Support Vector Machines

Lecture 10

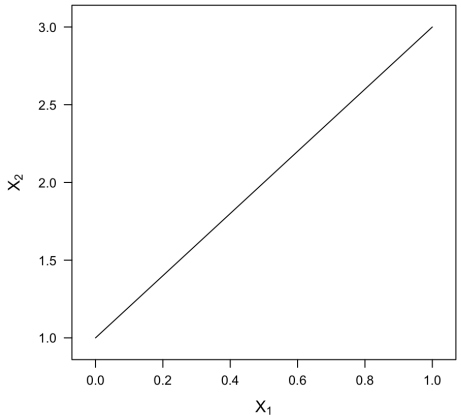
Termeh Shafie

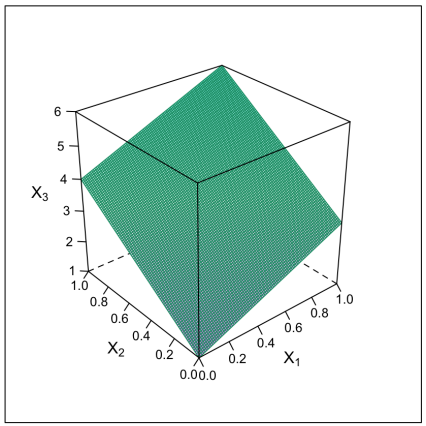
Support Vector Machine (SVM) is a supervised learning algorithm used to learn a **hyperplane** that can solve the **binary classification problem**

Hyperplanes

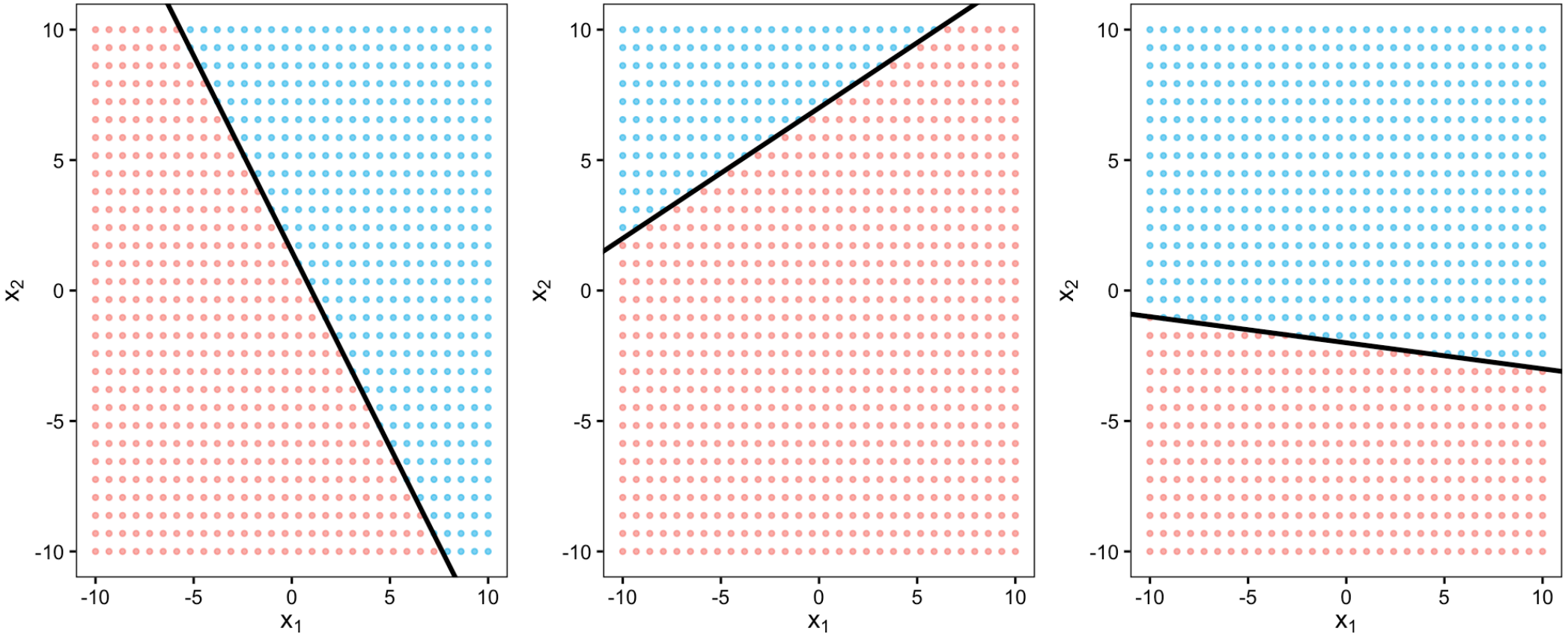
"a flat affine subspace" 😐

- **Flat:**
  - hyperplane is not curved, it increases/decreases constantly in each direction
- **Affine:**
  - the hyperplane doesn't need to pass through the origin
  - it can have an "offset" or be shifted (may have intercept)
- **Subspace:**
  - a subset of vectors in a larger vector space
  - in a $d-$dimensional space, a hyperplane has dimension $d-1$
  - in 2D it is a **line**, in 3D it is a **plane**



