Fall 2019: Advanced Topics in Numerical Analysis: Finite Element Methods Assignment 1 (due Oct. 4, 2019)

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1. **Weak forms and boundary conditions.** Consider a bounded open domain $\Omega \subset \mathbb{R}^n$, $n \in \{1, 2, 3\}$ with sufficiently smooth boundary $\partial\Omega$ and unit-length, outside-pointing boundary normal $\boldsymbol{n}$. So far, we have mostly focused on weak forms for problems with zero Dirichlet boundary conditions. Such Dirichlet conditions are also called *essential* boundary conditions, and they must be built into the function space, which we did by using the space $V = H_0^1(\Omega)$. Let us now consider Neumann and Robin boundary conditions, which are also called *natural* boundary conditions as they usually appear as part of the weak forms and are not part of the function space V.

   Consider first the problem of finding the solution $u : \Omega \to \mathbb{R}$ that satisfies

$$-\Delta u + cu = f \text{ on } \Omega, \tag{1}$$

$$\frac{\partial u}{\partial \boldsymbol{n}} = g \text{ on } \partial\Omega, \tag{2}$$

   where $f \in L_2(\Omega)$, $g \in L_2(\partial\Omega)$ and $c \geq 0$ is a constant.

   (a) By multiplying with a sufficiently smooth function[1] v and using integration by parts, derive the weak form for this problem, i.e., detail the function space V and the bilinear and linear forms $a(\cdot\,,\cdot)$ and $\ell(\cdot)$ of the weak form.

   Solution
   Multiplying by a test function v and integrating gives

$$\int_\Omega -\Delta uv + cuv \; dx = \int_\Omega fv \; dx$$

   Integrating by parts once on the left hand side, we get

$$\int_\Omega \nabla u \cdot \nabla v + cuv \; dx - \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}} v \; ds = \int_\Omega fv \; dx$$

   Using the boundary condition, this becomes

$$\int_\Omega \nabla u \cdot \nabla v + cuv \; dx = \int_\Omega fv \; dx + \int_{\partial\Omega} gv \; ds$$

   We need to take $u, v \in V = H^1(\Omega)$ so that the integrals are well defined. We do not want to take $V = H_0^1(\Omega)$ because then we will lose boundary information. The left hand side defines a symmetric bilinear form on $V \times V$.

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v + cuv \; dx$$

[1] This is usually called the *test function* in the finite element context.

Moreover, the right hand side defines a linear form

$$\ell(v) := \int_\Omega fv \; dx + \int_{\partial\Omega} gv \; ds$$

on V. The weak form then is just to find $u \in V$ such that $a(u,v) = \ell(v)$ for all $v \in V$.

(b) For $c = 1$, compute the coercivity constant c_0 of the bilinear form. Is the bilinear form coercive for $c = 0$ and what can you say about the solution of the problem, i.e., when does it have a solution?

Solution
If $c = 1$, then notice that $a(v,v) = \|v\|^2_{H^1(\Omega)}$ and so the coercivity constant is just $c_0 = 1$. If $c = 0$, then $a(v,v) = |v|_{H^1(\Omega)}$ is the H^1 semi-norm and the bilinear form is no longer coercive. To see this suppose that u solves the PDE, then $u + C$ also solves the PDE for every constant C. But the H^1 norm of $u + C$ can become arbitrarily large so $a(\cdot,\cdot)$ cannot be coercive. If we instead impose the homogeneous Dirichlet boundary condition $u \equiv 0$ on $\partial\Omega$, then the Poincaré-Friedrichs inequality implies that $|\cdot|_{H^1_0(\Omega)}$ is a norm and $a(\cdot,\cdot)$ will be coercive with coercivity constant $c_0 = 1$, implying a unique solution.

Let us now generalize the Neumann condition (2) to a Robin condition[2] (or condition of third kind):

$$\frac{\partial u}{\partial \boldsymbol{n}} + ku = g \text{ on } \partial\Omega, \tag{3}$$

for a constant $k \geq 0$.

(c) Derive the weak form for the problem (1), (3).

