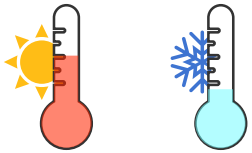
Classification I

Lecture 4

Termeh Shafie

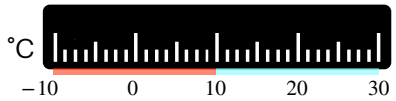
Regression vs. Classification



- What is the temperature going to be tomorrow?
- Will it be hot or cold tomorrow?

continuous variable
(from $-\infty$ to ∞)

categorical variable
(0 to 1)



Generative vs. Discriminative Classification Methods

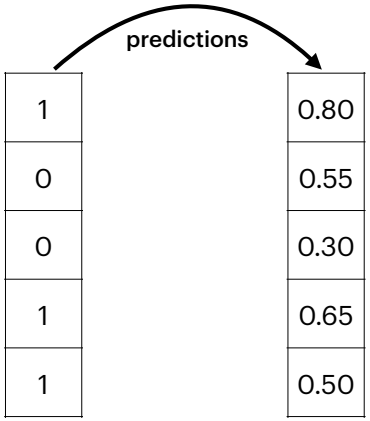
Generative model how data is produced	**Discriminative** model how to separate classes
• Learns a "recipe" for each class • Calculate probability based on recipe of a new point belonging to each class • Example: Naive Bayes, LDA, QDA	• Focus directly on distinguishing classes and not the data generating process • Draw the best possible boundary to separate the classes based on the data • Example: Logistic regression, KNN*, SVM, Tree based methods

