MATH 240: Discrete Structures - Assignment 3

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Problem 1

(a)

Proof. Suppose $P(x) = 2x^2 - 4x + 3$ where $x \in \mathbb{R}$.

$$2x^2 - 4x + 3 = 2(x^2 - 2x + 1) + 1$$
$$= 2((x - 1)(x - 1)) + 1$$
$$= 2(x - 1)^2 + 1$$

Notice that $x^2 \geq 0 \,\forall\, x \in R$.

It follows that $2(x - 1)^2 + 1 > 0 \,\forall\, x \in R$.

So $2x^2 - 4x + 3 > 0 \,\forall\, x \in \mathbb{R}$. □

(b)

Proof. Suppose $x \in \mathbb{Z}$ is odd.

$$x = 2k + 1, \; k \in Z$$
$$= 1 \times (2k + 1)$$
$$= (k + 1 + k)(k + 1 - k)$$

Let $a = k + 1$, $b = k$.

Then $2k + 1 = (a + b)(a - b) = a^2 - b^2$. □

Problem 2

(a)

Proof. Suppose $x \in \mathbb{Z}$ is even.

Then we can write $x = 2k$, $k \in \mathbb{Z}$.

$$x^3 - 2x + 3 = (2k)^3 - 2(2k) + 3$$
$$= 8k^3 - 4k + 3$$
$$= 8k^3 - 4k + 2 + 1$$
$$= 2(4k^3 - 2k + 1) + 1$$

Let $l = 4k^3 - 2k + 1$, $l \in \mathbb{Z}$.

Then $x^3 - 2x + 3 = 2l + 1$, and is odd by definition. $\square$

(b)

Proof. Suppose $log_2 5$ is rational.

Then we can write $log_2 5 = \frac{m}{n}$, where $m, n \in \mathbb{Z}$.

$$5 = 2^{\left(\frac{m}{n}\right)}$$
$$5^n = 2^m$$

Since all 5^n, $n \in Z$ are odd and all 2^m, $m \in Z$ are even, we've reached a contradiction. $\square$

Problem 3

(a)

Proof. Let $P(n) = \frac{1}{1\times3} + \frac{1}{3\times5} + \frac{1}{5\times7} + ... + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$.

<u>Base case:</u> $n = 1$
$$\frac{1}{(1 \times 3)} = \frac{1}{3}$$
$$\frac{n}{2n+1} = \frac{1}{2(1)+1} = \frac{1}{3}$$

<u>Induction step:</u> Assume $P(n)$ holds for some $n \leq k$.
$$\text{Then } \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + ... + \frac{1}{(2k-1) \times (2k+1)} = \frac{k}{2k+1}$$

$$\frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} = \frac{k(2k+3)+1}{(2k+1)(2k+3)}$$
$$= \frac{2k^2 + 3 + 1}{(2k+1)(2k+3)}$$
$$= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$
$$= \frac{k+1}{2k+3}$$

Notice that $\frac{k+1}{2k+3}$ is just $P(k+1)$, $\therefore$ we've shown that $k \implies k+1$. $\square$

(b)

Proof. Let $P(n) = \overline{A_1 \cup A_2 \cup ... \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_n}$.

<u>Base case:</u>
$$\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$$

Which is true by DeMorgan's Law.

Induction step: Assume $P(n)$ holds for some $n \leq k$.

$$\text{Then } \overline{A_1 \cup A_2 \cup ... \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_k}$$

$$
\begin{aligned}
\overline{(A_1 \cup A_2 \cup ... \cup A_k) \cup A_{k+1}} &= \overline{(A_1 \cup A_2 \cup ... \cup A_k)} \cap \overline{A_{k+1}} && \text{[De Morgan]} \\
&= (\overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_k}) \cap \overline{A_{k+1}} && \text{[Induction Hypothesis]} \\
&= \overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_k} \cap \overline{A_{k+1}}
\end{aligned}
$$

Therefore we've shown that we can reach $k+1$ from k. $\qquad\square$

Problem 4

Proof. Let a, b, $c \in \mathbb{N}$.

Suppose $a \mid (b+c)$ and $gcd(b,c) = 1$.

We can then write $ma = b + c$, $m \in \mathbb{N}$, and by Bézout's Identity we have $1 = bu + cv$ where $b, c \in \mathbb{N}$.

Case 1: Showing $gcd(a,b) = 1$

$$
\begin{aligned}
1 &= bu + cv \\
1 &= bu + (ma - b)v \\
1 &= mav - bv + bu \\
1 &= a(mv) + b(u - v)
\end{aligned}
$$

Since mv, $u - v \in \mathbb{N}$, by Bézout's Identity $gcd(a,b) = 1$.

Case 2: Showing $gcd(a,c) = 1$

$$
\begin{aligned}
1 &= bu + cv \\
1 &= (ma - c)u + cv \\
1 &= uma - cu + cv \\
1 &= a(um) + c(v - u)
\end{aligned}
$$

Since um, $v - u \in \mathbb{N}$, by Bézout's Identity $gcd(a,c) = 1$. $\qquad\square$