Orthogonality

CS 556

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Orthogonal Vectors

Two vectors v and w are orthogonal when their dot product is 0.

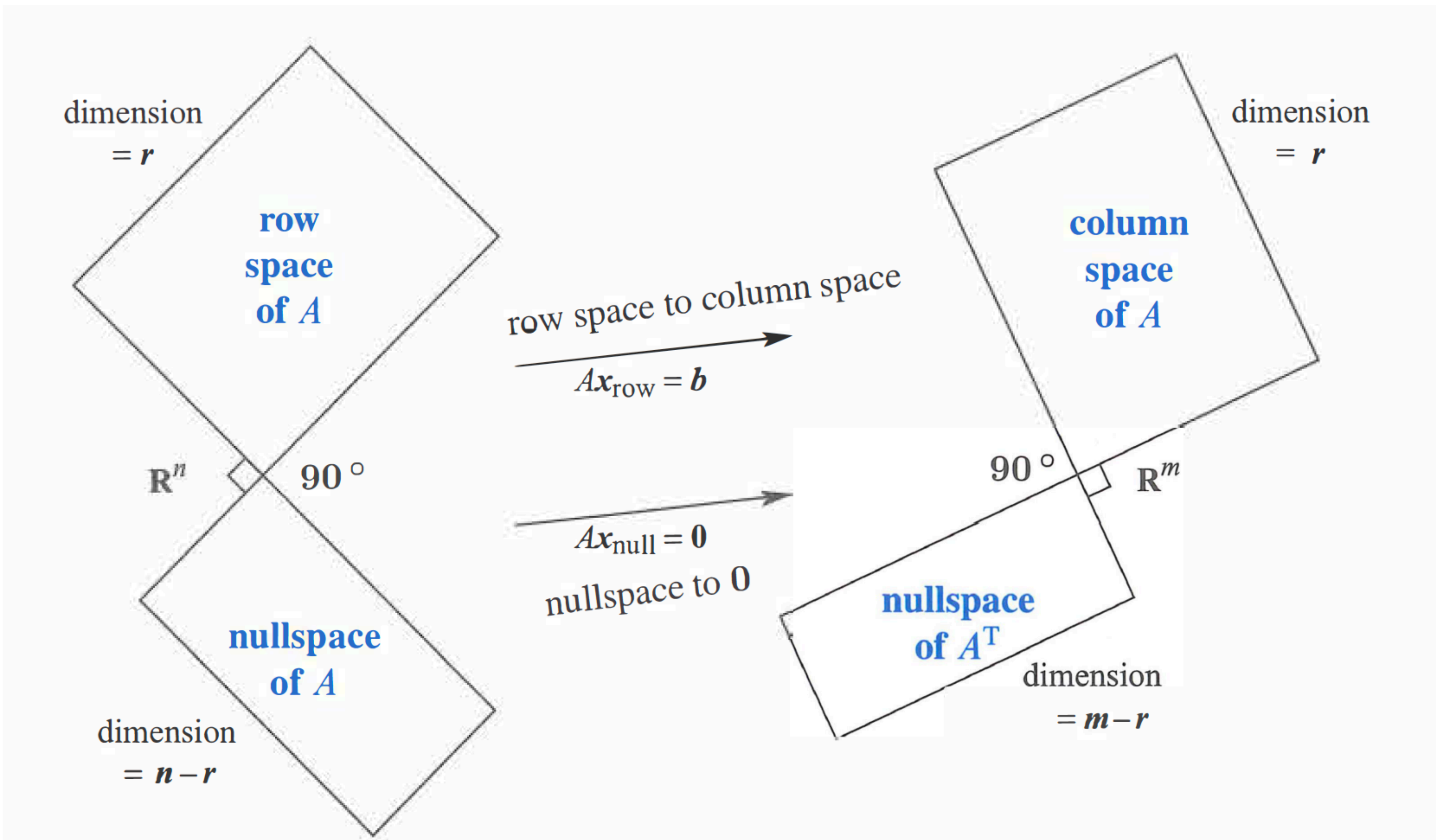
$$v^T w = 0, \qquad ||v||^2 + ||w||^2 = ||v + w||^2$$

Orthogonal Subspaces

Two subspaces V and W of a vector space are orthogonal if every vector $v \in V$ is perpendicular to every vector $w \in W$.

$$v^T w = 0 \; for \; all \; v \in V \; and \; w \in W$$

Orthogonality of the Four Spaces



Orthogonality of the Four Spaces

The null space $N(A)$ and the row space $C(A^T)$ are orthogonal subspaces of R^n.

Proof:

$$A\mathbf{x} = \begin{bmatrix} row\ 1 \\ \vdots \\ row\ m \end{bmatrix} [\mathbf{x}] = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} \rightarrow \begin{matrix} (row\ 1) \cdot \mathbf{x} = 0 \\ \vdots \\ (row\ m) \cdot \mathbf{x} = 0 \end{matrix}$$

The left null space $N(A^T)$ and the column space $C(A)$ are orthogonal in R^m.

