ExaMath Groups	Mathematics Exam Class: GS	Prepared by: Hassan Ahmad
Number of questions: 5	Sample 01 – year 2023 Duration: 3 hours	Name: N°:

- *This exam includes five problems on four pages.*
- *The use of a non-programmable calculator is allowed.*
- *Always show the steps of the calculation.*
- *Any unjustified answers will not be graded.*

I- (2 points)

In the table below, only one of the proposed answers is correct.

Choose the correct answer and justify your choice.

N°	Question	Proposed responses A	Proposed responses B	Proposed responses C
1)	The value of the limit $\lim\limits_{x\to+\infty}\dfrac{e^x+1}{xe^x+2x}$ is	$+\infty$	1	0
2)	The imaginary part of the complex number z such that $\left\lvert\dfrac{z-2i}{z+i}\right\rvert=1$ is	$\dfrac{1}{2}$	$-\dfrac{3}{2}$	0
3)	The solution set of the inequality $\ln\left(e^{2x}-2e^x+1\right)\le 0$ is $S=$	$]-\infty\ ;\ 0[\cup]0\ ;\ \ln 2]$	$]-\infty\ ;\ \ln 2]$	$[\ln 2\ ;\ +\infty[$
4)	Let A and B be two events of a sample space Ω and P a probability. Given $P(A)=0.3$, $P(B)=0.5$ and $P(A\cup B)=0.65$. Then $P(A/B)=$	0.8	0.3	0.5

II- (3½ points)

The complex plane is referred to an orthonormal system $(O\ ;\ \vec{u}\ ;\ \vec{v})$.

Consider the points A and B with respective affixes $z_A = i$ and $z_B = 1$.

For every point M of affix z we associate the point M' of affix z' such that: $z' = \dfrac{i\,\overline{z} - 1}{\overline{z} - 1}$ with $z \neq 1$.

1) In case where $z' = -1$ prove that z^{12} is a negative real number.

2) **a)** Show that for every point M distinct from B we have: $|z'| = \dfrac{AM}{BM}$.

b) Deduct the set of points M' when M describes the perpendicular bisector of $[AB]$.

3) **a)** Show that $\arg(z') = \dfrac{\pi}{2} + (\overrightarrow{AM}\ ;\ \overrightarrow{BM})[2\pi]$.

b) Determine the set of points M when z' is a strictly negative real number.

4) In this part suppose that $z = 1 + \sqrt{2}e^{i\theta}$ where θ is a real number.

a) Show that M describes the circle $(\mathscr{C})$ of center B and radius $\sqrt{2}$.

b) Calculate $(z' - i)(\overline{z} - 1)$.

c) Deduct the set of points M' when M describes the circle $(\mathscr{C})$.

III- (3½ points)

Consider two urns:

- The urn U_1 containing three balls carrying the numbers 2, 2, and 3;
- The urn U_2 containing 9 balls of which 4 are green and each carrying the number 3, and 5 balls are red carrying the numbers 1, 2, 2, 3 and 4.

We draw a ball from the urn U_1 and we note n the number carried by this ball:

- If $n = 2$, two balls are drawn at random and successively without replacement from the urn U_2;
- If $n = 3$, three balls are drawn at random and simultaneously from the urn U_2.

Consider the events:

E: "The ball drawn from the urn U_1 carries the number 2";

F: "The ball drawn from the urn U_1 carries the number 3";

A: "The balls drawn from the urn U_2 have the same color";

B: "The balls drawn from the urn U_2 carry the same number".

1) **a)** Calculate the probabilities $P(E)$ and $P(F)$.

b) Calculate $P(A/E)$, and justify that $P(A \cap E) = \dfrac{8}{27}$.

c) Calculate $P(A \cap F)$ and deduce that $P(A) = \dfrac{19}{54}$.

