Sensing Matrices for KAGRA Main Optics Optical Lever (OpLev) Displacement Sensing System.

Tsang Terrence Tak Lun
The Chinese University of Hong Kong

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Contents

1 Introduction

Optical lever is a device used to measure angular displacement of a reflective surface. [1] It consists of a light source (e.g. superluminescent LED), the reflective surface (e.g. the suspended main optics), and beam position sensing device (e.g. Quadphotodiode (QPD)). In KAGRA, the beam used by the optical lever can also be used to measure the longitudinal displacement (along the reflective normal) of the reflective surface. This is done by sensing the beam position behind a convex lens. Although the phrase "optical lever" refers to the angular sensing part of the whole device, we refer the term "optical lever" in KAGRA to the whole device which senses all three displacements, longitudinal, pitch, and yaw.

There are two types of optical levers that are used as displacement sensors in KAGRA, regular and folded (the one used in MCo). The regular optical lever system can be subdivided into two types, horizontal (e.g. those for Type-A and Type-Bp suspensions) and vertical (e.g. Those for Type-B suspensions). Without loss of generality, we will derive the optical lever sensing matrix with a tilted plane of incidence. The derivation of the sensing matrix of a regular type is slightly different then that of a folded optical lever system. We will first derive the sensing matrix of a regular type and then modify it to fit a folded optical lever system.

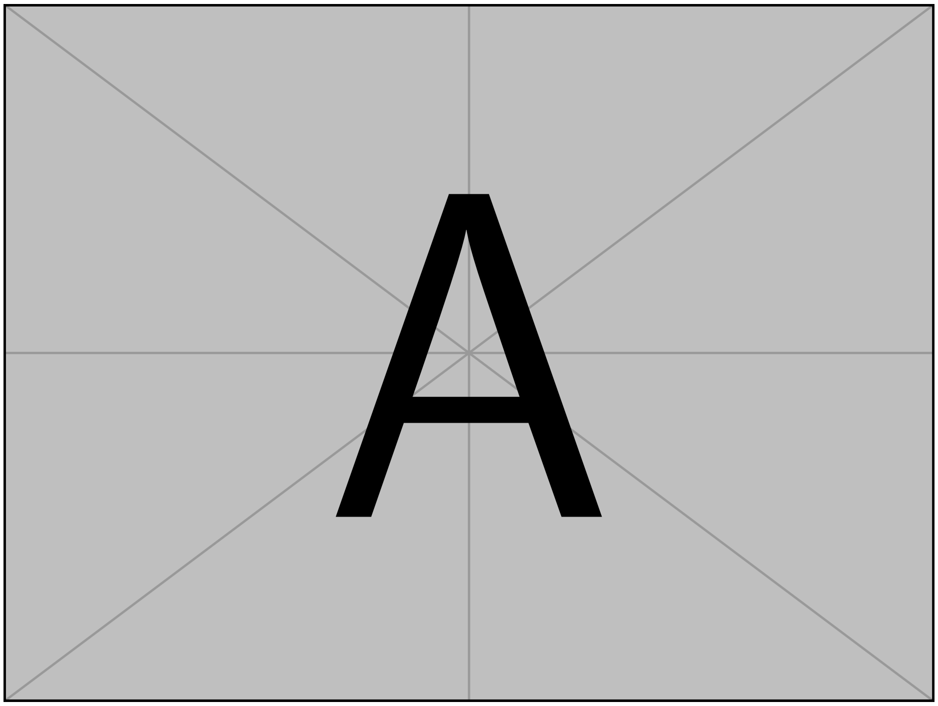

Figure 1: Different types of optical lever displacement sensing systems in KAGRA

2 Derivation

This section is organized as follows, we will first derive a very basic sensing matrix[1], assuming there's no misalignment in the optical system. And then, we will derive the general sensing matrix that includes all sorts of misalignment. Lastly, we will modify the matrix for a folded optical lever configuration.

