Question 1

Consider the sequence (U_n) defined as $U_0 = 2$ and $U_{n+1} = \frac{1}{2}U_n + \frac{1}{2}$ for all $n \in \mathbb{N}$.

Let (V_n) be the sequence defined as $V_n = U_n - 1$.

1) Calculate U_1 and U_2. Prove that (U_n) is neither arithmetic nor geometric.
2) Prove that (V_n) is a geometric sequence whose common ratio r and first term V_0 are to be determined.
3) Calculate, in terms of n, U_n and deduce $\lim\limits_{n\to+\infty} U_n$.
4) Determine, in terms of n, $\sum_{i=0}^{n} V_i$ and $\sum_{i=0}^{n} U_i$.

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1) $U_1 = \frac{1}{2}U_0 + \frac{1}{2} = 1.5$ $\quad U_2 = \frac{1}{2}U_1 + \frac{1}{2} = 1.25$

$U_1 - U_0 \neq U_2 - U_1$ ($1.5-2 \neq 1.25-1.5$) thus U_n is not an arithmetic sequence.

$\frac{U_1}{U_0} \neq \frac{U_2}{U_1}$ ($\frac{1,5}{2} \neq \frac{1,25}{1,5}$) thus U_n is not a geometric sequence

2) $\frac{V_{n+1}}{V_n} = \frac{U_{n+1}-1}{U_n-1} = \frac{\frac{1}{2}U_n+\frac{1}{2}-1}{U_n-1} = \frac{\frac{1}{2}(U_n-1)}{U_n-1} = \frac{1}{2}$, thus V_n is a geometric sequence of common ratio $r = \frac{1}{2}$ and first term $V_0 = U_0 - 1 = 1$

3) $V_n = V_0 r^n = (\frac{1}{2})^n$ then $U_n = V_n + 1 = (\frac{1}{2})^n + 1$.

$$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} (\frac{1}{2})^n + 1 = 0 + 1 = 1$$

4) $\sum_{i=0}^{n} V_i = V_0 + V_1 + V_2 + \dots\dots\dots + V_{n+1} = V_0 \frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}} = 2(1-(\frac{1}{2})^{n+1})$

$\sum_{i=0}^{n} U_i = U_0 + U_1 + U_2 + \dots\dots\dots + U_{n+1} =$

$V_0+1 + V_1+1 + V_2+1 + \dots\dots\dots + V_{n+1}+1 = 2(1-(\frac{1}{2})^{n+1}) + n+1$.

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$$\frac{V_{n+1}}{V_n}=\frac{U_{n+1}+24}{U_n+24}=\frac{\frac{2}{3}U_n-8+24}{U_n+24}=\frac{\frac{2}{3}U_n+16}{U_n+24}=\frac{\frac{2}{3}(U_n+24)}{U_n+24}=\frac{2}{3}$$

Thus (V_n) is a geometric sequence of common ratio $\frac{2}{3}$ and first term

$V_0=U_0+24=3+24=27$

3.b)

$V_n=U_n+24$, V_n is a geometric sequence, thus:

$V_n=V_0 r^n=27\left(\frac{2}{3}\right)^n$, and $U_n=V_n-24=27\left(\frac{2}{3}\right)^n-24$

4. a)

$$U_{n+1}-U_n=27\left(\frac{2}{3}\right)^{n+1}-24-\left[27\left(\frac{2}{3}\right)^n-24\right]=27\left(\frac{2}{3}\right)^{n+1}-27\left(\frac{2}{3}\right)^n=$$

$$27\left(\frac{2}{3}\right)^n\left[\left(\frac{2}{3}\right)-1\right]=-27\left(\frac{2}{3}\right)^n\left(\frac{1}{3}\right)<0$$

Thus, U_n is a decreasing sequence.

Question 2

Let (U_n) be the sequence defined by $U_0 = 3$ and $U_{n+1} = \frac{2}{3}U_n - 8$, for every $n \in \mathbb{N}$.