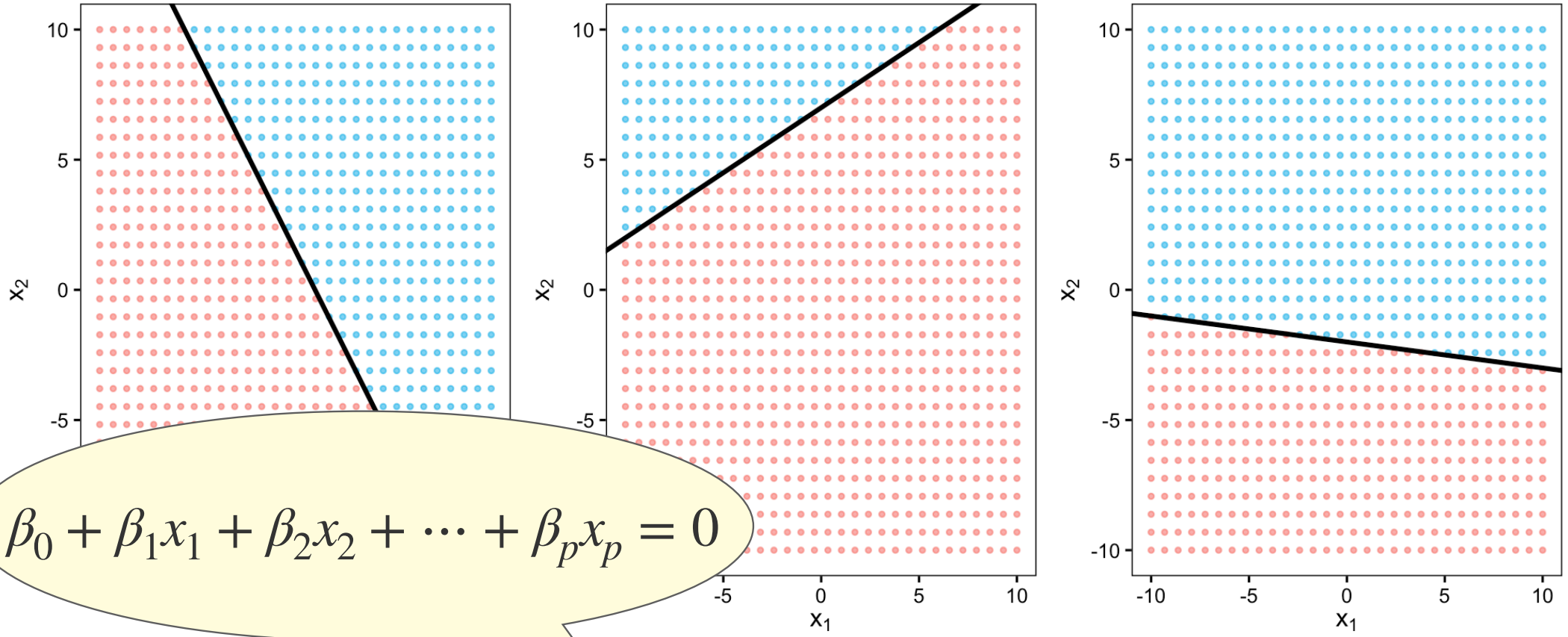


Hyperplanes Divide the Space in Half



$$\beta_0 + \beta^T x = 0$$

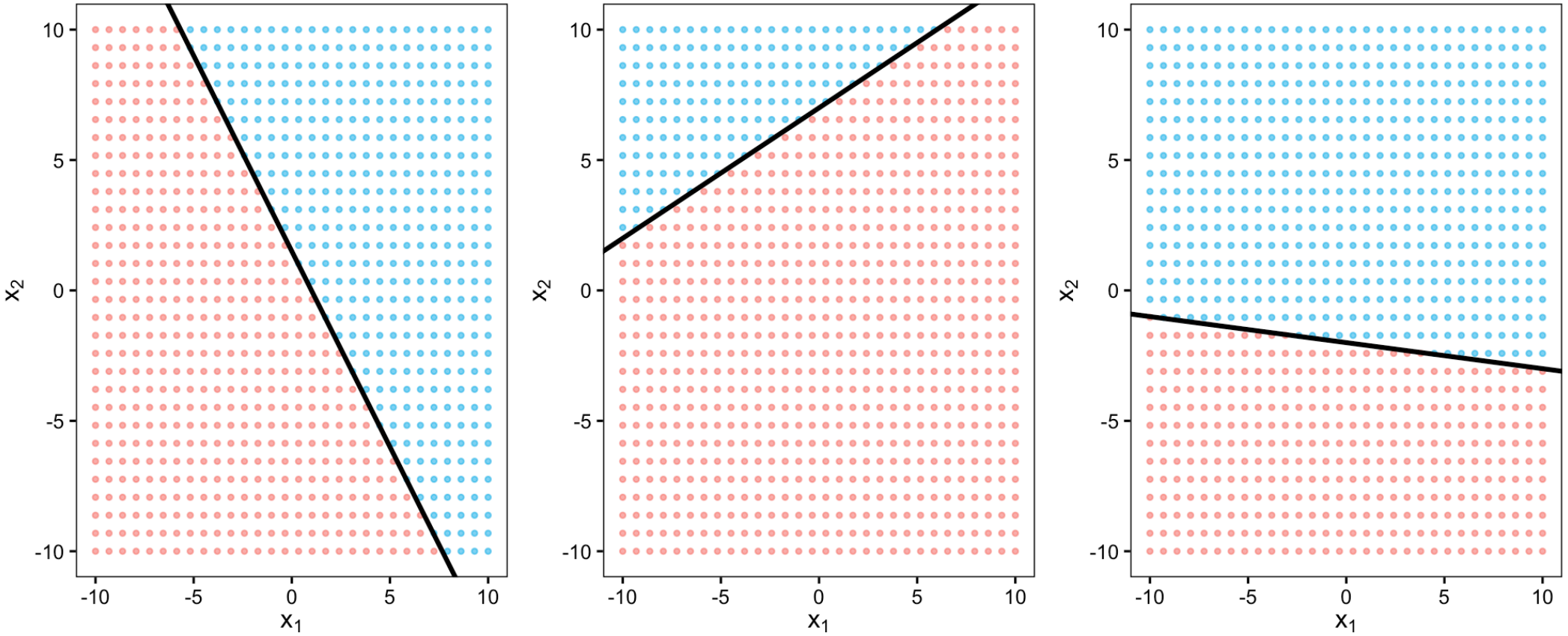
Hyperplanes Divide the Space in Half



$$\beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p = 0$$

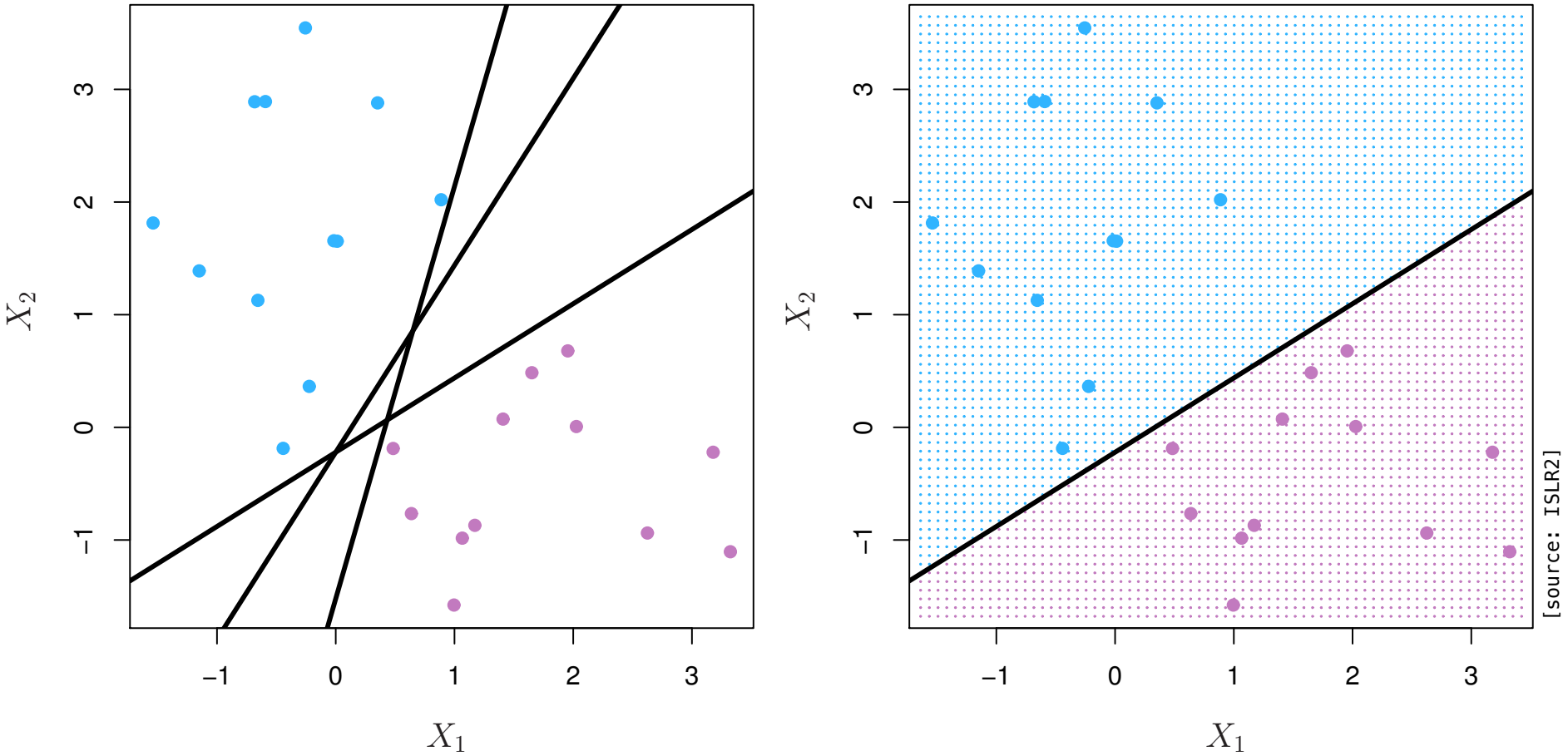
$$\beta_0 + \beta^T x = 0$$

Hyperplanes Divide Spaces in Half



$$\beta_0 + \beta^T x > 0 \text{ if } y_i = 1$$
$$\beta_0 + \beta^T x < 0 \text{ if } y_i = -1$$
$$\Longrightarrow \quad y_i(\beta_0 + \beta^T x) > 0$$

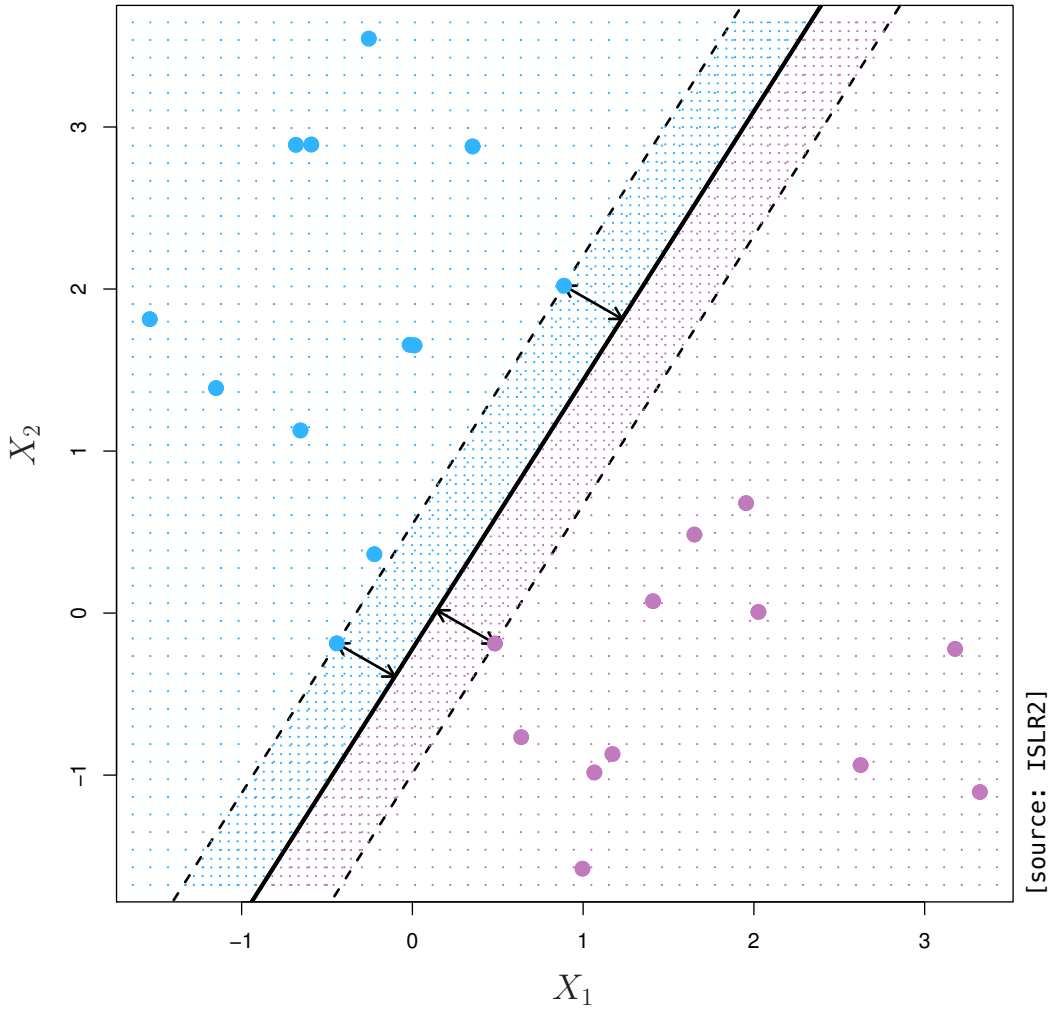
Hyperplanes Divide Spaces in Half



if a separating hyperplane exists, there will be ∞ many of them

how do we choose just one?

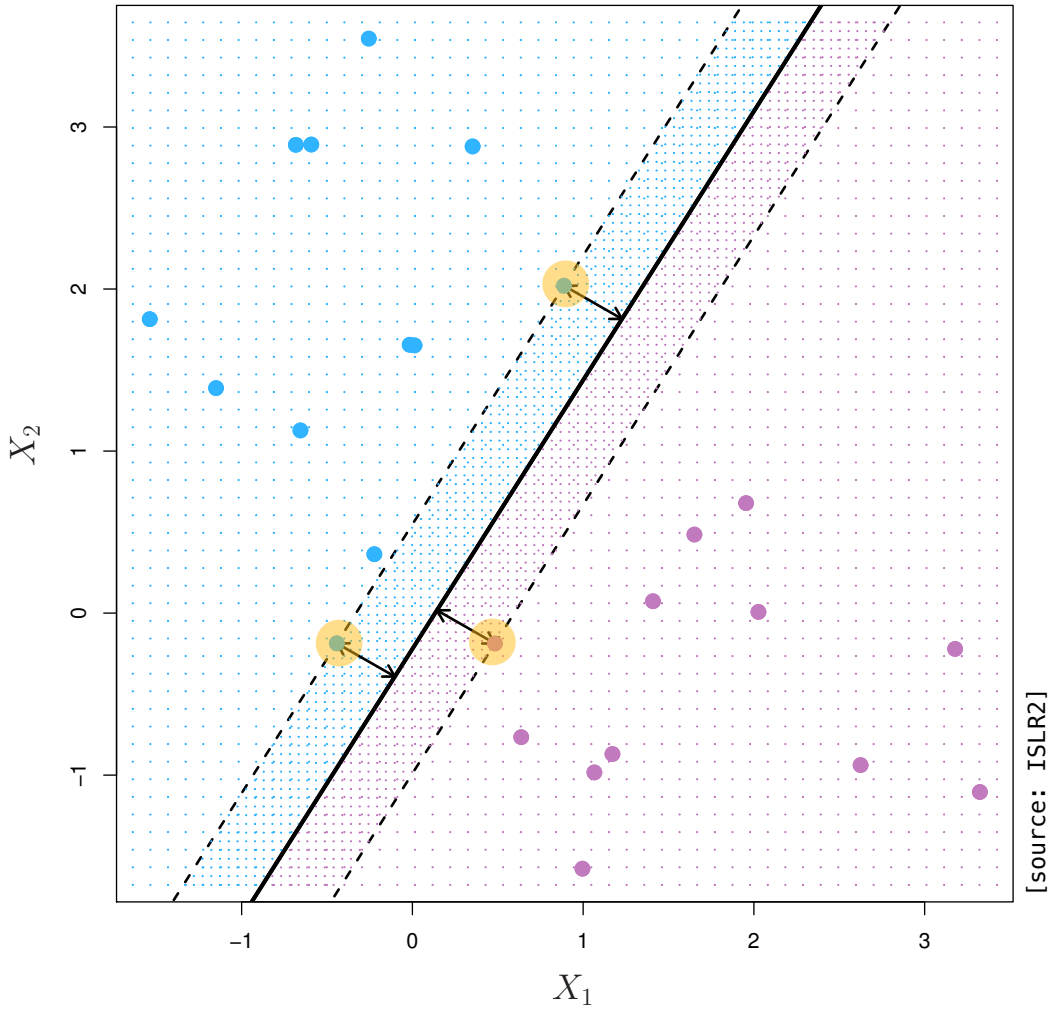
Maximal Margin Classifier



[source: ISLR2]

choose the one that's furthest from the training examples

Maximal Margin Classifier

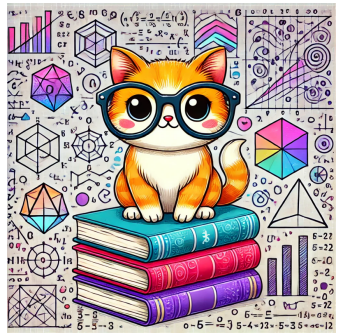


[source: ISLR2]

support vectors
the closest points from both classes

the margin
the distance from hyperplane to support vectors

Maximal Margin Classifier: The Math



The maximal margin classifier solves a constrained optimization problem:

$$\max_{\beta_0,\beta_1,\ldots,\beta_p} \quad M$$

subject to:

$$\|\beta\| = 1$$

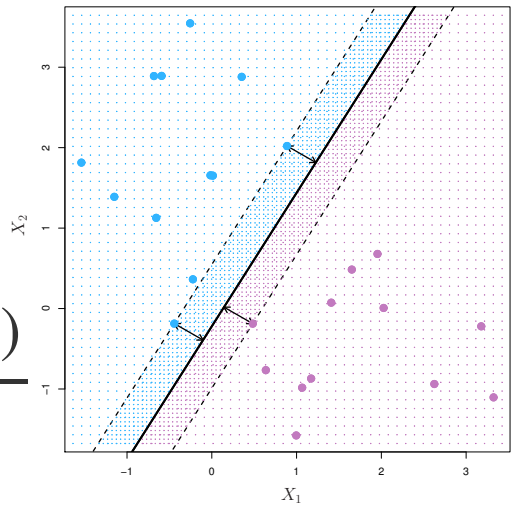
$$y_i(\beta_0 + \beta^T x_i) \geq M, \quad \forall i = 1,\ldots,n$$

ensured each observation is on the correct side of the hyperplane
and at least a distance M from the hyperplane,
i.e., M is the margin of the hyperplane

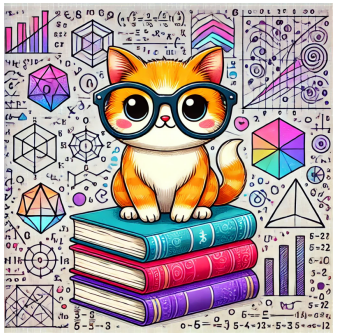
distance between x_i and line where

$\|\beta\| = \sqrt{\sum_{j=1}^{p} \beta_j^2}$ is the Euclidean norm of β

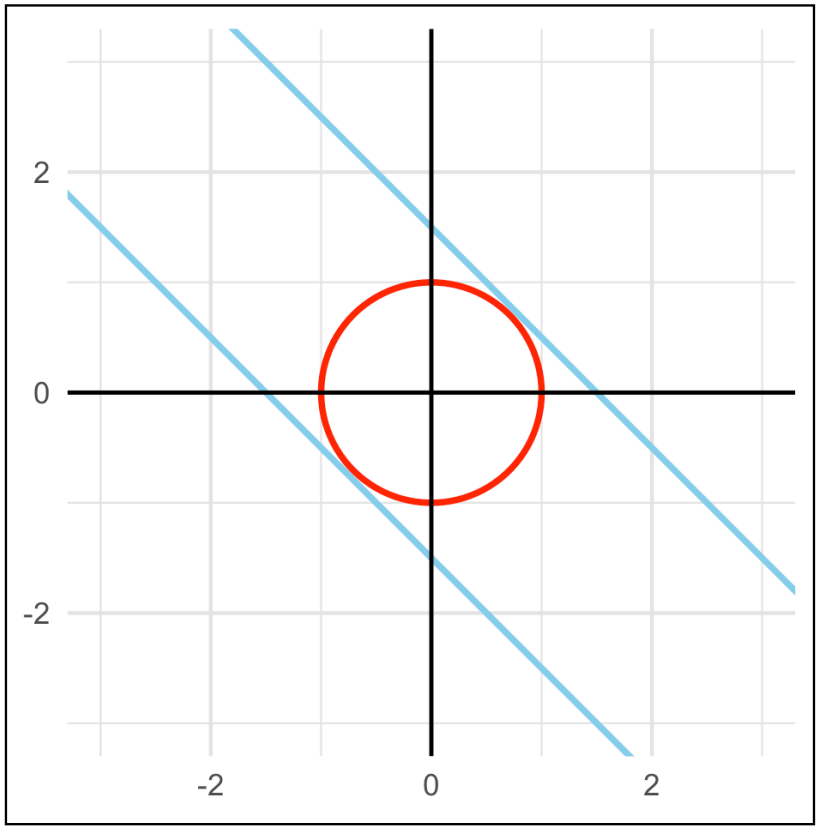
$$\left\{ \frac{y_i(\beta_0 + \beta^T x_i)}{\|\beta\|} \right.$$



What is a Constrained Optimization Problem?



