Solved Problems

N°1.

Determine a first order differential equation whose general solution is $y(x)$ in each of the following cases:

1) $y(x)=Ce^{x}+2x-5$. (C is a constant)
2) $y(x)=\ln x+Cx+4$. (C is a constant)
3) $y(x)=Cxe^{-x}+1$. (C is a constant)

N°2.

Determine a second order differential equation whose general solution is $y(x)$ in each of the following cases:

1) $y(x)=Ae^{-x}+Be^{2x}$. (A and B are two constants)
2) $y(x)=C_1e^{-x}+C_2e^{2x}+3x-1$. (C_1 and C_2 are two constants)
3) $y(x)=(C_1x+C_2)e^{2x}+x-2$. (C_1 and C_2 are two constants)
4) $y(x)=C_1e^{x}+C_2x+4$. (C_1 and C_2 are two constants)

N°3.

Solve each of the following differential equations:

1) $y'=\ln x$ $\quad x>0$
2) $y'=xe^{-x}$ $\quad x\in IR$
3) $y'=\dfrac{1}{x\ln x}$ $\quad 0<x<1$
4) $y'=\dfrac{1}{x(x+1)}$ $\quad x>0$

N°4.

Solve each of the following differential equations:

1) $y'+y=2e^{x}$
2) $y'+y=2x+1$
3) $y'+2y=2\cos x$
4) $y''+4y=0$

N°5.

Solve each of the following differential equations:

1) $x^2y'+y=0$
2) $x+yy'=2$
3) $(1+x)y'e^{y}=1$ with $x>-1$
4) $\left(1+x^2\right)y'-2xy=0$

N°6

Solve each of the following differential equations:

1) $y''-4y'+3y=0$ 2) $y''+4y'+5y=0$ 3) $y''-4y'+4y=0$

N°7

Consider the differential equation $(E): y'+2y=2x^2+1$.

Let $y=z+x^2-x+1$.

1) Form a differential equation (F) satisfied by z.
2) Solve (F) and deduce the general solution of (E).

N°8

Consider the differential equation $(E): y'+y=e^{-x}\ln x$.

Suppose that $y=ze^{-x}$. $(x>0)$

1) Form the differential equation (F) satisfied by z.
2) Solve (F) and deduce the general solution of (E).
3) Determine, from the solutions of (E), the one verifying $y(1)=\frac{1}{e}$.

N°9

Given the differential equation $(E): y'-(\tan x)y=\cos x$.

1) Find the particular solution of the equation $y'-(\tan x)y=0$, verifying $y(0)=1$.
2) Suppose $y=\frac{z}{\cos x}$.
   a- Form the differential equation (F) satisfied by z.
   b- Solve (F) and deduce the general solution of (E).

N°10

Consider the differential equation $(E):(x-1)y''-xy'+y=0$, $x\neq 1$.

1) Show that $y'''=y''$.
2) Deduce that $y''=Ce^{x}$.
3) Determine the general solution of (E).

N°11

Consider the differential equation $(E): y''+4y=3\cos x$.

Consider the differential equation (E) … Determine a and b so that $Y=a\cos x+b\sin x$ is solution of (E).

2) a- Solve the equation $y''+4y=0$ and deduce the general solution of (E).

b- Find, among the solutions of (E), the one that satisfies the following conditions $y(0)=0$ and $y'\left(\frac{\pi}{2}\right)=0$.

N°12

Consider the differential equation $(E): xy'+y+\frac{1}{x}=0$ with $x>0$ and let $z=xy$.

1) Form the differential equation (F) satisfied by z.

2) Solve (F) and deduce the general solution of (E).

N°13

Part A.

Consider the differential equation $(E): y''+2y'+y=-2e^{-x}+1$ and let $z=y+x^2e^{-x}-1$.

1) Determiner a differential equation (F) satisfied by z.

2) Solve the equation (F) and deduce the general solution of (E).

3) Let f be the function defined over IR by $f(x)=(-x^2+ax+b)e^{-x}+1$ with $f'(0)=0$.

a- Show that $a=b$.

b- Suppose that $a\neq-2$, show that the function $f(x)=(-x^2+ax+a)e^{-x}+1$ admits two extrema one whose abscissa is 0 and the other M whose abscissa is $a+2$.

c- Determine the set of points M as a varies.

d- determine a and b such that $f'(0)=0$ and $f(0)=1$.

Part B.

