Introduction to Linear Algebra

Lecture 11

Termeh Shafie

what is linear algebra?

- Linear
  - having to do with lines, planes, space, etc.
  - example: $x + y + 3z = 7$ (and not $\sin, \log, x^2, \ldots$)
- Algebra
  - solving equations involving numbers and symbols
  - "*reunion of broken parts*"

Linear algebra is the math of vectors and matrices, so we'll start by definitions and the mathematical operations we can perform on vectors and matrices.

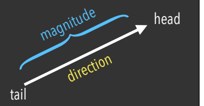
scalars

- any single element of some set, for example elements of the real numbers $\mathbb{R}$
- all quantities that have a magnitude but no direction, other than perhaps positive or negative
- scalars are easy to use: treat them as normal numbers
- examples of scalars are temperature, distance, speed, or mass
- notation: lowercase letters (can be both Latin and Greek)

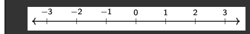
vectors

- a list of numbers or an object that has both a magnitude and a direction
- components: the entries of a vector
- the dimension of a vector is the number of components in the vector.
- notation: bold face ($\mathbf{x}, \mathbf{y}, \mathbf{z}$) or arrow over ($\vec{x}, \vec{y}, \vec{z}$)
- intuitively: Let n be a positive whole number. We define

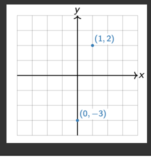
$\mathbb{R}^n =$ all ordered n-tuples of real numbers $(x_1, x_2, x_3, \ldots, x_n)$



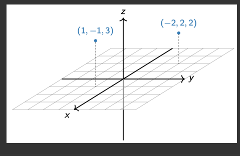
$n = 1$



$n = 2$



$n = 3$



magnitudes and direction

example

What is the magnitude of $\vec{a} = [4,3]$?

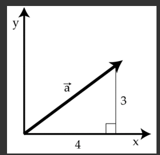
The "magnitude" of a vector is the distance from the endpoint of the vector to the origin, i.e. its length

This vector extends 4 units along the x-axis, and 3 units along the y-axis.

Magnitude $||\vec{a}||$ is compared using Pythagorean Theorem ($x^2 + y^2 = z^2$):

$$||\vec{a}|| = \sqrt{x^2 + y^2} = \sqrt{4^2 + 3^2} = 5$$

The magnitude of a vector is a scalar value.



magnitudes and direction

example

What is the direction of $\vec{a} = [4,3]$?

A way of representing the direction of a vector independent of its length is using unit vectors

> A unit vector is a vector of magnitude 1 and is obtained by diving a vector by its length (normalizing)
> Unit vectors can be used to express the direction of a vector independent of its magnitude.

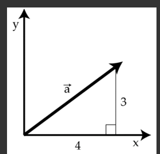
$$\frac{\vec{a}}{||a||} = \frac{[4,3]}{5} = \left[\frac{4}{3}, \frac{3}{5}\right]$$

by dividing each component of the vector by the same number,

we leave the direction of the vector unchanged, while we change the magnitude.

If done correctly, the magnitude of the unit vector must be equal to 1.

How do we verify this? By calculating the magnitude of the unit vector.



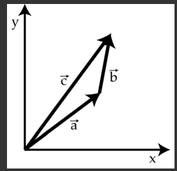
vector addition and subtraction

- Graphically
  - think of adding two vectors together as placing two line segments end-to-end, thus maintaining distance and direction
  - $\vec{a} + \vec{b} = \vec{c}$
- Numerically
  - we add vectors component-by-component
  - example: $\vec{a} = [4,3]$ and $\vec{b} = [1,2]$ then

    $\vec{c} = [4,3] + [3,2] = [4+1, 3+2] = [5,5]$
  - similarly for vector subtraction:

    $\vec{a} = [4,3]$ and $\vec{b} = [1,2]$ then

    $\vec{c} = [4,3] - [3,2] = [4-1, 3-2] = [3,1]$
- Vector addition has a very simple interpretation in the case of things like displacement (ex ship)



