1 Basics of Calculus

1.1 Derivatives

Chain Rule	Product Rule	Quotient Rule
$y = f(g(x))$ $g'(x)f'(g(x))$	$y = f(x)g(x)$ $g'(x)f(x) + f'(x)g(x)$	$y = \frac{f(x)}{g(x)}$ $\frac{g'(x)f(x) - f'(x)g(x)}{g(x)^2}$

1.2 Integrals

$$\int x^{-1}dx = \ln|x| + c \qquad \int e^x dx = e^x + c$$
$$\int \cos x dx = \sin x + c \qquad \int \sin x dx = -\cos x + c$$
$$\int \tan x dx = \ln|\sec^2 x| + c \qquad \int \sec x \tan x dx = \sec x + c$$
$$\int \sec^2 x dx = \tan x + c$$

1.2.1 u Substitution

$$\int \cos x e^{\sin x} dx$$

let $u = \sin x$

$$\frac{du}{dx} = \cos x$$

$$du = \cos x dx$$
$$\int e^u du$$
$$= e^u + c$$
$$= e^{\sin x} + c$$

1.2.2 Integration By Parts

$$\int udv = uv - \int vdu$$

let $u = 2t$ let $dv = e^t dt$

let $du = 2dt$ let $v = e^t$

$$\int 2te^t dt$$

$$\int 2te^t dt = 2te^t - \int 2e^t dt$$
$$= 2te^t - 2e^t + c$$

1.2.3 Trig Subsitution

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$\sec^2\theta - 1 = \tan^2\theta$

$$\int \frac{2}{x^2\sqrt{x^2 - 16}}$$

let $x = 4\sec\theta$

$$dx = 4\sec\theta\tan\theta d\theta$$
$$= \int \frac{8\sec\theta\tan\theta}{16\sec^2\theta\sqrt{16\sec^2\theta - 16}}d\theta$$
$$\int \frac{8\sec\theta\tan\theta}{16\sec^2\theta 4\tan\theta}$$

$$= \int \frac{1}{8\sec\theta}d\theta$$
$$\frac{1}{8}\int \cos\theta d\theta$$
$$= \frac{1}{8}\sin\theta + c$$
$$= \frac{1}{8}\frac{\sqrt{x^2 - 16}}{x} + c$$

1.3 Trig Identities

Pythagorean identities	Negative identities
$\sin^2\theta + \cos^2\theta = 1$	$\sin(-\theta) = -\sin\theta$
$1 - \sin^2\theta = \cos^2\theta$	$\cos(-\theta) = \cos\theta$
$1 - \cos^2\theta = \sin^2\theta$	$\tan(-\theta) = -\tan\theta$
$1 + \tan^2\theta = \sec^2\theta$	$\cot(-\theta) = -\cot\theta$
$1 + \cot^2\theta = \csc^2\theta$	$\sec(-\theta) = \sec\theta$
$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$	$\csc(-\theta) = -\csc\theta$
Double Angle formulas	**Sum & difference formulas**
$\sin(2\theta) = 2\sin\theta\cos\theta$	$\sin(x \pm y) = \sin x \cos y \pm \sin y \cos x$
$\cos(2\theta)$ $= \cos^2\theta - \sin^2\theta$ $= 1 - \sin^2\theta$ $= 2\cos^2\theta - 1$	$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$
$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}$	$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$

1.4 Linear trick

If integrating a linear to a power and there is nothing outside...

1. Add 1 to the power
2. Divide the whole thing by the power
3. Divide by the derivative of the inside
4. Add +C if indefinite integral
5. Simplify

For a trigonometric function, the linear trick allows:

$$\int \cos(5x)dx = \frac{1}{5}\sin(5x) + c$$

2 Differential Equations

2.1 Separable Differential Equations

If the De is in the form

$$\frac{dy}{dx} = f(x) \cdot g(y)$$

Then there exists a solution

$$\int \frac{1}{g(y)}dy = \int f(x)dx$$

2.2 First Order Linear Differential Equations

$$y' + yP(x) = Q(x)$$

The first stage of solving a First Order Linear DE, is finding the integrating factor ($I(x)$)

$$I(x) = Ce^{\int P(x)dx}$$

Then multiply everything by $I(x)$

$$I(x)y' + I(x)yP(x) = I(x)Q(x)$$
$$(yI(x))' = I(x)Q(x)$$
$$yI(x) = \int I(x)Q(x)dx$$

