Applications of Derivatives

CS 556

Absolute Extrema

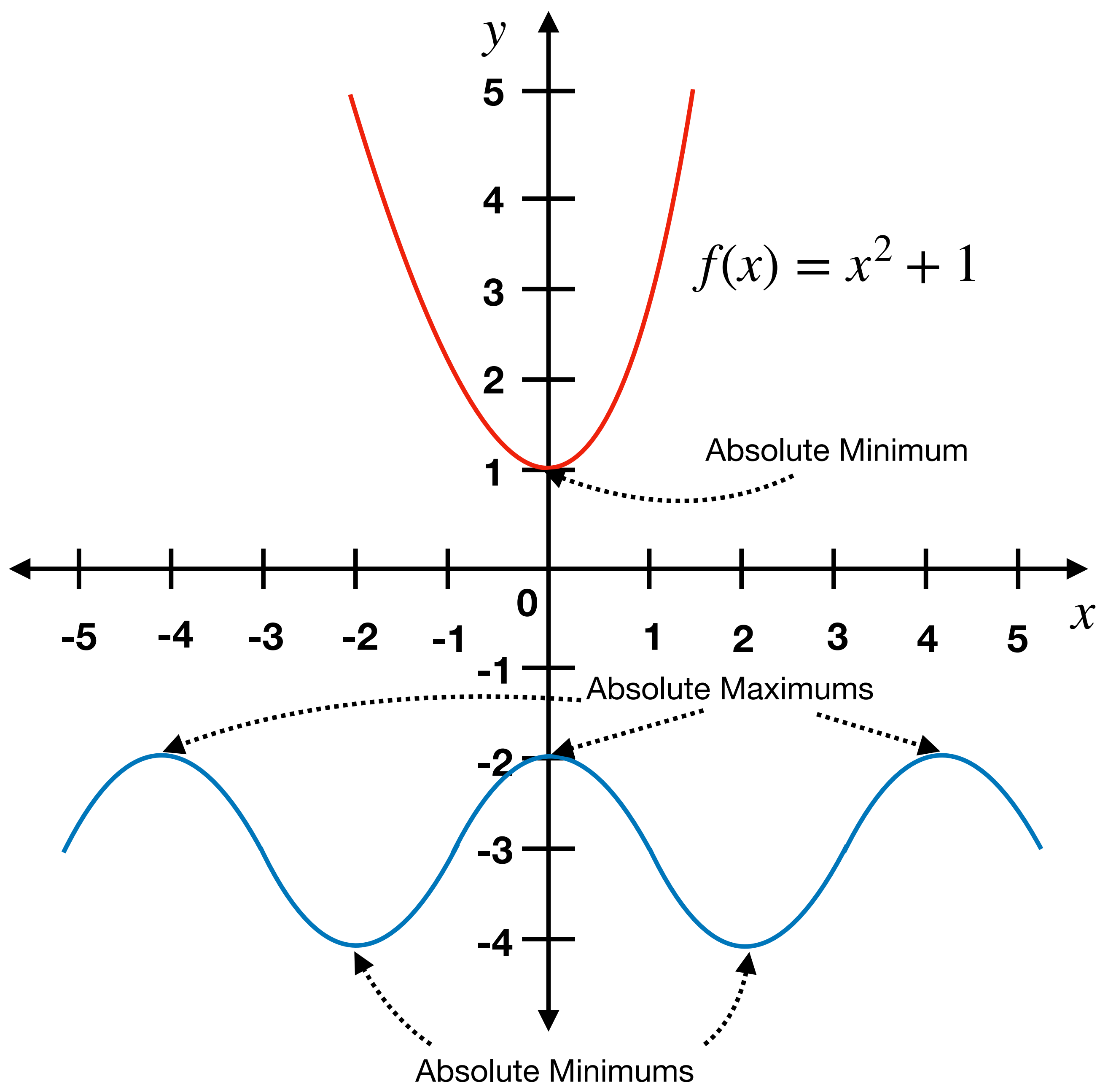
Let f be a function over an interval I and let $c \in I$.

