Sequences

N°1) Determine the domain of definition and calculate the first three terms of each of the following sequences

1) $U_n = 2n - 3$ 2) $U_n = \sqrt{n-2}$ 3) $U_n = \frac{n-1}{n+2}$ 4) $U_n = \frac{2^n}{n}$

N°2) For each of the following recursive sequences, calculate the first four terms and express U_{n+2} in terms of U_n.

1) $\begin{cases} U_0 = 0 \\ U_{n+1} = 2U_n - 3 \end{cases}$ 2) $\begin{cases} U_1 = -2 \\ U_{n+1} = \sqrt{U_n + 3} \end{cases}$ 3) $\begin{cases} U_0 = 0 \\ U_{n+1} = U_n - \frac{1}{n+1} \end{cases}$

N°3) Discuss the sense of variation of the following sequences:

1) $U_n = 5n^3 - 3$ 2) $U_n = n + \frac{2}{n}$ 3) $U_n = \frac{n+1}{n-1}$ 4) $U_n = \sqrt{n} - 1$

N°4) Let the sequence (U_n) defined by $\begin{cases} U_0 = 2 \\ U_{n+1} = 2U_n - 3 \end{cases}$ for any $n \in \mathbb{N}$

show, by induction, that $U_n = 3 - 2^n$

N°5) Let the sequence (U_n) defined by $\begin{cases} U_0 = 1 \\ U_{n+1} = \frac{1}{2}U_n - 1 \end{cases}$ for any $n \in \mathbb{N}$

show, by induction, that $U_n > -2$

N°6) Let (U_n) be the sequence defined by $\begin{cases} U_1 = 2 \\ U_{n+1} = 2U_n - 1 \end{cases}$ for any n

show, by induction that $U_n = 2^{n-1} + 1$

N°7) show, by induction over n, each of the following relations

a) $2^n > n$ for every $n \in \mathbb{N}$

b) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ $(n \in \mathbb{N}^*)$

c) $3^{2n} - 2^n$ is divisible by 7 for every $n \in \mathbb{N}$

N°8) a) Calculate x so that the terms x^2+1, $2x$, x^2-1, taken in this order, form an Arithmetic sequence

b) Same question for $2x$, $2x+1$, $6x-2$

N°9) Determine an Arithmetic sequence if its 11^{th} term is 2 and its 21^{th} term is 50

N°10) the 5^{th} and the 11^{th} terms of an A.S are -3 and 15. What is the sum of the first fourty terms of this sequence

N°11) the sum of 3 consecutive terms of an A.S is 36 and their product is 276. What are these terms?

N°12) Consider the sequence (U_n) defined by $\begin{cases} U_0 = -1 \\ U_{n+1} = \dfrac{9}{6-U_n} \end{cases}$, $n \in \mathbb{N}$

consider the sequence (V_n) defined by $V_n = \dfrac{1}{U_n - 3}$

1) show that the sequence (V_n) is an arithmetic sequence and define precisely the first term and the difference
2) Express V_n, then U_n in terms of n

N°13) Let (U_n) be the sequence defined by $\begin{cases} U_0 = -2 \\ U_{n+1} = \dfrac{U_n}{1-U_n} \end{cases}$ $n \in \mathbb{N}$.

1) show, by induction, that $U_n < 0$, for every natural number n
2) suppose that $V_n = 1 + \dfrac{1}{U_n}$, for all n of $\mathbb{N}$
   a) show that the sequence (V_n) is arithmetic and define precisely its difference and its first term
   b) Express V_n then U_n in terms of n.

N°14) a) the first two terms of a G.S is 2 and $\frac{1}{2}$. What is the 10^{th} term?
b) the 3^{rd} and the 9^{th} terms of a G.S are 3 and 27. What is the first term of this sequence?
c) the sum of 3 consecutive terms of a G.S is 43 and their product is 216. What are these terms?
d) Determine the real number x so that $4x$, $2x-1$, $x+4$ are three consecutive terms of a G.S.

