MATH 240: Discrete Structures - Assignment 3

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Problem 1

Proof. Assume $2^n - 1$ is prime.

Let $\sigma(n)$ be a function that computes the sum of all divisors of a given number n.

Then $\sigma(2^n - 1) = 2^n$ since the only divisors of $2^n - 1$ are 1 and $2^n - 1$ as a result of $2^n - 1$ being prime.

Moreover, we have $\sigma(2^{n-1}) = 2^n - 1$ as the divisors of 2^{n-1} are all even numbers up to and including 2^{n-1}.

Putting it all together:

$$\begin{aligned}\sigma(2^{n-1}(2^{n-1}))) &= \sigma(2^{n-1})\sigma(2^n - 1) \\ &= (2^n - 1)(2^n) \\ &= 2(2^{n-1})(2^n - 1)\end{aligned}$$

Therefore $2^n - 1(2^n - 1)$ is a perfect number. $\square$

Problem 2

Step 1: Find $148^{-1} \pmod{421}$

Compute the steps of the Euclidean GCD algorithm:

$$\begin{aligned}
421 &= 2 \times 148 + 125 \\
148 &= 1 \times 125 + 23 \\
125 &= 5 \times 23 + 10 \\
23 &= 2 \times 10 + 3 \\
10 &= 3 \times 3 + 1
\end{aligned}$$

Roll back the steps to find $s, t \in \mathbb{Z}$ such that $1 = 421s + 148t$

$$\begin{aligned}
1 &= 1(10) - 3(3) \\
&= 1(10) - 3(23 - 2(10)) \\
&= 7(10) - 3(23) \\
&= 7(125 - 5(23)) - 3(23) \\
&= 7(125) - 38(23) \\
&= 7(125) - 38(148 - 1(125)) \\
&= 45(125) - 38(148) \\
&= 45(421 - 2(148)) - 38(148) \\
&= 45(421) - 128(148)
\end{aligned}$$

So $1 = 45(421) - 128(148)$, we then get $1 = -128(148)$, hence $148^{-1} \pmod{421} \equiv -128 \equiv 293 \pmod{421}$.

Step 2: Solve $148x \equiv 12 \pmod{421}$

$$\begin{aligned}
148x &\equiv 12 \pmod{421} \\
x &\equiv 12 \times 293 \pmod{421} \\
&\equiv 3516 \pmod{421} \\
&\equiv 148 \pmod{421}
\end{aligned}$$

Problem 3

Problem 4