

# Matrix Decomposition

CS 556

Erisa Terolli

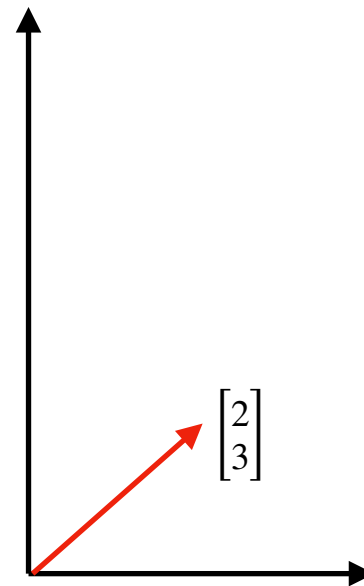
# Outline

- Eigen Decomposition
- Singular Value Decomposition (SVD)
- Dimensionality Reduction with SVD
- Movie Recommender System with SVD

# 2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

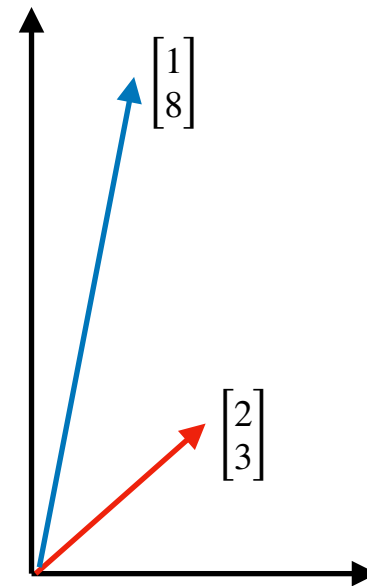
$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = ?$$



# 2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

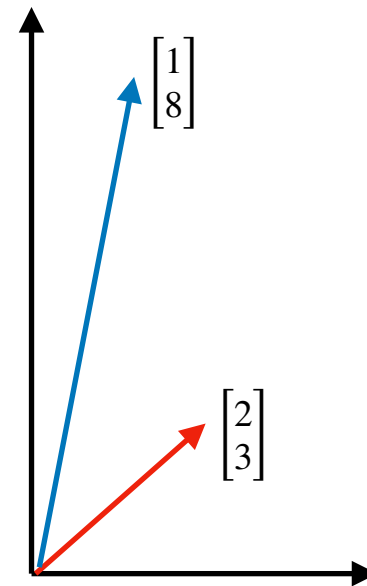
$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$



# 2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$



**The matrix will transform the vector by rotating and stretching/shortening it**

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

*Rotation by  $\theta$*

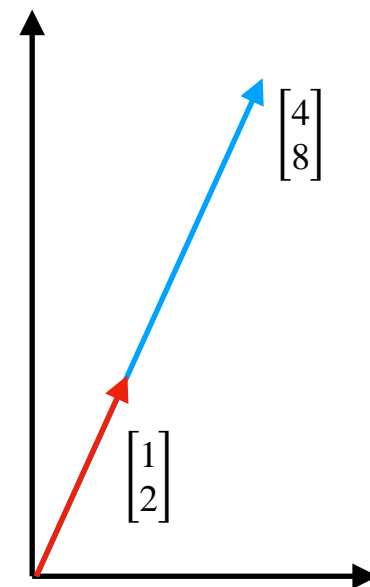
$$S = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}$$

*Stretching by  $\alpha$*

# 2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

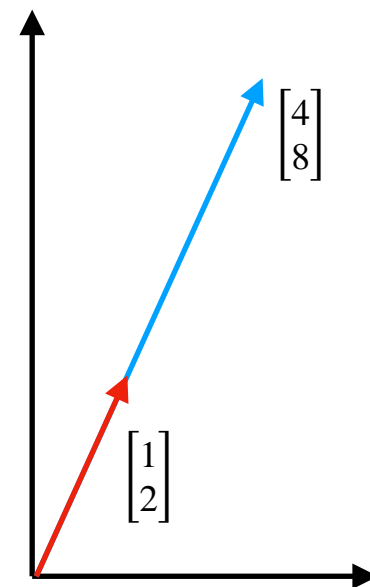


# 2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

$$4 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

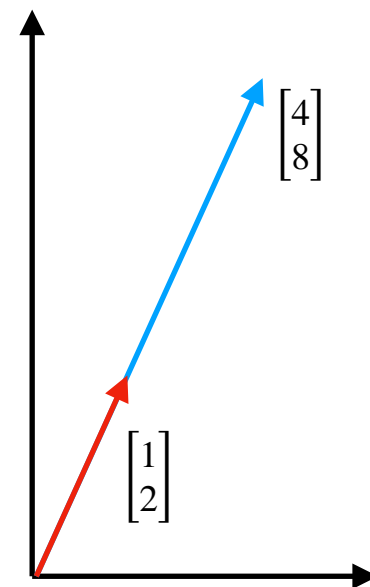


# 2D Transformation Matrix

What happens when we multiply a 2D matrix with a vector?

$$\begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

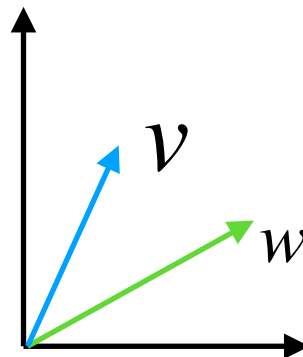
$$\begin{array}{c} \text{Eigen value} \nearrow 4 \\ \text{Eigen vector} \nwarrow \end{array} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$





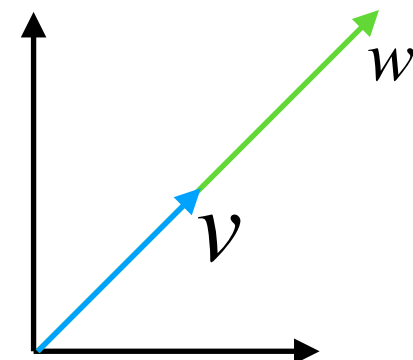
# Eigenvectors & Eigenvalues

$$A \mathbf{v} = \mathbf{w}$$



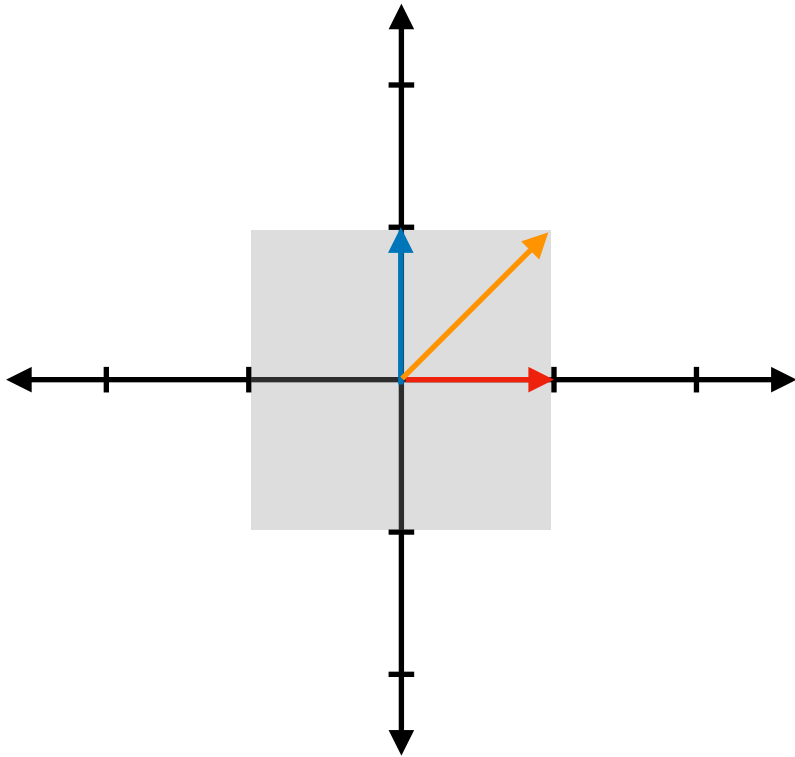
Transformation matrix  $A$  is applied to a vector  $\mathbf{v}$  and outputs a vector  $\mathbf{w}$ . If  $\mathbf{w}$  points in the same direction as  $\mathbf{v}$  (a.k.a. lies on the same 1 dimensional subspace), then  $\mathbf{v}$  is an eigenvector of matrix  $A$ .

