Calculus Fundamentals: Differentiation

Lecture 5

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welcome to calculus

- calculus allows us to deal with continuity in a consistent and productive way
- differentiation is the process of taking a derivative

notation

- with one variable:

  "f prime x" $f'(x)$

- with multiple variables:

  "the derivative of f of x with respect to x" $\dfrac{df(x)}{dx}$ or $\dfrac{\partial f(x,y)}{\partial x}$

welcome to calculus

Discrete change

- change between two measures of a concept at two distinct, discrete moments in time

  → first difference between two observations over a discrete interval

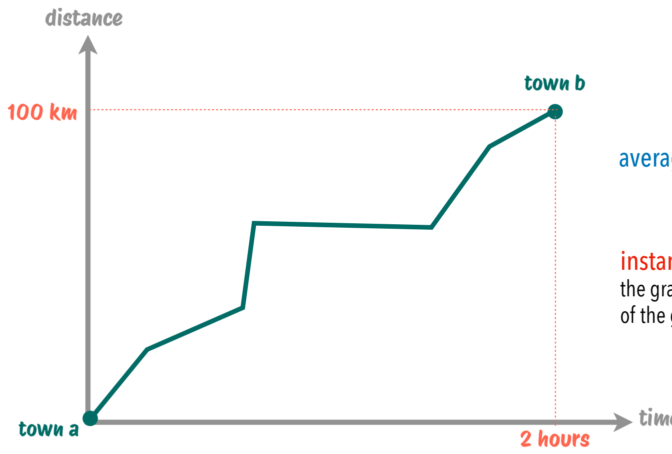
Instantaneous change

- change at a specific point in time
- derivative of function $f(x)$ with respect to x tells us the instantaneous rate of change of the function at each point

(First) derivative

- describes reactivity to change in function's output based on input argument x

rates of change



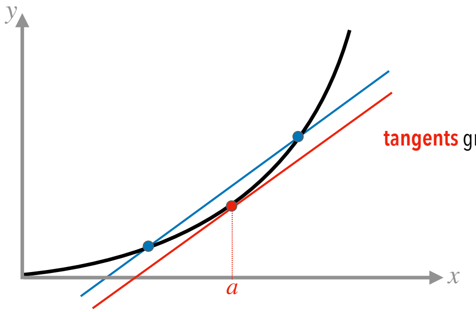
average speed: $\dfrac{100}{2} = 50$km/h

instantaneous speed:
the gradient (slope) "at a point" of the graph

secants and tangents

secants graphically represent discrete change

tangents graphically represent instantaneous change at $x = a$



Note: "secare" means to intersect and "tangere" means to touch in Latin

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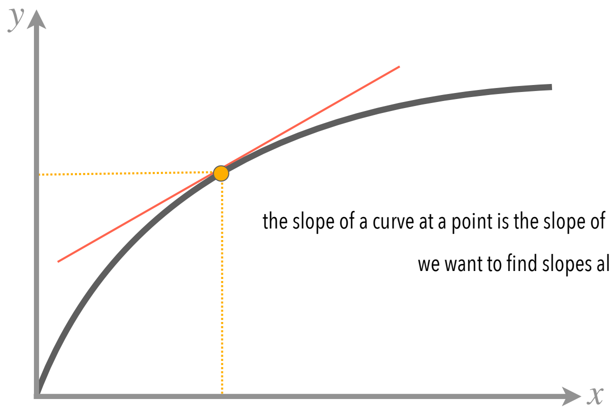
tangents



the slope of a curve at a point is the slope of the **tangent** line at that point

we want to find slopes algebraically

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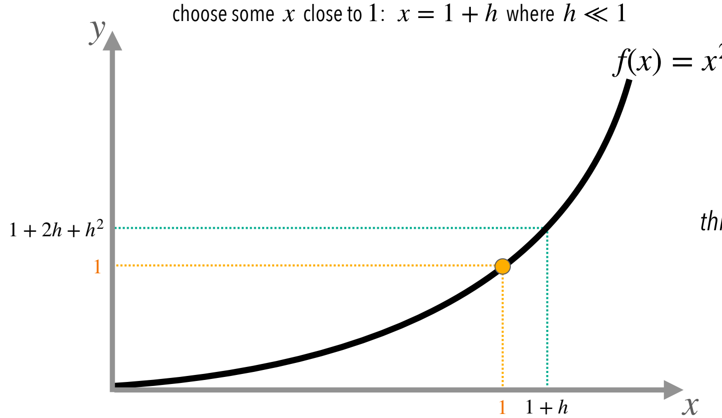
finding the slope at a point

how does this function behaves when x is close to 1?