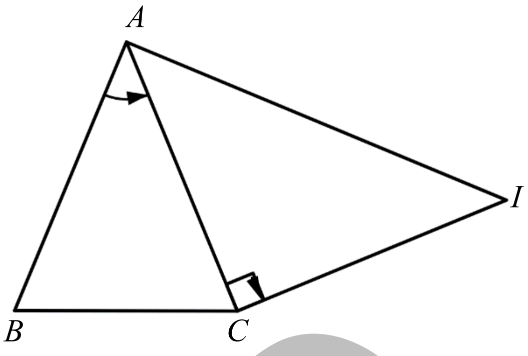
d) The balls drawn from the urn U_2 are of different colors, calculate the probability that the ball drawn from U_1 carries the number 3.

2) Calculate $P(B)$.

3) Justify that $P(A \cap B) = \dfrac{55}{378}$. Are the events A and B independent? Justify.

4) Calculate the probability of the event C: "Among the balls drawn from the urn U_2 there is exactly one red ball".

IV- (3½ points)



ABC is an isosceles triangle such that $AB = AC$ and $\left(\overrightarrow{AB}\ ;\ \overrightarrow{AC}\right) = \frac{\pi}{4}(2\pi)$.

I is the point such that the triangle CAI is isosceles and right with $\left(\overrightarrow{CA}\ ;\ \overrightarrow{CI}\right) = -\frac{\pi}{2}(2\pi)$.

1) Let R_A be the rotation of center A which transforms B into C and R_C the rotation of center C and angle $-\frac{\pi}{2}$.

Let $f = R_C \circ R_A$.

- **a)** Determine $f(A)$ and $f(B)$.
- **b)** Prove that f is a rotation whose center O and angle to be specified.
- **c)** What is the nature of the quadrilateral $ABOC$? Justify.

2) Let S be the direct plane similitude of center O which transforms A into B. Let H be the midpoint of $[BC]$, $C' = S(C)$ and $H' = S(H)$.

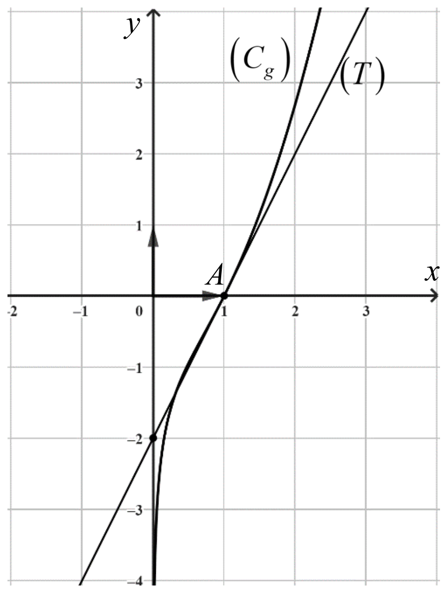
- **a)** Give a measure of the angle of S and show that C' belongs to the straight line (OA).
- **b)** Give $S([OA])$ and show that H' is the midpoint of $[OB]$.
- **c)** Show that $(C'H') \perp (OB)$. Deduce that C' is the center of the circle circumscribed about the triangle OBC.

3) Consider the transformation $S_n = \underbrace{S \circ S \circ \ldots \circ S}_{n \text{ fois}}$ where n is a natural number such that $n \geq 2$.

Determine the set of values of n for which S_n is a negative dilation.

V- (7½ points)

Part A



Let g be the function defined over $]0\ ;+\infty[$ by $g(x) = ax^2 + bx + \ln x$, where a and b are two real numbers.

The opposite curve (C_g) is the representative curve of g in an orthonormal system.

The straight line (T) is tangent to (C_g) at the point $A(1\ ;\ 0)$.

1) By graphical reading:

- **a)** Determine $g(1)$ and $g'(1)$.
- **b)** Determine the sign of $g(x)$ over $]0\ ;+\infty[$.

2) Show that $a = 1$ and $b = -1$.

3) Is the point A an inflection point for the curve (C_g)? Justify.