$$\begin{aligned} A\mathbf{v} &= \mathbf{w} \\ \lambda\mathbf{v} &= \mathbf{w} \\ A\mathbf{v} &= \lambda\mathbf{v} \end{aligned}$$



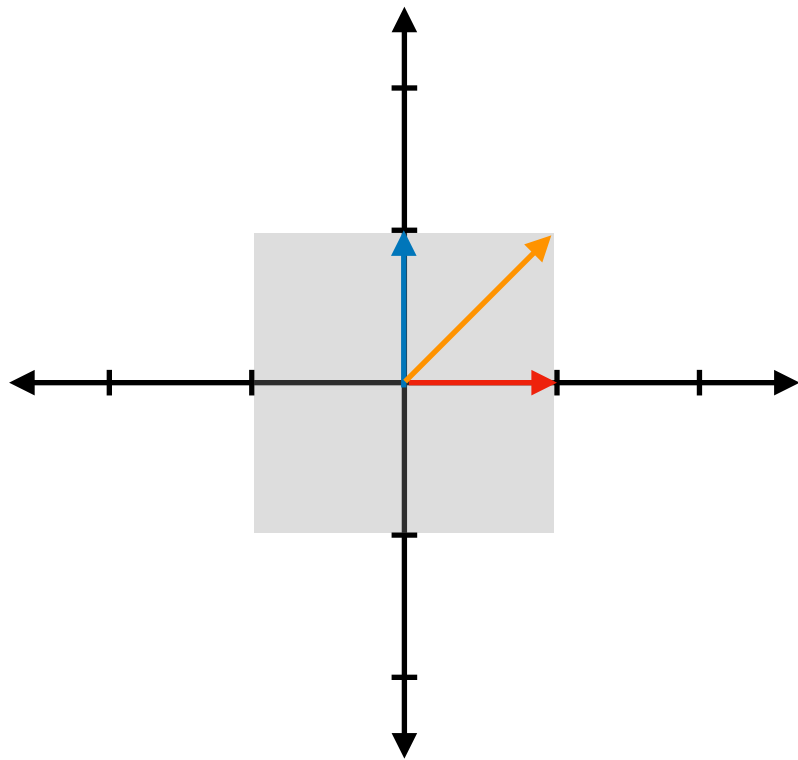
$\lambda$  is an eigenvalue associated with eigenvector  $\mathbf{v}$  of  $A$

# Eigenvectors & Eigenvalues

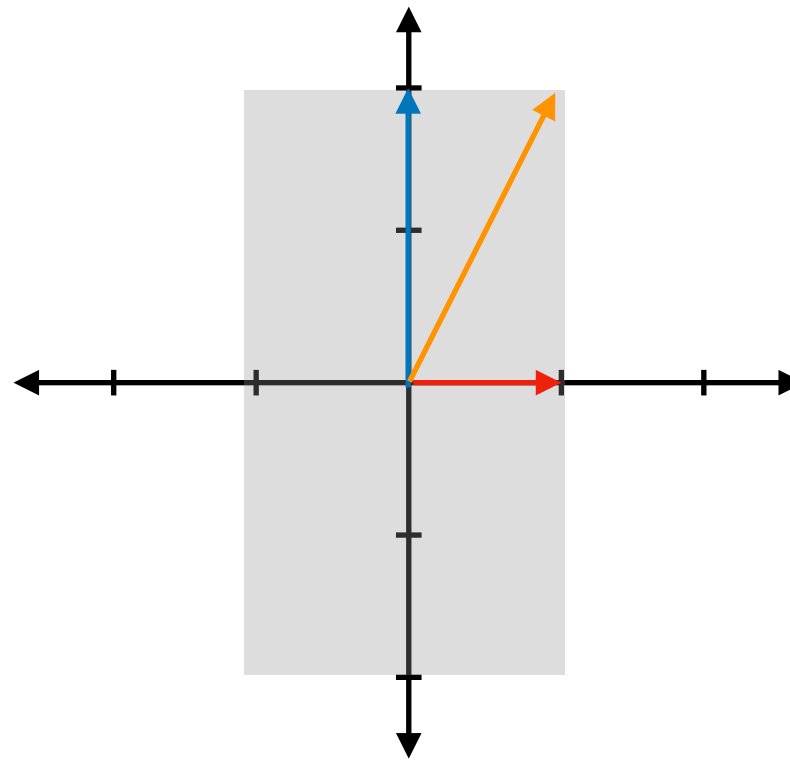


$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

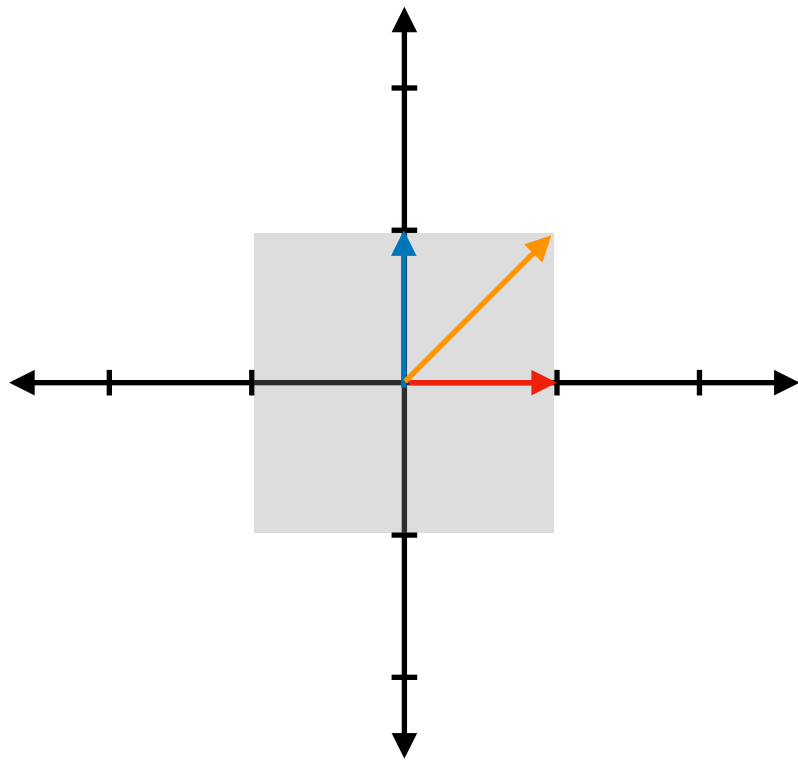
# Eigenvectors & Eigenvalues



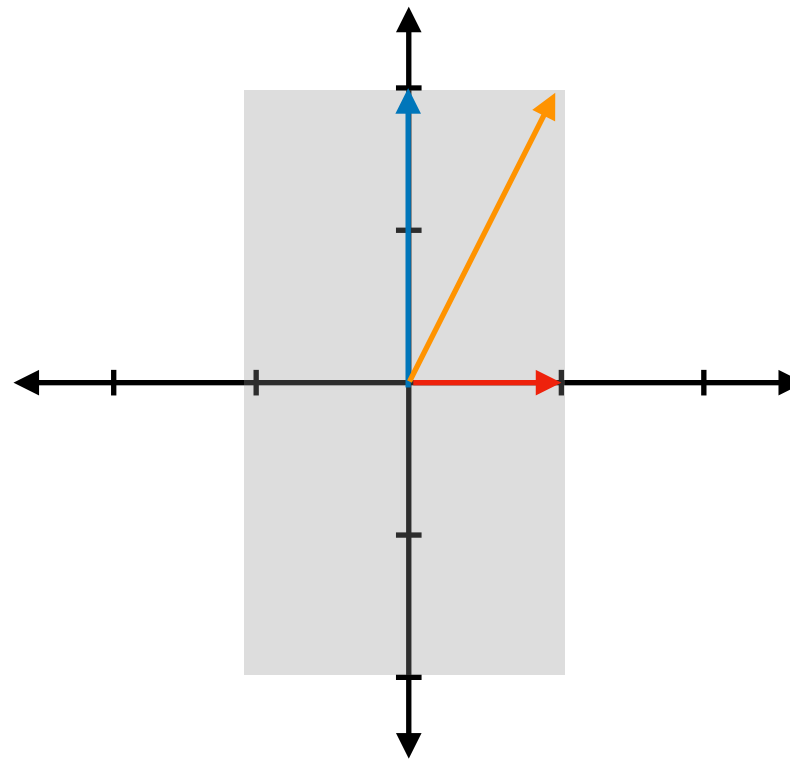
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# Eigenvectors & Eigenvalues

