CS 556 Homework 4 - Orthogonality and Projections

Solve the following questions. Type your solutions using Latex or any other program you prefer and submit your typed solutions as a PDF file.

Question 1 *(20 points)* Consider the vectors $b = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$, $a_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

a) Find the projection p of b onto the subspace spanned by a_1 and a_2.

b) Find the error vector e = b - p and show that it is orthogonal to both a_1 and a_2.

Question 2 *(15 points)* Find the line $y = C + Dx$ that best fits the data (x,y) = {(0,1), (1,8), (2,8), (3,20)}.

Question 3 *(20 points)* [Understanding projections and projection matrix] Assume $P = A(A^T A)^{-1} A^T$ is a projection matrix

a) Show that $P^2 = P$ by multiplying $P = A(A^T A)^{-1} A^T$ by itself and canceling.

b) Prove (a) geometrically by showing that for any vector b, Pb is a vector in the column space of A closest to b and then use this fact to show $p^2 = P(Pb) = Pb$ for any vector b.

c) The matrix P as above projects onto the column space of A. Is $1 - P$ a projection matrix? To which subspace does it project onto?

Question 4 *(20 points)* Use Gram-Schmidt Process to find an orthogonal basis for the subspace spanned by $a = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$, $b = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $c = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

Question 5 *(20 points)* Answer each part.

a) If Q_1 and Q_2 are orthogonal matrices, show that $Q_1 Q_2$ is an orthogonal matrix. [Hint: Use $Q^T Q = I$]

b) Show that if for orthogonal vectors, q_1, q_2, q_3, if

$$x_1 q_1 + x_2 q_2 + x_3 q_3 = b$$

then for each i, $x_i = q_i = b$. [Hint: Take the dot product of the two sides with each q_i at a time and use orthonormality conditions, $q_i . q_i = 1$ and $q_i . q_j = 0$ if $i \neq j$, to prove the statement].

c) The vectors $q_1 = 1/\sqrt{2} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $q_2 = 1/\sqrt{6} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$,$q_3 = 1/\sqrt{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ are orthogonal. Use Part (b), to solve

$$x_1 q_1 + x_2 q_2 + x_3 q_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$