Part B

Let f be the function defined over $]0\ ;+\infty[$ by $f(x) = (x-1)^2 + (\ln x)^2$.

Denote by (C_f) the representative curve of f in an orthonormal system $\left(O\ ;\ \vec{i}\ ;\ \vec{j}\right)$.

1) Calculate $\lim\limits_{x \to 0^+} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.

2) **a)** Show that for every $x \in]0\,;+\infty[$, $f'(x) = 2\dfrac{g(x)}{x}$.

b) Set up the table of variations of the function f.

3) Let u be the function defined over $]0\,;\,1]$ by $u(x) = f(x) - x$.

a) Set up the table of variation of u over $]0\,;\,1]$. Deduce that there exists a unique real number α in $]0\,;\,1]$ such that $f(\alpha) = \alpha$ and that $\dfrac{1}{2} < \alpha < 1$.

b) Show that (C_f) admits at $+\infty$ a parabolic branch of direction $(O\,;\,\vec{j})$.

4) Draw the straight line (d) of equation $y = x$ and the curve (C_f).

5) Let h be the function defined by $h(x) = e^{f(x)}$.

a) Determine the domain of definition of h.

b) Set up the table of variations of h.

c) Solve in the interval $]0\,;\,1]$ the equation $h(x) = e^x$.

QI	Answers	2 pts
1)	$\lim\limits_{x\to+\infty}\frac{e^x+1}{xe^x+2x}=\lim\limits_{x\to+\infty}\frac{e^x\left(1+\frac{1}{e^x}\right)}{e^x\left(x+\frac{2x}{e^x}\right)}=\frac{1+0}{+\infty+0}=0$ **OR** we apply the Hospital's rule. The correct answer is **C**.	½
2)	Let $z=x+iy$ $(x\in\mathbb{R},\ y\in\mathbb{R})$; $\left\lvert\frac{z-2i}{z+i}\right\rvert=1$; $\frac{\lvert x+i(y-2)\rvert}{\lvert x+i(y+1)\rvert}=1$ $(x\neq0 \text{ and } y\neq-1)$; $\frac{\sqrt{x^2+(y-2)^2}}{\sqrt{x^2+(y+1)^2}}=1$; $(y-2)^2=(y+1)^2$; $-6y=-3$; $y=\frac{1}{2}$. The correct answer is **A**.	½
3)	Condition of existence: $e^{2x}-2e^x+1>0$; $(e^x-1)^2>0$, then $x\in\left]-\infty\,;\,0\right[\cup\left]0\,;\,+\infty\right[$; $\ln\left(e^{2x}-2e^x+1\right)\le0$ is equivalent to $\ln\left(e^{2x}-2e^x+1\right)\le\ln1$, $e^{2x}-2e^x+1\le1$, $e^{2x}-2e^x\le0$ so $x\in\left]-\infty\,;\,\ln2\right]$. The solution set is therefore: $S=\left(\left]-\infty\,;\,0\right[\cup\left]0\,;\,+\infty\right[\right)\cap\left(\left]-\infty\,;\,\ln2\right]\right)=$ $\left]-\infty\,;\,0\right[\cup\left]0\,;\,\ln2\right]$. The correct answer is **A**.	½
4)	$P(A\cup B)=P(A)+P(B)-P(A\cap B)$; $0.65=0.3+0.5-P(A\cap B)$; $P(A\cap B)=0.15$; $P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{0.15}{0.5}=0.3$. The correct answer is **B**.	½