- We say f has an absolute maximum on I at c if $f(c) \geq f(x)$ for all $x \in I$.

- We say f has an absolute minimum on I at c if $f(c) \leq f(x)$ for all $x \in I$.

If f has an absolute maximum on I at c or an absolute minimum on I at c, we say f has an absolute extremum on I at c.

Absolute Extrema



Local Extrema

A function f has a local maximum at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \geq f(x)$ for all $x \in I$.

A function f has a local minimum at c if there exists an open interval I containing c such that I is contained in the domain of f and $f(c) \leq f(x)$ for all $x \in I$.

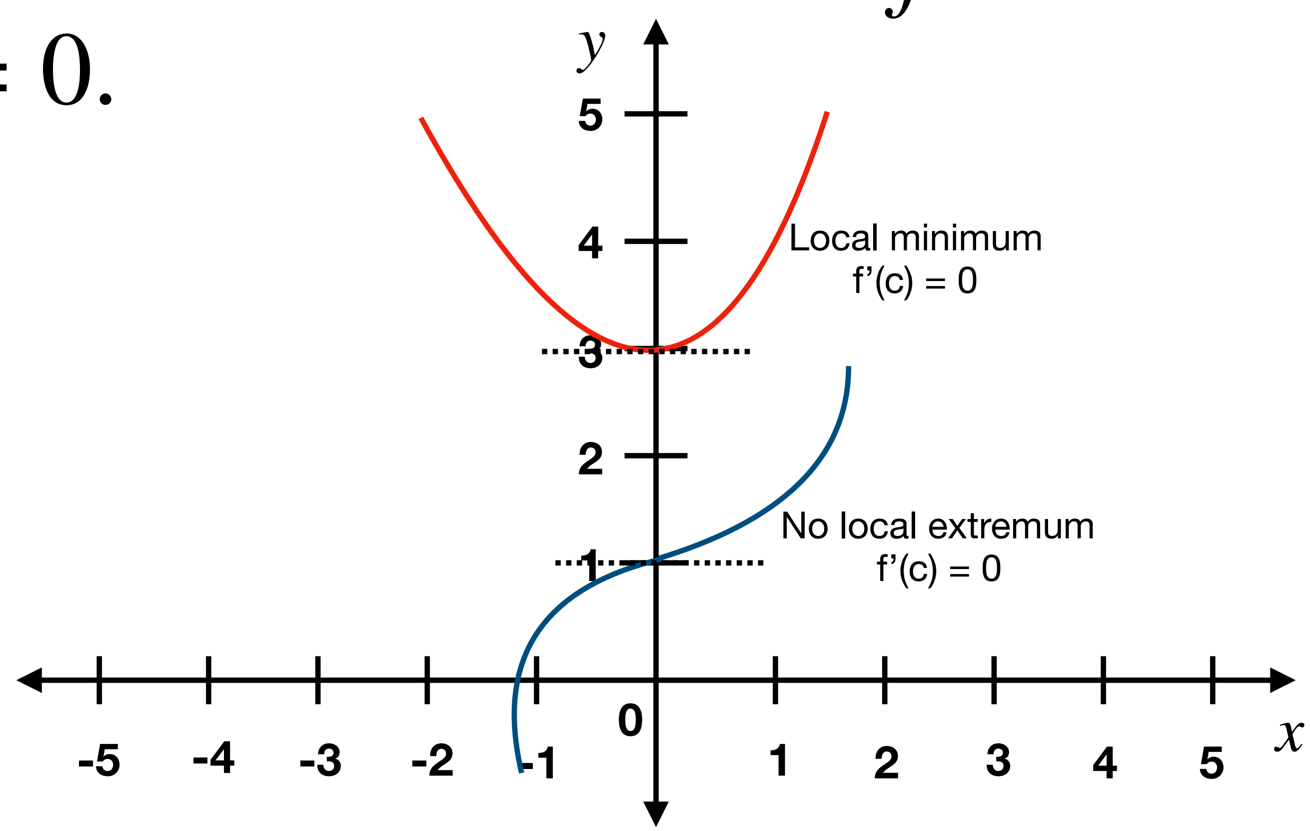
A function f has a local extremum at c if f has a local maximum at c or f has a local minimum at c.

