Solved Problems

N°1.

Solve, in IR, the following equations and inequalities:

1) $e^{2x}+e^{x}-2=0$
2) $e^{3x}-5e^{2x}-6e^{x}=0$
3) $e^{3x}-21e^{x}+20=0$
4) $e^{2x}+3e^{x}-4\leq 0$

N°2.

Solve the following systems:

1) $\begin{cases} e^{2x}.e^{y}=e^{-3} \\ xy=-2 \end{cases}$

2) $\begin{cases} e^{x}+e^{y}=2 \\ e^{2x}+e^{2y}=\dfrac{5}{2} \end{cases}$

3) $\begin{cases} e^{x}=3e^{y} \\ x+y=2-\ln 3 \end{cases}$

4) $\begin{cases} \ln(y+6)-\ln x=3\ln 2 \\ e^{5x}.e^{-y}=e^{-6} \end{cases}$

N°3.

Consider the function f defined over IR by: $f(x)=\left(x^{2}-4\right)e^{2x}$.

Determine the real numbers a, b and c so that the function F defined over IR by: $F(x)=\left(ax^{2}+bx+c\right)e^{2x}$ is an antiderivative of f over IR.

N°4.

Calculate the following integrals:

1) $\displaystyle\int_{0}^{1}\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}dx$

2) $\displaystyle\int_{0}^{\frac{\pi}{2}}\sin 2x\, e^{\sin^{2}x}dx$

3) $\displaystyle\int_{0}^{1}xe^{x^{2}-1}dx$

4) $\displaystyle\int_{0}^{1}xe^{x}dx$

5) $\displaystyle\int_{0}^{1}x^{2}e^{x}dx$

N°5.

Given $I=\displaystyle\int_{0}^{\frac{\pi}{2}}e^{x}\cos x dx$ and $J=\displaystyle\int_{0}^{\frac{\pi}{2}}e^{x}\sin x dx$.

1) By integrating I by parts, prove that $I=e^{\frac{\pi}{2}}-J$.
2) By integrating J by parts, find a relation between I and J.
3) Calculate I and J.

N°6

Show that the equation $e^{2x}+2x-m=0$ admits a unique solution in the set IR for all real numbers m.

N°7

Part A.

Consider the function g defined over IR by $g(x)=2e^{x}+2x-7$.

(Γ) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to-\infty} g(x)$.
2) Show that the straight line (d) of equation $y=2x-7$ is an asymptote to (Γ).
3) Study the variations of g over IR and set up its table of variations.
4) Justify that the equation $g(x)=0$ admits a unique solution α such that $0.94<\alpha<0.941$ and deduce the sign of $g(x)$.
5) Draw (Γ).

Part B.

Consider the function f defined over IR by $f(x)=(2x-5)(1-e^{-x})$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Unit: 2 cm.

1) Study the sign of f over IR.
2) Calculate $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
3) a- Verify that $f'(x)$ and $g(x)$ have the same sign.
   b- Set up the table of variations of f.
   c- Prove that the straight line (D) of equation $y=2x-5$ is an asymptote to (C) at $+\infty$ and study the relative position of (C) with respect to (D).
   d- Taking $\alpha=0.94$, draw (C).
4) Calculate, in cm^2, the area of the region bounded by (C), the axis of abscissa and the straight lines of equations $x=0$ and $x=\frac{5}{2}$.

N°8

Part A.

Consider the function g defined over IR by $g(x)=1-e^{2x}-2xe^{2x}$.

1) Calculate $\lim\limits_{x\to+\infty} g(x)$ and $\lim\limits_{x\to-\infty} g(x)$.
2) Calculate $g'(x)$ and set up the table of variations of g.
3) Calculate $g(0)$ and deduce the sign of $g(x)$.

Part B.

Let (C) be the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ of the function f defined over IR by $f(x)=x+3-xe^{2x}$.