Get y by itself

2.3 Bernoulli's Differential Equation

$$y' + P(x)y = Q(x)y^n$$

1. Multiply everything by $(1-n)y^{-n}$
2. let $u = y^{1-n}$, $\frac{du}{dx} = (1-n)y^{-n} \cdot \frac{dy}{dx}$

$$xy' + y = -xy^2$$
$$y' + \frac{1}{x}y = -y^2$$
$$-y^2y' - \frac{1}{x}\frac{1}{y} = 1$$

let $u = y^{-2}$

$$\frac{du}{dx} = -y^{-2}\frac{dy}{dx}$$
$$u' - \frac{1}{x}u = 1$$

Solve as if 2.2

2.4 Second Order Homogenous Linear DE

$$ay'' + by' + cy = 0$$

Has Solutions:

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

Using the Characteristic Equation:

$$ar^2 + br + c = 0$$

Solve for r which gives three possible cases:

1. Real roots r_1, r_2: $y_1(x) = e^{r_1x}, y_2(x) = e^{r_2x}$
2. Repeated Roots r, r: $y_1(x) = e^{rx}, y_2(x) = xe^{rx}$
3. Complex Conjugate $\alpha + \mathrm{B}i, \alpha - \mathrm{B}i$:

$$y(x) = e^{\alpha x}(C_1\cos(\mathrm{B}x) + C_2\sin(\mathrm{B}x))$$

2.5 Second Order non-Homogeneous Linear DE

A Second order non-Homogeneous DE takes the form of

$$ay'' + by' + cy = f(x)$$

The complementary solution still exists

$$y_{c(x)} = C_1y_1(x) + C_2y_2(x)$$

But there is a particular solution which helps form the exact solution, which is found using 2 methods

$$y(x) = y_{c(x)} + y_{p(x)}$$

> !
>
> Remember to use the method asked for in the question even if there is an easier way

2.5.1 Undetermined Coefficients

This method is suitable for polynomials, exponents, sin or cos

$f(x)$: Polynomial - Guess polynomial of the same degree; e.g. $f(x) = x^2, y_{p(x)} = Ax^2 + Bx + C$

$f(x)$: Exponent - Guess $y_{p(x)} = Ae^{bx}$ e.g. $f(x) = 6e^{2x+1}, y_{p(x)} = Ae^{2x+1} = Ae^{2x}$

$f(x)$: $\sin(bx)$ or $\cos(bx)$ - Guess $y_{p(x)} = A\sin(bx) + B\cos(bx)$

If two terms in $f(x)$ are added, add the guesses, similarly if they are multiplied, multiply the guesses, e.g.

$$f(x) = x^2\sin(x) + e^{2x}$$
$$y_{p(x)} = (Ax^2 + Bx + C)(D\sin(x) + E\cos(x)) + Fe^{2x}$$

Finally find y_p' and y_p'' and equate the coefficients - Some expansion and simplification will be required.

2.5.2 Variation of Parameters

Given $ay'' + by' + cy = f(x)$ and the complementary solution $y_{c(x)} = C_1y_1(x) + C_1y_2(x)$ the particular solution is $y_{p(x)} = u_1(x)y_1(x) + u_2(x)y_2(x)$

$$u_1(x) = -\frac{1}{a}\int \frac{y_2(x)f(x)}{W(y_1, y_2)}dx$$
$$u_2(x) = \frac{1}{a}\int \frac{y_1(x)f(x)}{W(y_1, y_2)}dx$$
$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

3 Statistics

3.1 Types of Data

Qualitative
- Nominal - Unordered
- Ordinal - Ordered

Quantitative
- Discrete - Counts
- Continous - Measures

3.2 Probability

A number between 0 and 1, that describes how likely an event is to happen

If probability of an even t is p then the probability of that even not happening is $1 - p$

3.3 Frequency

Favourite ice cream flavours:

C, S, C, C, V, V, S, C, M

Chocolate	4
Strawberry	2
Vanilla	2
Mint	1
Total	9

Put into bar graph (write numbers into bars)

^ For Discrete data

For Continous data:

Break into "bins" e.g. $10 < X \leq 15$, Then produce a frequency table, and graph with histogram (the bars must touch)

3.4 Averages

When mean > mode, Right Skew. mean < mode, Left Skew

3.4.1 Mean

Arithmatic Mean:

$$\overline{X} = \frac{\sum X}{n}$$

Sometimes can't be possible, e.g mean goals in a game 3.4 is not possible. Not great, susceptible to outliars.

