Sensing Matrices for Optical Levers of the KAGRA Main Optics

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Last update: May 3, 2021

Abstract

We have derived the general sensing matrix (16) for an optical lever that has an incidence plane tilted arbitrarily compared to the horizontal plane or the vertical plane. In KAGRA, the incidence plane is either roughly on the horizontal plane or the vertical plane. We have reduced the general sensing matrix down for horizontal optical levers and vertical optical levers as initial sensing matrice for KAGRA, Eqn. (18), and Eqn. (19), respectively. We have discussed three misalignment mechanisms, namely, rotation of the QPD frame, miscentering of the QPD beam at the optics plane, and misplacement of the length-sensing QPD. And, we have modified the general sensing matrix to account for these types of misalignement. The most general sensing matrix is written in Eqn. (26). Again, it's reduced down to a horizontal configuration and a vertical configuration, Eqn. (34) and Eqn. (34), respectively. At last, we have discussed some possible methods to obtain the parameters in the sensing matrix, and hence the sensing matrix. We note that this is neither the easiest nor the best way, to obtain the sensing matrix. There are faster and better ways to do that, e.g. measuring coupling ratios and compute the inverse directly.

Github repository of this document: `https://www.github.com/terrencetec/kagra-optical-lever`. Please feel free to submit an issue if you have any comments or questions.

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1 Introduction

Optical lever (OpLev) is a device used to measure angular displacement of a reflective surface. It consists of a light source (e.g. superluminescent LED), the reflective surface (e.g. the suspended main optics), and beam position sensing device (e.g. Quadphotodiode (QPD)). In KAGRA, the beam used by the optical lever can also be used to measure the longitudinal displacement (along the reflective normal) of the reflective surface. This is done by sensing the beam position behind a convex lens. Although the phrase "optical lever" refers to the angular sensing part of the whole device, we refer the term "optical lever" in KAGRA to the whole device which senses all three displacements, longitudinal, pitch, and yaw.

There are two types of optical levers that are used as displacement sensors in KAGRA, regular and folded (the one used in MCo). The regular optical lever system can be subdivided into two types, horizontal (e.g. those for Type-A and Type-Bp suspensions) and vertical (e.g. Those for Type-B suspensions). In this document, we will focus mainly on the regular one and derive the sensing matrices for both horizontal and vertical configuration. We will also discuss some misalignment of the optical lever, which leads to cross-coupling between different degrees of freedom in the sensing readout. At last, we will provide some methods to obtain the sensing matrices.

This document is organized as follows. In Sec. 2, the derivation of the sensing matrices for KAGRA's optical lever is provided. In Sec. 3, we provided some ways to obtain the parameters in the sensing matrix that we have derived in Sec. 2.

2 Derivation

This section is organized as follows, we will first derive a very basic sensing matrix[1], assuming there's no misalignment in the optical system. And then, we will derive the general sensing matrix that includes all sorts of misalignment. Lastly, we will modify the matrix for a folded optical lever configuration.

In Sec. 2.1, we will first derive the optical lever beam displacement due to a rotate and longitudinal shift of the sensing surface. In Sec. 2.2, we will derive the optical lever sensing matrix with a tilted plane of incidence, and

[1]Here, we define the sensing matrix to be a matrix that maps sensor readouts to the displacements that we want to measure.

reduce it to a horizontal and vertical configuration. In Sec. 2.3, we discuss some cross-coupling mechanism due to misalignment and correct the sensing matrice accordingly. We also briefly touched on the folded optical lever configuration and discussed a cross-coupling mechanism exclusive to this setup, and provided a solution for this.

2.1 Very basic derivation

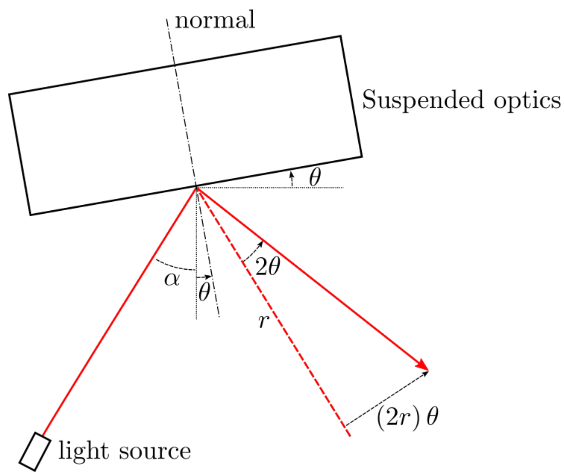

Figure 1: A rotated reflective surface (the suspended optics) caused the optical lever beam to displace.

In the simplest case, as shown in Fig. 1, the beam position x_1 (along the incidence plane) is related to the angular displacement θ by

$$x_1 = (2r)\,\theta, \tag{1}$$

where r is the lever arm defined by the distance between the reflective surface and the sensing device. The same beam can be used to measure the longitudinal displacement of the reflective surface, if the light beam has an angle of incidence α, as shown in Fig. 2. In this case, the beam displacement reads

$$x_1 = (2r)\,\theta + (2\sin\alpha)\,x_L, \tag{2}$$

where x_L is the longitudinal displacement of the reflective surface. As can be seen, equation. (2) shows a coupled

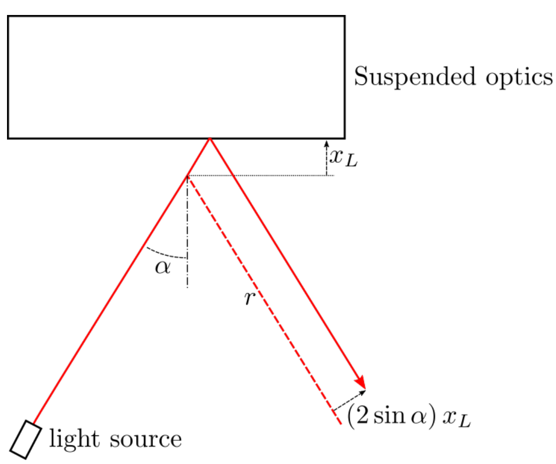

Figure 2: A shift of the optical lever beam due to longitudinal shift of the reflective surface (the suspended optics).

sensor where it reads both the angular displacement and the longitudinal shift.

In KAGRA, some optical levers have a second sensor measuring the beam displacement x_2 some distance d behind a convex lens with focal length f, as shown in Fig. 3. In this case, we obtain the second beam displacement x_2 via

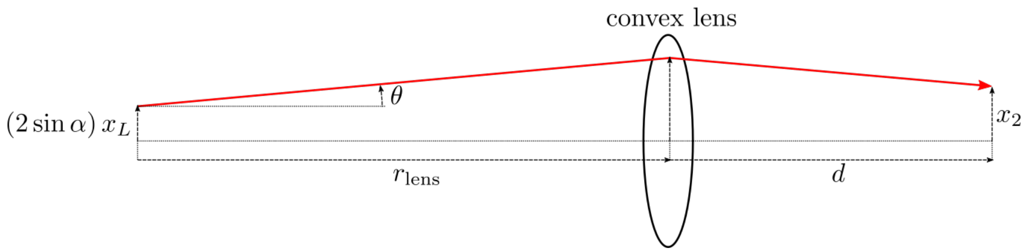

Figure 3: A second beam displacement sensor sensing the beam postion behind a lens.

ray transfer matrices [1]

$$\begin{pmatrix}x_2\\ \cdot\end{pmatrix} = \begin{bmatrix}1 & d\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ -1/f & 1\end{bmatrix}\begin{bmatrix}1 & r_{\rm lens}\\ 0 & 1\end{bmatrix}\begin{pmatrix}(2\sin\alpha)\,x_L\\ 2\theta\end{pmatrix}, \tag{3}$$

where $r_{\rm lens}$ is the distance between the reflective surface and the lens. This gives

$$x_2 = (2\sin\alpha)\left(1-\frac{d}{f}\right)x_L + 2\left[\left(1-\frac{d}{f}\right)r_{\rm lens}+d\right]\theta. \tag{4}$$

Furthermore, we can place the second beam displacement sensor distance behind the lens. So, if we set

$$d = \frac{r_{\rm lens}f}{r_{\rm lens}-f}, \tag{5}$$

then the angular coupling, i.e. the second term in Eqn. (4), becomes zero, effectively making the second beam displacement sensor a "length" (length as in longitudinal displacement) sensing device. If Eqn. (5) is satisfied, then the beam displacement measured by the second senor reads

$$x_2 = \left(\frac{-2f\sin\alpha}{r_{\rm lens}-f}\right)x_L. \tag{6}$$

Now, if we put Eqn. (2) and Eqn. (6) in a matrix form, we can obtain the sensing matrix, i.e.

$$\begin{pmatrix}x_L\\ \theta\end{pmatrix} = \begin{bmatrix}2\sin\alpha & 2r\\ \frac{-2f\sin\alpha}{r_{\rm lens}-f} & 0\end{bmatrix}^{-1}\begin{pmatrix}x_1\\ x_2\end{pmatrix} \tag{7}$$

Then, from here, we can diagonalize[2] the sensors.