Critical Point

Let c be an interior point in the domain of f. We say that c is a critical point of f if $f'(c) = 0$ or $f'(c)$ is undefined.

Fermat's Theorem

If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$.



Fermat Theorem Proof

To prove that $f'(c) = 0$, we will show that $f'(c) \geq 0$ and $f'(c) \leq 0$, and therefore $f'(c) = 0$. Since f has a local extremum at c, f has a local maximum or minimum at c. Supposing that f has a local maximum at c, then there exists an open interval I such that $f(c) \geq f(x)$ for all x in I. Since f is differentiable at c, we have $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$ .Since this limit exists, both one-sided limits also exists and are equal to f'(c).

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \qquad f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c}$$

Since f(c) is a local maximum, we have $f(x) - f(c) \leq 0$ for x near c. Therefore, for x near c but $x > c$ we have $\frac{f(x) - f(c)}{x - c} \leq 0$. From $f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}$ we conclude that $f'(c) \leq 0$. Similarly, it can be showed that $f'(c) \geq 0$. Therefore, $f'(c) = 0$.

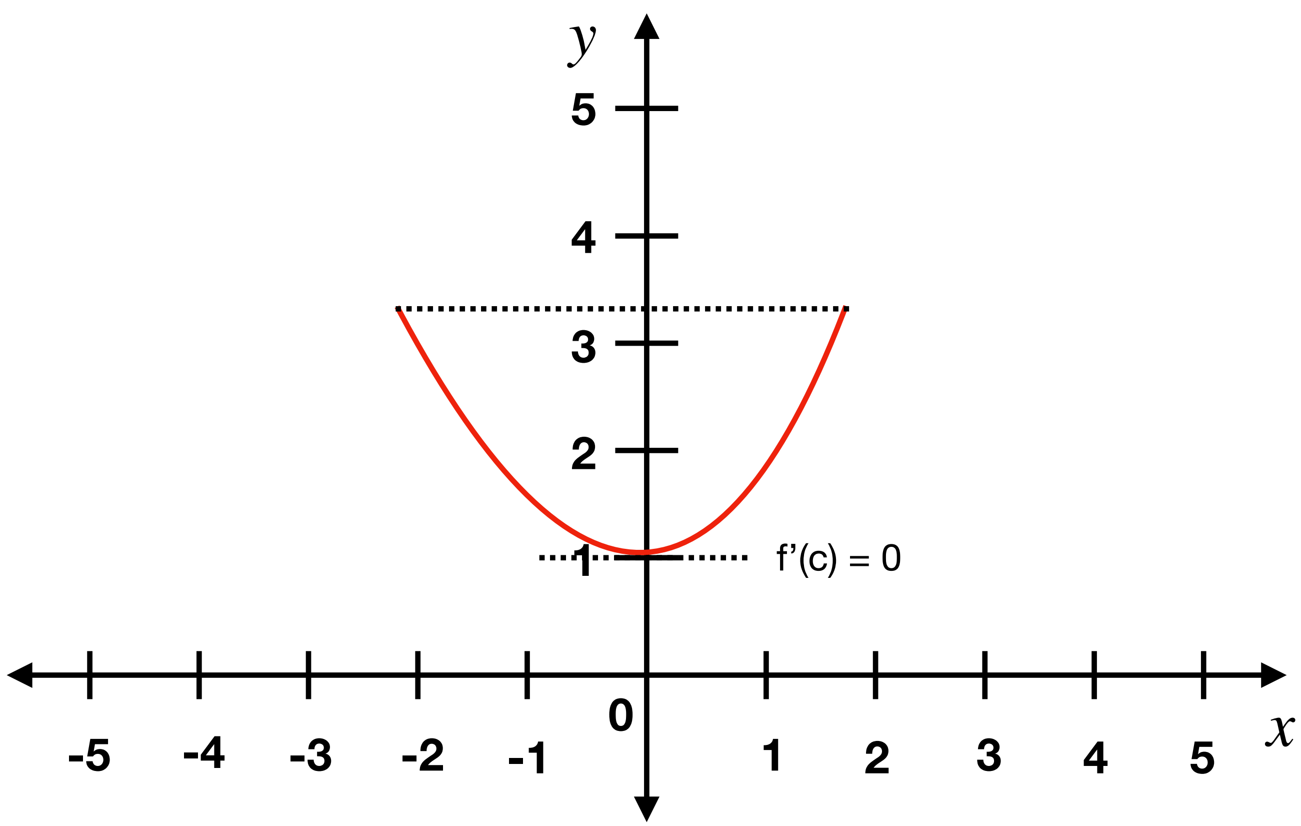
The case were f has a local minimum at c, can be handled similarly.