1) Calculate $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
2) a- Calculate $\lim\limits_{x\to-\infty} \frac{f(x)}{x}$ and deduce that the straight line (d) of equation $y=x+3$ is an asymptote to (C) at $-\infty$.
   b- Study the relative positions of (C) and (d).
3) a- Show that f is increasing for $x<0$.
   b- Show that the equation $f(x)=0$ has two roots α and β such that $-4<\alpha<-3$ and $\frac{1}{2}<\beta<1$.
4) Draw (d) and (C).
5) Let λ be a real negative number.
   a- Calculate the area S_λ of the region limited by (C), the straight line (d) and the straight lines of equations $x=\lambda$ and $x=0$.
   b- Calculate $\lim\limits_{\lambda\to-\infty} S_\lambda$.

N°9

Let (C) be the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ of the function f defined over $I=]1;+\infty[$ by $f(x)=x+1-\frac{3e^x}{e^x-e}$.

1) a- Show that the straight line of equation $x=1$ is an asymptote to (C).
   b- Calculate $\lim\limits_{x\to+\infty} f(x)$ and show that the straight line (d) of equation $y=x-2$ is an asymptote to (C).

c- Study the relative positions of (C) and (d).

2) Show that $f'(x)>0$ for all real numbers x of I and set up the table of variations of f.

3) Show that the equation $f(x)=0$ admits a unique solution α and verify that $2.6<\alpha<2.7$.

4) Draw (C).

5) Designate by (D) the region of the plane limited by (C), (d) and the straight lines of equations $x=3$ and $x=4$.

Calculate $\int_3^4 \frac{e^x}{e^x-e}dx$ and deduce the area of (D)

6) a- Show that f has an inverse function g in the interval I.

b- Show that the equation $f(x)=g(x)$ has no solutions in I.

N°10

f is the function f defined over IR by $f(x)=2x+1-x\,e^{x-1}$, designate by (C) its representative curve in a orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

2) Show that the straight line (d) of equation $y=2x+1$ is an asymptote to (C) at $-\infty$ and study the relative position of (C) and (d).

3) The table below is the table of variations of the function f' derivative of f.

x	$-\infty$		-2		1		$+\infty$
$f''(x)$		$+$	0	$-$		$-$	
$f'(x)$	2	$\nearrow$	$2+e^{-3}$	$\searrow$	0	$\searrow$	$-\infty$

a- Study the sign of f' and set up the table of variations of f.

b- Show that f admits a point of inflection L.

4) Show that the equation $f(x)=0$ admits two roots α and β such that $-1<\alpha<-\frac{1}{2}$ and $1<\beta<2$.

5) Draw (d) and (C).

6) a- Calculate the integral $J = \int_1^{\beta} x e^{x-1} dx$.

b- Calculate the area S_β of the region limited by (C) the axis $x'x$ and the straight lines of equations $x = 1$ and $x = \beta$.

c- Prove that $S_\beta = (\beta - 1)(\beta - \frac{1}{\beta})$.

N° 11

Part A.

h is the function defined over IR par $h(x) = (2 - x)e^x - 2$, designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim_{x\to+\infty} h(x)$ and $\lim_{x\to-\infty} h(x)$ and deduce an asymptote to (C).
2) Calculate $h'(x)$ and set up the table of variations of h.
3) Draw (C).
4) The equation $h(x) = 0$ admits two roots 0 and α, show that $1.5 < \alpha < 1.6$.

Part B

Consider the function f defined by $f(x) = \dfrac{e^x - 2}{e^x - 2x}$. Designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that the function defined over IR by $g(x) = e^x - 2x$ is positive and deduce the domain of definition of f.
2) Show that the straight lines of equations $y = 0$ and $y = 1$ are asymptotes to (Γ).
3) Show that $f'(x)$ and $h(x)$ have the same sign and set up the table of variations of f.
4) Show that $f(\alpha) = \dfrac{1}{\alpha - 1}$.

