Transformation

- Generalities about transformation:

→ Definition: A transformation T is a mapping from a plane (P) onto itself, that associates to every point M in (P) a unique image M' in (P) such that $T(M) = M'$

→ M' is called the image of M by T.

→ M is called the preimage / antecedent of M' by T

→ Invariant / Double point: T has an invariant point B iff $T(B) = B$

→ Identity transformation: is a transformation that associates for every point M, the point M itself $T(M) = M = Id$

→ Inverse transformation: is a transformation denoted by T^{-1} and defined by $T(M) = M' \Rightarrow T^{-1}(M') = M$

→ Composite of 2 transformations: $T \circ T'(M) = T(T(M')) = T(M') = M''$

- Translation:

→ Definition: A translation of vector $\vec{v}$ is the transformation that associates to each point M, a point M' such that $\overrightarrow{MM'} = \vec{v} \Rightarrow t_{\vec{v}}(M) = M'$

→ Invariant point: → $\vec{v} = \vec{0} \Rightarrow t_{\vec{0}}(M) = M \Rightarrow t$ is identical transformation $\Rightarrow$ all pts are invariant

→ $\vec{v} \neq 0 \Rightarrow t_{\vec{v}}$ has no invariant points

→ Inverse translation: $t_{\vec{v}}^{-1} = t_{-\vec{v}}$

→ Composite of 2 translations: $t_{\vec{v}} \circ t_{\vec{u}} = t_{\vec{v}+\vec{u}}$

→ Translation preserves: collinearity, midpoint of a segment, parallelism, orthogonality, measures of oriented angles, areas, distances, shapes

- Rotation:

→ Definition: It is a circular motion of an object around a fixed point where this point is called the "center of rotation" and the amount of rotation is measured in degrees or radians is called the "angle of rotation". It is the transformation that associates to every point M, a point M' such that $OM = OM'$ and $(\overrightarrow{OM}, \overrightarrow{OM'}) = \theta\,[2\pi]$ where $R(M) = M'$

→ Identity of rotation: $R(O, 0)$

→ Invariant point: center of rotation

→ Inverse of rotation: $R(O, \theta) \Rightarrow R^{-1}(O, -\theta)$

→ Composite of 2 rotations:

→ same center: → $\theta_1 + \theta_2 = 0 \Rightarrow$ Identity

→ $\theta_1 + \theta_2 \neq 0 \Rightarrow R(O, \theta_1 + \theta_2)$

→ distinct centers: → $\theta_1 + \theta_2 = 0 \Rightarrow t_{\vec{v}}$

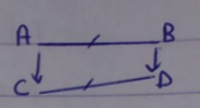
→ $\theta_1 + \theta_2 \neq 0 \Rightarrow R(L, \theta_1 + \theta_2)$

→ the composite of a rotation $r(A, \alpha)$ and a translation $t_{\vec{v}}$ is $R(L, \alpha)$

→ Properties:

→ $\left.\begin{matrix} A \xrightarrow{r} B \\ C \xrightarrow{r} D \end{matrix}\right\}$ $[AC] \xrightarrow{r} [BD]$ $\Rightarrow AC = BD$ and $(\overrightarrow{AC}, \overrightarrow{BD}) = \alpha$

→ if $AB = CD$ and $(\overrightarrow{AB}, \overrightarrow{CD}) \neq 0$ (not parallel), then there is one rotation taking $A \xrightarrow{r} C$ and $B \xrightarrow{r} D$



→ if $\theta = \pi \Rightarrow R$ is central symmetry

→ to find center of rotation: → intersection of ⊥ bisectors of 2 segments $(OA = OB)$

→ intersections of 2 semi-circles $(\overrightarrow{OA}, \overrightarrow{OB}) = \pi/2$

→ Rotation preserves: collinearity, midpoint of a segment, parallelism, orthogonality, measures of oriented angles, areas, distances, shapes, intersections.

- Dilation / Homothecy:

→ Definition: is the transformation that associates to every point M, a point M' such that $\overrightarrow{OM'} = k\overrightarrow{OM}$; it is denoted by $h(O; k)$ for center O and scale factor / ratio k. $h(M) = M'$ iff $\overrightarrow{OM'} = k\overrightarrow{OM}$.

