Vector Spaces and Subspaces

CS 556
Erisa Terolli

Vector Spaces and Subspaces

Vector Spaces

- The space $\mathbb{R}^n$ consists of all columns vectors **v** with n components.

- We can add any two vectors in $\mathbb{R}^n$, and we can multiply any vector **v** by any scalar c.

- Vector spaces must be closed under addition and multiplication

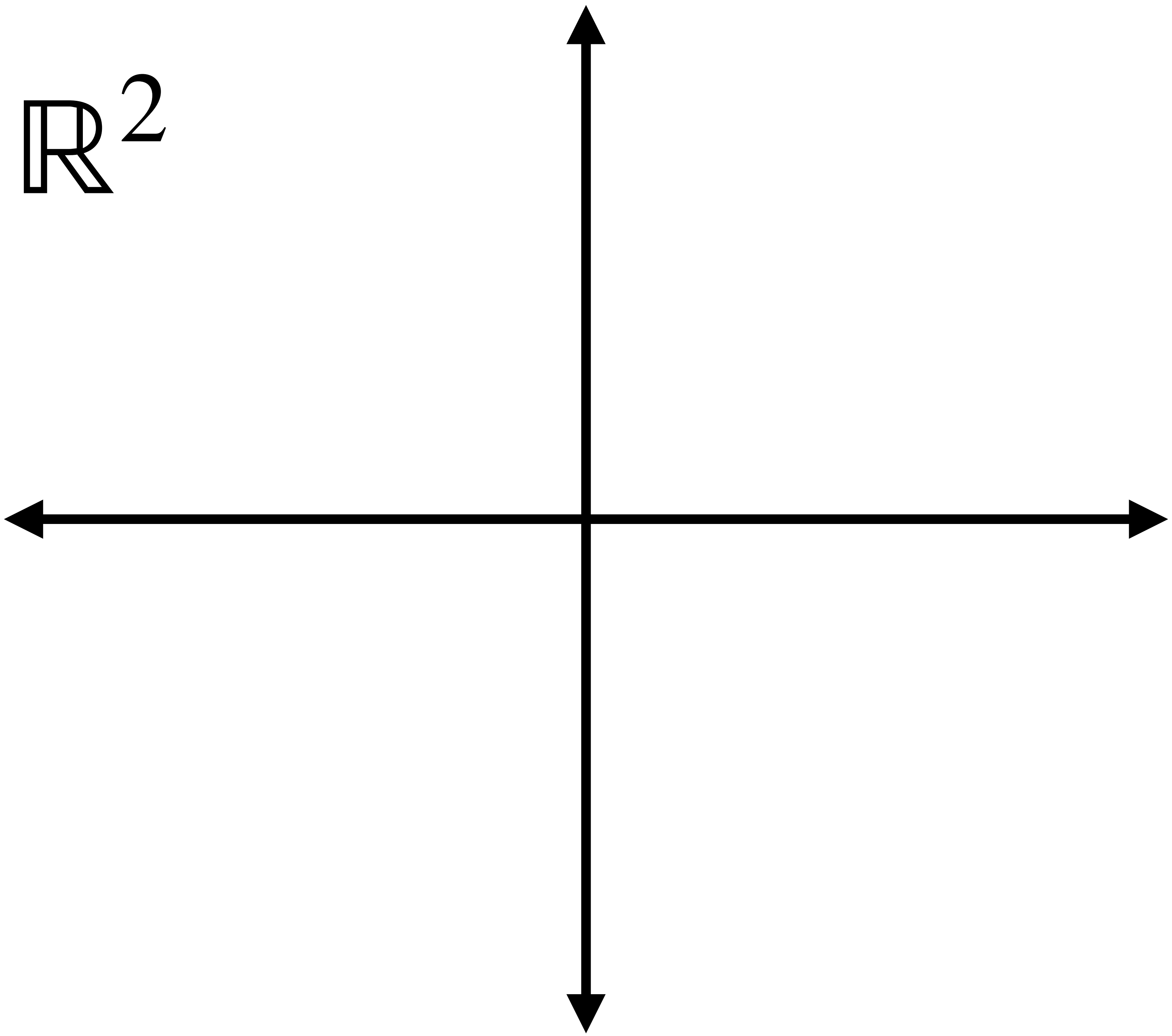
Examples:

- The vector space $\mathbb{R}^2$ is represented by the xy plane. Vector examples in this space: (3, 2), (0, 0) etc.

- The vectors space $\mathbb{R}^3$ is represented by the xyz 3-dimensional space. Vector examples in this space: (1, 2, 3), (0, 0, 0) etc.

