Solved Problems

Part I - The Parabola :

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.

N°1.

Find an equation of the parabola of focus $F(3;0)$ and directrix the straight line (d) of equation $x=-3$.

N°2.

1) Find , in two methods , an equation of the parabola of focus the point $F(2;0)$ and of directrix the straight line (d) of equation $x=-1$.
2) Find an equation of the parabola of focus the point $F(-1;1)$ and of directrix the straight line (d) of equation $x=1$.
3) Find an equation of the parabola of focus the point $F(2;1)$ and of vertex the point $S(0;1)$.
4) Find an equation of the parabola of vertex the point $S(1;1)$ and of directrix the straight line (d) of equation $y=4$.
5) Find an equation of the parabola of focus $F(1;0)$ and directrix the straight line (d) of equation $y=x$.

N°3.

Determine the elements and trace the representative curves of the parabolas whose equations are given below:

1) $y^2=2x-4$
2) $y^2-2y=3x-2$.
3) $x^2+4x-2y+2=0$.

N°4.

Consider the parabola (P) of equation $y^2=4x$ and the point $B(0;2)$.

1) Write an equation of the straight line (δ) passing through the point

B and of slope m.

2) a- Show that the relation between the abscissas of the points of intersection of (P) and (δ) is $m^2x^2+4(m-1)x+4=0$.

b- Study according to the values of m the intersection of (δ) and (P).

c- In the case where (δ) is tangent to (P), find the coordinates of the point of tangency and deduce an equation of the tangent to (P) through the point $C(0;-2)$ other than the y-axis .

N°5.

Consider the parabola (P) of equation $y^2=2px$, of focus F and directrix (d).

Let $M(x_0;y_0)$ be a variable point of (P) and H its orthogonal projection on the axis $y'y$.

The perpendiculars through O and H to the straight line (OM) intersect the straight line (MH) in K and (OF) in I.

1) a- Prove that the slope of the straight line (IH) is $-\frac{y_0}{2p}$.

b- Deduce that the point I is fixed .

2) What is the set of points K as M varies ?

N°6.

Consider the parabolas (P) and (P') of respective equations $y^2=2x+1$ and $y^2=-2x+1$.

1) Determine the points of intersection of (P) and (P').
2) Show that (P) and (P') are symmetric to each other with respect to the axis $y'y$.
3) a- Determine the vertex , the focus and the directrix of (P).
   b- Deduce the elements of (P').
4) Find the equations of the tangents to (P) and (P') at the point of abscissa 0 and of positive ordinate .
5) Trace (P) and (P').
6) Calculate the area of the domain (D) limited by (P) and (P').
7) Calculate the volume of the solid obtained by rotating (D) about

the axis $x'x$.

N°7.
In an oriented plane, consider a fixed straight line (d) and a fixed point A not belonging to (d) and designate by (C) the circle of center A and tangent to (d) at the point H.

1) Determine the set of points of the foci of all the parabolas (P) that pass through the point A and of directrix (d).
2) The plane is referred to an orthonormal system $(H;\vec{i},\vec{j})$ with $\vec{i}=\overrightarrow{HA}$.
   Suppose that the point $F(x;y)$ belongs to (C) and designate by S the vertex of (P).
   a- Express X and Y, the coordinates of S in terms of x and y
   b- Deduce the set of points S as F varies.

N°8.
In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the parabola (P) of equation $y^2=4x$.
Designate by F the focus of (P) and by (d) its directrix.
Let M be a variable point of (P) and (T) the tangent to (P) at M.

1) Find the set of points N orthogonal projection of F on (T) as M varies.
2) Find the set of points N' orthogonal projection of F on the normal to (P) at M.

Part II - The Ellipse:

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.

N°9.
1) Find an equation of the ellipse of focus the point $F(3;0)$ and of directrix the axis $y'y$ and eccentricity $e=\frac{1}{2}$.
2) Find an equation of the ellipse of foci $F'(2;0)$ and $F(6;0)$ knowing that the origin O is one of the vertices of this ellipse.
3) Find an equation of the ellipse of foci $O(0;0)$ and $F(0;6)$

knowing that the straight line $y=-1$ is a directrix of this ellipse.

4) Find an equation of the ellipse of vertices $A(4;1)$ and $A'(-2;1)$ knowing that the distance between the directrices of this ellipse is equal to 8.

