b- : $B'C' = A'C$ and $\left(\overrightarrow{A'C};\overrightarrow{B'C'}\right) = \frac{\pi}{2} (\bmod 2\pi)$.

$h(A) = A'$ and $h(M) = C$ then $\overrightarrow{A'C} = 2\overrightarrow{AM}$ hence $(B'C')$ is perpendicular to $(A'C)$ and since $(AM)$ is parallel to $(A'C)$ then $(AM)$ and $(B'C')$ are perpendicular.
$B'C' = A'C$ and $A'C = 2AM$ then $B'C' = 2AM$.

N° 12.

1) Triangle $OMN$ is isosceles since $OM = ON = R$ , the median $[OI]$ is at the same time the height .
Triangle $OIC$ is right at $I$ with $O$, $C$ being fixed, then the point $I$ describes circle $(T)$ of diameter $[OC]$ .




2) a- $h : I \longrightarrow H$ , then $\overrightarrow{BH} = 2\overrightarrow{BI}$ and consequently $I$ is the midpoint of $[BH]$ .
$I$ being the midpoint of diagonals $[BH]$ and $[MN]$ of quadrilateral $BMHN$ then it is a parallelogram .

b- To prove that $H$ is the orthocenter of triangle $AMN$ , we need to prove that $(AH) \perp (MN)$ and $(MH) \perp (AN)$ .

In triangle $BAH$ , we have $\overrightarrow{OI} = \dfrac{1}{2}\overrightarrow{AH}$ .

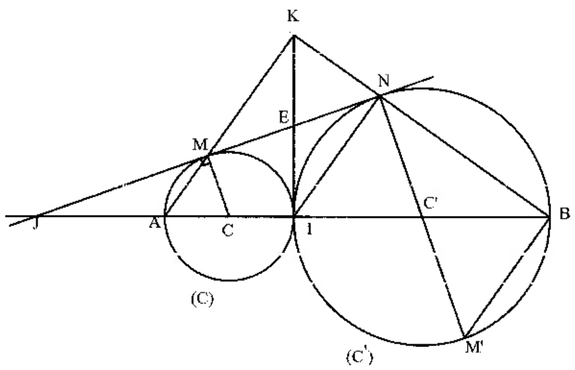
Midline Theorem
$(OI)$ is perpendicular to $(MN)$ , then its parallel $(AH)$ is perpendicular to $(MN)$ .
$(AN)$ is perpendicular to $(NB)$ since triangle $ANB$ is inscribed in a semi-circle.
Since $(BN)$ is parallel to $(MH)$ then $(AN)$ is perpendicular to $(MH)$.

c- $H$ is the image of $I$ by $h(B;2)$ then as $I$ describes the circle of diameter $[OC]$ , its image $H$ describes the circle $(T')$ image of $(T)$ by this dilation .

$h(O) = A$ and $h(C) = C'$ then $H$ describes the circle of diameter $[AC']$.

**N° 13.**

1) a- The negative dilation that transforms $(C)$ onto $(C')$ has a center $I$ and ratio $-\dfrac{R'}{R} = -2$ .




b- The image of point $A$ by $h_1$ is the point $B$, since $\overrightarrow{IB} = -2\overrightarrow{IA}$ .
$h_1(C) = C'$ and $h_1(M) = M'$ then $\overrightarrow{C'M'} = -2\overrightarrow{CM}$ , therefore : $C'M' = R'$ hence $M'$ is diametrically opposite to $N$ .

c- We have $h_1 : \begin{cases} A \longrightarrow B \\ M \longrightarrow M' \end{cases}$ then $\overrightarrow{BM'} = -2\overrightarrow{AM}$
then $(BM')$ is parallel to $(AM)$ ,but $(BM')$ is perpendicular to $(BN)$ since $[NM']$ is a diameter then $(AM)$ is perpendicular to $(BN)$ and consequently $A\hat{K}B = 90°$ .
The locus of $K$ is the circle of fixed diameter $[AB]$ .

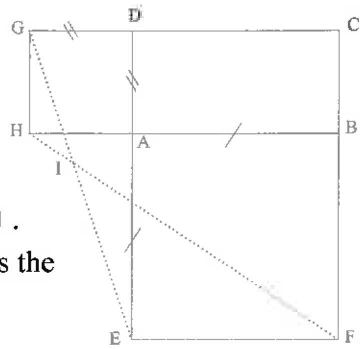
2) Let $J$ be the center of positive dilation $h'$ , $J$ is the point of intersection of straight lines $(CC')$ and $(MN)$ .

$h'(M) = N$ and $h'(A) = I$ then $\overrightarrow{IN} = 2\overrightarrow{AM}$ .
$h'(A) \in (C')$ hence $h'(A) = I$ or $h'(A) = B$ .
If $h'(A) = B$ and since $h'(M) = N$ then $(BN)$ and $(AM)$ are

parallel which is impossible hence $h'(A)=I$.
Therefore, $h'(M)=N$ and $h'(A)=I$ then $(IN)$ and $(AM)$are parallel , but $(IN)$ is perpendicular to $(BN)$ since $[IB]$ is a diameter , then $(AM)$ is perpendicular to $(BN)$ and consequently $A\hat{K}B=90°$.

3) Quadrilateral $IMKN$ is a rectangle since the 3 angles $I\hat{M}K$, $I\hat{N}K$ and $M\hat{K}N$ are right, there results that $(IK)$ and $(MN)$ bisect each other at $E$.
$EM=EI=EN$ implies that $(EI)$ is the common internal tangent to the two circles , and consequently $(IK)\perp(AB)$.

**$N°14.$**



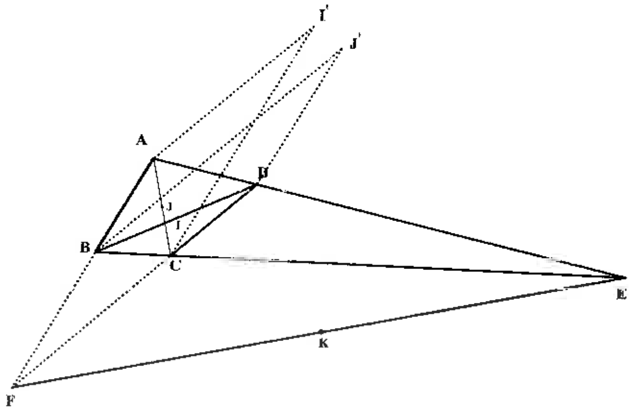

1) $(CG)$passes through $G$ then the image of the straight line $(CG)$ by $h$ is a straight line passing through $E=h(G)$ and parallel to $(CG)$ hence it is the straight line$(EF)$, $h(CG)=(EF)$.
The image of $(CG)$ by $h'\circ h$ is the image of the straight line $(EF)$ by $h'$ since $h(CG)=(EF)$.
The image of the straight line $(EF)$ by $h'$ is a straight line parallel to $(EF)$ and passing through $H$ since $h'(F)=H$.
Hence the image of $(CG)$ by $h'\circ h$ is the straight line $(AB)$, $h'\circ h(CG)=(AB)$.

2) The image of the straight line $(CF)$ by $h'$ is a straight line passing through $H$ and parallel to $(CF)$, hence it is the straight line $(HG)$, $h'(CF)=(HG)$.
The image of the straight line $(HG)$ by $h$ is a straight line parallel to $(HG)$ and passing through $E$, hence it is the straight line $(AD)$, $h'\circ h(CF)=(AD)$.
Therefore, the image of $(CF)$ by $hoh'$ is the straight line $(AD)$.

3) $hoh' = h'oh$ since they have the same center $I$ and the same ratio.
We have proved that $h' \circ h(CG) = (AB)$ and $h \circ h'(CF) = (AD)$.
Then, the image of $C$ , the common point of $(CG)$ and $(CF)$ is $A$, the point common to $(AB)$ and $(AD)$ .
Therefore, $h' \circ h(C) = A$ which gives that the center $I \in (AC)$.
Hence, $(AC)$ passes through $I$ .

N° 15.

1)




2) a- $h_1(B) = C$, then the image of the straight line $(BJ')$ by $h_1$ is the straight line passing through $C$ and parallel to $(BJ')$, then it is the straight line $(DC)$.
Since $h_2(D) = A$ then the image of the straight line $(DC)$ by $h_2$ is the straight line passing through $A$ and parallel to $(DC)$, hence it is the straight line $(AI')$ so $h_2 \circ h_1((BJ')) = (AI')$

b- The image of the straight line $(DJ')$ by $h_1 \circ h_2$ is $(CI')$ .

c- $h_2 o h_1 = h_1 o h_2$ since $h_1$ and $h_2$ have the same center $E$ and same ratio.
The image of $(BJ')$ is $(AI')$ .
The image of $(DJ')$ is $(CI')$ .
The image of point $J'$ common to $(BJ')$ and $(DJ')$ is the point $I'$ common to $(AI')$ and $(CI')$ .
Hence, the center $E$ belongs to $(I'J')$ and the three points $E$ , $I'$ and $J'$ collinear.

3) Consider the dilation $h\left(F;\frac{1}{2}\right)$.

The image of $E$ is $K$ since $\overrightarrow{FK}=\frac{1}{2}\overrightarrow{FE}$.

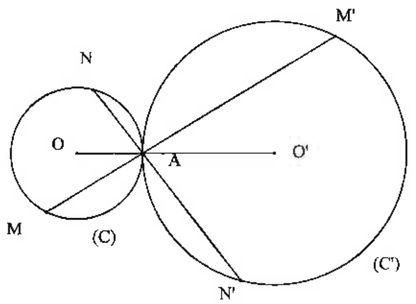
$BFDJ'$ is a parallelogram since $I$ is the midpoint of $[FI']$, hence $\overrightarrow{FJ}=\frac{1}{2}\overrightarrow{FI'}$ and consequently $h(I')=J$.

$E$ , $I'$ and $J'$ being collinear then $K$ , $J$ and $I$ are collinear since dilations preserve collinearity.

N° 16.

1) a- $h$ is the negative dilation of negative ratio $-\frac{R'}{R}$.

$M\in(C)$ then its image belongs to $(C')$ and at the same time to $(AM)$, hence it is the point of intersection other than $A$ between $(C')$ and $(AM)$ so it is point $M'$.




A similar reasoning shows that $h(N)=N'$.

b- $h(M)=M'$ and $h(N)=N'$ then the two straight lines $(M'N')$ and $(MN)$ are parallel.

2) a- If $MN=2R$ then $M'N'=\frac{R'}{R}\times 2R=2R'$ and since $M'$ and $N'$ belong to $(C')$ then $[M'N']$ is a diameter of $(C')$.

b- $\overrightarrow{M'N'}=\frac{R'}{R}\overrightarrow{NM}$ then there exists a positive dilation of ratio

$\frac{R'}{R}$ that transforms $N$ onto $M'$ and $M$ onto $N'$ of fixed center $I$, point of intersection of $(OO')$ with the exterior tangent to $(C)$ and $(C')$, hence $(MN')$ and $(NM')$ intersect at a fixed point $I$.

# Indications

N° 1.

1) a- Remark that $A\hat{O}M + A\hat{O'}N = 180°$.

2) a- If $J$ is the midpoint of $[OO']$ then $JK = \frac{R+R'}{2}$, hence $K$ varies on the circle of center $J$ and radius $\frac{R+R'}{2}$.

N° 2.

1) a- In triangles $IwP$ and $IOM$, $\frac{Iw}{IO} = \frac{Pw}{MO} = \frac{IP}{IM}$.

In triangle $AOM'$ we have $\frac{Aw}{AO} = \frac{Pw}{OM'}$ and $OM = OM'$.

b- $\frac{Iw}{2} = \frac{Aw}{6} = \frac{Iw + Aw}{8} = \frac{1}{2}$.

2) a- $\overrightarrow{IP} = -\frac{1}{3}\overrightarrow{IM}$.

N° 4.

1) Use the idea that triangle $BOA$ is isosceles and $POQ$ is isosceles.

2) $\overrightarrow{AI} = \frac{1}{2}\overrightarrow{AB}$.

3) $OG = \frac{1}{3}OA = \frac{1}{3}R$.

N° 5.

2) Remark that $\overrightarrow{CO} = \frac{1}{2}\overrightarrow{CM}$ then $O$ is the image of $M$ by $h\left(C;\frac{1}{2}\right)$, $M$ describes the segment $[A'B']$, deprived of points $A'$ and $B'$, image of $[AB]$.

3) In triangle $ABC$, the perpendicular bisector of $[AC]$ is the perpendicular bisector of $[MQ]$, similarly the perpendicular bisector of $[BC]$ is the perpendicular bisector of $[MP]$.
The perpendicular bisector of $[PQ]$ passes through the center of gravity $G$ of triangle ABC.

4) a- $I$ is the image of $O$ by the dilation $h(G;2)$.

# CHAPTER 9

# Direct Plane Similitude

**Chapter Review:**

- A direct plane similitude is any mapping of the plane that is a translation or the composite of a rotation and a positive dilation with the same center .
- Definition :
  $\alpha$ being a given real number , $O$ a given point of the plane and $k$ a non-zero positive real number .
  We call the direct plane similitude of center $O$ , ratio $k$ and angle $\alpha$ , denoted by $S(O;k;\alpha)$ all mappings of the plane that to every point $M$ of the plane associates the point $M'$ such that :
  $OM' = kOM$ and $(\overrightarrow{OM};\overrightarrow{OM'}) = \alpha \pmod{2\pi}$ .
  * $O$ is invariant under $r$ .
  * $S^{-1}(O;k;\alpha) = S\left(O;\frac{1}{k};-\alpha\right)$.
  * $S(O;1;0)$ is the identity mapping .
    $S(O;1;\alpha) = r(O;\alpha)$.
    $S(O;k;0) = h(O;k)$.
- Characteristic properties:
  A mapping is a similitude $S(O;k;\alpha)$ if and only if for all points $M$ and $N$ of respective images $M'$ and $N'$ we have:
  $M'N' = kMN$ and $(\overrightarrow{MN};\overrightarrow{M'N'}) = \alpha \pmod{2\pi}$.
- $r(O;\alpha) \circ h(O;-k) = S(O;k;\alpha+\pi)$.
- The image of a straight line $(d)$ by a direct plane similitude is a straight line $(d')$.

## Chapter Review

- A similitude of ratio $k$ multiplies distances by $k$, areas by $k^2$ and preserves geometric figures.
- $S(O;k;\alpha)\circ S(O';k';\alpha') = S(w;k\times k';\alpha+\alpha')$.
- The image of a circle $C(w;R)$ is a circle $C'(w';R')$ where $w'$ is the image $w$ by $S$ and $R' = k\times R$.

# Solved Problems

## Construction

**N°1.**

$ABCD$ is a square of center $O$ such that $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{2}(\text{mod }2\pi)$.

1) Construct the image of $ABCD$ by $S\left(A;\sqrt{2};\frac{\pi}{4}\right)$.

2) Construct the image of $ABCD$ by $S\left(O;\frac{\sqrt{2}}{2};\frac{\pi}{4}\right)$.

**N°2.**

$ABD$ is a triangle right at $B$ such that: $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{3}(\text{mod }2\pi)$.

1) a- Construct the point $C$ image of $B$ by $S\left(D;\frac{\sqrt{3}}{3};\frac{\pi}{2}\right)$.

   b- Determine the nature of $ABCD$.

2) Construct the image of $ABCD$ by $S\left(B;\frac{1}{2};\frac{2\pi}{3}\right)$.

**N°3.**

$ABCD$ is a rectangle such that $AD=4$, $AB=2$ and $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{2}(\text{mod }2\pi)$.

Let $I$ be the midpoint of $[AD]$ and $J$ the symmetric of $I$ with respect to $(AC)$.

1) Let $S$ be the direct plane similitude $I$, of ratio $\sqrt{2}$ and angle $\frac{3\pi}{4}$. Determine $S(A)$.

2) Let $S'$ be the direct plane similitude of center $J$ that transforms $A$ onto $C$.

Precise the ratio $k$ of $S'$ and a measure of its angle.

**N° 4.**

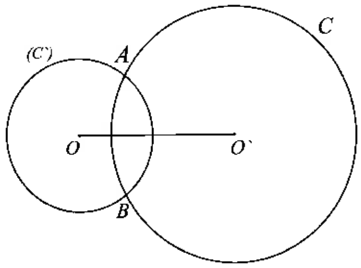
$ABC$ is a triangle whose center of gravity is $G$.
$M$, $N$ and $P$ are the midpoints of the segments $[BC]$, $[AC]$ and $[AB]$.

1) Determine the image of triangle $ABC$ by the direct plane similitude $S\left(G;\frac{1}{2};\frac{\pi}{3}\right)$.
2) Construct the image of triangle $ABM$ by the direct plane similitude $S\left(A;\frac{\sqrt{3}}{3};\frac{\pi}{6}\right)$.

**N° 5.**

In an oriented plane, consider the two circles $(C)$ and $(C')$ of centers $O$ and $O'$ respectively, and of respective radii $r$ and $r'$.
The two circles intersect in two points $A$ and $B$.

1) Prove that there exists a direct plane similitude $S$ of center $A$ that transforms $(C)$ onto $(C')$ and precise its angle and ratio.

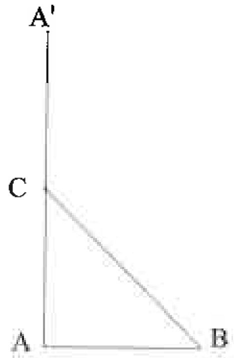



2) Let $M$ be a point of $(C)$ and $M'$ its image by $S$.
   a- Compare the angles $\left(\overrightarrow{OA};\overrightarrow{OM}\right)$ and $\left(\overrightarrow{O'A};\overrightarrow{O'M'}\right)$.
   b- Prove that the points $M$, $B$ and $M'$ are collinear.

**N° 6.**

In an oriented plane, consider a triangle $ABC$ right and isosceles of vertex $A$ such that: $AB = a$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \frac{\pi}{2}(\text{mod}\,2\pi)$.

Denote by $A'$ the symmetric of $A$ with respect to $C$.




1) Determine the ratio $k$ and the angle $\alpha$ of the direct plane similitude that transforms $A'$ onto $C$ and $C$ onto $B$.
2) What is the image of the straight line $(AC)$ by $S$?
3) Let $I$ be the center of $S$.
   a- Prove that triangle $ICB$ is right isosceles.
   b- Deduce a construction of $I$.

**N°7.**

Consider, in an oriented plane, a triangle $OAB$ right isosceles such that $OA = OB = a$ and $(\overrightarrow{OA};\overrightarrow{OB}) = \frac{\pi}{2}(\bmod 2\pi)$.

Designate by $I$ the midpoint of segment $[AB]$.

Let $M$ be a point on the straight line $(OA)$, $\lambda$ the real number such that $\overrightarrow{MA} = \lambda\,\overrightarrow{OA}$ and let $N$ be the point defined by $\overrightarrow{NB} = -\lambda\,\overrightarrow{OB}$.

1) In this question $M$ is distinct from $A$.
   Consider the rotation $r$ that transforms $A$ onto $B$ and $M$ onto $N$.
   a- Precise the angle of this rotation.
   b- Denote by $\Omega$ the center of $r$, prove that $OA\Omega B$ s a square.
2) Denote by $J$ the midpoint of $[MN]$.
   a- Determine the angle and ratio of the direct plane similitude $S$ of center $\Omega$ that transforms $M$ onto $J$.
   b- Determine the set $(E)$ of points $J$ as $M$ describes the straight line $(OA)$. Represent $(E)$.

**N°8.**

$ABCD$ is a rhombus of center $O$ and side $a$ such that :

$\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{3}(\bmod 2\pi)$.

1) Calculate $AO$ and $AC$.
2) Let $S$ be the direct similitude of center $C$, of angle $\frac{\pi}{6}$ and ratio $\frac{\sqrt{3}}{3}$.
   a- Prove that $S$ transforms $A$ onto $B$.
   b- Show that the image $O'$ of point $O$ is the midpoint of $[BC]$.
3) Denote by $D'$ the image of $D$ by $S$.
   a- Prove that $D'$ belongs to the semi straight line $[CA)$.
   b- What is the measure of angle $\left(\overrightarrow{OD};\overrightarrow{O'D'}\right)$ ?
   c- Deduce a measure of angle $\left(\overrightarrow{BC};\overrightarrow{O'D'}\right)$.
4) Prove that $D'$ is the center of the circle circumscribed about triangle $BCD$.

**N° 9.**

Consider a circle $(C)$ of diameter $[OB]$.
$A$ is a point of the segment $[OB]$, distinct of $O$ and of $B$, and $I$ the midpoint of $[AB]$.
The perpendicular bisector of the segment $[AB]$ cuts the circle at $M$ and $M'$ such that : $\left(\overrightarrow{MO};\overrightarrow{MB}\right)=\frac{\pi}{2}(\bmod 2\pi)$.
Designate by $N$ the orthogonal projection of $A$ on $(OM)$.

1) a- Precise the nature of quadrilateral $AMBM'$
   b- Deduce that the straight line $(AM')$ is perpendicular to $(OM)$ and that the points $N$, $A$ and $M'$ are collinear.
   In what follows, let $S$ be the direct plane similitude of center $N$ such that: $S(M)=A$.
2) a- Precise the angle of $S$.
   b- Determine the images of the straight lines $(MI)$ and $(NA)$

by $S$.

c- Deduce the image of $M'$ by $S$.

3) a- $I$ is the midpoint of $[MM']$, determine the position of point $I' = S(I)$.

b- Deduce that the straight line $(NI)$ is tangent at $N$ to the circle $(C')$ of diameter $[OA]$.

**N° 10.**

$ABC$ is a direct triangle, $A'$, $B'$ and $C'$ are the points situated on the exterior of triangle such that $A'BC$, $B'CA$ and $C'AB$ are equilateral triangles. $J$, $K$ and $L$ are the centers of gravity of triangles $A'BC$, $B'CA$ and $C'AB$ respectively.
We need to prove that triangle $JKL$ is equilateral.
Designate by $S_A$ the direct plane similitude of center $A$ that transforms $K$ onto $C$ and by $S_B$ that of center B that transforms $C$ onto $J$.

1) Determine the ratio and angle of $S_A$.
2) Determine the ratio and angle of $S_B$.
3) Show that $S_B \circ S_A$ is a rotation whose angle is to be determined.
Prove that $L$ is the center of $S_B \circ S_A$.
4) Deduce from part 3) that triangle $JKL$ is direct equilateral.

**N° 11.**

$ABCD$ is a square such that $\left(\overrightarrow{AB};\overrightarrow{AD}\right) = \frac{\pi}{2}(\text{mod } 2\pi)$.

Let $r$ be the rotation of center $A$ and angle $\frac{\pi}{2}$, $t$ is the translation of vector $\overrightarrow{AB}$ and $h = h(C;\sqrt{3})$.

1) a- Show that $t \circ r$ is a rotation whose angle is to be determined.
Designate by $r' = t \circ r$.

b- Determine the images of $A$ and $B$ by $r'$.
Deduce the center of $r'$.

2) Let $f = r' \circ h$.

a- Find the nature of $f$ and precise its angle and ratio.

b- Let $I$ be the center of $f$.
Determine the image of $C$ by $f$, and show that

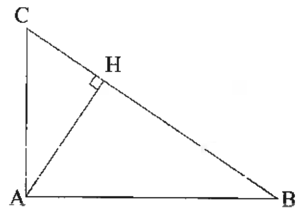
$\left(\overrightarrow{IC};\overrightarrow{ID}\right)=\frac{\pi}{2}(\bmod 2\pi)$ and that $ID=\sqrt{3}\ \ IC$ .

c- Find a measure of the angle $\left(\overrightarrow{CD};\overrightarrow{CI}\right)$ then construct $I$.

**N° 12.**

$ABC$ is a right triangle such that $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}(\bmod 2\pi)$.

$H$ is the foot of the perpendicular issued from $A$. Let $D$ be the point so that $ACD$ is a direct right isosceles triangle of vertex $A$. $O$ is the foot of the perpendicular issued from $D$ in triangle $DBC$ and by $K$ the foot of the perpendicular issued from $A$ in triangle $DAO$.




1) a- Show that the rotation $r$ of center $A$ and angle $\frac{\pi}{2}$ transforms the straight line $(CB)$ onto the straight line $(DO)$.

b- Determine the image of triangle $AHC$ by $r$ .

c- Deduce that $AHOK$ is a square .

2) Designate by $\Omega$ the point of intersection of $(AB)$ and $(KH)$.
Prove that there exists a dilation $h$ that transforms triangle $AKD$ onto triangle $BHA$ .

3) Consider the transformation $S=h\circ r$ .

a- Determine the image of the points $H$ , $C$ and $A$ by $S$.

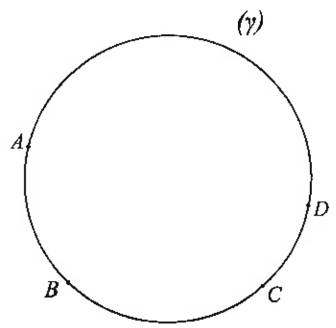
b- Determine the nature of $S$ and precise its elements.

**N° 13.**

$A$ , $B$ , $C$ and $D$ are four distinct points belonging to the circle $(\gamma)$.

1) Consider the direct plane similitude $S$ of center $A$ that transforms $C$ onto $D$ . Designate by $E$ the image of point $B$ by $S$.

a- Prove that $\left(\overrightarrow{CB};\overrightarrow{DE}\right)=\left(\overrightarrow{AC};\overrightarrow{AD}\right)$ (mod $2\pi$).




b- Prove that $E$ belongs to the straight line $(BD)$.

c- Prove that : $AD \times BC = DE \times AC$ .

2) a- Prove that : $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \left(\overrightarrow{AE};\overrightarrow{AD}\right) \pmod{2\pi}$

and that $\dfrac{AC}{AB} = \dfrac{AD}{AE}$ .

b- Let $S'$ be the similitude of center $A$ that transforms $B$ onto $C$.
Prove that $S'(E) = D$ .

c- Prove that $AB \times CD = AC \times BE$ .

3) Prove that $AC \times BD = AB \times CD + AD \times BC$ .

**N° 14.**

In the oriented plane , consider an isosceles triangle $ABC$ such that

$AB = AC$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \dfrac{\pi}{4} \pmod{2\pi}$.

Let $I$ be the point such that $CAI$ is right isosceles with

$\left(\overrightarrow{CA};\overrightarrow{CI}\right) = -\dfrac{\pi}{2} \pmod{2\pi}$ .

1) We denote by $r_A$ the rotation of center $A$ that transforms $B$ onto $C$ and $r_C$ the rotation of center $C$ and angle $-\dfrac{\pi}{2}$.

Let $f = r_C \circ r_A$ .

a- Determine the images of $A$ and of $B$ by $f$.

b- Determine the nature of $f$ and construct its center $O$ .

c- What is the nature of quadrilateral $ABOC$ ?

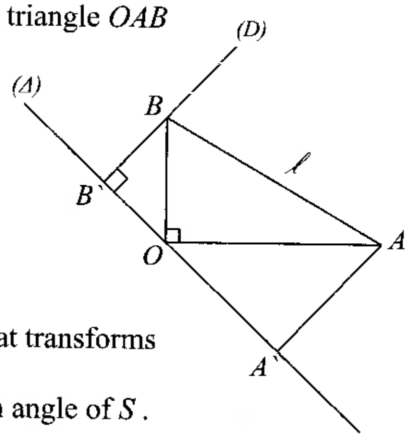
2) Let $S$ be the direct similitude of center $O$ that transforms $A$ onto $B$. Denote by $C'$ the image of $C$ by $S$, $H$ is the midpoint of $[BC]$ and $H'$ its image by S .
   a- Determine the measure of the angle of $S$.
   Show that $C'$ belongs to the straight line $(OA)$.
   b- Find the image of segment $[OA]$ by $S$ and show that $H'$ is the midpoint of $[OB]$.
   c- Show that $(C'H')$ is perpendicular to $(OB)$.
   Deduce that $C'$ is the center of the circle circumscribed about triangle $OBC$.

N° 15.

In an oriented plane, consider a triangle $OAB$ right angled at $O$ such that $AB = \ell$ and :

$$\left(\overrightarrow{OA};\overrightarrow{OB}\right) = \frac{\pi}{2} \pmod{2\pi}$$

$$\left(\overrightarrow{AB};\overrightarrow{AO}\right) = \frac{\pi}{6} \pmod{2\pi}$$




Let $S$ be the direct similitude that transforms $B$ onto $O$ and $O$ onto $A$.

1) Determine the ratio $k$ and an angle of $S$.
2) Let $I$ be the center of $S$.
   a- Construct geometrically $I$.
   b- Prove that $I$ is the foot of the perpendicular issued from $O$ in triangle $OAB$.
3) $(\Delta)$ is a variable straight line passing through $O$ and $(D)$ is a straight line passing through $B$ and perpendicular to $(\Delta)$, designate by $A'$ and $B'$ the orthogonal projections of $A$ and $B$ respectively on $(\Delta)$.
   a- Determine the image by $S$ of $(D)$ and of $(\Delta)$.
   b- Deduce the image of $B'$ by $S$.
   c- Prove that the circle of diameter $[A'B']$ passes through a fixed point when $(\Delta)$ varies.

**N° 16.**

$OIJ$ is a right isosceles triangle such that $\left(\overrightarrow{OI};\overrightarrow{OJ}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$.

Let $M$ be a variable point of $(IJ)$ and $N$ the point such that $\left(\overrightarrow{NO};\overrightarrow{NM}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$ and $NO=NM$. $I'$ is the midpoint of $[IJ]$.

1) Determine the ratio $k$ and an angle $\alpha$ of the direct plane similitude $S$ of center $O$ that transforms $M$ onto $N$.
2) a- Determine $S(I)$ and $S\circ S(I)$.
   b- Determine the image $(\Delta)$ of $(IJ)$ by $S$.
3) Let $S'$ be the direct plane similitude of center $I'$, ratio $\sqrt{2}$ and angle $-\frac{\pi}{4}$.
   Precise the nature of the transformations $t_1=S'\circ S$ and $t_2=S\circ S'$.

**N° 17.**

In an oriented plane, consider a triangle $ABC$ such that :

$AB=2$, $AC=1+\sqrt{5}$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$.

1) Let $S$ be the direct plane similitude that transforms $B$ onto $A$ and $A$ onto $C$.
   Determine the ratio $k$ and a measure $\alpha$ of the angle of $S$.
2) Designate by $\Omega$ the center of $S$, construct $\Omega$ geometrically.
3) Let $D$ be the image of $C$ by $S$.
   a- Prove that the points $A$, $\Omega$ and $D$ are collinear, and that the straight lines $(CD)$ and $(AB)$ are parallel.
   b- Construct the point $D$.
   c- Show that $CD=3+\sqrt{5}$.
4) Let $E$ be the orthogonal projection of $B$ on $(CD)$.
   a- Explain the construction of the point $F$ image of point $E$ by $S$ and place $F$ on the figure.
   b- What is the nature of quadrilateral $BFDE$ ?

# Supplementary Problems

N° 1.

$OABC$ is a square of center $I$ such that $\left(\overrightarrow{OA};\overrightarrow{OC}\right)=\frac{\pi}{2}(\bmod 2\pi)$ and $OA=4$.

Let $H$ be the midpoint of $[BC]$, $C'$ that of $[IA]$ and $J$ that of $[OI]$.

Let $S$ be the direct plane similitude such that $S(O)=I$ and $S(A)=J$.

1) Determine the ratio and an angle of $S$.
2) Show that $S(C)=C'$.
3) Determine the point $B'$ image of $B$ by $S$.

N° 2.

In the oriented plane, consider a triangle $ABC$ right at $A$ and such that $AB=2AC$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{2}(\bmod 2\pi)$.

$(C_B)$ and $(C_C)$ are two circles passing through $A$ and of respective centers $B$ and $C$.

1) Determine the characteristic elements of the similitude $S$ of center $A$ that transforms $B$ onto $C$.
2) What is the image of circle $(C_B)$ by $S$?
3) Let $D$ be the point of intersection, other than $A$ of circles $(C_B)$ and $(C_C)$.
   a- Let $E$ be the point such that $S(E)=D$, justify that $E$ belongs to $(C_B)$ and show that $E$ is diametrically opposite to $D$.
   b- Let $F$ be the image of $D$ by $S$, determine the position of $F$.
4) Determine $S\circ S$ and deduce that the points $E$, $A$ and $F$ are collinear and express $\overrightarrow{AF}$ in terms of $\overrightarrow{AE}$.

N° 3.

In the oriented plane, consider a direct triangle $ABC$.

$AEDB$, $BGFC$ and $CIHA$ are three direct squares constructed on the sides of the triangle exterior to triangle $ABC$ and of respective centers $C'$, $A'$ and $B'$.

Consider the two direct plane similitudes :

$S_A = S\left(A;\sqrt{2};\frac{\pi}{4}\right)$ and $S_C = S\left(C;\frac{1}{\sqrt{2}};\frac{\pi}{4}\right)$

1) Determine the images of $C'$ and $B'$ by $S_C \circ S_A$.
2) Show that $AA' = B'C'$ and that $(AA')$ and $(B'C')$ are perpendicular.

**N°4.**

$ABCD$ is a square of center $O$ such that $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$.

Let $P$ be a point of $[BC]$ distinct of $B$.

Denote by $Q$ the point of intersection of $(AP)$ and $(CD)$.

The perpendicular $(\Delta)$ to $(AP)$ and passing through $A$ cuts $(BC)$ in $R$ and $(CD)$ in $S$.

1) Let $r$ be the rotation of center $A$ and angle $\frac{\pi}{2}$.
   a- Precise the image of the straight line $(BC)$ by $r$.
   b- Determine the images of $R$ and of $P$ by $r$.
   c- What is the nature of each of these triangles $ARQ$ and $APS$ ?
2) Denote by $N$ the midpoint of $[PS]$ and $M$ that of $[QR]$.

   Let $S$ be the direct similitude of center $A$, and angle $\frac{\pi}{4}$ and of ratio $\frac{1}{\sqrt{2}}$.
   a- Determine the images of $R$ and of $P$ by $S$.
   b- What is the locus of point $N$ as $P$ describes the segment $[BC]$ deprived of $B$.
   c- Prove that the points $M$, $B$, $N$ and $D$ are collinear.

**N°5.**

$ABC$ is a direct triangle in the oriented plane.

Let $S_A$ be the direct plane similitude of center $A$ that transforms $B$ onto $C$.

Let $S_B$ be the direct plane similitude of center $B$ that transforms $C$ onto $A$.

Let $S_C$ be the direct plane similitude of center $C$ that transforms $A$ onto $B$.

Designate by $S = S_C \circ S_B \circ S_A$.

1) a- Determine $S(B)$.
   b- Prove that $S$ is the central symmetry of center $B$.
2) Let $C' = S_B \circ S_A(C)$ and $C'' = S_C(C')$.
   a- What is the relative position of points $B$, $C$ and $C''$ ?
   b- Deduce that $A$ is the midpoint of $[CC']$.

**N° 6.**

$ABC$ is an equilateral triangle such that $(\overrightarrow{AB};\overrightarrow{AC}) = \frac{\pi}{3} (\text{mod } 2\pi)$.

Let $S$ be the direct plane similitude of center $C$, of ratio $\sqrt{3}$ and angle $\frac{\pi}{6}$ and let $G$ be the center of gravity of triangle $ABC$, let $I$ be the midpoint of $[AB]$ and $A'$ the symmetric of $C$ with respect $I$.

1) Determine $S(A)$.
2) Construct the point $B' = S(B)$ and determine $S(G)$.
3) Let $(C)$ be the circle circumscribed about triangle $ABC$ and $(C')$ its image by $S$.
   Show that $B$ is the center of $(C')$.

**N° 7.**

In the oriented plane, consider an equilateral triangle $ABC$ such that $(\overrightarrow{AB};\overrightarrow{AC}) = \frac{\pi}{3} (\text{mod } 2\pi)$.

$O$ is the midpoint of $[BC]$, the perpendicular at $B$ to the straight line $(AB)$ cuts $(AO)$ in $D$.

Let $f$ be the similitude verifying $f(C) = A$ and $f(B) = D$ and let $r$ be the rotation of center $B$ and angle $-\frac{\pi}{3}$.

1) Determine the ratio $k$ and an angle $\alpha$ of $f$.
2) Determine $f(O)$ and construct geometrically the center $w$ of $f$.
3) Consider the transformation $f \circ r$
   a- Determine $f \circ r(A)$, $f \circ r(B)$ and $f \circ r(C)$.
   b- Determine the nature of $f \circ r$ and precise its elements.

**N° 8**

In the oriented plane, consider the isosceles triangle $ABC$ of vertex $A$ and such that $(\overrightarrow{AB};\overrightarrow{AC}) = \alpha(\text{mod}\,2\pi)$ and $AB = AC = a$.

Let $[BH]$ be the height relative to $(AC)$.

Consider the similitude $S$ that transforms $A$ onto $B$ and $C$ onto $H$.

1) Determine the angle of $S$ and show that its ratio $k$ is equal to $\sin\alpha$.
2) a- Determine and construct the straight lines $(d)$ and $(d')$ respective images of the straight lines $(AB)$ and $(AC)$ by $S$.
   b- Deduce that the point $B'$ is the image of $B$ by $S$.
   c- What is the nature of triangle $BB'H$ ?
3) Let $Q$ be the symmetric of $B$ with respect to $H$, determine the point $P$ such that $S(P) = Q$.
4) Calculate the area of triangle $BB'Q$ in terms of $a$ and $\alpha$.

**N° 9**

In the oriented plane, consider an equilateral triangle $ABC$ of center $G$ and such that $(\overrightarrow{AB};\overrightarrow{AC}) = \frac{\pi}{3}(\text{mod}\,2\pi)$.

Let $S$ be the similitude that transforms $A$ onto $B$ and $G$ onto $C$.

1) Determine the ratio and an angle of $S$.
2) Determine the point $E$ whose image by $S$ is the point $F$ the midpoint of $[BC]$.
3) Determine the center $w$ of $S$.
4) Let $h = S \circ S$.
   a- Determine the nature of $h$ and precise its elements.
   b- Deduce that $h(A)$ is the symmetric of $A$ with respect to $C$.

**N° 10**

In the oriented plane, consider an isosceles triangle $ABC$ of vertex $A$ and such that : $(\overrightarrow{AB};\overrightarrow{AC}) = \frac{2\pi}{3}(\text{mod}\,2\pi)$.

$D$ and $E$ are two points on the segment $[BC]$ such that $\overrightarrow{BD} = \overrightarrow{DE} = \overrightarrow{EC}$ and $O$ be the midpoint of $[BC]$.

Let $f$ be the similitude verifying $f(B) = D$ and $f(E) = A$.

1) Prove that triangle $ADE$ is equilateral.

2) Determine the ratio $k$ and an angle $\alpha$ of $f$.

3) Let $(\Gamma)$ be the circle circumscribed about triangle $ADE$, $(\Gamma)$ cuts $(AB)$ at a point $w$ other than $A$.

   a- Show that $w$ is the center of $f$.

   b- Show that $f(D)$ is the point $F$ midpoint of $[AD]$.

   c- Determine the image $(\Gamma')$ of $(\Gamma)$ by $f$ and construct $(\Gamma')$.

4) Let $r$ be the rotation of center $D$ and angle $-\frac{\pi}{3}$ and let $g = r \circ f$.

   a- Determine $g(E)$ and $g(B)$.

   b- Determine the nature of $g$ and precise its characteristic elements.

**N° 11.**

In the oriented plane, consider a rectangle $ABCD$ such that:

$(\overrightarrow{AB};\overrightarrow{AD}) = \frac{\pi}{2}(\text{mod}\,2\pi)$, $AB = 4$ and $AD = 3$.

Let $H$ be the foot of the perpendicular issued through $A$ to $(BD)$.

Consider the dilation $h$ of center $H$ that transforms $D$ onto $B$.

1) a- Determine and trace the image of the straight line $(AD)$ by $h$.

   b- Deduce the image $E$ of point $A$ by $h$.

   c- Calculate the ratio of $h$.

   d- Construct the point $F$ image of $B$ by $h$ and the point $G$ image of $C$ by $h$ and determine the image of rectangle $ABCD$ by $h$.

   e- Calculate the area of the image of $ABCD$ by $h$.

2) Let $S$ be the direct similitude that transforms $A$ onto $B$ and $D$ onto $A$.

   a- Determine an angle of $S$.

   b- Determine the image of the straight line $(AH)$ by $S$ and the image of the straight line $(BD)$ by $S$.

   c- Deduce that $H$ is the center of $S$.

3) a- Show that $S(B) = E$ and deduce that $S \circ S(A) = h(A)$.

   b- Show that $S \circ S = h$ and deduce the point $F = S(E)$.

# Solution of Problems

**N° 1.**

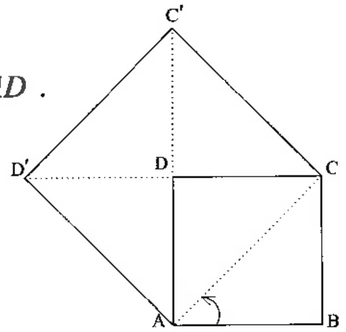
1) $S(A) = A$.

If $B' = S(B)$ then $\left(\overrightarrow{AB};\overrightarrow{AB'}\right) = \frac{\pi}{4} (\text{mod } 2\pi)$, hence $B' \in [AC)$.

$AB' = \sqrt{2}AB = AC$ then $B'$ is $C$, hence $S(B) = C$.

$C' = S(C)$ then $\left(\overrightarrow{AC};\overrightarrow{AC'}\right) = \frac{\pi}{4} (\text{mod } 2\pi)$, hence $C' \in [AD)$.

$AC' = \sqrt{2}AC = \sqrt{2} \times \sqrt{2}\ AD = 2AD$.

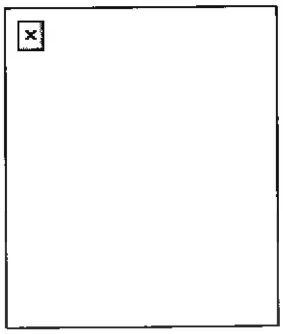



Therefore $C'$ is the symmetric of $A$ with respect to $D$.
If $D' = S(D)$ then $D'$ is the 4ths vertex of square $ACC'D'$.

2) Let $A' = S(A)$, we have:

$\left(\overrightarrow{OA},\overrightarrow{OA'}\right) = \frac{\pi}{4} (\text{mod } 2\pi)$

then $A' \in (ox)$ bisector of the angle $\left(\overrightarrow{OA},\overrightarrow{OB}\right)$.



$OA' = \frac{\sqrt{2}}{2}OA = \frac{\sqrt{2}}{2}\frac{1}{2}AC = \frac{\sqrt{2}}{4}\sqrt{2} = \frac{1}{2}$,

$OA' = \frac{1}{2}AB$ and consequently $A'$ is the midpoint of $[AB]$.

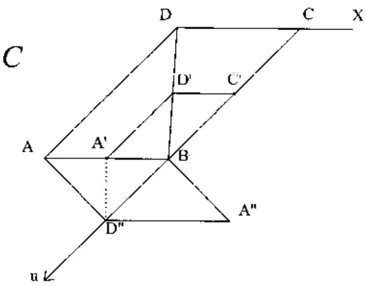
Similarly, the image of $B$ is the point $B'$ midpoint of $[BC]$.
The image of $C$ is the point $C'$ midpoint of $[CD]$ and the image of $D$ is the point $D'$ midpoint of $[AD]$.

Hence, the image of square $ABCD$ is the square $A'B'C'D'$.

N° 2.

1) a- $S(B)=C$, then:

$\left(\overrightarrow{DB};\overrightarrow{DC}\right)=\frac{\pi}{2}\ (\text{mod}\,2\pi)$, so $C$ belongs to the semi-straight line $[Dx)$ passing through $D$ and parallel to $(AB)$.




Taking $AD=\ell$.

$$DC=\frac{\sqrt{3}}{3}DB=\frac{\sqrt{3}}{3}\times\frac{\ell\sqrt{3}}{2}=\frac{\ell}{2}=AB.$$

Then $C$ belongs to circle $(\gamma)$ of center $D$ and radius $R=AB$.
The point $C$ is the intersection of the semi-straight line $[Dx)$ and of circle $(\gamma)$.

b- $DC=AB$ and $(CD)$ and $(AB)$ are parallel, then $ABCD$ is a parallelogram.

2) We know that :

$$S\left(B,\frac{1}{2},\frac{2\pi}{3}\right)=r\left(B,\frac{2\pi}{3}\right)\circ h\left(B,\frac{1}{2}\right).$$

Finding the image of $ABCD$ by $h\left(\text{B},\frac{1}{2}\right)$, we notice that

$h(B)=B$.
$h(C)=C'$ where $C'$ is the midpoint of $[BC]$.
$h(D)=D'$ where $D'$ the midpoint of $[BD]$.
$h(A)=A'$ where $A'$ is the midpoint of $[AB]$.

Hence, the image of the parallelogram $ABCD$ by $h\left(\text{B},\frac{1}{2}\right)$ is the parallelogram $A'BC'D'$.

Finding now the image of $A'BC'D'$ par $r\left(B,\frac{2\pi}{3}\right)$.

We notice that $r(B)=B$ .

If $C''=r(C')$ then $\left(\overrightarrow{BC'};\overrightarrow{BC''}\right)=\frac{2\pi}{3}$ (mod $2\pi$) . Hence $C''$ belongs to the semi-straight line $[BA)$ .

$BC''=BC'=\frac{1}{2}BC=\frac{1}{2}AD=\frac{\ell}{2}=BA$ .

Then $C''$ is the point $A$ and consequently $r(C')=A$ .

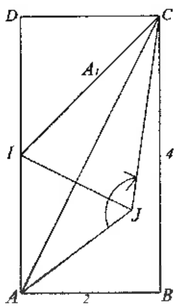
Let $D''=r(D')$ ,then:

$\left(\overrightarrow{BD'};\overrightarrow{BD''}\right)=\frac{2\pi}{3}$ (mod $2\pi$) , hence $D''$ belongs to the semi-straight line $[Bu)$ verifying $\left(\overrightarrow{BD};\overrightarrow{Bu}\right)=\frac{2\pi}{3}(\text{mod}\,2\pi)$.

$BD'=BD''$, then $D''$ belongs to circle $(\Gamma)$ of center $B$ and of radius $BD'$ hence $D''$ is the point of intersection , other than $B$, of $[Bu)$ and $(\Gamma)$ .

$A''$ is the 4$^{\text{th}}$ vertex of parallelogram $BAD''A''$ .

$N°3.$




1) If $A_1$ the image of $A$ by $r\left(I,\frac{3\pi}{4}\right)$ then

$\left(\overrightarrow{IA};\overrightarrow{IA_1}\right)=\frac{3\pi}{4}$ (mod $2\pi$) .

But $\left(\overrightarrow{IC};\overrightarrow{ID}\right)=\frac{\pi}{4}$ (mod $2\pi$) since triangle $DIC$ is right isosceles at $D$, so $\left(\overrightarrow{IA};\overrightarrow{IC}\right)=\frac{3\pi}{4}$ (mod $2\pi$) ,

hence, $A_1$ belongs to the semi-straight line $[IC)$ .

If $A'$ is the image of $A_1$ by $h(I,\sqrt{2})$ then $\overrightarrow{IA'}=\sqrt{2}\,\overrightarrow{IA_1}$

therefore $IA'=\sqrt{2}\,IA_1=\sqrt{2}\,IA=2\sqrt{2}$ and since

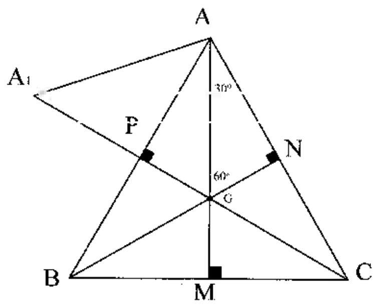
$IC^2 = ID^2 + DC^2 = 4 + 4 = 8$, then $IC = 2\sqrt{2}$ hence $IA' = IC$.
As a result, $A' = C$ and $S(A) = C$.

2) $S': A \longrightarrow C$, then $\beta = \left(\overrightarrow{JA};\overrightarrow{JC}\right) \pmod{2\pi}$ but

$\left(\overrightarrow{JA};\overrightarrow{JC}\right) = \left(\overrightarrow{IC};\overrightarrow{IA}\right) \pmod{2\pi}$ then $\beta = -\dfrac{3\pi}{4} \pmod{2\pi}$.

$k' = \dfrac{JC}{JA} = \dfrac{IC}{IA} = \sqrt{2}$.

**N°4.**




1) Determining $S(A)$:
$A_1$ is the image of $A$ by $r\left(G;\dfrac{\pi}{3}\right)$ then $GA_1 = GA$ and $\left(\overrightarrow{GA};\overrightarrow{GA_1}\right) = \dfrac{\pi}{3} \pmod{2\pi}$,
so $A_1 \in [GP)$.
Note that triangle $GAA_1$ is equilateral since $GA_1 = GA$
and $\left(\overrightarrow{GA};\overrightarrow{GA_1}\right) = \dfrac{\pi}{3} \pmod{2\pi}$ ,hence $P$ is the midpoint of $[GA_1]$

and consequently, $\overrightarrow{GP} = \dfrac{1}{2}\overrightarrow{GA_1}$, therefore $P$ is the image of $A_1$ by $h\left(G,\dfrac{1}{2}\right)$. Hence, $S(A) = P$.
Similarly, we have that $S(B) = M$ and $S(C) = N$.
Then the image of triangle $ABC$ by $S\left(G,\dfrac{1}{2},\dfrac{\pi}{3}\right)$ is triangle $PMN$.

2) $S(A) = A$.

Let $B_1 = S(B)$ by $S\left(A;\frac{\sqrt{3}}{3};\frac{\pi}{6}\right)$, we have $\left(\overrightarrow{AB};\overrightarrow{AB_1}\right)=\frac{\pi}{6}(\text{mod}\,2\pi)$

then $B_1$ belongs to the semi-straight line $[AM)$.

$AM = \frac{\sqrt{3}}{2}AB$ then $AB = \frac{2}{\sqrt{3}}AM$, therefore

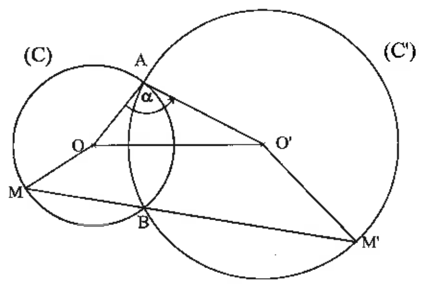
$AB_1 = \frac{\sqrt{3}}{3}AB = \frac{\sqrt{3}}{3}\times\frac{2AM}{\sqrt{3}} = \frac{2}{3}AM$, hence $S(B) = G$.

Triangle $ABM$ is a direct semi-equilateral triangle, then its image by $S$ is a direct semi-equilateral triangle, and since $S(A)=A$, $S(B)=G$ then the image of $ABM$ is the triangle $AGN$.

$\boxed{N^{\circ}5.}$

1) The similitude $S$ of center $A$ and that transforms $(C)$ onto $(C')$ should satisfy the relations: $S(A)=A$ and $S(O)=O'$.
The image of the segment $[OA]$ is the segment $[O'A]$.

The angle of this similitude is $\theta = \left(\overrightarrow{OA};\overrightarrow{O'A}\right)$ $(\text{mod}\,2\pi)$ and its ratio is $k = \frac{O'A}{OA} = \frac{r'}{r}$.




2) a- $S\begin{cases} O \longrightarrow O' \\ M \longrightarrow M' \\ A \longrightarrow A \end{cases}$ and since similitude preserves oriented angles, we deduce that $\left(\overrightarrow{OA};\overrightarrow{OM}\right)=\left(\overrightarrow{O'A};\overrightarrow{O'M'}\right)(\text{mod}\ 2\pi)$.

b- $M$ belongs to $(C)$ so :

$$2\left(\overrightarrow{BA};\overrightarrow{BM}\right)=\left(\overrightarrow{OA};\overrightarrow{OM}\right)\ (\text{mod } 2\pi).$$

(The measure of the inscribed angle is half the measure of the corresponding central angle ) then,

$$2\left(\overrightarrow{BM'};\overrightarrow{BA}\right)=\left(\overrightarrow{O'M'},\overrightarrow{O'A}\right)\ (\text{mod } 2\pi)$$

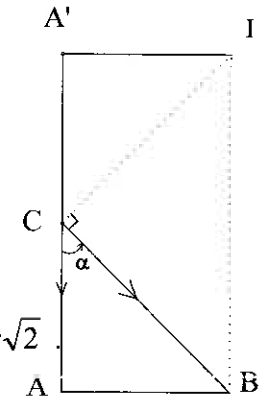
$$\left(\overrightarrow{BM'};\overrightarrow{BM}\right)=\left(\overrightarrow{BM'};\overrightarrow{BA}\right)+\left(\overrightarrow{BA};\overrightarrow{BM}\right)=$$

$$\frac{1}{2}\left(\overrightarrow{O'M'};\overrightarrow{O'A}\right)+\frac{1}{2}\left(\overrightarrow{OA};\overrightarrow{OM}\right)=$$

$$\frac{1}{2}\left[\left(\overrightarrow{O'M'};\overrightarrow{O'A}\right)+\left(\overrightarrow{O'A};\overrightarrow{O'M'}\right)\right](\text{mod } 2\pi)=0(\text{mod } 2\pi)$$

Hence, $\left(\overrightarrow{BM'};\overrightarrow{BM}\right)=0\ (\text{mod } 2\pi)$, consequently the points $M$, $B$ and $M'$ are collinear .

N°6.




1) $S:\begin{cases} A'\longrightarrow C \\ C \longrightarrow B\end{cases}$

From the characteristic properties of the similitude, we have : $k=\dfrac{CB}{A'C}$ .

$ABC$ is a right isosceles triangle of vertex at $A$, then:

$BC^2=AB^2+AC^2=2a^2$ which gives $BC=a\sqrt{2}$ .

Therefore, $k=\dfrac{CB}{A'C}=\dfrac{a\sqrt{2}}{a}=\sqrt{2}$

$$\alpha=\left(\overrightarrow{A'C};\overrightarrow{CB}\right)=\left(\overrightarrow{CA};\overrightarrow{CB}\right)=\frac{\pi}{4}\ (\text{mod } 2\pi).$$

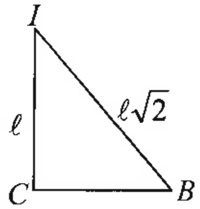
2) The straight line $(AC)$ and the straight line $(A'C)$ are confounded .

The image of a straight line by a similitude is a straight line .
Then, the image of straight line $(AC)$ is straight line $(BC)$ .

3) a- $I$ is the center of $S$ and since $S(C)=B$ we get :

$$IB=\sqrt{2}IC \text{ and } \left(\overrightarrow{IC};\overrightarrow{IB}\right)=\frac{\pi}{4} \pmod{2\pi} .$$

In triangle $IBC$ we have:




$BC^2=IB^2+IC^2-2IB\times IC\times\cos\hat{BIC}$.

If we take $IC=\ell$ then $IB=\ell\sqrt{2}$ so :

$BC^2=\ell^2+2\ell^2-2\ell\times\ell\sqrt{2}\times\cos\frac{\pi}{4}$ .

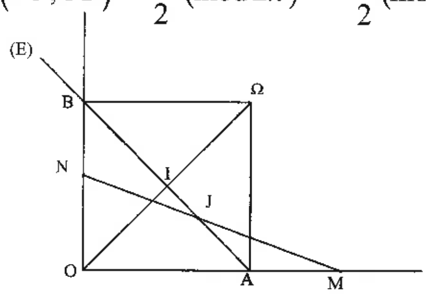
Hence, $BC^2=\ell^2$ and consequently $BC=\ell$ .
Therefore, triangle $IBC$ is isosceles of vertex $C$ ,
then $\hat{BIC}=\hat{IBC}=45^\circ$ and $\hat{ICB}=90^\circ$ , and consequently triangle $IBC$ is right isosceles at $C$ .

b- $\hat{ICB}=90^\circ$ , then the two straight lines $(IC)$ and $(CB)$ are perpendicular , then $I$ belongs to the straight line $(d)$ drawn through $C$ and perpendicular to $(CB)$.

Similarly $\hat{IBA}=\hat{IBC}+\hat{CBA}=45^o+45^o=90^o$ .
Then the point $I$ belongs to the straight line $(d')$ drawn through $B$ and perpendicular to $(AB)$.
Hence, the point $I$ is the point of intersection of the two straight lines $(d)$ and $(d')$ that are not distinct .

**N°7.**

1) a- $\left(\overrightarrow{MA};\overrightarrow{NB}\right)=\left(\overrightarrow{AO};\overrightarrow{OB}\right)=\frac{3\pi}{2}(\bmod 2\pi)=-\frac{\pi}{2}(\bmod 2\pi)$.



$\begin{cases} A \xrightarrow{r} B \\ M \xrightarrow{r} N \end{cases}$ then $\left(\overrightarrow{MA};\overrightarrow{NB}\right)$ is an angle of the rotation

$r$, whose angle has a measure $-\dfrac{\pi}{2}$.

b- The center $\Omega$ satisfies the following relations :

$$\begin{cases} \Omega A = \Omega B \\ \left(\overrightarrow{\Omega A};\overrightarrow{\Omega A}\right) = -\dfrac{\pi}{2} \pmod{2\pi} \end{cases}$$

As a result, we get that $A\Omega B$ is a right isosceles triangle at

$\Omega$ with $\left(\overrightarrow{\Omega A};\overrightarrow{\Omega B}\right) = -\left(\overrightarrow{OA};\overrightarrow{OB}\right) = -\dfrac{\pi}{2} \pmod{2\pi}$.

$OAB$ and $A\Omega B$ are two right isosceles triangles of a common hypotenuse $[AB]$, as a result, we get:

$\Omega A = \Omega B = OA = OB = AB\dfrac{\sqrt{2}}{2}$.

Quadrilateral $OA\Omega B$ having its four sides equal in length and a right angle is a square.

2) a- $M\Omega N$ is a right isosceles triangle at $\Omega$ since $r(M) = N$.

$J$ being the midpoint of $[MN]$, $\Omega JM$ is a right isosceles

triangle at $J$ with $\left(\overrightarrow{\Omega M};\overrightarrow{\Omega J}\right) = -\dfrac{\pi}{4} \pmod{2\pi}$.

Hence $\dfrac{\Omega J}{\Omega M} = \cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$.

$S$ is then the direct similitude of center $\Omega$, and angle $-\dfrac{\pi}{2}$

and ratio $\dfrac{\sqrt{2}}{2}$.

b- $S(A) = I$ since $\left(\overrightarrow{\Omega A};\overrightarrow{\Omega I}\right) = -\dfrac{\pi}{4} \pmod{2\pi}$ and

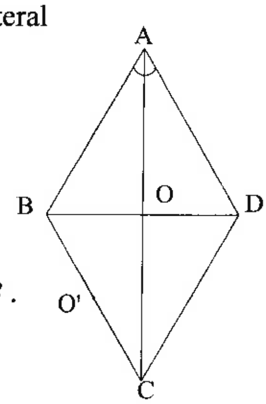
$\Omega I = \dfrac{1}{2}\Omega O = \dfrac{1}{2}\Omega A \times \sqrt{2} = \dfrac{\sqrt{2}}{2}\Omega A$.

Similarly, we have $S(O)=B$.
$M$ describes the straight line $(OA)$ which is the straight line $(AB)$, hence $J$ describes the straight line $(\delta)$, image of $(OA)$ by $S$.
The set $(E)$ of points $J$ as $M$ describes the straight line $(OA)$ is the straight line $(AB)$.

**N°8.**

1) $ABCD$ being a rhombus, triangle $ABD$ is isosceles, but since $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{3}\ (\text{mod}\ 2\pi)$, it is equilateral.

Hence $AO=a\frac{\sqrt{3}}{2}$,(height of an equilateral triangle of side of length a ).
$AC=2AO=a\sqrt{3}$.




2) a- Since $\left(\overrightarrow{CA};\overrightarrow{CB}\right)=\frac{\pi}{6}(\text{mod}\ 2\pi)$ and $CB=a=\frac{CA\sqrt{3}}{3}$ then $S(A)=B$.
b- $O$ is the midpoint of $[AC]$ since $ABCD$ is a rhombus, hence $O'=S(O)$ is the midpoint of the image segment.
But $S(A)=B$ and $S(C)=C$ then $O'$ is the midpoint of $[BC]$.
3) a- Let $D'=S(D)$.
From the definition of $S$ we know that $\left(\overrightarrow{CD};\overrightarrow{CD'}\right)=\frac{\pi}{6}(\text{mod}\ 2\pi)$,
but $\left(\overrightarrow{CD};\overrightarrow{CA}\right)=\frac{\pi}{6}(\text{mod}\ 2\pi)$ since triangle $BCD$ is direct equilateral direct, and $(CA)$ bisector issued from $C$, hence

$\left(\overrightarrow{CD'};\overrightarrow{CA}\right) = 0 \pmod{2\pi}$ and $D'$ belongs to $[CA)$.

b- $S(O) = O'$ and $S(D) = D'$ then $\left(\overrightarrow{OD};\overrightarrow{O'D'}\right) = \frac{\pi}{6}\pmod{2\pi}$.

c- $\left(\overrightarrow{BC};\overrightarrow{O'D'}\right) = \left(\overrightarrow{BC};\overrightarrow{OD}\right) + \left(\overrightarrow{OD};\overrightarrow{O'D'}\right)\pmod{2\pi}$.

$O$ is the midpoint of $[BD]$ hence:

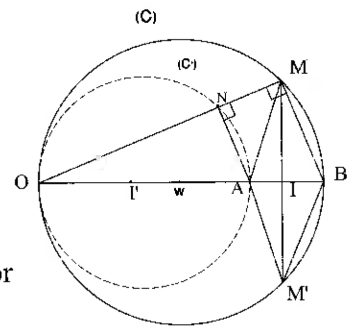
$\left(\overrightarrow{BC};\overrightarrow{OD}\right) = \left(\overrightarrow{BC};\overrightarrow{BD}\right) = \frac{\pi}{3}\pmod{2\pi}$ since $BCD$ is direct equilateral, hence

$\left(\overrightarrow{BC};\overrightarrow{O'D'}\right) = \frac{\pi}{3} + \frac{\pi}{6}\pmod{2\pi} = \frac{\pi}{2}\pmod{2\pi}$

consequently $(O'D')$ is perpendicular to $(BC)$.

4) The point $D'$ is on the straight line $(OC)$ that is the perpendicular bisector of $[BD]$.
$(O'D')$ is perpendicular to $(BC)$ and $O'$ is the midpoint of $[BC]$, then $D'$ belongs to the perpendicular bisector of $[BC]$.
$D'$ is then the point of intersection of the perpendicular bisectors of triangle $BCD$, It is hence the circumcenter of triangle $BCD$.

$N°9.$




1) a- $(MM')$ is the perpendicular bisector of $[AB]$ then $MA = MB$ and $M'A = M'B$. Triangle $\omega MM'$ is isosceles at $\omega$ and since $(\omega I)$ is perpendicular to $(MM')$ then $(\omega I)$ is the perpendicular bisector of $[MM']$, therefore $AM = AM'$ and consequently $AMBM'$ is a rhombus.

b- $(AM')$ is parallel to $(MB)$ that is perpendicular to $(OM)$.
Hence, $(AM')$ is perpendicular to $(OM)$.
But since the straight lines $(AN)$ and $(AM')$ are perpendicular to $(OM)$ and have a common point $A$ then they are confounded and consequently, the points $M'$, $A$ and $N$ are collinear.

2) a- $S(N) = N$ and $S(M) = A$, then the angle of $S$ is
$$\left(\overrightarrow{NM};\overrightarrow{NA}\right) = -\frac{\pi}{2} \pmod{2\pi}$$

b- Since $S(M)$ is $A$ and the angle of $S$ is $-\frac{\pi}{2}$, the image by $S$ of the straight line $(MI)$ is the straight line perpendicular to $(MI)$ passing through $A$, it is the straight line $(OA)$.
Similarly, since $N$ is the center of $S$, the image by $S$ of the straight line $(NA)$ is the straight line perpendicular to $(NA)$ passing through $N$, which is $(ON)$.

c- $M'$ is the point of intersection of the straight lines $(MI)$ and $(NA)$. Hence, its image by $S$ is the point of intersection of the straight lines images of $(MI)$ and $(NA)$, that is the point of intersection of $(OA)$ and $(ON)$ therefore $S(M') = O$.

3) a- $I$ is the midpoint of segment $[MM']$, then its image by $S$ is the midpoint of the image of $[MM']$, that is the midpoint $I'$ of $[AO]$.

b- $S(N) = N$ and $S(I) = I'$ then $S((NI)) = (NI')$.
$S$ has an angle $-\frac{\pi}{2}$ consequently, the straight lines $(NI)$ and $(NI')$ are perpendicular at $N$ but since $I'$ is the center of the circle $(C')$ of diameter $[OA]$ and $N$ is a point of $(C')$, then the straight line $(NI)$ is tangent to $(C')$.

N° 10.

1) The similitude $S_A$ has a ratio $\frac{AC}{AK}$ and an angle

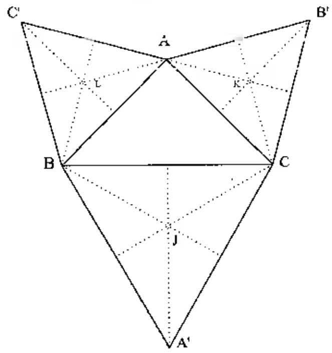
$$\left(\overrightarrow{AK};\overrightarrow{AC}\right) = -\frac{\pi}{6}(\text{mod}\,2\pi).$$

$AB'C$ being equilateral, each of its three heights has a length of $\frac{\sqrt{3}}{2}AC$ , also, $K$ is at the same time the orthocenter and the center of gravity of the triangle, hence $AK = \frac{2}{3}\times\frac{\sqrt{3}}{2}AC = \frac{1}{\sqrt{3}}AC$,

therefore $\frac{AC}{AK} = \sqrt{3}$ .

Hence, the ratio of $S_A$ is $\sqrt{3}$ .

Finally, $S_A$ is the similitude of center $A$ , ratio $\sqrt{3}$ and angle $-\frac{\pi}{6}$ .




2) Following the same reasoning as in the previous question on triangle $A'BC$, equilateral indirect and of centroid $J$, we get :

$$\left(\overrightarrow{BC};\overrightarrow{BJ}\right) = -\frac{\pi}{6}(\text{mod}\,2\pi) \text{ and } BJ = \frac{1}{\sqrt{3}}BC .$$

Consequently, $S_B$ is the similitude of center $B$ , ratio $\frac{\sqrt{3}}{3}$ and angle $-\frac{\pi}{6}$ .

3) $S_A\left(A;\sqrt{3};-\frac{\pi}{6}\right)\circ S_B\left(B;\frac{1}{\sqrt{3}};-\frac{\pi}{6}\right) = S\left(w;1;-\frac{\pi}{3}\right) = r\left(w;-\frac{\pi}{3}\right).$

It is left to prove that $L$ is the center of $S_B \circ S_A$; it is sufficient

Thus to prove that $L$ is invariant under $S_B \circ S_A$.

$S_A(L) = C'$ since $\left(\overrightarrow{AL};\overrightarrow{AC'}\right) = -\frac{\pi}{6}(\text{mod}\,2\pi)$ and $AC' = \sqrt{3}\,AL$.

Similarly $S_B(C') = L$ since $\left(\overrightarrow{BC'};\overrightarrow{BL}\right) = -\frac{\pi}{6}(\text{mod}\,2\pi)$ and

$BL = \frac{1}{\sqrt{3}}BC'$ , which gives that $S_B \circ S_A(L) = S_B(C') = L$.

Hence $L$ is the center of the rotation $S_B \circ S_A$.

4) $S_B \circ S_A(K) = S_B(C) = J$ and since $S_B \circ S_A$ is the rotation of center $L$ and angle $-\frac{\pi}{3}$ we get :

$LK = LJ$ and $\left(\overrightarrow{LK};\overrightarrow{LJ}\right) = -\frac{\pi}{3}(\text{mod}\,2\pi)$ .

This proves that triangle $LKJ$ is equilateral indirect, and that triangle $JKL$ is equilateral direct.

**N°11.**

1) a- Let $M$ and $N$ be two variable points of the plane , we have:

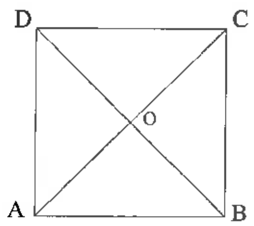
$M \xrightarrow{r} M' \xrightarrow{t} M''$

$N \xrightarrow{r} N' \xrightarrow{t} N''$

then $\begin{cases} M'N' = MN \\ \left(\overrightarrow{MN};\overrightarrow{M'N'}\right) = \frac{\pi}{2}\ (\text{mod}\,2\pi) \end{cases}$ and $\overrightarrow{M''N''} = \overrightarrow{M'N'}$ ;

Therefore : $M''N'' = MN$ and $\left(\overrightarrow{MN};\overrightarrow{M''N''}\right) = \frac{\pi}{2}\ (\text{mod}\,2\pi)$ .

Hence $t \circ r$ is a rotation of angle $\frac{\pi}{2}$.




b- $A \xrightarrow{r} A \xrightarrow{t_{\overrightarrow{AB}}} B$

$B \xrightarrow{r} D \xrightarrow{t_{\overrightarrow{AB}}} C$

then $A \xrightarrow{r'} B$

$B \xrightarrow{r'} C$

Hence the center of $r'$ is on the perpendicular bisector of $[AB]$ and on the perpendicular bisector of $[BC]$, hence it is $O$.

Therefore $r' = r'\left(O; \frac{\pi}{2}\right)$.

2) a- The composite of a dilation of positive ratio and a rotation is a direct plane similitude whose ratio is $\sqrt{3}$, the ratio of $h$ and angle that of $r$ hence it is $\frac{\pi}{2}$.

b- $h(C) = C$.

$r'(C) = D$ since $OC = OD$ and $\left(\overrightarrow{OC}; \overrightarrow{OD}\right) = \frac{\pi}{2} \pmod{2\pi}$.

Then $C \xrightarrow{S\left(I, \sqrt{3}, \frac{\pi}{2}\right)} D$ hence $\left(\overrightarrow{IC}; \overrightarrow{ID}\right) = \frac{\pi}{2} \pmod{2\pi}$

and $ID = \sqrt{3}\, IC$.

c- $\left(\overrightarrow{IC}; \overrightarrow{ID}\right) = \frac{\pi}{2} \pmod{2\pi}$, then $I$ belongs to the semi-circle of diameter $[CD]$ containing the point $O$ since $\left(\overrightarrow{OC}; \overrightarrow{OD}\right) = \frac{\pi}{2} \pmod{2\pi}$.

Triangle $ICD$ is right at $I$ then $\tan\left(D\hat{C}I\right) = \frac{DI}{IC} = \sqrt{3}$, hence $\widehat{DCI} = \frac{\pi}{3}$ and consequently $I$ belongs to the semi-straight line $[Cx)$ such that $\left(\overrightarrow{CD}; \overrightarrow{Cx}\right) = \frac{\pi}{3} \pmod{2\pi}$ hence $I$ is the point of intersection of $[Cx)$ with the semi- circle.

N° 12.

1) a- The image of a straight line by a rotation is a straight line .

Since $AC = AD$ and $\left(\overrightarrow{AC};\overrightarrow{AD}\right) = \frac{\pi}{2} \pmod{2\pi}$ then

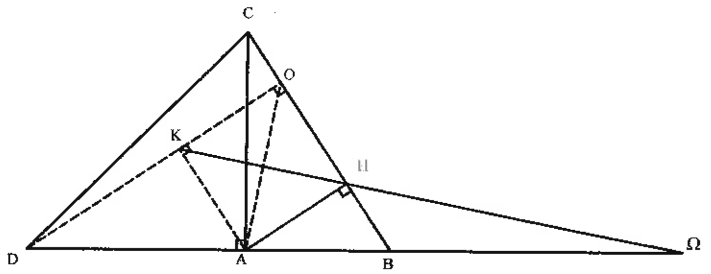
$r(C) = D$, hence the image of $(CB)$ is the straight line passing through $D$ and perpendicular to $(CB)$ hence it is the straight line $(DO)$.

b- $r(A) = A$ and $r(C) = D$.

Finding $r(H)$.

$H \in (BC)$ then its image $r(H)$ belongs to the straight line $(OD)$. $H$ is the orthogonal projection of $A$ on $(BC)$ then $r(H)$ is the orthogonal projection of $A$ on $(DO)$ hence it is the point $K$.

Consequently, the image of triangle $AHC$ is triangle $AKD$.




c- quadrilateral $AHOK$ is a rectangle .

$r(H) = K$ gives $AH = AK$ ; then $AHOK$ is a square .

2) In triangle $\Omega AH$, the straight lines $(KD)$ and $(HA)$ are parallel . There exists a unique dilation $h$ of center $\Omega$ such that

$h(D) = A$ and $h(K) = H$ of ratio $\lambda = \frac{HA}{KD}$.

But since $(AK)$ is parallel to $(BH)$, there exists a unique dilation $h'$ of center $\Omega$ such that $h'(A) = B$ and $h'(K) = H$ of ratio

$\lambda' = \frac{BH}{KA}$.

But $\frac{BH}{AK}=\frac{\Omega B}{\Omega A}=\frac{\Omega H}{\Omega K}$ and $\frac{HA}{KD}=\frac{\Omega H}{\Omega K}=\frac{\Omega A}{\Omega D}$ then: $\frac{BH}{AK}=\frac{HA}{KD}$

and consequently $h=h'$, hence by this dilation image of triangle $AKD$ is triangle $BHA$.

3) a- $S(H)=(h\circ r)(H)$ but $r(H)=K$, then $S(H)=h(K)$ and since $h(K)=H$ then $S(K)=H$.

$S(C)=(h\circ r)(C)$, but $r(C)=D$, then $S(C)=h(D)$ and since $h(D)=A$ then $S(C)=A$.

$S(A)=(h\circ r)(A)$ but $r(A)=A$, therefore $S(A)=h(A)$ and since $h(A)=B$ then $S(A)=B$.

b- The transformation $S$, composite of a rotation and a dilation is a direct plane similitude.

But since $S(H)=H$ then $H$ is the center of $S$.

$S(C)=A$ and $S(A)=B$, then $AB=kCA$ and

$\left(\overrightarrow{CA};\overrightarrow{AB}\right)=\alpha \pmod{2\pi}$ ; therefore:

$k=\frac{AB}{CA}$ and $\alpha=\frac{\pi}{2} \pmod{2\pi}$.

$S$ is the similitude of center $H$, ratio $\frac{AB}{AC}$ and angle $\frac{\pi}{2}$.

**$N^\circ$ 13.**

1) a- Let $\alpha$ be a measure of the angle of $S$, we have :

$S:\begin{cases}A\longrightarrow A\\ C\longrightarrow D\end{cases}$ then $\left(\overrightarrow{AC};\overrightarrow{AD}\right)=\alpha \pmod{2\pi}$.

$S:\begin{cases}C\longrightarrow D\\ B\longrightarrow E\end{cases}$ then $\left(\overrightarrow{CB};\overrightarrow{DE}\right)=\alpha \pmod{2\pi}$.

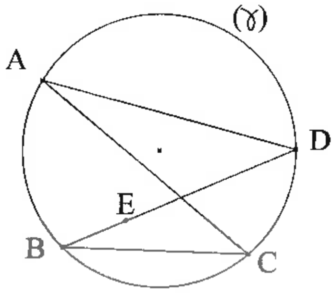
Hence $\left(\overrightarrow{CB};\overrightarrow{DE}\right)=\left(\overrightarrow{AC};\overrightarrow{AD}\right) \pmod{2\pi}$.

b- $\left(\overrightarrow{AC};\overrightarrow{AD}\right)=\left(\overrightarrow{BC};\overrightarrow{BD}\right) \pmod{2\pi}$ ,( inscribed angles intercepting the same arc).

$\left(\overrightarrow{CB};\overrightarrow{DE}\right)=\left(\overrightarrow{AC};\overrightarrow{AD}\right)$ (mod $2\pi$) we get

$\left(\overrightarrow{BC};\overrightarrow{BD}\right)=\left(\overrightarrow{CB};\overrightarrow{DE}\right)=\left(\overrightarrow{BC};\overrightarrow{ED}\right)$ (mod $2\pi$) ;

Hence $\overrightarrow{BD}$ and $\overrightarrow{ED}$ are collinear and of the same sense.
Consequently, $E \in (BD)$.




c- Let $k$ be the ratio of $S$.

We have $S(A)=A$ and $S(C)=D$, then $\frac{AD}{AC}=k$, similarly

$S(C)=D$ and $S(B)=E$ then $\frac{DE}{CB}=k$.

Therefore $\frac{AD}{AC}=\frac{DE}{CB}$, which gives $AD\times BC=DE\times AC$.

2) a- We have:

$S:\begin{cases} A\longrightarrow A \\ B\longrightarrow E \\ C\longrightarrow D \end{cases}$ then $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\left(\overrightarrow{AE};\overrightarrow{AD}\right)$ (mod $2\pi$)

since similitudes preserve measures of oriented angles.

Similarly, we have $\frac{AC}{AB}=\frac{AD}{AE}$ since similitudes preserve ratio of distances.

b- $S':\begin{cases} A \longrightarrow A \\ B \longrightarrow C \end{cases}$ then the ratio of S' is $k' = \dfrac{AC}{AB}$,

and a measure of the angle of $S'$ is $\beta = \left(\overrightarrow{AB};\overrightarrow{AC}\right)(\text{mod}\,2\pi)$.

$\dfrac{AD}{AE} = \dfrac{AC}{AB} = k'$, $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \left(\overrightarrow{AE};\overrightarrow{AD}\right) = \beta\,(\text{mod}\,2\pi)$

Hence, $S'(E) = D$.

c- $S':\begin{cases} A \longrightarrow A \\ B \longrightarrow C \\ E \longrightarrow D \end{cases}$, then $\dfrac{AC}{AB} = k'$ and $\dfrac{CD}{BE} = k'$, which gives

$\dfrac{AC}{AB} = \dfrac{CD}{BE}$ and consequently $AB \times CD = AC \times BE$.

3) $AC \times BD = AC \times (BE + ED) = AC \times BE + AC \times ED$.
Therefore : $AC \times BD = AB \times CD + AD \times BC$.

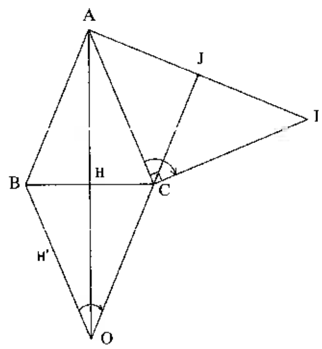
**N° 14.**

1) a- $f(A) = r_C \circ r_A(A) = r_C(r_A(A)) = r_C(A)$

but $r_C(A) = I$ since $CA = CI$ and $\left(\overrightarrow{CA};\overrightarrow{CI}\right) = -\dfrac{\pi}{2}$ (mod $2\pi$),

therefore $f(A) = I$.

$f(B) = r_C \circ r_A(B) = r_C(r_A(B)) = r_C(C) = C$ then $f(B) = C$.




b- $f$ is the composite of two rotations whose angle is the sum of the angles , $-\frac{\pi}{2}+\frac{\pi}{4}=-\frac{\pi}{4}$ .

As a result, $f$ is a rotation of angle $-\frac{\pi}{4}$ .
Since $f(A)=I$ then $OA=OI$, hence $O$ belongs to the perpendicular bisector of $[AI]$ .
Similarly, $f(B)=C$ then $OB=OC$ and $O$ belongs to the perpendicular bisector of$[BC]$ .
Consequently, $O$ is the point of intersection of these two perpendicular bisectors .

c- Triangle $ACI$ is isosceles at $C$ then the perpendicular bisector of $[AI]$ passes through $C$ and $[CJ)$is a bisector of angle $\hat{ACI}$ which gives that $\hat{ACJ}=\frac{\pi}{4}$ .

Angle $\hat{ACB}=\frac{1}{2}\left(\pi-\hat{CAB}\right)=\frac{1}{2}\left(\pi-\frac{\pi}{4}\right)=\frac{3\pi}{8}$ , then angle $\hat{OCB}=\pi-\left(\frac{\pi}{4}+\frac{3\pi}{8}\right)=\frac{3\pi}{8}$ .

In triangle $ACO$ the straight line$(BC)$ is at the same time the bisector then the triangle is isosceles and consequently $CA=OC$ and since $OC=OB$ , $(AO)$ is the perpendicular bisector of $[BC]$ , we deduce that $AB=AC=OC=OB$ and consequently $ACOB$ is a rhombus .

2) a- $S(A)=B$ then the angle of $S$ is $\left(\overrightarrow{OA};\overrightarrow{OB}\right)=\frac{\pi}{8}\ (\mathrm{mod}\ 2\pi)$

since $[OA)$ is the bisector of angle $\hat{COB}=\hat{BAC}=45^o$ .
$(OC)$ passes through $O$, center of $S$, and since the image of a straight line is a straight line then the image of $(OC)$is the straight line passing through $O$ and making an angle of $\frac{\pi}{8}$ with $(OC)$ then it is the straight line $(OA)$ and consequently $C'$ belongs to $(OA)$ .

b- $S(O)=O$ and $S(A)=B$, then the image of segment $[OA]$ by $S$ is $[OB]$.
$ABOC$ is a rhombus then the midpoint $H$ of $[BC]$ is also the midpoint of $[OA]$.
Similitude preserve midpoints , consequently $H'=S(H)$ is the midpoint of $[OB]$.

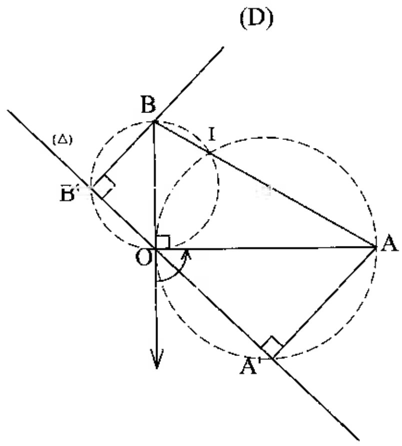
c- $(CH)$ and $(OA)$ are perpendicular then their images by $S$ are perpendicular, consequently $(C'H')$ and $(OB)$ are perpendicular .
$H'$ being the midpoint of $[OB]$ , the straight line $(C'H')$ is the perpendicular bisector of $[OB]$.
On the other hand, $C'$ belongs to $(OA)$ perpendicular bisector of $[BC]$. Hence the point $C'$ is the point of intersection of the two perpendicular bisectors of triangle $OBC$ , it is then the center of the circle circumscribed about triangle $OBC$ .

N°15.

1)

$$S:\begin{cases} B \longrightarrow O \\ O \longrightarrow A \end{cases} \text{ , so : } k=\frac{OA}{BO}=\tan\hat{B}=\tan\frac{\pi}{3}=\sqrt{3}$$

and $\alpha=\left(\overrightarrow{BO};\overrightarrow{OA}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$ .



2) a- $S\left(I;\sqrt{3};\frac{\pi}{2}\right)$ : $\begin{cases} B \longrightarrow O \\ O \longrightarrow A \end{cases}$ so :

$\left(\overrightarrow{IB};\overrightarrow{IO}\right)=\frac{\pi}{2}\ (\text{mod}\,2\pi)$ and $\left(\overrightarrow{IO};\overrightarrow{IA}\right)=\frac{\pi}{2}\ (\text{mod}\,2\pi)$.

Hence, the point $I$ belongs to the circle of diameter $[OB]$ and circle of diameter $[OA]$ , these two circles intersect at two points of which one of them is $O$ , and since $S(O)=A$, then, the center is the second point of intersection of the two circles.

b- $\hat{OIB}=90^o$ and $\hat{OIA}=90^o$ (angles inscribed in a semi-circle ). Then the points $B$ ,$I$ and $A$ are collinear , and consequently, $I$ is the foot of the perpendicular issued from $O$ in triangle $OAB$ .

3) a- The image of a straight line by a similitude is a straight line. The straight line $(D)$ passing through $B$, and since $S(B)=O$, then the image of $(D)$ is the straight line passing through $O$ and perpendicular to $(D)$ so this straight line is $(\Delta)$ .
The straight line $(\Delta)$ passes through $O$ and since $S(O)=A$, then the image of $(\Delta)$ by $S$ is the straight line passing through $A$ and perpendicular to $(\Delta)$ , it is then the straight line $(AA')$.

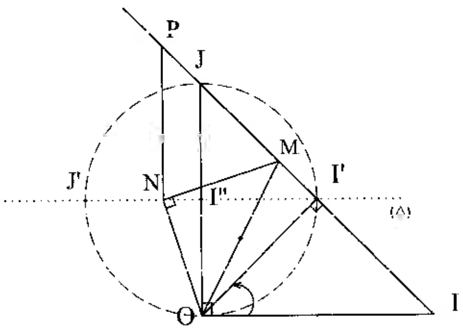
b- The point $B'$ is the point of intersection of the two straight lines $(D)$ and $(\Delta)$, then $S(B')$ is the point of intersection of the two straight lines $(\Delta)$ and $(AA')$ so it is the point $A'$.

c- $S(B')=A'$, then $\left(\overrightarrow{IB'};\overrightarrow{IA'}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$ , so $I$ belongs to the circle of diameter $[A'B']$ and consequently the circle of $[A'B']$ passes through the fixed point $I$ that is the center of $S$.

**N°16.**

1) $k=\frac{ON}{OM}=\sin \hat{OMN}=\sin 45^o=\frac{\sqrt{2}}{2}$,

$\alpha=\left(\overrightarrow{OM};\overrightarrow{ON}\right)=\frac{\pi}{4}\ (\text{mod}\ 2\pi)$ ,




2) a- We have already proved that if $N = S(M)$ then triangle $OMN$ is direct right isosceles of vertex $N$ .Then, if $I'$ is the image of $I$ by $S$ then, triangle $OII'$ is direct right isosceles of vertex $I'$, Therefore,

$\left(\overrightarrow{OI};\overrightarrow{OI'}\right) = \left(\overrightarrow{II'};\overrightarrow{IO}\right) = \frac{\pi}{4}(\text{mod } 2\pi)$. So, $I' \in [IJ)$ and

$I' \in [Ox)$ bisector of angle $\left(\overrightarrow{OI};\overrightarrow{OJ}\right)$ and consequently $I'$ is the midpoint of $[IJ]$ .

$S \circ S(I) = S(I') = I''$ where $I''$ is the midpoint of $[OJ]$ .

b- $J'$ is the image of $J$ by $S$ . The triangle $OJJ'$ is direct right isosceles of vertex $J'$. Therefore, $J'$ belongs to circle $(C)$ of diameter $[OJ]$ and $J'$ belongs to the perpendicular bisector $(\Delta)$ of $[OJ]$ , consequently $J'$ is the point of intersection of $(C)$ and $(\Delta)$ and verifies the relation $\left(\overrightarrow{OI};\overrightarrow{OJ'}\right) = \frac{\pi}{4}(\text{mod } 2\pi)$ .

The straight line $(\Delta)$ is then the straight line $(I'J')$.

3) $S' \circ S$ is a similitude of ratio $\sqrt{2} \times \frac{\sqrt{2}}{2} = 1$ and angle

$-\frac{\pi}{4} + \frac{\pi}{4} = 0(\text{mod } 2\pi)$ , then $S' \circ S$ is a translation .

but $I \xrightarrow{S} I' \xrightarrow{S'} I'$ then $S' \circ S$ is the translation of vector $\overrightarrow{II'}$.

Similarly, $S \circ S'$ is a translation , and since :
$I' \xrightarrow{S'} I' \xrightarrow{S} I''$ then $S \circ S'$ is the translation of vector $\overrightarrow{I'I''}$.

N° 17.

1) $S(B) = A$ and $S(A) = C$ then:

$$k = \frac{AC}{BA} = \frac{1+\sqrt{5}}{2} \text{ and } \alpha = \left(\overrightarrow{BA}; \overrightarrow{AC}\right) = -\frac{\pi}{2} \ (\text{mod } 2\pi).$$

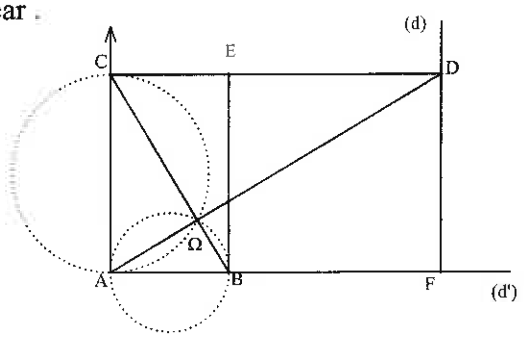
2) $\Omega$ is the center of $S$.

$S(B) = A$ gives $\left(\overrightarrow{\Omega B}; \overrightarrow{\Omega A}\right) = -\frac{\pi}{2}$ (mod $2\pi$) then $\Omega$ belongs to circle $(C_1)$ of diameter $[AB]$.

Similarly, $S(A) = C$ gives $\left(\overrightarrow{\Omega A}; \overrightarrow{\Omega C}\right) = -\frac{\pi}{2}$ (mod $2\pi$), then $\Omega$ belongs to circle $(C_2)$ of diameter $[AC]$.

$(C_1)$ and $(C_2)$ intersect in two points of which one is $A$.
$A$ is not invariant under $S$ since $S(A) = C$, then the center $\Omega$ of $S$ is the second point of intersection of $(C_1)$ and $(C_2)$.

Note that $\widehat{A\Omega B} = \widehat{A\Omega C} = 90^o$ , hence the points $B$ , $\Omega$ , and $C$ are collinear .




3) a- $S(C) = D$ then $\left(\overrightarrow{\Omega C}; \overrightarrow{\Omega D}\right) = -\frac{\pi}{2} (\text{mod}\, 2\pi)$.

$$\left(\overrightarrow{\Omega A}; \overrightarrow{\Omega D}\right) = \left(\overrightarrow{\Omega A}; \overrightarrow{\Omega C}\right) + \left(\overrightarrow{\Omega C}; \overrightarrow{\Omega D}\right) = -\frac{\pi}{2} - \frac{\pi}{2} (\text{mod}\, 2\pi) ;$$

Therefore, $\left(\overrightarrow{\Omega A};\overrightarrow{\Omega D}\right)=-\pi\,(\text{mod}\,2\pi)$ and consequently,

the points $\Omega$ , $A$ and $D$ are collinear .
The similitude $S$ transforms the straight line $(AB)$ onto the straight line $(AC)$ and the straight line $(AC)$ onto the straight line $(CD)$ hence $(AB)$ and $(AC)$ are perpendicular and $(AC)$ and $(CD)$ are perpendicular. Consequently, $(AB)$ and $(CD)$ are parallel .

b- $D$ is the point of intersection of the straight line $(\Omega A)$ and the straight line parallel to $(AB)$ passing through $C$ .

c- $S(A)=C$ and $S(C)=D$ , then $CD = k\,AC$ where $k$ is the ratio of $S$ .

Therefore, $CD=\dfrac{1+\sqrt{5}}{2}\left(1+\sqrt{5}\right)=3+\sqrt{5}$ .

4) a- The point $E\in(CD)$ then $F$ belongs to the straight line $(d)$ image of $(CD)$ by $S$ .
But $(CD)$ passes through $C$ hence $(d)$ passes through the point $D=S(C)$ and $(d)$ is perpendicular to $(CD)$.
On the other hand, $E\in(BE)$ then $F$ belongs to the straight line $(d')$ image of $(BE)$ by $S$ .
$(d')$ passes through the point $A$ , image of $B$ by $S$, and it is perpendicular to $(BE)$ hence $(d')$ is the straight line $(AB)$.
Therefore, $F$ is the point of intersection of $(d)$ and $(AB)$.

b- The straight lines $(BF)$ and $(ED)$ are parallel, the straight lines $(BE)$ and $(FD)$ are perpendicular to $(BF)$ then $BFDE$ is a rectangle .
Since $BE = AC = 1+\sqrt{5}$
$DE = CD - CE = CD - AB = 3+\sqrt{5}-2 = 1+\sqrt{5}$ then $BEDF$ is a square .

# Indications

**N°1.**

1) The ratio $k = \frac{IJ}{OA} = \frac{\sqrt{2}}{4}$ and the angle of $S$ is $\left(\overrightarrow{OA};\overrightarrow{IJ}\right) = -\frac{3\pi}{4}(\text{mod}\,2\pi)$.
2) Remark that $\frac{IC'}{OC} = \frac{\sqrt{2}}{4}$ and that $\left(\overrightarrow{OC};\overrightarrow{IC'}\right) = -\frac{3\pi}{4}(\text{mod}\,2\pi)$.

**N°2.**

1) The ratio $k = \frac{1}{2}$ and the angle of $S$ is $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \frac{\pi}{2}(\text{mod}\,2\pi)$.
2) The image of circle $(C_B)$ by $S$ is $(C_C)$ since $S(B) = C$ and the radius of $(C_C)$ is half the radius of $(C_B)$.
3) a- $S(E) = D$ then $E = S^{-1}(D)$, $D$ belongs to $(C_C)$ then $E$ belongs to $(C_B)$.

**N°3.**

1) $S_C \circ S_A(C') = A'$. $S_C \circ S_A(B') = A$.
2) $S_C \circ S_A = r\left(w;\frac{\pi}{2}\right)$ then $AA' = B'C'$ and $(AA')$ and $(B'C')$ are perpendicular .

**N°6.**

1) $S(A) = A'$.
2) $S(G) = B$ since $\left(\overrightarrow{CG};\overrightarrow{CB}\right) = \frac{\pi}{6}(\text{mod}\,2\pi)$ and $CG = \frac{2}{3}CI = \frac{2}{3} \times \frac{CB\sqrt{3}}{2}$.

## Indications

**N° 7.**

1) $k=\frac{AD}{AB}=\frac{1}{\cos 30^{\circ}}=\frac{2}{\sqrt{3}}$, $\alpha=(\overrightarrow{CB};\overrightarrow{AD})=\frac{\pi}{2}(\text{mod }2\pi)$.

2) Prove that $f(O)$ is the symmetric of $O$ with respect to $D$.

3) a- $f\circ r(A)=f(C)=A$.

b- $f\circ r=S\left(w;k;\frac{\pi}{2}\right)\circ S\left(B;1;-\frac{\pi}{3}\right)$.

**N° 8.**

2) a- $(AB)$ passes through $A$ then its image $(d)$ passes through $B$ and perpendicular to $(AB)$.

$(AC)$ passes through $C$ then its image $(d')$ passes through $H$ and perpendicular to $(AC)$ then it is $(BH)$.

b- $B$ belongs to $(AB)$ then $B'$ belongs to $(d)$.

$B'=S(B)$ and $H=S(C)$ then $(B'H)$ is perpendicular to $(BC)$. Consequently, $B'$ is the intersection of $(d)$ and of the perpendicular through $H$ to $(BC)$.

**N° 10.**

2) $\frac{DA}{BE}=\frac{DE}{BE}=\frac{1}{2}$, $(\overrightarrow{BE};\overrightarrow{DA})=\frac{\pi}{3}(\text{mod }2\pi)$.

3) a-Show that $(\overrightarrow{wB};\overrightarrow{wD})=\frac{\pi}{3}(\text{mod }2\pi)$ and $(\overrightarrow{wE};\overrightarrow{wA})=\frac{\pi}{3}(\text{mod }2\pi)$.

b- $D$ is the midpoint of $[BE]$ , then $f(D)$ is the point $F$ midpoint of $[AD]$.

4) a- $g(E)=E$.

# CHAPTER 10

# Complex Forms

**Chapter Review :**

Let $T$ be the transformation that to each point $M$ of affix $z$ associates the point $M'$ of affix $z'$ such that $z' = a\,z + b$ .

Cases to be considered:

- If $a = 1$.

  $T$ is a translation of vector $\vec{v}$ of affix $b$.
- If $a \neq 1$.

  $T$ is a similitude $S(I;k;\alpha)$ where:

  * $I$ has an affix $z_0 = \dfrac{b}{1-a}$.
  * $k = |a|$.
  * $\alpha = \arg(a)\ (\text{mod}\,2\pi)$.

* ***Ex :*** $z' = (1+i)z + 1 - i$.

  $T$ is the similitude of center the point $I$ of affix $z_0 = \dfrac{1-i}{-i} = 1+i$ ,of ratio $k = |a| = \sqrt{2}$ and of angle $\alpha = \arg(a) = \dfrac{\pi}{4}(\text{mod}\,2\pi)$.

- Particular Cases :
  * If $|a| = 1$ then $T$ is a rotation of center $I$ of affix $z_0 = \dfrac{b}{1-a}$ and of angle $\alpha = \arg(a)(\text{mod}\,2\pi)$, $T = r(I;\alpha)$.
  * ***Ex :*** $z' = i\,z + 1 - i$.

    $T$ is a rotation of center the point $I$ of affix

$z_0 = \dfrac{1-i}{1-i} = \text{i}$ and of angle $\alpha = \arg(a) = \dfrac{\pi}{2}(\text{mod } 2\pi)$.

* If $a$ is a real number, then $T$ is a dilation of center the point $I$ of affix $z_0 = \dfrac{b}{1-a}$ and of ratio $k = a$, so $T = h(I;a)$.
* ***Ex :*** $z' = -2z + 1 - i$.

  $T$ is a dilation of center the point $I$ of affix $z_0 = \dfrac{1-i}{3} = \dfrac{1}{3} - \dfrac{1}{3}i$ and of ratio $k = -2$.

***Inversion:***
Let $T$ be the transformation that to each point $M$ of affix $z \neq 0$ associates the point $M'$ of affix $z'$ such that $z' = \dfrac{1}{\bar{z}}$.

$T$ is the inversion of center $O$ and power 1.

***Geometrically :***
$I(O;1)$ : $M \longrightarrow M'$ such that $\overrightarrow{OM} \cdot \overrightarrow{OM'} = 1$, where $O, M$ and $M'$ are collinear .

* The set of invariant points is the circle $(C)$ of center $O$ and of radius 1.
* The inversion is an involution , $I(O;1) \circ I(O;1) = I_d$ .

***Analytically :***
$I(O;1)$ : $I(O;1) : M(x;y) \longrightarrow M'(x';y')$ such that :

$x' = \dfrac{x}{x^2 + y^2}$ and $y' = \dfrac{y}{x^2 + y^2}$

* All the straight lines passing through $O$ are invariant as a whole .
* The image of a straight line not passing through $O$ is a circle passing through $O$.
* The image of a circle passing through $O$ is a straight line passing through $O$ and perpendicular to the diameter.
* The image of a circle not passing through $O$ is a circle not passing through $O$ .

# *Solved Problems*

## $N°1.$

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Check , with justification , if the following statements are true or false?

1) Let $f$ be the mapping of the plane whose complex form is: $z' = (-2 + 2i)z + 1 - i$ .Then, $f$ is a direct plane similitude of ratio $2\sqrt{2}$ and angle $\frac{\pi}{4}$.
2) The composite of the rotation $r\left(O;\frac{\pi}{4}\right)$ and the dilation $h(O;-2)$ is a direct plane similitude of center $O$ and angle $\frac{\pi}{4}$ and of ratio 2 .
3) A direct plane similitude of ratio $k$ multiplies the areas by $k$ .

## $N°2.$

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Let $f$ be the mapping in the complex plane that associates to every point $M$ of affix $z$ the point $M'$ of affix $z'$ defined by $z' = \frac{1+i}{\sqrt{2}}z$ .

Are the following statements true ?

Justify your answers :

1) $f$ is the composite of a dilation of ratio $\frac{1}{\sqrt{2}}$ and a rotation of angle $\frac{\pi}{4}$ .
2) The image of a circle of center $O$ and radius $R$ by $f$ is a circle

of center $O$ and radius $\frac{R}{\sqrt{2}}$ .

3) The image of a circle $(C)$ of center $A(1;-2)$ and radius 2 by $f$ is a circle $(C')$ of equation $x^2+y^2-3\sqrt{2}\,x+\sqrt{2}y+3=0$.

**N° 3.**

Complete the following table :

| Complex Form | Affix of the center | Ratio | angle |
|---|---|---|---|
| $z'=iz+1$ | | | |
| | $1+i$ | 2 | $\frac{\pi}{3}$ |

**N° 4.**

The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$ .

Let $T$ be the mapping of the plane defined by :

$$M\begin{cases}x\\y\end{cases}\xrightarrow{T}M'\begin{cases}x'=x+y\sqrt{3}+4\\y'=-x\sqrt{3}+y-2\end{cases}$$

Let $z'=x'+iy'$ and $z=x+iy$ .

1) Express $z'$ in terms of $z$ .
2) Determine the nature and characteristic elements of $T$ .

**N° 5.**

The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

1) Let $S$ be the transformation of the plane that associates to every point $M(z)$ the point $M'(z')$ such that $z'=(1+i)z+3i$.
Determine the nature and the elements of $S$ .
2) Consider the rotation $r$ of center $O$ and angle $-\frac{\pi}{2}$ , let $f=r\circ S$ .
Determine the nature and the elements of $f$ .

**N°6.**

The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

Given the two fixed points $A$ and $B$ of respective affixes 12 and 9i
Designate by $S$ the transformation that to each point $M(z)$ associates the point $M'(z')$ such that $z'=-\frac{3}{4}iz+9i$ .

1) Determine the nature and the elements of $S$.
2) Determine the images of the points $A$ and $O$ by $S$.
3) Denote by $\Omega$ the center of $S$.
   a- Show that $\Omega$ is a common point to the circles $(C_1)$ and $(C_2)$ of respective diameters $[OA]$ and $[OB]$.
   b- Prove that $\Omega$ is the foot of the perpendicular drawn through $O$ in triangle $AOB$.
   c- Using $S$, show that $\Omega A\times\Omega B=\Omega O^2$.

**N°7.**

The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

Consider the points $A_0$, $A_1$ and $A_2$ of respective affixes $z_0=5-4i$, $z_1=-1-4i$ and $z_2=-4-i$ .

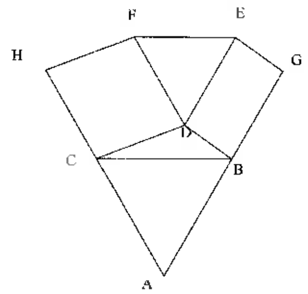
1) a- Determine the complex form of the direct plane similitude $S$ that transforms $A_0$ onto $A_1$ and $A_1$ onto $A_2$.
   b- Deduce the affix $\omega$ of the point $\Omega$ center of $S$, as well as the ratio and an angle of $S$.
2) Let $M$ be the point of affix $z$ and $M'(z')$ the image of $M$ by $S$.
   Verify that $\omega-z'=i(z-z')$, and deduce the nature of triangle $\Omega MM'$.

**N°8.**

The complex plane is referred to a direct orthonormal system $\left(O,\vec{u},\vec{v}\right)$.

In the figure below, triangles $ABC$ and $DEF$ are two equilateral triangles such that $\left(\overrightarrow{AB};\overrightarrow{AC}\right)=\frac{\pi}{3}(\bmod 2\pi)$ and $\left(\overrightarrow{DE};\overrightarrow{DF}\right)=\frac{\pi}{3}(\bmod 2\pi)$.

Let $G$ and $H$ be two points such that $EDBG$ and $CDFH$ are two parallelograms.




***Part A .***

$t_1$ is the translation of vector $\overrightarrow{BD}$ and $t_2$ the translation of vector $\overrightarrow{DC}$.

$r$ is the rotation of center $D$ and angle $\frac{\pi}{3}$.

Let $f=t_2 \circ r \circ t_1$.

1) a- Show that $f$ is a rotation whose angle is to be determined.
   b- Determine $f(B)$ and deduce the center of $f$.
2) Determine the image of $G$ by $f$ and show that triangle $AGH$ is equilateral.

***Part B .***

Designate by $a$, $b$, $c$, $d$, $e$, $f$, $g$ and $h$ the respective affixes of the points $A$, $B$, $C$, $D$, $E$, $F$, $G$ and $H$.

1) a- Show that $c-a=e^{i\frac{\pi}{3}}(b-a)$.
   b- Express $f-d$ in terms of $e-d$.
2) a- Express $g$ in terms of $b$, $d$ and $e$.
   b- Express $h$ in terms of $c$, $d$ and $f$.
3) Show that $h-a=e^{i\frac{\pi}{3}}(g-a)$ and deduce the nature of triangle $AGH$.

**N° 9.**

The complex plane is referred to a direct orthonormal system $\left(O, \vec{u}, \vec{v}\right)$.

Consider the points $M_0(1+5i)$, $M_1(1+i)$ and $M_2(-1-i)$.

1) Show that there exists a direct plane similitude $S$ such that $S(M_0)=M_1$ and $S(M_1)=M_2$ and determine the elements of $S$.
2) Let $S^n = S \circ S \circ \cdots\cdots \circ S$, $n$ times, where $n$ is an integer greater than 1.
   a- Precise the nature and elements of $S^n$.
   b- For what values of $n$, is $S^n$ a dilation ?
3) Let $M$ be a point of affix $z$ and $M_n = S^n(M)$.
   We define the sequence $(u_n)$ by $u_0 = \left\|\overrightarrow{\Omega M_0}\right\|$ and for all natural numbers $n$, $u_n = \left\|\overrightarrow{\Omega M_n}\right\|$ where $\Omega$ is the center of $S^n$.
   a- Show that the sequence $(u_n)$ is a geometric sequence whose ratio is to be determined.
   b- Express $u_n$ in terms of $n$ and calculate $\lim\limits_{n\to+\infty} u_n$.

**N° 10.**

$ABCD$ is a rectangle such that $AB = 2$, $AD = 4$ and $\left(\overrightarrow{AB}; \overrightarrow{AD}\right) = \frac{\pi}{2} (\text{mod } 2\pi)$, $E$ is a point of $[BC]$ such that $BE = 1$.

Let $S$ be the similitude that transforms $A$ onto $B$ and $D$ onto $A$.

1) Determine the ratio $k$ and the angle $\alpha$ of $S$.
2) a- Show that $S(B) = E$ and deduce that $(AE)$ and $(BD)$ are perpendicular.
   b- Let $H$ be the point of intersection of $(AE)$ and $(BD)$, show that $H$ is the center of $S$.
   c- Deduce that $HB^2 = HA \times HE$.
3) The plane is referred to the system $\left(A; \vec{u}, \vec{v}\right)$ such that $\overrightarrow{AB} = 2\vec{u}$.
   a- Determine the complex form of $S$ and deduce the affix of $H$.
   b- $S^{-1}$ is the inverse of $S$, write the complex form of $S^{-1}$.

c- Let $h = S \circ S$, determine the nature of $S$.

d- Show that $\overrightarrow{HB} = -\frac{1}{4}\overrightarrow{HD}$.

e- Write the complex form of $h$.

4) $(A_n)$ is the sequence of points defined by $A_0 = A$ and $A_{n+1} = S(A_n)$.

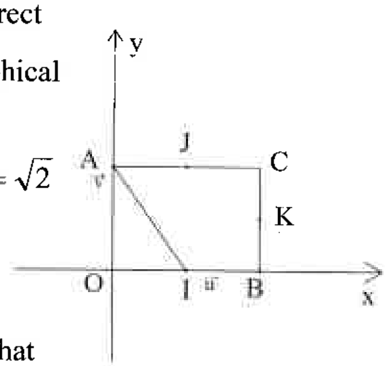
Let $\ell_n = A_n A_{n+1}$ where $n$ is a natural integer.

a- Show that the sequence $(\ell_n)$ is geometric.

b- Calculate $\lim_{n \to +\infty} \ell_n$.

N° 11.

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$ of graphical unit 5 cm, consider the points $A$, $B$ and $C$ of affixes: $z_A = i$, $z_B = \sqrt{2}$ and $z_C = \sqrt{2} + i$.




$I$, $J$ and $K$ are the respective midpoints of $[OB]$, $[AC]$ and $[BC]$.

Let $S$ be the direct plane similitude that transforms $A$ onto $I$ and $O$ onto $B$.

1) a- Determine the ratio $k$ and the angle $\alpha$ of $S$.

b- Show that the complex form of $S$ is $z' = \frac{\sqrt{2}}{2} i z + \sqrt{2}$.

c- Deduce the affix of the center $\Omega$ of $S$.

d- What is the image of rectangle $AOBC$ by $S$ ?

2) Let $h = S \circ S$.

a- What are the images of the points $O$, $B$ and $A$ by $S \circ S$?

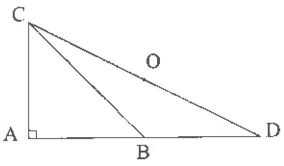
b- Show that $h$ is a dilation whose center and ratio are to be determined.

c- Deduce that the straight lines $(OC)$, $(BJ)$ and $(AK)$ are concurrent.

**N° 12.**

In the figure to the right, $ABC$ is an isosceles triangle such that :

$AB = AC = \ell$ and $\left(\overrightarrow{AB};\overrightarrow{AC}\right) = \frac{\pi}{2} \pmod{2\pi}$.




D is the symmetric of $A$ with respect to $B$, and $O$ is the midpoint of $[CD]$.

Let $(C)$ be the circle of diameter $[CD]$.

$S$ is the direct plane similitude that transforms $D$ onto $B$ and $B$ onto $C$.

1) a- Determine the ratio $k$ and angle $\alpha$ of $S$.

   b- Let $J$ be the center of $S$, show that

   $\left(\overrightarrow{JD};\overrightarrow{JC}\right) = -\frac{\pi}{2} \pmod{2\pi}$ and that $JC = 2\,JD$.

   c- Deduce that $J$ belongs to $(C)$ and that $JD = \ell$.

   d- Show that $(OB)$ is the perpendicular bisector of $[JC]$.
   Determine the nature of quadrilateral $CADJ$ and place $J$.

2) The plane is referred to the direct orthonormal system $\left(A;\overrightarrow{AB},\overrightarrow{AC}\right)$.

   a- Determine the complex form of $S$.

   b- Deduce the affix of $J$.

   c- Let $M$ be a variable point of $(C)$, what is the set of points $M'$ image of $M$ by $S$ ?

   d- $S^{-1}$ is the inverse transformation of $S$.
   Determine the nature, the elements and the complex form of $S^{-1}$.

3) a- Determine the image of circle $(C)$ by the inversion $I(A;1)$.

   b- Determine the image of the straight line $(BC)$ by the inversion $I(A;1)$.

**N° 13.**

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Consider the points $A$, $B$, $C$ and $D$ of respective affixes

$z_A = 2 + i$, $z_B = 1 + 2i$, $z_C = 6 + 3i$ and $z_D = -1 + 6i$.

1) a- Show that there exists a direct similitude $f$ such that : $f(A) = B$ and $f(C) = D$.

   b- Show that $f$ is a rotation and precise its characteristic elements.

2) Let $J$ be the point of affix $3 + 5i$.

   Show that the rotation $R$ of center $J$ and angle $-\frac{\pi}{2}$ transforms $A$ onto $D$ and $C$ onto $B$.

3) $I$ is the point of affix $1 + i$, $M$ and $N$ are the respective midpoints of segments $[AC]$ and $[BD]$.
   Determine the nature of quadrilateral $IMJN$.

4) Consider the points $P$ and $Q$ such that quadrilaterals $IAPB$ and $ICQD$ are direct squares.

   a- Calculate the affixes $z_P$ et $z_Q$ of points $P$ and $Q$.

   b- Determine $\frac{IP}{IA}$ and $\frac{IQ}{IC}$ as well ax a measure of the angles $\left(\overrightarrow{IA};\overrightarrow{IP}\right)$ and $\left(\overrightarrow{IC};\overrightarrow{IQ}\right)$.

   c- Deduce the characteristic elements of the direct similitude $g$ such that $g(A) = P$ and $g(C) = Q$.

   d- Show that $J$ is the image of $M$ by $g$.
   What can you deduce about the point $J$ ?

**N° 14.**

The complex plane is referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Consider the sequence of points $A_n$ of affix $z_n$ defined by :

$A_0 = O$ and $z_{n+1} = \frac{i}{1+i} z_n + i$ for all $n \in IN$.

1) Show that, for all $n \in IN$ the point $A_{n+1}$ is the image of $A_n$ by a direct similitude whose center $\Omega$, ratio and angle are to be determined.

2) Prove that, for all $n \in IN$, the triangle $\Omega A_n A_{n+1}$ is right at $A_{n+1}$.

3) Consider the sequence $(\ell_n)$ defined by :

$\ell_0 = \Omega A_0$ and $\ell_n = \Omega A_n$.

a- Prove that the $(\ell_n)$ is a geometric sequence whose first term and ratio are to be determined.

b- What is the smallest value of $n$ for which $\ell_n \le 0.4$?

4) Designate by $a_k$ the area of triangle $\Omega A_k A_{k+1}$ and consider the sequence $(a_k)$, $k \in IN$ .

a- Prove that the sequence $(a_k)$ is a geometric sequence whose first term and ratio are to be determined .

b- Let $S_n = a_0 + a_1 + a_2 + \cdots\cdots a_n$ .

Express $S_n$ in terms of $n$ and determine $\lim\limits_{n\to+\infty} S_n$.

**N° 15.**

$A$ and $C$ are two distinct points of the plane, designate by $(\Gamma)$ the circle of diameter $[AC]$ and center $O$, $B$ is a point of $(\Gamma)$ distinct of the points $A$ and $C$.

The point $D$ is such that triangle $BCD$ is equilateral with $\left(\overrightarrow{BC},\overrightarrow{BD}\right)=\frac{\pi}{3}(\text{mod}\, 2\pi)$ .

The point $G$ is the centroid of triangle $BCD$, the straight lines $(AB)$ and $(CG)$ intersect at $M$.

***Part A .***

1) Prove that the points $O$, $D$ and $G$ belong to the perpendicular bisector of $[BC]$ and that the point $G$ is the midpoint of $[CM]$ .
2) Determine the ratio $k$ and angle $\alpha$ of the direct similitude $S$ of center $C$ that transforms $B$ onto $M$.

***Part B .***

The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$ in such a way that the points $A$ and $C$ have affixes -1 and 1 respectively . Let $E$ be the point such that $ACE$ is equilateral with $\left(\overrightarrow{AC};\overrightarrow{AE}\right)=\frac{\pi}{3}(\text{mod}\, 2\pi)$.

1) Calculate the affix of $E$.
2) $\sigma$ is the direct plane similitude of complex form:

$$z' = \frac{3+i\sqrt{3}}{4}z + \frac{1-i\sqrt{3}}{4}.$$

Determine the characteristic elements of $\sigma$ , and deduce that $\sigma$ is the inverse of $S$ .

3) Find the affix of point $E'$ image of $E$ by $\sigma$ .
4) Denote by $(C)$ the set of points $M$ as $B$ traces $(\Gamma)$ deprived of the points $A$ and $C$ .
   a- Show that $E$ belongs to $(C)$.
   b- Let $O'$ be the image of $O$ by $S$ .
      Prove that $O'$ is the center of gravity of triangle $ACE$ .

N° 16.

The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

$A$ , $A'$ , $B$ and $B'$ are the points of respective affixes $z_A = 1-2i$ , $z_{A'} = -2+4i$ , $z_B = 3-i$ and $z_{B'} = 5i$ .

1) Place the points $A$ , $A'$ , $B$ and $B'$ in the plane and prove that $ABB'A'$ is a rectangle .
2) Let $S$ be the reflection such that $S(A) = A'$ and $S(B) = B'$ and denote by $(\Delta)$ its axis.
   Find an equation of $(\Delta)$ .
3) Let $z'$ be the affix of point $M'$ image of point $M$ of affix $z$ by $S$ .
   Knowing that the complex form of $S$ is $z' = a\bar{z} + b$ , show that
   $$z' = \left(\frac{3}{5}+\frac{4}{5}i\right)\bar{z} + 2i - 1.$$
4) Let $g$ be the mapping of the plane that to each point $M$ of affix $z$ associates the point $P$ of affix $z'$ defined by:
   $$z' = \left(-\frac{6}{5}-\frac{8}{5}i\right)\bar{z} + 5 - i.$$
   a- Designate by $C$ and $D$ the images of $A$ and $B$ by $g$ respectively.
      Determine the affixes of the points $C$ and $D$ .
   b- $\Omega$ is the point of affix $1+i$ and let $h$ be the dilation of center $\Omega$ and ratio -2 , show that $C$ and $D$ are the respective images of $A'$ and $B'$ by $h$ .

c- $M_1$ is the point of affix $z_1$ image of $M$, of affix $z$ by $h$.
Find the characteristic elements of $h^{-1}$ and express $z$ in terms of $z_1$.

5) Let $f = h^{-1} \circ g$.
   a- Determine the complex form of $f$.
   b- Identify $f$.

**N° 17.**

The complex plane is referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$.

$T$ is the mapping of the plane defined by :

$$M\begin{cases} x \\ y \end{cases} \xrightarrow{T} M'\begin{cases} x' = -3y + 2 \\ y' = -3x + 6 \end{cases}$$

1) Show that $T$ admits only one invariant point $\Omega$.
2) Let $z' = x' + i\,y'$ and $z = x + i\,y$.
   Show that $z' = a\bar{z} + b$ where $a$ and $b$ are two complex numbers to be determined.
3) Show that $T$ is the composite of a reflection of axis $x'x$ and a similitude to be determined.
4) Prove that $T$ is the composite of a dilation $h(\Omega; -3)$ and of a reflection of axis $(\Delta)$ passing through the point $\Omega$ and of slope 1.

**N° 18.**

The plane is referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$.

Let $f$ be the mapping of the plane that to each point $M$ of affix $z$ associates the point $M'$ of affix $z'$ defined by $z' = \frac{1}{z}$.

1) Show that $f$ is the composite of an inversion and of a reflection to be determined.
2) $(C)$ is the circle of equation $x^2 + y^2 - 4x - 2y = 0$.
   Construct the image of $(C)$ by $f$ geometrically.

# Supplementary Problems

**N° 1.**

Complete the following table :

| Complex Form | Nature of the transformation |
|---|---|
| $z' = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)z + 1 + i\sqrt{3}$ | |
| $z' = (1+i)z + 3 - 2i$ | |
| $z' = 2z + 1 + i$ | |
| $z' = -2z + 3i$ | |
| $z' = \left(\frac{4}{1+i}\right)z - 2 + 6i$ | |
| $z' = -z + 2 - 4i$ | |

**N° 2.**

The complex plane is referred to a direct orthonormal system $\left(O; \vec{u}, \vec{v}\right)$.

Consider the dilations :

$h_1$ of center $I(-1 ; 2)$ and ratio $k_1 = 2$.

$h_2$ of center $J(1 ; 3)$ and ratio $k_2 = -4$.

