Some ND-style rules for set-theoretic reasoning

Assertions vs Denials

$\widetilde{+\,\chi} := -\,\chi$

$\widetilde{-\,\chi} := +\,\chi$

Primitive meta-rules

$$\dfrac{\begin{matrix}\vdots\\ \varphi\end{matrix}\qquad\begin{matrix}\vdots\\ \widetilde{\varphi}\end{matrix}}{\divideontimes}\ \mathbf{PPS}(\varphi)
\qquad\qquad
\dfrac{\begin{matrix}\overline{\varphi}\ [i]\\ \vdots\\ \psi\end{matrix}\qquad\begin{matrix}\overline{\widetilde{\varphi}}\ [j]\\ \vdots\\ \psi\end{matrix}}{\psi}\ \mathbf{PEM}(\varphi)[i,j]$$

Derived meta-rules

$$\dfrac{\begin{matrix}\overline{\varphi}\ [i]\\ \vdots\\ \widetilde{\varphi}\end{matrix}}{\varphi}\ \mathbf{MIR}[i]
\qquad\qquad
\dfrac{\begin{matrix}\overline{\widetilde{\varphi}}\ [i]\\ \vdots\\ \divideontimes\end{matrix}}{\varphi}\ \mathbf{PBC}[i]
\qquad\qquad
\dfrac{\begin{matrix}\overline{\varphi}\ [i]\\ \vdots\\ \psi\end{matrix}\qquad\begin{matrix}\vdots\\ \widetilde{\psi}\end{matrix}}{\widetilde{\varphi}}\ \mathbf{CPS}(\psi)[i]$$

Set-Builder: $\{\bigcirc_1 \in \bigcirc_2 \mid \bigcirc_3[\bigcirc_4 \mapsto \bigcirc_1]\}$

$$\dfrac{\begin{matrix}\vdots\\ +\ c_2 \in c_1\end{matrix}\qquad\begin{matrix}\vdots\\ \varphi[x \mapsto c_2]\end{matrix}}{+\ c_2 \in \{c \in c_1 \mid \varphi[x \mapsto c]\}}\ +\mathbf{spec}\,\mathbf{I}_1
\qquad\qquad
\dfrac{\begin{matrix}\vdots\\ +\ c_2 \in c_1\end{matrix}\qquad\begin{matrix}\overline{\widetilde{\varphi}[x \mapsto c_2]}\ [i]\\ \vdots\\ \divideontimes\end{matrix}}{+\ c_2 \in \{c \in c_1 \mid \varphi[x \mapsto c]\}}\ +\mathbf{spec}\,\mathbf{I}_2[i]$$

$$\dfrac{\begin{matrix}\vdots\\ +\ c_2 \in \{c \in c_1 \mid \varphi[x \mapsto c]\}\end{matrix}}{+\ c_2 \in c_1}\ +\mathbf{spec}\,\mathbf{E}_2
\qquad\qquad
\dfrac{\begin{matrix}\vdots\\ +\ c_2 \in \{c \in c_1 \mid \varphi[x \mapsto c]\}\end{matrix}}{\varphi[x \mapsto c_2]}\ +\mathbf{spec}\,\mathbf{E}_1$$

$$\dfrac{\begin{matrix}\vdots\\ -\ c_2 \in c_1\end{matrix}}{-\ c_2 \in \{c \in c_1 \mid \varphi[x \mapsto c]\}}\ -\mathbf{spec}\,\mathbf{I}_1
\qquad\qquad
\dfrac{\begin{matrix}\vdots\\ \widetilde{\varphi}[x \mapsto c_2]\end{matrix}}{-\ c_2 \in \{c \in c_1 \mid \varphi[x \mapsto c]\}}\ -\mathbf{spec}\,\mathbf{I}_2$$

$$\dfrac{\begin{matrix}\vdots\\ -\ c_2 \in \{c \in c_1 \mid \varphi[x \mapsto c]\}\end{matrix}\qquad\begin{matrix}\vdots\\ +\ c_2 \in c_1\end{matrix}}{\widetilde{\varphi}[x \mapsto c_2]}\ -\mathbf{spec}\,\mathbf{E}_1
\qquad\qquad
\dfrac{\begin{matrix}\vdots\\ -\ c_2 \in \{c \in c_1 \mid \varphi[x \mapsto c]\}\end{matrix}\qquad\begin{matrix}\vdots\\ \varphi[x \mapsto c_2]\end{matrix}}{-\ c_2 \in c_1}\ -\mathbf{spec}\,\mathbf{E}_2$$

Inclusion [rel]: $\bigcirc_1 \subseteq \bigcirc_2$

Axiom: $c_1 \subseteq c_2$ iff $(\forall c)(c \in c_1 \to c \in c_2)$

$$\dfrac{\dfrac{}{\texttt{Set}\ c}[i] \quad \dfrac{}{+\ c \in c_1}[j] \quad \dfrac{}{-\ c \in c_2}[k] \\ \vdots \\ \divideontimes}{+\ c_1 \subseteq c_2} +\!\subseteq\!\mathbf{I}[i,j,k] \qquad \dfrac{\dfrac{}{\texttt{Set}\ c}[i] \quad \dfrac{}{+\ c \in c_1}[j] \\ \vdots \\ +\ c \in c_2}{+\ c_1 \subseteq c_2} +\!\subseteq\!\mathbf{I}_1[i,j] \qquad \dfrac{\dfrac{}{\texttt{Set}\ c}[i] \quad \dfrac{}{-\ c \in c_2}[j] \\ \vdots \\ -\ c \in c_1}{+\ c_1 \subseteq c_2} +\!\subseteq\!\mathbf{I}_2[i,j]$$

$$\dfrac{\begin{matrix}\vdots \\ +\ c_1 \subseteq c_2\end{matrix} \quad \begin{matrix}\vdots \\ +\ c_0 \in c_1\end{matrix}}{+\ c_0 \in c_2} +\!\subseteq\!\mathbf{E}_1 \qquad \dfrac{\begin{matrix}\vdots \\ +\ c_1 \subseteq c_2\end{matrix} \quad \begin{matrix}\vdots \\ -\ c_0 \in c_2\end{matrix}}{-\ c_0 \in c_1} +\!\subseteq\!\mathbf{E}_2$$

$$\dfrac{\begin{matrix}\vdots \\ +\ c_0 \in c_1\end{matrix} \quad \begin{matrix}\vdots \\ -\ c_0 \in c_2\end{matrix}}{-\ c_1 \subseteq c_2} -\!\subseteq\!\mathbf{I} \qquad \dfrac{\begin{matrix}\vdots \\ -\ c_1 \subseteq c_2\end{matrix} \quad \begin{matrix}\dfrac{}{\texttt{Set}\ c}[i] \quad \dfrac{}{+\ c \in c_1}[j] \quad \dfrac{}{-\ c \in c_2}[k] \\ \vdots \\ \varphi\end{matrix}}{\varphi} -\!\subseteq\!\mathbf{E}[i,j,k]$$