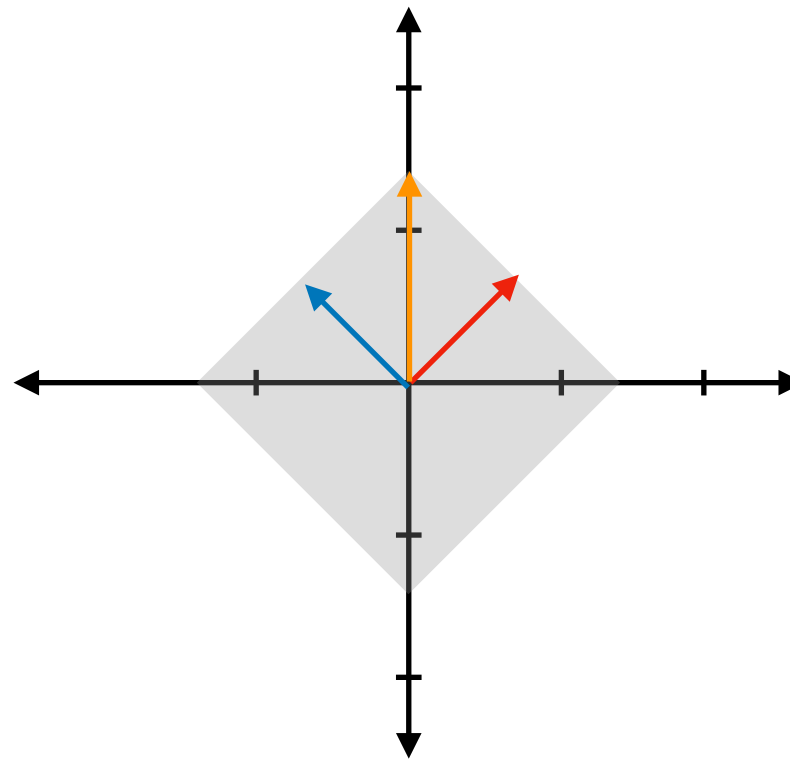
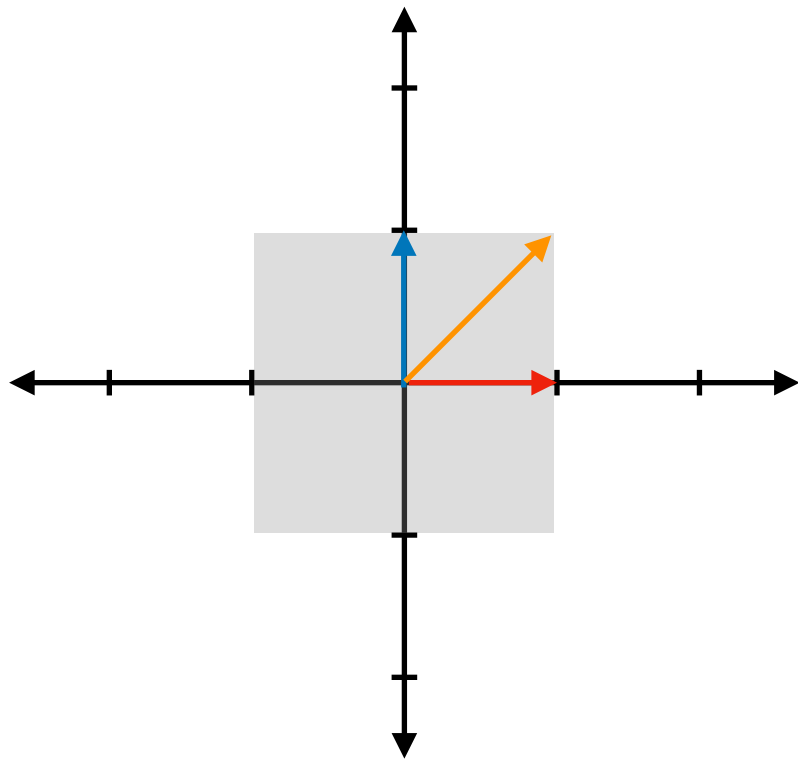


$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

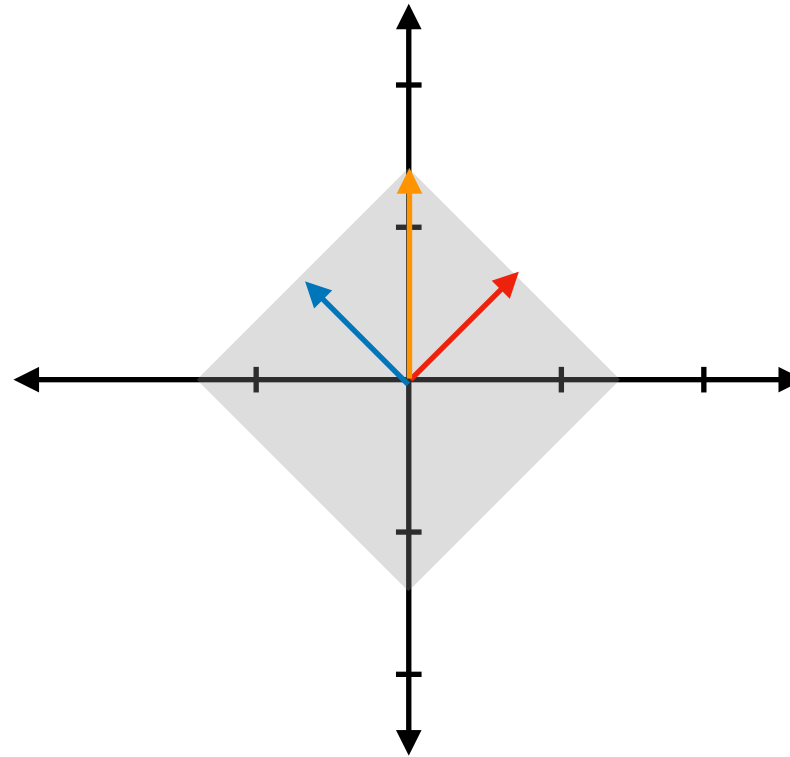
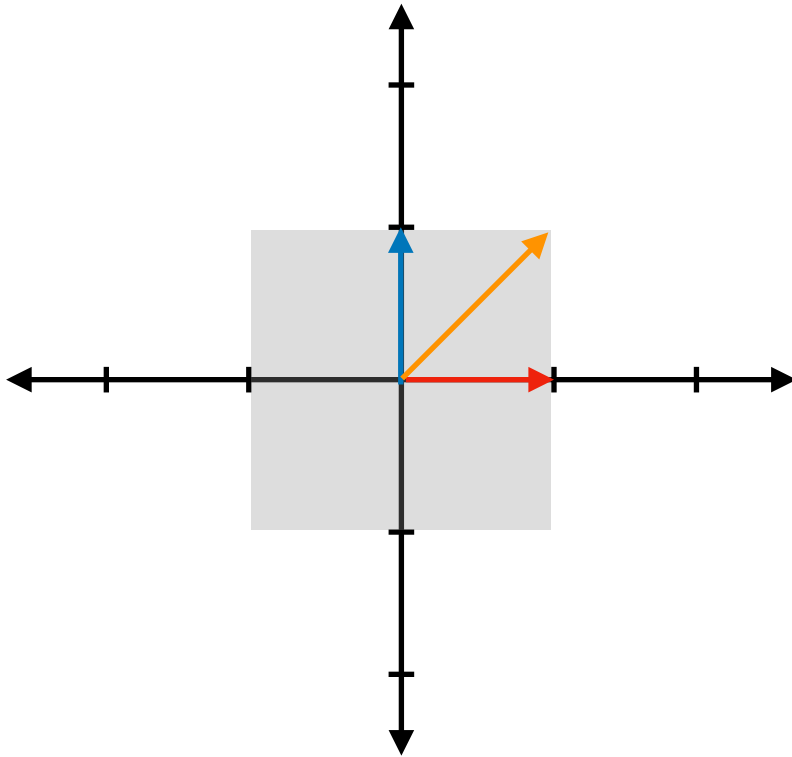