N°10.

1) Consider the ellipse (E) of equation $4x^2+9y^2-8x-32=0$.
Determine the center, the focal axis, the vertices, the foci and the directrices of (E) then trace (E).
2) Given the ellipse (E') of equation $9x^2+y^2+18x-2y+1=0$.
   a- Trace (E').
   b- Write an equation of the principal circle (C) of (E').
   c- Calculate the area of the domain limited by (C) and (E').

N°11.

In the complex plane referred to an orthonormal system $(O;\vec{u},\vec{v})$ consider the ellipse (E) of center O, focus the point $F(3;0)$ and vertex the point $A(5;0)$.

1) a- Find an equation of (E) as well as the equation of the directrix associated with the focus F.
   b- Trace (E).
2) Let B be the point of affix $4i$ and M a variable point of (E) of affix z.
   Let G be the point defined by $\overrightarrow{GA}+\overrightarrow{GB}+2\overrightarrow{GM}=\vec{0}$ and let Z be the affix of G.
   a- Determine Z in terms of z.
   b- Show that G traces an ellipse (E') as M traces (E).
   c- Prove that (E') is interior to (E), draw (E') and calculate the area of the domain limited by the two ellipses (E) and (E').

N°12.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the ellipse (E) of equation $9x^2+4y^2=36$.
P is a variable point of (E), designate by M the midpoint of the segment joining P to the vertex of (E) of negative abscissa.

Find the set (γ) of points M as P varies on (E) and write an equation of (γ).

N° 13.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the ellipse (E) of equation $\frac{x^2}{10}+\frac{y^2}{5}=1$ and let (d) be the straight line of equation $3x+2y+7=0$.

1) Trace (E) and (d).
2) Write the equations of the tangents to (E) that are parallel to (d).
3) Let F and F' be the foci of (E) and $M(x;y)$ a variable point of (E).
   a- Prove that $MF^2=\frac{1}{2}\left(x^2-4\sqrt{5}x+20\right)$ and $MF'^2=\frac{1}{2}\left(x^2+4\sqrt{5}x+20\right)$.
   b- Deduce that $MF+MF'=2\sqrt{10}$.

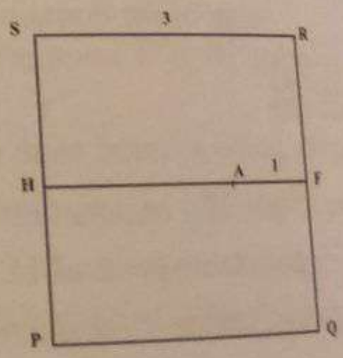
N° 14.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, given the circle (C_1) of center $I(1;0)$ and radius $R_1=8$ and the circle (C_2) of center $J(3;0)$ and radius $R_2=4$.

A variable circle (γ) of center M varies remaining tangent internally to (C_1) and externally to (C_2).

1) Prove that as (γ) varies, the point M traces a conic whose nature is to be determined.
2) Write an equation of the conic obtained.

N° 15.



$PQRS$ is a square of side 3.

H and F are the respective midpoints of $[SP]$ and $[QR]$.

Let $(\mathcal{E})$ be the ellipse of focus F, and of directrix the straight line (PS) and eccentricity $e=\frac{1}{2}$.

A is a point on $[HF]$ such that $AF=1$.

1) Show that R and Q belong to $(\mathcal{E})$

2) Show that A is a vertex of $(\mathcal{E})$

3) The plane is referred to a direct orthonormal system $(H;\vec{i},\vec{j})$ with $\overrightarrow{HF}=3\vec{i}$.

a- Prove that an equation of $(\mathcal{E})$ in the system $(H;\vec{i},\vec{j})$ is $3x^2+4y^2-24x+36=0$.

b- Determine the second vertex A' of $(\mathcal{E})$ lying on the focal axis.

c- Determine the second focus F' of $(\mathcal{E})$

d- Determine the vertices of $(\mathcal{E})$ lying on the non-focal axis

e- Determine an equation of the tangent (T) at R to $(\mathcal{E})$

f- Trace $(\mathcal{E})$ in the system $(H;\vec{i},\vec{j})$.

g- Let $(\mathcal{D})$ be the domain interior to $(\mathcal{E})$.
Calculate the volume of the solid obtained by rotating $(\mathcal{D})$ about the focal axis $(\mathcal{E})$.

