Logarithmic Functions

Properties:

- y = f (x) = ln (x); The domain of definition of f is]0;+∞[.
- ln (0^+) = - ∞.
- ln (+∞) = + ∞.
- x attends ln (x) when x is large.
- ln (1) = 0.
- ln (e ≈2.718) = 1.
- $\lim\limits_{x \to +\infty} \frac{\ln(x)}{x} = 0 \quad and \quad \lim\limits_{x \to 0^+} x \ln(x) = 0.$
- If f (x) = ln (x) then, f '(x) = $\frac{1}{x}$.
- If f (x) = ln (u (x)) then, f '(x) = $\frac{u'(x)}{u(x)}$.
- If 0 < x < 1 then, ln (x) < 0.
- If x > 1 then, ln (x) > 0.
- ln (x) = a then, x = e^a.
- a > 0 and b > 0 then, ln (a × b) = ln (a) + ln (b).
- a > 0 and b > 0 then, $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$.
- ln (a^n) = n ln (a) for all n.
- $\ln\left(\frac{1}{a}\right) = -\ln(a)$.
- ln $(\frac{1}{e}) = -1$.
- $\int \frac{1}{x} dx = \ln(x) + C;\ where\ x > 0.$
- $\int \frac{1}{x} dx = \ln(-x) + C\ where\ x < 0.$
- $\int \frac{1}{x} dx = \ln|x| + C.$
- $\int \frac{u'(x)}{u(x)} dx = \ln(u(x)) + C\ where\ u(x) > 0.$
- $\int \ln(x) dx = x \ln(x) - x + C.$
- $\int \frac{\ln(x)}{x} dx = \frac{\ln^2(x)}{2} + C.$

Exercise 1

Part A:

Consider the function g that is defined over $[1;+\infty[\ as: g(x) = \ln(x) - \frac{1}{2}.$

1) Study the variations de g over $[1;+\infty[$.
2) Solve the equation g (x) = 0 in the interval $[1;+\infty[$.
3) Deduce that g (x) > 0 if and only if $x > \sqrt{e}$.

Part B:

Consider the function f that is defined over $[1;+\infty[$ as: $f(x) = 2x^2[\ln(x) - 1] + 2$.

1) **Calculate** $\lim_{x\to+\infty} f(x)$ *and* $f(1)$.

2) **a- Prove that for all real number x of the interval** $[1; +\infty[$, $f'(x) = 4xg(x)$.

   b- Study the sign of $f'(x)$ **over** $[1; +\infty[$ **and then deduce the table of variations of f over** $[1; +\infty[$.

3) **a- Prove that the equation f (x) = 0 has a unique solutionα so that: 2.2 <α< 2.4.**

   b- Draw (C) the representative curve of f in an orthonormal system $(O; \vec{i}, \vec{j})$.

Solution of Exercise 1

Part A :

1. $g'(x) = \frac{1}{x} > 0$ **for all x** $\in [1; +\infty[$, **So g is strictly increasing over** $[1; +\infty[$.

2. $g(x) = 0 \Leftrightarrow \ln(x) = \frac{1}{2} \Leftrightarrow x = e^{\frac{1}{2}} \Leftrightarrow x = \sqrt{e}$.

x	1	$\sqrt{e}$	+∞
g '(x)		+	
g (x)	-0.5	0 ↗	+∞

According to the tableau of variations of g we have: g (x) > 0 for all x > $\sqrt{e}$.
And we have g (x) = 0 for x = $\sqrt{e}$, **g (x) < 0 for all:** $1 < x < \sqrt{e}$.

Part B :

1. $\lim_{x\to+\infty} f(x) = 2(+\infty)[\ln(+\infty) - 1] + 2 = (+\infty)(+\infty) + 2 = +\infty + 2 = +\infty$.

   $f(1) = 2(1)[\ln(1) - 1] + 2 = 2(-1) + 2 = -2 + 2 = 0$.

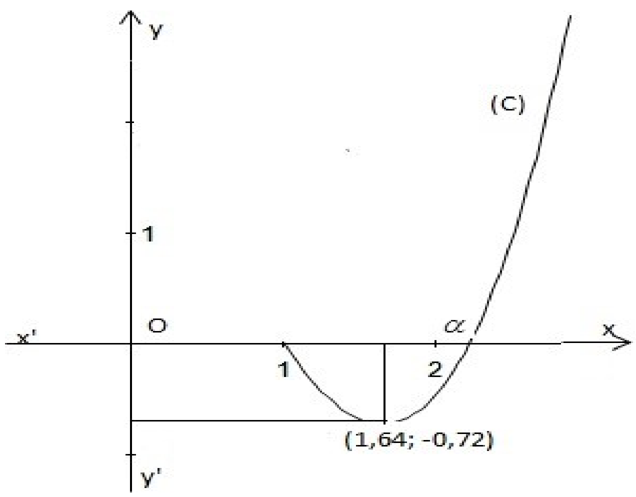
2. a) $f'(x) = 4x[\ln(x) - 1] + \left(\frac{1}{x}\right)(2x^2) = 4x\left[\ln(x) - \frac{1}{2}\right] = 4xg(x)$.

   b) $f'(x) < 0 \Leftrightarrow 1 < x < \sqrt{e}$, $f'(x) = 0 \Leftrightarrow x = \sqrt{e}$, $f'(x) > 0 \Leftrightarrow x > \sqrt{e}$.

x	1	$\sqrt{e}$	+∞
f '(x)	----	0 +	
f (x)	0 ↘	2 – e ↗	+∞

3. a) **We have: f is continuous over] 2.2; 2.4 [, f is strictly increasing over]2.2; 2. 4[,**
   f (2.2) x f (2.4) = (4.0328)× (-0.4192) = - 1. 69 < 0,
   So the equation f (x) = 0 has a unique solutionα so that: 2.2 <α< 2.4.

   b)



Exercise 2

Part A:

Consider the function g that is defined over $]0;+\infty[$ as: $g(x)=x^2+3-2\ln(x)$.

1) Study the variations of g and prove that g admits a minimum to be determined.
2) Deduce the sign of g (x) for all real x >0.

Part B:

Consider the function defined over $]0;+\infty[$ by: $f(x)=\frac{\ln(x)}{x}+\frac{x^2-1}{2x}$. (C) is the representative curve of the function f in an orthonormal system graphic unit 2 cm.

