MATH 240: Discrete Structures - Assignment 4

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Problem 1

(a)

(b)

Problem 2

The bulk of the work for this problem lay in figuring out what our 'pigeonholes' will be.

Since the we have points of the form (x, y, z), the mid-point, or 'average', of any two points will be of the form $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$. We want the mid-point to be integral, that is, $\frac{x_1+x_2}{2} \in \mathbb{Z}$, $\frac{y_1+y_2}{2} \in \mathbb{Z}$ and $\frac{z_1+z_2}{2} \in \mathbb{Z}$, so we notice that the numerator values must be of the same parity.

Our pigeonholes can thus be the sets of points $(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ according to their parity:

$$
\begin{aligned}
H_{EOO} &= \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ even, } y \text{ odd, } z \text{ odd})\} \\
H_{EEO} &= \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ even, } y \text{ even, } z \text{ odd})\} \\
H_{EEE} &= \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ even, } y \text{ even, } z \text{ even})\} \\
H_{OEE} &= \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ odd, } y \text{ even, } z \text{ even})\} \\
H_{EOE} &= \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ even, } y \text{ odd, } z \text{ even})\} \\
H_{OOE} &= \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ odd, } y \text{ odd, } z \text{ even})\} \\
H_{OOO} &= \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ odd, } y \text{ odd, } z \text{ odd})\} \\
H_{OEO} &= \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ odd, } y \text{ even, } z \text{ odd})\}
\end{aligned}
$$

Since we take 9 points of the form $(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, note that at least 2 of them will land in the same 'pigeonhole' by the Pigeonhole Principle, and thus whose mid-point will be integral, giving us our answer.

Problem 3

(a)

We have two base cases:

$$B_1 = B_2 = 1$$

For which we can deduce the recurrence as being:

$$B_n = B_{n-1} + B_{n-2}, \text{ for } n \geq 3$$

(b)

Proof. (By Induction)

Base case: $n = 4$

$$B_4 = B_3 + B_2 = 2 + 1 = 3$$

which is divisible by 3.

Inductive step:

Assume B_{4k} is divisible by 3 for some $k > n \in \mathbb{N}$.

We have $B_{4k} = 3l$ for some $l \in Z$.

$$\begin{aligned}
B_{4k} = B_{4(k+1)} &= B_{4k+4} \\
&= B_{4k+3} + B_{4k+4} && \text{[By RR]} \\
&= 2B_{4k+2} + B_{4k+1} \\
&= 2(B_{4k+1} + B_{4k}) + B_{4k+1} \\
&= 3B_{4k+1} + 2B_{4k} \\
&= 3B_{4k+1} + 2(3l) && \text{[By IH]} \\
&= 3(B_{4k+1} + 2l)
\end{aligned}$$

Which is divisible by 3, therefore proving that if n is divisible by 4, B_n is divisible by 3, by mathematical induction. $\square$

Problem 4

(a)

(b)