Functions & Relations

Lecture 3

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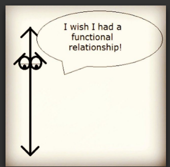
relations and functions

used to compare concepts and uncover relationships between them

- a <u>relation</u> is a relationship between sets of information
- a <u>function</u> is a well-behaved relation



all functions are relations
but not all relations are functions

relations and functions

example

Consider the set of fruits and the set of colors. We associate fruits with their colors, e.g.,

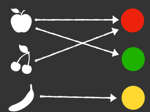
"apple may have the color red"

association

element of first set — element of second set

…so the following set of ordered pairs is a **<u>relation</u>**:

{(apple, red), (apple, green), (cherry, red), (banana, yellow)}



…but is this a function?

relations

- a relation R from the set A to the set B is a subset $A \times B$
  - relation R consists of ordered pairs (a, b) where $a \in A$ and $b \in B$
  - we say ***is related to*** and can write $a\,R\,b$

can be replaced by familiar symbols such as $<, \nless, >, \ngtr, =, \neq$

exercise 1

Suppose there are two sets A= {4, 36, 49, 50} and B = {1, -2, -6, -7, 7, 6, 2}
Define "*(a, b) is in the relation R if a is a square of b*"

exercise 2

Suppose there are two sets A= {1,2,3,4,5,6} and B = {1,2,3,4}
Define the relation $(a, b) \in R$ iff $(a - b) \mod 2 = 0.$

functions

input-output

we define the function f as $f(x) : A \rightarrow B$ which often is read as as *"f maps A into B"*

we assign this value to a variable y as in $y = f(x)$

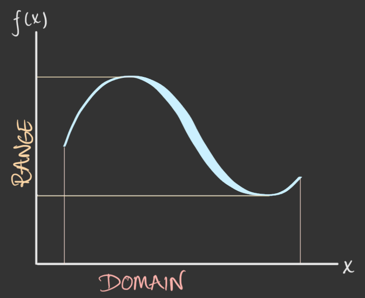
what is the rule we follow to obtain $f(x)$?

- computational
- table
- algorithm
- verbal or text
- ⋮

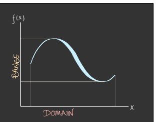
$$x \downarrow \boxed{\;f\;} \rightarrow f(x)$$

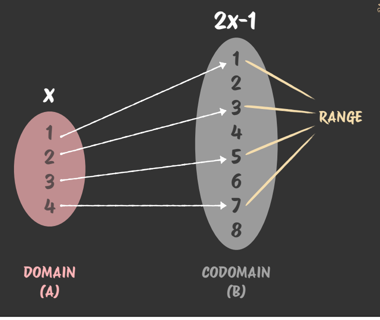
Note: a function must be single valued and can't give back 2 or more results for the same input
So *"$f(16) = 4$ or -4"* is not right!

domain, range and codomain



domain, range and codomain





visual representations of functions and relations

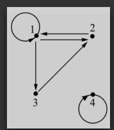
when domain or range of a relation is
- infinite $\Longrightarrow$ not possible to visualize the entire relation
- finite $\Longrightarrow$ lists and graphs can be used

example

let A be a small finite set: $A = \{1,2,3,4\}$

the relation R is defined as $R = \{(1,1), (4,4), (1,3), (3,2), (1,2), (2,1)\}$

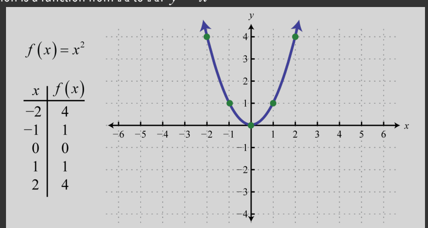
graph representation:



visual representations of functions and relations

example

Assume the relation is a function from $\mathbb{R}$ to $\mathbb{R}$: $y = x^2$

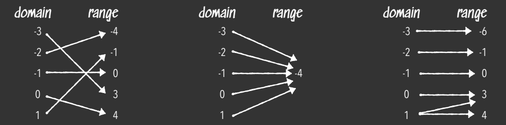


x	$f(x)$
-2	4
-1	1
0	0
1	1
2	4

Which points represent the relation $R = \{x, y) \in \mathbb{R} \times \mathbb{R} \mid y \geq x^2\}$?

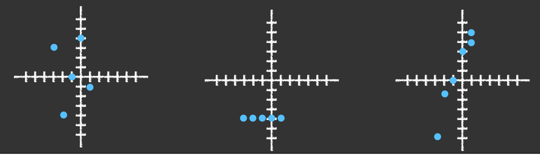
Is it bounded?

visual representations of functions and relations



are all of these functions?

