Optimal Sensor Fusion using $\mathcal{H}_\infty$ methods

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Introduction: Complementary Filters at KAGRA

- The pre-isolators of Type-A and Type-B suspensions in KAGRA are equipped with relative sensors and inertial sensors.
- A pair of complementary filters (low-pass and high-pass) can be used to combine the sensors into a virtual "super-sensor" that has superior noise properties.
- In this work, we discuss a scheme that uses $\mathcal{H}_\infty$ methods to synthesize complementary filters that optimally blend the sensors according to the sensor noises.

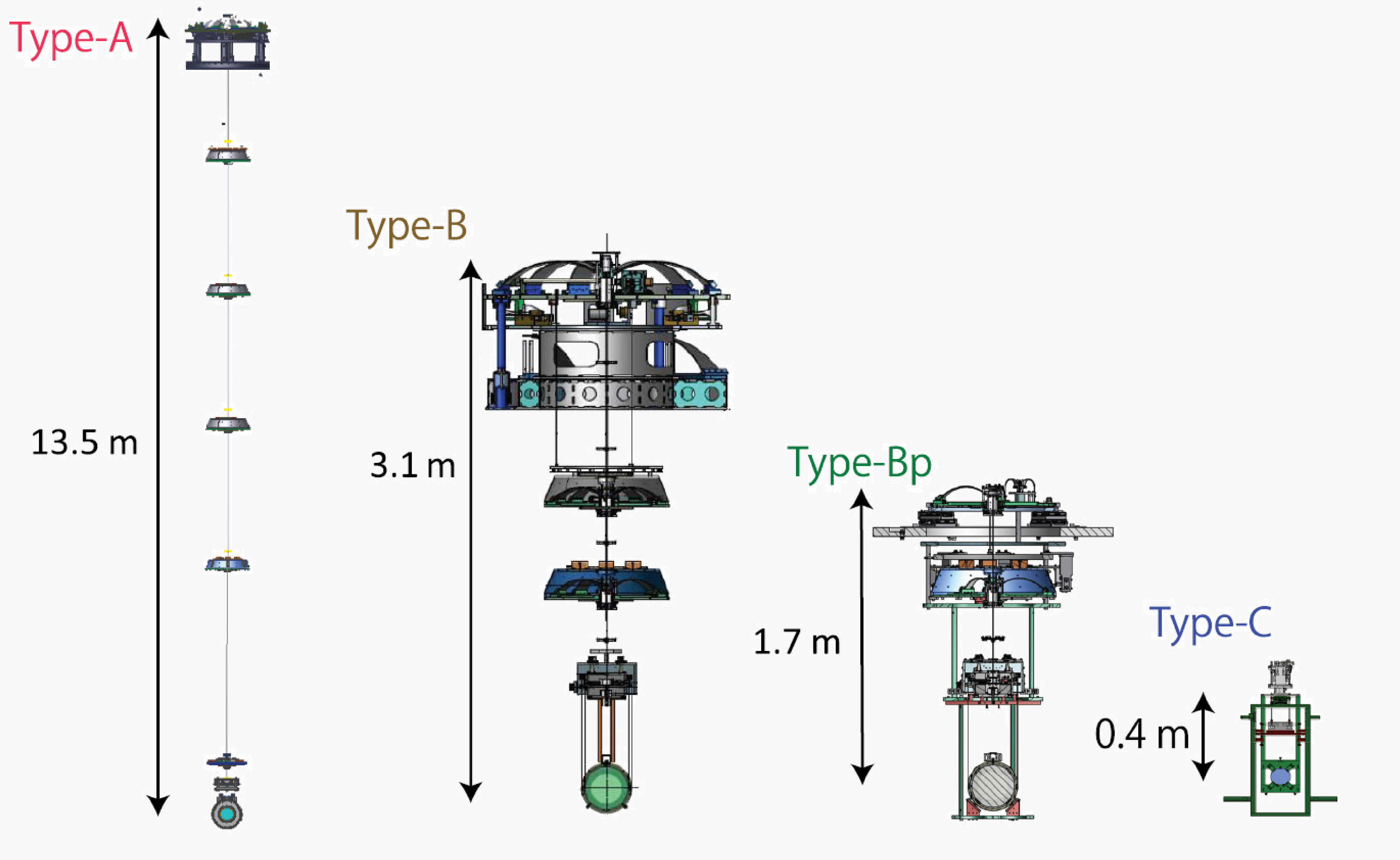

Figure 1: Type-A suspensions: input/end test masses, Type-B suspensions: beamsplitter and signal-recycling mirrors, Type-Bp suspensions: power-recycling mirrors, and Type-C suspensions: input/output mode cleaners [1].