   Deduce to the nearest 10^{-2} the value of $f(\alpha)$ for $\alpha = 1.55$.
5) Draw (Γ), precise the position of (Γ) with respect to the straight line of equation $y = 1$.

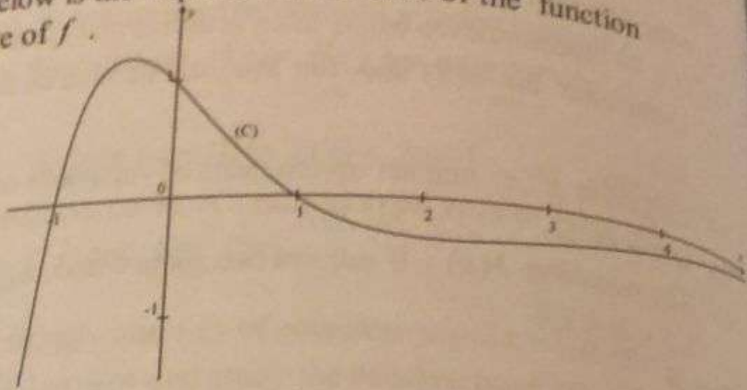
6) Let A_n be the region limited by (Γ'), the straight line of equation $y=1$ and the two straight lines of equations $x=1$ and $x=n$ with $n>1$. Calculate A_n and $\lim\limits_{n\to+\infty} A_n$.

N°12

Part A.

Let (Γ) be the representative curve in an orthonormal system $(O;\vec{i},\vec{j})$ of the function f defined over IR by $f(x)=(x+1)^2e^{-x}$.

1) Determine $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$ and deduce an asymptote to (Γ).
2) The curve below is the representative curve of the function f', derivative of f.



a- Set up the table of variations of f.

b- Write an equation of the tangent (T) to (Γ) at the point of abscissa 0.

c- Draw (Γ) and (T).

3) a- Show that the equation $f(x)=2$ admits a unique solution $\alpha\in[-2;-1]$.

b- Show that $\alpha=-1-\sqrt{2}e^{\frac{\alpha}{2}}$.

4) Calculate the area of the region limited by (Γ), by the axes $x'x$ and straight lines of equations $x=0$ and $x=1$.

Part B.

Consider the function g defined by $g(x)=\ln(f(x))$.

1) Show that the domain of definition of g is $]-\infty;-1[\cup]-1;+\infty[$.
2) Study the variations of g and set up its table of variations

1) Solve the equation $g(x) = -x$.

N° 13.

Part A.

1) Show that for all real numbers $x > 0$:
$e^x \geq x+1$ and that $\ln x \leq x-1$.
2) Deduce that $e^x - \ln x > 2$ for $x > 0$.

Part B.

Consider that the function g defined over $[0;+\infty[$ by
$g(x) = e^x - \ln x - xe^x + 1$.

1) Study the variations of g and set up its table of variations.
2) Show that the equation $g(x) = 0$ admits a unique root α and that $1.23 < \alpha < 1.24$.
3) Deduce the sign of $g(x)$.

Part C.

Consider the function f defined over $[0;+\infty[$ by $f(x) = \dfrac{x}{e^x - \ln x}$.

Let (C) be its representative curve in orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine the limit of f at $+\infty$ and the limit of f at 0.
2) Show that $f'(x)$ and $g(x)$ have the same sign.
3) Find a value to the nearest 10^{-2} of $f(\alpha)$ for $\alpha = 1.23$ and draw (C).

N° 14.

Consider the function f defined over IR by $f(x) = \ln(1+e^x) - x$.

Let (C) be its representative curve in orthonormal system $(O;\vec{i},\vec{j})$.