N°15) Let the sequence (U_n) defined by $\begin{cases} U_0 = 1 \\ U_{n+1} = 2U_n + 1 \end{cases}$; $n \in \mathbb{N}$

1) a) Calculate U_1, U_2, U_3
   b) show that the sequence (U_n) is neither Arithmetic nor geometric
2) suppose that $V_n = U_n + 1$
   a) show that (V_n) is a G.S and determine the ratio r and the first term V_0
   b) Calculate V_n then U_n in terms of n
3) Calculate $S_n = V_0 + V_1 + \dots + V_n$ then deduce $T_n = U_0 + U_1 + \dots + U_n$ in terms of n

I) Consider the sequence (I_n) defined, for all integers $n \geq 1$, as $I_n = \int_1^e \frac{(\ln x)^n}{x^2}\,dx$.

1) Prove that $I_n \geq 0$.
2) Show that $I_{n+1} \leq I_n$ and deduce the sense of variations of (I_n).
3) Justify that the sequence (I_n) is convergent.
4) Using integration by parts, prove that: $I_{n+1} = -\frac{1}{e} + (n+1)I_n$.
5) a- Using parts 2) and 4), prove that $I_n \leq \frac{1}{ne}$.

   b- Determine $\lim\limits_{n \to +\infty} I_n$.

II) Consider the sequence (U_n) defined, for all integers $n \geq 1$, by $U_n = \int_1^e x^2 (\ln x)^n\,dx$.

1) Using integration by parts, prove that $U_{n+1} = \frac{1}{3}\left(e^3 - (n+1)U_n\right)$.
2) Prove that all terms of the sequence (U_n) are positive, and show that $U_n \leq \frac{e^3}{n+1}$.
3) a- Prove, for all x in the interval $[1\,;\,e]$, that $(\ln x)^{n+1} \leq (\ln x)^n$.

   b- Prove that $U_{n+1} \leq U_n$ and that $U_n \geq \frac{e^3}{n+4}$.
4) Calculate $\lim\limits_{n \to +\infty} U_n$ and $\lim\limits_{n \to +\infty} n\,U_n$.

III) Consider the sequence (U_n) defined as : $U_n = \int_0^1 \frac{x^{2n}}{1+x^2}\,dx$ where $n \in \mathbb{N}$.

1) ~~a- Calculate U_0.~~ Knowing that $U_0 = \frac{\pi}{4}$.

   b- Calculate $U_0 + U_1$ and deduce U_1.
2) a- For all $n \in \mathbb{N}$, show that $U_n \geq 0$.

   b- For all $0 \leq x \leq 1$, prove that (U_n) is decreasing.

   c- Deduce that (U_n) is convergent.
3) a- For all $n \in \mathbb{N}$, show that $U_{n+1} + U_n = \frac{1}{1+2n}$.

   b- Deduce the limit of U_n as n tends to $+\infty$.

IV) Consider the sequence (u_n) defined by $u_1 = \frac{1}{2}$ and for all natural numbers $n \geq 1$: $u_{n+1} = \frac{n+1}{2n} u_n$.

1) a- Use mathematical induction to prove that $u_n > 0$ for all $n \geq 1$.

   b- Prove that the sequence (u_n) is decreasing. Deduce that (u_n) is convergent.
2) Let (v_n) be the sequence defined, for all $n \geq 1$, by $v_n = \ln\left(\frac{u_n}{n}\right)$.

   a- Prove that (v_n) is an arithmetic sequence whose common difference $d = -\ln 2$ and determine its first term.

   b- Express v_n in terms of n, then verify that $u_n = \frac{n}{2^n}$.

(U_n) and (V_n) are two sequences defined for every $n \in N$ by:

$U_n = \dfrac{3\times 2^n - 4n + 3}{2}$ and $V_n = \dfrac{3\times 2^n + 4n - 3}{2}$

1) Let (W_n) be the sequence defined by $W_n = U_n + V_n$
   a) Calculate W_n in terms of n and show that (W_n) is a geometric sequence whose first term and common ratio are to be determined.
   b) Calculate $S = W_0 + W_1 + \dots\dots\dots + W_n$
2) Let (t_n) be the sequence defined by $t_n = U_n - V_n$
   a) Calculate t_n in terms of n and show that (t_n) is an arithmetic sequence whose first term and common difference are to be determined.
   b) Calculate $S' = t_0 + t_1 + \dots\dots\dots + t_n$.
3) Express U_n in terms of W_n and t_n then deduce the sum $S_n = U_0 + U_1 + \dots\dots\dots + U_n$

V. A sequence (U_n) is defined by $\begin{cases} U_0 = 3 \\ U_{n+1} = \dfrac{2U_n + 6}{5} \end{cases}$ where $n \in \mathbb{N}$.