3.4.2 Median

The middle value of the ordered set of values. If there is no middle (e.g. even number of values) you take the mean of the 2 middle values

Is not as susceptible to outliars, good way at assessing real world situations, and for skewed data.

3.4.3 Mode

Most common value.

Can be useful but its often not. Can be multiple Modes, does tell you the nature of the data, e.g. "bimodal" if there are 2 modes.

3.5 Variability

Average doesn't always give the full story

Range: biggest - smallest

Range can be easily thrown off by outliar data

IQR = Q3 - Q1

Q1 is the median of the first half

Q3 is the median of the second half

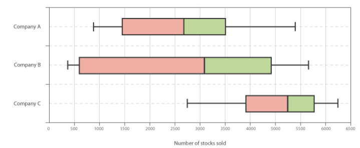
3.5.1 Box and Whisker plot

Take 5 number summary:

min, Q1, median, Q3, max

Draw a line where these values are on a number line

Draw a box between Q1 & Q3



3.6 Standard Deviation

$$\sigma = \sqrt{\frac{\Sigma(x_i - \mu)^2}{N}}$$

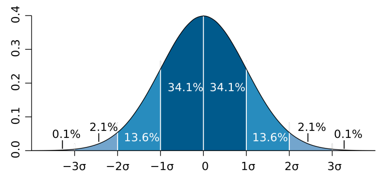
Variance is σ^2

Larger variance = the data is more varied

3.7 Normal Distribution

μ Defines where the middle of the bell is

σ Defines how wide/narrow the bell is



4 Vectors

4.1 Dot Product

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = a_1b_1 + a_2b_2$$
$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = |a| \cdot |b| \cos\theta$$

If a dot product is zero the vectors are othogonal

4.2 Cross Product (Vector Product)

Only works in 3 Dimensional Vector, produced a vector perpendicular to two vectors

$$\vec{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}, \vec{a} = \begin{pmatrix} 2 \\ 7 \\ -5 \end{pmatrix}$$
$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix}$$
$$= \vec{\imath}\begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix} - \vec{\jmath}\begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}$$
$$= \vec{\imath}(-15 - 28) - \vec{\jmath}(-5 - 8) + \vec{k}(7 - 6)$$
$$= -43\vec{\imath} + 13\vec{\jmath} + \vec{k}$$
$$\vec{a} \times \vec{b} = \begin{pmatrix} -43 \\ 13 \\ 1 \end{pmatrix}$$

4.3 Vector Functions

A function where the output is a vector eg

$$\vec{r}(t) = \begin{pmatrix} 2t \\ t^3 \end{pmatrix}$$

13.1 Ex (3)

$$\vec{r}(t) = \begin{pmatrix} 1 + t \\ 2 + 5t \\ -1 + 6t \end{pmatrix}$$

This defines a line within the 3D space

Ex (4)

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

This graph spirals upwards around z

4.3.1 Derivatives

13.2 Ex(1)

$$\vec{r}(t) = \begin{pmatrix} 1 + t^3 \\ te^{-t} \\ \sin 2t \end{pmatrix}$$
$$\vec{r}'(t) = \begin{pmatrix} 3t^2 \\ e^{-t} - te^{-t} \\ 2\cos 2t \end{pmatrix}$$

This defines the vector that describes the change of the function, but we apply the "unit tangent vector" as the magnitude needs to be normalised

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

4.3.2 Integrals

Ex (5)

$$\vec{r}(t) = \begin{pmatrix} 2\cos t \\ \sin t \\ 2t \end{pmatrix}$$
$$\int \vec{r}(t)dt = \begin{pmatrix} 2\sin t \\ -\cos t \\ t^2 \end{pmatrix} + \vec{C}$$

Definite integrals work the same

4.4 Arc Length

$\vec{r}(t)$ is a vector function

$$L = \int_a^b |\vec{r}'(t)| \, dt$$

This often doesn't work out nicely

13.3 Ex (1)

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} \text{ From } (1, 0, 0) \text{ to } (1, 0, 2\pi)$$
$$\vec{r}'(t) = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} \quad |\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$$
$$L = \int_0^{2\pi} \sqrt{2}dt = 2\sqrt{2}\pi$$

4.5 Re Parameterization

$$\vec{r}(t) = \begin{pmatrix} t \\ 2t \end{pmatrix}$$
$$x = 2t$$
$$\vec{r}(x) = \begin{pmatrix} \frac{1}{2}x \\ x \end{pmatrix}$$

