Discrete Distributions

Lecture 9

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random variables

- Many random experiments have outcomes that are numerical

  For example:
  - number of people on a train
  - the time a customer will spend in line at the post office
  - the number of people voting for a candidate in a political election

- In random experiments with outcomes not numerical, we map outcomes to numerical values

  For example:
  - in a coin toss experiment : H → 1 and T → 0

A random variable associates a number with each outcome of a random experiment

random variables

Given an experiment and the sample space Ω, a random variable is a function mapping an outcome ($\omega \in \Omega$) into a real number, i.e.

$$X : \omega \in \Omega \to X(\omega) \in (-\infty, \infty)$$

- We use a capital letter X to denote a random variable
- The values of a random variable will be denoted with a lower case letter x
- The range of a random variable is the set of values it can take
- a function of a random variable is another mapping from the sample space to real numbers, so another random variable

random variables

example

Toss a coin 3 times: the sample space is Ω : {H,T} $\times$ {H,T} $\times$ {H,T}
Define the random variable: $X =$ the number of heads
What is the probability of each outcome of X?

Outcome (ω)	HHH	HTH	THH	HHT	HTT	THT	TTH	TTT
$X(\omega)$	3	2			1			0

$$P(X=3) = \frac{1}{8} \qquad P(X=3) = \frac{3}{8}$$

$$P(X=2) = \frac{3}{8} \qquad P(X=3) = \frac{1}{8}$$

random variables

exercise 1

Toss two dice, the sample space is given by $\Omega : \{1,2,3,4,5,6\} \times \{1,2,3,4,5,6\}$
Let the random variable X denote the sum of the two dice.
What is the probability of each outcome X?

	1	2	3	4	5	6
1						
2						
3						
4						
5						
6						

discrete random variables: probability mass function

A random variable is discrete if its range is a countable (finite or infinite) set.

If X is a discrete random variable, the function given by $f(x) = P(X = x)$ for each x within the range of X is called the probability distribution of X, also called probability mass function (pmf)

A function can serve as the probability distribution of a discrete random variable X if and only if its values, $f(x)$, satisfy the conditions:

- $f(x) \geq 0$ for each value within its domain
- $\sum_x f(x) = 1$ where the sum if over all the x values within its domain.

discrete random variables: probability mass function

example (cont'd...)

Toss a coin 3 times: the sample space is $\Omega : \{H,T\} \times \{H,T\} \times \{H,T\}$

Define the random variable: $X =$ the number of heads

What is the probability distribution of X?

Outcome (ω)	HHH	HTH	THH	HHT	HTT	THT	TTH	TTT
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discrete random variables: probability mass function

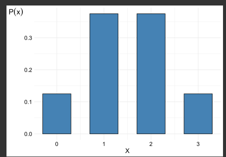
example (cont'd...)

Toss a coin 3 times: the sample space is $\Omega : \{H,T\} \times \{H,T\} \times \{H,T\}$

Define the random variable: $X =$ the number of heads

What is the probability distribution of X?

x	$f(x) = P(X=x)$
0	1/8
1	3/8
2	3/8
3	1/8



This function can be written as: $f(x) = \dfrac{4 - |3 - 2x|}{8}$

expected value

The expected value, is the (probability) weighted average of the possible outcomes

$$E(X) = \sum_x x \cdot P(X = x)$$

the center of gravity of the PMF

The expected value rule:

Let X be a random variable with PMF $f(x)$ and let $g(X)$ be a function of X. Then,

$$E[g(X)] = \sum_x g(x) \cdot f(x)$$

variance

The variance is given by

$$V(X) = E[(X - E(X))^2] = \sum_x (x - E(X))^2 \cdot P(X = x) = E[X^2] - E[X]^2$$

the standard deviation $\sqrt{V(X)}$ is usually easier to interpret

- The variance is always nonnegative
- We can find $V(X)$ by calculating the mean of $Z = (X - E[X])^2$ via the expected value rule
- When computing the variance often we use a different (equivalent) form of the variance equation:

$$V(X) = E[X^2] - E[X]^2$$

exercise 2

Prove this

expected value and variance

exercise 3

Toss a coin 3 times. Define the random variable: $X =$ the number of heads

What is the expected value and variance of X?

X	$f(x) = P(X = x)$
0	1/8
1	3/8
2	3/8
3	1/8

cumulative distribution function

If X is a discrete random variable, the function given by

$$F(x) = P(X \leq \leq x) = \sum_{t \leq x} f(t)$$

for $-\infty < x < \infty$ is the cumulative distribution of X

The values $F(x)$ of the cumulative distribution of a discrete random variable X satisfies the conditions:

- $f(-\infty) = 0$ and $f(\infty) = 1$
- If $a < b$, then $F(a) \leq F(b)$ for any real numbers a and b

cumulative distribution function

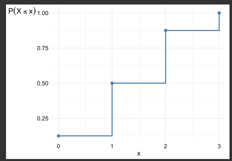
example (cont'd...)

Toss a coin 3 times: the sample space is Ω : {H,T} $\times$ {H,T} $\times$ {H,T}

Define the random variable: $X =$ the number of heads

What is the cdf of X?

x	$f(x) = P(X = x)$	$F(x) = P(X \leq x)$
0	1/8	1/8
1	3/8	4/8
2	3/8	7/8
3	1/8	8/8



exercise 4

What about the conditions for cdf, are they satisfied?

joint, marginal and conditional distributions

Contingency table based on relative frequencies

example

Suppose we are interested in the relationship between an individual's hair (X) and eye (Y) color.