2.1 Very basic derivation

In the simplest case, the beam position x_1 (along the incidence plane) is related to the angular displacement θ by

$$x_1 = (2r)\,\theta, \tag{1}$$

where r is the lever arm defined by the distance between the reflective surface and the sensing device. The same beam can be used to measure the longitudinal displacement of the reflective surface, if the light beam has an angle of incidence α. In this case, the beam displacement reads

$$x_1 = (2r)\,\theta + (2\sin\alpha)\,x_L, \tag{2}$$

where x_L is the longitudinal displacement of the reflective surface. As can be seen, equation. (2) shows a coupled sensor where it reads both the angular displacement and the longitudinal shift. In KAGRA, some optical levers have a second sensor measuring the beam displacement x_2 some distance d behind a convex lens with focal length f. In this case, we obtain the second beam displacement x_2 via ray transfer matrices [2]

$$\begin{pmatrix} x_2 \\ \cdot \end{pmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix} \begin{bmatrix} 1 & r_\mathrm{lens} \\ 0 & 1 \end{bmatrix} \begin{pmatrix} (2\sin\alpha)\,x_L \\ 2\theta \end{pmatrix}, \tag{3}$$

where r_lens is the distance between the reflective surface and the lens. This gives

$$x_2 = (2\sin\alpha)\left(1-\frac{d}{f}\right)x_L + 2\left[\left(1-\frac{d}{f}\right)r_\mathrm{lens} + d\right]\theta. \tag{4}$$

[1] Here, we define the sensing matrix to be a matrix that maps sensor readouts to the displacements that we want to measure.

Furthermore, we can place the second beam displacement sensor distance behind the lens. So, if we set

$$d = \frac{r_{\mathrm{lens}} f}{r_{\mathrm{lens}} - f}, \tag{5}$$

then the angular coupling, i.e. the second term in Eqn. (4), becomes zero, effectively making the second beam displacement sensor a "length" (length as in longitudinal displacement) sensing device. If Eqn. (5) is satisfied, then the beam displacement measured by the second senor reads

$$x_2 = \left(\frac{-2f\sin\alpha}{r_{\mathrm{lens}} - f}\right) x_L. \tag{6}$$

Now, if we put Eqn. (2) and Eqn. (6) in a matrix form, we can obtain the sensing matrix, i.e.

$$\begin{pmatrix} x_L \\ \theta \end{pmatrix} = \begin{bmatrix} 2\sin\alpha & 2r \\ \frac{-2f\sin\alpha}{r_{\mathrm{lens}}-f} & 0 \end{bmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{7}$$

Then, from here, we can diagonalize[2] the sensors.

2.2 Optical levers in KAGRA

Before diving into the discussion of misalignment, let's rewrite the matrix so it's closer to what we see in KAGRA.

In KAGRA, we have two beam displacement sensors[3], tilt-sensing QPD and length-sensing QPD. They are analogous to the first and second beam position sensors in Sec. 2.1, respectively. Each QPD has two readouts, the horizontal and the vertical displacement of the beam spot, denoted $(x_{\mathrm{tilt}}, y_{\mathrm{tilt}})$, and $(x_{\mathrm{len}}, y_{\mathrm{len}})$ for tilt-sensing QPD and length-sensing QPD respectively. Here, note that the displacements sensed by the QPDs are not, in general, parallel to the global horizontal or vertical plane, as the QPDs are virtually placed orthogonal to the beam. We are particularly interested in the optics' longitudinal x_L, pitch θ_P, and yaw θ_Y displacements. Therefore, the goal is to find a matrix that maps $\vec{x} = (x_{\mathrm{tilt}}, y_{\mathrm{tilt}}, x_{\mathrm{len}}, y_{\mathrm{len}})^T$ to longitudinal displacement, pitch angle, and yaw angle $(x_L, \theta_P, \theta_Y)^T$.