Exclusion / Disjointness [rel]: $\bigcirc_1 \mathrel{)\!(} \bigcirc_2$

OBS: $a \mathrel{)\!(} b$ iff $a \cap b = \varnothing$ iff $a \setminus \overline{b} = \varnothing$ iff $b \setminus \overline{a} = \varnothing$ iff $a \subseteq \overline{b}$ iff $b \subseteq \overline{a}$

Axiom: $c_1 \mathrel{)\!(} c_2$ iff $(\forall c)(\neg(c \in c_1 \land c \in c_2))$

$$\dfrac{\dfrac{}{\texttt{Set}\ c}[i] \quad \dfrac{}{+\ c \in c_1}[j] \quad \dfrac{}{+\ c \in c_2}[k] \\ \vdots \\ \divideontimes}{+\ c_1 \mathrel{)\!(} c_2} +\!\mathrel{)\!(}\!\mathbf{I}[i,j,k] \qquad \dfrac{\dfrac{}{\texttt{Set}\ c}[i] \quad \dfrac{}{+\ c \in c_1}[j] \\ \vdots \\ -\ c \in c_2}{+\ c_1 \mathrel{)\!(} c_2} +\!\mathrel{)\!(}\!\mathbf{I}_1[i,j] \qquad \dfrac{\dfrac{}{\texttt{Set}\ c}[i] \quad \dfrac{}{+\ c \in c_2}[j] \\ \vdots \\ -\ c \in c_1}{+\ c_1 \mathrel{)\!(} c_2} +\!\mathrel{)\!(}\!\mathbf{I}_2[i,j]$$

$$\dfrac{\begin{matrix}\vdots \\ +\ c_1 \mathrel{)\!(} c_2\end{matrix} \quad \begin{matrix}\vdots \\ +\ c_0 \in c_1\end{matrix}}{-\ c_0 \in c_2} +\!\mathrel{)\!(}\!\mathbf{E}_1 \qquad \dfrac{\begin{matrix}\vdots \\ +\ c_1 \mathrel{)\!(} c_2\end{matrix} \quad \begin{matrix}\vdots \\ +\ c_0 \in c_2\end{matrix}}{-\ c_0 \in c_1} +\!\mathrel{)\!(}\!\mathbf{E}_2$$

$$\dfrac{\begin{matrix}\vdots \\ +\ c_0 \in c_1\end{matrix} \quad \begin{matrix}\vdots \\ +\ c_0 \in c_2\end{matrix}}{-\ c_1 \mathrel{)\!(} c_2} -\!\mathrel{)\!(}\!\mathbf{I} \qquad \dfrac{\begin{matrix}\vdots \\ -\ c_1 \mathrel{)\!(} c_2\end{matrix} \quad \begin{matrix}\dfrac{}{\texttt{Set}\ c}[i] \quad \dfrac{}{+\ c \in c_1}[j] \quad \dfrac{}{+\ c \in c_2}[k] \\ \vdots \\ \varphi\end{matrix}}{\varphi} -\!\mathrel{)\!(}\!\mathbf{E}[i,j,k]$$

Empty set [op]: $\varnothing$

Axiom: $c_0 \in \varnothing$ iff $\bot$

$$\dfrac{\begin{matrix}\vdots \\ +\ c_0 \in \varnothing\end{matrix}}{\divideontimes} +\!\varnothing\mathbf{E}$$

$$\dfrac{}{-\ c_0 \in \varnothing} -\!\varnothing\mathbf{I}$$

Universal set [op]: $\mathcal{U}$

Axiom: $c_0 \in \mathcal{U}$ iff $\top$

$$\dfrac{\begin{matrix}\vdots \\ -\ c_0 \in \mathcal{U}\end{matrix}}{\divideontimes} -\!\mathcal{U}\mathbf{E}$$

$$\dfrac{}{+\ c_0 \in \mathcal{U}} +\!\mathcal{U}\mathbf{I}$$

Intersection [op]: $\bigcirc_1 \cap \bigcirc_2$

Axiom: $c_0 \in c_1 \cap c_2$ iff $c_0 \in c_1 \wedge c_0 \in c_2$

$$\dfrac{\begin{matrix}\vdots\\ + c_0 \in c_1\end{matrix} \quad \begin{matrix}\vdots\\ + c_0 \in c_2\end{matrix}}{+ c_0 \in c_1 \cap c_2}\ {+\cap\mathbf{I}} \qquad \dfrac{\begin{matrix}\vdots\\ + c_0 \in c_1 \cap c_2\end{matrix}}{+ c_0 \in c_2}\ {+\cap\mathbf{E}_1} \qquad \dfrac{\begin{matrix}\vdots\\ + c_0 \in c_1 \cap c_2\end{matrix}}{+ c_0 \in c_1}\ {+\cap\mathbf{E}_2}$$

$$\dfrac{\begin{matrix}\vdots\\ - c_0 \in c_2\end{matrix}}{- c_0 \in c_1 \cap c_2}\ {-\cap\mathbf{I}_1} \qquad \dfrac{\begin{matrix}\vdots\\ - c_0 \in c_1\end{matrix}}{- c_0 \in c_1 \cap c_2}\ {-\cap\mathbf{I}_2}$$

$$\dfrac{\begin{matrix}\dfrac{}{+ c_0 \in c_1}\,[i]\\ \vdots\\ - c_0 \in c_2\end{matrix}}{- c_0 \in c_1 \cap c_2}\ {-\cap\mathbf{I}_1[i]} \qquad \dfrac{\begin{matrix}\dfrac{}{+ c_0 \in c_2}\,[i]\\ \vdots\\ - c_0 \in c_1\end{matrix}}{- c_0 \in c_1 \cap c_2}\ {-\cap\mathbf{I}_2[i]}$$

$$\dfrac{\begin{matrix}\vdots\\ - c_0 \in c_1 \cap c_2\end{matrix} \quad \begin{matrix}\vdots\\ + c_0 \in c_1\end{matrix}}{- c_0 \in c_2}\ {-\cap\mathbf{E}_1} \qquad \dfrac{\begin{matrix}\vdots\\ - c_0 \in c_1 \cap c_2\end{matrix} \quad \begin{matrix}\vdots\\ + c_0 \in c_2\end{matrix}}{- c_0 \in c_1}\ {-\cap\mathbf{E}_2}$$

