Mathematics **12 LS** **Counting and Probability** **2021-2022**

Multiplication Rule of Counting

If a task consists of a sequence of choices in which there are p ways to make the first choice, q ways to make the second, etc., then the task can be done in

$$p \bullet q \bullet r \bullet \ldots$$

different ways.

Permutations with Non-distinct Items

The number of permutations of n objects, where there are n1 of the 1st type, n2 of the 2nd type, etc, is

$$\frac{n!}{n_1! \cdot n_2! \cdot \cdots \cdot n_k!}$$

The number of different ways that n distinct things may be ordered (or arranged along a line is:

$$n! = (n)(n-1)\ldots(3)(2)(1)$$

That is, **<u>there are $n!$ ways to order n items along a line.</u>**

When repetition of objects is allowed The number of permutations of n things taken all at a time, when repetion of objects is allowed is n^n.

The number of permutations of n objects, taken r at a time, when repetition of objects is allowed, is n^r.

Permutation Formula

The number of ways to choose and arrange k objects from a group of n objects is

$$_nP_k = \frac{n!}{(n-k)!}.$$

Combinations of n Distinct Objects Taken r at a Time

The number of arrangements of n objects using $r \leq n$ of them, in which

1. the n objects are distinct,
2. repeats are not allowed,
3. order does not matter,

is given by the formula $_nC_k = \dfrac{n!}{r!\,(n-r)!}$.

$$ {}_nP_k = \frac{n!}{(n-k)!} \qquad {}_nC_k = \binom{n}{k} = \frac{n!}{(n-k)!k!} $$

$0! = 1$ (by definition)

$1! = 1$

$2! = 2 * 1 = 2$

$n! = n * (n-1) * (n-2) * \ldots (3) * (2) * (1)$

Counting exercises

Exercise 1.

Four women and three men are participating in a televised debate.
a) In how many ways can these people be seated in 7 chairs?
b) In how many ways can they be seated with the men and women seated in alternate seats?
c) In how many different ways can they be seated with women are not separated?

Exercise 2.

How many plates can be made up using 2 letters followed by 3 digits if:
a) The letters and numbers can be repeated.
b) The letters can be repeated but not the numbers.
c) Neither the letters nor the numbers can be repeated.

Exercise 3.

a) How many different words can be formed using the letters of the word "BONUS" without repeating any letter?
b) How many of them have the letter B in the first place?
c) How many of them have B in the first place and S in the last?

Exercise 4.

A four-digit PIN number is needed to access a bank account (digits are: 0,1,2,3,4,5,6,7,8,9).
How many different four-digit PIN numbers are there:
a) If repetition is allowed?
b) If repetition is not allowed?
c) If repetition is allowed and the first and third digits must be odd?

Exercise 5.

Six people, all different ages, are seated in a row of six seats.
How many different arrangements are there:
a) If the youngest two persons are always on adjacent seats?
b) If the oldest person and the youngest person are always on the ends?

Exercise 6.

The digits of the number 32414 are rearranged to make different numbers.
a) How many different numbers can be formed?
b) How many of these numbers are: i) odd? ; ii) less than 30000?

Exercise 7.

In some states, the license plate of a car consists of three letters followed by three digits.
a) If repetition is not allowed, how many possibilities are there?
b) If repetition is allowed, how many possibilities are there?

Exercise 8.

In how many ways can one arrange the letters AMERICA …
a) if there are no restrictions?
b) if the rearrangement must begin with M?
c) if the rearrangement must begin with M and end with C?
d) if the rearrangement must begin with A?

Exercise 9.

Five pink marbles, two red marbles, and three rose marbles are to be arranged in a row.
If marbles of the same colour are identical, in how many different ways can these marbles be arranged?

Exercise 10.

1) In how many ways can 9 different colour balls be arranged in a row?
2) In how many ways can these balls be arranged so that black, white, red and green balls are i) always together ; ii) never together?

Exercise 11.

A box contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the box, if at least one black ball is to be included in the draw?

Exercise 12.

In a group of 6 boys and 4 girls, four children are to be selected. In how many different ways can they be selected such that at least one boy should be there?