→ Invariant point: center of dilation

→ Inverse of dilation: $h(\Omega, k) \Rightarrow h^{-1}(\Omega, -k)$

→ Identity of dilation: $h(\Omega, 1)$

→ Composite of 2 dilations:

→ same center: → $k_1 \times k_2 = 1 \Rightarrow$ Identity

→ $k_1 \times k_2 \neq 1 \Rightarrow h(O, k_1 \times k_2)$

→ distinct centers: → $k_1 \times k_2 = 1 \Rightarrow t_{\vec{v}}$

→ $k_1 \times k_2 \neq 1 \Rightarrow h(??, k_1 k_2)$

→ the composite of a dilation $h(A, k)$ and a translation $t_{\vec{v}}$ is $h(??, k)$

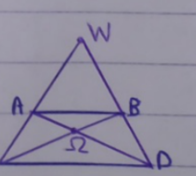
→ Properties:

→ $(d) \xrightarrow{h} (d')$ such that $(d) \parallel (d')$

→ $(d) \xrightarrow{h} (d)$ if center $W \in (d)$

→ If $\overrightarrow{CD} = k\,\overrightarrow{AB}$ $(k \neq 1)$

- there is a positive dilation $h: [AB] \xrightarrow{h} [CD]$ where $W = (AC) \cap (BD)$ and $k = \frac{CD}{AB}$
- there is a negative dilation $h': [AB] \xrightarrow{h'} [C]$ where $\Omega = (BC) \cap (AD)$ and $k = \frac{-CD}{AB}$



→ Dilation preserves: collinearity, midpoint of a segment, parallelism, orthogonality, measures of oriented angles, shapes, intersection

→ Dilation doesn't preserve: distances (multiplied by $|k|$), areas (multiplied by k^2)

note: angle: dilation: $k\pi$, pos dilation: $2k\pi$, neg dilation: $(2k+1)\pi$

Direct Plane Simultude :

→ Definition: it is the transformation that associates for every point M, a point M' such that $OM' = kOM$ and $(\vec{OM}, \vec{OM'}) = \theta \ [2\pi] \Rightarrow S(M) = M'$

→ Invariant point: center of similitude

→ Identity of similitude: $S(O; 1, 0)$

→ Inverse of similitude: $S(O; k; \theta) \Rightarrow S(O; 1/k, -\theta)$

→ Composite of 2 similitudes: $S_1 \circ S_2 = S(O; k_1 \times k_2, \theta_1 + \theta_2)$

→ Properties:

→ $\left.\begin{array}{l} S(A) = A' \\ S(B) = B' \end{array}\right\}$ $[AB] \xrightarrow{S} [A'B']$
$\Rightarrow A'B' = kAB$ and $(\vec{AB}, \vec{A'B'}) = \theta \ [2\pi]$

→ $S(O; -k, \theta) = S(O; k, \pm\pi + \theta)$ if $k > 0$

→ A direct plane similitude is the composite of dilation $h(O; k)$ and a rotation $R(O, \theta)$ or a translation

→ $h(O, -k) = S(O; k, \pi)$; $h(O, k) = S(O; k, 0)$; $S(O; 1, \alpha) = R(O, \alpha)$

→ Similitude preserves: collinearity, midpoint of a segment, parallelism, orthogonality, measures of oriented angles, shapes, intersection

→ Similitude doesn't preserve: distances (multiplied by $|k|$), areas (multiplied by k^2)

→ notes:

→ A, B, C and D are points such that $A \neq B$ and $C \neq D$, there exists one and only one similitude S that transforms $A \xrightarrow{S} B$ and $C \xrightarrow{S} D$

→ $S(O, 1, \pi)$ is central symmetry

→ The image of the line (D) by S is (D') passing throug $A' = S(A) \in (D)$ such that $((D), (D')) = \alpha$

Axial Symmetry / Reflection:

→ Definition: is the transformation that associates to every point M, a point M' such that (d) is perpendicular bisector of [MM']. $S_{(d)}(M) = M'$ iff (d) is ⊥ bis. of [MM']

→ Invariant point: all points that belongs to axis (d)

→ Invariant lines: axis (d) and any line ⊥ (d)

→ Composite of 2 reflections:

→ (d) and (L) are parallel $\Rightarrow S_{(L)} \circ S_{(d)} = t_{2\vec{AB}}$

→ (d) and (L) are intersecting $\Rightarrow S_{(L)} \circ S_{(d)} = R(A, 2\theta)$ where $A = (d) \cap (L)$

→ Reflection preserves: collinearity, midpoint of a segment, parallelism, orthogonality, areas, distances, shapes

→ Reflection doesn't preserve: measure of oriented angles (opposite)

Complex form:

$z' = az + b$

→ translation: $z' = z + z_{\vec{v}}$

→ rotation: $a = e^{i\theta}$, $z_\Omega = \frac{b}{1-a}$

→ dilation: $a = k$, $z_\Omega = \frac{b}{1-k}$

→ Similitude: $a = ke^{i\theta}$, $z_\Omega = \frac{b}{1-a}$