Projections

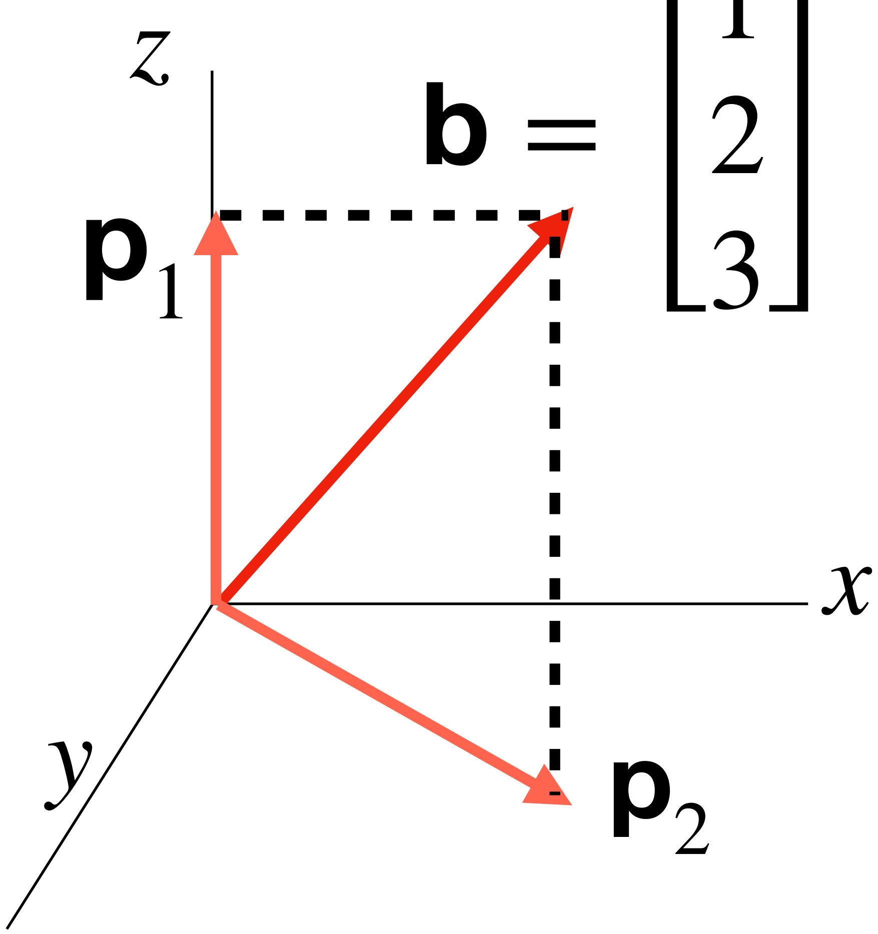
Projections

The projection of b onto a subspace S is the closest vector p in S. For example, when a vector b is projected onto a line, its projection p is the part of b along that line. When b is projected into a plane, p is the part in that plane.

A projection matrix P is a symmetric matrix with $P^2 = P$. The projection of b is Pb.

Projections

What is the projection of vector b onto the *z-axis* line and *xy* plane?



$$\mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\mathbf{p}_1 = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \quad \mathbf{P}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} \end{bmatrix}$$

$$\mathbf{p}_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \quad \mathbf{P}_2 = \begin{bmatrix} \mathbf{1} & 0 & 0 \\ 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

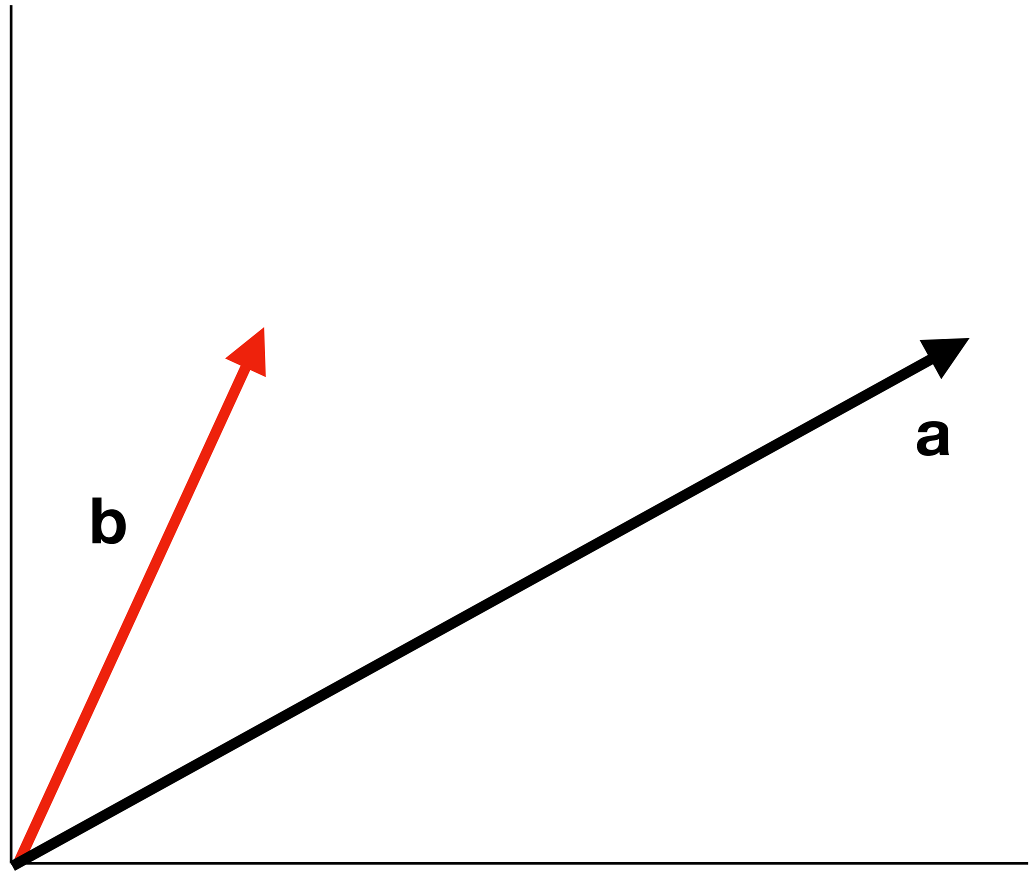
$$\mathbf{p}_1 + \mathbf{p}_2 = b,$$
$$\mathbf{P}_1 + \mathbf{P}_2 = I$$
the line and plane are orthogonal complements.

Why do we need projections?

