Solved Problems

N°1.
Solve in $\mathbb{N}$ each of the following equations:
1) $C_n^2 = 3$ 2) $P_n^3 = 6n$ 3) $C_{2n}^1 + C_{2n}^2 + C_{2n}^3 = 35n$

N°2.
Calculate each of the following sums:
1) $S_1 = C_n^0 + C_n^1 + C_n^2 + \dots\dots\dots\dots + C_n^n$ and deduce the sum of
$$S'_1 = \frac{1}{0!\,n!} + \frac{1}{1!(n-1)!} + \dots + \frac{1}{n!\,0!}.$$
2) $S_1 = 1 + 3C_n^1 + 3^2 C_n^2 + \dots\dots\dots\dots + 3^n$
3) $S_3 = C_n^1 + 2C_n^2 + 3C_n^3 + \dots\dots\dots\dots + nC_n^n$

N°3.
Consider n points $A_1, A_2, \dots, A_n$ of the plane where no three are collinear.
1) How many straight lines can be drawn using these points?
2) How many triangles can be formed using these points?

N°4.
Calculate the number of diagonals in a convex polygon of n sides.

N°5.
An urn contains 6 identical balls:
Four green balls (G) and two yellow balls (Y).
1) We draw, at random and simultaneously, two balls from the urn.
   We note by A, B, C the following events:
   A: "The two drawn balls are yellow"
   B: "The two drawn balls have different colors"
   C: "The two drawn balls are green"
   Calculate $p(A)$, $p(B)$ and $p(C)$.
2) The two drawn balls are not replaced back in the urn, we draw again two balls from the urn and we designate by D the event:
   D: A green ball and a yellow ball are drawn .
   a- Calculate $p(D/A)$; $p(D/B)$ and $p(D/C)$.
   b- Deduce $p(D\cap A)$; $p(D\cap B)$ and $p(D\cap C)$.
   c- Calculate $p(D)$.

N°6.
An urn contains 5 identical balls, two white and three black.
We draw, at random and simultaneously, two balls from the urn.

1) Calculate the probability of drawing:
   a- Two white balls.
   b- Two balls of the same color.
2) In this part, we draw the two balls according to the following strategy:
   We draw the first ball from the urn and we note its color; we then replace it back in the urn and we add another ball having the same color of the drawn ball.
   (Hence, there are six balls in the urn before the second draw).
   We, then, draw another ball. Consider the following events:
   w_1 : Getting a white ball in the first draw.
   B_1 : Getting a black ball in the first draw.
   w_2 : Getting a white ball in the second draw.
   a- Calculate $p(w_2 / w_1)$ and $p(w_2 / B_1)$.
   b- Calculate $p(w_2)$.

N°7.
In a factory, there are three machines M_1, M_2 and M_3.
Machine M_1 assures 20% of the total production of the factory of which 5% are defective.
Machine M_2 assures 30% of the total production of the factory of which 4% are defective.
Machine M_3 assures 50% of the total production of the factory of which 5% are defective.
We choose, at random, an article from the factory.

1) Draw the tree diagram showing all the probabilities.
2) Calculate the probability of choosing a defective article produced by M_1.
3) Calculate the probability of choosing a defective article.
4) The article chosen is defective, calculate the probability that it is produced by M_1.

N°8.
An urn contains six identical balls:

Three black, two green and one white.
A player draws one ball from the urn.
If the ball drawn is white, he gains 3 points.
If the ball drawn is green, he gains 2 points.
If the ball drawn is black, he replaces the ball in the urn and draws a new ball from the urn.
In the second drawing:
If the ball drawn is white, he gains 1 point, if the ball drawn is green he loses 1 point and if the ball drawn is black, he loses 2 points.
Calculate the probabilities of each of the following events :
E : The player gains 2 points. F : The player gains 1 point.
G : The player loses 1 point. H : The player loses 2 points.

N°9.