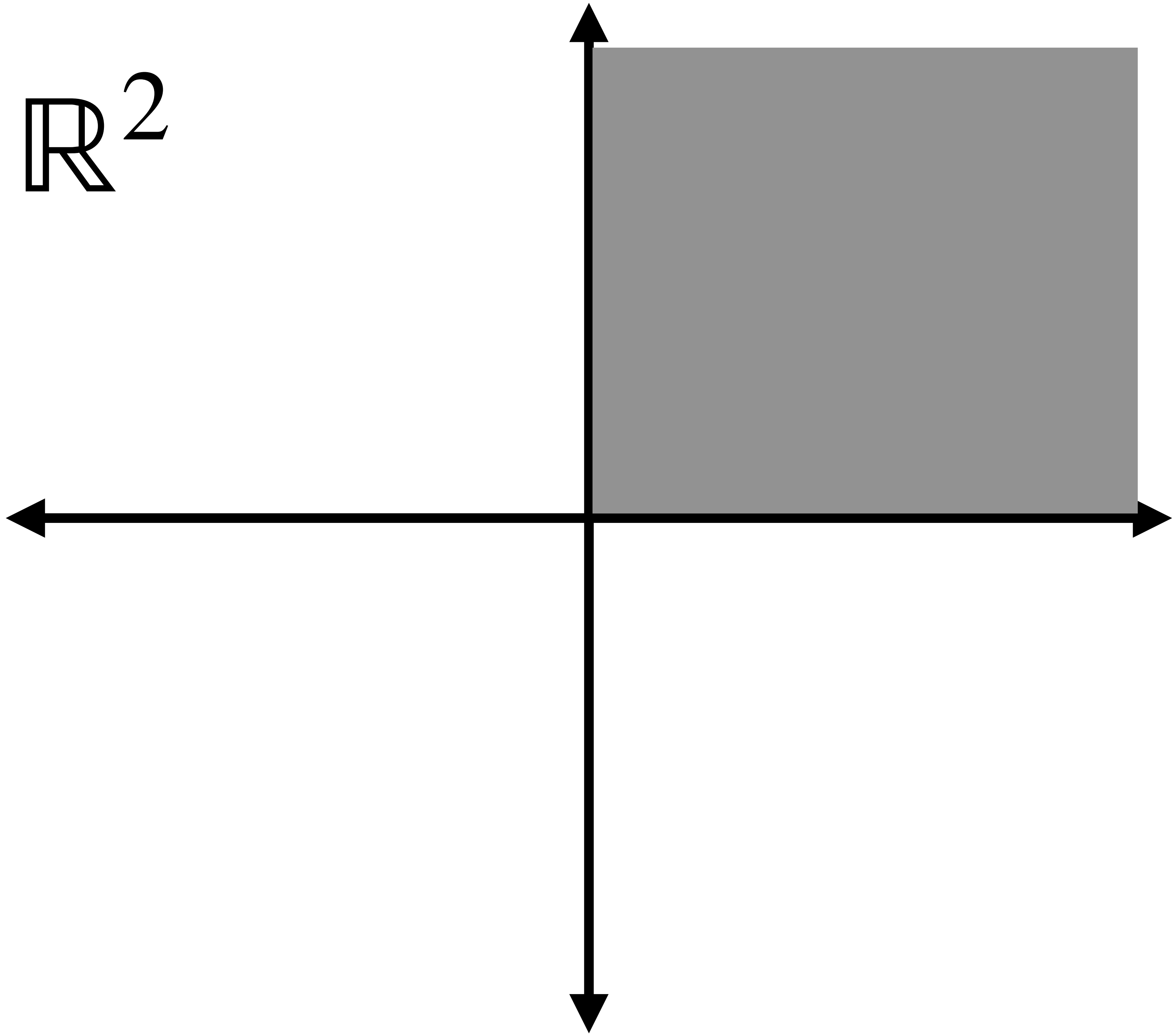
Vector Spaces

$\mathbb{R}^2$

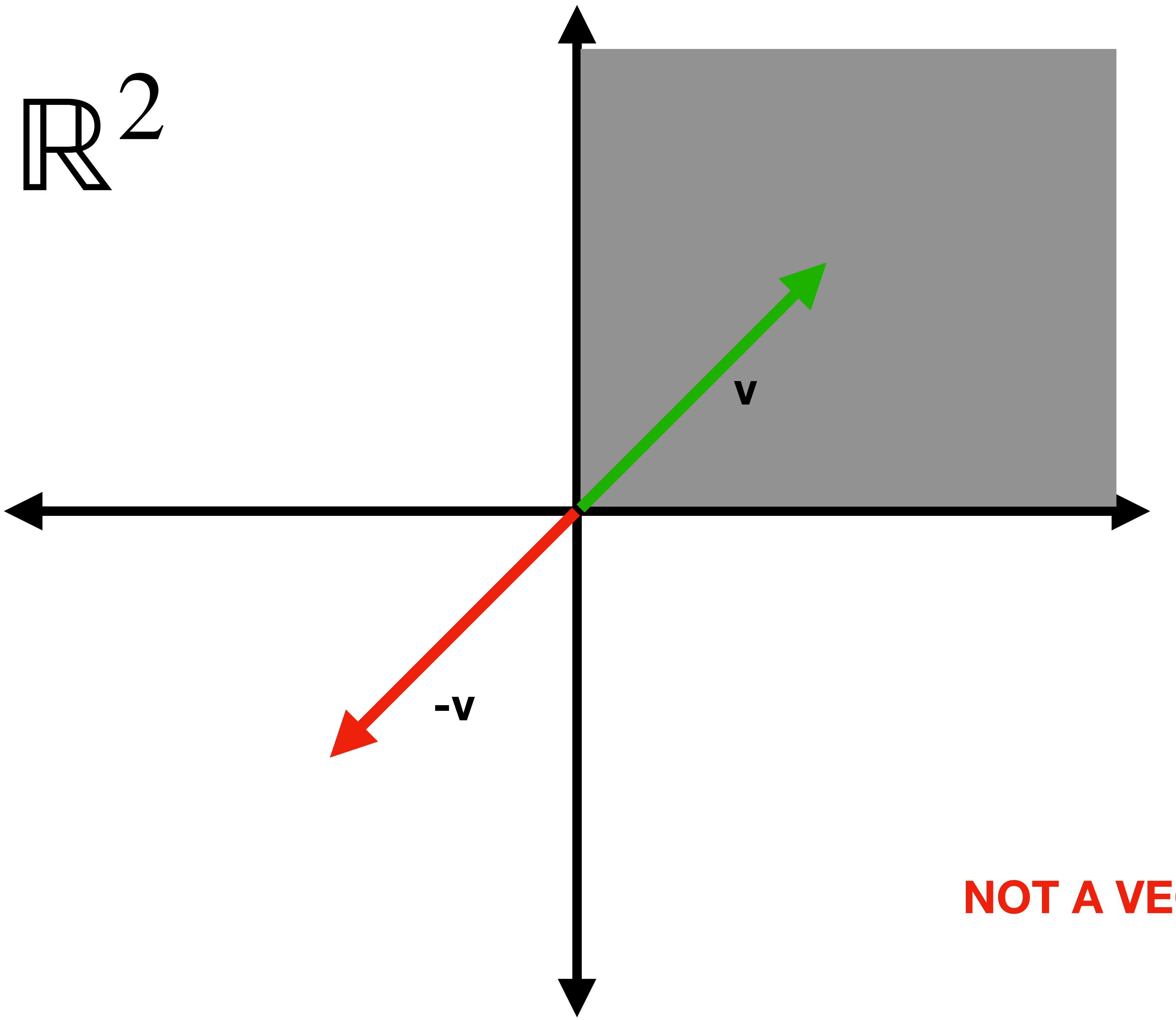


Vector Spaces

$\mathbb{R}^2$



Vector Spaces



NOT A VECTOR SPACE!!!!

