Problem 3.2.1(a) Find R for $A = \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 1 & 2 & 3 & 6 & 9 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{Row 2}-\text{Row 1}} \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{\text{Row 3}-\text{Row 2}} \begin{bmatrix} 1 & 2 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

leading 1's

This is U; lower-left variables are eliminated.

$$\xrightarrow{\text{Row 1}-2\,\text{Row 2}} \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ This is R.

leading 1's in Cols 1 and 3 $\rightarrow$ these are the "pivot columns"
columns 2,4,5 give free variables.

We can easily solve $A\vec{x} = \vec{0}$ using R : $[A \mid \vec{0}] \longrightarrow [R \mid \vec{0}]$

$$\begin{cases} x_1 + 2x_2 = 0 \\ x_3 + 2x_4 + 3x_5 = 0 \end{cases} \xrightarrow[\text{in terms of free variables}]{\text{solve for } x_1, x_3} \begin{aligned} x_1 &= -2x_2 \\ x_3 &= -2x_4 - 3x_5 \end{aligned}$$

from R

All solutions: $$\vec{x} = \begin{bmatrix} -2x_2 \\ x_2 \\ -2x_4 - 3x_5 \\ x_4 \\ x_5 \end{bmatrix} = \underbrace{x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{\text{Vectors in } N(A)\text{: Three free variables, three special solutions}}$$

For an $m \times n$ A, the number of leading 1's in R is important. It is called the rank of the matrix. (Use r for rand of A)
since A has n total columns, the number of free variables is $n - r$ (also the number of special solutions in N(A)).

Interesting question: What matrices have $r = 1$?
Answer: "Outer products" of vectors, $\underset{m \times n}{A} = \underset{m \times 1}{\vec{u}}\, \underset{1 \times n}{\vec{v}^T}$

Example: $A = \begin{bmatrix}1\\-1\\2\end{bmatrix}\begin{bmatrix}3 & 1 & 4\end{bmatrix} = \begin{bmatrix}1(3) & 1(1) & 1(4)\\ -1(3) & -1(1) & -1(4)\\ 2(3) & 2(1) & 2(4)\end{bmatrix}$

Every row is a multiple of the 1st row.

$$A \xrightarrow[\text{Row3}-2\text{Row1}]{\text{Row2+Row1}} \begin{bmatrix}3 & 1 & 4\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \xrightarrow{\frac{1}{3}\text{Row 1}} \begin{bmatrix}1 & 1/3 & 4/3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = R$$

one pivot column, one leading 1

Fun fact: You can write any A as a linear combination of "outer products": $A = \vec{u}_1\vec{v}_1^T + \vec{u}_2\vec{v}_2^T + \cdots + \vec{u}_r\vec{v}_r^T$

The rank r is the minimum number of outer products required to add up to r.

Let's see how to do this using LU decomposition:

$$A = \begin{bmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{bmatrix} \xrightarrow[\text{Row 3}-\boxed{7}\text{Row1}]{\text{Row 2}-\boxed{4}\text{Row 1}} \begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & -6 & -12\end{bmatrix} \xrightarrow{\text{Row 3}-\boxed{2}\text{Row 2}} \begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & 0 & 0\end{bmatrix} = U$$

$$A = \underbrace{\begin{bmatrix}1 & 0 & 0\\ +4 & 1 & 0\\ +7 & 2 & 1\end{bmatrix}}_{L}\begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & 0 & 0\end{bmatrix}$$

U has rank $=2 \rightarrow A$ has rank $2 \rightarrow A \stackrel{?}{=} \vec{u}_1\vec{v}_1^T + \vec{u}_2\vec{v}_2^T$?

In fact, can write $U = \begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0 & -3 & -6\end{bmatrix}$ (no need for $\begin{bmatrix}0\\0\\1\end{bmatrix}$(something) since 3rd row is all 0)

$$\rightarrow A = LU = \begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 7 & 2 & 1\end{bmatrix}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0 & -3 & -6\end{bmatrix}\right)$$

$$= \left(\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 7 & 2 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\right)\begin{bmatrix}1 & 2 & 3\end{bmatrix} + \left(\begin{bmatrix}1 & 0 & 0\\ 4 & 1 & 0\\ 7 & 2 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}\right)\begin{bmatrix}0 & -3 & 6\end{bmatrix}$$

$$= \begin{bmatrix}1\\4\\7\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix} + \begin{bmatrix}0\\1\\2\end{bmatrix}\begin{bmatrix}0 & -3 & -6\end{bmatrix}$$