1) Calculate U_1 and U_2.
2) Deduce that the sequence (U_n) is neither arithmetic nor geometric.
3) Let (V_n) be the sequence defined by $V_n = U_n + 24$, for every $n \in \mathbb{N}$.
   a) Prove that the sequence (V_n) is a geometric sequence whose common ratio and first term are to be determined.
   b) Find V_n in terms of n, then deduce that $U_n = 27(\frac{2}{3})^n - 24$
4) a) Prove that the sequence (U_n) is decreasing.
   b) Let S be the sum $S = V_0 + V_1 + \cdots + V_n$.

   and $T_n = U_0 + U_1 + ... + U_n$, where $n \in \mathbb{N}$.

   Express S in terms of n, then deduce T_n in terms of n.

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1)

$U_0 = 3$, for $n = 0$, $U_1 = \frac{2}{3}U_0 - 8 = \frac{2}{3}(3) - 8 = -6$

for $n = 1$, $U_2 = \frac{2}{3}U_1 - 8 = \frac{2}{3}(-6) - 8 = -12$

2)

$U_1 - U_0 = -6 - 3 = -9$, $U_2 - U_1 = -12 + 6 = -6$

thus (U_n) is not arithmetic since $U_1 - U_0 \neq U_2 - U_1$.

$\frac{U_2}{U_1} \neq \frac{U_1}{U_0}$ since $\frac{-12}{-6} \neq \frac{-6}{3}$ thus (U_n) is not a geometric sequence.

3. a) $V_n = U_n + 24$ and $U_{n+1} = \frac{2}{3}U_n - 8$,

$$\frac{V_{n+1}}{} = \frac{U_{n+1} + 24}{} = \frac{\frac{2}{3}U_n - 8 + 24}{} = \frac{\frac{2}{3}U_n + 16}{} = \frac{\frac{2}{3}(U_n + 24)}{} = \frac{2}{}$$

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Thus, U_n is a decreasing sequence.

4b) $S = V_0 + V_1 + \cdots + V_n$ is a geometric sum since V_n is a geometric sequence thus :

$$S = 27[\frac{1-\left(\frac{2}{3}\right)^{n+1}}{1-\left(\frac{2}{3}\right)}] = 27[\frac{1-\left(\frac{2}{3}\right)^{n+1}}{\frac{1}{3}}] = 81[1-\left(\frac{2}{3}\right)^{n+1}]$$

$$T_n = U_0 + U_1 + \dots + U_n =$$

$$= V_0 - 24 + V_1 - 24 + V_2 - 24 + \dots\dots + V_n - 24$$

$$= -24(n+1) + V_0 + V_1 + V_2 + \dots\dots + V_n$$

$$= -24(n+1) + 81[1-\left(\frac{2}{3}\right)^{n+1}]$$

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Question 3

Let (U_n) be the sequence defined as: $U_0 = \frac{1}{2}$ and $U_{n+1} = \frac{2U_n}{U_n+1}$ such that $\frac{1}{2} \leq U_n \leq 1$.

1) Show that the sequence (U_n) is not geometric.

2) Study the sense of variation of (U_n).

3) Show that the sequence (U_n) converges and determine its limit L.

4) Let $V_n = \frac{U_n + k}{U_n}$; k is a given real number.

a) Calculate k to get (V_n) as a geometric sequence with common ratio $r = \frac{1}{2}$

b) Express V_n in terms of n

c) find the limit of V_n and deduce the limit of (U_n).

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Sequence

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1)

$U_{n+1} = \frac{2\,U_n}{U_n + 1}$. for $n = 0$, $U_1 = \frac{2\,U_0}{U_0 + 1} = \frac{2\,(0.5)}{0.5 + 1} = \frac{2}{3}$

for $n = 1$, $U_2 = \frac{2\,U_1}{U_1 + 1} = \frac{2\left(\frac{2}{3}\right)}{\frac{2}{3} + 1} = \frac{4}{5}$

$\frac{U_2}{U_1} \neq \frac{U_1}{U_0}$ since $\frac{4}{5} \div \frac{2}{3} = \frac{6}{5}$, however $\frac{2}{3} \div \frac{1}{2} = \frac{4}{3}$; so, they are not equal.