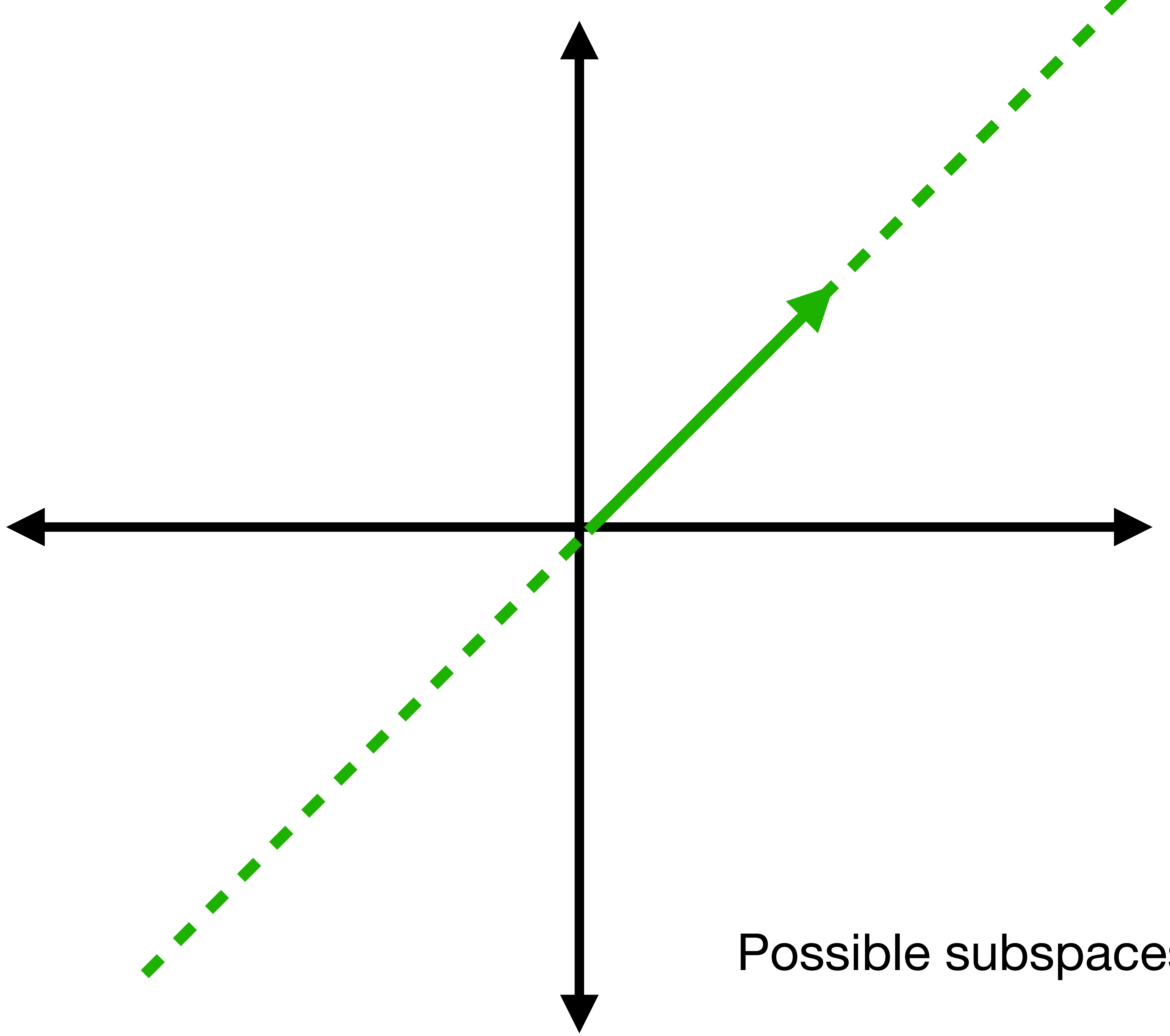
Subspace

A subspace is defined as a set of all vectors that can be created by taking linear combinations of some vectors or a set of vectors.

Formally, a subspace is the set of all vectors that satisfy the following conditions:

- Must be closed under addition and multiplication
- Must contain the zero vector

$$\forall v, w \in \mathbf{V}, \forall \lambda,\ \mu \in \mathbb{R};\ \ \lambda\mathbf{v} + \mu\mathbf{w} \in \mathbf{V}$$



