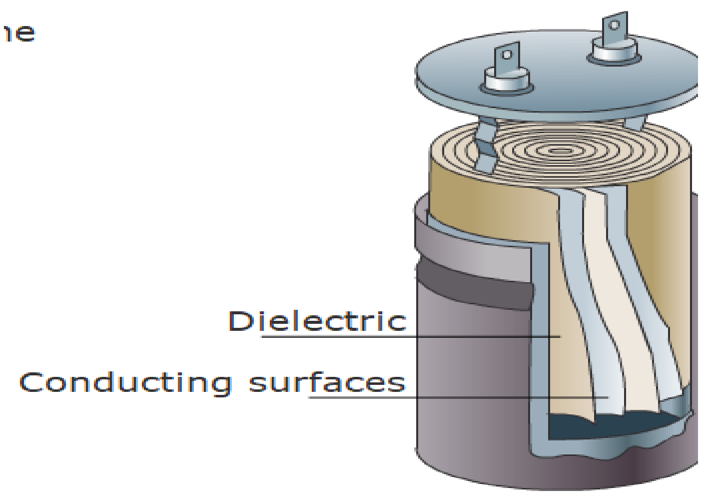
capacitors

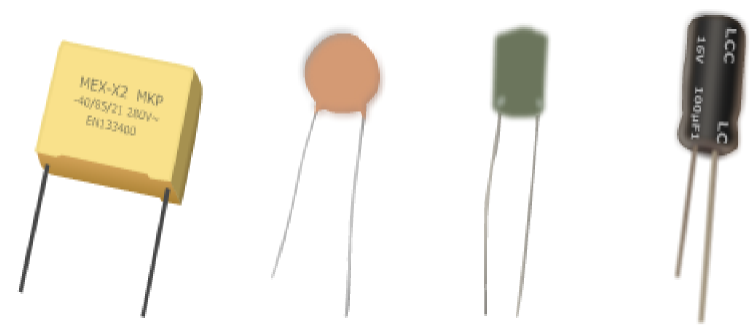
Learning objectives

- Define a capacitor
- Define the capacitance of a capacitor.
- Give the expression of the capacitance of a parallel capacitor.
- Know the expression of the energy stored in a charged capacitor.
- Define the breakdown potential.
- Explain the charging and discharging of a capacitor (R-C series circuit, time constant $\tau = RC$)

Definition

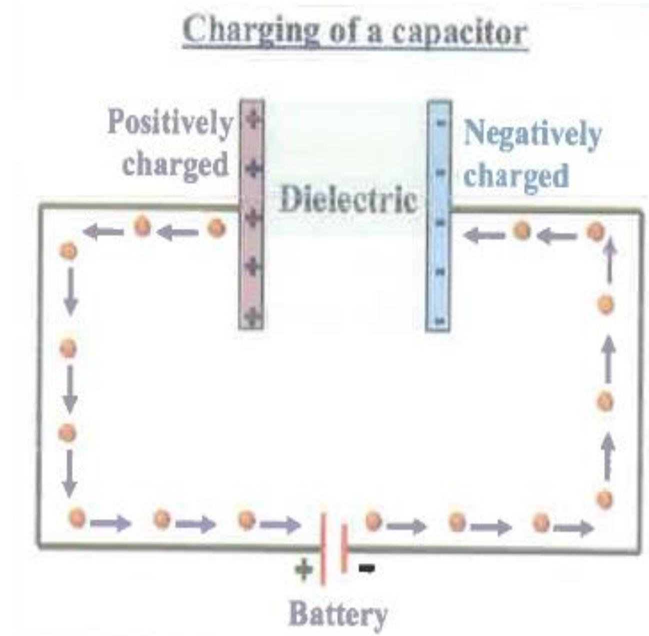
* A capacitor is an electric device formed of two conducting plates (called armatures) facing each other and separated by an insulator (dielectric).
* when the armatures are plane and parallel then the capacitor is called parallel plane capacitor