4.5.1 Reparameterisation of a vector function in terms of arc length

$$L = \int_a^b |\vec{r}'(t)| \, dt$$
$$S = \int_0^t |\vec{r}'(u)| \, du \text{ \{Arc length function\}}$$

Ex (2)

$$\vec{r}(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix} \quad |\vec{r}'(t)| = \sqrt{2} \text{ Starting point } (1, 0, 0)$$
$$S = \int_0^t \sqrt{2}du \quad S = \sqrt{2}t - 0$$
$$t = \frac{S}{\sqrt{2}} \quad \vec{r}(s) = \begin{pmatrix} \cos\frac{S}{\sqrt{2}} \\ \sin\frac{S}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{pmatrix}$$

4.6 Curvature of a Vector Function

Change in the unit tangent vector $T(t)$

Suppose $\vec{r}(t), \vec{r}(s) \to \vec{r}'(s)$, then $\vec{T}(s) = \vec{r}'(s)$, curvature uses κ

$$\kappa = |\vec{T}'(s)| \quad \vec{r}(s) = \begin{pmatrix} \cos\frac{S}{\sqrt{2}} \\ \sin\frac{S}{\sqrt{2}} \\ \frac{2}{\sqrt{2}} \end{pmatrix}$$
$$\vec{r}'(s) = \vec{T}(s) = \begin{pmatrix} -\frac{1}{\sqrt{2}}\sin\left(\frac{S}{\sqrt{2}}\right) \\ \frac{1}{\sqrt{2}}\cos\left(\frac{S}{\sqrt{2}}\right) \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$
$$\therefore \vec{T}'(s) = \begin{pmatrix} -\frac{1}{2}\cos\left(\frac{S}{\sqrt{2}}\right) \\ -\frac{1}{2}\sin\left(\frac{S}{\sqrt{2}}\right) \\ 0 \end{pmatrix}$$
$$\therefore \kappa = \sqrt{\frac{1}{4}\cos^2\left(\frac{S}{\sqrt{2}}\right) + \frac{1}{4}\sin^2\left(\frac{S}{\sqrt{2}}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

This is usually very difficult, the stars must align for this to work. There are other options:

$$\kappa = |\vec{T}'(s)|$$
$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$
$$\kappa = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{\left(|\vec{r}'(t)|\right)^3}$$

Basically just default to the third option, but if you have $\vec{T}'(S)$, or if it simple (just a constant or something) maybe use $\frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$.

5 Partial Derivatives

5.1 Multivariable Functions

Effectively a vector in and a scalar out:

$$f(x,y)=\sqrt{x+2y}$$

The $f'(x)$ notation is meaningless for this, instead

$$\frac{df}{dx}=\frac{1}{2}(x+2y)^{-\frac{1}{2}}\left(1+2\frac{dy}{dx}\right)$$

When we do this, we often just want to observe what happens when we change one thing, which results in a partial derivative

$$\frac{\partial f}{\partial x}=f_x=\frac{1}{2}(x+2y)^{-\frac{1}{2}}(1)$$

$$f_y=\frac{1}{2}(x+2y)^{-\frac{1}{2}}(2)$$

$f_{xy}=f_{xx}$ When f is "nice", $f_{xxy}=f_{yxx}=f_{xyx}$, polynomials are always nice

5.1.1 Product Rule

$$\frac{\partial}{\partial x}(f\cdot g)=f_x\cdot g+g_xf$$

5.1.2 Compound Functions

Suppose:

$$f(x,y), x=g(t), y=h(t)$$

Therefore you know that $f(t)$ exists but you don't/can't find it, how do we find $\frac{df}{dt}$

$$\frac{df}{dt}=f_x\frac{dx}{dt}+f_y\frac{dy}{dt}$$

If

$$f(x,y), x=g(u,v), y=h(u,v)$$

Then we know that $f(u,v)$

$$f_u=f_xx_u+f_yy_u$$

$$f_v=f_xx_v+f_yy_v$$

5.2 Optimisation

Find critical points - $f_x=0, f_y=0$ at the same time

Classify critical points - Use second derivative test

$$D=f_{xx}\cdot f_{yy}-\left[f_{xy}\right]^2$$

Check $D>0$, check $f_{xx}>0\Rightarrow$ minimum, $f_{xx}<0\Rightarrow$ maximum

$D<0\Rightarrow$ Saddle point

$D=0\Rightarrow$ No useful information $D=\begin{vmatrix}f_{xx}&f_{xy}\\f_{yx}&f_{yy}\end{vmatrix}$

