Solve $A^T\vec{x}=\vec{0}$: $\begin{bmatrix}1&1&0\\0&1&1\end{bmatrix} \xrightarrow{\text{Row 1 - Row 2}} \begin{bmatrix}1&0&-1\\0&1&1\end{bmatrix} \rightarrow \begin{matrix}x_1=x_3\\x_2=-x_3\end{matrix}$

So $\vec{x}=\vec{x}_3\begin{bmatrix}1\\-1\\1\end{bmatrix}$. $\vec{u}_3$ needs to be a unit vector: $\vec{u}_3=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$

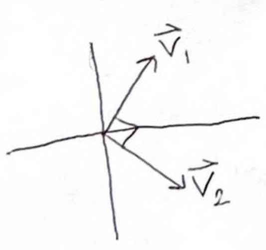
We can now write $U=\begin{bmatrix}1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3}\\ 2/\sqrt{6} & 0 & -1/\sqrt{3}\\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3}\end{bmatrix}$ ← Note that the columns are $\perp$

Finally, the SVD is $A=U\Sigma V^T=\begin{bmatrix}1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3}\\ 2/\sqrt{6} & 0 & -1/\sqrt{3}\\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3}\end{bmatrix}\begin{bmatrix}\sqrt{3}&0\\0&1\\0&0\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\-1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}$

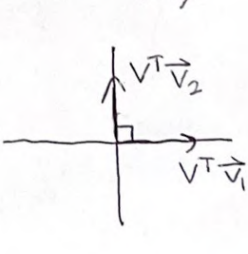
Check this is correct: $U\Sigma V^T=\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 2/\sqrt{2} & 0\\ 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2}&1/\sqrt{2}\\-1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}=A$ ✓

What are some things we can do with SVD?

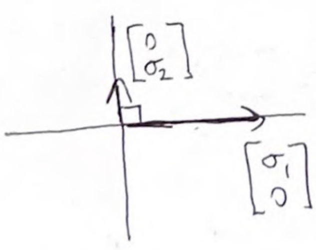
Geometry $A=U\Sigma V^T$



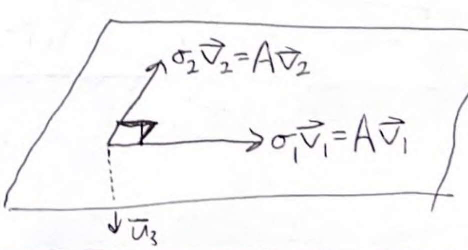
V^T, orthogonal, doesn't change length or angle



Σ, just scales the x and y-axis vectors



U, orthogonal, doesn't change length or angle



So SVD breaks A up into three pieces, and only Σ changes the lengths of vectors.

This gives us a way to measure the "size" of A, i.e., what is the maximum possible amount that A can stretch a vector.

Remember: length of a vector $\vec{x}$: $\|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$

The size of A, or norm of A, $\|A\|$, is the maximum possible ratio $\|A\vec{x}\| / \|\vec{x}\|$ (where $\vec{x} \neq \vec{0}$)

This is the factor that A stretches $\vec{x}$.

Theorem $\|A\| =$ largest singular value σ_1.

Proof: First let's show $\|A\| \geq \sigma_1$ by showing $\|A\vec{v}_1\| / \|\vec{v}_1\| = \sigma_1$:

$$\|A\vec{v}_1\| / \underbrace{\|\vec{v}_1\|}_{1} = \|U\Sigma V^T \vec{v}_1\| = \left\|U\Sigma \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\| = \left\|U \begin{bmatrix} \sigma_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\| = \|\sigma_1 \vec{u}_1\|$$
$$= \sigma_1 \|\vec{u}_1\| = \sigma_1.$$

Now let's show $\|A\| \leq \sigma_1$ by showing $\|A\vec{x}\| / \|\vec{x}\| \leq \sigma_1$ for all $\vec{x} \neq \vec{0}$:

$$\frac{\|A\vec{x}\|}{\|\vec{x}\|} = \frac{\|U\Sigma V^T \vec{x}\|}{\|\vec{x}\|} = \frac{\|\Sigma V^T \vec{x}\|}{\|\vec{x}\|} = \frac{\sqrt{(\sigma_1 (V^T\vec{x})_1)^2 + \dots + (\sigma_r (V^T\vec{x})_r)^2}}{\|\vec{x}\|}$$

Because U is orthogonal, doesn't change length.

