Extrema in One Dimension

Lecture 7

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extrema of a function

one of the most important applications of calculus is optimization of functions

This is basically understanding the behavior of a function f on a given interval I

- Does f have a maximum?
- Does f have a minimum?
- Where is the function increasing?
- Where is the function decreasing?

We use derivatives to answer these questions and at the end of today's lecture you'll see how we even can use derivatives to approximate a function…

subclasses of extrema

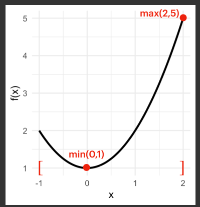
Extrema can be divided in the following subsections

- Maxima and minima
- Absolute (or global) and local (or relative) extrema

Note: Extrema, Maxima and Minima are the plural form of Extremum, Maximum and Minimum, respectively

example

$f(x) = x^2 + 1$



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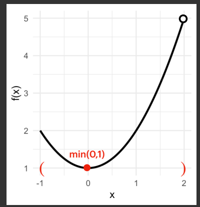
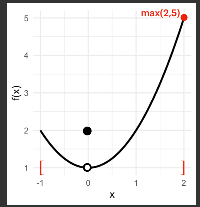


$f(x) = \begin{cases} x^2 + 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$



extrema can occur in interior points or endpoint of an interval,
extrema occurring at endpoints are called endpoint extrema

definition: absolute extrema

Let $f(x)$ be a function defined on interval I and let $a \in I$

- We say $f(x)$ has an absolute maximum at $x = a$ if $f(a)$ is the maximal value of $f(x)$ on I:

$$f(a) \geq f(x) \text{ for all } x \in I$$

- We say $f(x)$ has an absolute minimum at $x = a$ if $f(a)$ is the minimal value of $f(x)$ on I:

$$f(a) \leq f(x) \text{ for all } x \in I$$

the extreme value theorem

If f is continuous on a closed interval $[a, b]$ then f has both a minimum and a maximum

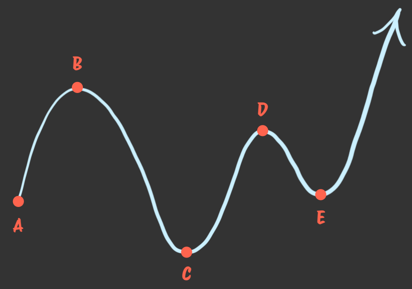
definition: local extrema

Let $f(x)$ be a function.

- We say that $f(x)$ has an local maximum at $x = a$ if $f(a)$ is the maximal value of $f(x)$ on some open interval I inside the domain of f containing a.
- We say that $f(x)$ has an local minimum at $x = a$ if $f(a)$ is the minimal value of $f(x)$ on some open interval I inside the domain of f containing a.

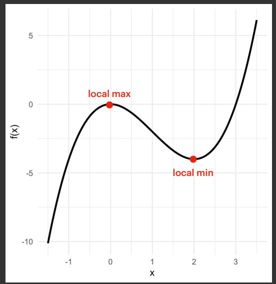
we look for "valleys" and "peaks" of a function

example



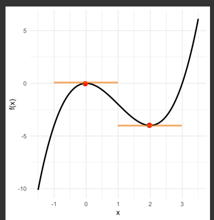
example

$f(x) = x^3 - 3x^2$



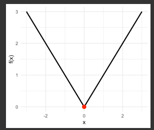
the first derivative test for local extreme values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.



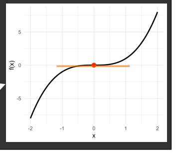
critical point

An interior point of the domain of a function f where f' is zero or undefined is a critical point of f.



If $f(c)$ is a local maximum or minimum, then c is a critical point of $f(x)$

Note: the reverse does not hold, i.e., if $f'(c) = 0$ then $f(c) = 0$ is not necessarily a maximum or minimum.



finding the absolute extrema

Suppose that $f(x)$ is continuous on the closed interval $[a, b]$. Then $f(x)$ attains its absolute maximum and minimum values on $[a, b]$ at either:

- a critical point
- one of the end points

How to find the absolute extrema of a continuous function f on a finite closed interval:

1. Evaluate f at all critical points and endpoints
2. Take the largest and smallest of these values

finding the absolute extrema

example

Find the absolute maximum and minimum values of $f(x) = 3x - x^3$ on the interval $[-1,3]$.

1. Find the critical points:

   $f'(x) = 3 - 3x^2 = 3(1 - x^2)$

   $f'(x) = 0$ when $x = \pm 1$, these are the critical points

2. Make a table with the critical points inside the interval and its endpoints:

x	$3x - x^3$	
1	$3 - 1 = 2$	maximal value is at $x = 1$
-1	$-3 - (-1) = -2$	
3	$3 \cdot 3 - 3^3 = -18$	minimal value is at $x = 3$

finding the absolute extrema

exercise 1

Find the absolute maximum and minimum value of $f(x) = 10x(2 - \ln x)$ on the interval $[1, e^2]$.

recall: the second derivative

- To characterize troughs and humps we need the second derivative
- We can view $f'(x)$ itself as a function that we differentiate it again:

$$\frac{d}{dx}\left(f'(x)\right) = \frac{d^2}{dx^2}\left(f(x)\right) = f''(x)$$

The geometric interpretation of f'':

1. $f''(x) > 0$
2. $f''(x) < 0$
3. $f''(c) = 0$

the second derivative test for local extreme values.