choose some x close to 1: $x = 1 + h$ where $h \ll 1$



this tells us two things ➞

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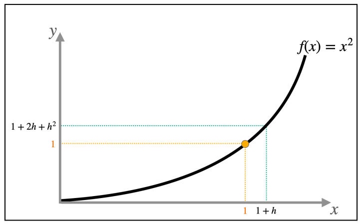
finding the slope at a point

1. Near 1 the value of $f(x)$ is also near 1, since if h is small, then $2h + h^2$ is also small (this is the concept of continuity)
2. Increasing x from 1 to $1 + h$ will increase the value of $f(x)$ from $f(1)$ to $f(1) + 2h + h^2$ which is an increase of approximately $2h$

$\implies$ the rate of change at 1,
defined as the derivative of the function $f(x) = x^2$,
is $f'(1) = 2$

$\boxed{\text{What about when } x = c?}$



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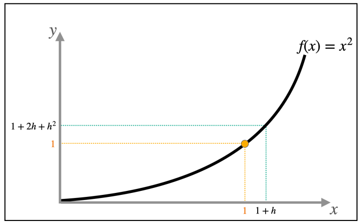
finding the slope at a point

What about when $x = c$?

As x changes from c to $c + h$,
$f(x) = x^2$ changes from c^2 to $(c+h)^2 = c^2 + \underbrace{2ch}_{} + h^2$

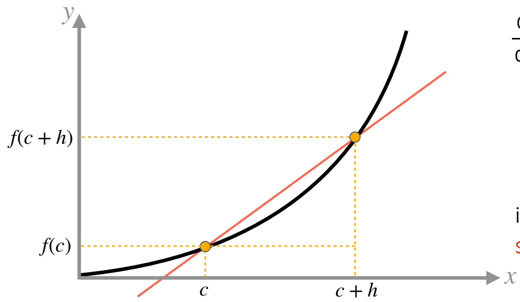
the change = the coefficient of h

$\Longrightarrow$ the derivative of f at c is $2c$
and in general $f'(x) = 2x$



secants

geometrically, we calculate the slope of the secant joining points on the curve:



$$\frac{\text{change in } f}{\text{change in } x} = \frac{f(c+h) - f(c)}{(c+h) - c} = \frac{f(c+h) - f(c)}{h}$$

and examine what happens as h approaches 0

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$$

in the hopes that the slope of the secant will approach the slope of the tangent line

derivative of a function

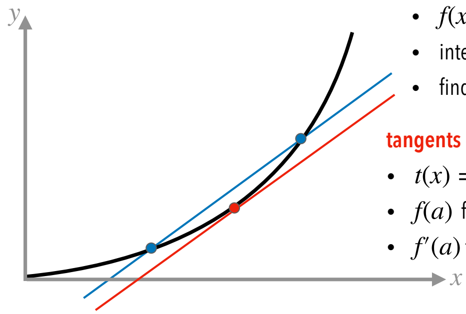
We define the derivative of a function f at x as

$$f'(x) = \frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided that the limit exists. If the limit exists, we say f is differentiable at x.

If we simply say f is differentiable, we mean f is differentiable at all values of x.
In this case, $f'(x)$ is also a function of x.

summary: secants and tangents



secants graphically represent discrete change

- $f(x) = mx + b$
- intercept b as point 1
- find mx to reach point 2

slope of the secant
$$\frac{f(x+h) - f(x)}{h}$$

tangents graphically represent instantaneous change at $x = a$

- $t(x) = f'(a) \cdot (x - a) + f(a)$
- $f(a)$ function f evaluated at a
- $f'(a)$ first derivative of function f evaluated at a

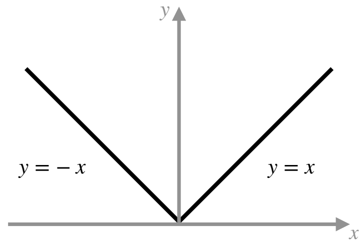
slope of the tangent
$$\frac{d}{dx} f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Note: "secare" means to intersect and "tangere" means to touch in Latin

derivative of a function

example

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$



is not differentiable at a certain point because it is not continuous at (0,0), we can draw many many many tangent lines there. Also:

for $h > 0$, $\dfrac{f(0+h) - f(0)}{h} = \dfrac{h-0}{h} = 1$

for $h < 0$, $\dfrac{f(0+h) - f(0)}{h} = \dfrac{-h-0}{h} = -1$

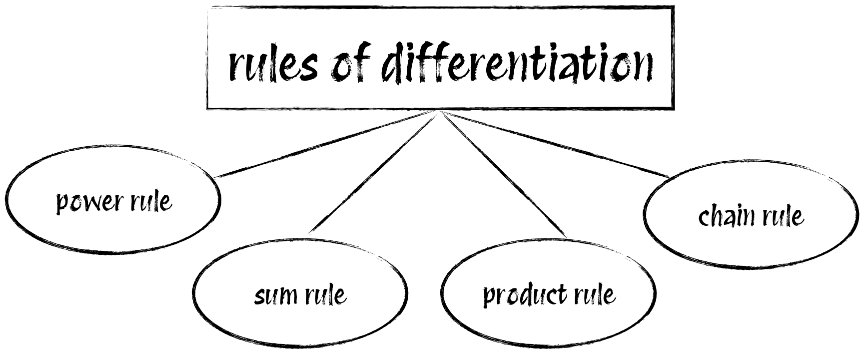
i.e. $\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$ does not exist since taking the limit from both sides must give the same answer.

derivative of a function

exercise 1

Solve $\dfrac{d}{dx}3x^2$ using the definition of derivative $\dfrac{d}{dx}f(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

In theory, we could now go on and find derivatives of "all" algebraic functions by definition. But that's too time consuming and impractical…



we build a "machine" with these three bits to help us to help us differentiate functions

the power rule

Let f be a function $f(x) = x^r$. Then,

$$\frac{d}{dx}f(x) = f'(x) = rx^{r-1}$$

exercise 2

Show that the derivative of x^n for a positive whole number n is equal to nx^{n-1} (use the definition of derivative).

the sum rule

If f and g are differentiable, then

$$\frac{d}{dx}\big(f(x)+g(x)\big) = \frac{d}{dx}\big(f(x)\big) + \frac{d}{dx}\big(g(x)\big) = f'(x)+g'(x)$$

If you repeatedly apply the sum rule, you have

$$\frac{d}{dx}\big(f_1(x)+f_2(x)+\cdots+f_n(x)\big) = \frac{d}{dx}\big(f_1(x)\big) + \frac{d}{dx}\big(f_2(x)\big)+\cdots+\frac{d}{dx}\big(f_n(x)\big)$$

example

Differentiate the function $f(x) = x^3 + x^4$.

$$\frac{d}{dx}\big(x^3+x^4\big) = \frac{d}{dx}\big(x^3\big) + \frac{d}{dx}\big(x^4\big) = 3x^2+4x^3$$

a special case

If $g(x) = c$ is constant then $\frac{d}{dx}\big(cf(x)\big) = c\frac{d}{dx}\big(f(x)\big) = cf'(x)$

The sum rule can then be generalized as:

If $f_1, f_2, \cdots, f_n$ are differentiable and $a_1, a_2, \cdots, a_n$ are constants, then

$$\frac{d}{dx}\big(a_1f_1(x)+a_2f_2(x)+\cdots+a_nf_n(x)\big) = a_1f'(x)+a_2f_2'(x)+\cdots+a_nf_n'(x)$$

example

Differentiating polynomial of degree n with constant coefficients given by

$$f(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n$$

yields a polynomial of degree $n-1$ with constant coefficients given by

$$f'(x) = a_1 + 2a_2x + 3a_3x^2 + \cdots + na_nx^{n-1}$$

the product rule

If f and g are differentiable, then

$$\frac{d}{dx}\big(f(x)g(x)\big) = f'(x)g(x) + f(x)g'(x)$$

example

Differentiate the function $f(x) = (x^2+1)(x^3-1)$.

Here we set $f(x) = x^2+1 \implies f'(x) = 2x$

$g(x) = x^3 - 1 \implies g'(x) = 3x^2$

and use the product rule:

$$\begin{aligned}\frac{d}{dx}\big((x^2+1)(x^3-1)\big) &= 2x(x^3-1) + (x^2+1)3x^2 \\ &= 2x^4 - 2x + 3x^4 + 3x^2 \\ &= 5x^4 + 3x^2 - 2x\end{aligned}$$

the chain rule

Let's now consider differentiating "compositions" of functions:

$$f \circ g(x) = f(g(x)) \quad \text{"do g then f"}$$

or

$$f \circ g \circ h(x) = f(g(h(x))) \quad \text{"do h then g then f"}$$

If f and g are differentiable, then

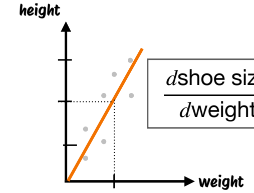
$$\frac{d}{dx}\big(f(g(x))\big) = f'(g(x)) \cdot g'(x)$$

the chain rule

a change in weight leads to a change in shoe size

for every one unit increase in **weight** there is a 2 unit increase in **height**

$$\frac{\Delta\text{height}}{\Delta\text{weight}} = \frac{d\text{height}}{d\text{weight}} = \frac{2}{1} = 2 \implies \text{height} = 2\cdot\text{weight}$$