$h_3$ of center $L(0 ; 1)$ and ratio $k_3 = \frac{1}{2}$.

1) Write a complex form of each of the dilations $h_1$, $h_2$ and $h_3$.
2) a- Determine the nature and characteristic elements of $h_2 \circ h_1$.
   b- Determine $h_3 \circ h_2$ and its characteristic elements.
   c- Show that $h_3 \circ h_1$ is a translation whose vector is to be determined.

**N° 3.**

The complex plane is referred to a direct orthonormal system

$\left(O;\vec{u},\vec{v}\right)$.

Let $f$ be the mapping that to every point $M(x;y)$ associates the point $M'(x';y')$ $M'(x',y')$ defined by $x' = x - y + 2$ and $y' = x + y - 1$.

1) Let $z = x + iy$ and $z' = x' + iy'$.
   Express $z'$ in terms of $z$.
2) Determine the nature of $f$ and its characteristic elements.
3) Let $\Omega$ be the center of $f$, prove that triangle $\Omega MM'$ is right isosceles.

**N°4.**

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$. $S$ is the transformation that to every point $M(z)$ associates the point $M'(z')$ such that $z' = (-1+i)z + 2 - i$.

1) Determine the nature and characteristic elements of $S$.
2) Let $G$ be the center of gravity of triangle $MM'M''$ where $M' = S(M)$ and $M'' = S(M')$.
   Designate by $g$ the transformation that to every point $M$ associates the point $G$.
   a- Calculate, in terms of the affix $z$ of $M$, the affix of point $G$.
   b- Show that $g$ is a similitude and determine its characteristic elements.

**N°5.**

In an oriented plane, consider a triangle $OAB$ right at $O$ and such that $\left(\overrightarrow{OA};\overrightarrow{OB}\right) = \frac{\pi}{2}(\text{mod } 2\pi)$ and $\left(\overrightarrow{AO};\overrightarrow{AB}\right) = -\frac{\pi}{3}(\text{mod } 2\pi)$.

$I$ is the midpoint of $[AB]$.

Consider the direct plane similitude that transforms $O$ onto $A$ and $I$ onto $B$.

1) Calculate the ratio and an angle $\theta$ of $S$.
2) Denote by $r$ the rotation of center $I$ and angle $\alpha = -\frac{\pi}{3}$.
   Designate by $f$ the transformation $r \circ S$ $(f = r \circ S)$.
   a- Determine $f(O)$ and $f(I)$.

b- Prove that $f$ is a dilation whose center and ratio are to be determined.

3) The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$ such that $\overrightarrow{OA}=2\vec{u}$ .
   a- Find the affixes of the points $A$ , $B$ and $I$.
   b- Determine the complex form of $S$ and that of $r$.
   c- Deduce the complex form of $f$ then find its nature and characteristic elements.

**N° 6.**

$ABCD$ is a rectangle such that $AB=\sqrt{2}$ , $AD=1$ and $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{2}(\text{mod}\,2\pi)$.

$I$ and $J$ are the midpoints of $[AB]$ and $[CD]$.

***Part A:***

Let $S$ be the similitude that transforms $D$ onto $I$ and $A$ onto $B$.

1) Determine the ratio and angle of $S$.
2) Show that $S(B)=C$ and that $S(C)=J$.
3) Let $w$ be the center of $S$, show that $w$ is the point of intersection of two circles to be determined.
4) Let $h=S\circ S$ , determine the nature of $h$ and deduce that $\overrightarrow{wJ}=-\frac{1}{2}\overrightarrow{wB}$ .

***Part B:***

The plane is referred to a direct orthonormal system $\left(A;\vec{u},\vec{v}\right)$ such that $z_B=\sqrt{2}$ and $z_D=i$.

1) Determine the complex form of $S$ and deduce the affix of its center $w$.
2) Calculate $\frac{z_B-z_D}{z_C-z_I}$ and deduce that the two straight lines $(BD)$ and $(IC)$ are perpendicular.
3) $(A_n)$ is the sequence of points defined by $A_0=A$ and

$A_{n+1} = S(A_n)$.

Suppose also that $\ell_n = A_n A_{n+1}$.

Show that the sequence $(\ell_n)$ is geometric and deduce $\lim\limits_{n \to +\infty} \ell_n$.

**N° 7.**

$ABC$ is a right isosceles triangle of vertex $A$.
$D$ is the symmetric of $A$ with respect to $C$.
Let $S$ be the direct plane similitude that transforms $D$ onto $C$ and $A$ onto $B$.

***Part A .***

1) Calculate the ratio and angle of $S$.
2) Let $J$ be the center of $S$.
   Show that triangle $JCB$ is right at $C$ and deduce a geometrical construction of point $J$.

***Part B:***

The plane is referred to the system $\left(A;\overrightarrow{AB},\overrightarrow{AC}\right)$.

1) Determine the complex form of $S$ and deduce the affix of $J$.
2) Let $T$ be the transformation that to every point $M(z)$ associates the point $M'(z')$ such that $z' = \dfrac{1}{z}$.
   a- Show that $T = I \circ S$ where $I$ and $S$ are a reflection and an inversion to be determined.
   b- Let $(C)$ be the circle of diameter $[BD]$, determine the image of $(C)$ by $I$ and deduce its image by $T$.
   c- Determine the image of the straight line $(BC)$ by $I$.

**N° 8.**

Given a square $ABCD$ such that $\left(\overrightarrow{DA};\overrightarrow{DC}\right) = \dfrac{\pi}{2}(\text{mod } 2\pi)$.

Let $E$ be the midpoint of $[CD]$.

$DEFG$ is also a square such that $\left(\overrightarrow{DE};\overrightarrow{DG}\right) = \dfrac{\pi}{2}(\text{mod } 2\pi)$.

1) Let $S$ be the similitude of center $D$ that transforms $A$ onto $B$.
   a- Determine the ratio and an angle of $S$.

b- Precise the image of $E$ by $S$ and deduce a measure of the angle $\left(\overrightarrow{AE};\overrightarrow{BF}\right)$.

2) $(\Gamma)$ is the circle circumscribed about the square $ABCD$ and $(\Gamma')$ the circle circumscribed about square $DEFG$.
The two straight lines $(AE)$ and $(BF)$ intersect at a point $I$.
a- Show that $I$ belongs to $(\Gamma)$ and $(\Gamma')$.
b- Prove that the points $C$, $G$ and $I$ are collinear.

3) The plane is referred to the system $\left(D;\overrightarrow{DA},\overrightarrow{DC}\right)$.
a- Determine the complex form of $S$.
b- Determine the image of $B$ by $S$.
c- Deduce the image of the square $ABCD$ by $S$.

**N°9.**

The complex plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$.

Consider the points $A$, $B$, $C$ and $D$ of respective affixes:
$z_A = 3$, $z_B = 1+\frac{2}{3}i$, $z_C = 3i$ and $z_D = -\frac{1}{3}i$.

1) a- Determine the complex form of the direct plane similitude $S$ that transforms $A$ onto $B$ and $C$ onto $D$.
b- Deduce the affix of the center $I$ of $S$.

2) Let $M$ be the point of coordinates $(x;y)$ and $M'(x';y')$ its image by $S$.

Show that : $\begin{cases} x' = -\frac{1}{3}y+1 \\ y' = \frac{1}{3}x - \frac{1}{3} \end{cases}$

3) Consider the points $(M_n)$ of the plane such that:
$M_0 = A$ and for all natural numbers $n$, $M_{n+1} = S(M_n)$.
For all natural numbers $n$, denote by $z_n$ the affix of point $M_n$ and suppose $r_n = |z_n - 1|$.

a- Show that $(r_n)$ is a geometric sequence whose first term and ratio are to be determined .

b- Find the smallest integer $k$ such that: $IM_k \leq 10^{-3}$ .

N° 10.

$ABCD$ is a square of center $I$ such that $\left(\overrightarrow{AB};\overrightarrow{AD}\right)=\frac{\pi}{2}(\mathrm{mod}\,2\pi)$.

Let $K$ be the midpoint of $[CD]$.

Let $S$ be the direct plane similitude such that $S(A)=I$ and $S(C)=K$.

1) Determine the ratio and angle of $S$.
2) Let $\Omega$ be the center of $S$. Construct $\Omega$ geometrically.
3) The plane is referred to a direct orthonormal system $\left(A;\overrightarrow{AB},\overrightarrow{AD}\right)$.

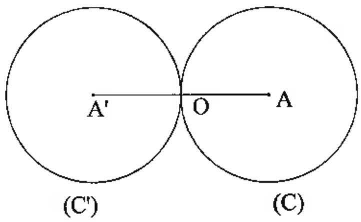
a- Find the complex form of $S$.

b- Deduce the affix of $\Omega$.

N° 11.

$(C)$ and $(C')$ are two circles of the same radius, of respective centers $A$ and $A'$ and tangent externally at point $O$.

Let $M$ be a variable point of $(C)$ and $M'$ a variable point of $(C')$ such that $\left(\overrightarrow{AM};\overrightarrow{A'M'}\right)=\frac{\pi}{2}\,(\mathrm{mod}\,2\pi)$.




1) a- Show that there exists a rotation $r$ of center $w$ that transforms $M$ onto $M'$.

b- Construct the point $w$ geometrically.

2) Let $I$ be the midpoint of $[MM']$.

a- Show that $I$ is the image of $M$ by a direct plane similitude $S$ of center $w$.

b- Deduce the locus $(\gamma)$ of $I$ as $M$ traces $(C)$, then trace $(\gamma)$.

c- Determine the image of $O$ by $S$.

3) The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$

with $\vec{u}=\overrightarrow{OA}$ .

a- Determine the complex form of $r$ and the complex form of $S$.

b- Deduce the affix of $w$.

**N° 12.**

In the complex plane referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$, consider the points $A$ and $B$ of respective affixes 6 and 3i.

Let $S$ be the direct plane similitude that transforms $A$ onto $O$ and $O$ onto $B$.

1) Determine the ratio $k$ and an angle $\alpha$ of $S$.
2) Let $(C)$ be the circle of diameter $[OB]$ and $(C')$ the circle of diameter $[OA]$.
   a- Prove that the center $I$ of $S$ is the point of intersection, other than $O$, of $(C)$ and $(C')$.
   b- What does $[OI]$ represent for the triangle $AOB$?
3) Let $w$ be the point of affix $z_w=-3+3i$ and $h$ the positive dilation that transforms $(C)$ onto $(C')$.
   a- Calculate the ratio of the dilation $h$ and prove that $w$ is its center.
   b- Write the complex form of $h$.

**N° 13.**

In an oriented plane, consider the square $OABC$ such that

$$\left(\overrightarrow{OA};\overrightarrow{OC}\right)=\frac{\pi}{2}(\text{mod }2\pi) \ .$$

Let $E$ be the midpoint of $[BC]$ and $F$ that of $[AB]$.

Let $T$ be the transformation verifying $T(O)=B$ and $T(A)=E$.

Let $r$ be the rotation of center $B$ and of angle $\frac{\pi}{2}$ .

1) a- Prove that $T$ is a dilation whose ratio is to be determined.
   b- Deduce the point $D$ such that $T(D)=C$.
   c- Determine, geometrically, the center $Q$ of $T$ then, determine

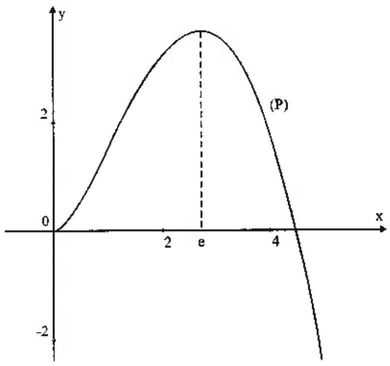
$T(C)$ .

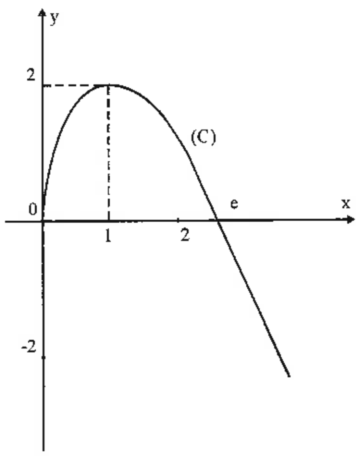
2) Let $f = r \circ T$ .
   a- Determine $f(O)$ and $f(A)$.
   b- Prove that $f$ is a similitude whose ratio and angle are to be determined.
   c- Determine, geometrically, the center $\Omega$ of $f$ .
3) Let $g = T \circ r$ .
   a- Determine $g(O)$ and $g(A)$ and deduce that $g \neq f$ .
   b- Determine the nature of $g$.
   c- Construct, geometrically, the center $\Omega'$ of $g$.
4) The plane is referred to a direct orthonormal system $\left(O;\vec{u},\vec{v}\right)$ such that $\overrightarrow{OA} = 3\vec{u}$ .
   a- Find the complex form of $T$ and that of $r$.
   b- Write the complex form of $f$ and determine the affix of its center $\Omega$.
   c- Give the complex form of $g$ and determine the affix of its center $\Omega'$.

**N° 14.**

The curves given below are the graphical representations of a function $f$ defined over $]0;+\infty[$ and an antiderivative $F$ of $f$ .

1) Determine the representative curve of $f$ and that of $F$ .



2) Suppose that the expression of $F$ is $F(x) = ax^2 + bx^2 \ln x$ where $a$ and $b$ are two real numbers, determine $a$ and $b$ and deduce the expression of $f(x)$ .
3) Discuss, according to the values $m$ , the existence of the roots of the equation $x = e^{1-\frac{m}{2x}}$.
4) The function $f$ admits for $x > 1$ an inverse function $f^{-1}$.
   a- Calculate the coordinates of the point $A$ intersection of $(C)$ and $(C')$, the representative curve of $f^{-1}$.
   b- Show that the two curves $(C)$ and $(C')$ have the same tangent $(T)$ at A .

***Part B:***

Let $T$ be the transformation of the plane that to each point $M$ of affix $z$ associates the point $M'$ of affix $z'$ such that $z' = \frac{-1}{e^2}\bar{z}$ .

1) Show that $T$ is the composite of a reflection and a dilation to be determined.
2) Let $z = x + iy$ and $z' = x' + iy'$ .
   Calculate $x$ and $y$ in terms of $x'$ and $y'$ .
3) Deduce that as $M$ traces $(C)$ ; $M'$ traces the curve $(\gamma)$ of equation $g(x) = 2x(1 + \ln(-x))$.
4) Calculate $\lim\limits_{\substack{x \to 0 \\ x<0}} g(x)$ , $\lim\limits_{x \to -\infty} g(x)$ and $\lim\limits_{x \to -\infty} \frac{g(x)}{x}$ .
5) Study the variations of $g$ and draw its table of variations .
6) Find an equation of the tangent $(T_1)$ to $(\gamma)$ at the point of intersection of $(\gamma)$ with $x'x$ .
7) Trace $(\gamma)$ and $(T_1)$ on the same figure .

# *Solution of Problems*

**N°1.**

1) $z' = az + b$ with $a = -2+2i$ and $b = 1-i$.
   Then, $f$ is a similitude of ratio $k = |a|$ and angle $\theta = \arg(a)(\text{mod}\, 2\pi)$.
   But $|a| = \sqrt{4+4} = 2\sqrt{2}$ therefore $a = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$.
   Hence, $a = 2\sqrt{2}e^{i\left(3\frac{\pi}{4}\right)}$.
   So $f$ is a similitude of ratio $k = 2\sqrt{2}$ and whose angle has a measure $3\frac{\pi}{4}$.
   We conclude that the given statement is false.
2) $r\left(O,\frac{\pi}{4}\right) \circ h(O;-2) = S\left(O;1;\frac{\pi}{4}\right) \circ S(O;2;\pi) = S\left(O;2;\pi+\frac{\pi}{4}\right)$.
   Hence, the given statement is false.
3) This statement is false since the similitude multiplies the areas by $k^2$.

**N°2.**

1) $a = \frac{1}{\sqrt{2}}(1+i) = e^{i\frac{\pi}{4}}$.
   Then, $f$ is a rotation of center $O$ and angle $\frac{\pi}{4}$.
   So, the given statement is false.
2) The image of a circle by rotation is a circle with the same radius so the statement is false.
3) The image of $A$ by $f$ is the point $A'$ of affix
   $\frac{1+i}{\sqrt{2}} \times (1-2i) = \frac{3\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}$.
   Then, the image of circle $(C)$ by $f$ is the circle $(C')$ of center A'

and radius 2 , its equation is $\left(x-\frac{3\sqrt{2}}{2}\right)^2+\left(y+\frac{\sqrt{2}}{2}\right)^2=4$

which gives $x^2+y^2-3\sqrt{2}\,x+\sqrt{2}\,y+1=0$ .
Hence, the statement is false.

**N° 3.**

- $z'=iz+1$ .

  $a=i=e^{i\frac{\pi}{2}}$ , so it represents a rotation of angle $\frac{\pi}{2}$ and center point $A$ of affix $z_0=\frac{b}{1-a}=\frac{1}{1-i}\times\frac{1+i}{1+i}=\frac{1+i}{2}=\frac{1}{2}+\frac{1}{2}i$ .

- $k=2$ and the angle is $\frac{\pi}{3}$ then $a=2e^{i\frac{\pi}{3}}=1+i\sqrt{3}$ .

  So $z'=\left(1+i\sqrt{3}\right)z+b$ .

  $1+i=\frac{b}{1-\left(1+i\sqrt{3}\right)}$ , therefore $b=(1+i)\left(-i\sqrt{3}\right)$ which gives

  $b=\sqrt{3}-i\sqrt{3}$ .Consequently
  $z'=\left(1+i\sqrt{3}\right)z+\sqrt{3}(1-i)$ .

| Complex Form | Affix of the center | Ratio | Angle |
|---|---|---|---|
| $z'=iz+1$ | $I\left(\frac{1}{2};\frac{1}{2}\right)$ | 1 | $\frac{\pi}{2}$ |
| $z'=\left(1+i\sqrt{3}\right)z+\sqrt{3}(1-i)$ | $1+i$ | 2 | $\frac{\pi}{3}$ |

**N° 4.**

1) $z'=x'+iy'=\left(x+y\sqrt{3}+4\right)+i\left(-x\sqrt{3}+y-2\right)$
$z'=(x+iy)+\left(y\sqrt{3}-ix\sqrt{3}\right)+4-2i$
$z'=(x+iy)-i\sqrt{3}(x+iy)+4-2i$
$z'=\left(1-i\sqrt{3}\right)(x+iy)+4-2i$ therefore $z'=\left(1-i\sqrt{3}\right)z+4-2i$ .

2) $z' = az + b$ with $a = 1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}}$ and $b = 4 - 2i$.

So, $T$ is a similitude of ratio $k = |a| = 2$, and angle $-\frac{\pi}{3}$ and of center the point $\Omega$ of affix $z_0 = \frac{b}{1-a} = \frac{4-2i}{i\sqrt{3}} \times \frac{i\sqrt{3}}{i\sqrt{3}}$ which gives

$z_0 = -\frac{2\sqrt{3}}{3} - \frac{4i\sqrt{3}}{3}$.

N° 5.

1) $z' = az + b$ with $a = 1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$ and $b = 3i$.

$S$ is a direct plane similitude of ratio $k = \sqrt{2}$, and angle $\alpha = \frac{\pi}{4}$ and of center the point $\Omega$ of affix $w = \frac{b}{1-a} = \frac{3i}{-i} = -3$.

Therefore, $S\left(\Omega; \sqrt{2}; \frac{\pi}{4}\right)$.

2) a- The complex form of $r$ is $z' = az + b$ with $a = e^{-i\frac{\pi}{2}} = -i$.

$O$ is the center of $r$ then $z_O = \frac{b}{1-a}$, which gives $b = 0$ and consequently $z' = -iz$.

$r \circ S : z \xrightarrow{S} (1+i)z + 3i \xrightarrow{r} z' = -i[(1+i)z + 3i]$.

Therefore $r \circ S$ is the transformation that to all points $M(z)$ associates the point $M'(z')$ such that $z' = (1-i)z + 3$.

The complex form of $f$ is:

$z' = az + b$ with $a = 1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$ and $b = 3$, so $f$ is a similitude of ratio $\sqrt{2}$ and angle $-\frac{\pi}{4}$ and of center the point $\Omega'$ of affix $\omega' = \frac{b}{1-a} = \frac{3}{i} = -3i$, then $\Omega'(0;-3)$. Therefore :

$f = S\left(\Omega'; \sqrt{2}; -\frac{\pi}{4}\right)$.

N° 6.

1) $z' = az + b$ with $a = -\frac{3}{4}i = \frac{3}{4}e^{-i\frac{\pi}{2}}$ and $b = 9i$.

Then, $S$ is a similitude of ratio $k=\frac{3}{4}$, and angle $\alpha=-\frac{\pi}{2}$ and of center the point $\Omega$ of affix :

$$\omega=\frac{b}{1-a}=\frac{9i}{1+\frac{3}{4}i}\times\frac{1-\frac{3}{4}i}{1-\frac{3}{4}i}=\frac{108}{25}+\frac{144}{25}i.$$

2) The point $A$ has an affix 12 , its image by S is the point $A'$ of affix $z_{A'}=-\frac{3}{4}i\times 12+9i=0$ .So, $S(A)=O$.

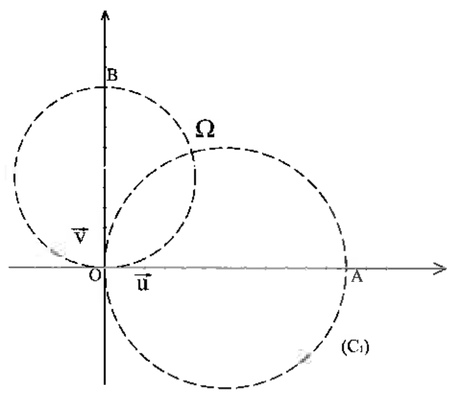
The image of $O$ by $S$ is the point of affix $z'=9i$ , then $S(O)=B$.

3) a- $S(A)=O$ , then $\left(\overrightarrow{\Omega A};\overrightarrow{\Omega O}\right)=-\frac{\pi}{2}$ (mod $2\pi$).

Therefore $\Omega$ belongs to circle $(C_1)$ of diameter $[OA]$ .

Similarly $S(O)=B$ , therefore, $\left(\overrightarrow{\Omega O};\overrightarrow{\Omega B}\right)=-\frac{\pi}{2}$ (mod $2\pi$).

Hence, $\Omega$ belongs to circle $(C_2)$ of diameter $[OB]$.
So $\Omega$ is a point common to circles $(C_1)$and $(C_2)$ .
But, $(C_1)$and $(C_2)$ intersect at $O$ also .But, $O$ is not invariant under $S$ since $S(O)=B$ , therefore, $\Omega$ is the second point of intersection of $(C_1)$ and $(C_2)$.



b- $O\hat{\Omega}B = O\hat{\Omega}A = 90^o$.
Then the points $A$, $\Omega$ and $B$ are collinear and $\Omega$ is the foot of the perpendicular issued from $O$ in triangle $AOB$.

c- $S(A) = O$ then $\Omega O = \frac{3}{4}\Omega A$.

$S(O) = B$ then $\Omega B = \frac{3}{4}\Omega O$.

Therefore, $\frac{\Omega O}{\Omega A} = \frac{\Omega B}{\Omega O}$, which gives $\Omega O^2 = \Omega A \times \Omega B$.

**N° 7.**

1) a- The complex form of $S$ is $z' = az + b$.
$S(A_0) = A_1$ gives $-1-4i = a(5-4i)+b$
$S(A_1) = A_2$ gives $-4-i = a(-1-4i)+b$.
Subtracting, we get :
$-1-4i+4+i = a(5+1-4i+4i)$, which gives $a = \frac{1}{2} - \frac{1}{2}i$

then $b = -\frac{3}{2} + \frac{1}{2}i$ and consequently, the complex form of $S$ is :

$z' = \left(\frac{1}{2} - \frac{1}{2}i\right)z - \frac{3}{2} + \frac{1}{2}i$.

b- $a = \frac{1}{2} - \frac{1}{2}i = \frac{\sqrt{2}}{2}e^{-i\frac{\pi}{4}}$, then the ratio of $S$ is $\frac{\sqrt{2}}{2}$ and its angle is $-\frac{\pi}{4}$.

The affix of $\Omega$ is $\omega = \frac{b}{1-a} = \frac{-\frac{3}{2} + \frac{1}{2}i}{1 - \frac{1}{2} + \frac{1}{2}i}$, so

$\omega = \frac{-3+i}{1+i} \times \frac{1-i}{1-i} = -1+2i$, therefore $\Omega(-1+2i)$.

2) $\omega - z' = -1+2i - \left(\frac{1}{2} - \frac{1}{2}i\right)z + \frac{3}{2} - \frac{1}{2}i$ then :

$\omega - z' = \frac{1}{2} + \frac{3}{2}i - \left(\frac{1}{2} - \frac{1}{2}i\right)z$.

$$i(z-z')=i\left(z-\left(\frac{1}{2}-\frac{1}{2}i\right)z+\frac{3}{2}-\frac{1}{2}i\right)$$ which gives :

$$i(z-z')=\frac{1}{2}+\frac{3}{2}i-\left(\frac{1}{2}-\frac{1}{2}i\right)z$$, then $\omega-z'=i(z-z')$.

$\frac{\omega-z'}{z-z'}=i$ then $\frac{z_{\overrightarrow{M'\Omega}}}{z_{\overrightarrow{M'M}}}=i=e^{i\frac{\pi}{2}}$.

Hence, $\frac{M'\Omega}{M'M}=1$ and $\left(\overrightarrow{M'M};\overrightarrow{M'\Omega}\right)=\frac{\pi}{2}\pmod{2\pi}$.

Consequently, triangle $M'M\Omega$ is right isosceles at $M'$.

**N° 8.**

***Part A:***

1) a- $f$ is the composite of a translation , a rotation and a translation then it is a rotation of angle $\frac{\pi}{3}$.

b- $t_1(B)=D$, $r(D)=D$ and $t_2(D)=C$, then $f(B)=C$.

But there exists a unique rotation of angle $\frac{\pi}{3}$ that transforms $B$ onto $C$ and since the rotation of center $A$ and angle $\frac{\pi}{3}$ transforms $B$ onto $C$ then the center of $f$ is the point $A$ and consequently $f=r\left(A;\frac{\pi}{3}\right)$.

2) $EDBG$ is a parallelogram so $\overrightarrow{BD}=\overrightarrow{GE}$ then $t_1(G)=E$.
$r(E)=F$ since $DEF$ is equilateral .
Therefore, $f(G)=t_2(F)=H$ since $CDFH$ is a parallelogram.
$f(G)=H$ gives $AG=AH$ and $\left(\overrightarrow{AG};\overrightarrow{AH}\right)=\frac{\pi}{3}\pmod{2\pi}$ then triangle $AGH$ is equilateral .

***Part B:***

1) a- $\frac{z_{\overrightarrow{AC}}}{z_{\overrightarrow{AB}}}=\frac{c-a}{b-a}=\frac{AC}{AB}e^{i\left(\overrightarrow{AB};\overrightarrow{AC}\right)}=e^{i\frac{\pi}{3}}$ then $c-a=e^{i\frac{\pi}{3}}(b-a)$.

**Another method :**

The point $C$ is the image of point $B$ by the rotation of center $A$ and angle $\frac{\pi}{3}$, then $z_{\overrightarrow{AC}} = e^{i\frac{\pi}{3}} z_{\overrightarrow{AB}}$, which gives

$c - a = e^{i\frac{\pi}{3}}(b-a)$.

b- The point $F$ is the image of point $E$ by the rotation of center $D$ and angle $\frac{\pi}{3}$, therefore $f - d = e^{i\frac{\pi}{3}}(e-d)$.

2) a- $EDBG$ is a parallelogram then $\overrightarrow{BG} = \overrightarrow{DE}$ which gives $g - b = e - d$ and consequently, $g = b + e - d$.

b- Similarly, $CDFH$ is a parallelogram which gives $h = c + f - d$.

3) $h - a = e^{i\frac{\pi}{3}}(b-a) + e^{i\frac{\pi}{3}}(e-d) = e^{i\frac{\pi}{3}}(g-a)$.

So $\dfrac{h-a}{g-a} = e^{i\frac{\pi}{3}}$ which gives $\dfrac{z_{\overrightarrow{AH}}}{z_{\overrightarrow{AB}}} = e^{i\frac{\pi}{3}}$, and consequently,

triangle $AGH$ is equilateral

$\boxed{N°9.}$

1) If there exists a direct similitude $S$ such that $S(M_0) = M_1$ and $S(M_1) = M_2$ and since the complex form of $S$ is $z' = az + b$, then: $z_{M_1} = a z_{M_0} + b$ and $z_{M_2} = a z_{M_1} + b$, so we get the system :

$\begin{cases}(1+5i)a + b = 1 + i \\ (1+i)a + b = -1 - i\end{cases}$ that gives the solutions $a = \frac{1}{2} - \frac{1}{2}i$ and

$b = -2 - 2i$, therefore $z' = \left(\frac{1}{2} - \frac{1}{2}i\right)z - 2 - i$.

$a = \frac{1}{2} - \frac{1}{2}i = \frac{\sqrt{2}}{2}e^{-i\frac{\pi}{4}}$, then the ratio of $S$ is $k = \frac{\sqrt{2}}{2}$ and its angle is $\alpha = -\frac{\pi}{4}$.

The center of $S$ is the point $\Omega$ of affix

$$\omega = \frac{b}{1-a} = \frac{-2-i}{1-\frac{1}{2}+\frac{1}{2}i} = -3+i.$$

2) a- $S^n$ is the composite of $n$ similitudes , then $S^n$ is a similitude of center $\Omega$ , of ratio $\left(\frac{\sqrt{2}}{2}\right)^n$ and angle $n\alpha = -n\frac{\pi}{4} \pmod{2\pi}$.

b- $S^n$ is a dilation when $-n\frac{\pi}{4} = k\pi$ with k being a non-zero integer then $n = -4k$ but $n \geq 1$, so $-4k \geq 1$ which gives $k \leq -\frac{1}{4}$ with $k \in Z$ then $k \leq -1$.

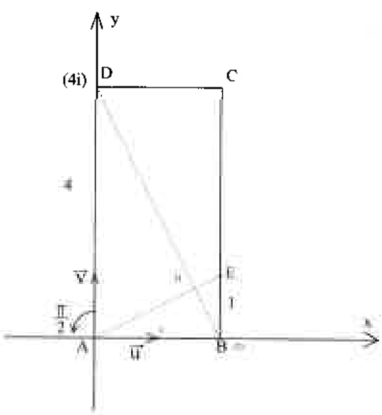
3) a- $$\frac{u_{n+1}}{u_n} = \frac{\left\|\overrightarrow{\Omega M_{n+1}}\right\|}{\left\|\overrightarrow{\Omega M_n}\right\|} = \frac{\left(\frac{\sqrt{2}}{2}\right)^{n+1}}{\left(\frac{\sqrt{2}}{2}\right)^{n}} = \frac{\sqrt{2}}{2}$$

then $(u_n)$ is a geometric sequence of ratio $r = \frac{\sqrt{2}}{2}$ and first term $u_0 = \left\|\overrightarrow{\Omega M_0}\right\| = 4\sqrt{2}$ , since $z_{\overrightarrow{\Omega M_0}} = 1+5i+3-i = 4+4i$.

b- $u_n = u_0 \times r^n = 4\sqrt{2} \times \left(\frac{\sqrt{2}}{2}\right)^n$.

$\lim_{n\to+\infty} u_n = 0$ since $0 < \frac{\sqrt{2}}{2} < 1$.

**N°10.**




1) $S(A) = B$ and $S(D) = A$, then

$k = \frac{BA}{AD} = \frac{2}{4} = \frac{1}{2}$ and

$\alpha = \left(\overrightarrow{AD};\overrightarrow{BA}\right) = \frac{\pi}{2} \pmod{2\pi}$.

2) a- Let $B' = S(B)$, since $B = S(A)$ then we get:

$\left(\overrightarrow{AB};\overrightarrow{BB'}\right)=\frac{\pi}{2}(\bmod 2\pi)$ ,which gives that $B'$ belongs to the semi-straight line $[BC)$ .

$BB'=\frac{1}{2}AB=\frac{1}{2}\times 2=1$ .

Hence, $B'$ is the point $E$, so $S(B)=E$.

$S(B)=E$ and $S(D)=A$ then $\left(\overrightarrow{BD};\overrightarrow{AE}\right)=\frac{\pi}{2}(\bmod 2\pi)$,

hence the two straight lines $(AE)$ and $(BD)$ are perpendicular.

b- Since $\left(\overrightarrow{HA};\overrightarrow{HB}\right)=\frac{\pi}{2}(\bmod 2\pi)$ and

$\left(\overrightarrow{HB};\overrightarrow{HE}\right)=\frac{\pi}{2}(\bmod 2\pi)$ then $H$ is the center of $S$.

c- $S(A)=B$ gives $HB=\frac{1}{2}HA$

$S(B)=E$ gives $HE=\frac{1}{2}HB$ .

Therefore $HB^2=HB\times HB=\frac{1}{2}HA\times 2HE=HA\times HE$ .

3) a- The complex form of $S$ is $z'=az+b$ with

$a=ke^{i\alpha}=\frac{1}{2}e^{i\frac{\pi}{2}}=\frac{1}{2}i$ , then: $z'=\frac{1}{2}iz+b$ .

$S(A)=B$ , therefore $z_B=az_A+b$ which gives $2=b$ and consequently, the complex form of $S$ is $z'=\frac{1}{2}iz+2$ .

The affix of $H$ is $z_H=\frac{b}{1-a}=\frac{2}{1-\frac{1}{2}i}=\frac{4}{2-i}\times\frac{2+i}{2+i}$ , so :

$z_H=\frac{8}{5}+\frac{4}{5}i$ .

b- $z' = \frac{1}{2}iz + 2$ , then $z' - 2 = \frac{1}{2}iz$, which gives :

$i(z'-2) = -\frac{1}{2}z$ and consequently $z = -2iz' + 4i$.

The complex form of $S^{-1}$ is then $z' = -2iz + 4i$.

**Another method :**

$S^{-1} = S\left(H;2;-\frac{\pi}{2}\right)$ , then the complex form of $S^{-1}$ is

$z' = az + b$ with $a = 2e^{-i\frac{\pi}{2}} = -2i$ , then : $z' = -2iz + b$.

$S^{-1}(B) = A$ gives $z_A = -2iz_B + b$, so $0 = -4i + b$

consequently, $b = 4i$, hence $z' = -2iz + 4i$.

c- We know that $S\left(H;\frac{1}{2};\frac{\pi}{2}\right) \circ S\left(H;\frac{1}{2};\frac{\pi}{2}\right) = S\left(H;\frac{1}{4};\pi\right)$.

Then $S \circ S$ is the dilation of center $H$ and ratio $-\frac{1}{4}$ , then:

$h = S \circ S = h\left(H;-\frac{1}{4}\right)$.

d- $D \xrightarrow{S} A \xrightarrow{S} B$ , then $S \circ S(D) = B$,

so $h(D) = B$, therefore $\overrightarrow{HB} = -\frac{1}{4}\overrightarrow{HD}$.

e- The complex form of $h$ is $z' = az + b$ with $a = -\frac{1}{4}$,

then $z' = -\frac{1}{4}z + b$.

$h(D) = B$ gives $z_B = -\frac{1}{4}z_D + b$,then $2 = -\frac{1}{4}(4i) + b$ , which

gives $b = 2 + i$ and consequently, $z' = -\frac{1}{4}z + 2 + i$.

4) a- $\frac{\ell_{n+1}}{\ell_n} = \frac{A_{n+1}A_{n+2}}{A_nA_{n+1}}$ but $A_{n+2} = S(A_{n+1})$ and $A_{n+1} = S(A_n)$,

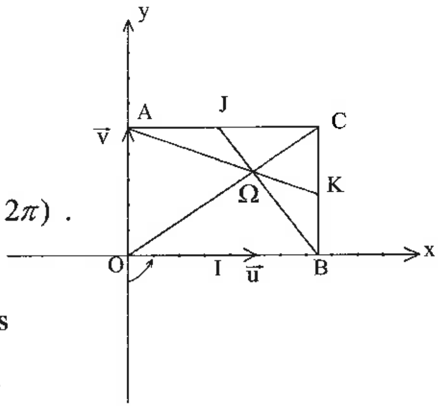
so, $\frac{A_{n+2}A_{n+1}}{A_{n+1}A_n} = \frac{1}{2}$ , ratio of similitude.

Therefore: $\frac{\ell_{n+1}}{\ell_n} = \frac{1}{2}$ and consequently, $(\ell_n)$ is a geometric sequence of ratio $r = \frac{1}{2}$ and first term $\ell_0 = A_0A_1 = AB = 2$.

b- $\ell_n = \ell_0 \times r^n = 2\times\left(\frac{1}{2}\right)^n$ , $\lim\limits_{n\to+\infty} \ell_n = 0$ since $0 < r < 1$.

**N° 11.**

1) a- $S:\begin{cases} A \longrightarrow I \\ O \longrightarrow B \end{cases}$ , then :




$k = \frac{IB}{AO} = \frac{\frac{\sqrt{2}}{2}}{1} = \frac{\sqrt{2}}{2}$

$\alpha = \left(\overrightarrow{AO};\overrightarrow{IB}\right) = \frac{\pi}{2}$ (mod $2\pi$) .

b- The complex form of $S$ is $z' = az + b$ with

$a = ke^{i\alpha} = \frac{\sqrt{2}}{2}e^{i\frac{\pi}{2}} = \frac{\sqrt{2}}{2}i$ .

$S(O) = B$ gives $z_B = az_O + b$, so $\sqrt{2} = b$ .

Therefore, $z' = \frac{\sqrt{2}}{2}iz + \sqrt{2}$ .

c- $z_\Omega = \frac{b}{1-a} = \frac{\sqrt{2}}{1-\frac{\sqrt{2}}{2}i} = \frac{2\sqrt{2}}{2-\sqrt{2}i} \times \frac{2+\sqrt{2}i}{2+\sqrt{2}i}$.

So, $z_\Omega = \frac{2\sqrt{2}}{3} + \frac{2}{3}i$.

d- The image of a rectangle by a similitude is a rectangle .
$S(A) = I$ and $S(O) = B$ .
The image of point $B$ by $S$ is the point of affix:
$z' = \frac{\sqrt{2}}{2}iz_B + \sqrt{2} = \frac{\sqrt{2}}{2}i\times\sqrt{2} + \sqrt{2} = i + \sqrt{2}$ , then $S(B) = C$ .

$AOBC$ is a direct rectangle, hence its image by $S$ is a direct rectangle having $I$, $B$ and $C$ as vertices .

Hence, the fourth vertex should be the point $J$, consequently the image of $AOBC$ is the rectangle $IBCJ$.

2) a- $h(O) = S \circ S(O) = S(S(O)) = S(B) = C$

$h(B) = S \circ S(B) = S(S(B)) = S(C) = J$

$h(A) = S \circ S(A) = S(S(A)) = S(I)$.

But $S(I)$ is the point of affix :

$z' = \frac{\sqrt{2}}{2} i z_I + \sqrt{2} = \frac{\sqrt{2}}{2} i \times \frac{\sqrt{2}}{2} + \sqrt{2}$ so $z' = \frac{1}{2} i + \sqrt{2} = z_k$.

Then, $h(A) = K$.

b- $h = S\left(\Omega; \frac{\sqrt{2}}{2}; \frac{\pi}{2}\right) \circ S\left(\Omega; \frac{\sqrt{2}}{2}; \frac{\pi}{2}\right) = S\left(\Omega; \frac{1}{2}; \pi\right)$.

So, $h$ is the dilation of center $\Omega$ and ratio $-\frac{1}{2}$.

c- $h(O) = C$ , $h(B) = J$ and $h(A) = K$.

Then, the three straight lines $(OC)$, $(BJ)$ and $(AK)$ are concurrent at point $\Omega$, center of $h$ and of S .

N° 12.

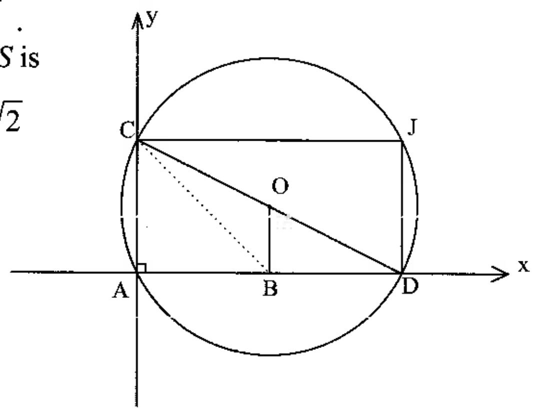
1) a- In the right triangle $ABC$ we have:

$BC^2 = AB^2 + AC^2 = 2\ell^2$

then $BC = \ell\sqrt{2}$ .

The ratio $k$ of $S$ is

$\frac{BC}{DB} = \frac{\ell\sqrt{2}}{\ell} = \sqrt{2}$



The angle of $S$ is $\alpha = \left(\overrightarrow{DB};\overrightarrow{BC}\right) = -\frac{\pi}{4} \pmod{2\pi}$ .

b- We haves : $D \xrightarrow{S} B \xrightarrow{S} C$ then :
$D \xrightarrow{S\circ S} C$

But $S \circ S = S\left(J;2;-\frac{\pi}{2}\right)$ so $\left(\overrightarrow{JD};\overrightarrow{JC}\right) = -\frac{\pi}{2} \pmod{2\pi}$

and $JC = 2\,JD$.

c- Triangle $JDC$ is right at $J$, so
$DC^2 = JD^2 + JC^2 = JD^2 + 4JD^2 = 5JD^2$ .
In the right triangle $ADC$ we have :
$DC^2 = AD^2 + AC^2 = 4\ell^2 + \ell^2 = 5\ell^2$ , then :
$5\ell^2 = 5JD^2$ , therefore $JD = \ell$ and $JC = 2\ell$.

d- $S(D) = B$, so $JB = \sqrt{2}JD = \ell\sqrt{2} = BC$ then $B$ belongs to the perpendicular bisector of $[JC]$.
$OC = OJ = R$ then $O$ belongs to the perpendicular bisector of $[JC]$.
Then $(OB)$ is the perpendicular bisector of $[JC]$ .
Similarly $(OB)$ is the perpendicular bisector of $[AD]$.
Hence $(JC)$ and $(AD)$ are parallel and since $JC = AD = 2\ell$ and $\hat{CAD} = 90^o$, then $ACJD$ is a rectangle .

2) a- Let $z' = az + b$ be the complex form pf $S$ .
We have $a = ke^{i\alpha} = \sqrt{2}\,e^{-i\frac{\pi}{4}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right) = 1 - i$ .

$S(D) = B$ gives $z_B = az_D + b$ , so $1 = 2a + b$ , which gives $1 = 2 - 2i + b$ and consequently $b = -1 + 2i$.
Then the complex form of $S$ is $z' = (1 - i)z - 1 + 2i$

b- $z_J = \frac{b}{1-a} = \frac{-1+2i}{1-1+i} = \frac{-1+2i}{i} \times \frac{i}{i} = \frac{-i-2}{-1} = 2 + i$.

c- The image of a circle by a similitude is a circle , then when $M$ traces circle $(C)$, $M'$ traces the circle $(C')$ image of $(C)$ by $S$, of center $O' = S(O)$ and radius $R' = R\sqrt{2}$, since

$z_O = 1+\frac{1}{2}i$ then $z_{O'} = (1-i)\left(1+\frac{1}{2}i\right) - 1 + 2i$ which gives

$z_{O'} = \frac{1}{2}+\frac{3}{2}i$, therefore $O'\left(\frac{1}{2};\frac{3}{2}\right)$.

$R = OA = \frac{1}{2}CD = \frac{\sqrt{5}}{2}$, so $R' = \frac{\sqrt{10}}{2}$.

d- $S^{-1}$ is a similitude of same center as $S$, of ratio $\frac{1}{k} = \frac{1}{\sqrt{2}}$ and angle $-\alpha = \frac{\pi}{4}$, then its complex form is

$z' = \frac{1}{\sqrt{2}}e^{i\frac{\pi}{4}}z + b$ or $z' = \left(\frac{1}{2}+\frac{1}{2}i\right)z + b$.

$z_J = \frac{b}{1-a}$ so $2+i = \frac{b}{1-\frac{1}{2}-\frac{1}{2}i}$, which gives

$b = \frac{3}{2}-\frac{1}{2}i$ therefore $z' = \left(\frac{1}{2}+\frac{1}{2}i\right)z + \frac{3}{2}-\frac{1}{2}i$.

3) a- $(C)$ passes by the center of the inversion, its image by $I(A;1)$ is a straight line.

Let $D'$ be the image of $D$ by $I(A;1)$, we have $\overline{AD}\times\overline{AD'} = 1$ which gives $\overline{AD'} = \frac{1}{\overline{AD}} = \frac{1}{2}$, then $D'$ is the midpoint of $[AB]$.

Let $C'$ be the image of $C$ by $I(A;1)$, we have $\overline{AC}\times\overline{AC'} = 1$ which gives $\overline{AC'} = \frac{1}{\overline{AC}} = 1$, then $C'$ is confounded with $C$, hence the image of $(C)$ by $I(A;1)$ is the straight line $(CD')$.

b- We have $\overline{AB}\times\overline{AB} = 1$, then $B$ is an invariant point by $I(A;1)$.
Similarly $C$ is an invariant point by $I(A;1)$.
Then the image of straight line $(BC)$ by $I(A;1)$ is the circle passing through points $B$ and $C$ and by the center $A$, then it's the circle circumscribed about triangle $ABC$.

**Another method :**
An equation of $(BC)$ is $x+y-1=0$.

We know that if $M(x;y)$ is a point on $(BC)$ and $M'(x';y')$ its image by $I(A;1)$ then $x = \dfrac{x'}{x'^2 + y'^2}$ and $y = \dfrac{y'}{x'^2 + y'^2}$, replacing $x$ and $y$ by their values in equation $x + y - 1 = 0$ we get $\dfrac{x'}{x'^2 + y'^2} + \dfrac{y'}{x'^2 + y'^2} - 1 = 0$ then $x'^2 + y'^2 - x' - y' = 0$.

Hence the image of $(BC)$ is the circle of center $w\left(\dfrac{1}{2};\dfrac{1}{2}\right)$ and radius $R = \dfrac{\sqrt{2}}{2}$.

$\boxed{N°13.}$

1) a- The points $A$, $B$, $C$ and $D$ are distinct, then there exists a unique direct similitude $f$ that transforms $A$ onto $B$ and $C$ onto $D$.

$f(A) = B$ gives $z_B = az_A + b$

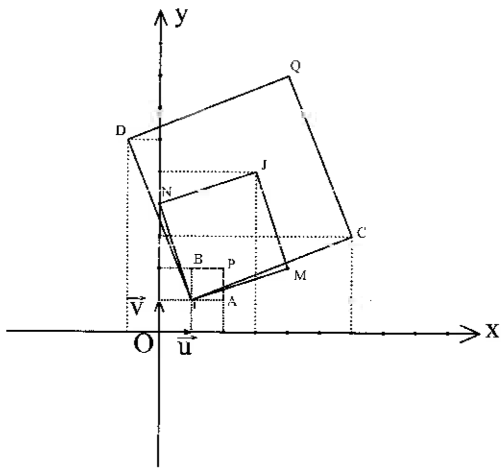
$f(C) = D$ gives $z_D = az_C + b$.

So, we get the system $\begin{cases} a(2+i) + b = 1 + 2i \\ a(6+3i) + b = -1 + 6i \end{cases}$ that admits $a = i$ and $b = 2$, so the complex form of the similitude is $z' = iz + 2$.

b- The angle of the similitude is given by an argument of i, which is $\dfrac{\pi}{2}$.

The ratio of $S$ is $|i| = 1$, consequently the similitude is a rotation $r$.

The center of $r$ is the point of affix $\dfrac{b}{1-a} = \dfrac{2}{1-i} \times \dfrac{1+i}{1+i}$, which gives $\dfrac{b}{1-a} = 1 + i$, then it is the point $I(1;1)$.




2) The complex form of the rotation $R$ is $z' = e^{i\frac{\pi}{2}} z + b$.

$z_J = \dfrac{b}{1-a}$ gives $3+5i = \dfrac{b}{1+i}$ therefore : $b = (3+5i)(1+i)$ .

So $b = -2+8i$, consequently the complex form of $R$ is :

$z' = -iz - 2 + 8i$.

The image of $A$ by $R$ is the point of affix $z' = -iz_A - 2 + 8i$, so

$z' = -i(2+i) - 2 + 8i = -1 + 6i$ , hence $R(A) = D$.

The image of $C$ by $R$ is the point of affix

$z' = -iz_C - 2 + 8i = -i(6+3i) - 2 + 8i$ , so $z' = 1+2i$

hence $R(C) = B$.

3) From the previous question, the image by $R$ of the segment $[AC]$ is the segment $[DB]$.

Since rotations preserve midpoints, the point $N$ is the image of $M$ by $R$.

So, triangle $MJN$ is direct right isosceles at $J$.

Similarly, the image by $f$ of segment $[AC]$ is the segment $[DB]$, then the point $N$ is the image of $M$ by $f$, so triangle $MIN$ is indirect right isosceles at $I$.

Quadrilateral $IMJN$ is hence a square .

4) a- $IAPB$ is a square then $\overrightarrow{AP} = \overrightarrow{IB}$, so $z_P - z_A = z_B - z_I$,

which gives $z_P = 1+2i-1-i+2+i = 2+2i$ .

Similarly $ICQD$ is a square then $\overrightarrow{DQ} = \overrightarrow{IC}$ which gives $z_Q = 4+8i$ .

b- The ratio of the diagonal of a square to its side is $\sqrt{2}$ so $\frac{IP}{IA} = \frac{IQ}{IC} = \sqrt{2}$ .

The measure of the angle between a side of a square and its diagonal is $\frac{\pi}{4}$ , so $\left(\overrightarrow{IA};\overrightarrow{IP}\right) = \left(\overrightarrow{IC};\overrightarrow{IQ}\right) = \frac{\pi}{4}\ (\text{mod } 2\pi)$.

c- Since $A$ and $C$ are distinct, there exists a unique similitude $g$ such that $g(A) = P$ and $g(C) = Q$.

From what preceded, this similitude has $I$ as a center, an angle of measure $\frac{\pi}{4}$ and a ratio of $\sqrt{2}$ .

d- $IMJN$ is a direct square, so $g(M) = J$ .

Similitudes preserve midpoints so $J$ is the midpoint of segment $[PQ]$ .

$N°14$

1) $z_{n+1} = \frac{1}{1+i}z_n + i = \frac{1-i}{2}z_n + i$ , which is the complex form of a similitude .

$a = \frac{1-i}{2} = \frac{\sqrt{2}}{2}e^{-i\frac{\pi}{4}}$ , and $z_\Omega = \frac{b}{1-a} = \frac{i}{1-\frac{1}{1+i}} = 1+i$ .

Hence $A_{n+1}$ is the image of $A_n$ by the direct similitude of center $\Omega(1+i)$, of ratio $\frac{\sqrt{2}}{2}$ and angle $-\frac{\pi}{4}$ .

2) $\left(\overrightarrow{\Omega A_n};\overrightarrow{\Omega A_{n+1}}\right) = \frac{-\pi}{4}(\text{mod}\,2\pi)$ and $\Omega A_{n+1} = \frac{\sqrt{2}}{2}\Omega A_n$ .

Let $\Omega A_n = \ell$ then $\Omega A_{n+1} = \ell\frac{\sqrt{2}}{2}$, using first system of fundamental relations we get

$A_nA_{n+1}{}^2 = \Omega A_n{}^2 + \Omega A_{n+1}{}^2 - 2\times\Omega A_n \times \Omega A_{n+1}\times\cos\frac{\pi}{4}$, so

$A_nA_{n+1}{}^2 = \ell^2 + \frac{\ell^2}{2} - 2\ell\times\ell\frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2} = \frac{\ell^2}{2}$, which gives

$A_nA_{n+1} = \ell\frac{\sqrt{2}}{2}$, consequently $\Omega A_nA_{n+1}$ is right at $A_{n+1}$.

3) a- $\frac{\ell_{n+1}}{\ell_n} = \frac{\Omega A_{n+1}}{\Omega A_n} = \frac{\sqrt{2}}{2}$, ratio of similitude , so $(\ell_n)$ is a

geometric sequence of common ratio $\frac{\sqrt{2}}{2}$ and first term

$\ell_0 = \Omega A_0 = \Omega O = \sqrt{2}$.

b- $\ell_n = \ell_0 \times r^n = \sqrt{2}\times\left(\frac{\sqrt{2}}{2}\right)^n = \frac{1}{\left(\sqrt{2}\right)^{n-1}}$.

$\ell_n \leq 0.4$ gives $\frac{1}{\left(\sqrt{2}\right)^{n-1}} \leq 0.4$ so $\left(\sqrt{2}\right)^{n-1} \geq 2.5$ then

$(n-1)\ln 2 \geq \ln 2.5$, since $\ln 2 > 0$ we get $n-1 > \frac{\ln 2.5}{\ln 2}$

therefore $n > 3.6$ hence 4 is the smallest value of $n$ for which $\ell_n \leq 0.4$

4) a- $a_k =$ Area of $(\Omega A_kA_{k+1}) = \frac{1}{2}\Omega A_k \times \Omega A_{k+1}\times \sin\left(\frac{\pi}{4}\right) = \frac{1}{4}\Omega A_k^2$

$= \frac{1}{4}\left(\frac{\sqrt{2}}{2}\Omega A_{k-1}\right)^2 = \frac{1}{4}\times\frac{1}{2}\times\Omega A_{k-1}^2 = \frac{1}{2}a_{k-1}$

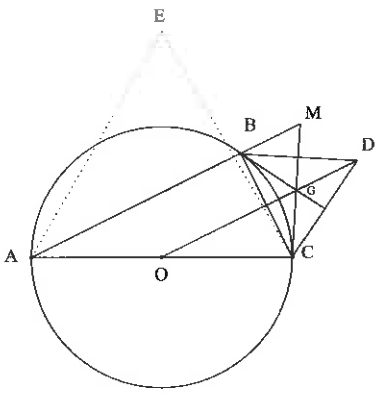
Hence $a_k$ is the general term of a geometric sequence of first term $a_0 = \frac{1}{4}\Omega A_0{}^2 = \frac{1}{2}$ and of ratio $r = \frac{1}{2}$.

b- $S_n = a_0\frac{1-r^{n+1}}{1-r} = \frac{1}{2}\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}$ and $\lim\limits_{n\to+\infty} S_n = 1$.

N° 15

***Part A:***

1) $B$ and $C$ belong to $(\Gamma)$ of center $O$ so $OB = OC$, consequently $O$ is a point on the perpendicular bisector of $[BC]$.
   $BCD$ is an equilateral triangle so $DB = DC$ and consequently $D$ is a point on the perpendicular bisector of $[BC]$.
   $G$ is the centroid of triangle $BCD$ so $G$ is the point of intersection of the perpendicular bisectors of triangle $BCD$.
   Consequently, $G$ is a point on the perpendicular bisector of $[BC]$.
   The straight lines $(AM)$ and $(BC)$ are perpendicular.
   The straight lines $(OG)$ and $(BC)$ are perpendicular then $(AM)$ and $(OG)$ are parallel.
   $(OG)$ passes through the midpoint of $[BC]$ so it cuts $[CM]$ at its midpoint.




2) Triangle $MBC$ is right at $B$ with
   $\left(\overrightarrow{CB};\overrightarrow{GM}\right)=\left(\overrightarrow{CB};\overrightarrow{CG}\right)=-\frac{\pi}{6}(\operatorname{mod} 2\pi)$
   since $[CG)$ is the internal bisector of angle $\left(\overrightarrow{CB};\overrightarrow{CD}\right)$ whose measure is $-\frac{\pi}{3}$.
   $\cos\frac{\pi}{6}=\frac{CB}{CM}=\frac{\sqrt{3}}{2}$, therefore $S = S\left(C;\frac{2\sqrt{3}}{3};-\frac{\pi}{6}\right)$.

***Part B:***

1) $ACE$ is equilateral and $(EO)$ is the perpendicular bisector of $[AC]$ then $EO = AC\frac{\sqrt{3}}{2} = 2\times\frac{\sqrt{3}}{2} = \sqrt{3}$ , therefore the affix of $E$ is $e = i\sqrt{3}$ .

2) The complex form is $z' = az + b$ with $z' = \frac{3+i\sqrt{3}}{4}z + \frac{1-i\sqrt{3}}{4}$

   $a = \frac{3+i\sqrt{3}}{4} = \frac{\sqrt{3}}{2}e^{i\frac{\pi}{6}}$ , hence $\sigma$ is a similitude of ratio

   $k = |a| = \frac{\sqrt{3}}{2}$ and angle $\frac{\pi}{6}$, the center of $\sigma$ has an affix

   $\frac{b}{1-a} = 1$ hence it is the point $C$ .

   Therefore $\sigma = S\left(C;\frac{\sqrt{3}}{2};\frac{\pi}{6}\right)$ so $\sigma = S^{-1}$ .

3) The point $E$ has an affix $e = i\sqrt{3}$ ,the affix $e'$ of point $E'$ is

   $e' = \frac{3+i\sqrt{3}}{4}\left(i\sqrt{3}\right) + \frac{1-i\sqrt{3}}{4} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.

4) a- $|e'| = 1$ so $E'$ belongs to $(\Gamma)$ and since $E' = \sigma(E)$ then $E = S(E')$, so $E$ belongs to the circle image of $(\Gamma)$ by $S$ ,$(C)$ is the image of $(\Gamma)$ by $S$ since $M = S(B)$ then $E \in (C)$.

   b- Taking $S(O) = O'$ so $O = \sigma(O')$, which gives

   $0 = \frac{3+i\sqrt{3}}{4}z + \frac{1-i\sqrt{3}}{4}$ therefore $z = \frac{i\sqrt{3}}{3} = \frac{1}{3}e$ hence

   $\overrightarrow{OO'} = \frac{1}{3}\overrightarrow{OE}$ and consequently $O'$ is the centroid of triangle $ACE$ .

**N° 16.**

1) Vector $\overrightarrow{AB}$ has an affix $z_B - z_A = 2 + i$ .

   Vector $\overrightarrow{A'B'}$ has an affix $z_{B'} - z_{A'} = 2 + i$ .

   Hence $\overrightarrow{AB} = \overrightarrow{A'B'}$ , consequently $ABB'A'$ is a parallelogram.

$A'B = 5\sqrt{2} = AB'$ then $ABB'A'$ is a rectangle .

2) $(\Delta)$ is the perpendicular bisector of $[BB']$, so $M$ is a point on $(\Delta)$, $MB = MB'$.

If $z = x + iy$ then $M(x; y)$:

$(3-x)^2 + (-1-y)^2 = (-x)^2 + (5-y)^2$, therefore $2x - 4y + 5 = 0$.

3) $S(A) = A'$ and $S(B) = B'$ give $z_{A'} = a\overline{z_A} + b$ and $z_{B'} = a\overline{z_B} + b$.

So we get the following system $\begin{cases} -2+4i = a(1+2i)+b \\ 5i = a(3+i)+b \end{cases}$ that

admits $a = \frac{3}{5} + \frac{4}{5}i$ and $b = 2i - 1$ as solutions.

