Physics Guide

GRAVITY

Life & General Sciences

Book 1

Mechanics & Electricity

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Third Year Secondary
Scientific Section
Life & General Sciences (LS & GS)

Book 1
Mechanics & Electricity

Hassan Kamar

For My Little Angel

First edition
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Preface

Physics is one of the most interesting sciences that is strongly connected to our daily life. Studying it becomes more interesting when both students and teachers attempt to interpret and analyze physical phenomena. One of the most common complaints of students of physics is that they cannot see the purpose or the real life applications of some physical concepts. For example, it is interesting to know that the seismometer make use of simple harmonic oscillator to detect seism and Earthquakes, the capacitor is used in emergency rooms to the revitalize patients experiencing cardiac arrest and used in camera's flash and also as a humidity sensor, the emission of light by lamps is due to dis-excitation of the atom, the radioactivity is used to date fossils and bones and also as pacemaker to help control abnormal heart rhythms, the new generation of power plant produces energy by nuclear fission and the promising resource of nuclear,...

This guide is designed for grade 12 Sciences students for both scientific sections Life and General Sciences (LS & GS), and it is divided into two books each one two units and 18 chapters accompanied with set of physics sessions for 2015, 2016 & 2017.

Each chapter is divided into four parts; the first is the essentials which is formed of a brief summary of the chapter, second "Applications" which is a set of easy and medium difficulty exercises that focus on independent concepts; the third is the "Problems" that target a most general view that examines whether the concepts are acquired, this part provides most comprehensive view of the chapter. The fourth entitled "Supplementary Problems" is formed of practice problems and the set of sessions exercises proposed before 2005 accompanied with key answer while the answers will be available on the Website **"emti7anat.com"**. Finally, the section titled "Sessions"; it is formed of set of sessions problems that targets the considered topic provided with detailed solutions. This guide is also accompanied with set of appendixes at the end of each book with some important notes, units and geometrical properties. Notice that, when you find questions **shaded** then they are devoted for **GS** students only.

Special thanks go to a group of talented students who create challenge, excitement and motivation in me to design a new and diverse set of problems. While others trigger my ability to adapt tools and perspective for an easier and clearer presentation.

I am thankful for supporting friend **Mr. Ali Al Moussawi**, and **Mr. Hassan Darwish**, for their notes and revisions. I am also grateful for my wife **Marwa Fakher Eldine Kamar** for her patience.

Finally, I hope that this guide will eliminate many of the difficulties experienced while studying physics and further provide an interestingly new perspective for students from where they can better view physics and apply physical principles.

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List of some of the action verbs and their requirements

This list does not include all the action verbs that could be used in the formulation of questions.

1. **Analyze:** Decompose a whole into its constituent elements to make evident the variations.
2. **Calculate**: (Compute) Perform mathematical operations.
3. **State:** Express written without explaining.
4. **Compare:** Indicate the similarities and/or differences between two or more entities.
5. **Complete**: Add what is missing.
6. **Conclude:** Reach to a decision.
7. **Determine:** Reach to a decision or a result through logical reasoning, calculation...
8. **Describe:** Express, using scientific language, to give the details of an observation, an experiment, a schema, an apparatus...
9. **Show:** Prove something is evident by logical reasoning, experimenting, calculating...
10. **Deduce:** Draw using logical reasoning new information from given or existing information.
11. **Draw out:** Draw from a set of given a relation, a role, a law ... without reasoning.
12. **Distinguish:** Recognize or discern one thing from another according to particular traits.
13. **Explain**: Clarify, make understandable a phenomenon, a result...
14. **Identify:** Recognize something based on its characteristics or its properties.
15. **Interpret:** Analyze and give significance to the result.
16. **Indicate:** Designate or state something without justification.
17. **Justify:** Prove something as true and real.
18. **Specify:** Indicate and justify.
19. **Pick out (Extract):** Select one or more information from a document.
20. **Verify:** Confirm using arguments, logical reasoning....whether something is true or false.

The international system of units is based on seven fundamental units

The value of a quantity is generally expressed as the product of a number and a unit.

Base Quantity		SI base unit	
Name	**Symbol**	**Name**	**Unit**
length	l, x, r, ...	metre	m
mass	M	kilogram	kg
time, duration	t	second	s
electric current	I, i	ampere	A
temperature	T	kelvin	K
amount of substance	N	mole	Mol
luminous intensity	I_v	candela	cd

The <u>SI base units</u> are a choice of seven well-defined units which by convention are regarded as dimensionally independent:

1. The metre is the length of the path travelled by light in vacuum during a time interval of 1/299792458 of a second.

2. The kilogram is the unit of mass; it is equal to the mass of the international prototype of the kilogram.

3. The second is the duration of 9 192 631 770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.

4. The ampere is that constant current which, if maintained in two straight parallel conductors of infinite length, of negligible circular cross-section, and placed 1 metre apart in vacuum, would produce between these conductors a force equal to 2×10^{-7} newton per metre of length

5. The kelvin, unit of thermodynamic temperature, is the fraction 1/273.16 of the thermodynamic temperature of the triple point of water.

6. The mole is the amount of substance of a system which contains as many elementary entities as there are atoms in 0.012 kilogram of carbon 12; its symbol is "mol".

7. The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency 540×10^{12} hertz and that has a radiant intensity in that direction of 1/683 watt per steradian

<u>http://www.bipm.org/en/measurement-units/</u>

Unit I

Mechanics

Chapter 1

Mechanical Energy

LS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Mechanical Energy		-	-	-	-	-	-	2nd	2nd
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Mechanical Energy	-	-	-	-	1st	-	2nd	-	-

Essentials

I- Energy

A system possesses energy if it is able to perform work.

1. Translational Kinetic energy «KE»

The kinetic energy of a body is the energy due to its motion.

The kinetic energy of a particle (taken as a system) of mass m moving with a speed v is $KE = \frac{1}{2} m v^2$

	SI Units	Subunits
Mass m	kilogram (kg)	$1g = 10^{-3} kg$
Speed v	Meter per second m/s	$\frac{(km/h)}{3.6} = m/s$
Kinetic energy KE	Joules J	$J = kg.m^2/s^2$

2. Potential energy «PE»

It is the energy that the system stores and it has many forms (gravitational, elastic, chemical).

a) Gravitational potential energy «GPE»

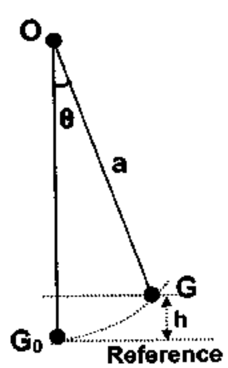
The gravitational potential energy of the system (body, Earth) where m is the mass of the body, whose center is at an altitude h, with respect to a horizontal plane taken as the reference is given by $GPE = \pm m g h$.

body above reference	$GPE = +m g h$
body on reference	$GPE = 0 \ (h = 0)$
body below reference	$GPE = -m g h$

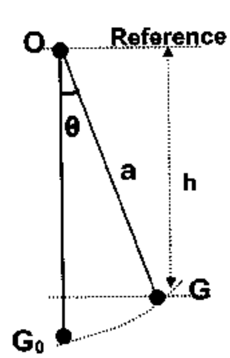
Notes:

✖ For a **simple pendulum** of length ℓ, to calculate the height we use $\cos\theta$ where θ is the angle that makes the pendulum at an instant t, with respect to the equilibrium position.

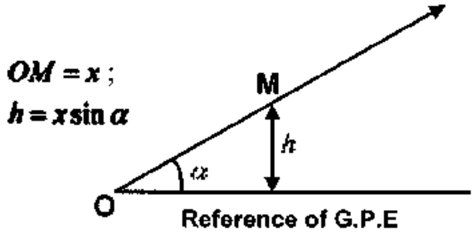
If the reference is the horizontal plane passing through equilibrium position G_0, then $h = \ell(1 - \cos\theta)$.

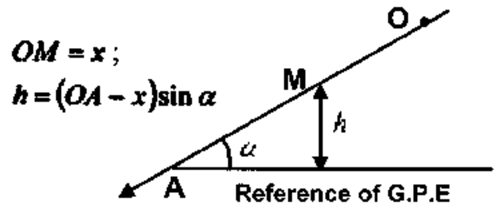


If the reference is the horizontal plane passing through the support O, then $h = -\ell \cos\theta$.



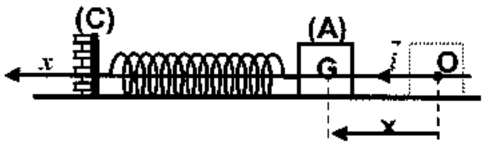
✖ For a body on a ramp (inclined plane), to calculate the height usually we use $\sin\alpha$ (where α is the angle that makes this plane with the horizontal).





b) Elastic potential energy «PEₑ»

The elastic potential energy stored in the system (spring, box) whose constant (stiffness) k, shifted (stretched or compressed) by a distance x with respect to its free end (equilibrium position), is $PE_e = \frac{1}{2}k\,x^2$. (The tension in the spring is given by Hooke's law $T = -k\,x$).



3. Mechanical energy «ME»

The mechanical energy of a system (body, spring, Earth) is the sum of its macroscopic kinetic and potential energies $ME = KE + PE = KE + PE_e + GPE$.

If the system is energetically isolated (no exchange with the surrounding, no friction…), then its mechanical energy is conserved.

4. Microscopic energy (Internal energy) «U»

The internal energy U of a system is the sum of the microscopic kinetic (vibrations and translations of atoms inside the matter) and potential (binding energy between the particles) energies.

5. Total energy «E_{total}»

The total energy of a system is the sum its mechanical and internal energies $E_{\text{total}} = ME + U$.

✖ If no friction nor chemical reactions take place within an energy-isolated system, then both its internal and mechanical energy are conserved (U and ME are constants).

✖ If the system is energetically isolated, then E_{total} remains constant and $\Delta(ME) = -\Delta U$.

Notes:

If the change in the mechanical energy is only due to a force of friction of constant magnitude f, so $\Delta(ME) = W_{\vec{f}} = -f \times d$ where d is the distance traveled.

The instantaneous expression $\frac{d(ME)}{dt} = P_{\vec{f}}$ where $P_{\vec{f}} = \vec{f}.\vec{v}$ (power due to the force of friction).

Work energy-theorem states that the variation in the kinetic energy of a system between two given instants is equal to the sum of work of all external forces acting on it $\Delta(KE) = \sum W_{\overrightarrow{F_{ext}}}$

II-GS

Energy & Rotation

1. Rotational kinetic energy «KE»

The rotational kinetic energy of a body rotating around a fixed axis with an angular velocity θ' (rad/s) and having I as moment of inertia with respect to this axis, is given by $KE = \frac{1}{2} I \theta'^2$.

Note:

The moment of inertia I of a point particle of mass m, at a distance d from an axis (usually taken as axis of rotation), is given by $I = m d^2$.

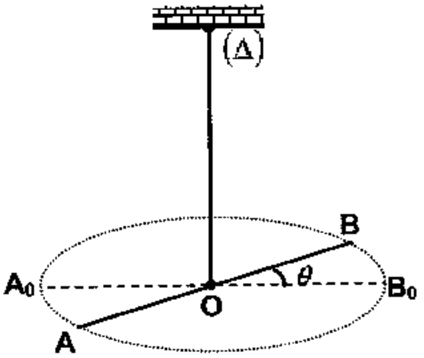
The moment of inertia for a disc of mass m & radius R, that rotates around an axis (Δ) passing through its center and perpendicular to its plane is $I_{\text{disc}} = \frac{1}{2} m R^2$ & for a hoop $I_{\text{hoop}} = m R^2$.

The moment of inertia is additive, for many particles or systems $I_{\text{total}} = I_1 + I_2 \dots$

2. Elastic potential energy of torsion «PEe»

The elastic potential energy stored in a torsion wire whose constant C twisted by an angle θ (rad), with respect to the equilibrium position, is $PE_e = \frac{1}{2} C \theta^2$.

(the moment of the restoring couple in the torsion wire, $\mathrm{M} = -C\theta$).



Note:
Analogy between physical quantities

Spring	Torsion wire
x	θ
k	C
$T = -k x$	$\mathrm{M} = -C\theta$

In rotational dynamics the work done by a force of moment M over an angle θ is $W_{\vec{f}} = \mathrm{M} \times \theta$ & the power $P_{\vec{f}} = \mathrm{M} \times \theta'$.

III-

Center of Gravity

The position of the center of gravity G with a respect to an origin O, for a system of many particles A, B, ... of respective masses m_A, m_B, ... is given by: $a = \overline{OG} = \dfrac{m_A \overline{OA} + m_B \overline{OB} + \dots}{m_A + m_B + \dots}$.

Applications

I- Kinetic Energy of a Particle

A point particle (P), of mass $m = 500\,g$, in rectilinear motion along an axis $(O;\vec{i})$. (P) is at the origin O at an instant taken as origin of time $t_0 = 0$ and its position at an instant t, is defined by its abscissa $x = OM$ as shown in figure 1.

Figure 1

The curve shown in figure 2, represents the variation of the kinetic energy of (P) in terms of x.

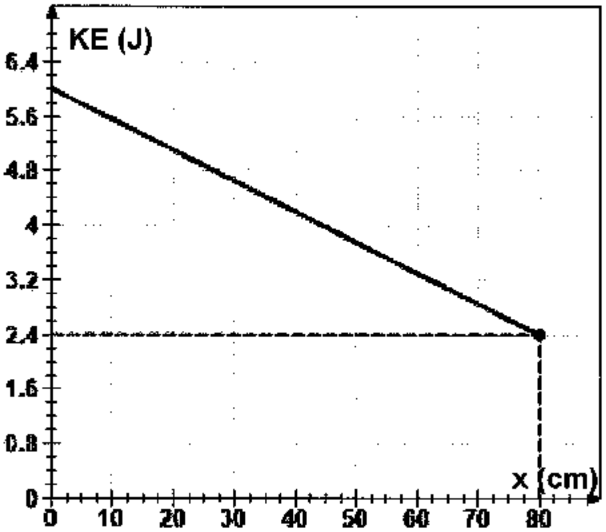


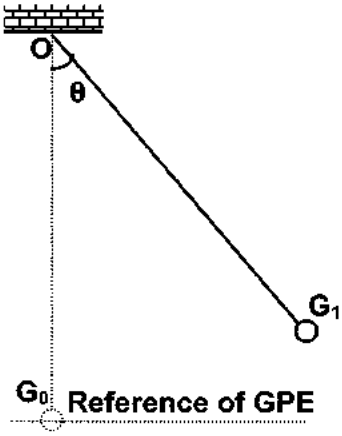
Figure 2

1. Justify that the expression of the kinetic energy KE can be written in the form $KE = Ax + B$ where A & B are constants whose values and units to be determined.
2. Calculate, in km/h, the speed of this particle after covering a distance of $40\,cm$.
3. a) Determine the acceleration of (P).
   b) Deduce the expression of the velocity in terms of time.
   c) Determine the instant at which this particle comes to rest.

II- Simple Pendulum

A simple pendulum of length $\ell = 40\,cm$ carrying by one of its ends a point particle of mass $m = 200\,g$ while to the other end is connected to a fixed horizontal support as shown in the adjacent figure. This pendulum is shifted by an angle $\theta_m = 60^0$ and then released at an instant t taken as origin of time $t_0 = 0$.

The horizontal plane passing through the equilibrium position G_0 as a reference for the gravitational potential energy for the system (pendulum, Earth) and $g = 10\,m/s^2$.



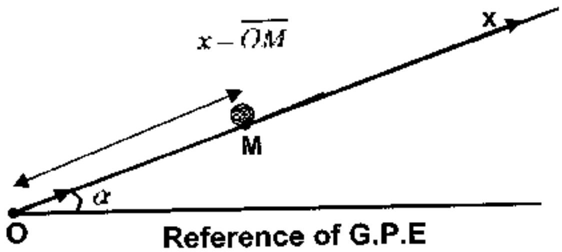
1. Calculate the gravitational potential energy at G_1 & G_0.
2. At an instant t, the position M of the pendulum is defined by the angle θ.
   a) Show the expression of the gravitational potential energy at M is given by $GPE = A + B\cos\theta$ where A and B are constants whose values are to be determined.

b) Plot the graph representing the variations of GPE in terms of θ for $\theta \in [0;60^0]$.

c) Determine the position at which at which GPE is equal to half its value at $t=0$.

III- Conservation of Mechanical Energy

A point particle (P) of mass $m = 45\,g$ is launched with initial velocity $\overrightarrow{v_0}$ of magnitude $v_0 = 6.4\ m/s$ from the bottom O, of a ramp making an angle α with the horizontal such that $\sin\alpha = 0.8$, and slides without friction.



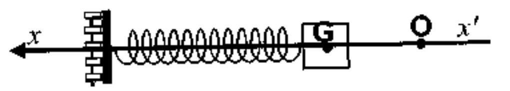
We study the motion of this particle with respect to the axis $(O;\vec{i})$, and the particle is referred at any instant t by its abscissa $OM = x$. **Take:** $g = 10\,m.s^{-2}$.

The horizontal plane passing through O is taken as reference of gravitational potential energy for the system [particle, Earth].

1. Calculate the mechanical energy ME of the system [particle, Earth].
2. Determine the maximum height reached by (P) on the ramp.
3. Determine the expression of the gravitational potential energy at M in terms of x.
4. Deduce the position at which the gravitational potential energy of (P) is twice its kinetic energy.

IV- Elastic Potential Energy

Consider a solid (S) of mass $m = 200\,g$ and whose center of inertia is G and a spring of negligible mass of un-jointed turns whose stiffness is $k = 80\,N/m$.



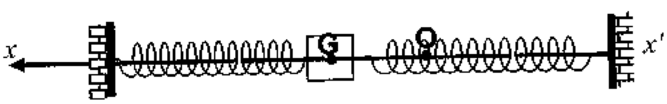
(S) may slide on a horizontal rail; the position of G on the horizontal axis $(O;\vec{i})$ is defined by its abscissa $x = \overline{OG}$ where O is the equilibrium position. The horizontal plane through G is taken as a gravitational potential energy reference. Suppose that the forces of friction are negligible.

At the instant $t_0 = 0$, G initially at O, is launched with a velocity $\overrightarrow{v_0} = v_0\,\vec{i}$ consequently the spring undergoes a maximum compression of $3.2\,cm$.

1. Calculate the mechanical energy ME of the system ((S), Spring, Earth).
2. Deduce the value v_0.
3. a) Determine the expression of the kinetic energy KE of (S) in terms of the abscissa for $x \in [0;3.2cm]$.

   b) Deduce the position at which the kinetic energy is equal to that of its potential elastic energy.

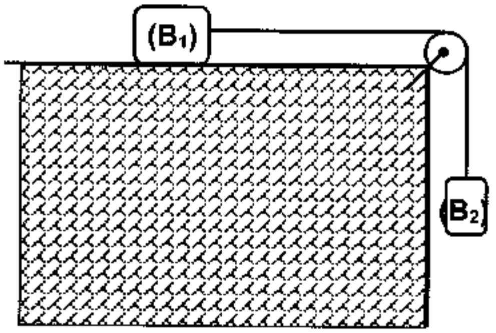
   c) Draw the curve representing the variations of ME & KE in terms of abscissa for $x \in [0;3.2cm]$
4. Repeat questions 1 & 2 if the same solid is connected to two identical spring each of constant $k = 80\,N/m$.



V-

Dynamics of a System

Consider the system shown in the adjacent figure.



- ✖ The block (B_1) of mass $m_1 = 1.8kg$.
- ✖ The block (B_2) of mass $m_2 = 400\,g$ is moving vertically.
- ✖ The pulley (P) is massless.

The string connecting the blocks being massless, inextensible and does not slide on the groove of the pulley.

We designate by (S) the system formed of $[(B_1), (B_2), (P)]$.

Take: $g = 10\,m/s^2$ and neglect friction.

The horizontal plane passing through the center of gravity of (B_1) is taken as reference for the gravitational potential energy for the system $[(S)$, Earth$]$.

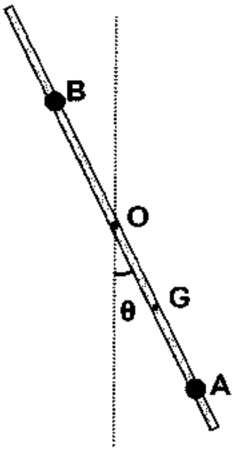
At an instant considered as origin of time $t_0 = 0$, (S) is released from rest. (B_2) is at a height h from reference.

1. Determine, at $t_0 = 0$, the expression of the mechanical energy of (S) in terms of m_2, g and h.
2. Applying the principle of conservation of mechanical energy, show that the speed of the blocks after traveling a distance x is given by $v^2 = \dfrac{2m_2}{m_1 + m_2} gx$.
3. Calculate the speed of (B_1) after traveling $80cm$.

VI-

Center of Gravity of a System (S)

A system (S) is formed of a uniform rod of mass $m = 100\,g$ and length $\ell = 15cm$ and two points particles A & B placed on the rod as shown in the adjacent figure. The system thus formed rotates in a vertical plane around a horizontal axis passing through O.



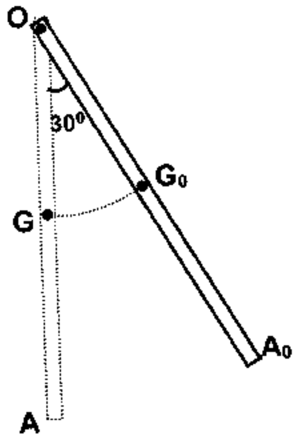
- ✖ The moment of inertia of the rod with respect to O is: $I = \dfrac{m\,\ell^2}{12}$.
- ✖ A: $m_A = 20\,g$ & $OA = 5\,cm$; B: $m_B = 10\,g$ & $OB = 4\,cm$.

1. Show that, with respect to O, the position of the center of gravity G of the system (S) is $a = OG = \dfrac{6}{13}\,cm$.
2. Calculate the gravitational potential energy of the system with respect to a horizontal plane passing through the axis of rotation at O $GPE_t = \sum GPE = GPE_A + GPE_B + GPE_{rod}$ and compare it to that of the center of gravity for an angular deviation of 30^0 with respect to the vertical.
3. Calculate the moment of inertia of the system $I_t = \sum I_i = I_A + I_B + I_{rod}$ and compare it to that of the center of gravity.

4. The system (S) is shifted by an angle $\theta_0 = 20^0$ with respect to the equilibrium position. The forces of friction are supposed negligible. The horizontal plane passing through G when it passes through the equilibrium position is considered as reference for the gravitational potential energy.
   a) Calculate the value of the mechanical energy of the system [(S), Earth].
   b) Determine the angular velocity when it passes through the equilibrium position.
   c) Deduce the speed of A at this position.

VII- Mechanical Energy and Rotation

A rod (OA) of mass $m = 250\ g$ and length $\ell = 40cm$ free to rotate without friction around an axis passing through its extremity O. The rod is shifted from the equilibrium position by an angle θ_0 such that $\cos\theta_0 = 0.8$ and then released with an initial velocity $\theta'_0 = 6$ rad/s. The moment of inertia of the rod with respect to an axis passing through O is $I = \dfrac{m\ell^2}{3}$.

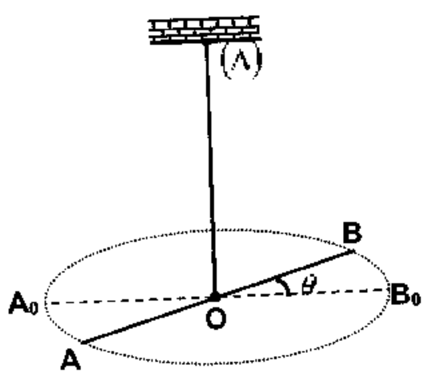


The horizontal plane passing through O is taken as reference for the system [rod, Earth].

1. Calculate the mechanical energy of the system [rod; Earth].
2. Determine the angular velocity of the rod when it passes through the equilibrium position.
3. Determine the expression of the kinetic energy of the rod in terms of the elongation θ.
4. Draw the curve representing the variation of the mechanical & kinetic energies in terms of θ.

VIII- Torsion Wires

A disc of mass $m = 300\ g$ and radius $R = 8\ cm$, is connected to the free end of vertical wire (Δ), whose constant of torsion is C, while its other extremity is fixed a horizontal support as shown in the adjacent figure. The moment of inertia of the disc with respect to (Δ) is $I_{(\Delta)} = (1/2)mR^2$.



The horizontal plane passing through the disc is taken as reference of gravitational energy of the system (S) [disc, wire, Earth].

The disc is then twisted in the horizontal plane by an angle $\theta_m = 15^0$ from the equilibrium position and then released without initial velocity. Its angular velocity when it passes through the equilibrium position is $2\text{rad}/s$.

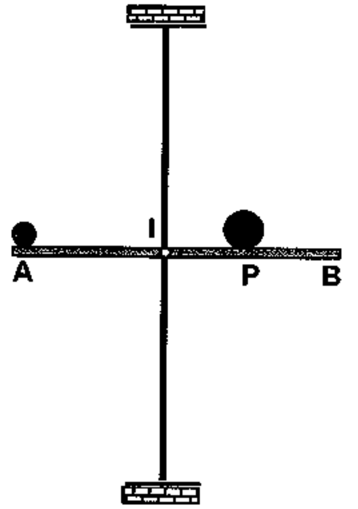
The forces of frictions are supposed to be negligible.

1. Calculate the value of the mechanical energy of (S).
2. Deduce the value of the constant of torsion C.
3. Draw on two separate graphs the shape of the curves representing the elastic potential energy in terms of θ and then in terms θ^2.
4. At any instant t, the position of the diameter AB is referred by the angle θ that makes with the equilibrium position and its angular velocity is θ'.
   a) Justify that θ and θ' are related by a relation of the form $\theta'^2 = A\theta^2 + B$ where A and B are constants whose values are to be determined.

b) Deduce the angular velocity of the diameter when it makes an angle of 10^0 with the equilibrium position.

IX- Two Torsion Wires

Consider a system (S) formed of a rod (AB) of length ℓ and negligible mass. We fix two point particles: on the extremity A a mass m and on P situated at a distance $x = IP$ another mass $3m$ where I is the midpoint of the rod.



1. The center of gravity of the system formed is at I.

   Show that $x = \dfrac{\ell}{6}$.
2. Determine the expression of the moment of inertia of the system thus formed in terms of m & ℓ.
3. The system formed placed horizontally is connected to two identical vertical torsion wires of constant $C = 0.4\,SI$ as shown in the adjacent figure and then twisted by an angle $\theta_0 = 0.8\,\text{rad}$ and then released from rest.

   Take: $m = 200\,g$ & $\ell = 50cm$.

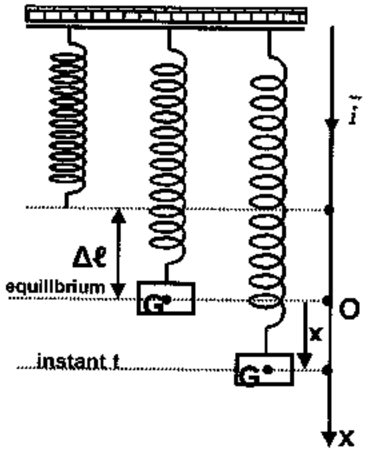
   Neglect friction and take the horizontal plane passing through the rod as reference of gravitational potential energy for the system [particles, rod, wires, Earth].

   a) Determine the mechanical energy of the system thus formed.

   b) Deduce the angular velocity of the rod when it passes through the equilibrium position.
4. Show that the kinetic energy of the system can be written $KE = A - C\,\theta^2$ where A is a constant.
5. Draw the curve representing the variations of KE in terms of $\theta \in [-0.8\,\text{rad}; +0.8\,\text{rad}]$.

X- Vertical Spring

We consider a spring of stiffness $k = 100\,N/m$ and of negligible mass, connected from its upper end to a fixed support while its other end carries a solid (S) of mass $m = 800\,g$. At equilibrium the center of mass G of (S) coincides with a point O taken as origin of an abscissa axis $(O;\vec{i})$ and the spring elongates by $\Delta\ell$ (adjacent figure).



The spring is stretched by pulling (S) vertically downwards from its equilibrium position, then releasing it without initial velocity at instant $t_0 = 0$. G oscillates around its equilibrium position O. At an instant t, G is defined by its abscissa $x = \overline{OG}$ and the algebraic value of its velocity is v. The horizontal plane passing through O is taken as a reference of gravitational potential energy.

1. Applying the equilibrium condition, determine $\Delta\ell$.
2. From the equilibrium positon, (S) is given an initial velocity $\vec{v_0} = 50\,\vec{i}\,(cm/s)$.

   a) Calculate the mechanical energy of the system thus formed.

   b) Determine the position at which (S) comes to rest.

Solutions

I-

1. The curve representing the variation of the kinetic energy KE in terms of abscissa x is a straight line then its equation is of the form $KE = Ax + B$;

   where $B = KE|_{x=0} = 6J$ & $A = \dfrac{\Delta(KE)}{\Delta x} = \dfrac{(2.4-6)J}{(80-0)\times 10^{-2}\,m} = -4.5J/m$;

   Then $KE = -4.5\,x + 6$ where x in m and KE in J .

2. For $x = 40cm = 0.4m$; we have $KE = -4.5\times 0.4 + 6 = 4.2J$;

   But $KE = \dfrac{1}{2} m v^2$, so $v = \sqrt{\dfrac{2\times 4.2}{0.5}} \approx 4.1m/s$; then $v = 4.1\times 3.6 = 14.76\,km/h$.

3. a) We have $KE = \dfrac{1}{2} m v^2 = -4.5\,x + 6$;

   Deriving both sides with respect to time, we get: $-4.5\,x' + 0 = \dfrac{1}{2} m(2vv')$; $-4.5\,x' = 0.5\times v v'$

   But $v = x' \neq 0$ (system is in motion) & $a = v'$;

   Thus $a = \dfrac{-4.5}{0.5} = -9\,m/s^2$.

   b) The acceleration is constant and negative then the motion is uniformly decelerated, so

   $v = a\,t + v_0$; but $KE_0 = \dfrac{1}{2} m v_0^2 = 6J$, then $v_0 = \sqrt{\dfrac{6\times 2}{0.5}} = 2\sqrt{6}\,m/s$;

   Thus $v = -9t + 2\sqrt{6}$ where t in s and v in m/s .

   c) The particle comes to rest, so $v = -9t + 2\sqrt{6} = 0$, then $t = \dfrac{2\sqrt{6}}{9} \approx 0.54s$.

II-

1. We have $GPE_1 = m\,g\,h_1 = m\,g\,\ell(1-\cos 60^0)$;

   So $GPE_1 = 0.2\times 10\times 0.4(1-\cos 60^0) = 0.4J$.

   & $GPE_0 = 0$ (on reference).

2. a) $GPE = m\,g\,\ell(1-\cos\theta) = 0.8 - 0.8\cos\theta$;

   GPE is of the form $GPE = A + B\cos\theta$

   where $A = 0.8J$ & $B = -0.8J$.

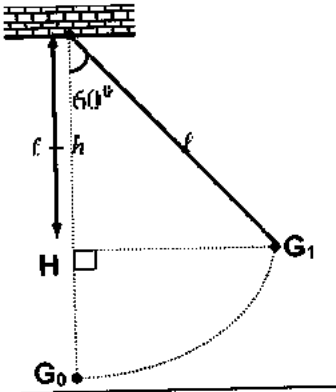
   b) Scales: On abscissa axis $1\,div \equiv 10^0$;

   On ordinate axis $1\,div \equiv 0.1J$;

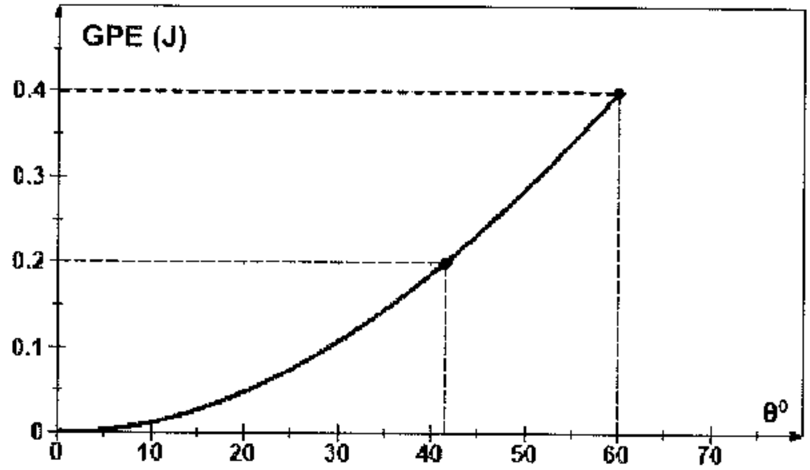
   c) We have $GPE = 0.8 - 0.8\cos\theta = 0.2$;

   $\cos\theta = \dfrac{0.6}{0.8} = \dfrac{3}{4}$;

   Then $\theta = \cos^{-1}(0.75) = 41.4^0$.



$$\cos\theta = \frac{\ell - h}{\ell},\ h = \ell(1-\cos\theta)$$



III-

1. The system slides without friction, then its mechanical energy is conserved:

$ME = ME|_{x=0} = KE|_{x=0} + GPE|_{x=0}$ but $GPE|_{x=0} = 0$ (on reference)

$ME = \frac{1}{2} m v_0^2 + 0 = \frac{1}{2}(0.045) \times 6.4^2 = 0.9216\,J$.

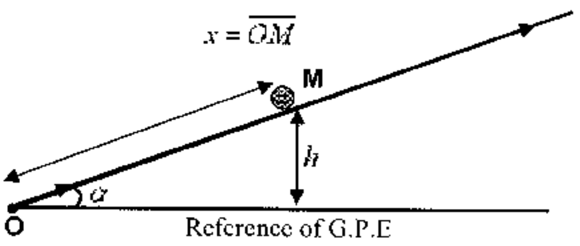
2. Conservation of the mechanical energy:

$ME|_{h_{max}} = 0.9216\,J$;

$KE|_{h_{max}} + GPE|_{h_{max}} = 0.9216\,J$;

But $KE|_{h_{max}} = 0$; so $m\,g\,h_{max} = 0.9216\,J$

$h_{max} = \frac{0.9216}{0.045 \times 10} = 2.048\,m = 204.8\,cm$.



3. At any point of abscissa x, we have $GPE = m\,g\,h$; but $\sin\alpha = \frac{h}{x}$, so $h = x\sin\alpha$;

Then $GPE = m\,g\,x\sin\alpha = 0.045 \times 10 \times x \times 0.8 = 0.36x$ (where x in m & GPE in J)

4. $GPE = 2KE$; but $KE = ME - GPE$; so $0.36\,x = 2(0.9216 - 0.36\,x)$; thus $x = 1.71\,m = 171\,cm$.

IV-

1. In the absence of the forces of friction, the mechanical energy of the system (solid, Spring, Earth) is conserved: $ME = ME|_{x_{max}} = PE_e|_{x_{max}} + KE|_{x_{max}}$;

But $KE|_{x_{max}} = 0$ (it comes to rest at maximum compression);

Then $ME = \frac{1}{2} k\,x_{max}^2 + 0 = \frac{1}{2} \times 80 \times (3.2 \times 10^{-2})^2 = 0.041\,J$.

2. Conservation of mechanical energy $ME = ME|_{x=0} = PE_e|_{x=0} + KE|_{x=0}$ but $PE_e|_{x=0} = 0$;

So $\frac{1}{2} m v_0^2 = 0.041$, then $v_0 = \sqrt{\frac{2 \times 0.041}{0.2}} = 0.64\,m/s$.

3. a) Conservation of mechanical energy $ME = ME|_x = PE_e|_x + KE|_x = 0.041\,J$;

Then $KE = ME - PE_e|_x = 0.041 - 40x^2$ where x in m & KE in J.

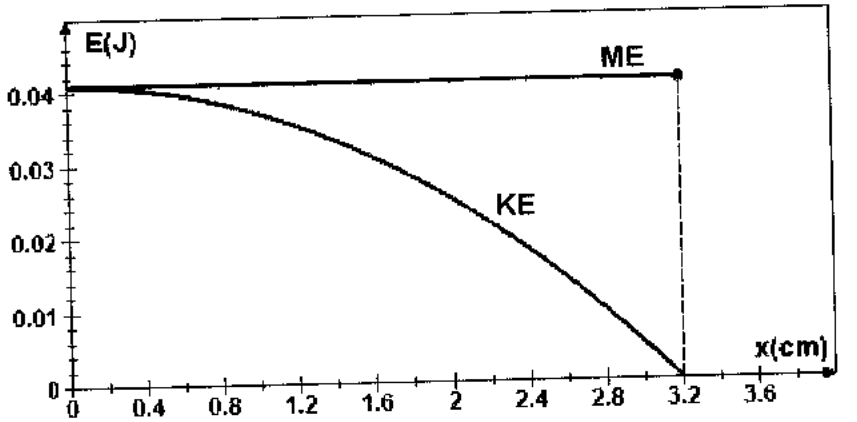
b) We have $KE = PE_e$, $0.041 - 40x^2 = 40x^2$; so $x = \sqrt{\frac{0.041}{80}} = 0.023m = 2.3cm$.

c) We have $ME - 0.041J$ is constant then its graphical representation is a horizontal straight line;

& $KE = 0.041 - 40x^2$ has a parabolic shape which concaves downwards.

Scales: On the abscissa axis $1\,div \equiv 0.4cm$;

On the ordinate axis $1\,div \equiv 0.01J$.



4. We have : $ME = ME|_{x_{max}} = PE_{e1}|_{x_{max}} + PE_{e2}|_{x_{max}} + KE|_{x_{max}}$;

(but $KE|_{x_{max}} = 0$ it comes to rest at maximum compression);

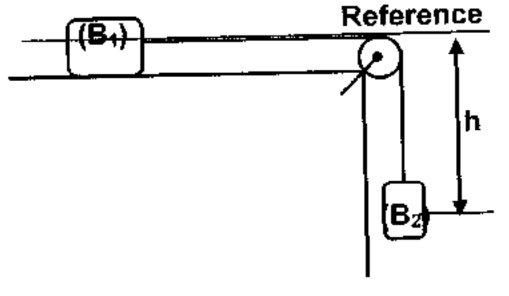
Then $ME = \frac{1}{2}k\,x_{max}^2 + \frac{1}{2}k\,x_{max}^2 + 0 = k\,x_{max}^2 = 80 \times (3.2 \times 10^{-2})^2 = 0.082J$.

Conservation of mechanical energy $ME = ME|_{x=0} = PE_{e1}|_{x=0} + PE_{e2}|_{x=0} + KE|_{x=0}$

So $\frac{1}{2}m\,v_0^2 = 0.082$, then $v_0 = \sqrt{\frac{2 \times 0.082}{0.2}} = 0.91\,m/s$.

V-

1. The pulley and the string are massless then mechanical energy of the system (S) is that of (B_1) and (B_2).



At $t = 0$, the system is released from rest so its kinetic energy is zero.

$ME|_{t=0} = KE|_{t=0} + GPE|_{t=0} = GPE_{(B_1)} + GPE_{(B_2)}$;

But $GPE_{(B_1)} = 0$ (on reference);

Then $ME|_{t=0} = -m_2\,g\,h$.

2. After traveling a distance x, $ME|_x = KE|_x + GPE|_x$;

The string does not slide then the two blocks have the same speed.

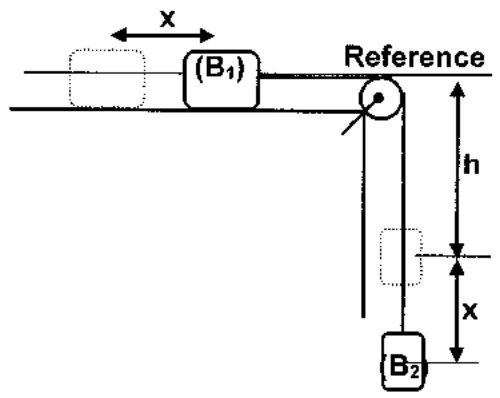
$KE|_x = \frac{1}{2}m_1\,v^2 + \frac{1}{2}m_2 v^2$; and $GPE|_x = GPE_{(B_1)} + GPE_{(B_2)} = 0 - m_2 g(h + x)$.

Then $ME|_x = \frac{1}{2}(m_1 + m_2)v^2 - m_2 g(h + x)$;

In the absence of friction, the mechanical energy is conserved, so $ME|_x = ME|_{t=0}$;

So, $\frac{1}{2}(m_1 + m_2)v^2 - m_2 g(h + x) = -m_2 gh$;

$\frac{1}{2}(m_1 + m_2)v^2 = m_2\, g\, x$; then $v^2 = \frac{2\,m_2}{m_1 + m_2} g\, x$.



3. If $x = 0.8m$, we get:

$$v = \sqrt{\frac{2\,m_2}{m_1 + m_2} g\, x} = \sqrt{\frac{2 \times 0.4}{1.8 + 0.4} \times 10 \times 0.8} = 1.7\, m/s.$$

VI-

1. The position of the center of gravity is:

$$a = \overline{OG} = \frac{m_A \overline{OA} + m_B \overline{OB} + m_r \overline{OG_r}}{m_A + m_B + m_r} = \frac{20 \times 5 - 10 \times 4}{100 + 20 + 10} = \frac{60}{130} = \frac{6}{13} cm.$$

2. $GPE_t = \sum GPE = GPE_A + GPE_B + GPE_{\text{rod}} = -m_A\, g\, h_A + m_B\, g\, h_B + m_r g\, h_r$;

$= -m_A\, g\ OA\ \sin\alpha + m_B\, g\ OB\ \sin\alpha + 0$.

$= -20 \times 10^{-3} \times 10 \times 5 \times 10^{-2} \sin 30^0 + 10 \times 10^{-3} \times 10 \times 4 \times 10^{-2} \sin 30^0 = -3 \times 10^{-3} J$.

$GPE_g = -m_t\, g\, a \sin\alpha = -130 \times 10^{-3} \times 10 \times \frac{6}{13} \times 10^{-2} \times \frac{1}{2} = -3 \times 10^{-3} J$.

Thus, $GPE_g = \sum GPE_i$.

3. The moment of inertia of the system $I_S = \sum I_i = I_{rod} + I_A + I_B = \frac{m\ell^2}{12} + m_A OA^2 + m_B OB^2$;

$$I_S = 100 \times 10^{-3} \times \frac{(15 \times 10^{-2})^2}{12} + 20 \times 10^{-3} \times (5 \times 10^{-2})^2 + 10 \times 10^{-3} \times (4 \times 10^{-2})^2;$$

We get $I_S = 2.535 \times 10^{-4} kg.m^2$.

The moment of inertia of the center of gravity

$I_G = m_t\, OG^2 = 130 \times 10^{-3} \times \left(\frac{6}{13} \times 10^{-2}\right)^2 = 2.77 \times 10^{-6} kg.m^2$, thus in general $I_G \neq \sum I_i$.

4. a) The following diagram is a simplified representation of the system.

In the absence of the forces of friction, the mechanical energy is conserved.

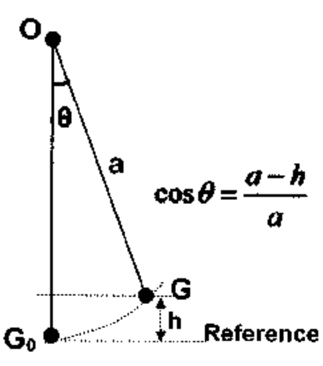
$ME = ME|_{\theta=20^0} = KE|_{\theta=20^0} + GPE|_{\theta=20^0}$;

(but $KE|_{\theta=20^0} = 0$ released from rest);

Then $ME = m_t\, g\, a\,(1 - \cos(20^0))$.

$ME = 130 \times 10^{-3} \times 10 \times \frac{6}{13} \times 10^{-2}\,(1 - \cos(20^0))$;

We get $ME = 3.62 \times 10^{-4} J$.



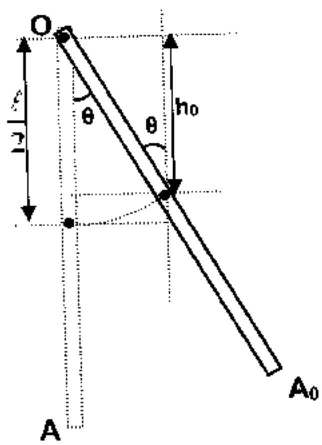
b) Conservation of mechanical energy:

$ME = KE|_{\theta=0} + GPE|_{\theta=0} = 3.62\times10^{-4}\,J$, (but $GPE|_{\theta=0}=0$ on reference);

So $\frac{1}{2}I\,\theta'^2 = 3.62\times10^{-4}\,J$; then $\theta' = \sqrt{\frac{2\times3.62\times10^{-4}}{2.535\times10^{-4}}} \approx 1.7\ rad/s$.

c) The speed of A is $v_A = OA\times\theta' = \frac{\ell}{2}\times\theta' = \frac{15}{2}\times10^{-2}\times1.7 = 0.13\ m/s$.

VII-



1. In the absence of friction, the mechanical energy is conserved:

so $ME = ME|_{\theta_0} = KE|_{\theta_0} + GPE|_{\theta_0}$;

But $KE|_{\theta_0} = \frac{1}{2}I\,\theta'^2 = \frac{1}{2}\times\frac{0.25(0.4)^2}{12}\times6^2 = 0.03\,J$.

$GPE|_{\theta_0=0} = -m\,g\,h_0$ where $h_0 = \frac{\ell}{2}\cos\theta_0$;

Then $GPE = -m\,g\,\frac{\ell}{2}\cos(\theta_0) = -0.25\times10\times\frac{0.4}{2}\times0.8 = -0.4J$;

So, $ME = -0.4 + 0.03 = -0.37\,J$.

2. Conservation of mechanical energy

$ME|_{\theta=0} = KE|_{\theta=0} + GPE|_{\theta=0} = -0.37J$;

$\frac{1}{2}I\,\theta'^2 - m\,g\,\frac{\ell}{2} = -0.37\,J$;

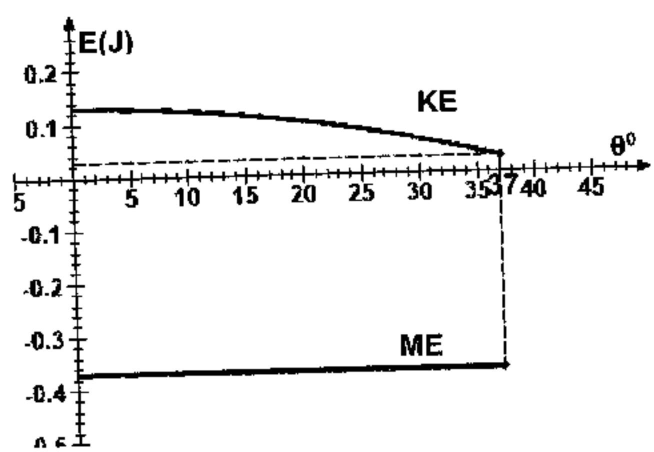
$\frac{1}{2}\frac{0.25(0.4)^2}{12}\theta'^2 - 0.5 = -0.24J$;

Then $\theta' = 12.5$ rad/s.

3. At any instant $KE = ME - GPE$;

$KE = -0.37 - \left(-m\,g\,\frac{\ell}{2}\times\cos\theta\right)$;

$KE = -0.37 + 0.5\cos\theta$ where θ in degree and KE in J.



4. Graphs.

The initial angle $\theta_0 = \cos^{-1}(0.8) \approx 37^0$.

VIII-

1. In the absence of friction, the mechanical energy is conserved $ME = ME|_{\theta=0} = KE|_{\theta=0} + PE_e|_{\theta=0}$

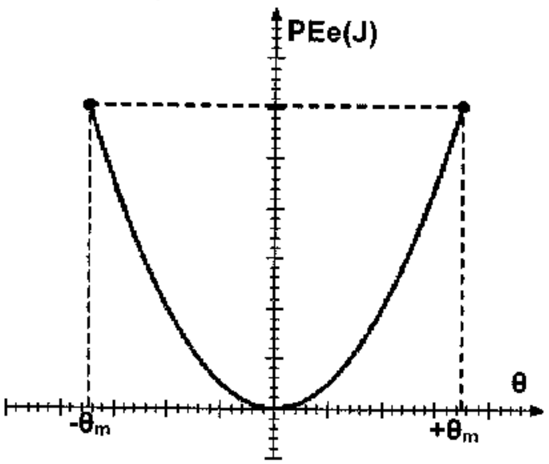
But $PE_e|_{\theta=0} = 0$, so $ME = KE|_{\theta=0} = \frac{1}{2}I\,\theta_0'^2 = \frac{1}{2}\times\frac{1}{2}m\,R^2\,\theta_0'^2 = \frac{0.3\times0.08^2\times2^2}{4} = 1.92\times10^{-3}\,J$.

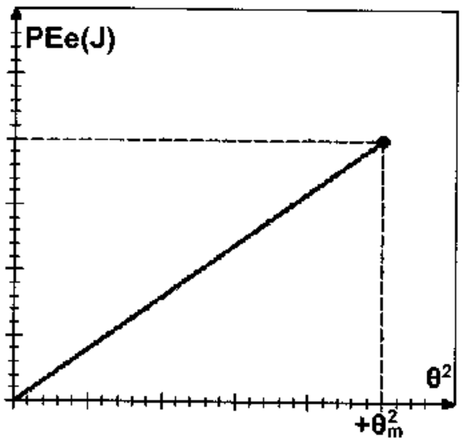
2. Conservation of mechanical energy: $ME = ME|_{\theta_m} = KE|_{\theta_m} + PE_e|_{\theta_m}$;

But $KE|_{\theta_m} = 0$, so $PE_e|_{\theta_m} = \frac{1}{2}C\,\theta_m^2 = 1.92\times10^{-3}\,J$; then $C = \frac{2\times1.92\times10^{-3}}{\left(\frac{15\times\pi}{180}\right)^2} = 0.056\ SI$.

3. We have $PE_e = \frac{1}{2}C\theta^2$;

The curve representing PE_e in terms of θ has a parabolic shape & that of $PE_e = f(\theta^2)$ is an increasing straight line.





4. a) Conservation of mechanical energy: $ME = KE|_\theta + PE_e|_\theta$; $\frac{1}{2}C\,\theta^2 + \frac{1}{2}I\,\theta'^2 = \frac{1}{2}I\,\theta_0'^2$;

We get $\theta'^2 = \theta_0'^2 - \frac{C}{I}\theta^2$ & $I = \frac{0.3\times 0.08^2}{2} = 9.6\times 10^{-4}\ kg.m^2$;

Then $\theta'^2 = -58.3\theta^2 + 4$ is of the form $\theta'^2 = A\ \theta^2 + B$ with $A = -58.3\,s^{-2}$ & $B = 4(rad/s)^2$.

b) We have $|\theta'| = \sqrt{-58.3\theta^2 + 4} = \sqrt{-58.3\times\left(\frac{10\times\pi}{180}\right)^2 + 4} = 1.49\ \text{rad}/s$.

IX-

1. The position of the center of gravity with respect to an origin taken at I is given by:

$$x_I = \frac{m_A\,x_A + m_P\,x_P}{m_A + m_P} = \frac{m\left(-\frac{\ell}{2}\right) + 3m(x)}{m + 3m} = 0\text{, then } m\left(-\frac{\ell}{2}\right) + 3m(x) = 0\text{; thus } x = \frac{\ell}{6}.$$

2. The moment of inertia of the system is $I = I_A + I_P = m\left(\frac{\ell}{2}\right)^2 + 3m\left(\frac{\ell}{6}\right)^2 = \frac{12}{36}m\ell^2 = \frac{m\ell^2}{3}$.

3. a) The mechanical energy of the system is conserved: $ME = ME|_{\theta_0} = KE|_{\theta_0} + PE_{e1}|_{\theta_0} + PE_{e2}|_{\theta_0}$;

(But $KE|_{\theta_0} = 0$, released from rest), then $ME = \frac{1}{2}C\,\theta_0^2 + \frac{1}{2}C\,\theta_0^2 = C\,\theta_0^2$;

We get $ME = 0.4\times 0.8^2 = 0.256\,J$.

b) Conservation of mechanical energy:

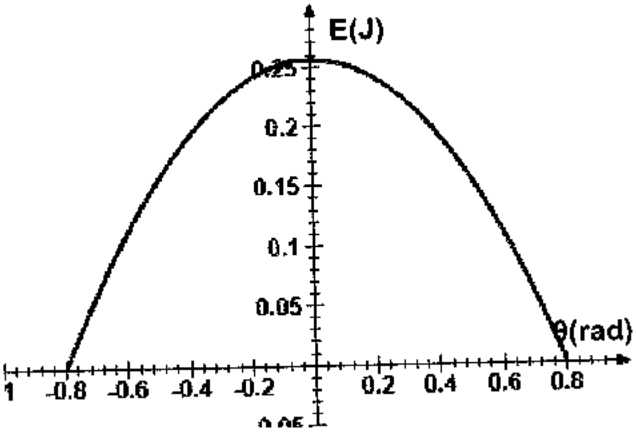
$ME = ME|_{\theta=0} = KE|_{\theta=0} + PE_e|_{\theta=0}$; $ME = \frac{1}{2}I\,\theta'^2 = 0.256\,J$.

We get $\theta' = \sqrt{\frac{2\times 0.256\times 3}{0.2\times 0.5^2}} = 5.54\ \text{rad}/s$.

4. For an elongation θ, we have $ME = KE + C\,\theta^2$;

Then $KE = 0.256 - 0.4\theta^2$ where θ in rad & KE in J.

5. Graph.



X-

1. The solid is subjected to two forces: its weight $\vec{w}$ & the tension $\vec{T_0}$;

   The solid is in equilibrium $\vec{w} + \vec{T_0} = \vec{0}$, then $m\,g = k\,\Delta\ell$;

   Thus $\Delta\ell = \dfrac{m\,g}{k} = \dfrac{0.8 \times 10}{100} = 0.08m = 8\,cm$.

2. a) The mechanical energy is conserved: $ME = ME|_{x=0} = KE|_{x=0} + PE_e|_{x=0} + GPE|_{x=0}$;

   But $GPE|_{x=0} = 0$ (on reference),

   Then $ME = \dfrac{1}{2} m\,v_0^2 + \dfrac{1}{2} k\,(\Delta\ell)^2 = \dfrac{1}{2} \times 0.8 \times 0.5^2 + \dfrac{1}{2} \times 100 \times 0.08^2 = 0.42\,J$.

   b) Let x_m be the position where (S) comes to rest: $ME|_{x_m} = KE|_{x_m} + PE_e|_{x_m} + GPE|_{x_m}$;

   $0 + \dfrac{1}{2} k\,(x_m + \Delta\ell)^2 - m\,g\,x_m = 0.42$; then $50\,x_m^2 + 12\,x_m - 0.1 = 0$;

   Thus, $x_m = 8.1 \times 10^{-3}\,m$.

Problems

I-

Energy Exchange and Horizontal Spring

A point particle (M) of mass $m = 100\,g$, is released ***without initial velocity*** from the top A, of a frictionless inclined plane making an angle $\alpha = 30^0$ with the horizontal, and of length $AB = 2.5\,m$.

In the horizontal plane passing through B, a spring of un-joined loops (S) is fixed, having a constant $k = 100\,N/m$, is fixed from one extremity to an obstacle, as shown in figure 1; while to the other extremity a mass-less plate (P) is attached to the plate can slide, without friction, along a horizontal axis $x'x$ of origin O.

The horizontal plane passing through B is taken as reference of the gravitational potential energy for the system ((M), Earth).

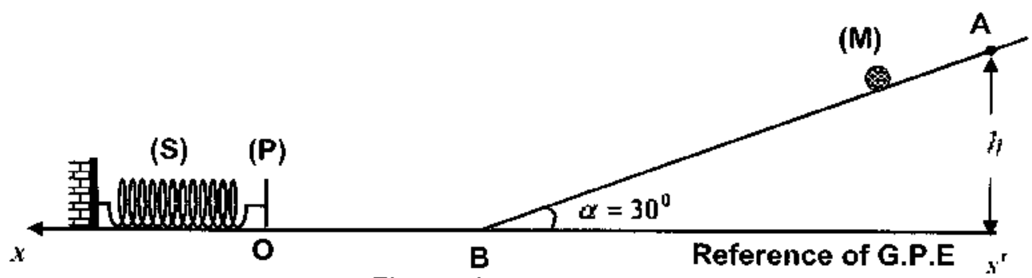


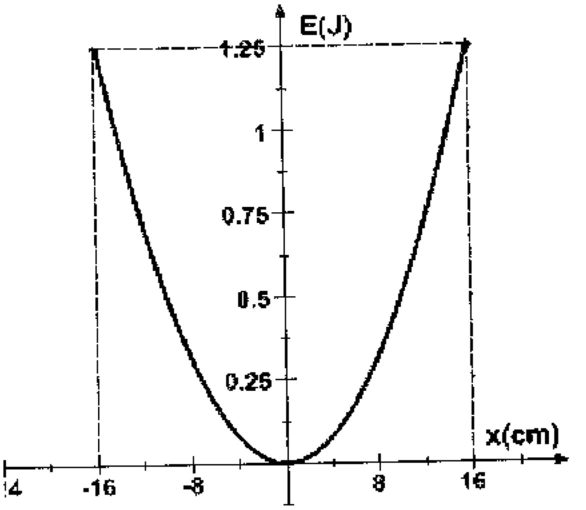
Figure 1

1. a) Calculate the mechanical energy of the system (Particle, Earth).
   b) Show that the speed of the ball when it reaches B is $v_B = 5\,m/s$.
2. Verify that the particle reaches the plate with the speed $v_0 = v_B$.
3. If the forces of friction are not negligible, the particle (M) reaches the point B with a speed of $v'_B = 3.60\,m/s$.
   Calculate the magnitude of the force of friction assumed constant along AB.

In the absence of friction, the particle (M) reaches the plate (P), with the speed $v_0 = 5\,m/s$ it is then stuck into it and the spring is set in motion, the position of the system (Plate, Particle) at any instant t is referred by the abscissa x (the distance that separates it from the equilibrium position O) and its speed is v.

4. a) Determine the mechanical energy of the system (Plate, Particle, Spring).
   b) Calculate the maximum compression of the spring x_m.
   c) Determine the position in which the kinetic energy is equal to the elastic potential energy.
5. a) Determine at an instant t, the expression of the mechanical energy of the system as a function of x, v, m & k.
   b) Justify that $\frac{d(ME)}{dt} = 0$.
   Deduce that the abscissa x of the system (Plate, Particle) satisfies the relation $x'' + w_0^2 x = 0$, where w_0 is a constant whose expression is to be determined as a function of m & k.

6. The adjacent figure represents the variations of the elastic potential energy of the system (Spring, Particle, Plate) as a function of the abscissa x.
   a) Justify graphically the value of the constant of spring (stiffness) k.
   b) Verify that the expression of the kinetic energy is $KE = 1.25 - 50x^2$ where x in meters and KE in joules.
   c) Construct on the same diagram the graphs that represent the variations of the mechanical and the kinetic energies of this system.



II- Energetic Study

A puck (A) of mass $m = 400g$ fixed to one end of a massless spring of unjointed turns and of stiffness $k = 10N/m$; the other end of the spring is attached to a fixed support (C) figure 1.

(A) slides on a horizontal rail and its center of inertia G can move on a horizontal axis $x'ox$. At equilibrium, G coincides with the origin O of the axis. At an instant t the position of G is defined, on the axis $(O;\vec{i})$, by its abscissa $x = \overline{OG}$; its velocity $\vec{v} = v\vec{i}$.

The horizontal plane through G is taken as reference level for the gravitational potential energy. The puck (A) is given at O, an initial velocity $\vec{v_0} = v_0\vec{i}$ where $v_0 = 0.4m/s$.

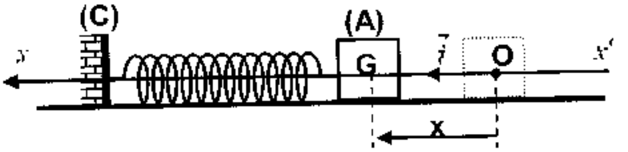


Figure 1

Part A
Theoretical study

In this part, we suppose that the puck slides without friction.

1. Calculate the mechanical energy ME of the system [(A), spring, Earth].
2. Deduce that the maximum compression of the spring is $8cm$.
3. a) Show that the expression of the kinetic energy at an instant t is given by $KE = a + bx^2$ where a and b are two constants whose values and units are to be determined.
   b) Determine the position at which the kinetic energy of the puck is equal to its elastic potential energy.
   c) Plot the curves representing the variations of the mechanical, kinetic and potential energies for $x \in [0; 8cm]$. Take as a scale:
   On the abscissa axis 1div $\equiv 2cm$ & on the ordinate axis 1div $\equiv 4mJ$.

Part B
Real motion

In reality, the maximum compression of the spring is only $6.4cm$.

1. Specify the form of energy which is not affected by the force of friction.

2. Determine the magnitude of the force of friction supposed constant.
3. Show that the expression of the mechanical energy at any instant t is given by $ME' = 32 - 180x$ (x in m and ME' in mJ).
4. Plot, using the same previous scale, the graphs representing the variations of the mechanical energy ME' and elastic potential energy in terms of the abscissa x.
5. Determine the speed with which G returns to the equilibrium position.

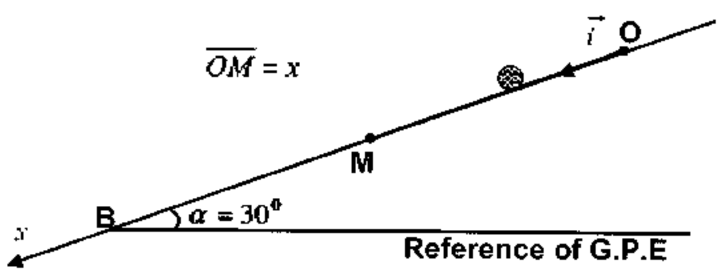
III-

Graphical Study of Energy

A point particle (P) of mass $m = 400\,g$ is released, without initial velocity from a point O, on an inclined plane making an angle $\alpha = 30^0$ with the horizontal. (P) slides without friction along the line of the greatest slope whose length $OB = \ell = 4\,m$.

The particle (P) is referred at any instant t, by its abscissa $OM = x$, with respect to the axis $(O;\vec{i})$ as shown in figure below. **Take:** $g = 10\,m.s^{-2}$.

The horizontal plane passing through B is taken as reference of gravitational potential energy for the system [Particle, Earth].



Part A

Absence of friction

1. Calculate the mechanical energy ME of the system [Particle, Earth].
2. Determine the speed of the particle upon reaching B.
3. Show that the expression of the gravitational potential energy, in a point whose abscissa x is given by: $GPE = 8 - 2x$ (x in m & GPE in J)
   a) Deduce the expression of the kinetic energy KE in terms of x.
   b) Deduce that the acceleration a is equal to $a = 5\,m.s^{-2}$.
4. Draw, on the same graph, the curves that represent the variations of ME, GPE & KE as a function of the abscissa x.

Part B

Presence of friction

In reality the forces of friction are not negligible but they are equivalent to a force whose direction is opposite to that of motion and having a magnitude $f = 0.5\,N$.

1. Which form of energy is not affected by the force of friction? Justify.
2. Show that the expression of the mechanical energy as a function of the abscissa x is $ME|_x = 8 - 0.5x$ (x in m & ME in J)

3. Deduce the expression of the kinetic energy as a function of x.
4. Draw, on the same graph, the curves that represent the variations of ME, GPE & KE as a function of the abscissa x. Use as scales:
   - ✗ on the abscissa axis: $1\text{div} \equiv 0.5m$.
   - ✗ on the ordinate (energy) axis: $1\text{div} \equiv 1J$;
5. Knowing that $\frac{d(ME)}{dt} = P_{\vec{f}}$, where $P_{\vec{f}}$ is the instantaneous power dissipated by the force of friction.
   Determine the new acceleration of motion and then compare it to the result found in Part A.

IV-
Motion of a Particle

We consider a marble (M) of mass $m = 0.65kg$ that is free to move on a semi-circular rail of radius $R = 50cm$ as shown in figure 1.

(M) is shifted from the vertical equilibrium position by an angle $\theta_m = 90^0$ and then released without initial velocity at an instant considered as origin of time $t_0 = 0$.

This marble slides without friction.

The horizontal plane passing through B is taken as reference of gravitational potential energy for the system [(M), Earth].

The position of (M) is referred by its angular abscissa θ which represents the angle that makes with the vertical. **Take:** $g = 10m/s^2$.

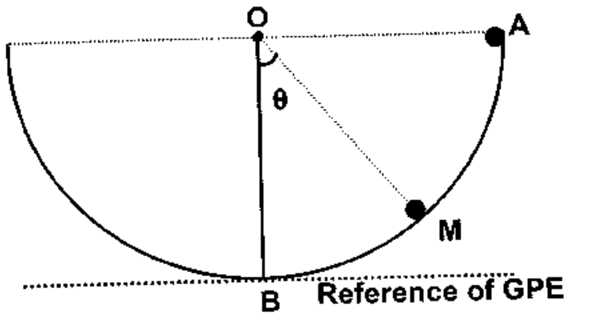


Figure 1

1. Calculate the mechanical energy of the system [(M), Earth].
2. Deduce the speed of this marble when it passes through B.
3. a) Determine the expression of the gravitational potential energy GPE at an instant t in terms of the angular abscissa θ.
   b) Deduce the speed of this particle when it makes an angle $\theta = 30^0$ with the vertical.
   c) Determine the position at which the kinetic of (M) is equal to its gravitational potential energy.
4. The curve of figure 2 represents the variation of an energy of the oscillator as a function of the angular abscissa θ.
   a) What form of energy is it? Justify.
   b) The energy of figure 2 is one of two terms of the mechanical energy ME of the system (marble, Earth). Redraw figure 2 and show on it the curve representing the variations of the mechanical energy ME and that of the other form of energy.

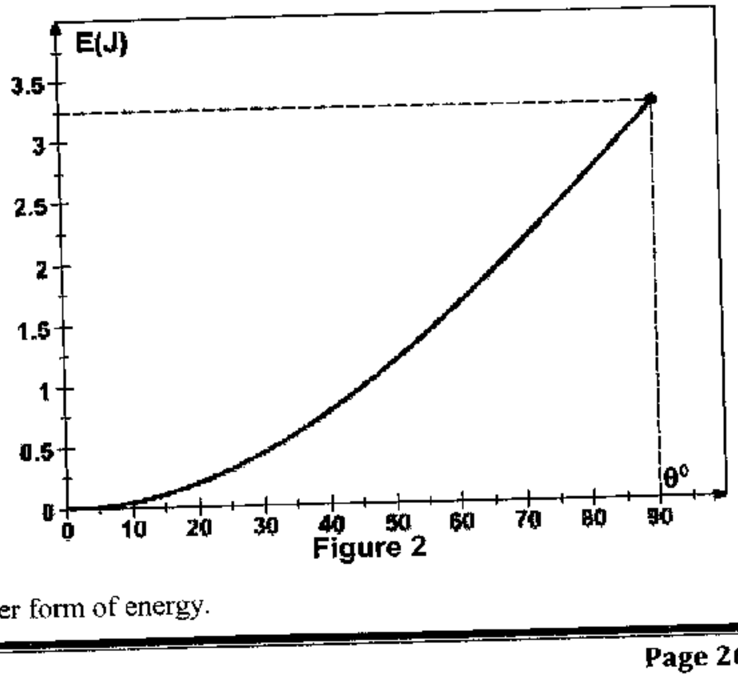
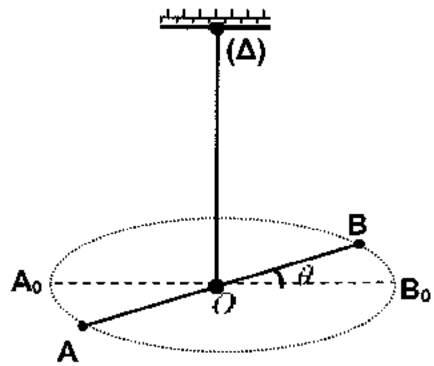


Figure 2

5. In reality, the forces of friction are not negligible and the marble reaches the equilibrium position with a speed of $v_1 = 2m/s$.
   a) Calculate the variations in the mechanical energy of the system (marble, Earth) between the points A and B.
   b) Deduce the magnitude of the force of friction supposed constant acting on the marble.

V-
Torsion Pendulum

A disc of mass $m = 20\,g$ and radius $R = 10\,cm$, is fixed to the free end of vertical wire (Δ), whose constant of torsion is $C = 0.024\,mN/\text{rad}$, while its other extremity is fixed a horizontal support.



The moment of inertia of the disc with respect to (Δ) is $I_{(\Delta)} = \frac{1}{2} m R^2$.

Two identical point particles A & B of respective masses $m_A = m_B = m = 20\,g$ are diametrically fixed on the circumference of the disc as indicated in the figure above.

Consider the horizontal plane passing through the disc as reference of gravitational energy of the system (disc, wire, Earth).

The forces of frictions are supposed to be negligible.

Under the action of a motive couple applied on the circumference at A_0 & B_0, the disc is then twisted by an angle of $\theta_m = 20^0$, then released without initial velocity.

1. a) Determine the expression of the moment of inertia I for the system (disc, wire, Earth) as a function of m & R and then verify that its value is $5 \times 10^{-4}\,kg.m^2$.
   b) Show that the numerical value of the mechanical energy of the system (disc, wire, Earth) is $1.46\,mJ$.
2. Determine the angular velocity of the disc when:
   a) it returns to the equilibrium position.
   b) it makes an angle $\theta_1 = 10^0$ with the equilibrium position.
3. At any instant t, the position of the diameter AB is referred by the angle θ that makes with the equilibrium position and its angular velocity is θ'.
   a) Determine the expression of the mechanical energy ME of the system (disc, wire, Earth) as a function of C, θ, θ', m & R.
   b) Justify that $\frac{d(ME)}{dt} = 0$, then show that the elongation θ satisfy the relation $\theta'' + \frac{2C}{5mR^2}\theta = 0$ where θ'' is the second derivative (angular acceleration) at an instant t.
   c) Deduce that the angular velocity is maximum when the diameter passes through the equilibrium position.
   d) Energetic Study:
      i- Show that the expression of the kinetic energy is given by: $KE = 1.46 \times 10^{-3} - 0.012\,\theta^2$ where θ in rad & KE in J.

ii- Construct, on the same graph, the curves that represent the variations of elastic potential energy PE_e, ME and the kinetic energy KE versus the elongation θ taking as scale:

On abscissa axis $1\text{div} \equiv 5^0$ & on ordinate axis $1\text{div} \equiv 0.2\,mJ$.

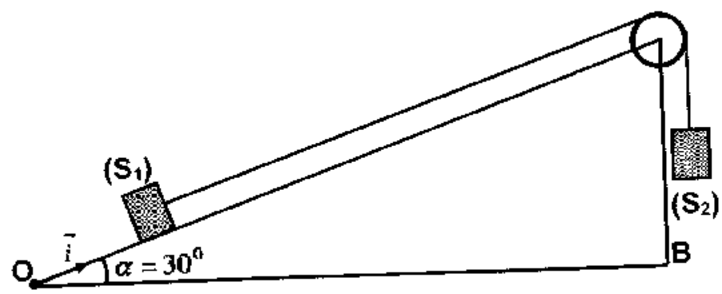
VI-

Acceleration of a System

The setup used is constituted of two particles (S_1) and (S_2) of respective masses $m_1 = 1.5kg$ and $m_2 = 3kg$, attached to the extremities of an inextensible string rolled on the hoop of a pulley. (S_1) and (S_2), the string and the pulley form a mechanical system (S).

The string and the pulley are massless.

(S_1) may slide along the line of the greatest slope on the inclined plane making an angle $\alpha = 30^0$ with respect to the horizontal OB and (S_2) hanged vertically.



At rest, (S_1) is found at the point O on the horizontal and (S_2) is found at an altitude h.

At the instant $t_0 = 0$, we release the system (S) starting from rest. (S_1) climbs on $(O;\vec{i})$ while (S_2) descends vertically. At an instant t, the position of (S_1) is referred by its abscissa $x = OS_1$ on an axis $x'Ox$. **Take:** $g = 10m/s^2$.

Consider the horizontal plane containing OB as reference for the gravitational potential energy.

Part A

Theoretical study

We neglect the forces of friction.

1. Write, at an instant t, the expression of the mechanical energy ME of the system [(S), Earth] as a function of m_1, m_2, h, x, v, α and g.
2. Justify that $\frac{d(ME)}{dt} = 0$, then show that the expression of the acceleration a of (S) is given by $a = \frac{(m_2 - m_1 \sin\alpha)}{m_2 + m_1} g$ and calculate its value.
3. Deduce the minimum value of m_2 required so that the system moves upwards.

Part B

Experimental study

In order to study the existence of a force of friction, we record the positions and the velocities of the center of gravity of (S_1) as shown in the table below.

t(s)	0.2	0.4	0.6	0.8	1
x(cm)	8	32	72	128	200
v(m/s)	0.8	1.6	2.4	3.2	4

1. Draw the curve representing the variation of the speed v as a function of time, using as a scale:
   - ✘ on the abscissas axis $1\,\text{div} \equiv 0.1\,s$;
   - ✘ on the ordinates axis $1\,\text{div} \equiv 0.8\,m/s$.
2. Show that the relation between the velocity $\vec{v} = v\vec{i}$ at a time t has the form $\vec{v} = kt\vec{i}$ where k is a constant.
3. What is the name of the constant k ? Calculate its value.
4. Justify the existence of a force of friction.
5. Applying Newton's 2nd law on (S_2), show that the magnitude T_2 of the tension $\overline{T_2}$ exerted by the string is $T_2 = 18\,N$.

Part C

Energetic study

The tension exerted by the string on (S_2) is equal to the magnitude of that exerted by the string on (S_1).

Applying the work energy theorem on (S_1) between two instants, determine the magnitude f of the force of friction supposed constant.

Solutions

I-

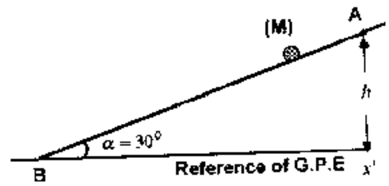
1. a) In the absence of friction, the mechanical energy is conserved: ($KE|_A = 0$, from rest)

$$ME = ME|_A = KE|_A + GPE|_A = GPE|_A = mgh_A = mg\,AB\sin\alpha = 1.25\,J.$$

b) The mechanical energy of the system (Particle, Earth) is conserved:

$ME = ME|_B = KE|_B + GPE|_B = KE_B$; ($GPE|_B = 0$, on reference)

So, $\frac{1}{2}mv_B^2 = 1.25\,J$; we get $v_B = 5\,m/s$.



2. The mechanical energy is conserved between B & O:

$ME|_B = ME|_O$; but $GPE|_O = GPE|_B = 0$ (on reference).

Then $KE|_B = KE|_0$, thus $v_0 = v_B = 5\,m/s$.

3. The new mechanical energy at B is $ME'|_B = GPE'|_B + KE'|_B = KE'|_B = \frac{1}{2}mv_B'^2 = 0.648\,J$.

The variation in the mechanical energy is $\Delta(ME) = ME'|_B - ME|_A = 0.648 - 1.25 = -0.6\,J$.

But $\Delta(ME) = -0.6\,J = -f \times AB$, we get $f = 0.24\,N$.

4. a) We have $ME = ME|_0 = \frac{1}{2}mv_0^2 = 1.25\,J$.

b) Conservation of mechanical energy between $x = 0$ & $x = x_m$:

$ME|_{x=0} = ME|_{x=x_m} = 1.25\,J$, $\frac{1}{2}kx_m^2 + 0 \Rightarrow x_m = 0.16\,m$ ($KE|_{x=x_m} = 0$, it comes to rest)

c) The kinetic is equal to the potential energy $KE = PE_e$, but $ME = KE + PE_e$;

Then, $PE_e = \frac{ME}{2}$, we get $\frac{1}{2}kx^2 = \frac{1.25}{2}$, thus $x = \pm\sqrt{\frac{1.25}{100}} = \pm 0.11\,m$.

5. a) At any position $ME = KE + PE_e = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$.

b) The mechanical energy is constant, since it is conserved then: $\frac{d(ME)}{dt} = 0$,

$mx'x'' + kxx' = 0$; $x'(mx'' + kx) = 0$, ($v = x' \neq 0$ the system is in motion)

Then $x'' + \frac{k}{m}x = 0$, by comparison with the general form $x'' + w_0^2 x = 0$; then $w_0 = \sqrt{\frac{k}{m}}$.

6. a) From graph, for $x = 0.16\,m$, we have $PE_e = 1.25\,J$.

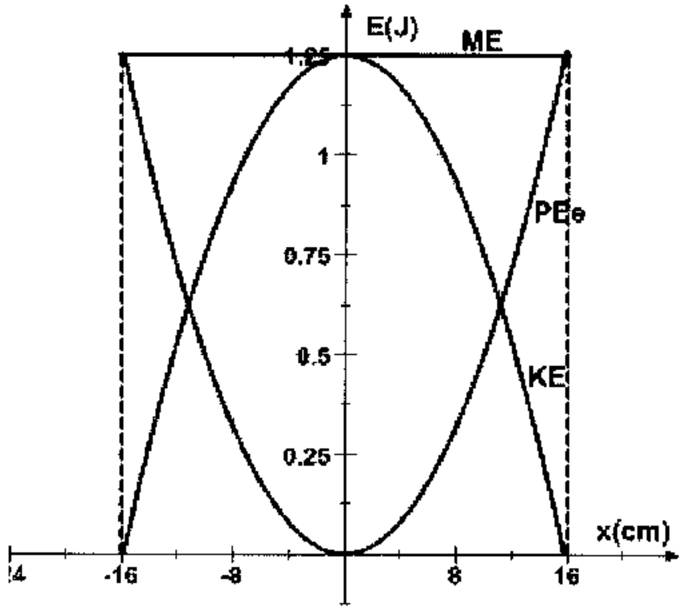
But $PE_e = \frac{1}{2}kx^2 = 1.25$; then $k = 97.6\,N/m$.

b) The mechanical energy is conserved: $ME = PE_e + KE = 1.25\,J$;

Then $KE = 1.25 - 50x^2$ where x in m & KE in J .

c) The curves are shown on the figure below.

The curve representing the elastic potential is a parabola concaving upwards while that of the kinetic energy is also a parabola concaving downwards.



II-

Part A

1. The forces of friction are negligible, then the mechanical energy of the system is conserved:

$$ME = ME|_{x=0} = KE|_{x=0} + PE_e|_{x=0} = \frac{1}{2} m v_0^2 + 0 = \frac{1}{2}(0.4)(0.4^2) = 0.032\,J.$$

2. Conservation of mechanical energy $ME|_{x_m} = 0.032\,J$; $KE|_{x_m} + PE_e|_{x_m} = 0.032\,J$;

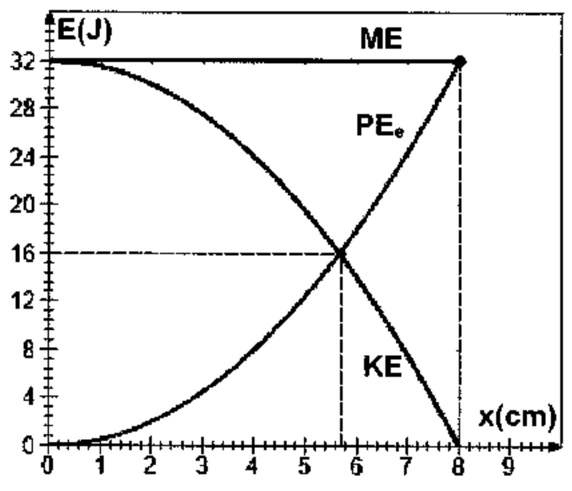
But $KE|_{x_m} = 0$; then $\frac{1}{2} k x_m^2 = 0.032$; $x_m^2 = 6.4\times10^{-3}$; thus $x_m = 0.08m = 8cm$.

3. a) Conservation of mechanical energy $ME = ME|_x = KE|_x + PE_e|_x = 0.032J$;

But $PE_e|_x = \frac{1}{2} k x^2 = 5x^2$;

Then $KE|_x = ME - PE_e|_x = 0.032 - 5x^2$

which is of the form $KE = a + bx^2$; where $a = 0.032\,J$ & $b = -5kg/s^2 = -5J/m^2$.



b) The kinetic energy is equal to the elastic potential energy then $0.032 - 5x^2 = 5x^2$;

So, $10x^2 = 0.032$;

We get $x = \sqrt{32\times10^{-4}} = 4\sqrt{2}\times10^{-2}m$;

Then $x = 4\sqrt{2}cm \approx 5.7cm$.

c) The mechanical energy is conserved, and then its graphical representation is a horizontal straight line; while the kinetic energy and potential energies have a parabolic shape.

Part B

1. The elastic potential energy is not affected by the force since it depends only on the elongation.
2. Non-conservation of mechanical energy: $\Delta(ME)=W_{\vec{f}}$;

   $ME|_{x'_m} - ME|_{x=0} = -f\times x'_m$; $\left(KE|_{x'_m} + PE_e|_{x'_m}\right) - ME|_{x=0} = -f\times x'_m$;

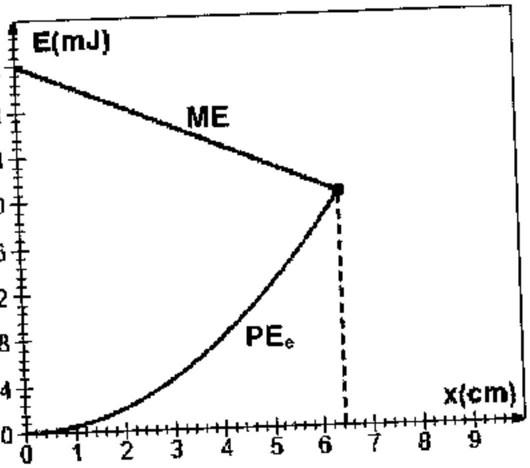
   We get $0+\frac{1}{2}(10)\left(6.4\times 0^{-2}\right)^2 - 0.032 = -f\times 6.4\times 10^{-2}$;

   Then $f=0.18N$.
3. Non-conservation of mechanical energy: $\Delta(ME)=W_{\vec{f}}$; $ME' - ME|_{x=0} = -f\times x$;

   $ME' = 0.032 - 0.18\times x$ (where x in m and ME' in J).

   $ME' = (0.032-0.18\times x)\times 10^3 = 32 - 180x$

   where x in m and ME' in mJ ;



4. The curve representing the variation of the mechanical energy is a straight line decreasing not passing through origin.
5. When it returns to the origin, the distance traveled is $d=6.4+6.4=12.8\,cm$;

   Non-conservation of mechanical energy: $ME'_{1(x=0)} - ME|_{x=0} = W_{\vec{f}} = -f\times d$;

   Then $\frac{1}{2}\times 0.4\times v_1'^2 - 0.032 = -0.18\times 0.128$; thus $v_1' = 0.39\,m/s$.

III-

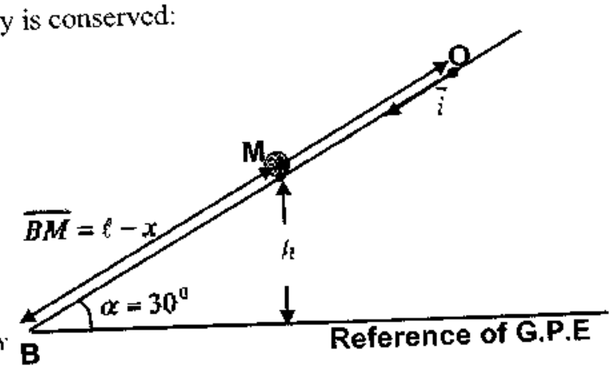
Part A

1. In the absence of friction, the mechanical energy is conserved:

   $ME = ME|_O = GPE|_O + KE|_O$;

   But $KE|_O = 0$ (released from rest);

   Then $ME = m\,g\,\ell\sin\alpha = 8J$.



2. In the absence of friction, the mechanical energy is conserved:

   $ME = ME|_B = KE|_B + GPE|_B$;

   But $GPE|_B = 0$ (on reference);

   Then $\frac{1}{2}m\,v_B^2 = 8J$,

   Thus $v_B = \sqrt{40} \approx 6.3m/s$.
3. a) At a point M : $GPE = m\,g\,h = mg(\ell - x)\sin\alpha = 8-2x$, where x in m & GPE in J .

   with $\sin\alpha = \frac{h}{BM} = \frac{h}{\ell - x} \Rightarrow h = (\ell - x)\sin\alpha$.

b) We have $KE = ME - GPE = 8 - (8 - 2x) = 2x$, where x in m & KE in J.

c) **1st method:** $ME = \frac{1}{2}mv^2 + 8 - 2x$ & $\frac{d(ME)}{dt} = 0$, we get $a = 5\,m/s^2$.

2nd method:

$KE = \frac{1}{2}mv^2 = 2x$;

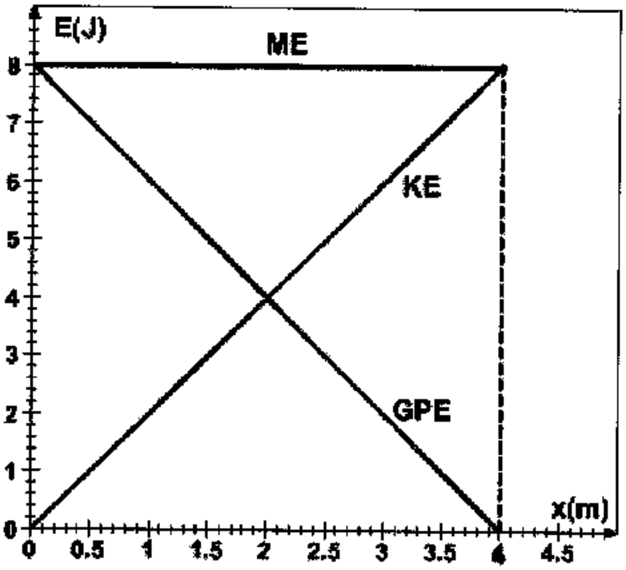
Deriving both sides with respect to time, we get: $mvv' = 2x'$;

But $v = x' \neq 0$ (in motion) & $a = v'$;

Then $a = 5\,m/s^2$.

4. Graphs.

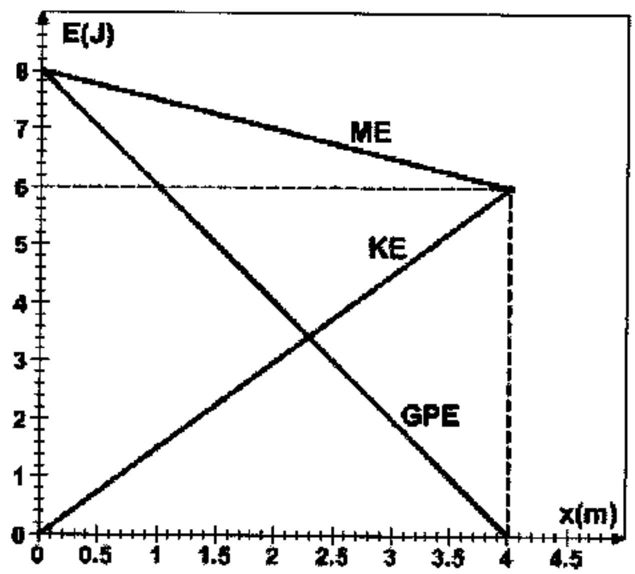
x	0	4
ME	8	8
GPE	8	0
KE	0	8



Part B

1. The gravitational potential energy is not affected by the friction since it depends on the height only and not on speed.
2. $\Delta(ME) = ME|_x - ME|_0 = -f \times x$.
   We get $ME|_x = 8 - 0.5x$
3. We have : $ME|_x = GPE|_x + KE|_x$;
   $KE|_x = 8 - 0.5x - 8 + 2x = 1.5x$;
   where x in m & KE in J.
4. Graphs.
   The expressions of three forms of energies are of degree 1 in terms of the abscissa, then their curves are straight line.

x	0	4
ME	8	6
GPE	8	0
KE	0	6



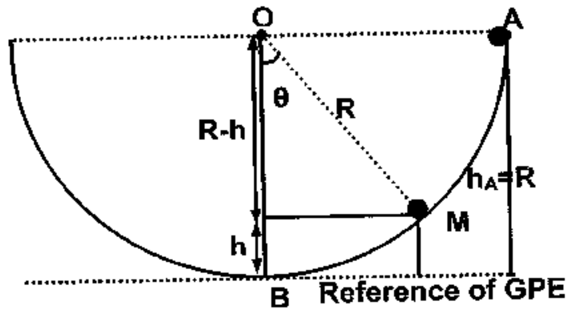
5. We have $\frac{d(ME)}{dt} = P_{\vec{f}}$; $\left(\frac{1}{2}mv^2 + 8 - 2x\right)' = -fv$;

$mvv' - 2x' = -0.5v$; but $v = x' \neq 0$ (in motion) & $a_1 = v'$, we get $ma_1 = 1.5$;

Thus, $a_1 = 3.75\,m/s^2 < a = 5\,m/s^2$, the decrease is due to the force of friction.

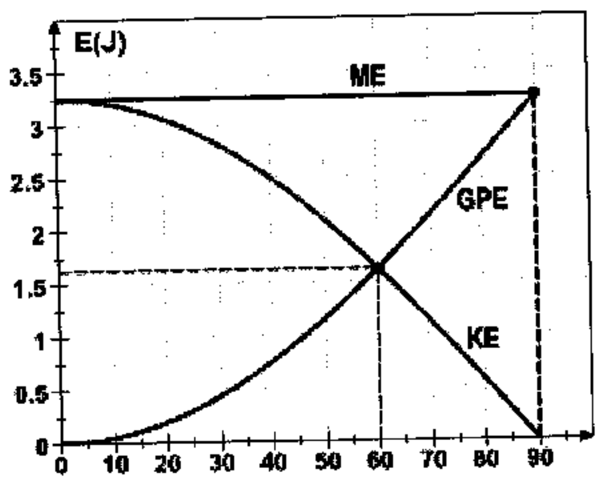
IV-

1. The marble slides without friction, then its mechanical energy is conserved:
   $ME = ME_A = KE_A + GPE_A$ but $KE_A = 0$ released from rest and $GPE_A = m\,g\,R$;
   Then $ME = 0 + m\,gR = 0.65 \times 10 \times 0.5 = 3.25J$.
2. Conservation of mechanical energy:
   $ME_B = 3.25J$; $KE_B + GPE_B = 3.25J$;
   But $GPE_B = 0$ (on reference);
   Then $KE_B = \frac{1}{2} m\,v_B^2 = 3.25J$;
   $v_B = \sqrt{\frac{2 \times 3.25}{0.65}} = \sqrt{10} \approx 3.1\,m/s$.



3. a) At M : $GPE = m\,g\,h_M = m\,g\,R(1 - \cos\theta)$.
   b) For $\theta = 30^0$, we get:
   $GPE = 3.25(1 - \cos 30^0) = 0.44J$;
   But the mechanical energy is conserved so : $KE = ME - GPE = 3.25 - 0.44 = 2.81J$.
   We have $KE = 2.81J = \frac{1}{2} m\,v^2$; then $v = 2.94\,m/s$.

   c) We have $KE = GPE$, but the mechanical energy is conserved then $KE = GPE = \frac{ME}{2}$;
   So $GPE = 3.25(1 - \cos\theta) = \frac{3.25}{2}$;
   $\cos\theta = \frac{1}{2}$, then $\theta = 60^0$.



4. a) The figure represents a non-conserved quantity so it represents either kinetic or gravitational.
   But for $\theta = 90^0$, $KE = 0$ (released from rest), then the curve of figure 2 represents GPE.
   b) The mechanical energy is conserved is constant so it is represented by a horizontal straight line.
   While the graph of the kinetic energy is increasing.
5. a) The variation in the mechanical energy is: $\Delta(ME) = ME'_B - ME|_{\theta=90^0} = \frac{1}{2} m\,v_1^2 - 3.25 = \frac{1}{2} \times 0.65 \times 2^2 - 3.25 = -1.95J$.
   b) We have $\Delta(ME) = W_{\vec{f}} = -1.95J$; $-1.95 = -f \times \ell$;
   where ℓ is the length of the arc AB is $\ell = R\theta_{AB} = R\left(\frac{\pi}{2}\right)$;
   Then $-1.95 = -f \times 0.5 \times \frac{\pi}{2}$; thus $f = \frac{1.95 \times 2}{0.5 \times \pi} \approx 2.48N$.

V-

1\. a) The moment of inertia of the system is:

$$I_S = I_D + I_A + I_B = \frac{1}{2} m R^2 + m R^2 + m R^2 = \frac{5}{2} m R^2 = 5 \times 10^{-4}\ kg.m^2 .$$

b) Since the dissipative forces are negligible, then the mechanical energy is conserved:

$$ME = ME\big|_{\theta=20^0} = KE\big|_{\theta=20^0} + GPE\big|_{\theta=20^0} + PE_e\big|_{\theta=20^0} = PE_e\big|_{\theta=20^0} ;$$

$$= \frac{1}{2} C \theta_m^2 = \frac{1}{2} \times 0.024 \times \left(20 \times \frac{\pi}{180}\right)^2 = 1.46 \times 10^{-3} J = 1.46\, mJ .$$

2\. a) Conservation of mechanical energy:

$$ME = ME\big|_{\theta=0^0} = KE\big|_{\theta=0^0} + PE_e\big|_{\theta=0^0} = KE\big|_{\theta=0^0} \Rightarrow \frac{1}{2} I \theta'^2 = 1.46 \times 10^{-3} \Rightarrow \theta' = \pm 2.42\ \text{rad}/s .$$

b) The mechanical energy is conserved: $ME = ME\big|_{\theta=10^0} = KE\big|_{\theta=10^0} + PE_e\big|_{\theta=10^0} = KE\big|_{\theta=0^0}$;

Then $\frac{1}{2} I \theta_1'^2 + \frac{1}{2} C \theta_1^2 = 1.45 \times 10^{-3} \Rightarrow \theta_1' = \pm 2.08\ \text{rad}/s$.

3\. a) At any instant t: $ME = ME\big|_\theta = KE\big|_\theta + PE_e\big|_\theta = \frac{1}{2} I \theta'^2 + \frac{1}{2} C \theta^2 = \frac{5}{4} m R^2 \theta'^2 + \frac{1}{2} C \theta^2$.

b) In the absence of friction, ME is conserved then it is constant, so $\frac{d(ME)}{dt} = 0$;

$\theta'\left(\frac{5}{2} m R^2 \theta'' + C\theta\right) = 0$, but $\theta' \neq 0$ (system in motion); then $\theta'' + \frac{2C}{5 m R^2}\theta = 0$.

c) The angular velocity θ' is maximum, then $\frac{d\theta'}{dt} = \theta'' = 0$, but $\theta'' + \frac{2C}{5 m R^2}\theta = 0$;

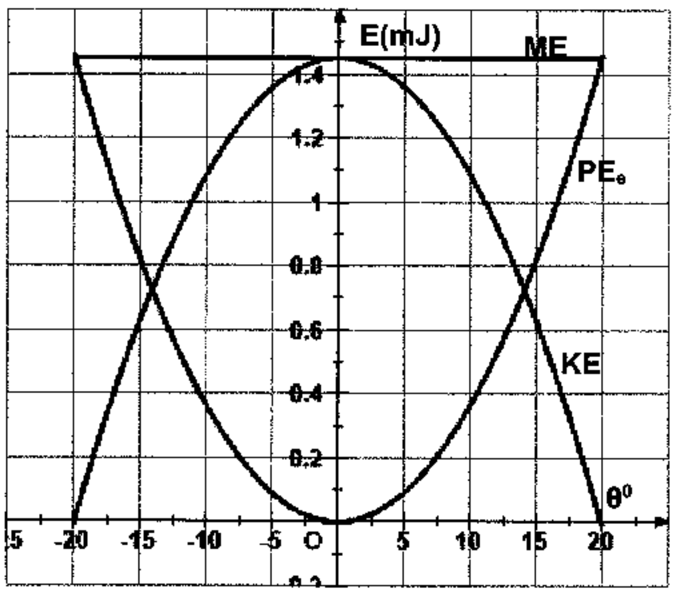
Thus $\theta = 0$ which corresponds to the equilibrium position.

d) Energetic Study:

i- $ME = 1.45 \times 10^{-3} J = 1.45\, mJ$ & $PE_e = \frac{1}{2} C \theta^2 = 0.012\, \theta^2$;

Then $KE = 1.45 \times 10^{-3} - 0.012\, \theta^2$ where θ in *rad* & KE in J .

ii- Graphs.



VI-

Part A

1\. At an instant t: $ME = ME_{(S_1)} + ME_{\text{string}} + ME_p + ME_{(S_2)}$;

But the string and the pulley are massless then $ME_{\text{string}} = ME_p = 0$;

Thus, $ME = KE_{(S_1)} + GPE_{(S_1)} + KE_{(S_2)} + GPE_{(S_2)}$;

The string is inextensible, then (S_1) and (S_2) have the same displacement $x_{(S_1)} = x_{(S_2)} = x$ &

$v_{(S_1)} = v_{(S_2)} = v$; thus $ME = \frac{1}{2} m_1 v^2 + m_1 g h_1 + \frac{1}{2} m_2 v^2 + m_2 g h_2$; $h_1 = x \sin\alpha$ and $h_2 = h - x$;

Therefore $ME = \frac{1}{2}(m_1 + m_2)v^2 + m_1\, g\, x \sin\alpha + m_2\, g\,(h - x)$.

2. The system slides without friction, then its mechanical energy is conserved:

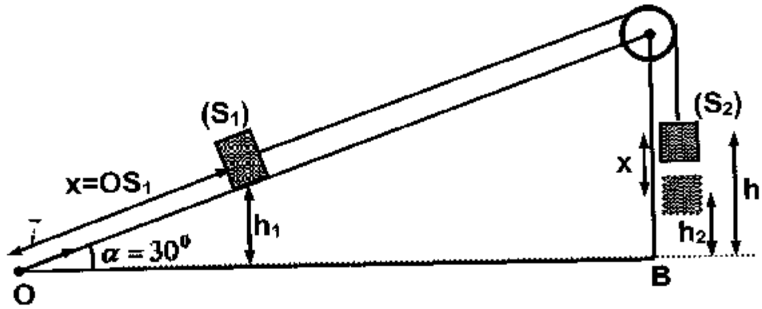
Then $\frac{d(ME)}{dt} = 0$; $(m_1 + m_2)v\, v' + m_1\, g\, x' \sin\alpha - m_2\, g\, x' = 0$;

But $v = x' \neq 0$ (the system is in motion) and $a = v'$;

So, $(m_1 + m_2)a + (m_1 \sin\alpha - m_2)g = 0$, we get $a = \frac{(m_2 - m_1 \sin\alpha)}{m_2 + m_1} g$.

$a = \left(\frac{3 - 1.5\sin 30^0}{3 + 1.5}\right) \times 10$;

Then $a = 5 m/s^2$.



3. In order to move upwards

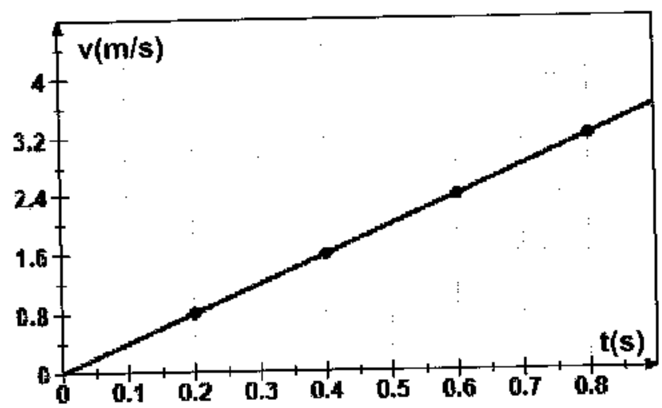
$a \geq 0$; $\frac{(m_2 - m_1 \sin\alpha)}{m_2 + m_1} g \geq 0$;

So, $m_2 - m_1 \sin\alpha \geq 0$;

Then $m_2 \geq m_1 \sin\alpha = 0.75 kg$; thus, the smallest value of the mass $m_{2\min} = 0.75 kg$.

Part B

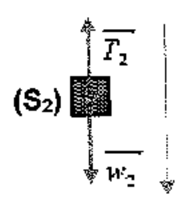
1. Graph.
2. The curve representing the variation of the velocity as a function of time is a straight line passing through origin, then its equation is $v = kt$.



3. The constant $k = \frac{dv}{dt}$ is the acceleration of motion, & since it is linear $k = \frac{\Delta v}{\Delta t} = \frac{(3.2 - 0)}{(0.8 - 0)} = 4 m/s^2$.
4. The empirical value of the acceleration $k = 4 m/s^2$ is less than the theoretical value $a = 5 m/s^2$ which justifies the existence of a force of friction.
5. Newton's 2[nd] law applied on (S_2): $\sum \overrightarrow{F_{ext}} = m_2 \vec{a}$; $\overrightarrow{T_2} + \overrightarrow{w_2} = m_2\, \vec{a}$;

Projection along the direction of motion: $m_2\, g - T_2 = m_2 a$;

Then $T_2 = m_2 g - m_2 a = 3 \times 10 - 3 \times 4 = 18N$.



Part C

We have $T_2 = T_1 = 18N$, work-energy theorem applied on (S_1) between $t = 0.2s$ & $t = 0.8s$:

$\Delta(KE) = \sum W$; $KE|_{t=0.8s} - KE|_{t=0.2s} = W_{\vec{N}} + W_{\vec{f}} + W_{\vec{T_1}}$;

But $W_{\vec{N}} = 0$ since it is perpendicular to the displacement, $W_{\vec{f}} = -f\,\Delta x$ & $W_{\vec{T_1}} = T_1\,\Delta x$;

$0.5 \times 1.5 \times 3.2^2 - 0.5 \times 1.5 \times 0.8^2 = 18 \times 1.2 - f \times 1.2$; then $f = 12N$.

Supplementary Problems

I-
Motion of a Particle

Consider an inclined plane that makes an angle α with the horizontal $(\sin\alpha = 0.15)$ and a marble (M), of mass $m = 800\,g$, taken as a particle. The particle (P) is referred at any instant t, by its abscissa $OM = x$, with respect to the axis $(O;\vec{i})$, as shown in figure 1.

The marble (M) is given, at the instant $t_0 = 0$, the velocity $\vec{v_0} = 5\vec{i}\,(m/s)$. The horizontal plane passing through O is taken as reference of gravitational potential energy for the system [(M), Earth]. A convenient apparatus is used to plot the graph representing the variations of the mechanical energy of the system as a function of the abscissa. **Given:** $g = 10\,m.s^{-2}$.

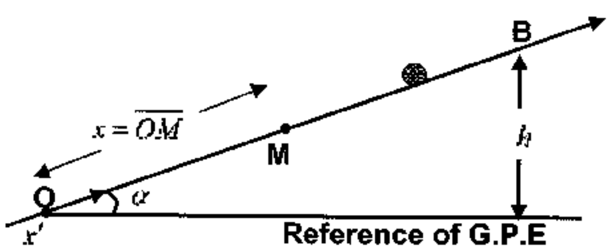


Figure 1

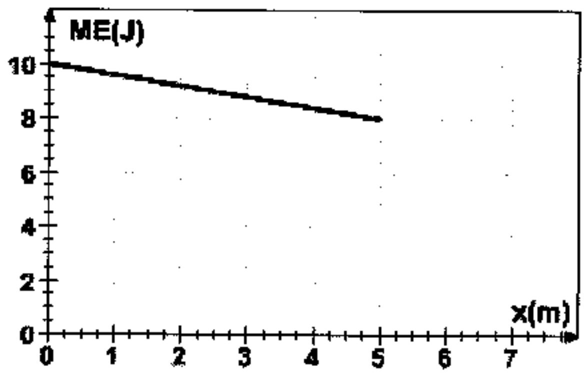


Figure 2

Part A
Energy exchange

1. Is the mechanical energy of this particle conserved? Justify.
2. Justify that the expression of the mechanical energy ME, of the system [(M), Earth], as a function of the abscissa x can be written in the form $ME = ax + b$ where a & b are constants.
3. Determine the values of a & b.
4. Knowing that the variations mechanical energy is related to the power $P_{\vec{f}}$ consumed by the force of friction by $\frac{d(ME)}{dt} = P_{\vec{f}}$. Calculate the magnitude f of the force of friction supposed constant.
5. Referring to the variation of the mechanical energy between two given positions, show that the particle comes to rest after covering a distance $d = 6.25m$.

Part B
Graphical study

1. Show that the expression of the gravitational potential energy at a point M of abscissa x is given by $GPE = 1.2x$ where x in m & GPE in J.
2. Recopy figure 2, then draw on it the graph representing the variation of the gravitational potential energy of the system [(M), Earth], deduce the position at which the particle comes to rest.

Answer Key
Part A 3. $ME = -0.4x + 10$; 4. $f = 0.4N$; **Part B** 2. $x \approx 6.25m$.

II- Variation of Elastic Potential Energy

A spring of unjointed turns is placed on a horizontal table and fixed by one extremity to a rigid support while the other end O is free. A box (B) of mass $m=100\ g$ is projected at the speed of $v_A=3\sqrt{2}\,(m/s)$, along the axis of the spring whose constant of elasticity is $k=80N/m$, to its free extremity O and reaches it with a speed $v_0=\sqrt{10}\,(m/s)$.

At the instant of launch, M is found at $40\,cm$ from O considered as an origin of the axis $(O;\vec{i})$ as shown in figure 1.

The plane passing through the center of inertia of the box is considered as a reference of the gravitational potential energy for the system (box, spring, Earth).

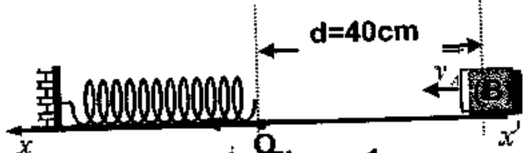


Figure 1

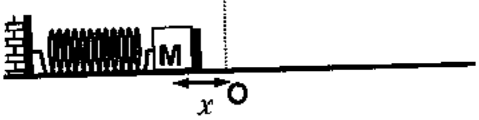


Figure 2

Part A

1. Calculate the mechanical energy of the system (box, Earth) at A.
2. Justify that the mechanical energy of the system (box, Earth) is not conserved between O and A.
3. *Deduce the variation in the internal energy between these points.*
4. Calculate the magnitude of the force of friction supposed constant.

Part B

The position of the box then is defined by its abscissa $x=\overline{OM}$. The box reaching the spring at an instant considered as an origin of time $t=0$ compresses the spring by x_m. The system (box, spring, Earth) is designated by (S). In what follows, we suppose that the magnitude of the force of friction is constant and is equal to $f=1N$.

1. a) Applying the theorem of non-conservation of mechanical energy, justify that the maximum elongation x_m satisfies the relation: $40\,x_m^2+x_m-0.5=0$.

   b) Deduce the numerical value of x_m.
2. Show that, at an instant t, after reaching the spring the expression of the mechanical energy of (S) is $ME=0.5-x$ (x in m & ME in J).
3. Verify that the expression of the kinetic energy can be written in the form $KE=ax^2+bx+c$ where a, b and c are constants to be determined specifying their units.
4. Figure 3, represents the variation of the kinetic energy KE as a function of the elongation x. Recopy figure 3 and represent on it the curves representing the variations of the mechanical energy ME of (S) and its elastic potential energy PE_e.
5. Determine the position at which the elastic potential energy is equal to the kinetic energy.

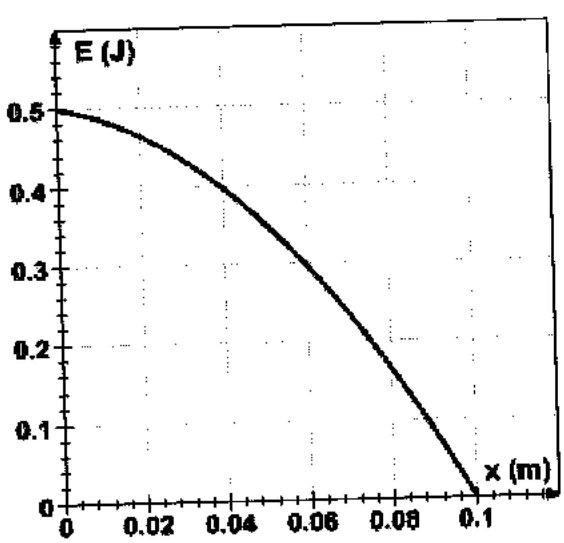


Figure 3

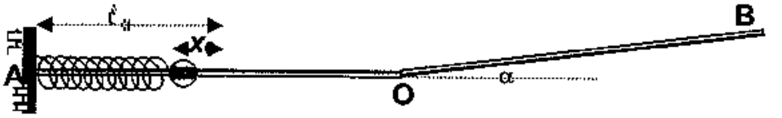
Answer Key

Part A 1. $0.9J$ 2. $0.5J$ **Part B** 3. $KE=-40x^2-x+0.5$

III-LS 2005 1st

Determination of the Value of a Force

A solid (S) of mass $m = 200\,g$ is free to move on a track AOB lying in a vertical plane. This rail is formed of two parts: the first one AO is straight and horizontal and the other OB is straight and inclined by an angle α with respect to the horizontal $(\sin\alpha = 0.1)$. Along the part AO, (S) moves without friction, and along the part OB, (S) is acted upon by a force of friction $\vec{f}$ that is assumed constant and parallel to the path.



The object of this exercise is to determine the magnitude f of the force $\vec{f}$ of friction.

Part A
Launching the solid

In order to launch this solid on the part AO, we use a spring of constant $k = 320\,N/m$ and of free length ℓ_0; one end of the spring is fixed at A to a support. We compress the spring by x_0; we place the solid next to the free end of the spring and then we release them. When the spring attains its free length ℓ_0, the solid leaves the spring with the speed $v_0 = 8\,m/s$; it thus slides along the horizontal part and then rises up at O the inclined part OB.

1. Determine the value of x_0.
2. The solid reaches O with the speed $v_0 = 8\,m/s$. Justify.

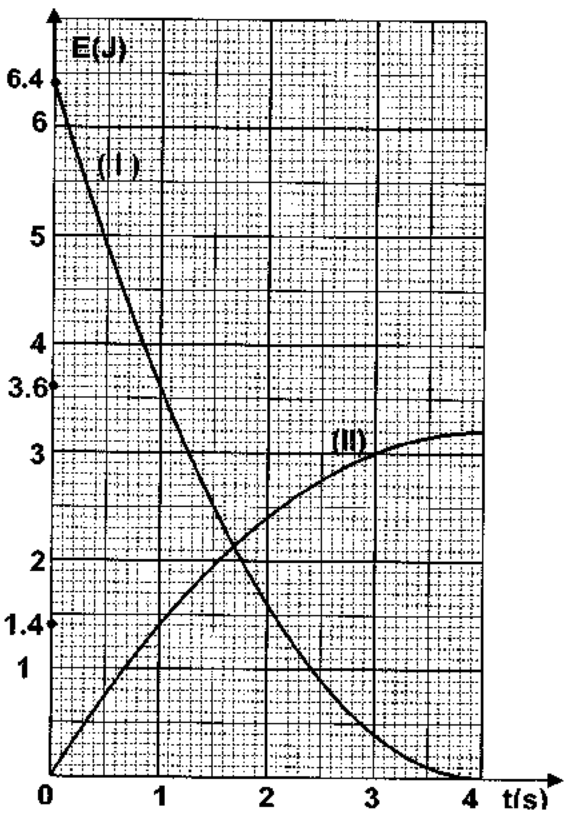
Part B
Motion of the solid along the inclined part OB

(S) moves, at O, up the inclined part OB with the speed V_0 at the instant $t_0 = 0$. A convenient apparatus is used to trace, as a function of time, the curves representing the variations of the kinetic energy KE of the solid and the gravitational potential energy GPE of the system (solid - Earth). These curves are represented in the adjacent figure between the instants $t_0 = 0$ and $t_4 = 4\,s$, according to the scale:

- 1 division on the time axis corresponds to $1\,s$;
- 1 division on the energy axis corresponds to $1\,J$

The horizontal plane through point O is taken as a gravitational potential energy reference.

Take: $g = 10\,m/s^2$.



1. The curve **(I)** represents the variation of the kinetic energy KE of (S) as a function of time. Why?
2. Using the curves:
   a) Specify the form of the energy of the system at the instant $t_4 = 4\,s$. Justify your answer.
   b) Determine the maximum distance covered by the solid along the part OB.

c) Graphical study

i- Complete the table with the values of the mechanical energy ME of the system at each instant t.

$t(s)$	0	1	2	3	4
$ME(J)$		5			

ii- Justify the existence of a force of friction $\vec{f}$.

iii- Calculate the variation in the mechanical energy of the system between the instants $t_0 = 0$ and $t_4 = 4s$.

iv- Determine f.

Answer Key

Part A 1. $x_0 = 20\,cm$; **Part B** 2.b) $d_{max} = 16\,m$ c) *iv*- $f = 0.2\,N$.

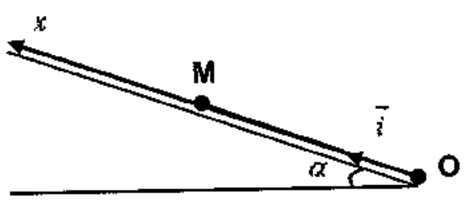
IV-LS 2003 2nd

Graphical Study of Energy Exchange

Consider an inclined plane that makes an angle α with the horizontal $(\sin\alpha = 0.2)$ and a marble (B), of mass $m = 100\,g$, taken as a particle.

We want to study, the energy exchange between the system [(B), Earth] with the surroundings.

To do that, the marble (B) is given, at the instant $t_0 = 0$, the velocity $\vec{v_0} = v_0\,\vec{i}$ along the line of the greatest slope Ox.



Given: $v_0 = 4\,m/s$ and $g = 10\,m s^{-2}$.

The horizontal plane passing through O is taken as reference of gravitational potential energy for the system [(B), Earth].

Part A

The forces of friction are supposed ***negligible***.

1. Determine the value of the mechanical energy ME of the system (marble, Earth).
2. At the instant t, the marble passes through a point M of abscissa $OM = x$.
   Determine as function of x, the expression of the gravitational potential energy GPE of the system (marble, Earth) when the marble passes through M.
3. a) Trace, on the same system of axes, the curves representing the variations of the energies ME & GPE as a function of x.
   On the axis of abscissa: $1\,cm$ represents $1\,m$ & on the axis of energies: $1\,cm$ represents $0.2\,J$.

   b) Determine using the graph, the speed of the marble for $x = 3\,m$.

   c) Determine using the graph, the value of x_m of x for which the speed of (B) is zero.

Part B

1. In reality the speed of the marble becomes zero at a point of abscissa $x = 3\,m$. The forces of friction are no longer negligible. Calculate then the work done by the forces of friction between $x = 0$ and $x = 3\,m$.
2. The system (marble, Earth) thus exchanges energy with its surroundings. In what form and how much?

Answer Key

Part A 1. $ME = 0.8\,J$. **Part B** 1. $-0.2\,J$.

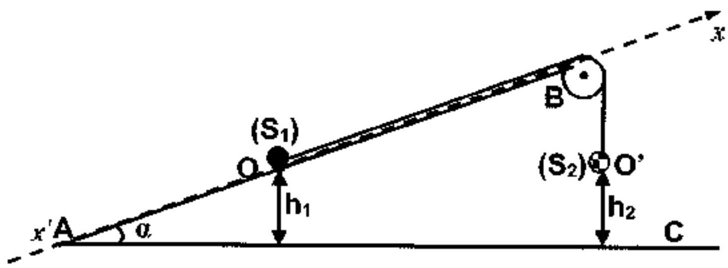
I-LS 2011 2nd

Acceleration of a Particle

The objective of this exercise is to determine the expression of the magnitude of the acceleration of a particle using two methods.

The apparatus used is formed of two particles (S_1) and (S_2) of respective masses m_1 and m_2, fixed at the extremities of an inextensible string passing over the groove of a pulley. (S_1), (S_2), the string and the pulley form a mechanical system (S). The string and the pulley have a negligible mass.

(S_1) may move on the line of the greatest slope AB on inclined plane that makes an angle α with respect to the horizontal AC and (S_2) hangs vertically. At rest, (S_1) is found at the point O at an altitude h_1 above AC and (S_2) is found at O' at a height h_2 (figure).



At the instant $t_0 = 0$, we release the system (S) from rest. (S_1) ascends on AB and (S_2) descends vertically. At an instant t, the position of (S_1) is defined by its abscissa $x = \overline{OS_1}$ on an axis $x'Ox$ confounded with AB, directed from A towards B.

Take the horizontal plane containing AC as a gravitational potential energy reference.

Neglect the forces of friction.

Part A
Energetic method

1. Write down, at the instant $t_0 = 0$, the expression of the mechanical energy of the system [(S), Earth] in terms of m_1, m_2, h_1, h_2 and g.
2. At the instant t, the abscissa of (S_1) is x and the algebraic value of the velocity is v.
   Determine, at that instant t, the expression of the mechanical energy of the system [(S), Earth] in terms of m_1, m_2, h_1, h_2, x, v, α and g.
3. Applying the principle of conservation of mechanical energy, verify that:
   $$v^2 = \frac{2(m_2 - m_1 \sin\alpha)}{m_2 + m_1} g\,x.$$
4. Deduce the expression of the value a acceleration of (S_1).

Part B
Dynamical method

1. Reproduce a diagram of the figure and show, on it, the external forces acting on (S_1) and (S_2). (The tension in the string acting on (S_1) is denoted $\vec{T_1}$ of magnitude T_1 and that acting on (S_2) is denoted $\vec{T_2}$ of magnitude T_2).
2. Applying the theorem of the center of inertia $\sum \overrightarrow{F_{ext}} = m\,\vec{a}$, on each particle, determine the expressions of T_1 and T_2 in terms of m_1, m_2, g, α and a.
3. Knowing that $T_1 = T_2$, deduce the expression of a.

II-LS 2010 2nd

Resistive Force on a Car

A car of mass $M = 1500\ kg$ moves on a straight horizontal road; its center of gravity G is moving on the axis $(O;\vec{i})$. The car is acted upon by the forces: its weight; the normal reaction of the road; a constant motive force $\overrightarrow{F_m} = F_m\,\vec{i}$ where $F_m = 3500\ N$; a resistive force $\overrightarrow{F_f} = -F_f\,\vec{i}$.

In order to determine F_f, we measure the speed v of the car at different instants, separated by equal time intervals each being $\tau = 1s$.

Part A

Value of F_f between the instants $t_0=0$ and $t_5=5s$

The results of the obtained recordings are tabulated as follows:

Instant	$t_0 = 0$	$t_1 = \tau$	$t_2 = 2\tau$	$t_3 = 3\tau$	$t_4 = 4\tau$	$t_5 = 5\tau$
Position	O	G_1	G_2	G_3	G_4	G_5
v(m/s)	0	2	4	6	8	10

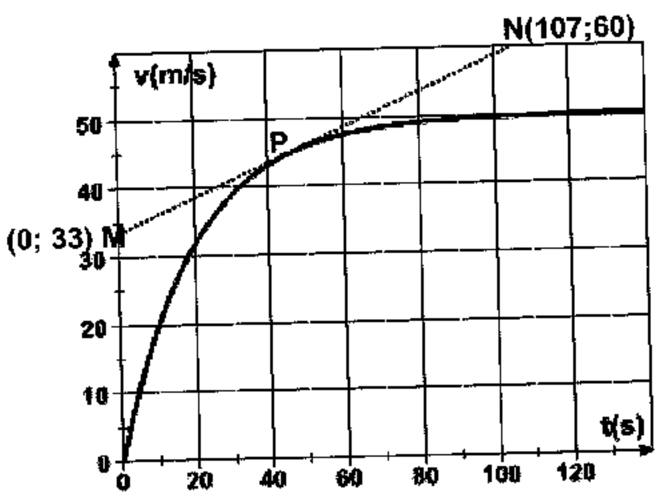
1. Using the scale below, draw the curve representing the variation of the speed v as a function of time. $1cm$ on the axis of abscissas represents $1s$ & $1cm$ on the axis of ordinates represents $1m/s$
2. Show that the relation between the velocity $\vec{v} = v\,\vec{i}$ at a time t has the form $\vec{v} = bt\,\vec{i}$ where b is a constant.
3. a) The constant b is a characteristic physical quantity of motion. Give its name.
   b) Calculate its value.
4. Applying Newton's second law:
   a) show that F_f is constant between $t_0 = 0$ and $t_5 = 5s$.
   b) calculate the value F_f of $\overrightarrow{F_f}$.

Part B

Variation of F_f between the instants $t_5=5s$ and $t_5=140s$

In reality, the measurement of the speed between the instants $t_5 = 0$ and $t = 140\ s$ allows us to plot the graph of the adjacent figure.

1. Show that the part of this graph between the instants $t_0 = 0$ and $t_5 = 5s$ is in agreement with the graph of part A.
2. We draw the tangent MN to the curve at the point P at the instant t_P where $v_P = 45\ m/s$.
   a) Determine the value of the acceleration at the instant t_P.
   b) Deduce the value of F_f at the instant t_P.
3. Starting from the instant $100\ s$, v attains a limiting value of $v_\ell = 50\ m/s$. Calculate then the value of F_f.
4. Indicate the time interval during which F_f increases.



Sessions Solutions

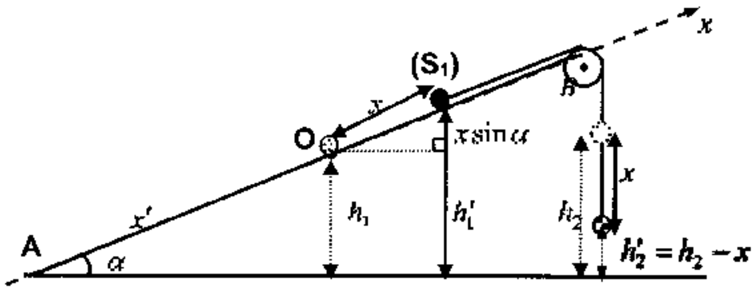
I-LS 2011 2nd

Part A

1. The mechanical energy $ME|_{t_0=0} = KE_1|_{t_0=0} + GPE_1|_{t_0=0} + KE_2|_{t_0=0} + GPE_2|_{t_0=0}$;

   $KE_1|_{t_0=0} = KE_2|_{t_0=0} = 0$ (released from rest).

   $ME|_{t_0=0} = m_1\, g\, h_1 + m_2\, g\, h_2$.



2. The string is inextensible then (S_1) and (S_2) should have the:

   ✖ same displacement $x_1 = x_2 = x$;

   ✖ same speed $v_1 = v_2 = v$.

   $ME|_t = KE_1|_t + GPE_1|_t + KE_2|_t + GPE_2|_t = \frac{1}{2}(m_1 + m_2)v^2 + m_1 g(h_1 + x\sin\alpha) + m_2 g(h_2 - x)$.

3. Since the friction is negligible then the mechanical energy is conserved:

   $ME|_{t_0=0} = ME|_t \Rightarrow m_1\, g\, h_1 + m_2\, g\, h_2 = \frac{1}{2}(m_1 + m_2)v^2 + m_1 g(h_1 + x\sin\alpha) + m_2 g(h_2 - x)$.

   Then $\frac{1}{2}(m_1 + m_2)v^2 = m_2 g x - m_1 g\, x\sin\alpha$; thus, $v^2 = \dfrac{2(m_2 - m_1\sin\alpha)x}{m_1 + m_2} g$.

4. We have $v^2 = \dfrac{2(m_2 - m_1\sin\alpha)x}{m_1 + m_2} g$ (deriving both sides with respect to time);

   $2vv' = \dfrac{2(m_2 - m_1\sin\alpha)\,x'}{m_1 + m_2} g$ but $v = x' \neq 0$ since the system is in motion;

   Then $a = \dfrac{(m_2 - m_1\sin\alpha)}{m_1 + m_2} g$.

Part B

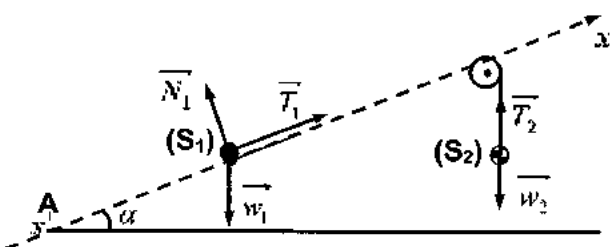
1. Schema.
2. Newton's 2nd law applied on (S_1):

   $\sum \overrightarrow{F_{ext}} = m_1 \vec{a}$, $\vec{w_1} + \vec{N_1} + \vec{T_1} = m_1\,\vec{a}$.

   Projection on the direction of motion:

   $T_1 - m_1\, g\sin\alpha = m_1 a$.

   Then, $T_1 = m_1 a + m_1\, g\sin\alpha$.



Newton's 2[nd] law applied on (S_2): $\sum \overrightarrow{F_{ext}} = m_2\vec{a}$; $\overrightarrow{w_2} + \overrightarrow{T_2} = m_2\vec{a}$.

Projection on the direction of motion: $-T_2 + m_2 g = m_2 a$; then, $T_2 = -m_2 a + m_2 g$.

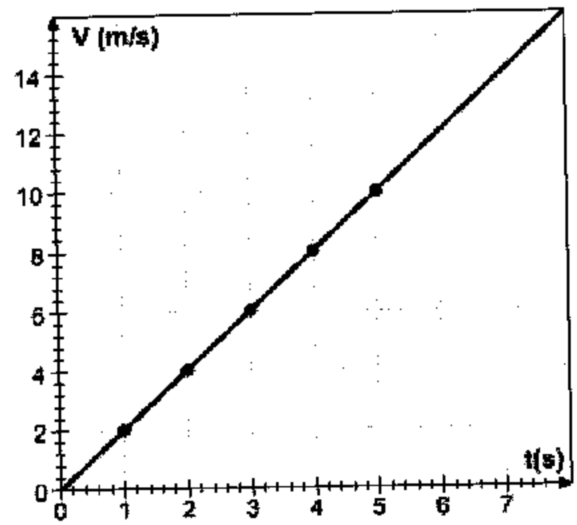
3. We know that $T_1 = T_2 \Rightarrow m_1 a + m_1 g \sin\alpha = m_2 g - m_2 a$.

Then $a = \dfrac{(m_2 - m_1 \sin\alpha)}{m_1 + m_2} g$.

II-LS 2010 2nd

Part A

1. Graph.
2. The curve representing the variation of the velocity in terms of time is a straight line passing through the origin, which is in agreement with the expression $\vec{v} = bt\vec{i}$ where b is a constant.



3. a) b is the acceleration of the motion.

b) $b = \dfrac{\Delta v}{\Delta t} = \dfrac{v|_{t=5s} - v|_{t=0}}{5-0} = \dfrac{(10-0)m/s}{5s} = 2\,m/s^2$.

4. a) According to Newton's 2[nd] law $\sum \overrightarrow{F_{ext}} = M\vec{a}$; so, $\vec{w} + \vec{N} + \overrightarrow{F_f} + \overrightarrow{F_m} = M\vec{a}$.

Projection along the positive direction we get: $F_m - F_f = M\dfrac{dv}{dt} = M\,a$;

Since F_m, M and a are constants then F_f is also constant.

b) $F_f = F_m - M\,a = 3500 - 1500 \times 2 = 500\,N$.

Part B

1. For $v < 10\,m/s$, the part of the curve is a straight line.
2. a) The acceleration at the considered instant is the slope of the tangent to the velocity's curve:

$$a = \frac{\Delta v}{\Delta t} = \frac{v|_{t=107s} - v|_{t=0}}{107-0} = \frac{(60-33)m/s}{(107-0)s} = 0.25\,m/s^2.$$

b) We have $F_f = F_m - M\,a = 3500 - 1500 \times 0.25 = 3125\ N$.

3. For $a = 0$; we get $F_f = F_m = 3500\,N$.
4. F_f increases for $5s < t < 100\,s$.

Unit I

Mechanics

Chapter 2

Linear Momentum

LS - Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Linear Momentum		2nd(A)	2nd	1st	-	-	2nd	-	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Linear Momentum	-	-	2nd	1st	-	2nd	-	1st & 2nd	2nd

Essentials

I- Definition

The linear momentum of an object of mass m moving with a velocity $\vec{v}$ is given by: $\vec{P} = m\vec{v}$.

In SI units P is measured in kg.m/s.

II- Newton's 2nd Law

The sum of the external forces applied on a system of particles is equal to the derivative with respect to time of its linear momentum: $\sum \vec{F} = \frac{d\vec{P}}{dt}$.

Note: We can verify Newton's 2nd law by calculating independently the resultant force acting on the direction of motion $\sum \vec{F}$, then compare it to the variation in the linear momentum $\frac{d\vec{P}}{dt} = \frac{\Delta \vec{P}}{\Delta t}$.

III- Conservation of Linear Momentum

If the resultant force acting on a given system is zero $\sum \vec{F} = \vec{0}$, then the linear momentum is conserved (which is usually observed in explosions and collisions), so $\vec{P}_{\text{just before}} = \vec{P}_{\text{just after}}$.

(S_1) & (S_2) are particles in motion enter in a collision, their velocities are described in the adjacent table.

	(S_1): m_1	(S_2): m_2
Just before	$\vec{v_1}$	$\vec{v_2}$
Just after	$\vec{v_1'}$	$\vec{v_2'}$

1. Conservation

The conservation of the linear momentum of the system [(S_1), (S_2)]: $m_1\vec{v_1} + m_2\vec{v_2} = m_1\vec{v_1'} + m_2\vec{v_2'}$

2. Forces of interaction

The force exerted by (S_1) on (S_2) through the interaction (collision or explosion) is given by Newton's 2nd law applied on (S_2): $\sum \vec{F} = \frac{d\vec{P_2}}{dt}$, $\overrightarrow{F_{1-2}} = \frac{\Delta \vec{P_2}}{\Delta t} = \frac{m_2\left(\vec{v_2'} - \vec{v_2}\right)}{\Delta t}$;

Similarly the force exerted by (S_2) on (S_1) is: $\overrightarrow{F_{2-1}} = \frac{\Delta \vec{P_1}}{\Delta t} = \frac{m_1\left(\vec{v_1'} - \vec{v_1}\right)}{\Delta t}$;

These two forces verify the principle of interaction (Newton's 3rd law) $\overrightarrow{F_{1-2}} = -\overrightarrow{F_{2-1}}$.

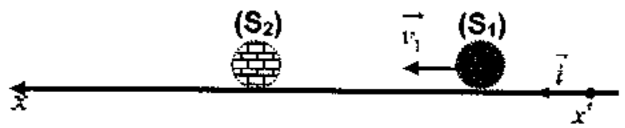
IV-

Types of Collisions

During a collision or explosion the linear momentum is conserved (in general).
If the **kinetic energy** of the system is also **conserved**, then the collision is called **elastic**.

Elastic collision

A particle (S_1) of mass m_1 moving with a velocity $\overrightarrow{v_1}$ enters with a head on elastic collision with another particle (S_2) of mass m_2 taken at rest (see table).



The physical quantity that remains conserved during a collision is the linear momentum.

$\vec{P}_{\text{just before collision}} = \vec{P}_{\text{just after collision}}$.

$m_1 \overrightarrow{v_1} + m_2 \vec{0} = m_1 \overrightarrow{v_1'} + m_2 \overrightarrow{v_2'}$

	$(S_1): m_1$	$(S_2): m_2$
Just before	$\overrightarrow{v_1}$	$\vec{0}$
Just after	$\overrightarrow{v_1'}$	$\overrightarrow{v_2'}$

The velocities are **collinear**, then we may use algebraic expression, $m_1(v_1 - v_1') = m_2 v_2'$ .....(1)

The collision is **elastic**, then the kinetic energy is conserved $KE_{\text{just before collision}} = KE_{\text{just after collision}}$.

$\frac{1}{2}m_1 v_1^2 + 0 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$, then $m_1(v_1 - v_1')(v_1 + v_1') = m_2 v_2'^2$ ....(2)

Dividing (2) by (1) we get: $v_1 + v_1' = v_2'$ ......(3)

Dividing (1) by m_1 (substitute if masses are given), and using (3) we get: $\begin{cases} v_1 - v_1' = \dfrac{m_2}{m_1} v_2' \\ v_1 + v_1' = v_2' \end{cases}$;

Then $v_2' = \dfrac{2m_1}{(m_1 + m_2)} v_1$ & $v_1' = \dfrac{(m_1 - m_2)}{(m_1 + m_2)} v_1$.

Discussion:

Masses	v_2'	v_1'
$m_1 > m_2$	Positive	Positive
$m_1 = m_2$ **(exchange of velocities)**	$v_2' = v_1$	$v_1' = 0$
$m_1 < m_2$	Positive	Negative (rebounds)

Applications

I- Newton's 2nd Law

A puck (P) of mass $m = 200\ g$ is released on a rough inclined plane making an angle α with the horizontal such that $\sin\alpha = 0.6$, from the origin O of the axis $(O;\vec{i})$ as shown in figure 1.

Take: $g = 10 m/s^2$.

The curve shown in figure 2 represents the variation of the linear momentum of the puck as a function of time.

1. Calculate the initial speed of the puck.
2. Justify that $\frac{dP}{dt}$ is constant and then determine its value.
3. State and then represent the forces acting on the puck.
4. Applying Newton's 2nd law, determine the magnitude of the force of friction supposed constant.

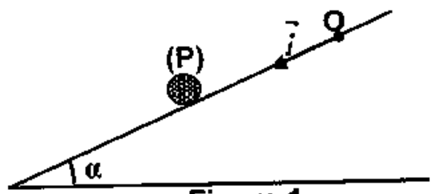


Figure 1

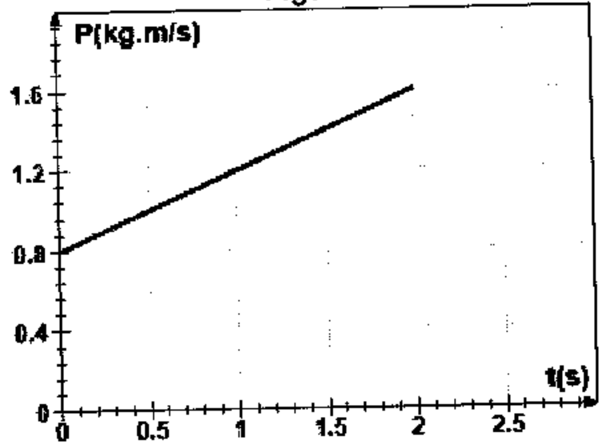


Figure 2

II- Collision

Consider two objects supposed points particles (P_1) & (P_2) of respective masses $m_1 = 0.5kg$ and $m_2 = 1.5\,kg$ in rectilinear motion and enters in a collision with the respective velocities $\vec{v_1} = 5\vec{i}$ (m/s) & $\vec{v_2} = -3\,\vec{i}$ (m/s).

Just after collision, the velocity of (P_1) is $\vec{v_1'} = -1\vec{i}$ (m/s).

The duration of the collision is $\Delta t = 4\,ms$ and neglect the forces of friction.

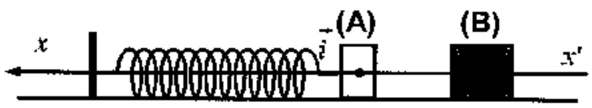
1. Determine the velocity of (P_2) just after collision.
2. State and then represent the forces acting on (P_2) during the collision.
3. Determine $\overrightarrow{F_{1/2}}$ the force exerted by (P_1) on (P_2) during collision and then deduce $\overrightarrow{F_{2/1}}$ the force exerted by (P_2) on (P_1).
4. Compare during the collision:
   a) the variation in the linear momentum of (P_1) to that of (P_2).
   b) the kinetic energies of the system $[(P_1), (P_2)]$.

III- Spring and Collision

On a table, we consider a puck (A), of mass $m = 50g$, fixed to one end of a massless spring of un jointed turns, and of stiffness $k = 80N/m$; the other end of the spring is attached to a fixed support. (A) slides on a horizontal rail along a horizontal axis $x'Ox$ and at equilibrium, coincides with the origin O of the axis $x'x$.

Another puck (B) of mass $M = 200g$ is launched towards (A) with a velocity $\vec{v_B} = 4\vec{i}\,(m/s)$. As a result the two pucks are **stuck** together. Neglect friction.

The horizontal plane containing $x'x$ is taken as a gravitational potential energy reference.



1. Determine the expression of the velocity $\vec{v_0}$ of the system [(A), (B)] just after collision.
2. Calculate the mechanical energy of the system [(A), (B), spring, Earth] after collision.
3. Deduce the maximum compression of the spring.
4. Determine the characteristics of the tension exerted by the spring at the maximum compression.
5. When the system is set in motion after collision, a time equation that describe the motion of the center of gravity G is given by $x = A\sin wt$ where w and A are constants.
   Determine, in terms of time and the given, the expression of the linear momentum, and then deduce the expression of its maximum value.

IV- Linear Momentum

A shark of mass $M = 150\,kg$ is moving in rectilinear motion with a speed of $36 km/h$, swallows a fish of mass $m = 2kg$.

Determine the velocity of the shark after interaction if the fish is:

1. stationary.
2. moving in opposite direction to that of shark with a speed of $10 km/h$
3. moving in same direction to that of shark with a speed of $18 km/h$.
   Compare the kinetic energy of the system (shark, fish) before and after interaction.

V- Explosion and Linear Momentum

An object of mass $m = 1kg$ explodes into two fragments (F_1) & (F_2) of respective masses $m_1 = 0.4kg$ and m_2.

(F_1) moves along the positive direction of an axis $(O; \vec{i})$ with a velocity $\vec{v_1} = 3\vec{i}\ (m/s)$.

1. Determine the velocity $\vec{v_2}$ of (F_2) just after collision.
2. Indicate the forms of energy in the system [(F_1), (F_2)] just after and just before explosion.
3. Calculate the value of the energy before explosion.
4. Specify the position of the center of gravity of the system of fragments after collision.

VI-
Linear Momentum and Basket Ball

A stationary basketball player of mass $M = 85 kg$ throws a ball of mass $m = 600 g$ with a horizontal velocity of magnitude $25 km/h$.

1. What is the physical quantity that remain conserved during this interaction?
2. Determine the recoil velocity $\vec{v}$ of the player.
3. Determine the magnitude of the force received by the ball during the interaction that lasts $20 ms$.
4. Compare the kinetic energy of the system (player-ball) just before and just after interaction.

VII-
Elastic Collision

A material point (A) of mass m_A, is moving along an axis $(O;\vec{i})$ with a velocity $\vec{V_A} = 10\vec{i}\ (m/s)$. It enters, in a head on elastic collision with a material point (B) of mass $m_B = 2m_A$, moving with a velocity $\vec{V_B} = -5\vec{i}\ (m/s)$ along the same axis.

1. Determine $\vec{V'_A}$ and $\vec{V'_B}$ the velocities of (A) and (B) after the collision.
2. Show that the variation in the linear momentum $\overrightarrow{\Delta P_A}$ of (A) is directly opposite to that of (B).
3. Determine the mass of (A) knowing that it receives a force $\vec{F} = -250\vec{i}\ (N)$ through the collision that lasts $16 ms$.

Solutions

I-

1. The value of the linear momentum at $t=0$, is $P_0 = 0.8\,kg.m/s = m v_0$;

   Then $v_0 = \frac{P_0}{m} = \frac{0.8\,kg.m/s}{0.2\,kg} = 4\,m/s$.

2. The curve representing the variation of the linear momentum P in terms of time is represented by a straight line, then its equation is of the form $P = At + B$ where A & B are constants.

   Thus, $\frac{dP}{dt} = A$ is the slope of the straight line.

   $\frac{dP}{dt} = \frac{\Delta P}{\Delta t} = \frac{(1.6-0.8)kg.m/s}{(2-0)s} = 0.4\,kg.m/s^2$.

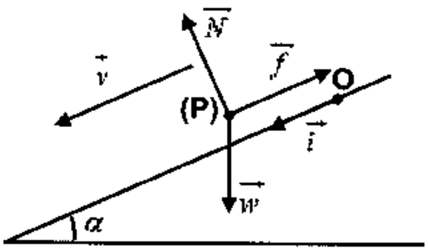


Figure 1

3. The forces acting on the puck (P) are:

   ✖ its weight $\vec{w}$;

   ✖ the normal reaction of the support $\vec{N}$;

   ✖ the force of friction $\vec{f}$.

4. Newton's 2nd law: $\sum \vec{F_e} = \frac{d\vec{P}}{dt}$;

   $\vec{w} + \vec{f} + \vec{N} = \frac{d\vec{P}}{dt}$; (projection along the direction of motion);

   $m\,g\sin\alpha - f = \frac{dP}{dt}$; $f = m\,g\sin\alpha - \frac{dP}{dt} = 0.2\times10\times0.6 - 0.4 = 0.8N$.

II-

1. During the collision, the linear momentum of the system [(P_1), (P_2)] is conserved:

   $\vec{P}_{\text{just before}} = \vec{P}_{\text{just after}}$;

   $m_1\vec{v_1} + m_2\vec{v_2} = m_1\vec{v_1'} + m_2\vec{v_2'}$;

   $0.5\times5\vec{i} + 1.5\times(-3)\vec{i} = 0.5\times(-1)\vec{i} + 1.5\vec{v_2'}$;

   Then $\vec{v_2'} = (-1)\vec{i}\,(m/s)$.

   The two particles are moving with same velocity.

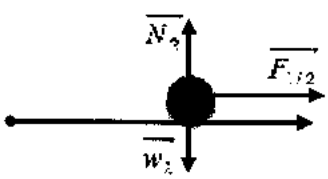
	(P_1): m_1	(P_2): m_2
Just before	$\vec{v_1} = 5\vec{i}$	$\vec{v_2} = -3\vec{i}$
Just after	$\vec{v_1'} = -1\vec{i}$	$\vec{v_2'}$

2. The forces acting on (P_2) are:

   ✖ its weight $\vec{w_2}$;

   ✖ the normal reaction $\vec{N_2}$ exerted by the support;

   ✖ the force exerted by $\vec{F_{1/2}}$ exerted by (P_1) on (P_2).



3. Newton's 2nd law applied on (P_2):

$\sum\vec{F_{ext}} = \frac{d\vec{P_2}}{dt}$, $\vec{w_2} + \vec{N_2} + \vec{F_{1/2}} = \frac{\Delta\vec{P_2}}{\Delta t}$; (the collision lasts for small duration $\frac{\Delta\vec{P_2}}{\Delta t} \approx \frac{d\vec{P_2}}{dt}$);

But $\overrightarrow{w_2} + \overrightarrow{N_2} = \vec{0}$,

Then $\overrightarrow{F_{1/2}} = \dfrac{m_2\left(\overrightarrow{v_2'} - \overrightarrow{v_2}\right)}{\Delta t} = \dfrac{1.5\left(-1\vec{i} - \left(-3\vec{i}\right)\right)}{4\times10^{-3}} = 750\ \vec{i}\ (N)$.

According to the principle of interaction $\overrightarrow{F_{2/1}} = -\overrightarrow{F_{1/2}} = -750\ \vec{i}\ (N)$;

4. a) $\Delta\overrightarrow{P_2} = m_2\overrightarrow{v_2'} - m_2\overrightarrow{v_2} = 1.5\left(-1\vec{i} - (-3)\vec{i}\right) = 3\vec{i}\ (kg.m/s)$;

$\Delta\overrightarrow{P_1} = m_1\overrightarrow{v_1'} - m_1\overrightarrow{v_1} = 0.5\left((-1)\vec{i} - 5\vec{i}\right) = -3\vec{i}\ (kg.m/s)$;

Then $\Delta\overrightarrow{P_2} = -\Delta\overrightarrow{P_1}$ (this result can be also derived basing on the principle of interaction).

b) $KE_{\text{just before}} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}\times0.5\times5^2 + \frac{1}{2}\times1.5\times3^2 = 13J$;

$KE_{\text{just after}} = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2 = \frac{1}{2}\times0.5\times1^2 + \frac{1}{2}\times1.5\times1^2 = 1J$;

$KE_{\text{just after}} < KE_{\text{just before}}$, then the kinetic energy decreases during the collision.

III-

1. During the collision, the linear momentum of the system [(A), (B)] is conserved:

	(B): M	(A): m
Just before	$\overrightarrow{v_B} = 4\ \vec{i}$	$\vec{0}$
Just after	$\overrightarrow{v_0}$	

$\vec{P}_{\text{just before collision}} = \vec{P}_{\text{just after collision}}$;

$m\ \vec{0} + M\overrightarrow{v_B} = (m+M)\overrightarrow{v_0}$;

$\overrightarrow{v_0} = \dfrac{M}{M+m}\overrightarrow{v_B} = \dfrac{200}{250}\times4\vec{i} = 3.2\ \vec{i}\ (m/s)$.

2. $ME|_{x=0} = KE|_{x=0} + PE_e|_{x=0}$;

(but $PE_e|_{x=0} = 0$ at the origin);

Then $ME|_{x=0} = \frac{1}{2}(m+M)v_0^2 + 0 = \frac{1}{2}\times250\times10^{-3}\times3.2^2 = 1.28J$.

3. The friction is neglected, then the mechanical energy is conserved: $ME|_{x=0} = ME|_{x_m} = 1.28J$;

So, $KE|_{x_m} + PE_e|_{x_m} = 1.28J$ ($KE|_{x_m} = 0$, it comes to rest);

Then $\frac{1}{2}k\,x_m^2 = 1.28$; thus $x_m = \sqrt{\dfrac{2\times1.28}{80}} = 0.18\,m = 18cm$.

4. According to Hooke's law:

$\vec{T} = -k\,x_m\,\vec{i} = -80\times0.18\vec{i} = -14.4\vec{i}\ (N)$;

The tension is acting in the negative direction and of magnitude $T = 14.4N$.

Reminder on derivatives:

$[A\cos(U)]' = -A\times U'\sin(U)$;

e.g $(2\cos7t)' = -2\times7\sin7t$.

$[A\sin(U)]' = A\times U'\cos(U)$;

e.g $(a\sin2t)' = 2\times a\cos2t$.

5. The linear momentum is given by $\vec{P} = (M+m)\vec{v}$;

But $\vec{v} = x'\vec{i} = A\,w\cos(wt)\vec{i}$;

Then $\vec{P} = (M+m)A\,w\cos(wt)$.

The maximum value of the linear momentum is $P_{\max} = (M+m)A\,w$.

IV-

The direction of motion of the shark is taken as positive direction.

1. During the collision, the linear momentum of the system [Shark, Fish] is conserved $\overrightarrow{P}_{\text{just before}} = \overrightarrow{P}_{\text{just after}}$;

   $M\overrightarrow{v_S} + m\,\vec{0} = (M+m)\overrightarrow{v_1}$;

   $\overrightarrow{v_1} = \dfrac{M}{M+m}\overrightarrow{v_S} = \dfrac{150}{150+2}\times 36\vec{i}$;

   Then $\overrightarrow{v_1} = 35.5\,\vec{i}\,(km/h)$.

	Shark: M	Fish: m
Just before	$\overrightarrow{v_S} = 36\vec{i}$ (km/h)	$\vec{0}$
Just after	$\overrightarrow{v_1}$	

2. During the collision, the linear momentum of the system [Shark, Fish] is conserved:

   $\overrightarrow{P}_{\text{just before}} = \overrightarrow{P}_{\text{just after}}$; $M\overrightarrow{v_S} + m\overrightarrow{v_f} = (M+m)\overrightarrow{v_2}$;

   $\overrightarrow{v_2} = \dfrac{M\overrightarrow{v_S} + m\overrightarrow{v_f}}{M+m} = \dfrac{150\times 36\vec{i} - 2\times 10\vec{i}}{150+2}$

   Then $\overrightarrow{v_2} = 35.4\,\vec{i}\,(km/h)$.

	Shark: M	Fish: m
Just before	$\overrightarrow{v_S} = 36\vec{i}$ (km/h)	$\overrightarrow{v_f} = -10\,\vec{i}$ (km/h)
Just after	$\overrightarrow{v_2}$	

3. During the collision, the linear momentum of the system [Shark, Fish] is conserved:

   $\overrightarrow{P}_{\text{just before}} = \overrightarrow{P}_{\text{just after}}$;

   $M\overrightarrow{v_S} + m\overrightarrow{v_f} = (M+m)\overrightarrow{v_3}$;

   $\overrightarrow{v_3} = \dfrac{M\overrightarrow{v_S} + m\overrightarrow{v_f}}{M+m} = \dfrac{150\times 36\vec{i} + 2\times 18\vec{i}}{150+2}$;

   Then $\overrightarrow{v_3} = 35.8\,\vec{i}\,(km/h)$.

	Shark: M	Fish: m
Just before	$\overrightarrow{v_S} = 36\vec{i}$ (km/h)	$\overrightarrow{v_f} = 18\vec{i}$ (km/h)
Just after	$\overrightarrow{v_1}$	

$KE_{\text{just before}} = KE_S + KE_f = \frac{1}{2}Mv_S^2 + \frac{1}{2}mv_f^2 = \frac{1}{2}\times 150\times\left(\frac{36}{3.6}\right)^2 + \frac{1}{2}\times 2\times\left(\frac{18}{3.6}\right)^2 = 7525\ J$.

$KE_{\text{just after}} = KE'_S + KE'_f = \frac{1}{2}(M+m)v_3^2 = \frac{1}{2}\times(150+2)\times\left(\frac{35.8}{3.6}\right)^2 = 7516\ J$.

$KE_{\text{just after}} < KE_{\text{just before}}$, then the kinetic energy decreases during the interaction.

V-

1. Conservation of linear momentum: $\overrightarrow{P}_{\text{just before}} = \overrightarrow{P}_{\text{just after}}$; $\vec{0} = m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2}$;

   But the total mass is conserved: $M_{\text{just before}} = M_{\text{just after}}$; $m = m_1 + m_2$;

   So $m_2 = m - m_1 = 1 - 0.4 = 0.6kg$; we get $\vec{0} = 0.4\times 3\vec{i} + 0.6\,\overrightarrow{v_2}$; then $\overrightarrow{v_2} = -2\vec{i}\,(m/s)$.

   The 2nd fragment (F_2) will move in the opposite direction to that of (F_1).
2. Just after collision the system possesses a kinetic energy KE while before collision this energy is stored as chemical potential energy PE_{chem}.
3. The PE_{chem} is converted into kinetic energy: $PE_{\text{chem}} = KE = KE_1 + KE_2 = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2$;

   $PE_{\text{chem}} = \frac{1}{2}(0.4)(3)^2 + \frac{1}{2}(0.6)(2)^2 = 3J$.
4. The linear momentum of the center of gravity $\overrightarrow{P}_{\text{just after}}$ is the sum of the linear momentums of the two fragments, and since the linear momentum is conserved through the collision.

   Then $\overrightarrow{P_G}_{\text{just after}} = \overrightarrow{P_G}_{\text{just before}} = \vec{0}$;

   Thus, $\overrightarrow{V_G} = \vec{0}$, therefore the center of gravity remains at rest.

VI-

1. During the interaction, the linear momentum of the system [Player, Ball] is conserved,

	Player: M	Ball: m
Just before	$\vec{0}$	$\vec{0}$
Just after	$\vec{v}$	$\vec{V}$

2. Conservation of linear momentum:

$\vec{P}_{\text{just before}} = \vec{P}_{\text{just after}}$;

$\vec{0} = M\vec{v} + m\vec{V}$; so $\vec{v} = -\dfrac{m}{M}\vec{V}$;

Then the recoil of the player is opposite to that of the displacement of the ball.

Thus, $v = \dfrac{0.6}{85}\times\left(\dfrac{25}{3.6}\right) = 0.05\,m/s$.

3. During this interaction, the external forces (weight and force exerted by the hand to hold the ball) are negligible compared to the internal force exerted by the player on the ball, then:

$\overrightarrow{F_{P/ball}} = \dfrac{d\overrightarrow{P_{ball}}}{dt} = \dfrac{\Delta\overrightarrow{P_{ball}}}{\Delta t} = \dfrac{m\vec{V} - 0}{\Delta t}$; in magnitude $F_{P/ball} = \dfrac{0.6}{20\times10^{-3}}\times\left(\dfrac{25}{3.6}\right) = 208.3N$.

4. $KE_{\text{just before}} = 0$;

$$KE_{\text{just after}} = KE_P + KE_b = \frac{1}{2}Mv^2 + \frac{1}{2}mV^2 = \frac{1}{2}\times85\times0.05^2 + \frac{1}{2}\times0.6\times\left(\frac{25}{3.6}\right)^2 = 14.57J;$$

VII-

1. During collision the linear momentum is conserved $\vec{P}_{\text{just before collision}} = \vec{P}_{\text{just after collision}}$;

So $\overrightarrow{P_A} + \overrightarrow{P_B} = \overrightarrow{P'_A} + \overrightarrow{P'_B}$; $m_A\overrightarrow{V_A} + m_B\overrightarrow{V_B} = m_A\overrightarrow{V'_A} + m_B\overrightarrow{V'_B}$;

The velocities are collinear then we can use algebraic expressions:

Then $V_A - V'_A = 2(V'_B - V_B)$.........(1)

The collision is elastic, then the kinetic energy is conserved:

$KE_{\text{just before collision}} = KE_{\text{just after collision}}$; $\left(V_A^2 - V_A'^2\right) = 2\left(V_B'^2 - V_B^2\right)$ .....(2)

$$\begin{cases} \dfrac{(2)}{(1)}: V_A + V'_A = V'_B + V_B \\ (1): V_A - V'_A = 2(V'_B - V_B) \end{cases};$$

Thus, $2\overline{V}_A = 3\overline{V}'_B - \overline{V}_B \Rightarrow \overline{V}'_B = \dfrac{1}{3}\left(2\overline{V}_A + \overline{V}_B\right) = 5\,m/s$ & using (1) we get $V'_A = -10\,m/s$.

2. The variation in the linear momentum of (A):

$\Delta\overrightarrow{P_A} = \overrightarrow{P'_A} - \overrightarrow{P_A} = m_A\left(\overrightarrow{V'_A} - \overrightarrow{V_A}\right) = m_A\left(-10\vec{i} - 10\vec{i}\right) = -20m_A\vec{i}$ (SI units).

The variation in the linear momentum of (B):

$\Delta\overrightarrow{P_B} = \overrightarrow{P'_B} - \overrightarrow{P_B} = m_B\left(\overrightarrow{V'_B} - \overrightarrow{V_B}\right) = +2m_A\left(5\vec{i} - \left(-5\vec{i}\right)\right) = +20m_A\vec{i}$ (SI units).

By comparison $\Delta\overrightarrow{P_A} = -20\,m_A\vec{i} = -\Delta\overrightarrow{P_B}$.

3. Newton's 2nd law: $\sum\overrightarrow{F_{ext}} = \dfrac{d\overrightarrow{P_A}}{dt}$; $\overrightarrow{w_A} + \overrightarrow{N_A} + \vec{F} = \dfrac{\Delta\overrightarrow{P_A}}{\Delta t}$;

But $\overrightarrow{w_A} + \overrightarrow{N_A} = \vec{0}$, then $\vec{F} = \dfrac{\Delta\overrightarrow{P_A}}{\Delta t}$; $-250\vec{i} = -\dfrac{20\,m_A}{16\times10^{-3}}\vec{i}$; thus, $m_A = 0.2\,kg = 200\,g$.

Problems

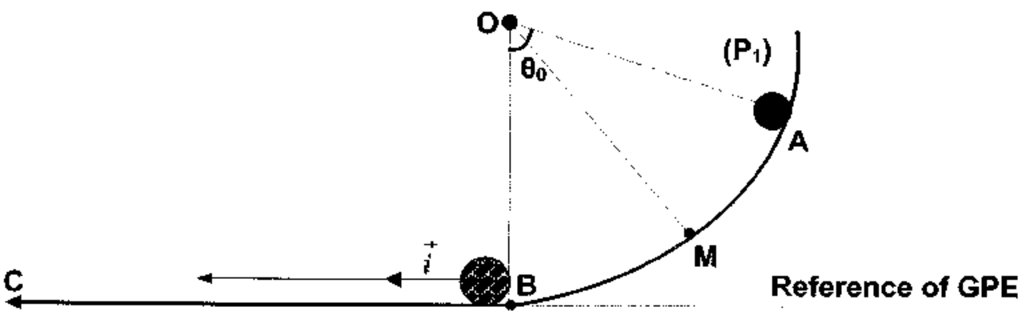
I- Collision Between two Particles

A point particle (P_1) of mass $m_1 = 200\,g$ is able to slide on a frictionless and circular path of radius $R = 1.2\,m$ of center O, as shown in the figure below.

(P_1) is placed on the circular path at a height $h_0 = 80\,cm$ then released from rest at an instant considered as origin of time $t_0 = 0$.

(P_2) is another point particle of mass $m_2 = 300\,g$ placed on the horizontal at B.

The horizontal passing through B is taken as reference of the gravitational potential energy of the system [(P_1), Earth]. **Take:** $g = 10\,m/s^2$.



Part A
Circular path

1. Calculate, to the nearest degree, the angle θ_0 that OA makes with the vertical at $t_0 = 0$.
2. Calculate the mechanical energy of the system [(P_1), Earth]
3. Show that the speed of (P_1) when it reaches the equilibrium position is $v = 4\,m/s$.
4. The position of (P_1) is defined by the angle that makes with the vertical $\theta = (OB, OM)$.
   a) Determine the expression of the gravitational potential energy in terms of m_1, g, R and θ.
   b) Deduce the position at which the kinetic energy of the system is equal the third of its mechanical energy.
5. Plot the graphs representing the variations of the mechanical energy and kinetic energy in terms of the elongation θ for $0 \le \theta \le \theta_0$.

   Scales : On the abscissa axis 1div $\equiv 10^0$; On the ordinate axis 1div $\equiv 0.4\,J$.

Part B
Collision

When (P_1) reaches the equilibrium position (vertical) with the speed $\vec{v} = v\vec{i}$ where $v = 4\,m/s$, it enters in a head on elastic collision with a particle (P_2) at rest.

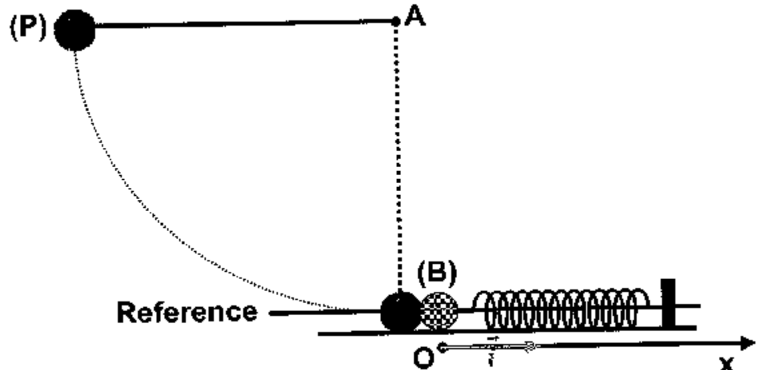
1. What are the physical quantities that remains conserved during this collision?
2. Determine the speeds v_1 and v_2 of (P_1) and (P_2) just after collision.
3. Calculate the average force exerted by (P_1) on (P_2) during the collision that lasts $\Delta t = 25\,ms$.

II-

Nature of a Collision

A pendulum is formed of a massless and inextensible string of length $\ell = 1.25m$, having one of its ends A fixed to a support while the other end carries a particle (P) of mass $m_1 = 100\,g$.

The pendulum is stretched horizontally and then released from rest.



When it passes through the equilibrium position, (P) enters in a head-on collision with another particle (B) of mass $m_2 = 300\,g$ initially at rest and connected to the free end of a spring whose constant of elasticity is $k = 120\,N/m$ as shown in the previous figure. $(O;\vec{i})$ is a horizontal axis whose origin is confounded with the center of gravity of (B) when it is a rest.

Neglect friction and take the horizontal plane passing through the center of (B) as a reference of gravitational potential energy.

Take: $g = 10m/s^2$.

Part A

Motion of the pendulum (P)

1. Calculate the mechanical energy ME of the system [(P); Earth].
2. Deduce that its speed when it passes through the equilibrium position is $v = 5\ m/s$.

Part B

Motion of the particle (B)

Due to collision, the spring is compressed to a maximum distance $x_m = 12.5cm$.

1. Calculate the mechanical energy ME' of the system [(B); Spring; Earth].
2. Deduce that its speed just after collision is $v_B = 2.5\ m/s$.
3. Determine the characteristics of the tension exerted by the spring on (B) for $x_m = 12.5cm$.

Part C

Nature of the collision

1. Determine the velocity of (P) just after collision.
2. Determine the angle to which (P) deviates after collision.
3. Specify the nature of the collision.

III-

Newton's 2nd Law

A puck (A) of mass $m = 400g$ is released from rest at the origin O of the axis $(O;\vec{i})$ as shown in figure 1, from the top of a rough inclined plane making an angle α with the horizontal where $\sin\alpha = 0.24$. Figure 1 represents the positions of the puck at different instants separated by equal intervals of time $\tau = 100ms$. **Take:** $g = 10m/s^2$.

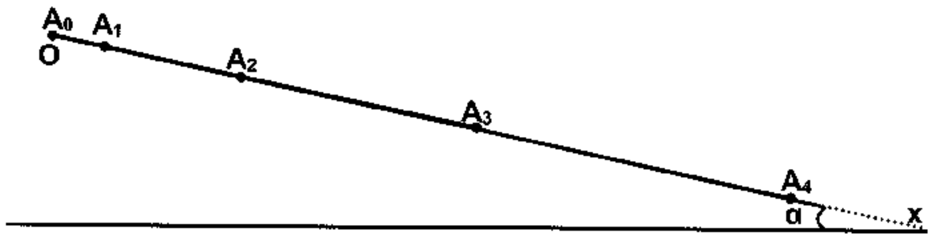


Figure 1

The recorded distances traveled are listed in the table below.

Position of the puck	O	A_1	A_2	A_3	A_4	A_5	A_6
t(ms)	0	100	200	300	400	500	600
x=OA in cm	0	0.8	3.2	7.2	12.8	20	28.8

Part A

Dynamic study

The forces of friction are not supposed negligible.

1. Verify that the velocity at A_2 is $\vec{v_2} = 0.32\vec{i}\ (m/s)$ then determine $\vec{v_4}$ at A_4.

2. a) Determine the linear momentums $\vec{P_2}$ and $\vec{P_4}$ at t_2 and t_4.

   b) Deduce the value of the ratio $\dfrac{\Delta\vec{P}}{\Delta t} = \dfrac{\vec{P_4} - \vec{P_2}}{2\tau}$.

3. a) What are the three forces acting on (A)? Represent them.

   b) Knowing that for small intervals $\dfrac{d\vec{P}}{dt} = \dfrac{\Delta\vec{P}}{\Delta t}$, by applying Newton's second law, determine the magnitude f of the force of friction.

Part B

Energetic study

The horizontal plane passing through A_2 is taken as reference for the gravitational potential energy for the system (Puck, Earth).

1. Show that the gravitational potential energy at O & A_4 are respectively $GPE_0 = 30.72mJ$ & $GPE_4 = -92.16mJ$.

2. a) Calculate the mechanical energy at the points O, A_2 & A_4.

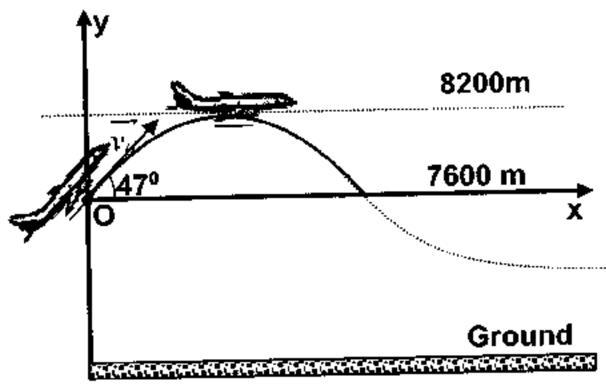
   b) Justify that the puck is subjected to a force of friction f.

3. Deduce the magnitude of the force of friction f.

IV- Physics of Airbus A310

Spring 2015, Airbus (A310 zero G) of mass $M = 1.5\times10^5\,kg$ performs its first parabolic flight. Passengers on board are in a state of weightlessness when the plane is set in the state of free fall which is achieved when the pilot succeed to maneuver the plane in the convenient parabolic trajectory.

Injection point



From a point O taken as origin of the frame $(O;\vec{i},\vec{j})$ and called injection point situated $7600\,m$ above ground the plane enters its parabolic trajectory with a speed of $527\,km/h$ making an angle of 47^0 with the horizontal as shown in the figure. It reaches its maximum altitude at $8200\,m$ with a speed of $355\,km/h$.

The horizontal plane passing through O is taken as reference of gravitational potential energy for the system (plane, Earth) and take $g = 9.81\,m/s^2$ as constant between the two heights.

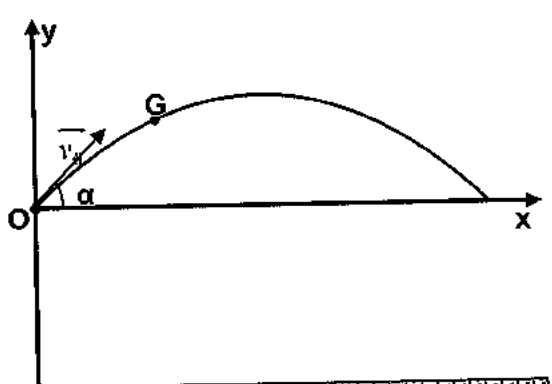
In what follows all the answers should be written using scientific notation.

Part A
Energetic study

1. Calculate the mechanical energy at the point O and at its maximum height.
2. Compare the two values and then draw a conclusion.

Part B
Linear momentum

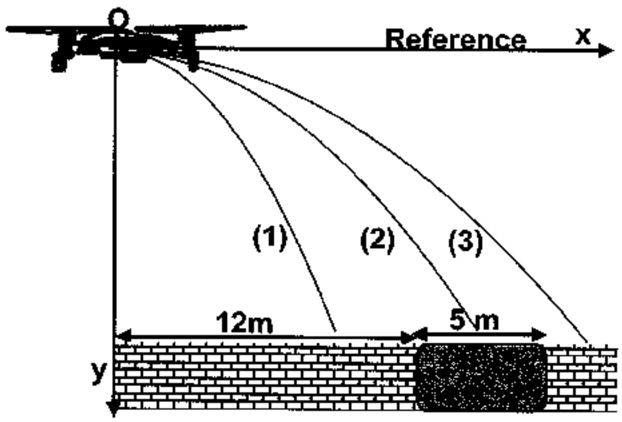
1. Justify that the resultant force acting on the plane is equal to its weight.
2. The linear momentum of the plane is denoted $\vec{P} = P_x\vec{i} + P_y\vec{j}$.
   a) Applying Newton's 2nd law, show that:
      i- the horizontal component P_x is constant.
      ii- the vertical component P_y can be written in the form $P_y = At + B$ where A & B are constants whose expressions are to be determined in terms of M, g, v_0 and α.
      iii-Deduce the values of P_x, A & B.



   b) Show that the ordinate of the plane is given by $y = -\frac{1}{2}gt^2 + v_0\sin\alpha\, t$.
3. Determine the expression of the gravitational potential energy in terms of time.
4. Deduce the time needed to perform the parabolic flight.

V- Drone Projectile

A drone plane of mass $m = 200\,g$ is flying horizontally at a height of $10\,m$ with a constant speed $v_0 = 4\,m/s$ loses its connection with the operator when it is at a horizontal distance of $12\,m$ from a pool of length $5m$, as a consequence it is in the state of free fall, neglecting friction and air resistance.



The purpose of this problem is to find whether the drone will fall in pool; trajectory (2) or on ground trajectories (1) & (3).

This position is taken as origin of the frame $(O;\vec{i},\vec{j})$ at an instant taken as origin of time $t_0 = 0$. The drone is reduced to a single point represented by its center of gravity G.

Take: $g = 10\,m/s^2$.

The horizontal plane passing through O is taken as a reference for the gravitational potential energy of the system [drone, Earth].

Part A
Energetic study

1. Calculate the mechanical energy of the system [drone, Earth].
2. Determine the speed of G when it hits the ground.
3. Specify the limits of the energetic study on the study of motion.

Part B
Point of impact

1. Applying Newton's 2nd law, show that the components of the linear momentum are given by:
   a) $P_x = 0.8\,kg.m/s$;
   b) $P_y = 2t$ where t in s & P_y in $kg.m/s$.
2. Derive the time equations of motion.
3. Show that the plane hits the ground at the instant $t = \sqrt{2}$, then justify whether the drone falls in the pool.
4. Determine the components of the velocity vector when it hits the ground.
5. Deduce the direction of the velocity when it hits the ground.

Part C
Energy during motion

1. Justify that the expression of the kinetic energy in terms of time can be written $KE = 1.6 + 10\,t^2$ where t in s & KE in J.
2. Draw the graphs representing the variations of the mechanical, kinetic and gravitational potential energies in terms of time using as scale on abscissa axis $1\,\text{div} \equiv 0.2s$ and on ordinate axis $1\,\text{div} \equiv 4J$.

VI-

Energy and Motion

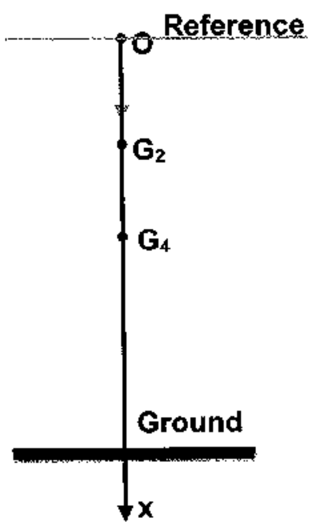
From a height $h=1.8m$, a small marble (M) of mass $m=50g$ is released from rest from a point O. The recording below showing different positions of the marble at equal intervals of time $\tau=80\,ms$.
At an instant t, the position of the marble is defined by its abscissa $x=OG$ along a vertically downwards axis $(O;\vec{i})$ and its velocity is v.
The horizontal plane passing through O is taken as reference for the gravitational potential energy of the system [marble, Earth].

Take: $g=10\,m/s^2$.

$t(ms)$	0	80	160	240	320	400	480	560
Position G_i	O	G_1	G_2	G_3	G_4	G_5	G_6	G_7
Abscissa $x(cm)$	0	3.2	12.8	28.8	51.2	80	115.2	156.8
Velocity $v(m/s)$	0	0.8		2.4			4.8	5.6

Part A

Motion before collision



1. Determine the velocities at the points G_2, G_4 & G_5.
2. Calculate the mechanical energy of the system (marble, Earth) at the points G_0, G_2 & G_6.
3. Deduce that the system is not subjected to a force of friction.
4. Show that the velocity of the marble when it hits the ground is $\vec{v_G}=6\vec{i}\,(m/s)$
5. Determine the expression of the mechanical energy at an instant t, in terms of m, g, v & x.
6. Deduce the acceleration of motion.

Part B

Collision

After collision with ground, the marble rebounds and rises up to a height of $1.25m$ where it comes to rest.

1. Show that the velocity of the marble just after collision with ground is $\vec{v'_G}=-5\,\vec{i}\,(m/s)$.
2. The duration of the collision is supposed to be $\Delta t=50ms$. Determine during Δt:

   a) the variation of the linear momentum $\Delta\vec{P}$ of the marble (M).

   b) the force $\overrightarrow{F_{G/M}}$ exerted by the ground on the marble, we can also consider that $\dfrac{\Delta\vec{P}}{\Delta t}\approx\dfrac{d\vec{P}}{dt}$.
3. Justify the following statement «during collision, the internal forces are very large to the external».

VII-(See Appendix 1 – Page 519)
Model of a Force of Friction

The purpose of this exercise is to identify whether the force of friction $\vec{f}$ acting on a marble is constant or proportional to the velocity ($\vec{f} = -h\vec{v}$ where h is a constant).

A small marble (M) of mass $m = 100\,g$ is given an initial velocity $\vec{v_0} = v_0\vec{i}$ from a point taken as origin of the axis $(O;\vec{i})$ on a horizontal surface. The position of the center of gravity G of (M) is defined by its abscissa x and its velocity is designated by $v = \frac{dx}{dt}$.

The horizontal plane passing through the center of (M) is taken as a reference of gravitational potential energy for the system [marble, Earth]. A convenient software is used to plot the graph representing the evolution of the linear momentum of the marble as a function of time as shown in figure 2.

Figure 1

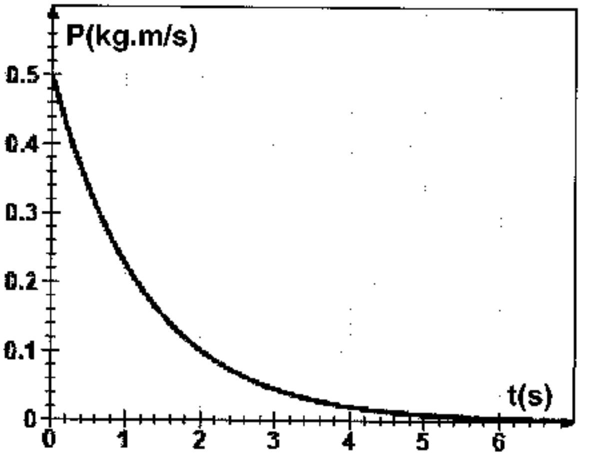


Figure 2

Part A
Interaction force

1. Verify that $v_0 = 5\,m/s$.
2. Determine the characteristics of the average force received by the marble at the instant of launch knowing that the interaction lasts $4\,ms$.

Part B
Constant force of friction

Suppose that the force of friction acting on the marble $\vec{f}$ is constant.

1. State and then represent the forces acting on the marble (M).
2. Applying Newton's 2nd law, show that the expression of the linear momentum P in terms of time t can be written in the form $\vec{P} = (At + B)\vec{i}$ where A and B are constants whose expressions to be determined in terms of f, v_0 & m.
3. Justify that the force of friction is not constant.

Part C
Variable force of friction

1. Applying Newton's 2nd law, show that the linear momentum P satisfy the differential equation: $\frac{dP}{dt} + \frac{h}{m}P = 0$.
2. Verify that $P = Ae^{-\frac{h}{m}t}$ is a solution of the previous differential equation where A is a constant whose expression to be determined in terms of m & v_0.
3. Justify that the curve drawn is compatible with the expression of P.
4. Referring to the graph, determine the value of the constant h.

Solutions

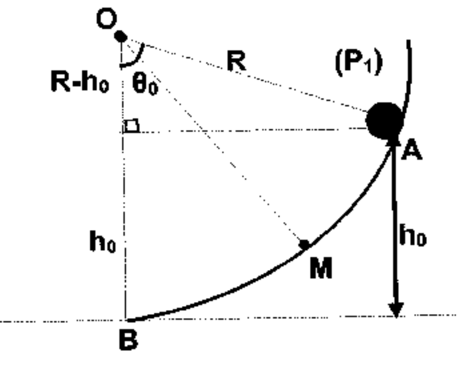
I-

Part A

1. According to the geometry of the figure:

$$\cos\theta_0 = \frac{R-h_0}{R} = \frac{1.2m - 0.8m}{1.2m} = \frac{0.4}{1.2} = \frac{1}{3};$$

$$\theta_0 = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^0.$$



2. In the absence of the forces of friction, the mechanical energy of (P_1) is conserved:

$ME = ME|_{\theta_0} = KE|_{\theta_0} + GPE|_{\theta_0}$;

$KE|_{\theta_0} = 0$ (from rest);

$ME = 0 + m\,g\,h_0 = 0.2\times10\times0.8 = 1.6J$.

3. Conservation of mechanical energy $ME = ME|_B = KE|_B + GPE|_B$; $GPE|_B = 0$ (on reference);

Then, $0 + \frac{1}{2}m v^2 = 1.6$; $v = \sqrt{16} = 4m/s$.

4. a) The gravitational potential energy is $GPE = m\,g\,h$;

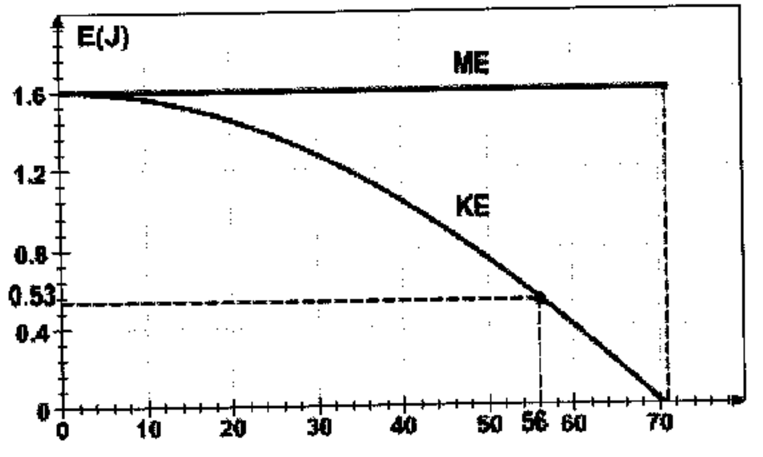
But $h = R(1-\cos\theta)$; then $GPE = m\,g\,R(1-\cos\theta)$.

b) We have $KE = \frac{ME}{3}$, but $KE = ME - GPE$, then $ME - GPE = \frac{ME}{3}$;

$GPE = \frac{2}{3}ME$, $m_1 g\,R(1-\cos\theta) = \frac{2}{3}ME$; we get $0.2\times10\times1.2(1-\cos\theta) = \frac{2}{3}\times1.6$;

$1-\cos\theta = \frac{4}{9}$; so, $\cos\theta = \frac{5}{9}$; thus, $\theta = \cos^{-1}\left(\frac{5}{9}\right) = 56^0$.

5. The mechanical energy is conserved, then it is represented by a horizontal straight line.



$KE = ME - GPE = 1.6 - 2.4(1-\cos\theta)$; so, $KE = 2.4\cos\theta - 0.8$.

The variation of the kinetic energy is not linear, it depends on $\cos\theta$.

Part B

1. The linear momentum and the kinetic energy are conserved.
2. Conservation of linear momentum:

	(P_1): m_1	(P_2): m_2
Just before	$\vec{v}$	$\vec{0}$
Just after	$\vec{v_1}$	$\vec{v_2}$

$m_1 \vec{v} + \vec{0} = m_1 \vec{v_1} + m_2 \vec{v_2}$;

The velocities are collinear,

$m_1 (v - v_1) = m_2 v_2 \ldots\ldots(1)$

Conservation of kinetic energy:

$\frac{1}{2} m_1 v^2 + 0 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$;

$m_1 (v^2 - v_1^2) = m_2 v_2^2$;

$m_1 (v - v_1) \times (v + v_1) = m_2 v_2^2$;

Dividing (2) by (1) we get : $v + v_1 = v_2 \ldots\ldots..(3)$

But, $m_1 = 0.2kg$ and $m_2 = 0.3kg$; so (1) gives $v - v_1 = \frac{3}{2} v_2 \ldots\ldots.(4)$

Adding the relations (3) and (4) we get: $2v = \left(1 + \frac{3}{2}\right) v_2$;

Then, $v_2 = \frac{4}{5} v = \frac{4}{5}(4) = \frac{16}{5} = 3.2\, m/s$;

But $v_1 = v_2 - v = 3.2 - 4 = -0.8 m/s$ [(P_1) rebounds after collision].

3. Newton's 2nd law applied on (P_2): $\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt} \approx \frac{\Delta \vec{P}}{dt}$;

$\overrightarrow{w_2} + \overrightarrow{N_2} + \overrightarrow{F_{1/2}} \approx \frac{\overrightarrow{P_2}_{\text{just after collision}} - \vec{0}}{\Delta t}$ but $\overrightarrow{w_2} + \overrightarrow{N_2} = \vec{0}$;

$\overrightarrow{F_{1/2}} \approx \frac{m_2 \overrightarrow{v_2}}{\Delta t} = \frac{0.3 \times 3.2 \vec{i}}{25 \times 10^{-3} s} = 38.4 \vec{i}\,(N)$.

II-

Part A

1. The mechanical energy of (P) in the horizontal position:

$ME_h = KE_h + GPE_h$ ($KE_h = 0$, released from rest);

$ME_h = m_1 g h = m_1 g \ell = 0.1 \times 10 \times 1.25 = 1.25J$.

In the absence of friction, the mechanical energy of (P) is conserved, then $ME = ME_h = 1.25J$

2. Conservation of mechanical energy: $ME = ME_{eq} = 1.25J$;

So, $KE_{eq} + GPE_{eq} = 1.25J$ (but $GPE_{eq} = 0$, on reference);

$\frac{1}{2} m_1 v^2 = 1.25J$; then $v = \sqrt{\frac{1.25 \times 2}{0.1}} = 5\, m/s$.

Part B

1. In the absence of friction, the mechanical energy of the system [(B); Spring; Earth] is conserved;

$ME' = ME_{x_m} = KE_{x_m} + GPE_{x_m} + PE_{e\,x_m}$; but $KE_{x_m} = 0$ (it comes to rest) & $GPE_{x_m} = 0$ (on reference); then $ME' = \frac{1}{2}k\,x_m^2 = \frac{1}{2}\times 120\times\left(12.5\times 10^{-2}\right)^2 = 0.9375\,J$.

2. The mechanical energy is conserved after collision:

   $ME' = ME_{\text{J after}} = KE_{\text{J after}} + PE_{e\,\text{J after}}$ but $PE_{e\,\text{J after}} = 0$ (no elongation);

   So, $\frac{1}{2}m_2\,v_B^2 = 0.9375\,J$, then $v_B = \sqrt{\frac{2\times 0.9375}{0.3}} = 2.5\,m/s$.

3. According to Hooke's law: $\vec{T} = -k\,x_m\,\vec{i}$;

   Direction: horizontal to the left;

   Magnitude: $T = k\,x_m = 120\times 0.125 = 15N$.

Part C

1. Conservation of linear momentum:

   $\vec{P}_{\text{just before}} = \vec{P}_{\text{just after}}$;

   $m_1\,\vec{v} + m_2\,\vec{0} = m_1\,\vec{v_P} + m_2\,\vec{v_B}$;

   $0.1\times 5\vec{i} = 0.1\times\vec{v_P} + 0.3\times 2.5\vec{i}$;

   Then $\vec{v_P} = -2.5\vec{i}\,(m/s)$.

	(P): m_1	(B): m_2
Just before	$\vec{v} = 5\vec{i}$	$\vec{v_2} = \vec{0}$
Just after	$\vec{v_P}$	$\vec{v_B} = 2.5\vec{i}$

2. The mechanical energy of the system [(P), Earth] is conserved:

   $ME_{\text{J after}} = ME_{\theta_{max}}$; $KE_{\text{J after}} + GPE_{\text{J after}} = KE_{\theta_{max}} + GPE_{\theta_{max}}$;

   (But $GPE_{\text{J after}} = 0$, on reference) & ($KE_{\theta_{max}} = 0$, it comes to rest);

   So, $\frac{1}{2}m_1\,v_P^2 = m_1\,g\,\ell(1-\cos\theta_{max})$; $1-\cos\theta_{max} = \frac{2.5^2}{2\times 10\times 1.25} = \frac{1}{4}$;

   Then $\theta_{max} = \cos^{-1}\left(\frac{3}{4}\right) \approx 41^0$.

3. The kinetic energy before collision is: $KE_{\text{just before}} = \frac{1}{2}m_1\,v^2 + 0 = \frac{1}{2}\times 0.1\times 5^2 = 1.25J$;

4. The kinetic energy after collision is:

   $KE_{\text{just after}} = \frac{1}{2}m_1\,v_P^2 + \frac{1}{2}m_2\,v_B^2 = \frac{1}{2}\times 0.1\times(2.5)^2 + \frac{1}{2}\times 0.3\times 2.5^2 = 1.25J$;

   The kinetic energy is conserved during the collision, then it is an elastic collision.

III-

Part A

1. $v_2 = \frac{A_1A_3}{2\tau} = \frac{x_3 - x_1}{2\tau} = \frac{(7.2-0.8)\times 10^{-2}\,m}{2\times 100\times 10^{-3}\,s} = 0.32m/s$;

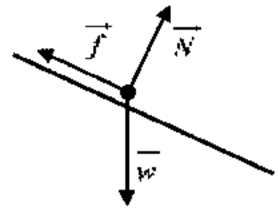
   Then $\vec{v_2} = 0.32\,\vec{i}\,(m/s)$.

   $v_4 = \frac{A_3A_5}{2\tau} = \frac{x_5 - x_3}{2\tau} = \frac{(20-7.2)\times 10^{-2}\,m}{2\times 100\times 10^{-3}\,s} = 0.64m/s$; then $\vec{v_4} = 0.64\,\vec{i}\,(m/s)$.

2. a) $\overrightarrow{P_2} = m\overrightarrow{v_2} = 0.4\,kg \times 0.32\,(m/s)\vec{i} = 0.128\vec{i}\;kg.m/s$;

$\overrightarrow{P_4} = m\overrightarrow{v_4} = 0.4\,kg \times 0.64\,(m/s)\vec{i} = 0.256\vec{i}\;kg.m/s$.

b) $\dfrac{\Delta\vec{P}}{\Delta t} = \dfrac{\overrightarrow{P_4} - \overrightarrow{P_2}}{2\tau} = \dfrac{(0.256 - 0.128)kg.m/s}{2 \times 100 \times 10^{-3}\,s}\vec{i} = 0.64\;\vec{i}\,(kg.m/s^2)$.



3. a) The forces acting are: the weight $\vec{w}$, the normal reaction $\vec{N}$ and the force of friction $\vec{f}$.

b) Newton's 2nd law: $\sum \overrightarrow{F_{ext}} = \dfrac{d\vec{P}}{dt}$; $\vec{w} + \vec{N} + \vec{f} = \dfrac{d\vec{P}}{dt}$;

By projection on the direction of motion, we get: $m\,g\sin\alpha - f = \dfrac{\Delta P}{\Delta t}$;

Then $f = m\,g\sin\alpha - \dfrac{\Delta P}{\Delta t} = 0.4 \times 10 \times 0.24 - 0.64 = 0.32\,N$.

Part B

1. $GPE_0 = +m\,g\,h_0 = m\,g\,A_0A_2\sin\alpha$ (above reference);

Then $GPE_0 = 0.4 \times 10 \times 3.2 \times 10^{-2} \times 0.24 = 0.03072\,J = 30.72\,mJ$.

$GPE_4 = -m\,g\,h_4 = -m\,g\,A_2A_4\sin\alpha$ (below reference);

Then $GPE_4 = -0.4 \times 10 \times (12.8 - 3.2) \times 10^{-2} \times 0.24 = -0.09216\,J = -92.16\,mJ$;

2. a) $ME_0 = KE_0 + GPE_0 = 0 + 0.03072 = 0.03072\,J$; ($KE_0 = 0$ released from rest)

$ME_2 = KE_2 + GPE_2 = \dfrac{1}{2}m\,v_2^2 + 0\,J = 0.5 \times 0.4 \times (0.32)^2 = 0.02048\,J$;

($GPE_2 = 0$ on reference);

$ME_4 = KE_4 + GPE_4 = 0.5 \times 0.4 \times 0.64^2 - 0.09216 = -0.01024\,J$.

b) $ME_4 < ME_2 < ME_0$, then the mechanical energy is not conserved;

Thus, the system is subjected to a force of friction.

3. According to the non-conservation of the mechanical energy: $\Delta(ME) = W_{\vec{f}}$;

$ME_4 - ME_0 = -f \times OA_4$; then $-0.01024 - 0.03072 = -f \times 0.128$; thus $f = 0.32\,N$.

IV-

Part A

1. The mechanical energy at O is:

$ME_O = KE_O + GPE_O$ (but $GPE_O = 0$, on reference);

Then $ME_O = \dfrac{1}{2}M\,v_0^2 = \dfrac{1}{2} \times 1.5 \times 10^5 \times \left(\dfrac{527}{3.6}\right)^2 = 1.6 \times 10^9\,J$.

& $ME|_{h_{max}} = KE|_{h_{max}} + GPE|_{h_{max}} = \dfrac{1}{2}M\,v^2 + M\,g\,h_{max}$;

Thus, $ME|_{h_{max}} = \dfrac{1}{2} \times 1.5 \times 10^5 \times \left(\dfrac{355}{3.6}\right)^2 + 1.5 \times 10^5 \times 9.81 \times (8200 - 7600) = 1.6 \times 10^9\,J$.

2. We have $ME|_{h_{max}} = ME_O = 1.6 \times 10^9\,J$;

Then the mechanical energy of the system (Plane, Earth) remains conserved during the flight.

Part B

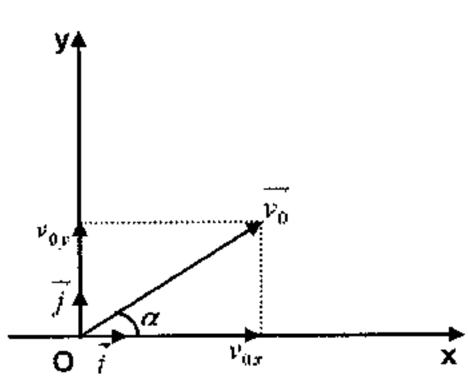
1. The plane is in the state of free fall, then the resultant force acting on it is equal to its weight.

2. a) Newton's 2nd law applied on the plane: $\sum \overrightarrow{F_{ext}} = \frac{d\overrightarrow{P}}{dt}$; $-M\, g\, \vec{j} = \frac{dP_x}{dt}\vec{i} + \frac{dP_y}{dt}\vec{j}$;

i- By identification $\frac{dP_x}{dt} = 0$, then P_x is constant.

ii- We have $\frac{dP_y}{dt} = -M\, g$; then $P_y = -M\, g\, t + P_{0y}$ which is of the form $P_y = A\,t + B$; where $A = -M\, g$ & $B = P_{0y} = M\, v_{0y}$.

But $v_{0y} = v_0 \sin\alpha$, then $B = P_{0y} = M\, v_0 \sin\alpha$.



iii- $P_x = P_{0x} = M\, v_{0x} = M\, v_0 \cos\alpha = 1.5\times 10^5 \times \left(\frac{527}{3.6}\right)\cos(47^0) = 1.5\times 10^7$ kg.m/s .

We have $A = -M\, g = -1.5\times 10^5 \times 9.81 = -1.5\times 10^6\, kg.m/s^2$;

& $B = P_{0y} = M\, v_{0y} = M\, v_0 \sin\alpha = 1.5\times 10^5 \times \left(\frac{527}{3.6}\right)\sin(47^0) = 1.6\times 10^7$ kg.m/s .

b) We have $P_y = -M\, g\, t + P_{0y}$, so $M \times v_y = -M\, g\, t + M\times v_0 \sin\alpha$;

$v_y = \frac{dy}{dt} = -\, g\, t + v_0 \sin\alpha$;

Then $y = -\frac{1}{2} g\, t^2 + v_0 \sin\alpha\, t + y_0$ (where $y_0 = 0$ origin).

3. The gravitational potential energy is given by: $GPE = M\, g\, y = -\frac{1}{2} M\, g^2\, t^2 + M\, g\, v_0 \sin\alpha\, t$;

Then $GPE = -\frac{1}{2}\times 1.5\times 10^5 \times 9.81^2\, t^2 + 1.5\times 10^5 \times 9.81\left(\frac{527}{3.6}\right)\times \sin(47^0)\, t$;

Thus $GPE = -\,7.2\times 10^6\, t^2 + 1.6\times 10^8\, t$ where t in s & GPE in J .

4. The plane achieves its parabolic flight, when it returns to the horizontal that passes through origin; So $GPE = -\,7.2\times 10^6\, t^2 + 1.6\times 10^8\, t = 0$; but $t \neq 0$;

Then $t = \frac{1.6\times 10^8}{7.2\times 10^6} = \frac{200}{9} \approx 22 s$.

V-

Part A

1. The mechanical energy at O is $ME_O = KE_O + GPE_O$; but $GPE_O = 0$ (on reference);

   Then $ME_O = \frac{1}{2} m v_0^2 + 0 = \frac{1}{2} \times 0.2 \times 4^2 = 1.6J$.

   The forces of friction are neglected, then the mechanical energy is conserved $ME = ME_O = 1.6J$

2. The mechanical energy is conserved so $ME_{\text{ground}} = ME_O = 1.6J$;

   $KE_{\text{ground}} + GPE_{\text{ground}} = 1.6J$; $\frac{1}{2} m v_G^2 - m g h_{\text{ground}} = 1.6J$;

   $\frac{1}{2} \times 0.2 \times v_G^2 - 0.2 \times 10 \times 10 = 1.6J$; then $v_G = 6\sqrt{6}\, m/s \approx 14.7\, m/s$.

3. The study of motion using the mechanical energy allows us to determine only the magnitude of the velocity without any indication concerning its direction nor trajectory.

Part B

1. The plane is moving horizontally, so $\overrightarrow{v_0} = v_0 \vec{i} = v_{0x} \vec{i} + v_{0y} \vec{i}$; then $v_{0x} = v_0 = 4m/s$ & $v_{0y} = 0$;

   The only force acting on the plane is its weight;

   Newton's 2nd law: $\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt}$, $\vec{w} = \frac{d\vec{P}}{dt}$; then $m g \vec{j} = \frac{dP_x}{dt} \vec{i} + \frac{dP_y}{dt} \vec{j}$;

   a) By identification we get: $\frac{dP_x}{dt} = 0$, so P_x is constant; then $P_x = P_{0x} = m v_{0x} = m v_0$;

      Thus, $P_x = 0.2 \times 4 = 0.8\, kg.m/s$.

   b) $\frac{dP_y}{dt} = m g = 0.2 \times 10 = 2\, kg.m/s^2$, then $P_y = 2t + P_{0y}$; where $P_{0y} = m v_{0y} = 0$;

      Thus $P_y = 2t$ where t in s & P_y in $kg.m/s$.

2. We have $P_x = 0.8 = m v_x$, so $v_x = 4m/s = \frac{dx}{dt}$; by integration we get $x = 4t + x_0$;

   But $x_0 = 0$ (at origin), so $x = 4t$ where t in s & x in m.

   We have $P_y = 2t = m v_y$, so $v_y = 10\, t = \frac{dy}{dt}$; by integration we get $y = 10 \frac{t^2}{2} + y_0$;

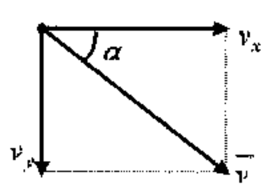
   But $y_0 = 0$, so $y = 5t^2$ where t in s & y in m.

3. At ground, $y = 5t^2 = 10$, so $t = \sqrt{2}\, s$;

   The horizontal distance until it hits the ground $d = x_G = 4t = 4\sqrt{2} \approx 5.6m \notin [12m; 17m]$;

   Then the drone will not fall in the pool.

4. At the instant of impact with ground $t = \sqrt{2}\, s$, $v_x = 4\, m/s$ & $v_y = 10\sqrt{2}\, m/s$.



5. α is the angle that makes the velocity when it hits the ground;

   Then $\tan\alpha = \frac{v_y}{v_x} = \frac{10\sqrt{2}}{4}$, thus $\alpha = \tan^{-1}(2.5\sqrt{2}) \approx 74^0$.