An urn contains 9 balls:
Three white balls numbered 1 to 3.
Three black balls numbered 1 to 3.
Three red balls numbered 1 to 3.
We draw, simultaneously and at random, two balls from the urn.
Consider the following events:
A: " The two drawn balls hold an odd number"
B: " The two drawn balls have the same color"
C: " The two drawn balls have different colors"
D: " The two drawn balls have different colors and hold odd numbers"

1) Calculate the following probabilities:
$p(A)$, $p(B)$, $p(A \cap B)$ and $p(A/B)$.
Are the events A and B independent?
2) a- Calculate $p(C)$ and prove that $p(D)=\frac{1}{3}$.
b- The two drawn balls have different colors, what is the probability that they hold odd numbers?

N°10.

For an oral exam, two teachers M_1 and M_2 prepare math exercises.
M_1 prepares 7 exercises distributed as follows:
2 exercises of probability, 1 exercise of statistics and 4 exercises of functio
M_2 prepares 7 exercises distributed as follows:

3 exercises of probability, 2 exercises of statistics and 2 exercises of functions.

Each exercise is written on a paper and placed in an envelope.

The exercises prepared by M_1 are placed in a box B_1 and the exercises prepared by M_2 are placed in a box B_2.

1) The candidate chooses, at random, a box and draws an envelope from this box. Calculate the probabilities of the following events:
   E : The candidate has drawn an exercise of probability knowing that he has chosen box B_1.
   F : The candidate has chosen an exercise of probability of box B_1.
   G : The candidate has chosen an exercise of probability.
2) In this question the candidate draws at random an envelope of each box. Calculate the probability of drawing two exercises of statistics.
3) In this question the candidate draws successively and without replacement three envelopes from box B_2.
   What is the probability that the third envelope drawn is the only exercise about functions?
4) In this question the candidate draws simultaneously 3 questions of box B_1.
   Let X be the random variable equal to the number of exercises of probability drawn. Determine the probability distribution of X.
5) In this question the candidate draws simultaneously 2 questions of box B_1 .and 1 question from box B_2
   Let X be the random variable equal to the number of exercises of functions drawn. Determine the probability distribution of X.

N°11.

An urn contains 12 white balls and 8 black balls.

1) We draw successively and with replacement 3 balls of this urn.
   Calculate the probability of each of the following events:
   D: the drawn balls are 2 white and 1 black in this order.
   E : the drawn balls are 2 white and 1 black in any order.
2) We draw successively 2 balls from the urn with respecting the following rule:
   If the ball drawn is white, we replace it back in the urn and we draw another ball. If not, we leave it outside and we draw another ball.

Calculate the probability of drawing:

a- Exactly one white ball. b- At least one white ball.

N° 12.

In a big store, a television and a *DVD* machine are on sale for a week. A person suggests that:

- The probability that he buys a television is $\frac{3}{5}$.
- The probability that he buys a *DVD* machine if he buys a television is $\frac{7}{10}$.
- The probability that he buys the *DVD* machine if he doesn't buy the television is $\frac{1}{10}$.

Designate by T and D the following events:

T : « The person buys the television »

D : «The person buys the *DVD* machine »

1) Use a tree diagram to translate the given.
2) Calculate the probabilities of each of the following events:
   a- E : « The person buys the two machines »
   b- F : « The person buys the *DVD* machine »
   c- G : « The person doesn't buy any of the two machines »
3) Before the sale, the television cost 500 000 LL and the *DVD* machine cost 200 000 LL.
   During the sale week, the store made an offer of 15 % discount for buying each of the two machines alone and 25 % for buying both machines together.
   Designate by X the possible expenses of this person (in LL).
   a- Determine the possible values of X.
   b- Determine the probability distribution of X and calculate $E(X)$.
4) On Monday morning, 30 clients visited the big store.
   Randomly we select three clients, what is the probability that the three clients bought only a television.

N° 13.

Consider two urns U and V such that:

U contains 4 **white balls and 3 black balls.**

V contains 2 **white balls and 1 black ball.**

We draw one ball from U and we place it in V then we draw one ball of V and we place it in U.

The set of these two drawings is called a « trial ».

1) Consider the event E: « After the trial, the two urns contain the same balls as in their initial state ».
Show that $p(E)=\frac{9}{14}$.
2) Consider the event F: After the trial, urn V contains only one white ball.
Calculate $p(F)$.
3) Consider the event H: After the trial, urn V contains 3 white balls.
Calculate $p(H)$.
4) If after the trial, urn V contains only one white ball, we gain 3 points.
If after the trial, urn V contains two white balls, we gain 1 point.
If after the trial, urn V contains three white balls, we lose 9 points.
Let X be the random variable equal to the algebraic gain of the player..
a- Determine the probability distribution of X.
c- Calculate the expected value $E(X)$. Is this game fair?