Because V^T is orthogonal, doesn't change lengths.

Because σ_1 is the biggest singular value

$$\leq \frac{\sqrt{\sigma_1^2 \left((V^T\vec{x})_1^2 + \dots + (V^T\vec{x})_n^2 \right)}}{\|\vec{x}\|} = \frac{\sigma_1 \|V^T \vec{x}\|}{\|\vec{x}\|} = \frac{\sigma_1 \|\vec{x}\|}{\|\vec{x}\|} = \sigma_1$$

So $\|A\| \geq \sigma_1$ and $\|A\| \leq \sigma_1 \rightsquigarrow \|A\| = \sigma_1$ ✓

This shows that σ_1 is the maximum amount that A stretches vectors, and that the vectors that get stretched the most are in span($\vec{v}_1$).

Example $A=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}$. We saw that the biggest singular vector is $\sigma_1=\sqrt{3}$, and $\vec{v}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$.

So for example $\begin{bmatrix}1\\1\end{bmatrix}$ gets stretched by a factor of $\sqrt{3}$:

$$\frac{\|A\vec{x}\|}{\|\vec{x}\|}=\frac{\left\|\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}\right\|}{\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|}=\frac{\left\|\begin{bmatrix}1\\2\\1\end{bmatrix}\right\|}{\left\|\begin{bmatrix}1\\1\end{bmatrix}\right\|}=\frac{\sqrt{1+4+1}}{\sqrt{1+1}}=\frac{\sqrt{6}}{\sqrt{2}}=\sqrt{3}$$

Another application: SVD gives a good way writing a rank r matrix A as a sum of r rank-1 matrices:

Theorem $A=\sigma_1\vec{u}_1\vec{v}_1^T+\sigma_2\vec{u}_2\vec{v}_2^T+\dots+\sigma_r\vec{u}_r\vec{v}_r^T$

"outer product," column vector × row vector, has rank = 1 because every row is a multiple of $\vec{v}_1^T$.

Proof Two matrices A and B are equal if $A\vec{x}=B\vec{x}$ for every $\vec{x}$ in $\mathbb{R}^n$.

Here $\{\vec{v}_1,\dots,\vec{v}_n\}$ is a basis, so can write $\vec{x}=c_1\vec{v}_1+c_2\vec{v}_2+\dots+c_n\vec{v}_n$.

Then $(\sigma_1\vec{u}_1\vec{v}_1^T+\dots+\sigma_r\vec{u}_r\vec{v}_r^T)\underbrace{(c_1\vec{v}_1+\dots+c_n\vec{v}_n)}_{\vec{x}}=$

$\sigma_1c_1\vec{u}_1\vec{v}_1^T\vec{v}_1+\dots+\sigma_rc_r\vec{u}_r\vec{v}_r^T\vec{v}_r+$ a bunch of 0's (because $\vec{v}_i^Tv_j=0$ if $i\neq j$.)