Thus, U_n is not a geometric sequence.

2) $U_{n+1} - U_n = \frac{2\,U_n}{U_n+1} - U_n = \frac{2\,U_n}{U_n+1} - U_n \times \frac{U_n+1}{U_n+1} = \frac{2U_n - U_n^2 - U_n}{U_n+1} = \frac{U_n - U_n^2}{U_n+1} = \frac{U_n(1-U_n)}{U_n+1}$

But $0.5 \leq U_n \leq 1$, thus $U_n > 0$ and $1 - U_n \geq 0$, hence $\frac{U_n(1-U_n)}{U_n+1} \geq 0$;

so (U_n) is an increasing sequence.

3) Since (U_n) is an increasing sequence and since its bounded above by 1 then it is convergent to a limit L.

And since $U_{n+1} = \frac{2U_n}{U_n+1}$ then $L = \frac{2L}{L+1}$ which yields $L^2 + L = 2L$, $L^2 - L = 0$

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And since $U_{n+1} = \frac{2U_n}{U_n+1}$ then $L = \frac{2L}{L+1}$ which yields $L^2 + L = 2L$, $L^2 - L = 0$

$L(L-1) = 0$, then $L = 0$ is rejected since $\frac{1}{2} \le U_n \le 1$ so $L=1$.

4a) $V_n = \frac{U_n + k}{U_n}$;

$$\frac{V_{n+1}}{V_n} = \frac{\frac{U_{n+1}+k}{U_{n+1}}}{\frac{U_n+k}{U_n}} = \frac{(U_{n+1}+k)U_n}{(U_n+k)U_{n+1}} = \frac{\left(\frac{2\,U_n}{U_n+1}+k\right)U_n}{(U_n+k)\frac{2\,U_n}{U_n+1}}$$

simplify by U_n, we get : $\frac{2\,U_n + k(U_n+1)}{2(U_n+k)}$

Then $\frac{2\,U_n + k(U_n+1)}{2(U_n+k)} = r = \frac{1}{2}$ (V_n is a geometric sequence with common ratio $r = \frac{1}{2}$)

$$\text{so} \quad 2\,U_n + k(U_n+1) = \frac{1}{2} \times 2(U_n+k)$$

$$\text{so} \quad 2\,U_n + kU_n + k = U_n + k$$

Then $\frac{2U_n+k(U_n+1)}{2(U_n+k)} = r = \frac{1}{2}$ (V_n is a geometric sequence with common ratio $r = \frac{1}{2}$)

$$\text{so} \quad 2U_n + k(U_n + 1) = \frac{1}{2} \times 2(U_n + k)$$

$$\text{so} \quad 2U_n + kU_n + k = U_n + k$$

$$\text{so} \quad U_n + kU_n = 0$$

$$\text{so} \quad U_n(k + 1) = 0$$

We have $U_n \neq 0$ so $k + 1 = 0$ then $k = -1$

4b) $V_n = \frac{U_n+k}{U_n}$, $V_0 = \frac{U_0+k}{U_0} = \frac{\frac{1}{2}-1}{\frac{1}{2}} = -1$

$V_n = V_0 r^n = -(0.5)^n$.

c) $\lim_{n \to +\infty} V_n = 0$ since $(0.5)^n$ tends to 0 because $-1 < 0.5 < 1$

Now using the relation between Un and V_n: $V_n = \frac{U_n-1}{U_n}$; and applying so $2U_n + kU_n + k = U_n + k$

so $U_n + kU_n = 0$

so $U_n(k+1) = 0$

We have $U_n \neq 0$ so $k + 1 = 0$ then $k = -1$

4b) $V_n = \frac{U_n + k}{U_n}$, $V_0 = \frac{U_0 + k}{U_0} = \frac{\frac{1}{2} - 1}{\frac{1}{2}} = -1$

$V_n = V_0 r^n = -(0.5)^n$.

