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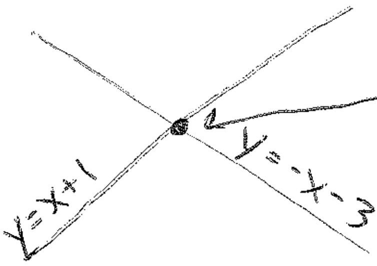
Lecture 1, [redacted]

Linear → Geometry (of lines, planes, higher-dimensional versions)

Algebra → numbers, vectors, variables, ...
Operations: Addition, multiplication, ...
Equations involving these

combine



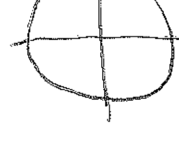
This point is on both lines. It is a solution to a system of linear equations:

$$\begin{cases} -x+y = 1 \\ x+y = -3 \end{cases}$$

Geometrically, these equations are linear because they describe lines in the plane
Algebraically, they are linear because they only involve first powers of the variables x, y and constants (no x^2, $\sin y$, e^{xy}, ...)

Why do we focus on linear algebra?

(For example, here's some non-linear algebra that could be interesting:



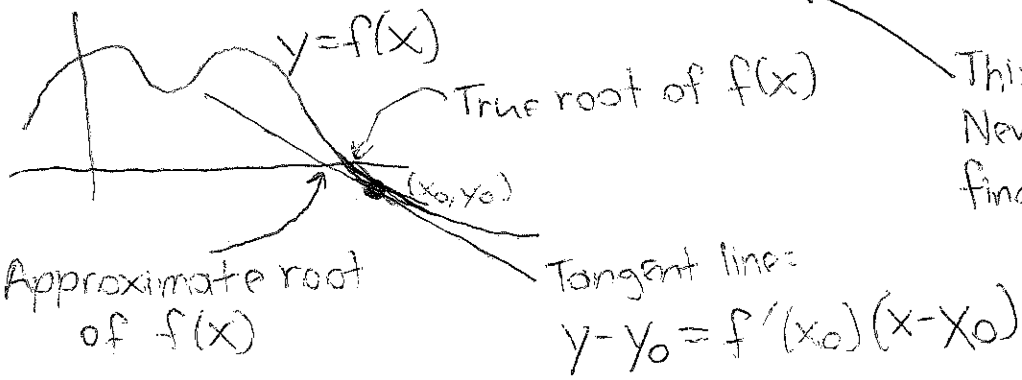
$x^2+y^2=1$ (eqn. of a unit circle

Reason 1: Linear algebra is easier. (It turns out we can always solve systems of linear equations exactly, at least as long as we have a big enough computer.)

Reason 2: If we have a non-linear problem to solve, we can approximate it with a linear problem.

This happens in calculus: Derivatives tell you how to find a linear approximation of a non-linear function.

Ex

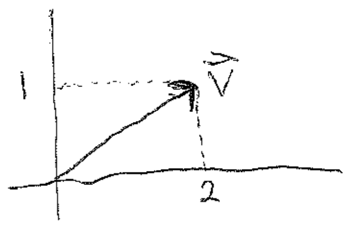


This is the basis of Newton's Method for finding roots of functions, for example.

To begin study of linear algebra, we need to introduce some basic mathematical objects:

Scalars: numbers (real numbers for now, maybe complex numbers later)

Vectors: Geometric picture

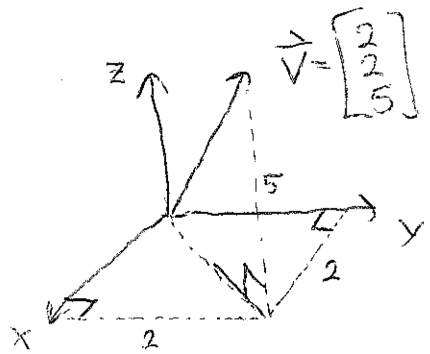


$\longleftrightarrow$

Algebraic picture

$\vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ (column vector notation)

Here's a vector in 3 dimensions:



Typical vector in n dimensions: $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$

(The algebraic pictures allows us to work with high-dimensional data in vector form, even if we can't visualize n dimensions.)