Solution
Multiply by a test function v and integrate to get

$$\int_\Omega -\Delta uv + cuv \; dx = \int_\Omega fv \; dx.$$

Integrating by parts results in

$$\int_\Omega \nabla u \cdot \nabla v + cuv \; dx - \int_{\partial\Omega} \frac{\partial u}{\partial \boldsymbol{n}} v \; ds = \int_\Omega fv \; dx.$$

Replace $\frac{\partial u}{\partial \boldsymbol{n}}$ with $g - ku$ to get

$$\int_\Omega \nabla u \cdot \nabla v + cuv \; dx + \int_{\partial\Omega} kuv \; ds = \int_\Omega fv \; dx + \int_{\partial\Omega} gv \; ds$$

We see that we now need to take $u, v \in V = H^1(\Omega)$ with

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + cuv \; dx + \int_{\partial\Omega} kuv \; ds$$

[2] Robin boundary conditions are, in their own right, useful in many applications. Additionally, it can be a theoretical as well as computational advantage of Robin conditions compared to Dirichlet conditions, that they can be incorporated in the weak form and not the space. Thus, one sometimes uses (3) with $g = 0$ and very large k to enforce approximate Dirichlet boundary conditions.

defining a symmetric bilinear form, and

$$\ell(v) := \int_\Omega fv\ dx + \int_{\partial\Omega} gv\ ds$$

defining a linear form. The weak form is to find $u \in V$ such that $a(u,v) = \ell(v)$ for all $v \in V$.

(d) Since the bilinear form is symmetric, the weak solution u can also be characterized as minimizer of an energy functional. Give that functional.

Solution
The energy functional that u minimizes is given by

$$u = \mathrm{argmin}_{v\in V}\ \frac{1}{2}a(v,v) - \ell(v)$$

where V, a, ℓ are as defined in the previous part.

(e) Is the bilinear form coercive for $c = 0$ when $k > 0$? Does the problem have a solution in that case, and is it unique? Is there problem with $k < 0$? (It's OK to not prove every statement you make rigorously in reply to this question.)

Solution
The bilinear form will only be coercive when

$$a(u,u) = |u|_{H^1(\Omega)} + k\int_{\partial\Omega} u^2\ ds > |u|^2_{H^1(\Omega)} + \|u\|^2_{L^2(\Omega)}$$

but this will not be true in general since we could choose u to be a very large constant that goes to zero smoothly but very quickly near the boundary. One example would be the mollification of a large constant function so that it is zero everywhere on the boundary. In this case,

$$0 = k\int_{\partial\Omega} u^2\ ds \ll \|u\|^2_{L^2(\Omega)}$$

Since we can find such a function for any k, the bilinear form is still not coercive. With $k < 0$ the problem is even worse because now

$$k\int_{\partial\Omega} u^2\ ds < 0 < \|u\|^2_{L^2(\Omega)}$$

for any $u \neq 0$. Thus, in both cases when $c = 0$ the bilinear form is not coercive.

2. **Properties of bilinear forms.** Let V be a real vector space. A bilinear form $a(\cdot,\cdot)$ over V is called skew-symmetric if $a(u,v) = -a(v,u)$ for all $u, v \in V$. It is called alternating if $a(u,u) = 0$ for all $u \in V$.

   (a) Show that every bilinear form on V can uniquely be written as the sum of a symmetric and a skew-symmetric bilinear form.

Solution
Let $a(u,v)$ be a bilinear form and define the symmetric and skew-symmetric parts to be

$$a_{\text{sym}}(u,v) := \frac{a(u,v)+a(v,u)}{2}, \quad a_{\text{skw}}(u,v) = \frac{a(u,v)-a(v,u)}{2}$$

We have

$$a_{\text{sym}}(u,v) + a_{\text{skw}}(u,v) = a(u,v)$$

Also, $a_{\text{sym}}(u,v) = a_{\text{sym}}(v,u)$ since addition is commutative and $a_{\text{skw}}(u,v) = -a_{\text{skw}}(v,u)$. Moreover, $a_{\text{sym}}(u,v), a_{\text{skw}}(u,v)$ are bilinear since they are just the composition of linear maps, and so we have written $a(u,v)$ as the sum of a symmetric and skew-symmetric bilinear form.

(b) Show that a bilinear form on V is alternating if and only if it is skew-symmetric.

Solution
Suppose that $a(u,v)$ is an alternating bilinear form. Then we have that

$$0 = a(u-v, u-v) = a(u,u) - a(v,u) - a(u,v) + a(v,v)$$

By assumption, $a(u,u) = a(v,v) = 0$ and we see that $a(u,v) = -a(v,u)$. Thus, a is skew-symmetric. On the other hand, suppose that $a(u,v)$ is a skew-symmetric bilinear form. Then,

$$a(u,u) = -a(u,u)$$

so $a(u,u) = 0$ and it is alternating.