1) Calculate $\lim\limits_{x\to 0} f(x)$ *and* $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote to (C).
2) Prove that for all x in $]0;+\infty[$ $f'(x)=\frac{g(x)}{2x^2}$. Deduce the sense of variations of f.
3) Consider the straight line (D) of equation: $y=\frac{1}{2}x$.

   Calculate $\lim\limits_{x\to+\infty}\left[f(x)-\frac{1}{2}x\right]$. Deduce an asymptote to the curve (C).
4) Study the relative position of (C) and (D).
5) Determine the coordinates of A point of intersection of (C) and (D).
6) Draw (D) and (C).
7) Calculate f (1) and then deduce the sign of f (x) over $]0;+\infty[$.
8) Deduce that for all real x > 1: $\frac{\ln(x)}{x}>\frac{1-x^2}{2x}$.

Solution of Exercise 2

Part A :

1. $g'(x)=2x-\frac{2}{x}=\frac{2(x^2-1)}{x}=\frac{2(x-1)(x+1)}{x}$.

   $g'(x)=0$ *for* $x=1$, $g'(x)>0$ *for* $x>1$ *and* $g'(x)<0$ *for* $x<1$.

   Thus g has a local minimum that is equal 4 for x = 1.

2.

X	0	1	+∞
g '(x)	------	0	+
g (x)	+∞ ↘	4	↗ +∞

According to the table of variations of g we have the local minimum of g is equal to 4 > 0 So g (x) > 0 for all x > 0

Part B :

1. $\lim\limits_{x\to 0} f(x)=\frac{\ln(0)}{0}+\frac{0-1}{0}=-\infty-\infty=-\infty \Rightarrow x=0$ is a V.A to (C)

   $\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\frac{f(x)}{x}+\lim\limits_{x\to+\infty}\frac{x^2-1}{2x}=0+\infty=+\infty.$

2. $f'(x)=\frac{1-\ln(x)}{x^2}+\frac{2x(2x)-2(x^2-1)}{4x^2}=\frac{1-\ln(x)}{x^2}+\frac{x^2+1}{2x^2}=\frac{g(x)}{2x^2}$.

   $f'(x)>0$ for all x > 0, So f is strictly increasing for all x > 0.

x	0	1	$+\infty$
f'(x)		+	
f (x)	0	$\nearrow$	$+\infty$

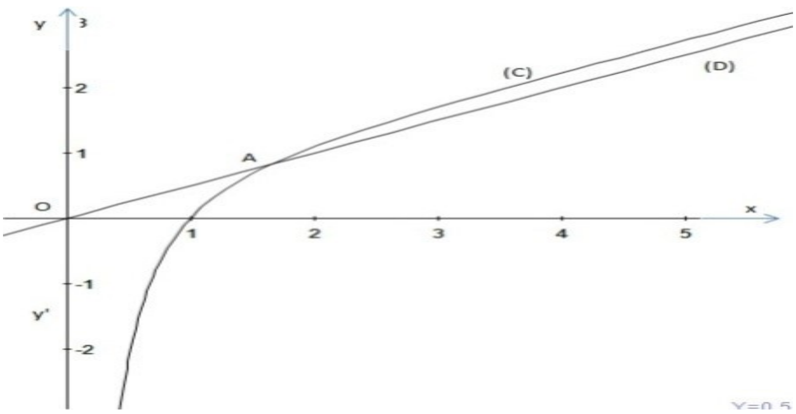
$$\lim_{x\to+\infty}\left[f(x)-\frac{1}{2}x\right]=\lim_{x\to+\infty}\left[\frac{\ln(x)}{x}+\frac{x}{2}-\frac{1}{2x}-\frac{1}{2}x\right]=0-0=0.$$

So the straight line (D) of equation y = $\frac{1}{2}x$ is an oblique asymptote to (C).

3. $f(x)-\frac{1}{2}x=\frac{\ln(x)}{x}-\frac{1}{2x}=\frac{2\ln(x)-1}{2x}$. (C) is below (D) for $0<x<\sqrt{e}$. (C) is above (D) for $x>\sqrt{e}$.

4. (C) cuts the line (D) at the point $A\left(\sqrt{e};f\left(\sqrt{e}\right)=\frac{\sqrt{e}}{2}\right)$.

5.



6. $f(1)=\frac{\ln 1}{1}+\frac{1-1}{2}=0+0=0,\ f(x)<0\ for\ 0<x<1\ and\ f(x)>0\ for\ x>1.$

7. f (x) > 0 for x > 1, $\frac{\ln(x)}{x}+\frac{x^2-1}{2x}>0\Rightarrow\frac{\ln(x)}{x}>\frac{1-x^2}{2x}$.

<u>Exercise 3</u>

Consider the function f that is defined over $]0;+\infty[$ as: $f(x)=x+3\frac{\ln x}{x}$ and let (C) be the representative curve of the function f in an orthonormal system $(O;\vec{i},\vec{j})$; unit: 2 cm.

1) a- Calculate $\lim_{x\to 0} f(x)$ and give a geometric interpretation.

   b- Determine $\lim_{x\to+\infty} f(x)$ and then verify that the straight line (d) of equation y = x is an asymptote to the curve (C).

   c- Study according to the values of x the relative positions of (C) and (d).

2) The table below is the table of variations of the function f ' the derivative of f.

x	0		e		$e\sqrt{e}$		$+\infty$
f ''(x)	‖	–	⋮	–	0	+	
f' (x)	‖ $+\infty$	$\searrow$	1	$\searrow$	$1-1.5e^{-3}$	$\nearrow$	1

   a- Prove that f is strictly increasing over its domain of definition and set up its table of variations.

   b- Write an equation of the tangent (D) to (C) at a point G of abscissa e.

c- Prove that the curve (C) has an inflection point L.

d- Prove that the equation f(x) = 0 has a unique root α and verify that:

$$0.8 < \alpha < 0.9$$

3) Draw (D), (d) and(C).
4) Calculate in cm^2 the area of the region bounded by (C) the straight line (d) and the two straight lines of equations x = 1and x = e.