Union [op]: $\bigcirc_1 \cup \bigcirc_2$

Axiom: $c_0 \in c_1 \cup c_2$ iff $c_0 \in c_1 \vee c_0 \in c_2$

$$\dfrac{\begin{matrix}\vdots\\ + c_0 \in c_2\end{matrix}}{+ c_0 \in c_1 \cup c_2}\ {+\cup\mathbf{I}_1} \qquad \dfrac{\begin{matrix}\vdots\\ + c_0 \in c_1\end{matrix}}{+ c_0 \in c_1 \cup c_2}\ {+\cup\mathbf{I}_2}$$

$$\dfrac{\begin{matrix}\dfrac{}{- c_0 \in c_1}\,[i]\\ \vdots\\ + c_0 \in c_2\end{matrix}}{+ c_0 \in c_1 \cup c_2}\ {+\cup\mathbf{I}_1[i]} \qquad \dfrac{\begin{matrix}\dfrac{}{- c_0 \in c_2}\,[i]\\ \vdots\\ + c_0 \in c_1\end{matrix}}{+ c_0 \in c_1 \cup c_2}\ {+\cup\mathbf{I}_2[i]}$$

$$\dfrac{\begin{matrix}\vdots\\ + c_0 \in c_1 \cup c_2\end{matrix} \quad \begin{matrix}\vdots\\ - c_0 \in c_1\end{matrix}}{+ c_0 \in c_2}\ {+\cup\mathbf{E}_1} \qquad \dfrac{\begin{matrix}\vdots\\ + c_0 \in c_1 \cup c_2\end{matrix} \quad \begin{matrix}\vdots\\ - c_0 \in c_2\end{matrix}}{+ c_0 \in c_1}\ {+\cup\mathbf{E}_2}$$

$$\dfrac{\begin{matrix}\vdots\\ - c_0 \in c_1\end{matrix} \quad \begin{matrix}\vdots\\ - c_0 \in c_2\end{matrix}}{- c_0 \in c_1 \cup c_2}\ {-\cup\mathbf{I}} \qquad \dfrac{\begin{matrix}\vdots\\ - c_0 \in c_1 \cup c_2\end{matrix}}{- c_0 \in c_2}\ {-\cup\mathbf{E}_1} \qquad \dfrac{\begin{matrix}\vdots\\ - c_0 \in c_1 \cup c_2\end{matrix}}{- c_0 \in c_1}\ {-\cup\mathbf{E}_2}$$

Relative Complement, or Set Difference [`op`]: $\bigcirc_1 \setminus \bigcirc_2$

Axiom: $c_0 \in c_1 \setminus c_2$ iff $c_0 \in c_1 \wedge \neg c_0 \in c_2$

$$\dfrac{\overset{\vdots}{+\, c_0 \in c_1} \qquad \overset{\vdots}{-\, c_0 \in c_2}}{+\, c_0 \in c_1 \setminus c_2}\ +\setminus\mathbf{I} \qquad\qquad \dfrac{\overset{\vdots}{+\, c_0 \in c_1 \setminus c_2}}{-\, c_0 \in c_2}\ +\setminus\mathbf{E}_1 \qquad\qquad \dfrac{\overset{\vdots}{+\, c_0 \in c_1 \setminus c_2}}{+\, c_0 \in c_1}\ +\setminus\mathbf{E}_2$$

$$\dfrac{\overset{\vdots}{+\, c_0 \in c_2}}{-\, c_0 \in c_1 \setminus c_2}\ -\setminus\mathbf{I}_1 \qquad\qquad \dfrac{\overset{\vdots}{-\, c_0 \in c_1}}{-\, c_0 \in c_1 \setminus c_2}\ -\setminus\mathbf{I}_2$$

$$\dfrac{\begin{array}{c}\dfrac{}{+\, c_0 \in c_1}\ [i] \\ \vdots \\ +\, c_0 \in c_2\end{array}}{-\, c_0 \in c_1 \setminus c_2}\ -\setminus\mathbf{I}_1[i] \qquad\qquad \dfrac{\begin{array}{c}\dfrac{}{-\, c_0 \in c_2}\ [i] \\ \vdots \\ -\, c_0 \in c_1\end{array}}{-\, c_0 \in c_1 \setminus c_2}\ -\setminus\mathbf{I}_2[i]$$

$$\dfrac{\overset{\vdots}{-\, c_0 \in c_1 \setminus c_2} \qquad \overset{\vdots}{+\, c_0 \in c_1}}{+\, c_0 \in c_2}\ -\setminus\mathbf{E}_1 \qquad\qquad \dfrac{\overset{\vdots}{-\, c_0 \in c_1 \setminus c_2} \qquad \overset{\vdots}{-\, c_0 \in c_2}}{-\, c_0 \in c_1}\ -\setminus\mathbf{E}_2$$

Absolute Complement [`op`]: $\overline{\bigcirc_1}$

Axiom: $c_0 \in \bar{c}_1$ iff $\neg c_0 \in c_1$

$$\dfrac{\overset{\vdots}{-\, c_0 \in c_1}}{+\, c_0 \in \overline{c_1}}\ +\overline{}\,\mathbf{I} \qquad\qquad \dfrac{\overset{\vdots}{+\, c_0 \in \overline{c_1}}}{-\, c_0 \in c_1}\ +\overline{}\,\mathbf{E}$$

$$\dfrac{\overset{\vdots}{+\, c_0 \in c_1}}{-\, c_0 \in \overline{c_1}}\ -\overline{}\,\mathbf{I} \qquad\qquad \dfrac{\overset{\vdots}{-\, c_0 \in \overline{c_1}}}{+\, c_0 \in c_1}\ -\overline{}\,\mathbf{E}$$

