Matrices and Vectors

CS 556
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Vectors

Linear Algebra

- Linear Algebra is the study of vectors and certain rules to manipulate vectors.
- We represent numerical data as vectors.

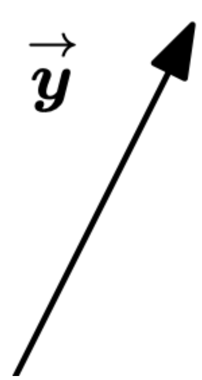
Vectors

- An algebraic vector is ordered list of elements, where the number of elements determine the dimensionally of the vector.

  Examples $\mathbf{v} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \in \mathbb{R}^2$, $\mathbf{w} = \begin{bmatrix} 1 \\ 7 \\ 3 \end{bmatrix} \in \mathbb{R}^3$

- A geometric vector is a straight line with some length and some direction.

  Example:



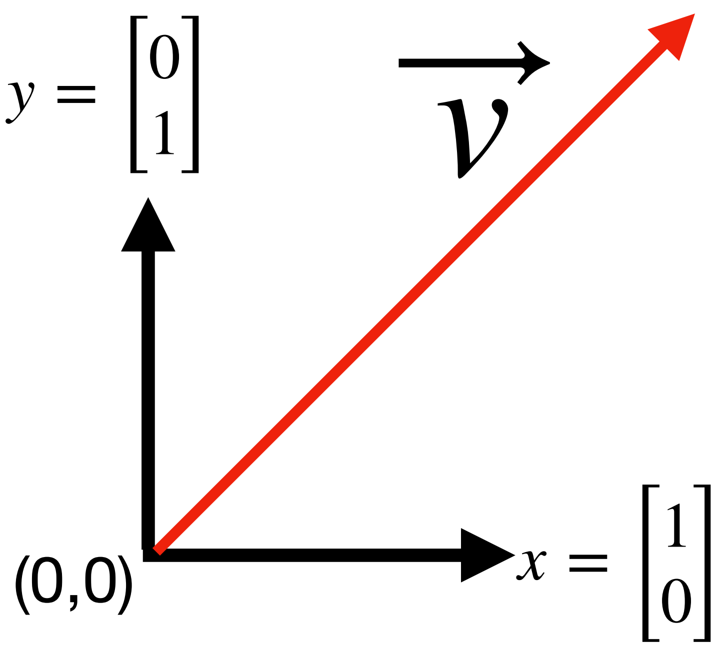
Vectors

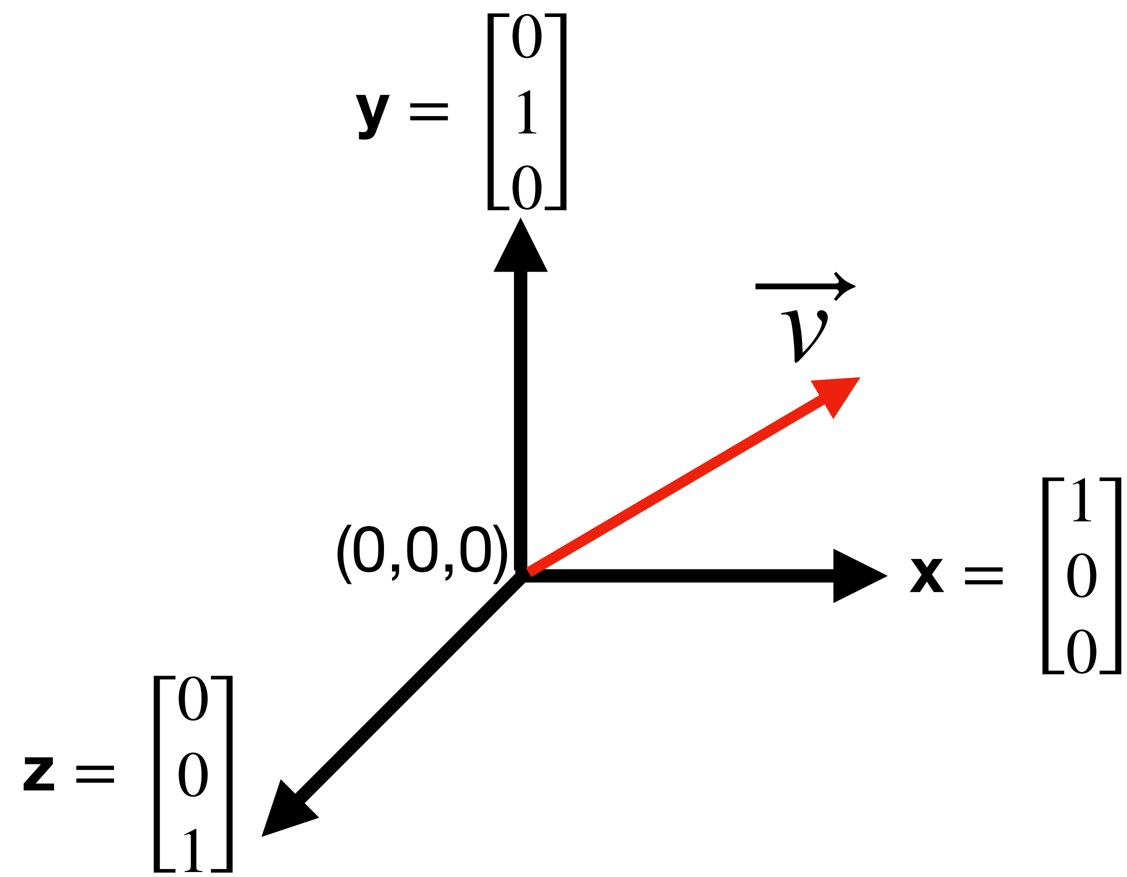
- Vectors - tuples *n* of real numbers $\mathbb{R}^n$

3 bedrooms
2 bathrooms
2 floors,
Year 2008
Price 150K

$$\longrightarrow \quad a = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 2008 \\ 150000 \end{bmatrix}$$

How to express vectors?





$$\mathbf{v} = ax + by = \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$$

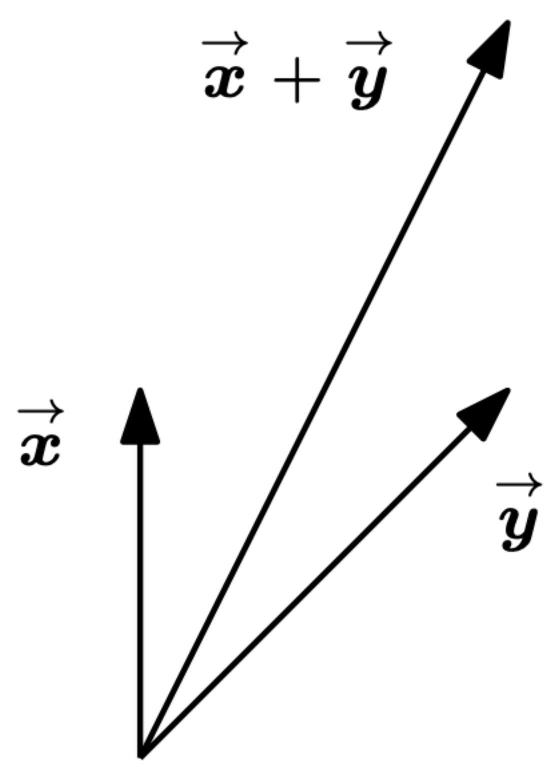
$$\mathbf{v} = ax + by + cz = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

Vectors Operations

- Addition
- Scalar Multiplication

Addition

- Add elements across corresponding dimensions.
- Put the tail of one vector at the head of the other vector.