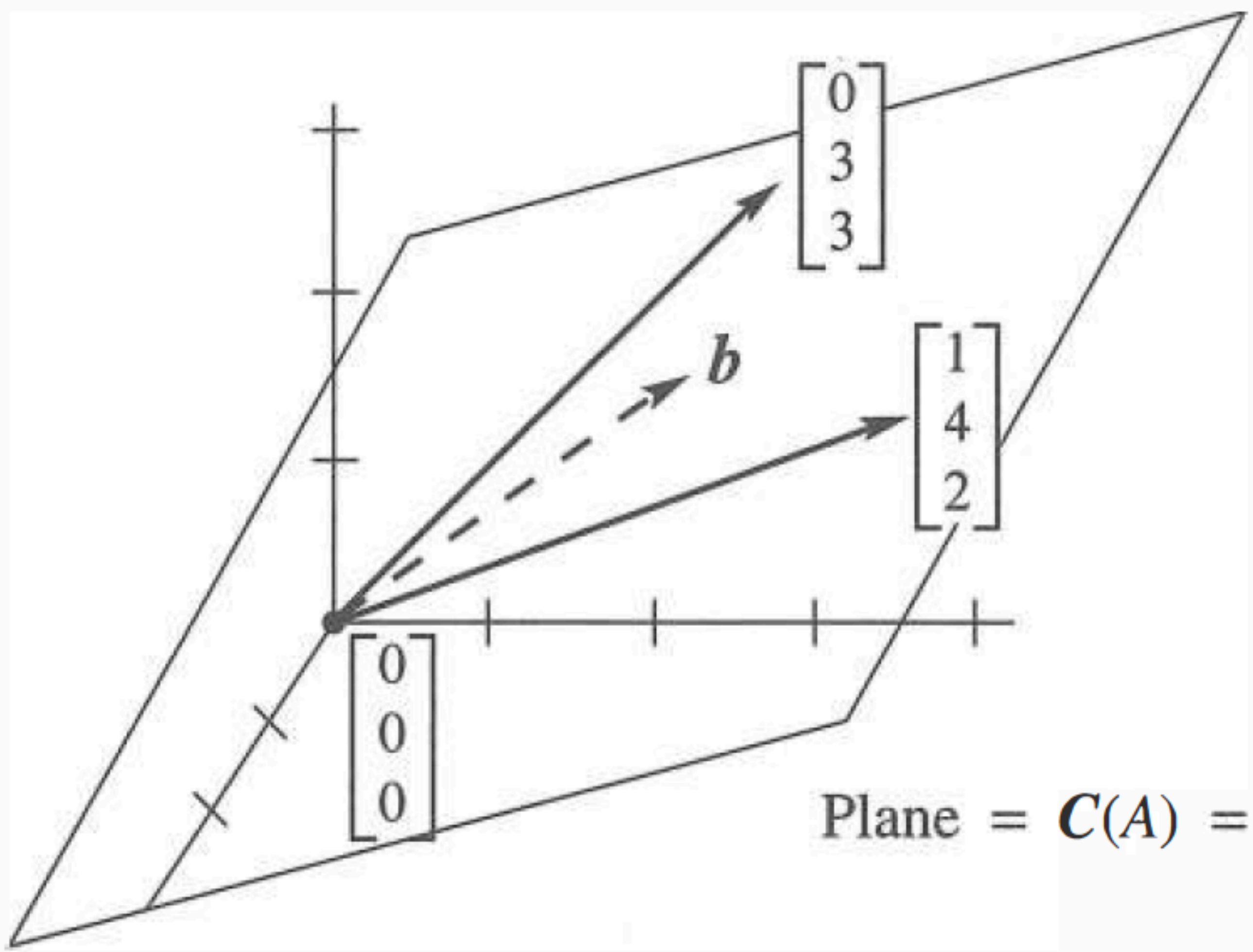
Possible subspaces in $\mathbb{R}^2$

- All of $\mathbb{R}^2$
- Lines that pass through the origin
- The zero vector $\mathbf{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Column Space

- The column space of a matrix A consists of all the linear combinations of the columns of A .
- The combinations are all possible vectors $A\mathbf{x}$, which fill the columns space denoted by $C(A)$.
- The system $A\mathbf{x} = b$ is solvable if and only if b is in the columns space of A.

Column Space Example



$$A = \begin{bmatrix} 1 & 0 \\ 4 & 3 \\ 2 & 3 \end{bmatrix}$$

$$\boldsymbol{b} = .4 \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} + .3 \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$$

$$A\boldsymbol{x} = \boldsymbol{b} \text{ has } \boldsymbol{x} = \begin{bmatrix} .4 \\ .3 \end{bmatrix}$$

Plane $= \boldsymbol{C}(A) =$ all vectors $A\boldsymbol{x}$

Credit: Intro to Linear Algebra. Strang Book

Null Space

- The null space of a matrix A_{mxn} consists of all the solutions to $A\mathbf{x} = 0$.

- The solution vectors x have n components. They are vectors in $\mathbb{R}^n$, so the null space is a subspace of $\mathbb{R}^n$. The columns space $C(A)$ is a subspace of $\mathbb{R}^m$.

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix}$$

$$\begin{array}{r} \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{array}{r} R_1 - R_2 \\ -1R_2 \\ R_3 + 2R_2 \end{array} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

REDUCED ROW-ECHELON FORM

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Pivot Columns **Free Column**

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} x + 1 &= 0 \\ y + 1 &= 0 \end{aligned}$$

$$\mathbf{s} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Special Solution

$$\mathbf{Z} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Null Space

Four Fundamental Subspaces

The four fundamental subspaces of A_{mxn}.

Name	Notation	Note
Column Space	$C(A) \in \mathbb{R}^m$	All combinations of the columns of matrix A.
Null Space	$N(A) \in \mathbb{R}^n$	
Row Space	$C(A^T) \in \mathbb{R}^n$	All combinations of the rows of matrix A.
Left Null Space	$N(A^T) \in \mathbb{R}^m$	

Complete Solution System of Linear Equations

Complete Solution to Ax = b

- Set all free variables to 0, then solve $Ax = b$ for pivot variables to find $x_{particular}$.
- Find the null space: $x_{null\ space}$.
- The compete solution is:

$$x = x_{particular} + x_{null\ space}$$

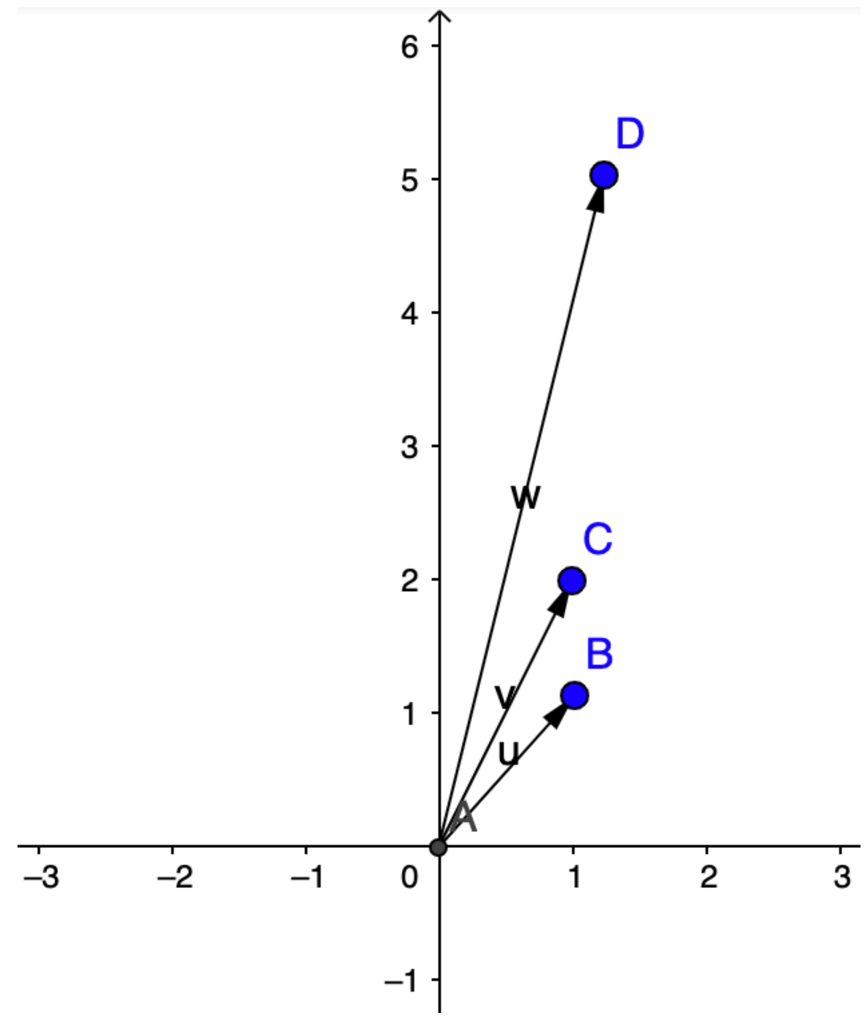
Matrix Rank

Matrix Rank

- The rank of a matrix denoted by **r** or **rk(A)** or **rank(A)** is the number of pivot columns.
- Single number the provides insights into the amount of information that is contained in the matrix.
- $r \in \mathbb{N}, s.t.\ 0 \leq r \leq min\{\#cols, \#rows\}$

Matrix Rank

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 2 & 1 & 0 & 5 \end{bmatrix}$$



Perform Gaussian elimination until the matrix is in row-echelon form and then count the number of pivot columns

Computing the rank

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -2 & -3 & 1 \\ 3 & 5 & 0 \end{bmatrix} \qquad rk(A) = ?$$

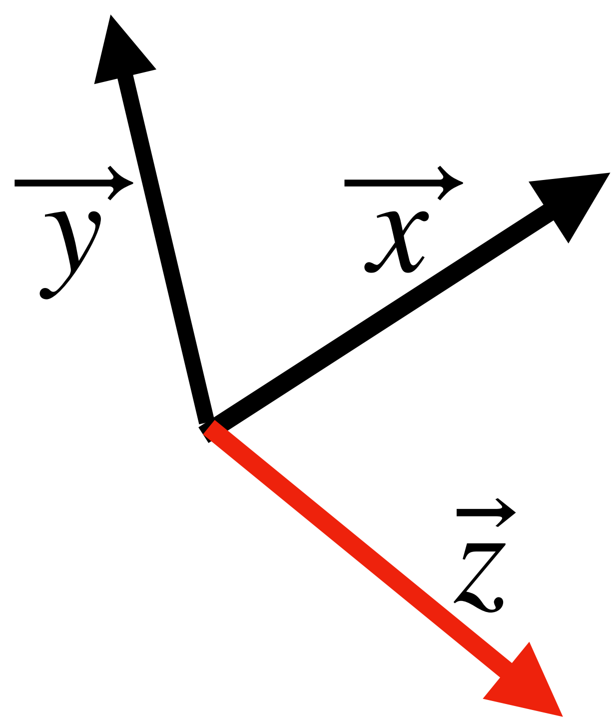
Linear Independence

Linear Independence

The sequence of vectors $v_1, \ldots, v_n$ is linearly independent if the only combination that gives the zero vectors is $0v_1 + 0v_2 + \ldots + 0v_n$.

$$x_1v_1 + x_2v_2 + \ldots + x_nv_n = 0$$

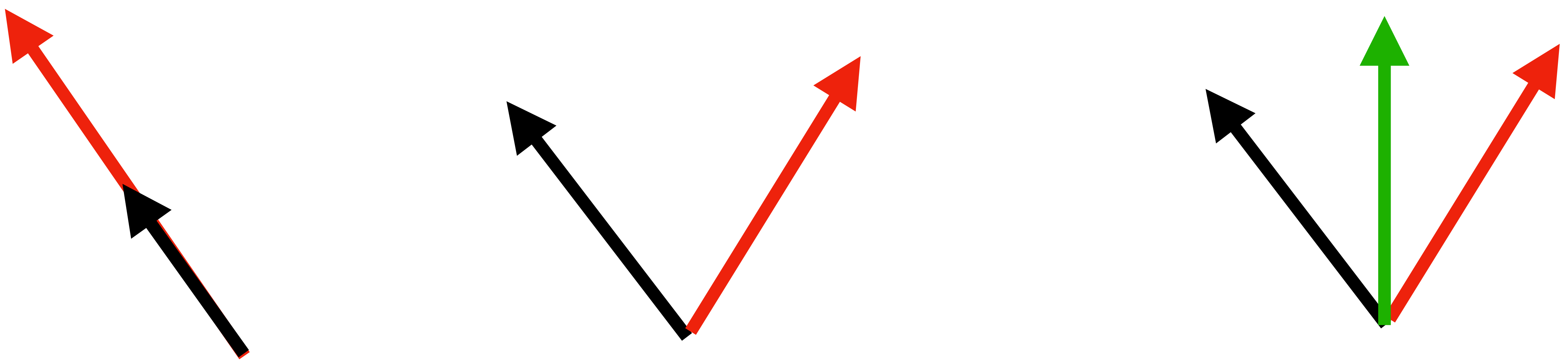
only happens when all x's are zero.



$$z \neq \alpha x + \beta y$$

z can not be express as a linear combination of *x* and *y*

Are these sets of vectors linearly independent?