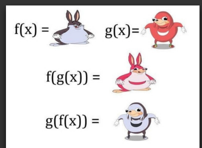
$\Longrightarrow$ *the vertical line test*



function composition

Let $f : A \to B$ and $g : C \to D$. The **composition** of g with f, denoted $g \circ f$, is the function from A to C defined by $g \circ f(x) = g(f(x))$.

- chaining multiple functions: *"g composed with f"*
- order matters!



function composition

example

Consider function decomposition of the following two functions

$f(x) = 2x + 3$

$g(x) = x^2$

Composition 1: $g \circ f$

1. First $f(x) = 2x + 3$
2. Then $g(f(x)) = (2x + 3)^2$

Composition 2: $f \circ g$

1. First $g(x) = x^2$
2. Then $f(g(x)) = 2(x^2) + 3 = 2x^2 + 3$

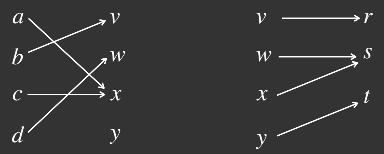
exercise 3

use specific value $x = 2$ in the example

function composition

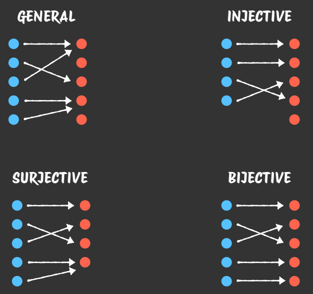
exercise 4

Let $A = \{a, b, c, d\}, B = \{v, w, x, y\}$ and $C = \{r, s, t\}$, and let $f : A \to B$ and $g : B \to C$ be defined as follows:

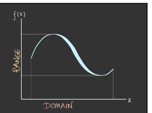


What is the composition $g \circ f$?

different classes of functions



different classes of functions



Let $f : A \to B$ be a function.

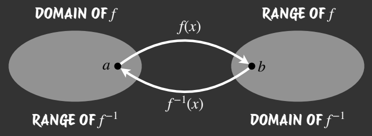
- The function f is said to be **injective** (or **one-to-one**) if for any $x, y \in A$, $f(x) = f(y)$ implies $x = y$.
  Or by contrapositive: $x \neq y$ implies $f(x) \neq f(y)$.
- The function f is said to be **surjective** (or **onto**) if range$(x) = B$.
- If f is both injective and surjective, we say that f is **bijective**.
  - a bijective function is invertible, and so has an inverse.

inverse functions

Suppose $f : A \to B$ is a bijection. Then the inverse of f, denoted

$$f^{-1} : B \to A$$

is the function defined by the rule $f^{-1}(y) = x$ if and only if $f(x) = y$



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inverse functions

example

Let $f(x) = 2x - 3$, then it's inverse is $f^{-1}(x) = \dfrac{x+3}{2}$.

We can check this both ways:

$$f^{-1}(f(x)) = \frac{(2x-3)+3}{2} = \frac{2x}{2} = x$$

$$f(f^{-1}(x)) = 2\left(\frac{x+3}{2}\right) - 3 = x + 3 - 3 = x$$

Since both compositions yield x, the functions are indeed inverses

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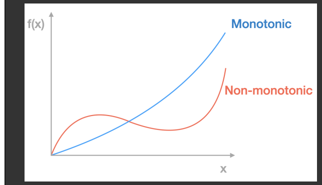
monotonic functions

Monotonicity is the characteristic of order preservation:
it preserves the order of elements from the domain in the range.



"strictly": <

A function is monotonic increasing if $f(x_1) \leq f(x_2)$ whenever $x_1 < x_2 \quad \forall x_1$ and $x_2 \in \mathbb{R}$

"strictly" >

A function is monotonic decreasing if $f(x_1) \geq f(x_2)$ whenever $x_1 > x_2 \quad \forall x_1$ and $x_2 \in \mathbb{R}$

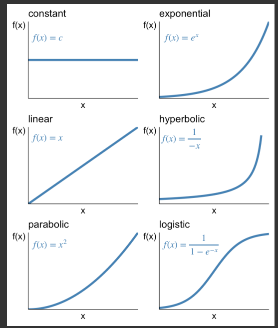
example

$f(x) = 2x + 3$ is monotonically increasing because for any two values x_1 and x_2, then $f(x_1) < f(x_2)$ always.

monotonic functions

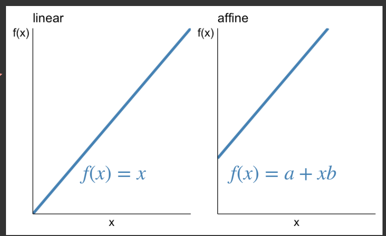
characteristics

- no local extrema
- continuity (when continuous)
- injective



linear function vs. linear equation

needs one point to be determined



needs two points to be determined

exercise 5

Which functions are the linear affine functions that have slope $m = 0$, and which are the ones with y-intercept $c = 0$?

identity function