3. **Poincare constants.** Let, as above, Ω be a bounded and sufficiently regular domain in $\mathbb{R}^n$. One very useful Poincare-Friedrichs inequality allows us to estimate functions in $H_0^1(\Omega)$ by their derivatives. Namely, there exists a $c > 0$ such that

$$\|u\|_{L_2} \le c\|\nabla u\|_{L_2} \quad \text{for all } u \in H_0^1(\Omega). \tag{4}$$

We call the smallest possible c in the above inequality the Poincare constant $c_\star$.

(a) Argue why (4) makes the H^1-seminorm an actual norm on $H_0^1(\Omega)$.

Solution
The only property of a norm that $|\cdot|_{H^1(\Omega)}$ does not satisfy a priori is that if $u \ne 0$ then we may still have $|u|_{H^1(\Omega)} = 0$, which can be seen by taking u to be constant. We now show that this is satisfied on $H_0^1(\Omega)$. By the Poincare-Friedrichs inequality, if $u \in H_0^1(\Omega)$ and $|u|_{H^1(\Omega)} = 0$, then

$$0 \le \|u\|_{L^2(\Omega)} \le c|u|_{H^1(\Omega)} = 0$$

However, $\|\cdot\|_{L^2(\Omega)}$ is a proper norm, so $u \equiv 0$. Thus, the H^1 semi-norm is a true norm on $H_0^1(\Omega)$.

(b) Show that $c_\star$ satisfies

$$\frac{1}{c_\star^2} = \min_{u \in H_0^1(\Omega),\, \|u\|_{L_2}^2 = 1} \|\nabla u\|_{L_2}^2. \tag{5}$$

Solution
Since $\|u\|_{L^2} \leq c\|\nabla u\|_{L^2}$ is satisfied for any c when $u \equiv 0$, we can write $c_\star$ as

$$\frac{1}{c_\star^2} = \min_{u \in H_0^1(\Omega),\, u \neq 0} \frac{\|\nabla u\|_{L_2}^2}{\|u\|_{L^2}^2}$$

Suppose that $\tilde{u} \in H_0^1(\Omega)$ achieves this minimum, then $u = \alpha\tilde{u} \in H_0^1(\Omega)$ as well for all α and

$$\frac{\|\nabla u\|_{L_2}^2}{\|u\|_{L^2}^2} = \frac{\|\nabla \tilde{u}\|_{L_2}^2}{\|\tilde{u}\|_{L^2}^2}$$

Since this holds for any α, we may choose α so that $\|u\|_{L^2} = 1$. Therefore, the minimization problem is equivalent to

$$\frac{1}{c_\star^2} = \min_{u \in H_0^1(\Omega),\, \|u\|_{L_2}^2 = 1} \|\nabla u\|_{L_2}^2$$

which was to be shown.

(c) Using a Lagrange multiplier to enforce the constraint $\|u\|_{L_2}^2 - 1 = 0$ find conditions that the minimizing function u in (5) necessarily needs to satisfy. That is, compute derivatives of the Lagrange function with respect to u and argue that u in (5) must satisfy an eigenvalue equation with an eigenvalue λ.

Solution
The Lagrangian of the optimization problem is

$$L(u;\lambda) = \|\nabla u\|_{L^2}^2 - \lambda\left(\|u\|_{L^2}^2 - 1\right)$$

who's first variation with respect to u in a direction $v \in H_0^1(\Omega)$ is

$$\delta_u L(u;\lambda)(v) = 2(\nabla u, \nabla v)_{L^2} - 2\lambda(u,v)_{L^2}$$

The stationary points of the Lagrangian are when this is zero for all $v \in H_0^1(\Omega)$, meaning

$$(\nabla u, \nabla v)_{L^2} - \lambda(u,v)_{L^2} = 0, \quad \forall v \in H_0^1(\Omega)$$

However, this is just the weak form to the PDE

$$\begin{aligned} -\Delta u &= \lambda u, \quad \text{in } \Omega \\ u &= 0, \quad \text{on } \partial\Omega \end{aligned}$$

which is just an eigenvalue problem in u for the minus Laplacian operator.

(d) Use this to show that $c_\star^2$ is $\lambda_{\min}^{-1}$, where $\lambda_{\min}$ is the minimal eigenvalue of that eigenvalue problem.