Methodology: Complementary Filter Problem as an $\mathcal{H}_\infty$ Problem

$\mathcal{H}_\infty$ method in a nutshell:

1. As shown in Fig. 2, define signals w, z, u, and v, and hence,
2. derive a generalized plant $\mathbf{P}(s)$.
3. $\mathcal{H}_\infty$ synthesis gives an $\mathcal{H}_\infty$ optimal controller $\mathbf{K}_\infty(s)$ that minimizes the $\mathcal{H}_\infty$-norm of the closed-loop plant $\|\mathbf{G}(s;\mathbf{K},\mathbf{P})\|_\infty$.

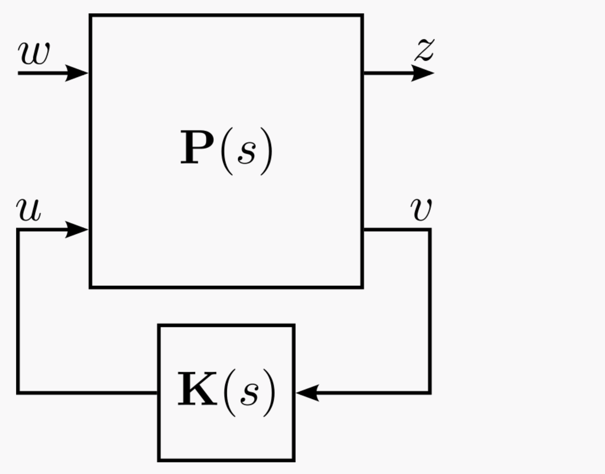

Figure 2: Generalized Plant Represenation.

- The close-loop transfer function $\mathbf{G}(s)$ is defined such that $z=\mathbf{G}(s)w$ and $u=\mathbf{K}(s)v$.
- The $\mathcal{H}_\infty$-norm is defined as

$$\|\mathbf{G}(s)\|_\infty = \sup_\omega \bar{\sigma}(\mathbf{G}(j\omega))\,, \tag{1}$$

where $\bar{\sigma}$ denotes the maximum singular value and ω is the angular frequency.

Methodology: Complementary Filter Problem as an $\mathcal{H}_\infty$ Problem

Formulating the complementary filter problem as an $\mathcal{H}_\infty$ problem:

Complementary filter configuration:
$N_1(s)$, $N_2(s)$: sensor noises,
$H_1(s)$, $H_2(s)$: complementary filters,
$N_{\mathrm{super}}(s)$: super sensor noise.

Step 1:
Apply constraint $H_1(s)+H_2(s)=1$.

Step 2:
Model the sensor noises with transfer functions $\hat{N}_1(s)$ and $\hat{N}_2(s)$.

Φ_1 and Φ_2 are white noises that has power spectrum with unitary amplitude.

Step 3:
Add weighting functions $W_1(s)$ and $W_2(s)$.
Define the generalized plant $\mathbf{P}(s)$ as Eqn. 2.

$W_1(s)$ and $W_2(s)$ are the inverse of the frequency dependent specification of the sensor noises.

Finally:
$\mathcal{H}_\infty$ synthesis gives optimal $H_1(s)$.

