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Homework 11

Solutions

5.1.2 A is a 3×3 matrix, $\det(A) = -1$

$$\det\left(\frac{1}{2}A\right) = \left(\frac{1}{2}\right)^3 \det(A) = \frac{1}{8}(-1) = -\frac{1}{8}$$

$$\det(-A) = (-1)^3 \det(A) = (-1)(-1) = 1$$

$$\det(A^2) = \det(A) \det(A) = (-1)(-1) = 1$$

$$\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{-1} = -1$$

5.1.7 Rotation: $\begin{vmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{vmatrix} = \cos^2 \theta - (-\sin \theta)(\sin \theta)$

$$= \cos^2 \theta + \sin^2 \theta = 1$$

Reflection: $\begin{vmatrix} 1-2\cos^2 \theta & -2\sin \theta \cos \theta \\ -2\cos \theta \sin \theta & 1-2\sin^2 \theta \end{vmatrix} =$

$$(1-2\cos^2 \theta)(1-2\sin^2 \theta) - (-2\cos \theta \sin \theta)^2 =$$

$$1-2\cos^2 \theta - 2\sin^2 \theta + 4\cos^2 \theta \sin^2 \theta - 4\cos^2 \theta \sin^2 \theta =$$

$$1-2(\cos^2 \theta + \sin^2 \theta) = 1-2 = -1$$

5.1.13

$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{vmatrix}$	Row 3 - Row 1	$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{vmatrix}$	Row 3 - Row 2	$\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix}$	$1(1)(1) = 1$
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$$\begin{array}{c|c|c} \begin{array}{ccc|l} 1 & 2 & 3 & \text{Row 2} - 2\text{Row 1} \\ 2 & 2 & 3 & \\ 3 & 3 & 3 & \text{Row 3} - 3\text{Row 1} \end{array} & \begin{array}{ccc|l} 1 & 2 & 3 & \\ 0 & -2 & -3 & \\ 0 & -3 & -6 & \end{array} & \begin{array}{ccc|l} 1 & 2 & 3 & \\ 0 & -2 & -3 & \\ 0 & 0 & -3/2 & \end{array} \end{array} \quad (2)$$

$$= 1(-2)\left(-\frac{3}{2}\right) = 3$$

$$\underline{5.1.18} \quad \begin{array}{c|c|c} \begin{array}{ccc|l} 1 & a & a^2 & \text{Row 2} - \text{Row 1} \\ 1 & b & b^2 & \\ 1 & c & c^2 & \text{Row 3} - \text{Row 1} \end{array} & \begin{array}{ccc|l} 1 & a & a^2 & \\ 0 & b-a & b^2-a^2 & \\ 0 & c-a & c^2-a^2 & \end{array} & \begin{array}{l} \text{Row 3} - \frac{c-a}{b-a} \text{Row 2} \\ = \\ \uparrow \\ \text{If } b \neq a \end{array} \end{array}$$

$$\begin{array}{c|c} \begin{array}{ccc|l} 1 & a & a^2 & \\ 0 & b-a & b^2-a^2 & \\ 0 & 0 & c^2 - \frac{(c-a)(b^2-a^2)}{b-a} & \end{array} & = 1(b-a) \left(c^2 - b^2 - \frac{(c-a)(b^2-a^2)}{b-a} \right) \end{array}$$

$$= (c^2 - \frac{a^2}{b-a})(b-a) - (c-a)(b^2-a^2) = (b-a)(c^2 - b^2 - (c-a)(b+a))$$

$$= (b-a)(c^2 - \frac{a^2}{b-a} - bc + ab - ac + a^2) = (b-a)(c(c-a) - b(c-a))$$

$$= (b-a)(c-a)(c-b)$$

$$\underline{5.1.30} \quad \frac{\partial f}{\partial a} = \frac{\partial}{\partial a} \ln(ad-bc) = \frac{1}{ad-bc} \cdot d = \frac{d}{ad-bc}$$

$$\frac{\partial f}{\partial b} = \frac{1}{ad-bc} (-c) \quad \frac{\partial f}{\partial c} = \frac{1}{ad-bc} (-b), \quad \frac{\partial f}{\partial d} = \frac{a}{ad-bc}$$