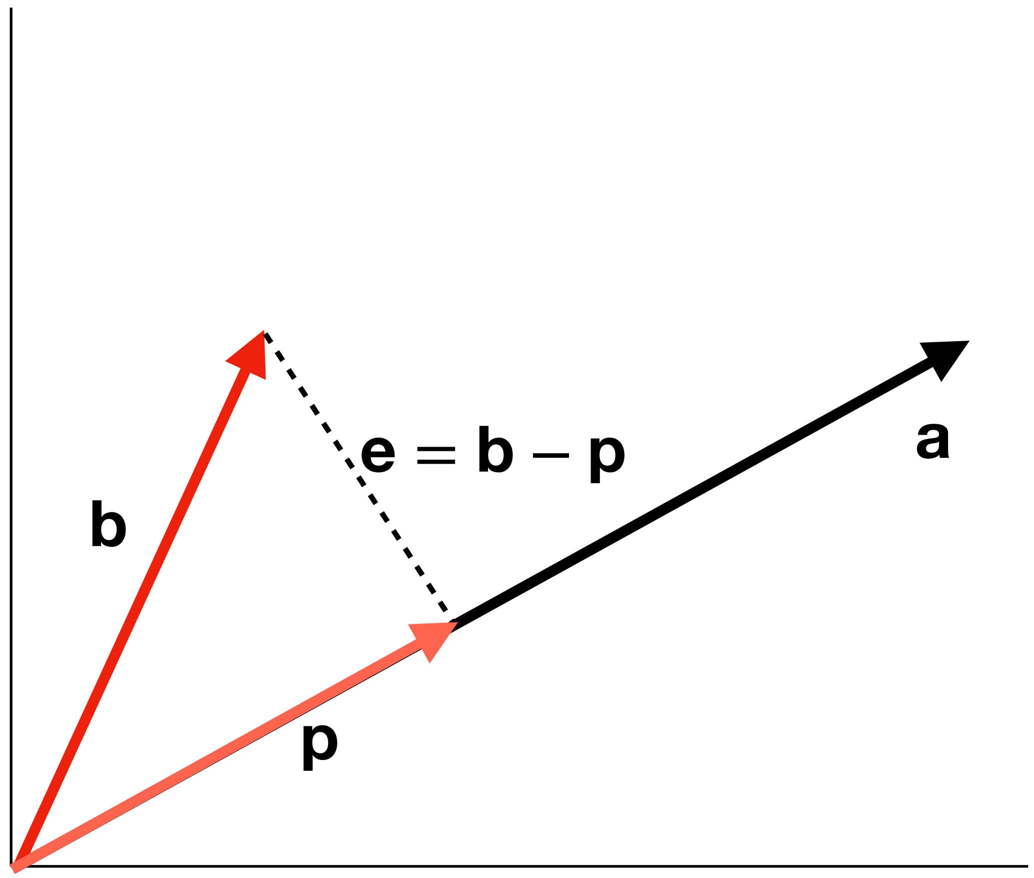
We need projections to cover cases when $A\mathbf{x} = \mathbf{b}$ does not have any solutions. In these cases we can solve the closest problem $A\hat{\mathbf{x}} = \mathbf{p}$.

If $A\mathbf{x} = \mathbf{b}$ does not have any solutions, $\mathbf{b}$ is not in the column space of A. We can solve $A\hat{\mathbf{x}} = \mathbf{p}$ instead where $\mathbf{p}$ is the projection of $\mathbf{b}$ onto the column space of A.

Projection onto a Line



Projection onto a Line



$$\mathbf{p} = \hat{x}\mathbf{a}, \ \ \mathbf{a} \perp (\mathbf{b} - \mathbf{p})$$

$$\mathbf{a} \cdot (\mathbf{b} - \hat{x}\mathbf{a}) = 0$$

$$\mathbf{a} \cdot \mathbf{b} - \hat{x}\mathbf{a} \cdot \mathbf{a} = 0$$

$$\mathbf{a}^T\mathbf{b} - \hat{x}\mathbf{a}^T\mathbf{a} = 0$$

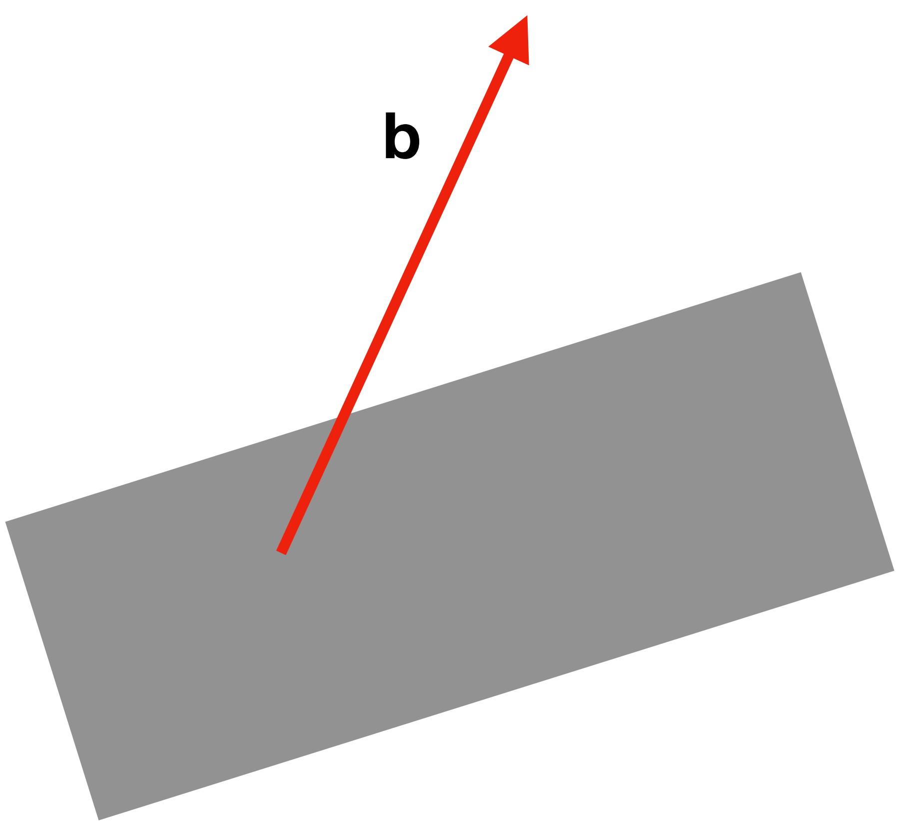
$$\hat{x} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}$$

$$\mathbf{p} = \frac{\mathbf{a}^T\mathbf{b}}{\mathbf{a}^T\mathbf{a}}\mathbf{a}$$

$$\mathbf{P} = \frac{\mathbf{a}\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}$$

