Now we can find $(A^{-1})^T$:

$$AA^{-1} = I \longrightarrow (AA^{-1})^T = I^T \longrightarrow (A^{-1})^T A^T = I$$

$$A^{-1}A = I \longrightarrow (A^{-1}A)^T = I^T \longrightarrow \underbrace{A^T (A^{-1})^T = I}_{(A^{-1})^T \text{ lookslike inverse of } A^T.}$$

So: $\boxed{(A^{-1})^T = (A^T)^{-1}}$ This means that A^T is invertible if A is, and A^T isn't invertible if A isn't.

Example $A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \xrightarrow{\text{Inverse}} A^{-1} = \frac{1}{\underbrace{(1)(4)-(1)(3)}_{1}} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix}$

$\downarrow$ Transpose $\qquad\qquad$ $\downarrow$ Transpose

$A^T = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \xrightarrow{\text{Inverse}} (A^T)^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = (A^{-1})^T$

Last time: Transpose of a matrix $\rightarrow$ switch rows and columns

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

Rules: $(A+B)^T = A^T + B^T$, $(AB)^T = B^T A^T$, $(A^{-1})^T = (A^T)^{-1}$, ...

special case: A is a matrix and B is a column vector:

(matrix)(vector) = column vector $A\vec{v}$

$\downarrow$ Transpose, switch order

$(\text{vector})^T(\text{matrix})^T$ = row vector $\vec{v}^T A^T = (A\vec{v})^T$

We usually prefer to think of matrices multiply column vectors on the left. But we could also have matrices multiply row vectors on the right. If we change column vectors to row vectors, we also have to switch A to A^T.

Example $\begin{cases} x+2y=3 \\ 4+5y=6 \end{cases}$

Two equations with matrix and vectors

$$\overset{A}{\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} x & y \end{bmatrix}\overset{A^T}{\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix}} = \begin{bmatrix} 3 & 6 \end{bmatrix}$$

Why care about matrix transposes? It helps us handle dot products (remember this is useful for relating algebra and geometry.

$\vec{v}^T \vec{w}$ = number (1x1 matrix) ← This number is just the dot product.

row vector $1 \times n$; column vector $n \times 1$

$$\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} \underbrace{v_1w_1 + v_2w_2 + v_3w_3}_{\vec{v} \cdot \vec{w}} \end{bmatrix}$$

So $\vec{v} \cdot \vec{w} = \vec{v}^T \vec{w}$ Dot product is also called the "innerproduct" of $\vec{v}$ and $\vec{w}$ (Transpose symbol ^{T} is on inside of $\vec{v}, \vec{w}$.)

We also have an "outer product" of two vectors:

$\vec{v}\,\vec{w}^T$ = $n \times n$ matrix

$n \times 1$; $1 \times n$

Example
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1(4) & 1(5) & 1(6) \\ 2(4) & 2(5) & 2(6) \\ 3(4) & 3(5) & 3(6) \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

Remember we had a "Key Property" of matrix multiplication (it was $(AB)\vec{x} = A(B\vec{x})$.)

We also have a Key Property of transposes:

$$\vec{v} \cdot (A^T\vec{w}) = (A\vec{v}) \cdot \vec{w}$$

When A "moves over" the dot product, it becomes A^T.

Why? Because $\vec{v}\cdot(A^T\vec{w}) = v^T(A^T\vec{w}) = (\vec{v}^TA^T)\vec{w}$
$= (A\vec{v})^T\vec{w} = (A\vec{v})\cdot\vec{w}$.

Test with $A = \begin{bmatrix}1&3\\1&4\end{bmatrix}$, $\vec{v} = \begin{bmatrix}1\\2\end{bmatrix}$, $\vec{w} = \begin{bmatrix}1\\-1\end{bmatrix}$

$A\vec{v} = \begin{bmatrix}7\\9\end{bmatrix} \longrightarrow (A\vec{v})\cdot\vec{w} = 7(1)+9(-1) = -2$ ✓

$A^T\vec{w} = \begin{bmatrix}1&1\\3&4\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} \longrightarrow \vec{v}\cdot(A^T\vec{w}) = 1(0)+2(-1) = -2$

Sometimes a matrix is the same as its transpose:

An $n\times n$ matrix is called symmetric if $A = A^T$

has to be square; otherwise A $m\times n$ and A^T $n\times m$ have different sizes, so can't be same.

Example $A = \begin{bmatrix}1&4&6\\4&2&5\\6&5&3\end{bmatrix} \xrightarrow[\text{diagonal}]{\text{flip over}} A^T = \begin{bmatrix}1&4&6\\4&2&5\\6&5&3\end{bmatrix} = A$

There are two ways to turn any $m\times n$ matrix into a symmetric one: AA^T and A^TA are both symmetric.

Check: $(AA^T)^T = \underbrace{(A^T)^T}_{\text{This is just } A}A^T = AA^T$, $(A^TA)^T = A^T(A^T)^T = A^TA$.