QII	Answers	3½ pts
1)	$z'=-1$; $\frac{i\,\bar{z}-1}{\bar{z}-1}=-1$ and $z\neq1$; $i\,\bar{z}-1=-\bar{z}+1$; $\bar{z}(i+1)=2$; $\bar{z}=\frac{2}{1+i}=1-i$ then $z=1+i$; $z^{12}=(1+i)^{12}=\left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{12}=\left(\sqrt{2}\right)^{12}e^{3i\pi}=-64$ which is a negative real number.	½
2) a)	$z'=\frac{i\,(\bar{z}+i)}{\bar{z}-1}=\frac{i\left(\overline{z-i}\right)}{\overline{z-1}}$ therefore $\lvert z'\rvert=\frac{\lvert i\rvert\left\lvert\overline{z-i}\right\rvert}{\left\lvert\overline{z-1}\right\rvert}=\frac{\lvert z-i\rvert}{\lvert z-1\rvert}=\frac{\lvert z_M-z_A\rvert}{\lvert z_M-z_B\rvert}=\frac{\lvert z_{\overline{AM}}\rvert}{\lvert z_{\overline{BM}}\rvert}=\frac{AM}{BM}$.	½
2) b)	M describes the perpendicular bisector of the segment $[AB]$ so $MA=MB$, therefore $\lvert z'\rvert=\frac{AM}{BM}=1$, then $OM'=1$; The set of points M' is the circle of center O and radius 1.	¼
3) a)	$z'=\frac{i\left(\overline{z-i}\right)}{\overline{z-1}}$; $\arg(z')=\arg\left[\frac{i\left(\overline{z-i}\right)}{\overline{z-1}}\right]$; $\arg(z')=\arg(i)+\arg\left(\overline{z-i}\right)-\arg\left(\overline{z-1}\right)$; $\arg(z')=\frac{\pi}{2}-\arg(z-i)+\arg(z-1)$; $\arg(z')=\frac{\pi}{2}-\arg\left(z_{\overline{AM}}\right)+\arg\left(z_{\overline{BM}}\right)$;	½

	$\arg(z')=\frac{\pi}{2}-(\vec{u}\,;\,\overrightarrow{AM})+(\vec{u}\,;\,\overrightarrow{BM})$; $\arg(z')=\frac{\pi}{2}+(\overrightarrow{AM}\,;\,\overrightarrow{BM})[2\pi]$.	
3) b)	z' is a strictly negative real number so $\arg(z')=\pi[2\pi]$, therefore, $\frac{\pi}{2}+(\overrightarrow{AM}\,;\,\overrightarrow{BM})=\pi[2\pi]$, $(\overrightarrow{AM}\,;\,\overrightarrow{BM})=\frac{\pi}{2}[2\pi]$, so the set of points M is the circle of diameter $[AB]$ deprived of A and B.	½
4) a)	$z-1=\sqrt{2}e^{i\theta}$, $\lvert z-1\rvert=\lvert\sqrt{2}e^{i\theta}\rvert$, $\lvert z-1\rvert=\sqrt{2}$, $BM=\sqrt{2}$, so M describes the circle $(\mathscr{C})$ of center B and radius $\sqrt{2}$.	¼
4) b)	$(z'-i)(\bar{z}-1)=\left(\frac{i\bar{z}-1}{\bar{z}-1}-i\right)(\bar{z}-1)=i\bar{z}-1-i\bar{z}+i=-1+i\ (z\neq 1)$.	½
4) c)	$\lvert(z'-i)(\bar{z}-1)\rvert=\lvert-1+i\rvert$; $\lvert z'-i\rvert\lvert\bar{z}-1\rvert=\sqrt{2}$; $\lvert z'-i\rvert\lvert z-1\rvert=\sqrt{2}$; $AM'\times BM=\sqrt{2}$; M describes the circle $(\mathscr{C})$ so $BM=\sqrt{2}$, so $AM'=1$ then the set of points M' is the circle of center A and radius 1.	½