Symmetric Difference [op]: $\bigcirc_1 \ominus \bigcirc_2$

Axiom: $c_0 \in c_1 \ominus c_2$ iff $(c_0 \in c_1 \wedge \neg c_0 \in c_2) \vee (c_0 \in c_2 \wedge \neg c_0 \in c_1)$

$$\dfrac{\overset{\vdots}{+\ c_0 \in c_1} \quad \overset{\vdots}{-\ c_0 \in c_2}}{+\ c_0 \in c_1 \ominus c_2}\ +\ominus\mathbf{I}_1 \qquad \dfrac{\overset{\vdots}{-\ c_0 \in c_1} \quad \overset{\vdots}{+\ c_0 \in c_2}}{+\ c_0 \in c_1 \ominus c_2}\ +\ominus\mathbf{I}_2$$

$$\dfrac{\overset{\vdots}{+\ c_0 \in c_1 \ominus c_2} \quad \overset{\vdots}{+\ c_0 \in c_1}}{-\ c_0 \in c_2}\ +\ominus\mathbf{E}_{1a} \qquad \dfrac{\overset{\vdots}{+\ c_0 \in c_1 \ominus c_2} \quad \overset{\vdots}{+\ c_0 \in c_2}}{-\ c_0 \in c_1}\ +\ominus\mathbf{E}_{2a}$$

$$\dfrac{\overset{\vdots}{+\ c_0 \in c_1 \ominus c_2} \quad \overset{\vdots}{-\ c_0 \in c_1}}{+\ c_0 \in c_2}\ +\ominus\mathbf{E}_{1b} \qquad \dfrac{\overset{\vdots}{+\ c_0 \in c_1 \ominus c_2} \quad \overset{\vdots}{-\ c_0 \in c_2}}{+\ c_0 \in c_1}\ +\ominus\mathbf{E}_{2b}$$

$$\dfrac{\overset{\vdots}{+\ c_0 \in c_1} \quad \overset{\vdots}{+\ c_0 \in c_2}}{-\ c_0 \in c_1 \ominus c_2}\ -\ominus\mathbf{I}_1 \qquad \dfrac{\overset{\vdots}{-\ c_0 \in c_1} \quad \overset{\vdots}{-\ c_0 \in c_2}}{-\ c_0 \in c_1 \ominus c_2}\ -\ominus\mathbf{I}_2$$

$$\dfrac{\overset{\vdots}{-\ c_0 \in c_1 \ominus c_2} \quad \overset{\vdots}{+\ c_0 \in c_1}}{+\ c_0 \in c_2}\ -\ominus\mathbf{E}_{1a} \qquad \dfrac{\overset{\vdots}{-\ c_0 \in c_1 \ominus c_2} \quad \overset{\vdots}{+\ c_0 \in c_2}}{+\ c_0 \in c_1}\ -\ominus\mathbf{E}_{2a}$$

$$\dfrac{\overset{\vdots}{-\ c_0 \in c_1 \ominus c_2} \quad \overset{\vdots}{-\ c_0 \in c_1}}{-\ c_0 \in c_2}\ -\ominus\mathbf{E}_{1b} \qquad \dfrac{\overset{\vdots}{-\ c_0 \in c_1 \ominus c_2} \quad \overset{\vdots}{-\ c_0 \in c_2}}{-\ c_0 \in c_1}\ -\ominus\mathbf{E}_{2b}$$

Ordered Pair [op]: $\bigcirc_1 \times \bigcirc_2$

Axiom: $(c_1, c_2) \in d_1 \times d_2$ iff $c_1 \in d_1 \wedge c_2 \in d_2$

$$\dfrac{\overset{\vdots}{+\ c_1 \in d_1} \quad \overset{\vdots}{+\ c_2 \in d_2}}{+\ (c_1, c_2) \in d_1 \times d_2}\ +\times\mathbf{I} \qquad \dfrac{\overset{\vdots}{+\ (c_1, c_2) \in d_1 \times d_2}}{+\ c_2 \in d_2}\ +\times\mathbf{E}_1 \qquad \dfrac{\overset{\vdots}{+\ (c_1, c_2) \in d_1 \times d_2}}{+\ c_1 \in d_1}\ +\times\mathbf{E}_2$$

$$\dfrac{\overset{\vdots}{-\ c_2 \in d_2}}{-\ (c_1, c_2) \in d_1 \times d_2}\ -\times\mathbf{I}_1 \qquad \dfrac{\overset{\vdots}{-\ c_1 \in d_1}}{-\ (c_1, c_2) \in d_1 \times d_2}\ -\times\mathbf{I}_2$$

$$\dfrac{\overset{\vdots}{-\ (c_1, c_2) \in d_1 \times d_2} \quad \overset{\vdots}{+\ c_1 \in d_1}}{-\ c_2 \in d_2}\ -\times\mathbf{E}_1 \qquad \dfrac{\overset{\vdots}{-\ (c_1, c_2) \in d_1 \times d_2} \quad \overset{\vdots}{+\ c_2 \in d_2}}{-\ c_1 \in d_1}\ -\times\mathbf{E}_2$$

Union of indexed family [op]: $\bigcup\limits_{\bigcirc_1 \in \bigcirc_2} \bigcirc_3$

Axiom: $c \in \bigcup\limits_{i\in I} c_i$ iff $(\exists j)(j \in I \wedge c \in c_j)$

$$\dfrac{\overline{\mathtt{Set}\ j}\,[i] \quad \overline{+\ j \in I}\,[j] \quad \overline{+\ c \in c_j}\,[k] \\ \vdots \\ \divideontimes}{-\ c \in \bigcup\limits_{i\in I} c_i}\ {-}{\cup}\mathbf{I}[i,j,k] \qquad \dfrac{\overline{\mathtt{Set}\ j}\,[i] \quad \overline{+\ j \in I}\,[j] \\ \vdots \\ -\ c \in c_j}{-\ c \in \bigcup\limits_{i\in I} c_i}\ {-}{\cup}\mathbf{I}_1[i,j] \qquad \dfrac{\overline{\mathtt{Set}\ j}\,[i] \quad \overline{+\ c \in c_j}\,[j] \\ \vdots \\ -\ j \in I}{-\ c \in \bigcup\limits_{i\in I} c_i}\ {-}{\cup}\mathbf{I}_2[i,j]$$

$$\dfrac{\begin{matrix}\vdots \\ -\ c \in \bigcup\limits_{i\in I} c_i\end{matrix} \quad \begin{matrix}\vdots \\ +\ j \in I\end{matrix}}{-\ c \in c_j}\ {-}{\cup}\mathbf{E}_1 \qquad \dfrac{\begin{matrix}\vdots \\ -\ c \in \bigcup\limits_{i\in I} c_i\end{matrix} \quad \begin{matrix}\vdots \\ +\ c \in c_j\end{matrix}}{-\ j \in I}\ {-}{\cup}\mathbf{E}_2$$

$$\dfrac{\begin{matrix}\vdots \\ +\ c \in c_j\end{matrix} \quad \begin{matrix}\vdots \\ +\ j \in I\end{matrix}}{+\ c \in \bigcup\limits_{i\in I} c_i}\ {+}{\cup}\mathbf{I} \qquad \dfrac{\begin{matrix}\vdots \\ +\ c \in \bigcup\limits_{i\in I} c_i\end{matrix} \quad \begin{matrix}\overline{\mathtt{Set}\ j}\,[i] \quad \overline{+\ j \in I}\,[j] \quad \overline{+\ c \in c_j}\,[k] \\ \vdots \\ \varphi\end{matrix}}{\varphi}\ {+}{\cup}\mathbf{E}[i,j,k]$$