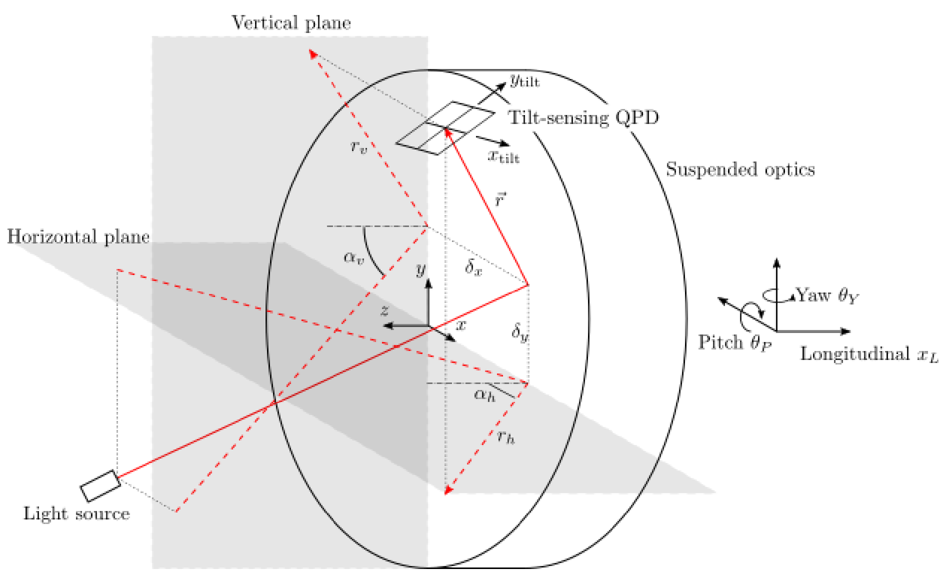

Figure 2: Tilt-sensing optical lever setup in KAGRA.

A 3D illustration of the a general tilt-sensing optical lever setup is shown in Fig. 2. For the current discussion, let's assume that the optical lever is well aligned so the beam spot miscentering at the optics δ_x and δ_y are zero.

[2]Consider a model $\vec{y} = \mathbf{C}\vec{x}$, where $\vec{y}$ are the measurements, and $\vec{x}$ are the states. The goal is to define another measurement basis $\vec{y'}$ such that $\vec{y'} = \mathbf{C}'\vec{x}$, where $\mathbf{C}'$ is a diagonal matrix. It's obvious that If we define $\vec{y'} \equiv \mathbf{C}^{-1}\vec{y}$, then $\mathbf{C}'$ becomes the identity, which is a diagonal matrix. Therefore, we define $\mathbf{C}^{-1}$ to be the sensing matrix, which maps sensor measurements to the displacements of the reflective surface.

[3]Some only has one, e.g. MCo.

Now, the lever arm of the optical lever can be an arbitrary vector, i.e. $\vec{r} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z}$. For the purpose of this discussion, let $\hat{x}$ be a direction aligned to the transverse direction of the main optics, $\hat{y}$ be a direction aligned to the vertical direction of the optics, and $\hat{z}$ be a direction aligned to the longitudinal direction of the optics. Optical levers in KAGRA are aligned on a horizontal plane (i.e. $r_y = 0$), or vertical plane (i.e. $r_x = 0$). But, let's assume that they are not zero.

If we project the beam onto a horizontal plane and a vertical plane (along the normal of the suspended optics), the beams have an incidence angle of α_h and α_v on the horizontal plane and the vertical plane, respectively. It follows that the lever arm that amplifies the pitch angle is the length of the projection of the lever arm $\vec{r}$ on the vertical plane, r_v[4]. Similarly, the lever arm amplifying the yaw angle is the length of the projection of the lever arm on the horizontal plane, r_h. Therefore, a rotation in yaw θ_Y and pitch θ_P would cause the beam spot at the tip of the lever arm to shift by $(2r_h)\,\theta_Y$ and $(2r_v)\,\theta_P$, on the horizontal plane and vertical plane respectively. Meanwhile, a longitudinal shift x_L would cause the beam spot to shift by $(2\sin\alpha_h)\,x_L$ and $(2\sin\alpha_v)\,x_L$ on the horizontal plane and vertical plane, respectively. From here, we can write the displacement of the beam spot as measured by the tilt-sensing QPD, placed at some distance $\vec{r}$ from the beam spot at the suspended optics plane. The beam spot displacement is simply a superposition of that caused by a rotation and a longitudinal shift,

$$x_{\text{tilt}} = (2r_h)\,\theta_Y + (2\sin\alpha_h)\,x_L, \tag{8}$$

and

$$y_{\text{tilt}} = (2r_v)\,\theta_P + (2\sin\alpha_v)\,x_L. \tag{9}$$

As for the length-sensing QPD, let's assume that beam travels some displacement $\vec{r}_{\text{lens}}$ from the suspended optics to a convex lens with focal length f. Fig. 3 shows the length-sensing optical lever setup. Here, note that the beam in Fig. 3 is common to that in Fig. 2[5].