Part C

1. The expression of the kinetic energy is given by: $KE = \frac{1}{2} m\left(v_x^2 + v_y^2\right)$

   $KE = \frac{1}{2} \times 0.2\left(4^2 + (10t)^2\right) = 10\,t^2 + 1.6$

   where t in s & KE in J.
2. According to the conservation of the mechanical energy:

   $GPE = ME - KE$;

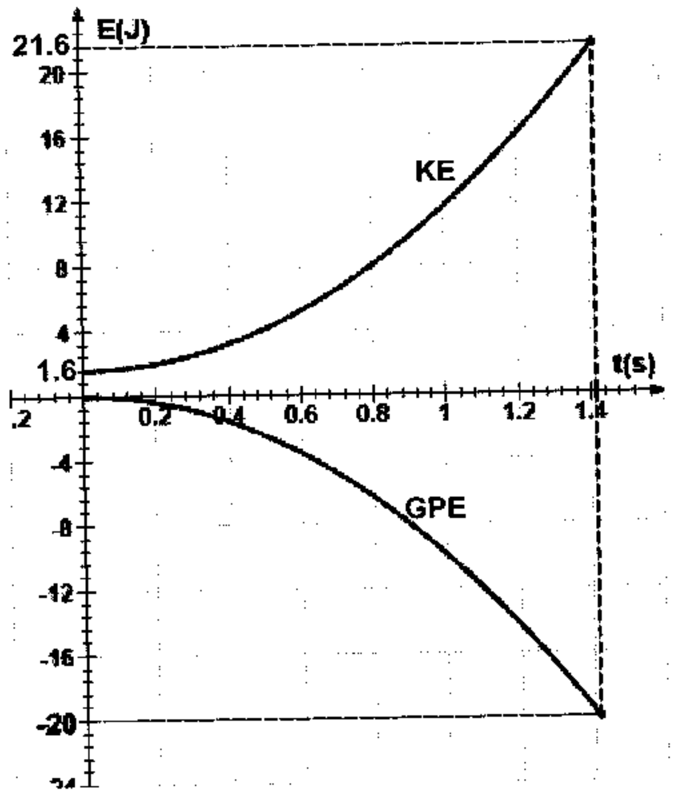
   $GPE = 1.6 - \left(10\,t^2 + 1.6\right) = -10\,t^2$;

   where t in s & KE in J;

   The curves representing KE & GPE have a parabolic shape.

   Particular values

t	0	$\sqrt{2}$
KE	1.6	21.6
GPE	0	20



VI-

Part A

1. $v_2 = \frac{G_1G_3}{2\tau} = \frac{OG_3 - OG_1}{2\tau} = \frac{(28.8 - 3.2)\times 10^{-2}\,m}{2\times 80\times 10^{-3}\,s} = 1.6\,m/s$;

   $v_4 = \frac{G_3G_5}{2\tau} = \frac{OG_5 - OG_3}{2\tau} = \frac{(80 - 28.8)\times 10^{-2}\,m}{2\times 80\times 10^{-3}\,s} = 3.2\,m/s$;

   $v_5 = \frac{G_4G_6}{2\tau} = \frac{OG_6 - OG_4}{2\tau} = \frac{(115.2 - 51.2)\times 10^{-2}\,m}{2\times 80\times 10^{-3}\,s} = 4\,m/s$.
2. $ME_0 = KE_0 + GPE_0 = 0$; where $KE_0 = 0$ (from rest) & $GPE_0 = 0$ (on reference);

   $ME_2 = KE_2 + GPE_2 = \frac{1}{2} m\,v_2^2 - m\,g\,x_2$;

   $ME_2 = \frac{1}{2}\times 50\times 10^{-3}\times 1.6^2 - 50\times 10^{-3}\times 10\times 12.8\times 10^{-2} = 0$.

   $ME_6 = KE_6 + GPE_6 = \frac{1}{2} m\,v_6^2 - m\,g\,x_6$;

   $ME_6 = \frac{1}{2}\times 50\times 10^{-3}\times 4.8^2 - 50\times 10^{-3}\times 10\times 115.2\times 10^{-2} = 0$.
3. We have $ME_6 = ME_2 = ME_0 = 0$, so the mechanical energy remains conserved, then the marble is not subjected to a force of friction.
4. Conservation of mechanical energy: $ME_G = KE_G + GPE_G = 0$;

   $\frac{1}{2}\times m\times v_G^2 - m\times g\times h_0 = 0$; then $v_G = \sqrt{2\,g\,h_0} = \sqrt{2\times 10\times 1.8} = 6\,m/s$.

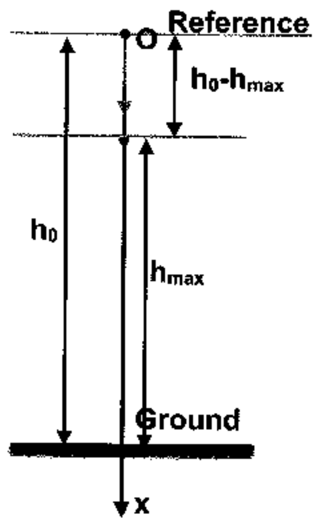
5. We have $ME = KE + GPE = \frac{1}{2} m v^2 - m g x$.

6. The mechanical energy is conserved, so $\frac{d(ME)}{dt} = 0$, $m v v' - m g x' = 0$;

But $v = x' \neq 0$ (the marble is in motion), then $a = v' = g = 10\, m/s^2$.

Part B

1. Conservation of mechanical energy (just after collision and the maximum altitude reached);



$ME'_G = ME_{h_{max}}$; $KE'_G + GPE'_G = KE_{h_{max}} + GPE_{h_{max}}$; (but $KE_{h_{max}} = 0$, it comes to rest);

$\frac{1}{2} m v'^2_G - m g h_0 = -m g (h_0 - h_{max})$;

We get $v'_G = \sqrt{2 g h_{max}} = \sqrt{2 \times 10 \times 1.25} = 5\, m/s$.

2. a) The variation in the linear momentum :

$\Delta\vec{P} = \vec{P'_G} - \vec{P_G} = m\left(\vec{v'_G} - \vec{v_G}\right)$;

So, $\Delta\vec{P} = 50 \times 10^{-3}\left(-5\vec{i} - 6\vec{i}\right) = -0.55\vec{i}\ (kg.m/s)$.

b) Newton's 2nd law: $\sum \vec{F_{ext}} = \frac{d\vec{P}}{dt}$;

$\vec{F_{G/M}} = \frac{\Delta\vec{P}}{\Delta t} = \frac{-0.55\,\vec{i}\ kg.m/s}{50 \times 10^{-3} s} = -11\,\vec{i}\ (N)$.

3. The magnitude of the resultant internal forces acting on the marble is: $F_{G/M} = 11\, N$;

The weight is the only external force acting on the marble $F_{ext} = w = m g = 50 \times 10^{-3} \times 10 = 0.5N$

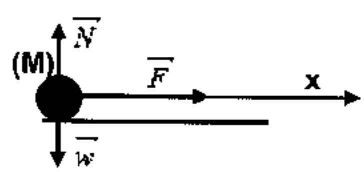
$\frac{F_{G/M}}{w} = \frac{11}{0.5} = 22 >> 1$; then, the statement is confirmed.

VII-

Part A

1. Graphically, we have $P_0 = m v_0$;

So $v_0 = \frac{P_0}{m} = \frac{0.5\, kg.m/s}{0.1 kg} = 5\, m/s$.



2. The forces acting on the marble are its weight $\vec{w}$, the normal reaction exerted by the support $\vec{N}$ and the force exerted by the operator on the marble $\vec{F}$

Newton's 2nd law applied on the marble during collision $\sum \vec{F_{ext}} = \frac{d\vec{P}}{dt}$;

$\vec{w} + \vec{N} + \vec{F} = \frac{d\vec{P}}{dt}$ but $\vec{w} + \vec{N} = \vec{0}$; then $\vec{F} = \frac{d\vec{P}}{dt} = \frac{\Delta\vec{P}}{\Delta t} = \frac{\vec{P_0} - \vec{0}}{\Delta t}$;

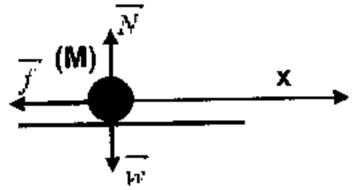
Thus, $\vec{F} = \frac{\vec{P_0}}{\Delta t} = \frac{m\vec{v_0}}{\Delta t} = \frac{0.1 \times 5\,\vec{i}}{4 \times 10^{-3}} = 125\,\vec{i}\ (N)$.

The force is horizontal to the right of magnitude $F = 125\, N$.

Part B

1. The forces acting on the marble are its weight $\vec{w}$, the normal reaction exerted by the support $\vec{N}$ and the force of friction $\vec{f}$.

2. Newton's 2nd law applied on the marble: $\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt}$;



$\vec{w} + \vec{N} + \vec{f} = \frac{d\vec{P}}{dt}$ but $\vec{w} + \vec{N} = \vec{0}$; then $\frac{dP}{dt}\vec{i} = -f\,\vec{i}$;

But f is constant, so $P - P_0 = -\int f\,dt = -f \times t$;

Thus, $P = -f \times t + P_0$ which is of the form $\vec{P} = (At + B)\vec{i}$ where $A = -f$ & $B = P_0$.

3. If f is constant, then the graphical representation of the linear momentum in terms of time should be a decreasing straight line.

Part C

1. Newton's 2nd law applied on the marble: $\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt}$; $-hv\vec{i} = \frac{dP}{dt}\vec{i}$ & $v = \frac{P}{m}$;

Then $\frac{dP}{dt} + \frac{h}{m}P = 0$.

2. We have $P = Ae^{-\frac{h}{m}t}$, so $\frac{dP}{dt} = -A\frac{h}{m}e^{-\frac{h}{m}t}$;

We replace in the differential equation: $\frac{dP}{dt} + \frac{h}{m}P = -A\frac{h}{m}e^{-\frac{h}{m}t} + A\frac{h}{m}e^{-\frac{h}{m}t} = 0$ (verified).

At $t = 0$, $P = P_0 = mv_0 = Ae^0$, then $A = mv_0$.

3. The linear momentum is an exponential decreasing function, starting from an initial value and decreases to reach a steady state.

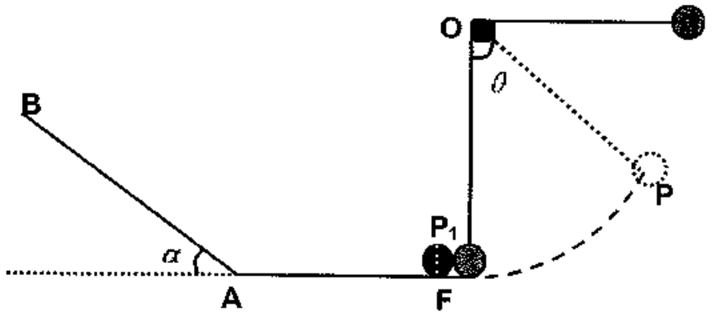
4. Graphically, if $t = 2s$ we have $P = 0.1\,kg.m/s$; but $P = P_0\,e^{-\frac{h}{m}t}$;

So, $0.1 = 0.5\,e^{-\frac{h}{0.1}\times 2}$, $0.2 = e^{-20h}$, then $h = -\frac{1}{20}\ell n(0.2) = 0.08\,kg/s$.

Supplementary Problems

I-LS & GS 2002 1st
Mechanical Energy and Collisions

Consider a material system (S) formed of an inextensible and mass less string of length $\ell = 0.45\,m$, having one of its end O fixed while the other end carries a particle (P) of mass $m = 0.1\,kg$. $g = 10\,m.s^{-2}$.



1. (S) is shifted from its equilibrium position by $\theta_m = 90^0$, while the string is under tension, and then released without initial velocity.

   Take the horizontal plane containing FA as a gravitational potential energy reference for the system [(S), Earth]. We neglect the force of friction on the axis through O and air resistance.

   a) Calculate the initial mechanical energy of the system [(S), Earth].

   b) Determine the expression of the mechanical energy of the system [(S), Earth] in terms of ℓ, m, g, v and θ where v is the speed of (P) when the string passes through a position making an angle θ with the vertical.

   c) Determine the value of $\theta\left(0 < \theta < 90^0\right)$, for which the kinetic energy of the particle (P) is equal to the gravitational potential energy of the system [(S), Earth].

   d) Calculate the magnitude of the velocity $\overrightarrow{V_0}$ of (P) as it passes through its equilibrium position.

2. Upon passing through the equilibrium position, the string is cut, and (P) enters in a head collision with a stationary particle (P_1) of mass $m_1 = 0.2\,kg$.

   As a result (P_1) is projected with a velocity $\overrightarrow{V_1}$ of magnitude $V_1 = 2\,m/s$.

   a) Determine the magnitude V of the velocity vector $\vec{V}$ of (P) right after the impact knowing that $\vec{V}$, $\overrightarrow{V_1}$ and $\overrightarrow{V_0}$ are collinear.

   b) Is the collision elastic? Justify your answer.

3. (P_1) being projected, after reaching the vertical, at F, with a velocity $V_1 = 2\,m/s$, moves along the frictionless horizontal track FA, and rises to B along the line of greatest slope of the inclined plane AB that makes an angle $\alpha = 30^0$ with the horizontal.

   a) Suppose that the friction along AB is negligible. Determine the velocity of this particle at A. Determine the position of the point M at which (P) turns back.

   b) In fact AB is not frictionless, (P) reaches a point N and turns back, where $AN = 20\,cm$. Calculate the variation of the mechanical energy of the system [(P), Earth] between A and N, and deduce the magnitude of the force of friction (assumed constant along AN).

Answer Key

1.d) $v_0 = 3\,m/s$ 2. $v = -1\,m/s$ 3.a) $AM = 0.4\,m$ 3.b) $f = 1\,N$.

II-LS & GS 2002 2nd

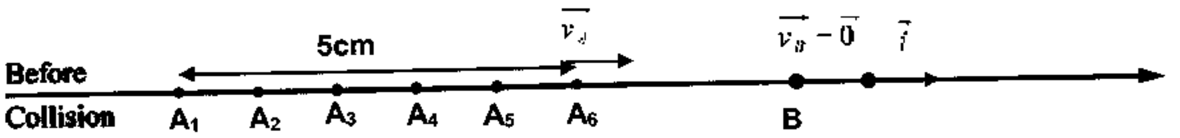
Collision and Laws of Conservation

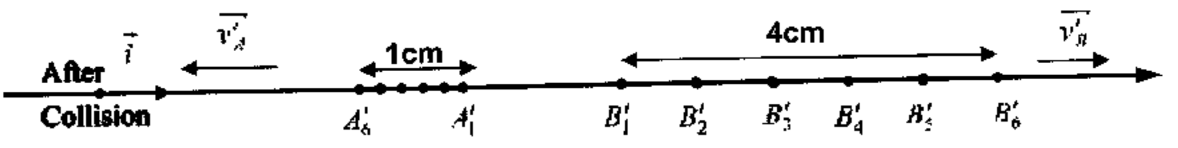
In order to study the collision between two bodies, we use a horizontal air table that is equipped with a launcher and two pucks (A) and (B) of respective masses $m_A = 0.2\,kg$ and $m_B = 0.3\,kg$.

(A), thrown with the velocity $\vec{v_A} = v_A\vec{i}$, enters in a head-on collision with (B), initially at rest.

(A) rebounds with the velocity $\vec{v'_A} = v'_A\vec{i}$ and (B) projected with the velocity $\vec{v'_B} = v'_B\vec{i}$.

The figure below shows, in real dimensions, a part of the dot-prints that register the positions of the centers of masses of (A) and (B), obtained when the time interval separating two successive dots is $\tau = 20\,ms$.





Part A
Law related to the linear momentum
Experimental study

1. Show, using the above dot-prints, that the velocities $\vec{v_A}$, $\vec{v'_A}$ and $\vec{v'_B}$ are constant and calculate the algebraic values v_A, v'_A and v'_B.
2. Determine the linear momentums $\vec{p_A}$ and $\vec{p'_A}$ of the puck (A), before and after collision respectively and $\vec{p'_B}$ of the puck (B) after collision.
3. Deduce the linear momentums $\vec{p}$ and $\vec{p'}$, of the center of mass of the system [(A) and (B)] before and after collision.
4. Compare $\vec{p}$ and $\vec{p'}$ then conclude.

Theoretical study

1. Name the forces acting on the system [(A),(B)].
2. What is the value of the resultant forces?
3. This result agrees with the conclusion of $(I-4)$. Why?

Part B
Law related to the kinetic energy

1. Calculate the kinetic energy of the system [(A),(B)] before and after collision.
2. Deduce the nature of this collision.

Answer Key

Part A 4. $\vec{p} = \vec{p'} = 0.1\vec{i}$ (kg.m/s)

Part B 1. $KE_{\text{just after collision}} = 0.025\,J$ & 2. Elastic collision.

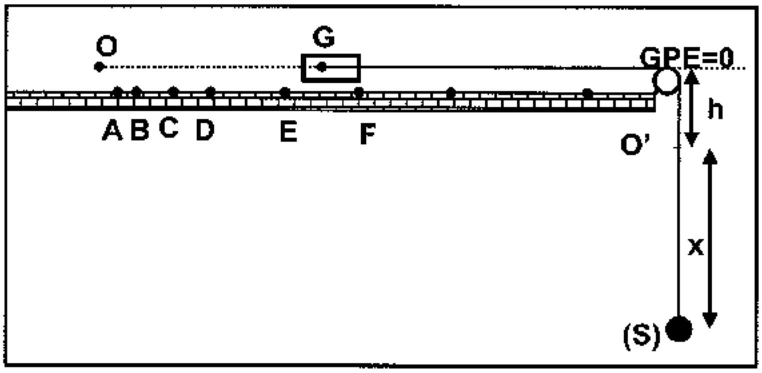
III-LS & GS 2001 2nd

Verification of Newton's Second Law

In order to verify Newton's second law relate to the dynamics of a solid in translation, we consider a puck of center of inertia G and of mass $M = 200\, g$, a horizontal air table, a solid (S) of mass $m = 50\, g$, an inextensible string and a pulley of negligible mass.

We build the set up represented in the adjacent figure. The part of the wire on the side of the puck is taut horizontally and the other part to the side of (S) is vertical.

The horizontal plane passing through G is taken as the gravitational potential energy reference.



At the instant $t = 0$, G is at O and the center of mass of (S) is at O', at a distance h below the reference.

We release (S) without initial velocity, and, at the same time, the positions of G are recorded at successive instants separated by a constant time interval $\tau = 50\, ms$.

At the instant t, G acquires a velocity $\vec{v}$ and (S) is found at a distance x below O'. Neglect all frictions and take $g = 10\, m s^{-2}$.

Part A

1. Give the expression of the mechanical energy of the system [Puck, string, (S), Earth] in terms of M, m, x, h, v and g. This energy is conserved. Why?
2. Deduce the expression of the acceleration of (S) in terms of g, m and M, then calculate its value.
3. Draw a diagram showing the forces acting on the puck and determine, using the relation $\sum \vec{F} = M\vec{a}$, the force $\vec{T}$ exerted by the string on the puck.

Part B

By means of a convenient method, we determine the speed v of the puck. The results are tabulated as shown below.

Point	A	B	C	D	E
t in ms	50	100	150	200	250
v in cm/s	10	20	30	40	50

Determine using the table, the linear momentum $\vec{P_B}$ at B and $\vec{P_D}$ at D and determine the ratio: $\frac{\Delta \vec{P}}{\Delta t} = \frac{\vec{P_D} - \vec{P_B}}{\Delta t}$.

Part C

Compare $\frac{\Delta \vec{P}}{\Delta t}$ and $\vec{T}$. Is Newton's second law thus verified? Justify.

Answer Key

Part A 2. $a = \frac{50}{200 + 50} \times 10 = 2\, m/s^2$.

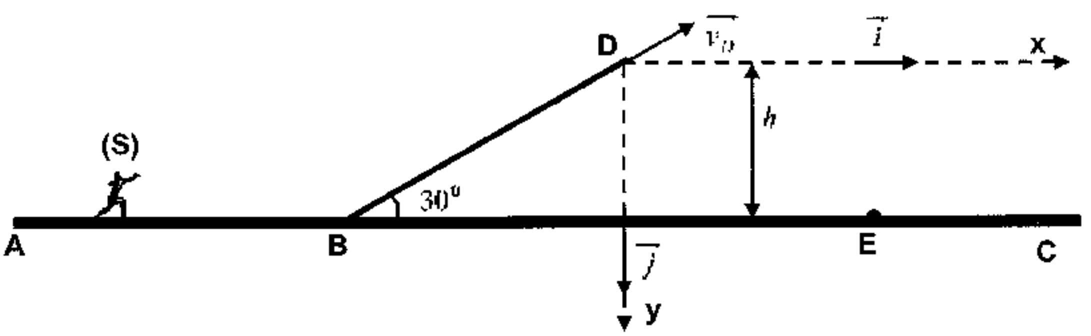
Part B 0.4 kg.m/s²

I-LS 2012 2nd

Study of the Motion of a Skier

A skier (S), of mass $m = 80\,kg$, is pulled by a boat using a rope parallel to the surface of water. He starts from point A at the instant $t_0 = 0$ without initial velocity.

The skier passes point B at the instant $t = 60\,s$ with a speed $v_B = 6\,m/s$, then he releases the rope. He continues his motion along a board BD inclined by an angle of 30^0 with respect to the horizontal surface of water. Suppose that during the passage from AB to BD the speed at point B does not change. The skier arrives point D, situated at an altitude $h = 1.6\,m$ from the water surface, with a velocity $\overrightarrow{v_D}$ then he leaves the board at point D to hit the water surface at point E (see figure below).



Given:

- ✗ The skier is considered as a particle;
- ✗ On the path AB, the force of traction $\vec{F}$ exerted by the rope on the skier has a constant magnitude F and the whole forces of friction are equivalent to a single force $\vec{f}$ opposite to the displacement, of magnitude $f = 100\,N$;
- ✗ Friction is negligible along the path BDE;
- ✗ After leaving point D the motion of the skier takes place in the vertical plane Dxy containing $\overrightarrow{v_D}$;
- ✗ The horizontal plane passing through AB is the reference level of the gravitational potential energy and $g = 10\,m/s^2$.

Part A

Motion of the skier between A and B

1. What are the external forces acting on (S) along the path AB? Draw, not to scale, a diagram of these forces.
2. Applying Newton's second law $\sum \overrightarrow{F_{ext}} = \dfrac{d\overrightarrow{P}}{dt}$ on the skier, between the points A and B, express the acceleration « a » of the motion of the skier in terms of F, f and m.
3. Determine the expression of the speed v of the skier in terms of F, f, m and the time t.
4. Deduce F.

Part B
Motion of the skier on the board BD

1. Why can we apply the principle of conservation of the mechanical of energy of system [(S), Earth] on the path BD?
2. Deduce that $v_D = 2\,m/s$.

Part C
Motion of the skier between D and E

The skier leaves the board at point D, at an instant t_0, taken as a new origin of time.

1. Apply Newton's second law on the skier to show that, at an instant t, the vertical component P_y of the linear momentum of the skier is of the form $P_y = 800\,t - 80$ (In SI unit).
2. Deduce the parametric equation $y(t)$ of the motion of the skier in the frame of reference Dxy.
3. Determine the duration taken by the skier to pass from D to E.

II-LS 2007 2nd
Mechanical Interaction

The object of this exercise is to study some physical quantities of a system whose parts are in mechanical interaction.

For that, we use two pucks (A) and (B), of respective masses $m_A = 100\,g$ and $m_B = 120\,g$, that may move without friction on a horizontal table. Each puck is surrounded by an elastic steel shock ring of negligible mass. The two pucks are connected by a massless and inextensible taut thread thus compressing the steel shock rings. The system (S) thus formed is at rest. (Figure 1)

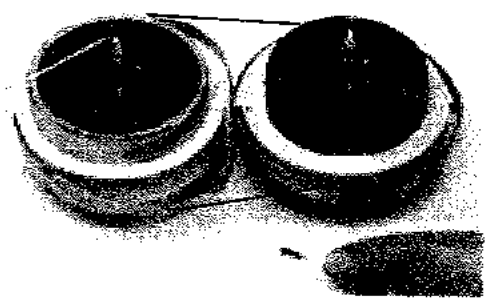


Figure 1

We burn the thread; the shock rings stretch and the pucks repel each other. The system (S) thus formed of the two pucks and the shock rings is said to "*explode*".

The positions of the center of mass of each puck are registered at successive instants separated by a constant time interval $\tau = 50\,ms$.

Figure (2) represents, on the axis $x'x$, the dot-prints of the positions of the centers of masses G_A and G_B of the two pucks after the «explosion».

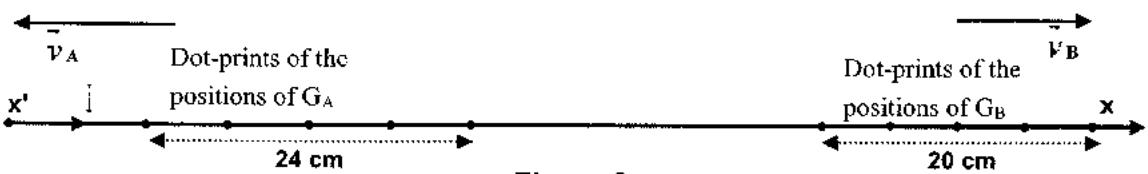


Figure 2

1. Using the document of figure (2), show that, after explosion:
   a) the motion of each puck is uniform;
   b) the speeds of (A) and (B) are $v_A = 1.2\,m/s$ and $v_B = 1\,m/s$ respectively.
2. Verify the conservation of the linear momentum of the system (S) during explosion.
3. Applying Newton's second law $\sum \overrightarrow{F_{ext}} = \frac{d\overrightarrow{P}}{dt}$ on each puck and assuming that the time interval of the explosion $\Delta t = 0.05\,s$ is so small that $\frac{\Delta \overrightarrow{P}}{\Delta t}$ has the same value as $\frac{d\overrightarrow{P}}{dt}$.

a) Determine the forces $\overrightarrow{F_{A/B}}$ and $\overrightarrow{F_{B/A}}$ exerted respectively by (A) on (B) and by (B) on (A)

b) Verify the principle of interaction.

4. The system (S) possesses a certain energy before the explosion.

a) Specify the part of (S) storing this energy.

b) In what form is this energy stored?

c) Determine the value of this energy.

III-LS 2006 1st
Verification of Newton's Second Law

A puck (S) of mass $M = 100\ g$ and of center of mass G, may slide along an inclined track that makes an angle α with the horizontal so that $\sin\alpha = 0.40$. Thus G moves along an axis $x'x$ parallel to the track as shown in figure 1.

Take: $g = 10\, m/s^2$.

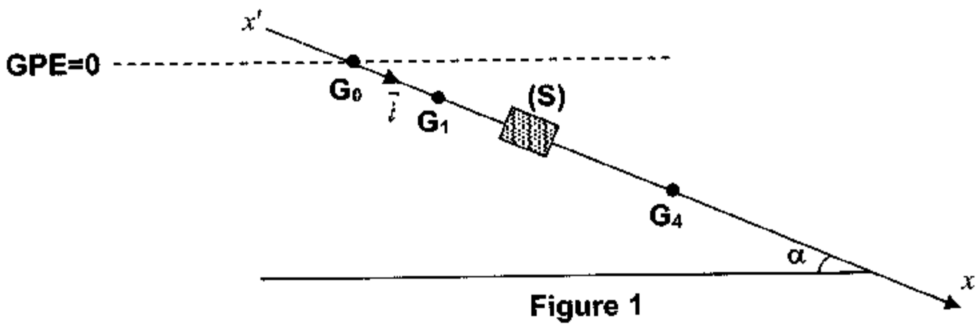


Figure 1

We release (S) without initial velocity at the instant $t_0 = 0$ and at the end of each interval of time $\tau = 50\, ms$, some positions G_0, G_1, G_2,..., G_5 of G are recorded at the instants t_0, t_1, t_2,...,t_5 respectively. The values of the abscissa x of G $\left(x = \overline{G_0 G}\right)$ are given in the table below.

t	0	τ	2τ	3τ	4τ	5τ
$x(cm)$	0	$G_0G_1 = 0.50$	$G_0G_2 = 2.00$	$G_0G_3 = 4.50$	$G_0G_4 = 8.00$	$G_0G_5 = 12.50$

1. Verify that the speed of the puck at the instants $t_2 = 2\tau$ and $t_4 = 4\tau$ are $v_2 = 0.40\, m/s$ and $v_4 = 0.80\, m/s$ respectively.

2. a) Calculate the mechanical energy of the system (puck-Earth) at the instants t_0, t_2 & t_4 knowing that the horizontal plane through G_0 is taken as a gravitational potential energy reference.

b) Why can we suppose that the puck moves without friction along the rail?

3. Determine the variation in the linear momentum $\Delta\vec{P} = \overrightarrow{P_4} - \overrightarrow{P_2}$ of (S) during $\Delta t = t_4 - t_2$.

4. a) Name the forces acting on (S) during its motion.

b) Show that the resultant $\sum\vec{F}$ of these forces may be written as $\sum\vec{F} = (Mg\sin\alpha)\,\vec{i}$.

5. Assuming that Δt is very small, $\dfrac{\Delta\vec{P}}{\Delta t}$ may be considered equal to $\dfrac{d\vec{P}}{dt}$. Show that Newton's second law is verified between the instants t_2 and t_4.

IV-LS 2004 2nd
Speed of a Bullet

A gun is used to shot bullet, each of mass $m = 20g$, with a horizontal velocity $\overrightarrow{v_0}$ of value v_0.

In order to determine v_0, we consider a set up formed of a wooden block of mass $M = 1kg$, suspended from the ends of two inextensible strings of negligible mass and of the same length (figure 1).

This set up can be taken as a block of wood suspended from the free end of a string of length $\ell = 1m$, initially at rest in the equilibrium position at G_1.

A bullet having the velocity $\overrightarrow{v_0}$ hits the block and is embedded in it at the level of the center of mass G of the block.

Just after impact, the system (block, bullet) moves with a horizontal velocity $\overrightarrow{v_1}$. The pendulum thus attains a maximum angular deviation $\alpha = 37^0$.

G_1 and G_2 are the respective positions of G in the equilibrium and in the highest position (figure 2). Take the horizontal plane through G_1 as a gravitational potential energy reference.

Neglect friction with air. **Take:** $g = 9.8\,m.s^{-2}$

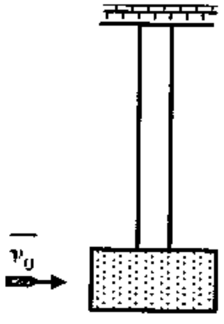


Figure 1

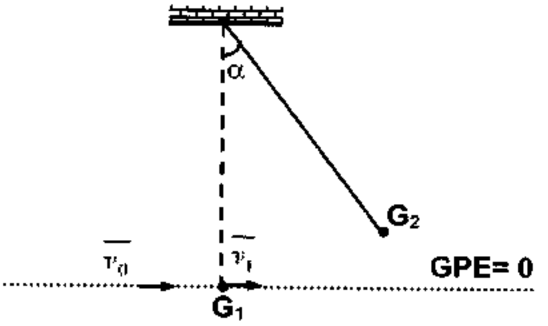


Figure 2

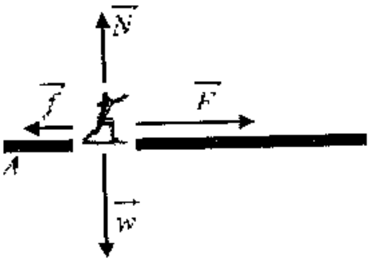
1. During a collision, which one of two physical quantities, the linear momentum or kinetic energy of the system does not remain always conserved?
2. Determine the expression of the value v_1 of the velocity $\overrightarrow{v_1}$ in terms of M, m and v_0.
3. a) Determine, just after impact, the mechanical energy of the system (pendulum, Earth) in terms of v_0, M and m.
   b) Determine, in terms of M, m, g, ℓ and α, the mechanical energy of the system (pendulum, Earth) at point G_2.
   c) Deduce the value of v_0.
4. Verify the answer of the question (1).

Sessions Solutions

I-LS 2012 2nd

Part A

1. The forces acting on (S) are:



✘ its weight $\vec{w}$;

✘ the normal reaction of the surface of water $\vec{N}$;

✘ the force of traction $\vec{F}$;

✘ the forces of friction $\vec{f}$.

2. Newton's 2nd law:

$$\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt}\text{; then } \vec{F} + \vec{f} + \vec{N} + \vec{w} = \frac{d\vec{P}}{dt} \text{ and } \vec{P} = m\vec{v}$$

Projection along the direction of motion: $F - f = ma$; then $a = \dfrac{F-f}{m}$.

3. We have: $a = \dfrac{dv}{dt} = \dfrac{F-f}{m} \Rightarrow v = \displaystyle\int \left(\frac{F-f}{m}\right)dt = \left(\frac{F-f}{m}\right)t + v_0$;

But the system is starting from rest; then $v_0 = 0$; thus $v = \left(\dfrac{F-f}{m}\right)t$.

4. For $t = 60\,s$, $v = v_B = 6\,m/s$; then $F = f + \dfrac{mv_B}{t_B} = 100 + \dfrac{80 \times 6}{60} = 108\,N$.

Part B

1. Because the forces of friction are negligible.

2. Conservation of mechanical energy $ME_B = ME_D \Rightarrow \dfrac{1}{2}mv_B^2 + 0 = \dfrac{1}{2}mv_D^2 + mgh_D$.

$$\frac{1}{2}(80)\times 6^2 + 0 = \frac{1}{2}(80)v_D^2 + 80\times 10\times 1.6\text{; then } v_D = 2\,m/s.$$

Part C

1. Newton's 2nd law $\sum \overrightarrow{F_{ext}} = \dfrac{d\vec{P}}{dt}$; $\dfrac{d\vec{P}}{dt} = mg\,\vec{j}$.

Projection along the y-axis $\dfrac{dP_y}{dt} = mg \Rightarrow P_y = \displaystyle\int mg\,dt = mgt + P_{0y}$;

Where $P_{0y} = mv_{0y} = -mv_D\sin\alpha = -80\times 2\times \sin 30^0 = -80\,kg.m.s^{-1}$;

Then $P_y = 800t - 80$ where t in s and P_y in $kg.m.s^{-1}$.

2. We have $P_y = 800t - 80 = mv_y = m\dfrac{dy}{dt} \Rightarrow y = \displaystyle\int\left(\frac{800t-80}{80}\right)dt = \int(10t-1)dt = 5t^2 - t + y_0$.

But $y_0 = 0$ then $y = 5t^2 - t$; (where t in s & y in m).

3. When the skier reaches E ; then $y=5t^2-t=1.6 \Rightarrow 5t^2-t-1.6=0$.

Solving this quadratic equation we get: $t_1=\frac{1-\sqrt{33}}{10}<0$ rejected and $t_2=\frac{1+\sqrt{33}}{10}\approx 0.67\,s$.

II-LS 2007 2nd

1. a) The distances covered by each puck, during the same intervals of time, are equal.

b) $v_B=\frac{d}{\Delta t}=\frac{d}{4\tau}=\frac{20\times 10^{-2}m}{4\times 50\times 10^{-3}s}=1\,m/s$ & $v_A=\frac{d}{\Delta t}=\frac{d}{4\tau}=\frac{24\times 10^{-2}m}{4\times 50\times 10^{-3}s}=1.2\,m/s$.

2. The system was at rest before explosion so $\overrightarrow{P}_{\text{just before}}=\vec{0}$;

& $\overrightarrow{P}_{\text{just after}}=m_A\overrightarrow{v_A}+m_B\overrightarrow{v_B}=0.1\left(-1.2\,\vec{i}\right)+0.12(1)\,\vec{i}=\vec{0}$; then $\overrightarrow{P}_{\text{just before}}=\overrightarrow{P}_{\text{just after}}$;

The linear momentum is conserved for the system formed of the two pucks.

3. a) Newton 2nd Law applied on (A) gives:

$$\sum\overrightarrow{F_A}=\frac{d\overrightarrow{P_A}}{dt}\ ;\ \overrightarrow{F_{B\to A}}+\overrightarrow{w_A}+\overrightarrow{N_A}=\frac{\Delta\overrightarrow{P_A}}{\Delta t}\Rightarrow\overrightarrow{F_{B\to A}}=\frac{0.1(-1.2-0)}{0.05}\vec{i}=-2.4\,\vec{i}\,(N).$$

$$\text{On }(B):\ \sum\overrightarrow{F_B}=\frac{d\overrightarrow{P_B}}{dt}\ ;\ \overrightarrow{F_{A\to B}}+\overrightarrow{w_B}+\overrightarrow{N_B}=\frac{\Delta\overrightarrow{P_B}}{\Delta t}\Rightarrow\overrightarrow{F_{A\to B}}=\frac{0.12\,(1-0)}{0.05}\vec{i}=2.4\,\vec{i}\,(N).$$

b) Principle of interaction: $\overrightarrow{F_{A\to B}}=-\overrightarrow{F_{B\to A}}$.

4. a) The deformed elastic shock ring.

b) Elastic potential energy.

c) The mechanical energy of the system is conserved because the system is isolated (the system does not exchange energy with the surroundings); (elastic potential energy is transformed into kinetic energy): $ME=KE+PE_e$; $ME_{\text{before}}=PE_e=ME_{\text{after}}=KE$;

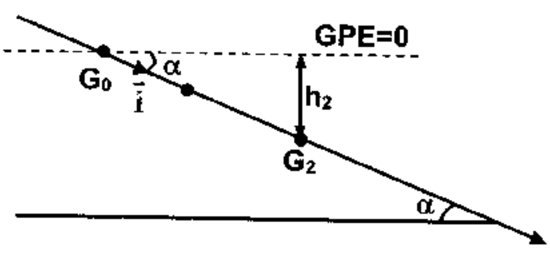
$$PE_e=\frac{1}{2}m_Av_A^2+\frac{1}{2}m_Bv_B^2=0.132\,J.$$

III-LS 2006 1st

1. $v_2=\frac{G_1G_3}{2\tau}=\frac{G_0G_3-G_0G_1}{2\tau}=\frac{(4.5-0.5)\times 10^{-2}m}{0.1s}=0.40\,m/s$;

$v_4=\frac{G_3G_5}{2\tau}=\frac{G_0G_5-G_0G_3}{2\tau}=\frac{(12.5-4.5)\times 10^{-2}m}{0.1s}=0.80\,m/s$.

2. a) The mechanical energy:



✖ $ME_0=KE_0+GPE_0=0+0=0$;

✖ $ME_2=KE_2+GPE_2=\frac{1}{2}M\,v_2^2-M\,g\,h_2$;

$h_2=G_0G_2\times\sin\alpha=0.8\,m$, so $ME_2=0$;

✖ $ME_4=KE_4+GPE_4=\frac{1}{2}M\,v_4^2-M\,g\,h_4$;

$h_4=G_0G_4\times\sin\alpha=8\times 0.4=3.2\,m\Rightarrow ME_4=0$.

b) $ME_0=ME_2=ME_4=0\,J$, then the mechanical energy is conserved during motion; Thus the system is not subjected to forces of friction between these considered instants.

3. $\Delta\overrightarrow{P}=\overrightarrow{P_4}-\overrightarrow{P_2}=M\left(v_4\,\vec{i}-v_2\,\vec{i}\right)=0.04\,\vec{i}\,\left(kg.m.s^{-1}\right)$.

4. a) The forces acting on (S) are its weight $\vec{w}$ and the normal reaction $\overrightarrow{N}$ of the support.

b) The resultant force: $\sum\overrightarrow{F}=\vec{w}+\overrightarrow{N}$.

Where $w_x=M\,g\sin\alpha$ & $N_x=0$ (perpendicular to the direction of motion);

Thus, $\sum\overrightarrow{F}=M\,g\sin\alpha\,\vec{i}$.

5. We have: $\sum\overrightarrow{F}=M\,g\sin\alpha\;\vec{i}=0.4\,\vec{i}\,(N)$ and $\frac{\Delta\overrightarrow{P}}{\Delta t}=\frac{0.04\,\vec{i}}{0.1}=0.4\,\vec{i}\,\left(kg.m/s^2\right)$.

Then $\sum\overrightarrow{F}=\frac{d\overrightarrow{P}}{dt}=\frac{\Delta\overrightarrow{P}}{\Delta t}$; thus Newton's second law is verified.

IV-LS 2004 2nd

1. The kinetic energy of the system (bullet, block).

2. Conservation of linear momentum:

	Bullet (m)	Block (M)
Just before	$\vec{v}_0$	$\vec{0}$
Just after	$\vec{v}_1$	

$\overrightarrow{P}_{\text{Just before collision}}=\overrightarrow{P}_{\text{Just after collision}}$

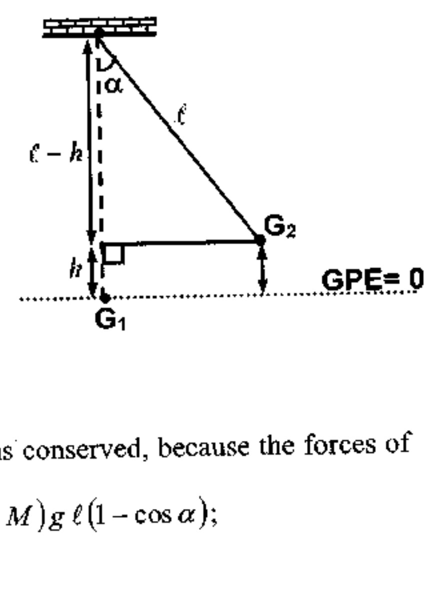
$m\,\overrightarrow{v_0}=(m+M)\overrightarrow{v_1}$; then, $v_1=\frac{m}{(M+m)}v_0$.

3. a) The mechanical energy at G_1 is given by:

$ME_{G_1}=GPE_{G_1}+KE_{G_1}$; ($GPE_{G_1}=0$, on reference)

$ME_{G_1}=0+KE_{G_1}=\frac{1}{2}(m+M)v_1^{\,2}$;

$ME_{G_1}=\frac{1}{2}(m+M)\left[\frac{m\,v_0}{(M+m)}\right]^2=\frac{1}{2}\frac{m^2v_0^{\,2}}{(M+m)}$.



b) $ME_{G_2}=GPE_{G_2}+KE_{G_2}$; $KE_{G_2}=0$ (maximum deviation);

The gravitational potential energy at G_2 is given by:

$GPE_{G_2}=(m+M)g\,h$;

where $\cos\alpha=\frac{\ell-h}{\ell}$; so $h=\ell-\ell\cos\alpha=\ell(1-\cos\alpha)$;

Then, $ME_{G_2}=(m+M)g\,\ell\,(1-\cos\alpha)$.

c) The mechanical energy of the system (pendulum, ground) is conserved, because the forces of friction are neglected, so $ME_{G_1}=ME_{G_2}$; $\frac{1}{2}\frac{m^2v_0^{\,2}}{(M+m)}=(m+M)g\,\ell\,(1-\cos\alpha)$;

Then $v_0=\frac{(M+m)}{m}\sqrt{2g\ell(1-\cos\alpha)}$; thus $v_0=101.3\,m/s$.

4. The kinetic energy just before collision $KE_{\text{just before}}=\frac{1}{2}m\,v_0^{\,2}=102.6\,J$;

The kinetic energy just after collision $KE_{\text{Just after}}=\frac{1}{2}(m+M)v_1^{\,2}=\frac{1}{2}\frac{m^2v_0^{\,2}}{(M+m)}\approx 2J$;

$KE_{\text{Just before}}\neq KE_{\text{Just after}}$; then the kinetic energy is not conserved.

Unit I

Mechanics

Chapter 3

Linear Oscillations

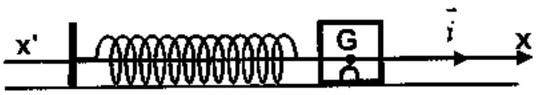
LS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Linear Oscillations		1st & 2nd(B)	1st & 2nd	2nd	1st	1st & 2nd	1st	1st	1st
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Linear Oscillations	1st & 2nd	1st & 2nd	1st	2nd	1st	1st	1st	-	1st

Essentials

I-
Free Undamped Oscillations

The motion of a simple harmonic mechanical oscillator can be described by:

- ✘ conservation of mechanical energy or $\frac{d(ME)}{dt} = 0$.
- ✘ differential equation of 2nd order of the form $x'' + \frac{k}{m}x = 0$.
- ✘ time equation of the forms
  $x = x_m \sin(w_0 t + \varphi)$ or
  $x = x_m \cos(w_0 t + \varphi)$.

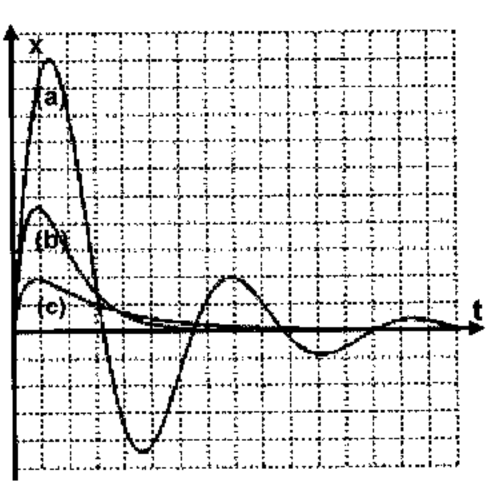


- ✘ oscillations with constant amplitude.
- ✘ a sinusoidal motion of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}}$ (this period is equal to that of velocity and acceleration of the pendulum).
- ✘ periodic elastic potential energy PE_e and kinetic energy KE of period $T_{KE} = T_{PE_e} = \frac{T_0}{2}$.

II-
Free Damped Oscillations

The motion of a simple harmonic oscillator subjected to friction can be described by:

- ✘ the differential equation $x'' + \frac{h}{m}x' + \frac{k}{m}x = 0$ where the force of friction is taken as proportional to the velocity and modeled by $\vec{f} = -h\vec{v}$.
- ✘ the decrease in the mechanical energy which is written as $\frac{d(ME)}{dt} = P_{\vec{f}}$; where $P_{\vec{f}}$ is the power dissipated by the force of friction given by: $P_{\vec{f}} = \vec{f}.\vec{v}$.
- ✘ decrease in the amplitude of oscillations with time.
- ✘ pseudo-periodic oscillations of pseudo-period T.
  - ➢ The pseudo-period is always larger than the proper period.
  - ➢ But if the system is slightly damped then T is slightly greater than T_0 and these quantities are related by: $\left(\frac{2\pi}{T}\right)^2 = \left(\frac{2\pi}{T_0}\right)^2 - \left(\frac{h}{2m}\right)^2$.



According to the magnitude of the force of friction, we can distinguish three cases:

- ✘ the pseudo-periodic mode curve (a).
- ✘ the critical mode curve (b), it is the best for the fastest return to rest.
- ✘ the non-periodic mode curve (c).

III-
Driven Oscillations

The oscillations are called driven if the system is given enough energy with its proper period in order to compensate the loss due to friction and retrieve the amplitude of oscillations.

The system is provided by a power equal to that dissipated by the force of friction $P = \frac{|\Delta(ME)|}{\Delta t}$.

IV-
Forced Oscillations

In this case we must differentiate between:

- ✘ the exciter or the external agent of adjustable frequency f.
- ✘ the resonator which is the system that is subjected to the effect of the external agent.

The period (frequency) of the system is imposed by the exciter.

The differential equation that governs the motion of the pendulum $x'' + \frac{h}{m}x' + \frac{k}{m}x = \frac{F_0}{m}\sin wt$

where $F = F_0 \sin wt$ is the force exerted by the exciter.

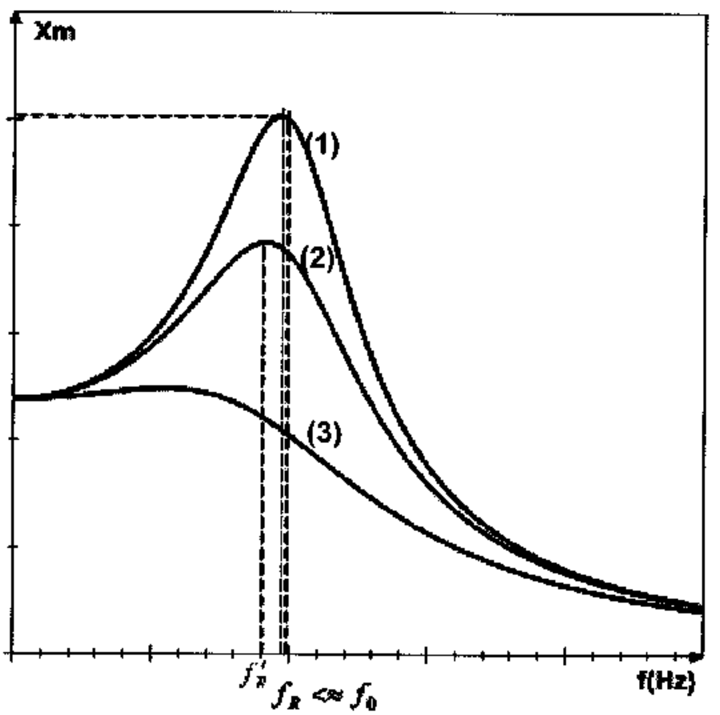
Whenever the period of the exciter becomes equal to the proper period, the pendulum is :

- ✘ the seat of the physical phenomenon called amplitude resonance.
- ✘ the system oscillates with maximum amplitude.
- ✘ $T = T_0$ under weak friction.

Under weak friction, the system is slightly damped, curve (1), the resonator enters into resonance when the frequency of the exciter f is almost equal to the proper frequency of the resonator $f = f_R \approx f_0$ (staying less than f_0).

Otherwise the system performs forced oscillations imposed by the exciter.

If the damping is strong, curve (2), the system enters into resonance for a frequency less than the proper frequency $f'_R < f_0$.



While, if the oscillations are strongly damped, curve 3, the resonance vanishes and the system performs only forced oscillations.

Applications

I- Time Equation of Motion

A time equation that describes a sinusoidal rectilinear motion is given by $x = 2\cos(4\pi t)$ where t in s & x in cm.

1. Determine the period of the oscillations performed.
2. Write the initial conditions of motion.
3. Draw the curve representing the variations of the abscissa x in terms of time.
4. Determine x'', and show that $x'' + \alpha x = 0$ where α is a constant whose value is to be determined.

II- Characteristics of Motion

The time equation that describes a sinusoidal rectilinear motion of amplitude $8\,cm$ and period $0.2\,s$ is given by $x = a\cos(w_0 t + \varphi)$ where t in s & x in cm.

1. Determine the extreme positions of motion and the values of a & w_0.
2. Determine the expression of the velocity v in terms of time.
3. Show that $v^2 = w_0^2\left(a^2 - x^2\right)$.
4. Deduce the value of v for $x = 0$, $x = +a$ or $x = -a$.
5. Give the name of the constant φ and calculate its value, knowing that at the origin of time t_0 it was at the point of abscissa $x = +a$.

In what follows:

Consider a mechanical oscillator constituted of a spring, of negligible mass, and of un-joined loops of stiffness k and a solid (S) of mass m.

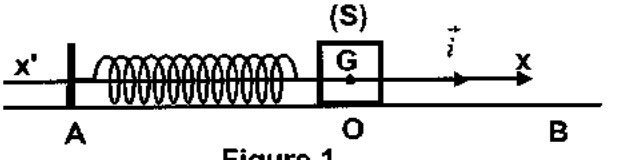


Figure 1

The spring, placed horizontally, is fixed from one of its extremities to a fixed support and (S) is attached to the other extremity. (S) may slide on a horizontal rail AB and its center of inertia G can move along a horizontal axis $x'Ox$. At equilibrium, G coincides with the origin O of the axis $x'x$ (figure 1).

At an instant t, the position of G is defined on the axis $(O;\vec{i})$, by its abscissa $x = \overline{OG}$; its velocity $\vec{v} = v\,\vec{i}$ where $v = \dfrac{dx}{dt}$.

The horizontal plane containing G is taken as a gravitational potential energy reference.

III- Graphical Study

At the instant $t_0 = 0$ taken as origin of time, G initially at the origin O and launched with a velocity $\overrightarrow{v_0} = v_0 \vec{i}$ and $v_0 > 0$.

The curves below shown in figures 1 & 2 are two possible representations of the variations of the abscissa of the center of gravity in terms of time.

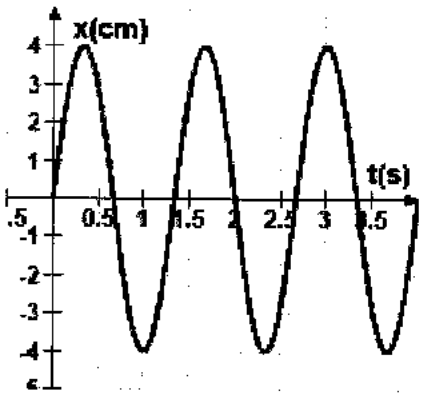


Figure 1

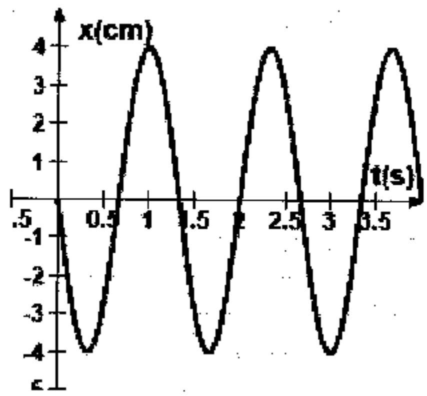


Figure 2

1. Specify the curve that describes the motion of the center of gravity under the given conditions.
2. The time equation of motion is of the form $x = a\sin(w_0 t + \varphi)$ t in s.
   a) What is the nature of motion of G? and the value of a?
   b) Give the name of the constant w_0 and its unit.
   c) Determine the values of w_0 & φ.
3. Determine the expression of the velocity in terms of time.
4. Deduce the position at which the velocity is maximum and then calculate the value of this maximum.

IV- Parameters of Period

We consider a horizontal mechanical oscillator undergoing free un-damped oscillations.

1. Write, at an instant t, the expression of the mechanical energy of the system [(S), spring, Earth] in terms of x, k, m and v.
2. Derive the differential equation, in x, that describes the motion of G.
3. The solution of the previous differential equation is of the form $x = x_m \cos(w_0 t + \varphi)$.
   a) Determine the expression of the proper angular frequency w_0 of the oscillations in terms of k and m.
   b) Deduce the expression of the proper period T_0.
4. Describe the change in the proper period in each of the following cases:
   a) the amplitude is doubled.
   b) the mass is doubled.
   c) the stiffness is doubled.

V- Oscillations and Energy

At the instant $t_0 = 0$, G initially at O, is launched with a velocity $\overrightarrow{v_0} = v_0\,\vec{i}$ and performs oscillations of constant amplitude x_m.

1. Give the expression of the proper period T_0 of the oscillations in terms of m & k.
2. Applying the principle of the conservation of mechanical energy and the expression of T_0, show that $T_0 = 2\pi \dfrac{x_m}{v_0}$.
3. Justify that the shape of the curve obtained in figure 1, agrees with the variation of elastic potential energy as a function of the square of the elongation x.

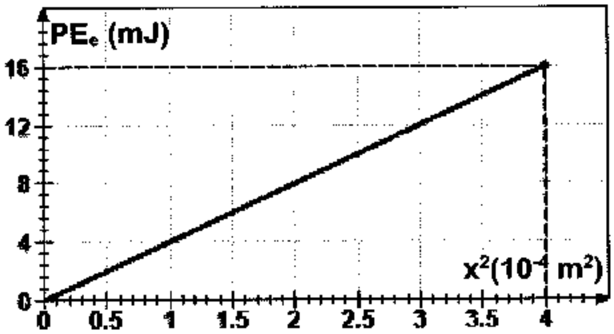


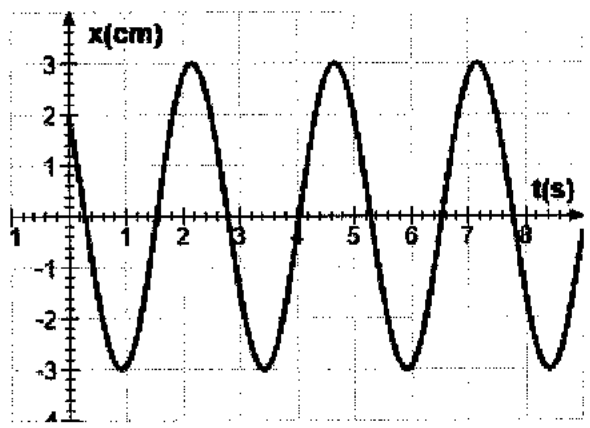
Figure 1

4. Referring to the curve of figure 1:
   a) give the value of x_m.
   b) determine:
      i- the constant of the spring k.
      ii- the value of the mechanical energy of the system.
5. Knowing that $m = 100\,g$, deduce the value of T_0 and then that of v_0.

VI- Characteristics of the Motion

We neglect friction and take $m = 1.6\ kg$ & $k = 10\ N/m$

1. Applying Newton's 2nd law, derive the differential equation of 2nd order that governs the motion of the pendulum.
2. Verify that $x = x_m \sin\left(\sqrt{\dfrac{k}{m}}t + \varphi\right)$ is a solution of the previous differential equation.
3. The curve representing the evolution of the abscissa x in terms of time t is shown in the adjacent figure.
   Determine x_m, m & φ.



4. Knowing that this box is subjected to a weak force of friction, draw a sketch showing the variations of the abscissa in terms of time if the system is released from rest when shifted by $3\,cm$ in the positive direction.
   Scales: on abscissa axis $1\,\text{div} \equiv 0.4\,s$;
   & on ordinate axis $1\,\text{div} \equiv 1\,cm$.

VII-
Energy and Oscillations

The center of gravity G is shifted by $4\,cm$ and then released from rest, on a frictionless surface.

We study during the oscillations the three forms of energy: mechanical, kinetic and elastic potential.

1. Associate to each of the curves shown in the adjacent figure it corresponding form of energy.
2. Give the period of curve 2.
3. Deduce the value of the proper period T_0 of the oscillations.
4. Determine the constant of the spring k.
5. Deduce that the mass $m \approx 255\,g$.
6. Determine the speed of the puck when it passes through the equilibrium position.

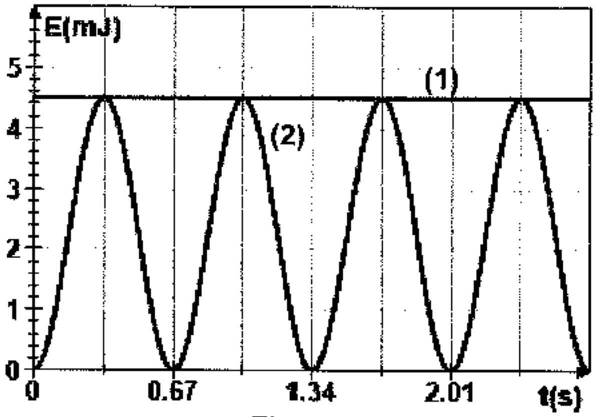
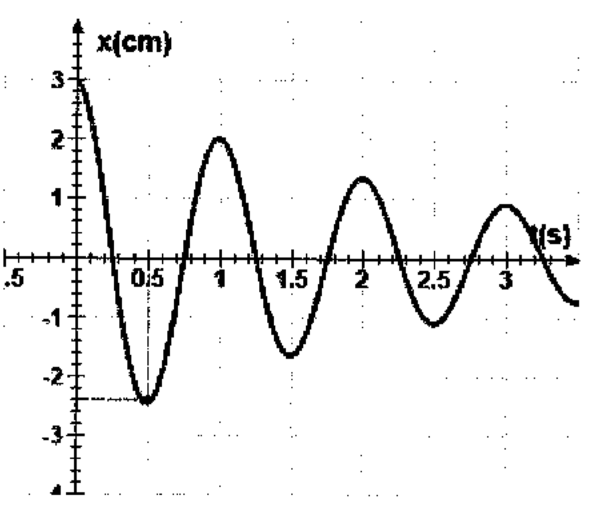


Figure 1

VIII-
Effect of Friction

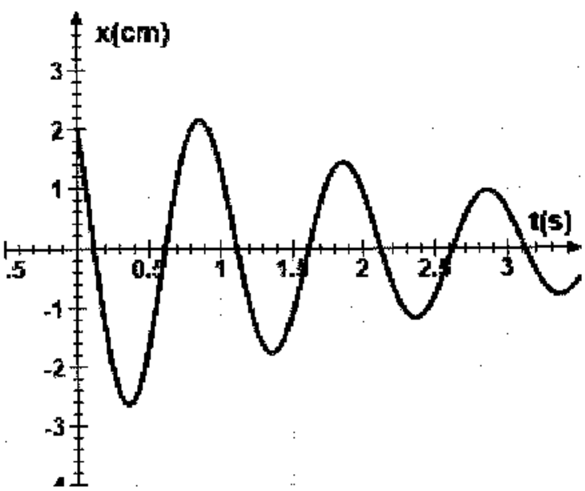
The adjacent curve represents the evolution of the abscissa of the puck in terms of time.

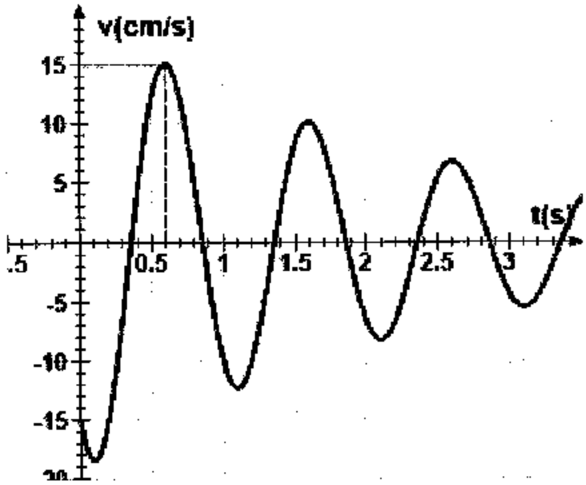
1. Give the mode of oscillations performed by the system? the pseudo-period T?
2. Calculate the variation in the mechanical energy between the instants $t_0 = 0$ & $t = 1s$ knowing that $k = 50\,N/m$.
3. Deduce the average power dissipated between the considered instants.
4. Calculate the value of the force of friction supposed constant.
5. Suppose that the force of friction $\overrightarrow{f}$ acting is modeled by a force whose expression is of the form $\overrightarrow{f} = -h\overrightarrow{v}$ where h is a constant & $\overrightarrow{v}$ its instantaneous velocity.
   a) Specify the SI units of the constant h.
   b) State and then represent the forces acting on the box, taking into considerations the origin and the direction of motion.
   c) Applying Newton's 2nd law, determine the differential equation that governs the oscillations of this system.



IX-
Friction and Acceleration

The curves below represent the variations of the abscissa x and velocity v of a system that oscillates with a mass $m = 252\,g$ connected to a spring of stiffness $k = 10\,N/m$.

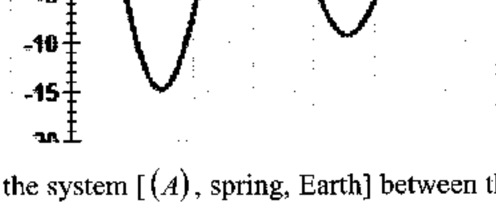




1. Write the initial conditions of motion.
2. Calculate the mechanical energy of the system [(S), spring, Earth] at the instant $t_0 = 0$.
3. Calculate the decrease in the mechanical energy between the instants $t_0 = 0$ & $t_1 = 0.6s$.
4. Knowing $\frac{d(ME)}{dt} = P_{\vec{f}}$ where ME is the mechanical energy & $P_{\vec{f}}$ is the power due to the force of friction and that this force is proportional to the velocity $\vec{f} = -b\vec{v}$.
   Derive the differential equation that governs the motion of the oscillator.
5. If $b = 0.4\,kg/s$, deduce the acceleration of (S) at $t_0 = 0$.

X-
A Mode of Oscillations

The mass of (S) is $m = 400\,g$ and the spring constant $k = 7.6\,N/m$. The adjacent curve represents the variations of the velocity in terms of time.



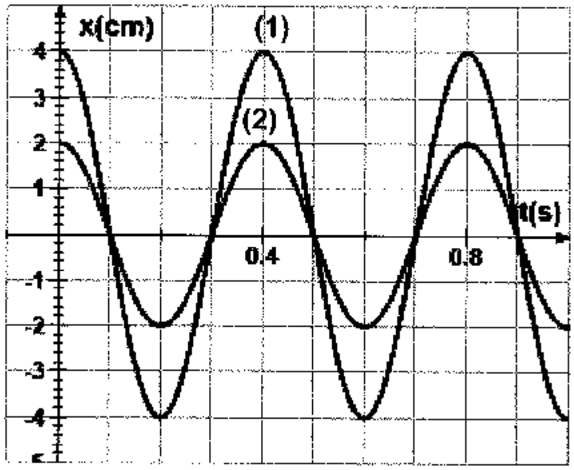
1. Give the value of the velocity at the origin of time.
2. What is the nature of motion of this pendulum?
3. We admit that the differential equation that governs the motion of the pendulum in this case is given by $x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$; with $b = 0.24\,kg/s$ is constant related to the force of friction by $\vec{f} = -b\vec{v}$.
   a) Justify that the velocity is maximum at a point away from origin.
   b) Deduce the positions where the velocity is $18cm/s$ & $12\,cm/s$.
4. Determine the variation in the mechanical energy of the system [(A), spring, Earth] between the instants $t = 0$ and $t = 1.5\,s$.
5. In order to drive the oscillations of (A), an appropriate set-up supplies the oscillator an average power P_{av}.
   a) What is meant by «drive the oscillations»?
   b) Calculate P_{av} between the instants $t = 0$ and $t = 1.5\,s$.

XI-
Influence of Amplitude

The spring constant $k = 8N/m$.

The curves shown in figure below, represent the recording of the elongation in terms of time a horizontal mechanical oscillator when we increase the amplitude of its oscillations.

The time equation of motion can be written in the form $x = x_{m1}\cos(w_{01}t + \varphi_1)$.

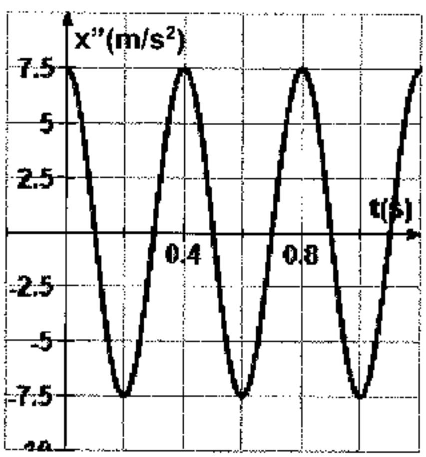


1. Referring to curve (1):
   a) give the initial conditions of motion and the value of x_m;
   b) determine w_{01} & φ.
2. Compare the periods T_{01} & T_{02} of the curves (1) & (2) respectively.
3. Deduce the mass m.
4. Compare the mechanical energies ME_1 & ME_2 corresponding respectively to the motion described by curves (1) & (2).
5. Draw a conclusion concerning the dependence of the amplitude on the period of oscillations and its mechanical energy.

XII-
Acceleration of Motion

The adjacent curve represents the variations of the acceleration of a puck undergoing free undamped oscillations when a mass $m = 100g$ is connected to the free end of the spring.

The differential equation that describes the oscillations of the pendulum is given by $x'' + \frac{k}{m}x = 0$ & take $\pi^2 = 10$.



1. The expression of the acceleration is of the form $a = x'' = A\cos(w_0t + \varphi)$.
   Referring to the adjacent curve, determine the values of A, w_0 & φ.
2. Determine the expression of the proper period T_0 of the oscillations in terms of m & k.
   Deduce the value of k.
3. Show that the expression of the amplitude of the oscillations is given by $x_m = A\left(\frac{T_0}{2\pi}\right)^2$.

   Deduce the value of x_m.

Solutions

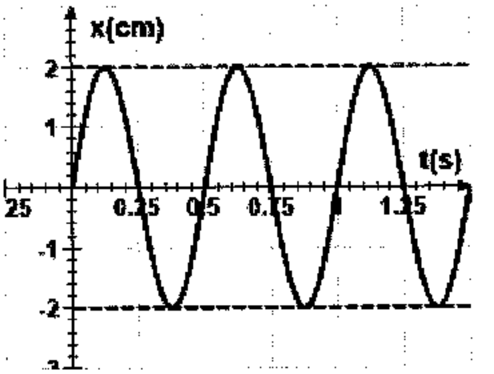
I-

1. The time equation of motion is of the form $x = x_m \cos(w_0\, t)$ where $w_0 = 4\pi\,(\text{rad}/s)$;

   The proper period is $T_0 = \dfrac{2\pi}{4\pi} = 0.5\,s$.

   Initial conditions of motion:
   At $t_0 = 0$, $x_0 = 2\cos(0) = 2cm = x_m$;

   & the velocity
   $v = x' = 2\times(-4\pi)\cos(4\pi t) = -8\pi\sin(4\pi t)$;
   then $v_0 = -8\pi\sin(0) = 0$.



2. The amplitude is $x_m = 2\,cm$ & period $T_0 = 0.5\,s$
3. Curve.
4. We have $x = 2\cos(4\pi t)$, so $v = x' = -8\pi\sin(4\pi t)$, then $x'' = -32\pi^2\cos(4\pi t)$;

   $x'' = -16\pi^2[2\cos(4\pi t)] = -16\pi^2 \times x$;

   Then $x'' + 16\pi^2 \times x = 0$ which is of the form $x'' + \alpha\, x = 0$ where $\alpha = +16\pi^2$.

II-

1. We have $x = a\cos(w_0\, t + \varphi)$ & $-1 \le \cos(w_0\, t + \varphi) \le +1$; so $-a \le a\cos(w_0\, t + \varphi) \le +a$;

   Then $-a \le x \le +a$;

   Thus, the extreme values of x are $-a = -2cm$ & $+a = +2cm$.

   We have $w_0 = \dfrac{2\pi}{T_0} = \dfrac{2\pi}{0.2} = 10\pi\,(\text{rad/s})$.

   Hence, $x = 2\cos(10\pi t + \varphi)$ (t in s & x in cm)
2. The velocity $v = x' = -2\times(10\pi)\sin(10\pi t + \varphi) = (-20\pi)\sin(10\pi t + \varphi)$ (t in s & v in cm/s).

   Its maximum value is $v_{\max} = |-20\pi| = 20\pi\,(cm/s)$.
3. We have $\cos^2(w_0\, t + \varphi) + \sin^2(w_0\, t + \varphi) = 1$; $\left(\dfrac{x}{a}\right)^2 + \left(\dfrac{-v}{a\,w_0}\right)^2 = 1$;

   Then $v^2 = w_0^2\left(a^2 - x^2\right)$.
4. For $x = 0$; $v^2 = a^2 w_0^2$; then $|v| = a\,w_0 = 20\pi\,(cm/s)$;

   For $x = a$, $v^2 = 0$; then $v = 0$ (it comes to rest);

   For $x = -a$, $v^2 = 0$; then $v = 0$ (it comes to rest).
5. Referring to the initial condition at $t = 0$, $x = +a$;

   But the time equation is given by $x = 2\cos(10\pi t + \varphi)$;

   So $2 = 2\cos\varphi$, then $\varphi = 0$ (rad);

   Thus, $x = 2\cos(10\pi t)$ (t in s & x in cm).

III-

1. Referring to the initial conditions, at $t_0 = 0$, $x_0 = 0$ & $v_0 > 0$.

   The two curves pass through origin which verify the condition for $t_0 = 0$, $x_0 = 0$; and we have $v_0 = \left.\frac{dx}{dt}\right|_{t_0=0} > 0$, then the slope of the tangent to the curve representing the variations of the abscissa x, in terms of time should be positive at the instant $t_0 = 0$.

   Thus, the curve of figure 1, verifies these condition.

2. a) The motion of G is sinusoidal (periodic or simple harmonic).

   b) The constant w_0 is called angular frequency or pulsation.

   c) a is the amplitude of motion; so $a = 4\,cm$.

   We have $w_0 = \frac{2\pi}{T_0}$ where $3T_0 = 4\,s$; then $T_0 = \frac{4}{3}s$; thus, $w_0 = \frac{2\pi}{4/3} = \frac{3\pi}{2}\text{rad}/s$.

   Initial conditions: at $t_0 = 0$, $x_0 = 0$ & $x = a\sin(w_0 t + \varphi)$; so $a\sin(\varphi) = 0$;

   Then $\varphi = 0$ & $\varphi = \pi(\text{rad})$;

   But $v = x' = a\,w_0\cos(w_0 t + \varphi)$; so $v_0 = a\,w_0\cos(\varphi) > 0$, then $\varphi = 0$.

   Thus, $x = 4\sin\left(\frac{3\pi}{2}t\right)$ (t in s & x in cm).

3. We have $v = x' = 4 \times \frac{3\pi}{2}\cos\left(\frac{3\pi}{2}t\right) = 6\pi\cos\left(\frac{3\pi}{2}t\right)$ (t in s & v in cm/s).

4. The velocity is maximum if $\left|\cos\left(\frac{3\pi}{2}t\right)\right| = 1$, so $v_{max} = 6\pi\,(cm/s)$;

   But if $\left|\cos\left(\frac{3\pi}{2}t\right)\right| = 1$ & $\cos^2\left(\frac{3\pi}{2}t\right) + \sin^2\left(\frac{3\pi}{2}t\right) = 1$; then $\sin\left(\frac{3\pi}{2}t\right) = 0$;

   & $x = 4\sin\left(\frac{3\pi}{2}t\right)$, so $x = 0$; then the velocity is maximum when it passes through origin.

IV-

1. At any instant t, the mechanical energy $ME = KE + PE_e = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$.

2. The system performs free un-damped oscillations, so its mechanical energy is conserved, then $\frac{d(ME)}{dt} = 0$, $m v v' + k x x' = 0$; but $v = x' \neq 0$ (in motion) and $v' = x''$;

   Then $v(mx'' + kx) = 0$; thus $x'' + \frac{k}{m}x = 0$.

3. a) We have $x = x_m\cos(w_0 t + \varphi)$; then $x'' = -x_m w_0^2\cos(w_0 t + \varphi)$;

   Replace in the differential equation: $-x_m w_0^2\cos(w_0 t + \varphi) + \frac{k}{m}x_m\cos(w_0 t + \varphi) = 0$;

   $x_m\cos(w_0 t + \varphi)\left(-w_0^2 + \frac{k}{m}\right) = 0$; (but $x_m\cos(w_0 t + \varphi) \neq 0$, in motion);

   So $-w_0^2 + \frac{k}{m} = 0$; thus, $w_0 = \sqrt{\frac{k}{m}}$.

b) The motion is periodic of proper period $T_0 = \dfrac{2\pi}{w_0} = 2\pi\sqrt{\dfrac{m}{k}}$.

4. a) The expression of the proper period is independent of the amplitude, then it is unchanged if we double the amplitude $T_a = T_0$.

b) The mass is doubled, the period will increase: $m_b = 2m$, so $T_b = 2\pi\sqrt{\dfrac{m_b}{k}} = 2\pi\sqrt{\dfrac{2m}{k}}$;

Then $T_b = \sqrt{2}\times\left(2\pi\sqrt{\dfrac{m}{k}}\right) = \sqrt{2}\,T_0 > T_0$.

c) The stiffness is doubled, the period will decrease: $k_c = 2k$, so $T_c = 2\pi\sqrt{\dfrac{m}{k_c}} = 2\pi\sqrt{\dfrac{m}{2k}}$;

Then $T_c = \dfrac{1}{\sqrt{2}}\times\left(2\pi\sqrt{\dfrac{m}{k}}\right) = \dfrac{T_0}{\sqrt{2}} < T_0$.

V-

1. The expression of the proper period is $T_0 = 2\pi\sqrt{\dfrac{m}{k}}$.
2. The amplitude of the oscillations is constant, then the system oscillates without friction. Conservation of mechanical energy applied between the origin $x = 0$ and $x = x_m$;

   $ME|_{x=0} = ME|_{x=x_m}$; $KE|_{x=0} + PE_e|_{x=0} = KE|_{x=x_m} + PE_e|_{x=x_m}$;

   (but $KE|_{x=x_m} = 0$, it comes to rest) & ($PE_e|_{x=0} = 0$, at origin);

   So $\dfrac{1}{2}m v_0^2 = \dfrac{1}{2}k x_m^2$; then $\dfrac{m}{k} = \dfrac{x_m^2}{v_0^2}$; thus $T_0 = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{x_m^2}{v_0^2}} = 2\pi\dfrac{x_m}{v_0}$.

3. The expression of the elastic potential energy is given by $PE_e = \dfrac{1}{2}k x^2$;

   Then this energy is proportional to the square of the elongation; thus its graphical representation $PE_e = f(x^2)$ should be a straight line passing through origin which is verified.

4. a) The maximum value of PE_e, corresponds to the greatest value of the elongation, then $x_m^2 = 4\times10^{-4}\,m^2$; thus $x_m = 2\times10^{-2}\,m = 2cm > 0$.

   b) Graphically:

   i- $\dfrac{1}{2}k$ is the slope of the straight line $PE_e = f(x^2)$;

   Then $\dfrac{1}{2}k = \dfrac{\Delta(PE_e)}{\Delta(x^2)} = \dfrac{(16-0)m\,J}{(4-0)\times10^{-4}\,m^2} = \dfrac{16\times10^{-3}\,J}{4\times10^{-4}\,m^2} = 40\,J/m^2$; thus $k = 80N/m$.

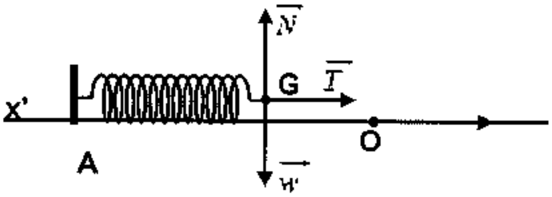
   ii- The mechanical energy is conserved $ME = ME|_{x=x_m} = PE_e|_{x=x_m} = 16\,mJ$.

5. We have $T_0 = 2\pi\sqrt{\dfrac{m}{k}} = 2\pi\sqrt{\dfrac{100\times10^{-3}}{80}} = 0.22\,s$;

   & $T_0 = 2\pi\dfrac{x_m}{v_0}$; then $v_0 = 2\pi\dfrac{x_m}{T_0} = 2\pi\dfrac{0.02}{0.22} = 0.57\,m/s$.

VI-

1. The forces acting on the system are:
   - its weight $\vec{w}$;
   - the normal reaction of the support $\vec{N}$;
   - the tension of the spring $\vec{T}$.



Newton's 2nd law $\sum \vec{F} = \frac{d\vec{P}}{dt}$;

$\vec{w} + \vec{N} + \vec{T} = m\vec{v}'$;

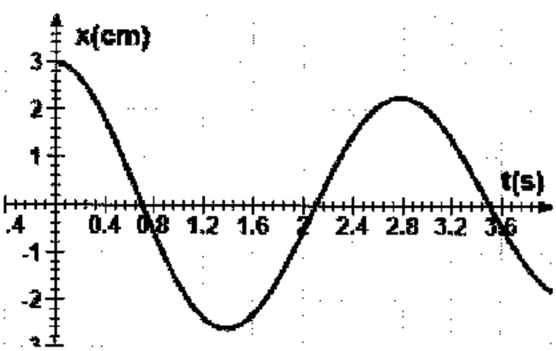
But $\vec{w} + \vec{N} = \vec{0}$; so $-k\,x\vec{i} = m\vec{x''}$; then $x'' + \frac{k}{m}x = 0$.

2. We have $x = x_m \sin\left(\sqrt{\frac{k}{m}}t + \varphi\right)$, then $x'' = -x_m\left(\frac{k}{m}\right)\sin\left(\sqrt{\frac{k}{m}}t + \varphi\right)$;

   Replacing in the differential equation, we get :

   $x'' + \frac{k}{m}x = x_m\left(\frac{k}{m}\right)\sin\left(\sqrt{\frac{k}{m}}t + \varphi\right) + x_m\left(\frac{k}{m}\right)\sin\left(\sqrt{\frac{k}{m}}t + \varphi\right) = 0$; (verified).

3. From graph $x_m = 3cm$;



   Referring to the initial conditions: at $t_0 = 0$, $x_0 = 2cm$ & $v_0 < 0$ (the curve is decreasing at this instant);

   So $2 = 3\sin\varphi$, $\sin(\varphi) = \frac{2}{3}$,

   Then $\varphi = \sin^{-1}\left(\frac{2}{3}\right) = 0.73$ rad ;

   or $\varphi = \pi - 0.73 = 2.41$ rad ;

   But $v_0 = x_m\sqrt{\frac{k}{m}}\cos(\varphi) < 0$, then $\cos(\varphi) < 0$; thus $\cos(\varphi) = 2.41$ rad ;

   We get $x = 3\sin\left(\sqrt{\frac{10}{1.6}}t + 2.41\right) = 3\sin(2.5t + 2.41)$ (t in s & x in cm).

4. Under weak friction, the amplitude will decrease and the pseudo-period becomes greater than the proper period.

VII-

1. The curve labelled (1) represents a constant quantity, which is the mechanical energy because it is conserved in the absence of friction.

   Referring to the initial conditions, at $t = 0$ we have $x_0 = 4cm$ & $v_0 = 0$ (released from rest).

   Then the initial kinetic energy is $KE_0 = 0$;

   Thus, the curve (2) represents the kinetic energy.

2. The period of energies is $T_{KE} = 0.67\,s$.
3. The proper period $T_0 = 2T_{KE} = 2 \times 0.67 = 1.34s$.

4. The mechanical energy is conserved $ME = KE_0 + PE_{e0}$; $4.5\times10^{-3} = \frac{1}{2}k\,x_0^2 = \frac{1}{2}k\left(4\times10^{-2}\right)^2$;

Then $k = \frac{4.5\times10^{-3}\times2}{\left(4\times10^{-2}\right)^2} = 5.6\,N/m$.

5. The expression of the proper period is given by $T_0 = 2\pi\sqrt{\frac{m}{k}}$, then $m = \frac{T_0^2\times k}{4\pi^2}$;

Thus, $m = \frac{1.34^2\times5.6}{4\pi^2} = 0.255\,kg = 255\ g$.

6. The mechanical energy is conserved $ME = ME\big|_{x=0} = KE\big|_{x=0} + PE_e\big|_{x=0}$;

$4.5\times10^{-3} = \frac{1}{2}m\,v_0^2$; then $v_0 = \sqrt{\frac{2\times4.5\times10^{-3}}{0.255}} = 0.19\,m/s$.

VIII-

1. The oscillations are called free damped of pseudo-period $T = 1s$.
2. The variation $\Delta(ME) = ME\big|_{t=1s} - ME\big|_{t=0} = \left(KE\big|_{t=1s} + PE_e\big|_{t=1s}\right) - \left(ME\big|_{t=0} + PE_e\big|_{t=0}\right)$;

   But $KE\big|_{t=1s} = KE\big|_{t=0} = 0$ (at the local maximum it comes to rest).

   $\Delta(ME) = PE_e\big|_{t=1s} - PE_e\big|_{t=0} = \frac{1}{2}k\left(x_m\big|_{t=1s}\right)^2 - \frac{1}{2}k\left(x_m\big|_{t=0}\right)^2$;

   Then $\Delta(ME) = \frac{1}{2}\times50\left[\left(2\times10^{-2}\right)^2 - \left(3\times10^{-2}\right)^2\right] = -0.0125\,J$.
3. The power dissipated is $P_{dis} = \frac{\Delta(ME)}{\Delta t} = \left(\frac{-0.0125}{1-0}\right) = -0.0125\,W$.
4. We have $\Delta(ME) = W_{\vec{f}}$ where $W_{\vec{f}} = -f\times d$; with $d = 3cm + 2\times2.4cm + 2cm = 9.8\,cm$;

   Then $f = -\frac{\Delta(ME)}{d} = \frac{-0.0125}{9.8\times10^{-2}} = 0.12\,N$.
5. a) We have $\vec{f} = -h\vec{v}$;

   So $[h] = \frac{[f]}{[v]} = \frac{N}{m/s} = \frac{kg.m/s^2}{m/s} = kg/s$.

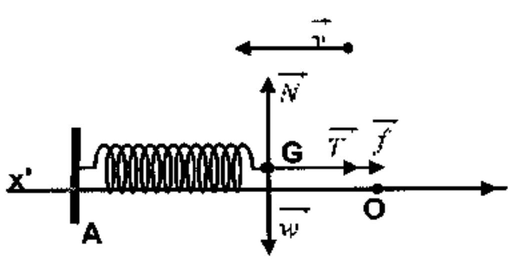
   b) The forces acting on the system are:

   ✖ its weight $\vec{w}$;

   ✖ the normal reaction of the support $\vec{N}$;

   ✖ the tension of the spring $\vec{T}$;

   ✖ the force of friction $\vec{f}$.



   c) Newton's 2nd law $\sum\vec{F} = \frac{d\vec{P}}{dt}$ $\vec{w} + \vec{N} + \vec{T} + \vec{f} = m\vec{v'}$; but $\vec{w} + \vec{N} = \vec{0}$;

   So $-kx\vec{i} - hv\vec{i} = m\vec{x''}$; then $x'' + \frac{h}{m}x' + \frac{k}{m}x = 0$.

IX-

1. The initial conditions are: at $t_0 = 0$, $x_0 = 2cm$ & $v_0 = -15\,cm/s$.

2. The mechanical energy $ME|_{t_0=0} = KE|_{t_0=0} + PE_e|_{t_0=0} = \frac{1}{2}m v_0^2 + \frac{1}{2}k x_0^2$;

$$ME|_{t_0=0} = \frac{1}{2} \times 252 \times 10^{-3} \times \left(-15 \times 10^{-2}\right)^2 + \frac{1}{2} \times 10 \times \left(2 \times 10^{-2}\right)^2 = 4.84 \times 10^{-3} J.$$

3. $ME|_{t_1} = KE|_{t_1} + PE_e|_{t_1} = \frac{1}{2}m v_1^2 + 0 = \frac{1}{2} \times 252 \times 10^{-3} \times \left(15 \times 10^{-2}\right)^2 = 2.84 \times 10^{-3} J$.

   The decrease in the mechanical energy is:

   $\Delta(ME) = ME|_{t_1} - ME|_{t_0} = 2.84 \times 10^{-3} J - 4.84 \times 10^{-3} J = -2 \times 10^{-3} J$.

4. At any instant t, the mechanical energy is $ME = KE + PE_e = \frac{1}{2}m v^2 + \frac{1}{2}k x^2$;

   and $P_{\vec{f}} = \vec{f}.\vec{v} = -b\,\vec{v}.\vec{v} = -b\,v^2$ & $\frac{d(ME)}{dt} = P_{\vec{f}}$; so $\frac{d}{dt}\left(\frac{1}{2}m v^2 + \frac{1}{2}k x^2\right) = -b v^2$;

   $m v v' + k x x' = -b x'^2$; (but $v = x' \neq 0$, in motion) & $v' = x''$; then $x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$;

5. The differential equation is verified at any instant, $a + \frac{b}{m}v + \frac{k}{m}x = 0$;

   Then $a|_{t_0=0} = -\frac{b}{m}v|_{t_0=0} - \frac{k}{m}x|_{t_0=0} = \frac{-0.4}{0.252} \times (-15) - \frac{10}{0.252} \times 2 = -56\,cm/s^2$.

X-

1. The initial conditions (at $t_0 = 0$) is $v_0 = v_{max} = 18\,cm/s$ (maximum).
2. The motion is pseudo-periodic.
3. a) We have: $x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$, the velocity is maximum so $v' = x'' = 0$;

   We get $+\frac{b}{m}v_{max} + \frac{k}{m}x = 0$, then $v_{max} = -\frac{k}{b}x \neq 0$, the maximum is not reached at origin.

   b) At $t = 0$, $v = v_{max1} = 0.18\,m/s$, then $x_1 = -\frac{b}{k}v_{max1} = -\frac{0.24 \times 0.18}{6.7} = -6.4 \times 10^{-3} m$;

   At $t = 1.5$, $v = v_{max2} = 0.12\,m/s$, then $x_2 = -\frac{b}{k}v_{max2} = -\frac{0.24 \times 0.12}{6.7} = -4.3 \times 10^{-3} m$;

4. At the instants $t = 0$ and $t = 1.5\,s$, the velocity reaches a local maximum.

   $\Delta(ME) = ME|_{t=1.5s} - ME|_{t=0} = \left(KE|_{t=1.5s} + PE_e|_{t=1.5s}\right) - \left(KE|_{t=0} + PE_e|_{t=0}\right)$;

   So, $\Delta(ME) = \frac{0.4}{2}\left(12 \times 10^{-2}\right)^2 + \frac{6.7}{2}\left(6.4 \times 10^{-3}\right)^2 - \frac{0.4}{2}\left(18 \times 10^{-2}\right)^2 - \frac{6.7}{2} \times \left(4.3 \times 10^{-3}\right)^2$;

   Then $\Delta(ME) = -3.52 \times 10^{-3} J$.

5. a) The oscillations are called driven if the system receives amount of energy enough to compensate the loss due to friction in order to restore the amplitude and the proper period of oscillations.

   b) The power supplied is $P_{av} = \frac{|\Delta(ME)|}{\Delta t} = \frac{\left(3.52 \times 10^{-3}\right)J}{(1.5 - 0)s} = 2.4 \times 10^{-3} W = 2.4\,mW$.

XI-

1. a) At $t_0 = 0$, $x_0 = 4\,cm$, $v_0 = 0$ & $x_m = 4cm$.

   b) From curve $T_{01} = 0.4\,s$ & $w_{01} = \dfrac{2\pi}{T_{01}} = \dfrac{2\pi}{0.4} = 5\pi\,(\text{rad}/s)$;

   Referring to the initial conditions of motion, we get: $x_m = x_m \cos\varphi$, $\cos\varphi = 1$; then $\varphi = 0$.

   Thus, $x = 4\cos(5\pi t)$ (t in s & x in cm).

2. Graphically $T_{01} = T_{02} = 0.4\,s$.

3. We have $T_{01} = 2\pi\sqrt{\dfrac{m}{k}}$; so $m = \dfrac{T_{01}^2 \times k}{4\pi^2} = \dfrac{0.4^2 \times 8}{4\pi^2} = 0.032\,kg = 32\,g$.

4. The amplitude of each curve remains constant, then the mechanical energy is conserved.

   $ME_1 = ME_1\big|_{t=0} = PE_{el}\big|_{t=0} = \dfrac{1}{2}k\,x_{m1}^2$, so ME depends on the square of the amplitude.

   But $x_{m2} < x_{m1}$, then $ME_2 < ME_1$.

5. We can conclude that:
   - ✘ the proper period is independent of the amplitude since the period remains constant even if the amplitude is modified;
   - ✘ the mechanical energy depends on the amplitude since it changes when the amplitude is modified.

XII-

1. A is the amplitude of the acceleration $A = 7.5\,m/s^2$;

   The proper period $T_0 = 0.4\,s$ & the angular frequency $w_0 = \dfrac{2\pi}{T_0} = \dfrac{2\pi}{0.4} = 5\pi\,(\text{rad/s})$;

   Referring to the initial conditions, at $t = 0$, $a = 7.5\,m/s^2$, we get $\cos\varphi = 1$; then $\varphi = 0$.

   Thus, $a = 7.5\cos(5\pi t)$ where t in s & a in m/s^2.

2. The differential equation that governs the motion $x'' + \dfrac{k}{m}x = 0$ is of 2nd order of the form

   $x'' + w_0^2 x = 0$ where $w_0 = \sqrt{\dfrac{k}{m}}$; then its motion is periodic of proper period $T_0 = \dfrac{2\pi}{w_0} = 2\pi\sqrt{\dfrac{m}{k}}$.

   Then, $k = \dfrac{4\pi^2}{T_0^2} \times m = \dfrac{4 \times 10}{0.4^2} \times 0.1 = 25\,N/m$.

3. We have $x'' + \dfrac{k}{m}x = 0$ & $a = x'' = A\cos(w_0 t)$, then $x = -\dfrac{m}{k}x'' = -\dfrac{m}{k}A\cos(w_0 t)$;

   The amplitude $x_m = \dfrac{m}{k}A$ & $\dfrac{T_0}{2\pi} = \sqrt{\dfrac{m}{k}}$, then $x_m = \left(\dfrac{T_0}{2\pi}\right)^2 A$.

   Thus, $x_m = \dfrac{0.4^2}{4\pi^2} \times 7.5 = 0.03\,m = 3\,cm$.

Problems

I- Mechanical Oscillator

The aim of this exercise is to study two types of oscillations of a horizontal elastic pendulum.

On a table, we consider a puck (A), of mass $m = 200g$, fixed to one end of a massless spring of un jointed turns, and of stiffness $k = 80N/m$; the other end of the spring is attached to a fixed support (C).

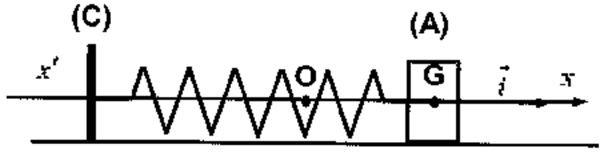


Figure 1

(A) slides on a horizontal rail and its center of inertia G can move along a horizontal axis $x'Ox$. At equilibrium, G coincides with the origin O of the axis $x'x$.

At an instant t, the position of G is defined on the axis $(O;\vec{i})$, by its abscissa $x = \overline{OG}$; its velocity $\vec{v} = v\vec{i}$. The horizontal plane containing G is taken as a gravitational potential energy reference. Suppose that, the forces of friction are negligible. At the instant $t_0 = 0$, G, initially at x_0, and launched with a velocity $\vec{v_0} = v_0\vec{i}$.

Part A
Theoretical study

1. Write, at an instant t, the expression of the mechanical energy of the system [(A), spring, Earth] in terms of x, k, m and v.
2. Derive the differential equation, in x, that describes the motion of G.
3. The solution of the previous differential equation is of the form $x = x_m \cos(2\pi f_0 t + \varphi)$. Determine the expression of the proper frequency f_0 of the oscillations in terms of k and m.

Part B
Graphical study

An appropriate device allows to obtain the variations with respect to time of the abscissa x of G (figure 2). Let $x = x_m \cos(2\pi f_0 t + \varphi)$. **Take:** $0.32\pi = 1$.

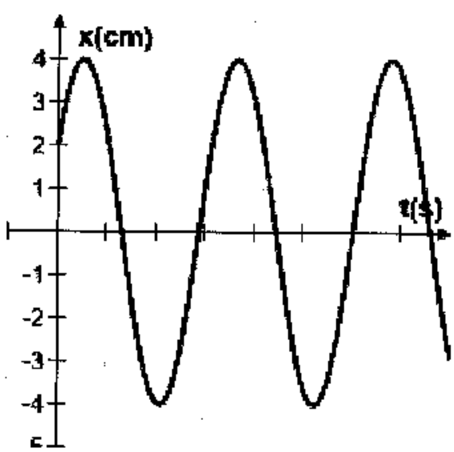


Figure 2

1. Referring to figure (2), indicate the value of:
   a) the initial abscissa x_0.
   b) the amplitude x_m.
2. Calculate the value of f_0 and then determine φ.
3. Calculate the value of the mechanical energy of the system.
4. Justify that the initial kinetic energy is $KE_0 = 48mJ$, and then deduce the value of v_0.
5. a) Determine the expression of the acceleration « a » as a function of time.
   b) Deduce the position (s) in which a is maximum, and then determine the value of this maximum.

Part C

Energetic study

The kinetic and potential energies are both periodic of period T.

1. What is the relation between T and T_0? Deduce the value of T.
2. Plot the curves representing the variations of the mechanical and elastic energies of the system (oscillator, Earth) as a function of time.
   (On the ordinate axis 1div $\equiv 16mJ$ & on the abscissa axis 1div $= T_0/4$).

II- Horizontal Elastic Pendulum

The horizontal elastic pendulum of the figure below is formed of a solid (S) of mass m and a spring of constant $k = 16N/m$.

The center of mass G of (S) may move along a horizontal axis $(O;\vec{i})$ as shown in figure 1. The abscissa of G at any instant t during oscillations is x and its velocity is $\vec{v} = v\vec{i}$.

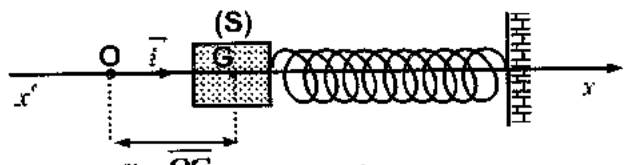


Figure 1

The horizontal plane containing G is taken as the gravitational potential energy reference. We neglect the forces of friction.

Part A

Theoretical study

The curve of figure 2 represents the evolution of the abscissa x of the algebraic value of the center of gravity as a function of time.

1. Write the expression of the mechanical energy of the pendulum [(S), spring] as a function of m, k, x and v.
2. Derive the differential equation that describes the motion of the center of mass G.
3. Knowing that $x = x_m \cos(2\pi f_0 t + \varphi)$ is a solution of the differential equation, and by referring to figure 2.
   a) Determine the expression of the proper frequency f_0 as a function of m and k.
   b) Determine the value of f_0 and deduce the mass m of the solid (S).
   c) Initial conditions.
      i- Justify that the solid is moving in the negative direction at $t = 0$.
      ii- Give the value of x_m and determine φ.

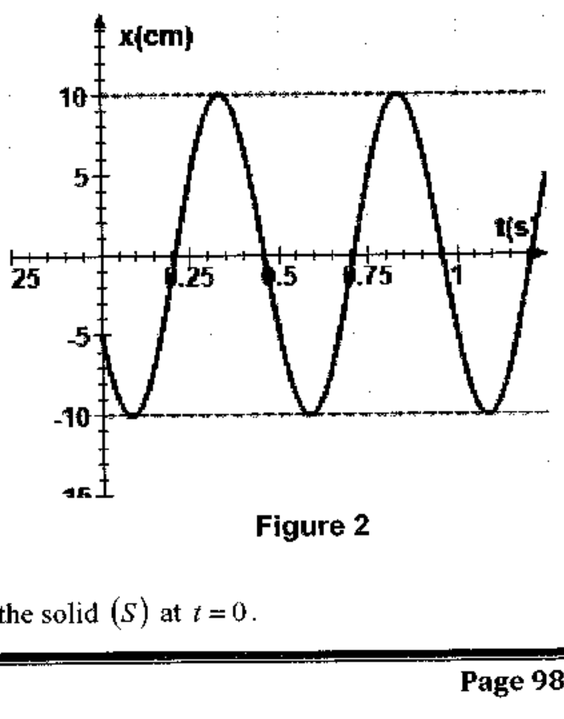


Figure 2

4. Show that the mechanical energy of the system is equal to $0.08J$.
5. Determine the velocity and the acceleration of the solid (S) at $t = 0$.

Part B
Mass of a virus

Scientists use a highly sensitive technique based on a system modeled by a simple harmonic oscillator to measure the mass of a vaccine virus. The procedure consists of measuring the frequency of oscillation of a silicon sliver with a laser, first without the virus and later after the virus had attached itself to the silicon.

1. Show that the ratio of the frequency of the system (virus + sliver) f_{S+V} to the frequency of sliver alone f_S is given by $\frac{f_{S+V}}{f_S} = \frac{1}{\sqrt{1+\left(\frac{m_V}{m_S}\right)}}$ where m_S is the mass of the silicon sliver and m_V is the mass of the virus.
2. If $f_S = 3\times10^{15}\,Hz$, the mass of silicon sliver $m_S = 3.8\times10^{-16}\,g$ and $f_{S+V} = 1.85\times10^{14}\,Hz$. Determine the mass of the virus.

III- Mechanical Oscillator

Consider a mechanical oscillator constituted of a spring, of negligible mass, and of un-joined loops of stiffness k and a solid (S) of mass $m = 400\,g$.

The spring, placed horizontally, is fixed from one of its extremities to a fixed support and (S) is attached to the other extremity. (S) may slide without friction on a horizontal rail AB and its center of inertia G can move along a horizontal axis $x'Ox$.

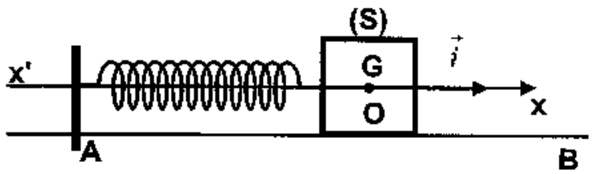


Figure 1

At equilibrium, G coincides with the origin O of the axis $x'x$ (figure 1). Thus, (S) performs mechanical oscillations along $x'Ox$. **Take:** $\pi^2 = 10$.

Part A
Experimental study

An appropriate device allows to obtain the variations with respect to time of the abscissa x of G (figure 2).

At the instant t, the abscissa of G is $x = OG$ and the algebraic measure of its velocity is $v = \frac{dx}{dt}$.

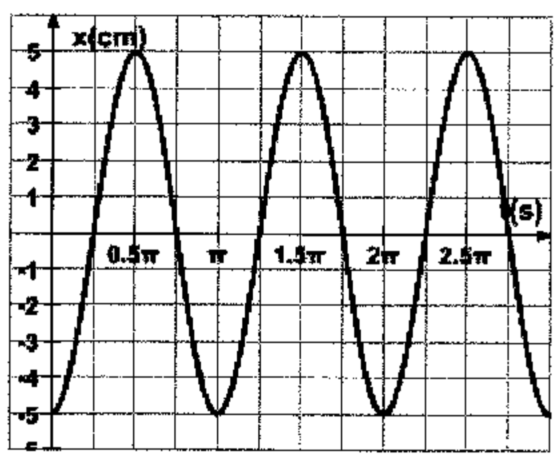


Figure 2

1. What is the mode of oscillations performed?
2. Referring to figure (2), indicate the value of:
   a) the initial abscissa x_0.
   b) the amplitude x_m.
   c) the period T_0.
3. The time equation of motion is of the form $x = x_m \sin(w_0 t + \varphi)$.
   Determine the values of the constants w_0 and φ.

4. a) Determine the instantaneous expression of the velocity v in terms of time.
   Deduce the maximum value v_m of the speed.

   b) Plot the graph representing the variations of v with respect to time, taking the same previous scale on the time axis.

Part B

Theoretical study

Take the horizontal plane passing through G as a reference level of gravitational potential energy.

1. Write, at an instant t, the expression of the mechanical energy ME of the system (oscillator, Earth) in terms of m, x, k and v.
2. Show that the second order differential equation in v that describes the motion of G is $v'' + \frac{k}{m}v = 0$.
3. The solution of this differential equation has the form $v = v_m \sin(2\pi f_0 t)$, where v_m & f_0 are constants.
   a) Determine the expression of the proper frequency f_0 in terms of m and k.
   b) Show that the value of the spring constant $k = 1.6\,N/m$.
4. Calculate the value of the mechanical energy ME.
5. Plot the graph representing the variations of the mechanical and kinetic energies with respect to time, taking the same previous scale on the time axis.

IV-

Forced Mechanical Oscillator

Consider a mechanical oscillator constituted of a spring, of negligible mass, and of un-joined loops of stiffness k and a solid (S) of mass $m = 160\,g$.

The spring, placed horizontally, is fixed from one of its extremities to a fixed support and (S) is attached to the other extremity. (S) may slide without friction on a horizontal rail AB and its center of inertia G can move along a horizontal axis $x'Ox$. At equilibrium, G coincides with the origin O of the axis $x'x$ (figure 1). **Take:** $\pi^2 = 10$

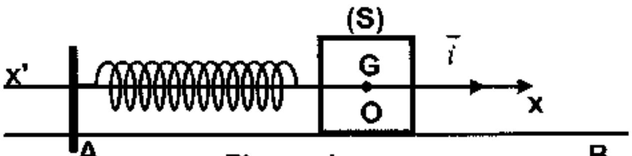


Figure 1

At the instant t, the abscissa of G is $x = OG$ and the algebraic measure of its velocity is $v = \frac{dx}{dt}$.

Part A

Free oscillations

The solid (S) is shifted, at an instant considered as origin of time $t_0 = 0$, by $2cm$ from the equilibrium then it is given an initial velocity v_0. The variations of the abscissa x of G in terms of time is shown in figure 2.

The time equation that describes the motion is of the form: $x = x_m \sin(w_0 t + \varphi)$ where w_0 and φ are constants.

1. Applying Newton's 2[nd] law derive the differential equation, in x, which describes the motion of the center of gravity G.
2. Determine the expression of w_0 in terms of m and k.

3. Referring to figure (2):

   a) indicate :

      i- the amplitude of motion x_m.

      ii- the direction of motion at $t_0 = 0$.

   b) Determine the values of the:

      i- angular frequency w_0.

      ii- initial phase φ.

   c) deduce the value of constant k.

4. a) Justify that the mechanical energy of the system (spring, solid (S)), is conserved then calculate its value.

   b) Determine the velocity of (S) at the instant $t_0 = 0$.

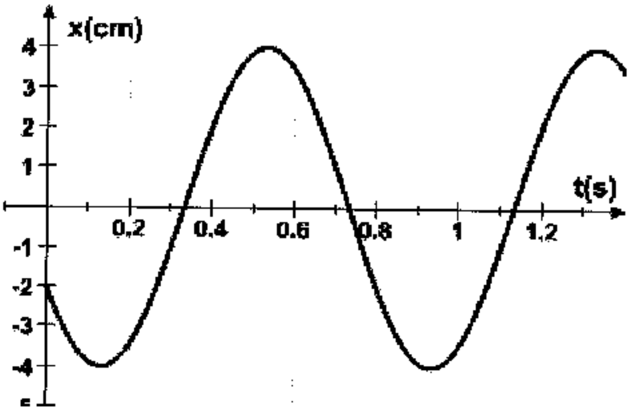


Figure 2

Part B

Forced oscillations

In reality the solid (S) is subjected to a force of friction $\vec{f}$ given by $\vec{f} = -h\vec{v}$ where h is a constant and an external force $\vec{F} = F_m \sin(wt)\vec{i}$.

The variations of the elongation $x(t) = x_m \sin(wt + \varphi_x)$ and the force $F(t)$ are given by the curves (C_1) & (C_2) shown in figure 3.

1. Identify, with justification, the curve that corresponds to the variations of $x(t)$.
2. Referring to figure (3) determine:

   a) the values of the amplitudes x_m & F_m.

   b) the initial phase φ_x of the elongation in terms of π and the angular frequency w of these oscillations.

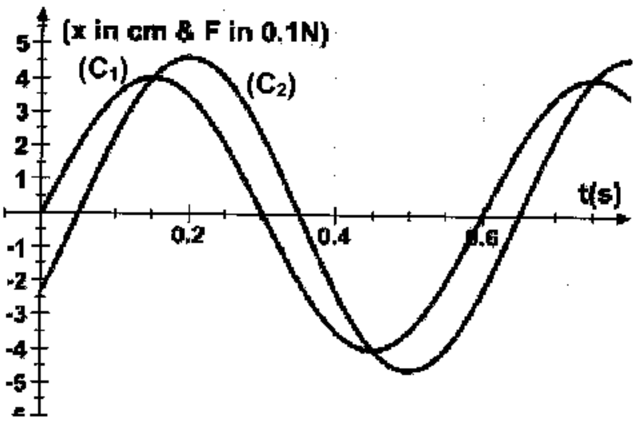


Figure 3

3. a) Derive the differential equation that governs the variations of the elongation x.

   b) Taking a particular value of time, determine the value of the constant h.

V-

Parameters of the Period

The aim of this problem is to find the parameters that effect the duration of oscillation of a mechanical oscillator (elastic pendulum) whose oscillation takes an interval of time T.

Part A

Experimental study

In order to show the effects of the amplitude of oscillations x_m, the mass m of the solid and the constant k of the spring on the duration of one oscillation of a free un-damped horizontal elastic pendulum.

We perform several experiments, in each one factor is only modified x_m in the first, m in the second and k in the third then we measure each time, the duration Δt for 10 oscillations using a stopwatch.

1. Why we measure the duration of 10 oscillations instead of 1 only directly?
2. Referring to the curves below shown in figures 1, 2 & 3. Specify the dependence of the proper period T on:

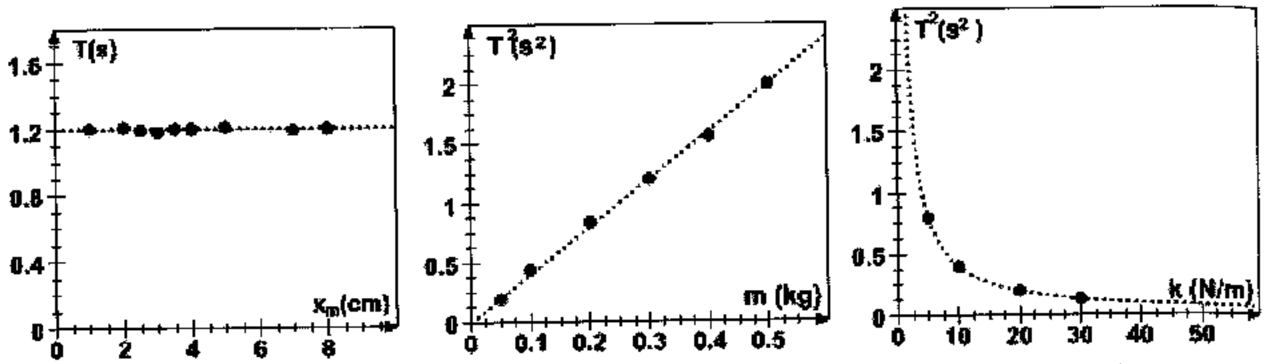


Figure 1 Figure 2 Figure 3

   a) the amplitude x_m from figure 1.

   b) the mass m from figure 2. Deduce the expression of T in terms of $\sqrt{m}$.

   c) the constant of elasticity k from figure 3.

3. Suppose that the proper period can be written in the form $T = A x_m^{\alpha} m^{\beta} k^{\gamma}$ where A, α, β & γ are real numbers.

   a) Determine the value of the constant α so that the expression of the period T is independent of the amplitude x_m?

   b) Verify that the SI unit of the constant of elasticity k is $kg.s^{-2}$.

   c) Basing on dimensional study (units study) and the expression of the proper period, show that $\gamma = -\dfrac{1}{2}$. Deduce the value of β?

   d) Knowing that the values of figure 2 are obtained when the constant of the spring $k = 5\,N/m$. Deduce the value of the constant A.

Part B
Theoretical study

Consider a horizontal elastic pendulum formed of a solid (S) of mass m attached to a spring of constant k and of negligible mass; the other end of the spring is fixed to a support. The forces of friction are supposed to be negligible and the solid of center of mass G can move on a horizontal axis Ox. When the solid is at rest, G coincides with the point O taken as origin of abscissa.

At any instant t, the abscissa of G is x and the algebraic measure of its velocity is $v = \dfrac{dx}{dt}$.

1. Derive the second order differential equation that governs the motion of (S).
2. Knowing that $x = x_m \cos\left(\dfrac{2\pi}{T_0}t + \varphi\right)$ is a solution of this differential equation, determine the expression of the natural period T as a function of m and k.
3. Compare this result to that obtained in A.3-d and that A.2.

VI-2009/2010
Oscillator – Accelerometer
Part A
Oscillator

A horizontal oscillator, formed of an object of mass $m = 2.5\,mg\left(2.5\times10^{-6}\,kg\right)$, is attached to two fixed points by means of two identical springs, each of natural length ℓ_0 and stiffness $k = 0.25\,N/m$. At equilibrium the length of each spring is $\ell_0 + \Delta\ell$.

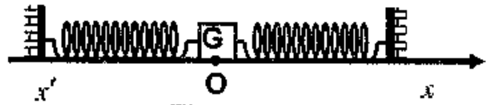


Figure 1

The center of inertia G of the object can move on the horizontal axis $\left(O;\vec{i}\right)$. O being the position of G at equilibrium. The oscillator, initially at rest, is set in oscillation starting from the instant $t_0 = 0$. At an instant t the position of G is located by its abscissa x. During motion, the springs remain always stretched and the forces of friction are negligible (Figure 1).

1. a) Show that, at the instant t, the elastic potential energy has the expression: $PE_e = k\,\Delta\ell^2 + k\,x^2$.
   b) Derive the differential equation that describes the variations of x.
   c) Deduce the value of the proper angular frequency w_0 of this oscillator.
2. Using an appropriate system we record the variations of the algebraic value of the acceleration x'' in terms of x (Figure 2).
   a) Show that this graph s in agreement with the obtained differential equation.
   b) Determine, using the graph, the experimental value w_{exp} of the proper angular frequency.

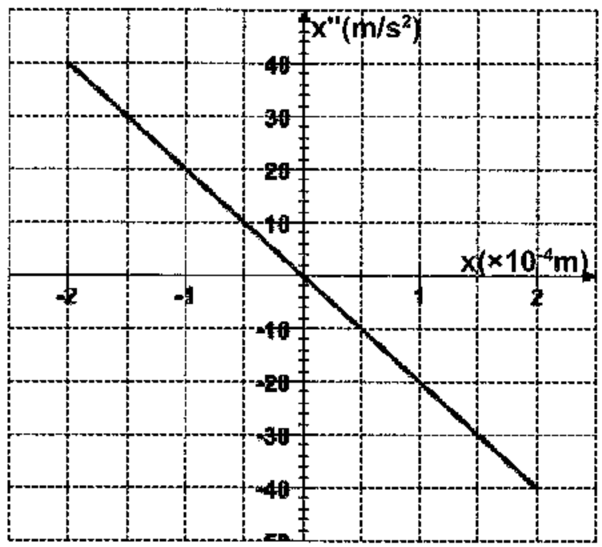
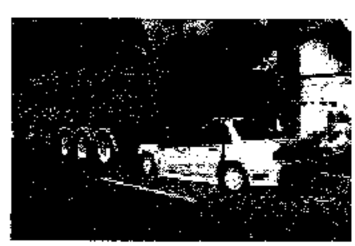


Figure 2

Part B
Accelerometer

A truck of mass $19\,000\,kg$ and moving with a speed $90\,km.h^{-1}$, is hit in the back by a car of mass $1500\,kg$ moving with a speed $130\,km.h^{-1}$. The collision is completely inelastic (soft collision) and the two vehicles are equipped with airbags.



1. Show that the speed of the set formed of the two vehicles right after the collision is $v = 25.8\,m/s$.
2. Knowing that the duration of collision is $40\,ms$, determine:
   a) the acceleration, supposed constant, of each vehicle.
   b) the force that the truck exerts on the car.
3. The functioning of an airbag is controlled by an accelerometer (A), which is the oscillator already considered, able to detect any acceleration the vehicle is subjected to, *i.e* any variation of capacitance ΔC of a grouping of suitably connected capacitors; the starting of (A) begins when ΔC, and consequently the acceleration, exceeds a certain threshold $\left(\Delta C_{\text{threshold}} = 3\times10^{-12}\,F\right)$.

The physical quantity ΔC is written in the form $\Delta C = 2\times10^{-4}x^2$, where x is the abscissa of G. The adjacent table gives estimation on the possible consequences of a collision on the passengers of a vehicle, each passenger having fastened his seat belt.

a) What will the state of each passenger be?

b) Tell, with justification, if the airbag will open.

Acceleration	Estimation of possible effects on the passengers
$100\, m/s^2$	Bearable for young people in good health
$150\, m/s^2$	Risked internal bleeding with lesions
$200\, m/s^2$	No chances of survival

VII-

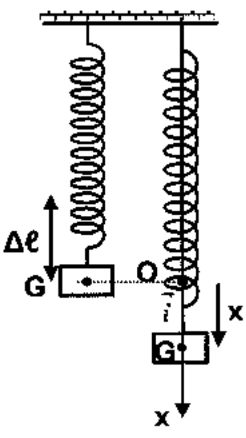
Vertical Mechanical Oscillator

The aim of this exercise is to study the seismometer, which is used to record the waves produced through seism and Earthquakes, one of the practical applications of the vertical mechanical oscillator.

Part A

Theoretical study

Consider a spring of stiffness k and of negligible mass, connected from its upper end to a fixed support while its other end carries a solid (S) of mass $m = 40\, g$. At equilibrium the center of mass G of (S) coincides with a point O and the spring elongates by $\Delta\ell$ (adjacent figure).



The spring is stretched by pulling (S) vertically downwards from its equilibrium position, then releasing it without initial velocity at instant $t_0 = 0$. G oscillates without friction around its equilibrium position O.

At an instant t, G is defined by its abscissa $x = \overline{OG}$ and the algebraic value of its velocity is $v = \frac{dx}{dt}$. **Take** $g = 10\, m/s^2$.

The horizontal plane passing through O is taken as a reference of gravitational potential energy.

1. Applying the equilibrium position on (S), determine a relation among m, g, k and $\Delta\ell$.
2. Show that, at an instant t, the mechanical energy of (S) is given by:

$$ME = \frac{1}{2}m v^2 - m g x + \frac{1}{2}k(x+\Delta\ell)^2.$$

3. Derive the differential equation in x, that governs the motion of the center of gravity G.
4. Deduce that the expression of the proper period of oscillations is given by $T_0 = 2\pi\sqrt{\frac{m}{k}}$.

Part B

Graphical study

The curve shown in figure 2, represents the evolution of the kinetic energy of (S) in terms of the abscissa x. Let x_m be the amplitude of the oscillations performed.

1. Applying the principle of conservation of mechanical energy between the positions of abscissas x & x_m, show that the kinetic energy is given by: $KE = \frac{1}{2}k\left(x_m^2 - x^2\right)$.
2. Referring to the graph, determine:
   a) the amplitude x_m of the oscillations.
   b) the stiffness k of the spring.
3. Deduce:
   a) the speed of (S) when it passes through the equilibrium position.
   b) the period of the oscillations.
   c) the elongation in the equilibrium position.

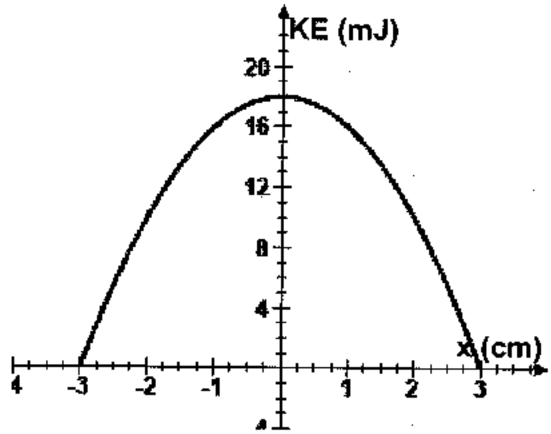


Figure 2

Part C
Functioning of a seismometer

A seismometer is a device used to detect and record any disturbance of Earth crust. It functions as a damped oscillator solid-spring connected to a building fixed to ground. The seismometers are sensitive to vertical and horizontal vibrations. We are interested to the functioning of a vertical seismometer.

In fact a seismic signal hold waves, that are produced at a point called epicenter, of which we study in particular two periodic types P and S-waves. The seismometer enters in resonance for a particular frequency. It acts equally as a damping system, necessarily to obtain a reliable restitution of the ground motion.

Note: without the phenomenon of resonance, the relative motion is very weak and the recording is not reliable.

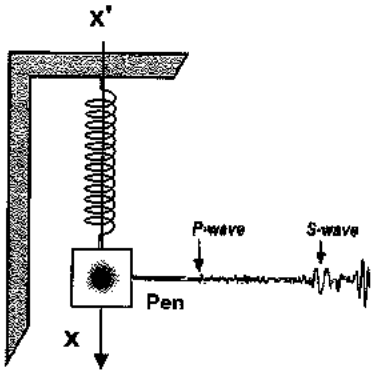
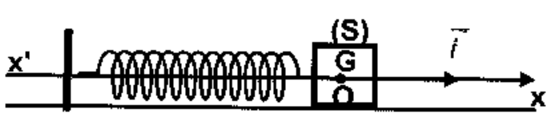


Figure 3

1. Indicate in this system the exciter? the resonator?
2. By admitting that the damping is weak, for what period of the exciter, the resonance phenomenon takes place?
3. If the mass of the solid used is $m = 40\,g$ and the spring constant $k = 40\,N/m$.
   Determine the frequency of these waves.
4. Draw a sketch showing the variations of the amplitude of oscillations in terms of the exciter frequency.
5. During the seism, certain waves have frequencies very small to that of P & S-waves.
   How should we modify the mass m in order to obtain a reliable recording?

VIII-Engineering 2006/2007
Mechanical Oscillator

A mechanical oscillator (C) is formed of a solid (S), of mass m, attached to the extremity A of a horizontal spring of stiffness $k = 80\,N/m$ whose other extremity B is fixed. The solid (S) can move on a horizontal rail.



The position of its center of gravity G is located, at an instant t, by its abscissa $x = \overline{OG}$, O being its equilibrium position.

Part A

Theoretical study

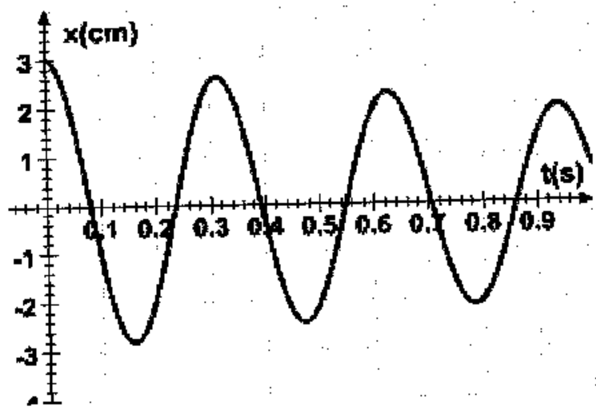
We neglect all frictional forces.

1. Derive the differential equation that describes the oscillation of (C).
2. Deduce the expression of the proper period T_0 of these oscillations.

Part B

Experimental study

1. Free oscillations



An appropriate device, connected to the computer, gives us the curve of the above figure representing the variation of x in terms of time.

a) Determine, using the above graph, the duration T of one oscillation.

b) Determine the average power dissipated between the instants 0 and $3T$.

2. Forced oscillations

Now the extremity B of the spring is connected to a vibrator of adjustable frequency f and for each frequency f, a recording is performed. The table below shows the amplitude x_m relative to each f.

$f(Hz)$	1.5	2.0	2.5	2.8	3.0	3.2	3.3	3.6	4.0	4.5
$x_m(cm)$	0.4	0.6	1.0	1.5	2.1	2.3	2.0	1.5	1.0	0.7

a) Using the table determine, with justification, an approximate value of T_0.

b) Determine then the value of m.

c) What do we obtain:

i- in the absence of all frictional forces?

ii- in the case where the magnitude of the frictional force is increased?

Part C

The hydrogen chloride molecule

A hydrogen chloride molecule $(HC\ell)$ can be represented by a harmonic oscillator of mass $m_H = 1.67\times10^{-27}\,kg$, and of stiffness k'. The potential energy due to the interaction between the two atoms can be reduced to:

$PE_e(x) = \dfrac{0.27\,e^2}{4\pi\varepsilon_0 r_0^{\,3}}x^2$; where $\dfrac{1}{4\pi\varepsilon_0} = 9\times10^9\,SI$, $e = 1.6\times10^{-19}\,C$ and $r_0 = 1.3\times10^{-10}\,m$, r_0 being the distance between the two atoms at equilibrium and x is the displacement of the hydrogen atom with respect to its equilibrium position with $x << r_0$.

This molecule, when excited with an electromagnetic wave of frequency υ, oscillates with a maximum amplitude x_m where $x_m << r_0$.

Determine with justification, the value of υ.

IX- Mechanical Oscillator

Consider a mechanical oscillator formed of a solid (S) of mass $m = 10\,g$ and whose center of inertia is G and a spring of negligible mass of un-jointed turns whose stiffness is k.

(S) may slide on a horizontal rail; the position of G on the horizontal axis $(O;\vec{i})$ is defined relative to the origin O, the position of G when (S) is in the equilibrium position (Figure 1).

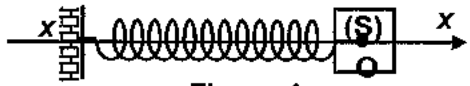


Figure 1

The horizontal plane through G is taken as a gravitational potential energy reference.

At $t_0 = 0$, the center of inertia G is shifted by $x = x_0$, and given an initial velocity $\overrightarrow{v_0}$, thus the system perform free undamped oscillations of amplitude x_m. (S) thus oscillates around O.

The abscissa of G at any instant t during oscillations is x and its velocity is $\vec{v} = v\vec{i}$.

Part A
Theoretical study

1. a) Write the expression of the mechanical energy of the system {pendulum -Earth} in terms of m, k, x & v.
   b) Derive the second order differential equation that describes the motion of the solid.
2. a) Knowing that $x = x_m \sin(w_0\, t + \varphi)$ is a solution of the previous differential equation, determine the expression of w_0, in terms m & k.
   b) Deduce the expression of its proper period T_0.
   c) Justify, using the expression of T_0, that it has a time dimension.
3. Indicate, with justification, how each parameter m, k & x_m should be modified and by how much in order to double the proper period T_0.

Part B
Graphical study

A convenient apparatus is used to trace the variations of the abscissa x of G and the algebraic measure v of its velocity as a function of time and the curves obtained are shown in figure 2.

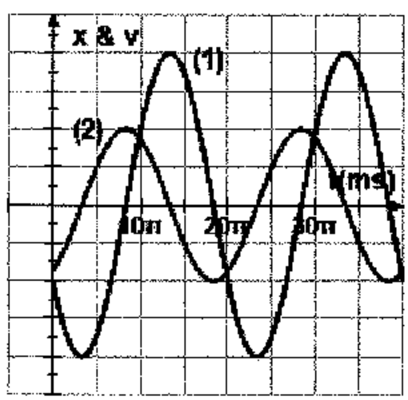


Figure 2

1. a) What does the velocity v represents with respect to the abscissa x?
   b) Justify, using a graphical study, that the curve (1) is associated to the elongation x and (2) to the velocity v.
2. Compare the periods T_0 & T_0' of the two curves and then deduce the value of the constant k.
3. Assuming that the time equation of the elongation is given by: $x = x_m \sin(w_0\, t + \varphi)$, where $x = x_0 = -2cm$ and $\overrightarrow{v_0} = -2\sqrt{3}\,\vec{i}\,(m/s)$.
   a) Determine the position of G for which the speed of (S) is zero.
   b) Graphical study:
      i- Applying the principle of conservation of mechanical energy, show that $v_0^2 = w_0^2\left(x_m^2 - x_0^2\right)$.
      ii- Deduce the value of the amplitude x_m.
      iii-Specify he scales that are used to represent the elongation x? the velocity v?
   c) Determine the value of φ.

Solutions

I-

Part A

1. The mechanical energy at any instant is $ME = KE + PE_e = \frac{1}{2}m\,v^2 + \frac{1}{2}k\,x^2$.

2. Since the mechanical energy is conserved then $\frac{d(ME)}{dt} = 0$, $m\,v\,v' + k\,x\,x' = 0$;

   (But $v = x' \neq 0$, the system is motion) & $v' = x''$, then $x'' + \frac{k}{m}x = 0$.

3. We have $x = x_m \cos(2\pi f_0 t + \varphi)$, so $x'' = -x_m (2\pi f_0)^2 \cos(2\pi f_0 t + \varphi)$;
   Replace x & x'' in the differential equation we get:

   $-x_m (2\pi f_0)^2 \cos(2\pi f_0 t + \varphi) + \left(\frac{k}{m}\right) x_m \cos(2\pi f_0 t + \varphi) = 0$;

   $x_m \cos(2\pi f_0 t + \varphi)\left[-(2\pi f_0)^2 + \left(\frac{k}{m}\right)\right] = 0$ (but $x_m \cos(2\pi f_0 t + \varphi) \neq 0$, system in motion);

   So $-(2\pi f_0)^2 + \left(\frac{k}{m}\right) = 0$, $2\pi f_0 = \sqrt{\frac{k}{m}}$; then $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

Part B

1. a) $x_0 = 2cm$.

   b) $x_m = 4cm$.

2. We have $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{0.32}{2}\sqrt{\frac{80}{0.2}} = 3.2Hz$.

   Initial conditions : at $t_0 = 0$, $x_0 = 2cm$ and $v_0 > 0$ (the curve is increasing);

   We have $x = x_m \cos(2\pi f_0 t + \varphi)$, $2 = 4\cos(\varphi)$, $\cos(\varphi) = \frac{1}{2}$ then $\varphi = +\frac{\pi}{3}$ (rad) or $\varphi = -\frac{\pi}{3}$ (rad);

   But $v = x' = -x_m 2\pi f_0 \sin(2\pi f_0 t + \varphi)$;

   $v_0 = -x_m 2\pi f_0 \sin(\varphi) > 0$; so $\sin(\varphi) < 0$ then $\varphi = -\frac{\pi}{3}$ (rad);

   Thus $x = 4\cos\left(20t - \frac{\pi}{3}\right)$ (t in s and x in cm)

3. The mechanical energy is conserved:

   $ME = ME|_{x=x_m} = KE|_{x=x_m} + PE_e|_{x=x_m} = 0 + \frac{1}{2}k\,x_m^2 = 0.5 \times 80 \times (4\times10^{-2})^2 = 6.4\times10^{-2} J = 64mJ$.

4. At $t_0 = 0$, $x_0 = 2cm$.

   The initial elastic potential energy $PE_{e0} = \frac{1}{2}k\,x_0^2 = \frac{1}{2}\times 80 \times (2\times10^{-2})^2 = 1.6\times10^{-2} J = 16mJ$;

The mechanical energy is conserved $ME = ME|_{x_0} = KE|_{x_0} + PE_e|_{x_0}$;

$KE|_{x_0} = ME - PE|_{x_0} = ME - PE_{e0} = 64mJ - 16mJ = 48mJ$.

We have $KE|_{x_0} = \frac{1}{2} m v_0^2 = 48mJ$, then $|v_0| = \sqrt{\frac{2\,KE|_{x_0}}{m}} = \sqrt{\frac{2 \times 48 \times 10^{-3}}{200 \times 10^{-3}}} = \frac{2\sqrt{3}}{5} m/s$;

But $v_0 > 0$, then $v_0 \approx 0.69\, m/s$.

5. a) We have $a = x'' = -x_m (2\pi f_0)^2 \cos(2\pi f_0 t + \varphi)$.

   b) The acceleration is maximum if $|\cos(2\pi f_0 t + \varphi)| = 1$;

   But $x = x_m \cos(2\pi f_0 t + \varphi)$, $|x| = x_m$, then $x = \pm x_m$;

   The maximum value is $a_{max} = x_m (2\pi f_0)^2 = 4 \times 10^{-2} \times 20^2 = 16 m/s^2$.

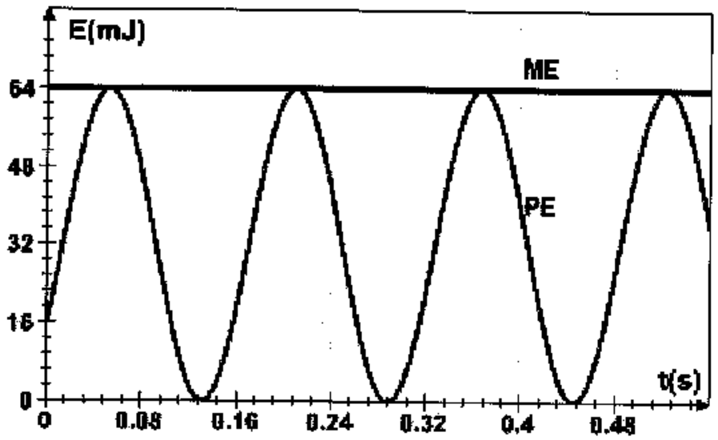
Part C

1. The period of the energies is:

   $T = \frac{T_0}{2} = \frac{1}{2 f_0}$;

   so $T = \frac{1}{2 \times 3.2} = \frac{1}{6.4} \approx 0.16s$.

2. The mechanical energy is conserved then its graphical representation is a horizontal straight line.



II-

Part A

1. The mechanical energy is: $ME = KE + PE_e = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$.

2. The mechanical energy is conserved $\frac{d(ME)}{dt} = 0$, $mx'x'' + kxx' = 0$,

   $x'(m x'' + k x) = 0$ (but $x' \neq 0$, the system is in motion); then $x'' + \frac{k}{m} x = 0$.

3. a) We have $x = x_m \cos(2\pi f_0 t + \varphi)$ & $x'' = -x_m (2\pi f_0)^2 \cos(2\pi f_0 t + \varphi)$

   By replacing in the differential equation, we get:

   $-x_m (2\pi f_0)^2 \cos(2\pi f_0 t + \varphi) + \frac{k}{m} x_m \cos(2\pi f_0 t + \varphi) = 0$;

   $x_m \cos(2\pi f_0 t + \varphi) \times \left(-(2\pi f_0)^2 + \frac{k}{m} \right) = 0$.

   Then $-(2\pi f_0)^2 + \frac{k}{m} = 0 \Rightarrow (2\pi f_0)^2 = \frac{k}{m}$; thus, $f_0 = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$.

b) Graphically, the period $T_0 = 0.5s$, then $f_0 = \frac{1}{T_0} = 2s$;

But $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \Rightarrow m = \frac{k}{4\pi^2 f_0^2} = \frac{16}{4\times10\times4} = 0.1\,kg = 100\,g$.

c) Properties of motion:

i- The curve representing the elongation is decreasing at $t=0$, so $v|_{t=0} = \left.\frac{dx}{dt}\right|_{t=0} < 0$

ii- The amplitude of the oscillations is $x_m = 10\,cm$.

At $t=0$, $x = 5cm \Rightarrow -5 = 10\cos\varphi$; $\cos\varphi = -\frac{1}{2} \Rightarrow \varphi = \pm\frac{2\pi}{3}$ (rad).

But at $t=0$, the curve is decreasing so $v = \left.\frac{dx}{dt}\right|_{t=0} = -x_m 2\pi f_0 \sin\varphi < 0 \Rightarrow \sin\varphi > 0$;

Therefore $\varphi = \frac{2\pi}{3}(\text{rad})$; thus, $x = 10\cos\left(4\pi t + \frac{2\pi}{3}\right)$ (x in cm and t in s).

4. The mechanical energy is conserved so:

$ME = ME|_{x_m} = PE_e|_{x_m} + KE|_{x_m} = \frac{1}{2}k\,x_m^2 + 0 = \frac{1}{2}\times16\times0.1^2 = 0.08\,J$.

5. Conservation of mechanical energy $ME = ME|_{t=0} = PE_e|_{t=0} + KE|_{t=0} = \frac{1}{2}k\,x_0^2 + \frac{1}{2}m\,v_0^2 = 0.08\,J$;

Then $\frac{1}{2}(16)\times0.05^2 + \frac{1}{2}\times0.1\times v_0^2 = 0.08J$;

But at $t_0 = 0$, the solid is moving in the negative direction $v_0 = -\sqrt{0.12} \approx -0.35\,m/s$.

Referring to the differential equation $x'' + \frac{k}{m}x = 0$, we get:

$a|_{t=0} = x''|_{t=0} = -\frac{k}{m}x|_{t=0} = -\frac{16}{0.1}\times(-0.05) = 8\,m/s^2$.

Part B

1. The frequency of the silicon without the virus is $f_S = \frac{1}{2\pi}\sqrt{\frac{k}{m_S}}$;

The frequency of the system (silicon + virus) is $f_{S+V} = \frac{1}{2\pi}\sqrt{\frac{k}{m_S + m_V}}$;

We get: $$\frac{f_{S+V}}{f_S} = \frac{\frac{1}{2\pi}\sqrt{\frac{k}{m_S+m_V}}}{\frac{1}{2\pi}\sqrt{\frac{k}{m_S}}} = \frac{\sqrt{\frac{1}{m_S+m_V}}}{\sqrt{\frac{1}{m_S}}} = \frac{1}{\sqrt{\frac{m_S+m_V}{m_S}}} = \frac{1}{\sqrt{1+\frac{m_V}{m_S}}}.$$

2. The mass of the virus is:

$$m_V = m_S\left[\left(\frac{f_S}{f_{S+V}}\right)^2 - 1\right] = 3.8\times10^{-16}\left[\left(\frac{3\times10^{15}}{1.85\times10^{14}}\right)^2 - 1\right] = 10^{-13}\,g.$$

III-

Part A

1. The oscillations are free un-damped.
2. a) The initial abscissa $x_0 = -x_m = -5\,cm$.

   b) The amplitude $x_m = 5\,cm$.

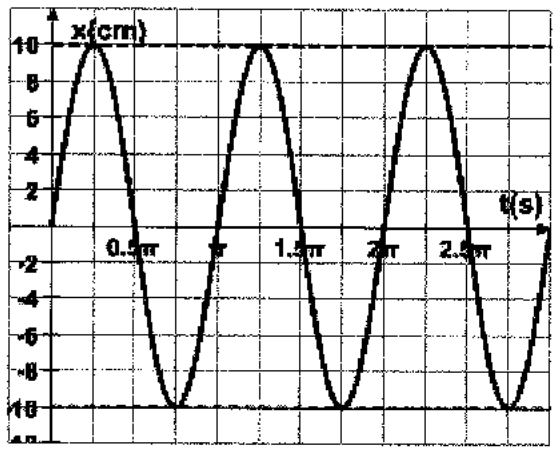
   c) The period $T = \pi \approx 3.14\,s$.
3. We have $x = x_m \sin(w_0 t + \varphi)$;

   The initial conditions at $t = 0$, $x = -x_m$; we get $-x_m = x_m \sin(\varphi)$;

   Then $\sin(\varphi) = -1 \Rightarrow \varphi = -\frac{\pi}{2}(\text{rad})$;

   And the angular frequency $w_0 = \frac{2\pi}{T_0} = \frac{2\pi}{\pi} = 2\,\text{rad}/s$.

   Thus $x = 5\sin\left(2t - \frac{\pi}{2}\right)$ (t in s and x in cm).



4. a) We have $v = \frac{dx}{dt} = x' = 5 \times 2\cos\left(2t - \frac{\pi}{2}\right)$;

   $v = 10\cos\left(2t - \frac{\pi}{2}\right)$ (t in s and v in cm/s)

   The maximum value of the velocity is $v_m = 10\,cm/s$.

   b) Graph.

Part B

1. The mechanical energy $ME = KE + PE_e = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$;
2. The mechanical energy is conserved $\frac{d(ME)}{dt} = 0$; $m v v' + k x x' = 0$;

   $v(mv' + kx) = 0$; $v = x' \neq 0$ since the system is in motion;

   Then $mv' + kx = 0$; deriving again with respect to time $mv'' + kx' = 0$; thus $v'' + \frac{k}{m}v = 0$.
3. a) We have $v = v_m \sin(2\pi f_0 t)$, then $v'' = -v_m(2\pi f_0)^2 \sin(2\pi f_0 t)$;

   Replacing in the differential equation we get:

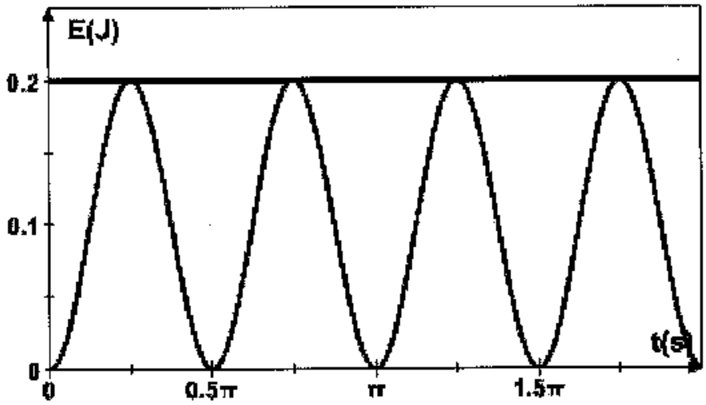
   $v_m \sin(2\pi f_0 t)\left(-(2\pi f_0)^2 + \frac{k}{m}\right) = 0$, (but $v_m \sin(2\pi f_0 t) \neq 0$, pendulum in motion);

   Then $-(2\pi f_0)^2 + \frac{k}{m} = 0$; thus, $f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$.

   b) We have $f_0 = \frac{1}{T_0} = \frac{1}{\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$; $\sqrt{\frac{k}{m}} = 2$; then $k = 4m = 4 \times 0.4 = 1.6\,N/m$.
4. The mechanical energy is conserved $ME = ME|_{x=x_m} = KE|_{x=x_m} + PE_e|_{x=x_m} = PE_e|_{x=x_m}$

   $ME = \frac{1}{2}k x_m^2 = \frac{1}{2} \times 1.6 \times (5 \times 10^{-2})^2 = 2 \times 10^{-3}\,J$

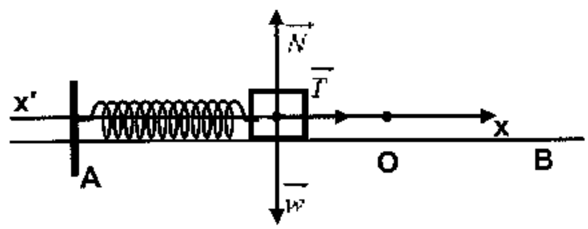
5. Graphs



IV-

Part A

1. Newton's 2nd law : $\sum \vec{F} = \frac{d\vec{P}}{dt}$;



$\vec{w} + \vec{N} + \vec{T} = m\vec{a}$, but $\vec{w} + \vec{N} = \vec{0}$;

So $-k\,x\vec{i} = m\,x''\vec{i}$; then $x'' + \frac{k}{m}x = 0$.

2. We have $x = x_m \sin(w_0 t + \varphi)$,

So $x' = x_m w_0 \cos(w_0 t + \varphi)$ and $x'' = -x_m w_0^2 \sin(w_0 t + \varphi)$;

Substitution in the differential equation: $-x_m w_0^2 \sin(w_0 t + \varphi) + \frac{k}{m} x_m \sin(w_0 t + \varphi) = 0$;

$x_m \sin(w_0 t + \varphi)\left(-w_0^2 + \frac{k}{m}\right) = 0$ but $x_m \sin(w_0 t + \varphi) \neq 0$ since the system is in motion;

So $-w_0^2 + \frac{k}{m} = 0$, then $w_0 = \sqrt{\frac{k}{m}}$.

3. a) Graphical study.

i- $x_m = 4cm$.

ii- The velocity is the slope of the tangent to the curve representing the abscissa x.

At $t_0 = 0$, the curve decreases $(v_0 < 0)$ then the object is moving is in the negative direction.

b) Properties of motion.

i- The proper period $T_0 = 0.8s$ and the angular frequency $w_0 = \frac{2\pi}{T_0} = \frac{2\pi}{0.8} = 2.5\pi\,(\text{rad/s})$.

ii- Initial conditions : at $t_0 = 0$, $x_0 = -2cm$ and $v_0 < 0$;

We have $x = x_m \sin(w_0 t + \varphi)$, so $-2 = 4\sin(\varphi)$, $\sin(\varphi) = -\frac{1}{2}$;

$\varphi = \sin^{-1}\left(-\frac{1}{2}\right)$ then $\varphi = -\frac{\pi}{6}(\text{rad})$ or $\varphi = \pi - \left(-\frac{\pi}{6}\right) = \frac{7\pi}{6}(\text{rad})$;

But $v = x_m w_0 \cos(w_0 t+\varphi)$ and $v_0 = x_m w_0 \cos(\varphi) < 0$ so $\varphi = 7\frac{\pi}{6}(\text{rad})$.

c) The period of motion $T_0 = 2\pi\sqrt{\frac{m}{k}}$, $k = 4\pi^2 \frac{m}{T_0^2} = 4\times 10 \times \frac{160\times 10^{-3}}{0.8^2} = 10N/m$.

4. a) The amplitude of motion is constant, then the mechanical energy of the system is constant.
Conservation of mechanical energy $ME = ME|_{x_m} = KE|_{x_m} + PE_e|_{x_m}$ but $KE|_{x_m} = 0$;

Then $ME = 0 + \frac{1}{2}k\,x_m^2 = \frac{1}{2}\times 10\times (4\times 10^{-2})^2 = 8\times 10^{-3}J$.

b) The elastic potential energy at $t_0 = 0$ is $PE_e|_{x_0} = \frac{1}{2}k\,x_0^2 = \frac{1}{2}\times 10 \times (-2\times 10^{-2})^2 = 2\times 10^{-3}J$;

The mechanical energy is conserved then :

$KE|_{x_0} = ME - KE|_{x_0} = 8\times 10^{-3}J - 2\times 10^{-3}J = 6\times 10^{-3}J$;

$KE|_{x_0} = \frac{1}{2}m\,v_0^2$, $|v_0| = \sqrt{\frac{KE|_{x_0}\times 2}{m}} = \sqrt{\frac{6\times 10^{-3}\times 2}{160\times 10^{-3}}} = 0.27m/s$;

But $v_0 < 0$, then $v_0 = -0.27m/s$.

Part B

1. We have $\vec{F} = F_m \sin(wt)\vec{i}$ and $x(t) = X_m \sin(wt+\varphi_x)$;
At $t_0 = 0$, $F = 0$, then the curve representing the variation of the force F should pass through origin; thus the curve of F is (C_1) and that of x is (C_2).

2. a) $x_m = 4.6cm$ & $F_m = 4\times 0.1 = 0.4N$.

b) The phase difference $|\varphi_x| = 2\pi\times\frac{d}{D} = 2\pi\times\frac{1}{12} = \frac{\pi}{6}$ (rad);

But (C_2) representing the elongation x lags (C_1), then $\varphi_x = -\frac{\pi}{6}(\text{rad})$;

The period of oscillations is $T = 0.6s$, the angular frequency $w = \frac{2\pi}{T} = \frac{2\pi}{0.6} = \frac{10\pi}{3}(\text{rad/s})$.

3. a) Newton's 2nd law: $\sum\vec{F} = \frac{d\vec{P}}{dt}$; $\vec{w}+\vec{N}+\vec{T}+\vec{F}+\vec{f} = m\vec{a}$ but $\vec{w}+\vec{N} = \vec{0}$;

$-k\,x\vec{i} + F_m \sin wt\vec{i} - hv\vec{i} = m x''\vec{i}$; then $x'' + \frac{h}{m}x' + \frac{k}{m}x = \frac{F_m}{m}\sin(wt)$;

But $x(t) = x_m \sin\left(wt - \frac{\pi}{6}\right)$ so $v = x' = x_m w\cos\left(wt-\frac{\pi}{6}\right)$ and $x'' = -x_m w^2 \sin\left(wt-\frac{\pi}{6}\right)$;

Then $x_m \sin\left(wt-\frac{\pi}{6}\right)\left(-w^2+\frac{k}{m}\right) + \frac{h}{m}w x_m \cos\left(wt-\frac{\pi}{6}\right) = \frac{F_m}{m}\sin(wt)$;

b) Let $wt - \frac{\pi}{6} = 0$, $\frac{10\pi}{3}t - \frac{\pi}{6} = 0$, then $t = \frac{\pi}{6}\times\frac{3}{10\pi} = \frac{1}{5} = 0.2s$;

So $-x_m w^2 \sin(0) + \frac{h}{m}w x_m \cos(0) + \frac{k}{m}x_m \sin(0) = F_m \sin\left(\frac{\pi}{6}\right)$;

$$\frac{h}{m} w x_m = \frac{F_m}{m}\sin\left(\frac{\pi}{6}\right);\ h\, w\, x_m = \frac{F_m}{2};$$

Thus, $h = \dfrac{F_m}{2 \times x_m \times w} = \dfrac{0.4 \times 3}{2 \times 4.6 \times 10^{-2} \times 10 \times \pi} \approx 0.42 kg/s$.

V-

Part A

1. In order to reduce the error of measurement, we should measure a greater number of periods.
2. a) The period is represented by a horizontal straight line, so it is constant, independent of the amplitude x_m.

   b) The square of the period T^2 is a straight line increasing passing through origin.

   Then T^2 is proportional to the mass m.

   So, $T^2 = a\, m \Rightarrow a = \dfrac{T^2}{m} = \dfrac{2}{0.5} = 4$; thus $T = 2\sqrt{m}$ where m in kg & T in s.

   c) The square of the period T^2 decreases as the constant of elasticity increases.
3. a) In order to be independent from the amplitude x_m, the value of α should be 0.

   b) According to Hooke's law : $[k] = \dfrac{[T]}{[x]} = \dfrac{N}{m} = \dfrac{kg.m.s^{-2}}{m} = kg.s^{-2}$.

   c) $[T] = [A] \times [x_m]^{\alpha=0} \times [kg]^{\beta} \left[kg\, s^{-2}\right]^{\gamma} = 1 \times 1 \times kg^{\beta+\gamma} s^{-2\gamma} = s$.

   By identification: $-2\gamma = 1 \Rightarrow \gamma = -\dfrac{1}{2}$; $\beta + \gamma = 0 \Rightarrow \beta = -\gamma = \dfrac{1}{2}$.

   Thus, $T = A\, x_m^0\, m^{\frac{1}{2}} k^{-\frac{1}{2}} = A\sqrt{\dfrac{m}{k}}$.

   d) We have $T = A\sqrt{\dfrac{m}{k}}$, then $A = T\sqrt{\dfrac{k}{m}} = 2\sqrt{\dfrac{5}{0.5}} = 2\sqrt{10} \approx 6.3$.

Part B

1. The mechanical energy at any instant is: $ME = KE + PE_e = \dfrac{1}{2} m v^2 + \dfrac{1}{2} k x^2$.

   Since friction is neglected, the mechanical energy is conserved $\dfrac{d(ME)}{dt} = 0$, $mx'x'' + kxx' = 0$;

   $x'(m x'' + k x) = 0$ (but $v = x' \neq 0$, since the system is in motion); then $x'' + \dfrac{k}{m} x = 0$.

2. We have $x = x_m \sin\left(\dfrac{2\pi}{T_0} t + \varphi\right)$, then $x'' = -x_m \left(\dfrac{2\pi}{T_0}\right)^2 \sin\left(\dfrac{2\pi}{T_0} t + \varphi\right)$;

   Replacing in the differential equation we get: $x_m \sin\left(\dfrac{2\pi}{T_0} t + \varphi\right)\left[-\left(\dfrac{2\pi}{T_0}\right)^2 + \dfrac{k}{m}\right] = 0$;

   But $x = x_m \sin\left(\dfrac{2\pi}{T_0} t + \varphi\right) \neq 0$, then $\left(\dfrac{2\pi}{T_0}\right)^2 = \dfrac{k}{m}$; thus, $T_0 = 2\pi\sqrt{\dfrac{m}{k}}$.

3. By comparison $T_0 = 2\pi\sqrt{\frac{m}{k}} = A\sqrt{\frac{m}{k}}$, we get $A = 2\pi$.

VI-

Part A

1. a) The potential energy stored in the 1st spring is $PE_{e_1} = \frac{1}{2}k(\ell - \ell_0 - x)^2 = \frac{1}{2}k(\Delta\ell - x)^2$;

In the 2nd spring $PE_{e_2} = \frac{1}{2}k(\ell - \ell_0 + x)^2 = \frac{1}{2}k(\Delta\ell + x)^2$;

The elastic potential energy stored in the system is

$PE_e = \frac{1}{2}k(\Delta\ell + x)^2 + \frac{1}{2}k(\Delta\ell - x)^2 = k\,\Delta\ell^2 + k\,x^2$.

b) The mechanical energy: $ME = PE_e + KE = k\,\Delta\ell^2 + k\,x^2 + \frac{1}{2}m\,v^2$;

Since ME is conserved then $\frac{d(ME)}{dt} = 0 \Rightarrow x'' + \frac{2k}{m}x = 0$.

c) The differential equation governing the motion is of the form $x'' + w_0^2 x = 0$ with $w_0 = \sqrt{\frac{2k}{m}}$;

Then $w_0 = \sqrt{\frac{0.5}{2.5\times 10^{-6}}} = 447.2$ rad/s.

2. a) The curve $x'' = f(x)$ is represented by:

- ✖ a straight line;
- ✖ decreasing;
- ✖ passing through the origin of equation $x'' = \alpha\,x$ with $\alpha < 0$.

These results are compatible with the differential equation which is written $x'' = -\frac{2k}{m}x$

b) The slope of this straight line is: $\alpha = \frac{\Delta x''}{\Delta x} = -2\times 10^5 = -w_{exp}^2$, then $w_{exp} = 447.2$ rad/s.

Part B

1. Conservation of linear momentum: $m_T\,\overrightarrow{v_T} + m_C\,\overrightarrow{v_C} = (m_T + m_C)\overrightarrow{v'}$

The velocities are collinear, we get $m_T\,v_T + m_C\,v_C = (m_T + m_C)v'$;

Then $v' = \frac{m_T\,v_T + m_C\,v_C}{m_T + m_C} = 25.83\,m/s$.

2. a) The acceleration is given by: $a = \frac{\Delta v}{\Delta t}$.

For the truck $a_T = \frac{\Delta v_T}{\Delta t} = \frac{(25.83 - 25)m/s}{40\times 10^{-3}s} \approx 20.3\,m/s^2$;

For the car $a_C = \frac{\Delta v_C}{\Delta t} = \frac{(25.83 - 36.11)m/s}{40\times 10^{-3}s} = -257.5\ m/s^2$.

b) Newton's 2nd law applied on the car during the collision $\overrightarrow{w_C} + \overrightarrow{N_C} + \overrightarrow{F_{T/C}} = \frac{d\overrightarrow{p_C}}{dt}$;

But $\overrightarrow{w_C}+\overrightarrow{N_C}=\vec{0}$; then $\overrightarrow{F_{T/C}}=\frac{\Delta\overrightarrow{p_C}}{\Delta t}=m_C\frac{\Delta v_C}{\Delta t}\vec{i}=-3.86\times10^5\vec{i}\ (N)$.

3. a) We have $\Delta C_{\text{threshold}}=2\times10^{-4}x^2=3\times10^{-12}\Rightarrow x_{\text{threshold}}=1.225\times10^{-5}m$;

The threshold acceleration $a_{\text{threshold}}=x''_{\text{threshold}}=-w_0^2\,x=-2\times10^5\times1.225\times10^{-5}=-2.45\,m/s^2$.

For the truck driver, nothing will occur, whereas for the car driver there's a risk of dying.

b) The airbag will be opened in the car since $a_C>a_{\text{threshold}}$ and nothing happens in the truck.

VII-

Part A

1. The forces acting on (S) are its weight $\vec{w}$ & the tension in the spring $\vec{T}$;

(S) is in equilibrium at O, then $\sum\vec{F}=\vec{0}$, $\vec{w}+\vec{T}=\vec{0}$; then $\vec{w}=-\vec{T}$;

We get $T=w$, thus $m\,g=k\,\Delta\ell$.

2. The elastic potential energy $PE_e=\frac{1}{2}k(\Delta\ell-x)^2$, the gravitational potential energy $GPE=-m\,g\,x$

Then, $ME=KE+PE_e+GPE=\frac{1}{2}m\,v^2+\frac{1}{2}k\,(\Delta\ell+x)^2-m\,g\,x$.

3. (S) oscillates without friction, it is energetically isolated, then its mechanical energy is conserved.

$\frac{d(ME)}{dt}=0$, we get $m\,v\,v'+k\,x'(\Delta\ell+x)-m\,g\,x'=0$;

But $v=x'\neq0$ (system in motion), so $v(m\ x''+k\,x+k\ \Delta\ell-m\,g\ )=0$;

Referring to the initial condition, we get: $x''+\frac{k}{m}x=0$.

4. The differential equation that governs the motion of (S) is of 2nd order of the form $x''+w_0^2\,x=0$,

where $w_0^2=\frac{k}{m}$, then the motion of (S) is sinusoidal of proper period $T_0=\frac{2\pi}{w_0}=2\pi\sqrt{\frac{m}{k}}$.

Part B

1. The mechanical energy of (S) is conserved, $ME|_x=ME|_{x_m}$ (but $KE|_{x_m}=0$, it comes to rest);

So, $KE+\frac{1}{2}k(\Delta\ell+x)^2-m\,g\,x=\frac{1}{2}k(\Delta\ell+x_m)^2-m\,g\,x_m$;

Then, $KE=\frac{1}{2}k\,\Delta\ell^2+\frac{1}{2}k\,x_m^2+k\,x_m\,\Delta\ell-m\,g\,x_m-\frac{1}{2}k\,\Delta\ell^2-\frac{1}{2}k\,x^2-k\,\Delta\ell\,x+m\,g\,x$;

$KE=\frac{1}{2}k\,x_m^2-\frac{1}{2}k\,x^2+x_m(k\,\Delta\ell-m\,g)+(-k\,\Delta\ell+m\,g)x$, (but $m\,g=k\,\Delta\ell$);

Thus, $KE=\frac{1}{2}k\left(x_m^2-x^2\right)$.

2. a) For $x=x_m$, $KE=0$; which corresponds to the abscissa of the point of intersection with the abscissa axis, then $x_m=3\,cm$.

b) For $x=0$, $KE=18\,mJ$ but $KE|_{x=0}=\frac{1}{2}k\,x_m^2$, then $k=\frac{2\times KE|_{x=0}}{x_m^2}=\frac{2\times18\times10^{-3}}{\left(3\times10^{-2}\right)^2}=40\,N/m$.

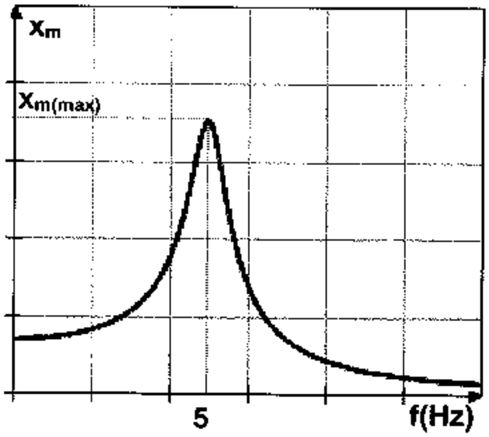
3. a) We have $KE|_{x=0} = \frac{1}{2} m v_O^2$, then $v_O = \sqrt{\frac{2 \times KE|_{x=0}}{m}} = \sqrt{\frac{2 \times 18 \times 10^{-3}}{40 \times 10^{-3}}} = \frac{3}{\sqrt{10}} = 0.95\, m/s$.

b) We have $T_0 = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{40 \times 10^{-3}}{40}} \approx 0.2\, s$.

c) Referring to the equilibrium condition, $\Delta \ell = \frac{m g}{k} = \frac{40 \times 10^{-3} \times 10}{40} = 10^{-2}\, m = 1\, cm$.

Part C

1. The exciter is the ground that shakes through Earthquakes while the resonator is the system mass–spring.
2. Under weak damping, the resonance phenomenon takes place when the period T of the exciter is equal to its proper period T_0.
3. At resonance $f_{waves} = f_0 = \frac{1}{T_0} \approx 5\, Hz$.
4. Curve.



5. The proper frequency is inversely proportional to the mass, for a smaller frequency we should increase the mass so that the system is the seat of amplitude resonance.

VIII-

Part A

1. Since friction is neglected, the mechanical energy is conserved $\frac{d(ME)}{dt} = 0$, $mx'x'' + kxx' = 0$;

$x'(m x'' + k x) = 0$; ($x' \neq 0$ since the system is in motion); then $x'' + \frac{k}{m} x = 0$.

2. The differential equation that governs the motion is of the form $x'' + w_0^2 x = 0$;

Then the motion is periodic of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi \sqrt{\frac{m}{k}}$.

Part B

1. a) We have $3T = 0.94\, s \Rightarrow T = 0.313\, s$.

b) The average power is given by: $P_{av} = \frac{\Delta(ME)}{\Delta t}$; where $\Delta t = 3T = 0.94\, s$.

$\Delta(ME) = \Delta(PE_{e\max}) = \frac{1}{2} k (x^2_{\max\, final} - x^2_{\max\, initial}) = \frac{1}{2}(80)(4-9) \times 10^{-4} = -2 \times 10^{-2}\, J$;

$P_{av} = \frac{-\Delta(ME)}{\Delta t} = \frac{2 \times 10^{-2}}{0.94} = 2.13 \times 10^{-2}\, W$.

2. a) According to the table, the amplitude of oscillations is maximum (amplitude resonance) when the frequency of the excitations is $f = 3.2\, Hz$.

According to the graph the free oscillations are slightly damped, then:

$$f \approx f_0 \Rightarrow T \approx T_0 = \frac{1}{f} = \frac{1}{3.2} = 0.3125\, s.$$

b) We have $T_0 = 2\pi\sqrt{\frac{m}{k}} \Rightarrow m = \frac{k\,T_0^2}{4\pi^2} = \frac{80 \times 0.313^2}{4\pi^2} = 0.199\, kg$.

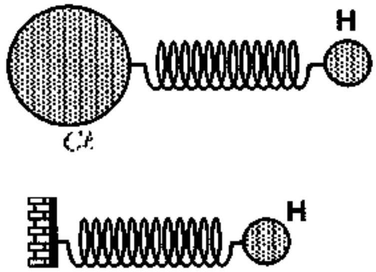
c) Study of motion

i- In the absence of all forces of friction, the energy delivered by the exciter will increase the amplitude x_m so it takes a large value $\left(x_m \longrightarrow \infty\right)$ for $T = T_0$ and there is a danger of damaging the spring.

ii- When we increase the intensity of the forces of friction, the amplitude x_m decreases and the pseudo-period of amplitude resonance is greater than T_0.

The resonance that was sharp, become less sharpness. (As long the critical state is not reached).

Part C[1]



The constant of elasticity of the equivalent spring is:

$$PE_e(x) = \frac{0.27\, e^2}{4\pi\varepsilon_0 r_0^{\,3}} x^2 = \frac{1}{2} k' x^2;$$

$$k' = \frac{2(0.27)e^2}{4\pi\varepsilon_0 r_0^{\,3}} = \frac{2(0.27)\left(1.6\times10^{-19}\right)^2 \times 9\times10^9}{\left(1.3\times10^{-10}\right)^3} = 56.63\, N/m.$$

The proper frequency of oscillation of the molecule is:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k'}{m}} = \frac{1}{2\pi}\sqrt{\frac{56.63}{1.67\times10^{-27}}} = 2.93\times10^{13}\ Hz.$$

The hydrogen acts a resonator that oscillates amplitude under the effect of the exciter (electromagnetic wave).

Thus, the system is resonating if $\upsilon = f_0 = 2.93\times10^{13}\, Hz$.

IX-

Part A

1. a) The mechanical energy $ME = KE + PE_e = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$.

b) The system performs free undamped oscillations, then the mechanical energy ME is conserved.

$\frac{d(ME)}{dt} = 0$, $mx'x'' + kxx' = 0$, $x'(m x'' + k x) = 0$; (but $v = x' \neq 0$, the system is in motion);

Then $x'' + \frac{k}{m} x = 0$.

2. a) We have $x = x_m \sin(w_0\, t + \varphi)$, then $x'' = -x_m\, w_0^2 \sin(w_0\, t + \varphi)$.

Replacing in the differential equation we get: $x_m \sin(w_0\, t + \varphi)\left(-w_0^2 + \frac{k}{m}\right) = 0$;

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[1] We know that the molecule $(H\,C\ell)$ is formed due to an ionic bond, the force of attraction can be simulated to the tension force. The mass of the chlorine is huge compared to that of the hydrogen, and then we may consider the chlorine as a support to a vibrating hydrogen atom.
For that also the mass of the oscillator is considered the mass of the hydrogen m_H only.

(But $x = x_m \sin(w_0 t + \varphi) \neq 0$ the system is in motion), then $-w_0^2 + \frac{k}{m} = 0$; $w_0 = \sqrt{\frac{k}{m}}$.

b) The expression of the proper period is given by: $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}}$.

c) Dimensional analysis: $[T_0] = \left[2\pi\sqrt{\frac{m}{k}}\right] = [2\pi]\sqrt{\frac{[m]}{[k]}} = 1\sqrt{\frac{kg}{N.m^{-1}}} = \sqrt{\frac{kg}{kg.m.s^{-2}m^{-1}}} = \sqrt{s^2} = s$.

3. The proper period is independent of the amplitude x_m of motion. There is no effect of the amplitude on the proper period T_0.

If $m' = 4m$, then $T_0' = 2\pi\sqrt{\frac{m'}{k}} = 2\pi\sqrt{\frac{4m}{k}} = 2\left(2\pi\sqrt{\frac{m}{k}}\right) = 2T_0$.

If $k' = \frac{k}{4}$, then $T_0'' = 2\pi\sqrt{\frac{m}{k'}} = 2\pi\sqrt{\frac{4m}{k}} = 2\left(2\pi\sqrt{\frac{m}{k}}\right) = 2T_0$.

Part B

1. a) $v = \frac{dx}{dt} = x'$.

b) $v = x'$; then the sign of the velocity v determine the variations of the elongation x. When $v > 0$ (respectively $v < 0$), x must be increasing (respectively decreasing); or if $v = 0$ (intersects the abscissa axis), x passes through an extremum (maximum or minimum). Thus, the curve (1) is associated to the elongation x and (2) to the velocity v.

2. Graphically $T_0 = 20\,\pi(ms) = T_0'$ & $w_0 = \frac{2\pi}{T_0} = \frac{2\pi}{20\,\pi \times 10^{-3}} = 100\ rad/s$.

Then $k = m w_0^2 = 10 \times 10^{-3} \times 100^2 = 100\ N/m$.

3. a) We have $x = x_m \sin(w_0 t + \varphi)$, we get $v = x' = x_m w_0 \cos(w_0 t + \varphi)$;
v is zero so $\cos(w_0 t + \varphi) = 0$, then $|\sin(w_0 t + \varphi)| = 1$; thus $|x| = x_m \Rightarrow x = \pm x_m$.

b) Properties of motion.

i- Conservation of mechanical energy :

$ME|_{x=x_0} = ME|_{x=x_m} \Rightarrow KE|_{x=x_0} + PE_e|_{x=x_0} = KE|_{x=x_m} + PE_e|_{x=x_m}$.

We get $\frac{1}{2}mv_0^2 + \frac{1}{2}kx_0^2 = \frac{1}{2}kx_m^2$; thus $v_0^2 = \frac{k}{m}(x_m^2 - x_0^2) = w_0^2(x_m^2 - x_0^2)$.

ii- We have: $x_0 = -2cm$ & $v_0 = -2\sqrt{3}cm/s$; by substitution we get $x_m = 4cm$.

iii-For the elongation $1div \equiv 1cm$ & for velocities $1div \equiv 2m/s$.

c) At $t = 0$; $x_0 = -2cm = 4\sin\varphi \Rightarrow \sin\varphi = -\frac{1}{2}$, then $\varphi = -\frac{\pi}{6}$ or $-\frac{5\pi}{6}$ rad.

But $v_0 < 0 \Rightarrow \cos\varphi < 0$; thus $\varphi = -\frac{5\pi}{6}$ rad;

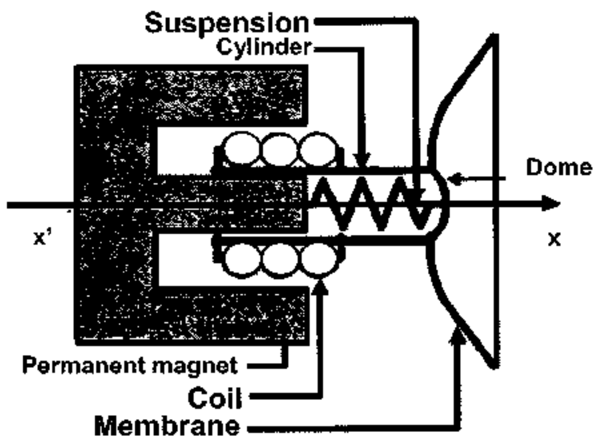
Thus, $x = 4\sin\left(100\,t - \frac{5\pi}{6}\right)$ (x in cm & t in s)

Supplementary Problems

I-Bac

Oscillations of a Loudspeaker

The aim of this exercise is to study the oscillations of the mobile part of a loudspeaker called «Woofer» specialized in the reproduction of deep sounds in order to determine some of its characteristics.



The mechanical part of a loudspeaker is constituted of a mobile membrane, connected to a hollowed cylinder on which a copper wire is wounded.

The set is called mobile equipment, of total mass m and is able to move along the axis $x'x$. The suspension is modeled by a spring whose constant of elasticity is k that functions during elongation and compression.

Figure 1: Simplified diagram of an electrodynamics loudspeaker

Part A

Theoretical study of the mobile part in the absence of friction

The mobile part shown in figure 1 is modeled by a solid (S), of mass m, and center of inertia G, able to slide without friction along a horizontal rail (figure 2). This solid is attached to a spring of unjointed turns, negligible mass and whose constant of elasticity is k.

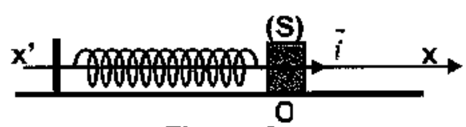


Figure 2

The position of the solid is referred by the abscissa x of its center of inertia G on the axis $(O;\vec{i})$.

The position of G at equilibrium corresponds to the origin of the abscissa axis.

The differential equation that describes the motion of the center of inertia G of the solid (S) at an instant t can be written in the form: $x'' + \frac{k}{m}x = 0$.

1. What is the mode of oscillations performed by the solid?
2. A solution of the previous differential equation is $x = x_m \sin\left(\frac{2\pi}{T_0}t\right)$ where x_m is a constant.

   Show that the expression of the proper period T_0 of the mobile part is given by $T_0 = 2\pi\sqrt{\frac{m}{k}}$.
3. By a dimensional study, verify that this expression has the time as a unit.

Part B
Experimental study of the mobile part

We intend in this part to determine experimentally the mass m of the mobile part and the constant of elasticity k of the spring that models the suspension system of the loudspeaker.
The membrane initially at rest is stroked slightly and quickly at the dome by a finger.
A convenient system of recording is used to represent the evolution of the abscissa x of the center of inertia G of the mobile part as a function of time.

1. Indicate:
   a) the mode of oscillations performed.
   b) the nature of motion.
2. Specify the sign of the initial velocity.
3. Interpret the evolution of the amplitude as a function of the time.
4. Determine the pseudo-period T of oscillations and deduce, in case of slightly damping oscillations, the proper frequency f_0 of the mobile part.

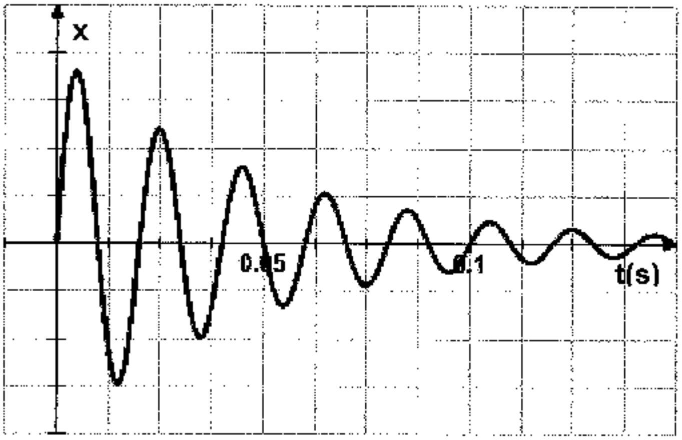


Figure 3

5. In order to determine the mass m of the mobile part, we fix at the dome of the membrane an additional mass $m' = 10\,g$.
   The proper frequency of the set (mobile part, additional mass) becomes $f_0' = 45\,Hz$.
   a) Give the expression the proper frequency f_0' as a function of m, m' and k.
   b) By comparing the literal expressions of the proper frequencies f_0 and f_0', show that the mass of the mobile part is given by: $m = \dfrac{m'}{\left(\dfrac{f_0}{f_0'}\right)^2 - 1}$.
   c) Take $\pi^2 = 10$, calculate the value of m.
   d) Deduce the value of k.

Part C
Forced oscillations

After disconnecting the system of recording, we connect across the loudspeaker a low frequency generator (L.F.G) that impose a sinusoidal signal of frequency f and non-zero amplitude.

1. Indicate the type of oscillations performed by the mobile part of the loudspeaker.
2. Identify in the system the exciter and the resonator.
3. Knowing that the oscillations are slightly damped.
   a) What do we observe when f is very close to f_0?
   b) Give the name of this phenomenon.

Answer Key

Part B 4. $f_0 = 50Hz$ 5. c) $m \approx 43\,g$.

II-LS & GS 2004 1st

Suspension System in a Car

Certain tracks present periodic variations of its level. A car moves in a uniform motion on such a track that has regularly spaced bumps. The distance between two consecutive bumps is d and the speed of the car is v.

In order to study the effect of the bumps on the car, we consider the car and the suspension system as a mechanical oscillator (elastic pendulum) whose oscillation takes a time T.

Part A

Study of T

1. Theoretical study

Consider a horizontal elastic pendulum formed of a solid of mass m attached to a spring of constant k and of negligible mass; the other end of the spring is fixed to a support. The forces of friction are supposed to be negligible and the solid of center of mass G can move on a horizontal axis Ox.

When the solid is at rest, G coincides with the point O taken as origin of abscissa.

The solid is pulled from its equilibrium position by a distance x_m, and then released without initial velocity at the instant $t_0 = 0$. The horizontal plane passing through G is taken as a gravitational potential energy reference.

At any instant t, the abscissa of G is x and the algebraic measure of its velocity is v.

a) Starting from the expression of the mechanical energy of the system {pendulum -Earth}, determine the second order differential equation that characterizes the motion of the solid.

b) Deduce the expression of its proper period T_0.

2. Experimental study

In order to show the effects of the mass m of the solid and the constant k of the spring on the duration of one oscillation of a horizontal elastic pendulum, we use four springs of different stiffness and four solids of different masses.

In each experiment, we measure the time Δt for 10 oscillations using a stopwatch.

a) Effect of the mass m of the solid

In a first experiment, the four solids are connected separately from the free end of the spring whose stiffness is $k = 10\,N/m$. The values of Δt are shown in the following table.

$m(g)$	50	100	150	200
$\Delta t(s)$	4.5	6.3	7.7	8.9

Determine, using the table, the ratio T^2/m. Conclude.

b) Effect of the stiffness k of the spring

In a second experiment, the solid of mass $m = 100\,g$ is connected successively from the free end of each of the four springs. The new values of Δt are shown in the following table.

$k(N/m)$	10	20	30	40
$\Delta t(s)$	6.3	4.5	3.7	3.2

Determine, using the table, the values of the product $T^2 \times k$. Conclude.

c) Expression of T

Deduce that T may be written in the form $T = A\sqrt{\frac{m}{k}}$ where A is a constant.

Part B

Oscillations of the car

1. The car, with the driver alone, form a mechanical oscillator whose proper period is around $1\,s$. It moves with a speed $v = 36 km/h$ on a path having equally spaced bumps. The distance between two consecutive bumps is $d = 10m$. The car enters then in resonance.
   a) Specify the exciter and the resonator.
   b) Explain why the car enters resonance.
   c) How can the driver avoid this resonance?
2. The driver, with four passengers, drives his car on the same path with the same speed of $v = 36km/h$. Would the car enter in resonance? Justify your answer.

Answer Key

Part A 2.a) T is proportional to $\sqrt{m}$. b) $T^2 k \approx 4.1$ **Part B** 1.b) $T = 1s$.

III-LS 2003 1st
Determination of a Spring's Constant

In order to determine the force constant k of a spring (R) of an un-jointed turns, we consider:

- ✗ a frictionless track ABC found in a vertical plane.
- ✗ a spring (R) having one of its end fixed to a support C and its another end is connected to a solid (S_2) of mass m_2 of negligible dimensions.
- ✗ a solid (S_1) of mass $m_1 = 0.1kg$ and of negligible dimensions held at A at height $h = 0.8m$ above the horizontal plane containing BC.

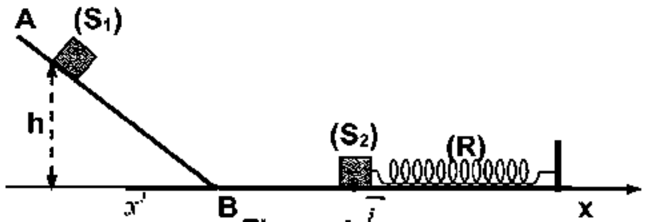


Figure 1

The horizontal plane containing BC is taken as the gravitational potential energy reference.

Take: $g = 10\,m\,s^{-2}$ and $\pi^2 = 10$.

1. (S_1), released from rest at A, reaches (S_2) with a velocity $\vec{v_1}$. Show that the magnitude of $\vec{v_1}$ is $v_1 = 4\,m/s$.
2. (S_1) collides with (S_2) and sticks to it, thus forming a particle (S). Determine in terms of m_2, the expression of V_0 the magnitude of the velocity $\vec{v_0}$ of (S) just after the impact.
3. The system $[(S),(R)]$ forms a horizontal elastic pendulum, (S) oscillating around its equilibrium position at O.
   a) Determine the differential equation that describes the motion of the oscillator.
   b) Deduce the expression of its proper period T_0.
4. Figure 2, represents the variation of the algebraic value of the velocity of (S) as a function of time. The origin of time corresponds to the instant when the velocity of (S) is $\vec{v_0}$.
   a) Give the value v_0 of $\vec{v_0}$.
   b) Deduce the value of m_2.
   c) Give the value of T_0.
   d) Calculate k.

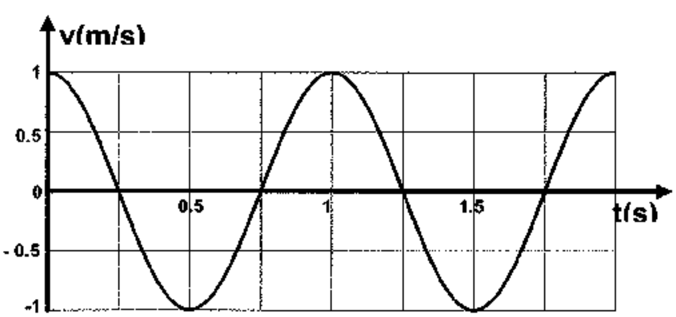


Figure 2

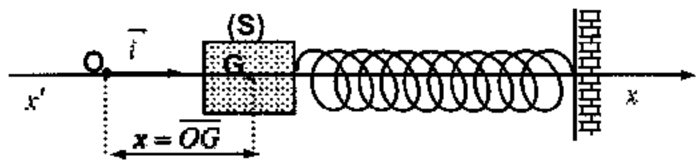
Answer Key

1. $v_1 = 4\,m/s$.

3.b) *ii-* $m_2 = 0.3\,kg$ *iv-* $k = 16\,N/m$.

IV-LS 2001 1st
Horizontal Elastic Pendulum

The horizontal elastic pendulum of the figure below is formed of a solid (S) of mass $m = 100\,g$ and a spring of constant $k = 80\,N/m$. The center of mass G of (S) may move along a horizontal axis $(O;\vec{i})$. At the instant $t_0 = 0$, G being at rest at O, (S) is given an initial velocity $\vec{v_0} = v_0\vec{i}$ $(v_0 = 3\,m/s)$. (S) thus oscillates around O.



The abscissa of G at any instant t during oscillations is x and its velocity is $\vec{v} = v\vec{i}$.
The horizontal plane containing G is taken as the gravitational potential energy reference.

Part A
Free undamped oscillations

In this part, we neglect the forces of friction.

1. a) Write the expression of the mechanical energy of the pendulum [(S), spring] as a function of x, m, k and v.
   b) Is the mechanical energy of the pendulum is conserved? Calculate its value.
2. Derive the differential equation that describes the motion of the center of mass G.
3. a) Verify that $x = x_m\cos(w_0t + \varphi)$ is a solution of this differential equation where $w_0 = \sqrt{\frac{k}{m}}$.
   b) Calculate the values of x_m, φ and the proper period T_0 of the pendulum.
   c) Determine the interval after which G passes through O for the first time.

Part B
Free damped oscillations

In this part, the forces of friction are not neglected and (S) performs damped oscillations of pseudo-period T.

1. Compare T to T_0.
2. At the instant $t = T$, the speed of (S) is $2.8\,m\,s^{-1}$.
   a) What is the position of G at this instant?
   b) Calculate the work done by the forces of friction between the two instants $t_0 = 0$ and $t = T$.

Answer Key

Part A 3.b) $x_m \approx 11\,cm$ 3.c) $\frac{T_0}{2} = 0.11\,s$.

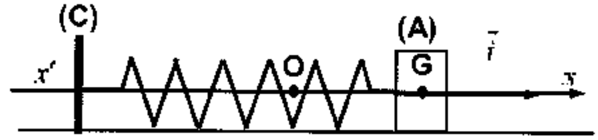
Part B 2.b) $-0.058\,J$.

LS Sessions

I-LS 2014 1st
Mechanical Oscillations

The aim of this exercise is to study two types of oscillations of a horizontal elastic pendulum.
On a table, we consider a puck (A), of mass $m = 200 g$, fixed to one end of a massless spring of un jointed turns, and of stiffness $k = 80 N/m$; the other end of the spring is attached to a fixed support (C) (adjacent figure).

(A) slides on a horizontal rail and its center of inertia G can move along a horizontal axis $x'Ox$. At equilibrium, G coincides with the origin O of the axis $x'x$.



At an instant t, the position of G is defined on the axis $(O;\vec{i})$, by its abscissa $x = \overline{OG}$; its velocity $\vec{v} = v\,\vec{i}$ where $v = \dfrac{dx}{dt}$.

The horizontal plane containing G is taken as a gravitational potential energy reference.

Part A
Free un-damped oscillations

Suppose that in this part, the forces of friction are negligible.

At the instant $t_0 = 0$, G initially at O, is launched with a velocity $\vec{v_0} = v_0\,\vec{i}$ $(v_0 = 2.5 m/s)$.

1. Determine, at $t_0 = 0$, the mechanical energy of the system [(A), spring, Earth].
2. Write, at an instant t, the expression of the mechanical energy of the system [(A), spring, Earth] in terms of x, k, m and v.
3. a) Derive the differential equation, in x, that describes the motion of G.
   b) Deduce the value of the proper angular frequency w_0 and that of the proper period T_0 of the oscillations.
4. The solution of the previous differential equation is of the form $x = x_m \cos(w_0 t + \varphi)$. Determine the values of x_m and φ.

Part B
Free damped oscillations

We suppose now that (A) is submitted to a force of friction $\vec{f}$ of average value f_{av}.

1. The center of inertia G is shifted by $x_{0m} = 12.5 cm$ from O. Then (A) is released at the instant $t_0 = 0$ without initial velocity. G passes through O, for the first time, at the instant $t_1 = 0.085 s$ with a speed $v_1 = 2 m/s$.
   a) Determine the variation of the mechanical energy of the system [(A), spring, Earth] between the instants t_0 and t_1.
   b) Deduce f_{av} between the instants t_0 and t_1.
2. In order to drive the oscillations of (A), an appropriate set-up supplies the oscillator an average power P_{av}.
   a) What is meant by «drive the oscillations»?
   b) Calculate P_{av} between the instants t_0 and t_1.

II-LS 2013 2nd

Mechanical Oscillator

Consider a mechanical oscillator constituted of a spring, of negligible mass, and of un-joined loops of stiffness k and a solid (S) of mass $m = 0.1kg$.

The spring, placed horizontally, is fixed from one of its extremities to a fixed support and (S) is attached to the other extremity. (S) may slide without friction on a horizontal rail AB and its center of inertia G can move along a horizontal axis $x'Ox$. At equilibrium, G coincides with the origin O of the axis $x'x$ (figure 1).

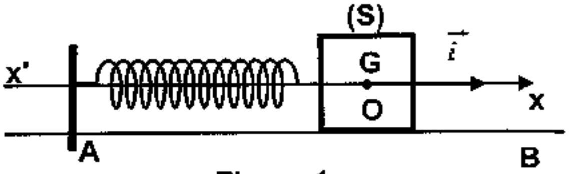


Figure 1

The solid (S) is displaced from its equilibrium position by a distance $x_0 = \overline{OG_0}$ and we give it, at the instant $t_0 = 0$, in the positive direction an initial velocity $\vec{v_0} = v_0\vec{i}$. Thus, (S) performs mechanical oscillations along $x'Ox$.

Part A
Theoretical study

At the instant t, the abscissa of G is $x = \overline{OG}$ and the algebraic measure of its velocity is $v = \dfrac{dx}{dt}$.

Take the horizontal plane passing through G as a reference level of gravitational potential energy.

1. Write, at an instant t, the expression of the mechanical energy ME of the system (oscillator, Earth) in terms of m, x, k and v.
2. Establish the second order differential equation in x that describes the motion of G.
3. The solution of this differential equation has the form: $x = x_m \sin\left(\dfrac{2\pi}{T_0}t + \varphi\right)$, where x_m, T_0 and φ are constants. Determine the expression of the proper period T_0 in terms of m and k.

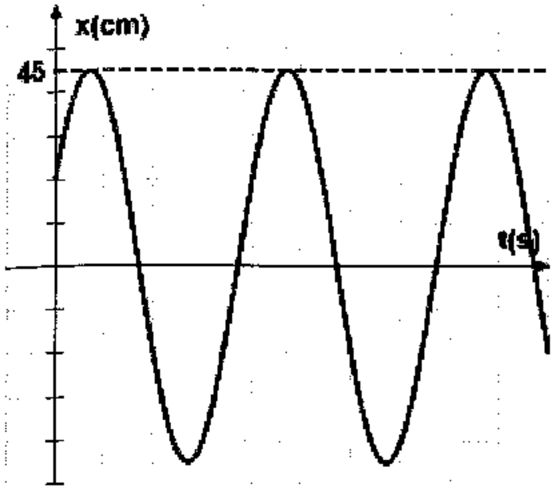
Part B
Graphical study of the motion

An appropriate device allows to obtain the variations with respect to time:

✘ of the abscissa x of G (figure 2);

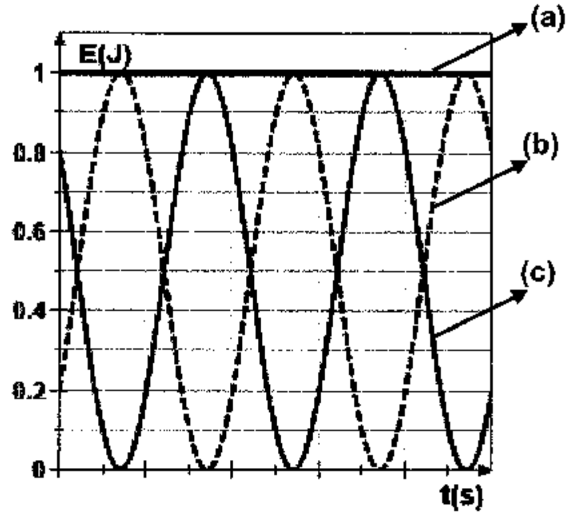
✘ of the kinetic energy KE, of the elastic potential energy PE_e and of the mechanical energy ME of the system (oscillator, Earth) (figure 3).

1. Referring to figure (2), indicate the value of:
   a) the initial abscissa x_0;
   b) the amplitude x_m;
   c) the period T_0.
2. Determine the values of k and φ.
3. The curves (a), (b), and (c) of figure 3 represent the variations of the energies of the system (oscillator, Earth) as a function of time. Using this figure:
   a) identify, with justification, the energy represented by each curve;
   b) determine the value of the initial velocity v_0;
   c) Comparison of periods
      i- indicate the value of the period T of the KE and PE_e;
      ii- deduce the relation between T and T_0.



1 horizontal division ⟶ 0.157s

Figure 2



2 horizontal divisions ⟶ 0.157s

Figure 3

III-LS 2013 1st
Collisions and Mechanical Oscillator

Part A

Collision

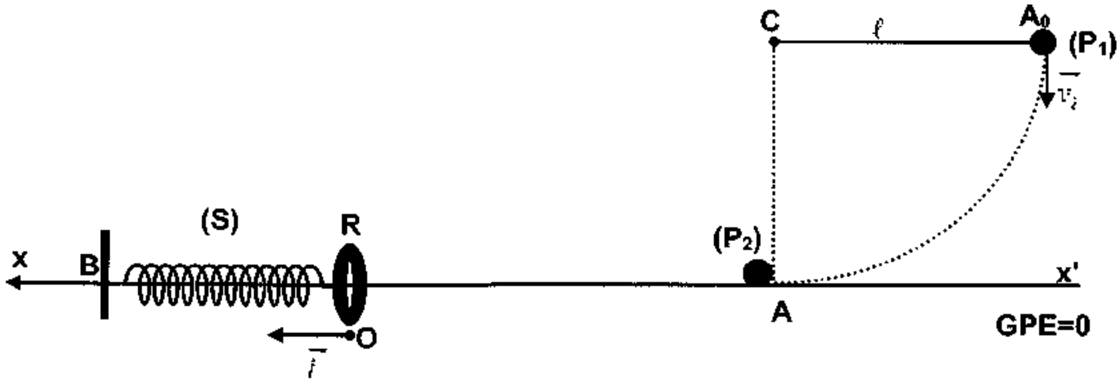
A pendulum is formed of a massless and inextensible string of length $\ell = 1.8m$, having one of its ends C fixed to a support while the other end carries a particle (P_1) of mass $m_1 = 200g$.

The pendulum is stretched horizontally. The particle (P_1) at A_0 is then launched vertically downward with a velocity $\overrightarrow{v_i}$ of magnitude $v_i = 8m/s$.

At the lowest position A, (P_1) enters in a head-on perfectly elastic collision with another particle (P_2) of mass $m_2 = 300g$ initially at rest. Neglect all frictional forces.

Take:

- ✖ the horizontal plane passing through A as a gravitational potential energy reference;
- ✖ $g = 10m/s^2$.



1. a) Calculate the mechanical energy of the system [pendulum, Earth] at the instant of launching (P_1) at A_0.

   b) Determine the magnitude v_1 of the velocity $\overrightarrow{v_1}$ of (P_1) just before colliding with (P_2).

2. a) Name the physical quantities that are conserved during this collision.

b) Show that the magnitude v_2' of the velocity $\overrightarrow{v_2'}$ of (P_2), just after collision, is $8m/s$.

Part B
Mechanical oscillator

A horizontal spring (S), of negligible mass and of stiffness $k = 120\ N/m$, is connected at one of its ends B to a fixed support while the other end is attached to a ring R.

(P_2) moves on the horizontal path AB until it hits the ring R at point O; (P_2) sticks to R forming a solid (P), considered as a particle, of mass $m = 1.2kg$. Thus (P) and the spring (S) form a horizontal mechanical oscillator of center of inertia G; G moves without friction on a horizontal axis $x'Ox$ along AB. Just after collision and at the initial instant $t_0 = 0$, G coincides with O, the equilibrium position of (P), and has a velocity $\overrightarrow{v_0} = v_0\vec{i}$ with $v_0 = 2\ m/s$.

At an instant t, the abscissa of G is x and the algebraic value of its velocity is $v = \dfrac{dx}{dt}$.

1. Write down the expression of the mechanical energy of the system (oscillator, Earth) at an instant t, in terms of k, m, x and v.
2. Derive the differential equation in x that describes the motion of G and deduce the nature of its motion.
3. Knowing that the solution of this differential equation is $x = x_m \cos\left(\sqrt{\dfrac{k}{m}}t + \varphi\right)$, determine the values of the constants x_m and φ.

IV-LS 2012 1st
Horizontal Elastic Pendulum

The aim of this exercise is to study the effect of the mass on the motion of a horizontal elastic pendulum. This pendulum is formed of:

- an elastic spring (R), of negligible mass and of stiffness $k = 400\ N/m$, wound around a horizontal rod;
- a solid (B) considered as a particle of mass $m = 100\ g$.

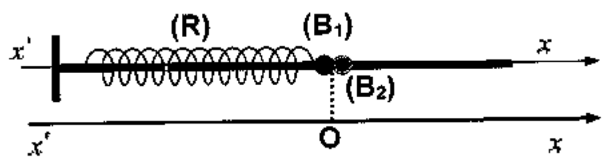


Figure 1

The solid (B) is formed of two particles (B_1) and (B_2) stuck together and of respective masses $m_1 = 25\ g$ and $m_2 = 75\ g$.

The solid (B) can slide, without friction, on the rod (Figure 1).

At equilibrium, (B) is at O, taken as an origin of abscissas of the axis $x'x$. (B) is displaced by a distance x_m, from O, in the positive direction, and then released without initial velocity at the instant $t_0 = 0$. The horizontal plane through (B) is taken as a gravitational potential energy reference.

At the end of two complete oscillations, (B_2) is detached from (B_1) and the system $[(R),(B_1)]$ continues its oscillations.

Figure 2 represents the variation of the abscissa x of the moving solid as a function of time in the two intervals $[0, 0.2\,s]$ and $[0.2\,s, 0.35\,s]$. **Take:** $\pi^2 = 10$.

Part A
Graphical study

Referring to figure 2, give in each of the intervals $[0, 0.2\,s]$ and $[0.2\,s, 0.35\,s]$:

1. the value of the amplitude of the motion;
2. the type of oscillations performed by the oscillator;
3. the value of the proper period of the oscillations.

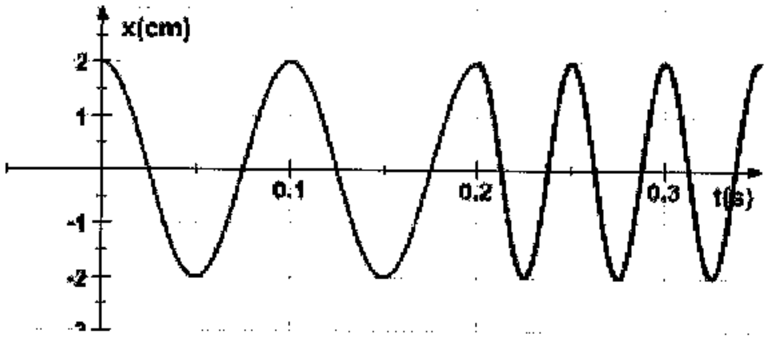


Figure 2

Part B
Theoretical study of the oscillations of (B)

Consider the system [(R), (B), Earth].

1. Calculate, at $t_0 = 0$, the value of the mechanical energy of the system.
2. At an instant t, (B) has an abscissa x and a velocity v of algebraic value $v = \frac{dx}{dt}$.
   Write, at an instant t, the expression of the mechanical energy of the system in terms of k, m, x and v.
3. a) Derive the second order differential equation in x that describes the motion of (B).
   b) Deduce the expression of the proper period T of the oscillations.
   c) Calculate the value of T, and then compare it to the result obtained in part **(A – 3)**.
4. The time equation of motion of (B) is of the form: $x = x_m \sin\left(\frac{2\pi}{T}t + \varphi\right)$.
   Determine the value of the constant φ.

Part C
Theoretical study of the oscillations of (B_1)

Consider the system [(R), (B_1), Earth].

1. Referring to figure 2, give the instant at which (B_2) is detached from (B_1).
2. The mechanical energy of the system [(R), (B_1), Earth] is equal to that of the system [(R), (B), Earth]. Justify.
3. When (B) passes through O, its speed is V and when (B_1) passes through O, its speed is V_1. Show that $V_1 = 2V$.

V-LS 2011 1st
Horizontal Elastic Pendulum

The aim of this exercise is to study some physical quantities associated to a horizontal elastic pendulum, formed of a spring of force constant $k = 20\,N/m$ and a solid (S) of mass $m = 500\,g$.

