the Hessian

The Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued function $f(x_1, x_2, \ldots, x_n)$ is

$$H = \begin{bmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- The Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued function
- The Hessian provides a way to classify critical points (where the gradient is zero):
  - If the Hessian is positive definite ($H > 0$), the critical point is a local minimum.
  - If the Hessian is negative definite ($H < 0$), the critical point is a local maximum.
  - If the Hessian has both positive and negative eigenvalues, the critical point is a saddle point.

global min/max

example

For Function: $f(x, y) = 4 - x^2 - y^2$ the Hessian is given by

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

Both eigenvalues are negative (-2, -2), so H is negative definite

$\Longrightarrow$ Local max at (0,0)

Is it global?

Since $f(x, y) \to -\infty$ as $|x|, |y| \to \infty$, the function is unbounded and has only one maximum, which must be global

$\Longrightarrow$ (0,0) is the global maximum, and f(0,0) = 4