Projection onto a Subspace

Find the projection of $\mathbf{b}$ in $\mathbb{R}^m$ onto the subspace spanned by columns of A.



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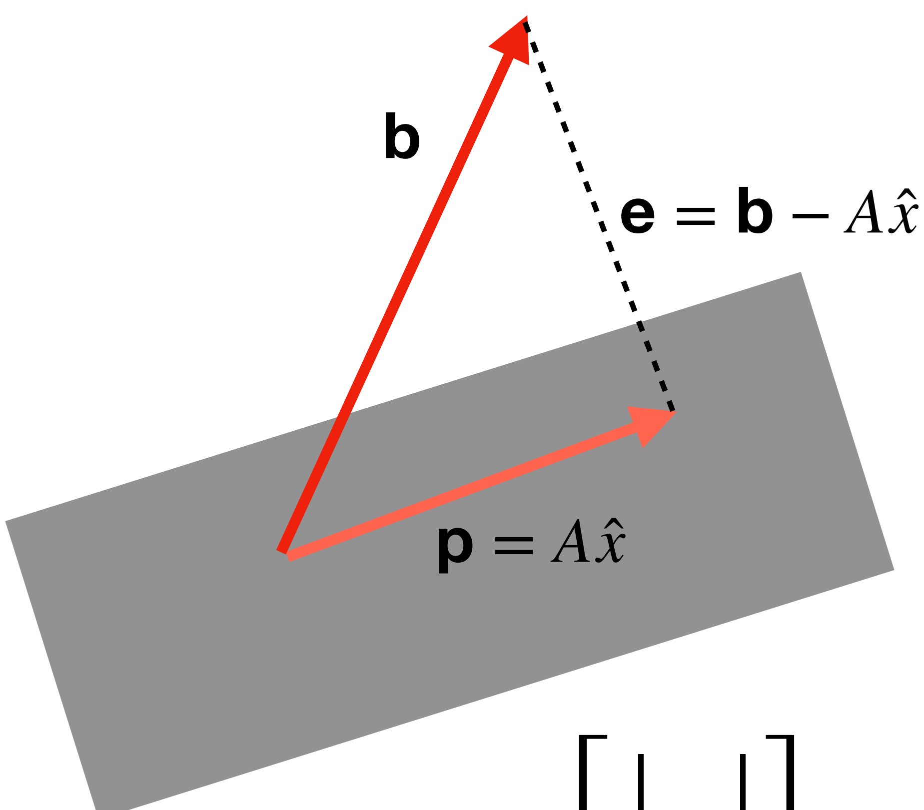

$$\mathbf{A} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix}$$

$a_1, and\ a_2\ are\ basis\ for\ plane$

$$\mathbf{p} = A\hat{x}$$

$$\begin{bmatrix} a_1^T \\ a_2^T \end{bmatrix} (\mathbf{b} - A\hat{x}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A^T(\mathbf{b} - A\hat{x}) = 0$$

$$A^T A\hat{x} = A^T\mathbf{b}$$

$$\hat{x} = (A^T A)^{-1} A^T \mathbf{b}$$

$$\mathbf{p} = A(A^T A)^{-1} A^T \mathbf{b}$$

$$\mathbf{P} = A(A^T A)^{-1} A^T$$

$A^T A$ Invertibility

Theorem

If A has linearly independent columns then $A^T A$ is invertible.

Proof:

Show that the the null space of $A^T A$ is only the zero vector.