Key concepts

- A function may have both an absolute maximum and an absolute minimum, have just one absolute extremum, or no absolute extremum.
- If a function has a local extremum, the point at which it occurs must be a critical point.
- A function might not have a local extremum at a critical point.
- A continuous function over a closed, bounded interval has an absolute maximum and minimum. Each extremum occurs at a critical point or an endpoint. [extreme value theorem]

Rolle's Theorem

Let f be a continuous function over the closed interval $[a, b]$ and differentiable over the open interval (a, b) such that $f(a) = f(b)$. There then exists at least one $c \in (a, b)$ such that $f'(c) = 0$.



Rolle's Theorem Proof

Let $k = f(a) = f(b)$. We consider 3 cases:

- $f(x) = k, \forall x \in (a, b)$
- There exits $x \in (a, b)$ such that $f(x) > k$
- There exits $x \in (a, b)$ such that $f(x) < k$

Case 1: If $f(x) = k$ for all $x \in (a, b)$, then $f'(x) = 0, \forall x \in (a, b)$

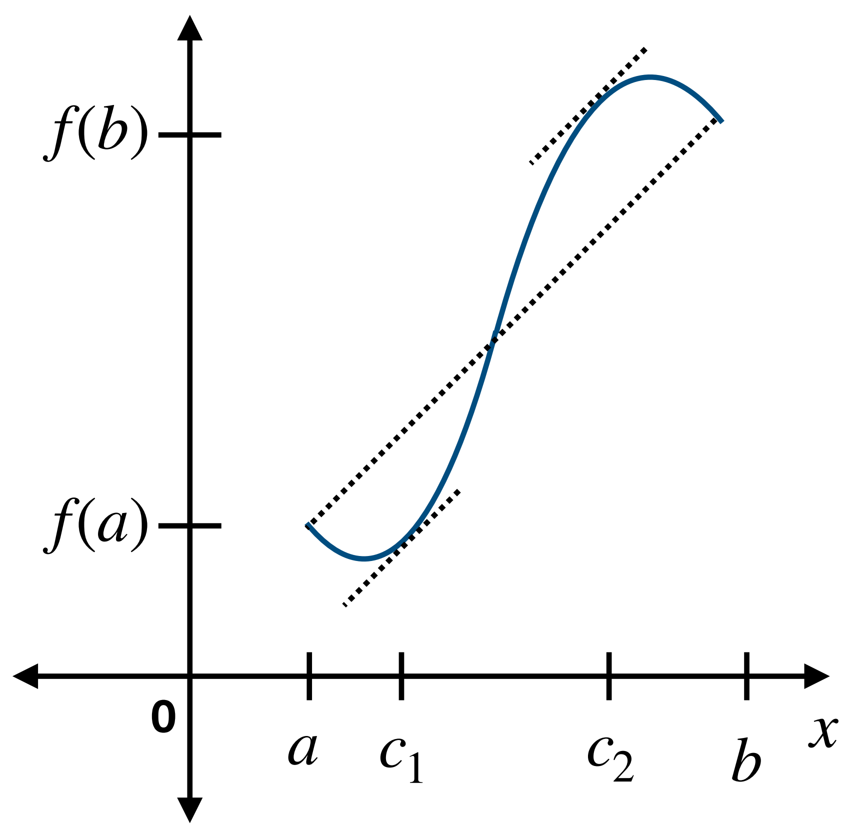
Case 2: Since f is continuous function over the closed, bounded interval $[a, b]$, by the extreme value theorem, it has an absolute maximum. Since there is a point $x \in (a, b)$ such that $f(x) > k$, then the absolute maximum is greater than k. We also know that the absolute maximum does not occur at either endpoints. As a result, the absolute maximum must occur at an interior point $c \in (a, b)$. Because f has a maximum at an interior point, and f is differentiable at c, then by Fermat's theorem, we say that $f'(c) = 0$.