2.2 Optical levers in KAGRA

Before diving into the discussion of misalignment, let's rewrite the matrix so it's closer to what we see in KAGRA.

In KAGRA, we have two beam displacement sensors[3], tilt-sensing QPD and length-sensing QPD. They are analogous to the first and second beam position sensors in Sec. 2.1, respectively. Each QPD has two readouts, the horizontal and the vertical displacement of the beam spot, denoted $(x_{\rm tilt},\,y_{\rm tilt})$, and $(x_{\rm len},\,y_{\rm len})$ for tilt-sensing QPD and length-sensing QPD respectively. Here, note that the displacements sensed by the QPDs are not, in general, parallel to the global horizontal or vertical plane, as the QPDs are virtually placed orthogonal to the beam. We are particularly interested in the optics' longitudinal x_L, pitch θ_P, and yaw θ_Y displacements. Therefore, the goal is to find a matrix that maps $\vec{x} = (x_{\rm tilt},\,y_{\rm tilt},\,x_{\rm len},\,y_{\rm len})^T$ to longitudinal displacement, pitch angle, and yaw angle $(x_L,\,\theta_P,\,\theta_Y)^T$.

A 3D illustration of the a general tilt-sensing optical lever setup is shown in Fig. 4. For the current discussion, let's assume that the optical lever is well aligned so the beam spot miscentering at the optics δ_x and δ_y are zero.

[2] Consider a model $\vec{y} = \mathbf{C}\vec{x}$, where $\vec{y}$ are the measurements, and $\vec{x}$ are the states. The goal is to define another measurement basis $\vec{y'}$ such that $\vec{y'} = \mathbf{C}'\vec{x}$, where $\mathbf{C}'$ is a diagonal matrix. It's obvious that If we define $\vec{y'} \equiv \mathbf{C}^{-1}\vec{y}$, then $\mathbf{C}'$ becomes the identity, which is a diagonal matrix. Therefore, we define $\mathbf{C}^{-1}$ to be the sensing matrix, which maps sensor measurements to the displacements of the reflective surface.

[3] Some only has one, e.g. MCo.

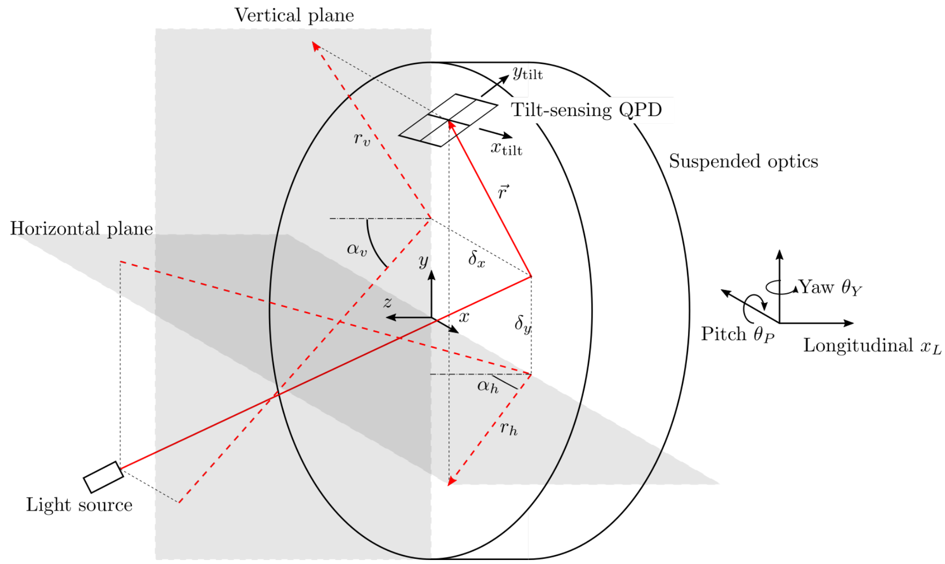

Figure 4: Tilt-sensing optical lever setup in KAGRA.

Now, the lever arm of the optical lever can be an arbitrary vector, i.e. $\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$. For the purpose of this discussion, let $\hat{x}$ be a direction aligned to the transverse direction of the main optics, $\hat{y}$ be a direction aligned to the vertical direction of the optics, and $\hat{z}$ be a direction aligned to the longitudinal direction of the optics. Optical levers in KAGRA are aligned on a horizontal plane (i.e. $r_y = 0$), or vertical plane (i.e. $r_x = 0$). But, let's assume that they are not zero.

If we project the beam onto a horizontal plane and a vertical plane (along the normal of the suspended optics), the beams have an incidence angle of α_h and α_v on the horizontal plane and the vertical plane, respectively. It follows that the lever arm that amplifies the pitch angle is the length of the projection of the lever arm $\vec{r}$ on the vertical plane, r_v[4]. Similarly, the lever arm amplifying the yaw angle is the length of the projection of the lever arm on the horizontal plane, r_h. Therefore, a rotation in yaw θ_Y and pitch θ_P would cause the beam spot at the tip of the lever arm to shift by $(2r_h)\,\theta_Y$ and $(2r_v)\,\theta_P$, on the horizontal plane and vertical plane respectively. Meanwhile, a longitudinal shift x_L would cause the beam spot to shift by $(2\sin\alpha_h)\,x_L$ and $(2\sin\alpha_v)\,x_L$ on the horizontal plane and vertical plane, respectively. From here, we can write the displacement of the beam spot as measured by the tilt-sensing QPD, placed at some distance $\vec{r}$ from the beam spot at the suspended optics plane. The beam spot displacement is simply a superposition of that caused by a rotation and a longitudinal shift,

$$x_{\text{tilt}} = (2r_h)\,\theta_Y + (2\sin\alpha_h)\,x_L, \tag{8}$$

and

$$y_{\text{tilt}} = (2r_v)\,\theta_P + (2\sin\alpha_v)\,x_L. \tag{9}$$

As for the length-sensing QPD, let's assume that beam travels some displacement $\vec{r}_{\text{lens}}$ from the suspended optics to a convex lens with focal length f. Fig. 5 shows the length-sensing optical lever setup. Here, note that the beam in Fig. 5 is common to that in Fig. 4[5].

Let's say the horizontal lever arm from the optics to the convex lens is $r_{\text{lens},h}$. Again, using ray transfer matrix, we can write down the beam spot displacement at the length-sensing QPD,

$$x_{\text{len}} = (2\sin\alpha_h)\left(1-\frac{d_h}{f}\right)x_L + 2\left[\left(1-\frac{d_h}{f}\right)r_{\text{lens},h}, +d_h\right]\theta_Y, \tag{10}$$

where d_h is the length between the lens and the length-sensing QPD on the horizontal plane, and f is the focal length of the lens. It follows that when $d_h = \frac{r_{\text{lens},h}f}{r_{\text{lens},h}-f}$, x_{len} is decoupled from yaw. But, d_h is a distance on the

[4] Think of it as a cross-product $\vec{\theta}\times\vec{r}$. For example, for pitch, $-\theta_P\hat{x}\times(r_x\hat{x}+r_y\hat{y}+r_z\hat{z}) = \theta_P\left(-r_y\hat{z}+r_z\hat{y}\right)$, which is a displacement on the horizontal plane, and $r_v = \sqrt{r_y^2+r_z^2}$ is the corresponding lever arm that amplifies pitch.

