Introduction to Probability

Lecture 8

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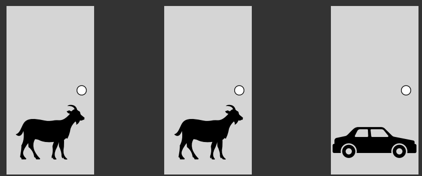
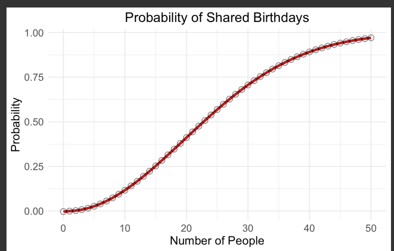
probabilities are rarely intuitive

example 1: Monty Hall



probabilities are rarely intuitive

example 2: happy birthday you two!



what is probability?

Frequentist interpretation of the probability

- probability as the long-run fraction of time that it would happen if the random process occurs over and over again under the same conditions
- many interesting random phenomena cannot be repeated over and over again, e.g., weather

Bayesian interpretation of the probability

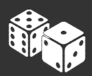
- probability as a subjective degree of belief: for the same event, two separate people could have different viewpoints and therefore assign different probabilities

> "Probability is orderly opinion... inference from data is nothing other than the revision of such opinion in the light of relevant new information."
> – Thomas Bayes (1701-1761)

both interpretations agree on the probability rules introduced!

terminology for probability theory

- experiment
  process of observation or measurement
- outcome
  result obtained through an experiments
- sample space
  set of all possible outcomes of an experiment
  - finite
  - infinite
- Events
  a subset of a sample space (what we are interested in)

terminology for probability theory

example

Flip a coin 3 times and record the side facing up each times

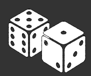
Sample space S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Events:

{all heads} = {HHH}
{get exactly one heads} = {HTT, THT, TTH}
{get at least two heads} = {HHT, HTH, THH, HHH}

exercise 1

Roll two dice, each with numbers 1–6.
Describe the event of rolling a total of 7 with the two dice.

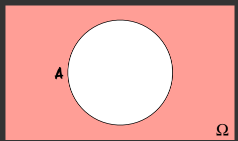


the algebra of events

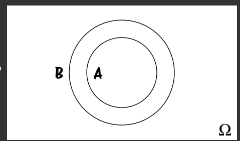
- Often we are interested in combinations of two or more events
- Events are sets (i.e. subsets of the sample space Ω) so we can do the usual set operations
- Assume sample space with two events A and B
  - complement $\overline{A}$ (also denoted A^c or A')
    all elements of S that are not in A
  - subset $A \subset B$
    all elements of A are also elements of B
  - union $A \cup B$
    all elements of Ω that are in A or B
  - intersection $A \cap B$
    all elements of Ω that are in A and B
- These operations can be represented graphically using Venn diagrams

venn diagrams

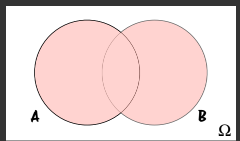
$\overline{A}$



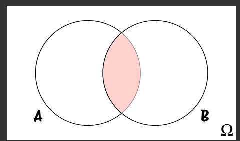
$A \subset B$



$A \cup B$



$A \cap B$



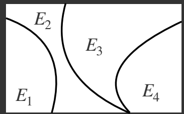
union and intersection: operator rules

Let E_1, E_2, E_3 denote events in Ω

- Commutative
  $E_1 \cup E_2 = E_2 \cup E_1$
  $E_1 \cap E_2 = E_2 \cap E_1$
- Associative
  $(E_1 \cup E_2) \cup E_3 = E_1 \cup (E_2 \cup E_3)$
  $(E_1 \cap E_2) \cap E_3 = E_1 \cap (E_2 \cap E_3)$
- Distributive
  $(E_1 \cup E_2) \cap E_3 = (E_1 \cap E_3) \cup (E_2 \cap E_3)$
  $(E_1 \cap E_2) \cup E_3 = (E_1 \cup E_3) \cap (E_2 \cup E_3)$

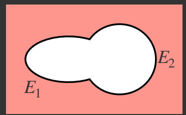
the complement

MECE = mutually exclusive and collectively exhaustive events

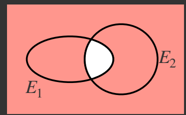


de Morgan's Rules

$\overline{E_1 \cup E_2} = \overline{E_1} \cap \overline{E_2}$



$\overline{E_1 \cap E_2} = \overline{E_1} \cup \overline{E_2}$



axioms of probability

1. The probability of an event is a nonnegative real number $P(A) \geq 0$ for any $A \subset \Omega$
2. $P(\Omega) = 1$ (also denoted $P(S) = 1$)
3. If $A_1, A_2, A_3, \ldots$ is a sequence of **mutually exclusive events** of Ω, then:

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots = P(A_1) + P(A_2) + P(A_3) + \cdots$$

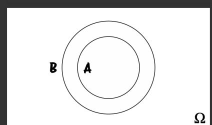
further properties

- $P(\varnothing) = 0$

  Proof:
  $1 = P(\Omega) + P(\Omega^c)$
  $1 = 1 + P(\varnothing) \implies P(\varnothing) = 0$
  Also evident from set theory:
  $\Omega \cup \varnothing = \Omega \implies P(\Omega) + P(\varnothing) = P(\Omega) \implies P(\Omega) = 0$

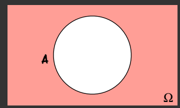
- if $A \subset B$ then $P(A) \leq P(B)$

  Proof:
  $B = A \cup (B \cap \overline{A})$
  $\implies P(B) = P(A) + P(B \cap \overline{A}) \geq P(A)$



further properties

- $P(A \cup \overline{A}) = P(A) + P(\overline{A}) = 1 \implies P(\overline{A}) = 1 - P(A)$
  (this is also referred to as the complement rule coming up shortly…)



- $0 \leq P(A) \leq 1$ for any event A
  Directly follows from axiom (1) and (2).
  Also directly evident from set theory:
  $\varnothing \subset A \subset \Omega$ for any event $A \implies P(\varnothing) \leq P(A) \leq P(\Omega) \implies 0 \leq A \leq 1$

probability of an event

If A is an event in a discrete sample space A and $O_1, O_2, O_3, \ldots$ are the individual outcomes comprising A, then $P(A) = P(O_1) + P(O_2) + P(O_3) + \ldots$

example
We flip a fair coin twice. What's the probability of obtaining at least one head?
The sample space $S = \{HH, HT, TH, TT\}$
As the coin is fair, all outcomes are equally likely: $P(HH) = P(HT) = P(TH) = P(TT) = \frac{1}{4}$
The event of obtaining at least one head is $A = \{HH, HT, TH\}$

$$\implies P(A) = P(HH) + P(HT) + P(TH) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = \frac{3}{4}$$

probability of an event

exercise 2
Roll two dice, each with numbers 1–6. Let X denote the first roll and Y the second roll.

(a) Find the probability $P(X = 1)$

(b) Let $Z = \min(X, Y)$ and find the probability $P(Z = 6)$

(c) Let $Z = \min(X, Y)$ and find the probability $P(Z = 3)$

6						
5						
4						
3						
2						
1						
	1	2	3	4	5	6

equally likely outcomes

If an experiment can result in N equally likely outcomes, and if n of these outcomes constitute an event A, then $P(A) = \frac{n}{N}$

This theorem is consistent with the frequency interpretation of probability theory: probability of an event is the proportion of the time events of the same kind occur in the long run.

exercise 3

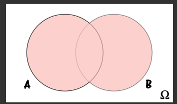
Assume all letters occur equally often in English. Then what's the probability of a three-letter word only consisting of vowels?

probability rules

The Addition Rule

If A and B are events in the sample space Ω, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Proof:

Let $a = P(A \cap \overline{B}), b = P(A \cap B), c = P(B \cap \overline{A})$

$P(A \cup B) = a + b + c$

$P(A) + P(B) - P(A \cap B) = [a + b] + [b + c] - b = a + b + c$

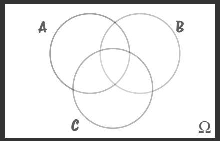
probability rules

Inclusion-Exclusion Rule

For three events $A,\ B,\ C$

$P(A \cup B \cup C) = P(A) + P(\overline{A} \cap B) + P(\overline{A} \cap \overline{B} \cap C)$

$$\begin{aligned} P(A \cup B \cup C) = P(A) &+ P(B) + P(C) \\ &- P(A \cap B) - P(A \cap C) - P(B \cup C) \\ &+ P(A \cap B \cap C) \end{aligned}$$



probability rules

Conditional Probability Rule

If A and B are events in the sample space Ω, then the conditional probability of A given B where $P(B) > 0$ is given by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$\Longrightarrow$ The Multiplication Rule

$$P(A \cap B) = P(A \mid B)P(B)$$

since $A \cap B = B \cap A$

$$\Longrightarrow \quad P(A \cap B) = P(B)P(A \mid B)$$

Independent Events and Their Complement

Two events A and B are independent if and only if

$$P(A \cap B) = P(A)P(B)$$

Two events A and B are independent then A and $\overline{B}$ are also independent.