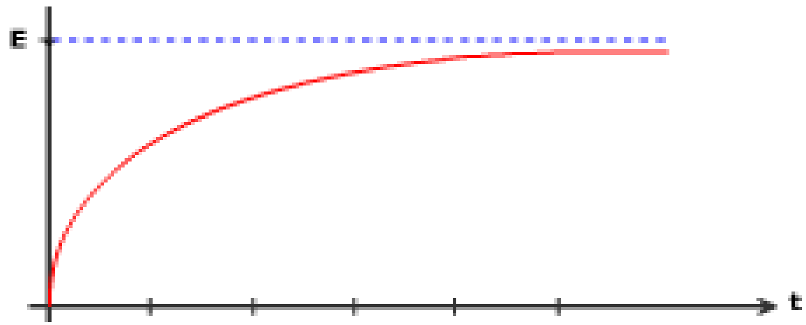


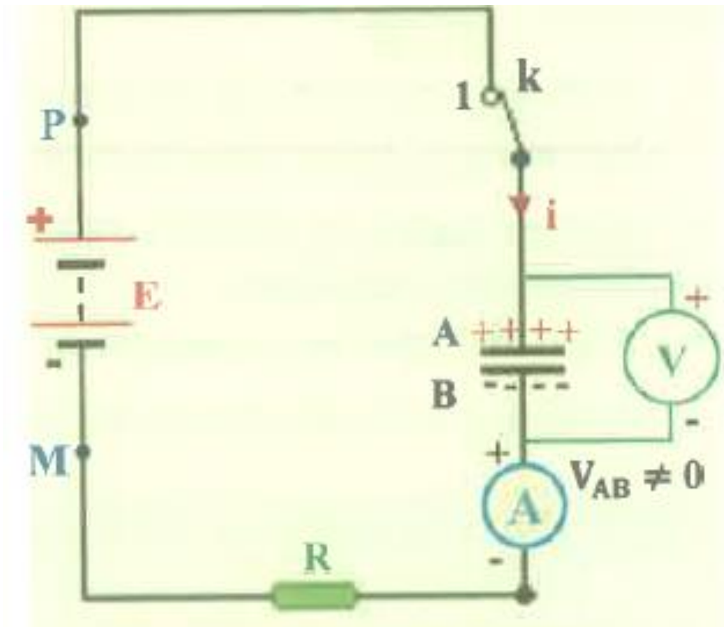
Charging of a capacitor

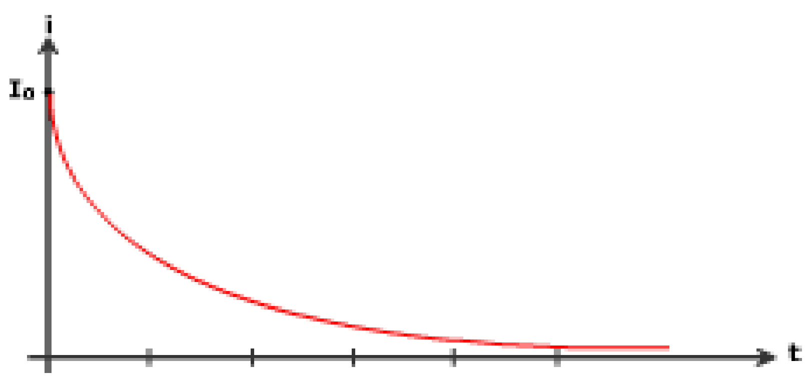
*If no excess or deficit of electrons on the plates of the capacitor, we say that the capacitor is neutral or uncharged.
*When a capacitor is connected across a battery, electrons flow from the negative pole of the battery to a plate of the capacitor connected to it. At the same rate, electrons flow from the other plate of the capacitor to the positive pole of the battery.



- As charges accumulate on the plates of the capacitor, the potential difference across the plates increases while the current decreases .
- When $U_C = U_{PM} = E$ (e.m.f),
  The current becomes zero and the capacitor is fully charged.
  The charge of the capacitor $Q = q_A = -q_B = N\,e$
- **Note**: No current passing through the capacitor since there is a dielectric between the plates.



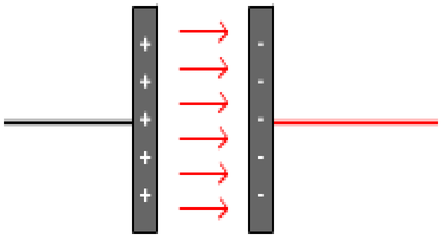




Capacitance of a capacitor

The charge of a capacitor is directly proportional to its voltage U where the ratio of proportionality between Q and U:
Q/U = C = constant > 0 it is called the **capacitance** of the capacitor that depends on the capacitor it self . Its unit in the SI units is the Farad (F)

<u>Capacitance of a parallel plate capacitor</u>



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* Dielectric is vacuum **or air**

Capacitance $\mathbf{C_0 = \frac{Q}{V} = \varepsilon_0 \frac{S}{d}}$

Where: S is the common area (m^2)

d distance between armatures (m)

ε_o is the permittivity of the medium(air)

* Dielectric is an insulating material : $\boldsymbol{C = \frac{Q}{V} = \varepsilon \frac{S}{d}}$