$$\text{So } \begin{bmatrix} \partial f / \partial a & \partial f / \partial c \\ \partial f / \partial b & \partial f / \partial d \end{bmatrix} = \begin{bmatrix} d/(ad-bc) & -b/(ad-bc) \\ -c/(ad-bc) & a/(ad-bc) \end{bmatrix} =$$

$$\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}.$$



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5.2.1

$$\det(A) = 1 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 2 \\ 3 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix}$$

$$= 1(1)(1) - 1(2)(2) - 2(3)(1) + 2(2)(3) + 3(3)(2) - 3(1)(3)$$

$$= 1 - 4 - 6 + 12 + 18 - 9 = 12 \neq 0 \rightarrow \text{independent rows}$$

$$\det(B) = 1 \begin{vmatrix} 4 & 4 \\ 6 & 7 \end{vmatrix} - 2 \begin{vmatrix} 4 & 4 \\ 5 & 7 \end{vmatrix} + 3 \begin{vmatrix} 4 & 4 \\ 5 & 6 \end{vmatrix}$$

$$= 1(4)(7) - 1(4)(6) - 2(4)(7) + 2(4)(5) + 3(4)(6) - 3(4)(5)$$

$$= 28 - 24 - 56 + 40 + 72 - 60 = 0 \rightarrow \text{dependent rows}$$

$$\det(C) = 1 \begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} - 1 \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= 1(1)(0) - 1(0)(0) - 1(1)(0) + 1(0)(1) + 1(1)(0) - 1(1)(1)$$

$$= 0 - 0 - 0 + 0 + 0 - 1 = -1 \rightarrow \text{independent rows}$$

5.2.15

(a)

$$E_n = \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & & & \\ 0 & 1 & 1 & & & \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix}$$

$$\text{1st row} \rightarrow 1 \begin{vmatrix} 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & \cdots & 1 \end{vmatrix}$$

$$-1 \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & & & \\ 0 & 1 & 1 & & & \\ \vdots & & & \ddots & & \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{vmatrix}$$

$$= E_{n-1} - 1(1) E_{n-2}$$

1st column

E_{n-1}

$E_{n-1} - E_{n-2}$

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$$(b) E_1 = 1, E_2 = 0, E_3 = E_2 - E_1 = -1, E_4 = E_3 - E_2 = -1 \quad (4)$$

$$E_5 = E_4 - E_3 = 0, E_6 = E_5 - E_4 = 1, E_7 = E_6 - E_5 = 1$$

$$E_8 = E_7 - E_6 = 0$$

~~NOTA~~ (c) Notice $E_7 = E_1$ and $E_8 = E_2$

So the values will repeat in cycles of 6:

$$1, 0, -1, -1, 0, 1,$$

$$\leftarrow E_n = E_{n-6} \text{ for any } n.$$

$$1, 0, -1, -1, 0, 1, \dots$$

$$\text{So } E_{100} = E_{94} = E_{88} = E_{82} = \dots = E_{10} = E_4 = -1$$

5.2.19 (a) Expand across 4th row:

$$V_4 = -1 \begin{vmatrix} a^2 & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + x \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - x^2 \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} + x^3 \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

The 3×3 determinants are just numbers, so V_4 is a V_3 cubic polynomial in x .

(b) $V_4 = 0$ when $x = a, b, c$ because then two rows of the matrix will be the same, so determinant will be 0.

$$\text{So } r_1 = a, r_2 = b, r_3 = c.$$

(c) Any cubic polynomial can be factored as $A(x-r_1)(x-r_2)(x-r_3)$

where A is the coefficient of x^3 and r_1, r_2, r_3 are the three roots (which could be complex numbers in general). For V_4 ,

$$V_4 = V_3(x-a)(x-b)(x-c) \text{ from parts (a) and (b)}$$



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(d) Using formula for V_3 from 5.1.18,

$$V_4 = (b-a)(c-a)(c-b)(x-a)(x-b)(x-c)$$

5.2.3 $\det(P) = 0 \left| \dots \right| - 0 \left| \dots \right| + 0 \left| \dots \right| - 1 \left| \begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix} \right|$

doesn't matter I

$$= -1(1) = -1$$

For "big formula," $\det P = \text{sum of 24 terms } \pm p_{1,i_1} p_{2,i_2} p_{3,i_3} p_{4,i_4}$