Take: $g = 10\,m/s^2$, $\pi^2 = 10$ and neglect all resistive forces.

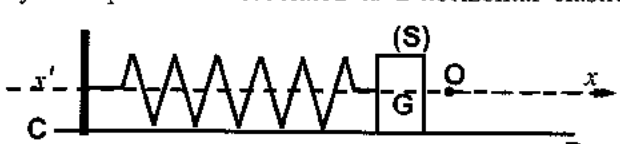


Figure 1

Part A
Theoretical study

The spring, placed horizontally, is fixed from one extremity to a fixed support. We attach the solid (S) to the other extremity. (S) can move along a horizontal rail CD and its center of inertia G an move along the horizontal axis $x'x$. At equilibrium, G coincides with the origin O of the axis $x'x$. We shift (S) to the left starting from O; G occupies then the position G_0 such that $x_0 = \overline{OG_0} = -10\,cm$.

At the instant $t_0 = 0$, (S) is released without velocity. At an instant t, the abscissa of G is x and the algebraic value of its velocity is $v = \frac{dx}{dt}$ (Figure 1).

The horizontal plane passing through G is considered as the reference level for the gravitational potential energy.

1. a) Derive the differential equation in x that describes the motion of G.
   b) Motion
      i- Deduce the expression of the proper angular frequency w_0 of this oscillator and that of its proper period T_0.
      ii- Calculate w_0 and T_0.
2. The time equation $x = x_m \cos(w_0 t + \varphi)$ is the solution of the previous differential equation, x_m and φ being constants. Determine the values of x_m and φ.
3. a) Determine the expression of v as a function of time.
   b) Deduce the maximum value of v.
4. Taking into consideration the initial conditions, trace the shape of the curve representing the variation of x as a function of time.
5. a) Calculate the value of the mechanical energy of the system (oscillator, Earth).
   b) Find again the maximum value of v.

Part B
Exploitation of the curves of the energies

An appropriate apparatus provides the curves giving the variation, as a function of time, of the kinetic energy and the elastic potential energy of the system (oscillator, Earth) (Figure 2).

1. Identify, with justification, the two curves a and b.
2. The kinetic energy and the elastic potential energy are periodic functions of period T. Determine the relation between T and T_0.

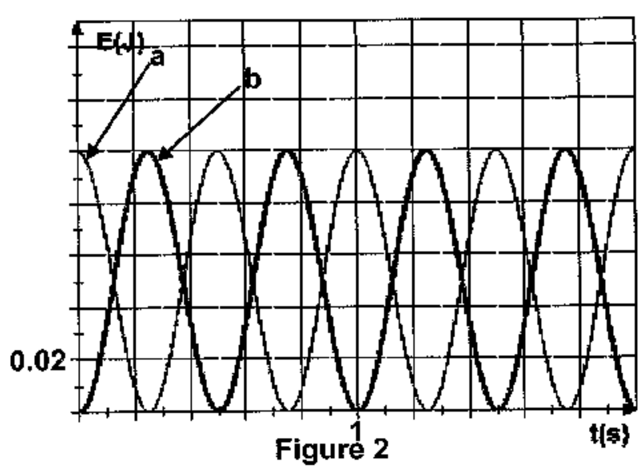
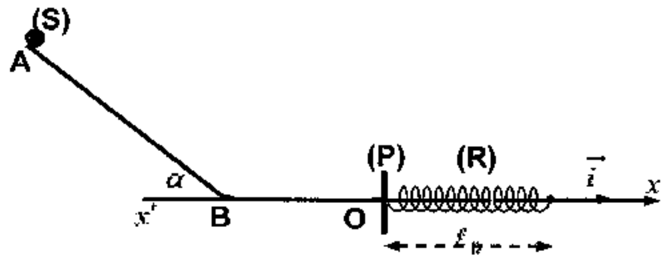


Figure 2

Horizontal Elastic Pendulum

A particle (S) of mass $m_1 = 100\,g$ can slide, without friction, on a track in a vertical plane, formed of a straight part AB, of length $10\,cm$, inclined by an angle $\alpha = 30^0$ with the horizontal and a straight horizontal part Bx.

A spring (R), of un-jointed turns and of negligible mass, of free length ℓ_0 and of stiffness $k = 10N/m$, is placed horizontally on the part Bx. One end of the spring is fixed to the track at point I and the other end is fixed to a plate (P).



(R) has a free length ℓ_0 and (P) is at point O of the horizontal part (figure below). The point O is taken as the origin of abscissas on the axis $x'Ox$.

The particle (S) is released from rest at point A. The horizontal plane containing Bx is taken as a gravitational potential energy reference.

Take: $g = 10m/s^2$.

Part A
Motion of the particle between A and O

1. Calculate the mechanical energy of the system [(S), Earth] at point A.
2. The mechanical energy of the system [(S), Earth] is conserved between the points A and O. Why?
3. (S) reaches point O with the velocity $\overrightarrow{V_0} = V_0\,\vec{i}$. Show that $V_0 = 1m/s$.

Part B
Motion of the oscillator in two situations
First situation

The plate (P) has a negligible mass.

(S) collides with (P) and sticks to it thus forming a single body $[(P), (S)]$ whose center of mass is G. At the instant $t_0 = 0$, G is at O. The system [(P), (S), spring] forms a horizontal mechanical oscillator. At an instant t, the abscissa of G is x and the algebraic measure of its velocity is v.

1. Write down the expression of the mechanical energy of the system [oscillator, Earth] in terms of m_1, x, v and k.
2. Derive the second order differential equation in x that governs the motion of G.
3. Deduce the nature of the motion of G and the expression of the period T_1 of this motion in terms of m_1 and k.
4. G, leaving O at the instant $t_0 = 0$, passes again through O for the first time at the instant t_1. Calculate the duration t_1.

Second situation

(P) is replaced by another plate (P') of mass $m_2 = 300\,g$ placed at O. Considering the initial conditions, (S) reaches (P'), just before collision, with the velocity $\overrightarrow{V_0} = V_0\,\vec{i}$ $(V_0 = 1m/s)$.

Just after the head-on collision (collinear velocities), (S) and (P') move separately, at the instant $t_0 = 0$, with the velocities V_1 and $\overrightarrow{V_2} = V_2\,\vec{i}$ respectively where $V_2 = 0.5\,m/s$.

1. Determine V_1.
2. Show that the collision is elastic.
3. (P') leaves O at the instant $t_0 = 0$ then passes again through point O for the first time at the instant t_2. We notice that the durations t_1 and t_2 are related by $t_2 > t_1$. Justify.

VII-LS 2009 2nd
Mechanical Oscillations

The aim of this exercise is to study different modes of oscillations of a horizontal elastic pendulum that is formed of a puck (A), of mass $m = 200\,g$, and a spring of un-jointed turns of negligible mass and of stiffness $k = 80\,N/m$.

The position of the center of mass G of (A) is defined, at an instant t, on an axis $x'x$, by its abscissa $x = OG$; the velocity of G is then $\vec{v} = v\vec{i}$ where $v = x' = \frac{dx}{dt}$.

Figure 1

The horizontal plane containing G is taken as a gravitational potential energy reference.

Part A
Free un-damped oscillations

At the instant $t_0 = 0$, the center of mass G of (A) being at O (origin of abscissa), (A) is launched with a velocity $\vec{V_0} = V_0\,\vec{i}$ $(V_0 = 2.5\,m/s)$. (A) thus moves along the support without friction.

1. Calculate the mechanical energy of the system [(A), spring, Earth].
2. a) Give, at the instant t, the expression of the mechanical energy of the system [(A), spring, Earth] in terms of m, k, x and v.
   b) Determine the differential equation that describes the motion of G.
   c) Determine the value of the proper angular frequency w_0 and that of the proper period T_0 of the oscillations.
3. The solution of the obtained differential equation has the form: $x = x_m \cos(w_0\,t + \varphi)$.
   Determine the values of the constants x_m and φ.

Part B
Free damped oscillations – Driving the oscillations

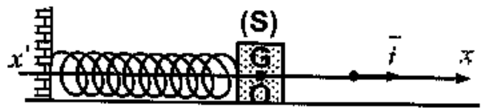
Now, G is at rest at O. We shift (A) by $12.5\,cm$ from O and then we release it from rest at the instant $t_0 = 0$. (A) thus performs pseudo-periodic oscillations of pseudo-period T. At the end of 10 oscillations, the amplitude of the motion becomes $12\,cm$.

1. Calculate the variation in the mechanical energy of the system during these 10 oscillations.
2. The value of T is very close to that of T_0. Why?
3. In order to drive the oscillations of (A), a convenient apparatus provides the oscillator with an energy E during these 10 oscillations.
   a) What does the term «driving the oscillations» mean?
   b) Calculate the average power P_{av} furnished during these 10 oscillations.

VIII-LS 2009 1st

Horizontal elastic pendulum

The free end of a spring of horizontal axis $(O;\vec{i})$, of negligible mass and of stiffness $k = 15 N/m$, is connected to a solid (S) of mass m. (S) is free to move on a horizontal table and G, center of mass of (S), may move along the horizontal axis $(O;\vec{i})$.



The horizontal plane through G is taken as a gravitational potential energy reference.

Part A
Theoretical study

G, shifted from its equilibrium position O by a distance x_0 in the positive direction along the axis $(O;\vec{i})$, is released from rest at the instant $t_0 = 0$. (S) thus performs simple harmonic oscillations of proper period T_0.

At an instant t, the abscissa of G is x and the algebraic value of its velocity is $v = \dfrac{dx}{dt}$.

1. Give the expression of the mechanical energy of the system [(S), spring, Earth] at the instant t in terms of k, m, x and v.
2. Derive the second order differential equation in x that governs the motion of G.
3. a) The solution of this equation has the form: $x = x_m \cos\left(\dfrac{2\pi}{T_0}t + \varphi\right)$. Determine, in terms of the given constants, the expressions of T_0 and x_m and calculate the value of φ.

   b) Write down the instantaneous expression of v. Deduce the relation between x_0, T_0 and the maximum value V_m of v.

Part B
Experimental study
First experiment

We record, as a function of time, the variations of the abscissa x of G (figure1) and that of v (figure 2 below).

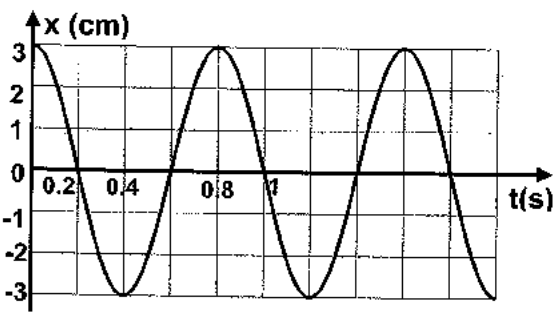


Figure 1

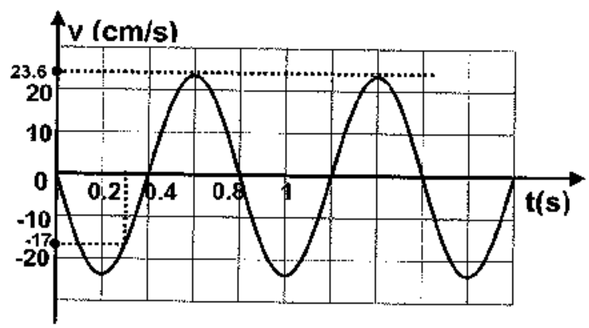


Figure 2

1. Referring to the graphs of figures 1 and 2, specify the value of T_0, that of V_m and the values of x and v at the instant $t_0 = 0$.
2. Determine the mass m of (S).

Second experiment

1. Copy and complete the table below, where KE is the kinetic energy of (S), PE_e is the elastic potential energy of the spring and ME is the mechanical energy of the system $[(S)$, spring, Earth]
2. Deduce from the table an indicator that confirms that the oscillations are simple harmonic.

t	0	0.2	0.3
$v(m/s)$		-0.236	-0.17
$KE(J)$		6.77×10^{-3}	
$x(m)$	0.030		-0.021
$PE_e(J)$	6.75×10^{-3}		
$ME(J)$			

IX-LS 2008 2nd
Mechanical Oscillator

A spring of un-jointed loops, of stiffness constant $k=10N/m$ and of horizontal axis, is fixed from one extremity to a fixed obstacle; the other extremity is attached to a puck M of mass $m=100\,g$. The center of inertia G of M can slide, without friction, along a horizontal axis $x'x$ of origin O and unit vector $\vec{i}$.

The horizontal plane passing through G is taken as a gravitational potential energy reference.

At the instant $t_0=0$, the puck M, initially at rest at O, is hit with another puck M' of mass $m'=\frac{m}{2}$ moving initially with a velocity $\overrightarrow{V'}=-V'\vec{i}$ $(V'>0)$.

After collision, the puck M' rebounds on M with a velocity $\overrightarrow{V_1'}$ and the puck M moves with a velocity $\overrightarrow{V_0}=V_0\,\vec{i}$, and performs oscillations with constant amplitude $x_m=10\,cm$.

1. Give the sign of V_0.
2. Let x and v be respectively the algebraic values of the abscissa and the velocity of G at an instant t after the collision.
   a) Write, in terms of x, m, k and v, the expression of the mechanical energy of the system (M, spring, Earth) at the instant t.
   b) Derive the differential equation of second order in x that describes the motion of M.
   c) The solution of this differential equation is of the form $x=A\sin(w_0t+\varphi)$. Determine the values of the positive constants A, w_0 and φ.
   d) Deduce that the magnitude of the velocity $\overrightarrow{V_0}$ of M just after the collision is $1\,m/s$.
3. Knowing that the collision between M' and M is supposed to be perfectly elastic, determine:
   a) the value V' of the velocity of M' before collision.
   b) the velocity $\overrightarrow{V_1'}$ of M' just after the collision.

X-LS 2008 1st
Horizontal Oscillator

Consider a mechanical oscillator that is formed of a spring (R) of stiffness k and a body (C), of mass m and of center of mass G.

Part A
Determination of k and m

In order to determine the values of m and k of this oscillator, we place it on a horizontal air table. The table functioning normally, we shift (C) from its equilibrium position and we then release it from rest at the instant $t_0 = 0$. (C) may move then without friction on the table, G moving along a horizontal axis. The origin O of this axis is the position of G when (C) is at equilibrium.

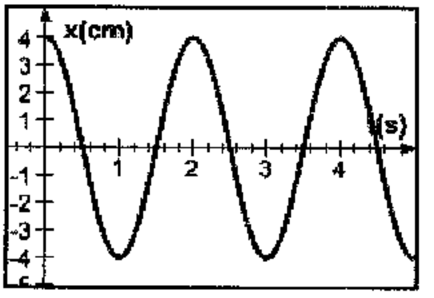


Figure 1

x and v are respectively the abscissa and the algebraic measure of the velocity of G at the instant t.

Convenient equipments allow us to record the variations of x, v and one form of the energies of the oscillator as a function of time. These variations are represented in the graphs of the figures 1, 2 and 3. The horizontal plane containing G is taken as a gravitational potential energy reference.

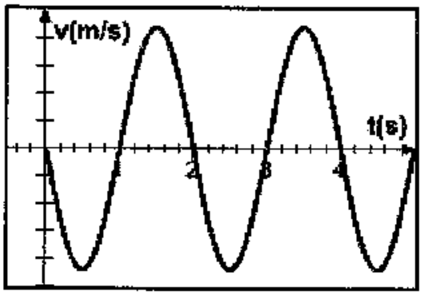


Figure 2

Take: $\pi^2 = 10$.

1. Referring to the graphs 1 and 2, give:
   a) the mode of the oscillations;
   b) the initial values x_0 and v_0 of the motion;
   c) the value of the proper period T_0 of the motion.
2. a) The figure 3 shows the variations of an energy E of the oscillator as a function of time. What form of energy is it? Justify.
   b) The energy E is one of two terms of the mechanical energy ME of the system (body, spring). Redraw figure 3 and show on it the shape of the variations of the mechanical energy ME and that of the other form of that energy.

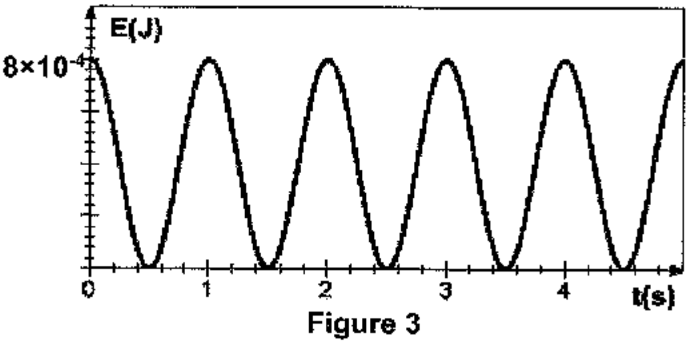


Figure 3

3. Deduce the values of m and k.

Part B
Driving the oscillations

The air table does not function normally anymore and the forces of friction can no longer be neglected. We repeat the experiment under the same initial conditions. The variations of x, as a function of time, are recorded by an apparatus thus giving the graph of figure 4.

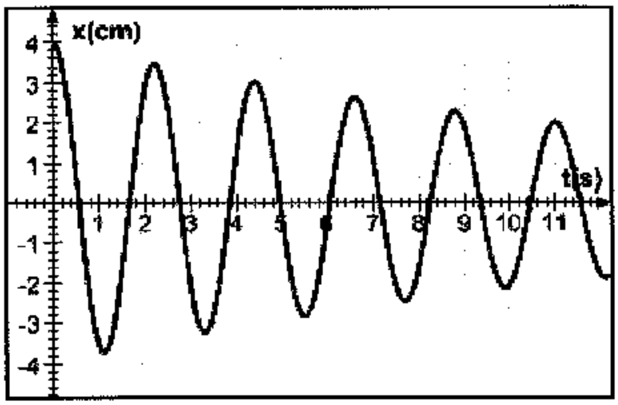


Figure 4

1. Specify the mode of oscillations performed by the oscillator.
2. Determine the value of the variation of the mechanical energy of the oscillator between the instants: $t_0 = 0$ and $t = 11s$.
3. A convenient apparatus allows us to drive these oscillations.
   a) What does the term «driving» the oscillations represent?
   b) Deduce the value of the average power of this apparatus between 0 and $11s$.

XI-LS 2007 1st
Mechanical Oscillator

Consider a mechanical oscillator formed of a solid (S) of mass m and whose center of inertia is G and a spring of negligible mass of un-jointed turns whose stiffness is k.

(S) may slide on a horizontal rail; the position of G on the horizontal axis $\overrightarrow{Ox}$ is defined relative to the origin O, the position of G when (S) is in the equilibrium position (Figure 1).

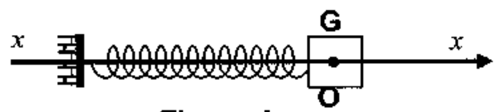


Figure 1

An apparatus is used to record the variations of the abscissa x of G and the algebraic measure v of its velocity as a function of time.

The horizontal plane through G is taken as a gravitational potential energy reference.

The object of this exercise is to compare the values of certain physical quantities associated with the motion of the oscillator in two situations.

Part A
First situation

The solid performs oscillations and the mechanical energy ME of the system (oscillator, Earth) keeps a constant value $ME = 64 \times 10^{-3} J$.

The recording apparatus gives the curves represented in figures (2) and (3).

1. Refer to figures (2) and (3).
   a) Indicate the type of oscillations of (S).
   b) Specify:
      i- the abscissa x_0 and the value v_0 of the velocity at the instant $t_0 = 0$.
      ii- the value of x_m, the amplitude of the oscillations and the maximum value v_m of the velocity.
      iii- the direction of motion of G when it passes through the origin O for the first time.

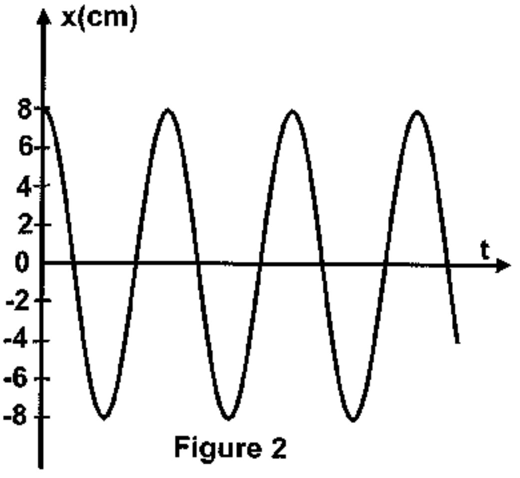


Figure 2

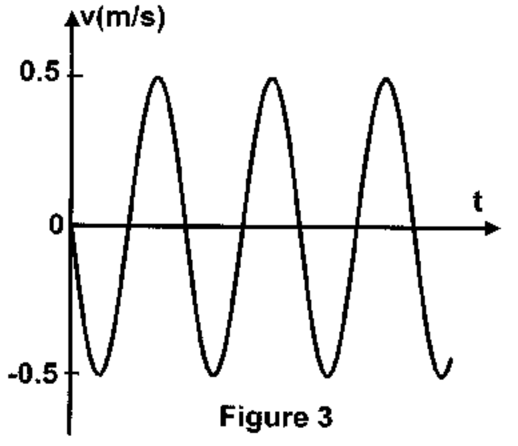


Figure 3

2. Applying the principle of conservation of mechanical energy, show that:
   a) the stiffness of the spring has a value $k = 20 N/m$.
   b) the mass of (S) has a value $m = 512\,g$.
3. a) Write the expression of the mechanical energy of the system (oscillator, Earth) in terms of m, v, k and x.
   b) Determine the second order differential equation in x which describes the motion of G.
   c) Deduce the expression of the proper angular frequency w_0 in terms of k and m.
   d) The solution of the second order differential equation in this situation is $x = x_m \cos(w_0 t + \varphi)$ where φ is a constant. Determine the value of φ.

Part B

Second situation

The solid (S), now shifted by a distance x_{01} from its equilibrium position, is launched, at the instant $t_0 = 0$, in the positive direction with an initial velocity of magnitude v_{01} the apparatus thus records the variations of the abscissa x as a function of time (figure 4).

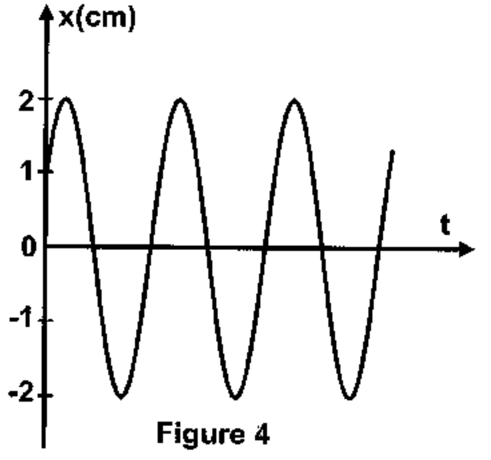


Figure 4

1. Referring to figure 4:
   a) give the value of x_{01} of G and that of the amplitude x_{m1} motion.
   b) show that the mechanical energy ME_1 of the system (oscillator, Earth) does not vary with time.
   c) show that the value of ME_1 is different from that of ME given in the first situation.
2. Calculate the value of the elastic potential energy of the oscillator at $t_0 = 0$ and determine the value of v_{01}.
3. The value of w_0 is the same in both situations. Why?
4. The solution of the second order differential equation in this situation is $x_1 = x_{m1} \cos(w_0 t + \varphi_1)$. Show that the value of φ_1 is different from that of φ.

XII-LS 2006 2nd

Horizontal Mechanical Oscillator

Consider a mechanical oscillator that is formed of a solid (S) of mass $m = 0.1kg$ and a spring whose stiffness (force constant) is k. (S) may move, without friction, on a horizontal track with its center of mass G on a horizontal axis $x'x$.

An apparatus is used to register the positions of the center of mass G at successive instants separated by a constant time interval $\tau = 20\,ms$.

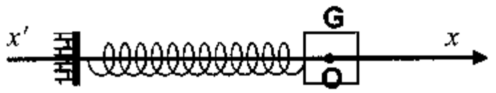
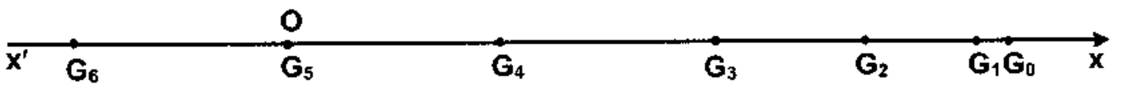


Figure 1

(S) is shifted, in the positive direction, from the equilibrium position O of G by a certain distance, and then is released without initial velocity at the instant $t_0 = 0$.

The above apparatus gives the positions G_0, G_1, G_2, G_3 and G_4 at the instants $t_0 = 0$, $t_1 = \tau$, $t_2 = 2\tau$, $t_3 = 3\tau$ ... respectively.



Some of the positions of G are given in the following table:

t	0	τ	2τ	3τ	4τ	5τ	6τ
OG=x(cm)	OG_0	OG_1= 9.53	OG_2 = 8.09	OG_3 = 5.88	OG_4 = 3.09	OG_5 = 0	OG_6= -3.09

1. At the instant t, the abscissa of G is x and the algebraic value of its velocity is v.
   Write the expression of the mechanical energy of the system (oscillator, Earth) in terms of x, v, m and k.

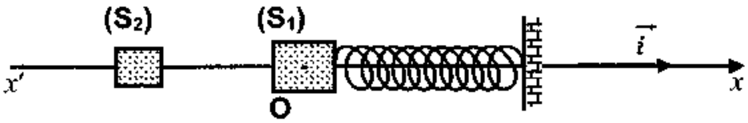
   Take the horizontal plane through G as a gravitational potential energy reference.

2. Derive the second order differential equation that governs the motion of G.
3. The solution of this differential equation may be written in the form: $x = x_m \sin(w_0 t + \varphi)$ where x_m, w_0 and φ are constants.
   a) Determine the expression of w_0 in terms of m and k.
   b) Determine the position of G for which the speed of (S) is maximum $(v_{\max})$.
   c) Applying the principle of conservation of mechanical energy, show that: $(v_{\max})^2 = v^2 + {w_0}^2 x^2$.
4. Using the above table, show that:
   a) the speed at the instant t_3 is $1.250\,m/s$;
   b) the maximum speed is $v_{\max} = 1.545\,m/s$.
5. Deduce the value of k.

XIII-LS 2005 2nd

Study of a Mechanical Oscillator

The object of this exercise is to determine the stiffness of the spring of a horizontal mechanical oscillator. This oscillator is formed of a solid (S_1) of mass $M = 400\,g$ and a spring of negligible mass and of stiffness k. The center of mass G of (S_1) may move on a horizontal straight axis $x'Ox$ The position of G is defined, at any instant t, by its abscissa $x = \overline{OG}$, O corresponding to the equilibrium position G_0 of G (figure).



Part A
Setting the oscillator in motion

(S_1) is initially at rest and G is at O. To set (S_1) in motion, a solid (S_2), of mass $m = \dfrac{M}{2}$, is launched towards (S_1) along the axis $x'Ox$. Just before collision, (S_2) was moving with the velocity $\overrightarrow{V_2} = V_2\,\vec{i}$ $(V_2 = 0.75\,m/s)$. The collision between (S_1) and (S_2) being elastic, (S_2) rebounds along $x'Ox$. Just after collision, (S_1) acquires the velocity $\overrightarrow{V_0} = V_0\,\vec{i}$.

1. What are the two physical quantities that remain conserved during this collision?
2. Write the equations that express the preceding conservations.
3. Deduce that $V_0 = 0.5 m/s$.

Part B
Energetic study of the oscillator

The graphical recordings show that the time equation of motion of G, after collision, may be written in the form: $x = x_m \sin\left(\sqrt{\dfrac{k}{M}}\,t\right)$ (x in m; t in s) where x_m is a positive constant.

The horizontal plane passing through G is taken as a gravitational potential energy reference.

1. a) Write the expression of the elastic potential energy PE_e of the mechanical oscillator in terms of k, x_m, M and t.
   b) Determine the expression of the kinetic energy KE of the oscillator in terms of k, x_m, M and t.
   c) Find the expression of the mechanical energy ME of the system (oscillator, Earth) in terms of k and x_m.
   d) Deduce that (S_1) is not subjected to any force of friction during its motion.
2. a) Determine the value of ME.
   b) During the motion of (S_1), G oscillates between two extreme positions A and B, $20\,cm$ apart.
   Determine the value of k.

Sessions Solutions

I-LS 2014 1st

Part A

1. The mechanical energy $ME|_{t_0=0} = KE|_{t_0=0} + PE_e|_{t_0=0} + GPE|_{t_0=0}$;

   Then $ME|_{t_0=0} = \frac{1}{2}mv_0^2 + 0 + 0 = \frac{1}{2}\times 0.2\times 2.5^2 = 0.625\,J$.

2. At an instant t: $ME|_t = KE|_t + PE_e|_t = \frac{1}{2}mv^2 + \frac{1}{2}k\,x^2$.

3. a) The forces of friction are negligible, then the mechanical energy is conserved, so $\frac{d(ME)}{dt} = 0$;

   $m\,v\,v' + k\,xx' = 0$; but $v = x' \neq 0$ (system in motion) and $v' = x''$; so, $v(m\,x'' + kx) = 0$;

   Thus, $x'' + \frac{k}{m}x = 0$.

   b) The differential equation that governs the motion of (A) is of 2nd order of the form $x'' + w_0^2 x = 0$

   Where $w_0^2 = \frac{k}{m}$; then $w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{80}{0.2}} = 20\,rad/s$;

   Then the motion of (A) is periodic of proper period $T_0 = \frac{2\pi}{w_0} = \frac{2\pi}{20} = 0.31\,s$.

4. Conservation of mechanical energy $ME|_{t_0=0} = ME|_{x_m} = 0.625\,J$;

   $KE|_{x_m} + PE_e|_{x_m} = 0.625\,J$; $0 + \frac{1}{2}k\,x_m^2 = 0.625\,J$; so $x_m = \sqrt{\frac{2\times 0.625}{80}} = 0.125\,m = 12.5\,cm$.

   The initial conditions:

   ✘ At $t = 0$, $x = 0$; so, $x = x_m\cos(\varphi) = 0$; then $\varphi = +\frac{\pi}{2}(rad)$ or $\varphi = -\frac{\pi}{2}(rad)$;

   ✘ But $v|_{t=0} = -x_m w_0 \sin\varphi > 0$; so, $\sin\varphi < 0$ then $\varphi = -\frac{\pi}{2}(rad)$;

   Then $x = 12.5\cos\left(20t - \frac{\pi}{2}\right)$ (where t in s and x in cm)

Part B

1. a) $ME|_{t_0=0} = KE|_{t_0=0} + PE_e|_{t_0=0} = 0 + \frac{1}{2}k\,x_{0m}^2 = 0 + \frac{1}{2}\times 80\times 0.125^2 = 0.625\,J$;

   $ME|_{t_1} = KE|_{t_1} + PE_e|_{t_1} = \frac{1}{2}m\,v_1^2 + 0 = \frac{1}{2}\times 0.2\times 2^2 = 0.4\,J$;

   The variation of the mechanical energy $\Delta(ME) = ME|_{t_1} - ME|_{t_0=0} = 0.4\,J - 0.625\,J = -0.225\,J$.

   b) The variation in the mechanical energy is equal to the work done by the force of friction:

   $\Delta(ME) = W_{\vec{f}_{av}} = -0.225\,J$; $-f_{av}\times x_{0m} = -0.225\,J$; then $f_{av} = \frac{0.225}{0.125} = 1.8\,N$.

2. a) To drive the oscillations is to give the system enough energy, with its proper period, in order to compensate the loss due to friction, and restore the undamped oscillations.

   b) The average power delivered to the system is $P_{av} = \frac{|\Delta(ME)|}{\Delta t} = \frac{0.225\,J}{0.085\,s} \approx 2.65W$.

II-LS 2013 2nd

Part A

1. The mechanical energy is $ME = KE + PE_e = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$.

2. (S) slides without friction, then the mechanical energy is conserved so $\frac{d(ME)}{dt} = 0$,

$mx'x'' + kxx' = 0$, $x'(mx'' + kx) = 0$ but $x' \neq 0$ since the system is in motion; then $x'' + \frac{k}{m}x = 0$.

3. $x = x_m \sin\left(\frac{2\pi}{T_0}t + \varphi\right)$, then $x'' = x_m\left(\frac{2\pi}{T_0}\right)^2 \sin\left(\frac{2\pi}{T_0}t + \varphi\right)$.

Replace in the differential equation we get $x_m \sin\left(\frac{2\pi}{T_0}t + \varphi\right)\left[-\left(\frac{2\pi}{T_0}\right)^2 + \frac{k}{m}\right] = 0$.

But $x = x_m \sin\left(\frac{2\pi}{T_0}t + \varphi\right) \neq 0$ then $\left(\frac{2\pi}{T_0}\right)^2 = \frac{k}{m}$; thus, $T_0 = 2\pi\sqrt{\frac{m}{k}}$.

Part B

1. a) $x_0 = 20cm$.

   b) $x_m = 45cm$.

   c) $T_0 = 4 \times 0.157 = 0.628s$.

2. We have $T_0 = 2\pi\sqrt{\frac{m}{k}}$, then $k = \frac{4\pi^2 m}{T_0^2} = \frac{4 \times 3.14^2 \times 0.1}{0.628^2}$; thus $k = 10\,N/m$.

Referring to the initial conditions at $t = 0$, we have $x = x_0 = 20\,cm$;

But $x = x_m \sin\left(\frac{2\pi}{T_0}t + \varphi\right)$, then $20 = 45\sin(\varphi)$; $\sin(\varphi) = \frac{4}{9}$;

So, $\varphi = 0.46\,\text{rad}$ or $\varphi = (\pi - 0.46)\text{rad}$;

And according to figure 2, the curve is increasing $t = 0$, so $v_0 = \left.\frac{dx}{dt}\right|_{t=0} = x_m\left(\frac{2\pi}{T_0}\right)\cos\varphi > 0$;

Then $\cos\varphi > 0$; thus, $\varphi = 0.46\,rad$ or $(\varphi = 26^0)$.

4. a) The amplitude of motion is constant then the mechanical energy is conserved; thus it corresponds to the curve (a);

   ✘ At $t = 0$, $x = x_0 = 20cm = 0.2\,m$, the elastic potential energy is:

   $PE_{e_0} = \frac{1}{2}kx_0^2 = \frac{1}{2} \times 10 \times 0.2^2 = 0.2\,J$; thus, it corresponds to the curve (b).

   ✘ Consequently the curve (c) corresponds to the kinetic energy.

b) At $t=0$; the kinetic energy $KE_0 = 0.8J = \frac{1}{2} m v_0^2$ then $v_0^2 = 16$ but $v_0 > 0$; thus, $v_0 = 4\,m/s$.

c) Comparison of periods:

i- The period of the kinetic and potential energies are equal to $T = 2 \times 0.157 = 0.314\,s$;

ii- We have $T_0 = 0.628\,s$, by comparison we get $T = \frac{T_0}{2}$.

III-LS 2013 1st

Part A

1. a) The mechanical energy at A_0 is: $ME_i = KE_i + GPE_i = \frac{1}{2} m_1 v_i^2 + m_1 g h_i = \frac{1}{2} m_1 v_i^2 + m_1 g \ell$;

$ME_i = 0.5 \times 0.2 \times 8^2 + 0.2 \times 10 \times 1.8 = 10J$.

b) In the absence of friction, the mechanical energy is conserved between the launch point and A $ME_i = 10J = KE_A + GPE_A$ but $GPE_A = 0$ on reference;

So $10J = \frac{1}{2} m_1 v_1^2 + 0 \Rightarrow v_1^2 = \frac{2 \times 10}{0.2} = 100$; then $v_1 = \sqrt{100} = 10\,m/s$.

2. a) The linear momentum and kinetic energy are conserved.

b) Conservation of linear momentum:

	Just before	Just after
(P_1): m_1	$\overrightarrow{v_1}$	$\overrightarrow{v_1'}$
(P_2): m_2	$\vec{0}$	$\overrightarrow{v_2'}$

$\vec{P}_{\text{just before collision}} = \vec{P}_{\text{just after collision}}$

$m_1 \overrightarrow{v_1} + m_2 \vec{0} = m_1 \overrightarrow{v_1'} + m_2 \overrightarrow{v_2'}$;

The collision is collinear, then we may use algebraic expression of the last result: $m_1 \bar{v}_1 = m_1 \bar{v}_1' + m_2 \bar{v}_2'$.

$m_1(\bar{v}_1 - \bar{v}_1') = m_2 \bar{v}_2'$ ....(1)

The collision is elastic, then the kinetic energy is conserved $KE_{\text{just before collision}} = KE_{\text{just after collision}}$

$\frac{1}{2} m_1 v_1^2 + 0 = \frac{1}{2} m_1 v_1'^2 + \frac{1}{2} m_2 v_2'^2 \Rightarrow \frac{1}{2} m_1 (v_1^2 - v_1'^2) = \frac{1}{2} m_2 v_2'^2$.

We know that: $\bar{v}^2 = v^2$; by simplifying, then factorizing we get:

$m_1 (v_1^2 - v_1'^2) = m_2 v_2'^2 \Rightarrow m_1 (\bar{v}_1 - \bar{v}_1')(\bar{v}_1 + \bar{v}_1') = m_2 \bar{v}_2'^2$ ....(2).

Dividing (2) by (1) we get $\bar{v}_1 + \bar{v}_1' = \bar{v}_2'$ ....(3)

Solving the system of equations (3) & (1), we find the velocities just after collision:

$$\begin{cases} \bar{v}_1 - \bar{v}_1' = \frac{m_2}{m_1} \bar{v}_2' \\ \bar{v}_1 + \bar{v}_1' = \bar{v}_2' \end{cases} ; \text{ so } 2\bar{v}_1 = \left(1 + \frac{m_2}{m_1}\right) \bar{v}_2' ; \text{ then } \bar{v}_2' = \frac{2 m_1}{m_1 + m_2} \bar{v}_1 = \frac{2 \times 0.2}{0.2 + 0.3} \times 10 = 8\,m/s.$$

Part B

1. The mechanical energy at any instant is $ME = KE + PE_e = \frac{1}{2} m v^2 + \frac{1}{2} k x^2$.

2. Since the mechanical energy is conserved then $\frac{d(ME)}{dt} = 0$; $m x' x'' + k x x' = 0$,

$x'(m x'' + k x) = 0$; but $v = x' \neq 0$ since the system is in motion; then $x'' + \frac{k}{m} x = 0$.

The motion of (S) is governed by a differential equation of 2[nd] order of the form $x''+w_0^2x=0$ its motion is sinusoidal (simple harmonic).

3. Conservation of the mechanical energy:

$ME|_{x=0}=ME|_{x=x_m}\Rightarrow KE|_{x=0}+PE_e|_{x=0}=KE|_{x=x_m}+PE_e|_{x=x_m}$;

But $PE_e|_{x=0}=0$ & $KE|_{x=x_m}=0$; so $\frac{1}{2}m v_0^2=\frac{1}{2}k x_m^2$;

Then $x_m=\sqrt{\frac{m}{k}}v_0=\sqrt{\frac{1.2}{120}}\times 2=0.2m=20cm$;

Referring to the initial conditions, at $t=0$, $x=0$ so $x_m\cos\varphi=0\Rightarrow\varphi=\pm\frac{\pi}{2}$ rad.

But $v|_{t=0}>0$ and $v=x'=-x_m\left(\frac{k}{m}\right)\sin(wt+\varphi)$; so $-x_m\sqrt{\frac{k}{m}}\sin(\varphi)>0$;

Thus, $\sin(\varphi)<0$ so $\varphi=-\frac{\pi}{2}$ rad.

IV-LS 2012 1[st]

Part A

1. In both intervals the amplitude of motion is $x_m=2\,cm$.
2. In both intervals the oscillations are free undamped.
3. In $[0,0.2\,s]$ the proper period is $T_{01}=0.1\,s$; and in $[0.2\,s,0.35\,s]$ is $T_{02}=0.05\,s$.

Part B

1. At $t=0$, the mechanical energy is

$$ME|_{t=0}=PE_e|_{t=0}+KE|_{t=0}=\frac{1}{2}k x_m^2+0=0.5\times 400\times\left(2\times 10^{-2}\right)^2=0.08\,J.$$

2. The mechanical energy at an instant t is $ME=KE+PE_e=\frac{1}{2}mv^2+\frac{1}{2}k x^2$.

3. a) Since the mechanical energy is conserved, then: $\frac{d(ME)}{dt}=0$; $mx'x''+kxx'=0$;

   $x'(m x''+k x)=0$; $v=x'\neq 0$ since the system is in motion; then $x''+\frac{k}{m}x=0$.

   b) The motion of (S) is governed by a differential equation of the form $x''+w_0^2x=0$; with $w_0^2=\frac{k}{m}$ then $w_0=\sqrt{\frac{k}{m}}$; thus its motion is simple harmonic of proper period $T_0=\frac{2\pi}{w_0}=2\pi\sqrt{\frac{m}{k}}$

   c) $T_0=2\pi\sqrt{\frac{m}{k}}=\sqrt{4\pi^2\times\frac{0.1}{400}}=\sqrt{4\times 10\times\frac{0.1}{400}}=\sqrt{0.01}=0.1s$ (Compatible with A-3).

4. Using the initial conditions at $t=0$, $x=x_m$ we get $x_m=x_m\sin(\varphi)$; $\sin(\varphi)=1\Rightarrow\varphi=\frac{\pi}{2}rad$ [1].

---

[1] If the unit is wrong, the answer is considered false.

Part C

1. (B_2) is detached from (B_1) at $t_1 = 0.2\,s$.
2. Since the amplitude x_m is constant in both cases is unchanged and we are using the same spring k also is not modified then the mechanical energy $ME = \frac{1}{2}k\,x_m^2$ will not be modified.[2]
3. The mechanical energy of both systems is the same then:

$ME_{B_1}\big|_{x=0} = ME_B\big|_{x=0} \Rightarrow KE_{B_1}\big|_{x=0} + PE_{e\,B_1}\big|_{x=0} = KE_B\big|_{x=0} + PE_{e\,B}\big|_{x=0}$.

But $PE_{e\,B_1}\big|_{x=0} = PE_{e\,B}\big|_{x=0} = 0$, $\frac{1}{2}m_1 V_1^2 = \frac{1}{2}mV^2$ and $m_1 = \frac{m}{4}$.

Then $\frac{m}{4}V_1^2 = mV^2$; $V_1^2 = 4V^2 \Rightarrow V_1 = 2V$.[3]

V-LS 2011 1st

Part A

1. a) The mechanical energy at any instant is $ME = KE + PE_e = \frac{1}{2}mv^2 + \frac{1}{2}k\,x^2$;

Since the mechanical energy is conserved then $\frac{d(ME)}{dt} = 0$; $mx'x'' + kxx' = 0$,

$x'(m\,x'' + k\,x) = 0$; but $v = x' \neq 0$ since the system is in motion; then $x'' + \frac{k}{m}x = 0$.

b) Motion

i- The motion of (S) is governed by a differential equation of the 2nd order of the form:

$x'' + w_0^2 x = 0$ with $w_0^2 = \frac{k}{m} \Rightarrow w_0 = \sqrt{\frac{k}{m}}$.

Then its motion is simple harmonic of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}}$.

ii- We have $w_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{0.5}} = 2\pi \approx 6.32\,rad/s$ and $T_0 = \frac{2\pi}{w_0} = \frac{2\pi}{2\pi} = 1\,s$.

2. At $t_0 = 0$, $v = 0$; $x_0 = -10\,cm$; then $x_m = -x_0 = 10\,cm$.

At $t_0 = 0$, $x = -x_m \Rightarrow -x_m = x_m \cos\varphi \Rightarrow \cos\varphi = -1$ then $\varphi = \cos^{-1}(-1) = \pi$ rad;

Thus, $x = 10\cos(2\pi t + \pi) = -10\cos(2\pi t)$ (t in s and x in cm).

3. a) The velocity $v = \frac{dx}{dt} = 10 \times 2\pi\sin(2\pi t) = 20\pi\sin(2\pi t)$ (t in s and v in cm/s).

b) The maximum speed is reached if $|\sin(2\pi t)| = 1$, then $v_m = 20\,\pi\,cm/s = 0.632\,m/s$.

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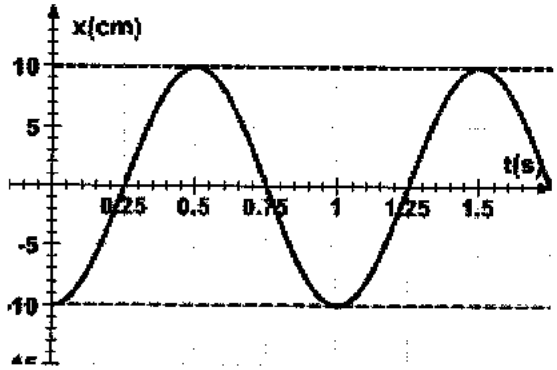
[2] If one factor X_m or k is mentioned only the answer is considered as false. Because if k is only unchanged then the mechanical energy will not have the same value.

[3] We can use another method $ME_{B_1} = \frac{1}{2}m_1 V_1^2 = 0.08J \Rightarrow V_1 = 8\sqrt{10}\,m/s$ &

$ME_B = \frac{1}{2}mV^2 = 0.08J \Rightarrow V = 4\sqrt{10}\,m/s$; comparing the two results we get $V_1 = 2V$

4. Graph.
Scales:
On abscissa axis 1div $\equiv 0.25s$;
On ordinate axis 1div $\equiv 5cm$;



5. a) The mechanical energy being conserved:
$ME = ME|_{x_0} = KE|_{x_0} + PE_e|_{x_0}$;
$ME = \frac{1}{2}k\,x_0^2 + 0 = \frac{1}{2}\times(20)\times 0.1^2 = 0.1\,J$.
b) According to the conservation of mechanical energy:
$ME = ME|_{x=0} = KE|_{x=0} + PE_e|_{x=0}$; so $\frac{1}{2}m\,v_m^2 + 0 = 0.1\,J$; then $v_m = \sqrt{\frac{0.1\times 2}{0.5}} = 0.632\,m/s$.

Part B

1. At $t=0$, $v_0 = 0$ so $KE_0 = 0$ then the curve (b) represents the kinetic energy KE because it starts from the origin, consequently the curve (a) represents the elastic potential energy.
2. From the curves of energies, we have $2T = 1s$ then $T = 0.5s$.
Comparing this value to the proper period we get $T = \frac{T_0}{2}$.

VI-LS 2010 1st

Part A

1. The mechanical energy at A is: $ME_A = KE_A + GPE_A$; $KE_A = 0$ (released from rest);
and $GPE_A = m\,g\,h_A$ but $\sin\alpha = \frac{h_A}{AB} \Rightarrow h_A = AB\sin\alpha$; then $GPE_A = m\,g\,AB\,\sin\alpha$.
Thus, $ME_A = 0 + m\,g\,AB\sin\alpha = 0.1\times 10\times 0.1\times 0.5 = 0.05\,J$.
2. Since the forces of friction are negligible; then the mechanical energy is conserved:
$ME_O = ME_A = 0.05\,J$.
3. $ME_O = KE_O + GPE_O = 0.05\,J$; $GPE_O = 0$ (on reference).
$KE_O = 0.05\,J = \frac{1}{2}m\,v^2 \Rightarrow v = \sqrt{\frac{2\times 0.05}{0.1}} = 1\,m/s$.

Part B

First situation

1. The mechanical energy $ME = KE + PE_e = \frac{1}{2}m_1 v^2 + \frac{1}{2}k\,x^2$.
2. Since the mechanical energy is conserved then: $\frac{d(ME)}{dt} = 0$; $m_1 x' x'' + k x x' = 0$,
$x'(m_1\,x'' + k\,x) = 0$ (but $v = x' \neq 0$ since the system is in motion), then $x'' + \frac{k}{m_1}x = 0$.
3. The motion of (S) is governed by a differential equation of the form $x'' + w_1^2 x = 0$, $w_1^2 = \frac{k}{m_1}$;

Then its motion is simple harmonic of proper period $T_1 = \frac{2\pi}{w_1} = 2\pi\sqrt{\frac{m_1}{k}}$.

4. It returns again to O after $t_1 = \frac{T_1}{2} = \pi\sqrt{\frac{m_1}{k}}$, then $t_1 = \pi\sqrt{\frac{0.1}{10}} = 0.314\,s$.

Second situation

1. The linear momentum is conserved: $\overrightarrow{P}_{\text{Just before}} = \overrightarrow{P}_{\text{Just after}}$;

   $m_1\vec{v} + \vec{0} = m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2}$; all the velocities are collinear.

   $m_1\overrightarrow{v_1} = m_1\vec{v} - m_2\overrightarrow{v_2}$; $\overrightarrow{v_1} = \vec{v} - \frac{m_2}{m_1}\overrightarrow{v_2}$; $\overrightarrow{v_1} = \vec{v} - \frac{m_2}{m_1}\overrightarrow{v_2} = 1\vec{i} - \frac{0.3}{0.1}\times 0.5\vec{i}$;

   Thus, $\overrightarrow{v_1} = -0.5\vec{i}\ \ (m/s)$.

2. Just before collision $KE_{\text{Just before}} = 0 + \frac{1}{2}m_1 v^2 = 0.5\times 0.1\times 1^2 = 0.05\,J$.

   Just after collision $KE_{\text{Just after}} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = 0.5\times 0.1\times 0.5^2 + 0.5\times 0.3\times 0.5^2 = 0.05\,J$.

   The kinetic energy is conserved thus the collision is elastic.

3. The duration needed to return is: $t_2 = \frac{T_2}{2} = \pi\sqrt{\frac{m_2}{k}} > t_1 = \pi\sqrt{\frac{m_1}{k}}\ \ (m_2 = 300\,g > m_1 = 100\,g)$.

VII-LS 2009 2nd

Part A

1. The mechanical energy $ME = KE|_{x=0} + PE_e|_{x=0} = \frac{1}{2}mV_0^2 = 0.625\,J$.

2. a) The mechanical energy $ME = KE + PE_e = \frac{1}{2}m\,v^2 + \frac{1}{2}k\,x^2$.

   b) Since friction is neglected, the mechanical energy is conserved and ME is constant.

   $\frac{d(ME)}{dt} = 0$, $mx'x'' + kxx' = 0$, $x'(m\,x'' + k\,x) = 0$; but $x' \neq 0$ since the system is in motion.

   Then $x'' + \frac{k}{m}x = 0$.

   c) The differential equation of motion is of the form $x'' + w_0^2 x = 0$ where $w_0 = \sqrt{\frac{k}{m}} = 20\,rad/s$.

   Then its motion is periodic of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}} = 0.314\,s$.

3. At maximum elongation $ME = PE_e|_{x=x_m} = \frac{1}{2}k\,x_m^2 = 0.625\,J$, then $x_m = 0.125\,m = 12.5\,cm$.

   We have: $x = x_m\cos(w_0 t + \varphi) \Rightarrow v = x' = -x_m w_0 \sin(w_0 t + \varphi)$;

   ✖ At $t = 0$, $x = 0 \Rightarrow \cos\varphi = 0$; then $\varphi = \pm\frac{\pi}{2}\,rad$.

   ✖ But at $t = 0$, $v_0 = -w_0\,x_m \sin\varphi > 0 \Rightarrow -\sin\varphi > 0$; so $\varphi = -\frac{\pi}{2}\,rad$.

Part B

1. The loss of mechanical energy is : $\Delta(ME)=\Delta(PE_{e\max})=\frac{1}{2}k(x_{m_2}^2-x_{m_1}^2)=-0.049\,J$.
2. The frictional forces are too weak.
3. a) Providing the system enough energy to compensate the loss due to friction and thus retrieving the harmonic oscillations of the system.

   b) The energy supplied to the system is : $E=|\Delta(ME)|=0.049\,J$.

   The average power supplied to the system is: $P_{av}=\frac{|\Delta(ME)|}{\Delta t}=\frac{0.049\,J}{10\times0.314\,s}=0.016\,W$.

VIII-LS 2009 1st

Part A

1. The mechanical energy is: $ME=KE+PE_e=\frac{1}{2}mv^2+\frac{1}{2}kx^2$.
2. The oscillations performed are simple harmonic, then mechanical energy is conserved so ME is constant, thus $\frac{d(ME)}{dt}=0$.

   $mx'x''+kxx'=0$, $x'(mx''+kx)=0$; but $v=x'\neq0$ since the system is in motion;

   Then $x''+\frac{k}{m}x=0$.

3. a) We have: $x=x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)\Rightarrow x''=-\left(\frac{2\pi}{T_0}\right)^2x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)$.

   Replace in the differential equation we get:

   $x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)\left(-\left(\frac{2\pi}{T_0}\right)^2+\frac{k}{m}\right)=0$; but $x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)\neq0$ (system in motion);

   Then $-\left(\frac{2\pi}{T_0}\right)^2+\frac{k}{m}=0$, thus $T_0=2\pi\sqrt{\frac{m}{k}}$.

   At x_0; $x=x_0$ & $v=0$, then $x_m=x_0$.

   At $t=0$; $x=x_0=x_m$, then $x_m=x_m\cos(\varphi)\Rightarrow\cos\varphi=1$; thus, $\varphi=0$.

   b) The velocity $v=x'=-\left(\frac{2\pi}{T_0}\right)x_0\sin\left(\frac{2\pi}{T_0}t\right)$;

   Then speed is maximum if $\left|\sin\left(\frac{2\pi}{T_0}t\right)\right|=1$, then $v_m=x_0\left(\frac{2\pi}{T_0}\right)$.

Part B

First experiment

1. Graphically $T_0=0.8\,s$ & $v_m=23.6\,cm/s=0.236\,m/s$.

   At $t_0=0$, $x_0=3\,cm$ & $v_0=0$.

2. We have $T_0=2\pi\sqrt{\frac{m}{k}}$, then $m=\frac{kT_0^2}{4\pi^2}=\frac{15\times(0.8)^2}{4\pi^2}\approx0.243\,kg$.

Second experiment

1. Complete the table:
2. We notice that the mechanical energy remains constant (despite the small difference that can be referred to experimental and calculations errors), then the motion is simple harmonic oscillator.

t	0	0.2	0.3
$v(m/s)$	0	-0.236	-0.17
$KE(J)$	0	6.77×10^{-3}	3.51×10^{-3}
$x(m)$	0.030	0	-0.021
$PE_e(J)$	6.75×10^{-3}	0	3.31×10^{-3}
$ME(J)$	6.75×10^{-3}	6.77×10^{-3}	6.82×10^{-3}

IX-LS 2008 2nd

1. $V_0 < 0$.

2. a) The mechanical energy of the system is $ME = PE_e + KE + GPE = \frac{1}{2}k\,x^2 + \frac{1}{2}m\,v^2$.

   But $GPE = 0$ (on reference);

   b) Since the system slides without friction, then the mechanical energy of the system is conserved:

   $\frac{d(ME)}{dt} = 0$; so $k\,x\,x' + m\,v\,v' = 0$, $v\left(x'' + \frac{k}{m}x\right) = 0$ but $v = x' \neq 0$ (system in motion);

   Then $x'' + \frac{k}{m}x = 0$.

   c) $x = A\sin(w_0 t + \varphi)$; $x' = A\,w_0\cos(w_0 t + \varphi)$ and $x'' = -A\,w_0^2\sin(w_0 t + \varphi)$;

   By replacing in the differential equation, we get:

   $-A\,w_0^2\sin(w_0 t + \varphi) + \frac{k}{m}A\sin(w_0 t + \varphi) = 0$; so $A\sin(w_0 t + \varphi)\times\left(-w_0^2 + \frac{k}{m}\right) = 0$.

   But $A\sin(w_0 t + \varphi) \neq 0$, $-w_0^2 + \frac{k}{m} = 0$; so $w_0 = \sqrt{\frac{k}{m}}$; we get $w_0 = \sqrt{\frac{10}{0.1}} = 10\,rad/s$.

   For $t_0 = 0$, $A\sin\varphi = 0$, then $\varphi = 0$ or π rad.

   But $v_0 = A\,w_0\cos\varphi < 0$, ($A > 0$); then $\cos\varphi < 0$; so $\varphi = \pi$ (rad); and $A = +10\,cm$.

   Thus $x = 10\sin(10t + \pi) = -10\sin(10t)$ where t in s & x in cm.

   d) The velocity $v = x' = -100\cos(10t)$ where t in s & v in cm/s.

   At $t_0 = 0$, $V_0 = -100\,cm/s = -1\,m/s$.

3. a) Conservation of linear momentum: $\vec{P}_{\text{just before}} = \vec{P}_{\text{just after}}$, so $m'\vec{V'} = m'\vec{V_1'} + m\vec{V_0}$

   In algebraic values $m'(V' - V_1') = m\,V_0$ ....(1)

   The collision is elastic, then the kinetic energy of the system is conserved:

   $\frac{1}{2}m'V'^2 = \frac{1}{2}m'V_1'^2 + \frac{1}{2}m\,V_0^2 \Rightarrow m'\left(V'^2 - V_1'^2\right) = m\,V_0^2$ ....(2);

   $\frac{(2)}{(1)} \Rightarrow V' + V_1' = V_0$; substituting in (1) we get: $V' = \frac{3}{2}V_0 = 1.5V_0 = -1.5\,m/s$.

   b) $V_1' = V_0 - V' = -1 - (-1.5) = 0.5\,m/s$, then $\vec{V_1'} = 0.5\vec{i}\ (m/s)$.

X-LS 2008 1st

Part A

1. a) Mode: **free undamped** oscillations.

   b) At $t=0$ we have: $x=x_0=4\,cm$;

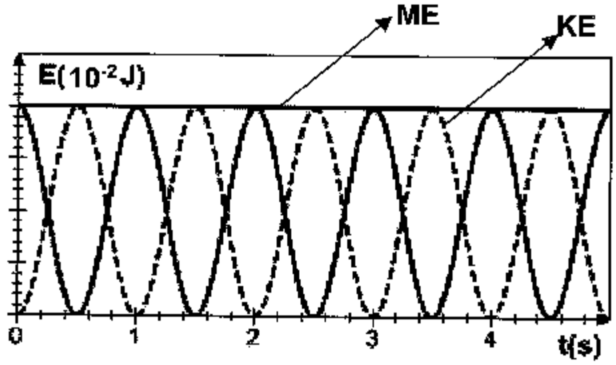
   $v=v_0=0$.

   c) From figure the proper period $T_0=2\,s$.

2. a) Elastic potential energy.

   But at $t=0$, $x=x_m$ and the value of that energy is maximum.

   b) Graphs of ME and KE.



3. Conservation of energy at $t=0$:

$$ME|_{t=0}=KE|_{t=0}+PE|_{t=0}=\frac{1}{2}m v_0^2+\frac{1}{2}k x_0^2\text{; then }\frac{1}{2}k x_m^2=8\times10^{-4}\Rightarrow k=1N/m.$$

The period $T_0=2\pi\sqrt{\frac{m}{k}}$, then $m=\frac{k\,T_0^2}{4\pi^2}=0.1\,kg$.

Part B

1. Mode: **Free damped** oscillations.
2. The variation of the mechanical energy between these instants:

$$\Delta(ME)=ME|_{t=11s}-ME|_{t=0s}=PE_e|_{t=11s}-PE_e|_{t=0s}=\frac{1}{2}k\,x_m^2(t)-\frac{1}{2}k\,x_m^2(0)=-6\times10^{-4}J.$$

3. a) The system is provided with energy enough to compensate the loss with the same period T_0.

   b) The average power dissipated $P_{av}=\frac{|\Delta(ME)|}{\Delta t}=5.45\times10^{-5}W$.

XI-LS 2007 1st

Part A

1. a) The oscillations are **free and undamped**.

   b) Initial conditions:

   i- At $t=0$, $x_0=8cm$ and $v_0=0$.

   ii- $x_m=8cm$ and $v_m=0.5\,m/s$.

   iii- When passes through O for the first time, $v<0$ so (S) is moving in the negative direction.

2. a) Conservation of mechanical energy $ME=ME|_{x=x_m}=KE|_{x=x_m}+PE|_{x=x_m}=\frac{1}{2}k\,x_m^2$;

   So, $64\times10^{-3}=0.5\times k\times(8\times10^{-2})^2$, then $k=20N/m$.

   b) Conservation of mechanical energy $ME=ME|_{x=0}=\frac{1}{2}m\,v_m^2$, then $m=0.512\,kg=512\,g$.

3. a) The mechanical energy is given by: $ME=\frac{1}{2}m\,v^2+\frac{1}{2}k\,x^2$ is constant.

   b) The mechanical energy is conserved $\frac{d(ME)}{dt}=0$; $m\,v\,v'+k\,x\,x'=x'(mx''+kx)=0$,

   but $v=x'\neq0$ (system in motion); then $mx''+kx=0\Rightarrow x''+\frac{k}{m}x=0$.

c) This differential equation is of 2^{nd} order of the form $x'' + w_0^2\, x = 0$ where $w_0 = \sqrt{\frac{k}{m}}$.

d) For $t = 0$, we have $x = x_0 = x_m \cos\varphi$; then $\cos\varphi = 1$, thus $\varphi = 0$.

Part B

1. a) $x_{01} = 1cm$ and $x_{m1} = 2\,cm$.

   b) The amplitude of the oscillations x_{m1} is constant, so its mechanical energy is conserved.

   c) The mechanical energy: $ME_1 = PE|_{x_{m1}} + KE|_{x_{m1}} = \frac{1}{2} k\, x_{m1}^2 = 4\times10^{-3}\,J \neq ME$.

   Then the mechanical energy of the system changes.

2. We have $PE_{e0} = \frac{1}{2} k\, x_{01}^2 = 10^{-3}\,J$;

   But $PE_{e01} + KE_{01} = 4\times10^{-3}\,J$, then $v_{01}^2 = 0.012$ but $v_{01} > 0$; thus $v_{01} = 0.108\, m.s^{-1}$.

3. Since the angular frequency does not depend on the initial conditions, it depends on m and k only.

4. Initial conditions in situation B: at $t = 0$, $x_{01} = 1\,cm$; then $\cos\varphi_1 = \frac{x_{01}}{x_{m1}} = \frac{1}{2}\left(\varphi_1 = \pm\frac{\pi}{3}\right)$;

   Thus, $\varphi_1 \neq \varphi$.

XII-LS 2006 2nd

1. The mechanical energy is given by $ME = KE + PE_e + GPE = \frac{1}{2} m v^2 + \frac{1}{2} k\, x^2 + 0$.

2. Since friction is neglected, the mechanical energy is conserved.

   $\frac{d(ME)}{dt} = 0$, $mx'x'' + kxx' = 0$, $x'(m\,x'' + k\,x) = 0$;

   (But $v = x' \neq 0$ since the pendulum is in motion), then $x'' + \frac{k}{m} x = 0$.

3. a) We have $x = x_m \sin(w_0 t + \varphi)$ so $x'' = -x_m\, {w_0}^2 \sin(w_0 t + \varphi)$.

   Replace x'' and x in the obtained differential equation, we get:

   $-\,x_m\, {w_0}^2 \sin(w_0 t + \varphi) + \frac{k}{m}\, x_m \sin(w_0 t + \varphi) = 0$;

   $x_m \sin(w_0 t + \varphi)\left(-\,{w_0}^2 + \frac{k}{m}\right) = 0$ but $x_m \sin(w_0 t + \varphi) \neq 0$, then $w_0 = \sqrt{\frac{k}{m}}$.

   b) $v = x' = x_m\, w_0 \cos(w_0 t + \varphi)$, $|v|$ is maximum when $|\cos(w_0\, t + \varphi)| = 1$;

   But at any instant $\cos^2(w_0\, t + \varphi) + \sin^2(w_0\, t + \varphi) = 1$, then $\sin(w_0\, t + \varphi) = 0$;

   But $x = x_m \sin(w_0\, t + \varphi)$; thus $x = 0$ which corresponds to the origin (equilibrium position).

   c) Conservation of mechanical energy:

   $ME|_{x=0} = ME|_t$ (at any point of abscissa x and speed v);

   So, $\frac{1}{2} m\,(v_{\max})^2 = \frac{1}{2} m\, v^2 + \frac{1}{2} kx^2$;

   Then $(v_{\max})^2 = v^2 + \frac{k}{m} x^2$; but $w_0^2 = \frac{k}{m}$, thus, $(v_{\max})^2 = v^2 + w_0^2\, x^2$.

4. a) $v_3 = \frac{G_4G_2}{2\tau} = \frac{OG_2 - OG_4}{2\tau} = \frac{(8.09-3.09)\times 10^{-2}\,m}{2\times 20\times 10^{-3}\,s} = 1.250\,m/s$.

b) $v_{\max} = v_O = \frac{G_6G_4}{2\tau} = \frac{(3.09+3.09)\times 10^{-2}\,m}{2\times 20\times 10^{-3}\,s} = 1.545\,m/s$.

5. Referring to the relation $(v_{\max})^2 = v^2 + w_0^2\,x^2$ at the point of abscissa $x_3 = 5.88\,cm$.

We get: $(v_{\max})^2 = v_3^2 + w_0^2\,x_3^2$, so $w_0^2 = \frac{(v_{\max})^2 - v_3^2}{x_3^2} = \frac{1.545^2 - 1.250^2}{(5.88\times 10^{-2})^2} = 238.5$

But $w_0^2 = \frac{k}{m}$; then $k = m\,w_0^2 = 0.1\times 238.5 = 23.85\,N/m$.

XIII-LS 2005 2nd

Part A

1. The linear momentum and the kinetic energy.
2. Conservation of linear momentum:

$m\overrightarrow{V_2} = m\overrightarrow{V_3} + M\overrightarrow{V_0}$;

Conservation of kinetic energy:

$\frac{1}{2}m\,V_2^2 = \frac{1}{2}m\,V_3^2 + \frac{1}{2}M\,V_0^2$.

	(P_1): m	(P_2): M
Just before	$\overrightarrow{V_2}$	$\vec{0}$
Just after	$\overrightarrow{V_3}$	$\overrightarrow{V_0}$

3. The velocities vectors are collinear, then: $m(V_2 - V_3) = M\,V_0$ ……(1)

But $m(V_2^2 - V_3^2) = M\,V_0^2$ ………(2)

Dividing (2) by (1) we get : $V_2 + V_3 = V_0$ …….(3)

But $M = 2m$; so (1) gives $V_2 - V_3 = 2V_0$ ……(4)

Adding the relations (3) and (4) we get: $2V_2 = 3V_0$; then $V_0 = 0.5\,m/s$; thus, $\overrightarrow{V_0} = 0.5\,\vec{i}\ (m/s)$.

Part B

1. a) The elastic potential energy: $PE_e = \frac{1}{2}k\,x^2 = \frac{1}{2}k\,x_m^2\sin^2\left(\sqrt{\frac{k}{M}}\,t\right)$.

b) The kinetic energy $KE = \frac{1}{2}M\,v^2 = \frac{1}{2}M\,(x')^2 = \frac{1}{2}M\,x_m^2\,\frac{k}{M}\cos^2\left(\sqrt{\frac{k}{M}}t\right) = \frac{1}{2}k\,x_m^2\cos^2\left(\sqrt{\frac{k}{M}}t\right)$

c) The mechanical energy: $ME = KE + PE_e = \frac{1}{2}k\,x_m^2\sin^2\left(\sqrt{\frac{k}{M}}\,t\right) + \frac{1}{2}k\,x_m^2\cos^2\left(\sqrt{\frac{k}{M}}\,t\right)$;

$$ME = \frac{1}{2}k\,x_m^2\left[\sin^2\left(\sqrt{\frac{k}{M}}\,t\right) + \cos^2\left(\sqrt{\frac{k}{M}}\,t\right)\right] = \frac{1}{2}k\,x_m^2.$$

d) Since k and x_m are constant, then ME is constant and the mechanical energy of the system is conserved thus the motion of (S_1) is performed without any force of friction.

2. a) Since ME is conserved $ME = ME|_{x=0} = KE|_{x=0} + PE|_{x=0} = \frac{1}{2}M\,V_0^2 = \frac{1}{2}(0.4)(0.5)^2 = 0.05\,J$.

b) The extreme positions A and B are $20\,cm$ apart; so, $AB = 2x_m \Rightarrow x_m = 0.1\,m$.

But the mechanical energy is conserved: $ME = ME|_{x=x_m} = KE|_{x=x_m} + PE|_{x=x_m}$.

$$ME = ME|_{x=x_m} = \frac{1}{2}k\,x_m^2 \Rightarrow k = \frac{2(ME)}{x_m^2} = 10\,N/m.$$

Unit I

Mechanics

Chapter 4

Angular Momentum

Session		2017	2016	2015	2014	2013	2012	2011	2010
Angular Momentum		1st(B)	-	-	-	-	1st(B)	1st(B)	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Angular Momentum	-	-	1st(B)	-	2nd(B)	-	-	-	-

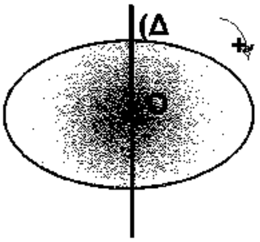
Essentials

I-

Angular Momentum

The angular momentum of a system (particle) in motion with respect to an axis of rotation (Δ) is given by $\sigma = I\ \theta'$ where I is the moment of the system around (Δ) measured in $kg.m^2$ & θ' is its angular velocity in rad/s.

In SI units, σ is measured in $kg.m^2.\text{rad}/s \equiv kg.m^2/s$.



Note 1: σ is an algebraic quantity whose sign depends on the direction of rotation.

Note 2: The angular momentum of a system of particles is additive so: $\sigma = \sum \sigma_i$

II-

Theorem of Angular Momentum

The resultant moment, about an axis of rotation (Δ), of all external forces applied is equal to the sum of moments of all the external forces acting on it and given by: $\sum \text{M}_{Fext} = \dfrac{d\sigma}{dt}$.

If the resultant moment is constant $\sum \text{M}_{Fext} = \text{M}$, then $\sigma = \text{M}\,t + \sigma_0$.

1. Conservation of angular momentum

When the resultant moment, of all the external forces about the axis of rotation is zero $\sum \text{M} = 0$.

The angular momentum is conserved, so $\sigma_{\text{just before}} = \sigma_{\text{just after}}$.

2. Analogy between physical quantities

Translation	Rotation
x, v & a	θ, θ' & θ''
Force F	Moment of the force M_F
Mass m	Moment of inertia I
Linear momentum $\vec{p} = m\vec{v}$	Angular momentum $\sigma = I\,\theta'$
Kinetic energy $KE = \frac{1}{2}mv^2$	Kinetic energy $KE = \frac{1}{2}I\,\theta'^2$
Newton's 2nd law $\sum \vec{F} = \frac{d\vec{P}}{dt}$	Theorem of angular momentum $\sum \text{M}_{Fext} = \frac{d\sigma}{dt}$

Applications

I- Rotation of a Disk

A disk of mass $m = 5kg$ and radius $R = 10\,cm$ is free to rotate in a vertical plane around a horizontal axis. On the rim of the disk, taken at rest, we apply a constant force $\overrightarrow{F}$ tangent to its circumference for $3\,s$ starting from an instant taken as origin of time $t_0 = 0$.

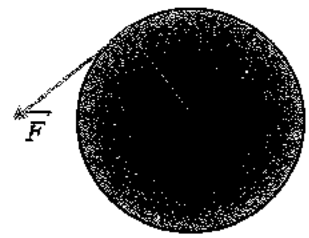


Figure 1

1. Show that the resultant moment acting on the disk is $\sum M_{\overrightarrow{F}} = +F \times R$.
2. Justify that the angular momentum of the disk at an instant t is given by $\sigma = F\,R\,t$.
3. A convenient apparatus is used to trace the curve representing the evolution of the angular momentum σ in terms of time over $[0\,;3s]$.
   a) Justify that the shape of the curve is compatible with the shape of the curve shown in figure 2.
   b) Deduce the value of F.
4. Determine the number of turns covered by the disk during this duration.
5. Applying work-energy theorem on the disk between $t = 0$ & $t = 3s$. Determine again the number of turns.

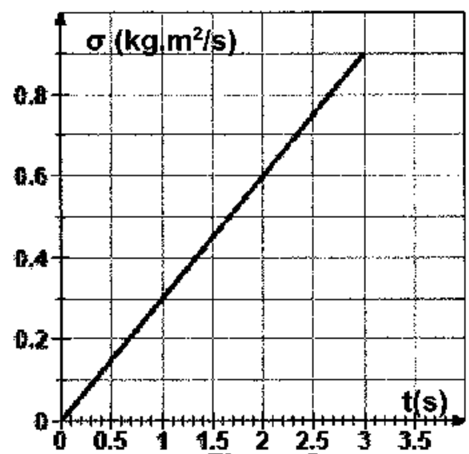
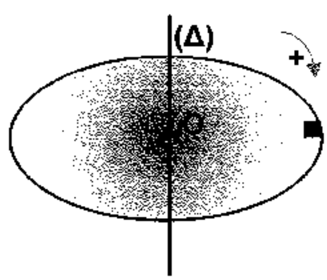


Figure 2

II- Turntable

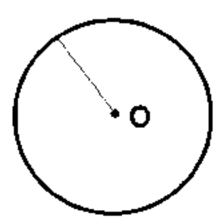
A turntable assimilated to a disk of mass $m = 10\,kg$ & radius $R = 40\,cm$ is rotating with a constant angular speed of $0.7\,\text{rad}/s$.



1. Calculate the angular momentum and the kinetic energy of the disk.
2. A point particle of mass $m_1 = 2kg$ is placed on the circumference of the disk.
   a) Show that the resultant moment acting on the system [disk, particle] is zero.
   b) Determine the angular velocity after placing the particle.
3. Compare the kinetic of the system [disk, particle] to that before placing the particle.

III- Stopping a Wheel

A wheel of mass $m = 600\,g$ and radius $R = 25\,cm$ assimilated to a hoop is rotating with an angular velocity of 90 revolutions per minute. At an instant taken as origin of time $t_0 = 0$, the wheel is subjected to a braking couple moment M that stops the disk within $1.4\,s$.



1. Calculate the angular momentum at $t_0 = 0$.
2. Applying the theorem of angular momentum, determine the value of M.
3. Draw the curve representing the variation of the angular momentum in terms of time.

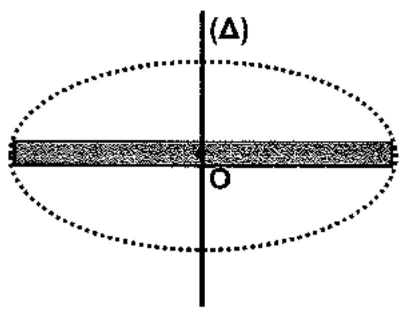
IV-
Ice Skater

An ice skater spins with an angular velocity of 2 revolutions per second when his hand are on his chest and his moment of inertia is taken $2.8\, kg.m^2$, is trying to stop by opening his arms and his moment of inertia becomes $3.2\, kg.m^2$.

1. Give a qualitative explanation to the increase in the moment of inertia.
2. Determine the angular velocity after opening his arms.
3. Compare the kinetic energy before and after opening his arms.
4. Interpret the change in the kinetic energy.

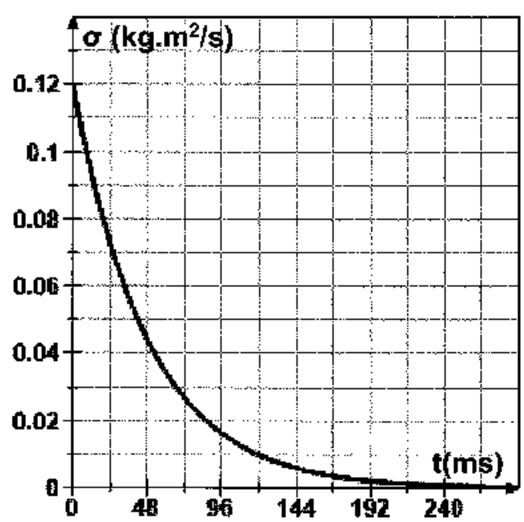
V-
Slowing Down a Rod

A rod of mass $m = 1kg$ and length $\ell = 24cm$, taken at rest is free to rotate in a horizontal plane around a vertical axis (Δ) passing through its midpoint as shown in the adjacent figure.



The moment of inertia of the rod around (Δ) is $I = \dfrac{m\,\ell^2}{12}$.

1. By a slight touch through a force exerted on one of the rod's extremities an angular velocity of $\theta'_0 = 25\,\text{rad}/s$ is communicated.
   Determine the average moment of the force exerted knowing that the contact lasts $\Delta t = 20ms$.
2. Specify the nature of motion of the rod in the absence of friction.
3. Let θ be the elongation of the rod which represents the angle that the rod travels at an instant t.
   In reality the angular velocity of the rod decreases, and moment of the force of friction acting on the rod is supposed proportional to the angular velocity θ' given by $M = -k\,\theta'$ where k is a constant.
   a) Show that the differential equation satisfied by the angular momentum σ, at any instant, is given by: $\dfrac{d\sigma}{dt} + \dfrac{k}{I}\sigma = 0$.
   b) Determine the expressions of A & τ, so that $\sigma = A\, e^{-\frac{t}{\tau}}$ is a solution of the previous differential equation.
   c) The adjacent curve represents the variation of the angular momentum in terms of time.
   Give the value of A and then determine the constant k.



Solution

I-

1. The forces acting on the disk are:

   its weight $\vec{w}$, whose moment $M_{\vec{w}/O} = 0$ (applied on axis);

   the reaction of axis $\vec{R}$, whose moment $M_{\vec{R}/O} = 0$ (on axis);

   the force $\vec{F}$, whose moment $M_{\vec{F}/O} = +F \times R$;

   The resultant moment is $\sum M_{\overrightarrow{F_{ext}}/O} = M_{\vec{R}/O} + M_{\vec{w}/O} + M_{\vec{F}/O}$;

   Then $\sum M_{\overrightarrow{F_{ext}}/O} = +F \times R$.

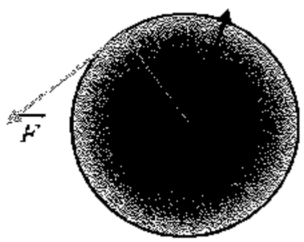


Figure 1

2. Theorem of angular momentum: $\sum M_{\overrightarrow{F_{ext}}/O} = \frac{d\sigma}{dt}$;

   so $\frac{d\sigma}{dt} = +F \times R$ (calculating the primitive);

   We get $\sigma = +F \times R\,t + \sigma_0$ but $\sigma_0 = I\,\theta'_0 = 0$ (from rest); then $\sigma = +F \times R\,t$.

3. a) The expression of the angular momentum σ is of the form $\sigma = a\,t$, then its graphical representation should be:

   ✖ a straight line (the variable t is of degree 1);
   ✖ increasing (slope $a = F\,R > 0$);
   ✖ passing through origin;
   These properties are satisfied graphically.

   b) Referring to the graph for $t = 2s$, we get $\sigma = 0.6\,kg.m^2/s$;

   But $F = \frac{\sigma}{R \times t} = \frac{0.6}{0.1 \times 2} = 3\,N$.

4. We have $\sigma = I\,\theta' = +F \times R\,t$, then $\theta' = \frac{d\theta}{dt} = \frac{+F \times R}{I}t$ & $I = \frac{1}{2}m\,R^2$ (disk);

   We get $\theta = \frac{F}{m\,R}t^2 + \theta_0$, the angle traveled $\Delta\theta = \frac{F}{m\,R}t^2$;

   The number of turns traveled $N = \frac{\Delta\theta}{2\pi} = \frac{F}{2\pi\,m\,R}t^2 = \frac{3 \times 3^2}{2\pi \times 5 \times 0.1} = 8.6$ revolutions.

5. Work energy theorem applied on the disk between $t = 0$ & $t = 3s$: $\Delta(KE) = \sum W_{F_{ext}}$;

   $KE|_{t=3s} - KE|_{t=0} = W_{\vec{w}} + W_{\vec{R}} + W_{\vec{F}}$; but $W_{\vec{w}} = W_{\vec{R}} = 0$ (no displacement) & $KE|_{t=0} = 0$;

   $\frac{1}{2}I\,\theta'^2 = M_{\vec{F}} \times \Delta\theta$; we get $\frac{1}{2} \times \frac{1}{2}m\,R^2\,\theta'^2 = F \times R \times \Delta\theta$;

   Then $\Delta\theta = \frac{m\,R^2\,\theta'^2}{4F\,R}$ where $\theta' = \frac{\sigma}{I} = \frac{0.9}{0.5 \times 5 \times 0.1^2} = 36\,rad/s$;

   Thus the number of turns is $N = \frac{\Delta\theta}{2\pi} = \frac{m\,R^2\,\theta'^2}{8\pi\,F\,R} \approx 8.6$ turns.

II-

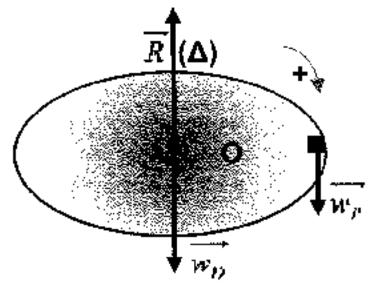
1. The angular momentum of the table: $\sigma_0 = I_{disc}\,\theta_0' = \frac{1}{2} m R^2 \times \theta_0'$;

   So, $\sigma_0 = \frac{1}{2} \times 10 \times 0.4^2 \times 0.7 = 0.56\, kg.m^2.rad/s$.

   The kinetic energy of the table: $KE_0 = \frac{1}{2} I\, \theta_0'^2$;

   So, $KE_0 = \frac{1}{4} m R^2 \times \theta_0'^2$;

   $KE_0 = \frac{1}{4} \times 10 \times 0.4^2 \times 0.7^2 = 0.196\, J$.



2. a) The forces acting on the disk:

   The weight of the disk whose moment is $M_{\vec{w_D}/O} = 0$ (applied on axis); the reaction of axis $\vec{R}$, whose moment $M_{\vec{R}/O} = 0$ (on axis) & the weight of the particle $\vec{w_P}$, whose moment $M_{\vec{w_P}/O} = 0$ (parallel to the axis).

   The resultant moment is $\sum M_{\vec{F_{ext}}/O} = M_{\vec{R}/O} + M_{\vec{w_D}/O} + M_{\vec{w_P}/O} = 0$.

   b) Theorem of angular momentum $\sum M_{\vec{F_{ext}}/O} = \frac{d\sigma}{dt} = 0$, the angular momentum is conserved.

   $\sigma_0 = \sigma$, we get $I_{disc}\,\theta_0' = (I_{disc} + I_{particle})\theta_1'$ but $I_{disc} = \frac{1}{2} m R^2$ & $I_{particle} = m_1 R^2$;

   Then $\theta_1' = \frac{I_{disc}}{I_{disc} + I_{particle}}\theta_0' = \frac{0.5 \times 10 \times 0.4^2}{0.5 \times 10 \times 0.4^2 + 2 \times 0.4^2} \times 0.7 = 0.5\, rad/s$.

3. $KE_{just\ before} = \frac{1}{2} I_{disc} \times \theta_0'^2 = 0.196\, J$;

   $KE_{just\ after} = \frac{1}{2}(I_{disc} + I_{particle}) \times \theta_1'^2 = \frac{1}{2}\left(\frac{1}{2} \times 10 \times 0.4^2 + 2 \times 0.4^2\right) \times 0.5^2 = 0.14\, J$.

   $KE_{just\ after} < KE_{just\ before}$, then the kinetic energy of the system decreases.

III-

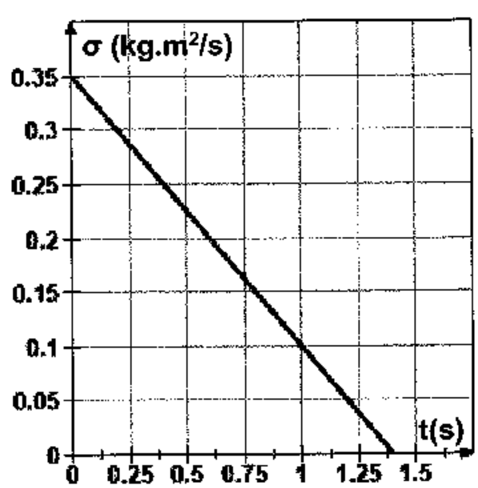
1. $\sigma_0 = I\,\theta_0' = m R^2 \times \theta_0' = 0.35\, kg.m^2/s$.
2. Theorem of angular momentum:

   $\sum M_{\vec{F_{ext}}/O} = \frac{d\sigma}{dt}$; $M_{\vec{R}/O} + M_{\vec{w}/O} + M = \frac{d\sigma}{dt}$;

   We get $M = \frac{d\sigma}{dt}$, then $\sigma = M\,t + \sigma_0$;

   The disk comes to rest, then $\sigma = I\,\theta' = 0$;

   Thus, $M = \frac{-\sigma_0}{t} = \frac{-0.35}{1.4} = -0.25\, N.m$.



3. The curve representing the variation of the angular momentum should be a decreasing straight line as shown in the adjacent curve.

IV-

1. The mass of the skater is unchanged, but we know that the moment of inertia of a point particle depends on the square of distance that separates this point from the axis of rotation $I_i = m_i\, d_i^2$.
   While opening his arms, the particles of his arms are moved away from the axis of rotation increasing its moment of inertia.

2. The skater is subjected to two forces: its weight $\vec{w}$ & the normal reaction $\vec{N}$ exerted by the support.
   Furthermore, the moment of these forces with respect to the spinning axis is zero, then the angular momentum of the skier is conserved.
   Conservation of angular momentum: $\sigma_1 = \sigma_2$, so $I_1\,\theta_1' = I_2\,\theta_2'$;
   Then $\theta_2' = \dfrac{I_1}{I_2}\theta_1' = \dfrac{2.8}{3.2} \times 2 = 1.75$ revolutions per second.
3. We have $KE_1 = \dfrac{1}{2} I_1 \times \theta_1'^2 = \dfrac{1}{2} \times 2.8 \times (4\pi)^2 = 224\,J$;
   & $KE_2 = \dfrac{1}{2} I_2 \times \theta_2'^2 = \dfrac{1}{2} \times 3.2 \times (1.75 \times 2\pi)^2 = 196\,J$;
4. The decrease in the kinetic energy is due to the mechanical work exerted by the skier in order to open his arms.

V-

1. The forces acting on the disk are: its weight $\vec{w}$, whose moment $\mathrm{M}_{\vec{w}/O} = 0$ (applied on axis); the reaction of axis $\vec{R}$, whose moment $\mathrm{M}_{\vec{R}/O} = 0$ (on axis); the force exerted of moment M.
   Theorem of angular momentum: $\sum \mathrm{M}_{\vec{F}_{ext}/O} = \dfrac{d\sigma}{dt}$, $\mathrm{M}_{\vec{R}/O} + \mathrm{M}_{\vec{w}/O} + \mathrm{M}_{\vec{F}/O} = \dfrac{d\sigma}{dt}$;
   $$\mathrm{M} = \frac{d\sigma}{dt} = \frac{\Delta\sigma}{\Delta t} = \frac{\sigma - \sigma_0}{\Delta t} = \frac{I\,\theta'}{\Delta t} = \frac{M\,\ell^2 \times \theta'}{12 \times \Delta t} = \frac{1 \times 0.24^2 \times 25}{12 \times 20 \times 10^{-3}} = 6\,N.m.$$
2. In the absence of a force of friction, the resultant moment acting on the rod is zero, the angular momentum is conserved.
   Thus, the motion of the rod is uniform circular with an angular velocity of $0.25\,\mathrm{rad}/s$.
3. a) Theorem of angular momentum: $\sum \mathrm{M}_{\vec{F}_{ext}/O} = \dfrac{d\sigma}{dt}$; $\mathrm{M}_{\vec{R}/O} + \mathrm{M}_{\vec{w}/O} + \mathrm{M}_{\vec{f}/O} = \dfrac{d\sigma}{dt}$;
   But $\mathrm{M}_{\vec{R}/O} = \mathrm{M}_{\vec{w}/O} = 0$ (applied on axis), $-k\,\theta' = \dfrac{d\sigma}{dt}$ & $\theta' = \dfrac{\sigma}{I}$; thus $\dfrac{d\sigma}{dt} + \dfrac{k}{I}\sigma = 0$.

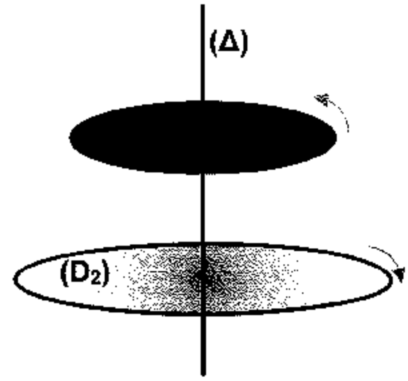
   b) We have $\sigma = A e^{-\frac{t}{\tau}}$, so $\dfrac{d\sigma}{dt} = -\dfrac{A}{\tau} e^{-\frac{t}{\tau}}$ (substitution in the differential equation);
   We get $A e^{-\frac{t}{\tau}}\left(-\dfrac{1}{\tau} + \dfrac{k}{I}\right) = 0$; but $A e^{-\frac{t}{\tau}} \neq 0$; then $\tau = \dfrac{I}{k}$.
   At $t = 0$, $A = \sigma_0 = I\,\theta_0'$.

   c) From graph: For $t = 0$, $A = \sigma_0 = I\,\theta_0' = 0.12\,kg.m^2/s$.
   for $t = 120\,ms$, we have $\sigma = 0.01\,kg.m^2/s$;
   Using solution, we get $k = -\dfrac{I}{t}\ell n\left(\dfrac{\sigma}{\sigma_0}\right) = -\dfrac{0.0192}{120 \times 10^{-3}}\ell n\left(\dfrac{0.01}{0.12}\right) \approx 0.4\,SI$.

Supplementary Problems

I- Clutching two Disks

Two disks (D_1) & (D_2) rotate in opposite directions around the same (Δ) as shown in the adjacent figure. (D_1) of mass $m_1 = 250\,g$, radius $R_1 = 8\,cm$ and rotates with an angular velocity of $2.5\,\text{rad/s}$; while (D_2) of mass $m_2 = 750\,g$ and radius $R_2 = 20\,cm$ and rotates with an angular velocity of $0.8\,\text{rad/s}$.



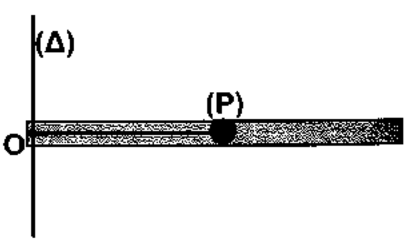
1. The two disks are clutched together at an instant taken as origin of time $t_0 = 0$ and rotate with the same angular velocity. Determine the angular velocity of the system $[(D_1), (D_2)]$.
2. Compare the kinetic energy of the system when they were independent and after being clutched together.
3. Applying the theorem of angular momentum, determine average moment of the force of interaction exerted by (D_1) on (D_2) through the interaction phase knowing that it lasts $5\,ms$ and assumed to occur without sliding.
4. Deduce the average moment of the force of interaction exerted by (D_2) on (D_1).

Answer Key

1. 0.73 rad/s. 2. Decrease in kinetic energy. 3. -0.051N.m.

II- Stopping a Rod in Motion

A rod (R) of mass m and length ℓ is free to rotate with an angular velocity 2revolutions per second in a horizontal plane around a vertical axis (Δ) passing through its extremity O as shown in the adjacent figure. A point particle (P) of mass $\frac{m}{2}$ is placed on the midpoint of the rod by the means of massless string.



The moment of inertia of the rod around (Δ) is $I = \frac{m\ell^2}{3}$.

At a certain instant, the string is cut and (P) reaches the extremity of the rod and stuck on it.

1. Show that the angular momentum of the system $[(P), (R)]$ is conserved.
2. Determine the angular velocity of the system when the particle reaches the extremity.
3. A constant force of magnitude F is applied perpendicular to the extremity of the rod in the horizontal plane to stop the rod within $3s$. **Take:** $m = 1.2\,kg$ & $\ell = 30\,cm$.
   a) Determine F.
   b) Calculate the work done by the force F to comes to rest.

Answer Key

2. 1.1rev per sec.

Unit I

Mechanics

Chapter 5

Angular Oscillations

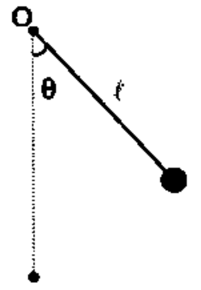
GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Angular Oscillations		1st(A) 2nd	1st & 2nd	-	1st	1st	1st	1st & 2nd	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Angular Oscillations	1st & 2nd	1st	1st & 2nd	2nd	2nd	-	2nd	2nd	1st

Essentials

We study many mechanical system whose motion is described as free undamped, and they are governed by a differential equation of 2^{nd} order of the form $y'' + w_0^2 y = 0$.

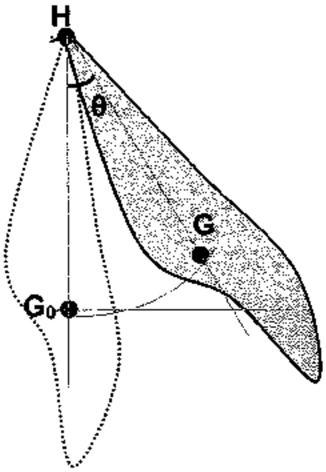
Consequently the motion is periodic of proper period $T_0 = \dfrac{2\pi}{w_0}$.

Simple pendulum (free undamped)



The length of the simple pendulum is ℓ, the differential equation that governs the motion is $\theta'' + \frac{g}{\ell}\theta = 0$ (for small angles $\theta \le 10^0$) then the expression of the proper period $T_0 = 2\pi\sqrt{\dfrac{\ell}{g}}$.

Compound pendulum (free undamped)

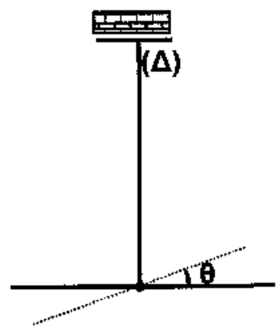


Let I be the moment of inertia of the pendulum around an axis (Δ) and a the distance between the axis of rotation and the center of gravity G, the differential equation that governs the motion is $\theta'' + \dfrac{m\,g\,a}{I}\theta = 0$ (for small angles $\theta \le 10^0$); then the expression of the proper period $T_0 = 2\pi\sqrt{\dfrac{I}{m\,g\,a}}$.

Note:

For a point particle $I = m\ell^2$ & $a = \ell$, then $T_0 = 2\pi\sqrt{\dfrac{I}{m\,g\,a}} = 2\pi\sqrt{\dfrac{\ell}{g}}$.

Torsion pendulum



Let I be the moment of inertia of the pendulum around an axis (Δ) and C be the torsion constant of the wire, the differential equation that governs the motion is $\theta'' + \dfrac{C}{I}\theta = 0$; then the expression of the proper period $T_0 = 2\pi\sqrt{\dfrac{I}{C}}$.

By Analogy

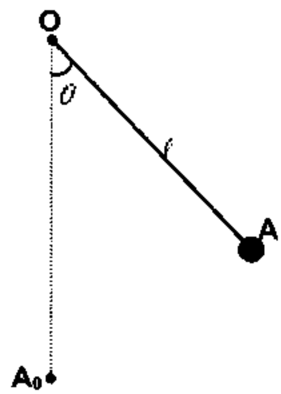
Linear Oscillators	Torsion Pendulum
x, x' & x''	θ, θ' & θ''
m & k	I & C
$x'' + \dfrac{k}{m}x = 0$	$\theta'' + \dfrac{C}{I}\theta = 0$

Applications

✘ In what follows, for small angles $\cos\theta = 1 - \frac{\theta^2}{2}$ and $\sin\theta \approx \theta$ where θ in rad.

Oscillations of a Simple Pendulum

A simple pendulum consists of a particle of mass $m = 200\ g$, fixed at the free end of a massless string OA of length $\ell = 30\ cm$.



The pendulum may oscillate in the vertical plane, around a horizontal axis (Δ) passing through the upper extremity O of the rod. At the instant $t_0 = 0$, the particle is shifted by an angle $\theta_0 = 0.16$ rad in the positive direction as indicated in figure 1, and then released from rest. At an instant t, the angular abscissa of the pendulum and the algebraic value of the angular velocity of the particle are θ and θ' respectively.

Take:

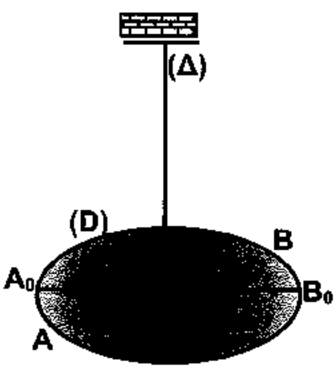
✘ The horizontal plane passing through A_0, the position of A at equilibrium, is taken as the reference level for the gravitational potential energy.

✘ $g = 10 m/s^2$ & neglect friction.

1. Show that the mechanical energy at $t_0 = 0$ is $ME_0 = 7.68\ mJ$.
2. Deduce the angular speed of the pendulum when it passes through the equilibrium position.
3. Determine, at the instant t, the expression of the mechanical energy of the system (pendulum, Earth) in terms of m, ℓ, g, θ and θ'.
4. Derive the differential equation in θ that describes the motion of the pendulum.
5. Deduce the expression of the proper period T_0 of these oscillations in terms of ℓ and g.
6. Determine, in terms of T_0, the expression of the period if:
   a) we double the length of the string.
   b) we double the mass suspended.
   c) the pendulum oscillates on moon.

II- Torsion Pendulum

Consider the adjacent system formed of a disk (D) connected to the free extremity O of a vertical torsion wire (Δ) of torsion constant C whose other extremity is fixed to a support as shown.



Let I be the moment of inertia of the disk about (Δ).

The horizontal plane passing through (D) is taken as the reference level for the gravitational potential energy.

AB is a diameter of the disk defined by the angle that makes with the equilibrium position by the elongation $\theta = \left(\overrightarrow{OB_0}, \overrightarrow{OB}\right)$ and the system performs free undamped oscillations.

1. Write the expression of the mechanical energy of the system (disk, wire, Earth) in terms of I, C, θ and the angular velocity θ'.
2. Derive the differential equation in θ that describes the motion of the pendulum.

3. The time equation that describes the elongation θ is given by $\theta = \theta_m \sin\left(\frac{2\pi}{T_0}t\right)$.

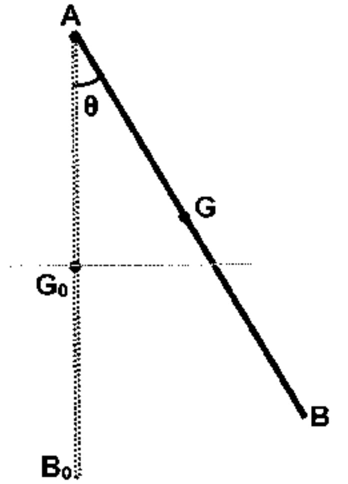
   Determine the expression of the proper period T_0 in terms of I & C.

4. Knowing that the torsion constant is inversely proportional to the length of the wire.
   By how much should modify the length in order to double the period of the oscillation?

III-

Moment of Inertia of a Rod

Consider a homogeneous and rigid rod AB of negligible cross-section, of length $\ell = 24\,cm$ and of mass $m = 200\,g$.



This rod may rotate about a horizontal axis (Δ) perpendicular to its extremity A. Let G be the midpoint of the rod. The vertical position AG_0 of the rod is considered as an origin of angular abscissa θ. Neglect all friction.

Let I be the moment of inertia of the rod about (Δ).

At the instant t, the angular abscissa of G is $\theta = \left(\overrightarrow{OG_0}, \overrightarrow{OG}\right)$ and the algebraic value of the angular velocity is $\theta' = \frac{d\theta}{dt}$.

The horizontal plane passing through G_0 is taken as a gravitational potential energy reference.

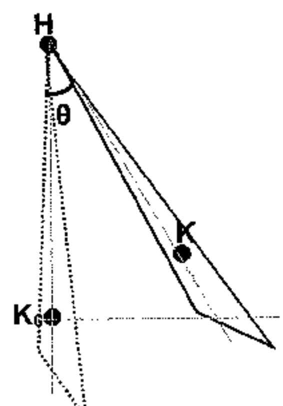
Take: $g = 10\,m/s^2$; $\pi^2 = 10$.

1. Determine the expression of the mechanical energy in terms of I, g, ℓ, θ and θ'.
2. Derive the differential equation in θ that describes the motion of the pendulum.
3. Deduce the expression of the proper period of the oscillations.
4. The duration of the 10 oscillations is $8s$. Determine I.
5. Knowing that $I = k\,m\,\ell^2$ where k is rational number. Deduce the value of k.

IV-

Moment of Inertia of a Pendulum

A pendulum (P) is formed of a rigid object of mass $m = 200\,g$ whose center of gravity is at K, rotates in a vertical plane about a horizontal axis (Δ) perpendicular to its extremity H, its moment of inertia is I. The vertical position HK_0 is considered as an origin of angular abscissa θ, and let $HK = HK_0 = \ell$. Neglect all friction.



At the instant t, the angular abscissa of K is $\theta = \left(\overrightarrow{OK_0}, \overrightarrow{OK}\right)$ and the algebraic value of the angular velocity is $\theta' = \frac{d\theta}{dt}$.

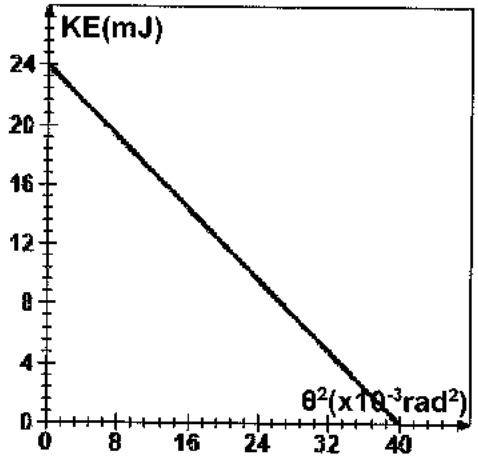
The horizontal plane passing through K_0 is taken as a gravitational potential energy reference.

Take: $g = 10\,m/s^2$, $\pi^2 = 10$ &

The expression of the period of oscillations for small angles is $T = 2\pi\sqrt{\frac{I}{m\,g\,\ell}}$.

The pendulum (P) is shifted by an angle θ_m and then released from rest.

1. Applying the principle of conservation of the mechanical energy, show that the expression of the kinetic energy of the pendulum is given $KE = \frac{1}{2} m g \ell \left(\theta_m^2 - \theta^2\right)$.



2. The curve shown in the adjacent figure represents the variations of the kinetic energy of (P) in terms of the square of the angular elongation.
   a) Justify that the curve is compatible with the expression previously derived.
   b) Referring the curve, determine:
      i- the amplitude θ_m of the oscillations.
      ii- the mechanical energy of (P).
      iii-the value of ℓ.
3. Knowing that the period of the oscillations of (P) is $0.45\,s$. Determine I.
4. Deduce the angular speed when it passes through the equilibrium position.

V- Time Equation of Motion

The time equation that describes the motion of a rod (MN) connected to a vertical torsion wire of constant C (figure 1) where (M_0N_0) corresponds to the equilibrium position, is $\theta = \theta_m \cos(w_0 t + \varphi)$

1. Show that $\theta'^2 = w_0^2\left(\theta_m^2 - \theta^2\right)$ where θ' is the angular velocity at an instant t.
2. Referring to the graph shown in figure 2, give the values of:
   a) the amplitude of the oscillations.
   b) the period of the oscillations.
3. Deduce the angular speed at $t_0 = 0$.
4. Show that the expression of the angular elongation, in SI units, is given by: $\theta = 0.1\cos\left(2.5\pi t + \frac{2\pi}{3}\right)$.

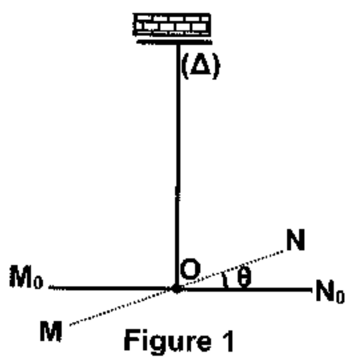


Figure 1

5. Deduce the angular velocity at $t_0 = 0$.
6. The moment of inertia of the rod is given by $I = \frac{m\ell^2}{12}$ where $m = 300\,g$ & $\ell = 25\,cm$ and that the horizontal plane passing through the rod as a reference of gravitational potential energy.
   a) Determine the mechanical energy of the system (rod, Earth).
   b) Deduce the torsion constant of the wire.
   c) Determine again the value of C, knowing that $T_0 = 2\pi\sqrt{\frac{I}{C}}$.

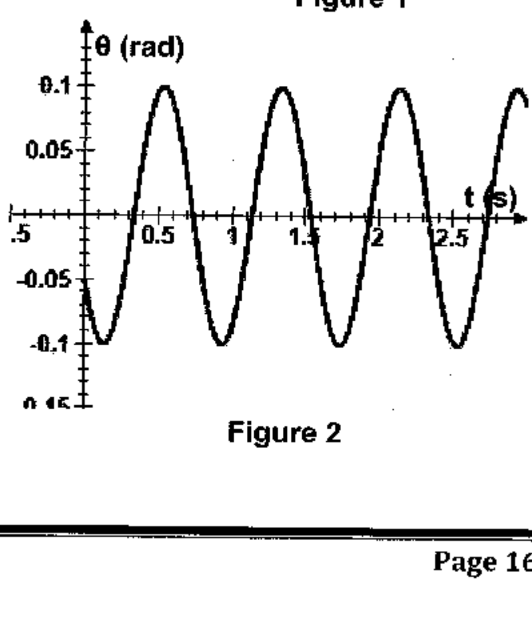


Figure 2

Solutions

I-

1. The mechanical energy $ME_0 = KE_0 + GPE_0$;

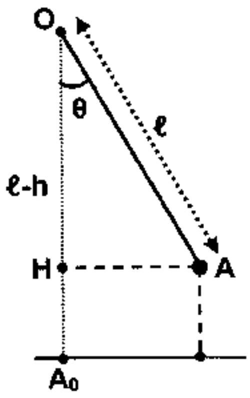
   But $KE_0 = 0$ (released from rest);

   & $GPE_0 = m\,g\,h_0 = m\,g\,\ell(1-\cos\theta_0)$;

   Then $ME_0 = m\,g\,\ell(1-\cos\theta_0)$;

   For small angles $1-\cos\theta_0 = \dfrac{\theta_0^2}{2}$;

   Thus, $ME_0 = \dfrac{1}{2} m\,g\,\ell\,\theta_0^2 = \dfrac{1}{2}\times 0.2\times 10\times 0.3\times 0.16^2 = 7.68\times 10^{-3}\,J$.



2. In the absence of friction, the mechanical energy is conserved:

   $ME_{A_0} = KE_{A_0} + GPE_{A_0} = 7.68\times 10^{-3}\,J$; $GPE_{A_0} = 0$ (on reference);

   So $\dfrac{1}{2} I\,\theta'^2 = 7.68\times 10^{-3}\,J$; then $\theta' = \sqrt{\dfrac{2\times 7.68\times 10^{-3}}{0.2\times 0.3^2}} \approx 0.92\ \mathrm{rad/s}$.

3. At an instant t, $ME = KE + GPE = \dfrac{1}{2} I\,\theta'^2 + m\,g\,\ell(1-\cos\theta)$.

   But $v = \ell\,\theta'$ & for small angles $1-\cos\theta = \dfrac{\theta^2}{2}$; then $ME = \dfrac{1}{2} m\,\ell^2\theta'^2 + \dfrac{1}{2} m\,g\,\ell\,\theta^2$.

4. In the absence of the forces of friction, the mechanical energy is conserved, $\dfrac{d(ME)}{dt} = 0$;

   We get $m\,\ell^2\theta'\times\theta'' + m\,g\,\ell\,\theta\times\theta' = 0$, so $m\,\ell^2\theta'\left(\theta'' + \dfrac{g}{\ell}\theta\right) = 0$;

   But $m\,\ell^2\theta' \neq 0$ (the system is in motion), thus $\theta'' + \dfrac{g}{\ell}\theta = 0$.

5. The differential equation that governs the motion of the pendulum is of 2^{nd} order of the form $\theta'' + \omega_0^2\,\theta = 0$ where $\omega_0^2 = \dfrac{g}{\ell}$;

   Then the motion of the pendulum is periodic of proper period $T_0 = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{\ell}{g}}$.

6. a) If the length is doubled, $\ell' = 2\,\ell$; then $T_0' = 2\pi\sqrt{\dfrac{\ell'}{g}} = \sqrt{2}\times 2\pi\sqrt{\dfrac{\ell}{g}} = \sqrt{2}\,T_0 > T_0$.

   b) If the mass is doubled, the period will remains unchanged because it is independent of mass.

   c) If the pendulum oscillates on moon, we know that $g_{\text{moon}} < g$,

   Then $T_{\text{moon}} = 2\pi\sqrt{\dfrac{\ell}{g_{\text{moon}}}} > T_0$, the period increases.

II-

1. The mechanical energy is given by: $ME = KE + GPE + PE_e$;

But $GPE=0$ (on reference); then $ME=\frac{1}{2}I\,\theta'^2+\frac{1}{2}C\,\theta^2$.

2. The system performs free undamped oscillations, then its mechanical energy is conserved, $\frac{d(ME)}{dt}=0$; we get $I\,\theta'\theta''+C\,\theta\,\theta'=0$, $I\,\theta'\left(\theta''+\frac{C}{I}\theta\right)=0$;

But $I\,\theta'\neq 0$ (system in motion), then $\theta''+\frac{C}{I}\theta=0$.

3. We have $\theta=\theta_m\sin\left(\frac{2\pi}{T_0}t\right)$, so $\theta''=-\theta_m\left(\frac{2\pi}{T_0}\right)^2\sin\left(\frac{2\pi}{T_0}t\right)$;

Substitution in the differential equation, we get: $-\theta_m\left(\frac{2\pi}{T_0}\right)^2\sin\left(\frac{2\pi}{T_0}t\right)+\frac{C}{I}\theta_m\sin\left(\frac{2\pi}{T_0}t\right)=0$;

So, $\theta_m\sin\left(\frac{2\pi}{T_0}t\right)\times\left(-\left(\frac{2\pi}{T_0}\right)^2+\frac{C}{I}\right)=0$, but $\theta_m\sin\left(\frac{2\pi}{T_0}t\right)\neq 0$;

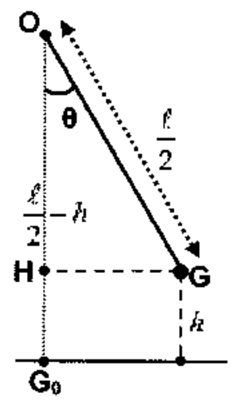
Then $-\left(\frac{2\pi}{T_0}\right)^2+\frac{C}{I}=0$, thus $T_0=2\pi\sqrt{\frac{I}{C}}$.

4. The period is doubled, so $T_0'=2\pi\sqrt{\frac{I}{C'}}=2T_0=2\times 2\pi\sqrt{\frac{I}{C}}$, we get $C'=\frac{C}{4}$;

But the torsion constant is inversely proportional to the length ℓ of the wire;

Then $C'=\frac{k}{\ell'}=\frac{1}{4}\times\frac{k}{\ell}$ where k is a constant that depends on the wire, thus $\ell'=4\,\ell$.

III-



1. The mechanical energy is given by: $ME=KE+GPE$;
   But $GPE=m\,g\,h$;

   where $\cos\theta=\frac{\frac{\ell}{2}-h}{\frac{\ell}{2}}$, so $h=\frac{\ell}{2}(1-\cos\theta)$ (see figure);

   Thus, $ME=\frac{1}{2}I\,\theta'^2+m\,g\,\frac{\ell}{2}(1-\cos\theta)$.

2. In the absence of friction, the mechanical energy is conserved, so $\frac{d(ME)}{dt}=0$, we get $I\,\theta'\theta''+m\,g\,\frac{\ell}{2}\theta'\sin\theta=0$;

   Then $I\,\theta'\left(\theta''+\frac{m\,g\,\ell}{2I}\sin\theta\right)=0$, but $I\,\theta'\neq 0$ (pendulum in motion) & $\sin\theta=\theta$ (small angles);

   Thus, $\theta''+\frac{m\,g\,\ell}{2I}\theta=0$.

3. The differential equation that governs the motion of the pendulum is of 2nd order of the form $\theta''+\omega_0^2\,\theta=0$ where $\omega_0^2=\frac{m\,g\,\ell}{2I}$;

Then the motion of the pendulum is periodic of proper period $T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{2I}{mg\ell}}$.

4. We have $10\,T_0 = 8\,s$, so $T_0 = 0.8\,s$;

But $T_0 = 2\pi\sqrt{\frac{2I}{mg\ell}}$, so $I = \frac{T_0^2\ mg\ell}{8\pi^2} = \frac{0.8^2 \times 0.2 \times 10 \times 0.24}{8 \times 10} = 3.84 \times 10^{-3}\ kg.m^2$.

5. If $I = k\,m\,\ell^2$, then $k = \frac{I}{m\,\ell^2} = \frac{3.84 \times 10^{-3}\ kg.m^2}{0.2\,kg \times (0.24\,m)^2} = \frac{1}{3}$;

Thus, the expression of the moment of inertia is $I = \frac{1}{3} m\,\ell^2$.

IV-

1. The mechanical energy is conserved between the positions of elongations θ & θ_m:

$ME|_{\theta_m} = ME|_{\theta}$, we get $KE|_{\theta_m} + GPE|_{\theta_m} = KE|_{\theta} + GPE|_{\theta}$;

But $KE|_{\theta_m} = 0$ (released from rest), & $GPE|_{\theta_m} = m\,g\,\ell(1-\cos\theta_m)$;

Then, $KE = GPE|_{\theta_m} - GPE|_{\theta} = m\,g\,\ell(1-\cos\theta_m) - m\,g\,\ell(1-\cos\theta) = m\,g\,\ell(\cos\theta - \cos\theta_m)$;

For small angles, $\cos\theta_m = 1 - \frac{\theta_m^2}{2}$, $KE = m\,g\,\ell\left(1 - \frac{\theta^2}{2} - 1 + \frac{\theta_m^2}{2}\right) = \frac{1}{2} m\,g\,\ell\left(\theta_m^2 - \theta^2\right)$.

2. a) The expression of the kinetic energy can be written $KE = B + A\theta^2$ where $B = \frac{1}{2} m\,g\,\ell\,\theta_m^2$ & $A = -\frac{1}{2} m\,g\,\ell$. Then its graphical representation in terms of θ^2 should be a decreasing straight line not passing through origin which are satisfied graphically.

b) Graphical study.

i- We have for $\theta = \theta_m$, $KE = 0$; then $\theta_m^2 = 40 \times 10^{-3}\ \text{rad}^2$; thus $\theta_m = 0.2\,\text{rad}$.

ii- For $\theta = 0$, $ME = KE = B = 24\,mJ$.

iii- For $\theta = 0$, $KE = B = 24 \times 10^{-3} = \frac{1}{2} m\,g\,\ell\,\theta_m^2$; thus $\ell = \frac{2 \times 24 \times 10^{-3}}{0.2 \times 10 \times 40 \times 10^{-3}} = 0.6\,m = 60\,cm$.

3. We have $T = 2\pi\sqrt{\frac{I}{mg\ell}}$, then $I = \frac{T^2}{4\pi^2} \times m\,g\,\ell = \frac{0.45^2}{4 \times 10} \times 0.2 \times 10 \times 0.6 = 6.1 \times 10^{-3}\ kg.m^2$.

4. When the pendulum passes through the equilibrium position, $KE = 24 \times 10^{-3}\,J = \frac{1}{2} I\,\theta_{eq}'^2$;

Then $\theta'_{eq} = \sqrt{\frac{24 \times 10^{-3} \times 2}{6.1 \times 10^{-3}}} = 2.8\,\text{rad}/s$.

V-

1. We have $\theta = \theta_m \cos(w_0 t + \varphi)$, so $\theta' = -\theta_m w_0 \sin(w_0 t + \varphi)$;

But $\cos^2(w_0 t + \varphi) + \sin^2(w_0 t + \varphi) = 1$, so $\left(\frac{\theta'}{\theta_m w_0}\right)^2 + \left(\frac{\theta}{\theta_m}\right)^2 = 1$; then $\theta'^2 = w_0^2\left(\theta_m^2 - \theta^2\right)$.

2. a) The amplitude $\theta_m = 0.1\,\text{rad}$.

b) The period $T_0 = 0.8\,s$.

3. We have $\theta_0'^2 = w_0^2(\theta_m^2 - \theta_0^2) = \left(\frac{2\pi}{T_0}\right)^2(\theta_m^2 - \theta_0^2) = \left(\frac{2\pi}{0.8}\right)^2(0.1^2 - 0.05^2)$, then $|\theta_0'| = 0.68\,\text{rad}/s$.

4. We have $\theta = \theta_m \cos(w_0 t + \varphi) = 0.1\cos\left(\frac{2\pi}{0.8}t + \varphi\right) = 0.1\cos(2.5\pi\, t + \varphi)$.

Referring to the initial conditions, at $t_0 = 0$, $\theta_0 = -0.05\,\text{rad}$; we get $0.1\cos(\varphi) = -0.05$;

So $\cos(\varphi) = -\frac{1}{2}$, then $\varphi = \frac{2\pi}{3}(\text{rad})$ or $\varphi = -\frac{2\pi}{3}(\text{rad})$;

But $\theta_0' = -0.1 \times 2.5\pi \sin(\varphi) < 0$ (the velocity represents the slope of the tangent at $t = 0$, which decreasing by referring to the curve);

We get $\varphi = +\frac{2\pi}{3}(\text{rad})$, thus $\theta = 0.1\cos\left(2.5\pi\, t + \frac{2\pi}{3}\right)$ (t in s & θ in rad).

5. The angular velocity $\theta' = -0.1 \times 2.5\pi \sin\left(2.5\pi\, t + \frac{2\pi}{3}\right)$;

At $t_0 = 0$, $\theta_0' = -0.1 \times 2.5\pi \sin\left(\frac{2\pi}{3}\right) = -0.68\,\text{rad}/s$.

6. a) The amplitude of the oscillations is constant, then the mechanical energy is conserved:

$ME = ME|_{\theta=0} = KE|_{\theta=0} + PE_e|_{\theta=0} = \frac{1}{2} I\,\theta_{\max}'^2$, but $PE_e|_{\theta=0} = 0$;

where $\theta_{\max}' = |-0.1 \times 2.5\pi| = 0.25\pi\,(\text{rad}/s)$;

Then $ME = \frac{1}{2} I\,\theta_{\max}'^2 = \frac{1}{2}\frac{m\ell^2}{12}\theta_{\max}'^2 = \frac{0.3 \times 0.25^2}{24}(0.25\pi)^2 = 4.82 \times 10^{-4}\,J$.

b) Conservation of mechanical energy $ME = ME|_{\theta_m} = KE|_{\theta_m} + PE_e|_{\theta_m} = +4.82 \times 10^{-4}\,J$;

But $KE|_{\theta_m} = 0$ & $\frac{1}{2} C\,\theta_m^2 = +4.82 \times 10^{-4}\,J$;

Then, $C = \frac{2 \times 4.82 \times 10^{-4}}{0.1^2} = 0.0964\,SI$.

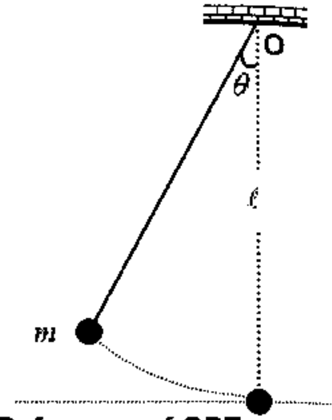
c) We have $T_0 = 2\pi\sqrt{\frac{I}{C}}$, then $C = I \times \left(\frac{2\pi}{T_0}\right)^2 = \frac{m\ell^2}{12} \times \left(\frac{2\pi}{T_0}\right)^2$;

Thus, $C = \frac{0.3 \times 0.25^2}{12}\left(\frac{2\pi}{0.8}\right)^2 = 0.096\,SI$.

Problems

I-Bac
Simple Pendulum and Period

We call a simple pendulum the system that is constituted from a heavy object of mass m, suspended by the means of an inextensible massless string of length ℓ and whose other end is fixed to a support as shown in the adjacent figure.



Part A
Historical study

In Galileo's (1564-1642) book «Discourse and demonstrations», we read «*To obtain a simple pendulum having the oscillation's duration is double to that of another one, it is convenient that the length of the first is quadrupled (multiplied by 4)*».

It is known that an oscillation corresponds to a to-and-fro motion around the equilibrium position.

1. Give the name of the physical quantity that is designated by the expression «duration of an oscillation».
2. Show that one of the following statements satisfies the preceding introduction.

Statement	1	2	3
T is proportional to	$\frac{1}{\ell}$	$\sqrt{\ell}$	ℓ^2

Part B
Experimental study

We intend to study, experimentally, the influence of different parameters on the duration of simple pendulum's oscillations. To do that, we use an inextensible, massless, string of length ℓ. The objects of mass m are suspended successively to the free end of the string.

This pendulum is shifted from the equilibrium position by a small angle $\theta_0 \left(\prec 10^0\right)$ then released without initial velocity. The system performs free un-damped oscillations whose duration or period noted as T. In each experiment, we measure the time Δt for 20 oscillations using a stopwatch.

1. Influence of the mass m of the solid

In a first experiment, four solids are connected separately from the free end of the string. The values of Δt are shown in the adjacent table.

$m(g)$	60	125	160	200
$\Delta t(s)$	19.9	19.8	19.9	19.7

Draw a conclusion concerning the influence of the mass m on the proper period of the pendulum.

2. Influence of the string's length ℓ

In a second experiment, the solid of mass $m = 125 g$ is connected successively from the free end of each of five strings. The new values of Δt are shown in the adjacent table.

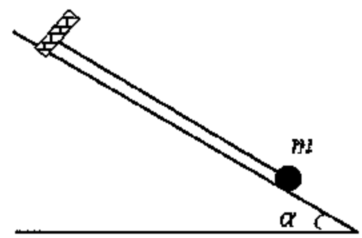
$\ell(cm)$	12.3	24.4	28.6	32.4	38.5
$\Delta t(s)$	14.1	19.8	21.4	22.8	24.9

a) Determine, using the table, the values of the product $T^2 \times \ell^{-1}$.

b) The literal expression of T and ℓ can be written in the form: $T = k\,\ell^a$. Determine then the values of a & k.

3. Influence of the gravitational field g

We can't modify the gravitational field g at the considered place. However, using the setup represented in the adjacent figure, which is equivalent to a vertical pendulum placed in gravitational field whose intensity is $g' = g \sin \alpha$ where $g = 9.8 m/s^2$.



Setup description

On an inclined making an angle α with the horizontal, a small puck of mass $m = 125\,g$ is suspended to the free end of a massless string of length $\ell = 24.4 cm$, whose other end is fixed.

For different values of α (we modify the value of g'), we measure Δt for 20 oscillations for small angles. The results obtained are shown in the following table:

α^{0}	90	70	50	30	20	10
$\Delta t(s)$	19.9	20.6	22.6	28.2	33.9	48
$\frac{1}{\sqrt{g'}}(m^{-1/2}\,s)$	0.32	0.33	0.36	0.45	0.55	0.77

a) Plot, on a graph paper, the curve representing the variations of T as a function of $\frac{1}{\sqrt{g'}}$.

Scales: On abscissa $1cm \equiv 0.1 m^{-1/2}s$ and on ordinate $1cm \equiv 0.2\,s$.

b) Give, in literal form, the equation of this curve.

4. Conclusion

The period can be written in the form: $T = C\sqrt{\frac{\ell}{g}}$ where C is a constant.

a) Show that C is a dimensionless physical quantity.

b) Determine the value of C using the result of **2.b)**.

Part C

Theoretical study

The simple pendulum (P) whose mass is m and length ℓ, is shifted by an angle θ_m with respect to the equilibrium position, and released then without initial velocity. At an instant t, (P) makes an angle θ with the equilibrium position, and we designate by θ' its angular velocity.

For small angles (θ in radian): $\cos\theta = 1 - \frac{\theta^2}{2}$, ($\theta$ in rad).

The lower position of the simple pendulum (P) is taken as reference of gravitational potential energy for the system [(P); Earth]. Neglect all frictional forces. **Take:** $g = 9.8\ m/s^2$.

1. Give, at the instant t, the expression of the mechanical energy ME of the system [(P); Earth] in terms of m, g, ℓ, θ & θ'.
2. a) Derive the differential equation that describes the motion of (P).

   b) Knowing that $\theta = \theta_m \cos\left(\frac{2\pi}{T_0}t + \varphi\right)$ is a solution of the previous differential equation, determine the expression of the proper period T_0 as a function of ℓ and g.

   c) Compare this result to that studied in **Part B** and show that they are confirmed.

II-

Torsion Pendulum

Consider the adjacent system formed of a rod (R) connected to the free extremity O of a vertical torsion wire (Δ) of torsion constant C whose other extremity is fixed to a support as shown in figure 1.

The moment of inertia of the rod about (Δ) is I.

The horizontal plane passing through (R) is taken as the reference level for the gravitational potential energy.

The position of the rod MN is defined by the angle that makes with the equilibrium position by the elongation $\theta = \left(\overrightarrow{ON_0}, \overrightarrow{ON}\right)$.

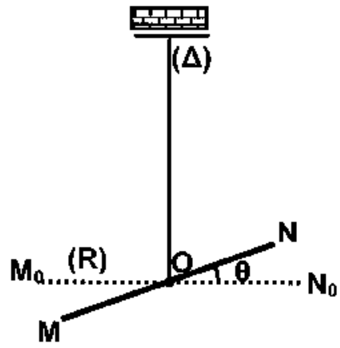


Figure 1

The rod is twisted in the horizontal plane by an angle θ_0 in the negative direction and then released from rest.

Take: $\pi^2 = 10$.

Part A

Theoretical study

1. Write the expression of the mechanical energy of the system (rod, wire, Earth) in terms of I, C, θ and the angular velocity θ'.
2. Derive the differential equation in θ that describes the motion of the pendulum.
3. Deduce the expression of the proper period T_0 in terms of I & C.
4. Show that the differential equation can be written in the form $\theta'' + \dfrac{4\pi^2}{T_0^2}\theta = 0$.

Part B

Graphical study

A convenient apparatus is used to represent the variations of the angular acceleration θ'' in terms of the elongation θ.

1. Justify that the curve is compatible with the expression previously obtained.
2. Referring to the graph:
   a) give the value of θ_0 and the angular acceleration θ'' for θ_0.
   b) determine the period of the oscillations.
3. If the torsion constant $C = 0.125$ SI.
   Deduce the value of I.
4. Calculate the mechanical energy of the pendulum.
5. Deduce the angular speed when it passes through the equilibrium position.

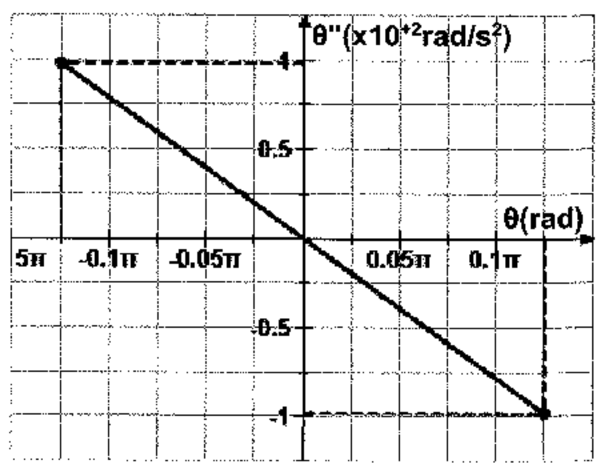
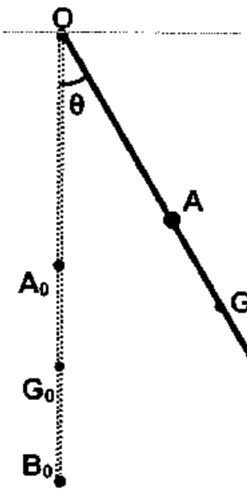


Figure 2

III- Compound Pendulum

A compound pendulum (adjacent figure) is formed of thin rod of length ℓ and negligible mass, two identical particles each of mass m, fixed on its midpoint and its free end as shown in the adjacent figure. The pendulum rotates around a horizontal axis (Δ) passing through the extremity O in a vertical plane.

The horizontal plane passing through O is taken as a reference of the gravitational potential energy.



Part A

Oscillations for small angles

At the instant t, the angular abscissa of the center of gravity G of the system is defined by the angular abscissa $\theta = \left(\overrightarrow{OG_0}, \overrightarrow{OG}\right)$ where G_0 is the position of the center of gravity in the equilibrium position and the algebraic value of the angular velocity is $\theta' = \frac{d\theta}{dt}$.

1. Show that:

   a) the position of its center of gravity is $a = OG = \frac{3}{4}\ell$.

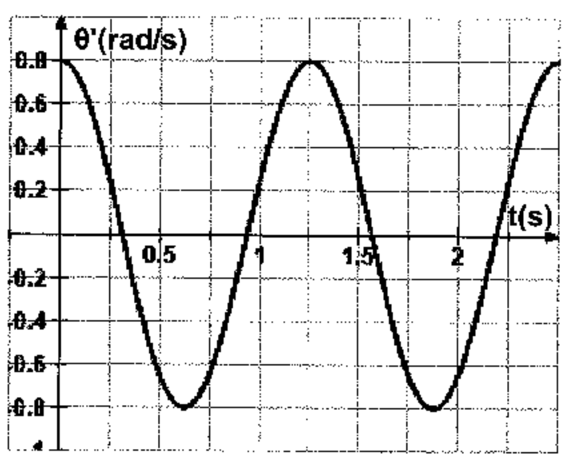
   b) its moment of inertia is $I = \frac{5}{4} m\,\ell^2$.

2. Determine the expression of the mechanical energy of the system (rod, particles, Earth) in terms of ℓ, m, g, θ and the angular velocity θ'.
3. Derive the differential equation in θ that describes the motion of the pendulum for oscillations of small angles.
4. Show that the expression of the period is $T_0 = 2\pi\sqrt{\frac{5\ell}{6g}}$.

Part B

Time equation of motion

A convenient apparatus is used to trace the curve representing the variations of the angular velocity of the pendulum in terms of time.



1. Give the value of:

   a) the maximum value of the angular velocity.

   b) the proper period of motion.

2. Deduce that the value of ℓ is $28\,cm$.
3. The expression of the angular velocity is given by $\theta' = A\cos(w_0 t + \varphi)$.

   Determine the expression of θ' in terms of time.

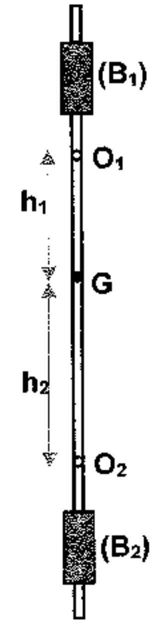
4. Show that amplitude of these oscillations is 0.16rad.
5. In reality the forces of friction are not negligible and the amplitude becomes 0.15 rad after the first oscillation. Knowing that $m = 250\ g$.
   a) Calculate the variation in the mechanical energy of the pendulum at the end of the 1st oscillation.
   b) Determine the average power that should be provided to this pendulum in order to drive these oscillations.

Huyghens principle (theorem of parallel axis) – Problems IV & V

If I_0 be the moment of inertia of a system with respect to an axis (Δ_0) passing through its center of mass. Then its moment of inertia about another axis (Δ) parallel to (Δ_0) at a distance x is given by $I = I_0 + m x^2$.

IV-
Kater's Pendulum

Kater's pendulum, is a reversible pendulum it has the same period for oscillations about two different axis, invented by the British captain Henry Kater in 1871, to be used as a gravimeter instrument in order to measure the local gravitational acceleration till 1950.



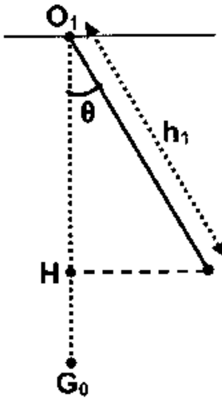
This pendulum (P) of mass m is formed of a thin homogenous rod, on which two similar shaped blocks are placed, a metallic block (B_1) and a wooden block (B_2). This pendulum may oscillates in a vertical plane around two horizontal axis passing through O_1 & O_2 as shown in the adjacent figure. Let I_1 & I_2 be the moment of inertia of (P) when it rotates about O_1 & O_2 respectively.

At the instant t, the angular abscissa of the center of gravity G is $\theta = \left(\overrightarrow{O_1G_0}, \overrightarrow{O_1G}\right)$ where G_0 its equilibrium position and the algebraic value of the angular velocity is $\theta' = \dfrac{d\theta}{dt}$.

The horizontal plane passing through O_1 is taken as a gravitational potential energy reference.

For small angles $\cos\theta = 1 - \dfrac{\theta^2}{2}$ and $\sin\theta \approx \theta$ where θ in rad.

Part A
Oscillations about O_1 & O_2



1. Determine the expression of the mechanical energy of the system (rod, wire, Earth) in terms of I_1, m, g, h_1, θ and the angular velocity θ'.
2. Derive the differential equation in θ that describes the motion of the pendulum for oscillations of small angles.
3. Show that the expression of the period is $T_1 = 2\pi\sqrt{\dfrac{I_1}{m g h_1}}$.
4. The pendulum now rotates about O_2, deduce the expression of the period T_2 in terms I_2, g, h_2 and m.

Part B
Measurement of g

According to theorem of parallel axis $I_1 = I_0 + m h_1^2$ & $I_2 = I_0 + m h_2^2$ where I_0 is the moment of inertia of the pendulum about G.

1. Show that $h_1 T_1^2 - h_2 T_2^2 = \frac{4\pi^2}{g}\left(h_1^2 - h_2^2\right)$.

   Let $h_1 = 48cm$, $h_2 = 52\,cm$, $\Delta t_1 = 100\,T_1 = 106\,s$ & $\Delta t_2 = 100\,T_2 = 116\,s$.

2. Deduce the value of g.
3. Knowing that the pendulum is reversible, determine the relation among I_0, m, h_1 & h_2.

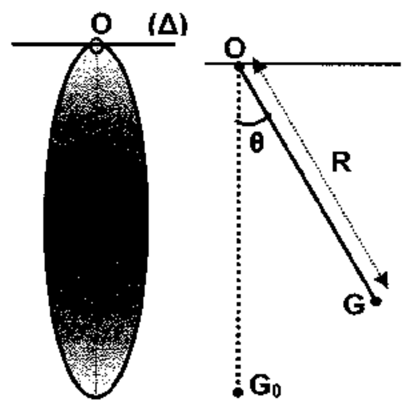
   If $m = 2\,kg$, deduce the value of I_0.

V-
Period and Position of the Axis

The aim of this exercise is to study the effect of the position of the axis of rotation on the variations of the period of a compound pendulum.

In what follows, the horizontal plane passing through the axis (Δ) is taken as a gravitational potential energy reference.

Part A
Oscillations of a disk

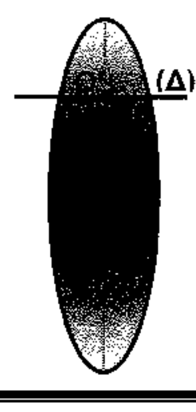


Consider a disk of center G, mass m and radius R, free to oscillate around an axis (Δ) perpendicular to its plane and passing through its circumference at O.

Take: $g = 10\,m/s^2$, $\pi^2 = 10$ & neglect dissipative forces.

1. Verify that the moment of inertia of the disk around (Δ) is $I = \frac{3}{2} mR^2$ (use theorem of parallel axis).
2. Determine the expression of the mechanical energy of the system [disk, Earth] in terms of m, g, R, θ and θ'.
3. Derive the differential equation in θ that describes the motion of the pendulum.
4. Deduce that expression of the proper period of the oscillations is $T_0 = 2\pi\sqrt{\frac{3R}{2g}}$.

Part B
Displacement of axis



The axis (Δ) is still perpendicular to the plane of the disk but passes through a point I at a distance x from the center G.

1. Show that the expression of the mechanical energy at an instant t is given by

   $$ME = \frac{1}{2}\left(\frac{1}{2} m R^2 + m x^2\right)\theta'^2 - m g x \cos(\theta).$$

2. Derive the differential equation in θ that describes the motion of the pendulum.

3. Deduce that expression of the proper period of the oscillations is $T_0' = 2\pi\sqrt{\frac{R^2+2x^2}{2gx}}$.

4. Compare T_0' to T_0 for $x = \frac{R}{2}$. Interpret the result obtained.

5. Explain how the period T_0 is obtained for four different positions of the axis (Δ).

Part C

Comparison of periods

A convenient apparatus is used to plot the curve representing the variations of the period in terms of the abscissa x, the radius of the disk is $R = 20\,cm$.

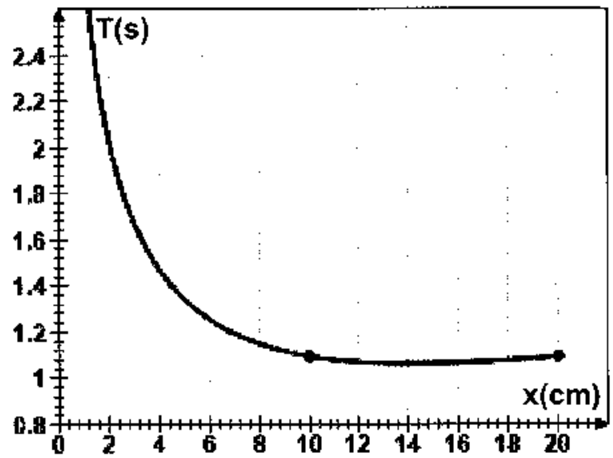


Figure 3

1. Give the value of the period for:
   a) $x = 2\,cm$.
   b) $x = 20\,cm$.
2. Specify the set of values of x for which the same period T_0' is obtained for two positions of the axis.
3. For what value of x the period is minimum?
4. Verify this result using the expression of T_0'.

VI-
Oscillations of a Pendulum

The aim of this exercise is to determine the conditions for which the oscillations of a particular pendulum are simple harmonic.

Consider a rod (R) of negligible mass, and length ℓ that may rotate, without friction, about a horizontal axis (Δ) perpendicular to its plane through the center O of the homogeneous rod.

On this rod, we fix a point particle of mass m on its lower extremity B, while another particle of mass $2m$ may slide on the upper part whose position is defined by $OP = x$ as shown in the adjacent figure.

The horizontal plane passing through O is taken as a gravitational potential energy reference.

Starting from its stable equilibrium position, we turn (R) by a small angle θ_m around AB, the two wires are twisted, in the same direction, by the same angle θ_m.

At an instant t, the position of the pendulum is defined by the angular abscissa θ, the angle that the vertical through O makes with OG, and its angular velocity is $\theta' = \frac{d\theta}{dt}$.

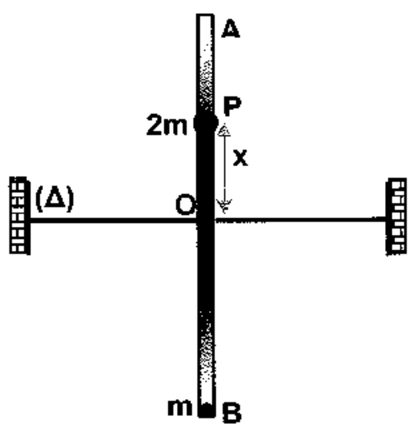


Figure 1

Let $a = OG$ be the distance between O and the center of gravity G of the pendulum.

Part A
Differential equation

1. Show that the position of G is given by $a = \frac{\ell - 4x}{6}$.
2. Determine the expression of the moment of inertia I of the pendulum about the axis (Δ).
3. Show that the expression of the mechanical energy of the system (pendulum, Earth) is given by $ME = \frac{1}{2} I \theta'^2 + C\theta^2 - \frac{mg}{2}(\ell - 4x)\cos\theta$.
4. Derive the second order differential equation in θ that governs the motion of the pendulum.

Part B
Oscillations

1. Justify that for $x = \frac{\ell}{4}$, the pendulum performs harmonic oscillations independent of the amplitude. Interpret the result obtained.

Take: $\sin\theta = \theta$ and $\cos\theta = 1 - \frac{\theta^2}{2}$ for small angles, θ being in radian.

2. Show that the differential equation for small angles can be written $\theta'' + \frac{4C + mg(\ell - 4x)}{2I}\theta = 0$.
3. Deduce the set of values of x for which the oscillations are simple harmonic.

VII-
Torsion Pendulum

Consider the adjacent system formed of a rod (R) connected to the free extremity O of a vertical torsion wire (Δ) of length $\ell = 40\,cm$ of torsion constant C_0 whose other extremity is fixed to a support as shown in figure 1.

The moment of inertia of the rod about (Δ) is $I = 1.6 \times 10^{-3}\ kg.m^2$.

The horizontal plane passing through (R) is taken as the reference level for the gravitational potential energy.

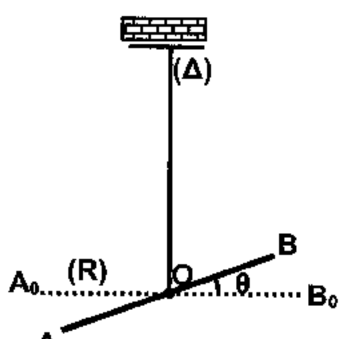


Figure 1

AB is a diameter of the disk defined by the angle that makes with the equilibrium position by the elongation $\theta = \left(\overrightarrow{OG_0}, \overrightarrow{OG}\right)$.

Part A
Single torsion wire

The rod is twisted in the horizontal plane by an angle θ_m in the positive direction and then released from rest.

1. Write the expression of the mechanical energy of the system (rod, wire, Earth) in terms of I, C_0, θ and the angular velocity θ'.
2. Derive the differential equation in θ that describes the motion of the pendulum.

3. Deduce the expression of the proper period T_0 in terms of I & C_0.
4. Knowing that the duration of 10 oscillations is $8\,s$.

   Show that $C_0 = 0.1\,N.m/\text{rad}$.

Part B
Two torsion wires

The previous torsion wires of length ℓ is divided into two wires (w_1) & (w_2) as shown in figure 2 of respective lengths x & $\ell - x$, and whose torsion constants are C_1 & C_2.

Note: The torsion constant is inversely proportional to the length of wire used.

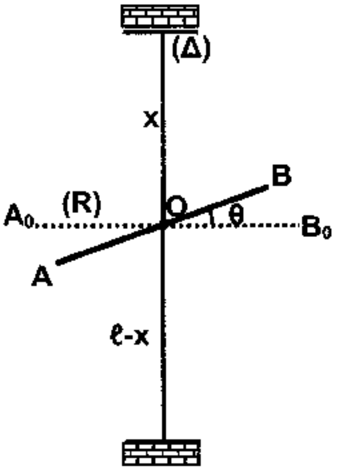


Figure 2

1. Determine the expressions of C_1 & C_2 in terms of C_0, x & ℓ.
2. Show that the elastic potential energy for the system of two wires is given by $PE_e = \frac{1}{2}\frac{C_0\ell^2}{x(\ell - x)}\theta^2$.
3. Derive the differential equation in θ that describes the motion of the pendulum.
4. Deduce that expression of the proper period of the oscillations is $T_0' = 2\pi\sqrt{\frac{x(\ell - x)I}{C_0\,\ell^2}}$.

Part C
Graphical study

1. Deduce that the expression of T_0' can be written $T_0' = 2\sqrt{x(0.4 - x)}$ (x in m & T_0' in s).
2. A convenient software is used to plot the curve representing the variations of the period T_0' in terms of x and the curve obtained is shown on the adjacent graph.
   a) Referring to the curve, give:
      i- the maximum value of the period of oscillations.
      ii- the value of x for which the period is maximum.
      iii- the values of x so that the period is $0.3\,s$.

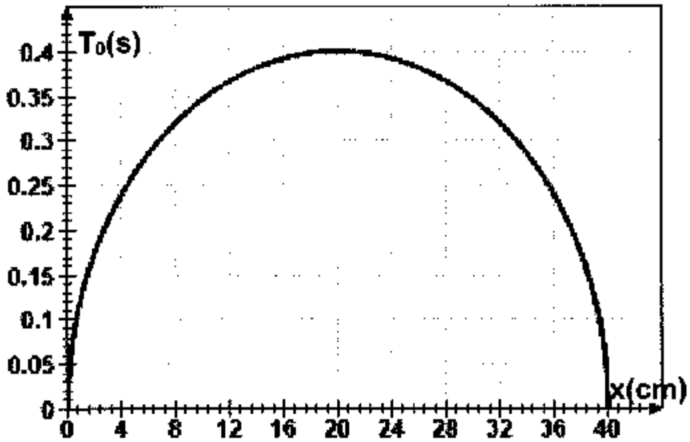


Figure 3

   b) Give a physical interpretation concerning the symmetry of the curve.
3. Justify by calculation the value of x so that T_0' is maximum.

VIII- Simple Pendulum and Mass of Earth

The object of this exercise is to determine the mass of the Earth M_E and its radius R_E knowing some physical characteristics of the simple pendulum and by measuring its period at two different altitudes.

Consider a simple pendulum (P) of mass is m and length ℓ, shifted by a small angle θ_m with respect to the equilibrium position, and released then without initial velocity.

Part A
Experimental study

Using an appropriate device, we record the three curves shown in the figures below and representing the variations of the pendulum's proper period, by changing successively its mass m, amplitude θ_m measured in degrees & the pendulum's length ℓ.

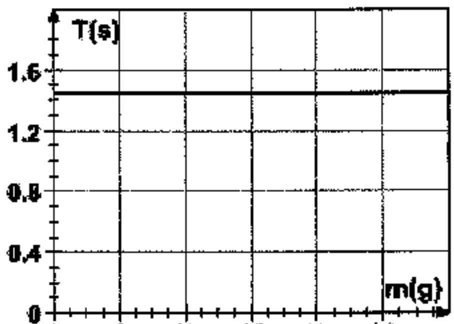
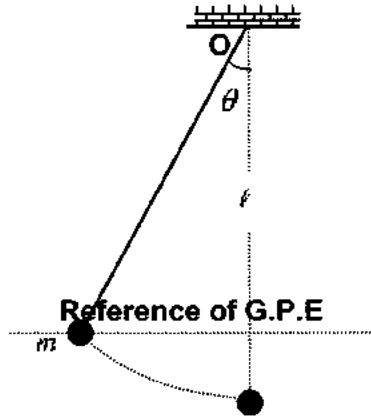


Figure 1



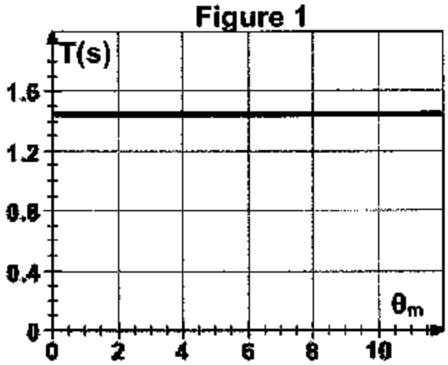


Figure 2

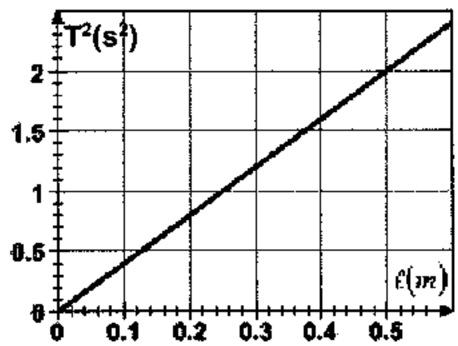


Figure 3

1. Draw a conclusion concerning the dependence of the proper period on the mass and amplitude of oscillation by referring to figures 1 & 2. Deduce the value of its proper period T_0.
2. Determine, from curve 3, the expression of the period T of this pendulum as a function ℓ.

Part B
Dimensional study

To Determine the expression of the period of this simple pendulum, we suppose that the period can be written in the form: $T = A m^{\alpha} g^{\beta} \ell^{\gamma}$ where A, α, β & γ are constants and g is the gravitational acceleration.

Using a dimensional study (by the means of units), determine the values of the constants α, β & γ, and then show this result is in coherence with the experimental study made in part **A**.

Part C
Theoretical study

For small angles (θ in radian): $\sin\theta = \theta$.

The higher position of the simple pendulum (P) is taken as reference of gravitational potential energy for the system [(P), Earth]. Neglect all frictional forces.

Take: $g = 9.83\ m/s^2$.

1. Show that the expression of the mechanical energy ME of the system [(P), Earth] at any instant t, referred by the angle θ, that makes with the vertical is given by :
   $ME = \frac{1}{2} m\ell^2\theta'^2 - mg\ell(\cos\theta - \cos\theta_m)$.
2. a) Determine the differential equation that describes the motion of (P).
   b) Knowing that $\theta = \theta_m \cos(2\pi f_0 t + \varphi)$ is a solution of the previous differential equation, express the proper frequency f_0 as a function of ℓ and g then deduce that of proper period T_0.
3. Calculate the phase φ knowing that at $t = 0$, $\theta = \frac{\theta_m}{2}$ and it was moving in the positive direction.
4. a) Compare this result to that studied in **A** and show that they are compatible.
   b) Verify that the length of the simple pendulum $\ell = 52.3cm$.

Part D
Experimental study

The intensity of the gravitational acceleration g_z at an altitude z of the Earth's surface taken as reference $(z = 0)$, is given by $g_z = \frac{GM_E}{(R_E + z)^2}$ where G is universal gravitational constant $G = 6.67\times10^{-11}\ SI$, M_E the Earth's mass and R_E its radius.

g_0 designate the intensity of the gravitational acceleration on the surface of the Earth.

1. Show that g_z is given by $g_z = g_0\left(\frac{R_E}{R_E + z}\right)^2$.
2. a) Show that the simple pendulum's period T_z whose expression $T_z = 2\pi\sqrt{\frac{\ell}{g_z}}$ at the altitude z is related to that on surface of Earth T_0 by $T_z = \left(1 + \frac{z}{R_E}\right)T_0$.
   b) We measure successively the duration of 20 oscillations of the pendulum whose length is ℓ, on the Earth's surface where $g_0 = 9.83\ m/s^2$, we find that it is equal to $\Delta t_0 = 29\,s$ and on board on a space station revolving at an altitude $z = 2000\ km$ it is equal to $\Delta t' = 38.1s$.
   Calculate the radius of the Earth R_E.
3. Deduce the mass of the Earth M_E.

Swing

A swing, on which sits a child of mass m and of center of inertia G, is considered as a simple pendulum (S) of adjustable length $\ell = OG$ and of mass m at G. (S) can oscillate in a vertical plane about a horizontal axis (Δ) passing through O.

We assume the forces of friction negligible and the oscillations of small amplitude.

At an instant t, the position of (S) is identified by its angular elongation θ relative to the vertical, $\theta' = \frac{d\theta}{dt}$ being its angular velocity.

The horizontal plane passing through O is the reference level for the gravitational potential energy.

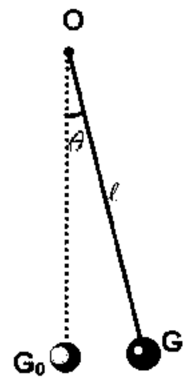


Figure 1

Take: $g = 9.8 m/s^2$, $\cos\theta = 1 - \frac{\theta^2}{2}$ and $\sin\theta = \theta$ (θ in rad).

Part A
Equation of motion

(S) is shifted in the positive direction from the equilibrium position, by a small angle θ_0, then left without speed at the instant $t_0 = 0$. (S) starts to oscillate with an amplitude θ_m supposed small.

1. a) Show that, at the instant t, the mechanical energy of the system [(S), Earth] is written:

$$ME = \frac{1}{2} m \ell^2 \theta'^2 - m g \ell \cos\theta .$$

b) Derive the differential equation in θ that describes the motion of (S).

c) Deduce the expression of the natural period T_0 of oscillations of (S) in terms of ℓ and g.

d) Show that the expression of the instant τ when the elongation of (S) reaches, for the first time, the value $\theta = 0$ rad is given by: $\tau = \frac{\pi}{2}\sqrt{\frac{\ell}{g}}$.

2. a) The solution of the differential equation is of the form: $\theta = \theta_m \cos\left(\frac{2\pi}{T_0} t + \varphi\right)$.

Determine the expression of θ_m and calculate the value of φ.

b) Show that the expression of the amplitude θ'_m of the angular velocity is given by: $\theta'_m = \sqrt{\frac{g}{\ell}}\,\theta_m$

c) Calculate, for $\ell = 2.0 m$ and $\theta_0 = 0.05$ rad, the values of θ_m, θ'_m and T_0.

Part B
Motion amplification

The child wishes, now, to increase (amplify) the amplitude θ_m of the oscillations starting from the instant $t_0 = 0$ where $\theta_0 = 0.05$ rad and $\theta'_0 = 0$.

✘ During the first passage by $\theta = 0$, at the instant t_1, the child instantaneously stands up (Figure 2), thus moving up his center of inertia G from G_1 to G_2, with $\ell_2 = OG_2$.

✗ When passing by $\theta = -\theta_{1m}$ (maximum deviation in the negative direction), at the instant t_2 (Figure 3), the child sits down again instantaneously, lowering his center of inertia from G_2 to G_1.

✗ When passing again by $\theta = 0$ at the instant t_3, the child stands up again instantaneously, shifting his center of inertia from G_1 to G_2 (Figure 2).

✗ When passing by $\theta = +\theta_{2m}$ (maximum deviation in the positive direction) at the instant t_4, the child sits down again instantaneously (Figure 4), lowering his center of inertia G_2 to G_1.

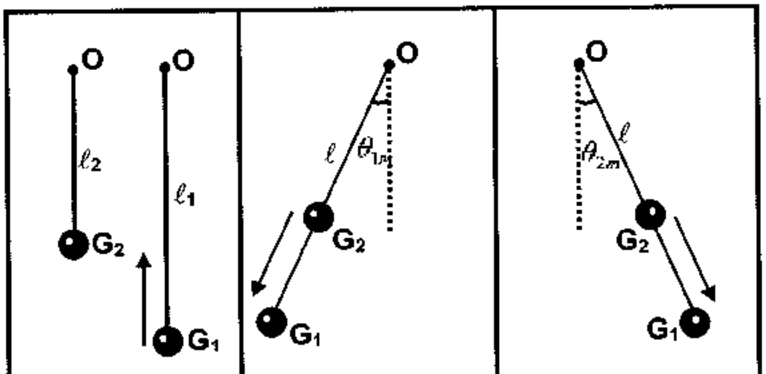


Figure 2	Figure 3	Figure 4
Passage by θ=0	Passage by θ=θ1m	Passage by θ=θ2m

1. Show that the expression of the period T of the oscillations of (S) is given by: $T = \pi\left(\dfrac{\sqrt{\ell_1}+\sqrt{\ell_2}}{\sqrt{g}}\right)$.

2. Just before passing by $\theta = 0$, at the instant t_1^-, the child begins to rise, which means that $r(t_1^-) = \ell_1$ and just after, at the instant t_1^+, the child completes its rise, which means that $r(t_1^+) = \ell_2$.

   a) Consider the physical quantity $\sigma(t) = m\, r^2\, \theta'$ where $r = OG$, r being equal to ℓ_1 or ℓ_2. What does $\sigma(t)$ represent? Explain why $\sigma(t_1^-) = \sigma(t_1^+)$.

   b) Let $\theta_0'^-$ and $\theta_0'^+$ be respectively the angular velocities of (S) at the instants (t_1^-) and (t_1^+). Determine, in terms of ℓ_1 and ℓ_2, the expression of the ratio $\theta_0'^+ / \theta_0'^-$.

   c) Deduce that $\dfrac{\theta_{2m}}{\theta_{0m}} = \dfrac{\ell_1^3}{\ell_2^3}$, where θ_{0m} and θ_{2m} are respectively the angular amplitudes of (S) at the instants t_0 and $t = T$.

3. a) After n oscillations, the amplitude is written as θ_{2nm}.

   Determine the expression of $\dfrac{\theta_{2nm}}{\theta_{0nm}}$ in terms of ℓ_1 and ℓ_2.

   b) Calculate n so that $\theta_{2nm} = 10^0$, with $\ell_1 = 2.0m$, $\ell_2 = 1.8m$ and $\theta_{0m} = 0.05$ rad.

Solutions

I-

Part A

1. Period of oscillations.
2. We have $T_2 = 2T_1$ & $\ell_2 = 4\ell_1$, then $\sqrt{\ell_2} = 2\sqrt{\ell_1}$;

   But $\frac{T_2}{T_1} = \frac{\sqrt{\ell_2}}{\sqrt{\ell_1}} = 2$, thus T is proportional to $\sqrt{\ell}$.

Part B

1. The proper period is independent of the mass of the pendulum.
2. a) Table

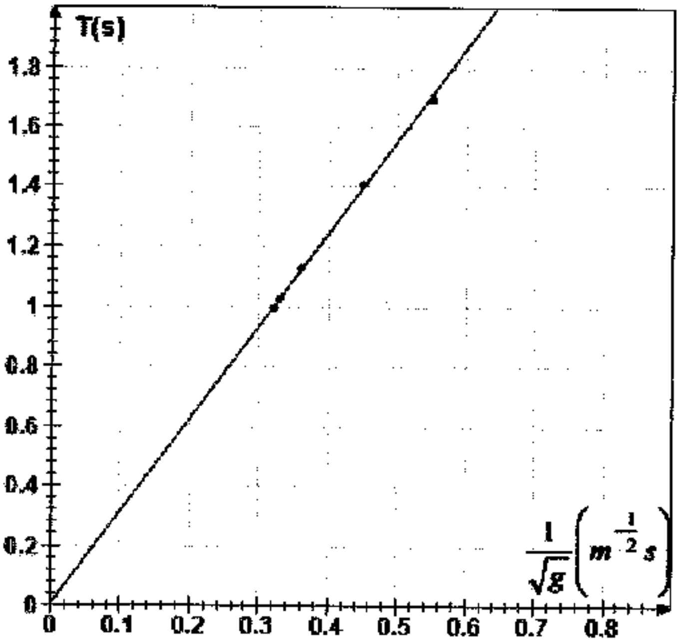
$\ell(cm)$	12.3	24.4	28.6	32.4	38.5
$\Delta t(s)$	14.1	19.8	21.4	22.8	24.9
$T = \frac{\Delta t}{20}$	0.705	0.99	1.07	1.14	1.245
$T^2 \times \ell^{-1}$	4.041	4.057	4.003	4.011	4.026

b) We conclude that: $T^2 \times \ell^{-1}$ is constant, T^2 is proportional to ℓ; $T^2 = 4.0276\,\ell$.

Then $T = 2.007\,\ell^{1/2} = k\ell^{\alpha}$;

where $k = 2.007$ & $\alpha = +\frac{1}{2}$.

3. a) Graph.



b) The curve representing the variation of T in terms of $\frac{1}{\sqrt{g}}$ is a straight line passing through origin, $T = k'\left(\frac{1}{\sqrt{g}}\right)$

with $k' = \frac{\Delta T}{\Delta\left(\frac{1}{\sqrt{g}}\right)} = \frac{0.995}{0.32} = 3.12$;

Thus, $T = 3.12\left(\frac{1}{\sqrt{g}}\right)$.

4. a) By a dimensional study $[T] = [C]\left[\sqrt{\frac{\ell}{g}}\right] = [C]\sqrt{\frac{[\ell]}{[g]}} = [C]\sqrt{\frac{m}{m\,s^{-2}}} = [C]s = s$, then $[C] = 1$;

   Thus, C is a dimensionless quantity.

b) We have $T = C\sqrt{\dfrac{\ell}{g}} = 2.007\sqrt{\ell} = C\sqrt{\dfrac{\ell}{g}} \Rightarrow C = 2.007\sqrt{g} = 2.007 \times \sqrt{9.8} = 6.2829$.

Part C

1. $ME = KE + GPE = \dfrac{1}{2} I\,\theta'^2 + mg\,\ell(1-\cos\theta) = \dfrac{1}{2} m\,\ell^2\,\theta'^2 + mg\,\ell(1-\cos\theta)$.

2. a) The forces of friction are negligible, ME is conserved $\dfrac{d(ME)}{dt} = 0$.

$\theta'\left(\theta'' + \dfrac{g}{\ell}\sin\theta\right) = 0$, ($\theta' \neq 0$, the pendulum is in motion).

(For small angles $\sin\theta = \theta$); we get $\theta'' + \dfrac{g}{\ell}\theta = 0$.

b) We have $\theta = \theta_m \cos\left(\dfrac{2\pi}{T_0}t + \varphi\right) \Rightarrow \theta'' = -\theta_m\left(\dfrac{2\pi}{T_0}\right)^2 \cos\left(\dfrac{2\pi}{T_0}t + \varphi\right)$.

Replacing in the previous differential equation: $\theta_m \cos\left(\dfrac{2\pi}{T_0}t + \varphi\right)\left(-\left(\dfrac{2\pi}{T_0}\right)^2 + \dfrac{g}{\ell}\right) = 0$.

(But $\theta_m \cos\left(\dfrac{2\pi}{T_0}t + \varphi\right) \neq 0$, in motion), then $-\left(\dfrac{2\pi}{T_0}\right)^2 + \dfrac{g}{\ell} = 0 \Rightarrow T_0 = 2\pi\sqrt{\dfrac{\ell}{g}}$.

c) The expression of the proper period is compatible with the results in **Part B**: $T_0 = 2\pi\sqrt{\dfrac{\ell}{g}}$;

Since $C = 6.2829 = 2\pi$.

II-

Part A

1. The mechanical energy is given by: $ME = KE + GPE + PE_e$;

(But $GPE = 0$, on reference); then $ME = \dfrac{1}{2} I\,\theta'^2 + \dfrac{1}{2} C\,\theta^2$.

2. The system performs free undamped oscillations, then its mechanical energy is conserved, so $\dfrac{d(ME)}{dt} = 0$; we get $I\,\theta'\theta'' + C\,\theta\,\theta' = 0$, $I\,\theta'\left(\theta'' + \dfrac{C}{I}\theta\right) = 0$;

($I\,\theta' \neq 0$, system in motion), then $\theta'' + \dfrac{C}{I}\theta = 0$.

3. The differential equation that governs the motion of the pendulum is of 2nd order of the form $\theta'' + \omega_0^2\,\theta = 0$ where $\omega_0^2 = \dfrac{C}{I}$;

Then the motion of the pendulum is periodic of proper period $T_0 = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{I}{C}}$.

4. We have $T_0 = 2\pi\sqrt{\dfrac{I}{C}}$, then $\dfrac{C}{I} = \left(\dfrac{2\pi}{T_0}\right)^2$;

Replacing in the differential equation, we get: $\theta'' + \left(\dfrac{2\pi}{T_0}\right)^2 \theta = 0$.

Part B

1. The curve representing the variations of θ'' in terms of θ, is of the form $\theta'' = a\,\theta$, then its graphical representation $\theta'' = f(\theta)$ should be:

✗ a straight line, the variable θ is of degree 1.

✗ passing through the origin.

✗ decreasing since it has a negative slope $a = -\frac{4\pi^2}{T_0^2} < 0$.

Which are satisfied graphically.

2. a) Graphically, $\theta_0 = -0.125\,\pi\,(\text{rad}) < 0$;

For $\theta = \theta_0$, we get $\theta'' = 10^{+2}\ \text{rad}/s^2$.

b) We have $\theta'' = -\left(\frac{2\pi}{T_0}\right)^2 \theta$, then $T_0 = 2\pi\sqrt{\frac{-\theta_0}{\theta''|_{\theta_0}}} = 2\pi\sqrt{\frac{0.125\,\pi}{1\times 10^{+2}}} \approx 0.4\,s$.

3. We have $T_0 = 2\pi\sqrt{\frac{I}{C}}$, so $I = \frac{T_0^2}{4\pi^2}\times C = \frac{0.4^2}{4\pi^2}\times 0.125 = 5.1\times 10^{-4}\,kg.m^2$.

4. The mechanical energy is conserved: $ME = ME|_{\theta_0} = KE|_{\theta_0} + PE_e|_{\theta_0}$, but $KE|_{\theta_0} = 0$;

Then $ME = \frac{1}{2}C\,\theta_0^2 = \frac{1}{2}\times 0.125\times(-0.125\,\pi)^2 = 9.6\times 10^{-3}\,J$.

5. Conservation of mechanical energy: $ME = ME|_{eq} = KE|_{eq} + PE_e|_{eq}$, but $PE_e|_{eq} = 0$;

Then $ME = \frac{1}{2}I\,\theta'^2_{eq} = 9.6\times 10^{-3}\,J$, then $\theta'_{eq} = \sqrt{\frac{2\times 9.6\times 10^{-3}}{5.1\times 10^{-4}}} = 6.1\,\text{rad}/s$.

III-

Part A

1. a) The position of the center of gravity is given by: $a = \overline{OG} = \frac{m_A\times\overline{OA} + m_B\times\overline{OB}}{m_A + m_B}$;

Then $a = \frac{m\times(\ell/2) + m\times\ell}{m+m} = \frac{3}{4}\ell$.

b) $I_{\text{system}} = I_{\text{rod}} + I_A + I_B = 0 + m\left(\frac{\ell}{2}\right)^2 + m\,\ell^2 = \frac{5}{4}m\,\ell^2$.

2. The mechanical energy $ME = KE + GPE = \frac{1}{2}I\,\theta'^2 - m_t\,g\,a\cos\theta$;

Then, $ME = \frac{1}{2}\times\frac{5}{4}m\,\ell^2\,\theta'^2 - 2\,m\,g\,\frac{3}{4}\ell\cos\theta = \frac{5}{8}m\,\ell^2\,\theta'^2 - \frac{3}{2}m\,g\,\ell\cos\theta$.

3. In the absence of the forces of friction, the mechanical energy is conserved so $\frac{d(ME)}{dt} = 0$;

$\frac{5}{4}m\,\ell^2\,\theta'\times\theta'' + \frac{3}{2}m\,g\,\ell\times\theta'\sin\theta = 0$; so $\frac{5}{4}m\,\ell^2\,\theta'\times\left(\theta'' + \frac{6\,g}{5\,\ell}\sin\theta\right) = 0$;

(but $\frac{5}{4}m\,\ell^2\,\theta' \neq 0$, pendulum in motion) & ($\sin\theta = \theta$, for small angles); thus, $\theta'' + \frac{6\,g}{5\,\ell}\theta = 0$.

4. The differential equation that governs the motion of the pendulum is of 2nd order of the form $\theta'' + \omega_0^2\,\theta = 0$ where $\omega_0^2 = \dfrac{6\,g}{5\,\ell}$;

Then the motion of the pendulum is periodic of proper period $T_0 = \dfrac{2\pi}{\omega_0} = 2\pi\sqrt{\dfrac{5\,\ell}{6\,g}}$.

Part B

1. a) $\theta'_{\max} = 0.8\,\text{rad}/s$.

   b) $T_0 = 1.25\,s$.

2. We have $T_0 = 2\pi\sqrt{\dfrac{5\,\ell}{6\,g}}$, then $\ell = \dfrac{T_0^2 \times 6\,g}{4\pi^2 \times 5} = \dfrac{1.25^2 \times 6 \times 10}{20\,\pi^2} = 0.48\,m = 48\,cm$.

3. We have $w_0 = \dfrac{2\pi}{T_0} = \dfrac{2\pi}{1.25} = 1.6\,\pi\,(\text{rad}/s)$.

   At $t_0 = 0$, we have $\theta' = 0.2\,\text{rad}/s$, we get $\cos\varphi = 1$, then $\varphi = 0$.

   Thus, $\theta' = 0.8\cos(1.6\,\pi\,t)$ (t in s & θ' in rad/s).

4. We have $\theta' = 0.8\cos(1.6\,\pi\,t) = \dfrac{d\theta}{dt}$, then $\theta = \int \theta'\,dt = \int 0.8\cos(1.6\,\pi\,t)\,dt = \dfrac{0.8}{1.6\pi}\sin(1.6\,\pi\,t)$;

   Then the amplitude of these oscillations is $\theta_m = \dfrac{0.8}{1.6\pi} \approx 0.16\,\text{rad}$.

5. a) The variation in the mechanical energy is: $\Delta(ME) = ME|_{\theta_{m2}} - ME|_{\theta_{m1}}$;

   Then $\Delta(ME) = (-2\,m\,g\,a\cos\theta_{m2}) - (-2\,m\,g\,a\cos\theta_{m1}) = 2\,m\,g\,a(\cos\theta_{m1} - \cos\theta_{m2})$;

   Thus, $\Delta(ME) = 2 \times 0.25 \times 10 \times \dfrac{3 \times 0.48}{4}(\cos 0.16 - \cos 0.15) = -2.8 \times 10^{-3}\,J$.

   b) We can suppose that the oscillations are slightly damped, $\Delta t = T = 1.25\,s$;

   The energy delivered to the pendulum is $P_{av} = \dfrac{|\Delta(ME)|}{\Delta t} = \dfrac{2.8 \times 10^{-3}}{1.25} = 2.24 \times 10^{-3}\,W$.

IV-

Part A

1. The mechanical energy is given by: $ME = KE + GPE$; where $GPE = -m\,g\,h = -m\,g\,h_1\cos\theta$;

   Thus, $ME = \dfrac{1}{2} I_1\,\theta'^2 - m\,g\,h_1\cos\theta$.

2. In the absence of friction, the mechanical energy is conserved, so $\dfrac{d(ME)}{dt} = 0$;

   We get $I_1\,\theta'\theta'' + m\,g\,h_1\theta'\sin\theta = 0$; then $I_1\,\theta'\left(\theta'' + \dfrac{m\,g\,h_1}{I_1}\sin\theta\right) = 0$;

   (but $I_1\,\theta' \neq 0$, pendulum in motion) & ($\sin\theta = \theta$, small angles); thus, $\theta'' + \dfrac{m\,g\,h_1}{I_1}\theta = 0$.

3. The differential equation that governs the motion of the pendulum is of 2nd order of the form $\theta'' + \omega_0^2\,\theta = 0$ where $\omega_0^2 = \dfrac{m\,g\,h_1}{I_1}$;

Then the motion of the pendulum is periodic of proper period $T_1 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{I_1}{m\,g\,h_1}}$.

4. Similarly we get $T_2 = 2\pi\sqrt{\frac{I_2}{m\,g\,h_2}}$.

Part B

1. Referring to the previous results, we get: $h_1\,T_1^2 = \frac{4\pi^2 I_1}{m\,g} = \frac{4\pi^2}{m\,g}\left(I_0 + m_1 h_1^2\right)$

Then, $h_1\,T_1^2 - h_2 T_2^2 = \frac{4\pi^2}{m\,g}\left(I_0 + m h_1^2\right) - \frac{4\pi^2}{m\,g}\left(I_0 + m h_2^2\right) = \frac{4\pi^2}{m\,g}\left(I_0 + m h_1^2 - I_0 - m\,h_2^2\right)$;

Thus, $h_1\,T_1^2 - h_2 T_2^2 = \frac{4\pi^2}{g}\left(h_1^2 - h_2^2\right)$.

2. We have $\Delta t_1 = 100\,T_1 = 106\,s$, so $T_1 = 1.06\,s$ & $\Delta t_2 = 100\,T_2 = 116\,s$, so $T_2 = 1.16\,s$;

Then $g = \frac{4\pi^2\left(h_1^2 - h_2^2\right)}{h_1\,T_1^2 - h_2 T_2^2} = \frac{4\pi^2\left(0.48^2 - 0.52^2\right)}{0.48\times 1.06^2 - 0.52\times 1.16^2} = 9.8\,m/s^2$.

3. The pendulum is reversible, then $T_1 = 2\pi\sqrt{\frac{I_1}{m\,g\,h_1}} = T_2 = 2\pi\sqrt{\frac{I_2}{m\,g\,h_2}}$;

$\frac{I_0 + m_1 h_1^2}{m\,g\,h_1} = \frac{I_0 + m_2 h_2^2}{m\,g\,h_2}$; we get $I_0 h_2 + m_1 h_1^2 h_2 = I_0 h_1 + m_1 h_2^2 h_1$;

So, $I_0\left(h_2 - h_1\right) = m\,h_1 h_2\left(h_2 - h_1\right)$ but $h_2 - h_1 \neq 0$ (blocks of different masses);

Then $I_0 = m\,h_1\,h_2 = 2\times 0.48\times 0.52 = 0.4992\ kg.m^2$.

V-

Part A

1. According to the theorem of parallel axis $I = I_0 + m\,R^2 = \frac{1}{2}m\,R^2 + m\,R^2 = \frac{3}{2}m\,R^2$.

2. The mechanical energy is given by: $ME = KE + GPE$; where $GPE = -m\,g\,h = -m\,g\,R\cos\theta$;

Thus, $ME = \frac{1}{2}I\,\theta'^2 - m\,g\,R\cos\theta$.

3. In the absence of friction, the mechanical energy is conserved, so $\frac{d(ME)}{dt} = 0$;

We get $I\,\theta'\theta'' + m\,g\,R\theta'\sin\theta = 0$; then $I\,\theta'\left(\theta'' + \frac{m\,g\,R}{\frac{3}{2}mR^2}\sin\theta\right) = 0$;

(but $I\,\theta' \neq 0$, pendulum in motion) & ($\sin\theta = \theta$, small angles); thus, $\theta'' + \frac{2\,g}{3R}\theta = 0$.

4. The differential equation that governs the motion of the pendulum is of 2nd order of the form $\theta'' + \omega_0^2\,\theta = 0$ where $\omega_0^2 = \frac{2g}{3\,R}$;

Then the motion of the pendulum is periodic of proper period $T = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{3R}{2g}}$.

Part B

1. The moment of inertia of the disk $I = I_0 + m x^2 = \frac{1}{2} m R^2 + m x^2$;

   The mechanical energy is given by $ME = KE + GPE = \frac{1}{2} I \theta'^2 - m g h$ where $h = x\cos\theta$;

   Then, $ME = \frac{1}{2}\left(\frac{1}{2} m R^2 + m x^2\right)\theta'^2 - m g x \cos(\theta)$.

2. In the absence of friction, the mechanical energy is conserved, so $\frac{d(ME)}{dt} = 0$;

   We get $\left(\frac{1}{2} m R^2 + m x^2\right)\theta'\theta'' + m g x \theta' \sin(\theta) = 0$, & $\sin\theta = \theta$ (small angles);

   (but $m\theta' \neq 0$, pendulum in motion); thus, $\theta'' + \frac{2 g x}{R^2 + 2x^2}\theta = 0$.

3. The differential equation that governs the motion of the pendulum is of 2nd order of the form $\theta'' + \omega_0'^2 \theta = 0$ where $\omega_0'^2 = \frac{2 g x}{R^2 + 2x^2}$;

   Then the motion of the pendulum is periodic of proper period $T_0' = \frac{2\pi}{\omega_0'} = 2\pi\sqrt{\frac{R^2 + 2x^2}{2 g x}}$.

4. For $x = \frac{R}{2}$, $T_0' = \frac{2\pi}{\omega_0'} = 2\pi\sqrt{\frac{R^2 + 2\left(\frac{R}{2}\right)^2}{2g\left(\frac{R}{2}\right)}} = 2\pi\sqrt{\frac{\frac{3R^2}{2}}{g R}} = 2\pi\sqrt{\frac{3R}{2g}} = T_0$.

   The same period is obtained for two different positions of the axis at $x = R$ & $x = \frac{R}{2}$.

5. By symmetry we can invert the disk and the same period is obtained for $x = \pm R$, $x = \pm\frac{R}{2}$.

Part C

1. a) Graphically, for $x = 2\,cm$, we get $T_0' = 2\,s$.

   b) Graphically, for $x = 20\,cm$, we get $T_0' = 1.1\,s$.

2. For $x \in [2\,cm; 20\,cm]$, the same period is obtained for two different positions of the axis.

3. Graphically, the period is minimum for $x = 14\,cm$, & this minimum is $T'_{0\min} = 1.04\,s$.

4. For $R = 20\,cm = 0.2\,m$, we get $T_0' = 2\pi\sqrt{\frac{0.04 + 2x^2}{20x}}$;

   The period is minimum for $\frac{dT_0'}{dx} = \frac{2\pi}{\sqrt{\ldots} \times 400\,x^2}\left(40\,x^2 - 0.8\right)$;

   $\frac{dT_0'}{dx} = 0$, we get $x = \sqrt{\frac{0.8}{40}} = 14.1\,cm$ & $T'_{0\min} = 1.04\,s$.

2^{nd} method:

We have $T_0' = 2\pi\sqrt{\frac{1}{20}\left(\frac{0.04}{x}+2x\right)}$, furthermore the product $\frac{0.04}{x}\times 2x = 0.08$ is constant;

Then T_0' is minimum when $\frac{0.04}{x} = 2x$, so $x=\sqrt{0.02}\approx 14.1\,cm$ & this minimum is $T'_{0\,\min} = 1.04\,s$.

VI-

Part A

1. The position of the center of gravity is given by: $a=\overline{OG}=\frac{m_B\,\overline{OB}+m_P\,\overline{OP}}{m_B+m_P}$;

So, $a=\frac{M(\ell/2)-2M\,x}{M+2M}=\frac{\ell-4x}{6}$.

2. The moment of inertia: $I=I_P+I_B+I_{\text{rod}}=M\left(\frac{\ell}{2}\right)^2+2M\,x^2=\frac{M\left(\ell^2+4x^2\right)}{4}$.

3. The mechanical energy $ME=KE+GPE+PE_e$ where $PE_e=\frac{1}{2}C\theta^2+\frac{1}{2}C\theta^2=C\theta^2$;

Then $ME=\frac{1}{2}I\,\theta'^2+C\,\theta^2-3M\,g\,a\,\cos\theta=\frac{1}{2}I\,\theta'^2+C\,\theta^2-\frac{1}{2}M\,g(\ell-4x)\cos\theta$.

4. In the absence of friction, the mechanical energy is conserved, then $\frac{d(ME)}{dt}=0$;

$I\,\theta'\theta''+2C\,\theta\,\theta'+\frac{1}{2}M\,g(\ell-4x)\,\theta'\sin\theta=0$, but $\theta'\neq 0$ (pendulum in motion);

Then $\theta''+\frac{2C}{I}\theta+\frac{M\,g(\ell-4x)}{2I}\sin\theta=0$.

Part B

1. For $x=\frac{\ell}{4}$, $\ell-4x=0$, the differential equation previously derived becomes $\theta''+\frac{2C}{I}\theta=0$;

It is of 2^{nd} order of the form $\theta''+w_0^2\,\theta=0$, where $w_0^2=\frac{2C}{I}$;

Then the motion of pendulum is simple harmonic independent of the amplitude of oscillations.

Interpretation:

For $x=\frac{\ell}{4}$, the position of the center of gravity $a=OG=\frac{\ell-4x}{6}=0$, it becomes on origin.

Consequently there is no effect of the gravitational potential energy on the oscillations.

2. For small angles $\sin\theta\approx\theta$, the differential equation becomes $\theta''+\frac{2C}{I}\theta+\frac{M\,g(\ell-4x)}{2I}\theta=0$;

Then $\theta''+\frac{4C+M\,g(\ell-4x)}{I}\theta=0$.

3. The oscillations are simple harmonic if $w_0^2=\frac{4C+M\,g(\ell-4x)}{I}>0$

Then, $x<\frac{\ell}{4}+\frac{C}{M\,g}$.

VII-

Part A

1. The mechanical energy is given by: $ME = KE + GPE + PE_e$;

   (But $GPE = 0$, on reference); then $ME = \frac{1}{2} I \theta'^2 + \frac{1}{2} C_0 \theta^2$.

2. The system performs free undamped oscillations, then its mechanical energy is conserved, $\frac{d(ME)}{dt} = 0$; so $I\theta'\theta'' + C_0 \theta\theta' = 0$, $I\theta'\left(\theta'' + \frac{C_0}{I}\theta\right) = 0$;

   (but $I\theta' \neq 0$, system in motion), then $\theta'' + \frac{C_0}{I}\theta = 0$.

3. The differential equation that governs the motion of the pendulum is of 2^{nd} order of the form $\theta'' + w_0^2 \theta = 0$ where $w_0^2 = \frac{C_0}{I}$;

   Then the motion of the pendulum is periodic of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{I}{C_0}}$.

4. We have $10\,T_0 = 8\,s$, then $T_0 = 0.8\,s$;

   But $T_0 = 2\pi\sqrt{\frac{I}{C_0}}$, we get $C_0 = \frac{4\pi^2 I}{T_0^2} = \frac{4 \times 10 \times 1.6 \times 10^{-3}}{0.8^2} = 0.1\,N.m/\text{rad}$.

Part B

1. We have $C_1 = \frac{k}{x}$ & $C_1 = \frac{k}{\ell - x}$, but $C_0 = \frac{k}{\ell}$, so $k = C_0\,\ell$; thus $C_1 = \frac{\ell}{x}C_0$ & $C_1 = \frac{\ell}{\ell - x}C_0$.

2. The elastic potential energy: $PE_e = PE_{e_1} + PE_{e_2} = \frac{1}{2}C_1\theta^2 + \frac{1}{2}C_2\theta^2 = \frac{1}{2}(C_1 + C_2)\theta^2$;

   Then, $PE_e = \frac{1}{2}\left(\frac{\ell}{x}C_0 + \frac{\ell}{\ell - x}C_0\right)\theta^2 = \frac{C_0\,\ell^2}{2x(\ell - x)}\theta^2$.

3. The mechanical energy $ME = KE + PE_e = \frac{1}{2}I\theta'^2 + \frac{C_0\,\ell^2}{2x(\ell - x)}\theta^2$;

   The mechanical energy is conserved, $\frac{d(ME)}{dt} = 0$, so $I\theta'\theta'' + \frac{C_0\,\ell^2}{x(\ell - x)}\theta\theta' = 0$;

   (But $\theta' \neq 0$, pendulum in motion); $\theta'' + \frac{C_0\,\ell^2}{x(\ell - x)I}\theta = 0$.

4. The differential equation that governs the motion of the pendulum is of 2^{nd} order of the form $\theta'' + w_0'^2\theta = 0$ where $w_0'^2 = \frac{C_0\,\ell^2}{x(\ell - x)I}$.

   Then its motion is periodic of proper period $T_0' = \frac{2\pi}{w_0'} = 2\pi\sqrt{\frac{x(\ell - x)I}{C_0\,\ell^2}}$.

Part C

1. We have $C_0 = 0.1\,SI$ & $\ell = 0.4\,m$, then $T_0' = \sqrt{4\pi^2\,\frac{x(\ell - x)I}{C_0\,\ell^2}} = \sqrt{4 \times 10\,\frac{x(0.4 - x)1.6 \times 10^{-3}}{0.1 \times 0.4^2}}$;

   Then, $T_0' = \sqrt{4x(0.4 - x)} = 2\sqrt{x(0.4 - x)}$ where x in m & T_0' in s.

2. a) Graphically $T'_{0\max} = 0.4\,s$.

b) The maximum is attained for $x_0 = 20\,cm$.

c) Graphically if $T'_0 = 0.3s$, then $x_1 = 6.5\,cm$ & $x_2 = 33.5\,cm$.

d) Since the wire is divided into two parts of respective lengths x & $(\ell - x)$, the period will be unchanged if they are inverted $(\ell - x)$ & x.

3. T'_0 is maximum if $\frac{dT'_0}{dx} = \frac{2}{2\sqrt{....}}(0.4 - 2x)$, then $x_0 = 0.2m = 20\,cm$.

2nd method: We have $T'_0 = 2\sqrt{x(0.4 - x)}$, furthermore $x + (0.4 - x) = 0.4$ is constant;

Then T'_0 is maximum if $x = 0.4 - x$, then $x = 0.2\,m = 20cm$.

VIII-

Part A

1. When the mass and amplitude are modified, the period T is invariant; then the period of a simple pendulum is independent of its mass and amplitude θ_m.

Graphically, the proper period $T_0 = 1.45s$.

2. The graph of figure 3, representing the variations of the square of period T^2 in terms of its length ℓ is a straight line passing through origin whose equation is $T^2 = a\,\ell$;

Where $a = \frac{\Delta(T^2)}{\Delta(\ell)} = \frac{2-0}{0.5-0} = 4s^2/m$; then $T^2 = 4\ell$.

Thus, $T = 2\sqrt{\ell}$ where ℓ in m and T in s.

Part B

We have $T = A\,m^{\alpha} g^{\beta}\,\ell^{\gamma}$;

By a dimensional study $[T] = [A\,m^{\alpha} g^{\beta}\,\ell^{\gamma}] = [A] \times [m^{\alpha}] \times [g^{\beta}][\ell^{\gamma}] = 1 \times kg^{\alpha}(m/s^2)^{\beta} m^{\ell}$;

Then $[T] = 1 \times kg^{\alpha} m^{\beta+\gamma} s^{-2\beta} = s$;

By identification $\alpha = 0$; $\beta + \gamma = 0$ & $\gamma = \frac{1}{2}$; then $\beta = -\gamma = -\frac{1}{2}$; thus $T = A\,m^0 g^{-\frac{1}{2}}\,\ell^{\frac{1}{2}} = A\sqrt{\frac{\ell}{g}}$;

The result shows that the proper period T is proportional to $\sqrt{\ell}$ which is compatible part A.

Part C

1. In the triangles (OA_0H_0) & (OAH_1);

We have $\cos\theta_m = \frac{OH_0}{OA_0} = \frac{OH_0}{\ell}$;

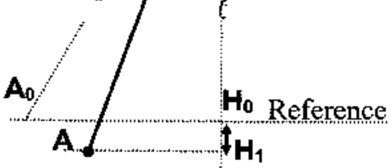
$\cos\theta = \frac{OH}{OA} = \frac{OH}{\ell}$; So $h_0 = \ell\cos\theta_m$ & $h = \ell\cos\theta$;

But $H_0H = \ell\cos\theta - \ell\cos\theta_m = \ell(\cos\theta - \cos\theta_m)$;

Then $GPE = -m\,g(\cos\theta - \cos\theta_m)$;

But $KE = \frac{1}{2}I\,\theta'^2$ where $I = m\ell^2$ (particle);

Thus, $ME = \frac{1}{2}m\,\ell^2\theta'^2 - mg\ell(\cos\theta - \cos\theta_m)$.



2. a) In the absence of friction, the mechanical energy is conserved, so $\dfrac{d(ME)}{dt} = 0$;

$$\frac{1}{2} m\ell^2 \times 2 \times \theta' \times \theta'' - mg\ell(-\theta' \sin\theta) = 0,\ m\ell^2\theta'\left(\theta'' + \frac{g}{\ell}\sin\theta\right) = 0;$$

($m\ell^2\theta' \neq 0$, the pendulum is in motion) and for small angles $\sin\theta \approx \theta$; then $\theta'' + \dfrac{g}{\ell}\theta = 0$

b) We have $\theta = \theta_m \cos(2\pi f_0 t + \varphi)$, so $\theta'' = -\theta_m (2\pi f_0)^2 \cos(2\pi f_0 t + \varphi)$;

By replacing in the differential equation we get: $\theta_m \cos(2\pi f_0 t + \varphi)\left(-4\pi^2 f_0^2 + \dfrac{g}{\ell}\right) = 0$;

But $\theta_m \cos(2\pi f_0 t + \varphi) \neq 0$; so $\dfrac{g}{\ell} - 4\pi^2 f_0^2 = 0$; then $f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{g}{\ell}}$

The proper period $T_0 = \dfrac{1}{f_0} = 2\pi\sqrt{\dfrac{\ell}{g}}$.

3. At $t = 0$, $\theta = \dfrac{\theta_m}{2}$ where $\theta = \theta_m \cos(2\pi f_0 t + \varphi)$;

So $\dfrac{\theta_m}{2} = \theta_m \cos(\varphi)$; $\cos\varphi = \dfrac{1}{2}$; then $\varphi = \pm\dfrac{\pi}{3}$ (rad);

The pendulum was moving in the positive direction, then $\theta' = -\theta_m 2\pi f_0 \sin(2\pi f_0 t + \varphi) > 0$;

So $\sin\varphi < 0$, then $\varphi = -\dfrac{\pi}{3}$ (rad).

4. a) The obtained expression of the proper period is $T_0 = 2\pi\sqrt{\dfrac{\ell}{g}}$ is proportional to the square root of the length of the pendulum which is compatible again with the result obtained in Part A.

b) We have $T_0 = 2\pi\sqrt{\dfrac{\ell}{g}}$, so $\ell = \dfrac{T_0^2 \times g}{4\pi^2} = \dfrac{1.45^2 \times 9.83}{4\pi^2} = 0.523\,m = 52.3\,cm$.

Part D

1. We have $g_z = \dfrac{GM_E}{(R_E + z)^2}$; on the surface of the Earth $z = 0$, then $g_0 = \dfrac{GM_E}{R_E^2}$; so $GM_E = g_0 R_E^2$

Then $g_z = \dfrac{g_0 R_E^2}{(R_E + z)^2} = g_0\left(\dfrac{R_E}{R_E + z}\right)^2$.

2. a) We have $T_z = 2\pi\sqrt{\dfrac{\ell}{g_z}}$ where $g_z = g_0\left(\dfrac{R_E}{R_E + z}\right)^2$;

$$\text{So } T_z = 2\pi\sqrt{\frac{\ell}{g_0\left(\dfrac{R_E}{R_E + z}\right)^2}} = 2\pi\sqrt{\frac{\ell}{g_0}} \times \frac{R_E + z}{R_E} = \left(1 + \frac{z_E}{R}\right)2\pi\sqrt{\frac{\ell}{g_0}} = \left(1 + \frac{z_E}{R}\right)g_0.$$

b) We have $T_z = \left(1 + \dfrac{z}{R_E}\right)T_0$ (multiplying by 20 both sides); so $\Delta t' = \left(1 + \dfrac{z}{R_E}\right)\Delta t$;

Then $38.1 = \left(1 + \dfrac{z}{R_E}\right)29$; so $\dfrac{z}{R_E} = \dfrac{38.1}{29} - 1 = \dfrac{91}{290}$;

Thus $R_E = \frac{290}{91} \times z = \frac{290}{91} \times 2000\,km = 6373\,km$.

3. We have $g_0 = \frac{G M_E}{R_E^2}$, then $M_E = \frac{g_0 R_E^2}{G} = \frac{9.83 \times \left(6373 \times 10^3\right)^2}{6.67 \times 10^{-11}} = 5.99 \times 10^{24}\,kg$.

IX-

Part A

1. a) The mechanical energy is given by: $ME = KE + GPE = \frac{1}{2} I\,\theta'^2 - m\,g\,z$;

But $\cos\theta = \frac{z}{\ell}$, so $z = \ell\cos\theta$ & $I = m\,\ell^2$ (for a point particle);

Then $ME = \frac{1}{2} m\,\ell^2\,\theta'^2 - m\,g\,\ell\cos\theta$.

b) The forces of friction are assumed to be negligible, then the mechanical energy of the pendulum is conserved, $\frac{d(ME)}{dt} = 0$; $\frac{1}{2} m\,\ell^2 \times 2\,\theta'\theta'' + m\,g\,\ell\;\theta'\sin\theta = 0$;

(But $\theta' \neq 0$, pendulum in motion) & $\sin\theta = \theta$; then $m\,\ell^2\,\theta'' + m\,g\,\ell\,\theta = 0$; thus, $\theta'' + \frac{g}{\ell}\theta = 0$.

c) The differential equation that governs the motion of this pendulum is of 2nd order of the form $\theta'' + \omega_0^2\,\theta = 0$ where $\omega_0^2 = \frac{g}{\ell}$;

Then the motion of this pendulum is periodic of proper period $T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{\ell}{g}}$.

d) (S) is shifted in the positive direction from the equilibrium position, by a small angle θ_0, then left without speed at the instant $t_0 = 0$, so it speed a quarter of period to return to this position, thus $\tau = \frac{T_0}{4} = \frac{\pi}{2\omega_0} = \frac{\pi}{2}\sqrt{\frac{\ell}{g}}$.

2. a) We have $\theta = \theta_m \cos\left(\frac{2\pi}{T_0}t + \varphi\right)$;

Referring to the initial conditions, for $t = 0$, $\theta = \theta_m \cos\varphi = \theta_0 > 0$, then $\cos\varphi > 0$;

And $\theta' = -\theta_m \frac{2\pi}{T_0}\sin\left(\frac{2\pi}{T_0}t + \varphi\right)$, then $\theta'_0 = -\theta_m \frac{2\pi}{T_0}\sin(\varphi) = 0$;

So $\sin(\varphi) = 0$; then $\varphi = 0$ & $\theta_m = \theta_0$; thus $\theta = \theta_0 \cos\left(\sqrt{\frac{g}{\ell}}\,t\right)$ where t in s and θ in rad.

b) $\theta' = -\theta_m \frac{2\pi}{T_0}\sin\left(\frac{2\pi}{T_0}t + \varphi\right)$; so the amplitude of the angular velocity is $\theta'_m = \theta_m \frac{2\pi}{T_0} = \theta_m\sqrt{\frac{g}{\ell}}$.

c) We have $\theta_m = \theta_0 = 0.05$ rad; $\theta'_m = \theta_m\sqrt{\frac{g}{\ell}} = 0.05\sqrt{\frac{9.8}{2}} = 0.111\,rad/s$;

& $T_0 = 2\pi\sqrt{\frac{\ell}{g}} = 2\pi\sqrt{\frac{2}{9.8}} = 2.84\,s$.

Part B

1. From $\theta_0 = 0.05\,\text{rad} \to \theta = 0$; the length of the pendulum is ℓ_2 (sitting down); it needs

$\Delta t_1 = \dfrac{\pi}{2}\sqrt{\dfrac{\ell_1}{g}}$;

From $\theta = 0\,\text{rad} \to -\theta_{1m}$; the length of the pendulum is ℓ_2 (standing up); it needs $\Delta t_2 = \dfrac{\pi}{2}\sqrt{\dfrac{\ell_2}{g}}$;

and similarly from $-\theta_{1m}\,\text{rad} \to \theta = 0$. From $\theta = 0\,\text{rad} \to \theta_{2m}$; the length of the pendulum is ℓ_2;

it needs $\Delta t_1' = \dfrac{\pi}{2}\sqrt{\dfrac{\ell_1}{g}}$ (independent of its amplitude);

The expression of the period T of the oscillations is $T = 2\Delta t_1 + 2\Delta t_2 = 2\times\dfrac{\pi}{2}\sqrt{\dfrac{\ell_1}{g}} + 2\times\dfrac{\pi}{2}\sqrt{\dfrac{\ell_2}{g}}$;

Then $T = \pi\left(\dfrac{\sqrt{\ell_1}+\sqrt{\ell_2}}{\sqrt{g}}\right)$.

2. a) $K(t) = m r^2 \theta'$ is the angular momentum of (S) with respect to (Δ) at an instant t.