Consequently, the complex form of $S$ is $z' = \left(\frac{3}{5} + \frac{4}{5}i\right)\overline{z} + 2i - 1$.

4) a- $z_C = \left(-\frac{6}{5} - \frac{8}{5}i\right)(1+2i) + 5 - i = 7 - 5i$.

$z_D = \left(-\frac{6}{5} - \frac{8}{5}i\right)(3+i) + 5 - i = 3 - 7i$.

b- The complex form of $h$ is $z' = -2z + b$

$z_\Omega = 1 + i = \frac{b}{1-a} = \frac{b}{3}$ which gives $b = 3 + 3i$.

Consequently the complex form of $h$ is $z' = -2z + 3 + 3i$ .

$h(A')$ is the point of affix :

$z' = -2z_{A'} + 3 + 3i = -2(-2+4i) + 3 + 3i = 7 - 5i = z_C$.

Hence $h(A') = C$, similarly we have $h(B') = D$ .

c- $h^{-1} = h\left(\Omega; -\frac{1}{2}\right)$.

The complex form of $h$ is $z' = -2z + 3 + 3i$ therefore

$z = \frac{z' - 3 - 3i}{-2}$ hence the complex form of $h^{-1}$ is

$z' = -\frac{1}{2}z + \frac{3}{3} + \frac{3}{2}i$.

5) a- $z \xrightarrow{g} z' = \left(-\frac{6}{5} - \frac{8}{5}i\right)\overline{z} + 5 - i \xrightarrow{h^{-1}} z'' = -\frac{1}{2}z' + \frac{3}{2} + \frac{3}{2}i$.

Which gives $z'' = \left(\frac{3}{5}+\frac{4}{5}i\right)\overline{z}-1+2i$.

b- $f = S$.

$N°17.$

1) $M$ is invariant if $M = T(M)$ which gives $x = x'$ and $y = y'$

so $x = -3y+2$ and $y = -3x+6$, we get the system $\begin{cases} x+3y=2 \\ 3x+y=6 \end{cases}$

that admits $x = 2$ and $y = 0$ as solutions so the invariant point is $\Omega(2;0)$.

2) $z' = x'+iy' = -3y+2+i(-3x+6) = -3ix-3y+2+6i$

$z' = -3ix+3i^2y+2+6i = -3i(x-iy)+2+6i = -3i\overline{z}+2+6i$.

3) $T \ : \ z \xrightarrow{S_{x'x}} \overline{z} \xrightarrow{S\left(\Omega;2;-\frac{\pi}{2}\right)} z' = -3i\overline{z}+2+6i$.

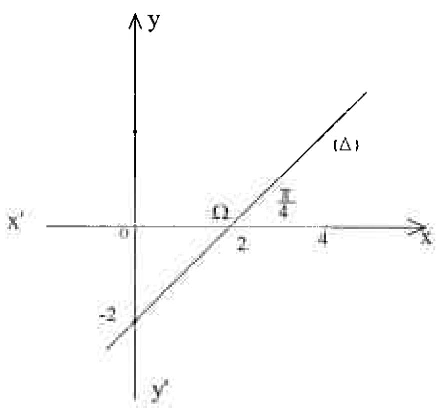
Then $T = S\left(\Omega;2;-\frac{\pi}{2}\right)\circ S_{x'x}$.

4) We know that $S_\Delta \circ S_{x'x} = r(\Omega;2(x'x;\Delta))$.

So $S_\Delta \circ S_{x'x} = r\left(\Omega;\frac{\pi}{2}\right)$.

Therefore $T = S\left(\Omega;3;-\frac{\pi}{2}\right)\circ S_{x'x} = h(\Omega;-3)\circ r\left(\Omega;\pi-\frac{\pi}{2}\right)\circ S_{x'x}$

$T = h(\Omega;-3)\circ S_\Delta \circ S_{x'x}\circ S_{x'x} = h(\Omega;-3)\circ S_\Delta$.

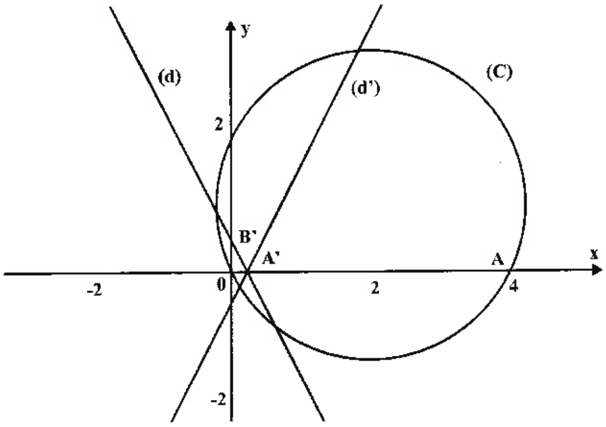


N° 18.

1) $f : z \xrightarrow{S_{x'x}} \bar{z} \xrightarrow{I(O;1)} z' = \frac{1}{\bar{\bar{z}}} = \frac{1}{z}$.

Hence $f = I(O;1) \circ S_{x'x}$.

2) Circle $(C)$ passes through the center of the inversion $O$, so its image is a straight line.




$(C)$ intersects $x'x$ at point $A(4;0)$, the image of $A$ by $I(O;1)$ is the point $A'$ of $(OA)$ such that $\overline{OA} \times \overline{OA'} = 1$, which gives $\overline{OA'} = \frac{1}{4}$.

$(C)$ intersects $y'y$ at point $B(0;2)$, the image of $B$ by $I(O;1)$ is the point $B'$ such that $\overline{OB} \times \overline{OB'} = 1$, which gives $\overline{OB'} = \frac{1}{2}$.

The image of $(C)$ by $I(O;1)$ is the straight line $(d)$ passing through the points $A'$ and $B'$.

The image of $(d)$ by the symmetry $S_{x'x}$ is the straight line $(d')$ passing through the points $A'$ and $B''\left(0;-\frac{1}{2}\right)$.

Hence $(d')$ is the image of $(C)$ by $f$.

# Indications

**N° 3.**

1) $z' = (1+i)z + 2 - i$.

2) $S\left(\Omega(1;2)\ ;\ \sqrt{2}\ ;\ \dfrac{\pi}{4}\right)$

3) $\Omega M' = \sqrt{2}\,\Omega M$ and $\left(\overrightarrow{\Omega M};\overrightarrow{\Omega M'}\right) = \dfrac{\pi}{4} \pmod{2\pi}$

Taking $\Omega M = \ell$.

$$MM'^2 = \Omega M^2 + \Omega M'^2 - 2\Omega M \times \Omega M' \times \cos\left(\frac{\pi}{4}\right)$$

$$MM'^2 = \ell^2 + 2\ell^2 - 2\ell^2 \times \frac{\sqrt{2}}{2} = \ell^2.$$

**N° 4.**

1) $S$ is a direct plane similitude of ratio $\sqrt{2}$, angle $\dfrac{3\pi}{4}$ and center the point $\Omega$ of affix $\omega = \dfrac{b}{1-a} = \dfrac{2-i}{2-i} = 1$.

Then: $S\left(\Omega;\sqrt{2};\dfrac{3\pi}{4}\right)$.

2) a- The affix of $M'$ is $z' = (-1+i)z + 2 - i$.

The affix of $M''$ is $z'' = -2iz + 1 + 2i$.

$G$ is the center of gravity of triangle $MM'M''$ therefore :

$$z_G = \frac{z_M + z_{M'} + z_{M''}}{3} = -\frac{1}{3}iz + 1 + \frac{1}{3}i.$$

**N° 5.**

1) $k = \dfrac{AB}{OI} = 2$, $\theta = \left(\overrightarrow{OI};\overrightarrow{AB}\right) = \dfrac{\pi}{3} \pmod{2\pi}$.

2) a- $f(O) = O$.

b- $r \circ S = S\left(A;1;-\frac{\pi}{3}\right) \circ S\left(\Omega;2;\frac{\pi}{3}\right) = S(\Omega';2;0) = h(\Omega';2)$.

N° 6.

***Part A:***

1) $k = \frac{\sqrt{2}}{2}$, $\alpha = \frac{\pi}{2}$.

2) $\frac{AB}{BC} = \frac{\sqrt{2}}{2}$ and $\left(\overrightarrow{AB};\overrightarrow{BC}\right) = \frac{\pi}{2}(\text{mod}\,2\pi)$ then $S(B) = C$.

4) $S \circ S = S\left(w;\frac{1}{2};\pi\right) = h\left(w;-\frac{1}{2}\right)$.

***Part B:***

3) $\frac{\ell_{n+1}}{\ell_n} = \frac{\sqrt{2}}{2}$, ratio of $S$.

N° 7.

***Part A:***

2) $JB = \sqrt{2}JC$ and $\left(\overrightarrow{JC};\overrightarrow{JB}\right) = \frac{\pi}{4}(\text{mod}\,2\pi)$.

Taking $JC = \ell$, we get:

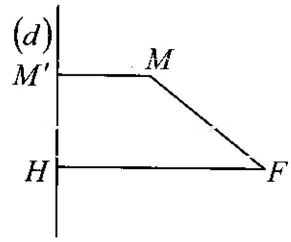
$BC^2 = JC^2 + JB^2 - 2JC \times JB \times \cos\frac{\pi}{4} = \ell^2$.

# CHAPTER 11

# Conics

**Chapter Review :**

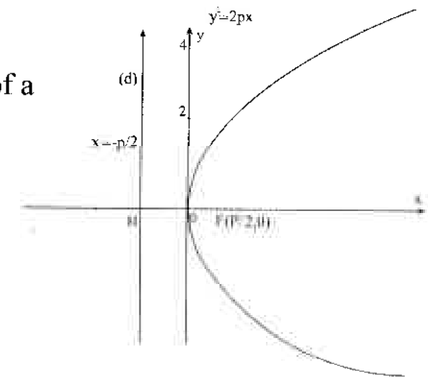
- Definition:
  Let $F$ be a fixed point of the plane , $(d)$a fixed straight line such that $F \not\subset (d)$ and $e$ a strictly positive number .
  We define the conic of focus $F$ , directrix $(d)$ and eccentricity $e$, the set $(C)$ of points $M$ of the plane such that $\frac{MF}{MM'} = e$ where $M'$ is the orthogonal projection of $M$ on $(d)$.



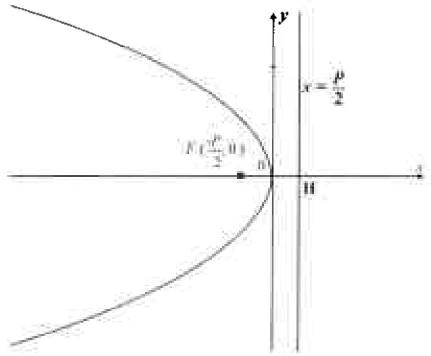

  * If $e = 1$ then this conic is a parabola .
  * If $e < 1$ then this conic is an ellipse .
  * If $e > 1$ then this conic is a hyperbola .
  * The straight line passing through $F$ and perpendicular to $(d)$ is called the focal axis of the conic.

## *I- Parabola:*

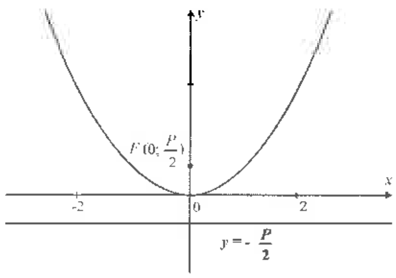
- The reduced equation of a parabola is $y^2 = 2px$ .



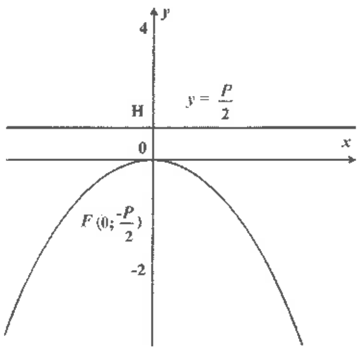
- Other forms of the parabola:




$y^2 = -2px$

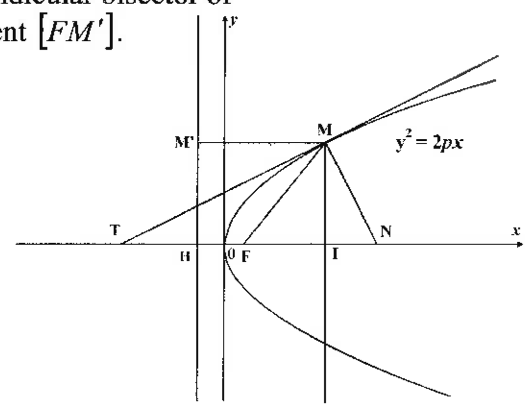



$x^2 = 2py$




$x^2 = -2py$

- Tangent and normal to a parabola:
  * The tangent $[MT)$ is the internal bisector of angle $F\hat{M}M'$ and perpendicular bisector of segment $[FM']$.



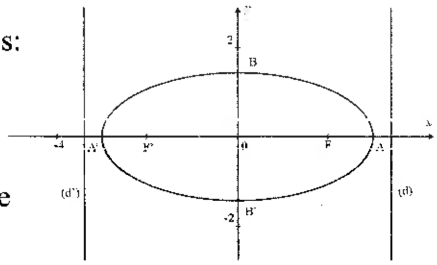
* The normal $[MN)$ is the external bisector of angle $F\hat{M}M'$.
* $[IT]$ is the sub-tangent , its midpoint is the vertex of $(P)$.
* $[IN]$ is the sub-normal , $IN = p$ .
* An equation of the tangent at $M(x_0;y_0)$ to $(P)$ is:

$$y - y_0 = \frac{p}{y_0}(x - x_0) .$$

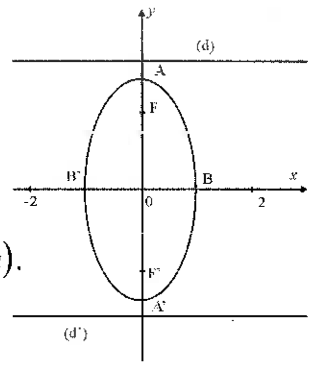
* An equation of the normal at $M(x_0;y_0)$ to $(P)$ is:

$$y - y_0 = -\frac{y_0}{p}(x - x_0) .$$

***II- Ellipse :***



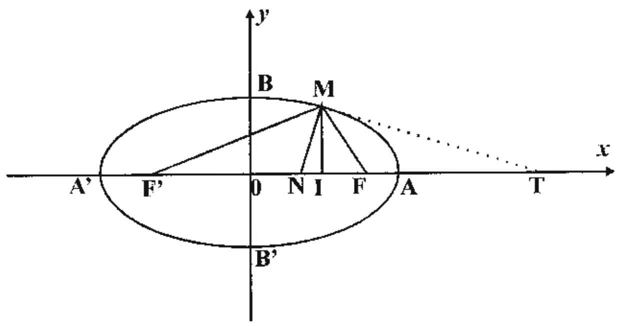

* The axis $x'Ox$ is the focal axis:
  * The reduced equation of $(E)$ is: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
  * The vertices of the ellipse on the focal axis are: $A(a;0)$, $A'(-a;0)$.
  * The vertices of the ellipse on the non-focal axis are: $B(0;b)$, $B'(0;-b)$.
  * The foci are: $F(c;0)$ , $F'(-c;0)$ with $c^2 = a^2 - b^2$.
  * The directrices are the straight lines of equations $x = \frac{a^2}{c}$ and $x = -\frac{a^2}{c}$.
  * The eccentricity is $e = \frac{c}{a}$.




* The axis $y'Oy$ is the focal axis :
  * The reduced equation of $(E)$ is: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$.
  * The vertices of the ellipse on the focal axis are : $A(0;a)$, $A'(0;-a)$.
  * The vertices on the non-focal axis are : $B(b;0)$, $B'(-b;0)$.

* The foci are: $F(0;c)$ , $F'(0;-c)$ with $c^2 = a^2 - b^2$.
* The directrices are the straight lines of equations $y = \dfrac{a^2}{c}$ and $y = -\dfrac{a^2}{c}$.
* The eccentricity is $e = \dfrac{c}{a}$.

- Tangent and normal to an ellipse :



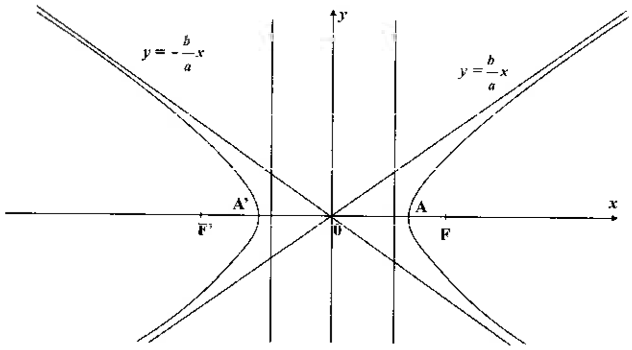

* The tangent $[MT)$ is the external bisector of angle $F\hat{M}F'$
* The normal $[MN)$ is the internal bisector of angle $F\hat{M}F'$.

- The principal circle is the circle of center , the center of symmetry of the ellipse and of radius $R = a$.
- The area of an ellipse is $A = \pi ab$ square units.
- $M$ belongs to an ellipse of foci $F$ and $F'$ if and only if $MF + MF' = 2a$ where $2a$ equals the length of the major axis.
- If $(C)$ is a circle of plane $(P)$, of center $w$ and of radius $R$ and if $(Q)$ is a plane , not parallel and not perpendicular to $(P)$, then the orthogonal projection of $(C)$ on $(Q)$ is an ellipse.
  The center of the ellipse is the orthogonal projection of $w$ on $(Q)$.
  $a = R$, $b = R\cos\alpha$ where $\alpha$ is the acute angle between $(P)$ and $(Q)$.

***III- Hyperbola.***

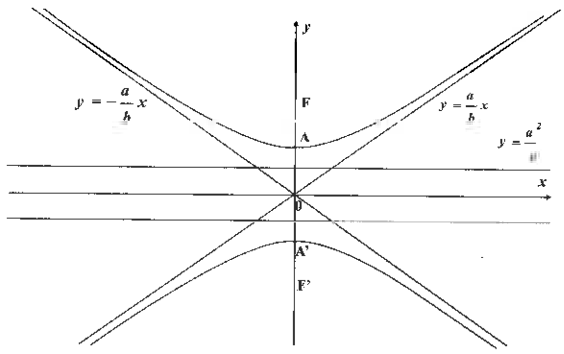
- The axis $x'Ox$ is the focal axis.
  * The reduced equation of $(H)$ is: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

* The vertices on the focal axis are : $A(a;0)$, $A'(-a;0)$.
* The foci are: $F(c;0)$ , $F'(-c;0)$ where $c^2 = a^2 + b^2$.



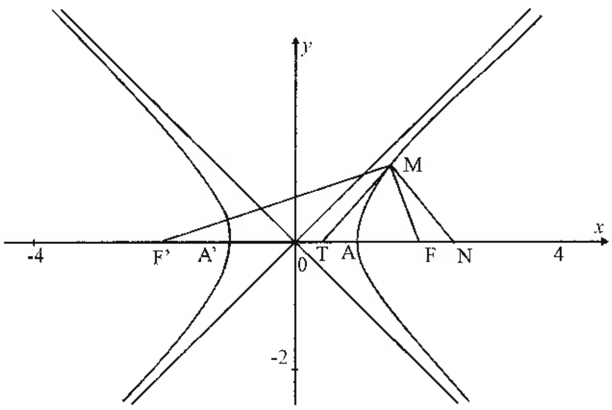

* The directrices are the straight lines of equations: $x = \frac{a^2}{c}$ and $x = -\frac{a^2}{c}$.
* The asymptotes are the straight lines of equations: $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.
* The eccentricity is $e = \frac{c}{a}$.

- The axis $y'Oy$ is the focal axis.
    * The reduced equation of $(H)$ is: $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.



* The vertices on the focal axis are: $A(0;a)$, $A'(0;-a)$
* The foci are : $F(0;c)$ , $F'(0;-c)$ where $c^2 = a^2 + b^2$.
* The directrices are the straight lines of equations:
  $y = \frac{a^2}{c}$ and $y = -\frac{a^2}{c}$.
* The asymptotes are the straight lines of equations $y = \frac{a}{b}x$ and
  $y = -\frac{a}{b}x$.
* The eccentricity is $e = \frac{c}{a}$.

- Tangent and normal to a hyperbola:




* The tangent $[MT)$ is the internal bisector of the angle $F\hat{M}F'$
* The normal $[MN)$ is the external bisector of the angle $F\hat{M}F'$.

- A hyperbola is said to be rectangular if $a = b$.
  In this case the asymptotes are the bisectors of the axes of symmetry.
  $c = a\sqrt{2}$ and $e = \sqrt{2}$.
- $M$ belongs to a hyperbola of foci $F$ and $F'$ if and only if
  $|MF - MF'| = 2a$ where $2a$ is the distance between the vertices.

# Solved Problems

### Part I - The Parabola :

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.

N° 1.

Find an equation of the parabola of focus $F(3;0)$ and directrix the straight line $(d)$ of equation $x=-3$.

N° 2.

1) Find , in two methods , an equation of the parabola of focus the point $F(2;0)$ and of directrix the straight line $(d)$ of equation $x=-1$.
2) Find an equation of the parabola of focus the point $F(-1;1)$ and of directrix the straight line $(d)$ of equation $x=1$.
3) Find an equation of the parabola of focus the point $F(2;1)$ and of vertex the point $S(0;1)$.
4) Find an equation of the parabola of vertex the point $S(1;1)$ and of directrix the straight line $(d)$ of equation $y=4$.
5) Find an equation of the parabola of focus $F(1;0)$ and directrix the straight line $(d)$ of equation $y=x$.

N° 3.

Determine the elements and trace the representative curves of the parabolas whose equations are given below:

1) $y^2=2x-4$
2) $y^2-2y=3x-2$.
3) $x^2+4x-2y+2=0$.

N° 4.

Consider the parabola $(P)$ of equation $y^2=4x$ and the point $B(0;2)$.

1) Write an equation of the straight line $(\delta)$ passing through the point

$B$ and of slope $m$.

2) a- Show that the relation between the abscissas of the points of intersection of $(P)$ and $(\delta)$ is $m^2x^2+4(m-1)x+4=0$.

   b- Study according to the values of $m$ the intersection of $(\delta)$ and $(P)$.

   c- In the case where $(\delta)$ is tangent to $(P)$, find the coordinates of the point of tangency and deduce an equation of the tangent to $(P)$ through the point $C(0;-2)$ other than the y-axis

$D=0$
$m=1$

**N°5.**

Consider the parabola $(P)$ of equation $y^2=2px$, of focus $F$ and directrix $(d)$.

Let $M(x_0;y_0)$ be a variable point of $(P)$ and $H$ its orthogonal projection on the axis $y'y$.

The perpendiculars through $O$ and $H$ to the straight line $(OM)$ intersect the straight line $(MH)$ in $K$ and $(OF)$ in $I$.

1) a- Prove that the slope of the straight line $(IH)$ is $-\dfrac{y_0}{2p}$.

   b- Deduce that the point $I$ is fixed.

2) What is the set of points $K$ as $M$ varies ? vectors we calculate $x_K = ??$

**N°6.**

Consider the parabolas $(P)$ and $(P')$ of respective equations $y^2=2x+1$ and $y^2=-2x+1$.

1) Determine the points of intersection of $(P)$ and $(P')$.
2) Show that $(P)$ and $(P')$ are symmetric to each other with respect to the axis $y'y$.
3) a- Determine the vertex, the focus and the directrix of $(P)$.
   b- Deduce the elements of $(P')$.
4) Find the equations of the tangents to $(P)$ and $(P')$ at the point of abscissa 0 and of positive ordinate.
5) Trace $(P)$ and $(P')$.
6) Calculate the area of the domain $(D)$ limited by $(P)$ and $(P')$.
7) Calculate the volume of the solid obtained by rotating $(D)$ about

the axis $x'x$.

**N°7.**

In an oriented plane, consider a fixed straight line $(d)$ and a fixed point $A$ not belonging to $(d)$ and designate by $(C)$ the circle of center $A$ and tangent to $(d)$ at the point $H$.

1) Determine the set of points of the foci of all the parabolas $(P)$ that pass through the point $A$ and of directrix $(d)$.
2) The plane is referred to an orthonormal system $(H;\vec{i},\vec{j})$ with $\vec{i}=\overrightarrow{HA}$.
   Suppose that the point $F(x;y)$ belongs to $(C)$ and designate by $S$ the vertex of $(P)$.
   a- Express $X$ and $Y$, the coordinates of $S$ in terms of $x$ and $y$
   b- Deduce the set of points $S$ as $F$ varies.

**N°8.**

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the parabola $(P)$ of equation $y^2=4x$.
Designate by $F$ the focus of $(P)$ and by $(d)$ its directrix.
Let $M$ be a variable point of $(P)$ and $(T)$ the tangent to $(P)$ at $M$.

1) Find the set of points $N$ orthogonal projection of $F$ on $(T)$ as $M$ varies.
2) Find the set of points $N'$ orthogonal projection of $F$ on the normal to $(P)$ at $M$.

***Part II - The Ellipse:***

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.

**N°9.**

Qu

1) Find an equation of the ellipse of focus the point $F(3;0)$ and of directrix the axis $y'y$ and eccentricity $e=\dfrac{1}{2}$.
2) Find an equation of the ellipse of foci $F'(2;0)$ and $F(6;0)$ knowing that the origin $O$ is one of the vertices of this ellipse.
3) Find an equation of the ellipse of foci $O(0;0)$ and $F(0;6)$

knowing that the straight line $y=-1$ is a directrix of this ellipse .

4) Find an equation of the ellipse of vertices $A(4;1)$ and $A'(-2;1)$ knowing that the distance between the directrices of this ellipse is equal to 8.

N° 10.

1) Consider the ellipse $(E)$ of equation $4x^2+9y^2-8x-32=0$.
Determine the center , the focal axis , the vertices , the foci and the directrices of $(E)$ then trace $(E)$.

2) Given the ellipse $(E')$ of equation $9x^2+y^2+18x-2y+1=0$.
   a- Trace $(E')$.
   b- Write an equation of the principal circle $(C)$ of $(E')$.
   c- Calculate the area of the domain limited by $(C)$ and $(E')$.

N° 11.

In the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$ consider the ellipse $(E)$ of center $O$ , focus the point $F(3;0)$ and vertex the point $A(5;0)$ .

1) a- Find an equation of $(E)$ as well as the equation of the directrix associated with the focus $F$ .
   b- Trace $(E)$ .

2) Let $B$ be the point of affix $4i$ and $M$ a variable point of $(E)$ of affix $z$ .
   Let $G$ be the point defined by $\overrightarrow{GA}+\overrightarrow{GB}+2\overrightarrow{GM}=\vec{0}$ and let $Z$ be the affix of $G$ .
   a- Determine $Z$ in terms of $z$ .
   b- Show that $G$ traces an ellipse $(E')$ as $M$ traces $(E)$.
   c- Prove that $(E')$ is interior to $(E)$, draw $(E')$ and calculate the area of the domain limited by the two ellipses $(E)$ and $(E')$.

N° 12.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the ellipse $(E)$ of equation $9x^2+4y^2=36$.
$P$ is a variable point of $(E)$ , designate by $M$ the midpoint of the segment joining $P$ to the vertex of $(E)$ of negative abscissa .

Find the set $(\gamma)$ of points $M$ as $P$ varies on $(E)$ and write an equation of $(\gamma)$.

**N° 13.**

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the ellipse $(E)$ of equation $\frac{x^2}{10}+\frac{y^2}{5}=1$ and let $(d)$ be the straight line of equation $3x+2y+7=0$.

1) Trace $(E)$ and $(d)$.
2) Write the equations of the tangents to $(E)$ that are parallel to $(d)$.
3) Let $F$ and $F'$ be the foci of $(E)$ and $M(x;y)$ a variable point of $(E)$.

   a- Prove that $MF^2=\frac{1}{2}\left(x^2-4\sqrt{5}x+20\right)$ and

   $MF'^2=\frac{1}{2}\left(x^2+4\sqrt{5}x+20\right)$.

   b- Deduce that $MF+MF'=2\sqrt{10}$.

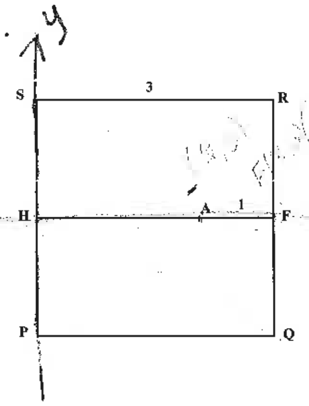
**N° 14.**

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, given the circle $(C_1)$ of center $I(1;0)$ and radius $R_1=8$ and the circle $(C_2)$ of center $J(3;0)$ and radius $R_2=4$ .

A variable circle $(\gamma)$ of center $M$ varies remaining tangent internally to $(C_1)$ and externally to $(C_2)$.

1) Prove that as $(\gamma)$ varies, the point $M$ traces a conic whose nature is to be determined.
2) Write an equation of the conic obtained . y

**N° 15.**




$PQRS$ is a square of side 3 .

$H$ and $F$ are the respective midpoints of $[SP]$ and $[QR]$.

Let $(\mathcal{E})$ be the ellipse of focus $F$ , and of directrix the straight line $(PS)$ and eccentricity $e = \frac{1}{2}$.

$A$ is a point on $[HF]$ such that $AF = 1$.

1) Show that $R$ and $Q$ belong to $(\mathcal{E})$.
2) Show that $A$ is a vertex of $(\mathcal{E})$.
3) The plane is referred to a direct orthonormal system $(H;\vec{i},\vec{j})$ with $\overrightarrow{HF} = 3\vec{i}$.
   a- Prove that an equation of $(\mathcal{E})$ in the system $(H;\vec{i},\vec{j})$ is $3x^2 + 4y^2 - 24x + 36 = 0$.
   b- Determine the second vertex $A'$ of $(\mathcal{E})$ lying on the focal axis.
   c- Determine the second focus $F'$ of $(\mathcal{E})$
   d- Determine the vertices of $(\mathcal{E})$ lying on the non-focal axis.
   e- Determine an equation of the tangent $(T)$ at $R$ to $(\mathcal{E})$.
   f- Trace $(\mathcal{E})$ in the system $(H;\vec{i},\vec{j})$.
   g- Let $(\mathcal{D})$ be the domain interior to $(\mathcal{E})$.
   Calculate the volume of the solid obtained by rotating $(\mathcal{D})$ about the focal axis $(\mathcal{E})$ .

***Part 3 - The Hyperbola:***

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.

$N°$ 16.

1) Find an equation of the hyperbola of focus the point $F(2;0)$, and directix the straight line of equation $x = -1$ and of eccentricity $e = \sqrt{3}$.
2) Find an equation of the hyperbola of foci $F'(1;0)$ and $F(5;0)$ and having $2\sqrt{2}$ as a length of its focal axis .
3) Find an equation of the hyperbola of focal axis $y'y$, of vertices the points $O(0;0)$ and $A(0;-2)$ and of eccentricity $e = 2$.
4) Find an equation of the hyperbola of foci $O(0;0)$ and $F(0;4)$ and passing through the point $M(12;9)$.

$N°$ 17.

1) Determine the center, the vertices, the foci, the asymptotes then trace the hyperbola of equation $\frac{(x-1)^2}{9}-\frac{(y+1)^2}{16}=1$.
2) Determine the reduced equation then trace the hyperbola of equation : $4y^2-9x^2+8y+18x-41=0$.

N° 18.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the family of curves $(C_m)$ of equation $\frac{x^2}{2m-4}+\frac{y^2}{m+1}=1$ where $m$ is a real parameter such that $m \in IR-\{-1;2\}$.
Study, according to the values of $m$, the nature of $(C_m)$.

N° 19.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the family of curves $(C_m)$ of equation: $(m-1)x^2+my^2-2x+2y-3=0$ where $m$ is a real parameter.

1) Study, according to the values of $m$, the nature of $(C_m)$.
2) Suppose that $m=1$ and consider the parabola $(P)$ of equation $y^2-2x+2y-3=0$.
   a- Show that the straight line of equation $y=-1$ is an axis of symmetry of $(P)$.
   b- Let $(\delta)$ be a variable straight line passing through the focus of $(P)$ and that cuts $(P)$ in two points $M$ and $N$.
      i- Prove that the tangents $(T_1)$ and $(T_2)$ at $M$ and $N$ to $(P)$ are perpendicular
      ii- Designate by $I$ the point of intersection of these two tangents. Prove that $I$ belongs to the directrix of $(P)$.

N° 20.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the points $F(8;0)$ and $F'(-2;0)$.
Let $(C)$ be the circle of center $F$ and radius $R=7$ cm and $(C')$ the

circle of center $F'$ and radius $R' = 1$ .
$(\omega)$ is a variable circle of center $M$ tangent externally at a point $P$ to $(C)$ and at a point $Q$ to $(C')$.

1) Prove that as $(\omega)$ varies, the point $M$ traces a branch of a hyperbola $(H)$ of equation $16x^2 - 9y^2 - 96x = 0$.
2) Determine the vertices and the asymptotes of $(H)$ and trace $(H)$.
3) $(E)$ is an ellipse of focal axis the straight line of equation $x = 3$, and having the same vertices as $(H)$ and of focal distance $2c = 2\sqrt{7}$.
   a- Find an equation of $(E)$ and trace $(E)$ in the same system as that of $(H)$.
   b- Deduce the drawing of the curve of equation: $16x^2 - 9y|y| - 96x = 0$.
4) Let $h_1$ be the negative dilation that transforms the circle $(C)$ to the circle $(\omega)$ and let $h_2$ be the negative dilation that transforms the circle $(\omega)$ to the circle $(C')$.
   Determine $h_2 \circ h_1$ and prove that $(PQ)$ passes trough a fixed point.

***General Problems :***

N°21.

In an oriented plane, consider a fixed straight line $(d)$ and a fixed point $A$ not belonging to $(d)$.
Let $(C)$ be a variable circle of center $M$ passing through $A$ and tangent to $(d)$.
Prove that as $(C)$ varies, the point $M$ traces a conic whose nature and elements are to be determined.

N°22.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

1) Let $M$ be a point of affix $z = x + iy$ .
   a- Determine the set $(d)$ of points $M$ such that $z + \bar{z} + 4 = 0$.
   b- Prove that for all points $M$ of the plane the distance of $M$ to

$(d)$ is equal to $\frac{1}{2}\left|z+\bar{z}+4\right|$.

2) Let $F$ be the point of affix $1+i$ and $(P')$ the plane deprived of the straight line $(d)$.

   Let $(E)$ be the set of points $M$ of affix $z$ of $(P')$ such that

   $\left|\frac{z-1-i}{z+\bar{z}+4}\right| = \frac{\sqrt{2}}{4}$.

   Prove that $(E)$ is a conic whose eccentricity and nature are to be determined.

$\boxed{N^\circ 23.}$

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

Designate by $(E)$ the ellipse of equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ , and by $S$ the direct plane similitude of center $O$, of ratio $\frac{\sqrt{2}}{2}$ and angle $\frac{\pi}{4}$.

Let $(E_1)$ be the image of $(E)$ by $S$.

1) a- Trace $(E)$.

   b- Let $F$ be one of the foci of $(E)$ and $(\Delta)$ the directrix associated with $F$, designate by $M$ a point of $(E)$ and by $H$ its orthogonal projection on $(\Delta)$.

   Designate by $(\Delta_1)$, $F_1$ , $H_1$ and $M_1$ the images of $(\Delta)$ , $F$ , $H$ and $M$ by $S$.

   i- Find the value of $\frac{M_1F_1}{M_1H_1}$.

   ii- Show that $(E_1)$ is an ellipse whose two axes of symmetry are to be determined.

2) a- $z$ and $z_1$ are the respective affixes of $M$ and of its image $M_1$ by $S$, show that $z_1=\frac{1}{2}(1+i)z$ and that if $M$ is distinct from $O$ then triangle $OMM_1$ is right isosceles at $M_1$.

   b- Deduce a construction of $M_1$ starting from a given point $M$ distinct from $O$.

   c- Construct $(E_1)$.

**N°24.**

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$. Let $S$ the direct plane similitude of center $\Omega(-2;3)$, of ratio $\sqrt{2}$ and angle $\frac{\pi}{4}$.

For all points $M$ of affix $z = x + iy$, we associate the point $M'$ of affix $z' = x' + iy'$ such that $M' = S(M)$.

1) a- Express $z'$ in terms of $z$.
   b- What is the inverse similitude $S'$ of $S$ ?
2) Write the complex form of $S'$.
3) Let $(E')$ be the ellipse of equation $9x^2 + 16y^2 = 144$ and designate by $(E)$ the image of $(E')$ by $S'$.
   Without finding an equation of $(E)$, precise the nature of $(E)$ and determine its center and vertices .

**N°25.**

$OABC$ is a square of center $I$ and such that $(\overrightarrow{OA};\overrightarrow{OC}) = \frac{\pi}{2}(\text{mod}\, 2\pi)$.

Let $r$ be the rotation of center $\Omega$ such that $r(O) = I$ and $r(I) = C$.

1) Determine the angle of $r$ and construct $\Omega$.
2) Construct the image of the square $OABC$ by $r$.
3) Let $(P)$ be the parabola of focus $A$ and of directrix $(OC)$.
   Designate by $(P')$ the image of $(P)$ by $r$.
   a- Verify that $B$ is a point of $(P)$.
   b- What does the straight line $(OB)$ represent for $(P)$ ?
   c- Determine the vertex of $(P)$.
   d- Show that the straight line $(AC)$ is tangent to $(P')$ at $B'$.
   e- Determine the vertex and the focus of $(P')$.

**N°26.**

The space is referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.
Consider the plane $(P)$ of equation $4x - 3z - 8 = 0$, the circle $(C)$ of plane $(P)$ of center $I(5;3;4)$ and radius $R = 5$ and designate by $(d)$ the line of intersection of the two planes $(P)$ and $(xOy)$.

1) Show that the orthogonal projection $(E)$ of $(C)$ on the plane $(xOy)$ is an ellipse .
2) Calculate the eccentricity of $(E)$ as well as the area of the domain interior to $(E)$.

N° 27.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the circle $(C)$ of center $I(6;0)$ and radius $R=2$.

Let $M$ be a variable point of $(C)$ and $M'$ the orthogonal projection of $M$ on $y'y$ and let $N$ be the midpoint of $[MM']$.

1) Show that $N$ describes an ellipse $(E)$ as $M$ describes $(C)$.
2) Trace $(E)$.
3) Let $L$ be a point defined by $\overrightarrow{ML}=\lambda\,\overrightarrow{M'M}$ where $\lambda$ is a non-zero real number .

   Prove that $L$ describes a conic $(\Gamma_\lambda)$ whose nature is to be determined.

N° 28.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the ellipse $(E)$ of equation $\dfrac{x^2}{25}+\dfrac{y^2}{9}=1$.

Let $F$ be the focus of $(E)$ of positive abscissa and $(d)$ the directrix associated to $F$.

1) Trace $(E)$, place $F$ and $(d)$and calculate the eccentricity $e$.
2) $(MM')$ is a focal cord passing through $F$ such that $(\vec{i};\overrightarrow{FM})=\theta(\mathrm{mod}\,2\pi)$ where $0<\theta<\pi$, suppose $FM=r$.
   a- Prove that the abscissa of $M$ is $x_M=4+r\cos\theta$.
   b- Calculate the distance of $M$ to $(d)$in terms of $r$ and $\theta$ and deduce that $r=\dfrac{9}{5+4\cos\theta}$.
   c- Prove that $\dfrac{1}{FM}+\dfrac{1}{FM'}$ remains constant as $\theta$ varies.

d- What is the minimal length of the focal cord ?
Construct this cord .

N°29.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.
Consider the points $A(m\,;0)$ and $B(0\,;n)$ where $m$ and $n$ are two real numbers.
Let $P$ be the point defined by $\overrightarrow{OA} = 2\,\overrightarrow{BP}$.

1) Determine the coordinates $x$ and $y$ of $P$ in terms of $m$ and $n$.
2) Suppose that $m$ and $n$ vary in such a way that $AB = 2$ .
Prove that $P$ varies on an ellipse $(E)$ of equation $4x^2 + y^2 = 4$.
3) Let $(C)$ be the curve of equation $5x^2 + 6xy + 5y^2 - 8 = 0$.
   a- Prove that $(C)$ is the image of $(E)$ by the rotation $r$ of center $O$ and angle $\dfrac{\pi}{4}$.
   b- Deduce the nature of $(C)$.
   c- Determine the focal axis and a focus of $(C)$.
   d- Calculate the eccentricity of $(C)$ and its area .

N°30.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the parabola $(P)$ of equation $y^2 = 4x + 4$.

1) Determine the vertex , the focus and the directrix of $(P)$.
2) Trace $(P)$.
3) $(P)$ cuts the axis $y'y$ in two points $A$ and $B$.
Prove that $OA = OB = 2$ .
4) a- Let $M$ be a point of $(P)$ of positive abscissa and $H$ the orthogonal projection of $M$ on $y'y$, show that $MO - MH = 2$.
   b- Deduce that the circle of center $M$ and radius $MH$ remains tangent to the circle of diameter $[AB]$.
5) Let $(d)$ be a straight line passing through $F$ and of slope $m$.
$(d)$ intersects $(P)$ in two points $M'$ and $M''$, designate by $I$ the midpoint of $[M'M'']$.
   a- Calculate the coordinates of $I$ in terms of $m$.

b- Deduce that $I$ varies on a parabola $(P')$.

c- $(T')$ and $(T'')$ are the tangents at $M'$ and $M''$ to $(P)$.
Show by two methods, algebraic and geometric that $(T')$ and $(T'')$ are perpendicular.

**N° 31**

R In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the fixed circle $(C)$ of center $w(3;0)$ and radius $R=1$.

1) Let $(\gamma)$ be a variable circle of center $M$ tangent externally to $(C)$ and to the axis $y'y$.
Prove that $M$ describes a conic $(\Gamma)$ whose focus and directrix are to be determined.
2) Write an equation of $(\Gamma)$.
3) Trace $(\Gamma)$.
4) a- Calculate the area of the domain limited by $(\Gamma)$ and the straight line of equation $x=3$.
b- Deduce the area of the domain limited by $(\Gamma)$, the axis $y'y$ and the two straight lines of equations $y=4$ and $y=-4$.

**N° 32**

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.
Consider the curve $(E)$ of equation $25(x^2+y^2)=(3x-16)^2$.

1) Interpret, geometrically, the equation of $(E)$ and show that $(E)$ is a conic of focus $O$ and directrix the straight line $(\Delta)$ of equation $x=\dfrac{16}{3}$.
2) Trace $(E)$.
3) Suppose that $(\vec{i};\overrightarrow{OM})=\theta$ where $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$ and consider a point $M(x;y)$ of $(E)$.
a- Express $OM$ in terms of $x$.
b- Deduce that $OM=\dfrac{16}{5+3\cos\theta}$.

4) The straight line $(OM)$cuts $(\Delta)$ at a point $I$ and cuts $(E)$ at a point $M'$.

a- Prove that $\frac{1}{OM}+\frac{1}{OM'}$ is a constant independent of the position of $M$ on $(E)$.

b- Prove that $\frac{1}{OM}-\frac{1}{OM'}=\frac{2}{OI}$ and deduce that $OI=\frac{16\times OM}{16-5\times OM}$.

5) Let $w$ be the center of$(E)$and $F$ the second focus of $(E)$, the straight line $(wM')$ cuts $(E)$at a point $M''$.

a- Prove that $\overrightarrow{FM''}$ and $\overrightarrow{OM}$ are collinear .

b- Calculate the coordinates of the point $M''$ in terms of $\theta$.

6) Let $N$ be the center of gravity of triangle $MM'M''$.

a- Using a dilation that transforms $M$ onto $N$, prove that $N$ describes a part of the conic $(E')$.

b- Calculate the area of the domain limited by $(E)$ and $(E')$.

c- Calculate $\theta$ in case $y_I=\frac{16\sqrt{3}}{3}$ and calculate the coordinates of $N$.

7) Suppose that $\theta=\frac{\pi}{3}$.

a- Write the equations of the tangents to $(E)$ at the points $M$, $M'$ and $M''$.

b- Prove that the tangents to$(E)$ at the points $M$ and $M'$ intersect at a point situated on the directrix $(\Delta)$ of $(E)$.

c- Prove that the tangents to $(E)$ at the points $M$ and $M''$ intersect at a point situated on the principal circle of$(E)$.

# Supplementary Problems

N° 1.

$(E)$ is the conic of equation $x^2+5y^2=1$.
Let $f$ be the mapping that to every point $M(z)$ of $(E)$ associates the point $M'(z')$ such that $z'=|z|+z-\bar{z}$ and let $z=x+iy$ and $z'=x'+iy'$.

1) Calculate $x'$ and $y'$ in terms of $x$ and $y$.
2) Let $(E')$ be the image of $(E)$ by $f$.
   Determine the nature of $(E')$.
3) Prove that $x'\geq 0$ and that $|y'|\leq 2x'$.
4) Deduce that $(E')$ is an arc of a circle.

N° 2.

The complex plane is referred to an orhonormal system $(O;\vec{u},\vec{v})$.
Let $M$ be a point of affix $z$ of the plane.

1) Determine the set $(H)$ of points $M$ if
   $\arg(z)+\arg(z-2)=(2k+1)\dfrac{\pi}{2}$ where $k$ is an integer.
2) Determine the center of $(H)$ as well as its asymptotes.
3) Trace $(H)$.

N° 3.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the points $A(-1;0)$, $B(3;0)$ and $C(5;0)$.
Let $(\gamma)$ be a variable circle tangent at $C$ to the axis $x'x$.
The tangents at $A$ and $B$ to $(\gamma)$ intersect at a point $M$.

1) Prove that as $(\gamma)$ varies, the point $M$ describes a conic whose nature is to be determined.
2) Determine an equation of this conic.

N°4.

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$ .

Q Let $(E)$ be the ellipse of focus $F(3;0)$, of directrix the axis of ordinates and eccentricity $e=\dfrac{1}{2}$.

1) Prove that $3x^2+4y^2-24x+36=0$ is an equation of $(E)$.
2) a- Without finding the reduced equation of $(E)$ , find the two vertices of the focal axis of $(E)$.
   b- Determine the center of $(E)$ , find the equation of the non-focal axis of $(E)$ and deduce the other two vertices of $(E)$.
   c- Trace $(E)$ .
3) $(H)$ is a hyperbola of focal axis $x'x$, having the same vertices as $(E)$ and of focal distance $2\sqrt{7}$ .
   a- Find an equation of $(H)$.
   b- Trace $(H)$ in the same system as that of $(E)$.
   c- Deduce the drawing of the curve $(C)$of equation: $3x^2+4y\,|\,y\,|-24x+36=0$ .

N°5.

Let $(d)$ and $(d')$be the straight lines of respective equations

Q $x=-4$ and $y=-4$ and let $A$ be their point of intersection.

1) Let $(P)$ be the parabola of focus $O$and directrix $(d)$. Find an equation of $(P)$.
2) Let $(P')$ be the parabola of focus $O$ and directrix $(d')$. Find an equation of $(P')$.
3) Prove that the perpendicular bisector of $[OA]$ is a common tangent to $(P)$ and $(P')$.
4) Trace $(P)$ and $(P')$ in the same system .

N°6.

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$ .

Let $(E)$ be the conic of equation $16x^2+25y^2=400$.

1) Trace $(E)$.
2) Let $\theta$ be a real number belonging to the interval $[0;2\pi]$.
Let $M$ be a point of the circle of center $O$ and radius $5$ and of coordinates $(5\cos\theta;5\sin\theta)$.
Let $N$ be the image of $M$ by the rotation of center $O$ and angle $\frac{\pi}{2}$.
To each point $M$ we associate the point $R$ of the conic $(E)$ that has the same abscissa as $M$ and whose ordinate has the same sign as that of $M$ and to point $N$ we associate the point $S$ of $(E)$ that has the same abscissa as $N$ and whose ordinate has the same sign as that of $N$.
a- Find the coordinates of $N$, $R$ and $S$ in terms of $\theta$.
b- Verify that $OR^2+OS^2=41$.
c- Calculate the area of triangle $ORS$.

N°7.

The space is referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.
Consider the points $A(-1;0;1)$ and $B(0;2;3)$ and the straight line $(d)$ of parametric equations $x=2m+1$ $y=-2m+2$ and $z=m+1$

1) Determine an equation of the plane $(P)$ passing through $A$ and $B$ and parallel to $(d)$.
2) Prove that the plane $(Q)$ of equation $2x-2y+z+1=0$ is perpendicular to $(P)$ and contains the straight line $(AB)$.
3) Let $E$ be the point of intersection of $(d)$ and $(Q)$, calculate the distance of $E$ to $(AB)$.
4) Let $(d')$ be the orthogonal projection of $(d)$ on $(P)$, calculate the distance of $B$ to $(d')$.
5) Determine the vertex of the parabola of plane $(Q)$ of focus $B$ and directrix the straight line $(D)$ passing through $E$ and perpendicular to $(P)$.

**N° 8.**

In the plane referred to a direct orthonormal system $(O;\vec{i},\vec{j})$ consider the parabola $(P)$ of equation $x^2 = 2ay$ where $a$ is a strictly positive real number.

1) Trace $(P)$.
2) Let $M_1$ and $M_2$ be two points of $(P)$ of respective abscissas $x_1$ and $x_2$. Find a relation between $x_1$ and $x_2$ for the tangents $(T_1)$ and $(T_2)$, at $M_1$ and $M_2$ to $(P)$ to be perpendicular.
3) Prove that the straight line $(M_1M_2)$ passes through a fixed point to be placed on the figure..
4) Determine the set of points of intersection of the tangents $(T_1)$ and $(T_2)$.

**N° 9.**

Consider the parabola $(P)$ of equation $y^2 = 4x$.

1) Determine the focus $F$ and the directrix $(d)$ of $(P)$ and trace $(P)$.
2) Write an equation of a straight line $(\delta)$ passing through the point $F$ and of slope $m \neq 0$.
3) a- Prove that the equation relating the ordinates of the points of intersection $M$ and $M'$ of $(P)$ and $(\delta)$ is
   $m^2y^2 - 4y - 4m = 0$.
   b- Prove that the tangents $(T)$ and $(T')$ at $M$ and $M'$ to $(P)$ are perpendicular.
   c- $(T)$ and $(T')$ intersect at a point $I$, determine the set of points of $I$.
   d- Let $J$ be the midpoint of $[MM']$, Find the set of points $J$.
4) a- Calculate the area of the domain $(D)$ limited by $(P)$ and the straight line of equation $x = 1$.
   b- Calculate the volume of the solid obtained by the rotation of $(D)$ about $x'x$.

**N° 10.**

In the plane referred to a direct orthonormal system $(O;\vec{i},\vec{j})$, consider the circle $(C)$ of center $O$ and radius 2 and the straight line $(d)$ of equation $x = -2$.

Let $M(x;y)$ be a variable point of the plane and designate by $H$ its orthogonal projection on $(d)$ , the tangent through $M$ to $(C)$ intersects this circle at a point $T$.

1) Show that $MT^2 = x^2 + y^2 - 4$.
2) Prove that the set of points $M$ such that $MH = MT$ is a parabola $(P)$ whose focus and directrix are to be determined.
3) Let $(\Omega)$ be a variable circle of center $M \in (P)$ and tangent to $(d)$. Show that $(\Omega)$ remains tangent to a fixed circle of center $F$ and whose radius is to be determined .

N° 11.

$ABCD$ is a rectangle such that $AB = 8$ , $AD = 4$ and $(\overrightarrow{AB};\overrightarrow{AD}) = \frac{\pi}{2}(\bmod 2\pi)$, $O$ and $O'$ are the respective midpoints of $[AD]$ and $[BC]$.

Let $(P)$ be the parabola of focus $F$ and directrix $(AD)$. Take $OF = 2$ .

1) Determine the vertex $S$ of $(P)$ and verify that $E$ and $E'$ belong to $(P)$.
2) The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$, such that $\vec{i} = \frac{1}{2}\overrightarrow{OF}$ and $\vec{j} = \frac{1}{2}\overrightarrow{OD}$.
   a- Write an equation of $(P)$ and show that $(OE)$ is tangent to $(P)$ at $E$.
   b- The straight line $(BC)$ cuts $(P)$ in $B'$ and $C'$, find the coordinates of $B'$ and $C'$.
   c- Calculate the area of the domain limited by $(P)$ and the straight line $(B'C')$.
3) Let $(H)$ be the hyperbola of focus $F$ ,directrix $(AD)$ and eccentricity $e = 3$.
   a- Determine the vertices $S_1$ and $S_2$ of $(H)$ knowing that $S_1 \in [OF]$.
   b- Determine the center of $(H)$ and the second focus $F'$.
   c- Find an equation of $(H)$ in the system $(O;\vec{i},\vec{j})$.

d- Determine the asymptotes to $(H)$ and trace $(H)$.

**N° 12.**

Consider the mapping $f$ of the plane defined by :

$$M\begin{cases} x \\ y \end{cases} \longrightarrow M'\begin{cases} x' = x - y - 2 \\ y' = x + y \end{cases}$$

Let $z = x + iy$ be the affix of $M$ and $z' = x' + iy'$ be the affix of $M'$.

1) Prove that $z' = (1+i)z - 2$.
2) Determine the nature and characteristic elements of $f$.
3) Let $\omega$ be the center of $f$, take $\omega M = \ell$, calculate $MM'$ in terms of $\ell$ and precise the nature of triangle $\omega MM'$.
4) Let $A(2;0)$ , designate by $A' = f(A)$ and $A'' = f(A')$. Calculate the coordinates of $A'$ and $A''$.
5) The parabola of focus $A$ and vertex $O$ has $(P')$ as its image by $f$. Determine the vertex , the focus and the directrix of $(P')$ and find an equation of $(P')$.

**N° 13.**

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

Consider the ellipse $(E)$ of equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and eccentricity $e$.

Let $M$ be a point of $(E)$ of affix $z$ such that $z = r\,e^{i\theta}$.

1) Prove that $r^2 = \dfrac{b^2}{1 - e^2\cos^2\theta}$.
2) Let $M'$ be a point of $(E)$ such that $(\overrightarrow{OM};\overrightarrow{OM'}) = \frac{\pi}{2} (\bmod 2\pi)$.

   Prove that $\dfrac{1}{OM^2} + \dfrac{1}{OM'^2} = \dfrac{1}{a^2} + \dfrac{1}{b^2}$.
3) Designate by $H$ the orthogonal projection of $O$ on $(MM')$. Determine the set of points $H$ as $\theta$ varies.

**N° 14.**

In the plane referred to a direct orthonormal system $(O;\vec{i},\vec{j})$, consider the parabola $(P)$ of equation $y^2 = 2x$.

1) Trace $(P)$.
2) Let $A(2a^2;2a)$ be a point of $(P)$ where $a$ is a strictly positive real numbers .
   a- Write an equation of the tangent $(d)$ and the normal $(\delta)$ at $A$ to $(P)$.
   b- $(\delta)$ intersect the focal axis of $(P)$ in $N$, designate by $H$ the orthogonal projection of $A$ on the focal axis of $(P)$, prove that $HN$ is independent of the position of $A$ on $(P)$.

**N° 15.**

In the plane referred to a direct orthonormal system $(O;\vec{i},\vec{j})$, consider the two conics $(E)$ and $(H)$ of equations :

$(E)$ : $4x^2+9y^2-8x-32=0$.

$(H)$ : $4x^2-9y^2-8x-32=0$.

1) Find the reduced equation of $(E)$ , determine its vertices and trace the curve of $(E)$ in the system $(O;\vec{i},\vec{j})$.
2) Find the reduced equation $(H)$, determine its vertices , its asymptotes and draw its curve $(H)$ in the same system .
3) Deduce a drawing of the curve $(C)$ of equation :
   $4x^2+9y\,|\,y\,|-8x-32=0$.

**N° 16.**

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.
Consider the following transformations :

The rotation $r\left(A;-\dfrac{\pi}{2}\right)$ with $z_A=1+i$, the dilation $h(B;k_1=2)$ with $z_B=-2$ and the similitude $S\left(I;k_2=\sqrt{2};\dfrac{\pi}{4}\right)$.

Let $f=S\circ h\circ r$.

1) Write the complex forms of $S$, $h$ and $r$.
2) a- Determine the complex form of $f$.
   b- Determine the nature of $f$ and precise its characteristic elements .
3) To each point $M$ of affix $z=x+iy$ we associate the point $M'$

of affix $z' = x' + iy'$ such that $M' = f(M)$.
Express $x'$ and $y'$ in terms of $x$ and $y$.

4) Let $(P)$ be the parabola of focus $F(1;1)$ and directrix the straight line $(d)$ of equation $y = -x$.
   a- Determine $f(F)$ as well as an equation of the straight line $(\delta)$ image of $(d)$ by $f$.
   b- Write an equation of $(P')$ image of $(P)$ by $f$.
   c- Trace $(P)$ and $(P')$.

**N° 17.**

In an oriented plane, consider the straight line $(\Delta)$ and a point $O$ not belonging to $(\Delta)$.
Designate by $H$ the orthogonal projection of $O$ on $(\Delta)$.
A variable point $P$ moves on $(\Delta)$.
The perpendicular through $P$ to $(\Delta)$ and the perpendicular through $O$ to $(OP)$ intersect at a point $M$. Let $I$ be the midpoint of $[PM]$.

1) Prove that as $P$ describes $(\Delta)$, the point $I$ moves on a conic $(C)$ whose focus and directrix are to be determined..
2) The plane is referred to a direct orthonormal system $(O;\vec{i},\vec{j})$ such that $\overrightarrow{OH} = \vec{i}$.
   a- Let $(x;y)$ be the coordinates of a point $M$, calculate the scalar product $\overrightarrow{OM}.\overrightarrow{OP}$ and deduce an equation of the curve $(C')$ set of points $M$ as $P$ moves on $(\Delta)$.
   b- Prove that $(C')$ is a parabola whose focus and directrix $(\Delta')$ are to be determined.
   c- Trace $(C)$ and $(C')$.

**N° 18.**

In an oriented plane, consider the square $ABFE$ of center $J$ such that $AB = 2$ and $(\overrightarrow{AB};\overrightarrow{AE}) = \frac{\pi}{2}(\text{mod}\,2\pi)$.
Let $D$, $C$ and $G$ be the respective midpoints of $[AE]$, $[BF]$ and $[AB]$. Let $I$ be a given point of $[DC]$.

The straight line $(AI)$ cuts $(EF)$ in $K$.
The perpendicular through $I$ to $(AI)$ cuts $(AE)$ in $N$ .
Designate by $M$ the symmetric of $N$ with respect to $I$ .

1) a- Precise the position of $M$ if $I$ is on $D$ then on $J$ .
   b- What is the nature of the quadrilateral $AMKN$ ?
   c- Prove that as $I$ moves on the segment $[DC]$ then the point $M$ moves on a conic $(P)$ whose focus and directrix are to be determined.
   d- What does $(MN)$ represent for $(P)$ ?
2) The plane is referred to a direct orthonormal system $\left(A;\overrightarrow{AG};\overrightarrow{AD}\right)$.
   a- Find an equation of $(P)$.
   b- Trace $(P)$.
   c- Calculate the area of the domain limited by $(P)$, the segment $[AB]$ and the segment $[AD]$.

$N^\circ 19.$

In the plane referred to a direct orthonormal system $(O;\vec{i},\vec{j})$, consider the points $A(3;0)$, $A'(-3;0)$ and $F(m;0)$ with $m \in \left]-3;3\right[$.
Let $(P)$ be the parabola of vertex $A$ and focus $F$ and $(P')$ the parabola of vertex $A'$ and focus $F$ .

