System of Linear Equations
Inverse
Determinants

CS 556
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System of Linear Equations

Matrix-Vector Multiplication

Three vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

The linear combinations in $\mathbb{R}^3$ are $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$.

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \longleftarrow$$

Matrix times vector Linear combination of columns in the matrix.

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b \longleftarrow$$

Which combinations of *u, v, w* produces a particular vector *b*?

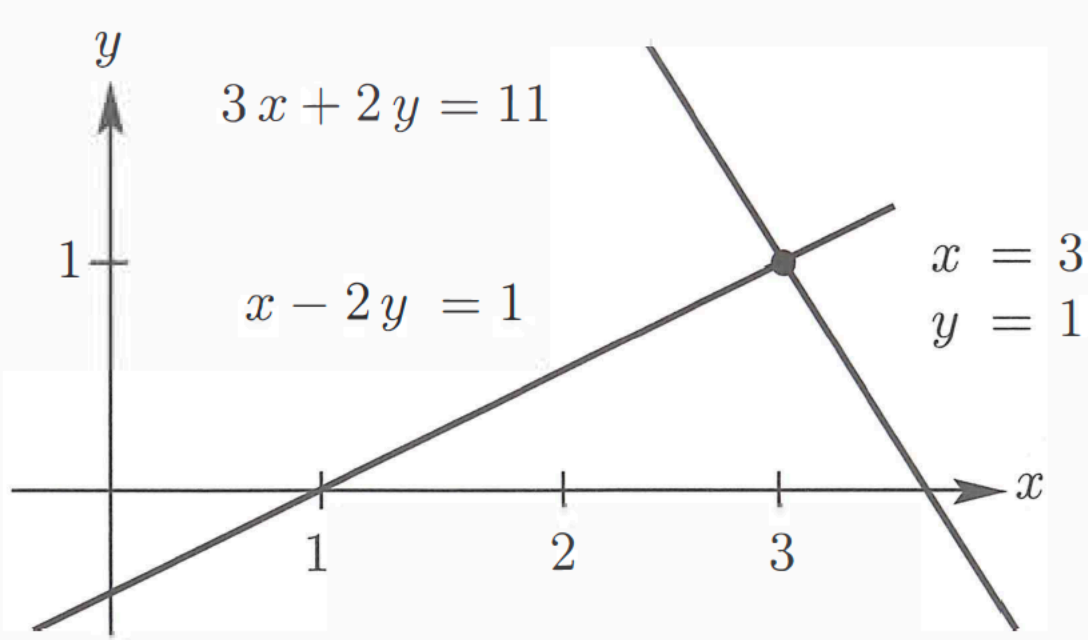
Solving Linear Equations

$$x - 2y = 1$$
$$3x + 2y = 11$$

Column Picture

$$\mathbf{x}\begin{bmatrix}1\\3\end{bmatrix} + \mathbf{y}\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

Row Picture



Matrix Equation

$$Ax = b$$

$$\begin{bmatrix}1 & -2\\3 & 2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}1\\11\end{bmatrix}$$

3 equations in 3 unknowns

$$x + 2y + 3z = 6$$
$$2x + 5y + 2z = 4$$
$$6x - 3y + z = 2$$

Row Picture

The row picture shows three planes meeting at a single point

Column Picture

$$\mathbf{x}\begin{bmatrix}1\\2\\6\end{bmatrix} + \mathbf{y}\begin{bmatrix}2\\5\\-3\end{bmatrix} + \mathbf{z}\begin{bmatrix}3\\2\\1\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

Matrix Equation

$$Ax = b$$

$$\begin{bmatrix}1 & 2 & 3\\2 & 5 & 2\\6 & -3 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}6\\4\\2\end{bmatrix}$$

Ax comes from dot products, each row times the column *x*.

$$\mathbf{Ax} = \begin{bmatrix}(row\ 1)\cdot x\\(row\ 2)\cdot x\\(row\ 3)\cdot x\end{bmatrix}$$

Ax is a combination of column vectors.

$$Ax = x(col\ 1) + y(col\ 2) + z(col\ 3)$$

Systems of Linear Equations SLE

$$x + y + 3z = 15$$
$$x + 2y + 4z = 21$$
$$x + y + 2z = 13$$

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 13 \end{bmatrix}$$

How to solve SLE?

