Solved Problems

N°1

Consider the sequence (u_n) defined over IN by:

$u_0 = 1$ and for all natural numbers n, $u_{n+1} = \dfrac{u_n}{\sqrt{1+u_n^2}}$.

Express the general term u_n in terms of n and deduce that it is convergent.

N°2

Consider the sequence (u_n) defined over IN by $u_n = \dfrac{3n+\cos n}{2n+1}$.

1) Show that $0 < \dfrac{3}{2} - u_n \leq \dfrac{5}{4n}$.
2) Deduce the limit of u_n as n tends to $+\infty$.

N°3

Let u_0 be a real number and (u_n) the sequence defined over IN by its first term u_0 and the recurring relation $u_{n+1} = \dfrac{u_n}{2+u_n^2}$.

1) Prove that $|u_{n+1}| \leq \dfrac{|u_n|}{2}$ for all natural numbers n.
2) Deduce that for all natural numbers n, $|u_n| \leq \dfrac{|u_0|}{2^n}$.
3) What is the limit of the sequence (u_n)?

N°4

For all natural numbers $n > 0$ define the sequence (u_n), by $u_n = \dfrac{n^2}{2^n}$.

1) For all natural numbers $n > 0$ let $v_n = \dfrac{u_{n+1}}{u_n}$.

a- Determine $\lim\limits_{n \to +\infty} v_n$.

b- Prove that, $v_n > \frac{1}{2}$ for all $n > 0$.

c- Find the smallest natural number N such that if $n \geq N$ then $v_n < \frac{3}{4}$.

d- Deduce that if $n \geq N$ then $u_{n+1} < \frac{3}{4} u_n$.

2) Suppose that , for $n \geq 5$, $S_n = u_5 + u_6 + \dots\dots\dots + u_n$.

a- Prove that $u_{n+1} \leq \left(\frac{3}{4}\right)^{n-5} u_5$.

b- Show that for all natural numbers $n \geq 5$,

$$S_n \leq \left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \dots\dots\dots\dots + \left(\frac{3}{4}\right)^{n-5}\right] u_5.$$

c- Deduce that , $S_n \leq 4u_5$ for all natural numbers $n \geq 5$.

d- Show that the sequence (S_n) is increasing then deduce that it converges.

N°5.

Let (u_n) be a sequence defined over IN by the relations:

$$\begin{cases} u_0 = 2 \\ u_{n+1} = \sqrt{1 + u_n^{\ 2}} \end{cases}$$

1) Show that (u_n) is a divergent sequence.

2) Let (v_n) be a sequence defined over IN by $v_n = u_n^2$.

a- Show that (v_n) is an arithmetic sequence whose common difference is to be determined .

b- Calculate v_n then u_n in terms of n.

c- Determine $\lim\limits_{n \to +\infty} u_n$.

d- Determine $\lim\limits_{n \to +\infty} \left(\frac{u_n}{\sqrt{n}}\right)$.

N°6.

Given the sequence (u_n), defined by:

$u_1 = \frac{1}{3}$ and for $n \geq 2$, $u_{n+1} = \frac{4-3u_n}{9(1-u_n)}$.

1) Let f be the function defined over $]1;+\infty[$ by $f(x) = \frac{-3x+4}{9(-x+1)}$.

   Study the variations of f.

2) a- Show that the sequence (u_n) is bounded above by $\frac{2}{3}$.

   b- Show that (u_n) is increasing.

   c- Deduce that (u_n) is convergent and calculate its limit.

3) Noting that $u_1 = \frac{1}{3}$, $u_2 = \frac{3}{6}$, $u_3 = \frac{5}{9}$ and $u_4 = \frac{7}{12}$,

   express u_n in terms of n.

N°7.

Consider the sequence (u_n) defined over IN by :

$$\begin{cases} u_0 = 1 \\ u_{n+1} = \frac{1}{3}u_n + n - 1 \end{cases}.$$

Let (v_n) be the sequence defined by $v_n = 4u_n - 6n + 15$ for all natural numbers n.

1) Show that (v_n) is a geometric sequence whose first term and ratio are to be determined.

2) Express v_n in terms of n and deduce that $u_n = \frac{19}{4} \times \frac{1}{3^n} + \frac{6n-15}{4}$.

3) Show that the sequence (u_n) can be written in the form $u_n = t_n + w_n$ where (t_n) is a geometric sequence and (w_n) is an an arithmetic sequence.

4) Calculate $T_n = t_0 + t_1 + \dots\dots\dots + t_n$ and $W_n = w_0 + w_1 + \dots\dots + w_n$ then deduce $U_n = u_0 + u_1 + \dots\dots\dots + u_n$.

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N°8.

Let f be the function defined for $x > \frac{1}{2}$ by $f(x) = \frac{x^2}{2x-1}$.

Define the sequence (u_n) by :

$$\begin{cases} u_0 = 2 \\ u_{n+1} = f(u_n) = \dfrac{u_n^{\,2}}{2u_n - 1} \end{cases}$$ for all natural numbers n.