Optimize $f(x, y)$ subject to $g(x, y) = k$



$$f(x, y) = 2x + y$$
$$g(x, y) = x^2 + y^2 = 1$$

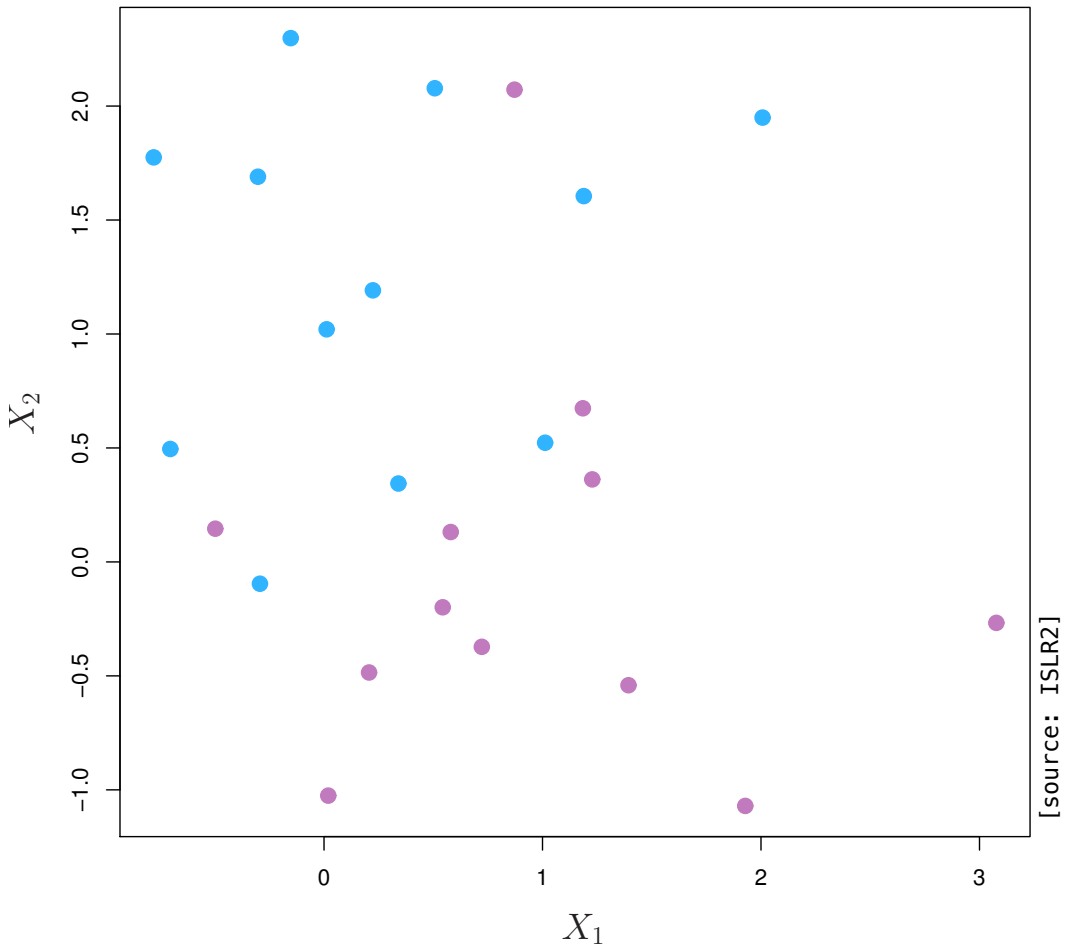
$$\max_{\beta_0, \beta_1, \ldots, \beta_p} M$$

subject to:

$$\|\beta\| = 1$$
$$y_i(\beta_0 + \beta^T x_i) \geq M$$

The Non-Separable Case

the optimization problem for the maximal margin classifier often has no solution with $M > 0$

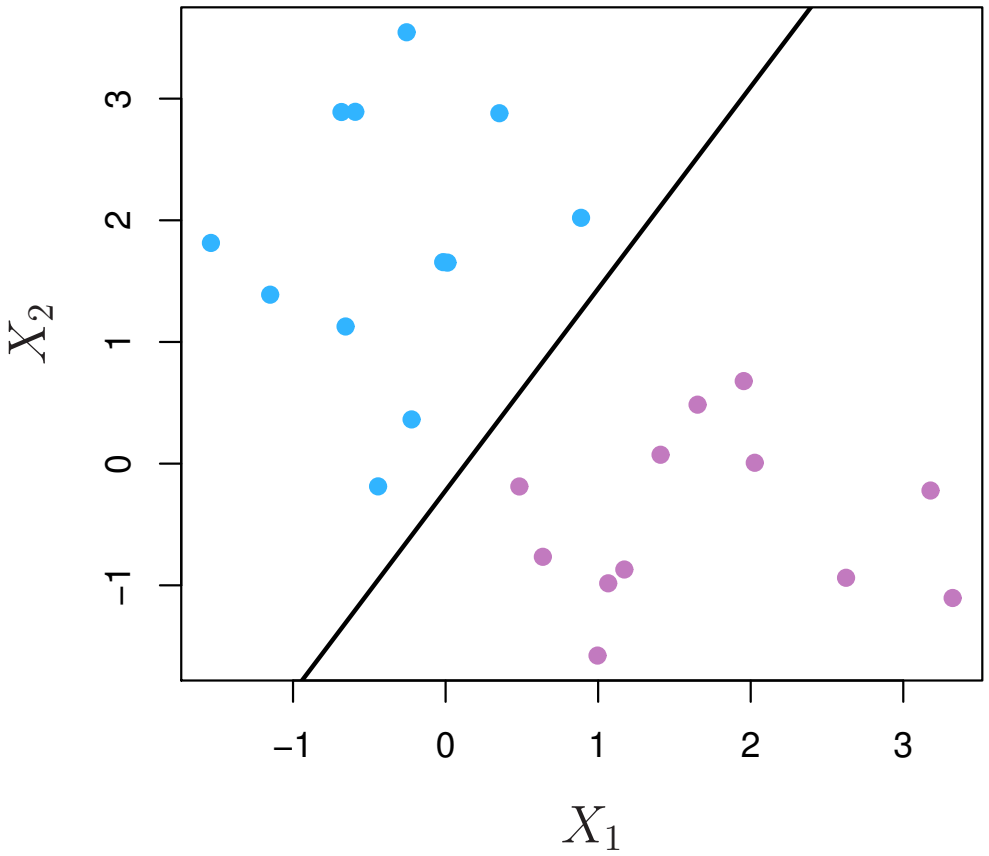


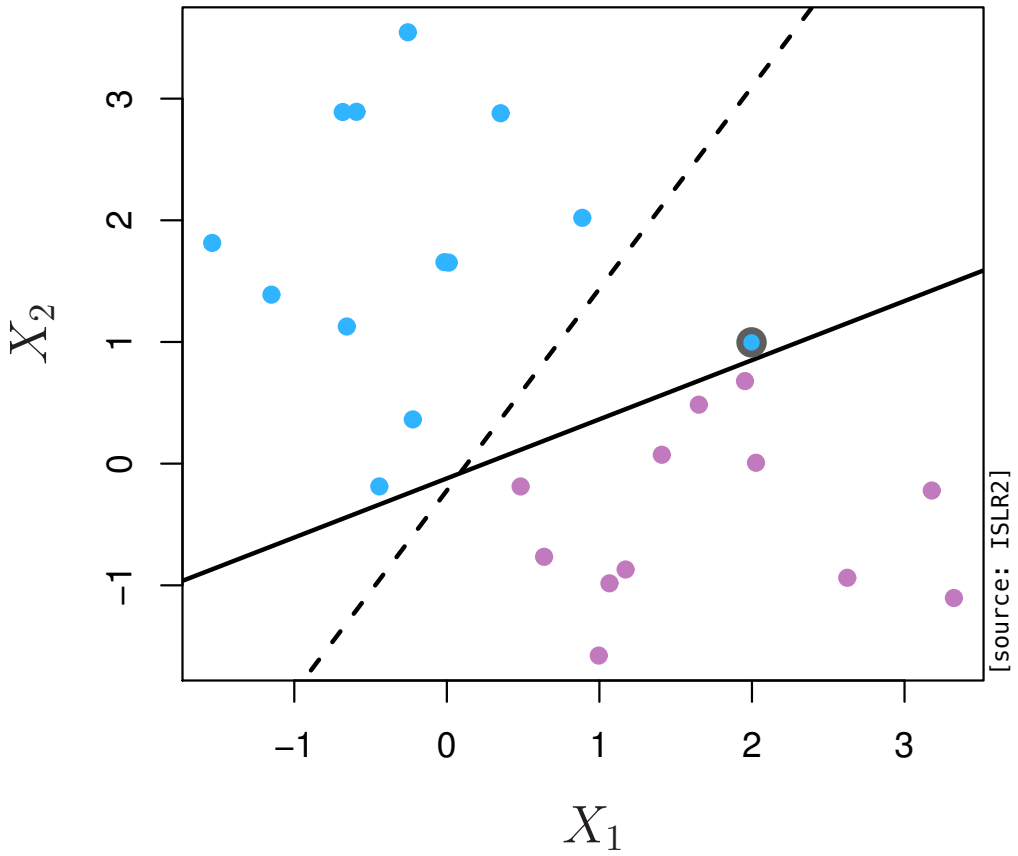
and it's very sensitive to outliers!