N° 14.

The 80 students of the third secondary classes of a school are distributed in three sections: GS, LS and ES according to the following table:

	GS	LS	ES
Girls	8	18	10
Boys	12	14	18

Part A.

The direction of the school chooses simultaneously and at random a group formed of 3 students of these classes to participate in a TV program.

1) What is the number of possible groups ?
2) Calculate the probability of choosing three students of the LS section.
3) The three chosen students are from the SE section, what is the probability that they are all boys ?
4) The chosen group is formed of 3 girls, what is the probability that they are from the same section.?
5) Show that the probability of choosing a group containing a girl of each section is $\frac{18}{1027}$.
6) Designate by X the random variable equal to the number of boys chose
Determine the probability distribution of X.

Part B.

The direction of the school chooses one student from each section.

1) Calculate the probability of choosing three girls.
2) Consider the two events:

   B : The student chosen of the LS class is a boy.

   H : The group chosen is formed of two girls and one boy.

   G : The student chosen of the LS class is a girl.

   Calculate $p(H/B)$ and $p(H/G)$.

N°15.

For an exam, each of ten teachers prepare 2 questions.
The 20 questions are placed in 20 identical envelopes.
Two candidates sit for the exam.
Each one chooses, at random, two questions.
The questions chosen by the first candidate are not offered for the second.
Denote by A_1 the event: The two questions obtained by the first candidate are written by the same teacher.
Denote by A_2 the event: The two questions obtained by the second candidate are written by the same teacher.

1) Show that $p(A_1)=\dfrac{1}{19}$.
2) a- Calculate $p(A_2/A_1)$.

   b- Calculate the probability that each of the two candidates get two questions written by the same teacher.
3) a- Calculate $p(A_2/\overline{A_1})$.

   b- Deduce $p(A_2)$ and $p(A_1 \cup A_2)$.

N°16.

An urn U_1 contains one ball holding the number 100 and three balls holding each the number 20 .
An urn U_2 contains three balls holding each the number 50 and two balls holding each the number 20 .

1) We draw at random one ball of each of the two urns.
   Let X be the random variable equal to the sum of the numbers on the balls thus drawn.
   Determine the probability distribution of X.
2) In this part, we choose one of these two urns and we draw simultaneously two balls at random. Consider the following events:

E : We choose urn U_1.
F : We choose urn U_2.
S : The sum obtained is less than 90.

Suppose that $p(E)=\frac{2}{3}$ and $p(F)=\frac{1}{3}$.

a- Calculate the probability $p(S/E)$.
b- Calculate the probability of each of the following events: $S\cap E$, $S\cap F$ and S.

N° 17.

In a school, there are two sections: English and French.
To participate in a marathon, the director of activities in the school chooses:
From the French section, a group formed of two students of the G.S. class and 3 students of the L.S. class.
From the English section, a group formed of three students of the G.S. class and 2 students from the L.S. class.
From the 10 representatives of the school chosen above, 3 members are chosen at random to represent the school in a conference.

1) Consider the following events:
   E : The 3 members chosen represent the French section.
   F : The three members chosen represent the English section.
   G : The three members chosen are two students of the G.S. class and 1 student of the L.S. class
   a- Show that $p(G)=\frac{5}{12}$.
   b- Calculate the following probabilities: $p(E)$, $p(G/E)$ and $p(G/F)$.
   c- What is the probability of the event H:
   The 3 chosen members are 2 students of the G.S. class and 1 student of the L.S. class all representing the French section?
2) Let X be the random variable equal to the number of students of the G.S. class chosen.
   a- Determine the probability distribution of X.
   b- Determine and trace the distribution function of X.

N° 18.

In a multiple choice questionnaire, for each question given, two answers are proposed of which one is correct.
The questionnaire is constituted of three questions.
A candidate answers, at random, each of the three questions.

