Preliminaries

Lecture 1

Termeh Shafie

roadmap of the course

Building blocks

I. Calculus in one dimension
II. Probability theory
III. Linear algebra
IV. Multivariate calculus and optimization

preliminaries

variables translating concepts into math

$\downarrow$

grouping of variables to sets

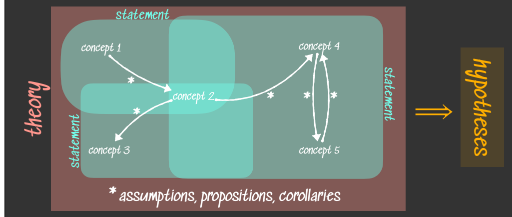
$\downarrow$

operators on variables and sets

$\downarrow$

methods of proof and generalization

building a toy model of the world in math



variables and constants

- theory: a set of statements that involve concepts
- concepts: helps understand the world and can be operationalized into mathematical expressions comprising
  - variables: take on different values in a given set (i.e. it can vary)
  - constants: take only one value for a given set (i.e. cannot vary)

what is a set?

A set is a collection of elements or members

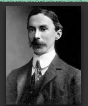
- curly braces { } used to list elements separated by comma ("Roster Method")
- Ellipsis (…) used within the braces to indicate that list continues in established pattern
- Cardinality of a set: the number of distinct elements in a set

example
set A: the natural numbers from 1 to 7
elements of A: 1,2,3,4,5,6,7
set notation: $A = \{1,2,3,4,5\} = \{1,2,3,\ldots,7\}$
cardinality: $|A| = 7$

What is a set?

- difficult to formally define sets: *what is the set of all sets?*

Russell's Paradox
Suppose a town's barber shaves every man who doesn't shave himself.
Who shaves the Barber?
Consider the set S of all sets which do not contain themselves.
Does S contain itself?



set notation

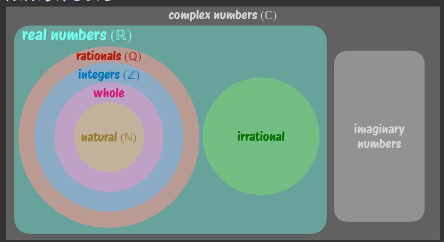
- To say an element belongs to a set we use a "funky E": $\in$
- $A \subseteq B$ or $B \supseteq A$ means set A is a subset of set B
- $A \subset B$ means that A is a proper subset of B

types of sets

- Finite/Infinite
- Countable/Uncountable
- Bounded/Unbounded
- Singleton
- Tuple
- Empty
- Universal
- Ordered/Unordered

...more on this in your tutorial

common sets



$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

basic operators

- addition $+$
- subtraction $-$
- multiplication $\times$
- division. $\div$
- exponentiation x^a
- *n*th root $\sqrt[n]{x}$
- factorial $!$
- sum $\sum_i x_i$
- product $\prod_i x_i$

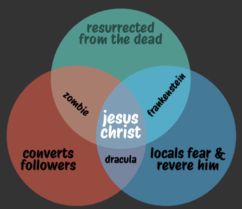
set operators

- difference $A \backslash B$
- complement A' or A^c or $\bar{A}$ or $\neg A$
- intersection $A \cap B$
- union $A \cup B$
- mutually exclusive $A \cup B = \varnothing$
- Cartesian product. $A \times B = \{(a,b) \,|\, a \in A, b \in B\}$
- symmetric difference $A \oplus B = (A - B) \cup (B - A)$
- partition: collection of subsets whose union forms the set

...more on this in your tutorial

Venn diagrams

popular "thanks" to social media but often used incorrectly



set operators with Venn diagrams

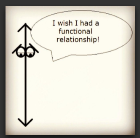
cardinality of the set union

$$|A \cup B| = |A| + |B| - |A \cap B|$$

relations and functions

used to compare concepts and uncover relationships between them

- a relation is a relationship between sets of information
- a function is a well-behaved relation



more on this in lecture 3...

mathematical proofs

a **proof** is an argument that demonstrates why a conclusion is true, subject to certain standards of truths

a **mathematical proof** is an argument that demonstrates why a statement is true, following the rules of mathematics

direct proofs

- proof by deduction
- proof by induction
- proof by exhaustion
- proof by construction

indirect proofs

- proof by contradiction
- proof by contrapositive

our first proof (by construction)

Theorem
For all integers n, if n is even, then n^2 is even.