Intersection of indexed family [op]: $\bigcap\limits_{\bigcirc_1 \in \bigcirc_2} \bigcirc_3$

Axiom: $c \in \bigcap\limits_{i\in I} c_i$ iff $(\forall j)(j \in I \rightarrow c \in c_j)$

$$\dfrac{\overline{\mathtt{Set}\ j}\,[i] \quad \overline{+\ j \in I}\,[j] \quad \overline{-\ c \in c_j}\,[k] \\ \vdots \\ \divideontimes}{+\ c \in \bigcap\limits_{i\in I} c_i}\ {+}{\cap}\mathbf{I}[i,j,k] \qquad \dfrac{\overline{\mathtt{Set}\ j}\,[i] \quad \overline{+\ j \in I}\,[j] \\ \vdots \\ +\ c \in c_j}{+\ c \in \bigcap\limits_{i\in I} c_i}\ {+}{\cap}\mathbf{I}_1[i,j] \qquad \dfrac{\overline{\mathtt{Set}\ j}\,[i] \quad \overline{-\ c \in c_j}\,[j] \\ \vdots \\ -\ j \in I}{+\ c \in \bigcap\limits_{i\in I} c_i}\ {+}{\cap}\mathbf{I}_2[i,j]$$

$$\dfrac{\begin{matrix}\vdots \\ +\ c \in \bigcap\limits_{i\in I} c_i\end{matrix} \quad \begin{matrix}\vdots \\ +\ j \in I\end{matrix}}{+\ c \in c_j}\ {+}{\cap}\mathbf{E}_1 \qquad \dfrac{\begin{matrix}\vdots \\ +\ c \in \bigcap\limits_{i\in I} c_i\end{matrix} \quad \begin{matrix}\vdots \\ -\ c \in c_j\end{matrix}}{-\ j \in I}\ {+}{\cap}\mathbf{E}_2$$

$$\dfrac{\begin{matrix}\vdots \\ -\ c \in c_j\end{matrix} \quad \begin{matrix}\vdots \\ +\ j \in I\end{matrix}}{-\ c \in \bigcap\limits_{i\in I} c_i}\ {-}{\cap}\mathbf{I} \qquad \dfrac{\begin{matrix}\vdots \\ -\ c \in \bigcap\limits_{i\in I} c_i\end{matrix} \quad \begin{matrix}\overline{\mathtt{Set}\ j}\,[i] \quad \overline{+\ j \in I}\,[j] \quad \overline{-\ c \in c_j}\,[k] \\ \vdots \\ \varphi\end{matrix}}{\varphi}\ {-}{\cap}\mathbf{E}[i,j,k]$$

Big union of sets [op]: $\bigcup \bigcirc_1$

Axiom: $c_1 \in \bigcup c_2$ iff $(\exists c)(c \in c_2 \wedge c_1 \in c)$

$$\frac{\begin{array}{c}\dfrac{}{\mathtt{Set}\ c}\,[i] \quad \dfrac{}{+\ c \in c_2}\,[j] \quad \dfrac{}{+\ c_1 \in c}\,[k] \\ \vdots \\ \divideontimes\end{array}}{-\ c_1 \in \bigcup c_2}\ -\cup\mathbf{I}[i,j,k] \qquad \frac{\begin{array}{c}\dfrac{}{\mathtt{Set}\ c}\,[i] \quad \dfrac{}{+\ c \in c_2}\,[j] \\ \vdots \\ -\ c_1 \in c\end{array}}{-\ c_1 \in \bigcup c_2}\ -\cup\mathbf{I}_1[i,j] \qquad \frac{\begin{array}{c}\dfrac{}{\mathtt{Set}\ c}\,[i] \quad \dfrac{}{+\ c_1 \in c}\,[j] \\ \vdots \\ -\ c \in c_2\end{array}}{-\ c_1 \in \bigcup c_2}\ -\cup\mathbf{I}_2[i,j]$$

$$\frac{\begin{array}{c}\vdots \\ -\ c_1 \in \bigcup c_2\end{array} \quad \begin{array}{c}\vdots \\ +\ c_0 \in c_2\end{array}}{-\ c_1 \in c_0}\ -\cup\mathbf{E}_1 \qquad \frac{\begin{array}{c}\vdots \\ -\ c_1 \in \bigcup c_2\end{array} \quad \begin{array}{c}\vdots \\ +\ c_1 \in c_0\end{array}}{-\ c_0 \in c_2}\ -\cup\mathbf{E}_2$$

$$\frac{\begin{array}{c}\vdots \\ +\ c_0 \in c_2\end{array} \quad \begin{array}{c}\vdots \\ +\ c_1 \in c_0\end{array}}{+\ c_1 \in \bigcup c_2}\ +\cup\mathbf{I} \qquad \frac{\begin{array}{c}\vdots \\ +\ c_1 \in \bigcup c_2\end{array} \quad \begin{array}{c}\dfrac{}{\mathtt{Set}\ c}\,[i] \quad \dfrac{}{+\ c \in c_2}\,[j] \quad \dfrac{}{+\ c_1 \in c}\,[k] \\ \vdots \\ \varphi\end{array}}{\varphi}\ +\cup\mathbf{E}[i,j,k]$$