Example: $A = \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}$ $AA^T = \begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}\begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix} = \begin{bmatrix}14&32\\32&77\end{bmatrix}$

$A^TA = \begin{bmatrix}1&4\\2&5\\3&6\end{bmatrix}\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix} = \begin{bmatrix}17&22&27\\22&29&36\\27&36&45\end{bmatrix}$ Two different symmetric matrices

Note: If $A = \vec{v}$, a column vector, these are just the inner and outer products of $\vec{v}$ with itself.

Permutation Matrices: These are the elimination matrices that do row switches. You get them by switching around rows in the identity matrix. They have exactly one 1 in each row and column.

Example: $\underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_4 \\ x_2 \\ x_1 \end{bmatrix}$

This moves:
- old Row 1 $\rightarrow$ new Row 4
- old Row 2 $\rightarrow$ new Row 3
- old Row 3 $\rightarrow$ new Row 1
- old Row 4 $\rightarrow$ new Row 2

What are PP^T and P^TP? $P^T = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$

$$PP^T = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Same with P^TP. So $P^T = P^{-1}$!

This works for any permutation matrix!

Why? Let's find (i,j)-entry of PP^T:

(i,j)-entry $=$ (ith row of P) $\cdot$ (jth column of P^T)

jth row of P

$=$ (ith row of P) $\cdot$ (jth row of P)

Diagonal entries $i=j$: 1's are in the same position $\rightarrow$ get 1

Off-diagonal, $i \neq j$: 1's are in different positions $\rightarrow$ dot product is 0.

So PP^T has 1's on the diagonal $(i=j)$ and 0's everywhere else $(i \neq j)$ $\rightsquigarrow$ $PP^T = I$. Same for P^TP.

Remember LU decomposition from last time:

$$A \xrightarrow[\text{elimination operations}]{\text{lower triangular}} U \text{ upper triangular} \rightsquigarrow A = LU$$

This doesn't work if we have to do row switches on A, because the elimination operations won't all be lower triangular. Some will be permutation matrices.

What we can do instead:

① Figure out what row switches we'll need to do and put them into a permutation matrix P.

② Reduce PA to upper triangular U. We've already done the row switches, so we'll get $PA = LU$

Or, $A = P^{-1}LU = P^T LU$

"Any $n \times n$ A can be factorized as a permutation matrix times lower triangular time upper triangular"

This form makes sense for solving linear equations, since $A\vec{x} = \vec{b}$ is really the same as $PA\vec{x} = P\vec{b}$

This system just rearranges the order that you write the equations.

Example: $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 5 & 7 \end{bmatrix} \xrightarrow[\text{Row 3} - 2\text{ Row 1}]{\text{Row 2} - \text{Row 1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ Oops! We need to do a row switch: Row 2 $\leftrightarrow$ Row 3

Instead: $PA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \\ 2 & 5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ 1 & 2 & 4 \end{bmatrix} \xrightarrow[\text{Row 3} - \boxed{1}\text{ Row 1}]{\text{Row 2} - \boxed{2}\text{ Row 1}}$

Switches Rows 2 and 3.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

Lower-diagonal entries of L

$$\underset{P}{\begin{bmatrix}1&0&0\\0&0&1\\0&1&0\end{bmatrix}}\underset{A}{\begin{bmatrix}1&2&3\\1&2&4\\2&5&7\end{bmatrix}}=\underset{L}{\begin{bmatrix}1&0&0\\2&1&0\\1&0&1\end{bmatrix}}\underset{U}{\begin{bmatrix}1&2&3\\0&1&1\\0&0&1\end{bmatrix}}$$

So if you want to solve: $\begin{cases}x+2y+3z=1\\x+2y+4z=2\\2x+5y+7z=4\end{cases}$

$A\vec{x} = \vec{b}$

It is best to first rewrite in a more "natural" order: $\begin{cases}x+2y+3z=1\\2x+5y+7z=4\\x+2y+4z=2\end{cases}$ ← of course this is really the same system of equations.

$PA\vec{x} = P\vec{b}$

Solve by LU: $\begin{bmatrix}1&0&0\\2&1&0\\1&0&1\end{bmatrix}\underbrace{\begin{bmatrix}1&2&3\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}}_{\text{call this unknown vector } \vec{y}}=\begin{bmatrix}1\\4\\2\end{bmatrix}$ ← $P\vec{b}$

LU decomposition of PA

$$\begin{bmatrix}1&0&0\\2&1&0\\1&0&1\end{bmatrix}\vec{y}=\begin{bmatrix}1\\4\\2\end{bmatrix} \leadsto \begin{aligned}y_1 &= 1\\ 2y_1+y_2 &= 4 \rightarrow y_2=4-2(1)=2\\ y_1+y_3 &= 2 \rightarrow y_3 = 2-(1)=1\end{aligned}$$

Now solve $U\vec{x}=\vec{y}$: $\begin{bmatrix}1&2&3\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix} \leadsto \begin{aligned}x+2y+3z&=1\\y+z&=2\\z&=1\end{aligned}$

So $\boxed{z=1}$, $y=2-(1)=\boxed{1}$, and $x=1-2(1)-3(1)=\boxed{-4}$

So $\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}-4\\1\\1\end{bmatrix}$ is the solution for the original system $A\vec{x}=\vec{b}$.