NETWORKS 2021

MANCHESTER 1824
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Goodness of Fit Tests for Random Multigraph Models

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multigraph representation of network data

graphs where multiple edges and self-edges are permitted

- can appear directly in applications (although scarce)
- can be constructed by different kinds of aggregations in graphs
  - aggregation based on vertex attributes
  - aggregation based on edge attributes

composition

structure

multigraph representation of network data

- multigraphs represented by their edge multiplicity sequence

$$\mathbf{m} = (m_{ij} : (i,j) \in R)$$

where R is the canonical site space for undirected edges

$$R = \{(i,j) : 1 \leq i \leq j \leq n\}$$

$$(1,1) < (1,2) < \ldots < (1,n) < (2,2) < (2,3) < \ldots < (n,n)$$

- the number of vertex pair sites is given by $r = \binom{n+1}{2}$
- edge multiplicities as entries in a matrix

$$\mathbf{m} = \begin{bmatrix} m_{11} & m_{12} & \ldots & m_{1n} \\ 0 & m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & m_{nn} \end{bmatrix} \qquad \mathbf{m}+\mathbf{m}' = \begin{bmatrix} 2m_{11} & m_{12} & \ldots & m_{1n} \\ m_{12} & 2m_{22} & \ldots & m_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ m_{1n} & m_{2n} & \ldots & 2m_{nn} \end{bmatrix}$$

Shafie, T. (2015). A Multigraph Approach to Social Network Analysis. *Journal of Social Structure*, 16.

random multigraph models

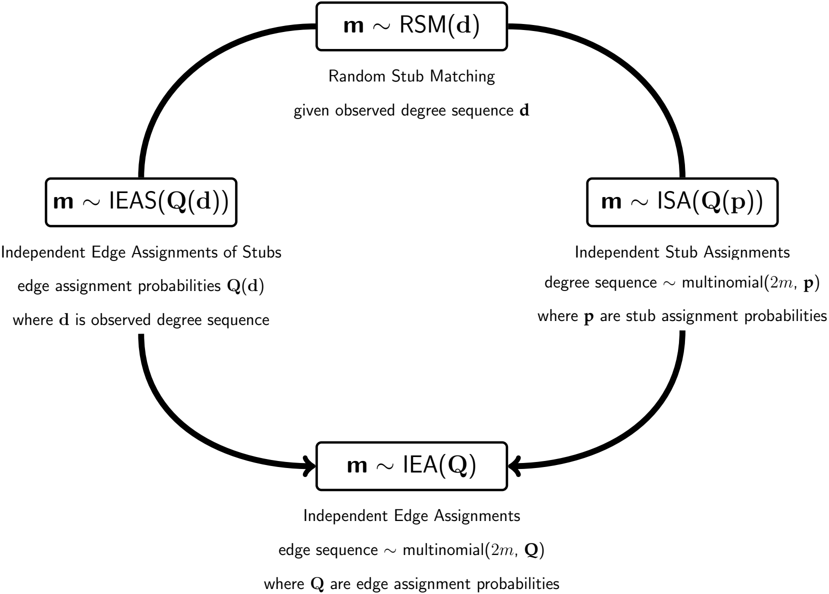
1. **random stub matching (RSM)**
   - edges are assigned to sites given fixed degree sequence $\mathbf{d} = (d_1, \ldots, d_n)$
   - probability that an edge is assigned to site $(i, j) \in R$

$$Q_{ij} = \begin{cases} \binom{d_i}{2} \Big/ \binom{2m}{2} & \text{for } i = j \\ d_i d_j \Big/ \binom{2m}{2} & \text{for } i < j \end{cases}$$

2. **independent edge assignment (IEA)**
   - edges are independently assigned to vertex pairs in site space R
   - edge assignment probabilities $\mathbf{Q} = (Q_{ij} : (i, j) \in R)$
   - $\mathbf{m}$ is multinomial distributed with parameters m and $\mathbf{Q}$
   - statistics for analysing local and global structure are easily derived
   - two variants:
     - **independent edge assignment of stubs (IEAS)**
     - **independent stub assignment (ISA)**

Shafie, T. (2016). Analyzing local and global properties of multigraphs. *The Journal of Mathematical Sociology*, 40(4), 239-264.