2 eigenvectors with values 1 and 2

# Eigenvectors & Eigenvalues

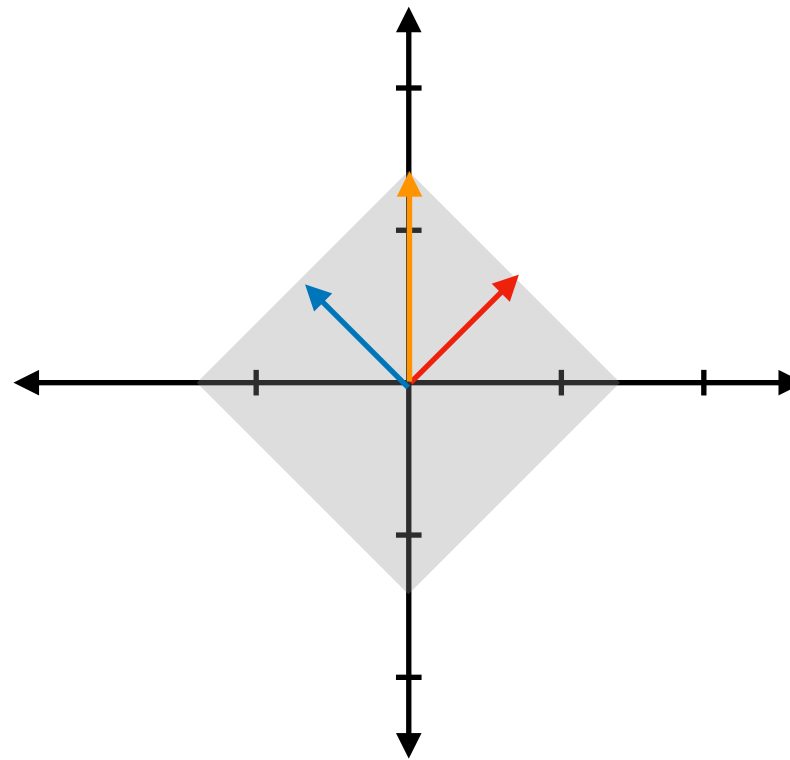
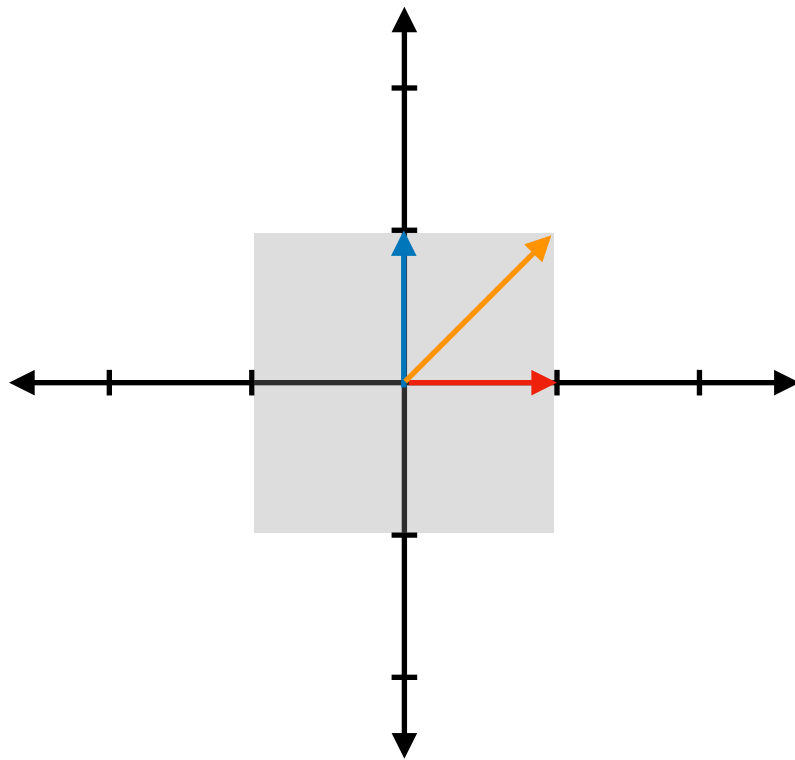


# Eigenvectors & Eigenvalues



0 Eigenvectors

# Eigenvectors & Eigenvalues



0 Eigenvectors

Check the vectors that lie on the same span after transformation  
and measure how much they magnitude change

# Finding Eigenvalues

$$Av = \lambda v$$

$$Av - \lambda v = 0$$

$$(A - \lambda)v = 0$$

Shift the matrix  $A$  by  $\lambda$   $\longrightarrow (A - \lambda I)v = 0$

It has nontrivial null space.  
It must be singular.  $\longrightarrow A - \lambda I$

$$\det(A - \lambda I) = 0$$

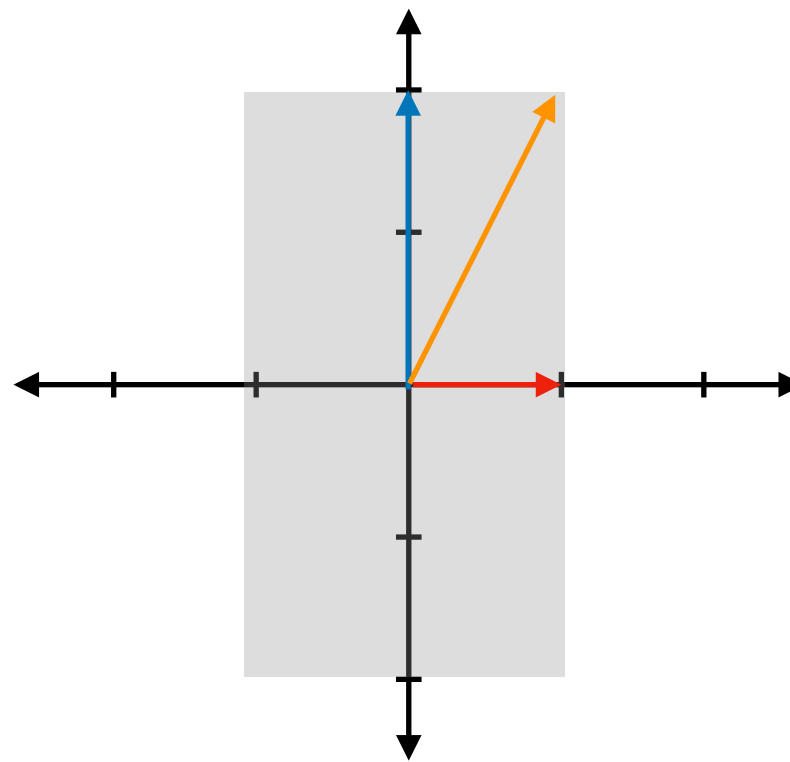
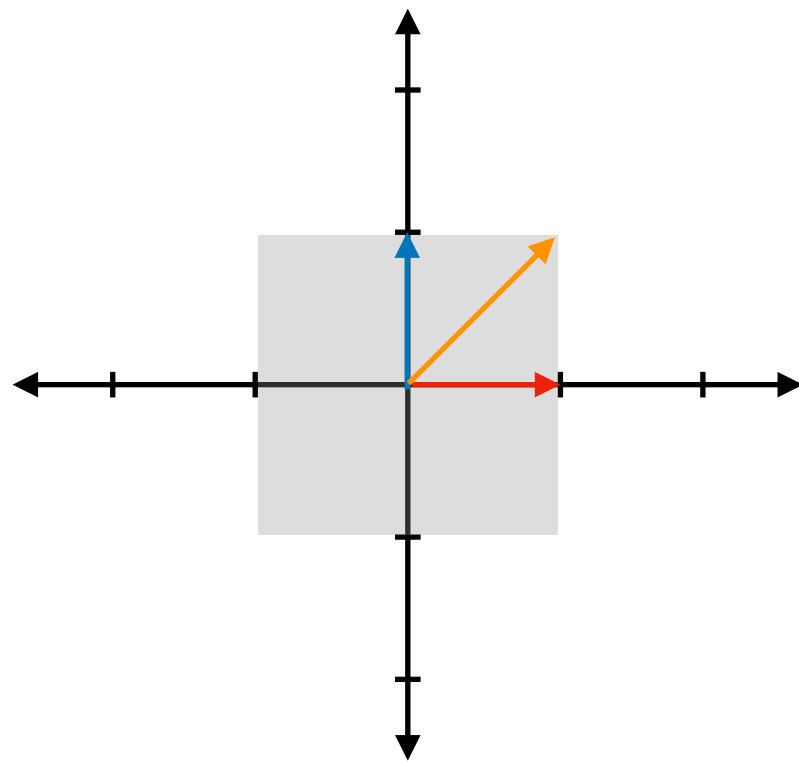
The Determinant of singular matrices is 0.