Which therefore allows you to find D, for more than 2 variable functions

6 Double Integrals

6.1 Double Integrals over Rectangles

The volume of a "slice of cake" where the top is defined by $z=f(x,y)$

$$\int_0^1\left(\int_0^1 xdx\right)dy$$

$$=\int_0^1\frac{1}{2}dy$$

$$=\frac{1}{2}$$

This is often written as

$$D:[a,b]\times[c,d]$$

$$\int\int_D f(x,y)dA=\int_c^d\int_a^b f(x,y)dxdy$$

6.2 Double Integrals over type I and type II region

Type I region: Bounded by a function on the top and bottom, and constants on the left and right

Type II region: Bounded by functions, left and right but constants top and bottom

These mean that the bounds of the integral are functions but **THE OUTSIDE INTEGRAL MUST NEVER HAVE THE FUNCTIONS**

(15.2)(EX1)

$$\int\int_D(x+2y)dA$$

D – The Region bounded by $y=2x^2, y=1+x^2$

$$2x^2=1+x^2$$

$$x=\pm1$$

Top func – $y=1+x^2$

Bottom func – $y=2x^2$

$$V=\int_{-1}^{1}\int_{2x^2}^{1+x^2}(x+2y)dydx \text{ \{Because inside func is } y\}$$

$$=\int_{-1}^{1}\left[xy+y^2\right]_{2x^2}^{1+x^2}dx$$

$$=\int_{-1}^{1}\left[\left(x(1+x^2)+(1+x^2)^2\right)-(2x^3+4x^4)\right]dx$$

$$=\int_{-1}^{1}x+x^3+1+x^4+2x^2-2x^3-4x^4dx$$

$$=\int_{-1}^{1}-3x^4-x^3+2x^2+x+1dx$$

$$=\left[-\frac{3}{5}x^5-\frac{1}{4}x^4+\frac{2}{3}x^3+\frac{1}{2}x^2+x\right]_{-1}^{1}$$

$$=\frac{32}{15}$$

6.3 Double Integrals in Polar Coordinates

Given (r,θ) the cartesian coordinates are $(r\cos\theta, r\sin\theta)$. $dA=rdrd\theta$

(15.3)(EX1)

$$\int\int_R(3x+4y^2)dA$$

R – Bounded by the circles $x^2+y^2=1, x^2+y^2=4$

Radii are: 1, 2

$$x=r\cos\theta, y=r\sin\theta, dA=rdrd\theta$$

$$=\int_0^\pi\int_1^2(3r\cos\theta+4r^2\sin^2\theta)rdrd\theta$$

$$=\int_0^\pi\int_1^2(3r^2\cos\theta+4r^3\sin^2\theta)drd\theta$$

$$=\int_0^\pi\left[r^3\cos\theta+r^4\sin^2\theta\right]_0^2d\theta \text{ \{Michael messed up the bounds here\}}$$

$$=\int_0^\pi 8\cos\theta+16\sin^2\theta d\theta$$

$$\cos 2\theta=1-2\sin^2\theta$$

$$2\sin^2\theta=1+\cos 2\theta$$

$$16\sin^2\theta=8+8\cos 2\theta$$

$$=\int_0^\pi 8\cos\theta+8+8\cos 2\theta d\theta$$

$$=\left[8\sin\theta+8\theta+4\sin 2\theta\right]_0^\pi$$

$$=(0+8\pi+0)-(0+0+0)$$

$$=8\pi$$

6.4 Line Integrals (Curve Integrals)

Suppose we have a curve C defined parametrically

$$\vec{r}(t)=(x(t),y(t)), t\in[a,b]$$

$$\text{Line Integral}: \int_C f(x,y)\,\mathrm{d}S$$

surface and $\mathrm{d}S$ Where $f(x,y)$ corresponds to a surface and $\mathrm{d}S$ corresponds to little bits of arc length

$$=\int_a^b f(x(t),y(t))\cdot\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2}\,\mathrm{d}t$$

6.5 Green's Theorem

Say we have a vector field $\vec{F}=(P,Q)$, C: Simple closed curve, C needs to be positively oriented. The curve is positively oriented if the inside of the curve is on the left side (if you imagine walking around the curve), and the area bounded by C is called D.

$$\int_C\vec{F}\cdot\mathrm{d}\vec{r}=\int P\,\mathrm{d}x\cdot Q\,\mathrm{d}y=\iint_D\left(Q_x-P_y\right)\mathrm{d}A$$

This is meaningless if the vector field is conservative

Ex 1

$$\int_C x^4\,\mathrm{d}x+xy\,\mathrm{d}y, C: \text{Triangle bounded by } (0,0),(1,0),(0,1)$$

$$\vec{F}=(x^4,xy)$$

$$\iint_D y-0\,\mathrm{d}A$$

Type I region

$$\int_0^1\int_0^{1-x}y\,\mathrm{d}y\,\mathrm{d}x$$

$$=\frac{1}{6}$$

It may be useful/required to draw a diagram of the bounding area.

