MATH 240: Discrete Structures - Assignment 2

Liam Scalzulli
liam.scalzulli@mail.mcgill.ca

October 12, 2022

Problem 1

(a)

Proof. Suppose $P(x) = 2x^2 - 4x + 3$ where $x \in \mathbb{R}$.

$$\begin{aligned} 2x^2 - 4x + 3 &= 2(x^2 - 2x + 1) + 1 \\ &= 2((x-1)(x-1)) + 1 \\ &= 2(x-1)^2 + 1 \end{aligned}$$

Notice that $x^2 \geq 0 \,\forall\, x \in \mathbb{R}$.

It follows that $2(x-1)^2 + 1 > 0 \,\forall\, x \in \mathbb{R}$.

So $P(x) > 0 \,\forall\, x \in \mathbb{R}$. $\square$

(b)

Proof. Suppose $x \in \mathbb{Z}$ is odd.

$$\begin{aligned} x &= 2k + 1,\ k \in Z \\ &= 1 \times (2k+1) \\ &= (k + 1 + k)(k + 1 - k) \end{aligned}$$

Let $a = k + 1$, $b = k$.

Then $2k + 1 = (a + b)(a - b) = a^2 - b^2$. $\square$

Problem 2

(a)

Proof. Suppose $x \in \mathbb{Z}$ is even.

Then we can write $x = 2k$, $k \in \mathbb{Z}$.

$$\begin{aligned} x^3 - 2x + 3 &= (2k)^3 - 2(2k) + 3 \\ &= 8k^3 - 4k + 3 \\ &= 8k^3 - 4k + 2 + 1 \\ &= 2(4k^3 - 2k + 1) + 1 \end{aligned}$$

Let $l = 4k^3 - 2k + 1$, $l \in \mathbb{Z}$.

Then $x^3 - 2x + 3 = 2l + 1$, and is odd by definition. $\square$

(b)

Proof. Suppose $log_2 5$ is rational.

Then we can write $log_2 5 = \frac{m}{n}$, where m, $n \in \mathbb{Z}$.

$$\begin{aligned} 5 &= 2^{(\frac{m}{n})} \\ 5^n &= 2^m \end{aligned}$$

Since 5 is odd and all 5^n, $n \in \mathbb{N}$ are odd by induction and 2 is even and all 2^m, $m \in \mathbb{N}$ are even by induction, an odd number can never equal an even number, we've reached a contradiction, hence $log_2 5$ is irrational. $\square$

Problem 3

(a)

Proof. Let $P(n) = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ... + \frac{1}{(2n-1)\cdot(2n+1)} = \frac{n}{2n+1}$.

Base case: $n = 1$

$$\frac{1}{(1\cdot 3)} = \frac{1}{3}$$

$$\frac{n}{2n+1} = \frac{1}{2(1)+1} = \frac{1}{3}$$

Induction step: Assume $P(n)$ holds for some $n \leq k$.

$$\text{Then } \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + ... + \frac{1}{(2k-1)\cdot(2k+1)} = \frac{k}{2k+1}$$

$$\begin{aligned}
\frac{k}{2k+1} + \frac{1}{(2(k+1)-1)(2(k+1)+1)} &= \frac{k(2k+3)+1}{(2k+1)(2k+3)} && \text{[Common Denominator]} \\
&= \frac{2k^2+3+1}{(2k+1)(2k+3)} && \text{[Expand } k(2k+3)+1\text{]} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} && \text{[Factor } 2k^2+3+1\text{]} \\
&= \frac{k+1}{2k+3}
\end{aligned}$$

Notice that $\frac{k+1}{2k+3}$ is just $P(k+1)$, $\therefore$ we've shown that $k \implies k+1$ by mathematical induction. $\square$

(b)

Proof. Let $P(n) = \overline{A_1 \cup A_2 \cup ... \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_n}$.

Base case: $n = 2$

$$\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$$

Which is true by DeMorgan's Law.

Induction step: Assume $P(n)$ holds for some $n \leq k$.

$$\text{Then } \overline{A_1 \cup A_2 \cup ... \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_k}$$

$$\begin{aligned}
\overline{(A_1 \cup A_2 \cup ... \cup A_k) \cup A_{k+1}} &= \overline{(A_1 \cup A_2 \cup ... \cup A_k)} \cap \overline{A_{k+1}} && \text{[De Morgan]} \\
&= (\overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_k}) \cap \overline{A_{k+1}} && \text{[Induction Hypothesis]} \\
&= \overline{A_1} \cap \overline{A_2} \cap ... \cap \overline{A_k} \cap \overline{A_{k+1}} && \text{[Associativity]}
\end{aligned}$$

Therefore we've shown that we can reach $k+1$ from k by mathematical induction. $\square$

Problem 4

Proof. Let a, b, $c \in \mathbb{N}$.

Suppose $a \mid (b+c)$ and $gcd(b,c) = 1$.

We can then write $ma = b + c$, $m \in \mathbb{N}$, and by Bézout's Identity we have $1 = bu + cv$ where $b, c \in \mathbb{N}$.

Then:

$$b = ma - c$$

$$c = ma - b$$

Case 1: Showing $gcd(a,b) = 1$

$$\begin{aligned} 1 &= bu + cv \\ 1 &= bu + (ma - b)v \\ 1 &= mav - bv + bu \\ 1 &= a(mv) + b(u - v) \end{aligned}$$

Since mv, $u - v \in \mathbb{N}$, by Bézout's Identity $gcd(a,b) = 1$.

Case 2: Showing $gcd(a,c) = 1$

$$\begin{aligned} 1 &= bu + cv \\ 1 &= (ma - c)u + cv \\ 1 &= uma - cu + cv \\ 1 &= a(um) + c(v - u) \end{aligned}$$

Since um, $v - u \in \mathbb{N}$, by Bézout's Identity $gcd(a,c) = 1$. □