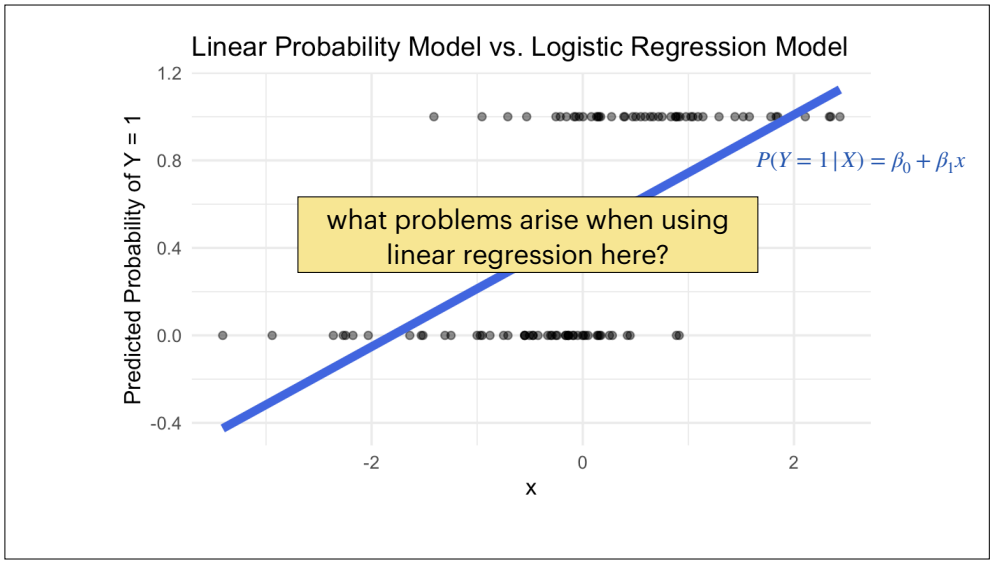
*non-probabilistic but boundary-based

Model $P(Y)$ and $P(X \mid Y)$, derive $P(Y \mid X)$

Model $P(Y \mid X)$ directly

Linear Probability Model





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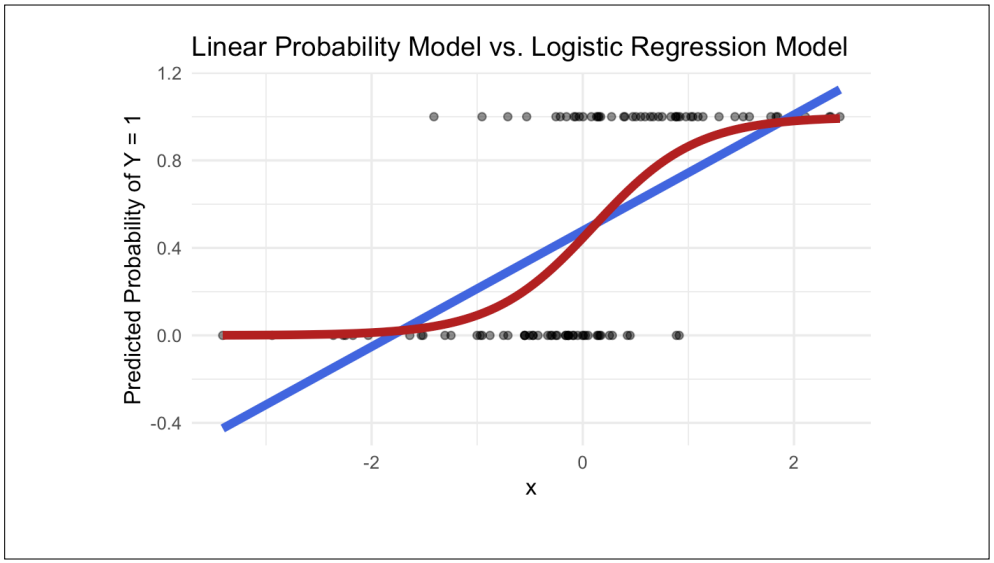

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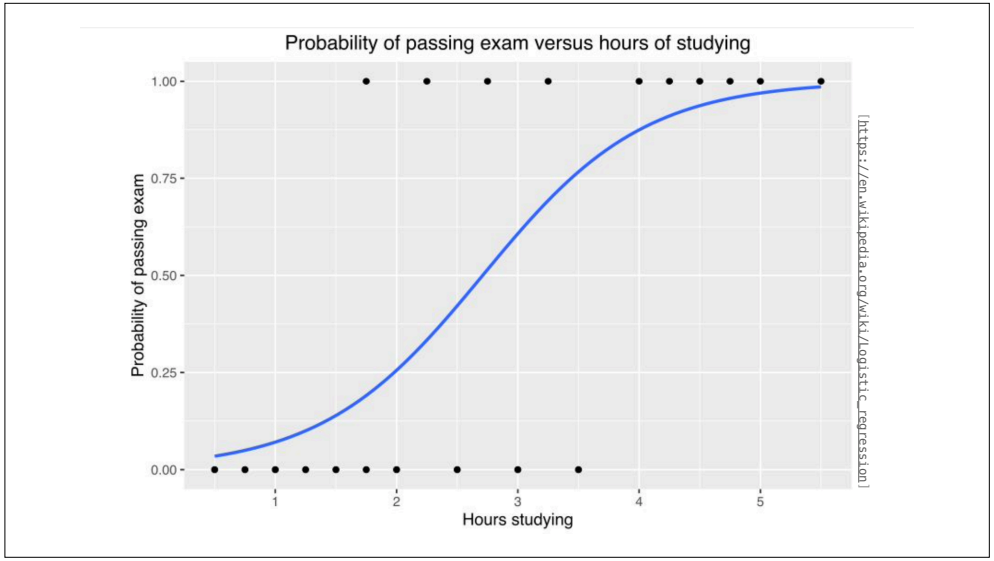

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Link Functions

a method to get "non-linear" linear regression
(more on this topic in a later lecture...)

$$y = X\beta$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ 1 & x_3 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

$$y = g^{-1}(X\beta)$$

the link function transforms the probabilities of the levels of a categorical response variable to a continuous scale that is unbounded

the link function transforms back the expectation of the response to the linear function