6.6 Vector Fields - What are they?

More general functions, vector input → vector output, somewhat like a wind map, at all points there is a vector representing the wind at a specific point.

Suppose we have a vector field $\vec{F}$, and a curve $C:\vec{r}(t)$

$$\int_C\vec{F}\cdot d\vec{r}=\int_a^b\vec{F}(\vec{r}(t))\cdot\vec{r}'(t)dt$$

16.2 Ex(7) - Find the work done by the force field $\vec{F}(x,y)=(x^2,-xy)$ moving along the semi circle $\vec{r}(t)=(\cos t,\sin t), t\in[0,\frac{\pi}{2}]$

$$\vec{F}(x,y)=(x^2,-xy)$$

$$\vec{r}(t)=(\cos t,\sin t), t\in\left[0,\frac{\pi}{2}\right]$$

$$\vec{r}'(t)=(-\sin t,\cos t)$$

$$\vec{F}(\vec{r}(t))=(\cos^2t,-\cos t\sin t)$$

$$\int_0^{\frac{\pi}{2}}(\cos^2t,-\cos t\sin t)\cdot(-\sin t,\cos t)dt$$

$$=\int_0^{\frac{\pi}{2}}-2\cos^2\sin tdt$$

let $u=\cos t$

$du=-\sin tdt$

$2du=-2\sin tdt$

when $t=0,u=1$, when $t=\frac{\pi}{2},u=0$

$$=\int_1^0u^2du$$

$$=2\left[\frac{u^3}{3}\right]_1^0$$

$$=-\frac{2}{3}\text{J}$$

6.7 Conservative Vector fields

$\vec{F}$ is conservative if some function f such that $\vec{F}=\nabla f$ then f is called the potential function for $\vec{F}$

$\vec{F}=(P,Q)$ on an open simply connected region D (exponentials, polynomials, trig functions all satisfy this), then if $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$ throughout D, then $\vec{F}$ is conservative.

16.3 Ex(2) $\vec{F}=(x-y,x-2)$, determine whether this function is conservative

$$\frac{\partial P}{\partial y}=-1,\frac{\partial Q}{\partial x}=1\therefore\text{Not conservative}$$

If $\vec{F}$ is conservative, $\vec{F}=\nabla f$

$$\int_C\vec{F}\cdot d\vec{r}, C:\vec{r}(t), a\le t\le b=f(\vec{r}(b))-f(\vec{r}(a))$$

6.8 Curl and Divergence

Curl is only relevant to 3D fields

$$\vec{F}=(P,Q,R)$$

$$\text{curl}\left(\vec{F}\right)=\left(R_y-Q_z,P_z-R_x,Q_x-P_y\right)$$

$$\vec{\nabla}=\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

$$\text{eg }\vec{\nabla}f=\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right)$$

$$\text{curl}\left(\vec{F}\right)=\vec{\nabla}\times\vec{F}=\vec{i}\begin{vmatrix}\partial_y&\partial_z\\Q&R\end{vmatrix}-\vec{j}\begin{vmatrix}\partial_x&\partial_z\\P&R\end{vmatrix}+\vec{k}\begin{vmatrix}\partial_x&\partial_y\\P&Q\end{vmatrix}$$

$$=\left(R_y-Q_z,P_z-R_x,Q_x-P_y\right)$$

16.5 Ex(1) $\vec{F}=(xz,xyz,-y^2)$

$$\text{curl}\left(\vec{F}\right)=\begin{vmatrix}\vec{i}&\vec{j}&\vec{k}\\\partial_x&\partial_y&\partial_z\\xz&xyz&-y^2\end{vmatrix}=\vec{i}\begin{vmatrix}\partial_y&\partial_z\\zyx&-y^2\end{vmatrix}-\vec{j}\begin{vmatrix}\partial_x&\partial_z\\xz&-y^2\end{vmatrix}+\vec{k}\begin{vmatrix}\partial_x&\partial_y\\xz&xyz\end{vmatrix}$$