Case 3: Identical to case 2, with maximum replaced by minimum

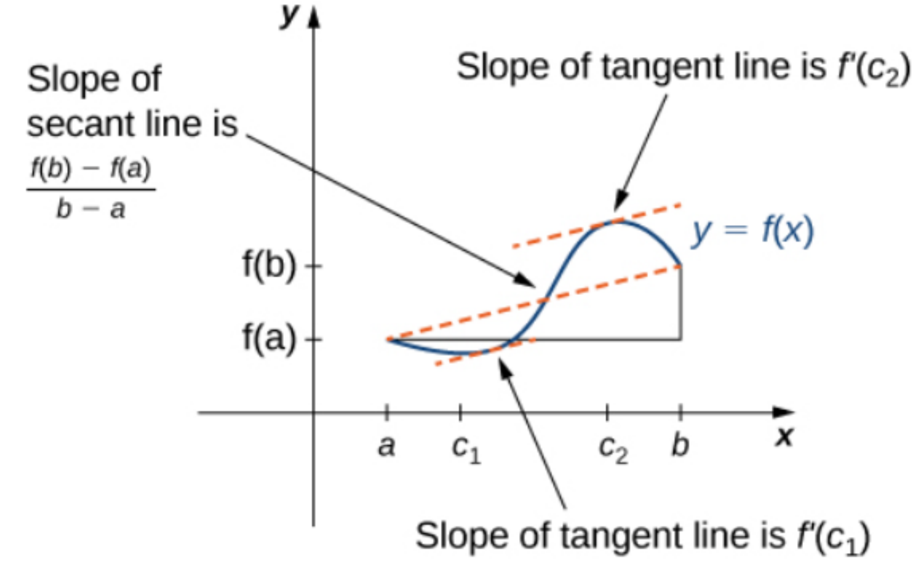
Mean Value Theorem

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b). Then, there exists at least one point $c \in (a, b)$, such that:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$



Proof



Consider the line connecting $(a, f(a))$ and $(b, f(b))$. The slope of this line is $\frac{f(b)-f(a)}{b-a}$. Since the line passes through $(a, f(a))$, the equation of the line can be written as: $y = \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$.

Let $g(x)$ denote the vertical difference between point $(x, f(x))$ and the point (x, y) on that line. Therefore, $g(x) = f(x) - [\frac{f(b)-f(a)}{b-a}(x-a)+f(a)]$

Since the graph of f intersect the secant line at x = a and x = b, we see that $g(a) = 0 = g(b)$. Since f is a differentiable function over (a, b), g is also a differentiable function over (a, b). Since f is continuous over $[a, b]$, so is function g. Therefore, g satisfies the criteria of Rolle's theory, which indicates that there exists a point $c \in (a, b)$ such that $g'(c) = 0$. Since $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \rightarrow g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$, we conclude that $f'(c) = \frac{f(b)-f(a)}{b-a}$

Mean Value Theorem

Corollary 1

Let f be differentiable over an interval I. If $f'(x) = 0$ for all $x \in I$, then $f(x) = constant$ for all $x \in I$.

Corollary 2

If f and g are differentiable over an interval I and $f'(x) = g'(x)$ for all $x \in I$, then $f(x) = g(x) + c$ for some constant c.