Consider the function f defined over IR by $f(x)=-x^2e^{-x}+1$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$ and deduce an asymptote to (C).

2) a- Calculate $f'(x)$ and set up the table of variations of f.

b- Deduce that the equation $x^2 = e^x$ has a unique solution α such that $-0.8 < \alpha < -0.7$.

3) Draw (C).

N°14

Consider the differential equation $(E): y' - y = 2xe^x$.

1) Let $y = ze^x$.

a- Form the differential equation (F) satisfied by z.

b- Solve (F) and deduce the particular solution f of (E) verifying $y(0) = 1$.

2) Let g be the function defined over IR by $g(x) = (x^2+1)e^x$.
Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

a- Determine $\lim\limits_{x \to +\infty} g(x)$ and $\lim\limits_{x \to -\infty} g(x)$.

b- Deduce an asymptote to (C).

c- Study the variations of g and set up its table of variations.

d- Show that g has two inflection points.

e- Draw (C).

3) a- Calculate the real numbers a, b and c so that $G(x) = (ax^2 + bx + c)e^x$ is an antiderivative of $g(x)$.

b- Calculate the area of the region limited by (C), the axis $x'x$, the axis $y'y$ and the straight line of equation $x = 1$.

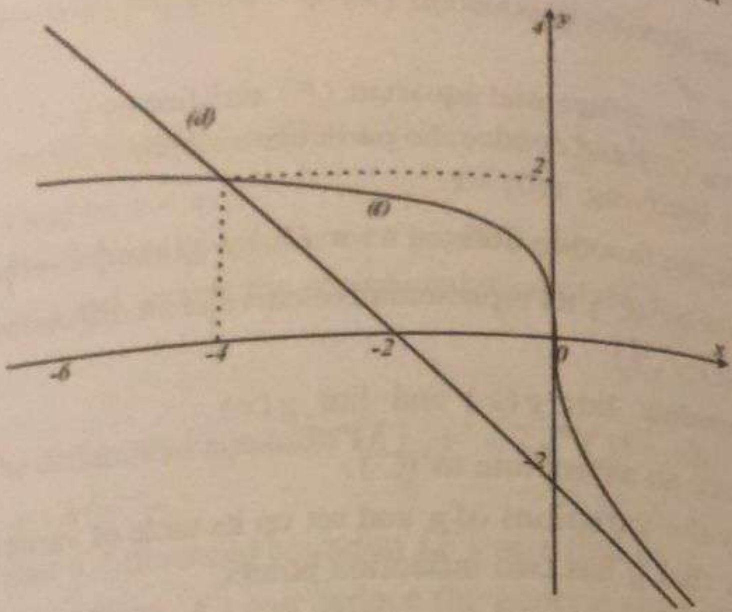
4) a- Show that g has an inverse function g^{-1} over IR and draw its representative curve (C').

b- Is g^{-1} differentiable at the point of abscissa $x = 2e^{-1}$?

N° 15

The plane is referred to an orthonormal system $(O;\vec{i},\vec{j})$.

A- The curve (ℓ) below represents a function h defined over IR.



1) Prove that h admits, over IR, an inverse function g.

2) (γ) is the representative curve of g.
   a- Determine the tangent to (γ) at the point O and deduce $g'(0)$.
   b- Prove that (d) is an asymptote to (γ) and determine the point of intersection of (γ) and (d).
   c- Draw (γ) in another system.

3) Let g be the function defined, over IR, by
   $g(x)=(ax+b)(1+e^{x})+c$, with a, b and c being real numbers.
   a- Calculate $g'(x)$.
   b- Using the values of $g(0)$, $g'(0)$ and $g(2)$, calculate a, b and c and verify that $g(x)=(2-x)e^{x}-x-2$.

B- Consider the differential equation $(E):(1+e^{x})y'-y=0$.

1) Noting that $\dfrac{1}{1+e^{x}}=\dfrac{e^{-x}}{1+e^{-x}}$, calculate $\displaystyle\int\frac{dx}{1+e^{x}}$.

2) Solve the differential equation (E) and deduce the particular

solution of (E) whose representative curve passes through the point $I(0;2)$.

N° 16

Part A.

Consider the differential equation (E) : $y'' + 2y' + y = x + 2$.

Let $z = y - x$.

1) Write a differential equation (E') satisfied by z.
2) Solve (E') and deduce the general solution of (E).
3) Determine the particular solution f of (E) verifying $f(0) = 1$ and $f'(0) = 1$.

Part B.

Consider the function f defined over IR by $f(x) = (x+1)e^{-x} + x$

Designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) a- Calculate $\lim\limits_{x \to +\infty} f(x)$ and show that the straight line (d) of equation $y = x$ is an asymptote to (C).
   b- Study the position of (C) with respect to (d).
   c- Calculate $\lim\limits_{x \to -\infty} f(x)$.
2) a- Study the variations of f' and deduce that the function f is strictly increasing over IR.
   b- Set up the table of variations of f.
3) Determine the point of (C) where the tangent (T) is parallel to (d).
4) Calculate $f(-2)$ and draw (d), (T) and (C).
5) a- Show that f admits an inverse function f^{-1} over IR.
   b- Draw the representative curve of f^{-1}.
6) Let (δ) be the straight line of equation $y = x + m$ where m is a real parameter. Study according to the values of m the number of points of intersection of (C) and (δ).
7) Calculate the area of the region limited by (C), (d) and the straight lines of equations $x = 0$ and $x = 1$.

N° 17

Part A

Consider the differential equation $(E): y' - 3y = \frac{-3e}{(1+e^{-3x})^2}$,

and suppose $z = ye^{3x}$.

1) Show that $z' - 3z = y'e^{3x}$.
2) Knowing that z is a solution of (E) and that $z(0) = \frac{e}{2}$.

   Determine the particular solution of (E).

Part B

f is the function defined over IR by $f(x) = \frac{e^{1-3x}}{1+e^{-3x}}$.

Denote by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to+\infty} f(x)$ and deduce the equations of the asymptotes to (C).
2) Study the variations of f and draw its table of variations.
3) Calculate $f(-x)+f(x)$ and deduce the center of symmetry of (C).
4) a- Calculate the area A_α of the region limited by (C), $x'x$ and the straight lines of equations $x = 0$ and $x = \alpha$ $(\alpha > 0)$.

   b- Calculate $\lim_{\alpha\to+\infty} A_\alpha$.
5) a- Show that f admits an inverse function f^{-1} over IR.

   b- Determine an equation of the tangent (T) to (C'), the representative curve of f^{-1}, at the point A of abscissa $\frac{1}{2}e$.

   c- Let g be the function defined over $]0;e[$ by $g(x) = \frac{1}{3}\ln\left(\frac{e-x}{x}\right)$, calculate $g \circ f(x)$ and deduce the expression of $f^{-1}(x)$.

N°18

Part A.

Consider the differential equation $(E): y'' - 2y' + y = -x + 1$.

Suppose $y = z - x - 1$.

1) Form the differential equation (F) satisfied by z.
2) Solve (F) and deduce the solution f of (E) verifying $f(0) = -1$ and $f'(0) = 0$.

Part B.

f is a function defined over IR by $f(x) = xe^x - x - 1$.

Denote by (C) its representative curve in an orthonormal system $(O; \vec{i}, \vec{j})$.

1) Determine $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$.
2) Show that the straight line (d) of equation $y = -x - 1$ is an asymptote to (C) as $x \to -\infty$.
3) Study the relative positions of (C) and (d).
4) The table below is the table of variations of the function f' the derivative of f.

x	$-\infty$		-2		0		$+\infty$
$f''(x)$		$-$	0		$+$		
$f'(x)$	-1	$\searrow$	$-1-e^{-2}$	$\nearrow$	0	$\nearrow$	$+\infty$

a- Draw the table of variations of f.

b- Show that (C) admits a point of inflection I.

c- Trace (C) and (d).

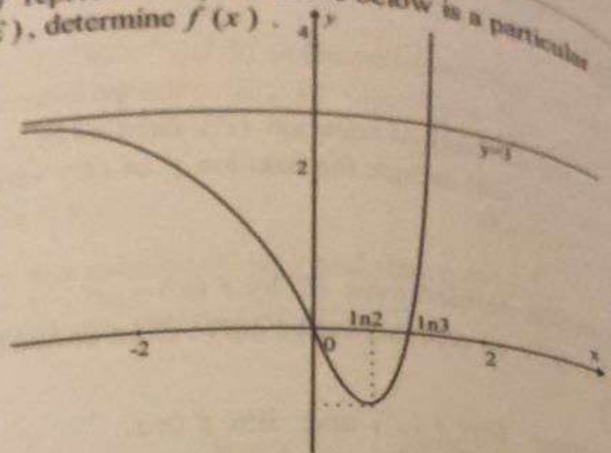
d- Calculate the area of the region bounded by (C), (d) and the two straight lines of equations $x = -1$ and $x = 0$.

N°19

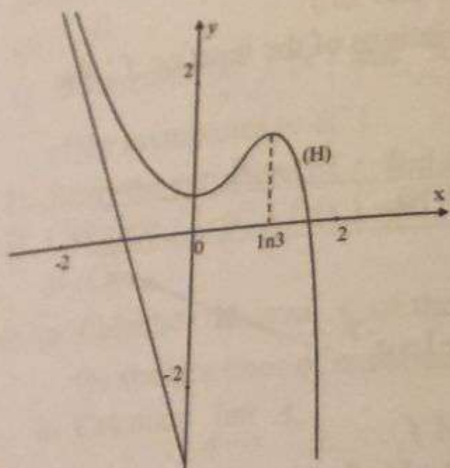
Given the differential equation $(E): y'' - 3y' + 2y = 2k$ where k is a real number and suppose that $y = z + k$.

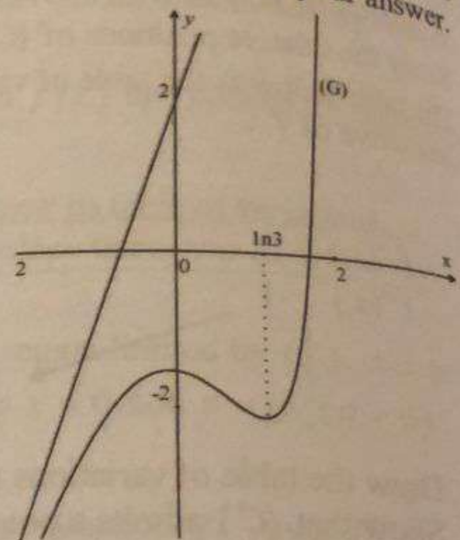
1) Form the differential equation (F) satisfied by z.
2) Solve (F) and deduce the general solution of (E).

3) The function f represented by the curve below is a particular solution of (E), determine $f(x)$.



2) One of the two curves given below is the representative curve of the primitive of f. Indicate which one and justify your answer.





N°20. *For the students of the GS section*

Part A.

Consider the differential equation $(E): y' + 2y^2e^x - y = 0$ where y is a function defined over IR such that, for all real numbers x, $y(x) \neq 0$.

Let $z = \frac{1}{y}$ and $u = z - e^x$ where z is a differentiable function defined over IR

1) Determine the differential equation (E') satisfied by z.

2) Solve (E') and deduce the general solution of (E).

3) Determine the particular solution of (E) verifying $y(0)=\frac{1}{2}$.

Part B.

Consider the function f defined over IR by $f(x)=\frac{1}{e^x+e^{-x}}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$. Graphical unit : 4 cm .

1) Show that f is an even function.
2) Calculate the limits of f at the boundaries of its domain of définition.
3) Calculate $f'(x)$ and set up the table of variations of f.
4) a- Set up the table of variations of the function g defined over $[0;+\infty[$ by $g(x)=f(x)-x$.
   b- Deduce that the equation $f(x)=x$ admits over $[0;+\infty[$ a unique solution α . Verify that $0.4<\alpha<0.5$.
   c- Draw (C).
5) a- Show that the restriction of f over $[0;+\infty[$ admits an inverse function f^{-1}.
   b- Determine the domain of definition of f^{-1} and find the expression of $f^{-1}(x)$ in terms of x .
   c- Draw the curve (γ) of f^{-1} in the same system as that of (C).

Part C.

Consider the function h defined by $h(x)=\ln[f(x)]$.

1) Justify that the domain of definition of h is IR .
2) a- Verify that $h(x)+x=-\ln(1+e^{-2x})$ and deduce that the straight line (d) of equation $y=-x$ is an asymptote to the representative curve (H) of the function h in the neighborhood of $+\infty$.
   b- Show that h is an even function and deduce an asymptote (d') to (H) in the neighborhood of $-\infty$.
   c- Study the variations of h and draw (H).

Part D.

Consider the sequence (v_n) defined over IN by $v_n = \int_0^n f(x)dx$.

1) a- Show that, for all $x \geq 0$, $f(x) < e^{-x}$.
   b- Deduce that, for all natural numbers n, $v_n \leq 1 - e^{-n}$.
2) a- Verify that $v_{n+1} - v_n = \int_n^{n+1} f(x)dx$.
   b- Deduce that the sequence (v_n) is strictly increasing.
   c- Show that the sequence (v_n) converges towards a limit ℓ such that $0 \leq \ell < 1$.
3) Verify that $f(x) = \dfrac{e^x}{1+e^{2x}}$. Calculate, then v_n in terms of n and determine ℓ.
4) Calculate, in cm^2, the area of the region bounded by (γ), $y'y$, $x'x$ and the straight line of equation $y = 2$.

For the students of the GS section

N°21.

Part A.

Consider the differential equation $(E): x^2y' - xy + y^2 = 0$ with $x > 0$.

Let $z = \dfrac{x}{y}$.

1) Determine a differential equation (F) satisfied by z.
2) Solve the equation (F) and deduce the general solution of (E).
3) Determine the particular solution of (E) verifying $y(1) = 1$.

Part B.

Consider the function f defined over $\left]0;\dfrac{1}{e}\right[\cup \left]\dfrac{1}{e};+\infty\right[$ by

$f(x) = \dfrac{x}{1+\ln x}$ and designate by (C) its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$; (Graphical unit : 2 cm).

1) a- Calculate the limits of $f(x)$ at the boundaries of its domain of definition.

b- Calculate $\lim_{x \to +\infty} \frac{f(x)}{x}$ and interpret the result graphically.

2) Calculate $f'(x)$ and set up the table of variations of f.

3) a- Prove that the curve (C) admits an inflection point I.

b- Write an equation of the tangent (d) to (C) at the point I.

4) Study, according to the values of x, the position of (C) and the straight line (D) of equation $y = x$.

5) Draw (d), (D) and (C).

Part C.

Consider the interval $I = [1; e]$.

1) a- Prove that $f(I)$ is included in I.

b- Study the sign of $f'(x) - \frac{1}{4}$ and deduce that, for all x of I, $0 \leq f'(x) \leq \frac{1}{4}$.

c- Prove that, for all x of I that $|f(x) - 1| \leq \frac{1}{4}|x - 1|$

2) Let (u_n) be the sequence defined by :

$u_0 = 2$ and for all $n \geq 0$, $u_{n+1} = f(u_n)$.

a- Prove, by mathematical induction over n, that u_n belongs to I.

b- Prove that $|u_{n+1} - 1| \leq \frac{1}{4}|u_n - 1|$.

c- Prove that $|u_n - 1| \leq \frac{1}{4^n}$ and deduce $\lim_{n \to +\infty} u_n$.

Part D.

Consider the function g defined over $]0; +\infty[$ by $g(x) = \frac{1}{f(x)}$.

1) Study the variations of g and set up its table of variations.

2) Draw the curve (G) representative of g in an orthonormal system $(O; \vec{i}, \vec{j})$.

3) Let (v_n) be the sequence defined by $v_n = \int_{e^n}^{e^{n+1}} g(x)dx$.

a- Calculate v_n in terms of n.

b- Show that the sequence (v_n) is an arithmetic sequence whose first term and common difference are to be determined.

c- Deduce the area a_n of the region bounded by (G), the axis $x'x$ and the straight lines of equations $x = 1$ and $x = e^n$.

For the students of the GS section.

N°22

Part A.

Consider the differential equation (E): $yy' - 2xy' - 2y = 0$.

Let $y = z + 2x$

1) Form the differential equation (E') satisfied by z.
2) Solve (E') and deduce the general solution of (E).
3) Find the particular solution of (E) whose representative curve, In an orthonormal system $(O;\vec{i},\vec{j})$, passes through the point $(0;1)$.

Part B.

Consider the function f defined over IR by $f(x) = 2x + \sqrt{4x^2 + 1}$.

(C) is its representative curve in an orthonormal system $(O;\vec{i},\vec{j})$.

1) Determine $\lim\limits_{x \to +\infty} f(x)$ and show that the straight line (d) of equation $y = 4x$ is an asymptote to (C) in the neighborhood of $+\infty$.
2) Calculate $\lim\limits_{x \to -\infty} f(x)$.
3) Prove that f is strictly increasing over IR.
4) Draw (C).
5) a- Show that f admits an inverse function f^{-1}.

   b- Determine $f^{-1}(x)$.

   c- Calculate $\left(f^{-1}\right)'(1)$ in two different ways.

   d- Draw the curve (C') of f^{-1} in the same system.