The Non-Separable Case

even if the data are separable, they are sometimes noisy

$\Longrightarrow$ poor solution for the maximum margin classifier





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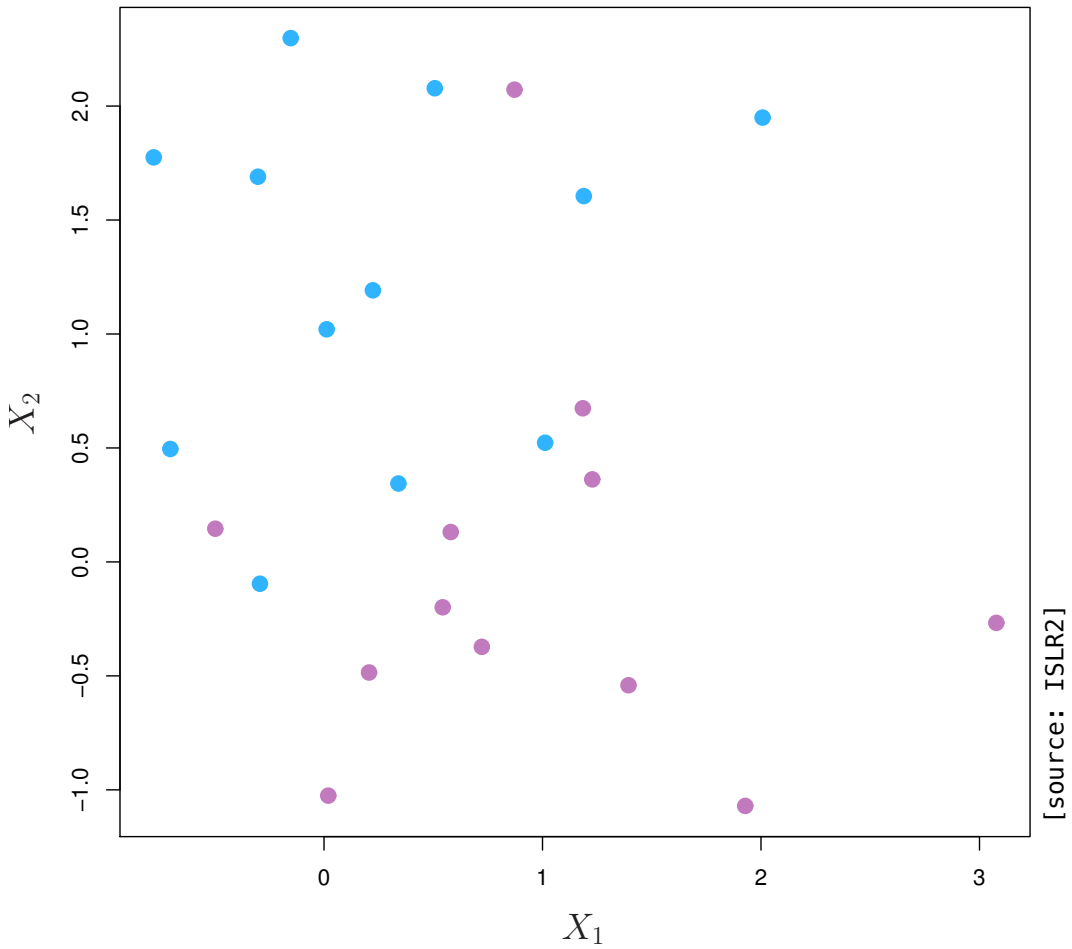
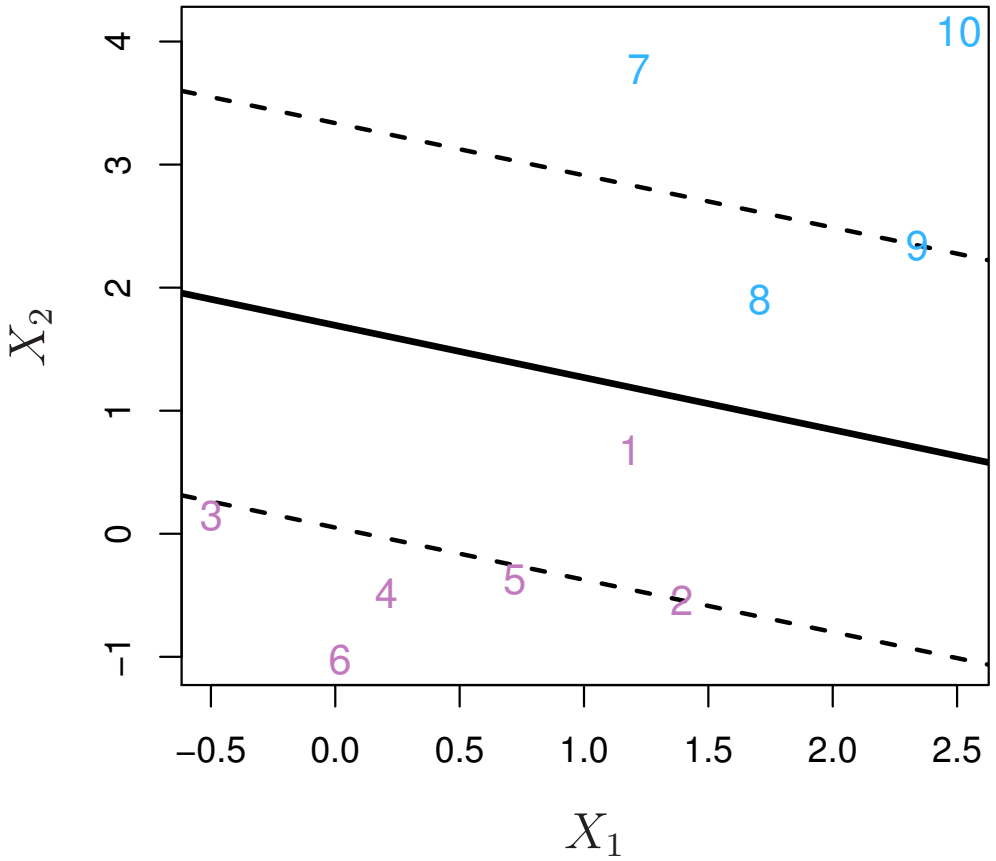


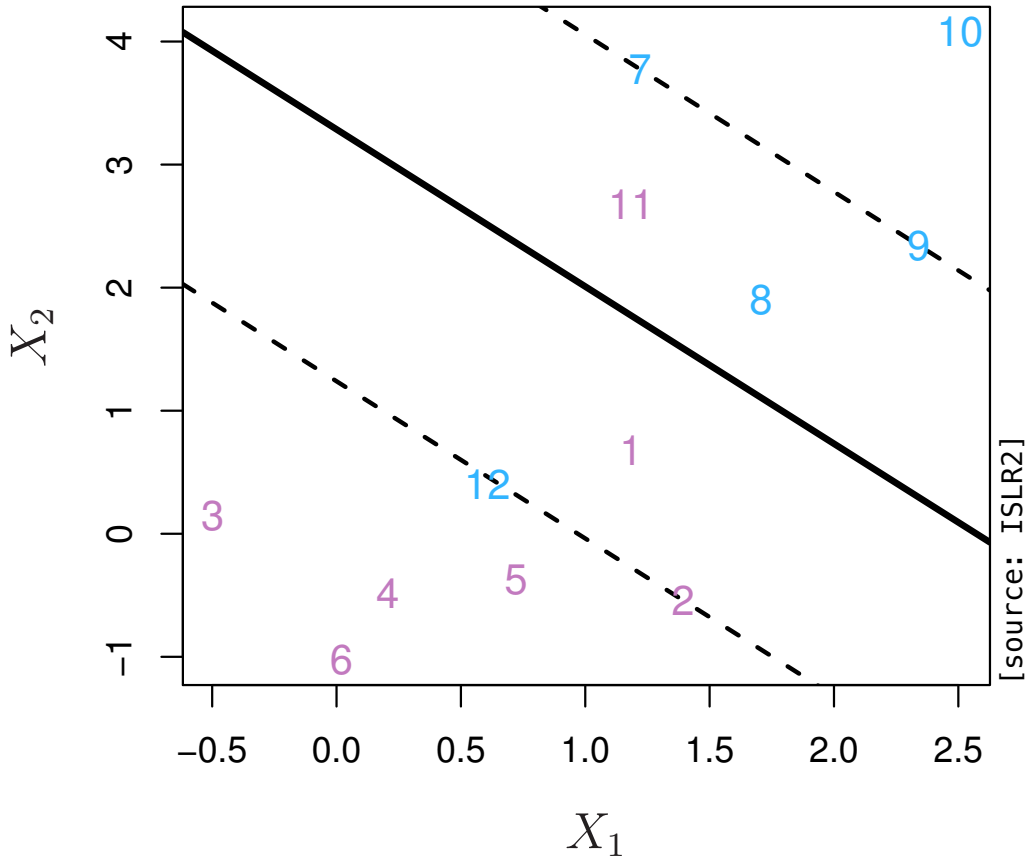
[source: ISLR2]

support vector classifier:
using a so-called **soft margin**
to **almost** separate the classes

Support Vector Classifier

allows us to classify data that is **not linearly separable**





Support Vector Classifier

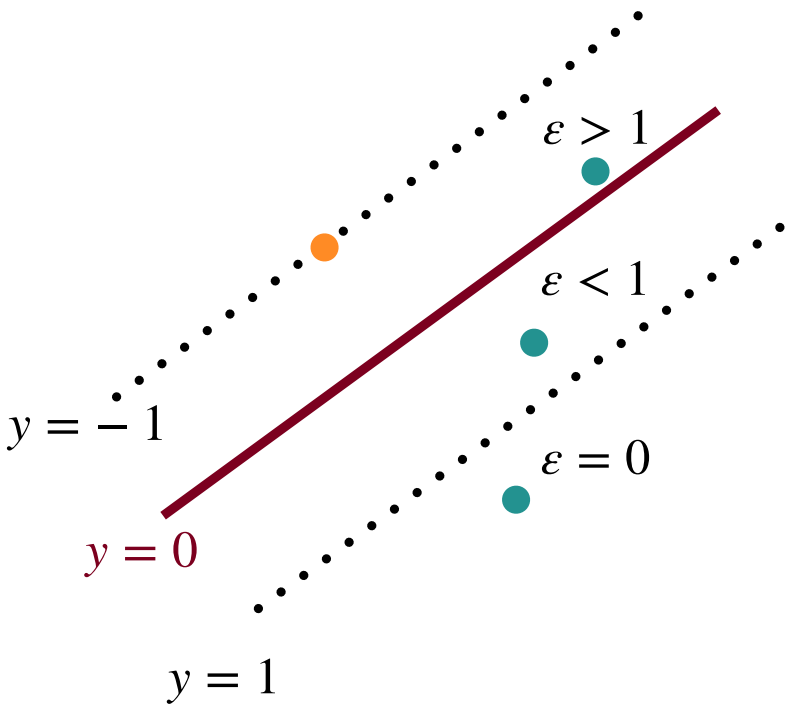
$$\max_{\beta_0,\beta_1,\ldots,\beta_p,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} M$$

subject to:

$$\|\beta\| = 1$$

$$y_i(\beta_0 + \beta^T x_i) \geq M(1 - \varepsilon_i)$$

$$\varepsilon_i \geq 0, \sum_{i=1}^{n} \varepsilon_i \leq C$$



$\varepsilon_1, \ldots, \varepsilon_n$ are **slack variables** where $\varepsilon_i = 0$ means i^{th} observation is on correct side of margin

< 1 means i^{th} observation is on wrong side of margin

> 1 means i^{th} observation is on wrong side of hyperplane

Support Vector Classifier

$$\max_{\beta_0,\beta_1,\ldots,\beta_p,\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n} M$$

subject to:

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$$y_i(\beta_0 + \beta^T x_i) \geq M(1 - \varepsilon_i)$$

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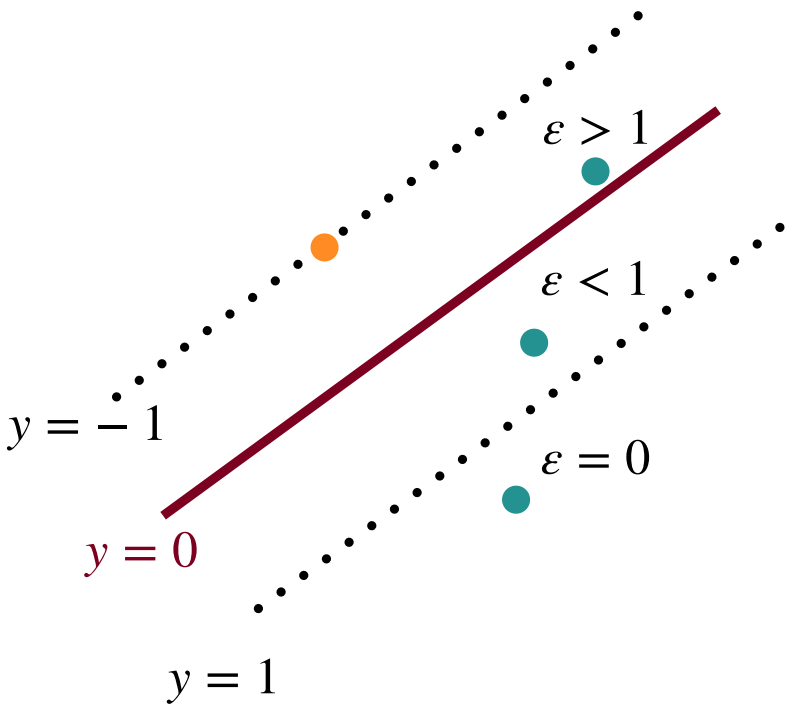


C is the tuning parameter/penalty on error:

$C = 0$ implies maximal margin hyperplane (superposed it exists)

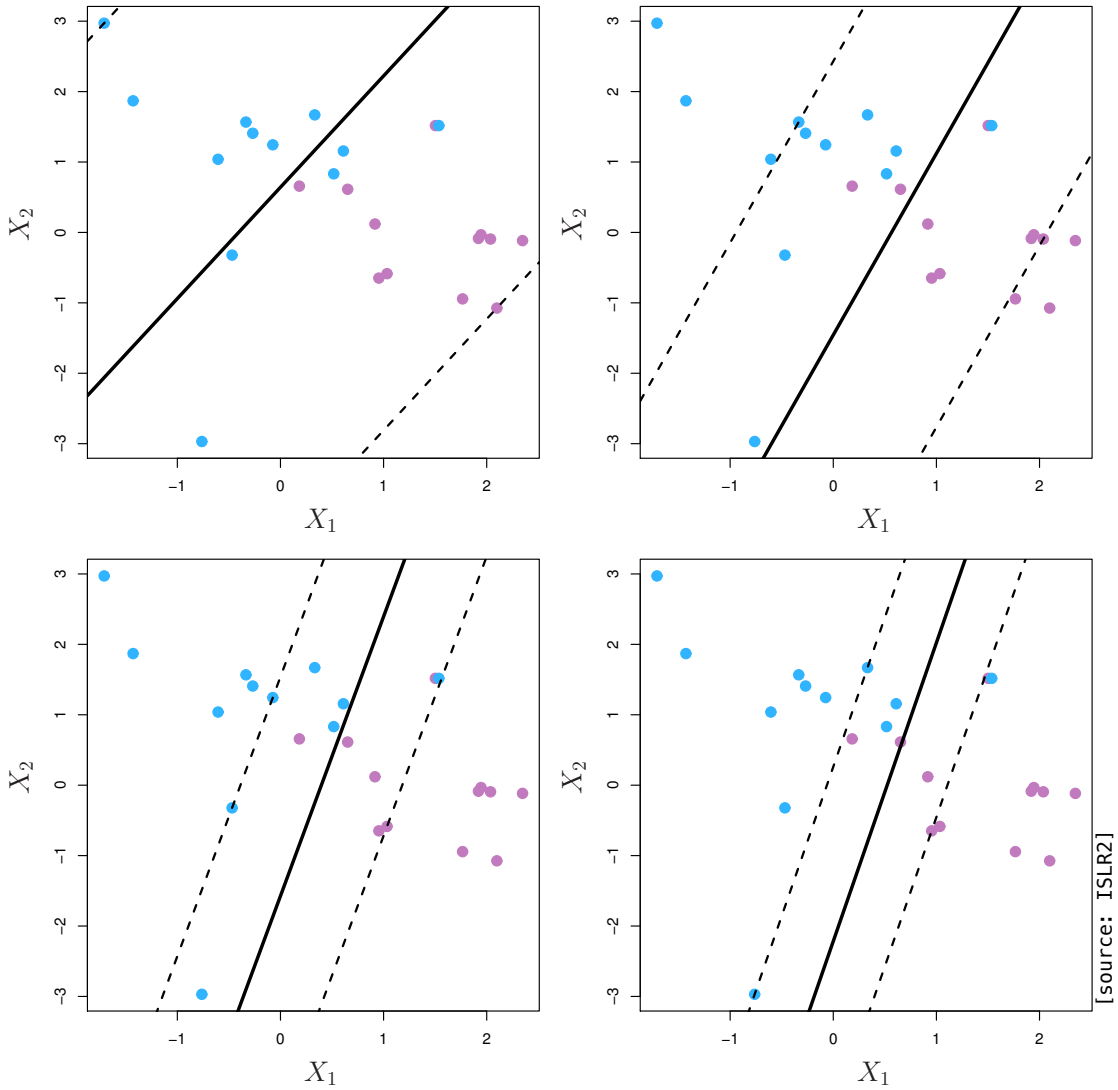
$C > 0$ is the total violations to the margin that we can tolerate