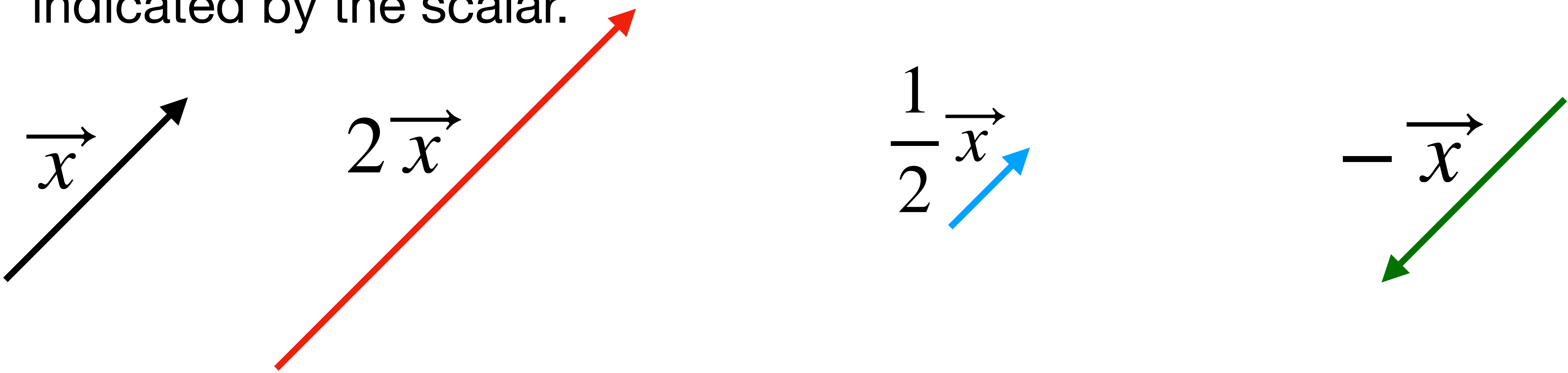
$$\mathbf{x} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \mathbf{y} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\mathbf{x} - \mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -2 \\ -2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Scalar Multiplication

- Scalar: A number, represented by a lower case greek letter such as α, β, λ.
- Algebraic: $\lambda\mathbf{x}$ multiply each element of the vector by the scalar.
- Geometric: Stretch or shrink the vector by the amount indicated by the scalar.



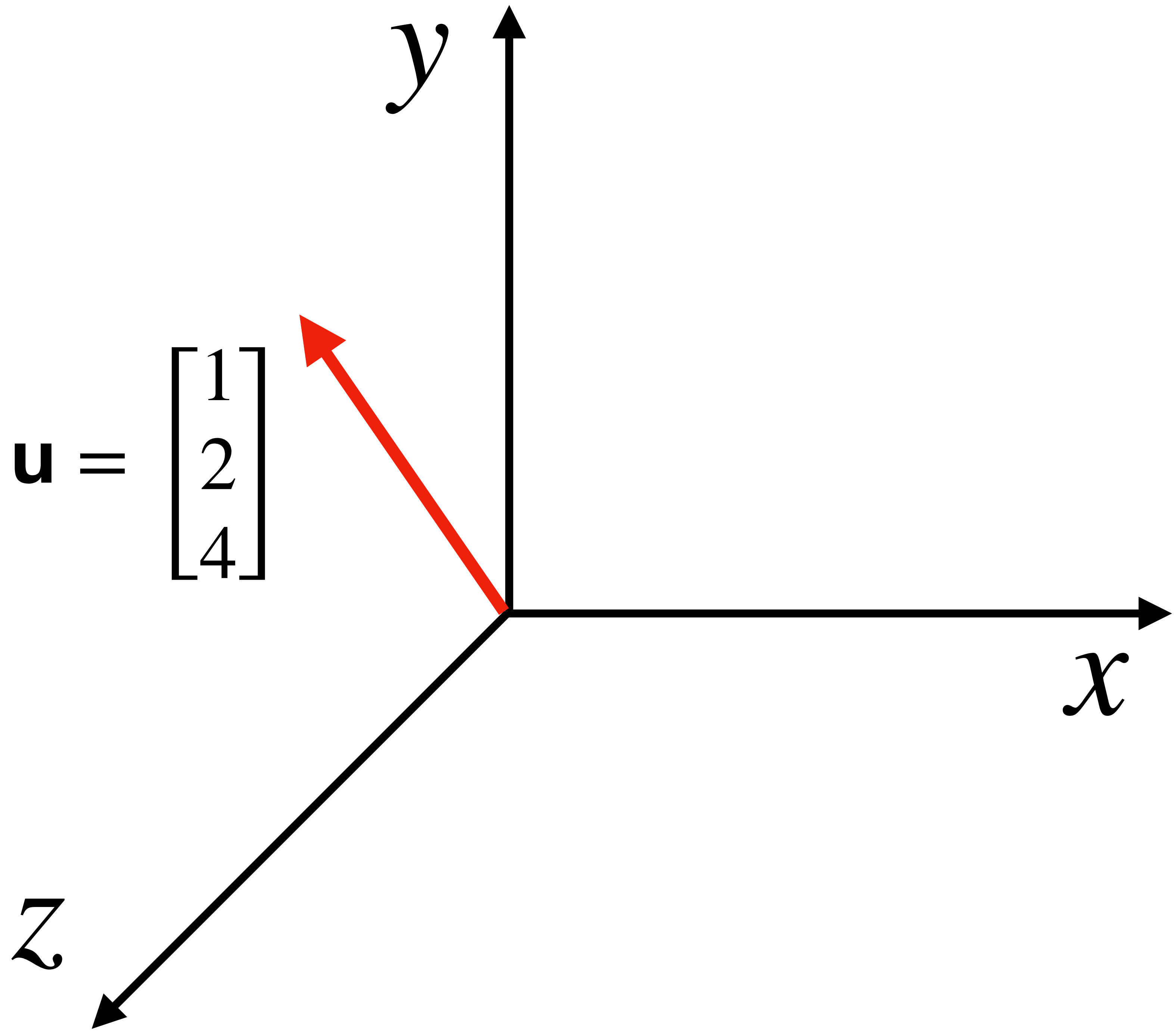
$$x = \begin{bmatrix} 2 \\ 2 \end{bmatrix} \qquad 2x = \begin{bmatrix} 4 \\ 4 \end{bmatrix} \qquad \frac{1}{2}x = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad -x = \begin{bmatrix} -2 \\ -2 \end{bmatrix}$$