1) Find an equation of $(P)$ and an equation of $(P')$.
2) Find the coordinates of $I$ and $J$ intersection of $(P)$ and $(P')$.
3) Prove , using two methods algebraic and geometric ,that the tangents at $I$ to $(P)$ and $(P')$ are perpendicular
   ($I$ is the point with positive ordinate) .

$N^\circ 20.$

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.
Consider the two points $P$ and $Q$ of respective affixes $z_P = a\,e^{i\,\theta}$ and $z_Q = b\,e^{i\,\theta}$ where $a$ and $b$ are two strictly positive real numbers with $a > b$ and $0 \le \theta \le \pi$ .
Let $(\Delta)$ be the straight line drawn through $P$ and parallel to $y'y$ and $(\Delta')$ the straight line drawn through $Q$ and parallel to $x'x$ .
$(\Delta)$ and $(\Delta')$ intersect at a point $M$ .

1) Calculate the coordinates of $M$ in terms of $a$ , $b$ and $\theta$.
2) Show that $M$ traces an ellipse $(E)$ as $\theta$ varies .
3) Suppose that $a = 5$ and $b = 3$ .
   Determine the foci , vertices , the directrices and eccentricity of $(E)$ and trace $(E)$.

**N°21.**

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the variable point $M(x;y)$ such that :

$x = \dfrac{\cos\theta}{2+\cos\theta}$ and $y = \dfrac{\sin\theta}{2+\cos\theta}$ where $\theta$ is a real number .

Let $(d)$ be the straight line of equation $x = 1$.

1) a- Calculate $OM$ in terms of $\theta$ then calculate the distance of $M$ to $(d)$.
   b- Deduce that $M$ describes an ellipse $(E)$.
   c- Trace $(E)$.
2) a- Calculate $\cos\theta$ in terms of $x$.
   b- Calculate $\tan\theta$ in terms of $x$ and $y$.
   c- Deduce an equation of $(E)$.
3) $(E)$ cuts the axis of ordinates in two points $P$ and $Q$.
   $P$ is the point of positive ordinate , write an equation of the tangent through $P$ to $(E)$.

**N°22.**

The space is referred to a direct orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.
Consider the points $A(1;0;0)$ , $B(-1;0;0)$ and $F(0;3;0)$.
Let $M$ be a variable point of the space such that $\left\|\overrightarrow{MA}\wedge\overrightarrow{MB}\right\| = 4\,MF$ .

1) Interpret geometrically $\left\|\overrightarrow{MA}\wedge\overrightarrow{MB}\right\|$ and deduce that $M$ describes a conic whose nature is to be determined .
2) Write an equation of this conic .

**N°23.**

In the plane referred to a direct orthonormal system $(O;\vec{i},\vec{j})$, consider the ellipse $(E)$ of equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and foci $F$ and $F'$.

Let $M$ and $M'$ be two points of $(E)$ such that $\overrightarrow{FM}$ and $\overrightarrow{F'M'}$ are parallel and having the same sense .

The tangents $(T)$ and $(T')$ at $M$ and $M'$ respectively to $(E)$ intersect at a point $P$.

$(T)$ cuts $(F'M')$ at a point $E$.

1) a- Prove that the triangle $MF'E$ is isosceles at $F'$.
   b- Deduce that $P$ is the orthogonal projection of $F'$ on the tangent $(T)$.
2) $(F'P)$ cuts $(FM)$at a point $Q$.
   a- Determine the set of points $Q$ as $M$ varies .
   b- Deduce that $(T)$ and $(T')$ intersect at a point situated on the principal circle of $(E)$.
3) Suppose that the equation of $(E)$ is $\frac{x^2}{25}+\frac{y^2}{16}=1$ and that $(\vec{i};\overrightarrow{FM})=\frac{\pi}{3}(\bmod 2\pi)$.
   a- Calculate the coordinates of the points $F$ and $M$. ( suppose that $x_F>0$ )
   b- Calculate the coordinates of $P$.

# *Solution of Problems*

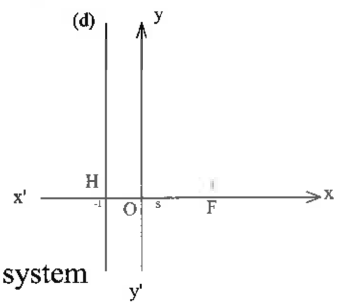
### *Part 1 – Parabola :*

**N° 1.**

1) Let $M(x;y)$ be a point of the parabola and $M'(-3;y)$ its orthogonal projection on $(d)$ , $\frac{MF}{MM'}=1$ , so $MF^2 = MM'^2$
which gives $(x-3)^2+(y-0)^2=(x+3)^2$ so
$x^2-6x+9+y^2=x^2+6x+9$
Hence $y^2=12x$ .
***N.B :*** Remark that $O$ is the vertex of this parabola then an equation of this parabola is $y^2=2px$ with $p=2\,OF=6$ , therefore $y^2=12x$ .

**N° 2.**

1) *1<sup>st</sup> Method :*
Let $M(x;y)$ be a point on the parabola and $M'(-1;y)$ its orthogonal projection on $(d)$, then $\frac{MF}{MM'}=1$ , so $MF^2 = MM'^2$
which gives $(x-2)^2+(y-0)^2=(x+1)^2$ so $y^2=6x-3$ .
*2nd method :*
The focal axis passes through $F$ and is perpendicular to $(d)$, then the focal axis is the axis of abscissas.
The vertex $S$ is the midpoint of $[FH]$
then $S\left(\frac{1}{2};0\right)$ .




The reduced equation of the parabola Is $Y^2=2pX$ with $p=HF=3$
therefore: $Y^2=6X$ with respect to the system $\left(S;\vec{i},\vec{j}\right)$ , taking $x=\frac{1}{2}+X$ and $y=Y$ hence an equation of $(P)$

in the system $\left(O;\vec{i},\vec{j}\right)$ is $y^2 = 6x - 3$ .

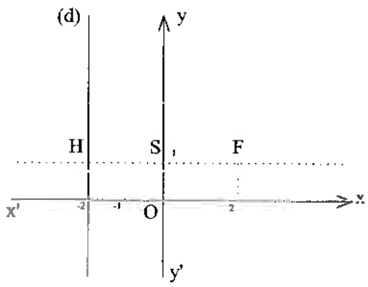
2) Let $M(x;y)$ be a point of $(P)$, we have $MF = d(M;(d))$,

so : $(x+1)^2 + (y-1)^2 = \dfrac{|x-1|^2}{\sqrt{1}}$ hence an equation of the parabola

is $y^2 - 2y + 4x + 1 = 0$ .

3) We have $F(2;1)$ and $S(0;1)$, so the straight line $(SF)$ is the focal axis of $(P)$.

In the system $\left(S;\vec{i},\vec{j}\right)$,




the reduced equation of$(P)$ is $Y^2 = 2pX$ .

$\dfrac{p}{2} = SF = 2$ hence $p = 4$, so

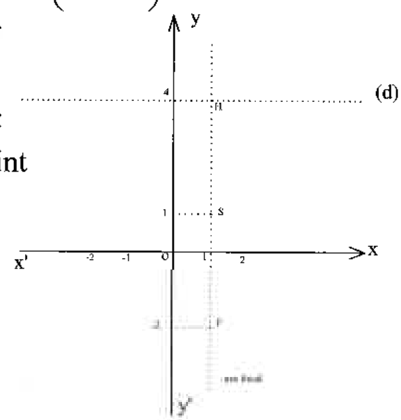
$Y^2 = 8X$ and since $x = X$

and $y = Y + 1$ we get :

$(y-1)^2 = 8x$ , therefore $y^2 - 2y - 8x + 1 = 0$

is an equation of $(P)$ in the system $\left(O;\vec{i},\vec{j}\right)$.

4) The vertex $S$ is the midpoint of the segment $[FH]$ and since $S(1;1)$ and $H(1;4)$ then we get: $F(1;-2)$ . Let $M(x;y)$ be a point on the parabola $(P)$ and let $M'(x;4)$be its orthogonal projection on $(d)$, we have :




$\dfrac{MF}{MM'} = 1$, so $MF^2 = MM'^2$,

which gives :

$(x-1)^2 + (y+2)^2 = (y-4)^2$ hence an equation of $(P)$ is

$x^2 - 2x + 12y - 11 = 0$.

5) Let $M(x;y)$ be a point of the parabola of focus $F(1;0)$ and

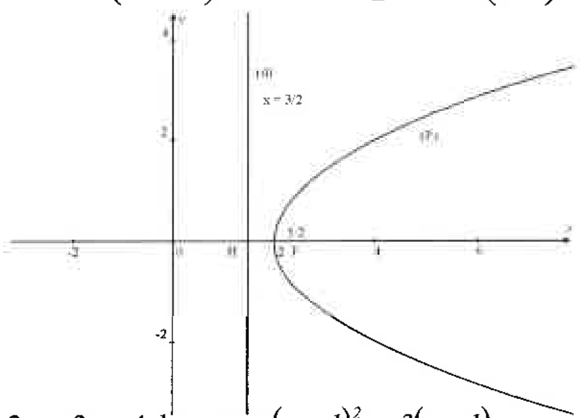
directrix the straight line $(d)$ of equation $y = x$ , hence

$d(M;F) = d(M;d)$ so $(x-1)^2 + y^2 = \left(\frac{|x-y|}{\sqrt{1+1}}\right)^2$ , we get the

equation $x^2 + y^2 + 2xy - 4x + 2 = 0$ .

**N°3.**

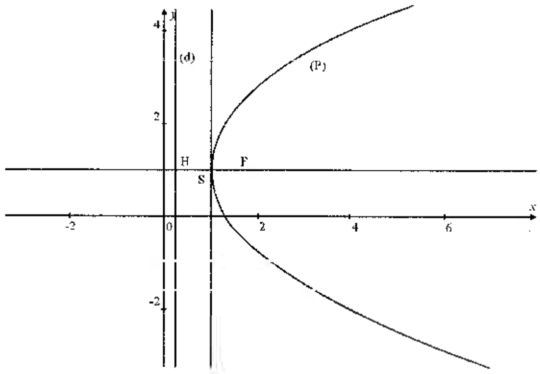
1) The equation $y^2 = 2x - 4$ can be written in the form
$y^2 = 2(x-2)$.
Taking $X = x - 2$ and $Y = y$, the vertex of $(P)$ is the point $S(2;0)$,
Its focal axis is the axis $x'x$, $p = 1$ and the reduced equation of $(P)$ is $Y^2 = 2X$.
In the system $\left(S;\vec{i},\vec{j}\right)$ an equation of the directrix $(d)$ is $X = -\frac{1}{2}$
and the focus is $F\left(\frac{1}{2};0\right)$.
In the system $\left(O;\vec{i},\vec{j}\right)$ $(d)$ : $x = \frac{3}{2}$ and $F\left(\frac{5}{2};0\right)$.




2) $y^2 - 2y = 3x - 4$ becomes $(y-1)^2 = 3(x-1)$ .
Letting $\begin{cases} x - 1 = X \\ y - 1 = Y \end{cases}$ the reduced equation becomes $Y^2 = 3X$ .
The vertex is the point $S(1;1)$, $2p = 3$ and the focal axis is the straight line of equation $y = 1$.

In the system $\left(S;\vec{i},\vec{j}\right)$ : an equation of the directrix $(d)$ is $X=-\frac{3}{4}$ and the focus is $F\left(\frac{3}{4};0\right)$.

In the system $\left(O;\vec{i},\vec{j}\right)$ : $(d)$ : $x=\frac{1}{4}$ and $F\left(\frac{7}{4};1\right)$.



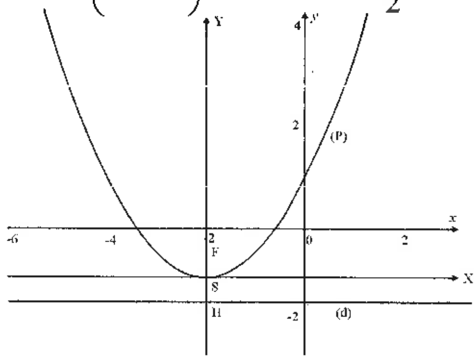

3) $x^2+4x-2y+2=0$

becomes $(x+2)^2=2(y+1)$, letting $x+2=X$ and $y+1=Y$ then the vertex is the point $S(-2;-1)$ and the reduced equation is : $X^2=2Y$.

The focal axis is the straight line of equation $x=-2$.

In the system $\left(S;\vec{i},\vec{j}\right)$ : $(d)$ : $Y=-\frac{1}{2}$ and $F\left(0;\frac{1}{2}\right)$.

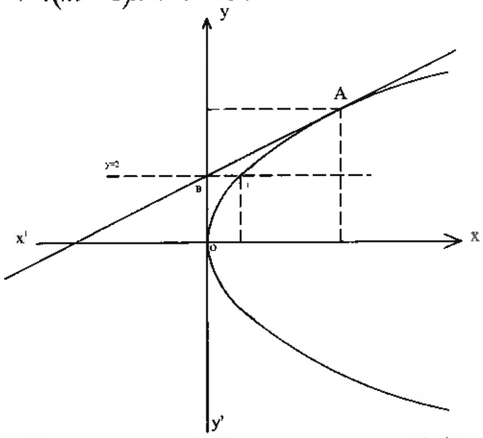
In the system $\left(O;\vec{i},\vec{j}\right)$ , : $(d)$ : $y=-\frac{3}{2}$ and $F\left(-2;-\frac{1}{2}\right)$.



N°4.

1) An equation of $(\delta)$ is t $y - y_B = m(x - x_B)$ , so $y = mx + 2$ .

2) a- $\begin{cases} y^2 = 4x & (P) \\ y = mx + 2 & (\delta) \end{cases}$ , the equation relating the abscissas of the points of intersection of $(P)$ and $(\delta)$ is $(mx+2)^2 = 4x$ with $x \geq 0$ which gives the equation $(E)$ :
$m^2x^2 + 4(m-1)x + 4 = 0$ .




b- If $m = 0$ then $y = 2$ is an equation of $(\delta)$.
In this case $(\delta)$ cuts $(P)$ in only one point $I(1;2)$ .
If $m \neq 0$ then the equation $(E)$ is quadratic .
$\Delta' = b'^2 - ac = 4(m-1)^2 - 4m^2 = -8m + 4$ .

- If $\Delta' < 0$, then $m > \frac{1}{2}$ so $(\delta)$ does not intersect $(P)$ .
- If $\Delta' > 0$ , then $m < \frac{1}{2}$ (with $m \neq 0$ ), so $(\delta)$ cuts $(P)$ in two distinct points .
- If $\Delta' = 0$ , then $m = \frac{1}{2}$ ,so $(\delta)$ is tangent to $(P)$.

3) For $m = \frac{1}{2}$ ,$(\delta)$ is tangent to $(P)$ at a point $A$ , an equation of

this tangent is $y = \frac{1}{2}x + 2$.

$$x_A = x' = x'' = -\frac{b'}{a} = \frac{-2(m-1)}{m^2} = \frac{-2\left(\frac{1}{2}-1\right)}{\frac{1}{4}} = 4$$

$y_A = \frac{1}{2}x_A + 2 = \frac{1}{2}(4) + 2 = 4$ then $A(4;4)$.

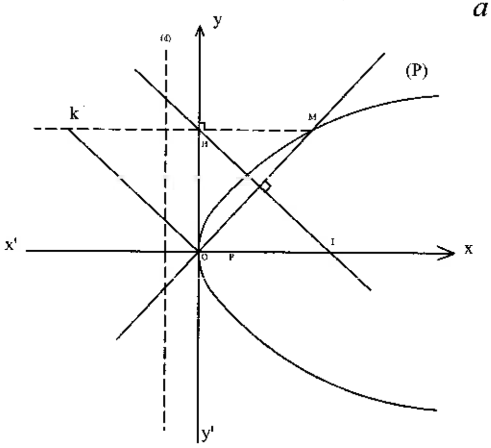
Let $(T)$ be the tangent to $(P)$ through $C(0;-2)$, since $C$ is symmetric to $B$ with respect to the focal axis $x'x$ of $(P)$, then $(T)$ is the symmetric of the straight line $(BA)$ with respect to $x'x$, hence an equation of $(T)$ is $y = -\frac{1}{2}x - 2$ and the point of tangency is the point $A'(4;-4)$ symmetric of $A$ with respect to the focal axis.

$\boxed{N°5.}$

1) a- The slope of $(OM)$ is $a = \frac{y_M}{x_M} = \frac{y_0}{x_0}$ but

$x_0 = \frac{y_0^2}{2p}$ then $a = \frac{y_0}{\frac{y_0^2}{2p}} = \frac{2p}{y_0}$. $(IH)$ is perpendicular

to $(OM)$ then the slope of $(IH)$ is : $a' = \frac{-1}{a} = \frac{-y_0}{2p}$.



b- $(IH)$ passes through the point $H(0;y_0)$ then an equation of $(IH)$ is : $y = \frac{-y_0}{2p}x + y_0$ .

This straight line cuts the axis $x'x$ at $I$ such that $y_I = 0$ therefore: $\frac{-y_0}{2p}x_I + y_0 = 0$, but $y_0 \neq 0$ , so $x_I = 2p$ and consequently $I(2p;0)$ is fixed .

2) The two straight lines $(OK)$ and $(HI)$ are parallel as well as the straight lines $(HK)$ and $(OI)$ hence $OIHK$ is a parallelogram, therefore $\overrightarrow{HK} = -\overrightarrow{OI}$ , hence $\overrightarrow{HK}(-2p;0)$ , which gives $x_K = -2p$ , constant , and consequently the locus of $K$ is the straight line $(\delta)$ of equation $x_K = -2p$ deprived of point $(-2p;0)$.

**N°6.**

1) The abscissas of the points of intersection of $(P)$ and $(P')$ are the roots of the equation $y^2 = y^2$ , so $2x+1 = -2x+1$ which gives $x = 0$ and consequently $y^2 = 1$ hence $(P)$ and $(P')$ intersect in two points $A(0;1)$ and $B(0;-1)$ .

2) If we replace $x$ by $-x$ in the equation of $(P)$ , we get : $y^2 = -2x+1$ that is the equation of $(P')$ hence $(P)$ and $(P')$ are symmetric with respect to the axis $y'y$ .

3) a- The equation $y^2 = 2x+1$ can be written $y^2 = 2\left(x+\frac{1}{2}\right)$, then the vertex of $(P)$ is the point $S\left(-\frac{1}{2};0\right)$ ,the reduced equation is $Y^2 = 2X$ and $p = 1$.

In the system $\left(S;\vec{i},\vec{j}\right)$ an equation of the directrix $(d)$ is $X = -\frac{1}{2}$ and the focus is $F\left(\frac{1}{2};0\right)$ .

In the system $\left(O;\vec{i},\vec{j}\right)$ ,: $(d)$ : $x=-1$ and $F(0;0)$.

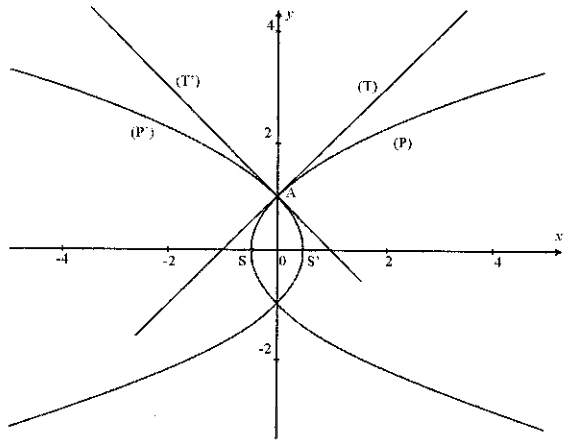
b- $(P')$ is symmetric to $(P)$ with respect to $y'y$ then the vertex of $(P')$ is $S'\left(\frac{1}{2};0\right)$ .

The focus of $(P')$ is $O(0;0)$ and the directrix is the straight line $(d')$ of equation $x=1$ .

4) $A(0;1)$ is a point common to $(P)$ and $(P')$.

$y^2=2x+1$ gives $2yy'=2$ then $y'=\frac{1}{y}$ and consequently the slope of the tangent at $A$ to $(P)$ is $y'_A=1$ consequently an equation of the tangent $(T)$ to $(P)$ at $A$ is $y-1=1(x-0)$, so $y=x+1$.
Let $(T')$ be the tangent to $(P')$ at $A$ then $(T')$ is symmetric to $(T)$ with respect to $y'y$ hence an equation of $(T')$ is $y=-x+1$.

5)




6) Designating by $A_1$ the area of the domain limited by $(P)$, the axis $x'x$ and the two straight lines of equations $x=-\frac{1}{2}$ and $x=0$.

$y^2=2x+1$ gives $y=\pm\sqrt{2x+1}$, then :

$$A_1 = \int_{-\frac{1}{2}}^{0} \sqrt{2x+1}dx = \frac{1}{2}\int_{-\frac{1}{2}}^{0} 2(2x+1)^{\frac{1}{2}}dx = \frac{1}{3}(2x+1)^{\frac{3}{2}}\Big|_{-\frac{1}{2}}^{0} = \frac{1}{3}$$ square units.

The required area is $A = 4\times A_1 = \frac{4}{3}$ square units.

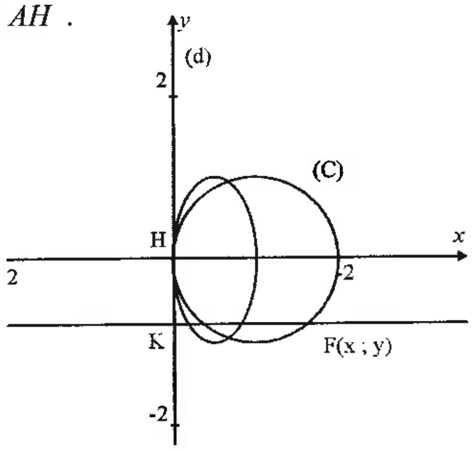
7) The required volume is :

$$v = 2\pi\int_{-\frac{1}{2}}^{0}(2x+1)dx = 2\pi\left[x^2+x\right]_{-\frac{1}{2}}^{0} = 2\pi\left[(0)-\left(\frac{1}{4}-\frac{1}{2}\right)\right]$$

$$= 2\pi\times\frac{1}{4} \text{ so } v = \frac{\pi}{2} \text{ cubic units.}$$

**N°7.**

1) Let $F$ be the focus of $(P)$ and $(d)$ its directrix, since $A\in(P)$, then $AF = AH$, hence $AF$ is constant and $A$ is fixed, consequently the locus of the focus $F$ is the circle $(C)$ of center $A$ and radius $AH$.




2) a- $K(0;y)$ is the orthogonal projection of $F$ on $(d)$, the vertex $S(X;Y)$ is the midpoint of $[KF]$ therefore $\begin{cases} X = \frac{x}{2} \\ Y = y \end{cases}$.

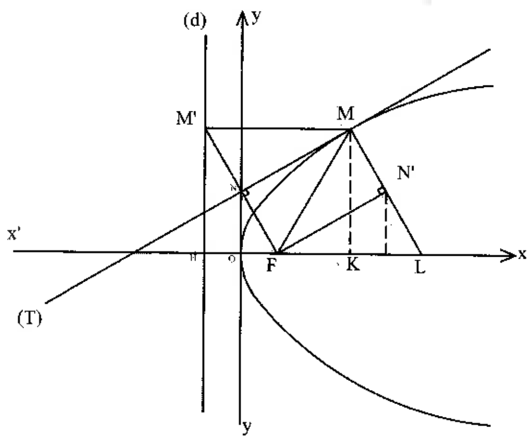
b- The point $F$ describes the circle $(C)$ of equation $(x-1)^2 + y^2 = 1$ therefore

$(2X-1)^2+Y^2=1$ , so $4\left(X-\frac{1}{2}\right)^2+Y^2=1$ and consequently the point $S$ describes the ellipse of equation :

$$\frac{\left(X-\frac{1}{2}\right)^2}{\frac{1}{4}}+Y^2=1.$$

N°8.

1) • Let $M'$ be the orthogonal projection of $M$ on $(d)$. We have $MF=MM'$ then triangle $FMM'$ is isosceles of vertex $M$, the tangent $(T)$ is a bisector of the angle $\widehat{FMM'}$ ,$(T)$ is also the perpendicular bisector of segment $[FM']$, then the straight line $(FM')$ passes through the point $N$ and $N$ is the midpoint of $[FM']$.




• In triangle $FHM'$ ,$O$ is the midpoint of $[HF]$ and $N$ is the midpoint of $[FM']$, hence $(ON)$ and $(HM')$ are parallel hence $N \in y'y$ consequently the locus of points $N$ is the axis $y'y$.

2) $FLMM'$ is a parallelogram ($L$ is the intersection of the normal with the focal axis) , hence $FL=MM'$ , but $MM'=MF$ then $FL=FM$, consequently $FLM$ is an isosceles triangle, in this triangle $[FN']$ is a height and a median at the same time , then

$N'$ is the midpoint of $[ML]$ , $[KL]$ is the sub-normal then $KL = 2$
If $N'(X;Y)$ then :

$$X = \frac{x_M + x_L}{2} = \frac{x_0 + x_0 + 2}{2} = x_0 + 1 \text{ and}$$

$$Y = \frac{y_M + y_L}{2} = \frac{y_0 + 0}{2} = \frac{y_0}{2}.$$

But $y_0^2 = 4x_0$, therefore $(2Y)^2 = 4(X-1)$ , so $4Y^2 = 4(X-1)$ and consequently $Y^2 = X-1$ , then the locus of $N'$ is the parabola $(P')$ of equation $Y^2 = X-1$ and of vertex the point $F(1;0)$ .

***Part 2 – Ellipse :***

**N°9.**

1) Let $M(x;y)$ be a point of the ellipse $(E)$ . $M'(0;y)$ is the orthogonal projection of $M$ on $(d)$, $\frac{MF}{MM'} = \frac{1}{2}$, then $MF^2 = \frac{1}{4}MM'^2$, which gives $(x-3)^2 + y^2 = \frac{1}{4}x^2$.
Hence, the equation of $(E)$ is $3x^2 + 4y^2 - 24x + 36 = 0$.
2) The foci $F$ and $F'$ are two points of the axis $x'x$ then $x'x$ is the focal axis , $2c = F'F = 4$, therefore $c = 2$.
Let $w$ be the center of the ellipse, $w$ is the midpoint of $[F'F]$ therefore, $w(4;0)$.
$wO = a$ then $a = 4$ and $a^2 - b^2 = c^2$ gives
$b^2 = a^2 - c^2 = 16 - 4 = 12$ .
Therefore an equation of the ellipse is : $\frac{(x-4)^2}{16} + \frac{y^2}{12} = 1$.
3) The foci $O$ and $F$ are two points of the axis $y'y$ then $y'y$ is the focal axis , $OF = 2c = 6$ gives $c = 3$.
The center $w$ of the ellipse is the midpoint of $[OF]$ therefore $w(0;3)$. $wH$ is the distance of $w$ to the directrix $(d)$, therefore

$wH = \dfrac{a^2}{c} = 4$ which gives $a^2 = 12$, $b^2 = a^2 - c^2 = 12 - 9 = 3$ .

Hence an equation of the ellipse is $\dfrac{x^2}{3} + \dfrac{(y-3)^2}{12} = 1$.

4) The vertices are $A(4;1)$ and $A'(-2;1)$ then the center $w(1;1)$ is the midpoint of $[AA']$ and the focal axis is parallel to the axis $x'x$ $AA' = 2a = 6$ therefore $a = 3$.

The distance between the two directrices is $2\dfrac{a^2}{c} = 8$ therefore

$c = \dfrac{2a^2}{8} = \dfrac{18}{8} = \dfrac{9}{4}$, or $c^2 = a^2 - b^2$ hence :

$b^2 = a^2 - c^2 = 9 - \dfrac{81}{16} = \dfrac{63}{16}$.

Consequently an equation of the ellipse is $\dfrac{(x-1)^2}{9} + \dfrac{(y-1)^2}{\frac{63}{16}} = 1$.

**N° 10.**

1) The equation $4x^2 + 9y^2 - 8x - 32 = 0$ is equivalent to :
$4(x^2 - 2x + 1 - 1) + 9y^2 - 32 = 0$, which leads to
$4(x-1)^2 + 9y^2 = 36$ which gives $(E)$ :
$\dfrac{(x-1)^2}{9} + \dfrac{y^2}{4} = 1$, which can be written as $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ with
$x = X + 1$ et $y = Y$.
The center of $(E)$ is the point $w(1;0)$ and the focal axis is the axis $x'x$. $a^2 = 9$, $b^2 = 4$, $c^2 = a^2 - b^2 = 5$, then $a = 3$ , $b = 2$ and $c = \sqrt{5}$.

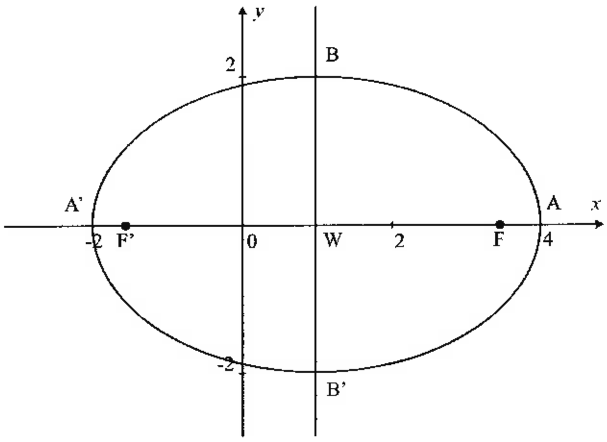
In the system $\left(w; \vec{i}, \vec{j}\right)$ we have :

Vertices on the focal axis : $A(3\,;0), A'(-3\,;\,0)$.
Vertices on the non-focal axis : $B(0\,;\,2), B'(0\,;\,-2)$.
Foci : $F(\sqrt{5}\,;0), F'(-\sqrt{5};0)$.

Directrices : $(d): X = \frac{9}{\sqrt{5}}$ , $(d'): X = \frac{-9}{\sqrt{5}}$.

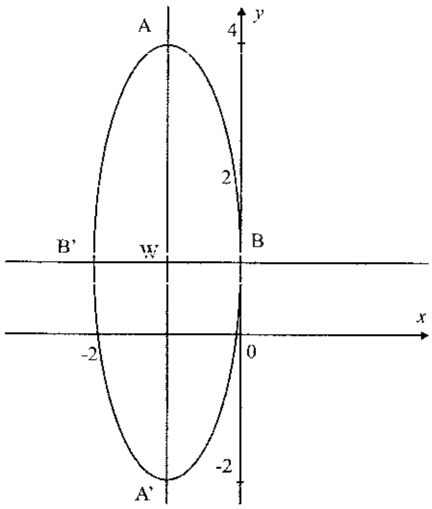
In the system $\left(O;\vec{i},\vec{j}\right)$ we have :

Vertices on the focal axis : $A(4\,;0)$, $A'(-2;0)$.
Vertices on the non-focal axis : $B(1\,;2)$, $B'(1;-2)$.
Foci : $F(\sqrt{5}+1;0)$, $F'(-\sqrt{5}+1;0)$.
Directrices : $(d)\ :\ x = \frac{9}{\sqrt{5}} + 1$, $(d')\!:\ x = \frac{-9}{\sqrt{5}} + 1$.




2) a- The equation $9x^2 + y^2 + 18x - 2y + 1 = 0$ is equivalent to :
$9(x^2 + 2x + 1 - 1) + y^2 - 2y + 1 = 0$, so $9(x+1)^2 + (y-1)^2 = 9$,
which gives $(E')$ : $\frac{(x+1)^2}{1} + \frac{(y-1)^2}{9} = 1$.
The center of the ellipse $(E')$ is $w(-1;1)$ and its focal axis is the straight line of equation $x = -1$, $a^2 = 9, b^2 = 1$,
$c^2 = a^2 - b^2 = 8$, then $a = 3$, $b = 1$ and $c = 2\sqrt{2}$.

| Elements | System $\left(w; \vec{i}, \vec{j}\right)$ | System $\left(O; \vec{i}, \vec{j}\right)$ |
|---|---|---|
| Vertices on the focal axis | $A(0;3)$, $A'(0;-3)$ | $A(-1;4)$, $A'(-1;-2)$ |
| Vertices on the non-focal axis | $B(1;0)$, $B'(-1;0)$ | $B(0;1)$, $B'(-2;1)$ |




b- The center of the principal circle is the center $w(-1;1)$ of the ellipse and its radius is $a = 3$, its equation is then $(x+1)^2 + (y-1)^2 = 9$.

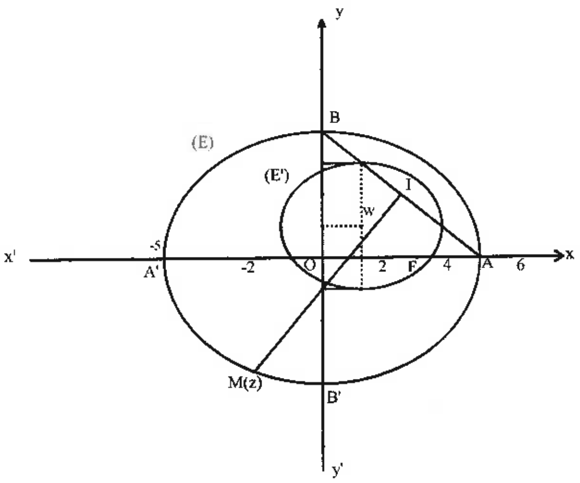
c- The area of the domain limited by $(C)$ and $(E')$ is $\pi a^2 - \pi ab = 9\pi - 3\pi = 6\pi$ square units .

**N° 11.**

1) a- $O(0;0)$ is the center of $(E)$ and $F(3;0)$ is a focus then $c = OF = 3$.
$A(5;0)$ is a vertex of $(E)$ hence $a = OA = 5$ .
$c^2 = a^2 - b^2$, therefore $b^2 = a^2 - c^2 = 25 - 9 = 16$.
The focal axis of $(E)$ is the axis $x'x$ hence an equation of the ellipse $(E)$ is : $\frac{x^2}{25} + \frac{y^2}{16} = 1$.

The directrix $(d)$ associated to $F$ is the straight line $(d)$ of equation $x = \frac{a^2}{c} = \frac{25}{3}$.

b- The vertices on the focal axis are : $A(5;0)$ and $A'(-5;0)$.
The vertices on the non-focal axis are: $B(0;4)$ and $B'(0;-4)$.




2) a- Remark that the point $B$ belongs to $(E)$.

$\overrightarrow{GA} + \overrightarrow{GB} + 2\overrightarrow{GM} = \vec{0}$ gives

$z_A - z_G + z_B - z_G + 2(z_M - z_G) = 0$ , therefore

$z_G = \frac{1}{4}(z_A + z_B + 2z_M) = \frac{1}{4}(5 + 4i + 2z)$.

b- Let $z = \alpha + i\beta$, since $M \in (E)$ then $\frac{\alpha^2}{25} + \frac{\beta^2}{16} = 1$.

$z_G = \frac{1}{4}[5 + 4i + 2(\alpha + i\beta)]$ , therefore:

$x_G = \frac{5 + 2\alpha}{4}$ and $y_G = \frac{4 + 2\beta}{4}$ , which gives :

$\alpha = \frac{4x_G - 5}{2}$ and $\beta = \frac{4y_G - 4}{2}$.

Hence $G$ describes the ellipse $(E')$ of equation :

$$\frac{\left(x-\frac{5}{4}\right)^2}{\frac{25}{4}}+\frac{(y-1)^2}{4}=1.$$

c- If $I$ is the midpoint of $[AB]$ then $\overrightarrow{GA}+\overrightarrow{GB}=2\overrightarrow{GI}$.

$\overrightarrow{GA}+\overrightarrow{GB}+2\overrightarrow{GM}=\vec{0}$ gives $2\overrightarrow{GI}+2\overrightarrow{GM}=\vec{0}$ , therefore

$\overrightarrow{GI}+\overrightarrow{GM}=\vec{0}$, hence $G$ is the midpoint of $[IM]$ that is interior to triangle $ABM$ and triangle $ABM$ is interior to $(E)$ then the point $G$ remains interior to $(E)$ and consequently $(E')$ is interior to $(E)$ .

The center of $(E')$ is the point $w\left(\frac{5}{4};1\right)$.

$\mathcal{A}$ = Area of $(E)$ – Area of $(E')$ , therefore
$A=\pi ab-\pi a'b'=20\pi-5\pi=15\pi$ square units.

**N° 12.**

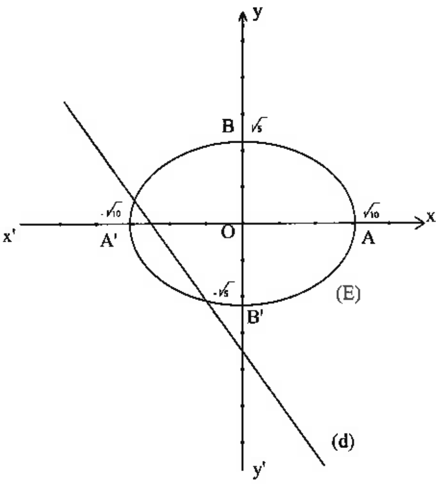
The vertex of negative abscissa is the point $B(-2;0)$ , $M$ being the midpoint of segment $[BP]$ then $\overrightarrow{BM}=\frac{1}{2}\overrightarrow{BP}$ and consequently $M$ is the image of $P$ by the dilation $h$ of center $B$ and ratio $k=\frac{1}{2}$.

Hence the locus of $M$ is the ellipse, image of $(E)$ by $h$.
The center of $(E')$ is the point $O'=h(O)$ then $O'(-1;0)$.
$a'=\frac{1}{2}a=\frac{3}{2}$ and $b'=\frac{1}{2}b=1$ , since dilation preserves length

hence an equation of $(E')$ is : $\frac{(x+1)^2}{1}+\frac{y^2}{\frac{9}{4}}=1$.

**N° 13.**

1) The vertices of the focal axis are : $A(\sqrt{10};0)$ and $A'(-\sqrt{10};0)$.
The vertices of the non-focal axis are : $B(0;\sqrt{5})$ and $B'(0;-\sqrt{5})$.




2) Let $(\delta)$ be a variable straight line parallel to $(d)$.

$(\delta)$ and $(d)$ have the same slope $a = \dfrac{-3}{2}$, an equation of $(\delta)$ is then $y = \dfrac{-3}{2}x + m$ where $m$ is a real parameter.

The abscissas of the points of intersection of $(E)$ and $(\delta)$ are solutions of the system:

$\begin{cases} x^2 + 2y^2 = 10 & (E) \\ y = -\dfrac{3}{2}x + m & (\delta) \end{cases}$ that gives the equation :

$x^2 + 2\left(-\dfrac{3}{2}x + m\right)^2 = 10$, which gives $\dfrac{11}{2}x^2 - 6mx + 2m^2 - 10 = 0$.

$\Delta' = b'^2 - ac = (-3m)^2 - \left(\dfrac{11}{2}\right)(2m^2 - 10) = -2m^2 + 55$.

For $(\delta)$ to be tangent to $(E)$, $\Delta' = 0$ then :

$m^2 = \dfrac{55}{2}$ and consequently $m = \sqrt{\dfrac{55}{2}}$ or $m = -\sqrt{\dfrac{55}{2}}$.

Hence there exist two tangents to $(E)$ parallel to $(d)$, of equations:

$$y=-\frac{3}{2}x+\sqrt{\frac{55}{2}} \text{ and } y=-\frac{3}{2}x-\sqrt{\frac{55}{2}}.$$

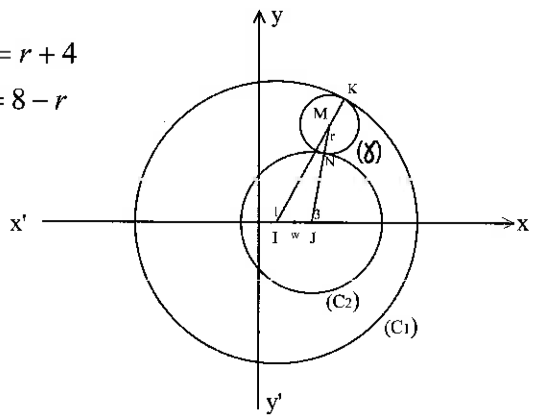
3) a- $c^2=a^2-b^2=10-5=5$, then the foci are :
$F\left(\sqrt{5};0\right)$ and $F'\left(-\sqrt{5};0\right)$.
$MF^2=\left(x-\sqrt{5}\right)^2+y^2=x^2-2x\sqrt{5}+5+y^2$ but
$y^2=\frac{10-x^2}{2}$ therefore $MF^2=x^2-2x\sqrt{5}+5+\frac{10-x^2}{2}$, so
$MF^2=\frac{1}{2}\left(x^2-4\sqrt{5}x+20\right)$.
$MF'^2=\left(x+\sqrt{5}\right)^2+y^2=x^2+2x\sqrt{5}+5+y^2$
$=x^2+2\sqrt{5}x+5+\frac{10-x^2}{2}=\frac{1}{2}\left(x^2+4\sqrt{5}x+20\right)$.

b- $MF=\frac{\sqrt{2}}{2}\left|x-2\sqrt{5}\right|$ and $MF'=\frac{\sqrt{2}}{2}\left|x+2\sqrt{5}\right|$.
But $M\in(E)$, then $-\sqrt{10}\le x\le\sqrt{10}$, which gives
$-2\sqrt{5}\le x\le 2\sqrt{5}$ then $x-2\sqrt{5}\le 0$, therefore
$MF=\frac{\sqrt{2}}{2}\left(2\sqrt{5}-x\right)$ and $MF'=\frac{\sqrt{2}}{2}\left(x+2\sqrt{5}\right)$, consequently :
$MF+MF'=\frac{\sqrt{2}}{2}\left(2\sqrt{5}-x\right)+\frac{\sqrt{2}}{2}\left(2\sqrt{5}+x\right)=2\sqrt{10}$.

**N° 14.**

1) $\begin{cases} MJ=MN+NJ=r+4 \\ MI=KI-KM=8-r \end{cases}$

Therefore $MI+MJ=12$. Hence $M$ traces an ellipse of foci $I$ and $J$ and $2a=12$.



2) $2c = IJ = 2$, therefore $c = 1$ and since $2a = 12$ we get $a = 6$.
The center $w$ is the midpoint of $[IJ]$, hence $w(2;0)$.
$c^2 = a^2 - b^2$, so $b^2 = a^2 - c^2 = 35$, therefore an equation of the ellipse is $\dfrac{(x-2)^2}{36} + \dfrac{y^2}{35} = 1$.

$N°15$.

1) We have $\dfrac{RF}{d(R;(SP))} = \dfrac{RF}{RS} = \dfrac{\frac{3}{2}}{3} = \dfrac{1}{2}$, hence $R \in (E)$.

$\dfrac{QF}{d(Q;(SP))} = \dfrac{QF}{QP} = \dfrac{\frac{3}{2}}{3} = \dfrac{1}{2}$, hence $Q \in (E)$.

2) $(HF)$ is the focal axis of $(E)$, since
$\dfrac{AF}{d(A;(SP))} = \dfrac{AF}{AH} = \dfrac{1}{2} = e$, then $A \in (E)$ but $A$ belongs to the focal axis hence $A$ is a vertex of $E$.

3) a- Let $M(x;y)$ be a point of $(E)$ and $M'(0;y)$ its orthogonal projection on the directrix which is the axis $y'y$, $\dfrac{MF}{MM'} = \dfrac{1}{2}$,
then : $MF^2 = \dfrac{1}{4}MM'^2$ which gives $(x-3)^2 + y^2 = \dfrac{1}{4}x^2$.
Hence an equation of $(E)$ is $3x^2 + 4y^2 - 24x + 36 = 0$.

b- The vertices are the points of intersection of $(E)$ and the focal axis $x'x$, for $y = 0$ we get $3x^2 - 24x + 36 = 0$, which leads to $x^2 - 8x + 12 = 0$, quadratic equation whose roots are $x' = 2$ and $x'' = 6$, hence the second vertex is the $A'(6;0)$.

c- The center of $(E)$ is the point $w$ midpoint of $[AA']$, so $w(4;0)$.
The second focus $F'$ is the symmetric of $F$ with respect to $w$,
Therefore $F'(5;0)$.

d- The non-focal axis is the straight line of equation $x = 4$, the vertices of the non-focal axis are the points of intersection of this conic with this line , replace $x = 4$ in the equation $3x^2 + 4y^2 - 24x + 36 = 0$ we get $y^2 = 3$ , therefore $B\left(4;\sqrt{3}\right)$ and $B'\left(4;-\sqrt{3}\right)$.

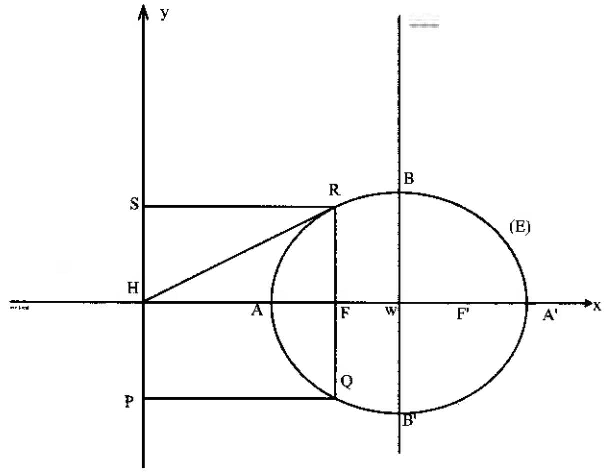
e- We have $R\left(3;\frac{3}{2}\right)$ , deriving $3x^2 + 4y^2 - 24x + 36 = 0$ with respect to $x$, so $6x + 8yy' - 24 = 0$, hence $y' = \frac{12-3x}{4y}$ then the slope of the tangent at $R$ to $(E)$ is

$$y'_R = \frac{12-3x_R}{4y_R} = \frac{12-9}{6} = \frac{1}{2}.$$

An equation of $(T)$ is $y - y_R = y'_R(x - x_R)$ , which gives :

$$y = \frac{1}{2}x.$$

f-




g- $v = \pi \int_2^6 y^2 dx$ but $y^2 = \frac{1}{4}\left(-3x^2 + 24x - 36\right) = -\frac{3}{4}x^2 + 6x - 9$

$$v = \pi \int_2^6 \left(-\frac{3}{4}x^2 + 6x - 9\right)dx = \pi\left[-\frac{x^3}{4} + 3x^2 - 9x\right]_2^6$$

$v = \pi\left[\left(-54 + 108 - 54\right) - \left(-2 + 12 - 18\right)\right] = 8\pi$ cubic units.

N° 16.

1) Let $M(x;y)$ be a point of $(H)$ and $M'(-1;y)$ its orthogonal projection on the directrix.. $\frac{MF}{MM'}=\sqrt{3}$, then $MF^2=3MM'^2$, which gives $2x^2-y^2+10x-1=0$.

2) The foci $F'(1;0)$ and $F(5;0)$ are two points of the axis $x'x$ then $x'x$ is the focal axis, the center $w$ is the midpoint of $[F'F]$ ; hence $w(3;0)$. $2c=F'F=4$ gives $c=2$, $2a=2\sqrt{2}$ gives $a=\sqrt{2}$.
$c^2=a^2+b^2$ gives $b^2=c^2-a^2=4-2=2$.
Hence an equation of the hyperbola is $\frac{(x-3)^2}{2}-\frac{y^2}{2}=1$.

3) The foci $O(0;0)$ and $A(0;-2)$ are two points on the axis $y'y$ hence $y'y$ is the focal axis, its center $w$ is the midpoint of $[OA]$ ,therefore $w(0;-1)$. $2a=OA=2$ gives $a=1$.
$e=\frac{c}{a}=2$ gives $c=2a$ then $c=2$.
$b^2=c^2-a^2=4-1=3$, hence an equation of the hyperbola is :
$\frac{(y+1)^2}{1}-\frac{x^2}{3}=1$.

4) The foci $O(0;0)$ and $F(0;4)$ are two points of the axis $y'y$ , then $y'y$ is the focal axis, then the equation of the hyperbola has the form $\frac{(y-\beta)^2}{a^2}-\frac{(x-\alpha)^2}{b^2}=1$ where $\alpha$ and $\beta$ are the coordinates of center $w$ that is the midpoint of $[OF]$, $w(0;2)$.
$2c=OF=4$ gives $c=2$.
$MO^2=(12)^2+(9)^2=144+81=225$, then $MO=15$.
$MF^2=(12)^2+(9-4)^2=144+25=169$, then $MF=13$.
Since $MO-MF=2a$ we get $2a=2$ and consequently $a=1$.
$b^2=c^2-a^2=4-1=3$ , hence the reduced equation of the hyperbola is : $\frac{(y-2)^2}{1}-\frac{x^2}{3}=1$.

**N° 17.**

1) $\frac{(x-1)^2}{9}-\frac{(y+1)^2}{16}=1$ is an equation of the hyperbola of center $w(1;-1)$.

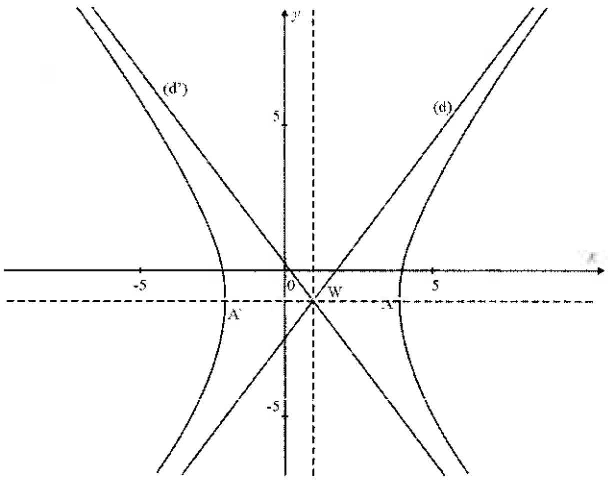
Letting $x-1=X$ and $y+1=Y$, the reduced equation is

$\frac{X^2}{9}-\frac{Y^2}{16}=1$.

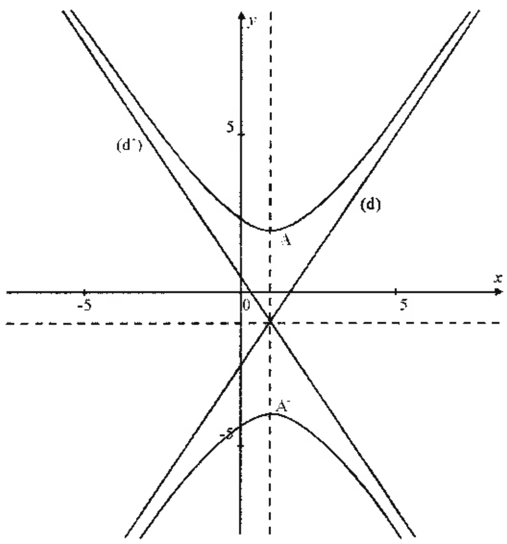
The focal axis is $X'X$, $a^2=9, b^2=16$ and

$c^2=a^2+b^2=25$ then $c=5$.

| Elements | System $\left(w; \vec{i}, \vec{j}\right)$ | System $\left(O; \vec{i}, \vec{j}\right)$ |
|---|---|---|
| Vertices | $A(3;0)$ , $A'(-3;0)$ | $A(4;-1)$ , $A'(-2;-1)$ |
| Foci | $F(5;0)$ , $F'(-5;0)$ | $F(6;-1)$ , $F'(-4;-1)$ |
| Asymptotes | $(d): Y=\frac{4}{3}X$<br>$(d'): Y=-\frac{4}{3}X$ | $(d): y+1=\frac{4}{3}(x-1)$<br>$(d'): y+1=-\frac{4}{3}(x-1)$ |



2) The equation $4y^2 - 9x^2 + 8y + 18x - 41 = 0$ is equivalent to :
$4(y^2 + 2y + 1 - 1) - 9(x^2 - 2x + 1 - 1) - 41 = 0$
$4(y+1)^2 - 9(x-1)^2 = 36$ and the equation becomes :
$\frac{(y+1)^2}{9} - \frac{(x-1)^2}{4} = 1$.
Letting $x - 1 = X$ and $y + 1 = Y$, the center of symmetry is the point $w(1;-1)$ and the reduced equation is $\frac{Y^2}{9} - \frac{X^2}{4} = 1$.
The focal axis is the straight line of equation $x = -1$ passing through $w$ and parallel to the axis $y'y$, $a^2 = 9, b^2 = 4$ and $c^2 = a^2 + b^2 = 13$ hence $c = \sqrt{13}$.
The vertices are $A(1;2)$ and $A'(1;-4)$.
The asymptotes are the straight lines of equations :
$(d): y + 1 = \frac{3}{2}(x-1)$ and $(d'): y + 1 = -\frac{3}{2}(x-1)$.



**N° 18.**

$\dfrac{x^2}{2m-4}+\dfrac{y^2}{m+1}=1$ , let $p=2m-4$ and $q=m+1$ .

| $m$ | $-\infty$ | -1 | | 2 | $+\infty$ |
|---|---|---|---|---|---|
| $p=2m-4$ | — | | — | | + |
| $q=m+1$ | — | | + | | + |
| Nature | impossible | | hyperbola | | ellipse |

Three cases to be considered :

- If $m<-1$ then $(C_m)$ is the void set .
- If $-1<m<2$ then the conic is a hyperbola .
- If $m>2$ then the conic is an ellipse .

**Remark that :** If $p=q$ then $2m-4=m+1$ which gives $m=5$ .
The equation becomes: $x^2+y^2=6$ .(the circle is a particular case of an ellipse)

**N° 19.**

1) The general equation of a conic is $ax^2+by^2+cx+dy+e=0$.
To study the nature of this conic we study the sign of $a\times b=m(m-1)$.

| $m$ | $-\infty$ | | 0 | | 1 | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $a\times b$ | | + | 0 | — | 0 | + | |

If $m=0$ or $m=1$ then the curve $(C_m)$ is a parabola .
If $0<m<1$ then the curve $(C_m)$ is a hyperbola .
If $m<0$ or $m>1$ then the curve $(C_m)$ is an ellipse.

2) a- For $m=1$ the equation of $(C_m)$ becomes
$y^2-2x+2y-3=0$ that is equivalent to
$(y+1)^2=2x+4=2(x+2)$.
Taking: $X=x+2$ and $Y=y+1$, we get the reduced equation of the parabola $Y^2=2X$ , of vertex $S(-2;-1)$ and focal axis the axis $X'X$ .
We know that the focal axis is an axis of symmetry for the

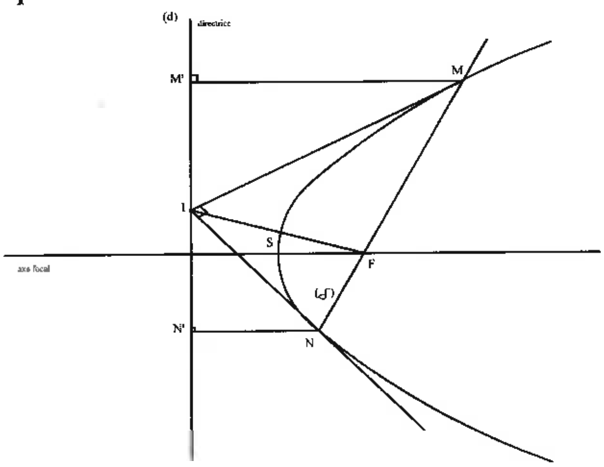
parabola and since the equation of the focal axis is $Y = 0$, we let $y+1=0$, and we deduce that the straight line of equation $y=-1$ is an axis of symmetry for $(C_1)$.

b- i- $(\delta)$ passes through the focus $F$ and cuts $(P)$ in $M$ and $N$, we designate by $M'$ and $N'$ the orthogonal projection of $M$ and $N$ respectively on the directrix $(d)$, the tangents $(T_1)$ and $(T_2)$ to $(P)$ at $M$ and $N$ intersect at a point $I$. The tangent $(T_1)$ is the interior bisector of angle $\widehat{FMM'}$ and the tangent $(T_2)$ is the interior bisector of angle $\widehat{FNN'}$.

But $\widehat{FMM'}+\widehat{FNN'} = 180°$, therefore :

$\widehat{FMI}+\widehat{FNI} = \dfrac{\widehat{FMM'}}{2}+\dfrac{\widehat{FNN'}}{2} = \dfrac{180}{2} = 90°$, consequently

$\widehat{MIN} = 90°$, hence the tangents $(T_1)$ and $(T_2)$ are perpendicular.




ii- $(T_1)$ is the perpendicular bisector of segment $[FM']$, but $I \in (T_1)$ then $IF = IM'$ and $\boxed{\widehat{FIM} = \widehat{MIM'}}$ (1).

Similarly, $(T_2)$ is the perpendicular bisector of segment $[FN']$, but $I \in (T_2)$ then $IF = IN'$ and $\boxed{\widehat{FIN} = \widehat{NIN'}}$ (2).

From (1) and (2), we get :

$$\widehat{FIM'}+\widehat{FIN'}=2\,\widehat{FIM}+2\,\widehat{FIN}$$
$$=2\left(\widehat{FIM}+\widehat{FIN}\right)=2\times 90^\circ=180^\circ.$$

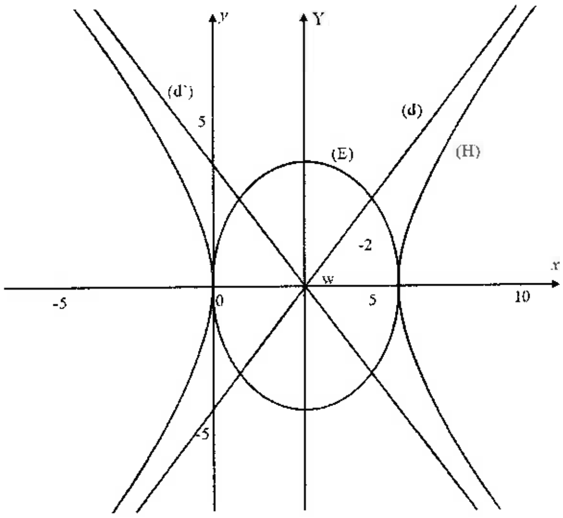
Hence, $\widehat{FIM'}+\widehat{FIN'}=\widehat{M'IN'}=180^\circ$ , consequently : $M'$ , $N'$ and $I$ are collinear hence $I$ belongs to the directrix $(d)$ and $I$ is the midpoint of segment $[M'N']$ since $IM'=IF=IN'$.

**N°20.**

1) $MF=MP+PF=r+7$ and $MF'=MQ+QF'=r+1$, then $MF-MF'=6$ , constant, then $M$ describes a branch of a hyperbola $(H)$ of foci $F$ and $F'$ and focal length $2a=6$ .
The center of the hyperbola is the midpoint $w(3;0)$ of segment $[FF']$.
$2c=F'F=10$ then $c=5$ and $2a=6$ gives $a=3$ .
$b^2=c^2-a^2=25-9=16$ hence an equation of the hyperbola is: $\dfrac{(x-3)^2}{9}-\dfrac{y^2}{16}=1$, so $16x^2-9y^2-96x=0$.