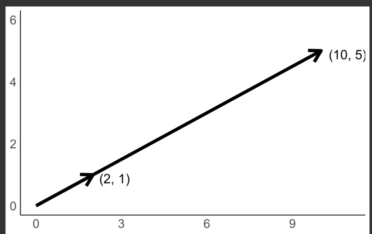
scalar multiplication

- An operation where a vector is multiplied by a scalar resulting in a new vector
- The new vector's direction remains the same (or reversed if the scalar is negative), and its magnitude is scaled by the absolute value of the scalar
- Geometric Interpretation (let k be a scalar)
  - Scaling: scalar multiplication changes the length (magnitude) of the vector
    - if k > 1, the vector is stretched
    - if 0 < k < 1, the vector is compressed.
    - if k = -1, the vector reverses direction but retains its magnitude.
    - The length of a scalar multiple of a vector is the absolute value of the scalar times the length of the vector, i.e. $||k\vec{a}|| = |k| \cdot ||\vec{a}||$
  - Direction:
    - if k > 0, the direction of the vector remains unchanged.
    - if k < 0, the direction of the vector is reversed

scalar multiplication

example

What is the scalar value here multiplied to $\vec{a} = [2,1]$?



vector multiplication

Two principal ways of multiplying vectors:

1. **Dot products** (a.k.a. scalar products)

   $d = \vec{a} \cdot \vec{b}$

   generates a scalar value from the product of two vectors
2. Cross products

   $\vec{c} = \vec{a} \times \vec{b}$

   The cross product generates a vector from the product of two vectors

dot product

The dot product is key for calculating vector projections, vector decompositions, and determining orthogonality

The dot product of two vectors $\vec{a}$ and $\vec{b}$ is

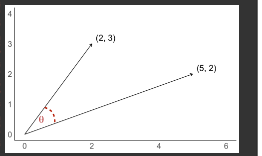
$$\vec{a} \cdot \vec{b} = \sum_{i=1}^{n} a_i b_i + a_2 b_2 + \cdots + a_n b_n$$

The angle θ of between two vectors is determined by the formula

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\| \cos\theta$$

where $\|\vec{a}\|$ is the length or norm or magnitude of a vector. Thus

$$\cos\theta = \frac{\vec{a} \cdot \vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{\vec{a}}{\|\vec{a}\|} \cdot \frac{\vec{b}}{\|\vec{b}\|}$$

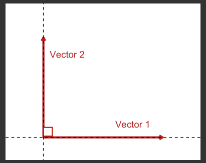


orthogonality

- $\cos 0° = 1$ the vectors point in exactly the same direction (they coincide)
- $\cos 90° = 0$ means the vectors are perpendicular (aka orthogonal) to each other in 2D or 3D

Two vectors are orthogonal to one another if the dot product of those two vectors is equal to zero

- Orthogonal vectors point in completely independent directions, meaning one vector cannot be expressed as a scalar multiple of the other



orthogonality

Two vectors are orthogonal to one another if the dot product of those two vectors is equal to zero

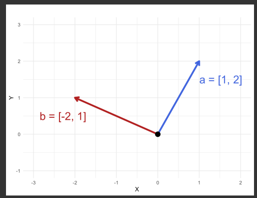
example

Let $\vec{a} = (1,2)$ and $\vec{b} = (-2,1)$

The dot product is

$$\vec{a} \cdot \vec{b} = (1)(-2) + (2)(1) = -2 + 2 = 0$$

Since their dot product is zero, $\vec{a}$ and $\vec{b}$ are orthogonal.



the matrix

- A matrix is a table of numbers rather than a list as is the case for vectors
- The size of a matrix: number of rows $\times$ number of columns = $m \times n$ (read "m *by* n")
- You can think of vectors as matrices that happen to only have one column or one row

