Optimal Sensor Fusion using $\mathcal{H}_\infty$ methods

Synthesizing Complementary Filters for Active Seismic Noise Isolation Systems in KAGRA

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Introduction: Vibration Isolation Systems and Complementary Filters

Ground-based gravitational-wave detectors require vibration isolation systems (Fig. 1) to attenuate seismic noise induced displacement for the interferometer main optics. Vibration isolation systems utilize local displacement sensors for feedback control to achieve active isolation at lower frequencies. The control performance is limited by the sensors, so it's desirable for sensors to be as low-noise as possible.

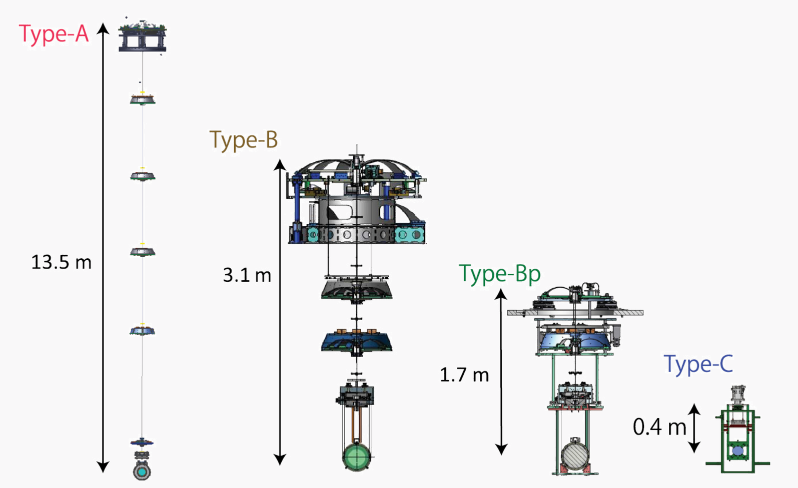

Figure 1: Type-A suspensions: input/end test masses, Type-B suspensions: beamsplitter and signal-recycling mirrors, Type-Bp suspensions: power-recycling mirrors, and Type-C suspensions: input/output mode cleaners [1].

Complementary filter is a sensor fusion method that combines two sensors with different noise characteristics to obtain a virtual "super sensor" that has overall better noise performance. Complementary filter designs were proposed previously [2, 3, 4], but were arguably suboptimal. Also, besides heuristics, it was not clear how exactly the filter shapes were constrained according to the sensor noises in question. Therefore, we propose to formulate the complementary filter problem as an $\mathcal{H}_\infty$ optimization problem and synthesize the filters, which optimally combine the sensors, using $\mathcal{H}_\infty$ method.

Methodology: Complementary Filter Problem as an $\mathcal{H}_\infty$ Problem

Fig. 2 shows the block diagram typical two-sensor configuration using complementary filters. The two sensors are each filtered with $H_1(s)$ and $H_2(s)$ respectively. The two sensors are measuring a common signal and the super sensor readout must contain the very same signal. Hence, the two filters $H_1(s)$ and $H_2(s)$ must be complementary, i.e.

$$H_1(s) + H_2(s) = 1\,. \tag{1}$$

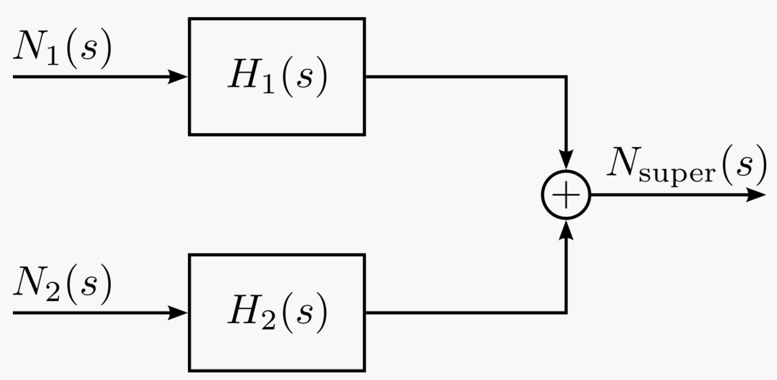

Figure 2: Two-sensor complementary filter configuration.

The super sensor noise then reads

$$N_{\mathrm{super}}(s) = H_1(s)N_1(s) + H_2(s)N_2(s)\,, \tag{2}$$

where $N_1(s)$ and $N_2(s)$ are the sensing noises of the two sensors. So, the goal is to design the complementary filters $H_1(s)$ and $H_2(s)$ such that $N_{\mathrm{super}}(s)$ is minimized in some sense, or that it exhibits desirable noise characteristics.

$\mathcal{H}_\infty$ method is used to synthesize regulator for feedback systems but is recently proposed for synthesizing complementary filters with frequency-dependent specifications [5]. And, It was shown that the method successfully reproduced one of the complementary filters at LIGO [6] using the same specifications. To use $\mathcal{H}_\infty$ method, the input-output system is first represented in the generalized plant as shown in Fig. 3.

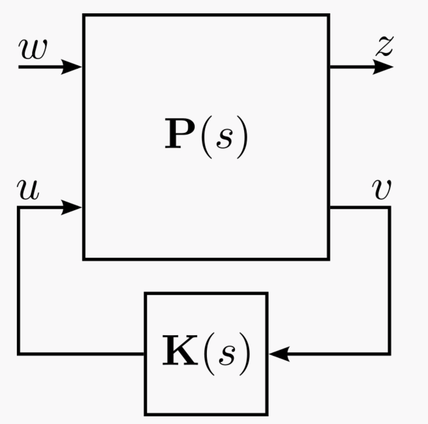

Figure 3: Generalized Plant Represenation

w are the inputs, z are the error signals to be minimized, u are the manipulated variables, v are the measurement signals, $\mathbf{P}(s)$ is the open loop plant, and $\mathbf{K}(s)$ is the closed-loop regulator. The close-loop response can be written as

$$z = \mathbf{G}(s)w\,, \tag{3}$$

where $\mathbf{G}(s)$ is the transfer function matrix from the inputs w to the errors z. $\mathcal{H}_\infty$ synthesis will then generate a regulator, which minimizes the $\mathcal{H}_\infty$ norm of the closed-loop transfer function $\mathbf{G}(s)$. For readers who are interested in the interpretation of the $\mathcal{H}_\infty$ norm, please refer to external resources such as [5, 7].

Methodology: Complementary Filter Problem as an $\mathcal{H}_\infty$ Problem (cont.)

Consider the generalized plant architecture as shown in Fig. 4, which is a slight modification from that of [5]. Here, Φ_1 and Φ_2 are some uncorrelated stochastic processes with unit magnitude. $\hat{N}_1(s)$ and $\hat{N}_2(s)$ are transfer function models of the ASD of the noises N_1 and N_2. $W_1(s)$ and $W_2(s)$ are some weighting functions, which can be used to specify the (inverse) frequency-dependent specification of the sensing noises N_1 and N_2 respectively.

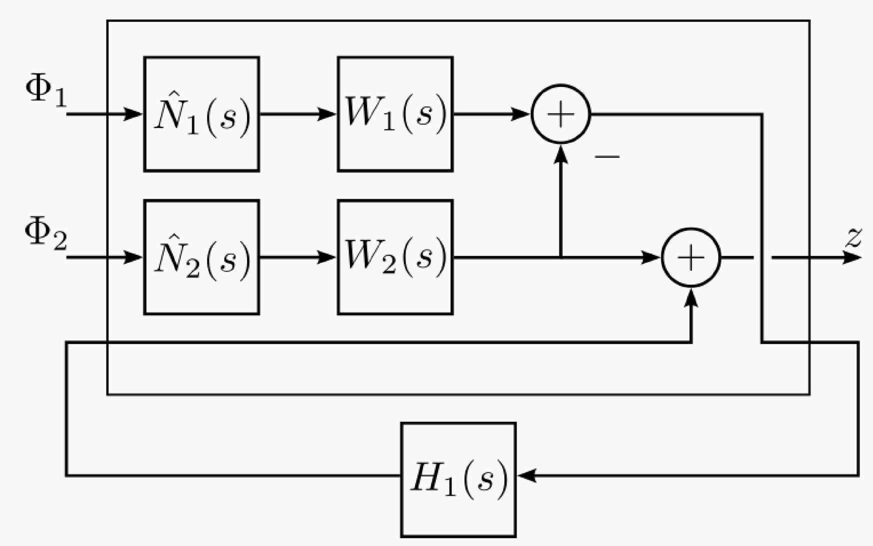

Figure 4: Generalized plant representation for complementary filter synthesis.

Minimizing the $\mathcal{H}_\infty$ norm of this plant will give optimal filters $H_1(s)$ and $H_2(s) \equiv 1 - H_1(s)$ that best filter the noises N_1 and N_2 according to the specifications. It follows that, by setting $W_1(s) = 1/\hat{N}_2(s)$ and $W_2(s) = 1/\hat{N}_1(s)$, the requirements of N_1 is set to N_2 when $N_1 \gg N_2$, and vice versa. These weights are reasonable specifications if there's no specific requirements for the sensing noises because over-suppressing one of the noises is not useful, i.e. there exists a lower bound defined by the either N_1 or N_2, whichever is lower.

Results: Synthesizing Complementary Filters for SRM in KAGRA

The proposed method is exemplified with sensing noises taken from the preisolator of the signal-recycling mirror (SRM) in KAGRA. The complementary filters previously designed in [2, 3] is also compared with that synthesized by the proposed method.

The amplitude spectral densities (ASDs) of the sensing noises N_1 and N_2 and the transfer function models $\hat{N}_1(s)$ and $\hat{N}_2(s)$ are shown in Fig. 5. Here, N_1 denotes quadrature sum of the relative displacement sensor (LVDT) self-noise and the mean seismic noise at KAGRA taken from [8], whereas N_2 is the geophone self-noise.

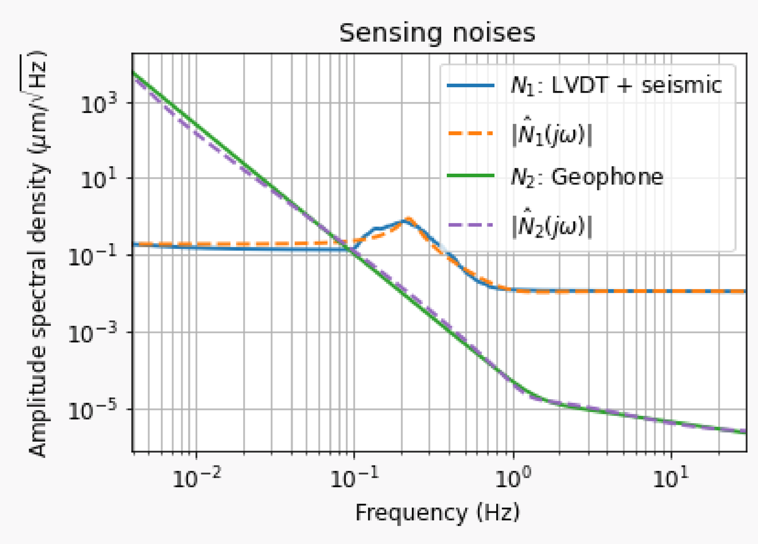

Figure 5: Sensing noises of the SRM preisolator sensors.

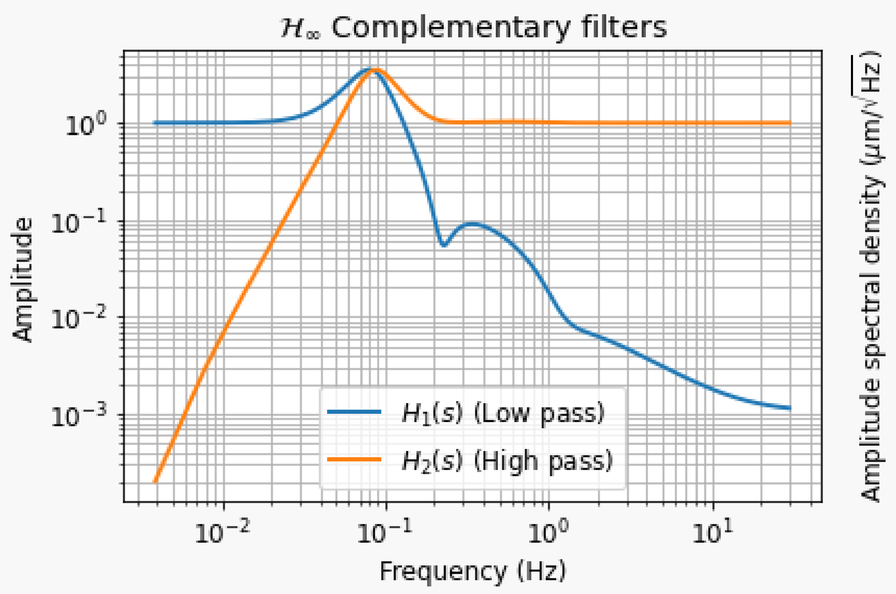

Figure 6: Filters synthesized using $\mathcal{H}_\infty$ method.

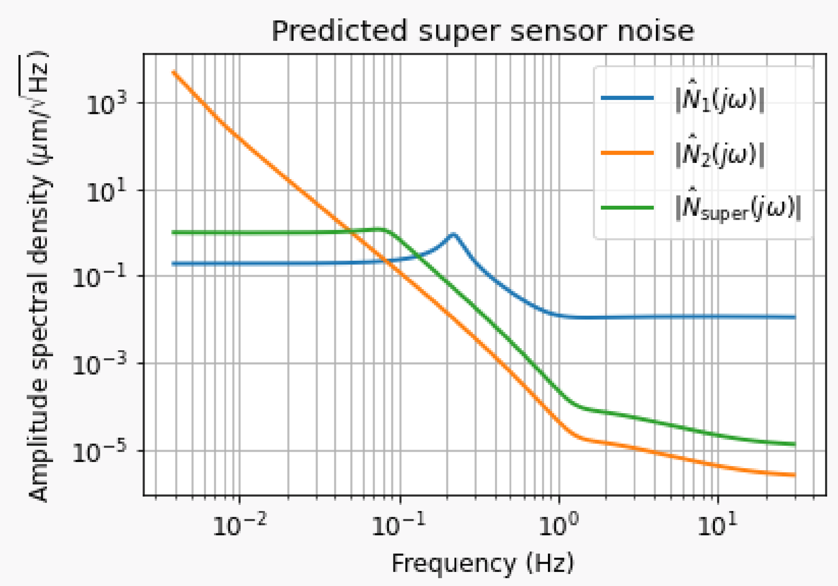

Figure 7: Predicted super sensor noise

With the noise models $\hat{N}_1(s)$ and $\hat{N}_2(s)$, complementary filters were synthesized using $\mathcal{H}_\infty$ method with no information other than the sensing noises themselves. The resulting complementary filters are shown in Fig. 6.

Fig. 7 shows the predicted ASD of the super sensor noise (in green) defined by

$$\left|\hat{N}_{\mathrm{super}}(j\omega)\right| = \left[\left|H_1(j\omega)\right|^2\left|\hat{N}_1(j\omega)\right|^2 + \left|H_2(j\omega)\right|^2\left|\hat{N}_2(j\omega)\right|^2\right]^{\frac{1}{2}}\,. \tag{4}$$

The super sensor noise here follows the shape of the lower bound of the sensing noises at all frequencies, which would indicate that the order of roll-off is critical at all frequencies.

Results: Synthesizing Complementary Filters for SRM in KAGRA (Cont.)

In Fig. 8, we compare noise performance of the complementary filters from [2], [3], and the proposed method, and the super sensor nosies are denoted $N_{\mathrm{super,\,1}}$, $N_{\mathrm{super,\,2}}$, and $N_{\mathrm{super},\mathcal{H}_\infty}$ respectively. The super sensor noises were calculated directly use the quadrature sum of the filtered noises $|H_1(j\omega)|N_1$ and $|H_2(j\omega)|N_2$.

The cross-over frequency of the filters in [2, 3] were set to be the cross-over frequency of the sensing noises in Fig. 5, which is 0.0898 Hz in this case, as recommended in [2]. The ASD of the super sensor noise from $\mathcal{H}_\infty$ filters is on par, if not lower, compared to the other two below 0.4 Hz but is slightly higher at higher frequencies. The shape of $N_{\mathrm{super},\mathcal{H}_\infty}$ at higher frequencies still follows that of the lower bound, which again, indicating that the sensing noises are critically roll-offed.

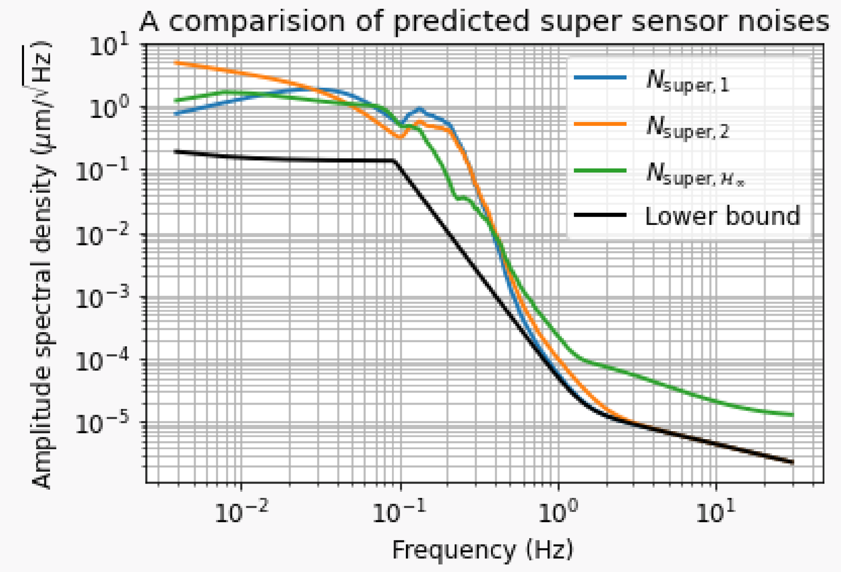

Figure 8: Comparison between the super sensor noises predicted using filter design from [2, 3] and the proposed method.

Performance indices are compared in table. 1, including the super sensor noises' RMS (overall noise performance), the band-limited RMS around 0.1 to 0.5 Hz (seismic noise attenuation performance at microseism), and the ASD at 10 Hz (potential feedback control limit).

	RMS (μm)	RMS (0.1-0.5 Hz) (μm)	ASD (10Hz) (μm/$\sqrt{\mathrm{Hz}}$)
$N_{\mathrm{super,\,1}}$	0.5895	0.2400	4.443e-6
$N_{\mathrm{super,\,2}}$	0.4726	0.1650	4.443e-6
$N_{\mathrm{super},\mathcal{H}_\infty}$	0.3631	0.1041	2.087e-5
Lower bound	0.0462	0.01422	4.443e-6

Table 1: RMS, band-limited RMS, and ASD values at 10 Hz of the super sensor noises predicted using filter design from [2, 3] and the proposed method (**lower the better**).

The $\mathcal{H}_\infty$ complementary filters performs better than the other two complementary filters s in terms of RMS value especially at the microseism band, which makes it a better candidate for active seismic noise isolation. While it performs worse at 10 Hz, there's still a 3 orders of magnitude reduction compared to that of the LVDT self-noise ($\approx 10^{-2}\,\mu$m/$\sqrt{\mathrm{Hz}}$).

Conclusion

- The complementary filter problem was formulated as an $\mathcal{H}_\infty$ optimization problem.
- Complementary filters can be synthesized using $\mathcal{H}_\infty$ method with no information other than the sensing noises themselves.
- The method was exemplified using SRM preisolator sensors and is shown to able to generate filters that better reduce the RMS of the super sensor noise especially around the microseism band.
- While the $\mathcal{H}_\infty$ filters perform worse at 10 Hz, 3 orders of magnitude reduction in super sensor noise ASD (compared to LVDT) can still be achieved.

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