Big intersection of sets [op]: $\bigcap \bigcirc_1$

Axiom: $c_1 \in \bigcap c_2$ iff $(\forall c)(c \in c_2 \rightarrow c_1 \in c)$

$$\frac{\begin{array}{c}\dfrac{}{\mathtt{Set}\ c}\,[i] \quad \dfrac{}{+\ c \in c_2}\,[j] \quad \dfrac{}{-\ c_1 \in c}\,[k] \\ \vdots \\ \divideontimes\end{array}}{+\ c_1 \in \bigcap c_2}\ +\cap\mathbf{I}[i,j,k] \qquad \frac{\begin{array}{c}\dfrac{}{\mathtt{Set}\ c}\,[i] \quad \dfrac{}{+\ c \in c_2}\,[j] \\ \vdots \\ +\ c_1 \in c\end{array}}{+\ c_1 \in \bigcap c_2}\ +\cap\mathbf{I}_1[i,j] \qquad \frac{\begin{array}{c}\dfrac{}{\mathtt{Set}\ c}\,[i] \quad \dfrac{}{-\ c_1 \in c}\,[j] \\ \vdots \\ -\ c \in c_2\end{array}}{+\ c_1 \in \bigcap c_2}\ +\cap\mathbf{I}_2[i,j]$$

$$\frac{\begin{array}{c}\vdots \\ +\ c_1 \in \bigcap c_2\end{array} \quad \begin{array}{c}\vdots \\ +\ c_0 \in c_2\end{array}}{+\ c_1 \in c_0}\ +\cap\mathbf{E}_1 \qquad \frac{\begin{array}{c}\vdots \\ +\ c_1 \in \bigcap c_2\end{array} \quad \begin{array}{c}\vdots \\ -\ c_1 \in c_0\end{array}}{-\ c_0 \in c_2}\ +\cap\mathbf{E}_2$$

$$\frac{\begin{array}{c}\vdots \\ +\ c_0 \in c_2\end{array} \quad \begin{array}{c}\vdots \\ -\ c_1 \in c_0\end{array}}{-\ c_1 \in \bigcap c_2}\ -\cap\mathbf{I} \qquad \frac{\begin{array}{c}\vdots \\ -\ c_1 \in \bigcap c_2\end{array} \quad \begin{array}{c}\dfrac{}{\mathtt{Set}\ c}\,[i] \quad \dfrac{}{+\ c \in c_2}\,[j] \quad \dfrac{}{-\ c_1 \in c}\,[k] \\ \vdots \\ \varphi\end{array}}{\varphi}\ -\cap\mathbf{E}[i,j,k]$$

Equality [`rel`]: $\bigcirc_1 = \bigcirc_2$

Axiom: $c_1 = c_2$ iff $(\forall c)(c \in c_1 \to c \in c_2) \wedge (\forall c)(c \in c_2 \to c \in c_1)$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{+\ c\in c_1}^{\,[j]} \quad \overline{-\ c\in c_2}^{\,[k]} \quad \overline{\texttt{Set}\ c}^{\,[l]} \quad \overline{+\ c\in c_2}^{\,[m]} \quad \overline{-\ c\in c_1}^{\,[n]} \qquad \begin{matrix}\vdots\\ \divideontimes\end{matrix} \qquad \begin{matrix}\vdots\\ \divideontimes\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_1[i,j,k,l,m,n]$$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{+\ c\in c_1}^{\,[j]} \quad \overline{-\ c\in c_2}^{\,[k]} \quad \overline{\texttt{Set}\ c}^{\,[l]} \quad \overline{+\ c\in c_2}^{\,[m]} \qquad \begin{matrix}\vdots\\ \divideontimes\end{matrix} \qquad \begin{matrix}\vdots\\ +\ c\in c_1\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_2[i,j,k,l,m]$$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{+\ c\in c_1}^{\,[j]} \quad \overline{-\ c\in c_2}^{\,[k]} \quad \overline{\texttt{Set}\ c}^{\,[l]} \quad \overline{-\ c\in c_1}^{\,[m]} \qquad \begin{matrix}\vdots\\ \divideontimes\end{matrix} \qquad \begin{matrix}\vdots\\ -\ c\in c_2\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_3[i,j,k,l,m]$$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{+\ c\in c_1}^{\,[j]} \quad \overline{\texttt{Set}\ c}^{\,[k]} \quad \overline{+\ c\in c_2}^{\,[l]} \quad \overline{-\ c\in c_1}^{\,[m]} \qquad \begin{matrix}\vdots\\ +\ c\in c_2\end{matrix} \qquad \begin{matrix}\vdots\\ \divideontimes\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_4[i,j,k,l,m]$$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{+\ c\in c_1}^{\,[j]} \quad \overline{\texttt{Set}\ c}^{\,[k]} \quad \overline{+\ c\in c_2}^{\,[l]} \qquad \begin{matrix}\vdots\\ +\ c\in c_2\end{matrix} \qquad \begin{matrix}\vdots\\ +\ c\in c_1\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_5[i,j,k,l]$$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{+\ c\in c_1}^{\,[j]} \quad \overline{\texttt{Set}\ c}^{\,[k]} \quad \overline{-\ c\in c_1}^{\,[l]} \qquad \begin{matrix}\vdots\\ +\ c\in c_2\end{matrix} \qquad \begin{matrix}\vdots\\ -\ c\in c_2\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_6[i,j,k,l]$$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{-\ c\in c_2}^{\,[j]} \quad \overline{\texttt{Set}\ c}^{\,[k]} \quad \overline{+\ c\in c_2}^{\,[l]} \quad \overline{-\ c\in c_1}^{\,[m]} \qquad \begin{matrix}\vdots\\ -\ c\in c_1\end{matrix} \qquad \begin{matrix}\vdots\\ \divideontimes\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_7[i,j,k,l,m]$$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{-\ c\in c_2}^{\,[j]} \quad \overline{\texttt{Set}\ c}^{\,[k]} \quad \overline{+\ c\in c_2}^{\,[l]} \qquad \begin{matrix}\vdots\\ -\ c\in c_1\end{matrix} \qquad \begin{matrix}\vdots\\ +\ c\in c_1\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_8[i,j,k,l]$$

$$\dfrac{\overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{-\ c\in c_2}^{\,[j]} \quad \overline{\texttt{Set}\ c}^{\,[k]} \quad \overline{-\ c\in c_1}^{\,[l]} \qquad \begin{matrix}\vdots\\ -\ c\in c_1\end{matrix} \qquad \begin{matrix}\vdots\\ -\ c\in c_2\end{matrix}}{+\ c_1 = c_2}\ +{=}\mathbf{I}_9[i,j,k,l]$$

$$\dfrac{\begin{matrix}\vdots\\ +\ c_1 = c_2\end{matrix} \quad \begin{matrix}\vdots\\ +\ c_0 \in c_1\end{matrix}}{+\ c_0 \in c_2}\ +{=}\mathbf{E}_1 \qquad\qquad \dfrac{\begin{matrix}\vdots\\ +\ c_1 = c_2\end{matrix} \quad \begin{matrix}\vdots\\ +\ c_0 \in c_2\end{matrix}}{+\ c_0 \in c_1}\ +{=}\mathbf{E}_2$$