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Odds and log Odds

odds are the ratio of something happening to something not happening

the odds of my team winning is 1 to 4: $\frac{1}{4} = \frac{\text{⚽}}{\text{⚽⚽⚽⚽}} = 0.25$

the odds of my team winning is 5 to 3: $\frac{5}{3} = \frac{\text{⚽⚽⚽⚽⚽}}{\text{⚽⚽⚽}} = 1.7$

the probability is the ratio of something happening to **everything that could happen**

$$\frac{\text{⚽}}{\text{⚽⚽⚽⚽⚽}} = \frac{1}{5} = 0.20$$

$$\frac{\text{⚽⚽⚽⚽⚽}}{\text{⚽⚽⚽⚽⚽⚽⚽⚽}} = \frac{5}{8} = 0.625$$

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Odds and log Odds

the probability is the ratio of something happening to **everything that could happen**

the probability of winning: $\frac{\text{⚽⚽⚽⚽⚽}}{\text{⚽⚽⚽⚽⚽⚽⚽⚽}} = \frac{5}{8} = 0.625$

the probability of losing: $\frac{\text{⚽⚽⚽}}{\text{⚽⚽⚽⚽⚽⚽⚽⚽}} = \frac{3}{8} = 1 - \frac{5}{8} = 0.375$

the ratio of probability of winning to the probability of losing $= \frac{p}{(1-p)} = \frac{5/8}{3/8} = \frac{5}{3} = \frac{\text{⚽⚽⚽⚽⚽}}{\text{⚽⚽⚽}}$

the ratio of probabilities are the same as the ratio of the raw counts resulting in the same odds

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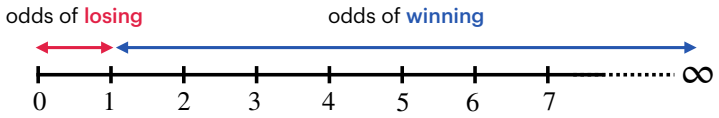
Odds and log Odds

why log odds?



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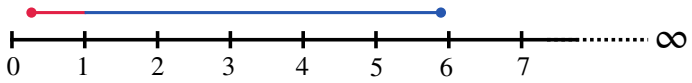
Odds and log Odds

why log odds?

example: odds against 1/6=0.17 but odds in favor 6/1=6

taking the log of the odds makes everything symmetrical



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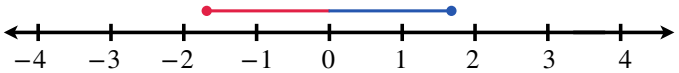
Odds and log Odds

why log odds?

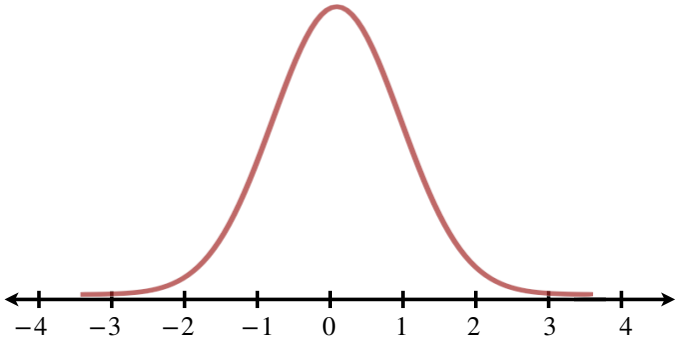
example: log odds against log(0.17)=-1.79 but log odds in favor log(6)=1.79

taking the log of the odds makes everything symmetrical



Odds and log Odds

why log odds?