QIII	**Answers**	**3½ pts**
1) a)	$P(E)=\frac{2}{3}$; $P(F)=\frac{1}{3}$.	½
1) b)	$P(A/E)=\frac{A_4^2+A_5^2}{A_9^2}=\frac{4}{9}$; $P(A\cap E)=P(A/E)\times P(E)=\frac{4}{9}\times\frac{2}{3}=\frac{8}{27}$.	½
1) c)	$P(A\cap F)=P(A/F)\times P(F)=\frac{C_4^3+C_5^3}{C_9^3}\times\frac{1}{3}=\frac{1}{6}\times\frac{1}{3}=\frac{1}{18}$; Using the formula of total probabilities: $P(A)=P(A\cap E)+P(A\cap F)=\frac{8}{27}+\frac{1}{18}=\frac{19}{54}$.	½
1) d)	$P(F/\bar{A})=\frac{P(F\cap\bar{A})}{P(\bar{A})}=\frac{P(\bar{A}/F)\times P(F)}{1-P(A)}$; $P(\bar{A}/F)=1-P(A/F)=1-\frac{1}{6}=\frac{5}{6}$; $P(F/\bar{A})=\frac{\frac{5}{6}\times\frac{1}{3}}{1-\frac{19}{54}}=\frac{3}{7}$.	½
2)	$P(B)=P(B\cap E)+P(B\cap F)=P(B/E)\times P(E)+P(B/F)\times P(F)$; $P(B)=\frac{A_5^2+A_2^2}{A_9^2}\times\frac{2}{3}+\frac{C_5^3}{C_9^3}\times\frac{1}{3}=\frac{46}{189}$.	½
3)	$P(A\cap B)=P$(the balls drawn from the urn U_2 have the same color and carry the same number) $=\frac{A_4^2+A_2^2}{A_9^2}\times\frac{2}{3}+\frac{C_4^3}{C_9^3}\times\frac{1}{3}=\frac{55}{378}$; $P(A)\times P(B)=\frac{19}{54}\times\frac{46}{189}=\frac{437}{5103}$, so $P(A\cap B)\neq P(A)\times P(B)$, so the events A and B are not independent.	½
4)	$P(C)=P(C\cap E)+P(C\cap F)=P(C/E)\times P(E)+P(C/F)\times P(F)$;	½

	$P(C)=\frac{A_5^1\times A_4^1}{A_9^2}\times\frac{2!}{1!\times 1!}\times\frac{2}{3}+\frac{C_5^1\times C_4^2}{C_9^3}\times\frac{1}{3}=\frac{185}{378}$.	
QIV	**Answers**	**3½ pts**
1) a)	• $f(A)=R_C\circ R_A(A)=R_C\left[R_A(A)\right]=R_C(A)=I$ since: $R_A(A)=A$ since A is the center of R_A; $R_C(A)=I$ since $CA=CI$ and $\left(\overrightarrow{CA}\,;\overrightarrow{CI}\right)=-\frac{\pi}{2}(2\pi)$. • $f(B)=R_C\circ R_A(B)=R_C\left[R_A(B)\right]=R_C(C)=C$ since: $R_A(B)=C$ by the definition of R_A; $R_C(C)=C$ since C is the center of R_A.	½
1) b)	Let $\alpha_1=\frac{\pi}{4}$ and $\alpha_2=-\frac{\pi}{2}$ be the angles of R_A and R_C respectively; $f=R_C\circ R_A$ is a rotation of angle $\alpha_1+\alpha_2=\frac{\pi}{4}-\frac{\pi}{2}=-\frac{\pi}{4}$ and center O: $f(A)=I$ then $OA=OI$ then O belongs to the perpendicular bisector of $[AI]$; $f(B)=C$ then $OB=OC$ then O belongs to the perpendicular bisector of $[BC]$; So O is the point of intersection of the perpendicular bisectors of $[AI]$ and $[BC]$.	½
1) c)	$AB=AC$ and $\left(\overrightarrow{AB}\,;\overrightarrow{AC}\right)=\frac{\pi}{4}(2\pi)$ (given); $f(B)=C$ and $f=r\left(O\,;-\frac{\pi}{4}\right)$ then $OB=OC$ and $\left(\overrightarrow{OB}\,;\overrightarrow{OC}\right)=-\frac{\pi}{4}(2\pi)$; So, the two isosceles triangles ABC and OBC are congruent so $AB=AC=OB=OC$, then the quadrilateral $ABOC$ is a rhombus.	½
2) a)	Let α be a measure of the angle of S; $S(A)=B$ therefore $\alpha=\left(\overrightarrow{OA}\,;\overrightarrow{OB}\right)=\frac{\pi}{8}(2\pi)$. $S(C)=C'$ so $\left(\overrightarrow{OC}\,;\overrightarrow{OC'}\right)=\alpha=\frac{\pi}{8}(2\pi)$ then C' belongs to the straight line (OA) since $\left(\overrightarrow{OC}\,;\overrightarrow{OA}\right)=\frac{\pi}{8}(2\pi)$.	½
2) b)	$S(O)=O$ and $S(A)=B$ therefore $S([OA])=[OB]$; H is the midpoint of $[OA]$ so $S(H)$ is the midpoint of $S([OA])$ since the similitude preserves the midpoints so H' is the midpoint of $[OB]$.	½
2) c)	$(CH)\perp(OH)$ so $S((CH))\perp S((OH))$ since the similitude preserves orthogonality, but $S((CH))=(C'H')$ and $S((OH))=S((OA))=(OB)$ then $(C'H')\perp(OB)$. In the isosceles triangle OCB, C' belongs to (OA) which is the perpendicular bisector of $[BC]$; $(C'H')$ is perpendicular to the segment $[OB]$ at its midpoint H' so $(C'H')$ is the perpendicular bisector of $[OB]$, so C' is the point of intersection of the perpendicular bisectors of the triangle OCB so C' is the center of the circle circumscribed about this triangle.	½