Why care about this? It might take less computer memory to store $\{\underbrace{\vec{u}_1}_{m \text{ entries}}, \underbrace{\vec{v}_1}_{n \text{ entries}}, \vec{u}_2, \vec{v}_2, \ldots, \vec{u}_r, \vec{v}_r\}$ than A itself

$$\underbrace{\{\vec{u}_1, \vec{v}_1, \ldots, \vec{u}_r, \vec{v}_r\}}_{r(m+n) \text{ entries}} \quad \text{vs.} \quad \underbrace{A}_{mn \text{ entries}}$$

$r(m+n)$ entries ← smaller if r is small and m, n are big

Example: Store a photograph as a matrix (1 pixel = 1 matrix entry, different numerical values for matrix entries represent different colors)

red	red	red	red	red	red
red	red	red	red	red	red
red	red	blue	blue	red	red
red	red	blue	blue	red	red
red	red	red	red	red	red
red	red	red	red	red	red

$$\rightsquigarrow \quad A = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}}_{36 \text{ matrix entries}} \xrightarrow[\substack{\text{Row 3} - \text{Row 1} \\ \text{Row 4} - \text{Row 1} \\ \text{Row 5} - \text{Row 1} \\ \text{Row 6} - \text{Row 1}}]{\text{Row 2} - \text{Row 1}}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-\text{Row 3}]{\text{Row 4}} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{U} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$A = LU = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

← 24 total vector entries

This idea of writing $A = \vec{u}_1 \vec{v}_1^{T} + \cdots + \vec{u}_r \vec{v}_r^{T}$ returns in Chapter 7 on <u>singular value decomposition</u>.

Solving Linear Equations $A\vec{x} = \vec{b}$

$$[A \mid \vec{b}\,] \xrightarrow{\text{elimination}} [R \mid \vec{d}\,]$$

"reduced row echelon" form of R

Possibilities for R:

1. R has a leading 1 in every row and every column.
2. R has a leading 1 in every row, but not every column.
3. R has a leading 1 in every column, but not every row.
4. R has leading 1's missing from both rows and columns.

Saw examples of case 4 last time:

Because some rows don't have leading 1's: For most $\vec{b}$, $A\vec{x}=\vec{b}$ has no solution.

Because some columns don't have leading 1's (there are free variables): For some $\vec{b}$, $A\vec{x}=\vec{b}$ has infinitely many solutions. (For example, a plane of solutions if there are two free variables, i.e., two columns don't have leading 1's.)

Case 1: If R has a leading 1 in every row and column, it has n rows and n columns, and $R = n \times n$ identity matrix. If $R = I$, then every $A\vec{x}=\vec{b}$ has a unique solution, and A is invertible.

Example: Find the unique solution to
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

in terms of $\vec{b}$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 1 & 2 & 3 & b_2 \\ 1 & 4 & 9 & b_3 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 2} - \text{Row 1}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 3 & 8 & b_3 - b_1 \end{array}\right] \xrightarrow{\text{Row 3} - 3\,\text{Row 2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & b_1 \\ 0 & 1 & 2 & -b_1 + b_2 \\ 0 & 0 & 2 & 2b_1 - 3b_2 + b_3 \end{array}\right]$$

$$\xrightarrow[\text{Row 2} - \text{Row 3}]{\text{Row 1} - \frac{1}{2}\text{Row 3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & \frac{3}{2}b_2 - \frac{1}{2}b_3 \\ 0 & 1 & 0 & -3b_1 + 4b_2 - b_3 \\ 0 & 0 & 2 & 2b_1 - 3b_2 + b_3 \end{array}\right] \xrightarrow[\frac{1}{2}\text{Row 3}]{\text{Row 1} - \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3b_1 - \frac{5}{2}b_2 + \frac{1}{2}b_3 \\ 0 & 1 & 0 & -3b_1 + 4b_2 - b_3 \\ 0 & 0 & 1 & b_1 - \frac{3}{2}b_2 + \frac{1}{2}b_3 \end{array}\right]$$

$R = I$

Solution to $A\vec{x}=\vec{b}$ is:

$$\vec{x}=\begin{bmatrix} 3b_1-\frac{5}{2}b_2+\frac{1}{2}b_3 \\ -3b_1+4b_2-b_3 \\ b_1-\frac{3}{2}b_2+\frac{1}{2}b_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -\frac{3}{2} & 1/2 \end{bmatrix}}_{\text{This is } A^{-1}.}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

only happens if $n>m$ (# columns > # rows)

Case 2: R has leading 1 in every row $\longrightarrow$ no row of 0's, so solutions to $A\vec{x}=\vec{b}$ always exist for all $\vec{b}$.