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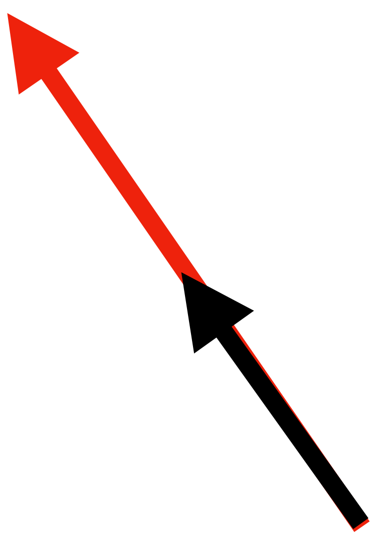
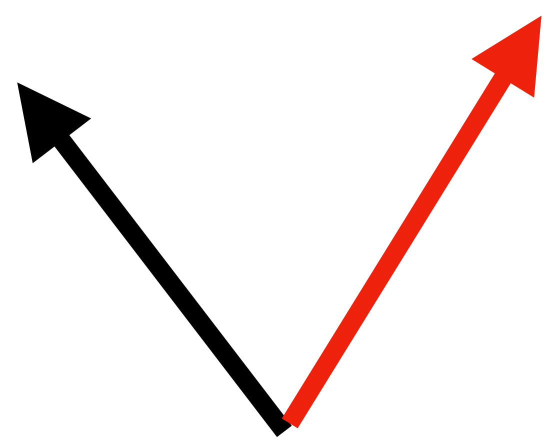
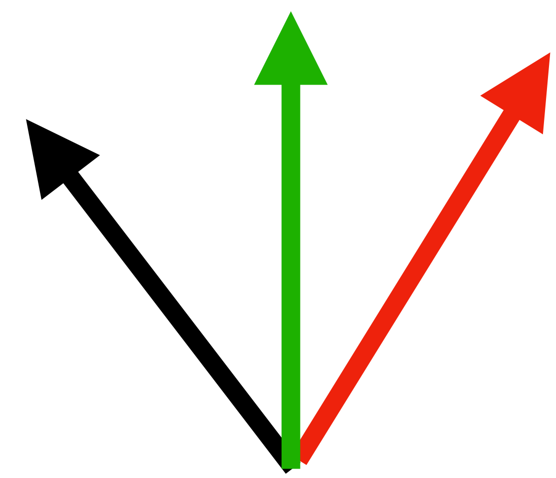


NO

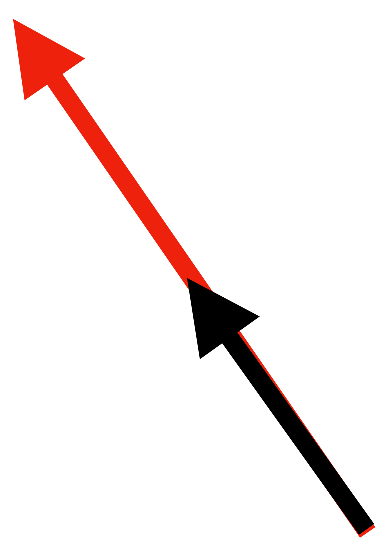


YES

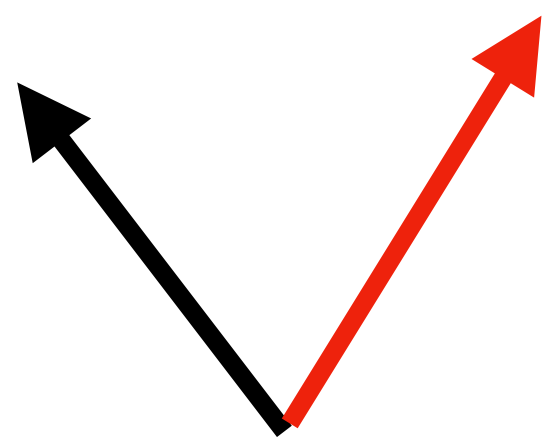


NO

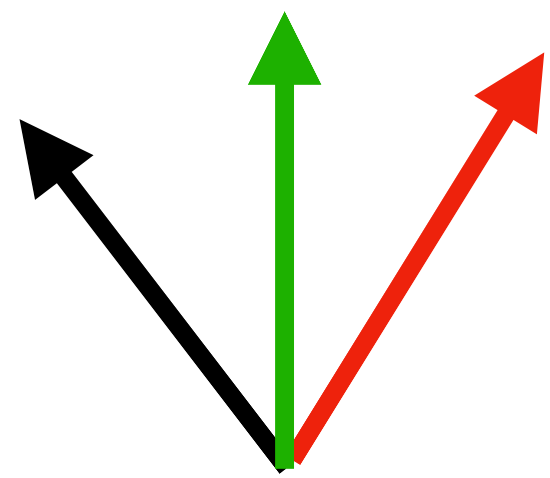
Are these sets of vectors linearly independent?



NO



YES



NO

There are a maximum of N independent vectors in $\mathbb{R}^N$.

Span

Span of a space is defined as all possible linear combinations of all the vectors in that space.

$$span(\{v_1, v_2, \ldots, v_n\}) = \alpha_1 v_1 + \ldots + \alpha_n v_n,\ \alpha_i \in \mathbb{R}$$

Span Example 1

- $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ span the full space $\mathbb{R}^2$.
- $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ span the full space $\mathbb{R}^2$.
- $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ span a line in $\mathbb{R}^2$.

Span Example 2

To determine if a vector $\mathbf{v}$ is in the span of a set S we need to check whether $\mathbf{v}$ can be expressed as a linear combination of vectors in S.