1) Show that the straight line (d) of equation $y = -x$ is an asymptote to (C) at $-\infty$ and the axis $x'x$ is an asymptote to (C) at $+\infty$.
2) a- Study the variations of f and set up its table of variations.
   b- Draw (C).
3) a- Show that the function f admits over IR an inverse function g.
   b- Draw, in the same system, the curve (C') representative of the function g.

c- Calculate $g'(\ln 2)$ and write an equation of the tangent to (C') at the point of (C') of abscissa $\ln 2$.

4) Consider the function h defined over $]0;+\infty[$ by $h(x) = -\ln(e^x - 1)$.

a- Show that $f \circ h(x) = x$.

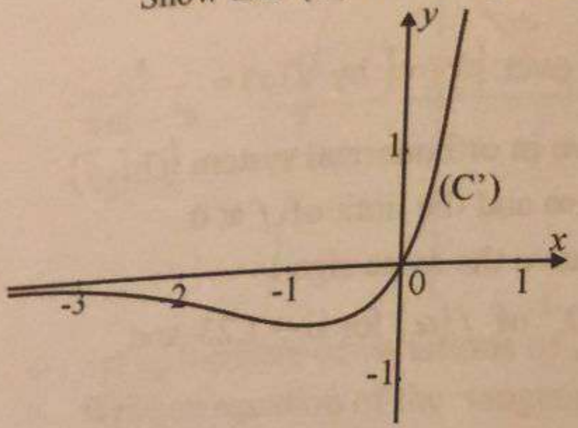
b- Deduce the expression $g(x)$ of the function g.

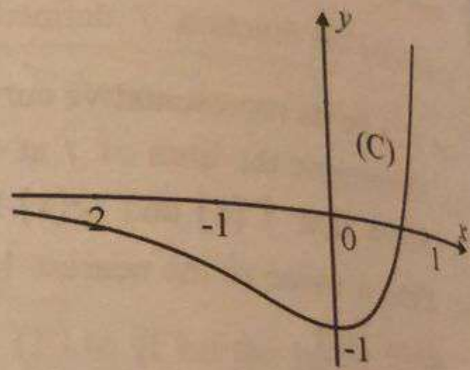
N°15

1) Consider the function f defined over IR by $f(x) = a\,e^{2x} + b\,e^{x}$ wh a and b are two non-zero real numbers and designate by f' the derivative function of f.

a- The curves are the representative curves of the function f and the function f'

Show that (C) is the representative curve of f.





b- Show that $f(x) = e^{2x} - 2\,e^{x}$.

2) Show that the function f admits an inverse function f^{-1} for $x \geq 0$.

Determine the domain of definition of f^{-1}, find the expression $f^{-1}(x)$.

3) Calculate the area of the region (D) bounded by the curves (C) and the semi-straight lines $[Ox)$ and the axis $y'y$.

4) Calculate the volume of the solid generated by rotating the region (D) about $x'x$.

N°16.

Consider the function f defined over IR by $f(x)=x+\ln 4+\dfrac{2}{e^x+1}$.

(C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Calculate $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.
2) Calculate for all real numbers x, $f(x)+f(-x)$, what can you say about the point $A(0;1+\ln 4)$?
3) Study the variations of f and set up its table of variations.
4) a- Show that for all real numbers x,

$$f(x)=x+2+\ln 4-\frac{2e^x}{e^x+1}.$$

b- Show that the straight lines (d) and (d') of respective equations $y=x+\ln 4$ and $y=x+2+\ln 4$ are asymptotes to (C). Study the position of (C) with respect to (d).
5) Show that f admits an inflection point whose coordinates are to be determined.
6) Draw (C), (d) and (d').
7) Show that the equation $f(x)=3$ admits a unique solution β and that $1.1<\beta<1.2$.
8) Show that f admits over IR an inverse function f^{-1} and calculate $\left(f^{-1}\right)'(\ln 4e)$.

N°17.