The only non-zero term chooses

col 4 for Row 1, col 1 for Row 2, col 2 for Row 3, and col 3 for Row 4.

different column for each row

Then $\det P = \pm (1)(1)(1)(1) + \text{a bunch of 0's}$

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 $(-1)^{\# \text{ row switches to reorder } 4,1,2,3 \text{ to } 1,2,3,4}$

we can do $4,1,2,3 \rightarrow 1,4,2,3 \rightarrow 1,2,4,3 \rightarrow 1,2,3,4$
 (3 row switches.

So $\det P = (-1)^3 (1)(1)(1)(1) = -1$, same as before.

Then $\det(P^2) = (\det P)(\det P) = (-1)(-1) = +1$.

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5,2,3,4 $A = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$

(6)

(a) ~~XXX~~ The last 3 rows are all contained in $\text{span}([0,0,0,1,0], [0,0,0,0,1])$, which is 2 dimensional. So one of the last three rows has to be a linear combination of the other 2.

(b) Each of the 120 terms in the big formula looks like $\pm a_{1,i_1} a_{2,i_2} a_{3,i_3} a_{4,i_4} a_{5,i_5}$, where columns i_1, i_2, i_3, i_4, i_5 are all different. ~~So in each term~~ For this to be non-zero, we'd have to choose either Col 4 or 5 for Row 3, and then the other one ~~of~~ Col 4 or 5 for Row 4. But then we have to choose one of Cols 1,2,3 for Row 5, so $a_{5,i_5} = 0 \rightarrow$ the whole term is 0.

Graded Problem 1.

$$\begin{array}{l|l} \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{matrix} & \begin{matrix} \text{Row 2-Row 1} \\ \text{Row 3-Row 1} \\ \text{Row 4-Row 1} \\ \text{Row 5-Row 1} \end{matrix} \end{array} \quad \begin{array}{l|l} \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 2 & 5 & 9 & 14 \\ 0 & 3 & 9 & 19 & 34 \\ 0 & 4 & 14 & 34 & 69 \end{matrix} & \end{array}$$

$$\begin{array}{l|l} \begin{matrix} \text{Row 3-2Row 2} \\ \text{Row 4-3Row 2} \\ \text{Row 5-4Row 2} \end{matrix} & \begin{matrix} \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 3 & 10 & 22 \\ 0 & 0 & 6 & 22 & 53 \end{matrix} \\ \text{Row 4-3Row 3} \\ \text{Row 5-6Row 3} \end{matrix} \end{array} \quad \begin{array}{l|l} \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 4 & 17 \end{matrix} & \end{array}$$

$$\begin{array}{l|l} \begin{matrix} \text{Row 5-4Row 4} \\ \text{Row 5-4Row 4} \end{matrix} & \begin{matrix} \begin{matrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 & 6 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{matrix} \\ \end{matrix} \end{array} = 1(1)(1)(1)(1) = 5$$



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Graded Problem 2

$$\left| \begin{array}{ccccc} 1 & -1 & 1 & -1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & 0 & 0 & 0 \\ 4 & 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 2 & 0 \end{array} \right| \xrightarrow{\text{Row 3}} = -(-1) \left| \begin{array}{cccc} 1 & 1 & -1 & 1 \\ 1 & 3 & 4 & 5 \\ 4 & -2 & 0 & 1 \\ 0 & 0 & 2 & 0 \end{array} \right|$$

$$\xrightarrow{\text{Row 4}} = 1(-2) \left| \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 4 & -2 & 1 \end{array} \right| \xrightarrow{\text{Row 1}} = -2 \left(1 \left| \begin{array}{cc} 3 & 5 \\ -2 & 1 \end{array} \right| - 1 \left| \begin{array}{cc} 1 & 5 \\ 4 & 1 \end{array} \right| + 1 \left| \begin{array}{cc} 1 & 3 \\ 4 & -2 \end{array} \right| \right)$$

$$= -2(13 + 19 - 14) = -36$$