Sequences & Series
Limits & Continuity
Lecture 4

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sequences

- A sequence is an list of numbers in a certain order

$$\{a_1, a_2, a_3, \ldots\}$$

a_1 is first term
a_2 is first term
a_3 is first term
etc.

- The sequence may be infinite
- The n-th term of the sequence is the n-th number on the list and given by $a_n = n$
- Some sequences have patterns, some do not

example
We roll a die repeatedly and generate a sequence of numbers which have no pattern.

example
The following two sequences have patterns:

$$\{1, 2, 3, 4, 5, 6, \ldots\}$$
$$\{-1, 1, -1, 1, -1, \ldots\}$$

sequences

- Sometime, we can use an algebraic expression for the n-th term of a sequence
- Factorials are commonly used in sequences $n! = n(n-1)(n-2)(n-3)\cdots 1$

example
The n-th term of the sequences $\{1, 2, 3, 4, 5, 6, \ldots\}$ is given by $a_n = n$
The n-th term of the sequences $\{-1, 1, -1, 1, -1, \ldots\}$ is given by $a_n = (-1)^{n+1}$

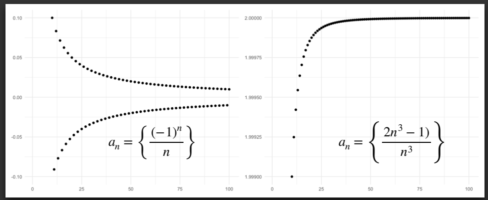
exercise 1
Find a formula for the n-th term in the following sequence $\left\{\frac{-1}{2}, \frac{1}{3}, \frac{-1}{4}, \frac{1}{5}, \frac{-1}{6}, \ldots\right\}$

exercise 2
Find a formula for the n-th term in the following sequence $\left\{\frac{2}{1}, \frac{4}{2}, \frac{8}{6}, \frac{16}{24}, \frac{32}{120}, \ldots\right\}$

sequences

- A sequence is a function from the positive integers to the real numbers, with $f(n) = a_n$
- We can draw a graph of this function as a set of points in the plane:

$$(1,a_1),\ (2,a_2),\ (3,a_3),\ \ldots,\ (n,a_n),\ldots$$



limits of a sequence

A sequence $\{a_n\}$ has limit L if we can make the terms an as close as we like to L by taking n sufficiently large. We denote this by

$$\lim_{n\to\infty} a_n = L$$

or

$$a_n \to L \text{ as } n \to \infty$$

- If $\lim_{n\to\infty} a_n$ exists (is finite), then the series converges, otherwise it diverges
- Graphically:
  If $\lim_{n\to\infty} a_n = L$ the graph of the sequence $\{a_n\}_{n=1}^{\infty}$ has a unique horizontal asymptote $y = L$

limits of a sequence

How do find the limits? Sequences are really just functions of the integers n…

Theorem

If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$, where n is an integer, then $\lim_{n\to\infty} a(n) = L$.

exercise 3

Determine if the following two sequences converge or diverge:

(a) $\left\{\frac{2^n-1}{2^n}\right\}_{n=1}^{\infty}$

(b) $\left\{\frac{2n^3-1}{n^3}\right\}_{n=1}^{\infty}$

rules of limits

If $\{a_n\}$ and $\{b_n\}$ are convergent sequences and c is any constant then

$$\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} ca_n = c \lim_{n\to\infty} a_n$$

$$\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty}(a_n b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} c = c$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$$

series

- A series is the sum of a sequence
- Finite series you have (probably encountered before)
- Infinite series are infinite sums

examples

$$\sum_{i=1}^{n} a_i$$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

series

Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_2 + \cdots$ we let s_n denote its n-th partial sum

$$s_n = a_1 + a_2 + a_3 + \cdots + a_n$$

If the sequence s_n is convergent and $\lim_{n\to\infty} s_n = S$ then the series $\sum_{n=1}^{\infty} a_n$ is convergent

and we let

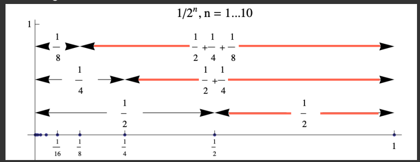
$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_n = \lim_{n\to\infty} s_n = S \longleftarrow \text{sum of the series}$$

Otherwise the series is divergent

determine convergence/divergence using limit of s_n

example

Find the partial sums $s_1, s_2, s_3, \dots, s_n$ of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Find the sum of the series.
Does the series converge?

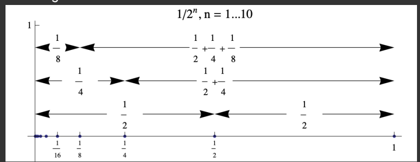


$s_1 = \frac{1}{2}$ $\quad s_2 = \frac{1}{2} + \frac{1}{4}$ $\quad s_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$ $\quad \cdots$

determine convergence/divergence using limit if s_n

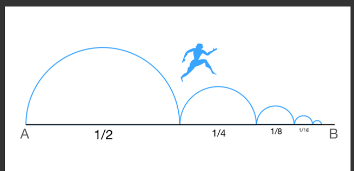
example

Find the partial sums $s_1, s_2, s_3, \dots, s_n$ of the series $\sum_{n=1}^{\infty} \frac{1}{2^n}$. Find the sum of the series.
Does the series converge?