random multigraph models



$\mathbf{m} \sim \mathsf{RSM}(\mathbf{d})$

Random Stub Matching

given observed degree sequence $\mathbf{d}$

$\mathbf{m} \sim \mathsf{IEAS}(\mathbf{Q}(\mathbf{d}))$

Independent Edge Assignments of Stubs

edge assignment probabilities $\mathbf{Q}(\mathbf{d})$

where $\mathbf{d}$ is observed degree sequence

$\mathbf{m} \sim \mathsf{ISA}(\mathbf{Q}(\mathbf{p}))$

Independent Stub Assignments

degree sequence $\sim$ multinomial($2m$, $\mathbf{p}$)

where $\mathbf{p}$ are stub assignment probabilities

$\mathbf{m} \sim \mathsf{IEA}(\mathbf{Q})$

Independent Edge Assignments

edge sequence $\sim$ multinomial($2m$, $\mathbf{Q}$)

where $\mathbf{Q}$ are edge assignment probabilities

goodness of fit

gof measures between observed and expected edge multiplicity sequence under simple or composite hypothesis

- test statistics: S of Pearson and A of information divergence type
- expected values of the Pearson statistic are derived
- exact distributions of the test statistics are numerically investigated

answers sought to the following:

- are significance levels of test statistics for small number of edges far from those of the asymptotic distribution?
- is the convergence of the cdf's of test statistics slow or rapid?
- does the convergence speed depend on specific parameters in models?
- can better approximations to the actual distributions be obtained using adjustments of the χ^2-distributions?
- can power approximations be made for small number of edges?
- how does RSM influence the distributions of statistics?
- how can RSM be tested?

tests of a simple multigraph hypothesis

edge multiplicities according to $\mathrm{IEA}(\mathbf{Q})$ and correct model $\mathbf{Q_0} = \mathbf{Q}$ tested:

- the Pearson statistic

$$S_0 = \sum\sum_{i \leq j} \frac{(m_{ij} - mQ_{0ij})^2}{mQ_{0ij}} = \sum\sum_{i \leq j} \frac{m_{ij}^2}{mQ_{0ij}} - m \overset{asymp}{\sim} \chi^2(r-1)$$

- the divergence statistic

$$D_0 = \sum\sum_{i \leq j} \frac{m_{ij}}{m} \log \frac{m_{ij}}{mQ_{0ij}} \quad \text{and} \quad A_0 = \frac{2m}{\log \mathrm{e}} D_0 \overset{asymp}{\sim} \chi^2(r-1)$$

tests of a composite multigraph hypothesis

the composite multigraph hypothesis

- ISA for unknown **p**
- IEAS for unknown **d**

parameters have to be estimated from data **m**

when correct model is tested:

- the Pearson statistic

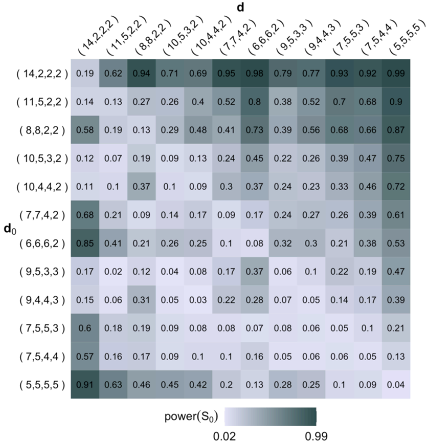
$$\hat{S} = \sum_{i \le j}\sum \frac{(m_{ij} - m\hat{Q}_{ij})^2}{m\hat{Q}_{ij}} = \sum_{i \le j}\sum \frac{m_{ij}^2}{m\hat{Q}_{ij}} - m \overset{asymp}{\sim} \chi^2(r-n)$$

- the divergence statistic

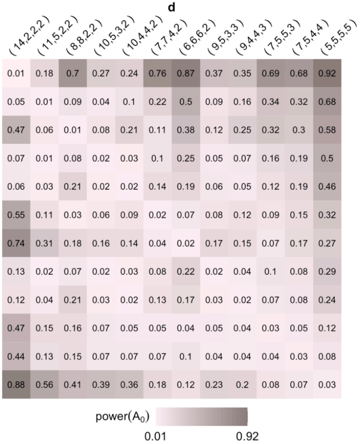
$$\hat{D} = \sum_{i \le j}\sum \frac{m_{ij}}{m} \log \frac{m_{ij}}{m\hat{Q}_{ij}} \qquad \text{and} \quad \hat{A} = \frac{2m}{\log \mathrm{e}}\hat{D} \overset{asymp}{\sim} \chi^2(r-n)$$

goodness of fit: test illustrations

power when simple IEAS($\mathbf{d}_0$) hypotheses are tested against IEAS($\mathbf{d}$) models for multigraphs with $n = 4$, $m = 10$ and $\alpha(\chi^2_9) = 0.04$