2) The vertices of $(H)$ are $A(6;0)$ and $O(0;0)$.
The asymptotes are the straight lines of equations :
$(d): y=\dfrac{4}{3}(x-3)$ and $(d'): y=-\dfrac{4}{3}(x-3)$ .
**Remark that :** The locus of $M$ is the branch of vertex $O$ $(x\le 0)$.

3) a- The point $w(3;0)$ is the center of $(E)$ .
$2c=2\sqrt{7}$ gives $c=\sqrt{7}$ , $2b=OA=6$ gives $b=3$.
$c^2=a^2-b^2$ then $a^2=b^2+c^2=9+7=16$ .
Hence an equation of $(E)$ is $\dfrac{(x-3)^2}{9}+\dfrac{y^2}{16}=1$.
This equation is equivalent to $16x^2+9y^2-96x=0$.

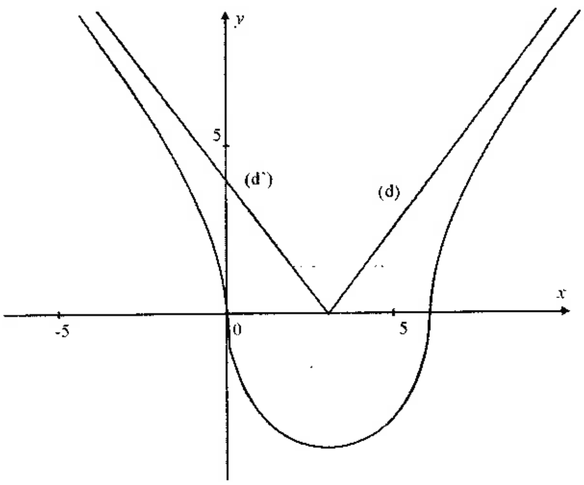



b- The equation $16x^2 - 9y|y| - 96x = 0$ is equivalent to :

For $y \leq 0$ , $16x^2 + 9y^2 - 96x = 0$, which is the part of the ellipse $(E)$ that is below $x'x$.

For $y \geq 0$, $16x^2 - 9y^2 - 96x = 0$ , which is the part of the hyperbola $(H)$ that is above $x'x$.

The curve $(C)$ is the union if these two parts .

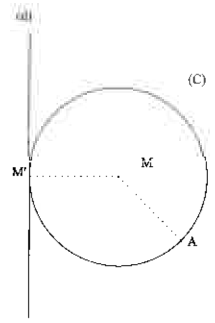


4) $(C) \xrightarrow{h_1\left(P;-\frac{r}{R}\right)} (\omega) \xrightarrow{h_2\left(Q;-\frac{R'}{r}\right)} (C')$ , then $h_2 \circ h_1$ is a positive dilation of center $I$ and ratio $k = \left(-\frac{r}{R}\right) \times \left(\frac{-R'}{r}\right) = \frac{R'}{R} = \frac{1}{7}$ transforming $(C)$ onto $(C')$, then $I$ is fixed .

But if a dilation is the composite of two other dilations then the three centers of the dilations are collinear, therefore $P$ , $Q$ and $I$ are collinear and consequently the straight line $(MP)$ passes through a fixed point $I$ .

**N°21.**

$M$ is the center of circle $(C)$ then $MA = MM' = r$ where $r$ is the radius of $(C)$ then $M$ is equidistant from the fixed point $A$ and the fixed straight line $(d)$ and consequently $M$ describes the parabola of focus $A$ and directrix $(d)$ .




**N°22.**

1) a- $z + \bar{z} + 4 = x + iy + x - iy + 4 = 2x + 4$ .

Hence $z + \bar{z} + 4 = 0$ gives $2x + 4 = 0$ , therefore the set of points $M$ is the straight line $(d)$ of equation $x = -2$ .

b- The distance of $M$ to $(d)$ is equal to :

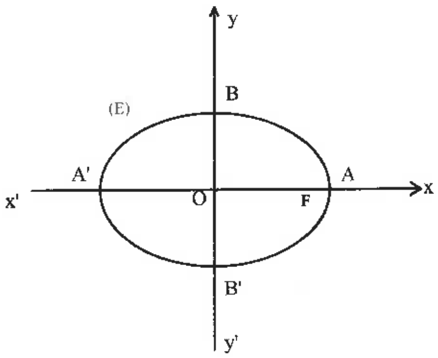
$$|x + 2| = \frac{1}{2}|2x + 4| = \frac{1}{2}|z + \bar{z} + 4| .$$

2) $\left|\frac{z - 1 - i}{z + \bar{z} + 4}\right| = \frac{|z - 1 - i|}{|z + \bar{z} + 4|} = \frac{|z_M - z_F|}{2d(M;(d))} = \frac{FM}{2d(M;(d))} = \frac{\sqrt{2}}{4}$ which

gives $\frac{FM}{d(M;(d))} = \frac{\sqrt{2}}{2}$ and consequently the set of points $M$ is the ellipse of focus $F$, directrix $(d)$ and eccentricity $e=\frac{\sqrt{2}}{2}$.

N°23.

1) a-




b- We have $c^2=a^2-b^2=9-4=5$ so $c=\sqrt{5}$ hence $F(\sqrt{5};0)$.
The directrix associated to $F$ is the straight line $(\Delta)$ of equation
$x=\frac{a^2}{c}=\frac{9}{\sqrt{5}}=\frac{9\sqrt{5}}{5}$ and the eccentricity is $e=\frac{c}{a}=\frac{\sqrt{5}}{3}$.

$M\in(E)$ therefore $\frac{MF}{MH}=e=\frac{\sqrt{5}}{3}$.

i- $\frac{M_1F_1}{M_1H_1}=\frac{\frac{\sqrt{2}}{2}MF}{\frac{\sqrt{2}}{2}MH}=\frac{MF}{MH}=\frac{\sqrt{5}}{3}$.

ii- $(\Delta_1)$ is fixed being the image by $S$ of the fixed straight line $(\Delta)$, similarly $F_1$ is fixed and since $\frac{M_1F_1}{M_1H_1}=\frac{\sqrt{5}}{3}$ then the point $M_1$ varies on an ellipse $(E_1)$, of focus $F_1$, directrix $(\Delta_1)$ eccentricity $\frac{\sqrt{5}}{3}$ and consequently, the image of $(E)$ by $S$ is the ellipse $(E_1)$.
The focal axis $x'x$ of $(E)$ has an image the straight line $(u'u)$ by $S$ the straight line of equation $y=x$.
The non-focal axis $y'y$ of $(E)$ has an image the straight

line of equation $y = -x$ .Remark that $O$ is an invariant point since $O$ is the center of similitude..

2) a- $z_1 = az + b$ with $a = \frac{\sqrt{2}}{2} e^{i\frac{\pi}{4}} = \frac{1}{2} + \frac{1}{2} i$ .

The point $O$ is invariant then $b = 0$ and consequently $z_1 = \frac{1}{2}(1+i)z$ . If we let $\ell = OM$ then $OM_1 = \frac{\sqrt{2}}{2}\ell$ , hence :

$$MM_1^{\ 2} = \ell^2 + \left(\frac{\sqrt{2}}{2}\ell\right)^2 - 2\ell \times \frac{\sqrt{2}}{2}\ell \times \cos\frac{\pi}{4} = \frac{\ell^2}{2}.$$

Therefore : $M_1M = OM_1 = \frac{\ell\sqrt{2}}{2}$ hence $OMM_1$ is an isosceles triangle of principal vertex $M_1$ having an angle $\frac{\pi}{4}$ , consequently $OMM_1$ is a right isosceles triangle at $M_1$ .

b- If $M$ is given then $M_1$ belongs to the perpendicular bisector of $[OM]$ and $M_1$ belongs to the semi-circle of diameter $[OM]$ with $\left(\overrightarrow{OM};\overrightarrow{OM_1}\right) = \frac{\pi}{4}(\text{mod } 2\pi)$ .

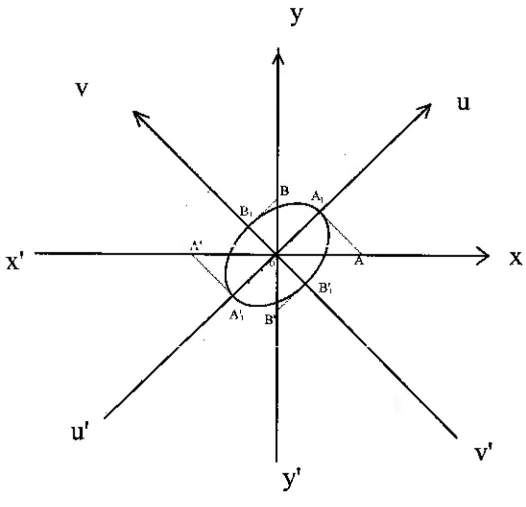
Then $M_1$ is the intersection of the perpendicular bisector and the semi-circle .

c- $A \xrightarrow{S} A_1$

$B \xrightarrow{S} B_1$

$A' \xrightarrow{S} A'_1$

$B' \xrightarrow{S} B'_1$



N°24.

1) a- $M' = S(M)$ then $z' = az + b$ with :

$$a = \sqrt{2}e^{i\frac{\pi}{4}} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i.$$

$\Omega$ is invariant under $S$ then $b = z_\Omega(1-a)$ therefore :
$b = (-2+3i)(1-1-i) = (-2+3i)(-i)$ then $b = 3+2i$
and consequently $z' = (1+i)z + 3 + 2i$.

b- The inverse of $S$ is the similitude $S'\left(\Omega;\frac{1}{\sqrt{2}};-\frac{\pi}{4}\right)$.

2) $z' = (1+i)z + 3 + 2i$, then $z' - 3 - 2i = (1+i)z$, which gives
$(1-i)(z'-3-2i) = (1-i)(1+i)z$, consequently
$(1-i)(z'-3-2i) = 2z$. Finally,

$$z = \left(\frac{1}{2} - \frac{1}{2}i\right)z' - \frac{5}{2} + \frac{1}{2}i \quad \text{or} \quad z' = \left(\frac{1}{2} - \frac{1}{2}i\right)z - \frac{5}{2} + \frac{1}{2}i.$$

3) $(E)$ is the image of $(E')$ by the similitude $S'\left(\Omega;\frac{\sqrt{2}}{2};-\frac{\pi}{4}\right)$ and since $(E')$ is an ellipse, then $(E)$ is also an ellipse .
$O(0;0)$ is the center of $(E')$ then the point $I = S'(O)$ is the center of $(E)$.

But $z_I = \left(\frac{1}{2} - \frac{1}{2}i\right)z_O - \frac{5}{2} + \frac{1}{2}i = -\frac{5}{2} + \frac{1}{2}i$, then $I\left(-\frac{5}{2};\frac{1}{2}\right)$.

The vertices of $(E')$ are $A(4;0)$ , $A'(-4;0)$ , $B(0;3)$ and $B'(0;-3)$ , their images by $S'$ are $A_1$ , $A_1'$, $B_1$ et $B_1'$ of respective affixes $z_{A_1} = 4\left(\frac{1}{2} - \frac{1}{2}i\right) - \frac{5}{2} + \frac{1}{2}i = -\frac{1}{2} - \frac{3}{2}i$ , $z_{A_1'} = -\frac{9}{2} + \frac{5}{2}i$
$z_{B_1} = -1 + 2i$ and $z_{B_1'} = -4 - i$.

N°25.

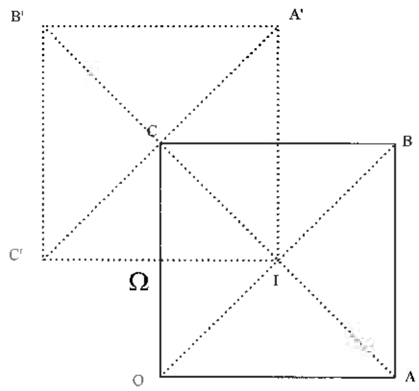
1) $r(O) = I$ and $r(I) = C$, then the angle of $r$ is

$\alpha = \left(\overrightarrow{OI};\overrightarrow{IC}\right) = \frac{\pi}{2}(\text{mod}\, 2\pi)$, the center $\Omega$ of $r$ belongs to the perpendicular bisector of $[OI]$ and the perpendicular bisector of $[IC]$ . The two perpendicular bisectors intersect at $\Omega$ that is the midpoint of $[OC]$.

2) $r(O)=I$ then $I$ is the first vertex of the image of the square $OABC$ and since $I$ is the center of $OABC$ then $C$ is the center of the image of $OABC$ by $r$ .

If $C'=r(C)$ then, $\Omega C'=\Omega C$ and $\left(\overrightarrow{\Omega C};\overrightarrow{\Omega C'}\right)=\frac{\pi}{2}(\bmod 2\pi)$

hence $C'$ is the symmetric of $I$ with respect to $\Omega$ .
If $A'=r(A)$ then $A'$ is the symmetric of $C'$ with respect to $C$.
Similarly, the point $B'=r(B)$ is the symmetric of $I$ with respect to $C$ so the image of square $OABC$ is square $IA'B'C'$.




3) a- $\frac{BA}{d(B;(OC))}=\frac{BA}{BC}=1$ then $B$ is a point of $(P)$.

b- $(OB)$ is the interior bisector of angle $A\hat{B}C$ then $(OB)$ represents a tangent to $(P)$ at $B$.

c- The vertex of $(P)$ is the point $S$ midpoint of segment limited by the focus $A$ and the orthogonal projection of $A$ on $(OC)$, hence $S$ is the midpoint of $[OA]$ .

d- The straight line $(OI)$ is the tangent $(T)$ to $(P)$ at $B$ .
Since $r(OI)=(IC)$ then the straight line $(IC)$ is the tangent to $(P')$ at $r(B)=B'$, but $(IC)$ is the straight line $(AC)$,

hence $(AC)$ is tangent to $(P')$ at $B'$.

e- Let $S'$ be the vertex of $(P')$, $S' = r(S)$, but $S$ is the midpoint of $[OA]$, then $S'$ is the midpoint of $r([OA])$ and consequently $S'$ is the midpoint of $[I\,A']$ and since $A$ is the focus of $(P)$, then $A'$ is the focus of $(P')$.

**N°26.**

1) $\vec{N_P}(4;0;-3)$ is a normal vector to plane $(P)$ and $\vec{k}(0;0;1)$ is a normal to plane $(xoy)$.

$\vec{N_P}\cdot\vec{k} \neq 0$ then $(P)$ is not perpendicular to plane $(xoy)$.

$\vec{N_P}$ and $\vec{k}$ are not collinear then $(P)$ and $(xoy)$ are not parallel, hence the projection of $(C)$ on $(xoy)$ is an ellipse $(E)$ of center $w(5;3;0)$ orthogonal projection of $I$ on plane $(xoy)$.

2) $\cos\theta = \dfrac{\left|\vec{N_P}\cdot\vec{k}\right|}{\left\|\vec{N_P}\right\|\times\left\|\vec{k}\right\|} = \dfrac{3}{5}$.

We know that $a = R = 5$ and $b = R\times\cos\theta = 5\times\dfrac{3}{5} = 3$.

$c^2 = a^2 - b^2 = 25 - 9 = 16$ then $c = 4$, therefore $e = \dfrac{c}{a} = \dfrac{4}{5}$.

The area interior to $(E)$ is $\mathrm{A} = \pi ab = 15\pi$ square units.

**N°27.**

1) $M(\alpha;\beta)$ is a variable point of $(C)$ whose equation is:

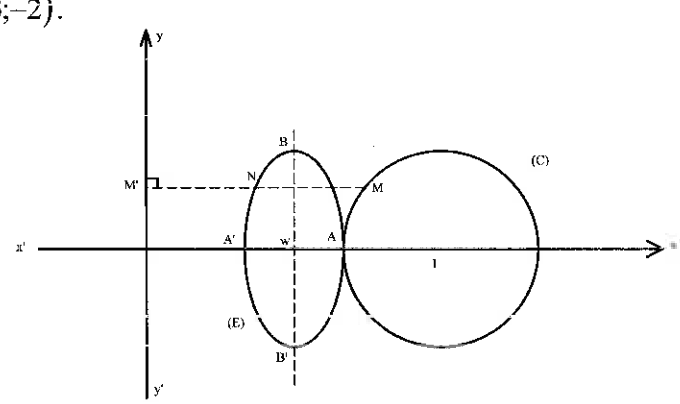
$(x-6)^2 + y^2 = 4$, we have then $(\alpha-6)^2 + \beta^2 = 4$.

If $M'$ is the orthogonal projection of $M$ on $y'y$ then $M'(0;\beta)$.

$N(x;y)$ is the midpoint of $[MM']$, therefore $x = \dfrac{\alpha}{2}$ and $y = \beta$.

Replacing $\alpha$ et $\beta$ by their values in the equation $(\alpha-6)^2 + \beta^2 = 4$ we get $(2x-6)^2 + y^2 = 4$ that is equivalent to the equation $\frac{(x-3)^2}{1}+\frac{y^2}{4}=1$ ,then $N$ describes the ellipse of center $w(3;0)$ and focal axis the straight line of equation $x=3$ .

2) The vertices of $(E)$ are the points $A(4;0)$ , $A'(2;0)$ $B(3;2)$ and $B'(3;-2)$.



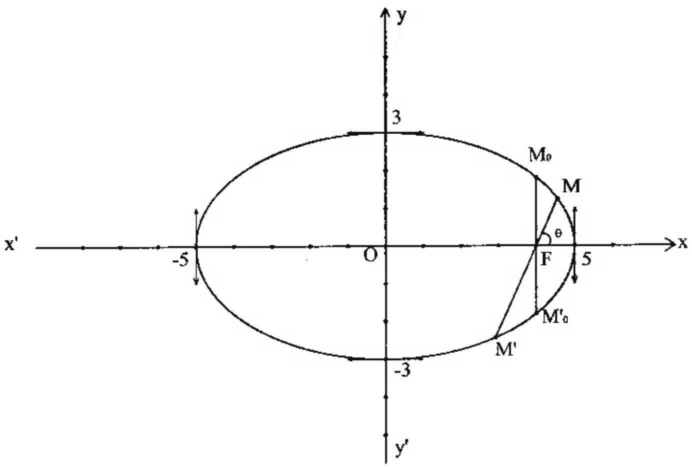

3) Let $L(x;y)$ be the point defined by $\overrightarrow{M'L}=\lambda\,\overrightarrow{M'M}$ ,
$x_L-0=\lambda(\alpha-0)$ and $y_L-\beta=\lambda(\beta-\beta)$, then $x=\lambda\alpha$ and $y=\beta$ and since $(\alpha-6)^2+\beta^2=4$ we get $\left(\frac{x}{\lambda}-6\right)^2+y^2=4$ ,

then $L$ describes the ellipse $(\Gamma_\lambda)$ of equation $\frac{(x-6\lambda)^2}{4\lambda^2}+\frac{y^2}{4}=1$.

**N°28.**

1) The vertices of $(E)$ are $A(5;0)$, $A'(-5;0)$ , $B(0;3)$ and $B'(0;-3)$.
The focus of positive abscissa is $F(4;0)$ and the directrix $(d)$ associated to $F$ has an equation $x=\frac{a^2}{c}=\frac{25}{4}$.

$e=\frac{c}{a}=\frac{4}{5}$.




2) a- $\overrightarrow{OM} = \overrightarrow{OF} + \overrightarrow{FM}$ , but $z_{\overrightarrow{FM}} = FM\, e^{i\left(\vec{u};\overrightarrow{FM}\right)} = re^{i\theta}$ then

$z_M = z_F + re^{i\theta}$, therefore $z_M = 4 + r(\cos\theta + i\sin\theta)$ and consequently, the abscissa of point $M$ is $x_M = 4 + r\cos\theta$.

b- Let $N$ be the orthogonal projection of $M$ on the directrix $(d)$, the distance of $M$ to $(d)$ is $MN = d_{(M,d)} = \left|x - \frac{25}{4}\right|$ , but

$M \in (E)$ hence $-5 \le x \le 5$ which gives $x < \frac{25}{4}$ and

consequently $MN = \frac{25}{4} - x = \frac{25}{4} - 4 - r\cos\theta$, then

$d_{(M,d)} = \frac{9}{4} - r\cos\theta$, $M \in (E)$ then $\frac{MF}{MN} = \frac{4}{5}$, therefore :

$r = MF = \frac{4}{5} MN = \frac{4}{5}\left(\frac{9}{4} - r\cos\theta\right)$, which gives

$r = \frac{9}{5 + 4\cos\theta}$.

c- Taking $FM' = r'$ and $\left(\overrightarrow{Ox}; \overrightarrow{FM'}\right) = \theta' (\text{mod}\, 2\pi)$, since

$$\left(\overrightarrow{Ox};\overrightarrow{FM'}\right)=\left(\overrightarrow{Ox};\overrightarrow{FM}\right)+\left(\overrightarrow{FM};\overrightarrow{FM'}\right)=\theta+\pi \quad (\text{mod } 2\pi)$$

then $\cos\theta'=-\cos\theta$, but from (b-) we have $r'=\dfrac{9}{5+4\cos\theta'}$

therefore $r'=\dfrac{9}{5-4\cos\theta}$, and consequently :

$\dfrac{1}{FM}+\dfrac{1}{FM'}=\dfrac{1}{r}+\dfrac{1}{r'}=\dfrac{5+4\cos\theta}{9}+\dfrac{5-4\cos\theta}{9}=\dfrac{10}{9}$ that is constant.

d- $MM'=FM+FM'=r+r'=\dfrac{9}{5+4\cos\theta}+\dfrac{9}{5-4\cos\theta}$

$$=\frac{45-36\cos\theta+45+36\cos\theta}{(5+4\cos\theta)(5-4\cos\theta)}=\frac{90}{25-16\cos^2\theta}$$

$MM'$ is minimal when $25-16\cos^2\theta$ is maximal, this is true when $\cos^2\theta=0$ but $0<\theta<\pi$ then $\theta=\dfrac{\pi}{2}$, hence the minimal length of $MM'$ is $\dfrac{90}{25}=\dfrac{18}{5}$ and $(MM')$ is perpendicular to the focal axis $x'x$.

Note that in this case $r=\dfrac{9}{5}$, $r'=\dfrac{9}{5}$, $z_{M_0}=4+\dfrac{9i}{5}$ and $z_{M'_0}=4-\dfrac{9i}{5}$.

**N°29.**

1) $A(m;0), B(0;n), P(x;y)$ and $\overrightarrow{OA}=2\overrightarrow{BP}$, then :

$2(x_P-x_B)=x_A$ et $2(y_P-y_B)=y_A$, which gives $x=\dfrac{m}{2}$ and $y=n$.

2) $AB^2=4$, then $m^2+n^2=4$ and consequently $(2x)^2+(y)^2=4$. Hence, the set of points $P$ is the ellipse $(E)$ of equation $\dfrac{x^2}{1}+\dfrac{y^2}{4}=1$.

3) a- The complex form $r$ is $z'=e^{i\frac{\pi}{4}}z$ which gives $z=e^{-i\frac{\pi}{4}}z'$.

If $M(x;y)$ is a point of $(E)$ and $M'(x';y')$ its image by $r$ then

$$x+iy=e^{-i\frac{\pi}{4}}(x'+iy')=\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right)(x'+iy')$$

$=\frac{\sqrt{2}}{2}(1-i)(x'+iy')=\frac{\sqrt{2}}{2}(x'+iy'-ix'+y')$ , which gives

$x=\frac{\sqrt{2}}{2}(x'+y')$ and $y=\frac{\sqrt{2}}{2}(y'-x')$ and since

$\frac{x^2}{1}+\frac{y^2}{4}=1$ , we get $\frac{1}{2}(x'+y')^2+\frac{1}{2}\frac{(y'-x')^2}{4}=1$, so

$4(x'+y')^2+(y'-x')^2=8$ and consequently

$5x'^2+6x'y'+5y'^2=8$.

Then, the image of $(E)$ by $r$ is the curve $(C)$ of equation :

$5x^2+6xy+5y^2=8$ .

b- $(E)$ is an ellipse and since rotation preserves geometric figures then $(C)$ is an ellipse .

c- The focal axis $(\Delta)$ of $(C)$ is the image of the focal axis of $(E)$ by $r$. But the focal axis of $(E)$ is the axis $y'y$ of equation $x=0$ .

$x=0$ gives $\frac{\sqrt{2}}{2}(x'+y')=0$ then the straight line $(\Delta)$ of equation . $y=-x$ is the focal axis of $(C)$.

$F(0;\sqrt{3})$ is a focus of $(E)$, $F$ is transformed onto point $F_1$ by the rotation $r$ .

$z_{F_1}=e^{i\frac{\pi}{4}}z_F=\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)(i\sqrt{3})=\frac{i\sqrt{6}}{2}-\frac{\sqrt{6}}{2}$ then

$F_1\left(-\frac{\sqrt{6}}{2};\frac{\sqrt{6}}{2}\right)$ is a focus of $(C)$.

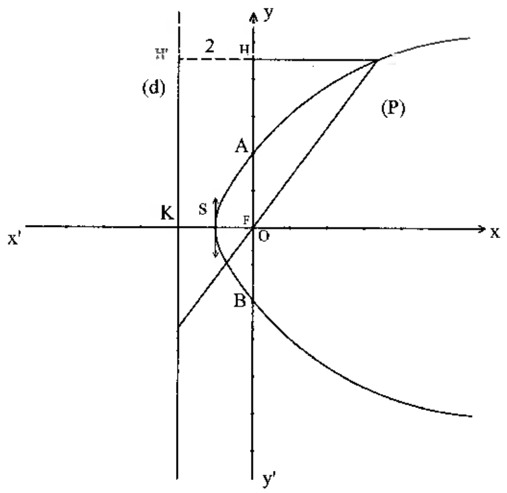
d- The eccentricity of $(C)$ is equal to that of $(E)$ , hence $e=\frac{\sqrt{3}}{2}$

Area of $(C)$ = Area of $(E)=\pi ab=2\pi$ square units .

N° 30.

1) The equation $y^2 = 4x + 4$ can be written as $y^2 = 4(x+1)$, letting $x + 1 = X$ and $y = Y$ so we get the reduced equation of $(P)$ : $Y^2 = 4X$ .
The vertex of $(P)$ is $S(-1;0)$ .
$p = 2$ then $\overrightarrow{SF}\left(\frac{p}{2};0\right) = (1;0)$ which gives $x_F - x_S = 1$ and $y_F - y_S = 0$, hence $F(0;0)$.
The directrix is parallel to $y'y$ and passes through the point $K$ symmetric of $F$ with respect to $S$ ,but $K(-2;0)$ then an equation of $(d)$ is $x = -2$ .

2)




3) For $x = 0$ we get $y = \pm 2$ then $OA = OB = 2$ .

4) a- The focus $F$ is confounded with the origin $O$,
$\frac{MO}{MH'} = e = 1$ gives $MO = MH'$ then $MO = MH + HH'$
and consequently $MO - MH = HH' = 2$ .

b- The parabola $(P)$ cuts the axis $y'y$ in two points $A(0;2)$ and $B(0;-2)$ ,the circle of diameter $[AB]$ is of centre $O$ and of radius $R = 2$ ,the circle of centre $M$ has a radius $R' = MH$ .
$R + R' = 2 + MH = MO$ since $MO - MH = 2$.

Then the sum of the two radii is equal to the distance between the two centers, hence the circles are tangent externally.

4) a- $y = mx$ is an equation of $(D)$. The abscissas of the points of $M'$ and $M''$ of intersection of $(D)$ and $(P)$ are the roots of the equation: $m^2x^2 - 4x - 4 = 0$ with $m \neq 0$ .

$I$ is the midpoint of $[M'M'']$ then $x_I = \dfrac{x'+x''}{2} = \dfrac{4}{2m^2} = \dfrac{2}{m^2}$

and since $I \in (D)$ then $y_I = mx_I = m\left(\dfrac{2}{m^2}\right) = \dfrac{2}{m^2}$, hence

$I\left(\dfrac{2}{m^2};\dfrac{2}{m}\right)$.

b- $x_I = \dfrac{2}{m^2}$ and $y_I = \dfrac{2}{m}$ therefore $y_I^2 = 2x_I$ and consequently $I$ varies on the parabola $(P')$ of equation $y^2 = 2x$.

c- ***Algebraic Method :***

Deriving $y^2 = 2x$ with respect to $x$, we get $2yy' = 4$ then

$y' = \dfrac{2}{y}$ , hence the slope of the tangent at $M'$ to $(P)$ is

$a' = \dfrac{2}{y_{M'}} = \dfrac{2}{mx'}$ and the slope of the tangent at $M''$ to $(P)$ is

$a'' = \dfrac{2}{y_{M''}} = \dfrac{2}{mx''}$, so $a \times a' = \dfrac{4}{m^2x'x''} = \dfrac{4}{m^2\left(-\dfrac{4}{m^2}\right)} = -1$, hence

the two tangents $(T')$ and $(T'')$ are perpendicular.

**Geometric Method :**

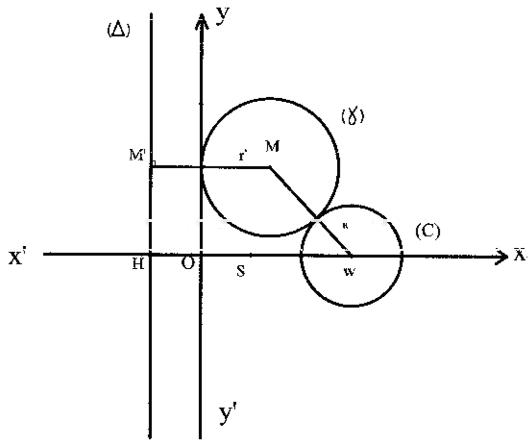
Let $H', H''$ be the orthogonal projections of $M'$ and $M''$ the directrix . $(T')$ is the bisector of angle $\widehat{FM'H'}$ and $(T'')$ is the bisector of angle $\widehat{FM''H''}$.

But $\widehat{FM'H'} + \widehat{FM''H''} = 180°$ then $\widehat{FM'I} + \widehat{FM''I} = \dfrac{180°}{2} = 90°$

which gives that in triangle $M'IM''$ the angle $\widehat{M'IM''} = 90°$

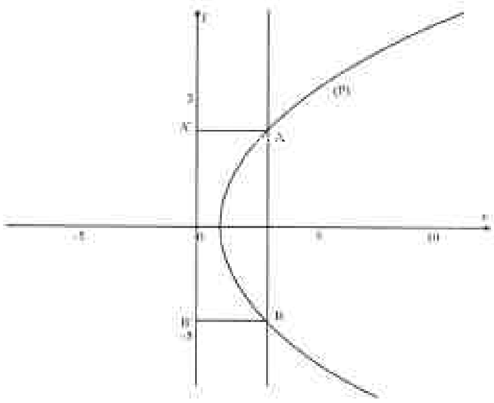
N°31.

1) Let $(\Delta)$ be the straight line parallel to $y'y$ at a distance 1 and on the other side of $M$ with respect to $y'y$ and let $M'$ be the orthogonal projection of $M$ on $(\Delta)$, then $MM' = 1 + R$ and $Mw = 1 + R$, hence $Mw = MM'$.
$M$ is equidistant of a fixed point $w$ and a fixed straight line $(\Delta)$, then $M$ traces a parabola of focus $w$ and directrix $(\Delta)$.




2) The directrix $(\Delta)$ is parallel to the axis $y'y$, $(\Delta)$ has as an equation $x = -1$ and $w(3;0)$ the focus, the vertex $S$ is the midpoint of $[Hw]$ therefore $S(1;0)$ and since the parameter is $p = Hw = 4$ then the reduced equation of the parabola is $(y-0)^2 = 2p(x-1)$, so: $y^2 = 8(x-1)$.

3)



4) $y = f(x) = \pm\sqrt{8(x-1)}$

- $\mathcal{A} = 2\int_1^3 \sqrt{8(x-1)}\,dx = 2\sqrt{8}\int_1^3 \sqrt{x-1}\,dx$

$= 2\sqrt{8} \times \frac{2}{3}(x-1)\sqrt{x-1}\ \Big|_1^3 = \frac{32}{3}$ square units .

- $\mathcal{A}' =$ (Area of rectangle $ABB'A'$) $- \mathcal{A}$

$= AB \times AA' - \mathcal{A} = (8\times 3) - \frac{32}{3} = 24 - \frac{32}{3}$

$= \frac{72-32}{3} = \frac{40}{3}$ square units.

**N°32.**

1) The equation $25(x^2+y^2) = (3x-16)^2$ is equivalent to

$25(x^2+y^2) = 9\left(x - \frac{16}{3}\right)^2$ then if $(\Delta)$ is the straight line of

equation $x = \frac{16}{3}$ and $M(x;y)$ then the equation gives

$25\,OM^2 = 9d^2_{(M,\Delta)}$ , therefore $\frac{OM}{d_{(M,\Delta)}} = \frac{3}{5}$ consequently $M$

describes an ellipse of focus $O$, directrix $(\Delta)$ and eccentricity

$e = \frac{3}{5}$ .

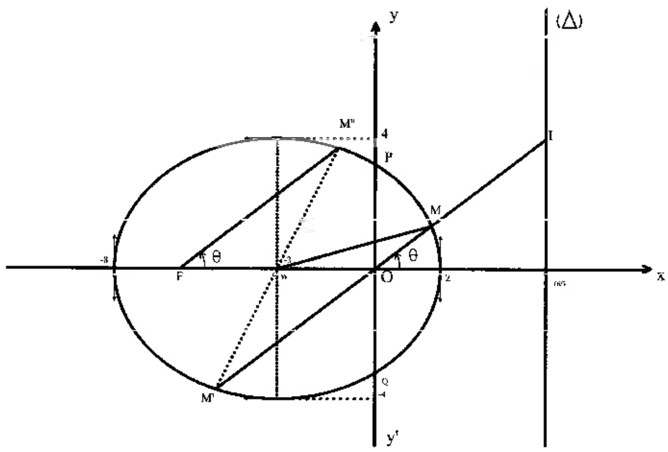
The equation $25(x^2+y^2) = (3x-16)^2$ is equivalent to

$16x^2 + 25y^2 + 96x - 256 = 0$ then to the equation

$\frac{(x+3)^2}{25} + \frac{y^2}{16} = 1$ .

2) The vertices of the ellipse are :

$A(2;0)$ , $A'(-8;0)$ , $B(-3;4)$ and $B(-3;-4)$.




3) a- $\dfrac{OM}{d_{(M,\Delta)}} = e = \dfrac{3}{5}$ ,then $5OM = 3d_{(M;\Delta)} = 3\times\left|x-\dfrac{16}{3}\right|$,

but $M \in (E)$ then $x < \dfrac{16}{3}$ therefore $d_{(M;\Delta)} = \dfrac{16}{3} - x$

consequently $OM = \dfrac{16-3x}{5}$.

b- $x_M = OM \times \cos\theta$ and since $5OM = 16 - 3x$ we get $5OM = 16 - 3OM\cos\theta$ then $OM(5+3\cos\theta) = 16$ and

consequently $OM = \dfrac{16}{5+3\cos\theta}$.

4) a- Let $z_{M'} = r'e^{i\theta'}$ on a :

$$\theta' = \left(\vec{u};\overrightarrow{OM'}\right) = \left(\vec{u};\overrightarrow{OM}\right) + \left(\overrightarrow{OM};\overrightarrow{OM'}\right) = \theta + \pi \pmod{2\pi}.$$

A reasoning similar to the previous one gives :

$OM' = \dfrac{16}{5+3\cos\theta'} = \dfrac{16}{5-3\cos\theta}$ then :

$\dfrac{1}{OM} + \dfrac{1}{OM'} = \dfrac{5+3\cos\theta}{16} + \dfrac{5-3\cos\theta}{16} = \dfrac{10}{16} = \dfrac{5}{8}$, so it is a constant .

b- $(OM)$ has an equation $y = (\tan\theta)x$, $(OM)$ cuts

$(\Delta)$ at a point $I$ of abscissa $x=\frac{16}{3}$ therefore $I\left(\frac{16}{3};\frac{16}{3}tan\,\theta\right)$.

Let $H$ be the point of intersection of $(\Delta)$ with the focal axis $x'x$, we have $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$ et $\cos\theta>0$ hence $cos\,\theta=\frac{OH}{OI}$

therefore $OI=\frac{OH}{cos\,\theta}=\frac{16}{3\,cos\,\theta}$, which gives

$\frac{2}{OI}=\frac{2}{\frac{16}{3\cos\theta}}=\frac{6\cos\theta}{16}$.

On the other hand we have :

$\frac{1}{OM}-\frac{1}{OM'}=\frac{5+3\cos\theta}{16}-\frac{5-3\cos\theta}{16}=\frac{6\cos\theta}{16}$ then

$\frac{1}{OM}-\frac{1}{OM'}=\frac{2}{OI}$.

We get the system $\begin{cases}\frac{1}{OM}+\frac{1}{OM'}=\frac{5}{8}\\ \frac{1}{OM}-\frac{1}{OM'}=\frac{2}{OI}\end{cases}$, Adding we get

$\frac{2}{OM}=\frac{5}{8}+\frac{2}{OI}$ which gives $\frac{2}{OI}=\frac{2}{OM}-\frac{5}{8}$, then

$\frac{2}{OI}=\frac{16-5\,OM}{8\,OM}$ and consequently $OI=\frac{16\,OM}{16-5\,OM}$.

5) a- The second focus $F$ is the symmetric of the focus $O$ with respect to the center $w$ then $w$ is the midpoint of $[OF]$ and $[M'M'']$. Consequently, $OM'FM''$ is a parallelogram hence

$(FM'')//(OM')$ therefore $\overrightarrow{FM''}$ and $\overrightarrow{OM'}$ are collinear.

b- $F(-6;0)$, $\left(\vec{u};\overrightarrow{FM''}\right)=\theta(\text{mod}\,2\pi)$ and

$FM''=OM'=\frac{16}{5-3\,cos\,\theta}$.

$z_{\overrightarrow{FM''}}=\frac{16}{5-3\cos\theta}(\cos\theta+i\sin\theta)$, then

$$z_{M''} - z_F = \left(\frac{16\cos\theta}{5-3\cos\theta}\right) + i\left(\frac{16\sin\theta}{5-3\cos\theta}\right) \text{which gives}$$

$$z_{M''} = \frac{16\cos\theta}{5-3\cos\theta} - 6 + i\left(\frac{16\sin\theta}{5-3\cos\theta}\right) \text{ and consequently}$$

$$M''\left(\frac{34\cos\theta-30}{5-3\cos\theta};\frac{16\sin\theta}{5-3\cos\theta}\right).$$

6) a- $N$ is the center of gravity of triangle $MM'M''$ then

$\overrightarrow{wN} = \frac{1}{3}\overrightarrow{wM}$ , hence $N$ is the image of $M$ by the dilation of center $w$ and of ratio $\frac{1}{3}$ and since $M$ describes the part of $(E)$ where $x \geq 0$ then $N$ describes a part of an ellipse $(E')$ image of $(E)$ by this dilation .

b- The dilation is of center $w$ (invariant point), The ellipse $(E')$ has an equation $\frac{(x+3)^2}{a'^2} + \frac{y^2}{b'^2} = 1$ with $a' = \frac{1}{3}a = \frac{1}{3}\times 5 = \frac{5}{3}$ and $b' = \frac{1}{3}b = \frac{1}{3}\times 4 = \frac{4}{3}$ .

The area of the domain limited by $(E)$ and $(E')$ is :

$$\pi ab - \pi a'b' = \pi\times 5\times 4 - \pi\times\frac{5}{3}\times\frac{4}{3}$$

$$= 20\pi - \frac{20\pi}{9} = \frac{160\pi}{9} \text{ square units .}$$

c- If $y_1 = \frac{16\sqrt{3}}{3}$ then $\tan\theta = \frac{y_1}{x_1} = \frac{\frac{16\sqrt{3}}{3}}{\frac{16}{3}} = \sqrt{3}$ , therefore :

$\theta = \frac{\pi}{3}$ In this case the length

$$OM = \frac{16}{5+3\cos\frac{\pi}{3}} = \frac{16}{5+\frac{3}{2}} = \frac{32}{13} \text{ then}$$

$$z_M = \frac{32}{13}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = \frac{32}{13}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \frac{16}{13} + i\frac{16\sqrt{3}}{13}.$$

Or $\overrightarrow{wN} = \frac{2}{3}\overrightarrow{wM}$ , therefore $x_N + 3 = \frac{2}{3}\left(\frac{16}{13} + 3\right)$

which gives $x_N = \frac{32}{39} + 2 - 3 = \frac{32}{39} - 1 = -\frac{7}{39}$

$y_N - 0 = \frac{2}{3}\left(\frac{16\sqrt{3}}{13} - 0\right)$ which gives $y_N = \frac{32\sqrt{3}}{39}$ and

consequently $N\left(-\frac{7}{39}; \frac{32\sqrt{3}}{39}\right)$.

7) a- For $\theta = \frac{\pi}{3}$ we have $OM = \frac{32}{13}$ and $OM' = \frac{16}{5 - \frac{3}{2}} = \frac{32}{7}$ then:

$$z_M = \frac{16}{13} + i\frac{16\sqrt{3}}{13}.$$

$$z_{M'} = \frac{32}{7}e^{i(\theta+\pi)} = \frac{32}{7}(-\cos\theta - i\sin\theta) = \frac{32}{7}\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) \text{ then}$$

$$Z_{M'} = -\frac{16}{7} - i\frac{16\sqrt{3}}{7}.$$

$w$ is the midpoint of $[M'M'']$ therefore

$$z_{M''} = 2z_w - z_{M'} = -6 + \frac{16}{7} + i\frac{16\sqrt{3}}{7} \quad , \quad z_{M''} = -\frac{26}{7} + i\frac{16\sqrt{3}}{7}$$

The equation of $(E)$ is $16x^2 + 25y^2 + 96x - 256 = 0$ then:

$32x + 50yy' + 96 = 0$ , we get : $y' = \frac{-16(x+3)}{25y}$.

- The slope of the tangent $(T)$ at $M$ to $(E)$ is

$$a = \frac{-16\left(\frac{16}{13} + 3\right)}{25\left(\frac{16\sqrt{3}}{13}\right)} = \frac{-11\sqrt{3}}{15} \text{ , hence an equation of } (T)$$

is: $y = \dfrac{-11\sqrt{3}}{15}\left(x - \dfrac{16}{13}\right) + \dfrac{16\sqrt{3}}{13}$

- The slope of the tangent $(T')$ at $M'$ to $(E)$ is :

$a' = \dfrac{-16\left(-\dfrac{16}{7}+3\right)}{25\left(\dfrac{-16\sqrt{3}}{7}\right)} = \dfrac{\sqrt{3}}{15}$ then the tangent $(T')$ has an equation $y = \dfrac{\sqrt{3}}{15}\left(x + \dfrac{16}{7}\right) - \dfrac{16\sqrt{3}}{7}$ ;

- The slope of the tangent $(T'')$ at $M''$ to $(E)$ is

$a'' = \dfrac{-16\left(-\dfrac{26}{7}+3\right)}{25\left(\dfrac{16\sqrt{3}}{7}\right)} = \dfrac{\sqrt{3}}{15}$ then an equation of

$(T'')$ is : $y = \dfrac{\sqrt{3}}{15}\left(x + \dfrac{26}{7}\right) + \dfrac{16\sqrt{3}}{7}$.

**Remark that:** $(T')$ and $(T'')$ are parallel since $M'$ and $M''$ are symmetric with respect to the center $w$ of the ellipse $(E)$ .

b- The two tangents $(T)$ and $(T')$ intersect at the point $K\left(\dfrac{16}{3};-\dfrac{16\sqrt{3}}{9}\right)$ of the directrix of equation $x = \dfrac{16}{3}$ .

c- The two tangents $(T)$ and $(T'')$ intersect at point $J\left(-\dfrac{1}{2};5\dfrac{\sqrt{3}}{2}\right)$ then :

$$wJ^2 = (x_J - x_w)^2 + (y_J - y_w)^2 = \left(-\frac{1}{2}+3\right)^2 + \left(\frac{5\sqrt{3}}{2} - 0\right)^2$$

$$=\left(\frac{5}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2=\frac{25}{4}+\frac{75}{4}=\frac{100}{4}=25\,.$$

then $wJ = 5 = a$ , and since $w$ is the center of the ellipse then $J$ belongs to the principal circle .

# Indications

**N°1.**

1) $x' = \sqrt{x^2 + y^2}$ and $y' = 2y$.
3) $y'^2 - 4x'^2 = -4x^2$.

**N°2.**

1) $\arg(z) + \arg(z-2) = \arg[(z)(z-2)] = (2k+1)\frac{\pi}{2}$ , then $z(z-2)$ is pure imaginary.
The equation of the set $(H)$ of points $M$ is $x^2 - y^2 - 2x = 0$

**N°3.**

1) $MA + MB = 8$, then $M$ describes an ellipse.
2) $\frac{(x-1)^2}{16} + \frac{y^2}{12} = 1$.

**N°4.**

2) a- The vertices of $(E)$ on the focal axis are $A(2;0)$ and $A'(6;0)$.
   b- The center of $(E)$ is $w(4;0)$, the equation of the non-focal axis of $(E)$ is $x = 4$, the vertices on the non-focal axis are $B(4;\sqrt{3})$ and $B'(4;-\sqrt{3})$.
3) a- An equation of $(H)$ is $\frac{(x-4)^2}{4} - \frac{y^2}{3} = 1$.

**N°5.**

1) An equation of $(P)$ is $y^2 = 8x + 16$.
2) An equation of $(P')$ is $x^2 = 8y + 16$.
3) The perpendicular bisector of $[OA]$ is the straight line of equation $y = -x - 4$.

N°6.

2) a- $N(-5\sin\theta;5\cos\theta)$, $R(5\cos\theta;4\sin\theta)$, $S(-5\sin\theta;4\cos\theta)$.

c- The area of triangle $ORS$ is $\frac{1}{2}\left\|\overrightarrow{OR}\wedge\overrightarrow{OS}\right\| = 10$ square units.

N°7.

1) An equation of plane $(P)$ is $2x + y - 2z + 4 = 0$.

3) $d(E;(AB)) = 2$, note that $(d)$ is perpendicular to $(Q)$.

$E(1;2;1)$, $H\left(-\frac{1}{3};\frac{4}{3};\frac{7}{3}\right)$ is the orthogonal projection of $E$ over $(P)$

5) The vertex of the parabola is the point $S$ midpoint of $[BH]$,

$S\left(-\frac{1}{6};\frac{5}{3};\frac{8}{3}\right)$.

N°8.

2) $x_1 \times x_2 = -a^2$.

3) The straight line $(M_1M_2)$ passes through the fixed point $F\left(0;\frac{a}{2}\right)$.

4) The locus of the points of intersection of the tangents is the

is the straight line of equation $y = -\frac{a}{2}$.

N°10.

2) The set of points $M$ is the parabola of equation $y^2 = 4x + 8$.

3) $(\Omega)$ remains tangent externally to the circle of center $F$ and radius 1.

N°11.

1) The vertex $S$ of $(P)$ is the midpoint of $[OF]$.

2) a- An equation of $(P)$ is $y^2 = 4x - 4$

3) a- $S_1$ verifies $\overrightarrow{S_1F} = -3\overrightarrow{S_1O}$, then $S_1\left(\frac{1}{2};0\right)$, similarly $S_2(-1;0)$.

**N° 16.**

1) $r\ :\ z' = -iz + 2i$ ; $S\ :\ z' = (1+i)z + 2$ ; $h\ :\ z' = 2z + 2$.

2) a- The complex form of $f$ is $z' = 2(1-i)z + 6i$.

4) b- An equation of $(P')\ :\ y^2 - 12y - 8x + 52 = 0$.

**N° 21.**

1) a- $OM = \dfrac{1}{2+\cos\theta}$, the distance of $M$ to $(d)$ is $= \dfrac{2}{2+\cos\theta}$.

c- The vertices corresponding to $\theta = 0$ , $\theta = \pi$.

2) c- $3x^2 + 4y^2 + 2x - 1 = 0$.

**N° 22.**

1) $\left\|\overrightarrow{MA} \wedge \overrightarrow{MB}\right\|$ represents $2 \times$ area of triangle $AMB$.

$2 \times \dfrac{AB \times d(M;(AB))}{2} = 4MF$ , then $\dfrac{MF}{d(M;(AB))} = \dfrac{AB}{4}$.

# CHAPTER 12

# Irrational Functions

**Chapter Review :**

- Definition :
  In this chapter, we are interested in irrational functions of the form: $f(x) = ax + b + c\sqrt{g(x)}$ where :
  * $g(x) = px^2 + qx + r$.
  * $g(x) = px + q$.
  * $g(x) = \dfrac{\alpha x + \beta}{\alpha' x + \beta'}$.

  ***Ex :***
  $f(x) = x - 2 + \sqrt{-x^2 + x + 2}$ defined over $[-1;+2]$.

  ***N.B :***
  $f$ is differentiable over $]-1;+2[$.
  The points of abscissas -1 and +2 , are called endpoints, the tangents to the representative curve of $f$ at these points are vertical.

- Asymptotes :
  * In the case the function has the Form $f(x) = ax + b + c\sqrt{x^2 + qx + r}$ the asymptotes are given by : $y = ax + b + c\left|x + \dfrac{q}{2}\right|$

    ***Ex :*** $f(x) = x + 2\sqrt{x^2 + 2x - 3}$.
    The asymptotes are given by $y = x + 2|x+1|$.
    In the neighborhood of $+\infty$, the asymptote is the straight line of equation: $y = x + 2(x+1) = 3x + 2$.

In the neighborhood of $-\infty$ the asymptote is the straight line of equation: $y = x + 2(-x-1) = -x-2$.

* In the case where the function has the form $f(x) = ax + b + c\sqrt{\dfrac{\alpha x + \beta}{\alpha' x + \beta'}}$ with $\alpha$ and $\alpha'$ having the same sign then, we have an oblique asymptote of equation: $y = ax + b + c\sqrt{\dfrac{\alpha}{\alpha'}}$ and a vertical asymptote of equation $x = -\dfrac{\beta'}{\alpha'}$.

  ***Ex :*** $f(x) = x + 2\sqrt{\dfrac{1+x}{x}}$ defined over $]-\infty;-1] \cup ]0;+\infty[$.

  The asymptotes are the straight lines of equations $y = x + 2$ and $x = 0$

* In the case where the function has the form $f(x) = ax + b + c\sqrt{qx + r}$ we have an asymptotic direction of slope $a$.

  ***Ex :*** $f(x) = x + 2\sqrt{x}$ defined over $[0;+\infty[$, has an asymptotic direction of slope 1 .

- General method for finding the oblique asymptote:
  An equation of the oblique asymptote, if it exists, is: $y = ax + b$ where $a = \lim\limits_{x\to\infty} \dfrac{f(x)}{x}$ and $b = \lim\limits_{x\to\infty}(f(x) - ax)$.

  ***N.B***

  If $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = \infty$, there is an asymptotic direction parallel to $y'y$.

  If $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = 0$, there is an asymptotic direction parallel to $x'x$.

  If $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = a$, finite , and $\lim\limits_{x\to\infty}(f(x) - ax) = \infty$ there is an asymptotic direction of slope $a$.

# Solved Problems

**N° 1.**

Consider the function $f$ defined over $]-\infty;1]$ by $f(x) = x + \sqrt{-x+1}$. and designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x \to -\infty} f(x)$ and show that $(C)$ admits an asymptotic direction to be determined.
2) Study the variations of $f$ and trace $(C)$.

**N° 2.**

Consider the function $f$ defined over $[-1;1]$ by $f(x) = x + \sqrt{-x^2+1}$ , and designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $f'(x) < 0$ for $\frac{\sqrt{2}}{2} < x < 1$.
2) Set up the table of variations of $f$.
3) Trace $(C)$.

**N° 3.**

Let $f$ be the function defined over $]-\infty;-1] \cup ]0;+\infty[$ by $f(x) = x - \sqrt{\frac{x+1}{x}}$.

Designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $(C)$ admits two asymptotes whose equations are to be determined.
2) Study the variations of $f$ and set up its table of variations.
3) Trace $(C)$.

N° 4.

Consider the function $f$ defined over $[0;1[$ by $f(x) = x + \sqrt{\frac{x}{1-x}}$,

and designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $(C)$ admits an asymptote whose equation is to be determined.
2) Study the variations of $f$ and draw its table of variations.
3) Trace $(C)$.

N° 5.

The plane is referred to a direct orthonormal system $(O;\vec{i},\vec{j})$.

***Part A:***

Let $f$ be the function defined over $[-2;+\infty[$ by $f(x) = -x + \sqrt{x+2}$.
Designate by $(C)$ its representative curve.

1) Determine $\lim\limits_{x\to+\infty} f(x)$, $\lim\limits_{x\to+\infty} \frac{f(x)}{x}$ and show that $f$ admits an asymptotic direction.
2) Study the variations of $f$ and draw its table of variations.
3) Trace $(C)$.
4) Is there a point of $(C)$ where the tangent to $(C)$ is parallel to the straight line of equation $y = -x$ ?
5) Find a point $B$ of $(C)$ at a distance equal to 1 of the straight line of equation $y = -x$.

***Part B :***

Let $g$ be the function defined over $[-2;+\infty[$ by $g(x) = -x - \sqrt{x+2}$.
Designate by $(C')$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Let $M(x; f(x))$ be a point of $(C)$ and $M'$ a point of $(C')$ of abscissa $x$, calculate $\frac{f(x)+g(x)}{2}$.

   b- Deduce a geometrical construction of $(C')$ starting from $(C)$.
2) Calculate the area of the domain limited by $(C)$, $(C')$ and the straight lines of equations $x = 0$ and $x = 1$.

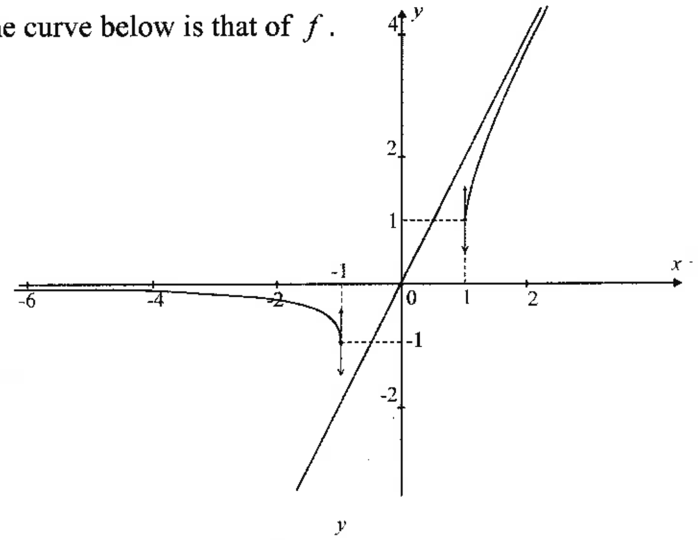
**N°6.**

Consider the function $f$ defined over $]-\infty;-1]\cup[1;+\infty[$ by

$f(x)=x+\sqrt{x^2-1}$ .

Designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty}[f(x)-2x]$ and deduce the asymptotes to $(C)$ .
2) Show that $f$ is decreasing for $x\in]-\infty;-1[$ .
3) For all non-zero real numbers $x_0$, designate by $M$ and $M'$ the points of $(C)$ of respective abscissas $x_0$ and $-x_0$.
   Show that $f(x_0)-f(-x_0)=2x_0$ and that the straight line $(MM')$ maintains a fixed direction .
4) The curve below is that of $f$ .




Let $g$ be the function defined over $]-\infty;-1]\cup[1;+\infty[$ by

$g(x)=x-\sqrt{x^2-1}$ .

Designate by $(C')$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

a- Compare $g(-x)$ and $f(x)$ and deduce the graph $(C')$ in the same system as that of $(C)$.

b- The curve $(\gamma)=(C)\cup(C'')$, of equation $y=x\pm\sqrt{x^2-1}$ is a hyperbola.
Find the equations of the axes of symmetry of this hyperbola.

5) a- Show that $f$ admits an inverse function $f^{-1}$ for $x\geq 1$.
b- Trace the curve $(C')$ of $f^{-1}$.
c- Determine $f^{-1}(x)$.

6) $(D)$ is the domain limited by $(C)$, the axis $x'x$ and the two straight lines of equations $x=1$ and $x=2$.
Calculate the volume obtained by rotating $(D)$ about $x'x$.

N° 7.

***Part A :***

Consider the function $f$ defined over $]-\infty;-1]\cup[2;+\infty[$ by $f(x)=\sqrt{x^2-x-2}$ and designate by $(C)$ its representative curve in a direct orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine $\lim\limits_{x\to+\infty} f(x)$, $\lim\limits_{x\to+\infty}\dfrac{f(x)}{x}$ and $\lim\limits_{x\to+\infty}(f(x)-x)$.
b- Deduce an asymptote to $(C)$ at $+\infty$.

2) a- Show that the straight line $(d)$ of equation $x=\dfrac{1}{2}$ is an axis of symmetry of $(C)$.
b- Deduce an asymptote to $(C)$ at $-\infty$.

3) Study the variations of $f$ and draw its table of variations.

4) Trace $(C)$.

5) a- Show that $f$ admits an inverse function $f^{-1}$.
b- Determine the domain of definition of $f^{-1}$, trace the curve $(C')$ representative of $f^{-1}$ and find $f^{-1}(x)$.

***Part B:***

Consider the function $g$ defined over $[-1;2]$ by $g(x)=\sqrt{-x^2+x+2}$ and designate by $(\gamma)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $(\gamma)$ is a semi-circle and trace $(\gamma)$.

2) Calculate $\int_{-1}^{2}\sqrt{-x^2+x+2}\,dx$.

***Part C:***

Let $h$ be the function defined over $IR$ by $h(x)=\sqrt{|x^2-x-2|}$.

Designate by $(\Gamma)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Trace $(\Gamma)$.
2) Let $(\delta)$ be the straight line of equation $y=a(x-2)$.
   Determine $a$ for $(\delta)$ to cut $(\Gamma)$ in three points.

$\boxed{N°8}$

In the plane referred to an orthonormial system $(O;\vec{i},\vec{j})$ consider the curve $(C_m)$ of equation $y^2=m(x^2-1)+2x$ where $m$ is a real parameter.

1) Let $f$ be the function defined over $]-\infty;-1-\sqrt{2}]\cup[-1+\sqrt{2};+\infty[$ by $f(x)=\sqrt{x^2+2x-1}$.
   Designate by $(C)$ its representative curve in the system $(O;\vec{i},\vec{j})$.
   a- Study the variations of $f$ and draw its table of variations.
   b- Show that $(C)$ admits an axis of symmetry.
   c- Tracer $(C)$.
   d- Prove that the curve $(C_1)$,corresponding to $m=1$,is the union of $(C)$ and a curve $(C')$ whose equation is to be determined.
   e- Trace $(C')$ in the same system as $(C)$ and show that $(C_1)$ admits a center of symmetry.
2) a- Trace the curve $(C_0)$ corresponding to $m=0$, in an new system.
   b- Let $(D)$ be the domain limited by $(C_0)$and the straight line of equation $x=1$, calculate the area of $(D)$.
3) For $m\neq 0$, the equation of $(C_m)$ can be written in the form:

$\frac{(x+a)^2}{\alpha}-\frac{y^2}{\beta}=1$ where $a$, $\alpha$ and $\beta$ are to be determined.

a- Study according to the values of $m$ the nature of $(C_m)$.

b- In the case where $(C_m)$ is a hyperbola, calculate its eccentricity.

N° 9.