Between the instants t_1^- and t_1^+, (S) is subjected to two forces: its weight $\vec{w}$ and tension of the wire $\vec{T}$; which are (at this position) two vertical forces whose prolongations pass through the origin O. Then the resultant moment is $\sum \mathrm{M}_F = \mathrm{M}_{\vec{w}/O} + \mathrm{M}_{\vec{T}/O} = 0$;

According to the theorem of angular momentum $\sum \mathrm{M}_{\vec{F}/O} = \dfrac{d\sigma(t)}{dt} = 0$;

Thus, the angular momentum is conserved $\sigma(t_1^-) = \sigma(t_1^+)$.

b) We have $\sigma(t_1^-) = \sigma(t_1^+)$, $m\ell_1^2\theta_0'^- = m\ell_2^2\theta_0'^+$; then $\dfrac{\theta_0'^+}{\theta_0'^-} = \dfrac{\ell_1^2}{\ell_2^2}$.

c) The amplitudes are related to the angular velocities in the equilibrium position $\theta_m' = \theta_m\sqrt{\dfrac{g}{\ell}}$,

$$\frac{\theta_{1m}}{\theta_{0m}} = \frac{\theta_0'^+\sqrt{\dfrac{\ell_2}{g}}}{\theta_0'^-\sqrt{\dfrac{\ell_1}{g}}} = \frac{\theta_0'^+}{\theta_0'^-}\sqrt{\frac{\ell_2}{\ell_1}} = \frac{\ell_1^2}{\ell_2^2}\sqrt{\frac{\ell_2}{\ell_1}} = \sqrt{\frac{\ell_1^3}{\ell_2^3}}\ ;\ \&\ \frac{\theta_{2m}}{\theta_{1m}} = \frac{\theta_0'^+}{\theta_0'^-}\sqrt{\frac{\ell_2}{\ell_1}} = \frac{\ell_1^2}{\ell_2^2}\sqrt{\frac{\ell_2}{\ell_1}} = \sqrt{\frac{\ell_1^3}{\ell_2^3}}\ ;$$

Then $\dfrac{\theta_{2m}}{\theta_{0m}} = \dfrac{\theta_{2m}}{\theta_{1m}}\times\dfrac{\theta_{1m}}{\theta_{0m}} = \sqrt{\dfrac{\ell_1^3}{\ell_2^3}}\times\sqrt{\dfrac{\ell_1^3}{\ell_2^3}} = \dfrac{\ell_1^3}{\ell_2^3}$.

3. a) After n oscillations $\dfrac{\theta_{2nm}}{\theta_{0nm}} = \left(\dfrac{\ell_1^3}{\ell_2^3}\right)^n = \left(\dfrac{\ell_1}{\ell_2}\right)^{3n}$.

b) We have $\left(\dfrac{\ell_1}{\ell_2}\right)^{3n} = \dfrac{\theta_{2nm}}{\theta_{0nm}}$; $\left(\dfrac{2}{1.8}\right)^{3n} = \dfrac{\frac{10\times\pi}{180}}{0.05}$; $3n\,\ell n\left(\dfrac{2}{1.8}\right) = \ell n\left(\dfrac{10\times\pi}{180\times 0.05}\right)$; thus $n = 3.95 \approx 4$.

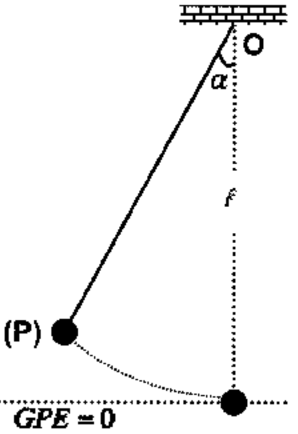
Supplementary Problems

Engineering (2005/2006)
The Simple Pendulum

The aim of this study is to find the condition to be satisfied so that a simple pendulum can be considered as a harmonic oscillator.

A simple pendulum (P) is formed of a small bob of mass $m = 200\ g$, suspended from a massless string of length $\ell = 1\ m$. (P) is shifted by an angle α_0 with respect to the equilibrium position, is released at $t_0 = 0$ without initial velocity. At an instant t, (P) makes an angle α and moves with a velocity $\vec{v}$.

Take: $g = 10\ m/s^2$ and neglect all frictional forces.



Part A
Theoretical study

1. The lower position of the bob is taken as zero gravitational potential energy. Give, at the instant t, the mechanical energy ME of the system [(P), Earth] in terms of m, g, ℓ, α & v.
2. Applying the conservation of ME, show that: $(\alpha')^2 = \left(\frac{d\alpha}{dt}\right)^2 = \frac{2g}{\ell}(\cos\alpha - \cos\alpha_0)$.
3. Show that the differential equation that describes the motion of (P) is given by: $\alpha'' + \frac{g}{\ell}\sin\alpha = 0$
4. a) What is the approximation that must be taken in order to consider (P) as a harmonic oscillator.
   b) Deduce then the expression of the proper period T_0 of this harmonic pendulum and calculate its value.

Part B
Experimental study

With an appropriate device, we record three curves shown in figures 1, 2 & 3. The curve shown in figure 1 gives the variation of the period of the pendulum in terms of α_0 and the curves of figures 2 and 3 give the variation of the angle α in terms of time for two different values of α.

1. Determine, from figure 1, the condition for which the pendulum behaves as a harmonic oscillator.
2. Determine, from figures 2 and 3, the periods T_1 & T_2 and the amplitudes α_{01} & α_{02} of oscillations of the pendulum in the two cases.
3. By comparing T_1 & T_2 with T_0, deduce that this pendulum does not always behave as a harmonic oscillator.
4. The condition in part A is in agreement with that of part B. Why?

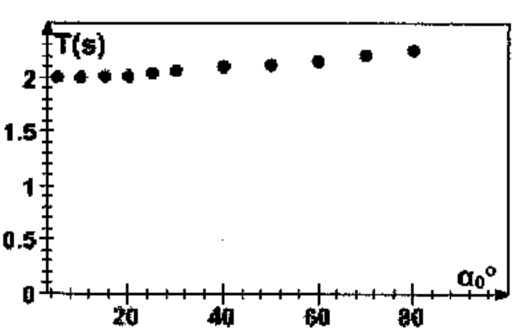


Figure 1

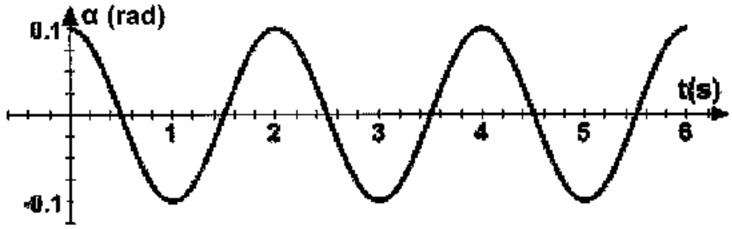


Figure 2

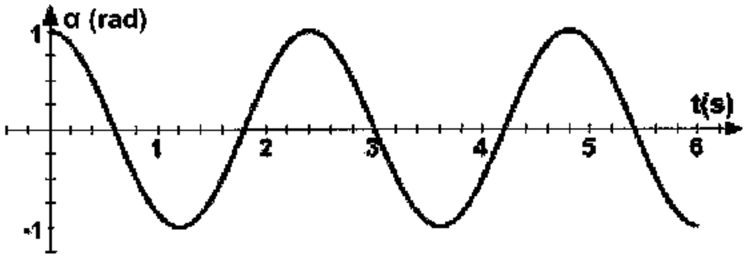


Figure 3

Part C

Damped oscillation

As long as the amplitude θ_m of a simple pendulum is not considered small, the motion of the simple pendulum is no more simple harmonic and the period T of oscillations may be given by the expression: $T = T_0\left(1+\frac{\theta_m{}^2}{16}\right)$. What is the maximum value which can be given to θ_m (in rad) for T to differ from T_0 by not more than 1%?

Answer Key

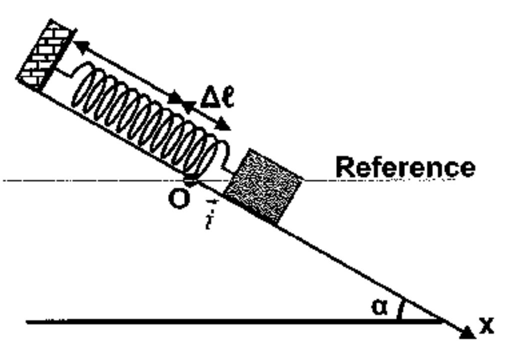
Part A 4.b) $T_0 \approx 2s$. **Part C** $\theta_m = 0.4\,\text{rad}$

II-

Mechanical Oscillator on an Inclined Plane

We consider a spring of stiffness $k = 50\,N/m$ and of negligible mass, connected from its upper end to a fixed support while its other end carries a solid (S) of mass m placed on an inclined plane as shown in the adjacent figure.

At equilibrium the center of mass G of (S) coincides with a point O and the spring elongates by $\Delta\ell$. At an instant t, G is defined by its abscissa $x = \overline{OG}$ and the algebraic value of its velocity is $v = \frac{dx}{dt}$.



At instant $t_0 = 0$ taken as origin of time, (S) is given an initial velocity $\overrightarrow{v_0} = v_0\vec{i}$ from its equilibrium position. G oscillates around its equilibrium position O.

The horizontal plane passing through O is taken as a reference of gravitational potential energy.

Part A

Static study

1. Name the external forces acting on (S) in the equilibrium position.
2. Determine a relation among m, g, $\Delta\ell$, α and k.

Part B

Energetic study

1. Show that the expression of the mechanical energy of the system [(S), spring, Earth] is given by: $ME = \frac{1}{2}m v^2 + \frac{1}{2}k(\Delta\ell + x)^2 - m g x \sin\alpha$.
2. Show that the differential equation in x that describes the motion of G has the form $x'' + w_0^2 x = 0$ where w_0^2 is a constant whose expression is to be determined.
3. The time equation that describes the motion of (S) is given by $x = x_m \sin\left(\frac{2\pi}{T_0}t + \varphi\right)$.

   Determine the expression of T_0 in terms of m & k and then the value of φ.
4. Show that the expression of the kinetic energy in terms of time is $KE = \frac{1}{2}k x_m^2 \cos^2\left(\frac{2\pi}{T_0}t\right)$.
5. Deduce the expression of the maximum kinetic energy.

Part C

Graphical Study

A convenient software is used to plot the graph representing the variations of the kinetic energy in terms of time.

Take: $\pi^2 = 10$.

1. Referring to the adjacent curve, give:
   a) the period of the kinetic energy T_E.
   b) the maximum value of the kinetic energy.
2. Deduce the amplitude x_m of the oscillations.
3. Show that $m = 1.8\,kg$.
4. Applying the principle of conservation of mechanical energy, determine the value of v_0.

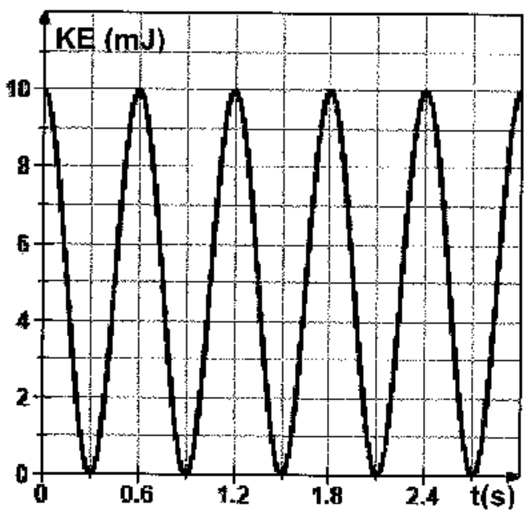


Figure 2

III-GS 2003 2nd

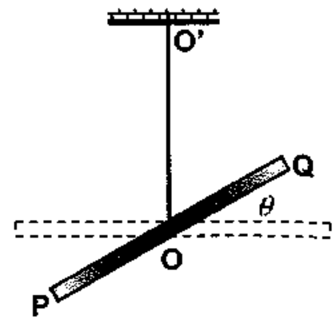
Moment of Inertia of a Rigid Rod

The object of this exercise is to determine, by two methods, the moment of inertia I_0 of a rigid and homogeneous rod PQ of negligible cross section, about an axis perpendicular to it through its midpoint O. In order to do that we consider the rod PQ of mass $M = 375\,g$ and of length $\ell = 20\,cm$. We neglect all frictions.

Take: $g = 10\,m.s^{-2}$ and $\pi^2 = 10$.

Part A
Case of a torsion pendulum

The rod PQ, being horizontal, its midpoint O is fixed to a vertical torsion wire OO' whose torsion constant is $C = 5\times10^{-4}\ SI$; the other end O' of the wire is fixed to a support. We thus obtain a torsion pendulum. The rod PQ, in the horizontal plane, is shifted from its equilibrium position around the vertical axis OO' by $\theta_m = 0.1$ rad in a direction taken as positive and is released from rest at the instant $t_0 = 0$. PQ thus oscillates around OO' in the horizontal plane around its equilibrium position.



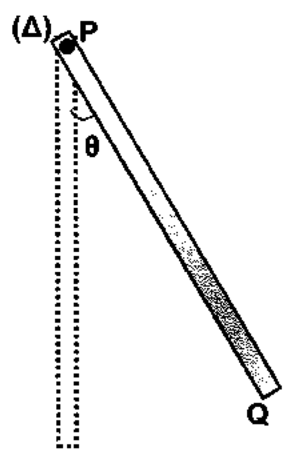
At any instant t during its motion, the position of the rod is defined by its angular elongation θ with its equilibrium position.

1. a) Write, at the instant t, the expression of the mechanical energy ME of the pendulum as a function of I_0, C, θ the angular speed θ'.
   b) Calculate the value of ME.
   c) Derive the second order differential equation that describes the motion of the pendulum.
   d) Prove that the expression of the proper period T_0 can be written as $T_0 = 2\pi\sqrt{\dfrac{I_0}{C}}$.
2. We measure the time t_1 for 10 oscillations; we find $t_1 = 100\,s$. Calculate I_0.

Part B
Case of a compound pendulum

The rod PQ, alone, is now free to rotate in the vertical plane around a horizontal axis (Δ) passing through its extremity P.

The rod PQ shifted by an angle $\theta_m = 0.1\,rad$ from its equilibrium position and is then released from rest at the instant $t_0 = 0$. The rod (Δ) thus oscillates around its equilibrium position.



At any instant t, the position of the rod is defined by the angular elongation θ with its equilibrium position. The horizontal plane containing (Δ) is taken as the gravitational potential energy reference.

Take: for $\theta \le 0.2$ rad, $\sin\theta = \theta$ and $1-\cos\theta = \dfrac{\theta^2}{2}$ (θ in rad).

1. a) Write the expression of the mechanical energy ME of the system (rod, Earth) at any instant t as a function of M, g, ℓ, θ the angular speed θ' and the moment of inertia I_1 of the rod about the axis (Δ).
   b) Calculate the value of the ME.
   c) Derive the differential equation that describes the motion of the rod.
   d) Prove that the expression of the proper period T_0' may be written as $T_0' = 2\pi\sqrt{\dfrac{2I_1}{Mg\ell}}$.
2. We measure the time t_2 of 10 oscillations of the rod. We find $t_2 = 7.3\,s$. Calculate I_1.
3. Knowing that $I_1 = I_0 + \dfrac{M\ell^2}{4}$, determine again the value of I_0.

Answer Key

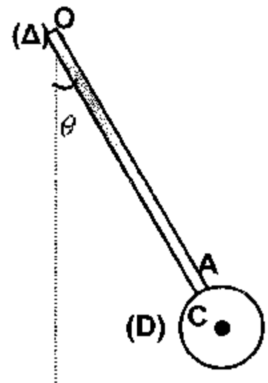
Part A 2. $I_0 = 1.25\times10^{-3}\ kg.m^2$ **Part B** 2. $I_1 = 5\times10^{-3}\ kg.m^2$.

GS 2002 2nd
Clock Pendulum

A clock pendulum may be represented by a homogeneous disk (D), of center C, fixed at the extremity A of a homogeneous rod OA.

Part A
Characteristics of the motion of the clock pendulum

In this part, friction is neglected.



The clock pendulum is a compound pendulum that may oscillate around a horizontal axis (Δ) passing through O (adjacent figure) during oscillations of small amplitude θ_m of proper period T_0, the pendulum passes through the equilibrium position with an angular speed $0.3\text{rad}/s$.

Given: $OA = 100\,cm$, $g = 10m/s^2$, radius of disk $AC = 10\,cm$;

$\pi = 3.14$; mass of rod $m' = 0.5\,kg$, mass of disk $m = 1\,kg$.

For small angles $\sin\theta = \theta\,\text{rad}$.

Moment of inertia of the clock pendulum about (Δ) is: $I = 1.38\,kg.m^2$.

Take the horizontal plane passing through G_0, the center of mass of the pendulum in its equilibrium position, as a gravitational potential energy reference.

1. a) Indicate the equilibrium position of the pendulum.
   b) Calculate the angular momentum of the pendulum while passing through the equilibrium position.
   c) Determine the sum of the moments of the forces acting on the pendulum while passing through the equilibrium position.
   d) Apply the theorem of angular momentum to determine the value of the angular acceleration of the pendulum while passing through the equilibrium position.
   Deduce that the maximum angular speed of the pendulum is 0.3 rad/s.
2. a) Show that the center of mass G of the pendulum is at a distance $a = OG = 90\,cm$ from O.
   b) Determine the mechanical energy of the system [Pendulum, Earth] for any angular elongation θ, in terms of m, m', I, a, g, θ & θ'.
   c) This mechanical energy is conserved. Why? Deduce the value of θ_m.
   d) Determine the differential equation that describes the periodic motion of the pendulum, knowing that for small angles $\sin\theta = \theta$. Calculate the value of T_0.

Part B
Driving the oscillations of the clock pendulum

In fact, the pendulum performs oscillations of pseudo-period T. If the motion of the pendulum is not driven, the oscillations tend to be damped.

1. Is the pseudo-period T greater, equal or smaller than T_0?
2. Why do the oscillations of the pendulum tend to be damped?
3. The driving of the oscillations is done by the very slow descending of a solid (S) of mass $M = 2\,kg$. Every week, (S) descends by a height $h = 1.5\,m$ and is raised back to its initial position within 10 seconds by means of an electric motor.
   Calculate the average power furnished by the electric motor.

Answer Key

Part A 1. b) $\sigma_0 = 0.414\, kg.m^2/s$. 2.c) $\theta_m \approx 5.5^0$

Part B 3. $3W$.

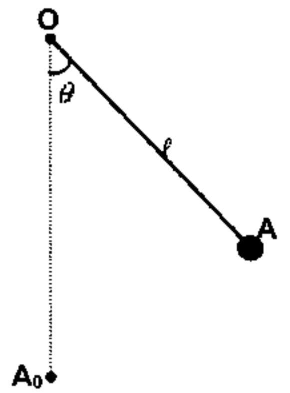
V-GS 2001 1st
Clock Pendulum

A simple pendulum is formed of a particle, of mass $m = 100\, g$, fixed at the end A of a rod OA of negligible mass and of length $OA = \ell = 25\, cm$.

This pendulum oscillates without friction about a horizontal axis (Δ) passing through O. The amplitude of oscillations is θ_m.

Take the reference level of the gravitational potential energy, the horizontal plane passing through A_0 the equilibrium position of A.

Take: $g = 10\, m s^{-2}$ & $\pi^2 = 10$.



Part A
Free undamped oscillations

1. Determine the expression of the mechanical energy of the system (pendulum, Earth) in terms of m, g, ℓ, θ & θ' where θ and θ' are respectively the angular abscissa and the angular velocity of the pendulum at any instant t.
2. Derive the second order differential equation that describes the motion of the given pendulum.
3. What condition must θ_m satisfy so that the motion of the pendulum is angular simple harmonic? Determine, in this case, the expression of the proper period T_0 of the pendulum and calculate its value.

Part B
Driven oscillations

The pendulum of a clock can be taken as the preceding pendulum. When the oscillations are not driven, we notice that the amplitude decreases from 10^0 to 8^0 within 5 oscillations.

1. What causes this decrease in the amplitude?
2. Is the motion of the pendulum periodic or pseudo-periodic?
3. The oscillations of the pendulum are now driven by means of a convenient apparatus. Calculate the average power of this apparatus.

Answer Key

Part A 3. $T_0 = 2\pi\sqrt{\frac{\ell}{g}} = 1s$.

Part B 3. $P = 2.74 \times 10^{-4}\, W$.

GS 2014 1st
Mechanical Oscillations

A simple pendulum consists of a particle of mass $m = 100g$, fixed at the free end of a massless rod OA of length $\ell = 0.45m$.

The pendulum may oscillate in the vertical plane, around a horizontal axis (Δ) passing through the upper extremity O of the rod.

The pendulum is initially at rest in its equilibrium position. At the instant $t_0 = 0$, the particle is launched horizontally in the positive direction as indicated in figure 1, with a velocity $\overrightarrow{v_0}$ of magnitude $v_0 = 0.3m/s$.

At an instant t, the angular abscissa of the pendulum and the algebraic value of the velocity of the particle are θ and v respectively.

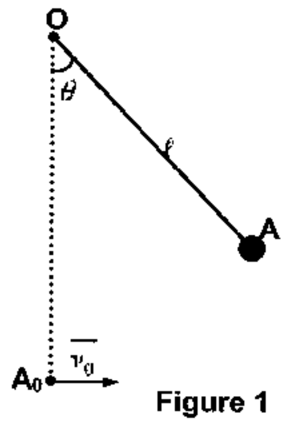


Figure 1

Take:

- ✘ The horizontal plane passing through A_0, the position of A at equilibrium, is taken as the reference level for the gravitational potential energy.
- ✘ $g = 10m/s^2$.
- ✘ For small angles $\cos\theta = 1 - \dfrac{\theta^2}{2}$ and $\sin\theta \approx \theta$ where θ in rad.

Part A

The forces of friction are negligible

1. a) Show that the mechanical energy of the system (pendulum, Earth) at the instant $t_0 = 0$ is $ME_0 = 4.5\,mJ$.
   b) Determine, at the instant t, the expression of the mechanical energy of the system (pendulum, Earth) in terms of m, v, ℓ, g and θ.
   c) Deduce the maximum angle of deviation θ_m performed by the pendulum.
2. a) Derive the differential equation in θ that describes the motion of the pendulum knowing that $v = \ell\dfrac{d\theta}{dt}$.
   b) Deduce the expression of the proper angular frequency w_0 and that of the proper period T_0 in terms of ℓ and g.
   c) Calculate the values of T_0 and w_0.
3. The time equation of motion of the pendulum is of the form $\theta = \theta_m \sin(w_0 t + \varphi)$. Determine φ.
4. Figure (2) shows three curves that represent the kinetic energy KE, the gravitational potential energy GPE and the mechanical energy ME of the system (pendulum, Earth).

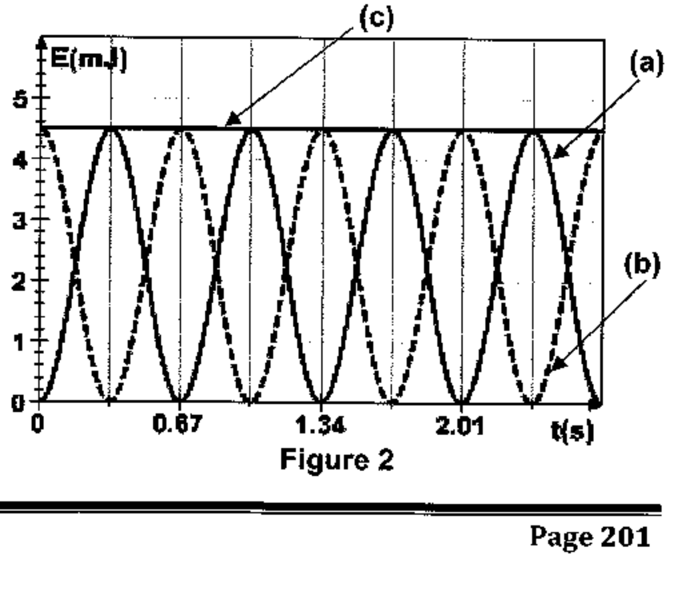


Figure 2

a) Identify each one of the curves a, b and c in the figure.

b) Pick up from figure 2 the value of the period T_E of the variations of the energy.

c) Deduce the relation between T_E and T_0.

Part B
Oscillations and friction

In reality, the forces of friction are not negligible. The variations of the angular abscissa θ of the pendulum as a function of time are represented by the graph of figure 3.

1. Referring to the graph:
   a) Indicate the type of oscillations performed by the pendulum.
   b) Determine the duration T of one oscillation.
   Compare T and T_0.
2. Knowing that the kinetic energy of the pendulum at the instant $t = 2T$ is $2.74\,mJ$, determine the average power furnished to the pendulum in order to compensate the loss in energy between 0 and $2T$.

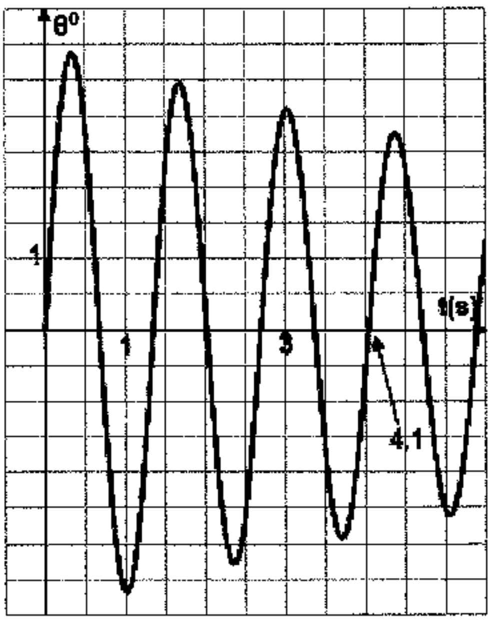


Figure 3

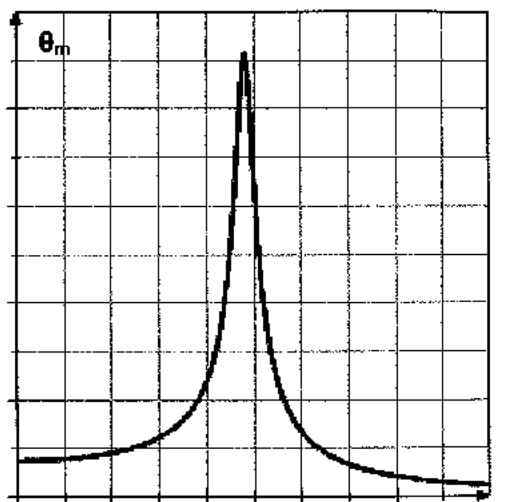


Figure 4

Part C
Oscillations and exciter

The pendulum undergoes periodic excitations of adjustable angular frequency ω_e.

We record for each value of ω_e the value of the amplitude θ_m of the oscillations of the pendulum, and we trace the graph of $\theta_m = f(\omega_e)$ represented in figure 4.

1. a) Name the phenomenon that takes place in the graph.
   b) Give the value of the angular frequency ω_e so that the amplitude of oscillations is maximum.
2. An appropriate system may increase slightly the forces of friction.
   Redraw figure 4 and draw roughly the shape of the curve giving the variations of the amplitude θ_m of oscillations of the pendulum in terms of the angular frequency ω_e of the excitations.

2013 1st
sion Pendulum

The aim of this exercise is to study the motion of a torsion pendulum in three different situations.

Consider a torsion pendulum that is constituted of a homogenous disk (D), of negligible thickness, suspended from its center of gravity O by a vertical torsion wire connected at its upper extremity to a fixed point O' (Figure 1).

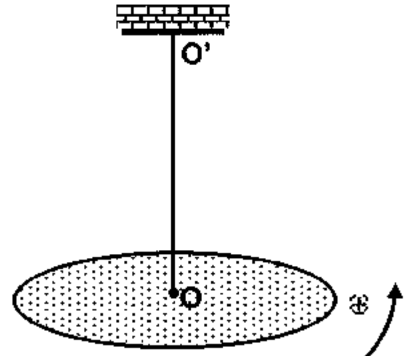


Figure 1

Given:

- ✗ torsion constant of the wire : $C = 0.16\, m.N/\text{rad}$;
- ✗ moment of inertia of the disk with respect to the axis OO': $I = 25\times10^{-4}\ kg.m^2$.

Part A
Free un-damped oscillations

The disk is in its equilibrium position. It is rotated around OO', in a direction considered positive, by an angle $\theta_m = 0.1\,\text{rad}$ (Figure 1). The disk is then released without initial velocity at the instant $t_0 = 0$. Take the horizontal plane passing through O as a gravitational potential energy reference.

At the instant t, the angular abscissa of the disk is θ and its angular velocity is $\theta' = \dfrac{d\theta}{dt}$.

1. Write the expression of the mechanical energy ME of the system (pendulum, Earth) in terms of I, θ, C and θ'.
2. Suppose that the forces of friction are negligible.
   a) Derive the differential equation, in θ, that describes the motion of the disk.
   b) The time equation of the motion of the disk has the form: $\theta = \theta_m \sin(w_0 t + \varphi)$.
   Determine w_0 and φ.
   c) Determine the angular velocity of the disk when it passes through its equilibrium position for the first time.

Part B
Free damped oscillations

In reality, the disk is subjected to a force of friction whose moment with respect to OO': $\mathrm{M} = -h\,\theta'$ where h is a positive constant.

1. Applying the theorem of angular momentum on the disk, show that the differential equation, in θ describing its motion is written as: $\theta'' + \dfrac{h}{I}\theta' + \dfrac{C}{I}\theta = 0$.
2. Determine, in terms of h and θ', the expression $\dfrac{d(ME)}{dt}$ (the derivative, with respect to time, of the mechanical energy ME of the system [pendulum, Earth]).
   Deduce the sense of the variation of ME.

Part C
Forced oscillations

The pendulum is at rest and at its equilibrium position. An exciter (E), coupled to the disk, provides it with periodic excitations of adjustable pulsation w_e. When we vary w_e of (E), the amplitude θ_m of motion of the disk takes a maximum value of $0.25\,\text{rad}$ for $w_e = w_r$.

1. Name the physical phenomenon that takes place.

2. Indicate the approximate value of w_r .

3. Sketch the shape of the curve that represents the variations of the amplitude θ_m as a function of w_e .

III-GS 2012 1st

Oscillation & Rotation of a Mechanical System

A rigid rod AB, of negligible mass and of length $\ell = 2\,m$, may rotate, without friction, around a horizontal axis (Δ) perpendicular to the rod through its midpoint O. On this rod, and on opposite sides of O, two identical particles (S) and (S'), each of mass $m = 100\,g$, may slide along AB.

Take: the gravitational acceleration on the Earth $g = 10\,m/s^2$;

For small angles: $\cos\theta = 1 - \frac{\theta^2}{2}$ and $\sin\theta = \theta$ in rad.

Part A

Oscillatory motion

The particle (S) is fixed on the rod at point C at a distance $OC = \frac{\ell}{4}$ and the particle (S') is fixed at point B (Figure 1). G is the center of gravity of the system (P) formed of the rod and the two particles. Let $OG = a$ and I_0 be the moment of inertia of (P) with respect to the axis (Δ).

We shift (P) by a small angle θ_m, about (Δ), from its stable equilibrium position, in the positive direction as shown on the figure, and then released without initial velocity at the instant $t_0 = 0$; (P) thus oscillates, around the axis (Δ) with a proper period T.

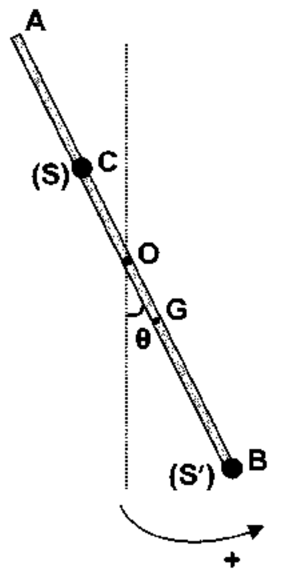


Figure 1

At an instant t, the angular abscissa of the compound pendulum, thus formed, is θ; (θ is the angle formed between the rod and the vertical passing through O), and its angular velocity is $\theta' = \frac{d\theta}{dt}$.

We neglect all frictional forces and take the horizontal plane through O as a gravitational potential energy reference.

1. Show that $a = \frac{\ell}{8}$.
2. Show that $I_0 = \frac{5}{16} m\ell^2$.
3. Write, at an instant t, the expression of the mechanical energy of the system [Earth, (P)] in terms of I_0, m, a, g, θ and θ'.
4. Derive the second order differential equation in θ that describes the motion of (P).
5. Deduce, in terms of ℓ and g, the expression of T. Calculate its value on the Earth.
6. The system (P) oscillates now on the Moon. In this case, the proper period, for small oscillations, is T'. Compare, with justification, T' and T.

Part B

Rotational motion

In this part, the particles (S) and (S') are fixed at A and B respectively (Figure 2).

At the instant $t_0 = 0$, we launch the system (P') thus formed, around (Δ) with an initial angular velocity $\theta'_0 = 2\,\text{rad}/s$; (P') then turns, in the vertical plane around (Δ). At an instant t, the angular abscissa of the rod, with respect to the vertical passing through O, is θ, and its angular velocity is $\theta' = \dfrac{d\theta}{dt}$.

During rotation, (P') is acted upon by a couple of forces of friction whose moment, with respect to (Δ) is $\text{M} = -h\,\theta'$, where h is a positive constant.

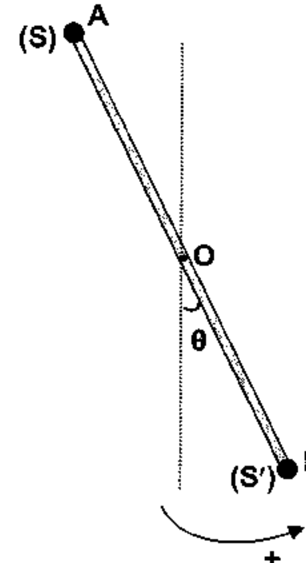


Figure 2

1. Give the name, at an instant t, of the couple and the forces acting on (P').
2. Show that the resultant moment of the couple and of the forces, with respect to (Δ), is equal to the moment $\text{M} = -h\,\theta'$.
3. Show that the moment of inertia of (P') about (Δ) is $I = 0.2\,kg.m^2$.
4. Using the theorem of angular momentum $\dfrac{d\sigma}{dt} = \sum \text{M}_{ext}$, show that the differential equation in σ is written as: $\dfrac{d\sigma}{dt} + \dfrac{h}{I}\sigma = 0$ where σ is the angular momentum of (P'), about (Δ).
5. Verify that $\sigma = \sigma_0\, e^{-\frac{h}{I}t}$ is a solution of the differential equation [σ_0 is the angular momentum of (P'), about (Δ), at the instant $t_0 = 0$].
6. The variation of σ as a function of time is represented by the curve of Figure 3. On this figure, we draw the tangent to the curve at point D at the instant $t_0 = 0$.
   a) The curve of Figure 3 is in agreement with the solution of the differential equation. Why?
   b) Determine the value of h.

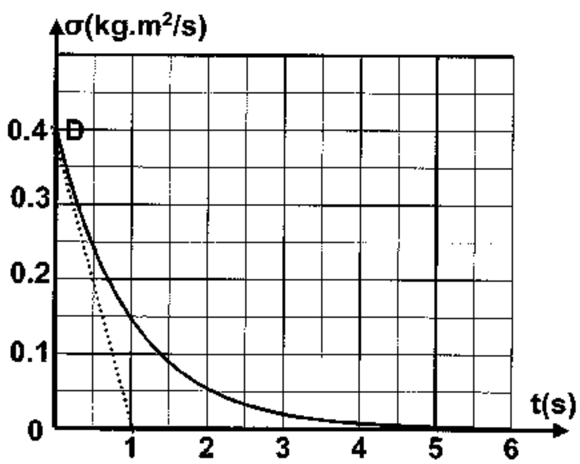


Figure 3

IV-GS 2011 2nd
Mechanical Oscillators

The parts A and B are independent. We neglect friction in all this exercise.

Part A
Simple pendulum

A simple pendulum (P) is formed of a particle G of mass m connected to one end of an inextensible string, of negligible mass and of length ℓ ; the other extremity is connected to a fixed point A. The pendulum is shifted by an angle θ_m from its equilibrium position AG_0 to the position AG_i, and then released from rest at the instant $t_0 = 0\,s$; thus it oscillates with the amplitude θ_m.

At an instant t, the position of AG is defined by θ, the angular abscissa relative to its equilibrium position, and v is the algebraic measure of the velocity of G (Figure 1).

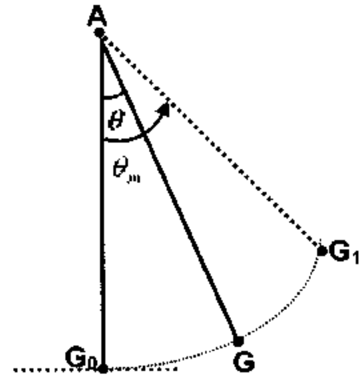


Figure 1

Take the horizontal plane through G_0 as a gravitational potential energy reference.

1. Determine the expression of the mechanical energy of the system [(P), Earth] in terms of m, g, ℓ, v and θ.
2. Derive the second order differential equation in θ that governs the motion of this pendulum.
3. a) What condition must θ satisfy in order to have a simple harmonic motion?
   b) Deduce, in this case, the expression of the proper period T_0 of the oscillations.
   c) Write down the time equation of motion, in the case $\theta_m = 0.1\,\text{rad}$.

Take: $g = 10\,m/s^2$; $\ell = 1\,m$ and $\pi^2 = 10$.

Part B
Horizontal elastic pendulum

A solid (S) of mass m may slide on a horizontal plane; it is connected to a spring (R) of stiffness $k = 4\,N/m$. When (S) is in equilibrium, its center of mass G is found vertically above the point O, taken as origin on the horizontal axis of abscissa.

(S), shifted from its equilibrium position, is released from rest at the instant $t_0 = 0$. At an instant t, the abscissa of G is x and the algebraic value of its velocity is v.

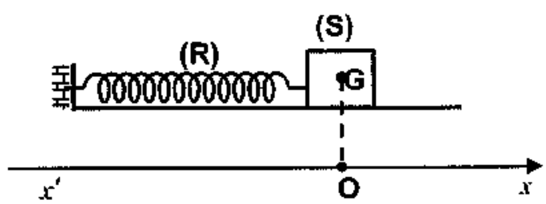


Figure 2

A convenient apparatus gives the variation of x as a function of time (Figure 3).

The horizontal plane containing G is taken as a gravitational potential energy reference.

1. Derive the second order differential equation in x that governs the motion of G.
2. The solution of this differential equation is of the form: $x = x_m \cos\left(\frac{2\pi}{T_0}t + \varphi\right)$, where x_m, T_0 and φ are constants.

   Referring to the graph of figure (3), give the values of x_m, T_0 and determine φ.

3. a) Determine the expression of the proper period T_0 in terms of m and k.
   b) Deduce m.
4. a) Referring to the graph of figure (3), give the instants at which the elastic potential energy is maximum.
   b) Calculate then the value of the mechanical energy of the system [(S), (R), Earth].

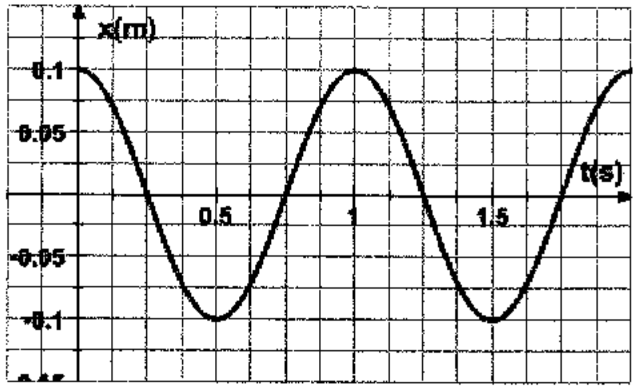


Figure 3

Part C

Behavior of the pendulums on the Moon

We suppose that the two preceding pendulums are now on the Moon.

Statement 1	Statement 2	Statement 3
T_0 does not vary	T_0 increases	T_0 decreases

Tell, with justification, for each pendulum, which of the statements in the following table is true.

V-GS 2011 1st
Moment of Inertia of a Rod

Consider a homogeneous and rigid rod AB of negligible cross-section, of length $\ell = 1\,m$ and of mass $m = 240\,g$. This rod may rotate about a horizontal axis (Δ) perpendicular to it through its midpoint O. The object of this exercise is to determine, by two methods, the moment of inertia I_0 of the rod about the axis (Δ). The vertical position CD of the rod is considered as an origin of angular abscissa. Neglect all friction.

Take: $g = 10\,m/s^2$; $\pi^2 = 10$; $\sqrt{3} = 1.732$;

$\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \dfrac{\theta^2}{2}$ for small angles θ measured in radians.

Part A

First method

The rod, starting from rest at the instant $t_0 = 0$, rotates around (Δ) under the action of a force F whose moment about (Δ) is constant of magnitude $\mathrm{M} = 0.1\,m.N$ (Figure 1).

At an instant t, the angular abscissa of the rod is θ and its angular velocity is θ'.

1. a) Show that the resultant moment of the forces acting on the rod about (Δ) is equal to M.
   b) Determine, using the theorem of angular momentum, the nature of the motion of the rod between t_0 and t.
   c) Deduce the expression of the angular momentum σ of the rod, about (Δ), as a function of time.

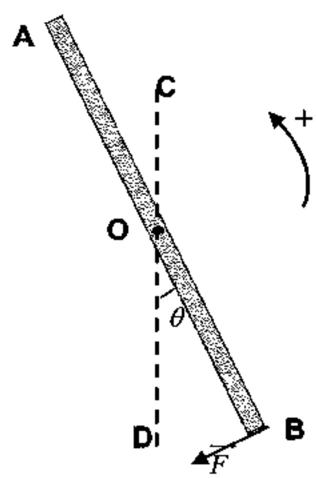


Figure 1

2. Determine the value of I_0, knowing that at the instant $t_1 = 10\,s$, the rotational speed of the rod is 8 turns/s.

Part B
Second method

We fix, at point B, a particle of mass $m' = 160\,g$. The system (S) thus formed constitutes a compound pendulum whose center of mass is G. (S) may oscillate freely, about the axis (Δ).

We shift (S), from its stable equilibrium position, by a small angle and we release it without velocity at the instant $t_0 = 0$.

At an instant t, the angular abscissa of the pendulum is θ and its angular velocity is $\theta' = \dfrac{d\theta}{dt}$.

The horizontal plane through O is taken as a gravitational potential energy reference.

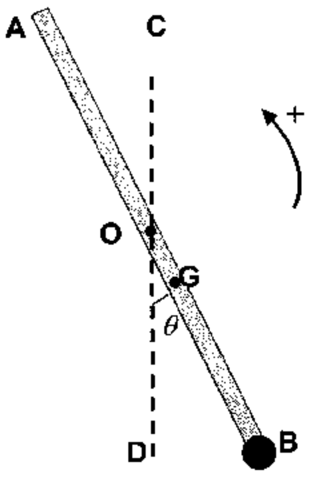


Figure 2

1. Determine:
   a) the position of G relative to O $(a = OG)$, in terms of m, m' and ℓ;
   b) the moment of inertia I of (S) about (Δ), in terms of I_0, m' and ℓ.
2. Determine, at the instant t, the mechanical energy of the system [(S), Earth], in terms of I, θ', θ, m, m', a and g.
3. a) Derive the second order differential equation that describes the motion of (S).
   b) Deduce the expression of the proper period T of the oscillations of (S), in terms of I_0, m', ℓ and g.
4. The duration of 10 oscillations of the pendulum is $17.32\,s$.
   Determine the value of I_0.

VI-GS 2009 2nd
Compound Pendulum

The aim of this exercise is to study the variation of the proper period T of a compound pendulum as a function of the distance a, of adjustable value, separating the axis of oscillation from the center of mass of this pendulum, and to show evidence of some properties associated to this distance a.

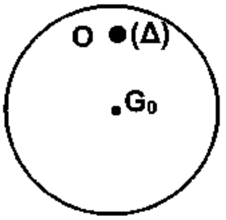


Figure 1

We consider a homogeneous disk (D) of mass $m = 200\,g$, free to rotate without friction around a horizontal axis (Δ) perpendicular to its plane through a point O (Figure 1).

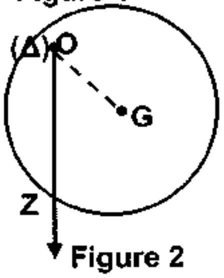


Figure 2

I_0 is the moment of inertia of (D) about the axis (Δ_0) parallel to (Δ) and passing through its center of mass G and I its moment of inertia about the axis (Δ), (Δ_0) being at a distance $a = OG$ from (Δ), so that: $I = I_0 + m a^2$.

The gravitational potential energy reference is the horizontal plane passing through the center of mass G_0 of (D) when (D) is in the position of stable equilibrium (Figure 1).

(D) is made to oscillate around (Δ) and we measure the value of the proper period corresponding to each value of a.

Take: $g = 10\, m/s^2$; $\pi^2 = 10$;

For small angles (in radian); $\sin\theta = \theta$ and $\cos\theta = 1 - \frac{\theta^2}{2}$, (θ in rad).

Part A

Theoretical study

(D) is shifted from its stable equilibrium position by a small angle θ_m and is then released from rest at the instant $t_0 = 0$. (D) thus oscillates around the axis (Δ) with a proper period T.

At an instant t, the angular abscissa of the pendulum is θ and its angular velocity is: $\theta' = \frac{d\theta}{dt}$ (Figure 2)

1. Write down, at the instant t, the expression of the mechanical energy of the system (pendulum, Earth) in terms of I, m, a, g, θ & θ'.
2. a) Derive the second order differential equation in θ that describes the motion of (D).

   b) Deduce that the expression of the period T of this pendulum is given by: $T = 2\pi\sqrt{\frac{I}{m\,ga}}$.
3. T_1 and T_2 are respectively the periods of the pendulum when it oscillates around (Δ) that passes successively through O_1 and O_2 where $O_1G = a_1$ and $O_2G = a_2$. The oscillations have the same period $(T_1 = T_2)$.

   I_1 and I_2 are respectively the moments of inertia of the pendulum around (Δ) that passes successively through O_1 and O_2.

   a) Relations between the position of the center of gravity and moment of inertia.

   i- Determine a relation among I_1, I_2, a_1 and a_2.

   ii- Deduce that: $I_0 = m\,a_1\,a_2$.

   b) The proper period T' of a simple pendulum of length ℓ, for oscillations of small amplitude, is given by the expression: $T' = 2\pi\sqrt{\frac{\ell}{g}}$.

   Show that, when the value of T' is equal to that of T_1, we obtain $\ell = a_1 + a_2$.

Part B

Experimental study

We measure the value of the period T of the pendulum for each value of a. The obtained measurements allow us to trace the curve giving the variation of T as a function of a. The straight line of equation $T = 1.1s$ intersects this curve in two points A and B. (Figure 3)

1. a) Referring to the curve, give the values of a_1 and a_2 corresponding to the period $T = 1.1s$.

b) Deduce the value of I_0 and that of ℓ.

2. According to the curve of figure 3, T takes a minimum value $(T_{min} = 1.05\,s)$ for a certain value a' of a.

   a) Give, using the curve, the value of a' corresponding to T_{min}.

   b) Determine, by calculation, the value of a' and that of T_{min}.

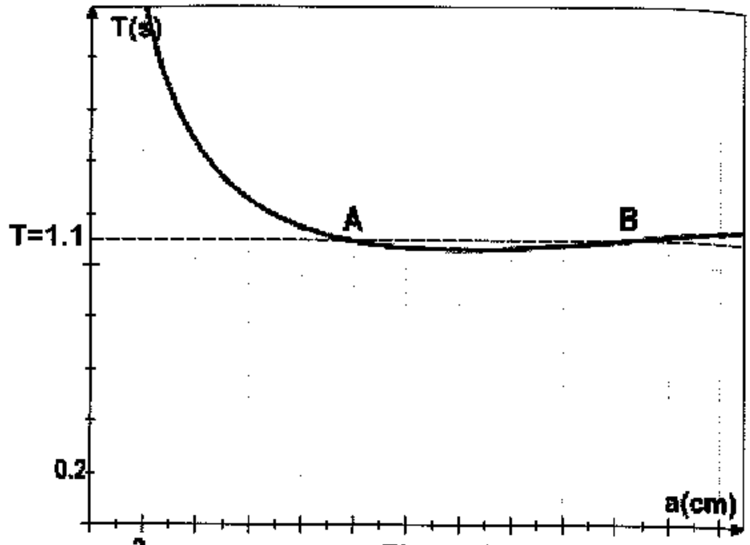


Figure 3

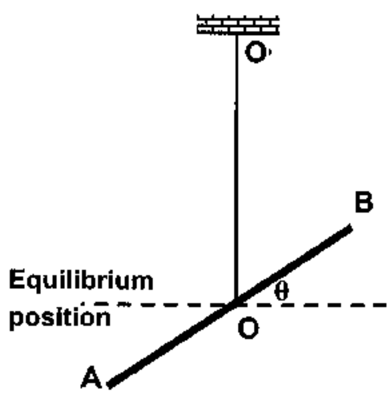
VII-GS 2009 1st
Torsion Pendulum

The object of this exercise is to determine the moment of inertia I of a homogeneous rod AB with respect to an axis perpendicular to the rod at its midpoint and the torsion constant C of a wire OO' of negligible mass.

The rod has a mass M and a length $AB = \ell = 60\,cm$.

A torsion pendulum $[P]$ is obtained by fixing the mid-point of AB to one end O of the wire while the other end O' is fixed to a support.

The rod is shifted, from its equilibrium position, by a small angle θ_m in the horizontal plane and it is released from rest at an instant $t_0 = 0$. The rod thus may turn in the horizontal plane about an axis (Δ) passing through OO'. At an instant t during motion, the angular abscissa of the rod is θ and its angular velocity is: $\theta' = \frac{d\theta}{dt}$.



The horizontal plane containing the rod is taken as a gravitational potential energy reference. We neglect any force of friction and take $\pi^2 = 10$.

Part A
Theoretical study

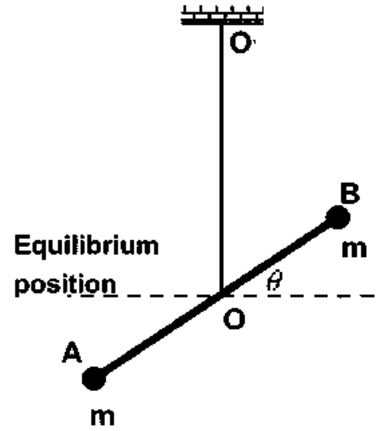
1. Give, at the instant t, the expression of the mechanical energy ME of the system [$[P]$, Earth] in terms of I, C, θ & θ'.
2. a) Write the expression of ME when $\theta = \theta_m$.

   b) Determine, in terms of C, θ_m & I, the expression of the angular speed of $[P]$ as it passes through its equilibrium position.

3. Derive the second order differential equation in θ that governs the motion of $[P]$.
4. Deduce that the motion of $[P]$ is sinusoidal.
5. Determine the expression of the proper period T_1 of the pendulum in terms of I & C.

Part B
Experimental study

1. By means of a stopwatch, we measure the duration t_1 of 20 oscillations and we obtain $t_1 = 20\,s$. Determine the relation between I and C.



2. At each extremity of the rod we fix a particle of mass $m = 25\,g$. We thus obtain a new torsion pendulum $[P']$ whose motion is also rotational sinusoidal of proper period T_2.
   a) Determine the moment of inertia I' of the system (rod + particles) with respect to the axis (Δ) in terms of I, m, and ℓ.
   b) Write down the expression of T_2 in terms of I, m, C and ℓ.
   c) By means of a stopwatch, we measure the duration t_2 of 20 oscillations and we obtain $t_2 = 40\,s$.
   Determine a new relation between I and C.
3. Calculate the values of I and C.

VIII-GS 2008 1st
Compound Pendulum

A compound pendulum is formed of a rod AB of negligible mass, which can rotate without friction in a vertical plane around a horizontal axis (Δ) passing through a point O of the rod so that $OB = d$. A particle of mass M is fixed at point B and another particle C of mass $m < M$, which can slide on the part OA of the rod is placed at a distance $OC = x$ of adjustable value. Let $a = OG$ be the distance between O and the center of gravity G of the pendulum (Figure 1). The gravitational potential energy reference is the horizontal plane containing O.

Given: $g = 10 m/s^2$; $\pi^2 = 10$;

$$\sin\theta = \theta \text{ and } \cos\theta = 1 - \frac{\theta^2}{2}, (\theta \text{ in rad}) \text{ for } \theta < 10^0.$$

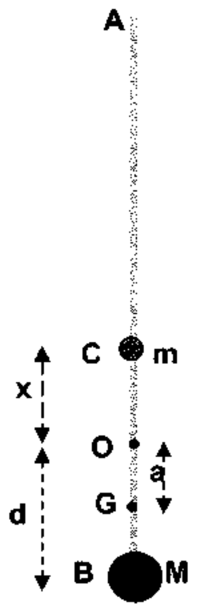


Figure 1

Part A
Theoretical study

1. Show that the position of G is given by: $a = \dfrac{M\,d - m\,x}{M + m}$.
2. Determine the expression of the moment of inertia I of the pendulum about the axis (Δ) in terms of m, x, M & d.

3. The pendulum thus formed is deviated by an angle θ_0 from its equilibrium position and then released from rest at the instant $t_0 = 0$. The pendulum then oscillates around the stable equilibrium position. At an instant t, the position of the pendulum is defined by the angular abscissa θ, the angle that the vertical through O makes with OG, and its angular velocity is $\theta' = \frac{d\theta}{dt}$.
   a) Write, at the instant t, the expression of the kinetic energy of the pendulum in terms of I & θ'.
   b) Show that the expression of the gravitational potential energy of the system (pendulum, Earth) is: $GPE = -(M+m)g\,a\,\cos\theta$.
   c) Write the expression of the mechanical energy of the system (pendulum, Earth) in terms of M, m, g, a, θ, I & θ'.
   d) Derive the second order differential equation in θ that governs the motion of the pendulum.
   e) Deduce that the expression of the proper period, for oscillations of small amplitude, has the form $T = 2\pi\sqrt{\frac{I}{(M+m)ga}}$.
   f) Deduce the expression of the period T, in terms of M, m, g, d & x.

Part B

Application: Metronome

A metronome is an instrument that allows adjusting the speed at which music is played.

The compound pendulum studied in part **A** represents a metronome where $M = 50\,g$, $m = 5\,g$ and $d = 2\,cm$. The graph of figure 2 represents the variations of the period T of this metronome as a function of the distance x.

Name	Indication	Period (in s)
Grave	very slow	$T = 1.5$
Lento	Slow	$1 \le T \le 1.1$
Moderato	Moderate	$0.6 \le T \le 0.75$
Prestissimo	very fast	$0.28 \le T \le 0.42$

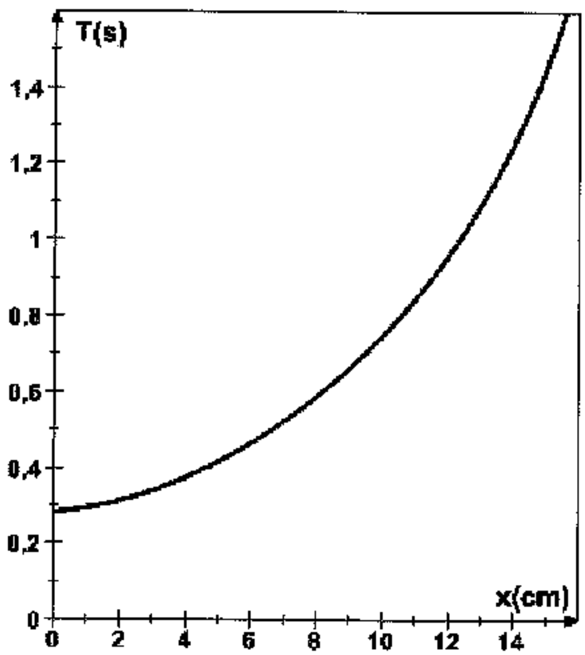


Figure 2

1. Determine, in this case, the expression of the period T of the metronome as a function of x.
2. The leader of the orchestra (conductor), using a metronome to play a distribution, changes the position of C along OA, to follow the rhythm of the musical piece. The rhythm is indicated by terms inherited from Italian for the classical distribution.

   Determine, using a method of your choice, the positions between which the leader of the orchestra may move C to adjust the speed to the rhythm **Lento**.

IX-GS 2007 2nd

Mechanical Oscillations

Consider a pierced disk (D), of mass $M = 59\,g$, that may rotate, without friction, about a horizontal axis (Δ) perpendicular to its plane through O, O being the center of the homogeneous disk before being pierced.

The center of mass G of the pierced disk (D) is at a distance a from O $(a = OG)$.

The object of this exercise is to determine the value of a and that of the moment of inertia I of the disk (D) with respect to the axis (Δ).

The horizontal plane through O is taken as a gravitational potential energy reference.

Take: $\sin\theta = \theta$ and $\cos\theta = 1 - \frac{\theta^2}{2}$ for small angles, θ being in radian; $g = 10 m/s^2$; $\pi^2 = 10$.

Part A

Compound pendulum

The disk (D) is at rest in its position of stable equilibrium. We shift it by a small angle θ_m and then we release it without velocity at the instant $t_0 = 0$.

The compound pendulum thus formed oscillates without friction on both sides of its equilibrium position with a proper period T_1 (Figure 1).

At an instant t, the position of (D) is defined by its angular abscissa θ that OG makes with the vertical OY, and its angular velocity is $\theta' = \frac{d\theta}{dt}$.

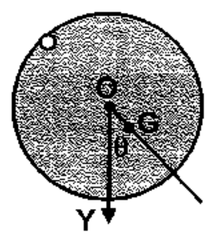


Figure 1

1. Write down, at the instant t, the expression of the kinetic energy of the pendulum in terms of I and θ'.
2. Show that the expression of the gravitational potential energy of the system (pendulum, Earth) is: $GPE = -M\,g\,a\cos\theta$.
3. Write down the expression of the mechanical energy of the system (pendulum, Earth) in terms of M, g, a, θ, θ' & I.
4. Derive the second order differential equation that governs the motion of (D).
5. Deduce that the expression of the proper period T_1, for small oscillations, can be written as : $T_1 = 2\pi\sqrt{\frac{I}{M\,g\,a}}$

Part B

Oscillating system

The disk (D) is now welded from its center to two identical and horizontal torsion wires OA and OB $(OA = OB)$ (Figure 2). The extremities A and B are fixed.

The torsion constant of each of the wires is $C = 2.8\times10^{-3}\,m.N$.

Starting from its stable equilibrium position, we turn (D) by a small angle θ_m around AB, confounded with (Δ); the two wires are twisted, in the same direction, by the same angle θ_m.

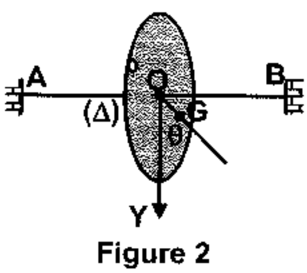


Figure 2

Released without velocity at the instant $t_0 = 0$, (D) starts to oscillate around the horizontal axis AB. At an instant t, the position of (D) is defined by its angular abscissa θ that OG makes with the vertical OY, (each wire is then twisted by θ) and its angular velocity is θ'. The oscillating system performs then a periodic motion of proper period T_2.

1. a) Write down, at an instant t, the expression of the torsion potential energy of the wires in terms of C & θ.
   b) Give then the expression of the potential energy of the system (oscillating system, Earth) in terms C, θ, M, g & a.
   c) Deduce the expression of the mechanical energy of the system (oscillating system, Earth).
2. Determine the expression of the proper period T_2 in terms of I, M, a, g & C.

Part C
Values of I and a

Knowing that the measured values of T_1 & T_2 are $T_1 = 4.77s$ and $T_2 = 2.45s$, use the results of parts **A** and **B.** Deduce the values of I and a.

X-GS 2007 1st
Solid In Rotation

Consider a rigid rod AB, of negligible mass and of length $AB = \ell = 80\,cm$. The rod may rotate around a horizontal axis (Δ), perpendicular to it through its midpoint O.

Two identical particles, each of mass $m = 10\,g$, may slide along this rod.

Take: $g = 10 m/s^2$ & $0.32\,\pi = 1$.

Part A
Work done by the couple of friction

We fix one of the two particles at the end A of the rod while the other particle is fixed at another point D, at a distance $\frac{\ell}{4}$ from O. G being the center of gravity of the system (S) formed of the rod and the two particles, we suppose $OG = a$.

Take as a gravitational potential energy reference, the horizontal plane through G when (S) is in the position of stable equilibrium (Figure 1).

Figure 1

1. Show that $a = \frac{\ell}{8}$.
2. (S) is in its stable equilibrium position. At the instant $t_0 = 0$, we communicate to (S) an initial kinetic energy $E_0 = 1.95 \times 10^{-4} J$; (S) oscillates then around (Δ), on both sides of its position of stable equilibrium. At an instant t, OG makes an angle θ with the vertical through O.
   a) Neglecting friction, show that:
      i- the expression of the gravitational potential energy of the system [(S), Earth] is: $GPE = 2mga(1-\cos\theta)$.
      ii- the value of the mechanical energy of the system [(S), Earth] is E_0.
      iii- the value of the angular amplitude of the motion of (S) is $\theta_m = 8^0$.

b) In reality, the forces of friction form a couple whose moment about the axis (Δ) is M. We suppose that M is constant. The measurement of the first maximum elongation of (S) is then $\theta_{1m} = 7^0$ at the instant t_1.

i- Determine the expression giving the variation of the mechanical energy of the system $[(S)$, Earth$]$ between t_0 and t_1 in terms of m, g, a, θ_{1m} and E_0.

ii- Deduce the value W of the work done by M between t_0 and t_1.

Part B

Moment of the couple of friction

We fix each particle on an extremity of the rod (figure 2). At the instant $t_0 = 0$, and we give (S), a rotational speed $N_0 = 1$ turn/s and we suppose that M keeps the same preceding value.

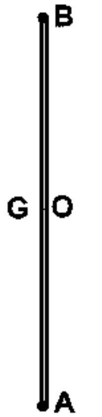


Figure 2

1. Show that the moment of inertia of (S) with respect to (Δ) is $I = 32\times10^{-4}\,kg.m^2$.
2. Show that the value of the angular momentum of (S) with respect to (Δ), at $t_0 = 0$, is $\sigma_0 = 2\times10^{-2}\,kg.m^2/s$.
3. a) Give the names of the external forces acting on (S).
   b) Show that the value of the resultant moment of these forces, with respect to (Δ), is M.
   c) Determine, applying the theorem of angular momentum, the expression of the angular momentum σ of (S) with respect to (Δ), in terms of M, t and σ_0.
4. Launched with the rotational speed $N_0 = 1$ turn/s, (S) stops at the instant $t' = 52.8\,s$.
   Determine then the value of M.

Part C

Relation between W and M

Referring to the parts **A** and **B**, verify that the work W is: $W = \mathrm{M}\times\theta_{1m}$.

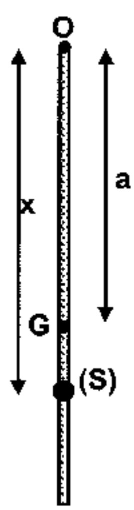
XI-GS 2006 2nd

Mechanical Oscillator

The purpose of this exercise is the study the response of a simple pendulum for the excitations imposed by a compound pendulum of an adjustable period.

In this purpose we consider a simple pendulum (R) and a compound pendulum (E). (R) is related into a plate, of negligible mass, allowing us to adjust the damping due to air. (E) is formed of a homogeneous rod of length $\ell = 1m$, of mass M and negligible section, on which a solid (S) can slide, supposed punctual and of mass $M' = M$.

(E) can oscillates around a horizontal axis (Δ) perpendicular to the rod with its upper extremity O (figure).



G is a center of gravity of the compound pendulum thus formed and I is the moment of inertia of this pendulum with respect to (Δ). We set $OG = a$ and we designate by x the distance between the position of (S) and O.

Take: $\sin\theta = \theta$, θ being in radian for $\theta \le 10^0$, $g = 10m/s^2$; $\pi^2 = 10$.

Part A
Theoretical study

The pendulum (E) is shifted from its stable equilibrium position by a small angle then released, without an initial velocity, at the instant $t_0 = 0$. (E) starts to oscillate around its stable equilibrium position. The forces of friction are negligible.

At an instant t, OG makes with the vertical passing through O an angle θ and (E) possess an angular velocity θ'. The horizontal plane passing through O is taken as a reference to gravitational potential energy.

1. Show that the expression of the mechanical energy of the system [(E), Earth] can be written: $ME = \frac{1}{2} I \theta'^2 - 2 M g a \cos\theta$.
2. Determine the differential equation of second order in θ that governs the motion of (E) for small angles $(\theta \leq 10^0)$.
3. a) Show that the expression of a is given by $a = \frac{\ell + 2x}{4}$.

   b) The moment of inertia of the rod alone with respect to (Δ) is $I_1 = M\frac{\ell^2}{3}$.

   Show that the expression of the moment of inertia I is written $I = \frac{M\left(\ell^2 + 3x^2\right)}{3}$.
4. Show that the expression of the proper period T of (E), in terms of x, is: $T = \sqrt{\frac{8\left(1 + 3x^2\right)}{3(1 + 2x)}}$.

Part B
Experimental study

1. The pendulum (R) is considered alone. We shift it by a small angle, from its equilibrium position, then released without initial velocity. The measure of time t_1 to perform 10 oscillations is given by $t_1 = 16.6\,s$.

   Calculate the duration T' of one oscillation.
2. We realize now, by the means of a spring, a coupling between (E) and (R) which are initially at rest, for each value of x, the pendulum (E) is shifted by a small angle from the equilibrium position then released without initial velocity; it oscillates (R). We suppose that (E) oscillates with a period equal to its proper period T.

   By changing x, we note that the amplitude θ_m of oscillations of (R) changes.

   a) The oscillations of (R) are called forced. Compare, then, for each value of x, the period of oscillations of (R) with respect to that of (E).

   b) Study of the period

   i- We give to x the value $0.3\,m$. In permanent regime, (R) performs oscillations of period T_1 and amplitude θ_{m1}. Calculate the value of T_1.

   ii- For $x = 0.65\,m$. In permanent regime, (R) performs oscillations of period $T_2 = 1.62\,s$ and amplitude θ_{m2}. Compare, with justification, θ_{m1} and θ_{m2}.

   c) For a given value of x, and in permanent regime, (R) oscillates with a maximal amplitude $\theta_{m\,max}$.

   i- Name the phenomenon that is placed in evidence.

   ii- Determine the value of x.

d) Trace the shape of the curve giving the variations of the amplitude θ_m of oscillations of (R) as a function of the period T of (E).

e) The plate of (R) is placed in a way to increase slightly the friction with the air. Trace, on the system of axis of the question (d), the shape of the curve giving the variations of the amplitude θ_m of oscillations of (R) in terms of the period T of (E).

XII-GS 2005 2nd

Moment of Inertia of a Disk

Consider a homogeneous disk (D) of mass $m = 400\,g$ and of radius $R = 10\,cm$.

The object of this exercise is to determine, by two methods, the moment of inertia I_0 of (D) about an axis (Δ_0) perpendicular to its plane through its center of mass. Neglect all frictions.

Take: $0.32\pi = 1$; $g = 10 m/s^2$; $\sin\theta = \theta_{rad}$ for small θ.

Part A

First method

The disk (D) is free to rotate about the horizontal axis (Δ_0) that is perpendicular to its plane through its center G (figure 1). This disk starts from rest, at the instant $t_0 = 0$, under the action of a force $\vec{F}$ of constant moment about (Δ_0) and of magnitude $\mathrm{M} = 0.2\,m.N$.

At the instant $t_1 = 5s$, (D) rotates then at the rotational speed $N_1 = 80$ turns/s.

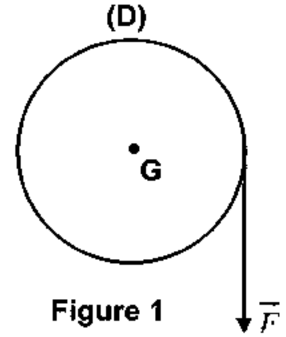


Figure 1

1. a) Give the names of all the external forces acting on (D) and represent them on a diagram.
   b) Show that the resultant moment of these forces, about (Δ_0), is equal to the moment M of the force $\vec{F}$.
   c) Specify, using the theorem of angular momentum, the nature of the motion of (D).
2. a) Deduce the expression of the angular momentum σ of the disk, about (Δ_0) as a function of t
   b) Calculate the value of I_0.

Part B

Second method

The disk (D) is free to rotate about a horizontal axis (Δ), perpendicular to its plane through a point O of its periphery.

We denote by I the moment of inertia of (D) about (Δ). We shift (D) from its equilibrium position, by a small angle θ_0 and then we release it without initial velocity, at the instant $t_0 = 0$.

The position of (D) is defined, at any instant t, by the angle θ that the vertical makes OZ with OG.

$\theta' = \dfrac{d\theta}{dt}$ represents the angular velocity of (D) at the instant t (figure 2).

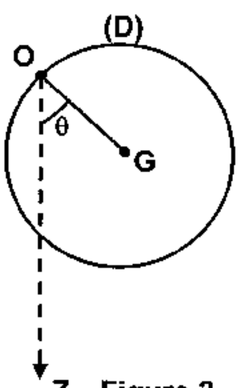


Figure 2

The horizontal plane passing through the point O is taken as a gravitational potential energy reference.

1. Determine, at the instant t, the mechanical energy of the system [(D), Earth], in terms of I , m , g , R, θ & θ'.

2. Derive the second order differential equation that describes the oscillatory motion of (D).

3. Deduce the expression of the period T of the oscillations of (D) in terms of I , m , g and R.

4. The time taken by the compound pendulum thus formed to perform 10 oscillations is $7.7\,s$.

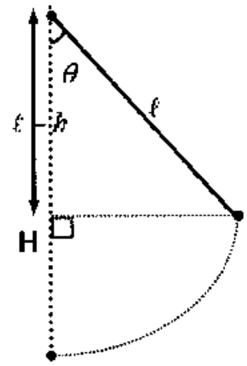
   Determine the value of I.

   Knowing that I_0 and I are related by the relation $I = I_0 + m R^2$, find again the value of I_0.

Sessions Solutions – GS

I-GS 2014 1st

Part A



1. a) $ME_0 = KE_0 + GPE_0$; (but $GPE_0 = 0$, on reference);

Then $ME_0 = \frac{1}{2} m v_0^2 + 0 = \frac{1}{2} \times 0.1 \times 0.3^2 = 4.5 \times 10^{-3} = 4.5\, mJ$.

b) At an instant t : $ME = KE + GPE = \frac{1}{2} m v^2 + mg\, h$;

But $\cos\theta = \frac{\ell - h}{\ell}$; then $h = \ell(1 - \cos\theta)$;

Thus, $ME = \frac{1}{2} m v^2 + m\, g\, \ell(1 - \cos\theta)$.

c) The forces of friction are negligible, then the mechanical energy is conserved $ME = ME|_{\theta_m} = KE|_{\theta_m} + GPE|_{\theta_m} = ME_0$;

At the maximum angle of deviation, the speed is zero, so $KE|_{\theta_m} = 0$;

Thus, $m\, g\, \ell(1 - \cos\theta_m) = ME_0$; $\theta_m = \cos^{-1}\left(1 - \frac{ME_0}{m\, g\, \ell}\right) = \cos^{-1}\left(1 - \frac{4.5 \times 10^{-3}}{0.1 \times 10 \times 0.45}\right) \approx 8.1^0$.

2. a) We have $v = \ell \frac{d\theta}{dt} = \ell\theta'$.

So $ME = \frac{1}{2} m\, \ell^2 \theta'^2 + m\, g\, \ell(1 - \cos\theta) = \frac{1}{2} m\, \ell^2 \theta'^2 + \frac{1}{2} m\, g\, \ell\, \theta^2$ (for small angles);

The mechanical energy is constant, then $\frac{d(ME)}{dt} = 0$; $\frac{1}{2} m\, \ell^2 \times 2\,\theta'\theta'' + \frac{1}{2} mg\, \ell \times 2\,\theta\theta' = 0$;

$m\, \ell^2 \theta' \left(\theta'' + \frac{g}{\ell}\theta\right) = 0$, (but $m\, \ell^2 \theta' \neq 0$, the system is in motion); thus $\theta'' + \frac{g}{\ell}\theta = 0$.

b) The differential equation that governs the motion of this particle is of 2nd order of the form $\theta'' + w_0^2\, \theta = 0$ where $w_0^2 = \frac{g}{\ell}$, then $w_0 = \sqrt{\frac{g}{\ell}}$.

The motion of this pendulum is periodic of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{\ell}{g}}$.

c) We have $w_0 = \sqrt{\frac{10}{0.45}} \approx 4.71\, \text{rad}/s$ & $T_0 = \frac{2\pi}{w_0} = \frac{2\pi}{4.71} \approx 1.33\, s$.

3. We have $\theta = \theta_m \sin(w_0 t + \varphi)$;

At $t = 0$, $\theta = 0$; $\theta_m \sin(\varphi) = 0$, then $\varphi = 0$ or π (rad);

But $\theta'|_{t=0} > 0$ & $\theta' = \theta_m w_0 \cos(w_0 t + \varphi)$, so $\theta_m w_0 \cos(\varphi) > 0$ then $\varphi = 0$.

4. a) The mechanical energy is conserved, then its graphical representation is a horizontal straight line, thus it corresponds to the curve (c).

At $t = 0$, the kinetic energy $KE_0 = 4.5mJ \neq 0$, then its curve is (b) ;

The graph of the gravitational potential energy corresponds to the curve (a).

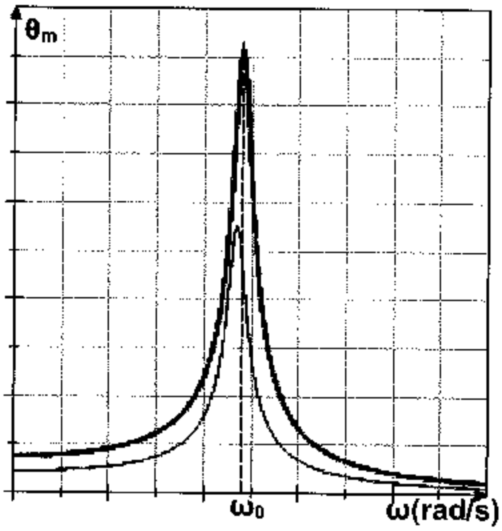
b) The energies are periodic of period $T_E = 0.67\, s$.

c) $T_E = 0.67\,s = \frac{T_0}{2}$ (half the proper period).

Part B

1. a) The oscillations are free damped.
   b) From figure 3, $3T = 4.1$, then
   $T = 4.1/3 = 1.37\,s$;
   By comparison $T = 1.37\,s > T_0 = 1.33\,s$.
2. The power furnished between 0 & $2T$ is:

$$P_{av} = \frac{|\Delta(ME)|}{\Delta t} = \frac{ME|_{t=2T} - ME|_{t=0}}{2T}$$
$$= \frac{|2.74 - 4.5|}{2\times 1.37} = 0.64\,mW$$



Part C

1. a) Amplitude (mechanical) resonance.
   b) The angular frequency of the exciter should be equal to the proper angular frequency $w_e \approx w_0 = 4.71\,\text{rad}/s$.
2. When we increase the force of friction, the resonance is shallow and takes place for an angular frequency slightly less than the proper angular frequency of the resonator.

II-GS 2013 1st
Part A

1. The mechanical energy at any instant is $ME|_t = KE|_t + PE_e|_t + GPE|_t = \frac{1}{2}I\,\theta'^2 + \frac{1}{2}C\theta^2 + 0$;
   ($GPE = 0$, on reference).
2. Study of oscillations
   a) The system performs free undamped oscillations, so ME is conserved.
   Then $\frac{d(ME)}{dt} = 0$; $I\,\theta'\theta'' + C\,\theta\,\theta' = 0$; $I\,\theta'\left(\theta'' + \frac{C}{I}\theta\right) = 0$;
   But $I\,\theta' \neq 0$ (the system is in motion); thus, $\theta'' + \frac{C}{I}\theta = 0$.
   b) We have $\theta = \theta_m \sin(w_0 t + \varphi)$, so $\theta'' = -\theta_m\, w_0^2 \sin(w_0 t + \varphi)$;
   Replacing in the differential equation we get : $-\theta_m\, w_0^2 \sin(w_0 t + \varphi) + \theta_m \frac{C}{I}\sin(w_0 t + \varphi) = 0$;
   $\theta_m \sin(w_0 t + \varphi)\left(-w_0^2 + \frac{C}{I}\right) = 0$, but $\theta_m \sin(w_0 t + \varphi) \neq 0$; then $-w_0^2 + \frac{C}{I} = 0$.
   Thus, $w_0 = \sqrt{\frac{C}{I}} = \sqrt{\frac{0.16}{2.5\times 10^{-3}}} = 8\,\text{rad}/s$.
   Initial conditions: for $t_0 = 0$, $\theta = \theta_m$, so $\theta_m = \theta_m \sin(\varphi)$; then $\varphi = \sin^{-1}(1) = \frac{\pi}{2}$ rad.

Thus, $\theta = \theta_m \sin(w_0 t + \varphi) = 0.1\sin\left(8t + \frac{\pi}{2}\right) = 0.1\cos(8t)$ (t in s & θ in rad).

c) When the disk passes through the equilibrium position
$\theta = \theta_m \sin(w_0 t + \varphi) = 0 \Rightarrow \sin(w_0 t + \varphi) = 0$;
Then $|\cos(w_0 t + \varphi)| = 1$; so $|\theta'| = \theta_m w_0 |\cos(w_0 t + \varphi)| = \theta_m w_0$;
But when the disk passes through the equilibrium position for the first time is in the negative direction : $\theta' = -\theta_m w_0 = -0.1 \times 8 = -0.8\,\text{rad}/s$.[1]

Part B

1. The forces acting on the disk and their moments are:
   - ✘ its weight $M_{\vec{w}} = 0$ (on axis);
   - ✘ the tension of the wire $M_{\vec{T}} = 0$ (on axis);
   - ✘ the restoring force $M_r = -C\theta$;
   - ✘ the force of friction $M = -h\theta'$.

   Theorem of angular momentum $\sum M = \frac{d\sigma}{dt} = I\theta''$; $M_{\vec{w}} + M_{\vec{T}} + M_r + M = I\theta''$;

   $-C\theta - h\theta' = I\theta''$; dividing by I we get: $\theta'' + \frac{h}{I}\theta' + \frac{C}{I}\theta = 0$.

2. We have $\frac{d(ME)}{dt} = I\theta'\left(\theta'' + \frac{C}{I}\theta\right)$;

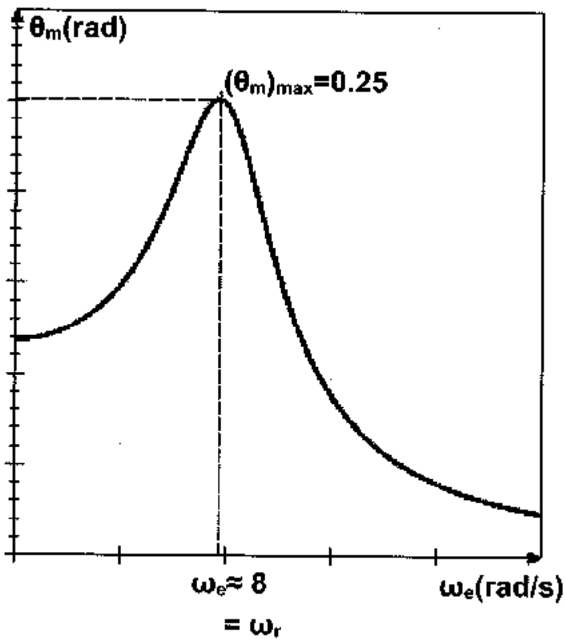
   But $\theta'' + \frac{C}{I}\theta = -\frac{h}{I}\theta'$;

   Then $\frac{d(ME)}{dt} = -h\theta'^2 < 0$;

   Thus, the mechanical energy decreases with time.

Part C

1. Amplitude (mechanical) resonance.
2. The resonance takes place for an angular frequency $w \approx w_r$.
3. Graph representing the evolution of the amplitude of the elongations as a function of the angular frequency.



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[1] Conservation of mechanical energy: $ME|_{\theta_m} = ME|_{\theta=0}$; $KE|_{\theta_m} + PE_e|_{\theta_m} = KE|_{\theta=0} + PE_e|_{\theta=0}$;

$KE|_{\theta_m} = 0$ (released from rest) & $PE_e|_{\theta=0} = 0$ (no elongation).

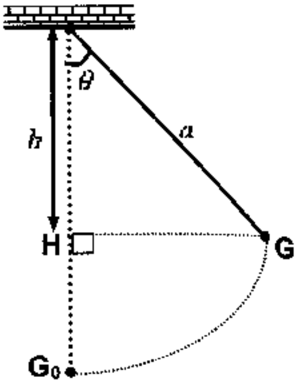
$\frac{1}{2}C\theta_m^2 = \frac{1}{2}I\theta_0'^2$; $\theta_0' = \sqrt{\frac{C}{I}}\,\theta_m = 8 \times 0.1 = 0.8\,rad/s$.

III-GS 2012 1st

Part A

1. The position of the center of gravity is given by:

$$a=\overline{OG}=\frac{m_C\overline{OC}+m_B\overline{OB}}{m_C+m_B}=\frac{m\times\frac{\ell}{2}-m\times\frac{\ell}{4}}{2m}=\frac{m\times\frac{\ell}{4}}{2m}=\frac{\ell}{8}.$$



(the mass of the rod is negligible).

2. The moment of inertia is given by:

$$I_0=m\left(\frac{\ell}{2}\right)^2+m\left(\frac{\ell}{4}\right)^2=m\frac{\ell^2}{4}+m\frac{\ell^2}{16}=\frac{5}{16}m\ell^2.$$

3. The mechanical energy $ME=KE+GPE=\frac{1}{2}I_0\theta'^2-m_tg\,h$.[2].

But $\cos\theta=\frac{h}{a}\Rightarrow h=a\cos\theta$, then $ME=\frac{1}{2}I_0\theta'^2-2mg\,a\cos\theta$.[3]

4. In the absence of friction, the mechanical energy is conserved, so $\frac{d(ME)}{dt}=0$;

$I_0\times\theta'\theta''+2\,m\,g\,a\,\theta'\sin\theta=0$, then $I_0\theta'\left(\theta''+\frac{2m\,ga}{I_0}\theta\right)=0$;

(but $I_0\neq0\;\&\;\theta'\neq0$, the system in motion); thus, $\theta''+\frac{2m\,g\,a}{I_0}\theta=0$.[4]

5. The differential equation is of second order of the form: $\theta''+w^2\theta=0$ where $w^2=\frac{2m\,g\,a}{I_0}$.

Then the motion of this pendulum is periodic whose proper period $T=\frac{2\pi}{w}=2\pi\sqrt{\frac{I_0}{2\,m\,ga}}$.

We get $T=2\pi\sqrt{\frac{\frac{5}{16}m\,\ell^2}{2\,m\,g\frac{\ell}{8}}}=2\pi\sqrt{\frac{5\,\ell}{4g}}$; thus, $T=2\pi\sqrt{\frac{5\times2}{4\times10}}=\pi\approx3.14\,s$.

6. We know that $g_{\text{moon}}<g_{\text{Earth}}$ and the period is inversely proportional to the gravitational acceleration, then, $T'=T_{\text{moon}}>T_{\text{Earth}}=T$.

Part B

1. The weight, the reaction of the axis and the couple of forces of friction.
2. The weight and the reaction of the axis meet the axis, their moments is zero.

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[2] The negative sign is due to the fact that the center of gravity is below the reference.

[3] The gravitational potential energy could be calculated also as sum of two energies: $GPE=GPE_C+GPE_B$.

$GPE=+m\,g\,h_C-m\,g\,h_B=+m\,g\frac{\ell}{4}\cos\theta-m\,g\frac{\ell}{2}\cos\theta=-m\,g\frac{\ell}{4}\cos\theta$, but $a=\frac{\ell}{8}\Rightarrow2a=\frac{\ell}{4}$.

Then: $GPE=2mga\cos\theta$.

[4] Pay attention that a negative sign in the differential equation $\left(\theta''-w_0^2\theta=0\right)$ will cancel the solutions of 4 & 5 immediately.

The resultant moment is: $\sum M = M_{\vec{w}} + M_{\vec{R}} + M = 0 + 0 - h\theta' = -h\theta'$;

3. The moment of inertia $I = m\left(\frac{\ell}{2}\right)^2 + m\left(\frac{\ell}{2}\right)^2 = 2m\frac{\ell^2}{4} = m\frac{\ell^2}{2} = 0.1\times\frac{2^2}{2} = 0.2\,kg.m^2$.

4. Theorem of angular momentum $\sum M_{ext} = \frac{d\sigma}{dt}$, $M = -h\theta'$ & $\sigma = I\theta' \Rightarrow \theta' = \frac{\sigma}{I}$.

Then $\frac{d\sigma}{dt} = -h\frac{\sigma}{I} \Rightarrow \frac{d\sigma}{dt} + \frac{h}{I}\sigma = 0$.

5. We have: $\sigma = \sigma_0\, e^{-\frac{h}{I}t} \Rightarrow \frac{d\sigma}{dt} = \sigma_0\left(-\frac{h}{I}\right)e^{-\frac{h}{I}t}$.

Replacing in the differential equation we get: $\frac{d\sigma}{dt} + \frac{h}{I}\sigma = \sigma_0\left(-\frac{h}{I}\right)e^{-\frac{h}{I}t} + \frac{h}{I}\sigma_0 e^{-\frac{h}{I}t} = 0$.

6. a) The curve shows that:
   - ✘ at $t = 0$, $\sigma_0 = I\,\theta'_0 = 0.2\times 2 = 0.4\,kg.m^2/s$;
   - ✘ it is decreasing;
   - ✘ as $t \longrightarrow \infty$, the curve tends to zero.

   So the curve is compatible with an exponential function.[5]

   b) Taking a particular value graphically: at $t = 1s$, $\sigma = 0.15\,kg.m^2/s$;

   Then $\sigma = \sigma_0\, e^{-\frac{h}{I}t} \Rightarrow 0.15 = 0.4\,e^{-\frac{h}{0.2}\times 1} \Rightarrow e^{-\frac{h}{0.2}\times 1} = 0.375$;

   $-\frac{h}{0.2} = \ell n(0.375) \Rightarrow h = -0.2\,\ell n(0.375) \approx 0.196\,SI$.

   Another method (using the slope of the tangent): Graphically: $\left.\frac{d\sigma}{dt}\right|_{t=0} = -\frac{0.4}{1} = -0.4\,SI$.

   Replacing in the differential equation: $\left.\frac{d\sigma}{dt}\right|_{t=0} = -h\frac{\sigma_0}{I} \Rightarrow -0.4 = -h\frac{0.4}{0.2} \Rightarrow h = 0.2\,SI$.

IV-GS 2011 2nd

Part A

1. $ME = KE + GPE = \frac{1}{2}mv^2 + mgh$, $h = \ell - \ell\cos\theta = \ell(1-\cos\theta)$;

   The pendulum is considered as a point particle then: $KE = \frac{1}{2}mv^2$.

   We have: $v = \ell\theta'$, the mechanical energy $ME = \frac{1}{2}m\,\ell^2\theta'^2 + m\,g\,\ell(1-\cos\theta)$.

2. The forces of friction are negligible $\frac{d(ME)}{dt} = 0 \Rightarrow m\,\ell^2\theta'\theta'' + m\,g\ell\theta'\sin\theta = 0$;

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[5] Another justification is used basing on the properties of an exponential function:

By analogy $\sigma = \sigma_0\, e^{-\frac{t}{\tau}}$, the tangent at the point of abscissa zero should cut the abscissa axis at the point of abscissa τ.

For $t = \tau$; $\sigma = 0.37\,\sigma_0$ (which is justified graphically) & σ_0 is verified.

$m\ell^2\theta'\left(\theta''+\frac{g}{\ell}\sin\theta\right)=0$, (but $m\ell^2\theta'\neq 0$); thus, $\theta''+\frac{g}{\ell}\sin\theta=0$.

3. a) In order so that the oscillations be simple harmonic, the differential equation must be of the form: $\theta''+w_0^2\theta=0$.

   This relation is obtained if $\sin\theta\approx\theta$; which is satisfied only for oscillations of small angles.

   b) The expression of the proper period is $T_0=\frac{2\pi}{w_0}$ where $w_0=\sqrt{\frac{g}{\ell}}$, then $T_0=2\pi\sqrt{\frac{\ell}{g}}$.

   c) The time equation is of the form: $\theta=\theta_m\sin(w_0t+\varphi)$.

   We have $\theta_m=0.1\,\text{rad}$ & $w_0=\sqrt{\frac{g}{\ell}}=\sqrt{\frac{10}{1}}=\sqrt{\pi^2}=\pi\,\text{rad/s}$.

   Initial conditions at $t=0$, $\theta=\theta_m$; so $\theta_m=\theta_m\cos\varphi\Rightarrow\cos\varphi=1$; then $\varphi=0$;

   Thus, $\theta=0.1\sin(\pi t)$ where t in s and θ in rad.

Part B

1. $ME=KE+PE_e=\frac{1}{2}mv^2+\frac{1}{2}kx^2$.

   Since the mechanical energy is conserved, then $mx'x''+kxx'=0$;

   $x'(mx''+kx)=0$, (but $x'\neq 0$ since the system is in motion); then $x''+\frac{k}{m}x=0$.

2. Graphically: $x_m=0.1m$ & $T_0=1s$;

   At $t=0$, $x=x_m$; so $x_m=x_m\sin\varphi$; then $\sin\varphi=1\Rightarrow\varphi=\frac{\pi}{2}\,\text{rad}$.

3. a) We have: $x=x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)\Rightarrow x''=-\left(\frac{2\pi}{T_0}\right)^2x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)$.

   Replace in the differential equation $-\left(\frac{2\pi}{T_0}\right)^2x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)+\frac{k}{m}x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)=0$;

   So, $x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)\left(-\left(\frac{2\pi}{T_0}\right)^2+\frac{k}{m}\right)=0$; (but $x_m\cos\left(\frac{2\pi}{T_0}t+\varphi\right)\neq 0$, in motion);

   Then: $-\left(\frac{2\pi}{T_0}\right)^2+\frac{k}{m}=0$; we get $T_0=2\pi\sqrt{\frac{m}{k}}$.

   b) We have $T_0=2\pi\sqrt{\frac{m}{k}}\Rightarrow m=\frac{T_0^2}{4\pi^2}\times k=\frac{1}{40}\times 4=0.1kg$.

4. a) $PE_e=\frac{1}{2}kx^2$ is maximum if $|x|$ is maximum then $x=\pm 0.1m$ at the instants 0, $0.5s$, $1s$, $1.5s$ & $2s$.

   b) The elastic potential energy PE_e is maximum if the kinetic energy is zero $KE=0$.

   The amplitude of motion is constant then the mechanical energy is conserved

   $ME=PE_{e_{max}}+0=\frac{1}{2}kx_m^2=0.5\times 4\times 0.1^2=0.02\,J$.

Part C

For the simple pendulum $T_0 = 2\pi\sqrt{\frac{\ell}{g}}$; but $g_{\text{moon}} < g_{\text{Earth}}$ so T_0 increases, "statement 2".

For the elastic pendulum $T_0 = 2\pi\sqrt{\frac{m}{k}}$; T_0 is independent from the gravitational acceleration, it does not vary on the Moon "statement 1".

V-GS 2011 1st

Part A

1. a) The forces acting on the rod are: its weight, the reaction of the axis and the force.
   The weight and the reaction meet the axis, their moment is zero, and the resultant moment is: $\sum \mathrm{M} = \mathrm{M}$.

   b) Theorem of angular momentum: $\sum \mathrm{M} = \frac{d\sigma}{dt} \Rightarrow \mathrm{M} = I_0\theta''$ is constant;
   θ'' is constant, then the motion of the rod is uniformly accelerated.

   c) $\theta'' = cst = \frac{\mathrm{M}}{I_0} \Rightarrow \theta' = \frac{\mathrm{M}}{I_0}t + \theta'_0$ (initial speed being zero $\theta'_0 = 0$);
   But the angular momentum $\sigma = I_0\theta' = \mathrm{M}\,t$.

2. We have $\theta' = 2\pi N$, then $2\pi N = \frac{\mathrm{M}}{I_0}t_1 \Rightarrow I_0 = \frac{\mathrm{M}}{2\pi N}t_1 = \frac{0.1}{2\pi \times 8} \times 10 = 0.02\, kg.m^2$.

Part B

1. a) The center of gravity is given by: $a = \overline{OG} = \frac{m\overline{OO} + m'\overline{OB}}{m+m'} = \frac{m \times 0 - m' \times \frac{\ell}{2}}{m+m'} = \frac{m'\ell}{2(m+m')}$.

   b) The moment of inertia is given by: $I = I_{rod} + I_B = I_0 + m'\left(\frac{\ell}{2}\right)^2 = I_0 + m'\frac{\ell^2}{4}$.

2. $ME = KE + GPE = \frac{1}{2}I\theta'^2 - m_t g\, h$, but $\cos\theta = \frac{h}{a} \Rightarrow h = a\cos\theta$;

   Then $ME = KE + GPE = \frac{1}{2}I\,\theta'^2 - (m+m')g\,a\cos\theta$.

3. a) In the absence of friction, the mechanical energy is conserved $\frac{d(ME)}{dt} = 0$.

   $\frac{1}{2}I \times 2\theta'\theta'' + (m+m')g\,a\,\theta'\sin\theta = 0 \Rightarrow I\theta'\left(\theta'' + \frac{(m+m')g\,a}{I}\theta\right) = 0$.

   (But $I_0\,\theta' \neq 0$, the system in motion), then we get: $\theta'' + \frac{(m+m')ga}{I}\theta = 0$.

   b) The differential equation of motion is of 2nd order of the form $\theta'' + w^2\theta = 0$, where $w^2 = \frac{(m+m')g\,a}{I}$.

Then the motion of this pendulum is periodic whose proper period $T=\frac{2\pi}{w}=2\pi\sqrt{\frac{I}{(m+m')ga}}$.

So, $T=2\pi\sqrt{\frac{I_0+m'\frac{\ell^2}{4}}{(m+m')g\frac{m'\ell}{2(m'+m)}}}=2\pi\sqrt{\frac{I_0+m'\frac{\ell^2}{4}}{g\frac{m'\ell}{2}}}=\sqrt{40\times\frac{I_0+m'\frac{\ell^2}{4}}{g\frac{m'\ell}{2}}}=\sqrt{\frac{8I_0+2m'\ell^2}{m'\ell}}$.

4. We have $10T_0=17.32\,s$, so $T_0=\frac{17.32}{10}=1.732\,s=\sqrt{3}\,s$.

$\sqrt{\frac{8I_0+2m'\ell^2}{m'\ell}}=\sqrt{3}$; then $\frac{8I_0+0.32}{0.16}=3\Rightarrow I_0=\frac{0.16}{8}=0.02\,kg.m^2$.

VI-GS 2009 2nd

Part A

1. $ME=KE+GPE=\frac{1}{2}I\theta'^2+m\,g\,a(1-\cos\theta)=\frac{1}{2}I\theta'^2+m\,g\,a\frac{\theta^2}{2}$.

2. a) In the absence of friction, the mechanical energy is conserved $\frac{d(ME)}{dt}=0$.

So, $I\theta'\theta''+m\,g\,a\,\theta'\theta=0$, $\theta'(I\theta''+m\,g\,a\,\theta)=0$; but $\theta'\neq 0$ since the system is in motion.

Thus, $\theta''+\frac{m\,g\,a}{I}\theta=0$

b) Differential equation of second order of the form: $\theta''+w_0^2\theta=0$ where $w_0^2=\frac{m\,g\,a}{I}$.

The period of motion is: $T_0=\frac{2\pi}{w_0}=2\pi\sqrt{\frac{I}{m\,g\,a}}$.

3. a) Moments of inertia.

i- We have $T_1=2\pi\sqrt{\frac{I_1}{m\,g\,a_1}}$ & $T_2=2\pi\sqrt{\frac{I_2}{m\,g\,a_2}}$;

The two pendulums have the same period: $T_1=T_2$, we get $\frac{I_1}{a_1}=\frac{I_2}{a_2}$.

ii- We have: $\frac{I_0+m\,a_1^2}{a_1}=\frac{I_0+m\,a_2^2}{a_2}$, then $I_0(a_2-a_1)=m\,a_2\,a_1(a_2-a_1)$.

But $a_2\neq a_1\Rightarrow a_2-a_1\neq 0$, so $I_0=ma_2a_1$.

b) We have $T'=2\pi\sqrt{\frac{I_1}{m\,g\,a_1}}=2\pi\sqrt{\frac{I_0+ma_1^2}{m\,g\,a_1}}=2\pi\sqrt{\frac{ma_2a_1+ma_1^2}{m\,g\,a_1}}=2\pi\sqrt{\frac{a_2+a_1}{g}}$;

But $T'=2\pi\sqrt{\frac{\ell}{g}}$, then $\ell=a_2+a_1$.

Part B

1. a) Graphically $a_1=10cm$ & $a_2=20cm$.

b) We have $I_0 = m\,a_1 a_2 = 4\times 10^{-3}\,kg.m^2$ & $\ell = a_2 + a_1 = 30\,cm$.

2. a) From graph $T_{\min}$ is reached if $a' = 14\,cm$.

b) We have $T = 2\pi\sqrt{\dfrac{I_0 + ma^2}{m\,g\,a}}$, T is minimum when $\dfrac{I_0}{mga} + \dfrac{a}{g}$ is min.[6]

Furthermore $\dfrac{I_0}{mga} \times \dfrac{a}{g} = \dfrac{I_0}{m\,g^2}$ is constant.

Then T is min, if $\dfrac{I_0}{mga} = \dfrac{a}{g}$, $a = \sqrt{\dfrac{I_0}{m}} = 14.1\,cm \Rightarrow T_{\min} = 1.05\,s$

2nd method (derivative):

We have $T = \dfrac{2\pi}{\sqrt{mg}}\sqrt{\dfrac{I_0}{a} + ma}$; $\dfrac{dT}{da} = \dfrac{2\pi}{\sqrt{mg}}\dfrac{-\dfrac{I_0}{a^2} + m}{\sqrt{\dfrac{I_0}{a} + ma}}$;

T is minimal when $\dfrac{dT}{da} = 0 \Rightarrow a = \sqrt{\dfrac{I_0}{m}} = 14.1\,cm$.

VII-GS 2009 1st

Part A

1. The mechanical energy $ME = KE + GPE + PE_e = \dfrac{1}{2} I\,\theta'^2 + 0 + \dfrac{1}{2} C\,\theta^2$.

2. a) For the maximum deviation $\theta = \theta_m$ & $\theta' = 0$ ($KE = 0$); then $ME = \dfrac{1}{2} C\,\theta_m^2$.

b) At the equilibrium position: $ME\big|_{\theta=0} = \dfrac{1}{2} I\,\theta_m'^2$;

The mechanical energy is conserved in the absence of the forces of friction then:

$ME\big|_{\theta=0} = ME\big|_{\theta=\theta_m}$; $\dfrac{1}{2} C\,\theta_m^2 = \dfrac{1}{2} I\,\theta_m'^2$; then $\theta_m' = \pm\theta_m\sqrt{\dfrac{C}{I}}$.

3. The mechanical energy is conserved, then $\dfrac{d(ME)}{dt} = 0$, so $I\,\theta'\theta'' + C\theta\theta' = 0$; $\theta'(I\,\theta'' + C\theta) = 0$;

(but $\theta' \neq 0$, since the system is in motion); then $\theta'' + \dfrac{C}{I}\theta = 0$.

4. The differential equation that governs the motion of this rod is of second order of the form: $\theta'' + w^2\theta = 0$, then the motion of $[P]$ is sinusoidal.

5. The period of the oscillations is $T_1 = \dfrac{2\pi}{w} = 2\pi\sqrt{\dfrac{I}{C}}$.

Part B

1. We know that the duration of an oscillation is equal to its period, then $t_1 = 20\,T_1 = 20\,s$;

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[6] If the product of two numbers is constant then their sum is minimal when they are equal.

So, $T_1 = 1s$; but $T_1 = 2\pi\sqrt{\dfrac{I}{C}}$; then $C = \dfrac{4\pi^2 I}{T_1} = 40I$ ....(1)

2. a) The moment of inertia is additive: $I' = I_{rod} + I_A + I_B = I + 2m\left(\dfrac{\ell}{2}\right)^2 = I + m\dfrac{\ell^2}{2}$.

b) The new expression of the period becomes (A-5) is: $T_2 = 2\pi\sqrt{\dfrac{I'}{C}} = 2\pi\sqrt{\dfrac{I + \dfrac{m\ell^2}{2}}{C}}$.

c) The duration of 20 oscillations is $20\,T_2 = 40\,s$, so $T_2 = 2\,s$.

But $T_2 = 2\pi\sqrt{\dfrac{2I + m\ell^2}{2C}} = 2$, then $C = 4\pi^2\,\dfrac{2I + 0.025\,(0.6)^2}{2^2}$;

Thus, $C = 10\left(I + 4.5\times 10^{-3}\right) = 10\,I + 0.045$ ....(2)

3. Using the obtained equations (1) & (2) we get: $\begin{cases} C - 40\,I = 0 \\ C - 10\,I = 0.045 \end{cases}$;

Then $C = 6\times 10^{-2}\ N.m/\text{rad}$ & $I = 1.5\times 10^{-3}\ kg.m^2$

VIII-GS 2008 1st

Part A

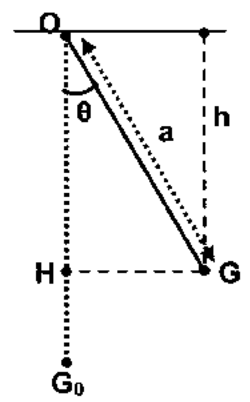
1. The position of the center of gravity G is : $\overline{OG} = \dfrac{M\times\overline{OB} + m\times\overline{OC}}{m+M}$, then $a = \dfrac{Md - mx}{M+m}$.

2. The moment of inertia I is $I = I_M + I_m = M\,d^2 + m\,x^2$.

3. a) $KE = \dfrac{1}{2}I\,\theta'^2$.

b) $GPE = -(M+m)g\,h = -(M+m)g\,a\cos\theta$ (below reference);

Where $\cos\theta = \dfrac{h}{a}$; so $h = a\cos\theta$.



c) $ME = KE + GPE = \dfrac{1}{2}I\,\theta'^2 - (M+m)g\,a\cos\theta$.

d) In the absence of friction, ME is conserved $\dfrac{d(ME)}{dt} = 0$;

$I\,\theta'\theta'' + (M+m)\,g\,a\,\theta'\sin\theta = 0$, $\theta'' + \dfrac{(M+m)ga}{I}\sin\theta = 0$.

e) For small θ, $\sin\theta = \theta$, we get $\theta'' + \dfrac{(M+m)ga}{I}\theta = 0$;

The differential equation that governs the motion of this rod is of the 2nd order of the form $\theta'' + w^2\,\theta = 0$ with $w = \sqrt{\dfrac{(M+m)ga}{I}}$;

The motion is simple harmonic of proper period $T = \dfrac{2\pi}{w}$, then $T = 2\pi\sqrt{\dfrac{I}{(M+m)ga}}$.

f) By replacing I by its expression we get $T = 2\pi\sqrt{\dfrac{M\,d^2 + m\,x^2}{g(Md - mx)}}$.

Part B

1. The period is given by: $T = 2\pi\sqrt{\dfrac{Md^2 + mx^2}{g(Md - mx)}} = \sqrt{\dfrac{40\left(50\times10^{-3}\times(0.02)^2 + 5\times10^{-3}\times x^2\right)}{10\left(50\times10^{-3}\times(0.02) - 5\times10^{-3}x\right)}}$;

   Thus, $T = \sqrt{\dfrac{0.08 + 20x^2}{1 - 5x}}$ where x in m & T in s.

2. **1st method: By calculation**

   For $T = \sqrt{\dfrac{0.08 + 20x^2}{1 - 5x}} = 1s$, we get $20x^2 + 5x - 0.92 = 0$;

   Then $x_1 = 0.123\,m$ (accepted) & $x_2 = -0.37\,m$ (rejected).

   For $T = \sqrt{\dfrac{0.08 + 20x^2}{1 - 5x}} = 1.1s \Rightarrow 20x^2 + 6.05x - 1.13 = 0$ then $x_1 = 0.13\,m$ (accepted).

   Thus $12.3 \le x(cm) \le 13$ (the curve shows that the period is an increasing function).

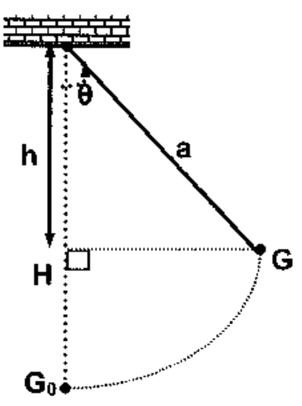
   2nd method: Graphically

   For $T = 1s$, $x = 12.3\,cm$.

   For $T = 1.1s$, $x = 13\,cm$; then $12.3 \le x(cm) \le 13$.

IX-GS 2007 2nd
Part A

1. The kinetic energy $KE = \dfrac{1}{2} I\,\theta'^2$.
2. The gravitational potential energy is given by
   $GPE = -M\,g\,h = -M\,g\,a\cos\theta$;



   According to the geometry of the figure: $\cos\theta = \dfrac{h}{a} \Rightarrow h = a\cos\theta$

3. The mechanical energy: $ME = KE + GPE = \dfrac{1}{2} I\,\theta'^2 - M\,g\,a\cos\theta$.
4. In the absence of friction, the mechanical energy is conserved;

   So, $\dfrac{d(ME)}{dt} = 0$, we get $I\theta'\theta'' + M\,g\,a\,\theta'\sin\theta = 0$

   But for small angles $\sin\theta = \theta$, then $I\theta'\theta'' + M\,g\,a\,\theta\theta' = 0$;

   $\theta'(I\theta'' + M\,g\,a\,\theta) = 0$; ($\theta' \neq 0$ since the system is in motion).

   Thus, $\theta'' + \dfrac{Mga}{I}\theta = 0$.

5. The differential equation of motion is of 2nd order of the form $\theta'' + w_1^2\theta = 0$ where $w_1 = \sqrt{\dfrac{M\,g\,a}{I}}$;

   Then the motion is simple harmonic of proper period $T_1 = \dfrac{2\pi}{w_1} = 2\pi\sqrt{\dfrac{I}{M\,g\,a}}$.

Part B

1. a) The torsion potential energy $PE_{torsion} = \dfrac{1}{2}C\theta^2 + \dfrac{1}{2}C\theta^2 = C\theta^2$.

b) The potential energy $PE = GPE + PE_{torsion} = -M\,g\,a\cos\theta + C\,\theta^2$.

c) The mechanical energy $ME = KE + PE = \frac{1}{2}I\,\theta'^2 - M\,g\,a\cos\theta + C\,\theta^2$.

2. In the absence of friction, the mechanical energy is conserved so, $\frac{d(ME)}{dt} = 0$;

Then $I\,\theta'\theta'' + M\,g\,a\,\theta'\sin\theta + 2\,C\,\theta'\theta = 0$; $\theta'(I\,\theta'' + M\,g\,a\sin\theta + 2\,C\theta) = 0$;

But $\theta' \neq 0$ (the system is in motion) & $\sin\theta \approx \theta$; thus, $\theta'' + \left(\frac{M\,g\,a + 2C}{I}\right)\theta = 0$.

The differential equation is of the 2nd order of the form $\theta'' + w_2^2\,\theta = 0$ where $w_2^2 = \frac{M\,g\,a + 2\,C}{I}$;

Then the period of oscillations is $T_2 = \frac{2\,\pi}{w_2} = 2\,\pi\sqrt{\frac{I}{M\,g\,a + 2\,C}}$.

Part C

We have $T_1 = 2\pi\sqrt{\frac{I}{M\,g\,a}}$, so, $I = \frac{T_1^2}{4\pi^2}M\,g\,a$.

But $T_2 = 2\,\pi\sqrt{\frac{I}{M\,g\,a + 2\,C}}$; so, $I = \frac{T_2^2}{4\pi^2}(2C + M\,g\,a)$.

We get $\frac{T_1^2}{4\pi^2}M\,g\,a = \frac{T_2^2}{4\pi^2}(2C + M\,g\,a)$; then $a = \frac{2\,C\,T_2^2}{M\,g\left(T_1^2 - T_2^2\right)} = 3.4\times10^{-3}\,m = 0.34\,cm$.

Thus, $I = \frac{T_1^2}{4\pi^2}M\,g\,a = \frac{4.77^2}{4\,\pi^2}\left(59\times10^{-3}\right)\times10\times\left(3.4\times10^{-3}\right) = 1.14\times10^{-3}\,kg.m^2$.

X-GS 2007 1st
Part A

1. The center of gravity G is $\overline{OG} = \frac{m\times\overline{OA} + m\times\overline{OD}}{m+m}$, then $a = \frac{m(\ell/2) - m(\ell/4)}{2m} = \frac{\ell}{8}$.

2. a) Study of motion:

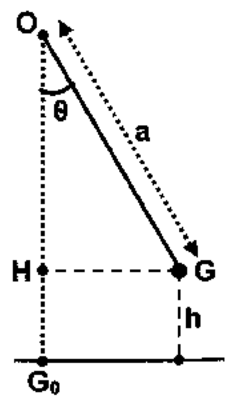
i- $GPE = M_t\,g\,h_G = 2\,m\,g\,(a - a\cos\theta) = 2\,m\,g\,a\,(1-\cos\theta)$.

ii- The mechanical energy is conserved, since friction is neglected.

$ME = ME|_{\theta=0} = KE|_{\theta=0} + GPE|_{\theta=0} = KE|_{\theta=0} + 0 = E_0$.

iii- Conservation of mechanical energy: $ME|_{\theta=\theta_0} = E_0$;

Then, $2\,m\,g\,a\,(1-\cos\theta_m) = E_0 \Rightarrow \theta_m = \cos^{-1}\left(1 - \frac{4E_0}{m\,g\,\ell}\right) = 8^0$.



b) Variation in energy:

i- The variation of ME is: $\Delta(ME) = ME|_{\theta_1m} - ME|_{\theta=0}$;

$\Delta(ME) = 2\,m\,g\,a\,(1-\cos\theta_{1m}) - E_0$.

ii- The work done is equal to the variation of ME:

$W = \Delta(ME) = 2\times0.01\times10\times0.1\left(1-\cos7^0\right) - 1.95\times10^{-4}$, then $W = -4.6\times10^{-5}\,J$.

Part B

1. The moment of inertia: $I = I_A + I_B = m\left(\frac{\ell}{2}\right)^2 + m\left(\frac{\ell}{2}\right)^2 = 2m\frac{\ell^2}{4} = 32\times10^{-4}\ kg.m^2$.

2. The angular momentum $\sigma_0 = I\,\theta'_2 = I\times 2\pi N_0 = 2\times10^{-2}\,kg.m^2/s$.

3. a) The forces applied on (S) are: the weight $\vec{w}$, the reaction $\vec{R}$ of axis (Δ), the couple of friction whose moment is M.

   b) $\mathrm{M}_{\vec{R}/(\Delta)} = \mathrm{M}_{\vec{w}/(\Delta)} = 0$ (forces applied on the axis);

   The resultant moment $\sum \mathrm{M} = \mathrm{M}_{\vec{R}/(\Delta)} + \mathrm{M}_{\vec{w}/(\Delta)} + \mathrm{M} = \mathrm{M}$.

   c) Theorem of angular momentum $\sum \mathrm{M} = \frac{d\sigma}{dt} = \mathrm{M}$; then $\sigma = \mathrm{M}\,t + \sigma_0$.

4. When the system stops $\sigma = \mathrm{M}\,t + \sigma_0 = 0$, then $\mathrm{M} = -\frac{\sigma_0}{t'} = -3.78\times10^{-4}\ m.N$.

Part C

$\mathrm{M}\times\theta_{1m} = -3.78\times10^{-4}\times\frac{7\pi}{180} = -4.6\times10^{-5}\,J$; then $W = \mathrm{M}\times\theta_{1m}$.

XI-GS 2006 2nd

Part A

1. The gravitational potential energy is given by

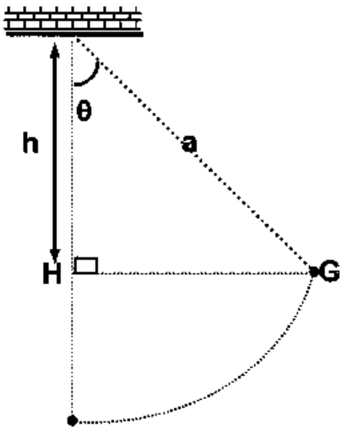
   $GPE = -M_{\text{total}}\,g\,h = -2M\,g\,a\cos\theta$;

   According to the geometry of the figure: $\cos\theta = \frac{h}{a}$;

   Then $h = a\cos\theta$;

   The mechanical energy is

   $ME = KE + GPE = \frac{1}{2}I\,\theta'^2 - 2M\,g\,a\cos\theta$.



2. The friction is negligible, then the mechanical energy is conserved, so $\frac{d(ME)}{dt} = 0$; we get $I\,\theta'\theta'' + 2M\,g\,a\theta'(\sin\theta) = 0$;

   But $\theta' \neq 0$ (system in motion) and $\sin\theta = \theta$ (small angles);

   Then $\theta'' + \frac{2M\,g\,a}{I}\theta = 0$.

3. a) The center of gravity is given by: $\overline{OG} = \frac{m_{\text{rod}}OG_{\text{rod}} + m_S OG_S}{m_{\text{rod}} + m_S} = \frac{M\times\frac{\ell}{2} + M\times x}{M+M} = \frac{\ell+2x}{4}$.

   b) The moment of inertia is additive $I = I_{\text{rod}} + I_{(S)} = \frac{M\,\ell^2}{3} + M\,x^2 = \frac{M(\ell^2+3x^2)}{3}$.

4. The differential equation that governs the motion of this pendulum is of 2nd order of the form $\theta'' + w^2\theta = 0$ where $w^2 = \frac{2M\,g\,a}{I}$.

The proper period $T=\frac{2\pi}{w}=2\pi\sqrt{\frac{I}{2Mga}}=2\pi\sqrt{\frac{M\left(\ell^2+3x^2\right)}{3}\times\frac{1}{2Mg\left(\frac{\ell+2x}{4}\right)}}$;

Thus, $T=\sqrt{\frac{4\pi^2\times 2\left(1+3x^2\right)}{3\times 10\left(1+2x\right)}}=\sqrt{\frac{8\left(1+3x^2\right)}{3\left(1+2x\right)}}$ (x in m & T in s)

Part B

1. The period of oscillations is $T'=\frac{16.6}{10}=1.66\,s$.[7]

2. a) Under forced oscillations, the period of the resonator (R) is imposed by the exciter (E), then $T'=T$

   b) Periods.

   i- We have $T_1=T=\sqrt{\frac{8\left(1+3x^2\right)}{3\left(1+2x\right)}}$ & for $x=0.3m$ & $T_1=\sqrt{\frac{8\left(1+3\times 0.3^2\right)}{3\left(1+2\times 0.3\right)}}=1.45\,s$.

   ii- The curve representing the variations of the amplitude in terms of the exciter's period is increasing for a period less than the proper period $T<T'$.
   But $T_1<T_2<T'$, then $\theta_{m1}<\theta_{m2}$.

   c) Oscillations.

   i- The resonator oscillates with its maximal amplitude, then the pendulum is in the state of mechanical amplitude resonance.

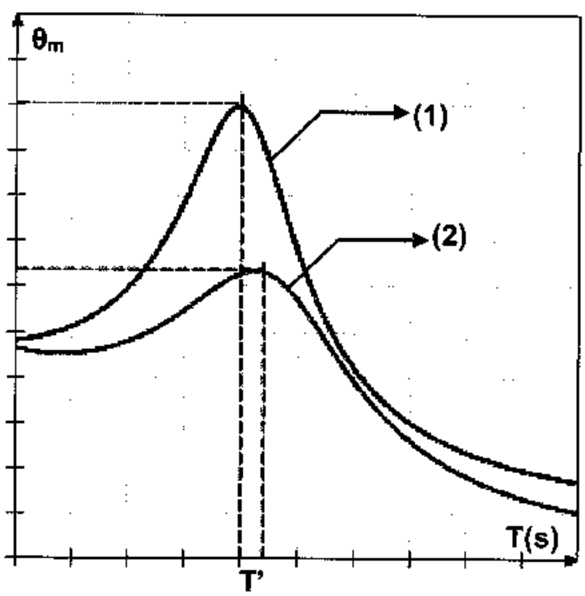
   ii- At resonance the period of the exciter is equal to that of the resonator,

   $T'=T=\sqrt{\frac{8\left(1+3x^2\right)}{3\left(1+2x\right)}}=1.66\,s$; $3T^2+6T^2x=8+24x^2$;

   So, $24x^2-16.5336\,x-0.2668=0$; thus, $x_1=0.7\,m$ (accepted) & $x_2=-0.015\,m$ (rejected).

   d) Graph 1.

   e) Graph 2.



[7]The compound pendulum (E) acts as exciter, having a variable period, while the simple pendulum whose period is constant is the resonator.

XII-GS 2005 2nd

Part A

1. a) The forces acting on the disk are its weight $\vec{w}$, the reaction of axis $\vec{R}$ and the force $\vec{F}$.

   b) The Moments of the external forces $M_{\vec{w}/(\Delta)} = M_{\vec{R}/(\Delta)} = 0$ (on axis) & $M_{\vec{F}/(\Delta)} = M$.

   c) Theorem of angular momentum: $\frac{d\sigma}{dt} = I_0\theta'' = M \Rightarrow \theta'' = \frac{M}{I_0}$ is constant.

   Then the motion is uniformly accelerated.

2. a) The primitive using the theorem of angular momentum $\frac{d\sigma}{dt} = M \Rightarrow \sigma = Mt + \sigma_0$;

   But $\sigma_0 = I_0\,\theta'_0 = 0$; thus, $\sigma = Mt$.

   b) We have $I_0\theta' = Mt \Rightarrow I_0 = \frac{Mt}{\theta'} = \frac{0.2\times 5}{2\times\pi\times 80} = 2\times 10^{-3}\,kg.m^2$.

Part B

1. The mechanical energy is given by: $ME = KE + GPE = \frac{1}{2}I\theta'^2 - m\,g\,R\cos\theta$.

2. Since the mechanical energy is conserved: $\frac{d(ME)}{dt} = 0 \Rightarrow I\,\theta'\theta'' + m\,g\,R\,\theta'\sin\theta = 0$,

   But $\sin\theta \approx \theta$, then $\theta'' + \frac{m\,g\,R}{I}\theta = 0$.

3. The differential equation that governs the motion of the pendulum is of 2nd order of the form

   $\theta'' + w^2\theta = 0$, where $w^2 = \frac{m\,g\,R}{I}$;

   Then the motion of the pendulum is periodic of proper period $T = \frac{2\pi}{w} = 2\pi\sqrt{\frac{I}{m\,g\,R}}$.

4. The period $T = \frac{7.7}{10} = 0.77\,s$.

   Then, $I = \frac{T^2 m\,g\,R}{4\pi^2} = \frac{(0.77)^2\times 0.4\times 10\times 0.1\times(0.32)^2}{4} = 6.1\times 10^{-3}\,kg.m^2$;

   We have $I = I_0 + mR^2$, then $I_0 = 6.1\times 10^{-3} - 0.4(0.1)^2 = 2.1\times 10^{-3}\,kg.m^2$.

Unit II

Electricity

Chapter 6

Electromagnetic Induction

LS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Electromagnetic Induction		-	-	-	-	-	$2^{nd(A)}$	-	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Electromagnetic Induction	1^{st}	-	$1^{st(A)}$	-	-	-	-	-	-

Essentials

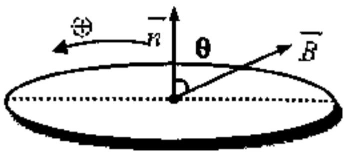
I-

Electromagnetic Induction

1. Magnetic flux

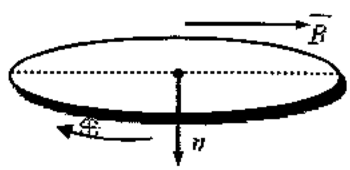
The magnetic flux ϕ crossing a given circuit formed of N turns, each of surface area S and subjected to a magnetic field B is given by $\phi = N\,B\,S\cos\theta$ where θ is the angle between the normal $\vec{n}$ to the surface and the magnetic field $\vec{B}$, $\theta = (\vec{n};\vec{B})$.

In SI units, ϕ is measured in weber Wb.



Note: The direction of the normal to the surface is given by right hand rule basing on the arbitrary positive direction chosen.

Particular case: If $\theta = (\vec{n};\vec{B}) = \pm\frac{\pi}{2}$ ($\vec{n}$ & $\vec{B}$ are orthogonal which means also that $\vec{B}$ is parallel to the surface of the circuit), then the magnetic flux is zero.



2. Definition

The electromagnetic induction phenomenon is the appearance of an induced electromotive force e in a given circuit due to the variations of flux in the circuit.

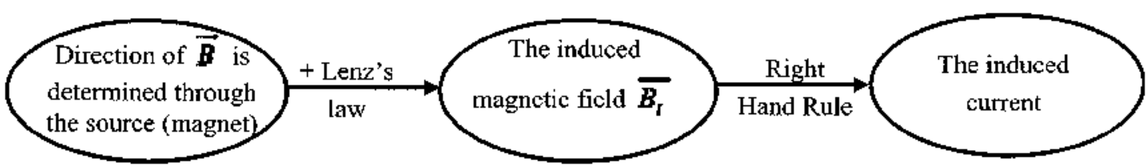
3. Induced electromotive force

If the magnetic flux crossing a given circuit is variable (variation of B, S or θ), then the circuit is the seat of an induced electromotive force.

4. Lenz's Law

The electromagnetic effect of the induced current $(\vec{B_i}, \vec{F_\ell})$ tends to oppose the variation of the cause creating it.

- ✘ If $\vec{B_i}$ is determined by Lenz's law, then the direction of the induced current is determined by **Right Hand Rule** (curly hand) usually when the magnitude of the magnetic field is variable.



- ✘ If $\vec{B}$ & the electromagnetic force (Laplace's force) $\vec{F_\ell}$ are defined, then we can determine the direction of the induced current by **Right Hand Rule** (three fingers), usually used when the surface of the circuit is variable.

5. Faraday's law

The induced electromotive force is given by $e = -\frac{d\phi}{dt}$.

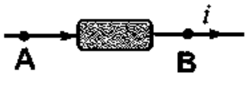
Note: If $e \neq 0$, and the circuit is closed then an induced current exists in the circuit given by $i = \frac{e}{R}$ where R is the resistance of the circuit.

II-

Voltage Across a Dipole

1. Resistor

The voltage (measured in Volts V) between the terminals of a resistor of resistance R (measured in Ohms Ω) is proportional to the current (measured in ampere A) passing through it, and given by Ohm's law: $u_{AB} = Ri$.

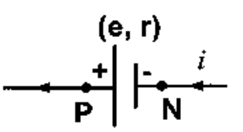


But $u_{BA} = -u_{AB} = -Ri < 0$;

Note: We conclude that the voltage across the resistor is positive if it measured in the direction of the current (negative in the opposite direction of current).

2. DC generator

A DC generator (dry cell) is characterized by its:



- ✗ electromotive force (*e.m.f*) e which represents the maximum voltage delivered by the source (voltage in open circuit);
- ✗ internal resistance r.

The voltage between its terminals by Ohm's law: $u_{PN} = e - ri$.

But $u_{NP} = -u_{PN} = ri - e < 0$;

Note: We conclude that the voltage across the generator is positive if it measured in the direction opposite to that of the current (negative in the direction of current).

III-

Electric Power

For a DC voltage, the power is the product of the voltage by the current $P = U \times I$.

For a resistor: $U = RI$, then $P = RI^2$ is the power dissipated due to joule's effect.

For a generator, the voltage is given by $u = e - ri$;

$P_u = u \times i$ is the useful power delivered to the circuit;

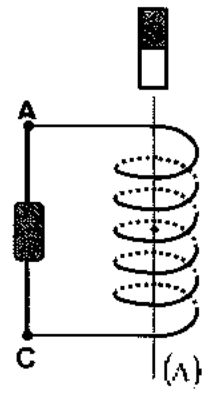
$P_t = e \times i$ is the total theoretical power;

$P_d = ri^2$ is the power dissipated due to Joule's effect.

Applications

I- Qualitative Description

In the adjacent figure, we consider a coil whose axis (Δ) is vertical. Along its axis a magnet is placed. (We suppose that the coil is wide enough so that the magnet penetrates from its upper side and exits from its lower part). A resistor is connected across the terminals of the coil.



Specify, illustrating with diagrams the direction of the induced current and the sign of the voltage u_{AC}, when the magnet:

1. approaches from the upper part of the coil.
2. exits from its lower part.

II- Variations of the Magnetic Field

The adjacent curve represent the evolution of the magnetic field through the solenoid when a magnet is in motion along the axis (Δ) as shown in the adjacent figure.

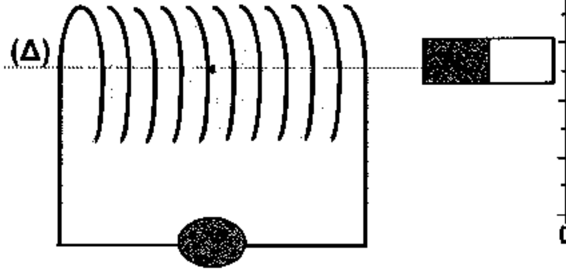


Figure 1

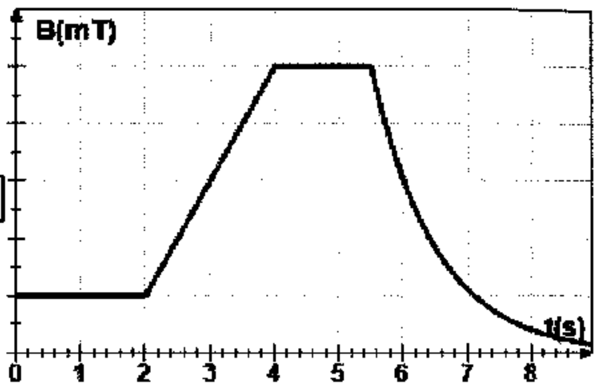
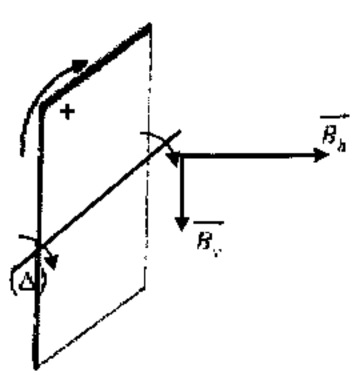


Figure 2

1. In which time intervals an induced current appears in the circuit? Justify.
2. Specify the time interval in which the current circulates in the clockwise direction.

III- Average Magnetic Flux

The Earth magnetic field vector $\vec{B}$, in a certain region has a horizontal component of magnitude $B_h = 20\,\mu T$ and a vertical component of magnitude $B_v = 10\,\mu T$ as shown in the adjacent figure. A square loop of area $S = 15\,cm^2$ formed of 50 turns, placed in a vertical plane perpendicular to $\overrightarrow{B_h}$ is rotated by 90^0 in the clockwise direction around a horizontal axis (Δ), until it becomes horizontal during a duration $\Delta t = 0.8s$.



1. Calculate the magnetic flux ϕ_{1h} & ϕ_{1v} due to the horizontal and vertical components of Earth magnetic field.

2. Calculate the magnetic flux ϕ_{2h} & ϕ_{2v} due to the horizontal and vertical components of Earth magnetic field after rotation.
3. Deduce the average value of the induced $e.m.f.$

IV- Electromagnetic Induction

A circular loop (L) of radius $a = 16\,cm$ and resistance $R = 2\Omega$ is placed in a vertical plane (figure 1) and perpendicular to a horizontal magnetic field $\vec{B}$ directed towards inside. The magnitude B varies with time as shown in figure 2.

Take $0.32\,\pi = 1$.

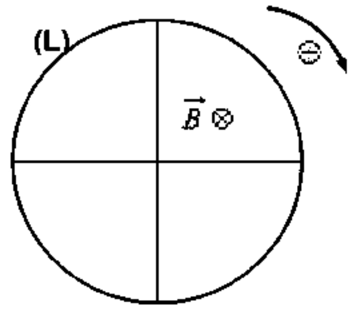


Figure 1

1. Applying Lenz's law, specify the:
   a) direction of the induced current in the time interval $[0;8s]$
   b) absence of induced current in $]8s;12\,s]$.
2. We are interested to study the variations of the magnetic flux in the time interval $[0;8s]$
   a) Expression of the magnetic flux.
      i- Determine the expression of magnitude of the magnetic field B as a function of time.
      ii- Deduce the expression of the magnetic flux crossing the loop.
   b) Induced current:
      i- Verify that the induced $e.m.f$ $e = 20\,\mu V$.
      ii- Determine the induced current and then specify its direction.
3. Draw the curve representing the variations of the induced $e.m.f$ e as a function of time.

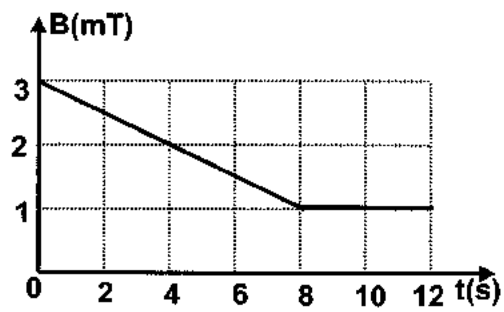


Figure 2

V- Motion of a Conductor in a Uniform Field

A homogeneous metallic rod HN of length $\ell = 4\,cm$, is moving on two horizontal and parallel metallic rails AA' and BB' with a constant speed of $1.2\,m/s$. The center of mass G moves along the axis $(O;\vec{i})$. At the instant $t_0 = 0$, G is at O, the origin of abscissa.

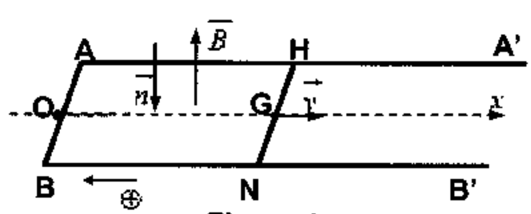


Figure 1

At an instant t, the abscissa of G is $x = \overline{OG}$ and v is the algebraic value of its velocity. The whole set-up formed of the rod and the rails is put within a uniform magnetic field perpendicular to the plane of the horizontal rails $B = 0.02\,T$ (Figure 1).

We neglect the forces of friction and the resistance of the circuit is $R = 10\,\Omega$.

1. Determine the expression of the magnetic flux crossing the circuit $(AHNB)$ in terms of time.
2. Deduce the value of the induced electromotive force.
3. Calculate the value of the induced current and then indicate its direction.
4. Determine the characteristics of the electromagnetic force acting on (HN).
5. Deduce the power dissipated by this force.

VI- Sinusoidal Magnetic Field

A coil formed of $n = 100$ loops of square shape whose side is $a = 4\,cm$, is placed in vertically downwards magnetic field $\vec{B}$ whose magnitude varies according to the relation $B = B_0 \sin(wt)$.

The terminals S and N of the coil are connected to the input Y and the ground M of an oscilloscope respectively as shown in figure 1.

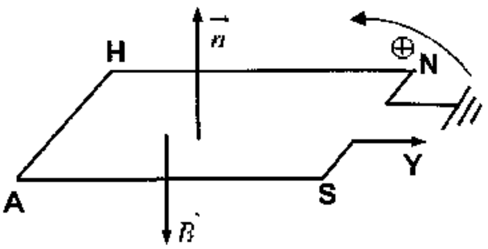
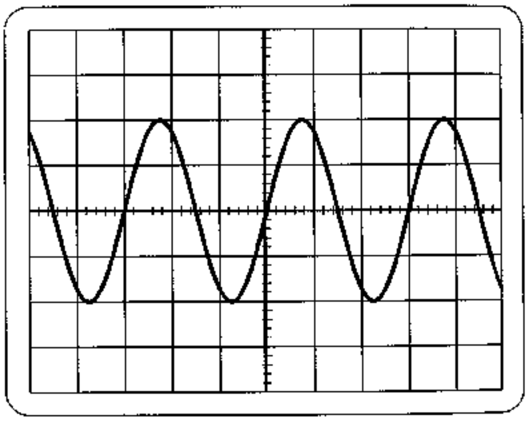


Figure 1

1. Show that the expression of the magnetic flux crossing the coil is given by $\phi = \phi_0 \sin(wt)$ where ϕ_0 is a constant whose expression is to be determined in terms of n, a, B_0.
2. Determine, in terms of n, a, B_0, w and t the expression of the induced $e.m.f$ « e ».
3. The coil does not carry a current. Why?
4. Deduce the expression of the voltage u_{SN} in terms of n, a, B_0, w and t supposing that the coil is oriented positively from N to H.
5. The waveform of figure 2 represents the variations of the voltage u_{SN} as a function of time. Referring to the waveform, determine:
   a) the angular frequency of the voltage displayed;
   b) the maximum value of the voltage u_{SN}.
6. Deduce the amplitude B_0 of the magnetic field B.



S_h=5 ms/div
S_v= 2 V/div

Figure 2

VII- Electromagnetic Induction

A square loop (S) of side $a = 4cm$ is placed horizontally in a uniform vertically upwards magnetic field $\vec{B}$ of magnitude B of length $4a$ as shown in figure 1 (below). (S) is moving with constant speed $v_0 = 10\,cm/s$.

The motion of the loop is studied along the axis $(O;\vec{i})$ where O is the midpoint of its right side (see figure 1 (as view from top)) and the abscissa $x = OM$.

The graph of figure 2, represents the evolution of the **electromotive force induced** $e.m.f$ « e » in the loop (S) in terms of the abscissa x

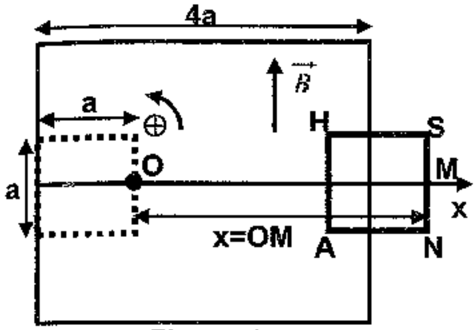


Figure 1

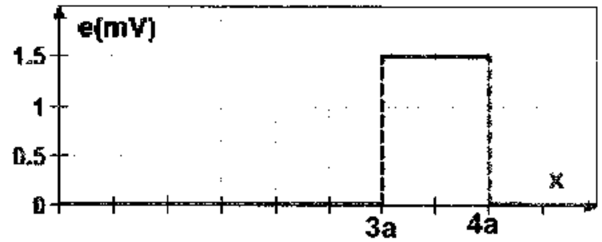


Figure 2

1. Justify that $e = 0$, for $x \leq 3a$.
2. For $3a < x < 4a$ (partially inside the magnetic field).
   a) Justify that the magnetic flux crossing (S) can be written in the form $\phi = k\,t + b$ where k & b are two constants whose expression are to be determined in terms of v_0, B & a.
   b) Show that the expression of the induced $e.m.f$ is $e = B\,a\,v_0$.
   c) Justify that the previous result is compatible with the graph of figure 2.
   d) Deduce B.
   e) Determine the characteristics of the magnetic force acting on the side HA and knowing that the resistance of the loop is 5Ω.
3. Explain why $e = 0$, for $x \geq 4a$.

Solutions

I-

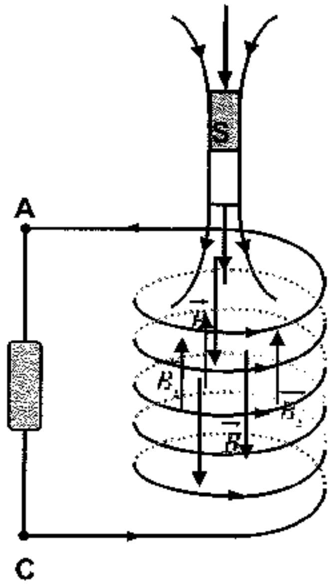
Let $\overrightarrow{B}$ be the magnetic field due to the magnet through the coil.

1. $\overrightarrow{B}$ is vertically downwards and the magnet is approaching from the coil, so its magnitude increases.

   According to Lenz's law, the induced magnetic field $\overrightarrow{B_i}$ tends to oppose this variation, then $\overrightarrow{B_i}$ is vertically upwards.

   By Right Hand Rule, the induced current circulates in the counterclockwise.

   The voltage $u_{AC} > 0$.

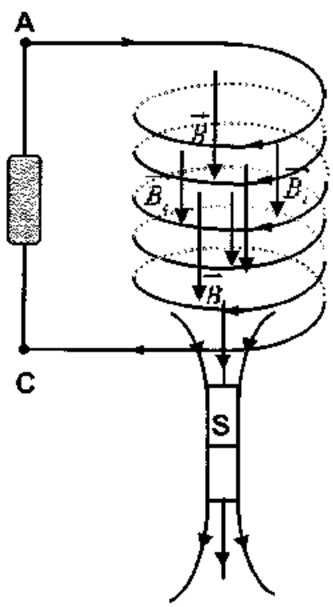


2. $\overrightarrow{B}$ is vertically downwards and the magnet is moving away from the coil, so its magnitude decreases.

   According to Lenz's law, the induced magnetic field $\overrightarrow{B_i}$ tends to oppose this variation, then $\overrightarrow{B_i}$ is vertically downwards.

   By Right Hand Rule, the induced current circulates in the clockwise direction.

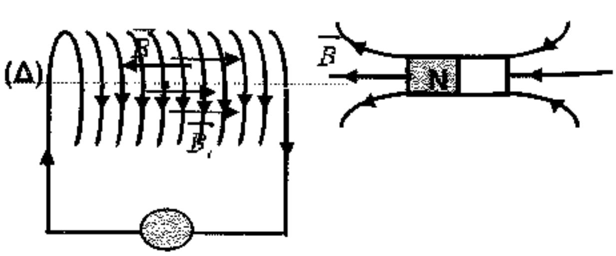
   The voltage $u_{AC} < 0$.



II-

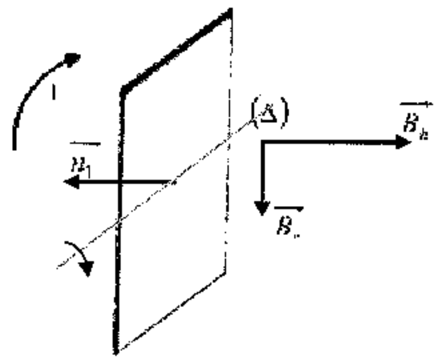
1. The circuit being closed, then the induced current appears when the magnetic flux (magnetic field) varies, which is obtained in the time intervals $2s \le t \le 4s$ & $t \ge 6s$.
2. The induced current circulates in the clockwise direction, by Right Hand Rule the induced magnetic field $\overrightarrow{B_i}$ is horizontal to the right.

   The magnetic field $\overrightarrow{B}$ due to the magnet is horizontal to the left. The direction of $\overrightarrow{B_i}$ is opposite to $\overrightarrow{B}$, then the magnetic field should be increasing according to Lenz's law; which corresponds to the interval $[2s;4s]$.



III-

1. If the circuit is placed vertical, the normal is horizontal to the left (Right Hand Rule).



The magnetic flux due to the horizontal component $\overrightarrow{B_h}$;

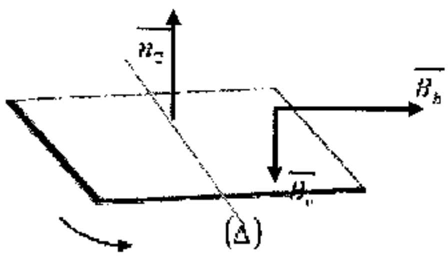
$\phi_{1h} = N\,B_h\,S\cos\theta_{1h}$ where $\theta_{1h} = \left(\overrightarrow{n_1}, \overrightarrow{B_h}\right) = 180^0$;

Then $\phi_{1h} = 1\times 20\times 15\times 10^{-4}\cos\left(180^0\right) = -1.5\,\mu Wb$.

The magnetic flux due to the vertical component $\overrightarrow{B_v}$;

$\phi_{1v} = N\,B_v\,S\cos\theta_{1v} = 0$ where $\theta_{1v} = \left(\overrightarrow{n_1}, \overrightarrow{B_v}\right) = 90^0$.

2. If the circuit is placed horizontal, the normal is vertically downwards (Right Hand Rule).



The magnetic flux due to the horizontal component $\overrightarrow{B_h}$

$\phi_{2h} = N\,B_h\,S\cos\theta_{2h} = 0$ where $\theta_{2h} = \left(\overrightarrow{n_2}, \overrightarrow{B_h}\right) = 90^0$;

The magnetic flux due to the vertical component $\overrightarrow{B_v}$;

$\phi_{2v} = N\,B_v\,S\cos\theta_{2v} = 0$ where $\theta_{2v} = \left(\overrightarrow{n_2}, \overrightarrow{B_v}\right) = 180^0$;

Then $\phi_{2v} = 1\times 10\times 15\times 10^{-4}\cos\left(180^0\right) = -0.75\,\mu Wb$.

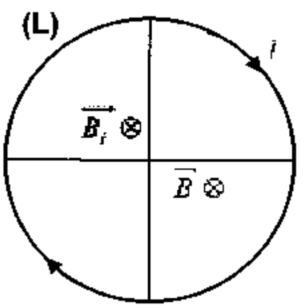
3. The total flux in the vertical position is: $|\phi_1| = |\phi_{1h} + \phi_{1v}| = +1.5\,\mu Wb$;

The total flux in the horizontal position is: $|\phi_2| = |\phi_{2h} + \phi_{2v}| = +0.75\,\mu Wb$;

The average induced *e.m.f* is given by Faraday's law: $e = \left|\frac{\Delta\phi}{\Delta t}\right| = \left|\frac{(0.75-1.5)\mu Wb}{0.5 s}\right| = 0.9375\,\mu V$.

IV-

1. a) For $t\in[0;8s]$ the magnetic field decreases, the magnetic flux $|\phi|$ crossing (L) is variable; thus an induced e.m.f appears in the circuit and since it is closed we have an induced current.



According to Lenz's law the induced magnetic field $\overrightarrow{B_i}$ tends to oppose the variations of the magnetic field $\overrightarrow{B}$. But B decreases and ⊗, then $\overrightarrow{B_i}$ is also ⊗. By Right Hand Rule, the induced current circulates in the clockwise.

b) For $t\in[8s;12s]$, the magnetic flux crossing (L) is constant, no induced current.

2. a) Expression of the magnetic flux

i- The curve representing B is a straight line, then its equation is $B = a\,t + b$;

with $a = \frac{\Delta B}{\Delta t} = \frac{(1-3)mT}{(8-0)s} = -0.25\,mT/s$ & $b = B|_{t=0} = 3\,mT$;

Then $B = -0.25\,t + 3$ (where t in s and B in mT)

ii- The magnetic flux $\phi = B\,S\cos\left(\vec{n}, \vec{B}\right) = B\times\left(\pi r^2\right)\cos(0) = (-0.25\,t+3)(1/0.32)\left(16\times 10^{-2}\right)^2$;

Then $\phi = -0.02t + 0.24$ (where t in s and ϕ in mWb).

b) Induced *e.m.f:*

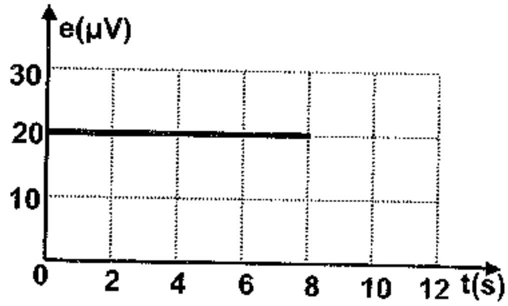
i- According to Faraday's law:

$$e=-\frac{d\phi}{dt}=0.02\ mV=20\ \mu V.$$

ii- The current $i=\frac{e}{R}=\frac{20}{2}=10\ \mu A$.

$i>0$, then the induced current flows in the positive direction.

3. Adjacent curve.



V-

1. The magnetic flux is given by: $\phi=BS\cos\theta$ where $\theta=(\vec{n},\vec{B})=180^0$ & $S=x\times\ell_{HN}$;
   The motion of the rod is uniform and starts from origin, then $x=vt$;
   We get, $\phi=B\times x\times\ell_{HN}\times\cos(180^0)=0.02\times(1.2t)\times0.04\times(-1)$;
   Thus, $\phi=-9.6\times10^{-4}t$ where t in s & ϕ in Wb.
2. Faraday's law $e=-\frac{d\phi}{dt}=9.6\times10^{-4}V$.
3. The induced current $i=\frac{e}{R}=9.6\times10^{-5}A>0$, then the induced current flows in the positive direction.
4. Let $\vec{F_\ell}$ be the electromagnetic force acting on the rod:
   - ✖ Point of application: midpoint of the rod;
   - ✖ Line of action: horizontal [perpendicular to the plane (field, rod)].
   - ✖ Direction: to the left (R.H.R).
   - ✖ Magnitude: $F_\ell=iB\times\ell_{HN}\sin(90^0)=9.6\times10^{-5}\times0.02\times0.04=7.68\times10^{-8}N$.
5. The power dissipated is: $P=\vec{F_\ell}.\vec{v}=F_\ell\times v\times\cos(180^0)=-9.22\times10^{-8}W$.

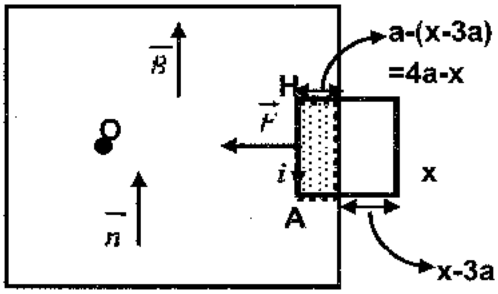
VI-

1. The magnetic field crossing the loop is given by $\phi=nBS\cos\theta$; where $\theta=(\vec{n};\vec{B})=180^0$;
   Then $\phi=nB_0\sin(wt)a^2\cos(180^0)=-nB_0a^2\sin(wt)$;
   So the expression of the magnetic flux is of the form, $\phi=\phi_0\sin(wt)$ where $\phi_0=-nB_0a^2$.
2. According to Faraday's law: $e=-\frac{d\phi}{dt}=nB_0a^2w\cos(wt)$.
3. Connecting an oscilloscope is equivalent to an opened circuit, so no induced current in circuit.
4. The voltage $u_{SN}=ri-e=-nB_0a^2w\cos(wt)$.
5. a) The period of the signal displayed is $T=S_h\times x=5\,ms/\text{div}\times3\text{div}=15\,ms$;
   The angular frequency $w=\frac{2\pi}{T}=\frac{2\pi}{15\times10^{-3}}=\frac{400\pi}{3}\text{rad}/s$.
   b) The maximum value $(U_{SN})_{\max}=S_v\times y_{\max}=2V/\text{div}\times2\text{div}=4V$.

6. The expression of the maximum voltage is $(U_{SN})_{\max} = \left|-n B_0 a^2 w\right| = n B_0 a^2 w$;

Then $B_0 = \dfrac{(U_{SN})_{\max}}{n \times a^2 \times w} = \dfrac{4}{100 \times (4\times 10^{-2})^2 \times \left(\dfrac{400\pi}{3}\right)} \approx 0.06T = 60mT$.

VII-



1. For $x \le 3a$, the loop is completely inside the region subjected to the magnetic field, so the magnetic flux crossing the loop ϕ remains constant.

   Then, no induced *e.m.f* in the loop.

2. a) For $3a < x < 4a$;

   The area subjected to the magnetic field (while crossing the region) is $S = a \times (4a - x) = 4a^2 - ax$;

   & $\theta = (\vec{n}\,;\vec{B}) = 0^0$;

   Then $\phi = B\,S\cos\theta = 4\,B\,a^2 - B\,a\,x$;

   But the motion is uniform, so $x = v_0\,t$;

   We get $\phi = 4\,B\,a^2 - B\,a\,v_0\,t$ which is of the form $\phi = \phi_0 + k\,t$ where $\phi_0 = 4\,B\,a^2$ & $k = -B\,a\,v_0$.

   b) According to Faraday's law: $e = -\dfrac{d\phi}{dt} = B\,a\,v_0$.

   c) The induced *e.m.f* is constant, so its graphical representation should be a horizontal straight line which is compatible with the curve drawn.

   d) Graphically $e = 1.5\,mV$, so $B = \dfrac{e}{a\,v_0} = \dfrac{1.5\times 10^{-3}}{4\times 10^{-2} \times 10 \times 10^{-2}} = 0.375\,T$.

   e) Characteristics of the force $\vec{F}$:

   Point of application: midpoint of HA;

   Direction: horizontal to the left (Right Hand Rule);

   Magnitude: $F = i \times B \times a \sin(90^0) = \dfrac{e}{R} \times B \times a = \dfrac{1.5\times 10^{-3}}{5} \times 0.375 \times 4 \times 10^{-2} = 4.5 \times 10^{-6}\,N$.

3. If $x > 4a$, the loop is no longer subjected to a magnetic field, then no magnetic flux crossing the loop; consequently no induced *e.m.f.*

Problems

I-
Magnetic Field and Circuit

Consider a horizontal square loop $(HNSA)$ of side $4cm$ in a vertically downwards magnetic field (figure 1) whose magnitude varies according to the graph shown in figure 2.

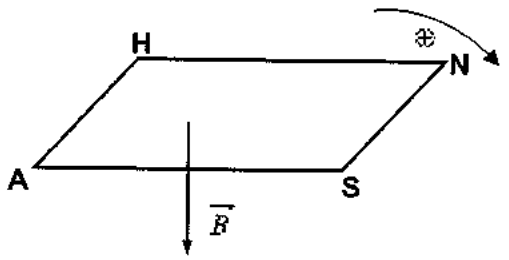


Figure 1

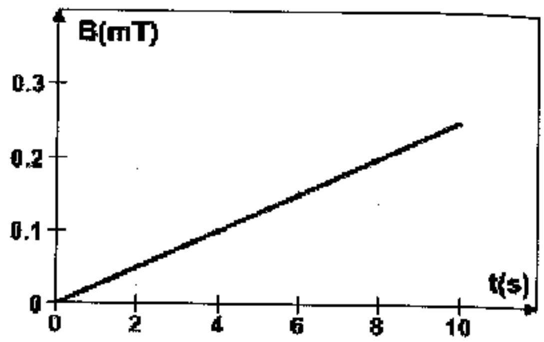


Figure 2

Part A
Direction of the induced current

1. Justify that the loop is the seat of the electromagnetic induction phenomenon.
2. Applying Lenz's law, determine the direction of the induced current.

Part B
Induced current

1. Justify that the expression of the magnetic field B (figure 2), in terms of time, can be written in the form $B = at + b$ where a and b are constants whose values are to be determined in SI units.
2. Considering the clockwise as positive direction (figure 1).
   a) Show the expression of the magnetic flux as a function of time is given by $\phi = 4\times10^{-8}t$ (t in s ; ϕ in Wb).
   b) Deduce the value of the induced $e.m.f$ « e ».
   c) Draw the graph representing the variations of « e » in terms of time.
3. A galvanometer is inserted in this circuit measures a current $i = 5\times10^{-8}A$.
   a) Determine the characteristics of the electromagnetic force acting on HN at the instant $t = 4s$.
   b) Calculate the resistance of the loop per unit of length.

Part C
Breathing monitor and induction

To monitor the breathing of a patient in a hospital, a thin belt is wrapped around the patient's chest. The belt is a 200 turns coil. When the patient breathes during a duration of $2.25s$, the area encircled by the coil increases by $\Delta S = 45\,cm^2$. The magnitude of the Earth's magnetic field is $B = 30\,\mu T$ and makes an angle of 20^0 with the plane of the coil.

Calculate of the average induced $e.m.f$ in the coil during this duration.

II- Electromagnetic Induction

A circular loop (L) placed in a vertical plane (figure 1) of radius[1] $a = 10\,cm$ and resistance $R = 10\,\Omega$ is placed in a horizontal magnetic field $\vec{B}$, directed towards outside of the loop's plane and perpendicular to it, whose magnitude varies with time as shown in figure 2. **Take** $\pi \approx 3.14$.

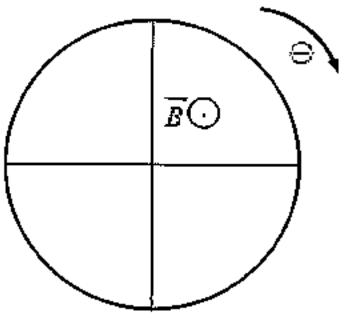


Figure 1

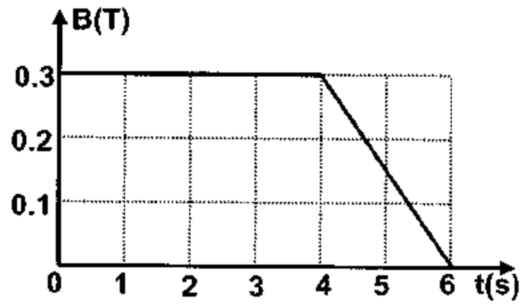


Figure 2

1. Determine, with justification, the existence and the direction of the induced current in the time intervals $[0;4s[$ and $[4\,s;6s]$.
2. By considering the clockwise as positive direction, all the following questions are to be answered, for the time interval $[4\,s;6s]$:
   a) Expression of the magnetic flux:
      i- Determine the expression of magnetic field B as a function of time.
      ii- Deduce the expression of the magnetic flux crossing the loop.
   b) Induced current:
      i- Verify that the induced $e.m.f\,(e)$ is $e = -4.71\,mV$.
      ii- Deduce the current and then its direction.
      iii- Is this result compatible with part 1?
3. Draw the graph representing the variations of the induced $e.m.f\;e$ as a function of time.
4. Recopy figure 1, and represent on it the direction of current and the equivalent generator in the interval $[4\,s;6s]$.
5. We consider the time interval $[4\,s;6s]$, when the loop is traversed by a constant current I_0 as shown in figure 3. On the circumference of the loop (L) we take two equal elements of lengths $d\ell$ and $d\ell'$ diametrically opposite centered at M and M' respectively $d\ell = d\ell'$.
   a) Represent the electromagnetic forces $\overrightarrow{dF}$ and $\overrightarrow{dF'}$ acting on two equal elements of lengths $d\ell$ and $d\ell'$.
   b) Determine the magnitudes of $\overrightarrow{dF}$ and $\overrightarrow{dF'}$ in terms of I_0, B and $d\ell$.
   c) Deduce that $\overrightarrow{dF} + \overrightarrow{dF'} = \vec{0}$.
   d) Show that the resultant electromagnetic force acting on the loop is zero.

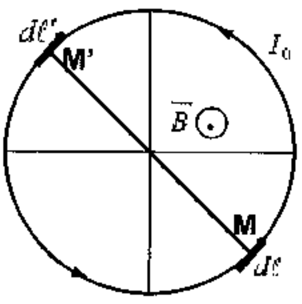


Figure 3

[1] Area of a circle is $S = \pi r^2$

III-(See Appendix 1 – page 519)

Motion of a Conductor in a Uniform Field

A homogeneous metallic rod HN of length $\ell = 20cm$, resistance $R = 5\Omega$ slides on two horizontal and parallel metallic rails AA' and EE'.

During its sliding, the rod remains perpendicular to the rails and its center of mass G moves along the axis $(O;\vec{i})$. At the instant $t_0 = 0$, G is at O, the origin of abscissa.

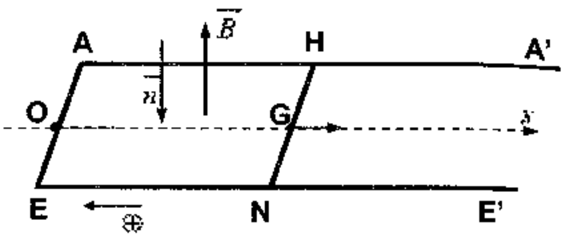


Figure 1

At an instant t, the abscissa of G is $x = \overline{OG}$ and v is the algebraic value of its velocity. The whole set-up formed of the rod and the rails is put within a uniform magnetic field perpendicular to the plane of the horizontal rails $B = 0.25T$ (figure 1).

We neglect the forces of friction and the resistance of the rest of circuit.

Part A
Theoretical study

1. Explain the existence of an induced current « i » in the rod.
2. Determine, at the instant t, the expression of the magnetic flux crossing the surface $AHNE$ in terms of B, ℓ and x taking into consideration the chosen arbitrary positive direction on Figure 1
3. Deduce the expression of the induced $e.m.f$ « e » in terms of B, ℓ and v.

Part B
Motion with constant velocity

The rod is pulled by a force $\vec{F} = F\vec{i}$ so that its velocity remains constant $\vec{v_0} = 0.8\vec{i}\ (m/s)$.

1. Justify that the induced current is constant and then calculate its value.
2. Determine the characteristics of the electromagnetic force (Laplace's force) acting on the rod.
3. State and then represent the forces acting on rod.
4. Deduce the value of F.
5. Justify that the power generated by the operator is $P = 320\,\mu W$.

Part C
Motion with initial velocity

The rod is given an initial velocity $\vec{v_0} = 0.8\vec{i}\ (m/s)$.

1. State and then represent the forces acting on the rod.
2. Show that the differential equation satisfied by the velocity is given by $\dfrac{dv}{dt} + \dfrac{B^2\ell^2}{mR}v = 0$.
3. Justify that $v = v_0 e^{-\frac{t}{\tau}}$ is a solution of the previous differential equation where τ is a constant whose expression to be determined.
4. Determine the instantaneous expression of the power dissipated by Laplace force.
   Deduce its value at the instant $t = 0$.
5. Draw a sketch representing the variations of the power as a function of time.

IV-

Electromagnetic Induction

A rectangular circuit $(HNAS)$ of resistance $R = 50\,\Omega$ is formed of $N = 150$ loops and of dimensions $(\ell = 10\,cm \times d = 5\,cm)$ is placed in a vertical plane and subjected to a horizontal inwards ⊗ magnetic field $\overrightarrow{B}$ (figure 1) whose magnitude varies in terms of time as shown in the curve below (figure 2).
The counterclockwise is taken as positive direction, so that the normal vector $\overrightarrow{n}$ to the loop is outwards ⊙.

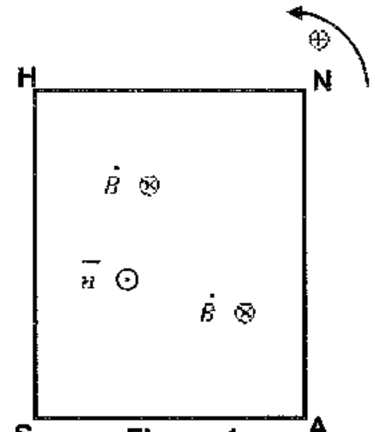


Figure 1

Part A

Induced electromotive force over [0; 3s]

1. Show that the magnetic flux crossing the loop at $t = 0$ is $\phi_0 = -4.5\,mWb$.
2. Applying Lenz's law, determine the direction of the induced current in the loop.
3. Justify that the expression of the induced $e.m.f$ can be written in the form $e_1 = N \times \ell \times d \times \frac{dB}{dt}$.
4. For $t \in [0; 3s]$, justify that $\frac{dB}{dt} = -2mT/s$, then deduce the values of the induced $e.m.f$, current and its direction.
5. Calculate the power dissipated in the loop.
6. Determine the characteristics of the electromagnetic force acting on the side $d = SA$ at the instant $t = 2\,s$.

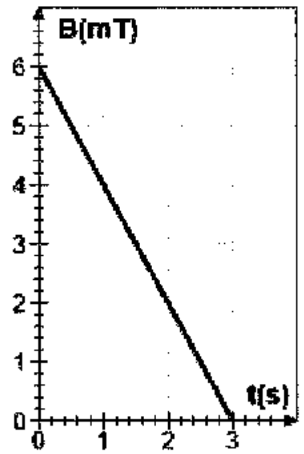


Figure 2

Part B

Periodic triangular magnetic field

The magnitude of the magnetic field is made to vary as shown in the curve of figure 3 below.

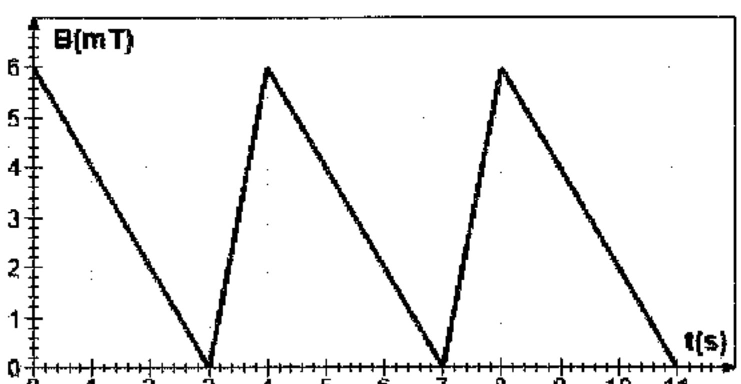


Figure 3

1. Justify that $e_2 = 4.5\,mV$ for $t \in [3s; 4s]$.
2. Give a physical interpretation for the signs of the $e.m.f$ and that $|e_2| > |e_1|$.
3. Justify that $e.m.f$ is periodic.
4. Draw the graph representing the variation of the induced $e.m.f$ in terms of time.

V-(See Appendix 1 – Page 519)

Electromagnetic Induction

A metallic rigid rod SN, of mass $m = 2g$, can slide without friction, on two parallel and horizontal metallic rails AP and HQ separated by a distance $\ell = 20\,cm$. During sliding, the rod remains perpendicular to the rails. These two rails, separated by the distance ℓ, are connected by a resistor of resistance $R = 5\Omega$ (Figure 1). Neglect the resistance of the rod and of the rails.

The whole setup is placed in an upward, uniform and vertical magnetic field $\vec{B}$ of magnitude B. The position of G, the center of inertia of the rod, is defined by its abscissa x on the horizontal axis $(O;\vec{i})$ with O corresponding to the position of G at $t_0 = 0$.

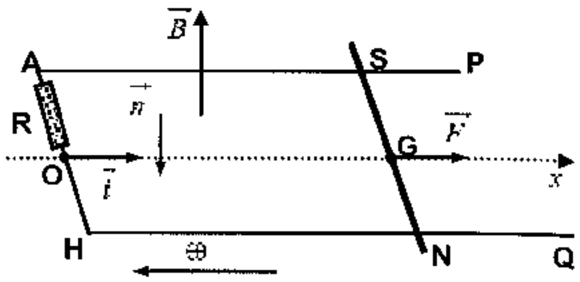


Figure 1

Taking into consideration the arbitrary positive direction chosen in figure 1.

At the instant $t_0 = 0$, taken as origin of time, the rod being at rest is submitted to a constant force $\vec{F} = F\vec{i}$ of magnitude F. At an instant t, G has an abscissa $OG = x$ and a velocity $\vec{v}$ of algebraic value v (Figure 1).

Part A

Theoretical study

1. Give the name of the physical phenomenon that will take place in the circuit.
2. Explain the existence of an induced *e.m.f* « e » between the terminals S and N of the rod.
3. Represent, with justification, the electromagnetic force (Laplace's force) acting on the rod.
4. Determine, at the instant t, the expression of the magnetic flux crossing the surface $ASNH$ in terms of B, ℓ, and x.
5. Show that the expression of the induced *e.m.f* « e » is given by $e = B\ell v$.

Part B

Variation of the induced *e.m.f* «e»

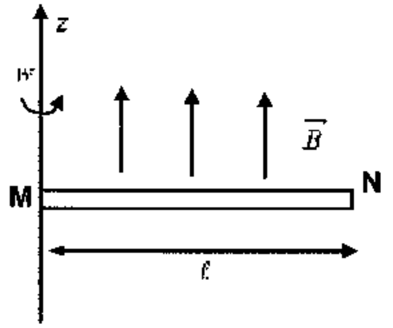
1. Applying Newton's 2nd law on the rod and taken into consideration the expression of e, show that the differential equation satisfied by the induced *e.m.f* « e » is given by: $\frac{de}{dt} + \frac{B^2\ell^2}{mR}e = \frac{FB\ell}{m}$.
2. Deduce the expression of E_0 the induced *e.m.f* reached in steady state.
3. The solution of this differential equation is $e = E_0\left(1 - e^{-\frac{t}{\tau}}\right)$.

   Determine the expression of the constant τ in terms of B, m, R & ℓ.
4. The steady state reached is found to be $4V$ within $2.5s$, deduce the magnitude of the magnetic field.
5. Draw the shape of the graph representing the variation of e as a function of time and show, on this graph, the points corresponding to E_0 & τ.

VI-(CSB 2008)

Electromagnetic Induction by Rotation

A conducting rod MN of length ℓ rotates in a horizontal plane about a vertical z-axis passing through its extremity M with a constant angular velocity w. The whole region is under the effect of a uniform magnetic field $\overrightarrow{B}$ along the vertical z-axis as shown in the adjacent figure.



Part A

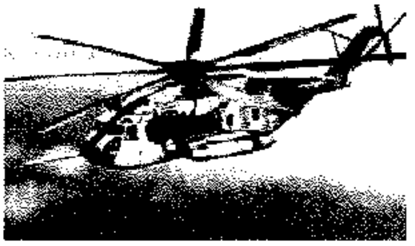
Theoretical study

1. Determine the angle of rotation and the area swept by the rod in a time element dt.
2. Deduce the magnetic flux swept by the rod during this interval dt.
3. Apply Faraday's law to determine the induced $e.m.f$ between M and N.

Part B

Application

A helicopter has blades with a length of $\ell = 3m$ extending outward from the center hub and rotating at an angular velocity of 10 revolutions per second. If the vertical component of the Earth's magnetic field is $50\,\mu T$.



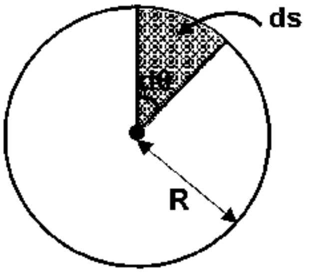
Deduce the $e.m.f$ induced between the blade tip and the center hub.

Area of a sector in a circle.

The area of a circle of radius $\boldsymbol{R}$, is $\boldsymbol{\pi R^2}$ corresponds to a central angle of $\boldsymbol{2\pi}$. Then, an elementary arc $\boldsymbol{d\theta}$ corresponds to an area $\boldsymbol{ds}$.

$\pi R^2 \longrightarrow 2\pi$

$ds \longrightarrow d\theta$; then $ds = \dfrac{1}{2}R^2 d\theta$



Solutions

I-

Part A

1. The loop is traversed by a variable magnetic field, then the magnetic flux crossing the loop is also variable. Thus, the loop is the seat of electromagnetic induction.
2. The magnitude of the magnetic field $\overline{B}$ is increasing and vertically downwards, according to Lenz's law the induced magnetic field $\overrightarrow{B_i}$ is vertically upwards.

   Referring to the right hand rule the induced current circulates in the counterclockwise direction.

Part B

1. The curve representing the variations of $\vec{B}$ in terms of time is a straight line passing through origin then $B = at + b$ where $b = 0$ & $a = \frac{\Delta B}{\Delta t} = \frac{(0.2-0)\times 10^{-3}T}{(8-0)s} = 2.5\times 10^{-5}T/s$.

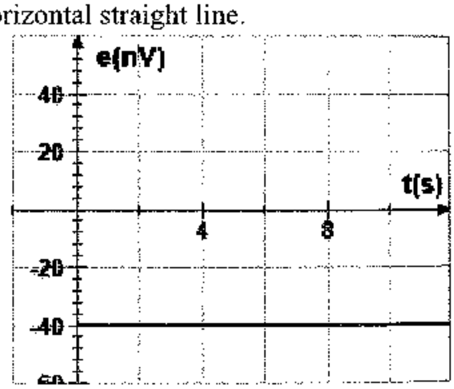
   Then $B = 2.5\times 10^{-5}\,t$ (where t in s and B in T).
2. a) The magnetic flux $\phi = NBS\cos\theta$ where $\theta = (\vec{n}, \vec{B}) = 0$;

   Then $\phi = 1(2.5\times 10^{-5}t)(16\times 10^{-4})(1) = 4\times 10^{-8}t$ (where t in s and ϕ in Wb).

   b) Faraday's law $e = -\frac{d\phi}{dt} = -4\times 10^{-8}V = -40nV < 0$.

   c) The induced $e.m.f$ is constant so it is represented by a horizontal straight line.

   Scale: on the ordinate axis $1\text{div} \equiv 20nV$.



3. a) Characteristics of the electromagnetic force:

   Point of application: Midpoint of HN;

   Line of action: Perpendicular to plane (wire, $\vec{B}$)

   Direction: outwards;

   Magnitude: (at $t = 4s$, $B = 0.1mT$)

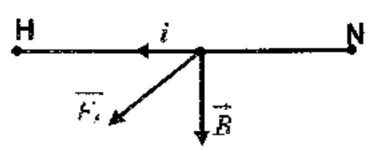
   $F_\ell = i\,B\,\ell_{HN}\sin\alpha$;

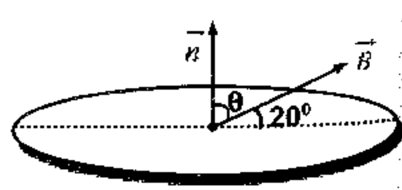
   $F_\ell = 5\times 10^{-8}\times 0.1\times 10^{-3}\times 0.04\times \sin(1) = 2\times 10^{-13}N$.

   b) Ohm's law $i = \frac{e}{R}$; then the resistance of the circuit is

   $R = \frac{4\times 10^{-8}V}{5\times 10^{-8}A} = 0.8\Omega$;

   But $r = \frac{R}{4a} = \frac{0.8\ \Omega}{4\times 4\times 10^{-2}m} = 5\Omega/m$.





Part C

The average induced $e.m.f$ $|e| = \left|-\frac{\Delta\phi}{\Delta t}\right| = \left|\frac{N\,B\,\Delta S\cos\theta}{\Delta t}\right|$;

Then, $|e| = \frac{200\times 30\times 10^{-6}\times 45\times 10^{-4}\times\cos(70^0)}{2} = 8.5\times 10^{-6}V$.

II-

1. For $t \in [0;4s[$, the magnitude of the magnetic field is constant; the magnetic flux crossing (L) is also constant. Then no induced $e.m.f$ and no induced current.

   For $t \in [4s\ ;\ 6s]$, the magnitude of the magnetic field decreases; the magnetic flux $|\phi|$ crossing (L) decreases also. Then the circuit is the seat of an induced $e.m.f$ and the circuit is closed. Thus, an induced circuit flows in the circuit.

   According to Lenz's law the induced magnetic field $\overrightarrow{B_i}$ must oppose to the variations of the magnetic field $\overrightarrow{B}$. Since $\left\|\overrightarrow{B}\right\|$ decreases and $\odot$ then $\overrightarrow{B_i}$ is also $\odot$.

   Referring to the right hand rule, the induced current flows in the counter-clockwise direction.

2. a) Expression of the magnetic flux

   i- $B = at + b$ with $a = \dfrac{\Delta B}{\Delta t} = -0.15\,T/s$ & $b = 0.9\,T$;

   Then $B = -0.15t + 0.9$ (where t in s & B in T)

   ii- The magnetic flux

   $$\phi = BS\cos(\vec{n},\overrightarrow{B}) = B \times (\pi r^2)\cos\pi;$$

   $$= 4.71 \times 10^{-3}\,t - 0.02826 \quad (t \text{ in } s \text{ \& } \phi \text{ in } Wb)$$

3. a) Induced $e.m.f$:

   i- According to Faraday's law:

   $$e = -\frac{d\phi}{dt} = -4.71 \times 10^{-3} = -4.71\,mV.$$

   ii- Ohm's law : $i = \dfrac{e}{R} = \dfrac{-4.71}{10} = -0.471\,mA$.

   $i < 0$, then the current flows in a direction opposite to the positive chosen.

   iii- Yes, they are compatible.

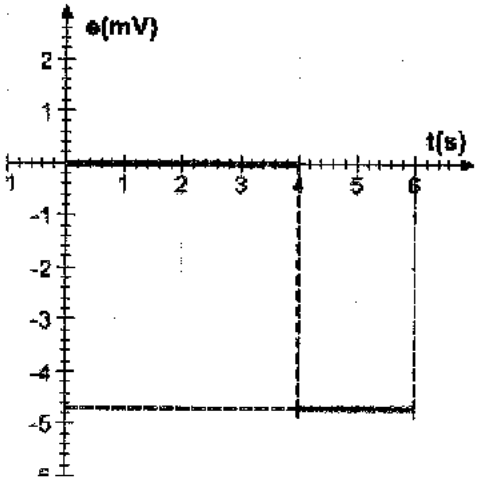


Figure 5

3. Figure 5.
4. Figure 6.

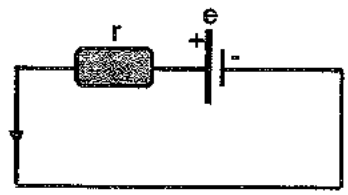


Figure 6

5. a) Figure 7.

   i- Point of application: Midpoint of the element.
   Along the corresponding radius but out from the circle (RHR).

   ii- Laplace's force: $dF = I_0\,B\,d\ell \sin 90^0 = I_0\,B\,d\ell$.

   $$dF' = I_0\,B\,d\ell' \sin 90^0 = I_0\,B\,d\ell.$$

   iii- $\overrightarrow{dF}$ & $\overrightarrow{dF'}$ have the same magnitude, same direction and opposite direction.

   Then $\overrightarrow{dF} = -\overrightarrow{dF'}$, thus $\overrightarrow{dF} + \overrightarrow{dF'} = \vec{0}$.

   b) The circle can be divided into pairs of elements that are diametrically opposite, having a zero sum by pairs.

   $$\sum \overrightarrow{F} = \left(\overrightarrow{dF_M} + \overrightarrow{dF'_{M'}}\right) + \left(\overrightarrow{dF_A} + \overrightarrow{dF'_{A'}}\right) + \ldots\ldots$$

   $$= \vec{0} + \vec{0} + \ldots\ldots = \vec{0}$$

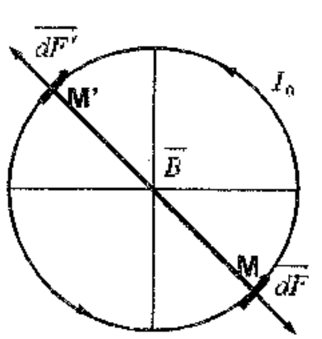


Figure 7

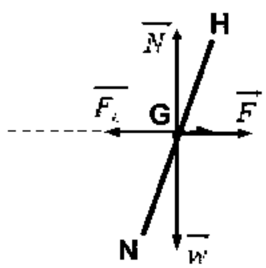
III-

Part A

1. When the rod is in motion, the magnetic flux $|\phi|$ crossing the circuit $(AHNE)$ is variable. Then the rod is the seat of an induced electromotive force & since the circuit is closed; thus an induced current flows in the circuit.
2. The magnetic flux $\phi = N\,B\,S\cos\left(\vec{n},\vec{B}\right)$ where the area $S=\ell\times x$ & $\left(\vec{n},\vec{B}\right)=\pi$ (opposite directions); Then $\phi = 1\times B\times\ell\times x\times\cos(\pi) = -B\times\ell\times x$.
3. According to Faraday's law: $e=-\dfrac{d\phi}{dt}=-\dfrac{d}{dt}\left(-B\times\ell\times x\right)=B\times\ell\times x'=B\times\ell\times v$.

Part B

1. The induced current is given by: $i=\dfrac{e}{R}=\dfrac{B\times\ell\times v_0}{R}$ which is constant since the velocity is uniform; so, $i=\dfrac{0.25\times0.2\times0.8}{5}=8\times10^{-3}\,A=8mA$.
2. $\overrightarrow{F_\ell}$ is Laplace's force acting on the rod :
   - Point of application: midpoint G of the rod.
   - Direction: referring to R.H.R, in the negative direction of $\left(O;\vec{i}\right)$.
   - Magnitude: $F_\ell = i\times B\times\ell\times\sin\left(90^0\right)=8\times10^{-3}\times0.25\times0.2\times\sin\left(90^0\right)=4\times10^{-4}\,N$.
3. The forces acting on the rod are: its weight $\vec{w}$, the normal reaction exerted by the rails $\vec{N}$, Laplace's force $\overrightarrow{F_\ell}$ and the force $\vec{F}$.



4. Newton's 2[nd] law applied on the rod: $\sum\overrightarrow{F_{ext}}=\dfrac{d\vec{P}}{dt}$;

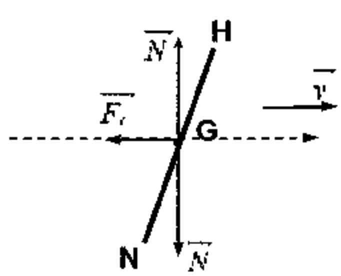
   But the rod is moving with constant velocity so $\dfrac{d\vec{P}}{dt}=\vec{0}$;

   We get $\overrightarrow{F_\ell}+\vec{F}+\vec{w}+\vec{N}=\vec{0}$, but $\vec{w}+\vec{N}=\vec{0}$;

   Then $\vec{F}=-\overrightarrow{F_\ell}$, thus $F=F_\ell=4\times10^{-4}\,N$.
5. The power due to the force is $P=\vec{F}.\vec{v}=F\,v_0\cos0=4\times10^{-4}\times0.8=3.2\times10^{-4}\,W=320\mu W$.

Part C

1. The forces acting on the rod are: its weight $\vec{w}$, the normal reaction exerted by the rails $\vec{N}$ and Laplace's force $\overrightarrow{F_\ell}$.



2. Newton's 2[nd] law applied on the rod: $\sum\overrightarrow{F_{ext}}=\dfrac{d\vec{P}}{dt}$;

   We get $\overrightarrow{F_\ell}+\vec{w}+\vec{N}=m\dfrac{dv}{dt}\vec{i}$, but $\vec{w}+\vec{N}=\vec{0}$;

   So, $-i\,B\,\ell=m\dfrac{dv}{dt}$ where $i=\dfrac{B\,\ell\,v}{R}$; we get $\dfrac{dv}{dt}+\dfrac{B^2\ell^2}{m\,R}v=0$.

3. At $t=0$, $v=v_0e^0=v_0$ (verified);

We have $v=v_0e^{-\frac{t}{\tau}}$, so $\frac{dv}{dt}=-\frac{v_0}{\tau}e^{-\frac{t}{\tau}}$, replacing in the differential equation we get :

$-\frac{v_0}{\tau}e^{-\frac{t}{\tau}}+\frac{B^2\ell^2}{mR}v_0e^{-\frac{t}{\tau}}=0$; $\left(-\frac{1}{\tau}+\frac{B^2\ell^2}{mR}\right)v_0e^{-\frac{t}{\tau}}=0$ is verified at any instant;

but $v_0e^{-\frac{t}{\tau}}\neq 0$, so $-\frac{1}{\tau}+\frac{B^2\ell^2}{mR}=0$, then $\tau=+\frac{mR}{B^2\ell^2}$.

4. The power dissipated is :

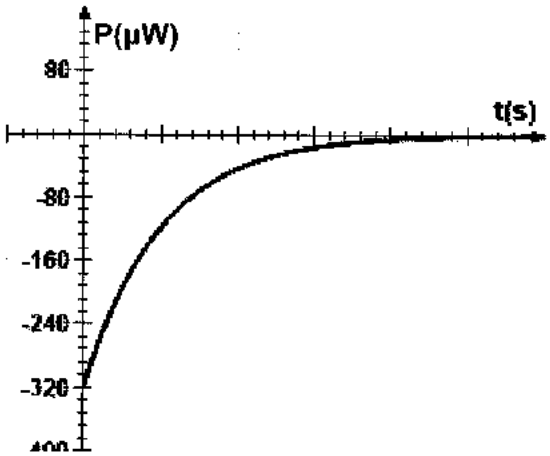
$P=\overrightarrow{F_\ell}.\vec{v}=F_\ell\times v\times\cos(\pi)$;

$P=-i\times B\times\ell\times v=-\frac{B^2\ell^2}{R}v^2$;

But $v=v_0e^{-\frac{t}{\tau}}$, then $P=-\frac{B^2\ell^2v_0^2}{R}e^{-\frac{2t}{\tau}}$;

At $t_0=0$, $P|_{t=0}=-\frac{B^2\ell^2v_0^2}{R}=-320\,\mu W$;

(Calculated in the previous part).



5. The power is an exponential function of the form $P=P_0\,e^{-\frac{2t}{\tau}}$ where $P_0<0$, and tends to zero in steady state.

IV-

Part A

1. The magnetic flux $\phi_0=N\,B_0\,S\cos\theta$ where $\theta=\left(\vec{n},\vec{B}\right)=180^0$;

So, $\phi_0=150\times6\times10^{-3}\times10\times10^{-2}\times5\times10^{-2}\cos\left(180^0\right)=-4.5\times10^{-3}\,Wb=-4.5\,mWb$

2. Over $[0;3s]$, the magnitude of the magnetic field $\vec{B}$ decreases with time.

According to Lenz's law, the induced magnetic field $\overrightarrow{B_i}$ tends to oppose this variation, so $\overrightarrow{B_i}$ is also inwards $\otimes$; referring to Right Hand Rule (R.H.R), the induced current circulates in the clockwise direction.

3. We have $\phi=N\,B\,S\cos\left(180^0\right)=-N\times\ell\times d\times B$;

Faraday's law: $e=-\frac{d\phi}{dt}=N\times\ell\times d\times\frac{dB}{dt}$.

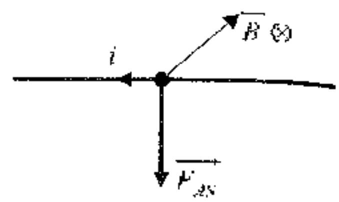
4. $\frac{dB}{dt}$ is the slope of the straight line; so $\frac{dB}{dt}=\frac{\Delta B}{\Delta t}=\frac{(0-6)mT}{(3-0)s}=-2mT/s$;

Then $e=150\times10\times10^{-2}\times5\times10^{-2}\times-2=-1.5\,mV$.

The induced current $i=\frac{e}{R}=\frac{-1.5}{50}=-0.03\,mA=-30\,\mu A$.

$i<0$, so the induced current circulates in a direction opposite to that taken positive; then it flows in the clockwise direction.

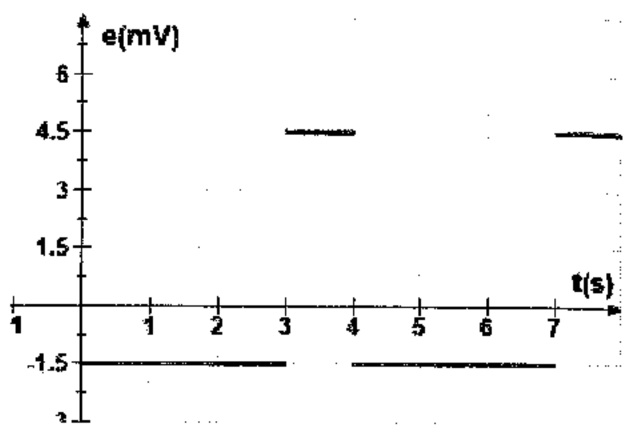
5. The power dissipated $P = R\,i^2 = 50 \times \left(30 \times 10^{-6}\right)^2 = 4.5 \times 10^{-8}\,W$.



6. Point of application: midpoint of $[AS]$;
   Direction: vertically downwards (Right Hand Rule);
   Magnitude: $F_{AS} = |i| \times B \times \ell_{AS} \sin\left(90^0\right)$;
   $$F_{AS} = 30 \times 10^{-6} \times 2 \times 10^{-3} \times 5 \times 10^{-2} \times 1 = 3 \times 10^{-9}\,N.$$

Part B

1. For $t \in [3s;4s]$, $\dfrac{dB}{dt} = \dfrac{\Delta B}{\Delta t} = \dfrac{(6-0)mT}{(3-2)s} = 6mT/s$;
   Then $e_2 = 150 \times 6 \times 10^{-3} \times 0.05 \times 0.1 \times 6 = 4.5\,mV$.
2. $e_2 > 0$, then the induced current circulates in the counterclockwise direction;
   $|e_2| > |e_1|$, is due to the fact that for $t \in [3s;4s]$, the absolute rate of change in the magnitude of the magnetic field is greater than that in $[0;3s]$.
3. $\dfrac{dB}{dt}$ is the same for $t \in [4s;7s]$ & $t \in [0;3s]$, (the straight lines are parallels); then $e_3 = e_1$ & similarly $e_4 = e_2$.
4. Graph.

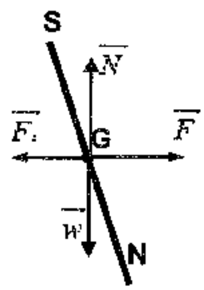


V-

Part A

1. Electromagnetic induction.
2. The surface area of the circuit increases, as the rod is moved; then the magnetic flux through the circuit increases. Thus, the circuit is the seat of an induced *e.m.f.*
3. According to Lenz's, the electromagnetic effect of Laplace's force should oppose the variations in the magnetic flux. Then it acts in a direction opposite to that of $\overrightarrow{F}$.
4. The magnetic flux crossing the circuit is given by $\phi = \overrightarrow{B}.\overrightarrow{S} = B\,S\cos\theta$;
   Where $\theta = \left(\overrightarrow{n};\overrightarrow{B}\right) = \pi$ & the circuit has a rectangular shape so $S = \ell\,x$;
   Then $\phi = B\,\ell\,x\cos(\pi) = -B\,\ell\,x$.
5. According to Faraday's law $e = -\dfrac{d\phi}{dt} = B\,\ell\,v$.

Part B



1. Newton's 2nd law: $\sum \overrightarrow{F_{ext}} = \dfrac{d\overrightarrow{P}}{dt}$, we get $\overrightarrow{F_\ell} + \overrightarrow{F} + \overrightarrow{w} + \overrightarrow{N} = \dfrac{d\overrightarrow{P}}{dt}$;
   where $\overrightarrow{F_\ell}$ is Laplace's force $F_\ell = i\,B\,\ell\sin\left(90^0\right) = \dfrac{e}{R}B\,\ell = \dfrac{B\,\ell}{R}e$;
   Projection along $\overrightarrow{F}$: $F - F_\ell = \dfrac{d(mv)}{dt}$ & $e = B\,\ell\,v$;

Then $F_\ell + m\frac{dv}{dt} = F$; $\frac{B\ell}{R}e + m\frac{d}{dt}\left(\frac{B\ell}{R}e\right) = F$; thus, $\frac{de}{dt} + \frac{B^2\ell^2}{mR}e = \frac{FB\ell}{m}$.

2. When the steady state is reached $e = E_0$ becomes constant, then $\frac{de}{dt} = 0$; thus $e = E_0 = \frac{Fm}{B\ell}$.

3. We have $e = E_0\left(1 - e^{-\frac{t}{\tau}}\right)$, then $\frac{de}{dt} = \frac{E_0}{\tau}e^{-\frac{t}{\tau}}$;

Replacing in the differential equation $\frac{E_0}{\tau}e^{-\frac{t}{\tau}} + \frac{B^2\ell^2}{mR}E_0\left(1 - e^{-\frac{t}{\tau}}\right) = \frac{FB\ell}{m}$;

$\left(\frac{1}{\tau} - \frac{B^2\ell^2}{mR}\right)E_0\, e^{-\frac{t}{\tau}} + \frac{B^2\ell^2}{mR}E_0 = \frac{FB\ell}{m}$

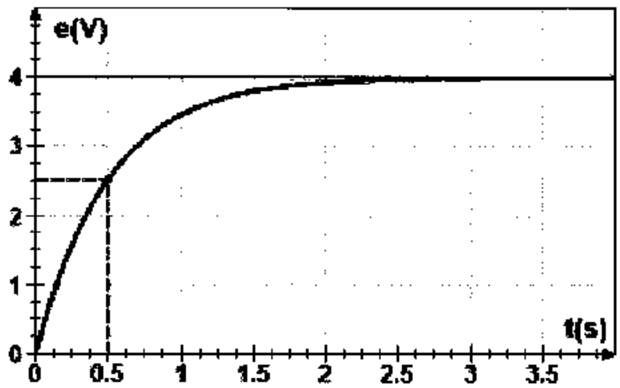
is verified at any instant t, & $e^{-\frac{t}{\tau}} \neq 0$;

Then $\frac{1}{\tau} - \frac{B^2\ell^2}{mR} = 0$; thus $\tau = \frac{mR}{B^2\ell^2}$.

4. The steady state is reached within $2.5s$ so

$5\tau = 2.5$, then $B^2 = \frac{mR}{\tau \times \ell^2}$.

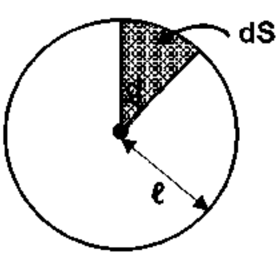
$B = \sqrt{\frac{2\times10^{-3}\times5}{0.5\times0.2^2}} = \sqrt{0.5} = 0.71T$.



5. Graph.

VI-

Part A



1. For a constant angular velocity w and during a duration dt, the elementary angle of traveled is $w = \frac{d\theta}{dt}$, so $d\theta = wdt$.

The elementary area swept is $dS = \frac{\ell^2}{2}d\theta = \frac{\ell^2}{2}w\,dt$.

2. The magnetic flux is given by $d\phi = B\,dS\cos(\vec{n},\vec{B})$ where $\vec{n}$ is taken in the same direction as $\vec{B}$, so $(\vec{n},\vec{B}) = 0$; then $d\phi = B\,dS\cos(\vec{n},\vec{B}) = B\frac{\ell^2}{2}d\theta\cos 0 = \frac{1}{2}B\ell^2\,d\theta$.

Thus, $\phi = \int d\phi = \int_0^\theta \frac{1}{2}B\ell^2\,d\theta = \frac{1}{2}B\ell^2\,\theta$.

3. According to Faraday's law: $e = -\frac{d\phi}{dt} = -\frac{d}{dt}\left(\frac{1}{2}B\ell^2\theta\right) = -\frac{1}{2}B\ell^2 w$.

Part B

$w = 10\text{rev}/s \times (2\pi\,\text{rad/rev}) = 20\pi\,\text{rad}/s$;

The induced e.m.f $|e| = \frac{1}{2}B\ell^2 w = 14.15\,mV$.

Supplementary Problems

I-(See Appendix 1 – Page 519)

(See Appendix 1 – Page 519)

Electromagnetic Induction

A metallic rigid rod SN, of mass $m = 16g$, can slide without friction, on two parallel and horizontal metallic rails AP and HQ separated by a distance $\ell = 10cm$. During sliding, the rod remains perpendicular to the rails. These two rails, separated by the distance ℓ, are connected by a resistor of resistance $R = 5\Omega$ (Figure 1). Neglect the resistance of the rod and of the rails. The whole setup is placed in an upward, uniform and vertical magnetic field $\vec{B}$ of value B.

The position of G, the center of inertia of the rod, is defined by its abscissa x on the horizontal axis $(O;\vec{i})$ with O corresponding to the position of G at $t_0 = 0$. Taking into consideration the arbitrary positive direction chosen in figure 1.

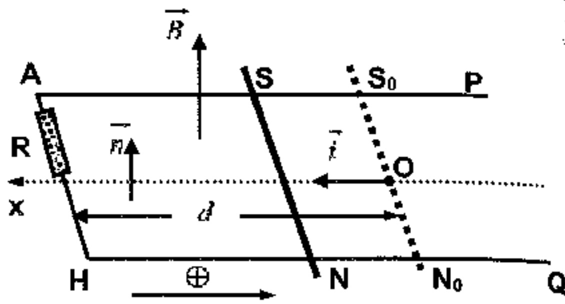


Figure 1

Part A
Static study

1. Determine, at the instant $t_0 = 0$, the expression of the magnetic flux crossing the surface $ASNH$ in terms of B, ℓ and d.
2. Explain the absence of an induced $e.m.f$ « e » across the ends S and N of the rod.

Part B
Dynamic study

At the instant $t_0 = 0$, taken as origin of time, the rod at rest is submitted to a constant force $\vec{F} = F\,\vec{i}$ of magnitude F. At an instant t, G has an abscissa $OG = x$ and a velocity $\vec{v}$ of algebraic value v.

First situation
Theoretical study

1. a) Determine, at the instant t, the expression of the magnetic flux crossing the surface $ASNH$ in terms of B, ℓ, d and x.
   b) Explain the existence of an induced $e.m.f$ « e » between the terminals S and N of the rod.
   c) Show that the expression of the induced $e.m.f$ « e » is given by $e = B\,\ell\,v$.
2. Applying Newton's 2nd law on the rod and taken into consideration the expression of e, show that the differential equation satisfied by the induced $e.m.f$ « e » is $\frac{de}{dt} + \frac{B^2\,\ell^2}{m\,R}e = \frac{F\,B\,\ell}{m}$.
3. The solution of this differential equation is $e = E_0\left(1 - e^{-\frac{t}{\tau}}\right)$.

   Determine the expressions of the constants E_0 and τ in terms of B, F, m, R & ℓ.
4. Show that, after a time $t = \tau$, the induced electromotive attains 63% of its maximum value E_0.

Second situation
Graphical study

The variation of e as a function of time is represented by the curve of figure 2.

1. Justify that the shape of the curve (Figure 2) is in agreement with the solution of the differential equation.
2. Referring to the curve of figure 2:
   a) give the value of E_0.
   b) using the result of part 4 determine the value of τ and deduce the magnitude of the magnetic field B.
3. Deduce the magnitude of the force acting on the rod F.

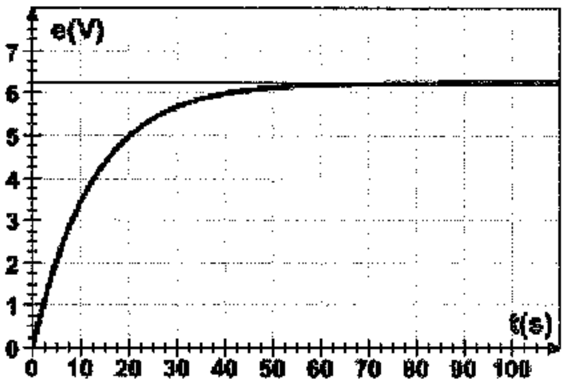


Figure 2

Answer Key

2\. b) $B = 0.8T$. 3. $F \approx 0.1N$.

Engineering 2009/2010
Phenomenon of Electromagnetic Induction

The device of Laplace's rails, of negligible resistance, lies in a horizontal plane. A rod CD, of resistance R, perpendicular to the rails, has a length $\ell = 10\,cm$ and a mass $m = 10g$ (figure 1). This device is placed in a uniform vertical magnetic field $\vec{B}$, of magnitude $B = 0.3T$.

At the instant $t_0 = 0$, the rod being C_0D_0, we apply at its center of inertia G a constant force $\vec{F} = F\vec{i}$.