$\Longrightarrow$ max C observations can be on the wrong side of hyperplane

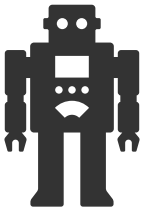
Support Vector Classifier

C: penalty on error

	Regularization	Margins	Bias/Variance
Small C	more	wider	prone to underfitting
Large C	less	narrower	prone to overfitting

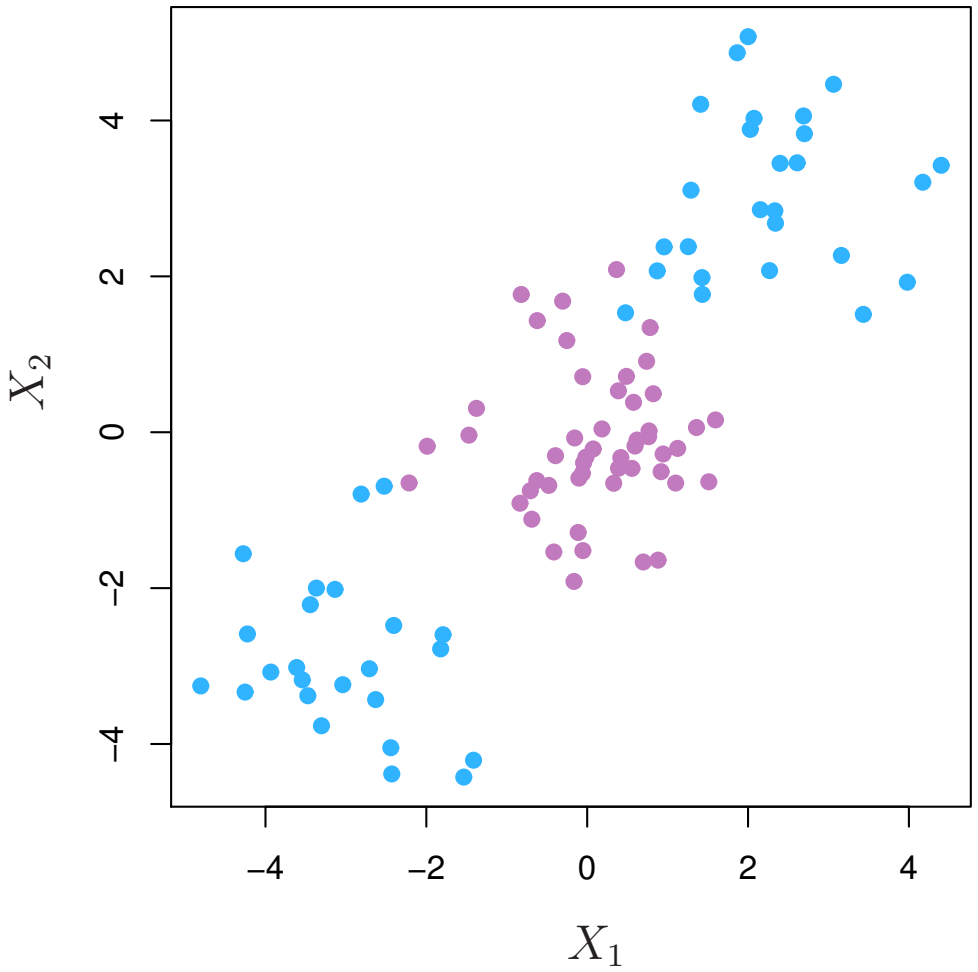


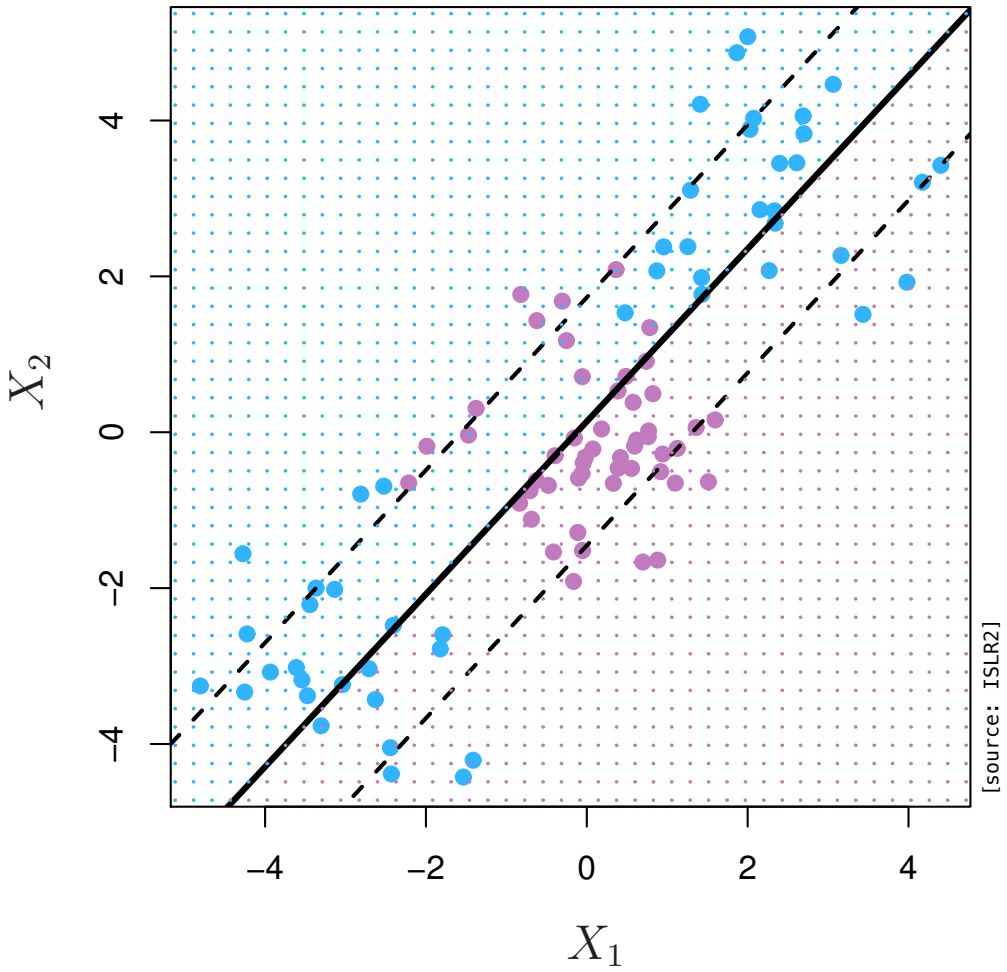
Support Vector Machines



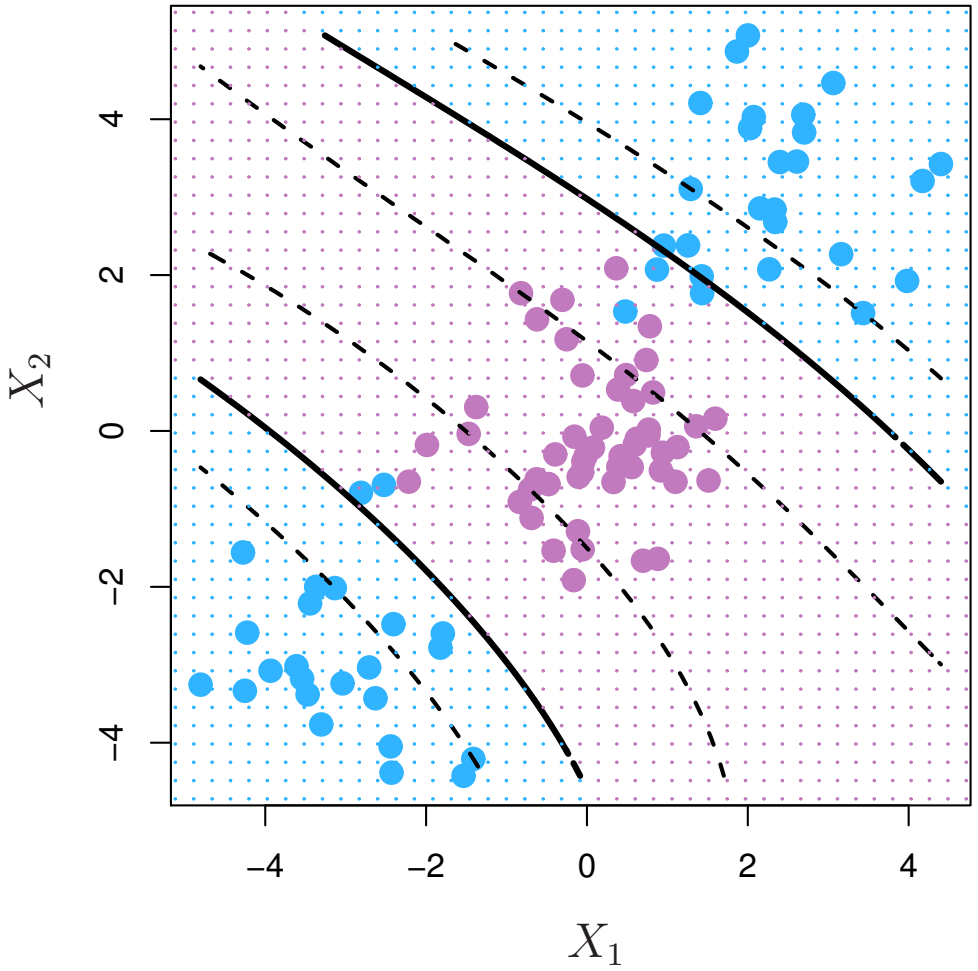
Support Vector Machine (SVM) use **Kernel Functions** to systematically find **Support Vector Classifiers** in higher dimensions