$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{S} = \{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} \}$. Check if $\mathbf{v} \in \mathbf{S}$

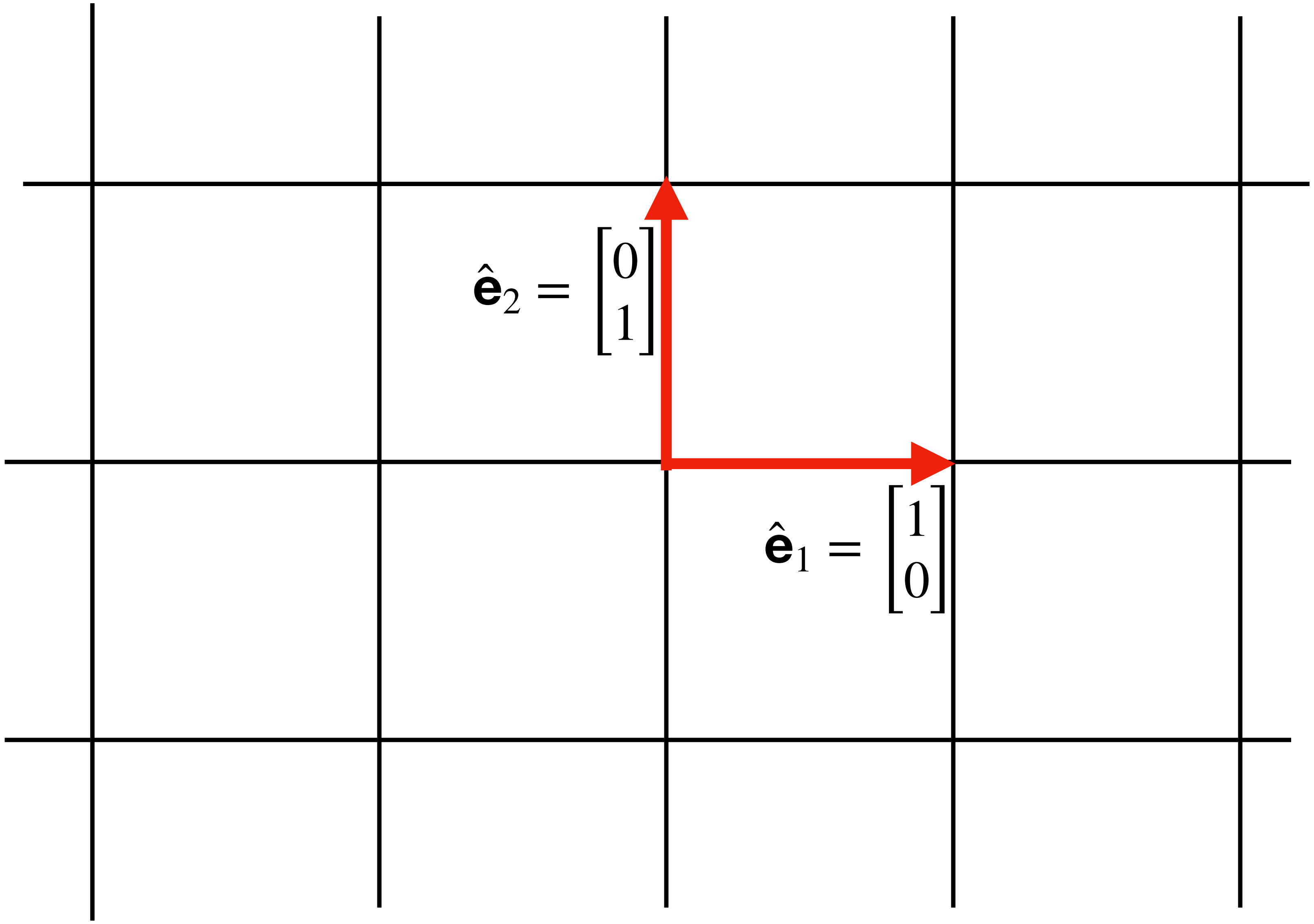
$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \frac{5}{6} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}$ Yes, $\mathbf{v} \in \mathbf{S}$