Consider the function f defined over IR by $f(x)=x+xe^x$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Calculate $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to+\infty} f(x)$.

b- Show that the straight line (d) of equation $y=x$ is an asymptote to (C) at $-\infty$.

c- Study the relative positions of (C) and (d).

d- Write an equation of the tangent (T) to (C) at the point of abscissa -1.

e- Show that f admits an inflection point whose coordinates are to be determined..

f- Study the variations of f' over IR and set up its table of variations.

g- Draw (d), (T), and (C).

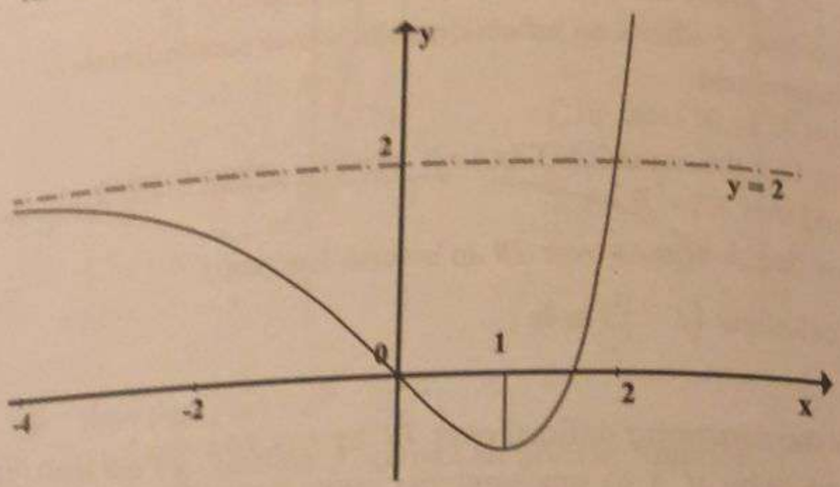
2) Calculate the area of the region limited by (C), (d) and the straight line of equations $x = 0$ and $x = -1$.

3) Study, according to the values of the real number α, the number of points of intersection of (C) and the straight line (δ) of equation $y = x + \alpha$

N°18 For the students of the G.S. section

Part A.

The curve (C) below is the representative curve of a function g defined over IR by $g(x) = (ax+b)e^x + c$.



1) Show that $g(x) = (x-2)e^x + 2$.

2) Show that the equation $g(x) = 0$ admits in the interval $\left[\frac{3}{2};2\right]$ has a unique solution α .

3) Calculate the area of the region limited by (C), its horizontal asymptote $y = 2$ and the two straight lines of equations $x = 0$ and $x = 1$.

Part B.

Consider the function f defined over $]0;+\infty[$ by $f(x)=\frac{e^x-1}{x^2}$.

Designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{\substack{x \to 0 \\ x>0}} f(x)$ and $\lim\limits_{x \to +\infty} f(x)$.

2) Justify that the table of variations of f is the following:

x	0		α		$+\infty$
$f'(x)$	‖	$-$	0	$+$	
$f(x)$	‖	↘	$f(\alpha)$	↗	

3) Taking $\alpha=1.75$, find an approximate value of $f(\alpha)$ to the nearest 10^{-2} then draw (Γ).

Part C.

1) Show that the equation $g(x)=0$ is equivalent to $2(1-e^{-x})=x$.

2) Consider the function h defined over $\left[\frac{3}{2};2\right]$ by $h(x)=2(1-e^{-x})$.

Show that $|h'(x)|\leq\frac{1}{2}$.

3) Consider the sequence (u_n) defined by $u_1=\frac{3}{2}$ and $u_{n+1}=h(u_n)$.

a- Show that $|u_{n+1}-\alpha|\leq\frac{1}{2}|u_n-\alpha|$.

b- Deduce that $|u_n-\alpha|\leq\left(\frac{1}{2}\right)^n$.

c- Determine the limit of the sequence (u_n).

N° 19 ***For the students of the G.S. section***

Consider the function f_n defined over IR by $f_n(x)=\frac{2e^{nx}}{1+e^x}-1$ where n is a natural number, and designate by (C_n) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Unit 2 cm.