$$A^T A x = 0$$

$$x^T A^T A x = 0$$

$$(Ax)^T A x = 0$$

$$Ax = 0$$

$$x = 0$$

Since columns of A are independent

$\blacksquare$

Least Squares Approximation

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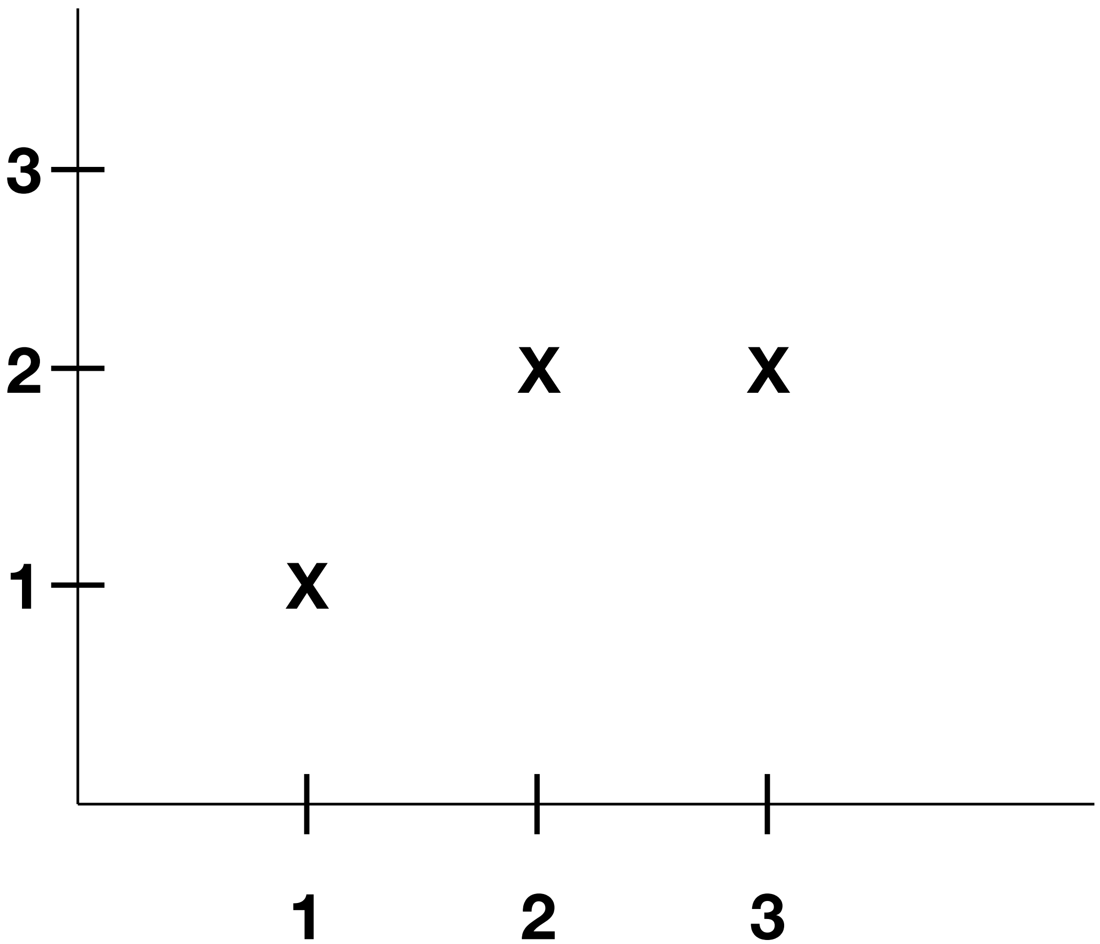
Find the closest line to the points (1,1), (2,2) and (3,2).



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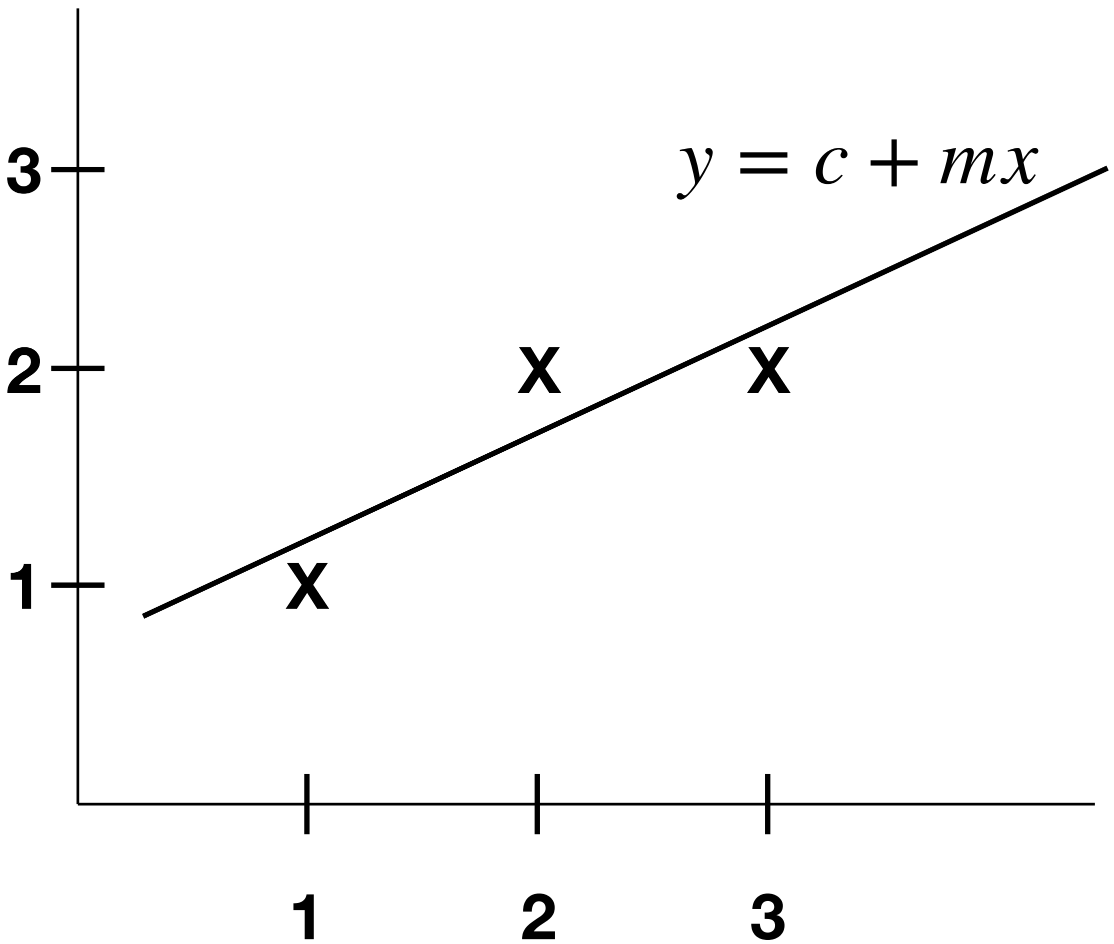