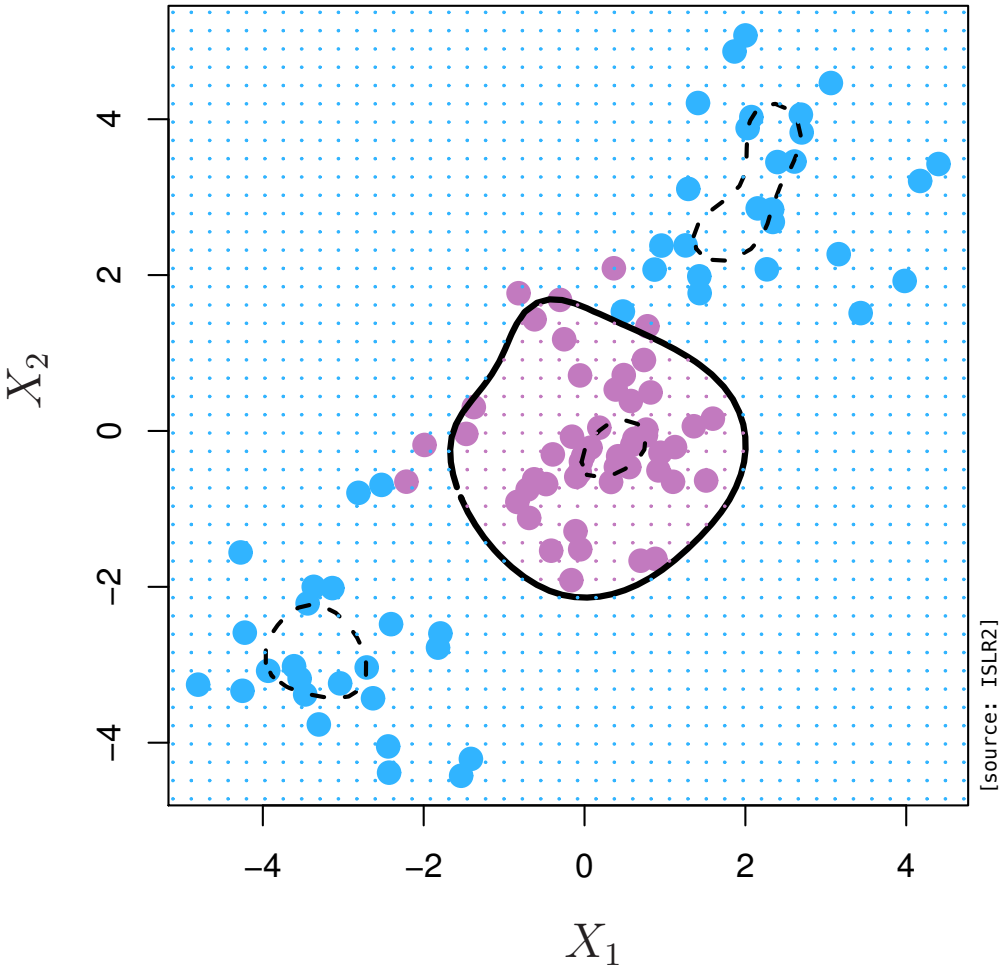
Not Linearly Separable Even With Error





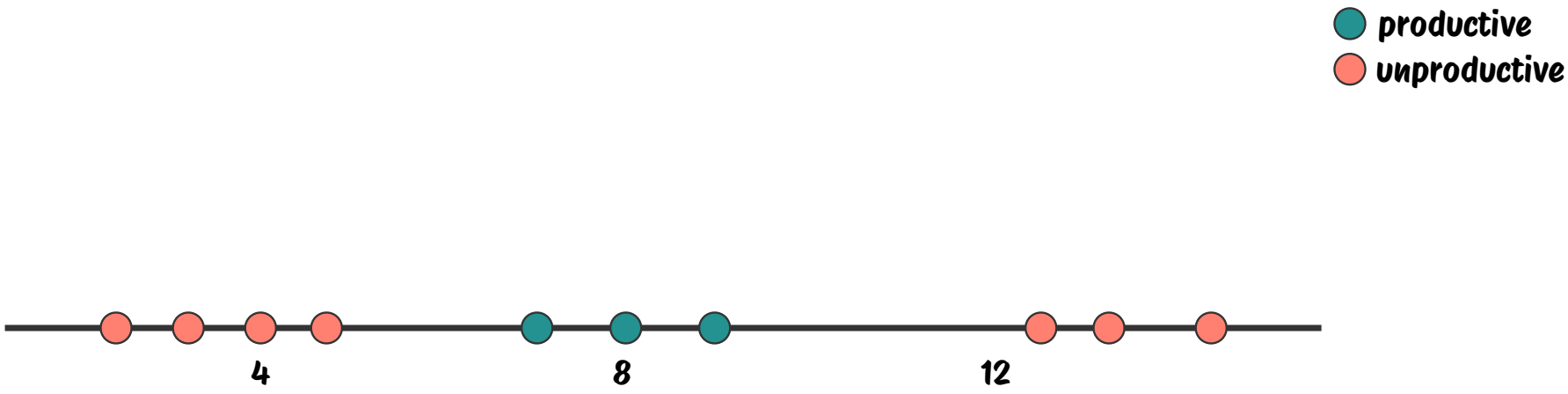
The Kernel Trick





The Kernel Trick

my productivity based on hours of sleep



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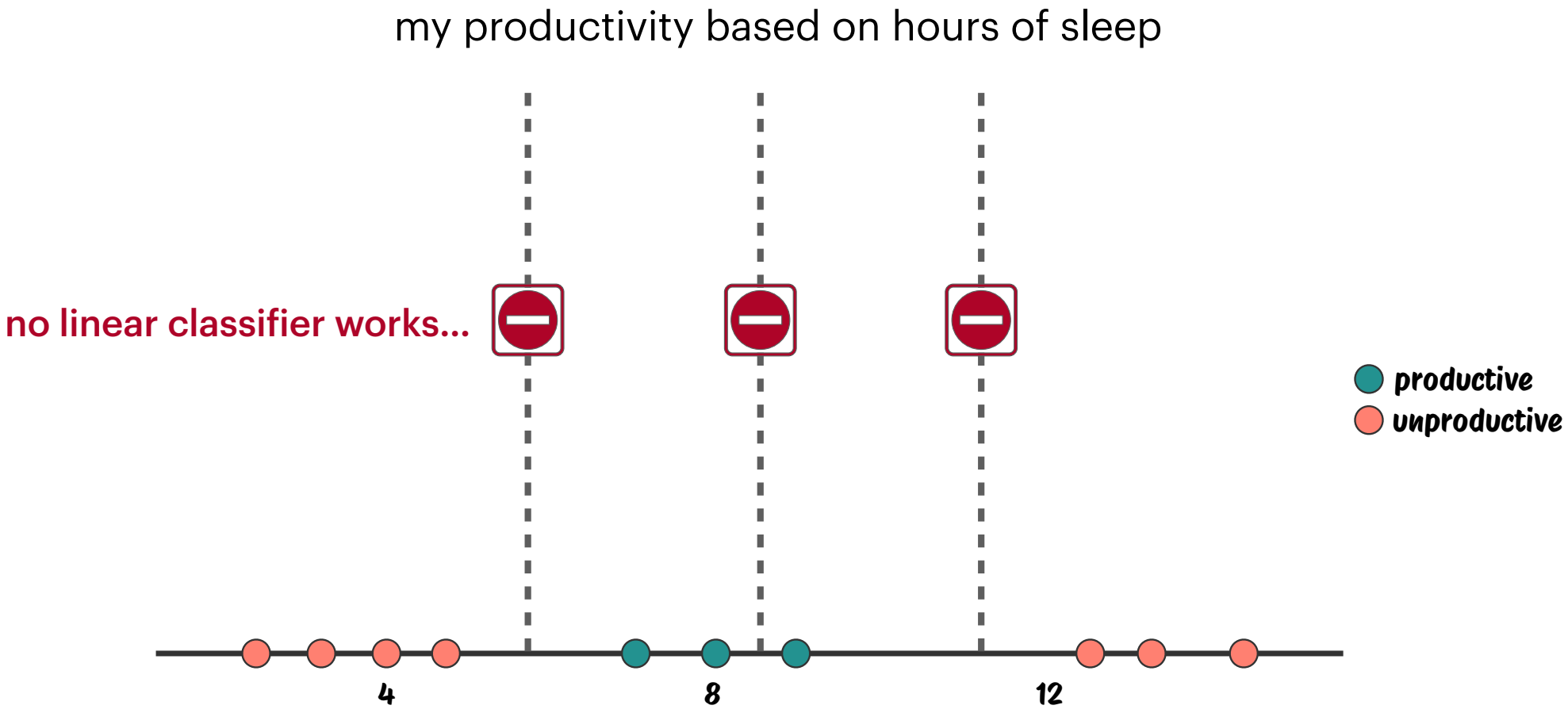

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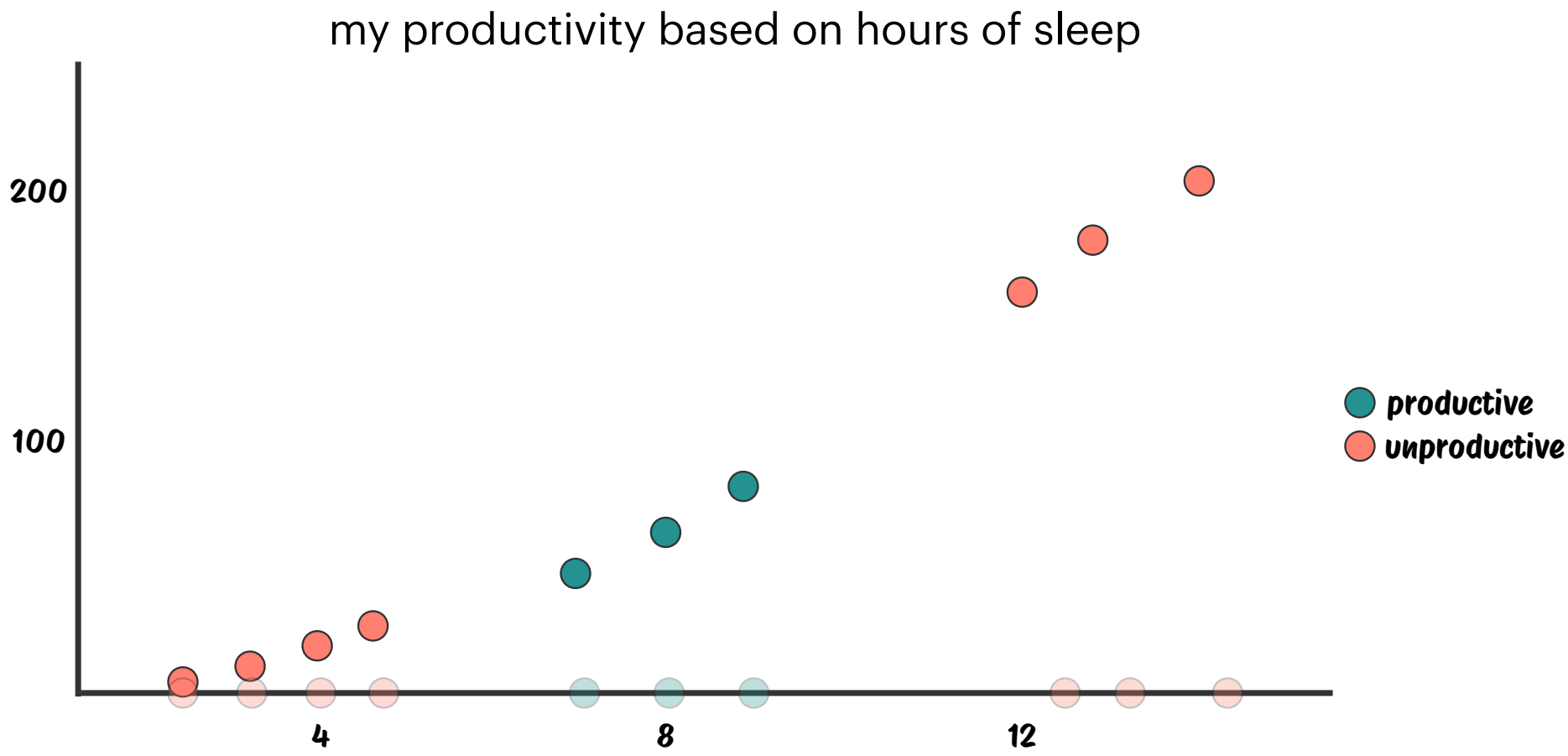