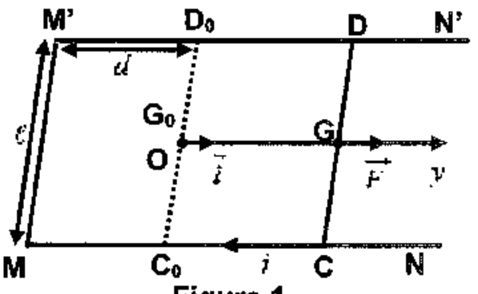


Figure 1

At an instant t, the abscissa of G is $x = OG$ its velocity is $\vec{v} = v\vec{i}$ and its linear momentum is: $\vec{P} = m\vec{v} = P\vec{i}$.

1. Knowing that the real direction of the induced current i is as shown in the figure 1, determine the direction of $\vec{B}$.
2. Determine the expression of i in terms of B, ℓ, R & v.
3. Show that the rod is under the action of the Laplace's force $\vec{F_e}$ of expression is $\vec{F_e} = -\dfrac{B^2\ell^2 v}{R}\vec{i}$.
4. a) Show that the differential equation that describes the variation of P as a function of time can be written: $\dfrac{dP}{dt} + \dfrac{B^2\ell^2}{mR}P = F$.
   b) Determine the expressions of the constants P_0 & τ, so that $P = P_0\left(1 - e^{-\frac{t}{\tau}}\right)$ is a solution of the previous differential equation.
   c) Give the physical interpretation of P_0 & τ.
   d) Knowing that the rod reaches a limit speed $v_\ell = 2\,m/s$ after a time very close to $20\,s$, determine the values of R and F.

Answer Key

1\. Vertically upwards. 4.d) $R = 0.36\,\Omega$; $F = 5\times10^{-3}\,N$.

LS – Sessions

I-LS 2009 1st
Measurement of the Speed of a Plane

The aim of this exercise is to measure the speed of a plane using the phenomenon of electromagnetic induction.

Part A
Motion of a conductor in a uniform magnetic field

A homogeneous metallic rod MN of length ℓ, slides on two horizontal and parallel metallic rails AA' and EE' at a constant velocity $\vec{v}$. During its sliding, the rod remains perpendicular to the rails and its center of mass G moves along the axis Ox.

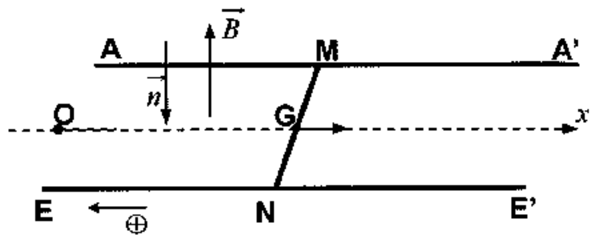


Figure 1

At the instant $t_0 = 0$, G is at O, the origin of abscissa. At an instant t, the abscissa of G is $x = \overline{OG}$ and $v = \frac{dx}{dt}$ is the algebraic value of its velocity. The whole set-up formed of the rod and the rails is put within a uniform magnetic field perpendicular to the plane of the horizontal rails (Figure 1).

1. Determine, at the instant t, the expression of the magnetic flux crossing the surface $AMNE$ in terms of B, ℓ and x taking into consideration the chosen arbitrary positive direction on Figure1.
2. Explain the existence of an induced e.m.f « e » across the ends M and N of the rod.
3. Determine the expression of the induced e.m.f « e » in terms of B, ℓ and v.
4. No current would pass in the rod. Why?
5. Deduce the polarity of the points M and N of the rod and give the expression of the voltage u_{NM} in terms « e ».

Part B
Measurement of the speed of a plane

A plane is flying horizontally along a straight path with a constant velocity $\vec{v_1}$ of magnitude v_1 within the uniform magnetic field B of the Earth.

The vector B, in the region of flight, has a horizontal component of magnitude $B_h = 2.3\times10^{-5}\,T$ and a vertical component of magnitude $B_v = 4\times10^{-5}\,T$ (Figure 2).

The wings of the plane, considered as a straight and horizontal conductor of length $\ell' = M'N' = 30\,m$, sweep with time a surface area (Figure 2).

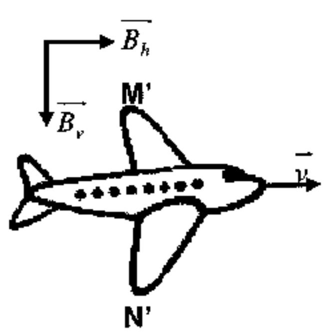


Figure 2

1. a) The magnetic flux of $\overrightarrow{B_h}$ through the swept surface area is zero. Why?

   b) Give the expression of the induced $e.m.f$ « e_1 » that appears between the ends M' and N' of the wings in terms of B_v, ℓ' and v_1.

2. Determine v_1, if the potential difference across the wings has a value $0.36\,V$.

II-LS 2007 1st
Usage of a Coil (Part A)

A bar magnet may be displaced along the axis of a coil whose terminals A and C are connected to a resistor of resistance R.

We approach the north pole of the magnet towards the face A of the coil (Figure 1). An induced current i is carried by the circuit.

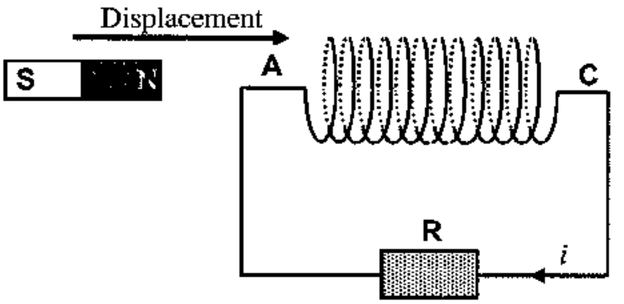


Figure 1

1. Give the name of the physical phenomenon that is responsible for the passage of this current.
2. Give, with justification, the name of each face of the coil.
3. The induced current passes from C to A through the resistor. Why?
4. Determine the sign of the voltage u_{AC}.

III-LS 2012 2nd
See Page 289 (Part A)

Sessions Solutions

I-LS 2009 1st

Part A

1. The magnetic flux $\phi = BS\cos(\vec{n}, \vec{B}) = BS\cos\pi = -BS = -B\ell x$.
2. The surface of the circuit varies, then the magnetic flux varies; therefore an induced *e.m.f* «e» appears across the extremities N and M of the rod.
3. According to Faraday's law $e = -\frac{d\phi}{dt} = B\ell\frac{dx}{dt} = B\ell v$.
4. The circuit is open so $i = 0$.
5. The induced *e.m.f* $e = B\ell v$ that appears between N & M is positive.

   So, M is the negative terminal of the induced generator (along the direction of the induced current) and consequently N is the positive.

   According to Ohm's law case of a generator $u_{NM} = e - ri$ where $(i = 0)$;

   So $u_{NM} = e > 0$; then $u_{NM} = e = B\ell v$.

Part B

1. a) $\overrightarrow{B_h}$ is perpendicular to the normal $\vec{n}$ to the surface of the circuit then $(\vec{n}, \vec{B}) = 90^0$,

   Then $\phi_h = B_h S\cos(90^0) = 0$.

   b) The flux swept by the vertical component of the magnetic field $|\phi_v| = B_v S|\cos(\theta)|$ where $\theta = 0^0$ or 180^0 then $|\cos(\theta)| = 1$; thus $|\phi_v| = B_v \ell' x$

   But $e = \left|-\frac{d\phi_v}{dt}\right| = B_v \ell' \frac{dx}{dt} = B_v \ell' v_1$.

2. The voltage $|u_{NM}| = B_v \ell' v_1 \Rightarrow v_1 = \frac{|u_{NM}|}{B_v \ell'} = \frac{0.36}{4\times10^{-5}\times30} = 300\ m/s$.

II-LS 2007 1st

1. Electromagnetic induction.
2. In order to oppose, by repulsion the approach of the N-pole of the magnet. (Lenz's law), A is the North face, B is the South face.
3. The induced magnetic field $\overrightarrow{B_i}$ opposes the increase of $\vec{B}$ in the magnet thus it has a sign opposite to that of $\vec{B}$, the induced current thus passes from C to A through the resistor.
4. «A» is the negative pole of the equivalent generator so $u_{AC} < 0$.

Unit II

Electricity

Chapter 7

Auto Induction

LS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Auto Induction		-	-	-	-	-	2nd(B)	-	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Auto Induction	-	-	1st(B)	-	-	-	1st(A)	2nd(A)	-

Essentials

I-

Auto Induction

Auto-induction (self-induction) is the appearance of a self-induced electromotive force in a circuit when traversed by a variable current.

The self-flux ϕ is proportional to the current i, $\phi = Li$ where L is called the inductance of the coil measured in SI units in Henry H.

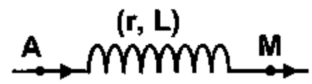
1. Voltage across the coil

The self-induced *e.m.f* is given by $e = -\frac{d\phi}{dt} = -L\frac{di}{dt}$ (Faraday's law).

For a positive current:

- ✘ during its decrease $e > 0$, so the coil acts as a generator.
  The direction of the induced current is the same direction as that of the main current.
- ✘ if it is constant $e = 0$, the coil acts as a resistor (wire).
  No induced current in the circuit.
- ✘ during its increase $e < 0$, so the coil acts as a generator in opposition (or receiver).
  The direction of the induced current is opposite to the main current.

Ohm's law across a coil : $u_{AM} = ri - e = ri + L\frac{di}{dt}$.



In general: If $e \times i > 0$ (same signs), the coil acts as a generator.

& if $e \times i < 0$ (opposite signs), the coil acts as a generator in opposition (receiver).

2. Magnetic energy stored in a coil

The coil stores, during the growth of the current, a magnetic energy given by $E_m = \frac{1}{2}Li^2$.

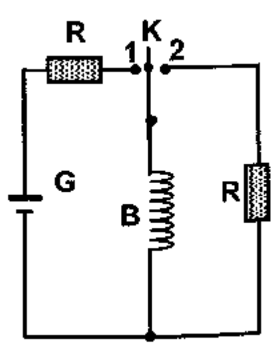
This energy is delivered to the circuit during the decay of the current.

II-

Coil in a Circuit

In a simple (R, L) circuit, formed of a DC generator of *e.m.f* E, connected in series with a coil and a resistor.

The time constant of the circuit is $\tau = \frac{L}{R}$.



	Growth of current – Switch on position (1)	Decay of current – Switch on position (2)
At $t = 0$	$i = 0$	$i = I_{max} = E/R$
Steady state after 5τ	$i = I_{max} = E/R$	$i = 0$
After $t = \tau$	$i = (63\%)I_{max}$	$i = (37\%)I_{max}$
Role	Coil acts as generator in opposition	Coil acts as a generator

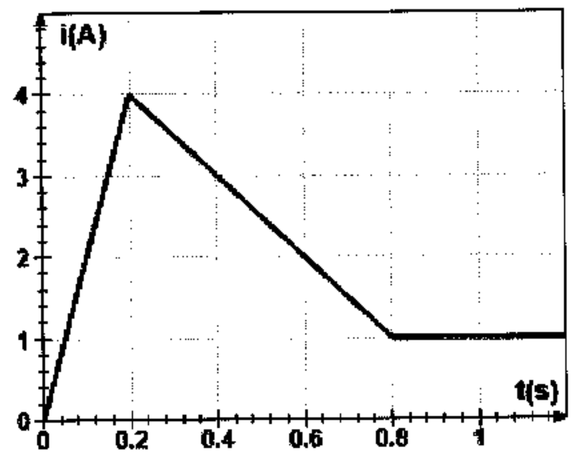
Applications

Coil in a Circuit

In order to study the influence of a coil, we connect in series the circuit formed of the following dipoles:

- ✗ an ideal DC generator G whose $e.m.f$ $E = 24V$;
- ✗ a coil of inductance $L = 0.1H$ and of internal resistance $r = 6\Omega$;
- ✗ a switch K.

1. Draw the diagram of the circuit thus formed.
2. We close the switch K, after a certain duration the current becomes constant I_0.
   a) Show that $I_0 = 4A$.
   b) Calculate the magnetic energy stored in the coil.
3. We open abruptly the switch of the circuit K. An electric spark appears between the terminals of the switch.
   a) Name the phenomenon responsible of this spark.
   b) The switch was opened during a duration $\Delta t = 5ms$. Calculate the average electric power emitted.
4. The switch again is closed. (G) is a L.F.G that delivers a variable voltage such that the circuit carries a variable current $i(t)$ represented in the adjacent figure.
   a) Justify in which intervals the coil is the seat of a self-induced $e.m.f$ and determine its value.
   b) Draw the graph representing the variation of the self-induced $e.m.f.$ in terms of time
   c) For $t \in [0;\ 0.2s]$.
   Determine the expression of the voltage u across the coil in terms of time.



II- Current and Self-Induction

The aim of this exercise is to study the effect of the coil for different types of currents.

Consider a coil of inductance $L = 50\,mH$ and internal resistance $r = 4\Omega$ connected between the terminals of a generator as shown in the adjacent circuit.

In what follows the self-induced electromotive force is denoted «e».

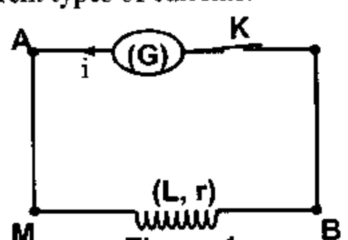


Figure 1

Part A

DC current

The circuit is traversed by a constant current $I_0 = 20\,mA$.

1. Referring to Lenz's law, justify that the coil is not the seat of a self-induced electromotive force.
2. What is the role of the coil in this case?
3. Calculate the voltage u_{AB}.

Part B

Linear current (triangular)

The circuit is traversed by a current that varies along the curve shown in figure 2.

1. Write the relation among L, i and e.
2. Determine the value of e.
3. Specify the role of the coil in this case. Draw an equivalent diagram.
4. Draw the curve representing e for $t \in [0; 4s]$.
5. Calculate the magnetic energy stored in the coil at $t = 0$.
6. Calculate the self-magnetic flux in the coil at $t = 0$.

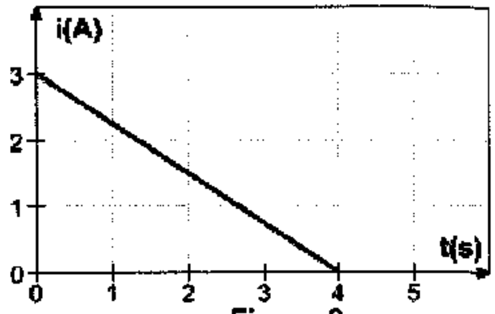


Figure 2

Part C

DC voltage

When the circuit is fed by a DC generator, the expression of the current in the circuit, in terms of time, is given by $i = I_0\left(1 - e^{-\frac{r}{L}t}\right)$.

1. Determine the expression of the self-induced electromotive force e_C.
2. Specify the role of the coil in this case.
3. a) Determine the equation of the tangent (T) to the curve representing e_C at $t = 0$.
   b) Deduce the abscissa of the point of intersection of (T) with the time axis.
4. Draw (T) and the curve representing e_C in the same plane.

Part D

Sinusoidal voltage

When an alternating sinusoidal generator feds the circuit, the instantaneous expression of the current is given by $i = I_m \sin(wt)$.

1. Give the value of I_m and calculate that of w.
2. Deduce the expression of the self-induced electromotive force e_D in terms of time.
3. Plot the graph representing the variations of e_D in terms of time.
4. Determine the magnetic energy stored in the coil at the instants $t = 0$ and $t = 0.5s$.
5. Draw the graph representing the evolution of the magnetic energy in terms of time.
   Interpret the graph obtained.

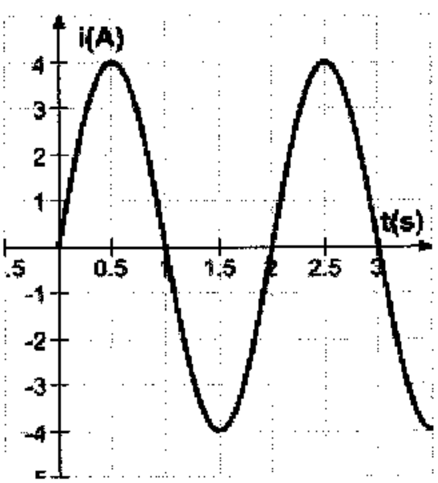


Figure 3

III- Bac 2013
Triangular Signal

A low frequency generator (LFG) delivering an alternating triangular voltage between the terminals A & B of a dipole formed of a coil of inductance L and negligible internal resistance and a resistor of resistance $R = 500\ \Omega$ connected in series as shown in the figure 1.

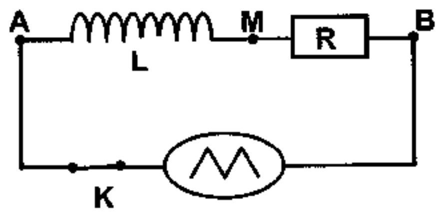


Figure 1

An oscilloscope conveniently connected, in order to display the voltage u_{AM} across a coil on the channel Y_1 and the voltage u_{BM} across the resistor on the channel Y_2.

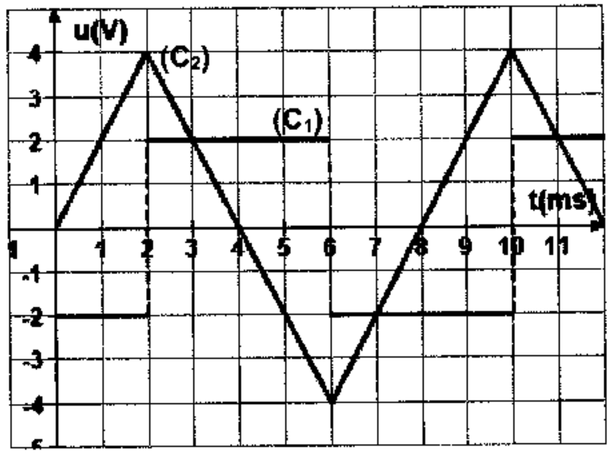
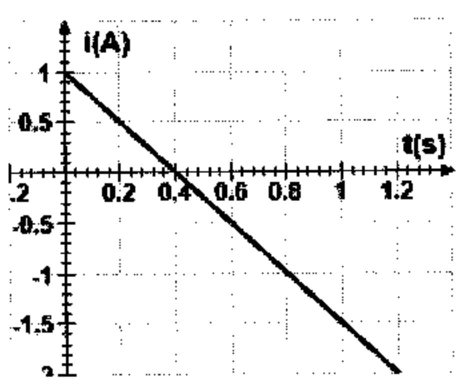


Figure 2

1. a) Determine the expressions of the voltages u_{AM} & u_{BM} in terms of the current i, L & R.
   b) Write u_{AM} in terms of u_{BM}, L & R.
2. Determine the frequency of the generator.
3. Show on the previous circuit the connections of the oscilloscope allowing us to display the curves shown in figure 2.
4. a) Identify, with justification, the curve that corresponds to the voltage displayed on channel Y_2.
   b) Justify, over a half period, the shape of the voltage u_{BM} displayed on Y_2.
   c) Determine the value of the inductance L.
5. In the interval $[0; 2ms]$.
   a) Specify the direction of the current in the circuit.
   b) Calculate the induced electromotive force and then deduce the role of the coil.

IV- Self-Induction

A pure inductive coil of inductance $L = 5\,mH$ (negligible internal resistance) is traversed by a current that varies in terms of time as shown in the adjacent figure.



1. Calculate the magnetic energy stored in the coil at the instant $t = 0$.
2. Determine the self-induced $e.m.f$ for $t \in [0; 1.2s]$.
3. Deduce the role of the coil.

V- The Phenomenon of Self-Induction

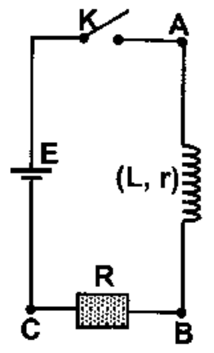
The set up represented in the figure below consists of an ideal generator of $e.m.f$ $E = 8V$, a coil of internal resistance $r = 4\,\Omega$ and inductance $L = 20\,mH$, a resistor of resistance $R = 16\,\Omega$ and a switch K. At the instant $t_0 = 0$, we close the switch K.

At an instant t, during the transient state, the circuit carries a current i.

1. Derive the differential equation that governs the variation of i as a function of time.

2. Calculate, at $t_0 = 0$, the value of $\frac{di}{dt}$ & then deduce the self-induced $e.m.f$ e.

3. The solution of the differential equation is of the form $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$.



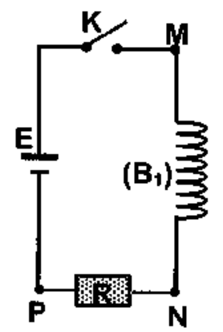
   a) Determine the expressions of τ & I_0 in terms of L, r and R then calculate their values.
   b) Give the physical significance of the constant τ.
   c) What is the minimum duration needed to reach the steady state?
   d) Deduce the current in the steady state.

4. At an instant t_1, the current reaches 80% of its value in the steady state I_0.
   a) Calculate t_1.
   b) Deduce the magnetic energy stored in the coil at this instant.

5. a) Determine the expression of the self-induced $e.m.f$ « e » as a function of time t.
   b) Calculate the algebraic value of e at the instant $t_0 = 0$, and the role of the coil at this instant.

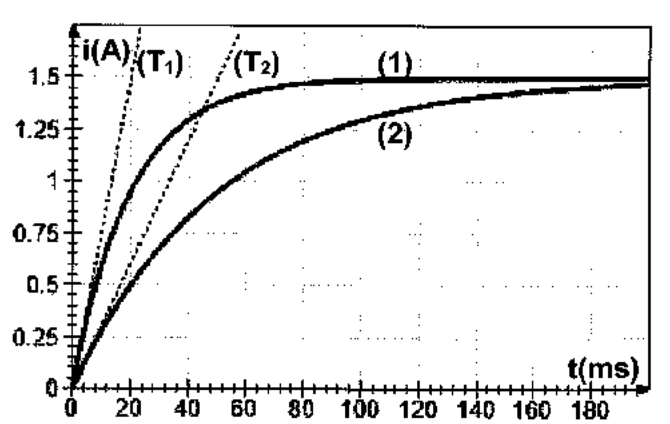
VI- Effect of Iron Core on the Current

Consider the circuit shown in the adjacent figure formed of an ideal generator of $e.m.f$ E, a coil [either (B_1) without iron core or (B_2) with an iron core of respective inductances L_1 & L_2] and negligible internal resistances, a resistor of resistance $R = 10\,\Omega$ and a switch K. At the instant $t_0 = 0$, we close the switch K. At an instant t, the circuit carries a current i.



The expression of the current is given by $i = \frac{E}{R}\left(1 - e^{-\frac{t}{\tau}}\right)$ where $\tau = \frac{L}{R}$ is the time constant of the circuit.

The curves (1) & (2) shown in the adjacent figure represents the variation of the current in terms of time while using the coils (B_1) & (B_2) successively while (T_1) & (T_2) are the tangents at $t_0 = 0$.

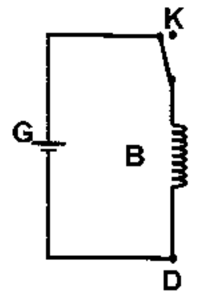


1. Determine the expression of the current I_0 in the steady state and then deduce the value of E.
2. Explain why the current in the steady state is not modified.
3. Show that τ is the abscissa of the point of intersection between the tangent at origin and the horizontal asymptote.
4. Determine L_1 & L_2 and then deduce the effect of the iron core on the inductance.
5. Compare the magnetic energy stored in the coil in both cases in steady state.
6. Interpret the result obtained.

Solutions

I-

1. Adjacent circuit.



2. a) Law of uniqueness of voltages: $u_G = u_{coil}$, $E = r\,I_0 + L\dfrac{dI_0}{dt}$;

   The current I_0 is constant, so $\dfrac{dI_0}{dt} = 0$, we get : $I_0 = \dfrac{E}{r} = \dfrac{24}{6} = 4\,A$.

   b) The magnetic energy stored is given by:
   $E_m = \dfrac{1}{2}L I_0^2 = \dfrac{1}{2}(0.1)(4^2) = 0.8\,J$.

3. a) The spark is due to self-induction.

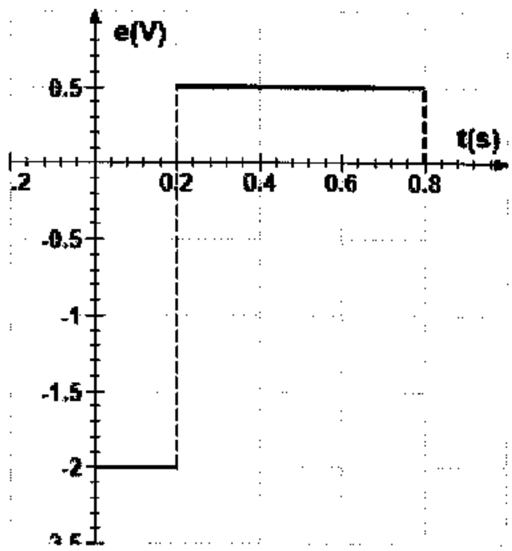
   b) The average power $P_{av} = \dfrac{E_m}{\Delta t} = \dfrac{0.8}{5\times10^{-3}} = 160\,W$.

4. a) The circuit is the seat of an induced $e.m.f$ when the current is variable which takes place in the intervals $[0;0.2s]$ & $[0.2s\,;0.8s]$.
   The expression of the self-induced $e.m.f$ is given by : $e = -L\dfrac{di}{dt}$.

   ✖ For $t \in [0;0.2s]$, we get:
   $e = -L\dfrac{di}{dt} = -100\times10^{-3}\,\dfrac{4-0}{0.2-0} = -2V$.

   ✖ For $t \in [0.2;0.8s]$, we get:
   $e = -L\dfrac{di}{dt} = -100\times10^{-3}\,\dfrac{1-4}{0.8-0.2} = 0.5V$.

   b) Graph.



   c) The voltage $u_{AB} = r\,i - e = 6i + 2$ where i in A & u_{AB} in V.
   The current is represented by a straight line passing through origin, then its equation is $i = at = 20t$;

   d) Thus, $u_{AB} = 120\,t + 2$ where t in s & u_{AB} in V.

II-

Part A

1. The current that flows in the circuit is constant, so the magnetic field in the coil is also constant[1]. Then the magnetic flux crossing it is constant, thus the coil is not the seat of a self-induced electromotive force.
2. The current is constant, then the coil acts as a resistor.
3. We have $u_{AB} = r\,I_0 + L\dfrac{dI_0}{dt} = 4\times20 = 80\,mV$.

[1] The magnetic flux crossing the coil is $\phi_0 = L I_0 = 50\times10^{-3}\times20\times10^{-3} = 10^{-3}Wb = 1mWb$.
Then the self-induced $e.m.f$ is given by Faraday's law $e = -\dfrac{d\phi}{dt}$.

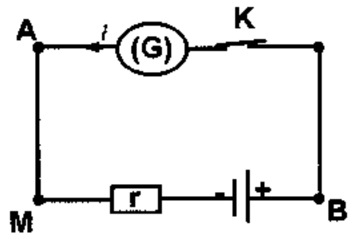
Part B

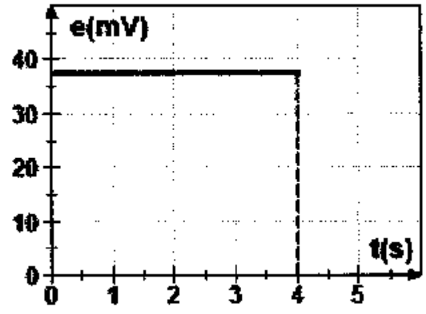
1. We have $e=-L\dfrac{di}{dt}$.

2. The current i varies linearly, then $\dfrac{di}{dt}=\dfrac{\Delta i}{\Delta t}=\dfrac{(0-3)A}{(4-0)s}=-0.75\,A/s$; So $e=50\times 0.75=37.5\,mV$.

3. We have $e>0$, then the coil acts as a generator (circuit (below).

4. $e=37.5\,mV$ is constant, then its graphical representation is a horizontal straight line.





5. The magnetic energy stored in the coil is $E_m=\dfrac{1}{2}LI_0^2=\dfrac{1}{2}\times 50\times 10^{-3}\times 3^2=0.225\,J$.

6. The self-magnetic flux $\phi_0=L\,I_0=50\times 10^{-3}\times 3=0.15\,Wb$.

Part C

1. We have $i=I_0\left(1-e^{-\frac{r}{L}t}\right)$ & $e_C=-L\dfrac{di}{dt}$;

   Then $e_C=-L\dfrac{di}{dt}=-L\,I_0\left(\dfrac{r}{L}\right)e^{-\frac{r}{L}t}=-r\,I_0e^{-\frac{r}{L}t}$.

2. We have $e_1<0$; then the coil acts as a generator in opposition (receiver).

3. a) Equation of the tangent (T) at $t_0=0$ is $u=a\,t+b$;

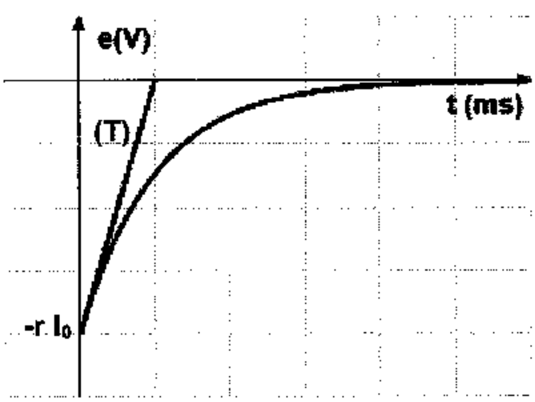
   Where $b=e_C\big|_{t_0=0}=-r\,I_0$ &

   $a=\left.\dfrac{de_C}{dt}\right|_{t_0=0}=-r\,I_0\left(\dfrac{-r}{L}\right)e^{-\frac{r}{L}t}\Big|_{t=0}=\dfrac{r^2I_0}{L}$;

   Thus, $u=\dfrac{r^2I_0}{L}t-r\,I_0$.

   b) Intersection of (T) with the time axis:

   $u=\dfrac{r^2I_0}{L}t-r\,I_0$; then $t=\dfrac{L}{r}=0.0125\,s$.



4. The self-induced $e.m.f$ has an exponential increasing shape.

   At $t=0$; $e_C=-r\,I_0$;

   and as $t\longrightarrow+\infty$, $e_C\longrightarrow 0$.

Part D

1. I_m is the maximum value of the current $I_m = 4A$;

   The period of this waveform is $T = 2s$ and its angular frequency $w = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi(\text{rad}/s)$;

   Then $i = 4\sin(\pi t)$ (t in s and i in A).

2. We have $e_D = -L\frac{di}{dt} = -0.2\pi\cos(\pi t)$

   where t in s and e_D in V.

3. $e_D = -0.2\pi\cos(\pi t)$ is a sinusoidal function of period $2s$ and amplitude 0.2π (V).

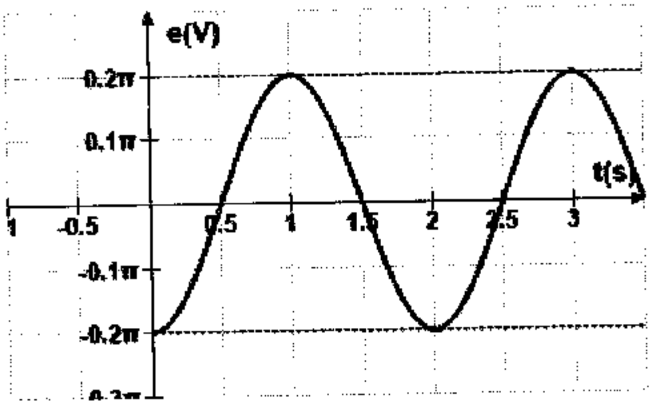
4. At $t = 0$, the current $i = 0$; then no magnetic energy is stored in the coil at this instant.

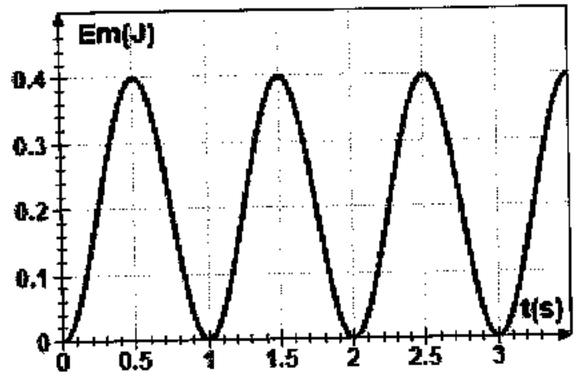
   At $t = 0.5s$, the current $i = 4A$; then the magnetic energy stored in the coil is:

   $E_m = \frac{1}{2}LI_0^2 = 0.4J$.

5. Graph of magnetic energy.

   The magnetic energy is also periodic but its period is half that of the current.



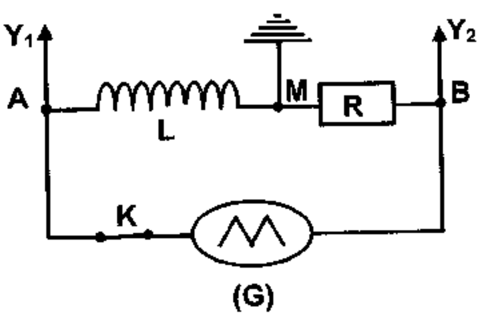


III-

1. a) Ohm's law across the coil $u_{AM} = L\frac{di}{dt}$;

   across the resistor $u_{BM} = -Ri$;

   b) We have $u_{AM} = L\frac{d}{dt}\left(-\frac{u_{BM}}{R}\right) = -\frac{L}{R}\frac{du_{BM}}{dt}$.



2. The frequency of the signal displayed is $f = \frac{1}{T} = \frac{1}{8\times10^{-3}} = 125\,Hz$.

3. Connections of the oscilloscope.

4. a) We have $u_{AM} = -\frac{L}{R}\frac{du_{BM}}{dt}$; one of the two curves is variable and the other is constant over certain interval.

   Then (C_2) corresponds to (Y_2) representing u_{BM}.

   b) Over a half period $[2ms\,;6\,ms]$, u_{BM} decreases linearly, so $\frac{du_{BM}}{dt} < 0$ and constant;

   But $u_{AM} = -\frac{L}{R}\frac{du_{BM}}{dt} > 0$ and constant, so it is represented graphically by a horizontal straight line above the abscissa axis.

c) We have $\frac{du_{BM}}{dt} = \frac{du_{BM}}{dt} = \frac{4-(-4)}{4\times10^{-3}} = 2000\ V/s$;

But $u_{AM} = -\frac{L}{R}\frac{du_{BM}}{dt}$, $2 = \frac{L}{500}\times 2000$; then $L = 0.5\,H$.

5. a) On $[0;2ms]$, $u_{AM} > 0$; the current circulates in the counter clockwise direction.

b) The self-induced $e = -u_{AM} = -2V < 0$, so the coil acts as a generator in opposition.(2)

IV-

1. The magnetic energy stored in the coil is: $E_{m0} = \frac{1}{2}L i_0^2 = \frac{1}{2}\times 5\times 1^2 = 2.5\,mJ$.

2. The self-induced *e.m.f* is given by: $e = -L\frac{di}{dt}$;

i varies linearly, so $\frac{di}{dt} = \frac{\Delta i}{\Delta t} = \frac{(-1.5-1)A}{(1-0)s} = -2.5\,A/s$; then $e = -5\times(-2.5) = +12.5mV$.

3. For $t\in[0;0.4s]$, $i>0$ (above the time axis) & $e\times i>0$; thus the coil acts as a generator.

For $t\in[0.4s;1.2s]$, $i<0$ (below the time axis) & $e\times i<0$; thus the coil acts as a generator in opposition.

V-

1. Law of addition of voltages: $u_{AC} = u_{AB} + u_{BC}$; $E = ri + L\frac{di}{dt} + Ri$; then $E = (r+R)i + L\frac{di}{dt}$.

2. The differential equation is verified at any instant, but at $t=0$, we have $i=0$;

$E|_{t=0} = (r+R)i|_{t=0} + L\frac{di}{dt}\Big|_{t=0}$, so $L\frac{di}{dt}\Big|_{t=0} = E$; then $\frac{di}{dt}\Big|_{t=0} = \frac{E}{L} = \frac{8}{20\times10^{-3}} = 400\ V/H$;

$e = -L\frac{di}{dt}\Big|_{t=0} = -L\times\frac{E}{L} = -E = -8V$.

3. a) We have $i = I_0\left(1-e^{-\frac{t}{\tau}}\right)$; so, $\frac{di}{dt} = I_0\left(\frac{1}{\tau}\right)e^{-\frac{t}{\tau}}$;

By replacing in the differential equation we get: $E = (r+R)I_0\left(1-e^{-\frac{t}{\tau}}\right) + \frac{L}{\tau}e^{-\frac{t}{\tau}}$;

Then $E = I_0 e^{-\frac{t}{\tau}}\left(\frac{L}{\tau} - (r+R)\right) + (r+R)I_0$;

This equation is verified at any instant t, we get $\frac{L}{\tau} - (r+R) = 0$ & $(r+R)I_0 = E$;

Then $\tau = \frac{L}{r+R}$ & $I_0 = \frac{E}{r+R}$.

Values: $\tau = \frac{L}{r+R} = 10^{-3}s = 1ms$ & $I_0 = \frac{E}{r+R} = \frac{8}{16+4} = 0.4A$.

2 The role of the coil is defined by the product $e\times i$ and not only by the sign of the self-induced *e.m.f*.

If $e\times i>0$, (e & i have same signs), the coil acts as a generator;

If $e\times i<0$, (e & i have opposite signs), the coil acts as a generator in opposition (receiver).

b) The time constant τ, during the growth of the current, of the (L, R) circuit is the duration after which the current reaches 63% of its value in the steady state.

c) The steady state is reached after a duration $\Delta t = 5\tau = 5\,ms$.

d) In the steady state $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right) = I_0\left(1 - e^{-5}\right) \approx I_0 = 0.4A$.

4. a) At the instant t_1, $i = 0.8 I_0$; so $0.8 I_0 = I_0\left(1 - e^{-\frac{t_1}{\tau}}\right)$; we get $e^{-\frac{t_1}{\tau}} = 0.2$;

Then $t_1 = -\tau\, \ell n(0.2) \approx 1.6ms$.

b) The magnetic energy stored at this instant is: $E_m = \frac{1}{2} L i_1^2 = \frac{1}{2}(20) \times (0.32)^2 = 1.024\,mJ$.

5. a) Faraday's law $e = -L\frac{di}{dt} = -L\frac{I_0}{\tau} e^{-\frac{t}{\tau}} = -L\frac{E/(R+r)}{L/(R+r)} e^{-\frac{t}{\tau}} = -E\, e^{-\frac{t}{\tau}}$.

b) At $t = 0$, $e = -Ee^0 = -E = -8V$; $e < 0$, the coil acts as a generator in opposition.

VI-

1. In steady state $t \to +\infty$, $i \to I_0 = \frac{E}{R}(1 - 0) = \frac{E}{R}$.

Graphically $I_0 = 1.5\,A$, then $E = R\, I_0 = 10 \times 1.5 = 15\,V$.

2. The expression of the current in steady state $I_0 = \frac{E}{R}$ is independent of the inductance of the coil, then it is the same as long the internal resistance of the coil is negligible.

3. The equation of tangent at the origin of time $i = a\,t$ where $a = \left.\frac{di}{dt}\right|_{t=0} = \frac{E}{R} \times \frac{1}{\tau} e^0 = \frac{E}{L}$;

Then (T): $i = \frac{E}{L} t$; & the equation of the horizontal asymptote $i = I_0 = \frac{E}{R}$;

The abscissa of the point of intersection is $i = \frac{E}{L} t = \frac{E}{R}$, then $t = \frac{L}{R} = \tau$.

4. From curve (1): we have $\tau_1 = 20\,ms = \frac{L_1}{R}$, then $L_1 = \tau_1 \times R = 20 \times 10^{-3} \times 10 = 0.2\,H$.

From curve (2): we have $\tau_2 = 50\,ms = \frac{L_2}{R}$, then $L_2 = \tau_2 \times R = 50 \times 10^{-3} \times 10 = 0.5\,H$.

We get $L_2 > L_1$; thus, the introduction of the iron core increases the inductance of the coil.

5. The magnetic energies $E_{m1} = \frac{1}{2} L_1\, I_0^2 = \frac{1}{2} \times 0.2 \times 1.5^2 = 0.225\,J$;

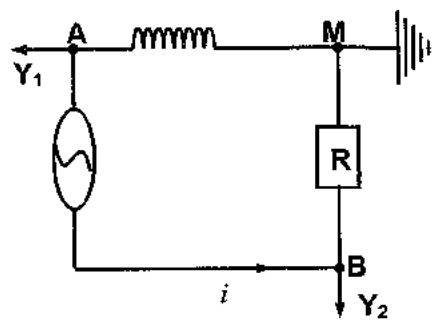
& $E_{m2} = \frac{1}{2} L_2\, I_0^2 = \frac{1}{2} \times 0.5 \times 1.5^2 = 0.5625\,J$.

6. We have $E_{m2} > E_{m1}$, the increase in the magnetic energy is due to the metallic core introduced storing larger energy.

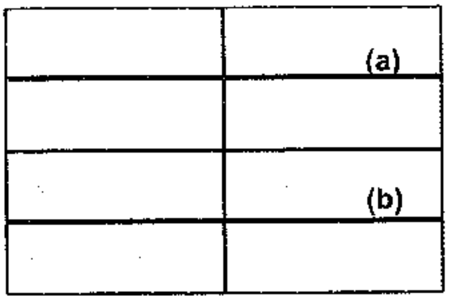
Problems

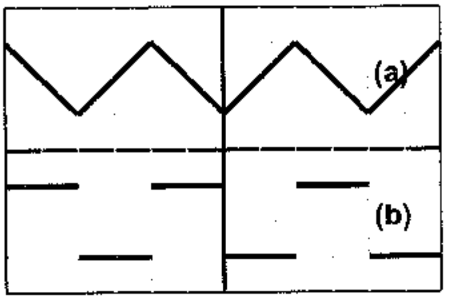
I-
Auto Induction

We consider a low frequency generator (LFG) delivering an alternating triangular signal coonected to a circuit formed a coil of inductance L and negligible internal resistance and a resistor R of resistance $R = 2.0 k\Omega$.



An oscilloscope is connected as shown on circuit.
The figures below show the zero level on the two channels (left) and the waveforms observed on the screen (right).





The adjustments of the oscilloscope are:

✘ vertical sensitivity on Y_1 : $S_{v_1} = 0.2V / \text{div}$.

✘ vertical sensitivity on Y_2 : $S_{v_2} = 5V / \text{div}$.

✘ horizontal sensitivity: $S_h = 1 ms / \text{div}$.

1. What is the voltage displayed on Y_1 ? on Y_2 ?
2. Determine the frequency of the voltage delivered by the generator.
3. a) Name the phenomenon that takes place in the previous experiment.
   b) Write down the expression of the voltage u_{AM} across the terminals of the coil.
   c) Show that u_{AM} can be written in the form $u_{AM} = -\frac{L}{R}\frac{du_{BM}}{dt}$ where u_{BM} is the voltage across the terminals of the resistor.
   d) Which of the waveforms (a) and (b) is that corresponding to the channel Y_1 ? to the channel Y_2 ?
4. a) Determine the maximum value of the voltage u_{AM} .
   b) Using the first half period of the waveform, calculate $\frac{du_{BM}}{dt}$.
5. a) Deduce the numerical value of the ratio $\tau = \frac{L}{R}$.
   b) Justify that τ is measured in seconds.
   c) Deduce the value of the inductance L .

Effect of a Coil

The set up represented in the adjacent figure consists of an ideal generator of $e.m.f$ E, a coil of resistance r and of inductance $L = 4\,mH$, a resistor of resistance R.

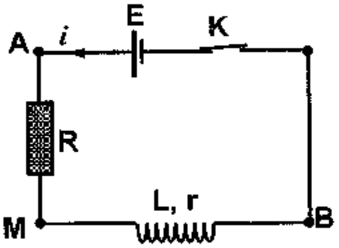
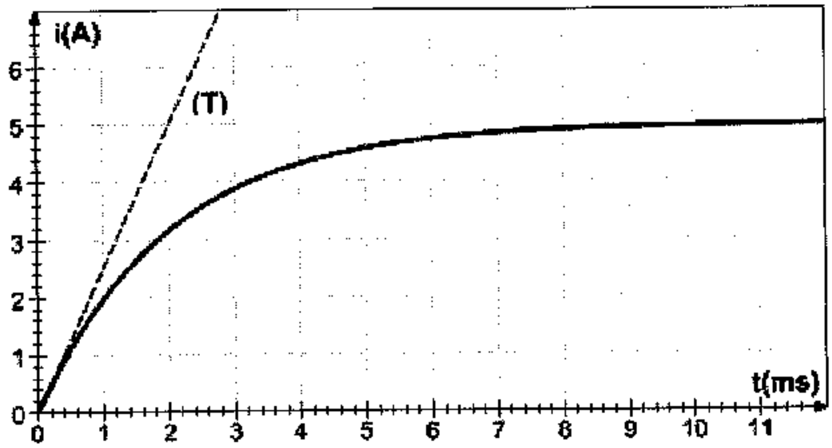


Figure 1

Part A

Graphical study

The adjacent curve represents the evolution of the current as a function of time. (T) is the tangent to this curve at the point of abscissa $t_0 = 0$.



1. Justify that the coil is the seat of auto-induction in $[0;10\,ms]$.
2. Calculate the self-induced $e.m.f$ e_1 at $t_0 = 0$.
3. Justify that there is no induced $e.m.f$ after $t = 10\,ms$.
4. Determine, giving all the necessary explanations, the value of the self-induced $e.m.f$ at $t = 3ms$.
5. The time constant τ is the abscissa of the point of intersection of the tangent at the origin to the curve representing the current and the asymptote to that curve. Determine the value of τ.
6. a) In what form does the coil stores energy?
   b) Calculate the energy stored in the coil when the steady state is reached.

Part B

Theoretical study

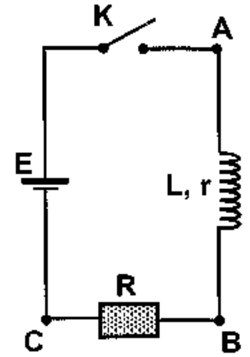
1. Write the relation between the self-induced electromotive force e_1, with L & $\frac{di}{dt}$.
2. Show that the differential equation that governs the evolution of the self-induced $e.m.f$ is given by:
   $$\frac{d\,e_1}{dt} + \left(\frac{R+r}{L}\right)e_1 = 0.$$
3. Applying the law of addition of voltages, show that at $t_0 = 0$ the induced $e.m.f$ $e_1 = -E$.
4. a) Determine the expressions of A & k so that $e_1 = Ae^{-kt}$ is a solution of the differential equation.
   b) Draw the curve representing the evolution of e_1 as a function of time.
   c) What is the role of the coil?
5. Show the expression of the current i as a function of time can be written $i = a\left(1 - e^{-\frac{t}{\tau}}\right)$ where a & τ are constants whose expressions are to be determined.

III- The Phenomenon of Self-Induction

In order to study the effect of the inductance L and the resistance R on the variations of the current in the circuit.

The set up represented by the adjacent figure consists of an ideal generator of $e.m.f$ E, a coil of inductance L and internal resistance r, a resistor of resistance R and a switch K. At the instant $t_0 = 0$, we close the switch K.

At an instant t, the circuit carries a current i in the transient state.



Part A

Theoretical study

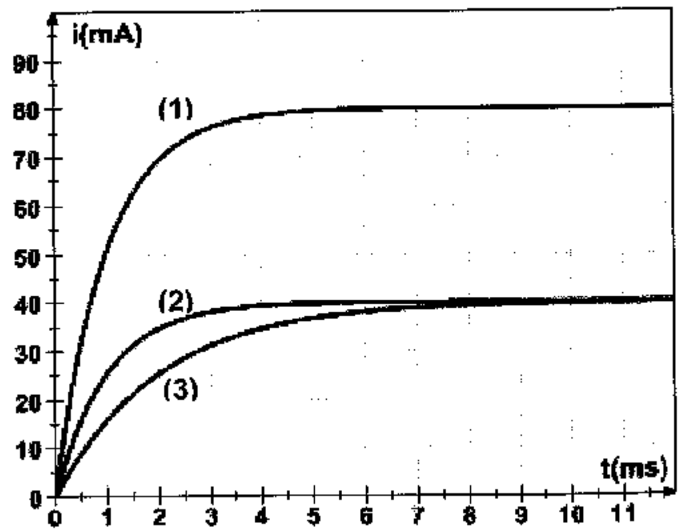
1. Explain why the voltage across the resistor is used to study the evolution of the current in the circuit.
2. Derive the differential equation that governs the variation of i as a function of time.
3. The solution of the differential equation is of the form $i = a + be^{-\frac{t}{\tau}}$.
   a) Determine the expressions of a, b & τ in terms of L, R and r.
   b) Explain the physical significance of the time constant τ.
4. a) Give the expression of the minimum duration needed to reach the steady state.
   b) Deduce the expression of the current I_0 in the steady state.

Part B

Influence of L & R

We perform three experiments using many couples of resistances and pure inductive coils listed in the table below.

	1st	2nd	3rd
$L(mH)$	100	200	400
$R(\Omega)$	80	180	180



1. Specify which physical quantity τ or I_0 that is not affected by the inductance of the coil.
2. Deduce that the curve (1) corresponds to the first experiment.
3. Explain why the current in the steady state is the same in the 2nd and 3rd experiments.
4. Determine the time constants for the curves (2) and (3), and then justify that (2) corresponds to the 2nd experiment.
5. Using the curves (1) & (2), determine the values of E and r.

Different Roles of a Coil

Part A

Opening and closing of the injector of a car

An electromagnet, formed of a coil, is used to control the opening and closing of the injector in the modern car engines. This coil can also be used as a metal detector. In this exercise, we are interested to determine the inductance L of the coil.

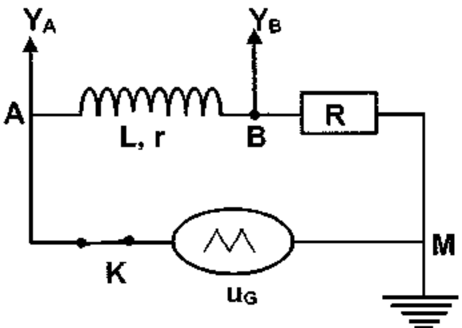


Figure 1

To determine the inductance L of the coil of negligible resistance, we carry out the circuit of figure 1. The used generator delivers, across its terminals, a triangular asymmetrical voltage u_G. The resistance of (R) is equal to $1.0 k\Omega$. A suitable system allows us to obtain the curves of figure 2 which represent the variation of the voltage $u_R = u_{BM}$ across (R) and that of the voltage $u_L = u_{AB}$ across the coil as a function of time.

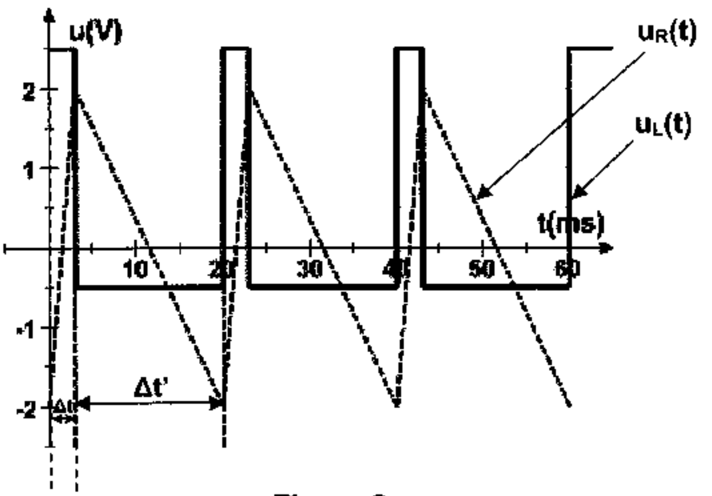


Figure 2

1. How did we obtain the curve u_L by using the recorded voltages on the channels Y_A and Y_B?
2. a) Give the expression of u_L in terms of u_R.

   b) The inductance:

   i- Referring to figure 2, determine the value of the inductance L in each of the two intervals Δt and $\Delta t'$.

   ii- The manufacturer announces $L \approx 2.0 H$.

   Comment briefly on the two values obtained of L by accepting a relative variation of absolute value 10%.

Part B

Effect of iron on the inductance

The set up used is that of figure 1, where we replace the generator by an ideal one of *e.m.f* $E = 3.2 V$

Using an appropriate device, we record the variation of the voltage $u_R = u_{BM}$ as a function of time.

The origin of time is taken at the instant when the switch K is closed.

1. Derive, at an instant t, the differential equation in u_R.
2. The solution of this differential equation is of the form $u_R = U_0\left(1 - e^{-\frac{t}{\tau}}\right)$. Determine the expressions of the constants U_0 and τ.

3. The recording of u_R is, initially, obtained in the absence of any metal placed near the coil (curve (a)), then in the presence of a piece of iron placed near the coil (curve (b)) (figure 3).

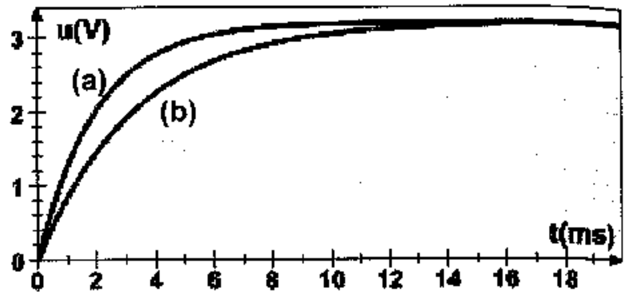


Figure 3

a) Determine the values of the constants τ_a and τ_b associated respectively with (a) and (b).

b) Comparison:

i- Compare the values L_a and L_b of the inductance of the coil in the absence and in the presence of iron.

ii- What can we deduce?

V-Engineering (2005-2006)
Electric and Mechanical Analogy

Part A
(R, L) series circuit

In the circuit of figure 1, $L=1H$, $R=1k\Omega$ and $E=10V$. At the instant $t_0=0$, we close the switch K. At the instant t, the circuit carries a current i.

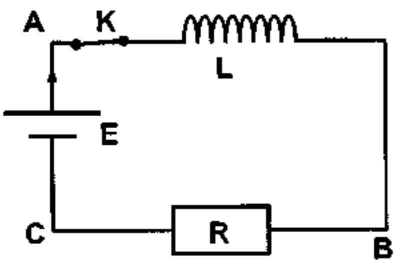


Figure 1

An oscilloscope, conveniently connected, allows to display the variations of the voltage $u=u_{BC}$ as a function of time (figure 2).

1. Derive the differential equation describing the variations of the current i in terms of R, L, E & t.
2. The solution of this equation is of the form $i=A_1-B_1\,e^{-\frac{t}{\tau_1}}$.
   Determine the values of the constants A_1, B_1 and τ_1 then give the physical significance of each.
3. Referring to figure 2, verify that the values of τ_1 and A_1 are equal to the calculated above.
4. Determine:
   a) the duration t_1 at the end of which the steady state is practically reached.
   b) the value of the energy stored in the coil starting from t_1.

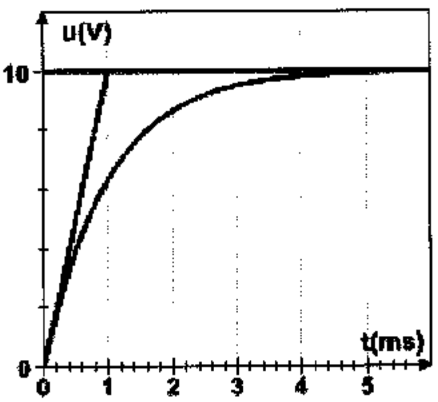
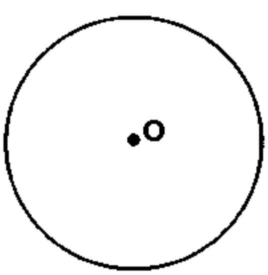


Figure 2

Part B
A disk in a rotational motion

A disk can rotate about a horizontal axis (Δ) that is perpendicular to its plane through its center O.



The moment of inertia I of the disk with respect to the axis (Δ) is $I=1.52\times10^{-5}\,kg.m^2$. When subjected to a motive couple, of constant moment $M=9.12\times10^{-3}\,m.N$, the disk starts to rotate from rest at the instant $t_0 = 0$. At the instant t, the physical quantities θ and θ' are respectively the angular abscissa and the angular velocity of the disk.

During its rotation, the disk undergoes a braking couple of moment $\mathrm{M}_F = -k\,\theta'$, where k is a positive constant of value $k = 3.04 \times 10^{-5}$ SI units.

Applying the theorem of angular momentum, show that the differential equation in θ' that describes the motion of the disk is written as: $I\dfrac{d\theta'}{dt} + k\,\theta' = \mathrm{M}$.

Part C

An analogy

1. Match each of the physical electric quantities E, R, L, i and $\dfrac{di}{dt}$ with the convenient mechanical physical quantity.
2. a) Determine the solution of the differential equation in θ'.

   b) Deduce the duration t_2 at the end of which the steady state is practically reached.

   c) Determine the angular velocity in the steady state.

Solutions

I-

1. The voltage u_{AM} across the coil is displayed on Y_1.

   The voltage u_{BM} across the terminals of the resistor is displaced on Y_2.

2. The period of the signal displayed is $T = S_h \times x = 1ms/\text{div} \times 4\text{div} = 4ms$.

   The frequency $f = \dfrac{1}{4\times10^{-3}} = 250\ Hz$.

3. a) Self or auto induction.

   b) $u_{AM} = u_L = ri + L\dfrac{di}{dt}$, the coil is pure inductive $r = 0$, then $u_{AM} = L\dfrac{di}{dt}$.

   c) We have $u_{BM} = -u_R = -Ri$, so $i = -\dfrac{u_{BM}}{R}$ & $\dfrac{di}{dt} = -\dfrac{1}{R}\dfrac{du_{BM}}{dt}$;

   Then $u_{AM} = L\dfrac{di}{dt} = -\dfrac{L}{R}\dfrac{du_{BM}}{dt}$.

   d) (a) corresponds to Y_2 represents i, u_{BM} is triangular.

   (b) corresponds to Y_1 the voltage u_{AM} (proportional to the derivative of u_{AM}).

4. a) $(u_{AM})_{\max} = 1\text{div} \times 0.2V/\text{div} = 0.2V$.

   b) During the first half period u_{BM} is decreasing,

   $$\frac{du_{BM}}{dt} = \frac{\Delta u_{BM}}{\Delta t} = \frac{-2\text{ div} \times 5V/\text{div}}{2\text{ div} \times 10^{-3}s/\text{div}} = -5000\ V/s.$$

5. a) We have $u_{AM} = -\dfrac{L}{R}\dfrac{du_{BM}}{dt} \Rightarrow 0.2 = -\dfrac{L}{R}(-5000) \Rightarrow \dfrac{L}{R} = 4\times10^{-5}\ SI$.

   b) $\tau = \dfrac{L}{R} = \dfrac{V.s.A^{-1}}{V.A^{-1}} = s$.

   c) We have $L/R = 4\times10^{-5}$; so $L = 0.08H$.

II-

Part A

1. For $t \in [0;10ms]$, the current is variable, then the circuit is the seat of a self induced *e.m.f.*

2. We have $e\big|_{t_0=0} = -L\dfrac{di}{dt}\Big|_{t_0=0}$ where $\dfrac{di}{dt}\Big|_{t_0=0}$ is the slope of the tangent at $t_0 = 0$;

   Graphically $\dfrac{di}{dt}\Big|_{t_0=0} = \dfrac{\Delta i}{\Delta t} = \dfrac{(5-0)A}{(2-0)\times10^{-3}s} = 2500\ A/s$.

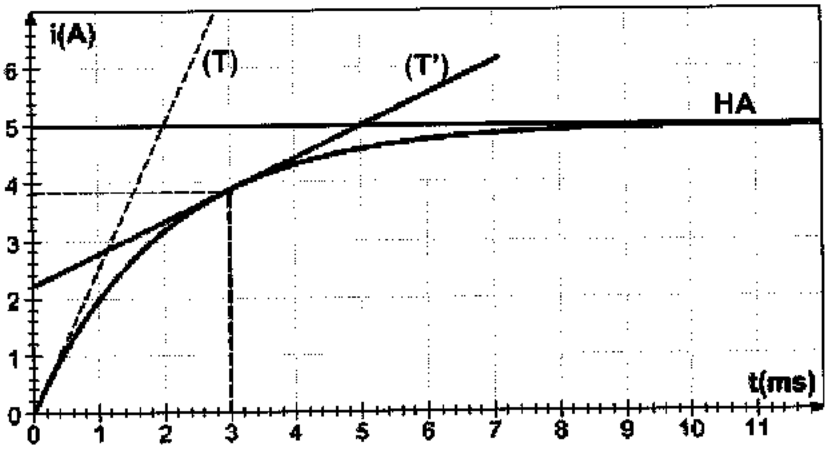
   Then $e\big|_{t_0=0} = -4\times10^{-3}\times2500 = -10V$.

3. For $t > 10ms$, the current is in the steady state then it is constant $i = I_0$;

Then $\left.\frac{di}{dt}\right|_{t>10ms} = 0$, thus $e|_{t>10ms} = -L\frac{di}{dt} = 0$ (no self induced $e.m.f$).

4. We have $e|_{t=3ms} = -L\left.\frac{di}{dt}\right|_{t=3ms}$ where $\left.\frac{di}{dt}\right|_{t=3ms}$ is the slope of the tangent at $t = 3ms$;

We construct the tangent to the curve at $t = 3ms$ which passes through $(3ms, 4A)$ & $(0, 2A)$;



$\left.\frac{di}{dt}\right|_{t=3ms} = \frac{\Delta i}{\Delta t} = \frac{(3.8-2.2)A}{(3-0)\times 10^{-3}s} = 533\ A/s$;

Then $e|_{t=3ms} = -4\times 10^{-3} \times 533 = -2.132V$.

5. The equation of the horizontal asymptote is $i = I_0 = 5A$, which intersects the tangent (T) at origin at the point of abscissa $\tau = 2ms$.

6. a) The coil stores magnetic energy.

b) In the steady state, the current is $i = I_0 = 5A$;

The magnetic energy stored is $E_m = \frac{1}{2}LI_0^2 = \frac{1}{2}(4\times 10^{-3})\times 5^2 = 0.05J$.

Part B

1. The self-induced $e.m.f$ is given by $e_1 = -\frac{d\phi}{dt}$ where ϕ is the magnetic flux crossing the coil;

But $\phi = Li$, then $e_1 = -L\frac{di}{dt}$.

2. Law of addition of voltages: $u_{AB} = u_{AM} + u_{MB} = Ri + ri - e_1$;

Then $E = (R+r)i - e_1$;

Deriving both sides with respect to time $\frac{dE}{dt} = (R+r)\frac{di}{dt} - \frac{de_1}{dt}$; but $\frac{dE}{dt} = 0$ & $\frac{di}{dt} = -\frac{e_1}{L}$;

Then $-(R+r)\frac{e_1}{L} - \frac{de_1}{dt} = 0$; thus, $\frac{de_1}{dt} + \frac{(R+r)}{L}e_1 = 0$.

3. We have $E = (R+r)i - e_1$ is verified at any instant; but at $t = 0$, $i = 0$; then $e_1|_{t_0=0} = -E$.

4. a) We have $e_1 = Ae^{-kt}$, then $\frac{de_1}{dt} = -Ak\,e^{-kt}$;

Replacing in the differential equation we get: $-Ak\,e^{-kt} + \left(\frac{R+r}{L}\right)Ae^{-kt} = 0$;

So, $Ae^{-kt}\left(-k + \frac{R+r}{L}\right) = 0$

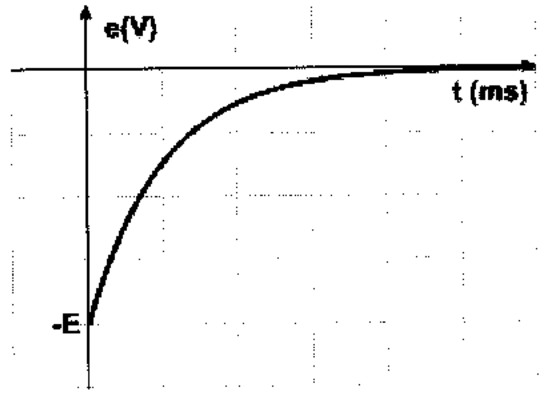
But $Ae^{-kt} \neq 0$, then $k = \frac{R+r}{L}$;

But $e_1\big|_{t_0=0} = -E$, then $A = -E$;

Thus, $e_1 = -E\,e^{-\left(\frac{R+r}{L}\right)t}$.

b) Graph, for $t = 0$, $e_1 = -E$.

As $t \longrightarrow +\infty$, $e_1 \longrightarrow 0$.



c) $e_1 < 0$, the coil acts as a generator in opposition.

5. We have $e_1 = -L\frac{di}{dt}$, $i = -\frac{1}{L}\int e_1\,dt = -\frac{1}{L}\int -E\,e^{-\left(\frac{R+r}{L}\right)t}\,dt = -\frac{E}{L} \times \frac{L}{R+r}e^{-\left(\frac{R+r}{L}\right)t} + C$;

But $t = 0$; $i = 0$; then $C = \frac{E}{R+r}$;

Thus, $i = \frac{E}{R+r}\left(1 - e^{-\left(\frac{R+r}{L}\right)t}\right)$ of the form $i = a\left(1 - e^{-\frac{t}{\tau}}\right)$, where $a = \frac{E}{R+r}$ & $\tau = \frac{L}{R+r}$.

2nd method: (refer to the differential equation)

We have $E = (R+r)i - e_1$ and $e_1 = -E\,e^{-\left(\frac{R+r}{L}\right)t}$;

Then $i = \frac{E + e_1}{R+r} = \frac{E - E\,e^{-\left(\frac{R+r}{L}\right)t}}{R+r} = \frac{E}{R+r}\left(1 - e^{-\left(\frac{R+r}{L}\right)t}\right)$.

III-

Part A

1. According to Ohm's law $u_R = Ri$, the voltage across the resistor u_R is proportional to the current i that flows through it.
   Then u_R is the image of the current in the circuit.

2. Law of addition of voltages: $u_{AC} = u_{AB} + u_{BC}$, $E = Ri + ri + L\frac{di}{dt}$, we get: $E = (R+r)i + L\frac{di}{dt}$.

3. a) We have $i = a + be^{-\frac{t}{\tau}}$, then $\frac{di}{dt} = -\frac{b}{\tau}e^{-\frac{t}{\tau}}$;

Replacing in the differential equation we get: $(R+r)\left(a + be^{-\frac{t}{\tau}}\right) - L\frac{b}{\tau}e^{-\frac{t}{\tau}} = E$;

Then $(R+r)a + be^{-\frac{t}{\tau}}\left(R+r-\frac{L}{\tau}\right) = E$ which is verified at any instant t;

But $be^{-\frac{t}{\tau}} \neq 0$, so $R+r-\frac{L}{\tau}=0$; then $\tau = \frac{L}{R+r}$ & $a = \frac{E}{R+r}$;

Referring to the initial conditions: at $t=0$, $i=0$; then $0=a+b$; $b=-a=-\frac{E}{R+r}$.

Thus, $i = \frac{E}{R+r}\left(1-e^{-\frac{t}{\tau}}\right)$ where $\tau = \frac{L}{R+r}$.

b) τ is called the time constant of the circuit, it represents the duration after which the current in the circuit reaches 63% of its value in the steady state.

4. a) The minimum duration needed to reach the steady state is $\Delta t_{\min} = 5\tau$.

b) After $\Delta t_{\min} = 5\tau$, $i = I_0 = \frac{E}{R+r}\left(1-e^{-5}\right) = 0.99\frac{E}{R+r} \approx \frac{E}{R+r}$.

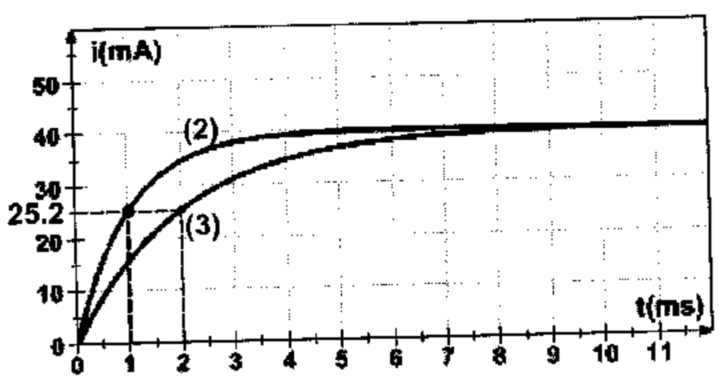
Part B

1. The current in steady state $I_0 = \frac{E}{R+r}$ is independent of the inductance L.
2. The current in steady state is inversely proportional to the resistance of the circuit and since $R_{(1)} < R_{(2)} = R_{(3)}$, then $I_{01} > I_{02}$.

   Thus, curve (1) corresponds to the first experiment.
3. We have $I_0 = \frac{E}{R+r}$, depends on the resistance of the resistor which are the same in the 2nd and 3rd experiments. Then, they have the same current in the steady state.
4. At $t=\tau$, $i = 0.63\, I_0 = 0.63 \times 40mA = 25.2mA$;

   From graph $\tau_{(2)} = 10ms$ & $\tau_{(3)} = 20ms$, but $\tau = \frac{L}{R+r}$ and $\tau_{(3)} > \tau_{(2)}$; then $L_{(3)} > L_{(2)}$.

   So, the curve (2) corresponds to the experiment which has the smallest L, then it corresponds to the 2nd experiment.



5. We have $I_{0(1)} = \frac{E}{R_1+r}$ &

   $I_{0(2)} = \frac{E}{R_2+r}$;

   $I_{0(1)} = 80mA$ & $I_{0(2)} = 40mA$;

   And $\frac{I_{0(2)}}{I_{0(1)}} = \frac{\frac{E}{R_2+r}}{\frac{E}{R_1+r}} = \frac{R_1+r}{R_2+r}$;

   $\frac{40}{80} = \frac{80+r}{180+r} = \frac{1}{2}$, we get, $160+2r = 180+r$; then $r = 20\Omega$.

   Thus, $E = I_{0(1)}(R_1+r) = 80\times10^{-3}(80+20) = 8V$.

IV-Engineering 2013/2014

Part A

1. The button «INV» changes the sign of the voltage displayed; while the button «ADD» is used to add the voltages displayed.

   The voltage displayed on Channel A is u_{AM} and on Channel B is u_{BM};

   According to the law of addition of voltages:

   $u_G = u_{AM} = u_{AB} + u_{BM}$; $u_{AB} = u_L = u_{AM} - u_{BM} = u_{AM} + (-u_{BM})$;

   So we should press the button «INV» on B followed by the button «ADD».

2. a) Ohm's law across the coil of negligible internal resistance $u_L = L\frac{di}{dt}$;

   Ohm's law across the resistor R is $i = \frac{u_R}{R}$;

   Then $u_L = L\frac{di}{dt} = L\frac{d}{dt}\left(\frac{u_R}{R}\right) = \frac{L}{R}\frac{du_R}{dt}$.

   b) The inductance:

   i- In the interval $\Delta t = [0\,;3ms]$, u_R varies linearly,

   So, $\frac{du_R}{dt} = \frac{\Delta u_R}{\Delta t} = \frac{(2)-(-2)}{(3-0)\times10^{-3}} = \frac{4}{3}\times10^3\ V/s$;

   and $u_L = 2.5V$, thus $2.5 = \frac{L}{1\times10^3}\frac{4}{3}\times10^3$; therefore $L = \frac{3\times2.5}{4} = 1.875\ H$.

   In the interval $\Delta t' = [3ms\,;20ms]$, u_R varies linearly,

   So, $\frac{du_R}{dt} = \frac{\Delta u_R}{\Delta t} = \frac{(-2)-(2)}{(20-3)\times10^{-3}} = \frac{-4}{17}\times10^3\ V/s$;

   and $u_L = -0.5V$; thus $-0.5 = \frac{L}{1\times10^3}\frac{-4}{17}\times10^3$; therefore $L = \frac{17\times0.5}{4} = 2.125\ H$.

   ii- The relative error is given $\frac{|\Delta L|}{L_0}$;

   Over Δt: $\frac{|\Delta L_{\Delta t}|}{L_0} = \frac{|1.875-2|}{2}\times100 = 6.25\,\% < 10\%$;

   Over $\Delta t'$: $\frac{|\Delta L_{\Delta t'}|}{L_0} = \frac{|2.125-2|}{2}\times100 = 6.25\,\% < 10\%$;

   Thus, these values are compatible with that provided by the manufacturer.

Part B

1. Law of addition of voltages: $u_G = u_R + u_L$ but $u_L = \frac{L}{R}\frac{du_R}{dt}$, then $E = u_R + \frac{L}{R}\frac{du_R}{dt}$.

2. We have $u_R = U_0\left(1 - e^{-\frac{t}{\tau}}\right)$ then $\frac{du_R}{dt} = \frac{U_0}{\tau}e^{-\frac{t}{\tau}}$;

   By substitution in the differential equation $E = U_0\left(1 - e^{-\frac{t}{\tau}}\right) + \frac{L}{R}\frac{U_0}{\tau}e^{-\frac{t}{\tau}} = U_0 + U_0e^{-\frac{t}{\tau}}\left(\frac{L}{R\tau} - 1\right)$;

This equation is verified at any instant t and $U_0 e^{-\frac{t}{\tau}} \neq 0$ so $\frac{L}{R\tau} - 1 = 0$;

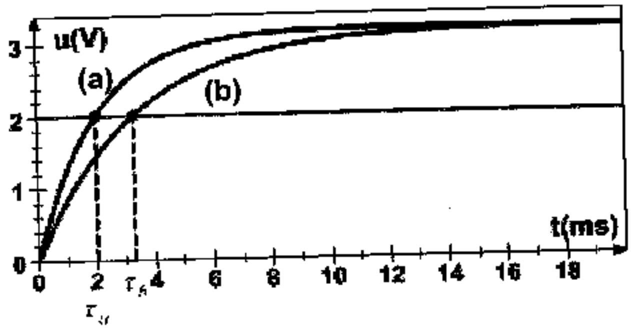
Then $\tau = \frac{L}{R}$ and $U_0 = E$.

3. a) For $t = \tau$, $u_L = 0.63 \times U_0 = 0.63 \times 3.2 = 2.02V$;

   Then graphically $\tau_a = 2\,ms$ and $\tau_b = 3.3\,ms$.

   b) Comparison

   i- $L_a = \tau_a \times R = 2 \times 10^{-3} \times 1 \times 10^3 = 2H$ & $L_b = \tau_b \times R = 3.3 \times 10^{-3} \times 1 \times 10^3 = 3.3H$.



   ii- $L_b > L_a$, thus the presence of iron nearby the coil leads to an increase in its inductance L.

V-

Part A

1. Law of addition of voltages: $E = Ri + L\frac{di}{dt}$ ......(1)

2. At $t = 0$, $i = 0$ then $A_1 = B_1$;

   We have: $i = A_1 - B_1\, e^{-\frac{t}{\tau_1}} \Rightarrow \frac{di}{dt} = \frac{B_1}{\tau_1} e^{-\frac{t}{\tau_1}}$.

   Substitution in the differential equation we get:

   $E = RA_1 - RB_1 e^{-\frac{t}{\tau_1}} + L\frac{B_1}{\tau_1} e^{-\frac{t}{\tau_1}} \Rightarrow E = RA_1 + B_1 e^{-\frac{t}{\tau_1}}\left(-R + \frac{L}{\tau_1}\right)$ is verified at any instant.

   By identification $A_1 = \frac{E}{R}$ which represents the maximum current; thus $A_1 = B_1 = 10^{-2}A = I_{max}$;

   & $\tau_1 = \frac{L}{R}$ (the time constant of the circuit), $\tau_1 = 10^{-3}s$.

3. From figure $I_{max} = \frac{(u_R)_{max}}{R} = 10^{-2}A$.

   But τ_1 represents the abscissa of the point of intersection of the tangent at the origin to the curve with the horizontal asymptote $\tau_1 = 1ms = 10^{-3}s$.

4. a) The steady state is reached after $t_1 = 5\tau_1 = 5 \times 10^{-3}s$.

   b) The magnetic energy stored is: $E_m = \frac{1}{2}LI_{max}^2 = 5 \times 10^{-4}J$.

Part B

Theorem of angular momentum $\sum M_{\overline{F_{ext}}} = \frac{d\sigma}{dt}$; $M_{\overline{w}} + M_{\overline{R}} + M_{\overline{F}} + M = I\frac{d\theta'}{dt}$;

But $M_{\overline{w}} = M_{\overline{R}} = 0$ (applied on axis), $I\frac{d\theta'}{dt} + k\theta' = M$ ....(2)

Part C

1. By comparison, we match:
   - ✘ $I \longrightarrow L$;
   - ✘ $\theta' \longrightarrow i$;
   - ✘ $k \longrightarrow R$;
   - ✘ $\frac{d\theta'}{dt} \longrightarrow \frac{di}{dt}$;
   - ✘ $M \longrightarrow E$.

2. a) By comparison $\theta' = \frac{M}{k}\left(1 - e^{-\frac{t}{\tau_2}}\right)$ where $\tau_2 = \frac{I}{k}$.

   b) The steady state is reached after $t_2 = 5\tau_2 = 5\frac{I}{k} = \frac{5 \times 1.52 \times 10^{-5}}{3.04 \times 10^{-5}} = 2.5\,s$.

   c) In steady state, the angular velocity by analogy with the current in steady state $I = \frac{E}{R}$, we get:

$$\theta'_{\max} = \frac{\tau_M}{k} = \frac{9.12 \times 10^{-3}}{3.04 \times 10^{-5}} = 300\,\text{rad}/s.$$

Supplementary Problems

The Phenomenon of Self-Induction

The set up represented by the adjacent figure consists of an ideal generator of *e.m.f* E, a coil of resistance r and of inductance L, a resistor of resistance $R = 50\,\Omega$ and a switch K.

At the instant $t_0 = 0$, we close the switch K.

At an instant t, the circuit carries a current i in the transient state.

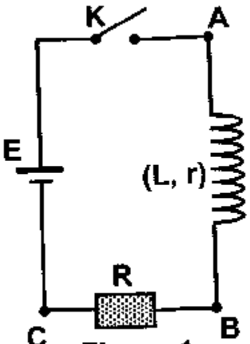


Figure 1

Part A

Theoretical study

1. Derive the differential equation that governs the variation of i as a function of time.
2. Verify that the solution of the differential equation is $i = \frac{E}{R+r}\left(1 - e^{-\frac{t}{\tau}}\right)$ where $\tau = \frac{L}{R+r}$.
3. An oscilloscope is used to display the voltages across the generator u_{AC} and that across the resistor u_{BC}.
   a) Recopy the previous circuit showing on it the connections of the oscilloscope.
   b) Justify how we can display the voltage u_{AB} across the coil without modifying the connections but by pressing specific knobs (buttons).
4. Give, in terms of τ, the minimum duration needed to reach the steady state.
5. Determine, in terms of R, r and E, the expression of the voltage U_0 across the capacitor in the steady state.

Part B

Graphical study

The curves of figure 2 represent the evolution of the voltages u_{AC} and u_{BC} as a function of time.

1. Referring to the curves of figure 2, give:
   a) the value of u_{BC} in steady state.
   b) the value of E.
2. Calculate the current and the energy stored in the coil in steady state.
3. Determine the time constant τ.
4. Deduce the values of r and L.

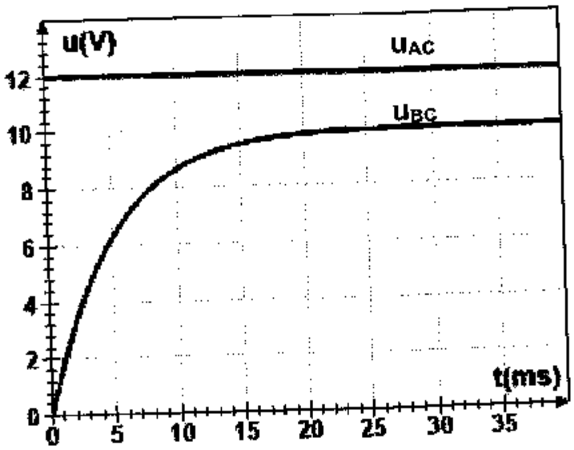


Figure 2

Answer Key

II- Self-Induction and Current

Consider the circuit represented in figure 1 where:

- ✖ (G) is a DC generator of $e.m.f\ E = 15\,V$ and of negligible internal resistance;
- ✖ a resistor of resistance $R = 80\,\Omega$;
- ✖ (B) is a coil of inductance L and of internal resistance r;
- ✖ (K) is a double switch.

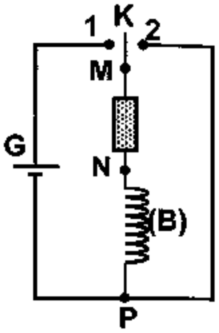


Figure 1

Part A
Growth of current

The switch is placed on position (1), the steady state is reached with a certain delay.

1. Give the name of the phenomenon responsible of the delay in the establishment of current.
2. Explain qualitatively the cause of this delay.
3. Applying the law of addition of voltages, determine the expression of the current in steady state I in terms of E, r & R.

Part B
Decay of current

At an instant taken as a new origin of time $t_0 = 0$, the switch is turned abruptly to position (2).

1. Applying the law of addition of voltages, derive the differential equation that governs the evolution of the current in the circuit.
2. Determine the expressions of I_0 & τ so that $i = I_0\, e^{-\frac{t}{\tau}}$ is a solution of the previous equation.
3. Show that the power dissipated by the resistor is given by $P = \dfrac{R E^2}{(R+r)^2} e^{-\frac{2t}{\tau}}$.

Part C
Experimental study

A convenient apparatus is used to plot the curve representing the variation of the power P dissipated in the resistor in terms of time.

1. Referring to the graph, give the value of P at $t_0 = 0$.
2. Deduce the value of r.
3. Determine, in terms of τ, the duration needed so that the power dissipated is 37% of its value at $t = 0$.
4. Determine the value of L.

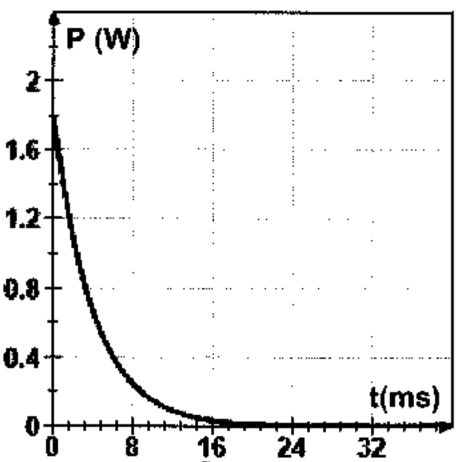


Figure 2

Answer Key

Part C $r = 20\,\Omega$ & $L = 0.8\,H$.

LS 2012 2[nd]
Electromagnetic and Self-Induction

Part A
Electromagnetic induction

A coil, of horizontal axis, is made up of $N = 500$ circular turns each of surface area $S = 10\,cm^2$. The normal $\vec{n}$ to the planes of the turns of the coil is directed as indicated in figure 1.

The coil rotates at a constant angular velocity w about a vertical axis (Δ) in a horizontal, constant and uniform magnetic field $\vec{B}$. The terminals A and C of the coil are connected to the input Y and the ground M of an oscilloscope respectively. Let θ be the angle between $\vec{n}$ and $\vec{B}$ at an instant t.

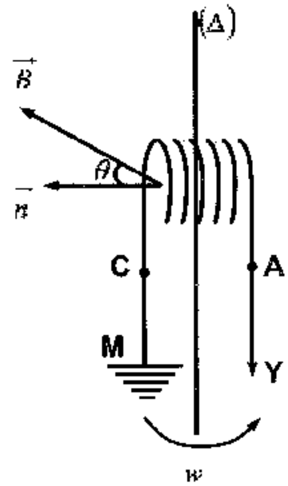
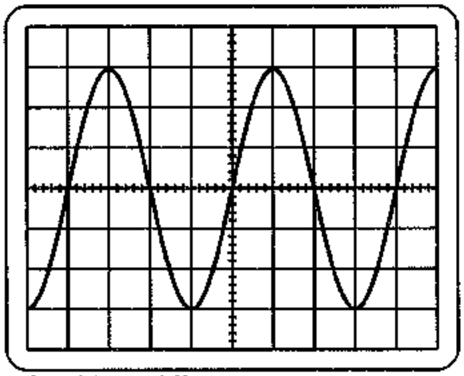


Figure 1

1. Knowing that $\theta = 0$ at the instant $t_0 = 0$, show that $\theta = wt$.
2. Deduce that the expression of the magnetic flux crossing the coil is given by $\phi = N\,B\,S\cos(wt)$.
3. Justify, qualitatively, the existence of an induced $e.m.f$ « e » during the rotation of the coil.
4. a) Determine, in terms of N, S, B, w and t the expression of the induced $e.m.f$ « e ».
   b) The coil does not carry a current. Why?
   c) Deduce the expression of the voltage u_{AC} in terms of N, S, B, w and t supposing that the coil is oriented positively from A to C.
5. The waveform of figure 2 represents the variation of the voltage u_{AC} as a function of time. Using this waveform, determine:
   a) the angular velocity w of the coil;
   b) the maximum value of the voltage u_{AC};
   c) the value B of the magnetic field $\vec{B}$.



S_h=10 ms/div
S_v= 1V/div
Figure 2

Part B
Self-induction

The coil is of negligible resistance and of inductance L. It is connected in series with a resistor of resistance $R = 1k\Omega$ and a generator G (figure 3).

The circuit of figure 3 thus carries a triangular current i. The positive orientation of the circuit is as that of the current.

With the aid of the oscilloscope, we visualize the variations of the voltages $u_1 = u_{BC}$ across the resistor and $u_2 = u_{AC}$ across the coil (figure 4).

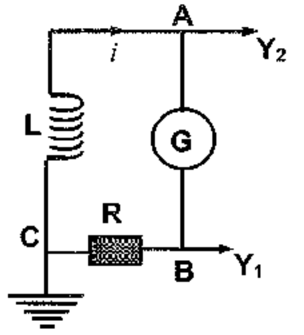
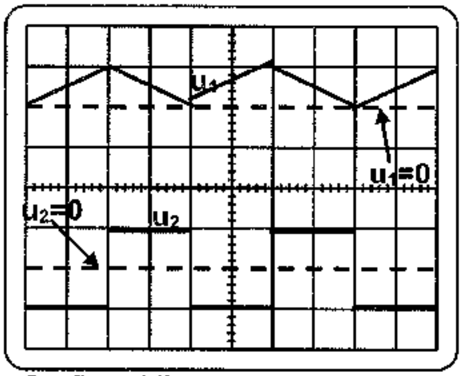


Figure 3



S_h=5 ms/div

S_{v1}= 1V/div; S_{v2}= 10mV/div

Figure 4

1. Show that $u_2 = -\frac{L}{R}\frac{du_1}{dt}$.
2. The shape of the waveform obtained on Y_2 is square. Justify this shape.
3. Determine the value of L.

II-LS 2007 1st
Usage of a Coil

A coil of inductance $L = 0.01H$ and of negligible resistance is connected in series with a resistor of resistance R across a generator G (figure 1). The coil thus carries a current i that varies with time as shown in figure 2.

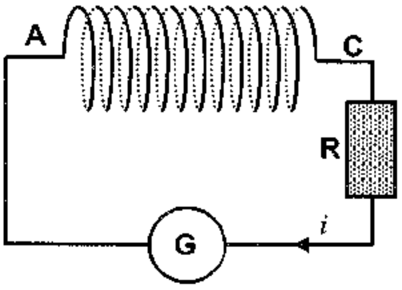


Figure 1

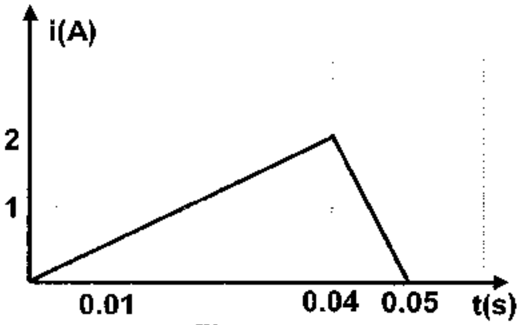


Figure 2

1. Give the name of the physical phenomenon that takes place in the coil.
2. Determine the voltage u_{AC} in each of the two intervals [0; 0.04s] & [0.04s; 0.05s].

III-LS 2003 1st
See Page 369 – Part A

IV-LS 2002 2nd
See Page 437 – Part A

Sessions Solutions

I-LS 2012 2nd

Part A

1. The system is rotating with constant angular velocity so $\theta = wt + \theta_0 = wt$.
2. The magnetic flux ϕ is given by $\phi = N.\vec{B}.\vec{S} = N\,B\,S\cos(\vec{n},\vec{B}) = N.B.S\cos(\theta) = N.B.S\cos(wt)$.
3. During the rotation of the coil, the angle θ varies then the magnetic flux ϕ also varies, therefore an induced electromotive « e » appears in the circuit.
4. a) According to Faraday's law : $e = -\dfrac{d\phi}{dt} = -\dfrac{d}{dt}(N.B.S\cos(wt)) = N.B.S.w\sin(wt)$.

   b) Since the circuit is not closed (the resistance of the oscilloscope is too large so the circuit is considered as open).

   c) Ohm's law case of a generator: $u_{AC} = r\,i - e$ but the internal resistance is negligible then $r = 0$

   Thus, $u_{AC} = -N.B.S.w\sin(wt)$.
5. a) The angular velocity of the coil is equal to the angular frequency of the displayed voltage:

   $T = S_h \times x = 10ms/\text{div} \times 4\text{div} = 40\,ms$.

   But $w = \dfrac{2\pi}{T} = \dfrac{2\pi}{40\times10^{-3}} = 50\,\pi\,\text{rad}/s \approx 157\,\text{rad}/s$.

   b) The maximum value of $(u_{AC})_{\max} = S_v \times y_{\max} = 1V/\text{div} \times 3\text{div} = 3V$.

   c) The maximum of $u_{AC} = -NBSw\sin(wt)$ is $(u_{AC})_{\max} = N\,B\,Sw$.

   Then $B = \dfrac{(u_{AC})_{\max}}{N\,S\,w} = \dfrac{3}{500\times10\times10^{-4}\times157} = 0.038\,T$.

Part B

1. According to Ohm's law: $u_2 = u_{AC} = e - r\,i = e = -L\dfrac{di}{dt}$ & $u_1 = R\,i \Rightarrow i = \dfrac{u_1}{R}$;

   Then $\dfrac{di}{dt} = \dfrac{1}{R}\dfrac{du_1}{dt}$; thus: $u_2 = -\dfrac{L}{R}\dfrac{du_1}{dt}$.
2. In the first half period, u_1 is a linear function of time and increasing, so $\dfrac{du_1}{dt}$ is constant and positive then $u_2 = -\dfrac{L}{R}\dfrac{du_1}{dt}$ is constant and negative.

   Thus whenever i is linear and increasing then u_2 is constant and negative and the vice versa is true[1].
3. In the first half period: $\dfrac{du_1}{dt} = \dfrac{\Delta u_1}{\Delta t} = \dfrac{1\text{div}\times 1V/\text{div}}{2\text{div}\times 5ms/\text{div}} = \dfrac{1V}{(2\times5\times10^{-3})s} = 100V/s$.

---

1 Otherwise if u_1 is constant so $\dfrac{du_1}{dt} = 0$, then $u_2 = -\dfrac{L}{R}\dfrac{du_1}{dt} = 0$;
thus the voltage u_2 will vanishes which is not the case.

and $u_2 = S_{v2} \times y_2 = 10mV / \text{div} \times (-1\text{div}) = -10mV = -10^{-2} V$;

But $u_2 = -\frac{L}{R}\frac{du_1}{dt} \Rightarrow L = \frac{-Ru_2}{\left(\frac{du_1}{dt}\right)} = -\frac{-10^{-2} \times 10^3}{100} = 0.1H = 100\ mH$.

II-LS 2007 1st

1. Self (auto) induction.

2. $u_{AC} = L\frac{di}{dt}$;

For $0 \le t \le 0.04s$, $\frac{di}{dt} = \frac{2}{0.04} = 50\,A/s$; $u_{AC} = 0.01 \times 50 = 0.5V$.

For $0.04\,s \le t \le 0.05s$, $\frac{di}{dt} = \frac{-2}{0.01} = -200\,A/s$; $u_{AC} = 0.01 \times (-200) = -2V$.

Unit II

Electricity

Chapter 8

Capacitor and DC Voltage

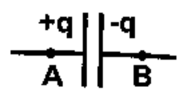
LS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Capacitor and DC Voltage		1st	1st	2nd	-	2nd	1st	1st	1st
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Capacitor and DC Voltage	-	1st	-	1st	-	2nd	2nd	-	1st

Essentials

I-

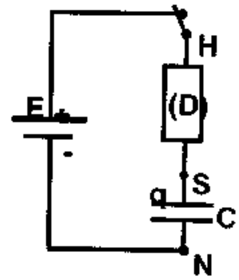
The Capacitor

A capacitor is formed of two conducting plates (armatures) separated by an insulator (dielectric) and of capacitance C measured in Farad F.



Let $u_C = u_{AB}$ be the voltage across the capacitor. The armature A carries a charge $q_A = +q$, while the armature B carries a charge $q_B = -q$ such that $q = C u_C$ & the current in the circuit is given by $i = \left|\frac{dq}{dt}\right|$.

A capacitor stores an electric energy given by $E_e = \frac{1}{2} C u_C^2 = \frac{1}{2}\frac{q^2}{C}$.



II-

Charge & Discharge of the Capacitor

In a simple circuit, formed of a DC generator of *e.m.f* E, connected in series with a capacitor and a resistor.

The time constant of the circuit is $\tau = RC$.

	Charging	Discharging
Current	$i = +\frac{dq}{dt} = C\frac{du_C}{dt}$	$i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$
At $t = 0$	$i = I_{\max} = \frac{E}{R}$ & $u_C = 0$	$i = I_{\max} = \frac{E}{R}$ & $u_C = E$
After $t = \tau$	$u_C = (63\%)E$	$u_C = (37\%)E$
Steady state after 5τ	$i = 0$ & $u_C = E$	$i = 0$ & $u_C = 0$
Role	Capacitor is storing charge	Capacitor acts as a generator

III-

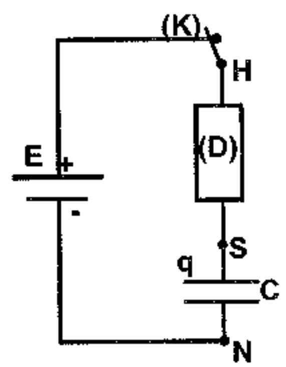
General Summary

Electric Dipole	Resistor	Coil	Capacitor
Characteristic physical quantity	Resistance R	Inductance L	Capacitance C
SI Unit	Ohms Ω	Henry H	Farad F
Law	$u_R = Ri$	$u = ri + L\frac{di}{dt}$	$q = C u_C$
Energy	Joule's effect $P = Ri^2$ & $E = P \times t$	Magnetic Energy $E_m = \frac{1}{2} L i^2$	Electric energy $E_e = \frac{1}{2} C u_C^2$

Applications

I- Charging the Capacitor

We set up the circuit whose diagram is represented in the adjacent figure. G is a generator of constant $e.m.f$ E and of negligible internal resistance, (C) is a capacitor, initially uncharged, of capacitance C, (D) is a resistor of resistance R, and K is a switch.



1. Show that the differential equation satisfied by the voltage u_C across the capacitor is given by $E = u_C + RC\dfrac{du_C}{dt}$.
2. Determine the expressions of the constants a & b so that $u_C = a + b\,e^{-\frac{t}{RC}}$ is the solution of this differential equation.
3. Calculate, in terms of E, the expressions of u_C at the following instants $t = 0$, $t = RC$ & $t = 5RC$.
4. Plot the graph representing the variations of the voltage u_C in terms of time.
5. Determine the percentage of electric energy stored in the capacitor at the instant $t = RC$ to that stored in steady state.

II- Current and Capacitor

Consider the adjacent circuit shown in figure 1 formed of:

- an ideal DC generator of $e.m.f$ E;
- a resistor of resistance $R = 2\,k\Omega$;
- a capacitor of capacitance C taken neutral.

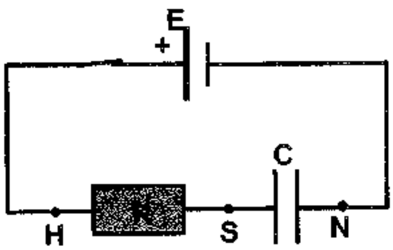


Figure 1

At an instant $t_0 = 0$, taken as origin of time, the switch is closed.

1. Name the phenomenon that takes place when the switch is closed.
2. Applying the law of addition of voltages and taking the positive direction along the circuit as that of the current, show that the differential equation that describes the variation of the voltage $u_R = u_{HS}$ across the resistor has the form: $u_R + RC\dfrac{du_R}{dt} = 0$.
3. The solution of this differential equation has the form: $u_R = \alpha\,e^{-\beta t}$ where α and β are constants.

   Show that $\alpha = E$ and $\beta = \dfrac{1}{RC}$.
4. An oscilloscope connected across the terminals of the resistor displays the voltage u_{HS} whose curve is shown in figure 2.

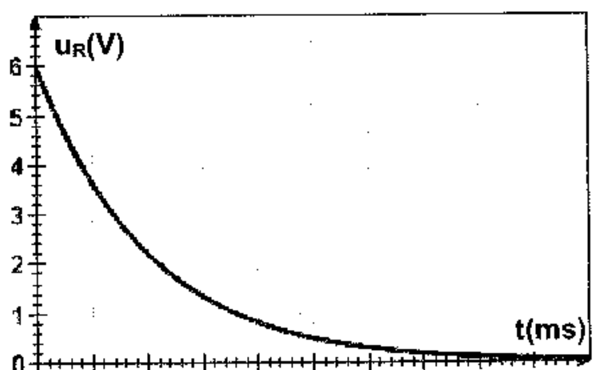


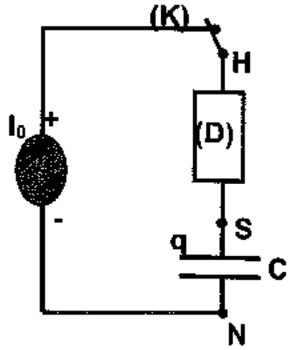
Figure 2

a) Redraw the circuit shown in figure 1 showing on it the connections of the oscilloscope.
b) Justify that u_{HS} is the image of the current.
c) Determine the values of E & C.

III-

Capacitor and Constant Current

We set the circuit whose diagram is represented in figure 1, G is a current generator of negligible internal resistance and delivering a constant current I_0, (C) is a capacitor, initially uncharged, of capacitance C, (D) is a resistor of resistance R, and K is a switch.

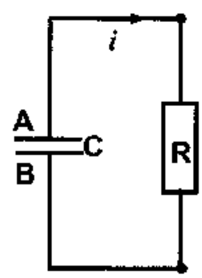


1. Show that the expression voltage across the capacitor is given by $u_C = \frac{I_0}{C}t$.
2. Write the expression of the voltage u_{HS} in terms of the given.
3. Describe the shapes of the curves representing voltages u_C & u_{HS} and then determine the coordinates of their point of intersection.
4. Draw the curves representing u_C & u_{HS} in terms of time.

IV-

Study of Discharging a Capacitor

A capacitor of capacitance C is initially charged and the voltage between its terminals is E. At $t_0 = 0$, we connect across the terminals of the capacitor a resistor of resistance R, at an instant t, the armature A carries the charge $q > 0$ and the circuit carries a current i.



1. Write the relation between i and q.
2. Show that the differential equation of the charge q across the capacitor is $\frac{dq}{dt} + \frac{1}{RC}q = 0$.
3. The solution of this differential equation is $q = De^{-\frac{t}{\tau}}$.
   Determine the expressions of the constants D and τ in terms of E, R and C.

V-

Linear Variations of a Voltage

We set up the circuit whose diagram is represented in figure 1, (C) is a capacitor initially charged under a voltage U_0, of capacitance C, and connected in series with a resistor (D) of resistance $R = 200\ \Omega$. The circuit is closed at an instant taken as origin of time $t_0 = 0$.

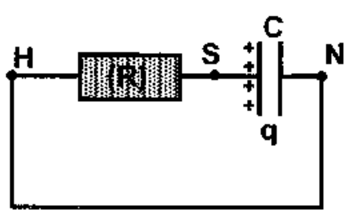


Figure 1

1. Recopy figure 1, showing the direction of the current.
2. Show that the differential equation satisfied by the voltage across the capacitor $u_C = u_{SN}$ is $u_C + RC\frac{du_C}{dt} = 0$.
3. Verify that $u_C = U_0\,e^{-\frac{t}{RC}}$ is a solution of the previous differential equation.

4. Justify that the expression of the natural logarithm of u_C as a function of time t is of the form $\ell n[u_C] = \alpha t + \beta$ where α & β are constants whose expressions to be determined in terms of the given.

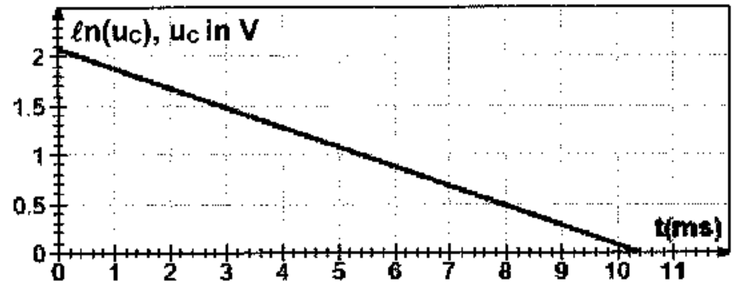


Figure 2

5. The variation of $\ell n[u_C]$ as a function of time is represented by figure 2.
Justify that the shape of the obtained graph agrees with the expression of $\ell n[u_C]$ as a function of time.
6. Deduce, using the graph, the values of U_0 & C.

VI-
Graphical Study

We set up the circuit whose diagram is represented in figure 1, G is a generator of constant $e.m.f$ E and of negligible internal resistance, (C) is a capacitor, initially uncharged, of capacitance $C = 2.5\,\mu F$, (D) is a resistor of resistance R.
A convenient software is used to display the voltage u_{SN} across the terminals of the capacitor and the curve obtained is shown in figure 2, (T) is the tangent to this curve at the point of abscissa 0.

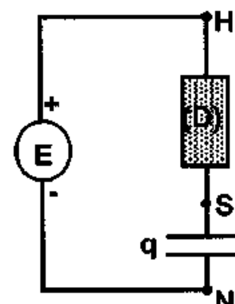


Figure 1

The expression of u_{SN} in terms of time is given by $u_{SN} = a + b e^{-\frac{t}{\tau}}$ where a, b & τ are constants.

1. Referring to the curve shown in figure 2:
   a) give the values of u_{SN} at the instants $t = 0$ & in steady state.
   b) determine the current at $t = 0$ & in steady state.
2. Deduce the values of E, a & b.
3. Give the expression of the time constant τ.
4. Deduce R.
5. Calculate the energy stored in the capacitor in the steady state.

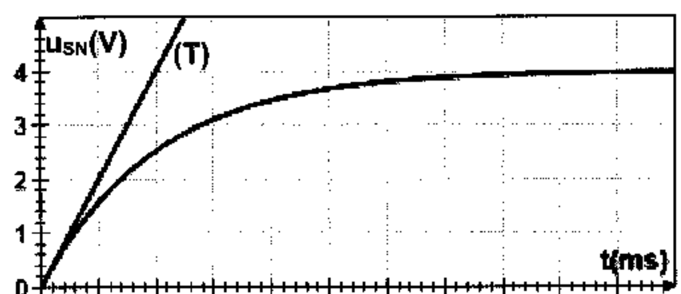


Figure 2

Solutions

I-

1. Law of addition of voltages: $u_{HN} = u_{HS} + u_{SN}$, so $E = Ri + u_C$;

   According to Ohm's law and taking the direction of the current as positive $i = +\frac{dq}{dt} = +C\frac{du_C}{dt}$

   (the capacitor is charging, so its charge q increases, then $\frac{dq}{dt} > 0$);

   Then $E = RC\frac{du_C}{dt} + u_C$.

2. We have $u_C = a + b^{-\frac{t}{RC}}$, $\frac{du_C}{dt} = -\frac{b}{RC}^{-\frac{t}{RC}}$; $E = RC\left(-\frac{b}{RC}e^{-\frac{t}{RC}}\right) + a + b^{-\frac{t}{RC}}$;

   Substitution in the differential equation, we get:

   Then, $E = -be^{-\frac{t}{RC}} + a + b^{-\frac{t}{RC}} = a$;

   Referring to initial conditions, at $t_0 = 0$, the capacitor is taken neutral so $u_C = a + b^0 = 0$;

   Then $b = -a = -E$; thus, $u_C = E - E\,e^{-\frac{t}{RC}} = E\left(1 - e^{-\frac{t}{RC}}\right)$.

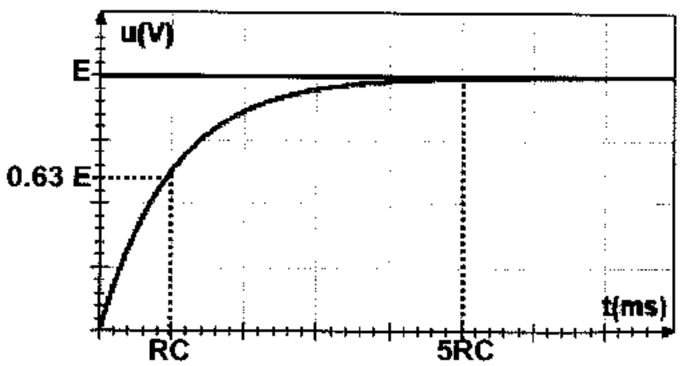
3. At $t_0 = 0$, $u_C = 0$;

   At $t = RC$,

   $u_C = E\left(1 - e^{-1}\right) = 0.63\,E = (63\%)E$;

   At $t = 5RC$,

   $u_C = E\left(1 - e^{-5}\right) = 0.99E = (99\%)E$;



4. We have $u_C = E\left(1 - e^{-\frac{t}{RC}}\right)$;

   u_C is an increasing exponential function, that tends to a horizontal asymptote of equation $u = E$.

5. The electric energy stored at the end of charging is $W_e = \frac{1}{2}Cu_C^2 = \frac{1}{2}CE^2$;

   At $t = RC$, $W_e(RC) = \frac{1}{2}Cu_C^2 = \frac{1}{2}C\left(1 - e^{-1}\right)^2 E^2$;

   The ratio is $\frac{W_e(RC)}{W_e} = \frac{\frac{1}{2}C\left(1 - e^{-1}\right)^2 E^2}{\frac{1}{2}CE^2} = \left(1 - e^{-1}\right)^2 \approx 0.4 = 40\%$;

   At $t = RC$, the electric energy stored is 40% of the total energy stored in the capacitor in the steady state.

II-

1. Charging of the capacitor.

2. We have $u_R = Ri$ & $i = +\frac{dq}{dt}$ where $q = Cu_C$, so $i = +C\frac{du_C}{dt}$; then $u_R = +RC\frac{du_C}{dt}$.

   Law of addition of voltages: $u_G = u_R + u_C$; $u_C = E - u_R$;

   We get: $u_R = +RC\frac{d(E-u_R)}{dt} = -RC\frac{du_R}{dt}$; then $u_R + RC\frac{du_R}{dt} = 0$.

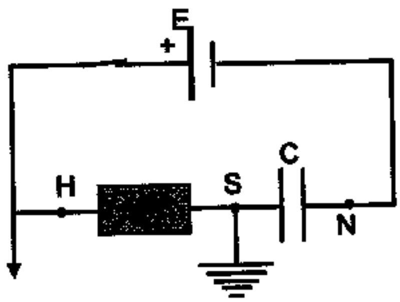
3. We have $u_R = \alpha e^{-\beta t}$, so $\frac{du_R}{dt} = -\alpha\beta e^{-\beta t}$;

   Substitution in differential equation: $\alpha e^{-\beta t} - RC\alpha\beta e^{-\beta t} = 0$, $\alpha e^{-\beta t}(1 - RC\beta) = 0$;

   But $\alpha e^{-\beta t} \neq 0$, we get $\beta = \frac{1}{RC}$.

   According to the initial conditions, at $t = 0$, $u_C = 0$ (the capacitor is taken neutral);

   Then $u_R = E - u_C = E$, we get $E = \alpha e^0$, $\alpha = E$.



4. a) Connections of adjacent figure.

   b) According to Ohm's law $u_R = Ri$, then u_R and the current are proportional.
   Thus, u_R is the image of the current in the circuit.

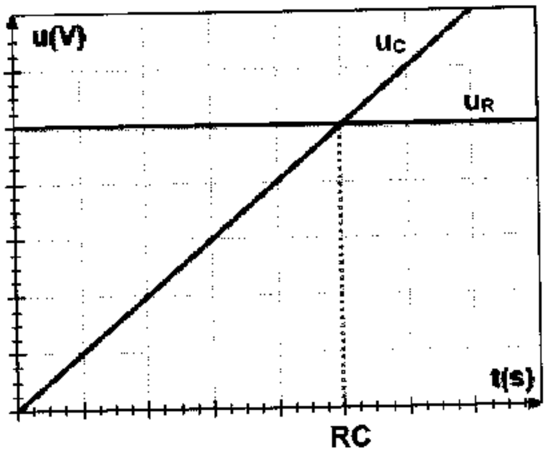
   c) Graphically at $t = 0$, $u_R = E = 6V$;
   For $t = 4ms$, we have $u_R = 2.2V$;
   Replacing in $u_R = Ee^{-\frac{t}{RC}}$; we get $2.2 = 6e^{-\frac{4\times10^{-3}}{2\times10^{3}\times C}}$;
   Then, $C = -\frac{2\times10^{-6}}{\ell n(2.2/6)} \approx 2\times10^{-6}F = 2\mu F$.

III-

1. We have $I_0 = \frac{dq}{dt} = C\frac{du_C}{dt}$, $du_C = \frac{I_0}{C}dt$; we get $\int du_C = \int \frac{I_0}{C}dt$;

   Then $u_C = \frac{I_0}{C}t + k$, but at $u_C = 0$, so $k = 0$;

   Thus, $u_C = \frac{I_0}{C}t$.



2. According to Ohm's law, $u_{HS} = RI_0$.

3. $u_{HS} = RI_0$ is constant, then its graphical representation is a horizontal straight line.

   $u_C = \frac{I_0}{C}t$ is represented by a straight line passing through origin.

   Point of intersection: $u_C = u_R$, so $\frac{I_0}{C}t = RI_0$;

   then $t = RC$.

4. Graphs.

IV-

1. During the discharge of the capacitor $i=-\frac{dq}{dt}$.
2. Law of uniqueness of voltages: $u_C = u_R = Ri = -R\frac{dq}{dt}$; but $q = C\,u_C$;

   So, $\frac{q}{u_C} = -R\frac{dq}{dt}$, then $\frac{dq}{dt}+\frac{1}{RC}q=0$.
3. We have $q = De^{-\frac{t}{\tau}}$, so $\frac{dq}{dt} = -\frac{D}{\tau}e^{-\frac{t}{\tau}}$;

   Substitution in the differential equation, we get: $-\frac{D}{\tau}e^{-\frac{t}{\tau}}+\frac{1}{RC}De^{-\frac{t}{\tau}}=0$;

   $De^{-\frac{t}{\tau}}\left(-\frac{1}{\tau}+\frac{1}{RC}\right)=0$, but $De^{-\frac{t}{\tau}}\neq 0$, then $\tau = RC$.

   At $t_0 = 0$, $u_C = E = \frac{Q_0}{C}$, so $Q_0 = CE$ & $Q_0 = De^0$; then $D = CE$;

   Thus, $q = C\,E\,e^{-\frac{t}{RC}}$.

V-

1. During discharging, the capacitor delivers the current from its positive armature.
2. Law of addition of voltages: $u_{SN} = u_{SH} + u_{HN}$;

   But $u_{HN} = 0$ (connecting wire);

   $u_{SH} = Ri$ where $i = -\frac{dq}{dt}$ & $q = Cu_C$; so $u_{SH} = -RC\frac{du_C}{dt}$;

   We get, $u_C = -RC\frac{du_C}{dt}$; then $u_C + RC\frac{du_C}{dt} = 0$.

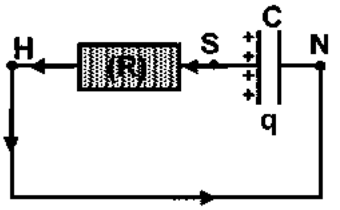


Figure 1

3. We have $u_C = U_0e^{-\frac{t}{RC}}$, so $\frac{du_C}{dt} = U_0\left(\frac{-1}{RC}\right)e^{-\frac{t}{RC}}$;

   Substitution in the differential equation we get: $u_C + RC\frac{du_C}{dt} = U_0e^{-\frac{t}{RC}} + RC\,U_0\left(\frac{-1}{RC}\right)e^{-\frac{t}{RC}} = 0$

   And at $t=0$, $u_C = U_0e^{-0} = U_0$.
4. We have $u_C = U_0e^{-\frac{t}{RC}}$, so $\ell n(u_C) = \ell n\left(U_0\,e^{-\frac{t}{RC}}\right) = \ell n U_0 + \ell n\left(e^{-\frac{t}{RC}}\right)$;[1]

   $\ell n(u_C) = \ell n(U_0) - \frac{t}{RC}$ which is of the form $\ell n(u_C) = \alpha t + \beta$ where $\alpha = -\frac{1}{RC}$ & $\beta = +\ell n(U_0)$.
5. The expression $\ell n(u_C) = \alpha t + \beta$ is of 1st degree in terms of time and of negative slope, so its graphical representation should be a decreasing straight line; which agrees with the curve.

[1] Pay attention: for any positive real numbers a & b, we have:
$\ell n(a\times b) = \ell n(a) + \ell n(a)$; & $\ell n(e^k) = k\times \ell n(e) = k(1) = k$.

6. At $t_0 = 0$, we have $\ell n(u_C) = \ell n(U_0) = 2.1$, then $U_0 = e^{2.1} \approx 8.2V$;

The slope of the curve $\alpha = \frac{\Delta \ell n(u_C)}{\Delta t} = \frac{0 - 2.1}{(10.4 - 0) \times 10^{-3} s} \approx -202\ s^{-1}$;

& $\alpha = -\frac{1}{RC} = -202$; then $C = \frac{1}{202 \times 200} \approx 2.5 \times 10^{-5} F = 25\ \mu F$.

VI-

1. a) At $t_0 = 0$, $u_C = u_{SN} = 0$ (passes through origin); and in steady state, $u_C = 4V$ (becomes constant).

b) The current i is given by $i = C\frac{du_C}{dt}$;

At $t_0 = 0$, $i|_{t_0=0} = C\frac{du_C}{dt}\Big|_{t_0=0}$;

But $\frac{du_C}{dt}\Big|_{t_0=0}$ is the slope of the tangent (T) to the curve representing u_C at $t_0 = 0$;

Then $i|_{t_0=0} = C\frac{\Delta u_C}{\Delta t} = 2.5 \times 10^{-6} \times \frac{(4-0)V}{(10-0) \times 10^{-3} s} = 10^{-3} A = 1 mA$.

In steady state, the voltage across the capacitor becomes constant, so $\frac{du_C}{dt} = 0$, thus $i_{\text{steady}} = 0$.

2. In steady state, the capacitor is completely charged, so $E = u_{SN}|_{\text{steady}} = 4V$;

We have $u_{SN} = a + be^{-\frac{t}{\tau}}$, in steady state $(t \longrightarrow +\infty)$; so $u_{SN} \longrightarrow a$, then $a = E = 4V$;

At $t_0 = 0$, $u_{SN} = a + be^0 = 0$; thus $b = -a = -4V$.

3. The time constant $\tau = RC$.

4. We know that τ is the abscissa of the point of intersection of the tangent at origin with the horizontal asymptote, then $\tau = 10\ ms$.

Thus, $R = \frac{\tau}{C} = \frac{10 \times 10^{-3}}{2.5 \times 10^{-6}} = 4 \times 10^3\ \Omega = 4\ k\Omega$.

5. The electric potential energy stored in the capacitor is given by:

$W_e = \frac{1}{2}C u_C^2 = \frac{1}{2}C E^2 = \frac{1}{2} \times 2.5 \times 4^2 = 20\ \mu J$.

Problems

I- Capacitors

During a lab session, students are trying to find the capacitance of a capacitor by two methods.

Part A

Charging a capacitor using constant current generator

The first method consists of charging the capacitor by the means of a generator delivering a constant current I_0 as shown in the circuit of figure 1.

At the instant $t = 0$, the switch K is closed; we record, using convenient software, the variations as a function of time of the voltage u_R across the terminals the resistor of resistance $R = 20\,\Omega$, the voltage «u_C» across the terminals of a capacitor and the curves obtained are shown in the following figures 2 & 3.

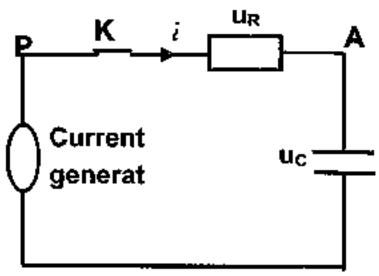


Figure 1

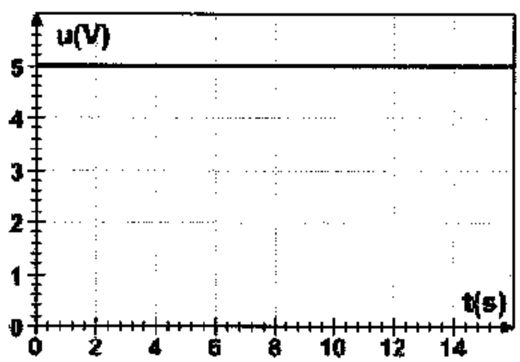


Figure 2

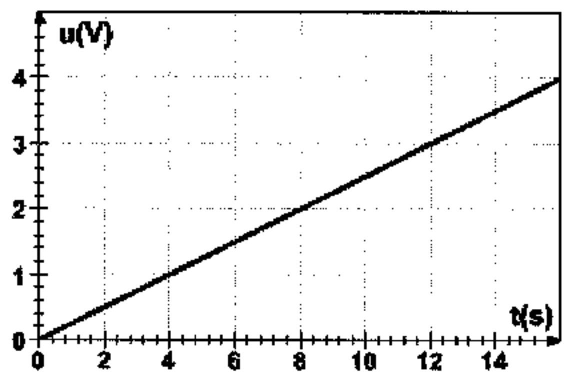


Figure 3

1. a) Redraw figure 1, showing the connections of an oscilloscope allowing us to display the voltage u_{PA} across resistor on channel 1 and that across the capacitor u_{AB} on channel 2.

   b) Justify that the graph of figure 2 corresponds to the voltage across the resistor $u_R(t)$.
2. Show that, in SI units, the expression of the charge q_A of the capacitor as a function of time is $q_A = 0.25\,t$ where q_A in C and t in s.
3. Referring to the graph of figure 3, determine the expression of the voltage u_C across the capacitor as a function of time.
4. Calculate the ratio $\frac{q_A}{u_C}$. What does it represent?
5. Calculate the time needed so that the voltage at the terminals of the capacitor becomes $2.5V$.

Part B
Charging the capacitor by constant voltage generator

Another method used to determine the value of the capacitor capacitance consists of charging it using a constant voltage generator $E = 5V$ connected in series with a resistor whose resistance is $R = 20\Omega$, as shown in figure 1.

At the instant $t = 0$, the switch K is closed. We record, using convenient software, the variations as a function of time of the voltage «u_C» across the terminals of a capacitor, the curve of figure 4 is obtained.

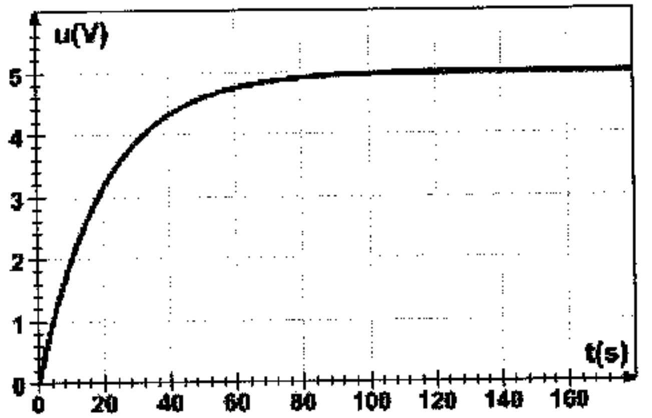


Figure 4

1. a) Show that the voltage u_C across the capacitor satisfies the relation $E = RC\dfrac{du_C}{dt} + u_C$.

   b) The solution of the previous differential equation is: $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$.

   Determine, in terms of E, R & C, the expressions of A & τ.
2. Referring to the graph, what are the values of u_C and i when the capacitor is completely charged?
3. What is the physical significance of the time constant τ of the circuit?
   Determine τ graphically and indicate the method used.
4. Deduce the capacitance C of the capacitor.
5. Calculate the time needed so that the voltage between the terminals of the capacitor becomes $2.5V$.

II- RC Series Circuit

We connect the circuit formed of a resistor of resistance $R = 100\,\Omega$, an ideal DC generator of *e.m.f* E, a capacitor of capacitance C, and a switch K (figure 1). At an instant taken as an origin of time $t_0 = 0$, we close the switch K.

We intend to study the charging of the capacitor through the variation of the voltage $u_{AB} = u_C$ as a function of time.

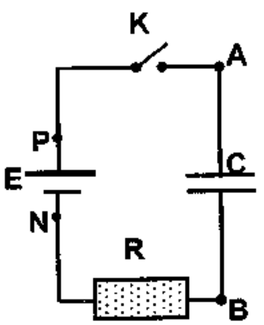


Figure 1

Part A
Theoretical study

1. Show that the differential equation satisfied by u_C is $E = u_C + \tau\dfrac{du_C}{dt}$ where τ is a constant whose expression is to be determined.
2. Verify that $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$ is the solution of the differential equation.

3. a) Give the expression of the voltage u_C at the end of charging.
   b) Determine, in terms of E, the expression of u_C at the instant $t=\tau$.
   c) Deduce the physical definition of the time constant τ.
4. Show that the expression of the natural logarithm of $(E-u_C)$ is given by:

$$\ell n(E-u_C)=-\frac{1}{\tau}t+\ell n(E).$$

Part B

Graphical study

The variation of $\ell n(E-u_C)$ as a function of time is represented by figure 2.

1. Justify that the shape of the obtained graph agrees with the expression of $\ell n(E-u_C)$ as a function of time.
2. Deduce, using the graph:
   a) the values of the capacitance C and that of E;
   b) the voltage u_C at the instant $t=6.6ms$.

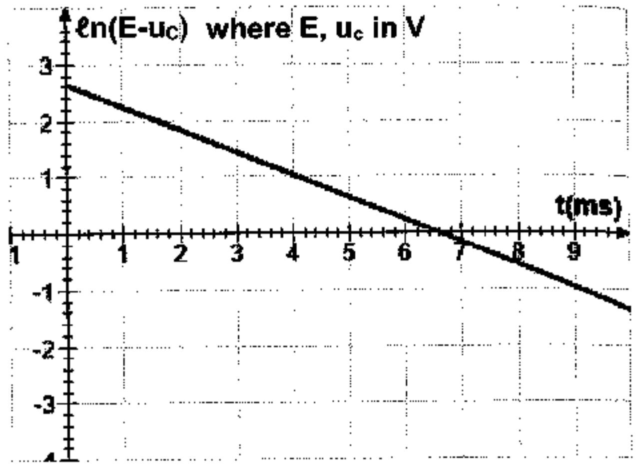


Figure 2

III-Engineering 2004/2005
Charging and Discharging of a Capacitor

In order to study the charging and discharging of a capacitor, we consider the circuit on the adjacent figure 1, where $u_{PN}=E$ (constant) and $r=1k\Omega$.

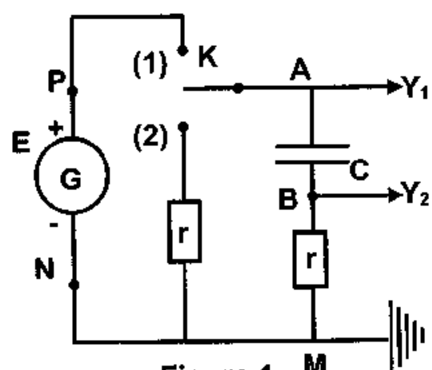


Figure 1

Part A

Charging of the capacitor

At the instant $t_0=0$, we put the switch (K) in position (1).

1. Derive the differential equation that is verified by the voltage $u_C=u_{AB}$.
2. The solution of this equation is of the form $u_C=D\left(1-e^{-\frac{t}{\tau}}\right)$. Deduce the expressions of D and τ in terms of r, C and E.
3. a) The waveform of figure 2, giving the variations of u_C in terms of time, is obtained by pushing the button «INV» of channel Y_2 and the button «ADD». Justify. The dashed line represents the tangent at $t_0=0$.
   b) Using the waveform, determine E and C.
4. Determine the instantaneous expression of the current i. Draw then the shape of the voltage u_{BM}.

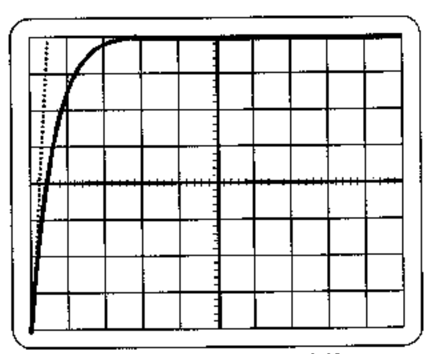


Figure 2 S_v=1V/div S_h=2ms/div

Part B
Discharging of the capacitor

The capacitor is completely charged and the buttons «INV» of channel Y_2 and «ADD» are always pushed. At the instant $t_0 = 0$, we put the switch into position (2).

We obtain the waveform of figure 3 which represents the variations of $u_C = u_{AB}$ as a function of time.

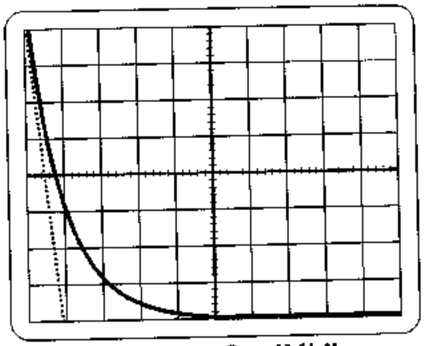


Figure 3 S_v=1V/div S_h=2ms/div

1. The variations of u_C is given by $u_C = E\,e^{-\frac{t}{\tau'}}$. Determine the expression of τ'. Verify the answer using the waveform of figure 3.
2. Draw the shape of the voltage u_{BM} and specify the used scale.

IV- Capacitor & Energy under DC Voltage

We set up the circuit whose diagram is represented in figure 1, G is a generator of constant $e.m.f$ E and of negligible internal resistance, (C) is a capacitor, of capacitance $C = 20\ \mu F$ initially uncharged, (D) is a resistor of resistance R, and K is a switch (Figure 1).

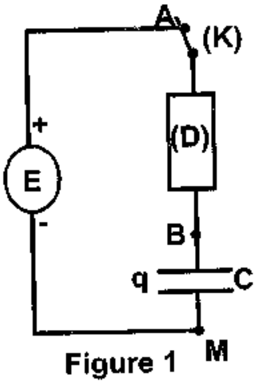


Figure 1

An oscilloscope is used to display the voltage u_{BM} across the terminals of the capacitor and the waveform obtained is shown on figure 2.

Part A
Charging of the capacitor

1. Show that the differential equation satisfied by u_C is given by $E = u_C + RC\dfrac{du_C}{dt}$.
2. Knowing that $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$ is the solution of this differential equation, determine the expression of the time constant τ in terms of R and C.
3. Determine the expression of the voltage across the capacitor in the steady state.
4. Referring to figure 2, show that $E = 16\,V$ & $R = 200\,\Omega$.

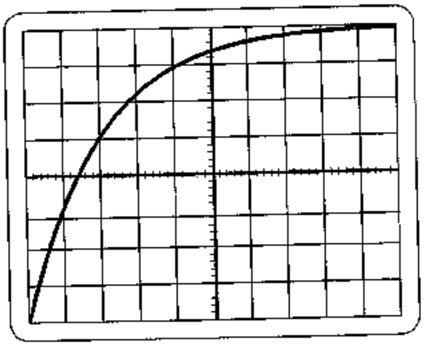


Figure 2
S_h=2 ms/div; S_v=2 V/div

Part B
Energy in the circuit

1. Calculate the energy stored in the capacitor at the end of charging.
2. Show that, at an instant t, the expression of the power dissipated (Joule's effect) is given by $P_d = \dfrac{E^2}{R}e^{-\frac{2t}{\tau}}$.

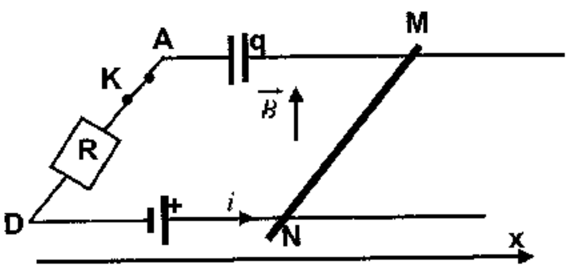
3. Knowing that $P_d = \dfrac{dW_J}{dt}$ where W_J is the energy dissipated.

   Determine the energy dissipated in the circuit due to Joules effect till the end of charging of the capacitor.

4. Show that the energy provided by the generator during the charging process is $5120\ \mu J$ and then calculate the percentage of the loss of energy.

V-Engineering 2012/2013
Charging a Capacitor and Motion of Rod

The circuit of the adjacent figure consists of two horizontal and parallel Laplace's rails connected to an ideal generator of $e.m.f$ $E = 6V$, a capacitor (C) of capacitance $C = 0.1F$ and a resistor of resistance $R = 5\Omega$. The rails, being horizontal and separated by a distance $\ell = 10\,cm$, are placed in an upward vertical magnetic field and of magnitude $B = 1.0T$.



A metallic rod MN, of mass $m = 0.10\,kg$, can move without friction on the rails while remaining perpendicular to these rails. The two rails and the rod are of negligible resistance.

At the instant $t_0 = 0$, (C) being discharged, we close K. At an instant t, the circuit carries a current i, (C) is charged by q and has, across its terminals, the voltage $u_{MA} = u_C$.

MN, located by its x-coordinate and undergoing the action of the Laplace's force, has a velocity $\vec{v}$ of algebraic value $v = \dfrac{dx}{dt}$. The circuit is thus oriented in the direction of i.

1. a) Give the direction of $\vec{F}$ and its magnitude F as a function of the current i.

   b) Show that the expression of the voltage across the terminals M and N of the rod is then written as $u_{MN} = +B\,\ell\,v$.

2. a) Applying Newton's second law, show that $v = k\,u_C$, and determine the positive constant k.

   b) Applying the law of addition of voltages, derive the differential equation: $E = RC\dfrac{du_C}{dt} + \left(\dfrac{B^2\ell^2C + m}{m}\right)u_C$.

3. a) The solution of this equation is of the form $u_C = a - b\,e^{-\frac{t}{\tau}}$.

   Determine the values of the constants a, b and τ.

   b) Deduce the expressions, as a function of time t, of v and i.

   c) Determine x as a function of time t knowing that, at the instant $t_0 = 0$, $x_0 = 0$.

   d) Nature of motion

   i- Determine the instant t_1 at which the steady state is practically reached.

   ii- Determine the charge Q of (C), the abscissa x_1 of MN and the nature of motion of the rod starting from t_1.

II- Capacitance of a Capacitor

We set up the circuit whose diagram is represented in figure 1, formed of:

- ✗ a generator G of constant $e.m.f$ E and of internal resistance r;
- ✗ a capacitor (C), initially uncharged, of capacitance C;
- ✗ a resistor (D) of resistance $R = 200\,\Omega$;
- ✗ a switch K.

At an instant taken as origin of time $t_0 = 0$, the switch is closed.

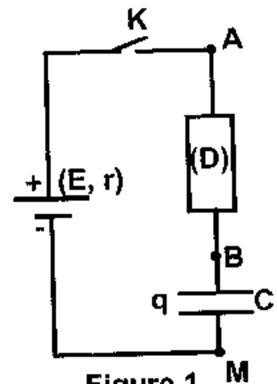


Figure 1

Part A
Differential equation

1. Show that the differential equation satisfied by the voltage across the capacitor $u_C = u_{BM}$ is given by $E = u_C + (R+r)C\dfrac{du_C}{dt}$.
2. Knowing that $u_C = a + b\,e^{-\frac{t}{\tau}}$ is the solution of this differential equation, determine the expressions of the constants a, b & τ in terms of E, R & C.
3. Give the expression of the minimum duration needed to reach the steady state.
4. Show that the expression of the current i in the circuit is $i = I_0\,e^{-\frac{t}{(R+r)C}}$ where $I_0 = \dfrac{E}{R+r}$.

Part B
Graphical study

A convenient software is used to trace the curves representing the variation of the voltage u_{AM} across the generator and that across the capacitor u_{BM} in terms of time respectively as shown in figure 2.

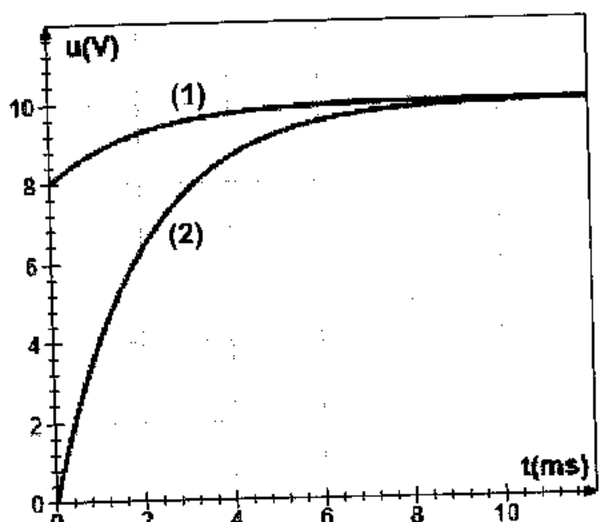


Figure 2

1. Justify that curve (2) represents the voltage across the capacitor.
2. Determine the expression of the voltage across the generator:
   a) in steady state in terms of E;
   b) at $t_0 = 0$ in terms of E, R, r.
3. Deduce that $E = 10\,V$ & $r = 50\,\Omega$.
4. a) Define the time constant τ of the circuit.
   b) Referring to figure, determine the value of τ.
5. Deduce the capacitance of the capacitor.

Solutions

I-

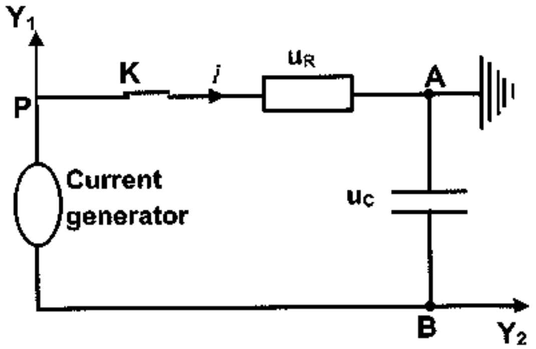
Part A

1. a) Connections of the oscilloscope.

The button "INV" should be pressed on channel (Y_2) in order to display u_{AB}.

b) According to Ohm's law, the voltage across the resistor is proportional to the current that flows through it.

Thus u_R is the image of the current i.



2. From figure 1, the current is constant of value $I_0 = 0.25A$;

But $I_0 = \frac{dq_A}{dt}$; $q_A = \int 0.25\, dt = 0.25\,t + q_0$;

From figure 2, at $t_0 = 0$ we have $u_C = 0$ but $q = Cu_C$ then $q_0 = 0$;

Thus $q_A = 0.25\,t$ (where t in s and q_A in C);

3. From graph of figure 3, u_C is represented by a straight line passing through origin so $u_C = at$;

where $a = \frac{\Delta(u_C)}{\Delta t} = \frac{2-1}{8-4} = 0.25\,V/s$;

Then $u_C = 0.25\,t$ $\qquad$ (t in s & u_C in V);

4. $\frac{q_A}{u_C} = \frac{0.25\,t}{0.25\ t} = 1C/V$.

But q_A and u_C are related by $q_A = Cu_C$, so $\frac{q_A}{u_C}$ represents the capacitance of the capacitor.

5. $u_C = 2.5V$; then $u_C = 0.25\,t = 2.5$; thus $t = 10s$.

Part B

1. a) Law of addition of voltages: $u_G = u_R + u_C$ but $i = \frac{dq}{dt}$ & $q = Cu_C$;

Then $E = Ri + u_C$; thus $E = u_C + RC\frac{du_C}{dt}$.

b) We have $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$, so $\frac{du_C}{dt} = A\left(0 - \left(-\frac{1}{\tau}\right)e^{-\frac{t}{\tau}}\right) = \frac{A}{\tau}e^{-\frac{t}{\tau}}$;

Replacing in the differential equation we get:

$E = A\left(1 - e^{-\frac{t}{\tau}}\right) + RC\frac{A}{\tau}e^{-\frac{t}{\tau}} = A - Ae^{-\frac{t}{\tau}} + RC\frac{A}{\tau}e^{-\frac{t}{\tau}}$;

Then $E = A + Ae^{-\frac{t}{\tau}}\left(-1 + \frac{RC}{\tau}\right)$ is verified at any instant t and $Ae^{-\frac{t}{\tau}} \neq 0$;

Thus, $\tau = RC$ & $A = E$.

2. From graph when the capacitor is completely charged $u_C = E = 5V$;

The current $i = C\dfrac{du_C}{dt} = 0$ (u_C becomes constant).

3. The time constant τ is the duration after which the voltage between the terminals of the capacitor reaches 63% of its maximum value.

From graph if $u_C = 0.63 \times 5 = 3.15V$;

The abscissa of the point, whose ordinate $3.15V$, is $\tau \approx 20s$.

4. We have $\tau = RC$; $C = \dfrac{\tau}{R} = \dfrac{20}{20} = 1F$ (huge capacitance).

5. We have $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$; $2.5 = 5\left(1 - e^{-\frac{t}{20}}\right)$; so $e^{-\frac{t}{20}} = \dfrac{1}{2}$; then $e^{\frac{t}{20}} = 2$; $t = 20\,\ell n2 \approx 14s$.

II-

Part A

1. Law of addition of voltages: $u_{PN} = u_{PA} + u_{AB} + u_{BN}$;

But $u_{PA} = 0$ (connecting wire); $u_{BN} = Ri$ where $i = +\dfrac{dq}{dt}$ & $q = Cu_C$;

Then $E = u_C + RC\dfrac{du_C}{dt}$ which is of the form $E = u_C + \tau\dfrac{du_C}{dt}$ where $\tau = RC$.

2. We have $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$, so $\dfrac{du_C}{dt} = \dfrac{E}{\tau}e^{-\frac{t}{\tau}}$;

Replacing in the differential equation, we get:

$$u_C + \tau\frac{du_C}{dt} = E\left(1 - e^{-\frac{t}{\tau}}\right) + \tau \times \frac{E}{\tau}e^{-\frac{t}{\tau}} = E - Ee^{-\frac{t}{\tau}} + Ee^{-\frac{t}{\tau}} = E \text{ (verified).}$$

3. a) At the end of charging, the voltage across the capacitor is $u_C = E$.

b) At $t = \tau$, we get $u_C = E(1 - e^{-1}) = 0.63E$.

c) The time constant τ is the duration after which the voltage across the capacitor reaches 63% of its value in the steady state E.

4. We have $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$, so $E - u_C = E - E\left(1 - e^{-\frac{t}{\tau}}\right) = E - E + Ee^{-\frac{t}{\tau}} = Ee^{-\frac{t}{\tau}}$;

Then $\ell n(E - u_C) = \ell n\left(E\,e^{-\frac{t}{\tau}}\right) = \ell nE + \ell n\left(e^{-\frac{t}{\tau}}\right) = -\dfrac{1}{\tau}t + \ell nE$.

Part B

1. The expression $\ell n(E - u_C) = -\dfrac{1}{\tau}t + \ell nE$ is of the form $f(t) = at + b$, then its graphical representation, should be:

✖ a straight line (the variable t is of degree 1);

✖ decreasing (its slope $a = -\dfrac{1}{\tau}$ is negative);

✖ not passing through origin ($b = \ell nE \neq 0$).

2. a) The slope of the line $a = \frac{\Delta \ell n(E - u_C)}{\Delta t} = \frac{2.6 - 1}{(4-0) \times 10^{-3} s} = -400 s^{-1}$;

But $a = -400 = -\frac{1}{RC}$; $C = \frac{1}{10^3 \times 400} = 2.5 \times 10^{-5} F = 25 \mu F$.

$\ell n(E)$ corresponds to the ordinate of the point of intersection with ordinate axis then $\ell n(E) = 2.6$; thus $E = e^{2.6} \approx 13.47 V$.

b) At $t = 6.6 ms$, $\ell n(E - u_C) = 0 = \ell n(1)$, then $E - u_C = 1$;

Thus, $u_C = E - 1 = 13.47 - 1 = 12.47 V$.

III-

Part A

1. Law of addition of voltages : $u_{PN} = u_{PA} + u_{AB} + u_{BM} + u_{MN}$;

but $u_{PA} = u_{MN} = 0$ (connecting wires);

We have $i = \frac{dq}{dt}$ & $q = C u_C$ so $i = C \frac{du_C}{dt}$; then $u_C + r C \frac{du_C}{dt} = E$.

2. We have $u_C = D\left(1 - e^{-\frac{t}{\tau}}\right)$, so $\frac{du_C}{dt} = D \frac{1}{\tau} e^{-\frac{t}{\tau}}$;

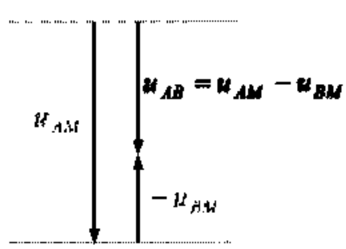
Replacing in the differential equation we get $E = D + D e^{-\frac{t}{\tau}}\left(1 - \frac{rC}{\tau}\right)$; which is verified at any instant, but $D e^{-\frac{t}{\tau}} \neq 0$, then $D = E$ and $\tau = rC$.

3. a) According to the connections of the oscilloscope: Y_1 displays $u_G = u_{AM}$ and Y_2 displays $u_r = u_{BM}$.

By pushing the button «INV», we get on Y_2 the voltage $-u_{BM} = u_{MB}$;

When the button «ADD» is pushed; the voltage displayed is the sum of those displayed on the two channels:

$u_{AM} + (-u_{BM}) = u_{AM} + u_{MB} = u_{AB} = u_C$.



b) The voltage across the capacitor in the steady state is: $E = (u_C)_{max} = 8 \text{div} \times 1 V/\text{div} = 8 V$.

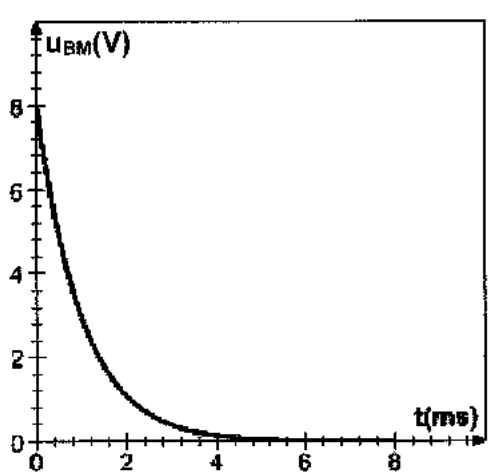
The tangent to the curve representing u_C at the point of abscissa $t = 0$, intersects with the horizontal asymptote at the point of abscissa $t = \tau = 0.5\text{div} \times 2 ms/\text{div} = 1 ms = 10^{-3} s$.

But $\tau = rC$, so $C = 10^{-6} F$.

4. The instantaneous current $i = C \frac{du_C}{dt} = \frac{E}{r} e^{-\frac{t}{\tau}}$;

We have $u_{BM} = r i = E e^{-\frac{t}{\tau}} = 8 e^{-1000 t}$

(t in s and u_{BM} in V).



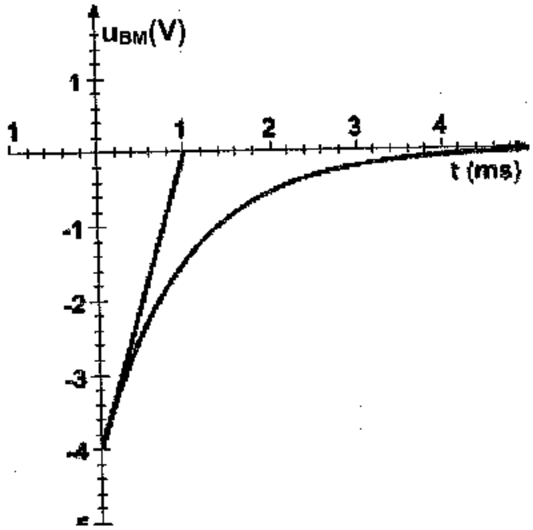
Part B

1. During the discharge: $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$, the capacitor acts as a generator so $u_{AB} = u_{AM} + u_{MB}$;

Then $u_C + 2rC\frac{du_C}{dt} = 0$;

But $u_C = E e^{-\frac{t}{\tau'}}$ so $\frac{du_C}{dt} = -\frac{E}{\tau'} e^{-\frac{t}{\tau'}}$;

Replacing in the differential equation we get $E e^{-\frac{t}{\tau'}}\left(1 - \frac{2rC}{\tau'}\right) = 0$ is verified at any instant t.



But $E e^{-\frac{t}{\tau'}} \neq 0$; so $\left(1 - \frac{2rC}{\tau'}\right) = 0$.

$\tau' = 2rC = 2\times10^3 \times 10^{-6} = 2\times10^{-3} s = 2ms$;

$\tau' \equiv 1\,\text{div} = 1\,\text{div} \times 2ms/\text{div} = 2ms = 2\times10^{-3} s$.

2. Ohm's law $u_{BM} = -ri = -\frac{rC}{\tau'}u_C = -\frac{E}{2}e^{-\frac{t}{\tau'}}$.

IV-

Part A

1. Law of addition of voltages: $u_{AM} = u_{AB} + u_{BM}$, $E = u_R + u_C$;

But $u_R = Ri$ where $i = +\frac{dq}{dt}$ & $q = Cu_C$, so $i = +C\frac{du_C}{dt}$

Then $E = u_C + RC\frac{du_C}{dt}$.

2. We have $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$, so $\frac{du_C}{dt} = \frac{E}{\tau}e^{-\frac{t}{\tau}}$;

Replacing in the differential equation, we get: $E\left(1 - e^{-\frac{t}{\tau}}\right) + RC\frac{E}{\tau}e^{-\frac{t}{\tau}} = E$;

So, $E + \left(-1 + \frac{RC}{\tau}\right)Ee^{-\frac{t}{\tau}} = E$, but $Ee^{-\frac{t}{\tau}} \neq 0$, then $-1 + \frac{RC}{\tau} = 0$; thus $\tau = RC$.

3. The steady state is reached as $t \longrightarrow +\infty$, then $u_C \longrightarrow E(1-0) = E$.

4. We know that $u_C = E$ in steady state;

From the graph $E \equiv 8\text{div} = S_v \times y = 2V/\text{div} \times 8\text{div} = 16V$;

At $t = \tau$, $u_C = (63\%)\times E = 0.63 \times E \equiv 0.63 \times 8\text{div} = 5.04\,\text{div}$;

Graphically $\tau \equiv 2\,\text{div} = S_h \times x = 2ms/\text{div} \times 2\text{div} = 4\,ms$;

But $\tau = RC$, then $R = \frac{\tau}{C} = \frac{4\times10^{-3}}{20\times10^{-6}} = 200\,\Omega$.

Part B

1. At the end of charging $u_C = E = 16\,V$.

   The electric energy stored is $W_e = \frac{1}{2} C u_C^2 = \frac{1}{2} C E^2 = \frac{1}{2} \times 20\,\mu F \times 16^2 = 2560\ \mu J$.

2. The power dissipated $P_d = R i^2$ where $i = +C\frac{du_C}{dt} = C\frac{E}{\tau} e^{-\frac{t}{\tau}} = \frac{E}{R} e^{-\frac{t}{\tau}}$;

   Then $P_J = R i^2 = R\left(\frac{E}{R} e^{-\frac{t}{\tau}}\right)^2 = \frac{E^2}{R} e^{-2\frac{t}{\tau}}$.

3. We have $P_d = \frac{dW_J}{dt}$, then $W_J = \int_0^{5\tau} P_d\, dt = \int_0^{5\tau} \frac{E^2}{R} e^{-2\frac{t}{\tau}}\, dt = \frac{E^2}{R} \times \frac{-\tau}{2} e^{-2\frac{t}{\tau}}\Big|_0^{5\tau}$;

   Then $W_J = -\frac{1}{2} C E^2 e^{-2\frac{t}{\tau}}\Big|_0^{5\tau} = -\frac{1}{2} C E^2 \left(e^{-10} - 1\right) \approx \frac{1}{2} C E^2 = \frac{1}{2} \times 20 \times 16^2 = 2560\ \mu J$;

4. The energy provided by the generator is equal to the sum of energies dissipated during the charging process with the end stored in the capacitor at the end of charging;
   Then $W_t = W_J + W_e = 2560 + 2560 = 5120\ \mu J$;

   The percentage of the loss of energy is $\eta = \frac{W_J}{W_t} \times 100 = 50\,\%$.

V-

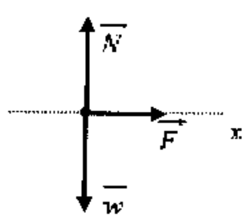
1. a) Referring to the right hand, the electromagnetic force $\vec{F}$ is horizontal to the right.
   The magnitude $F = i B \ell \sin\left(90^0\right) = i B \ell = i \times 1 \times 0.1 = 0.1 i$ (i in A and F in N)

   b) The magnetic flux is given by: $\phi = \vec{B}.\vec{S} = B S \cos\theta$ where $\theta = \left(\vec{n}, \vec{B}\right) = 0$;
   $\phi = B S \cos(0) = B \ell x$;

   According to Faraday's law: $e = -\frac{d\phi}{dt} = -B\ell\frac{dx}{dt} = -B\ell v$.

   The voltage between the terminals of the rod is: $u_{NM} = -e = B \ell v$ (along the direction of current in the rod which is considered as a generator, the terminal M is positive).

2. a) Newton's 2nd law: $\sum \vec{F} = \frac{d\vec{P}}{dt} \Rightarrow \vec{F} + \vec{N} + \vec{w} = \frac{d\left(m v \vec{i}\right)}{dt}$;



   Projection along the direction of motion:

   $F = m\frac{dv}{dt} \Rightarrow i B\ell = m\frac{dv}{dt}$; then $v = \frac{1}{m}\int (i B\ell) dt = \frac{B\ell}{m}\int i\, dt$;

   But $i = \frac{dq}{dt} = C\frac{du_C}{dt} \Rightarrow i\, dt = C\, du_C$;

   Then $v = \frac{B\ell}{m}\int i\, dt = \frac{B\ell}{m}\int C\, du_C = \frac{B\ell C}{m}\int du_C = \frac{B\ell C}{m} u_C + K$.

   But at $t_0 = 0$; the rod was at rest and the capacitor is discharged $v = 0\ \&\ u_C = 0$, so $K = 0$;

   Thus, $v = \frac{B\ell C}{m} u_C = k u_C$ where $k = \frac{B\ell C}{m}$.

b) Law of addition of voltages: $u_{ND} = u_{NM} + u_{MA} + u_{AD}$;

$E = B\ell v + u_C + Ri$, but $v = \frac{B\ell C}{m} u_C$ and $i = \frac{dq}{dt} = C\frac{du_C}{dt}$;

So $E = B\ell \frac{B\ell C}{m} u_C + u_C + RC\frac{du_C}{dt}$; thus, $E = RC\frac{du_C}{dt} + \left(\frac{B^2\ell^2 C}{m} + 1\right) u_C$.

3. a) We have $u_C = a - b e^{-\frac{t}{\tau}}$, but the capacitor is taken neutral; then $u_C = 0$ at $t = 0$;

$0 = a - b \Rightarrow a = b$; then: $u_C = a\left(1 - e^{-\frac{t}{\tau}}\right)$, so $\frac{du_C}{dt} = \frac{a}{\tau} e^{-\frac{t}{\tau}}$,

Replacing in the differential equation we get:

$$E = RC\frac{a}{\tau} e^{-\frac{t}{\tau}} + \left(\frac{B^2\ell^2 C}{m} + 1\right) a\left(1 - e^{-\frac{t}{\tau}}\right);$$

$$E = a e^{-\frac{t}{\tau}}\left(\frac{RC}{\tau} - \frac{B^2\ell^2 C + m}{m}\right) + \left(\frac{B^2\ell^2 C + m}{m}\right) a.$$

This equation is verified at any instant t and $a\, e^{-\frac{t}{\tau}} \neq 0$;

Then $\frac{RC}{\tau} - \frac{B^2\ell^2 C + m}{m} = 0$; so $\tau = \frac{mRC}{B^2\ell^2 C + m}$; thus $\tau = \frac{0.10 \times 0.5 \times 0.1}{1^2 \times 0.1^2 \times 0.1 + 0.10} = 0.495\ s$.

and $\left(\frac{B^2\ell^2 C + m}{m}\right) a = E$; so $a = \frac{mE}{B^2\ell^2 C + m}$; thus $a = \frac{0.10 \times 6}{1^2 \times 0.1^2 \times 0.1 + 0.10} = 5.94\ V$.

b) $u_C = 5.94\left(1 - e^{-2.02t}\right)$;

But $v = \frac{B\ell C}{m} u_C = \frac{0.1 \times 0.1 \times 1}{0.1} \times 5.94\left(1 - e^{-2.02t}\right) = 0.594\left(1 - e^{-2.02t}\right)$; ($v$ in m/s and t in s).

and $i = C\frac{du_C}{dt} = 0.1 \times 5.94\left(2.02 \times e^{-2.02t}\right) = 1.2\, e^{-2.02t}$ (i in A and t in s).

c) We have $v = \frac{dx}{dt} = 0.594\left(1 - e^{-2.02t}\right) \Rightarrow x = \int 0.594\left(1 - e^{-2.02t}\right)dt = 0.594\left(t + \frac{1}{2.02} e^{-2.02t}\right) + C_1$.

$x = 0.594\, t + \frac{0.594}{2.02} e^{-2.02t} + C_1$;

At $t = 0$; $x = 0 \Rightarrow C_1 = -\frac{0.594}{2.02} = -0.297$;

Then $x = 0.594\, t + 0.297 e^{-2.02t} - 0.297 = 0.594\, t + 0.297\left(e^{-2.02t} - 1\right)$; ($x$ in m and t in s).

d) Nature of motion

i- The steady state is reached after duration $t_1 = 5\tau = 5 \times 0.495 = 2.475\ s$.

ii- In the steady state $e^{-2.02t} \longrightarrow 0$; then $v = v_0 = 0.594\ (m/s)$;

Starting from the instant t_1, the speed v becomes constant and the motion of the rod is uniform. The charge of the capacitor is $Q = C u_C = C \times \frac{v}{k} = 0.1 \times \frac{5.94}{0.1} = 0.594\ C$;

The position of the rod $x_1 = 0.594 \times 2.475 + 0.297\,(0 - 1) = 1.17\ m$.

VI-

Part A

1. Law of addition of voltages $u_{AM} = u_{AB} + u_{BM}$, $E - ri = u_C + u_R$;

But $u_R = Ri$ & $i = \frac{dq}{dt}$ where $q = Cu_C$; so $E = u_C + (R+r)C\frac{du_C}{dt}$.

2. We have $u_C = a + be^{-\frac{t}{\tau}}$, so $\frac{du_C}{dt} = -\frac{b}{\tau}e^{-\frac{t}{\tau}}$;

Substitution in the differential equation, we get: $E = a + be^{-\frac{t}{\tau}} - (R+r)C\frac{b}{\tau}e^{-\frac{t}{\tau}}$;

Then $E = a + be^{-\frac{t}{\tau}}\left(1 - \frac{(R+r)C}{\tau}\right)$ is verified at any instant t & $be^{-\frac{t}{\tau}} \neq 0$

So, $1 - \frac{(R+r)C}{\tau} = 0$, we get $\tau = (R+r)C$ & $a = E$;

Referring to the initial condition at $t_0 = 0$, $u_C = 0$; so $0 = a + be^0$, then $b = -a = -E$;

Thus, $u_C = E - Ee^{-\frac{t}{\tau}} = E\left(1 - e^{-\frac{t}{\tau}}\right)$ where $\tau = (R+r)C$.

3. The steady state is reached after a minimum duration $\Delta t = 5\tau = 5(R+r)C$.

4. We have $i = C\frac{du_C}{dt} = CE\left(+\frac{1}{(R+r)C}\right)e^{-\frac{t}{(R+r)C}} = \frac{E}{R+r}e^{-\frac{t}{(R+r)C}} = I_0 e^{-\frac{t}{(R+r)C}}$ where $I_0 = \frac{E}{R+r}$.

Part B

1. At $t_0 = 0$, the capacitor is taken neutral; then the voltage across its terminals is zero;
Then $u_{BM} = u_C$ corresponds to the curve (2).

2. We have $i = \frac{E}{R+r}e^{-\frac{t}{(R+r)C}}$;

   a) The minimum duration needed to reach the steady state, $\Delta t = 5\tau = 5(R+r)C$, so $i = 0$;
   Then $u_G = E - r(0) = E$;

   b) At $t = 0$, $u_G\big|_{t_0=0} = E - ri = E - r\frac{E}{R+r}e^{-0} = E - \frac{RE + rE - rE}{R+r} = \frac{R}{R+r}E$.

3. Referring the curve (1), the value of voltage across the capacitor in steady state is $u_G = E = 10V$.

We have at $t_0 = 0$, $u_G\big|_{t_0=0} = 8V$, but $u_G\big|_{t_0=0} = \frac{R}{R+r}E$, then $8 = \frac{200}{200+r}\times 10$;

$1600 + 8r = 2000$; thus $r = \frac{2000 - 1600}{8} = 50\,\Omega$.

4. a) The time constant is the duration so that the voltage across the capacitor reaches 63% of its value in steady state.

   b) At $t = \tau$, $u_C = 0.63 \times E = 0.63 \times 10 = 6.3V$; graphically $\tau = 2\,ms$.

5. We have $\tau = (R+r)C = 2\,ms$; $C = \frac{2\times 10^{-3}}{R+r} = \frac{2\times 10^{-3}}{250} = 8\times 10^{-6}F = 8\,\mu F$.

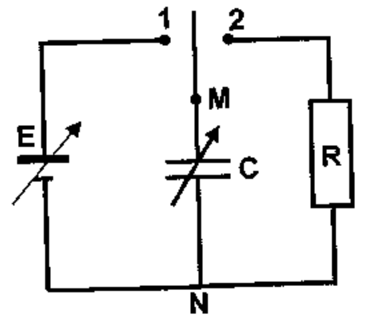
Supplementary Problems

I-LS 2004 1st

Saving Life Capacitor

A heart suffering from disordered muscular contractions is treated by applying electric shocks using a convenient apparatus.

In order to study the functioning of this apparatus, we use a source of DC voltage of adjustable value E, a double switch, a resistor of resistance R and a capacitor (initially neutral) of adjustable capacitance C. We connect the circuit represented in the adjacent figure.



Part A

Theoretical study

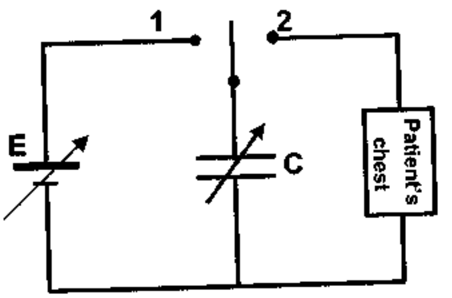
1. The switch is turned to position (1), give:
   a) the name of the physical phenomenon that takes place in the capacitor.
   b) the values of the current in the circuit and the voltage u_{MN} after few seconds.
2. The switch is now turned to position (2) at an instant taken as $t_0 = 0$.
   a) Derive, at the instant t, the differential equation giving the variation of the voltage $u_C = u_{MN}$ as a function of time.
   b) The expression $u_C = Ae^{-\frac{t}{\tau}}$, where A and τ are constants, is a solution of that equation. Determine the expressions of A and τ in terms of E, R & C.
   c) Derive the expression giving the current i during the discharging as a function of time.

Part B

Using the apparatus

The energy needed to save the life of a patient during an electric shock is $360\,J$. This energy is supplied by discharging the capacitor through the patient's chest (ribcage) considered as a resistor of resistance $50\,\Omega$ during a time t_1 that can be controlled by the switch.

The capacitance of the capacitor is adjusted on the value $C = 1\,mF$ and is charged under the voltage $E = 1810\,V$.



1. Determine the energy stored in the capacitor at the end of the charging process.
2. The discharging starts at the instant $t_0 = 0$. At the instant t_1, the energy delivered to the patient amounts to $360\,J$, the switch is then opened.
   a) Calculate the energy that remains in the capacitor at the instant t_1.
   b) Using the results of the above theoretical study; determine:
      i- the value of t_1;
      ii- the current at the end of the electric shock.

Answer Key

Part B 1. $1638\,J$. 2. a) $1278\,J$. b) *i-* $t_1 = 6.2\,ms$.

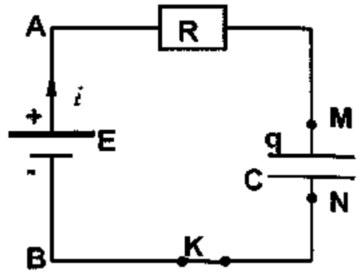
II-LS 2003 2nd

Response of an RC Circuit

The object of this exercise is to distinguish the response of an RC series circuit when we apply across its terminals a constant voltage, from its response when it carries a constant current.

Part A

Case of constant voltage



The circuit of the adjacent figure allows us to charge the capacitor of capacitance $C = 10\mu F$ through a resistor of resistance $R = 100\,k\Omega$ under a constant voltage $E = 9V$.

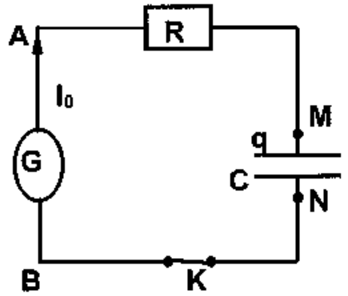
Take, the instant $t = 0$, the instant when the switch K is closed.

1. Denote by $u_C = u_{MN}$, the instantaneous value of the voltage across the terminals of the capacitor.
   a) Show that the differential equation in u_C is $u_C + RC\dfrac{du_C}{dt} = E$.
   b) Knowing that the solution of this equation has the form: $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$, determine the expressions of A and τ.
   c) Trace the shape of the curve that gives the variations of u_C as a function of time.
2. a) Determine the expression of the voltage $u_R = u_{AM}$ as a function of time.
   b) Trace, on the same system of axis, the shape of the curve giving the variation of u_R as a function of time.
   c) What is the value of the interval of time t_A at the end of which u_C becomes practically $9V$?

Part B

Case of constant current

The preceding capacitor being discharged is to be recharged through the same resistor by a generator giving a constant value $I_0 = 0.1\,mA$.



1. a) Show that the charge q can be written, in SI, is $q = 10^{-4}\,t$.
   b) The voltage $u_R = u_{AM}$ across the resistor remains constant. Determine its value.
   c) Trace the shape of the graph representing u_R.
2. a) Determine the expression of the voltage $u_C = u_{MN}$ as a function of time.
   b) Trace the shape of the graph of u_C.
   c) Determine the time interval t_B needed for the voltage u_C to attain the value $9V$.

Part C

Conclusions

1. Using the preceding graphs, specify the case where the voltage across the capacitor attains, in the steady state a limiting value.
2. A camera is equipped with a flash that is formed of the preceding RC circuit. We intend to take the largest number of photos in a given interval of time. To do we have to charge the capacitor. Which one of the two preceding charging modes is more convenient? Why?

Answer Key

Part A 2.c) $t_A = 5s$. **Part B** 1.b) $10V$. 2.c) $t_B = 0.9\,s$.

III-LS 2001 1st

Determination of Capacitance

In order to determine the capacitance C of a capacitor, we connect up the circuit of figure 1. This circuit is formed of the capacitor, a generator of *e.m.f* $E = 9V$ and negligible internal resistance, two resistors of resistances $R_1 = 200\,k\Omega$ and $R_2 = 100\,k\Omega$ two switches K_1 and K_2.

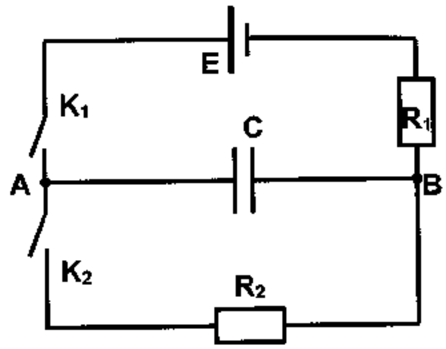


Figure 1

Part A
Charging of the capacitor

The capacitor being initially uncharged, we close K_1 and keep K_2 open. The capacitor will be charged.

1. Derive the differential equation that describes the variation of the voltage $u_C = u_{AB}$ across the capacitor.
2. Knowing that the solution of this differential equation has the form $u_C = E\left(1 - e^{-\frac{t}{\tau_1}}\right)$, deduce the expression of the constant τ_1 as a function of R_1 and C.
3. Knowing that at the instant $t_1 = 20\,s$, u_C has the value of $7.78\,V$, calculate the capacitance C of the capacitor.

Part B
Discharging the capacitor

The capacitor being charged under a voltage of $9V$, we open K_1 and close K_2. The capacitor is then discharged.

1. Draw a diagram of the circuit during that phase indicating the direction of the current.
2. Derive the differential equation that describes the variation of the voltage $u_C = u_{AB}$ across the capacitor.
3. Knowing that the solution of this differential equation is of the form $u_C = E e^{-\frac{t}{\tau_2}}$, deduce the expression of: (take the direction of the current as a positive direction)
   a) the current i as a function of time.
   b) the time constant τ_2 as a function of R_2 and C.
4. A convenient apparatus allows us to trace the graph of the variation of u_C as a function of time as shown in figure 2.
   ✘ Determine from the curve the value of τ_2.
   ✘ Deduce the value of C.

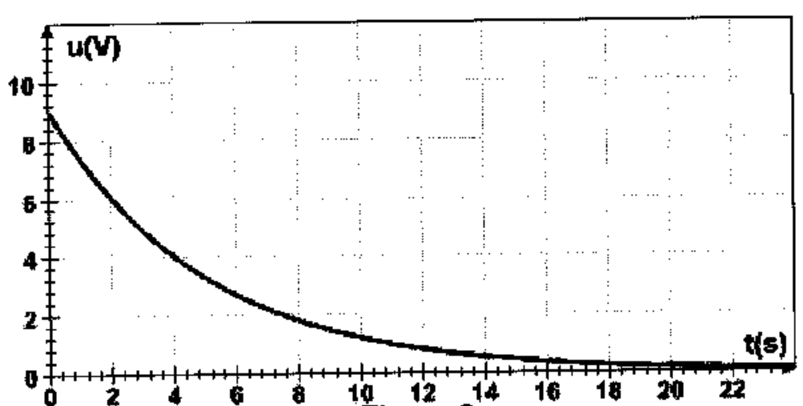


Figure 2

Part C

What conclusion can be drawn about the two values of C?

Answer Key

Part A $C = 50\,\mu F$. **Part B** $u_C = 3.33V$ & $\tau_2 \approx 4.9\,s$.

IV-Bac
Capacitors and Video Games

The last virgins of video games are evolving offering to the user a new whole new modes to play. In fact, the motion printed on the remote control leads to the response of persons on the screen: the gesture becomes a command.
This is made possible thanks of an integrated accelerometer in the joystick that converts these accelerations into electrical voltages.
During a player move, the mobile part of the accelerometer will displace without friction with respect to the frame (**figure 1**).

These nanometric displacements are realized in the space three dimensions (x, y, z) in order to transmit in the best possible manner the player gesture. How this displacement in converted into a measurable electrical voltage?

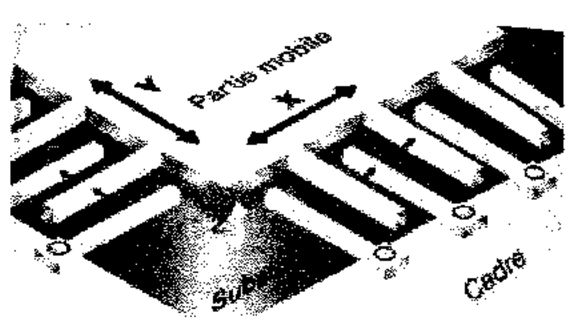


Figure 1 - Schema of the accelerometer
Micro Hebdo n° 619
Thursday February 25th 2010

When the player takes a move, the mobile part will displace without friction with respect to the frame. We are only interested to its displacement along Ox, its position is referred by its abscissa x, the distance between the frame and the mobile part is d (**figure 3**) at an instant t and it is equal to d_0 at equilibrium.

The two parts of the accelerometer constitute the armatures of a plane capacitor of capacitance C.

The capacitance is inversely proportional to the distance d between the armatures: $C = \frac{\alpha}{d}$ where α is a positive constant.

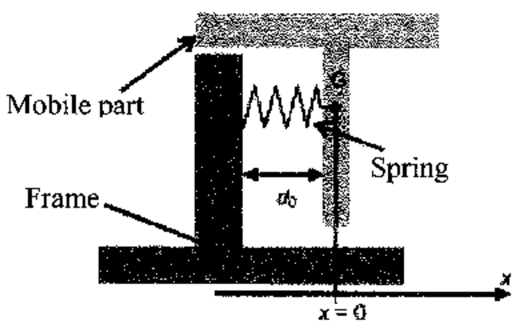


Figure 2
Accelerometer at rest

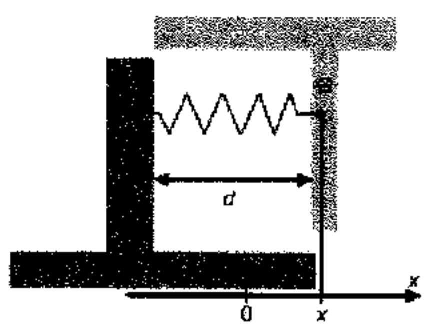


Figure 3
Accelerometer under acceleration

Part A
Motion of mobile part

Take the horizontal plane passing through G as a reference level of gravitational potential energy.

1. Write, at an instant t, the expression of the mechanical energy ME of the system (oscillator, Earth) in terms of m, x, k and v.
2. Derive the second order differential equation in x that describes the motion of G.

Part B

Variation of the capacitor's capacitance during the motion of a player

Given

✘ Distance between the armatures when the accelerometer is at rest: $d_0 = 1.5 \mu m$;

✘ Constant of spring: $k = 0.264\ N/m$;

✘ Mass of the mobile part: $m = 1.6 \times 10^{-9} kg$;

✘ Capacitance of two parts of the accelerometer at rest: $C_0 = 1.3 \times 10^{-14} F$.

The player imposes to the lever of the game, along Ox, an accelcration $a_{1x} = -4\, m/s^2$. The mobile armature will displace by $x = x_1$ with respect to the frame. The distance between the armatures is $d = d_1$ (**figure 3**). We denote by C_1 the new capacitance of the capacitor.

1. Calculate the value of x_1.
2. Compare, with justification, the values of C_1 & C_0.
3. Show that the capacitance C_1 can be written $C_1 = C_0 \dfrac{d_0}{d_1}$.
4. Calculate the value of C_1.
5. The structure of the accelerometer allows us to multiply its capacitance by a factor β that depends on the number of element. In the case where $\beta = 120$, calculate the value of $\Delta C_1^{tot} = \beta(C_1 - C_0)$

Part C

Variation of the voltage across the accelerometer

Consider the circuit shown in figure 4.

Given:

The total capacitance of the capacitor when the accelerometer is at rest $C_0^{tot} = 1.56 pF$, the resistance of the resistor $R = 100\ k\Omega$, the electromotive force of the generator $E = 3V$.

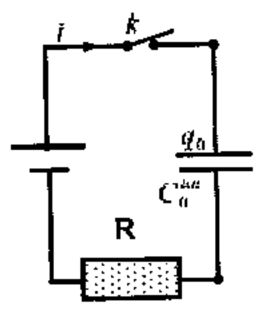


Figure 4

1. a) Calculate the value of the time constant τ of this circuit.
   b) Justify that the steady state is reached after $0.1 s$.
2. a) Show that the differential equation in i has the form: $i + RC\dfrac{di}{dt} = 0$.
   b) Determine the expressions of A, B & α so that the solution of the differential equation is $i = A + B e^{\alpha t}$.
   c) Deduce the value of the current in the steady state.
3. When the lever is in motion, the accelerometer is subjected to an acceleration a_{1x}. The capacitance of the capacitor is $C_1^{tot} = C_0^{tot} + \Delta C_1^{tot}$ and the voltage across its terminal is U_1. The circuit is open, the charge q_0 remains constant.
   a) Deduce the expression of the voltage U_1 across the capacitor is $U_1 = \dfrac{U_0\, C_0^{tot}}{C_0^{tot} + \Delta C_1^{tot}}$.
   b) Calculate the value of U_1 if $\Delta C_1^{tot} = -2.4 \times 10^{-14} F$.

Answer Key

Part B 1. 24.2 nm. 3. 1.28×10^{-14} F **Part C** 3.b) 3.05V.

V-

Capacitor and Current

We connect the circuit formed of a resistor of resistance $R = 800\,\Omega$, an ideal generator of *e.m.f* E, a capacitor of capacitance C & a switch K (Figure 1). At an instant taken as an origin of time $t_0 = 0$, we close the switch K.

Let $u_C = u_{MN}$ be the voltage across the capacitor.

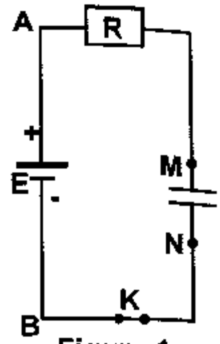


Figure 1

Part A
Theoretical Study

1. Recopy the adjacent circuit, showing the direction of the current and the signs of the armatures of the capacitor.
2. Show that the differential equation satisfied by u_C is $\frac{du_C}{dt} + \frac{1}{RC}u_C = \frac{E}{RC}$.
3. Verify that $u_C = E\left(1 - e^{-\frac{t}{RC}}\right)$ is the solution of the differential equation.
4. Determine the value of u_C in steady state.
5. Determine, in terms of R & C, the duration needed so that the voltage across the capacitor is half of its value in steady state.

Part B
Graphical Study

1. Show that the voltage across the resistor u_R is given by $u_R = \alpha \frac{du_C}{dt}$ where α is a constant whose expression is to be determined in terms of R & C.
2. A convenient software is used to plot the curve representing the variation of u_R in terms of $\frac{du_C}{dt}$ as shown in figure 2.
   a) Justify that the curve is compatible with the relation previously derived.
   b) Specify the position that corresponds to the instant:
      i- $t = 0$.
      ii- end of charging.
   c) Referring to the graph, determine:
      i- the value of E.
      ii- the current at $t = 0$.
      iii- the capacitance of the capacitor.

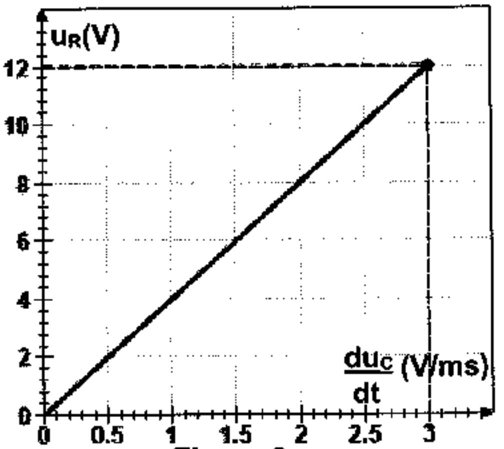


Figure 2

Answer Key
Part B

2. c) *i-* $E = 12V$ *iii-* $C = 5\,\mu F$.

LS 2013 2[nd]
Charging and Discharging of a Capacitor

The aim of this exercise is to determine, by two different methods, the value of the capacitance C of a capacitor. For this aim, we set-up the circuit of figure 1.

This circuit is formed of an ideal generator delivering a constant voltage of value $E = 10V$, a capacitor of capacitance C, two identical resistors of resistances $R_1 = R_2 = 10\,k\Omega$ and a double switch K.

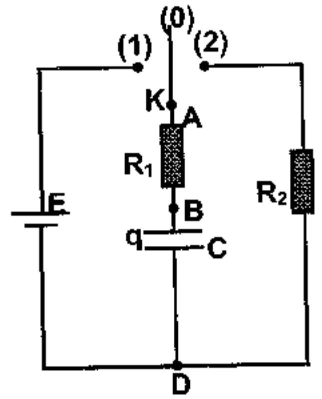


Figure 1

Part A
Charging the capacitor

The switch K is in the position (0) and the capacitor is neutral. At the instant $t_0 = 0$, we turn K to position (1) and the charging of the capacitor starts.

First situation
Theoretical study

1. Applying the law of addition of voltages and taking the positive direction along the circuit as that of the current, show that the differential equation that describes the variation of the voltage $u_C = u_{BD}$ across the capacitor has the form: $E = R_1 C\dfrac{du_C}{dt} + u_C$.
2. The solution of this differential equation has the form: $u_C = A\left(1 - e^{-\frac{t}{\tau_1}}\right)$ where A and τ_1 are constants. Show that $A = E$ and $\tau_1 = R_1 C$.
3. Show that at the end of charging $u_C = E$.
4. Show that the expression $u_{AB} = u_{R_1} = Ee^{-\frac{t}{R_1 C}}$.
5. Establish the expression of the natural logarithm of u_{R_1} $[\ell n(u_{R_1})]$ as a function of time.

Second situation
Graphical study

The variation of $[\ell n(u_{R_1})]$ as a function of time is represented by figure 2.

1. Justify that the shape of the obtained graph agrees with the expression of $[\ell n(u_{R_1})]$ as a function of time.
2. Deduce, using the graph, the value of the capacitance C.

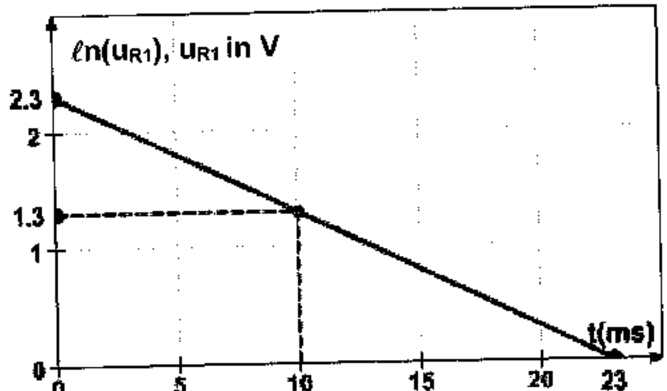


Figure 2

Part B

Discharging the capacitor

The capacitor being fully charged, we turn the switch K to position (2). At an instant $t_0 = 0$, taken as a new origin of time, the discharging of the capacitor starts.

1. During discharging, the current circulates from B to A in the resistor of resistance R_1. Justify.
2. Taking the positive direction along the circuit as that of the current, show that the differential equation in the voltage u_C across the capacitor has the form:

   $u_C + (R_1 + R_2)C\frac{du_C}{dt} = 0$.
3. The solution of the above differential equation has the form: $u_C = Ee^{-\frac{t}{\tau_2}}$ where τ_2 is the time constant of the circuit during discharging. Show that $\tau_2 = (R_1 + R_2)C$.
4. The variation of the voltage u_C across the capacitor and the tangent to the curve $u_C = f(t)$ at the instant $t_0 = 0$, are represented in figure 3.

   Deduce, from this figure, the value of the capacitance C.

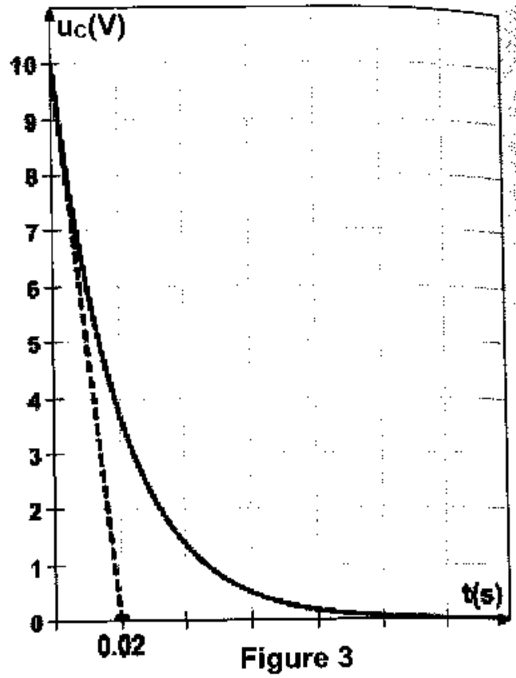


Figure 3

II-LS 2012 1st

Study of Discharging a Capacitor

A capacitor of capacitance C is initially charged under a voltage E.

At $t_0 = 0$, we connect across the terminals of the capacitor a resistor of resistance $R = 1k\Omega$ (Figure 1).

At an instant t, the armature A carries the charge $q > 0$ and the circuit carries a current i.

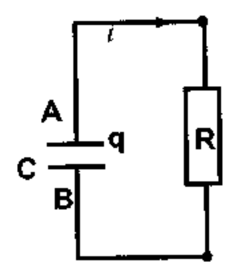


Figure 1

Part A

Theoretical study

1. Write the relation between i and q.
2. Show that the differential equation of the voltage $u_C = u_{AB}$ across the capacitor is

   $\frac{du_C}{dt} + \frac{1}{RC}u_C = 0$.
3. The solution of this differential equation is $u_C = De^{-\frac{t}{\tau}}$.

   Determine the expressions of the constants D and τ in terms of E, R and C.
4. Show that, after a time $t = \tau$, the voltage across the capacitor attains 37% of its maximum value E.

Part B

Determination of the capacitance C

In order to determine the value of C, we use a convenient apparatus, which traces, during the discharging of the capacitor, the curves representing $u_C = g(t)$ (figure 2), $\ell n(u_C) = f(t)$ (figure 3).

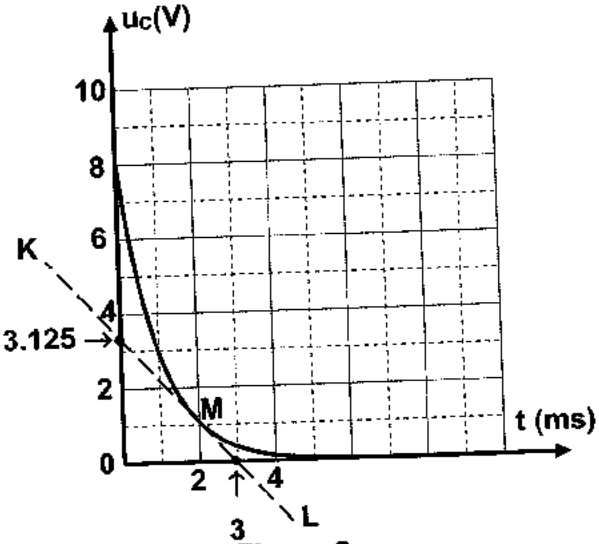


Figure 2

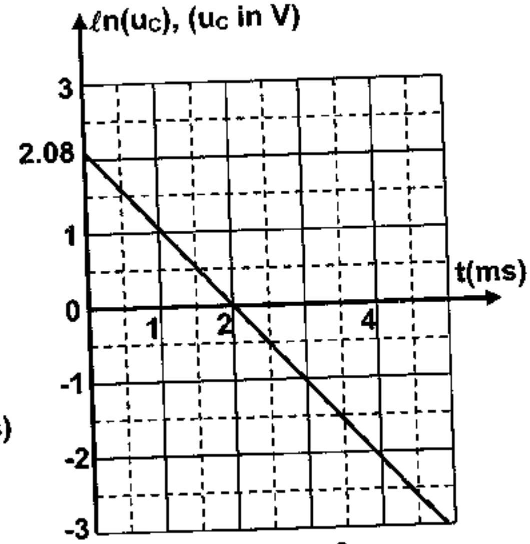


Figure 3

We proceed in the three following methods:

1. First method

Referring to the curve of figure 2:

a) give the value of E;

b) using the result of part (A-4) determine the value of τ and deduce the value of C.

2. Second method

The figure 2 shows also the tangent KL to the curve at point $M(2ms\ ,1V)$.

a) Referring to this figure determine the slope of the tangent at point M.

b) Determine the value of C.

3. Third method

a) Determine the expression of $\ell n(u_C)$ in terms of E, R, C and t.

b) Show that the shape of the curve in figure 3 is in agreement with the obtained expression of the function $\ell n(u_C) = f(t)$.

c) Referring to the curve of figure 3, determine again the values of E and C.

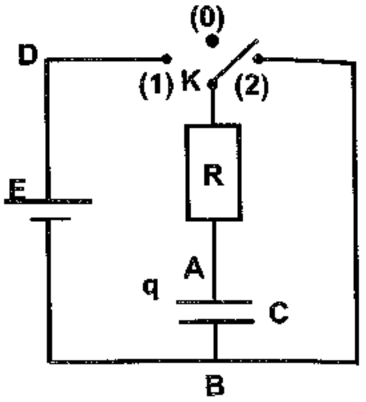
III-LS 2011 1st

Discharging of a Capacitor - The lightning

The electric circuit of the adjacent figure allows us to perform charging and discharging of a capacitor of capacitance C, through a resistor of resistance R. The used generator has a constant electromotive force E and is of negligible internal resistance.

Part A

Charging of the capacitor



The capacitor is initially uncharged and the switch K is in position (0).

1. To which position, (1) or (2), must we turn the switch K in order to charge the capacitor?
2. The variation of the voltage $u_C = u_{AB}$ across the terminals of the capacitor as a function of time is given by the expression: $u_C = E\left(1 - e^{-\frac{t}{RC}}\right)$. Deduce the value of u_C in terms of E, at the end of the charging of the capacitor.

Part B

Discharging of the capacitor

The charging of the capacitor being completed, the switch K is again in position (0).

1. To which position must we turn the switch K in order to discharge the capacitor?
2. The instant $t_0 = 0$ corresponds to the starting of the discharging. At an instant t, the circuit carries a current i.
   a) Draw the circuit of discharging and indicate on it the real direction of the current chosen as a positive direction.
   b) Discharging current:
      i- In this case, the current is written as $i = -\frac{dq}{dt}$ and not as $i = +\frac{dq}{dt}$. Why?
      ii- Show that the differential equation in i has the form: $i + RC\frac{di}{dt} = 0$.
   c) Verify that: $i = \frac{E}{R}e^{-\frac{t}{RC}}$ is the solution of this differential equation.
3. Trace the shape of the curve representing the variation of i as a function of time.
4. Give, in terms of R and C, the duration at the end of which the capacitor is practically completely discharged.

Part C

The lightning

In a cloud, the collisions between the water particles give rise to positive and negative charges: The lower part of the cloud becomes negatively charged while its upper part positively charged. Simultaneously, the ground is charged positively by induction. A capacitor of capacitance $C = 10^{-10}\,F$ is thus formed having the ground as the positive armature, the lower part of the cloud as the negative armature and the air between them being the insulator. The voltage across its armatures is $E = 10^{8}\,V$.

In certain conditions, the air between the armatures becomes a conductor of resistance $R = 5000\,\Omega$.

We suppose that the lightning corresponds to the complete discharging of this capacitor through air.

1. Calculate the duration of the lightning.
2. Determine the maximum current due to the lightning.

IV-LS 2010 1st
Determination of the Resistance of Resistor

We intend to determine the resistance R of a resistor (R). We thus connect up the circuit represented in figure (1) that is formed of an ideal generator of *e.m.f* $E = 5V$, the resistor (R), an uncharged capacitor (C) of capacitance $C = 33\,\mu F$ and a double switch (K).

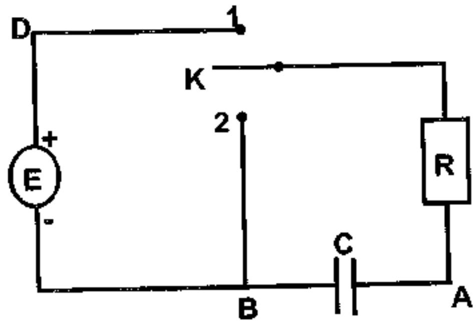


Figure 1

Part A
Charging of the capacitor

1. We intend to charge the capacitor. To what position, 1 or 2, must then (K) be moved?
2. The circuit reaches a steady state after a certain time. Give then the value of the voltage u_{AB} across (C) and that of the voltage across (R).

Part B
Discharging of the capacitor

1. Draw a diagram of the circuit during the discharging of the capacitor and show on it the direction of the current it carries.
2. Derive the differential equation in $u_C = u_{AB}$ during the discharging.
3. The solution of this differential equation has the form: $u_C = E\,e^{-\frac{t}{\tau}}$ (u_C in V, t in s) where τ is a constant.

   a) Determine the expression of τ in terms of R and C.

   b) Determine the value of u_C at the instant $t_1 = \tau$.

   c) Give, in terms of τ, the minimum duration needed at the end of which the capacitor is practically totally discharged.

   d) Derive the expression of $\ell n\, u_C$, the natural logarithm of u_C, in terms of E, τ and t.

   e) The diagram of figure 2 represents the variation of $\ell n\, u_C$ as a function of time. Referring to the graph of figure 2, determine the value of R.

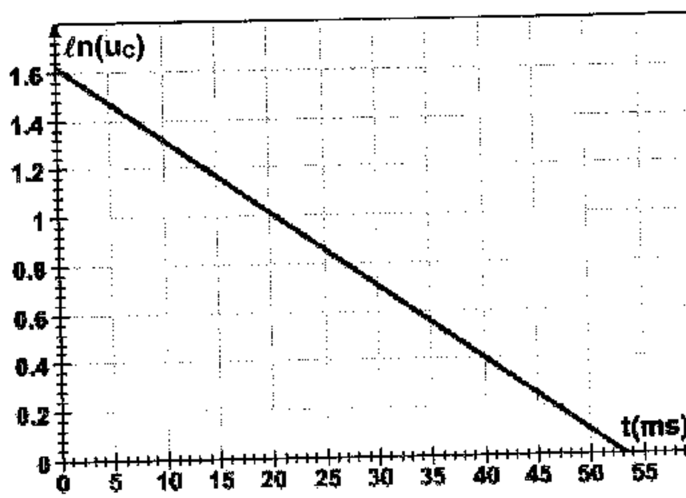


Figure 2

V-LS 2008 1st

Role of a Capacitor in a Circuit

The object of this exercise is to study the role of a capacitor in an electric circuit in two different cases. $\left(g = 10\, m/s^2\right)$.

Part A

Variation of the current in a circuit

1. Qualitative study

We connect the two circuits whose diagrams are represented in the diagram below; the two identical lamps L_1 and L_2 are fed respectively with two identical generators G_1 and G_2 each of constant voltage E, the component (D) being a capacitor that is initially uncharged (Figure 1).

We close the two switches simultaneously at the instant $t_0 = 0$. We notice initially that L_1 and L_2 glow with the same brightness, but the brightness of the lamp L_2 decreases progressively and finally its light goes out, L_1 keeping its same brightness.

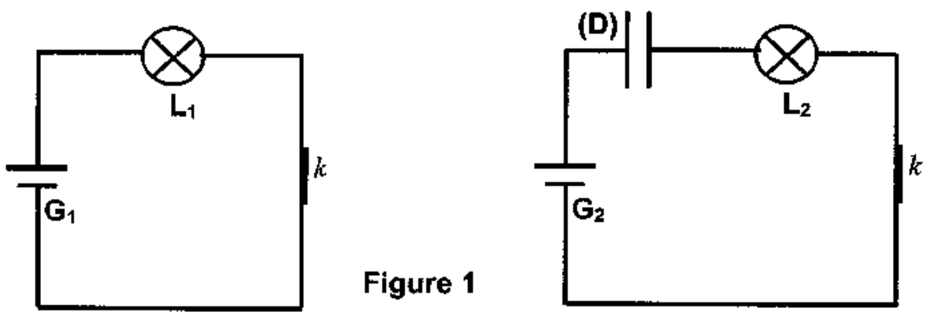


Figure 1

a) What can we say about the voltage across each of the lamps at the instant $t_0 = 0$? Justify.

b) Effect on the lamp:

i- How does the voltage across L_2 vary starting from the instant $t_0 = 0$?

ii- Deduce, when the light of L_2 goes out, the value of the voltage across the capacitor.

2. Quantitative study

We connect a series circuit formed of a resistor of resistance R, a capacitor of capacitance C and a switch k across an ideal generator of *e.m.f.* « E ».

At the instant $t_0 = 0$, the capacitor being uncharged, we close the switch k (Figure 2).

At the instant t, the charge of the armature B of the capacitor is q and the current carried by the circuit is i.

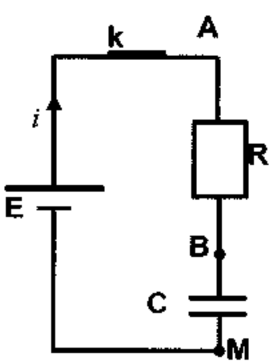


Figure 2

a) Write the relation between i and $\dfrac{dq}{dt}$.

b) Derive the differential equation in $u_{BM} = u_C$.

c) This differential equation has as solution: $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$.

i- Determine the expression of τ in terms of R and C.

ii- Determine the expression of the current i in the circuit as a function of time.

iii-Draw the shape of each curve representing the variations of u_C and of i as a function of time.

3. Conclusion

Deduce the role of the capacitor in the variation of the current i in an RC circuit fed by a DC voltage during the charging phase.

Part B

Energy stored in a capacitor

1. Qualitative study

Consider the experiment whose diagram is represented in figure (3), where (M) is a motor to which a body of mass m is suspended, a capacitor of large capacitance, G an ideal generator of constant voltage E, and K_1 and K_2 are two switches.