Linear Combinations

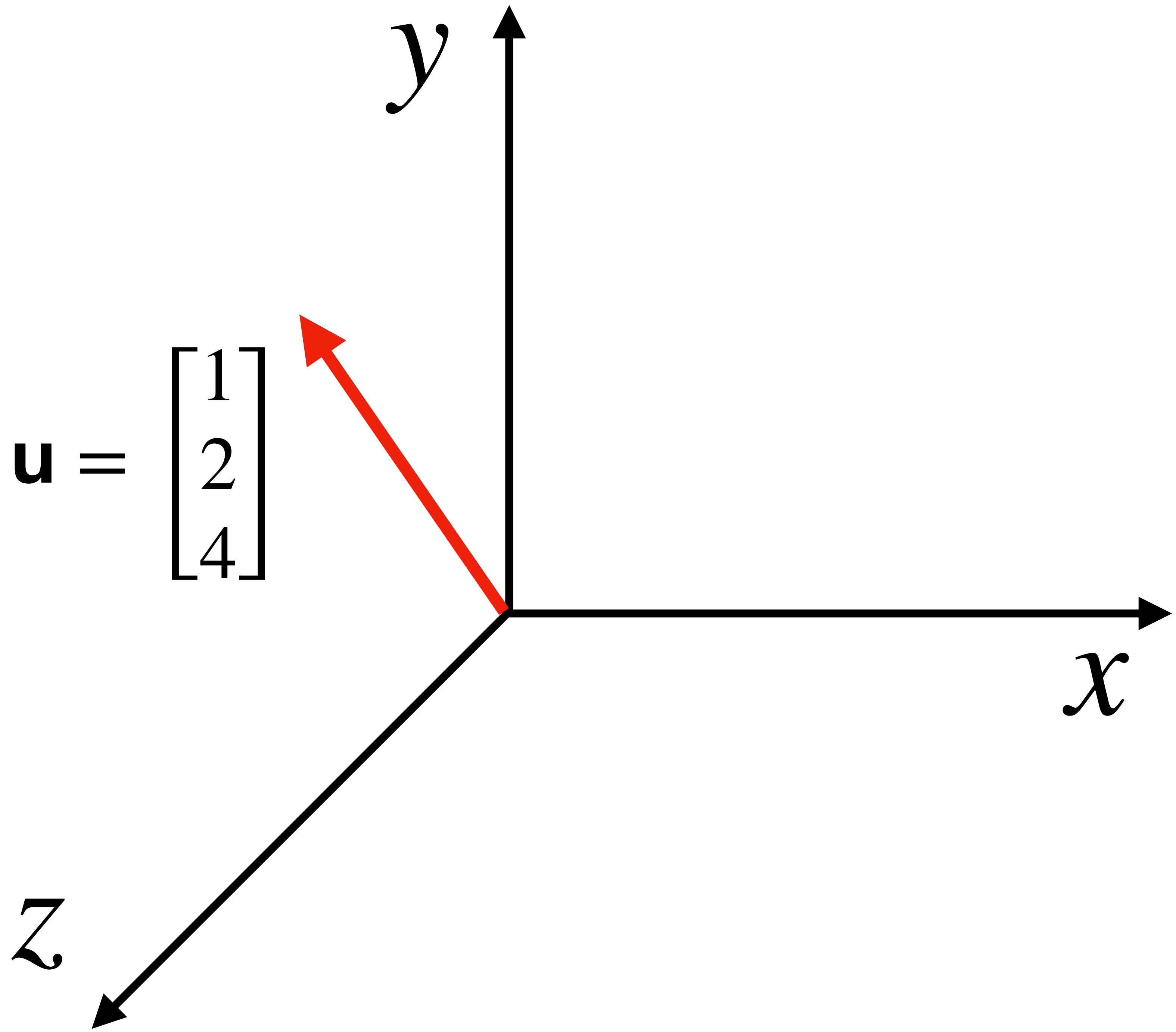
Linear Combinations

- Linear combinations of vectors are created by combining addition with scalar multiplication.

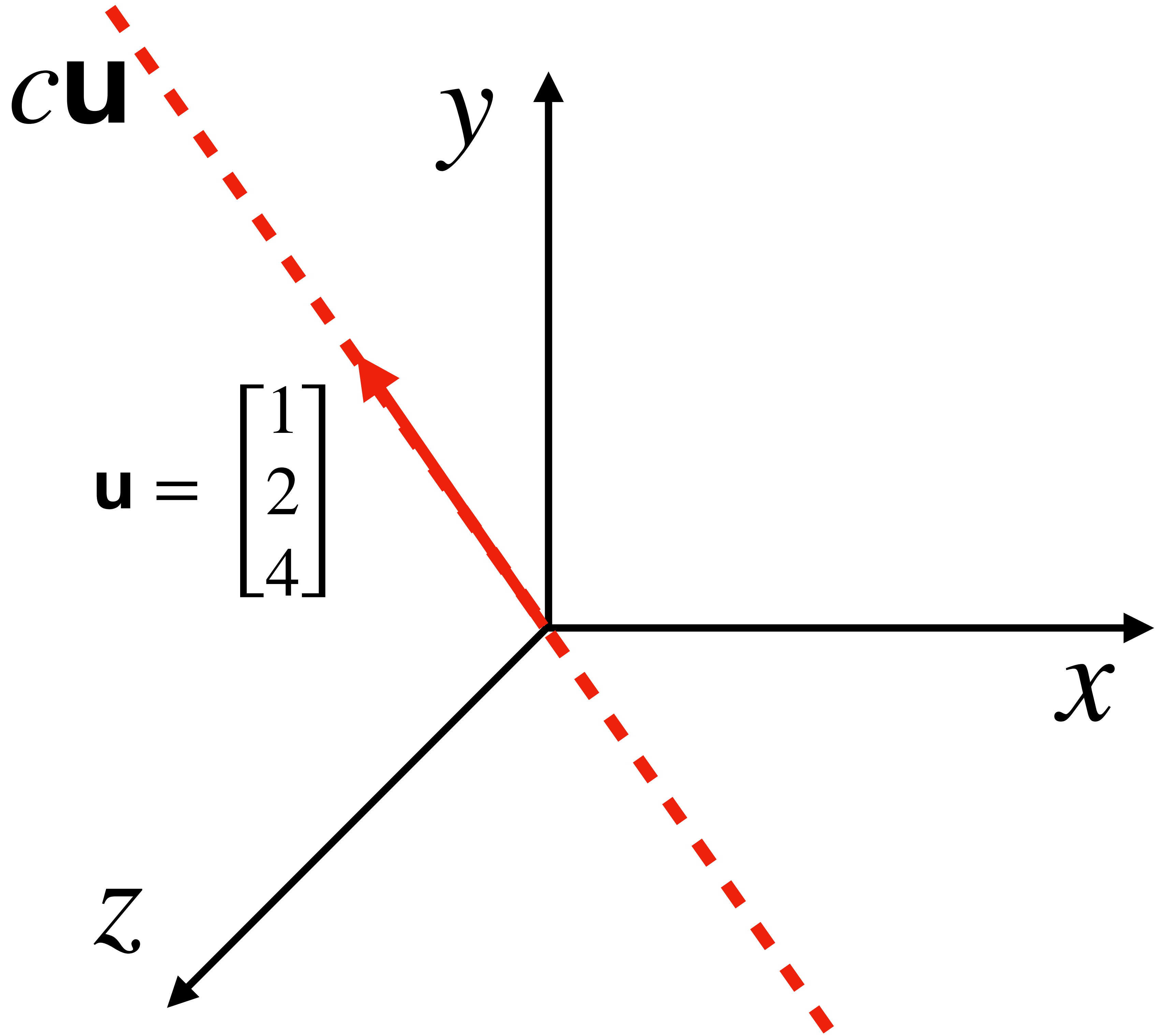
- For instance, assume we have two vectors $\mathbf{v}$ and $\mathbf{w}$ and c and d are two scalars. The sum of $c\mathbf{v}$ and $d\mathbf{w}$ is a linear combination $c\mathbf{v} + d\mathbf{w}$.



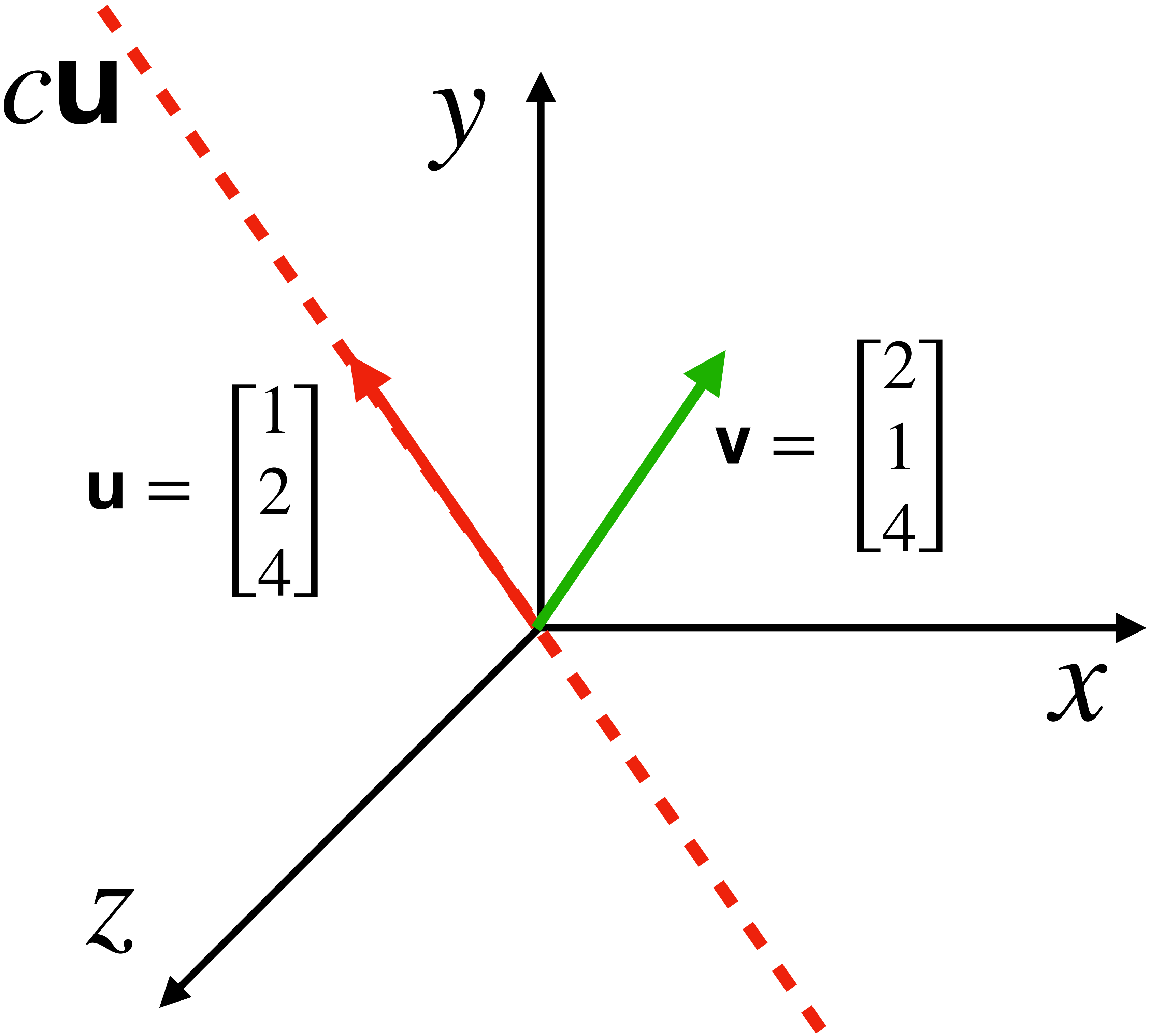
y
x
z
$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$



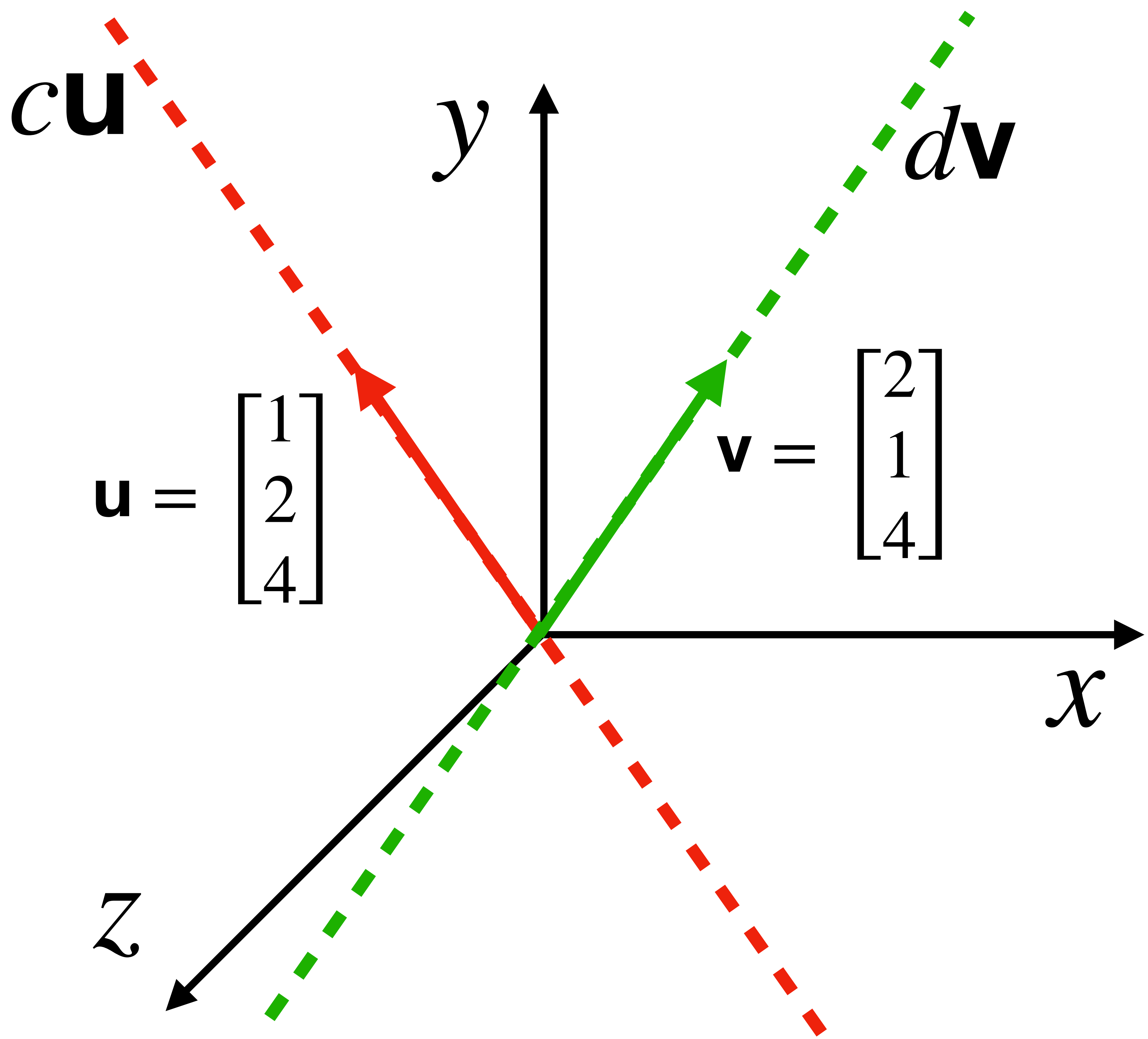
For one vector ***u***, the only linear combinations are the multiples *c**u***.