1) a- Prove that for all $x > 1$, $f(x) > 1$.

   b- Deduce that $u_n > 1$ for all natural numbers n.

2) Consider the two sequences (v_n) and (w_n) such that

   $v_n = \dfrac{-1 + u_n}{u_n}$ and $w_n = \ln(v_n)$.

   a- Verify that (v_n) and (w_n) are defined for all natural numbers n.

   b- Prove that (w_n) is a geometric sequence of common ratio $r = 2$ whose first term is to be determined.

   c- Express w_n then v_n in terms of n.

   d- Deduce the expression of u_n and calculate $\lim_{n \to +\infty} u_n$.

N°9.

Let (u_n) be the sequence defined over IN by its first term u_0 and for all natural numbers n, $u_{n+1} = \dfrac{2 + 4u_n}{3 + u_n}$.

1) Suppose that $u_0 = 2$, show that (u_n) is a constant sequence.

2) Suppose that $u_0 = 1$.

   a- Prove that $0 < u_n < 2$ for all natural numbers n.

   b- Show that this sequence is increasing for all natural numbers n.

   c- Deduce the limit of this sequence .

N°10

Consider the sequence (u_n) defined over IN by $u_0 = 1$ and $u_{n+1} = \frac{u_n + 8}{2u_n + 1}$.

1) Calculate u_1, u_2 and u_3.

2) Let h be the function defined over $\left]-\frac{1}{2};+\infty\right[$ by $h(x) = \frac{x+8}{2x+1}$ and let (H) be its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

a- Draw (H) and the straight line (d) of equation $y = x$ in the given system.

b- Construct the points of (H) and (d) of respective abscissas u_0, u_1, u_2 and u_3.

c- What do you notice about the convergence of the sequence (u_n)?

3) Let (v_n) be the sequence defined over IN by $v_n = \frac{u_n - 2}{u_n + 2}$.

a- Calculate: v_0, v_1 and v_2.

b- Show that (v_n) is a geometric sequence whose first term and common ratio are to be determined.

c- Determine $\lim\limits_{n\to+\infty} v_n$.

d- Express u_n in terms of v_n and deduce the limit of (u_n).

N°11

Consider the sequence (I_n) defined for all non-zero natural numbers n by $I_n = \int_0^1 x^n e^{-x} dx$.

1) Show that this sequence is bounded below by 0.
2) Show that this sequence is decreasing.
3) a- Prove that $I_{n+1} = (n+1)I_n - \frac{1}{e}$.

b- Calculate I_1

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N° 12.

For all natural numbers n of IN^*, consider the sequence (I_n) defined by $I_n = \int_1^e (\ln x)^n dx$.

1) Show that the sequence (I_n) is decreasing.
2) a- Calculate I_1.
   b- Prove, using integration by parts, that: $I_{n+1} = e-(n+1)I_n$.
3) a- Prove that for all natural numbers n of IN^*, $(n+1)I_n \leq e$.
   b- Deduce the limit of I_n.
   c- Determine the value of $nI_n + (I_n + I_{n+1})$ and deduce the limit of nI_n.

N° 13.

(u_n) and (v_n) are two sequences of real numbers such that $u_1 = 12$, $v_1 = 1$ and for all natural numbers $n \in IN^*$, $u_{n+1} = \dfrac{u_n + 2v_n}{3}$ and $v_{n+1} = \dfrac{u_n + 3v_n}{4}$.

1) For all natural numbers $n \in IN^*$, let $w_n = v_n - u_n$.
   a- Show that (w_n) is a geometric sequence whose first term and common ratio are to be determined.
   b- Express w_n in terms of n and determine $\lim_{n\to+\infty} w_n$.
2) Prove that the sequence (u_n) is decreasing and that (v_n) is increasing.
3) a- Prove that $u_n > v_n$ for all natural numbers $n \in IN^*$.
   b- Deduce that the two sequences are convergent.
4) For all natural numbers $n \in IN^*$, let $t_n = 3u_n + 8v_n$.
   a- Prove that the sequence (t_n) is a constant sequence.

b- Deduce the expressions of u_n and v_n in terms of n.

c- Show that (u_n) and (v_n) are convergent to the same limit.

N° 14.

For all natural numbers n, consider the sequence (u_n) defined by

$u_n = \int_0^1 \frac{e^x}{e^{nx}(1+e^x)}dx$.

1) Show that $u_0 = \ln\frac{1+e}{2}$.
2) Show that $u_0 + u_1 = 1$ then deduce u_1.
3) Prove that the sequence (u_n) is bounded below by 0.
4) Prove that the sequence (u_n) is decreasing for all natural numbers n.
5) a- Prove that $u_{n-1} + u_n = \frac{1-e^{1-n}}{n-1}$ for all natural numbers n.

   b- Calculate u_2.
6) Let (v_n) be the sequence defined by $v_n = \frac{u_{n-1}+u_n}{2}$.

   a- Calculate $\lim\limits_{n\to+\infty} v_n$.

   b- Prove that $0 \le u_n \le v_n$ for all natural numbers n.

   c- Deduce $\lim\limits_{n\to+\infty} u_n$.