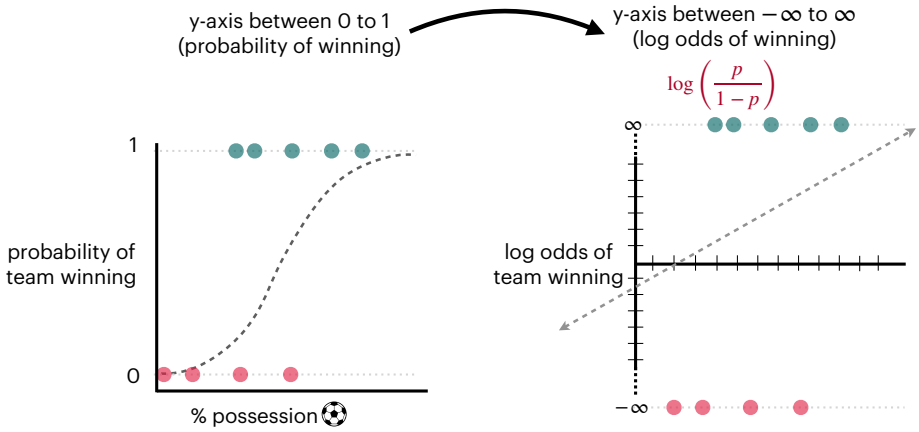
Logit Function

$$y = X\beta$$

$$\log\left(\underbrace{\frac{p}{1-p}}_{\text{odds}}\right) = \beta_0 + \beta_1 x_1$$

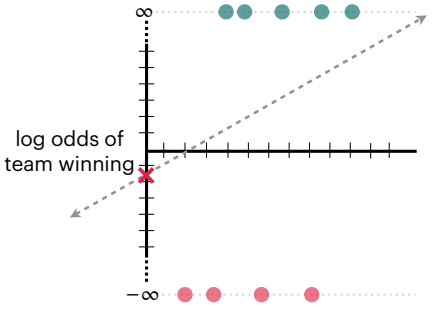
logit link function

Redefining The Response



Model Output

$y = -1.68 + 0.04 \times$ % possession

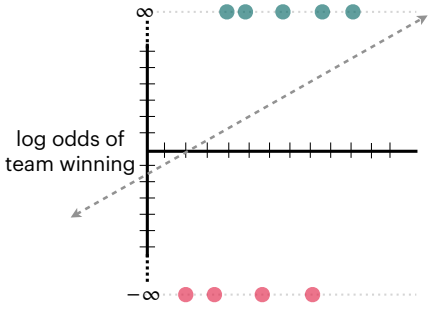



```
Call:
glm(formula = win ~ possession, family = binomial(), data = df)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -1.675714   0.414092  -4.047 5.19e-05 ***
possession   0.043467   0.007979   5.447 5.11e-08 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```


Model Output

$y = -1.68 + 0.04 \times$ % possession

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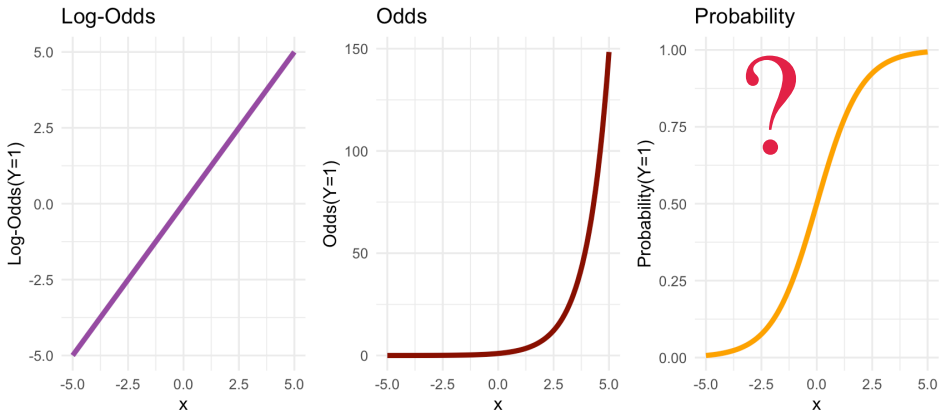
```

each +1% possession multiplies odds by $\exp(0.04)$

each +10% possession multiplies odds by $\exp(10 \times 0.04)$

⋮

Interpreting Coefficients



Link Functions

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$$y = g^{-1}(X\beta)$$

the **link function** transforms the probabilities of the levels of a categorical response variable to a continuous scale that is unbounded

the **link function** transforms back the expectation of the response to the linear function

Link Functions

$$y = X\beta$$

$$\log\left(\underbrace{\frac{p}{1-p}}_{\text{odds}}\right) = \beta_0 + \beta_1 x_1$$