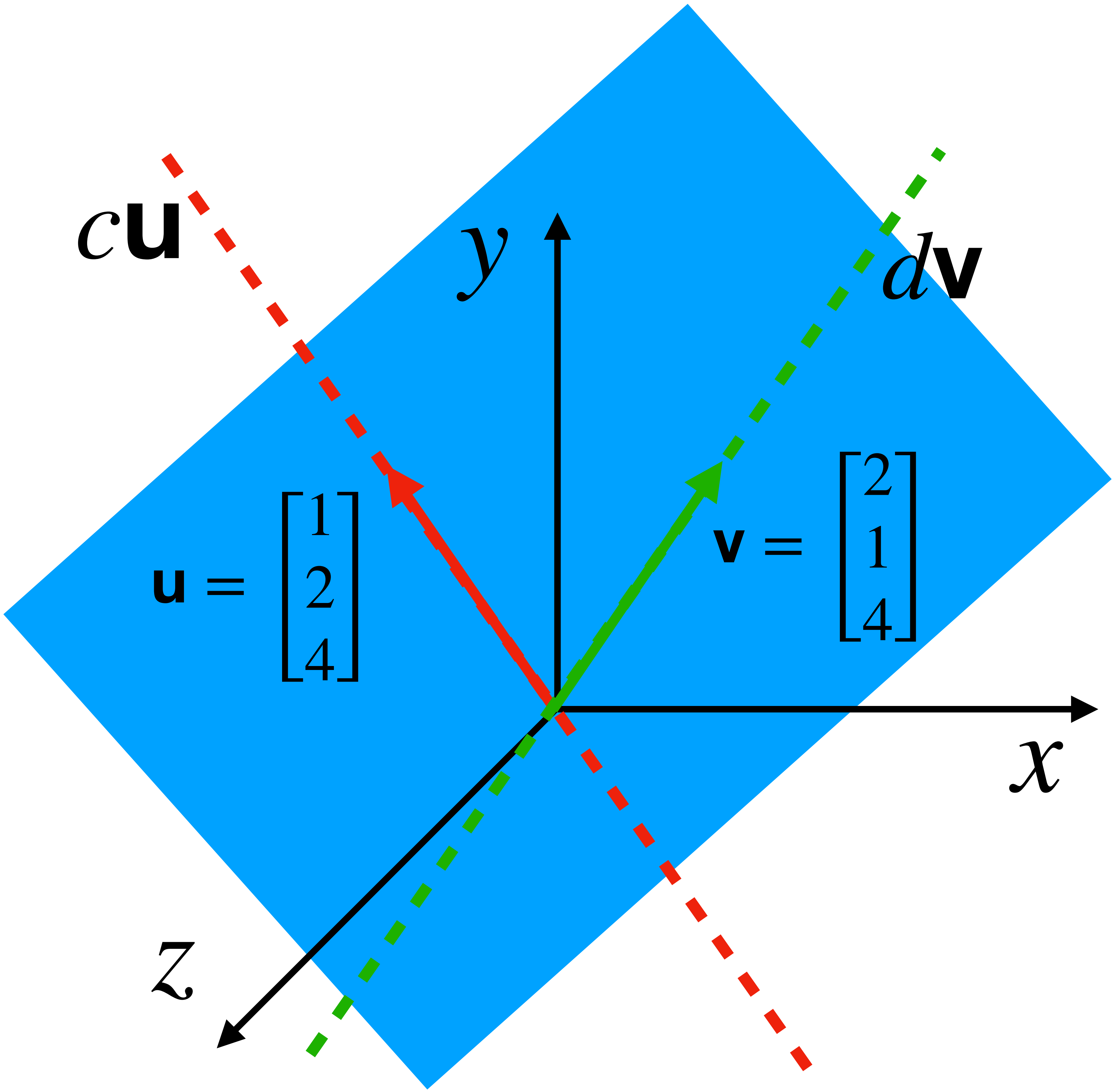
For one vector $\boldsymbol{u}$, the only linear combinations are the multiples $c\boldsymbol{u}$.
The combinations of $c\boldsymbol{u}$ fill a line through (0,0,0).



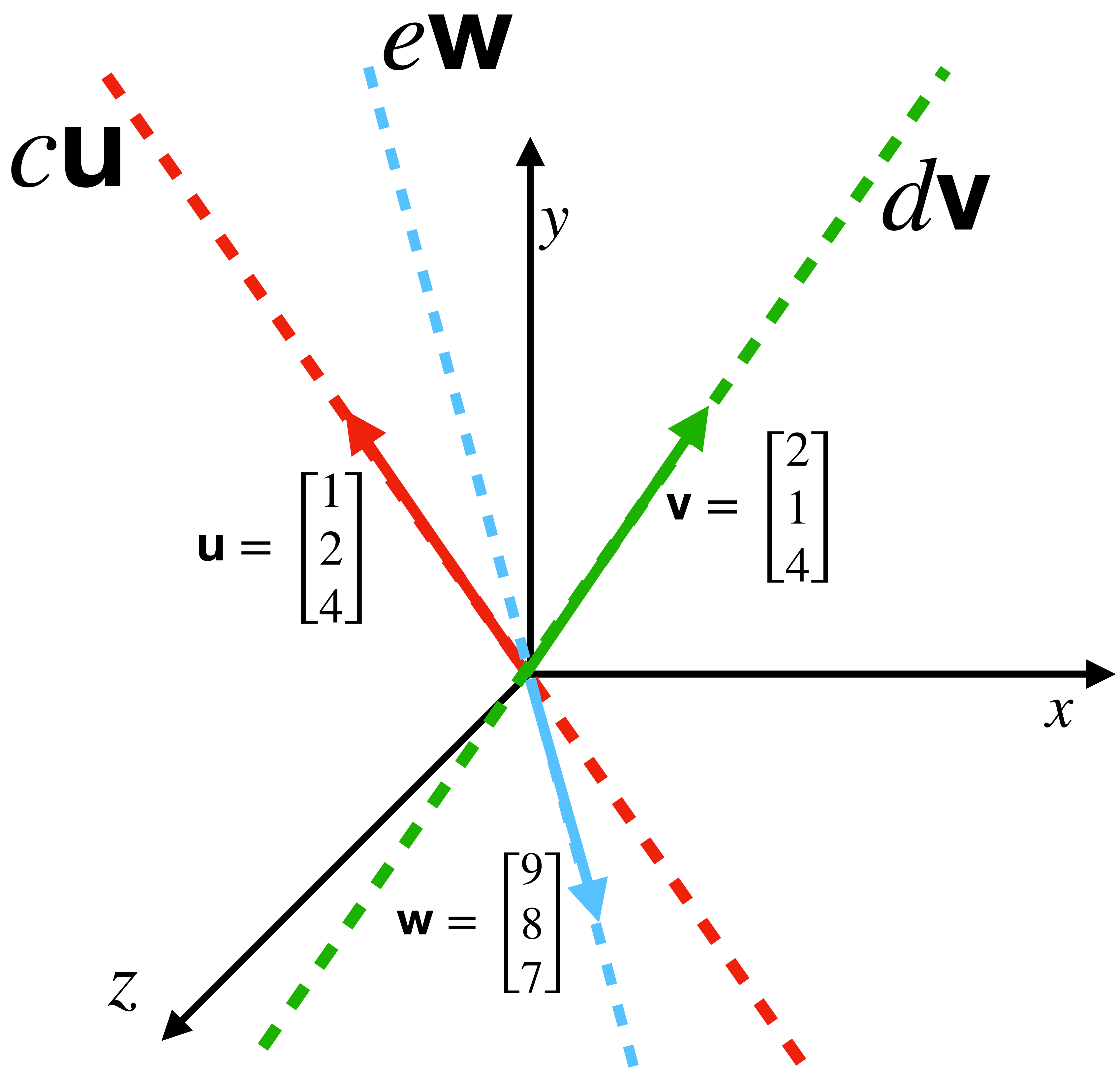
$c\mathbf{u}$
y
$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$
$\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$
x
z