$=c_1\underbrace{(\sigma_1\vec{u}_1)}_{A\vec{v}_1}+\dots+c_r\underbrace{(\sigma_r\vec{u}_r)}_{A\vec{v}_r}+c_{r+1}\underbrace{\vec{0}}_{Av_{r+1}}+\dots+c_n\underbrace{\vec{0}}_{A\vec{v}_n}$

$=A(c_1\vec{v}_1+\dots+c_r\vec{v}_r+c_{r+1}\vec{v}_{r+1}+\dots+c_n\vec{v}_n)=A\vec{x}$ ✓

Example If $A=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}$, then $A=\sigma_1\vec{u}_1\vec{v}_1^T+\sigma_2\vec{u}_2\vec{v}_2^T$

$$=\underbrace{\sqrt{3}}_{\sigma_1}\underbrace{\begin{bmatrix}1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{bmatrix}}_{\vec{u}_1}\underbrace{\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}}_{\vec{v}_1^T}+\underbrace{1}_{\sigma_2}\underbrace{\begin{bmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}}_{\vec{u}_2}\underbrace{\begin{bmatrix}-1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}}_{\vec{v}_2}$$

$$=\frac{1}{2}\begin{bmatrix}1\\2\\1\end{bmatrix}\begin{bmatrix}1&1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-1\\0\\1\end{bmatrix}\begin{bmatrix}-1&1\end{bmatrix}=\frac{1}{2}\left(\begin{bmatrix}1&1\\2&2\\1&1\end{bmatrix}+\begin{bmatrix}1&-1\\0&0\\-1&1\end{bmatrix}\right)=\begin{bmatrix}1&0\\1&1\\0&1\end{bmatrix}=A \checkmark$$

Application in image processing: Represent a digital photograph as an $m\times n$ matrix:

$$A=\underbrace{\begin{bmatrix} & & & \\ & \cdots\cdots & & \\ & & & \end{bmatrix}}_{n}\Big\}\, m$$

mn pixels in the image

n components

$$\text{Or } A=\sigma_1\vec{u}_1\vec{v}_1^T+\cdots+\sigma_r\vec{u}_r\vec{v}_r^T \qquad r(m+n) \text{ vector components}$$

m components

If r is much smaller than m,n, then it's more efficient to transmit or store $r(m+n)$ vector components than mn matrix entries.

<u>However</u>, even if r is <u>not</u> much smaller than m or n, many of the singular values σ_i are often very small. So we may be able to write

$$A\approx\sigma_1\vec{u}_1\vec{v}_1^T+\cdots+\sigma_s\vec{u}_s\vec{v}_s^T$$

where s <u>is</u> much smaller than m and n.

Image compression. We lose a little information by throwing out some terms, but it might not make a difference.

Today: (1) Review singular value decomposition, one more example.

(2) Briefly discuss Section 6.5 (won't be on exam) on positive-definite matrices.

(3) Review topics that will be covered on final exam

(1) Review SVD: A is any $m \times n$ matrix (don't need $m=n$)

Then A^TA and AA^T are both symmetric, have non-negative real number eigenvalues.

For A^TA: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = 0, \ldots, \lambda_n = 0$

orthonormal basis of eigenvectors: $\underbrace{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_r,}_{\text{Basis for row space } C(A^T)} \underbrace{\vec{v}_{r+1}, \ldots, \vec{v}_n}_{\text{Basis for } N(A^TA) = N(A)}$

Singular values: $\sigma_i = +\sqrt{\lambda_i}$, so $\boxed{A^TA\vec{v}_i = \sigma_i^2 \vec{v}_i}$

Orthonormal basis of eigenvectors for AA^T: $\underbrace{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_r,}_{\text{Basis for } C(A)} \underbrace{\vec{u}_{r+1}, \ldots, \vec{u}_m}_{\text{Basis for } N(AA^T) = N(A^T)}$

$\boxed{\vec{u}_i = \frac{1}{\sigma_i} A\vec{v}_i}$ $\boxed{AA^T\vec{u}_i = \sigma_i^2 \vec{u}_i}$

Then $A = U \Sigma V^T$,

$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_m \end{bmatrix}$ orthogonal $m \times m$

$\Sigma = \begin{bmatrix} \sigma_1 & & 0\text{'s} \\ & \ddots & \\ 0\text{'s} & & \end{bmatrix}$ or $\begin{bmatrix} 0 & \cdots & 0 \end{bmatrix}$ diagonal-like $m \times n$