# Finding Eigenvectors

1. Find all eigenvalues  $\lambda$
2. For each  $\lambda$ , find  $v \in N(A - \lambda I)$ 
  - Find a vector  $v$  that is in the null space of the matrix  $(A - \lambda I)$

# Example 1



$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

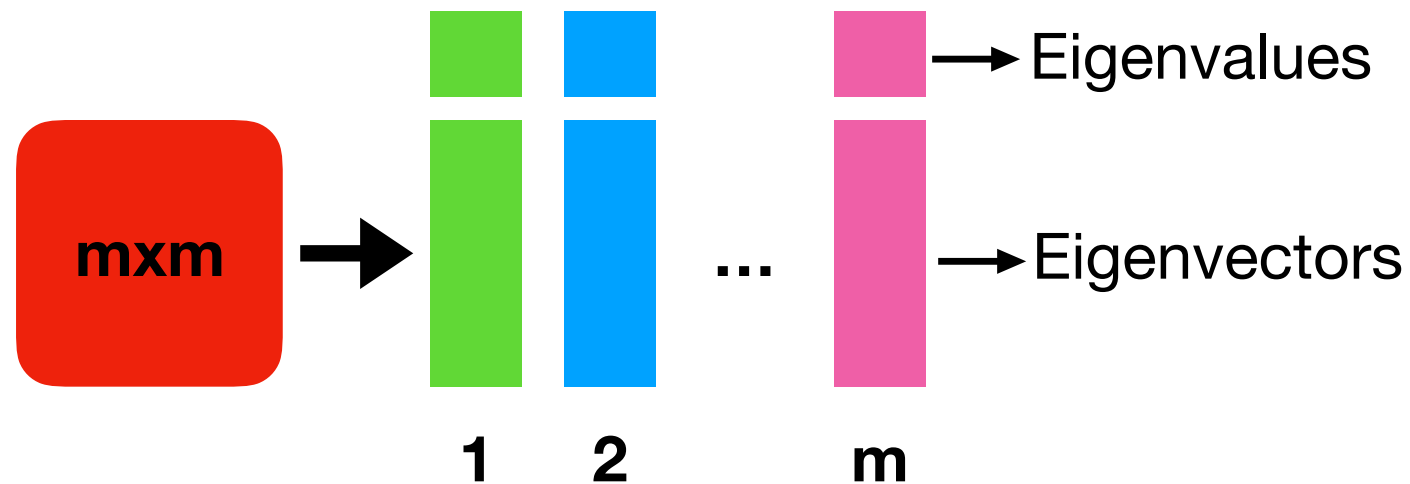
$$\det \left( \begin{bmatrix} 1 - \lambda & 0 \\ 0 & 2 - \lambda \end{bmatrix} \right) = 0$$

$$(1 - \lambda)(2 - \lambda) = 0$$

$$\lambda = 1, \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ v_2 \end{bmatrix} = 0, \quad v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

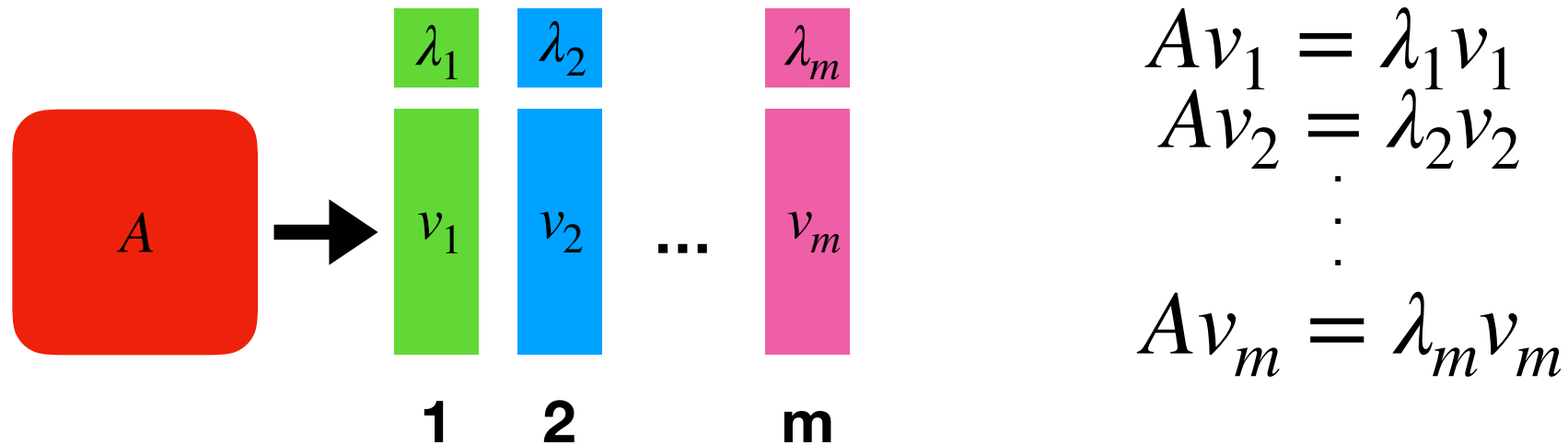
$$\lambda = 2, \quad \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_1 \\ 0 \end{bmatrix} = 0, \quad v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

# Eigen Decomposition



Eigendecomposition is the factorization of a matrix into a canonical form, whereby the matrix is represented in terms of its eigenvalues and eigenvectors.

# Diagonalization



$$\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \\ Av_3 &= \lambda_3 v_3 \end{aligned} \longrightarrow \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} v_{11}\lambda_1 & v_{12}\lambda_2 & v_{13}\lambda_3 \\ v_{21}\lambda_1 & v_{22}\lambda_2 & v_{23}\lambda_3 \\ v_{31}\lambda_1 & v_{32}\lambda_2 & v_{33}\lambda_3 \end{bmatrix} \longrightarrow AV = V\Lambda$$

$$AV = V\Lambda \begin{cases} \longrightarrow V^{-1}AV = \Lambda \\ \longrightarrow A = V\Lambda V^{-1} \end{cases}$$

Finding a set of basis vectors  $V$  such that the original matrix  $A$  is diagonal in that basis space, assuming that the columns in  $V$  are linearly independent.

# Spectral Theorem

Every symmetric matrix has the factorization  $S = Q\Lambda Q^{-1}$  with real eigenvalues in  $\Lambda$  and orthonormal eigenvectors in the columns of  $Q$  :  $S = Q\Lambda Q^{-1} = Q\Lambda Q^T$  with  $Q^{-1} = Q^T$ .

I.e. Eigenvectors of a real symmetric matrix are always perpendicular.