For two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ the linear combinations are $c\boldsymbol{u} + d\boldsymbol{v}$.



For two vectors ***u*** and ***v*** the linear combinations are $c\boldsymbol{u} + d\boldsymbol{v}$.
The combinations $c\boldsymbol{u} + d\boldsymbol{v}$ of two typical nonzero vectors fill a plane through (0,0,0).



For three vectors ***u, v*** and ***w*** the linear combinations are $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$.
The combinations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ of three typical non-zero vectors fill three dimensional space.

Example

- Describe the plane in $\mathbb{R}^3$ that is filled by the linear combinations of $v = (1,1,0)$ and $w = (0,1,1)$.

$$Combinations\ cv + dw = c\begin{bmatrix}1\\1\\0\end{bmatrix} + d\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}c\\c+d\\d\end{bmatrix} fill\ the\ plane$$

- Find a vector that is not a combination of v and w.

$$\mathbf{u} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

Dot Product

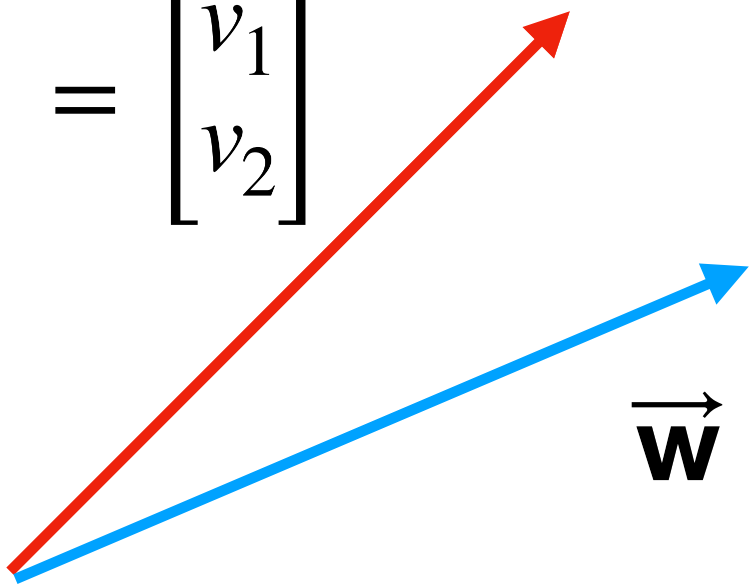
Dot Product

Dot product or **inner product** is an algebraic operation that takes two equal-length sequences of numbers and returns a single number.

$$v \cdot w = < v, w > = v^T w = \sum_{i=1}^{n} v_i w_i$$

Dot Product - Example

$$\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



$$\vec{\mathbf{w}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

$$v \cdot w = v_1 w_1 + v_2 w_2 = 3x4 + 4x3 = 24$$

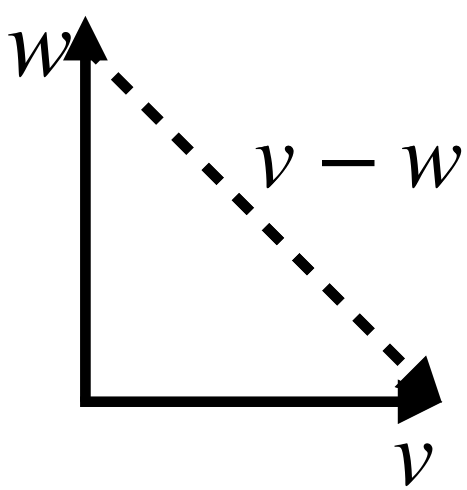
Properties of Dot Product

- Distributive $u^T(v+w) = u^Tv + u^Tw$
- Not Associative: $u^T(v^Tw) \neq (u^Tv)^Tw$
- Commutative: $u^Tv = v^Tu$

Angle between two vectors

The dot product is $v \cdot w = 0$ when v is perpendicular to w.

Proof: When v and w are perpendicular, they form the sides of a right triangle. The hypotenuse is $v - w$.



$$||v||^2 + ||w||^2 = ||v - w||^2$$

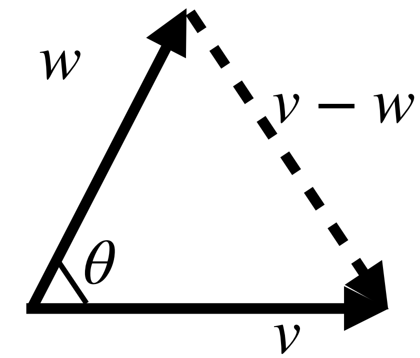
$$(v_1^2 + v_2^2) + (w_1^2 + w_2^2) = (v_1 - w_1)^2 + (v_2 - w_2)^2$$

$$-2v_1w_1 - 2v_2w_2 = 0$$

$$v_1w_1 + v_2w_2 = 0$$

$$v \cdot w = 0 \qquad \blacksquare$$

Cosine Formula for Dot Product



Let v, w be two non-zero vectors in $\mathbb{R}^n$, then:

$$v \cdot w = v^T w = ||v|| \; ||w|| \; cos(\theta)$$

Proof:

$$||v - w||^2 = ||v||^2 + ||w||^2 - 2||v|| \; ||w|| cos(\theta)$$ Cosine Law

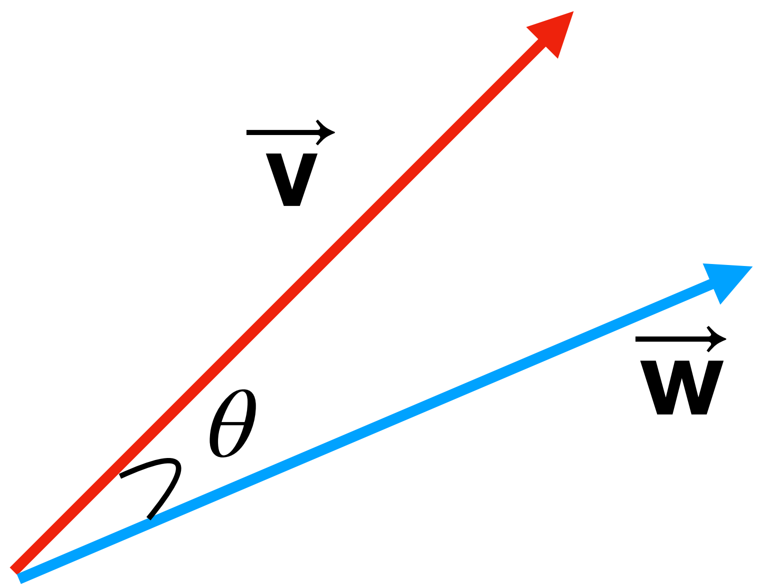
$$||v - w||^2 = (v - w) \cdot (v - w) = v \cdot v - 2(v \cdot w) + w \cdot w$$
$$= ||v||^2 - 2(v \cdot w) + ||w||^2$$

$$||v||^2 - 2(v \cdot w) + ||w||^2 = ||v||^2 + ||w||^2 - 2||v|| \; ||w|| cos(\theta)$$

$$v \cdot w = ||v|| \; ||w|| \; cos(\theta) \qquad \blacksquare$$

Dot Product

Cosine of the angle between the vectors scaled by the product of the lengths of these vectors.



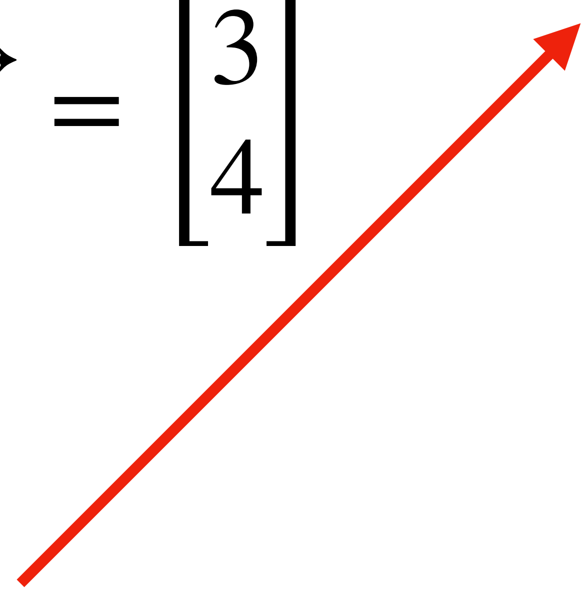
$$v \cdot w = v^T w = ||v||\ ||w||\ cos(\theta)$$

Length of Vectors
Unit Vectors

Vector Length/Magnitude/Norm

$$||\mathbf{v}|| = \sqrt{\mathbf{v}^T\mathbf{v}} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

$$\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$$\mathbf{v} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$||\mathbf{v}|| = \sqrt{3^2 + 4^2} = 5$$

Unit Vectors

Unit Vector: Vector with length of 1

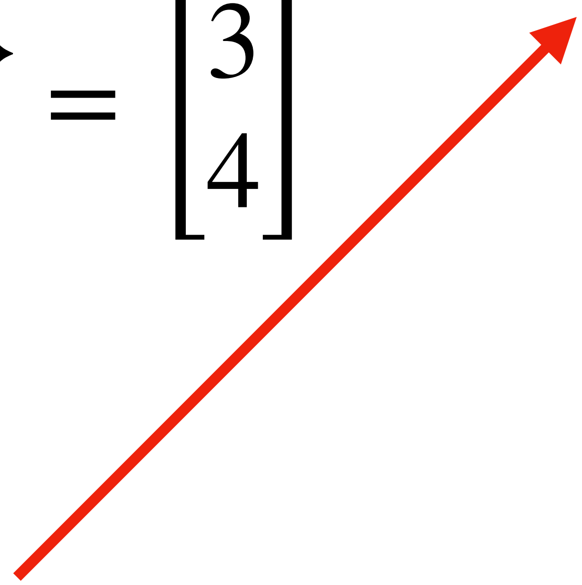
$$\mu \mathbf{v} \ s.t. \ ||\mu \mathbf{v}|| = 1$$

How to choose μ?

$$\mu = \frac{1}{||\mathbf{v}||}$$

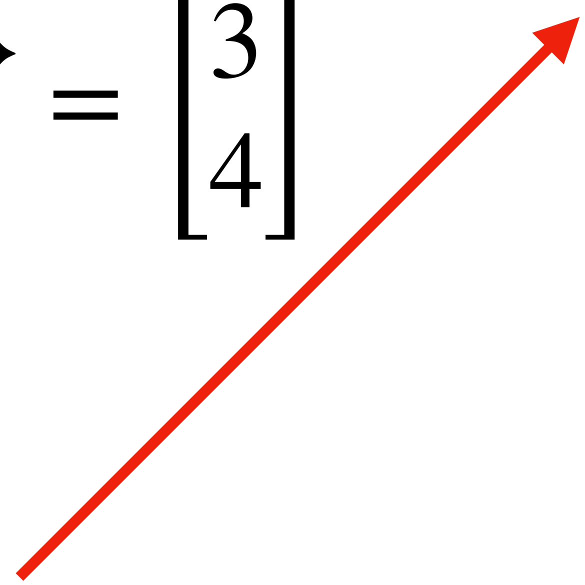
Convert to Unit Vectors

$$\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



Convert to Unit Vectors

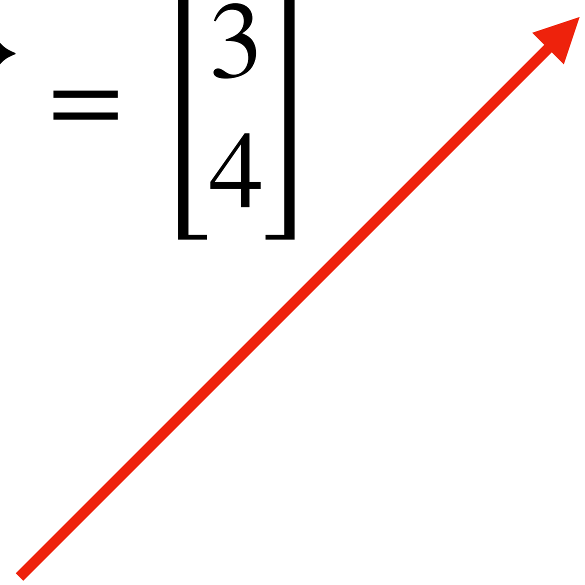
$$\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$$||\mathbf{v}|| = \sqrt{3^2 + 4^2} = 5$$