probability rules

Complement Rule

If A be an event in the sample space Ω, then the probability of its complement is given by

$$P(\overline{A}) = 1 - P(A)$$

exercise 4

What is the probability of at least one head (H) in four tosses of a coin?

probability rules

Rule of Total Probability

If events $A_1, A_2, \ldots, A_k$ constitute a partition of the sample space Ω and $P(A_i) \neq 0 \ \forall i$, then for any event B in Ω

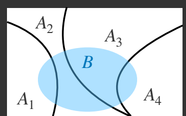
$$P(B) = \sum_{i=1}^{k} P(A_i)P(B \mid A_i)$$

Proof: $P(B) = P(B \cap (A_1 \cup A_2 \cup \cdots \cup A_k))$

$= P((B \cap A_1) \cup (B \cap A_2) \cup \cdots \cup (B \cap A_k))$ {events collectively exhaustive}

$= \sum_{i=1}^{k} P(B \cap A_i)$ {mutually exclusive}

$= \sum_{i=1}^{k} P(A_i)P(B \mid A_i)$ ■



so this rule only applies to MECE events!

probability rules

Bayes Rule

If events $A_1, A_2, \ldots, A_k$ constitute a partition of the sample space Ω and $P(A_i) \neq 0\ \forall i$, then for any event B in Ω such that $P(B) \neq 0$

$$P(A_i \mid B) = \frac{P(A_i)P(B \mid A_i)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \cdots + P(A_k)P(B \mid A_k)}$$

$$= \frac{P(A_i)P(B \mid A_i)}{\sum_{j=1}^{k} P(A_j \mid B)P(A_j)} \quad \text{(rule of total probability)}$$

$$= \frac{P(A_i)P(B \mid A_i)}{P(B)}$$

This theorem is consistent with the Bayesian interpretation of probability theory

probability rules

exercise 5

In an experiment on human memory, participants have to memorize a set of words (B_1), numbers (B_2), and pictures (B_3). These occur in the experiment with the probabilities $P(B_1) = 0.5,\ P(B_2) = 0.4,\ P(B_3) = 0.1$.

Then participants have to recall the items (where A is the recall event). The results show that $P(A \mid B_1) = 0.4,\ P(A \mid B_2) = 0.2,\ P(A \mid B_3) = 0.1$.

(a) Compute $P(A)$, the probability of recalling an item.

(b) What is the probability that an item that is correctly recalled (A) is a picture (B_3)?

counting outcomes

A permutation of items is an arrangement of the items in a certain order, where each item can be used only once in the sequence: $n! = n(n-1)(n-2)\cdots(2)(1)$

A permutation of n items taken k at of time is the number of ways to select k items from n distinct items and arranging them in order:

$$P(n,k) = \frac{n!}{(n-k)!}$$

A combination of n items taken k at of time any selection of k items from n elements where order is not important:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

counting outcomes

example

In a group of six men and four women, I select a committee of three at random. What is the probability that all three committee numbers are women?

The number of ways to select a three-women committee from four women: $\binom{4}{3} = \frac{3!}{3!(4-3)!} = 4$

The number of ways to select a three-person committee from the 10 people (the total number of outcomes):

$$\binom{10}{3} = \frac{10!}{10!(10-3)!} = 120$$

Probability of all female committee: $\frac{4}{120} = \frac{1}{30} = 0.03$

odds

- The odds of an event are the ratio of how likely the event is to occur and how likely it is to not occur.
- Let p denote probability that an event occurs, its complement that it doesn't occur is then $(1-p)$

$$\text{odds} = \frac{p}{1-p}$$

- Relationship odds and probability:
  - the odds are greater (less) than 1 if and only if the probability is greater (less) than 0.5
  - The odds are exactly 1 if and only if the probability is 0.5.

example

Let's say you attend a meeting of five people, including yourself. You each write your name on a piece of paper for a randomly drawn door prize. Your chance, or probability, of winning the prize is 1/5 (0.20). Your odds of winning, however, are 1 to 4 (1:4). There is one piece of paper with your name and four without, so you have one chance of winning and four chances to lose.

odds ratio and relative risk

- The odds ratio (OR) compares the odds of an event between two different groups.

$$\text{OR} = \frac{\frac{p_1}{1-p_1}}{\frac{p_2}{1-p_2}}$$

- OR > 1 (< 1) implies first event (numerator) has greater (smaller) risk of occurring
- An OR of 1 implies risk is equal for both events

example

For example, suppose the probability of disease is 0.35 for men and 0.25 for women. The OR is:

$$\frac{0.35/(1-0.35)}{0.25/(1-0.25)} = \frac{0.54}{0.33} = 1.62$$

How do we interpret this?