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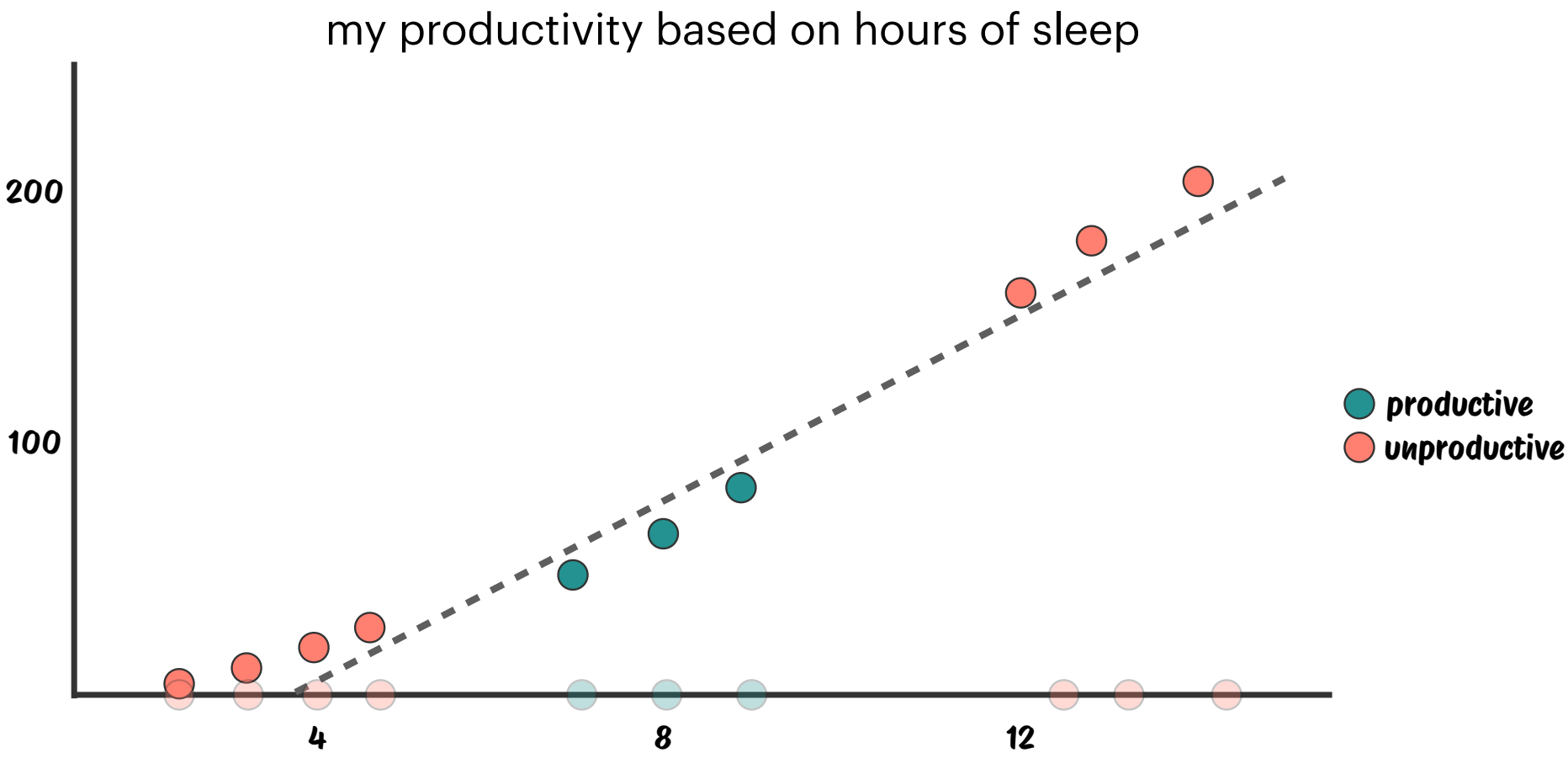
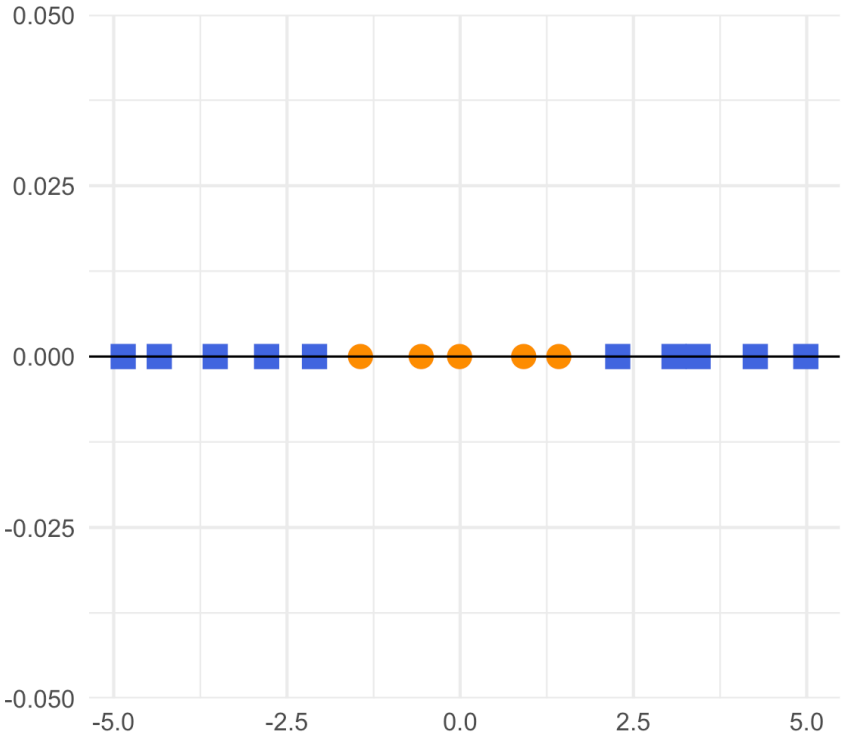
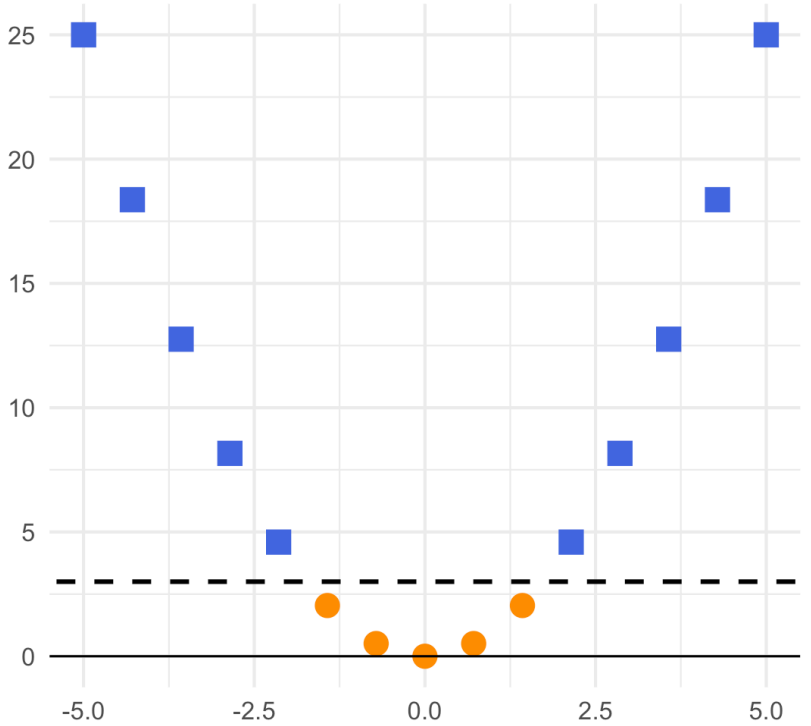


The Kernel Trick

what is an SVM Kernel?

A function that computes the relationship between vectors in multiple dimensions
(without actually having to calculate the coordinates for those dimensions)





The Polynomial Kernel

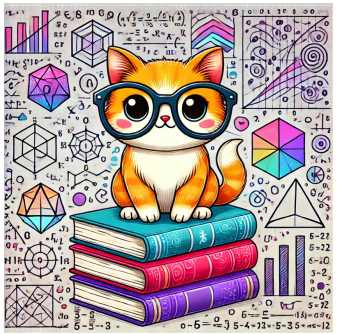
The **Polynomial Kernel** in the previous sleep vs. happiness example

$$\underbrace{K(a, b)}_{\text{Kernel}} = (a \cdot b + r)^d$$

where r is the coefficients and d the degree (determined by **cross validation**)

different observations

The Polynomial Kernel



The **Polynomial Kernel** in the previous sleep vs. happiness example

$K(a, b) = (a \cdot b + r)^d$ where r is the coefficients and d the degree

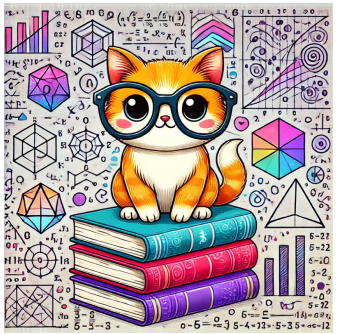
we set $r = \frac{1}{2}$ and $d = 2$:

$$(a \cdot b + \frac{1}{2})^2 = (a \cdot b + \frac{1}{2})(a \cdot b + \frac{1}{2})$$

$$+a^2b^2 + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{4}$$

$$= \boxed{ab + a^2b^2 + \frac{1}{4}}$$

The Polynomial Kernel



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$$+a^2b^2 + \frac{1}{2}ab + \frac{1}{2}ab + \frac{1}{4}$$

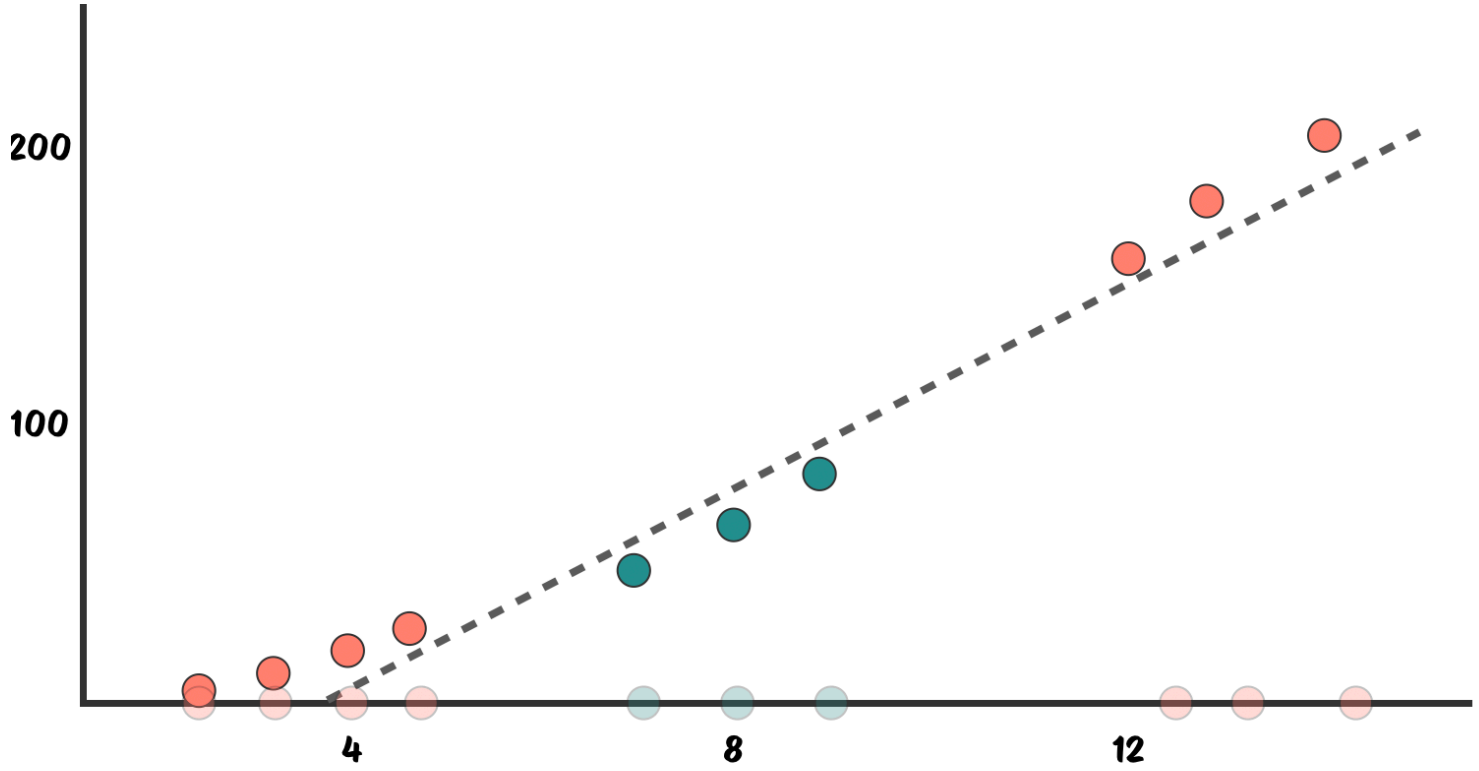
$$= \boxed{ab + a^2b^2 + \frac{1}{4}} = \underbrace{(a, a^2, \frac{1}{2}) \cdot (b, b^2, \frac{1}{2})}$$

dot product

gives us the high dimensional coordinates for the data

The Polynomial Kernel

$$(a, a^2, \frac{1}{2}) \cdot (b, b^2, \frac{1}{2})$$



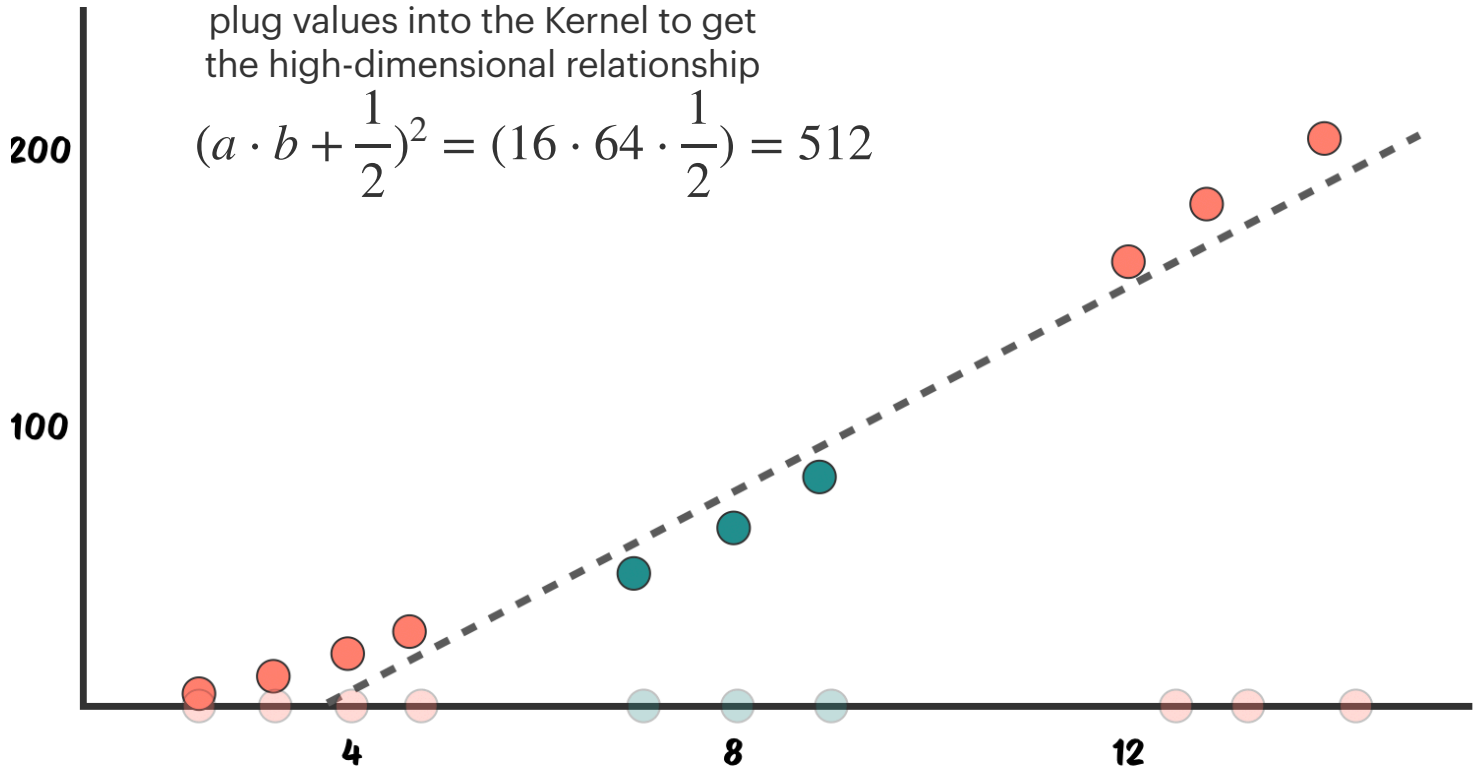
The Polynomial Kernel

A function that computes the relationship between vectors in multiple dimensions
(without actually having to calculate the coordinates for those dimensions)