[5] In reality, there's a beamsplitter in front of the tilt-sensing QPD to divide the beam into two.

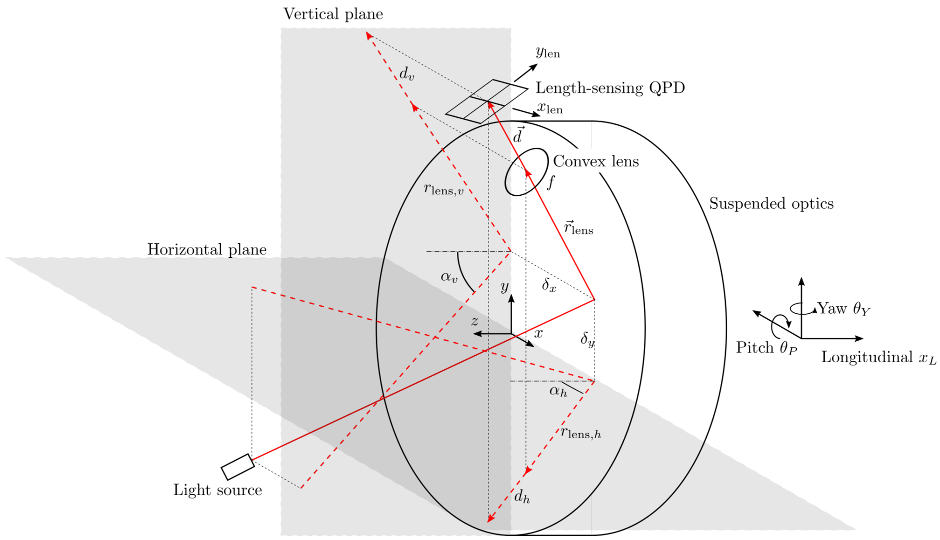

Figure 5: Length-sensing optical lever setup in KAGRA.

horizontal plane. To get back the length, d, we can simply use a similar triangle relationship, as shown in Fig. 6,

$$\begin{aligned} d_{\text{yaw}} + r_{\text{lens}} &= (d_h + r_{\text{lens}h})\,\frac{r_{\text{lens}}}{r_{\text{lens},h}}, \\ d_{\text{yaw}} &= (d_h + r_{\text{lens},h})\,\frac{r_{\text{lens}}}{r_{\text{lens},h}} - r_{\text{lens}} \end{aligned} \tag{11}$$

Here, if we set $d = d_{\text{yaw}}$, then the horizontal readout of the length-sensing QPD x_{len} will have no yaw coupling, i.e.

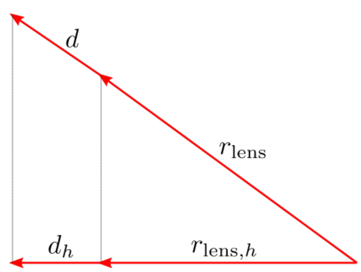

Figure 6: A similar triangle formed by the beam and it's projection on the horizontal plane (length-sensing QPD path).

$$x_{\text{len}} = \left(\frac{-2f\sin\alpha_h}{r_{\text{lens},h} - f}\right) x_L. \tag{12}$$

However, in this case, the vertical readout y_{len} is **not** decoupled from pitch, in general. Now, if we set $d = d_{\text{yaw}}$, then, on the vertical plane, d_v reads, again, from similar triangle relation,

$$\begin{aligned} d_v &= (d_{\text{yaw}} + r_{\text{lens}})\,\frac{r_{\text{lens},v}}{r_{\text{lens}}} - r_{\text{lens},v} \\ &= (d_h + r_{\text{lens},h})\,\frac{r_{\text{lens},v}}{r_{\text{lens},h}} - r_{\text{lens},v} \\ &= \left(\frac{r_{\text{lens},h} f}{r_{\text{lens},h} - f} + r_{\text{lens},h}\right)\frac{r_{\text{lens},v}}{r_{\text{lens},h}} - r_{\text{lens},v} \\ &= \left(\frac{r_{\text{lens},h}}{r_{\text{lens},h} - f} - 1\right) r_{\text{lens},v} \\ &= \frac{r_{\text{lens},v} f}{r_{\text{lens},h} - f} \end{aligned} \tag{13}$$

whereas if we want y_{len} to be decoupled from pitch, we need to set

$$d_v = \frac{r_{\text{lens},h} f}{r_{\text{lens},h} - f}, \tag{14}$$

In general, $r_{\text{lens},h} \neq r_{\text{lens},v}$. So, we cannot simultaneously decouple pitch and yaw from the length-sensing readout. And, if we choose to set $d = d_{\text{yaw}}$, i.e. to minimize yaw coupling, the vertical length-sensing QPD readout reads

$$\begin{aligned}
y_{\text{len}} &= (2\sin\alpha_v)\left(1 - \frac{d_v}{f}\right)x_L + 2\left[\left(1 - \frac{d_v}{f}\right)r_{\text{lens},v} + d_v\right]\theta_P \\
&= (2\sin\alpha_v)\left(\frac{r_{\text{lens},h} - r_{\text{lens},v} - f}{r_{\text{lens},h} - f}\right)x_L + 2\left[\left(\frac{r_{\text{lens},h} - r_{\text{lens},v} - f}{r_{\text{lens},h} - f}\right)r_{\text{lens},v} + \frac{r_{\text{lens},v} f}{r_{\text{lens},h} - f}\right]\theta_P \\
&= (2\sin\alpha_v)\left(\frac{r_{\text{lens},h} - r_{\text{lens},v} - f}{r_{\text{lens},h} - f}\right)x_L + 2\left[\left(\frac{r_{\text{lens},h} - r_{\text{lens},v}}{r_{\text{lens},h} - f}\right)r_{\text{lens},v}\right]\theta_P.
\end{aligned} \tag{15}$$

Here, note that if the length of the projections are equal, i.e. $r_{\text{lens},h} = r_{\text{lens},h}$, then vertical readout of the length-sensing QPD is decoupled from pitch. This refers to a rather interesting configuration, where the incidence plane of the beam is rotated 45° along the z-axis with respect to the vertical plane.

Without loss of generality, let's assume arbitrary[6] d_h and d_v, and write the sensing matrix $\mathbf{C}_{\text{align}}$ for a perfectly aligned optical lever. If we ensemble Eqn. (8), (9), (10), and the first line of (15) into a matrix form, we get

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \mathbf{C}_{\text{align}} \begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \\ x_{\text{len}} \\ y_{\text{len}} \end{pmatrix}, \tag{16}$$

where

$$\mathbf{C}_{\text{align}} = \begin{bmatrix} 2\sin\alpha_h & 0 & 2r_h \\ 2\sin\alpha_v & 2r_v & 0 \\ 2\sin\alpha_h\left(1 - \frac{d_h}{f}\right) & 0 & 2\left[\left(1 - \frac{d_h}{f}\right)r_{\text{lens},h} + d_h\right] \\ 2\sin\alpha_v\left(1 - \frac{d_v}{f}\right) & 2\left[\left(1 - \frac{d_v}{f}\right)r_{\text{lens},v} + d_v\right] & 0 \end{bmatrix}^+, \tag{17}$$

Here, $\left[\cdot\right]^+$ is the pseudoinverse of $\left[\cdot\right]$, and $\left[\cdot\right]^+\left[\cdot\right] = \mathbf{I}$.

2.2.1 Horizontal optical levers

Eqn. (16) gives a general relationship between the QPD readouts and the displacements of the optics without misalignments. It can be drastically simplified if we further assume that the incidence plane is aligned to the horizontal plane (e.g. Type-Bp and Type-A) or the vertical plane (e.g. Type-B). For horizontal optical levers, $\alpha_v = 0$. This gives $y_{\text{tilt}} = (2r_v)\,\theta_P$ and $y_{\text{len}} = 2\left[\left(1 - \frac{d_v}{f}\right)r_{\text{lens},v} + d_v\right]\theta_P$, which are not independent from each other. We can choose to omit the readout y_{len} and hence the sensing matrix for a horizontal optical lever is

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \begin{bmatrix} 2\sin\alpha_h & 0 & 2r_h \\ 0 & 2r_v & 0 \\ \frac{-2f\sin\alpha_h}{r_{\text{lens},h} - f} & 0 & 0 \end{bmatrix}^{-1} \begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \\ x_{\text{len}} \end{pmatrix}, \tag{18}$$

where we've assumed $d = d_h = \frac{r_{\text{lens},h} f}{r_{\text{lens},h} - f}$. Note that on a horitonzal setup, the beam is on the horizontal plane so the horizontal lever arm $r_h = r$. Furthermore, the projection on the vertical plane is simply $r_v = r\cos\alpha_h$. Similarly, the horizontal arm displacement from the optics to the lens $r_{\text{lens},h} = r_{\text{lens}}$.

[6] They can't be completely arbitrary. They must be related via the angle between the horizontal plane and the plane of incidence. But, I don't want to introduce that unnecessary parameter.

2.2.2 Vertical optical levers

Like in Sec. 2.2.1, without further derivation, we can write the sensing matrix for a vertical optical lever setup.

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \begin{bmatrix} 0 & 0 & 2r_h \\ 2\sin\alpha_v & 2r_v & 0 \\ \frac{-2f\sin\alpha_v}{r_{\mathrm{lens},v}-f} & 0 & 0 \end{bmatrix}^{-1} \begin{pmatrix} x_{\mathrm{tilt}} \\ y_{\mathrm{tilt}} \\ y_{\mathrm{len}} \end{pmatrix}, \tag{19}$$

where we've set $d = d_v = \frac{r_{\mathrm{lens},v}f}{r_{\mathrm{lens},v}-f}$. Again, for a vertical setup, $r_v = r$, $r_h = r\cos\alpha_v$, and $r_{\mathrm{lens},v} = r_{\mathrm{lens}}$.

2.2.3 Short summary

We have derived the sensing matrix, Eqn. (16), assuming no misalignment, for a general optical lever setup with a tilted incidence plane. From that, we reduced the general matrix to Eqn. (18) and (19), which corresponds to the sensing matrix for a horizontal optical lever setup and vertical optical lever setup, respectively. In general, the angle of incidence, arm lengths, and focal length are known. Therefore, Eqn. (18) and (19) can be used as initial sensing matrices for the Type-Bp/Type-A and Type-B optical levers, respectively.

2.3 Cross-coupling due to misalignment

Eqn. (18) and (19) give initial sensing matrices for converting the QPD readouts (x_{tilt}, y_{tilt}, x_{len}, y_{len}) to longitudinal x_L, pitch θ_P, and yaw θ_Y. However they are "initial" sensing matrix only There might still be residual cross-couplings between channels and the sensing matrices must be processed to minimize these couplings. In this section, we will discuss 3 mechanisms due to misalignment of the optical lever that could lead to cross-coupling between the three displacements, namely, rotation of the QPD frame, miscentering of the QPD beam at the optics plane, and misplacement of the length-sensing QPD. And, we will modify Eqn. (16) accordingly.

2.3.1 Rotated QPD frame

There's no reason why the QPD is exactly aligned to the a desired frame in the first place. In fact, the QPD frame can easily be rotated with respective to the desired frame. There are many ways how this could happen. For example, the QPD could be physically rotated. Or maybe there's a steering mirror in the middle that caused this rotation. Therefore, we must that this effect into account and make necessary corrections.

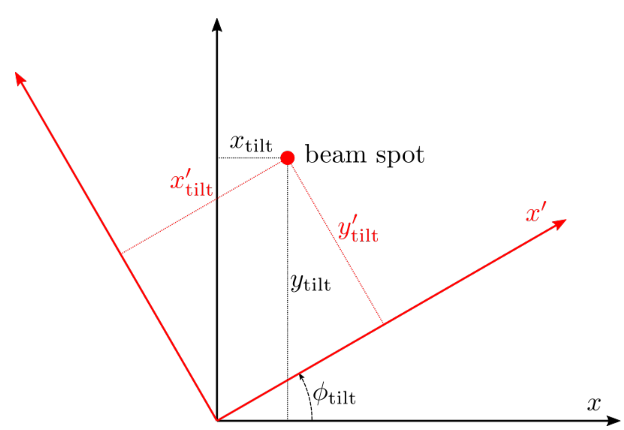

Figure 7: A rotational between the "yaw-pitch" frame (primed frame in red) and the tilt-sensing QPD frame (unprimed frame in black).

Take the tilt-sensing QPD for example. Fig. 7 shows a rotation between the "yaw-pitch" frame and the tilt-sensing QPD frame. In the figure, red axes indicates the direction of the beam spot displacement when subjected to a pure yaw displacement and pure pitch displacement. This means that, the primed horizontal displacement x'_{tilt} reads

$$x'_{\mathrm{tilt}} = (2r_h)\,\theta_Y + (2\sin\alpha_h)\,x_L, \tag{20}$$

and the primed vertical displacement reads

$$y'_{\mathrm{tilt}} = (2r_v)\,\theta_P + (2\sin\alpha_v)\,x_L. \tag{21}$$

These are the quantities that we should be reading. But instead, we are reading the QPD readouts x_{tilt} and y_{tilt}, which are not aligned with x'_{tilt} and y'_{tilt}.

Here, the primed frame is a rotated by an angle ϕ_{tilt} compared to the QPD frame. Hence, we can relate the two frames via

$$\begin{bmatrix} \cos\phi_{\mathrm{tilt}} & \sin\phi_{\mathrm{tilt}} \\ -\sin\phi_{\mathrm{tilt}} & \cos\phi_{\mathrm{tilt}} \end{bmatrix} \begin{pmatrix} x_{\mathrm{tilt}} \\ y_{\mathrm{tilt}} \end{pmatrix} = \begin{pmatrix} x'_{\mathrm{tilt}} \\ y'_{\mathrm{tilt}} \end{pmatrix}. \tag{22}$$

A similar operation can be done to the length-sensing readout x_{len} and y_{len}, but with a different angle ϕ_{len}. So, Eqn. (16) must be modified to

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \mathbf{C}_{\mathrm{align}}\mathbf{C}_{\mathrm{rotation}} \begin{pmatrix} x_{\mathrm{tilt}} \\ y_{\mathrm{tilt}} \\ x_{\mathrm{len}} \\ y_{\mathrm{len}} \end{pmatrix}, \tag{23}$$

where $\mathbf{C}_{\mathrm{align}}$ is the sensing matrix for the perfectly aligned optical lever, i.e. sensing matrix in Eqn. (16), and

$$\mathbf{C}_{\mathrm{rotation}} = \begin{bmatrix} \cos\phi_{\mathrm{tilt}} & \sin\phi_{\mathrm{tilt}} & 0 & 0 \\ -\sin\phi_{\mathrm{tilt}} & \cos\phi_{\mathrm{tilt}} & 0 & 0 \\ 0 & 0 & \cos\phi_{\mathrm{len}} & \sin\phi_{\mathrm{len}} \\ 0 & 0 & -\sin\phi_{\mathrm{len}} & \cos\phi_{\mathrm{len}} \end{bmatrix} \tag{24}$$

is the matrix to correct rotated QPDs.

There's a caveat in this correction, that is, we assume the QPD axes are orthogonal. While this is a very legitimate assumption, this may not be the case if the 4 photodiodes of the QPD has slightly different sensitivities. In this case, we should rotate the axes separately with different angles. However, we shouldn't allow such asymmetric rotation since the calibrations will not be preserved. So, it's meaningless to include this kind of correction into the sensing matrix. Instead, it should be corrected during the calibration stage, where the problem should have been discovered. Or we can simply discover this by observing the fluctuation in QPD sum. If the sensitivities don't match, then there will be large fluctuation in the QPD sum.