c) $\lim_{n \to +\infty} V_n = 0$ since $(0.5)^n$ tends to 0 because $-1 < 0.5 < 1$

Now using the relation between Un and V_n : $V_n = \frac{U_n - 1}{U_n}$; and applying limit to it we get : $\lim_{n \to +\infty} V_n = \lim_{n \to +\infty} \frac{U_n - 1}{U_n}$ which yields $0 = \frac{L-1}{L}$ thus $L = 1$.

for n = 0,

$$u_1 = f(u_0) = f\left(\frac{3}{2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\frac{3}{2}}\right) = \frac{1}{2}\left(\frac{3}{2} + 2 \times \frac{2}{3}\right) = \frac{1}{2}\left(\frac{9+8}{6}\right) = \frac{17}{12} \cong 1{,}416 < u_0$$

And we have $\sqrt{2} \cong 1.414$

then $\sqrt{2} < u_1 < u_0 \leq \frac{3}{2}$ so the inequality is true for n =0

Suppose it remains true up to order k then $\sqrt{2} < u_{k+1} < u_k \leq \frac{3}{2}$, and show that it is true for n = k + 1 ,

$\sqrt{2} < u_{k+2} < u_{k+1} \leq \frac{3}{2}$?

We have $\sqrt{2} < u_{k+1} < u_k \leq \frac{3}{2}$, and f is strictly increasing on $]\sqrt{2}, +\infty[$ (part 1.b)

So we can write $f(\sqrt{2}) < f(u_{k+1}) < f(u_k) \leq f(\frac{3}{2})$

So $\sqrt{2} < u_{k+2} < u_{k+1} \leq \frac{17}{12} \leq \frac{3}{2}$

So by induction, the inequality is true for all n.

b) according to question 2.a), we have $u_{n+1} < u_n$ for any natural number n, so (u_n) is strictly decreasing and it is lower bound by $\sqrt{2}$ so it converges.

		$\sqrt{2}$

$$f(\sqrt{2}) = \frac{1}{2}\left(\sqrt{2} + \frac{2}{\sqrt{2}}\right) = \sqrt{2}$$

$$f(-\sqrt{2}) = \frac{1}{2}\left(-\sqrt{2} + \frac{2}{-\sqrt{2}}\right) = -\sqrt{2}$$

2. a) $\sqrt{2} < u_{n+1} < u_n \leq \frac{3}{2}$?

We have $u_{n+1} = \frac{1}{2}\left(u_n + \frac{2}{u_n}\right)$, $u_0 = \frac{3}{2}$

for n = 0,

$$u_1 = f(u_0) = f\left(\frac{3}{2}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\frac{3}{2}}\right) = \frac{1}{2}\left(\frac{3}{2} + 2 \times \frac{2}{3}\right) = \frac{1}{2}\left(\frac{9+8}{6}\right) = \frac{17}{12} \cong 1{,}416 < u_0$$

And we have $\sqrt{2} \cong 1.414$

then $\sqrt{2} < u_1 < u_0 \leq \frac{3}{2}$ so the inequality is true for n =0

Suppose it remains true up to order k then $\sqrt{2} < u_{k+1} < u_k \leq \frac{3}{2}$, and show that it is true for n = k + 1.

1. a) Domain of definition : $\mathbb{R}^* =]-\infty, 0[\cup]0, +\infty[$.

For every $x \neq 0$ $f(x) = \frac{1}{2}\left(x + 2 \times \frac{1}{x}\right)$ then

$$f'(x) = \frac{1}{2}\left(1 + 2\left(\frac{-1}{x^2}\right)\right) = \frac{1}{2}\left(1 + \frac{-2}{x^2}\right) = \frac{1}{2}\left(\frac{x^2-2}{x^2}\right) = \frac{x^2-2}{2x^2} = \frac{x^2-\sqrt{2}^2}{2x^2}$$

$$= \frac{(x-\sqrt{2})(x+\sqrt{2})}{2x^2}$$

b)

x	$-\infty$		$-\sqrt{2}$		0		$\sqrt{2}$		$+\infty$
x^2-2		$+$	0	$-$	‖	$-$	0	$+$	
$2x^2$		$+$		$+$	‖	$+$		$+$	
$f'(x)$		$+$	0	$-$	‖	$-$	0	$+$	
$f(x)$		↗	$-\sqrt{2}$	↘	‖	↘	$\sqrt{2}$	↗	