example: $a = 4,\ b = 8$

plug values into the Kernel to get
the high-dimensional relationship

$$(a \cdot b + \frac{1}{2})^2 = (16 \cdot 64 \cdot \frac{1}{2}) = 512$$



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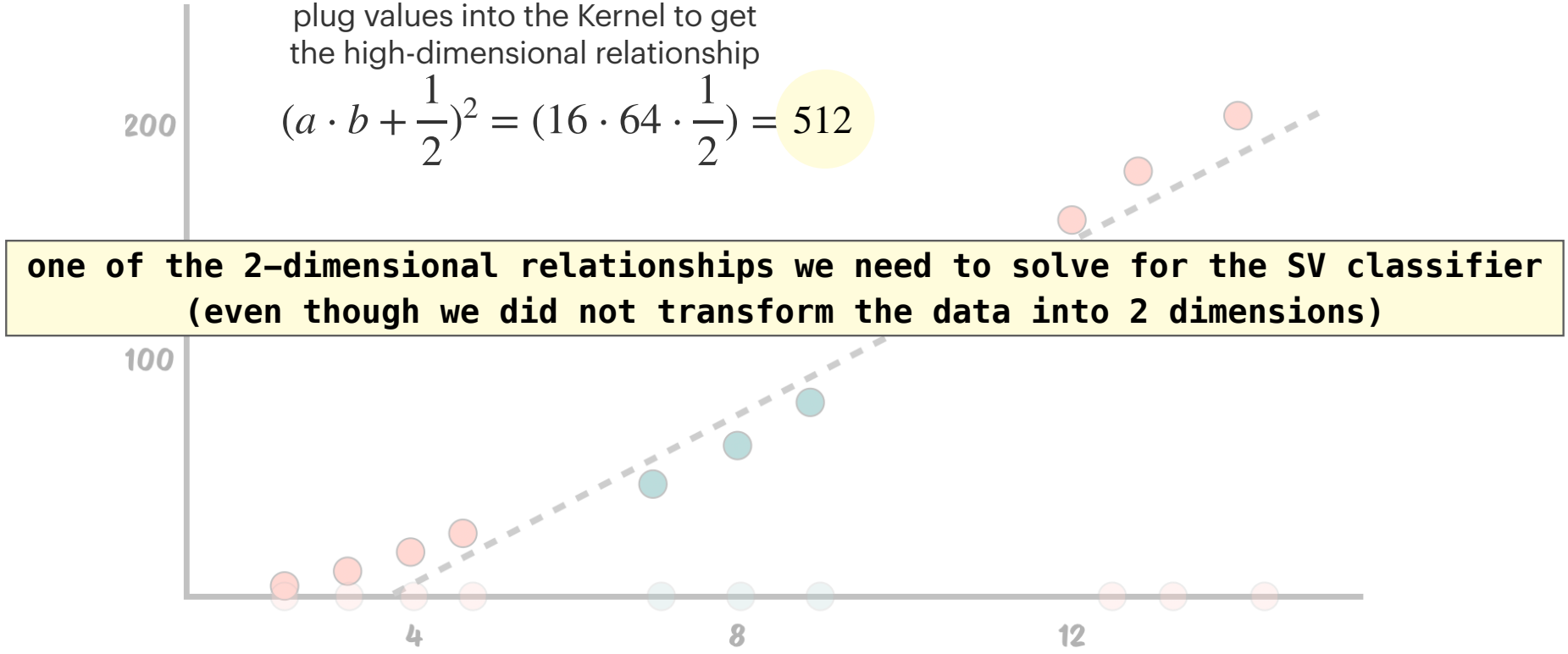


one of the 2-dimensional relationships we need to solve for the SV classifier (even though we did not transform the data into 2 dimensions)

The Radial Kernel (RBF)

The **Radial Kernel**

$$K(a, b) = e^{-\gamma}(a - b)^2$$

projects to **infinite dimensional** space

works similar to nearest neighbors classifier

The Radial Kernel (RBF)

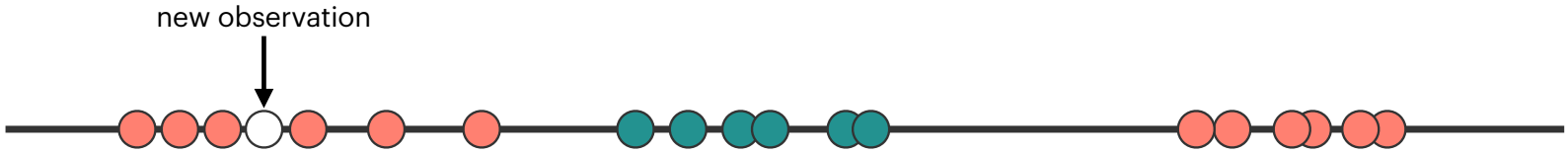
The **Radial Kernel**

$$K(a, b) = e^{-\gamma}(a - b)^2$$

projects to **infinite dimensional** space
works similar to nearest neighbors classifier

the amount of influence one observation has on another is a function of the squared distance

γ scales the squared distance to determine the strength of influence (determined by **cross validation)**



The Radial Kernel (RBF)

The **Radial Kernel**

$$K(a,b) = e^{-\gamma}(a-b)^2$$

projects to **infinite dimensional** space

works similar to nearest neighbors classifier

we can use the Polynomial Kernel to get the intuition
behind how Radial Kernel works in infinite dimensions

$$K(a,b) = (a \cdot b + r)^d$$

set $r = 0 \implies (a \cdot b)^d = a^d \cdot b^d$

set $d = 1 \implies (a) \cdot (b)$

The Radial Kernel (RBF)

The **Radial Kernel**

$$K(a,b) = e^{-\gamma}(a-b)^2$$

projects to **infinite dimensional** space

works similar to nearest neighbors classifier

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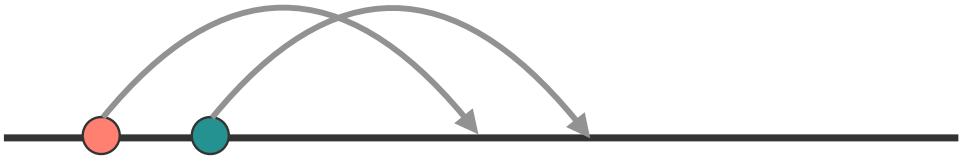

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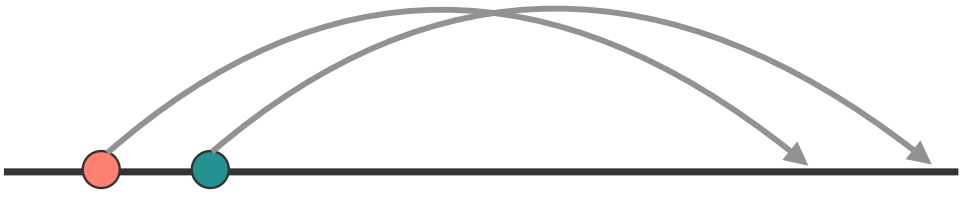

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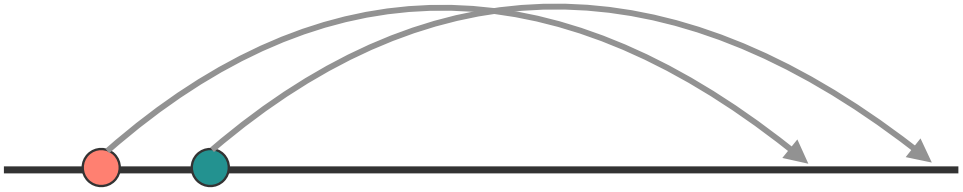

we stay in same dimension
but what if we took these polynomials as a sum?

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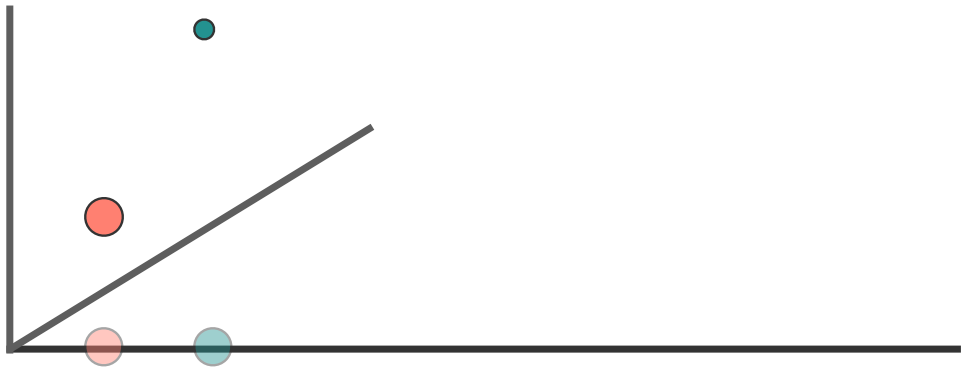

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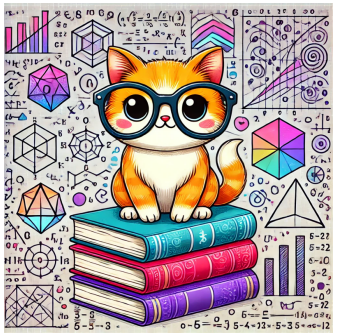
$$K(a,b) = (a \cdot b)^d$$

$$ab + a^2b^2 + a^3b^3 + \cdots + a^\infty b^\infty = (a, a^2, a^3, \ldots, a^\infty)(b, b^2, b^3, \ldots, b^\infty)$$

take sum for infinite terms gives dot product with infinite dimensions!

The Radial Kernel: Taylor Series Expansion



$$K(a,b) = e^{-\gamma}(a-b)^2 = e^{-\gamma(a^2+b^2-2ab)} = e^{-\gamma(a^2+b^2)}e^{\gamma 2ab}$$

set $\gamma = \frac{1}{2} \implies e^{-\frac{1}{2}\gamma(a^2+b^2)}\boxed{e^{ab}}$ Taylor expansion of this term

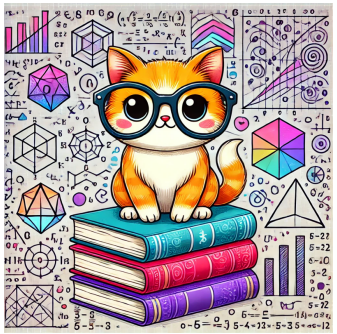
$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(\infty)}(a)}{\infty!}(x-a)^\infty$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^\infty}{\infty!}$$

$$e^{ab} = 1 + (ab) + \frac{(ab)^2}{2!} + \frac{(ab)^3}{3!} + \cdots + \frac{(ab)^\infty}{\infty!}$$

each term contains Polynomial Kernel with $r = 0$ and d from 0 to $d = \infty$

The Radial Kernel: Taylor Series Expansion



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$$e^{ab} = \boxed{1} + \boxed{(ab)} + \frac{1}{2!}\boxed{(ab)^2} + \frac{1}{3!}(ab)^3 + \cdots + \frac{1}{\infty!}\boxed{(ab)^\infty}$$

Radial Kernels have coordinates for infinite dimensions!

$$\boxed{a^0b^0} + \boxed{a^1b^1} + \boxed{a^2b^2} + a^3b^3 + \cdots + \boxed{a^\infty b^\infty} = (a, a^2, a^3, \ldots, a^\infty)(b, b^2, b^3, \ldots, b^\infty)$$

This Week's Practical

Support Vector Machines