$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}$ orthogonal $n \times n$

Example $A=\begin{bmatrix}1&3\\2&6\end{bmatrix}$. Then $A^TA=\begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}1&3\\2&6\end{bmatrix}=\begin{bmatrix}5&15\\15&45\end{bmatrix}$

Eigenvalues of A^TA: $\begin{vmatrix}5-\lambda&15\\15&45-\lambda\end{vmatrix}=(5-\lambda)(45-\lambda)-\underbrace{(15)(15)}_{225}$

$=\lambda^2-50\lambda+225-225=\lambda(\lambda-50)=0\rightarrow\lambda=50,0$

Only one non-zero singular value: $\sigma_1=\sqrt{50}=5\sqrt{2}$

Right singular vectors: $\vec{v}_1$ = unit vector basis of $\lambda=50$ eigenspace,

$\begin{bmatrix}5-50&15\\15&45-50\end{bmatrix}\rightarrow\begin{bmatrix}-45&15\\15&-5\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\rightarrow y=3x$

$\rightarrow\vec{x}=c\begin{bmatrix}1\\3\end{bmatrix}\rightarrow\vec{v}_1=\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$

$\vec{v}_2$ = unit vector basis of null space (of A or A^TA)

$\begin{bmatrix}1&3\\2&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\rightarrow x=-3y\rightarrow\vec{x}=c\begin{bmatrix}-3\\1\end{bmatrix}\rightarrow\vec{v}_2=\frac{1}{\sqrt{10}}\begin{bmatrix}-3\\1\end{bmatrix}$

Left singular vectors: $\vec{u}_1=\frac{1}{\sigma_1}A\vec{v}_1=\frac{1}{\sqrt{50}}\begin{bmatrix}1&3\\2&6\end{bmatrix}\frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$

$=\frac{1}{5\sqrt{20}}\begin{bmatrix}10\\20\end{bmatrix}=\frac{10}{10\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}=\frac{1}{\sqrt{5}}\begin{bmatrix}1\\2\end{bmatrix}$

$\vec{u}_2$ = unit vector basis of $N(A^T)$: $\begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$x+2y=0\rightarrow\vec{x}=c\begin{bmatrix}-2\\1\end{bmatrix}\rightarrow\vec{u}_2=\frac{1}{\sqrt{5}}\begin{bmatrix}-2\\1\end{bmatrix}$

So SVD is: $A=\underbrace{\begin{bmatrix}1/\sqrt{5}&-2/\sqrt{5}\\2/\sqrt{5}&1/\sqrt{5}\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}5\sqrt{2}&0\\0&0\end{bmatrix}}_{\Sigma}\underbrace{\begin{bmatrix}1/\sqrt{10}&3/\sqrt{10}\\-3/\sqrt{10}&1/\sqrt{10}\end{bmatrix}}_{V^T}$

Important geometric property of singular values:

σ_i = amount by which A rescales the length of $\vec{v}_i$,

because $\|A\vec{v}_i\| = \|\sigma_i \vec{u}_i\| = \sigma_i \|\vec{u}_i\| = \sigma_i = \sigma_i \|\vec{v}_i\|$

unit vector

In fact, the largest singular value σ_1 gives the maximum amount that A stretches the length of a vector:

σ_1 = maximum of $\|A\vec{x}\| / \|\vec{x}\|$ (where $\vec{x} \neq \vec{0}$)

this maximum is called the "operator norm," denoted $\|A\|$.