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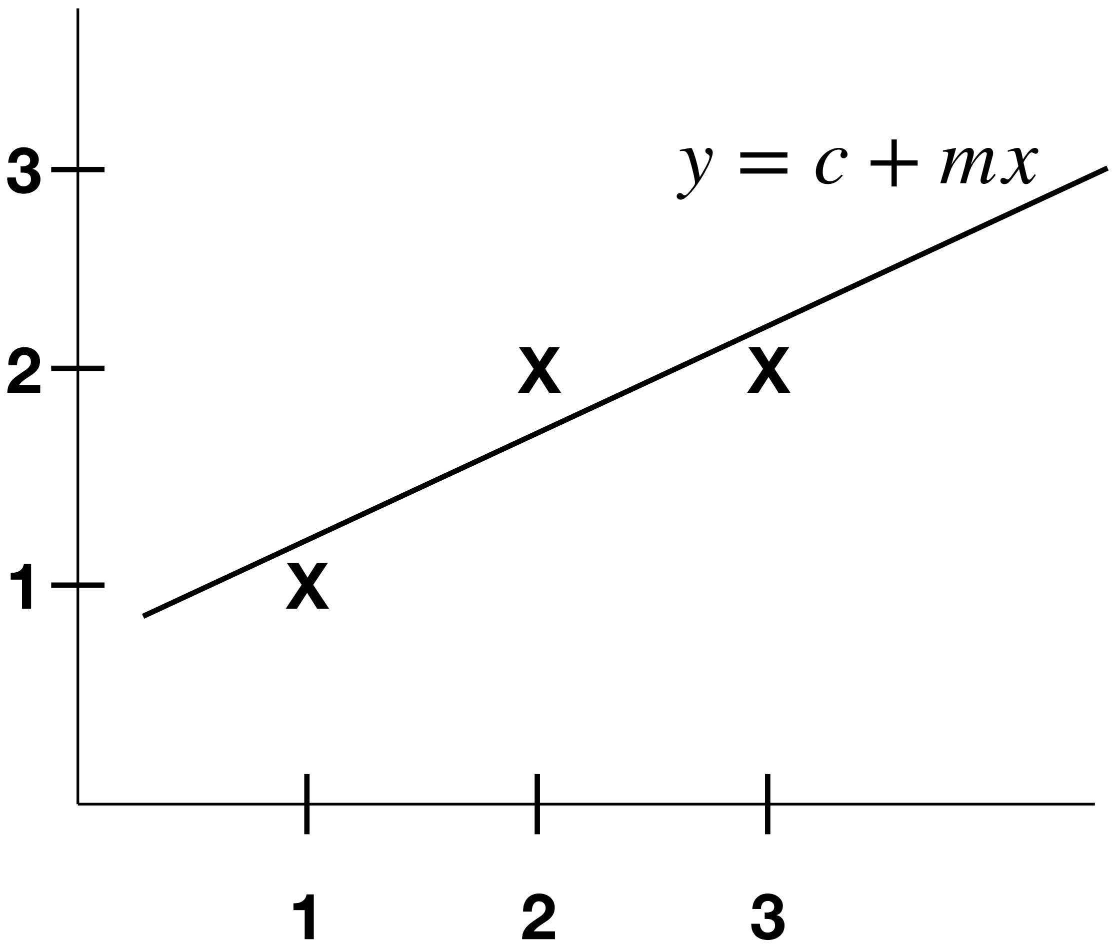

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

3 equation,
2 unknown
No solution

$$Ax = b$$

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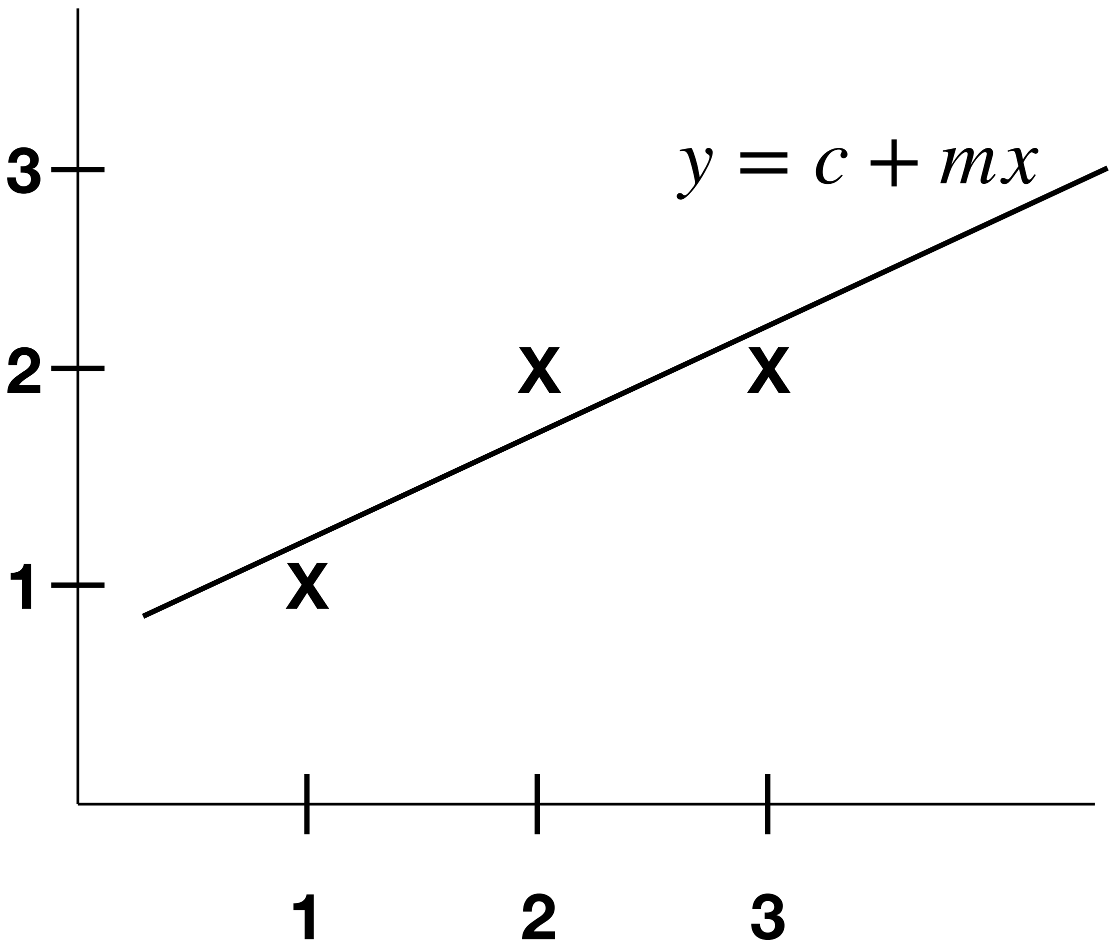