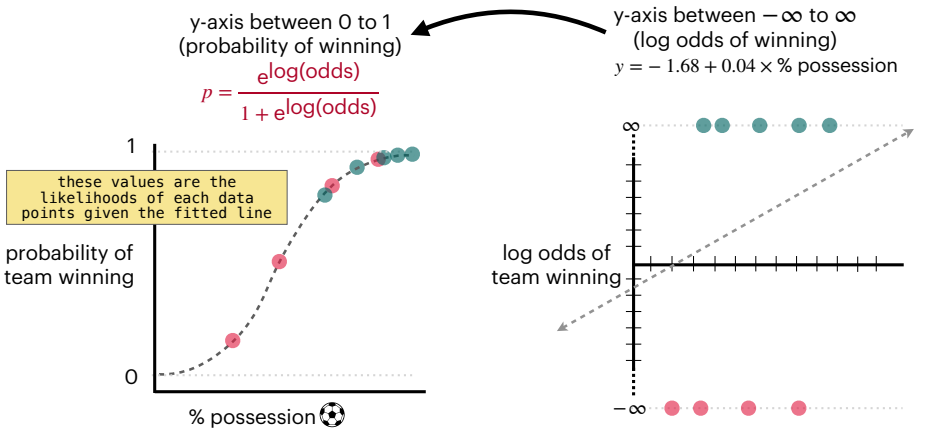
logit link function

$$y = g^{-1}(X\beta)$$

$$p = \frac{e^{\beta_0+\beta_1 x_1}}{1+e^{\beta_0+\beta_1 x_1}} \quad \text{= [a little algebra]}$$

$$= \frac{1}{1+e^{-(\beta_0+\beta_1 x_1)}}$$

Predicted Probabilities

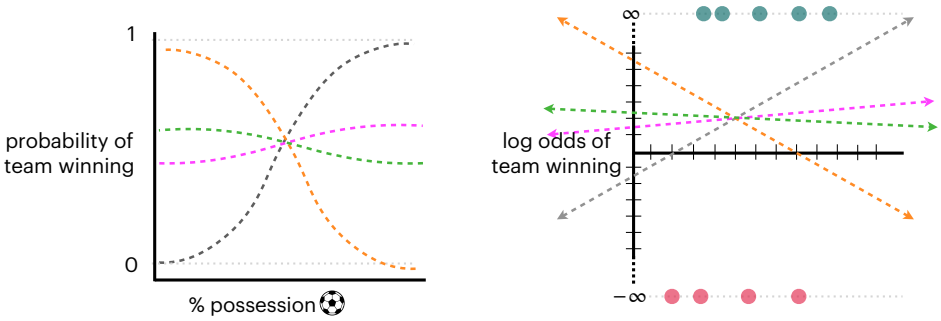


Interpreting Coefficients

Probability p	**Odds** p/(1-p)	**Log Odds** log[p/(1-p)]
0.1	0.1111	-2.1972
0.5	1	0
0.9	9	2.1972

Fitting the Best Line

we fit the line that maximizes the likelihood of the data



Estimating Coefficients: MLE

$$\prod_{i;y_i=1} p(x_i) \qquad \prod_{i;y_i=0} 1 - p(x_i)$$

$$L(\beta_0, \beta_1) = \prod_{i;y_i=1} p(x_i) \cdot \prod_{i;y_i=0} 1 - p(x_i)$$

$$L(\beta_0, \beta_1) = \prod_{i=1} p(x_i)^{y_i}(1 - p(x_i))^{1-y_i}$$

$$l(\beta_0, \beta_1) = \sum_{i=1} y_i \log(p(x_i)) + (1 - y_i)\log(1 - p(x_i))$$

[full proof: https://arunaddagatla.medium.com/maximum-likelihood-estimation-in-logistic-regression-f86ff1627b67