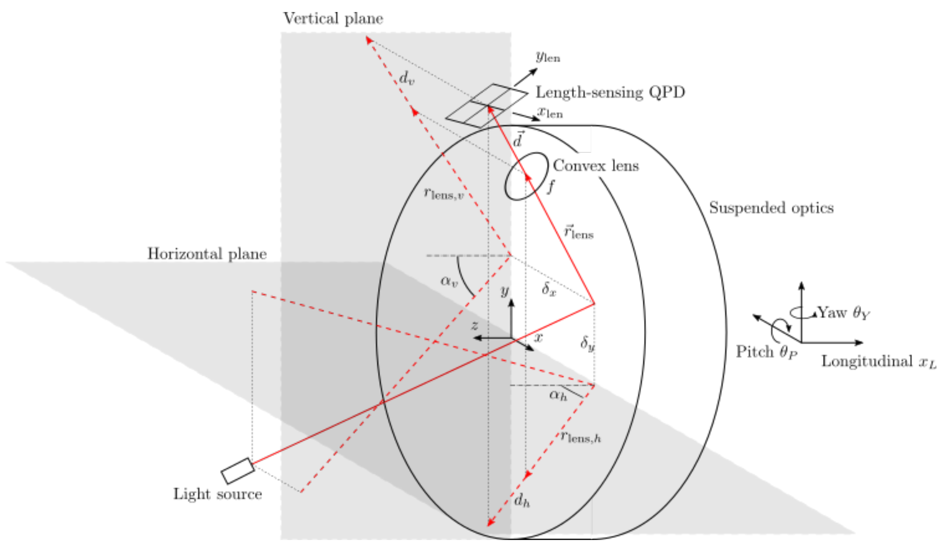

Figure 3: Length-sensing optical lever setup in KAGRA.

Let's say the horizontal lever arm from the optics to the convex lens is $r_{\text{lens},h}$. Again, using ray transfer matrix, we can write down the beam spot displacement at the length-sensing QPD,

$$x_{\text{len}} = (2\sin\alpha_h)\left(1 - \frac{d_h}{f}\right)x_L + 2\left[\left(1 - \frac{d_h}{f}\right)r_{\text{lens},h}, + d_h\right]\theta_Y, \tag{10}$$

where d_h is the length between the lens and the length-sensing QPD on the horizontal plane, and f is the focal length of the lens. It follows that when $d_h = \frac{r_{\text{lens},h} f}{r_{\text{lens},h} - f}$, x_{len} is decoupled from yaw. But, d_h is a distance on the

[4] Think of it as a cross-product $\vec{\theta} \times \vec{r}$. For example, for pitch, $-\theta_P\hat{x} \times (r_x\hat{x} + r_y\hat{y} + r_z\hat{z}) = \theta_P\,(-r_y\hat{z} + r_z\hat{y})$, which is a displacement on the horizontal plane, and $r_v = \sqrt{r_y^2 + r_z^2}$ is the corresponding lever arm that amplifies pitch.

[5] In reality, there's a beamsplitter in front of the tilt-sensing QPD to divide the beam into two.

horizontal plane. To get back the length, d, we can simply use a similar triangle relationship, as shown in Fig. 4,

$$\begin{aligned} d_{\mathrm{yaw}} + r_{\mathrm{lens}} &= \left(d_h + r_{\mathrm{lens}h}\right)\frac{r_{\mathrm{lens}}}{r_{\mathrm{lens},h}}, \\ d_{\mathrm{yaw}} &= \left(d_h + r_{\mathrm{lens},h}\right)\frac{r_{\mathrm{lens}}}{r_{\mathrm{lens},h}} - r_{\mathrm{lens}} \end{aligned} \tag{11}$$

Here, if we set $d = d_{\mathrm{yaw}}$, then the horizontal readout of the length-sensing QPD x_{len} will have no yaw coupling, i.e.

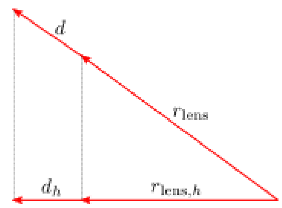

Figure 4: A similar triangle formed by the beam and it's projection on the horizontal plane (length-sensing QPD path).

$$x_{\mathrm{len}} = \left(\frac{-2f\sin\alpha_h}{r_{\mathrm{lens},h} - f}\right) x_L. \tag{12}$$

However, in this case, the vertical readout y_{len} is **not** decoupled from pitch, in general. Now, if we set $d = d_{\mathrm{yaw}}$, then, on the vertical plane, d_v reads, again, from similar triangle relation,

$$\begin{aligned} d_v &= \left(d_{\mathrm{yaw}} + r_{\mathrm{lens}}\right)\frac{r_{\mathrm{lens},v}}{r_{\mathrm{lens}}} - r_{\mathrm{lens},v} \\ &= \left(d_h + r_{\mathrm{lens},h}\right)\frac{r_{\mathrm{lens},v}}{r_{\mathrm{lens},h}} - r_{\mathrm{lens},v} \\ &= \left(\frac{r_{\mathrm{lens},h}f}{r_{\mathrm{lens},h} - f} + r_{\mathrm{lens},h}\right)\frac{r_{\mathrm{lens},v}}{r_{\mathrm{lens},h}} - r_{\mathrm{lens},v} \\ &= \left(\frac{r_{\mathrm{lens},h}}{r_{\mathrm{lens},h} - f} - 1\right) r_{\mathrm{lens},v} \\ &= \frac{r_{\mathrm{lens},v}f}{r_{\mathrm{lens},h} - f} \end{aligned} \tag{13}$$

whereas if we want y_{len} to be decoupled from pitch, we need to set

$$d_v = \frac{r_{\mathrm{lens},h}f}{r_{\mathrm{lens},h} - f}, \tag{14}$$

In general, $r_{\mathrm{lens},h} \neq r_{\mathrm{lens},v}$. So, we cannot simultaneously decouple pitch and yaw from the length-sensing readout. And, if we choose to set $d = d_{\mathrm{yaw}}$, i.e. to minimize yaw coupling, the vertical length-sensing QPD readout reads

$$\begin{aligned} y_{\mathrm{len}} &= (2\sin\alpha_v)\left(1 - \frac{d_v}{f}\right) x_L + 2\left[\left(1 - \frac{d_v}{f}\right) r_{\mathrm{lens},v} + d_v\right]\theta_P \\ &= (2\sin\alpha_v)\left(\frac{r_{\mathrm{lens},h} - r_{\mathrm{lens},v} - f}{r_{\mathrm{lens},h} - f}\right) x_L + 2\left[\left(\frac{r_{\mathrm{lens},h} - r_{\mathrm{lens},v} - f}{r_{\mathrm{lens},h} - f}\right) r_{\mathrm{lens},v} + \frac{r_{\mathrm{lens},v}f}{r_{\mathrm{lens},h} - f}\right]\theta_P \\ &= (2\sin\alpha_v)\left(\frac{r_{\mathrm{lens},h} - r_{\mathrm{lens},v} - f}{r_{\mathrm{lens},h} - f}\right) x_L + 2\left[\left(\frac{r_{\mathrm{lens},h} - r_{\mathrm{lens},v}}{r_{\mathrm{lens},h} - f}\right) r_{\mathrm{lens},v}\right]\theta_P. \end{aligned} \tag{15}$$

Here, note that if the length of the projections are equal, i.e. $r_{\mathrm{lens},h} = r_{\mathrm{lens},h}$, then vertical readout of the length-sensing QPD is decoupled from pitch. This refers to a rather interesting configuration, where the incidence plane of the beam is rotated 45° along the z-axis with respect to the vertical plane.

Without loss of generality, let's assume arbitrary[6] d_h and d_v, and write the sensing matrix. If we ensemble Eqn. (8),

[6] They can't be completely arbitrary. They must be related via the angle between the horizontal plane and the plane of incidence. But, I don't want to introduce that unnecessary parameter.