For our example: $\left\| \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix} \right\| = \sigma_1 = 5\sqrt{2}$, maximum stretching factor

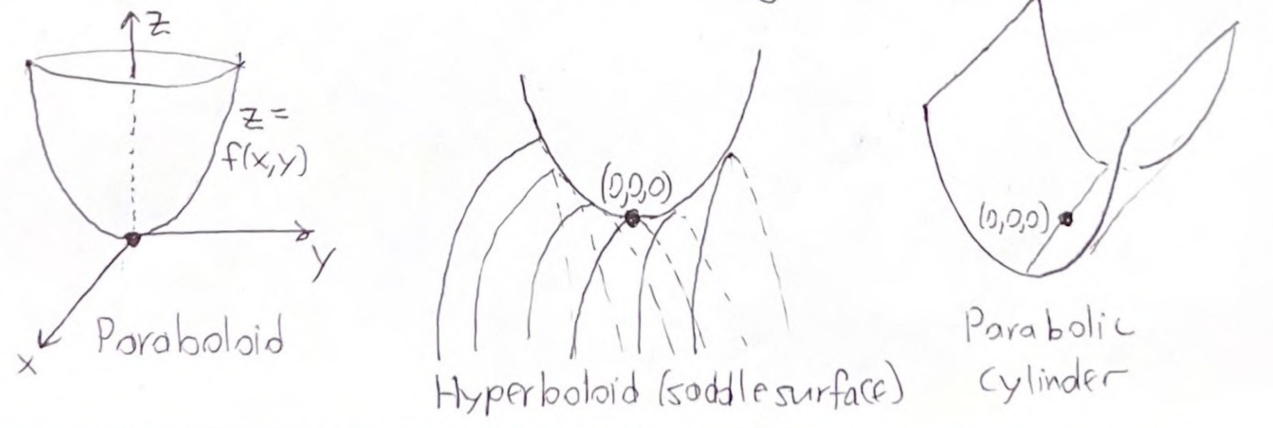
and vectors which get stretched the most are in span$(\vec{v}_1)$ = span$\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$ = Row space $C(A^T)$.

Section 6.5 Positive-definite matrices

Motivation: Understand quadratic multivariable functions.

$$f(x,y) = ax^2 + 2bxy + cy^2$$

Possible graphs of $f(x,y)$, depending on a, b, c:



Where does linear algebra come in?

$$f(x,y) = ax^2 + 2bxy + cy^2 = x(ax+by) + y(bx+cy)$$

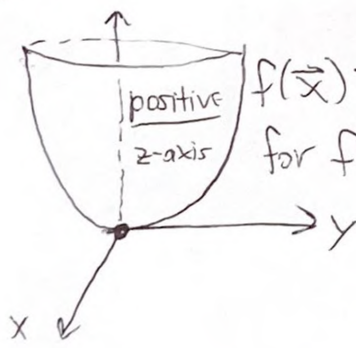
$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} ax+by \\ bx+cy \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} \underbrace{\begin{bmatrix} a & b \\ b & c \end{bmatrix}}_{\text{Symmetric matrix } S} \begin{bmatrix} x \\ y \end{bmatrix} = \vec{x}^T S \vec{x}$$

So a two-variable quadratic function has the form $f(\vec{x}) = \vec{x}^T S \vec{x}$ for some symmetric 2×2 matrix S.

Can express n-variable quadratic functions similarly, using symmetric $n\times n$ S.

<u>Definition</u>: A symmetric $n\times n$ matric S is <u>positive definite</u> if $\vec{x}^T S \vec{x} > 0$ for <u>all</u> <u>non-zero</u> $\vec{x}$ in $\mathbb{R}^n$.

For 2×2 S, this is the same as $f(x,y) = \vec{x}^T S \vec{x}$ having the graph of an <u>upward</u> opening paraboloid:



$f(\vec{x}) > 0$, except for $f(\vec{0}) = 0$.

Is there any practical way to test whether S is positive definite?

There are in fact a few equivalent ways to define positive definite:

<u>Theorem 1</u>: S is positive definite $\Longleftrightarrow$ All eigenvalues are > 0.