$f(\sqrt{2}) = \frac{1}{2}\left(\sqrt{2} + \frac{2}{\ldots}\right) = \sqrt{2}$

Question 5

f is the function defined on $\mathbb{R}^*$ by $f(x) = \frac{1}{2}\left(x + \frac{2}{x}\right)$

1. a) Show that for all real numbers $x \neq 0$, $f'(x) = \frac{(x-\sqrt{2})(x+\sqrt{2})}{2x^2}$

   b) Draw up the table of variation of f on $\mathbb{R}^*$

2. (u_n) is the sequence defined by $u_0 = \frac{3}{2}$ and for any natural number,

$u_{n+1} = f(u_n)$.

a) Prove by induction that for any natural number n, $\sqrt{2} < u_{n+1} < u_n \leq \frac{3}{2}$.

b) Deduce that the sequence (u_n) converges.

c) Prove that for any natural number n, $u_{n+1} - \sqrt{2} < \frac{1}{2}(u_n - \sqrt{2})$.

d) Deduce by induction that for any natural number n,

$$0 < u_n - \sqrt{2} \leq \frac{1}{2^n}(u_0 - \sqrt{2}).$$

e) Deduce the limit of the sequence (u_n).

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2) $V_0 = 1$ and $V_{n+1} = \frac{1}{2}V_n - 1$

$V_n > -2$?

for n=0 , $V_0 > -2$?

$1 > -2$ so the inequality is true for n =0

Suppose it remains true up to order k then $V_k > -2$, and show that it is true for n=k+1.

$V_{k+1} > -2$?

$V_k > -2$ then $\frac{1}{2}V_k > -1$ then $\frac{1}{2}V_k - 1 > -1 - 1$ thus $V_{k+1} > -2$.

So by induction, the inequality is true for all n.

3) $W_{n+1} > W_n$?

$W_1 = 3W_0 - 1 = 2$

For n=0 $2 > 1$ then $W_1 > W_0$. so the property is true for n =0

Suppose it remains true up to order k then $W_{k+1} > W_k$, and show that it is true for n=k+1.

$W_{k+2} > W_{k+1}$?

$W_{k+1} > W_k$ then $3W_{k+1} > 3W_k$ then $3W_{k+1} - 1 > 3W_k - 1$ thus $W_{k+2} > W_{k+1}$.

So by induction, the property is true for all n, and (Wn) is strictly increasing.

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Question 4

1) Consider the sequence (U_n) defined as $U_0 = 2$ and $U_{n+1} = 2U_n - 3$, where $n \in \mathbb{N}$. Prove, by mathematical induction, that $U_n = 3 - 2^n$.

2) Let (V_n) be the sequence defined as $V_0 = 1$ and $V_{n+1} = \frac{1}{2}V_n - 1$ for every natural number n.

Prove by mathematical induction that (V_n) is bounded below by -2.

3) (W_n) is a sequence defined as $W_0 = 1$ and $W_{n+1} = 3W_n - 1$, show by mathematical induction that (W_n) is strictly increasing, where $n \in \mathbb{N}$.

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b) according to question 2.a), we have $u_{n+1} < u_n$ for any natural number n, so (u_n) is strictly decreasing and it is lower bound by $\sqrt{2}$ so it converges.

c) $u_{n+1} - \sqrt{2} < \frac{1}{2}(u_n - \sqrt{2})$?

$\frac{1}{2}\left(u_n + \frac{2}{u_n}\right) - \sqrt{2} < \frac{1}{2}(u_n - \sqrt{2})$?

$\left(u_n + \frac{2}{u_n}\right) - 2\sqrt{2} < u_n - \sqrt{2}$?

