# Continuous Distributions

## Lecture 10

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## recall from last lecture: discrete random variables

A random variable is discrete if its range is a countable (finite or infinite) set.

If $X$ is a discrete random variable, the function given by $f(x) = P(X = x)$ for each $x$ within the range of $X$ is called the probability distribution of $X$, also called probability mass function (pmf)

the probability of an event $A$ associated with a discrete random variable $X$ is found by summing up its probability mass function over the values in that set: $P(X \in A) = \sum_{x \in A} f(x)$

this is not feasible when finding the probability of an event $A$ associated with a continuous random variable $X$

## continuous random variables

A continuous random variable is one that takes values over a continuous range: the whole real line; an interval on the real line, perhaps infinite; or a disjoint union of such intervals.

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A continuous random variable $X$ must have the property that no possible value has positive probability: $P(X = x) = 0$ for all $x \in \mathbb{R}$

## probability density function

A random variable $X$ is continuous if there is a nonnegative function $f(x)$, called the probability density function (pdf) of $X$, such that

$$P(X \in A) = \int_A f(x)dx$$

for every subset of $A$ of the real line. Specifically, the probability that $X$ is in an interval is

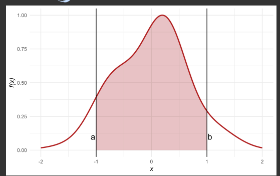
$$P(a \leq X \leq b) = \int_a^b f(x)dx$$

For any PDF we know that $f(x) \geq 0$ for all values of $x$ and the total area under the whole graph is 1

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

Note: $P(a \leq X \leq b) = P(a < X \leq b) = P(a \leq X < b) = P(a < x < b)$

## probability density function




For $f(x)$ to be a legitimate pdf, it must satisfy the following two conditions:

1. $f(x) \geq 0$ for all values of $x$
2. $\int_{-\infty}^{\infty} f(x)dx = 1$ i.e. area under the entire graph of $f(x) = 1$

## probability density function

*exercise 1*

Let $X$ be a continuous random variable with probability density function $f(x) = 3x^2, \quad 0 \leq x \leq 1$

(a) Verify that $f(x)$ is a valid probability function

(b) What is $P\,(1/2 \leq X \leq 1)$?

(c) What is $P(X = 1/2)$?

*exercise 2*

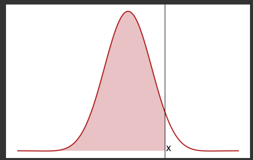
Let $X$ be a continuous random variable with probability density function $f(x) = \dfrac{x^3}{4}$ for $0 \leq X \leq c$.

What is the value of the constant $c$ that makes $f(x)$ a valid probability density function?

## cumulative distribution function

For a continuous random variable $X$ with pdf $f(x)$ its cumulative distribution function (cdf) is defined as follows

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)dy$$



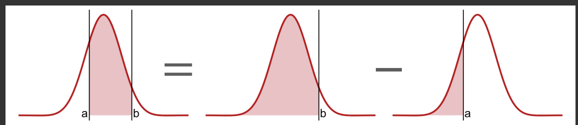

## computing probabilities with cdf

Let $X$ be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. Then for any value $a$ we have that

$$P(X \leq a) = F(a) \quad P(X > a) = 1 - F(a)$$

and for any two values $a < b$

$$P(a \leq X \leq b) = F(b) - F(a)$$




## computing probabilities with cdf

*exercise 3*

Random variable $T$ is distributed with the following probability density function:

$$f(t) = \begin{cases} ct(t-1) & \text{for } 0 \leq t \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Calculate the value of $c$.

(b) Calculate the cumulative distribution function $F(t)$.

(c) Use the cdf $F(t)$ to calculate $P(1/3 \leq T \leq 2/3)$.

## expected value of a continuous random variable

Let $X$ be a continuous random variable with pdf $f(x)$. The expected value $E(X)$ is calculated as a weighted integral

$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$

Let $X$ be a continuous random variable with pdf $f(x)$. If $h(X)$ is any real-valued function of $X$ then we can calculate an expected value for that as

$$E(h(X)) = \int_{-\infty}^{\infty} h(x)f(x)dx$$

## variance of a continuous random variable

Let $X$ be a continuous random variable with pdf $f(x)$ and mean $E(X) = \mu$. The variance $V(X)$ is is the expected value of the squared distance to the mean

$$V(X) = E((X-\mu)^2) = \int_{-\infty}^{\infty} (x-\mu^2) \cdot f(x)dx$$

The standard deviation is given by $\sqrt{V(X)}$.

## theoretical joint distributions

For two continuous random variables, we can write their joint pdf the same way: $f(x, y)$ "summing" the small bits of probability $f(x, y)dxdy$ over some region $X \in A, Y \in B$

Let $X, Y$ be a continuous random variables. The joint pdf for $X$ and $Y$ is $f(x, y) \geq 0$

The joint range is the set of pairs $(x, y)$ that have non-zero density.

The double integral over all values must be 1

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)dxdy = 1$$

# theoretical joint distributions

*exercise 4*

Let $X$ and $Y$ be two jointly continuous random variables with the following joint pdf

$$f(x, y) = \begin{cases} x + cy^2 & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find a sketch the joint range of $X$ and $Y$ (i.e. $\Omega_{X,Y}$).