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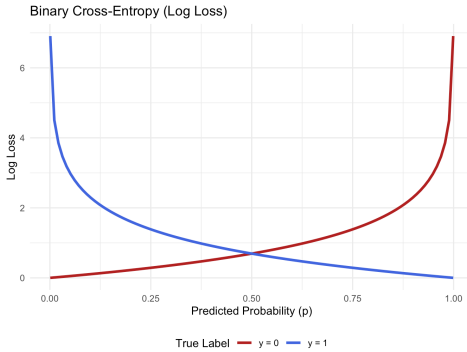
Binary Cross Entropy

$$l(\beta_0, \beta_1) = -\frac{1}{n}\sum_{i=1}^{n} y_i \log(p(x_i)) + (1 - y_i)\log(1 - p(x_i))$$

Loss function



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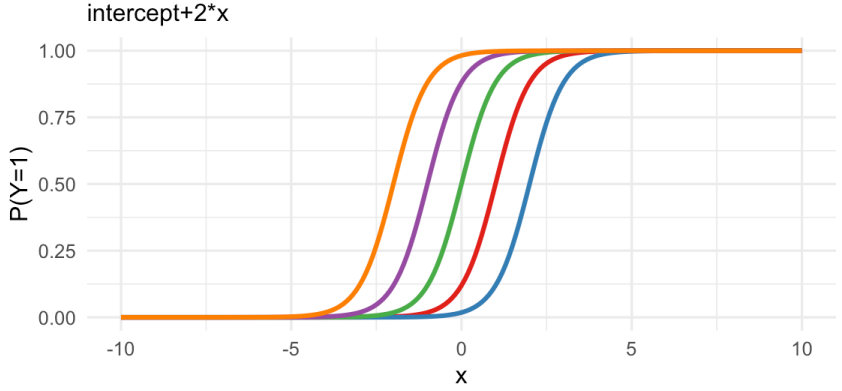

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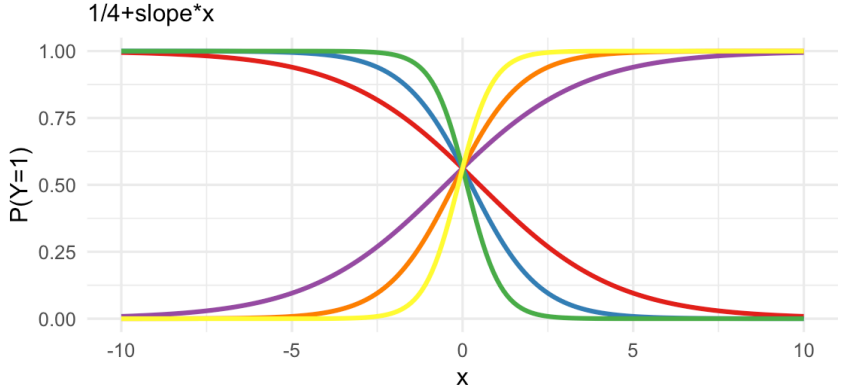

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Logistic Regression

$$y = X\beta$$

our link function is

$$g(x) = \log \frac{x}{1-x}$$

which has the inverse

$$g^{-1}(x) = \frac{e^x}{1+e^x}$$

$$y = g^{-1}(X\beta)$$ general case

$$p = \frac{e^{X\beta}}{1+e^{X\beta}}$$ specific case

Common distributions with typical uses and canonical link functions

Distribution	Support of distribution	Typical uses	Link name	Link function, $\mathbf{X}\beta = g(\mu)$	Mean function
Normal	real: $(-\infty, +\infty)$	Linear-response data	Identity	$\mathbf{X}\beta = \mu$	$\mu = \mathbf{X}\beta$
Exponential	real: $(0, +\infty)$	Exponential-response data, scale parameters	Negative inverse	$\mathbf{X}\beta = -\mu^{-1}$	$\mu = -(\mathbf{X}\beta)^{-1}$
Gamma					
Inverse Gaussian	real: $(0, +\infty)$		Inverse squared	$\mathbf{X}\beta = \mu^{-2}$	$\mu = (\mathbf{X}\beta)^{-1/2}$
Poisson	integer: $0, 1, 2, \ldots$	count of occurrences in fixed amount of time/space	Log	$\mathbf{X}\beta = \ln(\mu)$	$\mu = \exp(\mathbf{X}\beta)$
Bernoulli	integer: $\{0, 1\}$	outcome of single yes/no occurrence	Logit	$\mathbf{X}\beta = \ln\left(\frac{\mu}{1-\mu}\right)$	$\mu = \frac{\exp(\mathbf{X}\beta)}{1+\exp(\mathbf{X}\beta)} = \frac{1}{1+\exp(-\mathbf{X}\beta)}$
Binomial	integer: $0, 1, \ldots, N$	count of # of "yes" occurrences out of N yes/no occurrences		$\mathbf{X}\beta = \ln\left(\frac{\mu}{n-\mu}\right)$	
Categorical	integer: $[0, K)$	outcome of single K-way occurrence		$\mathbf{X}\beta = \ln\left(\frac{\mu}{1-\mu}\right)$	
	K-vector of integer: $[0, 1]$, where exactly one element in the vector has the value 1				
Multinomial	K-vector of integer: $[0, N]$	count of occurrences of different types $(1, \ldots, K)$ out of N total K-way occurrences			