Part A.

In this part, take $n=1$.

1) Calculate $\lim\limits_{x\to+\infty} f_1(x)$ and $\lim\limits_{x\to-\infty} f_1(x)$.

2) Calculate $f_1'(x)$ and set up the table of variations of f_1.

3) a- Prove that O is an inflection point of (C_1).

b- Write an equation of the tangent (d) at O to (C_1).

4) Draw (d) and (C_1).

Part B.

Let (C_0) be the representative curve of f_0, corresponding to $n=0$, in the same system $(O;\vec{i},\vec{j})$

1) Prove that the curve (C_0) is symmetric to the curve (C_1) with respect to the axis of ordinates.

2) Prove that (C_0) is symmetric to (C_1) with respect to the axis of abscissas.

3) Calculate, in cm^2, the area of the region limited by the curves (C_1), (C_0) and the straight lines of the equations $x=0$ and $x=1$.

Part C.

Consider the sequence (u_n) defined by $u_n=\int_0^1 f_n(x)dx$.

1) Prove that $u_{n+1}+u_n=2\frac{e^n-n-1}{n}$.

2) Calculate $\lim\limits_{n\to+\infty}(u_{n+1}+u_n)$ and deduce that the sequence (u_n) is not convergent.

20 ***For the students of the G.S. section***

Part A.

Consider the function g defined, over IR, by: $g(x)=e^{x}-2x-1$.

1) Calculate $g'(x)$ and set up the table of variations of g.
2) Show that the equation $g(x)=0$ admits two solutions 0 and α such that $1<\alpha<2$.
3) Deduce the sign of $g(x)$ according to the values of x.

Part B.

Consider the function f defined over IR by $f(x)=(2x+3)e^{-x}+x-1$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.
   b- Show that the straight line (d) of equation $y=x-1$ is an asymptote to (C).
   c- Study the relative positions of (C) and (d).
2) Show that $f'(x)=g(x)e^{-x}$ and set up the table of variations of f.
3) Take $\alpha=1.5$ and draw (C).
4) Calculate the area of the region limited by the curve (C), (d) and the two straight lines of equations $x=0$ and $x=1$.
5) Determine the set of values for which the equation $(2x+3)e^{-x}+x-1+m=0$ admits three distinct roots

Part C.

Consider the function h defined by $h(x)=\ln[f(x)]$.

1) Verify that the equation $f(x)=0$ has one root β such that $-1.2<\beta<-1.1$ and deduce the domain of definition of h.
2) Calculate $\lim\limits_{x\to-\infty} h(x)$ and $\lim\limits_{x\to\beta} h(x)$.
3) Study the variations of h and set up its table of variations.

For the students of the GS section

N° 21

Part A.

Consider the function f defined over IR by $f(x)=(x^2-3x+1)e^{x}$.

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Determine the limits of f at the boundaries of its domain of definition.

b- Deduce an asymptote (D) to (C).

2) a- Study the variations of f and set up its table of variations.

b- Draw (C).

3) Consider $I=\int_{-3}^{0} f(x)\,dx$, interpret I graphically and give its exact value.

Part B.

1) The table below is the table of variations of a function g defined over IR by $g(x)=e^{(x^2+ax+b)}$, where a and b are two real numbers. Determine a and b.

x	$-\infty$		$3/2$		$+\infty$
$g'(x)$		$-$	0	$+$	
$g(x)$	$+\infty$	$\searrow$	$e^{-5/4}$	$\nearrow$	$+\infty$

2) Consider the function h defined over IR by $h(x)=e^{(x^2-3x+1)}$.

Designate by (Γ) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

Prove that the straight line (D_1) of equation $x=\dfrac{3}{2}$ is an axis of symmetry of (Γ).

Part C.

Consider the function v defined over IR by $v(x)=e^{f(x)}$.

Study the variations of v and set up its table of variations.