Basis

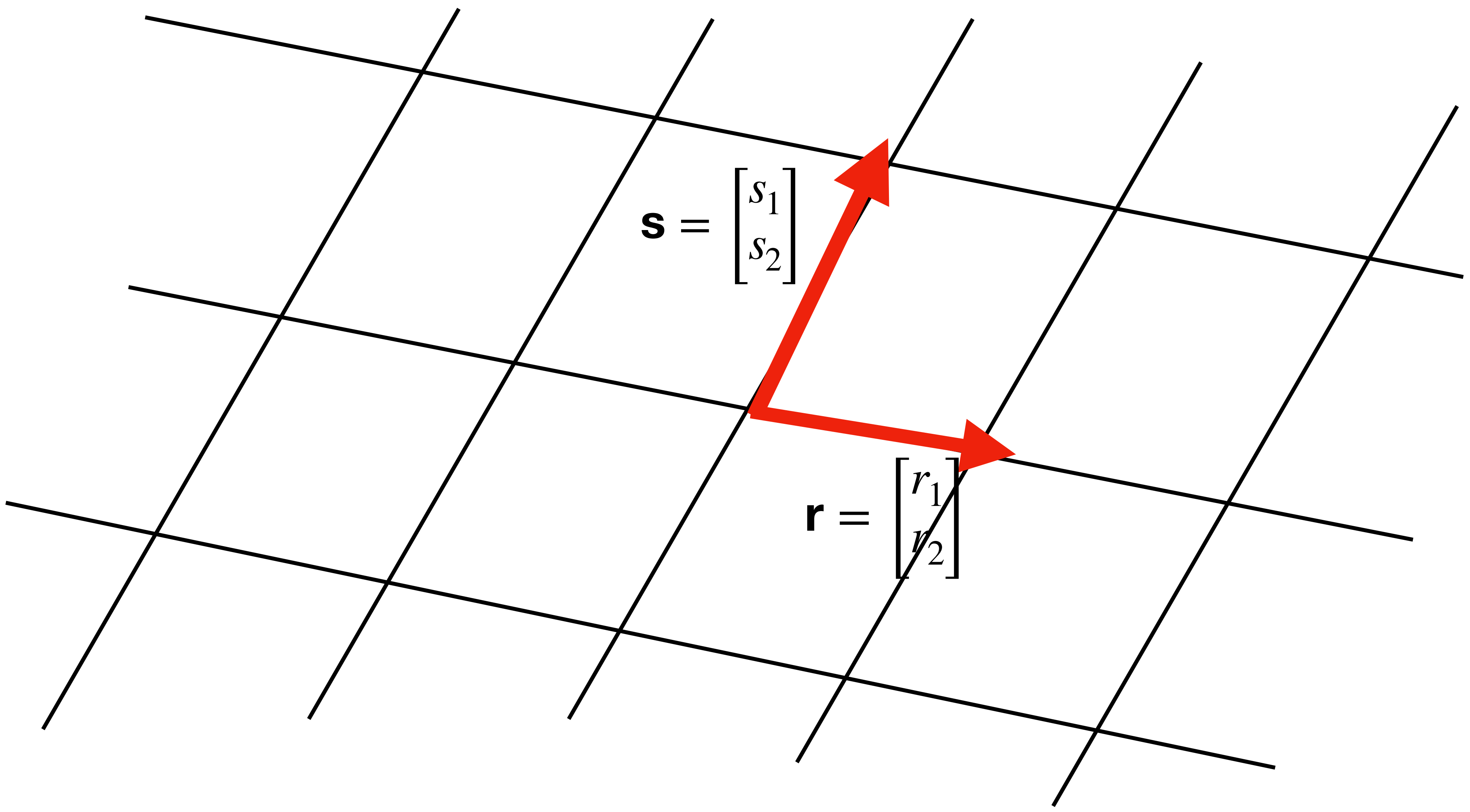
Basis and Dimension

- A basis for a vector space is a sequence of vectors with two properties:
  - The vectors are linearly independent.
  - The vectors span the space.
- Every vector in the space is a unique combination of the basis vectors.
- A space can be spanned by multiple basis.
- The dimension of a space is the number of vectors in every basis.

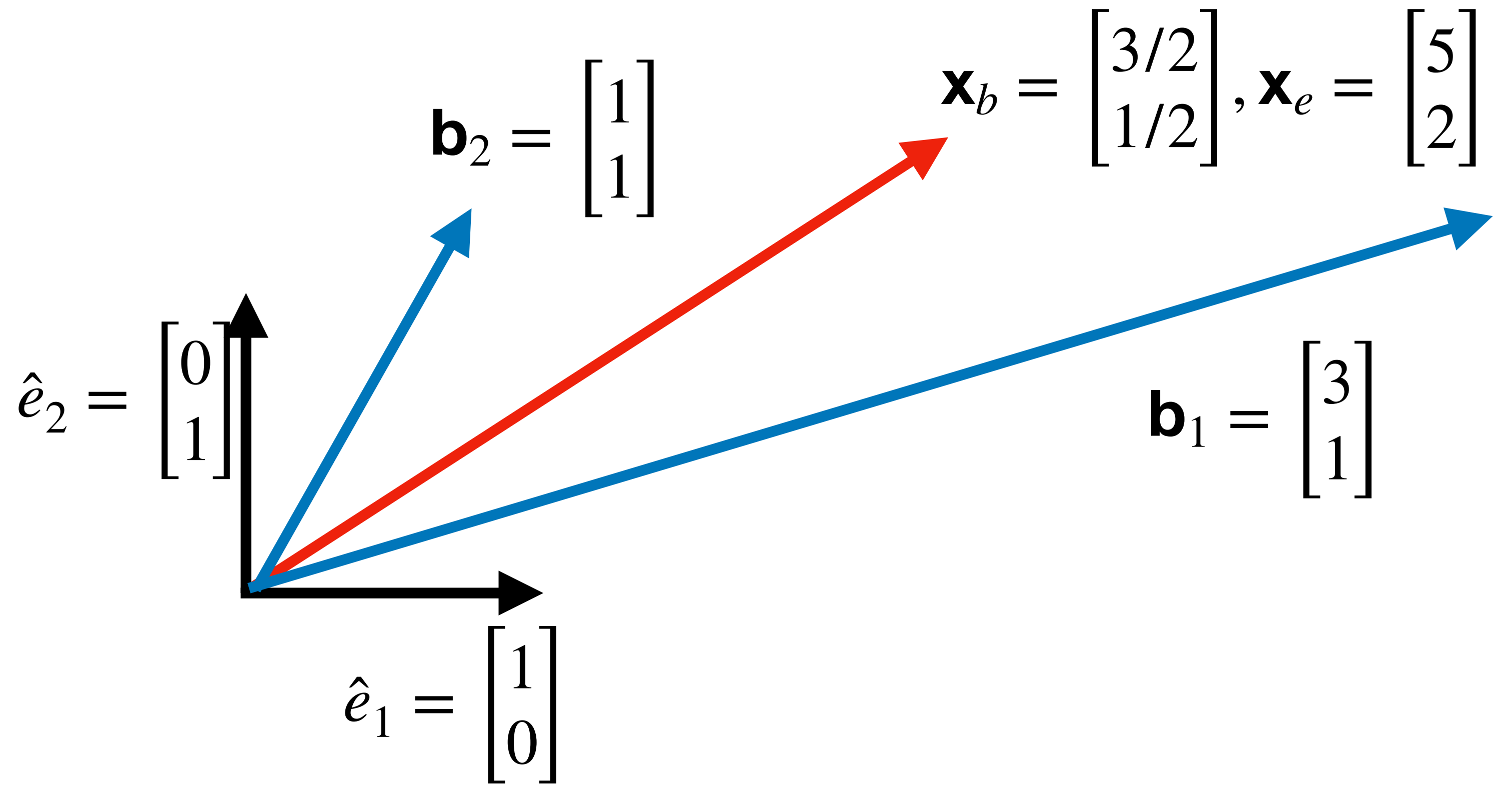
Natural Basis



Another Basis



Changing Basis



$$B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{x}_b = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} \qquad \mathbf{x}_e = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 9/2 + 1/2 \\ 3/2 + 1/2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix} \qquad \mathbf{x}_e = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \qquad \mathbf{x}_b = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

Thank you!