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 13 \end{bmatrix}$$

Key to solving SLE these elementary transformation, that keep the solution set the same, but that transform the equation system into a simpler form:

- Exchange of two rows
- Multiplication of a row with a constant
- Addition of two rows

Gaussian Elimination

Performs elementary transformation to bring a system of linear equation into reduced row-echelon form.

A matrix is in row-echelon form if:

- All rows that contain only zeros are at the bottom of the matrix
- Looking at nonzero rows only, the first nonzero number (aka pivot) is always strictly to the right of the pivot above it.

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

An matrix is in reduced row echelon form if:

- It is in row-echelon form
- Every pivot is 1
- The pivot is the only nonzero entry in its column

$$\begin{bmatrix} 1 & 1 & 3 \\ 1 & 2 & 4 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 15 \\ 21 \\ 13 \end{bmatrix}$$

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REDUCED ROW-ECHELON FORM

Determinant

Determinant

- The determinant of a square matrix $A \in \mathbb{R}^{nxn}$ is a function that maps A into a real number.

- Notation $det(A) = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$

- Only for square matrix

Basic Properties of Determinants

- The determinant of the identity matrix is 1.

- The determinant changes sign when two rows are exchanged.

- The determinant is a linear function of each row separately.

$$\begin{bmatrix} ta & tb \\ c & d \end{bmatrix} = t\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}$$

Derived Properties of Determinants

- If two rows of matrix A are equal, then det(A) = 0.
- Subtracting a multiple of one row from another row leaves the determinant of matrix A unchanged.
- A matrix with a row of zeros has det(A) = 0.
- If A is triangular then det(A) is the product of diagonal entries.
- If A is singular, then det(A) = 0. If A is invertible, then det(A) ≠ 0.
- $det(AB) = det(A)det(B)$
- $det(A^T) = det(A)$

Laplace Expansion

Consider a matrix $A \in \mathbb{R}^{nxn}$. Then for all $j = 1, \ldots, n$:

1. Expansion along column j

$$det(A) = \sum_{k=1}^{n} (-1)^{k+j}\ a_{kj}\ det(A_{k,j})\,.$$

2. Expansion along row j

$$det(A) = \sum_{k=1}^{n} (-1)^{k+j}\ a_{jk}\ det(A_{j,k})\,.$$

Here $A \in \mathbb{R}^{(n-1)x(n-1)}$ is the sub matrix of A

that we can obtain when deleting row k and column j.

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$det(A) = (-1)^{1+1} \cdot 1 \begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}$$

$$+(-1)^{1+2} \cdot 2 \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix}$$

$$+(-1)^{1+3} \cdot 3 \begin{vmatrix} 3 & 1 \\ 0 & 0 \end{vmatrix}$$

$$= 1(1-0) - 2(3-0) + 3(0-0) = -5$$

Determinant Applications

Geometry

- Area/Volume of shape specified by coordinates in the matrix

Matrix Inverse

- Divide by determinant. No inverse of matrix if determinant is zero

Matrix Inverse

Inverse Matrix

Consider a square matrix $A \in \mathbb{R}^{nxn}$. Let matrix $B \in \mathbb{R}^{nxn}$ have the property that $AB = I_n = BA$. B is called the inverse of A and denoted by A^{-1}. Not every matrix A possesses an inverse A^{-1}.

If this inverse exists, matrix A is called regular/invertible/nonsingular, otherwise singular/noninvertible.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix} \qquad \mathbf{A}^{-1} = \begin{bmatrix} -7 & -7 & 6 \\ 2 & 1 & -1 \\ 4 & 5 & -4 \end{bmatrix}$$