There's also an important note here. The rotational transformation should be applied directly to the QPD beam spot displacements readout, not the calibrated yaw-pitch readouts. This is as basic as knowing that matrices don't commute. If to mix the order of matrix multiplication, this can, of course, still decouple yaw and pitch. But, it will completely mess up the calibration because pitch and yaw don't necessarily have the same lever arm coupling.

2.3.2 Miscentered beam spot

Previous derivation assumes that the beam hits the optics at the center of rotation. But, in general, the beam will be miscentered by some small amount δ_x and δ_y in the horizontal and vertical direction, respectively, as shown in Fig. 4 and Fig. 5. In this case, rotation-to-longitudinal cross-couplings will be introduced.

Fig. 8 shows a horizontally miscentered optical lever beam. In Eqn. (2), the displacement $(2\sin\alpha)\,x_L$ is the shift of the beam from the original optical axis. If the beam spot is off-centered, say by an amount of δ_x in the horizontal direction, a rotation in the yaw direction will also introduce a longitudinal shift of the beam spot at the optics plane by an amount of $\delta_x\theta_Y$. This correspond to a parallel beam shift by an amount of $(2\sin\alpha_h)\,\delta_x\theta_Y$[7]. Similarly, if the beam spot is off-centered in the vertical direction by an amount δ_y, a rotation in the pitch direction will introduce a longitudinal shift of $\delta_y\theta_P$. This correspond to a parallel beam shift by an amount of $(2\sin\alpha_v)\,\delta_y\theta_P$.

[7]To the first order correction only. In reality, the angle of incidence is $\alpha_h + 2\theta_Y$, but not α_h. Here, we assume θ_Y is small, so $\sin\alpha_h \approx \sin(\alpha_h + 2\theta_Y)$

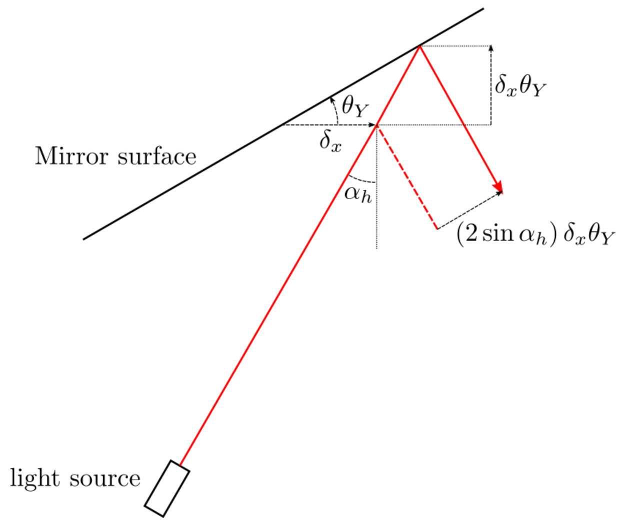

Figure 8: The beam spot of the optical lever beam at the optics plane is miscentered by an amount of δ_x horizontally with respect to the rotational axis. Note that the yaw angle θ_Y is largely exaggerated.

Note that the first row of Eqn. (16) longitudinal shift of the optics x_L. To correct for miscentered beam spot, we can simply replace x_L by the longitudinal shift of the beam spot at the optics plane, i.e. $x_L + \delta_y\theta_P + \delta_x\theta_Y$. This correspond to multiplying a matrix on the left-hand side of the equation, i.e.

$$\begin{pmatrix} x_L + \delta_y\theta_P + \delta_x\theta_Y \\ \theta_P \\ \theta_Y \end{pmatrix} = \begin{bmatrix} 1 & \delta_y & \delta_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix}. \tag{25}$$

Therefore, to account for beam miscentering, Eqn. 23 must be further modified to

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \mathbf{C}_{\text{miscenter}}\mathbf{C}_{\text{align}}\mathbf{C}_{\text{rotation}} \begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \\ x_{\text{len}} \\ y_{\text{len}} \end{pmatrix}, \tag{26}$$

where

$$\mathbf{C}_{\text{miscenter}} = \begin{bmatrix} 1 & \delta_y & \delta_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}. \tag{27}$$

2.3.3 Writing down the sensing matrix explicitly

Now, the sensing matrix

$$\mathbf{C}_{\text{sensing}} = \mathbf{C}_{\text{miscenter}}\mathbf{C}_{\text{align}}\mathbf{C}_{\text{rotation}} \tag{28}$$

is a 3×4 matrix with no zero element, as in, includes all longitudinal-to-pitch, longitudinal-to-yaw, pitch-to-longidinal, pitch-to-yaw, yaw-to-longitudinal, and yaw-to-pitch cross-couplings. And, it provides a correct calibration from beam spot displacements at the tilt-sensing and length-sensing QPDs to the suspended optics' longitudinal displacement, pitch angle, and yaw angle. And, in fact, Eqn. (26) is the most general sensing matrix for an optical lever setup in KAGRA. To write down the sensing matrix explicitly, we have

$$\mathbf{C}_{\mathrm{sensing}} = \left(\begin{bmatrix} \cos\phi_{\mathrm{tilt}} & -\sin\phi_{\mathrm{tilt}} & 0 & 0 \\ \sin\phi_{\mathrm{tilt}} & \cos\phi_{\mathrm{tilt}} & 0 & 0 \\ 0 & 0 & \cos\phi_{\mathrm{len}} & -\sin\phi_{\mathrm{len}} \\ 0 & 0 & \sin\phi_{\mathrm{len}} & \cos\phi_{\mathrm{len}} \end{bmatrix} \right.$$

$$\left. \begin{bmatrix} 2\sin\alpha_h & 0 & 2r_h \\ 2\sin\alpha_v & 2r_v & 0 \\ 2\sin\alpha_h\left(1-\frac{d_h}{f}\right) & 0 & 2\left[\left(1-\frac{d_h}{f}\right)r_{\mathrm{lens},h}+d_h\right] \\ 2\sin\alpha_v\left(1-\frac{d_v}{f}\right) & 2\left[\left(1-\frac{d_v}{f}\right)r_{\mathrm{lens},v}+d_v\right] & 0 \end{bmatrix} \begin{bmatrix} 1 & \delta_y & \delta_x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^{+}. \tag{29}$$

2.3.4 Length-sensing QPD misplacement

In this section, we will discuss angle-to-longitudinal cross-coupling due to a misplaced length-sensing QPD. This section is not as important since we add nothing to Sec. 2.3.3. Instead, we will discuss a degenerating angle-to-longitudinal coupling mechanism, that cannot be distinguished from that in Sec. 2.3.2.

In Sec. 2.3.2, we discussed an angle-to-longitudinal cross-coupling mechanism due to a miscentered beam. But, this is not the only way angular displacements can be cross-coupled to longitudinal readout. If fact, as we have already seen in Sec. 2.2, the length-sensing QPD readouts is only decoupled from pitch or yaw, when placed at a particular point. In particular, for a horizontal or vertical optical lever setup, we need to place the length-sensing QPD at

$$d = \frac{r_{\mathrm{lens}}f}{r_{\mathrm{lens}}-f} \tag{30}$$

behind the lens for a horizontal or vertical optical lever setup, so there exists axis x'_{len} (horizontal optical lever) or y'_{len} (vertical optical lever) that only reads longitudinal displacement. How can we place it so exactly at that point? The answer is we can't. Tradition wisdom says that we can do so by exciting yaw or pitch resonance and then adjust the position of the length-sensing QPD to minimize it. Even if we can excite the optics in pure yaw or pitch resonance, minimizing the coupling is not the correct thing to do. This is because we expect to see some angular-to-longitudinal cross-couplings due to a miscentered beam, as discussed in Sec. 2.3.2! Unless we know exactly how is the beam miscentered, there's no way to finely adjust the position of the length-sensing QPD. Therefore, we must expect some misalignment, say we placed the length-sensing QPD at

$$d_{\mathrm{misalign}} = d + \delta_d. \tag{31}$$

Consider a horizontal setup, in this case, the length-sensing QPD reads

$$x'_{\mathrm{len}} = -\left(\frac{2f\sin\alpha_h}{r_{\mathrm{lens}}-f}\right)x_L - \left(\frac{2\delta_d\sin\alpha_h}{f}\right)x_L - \left(\frac{2\delta_x f\sin\alpha_h}{r_{\mathrm{lens}}-f}\right)\theta_Y - 2\delta_d\left(\frac{r_{\mathrm{lens}}-f}{f}\right)\theta_Y., \tag{32}$$

where the second term and the forth term are additional terms due to misplacement of the length-sensing QPD, and the third term is due to miscentering of the optical lever beam at the optics plane. There are two things we can infer from Eqn. 32. Firstly, the longitudinal displacement will be miscalibrated, if we don't know the misplacement δ_d. Secondly, we cannot estimate the misplacement δ_d, unless we know the beam spot offset δ_x, or vice-versa. But, as we shall see later, there's a chance that we can estimate δ_d by from the inter-calibration between the optical lever and the displacements from sensors of upper stages.