1) a) Calculate U_1 and U_2.
   b) verify that (U_n) is neither arithmetic nor geometric sequence
2). Let $V_n = U_n - 2$ where $n \in \mathbb{N}$.
   a) show that (V_n) is a geometric sequence whose ratio r and first term V_0 are to be determined.
   b) Express V_n in terms of n, and deduce that $U_n = 2 + \left(\frac{2}{5}\right)^n$.
3) Calculate $U_{n+1} - U_n$ in terms of n and deduce the sense of variations of (U_n).
4) Calculate the sum: $S = V_2 + V_3 + \dots\dots + V_{19}$.
   then Deduce the sum: $S' = U_2 + U_3 + \dots\dots + U_{19}$

VI (U_n) and (V_n) are two sequences defined for every $n \in \mathbb{N}$ by:

$\begin{cases} U_0 = 1 \\ \text{and} \\ U_{n+1} = \frac{1}{3}(2U_n + V_n) \end{cases}$ and $\begin{cases} V_0 = 2 \\ \text{and} \\ V_{n+1} = \frac{1}{3}(U_n + 2V_n) \end{cases}$.

1) Calculate U_1 and V_1.
2) show, by induction, that $U_n - V_n < 0$ for every $n \in \mathbb{N}$.
3) show that the sequence (U_n) is strictly increasing and the sequence (V_n) is strictly decreasing.
4) suppose that $W_{n+1} = U_{n+1} - V_{n+1}$ where $n \in \mathbb{N}$.
   a) show that (W_n) is a geometric sequence whose first term W_1 and common ratio r are to be determined
   b) Calculate W_n and the sum $S_n = W_1 + W_2 + \dots + W_n$ in terms of n

I. Let (U_n) be the sequence defined by $\begin{cases} U_0 = 0 \ ; \ U_1 = 4 \\ U_{n+2} = \frac{1}{2}(U_{n+1} + U_n) \end{cases}$

1) a) calculate U_2 *and* U_3

b) Verify that (U_n) is neither arithmetic nor geometric sequence.

2) Suppose that $W_n = U_{n+1} - U_n$ where $n \in N$

a) Express W_{n+1} in terms W_n and deduce that (W_n) is a geometric sequence whose first term W_0 and the common ratio r are to be determined.

b) Calculate W_n in terms of n and deduce according to the values of n the sense of variations of the sequence (U_n).

c) Calculate the sum $S_n = W_0 + W_1 + \dots\dots\dots + W_n$ in terms of n.

3) Suppose in this part $U_0 = U_1 = 1$. Show that, for every $n \in N^*$, $U_n^{\ 2} = U_{n-1} \times U_{n+1}$.

II. Let (U_n) be a sequence defined by : $\begin{cases} U_0 = 2 \\ U_{n+1} = \frac{1}{2}U_n + 3; \quad n \in N \end{cases}$

1) Calculate U_1 and U_2 and verify that the sequence (U_n) is neither arithmetic nor geometric.
2) Show, by induction, that $U_n < 6$ for ever n
3) Consider the sequence (V_n) defined by: $V_n = U_n - 6$
   a) Show that (V_n) is a geometric sequence and determine its common ratio and its first term V_0.
   b) Calculate V_n in terms of n and deduce U_n in terms of n.
   c) Calculate $\lim_{n \to +\infty} V_n$
   d) Calculate the sum $S = V_3 + V_4 + V_5 + \dots\dots\dots V_{20}$ and deduce $S' = U_3 + U_4 + U_5 + \dots\dots\dots U_{20}$

III. Consider the sequence (I_n) defined by: $\begin{cases} I_0 = 1 \\ I_{n+1} = I_n + 2^{n+1} \end{cases}$

1) Calculate I_1, I_2, I_3, then deduce the sequence (I_n) is neither arithmetic nor geometric.

2) a) Show using mathematical induction that: $I_n = 2^{n+1} - 1$.

~~b) Study the sense of variation of the sequence I_n.~~

3) Consider the sequence: $J_n = I_n + 1$.

a) Show that the sequence (J_n) is a geometric sequence whose first term J_0 and common ratio q are to be determined.

b) Calculate the sum: $S_n = J_0 + J_1 + J_2 + \dots + \dots J_n$

c) Calculate the product: $P_n = J_0 \times J_1 \times J_2 \times \dots \dots J_n$