**Proof:** Let's assume that  $x$  and  $y$  are two eigenvectors of  $S$  with eigenvalues  $\lambda_1$  and  $\lambda_2$  where  $\lambda_1 \neq \lambda_2$ .

$$Sx = \lambda_1 x, Sy = \lambda_2 y$$

$$(Sx)^T y = (\lambda_1 x)^T y, x^T S^T y = x^T \lambda_1 y, x^T Sy = x^T \lambda_1 y$$

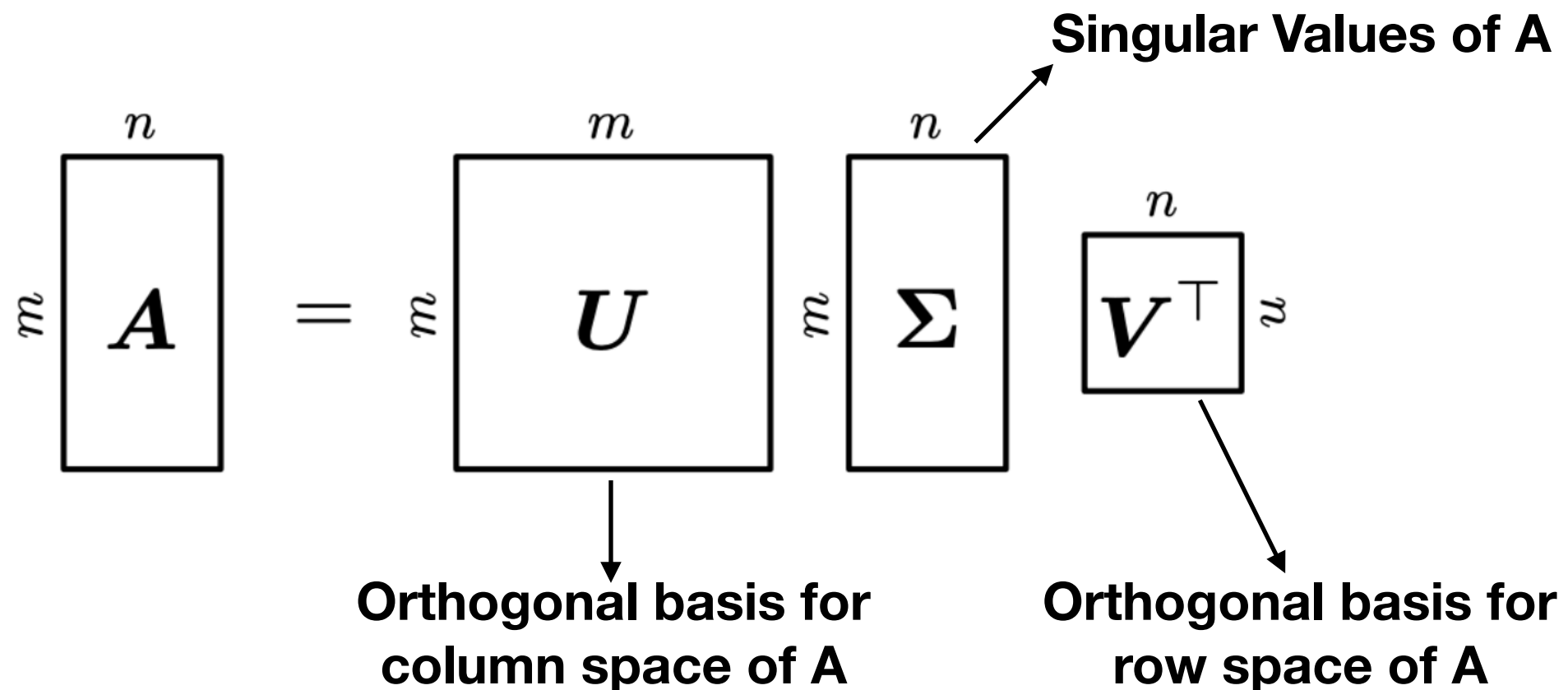
$$x^T \lambda_2 y = x^T \lambda_1 y$$

$$\text{Since } \lambda_1 \neq \lambda_2 \text{ then } x^T y = 0 \rightarrow x \perp y.$$

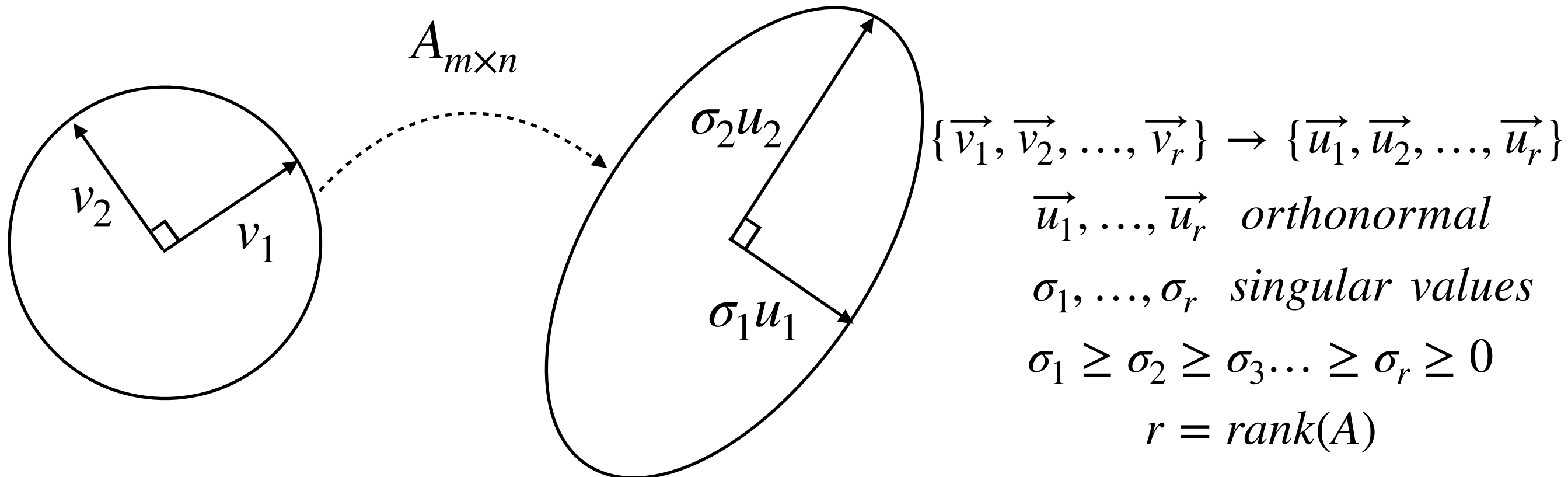
# Singular Value Decomposition

# SVD

The goal of SVD is to decompose a matrix  $A$  as the product of 3 other matrices  $A = U\Sigma V^T$ , where matrix  $V$  and  $U$  are orthogonal matrices and  $\Sigma$  is a diagonal matrix.



# Formulation



$$A\vec{v}_1 = \sigma_1\vec{u}_1, A\vec{v}_2 = \sigma_2\vec{u}_2 \rightarrow A\vec{v}_j = \sigma_j\vec{u}_j, \forall j = 1, 2, \dots, r$$

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} v_1 & v_2 & \dots & v_r \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & \dots & u_r \end{bmatrix} \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \cdot & \\ & & & \cdot \\ & & & & \sigma_r \end{bmatrix}$$

$$AV = U\Sigma$$

$$A = U\Sigma V^{-1} = U\Sigma V^T$$



# How to compute SVD?

$$A = U\Sigma V^T$$

$$A^T A = (U\Sigma V^T)^T U\Sigma V^T$$

$$A^T A = V\Sigma^T U^T U\Sigma V^T$$

$$A^T A = V\Sigma^T I \Sigma V^T$$

$$A^T A = V\Sigma^T \Sigma V^T$$

$$A^T A = V\Sigma^2 V^T$$

To find  $V$  compute eigendecomposition of  $A^T A$  where  $V$  will be the eigenvectors of  $A^T A$  and  $\Sigma^2$  are the eigenvalues of  $A^T A$ .