The geometric interpretation of f'':

1. If $f''(x) > 0$ then the slope of the tangent line is increasing in value
   $\Longrightarrow$ if $f'(c) = 0$ and $f''(c) > 0$, then around c, $f(x)$ is a trough/valley
   $\Longrightarrow$ we can expect a local minimum value of f at c

possible shapes:

the second derivative test for local extreme values.

The geometric interpretation of f'':

2. If $f''(x) < 0$ then the slope of the tangent line is decreasing in value
   $\Longrightarrow$ if $f'(c) = 0$ and $f''(c) < 0$, then around c, $f(x)$ is a hump/peak
   $\Longrightarrow$ we can expect a local maximum value of f at c

possible shapes:

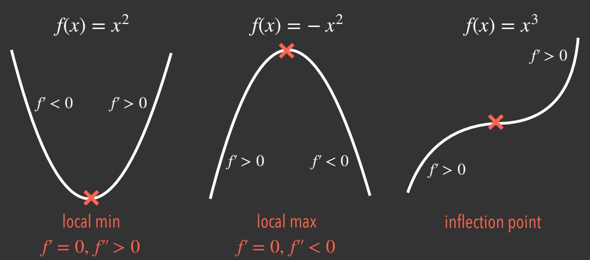
saddle or inflection points

The geometric interpretation of f'':

3. If $f'(c) = 0$ and $f''(c) = 0$ then the slope often doesn't change sign
   i.e. it goes from positive slope to zero to positive slope (decreasing to zero then increasing), or negative to zero to negative (increasing to zero then decreasing).

possible shapes:

example



hands on exercises in finding these points in your tutorial....

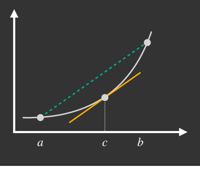
mean value theorem

Suppose $f(x)$ is a continuous function on closed interval $[a, b]$ and is differentiable on the interval's interior (a, b) at which a point c exists such that

slope of the tangent line

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

slope of the secant line between $x = a$ and $x = b$



note: there can be more than one such value of c

mean value theorem generalized

Suppose $f(x)$ is a continuous function on closed interval $[a, b]$ and is differentiable on the interval's interior (a, b) at which a point c exists such that

slope of the tangent line

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slope of the secant line between $x = a$ and $x = b$

we can re-write the above equation as

$$\begin{aligned} f(b) &= f(a) + f'(c)(b - a) \\ &\approx f(a) + f'(a)(b - a) \end{aligned}$$

this looks very much like the linear approximation for $f(b)$ using the tangent line at $x = a$

this is the idea behind Taylor's Theorem and using polynomials to approximate a smooth function

Taylor's theorem

A function $f(x)$ can be expressed as a sum of terms derived from its derivatives at a specific point, plus a remainder term. Mathematically, the theorem can be written as:

$$f(x) = \underbrace{f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n}_{\text{Taylor polynomial of degree } n} + R_n(x)$$

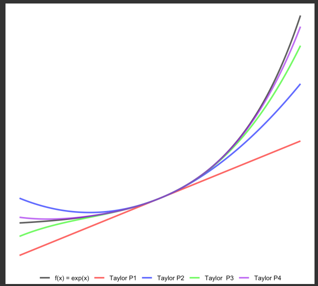
where

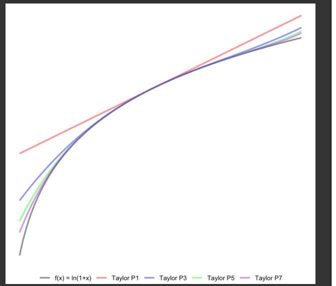
a is the point around which the function is approximated

n is the order of the polynomial approximation

$R_n(x)$ is the remainder term, representing the error in the approximation after n terms.

Taylor approximation





Taylor approximation

example

Find the Taylor polynomial of $f(x) = \dfrac{1}{1+x}$ of degree 3 at $x = 0$.

We have that $TP_3(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2}x^2 + \dfrac{f'''(0)}{3!}x^3$

Computing the successive derivatives we get

$$f(x) = \frac{1}{1+x}, \quad f'(x) = \frac{-1}{(1+x)^2}, \quad f''(x) = \frac{2}{(1+x)^3}, \quad f'''(x) = \frac{-6}{(1+x)^4}$$

Substituting $x = 0$ we get

$$f(0) = 1, \quad f'(0) = -1, \quad f''(x) = 2, \quad f'''(x) = -6$$

Therefore, the Taylor polynomial of f of degree 3 at $x = 0$ is equal to

$$TP_3(x) = 1 - x + x^2 - x^3$$

Taylor approximation

exercise 2

Approximate the function $f(x) = \sqrt{1+x}$ using a Taylor polynomial of degree 3 centered at $x = 0$. Compare it to the true value.