1) Consider the two following events:
   E: The candidate gives correct answers to the three questions.
   F: Among the three answers of the candidate, there are exactly two correct answers.
   Calculate $p(E)$ and $p(F)$.
2) The marking scheme gives $+5$ points for each correct answer and -3 points for each wrong answer.
   Let X be the random variable equal to the final grade of the candidate in this questionnaire.
   a- Determine the probability distribution of X.
   b- Calculate the expected value $E(X)$.
   c- Determine and trace the distribution function of X.

N°19

A university proposes to its students three options only: A, B and C.
Each student has to follow one and only one of the proposed options.
The students who chose option A are double those who have chosen B.
The students who have chosen B are triple those of C.
We know also that:
20% of the students who followed option A are girls.
30% of the students who followed option B are girls.
40% of the students who followed option C are girls.
We choose at random a student from this university.
Denote by:
A the event: the student follows option A.
B the event: the student follows option B.
C the event: the student follows option C.
G the event: the student is a girl.

1) Calculate the probabilities of each of the events A, B and C.
2) a- Calculate the probability that the student is a girl who has chosen option A.
   b- Calculate $p(G)$.
3) Calculate the probability that the student has chosen option A knowin[g] that she is a girl.
4) The student did not choose option A, calculate the probability that she a girl.

N°20.

For the maintenance of a central heating system, a company controls its radiators as follows:

20 % of the radiators are guaranteed.

From the radiators under guarantee, the probability that a radiator is defective is $\frac{1}{100}$.

From the non- guaranteed radiators, the probability that a radiator is defective is $\frac{1}{10}$.

We define by G the following event: The radiator is under guarantee.

1) Calculate the probabilities of the following events:
   A : The radiator is under guarantee and is defective.
   D : The radiator is defective.
2) The radiator is defective.
   Show that the probability that it is under guarantee is $\frac{1}{41}$.
3) The control is for free if the radiator is under guarantee.
   It costs 20 000 L.P if the radiator is not under guarantee and is not defective.
   It costs 50 000 L.P if the radiator is not under guarantee and is defective.
   Denote by X the random variable that represents the cost of control of the radiator.
   a) Determine the probability distribution of X as well as its expected value.
   b) Estimate the amount of money paid by the company that owns 50 radiators.

N°21.

Part A .

An urn contains n black balls ($n \geq 1$) and 2 white balls.
We draw at random, successively and without replacement, two balls from the urn.

1) What is the probability of drawing two white balls?
2) Denote by u_n the probability of drawing two balls of the same color.
   Calculate u_n in terms of n and calculate $\lim\limits_{n \to +\infty} u_n$.

Part B .

In this part, let $n=4$. A player draws successively and without replacement, two balls from the urn.

If the two balls are black, the player gains x Euros $(x>0)$ and the game stops.

If the two balls are white, the player gains $6x$ Euros and the game stops.

If the two balls drawn are of different colors, then he doesn't replace them in the urn and proceeds to a third drawing:

- If the third ball is black, he gains y Euros $(y>0)$ and the game stops
- If not he loses 3 Euros and the game stops.

Designate by G the random variable equal to the algebraic gain of the player.

1) What is the probability that the game stops after the first two drawings?
2) Determine the probability distribution of G.

N°22. ***For the students of the GS section***

Judy has a cellular for which she has subscribed in a monthly package of two hours.

Being concerned of regulating her expenses, she studies the succession of her calls.

- If during one month, she exceeds her subscription, the probability that she exceeds her subscription in the following month is $\frac{1}{5}$.
- If during one month, she doesn't exceed her subscription, the probability that she exceeds her subscription in the next month is $\frac{2}{5}$.

n being a natural number, designate by:

A_n the event: « Judy exceeds her subscription in month n »

$\overline{A_n}$ the contrary event.

Let $p_n = p(A_n)$ and suppose that $p_1 = \frac{1}{2}$.

1) Calculate the probability p_2
2) a- Find the probabilities of A_{n+1} given that A_n is realized and of the probability of A_{n+1} given that $\overline{A_n}$ is realized.

   b- Show that for all non zero natural numbers n , we have:

   $p(A_{n+1} \cap A_n) = \frac{1}{5} p_n$ and express $p(A_{n+1} \cap \overline{A_n})$ in terms of p_n

c- Show that $p_{n+1} = \frac{2}{5} - \frac{1}{5}p_n$.