Part 3 - The Hyperbola:

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.

N° 16.

1) Find an equation of the hyperbola of focus the point $F(2;0)$, and directix the straight line of equation $x=-1$ and of eccentricity $e=\sqrt{3}$.

2) Find an equation of the hyperbola of foci $F'(1;0)$ and $F(5;0)$ and having $2\sqrt{2}$ as a length of its focal axis .

3) Find an equation of the hyperbola of focal axis $y'y$, of vertices the points $O(0;0)$ and $A(0;-2)$ and of eccentricity $e=2$.

4) Find an equation of the hyperbola of foci $O(0;0)$ and $F(0;4)$ and passing through the point $M(12;9)$.

N° 17.

1) Determine the center, the vertices, the foci, the asymptotes then trace the hyperbola of equation $\frac{(x-1)^2}{9}-\frac{(y+1)^2}{16}=1$

2) Determine the reduced equation then trace the hyperbola of equation : $4y^2-9x^2+8y+18x-41=0$.

N° 18.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the family of curves (C_m) of equation $\frac{x^2}{2m-4}+\frac{y^2}{m+1}=1$

where m is a real parameter such that $m \in IR-\{-1;2\}$.

Study, according to the values of m, the nature of (C_m).

N° 19.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the family of curves (C_m) of equation: $(m-1)x^2+my^2-2x+2y-3=0$

where m is a real parameter.

1) Study, according to the values of m, the nature of (C_m).
2) Suppose that $m=1$ and consider the parabola (P) of equation $y^2-2x+2y-3=0$.
   a- Show that the straight line of equation $y=-1$ is an axis of symmetry of (P).
   b- Let (δ) be a variable straight line passing through the focus of (P) and that cuts (P) in two points M and N.
      i- Prove that the tangents (T_1) and (T_2) at M and N to (P) are perpendicular
      ii- Designate by I the point of intersection of these two tangents. Prove that I belongs to the directrix of (P).

N° 20.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the points $F(8;0)$ and $F'(-2;0)$.

Let (C) be the circle of center F and radius $R=7$ cm and (C') the

circle of center F' and radius $R' = 1$.
(ω) is a variable circle of center M tangent externally at a point P to (C) and at a point Q to (C').

1) Prove that as (ω) varies, the point M traces a branch of a hyperbola (H) of equation $16x^2 - 9y^2 - 96x = 0$.
2) Determine the vertices and the asymptotes of (H) and trace (H).
3) (E) is an ellipse of focal axis the straight line of equation $x = 3$, and having the same vertices as (H) and of focal distance $2c = 2\sqrt{7}$.
   a- Find an equation of (E) and trace (E) in the same system as that of (H).
   b- Deduce the drawing of the curve of equation: $16x^2 - 9y\,|\,y\,| - 96x = 0$.
4) Let h_1 be the negative dilation that transforms the circle (C) to the circle (ω) and let h_2 be the negative dilation that transforms the circle (ω) to the circle (C').
   Determine $h_2 \circ h_1$ and prove that (PQ) passes trough a fixed point.

General Problems :

N°21.

In an oriented plane, consider a fixed straight line (d) and a fixed point A not belonging to (d).
Let (C) be a variable circle of center M passing through A and tangent to (d).
Prove that as (C) varies, the point M traces a conic whose nature and elements are to be determined.

N°22.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.

1) Let M be a point of affix $z = x + iy$.
   a- Determine the set (d) of points M such that $z + \bar{z} + 4 = 0$.
   b- Prove that for all points M of the plane the distance of M to

(d) is equal to $\frac{1}{2}\left|z+\bar{z}+4\right|$.

2) Let F be the point of affix $1+i$ and (P') the plane deprived of the straight line (d).
Let (E) be the set of points M of affix z of (P') such that
$\left|\frac{z-1-i}{z+\bar{z}+4}\right|=\frac{\sqrt{2}}{4}$.
Prove that (E) is a conic whose eccentricity and nature are to be determined.