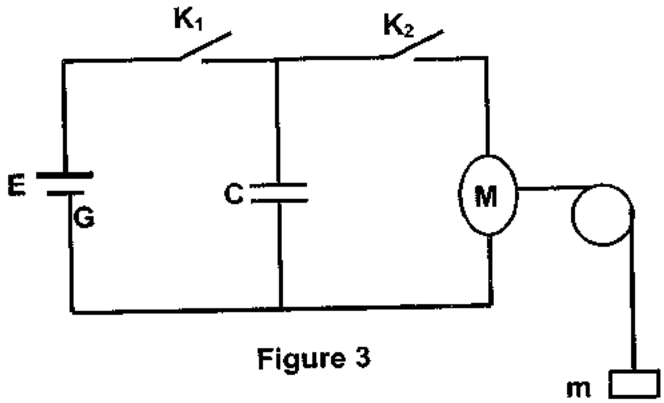


Figure 3

In the first step of the experiment, we open K_2, and we close K_1.

In the second step of the experiment, we open K_1 and we close K_2. We observe that the body rises.

Explain what happens in each step of the experiment and tell why the body rises

2. Quantitative study

The capacitor has a capacitance $C = 1F$, the body has a mass $m = 500\,g$ and the *e.m.f* of the generator is $E = 3V$.

a) Calculate the energy initially stored in the capacitor.
b) Calculate the height raised by the body neglecting all energy losses.
c) What type of energy transfer did take place?
d) In fact, the body rises $83\,cm$. Why?
e) Deduce the role of the capacitor in the previous circuit.

VI-LS 2006 1st

Measurement of the Speed of a Bullet

In order to measure the speed of a bullet, a convenient setup is used. The principle of functioning of this setup is based on the charging of a capacitor.

Part A

Study of the charging of a capacitor

We are going to study the charging of a capacitor using a series circuit formed of a resistor of resistance R, a switch K and a capacitor of capacitance C initially neutral across the terminals of a generator of constant *e.m.f* E and of negligible internal resistance (Figure 1).

The switch K is closed at the instant $t_0 = 0$. The capacitor starts to charge. At the instant t, the circuit carries a current i and the armature A of the capacitor carries the charge q.

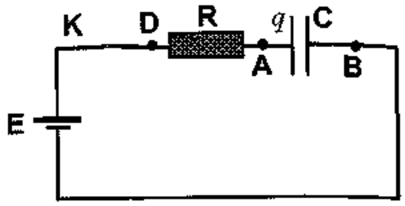


Figure 1

1. Applying the law of addition of voltages, determine the differential equation that describes the variation of the voltage $u_C = u_{AB}$ across the capacitor as a function of time.

2. a) Verify that $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$ is the solution of the differential equation where $\tau = RC$.

   b) What does the time interval τ represent?

3. After what time would the steady state be practically attained?

Part B

Measurement of the speed of a bullet

The setup used to measure the speed v of a bullet is represented in Figure 2.

AA' and BB' are two thin parallel connecting wires lying in a vertical plane and are of negligible resistance. AA' and BB' are separated by a distance ℓ.

Given: $E = 100\,V$, $R = 1000\,\Omega$, $C = 4\,\mu F$ & $\ell = 1m$.

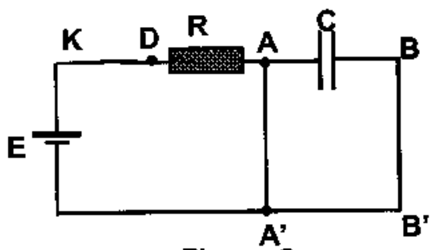


Figure 2

The capacitor being neutral, the switch K is closed.

1. a) The potential difference between A and A' is zero. Why?

   b) The charging of the capacitor did not start. Why?

2. K being closed, we shoot the bullet normally at AA' and BB' with a speed v (Figure 3). At the instant $t_0 = 0$, the bullet cuts the wire AA' and the capacitor starts to charge (Figure 4).

   The bullet continues its motion which is considered uniform rectilinear of the same speed v.

   At the instant t_1, the bullet cuts the wire BB' and the phenomenon of charging stops. The voltage across the capacitor is then $45.7V$.

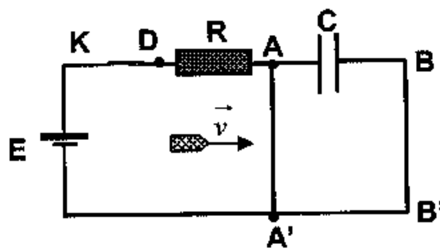


Figure 3

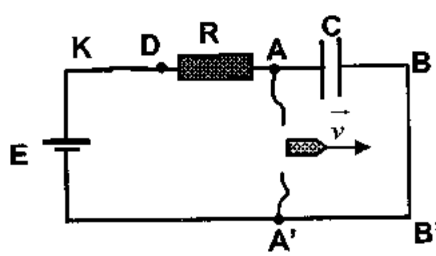


Figure 4

   a) Taking into consideration the study in part A, determine the time interval t_1 taken by the bullet to cover the distance ℓ.

   b) Calculate v.

3. In order to measure precisely the value of v, the distance ℓ between AA' and BB' must not exceed a maximum value ℓ_{max}.

   Determine the value of ℓ_{max}.

Sessions Solutions

I-LS 2013 2nd

Part A

Theoretical Study

1. Law of addition of voltages: $u_{AD} = u_{AB} + u_{BD} \Rightarrow E = R_1 i + u_C$;

   According to Ohm's law and taking the direction of the current as positive $i = +\frac{dq}{dt} = +C\frac{du_C}{dt}$;

   Then $E = R_1 C\frac{du_C}{dt} + u_C$.

2. We have $u_C = A\left(1 - e^{-\frac{t}{\tau_1}}\right)$, so $\frac{du_C}{dt} = \frac{A}{\tau_1}e^{-\frac{t}{\tau_1}}$;

   Replacing in the differential equation we get:

   $E = R_1 C\frac{A}{\tau_1}e^{-\frac{t}{\tau_1}} + A\left(1 - e^{-\frac{t}{\tau_1}}\right) = \left(\frac{R_1 C}{\tau_1} - 1\right)Ae^{-\frac{t}{\tau_1}} + A$; this equation is verified at any instant t.

   But $Ae^{-\frac{t}{\tau_1}} \neq 0$, so $A = E$ and $\frac{R_1 C}{\tau_1} - 1 = 0$; then $\tau_1 = R_1 C$.

3. At the end of charging $t \longrightarrow +\infty$; so $e^{-\frac{t}{\tau_1}} \longrightarrow 0$; then $u_C = E(1-0) = E$ [1].

4. Ohm's law $u_{AB} = R_1 i = R_1 C\frac{du_C}{dt} = R_1 C\frac{E}{\tau_1}e^{-\frac{t}{\tau_1}} = E\,e^{-\frac{t}{R_1 C}}$.

5. We have $u_{AB} = u_{R_1} = Ee^{-\frac{t}{R_1 C}}$;

   $\ell n\left(u_{R_1}\right) = \ell n\left(Ee^{-\frac{t}{R_1 C}}\right) = \ell n(E) + \ell n\left(e^{-\frac{t}{R_1 C}}\right) = \ell n(E) - \frac{t}{R_1 C}$.

Part A

Graphical Study

1. We have $\ell n\left(u_{R_1}\right) = -\frac{1}{R_1 C} \times t + \ell n(E)$ is of the form $at + b$, so its graphical representation is:

   ✘ a straight line (the variable t is of degree 1);

   ✘ decreasing (the slope $a = \frac{-1}{R_1 C} < 0$);

   ✘ and not passing through origin $b = \ell n(E) \neq 0$.

---

[1] In the steady state, the voltage u_C across the capacitor becomes constant then $\frac{du_C}{dt} = 0$, referring to the differential equation we get: $u_C + R_1 C(0) = E$, then $u_C = E$.

2. The slope is determined graphically: $a = \frac{\Delta(\ell n(u_{R_1}))}{\Delta t} = \frac{2.3 - 1.3}{0 - 0.01} = -100\ s^{-1}$;

But $a = \frac{-1}{R_1 C}$, then $C = \frac{1}{100\ R_1} = \frac{1}{100 \times 10 \times 10^3} = 10^{-6} F = 1 \mu F$.

Part B

1. Because the armature B of the capacitor is positively charged.
2. During the discharge, the capacitor acts as a generator $u_{BD} = u_{BA} + u_{AD}$;

   Then $u_C = u_{R_1} + u_{R_2} = (R_1 + R_2)i$; but $i = -C\frac{du_C}{dt}$ so $u_C = -(R_1 + R_2)C\frac{du_C}{dt}$;

   Thus $u_C + (R_1 + R_2)C\frac{du_C}{dt} = 0$.

3. We have $u_C = Ee^{-\frac{t}{\tau_2}} \Rightarrow \frac{du_C}{dt} = -\frac{E}{\tau_2}e^{-\frac{t}{\tau_2}}$; by replacing in the differential equation we get:

   $Ee^{-\frac{t}{\tau_2}} + (R_1 + R_2)C\left(-\frac{E}{\tau_2}\right)e^{-\frac{t}{\tau_2}} = 0 \Rightarrow Ee^{-\frac{t}{\tau_2}}\left(1 - \frac{(R_1 + R_2)C}{\tau_2}\right) = 0$;

   But $Ee^{-\frac{t}{\tau_2}} \neq 0$ so $1 - \frac{(R_1 + R_2)C}{\tau_2} = 0 \Rightarrow \tau_2 = (R_1 + R_2)C$.

4. We know that the time constant τ_2, is the abscissa of the point of intersection between the tangent at the point of abscissa $t = 0$ with the horizontal asymptote (abscissa axis), referring to the graph shown in figure 3 we get: $\tau_2 = 0.02s = (R_1 + R_2)C$;

   Then, $C = \frac{0.02s}{(10 + 10) \times 10^3 \Omega} = 10^{-6} F = 1 \mu F$.

II-LS 2012 1st

Part A

1. During discharge $i = -\frac{dq}{dt}$.
2. According to the law of uniqueness of voltages: $u_{AB} = u_{AB}$; but $u_{AB} = u_C$;

   $q = C\,u_C$ & $i = -C\frac{du_C}{dt}$ (the direction of current is taken as positive).

   $u_{AB} = u_R \Rightarrow u_C = R\,i = -R\,C\frac{du_C}{dt} \Rightarrow u_C + R\,C\frac{du_C}{dt} = 0$; then $\frac{du_C}{dt} + \frac{1}{RC}u_C = 0$.

3. We have $u_C = D\,e^{-\frac{t}{\tau}}$, we get $\frac{du_C}{dt} = -\frac{D}{\tau}e^{-\frac{t}{\tau}}$.

   Replacing in the differential equation we get:

   $D\,e^{-\frac{t}{\tau}} - RC\frac{D}{\tau}e^{-\frac{t}{\tau}} = 0 \Rightarrow De^{-\frac{t}{\tau}}\left(1 - \frac{RC}{\tau}\right) = 0$; but $Ee^{-\frac{t}{\tau}} \neq 0$ then $\tau = RC$.

   At $t = 0$, $u_C = E$ so $De^{-0} = E$, then $D = E$.

4. At $t = \tau$, $u_C = E\,e^{-\frac{\tau}{\tau}} = E\,e^{-1} \approx 0.37\,E = (37\%)E$.

Part B

1. a) At $t=0$, $u_C=E=8V$ (y-intercept).

b) At $t=\tau$, the voltage across the capacitor attains $u_C=0.37\,E=0.37\times 8=2.96\,V$.

Graphically for $u_C=2.96V$, then the abscissa of the corresponding point $t=\tau=1\,ms$.

But $\tau=RC=1ms \Rightarrow C=\frac{\tau}{R}=\frac{1\times10^{-3}}{1\times10^{3}}=10^{-6}\,F=1\,\mu F$.

2. a) slope $=\left.\frac{du_C}{dt}\right|_{t=2ms}=\frac{\Delta u_C}{\Delta t}=\frac{u_{C_2}-u_{C_1}}{t_2-t_1}=\frac{(3.125-1)V}{(0-2\times10^{-3})s}=-1062.5\,V/s$.[2]

b) The differential equation gives $\frac{du_C}{dt}=-\frac{1}{RC}u_C$,

for $t=2ms$: $\left.\frac{du_C}{dt}\right|_{t=2ms}=-\frac{1}{RC}u_C\big|_{t=2ms}=\frac{-1}{RC}$; so $\frac{-1}{RC}=1062.5$;

Then $C=\frac{1}{10^3\times1062.5}=9.41\times10^{-7}\,F=0.941\,\mu F\approx1\,\mu F$.

3. a) $u_C=E\,e^{-\frac{t}{RC}}\Rightarrow \ell n(u_C)=\ell n\left(E\,e^{-\frac{t}{RC}}\right)=\ell n(E)+\ell n\left(e^{-\frac{t}{RC}}\right)=\ell n(E)-\frac{t}{RC}$.

We have $\ell n(u_C)=f(t)=at+b$ satisfies the conditions:

✖ straight line (t is of degree 1);

✖ decreasing (its slope is $a=-\frac{1}{\tau}<0$);

✖ and not passing through origin ($b=\ell nE\neq1$).

b) At $t=0$; $\ell nu_C=2.08=\ell nE$ then $E=e^{2.08}\approx8\,V$.

But for $t=2\,ms$, $\ell n(u_C)=0$ so $0=\ell nE-\frac{1}{RC}\times2\times10^{-3}$.[3]

Then $C=\frac{2\times10^{-3}}{100\times2.08}\approx9.61\times10^{-7}\,F=0.961\,\mu F\approx1\,\mu F$.

III-LS 2011 1st

Part A

1. In order to charge, the switch should be placed in position 1.

2. The steady state is reached after duration $t=5\,RC\Rightarrow e^{-\frac{t}{RC}}=e^{-5}\approx0$;

Then $u_C=E(1-0)=E$.[4]

---

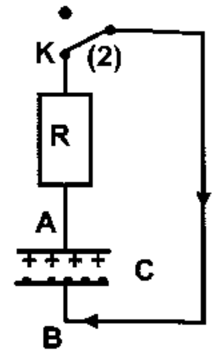
[2] A positive value of the slope is completely wrong answer.
If we choose other points the slope will be slightly different $\left.\frac{du_C}{dt}\right|_{t=2ms}=\frac{(3.125-0)V}{(0-3\times10^{-3})s}=-1041.6\,V/s$.

[3] We can use also another point for example at $t=1\,ms$ we have $\ell n(u_C)=1\Rightarrow C=0.96\times10^{-6}F$.

[4] Or as $t\longrightarrow+\infty$, $e^{-\frac{t}{RC}}\longrightarrow0$ then $u_C=E\left(1-e^{-\frac{t}{RC}}\right)\longrightarrow E$.

Part B

1. The discharge of the capacitor takes place when the switch is placed on position 2.



2. a) Circuit.
   b) Discharge phase

   i- During the discharge, the charge q decreases, then $\frac{dq}{dt} < 0$ and since the sense of current is taken as positive sense then: $i = -\frac{dq}{dt}$.

   ii- Law of uniqueness of voltages $u_{AB} = u_{AB}$ so $u_C = Ri$;

   But $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$;

   Deriving with respect to time of $u_C = Ri$ we get $\frac{du_C}{dt} = R\frac{di}{dt}$ (multiplying by $-C$) [5]

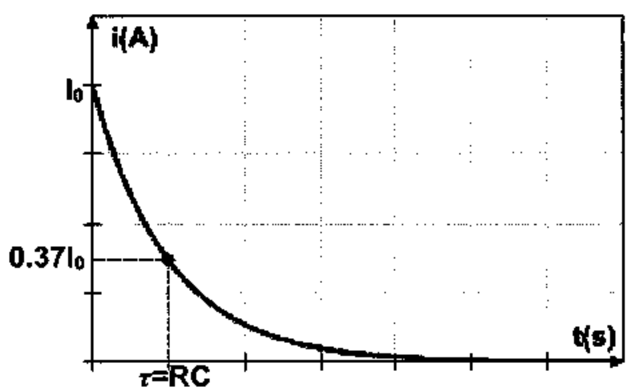
   $-C\frac{du_C}{dt} = -RC\frac{di}{dt}$, then $i + RC\frac{di}{dt} = 0$.[6]

   c) We have: $i = \frac{E}{R}e^{-\frac{t}{RC}} \Rightarrow \frac{di}{dt} = \frac{E}{R}\times\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}} = -\frac{E}{R^2C}e^{-\frac{t}{RC}}$

   Replacing in the differential equation we get : $i + RC\frac{di}{dt} = \frac{E}{R}e^{-\frac{t}{RC}} + RC\left(-\frac{E}{R^2C}e^{-\frac{t}{RC}}\right)$;

   $$= \frac{E}{R}e^{-\frac{t}{RC}} - \frac{E}{R}e^{-\frac{t}{RC}} = 0 \text{ (verified)}$$

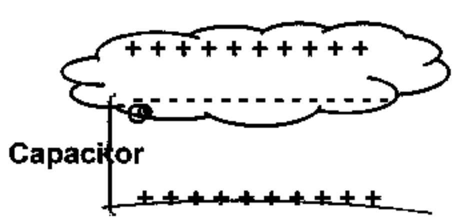
3. Graph of the current during the discharge is shown in the adjacent figure.



4. The discharge duration is $5RC$.

Part C

1. The duration of the complete discharging is $t = 5RC = 5\times 5000\times 10^{-10} = 2.5\times 10^{-6}\,s$.



2. The maximum current is delivered at $t = 0$ and equal to:

   $I_{max} = i\big|_{t=0} = \frac{E}{R}e^{-0} = \frac{E}{R} = \frac{10^8}{5000} = 20\,000\,A$.

---

[5] The new idea is the derivative of the equation with respect to time.

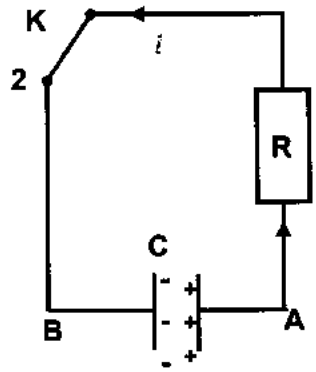
[6] We have $i = -C\frac{du_C}{dt}$ & $u_C = Ri$; so $i = -RC\frac{di}{dt}$, then $i + RC\frac{di}{dt} = 0$.

IV-LS 2010 1st

Part A

1. The switch (K) must be moved to position 1.
2. When the steady state is reached: $u_C = u_{MN} = E = 5V$ & $u_R = 0$.

Part B



1. Circuit.
2. According to the law of uniqueness of voltage:

   $u_{AB} = u_{AB}$; but $u_{AB} = u_C$; $q = C u_C$ & $i = -C\dfrac{du_C}{dt}$;

   (The sense of current is taken as positive sense).

   $u_{AB} = u_R \Rightarrow u_C = Ri = -RC\dfrac{du_C}{dt} \Rightarrow u_C + RC\dfrac{du_C}{dt} = 0$.

3. a) $u_C = E e^{-\frac{t}{\tau}} \Rightarrow \dfrac{du_C}{dt} = -\dfrac{E}{\tau} e^{-\frac{t}{\tau}}$.

   Replacing in the differential equation we get: $E e^{-\frac{t}{\tau}} - RC\dfrac{E}{\tau} e^{-\frac{t}{\tau}} = 0 \Rightarrow E e^{-\frac{t}{\tau}}\left(1 - \dfrac{RC}{\tau}\right) = 0$;

   But $E e^{-\frac{t}{\tau}} \neq 0 \Rightarrow \tau = RC$.

   b) For $t_1 = \tau$, $u_C = E e^{-1} = 1.85V$.

   c) The minimum duration for the complete discharging is $t_{\min} = 5\tau$.

   d) We have $u_C = E e^{-\frac{t}{\tau}}$, $\ell n(u_C) = \ell n\left(E e^{-\frac{t}{\tau}}\right) = \ell nE + \ell n\left(e^{-\frac{t}{\tau}}\right) = \ell nE - \dfrac{t}{\tau}\ell n(e)$;

   Then $\ell n(u_C) = \ell nE - \dfrac{t}{\tau} = -\dfrac{1}{\tau}t + \ell nE$. [7]

   e) The curve representing $\ell n(u_C) = f(t)$ is a straight line.

   Graphically, the slope is $a = \dfrac{\Delta \ell n(u_C)}{\Delta t} = \dfrac{\ell n(u_C)_2 - \ell n(u_C)_1}{t_2 - t_1} = \dfrac{1.61 - 1}{(0-20)\times 10^{-3} s} = -30\, s^{-1}$;

   But $a = \dfrac{-1}{\tau} = \dfrac{-1}{RC} = -30 s^{-1}$, then $R = \dfrac{1}{30 \times 33 \times 10^{-6}} = 1010\,\Omega$.

V-LS 2008 1st

Part A

1. a) The two voltages are equal because the lamps glow with the same brightness.

   b) Evolution of the voltage:

   i- u_2 decreases, in fact $E = u_2 + u_C$ where is constant and u_C increases.

---

[7] $\ell n(u_C) = f(t) = at + b$ is:

✖ a straight line since t is of degree 1;

✖ decreasing, since its slope is $a = -\dfrac{1}{\tau} < 0$;

✖ and not passing through origin since $b = \ell nE \neq 0$.

ii- When the light of L_2 goes out $u_2 = 0$ then $E = u_C + u_2 = u_C$;
The voltage across the generator G_2 becomes equal to that across the capacitor.

2. a) The relation is $i = \dfrac{dq}{dt}$.

b) Law of addition of voltages $E = u_R + u_C$ & $u_R = R\,i = R\dfrac{dq}{dt} = RC\dfrac{du_C}{dt}$;

Then $E = u_C + RC\dfrac{du_C}{dt}$.

c) Evolution of the voltage and the current

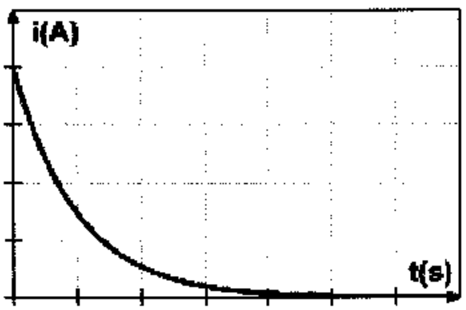
i- We have $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$ and $\dfrac{du_C}{dt} = E\left(\dfrac{1}{\tau}\right)e^{-\frac{t}{\tau}}$.

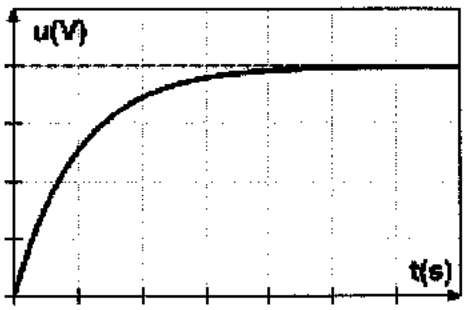
Replacing in the differential equation $E = E\left(1 - e^{-\frac{t}{\tau}}\right) + RC\dfrac{E}{\tau}e^{-\frac{t}{\tau}} = E + E\,e^{-\frac{t}{\tau}}\left(1 - \dfrac{RC}{\tau}\right)$;

By comparison, we get $1 - \dfrac{RC}{\tau} = 0 \Rightarrow \tau = RC$.

ii- The current $i = \dfrac{dq}{dt} = \dfrac{E}{R}e^{-\frac{t}{\tau}}$.

iii- The graphs are represented in the figures below:





iv- The capacitor does not allow the passage of the current except during a short time when the circuit is fed by a DC voltage.

Part B

1. In the first step, the capacitor is charged till it reaches the voltage $u_C = E$.
In the second step, the capacitor is discharged in the motor by providing across the motor a voltage u_C which decreases from the value E, thus it allows the lifting of the body.

2. a) The electric energy stored in the capacitor is $W = \dfrac{1}{2}CE^2 = \dfrac{1}{2}\times 1\times 9 = 4.5\,J$.

b) This energy is converted into gravitational potential energy:

$$W = m\,g\,h_{max} = 4.5\,J \Rightarrow h_{max} = \frac{4.5}{0.5\times 10} = 0.9\,m$$

c) The electric energy stored in the capacitor is transformed into mechanical energy.

d) This decrease is due to the dissipative factors (friction, air resistance and Joule's effect).

e) The capacitor stores electric energy and restitute this energy when needed.

VI-LS 2006 1st

Part A

1. Law of addition of voltages: $u_{DB} = u_{DA} + u_{AB}$; $E = u_R + u_C$ but $i = \frac{dq}{dt}$ and $q = Cu_C$;

   Then, $E = Ri + u_C \Rightarrow E = RC\frac{du_C}{dt} + u_C$.

2. a) We have $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$, so $\frac{du_C}{dt} = \frac{E}{\tau}e^{-\frac{t}{\tau}}$;

   Replacing in the differential equation: $RC\frac{du_C}{dt} + u_C = \frac{E}{\tau}RCe^{-\frac{t}{\tau}} + E - Ee^{-\frac{t}{\tau}} = E$.

   b) τ is the time taken for the voltage across the capacitor u_C to reach 63% of its maximum value E.

3. The steady state is practically reached after $t = 5\tau = 5RC$.

Part B

1. a) Because AA' is a connecting wire of negligible resistance.

   b) **1st method:**

   Law of addition of voltages: $u_{AA'} = u_{AB} + u_{BB'} + u_{B'A'}$ but $u_{AA'} = u_{BB'} = u_{B'A'} = 0$ (wires).

   Then $u_{AB} = u_C = 0$ (the current will not circulate in the branch containing the capacitor).

   2nd method:

   The electric current takes the easiest path so it will travel through AA' and not through the capacitor then: $u_{AB} = u_C = 0$.

2. The interval of time t_1 is:

   ✘ the duration needed by the bullet to travel the distance that separates AA' from BB';

   ✘ the duration needed by the capacitor to reach the value of $45.7\,V$.

   a) We have $u_C = E\left(1 - e^{\frac{-t}{\tau}}\right)$ so $1 - e^{\frac{-t}{\tau}} = \frac{u_C}{E}$, we get $e^{\frac{-t}{\tau}} = 1 - \frac{u_C}{E}$;

   $-\frac{t}{\tau} = \ell n\left(1 - \frac{u_C}{E}\right) \Rightarrow t_1 = -RC\ell n\left(1 - \frac{u_C}{E}\right)$;

   Then $t_1 = -RC\ell n\left(1 - \frac{u_C}{E}\right) = -0.004\,\ell n\left(1 - \frac{45.7}{100}\right) = 2.44 \times 10^{-3}\,s = 2.44\,ms$.

   b) The motion of the bullet is uniform, then its speed is $v = \frac{\ell}{t_1} = \frac{1m}{2.44 \times 10^{-3}s} = 410\,m/s$.

3. Because if this time exceeds $5RC$ the steady state is attained and the voltage will no longer vary.

   Then $t \le 5RC \Rightarrow \frac{\ell}{v} \le 5RC$; $\ell \le 5RC \times v \Rightarrow \ell \le 8.02\,m$, thus $\ell_{\max} = 8.02\,m$.

Unit II

Electricity

Chapter 9

Alternate Sinusoidal Voltage

LS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Alternate Sinusoidal		2nd	2nd	-	1st	1st	-	2nd	2nd
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Alternate Sinusoidal	2nd	2nd	-	2nd	2nd	2nd	1st(B)	1st	1st

Essentials

I- Generalities

The voltage displayed in figure 1 is alternating sinusoidal whose instantaneous expression can be written in the form: $u = U_m \sin(\omega t)$ or $u = U_m \cos(\omega t)$ where:

- U_m is the maximum voltage;
  $U_m(V) = S_v(V/\text{div}) \times y_{\max}$ where S_v is the vertical sensitivity
- ω is the angular frequency measured in rad/s and related to the period T by the relation $T = \frac{2\pi}{\omega}$ and to the frequency f by $f = \frac{1}{T}$.
  $T(ms) = S_h(ms/\text{div}) \times x(\text{div})$ where S_h is the horizontal sensitivity.

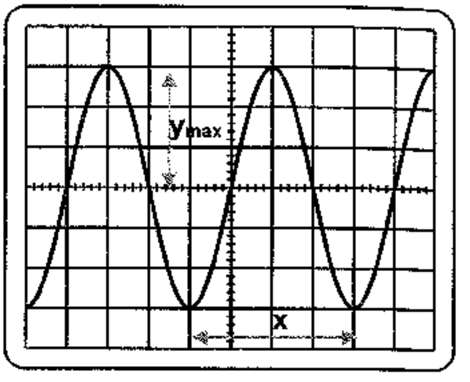


Figure 1

II- Phase Difference (Phase Shift)

In general, the instantaneous expression of an alternating sinusoidal voltage is $u = U_m \sin(wt + \varphi)$ or $u = U_m \cos(wt + \varphi)$ where φ is called the phase difference that is mainly used when we are comparing two signals or waveforms and given by $|\varphi| = 2\pi \frac{d}{D}$ (figure 2).

A waveform leads if it reaches its maximum first.

On figure 2, (a) leads (b) since it reaches maximum first, we can say also that (b) lags behind (a).

But if they reach their extremes (maximums & minimum) at the same instant, they are in phase.

Note: when a voltmeter (ammeter) in AC mode connected across (in series with) an alternating voltage it measures the effective voltage (current), $U = U_{\text{eff}}(V) = \frac{U_{\max}}{\sqrt{2}}$ & $I = I_{\text{eff}}(A) = \frac{I_{\max}}{\sqrt{2}}$.

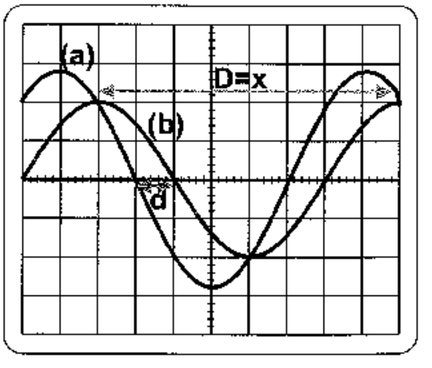


Figure 2

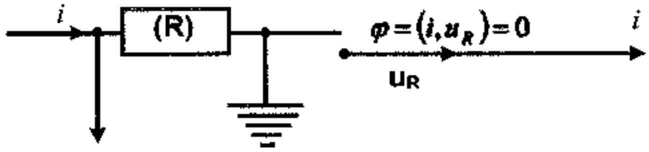
III- Phase Difference (Shift) Between a Voltage and the Current

Let i be the instantaneous current in a given circuit.

We are going to use Fresnel representation, in which each voltage is represented by a vector whose length is equal to the maximum value of the voltage and its direction is equal to the phase difference with respect to the current.

1. Resistor

The voltage across the resistor is in phase with the current flowing through it, so the vector representing u_R is collinear with the current axis.



2. Capacitive effect

a) Capacitor

The voltage across the capacitor u_C lags behind the current i by $\frac{\pi}{2}(\text{rad})$ as shown in figure 3.

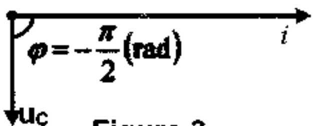


Figure 3

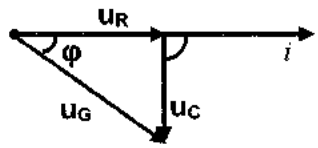


Figure 4

b) Generator

In (R, C) circuit, $u_G = u_R + u_C$; the voltage u_G should lag behind the current by an angle φ as shown in figure 4.

Furthermore $(u_G)_{\max} > (u_R)_{\max}$ &

$(u_G)_{\max} > (u_C)_{\max}$.

If the waveforms of figure 5 represents the voltages across generator and resistor in (R, C) circuit, then (2) corresponds to the voltage across generator while (1) to that across the resistor.

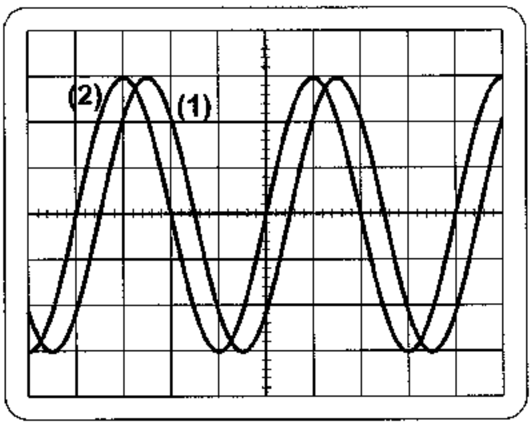


Figure 5

3. Inductive effect

a) Pure inductive coil $(r = 0,\ L \neq 0)$.

The voltage across a pure inductive coil u_L leads the current i by $\frac{\pi}{2}(\text{rad})$ as shown in figure 6.

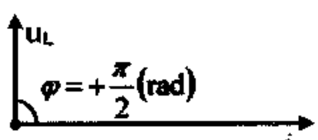


Figure 6

b) Real coil & (R, L) circuit

The voltage across a real coil could be modeled as the sum of the voltage across its internal resistance and its inductance, we may write $u_{\text{coil}} = u_r + u_L$; in this case u_{coil} <u>leads</u> the current i by $\varphi \neq \frac{\pi}{2}$ (rad) called <u>inductive effect</u> of the coil (figure 7).

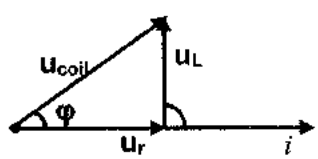


Figure 7

Similarly in (R, L) series circuit, the voltage across the generator leads the current due the inductive effect.

Furthermore $(u_G)_{\max} > (u_R)_{\max}$ & $(u_G)_{\max} > (u_{coil})_{\max}$.

IV-

Power Consumed

Let u & i be the instantaneous voltage across a given dipole (whatever its nature) and current flowing through it. The instantaneous power delivered (or consumed) by this dipole is given by $P = u \times i$.

For a resistor of resistance R and by referring to Ohm's law $u = Ri$, then $P = Ri^2$.

1. Constant DC voltage

If u & i are constant, we write $u = U$ & $i = I$, we get $P = UI$ & for a resistor $P = RI^2$.
These expressions are practically unchanged.

2. Alternate sinusoidal voltage

Under AC voltage, the average power consumed is given by $P = U \times I \times \cos\varphi$; where I & U, are the effective values of the current and voltage; and φ is the phase difference between the instantaneous voltage and current.
Note: $\cos\varphi$ is called power factor.

a) Resistor

u & i are in phase, $\varphi = 0$; then $P = U \times I = RI^2$.

b) Capacitor

u lags behind i by $|\varphi| = \frac{\pi}{2}(\text{rad})$, $\cos\varphi = 0$; $P_C = 0$ (the capacitor does not consume any power).

c) Coil

i- Ideal coil (pure inductive coil)

u leads i by $|\varphi| = \frac{\pi}{2}(\text{rad})$; $\cos\varphi = 0$; $P_L = 0$ (the coil does not consume any power).

ii- Real coil of internal resistance r

We have $P_{\text{coil}} = U_{\text{coil}} \times I \times \cos\varphi$, where $\varphi = (u_{\text{coil}}, i)$; but the resistor is only consuming energy then $P_{\text{coil}} = r \times I^2$.

d) (R, L, C) Series circuit

The total power consumed in such circuit is given by: $P = U_G \times I \times \cos\varphi$ where φ is the phase difference between the voltage across generator u_G of effective value $U_G = \frac{(U_G)_m}{\sqrt{2}}$ and the current i of effective value $I = \frac{I_m}{\sqrt{2}}$.

But the power is additive and only the resistors consume energy in the circuit.
Then $P = P_R + P_r + P_{\text{coil}} + P_C = (R + r)I^2$.

V- (R, L, C) Circuit

Under constant U_m, we modify the frequency.

We designate by φ the phase difference between the voltage across the generator u_G and the current in a circuit formed of a resistor R, capacitor C and a real coil (L, r) as shown in the adjacent circuit (in any order).

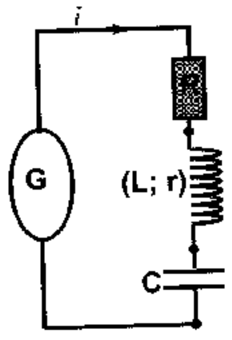


Figure 8

1. In the case where $\varphi = 0$, we say that u_G & i are in phase.
2. In this case the circuit is the seat of a particular electric phenomenon called current resonance.
3. The current resonance is attained if $L\omega_0^2 C = 1$; $T_0 = 2\pi\sqrt{LC}$ & $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$ (called proper frequency of the circuit).
4. The current takes its maximum value $(I_m)_{max} = \dfrac{U_m}{(R+r)}$.

   We say that the circuit acts as a resistor.
5. The curve representing the variations of the intensity of the current in terms of the frequency has an unsymmetrical bell-shape as shown in the adjacent curve (figure 9).
   The maximum corresponds to the state of current resonance and reached when the frequency of the generator is equal to the proper frequency of the circuit.
6. The power consumed by the circuit is maximum.

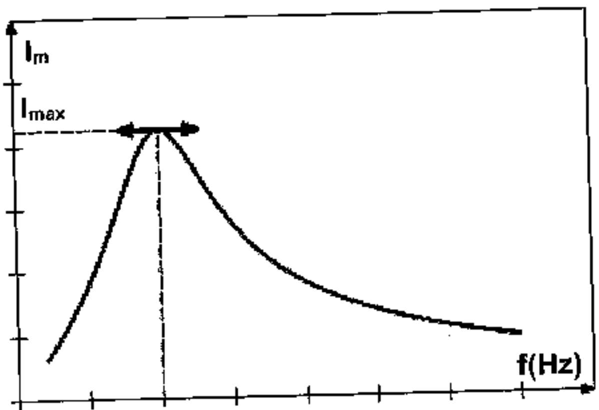


Figure 9

Note 1: In (R, L, C) circuit, the maximum value of the voltage across the generator $(u_G)_{max}$ is not always greater than that across the capacitor (nor that across the coil).
If $(u_C)_{max} > (u_G)_{max}$, we say that the capacitor is in the state of <u>overvoltage</u>.

Note 2: The power consumed in the (R, L, C) circuit is given by $P = U_G I \cos\varphi$ where φ is the phase difference between u_G & i, under **resonance** $\varphi = 0$, so $\cos\varphi = 1$.
The **power** consumed is then **maximum** $P_{max} = U_G I_{max}$.

Note 3: In (R, L, C) circuits, if the voltage u_G across the generator:

- ✗ leads the current, then the circuit is inductive (coil dominates);
- ✗ lags behind the current, then the circuit is capacitive (capacitor dominates);
- ✗ in phase with the current, the circuit is resistive (resonance).

Applications

Phase difference between two curves

A voltage (u_2) leads (u_1) if it reaches its maximum first, which is equivalent also to (u_1) lags behind (u_2).

Note: Maximum first is different than the higher maximum.

But if they reach their maximums at the same instant, then they are in phase.

I- Leads or Lags Behind?

The adjacent waveforms labelled (1) & (2) represent two voltages whose expressions are u_1 & u_2 respectively.

The adjustments of the oscilloscope $S_h = 8ms/\text{div}$ & $S_{v_1} = S_{v_2} = 2V/\text{div}$. Let $u_1 = U_m \sin(wt)$.

1. Calculate U_m.
2. Determine the expression of u_1 in terms of time.
3. Calculate the phase difference between (u_2) and (u_1).
4. Specify which signal leads the other and then deduce the instantaneous expression of u_2.

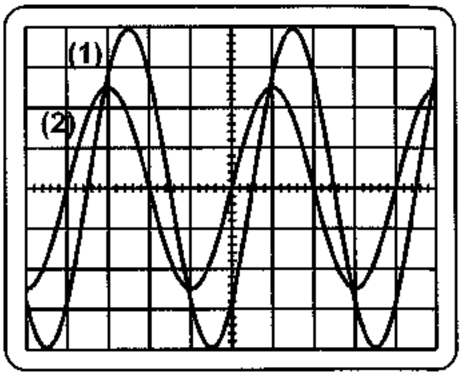
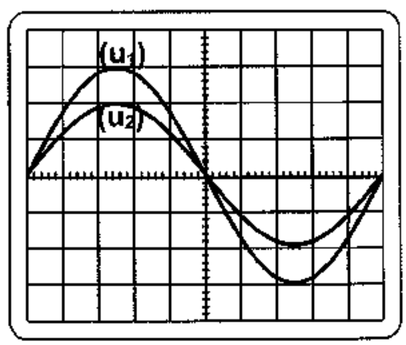


Figure 1

II- Signals in Phase

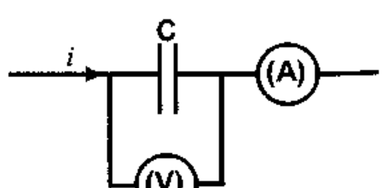
The adjacent waveforms are observed on the screen of an oscilloscope whose adjustments are $S_h = 2\,ms/\text{div}$, $S_{v_1} = 5V/\text{div}$ & $S_{v_2} = 2V/\text{div}$. If $u_1 = U_m \cos(wt)$.

1. The waveforms shown in the adjacent figure are in phase. Justify.
2. Determine the instantaneous expressions of u_1 & u_2.



III- Multimeters in AC Mode

An ammeter (A), and voltmeter (V) in AC mode are connected in a circuit containing among others a capacitor of capacitance C; and traversed by an alternating sinusoidal current whose instantaneous expression is given by: $i = 0.16\sin\left(100\,\pi t + \frac{\pi}{3}\right)$ where t in s & i in A.



1. Determine the frequency of the voltage delivered by the generator.
2. Calculate the value displayed by (A).

3. If (V) displays $3.8V$. Determine C.

4. Determine the phase difference between the voltage across the capacitor and the current.

IV- Capacitive Effect

A generator G delivering across its terminals an alternating sinusoidal voltage of maximum value U_m and of adjustable frequency f, in a circuit containing a resistor of resistance R, a capacitor of capacitance C, a switch and connecting wires connected as shown in figure 1.

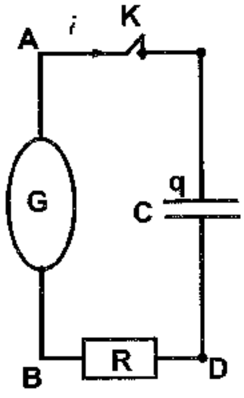


Figure 1

In the steady state, the current i carried by the circuit has the form: $i = I_m \sin(\omega t)$.

1. a) Determine the expression of u_{DB} in terms of R, I_m, ω and t.
   b) Specify the phase difference of u_{DB} with respect to the current.
2. a) Determine the expression of the voltage u_C in terms of I_m, C, ω and t.
   b) Deduce the expression of the maximum value U_C of u_C in terms of I_m, C and ω.
   c) Compare the phase difference between u_C and the current.
3. Let $u_{AB} = U_m \sin(\omega t + \varphi)$.
   a) Applying the law of addition of voltages and giving t two particular values, show that: $\tan\varphi = -\dfrac{1}{RC\omega}$.
   b) Specify the phase difference of u_{AB} with respect to the current.
   c) Show that $U_m = I_m\sqrt{R^2 + \dfrac{1}{C^2\omega^2}}$.

V- Inductive Effect

A generator G delivering across its terminals an alternating sinusoidal voltage of maximum value U_m and of adjustable angular frequency ω such that $u_G = U_m \sin(wt)$.

G is placed in a circuit containing a resistor of resistance R, a pure inductive coil of inductance L, a switch and connecting wires connected as shown in figure 1.

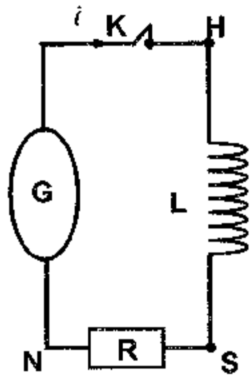


Figure 1

In the steady state, the current i carried by the circuit has the form: $i = I_m \sin(\omega t + \varphi)$.

1. a) Determine the expression of the voltage $u_L = u_{HS}$ in terms of I_m, L, ω and t.
   b) Deduce the expression of maximum value U_L of u_L in terms of I_m, L and ω.
   c) Specify the phase difference of u_L with respect to the current.

2. a) Applying the law of addition of voltages and giving t one particular value, show that $\tan\varphi = -\frac{L\omega}{R}$.

   b) Specifu the phase difference of u_{HN} with respect to the current.

3. An oscilloscope connected in order to displays the voltages across the generator and that across the resistor and the waveforms obtained are shown in figure 2.

   a) Recopy figure 1, showing on it the connections of the oscilloscope.

   b) Specify the waveform that corresponds to the voltage across the generator.

   c) Determine the phase difference between the voltages displayed.

   d) If $f = 125\,Hz$ & $R = 50\,\Omega$, deduce the value of L.

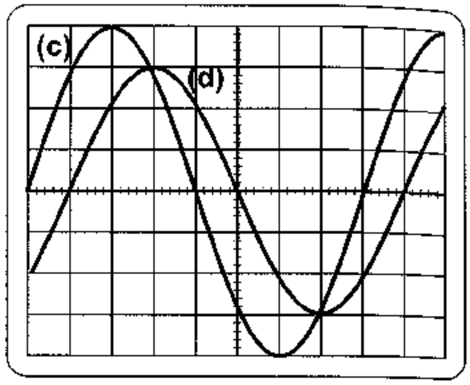


Figure 2

VI-
Physical Phenomenon

A generator G delivering an alternating sinusoidal voltage of adjustable frequency f, $u = 6\sqrt{2}\sin 2\pi f t$ (u in V and t in s), a coil of inductance L and internal resistance r, an ammeter, a capacitor of capacitance $C = 220\,\mu F$ and a resistor of resistance $R = 100\Omega$.

The effective value of u is kept constant throughout the experiment. f is modified and, for each value, the effective value of the intensity of current I is measured.

The graph of figure 2 represents the variations of the current I as a function of f.

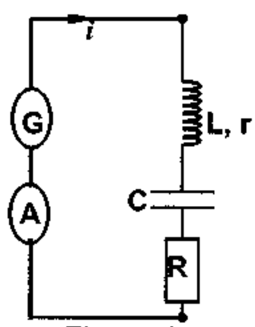


Figure 1

1. Name the physical phenomenon displayed in figure 2.
2. Determine the proper frequency f_0 of the circuit and deduce L.
3. Knowing that the frequency of the generator is $f = f_0$.

   a) Calculate the internal resistance r of the coil.

   b) Calculate the total power consumed in the circuit.

   c) Show that the expression of the current as a function of time is $i = 0.05\sqrt{2}\sin(200\pi t)$ (i in A and t in s).

   d) Deduce the expression of the voltage across the capacitor.

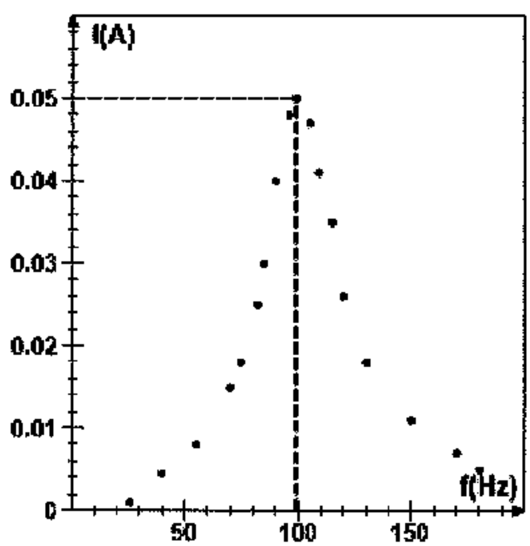


Figure 2

VII-
Dipoles Characteristics

The aim of this exercise is to determine the characteristics of a capacitor and a coil.

In order to determine these characteristics, we connect in series a capacitor of capacitance C, a coil of inductance L and of resistance r, a resistor of resistance $R = 100\,\Omega$ and a low frequency generator (G) delivering an alternating sinusoidal voltage u of constant maximum value U_m and of adjustable frequency f, $u = U_m \sin(\omega t)$ where u in V & t in s.

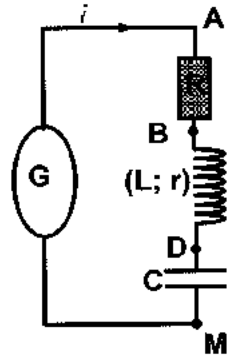


Figure 1

The circuit thus formed (figure 1), carries an alternating sinusoidal current i. An oscilloscope is connected to display the voltage $u = u_{AM}$ across the terminals of the generator (G) on channel (Y_1) and the voltage u_{DM} across the capacitor (C) on channel (Y_2) and the waveforms obtained are shown in figure 2.

An ammeter inserted in the previous circuit displays $61.2\,mA$.

The adjustments of the oscilloscope are as follows:

- ✘ horizontal sensitivity: $S_h = 1ms/\text{div}$.
- ✘ vertical sensitivity on (Y_1): $S_{v_1} = 5V/\text{div}$.
- ✘ vertical sensitivity on (Y_2): $S_{v_2} = 10V/\text{div}$.

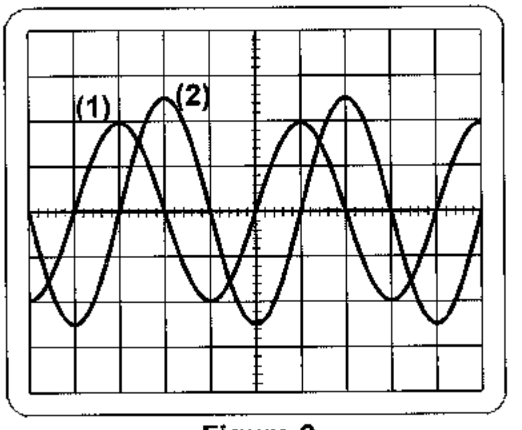


Figure 2

1. Recopy figure 1, showing on it the connections of the oscilloscope.
2. Justify that $u = 10\sin(500\,\pi\,t)$.
3. Determine the phase difference between the voltages u_G and u_C, and then deduce that the circuit is the seat of current resonance.
4. If the effective voltage across the capacitor is clearly greater than the effective voltage U across the generator, we say there is an over voltage across the terminals of the capacitor.
   Justify that the capacitor is in the state of over voltage.
5. Determine the expression of the current in terms of time.
6. Deduce that $C = 2.2\,\mu F$.
7. Determine the inductance and the internal resistance of the coil.

Solutions

I-

1. The maximum value $U_m = S_{v1} \times y_{1\max} = 2V/\text{div} \times 4\text{div} = 8V$.
2. The period of the signal displayed is $T = S_h \times x = 8ms/\text{div} \times 4\text{div} = 32\,ms$;

   The angular frequency $w = \dfrac{2\pi}{T} = \dfrac{2\pi}{32\times10^{-3}} = 62.5\pi\,\text{rad}/s$.

   Then $u_1 = 8\sin(62.5\pi t)$ where t in s & u_1 in V.
3. The phase difference $|\varphi| = 2\pi\dfrac{d}{D} = 2\pi \times \dfrac{0.5\,\text{div}}{4\text{div}} = \dfrac{\pi}{4}$ rad.
4. u_2 reaches its maximum before u_1, then u_2 leads u_1; so $\varphi = +\dfrac{\pi}{4}(\text{rad}) > 0$;

   Then $u_2 = U_{2m}\sin\left(62.5\pi t + \dfrac{\pi}{4}\right)$ where $U_{2m} = S_{v2} \times y_{2\max} = 2V/\text{div} \times 2.5\text{div} = 5V$.

   Thus, $u_2 = 5\sin\left(62.5\pi t + \dfrac{\pi}{4}\right)$ where t in s & u_2 in V.

II-

1. The waveforms reach their extremums at the same instants, then they are in phase.
2. We have $U_{1m} = S_{v1} \times y_{1\max} = 5V/\text{div} \times 3\text{div} = 15\,V$;

   The period of the signal displayed is $T = S_h \times x = 2\,ms/\text{div} \times 10\text{div} = 20\,ms$;

   The angular frequency $w = \dfrac{2\pi}{T} = \dfrac{2\pi}{20\times10^{-3}} = 100\,\pi\,\text{rad}/s$.

   Then $u_1 = 15\sin(100\,\pi t)$ where t in s & u_1 in V.

   u_1 & u_2 are in phase, then their phase difference is zero $\varphi = 0$;

   But $U_{2m} = S_{v2} \times y_{2\max} = 2\,V/\text{div} \times 2\,\text{div} = 4\,V$

   Then $u_2 = U_{2m}\sin(100\,\pi t) = 4\sin(100\,\pi t)$ where t in s & u_2 in V.

III-

1. The angular frequency $\omega = 100\,\pi = 2\pi f$, then $f = \dfrac{100\,\pi}{2\pi} = 50\,Hz$.
2. The ammeter displays the value of the effective current $I = \dfrac{I_m}{\sqrt{2}}$ where I_m is the maximum value

   of the current, then $I_m = 0.16\,A$; thus, $I = \dfrac{0.16}{\sqrt{2}} = 0.11\,A$.
3. The voltage across the capacitor is related to the current by the relation, $u_C = \dfrac{1}{C}\int i\,dt$;

   $u_C = \dfrac{1}{C}\int 0.16\sin\left(100\,\pi t + \dfrac{\pi}{3}\right)dt = \dfrac{0.16}{C} \times \dfrac{-1}{100\,\pi}\cos\left(100\,\pi t + \dfrac{\pi}{3}\right) = \dfrac{-0.16}{100\,\pi C}\cos\left(100\,\pi t + \dfrac{\pi}{3}\right)$;

The maximum value of u_C is $U_{C\max} = \dfrac{0.16}{100\,\pi C}$ & $U_{C\max} = U\sqrt{2} = 3.8\sqrt{2}\,V$;

Thus, $C = \dfrac{0.16}{100\,\pi \times 3.8\sqrt{2}} = 9.5\times 10^{-5}\,F \approx 95\,\mu F$.

4. We have $u_C = \dfrac{-0.16}{100\,\pi C}\cos\left(100\,\pi t + \dfrac{\pi}{3}\right) = 3.8\sqrt{2}\sin\left(100\,\pi t + \dfrac{\pi}{3} - \dfrac{\pi}{2}\right)$;

Thus, u_C lags behind the current i by $\dfrac{\pi}{2}$ (rad).

IV-

We have $i = I_m \sin(\omega t)$.

> **Pay attention (anti-derivative):**
> $\int A\sin(\omega t + \varphi)dt = -\dfrac{1}{\omega}A\cos(\omega t + \varphi)$
> $\int A\cos(\omega t + \varphi)dt = \dfrac{1}{\omega}A\sin(\omega t + \varphi)$

1. a) Ohm's law $u_{DB} = Ri = RI_m\sin(\omega t)$;

   b) There's no phase difference between $u_{DB} = Ri = RI_m\sin(\omega t)$ and the current.
   Then the voltage across the resistor is in phase with the current.

2. a) We have $i = C\dfrac{du_C}{dt}$, $u_C = \dfrac{1}{C}\int I_m\sin(\omega t)dt = \dfrac{I_m}{C}\int \sin(\omega t)dt = -\dfrac{I_m}{C\omega}\cos(\omega t)$.

   b) The maximum value of u_C is $U_{Cm} = \left|-\dfrac{I_m}{C\omega}\right| = \dfrac{I_m}{C\omega}$.

   c) We have $u_C = -\dfrac{I_m}{C\omega}\cos(\omega t) = \dfrac{I_m}{C\omega}\sin\left(\omega t - \dfrac{\pi}{2}\right)$;

   Then the voltage u_C lags behind the current i by $\dfrac{\pi}{2}$(rad).

3. a) Law of addition of voltages: $u_{AB} = u_{AD} + u_{DB}$;

   $U_m\sin(\omega t + \varphi) = RI_m\sin(\omega t) - \dfrac{I_m}{C\omega}\cos(\omega t)$;

   Let $\omega t = 0$, $U_m\sin(\varphi) = RI_m\sin(0) - \dfrac{I_m}{C\omega}\cos(0)$; $U_m\sin(\varphi) = -\dfrac{I_m}{C\omega}$ ....(1)

   Let $\omega t = \dfrac{\pi}{2}$, $U_m\sin\left(\dfrac{\pi}{2} + \varphi\right) = RI_m\sin\left(\dfrac{\pi}{2}\right) - \dfrac{I_m}{C\omega}\cos\left(\dfrac{\pi}{2}\right)$; $U_m\cos(\varphi) = RI_m$ ....(2)

   Divide (1) by (2), we get: $\dfrac{U_m\sin(\varphi)}{U_m\cos(\varphi)} = \dfrac{-\dfrac{I_m}{C\omega}}{RI_m}$, then $\tan\varphi = -\dfrac{1}{RC\omega}$.

   b) $\tan\varphi = -\dfrac{1}{RC\omega} < 0$, so $\varphi < 0$; then u_{AB} lags behind i by an angle φ.

   c) Squaring the relations, then adding $(1)^2 + (2)^2$:

   we get $U_m^2\sin^2(\varphi) + U_m^2\cos^2(\varphi) = \dfrac{I_m^2}{C^2\omega^2} + R^2I_m^2$;

   $U_m^2\left[\sin^2(\varphi) + \cos^2(\varphi)\right] = I_m^2\left(\dfrac{1}{C^2\omega^2} + R^2\right)$; then $U_m = I_m\sqrt{R^2 + \dfrac{1}{C^2\omega^2}}$.

Remember that

In a capacitive circuit (R, C)

The voltage across the capacitor lags behind the current by $\frac{\pi}{2}$ (rad).

The voltage across the generator lags behind the current by $\varphi < \frac{\pi}{2}$.

V-

1. a) We have $u_L = u_{HS} = r\,i + L\frac{di}{dt}$ (the coil is pure inductive so $r \approx 0$);

   But $i = I_m \sin(\omega t + \varphi)$, then $u_L = L\frac{di}{dt} = L\,I_m\,\omega\cos(\omega t + \varphi)$.

   b) The maximum value $U_{Lm} = L\,I_m\,\omega$.

   c) We have $u_L = U_{Lm}\cos(\omega t + \varphi) = U_{Lm}\sin\left(\omega t + \varphi + \frac{\pi}{2}\right)$;

   Then, the voltage across the pure inductive coil leads the current by $\frac{\pi}{2}$ (rad).

2. a) Law of addition of voltages: $u_{HN} = u_{HS} + u_{SN}$;

   $U_m \sin(\omega t) = L\,I_m\,\omega\cos(\omega t + \varphi) + R\,I_m \sin(\omega t + \varphi)$;

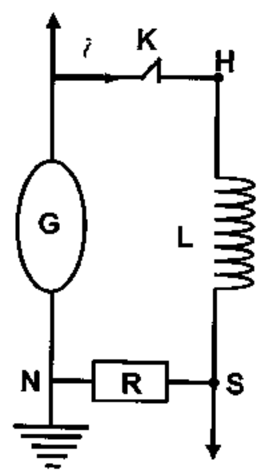
   Let $\omega t = 0$; we get $U_m \sin(0) = L\,I_m\,\omega\cos(\varphi) + R\,I_m \sin(\varphi)$;

   So, $R\,I_m \sin(\varphi) = -L\,I_m\,\omega\cos(\varphi)$; then $\tan(\varphi) = -\frac{L\omega}{R}$.

   b) We have $\tan(\varphi) = -\frac{L\omega}{R} < 0$, then $\varphi < 0$.

   But φ is the phase difference between the voltage across generator $u_G = U_m \sin(\omega t)$ & the current $i = I_m \sin(\omega t + \varphi)$;

   Thus, u_G leads the current i.



3. a) Connections are shown on circuit.

   b) The circuit is the seat of an inductive effect, then the voltage across the generator should lead the current.

   But the voltage across the resistor is the image of the current, then u_G leads u_R.

   Thus, u_G is represented by the waveform (c).

   c) The phase difference $|\varphi| = 2\pi\frac{d}{D} = 2\pi\frac{1\,\text{div}}{8\,\text{div}} = \frac{\pi}{4}$ (rad).

   But $\varphi < 0$, then $\varphi = -\frac{\pi}{4}$ (rad).

   d) We have $\tan(\varphi) = -\frac{L\omega}{R}$;

   So $L = -\frac{R\tan(\varphi)}{\omega} = \frac{R}{2\pi f} = \frac{50}{2\pi \times 125} \approx 0.04\,H$.

Remember that

In an inductive circuit (R, L)

The voltage across a pure inductive coil leads the current by $\frac{\pi}{2}$ (rad).

The voltage across the generator leads the current by $\varphi < \frac{\pi}{2}$.

VI-

1. Intensity current resonance.
2. When the effective current is maximum, the frequency of the generator is equal to the proper frequency of the circuit, from graph $f_0 = 100\,Hz$.

   But $f_0 = \frac{1}{2\pi\sqrt{LC}}$; then $L = \frac{1}{4\pi^2 f_0^2 C} = \frac{1}{4\pi^2 \times 100^2 \times (220\times10^{-6})} \approx 0.1H$.
3. We have $I_0 = 0.05A$ & $U = 6V$.

   But $I_0 = \frac{U}{R+r} \Rightarrow R + r = \frac{6}{0.05} = 110$, then $r = 110 - 100 = 10\Omega$.
4. Under resonance u & i are in phase so $\varphi = 0$;

   The power consumed is $P_{max} = U\,I_0 \cos\varphi = 6\times0.05\times1 = 0.3W$.
5. a) We have $i = I_m \sin(2\pi f_0 t + \varphi)$ where $\varphi = 0$ & $I_m = I_0\sqrt{2} = 0.05\sqrt{2}\,A$;

   So, $i = 0.05\sqrt{2}\sin(200\pi t)$ where i in A & t in s

   b) $u_C = \frac{1}{C}\int i dt = \frac{1}{C}\int 0.05\sqrt{2}\sin(200\pi t)dt$.

   Then $u_C = \frac{-0.05\sqrt{2}}{220\times10^{-6}} \times \frac{1}{200\pi}\cos(200\pi t) = -0.5\cos(200\pi t)$ where u_C in V & t in s.

VII-

1. Connections are shown on the adjacent circuit.
2. The waveform (1) corresponds to the voltage across the generator:

   $U_m = S_{v1} \times y_{1max} = 5V/\text{div} \times 2\,\text{div} = 10V$.

   The period of the signal displayed

   $T = S_h \times x = 1ms/\text{div} \times 4\,\text{div} = 4ms$.

   The angular frequency is $\omega = \frac{2\pi}{T} = \frac{2\pi}{4\times10^{-3}} = 500\pi\,(\text{rad}/s)$.

   Then $u = 10\sin(500\pi t)$ where u in V & t in s.

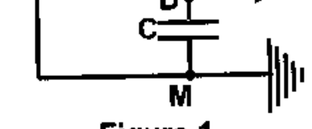


Figure 1

3. The phase difference is $|\varphi| = 2\pi\frac{d}{D} = 2\pi\frac{1\,\text{div}}{4\,\text{div}} = \frac{\pi}{2}(\text{rad})$;

   From the waveforms of figure 2: u_C lags behind u_G by $\frac{\pi}{2}(\text{rad})$;

   We know that: u_C lags behind i by $\frac{\pi}{2}(\text{rad})$;

   We conclude that u_G & i are in phase; then, the circuit is the seat of current resonance.
4. We have $U_{Cm} = S_{v2} \times y_{2max} = 10V/\text{div} \times 2.5\,\text{div} = 25V$;

$U_{Cm} = 25\,V > U_m = 10\,V$ or ($U_C = \dfrac{25\,V}{\sqrt{2}} > U = \dfrac{10\,V}{\sqrt{2}}$).

The capacitor is in the state of overvoltage.

5. Under resonance, the current in the circuit and the generator are in phase.
   Then, $i = I_m \sin(500\,\pi\,t)$, where I_m is the maximum current;
   The ammeter reads the effective value $I = 61.2\,mA$ of the current, so $I_{\max} = I\sqrt{2}$;
   Thus, $i = 61.2\sqrt{2}\times 10^{-3}\sin(500\,\pi\,t)$ where i in A & t in s.

6. We have $i = C\dfrac{du_C}{dt}$; $u_C = \dfrac{1}{C}\int i\,dt = \dfrac{1}{C}\int 61.2\sqrt{2}\times 10^{-3}\sin(500\,\pi\,t)dt$;

   Then, $u_C = \dfrac{61.2\sqrt{2}\times 10^{-3}}{C}\times\dfrac{-1}{500\,\pi}\cos(500\,\pi\,t)$.

   The expression of the maximum voltage across the capacitor is $U_{Cm} = \dfrac{61.2\sqrt{2}\times 10^{-3}}{C\times 500\,\pi} = 25$;

   Thus, $C = \dfrac{61.2\sqrt{2}\times 10^{-3}}{25\times 500\,\pi} = 2.2\times 10^{-6}\,F = 2.2\,\mu F$.

7. Under resonance, the expression of the proper angular frequency $\omega_0^2 = \dfrac{1}{LC}$;

   So, $L = \dfrac{1}{C\times\omega_0^2} = \dfrac{1}{2.2\times 10^{-6}\times(500\,\pi)^2} = 0.18\,H$.

   The circuit acts as resistor, so $U_m = (R + r)I_{\max}$;

   Then $R = \dfrac{U_m}{I_{\max}} - r = \dfrac{10}{61.2\times 10^{-3}\sqrt{2}} - 100 = 15.5\,\Omega$.

2nd method:

We have $u = 10\sin(500\,\pi\,t)$; $u_C = -25\cos(500\,\pi\,t)$ & $i = 61.2\sqrt{2}\times 10^{-3}\sin(500\,\pi\,t)$;

Law of addition of voltages:

We get, $10\sin(500\,\pi\,t) = -25\cos(500\,\pi\,t) + 100\times 61.2\sqrt{2}\times 10^{-3}\sin(500\,\pi\,t)$
$+ r\times 61.2\sqrt{2}\times 10^{-3}\sin(500\,\pi\,t) + L\times 61.2\sqrt{2}\times 10^{-3}\times 500\,\pi\cos(500\,\pi\,t)$;

For $500\,\pi\,t = 0$; $0 = -25\times 1 + 0 + 0 + L\times 61.2\sqrt{2}\times 10^{-3}\times 500\,\pi$;

We get, $L = \dfrac{25}{61.2\sqrt{2}\times 10^{-3}\times 500\,\pi} \approx 0.18\,H$.

For $500\,\pi\,t = \dfrac{\pi}{2}(\text{rad})$; $10 = 0 + 100\times 61.2\sqrt{2}\times 10^{-3} + r\times 61.2\sqrt{2}\times 10^{-3}$;

We get, $r = \dfrac{10}{61.2\sqrt{2}\times 10^{-3}} - 100 \approx 15.5\,\Omega$.

Problems

I- Inductance of a Coil

A coil of inductance L and negligible internal resistance is connected in series with a capacitor of capacitance $C = 13\,\mu F$ and a resistor of resistance $R = 100\Omega$ between the terminals of a L.F.G delivering an alternating sinusoidal voltage $u_{AM} = U_m \sin(wt)$.

An oscilloscope conveniently connected, in order to visualize the voltage $u_{DM} = u_R$ across the resistor the channel Y_1 and the voltage u_{AM} across the generator on channel Y_2. The adjustments of the oscilloscope are:

✘ horizontal sensitivity: $S_h = 5ms / \text{div}$;

✘ vertical sensitivity on both channels Y_1 and Y_2 : $S_v = 5V/\text{div}$.

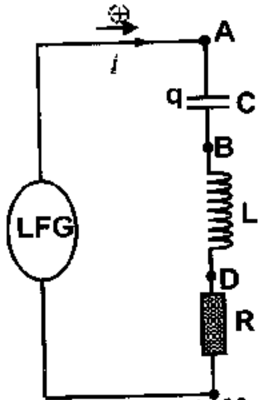


Figure 1

1. Redraw the circuit of figure 1 and show the connections of the oscilloscope.
2. Justify that the waveform (a) of figure 2, corresponds to the voltage across the generator.
3. Using figure, determine:
   a) the frequency of the voltage delivered.
   b) the maximum values of u_{AM} & u_R .
   c) the phase difference between the current i & u_{AM}
4. Show that the instantaneous expression of the current is given by $i = 0.1\sin\left(100\pi t + \frac{\pi}{4}\right)$ (where t in s & i in A).
5. a) Calculate the average power consumed by the circuit.
   b) Show that the average power consumed by the circuit is maximum, if the circuit is the seat of current resonance.
   Deduce the value of this power.
   c) Applying the law of addition of voltages and giving t a particular value, determine the inductance L.

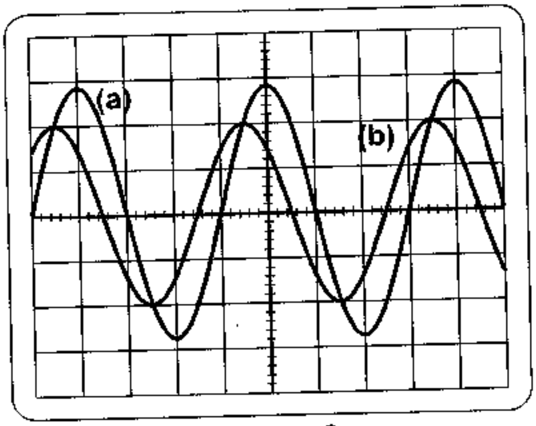


Figure 2

II- Study of an Electric Circuit

Part A

Identification of a dipole

Consider the circuit that is represented in figure 1, formed of:

✘ a generator of constant $e.m.f$ « E » and of negligible internal resistance;

✘ a resistor of resistance $R = 200\Omega$;

✘ a dipole (D) which could be:

a pure inductive coil of inductance X .

a capacitor of capacitance X taken neutral.

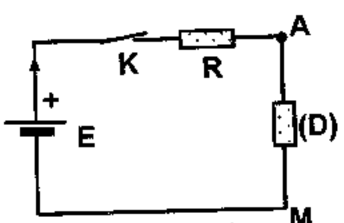


Figure 1

At the instant $t_0 = 0$, we close K.

The variation of the voltage $u_D = u_{AM}$, as a function of time, is represented by the curve of figure 2. The dashed line represents the tangent to the curve at the instant $t_0 = 0$.

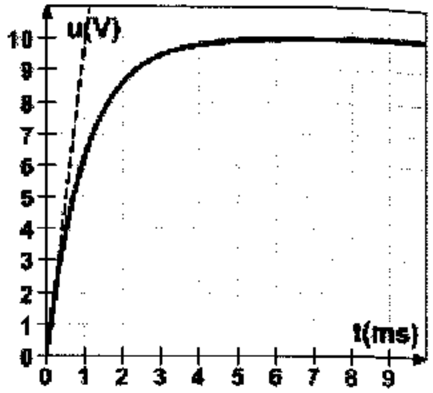


Figure 2

1. Specify the nature of the dipole.
2. Derive the differential equation that describes the variation of the voltage u_{AM} across the dipole (D).
3. The solution of the preceding differential equation is:

$$u_{AM} = U_0\left(1 - e^{-\frac{t}{\tau}}\right).$$

   a) Determine, in terms of E, R & X, the expressions of U_0 & τ.

   b) Determine graphically the values of U_0 & τ.

   c) Deduce the value of X.

Part B

Internal resistance and frequency

In order to determine the internal resistance of a coil, we consider the electric circuit represented in figure 3. This circuit is formed of a capacitor of capacitance $C = 5\mu F$, the coil of inductance $L = 0.5H$ and of resistance r, a resistor of resistance $R = 200\,\Omega$ and an ammeter (A) of negligible resistance, all connected in series across an LFG, of adjustable frequency f, that maintains across its terminals an alternating sinusoidal voltage: $u = U_m \sin(2\pi f t)$.

Thus the circuit carries an alternating sinusoidal current:

$$i = I_m \sin(2\pi f t + \varphi).$$

An oscilloscope, conveniently connected, displays the variations, as a function of time, of u_{AM} and u_{DM} on the channels (Y_1) and (Y_2) respectively.

These waveforms obtained are shown in figure 4.

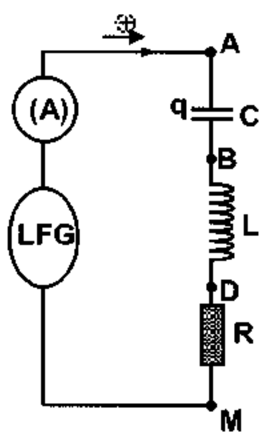
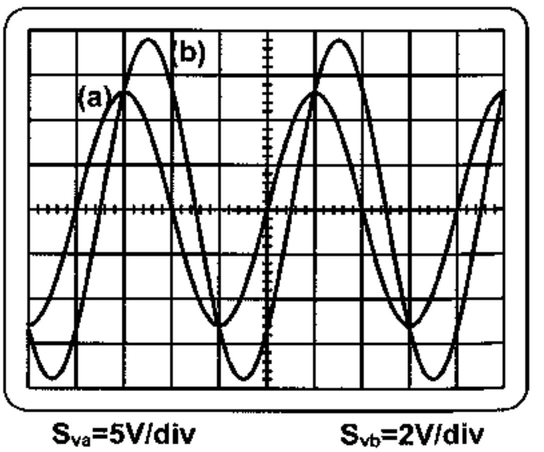


Figure 3

First situation

1. Show that the voltage across the terminals of the capacitor is $u_{AB} = -\dfrac{I_m}{2\pi f C}\cos(2\pi f t + \varphi)$.
2. Redraw Figure 3 showing the connections of the oscilloscope.
3. Show that the waveform (a) represents the voltage across the generator u_{AM}.
4. Justify that the waveform of u_{DM} represents the «image» of the current i.
5. a) Determine the phase difference φ between i and u_{AM}.

   b) Indicate, with justification, the nature (inductive, capacitive or resistive) of the circuit.



S_{va}=5V/div S_{vb}=2V/div

Figure 4

6. Applying the law of addition of voltages and giving t two particular values, show that:

a) $\frac{1}{2\pi f C} - 2\pi f L = R + r$.

b) $U_m = (R+r)\sqrt{2}\, I_m$.

7. Basing on the pervious results:

a) deduce the value of the internal resistance r.

b) show that the frequency $f \approx 70 Hz$.

Second situation

We vary the frequency f of the voltage delivered by G, keeping the same value of the maximum voltage U_m. We notice that for a certain frequency f_0, the ammeter measures the current having the maximum amplitude $I_0 = \frac{53.72}{\sqrt{2}} mA$.

1. Give the name of the physical phenomenon that takes place in the circuit.
2. Determine again the value of r.
3. The capacitor cannot withstand a voltage that overpass $20V$.
   Indicate whether there is a risk of burning it.

III-

Determination of the Characteristics of a Coil

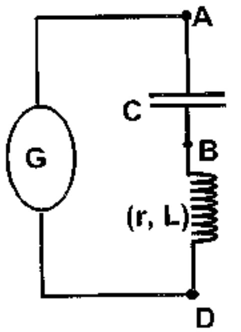


Figure 1

In order to determine the resistance r of a coil of inductance $L = 0.032\ H$, we connect it in series with a capacitor of capacitance $C = 160\ \mu F$ across the terminals of a low frequency generator (LFG) delivering an alternating sinusoidal voltage.

Take: $0.32\ \pi = 1$.

The circuit thus carries an alternating sinusoidal current i.

An oscilloscope is connected so as to display the voltage $u_G = u_{AD}$ on the channel Y_1, and the voltage across the coil $u_L = u_{BD}$ on the channel Y_2.

We see on the screen of the oscilloscope a display of the waveforms represented in figure 2.

✘ The vertical sensitivity on channel Y_1 is $S_{v_1} = 2.5 V/\text{div}$.

The voltage across the coil is given by: $u_L = 10 \sin(100\pi t)$ (u in V & t in s).

1. Redraw figure (1) showing on it the connections of the oscilloscope.
2. a) Calculate the vertical sensitivity S_{v_2} on channel Y_2.

   b) Determine the frequency of signal delivered by the LFG.
3. a) Calculate the phase difference between $u_G = u_{AD}$ and $u_L = u_{BD}$.

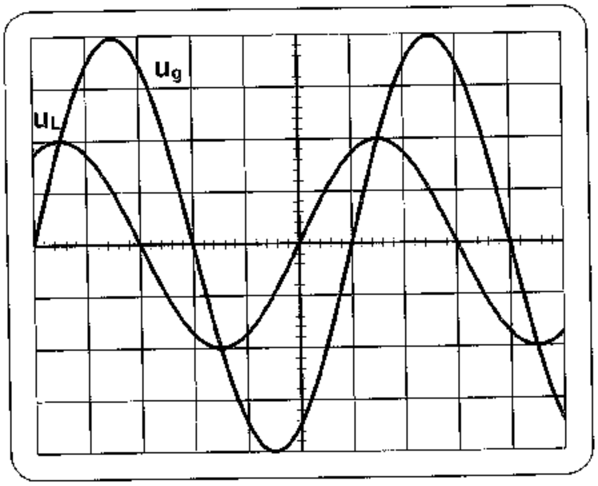


Figure 2

Which voltage leads the other?

b) Deduce the expression of the voltage u_{AD} across the terminals of the generator as a function of time.

4. By supposing that $u_C = u_{AB} = U_m \sin(100\,\pi t + \varphi)$.

Applying the law of addition of voltages and giving the time two particular values, determine the values of U_m and φ.

5. a) Deduce that the instantaneous expression of the current i as a function of time is given by:

$$i = 0.5\sin\left(100\pi t - \frac{\pi}{6}\right).$$

b) Using two expressions of the average power consumed by the coil, the expression justified in part 5 and that of u_L. Determine the value of r.

6. We keep the maximum value of u_g constant but we vary its frequency f, the effective value of the current in the circuit is maximum for certain value f_0.

a) Give the name of the physical phenomenon thus observed.

b) Determine the value of f_0.

c) The circuit is now equipped by a fuse that cannot withstand a current whose effective intensity exceeds $450\,mA$.

Specify if it is not possible to visualize the preceding physical phenomenon.

IV-Bac 2014
Characteristics of a Dipole

In order to determine the characteristics of a coil, we connect the circuit represented in figure 1.

This circuit is formed of:

- ✘ a resistor of resistance $R = 50\,\Omega$;
- ✘ a coil of inductance L and of internal resistance r;
- ✘ a capacitor of capacitance $C = 2.1\,\mu F$;
- ✘ a low frequency generator (LFG) of adjustable frequency f, delivering across its terminals an alternating sinusoidal voltage u_G whose instantaneous expression is given by $u_G = u_{AM} = 6\sin\left(2\pi f t + \frac{\pi}{4}\right)$ where t in s & u_G in V.
- ✘ two voltmeters (V_1) and (V_2) connected across the resistor and the set coil-capacitor as shown in figure 1, display the values $U_1 = 2.5\,V$ and $U_2 = 3.05\,V$.

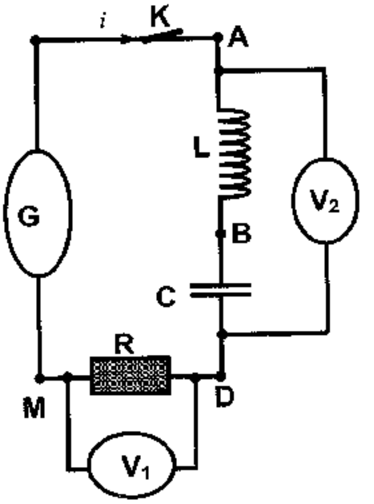


Figure 1

The instantaneous expression of the current is $i = I\sqrt{2}\sin(2\pi f t)$.

1. Determine the value of I.
2. Specify whether the circuit is capacitive, inductive or resistive.
3. a) Determine the expression of the voltage u_C across the capacitor in terms of time.

b) Deduce that the expression of the effective voltage of u_C is $U_C = \dfrac{I}{2\pi f C}$.

4. a) Write in two different forms, the expression of the power consumed in the circuit.
   b) Deduce the value of the internal resistance of the coil.
5. In what follows take $r = 10\,\Omega$, the instantaneous expression of the voltage u_{AD} can be written in the form $u_{AD} = U_2\sqrt{2}\sin(2\pi f t + \varphi)$.
   a) Write the expression of the voltage u_{AD} in terms of r, L, C, i and t.
   b) Taking two particular values of time t, show that $U_2 = I\sqrt{r^2 + \left(2\pi f L - \dfrac{1}{2\pi f C}\right)^2}$.
   c) Deduce the value of L knowing that $f = 330\,Hz$.
6. The frequency of the generator is modified so that $U_1 = 5U_2$.
   a) Referring the result (5.b), show that the circuit is the seat of current resonance.
   b) Deduce that $\dfrac{U_C}{U} = \dfrac{1}{R+r}\sqrt{\dfrac{L}{C}}$ where U is the effective voltage across the generator.
   c) Justify then that the capacitor is in the state of over voltage $(U_C > U)$.

V- Characteristics of two Dipoles

In order to determine the characteristics of a coil, we consider the electric circuit represented in figure 1.

This circuit is formed of:

- ✖ a capacitor of capacitance $C = 10\,\mu F$;
- ✖ a coil of inductance L and internal resistance r ;
- ✖ a resistor of resistance $R = 250\,\Omega$;
- ✖ and an ammeter (A) of negligible resistance,

connected in series across an LFG, of adjustable frequency f, that maintains across its terminals an alternating sinusoidal voltage: $u_{AM} = U_m \sin(2\pi f t + \varphi)$. Thus, the circuit carries an alternating sinusoidal current $i = I_m \sin(2\pi f t)$.

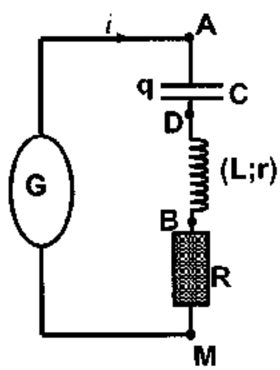


Figure 1

An oscilloscope is connected so as to display the voltage $u_G = u_{AM}$ on the channel Y_1 and the voltage $u_R = u_{BM}$ on the channel Y_2 (figure 1) and the waveforms obtained are shown in figure 2.

The adjustments of the oscilloscope are:

- ✖ horizontal sensitivity $S_h = 2.5\,ms/\text{div}$;
- ✖ vertical sensitivity on both channel $S_v = 2\,V/\text{div}$;

Part A
Studying the waveforms

1. Redraw the circuit of figure showing the connections of the oscilloscope.
2. Calculate the values of f and U_m.

3. Justify that u_{BM} is the image of the current in the circuit and then determine the maximum value I_m of the current that flows in the circuit.
4. Deduce the instantaneous expression of the current.
5. Calculate the total power consumed in the circuit and then determine again the internal resistance of the coil.
6. Applying the law of addition of voltages and giving the time two particular values, determine the values of L and r.

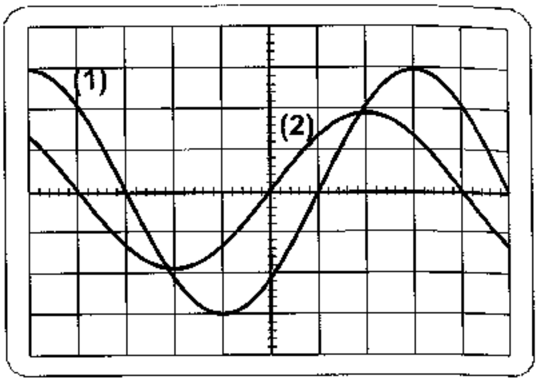


Figure 2

Part B
Particular phenomenon

The frequency is modified and we notify that for a certain frequency f_0, the voltage across the generator and that across the resistor become in phase.

1. Give the name of the physical phenomenon that takes place in the circuit.
2. Determine the value of f_0.
3. Calculate the power consumed in the circuit.

VI-
Determination of the Characteristics of Dipoles

The aim of this exercise is to determine the capacitance C of a capacitor (C), and the of inductance L a coil (L) whose internal resistance is negligible.

Part A
Alternate sinusoidal voltage

In order to determine these characteristics, we connect in series the capacitor (C), the coil (L), a resistor of resistance $R = 100\,\Omega$ across a low frequency generator (G) delivering an alternating sinusoidal voltage u_G of constant maximum value U_m and of adjustable frequency f given by: $u_G = U_m \sin(\omega t)$ where u in V & t in s.

The circuit thus formed, carries an alternating sinusoidal current i (figure 1).

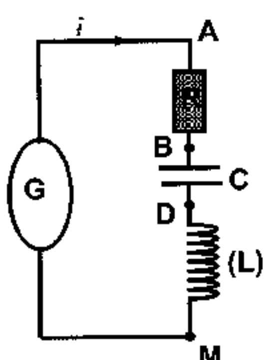


Figure 1

An oscilloscope is connected to display the voltage $u_G = u_{AM}$ across the terminals of (G) on channel (Y_1) and the voltage $u_L = u_{DM}$ across (L) on the channel (Y_2) and for a frequency $f = 159\,Hz$ the waveforms obtained are shown in figure 2.

The vertical sensitivity on both channels is $S_v = 3V/\text{div}$;

1. Recopy figure 1, showing on it the connections of the oscilloscope.

2. Calculate the phase difference between the voltages u_G and u_L, and then justify that the circuit is the seat of current resonance.
3. Show that the expression of the current in terms of time is given by $i = 0.09 \sin(318\pi t)$ where t in s & i in A.
4. Deduce the expression of the voltage u_{DM} in terms of L & t.
5. Determine L and then C.

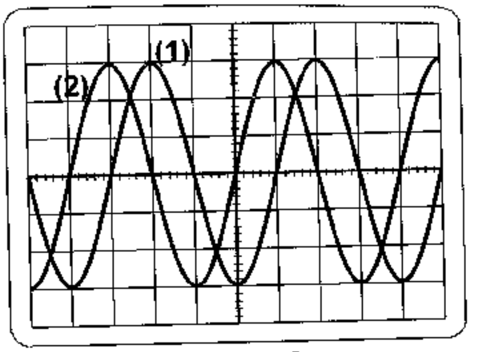


Figure 2

Part B

Triangular voltage

In order to verify the value of the inductance L of the coil, we insert it in series with the resistor across a generator delivering an alternating triangular signal as shown in figure 3. An oscilloscope is connected to display the voltage $u_R = u_{AB}$ across the terminals the resistor on channel (Y_1) and the voltage u_{BM} across the coil on the channel (Y_2) and the oscilloscope displays the signals shown in figure 4.

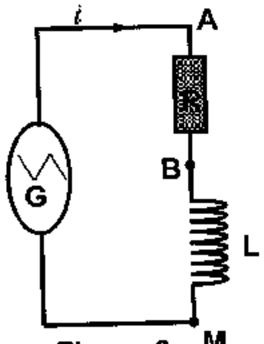


Figure 3

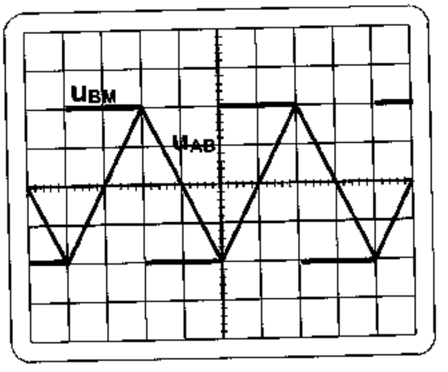


Figure 4

The vertical sensitivity on both channels is $S_v = 0.5\ V/\text{div}$;

The horizontal sensitivity is $S_h = 1 ms/\text{div}$.

1. Determine the frequency of the voltage delivered by the generator.
2. Recopy figure 3, showing the connections of the oscilloscope.
3. Show that $u_{BM} = \frac{L}{R}\frac{du_{AB}}{dt}$.
4. Consider a half-period and then calculate over it the values of u_{BM} & $\frac{du_{AB}}{dt}$.
5. Deduce the value of L.

Solutions

I-

1. On circuit of figure 1.

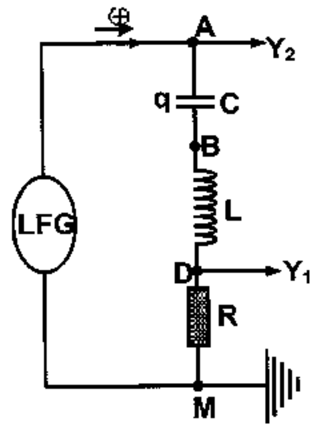


Figure 1

2. The waveform (a) corresponds to $u_G = u_{AM}$ and (b) to $u_R = u_{DM}$.
   The two channels have the same vertical sensitivities, so the signal which has the larger amplitude corresponds to that across the generator.
3. a) The period of the signal is:
   $T = S_h \times x = 5\,ms\,/\,\text{div} \times 4\text{div} = 20\,ms$;
   Its frequency is $f = \frac{1}{T} = \frac{1}{20 \times 10^{-3}} = 50\,Hz$.
   b) $U_m = S_{v_a} \times (y_a)_{\max} = 5V\,/\,\text{div} \times 2.8\text{div} = 14V$;
   $U_{Rm} = S_{v_b} \times (y_b)_{\max} = 5V\,/\,\text{div} \times 2\text{div} = 10V$;
   c) According to Ohm's law $u_R = u_{DM}$ & i are proportional then the phase difference between the waveforms (a) & (b) is that between u & i:
   Then $|\varphi| = 2\pi \frac{d}{D} = 2\pi \frac{0.5\,\text{div}}{4\,\text{div}} = \frac{\pi}{4}$ rad.
4. The maximum value of the current in the circuit: $I_m = \frac{(U_R)_m}{R} = \frac{10}{100} = 0.1A$;
   (b) leads (a) then $i = I_m \sin(wt + \varphi) = 0.1 \sin\left(100\pi t + \frac{\pi}{4}\right)$ (where t in s & i in A).
5. a) The power consumed in the circuit: $P = U\,I \cos\varphi = \frac{14}{\sqrt{2}} \times \frac{0.1}{\sqrt{2}} \cos\left(\frac{\pi}{4}\right) = \frac{7\sqrt{2}}{20} = 0.495\,W$.
   b) If the circuit is resonating then $\varphi_R = 0$; so $\cos\varphi_R = 1$ which is the maximum value of the power factor and the intensity of the current is also maximum.
   Then the power consumed by the circuit is maximum $P_{\max} = U\,I_R \cos(\varphi_R)$;
   Where $I_{R\max} = \frac{U_m}{R} = \frac{14}{100} = 0.14A$ & $I_R = \frac{I_{\max}}{\sqrt{2}} = \frac{0.14}{\sqrt{2}}A$;
   Then $P_{\max} = U\,I_R \cos(\varphi_R) = \frac{14}{\sqrt{2}} \times \frac{0.14}{\sqrt{2}} \cos 0 = 0.98\,W$.
6. Law of addition of voltages: $u_{AM} = u_{AB} + u_{BD} + u_{DM}$; then $U_m \sin wt = \frac{1}{C}\int i\,dt + L\frac{di}{dt} + Ri$;
   We have: $i = 0.1 \sin\left(100\pi t + \frac{\pi}{4}\right)$.
   Then $U_m \sin wt = -\frac{1}{C} \times \frac{1}{100\pi} \cos\left(wt + \frac{\pi}{4}\right) + L \times 0.1 \times 100\pi \cos\left(wt + \frac{\pi}{4}\right) + 10 \sin\left(wt + \frac{\pi}{4}\right)$;

$$U_m \sin wt = -\frac{10^{-3}}{\pi C}\cos\left(wt+\frac{\pi}{4}\right)+10\,\pi\,L\cos\left(wt+\frac{\pi}{4}\right)+10\sin\left(wt+\frac{\pi}{4}\right);$$

For $wt+\frac{\pi}{4}=0$; $U_m\sin\left(-\frac{\pi}{4}\right)=-\frac{10^{-3}}{\pi C}\cos(0)+10\,\pi\,L\cos(0)+10\sin(0)$; $10\,\pi\,L=\frac{10^{-3}}{\pi C}-7\sqrt{2}$;

Then $L=\frac{1}{10\,\pi}\left(\frac{1000}{13\pi}-7\sqrt{2}\right)=0.46\,H$.

II-

Part A

1. The dipole X is a capacitor, because the voltage between its terminals was zero (neutral) and increases until it reaches a steady state (complete charging).
2. Law of addition of voltages: $E=R\,i+u_{AM}$ & $i=+\frac{dq}{dt}=X\frac{du_{AM}}{dt}$; thus $E=u_{AM}+R\,X\frac{du_{AM}}{dt}$.
3. a) We have $u_{AM}=U_0\left(1-e^{-\frac{t}{\tau}}\right)$, so $\frac{du_{AM}}{dt}=\frac{U_0}{\tau}e^{-\frac{t}{\tau}}$;

   Substitution in differential equation, we get: $E=U_0+U_0\left(\frac{X\,R}{\tau}-1\right)e^{-\frac{t}{\tau}}$;

   But $U_0 e^{-\frac{t}{\tau}}\neq 0$, then $U_0=E$ & $\tau=X\,R$.

   b) U_0 is the value of u_{AM} in steady state, then $U_0=10\,V$;

   τ is the abscissa of the point of intersection of the tangent at the origin of time with the horizontal asymptote, then $\tau=1\,ms$.

   c) We have $\tau=1\,ms=X\,R$, then $X=\frac{\tau}{R}=\frac{1\times10^{-3}}{200}=5\times10^{-6}\,F=5\,\mu F$.

Part B

First situation

1. $u_{AB}=\frac{1}{C}\int i\,dt=\frac{1}{C}\int I_m\sin(2\,\pi\,f\,t+\varphi)dt$;

   $=-\frac{I_m}{2\,\pi\,f\,C}\cos(2\,\pi\,f\,t+\varphi)$.

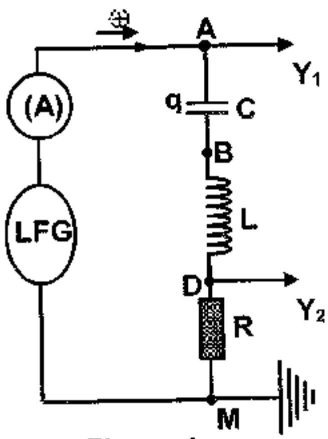


Figure 1

2. Connections on circuit.
3. $U_{a\max}=S_{va}\times y_{a\max}=5\,V/\text{div}\times2.6\,\text{div}=13\,V$;

   $U_{b\max}=S_{vb}\times y_{b\max}=2V/\text{div}\times3.8\text{div}=7.6V$;

   $U_{a\max}>U_{b\max}$; then the (a) corresponds to the voltage across the generator.
4. According to Ohm's law $u_{DM}=R\,i$, then u_{DM} & i are proportional then u_{DM} is an image of the current i.
5. a) The phase difference φ between u_{AM} and i is that between u_{AM} & u_{DM} then $\varphi=2\pi\frac{d}{D}=2\pi\frac{0.5}{4}=\frac{\pi}{4}$ rad.

   b) u_{AM} leads the current i, then the circuit is inductive.

6. Law of addition of voltages: $u_{AM} = u_{AB} + u_{BD} + u_{DM}$;

$$U_m \sin(2\pi f t) = -\frac{I_m}{2\pi f C}\cos\left(2\pi f t + \frac{\pi}{4}\right) + L 2\pi f I_m \cos\left(2\pi f t + \frac{\pi}{4}\right) + (R+r) I_m \sin\left(2\pi f t + \frac{\pi}{4}\right);$$

$$U_m \sin(2\pi f t) = \left(2\pi f L - \frac{1}{2\pi f C}\right) I_m \cos\left(2\pi f t + \frac{\pi}{4}\right) + (R+r) I_m \sin\left(2\pi f t + \frac{\pi}{4}\right)$$

a) Let $2\pi f t = 0$; we get $0 = \left(2\pi f L - \frac{1}{2\pi f C}\right) I_m \cos\left(\frac{\pi}{4}\right) + (R+r) I_m \sin\left(\frac{\pi}{4}\right)$;

$$\text{So } \left(2\pi f L - \frac{1}{2\pi f C}\right) I_m = -(R+r) I_m;\ \frac{1}{2\pi f C} - 2\pi f L = R + r.$$

b) Let $2\pi f t = \frac{\pi}{4}$; we get $U_m \frac{\sqrt{2}}{2} = (R+r) I_m$; then $U_m = (R+r)\sqrt{2}\, I_m$.

7. a) We have $U_m = (R+r)\sqrt{2}\, I_m$; $R + r = \frac{U_m}{I_m \sqrt{2}}$;

where $I_m = \frac{(u_{DM})_{\max}}{R} = \frac{7.6}{200} = 0.038\,A$;

Then $r = \frac{U_m}{I_m\sqrt{2}} - R = \frac{13}{0.038\sqrt{2}} - 200 = 42\,\Omega$.

b) We have also $\frac{1}{2\pi f C} - 2\pi f L = R + r$; $4\pi^2 LC f^2 - 2\pi(R+r)C f - 1 = 0$;

Then $4\pi^2 \times 0.5 \times 5 \times 10^{-6} f^2 - 2\pi \times 240 \times 5 \times 10^{-6} f - 1 = 0$;

We get $9.87 \times 10^{-5} f^2 + 7.603 \times 10^{-3} f - 1 = 0$;

By using a calculator we get: $f_1 = 69.3 Hz$ accepted & $f_2 = -147.3 Hz$ rejected.

Second situation

1. Current resonance.
2. Under resonance the circuit acts as a resistor so $U_m = (R+r) I_{\max}$;

where $I_{\max} = I_0\sqrt{2} = 53.72\, mA$;

Then $r = \frac{U_m}{I_{\max}} - R = \frac{13}{53.72 \times 10^{-3}} - 200 \approx 42\,\Omega$;

3. We have $u_{AB} = -\frac{I_m}{2\pi f C}\cos(2\pi f t + \varphi)$;

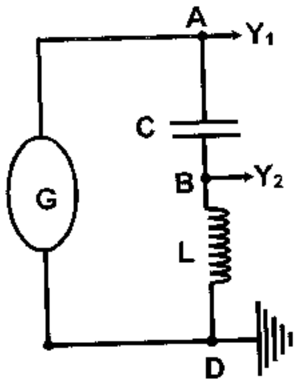
Then $U_{AB_{\max}} = \frac{I_m}{2\pi f_0 C}$;

where $f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{\sqrt{4\pi^2 LC}} = \frac{1}{\sqrt{4 \times 10 \times 0.5 \times 5 \times 10^{-6}}} = 100\ Hz$;

Thus, $U_{AB_{\max}} = \frac{I_{\max}}{2\pi f_0 C} = \frac{53.72 \times 10^{-3}}{2\pi \times 100 \times 5 \times 10^{-6}} = 17.1V < 20V$;

Therefore there is no risk of burning the capacitor.

III-



1. Connections are shown on the adjacent diagram.

2. a) The maximum value $U_{BD\max} = S_{v_2} \times y_2$;

So $S_{v_2} = \frac{U_{BD\max}}{y_{2\max}} = \frac{10\,V}{2\text{ div}} = 5\,V/\text{div}$.

b) We have $\omega = 100\,\pi\,(\text{rad}/s)$; then $f = \frac{\omega}{2\pi} = \frac{100\,\pi}{2\pi} = 50\,Hz$.

3. a) The phase difference $|\varphi| = 2\pi\frac{d}{D} = 2\pi\frac{1\text{ div}}{6\text{ div}} = \frac{\pi}{3}(\text{rad})$.

u_L reaches its maximum first, then u_L leads u_G.

b) The maximum value $U_m = S_{v_1} \times y_{1\max} = 2.5\,V/\text{div} \times 4\text{ div} = 10\text{ V}$.

Then $u_{AD} = 10\sin\left(100\,\pi t - \frac{\pi}{3}\right)$ where t in s & u_{AD} in V.

4. Law of addition of voltages $u_{AD} = u_{AB} + u_{BD}$;

$10\sin\left(100\,\pi t - \frac{\pi}{3}\right) = U_m \sin(100\,\pi t + \varphi) + 10\sin(100\,\pi t)$;

Let $100\,\pi t = 0$, we get $U_m \sin\varphi = -5\sqrt{3}$ ....(1)

Let $100\,\pi t = \frac{\pi}{2}$, $U_m \cos\varphi = -5$ ....(2)

$\frac{(1)}{(2)}$: $\tan\varphi = 1$ (but $\sin\varphi < 0$ & $\cos\varphi < 0$, so φ is the third quadrant);

So $\varphi = -\pi + \frac{\pi}{3} = -\frac{2\pi}{3}(\text{rad})$ & $U_m = 10V$;

Then $u_{BD} = 10\sin\left(100\,\pi t - \frac{2\pi}{3}\right)$ where t in s & u_{BD} in V.

5. a) We have $i = C\frac{du_C}{dt} = 160\times10^{-6}\times1000\,\pi\cos\left(100\,\pi t - \frac{2\pi}{3}\right)$.[1]

Thus, $i = 160\times10^{-6}\times1000\,\frac{1}{0.32}\sin\left(100\,\pi t - \frac{2\pi}{3} + \frac{\pi}{2}\right) = 0.5\sin\left(100\,\pi t - \frac{\pi}{6}\right)$

b) The phase difference between u_{BD} & i is $|\varphi| = \frac{\pi}{6}$ rad.

The average power consumed: $P_{av} = U_L \times I \times \cos\varphi = r \times I^2$;

Then, $\frac{10}{\sqrt{2}} \times \frac{0.5}{\sqrt{2}}\cos\left(\frac{\pi}{6}\right) = r\left(\frac{0.5}{\sqrt{2}}\right)^2$; thus $r = 10\sqrt{3}\,\Omega \approx 17.3\,\Omega$.

---

[1] If we use the other value $\varphi = +\frac{4\pi}{3}$ rad; $i = 0.5\sin\left(100\,\pi t + 4\frac{\pi}{3} + \frac{3\pi}{2}\right) = 0.5\sin\left(100\,\pi t - \frac{\pi}{6}\right)$.

6. a) The effective current is maximum then the circuit is the seat of electric current resonance.

b) Under resonance $f = f_0 = \dfrac{1}{2\pi\sqrt{LC}} = \dfrac{1}{2\pi\sqrt{0.032 \times 160 \times 10^{-6}}} \approx 70\ Hz$.

c) Under resonance the circuit as a resistor: $I_{\max} = \dfrac{U_m}{r} = \dfrac{10}{10\sqrt{3}} = 0.577\ A$.

$(I_{eff})_{\max} = \dfrac{0.577}{\sqrt{2}}\ A = 0.408\ A$; $(I_{eff})_{\max} = 408\,mA < 450\,mA$;

Thus, it is possible to visualize this phenomenon.

IV-

1. The voltmeter indicates the value of the effective voltage $U_R = 2.5\,V$;

According to Ohm's law $U_R = R\,I$, then $I = \dfrac{U_R}{R} = \dfrac{2.5}{50} = 0.05\ A$.

2. From the instantaneous expressions of the voltage across the generator u_G & the current i, we have $\varphi = +\dfrac{\pi}{4}(\text{rad})$; which indicates that u_G leads the current in the circuit.

Thus, the circuit is inductive.

3. a) The voltage across the capacitor is given by $u_C = \dfrac{1}{C}\int i\,dt$;

Then $u_C = \dfrac{1}{C}\int I\sqrt{2}\sin(2\pi f t)dt = \dfrac{-I\sqrt{2}}{C \times 2\pi f}\cos(2\pi f t)$.

b) The maximum value is $(U_C)_m = \dfrac{I\sqrt{2}}{C \times 2\pi f}$; then the effective voltage $U_C = \dfrac{I}{2\pi f C}$.

4. a) Expression «1» $P = U_G\, I\cos\varphi$ & expression «2» $P = (R + r)I^2$;

b) $P = U_G\, I\cos\varphi = (R + r)I^2$; $R + r = \dfrac{U_G\cos\varphi}{I} = \dfrac{\dfrac{6}{\sqrt{2}} \times \cos\left(\dfrac{\pi}{4}\right)}{0.05} = 60\,\Omega$;

Thus, $r = 60 - 50 = 10\,\Omega$.

5. a) We have $u_{AD} = u_C + u_{coil} = u_C + r\,i + L\dfrac{di}{dt}$;

Then, $u_{AD} = \dfrac{-I\sqrt{2}}{2\pi f C}\cos(2\pi f t) + r\,I\sqrt{2}\sin(2\pi f t) + LI\,2\pi f\sqrt{2}\cos(2\pi f t)$;

b) Law of uniqueness of voltages:

$u_{AD} = U_2\sqrt{2}\sin(2\pi f t + \varphi) = \dfrac{-I\sqrt{2}}{2\pi fC}\cos(2\pi f t) + r\,I\sqrt{2}\sin(2\pi f t) + LI\,2\pi f\sqrt{2}\cos(2\pi f t)$

Let $2\pi f t = 0$; $U_2\sqrt{2}\sin(\varphi) = -\dfrac{I\sqrt{2}}{2\pi f C} + LI\,2\pi f\sqrt{2}\cos(2\pi f t)$;

$$U_2\sin(\varphi) = \left(2\pi f L - \frac{1}{2\pi f C}\right)I \quad \text{.........(1)}$$

Let $2\pi f t = \frac{\pi}{2}$; $U_2\sqrt{2}\cos(\varphi) = r I\sqrt{2}$; thus $U_2 \cos(\varphi) = r I$ ......(2)

$(1)^2 + (2)^2$: $U_2^2\left(\cos^2\varphi + \sin^2\varphi\right) = r^2 I^2 + \left(2\pi f L - \frac{1}{2\pi f C}\right)^2 I^2$;

Then $U_2 = I\sqrt{r^2 + \left(2\pi f L - \frac{1}{2\pi f C}\right)^2}$.

c) Referring to the previous relation: $2\pi f L - \frac{1}{2\pi f C} = \sqrt{\left(\frac{U_2}{I}\right)^2 - r^2}$;

Then $660\,\pi L - \frac{1}{660\,\pi \times 2.1\times 10^{-6}} = \sqrt{\left(\frac{3.05}{0.05}\right)^2 - 10^2}$; thus $L = 0.14\,H$.

6. a) We have $U_1 = 5U_2$, $U_2 = \frac{U_1}{R}\sqrt{r^2 + \left(2\pi f L - \frac{1}{2\pi f C}\right)^2}$;

We get: $U_2 = \frac{5U_2}{R}\sqrt{r^2 + \left(2\pi f L - \frac{1}{2\pi f C}\right)^2}$; $R^2 = 25r^2 + 25\left(2\pi f L - \frac{1}{2\pi f C}\right)^2$;

So, $2500^2 = 2500 + 25\left(2\pi f L - \frac{1}{2\pi f C}\right)^2$; then $2\pi f L - \frac{1}{2\pi f C} = 0$;

Thus, the circuit is then the seat of current resonance.

b) Under resonance, the circuit acts as resistor $U = (R + r)I$ & $U_C = \frac{I}{2\pi f C}$.

So, $\frac{U_C}{U} = \frac{\frac{I}{2\pi f C}}{(R+r)I} = \frac{1}{2\pi f C(R+r)}$ & the frequency $2\pi f = \frac{1}{\sqrt{LC}}$;

Then $\frac{U_C}{U} = \frac{\sqrt{LC}}{C(R+r)} = \frac{1}{R+r}\sqrt{\frac{L}{C}}$.

c) $\frac{U_C}{U} = \frac{1}{50+10}\sqrt{\frac{0.14}{2.1\times 10^{-6}}} = 4.3 > 1$; then $U_C > U$;

Thus, the capacitor is in the state of overvoltage.

V-

Part A

1. Connections of the oscilloscope.
2. The period of the signal displayed: $T = S_h \times x$;

   Then, $T = 2.5\,ms/\text{div} \times 8\text{div} = 20\,ms$;

   Thus, the frequency $f = \frac{1}{T} = \frac{1}{20\times 10^{-3}} = 50\,Hz$.

   $U_m = S_v \times y_{1m} = 2V/\text{div} \times 3\text{div} = 6V$.
3. According to Ohm's law: $u_{BM} = Ri$, then u_{BM} is proportional to the current.

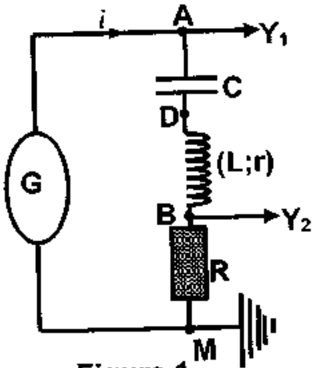


Figure 1

Thus u_{BM} is the image of the current in the circuit.

$$I_m = \frac{(u_{BM})_m}{R} = \frac{S_v \times y_{2m}}{R} = \frac{2V/\text{div} \times 1.9\text{div}}{250} = 0.0152\ A.$$

4. We have $i = I_m \sin(2\pi f t) = 0.0152 \sin(100\pi t)$ where t in s & i in A.
5. The phase difference between the voltage across generator and the current in the circuit is:

$$|\varphi| = 2\pi\frac{d}{D} = 2\pi\frac{1\,\text{div}}{8\,\text{div}} = \frac{\pi}{4}(\text{rad}).$$

Then the power consumed: $P = U_G I \cos\varphi = \frac{6}{\sqrt{2}} \times \frac{0.0152}{\sqrt{2}} \cos\left(\frac{\pi}{4}\right) = 0.032\,W$.

But $P = R_t I^2$; $R_t = \frac{P}{I^2} = \frac{0.032}{(0.0152/\sqrt{2})^2} = 277\,\Omega$.

Thus, $r = R_t - R = 277 - 250 = 27\,\Omega$.

6. We have $u_G = U_m \sin(2\pi f t + \varphi)$ but $\varphi = -\frac{\pi}{4}(\text{rad})$ since u_G lags behind i;

Then $u_G = 6\sin\left(100\pi t - \frac{\pi}{4}\right)$ where t in s & u_G in V.

& $u_{BM} = R\,i = 250 \times 0.0152 \sin(100\pi t) = 3.8\sin(100\pi t)$;

$$u_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int 0.0152\sin(100\pi t)dt = -\frac{0.0152}{10\times10^{-6}\times100\pi}\cos(100\pi t) = -\frac{15.2}{\pi}\cos(100\pi t);$$

$$u_{DB} = ri + L\frac{di}{dt} = 0.0152\,r\sin(100\pi t) + 1.52\pi L\cos(100\pi t);$$

Law of addition of voltages: $u_{AM} = u_{AB} + u_{BD} + u_{DM}$;

$$6\sin\left(100\pi t - \frac{\pi}{4}\right) = -\frac{15.2}{\pi}\cos(100\pi t) + 0.0152\,r\sin(100\pi t) + 1.52\pi L\cos(100\pi t) + 3.8\sin(100\pi t)$$

Let $100\pi t = 0$, we get $-3\sqrt{2} = -\frac{15.2}{\pi} + 0 + 1.52\pi L + 0$;

Then $L = \left(-3\sqrt{2} + \frac{15.2}{\pi}\right)/(1.52\pi) \approx 0.12\,H$;

Let $100\pi t = \frac{\pi}{2}$, we get $3\sqrt{2} = 0 + 0.0152\,r + 0 + 3.8$; then $r = \frac{(3\sqrt{2} - 3.8)}{0.0152} \approx 29\,\Omega$.

Part B

1. The circuit is the seat of current resonance, since the voltage across the generator is in phase with the current.
2. The frequency of the generator is equal to the proper frequency of the circuit;

$$f = f_0 = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi} \times \frac{1}{\sqrt{0.1\times10\times10^{-6}}} \approx 159\,Hz.$$

3. Under resonance the circuit acts as a resistor;

Then $I_{\max} = \frac{U_m}{R + r} = \frac{6}{277} \approx 0.02\,A = 20\,mA$.

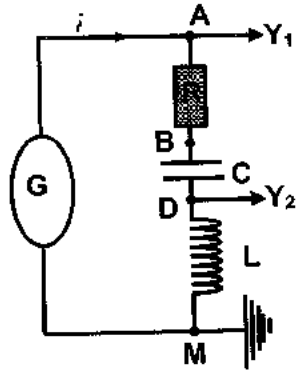
VI-

Part A

1. Connections are shown on circuit.

2. The phase difference $|\varphi| = 2\pi\frac{d}{D} = 2\pi\frac{1\text{div}}{4\text{div}} = \frac{\pi}{2}(\text{rad})$.

We know that the voltage across a pure inductive coil leads the current by $\frac{\pi}{2}(\text{rad})$, referring to the waveforms of figure 2, the voltage across the generator u_{AM} leads also the voltage across the coil u_{DM} by $\frac{\pi}{2}(\text{rad})$.



Then the voltage across the generator is in phase with the current in the circuit.
Thus, the circuit is the seat of current resonance.

3. Under resonance the current is in phase with the voltage across the generator then $i = I_m \sin(\omega t)$ where $\omega = 2\pi f = 2\pi \times 159 = 318\pi(\text{rad}/s)$;

The circuit, in this case, acts as a resistor; so $I_m = \frac{U_m}{R} = \frac{S_v \times y_{1\max}}{R} = \frac{3V/\text{div} \times 3\text{div}}{100} = 0.09\ A$;

Then $i = 0.09\sin(318\pi t)$ where t in s & i in A.

4. We have $u_L = u_{DM} = L\frac{di}{dt} = L \times 0.09 \times 318\pi\cos(318\pi t) = 28.62\pi L\cos(318\pi t)$ where t in s & u_L in V.

5. The maximum value of the voltage across the coil is $(u_L)_{\max} = 28.62\pi L\ (V)$;

From the waveforms, $(u_L)_{\max} = S_v \times y_{2\max} = 9V$;

Then $L = \frac{(u_L)_{\max}}{28.62\pi} = \frac{9}{28.62\pi} \approx 0.1H$.

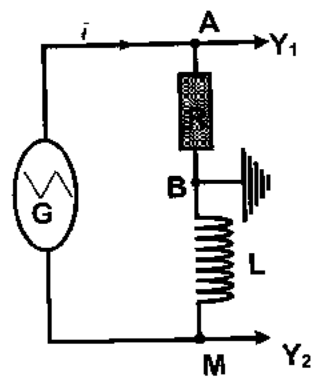
At resonance, the frequency of the voltage delivered by the generator is equal to the proper frequency of the circuit, so $f = f_0 = \frac{1}{2\pi\sqrt{LC}}$;

So, $C = \frac{1}{4\pi^2 L f_0^2} = \frac{1}{4\pi^2 \times 0.1 \times 159^2} = 1\times10^{-5}F = 10\ \mu F$.

Part B

1. The period $T = S_h \times x = 1ms/\text{div} \times 4\text{div} = 4ms$;

& the frequency $f = \frac{1}{T} = \frac{1}{4\times10^{-3}} = 250\ Hz$.



2. Connections are shown on the adjacent circuit.
Furthermore, the knob «INV» should be pressed on Channel 2 (the adjacent connections displays u_{BM} on channel 2).

3. According to Ohm's law: $u_{AB} = Ri$ & $u_{BM} = L\frac{di}{dt}$;

Then $u_{BM} = L\frac{d}{dt}\left(\frac{u_{AB}}{R}\right) = \frac{L}{R}\frac{du_{AB}}{dt}$.

4. We have $u_{BM} = S_v \times y_{BM} = 0.5V/\text{div} \times 2\text{div} = 1V$;

& $\frac{du_{AB}}{dt} = \frac{\Delta u_{AB}}{\Delta t} = \frac{4\text{ div} \times 0.5V/\text{div}}{2\text{div} \times 1ms/\text{div}} = \frac{2V}{2\times10^{-3}s} = 1000\ V/s$.

5. We have $u_{BM} = \frac{L}{R}\frac{du_{AB}}{dt}$, so $1 = \frac{L}{100} \times 1000$; thus $L = 0.1H$.

Supplementary Problems

I-LS & GS 2001 1st
Characteristics of an Electric Dipole

In order to determine the inductance L of a coil of negligible resistance, we connect this coil in series with a resistor of resistance $R = 10\,\Omega$ across a low frequency generator G (figure 1).

The generator G delivers an alternating sinusoidal voltage $u_G = U_m \cos wt$ (u in V and t in s).

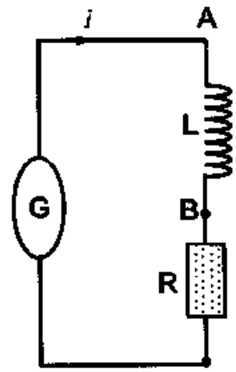


Figure 1

✘ Horizontal sensitivity: $5\,ms\,/\,\text{div}$;

✘ Vertical sensitivity on the two channels: $1V\,/\,\text{div}$.

1. Redraw the diagram of figure 1, showing the connections of the channels of an oscilloscope that allow us to display the voltages $u_G = u_{AD}$ across the generator and $u_R = u_{BD}$ across the resistor.
2. Which one of the two voltages u_G or u_R represents the image of the current in the circuit. Justify.
3. In figure 2, the waveform (1) displays the variations of the voltage u_G as a function of time.
   a) Justify that the waveform (1) corresponds to the voltage u_G.
   b) Determine the phase difference between the two waveforms.
4. Referring, to the waveforms, calculate the angular frequency w and the maximum value U_m of the voltage across the terminals of the generator G.
5. Show that the instantaneous expression of the current i as a function of time $i = 0.28\cos(80\pi t - 0.24\pi)$ and deduce the expression of the voltage u_L across the coil.
6. Applying the law of addition of voltages, and giving t a particular value, determine the value of L.

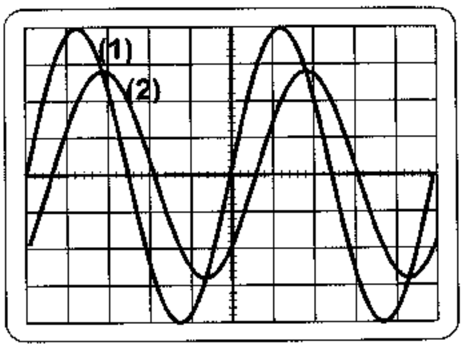


Figure 2

Answer Key

3. b) $|\varphi| = 0.24\pi\,(\text{rad})$ 4. $w = 80\pi\,(\text{rad/s})$ 5. $I_m = 0.28\,A$ 6. $L \approx 41\,mH$.

II-GS 2001 1st
Characteristics of a Coil

We want to determine, using two methods, the inductance L and the resistance r of a coil (B).

Part A
DC source

We place the coil (B) in a circuit formed of: a resistor of resistance $R = 50\,\Omega$, a battery whose *e.m.f* is $E = 6V$ and whose internal resistance is negligible, a switch K & an ammeter (A).

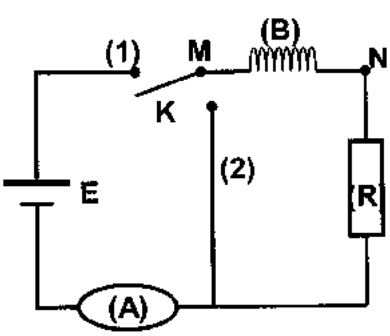


Figure 1

We close the circuit by pushing the switch into position (1). The ammeter indicates a current i_1.

1. a) In transient state, write the expression of the voltage u_{MN} between the terminals of the coil.

   b) In steady state, the ammeter indicates $I_0 = 100\ mA$. Which characteristic of the coil (L or r) can we determine? Justify. Determine its value.

At the instant $t_0 = 0$, taken as origin for time and in a very brief time, we turn K to position (2), assuming there was no loss of energy.

2. a) Derive the differential equation that governs the variation of the intensity i_2 of the current in this new circuit.

   b) Verify that $i_2 = I_0 e^{-\frac{t}{\tau}}$ where $\tau = \dfrac{L}{R+r}$ is a solution for this equation.
   Calculate then the value I of i_2 for $t = \tau$.

   c) The graph of figure 2 represents the variation of i_2 as a function of time.
   Referring to this graph, determine the value of τ.
   Deduce then the value of the other characteristic of the coil.

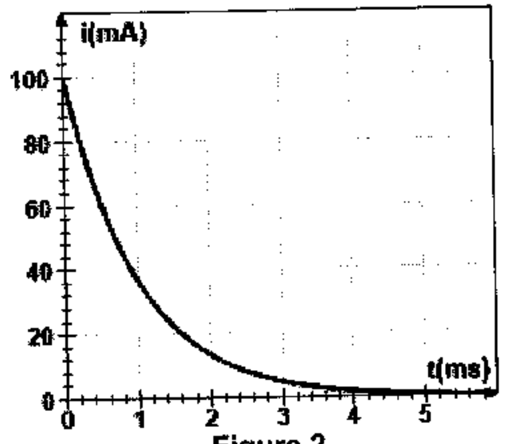


Figure 2

Part B
Sinusoidal voltage

To verify the values of r and L found in Part A above, we connect in series the coil (B), the resistor of resistance R and a capacitor of capacity $C = 47\ \mu F$ to the terminals of a L.F.G delivering an alternating sinusoidal signal as shown in figure 3.

✗ Horizontal sensitivity is 2ms/div;

✗ Vertical sensitivity on Y_1 is 2V/div & vertical sensitivity on Y_2 is 5V/div.

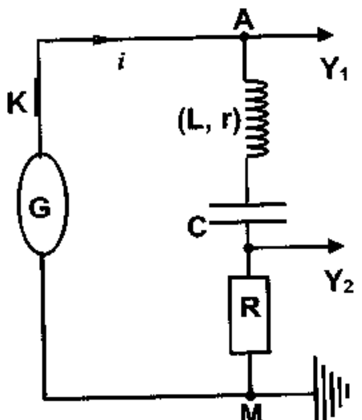


Figure 3

We display on the screen of an oscilloscope, the voltage $u_G = u_{AM}$ between the terminals of the generator on channel Y_1 and the voltage u_R across the terminals of the resistor on channel Y_2. For a specific value of f, we obtain the two waveforms shown in figure 4.

1. The two waveforms show a physical phenomenon. Which one? Justify?
2. Determine the corresponding value of f and deduce the value of L.
3. Determine the maximum value U_m of the voltage u_G and I_m of the current i.
   Deduce the value of r knowing that, in this case, we have $\dfrac{U_m}{I_m} = R + r$.

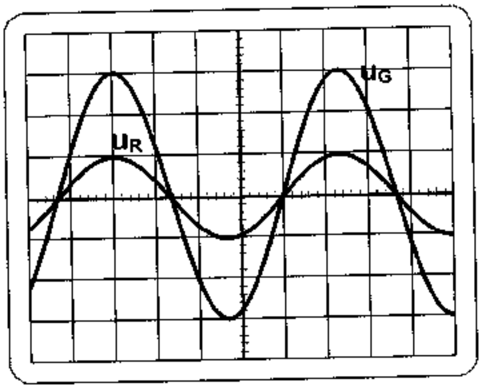


Figure 4

Answer Key

Part A 1.b) $r = 10\,\Omega$ 2.b) $I = 37\,mA$ 2.c) $L = 0.06H$.

Part B 2. $f_0 \approx 94Hz$ $L \approx 60\,mH$ 3. $I_m = 0.1A$.

III-LS & GS 2002 1st

Determination of the Capacitance of a Capacitor

In order to determine the capacitance C of a capacitor, we use the following components:

- ✘ a low frequency generator (L.F.G) delivering an alternating sinusoidal voltage: $u_G = U_m \sin wt$ (u in V and t in s);
- ✘ a resistor of resistance $R = 50\,\Omega$;
- ✘ a coil of inductance $L = 0.16\,H$ and of negligible resistance;
- ✘ an oscilloscope and connecting wires $(0.32\,\pi = 1)$.

Part A

First experiment

We connect the capacitor is series with the resistor across the L.F.G. The oscilloscope is used to display the voltage u_G across the L.F.G on the channel Y_1 and the voltage u_R across the resistor on the channel Y_2. The adjustments of the oscilloscope are:

- ✘ vertical sensitivity: $2V/\text{div}$ on both channels.
- ✘ horizontal sensitivity: $5ms/\text{div}$.

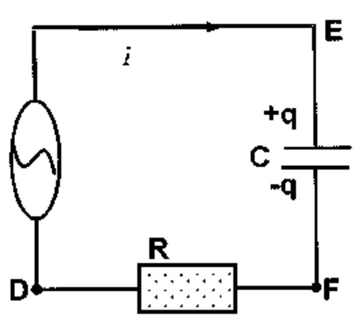


Figure 1

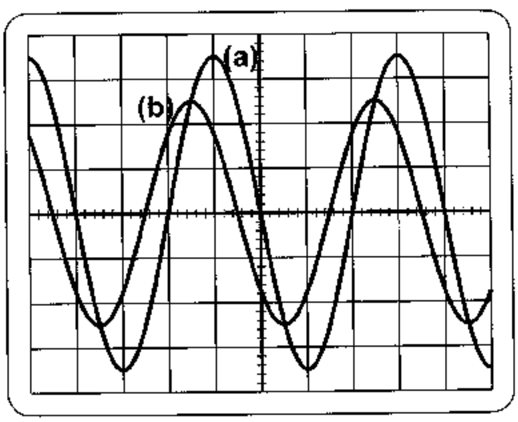


Figure 2

1. Redraw the circuit of figure 1, and show the connections of the oscilloscope in order to display the voltages $u_G = u_{ED}$ across the generator and $u_R = u_{FD}$ across the resistor.
2. The waveforms displayed are represented in figure 2.
   a) Show that the curve (a) represents u_G.
   b) Determine the frequency of the voltage u_G and the phase difference between u_G and u_R.
   c) Calculate the values of U_m and w, write the expressions of u_G and u_R as a function of time and deduce the expression of the instantaneous current i in the circuit.
   d) Knowing that u_C is the voltage across the capacitor, show that u_C is given by :
   $$u_C = \frac{3.2\times10^{-4}}{C}\sin\left(wt - \frac{\pi}{4}\right).$$
   e) Determine the value of C using the law of addition of voltages by taking a particular value of the time t.

Part B

Second experiment

We insert the coil in series with the above circuit. We thus obtain an RLC circuit and we keep the same connections of the oscilloscope. We observe only one waveform on the screen (the two waveforms are confounded). The above result shows evidence of an electric phenomenon that took place.

Name this phenomenon and calculate again the value of the capacitance C.

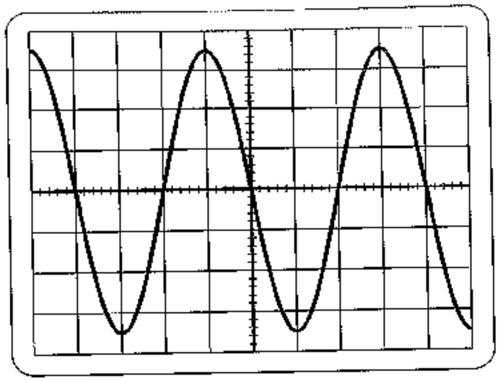


Figure 3

Answer Key

Part A 2. b) $f = 50\,Hz$. 2.c) $u = 7\sin(100\,\pi\,t)$ 2.e) $C = 64\,\mu F$

Part B $C = 64\,\mu F$

IV-LS & GS 2003 1st

Role and Characteristics of a Coil

Consider a coil (B) that bears the following indications $L = 65\,mH$ & $r = 20\,\Omega$.

Part A

Role of a coil

In order to show the role of a coil, we connect the coil across a generator G_1.

The variation of the current i carried by the coil as a function of time is represented in figure 1.

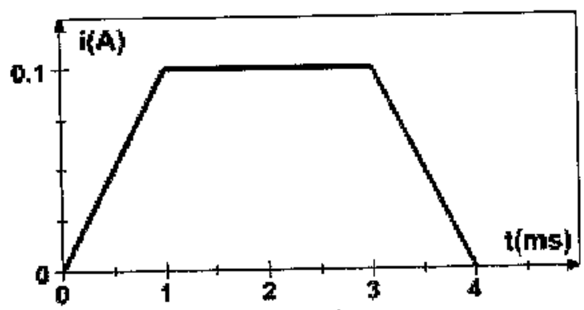


Figure 1

1. a) Give, in terms of L and i, the literal expression of the induced electromotive force e produced across the coil.
   b) Determine the value of e in each of the following time intervals $[0\,;1\,ms]$, $[1\,ms\,;3\,ms]$, $[3\,ms\,;4\,ms]$.
2. In what time would the coil act as a generator? Justify your answer.

Part B

Characteristics of a coil

In order to verify the values of L and r, we perform the two following experiment.

I-First experiment

The coil (B), a resistor of resistance $R = 20\,\Omega$ and an ammeter of negligible resistance are connected in series across a generator G_2 of electromotive force $E = 4\,V$ and negligible internal resistance (figure 2).

After a certain time, the ammeter reads $I = 0.1\,A$.

Deduce the value of r.

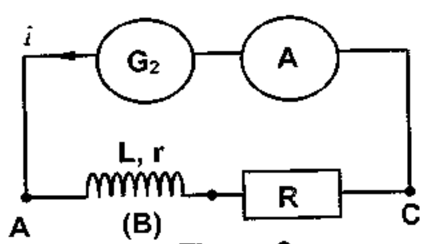


Figure 2

II-Second experiment

The ammeter is removed and G_2 is replaced by a generator G_3 delivering an alternating sinusoidal voltage.

1. Redraw figure (2) and show on it the connections of an oscilloscope that allows to display, on channel (1), the voltage u_G across the generator and, on channel (2), the voltage u_R across the resistor.
2. The voltages on the oscilloscope are represented on figure (3).
   - ✘ Vertical sensitivity on both channels: $2V$ / div.
   - ✘ Horizontal sensitivity: $1ms$ / div.

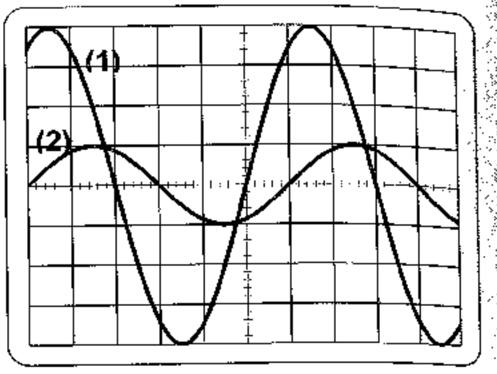


Figure 3

a) The waveform (1) represents u_G. Why?

b) The voltage across the generator has the form $u_G = U_m \cos wt$. Determine U_m and w.

c) Determine the phase difference φ between u_G and u_R.

d) Determine the expression of the instantaneous current i carried by the circuit.

e) Applying the law of addition of voltages and giving t a particular value, deduce the value of the inductance L.

Answer Key

Part A 1.b) $-6.5V$, 0.

Part B $r = 20\,\Omega$. 2. b) $u_G = 8\cos(1000\,\pi t/3)$ 2. e) $L \approx 66\,mH$.

V-LS & GS 2004 2nd

Identification of Electric Components

We intend to identify each of two electric components D_1 and D_2, one of them being a capacitor of capacitance C, and the other a coil of inductance L and of resistance r.

In order to do that, we consider a function generator (LFG) delivering an alternating sinusoidal voltage whose effective value is kept constant throughout the whole problem, an oscilloscope, a resistor of resistance $R = 10\,\Omega$, and connecting wires.

We connect up the circuit represented in figure 1, the component D may be either D_1 or D_2.

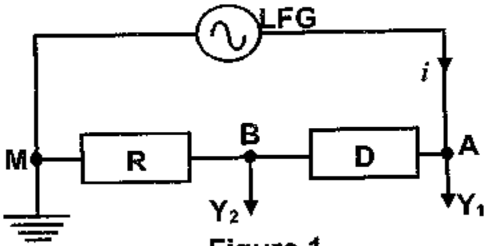


Figure 1

The figures (2) and (3) show the waveforms of each of the voltages u_{AM} and u_{BM}.

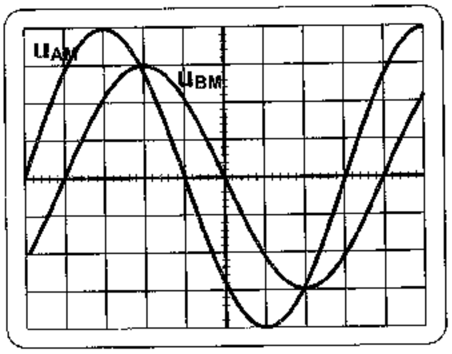


Figure 2

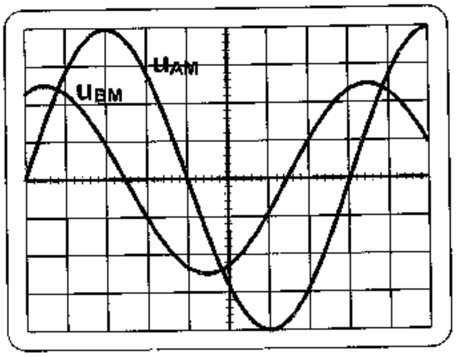


Figure 3

Given:

✘ Horizontal sensitivity: 1ms/div.

✘ Vertical sensitivity on (Y_1): 2V/div.

✘ Vertical sensitivity on (Y_2): 1V/div.

Part A

Nature of D_1

The waveform of figure (2) corresponds to the case when the component D is D_1.

Justify that D_1 is the coil.

Part B

Characteristics (L, r) of the coil

1. a) Determine the period of the voltage delivered by the LFG and deduce its angular frequency w.
   b) Determine the maximum values of the voltages u_{AM} & u_{BM}.
   c) Calculate the phase difference φ between the voltage u_{AM} and the current i carried by the circuit.
2. Knowing that the current i is given by the expression $i = I_{1m}\cos(wt)$, determine:
   a) the expression of u_{BM}, u_{AB} & u_{AM} as a function of time.
   b) the value of I_{1m}.
3. By applying the law of addition of voltages, and giving t two particular values determine the values of r and L.

Part C

Capacitance C of the capacitor

D_2 is now connected between A and B, the expression of the voltage u_{AB} is, in this case $u_{AB} = \dfrac{I_{2m}}{Cw}\sin wt$.

1. Verify that the expression of the current is: $i = I_{2m}\cos(wt)$.
2. Show that the expression of u_{AM} is given by $u_{AM} = 8\cos\left(wt - \dfrac{3\pi}{8}\right)$
3. Determine the value of C.

Answer Key

Part B 2.a) $u_{AM} = 8\cos\left(250\,\pi t + \dfrac{\pi}{4}\right)$ 3. $r = 8.85\,\Omega$ & $L = 24mH$.

Part C 3. $C = 43\,\mu F$.

I-LS 2014 1st

Identification of two Electric Components

Consider two electric components (D_1) and (D_2), a generator G delivering an alternating sinusoidal voltage of angular frequency $w = 100\,\pi$ (rad/s) and a resistor (R) of resistance $R = 100\,\Omega$. One of the components is a coil of inductance L and of negligible resistance; the other is a capacitor of capacitance C.

Take $\pi = \frac{1}{0.32}$.

Part A

Characteristics of the components (D$_1$)

We connect in series the component (D_1), the generator (G) and the resistor (R) (Figure 1).

An oscilloscope is used to display, on channel Y_1, the voltage u_{AM} across (D_1) and, on the channel Y_2, the voltage u_{MB} across the resistor, the button «INV» of channel Y_2 is pushed. The obtained waveforms are represented in figure 2.

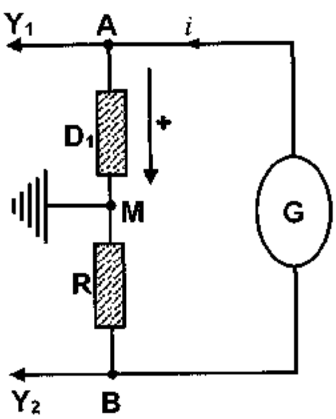
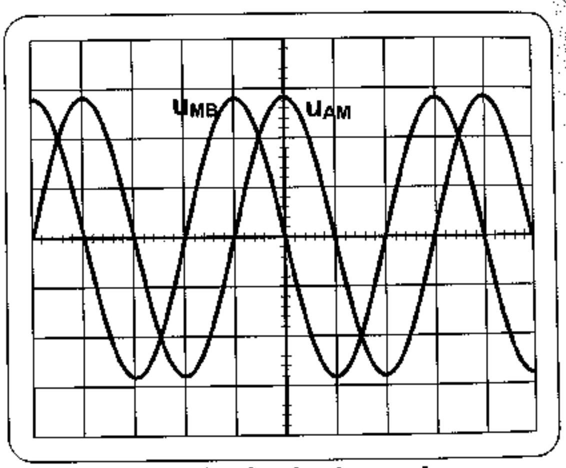


Figure 1



Sv=5V/div for both channels

Figure 2

1. Using the waveforms of figure 2, show that (D_1) is a capacitor.
2. Referring to the waveforms in figure 2. Determine:
   a) the maximum value $U_{m(R)}$ of the voltage u_{MB} and deduce the maximum value I_m of the current i carried by the circuit;
   b) the maximum value $U_{m(D_1)}$ of the voltage u_{AM}.
3. Knowing that the expression of i is $i = I_m \cos(wt)$, show that the expression of u_{AM} is of the form $u_{AM} = \frac{I_m}{Cw} \sin wt$.
4. Deduce the value of C.

Part B

Characteristics of the components (D₂)

(D_2) is then a coil. We connect the set-up of figure 3. We display the voltage $u_{AM} = u_G$ on channel Y_1 and the voltage u_{BM} on channel Y_2. The obtained waveforms are shown on figure 4.

1. Show that the curve (a) represents u_G.
2. Referring to the waveforms of figure 4, determine:
   a) the maximum value $U_{m(R)}$ of the voltage u_{BM} across the resistor and deduce the maximum value I_m of the current i carried by the circuit;
   b) the maximum value $U_{m(G)}$ of the voltage across the generator.
   c) the phase difference φ between the current i and the voltage u_G across the generator.

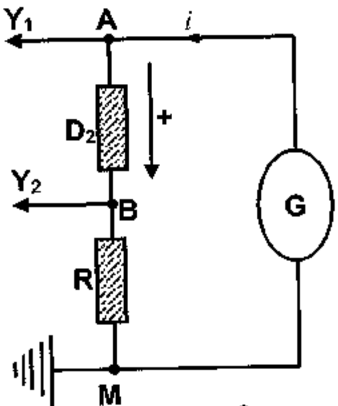
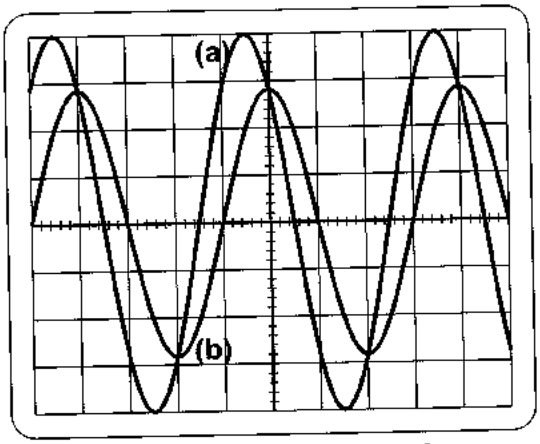


Figure 3



Sv=5V/div for both channels
Figure 4

3. Knowing that $i = I_m \cos(wt)$:
   a) determine the expression of the voltage u_{AB} across the coil in terms of L, I_m, w and t;
   b) write the expression of the voltage u_G as a function of time.
4. Applying the law of addition of voltages between A and M and giving t a particular value, determine the value of L.

II-LS 2013 1st

Determination of the Characteristics of a Coil and a Capacitor

The aim of this exercise is to determine the characteristics of a capacitor and a coil.

In order to determine these characteristics, we connect in series a capacitor of capacitance C, a coil of inductance L and of resistance r, a resistor of resistance $R = 20\,\Omega$ and a low frequency generator delivering an alternating sinusoidal voltage u of constant maximum value U_m and of adjustable frequency f.

The circuit thus formed, carries an alternating sinusoidal current i (figure 1).

An oscilloscope is connected to display the voltage $u = u_{AM}$ across the terminals of the (LFG) on channel (Y_1) and the voltage u_{BM} across the resistor (R) on the channel (Y_2).

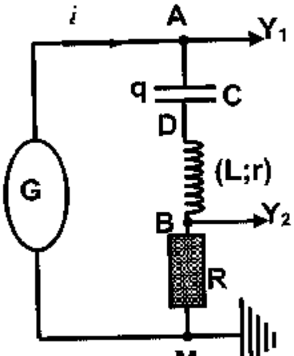


Figure 1

The adjustments of the oscilloscope are as follows:

✘ horizontal sensitivity: $S_h = 2ms/\text{div}$.

✘ vertical sensitivity on (Y_1): $S_{v_1} = 2V/\text{div}$.

✘ vertical sensitivity on (Y_2): $S_{v_2} = 0.25\ V/\text{div}$.

Part A

For a given value f_0 of the frequency f we observe on the screen of the oscilloscope he waveforms represented by figure 2.

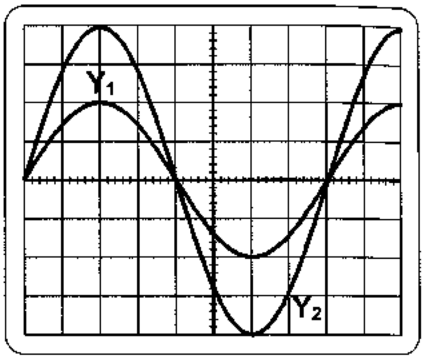


Figure 2

1. Determine f_0 and the proper angular frequency w_0.
2. Determine the maximum value U_m of u and the maximum current I_m of i.
3. a) The waveforms show that a physical phenomenon that takes place in the circuit.
   Name this phenomenon. Justify.
   b) Deduce the relation between L and C.
4. The circuit between A and M is equivalent to a resistor of resistance $R_t = R + r$. Determine R_t and deduce r.

Part B

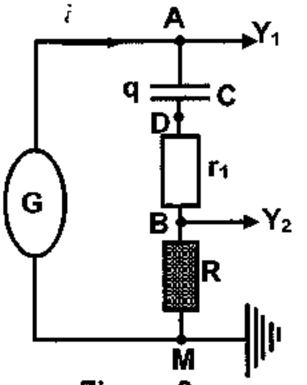


Figure 3

The coil is replaced by a resistor of resistance $r_1 = 60\,\Omega$.

The voltage between the terminals of the generator is given by: $u = u_{AM} = U_m \cos(w_0\, t)$. On the screen, we observed the oscillograms shown in figure 4.

The setting of the oscilloscope are not changed.

1. Using the waveforms of figure 4:
   a) tell why the voltage u_{AM} lags behind u_{BM}.
   b) calculate the phase difference φ between u_{AM} and u_{BM}.
   c) determine the expressions of u_{BM} and u_{AM} as a function of time t.
2. Write down the expression of i in terms of time t.
3. The voltage across the terminals of the capacitor is:

   $$u_C = u_{AD} = \frac{8.9\times10^{-5}}{C}\sin\left(125\pi t + \frac{\pi}{4}\right); (u \text{ in } V;\ t \text{ in } s).$$

   By applying the law of addition of voltage and giving t a particular value, calculate the value of C.

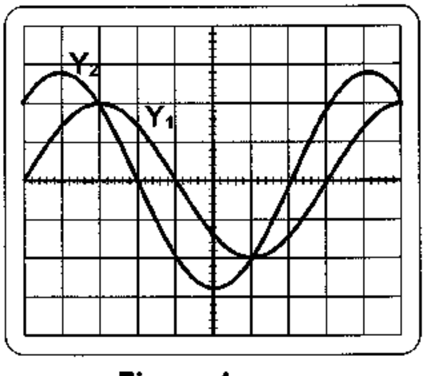


Figure 4

Part C

Using the relation found in [A-3(b)], calculate L.

III-LS 2011 2nd

Determination of the Inductance of a Coil

In order to determine the inductance L of a coil whose internal resistance is negligible, we connect this coil in series with a resistor of resistance $R = 10\,\Omega$ across a generator G (Figure 1).

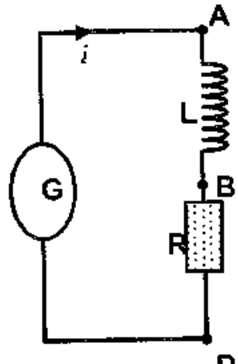


Figure 1

The generator G delivered an alternating sinusoidal voltage: $u_{AD} = u_G = U_m \cos(wt)$ (u_G in V, t in s).

The circuit is then traversed by a current whose intensity is i.

1. Reproduce the schema of figure (1), indicating the connections of the oscilloscope in order to visualize the voltage u_G across a generator and the voltage $u_R = u_{BD}$ across the resistor.
2. Which of the two voltages represents the image of i? Justify your answer.
3. In figure 2, the oscillogram (1) represents the evolution of u_G as a function of time.
   - ✘ Horizontal sensitivity: $5\,ms/\text{div}$.
   - ✘ Vertical sensitivity on the two channels: $1V/\text{div}$.

   a) Indicate, with justification, which of the oscillograms, (1) or (2), leads the other.

   b) Determine :
      i- the phase difference between the two oscillograms.
      ii- the pulsation w.
      iii-the maximum value U_m of the voltage across G.
      iv- the amplitude I_m of i.

   c) Write the expression of the intensity i as a function of time t.

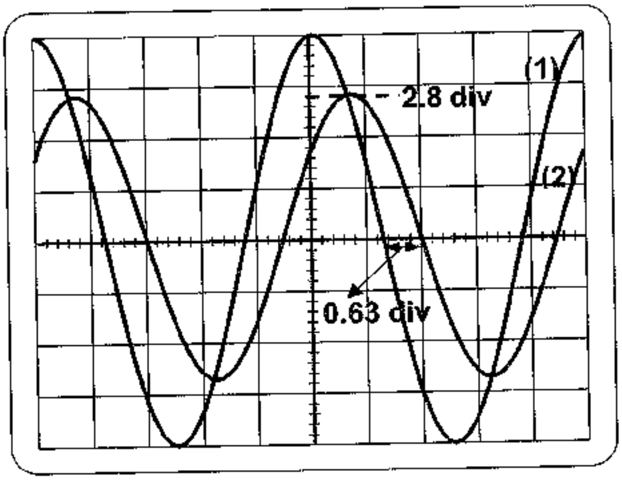


Figure 2

4. Determine the voltage $u_{AB} = u_L$ across the coil as a function of L and t.
5. Applying the law of addition of voltages, and giving t a particular value. Determine the value of L.

IV-LS 2010 2nd

Study of an RLC Series Circuit

The circuit of (figure1) is formed of a coil (L, r), a resistor of resistance $R = 50\,\Omega$ and a capacitor of capacitance $C = 64\,\mu F$ all connected in series across a generator G that maintains, across its terminals A and D, an alternating sinusoidal voltage of adjustable frequency f and of constant effective value U. The circuit thus carries an alternating sinusoidal current i whose expression as a function of time is given by:

$i = I_m \sin(2\pi f t)$ (i in A, t in s).

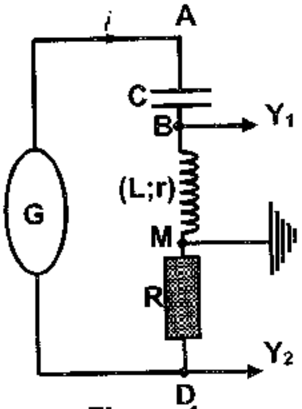


Figure 1

An oscilloscope, conveniently connected, allows us to display the voltage u_{BM} across the coil on channel Y_1, and the voltage u_{MD} across the resistor on channel Y_2. We obtain the waveforms (a) and (b) represented in figure 2.

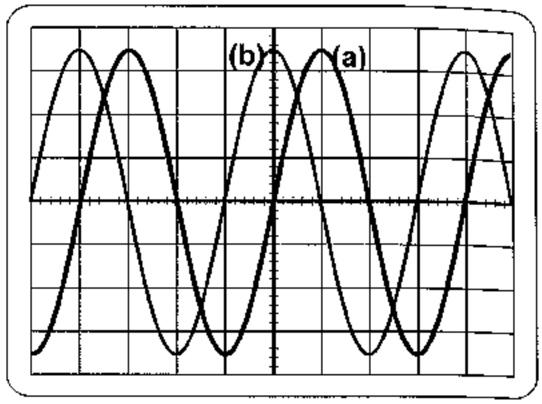


Figure 2

- ✘ The vertical sensitivity on both channels is $2V/\text{div}$.
- ✘ The horizontal sensitivity is $5\,ms/\text{div}$.

Take: $0.32\pi = 1$.

1. The button «INV» of channel Y_2 is pressed. Why?
2. Which one of the two waveforms represents the voltage u_{BM}? Why?
3. Referring to figure 2:
   a) calculate f.
   b) Characteristics of a circuit.
      i- calculate the phase difference between the voltages u_{BM} and u_{MD}.
      ii- deduce that the coil has no resistance.
   c) calculate the maximum voltage $U_{BM(\max)}$ across the coil;
   d) calculate the maximum voltage $U_{MD(\max)}$ across the resistor.
4. Show that the expression of the voltage u_{MD} is $u_{MD} = 7\sin(100\,\pi t)$ (u_{MD} in V, t in s)
5. Determine, as a function of time, the expression of:
   a) the current i;
   b) the voltage u_{BM};
   c) the voltage u_{AB} across the capacitor.
6. a) Applying the law of addition of voltages, determine the expression of the voltage u_{AD} across the generator as a function of time.
   b) Power.
      i- Deduce that the average electric power P consumed in the circuit is maximum.
      ii- Calculate P.

V-LS 2009 2nd

Response of an (r, L, C) Circuit to an Alternating Sinusoidal Voltage

Consider the series circuit that is represented in figure 1. This circuit is formed of a coil of inductance L and of resistance r, a capacitor of adjustable capacitance C, and a generator G delivering across its terminals an alternating sinusoidal voltage:

$$u_G = u_{AB} = 10\sqrt{3}\sin\left(\frac{200\pi}{3}t + \varphi\right), (u_G \text{ in } V,\ t \text{ in } s).$$

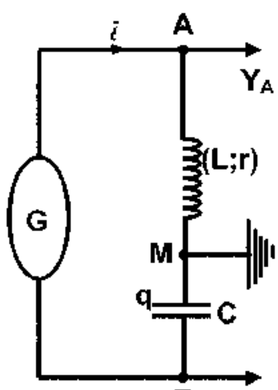


Figure 1

For a certain value of the capacitance C, the circuit carries an alternating sinusoidal current $i = \sin\left(\frac{200\pi}{3}t\right)$, ($i$ in A, t in s).

Take: $0.32\pi = 1$.

An oscilloscope, connected as shown in figure 1, displays the voltage u_{AM} across the coil on channel Y_A, and the voltage $u_{MB} = u_C$ across the capacitor on channel Y_B, the «INV» button of channel Y_B being pressed. On the screen of the oscilloscope, we observe the waveforms represented in figure 2.

The vertical sensitivity on both channels is: $S_v = 5V/\text{div}$.

1. Referring to figure 2:
   a) determine the horizontal sensitivity of the oscilloscope;
   b) determine the amplitudes $U_{AM\max}$ and $U_{C\max}$ of the voltages u_{AM} and u_C;
   c) show that the phase difference φ' between the voltages u_{AM} and u_C is $\frac{2\pi}{3}$ rad.
   Specify the voltage that leads the other.

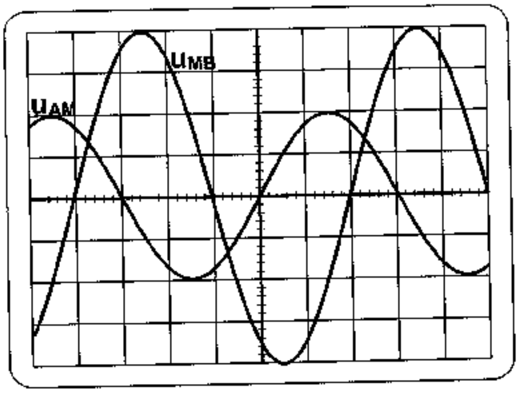


Figure 2

2. a) Studying the capacitor:
   i- Write down the relation among i, C & $\frac{du_C}{dt}$.
   ii- Show that the voltage u_C across the terminals of the capacitor is given by:
   $$u_C = \frac{3}{200\pi C}\sin\left(\frac{200\pi}{3}t - \frac{\pi}{2}\right).$$
   iii-Deduce that the value of C is $240\,\mu F$.
   b) Numerical values
   i- Using figure 2 and the expression of u_C, determine the expression of u_{AM} as a function of time.
   ii- Give the expression of u_{AM} in terms of r, i, L & $\frac{di}{dt}$.
   iii-Using the preceding results, and by giving t two particular values, show that: $r = 5\sqrt{3}\,\Omega$ and $L = 0.024\,H$.
3. The relation: $u_G = u_{AM} + u_{MB}$ is valid for any time t.
   Determine φ knowing that: $-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$ (rad).
4. The value of C is made to vary. We notice that, for a certain value C' of C, the amplitude of i attains a maximum value.
   a) Give the name of the physical phenomenon that thus took place.
   b) Determine C'.

VI-LS 2008 2nd

Determination of the Capacitance of a Capacitor

In order to determine the capacitance C of a capacitor, we connect it in series with a resistor of resistance $R = 10\sqrt{2}\,\Omega$ across the terminals of a low frequency generator (G) delivering across its terminals an alternating sinusoidal voltage: $u_G = U_m \cos wt$.

The circuit thus constructed carries an alternating sinusoidal current i (Figure 1).

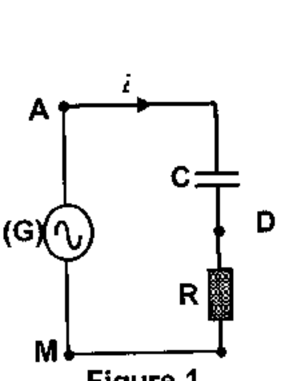


Figure 1

Take: $\sqrt{2} = 1.4$ and $0.32\pi = 1$.

1. Redraw the circuit of figure (1) and show the connections of the oscilloscope in order to display the voltages $u_G = u_{AM}$ across the generator and $u_R = u_{DM}$ across the resistor.
2. Which of the two voltages, u_G or u_R, represents the image of the current i? Justify your answer.
3. In figure 2, the waveform (1) represents the variation of the voltage u_G with time.
   a) Specify, with justification, which of the voltages u_G or u_R, leads the other.
   b) Determine the phase difference between the voltages u_G and u_R.

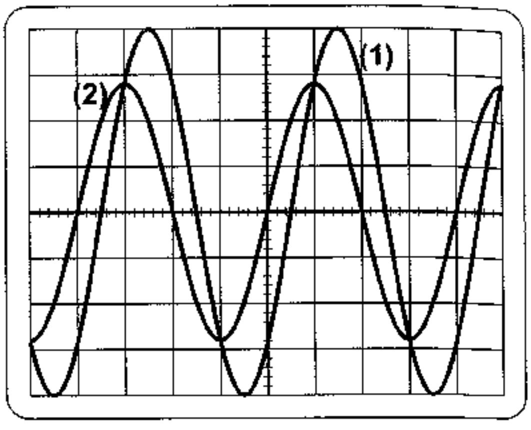


Figure 2

4. Using the waveforms of figure 2, determine the angular frequency w, the maximum value U_m of the voltage u_G and the maximum value I_m of the current i.
   Horizontal sensitivity: $5\,ms/\text{div}$.
   Vertical sensitivity on both channels: $1V/\text{div}$.
5. a) Write down the expression of the current i as a function of time t.
   b) Deduce the expression of the voltage $u_C = u_{AD}$ across the terminals of the capacitor as a function of C and t.
6. By applying the law of addition of voltages and giving the time t a particular value, determine the value of C.

I-LS 2006 2nd
The capacitor – A Humidity Sensor

In order to show evidence of the role of the capacitor in the humidity sensor, we connect up the circuit of Figure 1.

This circuit is formed of a function generator (LFG) delivering across its terminals an alternating sinusoidal voltage of frequency f, a coil of inductance $L = 0.07\,H$ and of negligible resistance, a resistor of resistance $R = 100\,k\Omega$ and a capacitor of capacitance C.

The voltage across the LFG is $u_{AM} = U_m \sin(wt)$, $(w = 2\pi f)$. The circuit thus carries an instantaneous current given by: $i = I_m \sin(wt + \varphi)$.

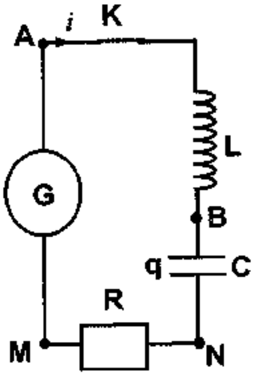


Figure 1

1. We denote by $u_C = u_{BN}$ the instantaneous voltage across the capacitor, by u_{AB} the voltage across the coil and by u_{NM} that across the resistor. Show that:
   a) $i = C\dfrac{du_C}{dt}$.
   b) u_C may be written in the form: $u_C = -\dfrac{I_m}{Cw}\cos(wt + \varphi)$.
   c) $u_{AB} = LwI_m \cos(wt + \varphi)$.
2. The relation $u_{AM} = u_{AB} + u_{BN} + u_{NM}$ is valid for any instant t.

Show, giving wt a particular value, that: $\tan\varphi = \dfrac{\dfrac{1}{Cw} - Lw}{R}$.

3. An oscilloscope, conveniently connected, displays the variations, as a function of time, of u_{AM} and u_{NM} on the channels (Y_1) and (Y_2) respectively.
   These variations are represented in the waveforms of figure 2.

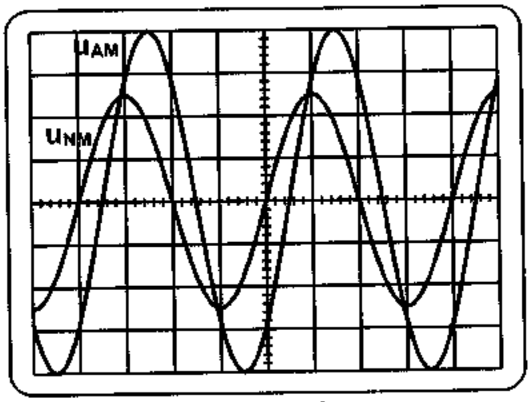


Figure 2

   a) Redraw Figure 1 showing the connections of the oscilloscope.
   b) The waveform of u_{NM} represents the «image» of the current i. Why?
   c) Find the value of f, knowing that the horizontal sensitivity is $5\,ms/div$.
   d) Determine the phase difference φ between i and u_{AM}.
4. Deduce the value of the capacitance C.
5. The frequency f is made to vary, keeping the same effective value of u_{AM}. It is noticed that, for a value f_1 of f, u_{AM} is in phase with i.
   a) Give the name of the phenomenon that appears in the circuit.
   b) Deduce, from what preceded, the relation among L, C and f_1.
6. A commercial humidity sensor can be considered as a capacitor whose capacitance C increases when the rate of relative humidity $H\%$ of air increases. The manufacturer provides the graph of the variation of C as a function of the rate of the relative humidity $H\%$ (Figure 3). $\left(1\,pF = 10^{-12}\,F\right)$.
   We replace the capacitor of the circuit of figure 1 by the sensor. In order to measure the value of C, the frequency f is made to vary; we notice that the voltage u_{AM} and the current i are in phase for a frequency $f = 5.20\times10^4\,Hz$.
   Deduce the rate of relative humidity of air under the atmospheric conditions of the experiment.

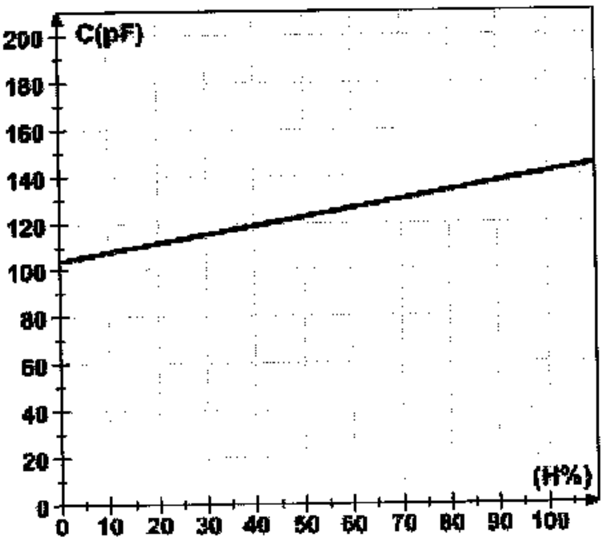


Figure 3

Sessions Solutions

I-LS 2014 1st

Part A

1. The voltage u_{MB} across the resistor is the image of the current in the circuit.
   The voltage u_{AM} lags u_{MB} so the component (D_1) is the seat of a capacitive effect. Then (D_1) is a capacitor.
2. a) We have $U_{m(R)} = S_v \times y_{\max} = 5V/\text{div} \times 2.8\text{div} = 14V$;

   According to Ohm's law $U_{m(R)} = R I_m$ so $I_m = \dfrac{U_{m(R)}}{R} = \dfrac{14}{100} = 0.14A$.

   b) $U_{m(D_1)} = S_v \times y_{\max(D_1)} = 5V/\text{div} \times 2.8\text{div} = 14V$.
3. We have $i = I_m \cos(wt)$ but $u_{AM} = \dfrac{1}{C}\int i\,dt = \dfrac{1}{C}\int I_m \cos(wt)dt = \dfrac{I_m}{Cw}\sin(wt)$.
4. We have $U_{m(D_1)} = \dfrac{I_m}{Cw}$ so $C = \dfrac{I_m}{U_{m(D_1)}w} = \dfrac{0.14}{14\times 100\pi} = \dfrac{10^{-4}}{\pi} = 0.32\times 10^{-4}F = 32\mu F$.

Part B

1. (D_2) is a coil, so the circuit is the seat of an inductive effect.
   So the voltage across the generator should lead the current i in the circuit whose image is u_{BM};
   Then (a) represents u_G.
2. a) We have $U_{m(R)} = S_v \times y_{\max} = 5V/\text{div} \times 2.8\text{div} = 14V$;

   According to Ohm's law $U_{m(R)} = R I_m$ so $I_m = \dfrac{U_{m(R)}}{R} = \dfrac{14}{100} = 0.14A$.

   b) $U_{m(G)} = S_v \times y_{\max(G)} = 5V/\text{div} \times 4\text{div} = 20V$.

   c) The phase difference $|\varphi| = 2\pi\dfrac{d}{D} = 2\pi\times\dfrac{0.5\,\text{div}}{4\,\text{div}} = \dfrac{\pi}{4}\text{rad}$.
3. a) Ohm's law for a coil $u_{AB} = r\,i + L\dfrac{di}{dt} = L\dfrac{di}{dt}$ (negligible internal resistance $r=0$);

   So $u_{AB} = -L\,w I_m \sin(wt)$.

   b) u_G leads the current i by φ so $u_G = u_{m(G)}\cos(wt+\varphi) = 20\cos\left(100\,\pi t + \dfrac{\pi}{4}\right)$ (t in s, u_G in V)
4. Law of addition of voltages : $u_G = u_{AB} + u_{BM}$;

   $20\cos\left(100\,\pi t + \dfrac{\pi}{4}\right) = 14\cos(100\,\pi t) - 100\,\pi\times 0.14\,L\sin(100\,\pi t)$;

   $20\cos\left(100\,\pi t + \dfrac{\pi}{4}\right) = 14\cos(100\,\pi t) - 14\,\pi\,L\sin(100\,\pi t)$;

   Let $100\,\pi t = \dfrac{\pi}{2}(\text{rad})$, then $20\cos\left(\dfrac{\pi}{2}+\dfrac{\pi}{4}\right) = 14\cos\left(\dfrac{\pi}{2}\right) - 14\,\pi\,L\sin\left(\dfrac{\pi}{2}\right)$;

   $20\left(-\dfrac{\sqrt{2}}{2}\right) = -14\,\pi\,L$ so $L = \dfrac{10\sqrt{2}}{14\,\pi} = \dfrac{10\times 1.4\times 0.32}{14} = 0.32\,H$.

II-LS 2013 1st

Part A

1. The frequency $f_0 = \frac{1}{T_0} = \frac{1}{S_h \times x} = \frac{1}{2ms/\text{div} \times 8\text{div}} = \frac{1}{16 \times 10^{-3} s} = 62.5\, Hz$.

   Then $w_0 = 2\pi f_0 = 125\pi\,(\text{rad/s})$.

2. The maximum voltage $U_m = S_{v_1} \times y_1 = 2V/\text{div} \times 2\,\text{div} = 4V$.

   $I_{m(\max)} = \frac{(U_R)_m}{R} = \frac{S_{v_2} \times y_2}{R} = \frac{0.25\,V/\text{div} \times 4\text{div}}{20\Omega} = \frac{1V}{20\Omega} = 0.05\,A$.

3. a) Current resonance, since u_{AM} and $u_{BM} = Ri$ are in phase.

   b) The proper frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$; so $LC = \frac{1}{4\pi^2 f_0^2} = \frac{1}{(125\pi)^2}$; then $LC = 6.48 \times 10^{-6}\, SI$

4. The circuit is equivalent to a resistor[1] so $R_t = \frac{U_m}{I_{m(\max)}} = \frac{4}{0.05} = 80\,\Omega$;

   and $R_t = R + r \Rightarrow r = R_t - R = 80 - 20 = 60\Omega$.

Part B

1. a) Since u_{AM} displayed on Y_1 reaches its maximum after u_{BM} displayed on Y_2.

   b) The phase difference $|\varphi| = 2\pi\frac{d}{D} = 2\pi \times \frac{1\text{div}}{8\text{div}} = \frac{\pi}{4}\text{rad}$.

   c) u_{BM} leads u_{AM} which is the image of the current i, so $\varphi = \frac{\pi}{4}\text{rad}$.

   $u_{BM} = (U_R)_m \cos\left(125\pi t + \frac{\pi}{4}\right)$; where $(U_R)_m = S_{v_2} \times y_2 = 0.25V/\text{div} \times 2.8\text{div} = 0.7V$.

   Then: $u_{BM} = 0.7\cos\left(125\pi t + \frac{\pi}{4}\right)$; ($u_{BM}$ in V; t in s).

   $u = u_{AM} = 4\cos(125\pi t)$ (u in V; t in s).

2. Referring to Ohm's law $u_{BM} = Ri$; $i = \frac{u_{BM}}{R} = 0.035\cos\left(125\pi t + \frac{\pi}{4}\right)$; ($i$ in A; t in s).

3. Law of addition of voltages: $u_G = u_R + u_{r_1} + u_C$;

   $4\cos(125\pi t) = \frac{8.9 \times 10^{-5}}{C}\sin\left(125\pi t + \frac{\pi}{4}\right) + (60+20) \times 0.035 \times \sin\left(125\pi t + \frac{\pi}{4}\right)$ [2]

   Let $125\pi t = 0$; $4\cos(0) = \frac{8.9 \times 10^{-5}}{C}\sin\left(\frac{\pi}{4}\right) + (60+20) \times 0.035 \times \sin\left(\frac{\pi}{4}\right)$;

   Then $C = \frac{8.9 \times 10^{-5}}{\left(4 - 1.4\sqrt{2}\right) \times \sqrt{2}} = 3.12 \times 10^{-5} F$.

---

[1] We can use also two expressions of the power $P = U_m I_m \cos\varphi = R I_m^2$; ($\varphi = 0$ since they are in phase).

So $U_m = R_t I_m \Rightarrow R_t = 80\Omega$.

[2] Let $125\pi t + \frac{\pi}{4} = \frac{\pi}{2}$; $4\cos\left(\frac{\pi}{4}\right) = \frac{8.9 \times 10^{-5}}{C}\sin\left(\frac{\pi}{2}\right) + (60+20) \times 0.035 \times \cos\left(\frac{\pi}{2}\right)$.

Part C

We have $LC = 6.48\times10^{-6}\ SI$, $LC = \dfrac{6.48\times10^{-6}}{3.12\times10^{-5}} = 0.21H$.

III-LS 2011 2nd

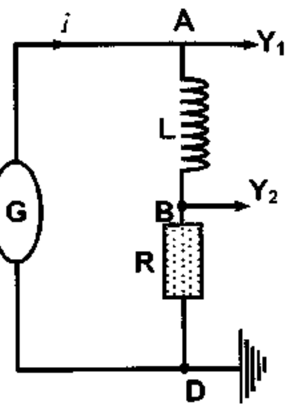


Figure 1

1. On $Y_1 \longrightarrow u_G = u_{AD}$; and $Y_2 \longrightarrow u_L = u_{BD}$;
   The connections are shown on circuit.
2. u_{BD} is the image of the current.
   According to Ohm's law: $u_{BD} = u_R = Ri$, then u_{BD} and i are proportional.
3. a) (1) leads (2), since it reaches its maximum first.
   b) Oscillogram:
   i- The phase difference is given by:
   $|\varphi| = 2\pi\dfrac{d}{D} = 2\pi\times\dfrac{0.63\text{div}}{5\text{div}} = \dfrac{1.26\,\pi}{5} \approx 0.79$ rad.
   ii- The period is given by:
   $T = S_h\times x = 5ms/\text{div}\times5\text{div} = 25ms = 25\times10^{-3}s$.
   The angular frequency: $w = \dfrac{2\pi}{T} = \dfrac{2\pi}{25\times10^{-3}} = 80\,\pi \approx 251.3\,\text{rad}/s$.
   iii- The maximum value $U_m = S_v\times y_{1\max} = 1V/\text{div}\times4\text{div} = 4V$.
   iv- We have: $I_m = \dfrac{(U_R)_{\max}}{R} = \dfrac{S_v\times y_{2\max}}{R} = \dfrac{1V/\text{div}\times2.8\text{div}}{10\Omega} = 0.28\,A$.
   c) We have: $i = I_m\cos(wt+\varphi) = 0.28\cos\left(80\pi t - \dfrac{1.26\pi}{5}\right)$. ($i$ in A; t in s).
4. Ohm's law across the coil: $u_L = ri + L\dfrac{di}{dt} = L\dfrac{di}{dt}$ (the internal resistance is negligible).
   $u_L = -L\times0.28\times80\pi\sin\left(80\,\pi t - \dfrac{1.26\pi}{5}\right)$;
   $= -22.4\,\pi L\sin\left(80\,\pi t - \dfrac{1.26\,\pi}{5}\right)$. ($u$ in V; t in s & L in H).
5. Law of addition of voltages:
   $u_G = u_L + u_R \Rightarrow 4\cos(80\pi t) = 2.8\cos\left(80\pi t - \dfrac{1.26\pi}{5}\right) - 22.4\,\pi L\sin\left(80\pi t - \dfrac{1.26\pi}{5}\right)$.
   Let $80\,\pi t = 0$;
   $4\cos(0) = 4 = 2.8\cos\left(-\dfrac{1.26\pi}{5}\right) - 22.4\,\pi L\sin\left(-\dfrac{1.26\pi}{5}\right)$;
   $$L = \frac{4 - 2.8\cos\left(\dfrac{1.26\pi}{5}\right)}{22.4\,\pi\sin\left(\dfrac{1.26\pi}{5}\right)} \approx 0.041H = 41mH.$$

IV-LS 2010 2nd

1. In order to display u_{MD} and not u_{DM}, and therefore comparing two voltages having identical signs.
2. On Y_1 we are displaying $u_{BM} = u_L$ which must lead the current (represented on Y_2) due to the inductive effect.
   Then u_{BM} is displayed by the graph (b).
3. a) The period $T = S_h \times x = 5\,ms / \text{div} \times 4\,\text{div} = 20\,ms$;

   The frequency $f = \dfrac{1}{T} = \dfrac{1}{20 \times 10^{-3}} = 50\ Hz$.

   b) Characteristics of the circuit.

   i- The phase difference $|\varphi| = 2\pi \dfrac{d}{D} = 2\pi \times \dfrac{1\text{div}}{4\text{div}} = \dfrac{\pi}{2}\,\text{rad}$.

   ii- Since $|\varphi| = \dfrac{\pi}{2}\,\text{rad}$, then the coil is pure inductive.

   c) $(U_{BM})_{\max} = S_v \times y_{\max} = 2V / \text{div} \times 3.5\,\text{div} = 7V$.

   d) $(U_{MD})_{\max} = S_v \times y_{\max} = 2V / \text{div} \times 3.5\,\text{div} = 7V$.
4. According to Ohm's law : $u_{MD} = Ri$, u_{MD} and the current i are in phase.
   Then $u_{MD} = (U_{MD})_{\max} \sin(2\pi f t) = 7 \sin(100\pi t)$.
5. a) Ohm's law: $i = \dfrac{u_{MD}}{R} = \dfrac{7\sin(100\pi t)}{50} = 0.147 \sin(100\pi t)$.

   b) From the oscillograms $u_{BM} = (U_{BM})_{\max} \sin(2\pi f t + \varphi) = 7\sin\left(100\,\pi t + \dfrac{\pi}{2}\right) = 7\cos(100\,\pi t)$.

   c) We have: $u_{AB} = \dfrac{1}{C}\int i\,dt = \dfrac{1}{C}\int 0.14 \sin(100\,\pi t)dt$.

   $$u_{AB} = \frac{1}{64 \times 10^{-6}} \times 0.14 \times \frac{-1}{100\,\pi}\cos(100\,\pi t) = \frac{-0.14 \times 0.32}{100 \times 64 \times 10^{-6}}\cos(100\,\pi t) = -7\cos(100\,\pi t).$$
6. a) Law of addition of voltages: $u_{AD} = u_{AB} + u_{BM} + u_{MD}$.
   $u_{AD} = -7\cos(100\,\pi t) + 7\cos(100\,\pi t) + 7\sin(100\,\pi t) = +7\sin(100\,\pi t)$.

   b) Power

   i- We have u_{AD} and i are in phase $(\varphi = 0)$, then the circuit is the seat of current resonance. The power factor between the voltage across the generator and the current is $\cos\varphi = 1$.
   Thus the power consumed is maximum.

   ii- The power consumed is: $P = (U_G)_{eff} \times (I_G)_{eff} = \dfrac{(U_G)_{\max}}{\sqrt{2}} \times \dfrac{I_m}{\sqrt{2}} = \dfrac{7}{\sqrt{2}} \times \dfrac{0.14}{\sqrt{2}} = 0.49\,W$.

V-LS 2009 2nd

1. a) The angular frequency is related to the period by: $w = \dfrac{200\,\pi}{3} = \dfrac{2\,\pi}{T}$; $T = \dfrac{3}{100} = 0.03s = 30\,ms$.

   But the period is given by $T = S_h \times x \Rightarrow S_h = \dfrac{T}{x} = \dfrac{30\,ms}{6\text{div}} = 5\,ms / \text{div}$.

b) We have $(U_{AM})_{\max} = S_v \times y_{\max} = 5V/\text{div} \times 2\,\text{div} = 10V$.

$(U_C)_{\max} = S_v \times y_{\max} = 5V/\text{div} \times 4\,\text{div} = 20V$.

c) The phase difference $|\varphi'| = 2\pi \dfrac{d}{D} = 2\pi \times \dfrac{2\text{div}}{6\text{div}} = \dfrac{2\pi}{3}$ rad. u_{AM} leads u_{MB}.

2. a) Numerical values

i- We have $i = \dfrac{dq}{dt}$ & $q = C u_C$; $i = C\dfrac{du_C}{dt}$.

ii- The voltage across the capacitor is related to the current by the relation:

$$u_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int \sin\left(\frac{200\,\pi}{3}t\right)dt = -\frac{1}{C}\times\frac{3}{200\,\pi}\cos\left(\frac{200\,\pi}{3}t\right).$$

$$u_C = \frac{3}{200\,\pi\,C}\sin\left(\frac{200\,\pi}{3}t - \frac{\pi}{2}\right).$$

iii- Basing on the previous part $(U_C)_{\max} = \dfrac{3}{200\,\pi\,C} = 20 \Rightarrow C = \dfrac{3\times 0.32}{20\times 200} = 2.4\times 10^{-4} = 240\,\mu F$

b) Numerical values

i- We can deduce that: $u_{AM} = (U_{AM})_{\max}\sin\left(\dfrac{200\,\pi}{3}t - \dfrac{\pi}{2} + \dfrac{2\pi}{3}\right) = 10\sin\left(\dfrac{200\,\pi}{3}t + \dfrac{\pi}{6}\right)$.

ii- Ohm's law case of a coil: $u_{AM} = r\,i + L\dfrac{di}{dt}$.

iii- Replacing the expression of the current in the previous relation we get:

$$u_{AM} = r\sin\left(\frac{200\,\pi}{3}t\right) + L\frac{200\,\pi}{3}\cos\left(\frac{200\,\pi}{3}t\right) = 10\sin\left(\frac{200\,\pi}{3}t + \frac{\pi}{6}\right).$$

Let $\dfrac{200\,\pi}{3}t = 0$, $r\sin 0 + \dfrac{200\,\pi}{3}L\cos 0 = 10\sin\left(\dfrac{\pi}{6}\right)$;

Then $L = \dfrac{5\times 3}{200\,\pi} = \dfrac{15\times 0.32}{200} = 0.024\,H$.

Let $\dfrac{200\,\pi}{3}t = \dfrac{\pi}{2}$, $r\sin\left(\dfrac{\pi}{2}\right) + \dfrac{200\,\pi}{3}L\cos\left(\dfrac{\pi}{2}\right) = 10\cos\left(\dfrac{\pi}{6}\right)$;

Then $r = 10\dfrac{\sqrt{3}}{2} = 5\sqrt{3}\,\Omega \approx 8.66\,\Omega$.

3. Law of addition of voltages:

$u_{AB} = u_{AM} + u_{MB}$; $10\sqrt{3}\sin\left(\dfrac{200\pi}{3}t + \varphi\right) = 20\sin\left(\dfrac{200\pi}{3}t - \dfrac{\pi}{2}\right) + 10\sin\left(\dfrac{200\pi}{3}t + \dfrac{\pi}{6}\right)$.

Let $\dfrac{200\,\pi}{3}t = 0 \Rightarrow 10\sqrt{3}\sin\varphi = 20\sin\left(-\dfrac{\pi}{2}\right) + 10\sin\left(\dfrac{\pi}{6}\right)$;

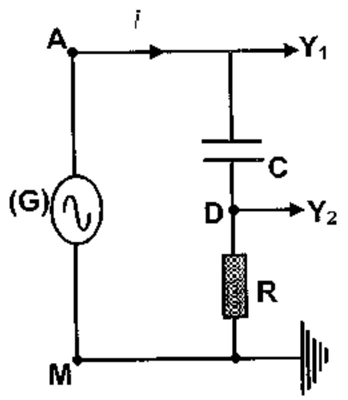
Then: $10\sqrt{3}\sin\varphi = -15 \Rightarrow \sin\varphi = -\dfrac{\sqrt{3}}{2}$ & $\varphi \in \left]-\dfrac{\pi}{2}; -\dfrac{\pi}{2}\right[\Rightarrow \varphi = -\dfrac{\pi}{3}$ rad.

4. a) Current resonance.

b) Under resonance $Lw_0^2C' = 1 \Rightarrow C' = \dfrac{1}{w_0^2 L} = \dfrac{3^2\times 0.32^2}{(200)^2\times 0.024} = 9.6\times 10^{-4}\,F = 960\,\mu F$.

VI-LS 2008 2nd

1. Connections of the oscilloscope are shown on figure 1.
2. According to Ohm's law : $u_R = Ri$; then u_R and i are proportional. Thus u_R is the image of the current i.



3. a) Referring to the oscillograms u_R reaches its maximum before u_G, then u_R leads u_G.

   b) We know that: $|\varphi| = 2\pi \frac{d}{D} = 2\pi \times \frac{0.5\text{div}}{4\text{div}} = \frac{\pi}{4}\text{rad}$.
4. The period $T = S_h \times x = 5ms / \text{div} \times 4\text{div} = 20\ ms$.

   The angular frequency $w = \frac{2\pi}{T} = \frac{2\pi}{0.02} = 100\,\pi\ (\text{rad/s})$.

   The maximum value of the voltage across the generator $U_m = S_{v_1} \times y_{1\max} = 1V / \text{div} \times 4\,\text{div} = 4V$;

   and $(U_R)_{\max} = S_v \times y_{2\max} = 1V / \text{div} \times 2.8\text{div} = 2.8V = 2\sqrt{2}\ V$;

   But $I_m = \frac{(U_R)_{\max}}{R} = \frac{2\sqrt{2}}{10\sqrt{2}} = 0.2\,A$.
5. a) The instantaneous expression of the current is $i = I_m \cos\left(100\,\pi\,t + \frac{\pi}{4}\right) = 0.2\cos\left(100\,\pi\ t + \frac{\pi}{4}\right)$.

   b) The voltage across the capacitor $u_C = \frac{1}{C}\int i\,dt = \frac{0.2}{100\,\pi\,C}\sin\left(100\,\pi\ t + \frac{\pi}{4}\right)$ (t in s ; u_C in V).

   c) Law of addition of voltages $u_G = u_C + u_R$ and $u_R = Ri = 2\sqrt{2}\cos\left(100\,\pi\,t + \frac{\pi}{4}\right)$;

   Then $4\cos 100\,\pi\,t = \frac{0.2}{100\,\pi\,C}\sin\left(100\,\pi\,t + \frac{\pi}{4}\right) + 2\sqrt{2}\cos\left(100\,\pi\,t + \frac{\pi}{4}\right)$;

   For $100\,\pi\,t = 0$, we get: $4 = \frac{0.2}{100\,\pi\,C} \times \frac{\sqrt{2}}{2} + 2\sqrt{2} \times \frac{\sqrt{2}}{2}$; then $C = 2.24 \times 10^{-4}\,F = 224\,\mu F$.

VIII-LS 2006 2nd

1. a) The current i is given by $i = \frac{dq}{dt}$ & $q = C\,u_C$ then $i = C\frac{du_C}{dt}$.

   b) The voltage across the capacitor: $u_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int I_m \sin(wt + \varphi)dt = \frac{-I_m}{C\,w}\cos(wt + \varphi)$.

   c) The voltage across the coil is: $u_{AB} = L\frac{di}{dt} = L\,w\,I_m \cos(wt + \varphi)$.
2. Law of addition of voltages $u_{AM} = u_{AB} + u_{BN} + u_{NM}$;

   $U_m \sin(wt) = L\,w I_m \cos(wt + \varphi) - \frac{I_m}{C\,w}\cos(wt + \varphi) + R\,I_m \sin(wt + \varphi)$;

   For $wt = 0$, $U_m \sin 0 = Lw\,I_m \cos\varphi - \frac{I_m}{Cw}\cos\varphi + RI_m \sin\varphi$;

$$R\,I_m \sin\varphi = -\left(Lw - \frac{1}{Cw}\right) I_m \cos\varphi \text{; then } \tan\varphi = \frac{\frac{1}{Cw} - Lw}{R}.$$

3. a) Connections of the oscilloscope.

b) According to Ohm's law: $u_R = R\,i$;

Then u_R and i are proportional thus u_R is the image of the current i.

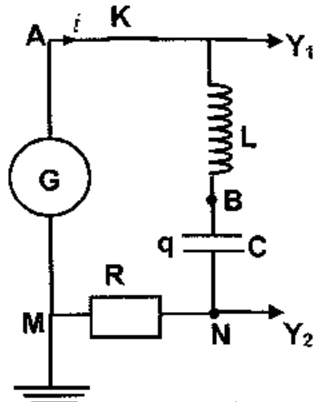


Figure 1

c) The period T is $T = S_h \times x = 5ms\,/\,\text{div} \times 4\,\text{div} = 20\,ms = 0.02\,s$.

The frequency $f = \frac{1}{T} = \frac{1}{0.02} = 50\,Hz$.

d) The phase difference $|\varphi| = 2\pi \times \frac{d}{D} = 2\pi \times \frac{0.5\text{div}}{4\text{div}} = \frac{\pi}{4}\,\text{rad}$,

But u_{NM} leads u_{AM} then $|\varphi| = +\frac{\pi}{4}\,\text{rad}$

4. We have $w = \frac{2\pi}{T} = 100\,\pi\,(\text{rad/s})$ and $\varphi = \frac{\pi}{4}\,\text{rad}$;

But $\tan\varphi = \frac{\frac{1}{Cw} - Lw}{R} = \tan\left(\frac{\pi}{4}\right) = 1$;

$$\frac{1}{C(100\pi)} - 0.07\,(100\pi) = 10^5,\ C = \frac{1}{100\pi\,(10^5 + 7\pi)} = 3.18 \times 10^{-8}\,F.$$

5. a) Current resonance.

b) Since u_{AM} and i are in phase then: $\varphi = 0 \Rightarrow \tan\varphi = 0$;

$L\,w_1^2 C = 1 \Rightarrow 4\pi^2\,LC\,f_1^2 = 1$.

6. We have: $4\pi^2\,L\,C\,f_1^2 = 1 \Rightarrow C = \frac{1}{4\pi^2 f_1^2 L} = \frac{1}{4\pi^2 (5.20 \times 10^4)^2 \times 0.07} = 1.33 \times 10^{-10}\,F$;

Then $C = 133\,pF$, graphically for $C = 133\,pF$, the percentage of relative humidity of air is 70%

Unit II

Electricity

Chapter 10

Electromagnetic Oscillations

GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Electromagnetic Oscillations		1st	-	1st(A)	1st(A)	-	1st	1st & 2nd	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Electromagnetic Oscillations	2nd	-	1st	2nd	-	1st	-	1st(C)	1st

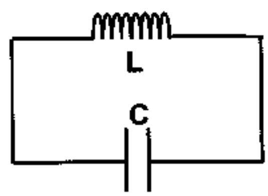
Essentials

I-
Modes of Oscillations

1. Free undamped oscillations

A charged capacitor is connected to a pure inductive coil (negligible internal resistance) as shown in the adjacent circuit.

Law of addition of voltages



The differential equation satisfied by u_C is $u''_C + \frac{1}{LC}u_C = 0$.

The voltage is sinusoidal of proper period $T_0 = 2\pi\sqrt{LC}$.

The oscillations are free undamped.

Conservation of energy

The total energy at any instant t is the sum of:

✖ magnetic energy stored in the coil $E_m = \frac{1}{2}Li^2$, or $E_m = \frac{1}{2}Lq'^2$;

✖ and electric energy $E_e = \frac{1}{2}Cu_C^2$ stored in the capacitor, $E_e = \frac{1}{2C}q^2$;

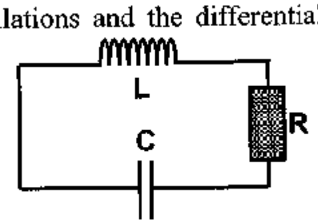
Then, $E_T = E_e + E_m = \frac{1}{2}Cu_C^2 + \frac{1}{2}Li^2$.

In the absence of dissipative factors (resistors), the energy is conserved $\frac{dE_T}{dt} = 0$, so $u''_C + \frac{1}{LC}u_C = 0$

The magnetic and electric energies are also periodic of period T_E is half that of the proper period then $T_{Em} = T_{Ee} = \frac{T_0}{2}$.

2. Free damped oscillations

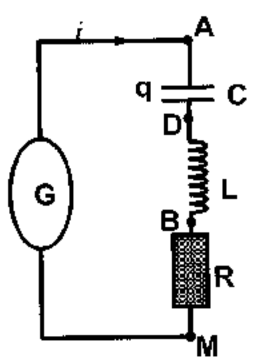
In the presence of a resistor, the system performs free damped oscillations and the differential equation will be $u''_C + \frac{R}{L}u'_C + \frac{1}{LC}u_C = 0$ which is obtained:



✖ either by the law of addition of voltages $u_R + u_L + u_C = 0$;

✖ or by deriving the total energy with respect to time $\frac{dE_T}{dt} = -Ri^2$.

The voltage across the capacitor u_C is pseudo-periodic of pseudo-period $T \geq T_0$



3. Forced oscillations under AC voltage

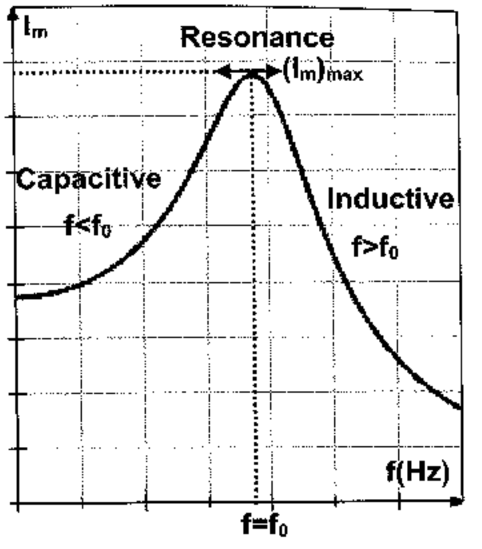
An alternating sinusoidal generator delivering the voltage $u_G = U_0\sin(2\pi f t)$, an alternating current circulates in the circuit and whose expression is given by $i = I_m\sin(2\pi f t + \varphi)$.

The proper frequency of the circuit is $f_0 = \frac{1}{2\pi\sqrt{LC}}$.

Applying the law of addition of voltages, we get:

$$\tan\varphi = \frac{\dfrac{1}{Cw} - Lw}{R} = \frac{4\pi^2 L}{Rw}\left(f_0^2 - f^2\right).$$

The curve shown in the adjacent figure represents the evolution of the amplitude of the current in the circuit as a function of the frequency of the L.F.G.

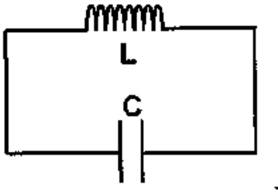


Discussion

- ✘ The circuit is the seat of a capacitive effect, if u_G lags i ($\varphi>0 \Rightarrow \tan\varphi>0$); thus $f<f_0$.
- ✘ The circuit/ is the seat of an inductive effect, if u_G leads i ($\varphi<0 \Rightarrow \tan\varphi<0$); thus $f>f_0$.
- ✘ The circuit acts as resistor if u_G is in phase with i ($\varphi=0$); thus $f=f_0$, the circuit is then the seat of current resonance.

II-
Mechanical and Electrical Analogies

Consider the following systems [(L, C) & simple horizontal oscillator], which are governed by the following differential equations:



Mechanical	Electrical
$x'' + \frac{k}{m}x = 0$	$q'' + \frac{1}{LC}q = 0$
$x'' + k\times\frac{1}{m}x = 0$	$q'' + \frac{1}{C}\times\frac{1}{L}q = 0$

We have a similarity between the roles of:

- ✘ a capacitor and the spring from energy point of view $E_e = \frac{1}{2}\times\frac{1}{C}q^2$, (the spring stores elastic potential energy while the capacitor stores electric potential energy $PE_e = \frac{1}{2}kx^2$) and then consequently between the kinetic and the magnetic energies.
- ✘ q is analog to x as variable, and the current i is analog to the velocity $v = x'$.

Taking the two previous differential equations we can consider also:

- ✘ the constant of spring k is analog to that related to the capacitor which is $\frac{1}{C}$;
- ✘ the constant related to kinetic energy which is the mass m is analog to the constant associated to the coil which is its inductance L;
- ✘ the expression of the proper period for free un-damped mechanical oscillations $T_0 = 2\pi\sqrt{\frac{m}{k}}$, by analogy the period free un-damped electromagnetic oscillations:

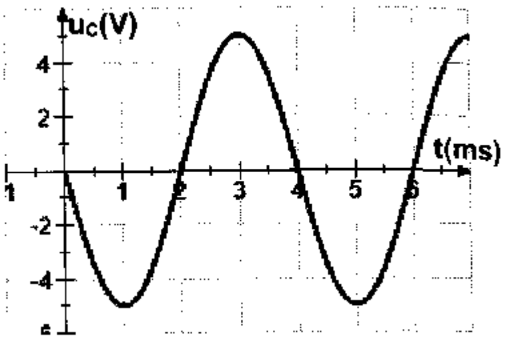
$$T_0 = 2\pi\sqrt{\frac{m}{k}} \longrightarrow 2\pi\sqrt{\frac{L}{1/C}} = 2\pi\sqrt{LC}.$$

- ✘ the resistance R is analog to the force of friction ($\frac{d(ME)}{dt} = P_{\vec{f}} = -h\,x'^2 \longrightarrow \frac{d(E_T)}{dt} = -Ri^2$).

Applications

I- Instantaneous Expression of u_C

The adjacent curve represents the variations of the voltage u_C across the capacitor in an circuit and whose expression is $u_C = U_m \cos\left(\frac{2\pi}{T_0}t + \varphi\right)$.



1. Determine the expression of u_C in terms of time.
2. By a dimensional study show that $\sqrt{LC}$ has a time dimension.

II- Study of a (L, C) Circuit

The (L, C) circuit of figure 1, is formed of a capacitor of capacitance C, of a coil of inductance L and of negligible resistance and a switch k. Initially the voltage between the terminals of the capacitor is $u_{AB} = U_0$. The switch is closed at $t_0 = 0$. At an instant t the charge of armature A is q and the current in the circuit is i.

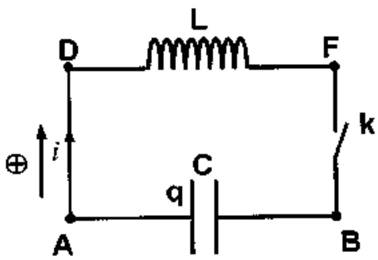
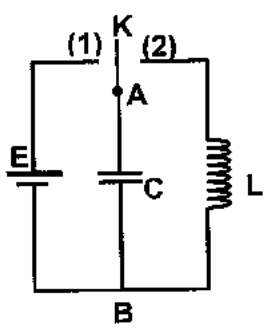


Figure 1

1. Applying the law of uniqueness of voltages, show that the differential equation that describes the variations of u_C has the form $u_C'' + \frac{1}{LC}u_C = 0$.
2. Specify the mode of oscillations performed by u_C.
3. The solution of this differential equation is of the form $u_C = A\cos(w_0 t)$ where A and w_0 are constants. Determine the expression of A in terms of U_0 and that of w_0 in terms of L and C.
4. Deduce the expression of the proper period of oscillations.

III- Electromagnetic Oscillations

The adjacent circuit is formed of:



✘ an ideal DC generator of $e.m.f\ E = 8V$;
✘ a capacitor of capacitance $C = 4\,\mu F$;
✘ a coil of inductance $L = 8\,mH$;
✘ a double switch K.

The capacitor is taken neutral.

1. The switch is placed on position (1).
   Calculate the electric charge Q_0 & the electric potential energy E_{e0} stored in the capacitor in steady state.
2. At an instant taken as origin of time, the switch is turned to position (2).

a) Applying the principle of conservation of the electromagnetic energy, derive the differential equation that governs the variations of the charge q stored in the capacitor.

b) Verify that $q = Q_0 \cos\left(\frac{1}{\sqrt{LC}}t\right)$ is a solution of the differential equation.

c) Deduce the value of the proper period of these oscillations.

IV- Electric Energy

Consider the circuit shown in figure 1, formed of a capacitor of capacitance $C = 6\mu F$ and a coil of inductance L and negligible internal resistance.

The curve shown in figure 2, represents the variations of the electric potential energy stored in the capacitor in terms of time.

Let $u_C = u_{HA}$ be the voltage between the terminals of the capacitor.

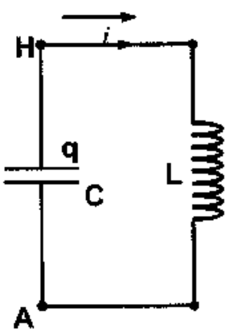


Figure 1

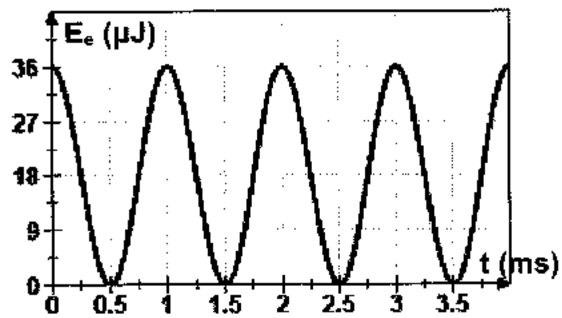


Figure 2

1. Choose, from the adjacent table, the differential equation in u_C.
2. Deduce the expression of the proper period of these oscillations.

Equation 1	Equation 2	Equation 3
$u'_C + \frac{1}{LC}u_C = 0$	$u''_C + \frac{1}{LC}u_C = 0$	$u''_C + \frac{1}{LC}u'_C = 0$

3. Referring to the curve shown in figure 2, determine:
   a) the proper period of the oscillations.
   b) the value of u_C at $t = 0$.
4. Deduce the inductance L of the coil.