Minimize

$$e^2 = ||Ax - \mathbf{b}||^2$$

Find $\hat{x} = \begin{bmatrix} \hat{c} \\ \hat{m} \end{bmatrix}$

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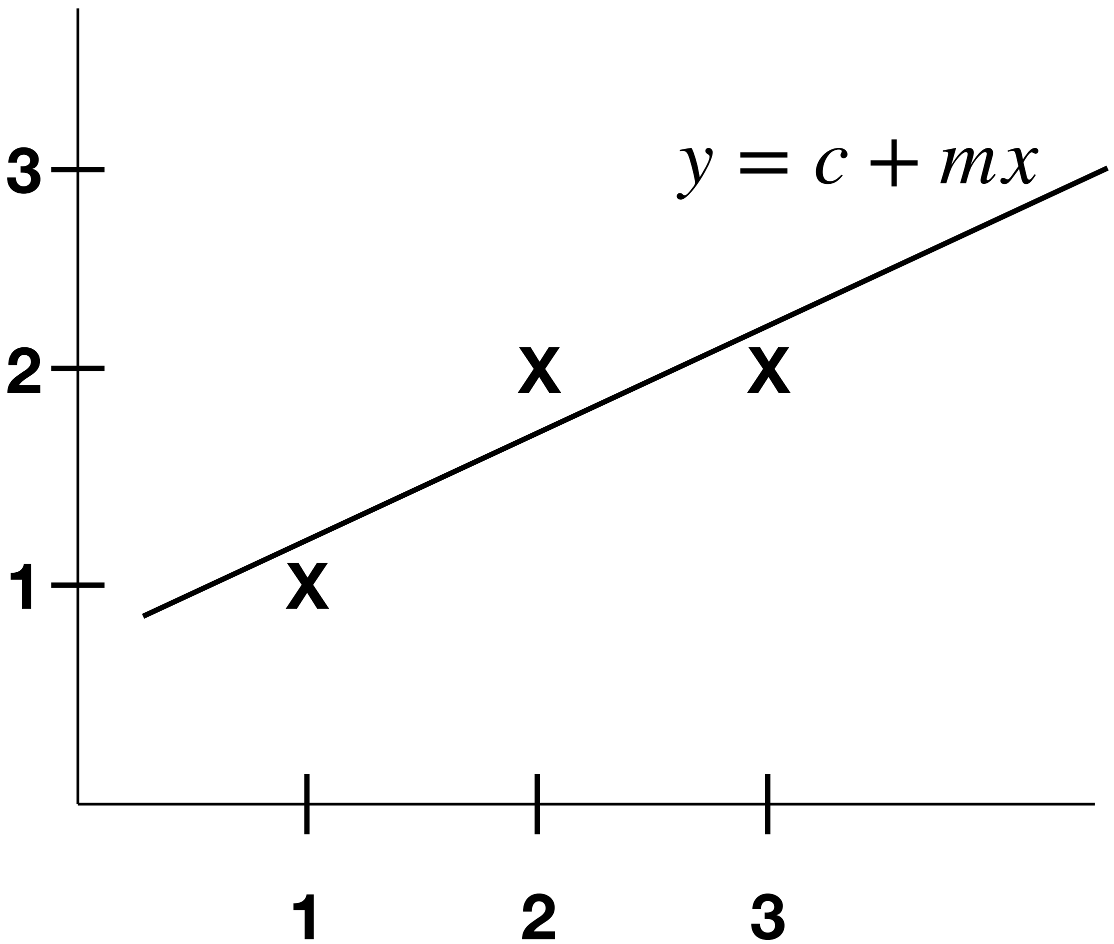

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$$A\hat{x} = p$$

$$A^T A\hat{x} = A^T\mathbf{b}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 2 \end{array}\right] = \left[\begin{array}{cc|c} 3 & 6 & 5 \\ 6 & 14 & 11 \end{array}\right]$$