- Find the **formal definitions** for any terms in the theorem:
  - an integer n is called **even** if there is an integer k where $n = 2k$
  - an integer n is called **odd** if there is an integer k where $n = 2k + 1$
- What is the grammatical structure of the theorem?
  - **For all** integers n, **if** n is even, **then** n^2 is even.

our first proof (by construction)

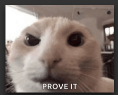
Theorem
For all integers n, if n is even, then n^2 is even.

- Pick some arbitrary even integer n and try some examples:
  - $2^2 = 4 = 2 \times 2$
  - $10^2 = 100 = 2 \times 50$
  - $0^2 = 0 = 2 \times 0$
  - $(-8)^2 = 64 = 2 \times 32$
  - $n^2 = \quad = 2 \times ?$

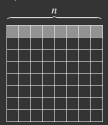
what's the pattern? can we predict this?

our first proof (by construction)

Theorem
For all integers n, if n is even, then n^2 is even.



- If possible, it's helpful to draw some pics



- an integer n is called **even** if there is an integer k where $n = 2k$

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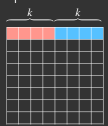

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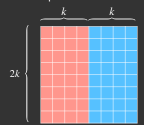


$n^2 = 2(2k^2)$

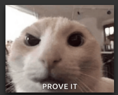
our first proof (by construction)

Theorem
For all integers n, if n is even, then n^2 is even.

Proof.

- Pick an arbitrary even integer n: we want to show that n^2 is even
- Since n is even, there is some integer such that $n = 2k$
- This means that $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$
- From this we see that there is an integer m (namely $2k^2$) where $n^2 = 2m$
- Therefore n^2 is even, which is what we wanted to show. ■

our first proof (by construction)



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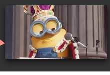
end of proof
"drop the mic"



let's try another

Theorem
For all integers m and n, if m and n are odd, then $m+n$ is even.

- Find the **formal definitions** for any terms in the theorem:
  - an integer n is called **even** if there is an integer k where $n = 2k$
  - an integer n is called **odd** if there is an integer k where $n = 2k + 1$
- What is the grammatical structure of the theorem?
  - **For all** integers m and n, **if** m and n are odd, **then** $m+n$ is even.

let's try another

Theorem
For all integers m and n, if m and n are odd, then $m+n$ is even.

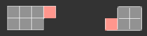
- Visual intuition

- an integer n is called odd if there is an integer k where $n = 2k + 1$

let's try another

Theorem
For all integers m and n, if m and n are odd, then $m+n$ is even.

- Visual intuition

$2k+1$  $2r+1$

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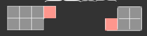
let's try another

Theorem
For all integers m and n, if m and n are odd, then $m+n$ is even.

- Visual intuition

$\overbrace{\quad}^{k}\overbrace{\ }^{1}\overbrace{\quad}^{r}$

$2k+1$  $2r+1$

$$\begin{array}{ccccc} (2k+1) & + & (2r+1) & = & 2(k+r+1) \\ m & + & n & = & 2(\quad s \quad) \end{array}$$

- an integer n is called **even** if there is an integer k where $n = 2k$

exercise: finish writing this proof by yourself

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the principle of mathematical induction

everybody do the wave!