Exercise 13.

From a group of 7 men and 6 women, five persons are to be selected to form a committee so that at least 3 men are there on the committee. In how many ways can it be done?

Exercise 14.

1) How many committees of 4 people can be chosen from 5 men and 3 women?
2) How many of these could be all men?
3) How many would consist of 2 men and 2 women?

Exercise 15.

An urn contains 9 balls: 2 blue balls, 3 red balls and 4 green balls1).
1) We draw simultaneously three balls from the urn.
   Prove that we have 84 possible drawings.
2) We draw, one by one and without replacement, three balls from the urn.
   Prove that we have 504 possible drawings.
3) We draw, one by one and with replacement, three balls from the urn.
   Prove that we have 729 possible drawings.

Exercise 16.

Solve the equations :

1) ${}_{n+1}P_2 = 5 \times {}_nP_1$

2) ${}_nP_3 = 6 \times {}_nC_5$

3) $2 \times {}_nP_2 + 50 = {}_{2n}P_2$

4) $3 \times {}_nC_4 = 14 \times {}_nC_2$

Probability

For any event A in a sample space Ω : $0 \leq P(A) \leq 1$; $P(\phi) = 0$ and $P(\Omega) = 1$.
For any event A in an equiprobable space Ω : $P(A) = \frac{n(A)}{n(\Omega)} = \frac{number\ of\ favorable\ outcomes}{total\ number\ of\ outcomes}$.

Addition rule : $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

If A and B are mutually exclusive: $P(A \cup B) = P(A) + P(B)$

Multiplication rule : $P(A \cap B) = P(A) * P(B \mid A)$ or $P(B) * P(A \mid B)$

If A and B are independent: $P(A \cap B) = P(A) * P(B)$

Complement rule : $P(\overline{A}) = 1 - P(A)$

A and B are mutually exclusive if $P(A \cap B) = 0$

A and B are independent if $P(A \mid B) = P(A)$ or $P(B \mid A) = P(B)$

Exercise 1.

For the events A and B, $p(A) = 0.6$, $p(B) = 0.8$ and $p(A \cup B) = 1$.
Find:
a) $p(A \cap B)$
b) $p(\overline{A} \cup \overline{B})$.

Exercise 2.

For events A and B, the probabilities are $P(A) = \frac{3}{11}$, $P(B) = \frac{4}{11}$.

Calculate the value of $P(A \cap B)$ if:

a) $P(A \cup B) = \frac{6}{11}$
b) events A and B are independent.

Exercise 3.

Let A and B be two events such that:
$p(A \cup B) = 0.8$; $p(A \cap B) = 0.25$; $p(\overline{A}) = 0.6$.
Calculate: $p(B)$; $p(A \cap \overline{B})$ and $p(\overline{A} \cap \overline{B})$.

Exercise 4.

I have 6 gold coins, 4 silver coins and 3 bronze coins in my pocket.
I take out **simultaneously three** coins at random. What is the probability that:

1) they are all of different material?
2) they are all of the same material?
3) only two of the coins are of the same material?

Exercise 5.

An urn contains 3 Red, 4 Blue, and 5 Green balls. Three balls are selected from the urn <u>**with replacement**</u>. What is the probability of the following events:

A: “the three balls will be of the same color”
B: “the balls will be Red, blue and Green in this order”
C: “the three balls will be of different colors”.

Exercise 6.

A bag contains 6 red, 4 white and 8 blue balls. If 3 balls are drawn at random, find the probability of:

1) Drawing simultaneously:
    a) All are red
    b) Two are white and 1 is red
    c) At least one is red
    d) One of each color is drawn
    e) All are of the same color
    f) Not all of the same color
    g) At least two balls are red
2) The balls are drawn in order: red, white and blue without replacement (then redo the same question with replacement).

Exercise 7.