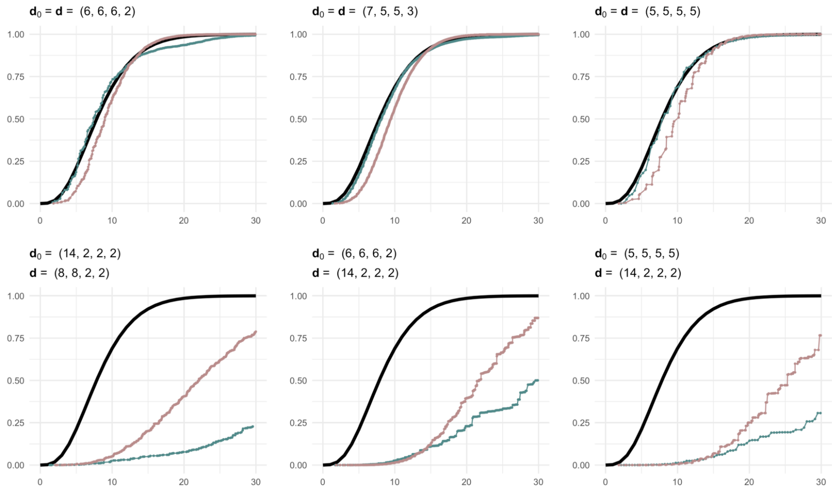
	(14,2,2,2)	(11,5,2,2)	(8,8,2,2)	(10,5,3,2)	(10,4,4,2)	(7,7,4,2)	(6,6,6,2)	(9,5,3,3)	(9,4,4,3)	(7,5,5,3)	(7,5,4,4)	(5,5,5,5)
(14,2,2,2)	0.19	0.62	0.94	0.71	0.69	0.95	0.98	0.79	0.77	0.93	0.92	0.99
(11,5,2,2)	0.14	0.13	0.27	0.26	0.4	0.52	0.8	0.38	0.52	0.7	0.68	0.9
(8,8,2,2)	0.58	0.19	0.13	0.29	0.48	0.41	0.73	0.39	0.56	0.68	0.66	0.87
(10,5,3,2)	0.12	0.07	0.19	0.09	0.13	0.24	0.45	0.22	0.26	0.39	0.47	0.75
(10,4,4,2)	0.11	0.1	0.37	0.1	0.09	0.3	0.37	0.24	0.23	0.33	0.46	0.72
(7,7,4,2)	0.68	0.21	0.09	0.14	0.17	0.09	0.17	0.24	0.27	0.26	0.39	0.61
(6,6,6,2)	0.85	0.41	0.21	0.26	0.25	0.1	0.08	0.32	0.3	0.21	0.38	0.53
(9,5,3,3)	0.17	0.02	0.12	0.04	0.08	0.17	0.37	0.06	0.1	0.22	0.19	0.47
(9,4,4,3)	0.15	0.06	0.31	0.05	0.03	0.22	0.28	0.07	0.05	0.14	0.17	0.39
(7,5,5,3)	0.6	0.18	0.19	0.09	0.08	0.08	0.07	0.08	0.06	0.05	0.1	0.21
(7,5,4,4)	0.57	0.16	0.17	0.09	0.1	0.1	0.16	0.05	0.06	0.06	0.05	0.13
(5,5,5,5)	0.91	0.63	0.46	0.45	0.42	0.2	0.13	0.28	0.25	0.1	0.09	0.04