$\varepsilon_r = \boldsymbol{\varepsilon/\varepsilon_0 > 1}$ where ε_r is the relative permittivity of a dielectric and

$$\boldsymbol{\varepsilon_o = \frac{1}{36\pi\times10^9} = 8.85 \times 10^{-12} \frac{C}{Nm^2}} \text{ or F/m}$$

Stored electric energy

Break down voltage:

Every dielectric has a limit of electric field E_{max} whenever the field exceeds the maximum value it causes damage for the dielectric (becomes conductor)

$E = \frac{U}{d} \rightarrow \ U_{max} = d \times E_{max}$ The break down potential

Stored electric energy:

The electric energy stored in a capacitor is given as

$W = \frac{1}{2} C U^2$ $= \frac{1}{2} Q U = \frac{1}{2} Q^2 / C$

Note: During charging, the capacitor acts as a receiver (electric device)

Grouping capacitors

* **Definition:** the equivalent Capacitor is a single capacitor that replaces grouping of capacitors and acquires the same charge Q under the same voltage U.

Parallel connection:

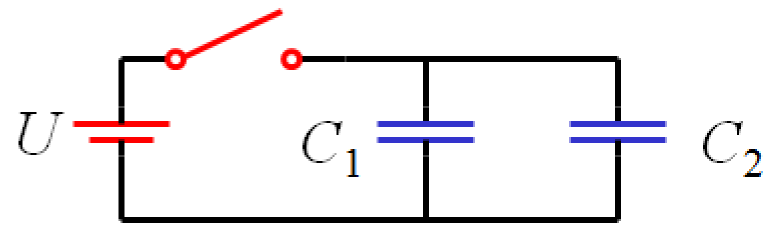
$Q_1 = C_1 U$ & $Q_2 = C_2 U$, $Q = CU$

but $\mathbf{Q = Q_1 + Q_2}$

→ $C_{eq} U = C_1 U + C_2 U$

→ $C_{eq} U = (C_1 + C_2)U$

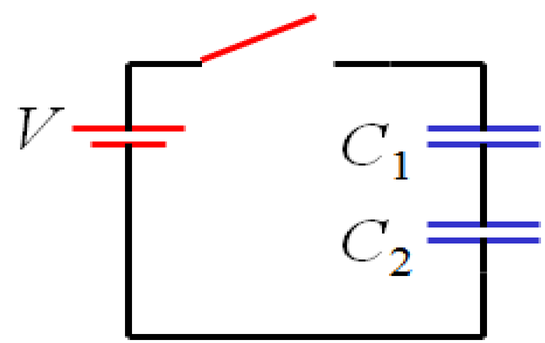
Thus $\mathbf{C_{eq} = C_1 + C_2 + \dots C_n}$



***Series connection**:* $\mathbf{Q_1 = Q_2 = Q}$

$U_1 = \frac{Q}{C_1}$ & $U_2 = \frac{Q}{C_2}$ $but\ \boldsymbol{U} = \boldsymbol{U_1} + \boldsymbol{U_2} then\ \frac{Q}{C_{eq}} = \frac{Q}{C_1} + \frac{Q}{C_2}$

$$\frac{\mathbf{1}}{\mathbf{C_{eq}}} = \frac{\mathbf{1}}{\mathbf{C_1}} + \frac{\mathbf{1}}{\mathbf{C_2}} + \dots \frac{\mathbf{1}}{\mathbf{C_n}}$$



Time constant of R-C circuit

The time constant (τ =RC) is the time taken by the capacitor to charge 63% of its maximum charge Q during charging phase.

Or

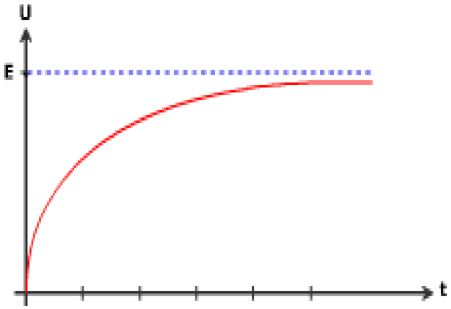
(τ =RC) It is the time taken by the capacitor to discharge 63% of its maximum charge Q during discharging phase.