$$AA^T = U\Sigma^2 U^T$$

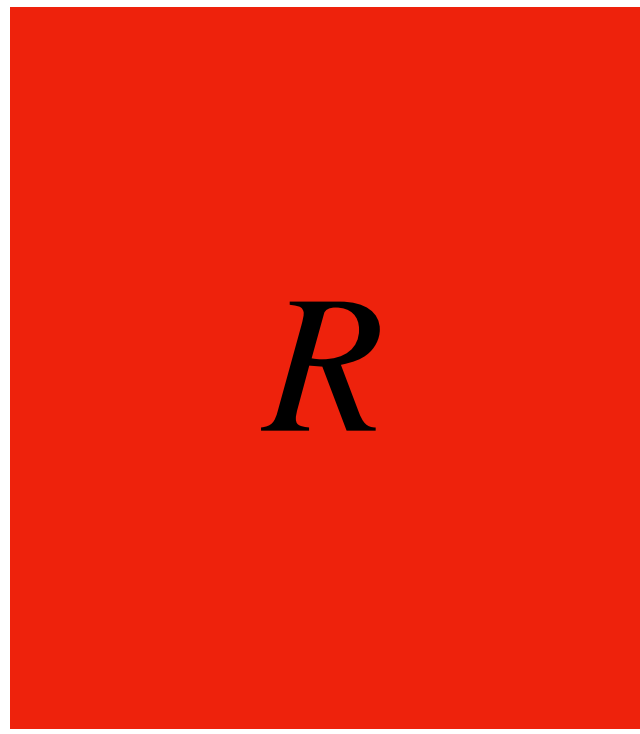
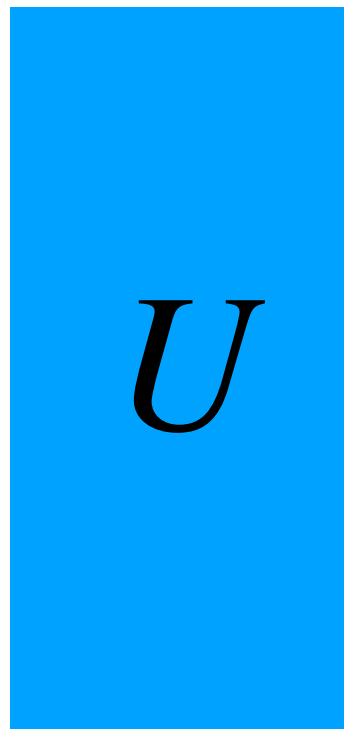
To find  $U$  compute eigendecomposition of  $AA^T$  where  $U$  will be the eigenvectors of  $AA^T$  and  $\Sigma^2$  are the eigenvalues of  $AA^T$ .

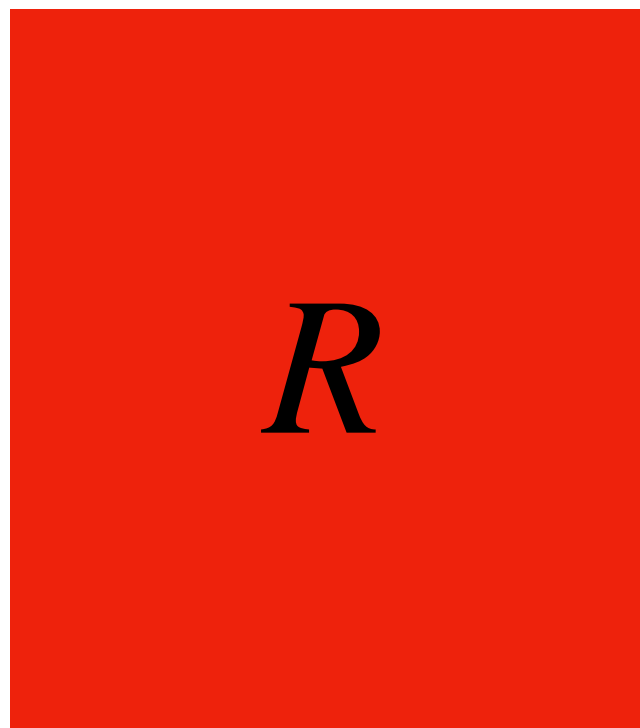
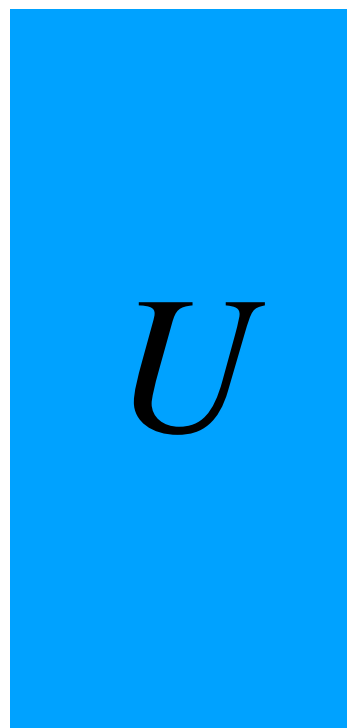
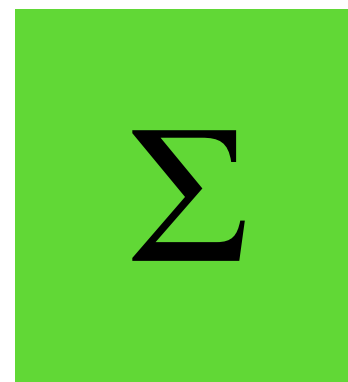
# **Dimensionality Reduction with SVD**



*R*

*m \times n*

 $R$  $m \times n$  $=$  $U$  $m \times r$

 $R$  $m \times n$  $=$  $U$  $m \times r$  $\Sigma$  $r \times r$

The diagram illustrates the Singular Value Decomposition (SVD) of a matrix  $R$ . It shows the equation  $R = U \Sigma V^T$  with dimensions indicated below each matrix:

- $R$  (red square) has dimensions  $m \times n$ .
- $U$  (blue square) has dimensions  $m \times r$ .
- $\Sigma$  (green square) has dimensions  $r \times r$ .
- $V^T$  (yellow square) has dimensions  $r \times n$ .

The dimensions are written in italics below each matrix:  $m \times n$ ,  $m \times r$ ,  $r \times r$ , and  $r \times n$ .

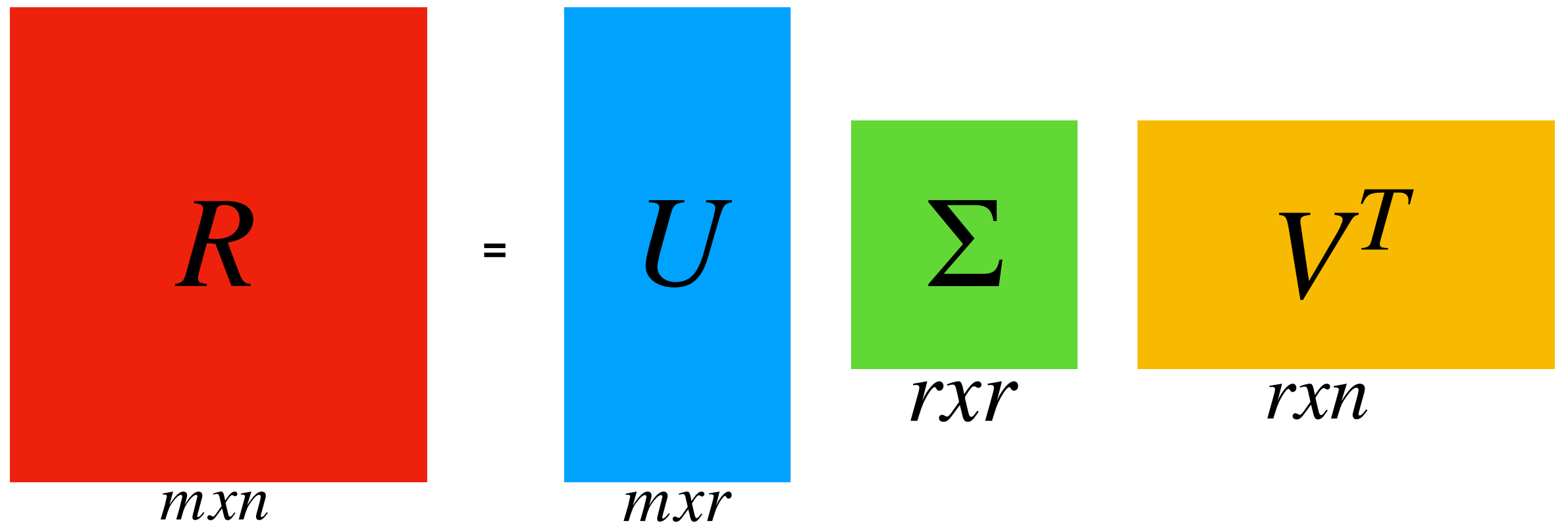


Diagram illustrating the Singular Value Decomposition (SVD) of a matrix  $R$  (size  $m \times n$ ):

$$R \quad m \times n = U \quad m \times r \quad \Sigma \quad r \times r \quad V^T \quad r \times n$$

The matrix  $R$  is represented by a red square. The matrix  $U$  is represented by a blue rectangle. The matrix  $\Sigma$  is represented by a green square. The matrix  $V^T$  is represented by a yellow rectangle.