2.3.5 Horizontal optical levers (with misalignment)

Without further derivation, we can right the sensing matrix for a horizontal setup,

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \mathbf{C}_{\mathrm{sensing},h} \begin{pmatrix} x_{\mathrm{tilt}} \\ y_{\mathrm{tilt}} \\ x_{\mathrm{len}} \\ y_{\mathrm{len}} \end{pmatrix}, \tag{33}$$

where

$$\mathbf{C}_{\text{sensing},h} = \begin{bmatrix} 2\sin\alpha_h & 2\delta_y\sin\alpha_h & 2\delta_x\sin\alpha_h + 2r_h \\ 2\sin\alpha_v & 2\delta_y\sin\alpha_v + 2r_v & 2\delta_x\sin\alpha_v \\ \frac{-2f\sin\alpha_h}{r_{\text{lens}}-f} - \frac{2\delta_d\sin\alpha_h}{f} & \frac{-2f\delta_y\sin\alpha_h}{r_{\text{lens}}-f} & \frac{-2f\delta_x\sin\alpha_h}{r_{\text{lens}}-f} - 2\delta_d\left(\frac{r_{\text{lens}}-f}{f}\right) \end{bmatrix}^{-1} \begin{bmatrix} \cos\phi_{\text{tilt}} & \sin\phi_{\text{tilt}} & 0 & 0 \\ -\sin\phi_{\text{tilt}} & \cos\phi_{\text{tilt}} & 0 & 0 \\ 0 & 0 & \cos\phi_{\text{len}} & \sin\phi_{\text{len}} \end{bmatrix}, \tag{34}$$

and we've further assumed a nonzero vertical angle of incidence α_v, as part of the misalignment. And again, for a horizontal setup, $r_h \approxeq r$, $r_v \approxeq r\cos\alpha_h$, $r_{\text{lens},h} \approxeq r_{\text{lens}}$, and that the length-sensing QPD is placed at $d_h \approxeq d = \frac{r_{\text{lens}}f}{r_{\text{lens}}-f} + \delta_d$.

2.3.6 Vertical optical levers (with misalignment)

Similarly, we can write the sensing matrix for a vertical setup,

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \mathbf{C}_{\text{sensing},v} \begin{pmatrix} x_{\text{tilt}} \\ y_{\text{tilt}} \\ x_{\text{len}} \\ y_{\text{len}} \end{pmatrix}, \tag{35}$$

where

$$\mathbf{C}_{\text{sensing},v} = \begin{bmatrix} 2\sin\alpha_h & 2\delta_y\sin\alpha_h & 2\delta_x\sin\alpha_h + 2r_h \\ 2\sin\alpha_v & 2\delta_y\sin\alpha_v + 2r_v & 2\delta_x\sin\alpha_v \\ \frac{-2f\sin\alpha_v}{r_{\text{lens}}-f} - \frac{2\delta_d\sin\alpha_v}{f} & \frac{-2f\delta_y\sin\alpha_v}{r_{\text{lens}}-f} & \frac{-2f\delta_x\sin\alpha_v}{r_{\text{lens}}-f} - 2\delta_d\left(\frac{r_{\text{lens}}-f}{f}\right) \end{bmatrix}^{-1} \begin{bmatrix} \cos\phi_{\text{tilt}} & \sin\phi_{\text{tilt}} & 0 & 0 \\ -\sin\phi_{\text{tilt}} & \cos\phi_{\text{tilt}} & 0 & 0 \\ 0 & 0 & -\sin\phi_{\text{len}} & \cos\phi_{\text{len}} \end{bmatrix}. \tag{36}$$

Note that Eqn. (34) and Eqn. (36) are not the same. And, for a vertical setup, $r_v \approxeq r$, $r_h \approxeq r\cos\alpha_v$, $r_{\text{lens},v} \approxeq r_{\text{lens}}$, and that the length-sensing QPD is placed at $d_v \approxeq d = \frac{r_{\text{lens}}f}{r_{\text{lens}}-f} + \delta_d$.

2.3.7 Misalignment summary

We have derived the sensing matrix Eqn. (26) for a general optical lever. We have included rotation misalignment, with tilt-sensing QPD and length-sensing QPD rotated by an amount of ϕ_{tilt} and ϕ_{len}, respectively, with respect to the "yaw-pitch" frame. We have also discussed the angle-to-longitudinal cross-coupling induced by with an optical lever beam offset by δ_x and δ_y horizontally and vertically, respectively, at the optics plane. We then discussed the effect of misplaced length-sensing QPD, by an amount of δ_d, which is the same as that of a miscentered beam. Lastly, we write down the sensing matrix with misalignment for a horizontal optical lever and vertical optical lever, in Eqn. (34) and Eqn. (36), respectively.

2.4 Folded optical lever

In this section, we will briefly introduce the folder optical lever layout. We will explain the main difference between a normal optical lever and a folded one. We will also introduce some cross-coupling mechanism that is exclusive to this setup. But we will not go into deep details and writing down equations like Eqn. (29). Fig. 9 shows a folded optical lever setup. This is a rare setup used for some optics, whose chamber doesn't have a viewport for the beam to exit after the first reflection. In this configuration, the light beam hits the optics with an incidence angle of α_1,

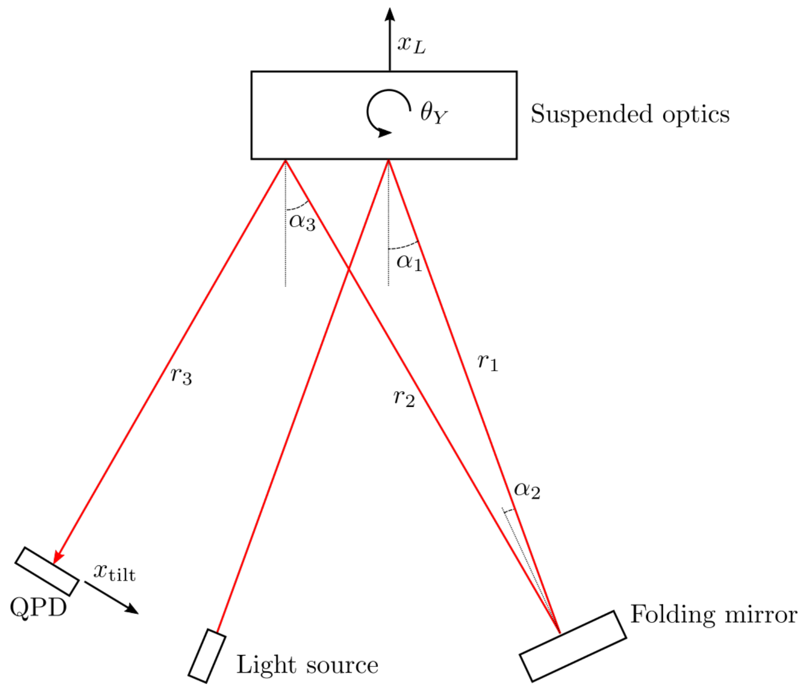

Figure 9: A schematic of the folded optical lever setup in KAGRA.

and travels some distances r_1 towards a folding mirror inside the chamber. The beam then hits the folding mirror with an incidence angle of α_2, and travels along r_2 back to the optics. At last, the beam hits the optics again with a difference angle of incidneces α_3, and then travels r_3 to reach the QPD, which measures the beam spot displacement x_{tilt}.