https://en.wikipedia.org/wiki/Generalized_linear_model#Link_function

Interpreting Logistic Regression Models

- we want to create a spam filter based on 3921 observations/emails
- simple model, one predictor: '`to_multiple`'

```
Call:
glm(formula = spam ~ to_multiple, family = binomial, data = email)

Deviance Residuals:
   Min      1Q  Median      3Q     Max
-0.477  -0.477  -0.477  -0.477   2.809

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept)    -2.11609    0.05618 -37.665  < 2e-16 ***
to_multipleyes -1.80918    0.29685  -6.095  1.1e-09 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 2437.2  on 3920  degrees of freedom
Residual deviance: 2372.0  on 3919  degrees of freedom
AIC: 2376

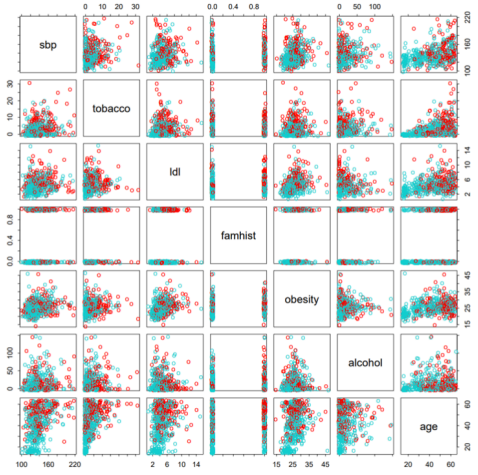
Number of Fisher Scoring iterations: 6
```


Example: South African Heart Disease

- From Western Cape,South Africa in early 80s
- Coronary Risk Factor Study (CORIS)
- High incidence of myocardial infarction (MI) in region: 5.1%
- Measurements on seven predictors (risk factors)
- 160 cases, 302 controls. Ages 15-64.
- Outcome is presence/absence of MI at time of survey
- Goal:
  - to identify relative strengths and directions of risk factors
  - intervention study aimed at educating the public on healthier diets

[For more info see ESL 4.4.2]

Example: South African Heart Disease



cases = red
controls = turquoise

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Example: South African Heart Disease

term	estimate	std.error	statistic	p.value
(Intercept)	-4.130	0.964	-4.283	0.000
sbp	0.006	0.006	1.023	0.306
tobacco	0.080	0.026	3.034	0.002
ldl	0.185	0.057	3.219	0.001
famhistPresent	0.939	0.225	4.177	0.000
obesity	-0.035	0.029	-1.187	0.235
alcohol	0.001	0.004	0.136	0.892
age	0.043	0.010	4.181	0.000