<u>Solution of Exercise 3</u>

1. a) $\lim_{x\to 0} f(x) = 0 + 3\frac{\ln(0)}{0} = 0 - \infty = -\infty$. So x = 0 is V.A to (C).

   b) $\lim_{x\to +\infty} f(x) = +\infty + 3\lim_{x\to +\infty} \frac{\ln(x)}{x} = +\infty + 3\times 0 = +\infty.$

   $\lim_{x\to +\infty} \left[f(x) - x\right] = 3\lim_{x\to +\infty} \frac{\ln(x)}{x} = 3\times 0 = 0.$

   Thus the line (d) of equation y = x is an O.A to (C).

   a- f (x) – x = $3\frac{\ln(x)}{x}$.

   $f(x) - x = 0 \Leftrightarrow x = 1,$ Thus (C) cuts (d) at the point (1; 1).

   $f(x) - x > 0 \Leftrightarrow x > 1,$ Thus (C) is above (d).

   $f(x) - x < 0 \Leftrightarrow 0 < x < 1,$ Thus (C) is below (d).

2. a) According to the table of variations of f' the local minimum of

   $f'(x) = 1 - 1.5e^{-3} > 0,$ Thus $f'(x) > 0$ for all x > 0

   implies f is strictly increasing for all x > 0.

x	0	+∞
f '(x)	+	
f (x)	-∞ →	+∞

   b) (D):

$$y - y_G = f'(x_G)(x - x_G) \Leftrightarrow y - e - \frac{3}{e} = 1(x - e) \Leftrightarrow y = x + \frac{3}{e}.$$

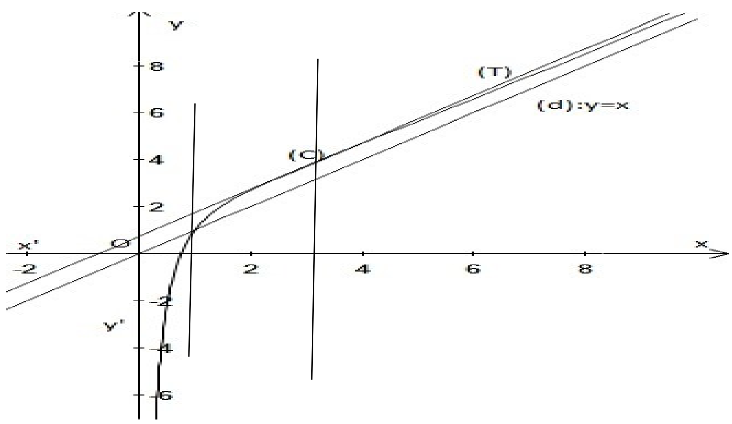
   c) $f''(x) = 0 \Leftrightarrow x = e\sqrt{e},\ f''(x) < 0 \Leftrightarrow x < e\sqrt{e},\ f''(x) > 0 \Leftrightarrow x > e\sqrt{e}.$

   $f''(x)$ changes its sign,

   Thus (C) has an inflection point L $\left(e\sqrt{e};\ e\sqrt{e} + 3\frac{\ln\left(e\sqrt{e}\right)}{e\sqrt{e}}\right)$.

   b- f is continuous over]0.8; 0.9[, f is strictly increasing over]0.8; 0.9[,
   f (0.8) x f (0.9) = (- 0.017) x (0.037) = - 0.00064 < 0.
   So the equation f (x) = 0 has a unique solution α so that: 0.8 <α< 0.9.

3.



4. $A = \int_1^e \left[f(x) - x\right]dx = 3\int_1^e \frac{\ln(x)}{x}dx = \frac{3}{2}\left[\ln^2(x)\right]_1^e = \frac{3}{2}\ln^2(e) - \ln^2(1) = \frac{3}{2} - 0 = \frac{3}{2}\times 4cm^2.$

Exercise 4

PART A : Consider the function g that is defined over $]0;+\infty[$ as: $g(x)=1-\frac{1}{x}+\ln(x)$.

Let (C) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. G.U = 2 cm.

1. Calculate $\lim_{x\to 0^+}[g(x)]$ *and* $\lim_{x\to+\infty}[g(x)]$. Deduce an asymptote to the curve (C).
2. Calculate g (1), g (2) and g (e).
3. Calculate $g'(x)$, then study the sense of variations of function g.
4. Set up the table of variations of the function g.
5. Write an equation of the tangent line (T) to (C) at a point A of abscissa 1.
6. Draw (T) and (C).

PART B: Consider the function f that is defined over $]0;+\infty[$ as: $f(x)=-1+(x-1)\ln(x)$.

The below table is the table of variations of the function f over $]0;+\infty[$:

x	0		1		$+\infty$
$f'(x)$		$-$	0	$+$	
$f(x)$	$+\infty$	↘	-1	↗	$+\infty$

1. Prove that the equation f (x) = 0 has exactly two roots α and β such that: 0.2 < α < 0.3 and 2.2 < B < 2.3.
2. Designate by (E) the region bounded by the curve (C) of the function g, the x-axis and the two straight lines x = α and x = β. Let A be the area of the region (E).
   a- Prove that for all x $\in]0;+\infty[$ we have: $f'(x)=g(x)$.
   b- Prove that: $A=\int_1^{\alpha} g(x)dx+\int_1^{\beta} g(x)dx$.
   c- Deduce the value of A in terms of α and β.

Solution of Exercise 4

Part A: $g(x)=1-\frac{1}{x}+\ln(x)$

1. $\lim_{x\to 0^+} g(x)=\lim_{x\to 0^+}\left(1-\frac{1}{x}+\ln(x)\right)=1-\frac{1}{0^+}+\ln(0^+)=1-\infty+(-\infty)=-\infty-\infty=-\infty$

   So x = 0 is Vertical asymptote to the curve (C) of the function g.