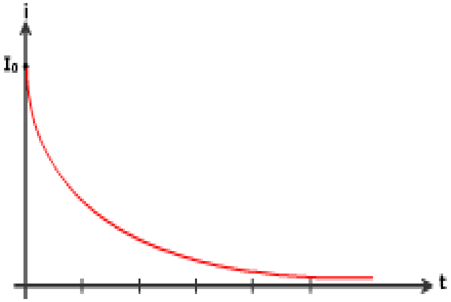
Charging of a capacitor:

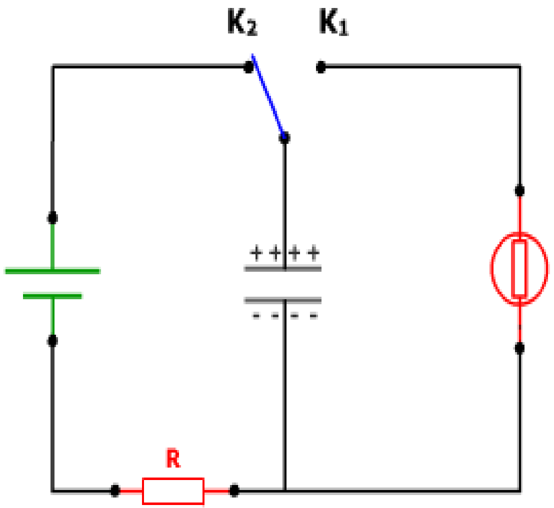
For $t_o = 0$,
$u_c = q_c = 0$
$i_o = I_{max} = E/R$

$t = \tau = RC$,
$q = 0.63Q$,
$u_c = 0.63U_{max}$

$t \approx 5\tau = 5RC$,
$u_c = U_{max} = E$ (emf of G)
$q = q_{max} = Q$
$i = 0$





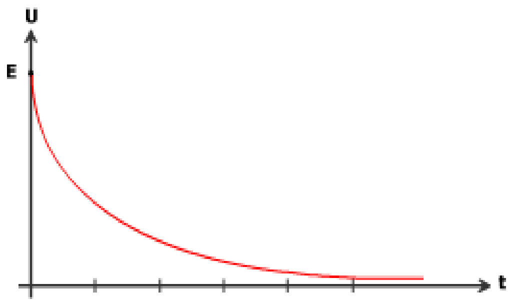


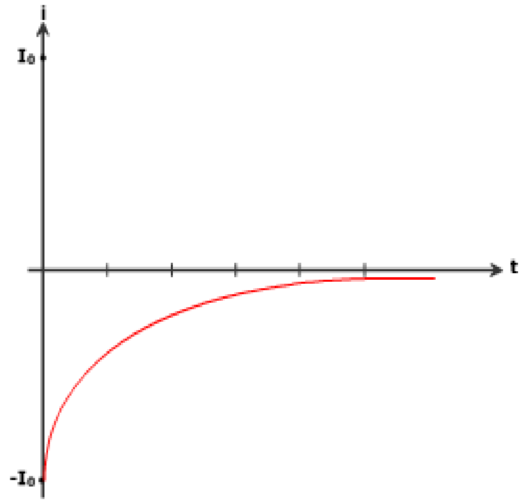
Discharging of a capacitor

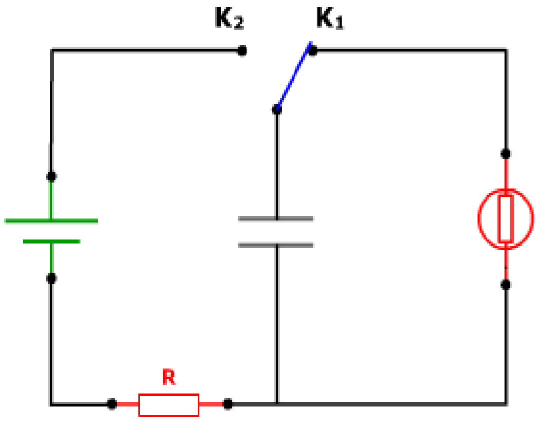
For $t_o = 0$: $u_c = U_{max} = E$ (emf of G), $qc = q_{max} = Q$, $i = -I_{max} = -E/R$

$t = \tau = RC$: $q = 0.37Q$ $u_c = 0.37U_{max}$

$t \approx 5\tau = 5RC$: $q = u = 0$ $i = 0$







Animation from Phet And Edulab

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charging phase of the capacitor

1. Differential equation in voltage u_C

According to the law of addition of voltage

$u_{AB} = u_{AM} + u_{MB}$

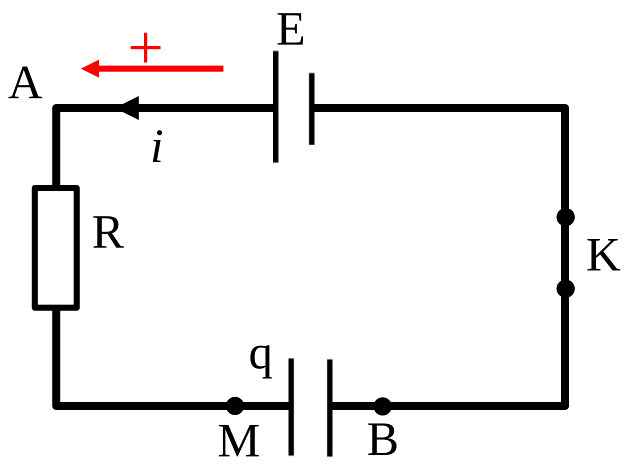
$u_G = u_R + u_C$ with: $u_C = u_{MB}$, $u_R = u_{AM}$