Let $f_\lambda$ be the function defined by $f(x)=x+\sqrt{\frac{x}{\lambda x+1}}$ where $\lambda$ is a real parameter and designate by $(C_\lambda)$ the representative curve of $f_\lambda$ in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine $\lambda$ so that $(C_\lambda)$ has no asymptotes.
   b- Study in this case the variations of $f_\lambda$ and trace its representative curve in the system $(O;\vec{i},\vec{j})$.
2) a- Determine $\lambda$ so that $(C_\lambda)$ admits only one asymptote.
   b- Find an equation of this asymptote.
3) a- Determine $\lambda$ so that $(C_\lambda)$ admits two asymptotes.
   b- Determine the equations of these asymptotes.
4) Suppose that $\lambda=-1$.
   a- Determine the domain of definition of $f_{-1}$.
   b- Study the variations of $f_{-1}$ and trace its representative curve $(C_{-1})$ in the system $(O;\vec{i},\vec{j})$.
5) Suppose that $\lambda=1$.
   a- Determine the domain of definition of $f_1$.
   b- Study the variations of $f_1$ and trace its representative curve $(C_1)$ in the system $(O;\vec{i},\vec{j})$.

N° 10.

Consider the function $f$ defined over $IR$ by

$f(x)=x+\sqrt{|x^2-1|}$ and designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the continuity of $f$ at the points of abscissas 1 and -1.

2) Study the differentiability of $f$ at the points of abscissas 1 and -1 .
3) The table below is the table of variations of $f$ .
   Justify all the elements of the table.

| $x$ | $-\infty$ | | $-1$ | | $\frac{\sqrt{2}}{2}$ | | $1$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | ‖ | $+$ | $0$ | $-$ | ‖ | $+$ | |
| $f(x)$ | $0$ | ↘ | $-1$ ‖ | ↗ | $\sqrt{2}$ | ↘ | $1$ ‖ | ↗ | $+\infty$ |

4) Determine $\lim\limits_{x\to+\infty}(f(x)-2x)$ and deduce an asymptote to $(C)$.
5) Trace $(C)$.
6) Let $(d)$ be the straight line passing through the origin $O$ and of slope $m$ .
   Study according to the values of $m$ the intersection of $(C)$ and $(d)$.

**N° 11.**

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the function $f$ defined over $[-3\sqrt{2};3\sqrt{2}]$ by $f(x)=2x+25-\sqrt{72-4x^2}$ .

1) Study the variations of $f$ ,draw its table of variations and show that $f(x)\geq 13$ .
2) Let $(E)$ be the ellipse of equation $\frac{x^2}{18}+\frac{y^2}{8}=1$ and $(d)$the straight line of equation $2x-3y+25=0$ .
   a- Trace $(E)$ and $(d)$ in the system $(O;\vec{i},\vec{j})$.
   b- Let $M(x;y)$ be a point of$(E)$ and let $MH$ be the distance from $M$ to $(d)$.
      Find the minimum of $MH$ and deduce the coordinates of the corresponding point $M$ .

N° 12.

Consider the function $f$ defined over $[-2;2]$ by $f(x)=\pm\frac{3}{2}\sqrt{4-x^2}$ and designate by $(E)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $f$ is an even function.

2) Let $g$ be the function defined over $[0;2]$ by $g(x)=\frac{3}{2}\sqrt{4-x^2}$.

   a- Study the variations of $g$ and trace its representative curve $(\gamma)$ in the system $(O;\vec{i},\vec{j})$.

   b- Deduce the graph $(E)$.

3) Let $M(x;y)$ be a point of $(E)$ and $H$ its orthogonal projection on $x'x$.

   Let $M'(X;Y)$ be a point of the straight line $(HM)$ such that $\overrightarrow{HM'}=\frac{2}{3}\overrightarrow{HM}$.

   a- Show that when $M$ traces $(E)$, the point $M'$ traces a circle $(C)$ whose equation is to be determined.

   b- Calculate the area of the domain $(D)$ limited by $(C)$, the semi straight line $[Ox)$ and the semi straight line $[Oy)$.

   c- Deduce the area of the domain limited by $(E)$.

N° 13.

Consider the function $f_\lambda$ defined by $f_\lambda(x)=x+\sqrt{\lambda x^2-4x+3}$ where $\lambda$ is a real parameter, and denote by $(C_\lambda)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine the set of values of $\lambda$ for $f_\lambda$ to be defined over $IR$.

   b- Determine in terms of $\lambda$ the equations of the asymptotes to $(C_\lambda)$ and determine $\lambda$ for the two asymptotes to be perpendicular.

2) Suppose $\lambda = 1$ and consider the function $f_1$ defined over $]-\infty;1] \cup [3;+\infty[$ by $f_1(x) = x + \sqrt{x^2 - 4x + 3}$. Designate by $(C_1)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
   a- Find the equations of the asymptotes to $(C_1)$.
   b- Study the variations of $f_1$ and draw its table of variations.
   c- Trace $(C_1)$.

N° 14.

Consider the function $f$ defined over $]-\infty;-2] \cup [1;+\infty[$ by $f(x) = x + \sqrt{x^2 + x - 2}$.
Designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $f(x)$ can be written in the form
$$f(x) = x + |x|\sqrt{1 + \frac{1}{x} - \frac{2}{x^2}}.$$

2) Suppose $t = \frac{1}{x}$ and consider the function $\varphi$ defined over $\left[-\frac{1}{2};1\right]$ by $\varphi(t) = \sqrt{-2t^2 + t + 1}$.
   a- Find the derivative of $\varphi$ at 0.
   b- Deduce the existence of a function $h$ such that
   $$\varphi(t) = \frac{1}{2}t + 1 + t\,h(t) \text{ with } \lim_{t \to 0} h(t) = 0.$$
   c- Deduce that $f(x) = x + \frac{|x|}{2x} + |x| + \frac{|x|}{x}h\left(\frac{1}{x}\right)$.
   d- Deduce the equations of the asymptotes to $(C)$ at $+\infty$ and at $-\infty$.
   e- Study the variations of $f$ and draw its table of variations.

N° 15.

Let $f$ be the function defined over $[-1;+1]$ by $f(x) = (1 - x)\sqrt{-x^2 + 1}$.
Designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the variations of $f$ and draw its table of variations.
2) Trace $(C)$.
3) Let $(\gamma)$ be the circle of center $O$ and radius 1, let $A(1;0)$ and $A'(-1;0)$ be two points of $(\gamma)$.
   Let $H$ be a point of the segment $[AA']$ distinct of $A'$ and of $A$.
   The perpendicular at $H$ to $[AA']$ cuts the circle at $M$ and $M'$,
   suppose $\overline{OH} = m$.
   Calculate, in terms of $m$, the area of the triangle $AMM'$ and show that the area of this triangle is maximal when the triangle is equilateral.

# Supplementary Problems

N° 1.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the curve $(C_m)$ of equation $y^2 = m(x^2-3)-2x$ where $m$ is a real parameter .

1) Let $f$ bet he function defined over $\left]-\infty;-1\right]\cup\left[3;+\infty\right[$ by $f(x)=\sqrt{x^2-2x-3}$ and designate by $(C)$ its representative curve in the system $(O;\vec{i},\vec{j})$.
   a- Study the variations of $f$ and draw its table of variations.
   b- Show that $(C)$ admits an axis of symmetry .
   c- Trace $(C)$.
   d- Show that the curve $(C_1)$,corresponding to $m=1$,is the union of $(C)$ and a curve $(C')$ whose equation is to be determined .
   e- Trace $(C')$ in the same system as that of $(C)$ and show that $(C_1)$admits a center of symmetry .
2) a- Trace the curve $(C_0)$ corresponding to $m=0$, in a new system.
   b- Let $(D)$ be the domain limited by $(C_0)$and the two straight lines of equations $x=-2$ and $x=-1$, calculate the volume of the solid generated by the rotation of $(D)$about $x'x$.
3) For $m \neq 0$, the equation of $(C_m)$ can be written in the form:
   $\dfrac{(x-\ell)^2}{\alpha}-\dfrac{y^2}{\beta}=1$ where $\ell=\dfrac{1}{m}$, $\alpha=\dfrac{3m^2+1}{m^2}$ and $\beta=\dfrac{3m^2+1}{m}$.
   a- Determine $m$ for $(C_m)$ to be an ellipse .
   b- Calculate in this case the eccentricity of this ellipse and find the coordinates of its foci.

N° 2.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the two functions $f$ and $g$ defined over $[0;+\infty[$ by $f(x)=x+\sqrt{x}$ and $g(x)=x-\sqrt{x}$.

Designate by $(C)$ and $(C')$ the representative curves of $f$ and $g$ in the system $(O;\vec{i},\vec{j})$.

***Part A :***

1) Study the variations $f$ and draw its table of variations.
2) Trace $(C)$.
3) Study the variations of $g$ and trace $(C')$ in the same system as $(C)$.
4) Designate by $(P)$ the curve of equation $x^2-2xy+y^2-x=0$.
   Calculate the area of the domain limited by $(P)$ and the two straight lines of equations: $x=\frac{1}{4}$ and $x=1$.

***Part B:***

Let $r$ be the rotation of center $O$ and angle $\frac{\pi}{4}$.

1) Write the complex form of $r$.
2) Let $M$ be a point of affix $z=x+iy$ and $M'$ the point of affix $z'=x'+iy'$ image of $M$ by $r$.
   a- Express $x$ and $y$ in terms of $x'$ and $y'$.
   b- Prove that the image of $(P)$ by $r$ is the parabola $(P')$ of equation: $4x^2-x-y=0$.
   c- Determine the vertex , focus and directrix of $(P')$.
   d- Deduce the nature of $(P)$.
   e- Write an equation of the focal axis and directrix of $(P)$.
   f- Find the coordinates of the vertex and focus of $(P)$.

N° 3.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the function $f$ defined over $IR$ by $f(x)=x+\sqrt{x^2+4}$ .

Designate by $(C)$ its representative curve in the system $(O;\vec{i},\vec{j})$.

1) Determine the equations of the asymptotes of $(C)$.
2) Show that $f$ is strictly increasing over $IR$.
3) Trace $(C)$.
4) Let $(H)$ be the curve of equation $y^2 - 2xy - 4 = 0$.
   a- Show that $(H)$ is the union of the curve $(C)$ and a curve $(C')$ whose equation is to be determined.
   b- Show that $(C')$ is symmetric to $(C)$ with respect to the origin and trace $(C')$.
5) Let $(d)$ be the straight line of equation $y = x + m$ where $m$ is a real parameter.
   Show that when $(d)$ cuts $(C)$ in two points $N_1$ and $N_2$ the abscissas of $N_1$ and $N_2$ are opposite and the tangents at $N_1$ and $N_2$ to $(C)$ intersect on the axis $y'y$.

**N°4.**

Consider the function $f$ defined over $IR$ by:

$f(x) = x + \sqrt{x^2+1}$ and designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that $f$ is strictly increasing over $IR$.
2) a- Show that the two straight lines of equations $y = 0$ and $y = 2x$ are asymptotes to $(C)$.
   b -Study the position of $(C)$ with respect to its asymptotes.
3) Trace $(C)$.
4) a- Show that $f$ admits an inverse function $f^{-1}$ for every $x \in IR$.
   b- Trace the representative curve of $f^{-1}$ in the same system.
   c- Find $f^{-1}(x)$.
5) Let $g$ be the function defined over $IR$ by $g(x) = x - \sqrt{x^2+1}$
   Compare $f(-x)$ and $g(x)$ and deduce the graph of the curve representative of $g$.

N° 5.

Consider the function $f$ defined over $IR$ by $f(x) = x - \sqrt{|x|}$ and designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Study the differentiability of $f$ at the point of abscissa $0$.
2) Show that $(C)$ admits an asymptotic direction in the neighborhood of $+\infty$.
3) Study the variations of $f$ for $x \geq 0$ and trace the corresponding part of $(C)$.
4) Trace $(C)$.
5) Designate by $(D)$ the domain limited by $(C)$, the axis $x'x$ and the two straight lines of equations $x = 0$ and $x = 1$.
   Calculate the volume of the solid generated by revolving $(D)$ about the axis $x'x$.

N° 6.

Consider the function $f$ defined over $]-\infty;-2]\cup[2;+\infty[$ by $f(x) = \frac{1}{2}\sqrt{x^2 - 4}$ and designate by $(C)$ its representative curve in the system $(O;\vec{i},\vec{j})$.

1) Determine the equations of the asymptotes to $(C)$.
2) Study the variations of $f$ and draw its table of variations.
3) Trace $(C)$.
4) Let $(C')$ be the curve symmetric of $(C)$ with respect to the axis $x'x$, write an equation of $(C')$ and trace $(C')$ in the same system.
5) Consider the curve $(H) = (C)\cup(C')$.
   Show that $(H)$ is a hyperbola having $F(\sqrt{5};0)$ as a focus, and an eccentricity $e = \frac{\sqrt{5}}{2}$ and a directrix the straight line of equation $x = \frac{4\sqrt{5}}{5}$.

6) Let $M(4;\sqrt{3})$ be a point of $(H)$ and $(d)$ the straight line of equation $\sqrt{3}y - x + 1 = 0$.
Show that $(d)$ is a bisector of the angle $F\hat{M}F'$ where $F'$ is the second focus of $(H)$.

N°7.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, given the conic $(\Gamma)$ of equation $4x^2 + 9y^2 - 8x - 32 = 0$.

1) Determine the nature, center and vertices of the focal axis of $(\Gamma)$.
2) Trace $(\Gamma)$.
3) The conic $(\Gamma)$ cuts the axis of ordinates at $G$ and $L$ ; write the equations of the tangents to $(\Gamma)$ at these points.
4) Let $f$ be the function defined over $[-2;4]$ by $f(x) = \pm\sqrt{-x^2 + 2x + 8}$ and designate by $(C)$ its representative curve in the system $(O;\vec{i},\vec{j})$.
   a- Study the variations of $f$ and trace $(C)$ in the same system as that of $(\Gamma)$.
   b- What does $(C)$ represent for $(\Gamma)$ ?
   c- Calculate the area of the domain limited by $(\Gamma)$ and $(C)$.
   d- Designate by $(D)$ the domain limited by $(C)$ and the axis of abscissas.
   Calculate the volume of the solid obtained by revolving $(D)$ about the axis of abscissas.

# Solution of Problems

N° 1.

1) $\lim\limits_{x\to-\infty} f(x) = -\infty + \infty$ , indeterminate, so

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} x\left(1+\frac{\sqrt{-x+1}}{x}\right) = \lim_{x\to-\infty} x\left(1-\sqrt{\frac{-x+1}{x^2}}\right)$$

Then $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to-\infty} x\left(1-\sqrt{\frac{-1}{x}}\right) = -\infty$.

$$a = \lim_{x\to-\infty} \frac{f(x)}{x} = \lim_{x\to-\infty}\left(1+\frac{\sqrt{-x+1}}{x}\right) = 1 .$$

$b = \lim\limits_{x\to-\infty}\left(f(x)-ax\right) = \lim\limits_{x\to-\infty}\sqrt{-x+1} = +\infty$ hence $(C)$ admits an asymptotic direction parallel to the straight line of equation $y = x$.

2) $f'(x) = 1 + \dfrac{-1}{2\sqrt{-x+1}} = \dfrac{2\sqrt{-x+1}-1}{2\sqrt{-x+1}}$

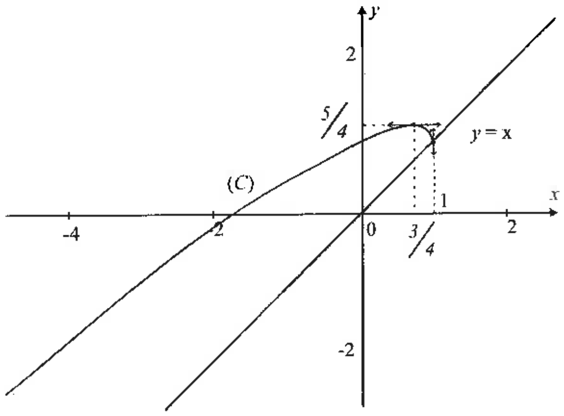
$f'(x) \geq 0$ for $\sqrt{-x+1} \geq \dfrac{1}{2}$ , which gives $-x+1 \geq \dfrac{1}{4}$ and consequently $x \leq \dfrac{3}{4}$ , therefore the table of variations of $f$ is as follows :

| $x$ | $-\infty$ | | $3/4$ | | $1$ |
|---|---|---|---|---|---|
| $f'(x)$ | | $+$ | $0$ | $-$ | $+\infty$ |
| $f(x)$ | $-\infty$ | $\nearrow$ | $5/4$ | $\searrow$ | $1$ |

$$f\left(\frac{3}{4}\right) = \frac{3}{4} + \sqrt{-\frac{3}{4}+1} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4} .$$

$f(1) = 1$ ; $f(0) = 1$ .

The tangent to $(C)$ at the point $(1;1)$ is vertical.




y

N°2.

1) $f'(x) = 1 + \dfrac{-x}{\sqrt{-x^2+1}}$

If $x \leq 0$ then $f'(x) > 0$.

If $x > 0$ then

$$f'(x) = \frac{\sqrt{-x^2+1} - x}{\sqrt{-x^2+1}} = \frac{\left(\sqrt{-x^2+1} - x\right)\left(\sqrt{-x^2+1} + x\right)}{\sqrt{-x^2+1}\left(\sqrt{-x^2+1} + x\right)}$$

So $f'(x) = \dfrac{-2x^2+1}{\sqrt{-x^2+1}\left(\sqrt{-x^2+1} + x\right)}$.

$f'(x) \leq 0$ for $-2x^2+1 \leq 0$, which gives $x^2 \geq \dfrac{1}{2}$ and consequently

$x \geq \dfrac{\sqrt{2}}{2}$ since $x > 0$, hence $f'(x) < 0$ for $\dfrac{\sqrt{2}}{2} < x < 1$.

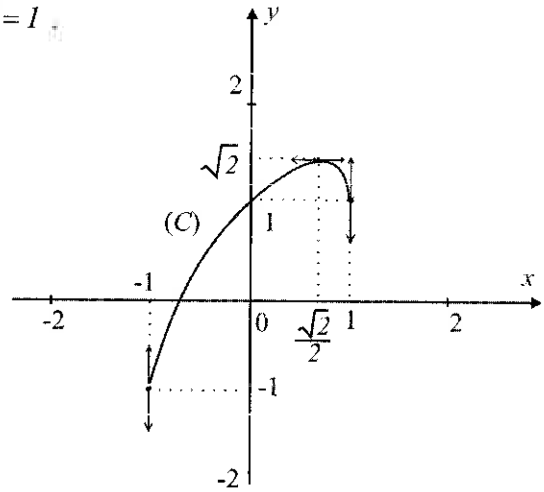
2) The table of variations of $f$ is :

| $x$ | $-1$ | | $\frac{\sqrt{2}}{2}$ | | $1$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+\infty$ | $+$ | $0$ | $-$ | $-\infty$ |
| $f(x)$ | $-1$ | $\nearrow$ | $\sqrt{2}$ | $\searrow$ | $1$ |

$$f\left(\frac{\sqrt{2}}{2}\right)=\frac{\sqrt{2}}{2}+\sqrt{-\frac{1}{2}+1}=\sqrt{2}\ .$$

3) At the points $(-1;1)$ and $(1;1)$ the tangents are vertical .

$f(0)=1$ .




N°3.

1) $\lim\limits_{x\to+\infty} f(x)=+\infty-1=+\infty$ since $\lim\limits_{x\to+\infty}\frac{x+1}{x}=1$.

In the neighborhood of $+\infty$ , the curve $(C)$ admits an oblique asymptote of equation $y=x-1$.

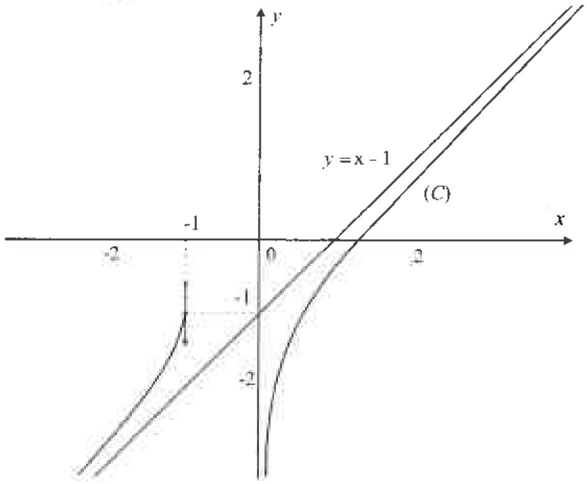
Similarly, $\lim\limits_{x\to-\infty} f(x)=-\infty-1=-\infty$ et $(C)$ admits an oblique asymptote of equation $y=x-1$.

$\lim\limits_{x\to 0} f(x)=-\infty$, hence the axis $y'y$ is an asymptote to $(C)$ , hence $(C)$ admits two asymptotes .

2) $f'(x)=1-\dfrac{\frac{-1}{x^2}}{2\sqrt{\frac{x+1}{x}}}=1+\dfrac{1}{2x^2\sqrt{\frac{x+1}{x}}}>0$ then the table of variations of $f$ is as follows :

| $x$ | $-\infty$ | | | $-1$ | $0$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $+$ | $+\infty$ | ‖ | | $+$ | |
| $f(x)$ | $-1$ | | | ‖ | | | $+\infty$ |
| | $-\infty$ | ↗ | | ‖ | $-\infty$ | ↗ | |

3) $f(1)=1-\sqrt{2}$ .



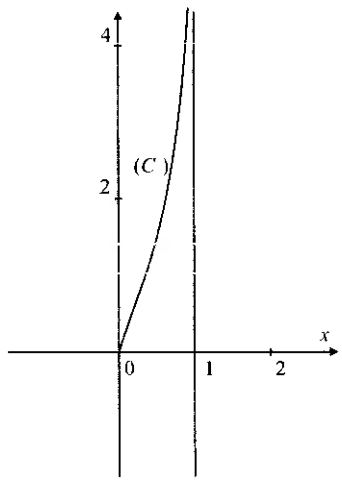

**N°4.**

1) $\lim\limits_{x\to 1} f(x)=+1+\infty=+\infty$ then the straight line of equation $x=1$ is a vertical asymptote to $(C)$.

2) $f'(x)=1+\dfrac{\dfrac{1}{(1-x)^2}}{2\sqrt{\dfrac{x}{1-x}}}>0$ for all $x\in[0;1[$ then the table of variations of $f$ is:

| $x$ | $0$ | | $1$ |
|---|---|---|---|
| $f'(x)$ | $\|\,+\infty$ | $+$ | $\|$ |
| $f(x)$ | $0$ | $\nearrow$ | $+\infty\,\|$ |

3)




**N°5.**

***Part A.:***

1) $\lim\limits_{x\to+\infty} f(x) = -\infty + \infty$ , indeterminate form , then

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} x\left(-1+\frac{\sqrt{x+2}}{x}\right) = \lim_{x\to+\infty} x\left(-1+\sqrt{\frac{x+2}{x^2}}\right)$$

So $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} x\left(-1+\sqrt{\dfrac{1}{x}}\right) = -\infty$ .

$$a = \lim_{x\to+\infty} \frac{f(x)}{x} = \lim_{x\to+\infty}\left(-1+\frac{\sqrt{x+2}}{x}\right) = -1 .$$

$b = \lim\limits_{x\to+\infty}(f(x)-ax) = \lim\limits_{x\to+\infty} \sqrt{x+2} = +\infty$ hence $(C)$ admits an asymptotic direction parallel to the straight line of equation: $y = -x$ .

2) $f'(x) = -1 + \dfrac{1}{2\sqrt{x+2}} = \dfrac{-2\sqrt{x+2}+1}{2\sqrt{x+2}}$ .

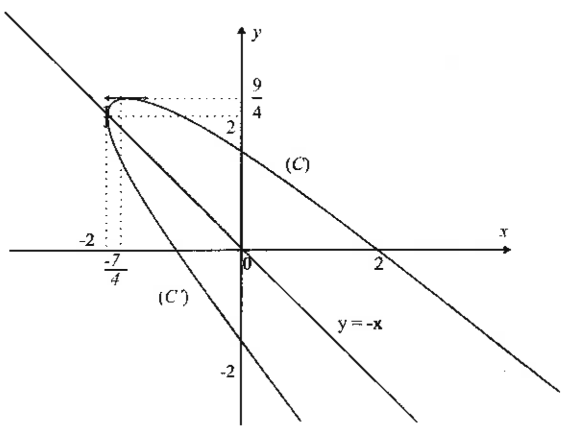
$f'(x) \geq 0$ for $-2\sqrt{x+2} \geq -1$ , which gives $\sqrt{x+2} \leq \dfrac{1}{2}$,so

$x+2 \leq \dfrac{1}{4}$ which gives $x \leq -\dfrac{7}{4}$ , therefore the table of variations of $f$ is:

| $x$ | $-2$ | | $\frac{-7}{4}$ | | $+\infty$ |
|---|---|---|---|---|---|
| $f'(x)$ | $+\infty$ | $+$ | $0$ | $-$ | $+\infty$ |
| $f(x)$ | $2$ | $\nearrow$ | $\frac{9}{4}$ | $\searrow$ | $-\infty$ |

$$f\left(-\frac{7}{4}\right) = \frac{7}{4} + \sqrt{-\frac{7}{4}+2} = \frac{7}{4} + \frac{1}{2} = \frac{9}{4}$$

3)




4) If there exists a point $M(x;y)$ at which the tangent is parallel to the straight line of equation $y=-x$ then $f'(x)=-1$ therefore

$-1 + \dfrac{1}{2\sqrt{x}+2} = -1$, which gives $\dfrac{1}{2\sqrt{x}+2} = 0$ which is impossible. So this point does not exist .

5) The distance of $B(x;y)$ to $(d)$ is 1 then $\frac{|x+y|}{\sqrt{2}}=1$ ,

so $|x+y|=\sqrt{2}$ which gives :

- $x+y=\sqrt{2}$ then $x-x+\sqrt{x+2}=\sqrt{2}$ so $\sqrt{x+2}=\sqrt{2}$ hence $x+2=2$ and consequently $x=0$ therefore $B(0;\sqrt{2})$.
- $x+y=-\sqrt{2}$ so $\sqrt{x+2}=-\sqrt{2}$ which is impossible .

***Part B:***

1) a- $\frac{f(x)+g(x)}{2}=\frac{-x+\sqrt{x+2}-x-\sqrt{x+2}}{2}=-x$

b- Let $I$ be the midpoint of $[MM']$ , we know that $x_I=x_M=x_{M'}$ and $y_I=\frac{f(x)+g(x)}{2}=-x=-x_I$ then the point $I$ belongs to the straight line of equation $y=-x$.
Consequently $M'$ is the point of $(C')$ having the same abscissa as $M$, such that the mid point of $[MM']$ remains on the line of equation $y=-x$. See figure .

2) $\mathsf{A}=\int_0^1[f(x)-g(x)]dx=\int_0^1\left(-x+\sqrt{x+2}+x+\sqrt{x+2}\right)dx$

$$=2\int_0^1\sqrt{x+2}\,dx=2\times\frac{2}{3}(x+2)\sqrt{x+2}\Big|_0^1=\frac{4}{3}\left[3\sqrt{3}-2\sqrt{2}\right]$$

square units.

**N°6.**

1) $\lim\limits_{x\to-\infty}f(x)=\lim\limits_{x\to-\infty}\left(x+\sqrt{x^2-1}\right)=-\infty+\infty$ , indeterminate form .

$$\lim_{x\to-\infty}f(x)=\lim_{x\to-\infty}\frac{\left(x+\sqrt{x^2-1}\right)\left(x-\sqrt{x^2-1}\right)}{\left(x-\sqrt{x^2-1}\right)}=\lim_{x\to-\infty}\frac{1}{\left(x-\sqrt{x^2-1}\right)}=0$$

$$\lim_{x\to+\infty}[f(x)-2x]=\lim_{x\to+\infty}\sqrt{x^2-1}-x=\lim_{x\to+\infty}\frac{-1}{\sqrt{x^2-1}+x}=0.$$

So the straight lines of equations: $y=0$ and $y=2x$ are asymptotes to $(C)$ .

2) $f'(x) = 1 + \dfrac{x}{\sqrt{x^2 - 1}} = \dfrac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}}$ .

If $x > 0$ then $f'(x) > 0$.

If $x < -1$ then $f'(x) < 0$ for $\sqrt{x^2 - 1} < -x$ so $x^2 - 1 < x^2$ ; which results in $-1 < 0$, which is always true.

Hence $f$ is strictly decreasing for $x \in \left]-\infty;-1\right[$ .

3) $f(x_0) - f(-x_0) = \left(x_0 + \sqrt{x_0^2 - 1}\right) - \left(-x_0 + \sqrt{x_0^2 - 1}\right) = 2x_0$ .

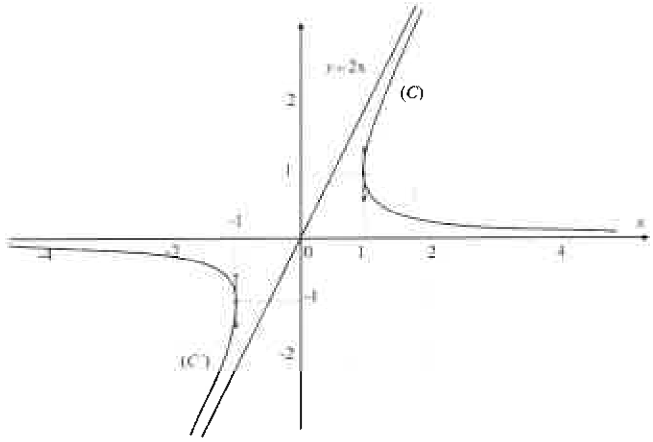
We have $M(x_0; f(x_0))$ and $M'(-x_0; f(-x_0))$ .

The slope of $(MM')$ is equal to :

$\dfrac{f(x_0) - f(-x_0)}{x_0 + x_0} = \dfrac{2x_0}{2x_0} = 1$, so $(MM')$ remains parallel to the straight line of equation $y = x$ .

4) a- $g(-x) = -x - \sqrt{x^2 - 1} = -f(x)$ .

Hence, $(C')$ is the symmetric of $(C)$ with respect to $O(0;0)$,




b The axes of symmetry of $(\gamma)$ are the bisectors of the angle formed by the asymptotes of equations $x - 2y = 0$ and $y = 0$.
If $M(x; y)$is a point on the axis of symmetry, then $M(x; y)$ is equidistant of these two straight lines , therefore :

$\frac{|x-2y|}{\sqrt{5}} = \frac{|y|}{1}$ , which gives $x-2y-\sqrt{5}=0$ or

$x-\left(2+\sqrt{5}\right)y=0$ consequently the axes of symmetry are

$x-2y+\sqrt{5}=0$ ; $x-\left(2-\sqrt{5}\right)y=0$ .

5) a- For $x \geq 1$ , $f$ being continuous and strictly increasing then it admits an inverse function $f^{-1}$.

b- $(C')$ is the symmetric of $(C)$ with respect to the first bisector of axes, see figure.

c- $y=x+\sqrt{x^2-1}$ gives $y-x=\sqrt{x^2-1}$ , then

$y^2+x^2-2xy=x^2-1$ so $y^2+1=2xy$ consequently

$x=\frac{y^2+1}{2y}$ , therefore $f^{-1}(x)=\frac{x^2+1}{2x}$.

6) $V=\int_1^2 \pi f^2(x)dx = \pi\int_1^2\left(x^2+x^2-1+2x\sqrt{x^2-1}\right)dx$

$$V=\pi\int_1^2\left(2x^2-1+2x\sqrt{x^2-1}\right)dx.$$

$$V=\pi\left[\frac{2x^3}{3}-x\right]_1^2+\int_1^2\left(x^2-1\right)'\left(x^2-1\right)^{\frac{1}{2}}dx$$

$$V=\pi\left[\frac{2x^3}{3}-x+\frac{2}{3}\left(x^2-1\right)\sqrt{x^2-1}\right]_1^2$$

$$V=\pi\left[\left(\frac{16}{3}-2+2\sqrt{3}\right)-\left(\frac{2}{3}-1\right)\right]=\pi\left[\frac{11}{3}+2\sqrt{3}\right]$$ cubic units.

## N°7.

***Part A :***

1) a- $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\sqrt{x^2-x-2}=+\infty$ .

$\lim\limits_{x\to+\infty}\frac{f(x)}{x}=\lim\limits_{x\to+\infty}\frac{\sqrt{x^2-x-2}}{x}=\lim\limits_{x\to+\infty}\sqrt{1-\frac{1}{x}-\frac{2}{x^2}}=1$.

$\lim\limits_{x\to+\infty}(f(x)-x)=\lim\limits_{x\to+\infty}\left(\sqrt{x^2-x-2}-x\right)$

$$\lim_{x\to+\infty}(f(x)-x)=\lim_{x\to+\infty}\frac{\left(\sqrt{x^2-x-2}-x\right)\left(\sqrt{x^2-x-2}+x\right)}{\left(\sqrt{x^2-x-2}+x\right)}$$

$$\lim_{x\to+\infty}(f(x)-x)=\lim_{x\to+\infty}\frac{\left(\sqrt{x^2-x-2}-x\right)\left(\sqrt{x^2-x-2}+x\right)}{\left(\sqrt{x^2-x-2}+x\right)}$$

$$\lim_{x\to+\infty}(f(x)-x)=\lim_{x\to+\infty}\frac{-x-2}{\left(\sqrt{x^2-x-2}+x\right)}=\lim_{x\to+\infty}\frac{-x}{x+x}=-\frac{1}{2}$$

b- The straight line of equation $y=x-\frac{1}{2}$ is an asymptote to $(C)$ at $+\infty$.

2) a- $x=\frac{1}{2}$ is an axis of symmetry since $f(1-x)=f(x)$.

But $f(1-x)=\sqrt{(1-x)^2-(1-x)-2}=\sqrt{1+x^2-2x-1+x-2}$

$=\sqrt{x^2-x-2}=f(x)$.

b- Using properties of symmetry, the straight line $(d')$ symmetric of $(d)$ with respect to the straight line of equation $x=\frac{1}{2}$ is an asymptote to $(C)$ at $-\infty$. The equation of $(d')$ can be found by replacing $x$ par $(1-x)$, an equation of $(d')$ is

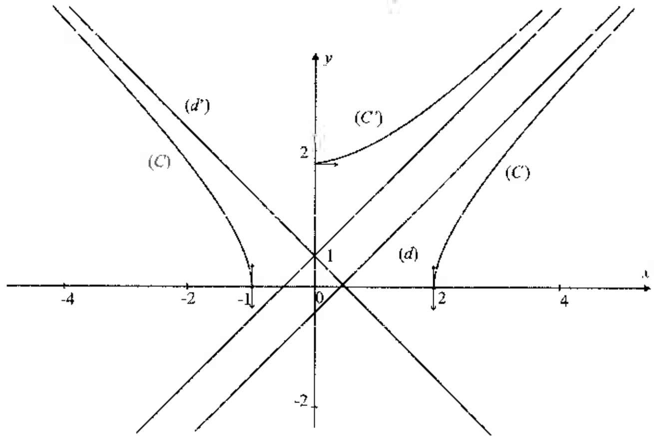
$y=(1-x)-\frac{1}{2}=-x+\frac{1}{2}$.

3) $f'(x)=\dfrac{2x-1}{2\sqrt{x^2-x-2}}$

$f'(x)>0$ for $x>\frac{1}{2}$ and $f'(x)<0$ for $x<\frac{1}{2}$, therefore the table of variations of $f$ is :

| $x$ | $-\infty$ | | $-1$ | $2$ | | $+\infty$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | $-\infty$ | | | $+\infty$ | $+$ |
| $f(x)$ | $+\infty \searrow$ | | $0$ | $0$ | | $\nearrow +\infty$ |

4)




5) a- $f$ admits an inverse function $f^{-1}$ over $[2;+\infty[$ since it is continuous and strictly increasing .

b- $D_{f^{-1}} = [0;+\infty[$ .

$(C')$ is symmetric of $(C)$ with respect to the straight line of equation $y = x$ .

$y = \sqrt{x^2 - x - 2}$ gives $y^2 = x^2 - x - 2$ , then

$y^2 = \left(x - \frac{1}{2}\right)^2 - \frac{9}{4}$, which gives $y^2 + \frac{9}{4} = \left(x - \frac{1}{2}\right)^2$ and

consequently $x - \frac{1}{2} = \sqrt{y^2 + \frac{9}{4}}$ , since $x \geq 2$ , so it is

$f^{-1}(x) = \frac{1}{2} + \sqrt{x^2 + \frac{9}{4}}$ .

***Part B:***

1) $y = \sqrt{-x^2 + x + 2}$ implies that $y^2 = -x^2 + x + 2$ .

$x^2 + y^2 - x - 2 = 0$ is equivalent to $\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{9}{4}$

with $y \geq 0$, hence $(\gamma)$ is a semi-circle of center $\left(\frac{1}{2};0\right)$ and radius $\frac{3}{2}$ .

2) $\int_{-1}^{2}\sqrt{-x^2+x+2}dx$ represents the area of half the disc limited by $(\gamma)$.

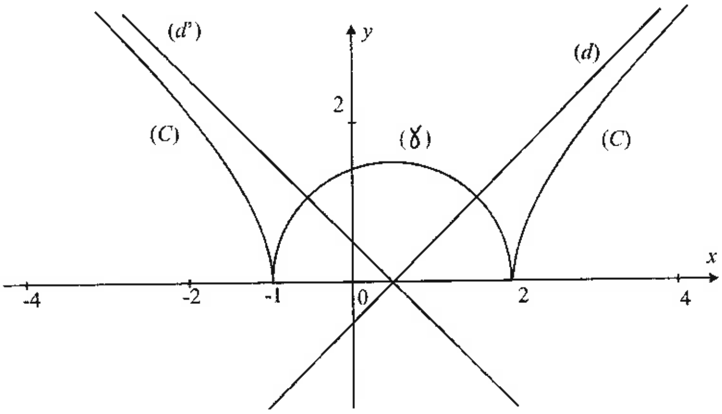
Therefore $\int_{-1}^{2}\sqrt{-x^2+x+2}dx = \frac{1}{2}\pi R^2 = \frac{1}{2}\pi\times\frac{9}{4} = \frac{9\pi}{8}$ square units.

***Part C:***

1) $h$ is defined over $IR$.

If $x^2-x-2\geq 0$ then $h(x)=f(x)$.

If $x^2-x-2\leq 0$ then $h(x)=g(x)$.




2) $(\delta)$ passes through the point $F(2;0)$ and the point $F\in(\Gamma)$.

Let $(\Delta)$ be the straight line passing through $F$ and parallel to $(d')$.

$(\delta)$ intersects $(\Gamma)$ in $3$ points when $(\delta)$ varies in the region of the plane limited by $(\Delta)$ and the axis $x'x$, therefore, $-1<a<0$.

**N°8.**

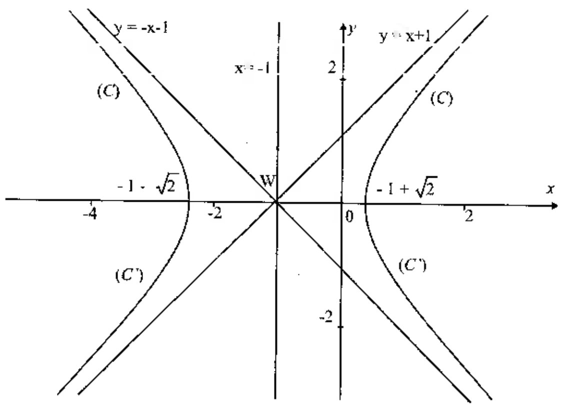
1) a- The curve $(C)$ admits two oblique asymptotes of equations $y=x+1$ in the neighborhood of $+\infty$ and $y=-x-1$ in the neighborhood of $-\infty$.

$f'(x)=\frac{2x+2}{2\sqrt{x^2+2x-1}}$, therefore the table of variations of $f$ is :

| $x$ | $-\infty$ | | $-1-\sqrt{2}$ | | $-1+\sqrt{2}$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | $-\infty$ ‖ | | ‖ $+\infty$ | $+$ | |
| $f(x)$ | $+\infty$ | ↘ | $0$ | | $0$ | ↗ | $+\infty$ |

b- Let $I(-1;0)$ be the midpoint of the segment joining the endpoints , the straight line of equation $x=-1$ is an axis of symmetry of $(C)$ since $f(-2-x)=\sqrt{(2-x)^2+2(2-x)-1}=\sqrt{x^2+2x-1}=f(x)$.

c-



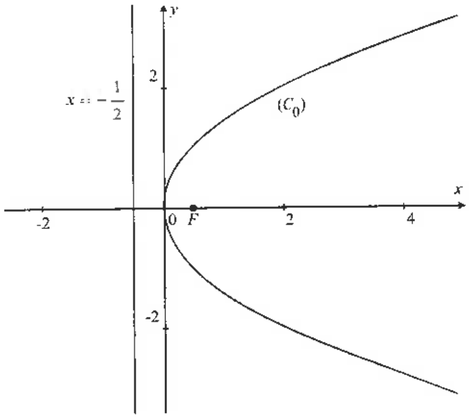

d- For $m=1$, the equation of $(C_1)$ will be $y^2=x^2-1+2x$, which gives $y=\sqrt{x^2+2x-1}$ or $y=-\sqrt{x^2+2x-1}$, hence $(C_1)$ is the union of $(C)$ and the curve $(C')$ of equation $y=-\sqrt{x^2+2x-1}$.

e- $(C')$ is the symmetric of $(C)$ with respect to the axis $x'x$. The origin $O$ is a center of symmetry of $(C_1)$.

2) a- For $m=0$, the equation of $(C_0)$ will be $y^2=2x$.

$(C_0)$ is a parabola whose vertex is the origin $O$, of focus $F\left(\frac{1}{2};0\right)$

and directrix the straight line of equation $x=-\frac{1}{2}$ and passing through the points $(2;2)$ and $(2;-2)$.




b- The area of the domain $(D)$ is equal to :

$$\int_0^1 2\sqrt{x}dx=\frac{4}{3}x\sqrt{x}\Big|_0^1=\frac{4}{3}\text{ square units.}$$

3) The equation $y^2=mx^2+2x-m$ can be written as :

$$y^2=m\left(x^2+\frac{2}{m}x+\frac{1}{m^2}-\frac{1}{m^2}\right)-m=m\left(x+\frac{1}{m}\right)^2-\frac{1}{m}-m.$$

$$m\left(x+\frac{1}{m}\right)^2-y^2=\frac{m^2+1}{m}\text{ then }\frac{\left(x+\frac{1}{m}\right)^2}{\frac{m^2+1}{m^2}}-\frac{y^2}{\frac{m^2+1}{m}}=1.$$

a- If $m>0$ then $(C_m)$ is a hyperbola of center $O'\left(-\frac{1}{m};0\right)$ and focal axis $x'x$.

If $m<0$ then $(C_m)$ is an ellipse of center $O'\left(-\frac{1}{m};0\right)$.

b- $a^2=\frac{m^2+1}{m^2}$, $b^2=\frac{m^2+1}{m}$, therefore :

$$c^2 = a^2 + b^2 = \frac{m^2+1}{m^2} + \frac{m^2+1}{m} = \frac{m^2+1}{m}\left(\frac{1}{m}+1\right)$$

$$c^2 = \frac{m^2+1}{m}\left(\frac{m+1}{m}\right)$$

$$e^2 = \frac{c^2}{a^2} = \frac{m^2+1}{m}\left(\frac{m+1}{m}\right)\times\frac{m^2}{m^2+1} = m+1$$ , consequently

$e = \sqrt{m+1}$.

**N°9.**

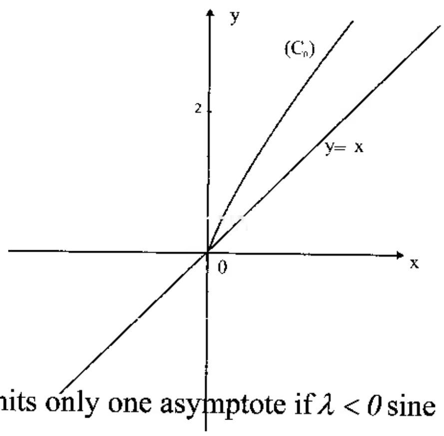
1) a- $(C_\lambda)$ has no asymptotes when $\lambda = 0$ since in this case $f(x) = x + \sqrt{x}$ .

$(C)$ admits an asymptotic direction parallel to the straight line of equation $y = x$.

b- The domain of definition of $f$ is $[0;+\infty[$.

$f'(x) = 1 + \dfrac{1}{2\sqrt{x}} > 0$ therefore the table of variations of $f$ :

| $x$ | $0$ | | $+\infty$ |
|---|---|---|---|
| $f'(x)$ | $\|$ $+\infty$ | $+$ | |
| $f(x)$ | $0$ | $\nearrow$ | $+\infty$ |




2) a- $(C_\lambda)$admits only one asymptote if $\lambda < 0$ sine in this case

$D_f = \left]-\dfrac{1}{\lambda};0\right]$.

b- $(C_\lambda)$ admits one asymptote only of equation $x = -\dfrac{1}{\lambda}$, since $\lim\limits_{x \to \frac{-1}{\lambda}} f(x) = +\infty$.

3) a- $(C_\lambda)$ admits two asymptotes when $\lambda > 0$ since in this case $D_f = \left]-\infty;\dfrac{-1}{\lambda}\right[ \cup [0;+\infty[$.

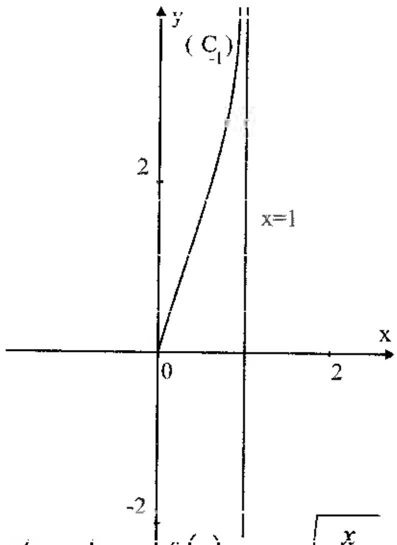
b- The straight lines of equations $y = x + \sqrt{\dfrac{1}{\lambda}}$ and $x = -\dfrac{1}{\lambda}$ are the asymptotes to $(C_\lambda)$.

4) For $\lambda = -1$ we have $f_{-1}(x) = x + \sqrt{\dfrac{x}{-x+1}}$.

a- The domain of $f_{-1}$ is $D_{f_{-1}} = [0;1[$.

b- $f'_{-1}(x) = 1 + \dfrac{\frac{1}{(-x+1)^2}}{2\sqrt{\frac{x}{-x+1}}} > 0$, therefore the table of variations of $f_{-1}$:

| $x$ | $0$ | | $1$ |
|---|---|---|---|
| $f'_{-1}(x)$ | $\|\; +\infty$ | | |
| $f_{-1}(x)$ | $0$ | $\nearrow$ | $+\infty$ |



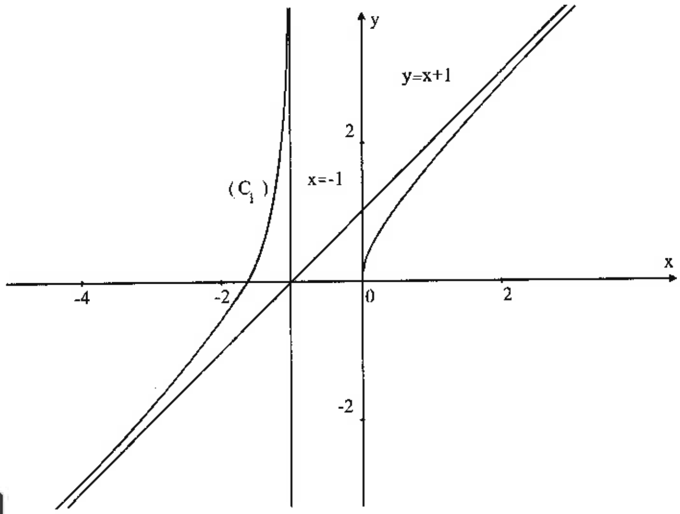

5) For $\lambda = 1$ we have $f_1(x) = x + \sqrt{\dfrac{x}{x+1}}$ .

a- The domain of definition of $f_1$ is $D_{f_1} = \left]-\infty;-1\right[ \cup \left[0;+\infty\right[$ .

b- $f_1'(x) = 1 + \dfrac{\dfrac{1}{(x+1)^2}}{2\sqrt{\dfrac{x}{x+1}}} > 0$ .

The asymptotes of $(C_1)$ are the straight lines of equations :
$y = x + 1$ and $x = -1$ , therefore the table of variations of $f_1$ :

| $x$ | $-\infty$ | | $-1$ | $0$ | | $+\infty$ |
|---|---|---|---|---|---|---|
| $f_1'(x)$ | | $+$ | ║ | ║ | $+$ | |
| $f_1(x)$ | $-\infty$ | $\nearrow$ | $+\infty$ ║ | ║ $0$ | $\nearrow$ | $+\infty$ |




N° 10.

1) We have $f(x)=\begin{cases} x+\sqrt{x^2-1} & x\in\left]-\infty;-1\right]\cup\left[1;+\infty\right[ \\ x+\sqrt{-x^2+1} & x\in[-1;1] \end{cases}$

$f$ is continuous at the point of abscissa 1 since
$f(1)=1$ and $\lim\limits_{\substack{x\to1\\x<1}} f(x)=\lim\limits_{\substack{x\to1\\x>1}} f(x)=1=f(1)$.

Similarly, $f$ is continuous at the point of abscissa -1 .

2) $f$ is not differentiable at the point of abscissa 1 since

$$\lim_{\substack{x\to1\\x>1}}\frac{f(x)-f(1)}{x-1}=\lim_{\substack{x\to1\\x>1}}\frac{x+\sqrt{x^2-1}-1}{x-1}=\lim_{\substack{x\to1\\x>1}}\left[1+\frac{\sqrt{x^2-1}}{x-1}\right]=\frac{0}{0}$$

Using L'Hopital's rule , we get:

$$\lim_{\substack{x\to1\\x>1}}\frac{f(x)-f(1)}{x-1}=\lim_{\substack{x\to1\\x>1}}\left[1+\frac{2x}{2\sqrt{x^2-1}}\right]=+\infty .$$

Similarly, $f$ is not differentiable at the point of abscissa -1 .

3) * For $x\in\left]-\infty;-1\right]\cup\left[1;+\infty\right[$ ; $f(x)=x+\sqrt{x^2-1}$ ;

$$f'(x)=1+\frac{x}{\sqrt{x^2-1}}=\frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}}$$

* For $x>1$ , $f'(x)>0$ .

* For $x < -1$ , $f'(x) = \dfrac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} < 0$ if $\sqrt{x^2 - 1} < -x$ which gives $x^2 - 1 < x^2$ , sot $-1 < 0$ which is always true .

* For $x \in [-1;1]$ , $f'(x) = 1 + \dfrac{-x}{\sqrt{1 - x^2}} = \dfrac{\sqrt{1 - x^2} - x}{\sqrt{1 - x^2}}$

  $f'(x) > 0$ for $\sqrt{1 - x^2} > x$ , so $1 - x^2 > x^2$ which gives $1 - 2x^2 > 0$ and consequently $-\dfrac{\sqrt{2}}{2} < x < \dfrac{\sqrt{2}}{2}$ and since $x \geq -1$ then $f'(x) > 0$ for $-1 < x < \dfrac{\sqrt{2}}{2}$.

  $f(-1) = -1$ ; $f(1) = 1$ .

  $f\left(\dfrac{\sqrt{2}}{2}\right) = \dfrac{\sqrt{2}}{2} + \sqrt{1 - \dfrac{2}{4}} = \dfrac{\sqrt{2}}{2} + \dfrac{\sqrt{2}}{2} = \sqrt{2}$.

  In the neighborhood of $-\infty$, $(C)$ admits an asymptote of equation:

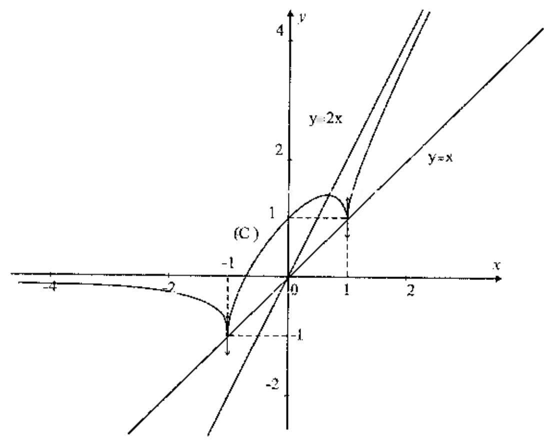
  $y = x + |x| = x - x = 0$.

  $\lim\limits_{x \to +\infty} f(x) = +\infty$, so all the elements of the table are justified .

4) $\lim\limits_{x \to +\infty} (f(x) - 2x) = \lim\limits_{x \to +\infty} \left(\sqrt{x^2 - 1} - x\right) = \lim\limits_{x \to +\infty} \dfrac{-1}{\sqrt{x^2 - 1} + x} = 0$

   Hence, the straight line of equation $y = 2x$ is an asymptote to $(C)$ at $+\infty$.

5)



6) An equation of $(d)$ is $y = mx$, graphically :

For $m \le 0$ $(d)$ cuts $(C)$ in one point .

For $0 < m \le 1$ $(d)$ cuts $(C)$ in two points .

For $1 < m < 2$ $(d)$ cuts $(C)$ in two points .

For $m \ge 1$ $(d)$ cuts $(C)$ in one point only .

**N°11.**

1) $f'(x) = 2 + \dfrac{4x}{\sqrt{72-4x^2}} = \dfrac{2\sqrt{72-4x^2}+4x}{\sqrt{72-4x^2}}$.

If $x \ge 0$ then $f'(x) > 0$.

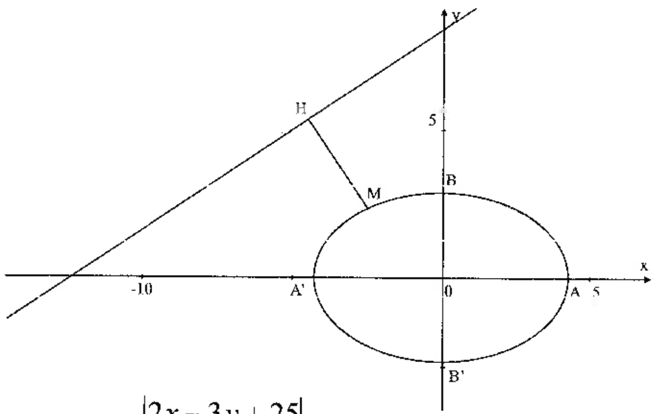
If $x < 0$ then $f'(x) > 0$ for $\sqrt{72-4x^2} > -2x$ which gives $72-4x^2 > 4x^2$ , so $8x^2 < 72$ then $x^2 < 9$.
And consequently $-3 < x < 3$ and since $x < 0$ then $-3 < x < 0$.
Hence $f'(x) > 0$ for $-3 < x < 3\sqrt{2}$ and $f'(x) < 0$ for $-3\sqrt{2} < x < -3$, therefore the table of variations of $f$ is :

| $x$ | $-3\sqrt{2}$ | | $-3$ | | $3\sqrt{2}$ |
|---|---|---|---|---|---|
| $f'(x)$ | $\Vert$ | $-$ | $0$ | $+$ | $\Vert$ |
| $f(x)$ | | $\searrow$ | $13$ | $\nearrow$ | |

The minimum of $f$ is 13 then $f(x) \ge 13$.

2) a- The vertices of $(E)$ are $A(3\sqrt{2};0)$, $A'(-3\sqrt{2};0)$, $B(0;2\sqrt{2})$ and $B'(0;-2\sqrt{2})$.

$(d)$ passes through the points $\left(0;\dfrac{25}{3}\right)$ and $\left(-\dfrac{25}{2};0\right)$.




b- $MH = \dfrac{|2x-3y+25|}{\sqrt{13}}$.

$M$ is a point of $(E)$ then $\dfrac{x^2}{18}+\dfrac{y^2}{8}=1$ which gives

$\dfrac{y^2}{8}=1-\dfrac{x^2}{18}$, therefore $y^2=\dfrac{72-4x^2}{9}$ and consequently

$y=+\dfrac{1}{3}\sqrt{72-4x^2}$ or $y=-\dfrac{1}{3}\sqrt{72-4x^2}$.

$MH$ is minimum when $y>0$, so $y=+\dfrac{1}{3}\sqrt{72-4x^2}$, therefore:

$MH=\dfrac{\left|2x+25-\sqrt{72-4x^2}\right|}{\sqrt{13}}=\dfrac{|f(x)|}{\sqrt{13}}$, hence $MH$ is minimum

When $f(x)$ is minimum, this minimum is equal to

$\dfrac{13}{\sqrt{13}}=\sqrt{13}$.

The coordinates of the corresponding point $M$ are $(-3;2)$.

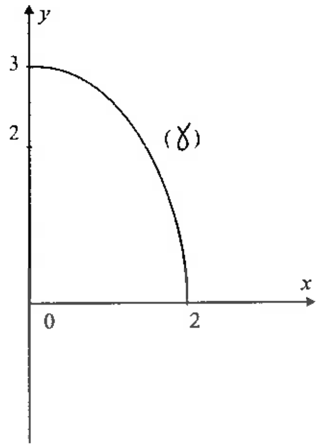
**N° 12.**

1) $f(-x)=\pm\dfrac{3}{2}\sqrt{4-x^2}=f(x)$ and $D_f$ is centered at $O$ then $f$ is an even function.

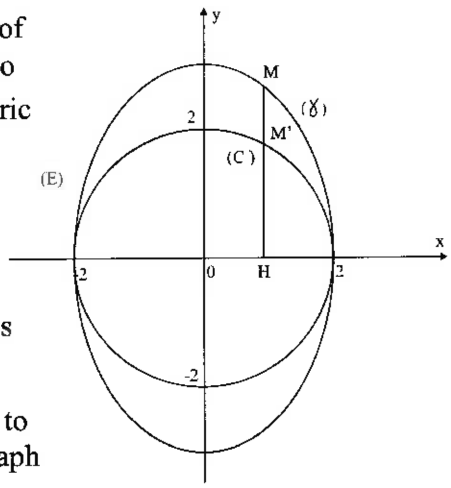
2) a $g'(x)=\dfrac{3}{2}\left(\dfrac{-x}{\sqrt{4-x^2}}\right)\leq 0$ over $[0;2]$ therefore the table of

variations of $g$ is :

| $x$ | $0$ | | $2$ |
|---|---|---|---|
| $g'(x)$ | $0$ | $-$ | $-\infty$ ‖ |
| $g(x)$ | $3$ | $\searrow$ | $0$ |




b- The part of the curve of $(E)$ that corresponds to $x \in [-2;0]$ is symmetric to $(\gamma)$ with respect to the axis $y'y$.
The part that corresponds to $f(x) = -\frac{3}{2}\sqrt{4-x^2}$ is symmetric to the part obtained with respect to the axis $x'x$ so the graph $(E)$ is as follows:




3) a- $\overrightarrow{HM'} = \frac{2}{3}\overrightarrow{HM}$ gives $\begin{cases} X = x \\ Y = \frac{2}{3}y \end{cases}$

Replacing $x$ and $y$ by their values in the equation

$y = \pm\frac{3}{2}\sqrt{4-x^2}$ we get $\frac{3}{2}Y = \pm\frac{3}{2}\sqrt{4-X^2}$ then

$Y = \pm\sqrt{4-X^2}$ ; so $Y^2 = 4-X^2$ and consequently

$Y^2 + X^2 = 4$ .