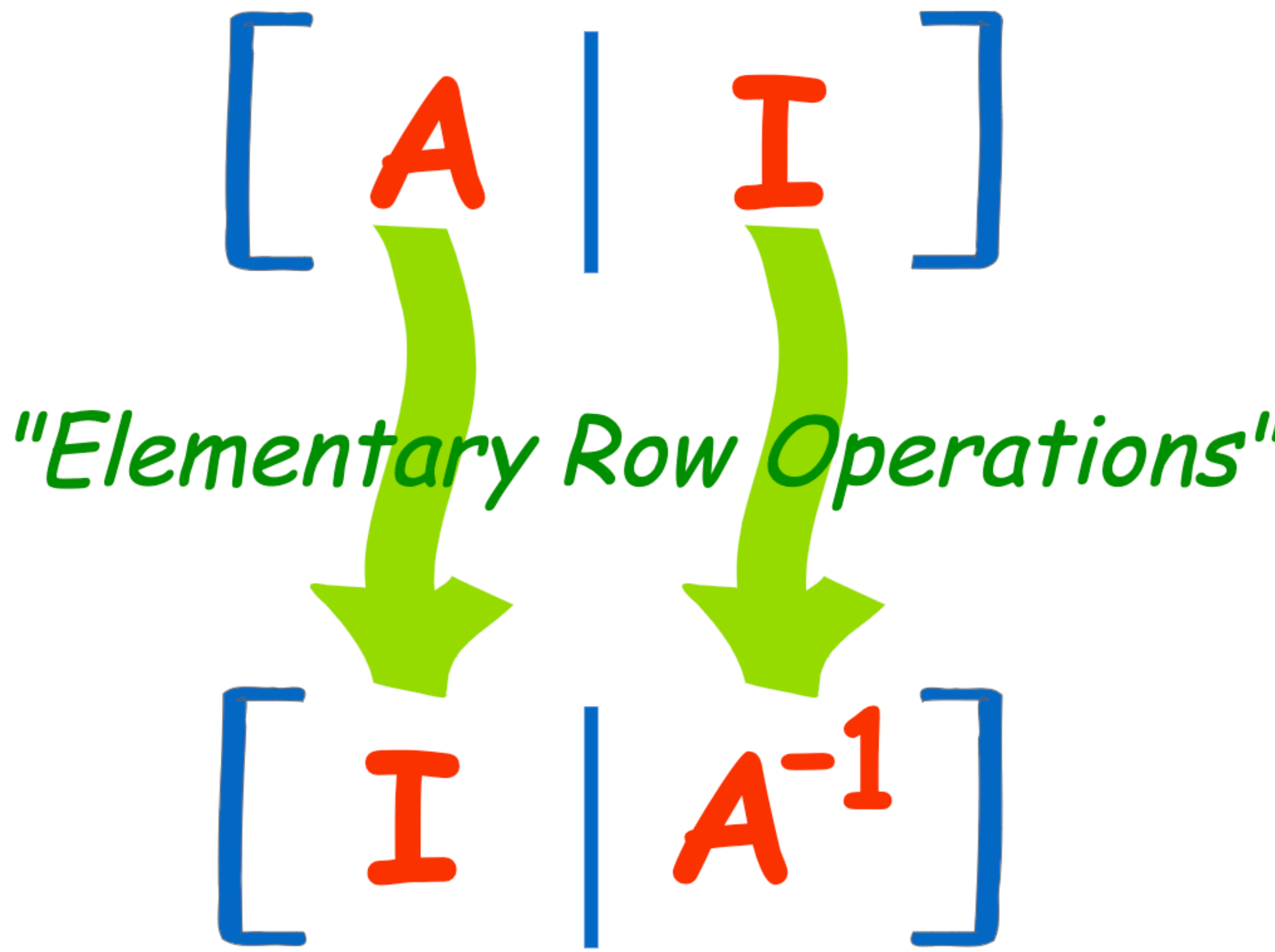
Invertibility Test

- The algorithm to test invertibility is elimination: Matrix A_{nxn} must have n(non-zero) pivots.

- The algebra test for invertibility is the determinant of $A : det(A)$ must be non zero.

- The equation that test for invertibility is $Ax = 0$: where $x = 0$ must be the only solution.

Gauss-Jordan Elimination



Cofactors Method

- The **minors matrix**: a matrix of determinants
- The **cofactors matrix**: the minors matrix element-wise multiplied by a grid of alternating +1 and -1.
- The **adjugate matrix**: the transpose of the cofactors matrix
- The **inverse matrix**: the adjugate matrix divided by the determinant

Inverse for a 2x2 Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} = \frac{1}{det(A)} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Inverse for a 3x3 Matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix}$$

$$\textbf{Minors} = \begin{bmatrix} -7 & -2 & +4 \\ +7 & +1 & -5 \\ +6 & +1 & -4 \end{bmatrix}$$

$$\textbf{Cofactors} = \begin{bmatrix} -7 & +2 & +4 \\ -7 & +1 & +5 \\ +6 & -1 & -4 \end{bmatrix}$$

$$\mathbf{A}^{-1} = \begin{bmatrix} -7 & -7 & 6 \\ 2 & 1 & -1 \\ 4 & 5 & -4 \end{bmatrix}$$

$$\textbf{Adjugate} = \begin{bmatrix} -7 & -7 & +6 \\ +2 & +1 & -1 \\ +4 & +5 & -4 \end{bmatrix}$$