N°23.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.
Designate by (E) the ellipse of equation $\frac{x^2}{9}+\frac{y^2}{4}=1$, and by S the direct plane similitude of center O, of ratio $\frac{\sqrt{2}}{2}$ and angle $\frac{\pi}{4}$.
Let (E_1) be the image of (E) by S.

1) a- Trace (E).
   b- Let F be one of the foci of (E) and (Δ) the directrix associated with F, designate by M a point of (E) and by H its orthogonal projection on (Δ).
   Designate by (Δ_1), F_1, H_1 and M_1 the images of (Δ), F, H and M by S.
   i- Find the value of $\frac{M_1F_1}{M_1H_1}$.
   ii- Show that (E_1) is an ellipse whose two axes of symmetry are to be determined.
2) a- z and z_1 are the respective affixes of M and of its image M_1 by S, show that $z_1=\frac{1}{2}(1+i)z$ and that if M is distinct from O then triangle OMM_1 is right isosceles at M_1.
   b- Deduce a construction of M_1 starting from a given point M distinct from O.
   c- Construct (E_1).

N°24.
The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$. Let S the direct plane similitude of center $\Omega(-2;3)$, of ratio $\sqrt{2}$ and angle $\frac{\pi}{4}$.
For all points M of affix $z = x + iy$, we associate the point M' of affix $z' = x' + iy'$ such that $M' = S(M)$.

1) a- Express z' in terms of z.
   b- What is the inverse similitude S' of S ?
2) Write the complex form of S'.
3) Let (E') be the ellipse of equation $9x^2 + 16y^2 = 144$ and designate by (E) the image of $(E')$by S'.
   Without finding an equation of (E), precise the nature of (E) and determine its center and vertices .

N°25.
$OABC$ is a square of center I and such that $(\overrightarrow{OA};\overrightarrow{OC}) = \frac{\pi}{2}(\text{mod}\, 2\pi)$.
Let r be the rotation of center Ω such that $r(O) = I$ and $r(I) = C$.

1) Determine the angle of r and construct Ω.
2) Construct the image of the square $OABC$ by r.
3) Let (P) be the parabola of focus A and of directrix (OC).
   Designate by $(P')$the image of (P) by r.
   a- Verify that B is a point of (P).
   b- What does the straight line (OB) represent for(P) ?
   c- Determine the vertex of (P).
   d- Show that the straight line (AC) is tangent to (P') at B'.
   e- Determine the vertex and the focus of (P').

N°26.
The space is referred to an orthonormal system $(O;\vec{i},\vec{j},\vec{k})$.
Consider the plane (P) of equation $4x - 3z - 8 = 0$, the circle(C) of plane $(P)$of center $I(5;3;4)$ and radius $R = 5$ and designate by (d) the line of intersection of the two planes (P) and (xOy).

1) Show that the orthogonal projection (E) of (C) on the plane (xOy) is an ellipse.
2) Calculate the eccentricity of (E) as well as the area of the domain interior to (E).

N°27.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the circle (C) of center $I(6;0)$ and radius $R=2$.
Let M be a variable point of (C) and M' the orthogonal projection of M on $y'y$ and let N be the midpoint of $[MM']$.

1) Show that N describes an ellipse (E) as M describes (C).
2) Trace (E).
3) Let L be a point defined by $\overrightarrow{M'L} = \lambda\,\overrightarrow{M'M}$ where λ is a non-zero real number.
Prove that L describes a conic (Γ_λ) whose nature is to be determined.

N°28.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$ consider the ellipse (E) of equation $\frac{x^2}{25}+\frac{y^2}{9}=1$.

Let F be the focus of (E) of positive abscissa and (d) the directrix associated to F.

1) Trace (E), place F and (d) and calculate the eccentricity e.
2) (MM') is a focal cord passing through F such that $(\vec{i};\overrightarrow{FM}) = \theta \pmod{2\pi}$ where $0<\theta<\pi$, suppose $FM=r$.
   a- Prove that the abscissa of M is $x_M = 4 + r\cos\theta$.
   b- Calculate the distance of M to (d) in terms of r and θ and deduce that $r = \frac{9}{5+4\cos\theta}$.
   c- Prove that $\frac{1}{FM}+\frac{1}{FM'}$ remains constant as θ varies.

d- What is the minimal length of the focal cord ? Construct this cord .

N°29.