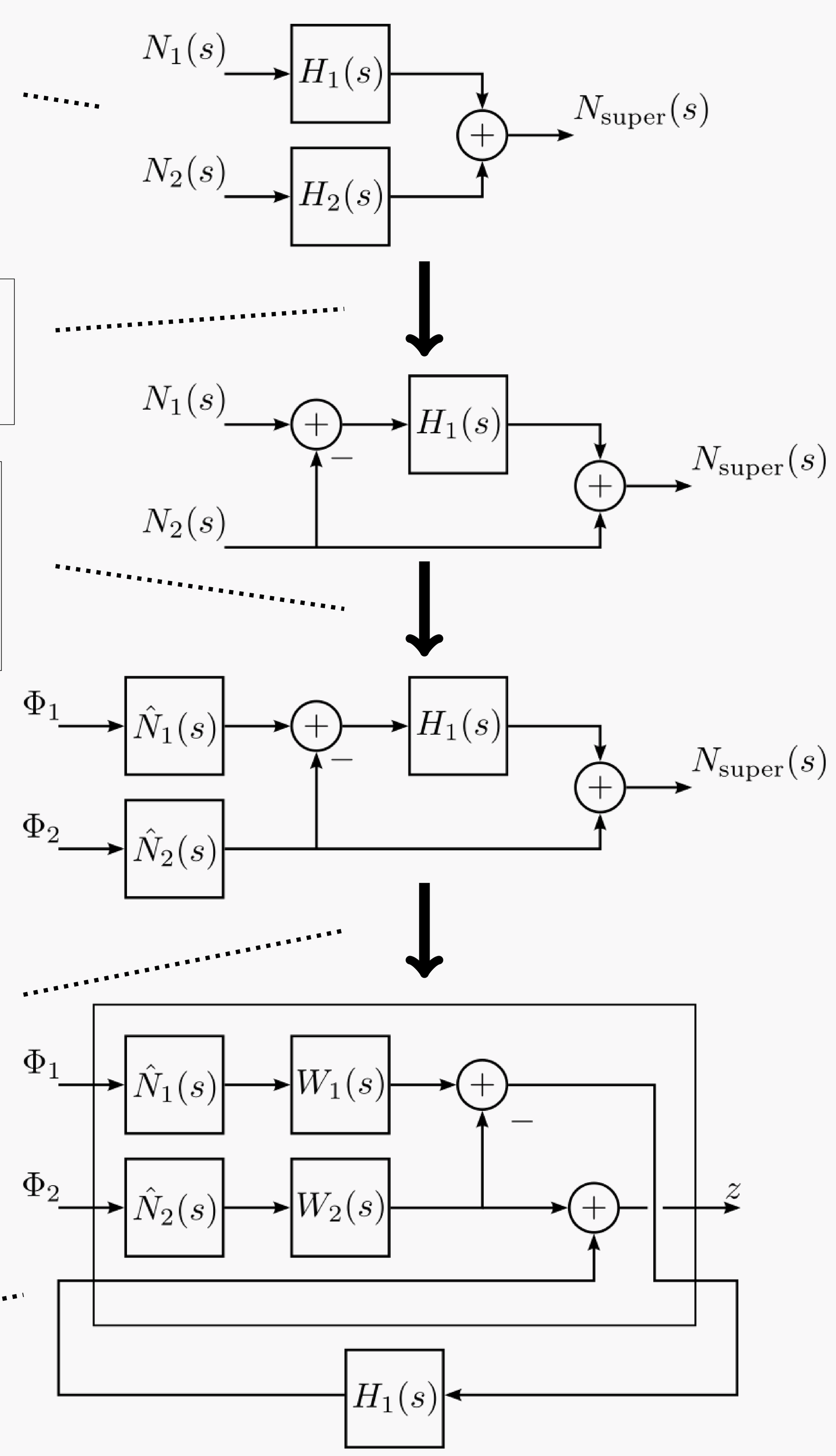

Figure 3: From a simple complementary filter configuration to generalized plant representation.

The generalized plant is given by

$$\mathbf{P}(s) = \begin{bmatrix} 0 & \hat{N}_2(s)W_2(s) & 1 \\ \hat{N}_1(s)W_1(s) & -\hat{N}_2(s)W_2(s) & 0 \end{bmatrix}\,. \tag{2}$$

If we set $W_1(s)=1/\hat{N}_2(s)$ and $W_2(s)=1/\hat{N}_1(s)$, then

$$\|\mathbf{G}(s)\|_\infty \approx \begin{cases} \sup_\omega \left|\frac{N_{\mathrm{super}}(j\omega)}{\hat{N}_2(j\omega)}\right|, & |N_{\mathrm{super}}(j\omega)| \gg |\hat{N}_2(j\omega)|\,, \\ \sup_\omega \left|\frac{N_{\mathrm{super}}(j\omega)}{\hat{N}_1(j\omega)}\right|, & |N_{\mathrm{super}}(j\omega)| \gg |\hat{N}_1(j\omega)|\,. \end{cases} \tag{3}$$

Minimizing $\|\mathbf{G}(s)\|_\infty$ is approximately equal to minimizing a cost function

$$J = \sup_\omega (\log|N_{\mathrm{super}}(j\omega)| - \log\min(|N_1(j\omega)|, |N_2(j\omega)|))\,, \tag{4}$$

where is the maximum difference between the log super sensor noise and the lower bound of the sensor noises.