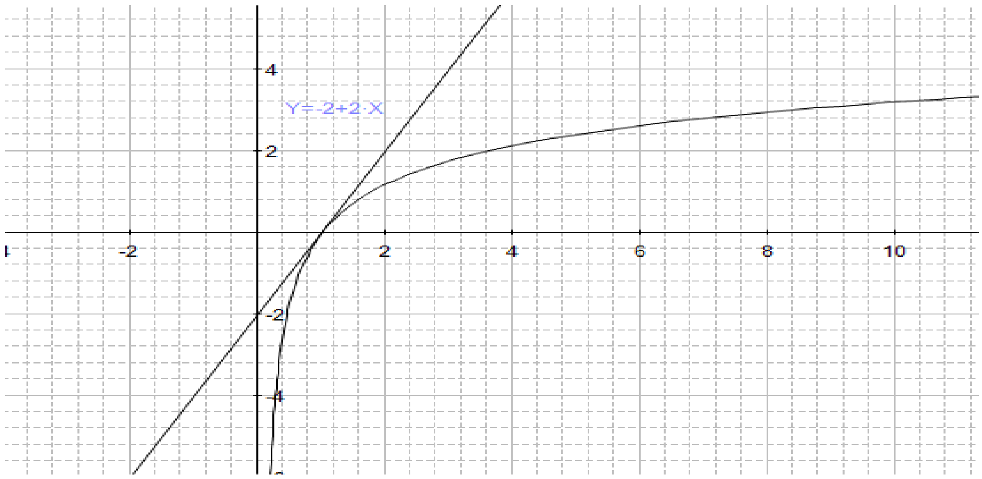
   $\lim_{x\to+\infty} g(x)=\lim_{x\to+\infty}\left(1-\frac{1}{x}+\ln x\right)=1-\frac{1}{+\infty}+\ln(+\infty)=1-0+\infty=+\infty.$
2. g (1) = 1 – 1 + ln (1) = 0 – 0 = 0 ; g (2) = 1 – 0.5 + ln (2) = 0.5 +ln (2);
   g (e) = 1 – 1/e + ln (e) = 1 – 1/e + 1 = 2 – 1/e
3. $g'(x)=0+\frac{1}{x^2}+\frac{1}{x}>0 \text{ for all } x>0$

   So g is strictly increasing for all x > 0
4.

x	0	+∞
g'(x)	+	
g (x)	-∞ ↗	+∞

5.

6. (T): y = f (1) + f '(1) (x – 1) = 0 + 2 (x-1) = 2 x – 2



Part B:

$$f(x) = -1 + (x-1)\ln(x)$$

1. **Since f is continous over the interval]0.2;0.3[included in]0;1[**
   Since f is strictly decreasing form + ∞ to -1.
   Since f (0.2) = ……. < 0 and f (0.3) = ….. < 0 then the equation f (x) = 0 has a unique root α
   Such that 0.2 < α < 0.3.
   Since f is continuous over]2.2;2.3[included in]1;+∞[
   Since f is strictly increasing from -1 to +∞
   Since f (2.2) =…..< 0 and f (2.3) =…….> 0 then the equation f (x) = 0 has a unique root
   β such that: 2.2 < β < 2.3.

Exercise 5

Part A:

Consider the function g that is defined over] 0 ; +∞ [as: $g(x) = x^2 - 1 + \ln x$.

1. Calculate $\lim_{x \to 0^+} g(x)$ and $\lim_{x \to +\infty} g(x)$.
2. Calculate g '(x) and set up the table of variations of g.
3. Calculate g (1) then determine the sign of $g(x)$ over $]0;+\infty[$.

Part B:

Consider the function f that is defined over $]0;+\infty[$ by :

f(x) = $f(x) = -\frac{1}{2}x + 1 + \frac{\ln x}{2x}$, and designate by (C) its representative curve in an orthonormal system $(O, \vec{i}, \vec{j})$.

1. Calculate $\lim_{x \to 0^+} f(x)$, $\lim_{x \to +\infty} f(x)$. Deduce an asymptote to (C).
2. Prove that the straight line (d) of equation y = $-\frac{1}{2}x + 1$ is an asymptote to (C).
3. Study the relative position between (C) and (d).
4. Prove that : $f'(x) = \frac{-2g(x)}{4x^2}$.
5. Deduce the sign of $f'(x)$ and set up the table of variations of f.

6. Prove that the equation $f(x)=0$ admits two solutions α and β such that: $\frac{1}{3}<\alpha<\frac{1}{2}$ *and* $2<\beta<\frac{5}{2}$.

7. Draw (C) and (d). (graphic unit: 2cm).

8. Calculate, in cm^2, the area of the region bounded by (C), (d) and the two straight lines of equations x = 1 and x = e.

Exercise 6

Part A:

Let g be a function defined over]0 ; + ∞[by: $g(x)=3-\frac{2}{x}+\ln\left(\frac{x}{2}\right)$.

Designate by (γ) its representative curve in an orthonormal system $\left(O;\vec{i},\vec{j}\right)$.

1. Calculate $\lim\limits_{x\to0+} g(x)$, $\lim\limits_{x\to+\infty} g(x)$, g (2) and g (4). Deduce an asymptote to (γ).
2. Calculate g '(x) and set up the table of variations of g.
3. Prove that the equation g (x) = 0 admits over]0; +∞[a unique root α. Verify that 0.9 < α < 0.91.
4. Write an equation of the tangent (T) to (γ) at the point A of abscissa 2.
5. Draw (T) and (γ).
6. Deduce the sign of g (x) for all real x > 0.
7. a – Let H be a function defined over]0 ; +∞[by H (x) = x ln $\left(\frac{x}{2}\right)$ - x.

   Prove that H is a primitive of the function h (x) = $\ln\left(\frac{x}{2}\right)$.

   b- Calculate the area A (α) of the region bounded by (γ), the x – axis and the straight line of equation x = 2. Prove that: A (α) = $\frac{(\alpha-2)^2}{\alpha}$.

Part B:

Let f be a function defined over]0 ; +∞[by f (x) = $(x-2)\left(2+\ln\left(\frac{x}{2}\right)\right)$. Designate by (C) its representative curve in a new system.

1. Calculate $\lim\limits_{x\to o+} f(x)$, $\lim\limits_{x\to+\infty} f(x)$. Deduce an asymptote to (C).
2. Determine the points of intersection of (C) and the x - axis.
3. Prove that : f (α) = $-\frac{(2-\alpha)^2}{\alpha}$.
4. Prove that: f '(x) = g (x) and set up the table of variations of f.
5. Draw (C). (Assume that: α = 0,91).