3)	The angle of S_n is $\frac{n\pi}{8}$. S_n is a negative dilation if $\frac{n\pi}{8}=\pi+2k\pi$ with $k\in\mathbb{N}$; So $n=8+16k$ with $k\in\mathbb{N}$, n is a multiple of 8 which is not a multiple of 16.	½

QV	Answers	7½ pts
A) 1) a)	By graphical reading: • $g(1)=0$ since the curve (C_g) passes through the point $A(1\ ;\ 0)$; • $g'(1)=$ slope of $(T)=\frac{0-(-2)}{1-0}=2$.	½
A) 1) b)	By graphical reading: $g(x)<0$ if $x\in\left]0\ ;1\right[$; $g(x)=0$ if $x=1$; $g(x)>0$ if $x\in\left]1\ ;\ +\infty\right[$.	½
A) 2)	$g(x)=ax^2+bx+\ln x$; $x\in\left]0\ ;+\infty\right[$; $g(1)=0$, $a+b=0$; $g'(x)=2ax+b+\frac{1}{x}$ therefore $g'(1)=2$ gives $2a+b=1$; We obtain the system: $\begin{cases}a+b=0\\2a+b=1\end{cases}$, so $a=1$ and $b=-1$.	¾
A) 3)	g is twice differentiable over $\left]0\ ;+\infty\right[$; $g'(x)=2x-1+\frac{1}{x}$ and $g''(x)=2-\frac{1}{x^2}=\frac{2x^2-1}{x^2}$; Sign of $g''(x)$: $\begin{array}{c\|ccccc} x & 0 & & \frac{\sqrt{2}}{2} & & +\infty \\ \hline g''(x) & \blacksquare & - & 0 & + & \end{array}$ $g''(x)$ is equal to 0 at $x=\frac{\sqrt{2}}{2}$ and changes the sign, so (C_g) admits an inflection point of abscissa $\frac{\sqrt{2}}{2}$, so the point A is not an inflection point of (C_g).	¾
B) 1)	$\lim\limits_{x\to0^+}f(x)=(0-1)^2+(-\infty)^2=+\infty$; The straight line of equation $x=0$ is a vertical asymptote to (C_f); $\lim\limits_{x\to+\infty}f(x)=(+\infty)^2+(+\infty)^2=+\infty$.	½
B) 2) a)	$f'(x)=2(x-1)+2\frac{\ln x}{x}=2\frac{x^2-x+\ln x}{x}=2\frac{g(x)}{x}$; $f'(x)$ have same sign as $g(x)$ for every $x\in\left]0\ ;+\infty\right[$.	½
B) 2) b)	Table of variations of: f $\begin{array}{c\|ccccc} x & 0 & & 1 & & +\infty \\ \hline f'(x) & \blacksquare & - & 0 & + & \\ \hline f(x) & \blacksquare & +\infty \searrow & 0 & \nearrow & +\infty \end{array}$	¾