R has no leading 1 in some columns $\longrightarrow$ free variables, so $N(A)$ is bigger than $\{\vec{0}\}$ $\longrightarrow$ every system has infinitely many solutions.

Example Find all solutions to $\begin{bmatrix} -1 & 1 & -1 & 1 \\ 1 & -2 & 3 & -4 \\ -1 & 3 & -6 & 10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

"wide" matrix

$$\left[\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 1 \\ 1 & -2 & 3 & -4 & 1 \\ -1 & 3 & -6 & 10 & 1 \end{array}\right] \xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}+\text{Row 1}} \left[\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & -3 & 2 \\ 0 & 2 & -5 & 9 & 0 \end{array}\right] \xrightarrow{\text{Row 3}+2\text{Row 2}} \left[\begin{array}{cccc|c} -1 & 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & -3 & 2 \\ 0 & 0 & -1 & 3 & 4 \end{array}\right]$$

$$\xrightarrow[\text{Row 2}+2\text{Row 3}]{\text{Row 1}-\text{Row 3}} \left[\begin{array}{cccc|c} -1 & 1 & 0 & -2 & -3 \\ 0 & -1 & 0 & 3 & 10 \\ 0 & 0 & -1 & 3 & 4 \end{array}\right] \xrightarrow{\text{Row 1}+\text{Row 2}} \left[\begin{array}{cccc|c} -1 & 0 & 0 & 1 & 7 \\ 0 & -1 & 0 & 3 & 10 \\ 0 & 0 & -1 & 3 & 4 \end{array}\right] \xrightarrow[\substack{-\text{Row 2} \\ -\text{Row 3}}]{-\text{Row 1}}$$

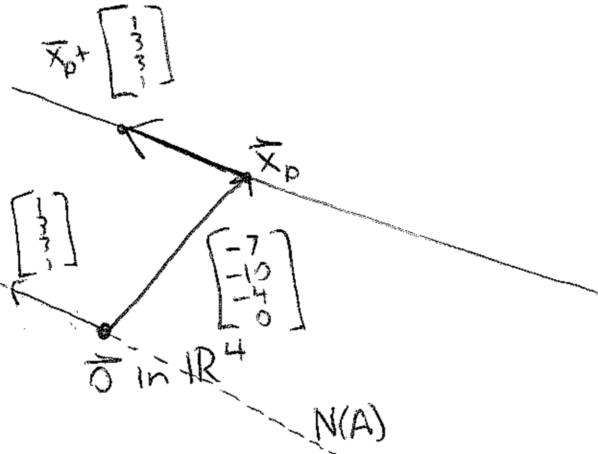
$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -7 \\ 0 & 1 & 0 & -3 & -10 \\ 0 & 0 & 1 & -3 & -4 \end{array}\right] \quad R\vec{x}=\vec{d} \qquad \begin{aligned} x_1-x_4&=-7 \\ x_2-3x_4&=-10 \\ x_3-3x_4&=-4 \end{aligned} \longrightarrow \begin{aligned} x_1&=-7+x_4 \\ x_2&=-10+3x_4 \\ x_3&=-4+3x_4 \\ x_4&=\text{anything (free)} \end{aligned}$$

Pivot variables x_1, x_2, x_3

free variable x_4

All solutions look like: $\vec{x}=\begin{bmatrix} -7+x_4 \\ -10+3x_4 \\ -4+3x_4 \\ x_4 \end{bmatrix}=\begin{bmatrix} -7 \\ -10 \\ -4 \\ 0 \end{bmatrix}+\underbrace{x_4\begin{bmatrix} 1 \\ 3 \\ 3 \\ 1 \end{bmatrix}}_{\substack{\text{null space vectors} \\ (\text{line in } \mathbb{R}^4)}}$

one particular solution ($x_4=0$)



line of solutions, parallel to $N(A)$

Case 3: R has leading 1 in each column $\longrightarrow$ no free variables
$\longrightarrow N(A)=\{\vec{0}\}$ and $A\vec{x}=\vec{b}$ has at most one solution.