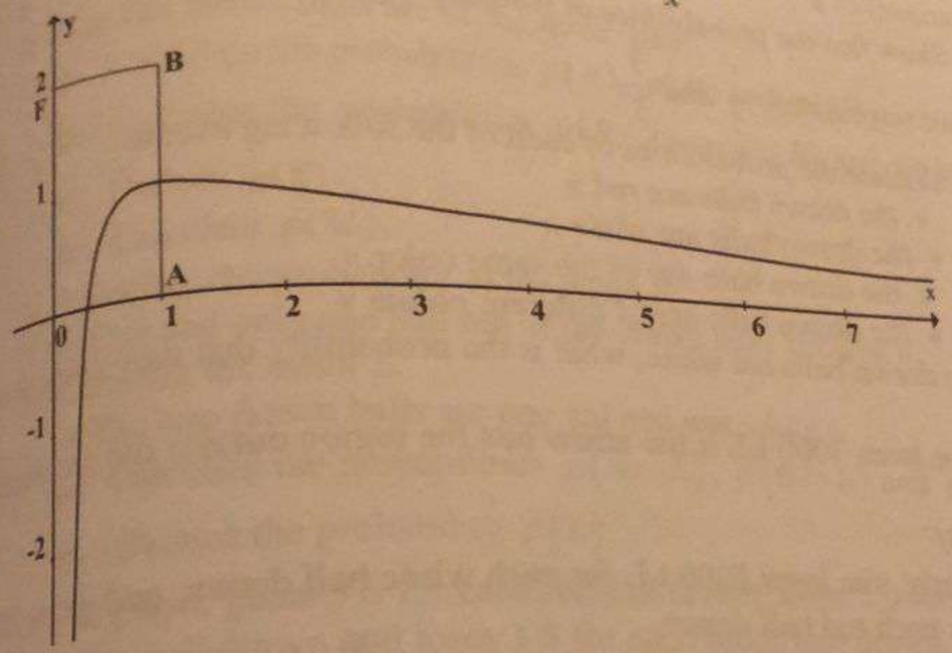
3) For all non zero natural numbers n, let $u_n = p_n - \frac{1}{3}$.

a- Show that (u_n) is a geometric sequence whose ratio and first term are to be determined.

b- Express u_n then p_n in terms of n, determine $\lim_{n \to +\infty} p_n$.

N° 23 **For the students of the G.S. section**

The curve (C) below is that, in an orthonormal system $(O;\vec{i},\vec{j})$, of the function f defined over $]0;+\infty[$ by $f(x) = \frac{1+\ln x}{x}$.



Part A.

1) Calculate the area A_1 of the region D_1 limited by par (C), the axis $x'x$ and the two straight lines of equations $x = \frac{1}{e}$ and $x = 1$.

2) Consider the rectangle $OABF$ with $A(1;0)$, $B(1;2)$ and $F(0;2)$.
D_2 is the domain inside the rectangle $OABF$ and excluding D_1.
Calculate the area A_2 of the region D_2.

Part B .

Reine throws an arrow on the rectangle $OABF$.
We admit that the probability of reaching the rectangle is 0.8 and that the probabilities of reaching D_1 and D_2 are proportional respectively to the two area of these domains.
Consider the two urns U and V.
U contains 3 red balls and 4 white balls.
V contains 4 red balls and 3 white balls .
If the arrow reaches region D_1, Reine draws two balls simultaneously and a random from U .
If the arrow reaches region D_2, Reine draws two balls simultaneously and at random of V.

1) Show that the probabilities of reaching regions D_1 and D_2 are respectively $\frac{1}{5}$ and $\frac{3}{5}$.
2) Calculate the probabilities of each of the following events:
   R : « the drawn balls are red »
   W: « the drawn balls are white »
   M : « the drawn balls are of the same color »
   D : « the drawn balls are of different colors »
3) The drawn balls are white, what is the probability that they come from U.
4) Reine loses 3000 LL if the arrow hits the region outside the rectangle

   $OABF$.
   Similarly, she loses 1000 LL for each white ball drawn, and gains 2000 LL for each red ball drawn.
   Let X be the random variable equal to the algebraic gain of Reine.
   a- Determine the probability distribution of X .
   b- Is the game fair ?