(b) Find the constant $c$ that makes $f(x, y)$ a valid joint pdf.

(c) Find $P(0 \le X \le 1/2,\ 0 \le Y \le 1/2)$.

# marginal distributions

Let $X, Y$ be jointly distributed continuous random variables with joint pdf $f(x, y)$.
The marginal pdf's of $X$ and $Y$ are respectively given by the following:

$$f(x) = \int_{-\infty}^{\infty} f(x, y)dy$$

$$f(y) = \int_{-\infty}^{\infty} f(x, y)dx$$

Note this is exactly like for joint discrete random variables, with integrals instead of sums.

# theoretical joint distributions

*exercise 5*

Find the marginal pdf $f(x)$ and $f(y)$ given the joint pdf:

$$f(x, y) = \begin{cases} x + \frac{3}{2}y^2 & \text{for } 0 \le x \le 1, 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

## some continuous random variables and their pdfs

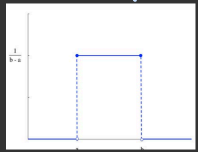
### uniform distribution $X \sim \mathrm{Unif}(a, b)$

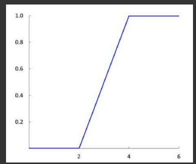
A continuous random variable $X$ has uniform distribution on the interval $[a, b]$ for values $a \leqslant b$ if it has the following pdf:

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

The cdf is given by

$$f(x \mid a, b) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$



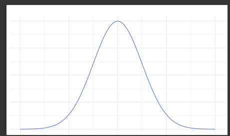




## some continuous random variables and their pdfs

### normal distribution $X \sim N(\mu, \sigma^2)$

A continuous random variable $X$ has normal distribution with parameters $\mu$ and $\sigma^2$ if it has the following pdf:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$



If continuous random variable $X \sim N(\mu, \sigma^2)$ then random variable $Z$ defined as

$$Z = \frac{X - \mu}{\sigma}$$

has standard normal distribution $Z \sim N(0,1)$

## some continuous random variables and their pdfs

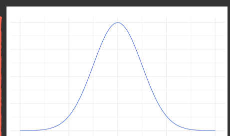
### standard normal distribution $Z \sim N(0,1)$

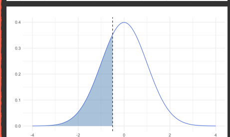
The normal distribution with parameters $\mu = 0$ and $\sigma = 1$ is the standard normal distribution and a random variable with that distribution is a standard normal random variable, usually named $Z$ and with the following probability density function.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$$

The corresponding cumulative distribution function is written $\Phi(z)$

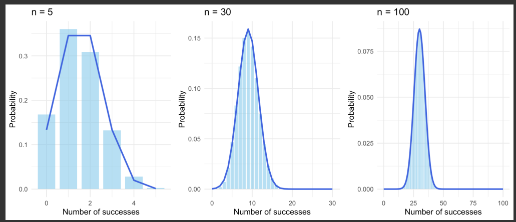
$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{x^2}{2}}\, dx$$





## some continuous random variables and their pdfs

### the importance of normal distribution…




$np \geq 10$

## some continuous random variables and their pdfs

### exponential distribution $X \sim \text{Exp}(\lambda)$

A continuous random variable $X$ has exponential distribution with parameter $\lambda$, for some $\lambda > 0$, if it has the following pdf

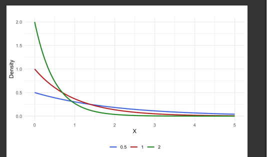
$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

and the following cdf

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

This distribution is **memoryless** i.e.

$$P(X \geq a \,|\, X \geq b) = P(X \geq a - b)$$




The exponential distribution is a specific version of gamma family distributions…

## some continuous random variables and their pdfs

### gamma distribution $X \sim \text{Gamma}(\alpha, \beta)$

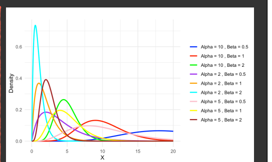
A continuous random variable $X$ has Gamma distribution with parameters $\alpha$ and $\beta$ (both positive) if

$$f(x) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{x/\beta} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$$

where $\Gamma(\alpha)$ is the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx,$$

which cannot be expressed in closed form analytical solution.




$\alpha = 1 \implies$ Exponential $\left(\beta = \frac{1}{\lambda}\right)$

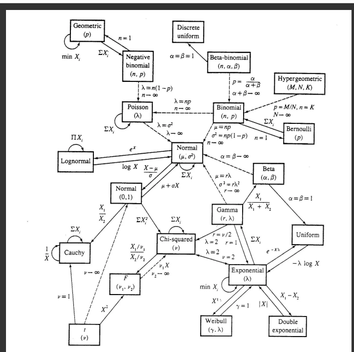
$\alpha = \frac{v}{2}, \beta = 2 \implies$ chi-square $\chi^2(v)$

# some continuous random variables and their pdfs

## important distributions for statistical hypothesis tests

- Chi-squared ($\chi^2$) distribution
- The (Student's) *t* Distribution (like normal but "thicker" tails)
- The F Distribution (ratio of two $\chi^2$ distributed variables)

*more on these in your tutorial this week…*

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