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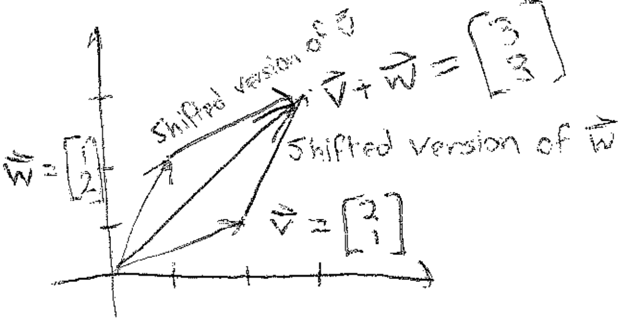
Now what can we do with vectors and scalars?

Vector addition:



$\vec{v}+\vec{w}$ is a <u>diagonal</u> of the parallelogram determined by $\vec{v}$ and $\vec{w}$.

Algebraic picture: $\underbrace{\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1+2\\2+1\end{bmatrix}}=\begin{bmatrix}3\\3\end{bmatrix}$

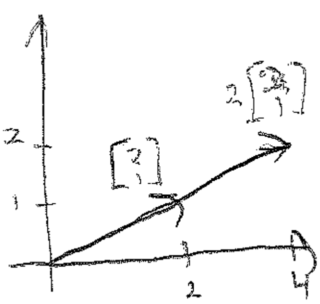
Just add up the individual components.

This works in any dimension: $\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}+\begin{bmatrix}w_1\\w_2\\\vdots\\w_n\end{bmatrix}=\begin{bmatrix}v_1+w_1\\v_2+w_2\\\vdots\\v_n+w_n\end{bmatrix}$

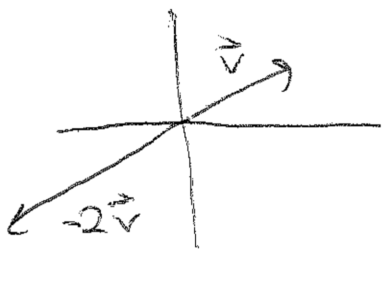
Next: We <u>don't</u> want to multiply two vectors to get another vector (It turns out $\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$ isn't interesting geometrically.)

But we do have:

<u>Scalar multiplication</u>: $2\cdot\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}2\cdot 2\\2\cdot 1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$



The length of $\vec{v}$ is <u>scaled</u> by a factor of 2.



<u>Negative</u> scalars <u>also</u> reverse the direction of the vector.

Now we can combine vector addition and scalar multiplication to get <u>linear combinations</u>:

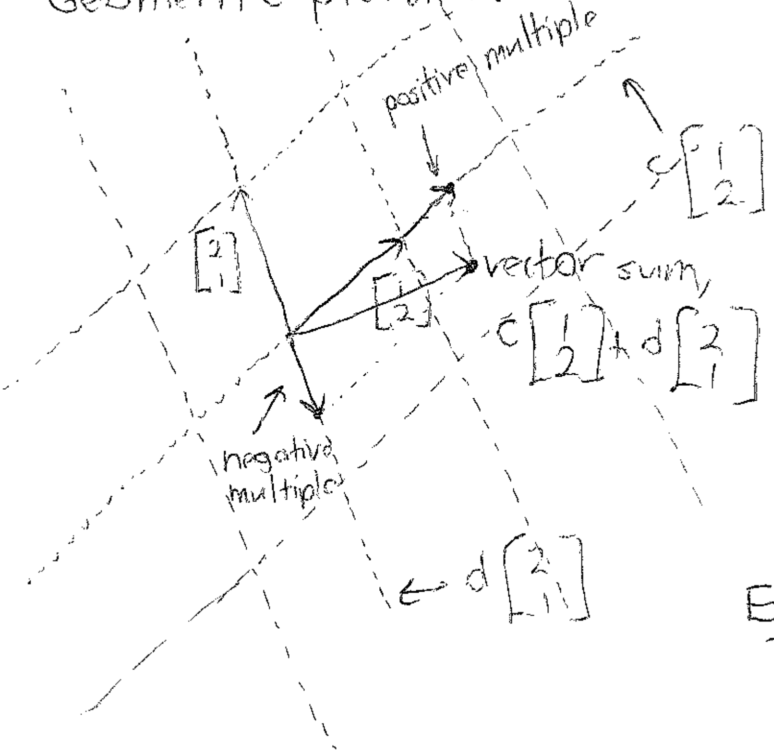
<u>Example</u> What are <u>all</u> the vectors we can create from $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$ using only vector addition and scalar multiplication?