$$\dfrac{\begin{matrix}\vdots\\ +\ c_1 = c_2\end{matrix} \quad \begin{matrix}\vdots\\ -\ c_0 \in c_2\end{matrix}}{-\ c_0 \in c_1}\ +{=}\mathbf{E}_3 \qquad\qquad \dfrac{\begin{matrix}\vdots\\ +\ c_1 = c_2\end{matrix} \quad \begin{matrix}\vdots\\ -\ c_0 \in c_1\end{matrix}}{-\ c_0 \in c_2}\ +{=}\mathbf{E}_4$$

$$\dfrac{\begin{matrix}\vdots\\ +\ c_0 \in c_1\end{matrix} \quad \begin{matrix}\vdots\\ -\ c_0 \in c_2\end{matrix}}{-\ c_1 = c_2}\ -{=}\mathbf{I}_1 \qquad\qquad \dfrac{\begin{matrix}\vdots\\ +\ c_0 \in c_2\end{matrix} \quad \begin{matrix}\vdots\\ -\ c_0 \in c_1\end{matrix}}{-\ c_1 = c_2}\ -{=}\mathbf{I}_2$$

$$\dfrac{\begin{matrix}\vdots\\ -\ c_1 = c_2\end{matrix} \qquad \overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{+\ c\in c_1}^{\,[j]} \quad \overline{-\ c\in c_2}^{\,[k]} \qquad \begin{matrix}\vdots\\ \varphi\end{matrix}}{\varphi}\ -{=}\mathbf{E}_1[i,j,k]$$

$$\dfrac{\begin{matrix}\vdots\\ -\ c_1 = c_2\end{matrix} \qquad \overline{\texttt{Set}\ c}^{\,[i]} \quad \overline{+\ c\in c_2}^{\,[j]} \quad \overline{-\ c\in c_1}^{\,[k]} \qquad \begin{matrix}\vdots\\ \varphi\end{matrix}}{\varphi}\ -{=}\mathbf{E}_2[i,j,k]$$

Powerset [op]: $\wp\bigcirc_1$

Axiom: $c_1 \in \wp\, c_2$ iff $(\forall c)(c \in c_1 \to c \in c_2)$

$$\begin{array}{c} \overline{\texttt{Set}\ c}\ [i] \quad \overline{+\ c \in c_1}\ [j] \quad \overline{-\ c \in c_2}\ [k] \\ \vdots \\ \dfrac{*}{+\ c_1 \in \wp\, c_2}\ {+}\wp\,\mathbf{I}[i,j,k] \end{array} \qquad \begin{array}{c} \overline{\texttt{Set}\ c}\ [i] \quad \overline{+\ c \in c_1}\ [j] \\ \vdots \\ \dfrac{+\ c \in c_2}{+\ c_1 \in \wp\, c_2}\ {+}\wp\,\mathbf{I}_1[i,j] \end{array} \qquad \begin{array}{c} \overline{\texttt{Set}\ c}\ [i] \quad \overline{-\ c \in c_2}\ [j] \\ \vdots \\ \dfrac{-\ c \in c_1}{+\ c_1 \in \wp\, c_2}\ {+}\wp\,\mathbf{I}_2[i,j] \end{array}$$

$$\dfrac{\begin{array}{c}\vdots\\ +\ c_1 \in \wp\, c_2\end{array} \quad \begin{array}{c}\vdots\\ +\ c_0 \in c_1\end{array}}{+\ c_0 \in c_2}\ {+}\wp\,\mathbf{E}_1 \qquad \dfrac{\begin{array}{c}\vdots\\ +\ c_1 \in \wp\, c_2\end{array} \quad \begin{array}{c}\vdots\\ -\ c_0 \in c_2\end{array}}{-\ c_0 \in c_1}\ {+}\wp\,\mathbf{E}_2$$

$$\dfrac{\begin{array}{c}\vdots\\ +\ c_0 \in c_1\end{array} \quad \begin{array}{c}\vdots\\ -\ c_0 \in c_2\end{array}}{-\ c_1 \in \wp\, c_2}\ {-}\wp\,\mathbf{I} \qquad \dfrac{\begin{array}{c}\vdots\\ -\ c_1 \in \wp\, c_2\end{array} \quad \begin{array}{c}\overline{\texttt{Set}\ c}\ [i] \quad \overline{+\ c \in c_1}\ [j] \quad \overline{-\ c \in c_2}\ [k] \\ \vdots \\ \varphi\end{array}}{\varphi}\ {-}\wp\,\mathbf{E}[i,j,k]$$

Exponentiation [op]: $\bigcirc_1^{\circ_2}$

OBS: $\circ_2$ is a nat

Axiom: $(c_1, c_2, \ldots, c_n) \in d^n$ iff $(\forall i)(1 \leq i \leq n \to c_i \in d)$

$$\dfrac{\begin{array}{c}\vdots\\ +\ c_1 \in d\end{array} \quad \begin{array}{c}\vdots\\ +\ c_2 \in d\end{array} \quad \ldots \quad \begin{array}{c}\vdots\\ +\ c_n \in d\end{array}}{+\ (c_1, c_2, \ldots, c_n) \in d^n}\ {+}{\times}\mathbf{I} \qquad \dfrac{\begin{array}{c}\vdots\\ +\ (c_1, c_2, \ldots, c_n) \in d^n\end{array}}{+\ c_i \in d}\ {+}{\times}\mathbf{E}_i \qquad \dfrac{\begin{array}{c}\vdots\\ -\ c_i \in d\end{array}}{-\ (c_1, c_2, \ldots, c_n) \in d^n}\ {-}{\times}\mathbf{I}_i$$

$$\dfrac{\begin{array}{c}\vdots\\ -\ (c_1, c_2, \ldots, c_n) \in d^n\end{array} \quad \begin{array}{c}\vdots\\ +\ c_1 \in d\end{array} \quad \begin{array}{c}\vdots\\ +\ c_2 \in d\end{array} \quad \ldots \quad \begin{array}{c}\vdots\\ +\ c_{i-1} \in d\end{array} \quad \begin{array}{c}\vdots\\ +\ c_{i+1} \in d\end{array} \quad \ldots \quad \begin{array}{c}\vdots\\ +\ c_n \in d\end{array}}{-\ c_i \in d}\ {-}{\times}\mathbf{E}_i$$