$E = Ri + u_C$

with $q = Cu_C$ and $i = +\frac{dq}{dt} = +C\frac{du_C}{dt}$

Then $E = RC\frac{du_C}{dt} + u_C$

or $$\frac{E}{RC} = \frac{du_C}{dt} + \frac{u_c}{RC}$$



This is first order differential equation that governs the variation of u_c with respect to time during charging phase

2. Solution of the differential equation

$$\mathbf{u_C = E\left(1 - e^{-\frac{t}{\tau}}\right)}$$

t = 0s:

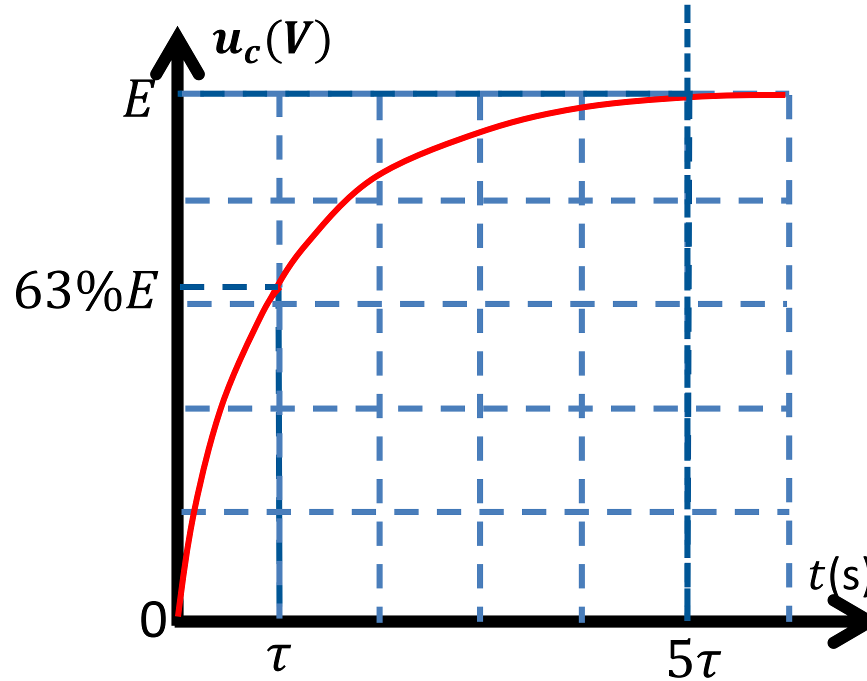
$$u_C = E\left(1 - e^{-\frac{0}{\tau}}\right) = E(1-1) = 0\text{V}.$$

t = τ :

$$u_C = E\left(1 - e^{-\frac{\tau}{\tau}}\right) = E(1 - 0.37) = 0.63\ \text{E}$$

t = 5τ :

$$u_C = E\left(1 - e^{-\frac{5\tau}{\tau}}\right) = E(1 - 0.007) = 0.99\ \text{E}$$



3. Verification of the solution :

$$u_C = E\left(1 - e^{-\frac{t}{\tau}}\right) \qquad u_C = E - Ee^{-\frac{t}{\tau}}$$

$$\frac{du_c}{dt} = \frac{E}{\tau}e^{-\frac{t}{\tau}} \qquad (e^u)' = u'e^u$$

Substitute in the differential equation: $\frac{E}{RC} = \frac{u_c}{RC} + \frac{du_C}{dt}$

$$\frac{E}{RC} = \frac{E\left(1-e^{-\frac{t}{\tau}}\right)}{RC} + \frac{E}{\tau}e^{-\frac{t}{\tau}}$$

$$\frac{E}{RC} = \frac{E}{RC} - \frac{E}{RC}e^{-\frac{t}{\tau}} + \frac{E}{RC}e^{-\frac{t}{\tau}}$$