Hence $M'$ describes a circle $(C)$ of center $O(0;0)$ and radius $R = 2$.

b- The domain $(D)$ is quarter of a disc , its area is

then $\frac{1}{4}\pi \times R^2 = \pi$ square units .

c- $\mathscr{A} = \int_0^2 \sqrt{4-x^2}\,dx = \frac{1}{4}$ the area of the disc is

$\frac{1}{4}\pi R^2 = \frac{1}{4}\pi \times 4 = \pi$ square units .

$\mathscr{A}' = 4\int_0^2 \frac{3}{2}\sqrt{4-x^2}\,dx = 6\int_0^2 \sqrt{4-x^2}\,dx = 6\pi$ square units .

**N° 13.**

1) a- $f_\lambda$ is defined over $IR$ if $\lambda x^2 - 4x + 3 \geq 0$ for all real numbers $x$ which is true when $\Delta' \leq 0$ and $\lambda > 0$ .

$\Delta' = 4 - 3\lambda \leq 0$ gives $\lambda \geq \frac{4}{3}$ .

Hence $f_\lambda$ defined over $IR$ if $\lambda \geq \frac{4}{3}$ .

b- For $\lambda \geq \frac{4}{3}$ the asymptotes are given by $y = x + \sqrt{\lambda}\left|x - \frac{2}{\lambda}\right|$. In the neighborhood of $+\infty$, the asymptote is the straight line of equation $y = \left(1+\sqrt{\lambda}\right)x - \frac{2\sqrt{\lambda}}{\lambda}$ and in the neighborhood of $-\infty$

the asymptote is the straight line of equation

$y = \left(1-\sqrt{\lambda}\right)x + \frac{2\sqrt{\lambda}}{\lambda}$.

The two asymptotes are perpendicular when $(1+\sqrt{\lambda})(1-\sqrt{\lambda})=-1$, which gives $1-\lambda=-1$ and consequently $\lambda=2$.

2) a- The asymptotes are :
in the neighborhood of $-\infty$ : $y=2$.
in the neighborhood of $+\infty$ : $y=2x-2$.

b- $f_1'(x)=1+\dfrac{x-2}{\sqrt{x^2-4x+3}}$.

If $x-2\geq 0$, then $f_1'(x)>0$.

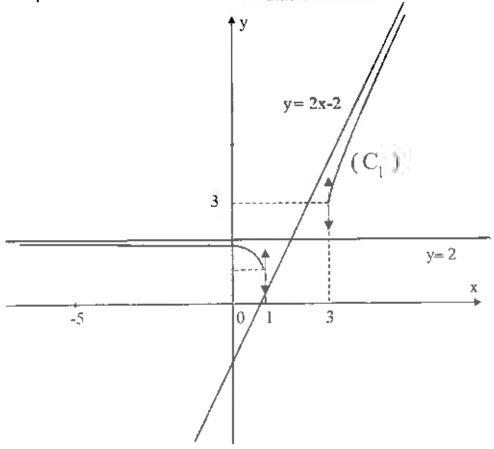
If $x-2<0$, then $x<2$ hence $f_1'(x)=\dfrac{\sqrt{x^2-4x+3}+(x-2)}{\sqrt{x^2-4x+3}}$

$$f_1'(x)=\frac{-1}{\sqrt{x^2-4x+3}\left(\sqrt{x^2-4x+3}-(x-2)\right)}<0.$$

Therefore the table of variations of $f_1$ is:

| $x$ | $-\infty$ | | $-1$ | $3$ | | $+\infty$ |
|---|---|---|---|---|---|---|
| $f_1'(x)$ | | $-$ | $\infty$ | $\infty$ | $+$ | |
| $f_1(x)$ | $2$ | $\searrow$ | $1$ | $3$ | $\nearrow$ | $+\infty$ |

c-



N° 14.

1) $f(x) = x + \sqrt{x^2\left(1 + \frac{1}{x} - \frac{2}{x^2}\right)} = x + |x|\sqrt{1 + \frac{1}{x} - \frac{2}{x^2}}$

2) a- $\varphi'(0) = \lim\limits_{t\to 0} \frac{\varphi(t) - \varphi(0)}{t - 0} = \lim\limits_{t\to 0} \frac{\sqrt{-2t^2 + t + 1} - 1}{t} = \frac{0}{0}$

$\varphi'(0) = \lim\limits_{t\to 0} \frac{-4t + 1}{2\sqrt{-2t^2 + t + 1}} = \frac{1}{2}$.

b- $\lim\limits_{t\to 0} \frac{\varphi(t) - 1}{t} = \frac{1}{2}$ then there exists a function $h$ such that

$\frac{\varphi(t) - 1}{t} = \frac{1}{2} + h(t)$ with $\lim\limits_{t\to 0} h(t) = 0$ .

Therefore $\varphi(t) = \frac{1}{2}t + 1 + t\,h(t)$ .

c- $f(x) = x + |x| \cdot \varphi(t) = x + |x| \cdot \left[\frac{1}{2}t + 1 + th(t)\right]$

$= x + |x|\left[\frac{1}{2x} + 1 + \frac{1}{x}h\left(\frac{1}{x}\right)\right] = x + \frac{|x|}{2x} + |x| + \frac{|x|}{x}h\left(\frac{1}{x}\right)$ .

d- For $x > 0$ ; $f(x) = x + \frac{1}{2} + x + h\left(\frac{1}{x}\right) = 2x + \frac{1}{2} + h\left(\frac{1}{x}\right)$ .

$\lim\limits_{x\to+\infty}\left[f(x) - \left(2x + \frac{1}{2}\right)\right] = \lim\limits_{x\to+\infty} h\left(\frac{1}{x}\right) = \lim\limits_{t\to 0} h(t) = 0$ .

Then the straight line of equation $y = 2x + \frac{1}{2}$ is an asymptote to $(C)$at $+\infty$ .

For $x < 0$ ; $f(x) = -\frac{1}{2} - h\left(\frac{1}{x}\right)$ .

Therefore $\lim\limits_{x\to-\infty} f(x) = -\frac{1}{2}$ so the straight line of equation $y = -\frac{1}{2}$ is an asymptote to $(C)$ at $-\infty$ .

e- $f'(x)=1+\dfrac{2x+1}{2\sqrt{x^2+x-2}}$ .

If $x \geq -\dfrac{1}{2}$ then $f'(x)>0$ .

If $x < -\dfrac{1}{2}$ then $f'(x)=\dfrac{2\sqrt{x^2+x-2}+2x+1}{2\sqrt{x^2+x-2}}$ which gives

$f'(x)=\dfrac{-9}{2\sqrt{x^2+x-2}\left(2\sqrt{x^2+x-2}-2x-1\right)}$.

Hence $f'(x)<0$ therefore the table of variations of $f$ is :

| $x$ | $-\infty$ | | $-2$ | | $1$ | | $+\infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | | $-$ | $-\infty$ ‖ | | ‖ $+\infty$ | $+$ | |
| $f(x)$ | $\frac{-1}{2}$ | $\searrow$ | $-2$ | | $1$ | $\nearrow$ | $+\infty$ |

N° 15.

1) $f'(x)=-\sqrt{-x^2+1}+\dfrac{-x}{\sqrt{-x^2+1}}(1-x)$

$f'(x)=\dfrac{-(1-x^2)-x(1-x)}{\sqrt{1-x^2}}=\dfrac{(x^2-1)+x(x-1)}{\sqrt{1-x^2}}$.

$=\dfrac{(x-1)(2x+1)}{\sqrt{1-x^2}}$ , then the table of variations of $f$ is :

| $x$ | $-1$ | | $-1/2$ | | $1$ |
|---|---|---|---|---|---|
| $f'(x)$ | ‖ $+\infty$ | $+$ | $0$ | $-$ | $0$ |
| $f(x)$ | $0$ | $\nearrow$ | $\frac{3\sqrt{3}}{4}$ | $\searrow$ | $0$ |

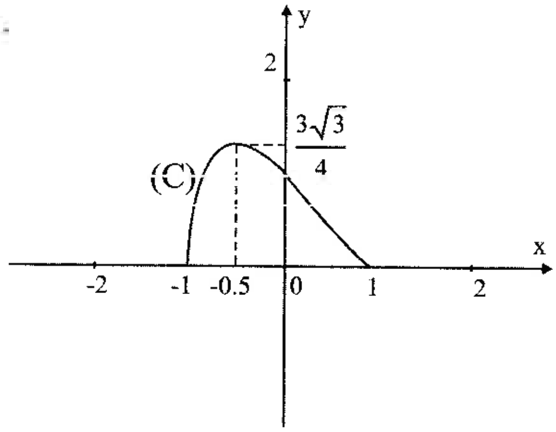
$$f\left(-\frac{1}{2}\right)=\left(1+\frac{1}{2}\right)\sqrt{1-\frac{1}{4}}=\frac{3}{2}\sqrt{\frac{3}{4}}=\frac{3\sqrt{3}}{4}.$$

$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}=\lim_{x\to 1}\frac{(1-x)\sqrt{-x^2+1}}{(x-1)}$$

$=\lim_{x\to 1}\left(-\sqrt{-x^2+1}\right)=0$ hence the tangent to $(C)$ at the point of abscissa 1 is horizontal.

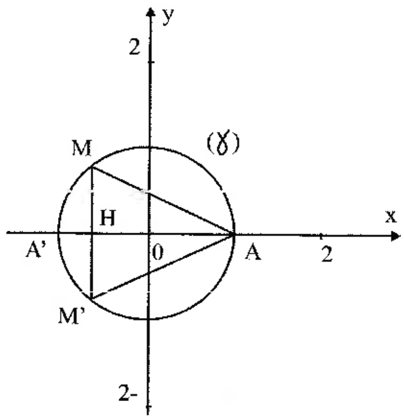
The tangent to $(C)$ at the point of abscissa $-1$ is vertical.

2) $f(0)=1$.




3) Area of triangle $AMM'=2\times$ Area of triangle $AHM$.

$$=2\times\frac{1}{2}HA\times HM=HA\times HM.$$



An equation of $(\gamma)$ is $x^2 + y^2 = 1$.

$HA = 1 - m$ with $-1 < m < 1$

$HM = \sqrt{1-m^2}$ is the ordinate of point $M$.

Hence the area is $\mathcal{A}_{(m)} = (1-m)\sqrt{1-m^2} = f(m)$.

$\mathcal{A}_{(m)}$ is maximal for $m = -\dfrac{1}{2}$.

In this case $M\left(-\dfrac{1}{2};\dfrac{\sqrt{3}}{2}\right)$ ; $M'\left(-\dfrac{1}{2};-\dfrac{\sqrt{3}}{2}\right)$ and $A(1;0)$.

Therefore $AM^2 = \left(-\dfrac{1}{2}-1\right)^2 + \left(\dfrac{\sqrt{3}}{2}-0\right)^2 = \dfrac{9}{4}+\dfrac{3}{4} = 3$.

$AM' = AM$ since $x'x$ is the perpendicular bisector of $[MM']$.

$MM' = \dfrac{\sqrt{3}}{2}+\dfrac{\sqrt{3}}{2} = \sqrt{3}$ .

Hence triangle $AMM'$ is equilateral.

# Indications

**N° 1.**

1) b- The straight line of equation $x = 1$ is an axis of symmetry of $(C)$.

d- For $m = 1$, we have $y^2 = x^2 - 2x - 3$, which gives $y = \pm\sqrt{x^2 - 2x - 3}$ the equation of $(C')$ is $y = -\sqrt{x^2 - 2x - 3}$.

2) a- For $m = 0$, we have $y^2 = -2x$, it is a parabola of focal axis $x'x$, vertex $O$ and focus $F\left(-\frac{1}{2};0\right)$.

3) a- $\alpha > 0$, for $m < 0$, $(C_m)$ is an ellipse of equation :

$$\frac{\left(x - \frac{1}{m}\right)^2}{\frac{3m^2+1}{m^2}} + \frac{y^2}{\frac{3m^2+1}{-m}} = 1.$$

b- $c^2 = \frac{3m^2+1}{m^2} + \frac{3m^2+1}{m} = \frac{(3m^2+1)(m+1)}{m^2}$,

therefore $e = \sqrt{m+1}$.

**N° 2.**

***Part A :***

4) $A = \int_{\frac{1}{4}}^{1} 2\sqrt{x}\,dx$.

***Part B :***

1) The complex form of $r$ is $z' = \left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)z$.

2) a- $z = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)z'$ which gives :

$x+iy=\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)(x'+iy')$ , so

$x=\frac{\sqrt{2}}{2}(x'+y')$ and $y=\frac{\sqrt{2}}{2}(-x'+y')$.

b- Replacing $x$ and $y$ by their values in the equation of $(P)$ we get $4x'^2-x'-y'=0$.

c- The reduced equation of $(P')$ is $X'^2=\frac{1}{4}Y'$ with $X'=x'-\frac{1}{8}$ and $Y'=y'+\frac{1}{16}$.

d- Rotation preserves geometric figures, then $(P)$ is a parabola.

e- The focal axis of $(P')$ is the straight line of equation $X'=0$ , then $x'=\frac{1}{8}$ and since $x'=\frac{\sqrt{2}}{2}(x-y)$ and $y'=\frac{\sqrt{2}}{2}(x+y)$ so the focal axis of $(P)$ is the straight line of equation $\frac{1}{8}=\frac{\sqrt{2}}{2}(x-y)$.

f- The focus of $(P')$ is $F'\left(\frac{1}{8};0\right)$ then the focus of $(P)$ is $F\left(\frac{\sqrt{2}}{16};-\frac{\sqrt{2}}{16}\right)$.

$\boxed{N^\circ 3.}$

2) $f'(x)=1+\frac{x}{\sqrt{x^2+4}}=\frac{\sqrt{x^2+4}+x}{\sqrt{x^2+4}}$.

If $x\geq 0$ then $f'(x)>0$.

If $x<0$ then $f'(x)=\frac{4}{\sqrt{x^2+4}\left(\sqrt{x^2+4}-x\right)}>0$.

4) $(H)$ is the curve of equation $y^2-2xy-4=0$.

a- $y^2 - 2xy - 4 = 0$ is a quadratic equation whose roots are:

$y = x \pm \sqrt{x^2 + 4}$ , then $(H)$ is the union of curve $(C)$ and curve $(C')$ of equation $g(x) = x - \sqrt{x^2 + 4}$ .

b- $g(-x) = -f(x)$.

5) The equation relating the abscissas of the points of intersection is $x + \sqrt{x^2 + 4} = x + m$ , which gives $x = \pm\sqrt{m^2 - 4}$ .
We find the equations of the tangents and note that for $x = 0$ the two tangents intersect at a point situated on the axis $y'y$ .

**N°4.**

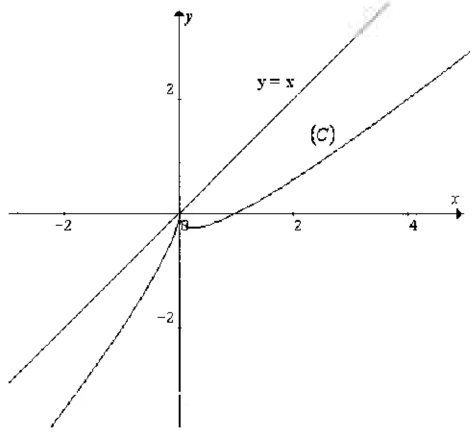
4) c- $y = x + \sqrt{x^2 + 1}$ gives $(y - x)^2 = x^2 + 1$ which gives

$y^2 - 2xy = 1$ and consequently $f^{-1}(x) = \dfrac{x^2 + 1}{2x}$.

5) $g(-x) = -f(x)$, hence the curve representative of $g$ is symmetric to $(C)$ with respect to the origin .

**N°5.**

4)



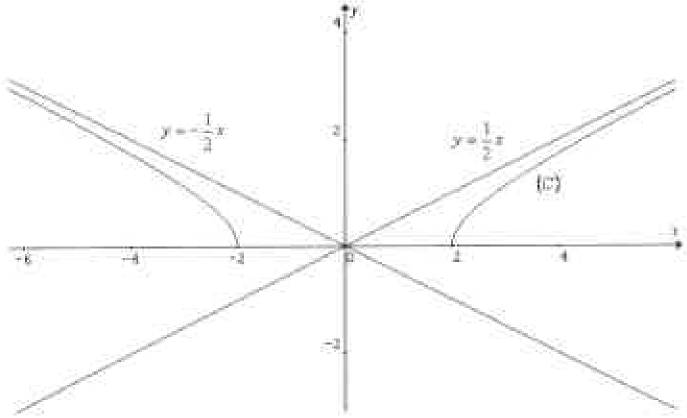

5) $v = \pi \int_0^1 (x - \sqrt{x})^2\, dx = \pi \int_0^1 (x^2 + x - 2x\sqrt{x})dx$

$v = \pi \left[ \dfrac{x^3}{3} + x^2 - \dfrac{4}{3} x\sqrt{x} \right]_0^1$ cubic units .

**N°6.**

1) The equations of the asymptotes to $(C)$ are $y=\frac{1}{2}x$ and $y=-\frac{1}{2}x$

3)




5) $y=\pm\frac{1}{2}\sqrt{x^2-4}$ gives $4y^2=x^2-4$, therefore $\frac{x^2}{4}-y^2=1$.

Then it is a hyperbola with $a^2=4$, $b^2=1$ and $c^2=5$.

**N°7.**

1) $4x^2+9y^2-8x-32=0$ is equivalent to $\frac{(x-1)^2}{9}+\frac{y^2}{4}=1$.

$(\Gamma)$ is an ellipse of center $O'(1;0)$.

3) For $y=0$ we get $G\left(0;\frac{4\sqrt{2}}{3}\right)$ and $L\left(0;-\frac{4\sqrt{2}}{3}\right)$.

Deriving with respect to $x$, the terms of the equation of $(\Gamma)$ we get $8x+18yy'-8=0$, which gives :

- Slope of the tangent at $G$ is $y'(x_G)=\frac{8}{18y_G}$.
- Slope of the tangent at $L$ is $y'(x_L)=\frac{8}{18y_L}$.

4) a- $(C)$ is a circle of center $I(1;0)$ and radius $R=3$.

b- $(C)$ is the principal circle of $(\Gamma)$.

c- The area of the domain limited by $(\Gamma)$ is $A_1 = \pi \times a \times b = 6\pi$ square units.
The area of the domain limited by $(C)$ is $A_2 = \pi R^2 = 9\pi$ square units .
The demanded area is $A_2 - A_1$ .

# Sample Test 1

### EXERCISE - I

Consider the functions $f$ and $g$ defined over $]-1;+1[$

by: $f(x)=\arctan\frac{2x}{1-x^2}$ and $g(x)=\arctan x$ .

1) Calculate $f\left(\frac{\sqrt{3}}{3}\right)$ and simplify $\tan(2g(x))$.

2) Show that $g(x)+g\left(\frac{1}{x}\right)=\frac{\pi}{2}$ for $x>0$.

3) Calculate $f'(x)$ and deduce that $f(x)=2\,g(x)$ .

4) Solve the equation $f(x)=\frac{\pi}{2}-2\arctan\frac{1}{2}$.

### EXERCISE - II

Given the two integrals : $I_n=\int_0^{\frac{\pi}{2}} x^n\cos x\,dx$ and $J_n=\int_0^{\frac{\pi}{2}} x^n\sin x\,dx$,

where $n$ is a natural number.

1) Calculate $I_0$ and $J_0$ .

2) a- Integrate $I_n$ by parts and find a relation between $I_n$ and $J_{n-1}$.

   b- Integrate $J_n$ by parts and find a relation between $J_n$ and $I_{n-1}$.

3) Deduce $I_n$ in terms of $I_{n-2}$ then find $I_2$ and $I_4$.

### EXERCISE - III

In the space referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the plane $(P)$ of equation $x+2y-z+1=0$, the plane $(Q)$ of equation $x-2y-z-1=0$ and the point $A(1;1;1)$.

Designate by $(d)$ the line of intersection of $(P)$ and $(Q)$.

1) Show that a system of parametric equations of $(d)$ is

   $x=m$ , $y=-\frac{1}{2}$, $z=m$ where $m$ is a real parameter .

2) Determine an equation of plane $(R)$ passing through $(d)$ and $A$ and verify that $(R)$ is one of the bisector planes of the dihedral formed by the two planes $(P)$ and $(Q)$.
3) Let $I$ be the orthogonal projection of $A$ on $(P)$ and $J$ the orthogonal projection of $A$ on $(Q)$.
   a- Without finding the coordinates of points $I$ and $J$ determine a normal vector of plane $(AIJ)$.
   b- Find the coordinates of the points $I$ and $J$ and find an equation of plane $(AIJ)$.
   c- Find the coordinates of point $B$, the point of intersection of $(AIJ)$ and $(d)$.
   d- Calculate the cosine of the angle $A\hat{B}I$ and deduce the cosine of the dihedral angle formed by the two planes $(P)$ and $(Q)$.

***EXERCISE - IV***

A goalkeeper has to face a certain number of direct shoots.
Previous experience urges us to think in the following manner:
If he rebounds the n$^{th}$ shoot then the probability that he rebounds the following one that is the $(n+1)^{th}$ shoot is 0.8
If he doesn't rebound the n$^{th}$ shoot then the probability that he rebounds the following is 0.6
Suppose that the probability that he rebounds the first shoot is 0.7
Designate by $A_n$ the event : the goalkeeper rebounds the n$^{th}$ shoot.

1) Calculate $p(A_2/A_1)$ and $p(A_2/\overline{A_1})$.
2) Deduce that $p(A_2)=0.74$
3) Designate by $p_n$ the probability of the event $A_n$ and by $p_{n+1}$ the probability of the event $A_{n+1}$.
   a- Prove that $p_{n+1}=0.2p_n+0.6$
   b- Define the sequence $(u_n)$ defined for $n \geq 1$ by
      $u_n = p_n - 0.75$
      Show that $(u_n)$ is a geometric sequence of ratio 0.2
      Deduce an expression of $p_n$ in terms of n and determine the limit of $(p_n)$.

***EXERCISE - V***

***Part A :***

Consider the differential equation $(E)$: $y''-2y'+y=2\,e^{x}$.

Let $z=y-x^{2}\,e^{x}$.

1) Find the differential equation $(F)$ satisfied by $z$ .
2) Solve $(F)$ and deduce the general solution of $(E)$ verifying $y(0)=1$ and $y(1)=0$.

***Part B :***

Let $f$ be the function defined over $IR$ by $f(x)=(x-1)^{2}\,e^{x}$.
Designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to-\infty} f(x)$ $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to+\infty} \dfrac{f(x)}{x}$.
2) Calculate $f'(x)$ and draw the table of variations of $f$.
3) Write an equation of the tangent $(T)$ to $(C)$ at the point of abscissa 0.
4) Trace $(C)$ and $(T)$ .
5) Calculate the area of the domain limited by $(C)$, the axis $x'x$ and the straight lines of equations $x=0$ and $x=-1$ .
6) Let $g$ be the function defined by $g(x)=\ln(f(x))$.
   a- Show that the domain of definition of $g$ is $\left]-\infty;1\right[\cup\left]1;+\infty\right[$.
   b- Study the variations of $g$ and draw its table of variations. .
   c- Prove that the curve $(G)$ of $g$ admits as an asymptotic direction the straight line $(d)$ of equation $y=x$.
   d- Show that the two curves $(G)$ and $(d)$ intersect in two points to be determined.
   e- Trace $(G)$ in a new system.

***EXERCISE - VI***

In an oriented plane, consider a rectangle $OPQF$ such that $OP=3$, $OF=4$ and $\left(\overrightarrow{FO};\overrightarrow{FQ}\right)=\dfrac{\pi}{2}(\bmod 2\pi)$.

$FOB$ is an equilateral triangle such that $\left(\overrightarrow{FO};\overrightarrow{FB}\right)=\dfrac{\pi}{3}(\bmod 2\pi)$.

***Part A :***

Let $(\mathcal{E})$ be the ellipse of foci $O$ and $F$ .

1) Show that $P$, $Q$ and $B$ belong to this ellipse.
2) Calculate the eccentricity of $(\mathcal{E})$.
3) Determine the vertices of $(\mathcal{E})$ and trace $(\mathcal{E})$.

***Part B:***

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$ with $\overrightarrow{OP} = 3\vec{v}$ .

1) Write an equation of $(\mathcal{E})$ and determine the directrix $(d)$ associated with the focus $O$.
2) Determine an equation of the tangent $(T)$ at $P$ to $(\mathcal{E})$ and deduce an equation of the tangent $(T')$ at $Q$ to $(\mathcal{E})$.
3) The tangents $(T)$ and $(T')$ intersect at a point $R$ and these tangents intersect the focal axis at $G$ and $L$ respectively.
4) Calculate the area of the domain limited by the triangle $RGL$ and the semi ellipse of $(\mathcal{E})$ situated above the focal axis
5) *Let* $M$ be a point of $(\mathcal{E})$ of affix $z = re^{i\theta}$ where $0 < \theta < \pi$ .
   a- Calculate the distance of $M$ to the directrix $(d)$ in terms of $r$ and $\theta$.
   b- Deduce that $OM = \dfrac{6}{2+\cos\theta}$.
   c- The straight line $(OM)$ cuts again $(\mathcal{E})$ at $M'$.
   Prove that $OM' = \dfrac{6}{2-\cos\theta}$.
   d- Determine $\theta$ and precise the position of $M$ for which the length $MM'$ is minimal.
6) Let $S$ be the similitude of center $O$, ratio $\dfrac{1}{2}$ and angle $\dfrac{\pi}{2}$.
   Designate by $(E')$ the image of $(\mathcal{E})$ by S .
   Construct $(E')$ in the same system as $(\mathcal{E})$.

# Sample Test 2

***EXERCISE – I***

In this exercise, one of the proposed answers to each question is correct .

Write the number of each question and give its answer with justification.

1) An urn $U$ contains 3 black balls and 5 white balls .
A player draws a ball from the urn, if it is white he gains 2 points, if it is black he leaves the ball outside the urn and draws a new ball.
Again, if the drawn ball is white he gains 1 point . If it is black he loses 1 point .
**The probability of gaining 1 point is :**
a- $\frac{5}{7}$ b- $\frac{15}{64}$ c- $\frac{15}{56}$

2) The following table shows the heights of 100 students in a school :

| Classes | [150;155[ | [155;160[ | [160;165[ | [165;170[ | [170;175] |
|---|---|---|---|---|---|
| Increasing Cumulativ Frequency | 5 | 23 | 65 | 92 | 100 |

**The mean and standard deviation of this distribution are:**
**N.B The answers are given to the nearest** $10^{-2}$ **.**
a- $\bar{x} = 167{,}04$ and $\sigma = 5{,}12$ .
b- $\bar{x} = 163{,}25$ and $\sigma = 4{,}86$ .
c- $\bar{x} = 167{,}04$ and $\sigma = 4{,}86$ .

3- $(C)$ is the representative curve of the function $g$ defined over $IR$ by $g(x) = (x-1)^2$ and $(d)$ is the straight line of equation $y = x+1$ .
**The area of the domain limited by $(C)$ , $(d)$ and the two straight lines of equations $x = -1$ and $x = 3$ is:**

a- $\frac{28}{3}$ square units b- $\frac{38}{6}$ square units

c- $\frac{21}{6}$ square units

***EXERCISE – II***

In the space referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the plane $(P)$ of equation $x+2y-z-1=0$ and the points $A(1,-6,6)$ , $B(2,-4,2)$ and $C(0,0,2)$.

1) a- Show that $2x+y+z-2=0$ is an equation of plane $(Q)$ passing through the points $A$ , $B$ and $C$.
   b- Calculate the acute angle of the two planes $(P)$ and $(Q)$.
2) Determine a system of parametric equations of the straight line $(d)$ intersection of the two planes $(P)$ and $(Q)$.
3) $D(0,0,-1)$ is a point in $(P)$.
   a- Calculate the distance of $D$ to $(Q)$.
   b- Deduce that the circle $(C)$ of plane $(P)$ of center $D$ and radius $R=\sqrt{5}$ cuts $(d)$ in two points $E$ and $F$.
   c- Calculate the area of triangle $DEF$.
4) Let $(R)$ be the mediator plane of $[EF]$.
   a- Determine an equation of $(R)$.
   b- Deduce the coordinates of the point $I$ midpoint of $[EF]$.
   c- Calculate the coordinates of the points $E$ and $F$.

***EXERCISE – III***

***Part A :***

Consider the function $g$ defined over $IR$ by

$$g(x)=\frac{e^x}{1+2e^x}-\ln\left(1+2e^x\right).$$

1) Show that $g$ is strictly decreasing over $IR$.
2) Determine $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to-\infty} g(x)$.
3) Draw the table of variations of $g$ and study the sign of $g$.

***Part B :***

Let $f$ be the function defined over $IR$ by $f(x)=e^{-2x}\ln\left(1+2e^{x}\right)$.
Designate by $(C)$ its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$. Graphical Unit : 2 cm .

1) Show that $f'(x)=2e^{-2x}g(x)$.
2) a- Calculate $\lim\limits_{x\to-\infty} f(x)$.

   b- Calculate $\lim\limits_{x\to+\infty} f(x)$.
3) Draw the table of variations of $f$ .
4) Knowing that $\lim\limits_{x\to-\infty}\dfrac{f(x)}{x}=-\infty$ , trace $(C)$.
5) Let $\alpha$ be a strictly positive real number .

   a- Verify that for all real numbers $x$, $\dfrac{e^{-x}}{1+2e^{x}}=e^{-x}-2\dfrac{e^{-x}}{e^{-x}+2}$.

   Deduce the value of the integral $\displaystyle\int_0^{\alpha}\frac{e^{-x}}{1+2e^{x}}dx$.

   b- Calculate, using integration by parts, the integral $J(\alpha)=\displaystyle\int_0^{\alpha}f(x)dx$ and give a graphical interpretation of $J(\alpha)$.

***Part C:***

Consider the differential equation $(E)$ : $y'+2y=2\dfrac{e^{-x}}{1+2e^{x}}$.

1) Verify that the function $f$ is a solution of $(E)$.
2) Let $y=z+f(x)$.

   a- Form a differential equation $(F)$ satisfied by $z$.

   b- Solve $(F)$ and deduce the general solution of $(E)$.

***EXERCISE – IV***

In the complex plane referred to an orthonormal system $\left(O;\vec{u},\vec{v}\right)$ consider the points $M'$ and $M''$ respective affixes of the complex numbers $z'$ and $z''$ roots of equation $(E)$:

$$z^2-\frac{2}{\cos\theta}z+\frac{5}{\cos^2\theta}-4=0 \quad \text{where } \theta\in\left]-\frac{\pi}{2};\frac{\pi}{2}\right[.$$

1) Solve$(E)$.
2) Designate by $M'$ and $M''$ the points of respective affixes $z'$ and $z''$ the roots of $(E)$.
   a- Show that $M'$ and $M''$ describe a branch of a hyperbola $(H)$.
   b- Trace$(H)$.

***EXERCISE – V***

In an oriented plane, consider a rectangle $ABCD$ such that $AB = 1$, $AD = 2$ and $(\overrightarrow{AB};\overrightarrow{AD}) = \frac{\pi}{2}(\text{mod } 2\pi)$.

Let $M$ be the midpoint of $[BC]$, the straight line $(DM)$ intersects $(AB)$ in $I$.

1) a- Prove that triangle $BIM$ is right isosceles.
   b- Deduce the center and radius of circle $(C)$ circumscribed about triangle $AMI$ and draw $(C)$.
2) $S$ is the similitude of center $O$ such that $S(A) = M$ and $S(B) = D$.
   a- Determine the ratio and an angle of $S$.
   b- Calculate $(\overrightarrow{OA};\overrightarrow{OM})$, $(\overrightarrow{IM};\overrightarrow{IA})$, and $(\overrightarrow{OA};\overrightarrow{OM}) + (\overrightarrow{IM};\overrightarrow{IA})$.
   Deduce that $O$ belongs to circle$(C)$.
3) a- Show that $DM = DO = \sqrt{2}$.
   b- Deduce that $O$ belongs to circle $(C')$ to be determined.
   c- Construct $O$.
4) The plane is referred to a direct orthonormal system$(A;\vec{u},\vec{v})$ with $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \frac{1}{2}\overrightarrow{AD}$.
   a- Calculate the affixes of points $A$, $C$, $D$, $M$ and $I$.
   b- Determine the complex form of $S$ and deduce the affix of its center $O$.
   c- Let $S^{-1}$ be the inverse of S.
   Determine the complex form of $S^{-1}$.

***EXERCISE – VI***

Consider the function $f$ defined over $[0;+\infty[$ by $f(x) = xe^{-x}$.
And designate by $(C)$ its representative curve in an orthonormal

system $(O;\vec{i},\vec{j})$. Graphical Unit : 10 cm .

***Part A:***

1) Construct $(C)$.
2) Show that for all real numbers $m \in \left]0;\frac{1}{e}\right[$, the equation $f(x)=m$ admits two solutions .
3) In the case where $m=\frac{1}{4}$, denote by $\alpha$ and $\beta$ the solutions (with $\alpha<\beta$).Show that $0.35<\alpha<0.36$.
4) Solve the equation $f(x)=m$ for $m=0$ and $m=\frac{1}{e}$.

***Part B:***

1) Consider the sequence $(u_n)$ defined by $u_0=\alpha$ and for all natural numbers $n$ $u_{n+1}=u_n e^{-u_n}$.
   a- Prove that $(u_n)$ is bounded below by 0.
   b- Prove that $(u_n)$is decreasing .
   c- Determine $\lim\limits_{n\to+\infty} u_n$.
2) Consider the sequence $(w_n)$ defined over $IN$ by $w_n=\ln(u_n)$ .
   a- Show that for all natural numbers $n$, $u_n=w_n-w_{n+1}$.
   b- Let $S_n=u_0+u_1+\dots\dots\dots\dots+u_n$, show that $S_n=w_0-w_{n+1}$.
   c- Deduce $\lim\limits_{n\to+\infty} S_n$.
3) Consider the sequence $(v_n)$ defined over $IN$ by its first term $v_0>0$ and by $v_{n+1}=v_n e^{-v_n}$.
   Does there exist a value of $v_0$ different of de $\alpha$ such that for $n\geq 1$, we have $u_n=v_n$ ? If yes find it .

# Sample Test 3

### EXERCISE - I

In the space referred to an orthonormal system $(O;\vec{i};\vec{j};\vec{k})$ consider the plane $(P)$ of equation $x+y+z-1=0$ , the straight line $(d)$ of parametric equations $x=-t-1$, $y=t+5$ and $z=3t+9$ and the point $E(2;2;0)$.

Designate by $H$ the orthogonal projection of $E$ on $(P)$.

1) Verify that $E$ is a point of $(d)$ and determine the point $A$, intersection of $(d)$ and $(P)$.
2) a-Determine an equation of plane $(Q)$ containing $(d)$ and perpendicular to $(P)$.
   b- Deduce a system of parametric equations of the straight line $(AH)$.
3) a- Calculate the coordinates of $H$.
   b- Write a system of parametric equations of the straight line $(d')$ symmetric of $(d)$ with respect to plane $(P)$.
4) Let $(C)$ be the circle of plane $(Q)$, of center $H$ and tangent to the straight line $(d)$ and let $M$ be a variable point of $(C)$.
   Calculate the greatest value of $AM$.

### EXERCISE - II

1) Consider the two functions $f$ and $g$ defined over $IR$ by:
   $f(x)=2\sin(\pi x)$ and $g(x)=2\sin(\pi x-\pi^2)$.
   Designate by $(C)$ and $(C')$ the representative curves of $f$ and $g$ respectively in an orthonormal system $(O;\vec{i},\vec{j})$.
   a- Show that $T=2$ is the period of the function $f$.
   b- Show that $f$ is odd and that the straight line of equation $x=\dfrac{1}{2}$ is an axis of symmetry of $(C)$.
   c- Trace $(C)$ in the interval [ -1 ; 1 ] .
2) Determine a vector of translation that maps $(C)$ onto $(C')$.

3) Deduce the graph of $(C')$ on the same figure of $(C)$.

***EXERCISE - III***

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.

Consider the family of curves $f(x)=\dfrac{2}{m}e^{2x}+x^2$ and designate by $(C_m)$ its representative curve.

Let $A$ be the point of intersection of $(C_m)$ and the axis $y'y$.

1) Show that the tangent to $(C_m)$ at the point $A$ passes through the fixed point $F\left(-\dfrac{1}{2};0\right)$.
2) a- Show that the set of points $S$ of $(C_m)$ where the tangent is parallel to the axis $x'x$ is a parabola $(P)$ of equation $y=x^2-x$.
   b- Trace $(P)$.
   c- Calculate the area of the domain limited by $(P)$ and the axis $x'x$.
3) a- Show that the function $f$ verifies the differential equation: $y''-4y=-4x^2+2$.
   b- Deduce the set of points of inflection of $(C_m)$.

***EXERCISE - IV***

An urn $U_1$ contains **three white balls and two black balls.**

Another urn $U_2$ contains **a white ball and a black ball.**

We draw at random and simultaneously two balls of $U_1$ and one ball of $U_2$ ,we get, then three balls , the drawings are considered to be equiprobable .

1) Designate by $E$ the following event:
   " From the three drawn balls, there are exactly two white balls "
   Show that $p(E)=\dfrac{9}{20}$.
2) Let $X$ be the random variable that to each drawing associates the number of white balls obtained.
   a- Determiner the possible values of $X$.

b- Determine the probability distribution of $X$.

3) Calculate the probability of drawing one and only one white ball of $U_1$ knowing that we have drawn two white balls.

***EXERCISE - V***

***Part A :***

Let $f$ be the function defined over $IR$ by $f(x)=\ln(e^x+e^{-x})$; and designate by $(C)$ its representative curve in an orthonormal system $(O,\vec{i},\vec{j})$.

1) a- Determine the limit of $f$ at $+\infty$.
   b- Show that for all $x$, $f(x)=x+\ln(1+e^{-2x})$.
   c- Deduce that the straight line $(d)$ of equation $y=x$ is an asymptote to $(C)$ at $+\infty$.
   d- Prove that $f$ is odd and deduce the equation of the asymptote $(d')$ to $(C)$ at $-\infty$.
2) Study the relative positions of $(C)$ with respect to $(d)$ and $(d')$.
3) Study the variations of $f$ and trace $(C)$, $(d)$ and $(d')$.

***Part B:***

For all real numbers $x\in[0,+\infty[$, let $F(x)=\int_0^x \ln(1+e^{-2t})dt$.

**It is not required to calculate F(x).**

1) Study the sense of variations of $F$ over $[0,+\infty[$.
2) a- $g$ is the function defined over $[0,+\infty[$ by

$$g(x)=\frac{x}{x+1}-\ln(1+x).$$

   Study the variations of g and deduce that $\frac{x}{1+x}\leq\ln(1+x)$.

   b- Show that for all positive real numbers $x$; $\ln(1+x)\leq x$.
   c- Deduce that for all positive real numbers $x$, we have:

   i- $\int_0^x \frac{e^{-2t}}{1+e^{-2t}}dt\leq F(x)\leq\int_0^x e^{-2t}dt$

   ii- $\frac{1}{2}\ln 2-\frac{1}{2}\ln(1+e^{-2x})\leq F(x)\leq\frac{1}{2}-\frac{1}{2}e^{-2x}$.

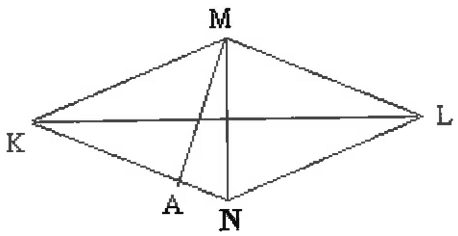
3) For all natural numbers $n$, let $u_n = \int_n^{n+1} \ln(1+e^{-2t})\,dt$.

   a- Prove that: $0 \le u_n \le \ln(1+e^{-2n})$.

   b- Deduce the limit of the sequence $(u_n)$.

***EXERCISE - VI***

$KMLN$ is a rhombus of center $O$.
$A$ is a given point of the segment $[KN]$ distinct of $K$ and $N$.




Let $B$ be the point of intersection of the straight line $(AM)$ and the straight line $(LN)$.The parallel drawn through $A$ to the straight line $(MN)$ cuts the straight line $(KM)$ in $P$ and the parallel through $B$ to the straight line $(MN)$ cuts the straight line $(LM)$ at $Q$.

1) Designate by $h$ the dilation of center $K$ and ratio $\frac{\overline{KA}}{\overline{KN}}$.

   a- Prove that $h(M) = P$.

   b- Deduce that the midpoint $I$ of segment $[AP]$ belongs to the straight line $(KL)$.

2) Find the dilation that helps to prove that $J$ the midpoint of $[BQ]$ belongs to the straight line $(KL)$.

3) Justify that the points $N$, $P$, $Q$ are the symmetrics of the points $M$ , $A$ and $B$ with respect to a straight line to be determined. Deduce that the points $N$, $P$, $Q$ are collinear.

# Sample Test 4

## EXERCISE - I

$HGF$ is an equilateral triangle of side having a length $\ell$ and such that $(\overrightarrow{HF};\overrightarrow{HG}) = \frac{\pi}{3} (\mathrm{mod}\, 2\pi)$. Let $K$ be the midpoint of $[HG]$.

Consider the hyperbola $(H)$ of focus $F$, directrix $(HG)$ and eccentricity 2.

1) a- Determine the vertices S and $S'$ of $(H)$, its center $O$ and the second focus $F'$.

   b- Calculate, in terms of $\ell$, the distance between the two vertices $2a$, and the focal distance $2c$.

2) The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$ where $O$ is the center of $(H)$ and $\vec{i}$ the unit vector of the semi-straight line $[OF[$. Write an equation of $(H)$.

## EXERCISE - II

$f$ is the function defined over $]0;+\infty[$ by $f(x) = \frac{\ln x}{\sqrt{x}} + 1 - x$.

Designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine $\lim\limits_{x\to 0} f(x)$ and deduce an asymptote to $(C)$.

   b- Determine $\lim\limits_{x\to +\infty} f(x)$ and show that the straight line $(d)$ of equation $y = -x+1$ is an asymptote to $(C)$.

   c- Study the position of $(C)$ with respect to $(d)$.

   d- Show that $f'(x)$ has the same sign as $g(x) = -(2x\sqrt{x} - 2x + \ln x)$.

   e- Show that 1 is the only root of the equation $g(x) = 0$ and deduce the sign of $g(x)$.

   f- Study the variations of $f$ and draw its table.

   g- Trace $(C)$.

2) a- Show that $f$ admits an inverse function $h$ for $0 < x \leq 1$.

b- Find the domain of definition of $h$ and trace its representative curve $(C_1)$ in the same system.

c- Calculate, to the nearest $10^{-1}$, the slope of the tangent to $(C_1)$ at the point of ordinate $\frac{1}{e}$.

3) a- Calculate the area $A(\alpha)$ of the domain limited by $(C)$, the axis $x'x$ and the straight line of equation $x=\alpha$ with $0<\alpha<1$.

b- Calculate $\lim\limits_{\alpha\to 0} A(\alpha)$.

4) Define the sequence $(u_n)$ by its first term $u_0 \in ]1;2[$ and for all natural numbers $n$, $u_{n+1} = \frac{\ln(u_n)}{\sqrt{u_n}} + 1$.

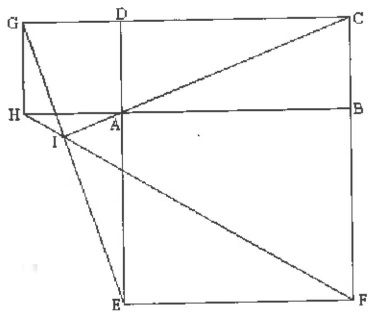
a- Prove that for all real numbers $x \in ]1;2[$; $0 < \frac{\ln x}{\sqrt{x}} < 1$.

b- Prove that for all natural numbers $n$; $u_n \in ]1;2[$.

c- Verify that $u_{n+1} = u_n + f(u_n)$ and deduce the sense of variations of $(u_n)$.

d- Show that $(u_n)$ is convergent to a limit to be determined.

***EXERCISE - III***




In the figure to the right:
$ABCD$ is a rectangle of sides: $AB=4$, $AD=2$ and $(\overrightarrow{AB};\overrightarrow{AD}) = \frac{\pi}{2} (\mathrm{mod}\, 2\pi)$.
$ADGH$ and $ABFE$ are two squares.
The straight lines $(GE)$, $(HF)$ and $(AC)$ intersect at a point $I$.

1) Let $S$ be the similitude that transforms $A$ onto $B$ and $D$ onto $A$. Determine the ratio and angle of $S$

2) The plane is referred to the system $(A;\vec{u},\vec{v})$ such that $\overrightarrow{AB} = 2\vec{u}$.

a- Find the affixes of the points $D$, $B$, $H$ and $E$.

b- Let $J$ be the midpoint of $[HE]$.

Calculate $\frac{z_B - z_D}{z_J}$ and deduce that the straight lines $(BD)$ and $(AJ)$ are perpendicular , denote by $O$ the point of intersection of $(BD)$ and $(AJ)$.

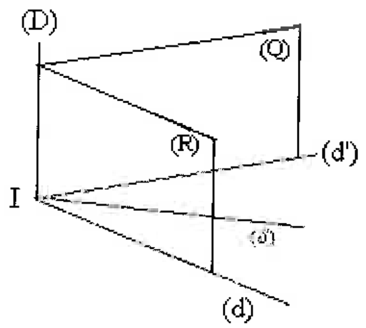
3) a- Determine the image of the straight line $(BD)$ by $S$ then the image of $(AJ)$ by $S$.

b- Deduce the center $\Omega$ of $S$.

4) Find the complex form of $S$ and deduce the affix of its center $\Omega$.

***EXERCISE - VI***

In the space referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the straight lines $(d)$, $(d')$, $(D)$ and $(\delta)$.




The straight line $(D)$ is perpendicular to the plane $(P)$ formed by $(d)$ and $(d')$.

The parametric equations of $(d)$ and $(d')$ are:

$$(d)\begin{cases} x = m-1 \\ y = 2m \\ z = m-1 \end{cases} \qquad (d')\begin{cases} x = -t \\ y = t+2 \\ z = -2t \end{cases}$$

$m$ and $t$ being two real parameters.

1) Find the coordinates of $I$.
2) Show that an equation of plane $(P)$ is: $-5x+y+3z-2=0$.
3) Find a system of parametric equations of $(D)$.

4) Suppose that $(\delta)\begin{cases} x=0 \\ y=3\lambda+2 \\ z=-\lambda \end{cases}$ where $\lambda$ is a real parameter .

a- Show that $(\delta)$ lies in $(P)$.

b- Show that all points of $(\delta)$ are equidistant of $(d)$ and $(d')$ .

c- Determine an equation of the bisector plane of the dihedral determined by $(R)$ and $(Q)$.

***EXERCISE - V***

A big store decides to organize a lottery to attract the attention of its clients.

They use two urns $U$ and $V$ .

Urn $U$ contains 2 white balls and 4 black balls.

Urn $V$ contains 5 identical counters: three counters hold the number 0

A counter holds the number 50 000 .

A counter holds the number 100 000 .

A customer chooses, at random, a ball from urn $U$ .

If the ball drawn is black, he gains nothing.

If the ball drawn is white, he draws simultaneously three counters from urn $V$ .

He, then receives the amount of money identical to the sum of numbers on the counters.

Let $X$ be the random variable that represents the amount of money received by the customer.

1) Determine the set of values of $X$ as well as its probability distribution.
2) Calculate the expected value $E(X)$.

# Sample Test 5

## EXERCISE - I

In the following table, only one of the proposed answers to each question is correct.

Write the number of each question and the corresponding answer with justification.

| N° | Questions | Answers | | |
|---|---|---|---|---|
| | | a | b | c |
| 1 | $C_{10}^{1}+C_{10}^{2}+...+C_{10}^{10}=$ | $2^{10}$ | $2^{10}-1$ | $3^{10}$ |
| 2 | Si $p(A)=0.2$ ; $p(B)=0.5$ $p(A\cup B)=0.6$ then $p(A/\overline{B})=$ | 0.1 | 0.2 | 0.5 |
| 3 | Si $z=1-e^{-i\frac{\pi}{3}}$ alors $\overline{z}=$ | $e^{-i\frac{\pi}{3}}$ | $e^{i\frac{\pi}{3}}$ | $1-\frac{\sqrt{3}}{2}+\frac{1}{2}i$ |
| 4 | $n$ is a non-zero natural number, then: $i^{n}+i^{2n}+i^{3n}+.....+i^{7n}=$ | -1 | 1 | 0 |

## EXERCISE - II

An urn contains one white ball, one red ball and 3 black balls.

1) We draw, at random, one ball from the urn.
   Calculate la probability of the following event: the balls left in the urn have different colors.
2) We draw, successively and without replacement, two balls from the urn.
   Calculate the probability of the following event:
   The balls left in the urn have different colors.

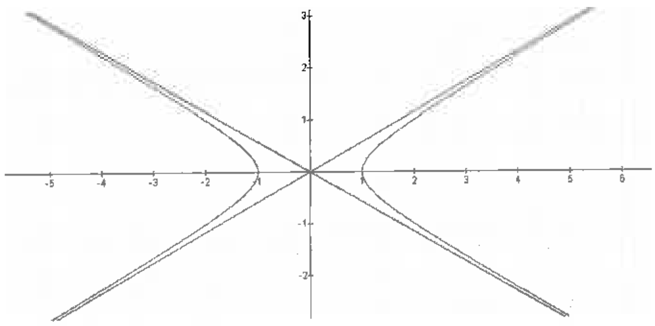
3) We draw, simultaneously, two balls from the urn. Let $X$ be the random variable equal the number of colors left in the urn.
   a- Determine the probability distribution of $X$.
   b- Calculate the expected value of $X$.

***EXERCISE - III***

The curve below is the graphical representation of a hyperbola $(H)$ in an orthonrmal system $(O;\vec{i},\vec{j})$

***Part A :***

1) Prove that an equation of $(H)$ is $x^2-3y^2=1$.
2) Calculate the eccentricity of $(H)$.



***Part B***

1) Calculate the derivative of the function $f$ defined over $]1;+\infty[$ by $f(x)=x\sqrt{x^2-1}-\ln\left(x+\sqrt{x^2-1}\right)$.
2) Calculate the area of the domain $(D)$ limited by $(H)$ and the two straight lines of equations $x=2$ and $x=4$.

***Part C:***

Let $S$ be the similitude that to every point $M(z)$ associates the point $M'(z')$ such that $z'=\left(1+i\sqrt{3}\right)z$.

1) Determine, geometrically, the images of the asymptotes and vertices of $(H)$ by $S$.
1) Let $z=x+iy$ and $z'=x'+iy'$.
   a- Calculate $x$ and $y$ in terms of $x'$ and $y'$.

b- $(G)$ is the image of $(H)$ by $S$, write an equation of $(G)$.

c- Calculate the area of the domain $(D')$ image of $(D)$ by S .

***EXERCISE - IV***

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the function $f$ defined over $IR$ by $f(x)=e^{-x}+x-1$ and the function $g$ defined over $]-\infty;2[$ by $g(x)=-x+1-2\ln(-x+2)$.

Designate by $(C)$ and $(G)$ the representative curves of $f$ and $g$ respectively.

***Part A:***

1) Study the variations of $f$ and draw its table of variations .
2) Deduce the sign of $f(x)$.
3) Find the coordinates of the point $B$ where the tangent is parallel to the straight line of equation $y=-x$.
4) Trace $(C)$.

***Part B:***

Given the similitude $S$ of center $I(0;1)$ and ratio $\sqrt{2}$ and angle $\frac{\pi}{4}$.

1) Write the complex form of $S$.
2) Let $M'$ be the image of a point $M$ by $S$.
   Determine the nature of triangle $IMM'$.
3) $M$ is a point of $(C)$, show that the point $M'$ belongs to $(G)$.
4) Study the variations of $g$.
5) Use $S$ in order to determine the asymptotic direction and the asymptote of $g$.
6) Trace $(G)$.

***EXERCISE - V***

In the space referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$, consider the points $A(1;1;1)$, $B(-1;2;2)$ and $C(0;-1;4)$.

1) Find an equation of plane $(P)$ determined by the points $A$, $B$ and $C$.
2) a- Determine an equation of plane $(Q)$ passing through $A$ and perpendicular to the straight line $(BC)$.

b- Deduce a system of parametric equations of the height $(AH)$ issued from $A$ in triangle $ABC$.

3) a- Determine a system of parametric equations of the straight line $(AC)$.

b- Determine the coordinates of point $H$, orthogonal projection of point $B$ on the straight line $(AC)$.

c- Deduce a system of parametric equations of the height issued from $B$ in triangle $ABC$.

# Sample Test 6

### EXERCISE - I

Answer by true or false with justification:

1) Consider the function $f$ defined over $IR-\{1\}$ by $f(x)=\dfrac{x+1}{x-1}$.
   Then $f$ admits an inverse function $g$ for $x>1$ defined by
   $g(x)=\dfrac{x+1}{x-1}$.

2) Consider the function $f$ defined over $IR^*$ by $f(x)=\dfrac{1}{x}$ and let $g$ be the function defined over $IR-\{1\}$ by $g(x)=\dfrac{x}{x-1}$.
   Then the domain of definition of $g\circ f$ is $IR^*$.

3) $\arccos\left[\cos\dfrac{7\pi}{6}\right]=\dfrac{7\pi}{6}$ .

4) $(P)$ is a plane parallel to the axis $z'z$.
   The vector $\vec{n}(0,\alpha,\beta)$ is normal to $(P)$.

5) The equation $(E)$: $\arcsin x+\arcsin\dfrac{x}{2}=\dfrac{\pi}{4}$ admits solutions for $-1<x<0$ .

### EXERCISE - II

The complex plane is referred to a direct orthonormal system $(O;\vec{u},\vec{v})$.

Consider the points $A$ and $B$ of respective affixes $a=1+i$ and $b=-4-i$.

Let $T$ be the transformation that to every point $M$ of affix $z$ associates the point $M'$ of affix $z'$ such that $\overrightarrow{OM'}=2\overrightarrow{AM}+\overrightarrow{BM}$.

1) Calculate the affixes of points $O'$, $A'$ and $B'$ respective images of points $O$, $A$ and $B$ by $T$.
2) a- Prove that the complex form of $T$ is $z'=3z+2-i$.
   b- Determine the nature of $T$ and precise its elements .
3) Let $r$ be the rotation of center $I(1;1)$ and angle $\dfrac{\pi}{2}$.

a- Find the complex form of $r$.
b- Let $S = T \circ r$.
Find the nature of $S$ and determine its elements.

*EXERCISE - III*

In the space referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$,
Consider the points $A(2;3;1)$, $B(1;2;3)$, $C(1;0;2)$ and $D(0;-3;0)$.

1) Show that an equation of plane $(P)$ passing through the points $B$, $C$ and $D$ is Prove that the straight line $(AB)$ is perpendicular to plane $(P)$.
3) a- Find a system of parametric equations of the straight line $(d)$ passing through the point $A$ and parallel to the straight line $(DC)$.
   b- $M$ is a variable point of $(d)$, calculate $\overrightarrow{BM}.\left(\overrightarrow{BC} \wedge \overrightarrow{BD}\right)$ and that the volume of the tetrahedron $MBCD$ is a constant to be determined.
4) a- Prove that the point $A'(0;1;5)$ is the symmetric of $A$ with respect to plane $(P)$.
   b- Find a system of parametric equations of the straight line $(d')$ symmetric of $(d)$ with respect to plane $(P)$.
5) Prove that $\dfrac{CB}{DB} = \dfrac{\left\|\overrightarrow{CA} \wedge \overrightarrow{CA'}\right\|}{\left\|\overrightarrow{DA} \wedge \overrightarrow{DA'}\right\|}$.

*EXERCISE - IV*

1) $(u_n)$ is the sequence defined by $u_1 = \dfrac{1}{2}$ and by the relation
$u_{n+1} = \dfrac{1}{6}u_n + \dfrac{1}{3}$.
   a- Let $(v_n)$ be the sequence defined for $n \geq 1$ by $v_n = u_n - \dfrac{2}{5}$, prove that $(v_n)$ is a geometric sequence whose first term an ratio are to be determined.
   b- Express $v_n$ then $u_n$ in terms of $n$.

2) Given two dice denoted by $A$ and $B$.
Die $A$ has three red faces and three white faces.
Die $B$ has four red faces and two white faces.
We choose a die at random and we throw it : if we obtain a red face, we keep the same die. If the face obtained is white, we exchange the die, we throw it again and we follow the same procedure.
Consider the following events :
$A_n$: We use die A in the $n^{th}$ throw.
$R_n$: We obtain a red face in the $n^{th}$ throw.
Denote by $a_n$ and $r_n$ be the probabilities of events $A_n$ and $R_n$ respectively.
a- Calculate $a_1$.
b- Calculate $r_1$.
c- Show that $r_n = -\frac{1}{6}a_n + \frac{2}{3}$.
d- Deduce that $a_{n+1} = \frac{1}{6}a_n + \frac{1}{3}$ then determine the expression of $a_n$ in terms of $n$
e- Deduce the expression of $r_n$ in terms of $n$ then the limit of $r_n$ as $n$ tends to $+\infty$.

***EXERCISE - V***

1) Consider the function g defined over $]0;+\infty[$ by $g(x) = 1 - x + 2\ln x$.
Designate by $(C')$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
a- Determine $\lim\limits_{x\to+\infty} g(x)$ and prove that $(C')$ admits an asymptotic direction.
b- Study the variations of g and show that the equation $g(x) = 0$ admits two roots of which one is 1 and the other is $\alpha \in ]3{,}51 ; 3{,}52[$ .
c- Deduce that $g(x) > 0$ for $1 < x < \alpha$.

2) Let $f$ be the function defined over $]0;+\infty[$ by $f(x)=\dfrac{1+2\ln x}{x^2}$.
Designate by $(C)$ its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.
    a- Study the limits of $f$ at 0 and at $+\infty$ and deduce that $(C)$ admits two asymptotes to be determined.
    b- Study the variations of $f$ and draw its table of variations.
    d- Trace $(C)$.
3) Let $(D)$ be the domain limited by $(C)$, the axis $x'x$ and the two straight lines of equations $x=1$ and $x=\alpha$.
    a- Using integration by parts , calculate the area $A(\alpha)$ of domain $(D)$.
    b- Show that $A(\alpha)=2-\dfrac{2}{\alpha}$.
4) Consider the sequence $(I_n)$ , defined for $n\geq 1$ by $I_n=\displaystyle\int_n^{n+1} f(x)dx$.
    a- Prove that $f(x)-\dfrac{1}{x}=\dfrac{g(x)}{x^2}$.
    b- Prove that for $x\geq 4$, : $0<f(x)\leq\dfrac{1}{x}$.
    c- Deduce that $0<I_n\leq\ln\left(\dfrac{n+1}{n}\right)$ , and calculate $\lim\limits_{n\to+\infty} I_n$.
    d- Let $S_n=I_1+I_2+I_3+..............+I_n$.
    Calculate $S_n$ then $\lim\limits_{n\to+\infty} S_n$.

*Published by :*

*DAR AL AMAL*

*Beirut / Tripoli – Lebanon.*

*Typed and Designed by:*

*Souraya Darwich*

*Printed and Packed by:*

*Chemaly and Chemaly*