Solution
Since $c_\star^{-2}$ is a minimum with minimizer $u_{\min}$ we know that this is a stationary point for the Lagrangian and therefore solves the eigenvalue problem. In other words, $c_\star^{-2}$ is an eigenvalue of $-\Delta$ with eigenfunction $u_{\min}$. Suppose now by way of a contradiction that there exists

a $\lambda < c_\star^{-2}$ such that λ is an eigenvalue of the PDE and let u denote its corresponding eigenfunction. Then it is is a stationary point and

$$(\nabla u, \nabla v)_{L^2} - \lambda(u, v)_{L^2} = 0, \quad \forall v \in H_0^1(\Omega)$$

Setting $v = u$ gives

$$\|\nabla u\|_{L^2}^2 = \lambda \|u\|_{L^2}^2$$

but this implies that

$$\lambda = \frac{\|\nabla u\|_{L^2}^2}{\|u\|_{L^2}^2} < c_\star^{-2}$$

and is a contradiction. Therefore, we must have $\lambda_{\min} = c_\star^{-2}$.

(e) Solve the eigenvalue problem explicitly to compute the Poincare constants for $\Omega = (0, 1)$ and for $\Omega = (0, 1) \times (0, 1)$.[3]

Solution

On the interval $(0, 1)$ with zero Dirichlet boundary conditions the eigenfunctions and values of $-\Delta$ are

$$u_k(x) = \sin(k\pi x), \quad \lambda_k = (k\pi)^2, \quad \text{for } k \geq 1$$

The minimal eigenvalue is just $\lambda_1 = \pi^2$, and thus $c_\star = \pi^{-1}$ for this problem.

For the unit square $(0, 1)^2$ with zero Dirichlet boundary conditions, the eigenfunctions of $-\Delta$ are

$$u_{kn}(x, y) = \sin(k\pi x)\sin(n\pi y), \quad \lambda_{kn} = (k\pi)^2 + (n\pi)^2, \quad \text{for } k, n \geq 1$$

Thus, the minimal eigenvalue is $\lambda_{1,1} = 2\pi^2$ and the Poincaré constant is $c_\star = (\sqrt{2}\pi)^{-1}$.

4. **Quadratic elements in 1D.** In class we discretized $\Omega = (0, 1)$ using piece-wise linear functions to solve the problem

$$\begin{aligned} -(pu')' + qu &= f \text{ on } (0, 1) \\ u(0) = u(1) &= 0, \end{aligned}$$

where $p > 0$ and $q \geq 0$ are constants. Let us generalize the computation to piece-wise quadratics, which will often give a better approximation. While we could use any basis in the space of piece-wise quadratics, we again choose a *nodal* basis, i.e., functions Φ_i that have the value of 1 at the node they correspond to, and zero at any other node. However, note that they are quadratic polynomials on each element $[x_{i-1}, x_i]$, $i = 1, \ldots, N$.

(a) Specify these nodal basis functions $\Phi_i(x)$ using the mid point in each interval as additional node, i.e.: $\Phi_1(x)$ corresponds to $x_1/2$, $\Phi_2(x)$ corresponds to x_1, $\Phi_3(x)$ corresponds to $(x_2 + x_1)/2$ etc. You should need overall $2N - 1$ basis functions for N intervals/elements, where the basis functions with the even indices $2i$ correspond to the element boundary points x_i, and the ones with odd indices to the element mid points. Note that the support of each basis function is local (either one or two neighboring elements).

[3] I might have mentioned wrong Poincare constants $c_\star$ in class.

Solution
On an individual element $[x_{i-1}, x_i]$, the basis functions will be the Lagrange interpolating polynomials for the points $\{x_{i-1}, (x_i + x_{i-1})/2, x_i\}$. The basis functions are therefore,

$$\Phi_{2i-1}(x) = \frac{(x - x_{i-1})(x - x_i)}{((x_i + x_{i-1})/2 - x_{i-1})((x_i + x_{i-1})/2 - x_i)}, \quad x \in [x_{i-1}, x_i], \quad \text{for } i = 1, \ldots, N$$

and are zero outside the element. We also have basis functions defined over two consecutive elements.

$$\Phi_{2i}(x) = \begin{cases} \dfrac{(x - x_{i-1})(x - (x_i + x_{i-1})/2)}{(x_i - x_{i-1})(x_i - (x_{i-1} + x_i)/2)} & x \in [x_{i-1}, x_i] \\ \dfrac{(x - (x_i + x_{i+1})/2)(x - x_{i+1})}{(x_i - (x_i + x_{i+1})/2)(x_i - x_{i+1})} & x \in [x_i, x_{i+1}] \end{cases}, \quad \text{for } i = 1, \ldots, N-1$$

and is zero outside of the two elements.