Convert to Unit Vectors

$$\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



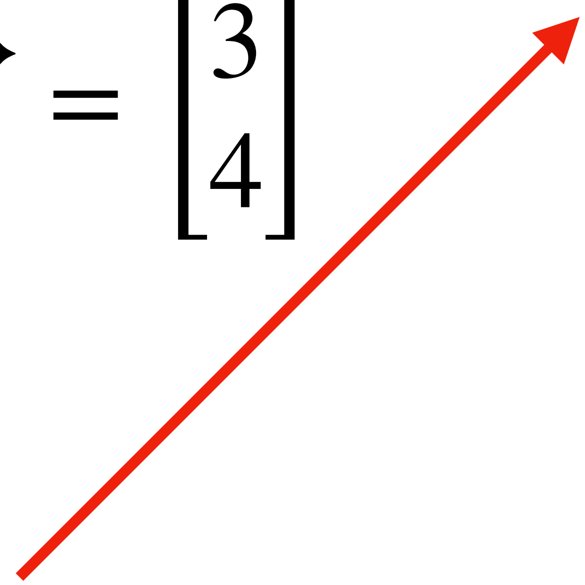
$$||\mathbf{v}|| = \sqrt{3^2 + 4^2} = 5$$

$$\mathbf{u} = \mu\mathbf{v} \ s.t. \ ||\mu\mathbf{v}|| = 1$$

$$\mu = \frac{1}{||\mathbf{v}||} = \frac{1}{5}$$

Convert to Unit Vectors

$$\vec{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$



$$||\mathbf{v}|| = \sqrt{3^2 + 4^2} = 5$$

$$\mathbf{u} = \mu\mathbf{v} \ s.t. \ ||\mu\mathbf{v}|| = 1$$

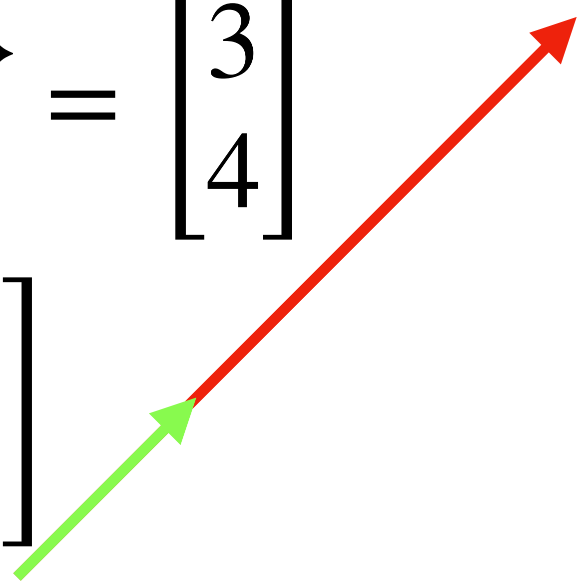
$$\mu = \frac{1}{||\mathbf{v}||} = \frac{1}{5}$$

$$\mathbf{u} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

Convert to Unit Vectors

$$\overrightarrow{\mathbf{v}} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\overrightarrow{\mathbf{u}} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$



$$||\mathbf{v}|| = \sqrt{3^2 + 4^2} = 5$$

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$$\mu = \frac{1}{||\mathbf{v}||} = \frac{1}{5}$$

$$\mathbf{u} = \frac{1}{5}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$$

Matrices

Matrices

With $m, n \in \mathbb{N}$, a real-valued (m, n) *matrix* A is a *m*n*-tuple of elements which is ordered according to a rectangle scheme consisting of *m* rows and *n* columns:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, \; a_{ij} \in \mathbb{R}$$

Diagonal Matrix

In $\mathbb{R}^{nxn}$ we define the diagonal matrix as the nxn matrix containing numbers on the diagonal and 0 elsewhere.

$$\mathbf{D} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 8 \end{bmatrix} \in \mathbb{R}_{3x3}$$

Identity Matrix

In $\mathbb{R}^{nxn}$ we define the identity matrix as the nxn matrix containing 1 on the diagonal and 0 elsewhere.

$$\mathbf{I}_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \in \mathbb{R}_{nxn}$$

Symmetric Matrix

A matrix where elements are mirrored around the diagonal.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 \\ 2 & 5 & 6 \\ 4 & 6 & 8 \end{bmatrix} \in \mathbb{R}_{3x3}$$

If the elements are mirrored around the diagonal with a flipped sign, the matrix is called Skew-symmetric.

Symmetric: $A = A^T$

Skew-symmetric: $A = -A^T$

Matrix Addition

The sum of two matrices $A \in \mathbb{R}^{mxn}$, $B \in \mathbb{R}^{mxn}$

is defined as the element wise sum, i.e.

$$\textbf{A+B} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix} \in \mathbb{R}^{mxn}$$

Matrix Scalar Multiplication

$$\delta\mathbf{A} = \begin{bmatrix} \delta a_{11} & \delta a_{12} & \dots & \delta a_{1n} \\ \delta a_{21} & \delta a_{22} & \dots & \delta a_{2n} \\ \vdots & \vdots & & \vdots \\ \delta a_{m1} & \delta a_{m2} & \dots & \delta a_{mn} \end{bmatrix} \in \mathbb{R}^{mxn}$$

Transpose

For $A \in \mathbb{R}^{mxn}$ the matrix $B \in \mathbb{R}^{nxm}$ with $b_{ij} = a_{ji}$ is called the transpose of A. We write $B = A^T$.

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

Matrix Trace

Sum of elements in the diagonal of the matrix in a square matrix.

$$\mathbf{A} = \begin{bmatrix} \mathbf{3} & 2 & 4 \\ 2 & \mathbf{5} & 6 \\ 4 & 6 & \mathbf{8} \end{bmatrix}$$

$$trace(A) = \sum_{i=1}^{3} A_{ii} = 16$$

Matrix Multiplication

Matrix-Vector Multiplication

Three vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^3$.

The linear combinations in $\mathbb{R}^3$ are $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w}$.

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Ax = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix} \longleftarrow$$

Matrix times vector Linear combination of columns in the matrix.

$$Ax = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b \longleftarrow$$

Which combinations of *u, v, w* produces a particular vector *b*?

Matrix Multiplication

For matrices $A \in \mathbb{R}^{mxn}$, $B \in \mathbb{R}^{nxk}$ the elements c_{ij} of the product $C = AB \in \mathbb{R}^{mxk}$ are computed as:

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}, \qquad i = 1, \ldots, m,\ j = 1, \ldots, k$$

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{AB} = \begin{bmatrix} 2 & 7 \\ 2 & 9 \end{bmatrix}$$

Matrix Multiplication

- Each column of AB is a combination of the columns of A.

Matrix *A* times every column of *B*

$$A[b_1\ b_2\ \ \dots\ \ b_k] = [Ab_1\ Ab_2\ \ \dots\ \ Ab_k]$$

- Every row of AB is a combination of the rows of B.

Every row of *A* times matrix *B*

$$[a_{1i}\ a_{2i}\ \ \dots a_{ni}]B = [row\ \ i\ \ of\ AB]$$

Multiplication Properties

- Associativity

$$\forall A \in \mathbb{R}^{mxn},\ B \in \mathbb{R}^{nxp},\ C \in \mathbb{R}^{pxq}:\ \ (AB)C = A(BC)$$

- Distributivity

$$\forall A, B \in \mathbb{R}^{mxn},\ C, D \in \mathbb{R}^{nxp}:$$
$$(A+B)C = AC + BC,\ A(C+D) = AC + AD$$

- Multiplication with the identity matrix

$$\forall A \in \mathbb{R}^{mxn}:\ \ I_m A = A I_n = A$$