N° 15.

In the plane of an orthonormal system $(O;\vec{i},\vec{j})$, consider the function f defined over $]0;+\infty[$ by $f(x) = \frac{\ln x + e}{x^2}$.

Consider the sequences defined by $I_n = \int_{e^n}^{e^{n+1}} \frac{\ln t}{t^2}dt$ and $A_n = \int_{e^n}^{e^{n+1}} f(t)dt$

where n is a natural number.

1) Prove that $I_n = \frac{n+1}{e^n} - \frac{n+2}{e^{n+1}}$.

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2) a- Show that $A_n = I_n + \frac{e-1}{e^n}$.

b- Calculate I_0 and A_0.

c- Give a graphical interpretation of A_0.

3) Show that the sequence (A_n) converges to 0.

N° 16.

Consider the function f defined over $[0;+\infty[$ by $f(x) = 1 - x^2 e^{1-x^2}$, and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

Part A:

1) Study the variations of f and show that $f(x) \geq 0$.
2) Trace (C).
3) Let k be a given real number, study according to the values of k the number of solutions of the equation $f(x) = k$ in the interval $[0;+\infty[$.
4) n is a non-zero natural number , determine the values of n for which the equation $f(x) = \frac{1}{n}$ admits two distinct solutions .

Part B.

1) Let n be a natural number greater than or equal to 2 .
   Show that the equation $f(x) = \frac{1}{n}$ admits two solutions u_n and v_n included in the intervals $[0;1]$ and $[1;+\infty[$ respectively.
2) Determine the sense of variations of (u_n) and (v_n).
4) Show that the sequence (u_n) is convergent and determine its limit. Proceed in a similar way for the sequence (v_n).

N° 17.

A student should answer questions in an exam successively .
We admit that if he answers the nth question right then the probability that he answers right to the question that follows that is to the $(n+1)^{th}$ question is 0.8 and that if he answers wrong to the nth question then the probability that he answers right to the question that follows is 0.6

Suppose that the probability that he answers the first question right is 0.7.

Designate by A_n the event:

« the student answered the n^{th} question right ».

1) Verify that $p(A_2)=0.74$

3) Designate by p_n the probability of the event A_n and by p_{n+1} the probability of the event A_{n+1}.

a- Show that $p_{n+1}=0.2p_n+0.6$

b- For $n\geq 1$, let $u_n=p_n-0.75$, show that (u_n) is a geometric sequence of common ratio 0.2 ,and deduce an expression of p_n in terms of n and determine $\lim\limits_{n\to+\infty} p_n$.

N° 18.

Part A:

Let f be the function defined over $]0;+\infty[$ by $f(x)=x+\ln\left(\frac{x}{2x+1}\right)$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine $\lim\limits_{x\to 0} f(x)$ and deduce an asymptote to (C).

b- Determine $\lim\limits_{x\to+\infty} f(x)$.

2) Study the variations of f and draw its table of variations .

3) a- Show that the straight line (d) of equation $y=x-\ln 2$ is an asymptote to (C) and study the position of (C) with respect to (d).

b- Trace (C) .

4) Show that the equation $f(x)=0$ admits over $]0;+\infty[$ a unique solution α and such that $1<\alpha<\frac{5}{4}$.

Part B:

Let g be the function defined over $[0;+\infty[$ by $g(x)=(2x+1)e^{-x}$.

and designate by (γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

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1) Determine $\lim_{x\to+\infty} g(x)$ and deduce an asymptote to (γ).
2) Study the variations of g and draw its table of variations.
3) Trace (γ).

Part C:

1) Show that α is a solution of the equation $g(x)=x$.
2) Prove that if $x\in\left[1;\frac{5}{4}\right]$, then $g(x)\in\left[1;\frac{5}{4}\right]$.
3) Study the variations of g' and show that for all $x\in\left[1;\frac{5}{4}\right]$ then $|g'(x)|\le\frac{1}{2}$.
4) Consider the sequence (u_n), defined over IN by $u_0=1$ and $u_{n+1}=g(u_n)$, show that:
   a- For all natural numbers n, $u_n\in\left[1;\frac{5}{4}\right]$.
   b- For all natural numbers n, $|u_{n+1}-\alpha|\le\frac{1}{2}|u_n-\alpha|$.
   c- For all natural numbers n, $|u_n-\alpha|\le\frac{1}{2^{n+2}}$.
   d- The sequence (u_n) is convergent to α.

N° 19.

Consider the sequences (u_n) and (v_n) defined over IN by

$$\begin{cases} u_0=1 \\ u_{n+1}=\left(1+i\sqrt{3}\right)u_n+3 \end{cases} \quad \text{and} \quad v_{n+1}=u_{n+1}-i\sqrt{3}.$$

1) Calculate v_0 and write it in exponential form.
2) Show that the sequence (v_n) is a geometric sequence of common ratio $r=1+i\sqrt{3}$.
3) Express v_n in terms of n.