(b) Compute the stiffness matrix corresponding to this system, i.e., the matrix $A \in \mathbb{R}^{(2N-1)\times(2N-1)}$ and the right hand side vector $F \in \mathbb{R}^{2N-1}$ such that the coefficients $U \in \mathbb{R}^{2N-1}$ in the solution expansion

$$u_h(x) = \sum_{i=1}^{2N-1} U_i \Phi_i(x)$$

can be found as solution to the linear system

$$AU = F.$$

Solve this system numerically for $p = 1$, $q = 0$ and $f(x) = \mathrm{sgn}(x - 0.5)$, and plot the solution for various N. Note that when using a nodal basis, you can plot the nodal values as a function similar as when using finite differences—the software you are using will connect these points with linear lines. That's visually usually fine, but the true finite element solution is piece-wise quadratic.

Solution
The stiffness matrix A is given by $A_{ij} = a(\Phi_i, \Phi_j)$ for $i, j = 1, \ldots, 2N-1$ where

$$a(u, v) = \int_0^1 pu'v' + quv \; dx.$$

Since the support Φ_i are supported on at most two neighboring elements, we have that $A_{ij} = 0$ if $|i - j| > 2$. The derivatives of the Φ_i are

$$\Phi'_{2i-1}(x) = \frac{(x - x_{i-1}) + (x - x_i)}{((x_i + x_{i-1})/2 - x_{i-1})((x_i + x_{i-1})/2 - x_i)}, \quad x \in [x_{i-1}, x_i], \quad \text{for } i = 1, \ldots, N$$

and

$$\Phi'_{2i}(x) = \begin{cases} \dfrac{(x - x_{i-1}) + (x - (x_i + x_{i-1})/2)}{(x_i - x_{i-1})(x_i - (x_{i-1} + x_i)/2)} & x \in [x_{i-1}, x_i] \\ \dfrac{(x - (x_i + x_{i+1})/2) + (x - x_{i+1})}{(x_i - (x_i + x_{i+1})/2)(x_i - x_{i+1})} & x \in [x_i, x_{i+1}] \end{cases}, \quad \text{for } i = 1, \ldots, N-1$$

and are zero elsewhere. We have that the diagonal entries are

$$A_{2i-1,2i-1} = \int_{x_{i-1}}^{x_i} p\Phi'_{2i-1}(x)\Phi'_{2i-1}(x) + q\Phi_{2i-1}(x)\Phi_{2i-1}(x)\ dx$$
$$A_{2i,2i} = \int_{x_{i-1}}^{x_i} p\Phi'_{2i}(x)\Phi'_{2i}(x) + q\Phi_{2i}(x)\Phi_{2i}(x)\ dx + \int_{x_i}^{x_{i+1}} p\Phi'_{2i}(x)\Phi'_{2i}(x) + q\Phi_{2i}(x)\Phi_{2i}(x)\ dx$$

Because p, q are constant, the integrand in all of these integrals is a degree 4 polynomial over the individual elements. If $q = 0$ then it is a polynomial of degree 2. Therefore, we can use Gaussian quadrature with $m = 3$ ($m = 2$ if $q = 0$) points to integrate this exactly (we can exactly integrate polynomials of degree $2m - 1$). The same will be true of the off-diagonal entries. For the load vector F, we can compute the entries as

$$F_{2i-1} = \int_{x_{i-1}}^{x_i} f(x)\Phi_{2i-1}(x)\ dx, \quad F_{2i} = \int_{x_{i-1}}^{x_i} f(x)\Phi_{2i}(x)\ dx + \int_{x_i}^{x_{i+1}} f(x)\Phi_{2i}(x)\ dx$$

We split the integration over the two different elements for the even entries so that the integrands will be smooth, assuming f is smooth as well. If f is piecewise constant, then we can further split the integrals into regions where f is continuous and then use Gaussian quadrature with $m = 2$ points to integrate exactly. For example, if $f(x) = \mathrm{sgn}(x - 0.5)$, then we can split the integration as

$$F_{2i-1} = \int_{x_{i-1}}^{0.5\wedge x_i} f(x)\Phi_{2i-1}(x)\ dx + \int_{0.5\wedge x_i}^{x_i} f(x)\Phi_{2i-1}(x)\ dx$$

and similarly for the other integrals. The Python file `FEM_solution.py` assembles and solves the system for $p = 1$, $q = 0$, and $f(x) = \mathrm{sgn}(x - 0.5)$.