We can get $c\begin{bmatrix}1\\2\end{bmatrix}$ for <u>any</u> scalar c; we can get $d\begin{bmatrix}2\\1\end{bmatrix}$ for any d.

We can then <u>add</u> any two of these scalar multiples:

$$c\begin{bmatrix}1\\2\end{bmatrix}+d\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}c+2d\\2c+d\end{bmatrix}$$ ← These vectors are called <u>linear combinations</u>.

Geometric picture:



It looks like linear combinations of $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$ fill up the whole plane.

But can we see this algebraically?

<u>Example</u> Is $\begin{bmatrix}1\\0\end{bmatrix}$ a linear comb. of $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\end{bmatrix}$?

This means: Can we find c, d so that $c\begin{bmatrix}1\\2\end{bmatrix}+d\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$?

Or, $\begin{bmatrix}c+2d\\2c+d\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix} \longrightarrow \begin{cases}c+2d=1\\2c+d=0\end{cases}$ System of linear equations!

Not hard to solve: $d=-2c \longrightarrow c+2(-2c)=1 \rightarrow -3c=1$

$\longrightarrow c=-\frac{1}{3} \longrightarrow d=-2\left(-\frac{1}{3}\right)=\frac{2}{3}$.

So we get $\begin{bmatrix}1\\0\end{bmatrix} = -\frac{1}{3}\begin{bmatrix}1\\2\end{bmatrix} + \frac{2}{3}\begin{bmatrix}2\\1\end{bmatrix}$, and $\begin{bmatrix}1\\0\end{bmatrix}$ is a lin. comb.

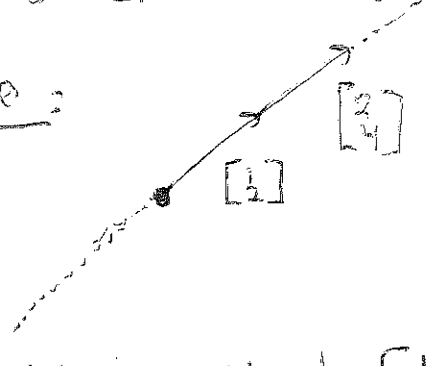
We often want to study the set of all linear combinations of some vectors in n-dimensional space.

In the example, we saw that linear combinations of the 2 vectors $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\1\end{bmatrix}$ form a plane (2-dimensional)

In general, we might expect: all linear combinations of m vectors form an m-dimensional space inside n dimensions.

But this doesn't always work!

Example:



Linear combinations of $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}$ only fill up a 1-dimensional line in the plane (all the multiples of $\begin{bmatrix}1\\2\end{bmatrix}$).

The problem is that $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\4\end{bmatrix}$ aren't "independent."

(one of them is a linear combination of the other.)

· What happens in three dimensions?

Usually, the linear combinations of 3 vectors $\vec{u}, \vec{v}, \vec{w}$ fill up all of 3-dimensional space.