Corollary 3

Let f be continuous over the closed interval $[a, b]$ and differentiable over the open interval (a, b)

1. If $f'(x) > 0$ for all $x \in (a, b)$, than f is an increasing function over $[a, b]$
2. If $f'(x) < 0$ for all $x \in (a, b)$, than f is an decreasing function over $[a, b]$

First Derivative Test

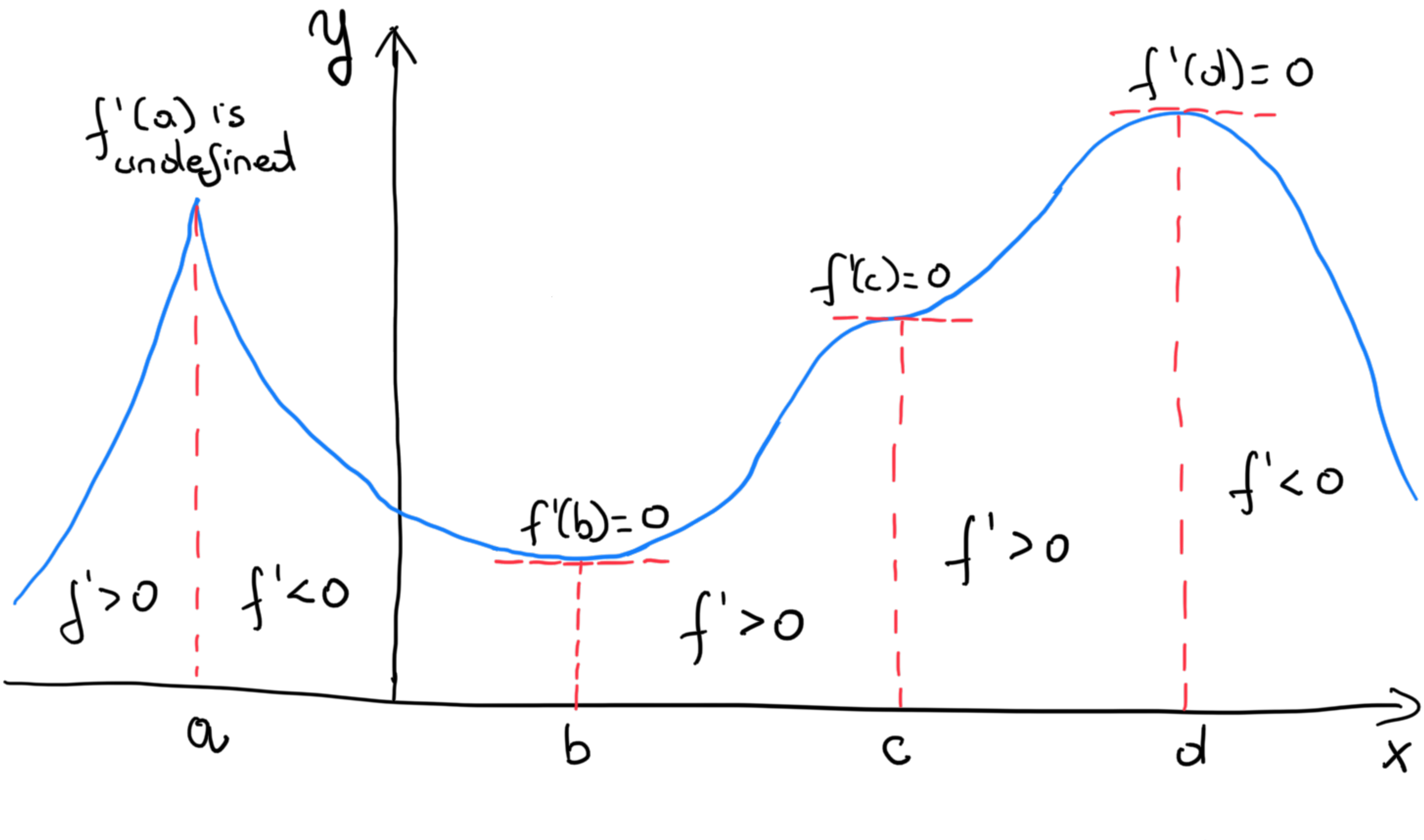
Suppose that f is a continuous function over an interval I containing a critical point c. If f is differentiable over I, except possible at point c, then $f(c)$ satisfies one of these statements:

If f' changes sign from + when $x < c$ to - when $x > c$, then $f(c)$ is a local maximum of f.