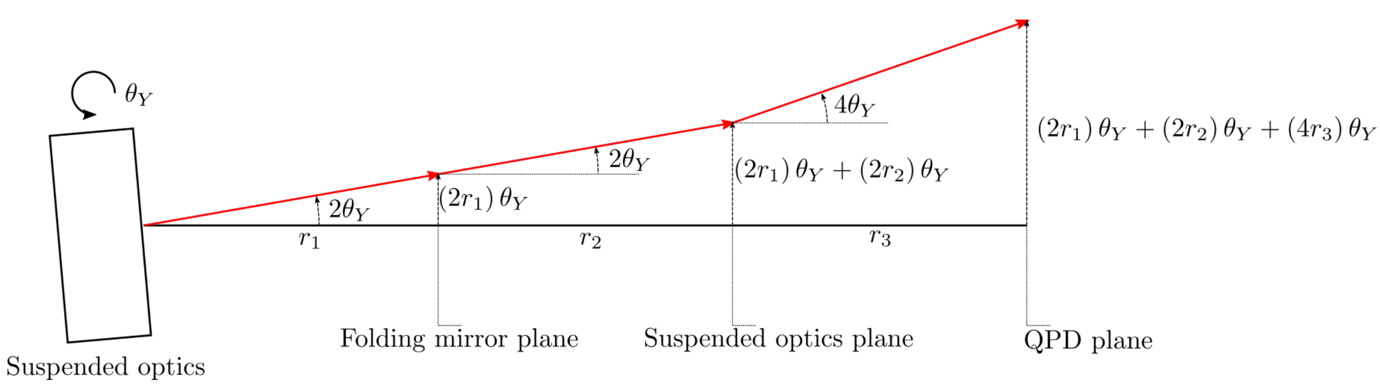

Figure 10: A yaw of the optics can be treated as three path segments r_1, r_2, and r_3 in a folded optical lever configuration.

Let's consider an angular displacement in yaw, as shown in Fig. 10. Black line formed by r_1, r_2, and r_3 is the original path of the beam. As can be inferred from Eqn. (1), a rotation of the optics, say by an amount θ_Y is equivalent to adding an angle $2\theta_Y$ to the beam path. Hence, the angles of the path r_1 and r_2 are $2\theta_Y$, and that of r_3 is $2\theta_Y + 2\theta_Y = 4\theta_Y$ (reflected twice). So, the displacement of the beam spot at the QPD plane can be straightforwardly derived via the ray transfer matrix,

$$\begin{pmatrix} x_{\text{tilt}} \\ \cdot \end{pmatrix} = \begin{bmatrix} 1 & r_1 + r_2 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 2\theta_Y \end{pmatrix} + \begin{bmatrix} 1 & r_3 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} 0 \\ 4\theta_Y \end{pmatrix}, \tag{37}$$

which gives

$$x_{\text{tilt}} = (2r_1 + 2r_2 + 4r_3)\,\theta_Y. \tag{38}$$

Now, let's consider a longitudinal shift of the optics. Similar to that of a regular optical lever, a longituidnal displacement is going to introduce a beam shift of $(2\sin\alpha_1)\,x_L$ at the first reflection. A second beam shift of $(-2\sin\alpha_3)\,x_L$

is introduced in the return path to the QPD. This gives a total beam spot displacement of $2\left(\sin\alpha_1 - \sin\alpha_3\right)x_L$ at the QPD plane. Without loss of generality, let's write down the beam spot displacement due to a rotation in yaw and a longitudinal shift,

$$x_{\text{tilt}} = \left(2r_1 + 2r_2 + 4r_3\right)\theta_Y + 2\left(\sin\alpha_1 - \sin\alpha_3\right)x_L. \tag{39}$$

Comparing Eqn. (2) and Eqn. (39), we see that the effective arm length of yaw is 4 times that of a regular optical lever, if we assume $r_1 = r_2 = r_3 = r$. Also, if the incidence angles α_1 and α_3 are exactly the same, then the QPD readout will be completely decoupled from longituindal displacement of the optics. However, this cannot be true if we consider a configuration as shown in Fig. 9. The angle of incidence α_3 is related to the other angle of incidences by

$$\alpha_3 = \alpha_1 + 2\alpha_2. \tag{40}$$

Having equal α_1 and α_3 would necessarily implies that $\alpha_2 = 0$. This means that the beam goes back to via the same path and returns to the light source, which is physically impossible. Therefore, there exists this irremovable longitudinal-to-yaw coupling. Not to mention, the cross-coupling due to miscentered beam, as discussed in Sec. 2.3.2, still exist and is also irremovable.

2.4.1 Augmented folded optical lever

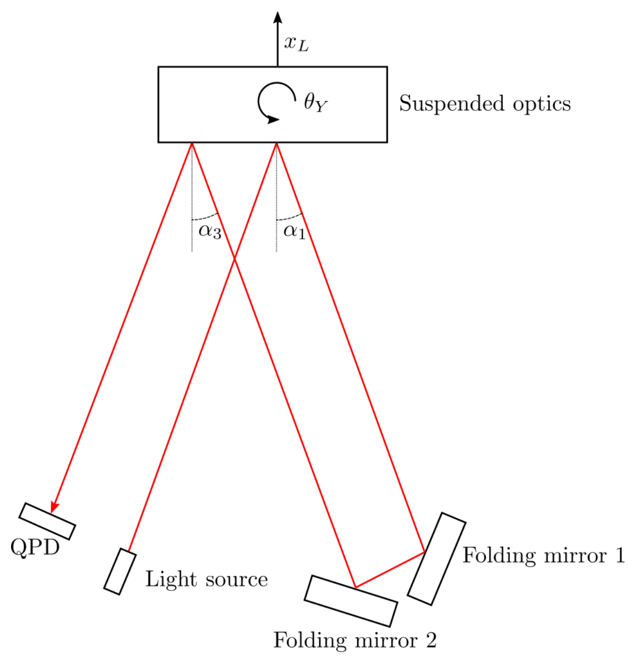

Figure 11: Folded optical lever augmented by adding a second folding mirror.

As is discussed in the Sec. 2.4, the QPD will sense both yaw and longitudinal displacement due to asymmetric angle of incidences between the forward and return path. Here, as shown in Fig. 11, we propose an augmented configuration where a second folding mirror is added. In this configuration, the we can finely tweak the orientation of the second folding mirror to equalize the angle of incidences α_1 and α_3. This way we can minimize the longitudinal coupling due to difference in angle of incidences. However, this cannot fix the problem as discussed in Sec. 2.3.2, so there will be residual uncertain coupling due to a miscentered beam.

3 Measuring the sensing matrix parameters

Before going into the details, I would like to mention that measuring the parameters is **not** important. It is not important anyway because all we need is the sensing matrix, but not the parameters. And there're simpler ways to obtain the sensing matrix. Also, we should say here, there's no way to obtain the parameters accurately. It's impossible. Instead, we are assuming that the misalignment parameters are small. From here, we can roughly estimate the parameters in the sensing matrix.

If we want to use the sensing matrix (34), we need to find a few parameters. Some are directly measurable or known, and some are not. In particular, we need to find the lever arm from the optics to the tilt-sensing QPD r, the lever arm from the optics to the lens r_{lens}, the angle of incidences α_h and α_v, the position of the beam spot at the optics plane δ_x and δ_y, the rotational angles ϕ_{tilt} and ϕ_{len}, the focal length f, and the length-sensing QPD misplacement δ_d. Table 1 summarizes the parameters that we need to find. For a vertical setup, we know α_v, but not α_h.