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$. Consider the points $A(m\,;0)$ and $B(0\,;n)$ where m and n are two real numbers.

Let P be the point defined by $\overrightarrow{OA} = 2\,\overrightarrow{BP}$.

1) Determine the coordinates x and y of P in terms of m and n.
2) Suppose that m and n vary in such a way that $AB = 2$. Prove that P varies on an ellipse (E) of equation $4x^2 + y^2 = 4$.
3) Let (C) be the curve of equation $5x^2 + 6xy + 5y^2 - 8 = 0$.
   a- Prove that (C) is the image of (E) by the rotation r of center O and angle $\dfrac{\pi}{4}$.
   b- Deduce the nature of (C).
   c- Determine the focal axis and a focus of (C).
   d- Calculate the eccentricity of (C) and its area .

N°30.

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the parabola (P) of equation $y^2 = 4x + 4$.

1) Determine the vertex , the focus and the directrix of (P).
2) Trace (P).
3) (P) cuts the axis $y'y$ in two points A and B. Prove that $OA = OB = 2$.
4) a- Let M be a point of (P) of positive abscissa and H the orthogonal projection of M on $y'y$, show that $MO - MH = 2$.
   b- Deduce that the circle of center M and radius MH remains tangent to the circle of diameter $[AB]$.
5) Let (d) be a straight line passing through F and of slope m. (d) intersects (P) in two points M' and M'', designate by I the midpoint of $[M'M'']$.
   a- Calculate the coordinates of I in terms of m.

b- Deduce that I varies on a parabola (P').

c- (T') and (T'') are the tangents at M' and M'' to (P).
Show by two methods, algebraic and geometric that (T') and (T'') are perpendicular .

N°31

In the plane referred to an orthonormal system $(O;\vec{i},\vec{j})$, consider the fixed circle (C) of center $w(3;0)$ and radius $R=1$.

1) Let (γ) be a variable circle of center M tangent externally to (C) and to the axis $y'y$.
Prove that M describes a conic (Γ) whose focus and directrix are to be determined.
2) Write an equation of (Γ).
3) Trace (Γ).
4) a- Calculate the area of the domain limited by (Γ) and the straight line of equation $x=3$.
b- Deduce the area of the domain limited by (Γ), the axis $y'y$ and the two straight lines of equations $y=4$ and $y=-4$.

N°32

The complex plane is referred to an orthonormal system $(O;\vec{u},\vec{v})$.
Consider the curve (E) of equation $25(x^2+y^2)=(3x-16)^2$.

1) Interpret, geometrically, the equation of (E) and show that (E) is a conic of focus O and directrix the straight line (Δ) of equation $x=\dfrac{16}{3}$.
2) Trace (E).
3) Suppose that $(\vec{i};\overrightarrow{OM})=\theta$ where $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$ and consider a point $M(x;y)$ of (E).
a- Express OM in terms of x.
b- Deduce that $OM=\dfrac{16}{5+3\cos\theta}$.

4) The straight line (OM) cuts (Δ) at a point I and cuts (E) at a point M'.

a- Prove that $\frac{1}{OM}+\frac{1}{OM'}$ is a constant independent of the position of M on (E).

b- Prove that $\frac{1}{OM}-\frac{1}{OM'}=\frac{2}{OI}$ and deduce that $OI=\frac{16\times OM}{16-5\times OM}$.

5) Let w be the center of (E) and F the second focus of (E), the straight line (wM') cuts (E) at a point M''.

a- Prove that $\overrightarrow{FM''}$ and $\overrightarrow{OM}$ are collinear.

b- Calculate the coordinates of the point M'' in terms of θ.

6) Let N be the center of gravity of triangle $MM'M''$.

a- Using a dilation that transforms M onto N, prove that N describes a part of the conic (E').

b- Calculate the area of the domain limited by (E) and (E').

c- Calculate θ in case $y_I=\frac{16\sqrt{3}}{3}$ and calculate the coordinates of N.

7) Suppose that $\theta=\frac{\pi}{3}$.

a- Write the equations of the tangents to (E) at the points M, M' and M''.

b- Prove that the tangents to (E) at the points M and M' intersect at a point situated on the directrix (Δ) of (E).

c- Prove that the tangents to (E) at the points M and M'' intersect at a point situated on the principal circle of (E).