$$=(-2y-xy,0+x,yz-0)=(-2y-xy,x,yz)$$

Conservative vector fields are irrotational

Divergence - Exists in any dimensions, how much is something moving towards ($\text{div}(\vec{F})<0$) or away ($\text{div}(\vec{F})>0$) from a point

$$\text{div}\left(\vec{F}\right)=P_x+Q_y+R_z=\vec{\nabla}\cdot\vec{F}$$

7 Sample Exams

7.1 SE 1 Q7 - Optimisation

$$f(x,y)=y^3+3x^2y-6y^2-6y^2+2$$

$$f_x=6xy-12x$$

$$f_y=3y^2+3x^2-12y$$

$$f_{xx}=6y-12$$

$$f_{yy}=6y-12$$

$$f_{xy}=6x$$

$$f_{xx}=6x(y-2)=0$$

$$x=0\text{ or }y=2$$

$$x=0: f_y=3y^2-12y=0$$

$$3y(y-4)=0$$

$$y=4,0$$

$$y=2: f_y=3x^2-12=0$$

$$3x^2=12$$

$$x^2=4$$

$$x=\pm2$$

$(0,0), D>0, f_{xx}<0,$ Maxima

$(0,4), D>0, f_{xx}>0,$ Minima

$(2,2), D<0,$ Saddle Point

$(-2,2), D<0,$ Saddle Point

$$D=f_{xx}f_{yy}-\left(f_{xy}\right)^2$$

$$D=(6y-12)^2-36x^2$$

7.2 SE2 Q2 (a) - First Order DE

$$2xy'+4y=12x$$

$$y'+\frac{2}{x}y=6$$

$$I(x)=e^{\int\frac{2}{x}\,\mathrm{d}x}=e^{2\ln x}=x^2$$

$$x^2y'+2xy=6x^2$$

$$\left(yx^2\right)'=6x^2$$

$$yx^2=\int 6x^2\,\mathrm{d}x$$

$$yx^2=2x^3+C$$

$$y=2x+\frac{C}{x^2}$$

7.3 SE2 Q3 - Coefficients

$$y''-4y'+4y=2xe^x+25\sin(x)$$

Characteristic Equation

$$r^2-4r+4=0$$

$$(r-2)^2=0$$

$$r=2$$

$$y_c(x)=C_1e^{2x}+C_2xe^{2x}$$

$$y_p(x)=(Ax+B)e^x+C\sin x+D\cos x$$

$$y_p'=Ae^x+(Ax+B)e^x+C\cos x-D\sin x$$

$$y_p'=(Ax+A+B)e^x+C\cos x-D\sin x$$

$$y_p''=Ae^x+(Ax+A+B)e^x-C\sin x-D\cos x$$

$$y_p''=(Ax+2A+B)e^x-C\sin x-D\cos x$$

Plug into original equation

$$(Ax+2A+B)e^x-C\sin x-D\cos x-4(Ax+A+B)e^x$$

$$-4(\cos x+4D\sin x+4(Ax+B)e^x+4C\sin x+4D\cos x)$$

$$=Axe^x+(2A+B-4A-4B)e^x+(-D-4C+4D)\cos x$$

$$+(-C+4D+4C)\sin x$$

$$=Axe^x+(B-2A)e^x+(3D-4C)\cos x+(3C+4D)\sin x$$

$$=2xe^x+25\sin x$$

$$A=2$$

$$B-2A=0, B=4$$

$$3C+4D=25$$

$$3D-4C=0$$

$$C=3, D=4$$

$$y_p(x)=(2x+4)e^x+3\sin x+4\cos x$$

$$y=y_c+y_p$$

$$y=C_1e^{2x}+C_2xe^{2x}+(2x+4)e^x+3\sin x+4\cos x$$

7.4 SE 1 Q3 - Variation of parameters

$$y''-2y'-3y=4e^{5x}$$

$$r^2-2r-3=0$$

$$(r+3)(r-1)=0$$

$$r=-3, r=1$$

$$y_{c(x)}=C_1e^{3x}+C_2e^{-x}$$

$$y_{p(x)}=u_1e^{3x}+u_2(x)e^{-x}$$

$$u_1=-\frac{1}{a}\int\frac{y_2(x)f(x)}{W(y_1,y_2)}\,\mathrm{d}x$$

$$u_1=\frac{1}{a}\int\frac{y_1(x)f(x)}{W(y_1,y_2)}\,\mathrm{d}x$$

$$W=\begin{vmatrix}y_1&y_2\\y_1'&y_2'\end{vmatrix}=y_1y_2'-y_2y_1'$$