$\frac{E}{RC} = \frac{E}{RC}$ So the equation is verified

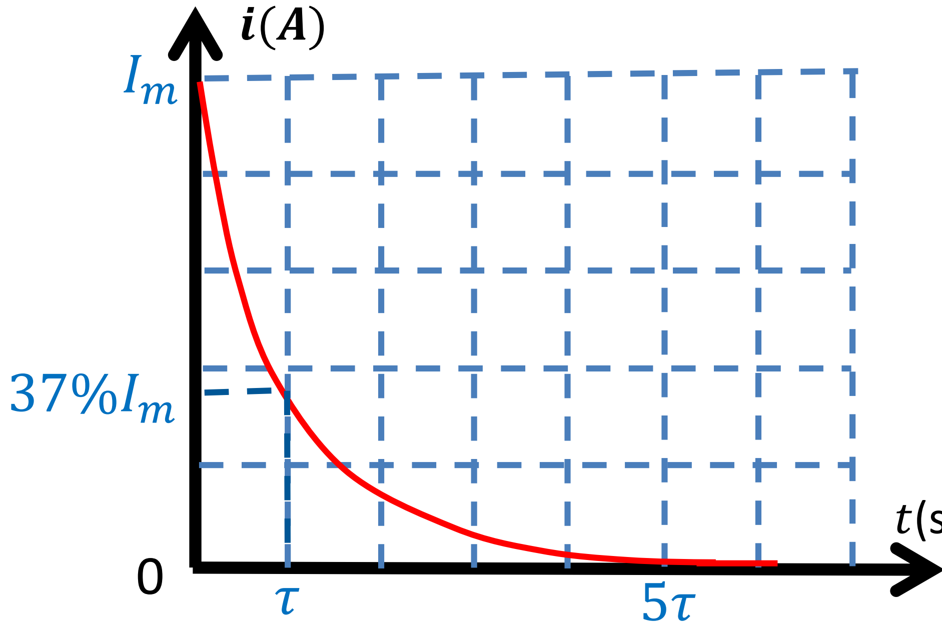
4. Expression of the current i as function of time:

$u_G = u_R + u_C \quad E = Ri + u_C$

$$i = \frac{E-u_C}{R} = \frac{E-E\left(1-e^{-\frac{t}{\tau}}\right)}{R}$$

$$= \frac{E-E+Ee^{-\frac{t}{\tau}}}{R} = \frac{+Ee^{-\frac{t}{\tau}}}{R}$$

$$\boldsymbol{i = I_m e^{-\frac{t}{\tau}}}$$



Or derive w.r.t time the eq. ($E = Ri + u_C$)

$$\boldsymbol{0 = R\frac{di}{dt} + \frac{du_c}{dt} = R\frac{di}{dt} + \frac{i}{C}}$$

$$\boldsymbol{\frac{di}{dt} + \frac{i}{RC} = 0 \quad i = I_m e^{-\frac{t}{\tau}}}$$

5. Calculation of the time constant τ :

Draw the tangent to the curve $u_C = \mathrm{f(t)}$ at $t_0 = 0$.

Charging phase: $u_C = E(1 - e^{-\frac{t}{\tau}})$

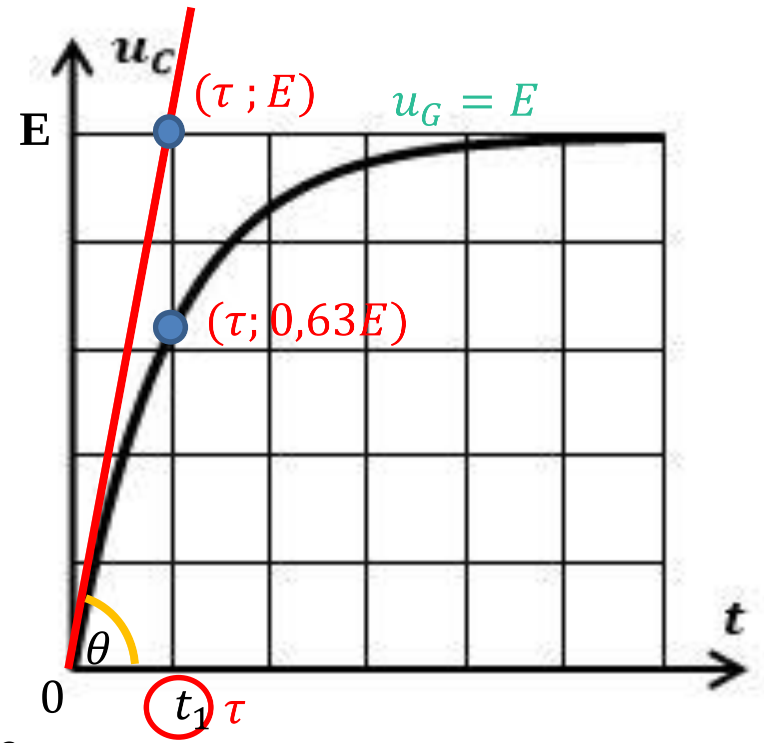
$\frac{du_c}{dt} = \frac{E}{\tau} e^{-\frac{t}{\tau}}$ $\quad\quad$ $\left(\frac{du_c}{dt}\right)_{t=0} = \frac{E}{\tau}$