(9), (10), and the first line of (15) into a matrix form, we get

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \begin{bmatrix} 2\sin\alpha_h & 0 & 2r_h \\ 2\sin\alpha_v & 2r_v & 0 \\ 2\sin\alpha_h\left(1-\frac{d_h}{f}\right) & 0 & 2\left[\left(1-\frac{d_h}{f}\right)r_{\mathrm{lens},h}+d_h\right] \\ 2\sin\alpha_v\left(1-\frac{d_v}{f}\right) & 2\left[\left(1-\frac{d_v}{f}\right)r_{\mathrm{lens},v}+d_v\right] & 0 \end{bmatrix}^+ \begin{pmatrix} x_{\mathrm{tilt}} \\ y_{\mathrm{tilt}} \\ x_{\mathrm{len}} \\ y_{\mathrm{len}} \end{pmatrix}, \tag{16}$$

where $\left[\cdot\right]^+$ is the pseudoinverse of $\left[\cdot\right]$, and $\left[\cdot\right]^+\left[\cdot\right] = \mathbf{I}$.

2.2.1 Horizontal optical levers

Eqn. (16) gives a general relationship between the QPD readouts and the displacements of the optics without misalignments. It can be drastically simplified if we further assume that the incidence plane is aligned to the horizontal plane (e.g. Type-Bp and Type-A) or the vertical plane (e.g. Type-B). For horizontal optical levers, $\alpha_v = 0$. This gives $y_{\mathrm{tilt}} = (2r_v)\,\theta_P$ and $y_{\mathrm{len}} = 2\left[\left(1-\frac{d_v}{f}\right)r_{\mathrm{lens},v}+d_v\right]\theta_P$, which are not independent from each other. We can choose to omit the readout y_{len} and hence the sensing matrix for a horizontal optical lever is

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \begin{bmatrix} 2\sin\alpha_h & 0 & 2r_h \\ 0 & 2r_v & 0 \\ \frac{-2f\sin\alpha_h}{r_{\mathrm{lens},h}-f} & 0 & 0 \end{bmatrix}^{-1} \begin{pmatrix} x_{\mathrm{tilt}} \\ y_{\mathrm{tilt}} \\ x_{\mathrm{len}} \end{pmatrix}, \tag{17}$$

where we've assumed $d = d_h = \frac{r_{\mathrm{lens},h}f}{r_{\mathrm{lens},h}-f}$. Note that on a horitonzal setup, the beam is on the horizontal plane so the horizontal lever arm $r_h = r$. Furthermore, the projection on the vertical plane is simply $r_v = r\cos\alpha_h$. Similarly, the horizontal arm displacement from the optics to the lens $r_{\mathrm{lens},h} = r_{\mathrm{lens}}$.

2.2.2 Vertical optical levers

Like in Sec. 2.2.1, we can write the sensing matrix without further derivation

$$\begin{pmatrix} x_L \\ \theta_P \\ \theta_Y \end{pmatrix} = \begin{bmatrix} 0 & 0 & 2r_h \\ 2\sin\alpha_v & 2r_v & 0 \\ \frac{-2f\sin\alpha_v}{r_{\mathrm{lens},v}-f} & 0 & 0 \end{bmatrix}^{-1} \begin{pmatrix} x_{\mathrm{tilt}} \\ y_{\mathrm{tilt}} \\ y_{\mathrm{len}} \end{pmatrix}, \tag{18}$$

where we've set $d = d_v = \frac{r_{\mathrm{lens},v}f}{r_{\mathrm{lens},v}-f}$. Again, for a vertical setup, $r_v = r$, $r_h = r\cos\alpha_v$, and $r_{\mathrm{lens},v} = r_{\mathrm{lens}}$.

2.2.3 Short summary

We have derived the sensing matrix, Eqn. (16), assuming no misalignment, for a general optical lever setup with a tilted incidence plane. From that, we reduced the general matrix to Eqn. (17) and (18), which corresponds to the sensing matrix for a horizontal optical lever setup and vertical optical lever setup, respectively. In general, the angle of incidence, arm lengths, and focal length are known. Therefore, Eqn. (17) and (18) can be used as initial sensing matrices for the Type-Bp/Type-A and Type-B optical levers, respectively.

2.3 Misalignment

References

[1] Tak Lun Terrence Tsang. BS and SR TM optical lever (oplev) diagonalization matrix. https://gwdoc.icrr.u-tokyo.ac.jp/cgi-bin/private/DocDB/ShowDocument?docid=10189.

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