Parameter	Physical meaning	Measurable?
r	lever arm from the optics to the tilt-sensing QPD plane	yes
r_{lens}	lever arm from the optics to the lens plane	yes
α_h	Angle of incidence of the beam projected on a horizontal plane	yes/no
α_v	Angle of incidence of the beam projected on a vertical plane	no/yes
δ_x	Beam spot horizontal offset from the rotational center at the optics plane	no
δ_y	Beam spot vertical offset from the rotational center at the optics plane	no
ϕ_{tilt}	Rotational angle between the "yaw-pitch" frame and the tilt-sensing QPD frame	no
ϕ_{len}	Rotational angle between the "yaw-pitch" frame and the length-sensing QPD frame	no
f	Focal length of the convex lens	known
δ_d	Misplacement of the length-sensing QPD from the $d = \frac{r_{\text{lens}} f}{r_{\text{lens}} - f}$ point	no

Table 1: Parameters of the sensing matrix for a horizontal/vertical optical lever layout.

3.1 Measuring the rotational angles ϕ_{tilt} and ϕ_{len}

First we need to align the QPD axes to the desired frame, i.e. finding ϕ_{tilt} and ϕ_{len}. For ϕ_{tilt}, we need to identify resonance frequencies of the pure pitch and pure yaw resonances, f_{pitch} and f_{yaw}. Then, we excite the optics using the coil magnet actuators. Note here, we must excite the optics with white noise only. The best way is to "kick" the suspensions and measure the free swing spectra. However, this is not always possible because the resonances might have very low Q-value and so the amplitude decays very quickly to a level lower than the sensor noise. We cannot use a single frequency line injection, or else we cannot distinguish cross-coupling between from the sensors and from the actuators.

If we excite the optics, we will see two peaks at resonances frequencies f_{pitch} and f_{yaw} in both x_{tilt} and y_{tilt} spectra. We can pick one of them. If we pick f_{pitch}, then we can calculate the angle by

$$\tan\phi_{\text{tilt}} = -\frac{x_{\text{tilt}}(f_{\text{pitch}})}{y_{\text{tilt}}(f_{\text{pitch}})}, \tag{41}$$

where $x_{\text{tilt}}(f_{\text{pitch}})$ is the amplitude spectral density at frequency f_{pitch}, and $y_{\text{tilt}}(f_{\text{pitch}})$ is the amplitude spectral density at frequency f_{pitch}. Note here, $\frac{x_{\text{tilt}}(f_{\text{pitch}})}{y_{\text{tilt}}(f_{\text{pitch}})}$ is in fact a transfer function and has relative phase. In theory, the phase should be either $0°$ or $\pm 180°$. We need to add an additional minus sign if the phase is at $\pm 180°$.

If we choose f_{yaw}, then we can calculate the angle by

$$\tan\phi_{\text{tilt}} = \frac{y_{\text{tilt}}(f_{\text{yaw}})}{x_{\text{tilt}}(f_{\text{yaw}})}. \tag{42}$$

It's normal that the resulting angle calculated by Eqn. (41) and Eqn. (42) are not matched. This can be explained by non-negligible δ_x and δ_y, which tilts the pure yaw and pure pitch axes asymmetrically. If they differ by a lot, then this method automatically fails here. If they are similar, then let's just pick one or simply pick the average, or even somewhere randomly between.

Now, for the angle ϕ_{len}, assuming small misalignment, let's excite the longitudinal mode at resonance frequency

$f_{\text{longitudinal}}$. Then, the angle can be calculated using

$$\tan\phi_{\text{len}} = \frac{x_{\text{len}}(f_{\text{longitudinal}})}{y_{\text{len}}(f_{\text{longitudinal}})}. \tag{43}$$

Again, note the phase between x_{len} and y_{len}. For a vertical setup, we can calculate the angle using

$$\tan\phi_{\text{len}} = -\frac{y_{\text{len}}(f_{\text{longitudinal}})}{x_{\text{len}}(f_{\text{longitudinal}})}. \tag{44}$$

3.2 Measuring misplacement of the length-sensing QPD δ_d

Now, for a horizontal optical lever, with the rotational angles obtained, we can read $x'_{\text{len}} = \cos\phi_{\text{len}} x_{\text{len}} + \sin\phi_{\text{len}} y_{\text{len}}$. Using Eqn. (32), we see that, without a yaw motion, the misplacement δ_d can be found by

$$\delta_d = -\left(\frac{x'_{\text{len}}}{x_L} + \frac{2f\sin\alpha_h}{r_{\text{lens}} - f}\right)\frac{f}{2\sin\alpha_h}. \tag{45}$$

And, for a vertical setup, assuming no pitch, we get

$$\delta_d = -\left(\frac{y'_{\text{len}}}{x_L} + \frac{2f\sin\alpha_v}{r_{\text{lens}} - f}\right)\frac{f}{2\sin\alpha_v}. \tag{46}$$

Here, we assume that r_{lens}, α_h (for horizontal setup), α_v (for vertical setup), and f are known. The problem is how to get x_L. For suspensions with an upper stage, such as preisolator, we can offset the whole suspension in longitudinal direction using the actuators at upper stages, and we can measure x_L at that stage. Doing so, we assume that the sensors at the upper stage are diagonalized and are better calibrated. In this sense, this approach is better for the vertical setup because a preisolator can never offset the suspension in pitch!

3.3 Measuring angle of incidences α_h and α_v

For a horizontal setup, α_h is known, and we need to measure α_v. To do so, we can use the same trick used in Sec. 3.2 and offset the whole suspension chain in longitudinal direction. Using Eqn. (21), we can straightforwardly write down the angle α_v, assuming no pitch,

$$\sin\alpha_v = \frac{y'_{\text{tilt}}}{2x_L}. \tag{47}$$

For a vertical setup, assuming no yaw,

$$\sin\alpha_h = \frac{x'_{\text{tilt}}}{2x_L}. \tag{48}$$

Again, x_L here is estimated from a sensor at the upper stages, which are diagonalized and calibrated.

3.4 Measuring the beam offsets δ_x and δ_y

Now, we have all the parameters except the beam offsets δ_x and δ_y. So, let's plug all the parameters into Eqn. (34) and Eqn. (36) to get some temporary longitudinal readout ($x_{L,\text{temp}}$, $\theta_{P,\text{temp}}$, $\theta_{Y,\text{temp}}$). Then, we can excite the pitch and yaw resonances at f_{pitch} and f_{yaw}, and measure the spectra using the temporary readouts. The offsets can be obtained straightforwardly by

$$\delta_x = \frac{x_{L,\text{temp}}(f_{\text{yaw}})}{\theta_{Y,\text{temp}}(f_{\text{yaw}})}, \tag{49}$$

and

$$\delta_y = \frac{x_{L,\text{temp}}(f_{\text{pitch}})}{\theta_{P,\text{temp}}(f_{\text{pitch}})}. \tag{50}$$

And this concludes all the measurements needed to obtain the whole sensing matrix.

4 Conclusion

We have derived the general sensing matrix (16) for an optical lever that has an incidence plane tilted arbitrarily compared to the horizontal plane or the vertical plane. In KAGRA, the incidence plane is either roughly on the horizontal plane or the vertical plane. We have reduced the general sensing matrix down for horizontal optical levers and vertical optical levers as initial sensing matrice for KAGRA, Eqn. (18), and Eqn. (19), respectively. We have discussed three misalignment mechanisms, namely, rotation of the QPD frame, miscentering of the QPD beam at the optics plane, and misplacement of the length-sensing QPD. And, we have modified the general sensing matrix to account for these types of misalignement. The most general sensing matrix is written in Eqn. (26). Again, it's reduced down to a horizontal configuration and a vertical configuration, Eqn. (34) and Eqn. (34), respectively. At last, we have discussed some possible methods to obtain the parameters in the sensing matrix, and hence the sensing matrix. We note that this is neither the easiest nor the best way, to obtain the sensing matrix. There are faster and better ways to do that, e.g. measuring coupling ratios and compute the inverse directly.

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