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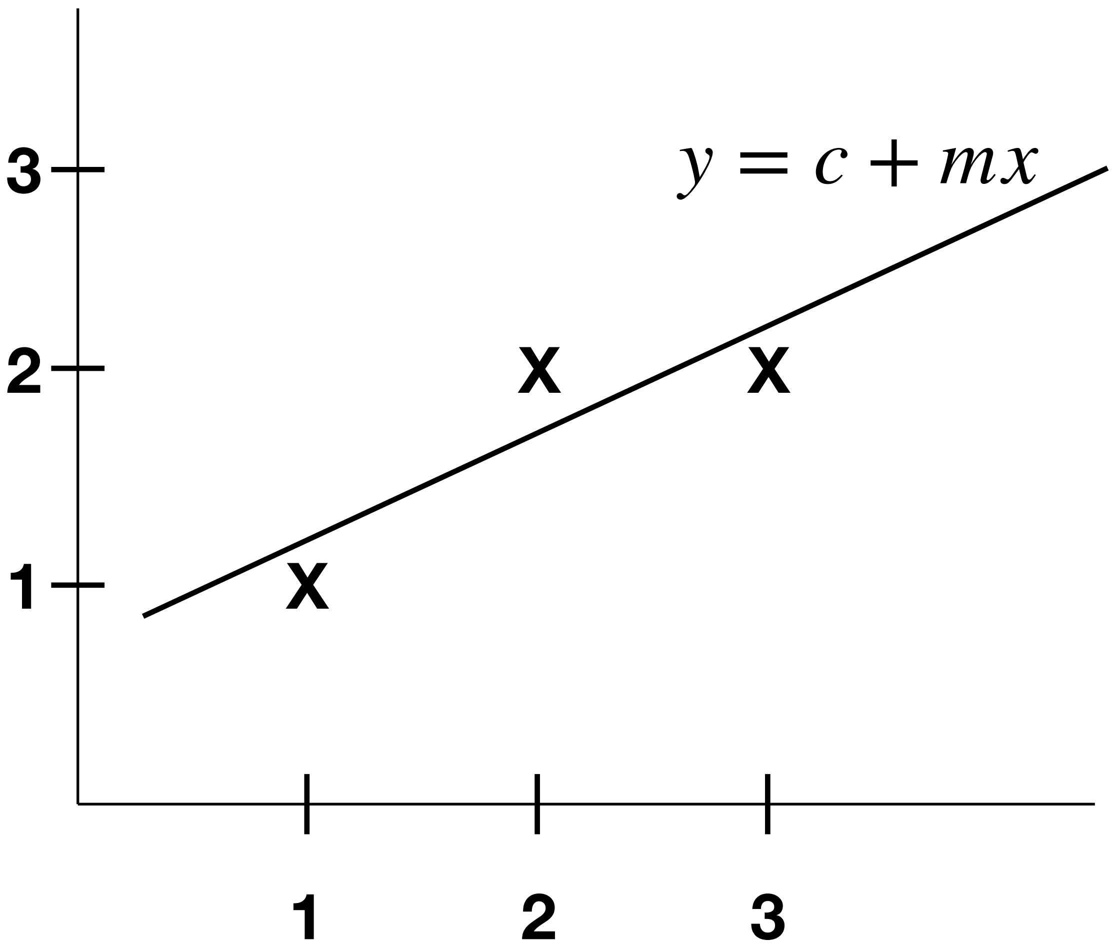

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$$\begin{cases} 3\hat{c} + 6\hat{m} = 5 \\ 6\hat{c} + 14\hat{m} = 11 \end{cases} \rightarrow \quad \hat{c} = \frac{2}{3} \quad \hat{m} = \frac{1}{2}$$

Orthonormal Vectors

Orthonormal Vectors

Vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$ are **orthonormal** if:

$$\mathbf{q}_i^T \mathbf{q}_j = \begin{cases} 0 & \textbf{when } i \neq j \ (\textbf{orthogonal vectors}\ ) \\ 1 & \textbf{when } i = j \ (\textbf{unit vectors: } ||\mathbf{q}_i|| = 1) \end{cases}$$

A matrix Q with orthonormal columns satisfies $Q^T Q = I$:

$$Q^T Q = \begin{bmatrix} - & \mathbf{q}_1^T & - \\ - & \mathbf{q}_2^T & - \\ - & \mathbf{q}_n^T & - \end{bmatrix} \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_n \\ | & | & | \end{bmatrix} = I$$

Examples:

$$Q_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Permutation

$$Q_2 = \begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$$

Rotation

Projections Using Orthonormal Bases

$$\hat{x} = (A^T A)^{-1} A^T \mathbf{b}, \quad \mathbf{P} = A(A^T A)^{-1} A^T$$

Assume that A has orthonormal columns.

$$\hat{x} = (Q^T Q)^{-1} Q^T \mathbf{b}$$

$$\hat{x} = Q^T \mathbf{b}$$

$$\mathbf{P} = Q(Q^T Q)^{-1} Q^T$$

$$\mathbf{P} = Q I^{-1} Q^T$$

$$\mathbf{P} = Q Q^T$$

P satisfies both $P^T = P$ and $P^2 = P$.

Gram-Schmidt Process

Gram-Schmidt Process

For independent vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$, Gram-Schmidt process constructs orthonormal vectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$.

- Start with three independent vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$
- Construct three orthogonal vectors $\mathbf{A}, \mathbf{B}, \mathbf{C}$ as follows:
  - Choose $\mathbf{A} = \mathbf{a}$
  - To construct $\mathbf{B}$, start with $\mathbf{y}$ and subtract its projection along $\mathbf{A}$.
    $$\mathbf{B} = \mathbf{y} - \frac{\mathbf{A}^T\mathbf{y}}{\mathbf{A}^T\mathbf{A}}\mathbf{A}$$
  - To get $\mathbf{C}$, subtract its component in directions $\mathbf{A}$ and $\mathbf{B}$.
    $$\mathbf{C} = \mathbf{z} - \frac{\mathbf{A}^T\mathbf{z}}{\mathbf{A}^T\mathbf{A}}\mathbf{A} - \frac{\mathbf{B}^T\mathbf{z}}{\mathbf{B}^T\mathbf{B}}\mathbf{B}$$
- Produce three orthonormal vectors: $\mathbf{q}_1 = \mathbf{A}/||\mathbf{A}||$, $\mathbf{q}_2 = \mathbf{B}/||\mathbf{B}||$, $\mathbf{q}_3 = \mathbf{C}/||\mathbf{C}||$

Gram-Schmidt Example

$$x = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}, y = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}, z = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$$B = y - \frac{A^T y}{A^T A} A = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} - \frac{\begin{bmatrix} 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix}}{\begin{bmatrix} 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ -4 \end{bmatrix} - \frac{8}{8} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix}$$

$$C = z - \frac{A^T z}{A^T A} A - \frac{B^T z}{B^T B} B = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix} - \frac{24}{8} \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} + \frac{24}{24} \begin{bmatrix} 2 \\ 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

$$q_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, q_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}, q_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Thank you!