MATH 240: Discrete Structures - Assignment 4

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Problem 1

(a)

Proof. (By Induction)

Base case: k = 0

$$\binom{n}{n} = \binom{n+1}{n+1} = 1$$

Inductive step:

Assume $\sum_{i=0}^{k} \binom{n+i}{n} = \binom{n+k+1}{n+1}$ for some k, then:

$$\begin{aligned}
\sum_{i=0}^{k+1} \binom{n+i}{n} &= \sum_{i=0}^{k} \binom{n+i}{n} + \binom{n+k+1}{n} \\
&= \binom{n+k+1}{n+1} + \binom{n+k+1}{n} \\
&= \binom{n+k+2}{n+1} \\
&= \binom{n+(k+1)+1}{n+1}
\end{aligned}$$

which is what we wanted to show. □

(b)

Proof. (By Combinatorial Proof)

Choose any integers n, k and let S be a set such that $|S| = n + k + 1$.

The RHS counts the number of $n + 1$ element subsets of S.

The LHS counts the same thing in a different way. Remove $k + 1$ elements from S, then count the number of n element subsets of S, repeat this until we've added each removed element back into S. We arrive at the same conclusion. □

Problem 2

The bulk of the work for this problem lay in figuring out what our 'pigeonholes' will be.

Since the we have points of the form (x, y, z), the mid-point, or 'average', of any two points will be of the form $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$. We want the mid-point to be integral, that is, $\frac{x_1+x_2}{2} \in \mathbb{Z}$, $\frac{y_1+y_2}{2} \in \mathbb{Z}$ and $\frac{z_1+z_2}{2} \in \mathbb{Z}$, so we notice that the numerator values must be of the same parity.

Our pigeonholes can thus be the sets of points $(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ according to their parity:

$$H_{EOO} = \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ even},\, y \text{ odd},\, z \text{ odd})\}$$
$$H_{EEO} = \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ even},\, y \text{ even},\, z \text{ odd})\}$$
$$H_{EEE} = \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ even},\, y \text{ even},\, z \text{ even})\}$$
$$H_{OEE} = \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ odd},\, y \text{ even},\, z \text{ even})\}$$
$$H_{EOE} = \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ even},\, y \text{ odd},\, z \text{ even})\}$$
$$H_{OOE} = \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ odd},\, y \text{ odd},\, z \text{ even})\}$$
$$H_{OOO} = \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ odd},\, y \text{ odd},\, z \text{ odd})\}$$
$$H_{OEO} = \{(x,\, y,\, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} : (x \text{ odd},\, y \text{ even},\, z \text{ odd})\}$$

Since we take 9 points of the form $(x, y, z) \in \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$, note that at least 2 of them will land in the same 'pigeonhole' by the Pigeonhole Principle, and thus whose mid-point will be integral, giving us our answer.

Problem 3

(a)

We have three base cases:

$$B_1 = 1$$
$$B_2 = 1$$
$$B_3 = 2$$

For which we can deduce the recurrence as being:

$$B_n = B_{n-1} + B_{n-2}, \text{ for } n \geq 3$$

(b)

Proof. (By Induction)

<u>Base case:</u> $n = 4$

$$B_4 = B_3 + B_2 = 2 + 1 = 3$$

which is divisible by 3.

<u>Inductive step:</u>

Assume B_{4k} is divisible by 3 for some $k > n \in \mathbb{N}$.

We have $B_{4k} = 3l$ for some $l \in Z$.

$$\begin{aligned}
B_{4k} = B_{4(k+1)} &= B_{4k+4} \\
&= B_{4k+3} + B_{4k+4} && \text{[By RR]} \\
&= 2B_{4k+2} + B_{4k+1} \\
&= 2(B_{4k+1} + B_{4k}) + B_{4k+1} \\
&= 3B_{4k+1} + 2B_{4k} \\
&= 3B_{4k+1} + 2(3l) && \text{[By IH]} \\
&= 3(B_{4k+1} + 2l)
\end{aligned}$$

Which is divisible by 3, therefore proving that if n is divisible by 4, B_n is divisible by 3, by mathematical induction. $\square$

Problem 4

(a)

I approach this by setting up the homegeneous recurrence relation, solving for the characteristic polynomial, solving for roots and then solving for the remaining constants a_1 and a_2.

$$\begin{aligned} g_n - 4g_{n-1} + 3g_{n-2} &= 0 \\ = r^n - 4r^{n-1} + 3r^{n-2} &= 0 \\ = r^2 - 4r + 3 &= 0 \\ = (r-1)(r-3) &= 0 \end{aligned}$$

So after this we have roots of the characteristic polynomial $= \{1, 3\}$, therefore we need to solve for constants a_1 and a_2 such that $g_n = a_1(1)^n + a_2(3)^n$.

We have base cases $g_0 = 1$ and $g_1 = 7$, so:

$$\begin{aligned} 1 &= a_1(1)^0 + a_2(3)^0 \\ 7 &= a_1(1)^1 + a_2(3)^1 \end{aligned}$$

Solving for a_1 and a_2:

$$\begin{aligned} 1 - a_2 &= a_1 \\ 7 = 1 - a_2 &+ 3a_2 \\ 2a_2 &= 6 \\ \underline{a_2} &\underline{= 3} \\ 1 - 3 &= a_1 \\ \underline{a_1} &\underline{= -2} \end{aligned}$$

So we have $g_n = -2(1)^n + 3(3)^n$.

(b)

In order to figure out what the recurrence relation might be, its useful to generate some base cases.

Let f_n be the function that computes the number of sequences of length n over the alphabet $\{0, 1, 2\}$ that do not contain consecutive even numbers.

For f_0, we know this to be 1, as the empty string is a valid string given the criteria. We also have $f_1 = 3$, as we can clearly see the only sequences of length 1 are $\{0, 1, 2\}$, which are valid sequences.

Moreover, we can see that $f_2 = 5$ since we remove the sequences $\{02, 22, 00, 20\}$ from the set $\{10, 11, 01, 02, 22, 00, 20, 12, 21\}$, giving us enough information to deduce a recurrence relation:

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 3 \\ f_2 &= 5 \\ f_n &= f_{n-1} + 2f_{n-2} \end{aligned}$$

Solving for the characteristic polynomial:

$$\begin{aligned} f_n - f_{n-1} - 2f_{n-2} &= 0 \\ = r^n - r^{n-1} - 2r^{n-2} &= 0 \\ = r^2 - r - 2 &= 0 \\ = (r+1)(r-2) &= 0 \end{aligned}$$

Thus, we have roots of the characteristic polynomial $= \{-1, 2\}$, therefore we need to solve for constants a_1 and a_2 such that $f_n = a_1(2)^n + a_2(-1)^n$.

We use the base cases $f_0 = 1$ and $f_0 = 3$, so:

$$\begin{aligned} 1 &= a_1(2)^0 + a_2(-1)^0 \\ 3 &= a_1(2)^1 + a_2(-1)^1 \end{aligned}$$

Solving for a_1 and a_2:

$$\begin{aligned} 1 - a_2 &= a_1 \\ 3 &= 2(1 - a_2) - a_2 \\ 3 &= 2 - 2a_2 - a_2 \\ 1 &= -3a_2 \\ \underline{a_2 = \frac{1}{3}} \\ a_1 + 1 + \frac{1}{3} \\ \underline{a_1 = \frac{4}{3}} \end{aligned}$$

So we have $f_n = \frac{4}{3}(2)^n - \frac{1}{3}(-1)^n$.