INTEGRALS
CHAPTER REVIEW

Definition and notations:

1. Let f be a continuous function over an interval I, F is a primitive of f over I, a and b two points of I. We say that the integral of f from a to b, the real number F (b)-F (a).
   This number is denoted by:

1. $\int_a^b f(x)dx = [F(x)]_a^b = F(b)-F(a).$ 2. $\int_a^a f(x)dx = 0.$

3. $\int_b^a f(x)dx = -\int_a^b f(x)dx.$ 4. $\int_a^b 1dx = [x]_a^b = b-a.$

5. $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx.$

6. Let f be a continuous function and positive over [a; b], $a \le b$.
   The area of the region bounded by the representative curve of f, (orthonormal system), the x – axis and the two straight lines of equations: x =a and x =b, expressed in unit square of areas is calculated by: A $= \int_a^b f(x)dx$.

   When the function f is negative then we consider– f, and in this case, $\int_a^b -f(x)dx$ is the given Area.

7. Let f and g be two continuous functions over an interval I. a and b two real's of I.
   for all real's α and β we have: $\int_a^b [\alpha f(x) + \beta g(x)]dx = \alpha\int_a^b f(x)dx + \beta\int_a^b g(x)dx.$

8. Let f be a continuous function over an interval I and a is a real of I.
   The defined function over I by: $x \to \int_a^x f(t)dt$ is a primitive of f over I, that vanishes at a.

9. Let f be a continuous function over an interval I and let a and b two real's of I.
   If $a \le b$ and if f≥ 0 over $[a,b]$ then: $\int_a^b f(x)dx \ge 0.$

10. Let f and g be two continuous functions over an interval I and let a< b be two real's of I.
    If $f(x) \le g(x)$ over $[a,b]$, then $\int_a^b f(t)dt \le \int_a^b g(t)dt.$

11. If f is a continuous function over an interval I of center O, and a is any real of I.
    a) If f is an even function then: $\int_{-a}^{+a} f(t)dt = 2\int_0^{+a} f(t)dt.$
    b) If f is an odd function then: $\int_{-a}^{+a} f(t)dt = 0.$

12. Let f be a differentiable function over IR and a is a any real number. if f is periodic function of period T, then : $\int_a^{a+T} f(t)dt = \int_0^T f(t)dt.$

13. Integration by change of variable:

Let f be a continuous function on an interval I, and ϕ a function whose derivative is continuous on an interval: $J=[\alpha,\beta]$ such that: $\phi(J)\subset I$.

We have: $\int_{\alpha}^{\beta} f\left(\phi(t)\right)\phi'(t)dt=\int_{a}^{b} f(x)dx$ where $a=\phi(\alpha)$ *et* $b=\phi(\beta)$.

Example: $I=\int_{0}^{2} t\,e^{t^2}dt$, suppose that: u= $\phi(t)=t^2 \Rightarrow u'=2t\,dt=\phi'(t)$. thus

$$I=\frac{1}{2}\int_{0}^{4} e^{u}du=\frac{1}{2}\left[e^{u}\right]_{0}^{4}=\frac{1}{2}\left(e^{4}-1\right).$$

14. Primitives:

- $\int u^{\alpha}(x)u'(x)dx=\frac{1}{\alpha+1}u^{\alpha+1}(x)+c.\quad \alpha\neq -1.$
- $\int e^{u(x)}u'(x)dx=e^{u(x)}+c.$
- $\int \frac{u'(x)}{u(x)}dx=\ln\left|u(x)\right|+c.\ \int x^{n}dx=\frac{x^{n+1}}{n+1}+C$

15. Integration by parts:

Let u and v be two differentiable functions on an interval I such that u' and v' are Continuous functions on I. We have, for all real numbers a and b in I:

$$\int_{a}^{b} u'(x)v(x)dx=\left[u(x)v(x)\right]_{a}^{b}-\int_{a}^{b} u(x)v'(x)dx.$$

Example:

To calculate $I=\int_{0}^{1} xe^{x}dx=\left[xe^{x}-e^{x}\right]_{0}^{1}=$

	Derivative	Primitive
+	x	e^{x}
-	1	e^{x}
	0	e^{x}

= (1 e^{1} – e^{1}) – (0× e^{0} – e^{0}) = 1.

Example: $\int_{0}^{1}\left(x^{2}+1\right)e^{2x}dx$

	Derivative	Primitive
+	$x^{2}+1$	e^{2x}
-	2 x	½ e^{2x}
+	2	¼ e^{2x}
-	0	1/8 e^{2x}

$=\left[(x^{2}+1)\left(\frac{1}{2}e^{2x}\right)-\frac{1}{2}xe^{2x}+\frac{1}{4}e^{2x}\right]_{0}^{1}$ = (e^{2}- ½ e^{2} +1/4 e^{2}) – (1/2 +1/4) = ½ e^{2} – ¾.

Example: Calculate $\int_{1}^{e}(x+1)\ln x dx$.

Let u = ln (x) implies u ' = 1/x and let V' = x +1 implies V = ½ x^{2} + x.

$\int_{a}^{b} u\times v'dx=\left[u\times v\right]_{a}^{b}-\int_{a}^{b} v\times u'dx=\left[\ln x\times\left(\frac{1}{2}x^{2}+x\right)\right]_{1}^{e}-\int_{1}^{e}\left[\frac{1}{2}x+1\right]dx$

= (ln(e)× (1/2 e^{2} + e) – (ln (1) ×(3/2)) - $\left[\frac{1}{4}x^{2}+x\right]_{1}^{e}$ =

= ½ e^{2} + e - ((1/4 e^{2} +e) – (5/4)) = ½ e^{2} + e -1/4 e^{2} – e + 5/4 = ¼ e^{2} + 5/4.

16. Linearization of trigonometric polynomials:

a) To calculate integrals such as:

$\int_a^b \sin(px)\sin(qx)dx, \int_a^b \sin(px)\cos(qx)dx \, and \int_a^b \cos(px)\cos(qx)dx$ where p and q are positive integers, we use the following identities:

$$\sin(px)\sin(qx) = \frac{1}{2}\left[\cos(p-q)x - \cos(p+q)x\right];$$

$$\sin(px)\cos(qx) = \frac{1}{2}\left[\sin(p-q)x + \sin(p+q)x\right];$$

$$\cos(px)\cos(qx) = \frac{1}{2}\left[\cos(p-q)x + \cos(p+q)x\right].$$

b) To calculate integrals of the form $\int_a^b \sin^p(x)\cos^q(x)dx$.