A cage in a laboratory contains 24 mice:
9 white mice (6 females and 3 males), and 15 grey mice (8 females and 7 males).
1) A mouse is selected at random from the cage.
Consider the following events:
A: "The selected mouse is grey".
B: "The selected mouse is a male".
C: "The selected mouse is a grey male".
a) Calculate the probability of the events A, B and C.
b) Calculate the probability of the events $A \cap \overline{B}$ and $B \cap \overline{A}$.
c) Calculate the probability that the selected mouse is grey or a male mouse.
2) The selected mouse is not replaced in the cage, another mouse is selected.
Calculate the probability of each of the following events:
D: "The first mouse is white and the second is grey".
E: "The first mouse is a male and the second is a female".
F: "The two mice have the same sex".

Exercise 8.

From an urn containing 2 white balls, 3 red balls and 5 black balls we draw <u>three</u> balls one after the other **with replacement**.
Calculate the probability of each of the following events:
A: "the three balls are white"
B: "the three balls have the same color"
C: "the three balls have different colors"
D: "no red ball is drawn"
E: "at least one of the balls is red"
F: "only one of the balls is red"
G: "only two of the balls have the same color".

Exercise 9.

An urn contains **5** red balls, **4** black balls and **3** green balls. Three balls are randomly selected from the urn. Consider the following events:

E: "The three balls have the same color"

F: "The three selected balls have three different colors"

G: "Only two of the three selected balls have the same color"

Part A:

In this part, the selection of the three balls is done **simultaneously**.
Calculate the probabilities $p(\text{E}), p(\text{F})$ and $p(\text{G})$.

Part B:

In this part, the selection of the three balls is done **successively and with replacement**.

1) Calculate $p(\text{E})$ and $p(\text{F})$. Deduce $p(\text{G})$.

2) Calculate the probability of each of the events:

   N: "No red ball is selected"

   T: The three selected balls are red".

3) Consider the event **S:** "two of the three selected balls are red".

   Prove that : $p(\text{S}) = \dfrac{175}{576}$.

Exercise 10.

A library has 100 textbooks such that:
• 20% of the books are in Arabic of which 80% are of the Intermediate level.
• 60% of the books are in English of which 75% are of the Secondary level.
• 60% of the books are of the secondary level.
Copy and complete the following table :

	Arabic	English	French	Total
Intermediate		15		
Secondary	4			
Total			20	100

Part A.
One book is randomly selected from the library.
Consider the following events :
A: "The chosen book is in English and of the Intermediate level".
B: "The chosen book is in Arabic or of the Secondary level".
C: "The chosen book is of the Secondary level and not in French".
1) Verify that $P(A) = \frac{3}{20}$.
2) Calculate $P(B)$ and $P(C)$.
Part B.
Three books are randomly and simultaneously selected from the library.
1) How many possible selections of the three books are there?
2) Consider the following events :
D: "The three chosen books are of the same level".
E: "Each of the three chosen books is of a different language".
F: "Only two from the three chosen books are in French".
G: "At least one of the three chosen books is of the Intermediate level".
a) Verify that : $P(D) = \frac{3}{11}$.
b) Calculate $P(E), P(F)$ and $P(G)$.
Part C.
Two books are selected from the library successively and with replacement.
1) How many possible selections of the two books are there?
2) Calculate the probability of each of following events :
M: "The two chosen books are of the Intermediate level".
N: "The first book is in English and the second is in French in this order".

Exercise 11.

An urn contains 9 balls: 2 blue balls, 3 red balls and 4 green balls1).
1) We draw, simultaneously and at random, three balls from the urn.
a) Prove that we have 84 possible drawings.
b) What is the probability of drawing exactly two red balls?

c) What is the probability of drawing at least one blue ball?

2) We draw, one by one and without replacement, three balls from the urn.

a) Prove that we have 504 possible drawings.

b) What is the probability of drawing a blue ball, a red ball and a green ball?

c) What is the probability of drawing at most one green ball?

3) We draw, one by one and with replacement, three balls from the urn.

a) Prove that we have 729 possible drawings.

b) What is the probability of drawing at least two blue balls?

c) What is the probability of drawing that the third drawn ball is the only green ball?

Exercise 12.

Consider two urns:

Urn ***A*** containing **6 balls**: 2 red balls and 4 white balls.