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- A square matrix is a matrix that has an equal number of columns and rows, i.e.,$m = n$
- A zero matrix is a square matrix in which all elements are 0

the matrix

- A diagonal matrix is a square matrix with non-zero elements only on the main diagonal
- An identity matrix is a diagonal matrix in which all elements on the main diagonal are 1:

$$D_{n\times n} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{44} \end{bmatrix} \qquad I_{n\times n} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- The identity matrix is special because, when multiplied by another matrix, it produces the original matrix back again (we'll return to this later after covering matrix multiplication)
- A lower triangular matrix has non-zero elements only on or below the main diagonal
- An upper triangular matrix has non-zero elements only on or above the main diagonal
- A symmetric matrix is a square matrix with elements symmetric such that $a_{ij} = a_{ji}$

the transpose of a matrix

Let A be an $m \times n$ matrix. The transpose of A, denoted A^T or A', is the $n \times m$ matrix whose columns are the respective rows of A.

- A matrix is symmetric if it doesn't change when you take its transpose

example

If you take the transpose of matrices $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

we get $A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}$ and $B^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$.

Note: matrix B is thus symmetric.

matrix arithmetic: addition and subtraction

Let A and B be $m \times n$ matrices. The sum of A and B, denoted $A + B$, is

$$\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{22} & a_{22}+b_{23} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}$$

exercise 1

Let $A = \begin{bmatrix} 2 & -1 \\ 3 & 6 \end{bmatrix}$.

Find the matrix X such that $2A + 3X = -4A$

matrix arithmetic: scalar multiplication

Let A be an $m \times n$ matrix and let k be a scalar. The scalar multiplication of k and A, denoted kA, is

$$\begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

exercise 2

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 1 \\ 5 & 5 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 2 \\ -1 & 0 & 4 \end{bmatrix}$

Simplify the following expression: $5(A + B)$

matrix arithmetic: matrix multiplication

Let A be an $m \times r$ matrix, and let B be an $r \times n$ matrix.

The matrix product of A and B, denoted $A \cdot B$ or AB, is the $m \times n$ matrix M whose entry in the ith row and jth column is the product of the ith row of A and the jth column of B.

- In order to multiply two matrices A and B, the number of columns of A must be the same as the number of rows of B (the inner dimensions must be the same)
- The resulting matrix has same number of rows as A and same number of columns as B (i.e. the outer dimensions)

$$\overbrace{(m \times \underbrace{r) \times (r}_{\text{inner dimensions must match}} \times n)}^{\text{final dimensions are outer dimensions}}$$

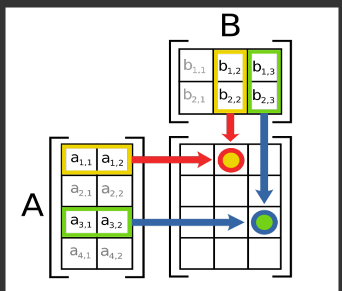
matrix arithmetic: matrix multiplication

Let matrix A have rows $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m \implies A = \begin{bmatrix} - & \vec{a}_1 & - \\ - & \vec{a}_2 & - \\ & \vdots & \\ - & \vec{a}_m & - \end{bmatrix}$

and let matrix B have columns $\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_m \implies B = \begin{bmatrix} | & | & & | \\ \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_n \\ | & | & & | \end{bmatrix}$

Then $AB = \begin{bmatrix} \vec{a}_1\vec{b}_1 & \vec{a}_1\vec{b}_2 & \cdots & \vec{a}_1\vec{b}_n \\ \vec{a}_2\vec{b}_1 & \vec{a}_2\vec{b}_2 & \cdots & \vec{a}_2\vec{b}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{a}_m\vec{b}_1 & \vec{a}_m\vec{b}_2 & \cdots & \vec{a}_m\vec{b}_n \end{bmatrix}$

matrix arithmetic: matrix multiplication



[source: https://commons.wikimedia.org/wiki/File:Matrix_multiplication_diagram_2.svg]

matrix arithmetic: matrix multiplication

example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$. What is AB?