We distinguish two cases:

i) If at least one of the two numbers p and q is an odd positive integer:

If, in the integral $I = \int_a^b \sin^p x \cos^q x dx$, $p = 2k+1$, where k is positive integer, we write é

$\sin^p x = \sin^{2k} x \sin x = \left(1-\cos^2 x\right)^k .\sin x$ and then we perform the change of variable t = cos(x).

If q is an odd positive integer, we interchange the roles of sin(x) and cos(x).

ii) If, in the integral $\int_a^b \sin^p x \cos^q x dx$, p and q are even positive integers, we use the following identities:

$$\sin^2(x) = \frac{1-\cos(2x)}{2} \quad and \quad \cos^2(x) = \frac{1+\cos(2x)}{2}.$$

Example:

$$\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^4(x)dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left(1-\cos(2x)\right)^2 dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} \left[1+\cos^2(2x) - 2\cos(2x)\right]dx$$

$$= \frac{\pi}{8} - \frac{1}{2} + \frac{1}{4}\int_0^{\frac{\pi}{4}} \left(1+\cos(4x)\right)dx = \frac{3\pi}{16} - \frac{1}{2}.$$

17. Area between two curves:

If f and g are two continuous functions on an interval [a, b], If $f \le g$ over $[a,b]$, the Area A of the region, bounded by the graphs of f and g and the two vertical line x = a and x = b, is given by: $A = \int_a^b \left[g(x) - f(x)\right]dx$.

18. Volume of a solid of revolution:

$$V = \int_a^b \pi\left[f(x)\right]^2 dx.$$

CHAPTER 7 INTERGRAL

SOLVED EXERCISES

EXERCISE 1

Calculate the following integrals:

1. $\int_{-3}^{2} \left(x^4 - 5x^2 + 3\right)dx$; 2. $\int_0^1 (x+3)\left(x^2+6x+4\right)^2 dx$; 3. $\int_1^2 \frac{4x^3 - 3x^2 + 1}{5x^2} dx$;

1. $\int_{-3}^{2}\left(x^4-5x^2+3\right)dx=[\frac{x^5}{5}-\frac{5}{3}x^3+3x]_{-3}^{2}$ = (32/5 -40/3 +6) – (-253/5 +45 -9) =

2. $\int_0^1 (x+3)\left(x^2+6x+4\right)^2 dx=\frac{1}{2}\int_0^1 (2x+6)\left(x^2+6x+4\right)^2 dx=\left[\frac{1}{2}\left(x^2+6x+4\right)^3\times\frac{1}{3}\right]_0^1$

= (1/6 (1+6+4)) – (1/2 (4)) = 11/6 – 2 = -1/6.

$\int u'\times u^n dx=\frac{u^{n+1}}{n+1}+C$. $\int\frac{1}{x^2}dx=-\frac{1}{x}+C$

3. $\int_1^2\frac{4x^3-3x^2+1}{5x^2}dx=\int_1^2\frac{4}{5}x-\frac{3}{5}+\frac{1}{5x^2}dx=\left[\frac{4}{5}\frac{x^2}{2}-\frac{3}{5}x-\frac{1}{5x}\right]_1^2$ = (8/5 -6/5 -1/10) – (2/5 -3/5 -1/5)

4. $\int_0^2 (2x+1)^3 dx$; 5. $\int_1^2\frac{3x+6}{\left(x^2+4x+3\right)^4}dx$; 6. $\int_0^1\frac{3}{(3-2x)^4}dx$;

4. $\int_0^2 (2x+1)^3 dx=\frac{1}{2}\int_0^2 2(2x+1)^3 dx$ = ½ $\left[\frac{(2x+1)^4}{4}\right]_0^2$ = 1/8 (625 - 1).

7. $\int_2^3\left(3x^2-\frac{4}{x^2}+\sqrt{x}\right)dx$; 8. $\int_{\pi/6}^{\pi/4}$; 9. $\int_1^e\frac{\ln^3 x}{x}dx$; 10. $\int_0^{\pi/4}\tan x dx$;

11. $\int_0^1 (x+1)^n dx$; 12. $\int_0^1 (3x+1)^5 dx$; 13. $\int_0^1\frac{x}{\sqrt{\left(x^2+1\right)^3}}dx$; 14. $\int\sin x\cos^3 x dx$;

15. $\int_0^1 x\left(x^2+1\right)^5 dx$; 16. $\int_{-2}^{1}\sqrt{x+3}dx$; 17. $\int_{-\pi/4}^{\pi/4}\tan^2 x dx$; 18. $\int_1^e\frac{1+\ln x}{x}dx$;

19. $\int_0^{\pi/3}\cos(2x+\pi/3)dx$; 20. $\int_0^1\left(x+e^x\right)dx$; 21. $\int_{-1}^{0}(2x+1)e^{x^2+x+1}dx$;

22. $\int_0^1\frac{3x}{x^2+4}dx$; 23. $\int_1^2\frac{10x+1}{\sqrt{5x^2+x+3}}dx$; 24. $\int_0^{\pi}\cos x\left(1-3\sin^2 x\right)dx$;

25. $\int_1^2\sqrt{3x-2}dx$; 26. $\int_1^2 x\left(x^2-3\right)dx$; 27. $\int_3^5\left(2x-1+\frac{1}{(x-1)^2}\right)dx$;

EXERCISE 2

Calculate the following integrals by using integration by parts:

1. $\int_0^{\pi/2} x\sin x dx$; 2. $\int_0^1 x^2e^x dx$; 3. $\int_0^1\left(-2x^2+x+1\right)e^x dx$; 4. $\int_1^2 x\sqrt{2x+1}dx$;

$\int_{-1}^{0}(2x+1)e^{x^2+x+1}dx=\left[e^{x^2+x+1}\right]_{-1}^{0}$ = (e ¹ – e) = 0.

Exercise 1

Part A:

Consider the function g that is defined over $\mathbb{R}$ by: $g(x)=(2x-1)e^x+1$.

1. Calculate $\lim_{x\to-\infty} g(x)$ *and* $\lim_{x\to+\infty} g(x)$.

Solution of part 1: $\lim\limits_{x\to-\infty} g(x) = \lim\limits_{x\to+\infty} \dfrac{(2x-1)'}{(e^{-x})'} + 1 = \lim\limits_{x\to-\infty} \dfrac{2}{-e^{-x}} + 1 = \dfrac{2}{-\infty} + 1 = 0 + 1 = 1.$

$$\lim_{x\to+\infty} g(x) = (+\infty)e^{+\infty} + 1 = +\infty + 1 = +\infty.$$

2. Calculate the exact value of $g(0)$.

Solution part 2: g (0) = (0-1) e ^{0} +1 = -1 +1 = 0.