Injectiveness [`rel`]: $\mathit{inj}(\bigcirc_1)$

Axiom: $\mathit{inf}(f)$ iff $(\forall x)(x \in \mathit{dom}(f) \to (\forall y)(y \in \mathit{dom}(g) \to (f(x) = f(y) \to x = y)))$

$$\begin{array}{c} \overline{\mathtt{Set}\ x}\,[i] \quad \overline{\mathtt{Set}\ y}\,[j] \quad \overline{+\ x \in \mathit{dom}(f)}\,[k] \quad \overline{+\ y \in \mathit{dom}(f)}\,[l] \quad \overline{+\ f(x) = f(y)}\,[m] \quad \overline{-\ x = y}\,[n] \\ \vdots \\ \dfrac{\text{※}}{+\ \mathit{inj}(f)}\ +\mathit{inj}\ \mathbf{I}[i,j,k,l,m,n] \end{array}$$

$$\begin{array}{c} \overline{\mathtt{Set}\ x}\,[i] \quad \overline{\mathtt{Set}\ y}\,[j] \quad \overline{+\ x \in \mathit{dom}(f)}\,[k] \quad \overline{+\ y \in \mathit{dom}(f)}\,[l] \quad \overline{+\ f(x) = f(y)}\,[m] \\ \vdots \\ \dfrac{+\ x = y}{+\ \mathit{inj}(f)}\ +\mathit{inj}\ \mathbf{I}_1[i,j,k,l,m] \end{array}$$

$$\begin{array}{c} \overline{\mathtt{Set}\ x}\,[i] \quad \overline{\mathtt{Set}\ y}\,[j] \quad \overline{+\ x \in \mathit{dom}(f)}\,[k] \quad \overline{+\ y \in \mathit{dom}(f)}\,[l] \quad \overline{-\ x = y}\,[m] \\ \vdots \\ \dfrac{-\ f(x) = f(y)}{+\ \mathit{inj}(f)}\ +\mathit{inj}\ \mathbf{I}_2[i,j,k,l,m] \end{array}$$

$$\dfrac{\begin{matrix}\vdots \\ +\ \mathit{inj}(f)\end{matrix} \quad \begin{matrix}\vdots \\ +\ x \in \mathit{dom}(f)\end{matrix} \quad \begin{matrix}\vdots \\ +\ y \in \mathit{dom}(f)\end{matrix} \quad \begin{matrix}\vdots \\ +\ f(x) = f(y)\end{matrix}}{+\ x = y}\ +\mathit{inj}\ \mathbf{E}_1$$

$$\dfrac{\begin{matrix}\vdots \\ +\ \mathit{inj}(f)\end{matrix} \quad \begin{matrix}\vdots \\ +\ x \in \mathit{dom}(f)\end{matrix} \quad \begin{matrix}\vdots \\ +\ y \in \mathit{dom}(f)\end{matrix} \quad \begin{matrix}\vdots \\ -\ x = y\end{matrix}}{-\ f(x) = f(y)}\ +\mathit{inj}\ \mathbf{E}_2$$

$$\dfrac{\begin{matrix}\vdots \\ +\ x \in \mathit{dom}(f)\end{matrix} \quad \begin{matrix}\vdots \\ +\ y \in \mathit{dom}(f)\end{matrix} \quad \begin{matrix}\vdots \\ +\ f(x) = f(y)\end{matrix} \quad \begin{matrix}\vdots \\ -\ x = y\end{matrix}}{-\ \mathit{inj}(f)}\ -\mathit{inj}\ \mathbf{I}$$

$$\dfrac{\begin{matrix}\vdots \\ -\ \mathit{inj}(f)\end{matrix} \quad \begin{matrix}\overline{\mathtt{Set}\ x}\,[i] \quad \overline{\mathtt{Set}\ y}\,[j] \quad \overline{+\ x \in \mathit{dom}(f)}\,[k] \quad \overline{+\ y \in \mathit{dom}(f)}\,[l] \quad \overline{+\ f(x) = f(y)}\,[m] \quad \overline{-\ x = y}\,[n] \\ \vdots \\ \varphi\end{matrix}}{\varphi}\ -\mathit{inj}\ \mathbf{E}[i,j,k,l,m,n]$$

Surjectiveness [`rel`]: $\mathit{surj}(\bigcirc_1)$

Axiom: $\mathit{surj}(f)$ iff $(\forall y)(y \in \mathit{cod}(f) \to (\exists x)(x \in \mathit{dom}(g) \wedge (f(x) = y)))$

$$\dfrac{\begin{matrix}\vdots \\ +\ x \in \mathit{dom}(f)\end{matrix} \quad \begin{matrix}\overline{\mathtt{Set}\ y}\,[i] \quad \overline{+\ y \in \mathit{cod}(f)}\,[j] \\ \vdots \\ +\ f(x) = y\end{matrix}}{+\ \mathit{surj}(f)}\ +\mathit{surj}\ \mathbf{I}[i,j] \qquad \dfrac{\begin{matrix}\vdots \\ +\ y \in \mathit{cod}(f)\end{matrix} \quad \begin{matrix}\overline{\mathtt{Set}\ x}\,[i] \quad \overline{+\ x \in \mathit{dom}(f)}\,[j] \\ \vdots \\ -\ f(x) = y\end{matrix}}{-\ \mathit{surj}(f)}\ -\mathit{surj}\ \mathbf{I}[i,j]$$

$$\dfrac{\begin{matrix}\vdots \\ +\ \mathit{surj}(f)\end{matrix} \quad \begin{matrix}\vdots \\ +\ y \in \mathit{cod}(f)\end{matrix} \quad \begin{matrix}\overline{\mathtt{Set}\ x}\,[i] \quad \overline{+\ x \in \mathit{dom}(f)}\,[j] \quad \overline{+\ f(x) = y}\,[k] \\ \vdots \\ \varphi\end{matrix}}{\varphi}\ +\mathit{surj}\ \mathbf{E}[i,j,k]$$

$$\dfrac{\begin{matrix}\vdots \\ -\ \mathit{surj}(f)\end{matrix} \quad \begin{matrix}\overline{\mathtt{Set}\ y}\,[i] \quad \overline{+\ y \in \mathit{cod}(f)}\,[j] \quad \overline{-\ x \in \mathit{dom}(f)}\,[k] \\ \vdots \\ \varphi\end{matrix} \quad \begin{matrix}\overline{\mathtt{Set}\ y}\,[l] \quad \overline{+\ y \in \mathit{cod}(f)}\,[m] \quad \overline{-\ f(x) = y}\,[n] \\ \vdots \\ \varphi\end{matrix}}{\varphi}\ -\mathit{surj}\ \mathbf{E}[i,j,k,l,m,n]$$