gradient: saddle point

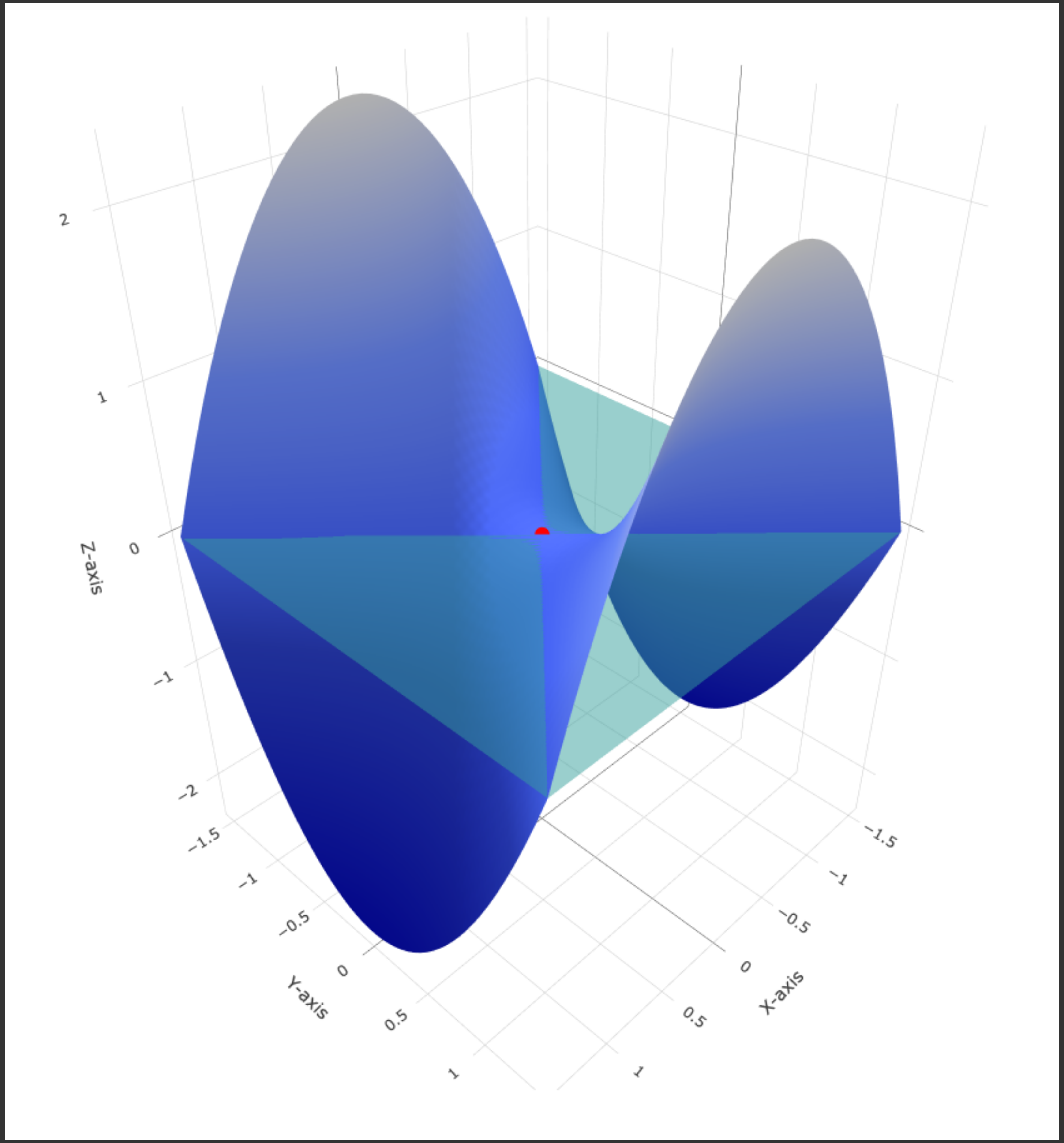
example

For $f(x, y) = x^2 - y^2$, the gradient is:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (2x, -2y)$$

$$\nabla f = \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \vec{0}$$

What's the solution? Is it max or min?



the Hessian

The Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued function $f(x_1, x_2, \ldots, x_n)$ is

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

- The Hessian matrix is a square matrix of second-order partial derivatives of a scalar-valued function
- The Hessian provides a way to classify critical points (where the gradient is zero):
  - If the Hessian is positive definite ($H > 0$), the critical point is a local minimum.
  - If the Hessian is negative definite ($H < 0$), the critical point is a local maximum.
  - If the Hessian has both positive and negative eigenvalues, the critical point is a saddle point.