Only happens if $m>n$

Some rows missing leading 1's $\longrightarrow$ row of 0's in R
$\longrightarrow$ most $A\vec{x}=\vec{b}$ have no solutions (for most $\vec{b}$).

Example "skinny" matrix $\begin{bmatrix}2&1\\3&1\\4&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix} \longrightarrow \left[\begin{array}{cc|c}2&1&b_1\\3&1&b_2\\4&3&b_3\end{array}\right] \xrightarrow[\text{Row 3}-2\text{Row 1}]{\text{Row 2}-\frac{3}{2}\text{Row 1}}$

$$\left[\begin{array}{cc|c}2&1&b_1\\0&-\frac{1}{2}&-\frac{3}{2}b_1+b_2\\0&1&-2b_1+b_3\end{array}\right] \xrightarrow[-2\text{Row 2}]{\frac{1}{2}\text{Row 1}} \left[\begin{array}{cc|c}1&\frac{1}{2}&\frac{1}{2}b_1\\0&1&3b_1-2b_2\\0&1&-2b_1+b_3\end{array}\right] \xrightarrow[\text{Row 3}-\text{Row 2}]{\text{Row 1}-\frac{1}{2}\text{Row 2}} \underbrace{\left[\begin{array}{cc|c}1&0&-b_1+b_2\\0&1&3b_1-2b_2\\0&0&-5b_1+2b_2+b_3\end{array}\right]}_{R}$$

both columns are pivot

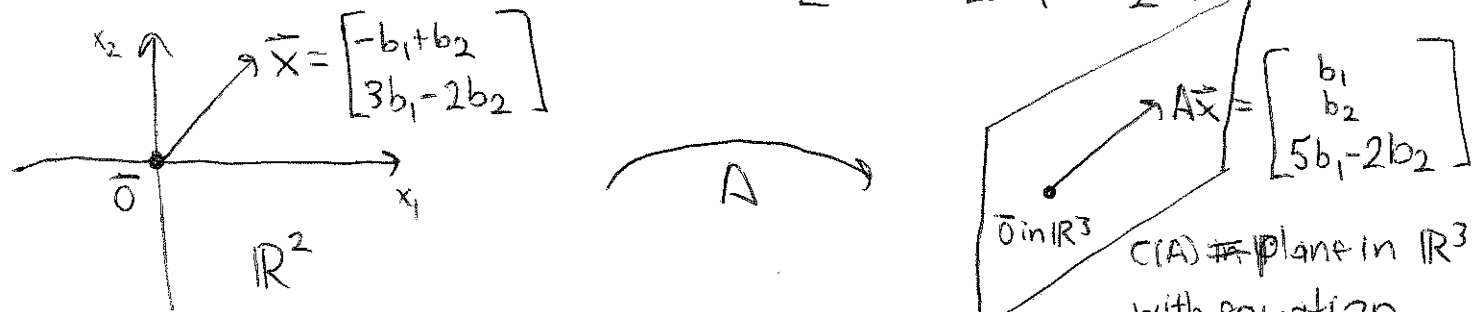
System $R\vec{x}=\vec{d}$: $\begin{cases}x_1=-b_1+b_2\\x_2=3b_1-2b_2\\0=-5b_1+2b_2+b_3\end{cases}$

No solution if $-5b_1+2b_2+b_3\neq 0$.

One solution if $-5b_1+2b_2+b_3=0$

It's $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}-b_1+b_2\\3b_1-2b_2\end{bmatrix}$

How to visualize this:



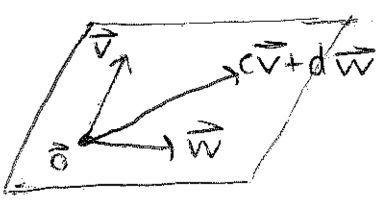
$C(A)$ ~~=~~ plane in $\mathbb{R}^3$ with equation $-5b_1+2b_2+b_3=0$

Section 3.4 Linear Independence, Basis, Dimension

Note: These are maybe the most important ideas in linear algebra, but warning: they are a little abstract.

Motivation: What do we mean by dimension?

Here's a plane in $\mathbb{R}^3$:



we think it is 2-dimensional, even though it is living in $\mathbb{R}^3$.