If f' changes sign from - when $x < c$ to + when $x > c$, then $f(c)$ is a local minimum of f.

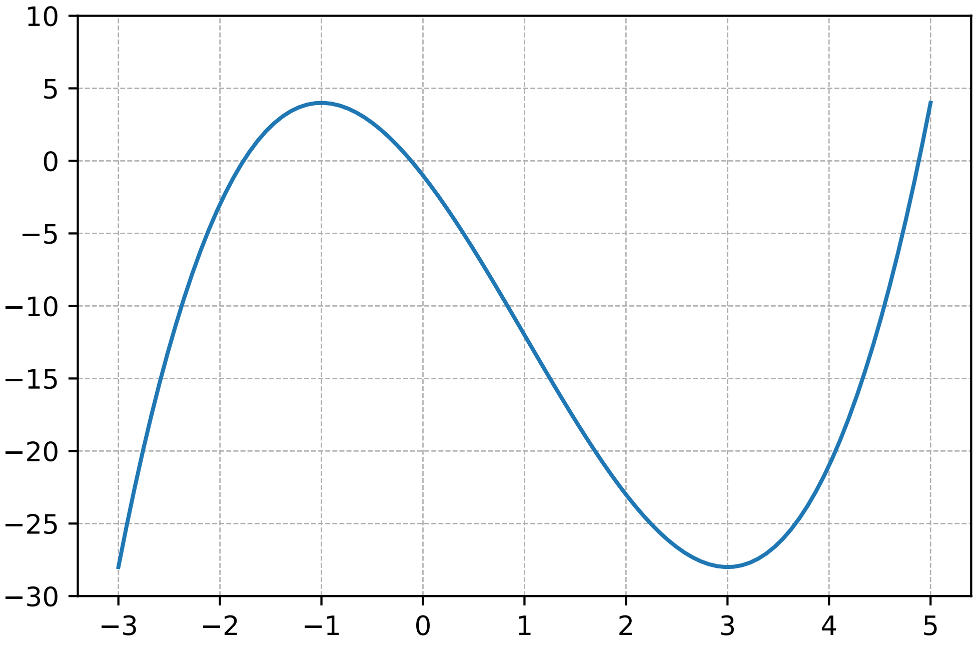
If f' has the same sign for $x < c$ and $x > c$, then $f(c)$ is neither a local maximum nor a local minimum of f.

First Derivative Test



Exercise

Use the first derivative test to find the location of all local extrema for $f(x) = x^3 - 3x^2 - 9x - 1$.



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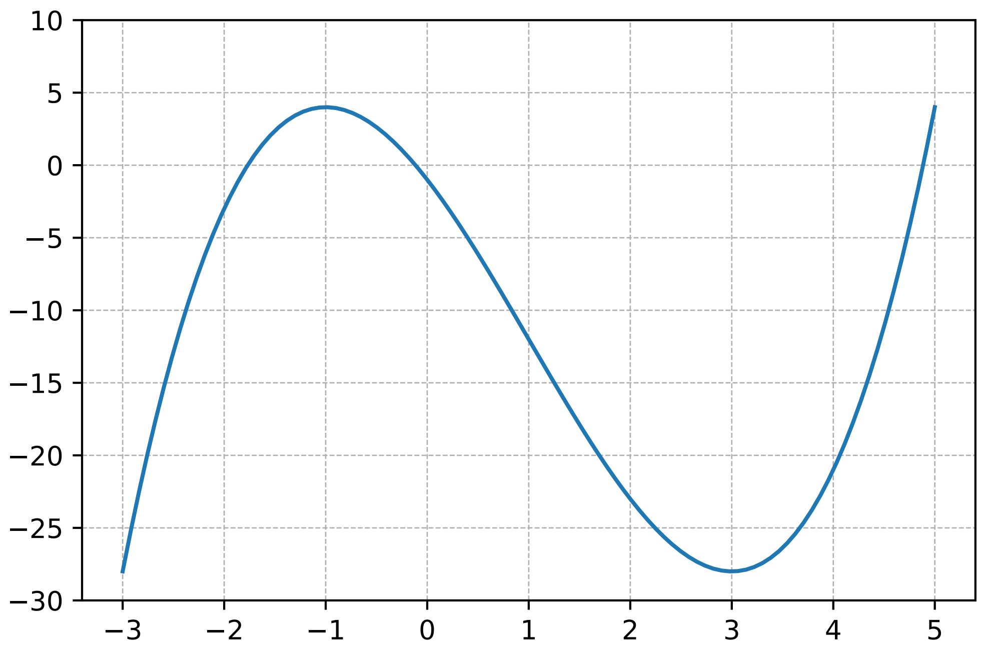