N°24.

Consider:

- two urns U and V such that :
  U contains 3 red balls, 2 white balls and 1 black ball.
  V contains 2 red balls and 4 white balls.
- a bag S containing 6 counters holding the letters:
  « a ; b ; b ; c ; c ; c »

A game rolls in the following manner:
A player draws a counter of sac S.
If he obtains a counter holding the letter a then he draws simultaneously 2 balls of U.
If he obtains a counter holding the letter b then he draws simultaneously 2 balls of V.
If he obtains a counter holding the letter c then he draws one ball from U and one ball from V .
Consider the following events:
A : The player obtains a counter holding the letter a.
B : The player obtains a counter holding the letter b.
C : The player obtains a counter holding the letter c
R : The two drawn balls are red.
M : The two drawn balls are of the same color.

1) a- Calculate the probabilities $p(A)$, $p(B)$ and $p(C)$
   b- Calculate the probabilities $p(R/A)$,, $p(R/B)$ and $p(R/C)$.
   c- Deduce $p(R)$.
   d- Calculate $p(M)$.
2) Calculate the probability of the event B:
   « One and only one ball out of the drawn balls is black »
3) Consider the event L:
   « The two drawn balls are one red and one white »
   a- Calculate the probabilities $p(A \cap L)$, $p(B \cap L)$ and $p(C \cap L)$.
   b- Deduce the probability $p(L)$
4) The player gains 5 \$ for each red ball drawn, loses 2 \$ for each white ball drawn and loses 1 \$ for each black ball drawn.
   Let X be the random variable equal to the algebraic gain of the player.
   a- Give the possible values of X.
   b- Determine the probability distribution of X.

N°25.

An urn U_1 contains 3 green balls, 4 red balls and n yellow balls where n is a natural number verifying $1 \leq n \leq 10$.
An urn U_2 contains 3 green balls , 4 red balls and $n-1$ yellow balls.

Solved Problems

1) A player draws simultaneously and at random two balls from the urn U_1.
Calculate, in terms of n, the probabilities of each of the following events:
R : the two drawn balls are red.
D : the two drawn balls are one red and one green.

2) The player decides to play a game, for this he draws one ball from urn U_1 :

- If the ball drawn is green, he gains 16 000 LL.
- If the ball drawn is red, he loses 12 000 LL.
- If the ball drawn is yellow, he puts it in urn U_2 and then draws one ball from U_2.
  - * If the ball drawn from U_2 is green he gains 8 000 LL
  - * If the ball drawn of U_2 is red, he loses 2 000 LL
  - * If the ball drawn from U_2 is yellow he neither gains or loses.

At the beginning of the game, the player has 12 000 LL.
Let X be the random variable equal to the amount that the player has at the end of the game.

a- Give the possible values of the random variable X.
b- Determine the probability distribution of X.
c- Prove that the expected value of X is $E(X)=4000\left[3+\frac{4n}{(n+7)^2}\right]$.

3) Consider the function defined over $[0;10]$ by $f(x)=\frac{x}{(x+7)^2}$.

a- Set up the table of variations of f.
b- Deduce the value of n for which $E(X)$ is maximal.

N°26.

A game consists of throwing an arrow on a target divided into three parts numbered 1, 2, 3.
Two players A and B are present.
We admit that each of the throws hits one part only and that the throws are independent.
For the player A, the probabilities of hitting 1, 2, 3 are in this order $\frac{1}{12};\frac{1}{3};\frac{7}{12}$.
For player B the three eventualities are equiprobable.

Part A.

Player A throws the target three times.

Designate by E, F and G the following events:

E : Player A reaches part 3.

F : Player A reaches parts 1, 2, 3 in this order.

G : Player A reaches parts 1, 2, 3.

Calculate $p(E)$, $p(F)$ and $p(G)$.

Part B.

We choose at random one of the two players.

The probability of choosing A is two times the probability of choosing B.

1) What is the probability of choosing A ?
2) Only one throw is made.
   What is the probability that part 3 is hit ?
3) Only one throw is made, and part 3 is hit.
   What is the probability that player A threw the arrow ?