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the principle of mathematical induction

let P be some predicate

$\boxed{\text{If } P(0) \text{ is true and } \forall k \in N\ P(k) \rightarrow P(k+1)\text{, then } \forall n \in N\ P(n)}$

if it starts true ***and it stays true*** ***then it's always true***

- it is true for 0
- since it's true for 0, it's true for 1
- since it's true for 1, it's true for 2
- since it's true for 2, it's true for 3
- since it's true for 3, it's true for 4

$\vdots$

proof by induction

- use the principle of mathematical induction to show that some result is true for all natural numbers n
- the proof, step by step:
  1. **The base case:** prove that $P(0)$ is true
  2. **Inductive step:** prove that if $P(k)$ is true then $P(k+1)$ is true
  3. Conclude by induction that $P(n)$ is true for all $n \in \mathbb{N}$

proof by induction

Theorem
The sum of the first n powers of two is $2^n - 1$.

Proof.

- Let $P(n)$ be the statement "the sum of the first n powers of two is $2^n - 1$."
- We prove by induction, that $P(n)$ is true for all $n \in \mathbb{N}$ from which the theorem follows
- The base case:
  - we need to show $P(0)$ is true, meaning that the sum of the first zero powers of two is $P^0 - 1$.
  - since the sum of the first zero powers of two is zero and $2^0 - 1 = 0$, we see that $P(0)$ is true. ✓

proof by induction

Theorem
The sum of the first n powers of two is $2^n - 1$.

Proof cont'd.

- The inductive step:
  - the goal here is to prove "if $P(k)$ then $P(k+1)$ is true"
  - to do this we choose an arbitrary k, assume $P(k)$ is true, then try to prove $P(k+1)$

    $\Longrightarrow$ assume that for some arbitrary $k \in \mathbb{N}$ that $P(k)$ holds, meaning that

    $$2^0 + 2^1 + \cdots + 2^{k-1} = 2^k - 1$$
  - we need to show that $P(k+1)$ holds, meaning the sum of the first $k+1$ powers of two is $2^{k+1} - 1$

    $$\begin{aligned} 2^0 + 2^1 + \cdots + 2^{k-1} + 2^k &= 2^k - 1 + 2^k \\ &= 2(2^k) - 1 \\ &= 2^{k+1} - 1 \ ✓ \end{aligned}$$
- Therefore, $P(k+1)$ is true, completing the induction. $\blacksquare$

indirect proofs

- Proof by contrapositive

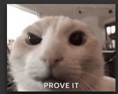
  to prove the statement
  "if P is true, then Q is true"
  you instead prove the equivalent statement
  "if Q is false, then P is false"

- Proof by contradiction

  to prove the statement
  "if P is true, then Q is true"
  and show that the following is not possible
  "if P is true, then Q is false"

- Proof by counterexample (not technically a proof)

indirect proofs: proof by contrapositive



Theorem
For any $n \in \mathbb{Z}$, if n^2 is even, then n is even.

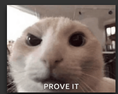
Proof.

- By contrapositive; we prove that if n is odd, then n^2 is odd
- Let n be an arbitrary odd integer.
- Since n is odd, there is some integer k such that $n = 2k + 1$.
- Squaring both sides of this equality and simplifying yields the following:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

- From this we see that there is an integer m (namely $2k^2 + 2k$) such that $n^2 = 2m + 1$.
- Therefore n^2 is odd. ■

indirect proofs: proof by contradiction



Theorem
For any $n \in \mathbb{Z}\ n$, if n^2 is even, then n is even.

Proof.

- Assume for the sake of contradiction that n is an integer and that n^2 is even, but that n is odd.
- Since n is odd, there is some integer k such that $n = 2k + 1$.
- Squaring both sides of this equality and simplifying yields the following:

$$\begin{aligned} n^2 &= (2k+1)^2 \\ &= 4k^2 + 4k + 1 \\ &= 2(2k^2 + 2k) + 1 \end{aligned}$$

- This tells us that n^2 is odd, which is impossible, by assumption n^2 is even.
- We have a contradiction so our assumption is incorrect
  $\Longrightarrow$ if n is an integer and n^2 is even then n is also even. ■