$$y_1=e^{3x}, y_2=e^{-x}, f(x)=4e^{5x}, y_1'=3e^{3x}, y_2'=-e^{-x}$$

$$W(y_1,y_2)=-4e^{2x}$$

$$u_1=-\int\frac{e^{-x}\cdot4e^{5x}}{-4e^{2x}}\,\mathrm{d}x$$

$$u_1=-\int e^{2x}\,\mathrm{d}x=\frac{1}{2}e^{2x}$$

$$u_2=\int\frac{e^{3x}\cdot4e^{5x}}{-4e^{2x}}\,\mathrm{d}x$$

$$u_2=\int-e^{6x}\,\mathrm{d}x=-\frac{1}{6}e^{6x}$$

$$y_p=\frac{1}{2}e^{5x}-\frac{1}{6}e^{5x}=\frac{1}{3}e^{5x}$$

$$y=C_1e^{3x}+C_2e^{-x}+\frac{1}{3}e^{5x}$$

7.5 SE 1 Q1 (b)

$$\frac{\mathrm{d}y}{\mathrm{d}x}=4x-5y$$

Using $u=y-ax$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=\frac{\mathrm{d}y}{\mathrm{d}x}-a$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}+a=4x-5(u-ax)$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=(4-5a)x-5u-a$$

Want to cancel x

$$(4-5a)=0$$

$$a=\frac{4}{5}$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=-\frac{4}{5}-5u$$

$$\frac{\mathrm{d}u}{\mathrm{d}x}=-5\left(u+\frac{4}{25}\right)$$

$$\int\frac{1}{u+\frac{4}{25}}\,\mathrm{d}u=-\int 5\,\mathrm{d}x$$

$$\ln\left(\left|u+\frac{4}{25}\right|\right)=-5x+C$$

$$u+\frac{4}{25}=Ce^{-5x}$$

$$u=Ce^{-5x}-\frac{4}{25}$$

$$y-\frac{4}{5}x=Ce^{-5x}-\frac{4}{25}$$

$$y=\frac{4}{5}x+Ce^{-5x}-\frac{4}{25}$$

7.6 SE 1 Q9 (b)

$$\oint y\,\mathrm{d}x-x\,\mathrm{d}y, \text{Bounded by the circle at the origin of radius 4}$$

$$\vec{F}=(y,-x)$$

$$\iint_D(-1-1)\,\mathrm{d}A$$

Switch to polar coords

$$\int_0^{2\pi}\int_0^4-2r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$\int_0^{2\pi}\left[-r^2\right]_0^4\mathrm{d}\theta$$

$$\int_0^{2\pi}-16\,\mathrm{d}\theta$$

$$\left[-16\theta\right]_0^{2\pi}$$

$$=-32\pi$$

7.7 SE 3 Q9 (a)

$$\vec{F}(x,y,z)=\left(\frac{1}{\sqrt{x+9}},\frac{1}{\sqrt{y+1}},\frac{1}{\sqrt{z+3}}\right)$$

Over $r(t)=(t^2,2t,6)$

$$r'(t)=(2t,2,0)\vec{F}(r(t))=\left(\frac{1}{\sqrt{t^2+9}},\frac{1}{\sqrt{2t+1}},\frac{1}{3}\right)$$

$$r'(t)\cdot\vec{F}(r(t))=\frac{2t}{\sqrt{t^2+9}}+\frac{2}{\sqrt{2t+1}}$$

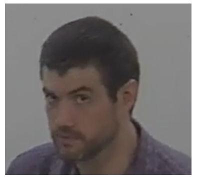
$$\int_0^4\frac{2t}{\sqrt{t^2+9}}+\frac{2}{\sqrt{2t+1}}\,\mathrm{d}t$$

let $u=t^2+9, \mathrm{d}u=2t\,\mathrm{d}t, a=9, b=25$

$$\int_9^{25}u^{-\frac{1}{2}}\,\mathrm{d}t+2\int_0^4\frac{1}{\sqrt{2t+1}}\,\mathrm{d}t$$

$$\left[2\sqrt{u}\right]_9^{25}+2\left[\sqrt{2t+1}\right]_0^4$$

$$10-6+6-2=8$$



Big D?