$$f'(x) = 3x^2 - 6x - 9$$

$$x = -1,\ x = 3$$

Interval	Test Point	Sign f'(x)	f(x)
$(-\infty,-1)$	$x=-2$	+	*Increasing*
$(-1, 3)$	$x=0$	-	*Decreasing*
$(3, \infty)$	$x=4$	+	*Increasing*

Concavity

Definition

Let f be a function that is differentiable over an open interval I.
If f' is increasing over I, we say f is concave up over I.
If f' is decreasing over I, we say f is concave down over I.

Test for Concavity

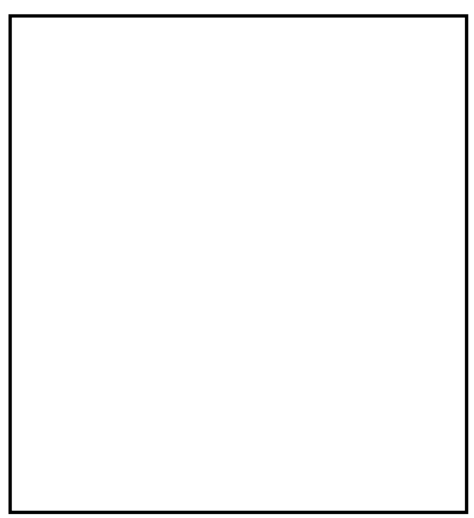
Let f be a function that is twice differentiable over an interval I.
If $f''(x) > 0$ for all $x \in I$, then f is concave up over I.
If $f''(x) < 0$ for all $x \in I$, then f is concave down over I.

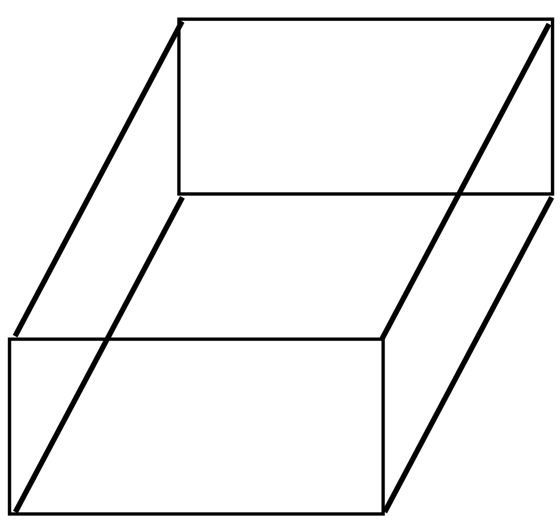
Optimization

Optimization is the branch of research-and-development that aims to solve the problem of finding the elements which maximize or minimize a given real-valued function, while respecting constraints. Many problems in engineering and machine learning can be cast as optimization problems, which explains the growing importance of the field. An optimization problem is the problem of finding the best solution from all feasible solutions.

Problem 1

An open-top box is to be made from a 24 inch by 36 inch piece of cardboard by removing a square from each corner of the box and folding up the flaps on each side. What side square should be put out of each corner to get a box with the maximum volume?





Problem 1