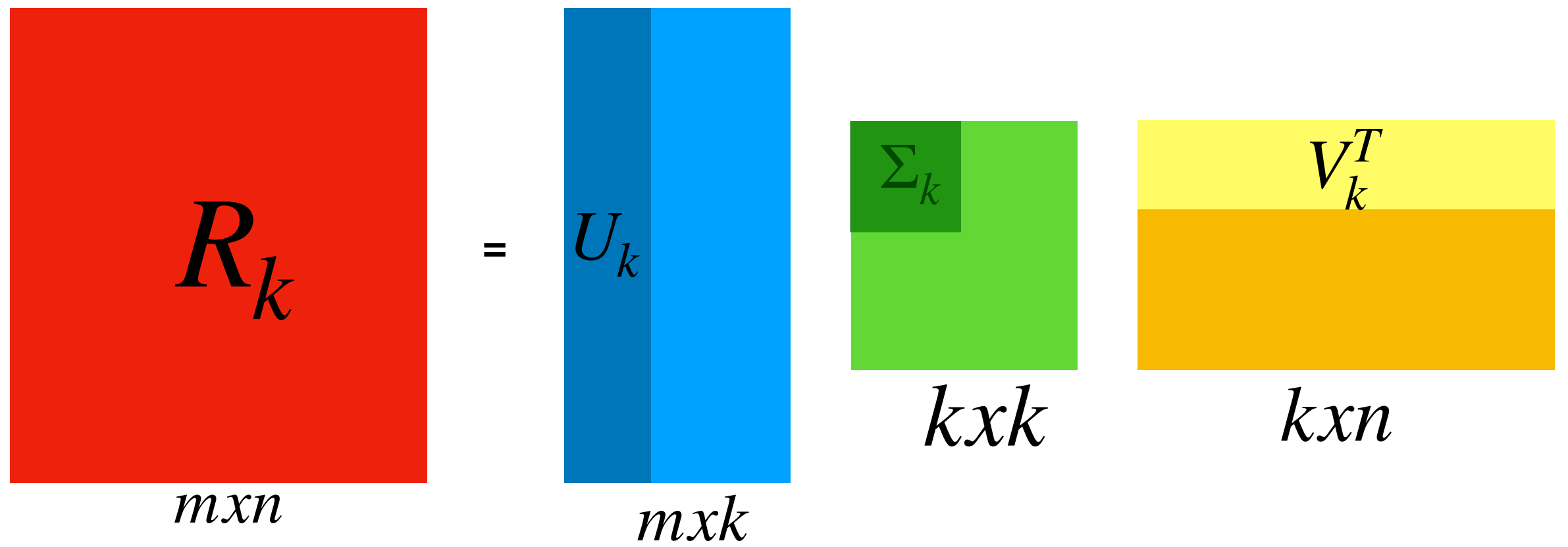


Diagram illustrating the Truncated Singular Value Decomposition (TSVD) of a matrix  $R_k$  (size  $m \times n$ ):

$$R_k \quad m \times n = U_k \quad m \times k \quad \Sigma_k \quad k \times k \quad V_k^T \quad k \times n$$

The matrix  $R_k$  is represented by a red square. The matrix  $U_k$  is represented by a blue rectangle with a dark blue vertical strip on the left. The matrix  $\Sigma_k$  is represented by a green square with a dark green top-left corner. The matrix  $V_k^T$  is represented by a yellow rectangle with a light yellow top section.

$$\begin{matrix}
 \text{Red Box} & = & \text{Blue Box} & \text{Green Box} & \text{Yellow Box} \\
 R & & U & \Sigma & V^T \\
 m \times n & & m \times r & r \times r & r \times n
 \end{matrix}$$

$$\begin{matrix}
 \text{Red Box} & = & \text{Blue Box} & \text{Green Box} & \text{Yellow Box} \\
 R_k & & U_k & \Sigma_k & V_k^T \\
 m \times n & & m \times k & k \times k & k \times n
 \end{matrix}$$

### Eckart - Young Theorem 1936 [1]

The reconstruction matrix  $R_k = U_k \Sigma_k V_k^T$  is the closest rank- $k$  matrix to  $R$ .



# Movie Recommendation with SVD

# User-Movie Rating



John



5

1

3

5

Tom



?

?

?

2

Alice



4

?

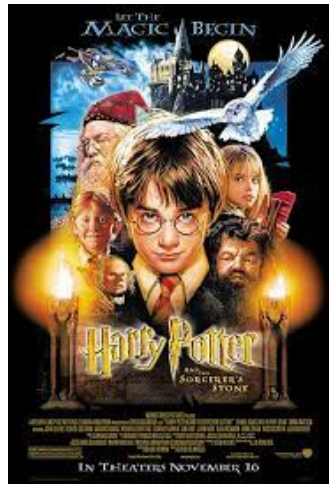
3

?

# User Movie Rating

- Overfit representation of user tastes

Alice



John



- Computational Complexity, potentially poor results
- Goal: A more compact representation of users tastes and items descriptions
  - Represent our tastes not in terms of the products we like and dislike, but in terms of higher level attributes

# How to create these representations?

- Singular Value Decomposition (SVD)
- Intuitive Description: Reduce space to a smaller taste space that is compact and robust

# Singular Value Decomposition

$$R = U\Sigma V^T$$

- *R is  $m \times n$  rating matrix*
- *U is  $m \times k$  user feature affinity matrix*
- *V is  $n \times k$  item feature relevance matrix*
- *$\Sigma$  is  $k \times k$  diagonal feature weight matrix*
- *Exists for any real R*

# Rating Matrix

	The Notebook	Captain America	Seven	Forrest Gump
User 1	5	-	-	5
User 2	-	5	-	3
...				
User m	1	5	5	-

## User Feature Matrix

	Drama	Thriller	Action	Fantasy
User 1	0.7	0.2	0.1	0
User 2	0	0.05	0.75	0.2
User m	0.4	0.3	0.2	0.1

## Feature Diagonal

	Drama	Thriller	Action	Fantasy
Drama	0.4	0	0	0
Thriller	0	0.3	0	0
Action	0	0	0.27	0
Fantasy	0	0	0	0.05

## Movie Feature Matrix

	Drama	Thriller	Action	Fantasy
The notebook	1.0	0	0	0
Captain America	0	0.1	0.7	0.2
Seven	0.5	0.5	0.1	0
Forrest Gump	0.98	0	0	0.03

## Rating Matrix

	The Notebook	Captain America	Seven	Forrest Gump
User 1	5	???	-	5
User 2	-	5	-	3
...				
User m	1	5	5	-

## User Feature Matrix

	Drama	Thriller	Action	Fantasy
User 1	0.7	0.2	0.1	0
User 2	0	0.05	0.75	0.2
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## Feature Diagonal

	Drama	Thriller	Action	Fantasy
Drama	0.4	0	0	0
Thriller	0	0.3	0	0
Action	0	0	0.27	0
Fantasy	0	0	0	0.05

## Movie Feature Matrix

	Drama	Thriller	Action	Fantasy
The notebook	1.0	0	0	0
Captain America	0	0.1	0.7	0.9
Seven	0.5	0.5	0.1	0
Forrest Gump	0.98	0	0	0.03

$$R(\text{User}_1, \text{Captain America}) = U_{\text{User}_1} S V_{\text{Captain America}}^T$$

# References

[1] <https://citeseer.ist.psu.edu/viewdoc/download;jsessionid=6B65A9A35513A03050594C88798801DB?doi=10.1.1.23.1831&rep=rep1&type=pdf>



**Thank you!**

# In-Class Exercise

Find all eigenvalues and eigenvectors of matrix A.

$$A = \begin{bmatrix} 3 & 1 \\ 4 & 6 \end{bmatrix}$$

# In Class Exercise

Compute SVD for matrix A:

$$A = \begin{bmatrix} 4 & 0 \\ 3 & -5 \end{bmatrix}$$