From graph $\boldsymbol{tan\theta = \frac{E}{t_1}}$

$by\ comparison, same\ slope\ \frac{E}{t_1} = \frac{E}{\tau} \Rightarrow \boldsymbol{t_1 = \tau}$

but $\frac{E}{RC} = \frac{du_C}{dt} + \frac{u_c}{RC}$

$\left(\frac{du_c}{dt}\right)_{t=0} = \frac{E}{\tau}$ **and uc = 0 then** $\frac{E}{RC} = \frac{E}{\tau}, \boldsymbol{\tau} = \mathrm{RC}$



Conclusion:
The straight line tangent to the curve uc at t = 0 cuts the horizontal asymptote u = E at point of abscissa $t_1 = \tau = RC$

Discharging of a capacitor

1- Differential equation in voltage u_C

According to the law of addition of voltage

$$u_{MB} = u_{MA} + u_{AB}$$

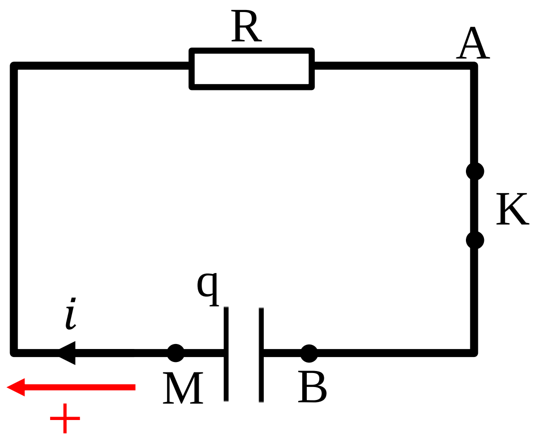
$$u_{MB} = u_{MA} \quad (u_C = u_{MB} \quad ; \quad u_{MA} = Ri)$$

$$u_C = Ri$$

with $q = Cu_C$ and $i = -\frac{dq}{dt} = -C\frac{du_C}{dt}$

$$u_C - Ri = 0$$

$$u_C + RC\frac{du_C}{dt} = 0$$



$$\boxed{\frac{du_C}{dt} + \frac{u_c}{RC} = 0}$$

First order differential equation that governs the variation of u_c with respect to time during discharging phase

2- Solution of the differential equation:

$$\boxed{u_C = E\,e^{-\frac{t}{\tau}}}$$

Where $\tau = \mathrm{RC}$ is the time constant

For t = 0s:

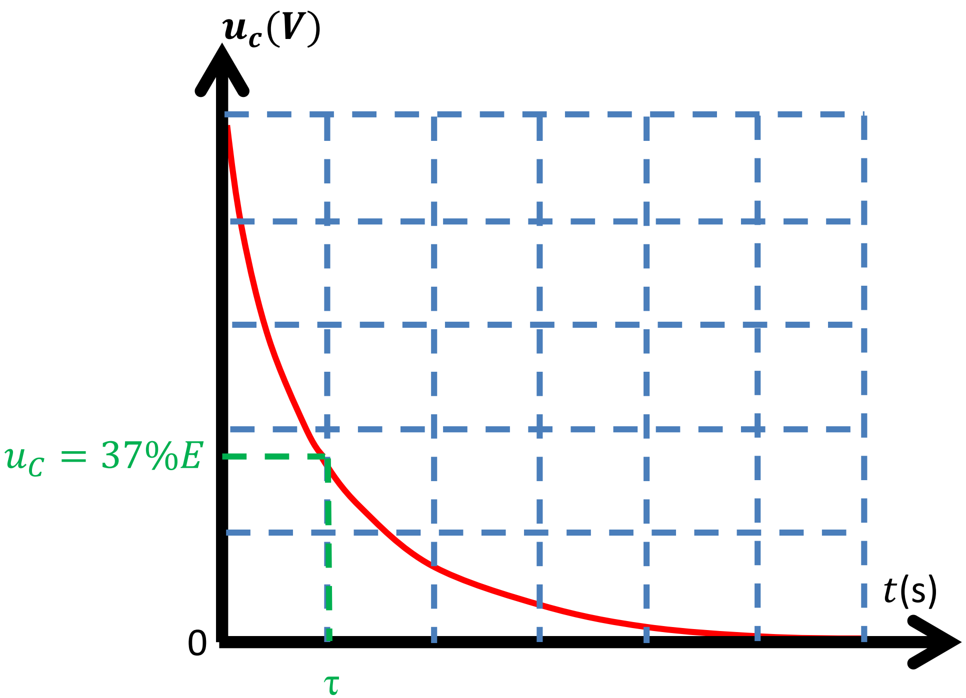
$$u_C = E\left(e^{-\frac{0}{\tau}}\right) = E(1) = E.$$