Results: Synthesizing Complementary Filters for SRM in KAGRA

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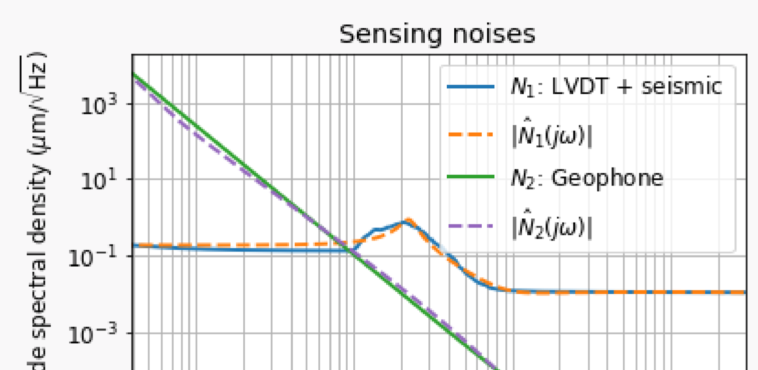

Results: Synthesizing Complementary Filters for SRM in KAGRA (Cont.)

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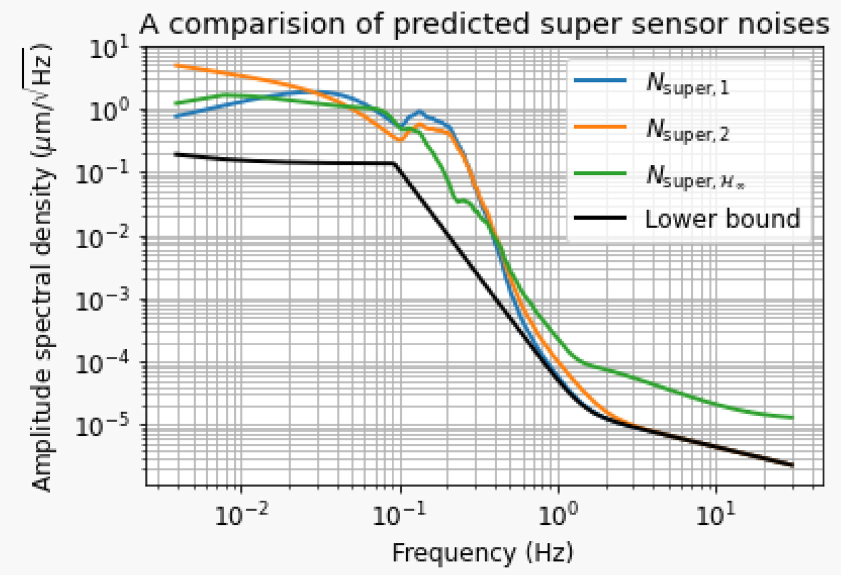

Figure 7: Comparison between the super sensor noises predicted using filter design from [2, 3] and the proposed method.

	RMS (μm)	RMS (0.1-0.5 Hz) (μm)	ASD (10 Hz) ($\mu\mathrm{m}/\sqrt{\mathrm{Hz}}$)
$N_{\mathrm{super},\,1}$	0.5895	0.2400	4.443e-6
$N_{\mathrm{super},\,2}$	0.4726	0.1650	4.443e-6
$N_{\mathrm{super},\mathcal{H}_\infty}$	0.3631	0.1041	2.087e-5
Lower bound	0.0462	0.01422	4.443e-6

Table 1: RMS, band-limited RMS, and ASD values at 10 Hz of the super sensor noises predicted using filter design from [2, 3] and the proposed method (**lower the better**).

Conclusion

- The complementary filter problem is formulated as an $\mathcal{H}_\infty$ optimization problem.
- Complementary filters can be synthesized using $\mathcal{H}_\infty$ method with no information other than the sensing noises themselves.
- The method is exemplified using SRM preisolator sensors and is shown to able to generate filters that better reduce the RMS of the super sensor noise especially around the microseism band.
- While the $\mathcal{H}_\infty$ filters perform worse at 10 Hz, 3 orders of magnitude reduction in super sensor noise ASD (compared to LVDT) can still be achieved.

References

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