Why is dimension = 2 for a plane?

Answer attempt: Every vector in the plane is a linear combination of 2 vectors, $\vec{v}$ and $\vec{w}$

But: Every vector in plane is also a linear combination of 3 vectors $\vec{v}, \vec{w}$, and $\vec{v}-\vec{w}$, because:

$$c\vec{v}+d\vec{w} = c\vec{v}+d\vec{w}+0(\vec{v}-\vec{w}), \text{ or } c\vec{v}+d\vec{w} = (c-1)\vec{v}+(d+1)\vec{w} + 1(\vec{v}-\vec{w}), \dots$$

So we should say that the dimension is the minimum number of vectors required to span the plane, which is 2, not 3.

Question: How can we tell if a spanning set is minimal.

Every vector = lin. comb. of these ones

Answer: The vectors in the spanning set should be independent: None of them is a linear combination of the others.

In our plane: $\vec{v} \neq c\vec{w}, \vec{w} \neq d\vec{v} \rightarrow \{\vec{v}, \vec{w}\}$ is an independent set of vectors

But: $\vec{v}-\vec{w} = 1\vec{v}+(-1)\vec{w}$

Also: $\vec{v} = 1\vec{w}+1(\vec{v}-\vec{w})$

$\vec{w} = 1\vec{v}+(-1)(\vec{v}-\vec{w})$

$\longrightarrow \{\vec{v}, \vec{w}, \vec{v}-\vec{w}\}$ is a dependent set of vectors

Definition: A set of vectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ in $\mathbb{R}^m$ is dependent if one of the vectors is a linear combination of the others. If none is a linear combination of the others, the set of vectors is independent.

Quick example: Is $\{\vec{v}, \vec{w}, \vec{0}\}$ dependent?

Yes! For example, $\vec{0} = 0\vec{v}+0\vec{w}$.

Another formulation: $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ dependent means something like

$\vec{v}_1 = c_2\vec{v}_2 + \dots + c_m\vec{v}_m$

Or: $1\vec{v}_1 + (-c_2)\vec{v}_2 + \dots + (-c_m)\vec{v}_m = \vec{0}$

Definitely not $\vec{0}$ — Could be 0's.

So if $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is dependent, then there is a non-zero linear combination adding up to $\vec{0}$:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m = \vec{0}$$

some of these could be 0, but not all are 0

Turn the logic around: If the only way to get $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m = \vec{0}$ is to set $x_1 = x_2 = \dots = x_m = 0$, then the vectors must be independent:

$\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ independent $\Longleftrightarrow$ Only solution to $x_1\vec{v}_1 + x_2\vec{v}_2 + \dots + x_m\vec{v}_m = \vec{0}$ is $\vec{0}$

$\Longleftrightarrow$ Only solution to $\begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \vec{0}$ is $\vec{x} = \vec{0}$.

$\Longleftrightarrow$ For $A = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{bmatrix}$, $N(A) = \{\vec{0}\}$

$\Longleftrightarrow$ R for A has no free variables (there's a leading 1 in every column.

means R look like $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ & & \boxed{} & \end{bmatrix}$ could be rows of 0's here if # vectors < # components in each vector

So how to tell if $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ is independent?

(1) Put $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$ as the columns of a matrix A

(2) Do elimination on A to find R.

(3) If R has → free variables → dependent
→ no free variables → independent

Let's do a 3x3 example for both cases.

Check if $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$ is independent:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 1} \leftrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row 3+Row 2}]{\text{Row 3+Row 1, then}} \underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}}_{} \xrightarrow[\text{elimination}]{\text{more}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

can already see: no free variables (vectors are independent)

So the only way to get

$$x_1 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

is to set $x_1 = x_2 = x_3 = 0$.

Check if $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \right\}$ is dependent or not.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \xrightarrow[\text{Row 3}-3\text{Row 1}]{\text{Row 2}-2\text{Row 1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{\text{Row 3}-2\text{Row 2}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

free variable; (vectors are dependent)

$$\xrightarrow[\text{Then: }-\text{Row 2}]{\text{Row 1}+2\text{Row 2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Null space equations: $x_1 = x_3$, $x_2 = -2x_3$, $x_3 = \text{anything}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Special solution tells us how to write the vectors as linear combinations of each other

$$1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-2) \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \qquad \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

Dependent because we can write at least one of the vectors as a linear combination of the other two.