$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{2^n}\right) = S = 1 \implies \text{converges}$$

Zeno's Paradox



- Consider a runner who is to complete a course from point A to point B.
- Imagine that the runner completes half the distance from A to B,
  and then completes half the remaining distance,
  and again half the remaining distance, and so on…
- Will the runner ever reach point B?

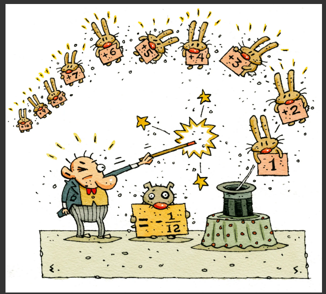
determine convergence/divergence using limit if s_n

exercise 4

Does the series $\sum_{n=1}^{\infty} n$ converge?

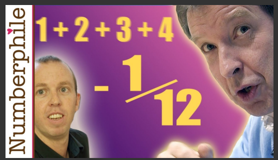
Hint: start by considering the partial sum of the first n natural numbers

in the end, it all adds up to – 1/12

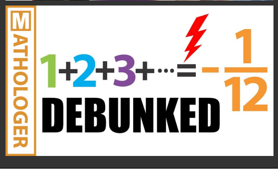


https://www.nytimes.com/2014/02/04/science/in-the-end-it-all-adds-up-to.html

in the end, it all adds up to – 1/12



…or does it?



geometric series

a geometric series is a series summing the terms of an infinite geometric sequence, in which the ratio of consecutive terms is constant

The geometric a series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{r-1} \qquad |r| < 1$$

If $|r| > 1$ the series is divergent.

geometric series

example

Find the sum of the series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 10}{4^{n-1}}$

- We identify the values of a and r
- The series expands to

first term $= a$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 10}{4^{n-1}} = -10 + \frac{10}{4} - \frac{10}{16} + \cdots$$

second term $= ar \implies r = \frac{\text{term 2}}{\text{term 1}} = \frac{10/4}{-10} = \frac{-1}{4}$

- Double check: $ar^{n-1} = \dfrac{(-1)^n 10}{4^{n-1}}$
- Since $|r| < 1 \implies \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 10}{4^{n-1}} = \frac{a}{1-r} = \frac{-10}{1 - \frac{(-1)}{4}} = -8$

harmonic series

the harmonic series is the infinite series formed by summing all positive unit fractions

The harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (does not have a finite limit).

Proof by contradiction.

- Suppose the series converges to S: $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \cdots$
- Then: $\frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2n} + \cdots$
- Therefore, the sum of the odd-numbered terms: $1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} + \cdots$ must be the other half of S
- However this is impossible since $\frac{1}{2n-1} > \frac{1}{2n}$ for each positive integer n. ■

limit of a function

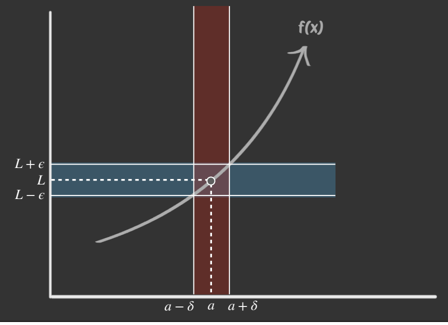
Let $f(x)$ be a function defined on some open interval that contains a, except possibly at a itself. We say that the limit of $f(x)$ as x approaches a is L, and we write:

$$\lim_{x \to a} f(x) = L$$

if, for every number $\epsilon > 0$ there exists a number $\delta > 0$ such that whenever $0 < |x - a| < \delta$, it follows that $|f(x) - L| < \epsilon$.

- ϵ: this represents how close we want $f(x)$ to be to L. We can choose ϵ to be any small positive number, indicating the "closeness" level we desire
- δ: this represents how close x needs to be to a in order for $f(x)$ to be within ϵ of L

limit of a function



continuity

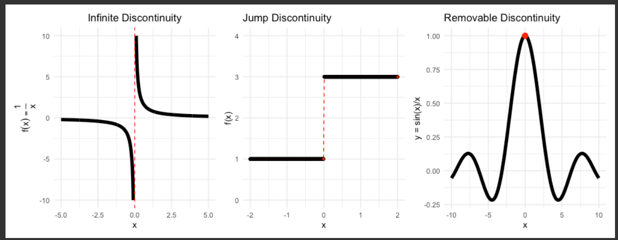
A function $f(x)$ is continuous at a point x_0 if the limit exists at x_0 and is equal to $f(x_0)$

Continuity test:

A function is continuous at $f(x)$ if it satisfies the following conditions:

1. $f(x)$ is defined at c, i.e. $f(c)$ exists
2. $f(x)$ approaches the same function value to the left and right of c, i.e. $\lim_{x \to c} f(x)$ exists
3. The function value that $f(x)$ approaches from each side of c is $f(c)$, i.e. $\lim_{x \to c} f(x) = f(c)$

discontinuity



discontinuity

exercise 5

Determine whether each function is continuous at the given x-values.
Justify using the continuity test. If discontinuous, identify the type of discontinuity.

(a) $f(x) = 3x^2 + x - 7$ at $x = 1$

(b) $f(x) = \dfrac{|2x|}{x}$ at $x = 0$

(c) $f(x) = \dfrac{x^2 - 4}{x + 2}$ at $x = -2$

(d) $f(x) = \dfrac{1}{3x^2}$ at $x = 0$

plenty more limit and continuity exercises in your tutorial...

continuity and differentiability

Differentiability $\Rightarrow$ Continuity

but

Continuity $\not\Rightarrow$ Differentiability