Compute each element of resulting matrix $C = A \times B$ by summing products of rows of A and columns of B:

$C = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$ where

$$\begin{aligned} c_{11} &= 1 \cdot 5 + 2 \cdot 7 = 5 + 14 = 19 \\ c_{12} &= 1 \cdot 6 + 2 \cdot 8 = 6 + 16 = 22 \\ c_{21} &= 3 \cdot 5 + 4 \cdot 7 = 15 + 28 = 43 \\ c_{22} &= 3 \cdot 6 + 4 \cdot 8 = 18 + 32 = 50 \end{aligned} \implies C = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

trace

Let A be an $n \times n$ matrix. The trace of A, denoted $tr(A)$, is the sum of the diagonal elements of A. That is,

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$$

Properties of trace:

- $tr(A + B) = tr(A) + tr(B)$
- $tr(A - B) = tr(A) - tr(B)$
- $tr(kA) = k \cdot tr(A)$
- $tr(AB) = tr(BA)$
- $tr(A^T) = tr(A)$

The trace will come up again in reference to eigenvalues.

determinant

Let A be an 2×2 matrix given as $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

The determinant of A, denoted by

$$det(A) \text{ or } |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

is given by $ad - bc$.

All good, but what if $n > 2$?

Then we need to define matrix minor and matrix cofactor.

matrix minor and matrix cofactor

Let A be an $n \times n$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The i, j minor of A, denoted $M_{i,j}$ is the determinant of the $(n-1) \times (n-1)$ matrix formed by deleting the ith row and jthcolumn of A.

The i, j-cofactor of A is the number. $C_{ij} = (-1)^{i+j} M_{i,j}$

Let A be an $n \times n$ matrix where $n > 2$. Then $det(A)$ is the number found by taking the cofactor expansion along the first row of A. That is,

$$\det(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \cdots + a_{1,n}C_{1,n}.$$

matrix minor and matrix cofactor

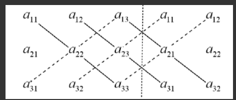
exercise 3

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(a) Find the cofactor expansions along the first column.

(b) Find the determinant of A.

Note: using your tutorial you will cover another way to find the determinant called the butterfly method (only works for 3×3 matrices)



properties of the determinant

Let A and B be $n \times n$ matrices and let k be a scalar

- $det(kA) = k^n \cdot det(A)$
- $det(A^T) = det(A)$
- $det(AB) = det(A)det(B)$
- If A is invertible then

$$det(A^{-1}) = \frac{1}{det(A)}$$

- A matrix A is invertible if and only if $det(A) \neq 0$
- A square matrix that has $det(A) = 0$ is called singular and is not invertible

inverse of a matrix

Let A be an $n \times n$ matrix. It is invertible if if one can find a second $n \times n$ matrix, X, such that the product AX and the product XA both produce the $n \times n$ the identity matrix $I_{n \times n}$.

X is then the inverse of A, denoted by $A^{-1} \implies A \cdot A^{-1} = A^{-1} \cdot A = I$

Let A be an 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. A is invertible if and only if $ad - bc \neq 0$, If it is invertible then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The general rule: $A^{-1} = \frac{1}{|A|} C^T$ where C^T is the transpose of the matrix of cofactors of A.

Each element of C is the cofactor of the corresponding element of A.

inverse of a matrix

exercise 4

Find the inverse of $A = \begin{bmatrix} 3 & 2 \\ -1 & 9 \end{bmatrix}$ if it exists.

more hands on exercises on the inverse during your tutorial

why do we need linear algebra?

- Coordinates and Geometry:
  Linear algebra provides a way to represent and understand geometric concepts like lines, planes, and transformations in higher dimensions.
- Equations Systems:
  It offers tools to solve systems of linear equations, which appear in countless areas of study.

very simple example

Imagine you go to a restaurant and order some food. Here's what happens:

1. Your friend orders 3 pizzas and 4 burgers, and the total cost is 38€.
2. Another friend orders 2 combo meals and 1 pizza, and the total cost is 34€.
3. You order 1 combo meal and 2 sodas, and the total cost is 14€.

How much does one pizza cost? How much does one burger cost? How much does one soda cost?

This is a system of two linear equations with two unknowns x and y:

$$3x + 4y = 38$$
$$2(x + 2y + z) + x = 34$$
$$x + 2y + 3z = 14$$