	(14,2,2,2)	(11,5,2,2)	(8,8,2,2)	(10,5,3,2)	(10,4,4,2)	(7,7,4,2)	(6,6,6,2)	(9,5,3,3)	(9,4,4,3)	(7,5,5,3)	(7,5,4,4)	(5,5,5,5)
(14,2,2,2)	0.01	0.18	0.7	0.27	0.24	0.76	0.87	0.37	0.35	0.69	0.68	0.92
(11,5,2,2)	0.05	0.01	0.09	0.04	0.1	0.22	0.5	0.09	0.16	0.34	0.32	0.68
(8,8,2,2)	0.47	0.06	0.01	0.08	0.21	0.11	0.38	0.12	0.25	0.32	0.3	0.58
(10,5,3,2)	0.07	0.01	0.08	0.02	0.03	0.1	0.25	0.05	0.07	0.16	0.19	0.5
(10,4,4,2)	0.06	0.03	0.21	0.02	0.02	0.14	0.19	0.06	0.05	0.12	0.19	0.46
(7,7,4,2)	0.55	0.11	0.03	0.06	0.09	0.02	0.07	0.08	0.12	0.09	0.15	0.32
(6,6,6,2)	0.74	0.31	0.18	0.16	0.14	0.04	0.02	0.17	0.15	0.07	0.17	0.27
(9,5,3,3)	0.13	0.02	0.07	0.02	0.03	0.08	0.22	0.02	0.04	0.1	0.08	0.29
(9,4,4,3)	0.12	0.04	0.21	0.03	0.02	0.13	0.17	0.03	0.02	0.07	0.08	0.24
(7,5,5,3)	0.47	0.15	0.16	0.07	0.05	0.05	0.04	0.05	0.04	0.03	0.05	0.12
(7,5,4,4)	0.44	0.13	0.15	0.07	0.07	0.07	0.1	0.04	0.04	0.04	0.03	0.08
(5,5,5,5)	0.88	0.56	0.41	0.39	0.36	0.18	0.12	0.23	0.2	0.08	0.07	0.03

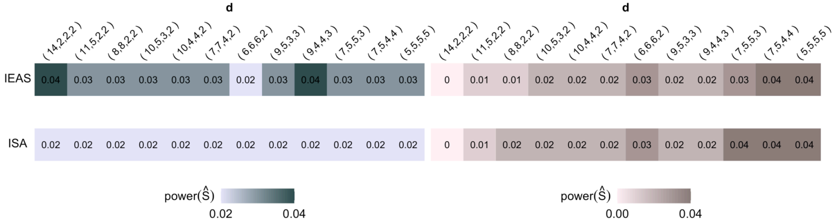
goodness of fit: test illustrations

null and non-null distributions of S_0 and A_0, and the χ^2_9-distribution when simple IEAS($\mathbf{d}_0$) hypotheses are tested against IEAS($\mathbf{d}$)



goodness of fit: test illustrations

probabilities of false rejection (top) and power (bottom) when composite IEAS and ISA hypotheses are tested against IEAS(**d**) models for multigraphs with $n = 4$, $m = 10$ and $\alpha(\chi^2_6) = 0.04$

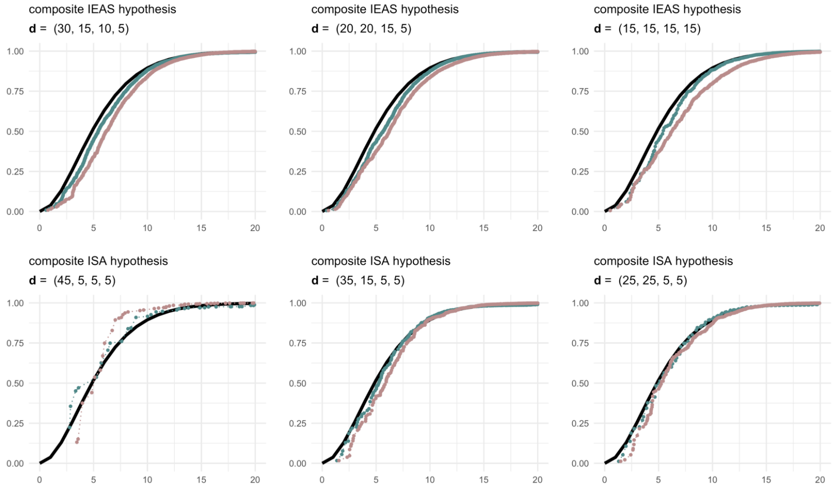


	d											
	(14,2,2,2)	(11,5,2,2)	(8,8,2,2)	(10,5,3,2)	(10,4,4,2)	(7,7,4,2)	(6,6,6,2)	(9,5,3,3)	(9,4,4,3)	(7,5,5,3)	(7,5,4,4)	(5,5,5,5)
IEAS	0.04	0.03	0.03	0.03	0.03	0.03	0.02	0.03	0.04	0.03	0.03	0.03
ISA	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02

	d											
	(14,2,2,2)	(11,5,2,2)	(8,8,2,2)	(10,5,3,2)	(10,4,4,2)	(7,7,4,2)	(6,6,6,2)	(9,5,3,3)	(9,4,4,3)	(7,5,5,3)	(7,5,4,4)	(5,5,5,5)
IEAS	0	0.01	0.01	0.02	0.02	0.02	0.03	0.02	0.02	0.03	0.04	0.04
ISA	0	0.01	0.02	0.02	0.02	0.02	0.03	0.02	0.02	0.04	0.04	0.04

goodness of fit: test illustrations

non-null distributions of $\hat{S}$, $\hat{A}$, and the χ^2_6-distribution when composite IEAS and ISA hypotheses tested against RSM($\mathbf{d}$) models


composite IEAS hypothesis

d = (30, 15, 10, 5)

composite IEAS hypothesis

d = (20, 20, 15, 5)

composite IEAS hypothesis

d = (15, 15, 15, 15)

composite ISA hypothesis

d = (45, 5, 5, 5)

composite ISA hypothesis

d = (35, 15, 5, 5)

composite ISA hypothesis

d = (25, 25, 5, 5)

summary of error probabilities

		simple IEAS($\mathbf{d}_0$) hypothesis			composite hypothesis	
model		$\mathbf{d}_0 = \mathbf{d}$	Flat $\mathbf{d}_0 \neq \mathbf{d}$	Skew $\mathbf{d}_0 \neq \mathbf{d}$	IEAS	ISA
IEAS	Flat $\mathbf{d}$	$\alpha_{S_0} > \alpha_{A_0}$	$\beta_{S_0} < \beta_{A_0}$	$\beta_{S_0} < \beta_{A_0}$	$\alpha_{\hat{S}} \leq \alpha_{\hat{A}}$	$\beta_{\hat{S}} > \beta_{\hat{A}}$
	Skew $\mathbf{d}$	$\alpha_{S_0} > \alpha_{A_0}$	$\beta_{S_0} < \beta_{A_0}$	$\beta_{S_0} < \beta_{A_0}$	$\alpha_{\hat{S}} > \alpha_{\hat{A}}$	$\beta_{\hat{S}} \geq \beta_{\hat{A}}$
		simple ISA($\mathbf{d}_0/2m$) hypothesis			composite hypothesis	
		$\mathbf{d}_0 = \mathbf{d}$	Flat $\mathbf{d}_0 \neq \mathbf{d}$	Skew $\mathbf{d}_0 \neq \mathbf{d}$	IEAS	ISA
ISA	Flat $\mathbf{d}$	$\alpha_{S_0} \geq \alpha_{A_0}$	$\beta_{S_0} \leq \beta_{A_0}$	$\beta_{S_0} < \beta_{A_0}$	inconclusive	$\alpha_{\hat{S}} \leq \alpha_{\hat{A}}$
	Skew $\mathbf{d}$	$\alpha_{S_0} > \alpha_{A_0}$	$\beta_{S_0} \leq \beta_{A_0}$	$\beta_{S_0} < \beta_{A_0}$	$\beta_{\hat{S}} < \beta_{\hat{A}}$	$\alpha_{\hat{S}} > \alpha_{\hat{A}}$
		simple IEAS($\mathbf{d}_0$) or ISA($\mathbf{d}_0/2m$) hypothesis			composite hypothesis	
		$\mathbf{d}_0 = \mathbf{d}$	Flat $\mathbf{d}_0 \neq \mathbf{d}$	Skew $\mathbf{d}_0 \neq \mathbf{d}$	IEAS	ISA
RSM	Flat $\mathbf{d}$	$\beta_{S_0} \geq \beta_{A_0}$	inconclusive	$\beta_{S_0} = \beta_{A_0}$	$\beta_{\hat{S}} > \beta_{\hat{A}}$	$\beta_{\hat{S}} > \beta_{\hat{A}}$
	Skew $\mathbf{d}$	$\beta_{S_0} \leq \beta_{A_0}$	$\beta_{S_0} = \beta_{A_0}$	$\beta_{S_0} < \beta_{A_0}$	$\beta_{\hat{S}} \geq \beta_{\hat{A}}$	$\beta_{\hat{S}} > \beta_{\hat{A}}$

conclusions

- even for very small m, the null distributions of the test statistics under IEA are fairly well approximated by their asymptotic distributions
- the convergence of the cdf's of test statistics are rapid and depend on the parameters in models
- approximations to the actual distributions can be obtained using adjustments of the χ^2-distributions yielding better power
- the influence of RSM on both test statistics is substantial for small m, implying a shift of their distributions towards smaller values compared to what holds true for the null distributions under IEA

the tests in R

https://github.com/termehs/multigraphr

```
# install.packages("devtools")
devtools::install_github("termehs/multigraphr")
```