V- Inductance of a Coil

The voltage between the terminals of a capacitor of capacitance C is $u_{AB} = u_C = U_0 = 8V$. The capacitor is connected successively to two coils (B_1) & (B_2) of inductances L_1 & L_2 (figure 1).

The curves shown in figures 2 & 3, represent the variations of the voltage across the capacitor while using the coils (B_1) & (B_2) respectively.

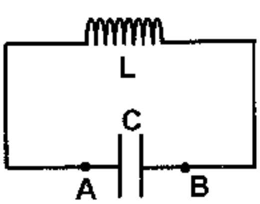


Figure 1

1. Explain why the internal resistances of the coils are negligible.
2. Justify that $L_1 > L_2$.

Knowing that the capacitance of the capacitor used is $C = 5\,\mu F$ & take $\pi^2 = 10$.

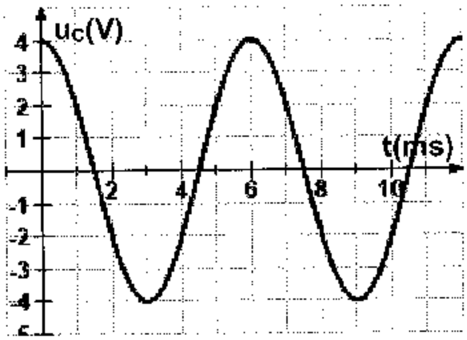


Figure 2

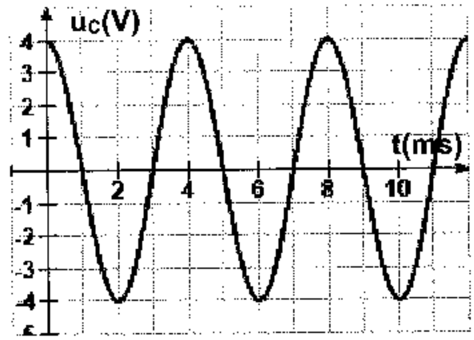


Figure 3

3. Determine L_1 & L_2.
4. Referring to curve 1:
   a) determine the electromagnetic energy stored.
   b) deduce the maximum intensity of the current.
5. Justify the electromagnetic energy when we use (B_2) is not modified.
6. Deduce the maximum intensity of the current in this case.
7. What can you conclude?

VI-

Free Damped Oscillations

The circuit shown in figure 1 is formed of:

✖ a coil of inductance $L = 10\,mH$ and internal resistance r;

✖ and a capacitor of capacitance $C = 5\,\mu F$.

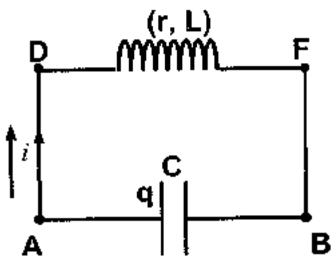


Figure 1

The curve shown in figure 2 represents the variations of the voltage u_C across the capacitor in terms of time.

1. Referring to figure 2:
   a) justify that the internal resistance of the coil is not negligible.
   b) give the value of the pseudo-period T of the oscillations.
   c) give the value of the voltage across the capacitor at $t = 0$.
2. Calculate the proper period T_0 of oscillations and then compare it to T.
3. Determine r knowing that $\frac{1}{T^2} = \frac{1}{T_0^2} - \frac{1}{4\pi^2}\left(\frac{r}{2L}\right)^2$.
4. Write the expression of the electromagnetic energy E_T stored in the circuit.
5. Knowing that $\frac{dE_T}{dt} = -r\,i^2$ where i is the current at an instant t, derive the differential equation that governs the variations of u_C.

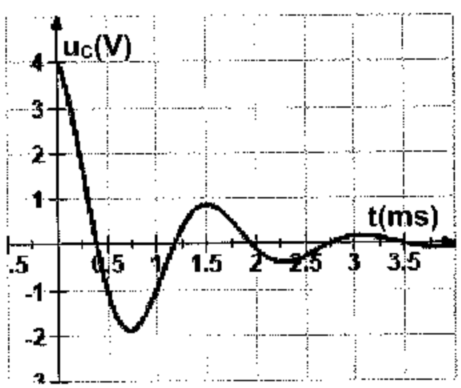


Figure 2

Solutions

I-

1. We have $u_C = U_m \cos\left(\frac{2\pi}{T_0}t + \varphi\right)$;

   The period of the voltage displayed is $T_0 = 4\,ms$.

   The amplitude of the voltage is $U_m = 5V$;

   At $t = 0$; $u_C = 0$ then $U_m \cos(\varphi) = 0$; thus $\varphi = \pm\frac{\pi}{2}(\text{rad})$;

   But $\left.\frac{du_C}{dt}\right|_{t=0} = -U_m \frac{2\pi}{T_0}\sin(+\varphi) < 0$ (decreasing at $t = 0$) then $\sin(+\varphi) > 0$; thus $\varphi = +\frac{\pi}{2}(\text{rad})$

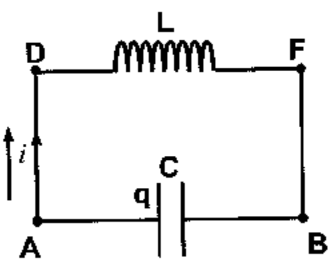
   Therefore $u_C = 5\cos\left(\frac{2\pi}{4\times10^{-3}}t + \frac{\pi}{2}\right) = 5\cos\left(500\,\pi\, t + \frac{\pi}{2}\right)$ (t in s and u_C in V).

2. We have $\left[\sqrt{LC}\right] = 1\sqrt{[L][C]}$

   But $u_L = L\frac{di}{dt}$ then $[L] = \frac{V\times s}{A}$ & $i = C\frac{du_C}{dt}$ then $[C] = \frac{A\times s}{V}$;

   Thus, $\left[\sqrt{LC}\right] = \sqrt{\frac{V\times s}{A}\times\frac{A\times s}{V}} = \sqrt{s^2} = s$ (time dimension).

II-



1. Law of addition of voltages: $u_{AB} = u_{AD} + u_{DF} + u_{FB}$;

   But $u_{AD} = u_{FB} = 0$ & $u_{DF} = L\frac{di}{dt}$ & $i = -C\frac{du_C}{dt}$;

   We get: $\frac{d^2u_C}{dt^2} + \frac{1}{LC}u_C = 0$.

2. The differential equation satisfied by u_C is of 2nd order of the form $u_C'' + w_0^2\, u_C = 0$, then the voltage performs free undamped oscillations.

3. We have $u_C = A\cos(w_0 t)$, so $u_C'' = -A\, w_0^2 \cos(w_0 t)$;

   Substitution in the differential equation we get: $A\cos(w_0 t)\left(-w_0^2 + \frac{1}{LC}\right) = 0$;

   (but $A\cos(w_0 t) \neq 0$), then $-w_0^2 + \frac{1}{LC}$; thus $w_0 = \frac{1}{\sqrt{LC}}$.

   At $t = 0$, we have $u_C = U_0$, then $U_0 = A\cos(0)$; thus $A = U_0$.

4. The proper period is given by $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{LC}$.

III-

1. In steady state $u_{AB} = E = 8V$, the electric charge $Q_0 = C\,u_{AB} = C\times E = 4\mu F\times 8V = 32\,\mu C$;

The electric potential energy stored $E_{e0} = \frac{1}{2} C \times E^2 = \frac{1}{2} 4\mu F \times (8V)^2 = 128\ \mu J$.

2. a) The electromagnetic energy stored in the system (coil, capacitor) is $E_T = E_e + E_m$;

Then $E_T = E_e + E_m = \frac{1}{2} C u_C^2 + \frac{1}{2} L i^2$ ($i = \frac{dq}{dt} = q'$ & $u_C = \frac{q}{C}$),

We get $E_T = \frac{1}{2C} q^2 + \frac{1}{2} L q'^2$.

The total energy is conserved so $\frac{dE_T}{dt} = 0$, $\frac{1}{C} q \times q' + L q' \times q'' = 0$;

But $i = q' \neq 0$, then $q'' + \frac{1}{LC} q = 0$.

b) We have $q = Q_0 \cos\left(\frac{1}{\sqrt{LC}} t\right)$, so $q'' = -Q_0 \left(\frac{1}{\sqrt{LC}}\right)^2 \cos\left(\frac{1}{\sqrt{LC}} t\right) = -\frac{Q_0}{LC} \cos\left(\frac{1}{\sqrt{LC}} t\right)$;

Substitution in the differential equation, we get:

$q'' + \frac{1}{LC} q = -\frac{Q_0}{LC} \cos\left(\frac{1}{\sqrt{LC}} t\right) + \frac{Q_0}{LC} \cos\left(\frac{1}{\sqrt{LC}} t\right) = 0$.

Furthermore, at $t = 0$, $q = Q_0 \cos(0) = Q_0$ (verified).

c) The expression of the proper angular frequency $w_0 = \frac{1}{\sqrt{LC}}$;

Then the proper period $T_0 = 2\pi\sqrt{LC} = 2\pi\sqrt{8 \times 10^{-3} \times 4 \times 10^{-6}} = 1.1 \times 10^{-3} s = 1.1 ms$.

IV-

1. The resistance of the coil is negligible, then the voltage across the capacitor should be alternating sinusoidal, which is satisfied by a differential equation of 2nd order of the form $u_C'' + w_0^2 u_C = 0$ (equation 2).
2. The angular frequency $w_0 = \frac{1}{\sqrt{LC}}$, then the proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{LC}$.
3. a) The period of the elastic potential energy is $T_{PE} = 1\,ms$, the proper period $T_0 = 2T_{PE} = 2\,ms$.

   b) The electric potential energy at $t_0 = 0$, is $E_{e0} = 36\ \mu J$;

   But $E_{e0} = \frac{1}{2} C u_{C0}^2$, then $u_{C0} = \sqrt{\frac{2 \times E_{e0}}{C}} = \sqrt{\frac{2 \times 36 \times 10^{-6}}{6 \times 10^{-6}}} \approx 3.5V$.
4. We have $T_0 = 2\pi\sqrt{LC}$, then $L = \frac{T_0^2}{4\pi^2 \times C} = \frac{(2 \times 10^{-3})^2}{4\pi^2 \times 36 \times 10^{-6}} = 2.8 \times 10^{-3} H = 2.8\,mH$.

V-

1. In the two cases the voltage across the capacitor is sinusoidal, then the internal resistances are negligible.
2. Referring to the curves, the proper periods are $T_{01} = 6ms$ & $T_{02} = 4\,ms$;

   But $T_0 = 2\pi\sqrt{LC}$, so $T_{01} = 2\pi\sqrt{L_1 C} > T_{02} = 2\pi\sqrt{L_2 C}$; then $L_1 > L_2$.

3. We have $L_1 = \frac{T_{01}^2}{4\pi^2 C} = \frac{(6\times10^{-3})^2}{4\times10\times5\times10^{-6}} = 0.18\,H = 180\,mH$;

& $L_2 = \frac{T_{02}^2}{4\pi^2 C} = \frac{(4\times10^{-3})^2}{4\times10\times5\times10^{-6}} = 0.08H = 80\,mH$.

4. a) The oscillations are free undamped, the electromagnetic energy is conserved:

$E_T = E_T|_{t=0} = E_m|_{t=0} + E_e|_{t=0}$ but at $t=0$, u_C is maximum then $i=0$;

Then, $E_T = \frac{1}{2}Cu_C^2 = \frac{1}{2}\times5\times4^2 = 40\,\mu J$.

b) The electromagnetic energy is conserved: $E_T = 40\,\mu J = \frac{1}{2}L_1 I_1^2$;

Then $I_1 = \sqrt{\frac{2\times40\times10^{-6}}{0.18}} = 0.021\,A = 21\,mA$.

5. The electromagnetic energy depends on the capacitance and the amplitude of the voltage across the capacitor which are not modified.

6. Similarly, $I_2 = \sqrt{\frac{2\times40\times10^{-6}}{0.08}} = 0.032\,A = 32\,mA$.

7. If the inductance of the coil is increased, the maximum value of the current will decrease while the electromagnetic energy remains unchanged.

VI-

1. a) The amplitude of the voltage u_C decreases with time, indicating a loss of energy which is due to the internal resistance of the coil.

b) Graphically $T = 1.5\,ms$.

c) At $t=0$, $u_C = 4V$.

2. The proper period $T_0 = 2\pi\sqrt{LC} = 2\pi\sqrt{10\times10^{-3}\times5\times10^{-6}} \approx 1.4\times10^{-3}s = 1.4\,ms$.

3. We have $\frac{1}{T^2} = \frac{1}{T_0^2} - \frac{1}{4\pi^2}\left(\frac{r}{2L}\right)^2$, so $r = 2\pi\times2L\sqrt{\frac{1}{T_0^2} - \frac{1}{T^2}}$

Then $r = 2\pi\times2\times10\sqrt{\frac{1}{(1.4)^2} - \frac{1}{(1.5)^2}} \approx 32\,\Omega$.

4. The electromagnetic energy $E_T = E_e + E_m = \frac{1}{2}Cu_C^2 + \frac{1}{2}Li^2$.

5. We have $\frac{dE_T}{dt} = -ri^2$, $Cu_C\times u'_C + L\times i\times i' = -ri^2$; but $i = C\frac{du_C}{dt} = C\times u'_C$ & $i' = C\times u''_C$;

Then $Cu_C\times u'_C + L\times C\times u'_C\times C\times u''_C = -rC^2u'^2_C$, (But $u'_C \neq 0$);

We get $u_C + L\times C\times u''_C = -rCu'_C$; divide by (LC)

Thus, $u''_C + \frac{r}{L}u'_C + \frac{1}{LC}u_C = 0$.

Problems

I- Electric Oscillations

An electric circuit is formed of a generator of constant *e.m.f.* $E = 6V$ and of internal resistance r_0, a capacitor, initially uncharged and of capacitance $C = 2.1\mu F$, a coil of inductance L and internal resistance $r = 10\Omega$ and a resistor of resistance $R = 50\Omega$.

In order to study the effect of R on the electric oscillations of an (R, L, C) circuit, we connect the circuit represented in figure 1.

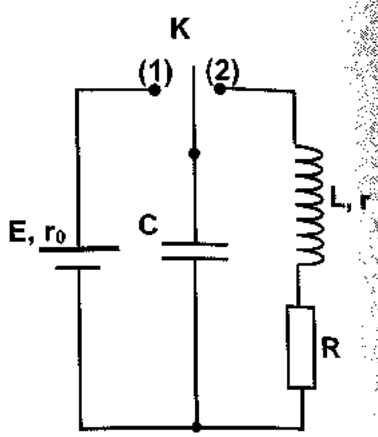


Figure 1

Part A
Charging of the capacitor

The switch is placed on position (1).

1. Derive the differential equation that describes the variations of $u_C = u_{AM}$ as a function of time.

2. The solution of this differential equation is $u_C = a + be^{-\frac{t}{\tau}}$.
   a) Determine the expressions of a, b & τ in terms of E, r_0 and C.
   b) Show that at the end of duration 5τ, the charging of the capacitor is practically completed.
   c) Calculate the value of r_0 if the minimum duration of charging is $4ms$.

3. Calculate the electric potential energy E_e stored in the capacitor at 5τ.

Part B
Electric oscillations

The capacitor being totally charged, we turn the switch to position (2) at an instant $t_0 = 0$ taken as an origin of time.

1. Derive the differential equation that describes the variations of $u_{AM} = u_C$ as a function of time.
2. The curves below (2-a, 2-b & 2-c) represent the variations of the voltages across the capacitor u_C, across the resistor u_R and that across the coil.

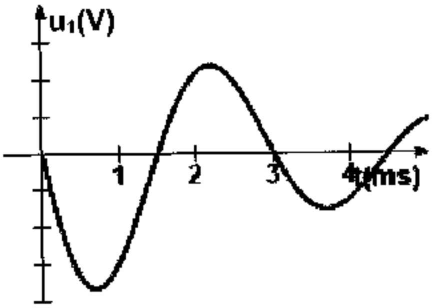


Figure 2-a

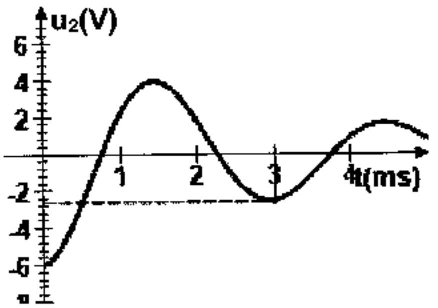


Figure 2-b

   a) Show that the curve (2-c) represents the voltage across the capacitor.

b) Specify to which component each of the other curves is associated.

c) Calculate the variation in the electromagnetic energy stored in the system between the instants $t_0 = 0$ and $t_1 = 3ms$.

3. Give:

a) the mode of oscillations performed.

b) the pseudo-period T of the oscillations.

4. Knowing that the ratio of two successive extremums $\frac{u_C(T)}{u_C(0)} = e^{-\left(\frac{R+r}{L}\right)T}$. Determine the value of L.

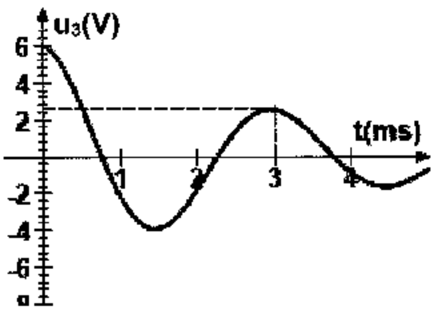


Figure 2-c

II- Modes of Oscillations

The aim of this exercise is to study the evolution of the voltage across the capacitor $u_C(t)$ in terms of time when connected to two dipoles (D_1) & (D_2).

✖ (D_1) is a coil of inductance L and negligible internal resistance;

✖ (D_2) is a coil of inductance L and internal resistance r.

We consider an ideal generator (G) of electromotive force is E, a capacitor of capacitance $C = 1\mu F$.

First experiment

✖ The capacitor is completely charged by the generator (G) then we connect it across the terminals of (D_1).

Second experiment

✖ The capacitor is completely charged by the generator (G) then we connect it across the terminals of (D_2).

An oscilloscope is connected across the capacitor in order to display its voltage $u_C(t) = u_{AB}$.

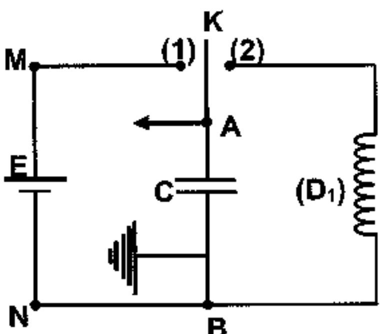


Figure 1

Part A

Theoretical study

First experiment

1. Applying the law of uniqueness of voltages, derive the differential equation that governs the variations of u_C.
2. Specify the mode of oscillations performed in this circuit.
3. The solution of the differential equation is $u_C = E\cos\left(\frac{2\pi}{T_0}t\right)$.

   Determine the expression of the proper period T_0 in terms of L & C.
4. Determine, at any instant t, the expression of :

   a) the electric potential energy stored in the capacitor.

   b) the magnetic energy stored in the coil.
5. Deduce that the total electromagnetic energy of the circuit remains constant.

Second experiment

1. Show that the differential equation that governs the variations of u_C is of the form $\frac{d^2u_C}{dt^2}+2\delta\frac{du_C}{dt}+\frac{1}{LC}u_C=0$ where δ is a constant whose expression is to be determined in terms of the given.
2. Specify the mode of oscillations performed in this circuit.
3. The pseudo period of these oscillations is given by $\frac{1}{T^2}=\frac{1}{T_0^2}-\left(\frac{\delta}{2\pi}\right)^2$.

   Compare, with justification, the values of T to that T_0.

Part B

Experimental study

The oscilloscope connected across the capacitor displays the curves shown in figures 1 & 2. Each of the oscillatory phenomena shown in figures 1 and 2 is characterized by its proper period T_0 or pseudo-period T.

1. Indicate, with justification, the curve that corresponds to each experiment.
2. Give the values of E, T & T_0.
3. Deduce the values of L and r.

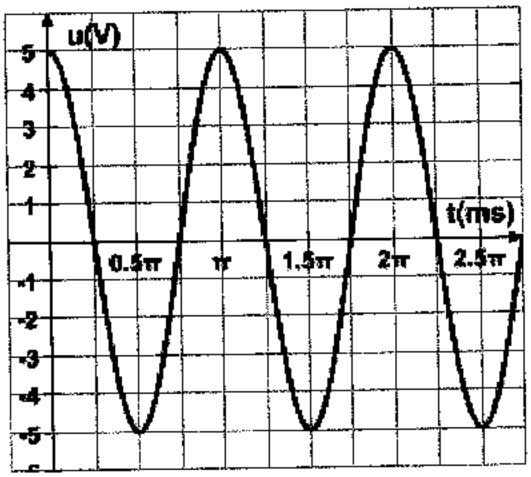


Figure 1

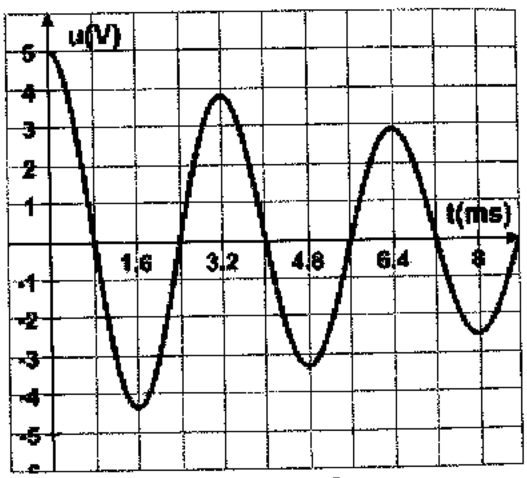


Figure 2

III- Capacitor in Different Circuits

In order to study the variations of the voltage across a capacitor, we connect the circuit shown in figure 1.

This circuit is formed of:

- ✖ a generator of constant *e.m.f* «E» and of negligible internal resistance;
- ✖ a resistor of resistance R of variable resistance;
- ✖ a coil of inductance L and internal resistance r;
- ✖ a capacitor of capacitance $C = 1\mu F$;
- ✖ a double switch K.

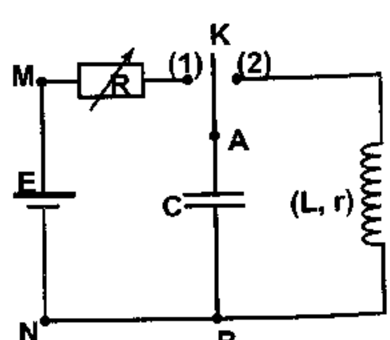


Figure 1

Part A
DC voltage

The capacitor is initially neutral. At the instant $t_0 = 0$, we turn K to position (1).

1. Applying the law of addition of voltages, show that the differential equation that describes the variation of the voltage u_C across the capacitor has the form: $E = RC\frac{du_C}{dt} + u_C$.

2. Verify that the solution of this differential equation $u_C = E\left(1 - e^{-\frac{t}{RC}}\right)$.

3. The graphs representing the evolution of u_C as a function of time, for two values of the resistances $R_1 = 1k\Omega$ and $R_2 = 5k\Omega$ are represented by the curves (C_1) & (C_2) shown in figure 2. (Δ_1) & (Δ_2) are the tangents to these curves respectively at $t_0 = 0$.
   a) Identify, with justification, the curve that describes the steady state in the circuit.
   b) Determine E, and deduce the maximum value Q_0 of the charge of the capacitor.

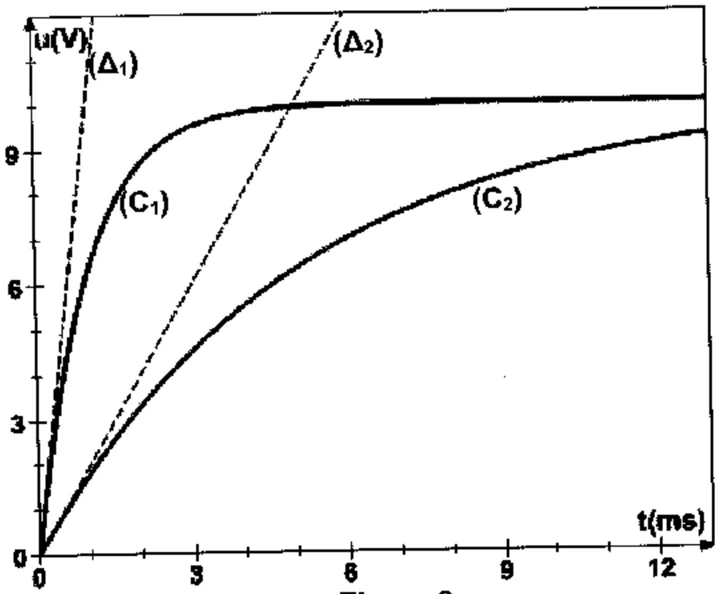


Figure 2

4. a) Define the time constant τ of an R-C circuit.
   b) Determine the time constants τ_1 & τ_2 corresponding respectively to R_1 & R_2.
   c) Verify that $\frac{\tau_1}{R_1} = \frac{\tau_2}{R_2}$ then compare this value to that of the capacitance C.

5. Specify, the influence of increasing the resistance on:
   a) the duration Δt needed to charge completely the capacitor.
   b) the value of the voltage across the capacitor in steady state.
   c) the current in the circuit at $t_0 = 0$.

Part B
Oscillations

The capacitor is completely charged, the switch is turned to position (2) at an instant taken as a new origin of time.

The curve representing the evolution of the voltage across the capacitor u_C as a function of time is represented in figure 3.

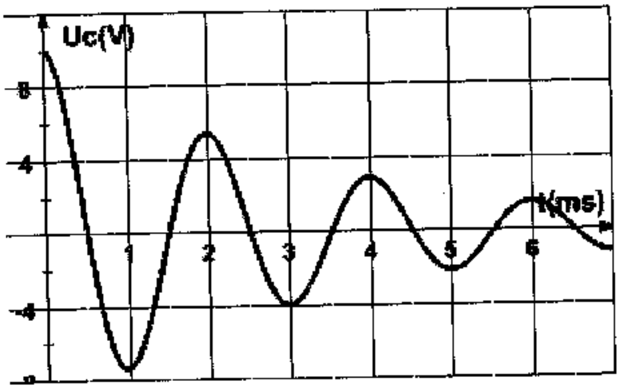


Figure 3

1. a) Give the mode of oscillations performed.
   b) Specify the cause of the damping observed.

2. a) Determine, graphically, the duration of one oscillation T.

b) Taking T as equal to the proper period T_0 of the circuit, calculate the value of the inductance L of the coil.

3. Calculate the average power dissipated between the instants $t_0 = 0$ & $t = 6ms$.

Part C

Driven oscillations

In order to drive the oscillations of the circuit, we add an electronic device (D)

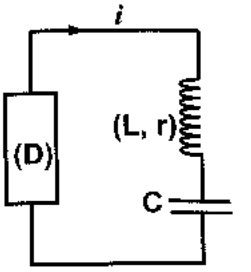


Figure 4

1. Explain the meaning of «drive the oscillations».
2. Write the voltage u_D across the device (D), as a function of the resistance r of the coil and the current i in the circuit, so that the differential equation that governs the evolution of the voltage across the capacitor u_C to be of the form $\frac{d^2u_C}{dt^2} + \frac{1}{LC}u_C = 0$.
3. Deduce that the device (D) is equivalent to a resistor of negative resistance.

IV-Engineering 2007/2008

Importance of an Oscillating (L, C) Circuit

Part A

Charging of the capacitor

In the circuit of figure 1, $C = 10\,\mu F$, $L = 1H$, the value of R is adjustable and $E = 20\,V$. An oscilloscope can display the variations of the voltage $u_C = u_{AM}$ and that of the voltage $u_R = u_{BM}$.

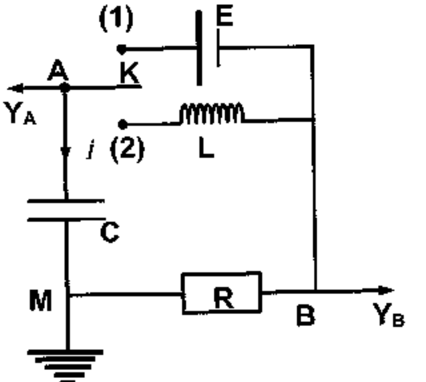


Figure 1

1. We adjust R to the value $R = 50\,\Omega$. At a given instant, the switch K is placed on the position (1).
   a) Give the expression of the RC series circuit time constant τ.
   b) Deduce the minimum duration at the end of which the capacitor can be considered practically charged.
2. Calculate, at the end of charging, the energy stored by the capacitor.

Part B

Ideal oscillating circuit

We adjust R to the value zero and, at the instant $t_0 = 0$, we place K on the position 2.

1. Derive the differential equation describing the variations of u_C in terms of time.
2. Determine the expression of T_0 so that $u_C = A\cos\left(\frac{2\pi}{T_0}t\right)$ is a solution of the above differential equation. Calculate A and T_0.

Part C

Exploiting a waveform

We adjust R to the value $R = 50\,\Omega$. The oscilloscope gives us the curves of figure 2.

1. Calculate, at the instant $t_1 = 5\,ms$:
   a) the value of the current carried by the circuit.
   b) the total energy stored in the circuit.
2. Deduce the average power lost between the instants t_0 and t_1.
3. Determine the duration T of one oscillation.
   Compare T with T_0.

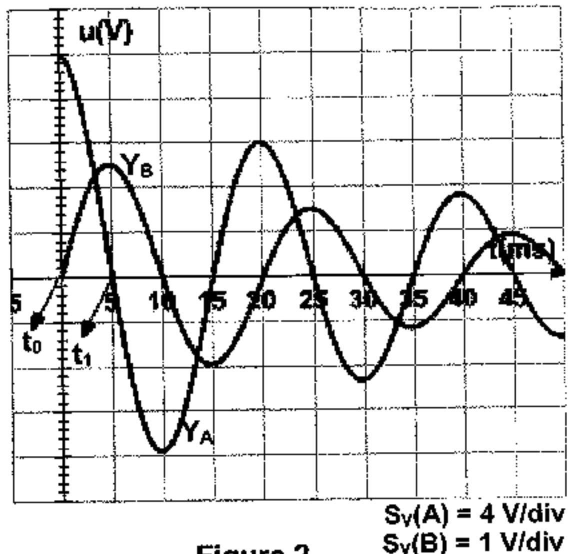


Figure 2

Part D

The gantry

To avoid merchandise robbery, we attach to each item a small LC oscillating circuit. At the exit of a store; everyone is obliged to pass through a security gantry.

This gantry always emits a radio wave of weak power and of frequency $f = 10\,MHz$, which is exactly equal to the natural frequency f_0 of small oscillator. In these conditions, the circuit picks-up the emitted energy, oscillates, and then emits a wave disturbing that emitted by the gantry. The detection of this disturbance triggers an alarm.

1. Why must f equal to f_0?
2. The capacitance C' of the capacitor is equal to $0.5\,nF$. Determine the coil's inductance L'.
3. a) Calculate the wavelength of the radio wave emitted by the gantry (given $c = 3\times10^8\,m/s$).
   b) This wave has to be emitted in many directions. It undergoes one of the physical phenomena: reflection, refraction or diffraction. Which one?

Solutions

I-

Part A

1. Law of uniqueness of voltages $u_G = u_C$, $E - r_0 i = u_C$; so $u_C + r_0\, i = E$;

   But the capacitor is charging so $i = C\dfrac{du_C}{dt}$, then $u_C + r_0\, C\dfrac{du_C}{dt} = E$.

2. a) We have $u_C = a + be^{-\frac{t}{\tau}}$, so $\dfrac{du_C}{dt} = -\dfrac{b}{\tau}e^{-\frac{t}{\tau}}$;

   Replacing in the differential equation, we get $a + be^{-\frac{t}{\tau}}\left(1 - \dfrac{r_0 C}{\tau}\right) = E$ verified at any instant t;

   (but $be^{-\frac{t}{\tau}} \neq 0$), then $a = E$ & $1 - \dfrac{r_0 C}{\tau} = 0$, we get $\tau = r_0 C$.

   The capacitor is taken neutral at the origin of time so $t = 0$, for $u_C = 0$; then $b = -a = -E$;

   Thus $u_C = E - E\,e^{-\frac{t}{\tau}} = E\left(1 - e^{-\frac{t}{\tau}}\right)$ where $\tau = r_0 C$.

   b) At $t = 5\tau$, $u_C = E\left(1 - e^{-5}\right) = 0.99\,E \approx E$.

   c) The minimum duration for the complete charging of the capacitor $5\tau = 4ms = 5\,r_0 C$

   Then $r_0 = \dfrac{4 \times 10^{-3}}{5 \times 2.1 \times 10^{-6}} = 381\,\Omega$.

3. The electric potential energy stored in the capacitor is given by $E_e = \dfrac{1}{2}Cu_C^2 = \dfrac{1}{2}C\,E^2$;

   Then $E_e = \dfrac{1}{2} \times 2.1 \times 6^2 = 37.8\mu J$.

Part B

1. Law of addition of voltages: $u_C = u_R + u_L$; so $u_C = ri + L\dfrac{di}{dt} + Ri = (r + R)i + L\dfrac{di}{dt}$;

   For a convenient positive direction $i = -C\dfrac{du_C}{dt}$;

   Then $u_C = -(r + R)C\dfrac{du_C}{dt} - LC\dfrac{d^2u_C}{dt^2}$; thus $\dfrac{d^2u_C}{dt^2} + \dfrac{(r + R)}{L}\dfrac{du_C}{dt} + \dfrac{1}{LC}u_C = 0$.

2. a) The capacitor was charged at $t = 0$, $u_C = 6V \neq 0$; then its representative curve is 2-c.

   b) At $t = 0$, the tangent to the curve representing u_C is horizontal, so $\left.\dfrac{du_C}{dt}\right|_{t=0} = 0$, then the voltage across the resistor $u_R = Ri = 0$ which corresponds to the curve 2.a)

   Thus, the curve 2.b) corresponds to the voltage across the coil.

   c) At $t_0 = 0$ & $t_1 = 3ms$, the current is zero then no magnetic energy stored in the coil at these instants $E_m\big|_{t_1} = E_m\big|_{t_0=0} = 0$;

$$\Delta E = E|_{t_1} - E|_{t_0=0} = E_e|_{t_1} - E_e|_{t_0=0} = \frac{1}{2}C\left(u_C|_{t_1}\right)^2 - \frac{1}{2}C\left(u_C|_{t_0=0}\right)^2;$$

Then $\Delta E = \frac{1}{2}2.1\times10^{-6}\times6^2 - \frac{1}{2}2.1\times10^{-6}\times2.6^2 = 7.1\times10^{-6} - 3.78\times10^{-5} = -3.17\times10^{-5}\,J$.

3. a) The oscillations are free damped.

   b) The pseudo-period is $T = 3ms$.

4. We have $\frac{u_C(T)}{u_C(0)} = e^{-\left(\frac{R+r}{L}\right)T}$, then $L = -\frac{(R+r)T}{\ell n\left(u_C(T)/u_C(0)\right)}$;

   Thus, $L = -\frac{(10+50)\times3\times10^{-3}}{\ell n(2.6/6)} = 0.215H$.

II-

Part A

First experiment

1. Law of uniqueness of voltages $u_C = u_{D_1}$; so $u_{D_1} = L\frac{di}{dt}$ (negligible internal resistance);

   But $i = -C\frac{du_C}{dt}$, then $\frac{di}{dt} = -C\frac{d^2u_C}{dt^2} = -Cu_C''$;

   Then $u_C + LC\,u_C'' = 0$; thus $u_C'' + \frac{1}{LC}u_C = 0$.

2. Free un-damped oscillations.

3. We have $u_C = E\cos\left(\frac{2\pi}{T_0}t\right)$; so $u_C'' = -E\left(\frac{2\pi}{T_0}\right)^2\cos\left(\frac{2\pi}{T_0}t\right)$;

   Replace u_C and u_C'' in the differential equation: $-E\left(\frac{2\pi}{T_0}\right)^2\cos\left(\frac{2\pi}{T_0}t\right) + \frac{E}{LC}\cos\left(\frac{2\pi}{T_0}t\right) = 0$;

   $E\cos\left(\frac{2\pi}{T_0}t\right)\left(-\left(\frac{2\pi}{T_0}\right)^2 + \frac{1}{LC}\right) = 0$; is verified at any instant t;

   But $E\cos\left(\frac{2\pi}{T_0}t\right) \neq 0$, then $-\left(\frac{2\pi}{T_0}\right)^2 + \frac{1}{LC} = 0$; thus $T_0 = 2\pi\sqrt{LC}$.

4. a) The electric potential energy $E_e = \frac{1}{2}Cu_C{}^2 = \frac{1}{2}CE^2\cos^2\left(\frac{2\pi}{T_0}t\right)$.

   b) The magnetic energy $E_m = \frac{1}{2}Li^2$; $i = -C\frac{du_C}{dt} = -Cu_C' = EC\frac{2\pi}{T_0}\sin\left(\frac{2\pi}{T_0}t\right)$;

   $E_m = \frac{1}{2}LE^2C^2\frac{4\pi^2}{T_0{}^2}\sin^2\left(\frac{2\pi}{T_0}t\right) = \frac{1}{2}LE^2C^2\frac{4\pi^2}{4\pi^2LC}\sin^2\left(\frac{2\pi}{T_0}t\right)$;

   Thus, $E_m = \frac{E^2C}{2}\sin^2\left(\frac{2\pi}{T_0}t\right)$;

5. The electromagnetic energy: $E_T = E_e + E_m$; then $E_T = \frac{E^2C}{2}\cos^2\left(\frac{2\pi}{T_0}t\right) + \frac{E^2C}{2}\sin^2\left(\frac{2\pi}{T_0}t\right)$;

Thus, $E_T = \frac{E^2C}{2}\left[\cos^2\left(\frac{2\pi}{T_0}t\right) + \sin^2\left(\frac{2\pi}{T_0}t\right)\right] = \frac{1}{2}CE^2$ which is constant;

Second experiment

1. Law of addition of voltages: $u_C = u_{D_2}$; $u_C = ri + L\frac{di}{dt}$; $u_C + rC\frac{du_C}{dt} + LC\frac{d^2u_C}{dt^2} = 0$;

$\frac{d^2u_C}{dt^2} + \frac{rC}{LC}\frac{du_C}{dt} + \frac{1}{LC}u_C = 0$; by comparison : $2\delta = \frac{r}{L} \Rightarrow \delta = \frac{r}{2L}$;

2. Free damped oscillations.

3. We have $\frac{1}{T^2} = \frac{1}{T_0^2} - \left(\frac{\delta}{2\pi}\right)^2$; so $\frac{1}{T^2} - \frac{1}{T_0^2} = -\left(\frac{\delta}{2\pi}\right)^2 < 0$;

$\frac{1}{T^2} - \frac{1}{T_0^2} < 0$; then $\frac{1}{T^2} < \frac{1}{T_0^2}$; thus $T > T_0$.

Part B

1. Figure 1, corresponds to the first experiment since the oscillations are un-damped; while figure 2, corresponds to the second experiment since the oscillations are damped.
2. At $t = 0$, $u_C = E = 5V$;

   From figure 1: $T_0 = \pi\,(ms) \approx 3.14\,ms$.

   From figure 2: $T = 3.2ms$.

3. We have $T_0 = 2\pi\sqrt{LC}$; then $L = \frac{T_0^2}{4\pi^2C} = \frac{(\pi\times10^{-3})^2}{4\pi^2\times1\times10^{-6}} = 0.25H$;

   And $\frac{1}{T^2} - \frac{1}{T_0^2} = -\left(\frac{\delta}{2\pi}\right)^2$; $\left(\frac{\delta}{2\pi}\right)^2 = \frac{T^2 - T_0^2}{T^2\times T_0^2}$; $\delta = \frac{2\pi}{T\times T_0}\sqrt{T^2 - T_0^2}$;

   Then $r = \frac{4\pi L}{T\times T_0}\sqrt{T^2 - T_0^2} = \frac{4\pi\times0.25}{3.2\times10^{-3}\times\pi\times10^{-3}}\sqrt{(3.2\times10^{-3})^2 - (\pi\times10^{-3})^2} \approx 190\Omega$.

III-

Part A

1. Law of addition of voltages: $u_{MN} = u_{MA} + u_{AB} + u_{BN}$;

   But $u_{AB} = Ri$ where $i = \frac{dq}{dt}$ with $q = Cu_C$, then $i = C\frac{du_C}{dt}$; thus $E = u_C + RC\frac{du_C}{dt}$.

2. We have $u_C = E\left(1 - e^{-\frac{t}{RC}}\right)$, so $\frac{du_C}{dt} = \frac{E}{RC}e^{-\frac{t}{RC}}$;

   Replacing in the differential equation, we get:

   $u_C + RC\frac{du_C}{dt} = E\left(1 - e^{-\frac{t}{RC}}\right) + RC\frac{E}{RC}e^{-\frac{t}{RC}} = E - Ee^{\frac{t}{RC}} + Ee^{-\frac{t}{RC}} = E$ (verified).

3. $R_1 = 1k\Omega$ and $R_2 = 5k\Omega$;

a) The curve (C_1) describes the steady state, because it has a horizontal asymptote.

b) In the steady state, the voltage between the terminals of the capacitor is $u_C = E = 10V$.

The charge carried $Q_0 = Cu_C = CE = 1\mu F \times 10V = 10\mu C$.

4. a) The time constant τ of the (R, C) circuit, is the duration required so that the voltage between the terminals of the capacitor reaches 63% of its value in the steady state.

b) The time constant τ is the abscissa of the point of intersection between the tangent (Δ) to the curve representing u_C with the horizontal asymptote.

From (C_1): $\tau_1 = 1ms$.

From (C_2): $\tau_2 = 5ms$.

c) $\frac{\tau_1}{R_1} = \frac{1\times10^{-3}s}{1\times10^{+3}\Omega} = 10^{-6}s/\Omega$ & $\frac{\tau_2}{R_2} = \frac{5\times10^{-3}s}{5\times10^{+3}\Omega} = 10^{-6}s/\Omega$;

The ratio is constant equal to the capacitance of the capacitor $\frac{\tau_1}{R_1} = \frac{\tau_2}{R_2} = C = 10^{-6}F$.

5. a) The minimum duration for the complete charging of the capacitor is $\Delta t = 5\tau = 5RC$ which is proportional to the resistance of the resistor. Then if we increase the resistance of the resistor, the charging duration increases also.

b) The voltage across the capacitor in the steady state is $u_C = E$, independent of the resistance.

c) We have $E = u_R + u_C$, at $t = 0$, $u_C = 0$; then $I_0 = \frac{E}{R}$;

The current at $t = 0$, is inversely proportional to the resistance of the resistor, then if we increase R the current I_0 decreases.

2nd method:

Observing the tangents (Δ_1) & (Δ_2), it is obvious that (Δ_1) is above (Δ_2);

So $\text{slope}(\Delta_1) > \text{slope}(\Delta_2)$; $\left.\frac{du_C}{dt}\right|_{t=0/C_1} > \left.\frac{du_C}{dt}\right|_{t=0/C_2}$; but $i = C\frac{du_C}{dt}$;

Then $\left.C\frac{du_C}{dt}\right|_{t=0/C_1} > \left.C\frac{du_C}{dt}\right|_{t=0/C_2}$; thus $I_{01} > I_{02}$.

Part B

1. a) The oscillations are free damped.

b) The damping is due to the internal resistance of the coil.

2. a) The pseudo-period $T = 2ms$.

b) The expression of the pseudo-period $T = T_0 = 2\pi\sqrt{LC}$;

Then $L = \frac{T^2}{4\pi^2 C} = \frac{(2\times10^{-3})^2}{4\pi^2\times10^{-2}} = 0.1H$.

3. At the instants $t = 0$ & $t = 6ms$, the voltage of the capacitor passes through a local maximum, then the total energy of the circuit is electric potential:

$\Delta(E_t) = \Delta(E_e) = \frac{1}{2}C\left(u_C|_{t=6ms}\right)^2 - \frac{1}{2}C\left(u_C|_{t=0}\right)^2 = \frac{1}{2}\times10^{-6}(2^2 - 10^2) = -4.8\times10^{-5}J$;

The average power dissipated is $P_{av} = \frac{\Delta(E_t)}{\Delta t} = \frac{-4.8\times10^{-5}}{6\times10^{-3}} = -0.008W = -8mW$.

Part C

1. To drive the oscillations is to give the system an energy enough to compensate the loss due to Joule's effect and restore its proper period of oscillations.
2. Law of addition of voltages $u_D = u_L + u_C$; $u_D = ri + L\frac{di}{dt} + u_C$;

   Then $u_D = ri + LC\frac{du_C^2}{dt^2} + u_C$; thus $LC\frac{du_C^2}{dt^2} + u_C = 0$ if $u_D = ri$;

   We have $u_D = +ri$, the component (D) should provide the system equal to that dissipated by the resistor.

IV-

Part A

1. a) The time constant $\tau = RC$.

   b) The capacitor is practically charged after $t = 5\tau = 5\times5\times10^{-4}\ s = 2.5ms$.
2. At the end of charging $u_C = E = 20V$.

   The electric energy stored is then: $E_e = \frac{1}{2}CE^2 = 2\times10^{-3}J$.

Part B

1. Applying the law of addition of voltages $u_C + LC\frac{d^2u_C}{dt^2} = 0 \Rightarrow \frac{d^2u_C}{dt^2} + \frac{1}{LC}u_C = 0$.
2. We have $u_C = A\cos\left(\frac{2\pi}{T_0}t\right) \Rightarrow \frac{d^2u_C}{dt^2} = -A\left(\frac{2\pi}{T_0}\right)^2\cos\left(\frac{2\pi}{T_0}t\right)$;

   Replacing in the differential equation: $A\cos\left(\frac{2\pi}{T_0}t\right) - LCA\left(\frac{2\pi}{T_0}\right)^2\cos\left(\frac{2\pi}{T_0}t\right) = 0$;

   $A\cos\left(\frac{2\pi}{T_0}t\right)\left[1 - LC\left(\frac{2\pi}{T_0}\right)^2\right] = 0$, (but $A\cos\left(\frac{2\pi}{T_0}t\right) \neq 0$); then $1 - LC\left(\frac{2\pi}{T_0}\right)^2 = 0$;

   Thus $T_0 = 2\pi\sqrt{LC}$.

   At $t = 0$, $u_C = A\cos 0 = A = E \Rightarrow A = E = 10V$ & $T_0 = 2\pi\sqrt{LC} = 19.87\,ms$.

Part C

1. a) At t_1, $U_R = S_{vB}\times y = 1V/\text{div}\times2.5\text{div} = 2.5V$; then $I_1 = \frac{U_R}{R} = 0.05\,A$.

   b) At the instant t_1, $u_C = 0$ and the electric energy stored in the capacitor $E_e = 0$;

   ✖ the magnetic energy stored is: $E_m = \frac{1}{2}LI_1^2 = 1.25\times10^{-4}J$;

   ✖ the total energy stored is: $E_t = E_e + E_m = 1.25\times10^{-4}J$.

2. At t_0, $u_C = 20\,V$ then $E_e = 2\times10^{-3}\,J$ & $E_m = 0\,J$, since $I_0 = 0$.

Thus, $E_{t_0} = 2\times10^{-3}\,J$.

The power dissipated is $P = \dfrac{\Delta(E_t)}{\Delta t} = \dfrac{E_{t_1} - E_{t_0}}{t_1 - t_0} = \dfrac{1.25\times10^{-4} - 2\times10^{-3}}{5\times10^{-3}} = -0.3375\,W$.

3. The duration of one oscillation is then: $T = 20\,ms \approx T_0$. (Slightly greater than the proper period)

Part D

1. In order so that the circuit resonates.

2. At current resonance $f = f_0 = \dfrac{1}{2\pi\sqrt{LC}}$;

Then $L' = \dfrac{1}{4\pi^2 f_0^2 C'} = 5.07\times10^{-7}\,H = 0.507\,\mu H$.

3. a) The wavelength $\lambda = \dfrac{c}{f} = 30\,m$.

b) Diffraction.

Supplementary Problems

I-GS 2002 1st

Effect of the Resistance of a Resistor

According to the value of the resistance of the resistor in a circuit, the steady state is attained slower or faster, or the circuit may be (or may be not) the seat of an ideal electric oscillations.

In this exercise, we intend to show the effect of the resistance of the resistor in some electric circuits, when a resistor (R) of adjustable resistance R, a coil (B) if inductance $L = 0.64\,H$ and of negligible resistance, a capacitor (C) of capacitance $C = 10^{-6}\,F$, a generator (G) of negligible internal resistance and of electromotive force E, a switch (K), an oscilloscope with a memory and connecting wires.

Part A

Case of an R-L circuit

We connect up the (R, L) series circuit of figure 1, the switch (K) is closed at $t = 0$.

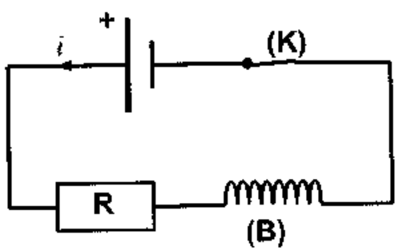


Figure 1

1. Derive, in the transient state, the differential equation in $u_R = Ri$ associated with the considered circuit.
2. The expression $u_R = U_0\left(1 - e^{-\frac{t}{\tau}}\right)$ is a solution of this differential equation. Deduce the expressions of U_0 and τ in terms of E, R & L.
3. Express, in terms of τ, the time t at the end of which the steady state is practically attained. Indicate then the value of the voltage across the coil.
4. a) Compare the values of τ and U_0 corresponding to:
   - *i-* $R_1 = 12\,\Omega$.
   - *ii-* $R_2 = 60\,\Omega$.
   - *iii-* $R_3 = 600\,\Omega$.

   b) Draw, in the system of axes (t, u_R), the shape of the curve that represents u_R for each value of R.

   c) Specify then the role of the value of R in the growth of the current towards the steady state.

Part B

Case of an R-C series circuit

We connect up the (R, C) series circuit of the figure 2. We close (K) at $t = 0$.

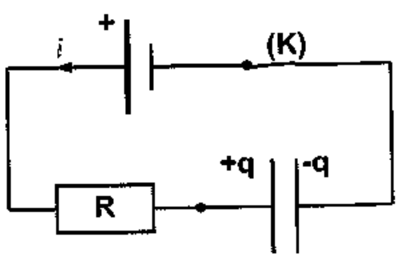


Figure 2 (C)

1. Derive, in the transient state the differential equation in $u_C = \frac{q}{C}$ associated with the considered circuit, q being the charge of the armature A of the capacitor.

2. The expression $u_C = U_C\left(1 - e^{-\frac{t}{\tau'}}\right)$ is a solution of this differential equation.

Deduce the expressions of U_C and τ' in terms of E , R & C .

3. a) Express in terms of τ', the time at the end of which the steady state is practically attained? Give then the value of the voltage across the terminals of the resistor.

b) Compare the values of τ' and U_C corresponding to:

i- $R_1 = 12\,\Omega$.

ii- $R_2 = 60\,\Omega$.

iii- $R_3 = 600\,\Omega$.

c) Draw, on the same system of axis (t, u_C) the shape of the curve that represents u_C for each value of R .

d) Specify then the role of the value of R in the growth of the current towards the steady state.

Part C

Case of an RLC series circuit

(C) Being charged is connected with (B) and (R) thus forming an RLC series circuit, this circuit is the seat of free electric oscillations. The oscilloscope, connected across the terminals of (C), would display the variation of u_C as a function of time. If (R) takes the value $R = 0$ and the switch (K) is closed at $t = 0$, we observe on the screen of the oscilloscope the waveform of figure 3.
If we give R a certain value and we close the switch at $t = 0$, we obtain the waveform of figure 4 without changing the adjustments of the oscilloscope .

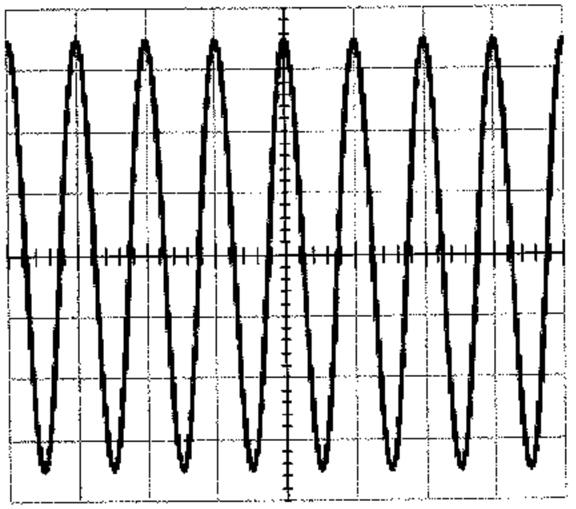


Figure 3

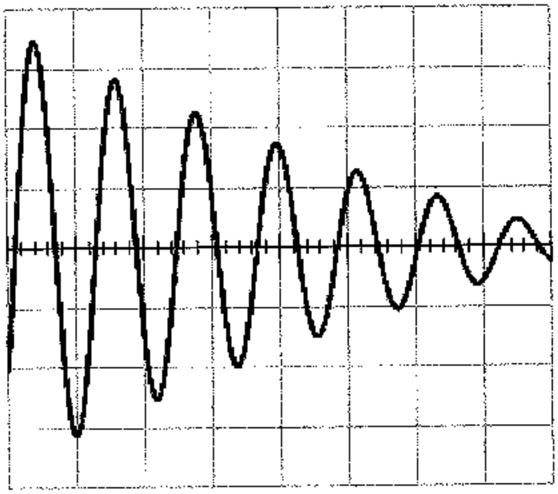


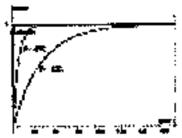
Figure 4

1. Give the expression of the proper (natural) period T_0 of the RLC series circuit thus formed and calculate its value.
2. Determine, using the waveforms of figure 3 the time base (horizontal sensitivity) used.
3. a) Specify the case in which the oscillations are un-damped

   b) Calculate the value of the pseudo-period T of oscillations.
4. Compare T & T_0. Conclude.

Answer Key

Part A



Part C

1. $T_0 \approx 5ms$.

II-GS 2001 1st
Electric Oscillations

Part A
Un-damped oscillations

Consider a capacitor of capacitance $C = 2\times10^{-10}\,F$ carrying a charge $Q = 2\times10^{-9}\,C$ and two coils (B_1) of inductance $L_1 = 5\times10^{-4}\,H$ & negligible resistance and (B_2) of inductance $L_2 = 5\times10^{-4}\,H$ & of resistance r . At the instant $t_0 = 0$, taken as the origin of time, we connect the capacitor across the terminals of (B_1) (Figure 1).

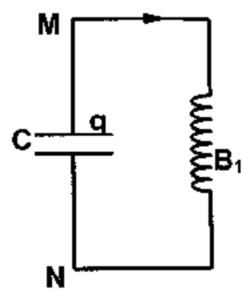


Figure 1

An ideal oscillating circuit is thus formed. **Take** $\pi^2 = 10$.

Denote by q the electric charge at the instant t of the armature of the capacitor that is connected to M and by i the electric current at the instant (taken positive when it circulates in the direction indicated on figure 1).

1. In this circuit, i and q are related by the expression: $i = -\dfrac{dq}{dt}$.

   Justify the $(-)$ sign in this expression.

2. Apply the law of uniqueness of potential difference to derive the differential equation that describes the variation of the charge q as a function of time.

   Deduce the natural (proper) frequency f_0 of this circuit.

3. The solution of the preceding differential equation has the form $q = Q\cos(2\pi f_0 t)$.

   a) Give the expression of the electric energy E_1 of the capacitor at the instant t .

   b) Give the expression of i as a function of time. Deduce the expression of the magnetic energy E_2 of the coil at the instant t .

   c) Show that the electromagnetic energy $E = E_1 + E_2$ of the circuit is constant and deduce its numerical value.

Part B
Damped oscillating circuit

Instead of connecting (B_1) to the capacitor at the instant $t_0 = 0$, we connect (B_2) (Figure 2).

1. Taking the same definitions for q and i , derive the differential equation that describes the variation of q with time.

2. Determine $\dfrac{dE}{dt}$ the derivative with respect to time of the electromagnetic energy E of the circuit.

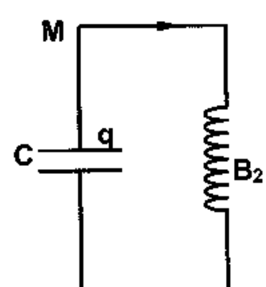


Figure 2

3. Derive the relation between $\frac{dE}{dt}$ and $r\,i^2$ then comment on this relation in terms of the energy transfer.
4. The circuit thus formed is used as a detector of radio waves. The most convenient wave to this circuit is the one whose frequency is equal to the natural frequency f_0 of the circuit.
   a) In what particular electric state would the circuit be when the most convenient wave is received?
   b) Calculate then the wavelength of the corresponding wave.

   Given: Speed of light in air: $c = 3\times10^8\ m/s$.

Answer Key

Part A 2. $f_0 = 5\times10^5\ Hz$. 3.c) $E = 10^{-8}\ J$.

Part B 4.b) $\lambda = 600\ m$.

GS – Sessions

I-GS 2014 1st
Capacitor and Coil

The aim of this exercise is to determine, by different methods, the characteristics of a capacitor and a coil.

Part A
RC circuit

Consider a series circuit formed, of a resistor of resistance $R = 100\Omega$, a neutral capacitor of capacitance C and a switch K, fed by a generator of negligible internal resistance and of $e.m.f$ E (figure 1).

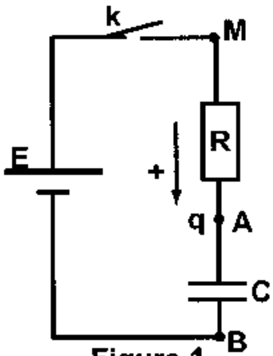


Figure 1

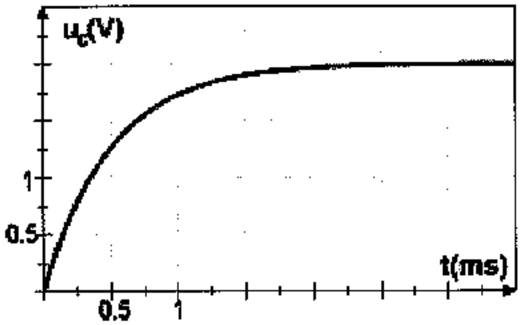


Figure 2

At the instant $t_0 = 0$, we close the switch K; then a current i flows in the circuit.

1. Derive the differential equation that describes the variations of $u_C = u_{AB}$ as a function of time.
2. The solution of this differential equation is $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$.
   a) Determine the expression of τ in terms of R and C.
   b) Show that at the end of duration 5τ, the charging of the capacitor is practically completely charged.
3. An appropriate system registers the variations of the voltage $u_C = u_{AB}$ across the terminals of the capacitor (figure 2).
   a) Referring to figure 2:
      i- indicate the value of E.
      ii- determine the value of τ.
   b) Deduce the value of C.

Part B
RL circuit

The capacitor is replaced by a coil of inductance L and of resistance r (figure 3).
At the instant $t_0 = 0$, the switch K is closed. An appropriate system records the variations of the current i in the circuit as a function of time (figure 4).

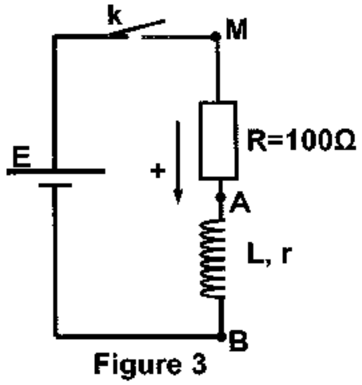


Figure 3

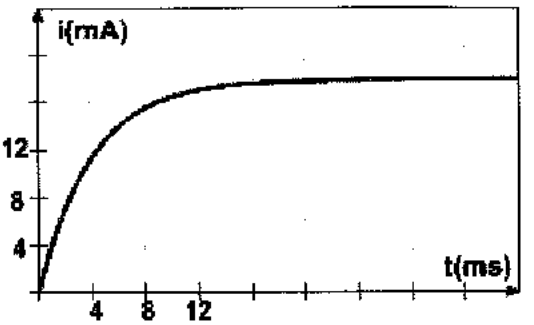


Figure 4

1. Derive the differential equation that describes the variations of i as a function of time.
2. Verify that $i = \frac{E}{R+r}\left(1 - e^{-\frac{t}{\tau}}\right)$ is a solution of the differential equation, where $\tau = \frac{L}{R+r}$.
3. Determine, in steady state, the expression of the current I in terms of E, R and r.
4. Referring to figure 4, indicate the value of I.
5. Determine the values of r and L.

Part C
RLC circuit

The previous capacitor of capacitance $C = 5\times10^{-6}F$, initially charged under the voltage E, is connected in series with the coil $(L, r = 11\,\Omega)$, the resistor of resistance $R = 100\,\Omega$ and the switch K as indicated in figure 5.

At $t_0 = 0$, the switch K is closed. The recording of the voltage $u_C = u_{AB}$ across the capacitor as a function of time is represented in figure 6.

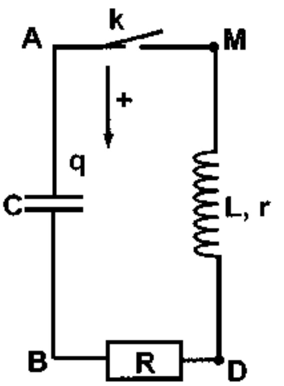


Figure 5

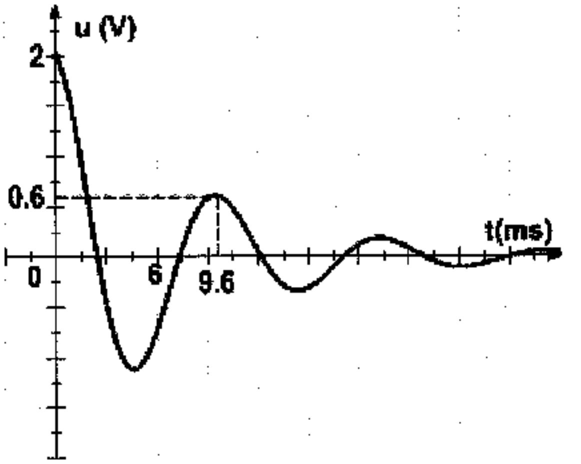


Figure 6

1. Derive the differential equation that describes the variation of u_C as a function of time.
2. The solution of this differential equation is $u_C = 2e^{-\left(\frac{R+r}{2L}\right)t}\cos\left(\frac{2\pi}{T}t\right)$.

   Use the graph of figure 6, to determine again the value of the inductance L found in part (B-5).

II-GS 2012 1st
Electromagnetic Oscillations

An electric circuit is formed of a generator of constant *e.m.f.* $E = 10\,V$ and of negligible internal resistance, a capacitor, initially uncharged and of capacitance $C = 10^{-3}\,F$, a coil of inductance $L = 0.1\,H$ and of negligible resistance and a rheostat of variable resistance R.
In order to study the effect of R on the electric oscillations of an (R, L, C) circuit, we connect the circuit represented in figure 1.

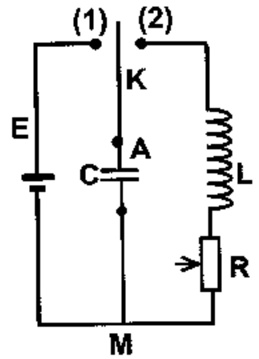


Figure 1

Part A
The switch is in position (1).
1. Give the name of the physical phenomenon that takes place in the electric circuit.
2. After closing the circuit for a sufficient time, specify the value of:
   a) the current;
   b) the voltage $u_{AM} = u_C$ across the capacitor;
   c) the electric energy W_{ele} stored in the capacitor.

Part B
The capacitor being totally charged, we turn the switch to position (2) at an instant $t_0 = 0$ taken as an origin of time.

First situation
The resistance of the rheostat is regulated at a value $R = 0$.
1. Derive the differential equation of the variation of $u_{AM} = u_C$ as a function of time.
2. The solution of the differential equation is of the form $u_C = E\cos\left(\frac{2\pi}{T_0}t\right)$.
   a) Determine, in terms of L and C, the expression of the proper period T_0 of the free electric oscillations that take place in the circuit.
   b) Calculate the value of T_0.
3. Express, as a function of time, the electric energy W_{ele} stored in the capacitor.
4. The electric energy W_{ele} is a periodic function of period T'. Write the relation between T' and T_0
5. Calculate the electric energy stored in the capacitor at the instant $t_0 = 0$.
6. Trace the shape of the graph of W_{ele} as a function of time.

Second situation
The rheostat is regulated at a small resistance R.
The variation of the electric energy W_{ele} as a function of time is represented in figure (2).
Referring to this figure:
1. give the name of the type of the electric oscillations;
2. determine the value of the pseudo-period T of the electric oscillations;
3. justify that at the instants: 0 ; $31.5\,ms$; $63\,ms$; $t_2 = 94.5\,ms$; $126\,ms$, the total energy stored in the circuit is electric;

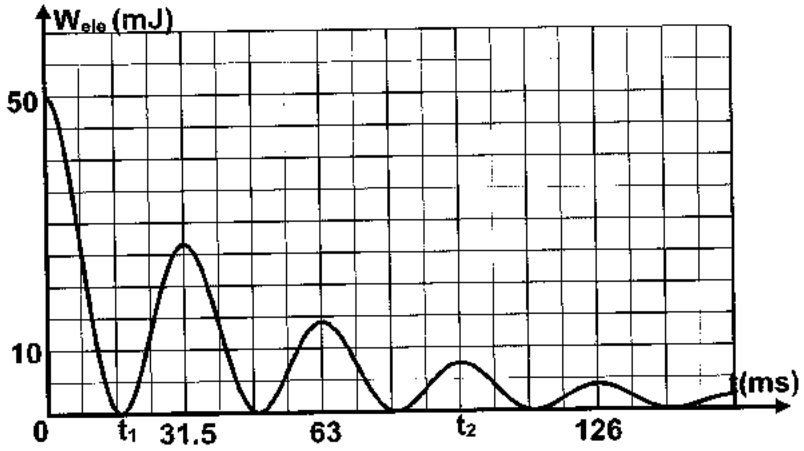


Figure 2

4. specify the form of the energy in the circuit at the instant t_1;
5. specify, between the instants $t_0 = 0$ and $t = 31.5\,ms$, the time interval during which the:
   - ✘ coil provides energy to the circuit;
   - ✘ capacitor provides energy to the circuit;
6. calculate the energy dissipated in the rheostat between the instants $t_0 = 0$ and t_2.

Third situation

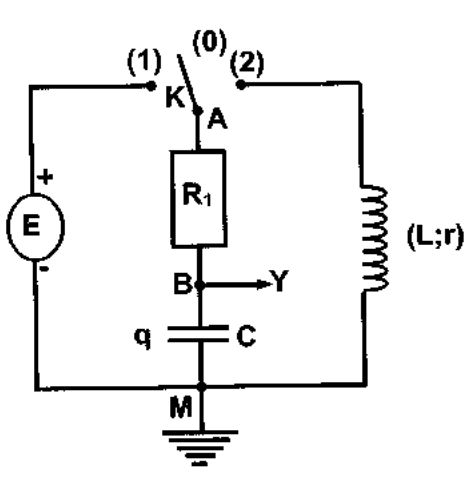
What will happen if the resistance of the rheostat is very large?

III-GS 2011 2nd
Studying Charging and Discharging Capacitor

The adjacent circuit allows to study the variation of the voltage $u_C = u_{BM}$ across a capacitor of capacitance C during charging.

We consider a generator delivering a constant voltage E, a resistor of resistance $R_1 = 25\,\Omega$.

Initially, the switch K is in position (0) and the capacitor is uncharged. An oscilloscope allows displaying the variation of u_C as a function of time.



Part A
Charging of a capacitor

At the instant $t_0 = 0$, the switch is in position (1) and the capacitor starts charging. At an instant t, the circuit carries a current « i » and the capacitor carries the charge q.

1. a) Redraw the diagram of the circuit indicating on it the real direction of « i ».
   b) Write down the relation between i and u_C.
2. a) Derive the differential equation in u_C.

b) The solution of this differential equation is of the form: $u_C = A + B e^{-\frac{t}{\tau_1}}$.

Determine the expressions of the constants A, B and τ_1.

c) Referring to the graph, determine:

i- the values of E and τ_1. Deduce that the value of C is $4\,\mu F$.

ii- the minimum duration at the end of which the capacitor is practically completely charged.

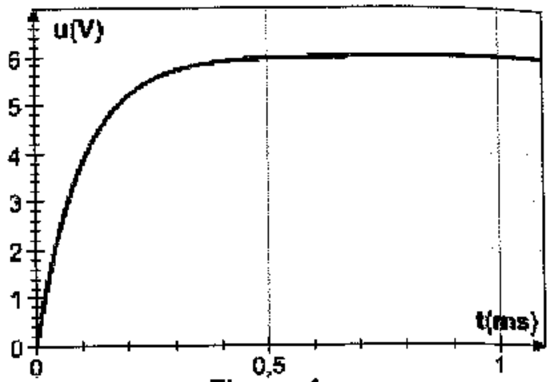


Figure 1

Part B

Discharging of the capacitor through a coil

K is moved from position (1) at the instant $t_1 = 0.6\,ms$ and becomes in position (2) at the instant $t_2 = 1\,ms$. Figure 2 shows the variation of u_C between the instants 0 and $17\,ms$.

1. The voltage u_C remains constant between the instants t_1 and t_2. Why?

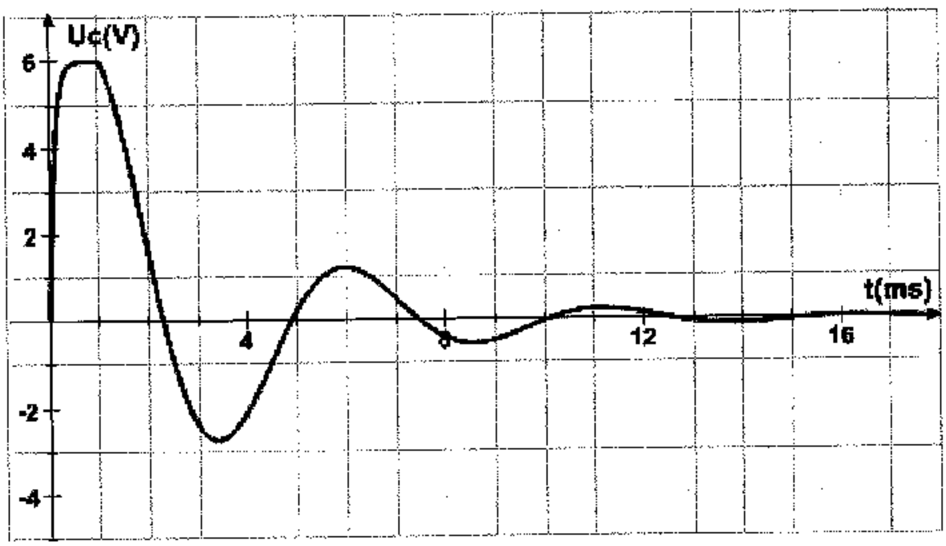


Figure 2

2. Starting from the instant $t_2 = 1\,ms$, the circuit is the seat of electric oscillations. Referring to the graph of Figure 2, give the value of the pseudo period T of the electric oscillations.
3. a) Write down the expression of the proper period T_0 in an LC circuit.

   b) Knowing that $L = 0.156\,H$ and $\left(\frac{2\pi}{T}\right)^2 = \left(\frac{2\pi}{T_0}\right)^2 - \left(\frac{R_1 + r}{2L}\right)^2$, calculate r.
4. a) Determine, referring to figure 2, the value of u_C at the instant $t = 6\,ms$.

   b) Calculate the value of the loss in electric energy in the circuit at the end of the first oscillation.

IV-GS 2011 1st

Electromagnetic Oscillations

The object of this exercise is to show evidence of the phenomenon of electromagnetic oscillations in different situations.

For this purpose, we consider an ideal generator G of $e.m.f$ $E = 3V$, an uncharged capacitor of capacitance $C = 1\mu F$, a coil of inductance $L = 0.1H$ and of resistance r, a resistor of resistance R, an oscilloscope, a double switch K and connecting wires.

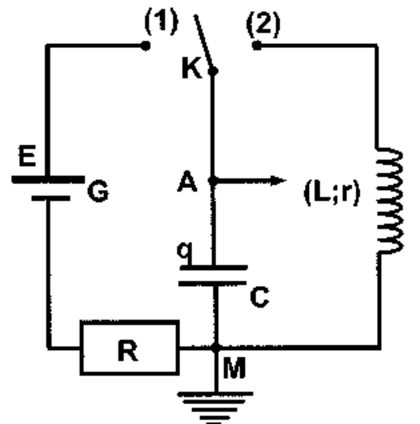


Figure 1

Part A

Charging of a capacitor

We connect up the circuit whose diagram is represented in figure 1. The oscilloscope is connected across the capacitor.

The switch K is in position (1). The capacitor is totally charged and the voltage across it is then $u_{AM} = U_0$.

1. Determine the value of U_0.
2. Calculate the electric energy W_0 stored in the capacitor at the end of charging.

Part B

Electromagnetic oscillations

The capacitor being totally charged, we turn the switch K to position (2) at the instant $t_0 = 0$. The circuit is then the seat of electromagnetic oscillations. At an instant t, the circuit carries a current i.

1. First situation (ideal circuit)

In the ideal circuit, we neglect the resistance r of the coil.

a) Redraw figure 1 showing on it an arbitrary direction of i.
b) Derive the differential equation that governs the variation of the voltage $u_{AM} = u_C$ across the capacitor as a function of time.
c) Deduce, then, the expression of the proper period T_0 of the electric oscillations in terms of L and C and calculate its value in ms with 2 digits after the decimal.
Take $\pi = 3.14$.
d) Draw a rough sketch of the curve representing the variation of the voltage u_C as a function of time.
e) Specify the mode of the electric oscillations that take place in the circuit.

2. Second situation (real circuit)

The variation of the voltage $u_{AM} = u_C$ is displayed on the screen of the oscilloscope as shown in the waveform of figure 2.

a) Specify the mode of the electric oscillations that take place in the circuit.
b) Give an energetic interpretation of the obtained phenomenon.
c) Referring to the waveform of figure 2:
i- give the duration T of one oscillation;

ii- compare T and T_0;

iii-specify the value around which the voltage u_C varies.

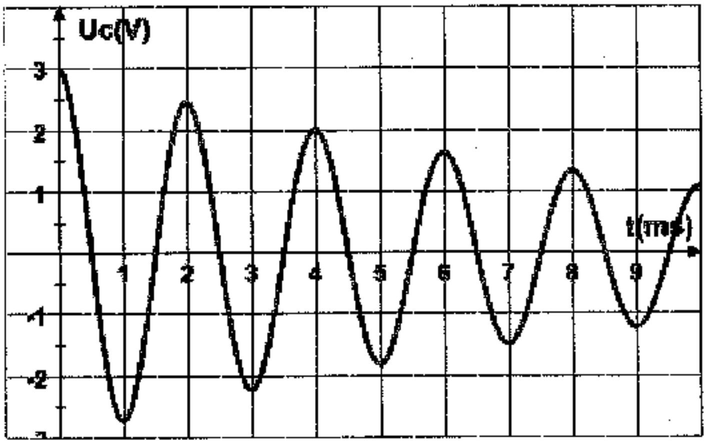


Figure 2

3. Third situation

We connect up a new circuit in which the coil, the uncharged capacitor and the switch K are connected in series across the generator G (Figure 3).

We close K at the instant $t_0 = 0$. At an instant t, the circuit carries a current i.

Figure 4 gives the variations, as a function of time, of i (Figure 4a) and u_C (Figure 4b).

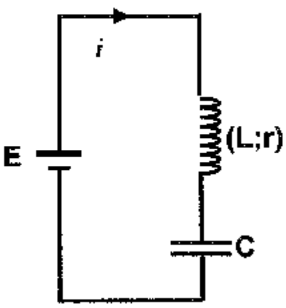


Figure 3

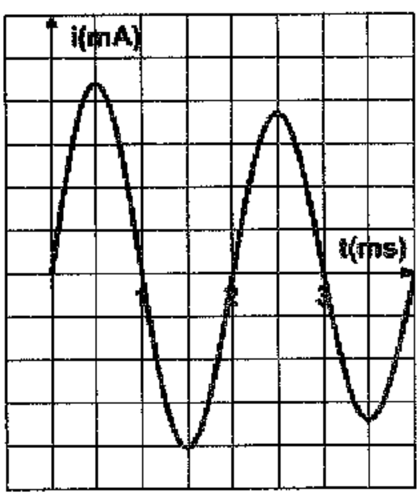


Figure 4a

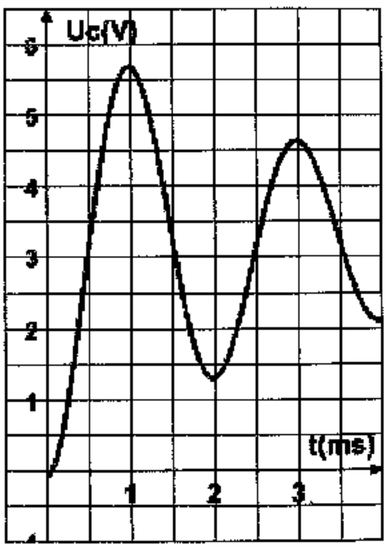


Figure 4b

a) Specify the value around which the voltage u_C varies.

b) Give the duration of one oscillation.

c) We consider the following 3 intervals of time: $0 \le t \le 0.5\,ms$; $0.5\,ms \le t \le 1\,ms$; $1\,ms \le t \le 1.5\,ms$.

Referring to the curves of figure 4, specify, with justification, the interval in which:

i- the coil supplies energy to the capacitor;

ii- the capacitor supplies energy to the coil;

iii-no energy exchange takes place between the coil and the capacitor.

V-GS 2009 2nd
Exchanged Energy

We connect up the circuit formed of a resistor of resistance $R = 2.2k\Omega$, an ideal generator of *e.m.f* $E = 8V$, a coil of inductance $L = 0.8H$ and of negligible resistance, a resistor of adjustable resistance r and two switches K_1 and K_2 (Figure 1).

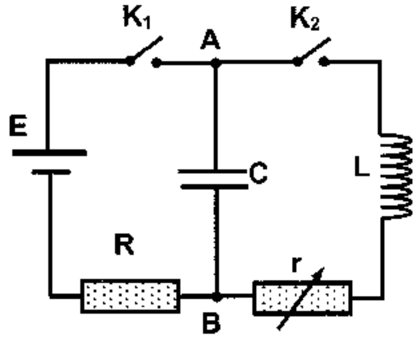


Figure 1

Part A
(R, C) series circuit

At an instant taken as an origin of time, $t_0 = 0$, we close the switch K_1, and K_2 remains open. We study the charging of the capacitor through the variation of the voltage $u_{AB} = u_C$ as a function of time.

1. Show that the differential equation in u_C is: $E = u_C + RC\dfrac{du_C}{dt}$.
2. Knowing that $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$ is the solution of this differential equation, determine the expression of the time constant τ in terms of R and C.
3. The curve of figure 2 shows the variation of u_C as a function of time.
   Using this curve, determine the time constant τ (indicating the method used).
4. Calculate the value of C.
5. a) Give, in ms, the duration t_1 at the end of which the voltage across the capacitor will no longer practically vary.
   b) Calculate the charge of the capacitor and the energy W_0 stored at the end of the duration t_1.

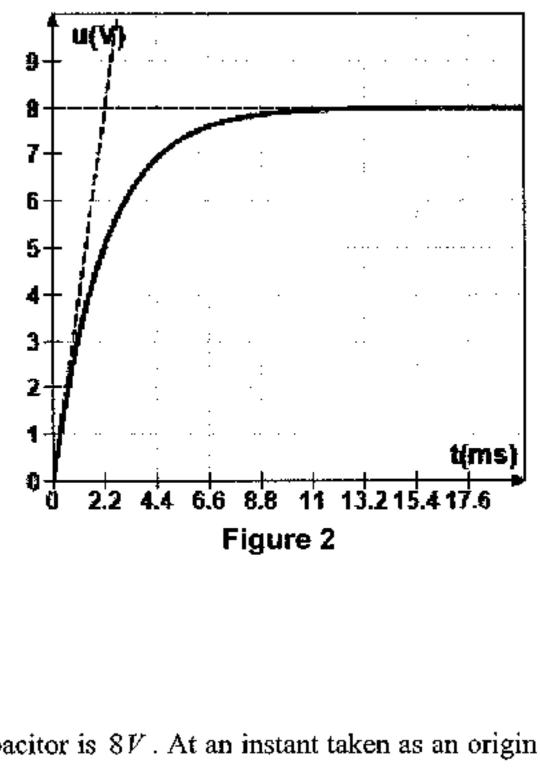


Figure 2

Part B
(L, C) series circuit

We give r the value zero. The voltage across the capacitor is $8V$. At an instant taken as an origin of time $(t_0 = 0)$, we open the switch K_1 and we close the switch K_2.

1. Derive the differential equation that describes the variation of the voltage u_C as a function of time.
2. The circuit is the seat of electric oscillations of proper period T_0.

   The solution of this differential equation is: $u_C = E\cos\left(\dfrac{2\pi}{T_0}t\right)$. Determine the value of T_0.
3. Trace the shape of the curve representing the variation of u_C as a function of time.
4. Specify the energy exchanges that take place in the circuit.

Part C
(r, L, C) series circuit

We give r a certain value. The voltage across the terminals of the capacitor is $8V$. We open K_1 and we close K_2 at the instant $t_0 = 0$.

The waveform of figure 3 shows the variation of the voltage u_C as a function of time.

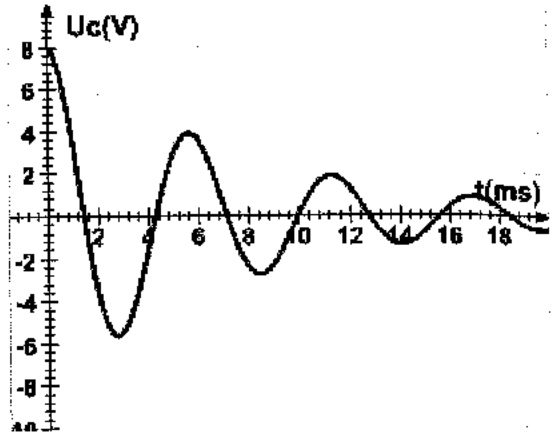


Figure 3

1. Specify the energy exchanges that take place in the circuit.
2. Referring to figure 3, find the pseudo-period T of the electric oscillations, then compare T and T_0.
3. At the end of the duration $t_n = nT$ (n being a whole number), the energy dissipated by Joule's effect is 98.6% of the energy W_0 initially stored in the capacitor.
   a) At the instant $t_n = nT$, the energy stored in the circuit is purely electric. Why?
   b) We denote by W_0 and W_n the electric energy of the oscillator at the instants t_0 and t_n respectively. Calculate W_n and determine n.

VI-GS 2007 1st

An Analogy

The object of this exercise is to show evidence of the analogy between a mechanical oscillator and an electric oscillator in the case of free oscillations.

Part A
Mechanical oscillator

A horizontal mechanical oscillator is formed of a solid (S) of mass $m = 0.546\,kg$ and a spring of un-jointed turns of stiffness $k = 5.70\,N/m$ and of negligible mass.

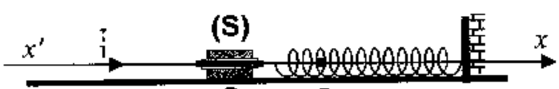


Figure1

The center of mass G of (S) is initially at the equilibrium position O on the axis $x'x$.

(S) shifted from O by a certain distance, is then released without initial velocity at the instant $t_0 = 0$, G thus performs a rectilinear motion along the axis $x'x$ (figure 1). At the instant t, its abscissa is x $\left(\overrightarrow{OG} = x\vec{i}\right)$ and its velocity is $\vec{v}$ $\left(\vec{v} = v\vec{i} = \frac{dx}{dt}\vec{i}\right)$.

The horizontal plane through the axis $x'x$ is taken as a gravitational potential energy reference.

First situation

General study

1. Write down the expression of the mechanical energy ME of the system [oscillator, Earth] in terms of m, k, x and v.
2. Determine the expression giving $\frac{d(ME)}{dt}$, the derivative of ME with respect to time.

Second situation

Free non-damped oscillations

We neglect all friction.

1. Derive the second order differential equation that governs the variations of x as a function of time.
2. Deduce the expression of the proper frequency f_0 of the oscillator and show that its value is $0.51\,Hz$.

Third situation

Free damped oscillations

In reality, the force $\vec{F}$ of friction is not negligible and its expression is given by: $\vec{F} = -\lambda\vec{v}$ at an instant t, λ being a positive constant.

1. Derive the second order differential equation describing the variations of x as a function of time knowing that $\frac{d(ME)}{dt} = \vec{F}.\vec{v}$
2. The adjacent figure 2 shows the variations of x as a function of time.
   a) How does the effect of the force of friction appear?
   b) Determine the pseudo-frequency f of the mechanical oscillations.
   c) Calculate the value of λ, knowing that f is given by the expression:

$$f^2 = f_0^2 - \frac{1}{4\pi^2}\left(\frac{\lambda}{2m}\right)^2.$$

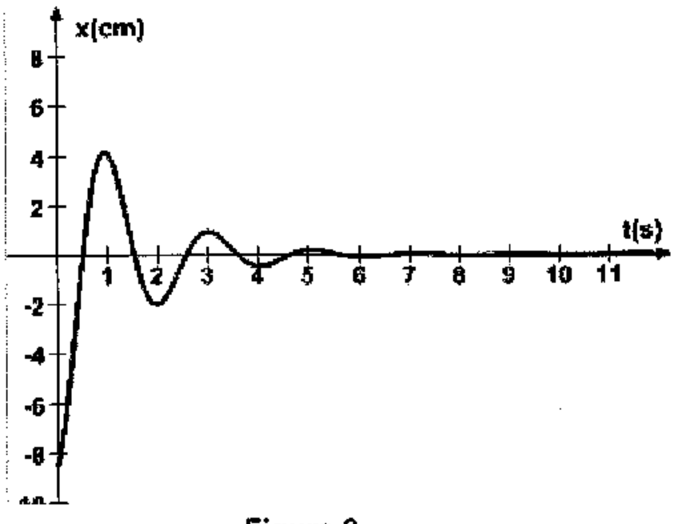


Figure 2

Part B

Electric oscillator

This oscillator is a series circuit formed of a coil of inductance $L = 43\,mH$ and of resistance $r = 11\,\Omega$, a resistor of adjustable resistance R, a switch K and a capacitor of capacitance $C = 4.7\,\mu F$ initially charged with a charge Q (Figure 3).

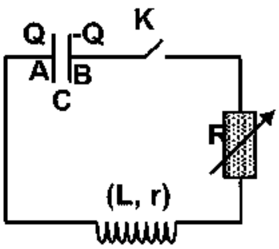


Figure 3

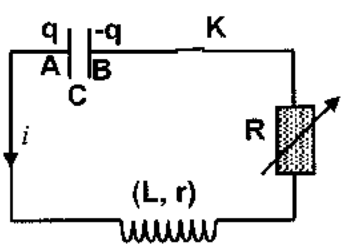


Figure 4

We close the switch K at the instant $t_0 = 0$. The circuit is thus the seat of electric oscillations. At the instant t, the armature A carries a charge q and the circuit carries a current i (Figure 4).

1. Write down the expression of the electromagnetic energy E of the circuit at the instant t (total energy of the circuit) as a function of L, i, q & C.
2. Knowing that $\frac{dE}{dt} = -(R+r)i^2$, derive the second order differential equation of the variations of q as a function of time.
3. Give the expression of the proper frequency f_0' of the electric oscillations and show that its value is $354.2\,Hz$.
4. The figure 5 gives the variations of q as a function of time.
   a) Due to what is the decrease with time in the amplitude of oscillations?
   b) Determine the pseudo-frequency f' of the electric oscillations.

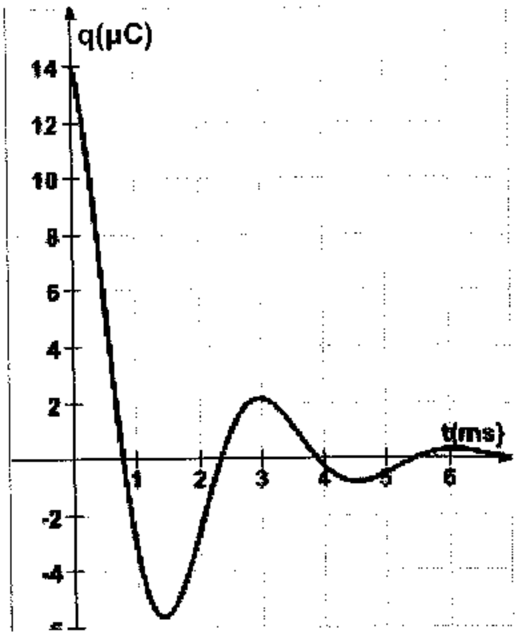


Figure 5

Part C

An analogy

1. Match each of the physical mechanical quantities x, v, m, λ and k with its corresponding convenient electric quantity.
2. a) Deduce the relation between f', f_0', L & $(R+r)$.
   b) Calculate the value of R.

VII-GS 2006 2nd

Electromagnetic Oscillations

An oscillating circuit is formed of a capacitor of capacitance $C = 1\,\mu F$ and a coil of inductance L and resistance r. In order to determine L and r, we connect up the circuit whose diagram is represented in figure 1. The connections of an oscilloscope are as indicated on this figure.

The $e.m.f$ of the generator is: $E = 10\,V$.

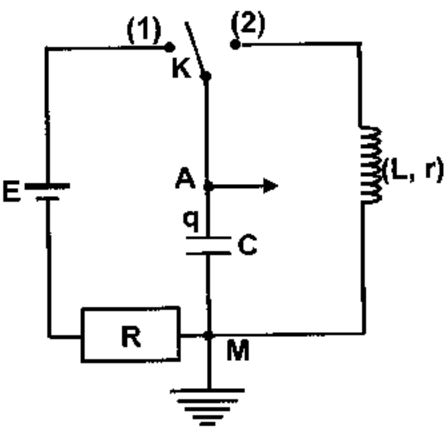


Figure 1

Part A

Charging the capacitor

The switch (K) is in position (1). The capacitor is totally charged and the voltage across its terminals is: $u_{AM} = U_0$.

1. Determine the value of U_0.
2. Calculate the electric energy stored in the capacitor.

Part B

Electromagnetic oscillations

The capacitor being charged, we move the switch (K) to the position (2) at the instant $t_0 = 0$.

At the instant t, the circuit carries a current i and the armature (A) carries the charge q.

First situation

Ideal circuit

In the ideal circuit, we neglect the resistance r of the coil.

1. Redraw figure 1 indicating an arbitrary direction of the current.
2. Derive the differential equation that governs the variation of the voltage $u_C = u_{AM}$ across the terminals of the capacitor as a function of time.
3. Deduce then the proper period T_0 of the electric oscillations in terms of L and C.
4. Give the shape of the curve representing u_C as a function of time.
5. Specify the mode of electric oscillations that is taking place in the circuit.

Second situation

Real circuit

The variation of the voltage u_C observed on screen of the oscilloscope is represented in the waveform of figure 2.

1. Specify the mode of electric oscillations that takes place in the circuit.
2. By referring to the waveform:
   a) give the value of the pseudo-period T of the electric oscillations.
   b) verify that the ratio of two successive positive extreme values of the voltage u_C is practically equal to a constant a (this is limited to the four extreme values).
3. We denote by E_n and E_{n+1} the electromagnetic energy of the electric oscillator at the instants nT and $(n+1)T$ respectively (n is a positive whole number).

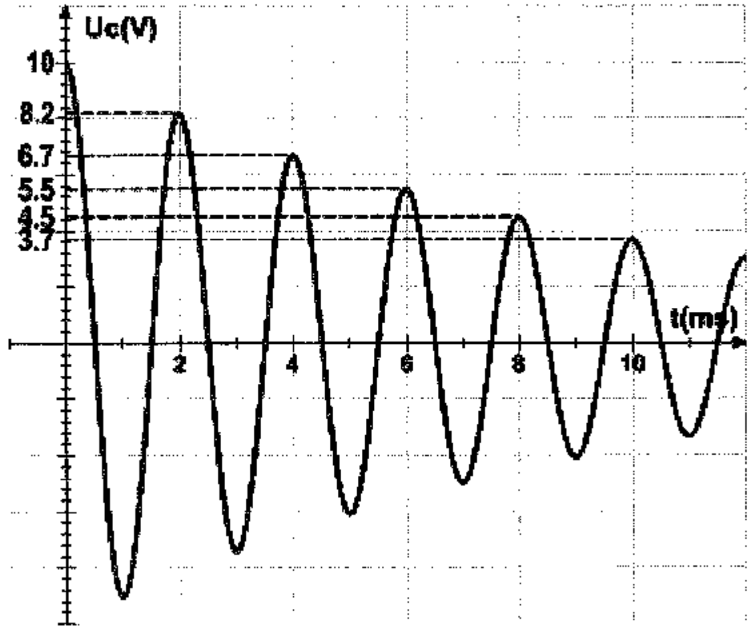


Figure 2

   a) The energy stored in the circuit at the instant when the voltage u_C is maximum is electric. Why?
   b) Derive the expression of the ratio $\dfrac{E_{n+1}}{E_n}$ in terms of a.
   c) Determine L and r knowing that $\dfrac{E_{n+1}}{E_n} = e^{-\frac{r}{L}T}$ and that the expression of the pseudo-period is:

$$\frac{4\pi^2}{T^2} = \frac{4\pi^2}{T_0^2} - \frac{1}{4}\left(\frac{r}{L}\right)^2.$$

Sessions Solutions – GS

I-GS 2014 1st

Part A

1. Law of addition of voltages $u_{MB} = u_{MA} + u_{AB}$;

So $E = Ri + u_C$ but $i = \frac{dq}{dt}$ and $q = Cu_C$ so $i = C\frac{du_C}{dt}$; then $u_C + RC\frac{du_C}{dt} = E$.

2. a) We have $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$ so $\frac{du_C}{dt} = \frac{E}{\tau}e^{-\frac{t}{\tau}}$;

Replacing in the differential equation we get $E = RC\frac{E}{\tau}e^{-\frac{t}{\tau}} + E - Ee^{-\frac{t}{\tau}} = E + Ee^{-\frac{t}{\tau}}\left(\frac{RC}{\tau} - 1\right)$

This equation is verified at any instant t so $\left(\frac{RC}{\tau} - 1\right) = 0$ then $\tau = RC$.

b) At $t = 5\tau$, $u_C = E\left(1 - e^{-5}\right) = 0.99E \approx E$ so it is completely charged.

3. a) Graphical study:

i- $u_C = E$ in the steady state so $E = 2V$.

ii- At $t = \tau$, $u_C = 0.63E = 0.63 \times 2 = 1.26V$; graphically $\tau = 0.5ms$.

b) We have $\tau = RC$ so $C = \frac{\tau}{R} = \frac{0.5 \times 10^{-3}}{100} = 5 \times 10^{-6}F = 5\mu F$.

Part B

1. Law of addition of voltages $u_{MB} = u_{MA} + u_{AB}$ so $E = Ri + ri + L\frac{di}{dt}$;

Then $E = (R + r)i + L\frac{di}{dt}$.

2. We have $i = \frac{E}{R+r}\left(1 - e^{-\frac{t}{\tau}}\right)$ so $\frac{di}{dt} = \frac{E}{R+r} \times \frac{1}{\tau}e^{-\frac{t}{\tau}}$ & $\tau = \frac{L}{R+r}$, then $\frac{di}{dt} = \frac{E}{L}e^{-\frac{t}{\tau}}$;

Substitution in the differential equation we get:

$(R + r)i + L\frac{di}{dt} = (R + r)\frac{E}{R+r}\left(1 - e^{-\frac{t}{\tau}}\right) + L\frac{E}{L}e^{-\frac{t}{\tau}} = Ee^{-\frac{t}{\tau}} + E - Ee^{-\frac{t}{\tau}} = E$ (verified).

3. In the steady state, the current i becomes constant so $\frac{di}{dt} = \frac{dI}{dt} = 0$;

Referring to the differential equation $E = (R + r)I + L\frac{dI}{dt} = (R + r)I$, we get $I = \frac{E}{R+r}$.

4. From graph $I = 18mA$.

5. We have $I = \frac{E}{R+r}$, then $r = \frac{E}{I} - R = \frac{2}{18 \times 10^{-3}} - 100 = 11\,\Omega$.

At $t = \tau$, $i = 0.63I = 0.63 \times 18 = 11.34\,mA$, referring to the graph $\tau = 4ms$;

But $\tau = \frac{L}{R+r}$, then $L = \tau(R+r) = 4\times10^{-3}(111) = 0.444\,H$.

Part C

1. Law of addition of voltages $u_{AB} = u_{AM} + u_{MD} + u_{DB}$; $u_C = 0 + ri + L\frac{di}{dt} + Ri = (r+R)i + L\frac{di}{dt}$;

But $i = -C\frac{du_C}{dt}$ so $u_C = -(r+R)C\frac{du_C}{dt} - LC\frac{d^2u_C}{dt^2}$

Then $\frac{d^2u_C}{dt^2} + \frac{(R+r)}{L}\frac{du_C}{dt} + \frac{1}{LC}u_C = 0$.

2. From graph, after a pseudo-period $t = T = 9.6ms$, $u_C = 0.6V$;

Replacing in the solution $u_C = 2e^{-\left(\frac{R+r}{2L}\right)t}\cos\left(\frac{2\pi}{T}t\right)$; $0.6 = 2e^{-\left(\frac{100+11}{2L}\right)9.6\times10^{-3}}\cos(2\pi)$;

$\frac{-111\times9.6\times10^{-3}}{2L} = \ell n(0.3)$ then $L = \frac{-111\times9.6\times10^{-3}}{2\,\ell n(0.3)} = 0.443\,H$.

II-GS 2012 1st

Part A

1. Charging of the capacitor.
2. a) No current $i = 0$.

   b) The voltage across the capacitor is $u_C = E$.

   c) The electric energy stored is $W_{ele} = \frac{1}{2}C u_C^2 = \frac{1}{2}C E^2 = \frac{1}{2}\times10^{-3}\times10^2 = 0.05\,J$.

Part B

First situation

1. Law of uniqueness of voltages: $u_C = u_L = L\frac{di}{dt}$ & $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$;

We get $u_C = -LC\frac{d^2u_C}{dt^2}$; then $u_C + LC\frac{d^2u_C}{dt^2} = 0$.

2. a) We have $u_C = E\cos\left(\frac{2\pi}{T_0}t\right) \Rightarrow \frac{d^2u_C}{dt^2} = -E\left(\frac{2\pi}{T_0}\right)^2\cos\left(\frac{2\pi}{T_0}t\right)$;

Replacing in the differential equation, we get $E\cos\left(\frac{2\pi}{T_0}t\right)\left[1 - LC\left(\frac{2\pi}{T_0}\right)^2\right] = 0$;

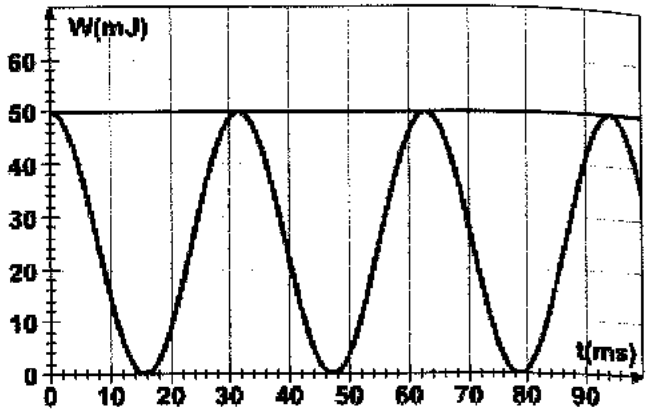
But $E\cos\left(\frac{2\pi}{T_0}t\right) \neq 0$, then $1 - LC\left(\frac{2\pi}{T_0}\right)^2 = 0 \Rightarrow T_0 = 2\pi\sqrt{LC}$.

b) We have $T_0 = 2\pi\sqrt{LC} = 2\pi\sqrt{10^{-3}\times0.1} = 0.02\,\pi\,s \approx 0.0628\,s = 62.8\,ms$.

3. The electric energy stored is:

$W_{ele} = \frac{1}{2}C u_C^2 = \frac{1}{2}\times10^{-3}\times\left(10\cos\left(\frac{2\pi}{0.02\pi}t\right)\right)^2 = 0.05\cos^2(100\,t)$ (t in s & W_{ele} in J)

4. The period of the energy to that of proper period by the relation $T' = \frac{T_0}{2}$.
5. At $t=0$; $W_{ele} = 0.05\cos^2(0) = 0.05J$.
6. Graph.



Second situation

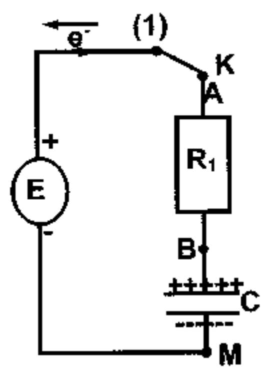
1. The oscillations are free damped.
2. The pseudo-period of oscillations is double that of energy, then $T = 63\,ms$.
3. At the considered instants the electric energy is maximum; consequently the magnetic energy is zero. Then the total energy stored in the circuit is electric.
4. At the instant t_1, the electric energy stored is zero.
   But the total energy is sum of its electric and magnetic energies so the energy is stored as a magnetic.
5. For $t \in [0;t_1]$, the electric energy decreases, then the capacitor provides energy to the circuit.
   For $t \in [t_1;31.5ms]$, the electric energy increases, the coil provides energy.
6. At the considered instants, the energy of the system is electrical.
   The loss of energy is: $\Delta E = E_T\big|_{t_2} - E_T\big|_{t_0=0} = W_{ele}\big|_{t_2} - W_{ele}\big|_{t_0=0} = 7.5mJ - 50mJ = -42.5mJ$.

Third situation

If the resistance is increased then the damping becomes stronger.

III-GS 2011 2nd

Part A



1. a) Circuit.
   b) $i = C\frac{du_C}{dt}$.
2. a) Law of addition of voltages : $u_{AM} = u_{AB} + u_{BM}$ and $u_{AB} = R_1 i$;
   Then $E = u_C + R_1C\frac{du_C}{dt}$.
   b) We have $u_C = A + Be^{-\frac{t}{\tau_1}}$, so $\frac{du_C}{dt} = -\frac{B}{\tau_1}e^{-\frac{t}{\tau_1}}$;
   By replacing in D.E we get: $E = A + Be^{-\frac{t}{\tau_1}} + R_1C\left(-\frac{B}{\tau_1}e^{-\frac{t}{\tau_1}}\right) \Rightarrow E = A + Be^{-\frac{t}{\tau_1}}\left(1 - \frac{R_1C}{\tau_1}\right)$;
   This equation is verified at any instant t, then: $A = E$ and $1 - \frac{R_1C}{\tau_1} = 0 \Rightarrow \tau_1 = R_1C$.
   But at $t=0$, $u_C = 0$ (the capacitor was neutral).

By replacing in the above solution $0 = A + Be^{-0} \Rightarrow B = -A = -E$; thus, $u_C = E\left(1 - e^{-\frac{t}{\tau_1}}\right)$.

c) Studying the voltage across the capacitor.

i- The limit value of u_C is $(u_C)_\ell = E = 6V$.

For $t = \tau_1$, $u_C = 0.63\,E = 3.78\,V$.

From the graph the abscissa of the point, whose ordinate $3.78\,V$ is $\tau_1 = 0.1\,ms$;

But $\tau_1 = R_1C \Rightarrow C = 4\times10^{-6}F$.

ii- The capacitor is $t_{\min} = 5\tau_1 = 0.5\,ms = 5\times10^{-4}s$.

Part B

1. When the switch is turned from position (1) to (2), the circuit is open and the voltage u_C does not vary and retains the value of $6V$ between $t_1 = 0.6ms$ and $t_2 = 1ms$.
2. The pseudo-period $T = 5\,ms = 5\times10^{-3}s$.
3. a) The proper period $T_0 = 2\pi\sqrt{LC}$.

   b) We have $T_0 = 2\pi\sqrt{LC} = 2\pi\sqrt{0.156\times4\times10^{-6}} \approx 4.96\times10^{-3}s$.

   But $\left(\frac{2\pi}{T}\right)^2 = \left(\frac{2\pi}{T_0}\right)^2 - \left(\frac{R_1 + r}{2L}\right)^2$, then $r = 2L\sqrt{\left(\frac{2\pi}{T_0}\right)^2 - \left(\frac{2\pi}{T}\right)^2} - R_1 \approx 23\,\Omega$.
4. a) At $t = 6\,ms$; we have $u_C = 1.25\,V$.

   b) The loss of energy: $E_{loss} = W_e\big|_{t_1} - W_e\big|_{t_0} = \frac{1}{2}C\left(u_C\big|_{t_1}\right)^2 - \frac{1}{2}CE^2$.

   $E_{loss} = \frac{1}{2}\times4\times10^{-6}\times\left(1.25^2 - 6^2\right) = 6.89\times10^{-5}J$.

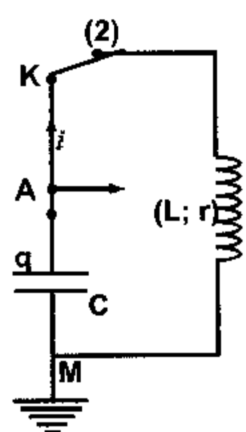
IV-GS 2011 1st

Part A

1. At the end of charging $u_C = E = U_0 = 3V$.
2. The electric energy stored $W_0 = \frac{1}{2}CE^2 = 4.5\times10^{-6}J$.

Part B

1. a) Circuit.



b) Law of uniqueness of voltage: $u_C = u_L = L\frac{di}{dt}$ & $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$;

We get $u_C = -LC\frac{d^2u_C}{dt^2}$; then $\frac{d^2u_C}{dt^2} + \frac{1}{LC}u_C = 0$.

c) The differential equation satisfied by u_C is of second order of the form

$u_C'' + w_0^2u_C = 0$, where $w_0^2 = \frac{1}{LC}$, then the voltage u_C is sinusoidal of

proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{LC}$.

Thus, $T_0 = 2\pi\sqrt{LC} = 2\times3.14\sqrt{0.1\times1\times10^{-6}} \approx 1.99\times10^{-3}s = 1.99\,ms$.

d) Graph.

e) The oscillations are free un-damped.

2. a) The decrease in amplitude indicates that the oscillations are free damped.

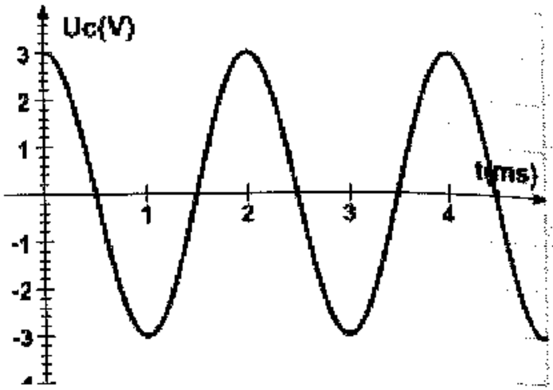
b) The total energy in the circuit is not constant because of resistance of the coil which dissipates energy in the form of heat.

c) Waveforms:

i- The pseudo-period $T = 2ms$.

ii- $T = 2ms > T_0 = 1.99\ ms$.

iii-Around zero.



3. a) Around $E = 3V$.

b) $T = 2ms$.

c) [1]Graphical study:

i- The coil supplies energy to the capacitor if the current decreases and the voltage across the capacitor increases; then $0.5\,ms \le t \le 1ms$.[2]

ii- The capacitor supplies energy to the coil; if the voltage across the capacitor decreases and the current increases; then $1\,ms \le t \le 1.5ms$.[3]

iii-For $0 \le t \le 0.5ms$; u_C increases so the capacitor is storing electrical energy and in this interval the current i is also increasing so the coil is storing magnetic energy.
Thus, no exchange of energy takes place between the capacitor and the coil, but both of them are storing energies from the generator.[4]

V-GS 2009 2nd
Part A

1. Law of addition of voltages: $u_G = u_C + u_R$ but $u_R = R\,i$ and $i = \dfrac{dq}{dt}$, then $i = C\dfrac{du_C}{dt}$.

Thus, $E = u_C + RC\dfrac{du_C}{dt}$.

2. We have $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$, then $\dfrac{du_C}{dt} = \dfrac{E}{\tau}e^{-\frac{t}{\tau}}$.

Replacing in the differential equation we get: $E = E\left(1 - e^{-\frac{t}{\tau}}\right) + \dfrac{RC}{\tau}Ee^{-\frac{t}{\tau}} = E + e^{-\frac{t}{\tau}}\left(-1 + \dfrac{RC}{\tau}\right)$.

---

[1] The magnetic energy stored in the coil is proportional to the square of the current and the electric energy stored is proportional to the square of the voltage across the capacitor.

[2] The energy of the coil decreases if $\dfrac{d(E_m)}{dt} = Lii' < 0$; then $i\,\&\,i'$ should have opposite signs which means that if the current i is positive (above the abscissa axis) and it is decreasing.
These two conditions are satisfied if $0.5\,ms \le t \le 1ms$.

[3] The energy of the capacitor decreases if $\dfrac{d(E_e)}{dt} = Lu_C u_C' < 0$; then $u_C\,\&\,u_C'$ should have opposite signs which means that if the voltage u_C is positive (above the abscissa axis) and it is decreasing.
These two conditions are satisfied if $0.5\,ms \le t \le 1ms$.

[4] In the interval $0 \le t \le 0.5ms$; $\dfrac{d(E_m)}{dt} = Lii' > 0$ & $\dfrac{d(E_e)}{dt} = Lu_C u_C' > 0$; the graphs of $u_C\,\&\,i$ are above the abscissa axis and increasing.

Then $Ee^{-\frac{t}{\tau}}\left(-1+\frac{RC}{\tau}\right)=0$; but $Ee^{-\frac{t}{\tau}}\neq 0$; thus, $-1+\frac{RC}{\tau}=0$ so $\tau=RC$.

3. **1st method:**

   The tangent to the curve at origin intersects the horizontal asymptote at the point of abscissa $\tau = 2.2\,ms$.

   2nd method:

   The abscissa of the point whose ordinate $u_C = 0.63\times 8 = 5.02\,V$, is $\tau = 2.2\,ms$.

4. We have $\tau = 2.2\,ms = RC \Rightarrow C = \frac{2.2\times 10^{-3}}{2.2\times 10^{3}} = 10^{-6}\,F$.

5. a) The voltage of the capacitor is practically constant after $t_1 = 5\tau = 11\,ms$.

   b) The charge stored at t_1 is $Q_1 = CE = 8\times 10^{-6}\,C$.

   The electric energy stored in the capacitor is $E_e = \frac{1}{2}CE^2 = \frac{1}{2}\times 10^{-6}\times 8^2 = 3.2\times 10^{-5}\,J$.

Part B

1. Law of uniqueness of voltages $u_C = u_L = L\frac{di}{dt}$ & $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$;

   Then $u_C = -LC\frac{d^2u_C}{dt^2}$; $u_C + LC\frac{d^2u_C}{dt^2} = 0 \Rightarrow \frac{d^2u_C}{dt^2} + \frac{1}{LC}u_C = 0$.

2. We have $u_C = E\cos\left(\frac{2\pi}{T_0}t\right) \Rightarrow \frac{d^2u_C}{dt^2} = -E\left(\frac{2\pi}{T_0}\right)^2\cos\left(\frac{2\pi}{T_0}t\right)$;

   Replacing in the differential equation:

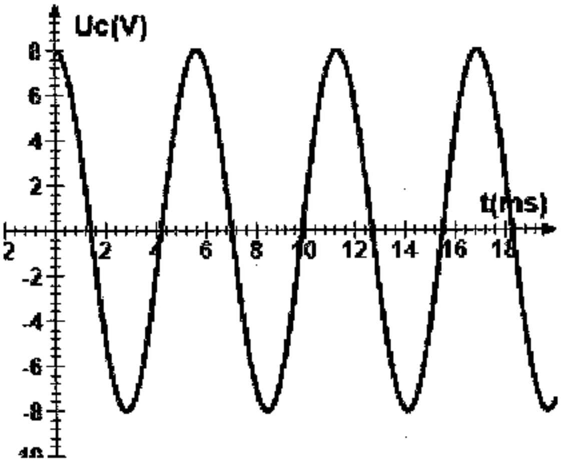
   $-\left(\frac{2\pi}{T_0}\right)^2 E\cos\left(\frac{2\pi}{T_0}t\right) + \frac{1}{LC}E\cos\left(\frac{2\pi}{T_0}t\right) = 0$, so $E\cos\left(\frac{2\pi}{T_0}t\right)\left[-\left(\frac{2\pi}{T_0}\right)^2 + \frac{1}{LC}\right] = 0$;

   (But $E\cos\left(\frac{2\pi}{T_0}t\right)\neq 0$),

   Then $1 - LC\left(\frac{2\pi}{T_0}\right)^2 = 0 \Rightarrow T_0 = 2\pi\sqrt{LC}$;

   $T_0 = 2\pi\sqrt{0.8\times 10^{-6}} = 5.62\,ms$.

3. Graph.



4. The electric energy W_0 of the capacitor passes to the coil that stores it in the form of magnetic energy and vice versa.

Part C

1. The electric energy W_0 of the capacitor passes (partially) to the coil that stores it in the form of magnetic energy and the rest of this energy is dissipated in the form of the thermal energy.

2. a) $3T = 17\,ms \Rightarrow T = 5.67\,ms$.

   b) T slightly greater than T_0.

3. a) At the instant $t_n = nT$, u_C is maximum then the current i is maximum thus the energy is pure electrical.

b) When the capacitor loses 98.6% of its energy; then the energy stored becomes:

$W_n = W_0 - 0.986W_0 = 0.014W_0$; then $W_n = 0.014 \times 3.2\times10^{-5} = 4.48\times10^{-7}\,J$.

c) We have $W_n = \frac{1}{2}C u_n^2 = 4.48\times10^{-7}\,J \Rightarrow u_n = 0.95V$;

This voltage is obtained at the instant $t_n = 17ms = nT \Rightarrow n = \frac{17}{5.67} \approx 3$.

VI-GS 2007 1st

First situation

1. The mechanical energy is $ME = KE + PE_e = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$.

2. The derivative $\frac{d(ME)}{dt} = m\,vv' + k\,x\,x'$ (but $v = x'$ and $v' = x''$); then $\frac{d(ME)}{dt} = v(m\,x'' + kx)$.

Second situation

1. In the absence of friction, the mechanical energy is conserved $\frac{d(ME)}{dt} = 0$, $x'(mx'' + kx) = 0$;

(But $v = x' \neq 0$, since the system is in motion), then $x'' + \frac{k}{m}x = 0$.

2. The differential equation that governs the motion is of 2nd order of the form $x'' + w_0^2 x = 0$ where $w_0^2 = \frac{k}{m}$, then the motion of the solid is sinusoidal of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}}$.

The proper frequency $f_0 = \frac{1}{T_0} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{0.57}{0.546}} \approx 0.51\,Hz$.

Third situation

1. We have $\frac{d(ME)}{dt} = x'(mx'' + kx) = \vec{F}.\vec{v} = -\lambda\,\vec{v}.\vec{v} = -\lambda\,x'\,x'$ (but $v = x' \neq 0$, in motion).

So $mx'' + kx = -\lambda\,x'$, then we get $x'' + \frac{\lambda}{m}x' + \frac{k}{m}x = 0$.

2. a) The amplitude of oscillations decreases with time.

b) The pseudo-period is $T = 2s$ & the pseudo-frequency $f = \frac{1}{T} = 0.5\,Hz$.

c) We have $f^2 = f_0^2 - \frac{1}{4\pi^2}\left(\frac{\lambda}{2m}\right)^2$, so $\frac{1}{4\pi^2}\left(\frac{\lambda}{2m}\right)^2 = f_0^2 - f^2$, then $\lambda = 4\pi m\sqrt{f_0^2 - f^2}$;

$\lambda = 0.69\,SI$, in SI units λ is measured in kg/s.

Part B

1. The electromagnetic energy stored in the system (L, C) is $E = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}$.

2. We have $\frac{dE}{dt} = -(R+r)i^2$; so $Li \times i' + \frac{1}{C} q \times q' = -(R+r)i^2$; but $i = q' \neq 0$ & $i' = q''$;

We get $q'' + \left(\frac{R+r}{L}\right)q' + \frac{1}{LC}q = 0$.

3. The proper frequency $f_0' = \frac{1}{2\pi}\sqrt{\frac{1}{LC}} = 354.2\,Hz$.

4. a) The decrease in the amplitude of oscillations is due to the loss of energy by Joule's effect.

b) The pseudo-period $T' = 3 \times 10^{-3} s$ and the pseudo frequency $f' = 333.3\,Hz$.

Part C

1. By comparing the differential equations in x and in q we obtain the analog quantities x to q, v to i, m to L, k to $1/C$ and λ to $(R+r)$.

2. a) We have $f'^2 = f_0'^2 - \frac{1}{4\pi^2}\left[\frac{R+r}{2L}\right]^2 \Rightarrow R + r = 4\pi L\sqrt{f_0'^2 - f'^2}$.

b) Using the data, we get $R = 54\,\Omega$.

VII-GS 2006 2nd

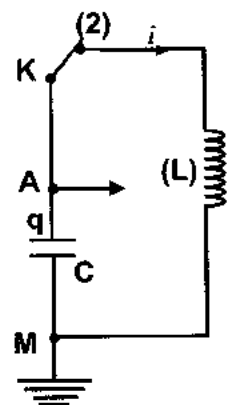
Part A

1. At the end of charging, the capacitor is fully charged $i = 0$, and $u_C = E - Ri = U_0 = 10\,V$.

2. The electric energy stored is given by : $W = \frac{1}{2}CE^2 = 5 \times 10^{-5} J$.

Part B

First situation

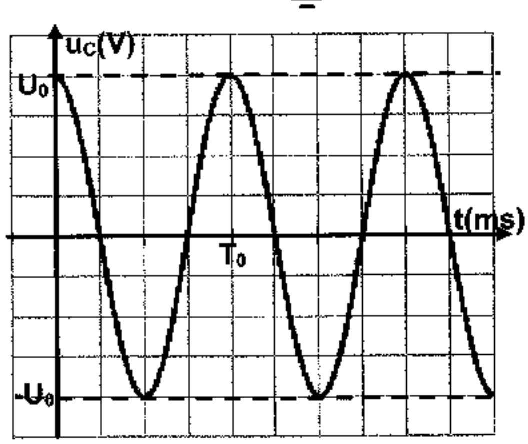


1. Circuit.

2. Law of addition of voltages: $u_{AM} + u_{MA} = 0$ where $u_{AM} = \frac{q}{C} = u_C$ and $u_{MA} = L\frac{di}{dt}$, but the current is directed towards the positive armature then: $i = -\frac{dq}{dt} = -C\frac{d^2u_C}{dt^2}$, thus $\frac{d^2u_C}{dt^2} + \frac{1}{LC}u_C = 0$.

3. The differential equation satisfied by u_C is of 2nd order of the form $u_C'' + w_0^2 u_C = 0$ where $w_0^2 = \frac{1}{LC}$.

Then the voltage u_C is sinusoidal of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{LC}$.



4. The variation of u_C is alternating sinusoidal as shown in the adjacent figure.

5. The system is the seat of free un-damped oscillations.

Second situation

1. Referring to the waveform, the circuit is the seat of free damped electric oscillations.
2. a) The pseudo-period $T = 2\,ms = 2\times10^{-3}\,s$.
   b) According to the waveform:
   $\frac{u_C(T)}{u_C(0)} = \frac{8.2}{10} = 0.82$, $\frac{u_C(2T)}{u_C(T)} = \frac{6.7}{8.2} = 0.82$ & $\frac{u_C(3T)}{u_C(2T)} = \frac{5.5}{6.7} = 0.82$;
   Then this ratio is constant equal to $a = 0.82$.
3. a) At any instant t, the energy stored is: $E = E_{el} + E_{mag}$;
   But when u_C is maximum, so $\frac{du_C}{dt} = 0$, then $i = -C\frac{du_C}{dt} = 0$; thus energy stored is electrical.
   b) The energies ratio $\frac{E_{n+1}}{E_n} = \frac{\frac{1}{2}Cu_{n+1}^2}{\frac{1}{2}Cu_n^2} = \left(\frac{u_{n+1}}{u_n}\right)^2 = a^2$.
   c) We have $\frac{E_{n+1}}{E_n} = e^{-\frac{r}{L}T} = a^2$, then $\frac{r}{L} = -\frac{2\,\ell n a}{T} = 198.45 \approx 200$.
   And $\frac{4\pi^2}{T^2} = \frac{4\pi^2}{T_0^2} - \frac{1}{4}\left(\frac{r}{L}\right)^2$ where $T = 2\times10^{-3}\,s$ & $T_0 = 2\pi\sqrt{LC} \Rightarrow \frac{4\pi^2}{T_0^2} = \frac{10^6}{L}$.
   By replacing we get: $\frac{4\pi^2}{4\times10^{-6}} = \frac{10^6}{L} - \frac{1}{4}\left(4\times10^4\right) \Rightarrow \frac{10^6}{L} = \pi^2\times10^6 + 10^4$;
   Then, $L \approx 0.1\,H$ and $r = 20\,\Omega$.

Unit II

Electricity

Chapter 11

Transformer

GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Transformer		-	-	-	-	-	-	-	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Transformer	-	2nd(A)	-	-	-	-	-	-	-

Essentials

I- Definition

The transformer is a static apparatus that is used to modify some physical quantities without modifying the frequency.

It is formed of:

✖ a primary coil (C_1) formed of N_1 turns;

✖ a secondary coil (C_2) formed of N_2 turns;

The coils are separated by laminated metallic sheets to facilitate the transmission of the magnetic field.

A simplified diagram is shown in figure 1.

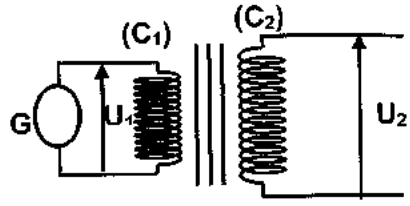


Figure 1

II- Principle of Functioning

The primary coil (C_1) is fed by a variable (sinusoidal) voltage which lead to a variable (sinusoidal) current, consequently a variable magnetic field $\overrightarrow{B_1}$ is present in the primary. By the means of the metallic sheets, the magnetic field is transmitted to the secondary and a variable magnetic field $\overrightarrow{B_2}$ appears in the secondary. The magnetic flux in the secondary $|\phi_2| = N_2\, B_2\, S_2 \cos\theta_2$ is variable, then an induced *e.m.f* appears in the secondary.

Note 1: If the secondary circuit is closed, an induced current flows.

Note 2: The transformer do not function if the primary is fed by a constant voltage. Then $\overrightarrow{B_1}$, $\overrightarrow{B_2}$ & $|\phi_2|$ are then constants. No induced *e.m.f* in the secondary.

III- Laws of Transformers

The ratio of the transformer is defined by $m = \dfrac{N_2}{N_1}$.

1. Law of voltages

The voltage is proportional to the number of turns $\dfrac{U_2}{U_1} = \dfrac{N_2}{N_1}$;

If $N_2 > N_1$, $m = \dfrac{N_2}{N_1} > 1$ $(U_2 > U_1)$; it is a step-up transformer.

If $N_2 < N_1$, $m = \dfrac{N_2}{N_1} < 1$ $(U_2 < U_1)$; it is a step-down transformer.

2. Efficiency of a transformer

If P_1 & P_2 are respectively the electric powers in the primary and secondary.

The efficiency is defined by $\eta = \dfrac{P_2}{P_1}$ where $P_i = U_i\, I_i \cos\varphi_i$;

For an ideal transformer: the power factor $\cos\varphi_i = 1$ & $\eta = 1$.

3. Law of currents $\dfrac{I_1}{I_2} = \dfrac{N_2}{N_1}$.

I-LS 2007 1st (Part C)
Usage of a Coil

The Figure 1 represents the diagram of a loaded transformer. The generator delivers an alternating sinusoidal voltage of frequency f. The coil (1) carries an alternating sinusoidal current i_1 of frequency f. The coil (2) thus carries an alternating sinusoidal current i_2 having the same frequency f.

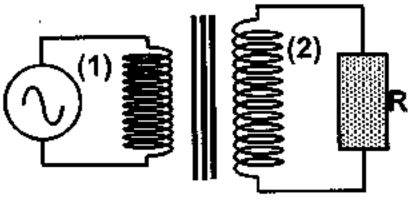


Figure 1

1. Explain the existence of the current in coil (2).
2. The object of this part is to show evidence of the role of a transformer in the transmission of electric energy.
   - ✘ An electric generator G delivers a power $P = 20\,kW$ under an alternating sinusoidal voltage of effective value U.
   - ✘ A transmission line of total resistance $r = 1\Omega$ feeds an electric installation (B).

   Let I be the effective current that passes in the line. The power factor of the system formed of the line and the installation is $\cos\varphi = 0.95$.

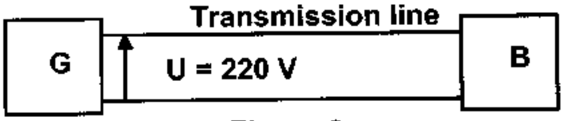


Figure 2

a) Give the expression of the power P in terms of U, I & $\cos\varphi$.

b) Power

i- Find the expression of the power P' lost in the line due to Joule's effect in terms of P, r, $\cos\varphi$ & U.

ii- Calculate P' in the case when $U = 220V$ (Figure 2).

iii- A transformer, connected across the generator, raises the effective value of the voltage across the transmission line. The transmission of the same power P through the line thus takes place under the new effective voltage $U = 10^4 V$ (Figure 3).
Calculate the new value of for P'.

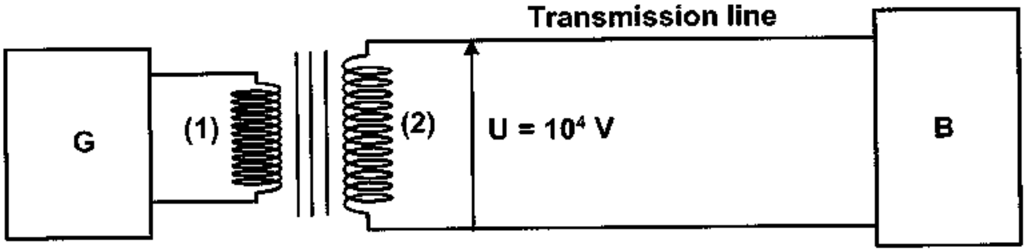


Figure 3

c) Draw a conclusion about the importance of the usage of the transformer in the transmission of electric energy over large distances.

Solutions

I-LS 2007 1st

1. The current i_1 in the primary coil is variable, a variable magnetic field $\vec{B}$ is produced in the primary. The magnitude field $\vec{B}$ transmitted to the secondary is also variable at any instant, then the magnetic flux in the secondary is variable.
Thus, the secondary is the seat of induced *e.m.f* and since the secondary circuit is closed, an induced current i_2 will pass in it.

2. a) The power is given by $P = U\,I\cos\varphi$.
   b) Power:
      i- The loss of power is given by $P' = r\,I^2 = r\left(\frac{P}{U\cos\varphi}\right)^2$.
      ii-If $U = 220\,V$, we get $P' = 1\left(\frac{20\times10^3}{220\times0.95}\right)^2 = 9157\,W$.
      iii-If $U = 10^4 V$, we get $P'' = 1\left(\frac{20\times10^3}{10^4\times0.95}\right)^2 = 4.4\,W$.
   c) We have $P'' = 4.4\,W < P' = 9157\,W$;
   The use of step-up transformers $U = 10^4 V$ for the transmission of electric power reduces considerably the loss of energy due to Joule's effect.

Supplementary Problems

I-LS 2002 2nd
The Transformer

The purpose of this exercise is to study the principle of functioning of an ideal transformer and its role.

Consider two coils, (C_1) of 1000 turns and (C_2) of 500 turns, the surface area of each turns of (C_1) and (C_2) is $100\,cm^2$.

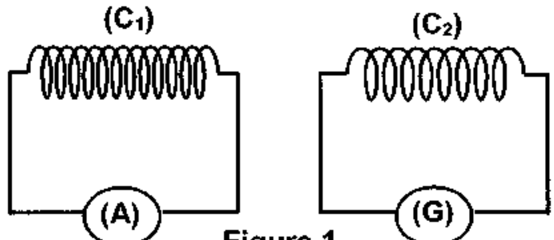


Figure 1

Part A
Principle of functioning

The coil (C_1) is connected to a sensitive ammeter (A) and the coil (C_2) is connected across a generator thus forming two closed circuits (figure 1).

The coil (C_2) carries then a current i that varies with time as shown in the graph of figure 2. As a result, (C_2) produces, through (C_1) a magnetic field $\overrightarrow{B}$ supposed uniform of magnitude $B = 2\times10^{-3}\,i$ (B in T and i in A).

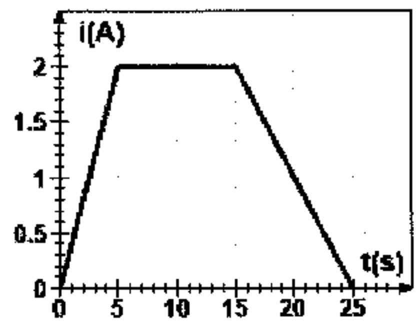


Figure 2

1. Give the expression of the magnetic flux crossing (C_1) in terms of i.
2. Give the expression of e, the $e.m.f$ induced in (C_1).
3. Determine the values of e for $0s \le t \le 25\,s$.
4. Trace the graph giving the variation of e as a function of time t for $0s \le t \le 25\,s$.

   Scale: on the abscissa axis: $1\,cm \longrightarrow 5\,s$; and on the ordinate axis: $1\,cm \longrightarrow 4\,mV$.
5. Draw again figure 1 and indicate, using Lenz's law, the direction of the induced current in (C_1), in the interval of time $0s \le t \le 5\,s$.

Part B
Role

The coils (C_1) and (C_2), disconnected from the preceding circuit, are used to construct an ideal transformer (T) using a convenient iron core. (C_1) and (C_2) are respectively the primary and secondary.

1. We connect across (C_1) a sinusoidal alternating voltage of effective value $U_1 = 220\,V$.

   A voltmeter, in AC mode connected across (C_2), reads a value U_2.

   a) Give a simplified diagram of (T).

   b) Does (T) act as a step-up or a step-down transformer? Justify your answer and calculate U_2.
2. A lamp, connected across the terminals of (C_2), carries a current of effective value $I_2 = 1\,A$. Calculate the effective current I_1 carried by the coil (C_1).

Answer Key
Part A 3. (8mV, 0, -4mV)
Part B 1.b) 110V

GS Sessions

I-GS 2008 2nd

(R, L, C) Series Circuit

Consider a capacitor of capacitance $C = 5\,\mu F$, a resistor of resistance $R = 40\,\Omega$ and a coil of inductance L and of resistance r, connected in series across the secondary of an ideal transformer.

Part A

Physical quantities of the transformer

The primary coil of the transformer is connected to the mains $(220\,V;\ 50\,Hz)$ (Figure 1).

The secondary of the transformer delivers across its terminals a voltage:

$$u_{NM} = 3\cos wt \ \ (u \text{ in } V,\ t \text{ in } s).$$

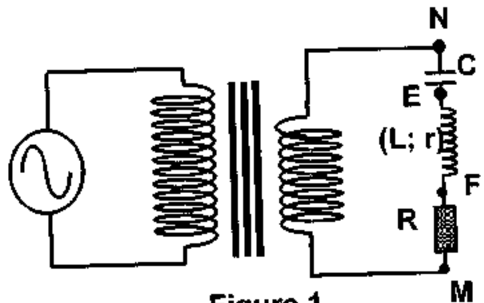


Figure 1

The circuit thus carries an alternating sinusoidal current $i = I_m \cos(wt + \varphi)$.

The secondary coil has 15 turns and cannot withstand a current of effective value greater than $10\,A$.

Take: $\sqrt{2} = 1.4$.

1. Give the value of the frequency of the alternating sinusoidal voltage across the secondary coil.
2. Determine the number of turns of the primary coil.
3. Calculate the maximum effective value of the current that the primary coil can withstand.

Part B

Determination of L and r

An oscilloscope, connected in the previous circuit, allows us to display on the channel Y_1 the voltage $u_1 = u_{NM}$ and on the channel Y_2 the voltage $u_2 = u_{FM}$ across the terminals of the resistor.

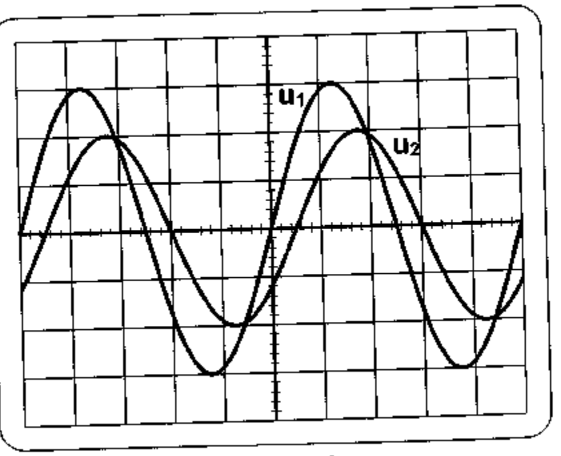


Figure 2

1. Redraw the circuit of figure 1 and show the connections of the oscilloscope.
2. The sensitivities of the oscilloscope are:
   Horizontal sensitivity: $4ms\,/\,\text{div}$.
   Vertical sensitivity is $1V\,/\,\text{div}$
   Using the waveforms of figure 2, show that:
   $i = 0.05\cos(100\,\pi t - 0.2\pi)$; ($i$ in A, t in s).
3. Calculate the average power consumed by the component NM.
4. Deduce the value of the resistance r of the coil.
5. Knowing that: $u_{NM} = u_{NE} + u_{EF} + u_{FM}$ is verified for any value of time t, determine the value of L.

GS Sessions Solutions

I-GS 2008 2nd

Part A

1. The frequency is conserved $f = 50\,Hz$.
2. Law of voltages $\frac{U_2}{U_1} = \frac{N_2}{N_1}$, we get $\frac{3/\sqrt{2}}{220} = \frac{15}{N_1}$; then $N_1 = 1540$ turns.
3. The transformer is ideal, law of currents $\frac{I_2}{I_1} = \frac{N_1}{N_2}$, $\frac{10}{I_1} = \frac{1540}{15}$; then $I_1 = 0.097\,A = 97\,mA$

Part B

1. Diagram of Figure 1.
2. The frequency $f = 50Hz$;

   The angular frequency $w = 2\pi f = 100\pi$ rad/s.

   Referring to the waveforms u_R lags u_{NM} by φ;

   Then $u_R = U_{R_{max}} \cos(wt - \varphi)$;

   But $U_{R_{max}} = S_v \times y_{2_{max}} = 1V/\text{div} \times 2\text{div} = 2V$;

   & $\varphi = 2\pi \times \frac{0.5\text{div}}{5\text{div}} = 0.2\pi$ div .

   So $u_R = 2\cos(100\pi t - 0.2\pi)$;

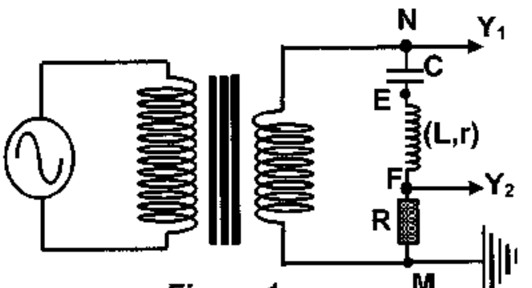


Figure 1

   Ohm's law $i = \frac{u_R}{R} = 0.05\cos(100\pi t - 0.2\pi)$ (i in A, t in s).

3. The power consumed $P = U\,I\cos\varphi = \frac{3}{\sqrt{2}} \times \frac{0.05}{\sqrt{2}} \cos(0.2\pi) = 0.061\,W$.
4. This power consumed in the circuit is due to the resistance of the circuit:

   $P = R_{\text{total}} I^2 \Rightarrow R_{\text{total}} = \frac{P}{I^2} = \frac{0.061}{(0.05/\sqrt{2})^2} = 48.8\Omega$; but $R + r = 48.8\Omega$, then $r = 8.8\Omega$.

5. The voltage across the capacitor: $u_{NE} = u_C = \frac{1}{C}\int i\,dt = \frac{100}{\pi}\sin(100\pi t - 0.2\pi)$.($u_C$ in V, t in s)

   But $u_{EF} = r\,i + L\frac{di}{dt} = 8.8 \times 0.05\cos(100\pi t - 0.2\pi) - L \times 5\pi\sin(100\pi t - 0.2\pi)$;

   $= 0.44\cos(100\pi t - 0.2\pi) - 5\pi L\sin(100\pi t - 0.2\pi)$

   and $u_{FM} = 2\cos(100\pi t - 0.2\pi)$ (u in V, t in s).

   Law of addition of voltages: $u_{NM} = u_{NE} + u_{EF} + u_{FM}$;

   So $3\cos wt = \frac{100}{\pi}\sin(wt - 0.2\pi) + 0.44\cos(wt - 0.2\pi) - 5\pi L\sin(wt - 0.2\pi) + 2\cos(wt - 0.2\pi)$

   Let $wt = 0$; $3\cos 0 = \frac{100}{\pi}\sin(-0.2\pi) + 0.44\cos(-0.2\pi) - 5\pi L\sin(-0.2\pi) + 2\cos(-0.2\pi)$;

   $3 + \frac{100}{\pi}\sin(0.2\pi) - 2.44\cos(0.2\pi) = 5\pi L\sin(0.2\pi)$; thus $L = 2.14\,H$.

Unit I – GS

Mechanics

GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Linear Momentum		-	-	-	-	1st	-	-	2nd
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Linear Momentum	-	-	-	-	-	2nd	-	-	-

GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Linear Oscillations		-	-	-	-	1st(S) & 2nd	-	-	1st
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Linear Oscillations	-	1st(S)	-	-	1st	-	-	-	-

GS Sessions – Linear Momentum

I-GS 2013 1st
Verification of Newton's Second Law

Consider an inclined plane that makes an angle $\alpha = 30^0$ with the horizontal plane.
An object (S), supposed as a particle, of mass $m = 0.5\,kg$ is launched from the bottom O of the inclined plane, at the instant $t_0 = 0$, with a velocity $\overrightarrow{v_0} = v_0\,\vec{i}$ along the line of the greatest slope (OB).
Let A be a point of OB such that $OA = 5m$ (figure 1). The position of (S), at the instant t, is given by $\overrightarrow{OM} = x\vec{i}$ where $x = f(t)$.

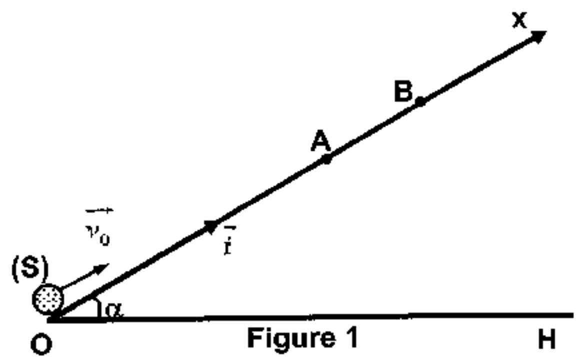


Figure 1

The variation of the mechanical energy of the system [(S), Earth], as a function of x, is represented by the graph of figure 2.

Take:

✘ The horizontal plane passing through OH as a gravitational potential energy reference;

✘ $g = 10\,m.s^{-2}$.

1. Using the graph of figure 2:
   a) show that (S) is submitted to a force of friction between the points of abscissas $x_0 = 0$ and $x_A = 5m$;
   b) Graphical study:
      i- Calculate the variation of the mechanical energy of the system [(S), Earth] between the instants of the passage of (S) through the points O and A;
      ii- Deduce the magnitude of the force of friction, supposed constant, between O and A;

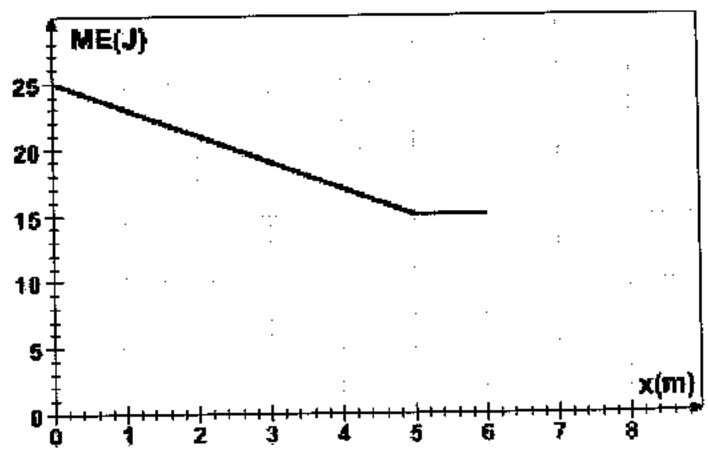


Figure 2

   c) Determine, for $0 \le x \le 5m$, the expression of the mechanical energy of the system [(S), Earth] as function of x;
   d) Determine the speed of (S) at the point of abscissa $x = 6m$.
2. Let v be the speed of (S) when it passes through the point M of abscissa x so that $0 \le x \le 5m$.
   a) Determine the relation between v and x.
   b) Deduce that the algebraic value of the acceleration of (S) is $a = -9\,m.s^{-2}$.
3. a) Determine the values of the speed of (S) at O and at A.

b) Calculate the duration $\Delta t = t - t_0 = t$ of the displacement of (S) from O to A, knowing that the algebraic value of the velocity of (S) is given by: $v = a\,t + v_0$.

c) Determine the linear momentums $\overrightarrow{P_O}$ and $\overrightarrow{P_A}$ of (S), at O and at A respectively.

4. Determine the resultant of the external forces $\sum \overrightarrow{F_{ext}}$ acting on (S).

5. Verify, using the previous results, Newton's second law knowing that $\dfrac{d\vec{P}}{dt} = \dfrac{\Delta \vec{P}}{\Delta t}$.

II-GS 2010 2nd

Moment of Inertia of a Pulley

In order to determine the moment of inertia of a pulley with respect to its axis of rotation, we use the system of the adjacent figure that is formed of a trolley (A), of mass $M = 1kg$, connected to a block (B), of mass $m = 0.18\,kg$, by means of an inextensible string of negligible mass. The string passes over a pulley of radius $r = 5\,cm$.

A convenient device can record, at equal and successive intervals of time $\tau = 50\,ms$, the abscissa $x = OC$ of the different positions of the center of inertia C of (B).

Neglect all forces of friction and $g = 10m/s^2$. The table below gives the abscissa x of the position of C and its speed v at different instants.

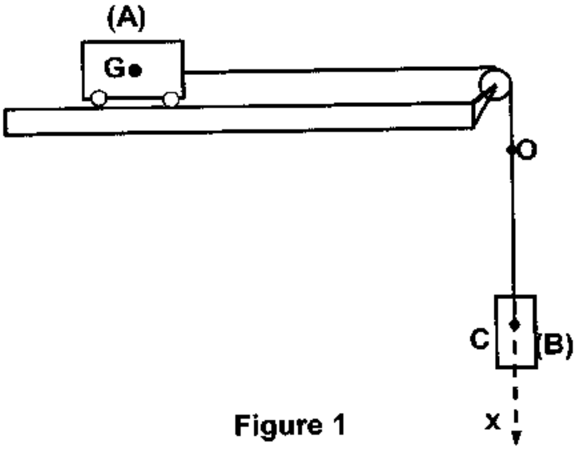


Figure 1

t(ms)	$t_0 = 0$	$t_1 = 50$	$t_2 = 100$	$t_3 = 150$	$t_4 = 200$
x(cm)	0	0.175	0.7	1.575	2.8
v(m/s)	0	0.07	0.14	0.21	0.28

Part A
Energetic study

1. Calculate the kinetic energy of (B) at the instant $t_4 = 200\,ms$.
2. Calculate the variation of the kinetic energy of (B) between the instants t_0 and t_4.
3. Applying the theorem of kinetic energy $\left(\Delta KE = \sum W\right)$, calculate the work done by the tension T_1 applied by the string on the block (B).
4. Show that the value T_1 of $\overrightarrow{T_1}$ supposed constant, is equal to $1.548\,N$.

Part B
Dynamical study

1. Calculate the values P_0, P_1, ... and P_4 of the linear momentum P of the trolley (A) at the instants t_0, t_1 ... and t_4 respectively.
2. a) Draw the graph representing the variation of P as a function of time.
   b) Show that the equation of the corresponding graph may be written in the form: $P = kt + b$ where k and b are constants to be determined.
3. Applying Newton's second law on trolley (A):

a) Determine the relation among the constants k , M and the algebraic value a of the acceleration of motion and deduce the value of a .

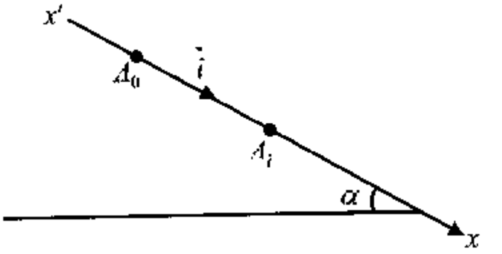
b) Show that the value T_2 of the tension $\overrightarrow{T_2}$ applied by the string on the trolley (A) is equal to $1.40\,N$.

Part C (only GS)
Determination of the moment of inertia of the pulley

1. Specify the forces acting on the pulley.
2. Applying the theorem of angular momentum, determine the moment of inertia of the pulley with respect to its axis of rotation.

III-GS 2004 2nd
Determination of a Force of Friction

In order to determine the value of the force of friction between a moving body of mass $M = 0.50\,kg$ and a table inclined by an angle $\alpha = 30^0$ with respect to the horizontal, we release the body from a point A_0 without initial velocity at the instant $t_0 = 0$ that is taken as the origin of time and we record the different positions A_i of the projection of its center of mass on the table at instants separated by a constant time interval $\tau = 60\,ms$, the points A_i being held by the axis $x'x$ of motion of unit vector $\vec{i}$.



Take $g = 9.8\,m/s^2$.

The recordings allow us to obtain the following table.

Instant	$t_0 = 0$	$t_1 = \tau$	$t_2 = 2\tau$	$t_3 = 3\tau$	$t_4 = 4\tau$	$t_5 = 5\tau$	$t_6 = 6\tau$
Position	A_0	A_1	A_2	A_3	A_4	A_5	A_6
Abscissa x(mm)	0	$A_0A_1 = 7.2$	28.8	64.8	115	180	259
Speed v(m/s)	0	0.24		0.72		1.20	
Linear momentum P(kg.m/s)	0	0.12		0.36		0.60	

1. Complete the above table by calculating, at the instants t_2 and t_4, the speeds v_2 and v_4, the values P_2 and P_4 of the linear momentum of the body.
2. Trace the curve representing the variation of P as a function of time, using the scale:
   - ✗ $1\,cm$ on the axis of abscissas represents $30\,ms$;
   - ✗ $1\,cm$ on the axis of ordinates represents $0.05\,kg.m/s$.
3. Show that the relation between the linear momentum $\vec{P} = P\,\vec{i}$ and the time t has the form $\vec{P} = b\,t\,\vec{i}$ where b is a constant.
4. Calculate b in SI units.
5. a) Show that the inclined table exerts on the body a force of friction $\vec{f}$ supposed constant and parallel to the axis $x'x$.

   b) Calculate the value f of $\vec{f}$.

Solutions

I-GS 2013 1st

1. a) Between the points of abscissas $x_0 = 0$ and $x_A = 5m$, the mechanical energy deceases so (S) is subjected to a force of friction.

b) Force of friction:

i- Variation of mechanical energy $\Delta(ME) = ME|_{x_A} - ME|_{x_O} = 15 - 25 = -10J$.

ii- We have $\Delta(ME) = W_{\vec{f}}$; so $-10J = -f \times (5-0)$; then $f = 2N$.

c) For $0 \le x \le 5m$, the curve representing the variation of the mechanical energy in terms of the abscissa is represented by a straight line; then its equation is $ME = Ax + B$;

$B = ME|_{x=0} = 25J$; and $A = \dfrac{\Delta(ME)}{\Delta x} = \dfrac{-10J}{5m} = -2J/m$;

Then $ME = -2x + 25$ (x in m & ME in J).

d) From the graph at $x = 6m$, we have $ME = 15J$;

But $ME = KE + GPE = \frac{1}{2}mv^2 + mgh = \frac{1}{2}mv^2 + mgx\sin\alpha$;

So, $ME = \frac{1}{2}mv^2 + 0.5 \times 10 \times 6 \times 0.5 = \frac{1}{2}mv^2 + 15$; then $15 = \frac{1}{2}mv^2 + 15$; thus, $v = 0$.

2. a) If $0 \le x \le 5m$; $ME = \frac{1}{2}mv^2 + mgx\sin\alpha = \frac{1}{2}0.5v^2 + 0.5 \times 10 \times x \times 0.5 = 0.25v^2 + 2.5$;

Then $0.25v^2 + 2.5x = -2x + 25$; $0.25v^2 + 4.5x - 25 = 0$; thus $v^2 = -18x + 100$).

b) Deriving with respect to time the result obtained we get: $0.5v \times \dfrac{dv}{dt} + 4.5\dfrac{dx}{dt} = 0$;

But $a = \dfrac{dv}{dt}$ & $v = \dfrac{dx}{dt} \neq 0$ (because the system is in motion); then $a = -9m/s^2$.

3. a) Refer to the relation previously derived: $v^2 = -18x + 100$;

At O, $x = 0$; $v_O^2 = -18x_0 + 100 = 100$; then $v_O = 10m/s$;

At A, $x_A = 0$; $v_A^2 = -18x_A + 100 = 10$; then $v_A = \sqrt{10}\ m/s \approx 3.16\ m/s$;

2nd method:

At O; $GPE = 0$ (on reference); $ME_O = KE_O + GPE_O = \frac{1}{2}mv_O^2 + 0$;

Then $\frac{1}{2} \times 0.5 \times v_O^2 = 25$; thus, $v_O = 10\ m/s$.

At A; $ME_A = KE_A + GPE_A = \frac{1}{2}mv_A^2 + mgx_A\sin\alpha = 15$;

Then $\frac{1}{2} \times 0.5 \times v_A^2 + 0.5 \times 10 \times 5 \times \sin 30^0 = 15J$; thus, $v_O = \sqrt{10}\ m/s \approx 3.16\ m/s$.

b) We have $v_A = at_A + v_0$; so $t_A = \dfrac{v_A - v_O}{a} = \dfrac{3.16 - 10}{-9} = 0.76\ s$.

c) $\overrightarrow{P_0} = m\overrightarrow{v_0} = 0.5\times 10\vec{i} = 5\vec{i}\ \left(kg.m.s^{-1}\right)$ & $\overrightarrow{P_A} = m\overrightarrow{v_A} = 0.5\times 3.16\vec{i} = 1.58\vec{i}\ \left(kg.m.s^{-1}\right)$.

4. The resultant force $\sum\overrightarrow{F_{ext}} = \vec{w} + \vec{f} + \vec{N}$;

But $w_x = -m\,g\sin\alpha$; $N_x = 0$ (perpendicular to the displacement) & $f_x = -f$;

Projection along the direction of motion: $\sum\overrightarrow{F_{ext}} = (-m\,g\sin\alpha - f)\vec{i} = -4.5\vec{i}\ (N)$.

5. $\dfrac{d\vec{P}}{dt} = \dfrac{\Delta\vec{P}}{\Delta t} = \dfrac{\overrightarrow{P_A} - \overrightarrow{P_o}}{t_A - t_O} = \dfrac{(1.58-5)\vec{i}}{(0.76-0)} = -4.5\vec{i}\ (N)$ & $\sum\overrightarrow{F_{ext}} = -4.5\vec{i}\ (N)$;

Then $\sum\overrightarrow{F_{ext}} = \dfrac{\Delta\vec{P}}{\Delta t} = -4.5\vec{i}\ (N)$; thus Newton's 2nd law is verified.

II-GS 2010 2nd

Part A

1. At the instant t_4 the kinetic energy is: $KE\big|_{t=t_4} = \dfrac{1}{2}m\,v_4^2 = 7.056\times 10^{-3}\,J$.

2. Variations of the kinetic energy: $\Delta(KE) = KE\big|_{t=t_4} - KE\big|_{t=0} = 7.056\times10^{-3}\,J - 0 = 7.056\times10^{-3}\,J$

3. The forces acting on (B) are its weight $\vec{w}$ and the tension $\overrightarrow{T_1}$.

   $\Delta(KE) = \sum W = W(\overrightarrow{T_1}) + W(\vec{w}) = W(\overrightarrow{T_1}) + mg(x_4 - x_0) = 7.056\times10^{-3}\,J$;

   So, $W(\overrightarrow{T_1}) + 0.18\times10\times(2.8\times10^{-2} - 0) = 7.056\times10^{-3}$; then $W(\overrightarrow{T_1}) = -4.33\times10^{-2}\,J$.

4. The work done by the tension is given by: $W(\overrightarrow{T_1}) = \overrightarrow{T_1}.\vec{x} = -T_1\times(x_4 - x_0)$;

   Then $-43.344\times10^{-3}\,J = -T_1\times 2.8\times10^{-2}$; thus $T_1 = 1.548\,N$.

Part B

1. We know that the linear momentum at a given instant is given by: $P_i = M\,V_i$

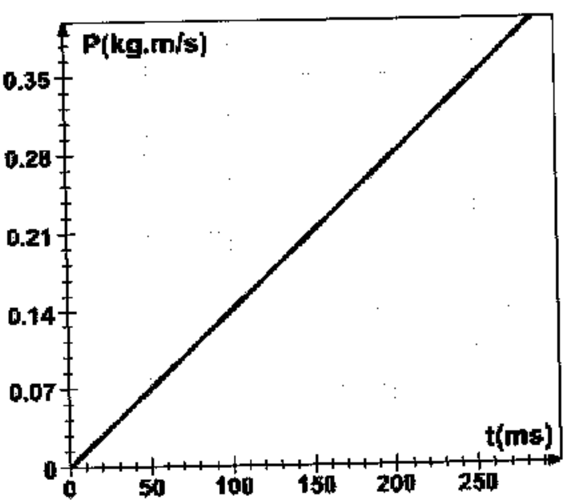
   Then $P_0 = 0$, $P_1 = 0.07\,kg.m.s^{-1}$, $P_2 = 0.14\,kg.m.s^{-1}$;

   $P_3 = 0.21\,kg.m.s^{-1}$ & $P_4 = 0.28\,kg.m.s^{-1}$.

2. a) Graph.

   On the abscissa axis $1\text{div} \equiv 50\,ms$;

   On the ordinate axis $1\text{div} \equiv 0.07\,kg.m/s$



   b) The graph representing the variation of the linear momentum P in terms of time is represented by a straight line passing through origin, then its equation is $P = k\,t$

   where $k = \dfrac{\Delta P}{\Delta t} = \dfrac{P_1 - P_0}{t_1 - t_0} = 1.40\,kg.m/s^2$

   Then $P = 1.4t$ (P in $kg.m./s$ & t in s)

3. a) We know that: $\dfrac{dP}{dt} = \dfrac{d(MV)}{dt} = M\,a$.

   Basing on the pervious result we have:

   $\dfrac{dP}{dt} = k$; so $M\,a = k$; thus $a = 1.40\,m/s^2$.

b) Newton's 2nd law applied on the trolley $\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt} \Rightarrow \overrightarrow{T_2} + \overrightarrow{w_2} + \vec{N} = \frac{d\vec{P}}{dt}$.

Projecting in the positive direction, we get: $T_2 = \frac{dP}{dt} = 1.40\,N$.

Part C

1. The forces acting on the pulley are: $\overrightarrow{T_2'}$, $\overrightarrow{T_1'}$, $\overrightarrow{w_p}$ & $\overrightarrow{N_R}$.

2. Theorem of angular momentum $\sum M_{\vec{F}} = I\,\theta'' = I\frac{a}{r}$, so $(T_1 - T_2)r = I\frac{a}{r}$;

   Thus $I = 2.643 \times 10^{-4}\,kg.m^2$.

III-GS 2004 2nd

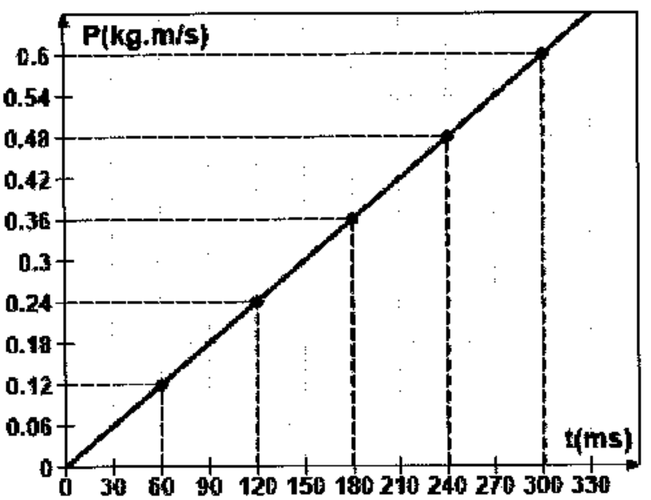
1. We know that:

$$v_2 = \frac{A_1A_3}{2\tau} = \frac{A_0A_3 - A_0A_1}{2\tau} = \frac{(64.9 - 7.2)\times 10^{-3}\,m}{2\times 60\times 10^{-3}\,s} = \frac{57.7m}{0.12s} = 0.48m/s\ ;$$

$$v_4 = \frac{A_3A_5}{2\tau} = \frac{A_0A_5 - A_0A_3}{2\tau} = \frac{(181 - 64.9)\times 10^{-3}\,m}{2\times 60\times 10^{-3}\,s} = 0.98m/s\,;$$

So $P_2 = m\,v_2 = 0.24\,kg.m/s$ & $P_4 = m\,v_4 = 0.48\,kg.m/s$.

2. Curve.



3. The curve representing the variation of the linear momentum as a function of time is represented by a straight line passing through the origin of equation: $P = bt$ but $\vec{P} = m\vec{V}$; thus, $\vec{P} = b\,t\,\vec{i}$.

4. b is the slope of the straight line, then:

$$b = \frac{\Delta P}{\Delta t} = \frac{P_5 - P_1}{4\tau} = 2\,kg.m/s^2.$$

5. a) $\frac{d\vec{P}}{dt} = b\vec{i} = 2\vec{i}\,(N)$.

   The resultant of the forces acting, if the force of friction is neglected, is: $\sum \vec{F} = \vec{w} + \vec{N}$.

   Along the axis $(O;\vec{i})$: $\sum \vec{F} = (w_x + N_x)\vec{i}$ where $w_x = m\,g\sin\alpha$ & $N_x = 0$,

   So, $\sum \vec{F} = m\,g\sin\alpha\,\vec{i}$; we get $\sum \vec{F} = 0.5\times 9.8\times 0.5\vec{i} = 2.45\vec{i}\ (N)$;

   Then, $\sum \vec{F} = m\,g\sin\alpha\,\vec{i} = 2.45\,\vec{i} \neq \frac{d\vec{P}}{dt} = 2\vec{i}$;

   Thus, the system is subjected to a force of friction $\vec{f}$.

   b) Newton 2nd law: $\sum \vec{F} = \frac{d\vec{P}}{dt}$; $\vec{w} + \vec{N} + \vec{f} = \frac{d\vec{P}}{dt}$;

   Along the axis $(O;\vec{i})$: $\sum \vec{F} = (m\,g\sin\alpha - f)\vec{i} = \frac{d\vec{P}}{dt} = 2\vec{i}$; $m\,g\sin\alpha - f = 2$;

   Then $f = 2.45 - 2 = 0.45\,N$.

GS Sessions – Linear Oscillations

I-Saida 2013 (Special)

Measurement of the Mass of an Astronaut

The aim of this exercise is to measure, in a spaceship, the mass of an astronaut using a horizontal mechanical oscillator.

Part A

Theoretical study

Consider a horizontal mechanical oscillator formed of a solid (S), of mass m, connected to two identical springs of negligible mass and each of stiffness k_1. The center of inertia G of (S) may slide along a horizontal axis $x'Ox$, where O is confounded with the equilibrium position of G.

At equilibrium, the two springs are neither compressed nor elongated (Figure 1).

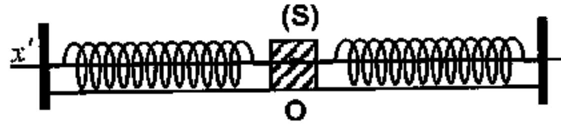


Figure 1

The solid (S), is displaced by a distance x_0 from its equilibrium position in the chosen positive direction, then released without initial velocity at the instant $t_0 = 0$. At an instant t, the abscissa of G is x and the algebraic value of its velocity $\vec{v}$ is v.

Neglect all frictional forces, and take the horizontal plane passing through G as a gravitational potential energy reference.

1. Show that the expression of the elastic potential energy of the system [(S), two springs, Earth] is $PE_e = \frac{1}{2}kx^2$ where $k = 2k_1$.
2. Write, as a function of k, m, v and x, at an instant t, the expression of the mechanical energy of the system [(S), two springs, Earth].
3. Derive the differential equation, in x, which describes the motion of G.
4. The solution of this differential equation is of the form: $x = A\cos(w_0 t + \varphi)$ where A, w_0 and φ are constants. Determine the expressions of A and w_0 in terms of x_0, m and k and determine the value of φ.
5. Deduce, in terms of m and k_1, the expression of the proper period T_0 of the oscillations of G.

Part B

Practical study

In spaceships, astronauts measure their masses using a mechanical oscillator as the one above. An astronaut sits in a chair attached to two identical massless springs each of stiffness $k_1 = 700\,N/m$ forming a horizontal oscillator (Figure 2).

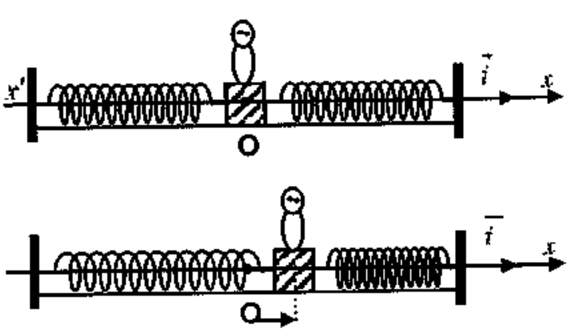


Figure 2

Let m be the total mass of the astronaut and the chair.

With an appropriate device, we record the variation of the abscissa x of the center of mass of the system [astronaut, chair, 2 springs] as a function of time (Figure 3).

1. Indicate:
   a) the type of the observed oscillations;
   b) the value of the pseudo-period T of these oscillations.
2. The pseudo-period T is approximately equal to the proper period T_0. Conclude.

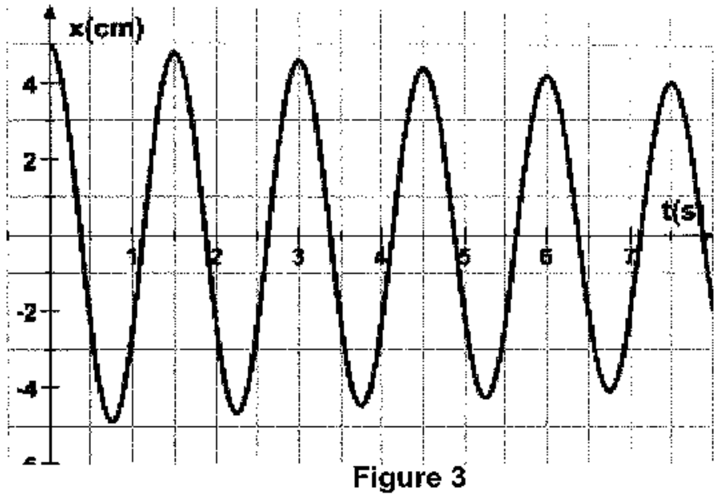


Figure 3

3. Deduce the mass of the astronaut knowing that the mass of the chair is $6.5\,kg$.

II-GS 2013 2[nd]
Horizontal Mechanical Oscillator

A horizontal mechanical oscillator is formed of a puck (S), of mass $m = 510\,g$, attached to two identical springs of un-jointed loops whose other extremities A and B are connected to two fixed supports.

Each spring is of negligible mass, natural length ℓ_0 and stiffness $k = 10\,N.m^{-1}$. (S) may slide along a horizontal air table and its center of inertia G can then move along a horizontal axis $x'Ox$.

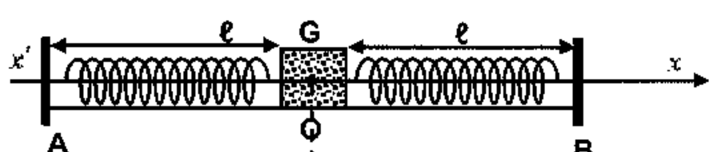


Figure 1

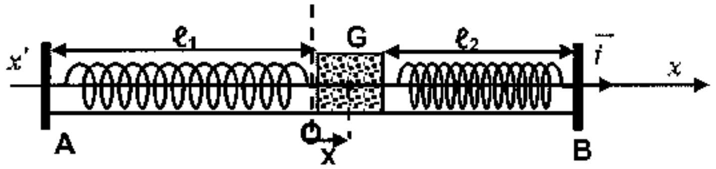


Figure 2

At equilibrium (Figure 1):

✖ G coincides with the origin O of the axis $x'x$;

✖ each spring is elongated by $\Delta\ell$ such that its length is $\ell = \ell_0 + \Delta\ell$.

The horizontal plane passing through G is taken as a reference level of gravitational potential energy.

Part A
Theoretical study

(S) is supposed to oscillate without friction. At an instant t, the abscissa of G is $x = OG$, the algebraic value of its velocity is $v = \frac{dx}{dt}$ and the two springs have lengths ℓ_1 and ℓ_2 (Figure 2).

1. a) Referring to figure 2, express ℓ_1 and ℓ_2 in terms of ℓ and x.
   b) Show that, at an instant t, the total elastic potential energy stored in the two springs is given by: $PE_e = k\left[(\Delta\ell)^2 + x^2\right]$.

2. Write down, at an instant t, the expression of the mechanical energy of the system (puck, two springs, Earth) in terms of v, m, k, $\Delta\ell$ and x.
3. Derive the second order differential equation in x that describes the motion of G.
4. The solution of the differential equation is of the form: $x = X_m \cos(w_0 t + \varphi)$ where X_m, w_0 and φ are constants.

   a) Determine, in terms of k and m, the expression of w_0.

   b) Deduce the value T_0 of the proper period of the oscillations of G.

Part B

Experimental study

An appropriate apparatus allows the recording of the abscissa x of G as a function of time (Figure 3).

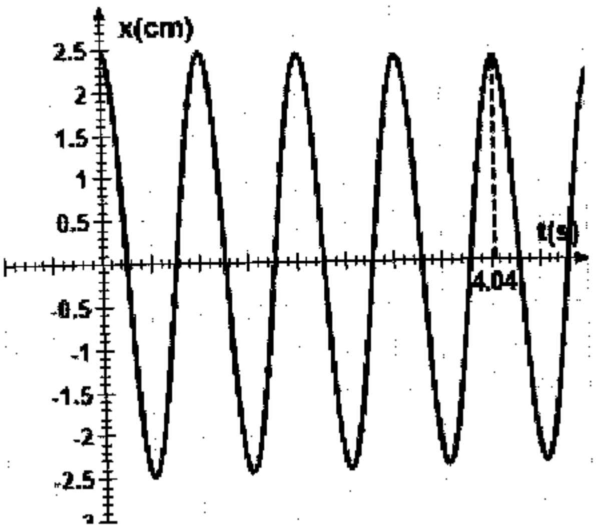


Figure 3

1. a) The experimental value of the period T is slightly different from the theoretical value T_0.

   Indicate the cause of this difference.

   b) Determine, referring to figure 3, the period T of the oscillations of G.
2. At $t = 4.04\,s$, the amplitude of the oscillations is $2.36\,cm$.

   a) Determine the mechanical energy lost by the system (puck, two springs, Earth) between the instants $t_0 = 0$ and $t = 4.04\,s$.

   b) Deduce the average power lost in this interval.
3. The extremity A of the left spring is coupled to an exciter of adjustable frequency « f » (Figure 4).

   With an appreciable amount of friction, the puck is forced to oscillate on the air table with a frequency equal to that of (E). The variations, as a function of time, of the abscissa x of G is represented for two values of « f » by figures 5 and 6.

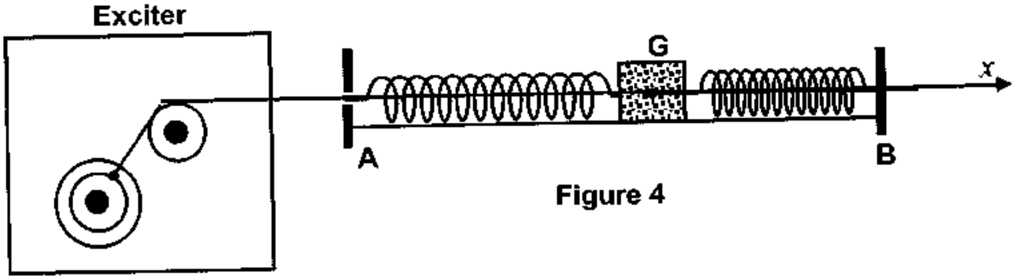


Figure 4

   a) Determine, in each case, the amplitude and the period of the oscillations of G.

   b) The amplitude of the oscillations represented in figure 6 is larger than that of the oscillations of figure 5. Interpret this increase.

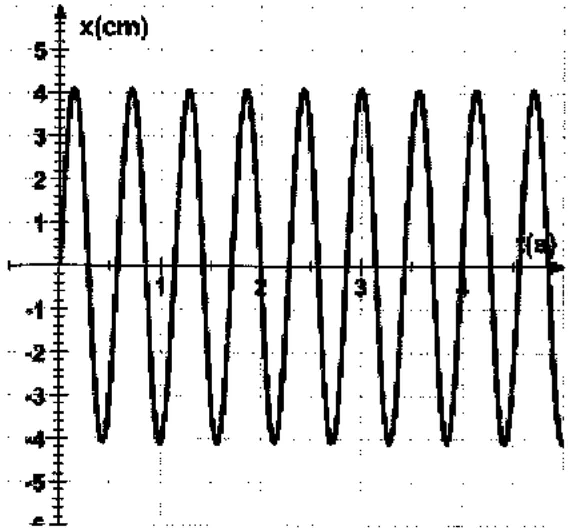


Figure 5

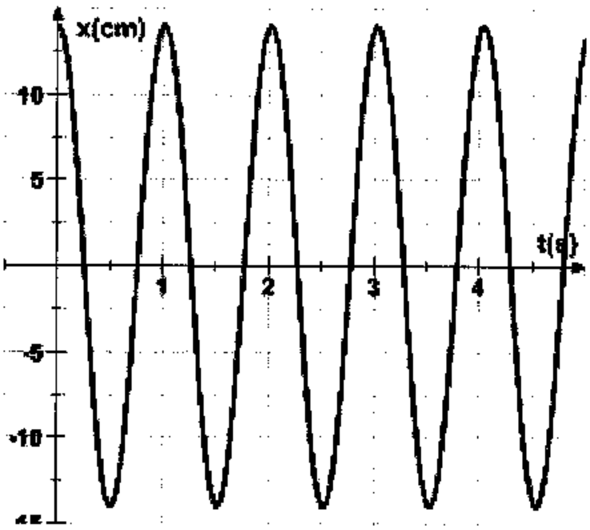
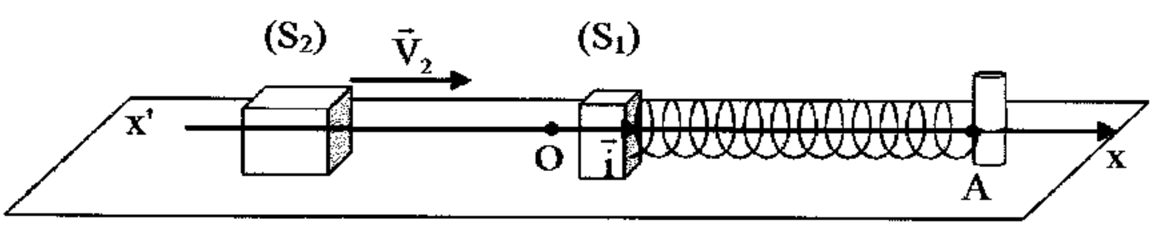


Figure 6

III-GS 2010 1st
Mechanical Oscillator

Two solids (S_1) and (S_2), of respective masses $m_1 = 100\,g$ and $m_2 = 500\,g$, can slide on a horizontal table. The solid (S_1) is fixed to one end of a spring with un-jointed turns and of negligible mass and of stiffness $k = 25\,N/m$, the other end A of the spring being fixed to a support as shown in the figure below. (S_2) is launched towards (S_1) and attains, just before impact, the velocity $\overrightarrow{V_2} = V_2\,\vec{i}$ where $V_2 = 0.48\,m/s$.



Due to collision, (S_2) sticks to (S_1) thus forming a single solid (S), just after collision, at the instant $t_0 = 0$, whose center of inertia G moves with the velocity $\overrightarrow{V_0} = V_0\,\vec{i}$.
The horizontal plane through G is taken as a gravitational potential energy reference.

Part A
Theoretical Study

Neglect all forces of friction.

1. Show that $V_0 = 0.40\,m/s$.
2. After collision, (S) still connected to the spring, continues its motion. At an instant t, we define the position of G by its abscissa x on the axis $(O,\vec{i})$, $v = \dfrac{dx}{dt}$ being the algebraic measure of the velocity of G. The origin O of abscissas is the position of G at the instant $t_0 = 0$.

a) Calculate the mechanical energy of the system [(S), spring, Earth] at the instant $t_0 = 0$.

b) Give, at the instant t, the expression of the mechanical energy of the system [(S), spring, Earth] in terms of m_1, m_2, k, x and v.

c) Deduce that the abscissa of G is $6.2\,cm$ when v is equal to zero for the first time.

3. a) Derive the second order differential equation of the motion of G.

b) The solution of this differential equation is of the form: $x = X_m \sin(w_0 t + \varphi)$.

i- Determine the values of the constants X_m, w_0 & φ.

ii- Calculate the value of the proper period T_0 of oscillations of G and deduce the time t_1 needed by G to pass from O to the position where v becomes zero for the first time.

Part B

Experimental study

In fact, (S), again shot with the velocity V_0 at the instant $t_0 = 0$, performs oscillations of pseudo-period very close to T_0. The velocity of G becomes zero for the first time at the instant t_1 but the abscissa of G is just $6\,cm$.

1. Determine the energy lost during t_1.
2. An apparatus (D), conveniently connected to the oscillator, provides energy in order to compensate for the loss. Calculate the average power provided by (D).
3. The oscillator is at rest. The apparatus (D) and the support are removed. The end A of the spring is connected to a vibrator, which vibrates along the spring, with an adjustable frequency f.

a) In steady state, (S) performs oscillations of frequency f. Why?

b) For a certain value f_1 of f, the amplitude of oscillations of (S) attains a maximum value.

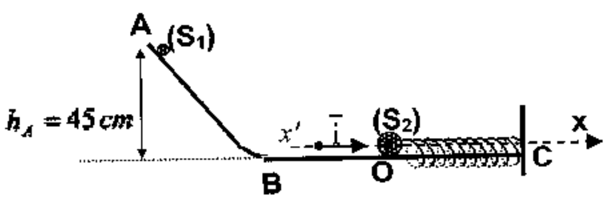
i- Give the name of the phenomenon that thus took place.

ii- Calculate the value of f_1.

IV-GS 2008 2nd

Determination of the Stiffness Constant of a Spring

To determine the stiffness constant k of a spring we attach to its extremity a solid (S_2), of mass $m_2 = 200\,g$, which can slide without friction on the horizontal part BC of a track ABC situated in a vertical plane, the other extremity of the spring is fixed at C.



Another solid (S_1), of mass $m_1 = 50\,g$, is released without initial velocity from a point A of the curved part of the track.

Point A is situated at a height $h_A = 45\,cm$ from the horizontal part of the track.

(S_2), initially at rest at point O, is thus hit by (S_1). (S_1) and (S_2) are supposed to be point masses.

The horizontal plane passing through BC is taken as a gravitational potential energy reference.

Take: $g = 10\,m.s^{-2}$, $0.32\pi = 1$.

Neglect all frictional forces.

1. Determine the value V_1 of the velocity $\overrightarrow{V_1}$ of (S_1) just before colliding (S_2).
2. After collision, (S_1) remains in contact with (S_2) and the two solids form a solid (S) of center of inertia G and of mass $M = m_1 + m_2$. Thus G performs oscillations around O with amplitude $3\,cm$ on the axis $x'Ox$ of origin O and unit vector $\vec{i}$.
   a) Show that the value of the velocity $\overrightarrow{V_0}$ of G just after the collision is equal to $0.6\,m/s$.
   b) Let x and v be respectively the abscissa and the algebraic value of the velocity of G at an instant t after the collision. The instant of collision at O is considered as an origin of time $t_0 = 0$.
      i- Write down, at an instant t, the expression of the mechanical energy of the system ((S), spring, Earth) in terms of k, x, M and v.
      ii- Deduce the second order differential equation in x that describes the motion of G.
      iii- The time equation of oscillation of (S) is given by: $x = x_m \sin(w_0 t + \varphi)$. Determine the value of φ and the expressions of the constants x_m and w_0 in terms of k, M & V_0.
      iv- Deduce the value of the stiffness constant k of the spring.
3. In reality friction is not neglected. To ensure the value of k, the extremity C of the spring is attached to a vibrator of adjustable frequency f and which can vibrate in the same direction of the spring. We notice that the amplitude of the oscillations of (S) varies with f and attains a maximum value for $f = 3.2\,Hz$.
   a) Name the physical phenomenon that takes place when $f = 3.2\,Hz$.
   b) Calculate the value of k.

V-GS 2008 1st Tripoli
Harmonic Oscillator

In order to study a harmonic oscillator, we consider a solid (S) taken as a particle of mass $m = 100\,g$ and two identical springs (S_1) and (S_2) of un-jointed turns each of stiffness k and of free length ℓ_0. The oscillator thus formed is represented in figure 1. At equilibrium, (S) is at the origin O of the axis $x'x$ on which $\vec{i}$ is a unit vector and the length of each spring is ℓ_0.

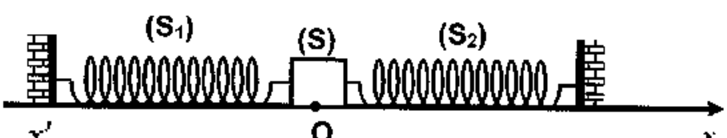


Figure 1

(S) is shifted from this position by a distance d to the right and then released without velocity at the instant $t_0 = 0$. At an instant t, the abscissa of (S) is x, the algebraic value of its velocity is $v = x'$ and its acceleration is x''. (S) would thus oscillate, without friction, on the axis $x'x$; the horizontal plane containing this axis is taken as a gravitational potential energy reference.

Part A
Differential equation

1. Write, at the instant t, the expression of the mechanical energy of the system [(S), springs].
2. Derive the differential equation that governs the motion of (S).
3. Deduce the expression of the proper angular frequency w_0 of the motion in terms of k and m.

Part B
Values of some physical quantities

A convenient apparatus is used to trace the curve representing the variations of the acceleration as a function of the abscissa $x'' = f(x)$, figure 2.

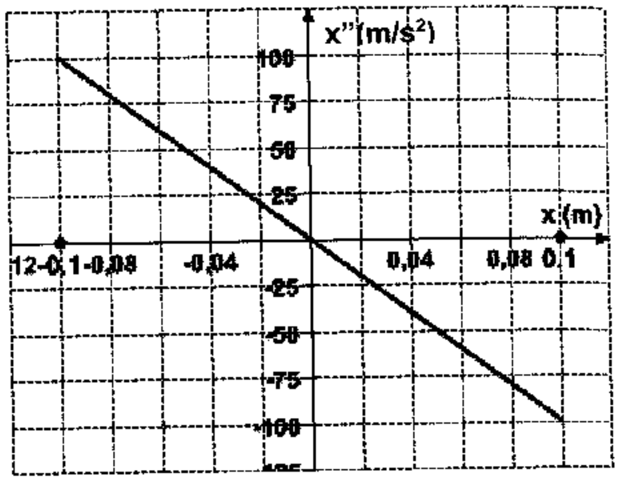
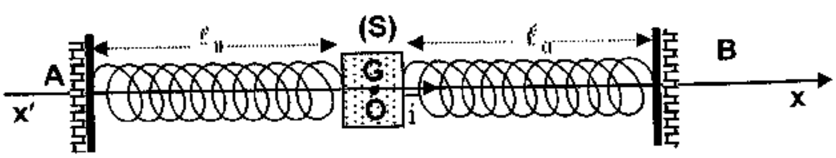


Figure 2

1. Show that the curve representing the acceleration $x'' = f(x)$ agrees with the differential equation just derived.
2. Referring to the graph:
   a) give the value of the amplitude x_m of the motion.
   b) give the value of the acceleration x'' for $x = -x_m$.
   c) find the value of the proper angular frequency w_0 of the motion.
3. a) Show, using the relation $x'' = f(x)$, that the speed (S) is maximum when it passes through its equilibrium position.
   b) Deduce the value v_{max} of the maximum speed.
4. Calculate the value of the spring constant k.

VI-GS 2005 1st
Study of a Horizontal Mechanical Oscillator

A solid (S), of mass $m = 140\ g$, may slide on a straight horizontal track. The solid is connected to two identical springs of un-jointed turns, of negligible mass, fixed between two supports A and B. Each of these springs has a stiffness (force constant) $k = 0.6 N/m$, and a free length ℓ_0.



We denote by O the position of the center of mass G of (S) when the oscillator [(S) + two springs] is in equilibrium, each spring having then the length ℓ_0 (figure).

The solid is shifted from this equilibrium position along the direction $x'x$, by a distance of $4.2\,cm$, and then released without initial velocity at the instant $t_0 = 0$. During its oscillations, at any instant t, the abscissa of G is x and the algebraic value of its velocity is v, O being the origin of abscissas.

The horizontal plane through G is taken as a gravitational potential energy reference.

Part A

Theoretical study

In this part, we neglect friction

The solid (S) performs, in this case, oscillations of amplitude $X_{m0} = 4.2\,cm$.

1. a) Show that the expression of the elastic potential energy of the oscillator is $PE_0 = k\,x^2$.
   b) Write the expression of the mechanical energy ME of the system [oscillator, Earth] as a function of m, v, x & k.
2. a) Derive the differential equation that governs the motion of (S).
   b) Deduce the expression of the proper period T_0 of the oscillator in terms of m & k.
   c) Calculate the value of T_0. Take $\pi = 3.14$.

Part B

Experimental study

In reality, the value of the amplitude X_m decreases during oscillations, each of duration T.

Some values of X_m are tabulated as below.

Instant	0	T	$2T$	$3T$	$4T$	$5T$
Amplitude $X_m(cm)$	$X_{m0} = 4.2\,cm$	$X_{m1} = 2.86\,cm$	$1.95\,cm$	$1.33\,cm$	$0.91\,cm$	$0.62\,cm$

1. Draw the shape of the curve representing the variation of the abscissa x of G as a function of time.

   <u>Scale</u>: on the axis of abscissas $1\,cm$ represents $T/4$;

   and on the axis of ordinates $1\,cm$ represents $1\,cm$.
2. The duration of 5 oscillations is measured and found to be $10.75\,s$.
   a) Calculate T.
   b) Compare T and T_0.
   c) What is then the type of oscillations?
3. The decrease in the mechanical energy of the system [oscillator, Earth] is due to the existence of a force of friction of the form $\vec{f} = -h\,\vec{v}$ where $\vec{v} = v\,\vec{i}$ and h is a positive constant.
   a) Determine the differential equation of motion of (S).
   b) From the above table of values, verify that: $\frac{X_{m1}}{X_{mo}} \approx \frac{X_{m2}}{X_{m1}} \approx \ldots.. \approx A$ where A is a positive constant.
   c) Knowing that A is given by the expression $A = e^{-\frac{hT}{2m}}$, calculate h.
4. In order to compensate for the loss in the mechanical energy of the system, an apparatus (D) allows, at regular time intervals, to provide energy to the oscillator.
   a) Determine the average power furnished by (D) between the instants $t = 0$ and $t = 5T$.
   b) What is then the type of oscillations?

Solutions

I-Saida 2013 (Special)

Part A

1. $PE_e = PE_{e1} + PE_{e2} = \frac{1}{2}k_1 x^2 + \frac{1}{2}k_1 x^2 = k_1 x^2 = \frac{1}{2}(2k_1)x^2 = \frac{1}{2}k x^2$ with $k = 2k_1$.

2. $ME = KE + PE_e = \frac{1}{2}m v^2 + \frac{1}{2}k x^2$.

3. The forces of friction are negligible, then its mechanical energy is conserved so $\frac{d(ME)}{dt} = 0$;

   $m v v' + k x x' = 0$ but $v = x'$ & $v' = x''$, then $m v\left(x'' + \frac{k}{m}x \right) = 0$;

   The system is in motion so $v \neq 0$, then $x'' + \frac{k}{m}x = 0$.

4. $x = A\cos(w_0 t + \varphi)$;

   ✘ At $t = 0$, $x = x_0 > 0$ & $v_0 = 0$ then $A = x_0$; and $A = A\cos(\varphi) \Rightarrow \cos\varphi = 1$ thus $\varphi = 0$.

   ✘ We have $x = A\cos(w_0 t + \varphi) \Rightarrow x'' = -A w_0^2 \cos(w_0 t + \varphi)$;

   Replacing in the differential equation, we get:

   $-A w_0^2 \cos(w_0 t + \varphi) + \frac{k}{m} A\cos(w_0 t + \varphi) = 0$;

   $A\cos(w_0 t + \varphi)\left(-w_0^2 + \frac{k}{m} \right) = 0$ but $A\cos(w_0 t + \varphi) = x \neq 0$ (the system is in motion);

   Then $-w_0^2 + \frac{k}{m} = 0$; $w_0 = \sqrt{\frac{k}{m}}$.

5. The period and the angular frequency are related by $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{2k_1}}$.

Part B

1. a) Free damped oscillations.

   b) From graph $T = 1.5\,s$.

2. The system is slightly damped so $T_0 \approx T = 1.5\,s$.

3. We have $T_0 = 2\pi\sqrt{\frac{m}{2k_1}} \Rightarrow m = \frac{2k_1 T_0^2}{4\pi^2} = \frac{700 \times 1.5^2}{2\pi^2} \approx 79.8\,kg$ where m is the total mass;

   But $m = m_{\text{chair}} + m_{\text{astronaute}}$; then $m_{\text{astronaute}} = m - m_{\text{chair}} = 79.8 - 6.5 = 73.3\,kg$.

II-GS 2013 2[nd]

Part A

1. a) $\ell_1 = \ell + x$ and $\ell_2 = \ell - x$.

   b) $PE_e = PE_{e1} + PE_{e1} = \frac{1}{2}k(\Delta\ell_1 - x)^2 + \frac{1}{2}k(\Delta\ell_2 - x)^2$ But $\Delta\ell_1 = \Delta\ell_2 = \Delta\ell$;

$$PE_e=\frac{1}{2}k\left[(\Delta\ell-x)^2+(\Delta\ell-x)^2\right]=\frac{1}{2}k\left[(\Delta\ell)^2-2x\Delta\ell+x^2+(\Delta\ell)^2-2x\Delta\ell+x^2\right];$$

$$=\frac{1}{2}k\left[2(\Delta\ell)^2+2x^2\right]=k\left[(\Delta\ell)^2+x^2\right].$$

2. The mechanical energy $ME=KE+PE_e=\frac{1}{2}mv^2+k\left[(\Delta\ell)^2+x^2\right]$.

3. The friction is negligible then the mechanical energy is conserved so $\frac{d(ME)}{dt}=0$;

   $m\,v\,v'+k[0+2\,x\,x']=0$; $v(m\,v'+2k\,x)=0$

   where $v=x'\neq 0$ (the system is in motion) & $v'=x''$;

   Then $m\,x''+2k\,x=0\Rightarrow x''+\frac{2k}{m}x=0$.

4. a) We have $x=A\cos(w_0t+\varphi)\Rightarrow x''=-A\,w_0^2\cos(w_0t+\varphi)$.

   Replacing in the differential equation we get:

   $x''+\frac{2k}{m}x=0\Rightarrow -A\,w_0^2\cos(w_0t+\varphi)+\frac{2k}{m}A\cos(w_0t+\varphi)=0$;

   $A\cos(w_0t+\varphi)\times\left(-w_0^2+\frac{2k}{m}\right)=0$;

   But $x=A\cos(w_0t+\varphi)\neq 0$ the system is in motion;

   Then $-w_0^2+\frac{2k}{m}=0\Rightarrow w_0=\sqrt{\frac{2k}{m}}$.

   b) The proper period and the angular frequency are related by: $T_0=\frac{2\pi}{w_0}=2\pi\sqrt{\frac{m}{2k}}$.

   Numerical values: $T_0=2\pi\sqrt{\frac{0.51}{2\times 10}}=1.003\,s\approx 1\,s$.

Part B

1. a) T is slightly different from the theoretical value T_0 due to friction.

   b) From graph $4T=4.04$, then $T=1.01\,s$.

2. a) $ME|_{t_0=0}=KE|_{t_0=0}+PE_e|_{t_0=0}=k\left[(\Delta\ell)^2+x_{m0}^2\right]$;

   and $ME|_{t=4.04s}=KE|_{t=4.04s}+PE_e|_{t=4.04s}=k\left[(\Delta\ell)^2+x_{m4}^2\right]$;

   The variation in the mechanical energy is :

   $\Delta(ME)=ME|_{t=4.04s}-ME|_{t_0=0}=k\left[(\Delta\ell)^2+x_{m4}^2\right]-k\left[(\Delta\ell)^2+x_{m0}^2\right]=k\left[x_{m4}^2-x_{m0}^2\right]$;

   $\Delta(ME)=10\left[5.57\times10^{-4}-6.25\times10^{-4}\right]=-6.8\times10^{-4}\,J$.

   b) The average power lost is: $P_\ell=\frac{\Delta(ME)}{\Delta t}=\frac{-6.8\times10^{-4}\,J}{4.04s}=-1.68\times10^{-4}\,W$.

3. From figure 5, $x_{m1}=4.1\,cm$ and $T_1=4/7\,s=0.57\,s$.

   From figure 6, $x_{m2}=14\,cm$ and $T_2=1.01\,s$.

4. $T_2=1.01\approx T_0$, so the system is resonating then it oscillates with its maximum amplitude.

   Then T_1 is away from its proper period thus $x_{m2}=14\,cm>x_{m1}=4.1\,cm$.

III-GS 2010 1st

Part A

1. The linear momentum is conserved $\overrightarrow{P}_{\text{Just before collision}} = \overrightarrow{P}_{\text{Just after collision}}$;

$$m_1\overrightarrow{V_2} + m_2\overrightarrow{0} = (m_1 + m_2)\overrightarrow{V_0} \Rightarrow \overrightarrow{V_0} = \frac{m_1}{m_1 + m_2}\vec{i} = \frac{500}{600} \times 0.48\vec{i} = 0.40\vec{i} \ (m/s).$$

2. a) $ME = ME|_{t_0=0} = KE|_{t_0=0} + PE_e|_{t_0=0} = \frac{1}{2}(m_1 + m_2)V_0^2 + 0 = 0.048\,J$.

b) The mechanical energy at any instant $ME = KE + PE_e = \frac{1}{2}(m_1 + m_2)v^2 + \frac{1}{2}k\,x^2$.

c) The forces of friction are negligible, then the mechanical energy is conserved:

$$ME|_{t_0=0} = KE|_{x=X_m} + PE_e|_{x=X_m} = \frac{1}{2}k\,X_m^2 = 0.048\,J;\ X_m = \sqrt{\frac{2 \times 0.048}{k}} = 0.062\,m = 6.2\,cm.$$

3. a) ME is conserved so $\frac{d(ME)}{dt} = 0$; $(m_1 + m_2)v\,v' + 2\,k\,x\,x' = 0$;

But $v = x' \neq 0$ (since the system is in motion), so $x'' + \frac{k}{m_1 + m_2}x = 0$; $\quad (M = m_1 + m_2)$

b) Properties of motion

i- $x = X_m \sin(w_0 t + \varphi)$, so $x'' = -X_m w_0^2 \sin(w_0 t + \varphi)$.

Replacing in the differential equation: $X_m \sin(w_0 t + \varphi)\left(-w_0^2 + \frac{k}{M}\right) = 0$;

But $X_m \sin(w_0 t + \varphi) \neq 0$, then $w_0 = \sqrt{\frac{k}{M}} = 6.45\,\text{rad}/s$;

The initial conditions of motion at $t_0 = 0$, $x = 0$; so $\sin\varphi = 0$; then $\varphi = 0$ or π.

But: $v|_{t_0=0} > 0 \Rightarrow X_m w_0 \cos\varphi > 0$, thus $\varphi = 0$.

Since $v|_{t_0=0} = V_0 = X_m w_0 \cos\varphi \Rightarrow X_m = \frac{0.40}{6.45} = 0.062\,m = 6.2\,cm$.

Thus, $x = 6.2\sin(6.45\,t)$ (where x in cm & t in s).

ii- The proper period $T_0 = \frac{2\pi}{w_0} = 0.974\,s$ & $t_1 = \frac{T_0}{4} = 0.243\,s$.

Part B

1. The loss of energy is : $E = |\Delta(ME)| = \frac{1}{2}k\left(X_m^2 - X_{m_1}^2\right) = 3.05 \times 10^{-3}\,J$.

2. The average power provided by (D) is : $P_{av} = \frac{E}{t_1} = 1.25 \times 10^{-2}\,W$.

3. a) The system undergoing forced oscillations, then the power of the exciter f is transmitted to the mechanical oscillator.

b) Amplitude and frequency:

i- Amplitude resonance.

ii- $T \approx T_0$ & $f_1 \approx \frac{1}{T_0} \Rightarrow f_1 = 1.03\,Hz$.

IV-GS 2008 2nd

1. The solid (S_1) slides without friction, then its mechanical energy is conserved between A and B:

   $ME_A = ME_B$; $KE_A + GPE_A = KE_B + GPE_B$;

   But, $KE_A = 0$ (from rest) & $GPE_B = 0$ (on reference);

   Then $m_1 g\, h_A + 0 = \frac{1}{2} m_1 V_1^2 + 0$; thus, $V_1 = \sqrt{2 g\, h_A} = \sqrt{2\times 10\times 0.45} = 3m/s$.

