Calculus A(1): Homework 6

The total is 100 points. When we refer to a paragraph number (e.g. §4.6), we refer to the PDF of the textbook *Thomas Calculus* that you can find on the weblearn. The bonus exercises are optional and more difficult. We may (or may not) decide to grade one of your bonus exercises and use it to replace one assigned exercise (if it improves your total grade).

Routine exercises (do not hand-in)

1. §4.2, Exercises 3, 8, 9, 23, 25, 51, 52
2. §4.3, Exercises 3, 8, 22, 36, 37, 43
3. §4.4, Exercises 3, 34, 38, 47, 50, 64, 65, 67, 70, 81, 82
4. §4.6, Exercises 2, 6, 10, 21, 28, 31, 34

Assigned exercises (hand-in)

1. (20pts) Sketch the graph of the following functions. (Including roots of the function, local/global extrema, inflection points, convexity, asymptotes, monotony, and symmetries.)

   a. $f(x) = x^3 + 3x^2 - 9x + 1.$ **b.** $f(x) = \dfrac{2x}{x^2+1}$

2. (20pts) Using L'Hopital's rule, find the following limits.

   a. $\displaystyle\lim_{x\to 1} \frac{x^3 - 3x + 2}{(x^2-1)^2}$ **b.** $\displaystyle\lim_{x\to 0} \frac{\sqrt{a(a+x)} - a}{x}$, $a > 0$

3. (20pts) Show that for all $x > 0$ we have the following inequality:

   $$\sqrt{x} - \frac{1}{2}x - \frac{1}{2} \leq 0$$

   and show that the equality holds if and only if $x = 1$.

4. (20pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Assume that f is convex (i.e. f' is increasing on $\mathbb{R}$). Prove that for any $x_1, x_2, x_3 \in \mathbb{R}$ with $x_1 < x_2 < x_3$, we have $\frac{f(x_3)-f(x_2)}{x_3-x_2} > \frac{f(x_2)-f(x_1)}{x_2-x_1}$.

5. (20pts) Let $f(x)$ be twice-differentiable on $[a, b]$. Assume that $f(a) = f(b) = 0$ and that there exists $c \in (a, b)$ such that $f(c) > 0$. Prove that there exists $\xi \in (a, b)$, $f''(\xi) < 0$.

Bonus exercises (optional)

1. Give an example of functions $f, g : I \to \mathbb{R}$, where I is an open interval contaning 0, and such that

   - f and g are differentiable on I (except possibly at $x = 0$),
   - $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$,
   - For all $x \in I$ and $x \neq 0$, we have $g'(x) \neq 0$,
   - $\lim_{x\to 0} \frac{f(x)}{g(x)}$ exists (in $\mathbb{R}$) but $\lim_{x\to 0} \frac{f'(x)}{g'(x)}$ does not exist.

   Explain why this does not contradict l'Hopital's rule (strong version).