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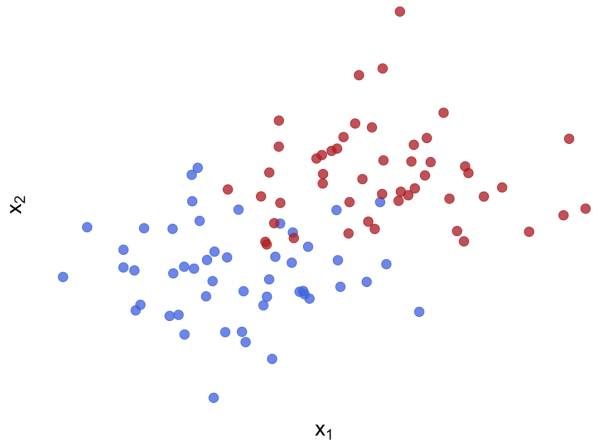
K-Nearest Neighbors (KNN)



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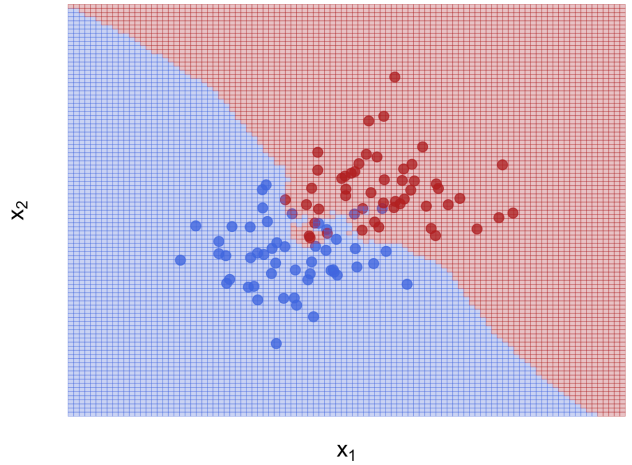
K-Nearest Neighbors (KNN)



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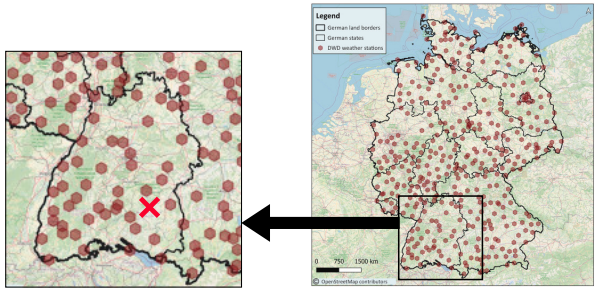
K-Nearest Neighbors (KNN)

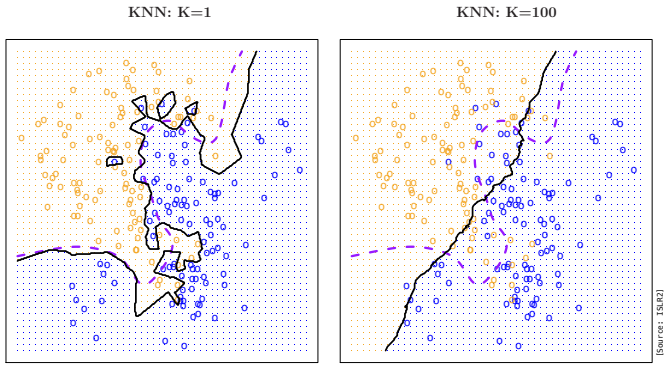
KNN regression tries predicting values of output variable by using a local average

KNN classification attempts predicting the class to which the output variable belong by computing the local probability



The Parameter K

- Smaller K $\Longrightarrow$ Less smooth and more sensitive to noise (more flexible)
- Larger K $\Longrightarrow$ More smooth (less flexible)



The Hyperparameter K

- Smaller K $\Longrightarrow$ Less smooth and more sensitive to noise (more flexible)
- Larger K $\Longrightarrow$ More smooth (less flexible)

how do we choose K?

- choose yourself
- let the data decide (hyperparameter tuning: more on this later...)

The Influence of the Similarity Function

The choice of the distance/similarity function is also important:

- Performance only be good if the distance function encodes "relevant information"

  Example: you want to classify mushrooms as "edible" or "not edible" and as distance function between mushrooms you use the difference in weight...

- Not so obvious sometimes how to define a good distance or similarity function

  Example: you want to classify the genre of songs but how do you compute a similarity between different songs?

This Week's Practical

Logistic Regression and KNN