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<tr><td>B) 3) a)</td><td>

u is differentiable over $]0\ ;\ 1]$ and $u'(x) = f'(x) - 1 < 0$ for every $x \in]0\ ;\ 1]$ because $f'(x) < 0$ for every $x \in]0\ ;\ 1]$.

Table of variations of u over $]0\ ;\ 1]$:

$$\begin{array}{c|cccc} x & 0 & & & 1 \\ \hline u'(x) & \| & & - & \\ \hline u(x) & \| & +\infty & \searrow & -1 \end{array}$$

The function u is continuous and strictly decreasing over $]0\ ;\ 1]$ and changes the sign, so using the intermediate value theorem there exists a unique real number $\alpha \in]0\ ;\ 1]$ such that $u(\alpha) = 0$ then $f(\alpha) = \alpha$.

In addition, $u\left(\frac{1}{2}\right) \approx 0.2 > 0$ and $u(1) = -1 < 0$ therefore $\frac{1}{2} < \alpha < 1$.

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<tr><td>B) 3) b)</td><td>

$\lim\limits_{x\to+\infty} f(x) = +\infty$ and $\lim\limits_{x\to+\infty} \frac{f(x)}{x} \overset{HR}{=} \lim\limits_{x\to+\infty}\left[2(x-1) + 2\frac{\ln x}{x}\right] = +\infty + 0 = +\infty$; therefore (C_f) admits at $+\infty$ a parabolic branch of direction $(O\ ;\ \vec{j})$.

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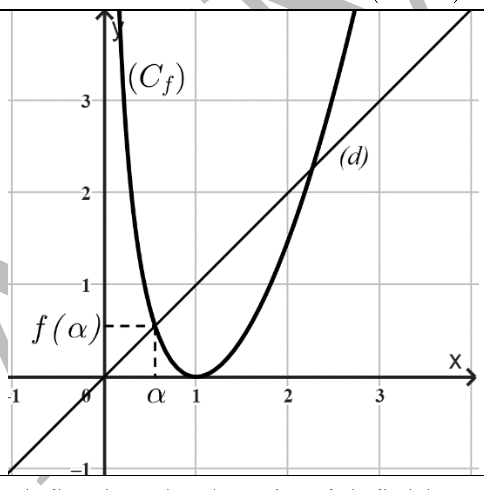
<tr><td>B) 4)</td><td>



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<tr><td>B) 5) a)</td><td>

h is defined when f is defined so the domain of definition of h is $D_h =]0\ ;+\infty[$.

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<tr><td>B) 5) b)</td><td>

h is differentiable over $]0\ ;+\infty[$ and $h'(x) = f'(x)e^{f(x)}$ which has the same sign of $f'(x)$ since $e^{f(x)} > 0$ for every $x \in]0\ ;+\infty[$;

Table of variations of h:

$$\begin{array}{c|cccccc} x & 0 & & 1 & & +\infty \\ \hline h'(x) & \| & - & 0 & + & \\ \hline h(x) & \| & +\infty \searrow & 1 & \nearrow & +\infty \end{array}$$

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<tr><td>B) 5) c)</td><td>

$h(x) = e^x$; $e^{f(x)} = e^x$; $f(x) = x$ since the function $x \mapsto \exp(x)$ is continuous and strictly increasing over $\mathbb{R}$, and as $x \in]0\ ;\ 1]$ then $x = \alpha$.

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