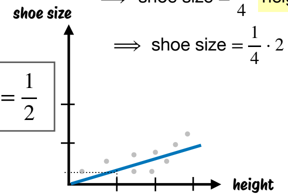
$$\boxed{\frac{d\text{shoe size}}{d\text{weight}} = \frac{d\text{shoe size}}{d\text{height}}\cdot\frac{d\text{height}}{d\text{weight}} = \frac{1}{2}}$$

for every one unit increase in **height** there is a 1/4 unit increase in **shoe size**

$$\frac{\Delta\text{shoe size}}{\Delta\text{height}} = \frac{d\text{shoe size}}{d\text{height}} = \frac{1}{4}$$

$$\implies \text{shoe size} = \frac{1}{4}\cdot\text{height}$$

$$\implies \text{shoe size} = \frac{1}{4}\cdot 2\cdot\text{weight}$$



the chain rule

example

To differentiate the function $y(x) = (x^3 + 2x)^{10}$ we can set $f(z) = z^{10}$ and $g(x) = x^3 + 2x$.

We then have that $f'(z) = 10z^9$ and $g'(x) = 3x^2 + 2$. Then

$$\begin{aligned} \frac{d}{dx}(y(x)) &= \frac{d}{dx}(f(g(x)) \\ &= \frac{d}{dx}(x^3+2x)^{10} \\ &= f'(g(x))g'(x) \\ &= 10(x^3+2x)^9\cdot(3x^2+2) \end{aligned}$$

exercise 3

Differentiate the function $y(x) = \frac{1}{x^3}$

a special case: quotient rule

If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)^2]}$$

This is simply the product rule for rational functions!

exercise 4

Write

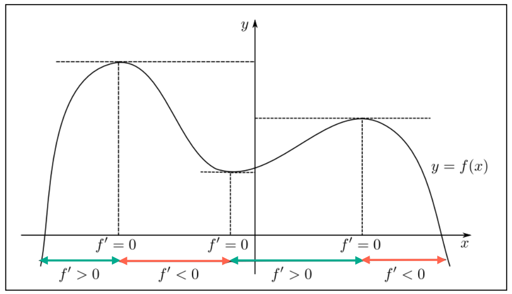
$$\frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$$

and show that the quotient rule is the product rule for rational functions.

maxima and minima

given some function f, where does it achieve its maximum or minimum values?

- if $f(x)$ is increasing, then $f'(x) > 0$
- if $f(x)$ is decreasing, then $f'(x) < 0$



- we have troughs and humps occur at places through which f' changes sign (where $f'(x) = 0$)
- The derivative gives us a way to look for maximum and minimum values of a function.

second derivative

- To characterize troughs and humps we need the second derivative
- We can view $f'(x)$ itself as a function that we differentiate it again:

$$\frac{d}{dx}\big(f'(x)\big) = \frac{d^2}{dx^2}\big(f(x)\big) = f''(x)$$

The geometric interpretation of f'':

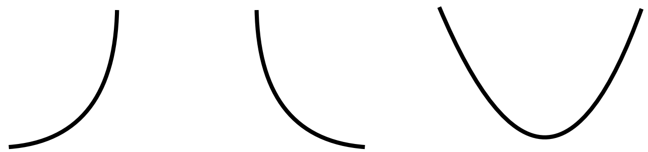
1. $f''(x) > 0$
2. $f''(x) < 0$
3. $f''(c) = 0$

second derivative

The geometric interpretation of f'':

1. If $f''(x) > 0$ then the slope of the tangent line is increasing in value
   $\Longrightarrow$ if $f'(c) = 0$ and $f''(c) > 0$, then around c, $f(x)$ is a trough
   $\Longrightarrow$ we can expect a local minimum value of f at c

possible shapes:

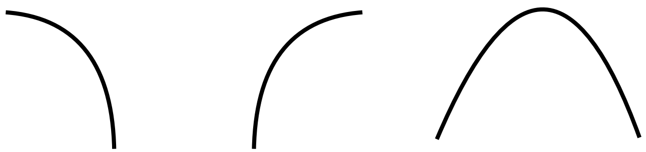


second derivative

The geometric interpretation of f'':

2. If $f''(x) < 0$ then the slope of the tangent line is decreasing in value
   $\Longrightarrow$ if $f'(c) = 0$ and $f''(c) < 0$, then around c, $f(x)$ is a hump
   $\Longrightarrow$ we can expect a local maximum value of f at c

possible shapes:



second derivative

The geometric interpretation of f'':

3. If $f'(c) = 0$ and $f''(c) = 0$ then the slope often doesn't change sign
   i.e. it goes from positive slope to zero to positive slope (decreasing to zero then increasing), or negative to zero to negative (increasing to zero then decreasing).

possible shapes: