powers of diagonalizable matrices

Multiplying diagonal matrices together just multiplies their diagonal entries:

$$\begin{bmatrix} x_1 & 0 & 0 \\ 0 & x_2 & 0 \\ 0 & 0 & x_3 \end{bmatrix} \begin{bmatrix} y_1 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & y_3 \end{bmatrix} = \begin{bmatrix} x_1y_1 & 0 & 0 \\ 0 & x_2y_2 & 0 \\ 0 & 0 & x_3y_3 \end{bmatrix}$$

so it is easy to take powers of a diagonal matrix:

$$\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}^n = \begin{bmatrix} x^n & 0 & 0 \\ 0 & y^n & 0 \\ 0 & 0 & z^n \end{bmatrix}$$

$\Longrightarrow$ if $A = QDQ^{-1}$ where D is the diagonal matrix, then $A^n = QD^nQ^{-1}$

diagonalization theorem

An $n \times n$ matrix A is diagonalizable if if and only if A has n linearly independent eigenvectors.

simply put: a matrix is diagonalizable if it has distinct eigenvalues or, if it has repeated eigenvalues, it still has enough independent eigenvectors to match its dimensionality

- the eigenvalues determine the entries of the diagonal matrix
- the eigenvectors form the columns of a matrix Q
- the transformation reflects how the original matrix A can be simplified, highlighting the intrinsic properties of A

$$A = QDQ^{-1} \quad \text{where} \quad Q = \begin{bmatrix} | & | & & | \\ \vec{v_1} & \vec{v_2} & \dots & \vec{v_1} \\ | & | & & | \end{bmatrix}, \quad D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

$\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ are linearly independent eigenvectors, and $\lambda_1, \lambda_2, \dots, \lambda_n$ are the corresponding eigenvalues