

# LINEAR ALGEBRA — HOMEWORK 6

1 Nov 2023  
Due: 9 Nov 2023

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**Textbook Problems.** These problems will not be graded, but you must submit solutions to receive full credit for the homework.

**Problem 2.7.5.**

- (a) The row vector  $\mathbf{x}^T$  times  $A$  times the column  $\mathbf{y}$  produces what number?

$$\mathbf{x}^T A \mathbf{y} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \underline{\hspace{2cm}}$$

- (b) This is the row  $\mathbf{x}^T A = \underline{\hspace{2cm}}$  times the column  $\mathbf{y} = (0, 1, 0)$ .  
 (c) This is the row  $\mathbf{x}^T = \begin{bmatrix} 0 & 1 \end{bmatrix}$  times the column  $A \mathbf{y} = \underline{\hspace{2cm}}$ .

**Problem 2.7.11.** Find a permutation matrix  $P$  such that  $PA$  is upper triangular. Find permutation matrices  $P_1$  and  $P_2$  such that  $P_1 A P_2$  is lower triangular. Multiplying  $A$  on the *right* by  $P_2$  exchanges the          of  $A$ .

$$A = \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}$$

**Problem 2.7.16.** If  $A = A^T$  and  $B = B^T$ , which of these matrices are certainly symmetric?

- (a)  $A^2 - B^2$   
 (b)  $(A + B)(A - B)$   
 (c)  $ABA$   
 (d)  $ABAB$

**Problem 2.7.22.** Find the  $PA = LU$  factorizations (and check them) for

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

**Problem 2.7.39.** Suppose  $Q^T$  equals  $Q^{-1}$  (transpose equals inverse, so  $Q^T Q = I$ ).

- (a) Show that the columns  $\mathbf{q}_1, \dots, \mathbf{q}_n$  of  $Q$  are unit vectors:  $\|\mathbf{q}_i\|^2 = 1$ .  
 (b) Show that every two different columns of  $Q$  are perpendicular:  $\mathbf{q}_1^T \mathbf{q}_2 = 0$ .  
 (c) Find a  $2 \times 2$  example with first entry  $q_{11} = \cos \theta$  for some angle  $\theta$ .

**Problem 3.1.4.** The matrix  $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$  is a “vector” in the space  $\mathbf{M}$  of all  $2 \times 2$  matrices. Write down the zero vector in this space, the vector  $\frac{1}{2}A$ , and the vector  $-A$ . What matrices are in the smallest subspace containing  $A$ ?

**Problem 3.1.10.** Which of the following subsets of  $\mathbf{R}^3$  are actually subspaces?

- (a) The plane of vectors  $(b_1, b_2, b_3)$  with  $b_1 = b_2$ .
- (b) The plane of vectors with  $b_1 = 1$ .
- (c) The vectors with  $b_1 b_2 b_3 = 0$ .
- (d) All linear combinations of  $\mathbf{v} = (1, 4, 0)$  and  $\mathbf{w} = (2, 2, 2)$ .
- (e) All vectors that satisfy  $b_1 + b_2 + b_3 = 0$ .
- (f) All vectors with  $b_1 \leq b_2 \leq b_3$ .

**Problem 3.1.15.**

- (a) The intersection of two planes through  $(0, 0, 0)$  is probably a \_\_\_\_\_ in  $\mathbf{R}^3$  but it could be a \_\_\_\_\_. It can't be  $\mathbf{Z}$ !
- (b) The intersection of a plane through  $(0, 0, 0)$  with a line through  $(0, 0, 0)$  is probably a \_\_\_\_\_ but it could be a \_\_\_\_\_.
- (c) If  $\mathbf{S}$  and  $\mathbf{T}$  are subspaces of  $\mathbf{R}^5$ , show that their intersection  $\mathbf{S} \cap \mathbf{T}$  is a subspace of  $\mathbf{R}^5$ . Here  $\mathbf{S} \cap \mathbf{T}$  consists of all vectors that lie in *both* subspaces. (Check that  $\mathbf{x} + \mathbf{y}$  and  $c\mathbf{x}$  are in  $\mathbf{S} \cap \mathbf{T}$  if  $\mathbf{x}$  and  $\mathbf{y}$  are in both spaces.)

**Problem 3.1.20.** For which right sides (find condition(s) on  $b_1, b_2, b_3$ ) are these systems solvable?

$$(a) \begin{bmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$(b) \begin{bmatrix} 1 & 4 \\ 2 & 9 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

**Problem 3.1.25.** Suppose  $A\mathbf{x} = \mathbf{b}$  and  $A\mathbf{y} = \mathbf{b}^*$  are both solvable. Then  $A\mathbf{z} = \mathbf{b} + \mathbf{b}^*$  is solvable. What is  $\mathbf{z}$ ? This translates into: If  $\mathbf{b}$  and  $\mathbf{b}^*$  are in the column space  $\mathbf{C}(A)$ , then  $\mathbf{b} + \mathbf{b}^*$  is in  $\mathbf{C}(A)$ .

**Graded Problems.** In these problems, you should check the three properties of a subspace:

1. Does the subset contain the zero vector  $\mathbf{0}$ ?
2. If  $\mathbf{v}$  and  $\mathbf{w}$  are in the subset, what about  $\mathbf{v} + \mathbf{w}$ ?
3. If  $\mathbf{v}$  is in the subset and  $c$  is any scalar, what about  $c\mathbf{v}$ ?

**Problem 1.**

- (a) Show that the subset of symmetric matrices is a subspace of the vector space  $\mathbf{M}$  of  $n \times n$  matrices.
- (b) Is the subset of symmetric matrices closed under *matrix* multiplication? That is, if  $A$  and  $B$  are both symmetric, does  $AB$  have to be symmetric?

**Problem 2.** Show that the subset  $\mathbf{P}_n$  of polynomials with degree  $\leq n$  is a subspace of the vector space  $\mathbf{F}$  of all real-valued functions. (Functions in  $\mathbf{P}_n$  look like  $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ , with  $a_0, a_1, \dots, a_n$  allowed to be any scalars, including 0.)