2. a) The linear momentum is conserved $\overrightarrow{P}_{\text{Just before collision}} = \overrightarrow{P}_{\text{Just after collision}}$;

   $m_1 \overrightarrow{V_1} + \vec{0} = (m_1 + m_2)\overrightarrow{V_0}$;

   Then $V_0 = \frac{m_1}{m_1 + m_2} V_1 = \frac{0.05\,kg}{(0.05+0.2)kg}(3\,m/s) = 0.6\,m/s$.

   b) Properties of the motion :

   i- The mechanical energy at any instant is $ME = \frac{1}{2} M\, v^2 + \frac{1}{2} k\, x^2$.

   ii- Since the mechanical energy is conserved then :

   $\frac{d(ME)}{dt} = 0$; $M\, v v' + k\, x x' = 0$ but $(v = x' \neq 0)$ since the system is motion & $v' = x''$;

   Thus, $x'' + \frac{k}{m_1 + m_2} x = 0$;

   iii-Initial conditions: at $t=0$, $x=0$; so $x = x_m \sin\varphi = 0$; then $\varphi = 0$ or π (rad);

   But, at $t=0$, $V = V_0 > 0$, $V_0 = x_m w_0 \cos\varphi > 0$, $\cos\varphi > 0$; then, $\varphi = 0$;

   Thus, $x = x_m \sin(w_0 t)$.

   ✖ But $v = x' = x_m\, w_0 \cos(w_0 t)$; and at $t=0$, $v = V_0$; then $V_0 = x_m w_0$.

   ✖ We have $x = x_m \sin(w_0 t)$, so $x' = x_m w_0 \cos(w_0 t)$ & $x'' = -x_m w_0^2 \sin(w_0 t)$;

   By replacing in the differential equation we get: $-x_m w_0^2 \sin(w_0 t) + \frac{k}{M} x_m \sin(w_0 t) = 0$;

   $x_m \sin(w_0 t)\times\left(-w_0^2 + \frac{k}{M}\right) = 0$, but $x_m \sin(w_0 t) \neq 0$ (in motion).

   Then, $w_0 = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{m_1 + m_2}}$.

   ✖ We get, $x_m = \frac{V_0}{w_0} = V_0\sqrt{\frac{M}{k}}$.

   iv- We have $x_m = V_0\sqrt{\frac{M}{k}}$; then, $k = \frac{V_0^2 \times M}{x_m^2} = \frac{0.36\times 0.25}{0.03^2} = 100\,N/m$.

3. a) Amplitude (mechanical) resonance.

   b) The frequency of the exciter is equal to the proper frequency of the resonator then $f = f_0 = \frac{w_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{M}}$;

   Then, $4\pi^2 f_0^2 = \frac{k}{M} \Rightarrow k = 4\pi^2 f_0^2 M = 4\times 10\times 3.2^2 \times 0.25 = 100\,N/m$.

V-GS 2008 1st Tripoli

Part A

1. The mechanical energy at any instant is $ME = \frac{1}{2}k\,x^2 + \frac{1}{2}kx^2 + \frac{1}{2}m\,v^2 = \frac{1}{2}m\,v^2 + k\,x^2$.

2. (S) oscillates without friction, then the mechanical energy is conserved.

   $\frac{d(ME)}{dt} = 0$, $m\,v\,v' + 2\,k\,x\,x' = 0$; $v(m\,x'' + 2k\,x) = 0$;

   (But $v = x' \neq 0$, since the system is in motion); then, $x'' + \frac{2\,k}{m}x = 0$.

3. The differential equation that governs the motion of the solid (S) is of 2nd order of the form

   $x'' + w_0^2 x = 0$ where $w_0 = \sqrt{\frac{2\,k}{m}}$.

Part B

1. The relation $x'' = -w_0^2\,x$ is satisfied; since its graphical representation $x'' = f(x)$ is:
   - ✘ a straight line, the variable x is of degree 1;
   - ✘ passing through the origin;
   - ✘ decreasing since it has a negative slope $-w_0^2$.

2. a) From graph the amplitude $x_m = 10\,cm$.

   b) The acceleration at $x = -x_m$ is $x'' = 100\,m.s^{-2}$.

   c) The differential equation $x'' + w_0^2 x = 0$ can be written as: $x'' = -w_0^2 x$.

   The slope of this straight line is $-w_0^2 = \frac{x''|_{x=-x_m} - x''|_{x=0}}{x|_{x=-x_m} - x|_{x=0}} = \frac{(100-0)}{(-0.1-0)} = -1000$;

   Then $w_0 = \sqrt{1000} = 10\sqrt{10}$ rad / s.

3. a) The maximum[1] of the speed is reached if $v' = \frac{dv}{dt} = 0$, then $x'' = 0$; but $x'' = -w_0^2\,x = 0$

   Thus $x = 0$ which corresponds to the equilibrium position taken as the origin of abscissas.

   b) According to the conservation of mechanical energy:

   $ME|_{x=x_m} = ME|_{x=0}$, so $k\,x_m^2 = \frac{1}{2}m\,v_{\max}^2$; then $|v_{\max}| = w_0\,x_m \Rightarrow v_{\max} = 3.16\,m\,s^{-1}$.

4. We have $w_0^2 = \frac{2k}{m}$; then $k = \frac{1}{2}m\,w_0^2 = \frac{1}{2}\times 0.1 \times 1000 = 50\,N/m$.

VI-GS 2005 1st

Part A

1. a) The potential energy of the two springs is given by: $PE_e = \frac{1}{2}k\,x^2 + \frac{1}{2}k\,x^2 = k\,x^2$.

   b) The mechanical energy of the system: $ME = PE_e + GPE + KE = kx^2 + \frac{1}{2}m\,v^2$.

   But $GPE = 0$ (moving on reference).

[1] The extremum of a function (maximum or minimum) is defined as the point in which the derivative changes its sign thus f'=0

2. a) The forces of friction are negligible, then the mechanical energy is conserved, so $\frac{d(ME)}{dt} = 0$;

We get $2k\,x\,x' + m\,v\,v' = 0$, but $v = x' \neq 0$ (system in motion) and $v' = x''$;

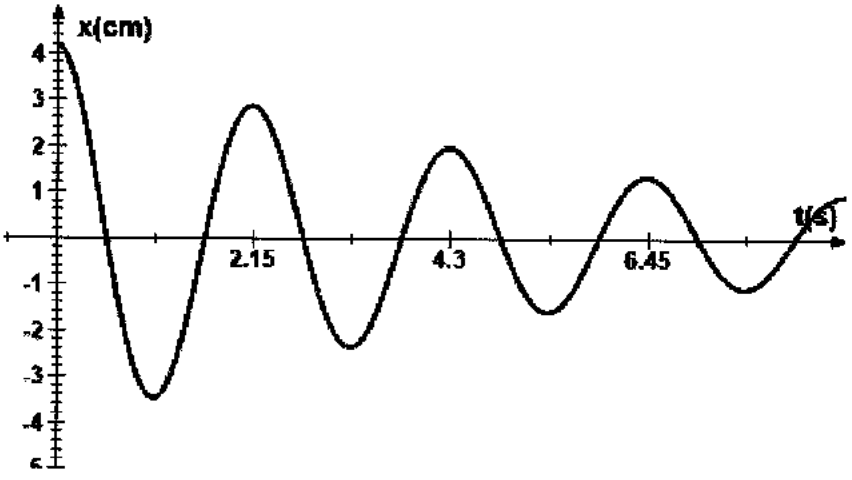
Then, $x'' + \frac{2k}{m}x = 0$.

b) The differential equation, that governs the motion of the oscillator, is of 2[nd] order of the form $x'' + w_0^2 x = 0$ where $w_0 = \sqrt{\frac{2k}{m}}$; then the motion of the pendulum is periodic, of proper period $T_0 = \frac{2\pi}{w_0} = 2\pi\sqrt{\frac{m}{2k}}$.

c) We have $T_0 = 2\pi\sqrt{\frac{m}{2k}} = 2 \times 3.14\sqrt{\frac{0.140}{2 \times 0.6}} = 2.145\,s$.

Part B

1. The curve representing the variations of the abscissa in terms of time is shown in the adjacent figure.



2. a) The motion is pseudo-periodic of pseudo period: $T = 10.75/5 = 2.15s$.

b) $T = 2.15s$ is slightly greater than $T_0 = 2.145s$.

c) The oscillator performs free damped oscillations.

3. a) According to Newton's 2[nd] law : $\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt}$; then $\vec{w} + \vec{N} + \vec{T} + \vec{f} = m\,v'\vec{i}$;

But $\vec{w} + \vec{N} = \vec{0}$ & $v' = x''$; so $-bv\vec{i} - kx\vec{i} = +mx''\vec{i}$; thus, $x'' + \frac{b}{m}x' + \frac{k}{m}x = 0$.

b) Referring to the table, we get $\frac{X_{m1}}{X_{m0}} = \frac{2.86}{4.20} = 0.681$; $\frac{X_{m2}}{X_{m1}} = \frac{1.95}{2.86} = 0.682$; .....

$\frac{X_{m5}}{X_{m4}} = \frac{0.62}{0.91} = 0.681$; then $A = \frac{X_{m_1}}{X_{m_0}} = \frac{X_{m_2}}{X_{m_1}} \approx 0.681$.

c) We have $A = e^{-\frac{ht}{2m}}$, then $\ell nA = -\frac{hT}{2m}$; thus $h = \frac{-2m\,\ell n(A)}{T} \approx 0.05$kg/s.

4. a) When the solid reaches a local maximum, it comes to rest; so for $x = \pm X_m$, we have $v = 0$; then the mechanical energy at these positions is equal to the elastic potential energy;

Then $\Delta(ME) = \Delta(PE_e) = k\,X_{m5}^2 - k\,X_{m0}^2 = 0.6\left[\left(0.62\times10^{-2}\right)^2 - \left(4.2\times10^{-2}\right)^2\right] = -1.04\times10^{-3}\,J$.

The average power furnished to the oscillator $P_{av} = \dfrac{|\Delta(ME)|}{\Delta t} = \dfrac{1.04\times10^{-3}\,J}{5\times2.15} = 9.6\times10^{-5}\,W$.

b) The system performs driven oscillations.

Unit II – GS

Electricity

GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Electromagnetic Induction		-	-	-	-	-	2nd	2nd	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Electromagnetic Induction	-	-	-	1st	-	-	-	-	-

GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Auto Induction		1st(A)	-	1st	-	-	-	-	-
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Auto Induction	1st	1st(S)	2nd	2nd	-	2nd	-	-	-

GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Capacitor and DC Voltage		2nd	2nd	2nd	-	-	1st	-	1st
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Capacitor and DC Voltage	-	2nd	2nd(B) & 1st	-	1st	-	-	-	-

GS – Sessions		2017	2016	2015	2014	2013	2012	2011	2010
Sinusoidal Voltage		-	1st & 2nd	1st & 2nd	-	1st & 2nd	2nd	1st	2nd
	2009	2008	2007	2006	2005	2004	2003	2002	2001
Sinusoidal Voltage	1st	1st	-	1st	2nd	-	-	-	-

I-GS 2012 2nd
Mechanical Oscillator

A metallic rigid rod MN, of mass $m = 0.25\,kg$, can slide without friction, on two parallel and horizontal metallic rails PP' and QQ'. During sliding, the rod remains perpendicular to the rails. These two rails, separated by a distance ℓ, are connected by a resistor of resistance R (Figure 1). Neglect the resistance of the rod and of the rails.

Part A
Electromagnetic induction

The whole setup is placed in an upward, uniform and vertical magnetic field $\vec{B}$ of value B. The position of G, the center of inertia of the rod, is defined by its abscissa x on the horizontal axis $(O,\vec{i})$ with O corresponding to the position of G at $t_0 = 0$. Let $O'O = d$.

At an instant t, G has an abscissa $OG = x$ and a velocity $\vec{v}$ of algebraic value v (Figure 1).

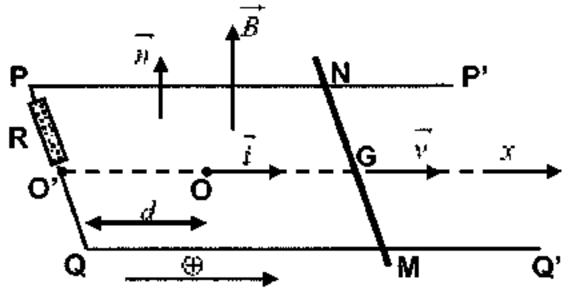


Figure 1

1. Show that, taking into consideration the arbitrary positive direction chosen in figure 1, the expression of the magnetic flux crossing the surface limited by the circuit $MNPQ$ is given by: $\phi = B(d + x)\ell$.
2. a) Derive the expression of the induced $e.m.f$ « e » across the terminals of the rod MN in terms of B, ℓ and v.
   b) Induced current
      i- Derive the expression of the induced current « i » in the circuit in terms of R, B, ℓ and v.
      ii- Deduce the direction of the induced current.
3. Show that the expression of the electromagnetic force $\vec{F}$ acting on the rod can be written as: $\vec{F} = -\dfrac{B^2\ell^2}{R}\vec{v}$.

Part B
Free un-damped oscillations

We remove the magnetic field $\vec{B}$.

The center of inertia G of the rod is attached to a horizontal massless spring, of un-stretched length $L_0 = O'O = d$ and stiffness $k = 50\,N/m$. Thus at equilibrium the abscissa of G is $x = 0$.

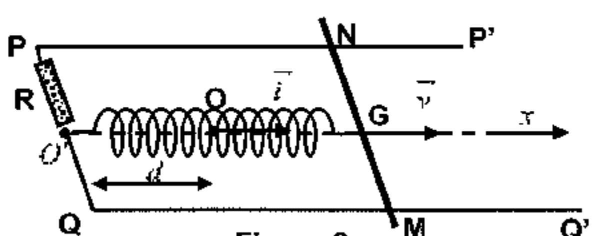


Figure 2

The rod, is displaced by a distance $X_m = 10\,cm$ in the positive direction, and is then released without initial velocity at the instant $t_0 = 0$; the rod thus oscillates around its equilibrium position.

At an instant t, G has an abscissa x and a velocity $\vec{v}$ of algebraic value v (Figure 2).

1. Write, in terms of m, v, k and x, the expression of the mechanical energy of the system (rod, spring, Earth).
   Take the horizontal plane through G as a gravitational potential energy reference.
2. Derive the differential equation of the second order in x, which describes the motion of G.
3. The solution of this differential equation is of the form: $x = A\cos(wt+\varphi)$.
   Determine the values of the constants w, φ and $A(A>0)$.

Part C
Free damped oscillations

The setup of figure 2 is placed now in the magnetic field B. The rod is displaced again by $x_m = 10\,cm$ in the positive direction, and is then released without initial velocity at the instant $t_0 = 0$; the rod thus oscillates around its equilibrium position.
At an instant t, G has an abscissa x and a velocity $\vec{v}$ of algebraic value v (Figure 3).

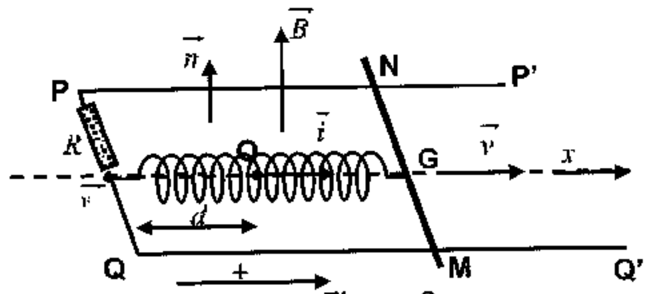
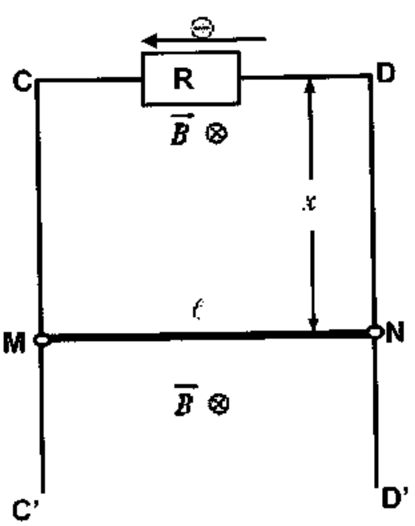


Figure 3

1. Calculate, at the instant $t_0 = 0$, the mechanical energy of the system (rod, spring, Earth).
   Take the horizontal plane through G as a gravitational potential energy reference.
2. During its motion, the oscillator loses mechanical energy.
   a) Show that the power of the electromagnetic force exerted on the rod, is given by:
   $$P = -\frac{B^2\ell^2 v}{R}.$$
   b) Determine the expression of the power lost due to Joule's effect in the resistor in terms of R, B, ℓ and v.
   c) Deduce in what form the energy of the oscillator is dissipated.
3. After a few oscillations, the rod stops.
   Give, in J, the value of the total energy dissipated by the oscillator during its motion.

II-GS 2011 2nd
Motion of a Conductor in Two Fields



Two vertical rails CC' and DD' are connected by a resistor of resistance R.

A conducting rod MN, of mass m and length ℓ, can slide without friction along these rails and remains horizontally in contact with these rails.

The whole set-up is placed within a uniform and horizontal magnetic field $\vec{B}$ that is perpendicular to the plane of the rails.

The rod MN, released from rest at the instant $t_0 = 0$, is found at an instant t at a distance x from CD, moving with a velocity whose algebraic value is $v(v>0)$ (adjacent figure).

1. Determine, at the instant t, the expression of the magnetic flux due to $\overrightarrow{B}$ through the circuit $CMND$ in terms of B, ℓ and x taking into consideration the arbitrary positive direction as shown on the figure.
2. a) Determine the expression of:
   i- the *e.m.f* « e » induced across the rod MN, in terms of v, B and ℓ.
   ii- the induced current i in terms of R, B, ℓ and v.
   b) Indicate, with justification, the direction of the current.
3. Show that the electric power dissipated by the resistor, at the instant t, is given by:
   $P_{e\ell} = \dfrac{B^2\ell^2}{R}v^2$.
4. The rod MN is acted upon by two forces: its weight $m\overrightarrow{g}$ and the Laplace's force $\overrightarrow{F}$ of magnitude $F = i\ell B$.
   a) Applying Newton's second law, show that the differential equation in v is given by:
   $\dfrac{dv}{dt} + \dfrac{B^2\ell^2}{mR}v = g$.
   b) The solution of this differential equation is: $v = A\left(1 - e^{-\frac{t}{\tau}}\right)$.
   Show that $A = \dfrac{mgR}{B^2\ell^2}$ and $\tau = \dfrac{mR}{B^2\ell^2}$.
   c) Show that v would attain a limiting value $V_{\ell im}$.
   d) Velocity
   i- Give the expression of v as a function of $V_{\ell im}$ at the instant $t = \tau$.
   ii- Deduce the time at the end of which v attains practically its limiting value.
   e) Calculate the value of $V_{\ell im}$ and that of τ, knowing that: $\ell = 20\,cm$, $m = 10\,g$, $R = 0.1\Omega$, $B = 0.5\,T$ and $g = 10\,m/s^2$.
5. In the steady state, starting from the instant when $v = V_{\ell im}$, the mechanical energy of the system (MN in the field $\overrightarrow{B}$, Earth) decreases.
   a) Explain this decrease.
   b) In what form is this energy dissipated?
   c) Calculate the power dissipated.

III-GS 2006 1st
Free Mechanical Oscillations

A horizontal elastic pendulum is formed of a homogeneous metallic rod MN of mass $m = 0.5\,kg$ and of length ℓ, and a spring of un-jointed turns of negligible mass having a stiffness $k = 50\,N/m$. The length of the spring, when free, is L_0. One of the ends of this spring is connected at I to a fixed support while the other end is connected to the midpoint G of the rod. This rod may slide without friction along the metallic rails AA' and CC', that are horizontal and parallel to the axis $x'x$ of the spring; during sliding, the rod remains perpendicular to the rails, and G moves on the axis $x'x$.

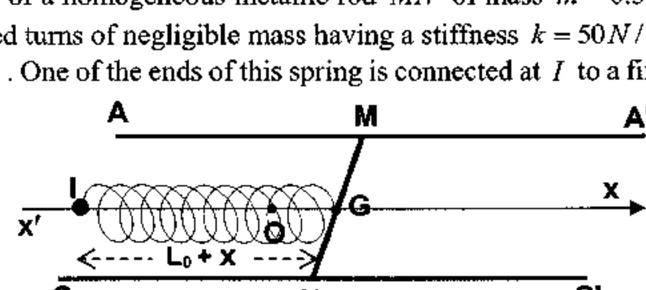


Figure 1

We move the rod, from its equilibrium position, kept parallel to itself, by $5cm$ in the positive direction and then we release it without initial velocity at the instant $t_0 = 0$.

At the instant t, the abscissa of G is $x = \overline{OG}$ and $v = \frac{dx}{dt}$ is the algebraic measure of its velocity; the origin of abscissa O, corresponds to the position of G at equilibrium when the length of the spring is L_0 (figure 1).

Part A
Free un-damped oscillations

1. Write, at the instant t, the expression of the mechanical energy ME of the system (pendulum, Earth) in terms of m, x, k and v taking the horizontal plane through G as a gravitational potential energy reference.
2. Derive the second order differential equation in x that describes the motion of G.
3. The solution of this differential equation is given by the expression: $x = x_m \cos(wt + \varphi)$ where x_m is the amplitude of oscillations.

   Determine the values of w, x_m and φ.

Part B
Free damped oscillations

The system formed of the pendulum and the rails is placed within a uniform magnetic field $\vec{B}$, perpendicular to the plane of the rails (figure 2).

We connect between A and C a resistor of convenient resistance; the resistance of the whole circuit is then R.

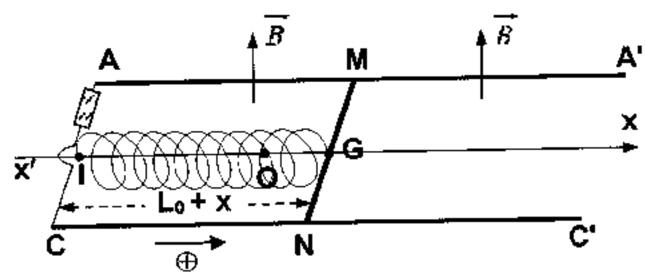


Figure 2

After shifting the rod by $5cm$ in the positive direction, we release it from rest at the instant $t_0 = 0$.

An induced current i passes in the circuit.

The horizontal pendulum performs few oscillations then comes to rest within an interval of time t_1.

1. During motion, an induced electromotive force «e» appears across the ends M and N of the rod. Explain why?
2. a) Determine, at the instant t, the expression of the magnetic flux through the surface limited by the circuit $AMNC$ in terms of B, L_0, x and ℓ, taking into account the arbitrary positive direction chosen in figure 2.

   b) Deduce the expression of the induced $e.m.f$ «e» in terms of B, ℓ and v.

   c) Determine the expression of i in terms of B, R, ℓ and v.

   d) Specify the direction of the current induced when the rod is moving in the positive direction.
3. a) Interpret the damping of the oscillations and the stopping of the rod.

   b) Calculate the mechanical energy of the oscillator at the instant $t_0 = 0$.

   c) Deduce the value of the energy dissipated in the circuit between the instants $t_0 = 0$ and t_1.

   d) In what form is this energy dissipated?

Solutions

I-GS 2012 2nd

Part A

1. The magnetic flux ϕ crossing the loop is given by: $\phi = \vec{B}.\vec{S} = B.S\cos(\vec{n},\vec{B}) = B.S\cos(0) = BS$.
   The circuit has a rectangular shape so its area $S = \ell \times (d+x)$; then $\phi = B.\ell(d+x)$.
2. a) According to Faraday's law: $e = -\frac{d\phi}{dt} = -\frac{d}{dt}(B\,\ell\,(d+x)) = -B\,\ell\frac{d}{dt}(d+x) = -B\,\ell\frac{dx}{dt} = -B\,\ell\,v$.
   b) Induced current:
      i- According to Pouillet's law: $i = \frac{e}{R} = -\frac{B\,\ell\,v}{R}$.
      ii- The sign of the current is negative so it flows in the direction opposite to the positive chosen or clockwise direction.
3. The magnitude of the electromagnetic force is given by:
   $$F = i\,B\,\ell\sin(\vec{i},\vec{B}) = \frac{B\,\ell\,v}{R}\times B\,\ell\times\sin(90^0) = \frac{B^2\,\ell^2 v}{R}.$$
   By Lenz's law this force is opposite to the direction of displacement so $\vec{F} = -\frac{B^2\,\ell^2}{R}\vec{v}$.

Part B

1. The mechanical energy is: $ME = KE + PE_e = \frac{1}{2}m\,v^2 + \frac{1}{2}k\,x^2$.
2. Since friction is neglected, the mechanical energy is conserved so $\frac{d(ME)}{dt} = 0$;
   $mx'x'' + kxx' = 0$, $x'(m\,x'' + k\,x) = 0$ ($x' \neq 0$, since the system is in motion);
   Then $x'' + \frac{k}{m}x = 0$.
3. $x = A\cos(wt+\varphi)$, then $x'' = -A\,w^2\cos(wt+\varphi)$.
   Replace in the differential equation we get: $A\cos(wt+\varphi)\left[-w^2 + \frac{k}{m}\right] = 0$.
   But $x = A\cos(wt+\varphi) \neq 0$ (in motion) then $w^2 = \frac{k}{m}$; $w = \sqrt{\frac{k}{m}} = 1.41\,\text{rad}/s$.
   Initial conditions: $v|_{t_0=0} = 0$ and $v = \frac{dx}{dt} = -A\,w\sin(w_0 t + \varphi)$, $v|_{t=0} = -Aw\sin\varphi = 0$; $\varphi = 0$ or π.
   But at $t_0 = 0$, $x = x_m = A\cos\varphi > 0$ and $A > 0$, then $\varphi = 0$ and $A = 10\,cm$.

Part C

1. The mechanical energy at $t_0 = 0$ is:
   $$ME|_{t_0=0} = KE|_{t_0=0} + PE_e|_{t_0=0} = 0 + \frac{1}{2}k\,x_m^2 = \frac{1}{2}\times 50\times 0.1^2 = 0.25\,J.$$

2. a) The power dissipated is $P=\vec{F}.\vec{v}=-\frac{B^2\ell^2}{R}\vec{v}.\vec{v}=-\frac{B^2\ell^2v^2}{R}$.

b) The power dissipated due to Joule's effect is: $P=Ri^2=R\left(\frac{B\ell v}{R}\right)^2=\frac{B^2\ell^2v^2}{R}$.

c) The power dissipated by the electromagnetic force is equal in magnitude to the power consumed by the resistor.
Therefore the mechanical energy is totally transformed into thermal energy in the resistor.

3. Since it stops after few oscillations then the total mechanical energy is converted into thermal energy then: $E_{diss}=\Delta(ME)=ME|_{t_0=0}-0=0.25\,J$.

II-GS 2011 2nd

1. Referring to Right Hand Rule the normal is outwards, then $\theta=(\vec{n};\vec{B})=\pi$;
The magnetic flux is given by $\phi=BS\cos(\vec{n};\vec{B})=BS\cos\pi=-BS=-B\ell x$.

2. a) Induction

i- According to Faraday's law $e=-\frac{d\phi}{dt}=B\ell\frac{dx}{dt}=B\ell v$.

ii- The induced current $i=\frac{e}{R}=\frac{B\ell v}{R}$.

b) The current $i>0$, then the induced current circulates in the positive direction chosen.
Then it circulates from M towards N through the rod, in the counter clockwise direction.

3. The power dissipated due to Joule's effect is given: $P_{e\ell}=Ri^2=R\left(\frac{B\ell v}{R}\right)^2=\frac{B^2\ell^2}{R}v^2$.

4. a) According to Newton's 2nd law: $\sum\overrightarrow{F_{ext}}=\frac{d\vec{P}}{dt}$, so $\vec{w}+\vec{F}=m\frac{d\vec{v}}{dt}$;

Then $-F\vec{i}+mg\vec{i}=m\frac{dv}{dt}\vec{i}$; (along a vertically downwards axis);

$-iB\ell+mg=m\frac{dv}{dt}$; $-\frac{B\ell}{R}vB\ell+mg=m\frac{dv}{dt}$; thus, $\frac{dv}{dt}+\frac{B^2\ell^2}{mR}v=g$.

b) We have $v=A\left(1-e^{-\frac{t}{\tau}}\right)$; then $\frac{dv}{dt}=\frac{A}{\tau}e^{-\frac{t}{\tau}}$;

By replacing in the differential equation: $\frac{A}{\tau}e^{-\frac{t}{\tau}}+A\left(\frac{B^2\ell^2}{mR}\right)\left(1-e^{-\frac{t}{\tau}}\right)=g$;

$\frac{B^2\ell^2}{mR}A+Ae^{-\frac{t}{\tau}}\left(\frac{1}{\tau}-\frac{B^2\ell^2}{mR}\right)=g$; but $e^{-\frac{t}{\tau}}\neq 0$; his equation is verified at any instant;

$\frac{B^2\ell^2}{mR}A=g$, so $A=\frac{mgR}{B^2\ell^2}$ and $\frac{1}{\tau}-\frac{B^2\ell^2}{mR}=0$, so $\tau=\frac{mR}{B^2\ell^2}$.

c) As the time increases $t\longrightarrow+\infty$, $e^{-\frac{t}{\tau}}\longrightarrow 0$ then $v\longrightarrow A$; thus, $v=V_{\ell im}=A=\frac{mgR}{B^2\ell^2}$.

d) Velocity

i- We have $v = V_\ell\left(1 - e^{-\frac{t}{\tau}}\right)$; then $v|_{t=\tau} = V_\ell\left(1 - e^{-1}\right) = 0.63\,V_\ell = 63\%V_\ell$.

ii- The limit velocity is reached after $t = 5\tau$.

e) We have $V_{\ell im} = \frac{m\,gR}{B^2\ell^2} = \frac{10\times10^{-3}\times10\times0.1}{0.5^2\times0.2^2} = 1\ m/s$ and $\tau = \frac{m\,R}{B^2\ell^2} = \frac{10\times10^{-3}\times0.1}{0.5^2\times0.2^2} = 0.1\,s$.

5. a) When the steady state is reached, the speed becomes constant thus the kinetic energy is also constant. But the rod is still falling so its gravitational potential energy decreases.
   Thus the mechanical energy decreases.

   b) The energy dissipated appears in the form of heat due to Joules effect.

   c) According to 3, the power dissipated is $P_{e\ell} = \frac{B^2\ell^2}{R}V_\ell^2 = \frac{0.5^2\times0.2^2}{0.1}\times1^2 = 0.1\,W$.

III-GS 2006 1st

Part A

1. The mechanical energy $ME = KE + PE_e = \frac{1}{2}m\,v^2 + \frac{1}{2}k\,x^2$.

2. In the absence of friction, the mechanical energy is conserved $\frac{d(ME)}{dt} = 0$, we get $x'' + \frac{k}{m}x = 0$.

3. We have: $x = x_m\cos(wt + \varphi)$; so $x'' = -w^2x_m\cos(wt + \varphi)$.

   Replace in the differential equation we get $-w^2x_m\cos(wt+\varphi) + \frac{k}{m}x_m\cos(wt+\varphi) = 0$;

   $x_m\cos(wt+\varphi)\left(-w^2 + \frac{k}{m}\right) = 0$; but $x_m\cos(wt+\varphi) \neq 0$ (the system is in motion);

   So $-w^2 + \frac{k}{m} = 0$; then $w = \sqrt{\frac{k}{m}} = 10\ \text{rad}/s$.

   At $t = 0$, $x = x_0 = 5\,cm$ and $v_0 = 0$ (without initial velocity), then $x_m = 5\,cm$;

   Referring to the initial conditions, we get: $x_m = x_m\cos\varphi$; $\cos\varphi = 1$; then $\varphi = 0$.

   Thus, $x = 5\cos(10t)$ (t in s & x in cm).

Part B

1. The rod MN is in motion, the surface swept varies with time; then magnetic flux ϕ is also variable.
   This variation creates an induced *e.m.f* between the terminals of the rod MN.

2. a) The magnetic flux crossing the circuit $\phi = B\,S\cos\theta$ with $\theta = (\vec{n};\vec{B}) = 0$;
   & $S = AM\times MN = (L_0 + x)\,\ell$; then $\phi = B\,\ell(L_0 + x)$.

   b) Faraday's law: $e = -\frac{d\phi}{dt} = -\frac{d}{dt}\left[B\,\ell(L_0 + x)\right] = -B\,\ell\,v$.

   c) The induced current: $i = \frac{e}{R} \Rightarrow i = -\frac{B\,\ell\,v}{R}$.

d) The rod is moving in the positive direction, the velocity v is positive & $i=-\frac{B\ell v}{R}$, then i & v have opposite signs. Thus, the induced current circulates in a direction opposite to that taken as positive.

3. a) According to Lenz's law, the rod is submitted to Laplace's force which acts in the opposite direction to that of motion of the rod. This force acts as a force of friction causing the damping in the oscillations and eventually it stops the rod.

   b) $ME|_{t_0} = KE|_{t_0} + PE_e|_{t_0} = \frac{1}{2}k\,x_0^2 = 0.0625\ J$. ($KE|_{t_0} = 0$ released from rest);

   c) At the instant t_1, the rod stops at the origin then $v_{t_1} = 0$ & $x_{t_1} = 0$; thus, $ME_{t_1} = 0$.

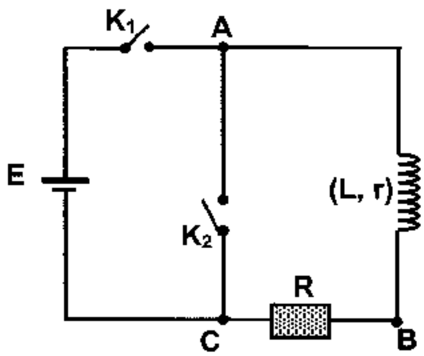
   The energy dissipated $\Delta(ME) = \left| ME|_{t_1} - ME|_{t_0} \right| = 0.0625\ J$.

   d) The energy dissipated is converted into thermal energy due to Joule's effect that appears in the form of heat.

I-GS 2009 1st

The Phenomenon of Self-Induction

The set up represented by the adjacent figure consists of an ideal generator of $e.m.f$ $E = 12\,V$, a coil of resistance $r = 10\,\Omega$ and of inductance $L = 40\,mH$, a resistor of resistance $R = 40\,\Omega$ and two switches K_1 and K_2.



Part A

Growth of current

At the instant $t_0 = 0$, we close the switch K_1 and we leave K_2 open. At an instant t, the circuit carries a current i_1 in the transient state.

1. Derive the differential equation that governs the variation of i_1 as a function of time.
2. I_0 is the current in the steady state.
   Determine the expression of I_0 in terms of E, r and R and calculate its value.
3. The solution of the differential equation is of the form: $i_1 = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$
   a) Determine the expression of τ in terms of L, r and R and calculate its value.
   b) Give the physical significance of τ.
4. a) Determine the expression of the self-induced $e.m.f$ e_1 as a function of time t.
   b) Calculate the algebraic value of e_1 at the instant $t_0 = 0$.

Part B

Decay of current

After a few seconds, the steady state being reached, we open K_1 and we close K_2 at the same instant. We consider the instant of closing K_2 as a new origin of time $t_0 = 0$. The circuit (L, R, r) thus carries an induced current i_2 at an instant t.

1. Determine the direction of i_2.
2. Derive the differential equation that governs the variation of i_2 as a function of time.
3. Verify that $i_2 = I_0\, e^{-\frac{t}{\tau}}$ is the solution of this differential equation.
4. Calculate the algebraic value of the self-induced $e.m.f$ e_2 at the instant $t_0 = 0$.

Part C

Comparison

Compare e_1 and e_2 and deduce the role of the coil in each of the two previous circuits.

II-GS 2008 1st (Special)
Electrical and Mechanical Analogy
Part A
R-L circuit
An electric circuit is formed of a coil of inductance L and of negligible resistance, a switch K and a resistor of resistance R all connected in series across an ideal generator of $e.m.f\ E$ as shown in the adjacent figure. At the instant $t_0 = 0$, we close K. At the instant t, the circuit carries a current i in the transient state.

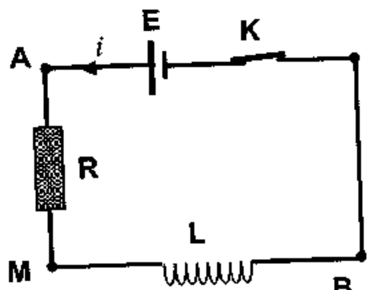


Figure 1

1. Show that the differential equation that governs the variation of the current i as a function of time is given by the expression $E = Ri + L\frac{di}{dt}$ ....(1).
2. Verify that the expression $i = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$ is a solution of this differential equation.
3. Find the expression of I_{max}, the current carried by the circuit in the steady state (after a long time).
4. Determine, in terms R and L, the expression of the duration t_1 at the end of which the current i becomes equal to $0.63\, I_{max}$.
5. Draw the shape of the curve that represents the variations of i as a function of time, and indicate on it t_1 and I_{max}.

Part B
Vertical fall in a liquid
A metallic ball, of mass m, released from rest, falls vertically in a liquid. We suppose that the only force resisting the motion of the ball in the liquid is given by the expression $\vec{f} = -h\vec{v}$, $\vec{v}$ being the instantaneous velocity of the ball and h is a positive constant.

1. Give a list of the forces acting on the ball during its fall.
2. Applying Newton's 2nd law, show that the differential equation that governs the variations of the algebraic value v of the velocity $\vec{v}$ is given by : $mg = hv + m\frac{dv}{dt}$ ....(2).

Part C
An analogy
1. By comparing the differential equations (1) and (2), give the convenient mechanical quantity corresponding to each of the electric quantities E, R, L, i & $\frac{di}{dt}$.
2. Deduce, using the analogy between the physical quantities:
   a) the solution of the differential equation (2).
   b) the expression $v_{\ell im}$ of the velocity after a very long time.
   c) determine, in terms of h and m, the expression of the duration t_1' at the end of which v becomes equal to $0.63\, v_{\ell im}$.
3. Draw the shape of the curve that represents the variations of v as a function of time, and indicate on it t_1' and $v_{\ell im}$.

III-GS 2007 2nd

Role of a Coil in a Circuit

Consider the circuit represented in figure 1 where:

- (G) is a DC generator of $e.m.f$ $E = 9V$ and of negligible internal resistance;
- (D_1) is a resistor of resistance $R_1 = 90\,\Omega$;
- (D_2) is a resistor of resistance R_2;
- (B) is a coil of inductance $L = 1H$ and of negligible resistance;
- (K) is a double switch.

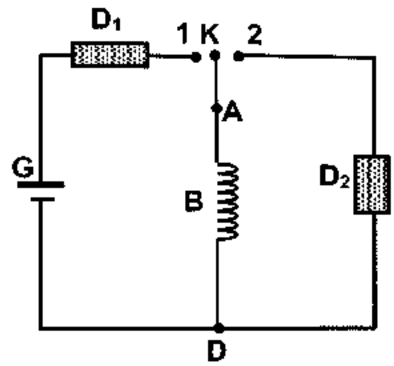


Figure 1

First situation

Growth of the current in the component (R₁, L)

We place the switch in position 1 at an instant taken as an origin of time $(t_0 = 0)$.

At an instant t, the circuit carries a current i_1.

1. Derive the differential equation in i_1.
2. Verify that $i_1 = \dfrac{E}{R_1}\left(1 - e^{-\frac{R_1}{L}t}\right)$ is a solution of the preceding differential equation.
3. a) Determine, in the steady state, the expression of the current I_0 in terms of E and R_1.
   b) Calculate I_0.

Second situation

Decay of the current in the component (R₂, L) and illumination of a lamp

Part A

Decay of the current in the component (R₂, L)

At an instant chosen as a new origin of time $(t_0 = 0)$, we turn the switch K to position 2.

At an instant t, the circuit carries thus a current i_2.

1. Determine the direction of this current.
2. Derive the differential equation in i_2.
3. The solution of this differential equation is of the form $i_2 = \alpha\, e^{-\beta t}$. Show that $\alpha = I_0$ and $\beta = \dfrac{R_2}{L}$.

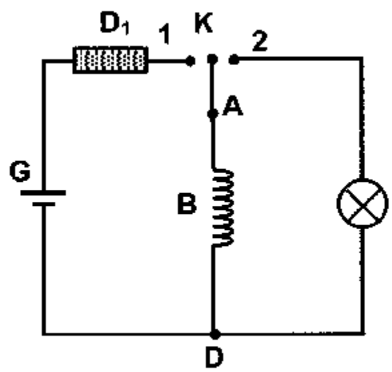


Figure 2

Part B

Duration of illumination of a lamp

The resistor D_2 is a lamp of resistance $R_2 = 400\,\Omega$ (figure 2).

This lamp gives light as long as the current it carries is not less than $20\,mA$.

1. Show that the lamp gives light at the instant when the circuit is closed.
2. Determine the duration of the illumination of the lamp.

IV-GS 2006 2nd
Ignition System in a Car

The study of the ignition system in certain cars is reduced to the study of a circuit formed of a coil (B) of inductance L and resistance r, a resistor R, an ammeter (A) and a switch K, all connected in series across a generator (G) that provides across its terminals M and N a voltage $u_{MN} = E = 12V$ (Figure 1).

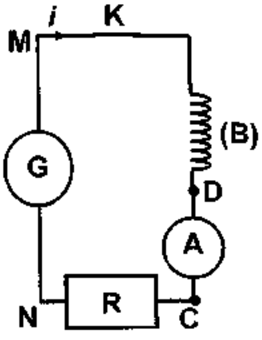


Figure 1

We close the switch K at the instant $t_0 = 0$. At the instant, the circuit carries a current i. We display, using an oscilloscope, the voltage u_{MN} on the channel Y_1 and the voltage u_{CN} on the channel Y_2.

The waveforms are represented on figure 2.

The vertical sensitivity on both channels is: $2V/\text{div}$

The horizontal sensitivity (time base) is: $1ms/\text{div}$.

In steady state the ammeter reads $I_0 = 0.2\,A$.

1. Redraw figure 1 showing on it the connections of the oscilloscope.
2. a) Derive the differential equation that governs the variation of the current i as a function of time.
   b) Resistances:
      i- Show that in steady state: $E = (r+R)I_0$ and $u_{MD} = r\,I_0$.
      ii- Determine R and r using the waveforms of the preceding results.
3. a) Inductance of the coil.
      i- Show, using the preceding differential equation, that the voltage u_{CN} satisfies the relation: $\dfrac{RE}{L} = \dfrac{du_{CN}}{dt} + \dfrac{R+r}{L}u_{CN}$.

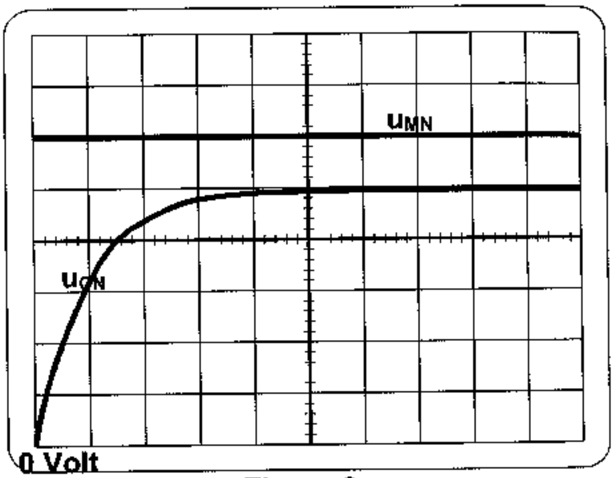


Figure 2

      ii- Deduce the expression of $\dfrac{du_{CN}}{dt}$, in terms of R, E & L at the instant $t_0 = 0$.
      iii- The time constant τ is the abscissa of the point of intersection of the tangent at the origin to the curve u_{CN} and the asymptote to that curve.
      Show that the expression of τ is $\tau = \dfrac{L}{R+r}$
   b) Show, using one of the waveforms, that the value of τ is $1ms$.
   c) Deduce the value of L.
4. Determine the maximum energy stored in the coil (B).
5. The above circuit (ignition system) helps, through an intermediately switch, to feed the spark plugs of the car at well determined instants, with the energy needed to make the engine function normally.

The expression of the current i, in the circuit, is given by: $i = I_0\left(1-e^{-\frac{t}{\tau}}\right)$.

We define the «rate of storage» of a coil as the ratio of the energy stored in the coil at a given instant to the maximum energy it can store.
Determine the minimum duration of closure of the switch so that the rate of storage of the coil is not less than 90.3%.

V-GS 2004 2nd
Coil in an Electric Circuit

Consider a generator of D.C. voltage (E, r), a coil (L, R_1), a resistor of resistance $R_2 = 100\,\Omega$, two lamps (C_1) and (C_2), an oscilloscope and a switch k.

Part A
Qualitative study

In order to study the role of a coil in an electric circuit, we connect up the circuit that is represented in figure 1.
We close k. One of the two lamps gives bright light first. Explain the phenomenon responsible for the delay in the brightness between the two lamps.

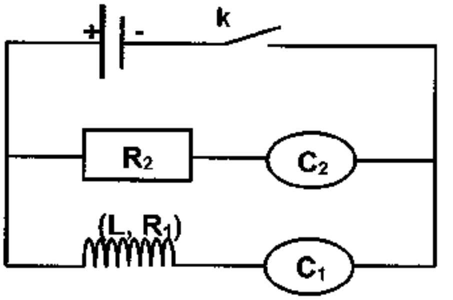


Figure 1

Part B
Quantitative study

In order to determine the characteristics (L, R_1) of the coil and (E, r) of the generator, we connect up the circuit represented in figure 2.

Take $R = R_1 + R_2 + r$ the total resistance of the circuit.

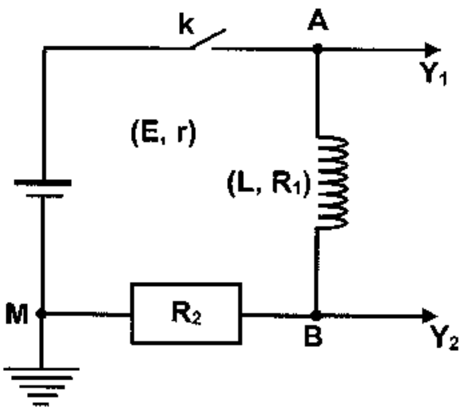


Figure 2

First situation
Analytical study of the growth of the current

We close the switch k at the instant $t_0 = 0$. At any instant t, the circuit carries an electric current i.

1. Applying the law of addition of voltages, derive the first order differential equation of the variation of the current as a function of time.
2. The solution of this differential equation is of the form $i = a + be^{-\frac{t}{\tau}}$ where a, b & τ are constants.
   Determine the expressions of a, b & τ in terms of E, R & L
3. Deduce that $i = \frac{E}{R}\left(1-e^{-\frac{t}{\tau}}\right)$.

Second situation

Determination of the values of E, r, R_1 and L

An oscilloscope, connected as shown in figure 2, allows us to display the variation of the two voltages represented in the curves (a) and (b) of figure 3.

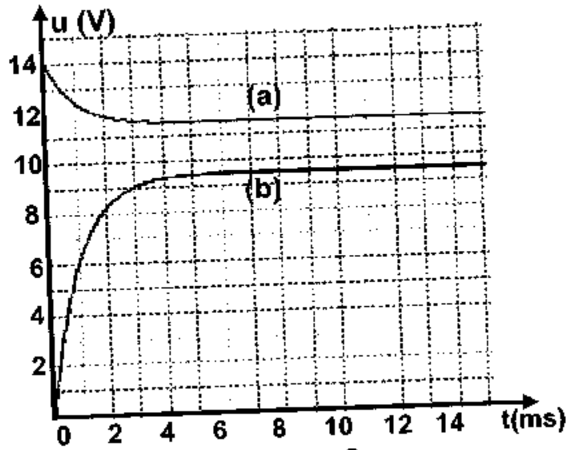


Figure 3

1. a) Specify the voltage u_1 displayed on channel 1.
   b) Determine the expression of u_1 as a function of t.
2. a) Specify the voltage u_2 displayed on channel 2.
   b) Give the expression of u_2 as a function of t.
3. a) Give the values of u_1 and u_2 at the instant $t_0 = 0$.
   b) Deduce the value of E.
4. Using the curves (a) and (b), determine:
   a) the value of τ.
   b) the values of r and R_1.
5. Calculate L.

Solutions

I-GS 2009 1st

Part A

1. Law of addition of voltages $u_{AC} = u_{AB} + u_{BC}$; $E = ri_1 + L\frac{di_1}{dt} + Ri_1 \Rightarrow E = (r+R)i_1 + L\frac{di_1}{dt}$.

2. When the steady state is reached, the intensity of the current becomes constant $i_1 = I_0$ so $\frac{di_1}{dt} = 0$

   Then, $E = (r+R)I_0$, we get $I_0 = \frac{E}{r+R}$;

   Thus, $I_0 = \frac{12V}{(10+40)\Omega} = 0.24\,A$.

3. a) We have $i_1 = I_0\left(1 - e^{-\frac{t}{\tau}}\right) \Rightarrow \frac{di_1}{dt} = \frac{I_0}{\tau}e^{-\frac{t}{\tau}}$.

   By replacing in the differential equation, we get $E = (r+R)I_0\left(1 - e^{-\frac{t}{\tau}}\right) + \frac{LI_0}{\tau}e^{-\frac{t}{\tau}}$;

   $E = (r+R)I_0 + \left(\frac{L}{\tau} - (R+r)\right)I_0 e^{-\frac{t}{\tau}}$ is verified at any instant & $I_0 e^{-\frac{t}{\tau}} \neq 0$;

   Then $\frac{L}{\tau} - (R+r) = 0$ so $\tau = \frac{L}{R+r}$; thus $\tau = \frac{0.04}{50} = 0.8\,ms$.

   b) τ is called the time constant of the $(R+r;L)$ circuit, which represents the duration after which the current reaches $63\,\%$ of its value in the steady state.

4. a) The induced $e.m.f$ $e_1 = -L\frac{di_1}{dt} = -\frac{LI_0}{\tau}e^{-\frac{t}{\tau}} = -\frac{L\left(\frac{E}{R+r}\right)}{\left(\frac{L}{R+r}\right)}e^{-\frac{t}{\tau}} = -E\,e^{-\frac{t}{\tau}}$.

   b) At $t = 0$, $e_1 = -E_1e^0 = -E_1 = -12V$.

Part B

1. According to Lenz's law, the induced current will oppose to the variation of the current. When we open K_1 and we close K_2, the coil acts as a generator delivering a current i_2 in the same direction as that of the current i_1.

2. Law of addition of voltages: $u_{AC} = u_{AB} + u_{BC} \Rightarrow 0 = ri_2 + L\frac{di_2}{dt} + Ri_2 \Rightarrow (r+R)i_2 + L\frac{di_2}{dt} = 0$.

3. We have $i_2 = I_0\,e^{-\frac{t}{\tau}}$ so $\frac{di_2}{dt} = -\frac{I_0}{\tau}e^{-\frac{t}{\tau}}$.

   Replacing in the differential equation we get $(r+R)i_2 + L\frac{di_2}{dt} = (r+R)I_0e^{-\frac{t}{\tau}} - L\frac{I_0}{\tau}e^{-\frac{t}{\tau}} = 0$.

4. At $t = 0$, $e_2 = E_1e^0 = E_1 = 12V$.

Part C

$e_1 = -e_2$

- ✗ When the switch K_1 is closed, the self-induced $e.m.f$ opposes to the growth of the current.

  $e_1 < 0$; the coil acts as a generator in opposition.

- ✗ When the switch K_1 is opened and K_2 is closed, the self-induced $e.m.f$ opposes to the decay of the current.

  $e_2 > 0$; the coil acts as a generator.

II-GS 2008 1st

Part A

1. Law of addition of voltages: $u_G = u_R + u_L$; so $E = Ri + L\dfrac{di}{dt}$.

2. We have $i = \dfrac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$, so $\dfrac{di}{dt} = \dfrac{E}{L}e^{-\frac{R}{L}t}$

   Replace in the previous differential equation we get: $Ri + L\dfrac{di}{dt} = E\left(1 - e^{-\frac{R}{L}t}\right) + E\,e^{-\frac{R}{L}t} = E$.

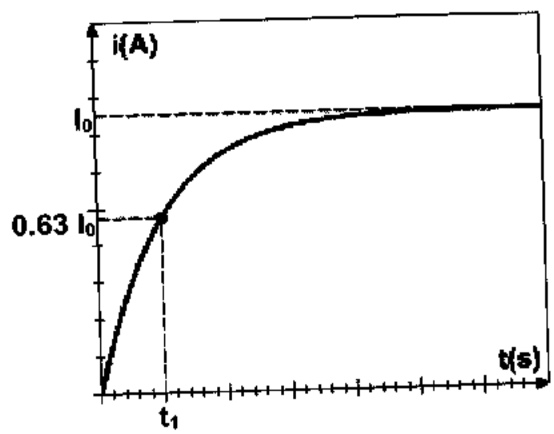
3. As $t \longrightarrow +\infty$, $e^{-\frac{R}{L}t} \longrightarrow 0 \Rightarrow i = \dfrac{E}{R} = I_{max}$.

4. $i = \dfrac{E}{R}\left(1 - e^{-\frac{R}{L}t_1}\right) = 0.63\, I_{max} = 0.63\,\dfrac{E}{R}$.

   We get $1 - e^{-\frac{R}{L}t_1} = 0.63 \Rightarrow e^{-\frac{R}{L}t_1} = 0.37$;

   $-\dfrac{R}{L}t_1 = \ell n(0.37) = -0.99 \approx -1$; $t_1 = \dfrac{L}{R}$.

5. Graph.



Part B

1. The forces acting on the particle are its weight $\vec{w}$ & the force of friction $\vec{f}$.

2. Newton's 2nd law: $\sum \overrightarrow{F_{ex}} = \dfrac{d\vec{P}}{dt}$; $\vec{w} + \vec{f} = \dfrac{d\vec{P}}{dt}$.

   Projection along a vertically downwards axis, we get: $mg - hv = m\dfrac{dv}{dt}$; then $mg = hv + m\dfrac{dv}{dt}$.

Part C

1. E corresponds to mg ; R corresponds to h ; i corresponds to v.

   L corresponds to m ; $\dfrac{di}{dt}$ corresponds to $\dfrac{dv}{dt}$.

2. a) By analogy $v = \dfrac{mg}{h}\left(1 - e^{-\frac{h}{m}t}\right)$.

   b) By analogy with I_{max} we get $v_{lim} = \dfrac{mg}{h}$.

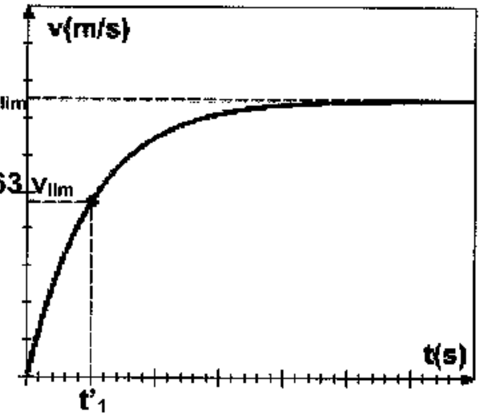
c) $v = 0.63\, v_{\ell im} = 0.63\dfrac{mg}{h} = \dfrac{mg}{h}\left(1 - e^{-\frac{h}{m}t_1'}\right)$;

Then $0.63 = 1 - e^{-\frac{h}{m}t_1'} \Rightarrow e^{-\frac{h}{m}t_1'} = 0.37$;

$-\dfrac{h}{m}t_1' = \ell n(0.37) = -0.99 \approx -1$;

Thus, $t_1' = \dfrac{m}{h}$.

3. Graph.



III-GS 2007 2nd
First situation

1. Law of addition of voltages $u_G = u_R + u_L \Rightarrow E = R_1\, i_1 + L\dfrac{di_1}{dt}$.

2. We have: $i_1 = \dfrac{E}{R_1}\left(1 - e^{-\frac{R_1}{L}t}\right)$, $\dfrac{di_1}{dt} = \dfrac{E}{R_1}\left(\dfrac{R_1}{L}e^{-\frac{R_1}{L}t}\right) = \dfrac{E}{L}e^{-\frac{R_1}{L}t}$.

   Replacing in the differential equation we get: $R_1\, i_1 + L\dfrac{di_1}{dt} = R_1\dfrac{E}{R_1}\left(1 - e^{-\frac{R_1}{L}t}\right) + L\dfrac{E}{L}e^{-\frac{R_1}{L}t} = E$.

3. a) In steady state the current $i_1 = I_0$ becomes constant then $\dfrac{di_1}{dt} = 0$;

   The differential equation in this case gives $E = R_1\, I_0 + 0$ then $I_0 = \dfrac{E}{R_1}$.

   b) We have $I_0 = \dfrac{E}{R_1} = \dfrac{9}{90} = 0.1A$.

Second situation
Part A

1. During the current decay, according to Lenz's law, the coil acts as a generator delivering a current in the same direction as before, from A to D through the coil.

2. Law of uniqueness of voltages $u_{coil} = u_{D_2}$, $L\dfrac{di_2}{dt} = -R_2 i_2$; then $L\dfrac{di_2}{dt} + R_2 i_2 = 0$.

3. We have: $i_2 = \alpha\, e^{-\beta t}$ then $\dfrac{di_2}{dt} = -\alpha\,\beta\, e^{-\beta t}$;

   Replacing in the differential equation we get: $\alpha\, e^{-\beta t}(R_2 - L\beta) = 0$;

   But $\alpha\, e^{-\beta t} \neq 0$, so $R_2 - L\beta = 0$; then $\beta = \dfrac{R_2}{L}$.

   At $t = 0$, $i = I_0 \Rightarrow \alpha = I_0 = \dfrac{E}{R_1}$.

Part B

1. Just after closing the circuit, a current I_0 passes through the lamp $I_0 = 0.1A > 0.02\cdot A$. Therefore the lamp illuminates.

2. The lamp illuminates as long $i_2 > 0.02$, so $e^{-400t} > 0.2$;

$-400\,t > \ell n(0.2) \Rightarrow t < \dfrac{\ell n(0.2)}{-400} \approx 4\times10^{-3}\,s = 4\,ms$.

The illumination of the lamp lasts for $4\,ms$.

IV-GS 2006 2nd

1. Circuit.

2. a) Law of addition of voltages: $u_{MN} = u_{MD} + u_{DC} + u_{CN}$; ($u_{DC} = 0$ wire).

$E = r\,i + L\dfrac{di}{dt} + Ri$, then $E = (r+R)i + L\dfrac{di}{dt}$.

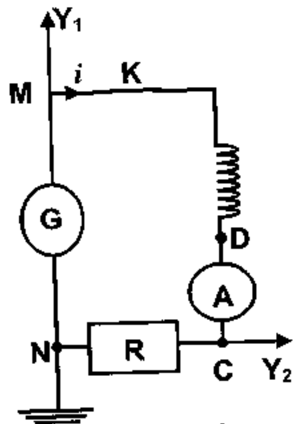


Figure 1

b) Resistance of the resistor :

i- In the steady state, the current becomes constant $i = I_0 \Rightarrow \dfrac{di}{dt} = 0$, referring to the differential equation we get: $E = (r+R)I_0 + 0 \Rightarrow E = (R+r)I_0$.

According to Ohm's law: $u_{MD} = rI_0 + L\dfrac{dI_0}{dt} = r\,I_0$.

ii- In steady state: $U_{CN} = S_v \times y_{\text{steady}} = 2V/\text{div} \times 5\text{div} = 10\,V$;

But at any instant $U_{MD} = E - U_{CN}$,

$U_{MD} = 12 - 10 = 2V = r\,I_0$, then $r = \dfrac{U_{MD}}{I_0} = \dfrac{2V}{0.2A} = 10\,\Omega$.

But $E = (R+r)I_0$, thus $R = \dfrac{E}{I_0} - r = \dfrac{12V}{0.2A} - 10\Omega = (60-10)\Omega = 50\Omega$.

3. a) Inductance of the coil.

i- We have $u_{CN} = R\,i$, so $i = \dfrac{u_{CN}}{R}$, replacing in the differential equation we get:

$E = (r+R)\dfrac{u_{CN}}{R} + \dfrac{L}{R}\dfrac{du_{CN}}{dt}$; then $\dfrac{du_{CN}}{dt} + \left(\dfrac{R+r}{L}\right)u_{CN} = \dfrac{RE}{L}$.

ii- At $t_0 = 0$, $u_{CN} = 0$, and the previous differential equation is satisfied at any instant;

Then, we get : $\left.\dfrac{RE}{L}\right|_{t=0} = \left.\dfrac{du_{CN}}{dt}\right|_{t=0} + \dfrac{R+r}{L}\,u_{CN}\big|_{t=0}$, thus $\left.\dfrac{du_{CN}}{dt}\right|_{t=0} = \dfrac{RE}{L}$.

iii- In the steady, the horizontal asymptote is $U_{CN} = RI_0$ and $I_0 = \dfrac{E}{R+r}$; $U_{CN} = \left(\dfrac{R}{R+r}\right)E$;

The equation of the tangent (T) at origin is: $u = a\,t$ where $a = \left.\dfrac{du_{CN}}{dt}\right|_{t=0} = \dfrac{RE}{L}$,

Then $u = \dfrac{RE}{L}t \ldots (T)$

The time constant τ is the abscissa of the point of intersection between the tangent and the horizontal asymptote so : $u = U_{CN}$ so $\dfrac{RE}{L}\tau = \dfrac{R}{R+r}E$; then $\tau = \dfrac{L}{R+r}$.

c) Basing on the definition stated:

✗ we construct the tangent at origin;

✗ we trace the horizontal asymptote to the curve of u_{CN}.

The time constant is the abscissa of their point of intersection so graphically $\tau \cong 1\text{div}$.

Then $\tau = S_h \times x = 1ms/\text{div} \times 1\text{div} = 1\,ms$.

d) We have $\tau = \dfrac{L}{R+r}$,

Then $L = \tau(R+r) = 1 \times 60 = 60\,mH$.

4. The maximum energy is stored in the steady $E_0 = \dfrac{1}{2}L I_0^2 = \dfrac{1}{2} \times 60 \times 10^{-3} \times 0.2^2 = 1.2 \times 10^{-3}\,J$.

5. At the considered instant:

$$\frac{E_m(t)}{E_0} = \frac{\frac{1}{2}Li^2}{\frac{1}{2}LI_0^2} = \frac{i^2}{I_0^2} \geq 0.903\text{ ; so } \frac{i}{I_0} \geq \sqrt{0.903}.$$

But $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$, so $\dfrac{i}{I_0} = 1 - e^{-\frac{t}{\tau}} \geq \sqrt{0.903}$;

Then $t \geq -\tau\,\ell n\left(1 - \sqrt{0.903}\right) \approx 3\,ms$.

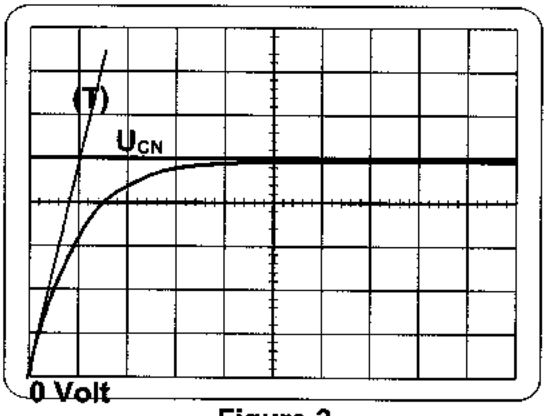


Figure 2

V-GS 2004 2nd

Qualitative study

When the switch k is closed, the current circulates through (C_1) and (C_2) simultaneously:

✗ in the branch containing R_2 and (C_2), the current will not undergo any delay. Thus, (C_2) shines immediately with maximum brightness.

✗ while the current through the coil is the seat of self-induction phenomenon. Thus, it increases gradually and (C_1) will reach its maximum brightness with a certain delay.

Quantitative study

Part A

1. Law of addition of voltages $u_{AM} = u_{AB} + u_{BM}$;

Then $E - ri = R_1 i + L\dfrac{di}{dt} + R_2 i \Rightarrow E = (R_1 + R_2 + r)i + L\dfrac{di}{dt}$;

Thus, $Ri + L\dfrac{di}{dt} = E$ where $R = R_1 + R_2 + r$.

2. We have $i = a + be^{-\frac{t}{\tau}}$ then $\dfrac{di}{dt} = -\dfrac{b}{\tau}e^{-\frac{t}{\tau}}$.

Replacing in the differential equation we get $R\left(a + be^{-\frac{t}{\tau}}\right) - L\dfrac{b}{\tau}e^{-\frac{t}{\tau}} = E$;

So, $Ra + \left(R - L\dfrac{1}{\tau}\right)be^{-\frac{t}{\tau}} = E$ is verified at any instant.

But $be^{-\frac{t}{\tau}} \neq 0$, so $Ra = E \Rightarrow a = \dfrac{E}{R}$ & $R - \dfrac{L}{\tau} = 0 \Rightarrow \tau = \dfrac{L}{R}$.

But at the instant $t=0$, $i=0$ (no current), so $0=a+b \Rightarrow b=-a=-\frac{E}{R}$.

3. We have $i=a+b\,e^{-\frac{t}{\tau}}$, so $i=\frac{E}{R}-\frac{E}{R}e^{-\frac{t}{\tau}}=\frac{E}{R}\left(1-e^{-\frac{t}{\tau}}\right)$.

Part B

1. a) On channel 1, the voltage displayed is $u_1=u_{AM}$ (across the generator).

   b) Ohm's law across the generator $u_1=u_{AM}=E-r\,i=E-\frac{r\,E}{R}\left(1-e^{-\frac{t}{\tau}}\right)$.

2. a) On channel 2, the voltage displayed is $u_2=u_{BM}$.

   b) Ohm's law across the resistor $u_2=u_{BM}=R_2\,i=E\frac{R_2}{R}\left(1-e^{-\frac{t}{\tau}}\right)$.

3. a) Using the expression previously derived: at $t=0$, $u_1=E$ & $u_2=0$.

   b) From graph (a): at $t=0$, $u_1=E=14V$.

4. a) At the instant τ, the voltage u_2 reaches 63% of its value in the steady state $(u_2)_{max}=9.5V$.

   The value of u_2 at the instant τ is $u_2=0.63\times 9.5\approx 6V$.

   The point of the curve (b), of ordinate $6V$, has an abscissa $\tau=1ms$.

   b) In the steady state, the current attains a maximum value I_0.

   From graph (a): $U_{R_2}=9.5V$; but $I_0=\frac{U_{R_2}}{R_2}=\frac{9.5}{100}=0.095A$.

   The voltage across the generator $u=u_{AM}=E-r\,i$;

   In steady state $U_{AM}=E-r\,I_0$;

   From the curve (a) we get $U_{AM}=11.5V$, then $r=\frac{E-U_{AM}}{I_0}=\frac{14-11.5}{0.095}=26.3\Omega$.

   1st method:

   We have $I_0=\frac{E}{R_1+R_2+r}$; so $0.095=\frac{14}{100+26.3+R_1}$;

   Then $R_1=\frac{14}{0.095}-126.3=21\Omega$.

   2nd method:

   The voltage across the generator in the steady state $U_{AM}=(R_1+R_2)I_0$;

   Then $R_1=\frac{11.5}{0.095}-100=21\Omega$.

5. We have $L=R\times\tau$ where $R=R_1+R_2+r=21+100+26=147\ \Omega$ & $\tau=10^{-3}s$;

   We get $L=147\times 10^{-3}H=147\,mH$.

GS Sessions – Capacitor

I-GS 2012 1st

Charging and Discharging of a Capacitor

We set up the circuit whose diagram is represented in figure 1, G is a generator of constant $e.m.f$ $E = 10\,V$ and of negligible internal resistance, (C) is a capacitor, initially uncharged, of capacitance $C = 1F$, (D) is a resistor of resistance $R = 10\,\Omega$, K is a switch and M is an electric motor whose axis is wrapped by a string of negligible mass and carrying a solid of mass $m = 1\,kg$ (Figure 1). **Take** $g = 10\,m/s^2$.

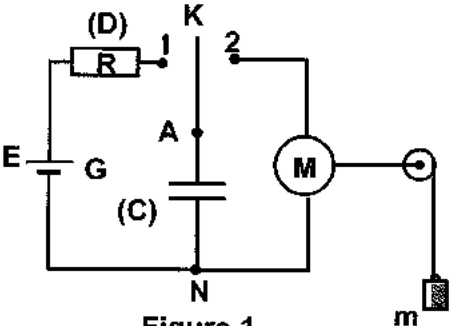


Figure 1

Part A

Charging of the capacitor

K is in position 1 at the instant $t_0 = 0$.

1. Determine the differential equation that describes the variation of the voltage $u_{AN} = u_C$ across the capacitor.
2. The solution of the differential equation is of the form $u_C = A + B e^{-\frac{t}{\tau}}$ where A, B and τ are constants. Determine the expressions of A, B and τ in terms of E, R and C.
3. At the end of charging:
   a) deduce the value of the voltage u_C;
   b) calculate, in J, the energy stored in the capacitor.

Part B

Discharging of the capacitor through the motor

The capacitor being totally charged, we turn the switch K to the position 2 at an instant taken as a new origin of time. During a time t_1, the solid is raised by height $h = 1.5\,m$. At the instant t_1, the voltage across the capacitor is $u_C = u_1$.

The variation of the voltage u_C across the capacitor during discharging through the motor between the instants 0 and t_1 is represented by the curve of figure 2.

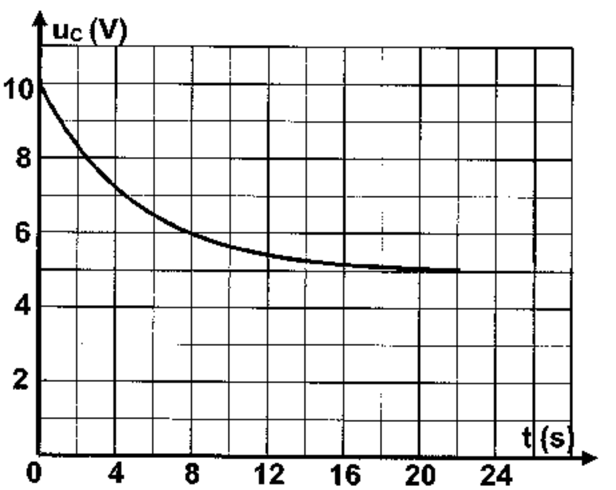


Figure 2

1. Referring to figure 2:
   a) give the value of t_1, at which the voltage u_C attains the minimum value u_1;

b) give the value of the voltage u_1.

2. At the instant t_1, the capacitor still stores energy W_1.
   a) Tell why.
   b) Calculate the value of W_1.
3. Assume that the energy yielded by the capacitor is received by the motor.
   a) Calculate the value of the energy W_2 yielded by the capacitor between the instants 0 and t_1.
   b) Indicate the forms of energy for which W_2 is transformed.
   c) Determine the efficiency of the motor.

II-GS 2010 1st
Duration of Charging & Discharging of a Capacitor

Consider the circuit whose diagram is shown in figure 1, where G is a generator delivering a square signal $(E;0)$ of period T (Figure 2), D a resistor of resistance $R = 10\,k\Omega$ and (C) a capacitor of capacitance $C = 0.2\,\mu F$. An oscilloscope displays the voltage $u_g = u_{AM}$ across G and the voltage $u_C = u_{BM}$ across (C).

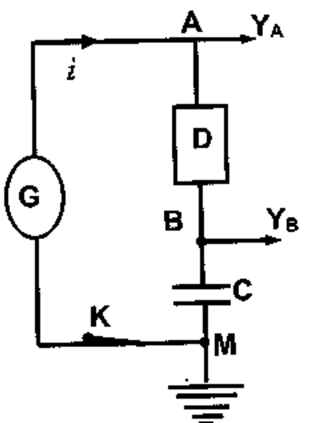


Figure 1

Part A
Theoretical study
Charging of (C)

During the charging of (C), the voltage u_g has the value E and at an instant t, the circuit carries a current i.

1. Give the expression of i in terms C and $\frac{du_C}{dt}$.
2. Derive, for $0 \le t \le \frac{T}{2}$, the differential equation in u_C.
3. The solution of this differential equation has the form: $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$, where A & τ are constants.

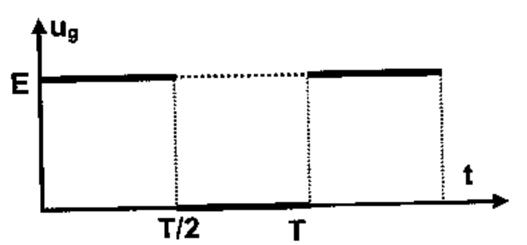


Figure 2

   a) Determine, in terms of E, R & C, the expressions of A & τ.
   b) Draw the shape of the graph representing the variation of u_C as a function of time and show, on this graph, the points corresponding to A & τ.

Discharging of (C)

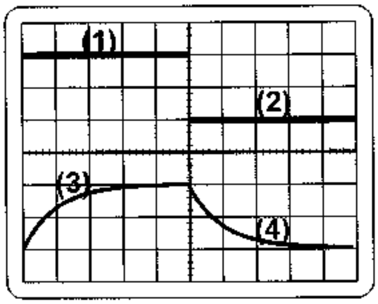
4. During discharging of (C) the voltage u_g. We consider the instant $\frac{T}{2}$ as a new origin of time.

   Verify that $u_C = E e^{-\frac{t}{\tau}}$.
5. a) What must the minimum duration of charging be so that u_C reaches practically the value E?
   b) What is then the minimum value of T?

Part B

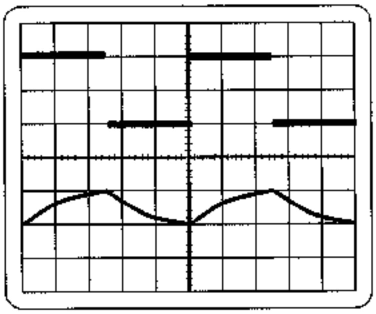
Experimental study

1. On the screen of the oscilloscope, we observe the waveforms of figure 3.
   a) Which curve corresponds to the charging of the capacitor? Justify the answer.
   b) Calculate the value of E and that of the period T of the square signal.
2. a) We increase the frequency of the voltage delivered by G. The waveforms obtained are as in figure 4.
   ✘ Determine the new period of the square signal.
   ✘ Justify the shape of the waveform of u_C displayed.
   b) We keep increasing the frequency of the voltage delivered by G.
   The waveform becomes almost triangular. Why?



S_v=5V/div S_h=2ms/div

Figure 3



S_v=5V/div S_h=1ms/div

Figure 4

III-GS 2008 2nd

Response of an Electric Component

In order to study the response of the current in an electric component when submitted to a DC voltage, we use a coil of inductance $L = 40\,mH$ and of resistance $r = 18\,\Omega$, a capacitor of capacitance $C = 100\,\mu F$, a resistor of resistance $R = 2\Omega$, a switch K and a DC generator delivering across its terminals a constant voltage $E = 8V$.

Part A

Response of the electric component (R, L)

We connect the coil in series with the resistor across the terminals of the generator (Figure 1). At the instant $t_0 = 0$, we close K. The circuit thus carries a current i. With an oscilloscope, we display the variation of the voltage u_{AM} across the terminals of the resistor as a function of time (Figure 2).

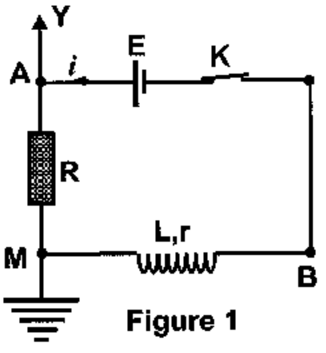


Figure 1

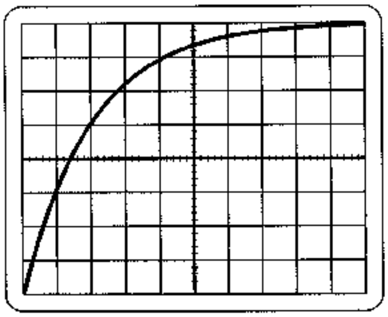


Figure 2
S_h=1ms/div S_v=0.1 V/div

1. Express the voltage u_{AM} across the resistor and the voltage u_{MB} across the coil in terms of R, L, r, i and $\frac{di}{dt}$.
2. Derive the differential equation in i.
3. The solution of this differential equation is of the form: $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$.

   a) Show that: $I_0 = \frac{E}{r+R}$ and $\tau = \frac{L}{r+R}$.

   b) Calculate the values of I_0 and τ.
4. Using figure 2, determine the values of I_0 and that of τ.

Part B

Response of the electric component (R, C)

We replace, in the previous circuit, the coil by the capacitor (Figure 3).

At $t_0 = 0$, we close K. The circuit thus carries a current i. With the oscilloscope, we display the variation of the voltage u_{AM} as a function of time (Figure 4).

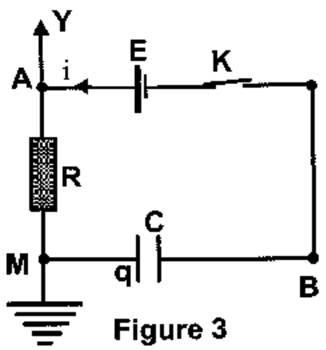


Figure 3

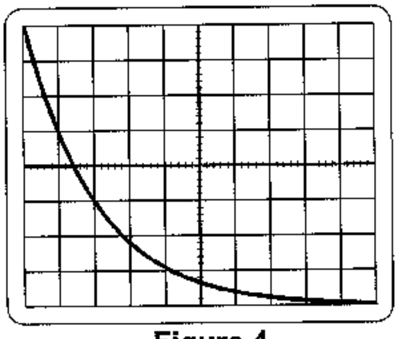


Figure 4

S_h= 0.1 ms/div S_v=1 V/div

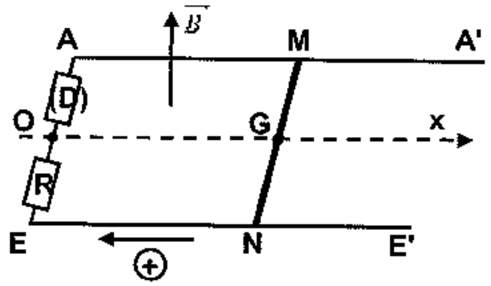
1. Express the current i in terms of C and $\frac{du_C}{dt}$, where u_C is the voltage u_{MB} across the terminals of the capacitor.
2. Using the law of addition of voltages, show that the differential equation in i is of the form: $RC\frac{di}{dt} + i = 0$.
3. The solution of this differential equation is of the form: $i = I_1 e^{-\frac{t}{\tau_1}}$.
   Determine, in terms of E, R & C, the expressions of the two constants I_1 and τ_1 and calculate their values.
4. Referring to figure 4, determine the value of I_1 and that of τ_1.

Part C

In each of the two previous circuits, we replace the resistor by a lamp. Explain the variation of the brightness of the lamp in each circuit.

IV-GS 2007 2nd

Mode of Charging a Capacitor



A metallic rod MN, of length $\ell = 1m$ and of negligible resistance, may slide without friction along two long parallel and horizontal rectilinear rails AA' and EE' of negligible resistance.
During its displacement, the rod remains perpendicular to the rails. An electric component (D) and a resistor of resistance $R = 100\,\Omega$ are connected to the rails with connecting wires. The whole set-up thus described is placed in a uniform vertically upwards magnetic field $\vec{B}$ of magnitude $B = 0.8T$ (adjacent figure).

At the instant $t_0 = 0$, the center of mass G of the rod is at O. A convenient apparatus causes the rod to move in a uniform translational motion from left to right with a speed $v = 0.5\,m/s$.

At an instant t, the position of G is defined by its abscissa $x = \overline{OG}$ on the axis $x'x$.

1. Determine, at the instant t, the expression of the magnetic flux that crosses the surface $AMNE$ in terms of B, ℓ and x taking into consideration the positive direction indicated on the figure.
2. a) Explain the existence of an induced $e.m.f$ « e » across the ends M and N of the rod and show that its value is $0.4V$.

   b) At the instant t, an induced current i passes in the circuit. Determine its direction.

   c) Draw a diagram showing the equivalent generator between M and N and specify its positive terminal.
3. The component (D) is a capacitor of capacitance $C = 10^{-2}F$.

   During the displacement of the rod, (D) undergoes the phenomenon of electric charging.

   a) Derive the differential equation that describes the variations of $u_C = u_{OA}$ as a function of time.

   b) Charging.

   i- Calculate the value of the time constant of the circuit thus formed.

   ii- After how long would the capacitor be practically charged completely?

   c) At the end of charging, the voltage across the capacitor is U and its charge is Q. Calculate U and Q.

   d) Determine the values of i at the instants $t_0 = 0$ and $t_1 = 6s$.

   e) At the instant $t_1 = 6s$, the rod is stopped. The circuit carries again a current.

   i- Due to what is this current?

   ii- Specify the duration of the passage of this current.

V-GS 2007 1st

Energy Dissipated During the Charging of a Capacitor

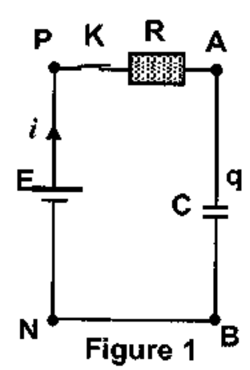


Figure 1

The object of this exercise is to determine the energy dissipated, by Joule's effect, during the charging of a capacitor.

We charge a capacitor of capacitance $C = 5\times10^{-3}F$, initially neutral, using an ideal generator of constant voltage of $e.m.f$ E through a resistor of resistance $R = 200\,\Omega$ (figure 1). At the instant $t_0 = 0$, the switch K is closed.

The circuit thus carries a current i at the instant t.

Part A
Exploiting a waveform

Using an oscilloscope, we display the variations of the voltage $u_R = u_{PA}$ across the resistor and that of $u_C = u_{AB}$ across the capacitor.

We obtain the waveforms of figure 2.

1. The curve (b) represents the variation of u_R as a function of time. Why?
2. Determine, using the waveforms:
   a) the value of E;
   b) the maximum value I of i;
   c) the time constant τ of the RC circuit.
3. Give the time at the end of which the capacitor will be practically completely charged.

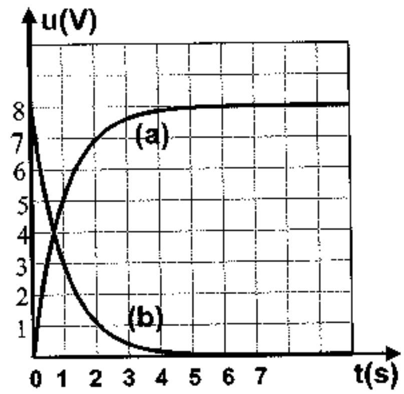


Figure 2

Part B
Theoretical study of charging

1. Show that the differential equation in u_C may be written as: $E = RC\dfrac{du_C}{dt} + u_C$.
2. The solution of this equation has the form $u_C = Ae^{-\frac{t}{\tau}} + B$ where A, B and τ are constants.
   a) Determine, starting from the differential equation, the expression of B in terms of E and that of τ in terms of R and C.
   b) Using the initial condition, determine the expression of A in terms E.
3. Show that $i = \dfrac{E}{R}e^{-\frac{t}{\tau}}$.

Part C
Energetic study of charging

1. Calculate the value of the electric energy W_C stored in the capacitor at the end of the charging process.
2. The instantaneous electric power delivered by the generator at the instant t is $p = \dfrac{dW}{dt} = Ei$ where W is the electric energy delivered by the generator between the instants t_0 and t.
   a) Show that the value of the electric energy delivered by the generator during the whole duration of charging is $0.32\,J$.
   b) Deduce the energy dissipated due to Joule's effect in the resistor.

VI-GS 2005 1st
Flash of a Camera

In this exercise, we intend to show evidence of the functioning of the flash of a camera.

The simplified circuit of the flash of a camera is formed of an apparatus taken as a source of DC voltage of $E = 300\,V$, a capacitor of capacitance $C = 200\,\mu F$, a resistor of resistance $R = 10\,k\Omega$, a lamp (L), considered as a resistor of resistance $r = 1\,\Omega$ and a double switch K as shown in Figure 1.

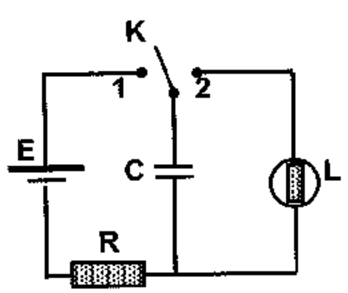


Figure 1

Part A

Charging the capacitor

The capacitor is initially neutral. The double switch is turned to position 1 at the instant $t_0 = 0$. The capacitor starts charging (figure 2)

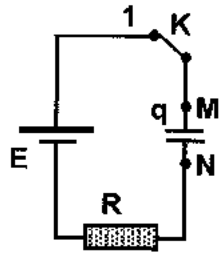


Figure 2

1. a) Derive, at an instant t, the differential equation that governs the variation of the voltage $u_C = u_{MN}$, as a function of time, during the charging of the capacitor.

   b) The solution of this equation, at an instant t, has the form: $u_C = A + B e^{-\frac{t}{\tau}}$ where A, B & τ are constants.

   Determine these constants in terms of E, R & C.

2. Calculate the energy W stored by the capacitor at the end of charging.

Part B

Discharging the capacitor

The capacitor being completely charged, the double switch is turned to position 2. The capacitor starts to discharge through the lamp (L).

The instant of closing the circuit is taken as an origin of time. At an instant t, the voltage across the capacitor is $u_C = u_{MN} = E e^{-\frac{t}{rC}}$ and the circuit carries then a current i (Figure 3).

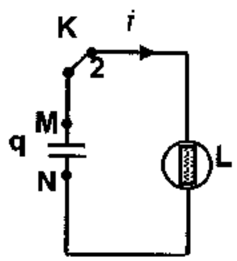


Figure 3

1. Justify the direction of the current in Figure 3.
2. Knowing that $i = -\frac{dq}{dt}$.

   a) Determine the expression of the current i as a function of time.

   b) Calculate the maximum value of i.

   c) Determine the duration t_1 at the end of which the current reaches 70 % of its maximum value.

   d) Calculate, at the instant t_1 the voltage u_C across the capacitor.

3. a) Assuming that the energy released by the capacitor by the end of the duration t_1 is converted totally into light in the lamp, determine the average power received by the lamp during t_1.

   b) The flash lamp emits light as long as the average power it receives is greater or equal to $6.4 \times 10^4 W$.

   Knowing that the duration of the flash is t_1, justify the emission of the flash between the instants 0 and t_1.

Solutions

I-GS 2012 1st

Part A

1. Law of addition of voltages : $u_G = u_R + u_C = Ri + u_C$;

But $i = \frac{dq}{dt} = C\frac{du_C}{dt}$ and $u_R = Ri$.

We get $E = u_C + RC\frac{du_C}{dt}$.

2. We have $u_C = A + Be^{-\frac{t}{\tau}}$ then $\frac{du_C}{dt} = -\frac{B}{\tau}e^{-\frac{t}{\tau}}$;

By replacing: $E = A + Be^{-\frac{t}{\tau}} + RC\left(-\frac{B}{\tau}e^{-\frac{t}{\tau}}\right) \Rightarrow E = A + Be^{-\frac{t}{\tau}}\left(1 - \frac{RC}{\tau}\right)$;

This equation is verified at any instant t : then $A = E$ and $1 - \frac{RC}{\tau} = 0 \Rightarrow \tau = RC$.

But at $t = 0$, $u_C = 0$ (the capacitor was taken neutral).

By replacing in the above solution we get $0 = A + Be^{-0} \Rightarrow B = -A = -E$;

Thus $u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)$.

3. a) The capacitor is completely charged at $t \to +\infty \Rightarrow u_C = E\left(1 - e^{-\frac{t}{\tau}}\right) \to E(1-0) = E$.[1].

b) The energy stored at the end of charging is $W = \frac{1}{2}CE^2 = 0.5 \times 1 \times 10^2 = 50J$.

Part B

1. a) The minimum voltage u_1 is reached at $t_1 = 20\,s$ [2].

b) $u_1 = 5V$.

2. a) At the instant t_1, the capacitor still stores energy because the voltage across it is not yet zero.

b) The energy that still stored at t_1 is $W_1 = \frac{1}{2}Cu_1^2 = 0.5 \times 1 \times 5^2 = 12.5\,J$.

3. a) The energy yielded by the capacitor between 0 and t_1 is: $W_2 = W - W_1 = 50 - 12.5 = 37.5J$.

---

[1] At the end of charging u_C becomes constant, $\frac{du_C}{dt} = 0$; replacing in DE, we get: $E = u_C + RC(0) \Rightarrow u_C = E$

The capacitor is completely charged at $t = 5\tau$; $e^{-\frac{t}{\tau}} = e^{-5} = 0.01 \approx 0 \Rightarrow u_C = E\left(1 - e^{-\frac{t}{\tau}}\right) \approx E$.

[2] Any value of $t_1 \in [19s, 22s]$ is also correct

b) W_2 is converted into mechanical energy[3] in order to raise up the object and thermal energy dissipated in the wires due to Joules effect.

c) The efficiency of the motor is given by: $\eta = \frac{W_{\text{useful}}}{W_{\text{total}}} = \frac{W_{\text{mechanical}}}{W_{\text{electrical}}}$.

But $W_{\text{mechanical}} = m\,g\,h = 1kg \times 10m/s^2 \times 1.5m = 15\,J$ & $W_{\text{electrical}} = W_2 = 37.5J$.

Then: $\eta = \frac{15}{37.5} = 0.4 = 40\,\%$.

II-GS 2010 1st
Part A

1. We have $i = \frac{dq_B}{dt} = C\frac{du_C}{dt}$.

2. Law of addition of voltages: $u_{AM} = u_{AB} + u_{BM}$, but $u_{AB} = u_R = Ri$ & $i = \frac{dq_B}{dt} = C\frac{du_C}{dt}$

Then $u_C + RC\frac{du_C}{dt} = E$.

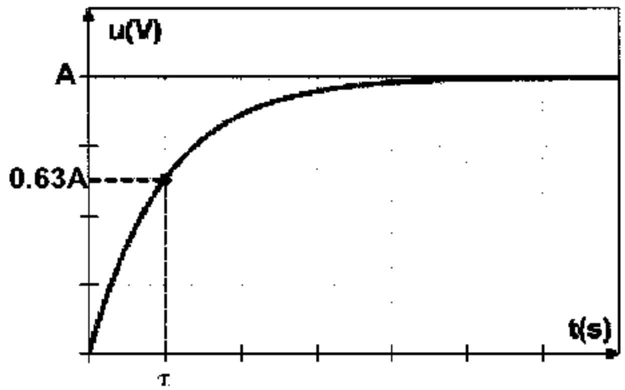
3. a) $u_C = A\left(1 - e^{-\frac{t}{\tau}}\right)$, so $\frac{du_C}{dt} = \frac{A}{\tau}e^{-\frac{t}{\tau}}$.

Replacing in the differential equation,

we get: $E = A + A\,e^{-\frac{t}{\tau}}\left(\frac{RC}{\tau} - 1\right)$;

which is verified at any instant t.

So, $\frac{RC}{\tau} - 1 = 0$, then $\tau = RC$ & $A = E$



b) Adjacent curve.

4. For $\frac{T}{2} \le t \le T$; the generator acts as a connecting wire;

Law of addition of voltages $u_{AM} = u_{AB} + u_{BM}$, $0 = -u_R + u_C$ & $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$;

Then $u_C + RC\frac{du_C}{dt} = 0$.

We have $u_C = E\,e^{-\frac{t}{\tau}}$ then $\frac{du_C}{dt} = E\,e^{-\frac{t}{\tau}}$ replacing in DE of 4.a);

$u_C + RC\frac{du_C}{dt} = E\,e^{-\frac{t}{\tau}} + RC\,E\left(-\frac{1}{\tau}\right)e^{-\frac{t}{\tau}} = 0$ & at $t_0 = 0$, $u_C = E\,e^0 = E$ (Verified)[4].

2. a) During the charging, the steady state is reached after a minimum duration of: $t_{\min} = 5\tau = 5RC$

b) To ensure the complete charge and discharge, the minimum value of the generator's period is $T_{\min} = 10\,\tau = 10\,R\,C = 10 \times 10^3 \times 0.2 \times 10^{-6} = 20 \times 10^{-3}s = 20\,ms$.

[3] Gravitational energy and thermal or kinetic and thermal is also correct but one form only is wrong.

[4] To verify that it is <u>a solution</u> we should replace in the differential equation and then we check the verification of the initial conditions.

Part B

1. a) The curve (3) corresponds to the charging of the capacitor. Since the voltage across the capacitor during this phase increases with time.

   b) The curve (2) is considered as reference for the voltage across the generator so:

   $E = S_v \times y = 5\,V/\text{div} \times 2\text{div} = 10\,V$;

   The period $T = S_h \times x = 2\,ms/\text{div} \times 10\text{div} = 20\,ms$.

2. a) The new period is $T' = S_h \times x' = 1ms/\text{div} \times 5\text{div} = 5ms$.[5]

   This period is not long enough $T' = 5ms < T_{\min} = 20ms$, to ensure the complete charge and discharge.

   b) When the frequency is increased, the period T of the voltage delivered by the generator becomes very small.

   Under this condition $T <<<< 10\,\tau$; the exchange between charge and discharge is very fast, the curve is reduced to the tangent at the origin of time.

   So these curves are reduced to straight lines, thus appears to have a triangular shape.

III-GS 2008 2nd

Part A

1. The voltages are given by: $u_{AM} = Ri$ & $u_{MB} = ri + L\dfrac{di}{dt}$.

2. Law of addition of voltages: $u_G = u_R + u_{coil}$, $(R+r)i + L\dfrac{di}{dt} = E$.

3. a) We have: $i = I_0\left(1 - e^{-\frac{t}{\tau}}\right) \Rightarrow \dfrac{di}{dt} = \dfrac{I_0}{\tau}e^{-\frac{t}{\tau}}$.

   Replacing in the differential equation: $(R+r)I_0 + I_0\left(-(R+r) + \dfrac{L}{\tau}\right)e^{-\frac{t}{\tau}} = E$.

   This equation is satisfied at any instant, so: $(R+r)I_0 = E$, then $I_0 = \dfrac{E}{R+r}$.

   & $-(R+r) + \dfrac{L}{\tau} = 0$, then $\tau = \dfrac{L}{R+r}$.

   b) Numerical values: $I_0 = \dfrac{8}{18+2} = 0.4\,A$; and $\tau = \dfrac{L}{R+r} = \dfrac{0.04}{18+2} = 2\times10^{-3}\,s = 2\,ms$.

4. From graph: $u_R(\max) = 0.1\times 8 = 0.8\,V$ but $u_R(\max) = R\,I_0 \Rightarrow I_0 = \dfrac{u_R(\max)}{R} = 0.4\,A$.

   Also for $t = \tau$, $u_R = 0.63\,u_R(\max)$, which corresponds to 2divisions then $\tau = 2\,ms$.

Part B

1. $i = \dfrac{dq}{dt} = C\dfrac{du_C}{dt}$.
2. Law of addition of voltages: $E = u_{AM} + u_{MB}$, then $E = u_C + Ri$.

---

[5] The voltage at the end of the half period is $u_C = S_v \times y = 5V/div \times 1div = 5V < E = 10V$.

Deriving this result with respect to time, we get: $0=\frac{du_C}{dt}+R\frac{di}{dt}\Rightarrow\frac{i}{C}+R\frac{di}{dt}=0$.

Then $i+RC\frac{di}{dt}=0$.

2nd method: $i=\frac{dq}{dt}=C\frac{du_C}{dt}$ & $u_C=E-Ri$ then: $i=-RC\frac{di}{dt}$, $i+RC\frac{di}{dt}=0$.

3. For $t=0$, $u_C=0$ and $i=I_1$, $E=RI_1$; then $I_1=\frac{E}{R_1}=\frac{8}{2}=4A$.

   We have: $i=I_1\,e^{-\frac{t}{\tau_1}}$, so $\frac{di}{dt}=-\frac{I_1}{\tau_1}e^{-\frac{t}{\tau_1}}$;

   Replacing in the differential equation we get $I_1\,e^{-\frac{t}{\tau_1}}\left(1-\frac{RC}{\tau_1}\right)=0$; but $I_1\,e^{-\frac{t}{\tau_1}}\neq0$

   So, $\tau_1=RC=2\times100\times10^{-6}=2\times10^{-4}\,s=0.2\,ms$.

4. From graph $u_R(\max)=8V=RI_1\Rightarrow I_1=\frac{u_R(\max)}{R}=\frac{8}{2}=4A$.

   And for $t=\tau_1$, $u_R=0.37\,u_R(\max)=3V\Rightarrow\tau_1=0.2\,ms$.

Part C

In A: after closing the switch the brightness of the lamp increases and reaches a stable brightness after a very short duration.

In B: at the instant of closing the switch the lamp shines then its brightness decreases and vanishes after a short duration.

IV-GS 2007 2nd

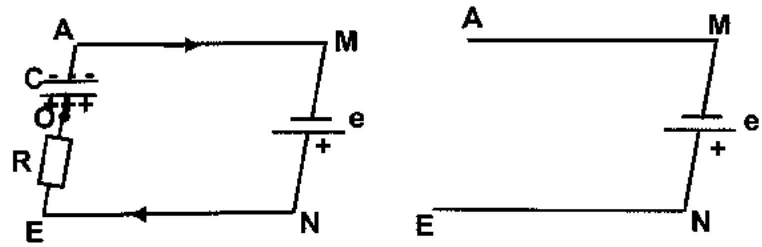
1. The magnetic flux crossing the circuit is $\phi=BS\cos\theta$ where $\theta=(\vec{n},\vec{B})=\pi$;
   Then $\phi=BS\cos\theta=BS\cos(\pi)=-B\ell x$.

2. a) The surface area of the circuit is variable S, then the flux is variable.
   Thus, the circuit is the seat of an induced electromotive force.

   According to Faraday's law $e=-\frac{d\phi}{dt}=-\frac{d(-B\ell x)}{dt}=B\ell v$;

   $e=B\ell v=0.8\times1\times0.5=0.4V$.

   b) $e=0.4V>0$, then the induced current circulates in the positive chosen direction (clockwise).



   c) Circuit.

3. a) Law of addition of voltages : $u_{MN}=u_{MA}+u_{AO}+u_{OE}+u_{EN}$;

But $u_{MA}=u_{EN}=0$ (Connecting wires); $-e=-u_C-Ri$ & $i=C\dfrac{du_C}{dt}$;

We get $e=RC\dfrac{du_C}{dt}+u_C$

b) Charging

i- The time constant $\tau=RC=100\times10^{-2}=1s$.

ii- The complete charge is practically attained after $5\tau=5s$.

c) At the end of charging $U=e=0.4V$.

The charge stored is : $Q=CU=10^{-2}\times0.4=4\times10^{-3}C$.

d) At any instant we have : $e=Ri+u_C$

At $t_0=0$; the capacitor is taken neutral $u_C=0$, so $e=RI_0$;

Then, $I_0=\dfrac{e}{R}=\dfrac{0.4}{100}=4\times10^{-3}A=4mA$.

At $t=6s$; the capacitor is completely charged $u_C=e$, so $Ri=0\Rightarrow i=0$

e) *Stopping the rod.*

i- This current is a result of discharging of the capacitor through the resistor.

ii- The duration of the passage of the current of discharging is $5\tau=5RC=5s$.

V-GS 2007 1st

Part A

1. The current i decreases with time, [at the end of charging $i=0$];
   But the voltage $u_R=Ri$ is represented by the curve (b).
2. a) At the end of charging $u_C=E$ then $E=0$.
   b) According to Ohm's law, the current is maximum when u_R is also maximum;
   Then $u_R=RI$, $I=\dfrac{u_R}{R}=\dfrac{8}{200}=0.04A$.
   c) The time constant τ is the duration after which the voltage across the capacitor reaches 63% of its value in the steady state.
   If $u_C=0.63E=0.63\times8=5.04V$, then $t=\tau=1s$.
3. The minimum duration needed to reach the steady state is $5\tau=5s$.

Part B

1. Law of addition of voltages: $u_{PN}=u_{PA}+u_{AB}+u_{BN}$;
   (but $u_{BN}=0$, connecting wires); $u_{PA}=Ri$ & $i=C\dfrac{du_C}{dt}$ then $u_{PA}=RC\dfrac{du_C}{dt}$;
   Then $E=RC\dfrac{du_C}{dt}+u_C$.
2. a) $u_C=Ae^{-\frac{t}{\tau}}+B$; $\dfrac{du_C}{dt}=-\dfrac{A}{\tau}e^{-\frac{t}{\tau}}$; by substitution in the differential equation we get:
   $$E=RC\left(-\frac{A}{\tau}e^{-\frac{t}{\tau}}\right)+Ae^{-\frac{t}{\tau}}+B;\ E=B+Ae^{-\frac{t}{\tau}}\left(1-\frac{RC}{\tau}\right)$$

This equation is verified at any instant t & $Ae^{-\frac{t}{\tau}} \neq 0$; then $1-\frac{RC}{\tau}=0$; so $\tau = RC$ & $B = E$.

b) The capacitor is taken neutral so at $t=0$, $u_C = 0$; so $0 = A + B$; $A = -B = -E$.

Then $u_C = -Ee^{-\frac{t}{\tau}} + E = E\left(1-e^{-\frac{t}{\tau}}\right)$.

3. We know that $i = C\frac{du_C}{dt}$ & $u_C = E\left(1-e^{-\frac{t}{\tau}}\right)$; then $i = CE\frac{1}{RC}e^{-\frac{t}{\tau}} = \frac{E}{R}e^{-\frac{t}{\tau}}$.

Part C

1. At the end of charging the voltage across the capacitor is $u_C = E$;

Then the electric energy stored is $W = \frac{1}{2}C u_C^2 = \frac{1}{2}C E^2 = \frac{1}{2}\times 5\times 10^{-3}\times 8^2 = 0.16J$.

2. a) We have $p = \frac{dW}{dt} = Ei$; so $W = \int_0^{5\tau} Ei\, dt = \int_0^{5\tau} E\frac{E}{R}e^{-\frac{t}{RC}}dt$;

$$W = -\frac{E^2}{R}RCe^{-\frac{t}{\tau}}\Big|_0^{5\tau} = -CE^2e^{-\frac{t}{\tau}}\Big|_0^{5\tau} = -CE^2e^{-5} - \left(-CE^2e^0\right) = CE^2 - CE^2e^{-5} = CE^2$$

Then $W = 5\times 10^{-3}\times 8^2 = 0.32J$.

b) The energy dissipated in the resistor is $W_R = W - W_e = 0.32J - 0.16J = 0.16J$.

VI-GS 2005 1st

Part A

1. a) Law of addition of voltages $E = u_C + u_R$; but $i = \frac{dq}{dt} = C\frac{du_C}{dt}$ and $u_R = Ri$;

Then $E = u_C + RC\frac{du_C}{dt}$.

b) We have $u_C = A + Be^{-\frac{t}{\tau}}$ then $\frac{du_C}{dt} = -\frac{B}{\tau}e^{-\frac{t}{\tau}}$;

By replacing in the differential equation: $E = A + Be^{-\frac{t}{\tau}} + RC\left(-\frac{B}{\tau}e^{-\frac{t}{\tau}}\right)$,

$E = A + Be^{-\frac{t}{\tau}}\left(1-\frac{RC}{\tau}\right)$; this equation is verified at any instant t & $Be^{-\frac{t}{\tau}} \neq 0$;

Then $A = E$ and $1-\frac{RC}{\tau} = 0 \Rightarrow \tau = RC$.

But at $t=0$, the capacitor was neutral so $u_C = 0$; then $0 = A + Be^{-0} \Rightarrow B = -A = -E$;

Thus $u_C = E\left(1-e^{-\frac{t}{\tau}}\right)$ where $\tau = RC$.

c) At the end of charging the voltage across the capacitor is $u_C = E = 300V$, the energy stored is

$W = \frac{1}{2}C u_C^2 = \frac{1}{2}C E^2 = \frac{1}{2}\times\left(200\times 10^{-6}\right)\times(300)^2 = 9J$.

Part B

1. When the witch is turned to position (2), the capacitor acts as a generator delivering the current from its positive armature M.

2. a) Knowing that $i=-\frac{dq}{dt}$ and $q=Cu_C$, then $i=-C\frac{du_C}{dt}=\frac{CE}{rC}e^{-\frac{t}{rC}}=\frac{E}{r}e^{-\frac{t}{rC}}$.

   b) The current $i=\frac{E}{r}e^{-\frac{t}{rC}}$ decreases exponentially, it takes its maximum value at $t=0$.

   Then the maximum intensity I_m of the current is $I_m=\frac{E}{r}e^0=\frac{300}{1}=300\,A$.

   c) At the instant t_1, the current becomes $i_1=0.7\,I_m$.

   But $i_1=\frac{E}{r}e^{-\frac{t_1}{rC}}=I_m e^{-\frac{t_1}{rC}}=0.7I_m$; so $e^{-\frac{t_1}{rC}}=0.7$; we get $-\frac{t_1}{rC}=\ell n(0.7)$;

   Then $t_1=-rC\,\ell n(0.7)=-1\times 200\times 10^{-6}\times \ell n(0.7)=7.13\times 10^{-5}\,s$.

   d) At the instant $t_1=7.13\times 10^{-5}\,s$, $u_C=E\,e^{-\frac{t_1}{rC}}=E(0.7)=300\times 0.7=210V$.

3. a) The electric energy remained in the capacitor at the instant t_1 is:

   $W_1=\frac{1}{2}Cu_C^2=\frac{1}{2}\times 200\times 10^{-6}\times (210)^2=4.41J$.

   The energy delivered by the capacitor at the end of t_1 is $|\Delta W|=|W_1-W_0|=|9-4.41|=4.59J$.

   The average electric power received by the lamp:

   $P_{aver}=\frac{|\Delta W|}{\Delta t}=\frac{4.59\,J}{7.13\times 10^{-5}\,s}=6.43\times 10^4\,W$.

   b) The average power received by the lamp between the instants 0 and t_1 is:

   $P_{aver}=6.43\times 10^4\,W>6.4\times 10^4\,W$, thus a flash is emitted.

I-GS 2013 2nd
Determination of the Characteristics of a Coil

In order to determine the characteristics of a coil, we consider the electric circuit represented in figure 1. This circuit is formed of a capacitor of capacitance C, the coil of inductance L and of resistance r, a resistor of resistance R and an ammeter (A) of negligible resistance, all connected in series across an LFG, of adjustable frequency f, that maintains across its terminals an alternating sinusoidal voltage : $u = u_{AM} = U\sqrt{2}\sin(2\pi f t + \varphi)$.

Thus the circuit carries an alternating sinusoidal current: $i = I\sqrt{2}\sin(2\pi f t)$ (Figure 1).

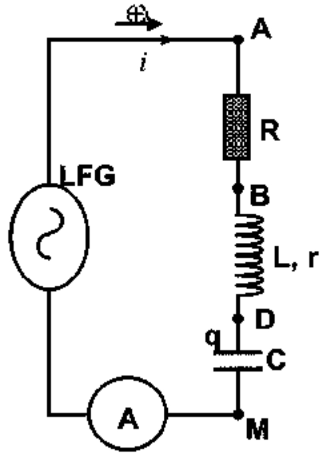


Figure 1

Part A

1. Write the expression of the voltage:
   a) u_{AB} across the terminals of the resistor in terms of R, I, f & t;
   b) u_{BD} across the terminals of the coil in terms of r, L, I, f & t.
2. Show that the voltage across the terminals of the capacitor is $u_{DM} = -\dfrac{I\sqrt{2}}{2\pi f C}\cos(2\pi f t)$.

Part B

1. Applying the law of addition of voltages and giving t two particular values, show that:
   a) the effective value of the current is: $I = \dfrac{U}{\sqrt{(R+r)^2 + \left(2\pi f L - \dfrac{1}{2\pi f C}\right)^2}}$
   b) the phase difference φ between the voltage u_{AM} and the current i is: $\tan\varphi = \dfrac{2\pi f L - \dfrac{1}{2\pi f C}}{R+r}$.
2. U is maintained constant and f is varied; the ammeter indicates a value I for each value of f. An appropriate device allows to plot the curve representing the variation of I as a function of f (Figure 2).[1]
   This curve shows an evidence of a physical phenomenon for $f = f_0 = 110\ Hz$.
   a) Name this phenomenon.

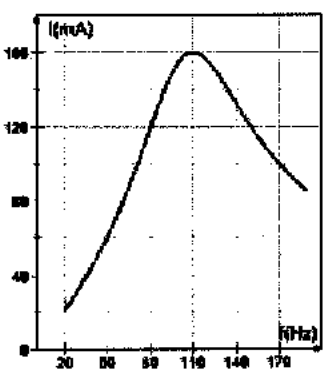


Figure 2

[1]Graphical error in the construction of the curve representing the variations of the current in terms of the frequency, it was taken symmetrical which is not true.

b) Indicate the value I_0 of I corresponding to the value f_0 of f.

c) For $f = f_0$, show that:

i- $4\pi^2 f_0^2 LC = 1$ using the relation given in part (B-1-b).

ii- the circuit is equivalent to a resistor of resistance $R_t = R + r$ using the relation given in part (B-1-a).

d) Calculate the value of L knowing that $C = 21\,\mu F$.

e) Calculate the resistance r knowing that $U = 8V$ and $R = 30\,\Omega$.

II-GS 2013 1st

Determination of the Characteristics of an Unknown Component

An electric component (D), of unknown nature, may be a resistor of resistance R', or a coil of inductance L and of resistance r or a capacitor of capacitance C.

To determine its nature and its characteristics, we connect it in series with a resistor of resistance $R = 10\,\Omega$ across a generator G as shown in figure 1. An oscilloscope is connected so as to display the voltage $u_G = u_{AM}$ across the generator and the voltage $u_R = u_{BM}$ across the resistor.

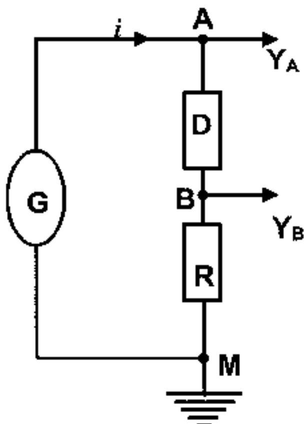


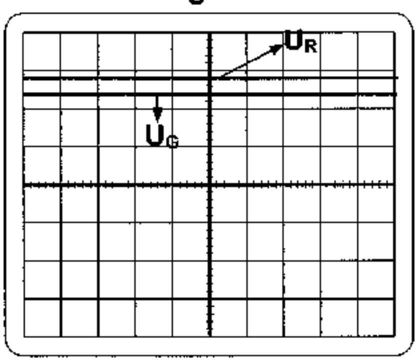
Figure 1

Part A

Case of a DC voltage

The generator G delivers a constant voltage U_0.

On the screen of the oscilloscope we observe the oscillograms of figure 2.



S_{VA} =5V/div S_{VB} = 2V/div

Figure 2

1. Prove that the voltage $U_0 = 12\,V$.
2. a) Determine, in the steady state, the value I of the current in the circuit.

   b) Deduce that (D) is not a capacitor.

   c) Determine the resistance of the component (D).

Part B

Case of an AC voltage

The generator G delivers now an alternating sinusoidal voltage. On the screen of the oscilloscope we observe the waveforms of figure 3.

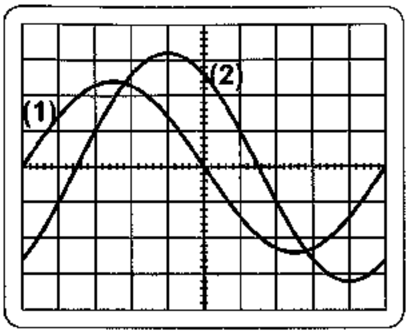
1. Referring to the waveforms of figure 3, show that:

   a) (D) is a coil;

   b) the waveform (2) represents the variation of the voltage u_R across the resistor.
2. The voltage across the generator is given by: $u_G = U_m \sin(wt)$.

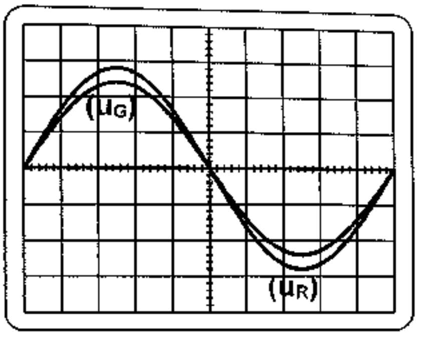
   Determine U_m and w.
3. Determine the expression of i as a function of time.

4. Applying the law of addition of voltages and giving t two particular values, determine the inductance L and the resistance r of (D).



S_{VA} = 5V/div S_{VB} = 1V/div
S_h = 2ms/div

Figure 3



S_{VA} = 5V/div S_{VB} = 2V/div
S_h = 2ms/div

Figure 4

5. To verify the values of L and r of (D), we add a capacitor of adjustable capacitance C in series to the previous circuit. For $C = 10^{-4}\,F$, we obtain the waveforms of figure 4.
   a) Name the observed phenomenon.
   b) Verify, using the waveforms of figure 4, the values of L and r.

III-GS 2012 2nd

Determination of the Characteristics of a Coil

In order to determine the inductance L and the resistance r of a coil, we connect the coil in series with a capacitor of capacitance $C = 160\,\mu F$ across the terminals of a low frequency generator (LFG) delivering an alternating sinusoidal voltage:

$u_g = u_{AD} = 20\sin(100\,\pi\,t)$; ($u_g$ in V, t in s).

The circuit thus carries an alternating sinusoidal current i. An oscilloscope is connected so as to display the voltage $u_g = u_{AD}$ on the channel Y_A and the voltage $u_C = u_{BD}$ on the channel Y_B (Figure 1).

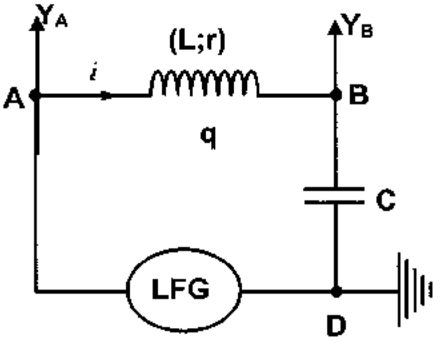


Figure 1

On the screen of the oscilloscope we observe a display of the waveforms represented in figure 2.
Take $0.32\,\pi = 1$.

1. Knowing that the vertical sensitivity S_v is the same on both channels, calculate its value.
2. Calculate the phase difference between u_g and u_C.
   Which of them lags behind the other?
3. Deduce the expression of the voltage u_C across the terminals of the capacitor as a function of time.
4. Using the relation between the current i and the voltage u_C, determine the expression of i as a function of time.

5. Applying the law of addition of voltages, and by giving the time t two particular values, determine r and L.
6. In order to verify the preceding calculated values of r and L, we proceed as follows:
   - ✘ we measure the average power consumed in the circuit for $w = 100\,\pi\,\text{rad}/s$ and we obtain $8.66\,W$.
   - ✘ we keep the maximum value of u_g constant but we vary its frequency f; for $f = 71\,Hz$ the effective value of the current in the circuit is maximum.

   Determine the values of r and L.

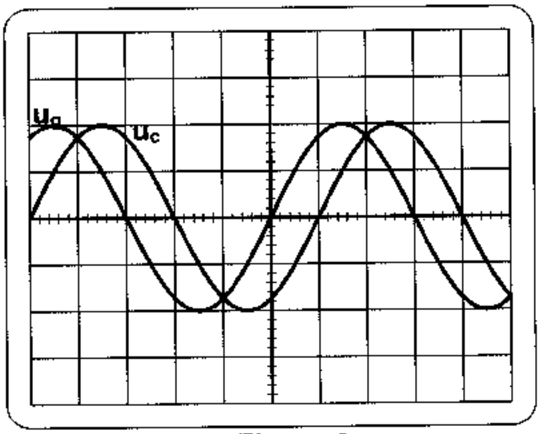


Figure 2

IV-GS 2011 1st
Determination of the Capacitance of a Capacitor

In order to determine the capacitance C of a capacitor, we consider the following components:

- ✘ a generator G delivering across its terminals an alternating sinusoidal voltage of effective value U and of adjustable frequency f;
- ✘ a resistor of resistance $R = 250\,\Omega$;
- ✘ an oscilloscope;
- ✘ two voltmeters V_1 and V_2;
- ✘ a switch
- ✘ connecting wires.

We connect up the circuit whose diagram is represented in figure 1.

Part A
Theoretical study

The voltage across the generator is $u_{AB} = U\sqrt{2}\sin(wt)$. In the steady state, the current i carried by the circuit has the form: $i = I\sqrt{2}\sin(wt + \varphi)$, where I is the effective value of i.

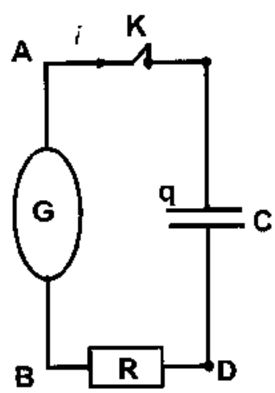


Figure 1

1. a) Give the expression of the current i in terms of C and $\frac{du_C}{dt}$ with $u_C = u_{AD}$.
   b) Determine the expression of the voltage u_C in terms of I, C, φ, w and t.[2]
   c) Deduce the expression of effective value U_C of u_C in terms of I, C and w.
2. Applying the law of addition of voltages and giving t a particular value, show that $\tan\varphi = \frac{1}{RCw}$

[2] φ is not figuring in the main official copy.

Part B

Determination of C

1. Using the oscilloscope

The oscilloscope, conveniently connected, displays on channel (Y_1) the voltage u_{AB} across the generator and on channel (Y_2) the voltage u_{DB} across the resistor.

On the screen of the oscilloscope, we obtain the waveforms represented in figure 2.

Time base [horizontal sensitivity]: $1\,ms\,/\,\text{div}$.

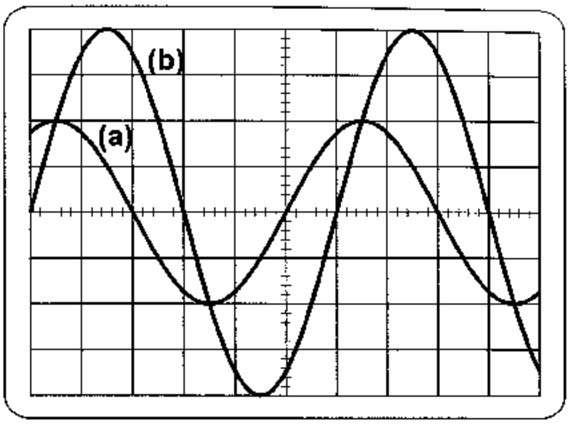


Figure 2

a) Redraw figure 1 showing on it the connections of the oscilloscope.

b) Referring to figure 2:

i- determine the frequency of u_{AB};

ii- which of the waveforms, (a) or (b), leads the other?

iii- the waveform (a) displays u_{DB}. Why?

iv- determine the phase difference between the voltages u_{AB} and u_{DB}.

c) Calculate the value of C.

2. Using the voltmeters

The oscilloscope is removed and the frequency f is adjusted to the value $200\,Hz$. We then connect V_1 across the resistor and V_2 across the capacitor. V_1 and V_2 reads then the values $2.20\,V$ and $3.20\,V$ respectively.

Using these obtained measured values and the results of part A, determine the value of C.

V-GS 2010 2nd

Identifying Two Electric Components

Consider a generator G that maintains across its terminals a constant voltage E, a generator G' that maintains across its terminals an alternating sinusoidal voltage of expression: $u = 5\sqrt{2}\sin(2\pi f t)$ (u in V, t in s) of adjustable frequency f, an ammeter (A) of negligible resistance, a switch K, connecting wires and two electric components (D_1) and (D_2) where one of them is a coil of inductance L and of resistance r, and the other a capacitor of capacitance C.

Take: $0.32\pi = 1$.

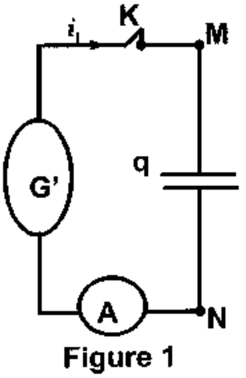


Figure 1

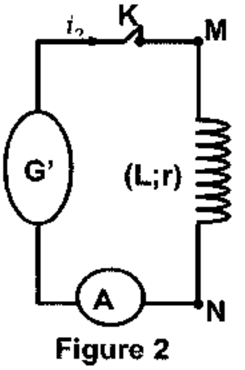


Figure 2

In order to identify each of these two components and to determine their characteristics, we perform the following experiments, the measurements being taken after attaining the steady state in the circuit.

Part A

First experiment

Each of the two components, taken separately, is fed by the generator G.

In the steady state:

- ✗ the circuit containing (D_1) does not carry any current;
- ✗ the circuit containing (D_2) carries a current $I = 1A$ and consumes a power of $5W$.

1. Determine the nature of (D_1).
2. Determine the resistance r of the coil.

Part B

Second experiment

Each of the two components, taken separately, is fed by the generator G', the voltage u being of frequency $f = 50\,Hz$. In the steady state:

- ✗ the circuit containing the capacitor carries an alternating sinusoidal current i_1 of effective value $I_1 = 50\,mA$ and does not consume any power (Figure 1);
- ✗ The coil carries an alternating sinusoidal current i_2 of effective value $I_2 = \frac{\sqrt{2}}{2}A$, and consumes an average power of $2.5W$ (Figure 2).

1. Determine the phase difference between i_1 & u and that between i_2 & u.
2. Write down, with justification, the expressions of i_1 and i_2 as a function of time.
3. a) Show that $i_1 = C\frac{du}{dt}$.
   b) Deduce the value of C.
4. a) Write down the relation among u, i_2, r & L.
   b) Using the expressions of u and i_2 as a function of time and giving t a particular value, determine the value of L.

Part C

Third experiment

In fact, the values of r and L are labeled on the coil. To verify the value of C, we connect the coil and the capacitor in series across G' by giving f different values; we notice that the effective current in the circuit attains a maximum value for $f = f_0 = 225\,Hz$.

1. For the frequency f_0, the circuit is the seat of a particular electric phenomenon.
   Give the name of this phenomenon.
2. Determine the value of C.

VI-GS 2009 1st

Characteristics of an (R, L, C) Circuit

In order to determine the characteristics of an (R, L, C) circuit, we connect the circuit represented in figure 1. This circuit is formed of a resistor of resistance $R = 650\,\Omega$, a coil of inductance L and of negligible resistance and a capacitor of capacitance C, all connected in series across a function generator (LFG) delivering across its terminals a sinusoidal alternating voltage u_G of the form: $u_G = u_{AM} = U_m \cos(2\pi f t)$.

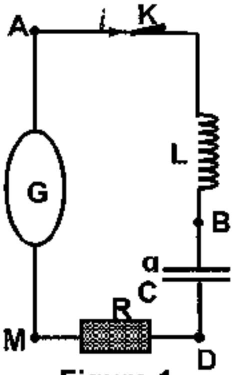


Figure 1

Part A

The frequency of the voltage u_G is adjusted on the value f_1.

We display, on the screen of an oscilloscope, the variations, as a function of time, of the voltage u_{AM} across the generator on the channel (Y_1) and the voltage u_{DM} across the resistor on the channel (Y_2)

The waveforms obtained are represented in figure 2.

- ✘ Vertical sensitivity on both channels is: $2V/\text{div}$;
- ✘ Horizontal sensitivity is: $0.1\,ms/\text{div}$.

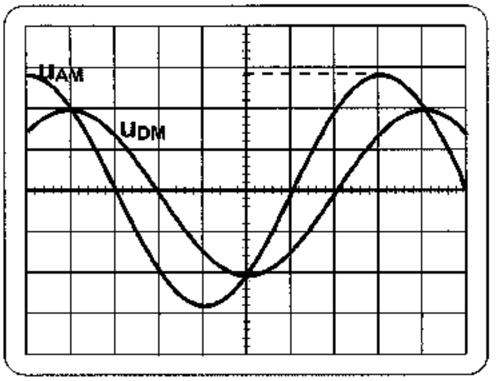


Figure 2

1. Redraw figure (1) showing on it the connections of the oscilloscope.
2. Referring to the waveforms, determine:
   a) the value of the frequency f_1;
   b) the absolute value of φ_1 the phase difference between u_{AM} and u_{DM}.
3. The current i carried by the circuit has the form: $i = I_m \cos(2\pi f_1 t - \varphi_1)$.
   a) Write down the expressions of the voltages: u_{AB}, u_{BD} and u_{DM} as a function of time, I_m, f_1, L, C, R & φ_1.
   b) The relation: $u_{AM} = u_{AB} + u_{BD} + u_{DM}$ is valid for any instant t.

   Show, by giving t a particular value, that: $\tan\varphi_1 = \dfrac{L(2\pi f_1) - \dfrac{1}{C(2\pi f_1)}}{R}$

Part B

Starting from the value f_1, we decrease continuously the frequency f. We notice that, for $f_0 = 500\,Hz$ the circuit is the seat a of current resonance phenomenon.

Deduce from what preceded a relation between L, C & f_0.

Part C

We keep decreasing the frequency f. For a value f_2 of f we find that the phase difference between u_{AM} & u_{DM} is φ_2 such that $\varphi_2 = -\varphi_1$.

1. Determine the relation among f_1, f_2 & f_0.
2. Deduce the value of f_2.

Part D

Deduce from what is preceded the values of L and C.

VII-GS 2008 1st

Determination of the Capacitance of a Capacitor

In order to determine the capacitance C of a capacitor, we consider two experiments.

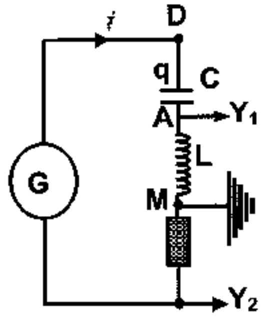


Figure 1

Part A

First experiment

We place the capacitor in series, with a coil of inductance $L = 0.32\,H$, a resistor of resistance $R = 100\,\Omega$ and a low frequency generator G (LFG) that delivers across its terminals an alternating sinusoidal voltage:

$u_G = u_{DB} = 8\sin\left(100\,\pi t - \frac{\pi}{3}\right)$ (u_G in V, t in s) (Figure 1).

As a result, the circuit carries an alternating sinusoidal current of value: $i = I_m \sin(100\,\pi t)$ (i in A, t in s).

An oscilloscope is connected so as to display, on channel Y_1, the voltage $u_{coil} = u_{AM}$ across the coil and, on channel Y_2, the voltage $u_R = u_{MB}$ across the resistor.

The knob «INV» (inverse) on channel Y_2 is pushed.

On the screen of the oscilloscope, we observe the waveforms (1) and (2) represented in Figure 2.

The vertical sensitivity S_v is the same on the two channels $S_v = 1V/\text{div}$.

Take: $0.32\,\pi = 1$.

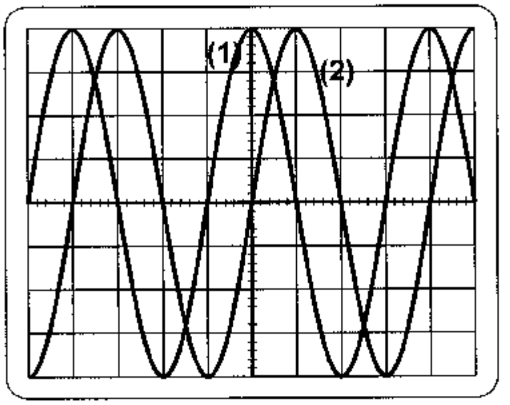


Figure 2

1. Why did we push in the knob «Inv»?
2. Referring to figure 2:
   a) determine the horizontal sensitivity S_h that is selected on the oscilloscope.
   b) determine the phase difference between u_{coil} and u_R.
   c) which of the two voltages leads the other?
   d) deduce that the coil has a negligible resistance.
   e) determine the value of I_m.
3. Determine the expression of u_L as a function of time t.
4. Show that expression of the voltage $u_C = u_{DA}$ across the capacitor is given by
   $u_C = -\dfrac{I_m}{100\,\pi C}\cos 100\,\pi t$.
5. Applying the law of addition of voltages, determine the value of C by giving t a particular value.

Part B

Second experiment

The capacitor, initially charged, is now connected across the coil of inductance $L = 0.32\,H$ (Figure 3).

The oscilloscope, adjusted on the horizontal sensitivity $S_h = 2\,ms/\text{div}$, allows to display the voltage u_C across the capacitor (Figure 4).

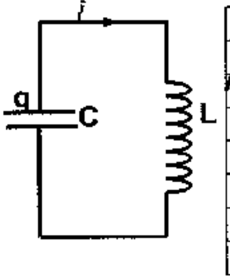


Figure 3

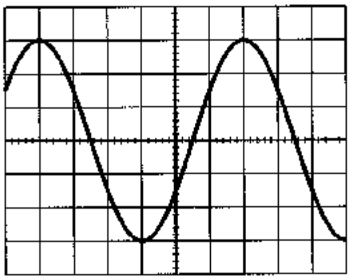


Figure 4

1. a) Show that the voltage u_C is sinusoidal of period T.

b) Determine T in terms of L and C.

2. Calculate the value of C.

VIII-GS 2006 1st
Determination of the Characteristics of a Coil

In order to determine the inductance L and the resistance r of a coil, we connect it in series with a capacitor of capacitance $C = 160\,\mu F$ across the terminals of a low frequency generator (LFG) delivering an alternating sinusoidal voltage:

$$u_g = u_{AD} = 20\sin(100\,\pi t) \quad (u \text{ in } V,\ t \text{ in } s)$$

The circuit thus carries an alternating sinusoidal current i.

An oscilloscope is connected so as to display the voltage $u_g = u_{AD}$ on the channel Y_A and the voltage $u_{Coil} = u_{BD}$ on the channel Y_B.

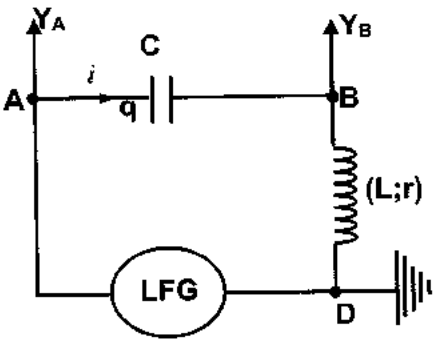


Figure 1

We see on the screen of the oscilloscope a display of the waveforms represented in figure 2.

1. Knowing that the vertical sensitivity S_v is the same on both channels, calculate its value.
2. Calculate the phase difference between u_{AD} and u_{BD}.

   Which of them lags behind the other?
3. Deduce the expression of the voltage u_{BD} across the terminals of the coil as a function of time.
4. Applying the law of addition of voltages, and giving the time t two particular values, verify that the voltage u_{AB} may be written as:

$$u_C = u_{AB} = 20\sin\left(100\,\pi t - \frac{\pi}{3}\right) \qquad (u_C \text{ in } V,\ t \text{ in } s).$$

5. Using the relation between the current i and the voltage u_C, determine the expression of i as a function of time.
6. a) Give the expression of the voltage u_{BD} across the terminals of the coil as a function of i.

   b) Calculate r and L by giving t two particular values.
7. In order to verify the preceding calculated values of L and r, we proceed in the following way:

   ✘ we measure the average power consumed in the circuit, for $w = 100\,\pi$ (rad/s) and we obtain $8.66\,W$;

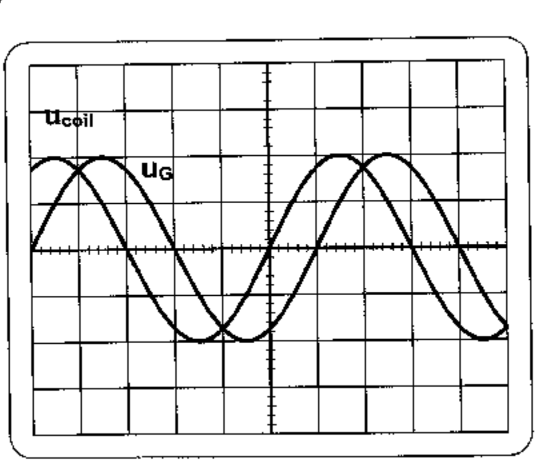


Figure 2

   ✘ we keep the maximum value of u_g constant but we vary its frequency f; for $f = 71\,Hz$, the effective value of the current in the circuit is maximum.

   Determine the values of r and L.

IX-GS 2005 2nd

Identification of an Electric Component

We intend to exploit a waveform and identify an electric component (D) of physical characteristic X. (D) may be:

- a resistor of resistance $X = R_1$;
- or a capacitor of capacitance $X = C$;
- or a coil of inductance $X = L$ and negligible resistance.

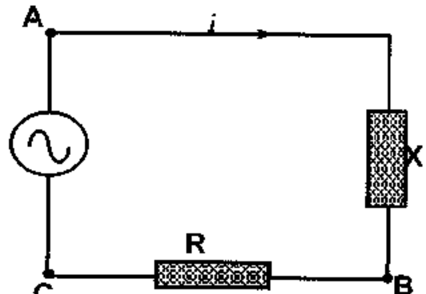


Figure 1

In order to do that, we connect (D) in series with a resistor of resistance $R = 40\,\Omega$ across a generator delivering across its terminals an alternating sinusoidal voltage:

$$u_g = u_{AC} = 4\sqrt{2}\cos(100\,\pi t)\ (u \text{ in } V,\ t \text{ in } s).$$

The circuit thus carries an alternating sinusoidal current i.

An oscilloscope, conveniently connected, displays the waveforms that represent the variation, as a function of time, of the voltage $u_{AC} = u_g$ on channel (1) and that of the voltage $u_{BC} = u_R$ across the resistor on channel 2 (figure 2).

The vertical sensitivity on channel 2 is $2\,V/\text{div}$.

1. a) Redraw the figure 1 showing the connections of the oscilloscope.
   b) Calculate the value of the period T of the voltage u_g.
2. a) Determine the horizontal sensitivity of the oscilloscope.
   b) The waveform of u_{BC} represents the «image» of the current i. Why?
3. a) Specify the nature of the component (D). Justify your answer.
   b) Determine the phase difference between u_{AC} and u_{BC}.
4. a) Determine the maximum value I_m of the current i.
   b) Write the expression of i as a function of time.
   c) Show that u_{AB} may be written in the form $u_{AB} = \dfrac{0.1}{100\,\pi X}\sin\left(100\,\pi t + \dfrac{\pi}{4}\right)$.
5. Applying the law of addition of voltages, determine X by giving t a particular value.

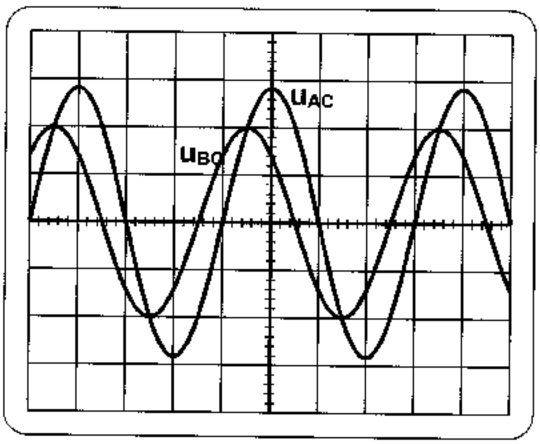


Figure 2

Solutions

I-GS 2013 2nd

Part A

1. a) According to Ohm's law $u_{AB} = Ri = RI\sqrt{2}\sin(2\pi f t)$.

b) $u_{BD} = ri + L\dfrac{di}{dt} = rI\sqrt{2}\sin(2\pi f t) + 2\pi L I\sqrt{2} f\cos(2\pi f t)$.

2. $u_{DM} = \dfrac{1}{C}\int idt = -\dfrac{I\sqrt{2}}{2\pi f C}\cos(2\pi f t)$.

Part B

1. Law of addition of voltages $u_{AM} = u_{AB} + u_{BD} + u_{DM}$;

$$RI\sqrt{2}\sin(2\pi f t) + rI\sqrt{2}\sin(2\pi f t) + 2\pi LI\sqrt{2}f\cos(2\pi f t) - \frac{I\sqrt{2}}{2\pi f C}\cos(2\pi f t)$$

$$= U\sqrt{2}\sin(2\pi f t + \varphi)$$

$$(R+r)I\sqrt{2}\sin(2\pi f t) + \left(2\pi Lf - \frac{1}{2\pi f C}\right)I\sqrt{2}\cos(2\pi f t) = U\sqrt{2}\sin(2\pi f t + \varphi);$$

Let $2\pi f t = 0$; $(R+r)I\sqrt{2}\sin(0) + \left(2\pi f L - \dfrac{1}{2\pi f C}\right)I\sqrt{2}\cos(0) = U\sqrt{2}\sin(\varphi)$

Then $\left(2\pi f L - \dfrac{1}{2\pi f C}\right)I = U\sin(\varphi)$ ....(1)

Let $2\pi f t = \dfrac{\pi}{2}$; $(R+r)I\sqrt{2}\sin\left(\dfrac{\pi}{2}\right) + \left(2\pi Lf - \dfrac{1}{2\pi f C}\right)I\sqrt{2}\cos\left(\dfrac{\pi}{2}\right) = U\sqrt{2}\sin\left(\dfrac{\pi}{2} + \varphi\right)$;

Then $(R+r)I = U\cos(\varphi)$.................................(2)

a) Squaring then adding $(1)^2 + (2)^2$ we get:

$$\left(2\pi f L - \frac{1}{2\pi f C}\right)^2 I^2 + (R+r)^2 I^2 = U^2\sin^2(\varphi) + U^2\cos^2(\varphi) = U^2\left(\sin^2(\varphi) + \cos^2(\varphi)\right);$$

Then $\left[\left(2\pi f L - \dfrac{1}{2\pi f C}\right)^2 + (R+r)^2\right]I^2 = U^2$;

Thus $I = \dfrac{U}{\sqrt{(R+r)^2 + \left(2\pi f L - \dfrac{1}{2\pi f C}\right)^2}}$.

b) The ratio $\dfrac{(1)}{(2)}$: $\dfrac{U\sin\varphi}{U\cos\varphi} = \dfrac{\left(2\pi f L - \dfrac{1}{2\pi f C}\right)I}{(R+r)I}$; then $\tan\varphi = \dfrac{2\pi f L - \dfrac{1}{2\pi f C}}{R+r}$.

2. a) Electrical current resonance.

b) $I = I_0 = 160\ mA$.

c) Interpretation

i- For $f = f_0$, u_{AM} & i are in phase then $\varphi = 0 \Rightarrow \tan\varphi = 0$; $2\pi f_0 L - \dfrac{1}{2\pi f_0 C} = 0$;

Thus, $4\pi^2 f_0^2 LC = 1$.

ii- Replacing $2\pi f_0 L - \dfrac{1}{2\pi f_0 C} = 0$ in (1) we get $I = \dfrac{U}{\sqrt{0^2 + (R+r)^2}} \Rightarrow U = (R+r)I$;

Then the circuit is equivalent to a resistor of resistance $R_t = R + r$.

d) We have $4\pi^2 f_0^2 LC = 1$; $L = \dfrac{1}{4\pi^2 f_0^2 C} = \dfrac{1}{4\pi^2 (110)^2 \times 21 \times 10^{-6}} = 0.1H$.

We have also $U = (R+r)I$; $r = \dfrac{U}{I} - R = \dfrac{8}{160 \times 10^{-3}} - 30 = 20\,\Omega$.

II-GS 2013 1st

Part A

1. The voltage across the generator $U_0 = U_G = S_{v_A} \times y_A = 5V/\text{div} \times 2.4\text{div} = 12\,V$.
2. a) The voltage across the resistor $U_R = S_{v_B} \times y_B = 2V/\text{div} \times 2.8\text{div} = 5.6V$;

   According to Ohm's law $U_R = RI \Rightarrow I = \dfrac{U_R}{R} = \dfrac{5.6V}{10\,\Omega} = 0.56\,A$.

   b) In the steady state $I = 0.56\,A \neq 0$, then (D) is not a capacitor.

   c) According to the law of addition of voltages:
   $U_G = U_R + U_D \Rightarrow U_D = U_G - U_R = 12 - 5.6V = 6.4V$;

   If (D) is a coil then it acts as a resistor in steady state $U_D = r_D i + L\dfrac{di}{dt} = r_D I$ or a resistor

   $U_D = r_D I$; then $U_D = r_D I \Rightarrow r_D = \dfrac{U_D}{I} = \dfrac{6.4V}{0.56\,A} = 11.43\,\Omega$.

Part B

1. a) If (D) is a resistor, then u_G and u_R would be in phase; then (D) is neither a resistor nor a capacitor. Thus, (D) is a coil.

   b) Due to the inductive effect of the coil, the voltage across the generator u_G should lead the current i, whose image is u_R.

   Then the waveform (2) represents the variations of the voltage across the resistor.
2. The maximum voltage of the generator $U_m = S_{v_A} \times y_{\max} = 5V/\text{div} \times 2.4\text{div} = 12\,V$;

   The period $T = S_h \times x = 2\,ms/\text{div} \times 10\text{div} = 20\,ms = 20 \times 10^{-3}\,s$;

   The angular frequency $w = \dfrac{2\pi}{T} = \dfrac{2\pi}{20 \times 10^{-3}} = 100\,\pi\,(\text{rad}/s)$.

   Then $u_G = 12\sin(100\,\pi t)$ (where t in s and u_G in V)
3. The voltage across the resistor (figure 3) lags that across the generator, then $u_R = (U_R)_m \sin(wt - \varphi)$.

   The maximum voltage across the resistor $(U_R)_m = S_{vB} \times y_{\max} = 1V/\text{div} \times 3.2\text{div} = 3.2V$;

   The phase difference $\varphi = 2\pi\dfrac{d}{D} = 2\pi \times \dfrac{1.5\,\text{div}}{10\,\text{div}} = \dfrac{3\pi}{10} \approx 0.94\,(\text{rad})$;

   Then $u_R = 3.2\sin\left(100\,\pi t - \dfrac{3\pi}{10}\right)$ (where t in s and u_R in V)

According to Ohm's law $u_R = R\,i \Rightarrow i = \frac{u_R}{R} = 0.32\sin\left(100\,\pi t - \frac{3\pi}{10}\right)$ (where t in s and i in A)

4. The voltage across the coil:

$$u_L = r\,i + L\frac{di}{dt} = 0.32\,r\sin\left(100\,\pi t - \frac{3\pi}{10}\right) + L \times 0.32 \times 100\,\pi\cos\left(100\,\pi t - \frac{3\pi}{10}\right);$$

$$= 0.32\,r\sin\left(100\,\pi t - \frac{3\pi}{10}\right) + 32\;\pi L\cos\left(100\,\pi t - \frac{3\pi}{10}\right)\ (t \text{ in } s\ \&\ u_L \text{ in } V)$$

Law of addition of voltages $u_G = u_L + u_R$;

$$u_G = 12\sin(100\,\pi t) = 3.2\sin\left(100\,\pi t - \frac{3\pi}{10}\right) + 0.32\,r\sin\left(100\,\pi t - \frac{3\pi}{10}\right) + 32\,\pi L\cos\left(100\,\pi t - \frac{3\pi}{10}\right)$$

✖ Let $100\,\pi t = 0$; $12\sin(0) = 3.2\sin\left(-\frac{3\pi}{10}\right) + 0.32\,r\sin\left(-\frac{3\pi}{10}\right) + 32\,\pi L\cos\left(-\frac{3\pi}{10}\right)$;

$$32\,\pi L\cos\left(\frac{3\pi}{10}\right) = 0.32(10 + r)\sin\left(\frac{3\pi}{10}\right);$$

✖ Let $100\,\pi t = \frac{3\pi}{10}$;[3] $12\sin\left(\frac{3\pi}{10}\right) = 3.2\sin 0 + 0.32\,r\sin 0 + 32\pi L\cos 0$;

$$L = \frac{12}{32\,\pi}\sin\left(\frac{3\pi}{10}\right) \approx 0.097\,H.$$

✖ $32\,\pi L\cos\left(\frac{3\pi}{10}\right) = 0.32(10 + r)\sin\left(\frac{3\pi}{10}\right) \Rightarrow 12\sin\left(\frac{3\pi}{10}\right)\cos\left(\frac{3\pi}{10}\right) = 0.32(10 + r)\sin\left(\frac{3\pi}{10}\right)$;

But $12\cos\left(\frac{3\pi}{10}\right) = 0.32(10 + r)$; then $r = \frac{12}{0.32}\cos\left(\frac{3\pi}{10}\right) - 10 = 12.04\,\Omega$;

5. a) The electric resonance phenomenon.

b) The circuit's proper period $T_0 = 20\,ms$ & $T_0 = 2\pi\sqrt{LC}$;

So $L = \frac{T_0^2}{4\pi^2 C} = \frac{(20 \times 10^{-3})^2}{4\pi^2 \times 10^{-4}} = 0.1H$

Under resonance the circuit acts as a resistor $(U_G)_m = (R + r)I_m$;

$$(U_G)_m = 12V \ \&\ I_m = \frac{(U_R)_m}{R} = \frac{S_v \times y}{R} = \frac{2V/\text{div} \times 2.8\,\text{div}}{10\,\Omega} = 0.56A;$$

$$R + r = \frac{(U_G)_m}{I_m} \Rightarrow R = \frac{12}{0.56} - 10 = 11.43\,\Omega.$$

III-GS 2012 2nd

1. From the waveforms $S_v = \frac{U_{max}}{y_{max}} = \frac{20\,V}{2\,\text{div}} = 10\,V/\text{div}$.

2. The phase difference $|\varphi| = 2\pi\frac{d}{D} = 2\pi \times \frac{1\text{div}}{6\text{div}} = \frac{\pi}{3}(\text{rad})$. u_C lags u_G.

3. The instantaneous expression $u_C = (U_C)_{max}\sin\left(100\,\pi t - \frac{\pi}{3}\right) = 20\sin\left(100\,\pi t - \frac{\pi}{3}\right)$.

---

[3] We can take also $100\,\pi t = \frac{\pi}{2}$, we get a 2nd equation with two unknowns.

4. We have: $i = C\dfrac{du_C}{dt} = C\times 20\times 100\,\pi \sin\left(100\,\pi\, t - \dfrac{\pi}{3}\right)$;

So $i = 160\times 10^{-6}\times 20\times 100\,\pi \cos\left(100\,\pi\, t - \dfrac{\pi}{3}\right) \Rightarrow i = \cos\left(100\,\pi\, t - \dfrac{\pi}{3}\right)$.

5. Law of addition of voltages: $u_{AD} = u_{AB} + u_{BD} \Rightarrow u_g = ri + L\dfrac{di}{dt} + u_C$

$$20\sin(100\,\pi\, t) = r\cos\left(100\,\pi\, t - \frac{\pi}{3}\right) - 100\,\pi\, L\sin\left(100\,\pi\, t - \frac{\pi}{3}\right) + 20\sin\left(100\,\pi\, t - \frac{\pi}{3}\right);$$

✖ For $100\,\pi\, t = \dfrac{\pi}{3}$; we get: $20\sin\left(\dfrac{\pi}{3}\right) = r\cos(0) - 100\,\pi\, L\sin(0) + 20\sin(0)$;

$$20\times\frac{\sqrt{3}}{2} = r \Rightarrow r = 10\sqrt{3}\,\Omega.$$

✖ For $100\,\pi\, t = 0$; we get: $20\sin(0) = r\cos\left(-\dfrac{\pi}{3}\right) - 100\,\pi\, L\sin\left(-\dfrac{\pi}{3}\right) + 20\sin\left(-\dfrac{\pi}{3}\right)$

$$0 = 10\sqrt{3}\times\frac{1}{2} + 100\,\pi\, L\times\frac{\sqrt{3}}{2} - 20\times\frac{\sqrt{3}}{2} \Rightarrow 10\,\pi L = 1\,;\, L = \frac{1}{10\pi} = 0.032\ H$$

6. The electric power consumed in the resistor of the coil is: $P = r\, I_{eff}^2 = r\left(\dfrac{I_m}{\sqrt{2}}\right)^2$;[4]

$$8.66 = \left(\frac{1}{\sqrt{2}}\right)^2 \times r \Rightarrow r = 17.3\,\Omega.$$

The circuit is the seat of current resonance so the frequency is equal to the proper frequency

$f_0 = \dfrac{1}{2\pi\sqrt{LC}}$, then $L = \dfrac{1}{4\pi^2 f_0^2 C} = 0.03\,H$.

IV-GS 2011 1st
Part A

1. a) We have $i = \dfrac{dq}{dt}$ & $q = C u_C$, so $i = C\dfrac{du_C}{dt}$.

b) $u_C = \dfrac{1}{C}\int i\,dt = \dfrac{1}{C}\int I\sqrt{2}\sin(wt+\varphi)dt = -\dfrac{I\sqrt{2}}{C\,w}\cos(wt+\varphi)$.

c) $(U_C)_{\max} = \dfrac{I\sqrt{2}}{C\,w} = \sqrt{2}\,U_C$, then $U_C = \dfrac{I}{C\,w}$.

---

[4]The total power consumed in the circuit can be calculated by: $P = (U_g)_{eff}\, I_{eff}\cos\varphi$; where φ is the phase difference between $u_g = u_{AD} = 20\sin(100\,\pi\, t)$ & $i = \cos\left(100\,\pi t - \dfrac{\pi}{3}\right)$;

To get φ the two expressions should be identical:

$i = \cos\left(100\,\pi t - \dfrac{\pi}{3}\right) = \sin\left(100\,\pi t - \dfrac{\pi}{3} + \dfrac{\pi}{2}\right) = \sin\left(100\,\pi t + \dfrac{\pi}{6}\right)$; then $|\varphi| = \dfrac{\pi}{6}$ rad;

Thus $P = (U_g)_{eff}\, I_{eff}\cos\varphi = \dfrac{20}{\sqrt{2}}\times\dfrac{1}{\sqrt{2}}\cos\left(\dfrac{\pi}{6}\right) = 5\sqrt{3}\,W \approx 8.66\,W$.

2. Law of addition of voltages:

$$u_{AB} = u_{AD} + u_{DB}\ ;\ U\sqrt{2}\sin(wt) = -\frac{I\sqrt{2}}{Cw}\cos(wt+\varphi) + RI\sqrt{2}\sin(wt+\varphi)\ ;$$

Let $wt = 0$; $U\sqrt{2}\sin(0) = -\frac{I\sqrt{2}}{Cw}\cos(\varphi) + RI\sqrt{2}\sin(\varphi) = 0$;

Then $\frac{I\sqrt{2}}{Cw}\cos(\varphi) = RI\sqrt{2}\sin(\varphi) \Rightarrow \frac{\sin\varphi}{\cos\varphi} = \frac{1}{RCw}$; thus, $\tan(\varphi) = \frac{1}{RCw}$.

Part B

1. a) The Connections are shown on the adjacent circuit.

b) Properties of the circuit:

i- The period $T = S_h \times x = 1ms/\text{div} \times 6\,\text{div} = 6\,ms = 6\times10^{-3}\,s$

The frequency $f = \frac{1}{T} = \frac{1}{6\times10^{-3}} = \frac{1000}{6} = \frac{500}{3}\,Hz$.

ii- (a) leads (b) since it reaches its maximum first.

iii-Due to the capacitive effect in the circuit, u_{AB} must lag behind the current i (whose image is u_{DB}), thus (a) displays u_{DB}.

iv- The phase difference $\varphi = 2\pi\frac{d}{D} = 2\pi\times\frac{1\text{div}}{6\text{div}} = \frac{\pi}{3}(\text{rad})$.

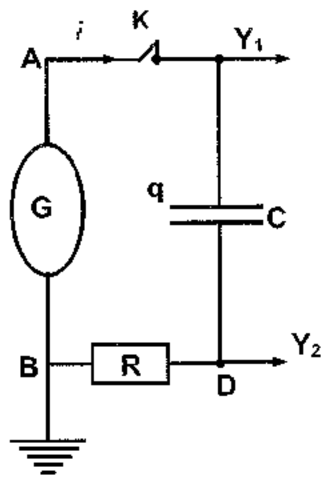


Figure 1

c) We have $\tan(\varphi) = \frac{1}{RCw} \Rightarrow C = \frac{1}{R\,2\pi f\tan\varphi}$;

Then $C = \frac{1}{R\,2\pi f\tan\varphi} = \frac{3}{250\times2\pi\times500\sqrt{3}} \approx 2.2\times10^{-6}\,F$.

Part C

The voltmeters measure the effective voltages.

1st method:

$U_R = RI \Rightarrow I = \frac{U_R}{R} = \frac{2.2}{250} = 8.8\times10^{-3}\,A$;

And according to part A- 1.c) $U_C = \frac{I}{Cw}$, so $C = \frac{I}{U_C 2\pi f} = \frac{8.8\times10^{-3}}{3.2\times2\times\pi\times200} = 2.19\times10^{-6}\,F$.

2nd method: (Without calculating I)

We have $U_R = RI$ and $U_C = \frac{I}{Cw}$; by dividing them we get $\frac{U_C}{U_R} = \frac{\frac{I}{Cw}}{RI} = \frac{1}{RCw} \Rightarrow C = \frac{U_R}{U_C R w}$;

Then $C = \frac{U_R}{U_C R w} = \frac{2.2}{3.2\times250\times2\pi\times200} \approx 2.19\times10^{-6}\,F = 2.2\mu F$.

V-GS 2010 2nd

Part A

1. (D_1) is a capacitor.

2. The power consumed is given by $P = rI^2$; $r = \frac{P}{I^2} = 5\Omega$.

Part B

1. The power consumed $P_1 = U_1 I_1 \cos\varphi_1 = 0 \Rightarrow \cos\varphi_1 = 0 \Rightarrow |\varphi_1| = \frac{\pi}{2}(\text{rad})$;

$P_2 = U_2 I_2 \cos\varphi_2 \Rightarrow \cos\varphi_1 = \frac{\sqrt{2}}{2} \Rightarrow |\varphi_2| = \frac{\pi}{4}(\text{rad})$.

2. In the capacitive circuit i_1 leads u, so $i_1 = I_{1m}\sin\left(100\,\pi t + \frac{\pi}{2}\right) = 0.05\sqrt{2}\sin\left(100\,\pi t + \frac{\pi}{2}\right)$;

In the inductive circuit i_2 lags behind u, so $i_2 = I_{2m}\sin\left(100\,\pi t - \frac{\pi}{4}\right) = 1\sin\left(100\,\pi t - \frac{\pi}{4}\right)$.

3. a) $i = \frac{dq}{dt}$ & $q = C u_C$ then : $i = C\frac{du_C}{dt}$;

But according to the law of uniqueness of voltage $u_C = u$; then $i = C\frac{du}{dt}$.

b) **1st method: (by integration)**

According to the law of uniqueness of voltage $u = u_C = \frac{1}{C}\int i_1\, dt$;

Then $u = 5\sqrt{2}\sin(100\,\pi t) = 0.05\sqrt{2} \times \frac{1}{C} \times \frac{-1}{100\,\pi}\cos\left(100\,\pi t + \frac{\pi}{2}\right)$;

$$\text{Thus, } C = \frac{0.05\sqrt{2}}{5\sqrt{2} \times 100\,\pi} = \frac{0.05\sqrt{2} \times 0.32}{5\sqrt{2} \times 100} = 32 \times 10^{-6} F = 32\,\mu F.$$

2nd method: (by derivative)

$u = u_C$ & $i_1 = C\frac{du}{dt}$; $0.05\sqrt{2}\sin\left(100\,\pi t + \frac{\pi}{2}\right) = C \times 5\sqrt{2} \times 100\,\pi t\cos(100\,\pi t)$;

$$0.05\sqrt{2}\cos(100\,\pi t) = C \times 5\sqrt{2} \times 100\,\pi t\cos(100\,\pi t);$$

$$\text{Then } C = \frac{0.05\sqrt{2}}{5\sqrt{2} \times 100\,\pi} = \frac{0.05\sqrt{2} \times 0.32}{5\sqrt{2} \times 100} = 32 \times 10^{-6} F = 32\,\mu F.$$

4. a) $u = r\,i_2 + L\frac{di_2}{dt}$.

b) $u = 5\sqrt{2}\sin(100\,\pi t) = 5\sin\left(100\,\pi t - \frac{\pi}{4}\right) + L \times 100\pi\cos\left(100\,\pi t - \frac{\pi}{4}\right)$;

Let $100\,\pi t = 0 \Rightarrow 5\sqrt{2}\sin\left(\frac{\pi}{4}\right) = 5\sin(0) + L \times 100\pi\cos(0)$;

$$L = \frac{5}{100\,\pi} = \frac{5 \times 0.32}{100} = 0.016\,H.$$

Part C

1. Current resonance.

2. Under resonance $f = f_0 = \frac{1}{2\pi\sqrt{LC}} \Rightarrow C = \frac{1}{4\pi^2 f_0^2 L} = \frac{0.32^2}{4 f_0^2 L} = 31.6 \times 10^{-6} F = 31.6\mu F$.

VI-GS 2009 1st

Part A

1. Connections on circuit.

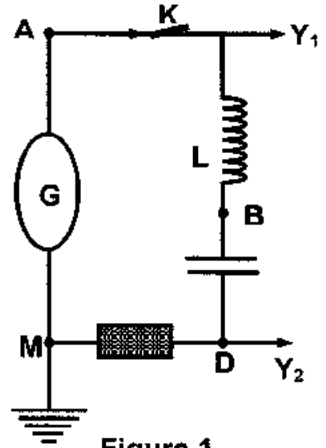


Figure 1

2. a) The period $T_1 = S_h \times x = 0.1 ms/\text{div} \times 8\text{div} = 0.8\, ms$

The frequency $f_1 = \frac{1}{T_1} = \frac{1}{0.8\times 10^{-3}} = 1250\ Hz$.

b) The phase difference $|\varphi_1| = 2\pi \frac{d}{D} = 2\pi \times \frac{1\text{div}}{8\text{div}} = \frac{\pi}{4}(\text{rad})$.

3. $i = I_m \cos(2\pi f_1 t - \varphi_1)$.

a) $u_{AB} = L\frac{di}{dt} = -2\pi f_1 L I_m \cos(2\pi f_1 t - \varphi_1)$;

$u_{BD} = \frac{1}{C}\int i dt = \frac{I_m}{C 2\pi f_1}\cos(2\pi f_1 t - \varphi_1)$;

& $u_{DM} = Ri = R I_m \sin(2\pi f_1 t - \varphi_1)$.

b) Referring to the relation $u_{AM} = u_{AB} + u_{BD} + u_{DM}$;

$$U_m \cos(2\pi f_1 t) = -2\pi f_1 L I_m \sin(2\pi f_1 t - \varphi_1) + \frac{I_m}{2\pi f_1 C}\sin(2\pi f_1 t - \varphi_1) + R I_m \cos(2\pi f_1 t - \varphi_1)$$

For $2\pi f_1 t = \frac{\pi}{2}$;

$$U_m \cos\left(\frac{\pi}{2}\right) = -2\pi f_1 L I_m \sin\left(\frac{\pi}{2} - \varphi_1\right) + \frac{I_m}{2\pi f_1 C}\sin\left(\frac{\pi}{2} - \varphi_1\right) + R I_m \cos\left(\frac{\pi}{2} - \varphi_1\right);$$

$$0 = -2\pi f_1 L I_m \cos(\varphi_1) + \frac{I_m}{2\pi f_1 C}\cos(\varphi_1) + R I_m \sin\varphi_1;$$

$$\left(2\pi f_1 L - \frac{1}{2\pi f_1 C}\right)\cos(\varphi_1) = R\sin\varphi_1; \text{ then } \tan\varphi_1 = \frac{2\pi f_1 L - \frac{1}{2\pi f_1 C}}{R}.$$

Part B

The circuit is the seat of current resonance if u_G & i are in phase, then $\varphi_1 = 0 \Rightarrow \tan\varphi_1 = 0$;

$2\pi f_0 L = \frac{1}{2\pi f_0 C}$; we get $4\pi^2 f_0^2 LC = 1$; thus $f_0 = \frac{1}{2\pi\sqrt{LC}}$.

Part C

1. We have $\varphi_2 = -\varphi_1 \Rightarrow \tan\varphi_2 = -\tan\varphi_1$; $\frac{2\pi f_2 L - \frac{1}{2\pi f_2 C}}{R} = -\frac{2\pi f_1 L - \frac{1}{2\pi f_1 C}}{R}$;

$$2\pi f_2 L - \frac{1}{2\pi f_2 C} = -2\pi f_1 L + \frac{1}{2\pi f_1 C} \Rightarrow 2\pi f_2 L + 2\pi f_1 L - \frac{1}{2\pi f_2 C} - \frac{1}{2\pi f_1 C} = 0;$$

$$2\pi L(f_2 + f_1) - \frac{1}{2\pi C}\left(\frac{1}{f_2} + \frac{1}{f_1}\right) = 0 \Rightarrow 2\pi L(f_2 + f_1) - \frac{1}{2\pi C}\left(\frac{f_2 + f_1}{f_2 f_1}\right) = 0;$$

But $f_2 + f_1 \neq 0$; then $2\pi L - \frac{1}{2\pi C}\times\frac{1}{f_2 f_1} = 0$, so $f_2 f_1 = \frac{1}{4\pi^2 LC}$; thus $f_2 f_1 = f_0^2$.

2. We have $f_2=\frac{f_0^2}{f_1}=\frac{500^2}{1250}=200\ Hz$.

Part D

We have $4\pi^2 f_0^2 LC=1 \Rightarrow 4\times 10\times 500^2 LC=1$; then $LC=10^{-7}(H.F)$;

When $\varphi_1=\frac{\pi}{4}$ & $\tan\varphi_1\times R=2\pi f_1 L-\frac{1}{2\pi f_1 C}$; $1\times 650=2\pi\times 1250\times L-\frac{1}{2\pi\times 1250\times C}$;

$$650=2500\,\pi L-\frac{1}{2500\,\pi C};\ 650=\frac{2500^2\pi^2 LC-1}{2500\,\pi C}=\frac{2500^2\times 10^{-6}-1}{2500\,\pi C}$$

Then $C=\frac{2500^2\times 10^{-6}-1}{2500\,\pi\times 650}=10^{-6}F=1\mu F$;

& $L\times 10^{-6}=\frac{10^{-6}}{0.1}$; thus $L=0.1H$.

VII-GS 2008 1st

Part A

1. The connections of the oscilloscope displays the voltage u_{BM}, but as we want to display $u_{MB}=-u_{BM}$, then we have to push the knob inversion «INV».
To eliminate the phase opposition obtained due to the connections of the oscilloscope.

2. a) We have $w=100\,\pi\,\text{rad}/s$ and the period is $T=\frac{2\pi}{w}=0.02\,s=20\,ms$.

Then the horizontal sensitivity $S_h=\frac{T}{x}=\frac{20\,ms}{4\,\text{div}}=5\,ms/\text{div}$.

b) We have $|\varphi|=2\pi\times\frac{1\text{div}}{4\text{div}}=\frac{\pi}{2}\text{rad}$.

c) u_L leads u_R.

d) The voltage across the coil of negligible resistance leads the current by an angle of $\frac{\pi}{2}$.

e) $(U_R)_{\max}=S_v\times y=4\text{div}\times 1V/\text{div}=4V$.

Ohm's law $(U_R)_{\max}=RI_m\Rightarrow I_m=\frac{4}{100}=0.04\,A$.

3. Ohm's law across the coil: $u_L=L\frac{di}{dt}=0.32\times 0.04\times 100\,\pi\cos(100\,\pi t)=4\cos(100\,\pi t)$.

4. We have $i=\frac{dq}{dt}=C\frac{du_C}{dt}$ then $u_C=\frac{1}{C}\int i\,dt=-\frac{I_m}{100\pi C}\cos(100\pi t)$;

$u_C=-\frac{1.28\times 10^{-4}}{C}\cos(100\,\pi t)$. $\quad$ (u_C in V, t in s)

5. Law of addition of voltages $u_{DB}=u_{DA}+u_{AM}+u_{MB}$;

$8\sin\left(100\,\pi t-\frac{\pi}{3}\right)=-\frac{1.28\times 10^{-4}}{C}\cos(100\,\pi t)+4\cos(100\,\pi t)+4\sin(100\,\pi t)$

Let $100\,\pi t=0$ we get: $-4\sqrt{3}=-\frac{1.28\times 10^{-4}}{C}+4+0$; then $C=11.7\times 10^{-6}F$.

Part B

1. We have $u_C = u_b \Rightarrow u_C = L\left(\frac{di}{dt}\right) = -L\,q'' = -LC\,u''_C \Rightarrow u''_C + \frac{1}{LC}u_C = 0$.

   The differential equation that governs the variations of u_C, is of 2nd order of the form $u''_C + w^2 u_C = 0$, then voltage of u_C is sinusoidal of period T.

2. a) The angular frequency w of these oscillations is $w = \frac{1}{\sqrt{LC}}$ & the period is $T = \frac{2\pi}{w} = 2\pi\sqrt{LC}$

   b) From the waveform of figure 4 we have: $T = 6\,\text{div} \times 2ms/\text{div} = 12\,ms = 0.012\,s$.

   Then $C = \frac{T^2}{4\pi^2 L} = \frac{144\times10^{-6}}{12.5} = 11.5\times10^{-6}\,F$.

VIII-GS 2006 1st

1. The vertical sensitivity $S_v = \frac{U_{\max}}{y_{\max}} = \frac{20\,V}{2\,\text{div}} = 10\,V/\text{div}$.

2. The phase difference $|\varphi| = 2\pi\times\frac{d}{D} = 2\pi\frac{1\text{div}}{6\,\text{div}} = \frac{\pi}{3}\,\text{rad}$.

   Referring to the waveforms u_g lags behinds u_{coil}.

3. We have $(U_m) = 20\,V$, $w = 100\,\pi(\text{rad/s})$ and u_{coil} leads u_g by $\frac{\pi}{3}\,\text{rad}$;

   Then $u_{\text{coil}} = 20\sin\left(100\,\pi t + \frac{\pi}{3}\right)$ (u in V, t in s)

4. Law of addition of voltages $u_{AD} = u_{AB} + u_{BD}$; let $u_{AB} = A\sin(100\,\pi t + \varphi)$.

   $20\,\sin(100\pi t) = A\sin(100\,\pi t + \varphi) + 20\,\sin\left(100\pi t + \frac{\pi}{3}\right)$;

   For $100\pi t = 0$, we get: $0 = A\sin\varphi + 20\frac{\sqrt{3}}{2} \Rightarrow A\sin\varphi = -10\sqrt{3}$;

   For $100\pi t = \frac{\pi}{2}$ (rad), we get: $20 = A\cos\varphi + 10 \Rightarrow A\cos\varphi = 10$.

   Dividing the previous results we get $\tan\varphi = -\sqrt{3} \Rightarrow \varphi = -\frac{\pi}{3}(\text{rad})$ & $A = 20\,V$.

   Then $u_{AB} = 20\sin\left(100\,\pi t - \frac{\pi}{3}\right)$ (u_{AB} in V, t in s).

5. $i = C\frac{du_C}{dt} = 160\times10^{-6}\left[20\times100\pi\cos\left(100\pi t - \frac{\pi}{3}\right)\right] = \cos\left(100\pi t - \frac{\pi}{3}\right)$;

   $i = \sin\left(100\pi t + \frac{\pi}{6}\right)$ where (i in A, t in s).

6. a) $u_{coil} = r\,i + L\frac{di}{dt} = 20\sin\left(100\,\pi t + \frac{\pi}{3}\right)$.

   b) Law of uniqueness of voltages, across the coil:

   $20\sin\left(100\,\pi t + \frac{\pi}{3}\right) = r\sin\left(100\,\pi t + \frac{\pi}{6}\right) + 100\,\pi L\sin\left(100\,\pi t + \frac{\pi}{6}\right)$;

For $100\pi t = 0$, we get $20\frac{\sqrt{3}}{2} = r(0.5) + 100\,\pi L\frac{\sqrt{3}}{2}$; $10\sqrt{3} = r + 100\,\pi L\sqrt{3}$;

For $100\pi t = \frac{\pi}{2}$ (rad), we get $20 = r\sqrt{3} - 100\,\pi L$;

Then $L = \frac{1}{10\pi} = 0.032\ H$ & $r = 10\sqrt{3}\,\Omega$.

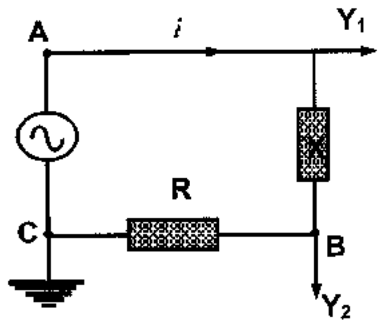
7. The electric power is consumed only in the resistor of the coil: $P = r\left(I_{eff}\right)^2 = 8.66\,W = \left(\frac{1}{\sqrt{2}}\right)^2 r$;

Then $r = 8.66\left(\sqrt{2}\right)^2 = 17.3\Omega = 10\sqrt{3}\,\Omega$.

The observed phenomenon is the current resonance, and we have:

$$LCw^2 = 1 \Rightarrow L = \frac{1}{Cw^2} = \frac{1}{160\times10^{-6}\times(142\,\pi)^2} = 0.032\,H.$$

IX-GS 2005 2nd



1. The connections are shown on circuit.

2. a) We have $w = 100\,\pi = \frac{2\pi}{T}$, so $T = 2\times10^{-2} = 20\,ms$.

b) The vertical sensitivity $S_v = \frac{T}{x} = \frac{20\,ms}{4\,\text{div}} = 5\,ms/\text{div}$.

3. a) According to Ohm's law $u_{BC} = Ri$, the voltage across the resistor is proportional to the current that flows through it.

Then u_{BC} is the image of the current.

b) According to the waveforms the current leads the voltage u_{AC}, then (D) is a capacitor.

4. a) The phase difference $|\varphi| = 2\pi\times\frac{0.5\,\text{div}}{4\,\text{div}} = \frac{\pi}{4}$ rad.

b) The maximum of the voltage across the resistor is: $U_{R\max} = S_v \times y_{\max} = 2\text{div}\times 2V/\text{div} = 4V$;

The maximum value of the current: $I_m = \frac{U_{R\max}}{R} = \frac{4}{40} = 0.1\,A$.

c) The expression of the current: $i = 0.1\cos\left(100\,\pi t + \frac{\pi}{4}\right)$.

5. We have $i = C\frac{du_C}{dt} \Rightarrow u_{AB} = u_C = \frac{1}{C}\int i\,dt = \frac{0.1}{100\,\pi C}\sin\left(100\,\pi t + \frac{\pi}{4}\right)$.

6. Law of addition of voltages: $u_{AC} = u_{AB} + u_{BC}$;

$$4\sqrt{2}\cos(100\,\pi t) = \frac{0.1}{100\,\pi C}\sin\left(100\,\pi t + \frac{\pi}{4}\right) + 4\cos\left(100\,\pi t + \frac{\pi}{4}\right);$$

Let $100\,\pi t = \frac{\pi}{4}$; $4\sqrt{2}\cos\left(\frac{\pi}{4}\right) = \frac{0.1}{100\,\pi C}\sin\left(\frac{\pi}{2}\right) + 4\cos\left(\frac{\pi}{2}\right)$;

$$4\sqrt{2}\left(\frac{\sqrt{2}}{2}\right) = \frac{0.1}{100\,\pi C} \Rightarrow C = \frac{0.1}{400\,\pi} = \frac{0.32\times0.1}{400} = 8\times10^{-5}F = 80\,\mu F.$$

Appendix 1

Exponential Function

We are interested to two particular types of exponential functions whose expressions are of the form $y = g(t) = A e^{-\frac{t}{\tau}}$ & $y = h(t) = A\left(1 - e^{-\frac{t}{\tau}}\right)$.

Derivatives

Let $U = U(t)$, and $y = e^{U}$, then $y' = \left(e^{U}\right)' = U' e^{U}$.

Examples:

$y = Ae^{-2t}$, then $y' = A \times (-2t)' e^{-2t} = -2Ae^{-2t}$;

$y = A\left(1 - e^{-\frac{t}{8}}\right)$, then $y' = A\left((1)' - \left(e^{-\frac{t}{8}}\right)'\right) = A\left(0 - \left(-\frac{t}{8}\right)'\right)e^{-\frac{t}{8}} = +\frac{A}{8}e^{-\frac{t}{8}}$.

Limits

Let $\lim\limits_{t \to +\infty} \left(e^{t}\right) = e^{+\infty} \longrightarrow +\infty$; $\lim\limits_{t \to +\infty} \left(e^{-t}\right) = e^{-\infty} \longrightarrow 0$.

Equations

If $e^{t} = a > 0$, then $t = \ell n(a)$.

1. First particular function $y = A\left(1 - e^{-\frac{t}{\tau}}\right)$

Consider the function $y = A\left(1 - e^{-\frac{t}{\tau}}\right)$ defined for $t \geq 0$ ($t \in [0; +\infty[$) where A & τ are constants.

For $t = 0$, $y = A\left(1 - e^{0}\right) = 0$ (it passes through origin);

$\lim\limits_{t \to +\infty} (y) = \lim\limits_{t \to +\infty} A\left(1 - e^{-\frac{t}{\tau}}\right) = A$, then the straight line of equation $y = A$ is a horizontal asymptote;

We can say that A is the limit value or value of y in the steady state.

Variation $\frac{dy}{dt} = y' = \frac{A}{\tau} e^{-\frac{t}{\tau}} > 0$, then it is increasing.

Physical interpretation

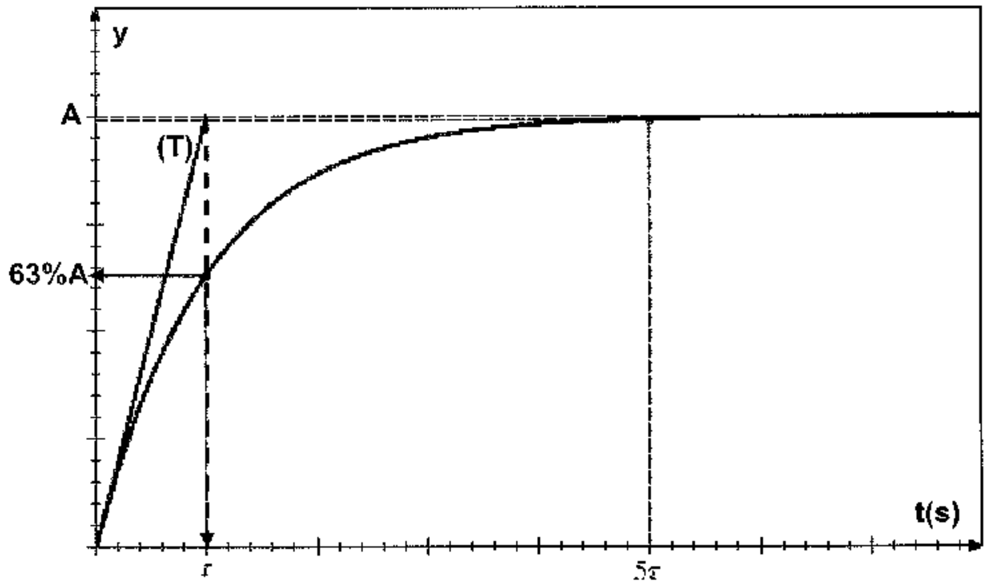
For $t = 5\tau$, $y = A\left(1 - e^{-\frac{5\tau}{\tau}}\right) = A\left(1 - e^{-5}\right) = 0.99\,A = 99\%A \approx A$;

Interpretation of 5τ

From physics perspective, after the duration $t = 5\tau$; y reaches 99% of its value in the steady state; thus we can say that $y = A$ is **practically** reached.

Time constant τ

For $t=\tau$, $y=A\left(1-e^{-\frac{\tau}{\tau}}\right)=A\left(1-e^{-1}\right)=0.63\,A=63\%A$;



Tangent

Equation of the tangent at the point of abscissa $t=0$ (at origin);

The equation of tangent is $y=at$ (it passes through origin) & $a=\left.\frac{dy}{dt}\right|_{t=0}=\left.\frac{A}{\tau}e^{-\frac{t}{\tau}}\right|_{t=0}=\frac{A}{\tau}$;

Then, the equation of the tangent (T) is: $y=\frac{A}{\tau}t$.

Intersection with the horizontal asymptote: $y=\frac{A}{\tau}t=A$, $t=\tau$.

Interpretation of τ

- Basing on the previous result we can define the time constant τ, as the duration needed by the variable y to reach 63% of its value in steady state A.
- The time constant τ, is the abscissa of the point of intersection of the tangent at origin with the horizontal asymptote $y=A$.

2. Second particular function $y=Ae^{-\frac{t}{\tau}}$

Consider the function $y=Ae^{-\frac{t}{\tau}}$, where A & τ are constants defined for $t\geq 0$ ($t\in[0;+\infty[$).

For $t=0$, $y=A\left(e^{0}\right)=A$;

$\lim\limits_{t\to+\infty}(y)=\lim\limits_{t\to-\infty}\left(Ae^{-\frac{t}{\tau}}\right)=0$, then the straight line of equation $y=0$ is a horizontal asymptote;

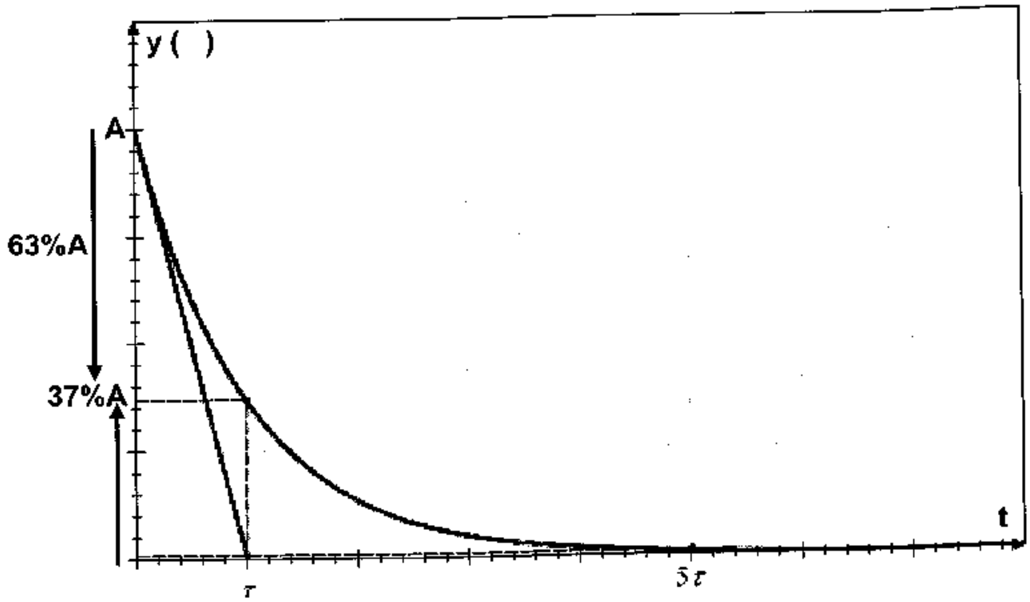
We can say that 0 is the **limit value** or value of y in the **steady state**.

Variation $\frac{dy}{dt}=y'=-\frac{A}{\tau}e^{-\frac{t}{\tau}}<0$, then it is decreasing.

Interpretation of 5τ

For $t = 5\tau$, $y = A\left(e^{-\frac{5\tau}{\tau}}\right) = A\left(e^{-5}\right) = 0.01\,A = (1\%)A \approx 0$;

From physics perspective, after the duration $t = 5\tau$; y reaches 1% of its value in the steady state 0; thus we can say that $y = 0$ is **practically** reached.



Time constant τ

For $t = \tau$, $y = A\left(e^{-\frac{\tau}{\tau}}\right) = A\left(e^{-1}\right) = 0.37\,A = (37\%)A$;

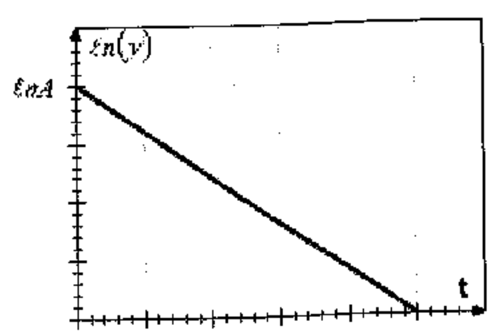
Interpretation of τ:

- ✗ Basing on the previous result we can define the time constant τ, as the duration needed by the variable y to reach 37% of its initial value A.
- ✗ The time constant τ, is the abscissa of the point of intersection of the tangent at origin with the horizontal asymptote $y = 0$.

Linear function & exponential

The function $y = Ae^{-\frac{t}{\tau}}$ could be represented by a linear function when we introduce the mathematical operator logarithm:

$$\ell n(y) = \ell n\left(Ae^{-\frac{t}{\tau}}\right) = \ell n(A) + \ell n\left(e^{-\frac{t}{\tau}}\right) = -\frac{1}{\tau}t + \ell n(A).$$



The expression $\ell n(y) = f(t)$ is represented:

- ✗ by a straight line (of the form $y = at + b$);
- ✗ decreasing slope $a = -\frac{1}{\tau}$;
- ✗ that intercepts the ordinate axis at $+\ell n(A)$.

Appendix 2

Trigonometric Functions

1. Derivatives

$f(t) = a\sin(wt+c)$, so $f'(t) = a \times w\cos(wt+c)$;

$f(t) = a\cos(wt+c)$, so $f'(t) = -a \times w\sin(bt+c)$;

2. Antiderivatives (primitives)

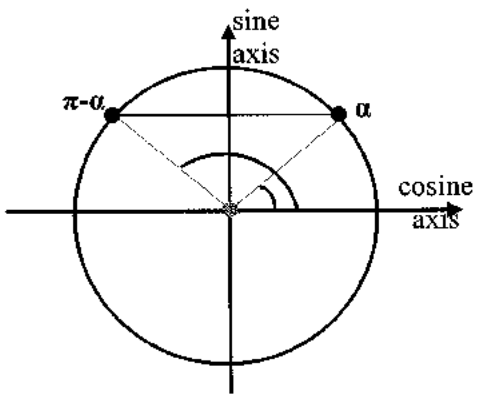
$$\int a\sin(wt+c) = a\frac{-1}{w}\cos(wt+c);$$

$$\int a\cos(wt+c) = a\frac{1}{w}\sin(wt+c);$$

3. Equations

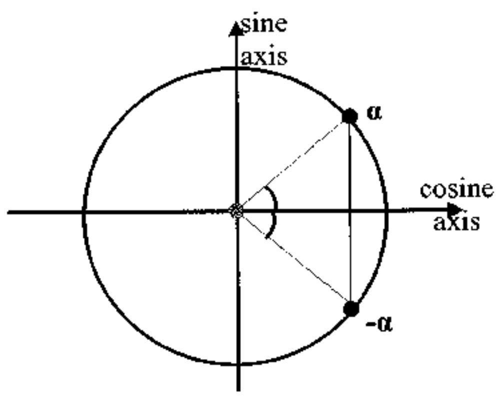
If $\sin x = \sin\alpha$;

Then $x = \alpha$ or $\pi - \alpha$ $[2\pi]$;



If $\cos x = \cos\alpha$;

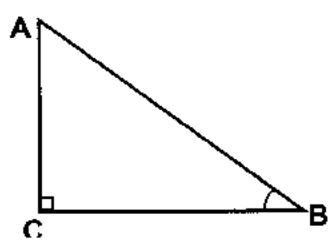
Then $x = +\alpha$ or $-\alpha$ $[2\pi]$;



Subunits

Prefix	Abbreviation	Power of ten
mega-	M	10^{6}
kilo-	k	10^{3}
centi-	c	10^{-2}
milli-	m	10^{-3}
micro-	μ	10^{-6}
nano-	n	10^{-9}
pico-	p	10^{-12}
femto-	f	10^{-15}

Geometrical Properties



$$\sin B = \frac{\text{opposite}}{\text{hypothenuse}} = \frac{AC}{AB}$$

$$\cos B = \frac{\text{adjacent}}{\text{hypothenuse}} = \frac{BC}{BA}$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}} = \frac{AC}{BC} \;\&\; \tan B = \frac{\sin B}{\cos B}$$

$$\cos^2 B + \sin^2 B = 1$$

Appendix 3

Suggested Annual Distribution – LS

Month	Week 1	Week 2	Week 3	Week 4
September			Mechanical Energy	Mechanical Energy
October	Linear Momentum	Linear Momentum	Linear Oscillations	Linear Oscillations
November	Linear Oscillations	Exams	Exams	Electromagnetic Induction
December	Electromagnetic Induction	Auto Induction	Capacitor and DC voltage	Vacation
January	Vacation	Capacitor and DC Voltage	Alternate Circuits	Alternate Circuits
February	Exams	Exams	Diffraction of light	Interference of light
March	Interference of Light	Photoelectric Effect	Photoelectric Effect	Energy Levels of the Atom
April	Energy Levels of the Atom	Vacation	Exams	The nucleus
May	Radioactivity	Radioactivity	Radioactivity	Fission & Fusion
June	Fission & Fusion	Revision		

Suggested Annual Distribution – GS

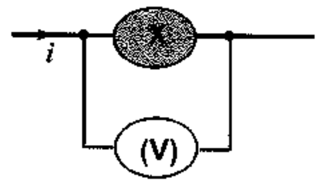
Month	Week 1	Week 2	Week 3	Week 4
September			Mechanical Energy	Mechanical Energy
October	Linear Momentum	Linear Momentum	Angular Momentum	Linear Oscillations
November	Linear Oscillations	Exams	Exams	Angular Oscillations
December	Electromagnetic Induction	Electromagnetic Induction	Auto Induction	Vacation
January	Vacation	Capacitor and DC Voltage	Alternate Circuits	Alternate Circuits + Transformer
February	Exams	Exams	Electromagnetic Oscillations	Electromagnetic Oscillations
March	Diffraction of Light	Interference of Light	Interference of Light	Photoelectric Effect
April	Energy Levels of the Atom	Vacation	Exams	Energy Levels of the Atom
May	The nucleus	Radioactivity	Radioactivity	Fission & Fusion
June	Fission & Fusion	Revision		

Appendix 4
Instrument of Measurement

The voltage or potential difference across a given dipole is measured using a voltmeter or an oscilloscope.

I- Voltmeter

A voltmeter should be connected in parallel across the dipole under study.



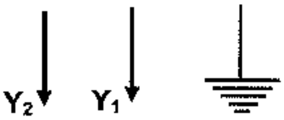
- In **DC Mode** it displays the value of the measured voltage.
- In **AC Mode** it displays the value of the effective value of the alternating voltage.

For an alternating sinusoidal voltage $U_{\text{effective}} = \dfrac{U_{\text{maximum}}}{\sqrt{2}}$.

Note:
The effective voltage is the value of the DC voltage that dissipate over a period the same amount of heat as the AC Voltage.

II- Oscilloscope

The oscilloscope is used to display the shape of the voltage across the dipole under study.



An oscilloscope holds two channels (equivalent to positive) denoted (Y_1) & (Y_2) and the ground (equivalent to negative).

The maximum value is given by $U_{\text{maximum}} = S_v \times y_{\max}$ where S_v is the vertical sensitivity.

1. Single dipole.

The oscilloscope is used to display the voltage across a single dipole that we call X, as shown in figures 2 & 3.

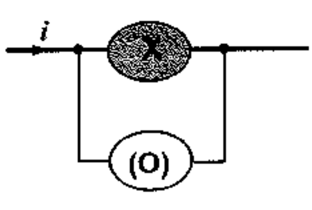


Figure 2

The oscilloscope connected in parallel has a very large resistance $R \longrightarrow +\infty$, so the current that passes through it could be taken zero, which is equivalent to an open circuit as shown in figure 3.

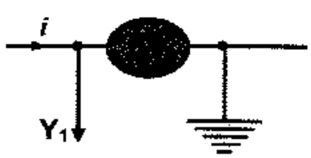


Figure 3

2. Two dipoles

Connect an oscilloscope in order to display the voltages across the generator u_{AC} & u_{BC} across the dipole Y on the channels Y_1 & Y_2 respectively.

The common letter of the voltages under study A$\underline{C}$ & B$\underline{C}$ is C, so the ground is placed on C; the channel (Y_1) is then placed on A and (Y_2) is placed on B.

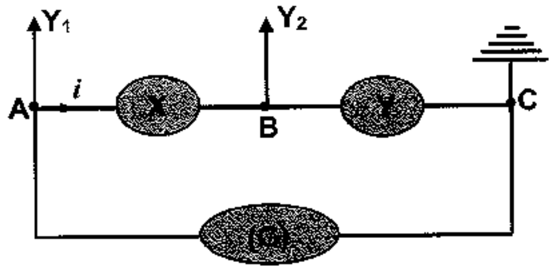


Figure 5

3. Particular buttons

a) INV Knob (button)

If we intend to display the voltages u_{AB} and u_{BC} across two dipoles X & Y respectively connected in series shown in figure 6.

The voltage across X is u_{AB} & the voltage across Y is u_{BC};

The common letter of A$\underline{B}$ & $\underline{B}$C is B, so the ground is placed on B;

The channel (Y_1) is then placed on A and (Y_2) is then placed on C.

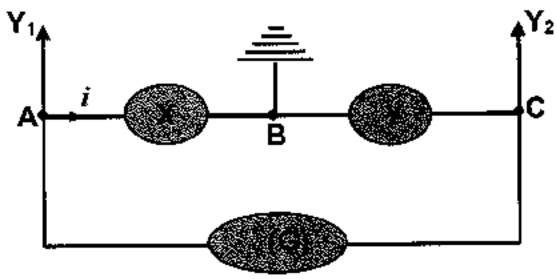


Figure 6

Thus, the connections of the oscilloscope in this case displays on (Y_1): u_{AB} & on (Y_2): u_{CB} (from channel to ground).

To display u_{BC} on channel (Y_2) we should the push the knob "INV" which multiply the voltage displayed by a sign minus.

Thus, u_{CB} **+ INV** $\longrightarrow u_{BC}$.

a) ADD Knob

With convenient connections we can display obtain a display of three voltages in a circuit and not to be limited to two.

The adjacent connections of the circuit displays u_{AB} & u_{CB}.

By pressing «INV» on channel Y_2 it displays u_{BC}.

The knob «ADD» is pressed to display the sum of the voltages visualized on the two channels.

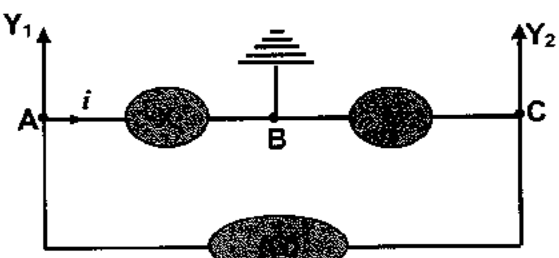


Figure 6

We consequently obtain a display of the voltage across the generator $u_{AC} = u_{AB} + u_{BC}$.

Appendix 5

Basic Physical Quantities in Mechanics

Physical quantity	Symbol	Unit	Formula – Relation
Distance/Abscissa – Angle	x - θ	m - rad	$x = R\theta$ & $\left(x^2\right)' = 2xv$
Velocity – Angular velocity	v - θ'	m/s - rad$/s$	$v = x'$ & $v = R\theta'$
Acceleration	a	m/s^2	$a = v' = x''$ & $\left(v^2\right)' = 2va$
Mass	m	kg	
Moment of inertia	I	$kg.m^2$	$I = md^2$ (point particle)
Force	F	N	
Stiffness of spring	k	N/m	
Torsion constant of a wire	C	$N.m/$rad	
Tension in spring	T	N	$T = -kx$ (Hooke's law)
Moment of restoring couple in torsion wire	M	$N.m$	$\mathrm{M} = -C\theta$
Moment (torque) of a force around an axis	$\mathrm{M}_{\vec{f}}$	$N.m$	$\mathrm{M}_{\vec{f}} = f \times d$ (distance between the force and axis of rotation)
Work	$W_{\vec{f}}$	Joules J	$W_{\vec{f}} = \vec{f}.\overrightarrow{AB}$ or $W_{\vec{f}} = \mathrm{M}_{\vec{f}}.\theta$
Linear Momentum	p	$kg.m/s$	$p = mv$
Angular momentum	σ	$kg.m^2/s$	$\sigma = I\theta'$
Newton's 2nd law			$\sum \vec{F} = \frac{d\vec{p}}{dt}$
Theorem of angular momentum			$\sum \mathrm{M} = \frac{d\sigma}{dt}$
Kinetic energy	KE	J	$KE = \frac{1}{2}mv^2$ or $KE = \frac{1}{2}I\theta'^2$
Gravitational potential energy	GPE	J	$GPE = \pm mgh$
Elastic potential energy	PE_e	J	$PE_e = \frac{1}{2}kx^2$ or $PE_e = \frac{1}{2}C\theta^2$
Mechanical energy	ME	J	$ME = KE + GPE + PE_e$
Power	P	W (Watt)	$P = \frac{dW_{\vec{f}}}{dt}$ or $P = \frac{\Delta(ME)}{\Delta t}$ $P_{\vec{f}} = \vec{f}.\vec{v}$ or $W_{\vec{f}} = \mathrm{M}_{\vec{f}}.\theta'$

Basic Physical Quantities in Electricity

Physical quantity	Symbol	Unit	Formula – Relation
Current	I	A	
Voltage	U	V	
Effective current – voltage (AC Only)	I_{eff} U_{eff}	A V	$I_{\text{eff}} = \dfrac{I_{\max}}{\sqrt{2}}$ & $U_{\text{eff}} = \dfrac{U_{\max}}{\sqrt{2}}$
Resistance (Ohm's law)	R	Ω	$U = RI$ (with current)
Capacitance	C	F	
Charge	q	C	$q = C u_C$ or $i = \left\lvert \dfrac{dq}{dt} \right\rvert$
Power	P	W	$P = UI$ (DC) $P = RI^2$ (Joules effect - DC) $P = UI\cos\varphi$ (AC)
Energy	E	J	$E = P \times \Delta t$
Electric potential energy	E_e	J	$E_e = \dfrac{1}{2} C u_C^2$ or $E_e = \dfrac{Q^2}{2C}$
Inductance of a coil	L	H	$u_{AB} = ri + L\dfrac{di}{dt}$
Magnetic energy of coil	E_m	J	$E_m = \dfrac{1}{2} L i^2$
Charge of an electron	e	C	$\lvert q \rvert = N \times e$
Magnetic field	B	T	
Magnetic flux	ϕ	Wb	$\phi = NBS\cos\theta$ or $\phi = Li$
Induced electromotive force	e	V	$e = -\dfrac{d\phi}{dt}$ or $e = -L\dfrac{di}{dt}$
Voltage across generator	u_G	V	$u_G = E - ri$ opposite to current $u_G = ri - E$ with current
Electromagnetic energy	E	J	$E = E_m + E_e$
Frequency - Angular frequency	f - w	Hz - rad$/s$	$w = 2\pi f$
Proper period of electromagnetic oscillations	T_0	s	$T_0 = 2\pi\sqrt{LC}$