X (columns), Y (rows)

$P(X,Y)$	blonde	red	brown	black	$\sum$
blue	0.12	0.05	0.12	0.01	0.30
green	0.12	0.07	0.09	0	0.28
brown	0.16	0.07	0.16	0.03	0.42
$\sum$	0.40	0.19	0.37	0.04	1.00

Bernoulli random variable

$$P(X = x \mid p) = \begin{cases} p, & \text{if } x = 1 \\ 1 - p, & \text{if } x = 0 \end{cases}$$

- A random variable for modeling binary events
- Two possible outcomes:
  - Success - value 1
  - Failure - value 0

X	$P(X = x)$
0	$1 - p$
1	p

- Single parameter p, probability of a success
- multiple Bernoulli r.v. can be combined to model more complex random variables
- Shorthand notation: $X \sim$ Bern(p)
- $E(X) = p,\ V(X) = p(1-p)$

geometric random variable

- A r.v. modeling the number of (identical) Bernoulli trials needed to obtain the first success
- Infinite outcomes $\{1,2,3\ldots,\infty\}$
- Single parameter p, probability of a success for each trial
- Shorthand notation: $X \sim$ Geo(p)
- $E(X) = \dfrac{1}{p}$, $V(X) = \dfrac{1-p}{(p^2)}$

$$P(X = x \,|\, p) = p(1-p)^{x-1}$$

X	$P(X = x)$
1	p
2	$p(1-p)$
3	$p(1-p)^2$
4	$p(1-p)^3$
$\vdots$	$\vdots$
∞	$p(1-p)^\infty \approx 0$

exercise 5

What is the probability of flipping a coin more than 4 times before getting a heads?

exercise 6

What is the expected number of rolls it will take to get a 7 when rolling two dice?

binomial random variable

- A r.v. modeling the number of successes in a fixed number of independent Bernoulli trials.
- Discrete outcomes $\{0,1,2,3\ldots,n\}$
- Two parameter
  - p - probability of a success for each trial
  - n - number of trials
- Shorthand notation: $X \sim$ Binom(n, p)
- $E(X) = np$, $V(X) = np(1-p)$

$$P(X = x \,|\, n, p) = \binom{n}{x} p^x (1-p)^{n-x}$$

X	$P(X = x)$
0	$\binom{n}{0} p^0(1-p)^n$
1	$\binom{n}{1} p^1(1-p)^{n-1}$
2	$\binom{n}{2} p^2(1-p)^{n-2}$
$\vdots$	$\vdots$
n-1	$\binom{n}{(n-1)} p^{n-1}(1-p)^1$
n	$\binom{n}{n} p^n(1-p)^0$

binomial random variable

example (cont'd...)

Toss a coin 3 times: the sample space is Ω : {H,T} $\times$ {H,T} $\times$ {H,T}

Define the random variable: $X =$ the number of heads

What is the probability distribution of X?

$X \sim Bin(n = 3, p = 0.5)$

$$\implies P(X = x) = \binom{x}{n} p^x (1-p)^{n-x} = \binom{x}{3} 0.5^x (0.5)^{3-x}$$

Poisson random variable

- A r.v. that expresses the probability of how many times an event occurs in a fixed period of time if these events
  - occur with known average rate of λ
  - and independently of each other
- Discrete outcomes $\{0,1,2,3\ldots\}$
- Shorthand notation: $X \sim$ Poisson(λ)
- $E(X) = V(X) = \lambda$
- If the data shows overdispersion (variance > mean) or underdispersion (variance < mean), other models like the Negative Binomial

$$P(X = x \,|\, \lambda) = e^{-\lambda}\frac{\lambda^x}{x!}$$

X	$P(X=x)$
0	$e^{-\lambda}$
1	$e^{-\lambda}\lambda$
2	$e^{-\lambda}\frac{\lambda^2}{2}$
$\vdots$	$\vdots$

negative binomial random variable

- A generalization of the geometric distribution Pascal(1,p)=Geometric(p)
- It relates to the random experiment of repeated independent trials until observing r successes
- Discrete outcomes $\{1,2,3\ldots\}$
- Two parameter
  - the number of success we are waiting for
  - the probability that a single experiment gives a "success"
- Shorthand notation: $X \sim$ NegBin(r, p)
- $E(X) = \frac{r}{p}$, $V(X) = \frac{r(1-p)}{p^2}$

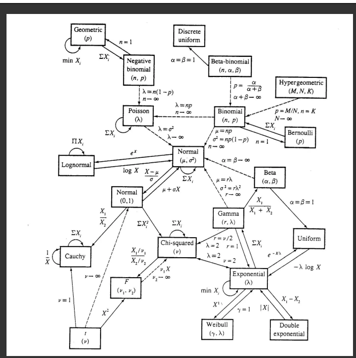
$$P(X = x \,|\, r, p) = \binom{x-1}{r-1} p^r (1-p)^{x-r}$$

negative binomial random variable

exercise 7

On a (American) roulette wheel, there are 38 spaces: 18 black, 18 red, and 2 green. You've been at the casino for a while now and decide to leave after you have won 3 bets on red. What is the probability that you leave the casino after placing exactly 5 bets on red?

so much more...



https://www.math.wm.edu/~leemis/chart/UDR/UDR.html