3. Calculate g '(x) and set up the table of g (x).

Solution of part 3:

g '(x) = (2) × e ^{x} + e ^{x} (2 x – 1) = e ^{x} (2 + 2 x – 1) = e ^{x} (2 x+1).

x	-∞	α		-1/2		0	+∞
g '(x)			-----	0	+		
g (x)	1	0	$\searrow$	-0.21	$\nearrow$	0	+∞

4. Show that the equation g (x) = 0 has a unique root α such that -2 < α < -0.5

Since g is continuous over]-2;-0.5[included in ℝ.

Since g is strictly decreasing from 1 > 0 to -0.21 < 0

Since g (-2) = 0.3233>0 and g (-0.5) = -0.21 < 0

Then the equation g (x) = 0 has a unique root α such that: -2 < α < -0.5.

5. Deduce the sign of $g(x)$ according to the values of x.

Solution of part 5:

g (x) > 0 for all x < α or x > 0 and g (x) < 0 for all α < x < 0.

<u>Part B</u>:

Consider the function f that is defined over ℝ by: $f(x) = (x-1)e^{2x} + e^{x}$.

Let (C) be the representative curve of the function f in the system $(O;\vec{i},\vec{j})$

1. Calculate $\lim\limits_{x\to+\infty} f(x)$.

Solution of part 1: $\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty} \left[(x-1)e^{2x} + e^{x}\right] = +\infty + \infty = +\infty$

2. Calculate $\lim\limits_{x\to-\infty} f(x)$. Deduce an asymptote to the curve (C).

Solution of part 2: $\lim\limits_{x\to-\infty} f\left(x\right)=\lim\limits_{x\to-\infty}\frac{\left(x-1\right)'}{\left(e^{-2x}\right)'}+e^{-\infty}=\lim\limits_{x\to\infty}\frac{1}{-2e^{-2x}}+0\ =0+0=0.$

So y = is H.A to (C).

3. Calculate $f'(x)$ and prove that, for all real number *x*, $f'(x)=g(x)\times e^{x}$.

4. a) By using the result of the question 4 of the part A, determine according to the values of x the sign of $f'(x)$.

b) Set up the table of variations of the function f over $\mathbb{R}$.

c) Take $\alpha\approx-1{,}3$, determine an approximate value of $f\left(\alpha\right)$.

5. a) Consider the function F that is defined over $\mathbb{R}$ by: $F(x)=\left(\frac{x}{2}-\frac{3}{4}\right)e^{2x}+e^{x}$

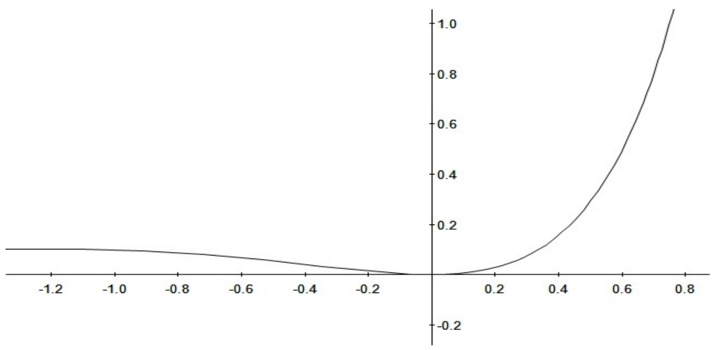
Prove that the function F is a primitive of the function f over $\mathbb{R}$.

b) Calculate the area of the domain bounded by (C), the x – axis and the two straight lines x = 0 and x = 1.

Figure 1

x	$-\infty$		$-1/2$		0		$+\infty$
$g'(x)$		$-$	0			$+$	
$g(x)$							

Figure 2



Exponential Functions

1. $y = f(x) = e^x$ is called exponential function
2. $y = e^x > 0$ for all real number x.
3. $e^0 = 1$; $e^1 = e$; $e^{-\infty} = 0$ and $e^{+\infty} = +\infty$.
4. $\left(e^x\right)' = e^x$; $(e^{ax})' = ae^{ax}$; $\left(e^{ax+b}\right)' = ae^{ax+b}$; $\left(e^u\right)' = u'e^u$.
5. $e^x = a > 0$ then $x = \ln a$.
6. $e^{\ln a} = a$ for all $a > 0$.
7. $\lim\limits_{x \to +\infty} \frac{e^x}{x} = +\infty$; $\lim\limits_{x \to +\infty} \frac{x}{e^x} = 0$.
8. $\int e^x dx = e^x + C$; $\int e^{ax} dx = \frac{1}{a} e^{ax} + C$; $\int u' e^u dx = e^u + C$.

Exercise 1

Part A:

The plane is referred to a direct orthonormal system $\left(O; \vec{i}, \vec{j}\right)$.

Consider the function g defined over $]0; +\infty[$ by : $g(x) = -3 - \ln x + \frac{1}{x}$ and designate by (C_g) its representative curve.

1) Study the variations of g and set up its table of variations.
2) Prove that the equation g (x) = 0 has a unique solution α and verify that: $0.45 < \alpha < 0.46$.
3) Deduce the sign of g (x) over $]0; +\infty[$.

Part B: Consider the function f defined over $]0; +\infty[$ by: $f(x) = e^{-x}(3 + \ln x)$ and designate by (C) its representative curve in an orthonormal system $\left(O; \vec{i}, \vec{j}\right)$.

1) Calculate the limits of f at the boundaries of its domain of definition and deduce the equations of the asymptotes to (C).
2) a – Prove that for all x of $]0; +\infty[$, $f'(x) = e^{-x}.g(x)$.

   b- Study, according to the values of x, the sign of $f'(x)$ and set up the tableau of variations of f.
3) Prove that $f(\alpha) = \frac{e^{-\alpha}}{\alpha}$. Suppose that $\alpha = 0.455$.
4) Calculate $f\left(e^{-3}\right)$ and draw (C).

Solution of Exercise 1

Part A:

1. g '(x) = $-\frac{1}{x} - \frac{1}{x^2} < 0$ for all x belongs to the interval $]0;+\infty[$ So g is strictly decreasing for all $x > 0$.