$\frac{2}{u_n} < \sqrt{2}$? (we have $0 < \sqrt{2} < u_n$)

$\frac{2}{\sqrt{2}} < u_n$?

$\sqrt{2} < u_n$?

And $\sqrt{2} < u_n$ is true for all natural numbers n (already proved in 2.a) so we have $u_{n+1} - \sqrt{2} < \frac{1}{2}(u_n - \sqrt{2})$ is true for all n.

d) $0 < u_n - \sqrt{2} \leq \frac{1}{2^n}(u_0 - \sqrt{2})$?

we have $\sqrt{2} < u_n$ so $0 < u_n - \sqrt{2}$.

1) $U_{n+1} = 2U_n - 3$

$U_n = 3 - 2^n$?

For n=0 $U_0 = 3 - 2^0$?

2=2 so the formula is true for n =0

Suppose it remains true up to order k then $U_k = 3 - 2^k$, and show that it is true for n=k+1.

$U_{k+1} = 3 - 2^{k+1}$?

$U_{k+1} = 2U_k - 3 = 2(3 - 2^k) - 3 = 6 - 2^{k+1} - 3 = 3 - 2^{k+1}$.

So by induction, the formula is true for all n.

2) $V_0 = 1$ and $V_{n+1} = \frac{1}{2}V_n - 1$

$V_n > -2$?

for n=0 , $V_0 > -2$?

1 > -2 so the inequality is true for n =0

Suppose it remains true up to order k then $V_k > -2$, and show that it is true for n=k+1.

$V_{k+1} > -2$?

d) $0 < u_n - \sqrt{2} \le \frac{1}{2^n}(u_0 - \sqrt{2})$?

we have $\sqrt{2} < u_n$ so $0 < u_n - \sqrt{2}$.

We want to show $u_n - \sqrt{2} \le \frac{1}{2^n}(u_0 - \sqrt{2})$?

for n=0 ,

$\frac{1}{2^0}(u_0 - \sqrt{2}) = u_0 - \sqrt{2}$ so $u_0 - \sqrt{2} \le \frac{1}{2^0}(u_0 - \sqrt{2})$ so the inequality is true for n =0

Suppose it remains true up to order k then $u_k - \sqrt{2} \le \frac{1}{2^k}(u_0 - \sqrt{2})$, and show that it is true for n=k+1 ,

$u_{k+1} - \sqrt{2} \le \frac{1}{2^{k+1}}(u_0 - \sqrt{2})$?

from 2.c) we have $u_{K+1} - \sqrt{2} < \frac{1}{2}(u_K - \sqrt{2})$

but we have $u_k - \sqrt{2} \le \frac{1}{2^k}(u_0 - \sqrt{2})$ so $u_{K+1} - \sqrt{2} < \frac{1}{2} \times \frac{1}{2^k}(u_0 - \sqrt{2})$ so

$u_{K+1} - \sqrt{2} < \frac{1}{2^{k+1}}(u_0 - \sqrt{2})$ so we can write $u_{K+1} - \sqrt{2} \le \frac{1}{2^{k+1}}(u_0 - \sqrt{2})$

So by induction, the inequality is true for all natural number n.

So by induction, the inequality is true for all natural number n.

e) $\lim_{x\to+\infty} u_n = ?$

We have $\lim_{x\to+\infty} (\frac{1}{2^n}(u_0 - \sqrt{2}) = 0$

It has been demonstrated that $0 < u_n - \sqrt{2} \leq \frac{1}{2^n}(u_0 - \sqrt{2})$

so $\lim_{x\to+\infty} 0 < \lim_{x\to+\infty} (u_n - \sqrt{2}) \leq \lim_{x\to+\infty} (\frac{1}{2^n}(u_0 - \sqrt{2}))$

$0 < \lim_{x\to+\infty} (u_n - \sqrt{2}) \leq 0$

Then according to the sandwich's theorem

$\lim_{x\to+\infty} u_n - \sqrt{2} = 0$ and so $\lim_{x\to+\infty} u_n = \sqrt{2}$

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