For t = τ :

$$u_C = E\left(e^{-\frac{\tau}{\tau}}\right) = E(0{,}37) = 0{,}37E$$

For t = 5τ :

$$u_C = E\left(e^{-\frac{5\tau}{\tau}}\right) = E(0{,}007) = 0{,}007E$$



3- Verification of the solution :

$$\boxed{u_C = E\,e^{-\frac{t}{\tau}}}$$

Where $\tau = RC$

$$\frac{du_c}{dt} = -\frac{E}{\tau}e^{-\frac{t}{\tau}}$$

$$(e^u)' = u'\,e^u$$

Substitute in the differential equation : $\frac{u_c}{RC} + \frac{du_C}{dt} = 0$

$$\frac{E\,e^{-\frac{t}{\tau}}}{RC} - \frac{E}{\tau}e^{-\frac{t}{\tau}} = 0$$

$$\frac{E\,e^{-\frac{t}{\tau}}}{RC} - \frac{E}{RC}e^{-\frac{t}{\tau}} = 0$$

$$0 = 0$$

So the solution is verified

4- Finding the time constant τ

The tangent to the curve u_C = f(t) at $t_0 = 0$

$$u_C = Ee^{-\frac{t}{\tau}}$$

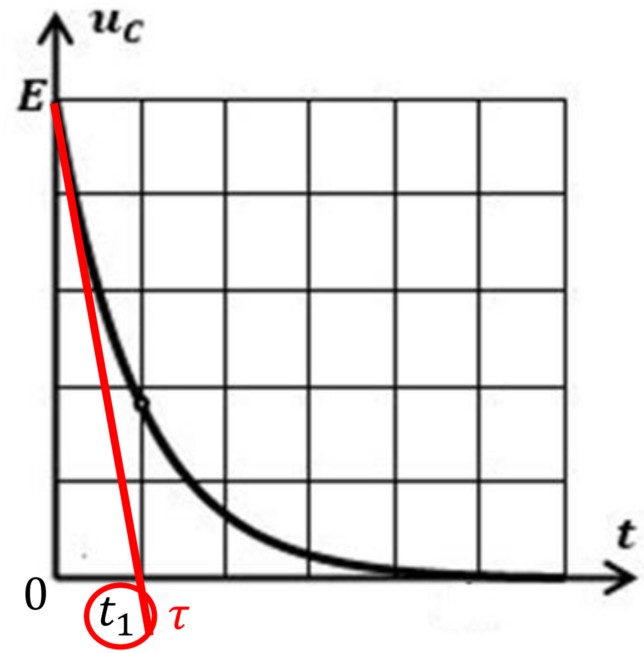
$$\frac{du_c}{dt} = -\frac{E}{\tau}e^{-\frac{t}{\tau}} \qquad \left(\frac{du_c}{dt}\right)_{t=0} = -\frac{E}{\tau}$$

Slope of the straight line: $\frac{\Delta u}{\Delta t} = \frac{0-E}{t_1-0} = -\frac{E}{t_1}$

by comparison: $-\frac{E}{t_1} = -\frac{E}{\tau} \Rightarrow t_1 = \tau$

Use diff. Equation $\frac{du_C}{dt} + \frac{u_c}{RC} = 0$

t = 0s, $\frac{du_C}{dt} + \frac{E}{RC} = 0 \quad \frac{du_C}{dt} = \frac{-E}{RC} = -\frac{E}{\tau} \qquad \boldsymbol{\tau = \mathrm{RC}}$



Conclusion:

The straight line tangent to the curve uc at t = 0 cuts the horizontal asymptote u = E at point of abscissa $t_1 = \tau = RC$

5- Expression of the electric current i during time :

$$u_{MB} = u_{MA} \quad u_C = Ri$$

$$i = \frac{u_C}{R} = \frac{Ee^{-\frac{t}{\tau}}}{R} \quad i = I_m e^{-\frac{t}{\tau}}$$

Or derive wrt time:

$$\frac{du_c}{dt} = R\frac{di}{dt}$$

But i = $-\frac{dq}{dt}$ **= --C** $\frac{du_c}{dt} \quad \frac{du_c}{dt} = \frac{-i}{C}$

$$\frac{di}{dt} + \frac{i}{RC} = 0 \quad i = I_m e^{-\frac{t}{\tau}}$$