X	0	$\alpha = 0.455$	$+\infty$
g '(x)	‖	---------	
g (x)	‖ $+\infty$	0	$\rightarrow -\infty$

2. Since g is continuous over the interval $]0.45; 0.46[$ included in $]0; +\infty[$.

Since g is strictly decreasing from + ∞ to - ∞ for all x in]0; +∞[.
Since g (0.45) = 0.02 > 0 and g (0.46) = -0.049 < 0.
Then the equation g (x) = 0 has a unique root **α** such that: 0.45 < **α** < 0.46.

3. We have: g (x) > 0 for all 0 < x < **α** and g (x) < 0 for all x > **α**.

Part B:

1. $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+}\left[e^{-x}(3+\ln(x))\right] = e^{0}\left(3+\ln\left(0^{+}\right)\right) = 1\times(-\infty) = -\infty.$ So x = 0 is V.A to (C).

$\lim\limits_{x\to+\infty} f(x) = \lim\limits_{x\to+\infty}\dfrac{(3+\ln(x))'}{\left(e^{x}\right)'} = \lim\limits_{x\to+\infty}\dfrac{1/x}{e^{x}} = \lim\limits_{x\to+\infty}\dfrac{1}{xe^{x}} = \dfrac{1}{+\infty} = 0$ So y = 0 is H.A to (C).

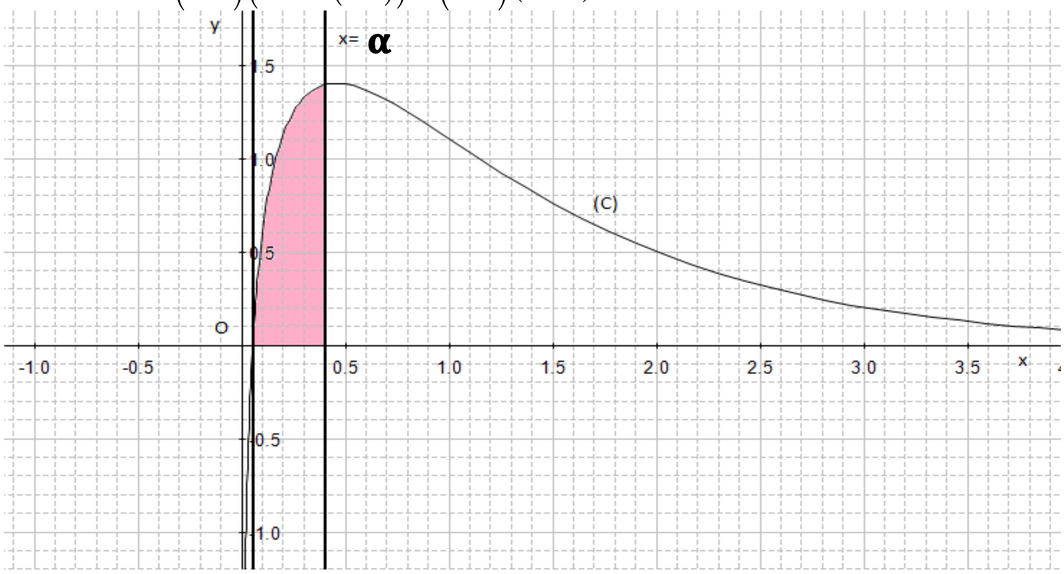
2. a - $f'(x) = \left(-e^{-x}\right)(3+\ln(x)) + \left(\dfrac{1}{x}\right)e^{-x} = e^{-x}\left[-3-\ln(x)+\dfrac{1}{x}\right]$ = $e^{-x}\times$ g (x).

b- If 0 < x < **α** then g (x) > 0 So f ‘(x) > 0 thus f is strictly increasing.
If x = **α** then g (x) = 0 So f ‘(x) = 0 thus f is constant.
If x > **α** then g (x) < 0 So f ‘(x) < 0 thus f is strictly decreasing.

x	0		α = 0.455		+∞
f ‘(x)	||	+	0	------	
f (x)	|| -∞	↗	1.403	↘	0

3. f (**α**) = $e^{-\alpha}$ (3 + ln (**α**)) =
But we have g (**α**) = 0 then – 3 – ln (**α**) + 1/**α** = 0 So ln (**α**) = - 3 + 1/**α**.
Thus f (**α**) = $e^{-\alpha}$ (3 – 3 + 1/**α**) = $\dfrac{e^{-\alpha}}{\alpha}$.

4. f (e^{-3}) = $\left(e^{e^{-3}}\right)\left(3+\ln\left(e^{-3}\right)\right) = \left(e^{e^{-3}}\right)(3-3) = 0.$



Exercise 2

Part A: Consider the function g that is defined over IR by: $g(x) = (4-x)e^{-\frac{x}{2}} - 1.$

1. Calculate $\lim\limits_{x\to-\infty} g(x)$ *and* $\lim\limits_{x\to+\infty} g(x).$
2. Calculate $g'(x)$, and then set up the table of variations of g.
3. Prove that the equation g (x) = 0 admits a unique solution α, and verify that: 1.6 < α < 1.8.
4. Deduce the sign of g (x) for all real x.

Part B:

Consider the function f that is defined on IR by: $f(x)=(2x-4)e^{-\frac{x}{2}}+2-x.$ Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Graphic unit = 2 cm.

1. Calculate $\lim_{x\to-\infty} f(x)$ *and* $\lim_{x\to+\infty} f(x)$.
2. Prove that the straight line (d) of equation y = 2 – x is an asymptote to (C) at + ∞.
3. Prove that: $f'(x)=g(x)$. study the sense of variations of f.
4. Prove that: $f(\alpha)=\dfrac{\alpha^2-4\alpha+4}{4-\alpha}$. Set up the tableau of variations of f.
5. a- Calculate the coordinates of the points of intersection of (C) and the axis of abscissas.
   b- Calculate the coordinates of E the intersection point of (C) and the axis of ordinates
   a- Write an equation of the tangent (T) to (C) at the point E.
6. Suppose that α = 1.7. Draw (d), (T) and (C).
7. Calculate, in cm^2, the area of the region bounded by the curve (C), (d) and the straight lines of equations: x = 0 and x = 2.