Urn ***B*** containing **9 bills**: 3 bills of 5000 LL; 4 bills of 10 000 LL; and 2 bills of 20 000 LL.

Part A.

One ball is randomly drawn from urn A:

- If the ball is red, then **two bills** are drawn randomly and simultaneously from urn B.
- If the ball is white then **three bills** are drawn randomly and simultaneously from urn B.

Consider the events:

R: "the ball drawn from A is red".

W: "the ball drawn from A is white".

S: "the total amount of money of the bills drawn from urn B is 30 000 LL".

1) Calculate the probabilities $P(W), P(R), P(S|W)$ and $P(S|R)$.

2) Prove that $P(S) = \dfrac{29}{189}$.

3) Assume that the amount of money of the bills drawn from B is not 30 000 LL, what is the probability that the ball drawn from urn A is white?

Part B.

Three bills are drawn randomly and simultaneously from urn B.

Let M be the event : the sum of money of the three drawn bills is equal to 30000.

Prove that $P(M) = \dfrac{5}{42}$.

Exercise 13.

A study, done on the students of grade 12 in a school, has shown that 60% of them are girls. Moreover, 40% of the girls are blond and 30% of the boys are blond.
We chose a student at random. Consider the following events:
A: « the chosen student is blond »
F: « the chosen student is a girl »

1) Prove that the probability of choosing a blond girl is $\frac{6}{25}$.
2) Calculate the probability of choosing a blond student.
3) The chosen student is blond. What is the probability that he is a boy?
4) The study allows us to know that:
   - Among the blond students, half of them have blond parents;
   - Among the not blond students, 65% have not blond parents.

   We note B the event « the chosen student has blond parents »

   a) Prove that $P(A \cap B) = \frac{9}{50}$.
   b) Calculate $P(\overline{A} \cap B)$. Deduce that $P(B) = 0.404$.
   c) Calculate $P(A/B)$ and prove that $P(A/\overline{B}) = \frac{45}{149}$.
   d) What can we deduce from comparing $P(A/B)$ and $P(A/\overline{B})$?

Exercise 14.

In order to prevent a certain disease, we vaccinated 40% of persons of a population. Then we noticed that 85% of the vaccinated persons were not affected by the disease and that 75% of the persons who were not vaccinated are affected by the disease. A person is chosen randomly from this population.
Consider the following events:
D: « the chosen person is affected by the disease».
V: « the chosen person is vaccinated ».

1) a) Verify that the probability of the event D ∩ V is equal to $\frac{6}{100}$.
   b) What is the probability that the chosen person is affected by the disease and is not vaccinated?
   c) Deduce the probability P(D).
2) The chosen person is not affected by the disease.
   Calculate the probability that he/she is vaccinated.
3) In this part, suppose that this population is formed of 300 persons.
   We choose randomly 3 persons from this population.
   What is the probability that at least one, among the 3 chosen persons, is vaccinated?

Exercise 15.

Consider two urns **U** and **V**.
Urn **U** contains eight balls: four balls numbered 1, three balls numbered 2 and one ball numbered 4.
Urn **V** contains eight balls: three balls numbered 1 and five balls numbered 2.

1) **Two balls are selected, simultaneously and randomly, from the urn U**.
   Consider the following events:
   - A: « the two selected balls have the same number »
   - B: « the product of the numbers on the two selected balls is equal to 4 ».

   Calculate the probability P(A) of the event A, and show that P(B) is equal to $\frac{1}{4}$.

2) **One of the two urns U and V is randomly chosen, and then two balls are simultaneously and randomly selected from this urn.**
   Consider the following events:
   - E: « the chosen urn is **V** »
   - F: « the product of numbers on the two selected balls is equal to 4 ».

   a) Verify that $P(F \cap E) = \frac{5}{28}$ and calculate $P\left(F \cap \overline{E}\right)$.

   b) Deduce P(F).

3) **One ball is randomly selected from U, and two balls are randomly and simultaneously selected from V.**
   Calculate the probability of the event H: « the product of the three numbers on the three selected balls is equal to 8 ».