Proof If S is positive definite: Suppose λ is an eigenvalue. (5)

$$S\vec{x}=\lambda\vec{x} \rightsquigarrow \underbrace{\vec{x}^T S\vec{x}}_{\substack{>0 \\ (S \text{ is positive definite})}} = \lambda \underbrace{\vec{x}^T\vec{x}}_{\substack{>0 \\ (\vec{x}\neq\vec{0})}} \to \lambda = \frac{\vec{x}^T S\vec{x}}{\vec{x}^T\vec{x}} > 0 \checkmark$$

If S has positive eigenvalues: Write $S = Q^T \Lambda Q$

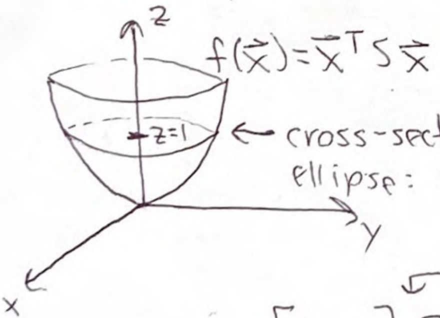
Λ: positive diagonal, Q: orthogonal

Then $\vec{x}^T S\vec{x} = \vec{x}^T Q^T \Lambda Q\vec{x} = (Q\vec{x})^T \Lambda (Q\vec{x})$

$$= \begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} \lambda_1 y_1, & \lambda_2 y_2 & \cdots & \lambda_n y_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2 > 0 \text{ since } \lambda_1, \lambda_2, \ldots, \lambda_n > 0. \checkmark$$

This expression might remind you of the equation of a circle or ellipse.



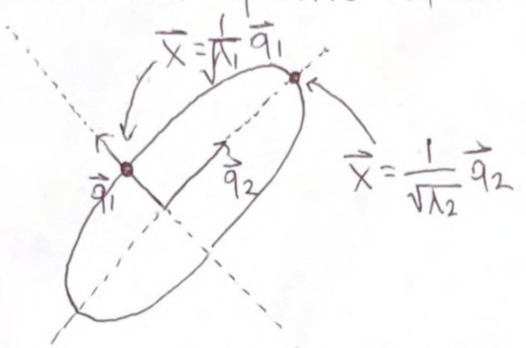
← cross-section is an ellipse: $\{(x,y) \text{ such that } \vec{x}^T S\vec{x} = 1\}$

$$1 = \begin{bmatrix} x & y \end{bmatrix} S \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix} Q \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} Q^T \begin{bmatrix} x \\ y \end{bmatrix}$$

Q: $[\vec{q}_1\ \vec{q}_2]$, eigenvector matrix

$$1 = \begin{bmatrix} \vec{x}\cdot\vec{q}_1 & x\cdot\vec{q}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} \vec{q}_1\cdot\vec{x} \\ \vec{q}_2\cdot\vec{x} \end{bmatrix} = \lambda_1 (\vec{q}_1\cdot\vec{x})^2 + \lambda_2 (\vec{q}_2\cdot\vec{x})^2$$

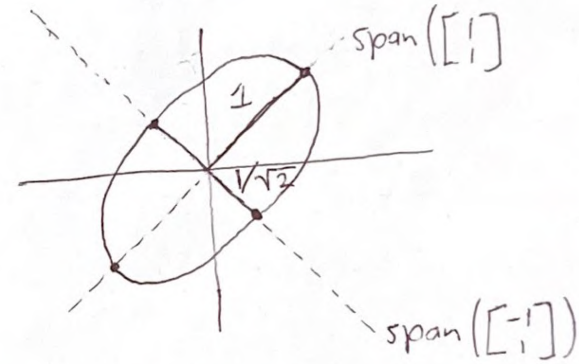
This is the equation of an ellipse: $\lambda_1(\vec{q}_1\cdot\vec{x})^2+\lambda_2(\vec{q}_2\cdot\vec{x})^2=1$ ⑥



<u>Example</u> $S=\frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}1&0\\0&2\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\-1&1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}1&1\\-2&2\end{bmatrix}=\begin{bmatrix}3/2&-1/2\\-1/2&3/2\end{bmatrix}$

$$1\left(\frac{x+y}{\sqrt{2}}\right)^2+2\cdot\left(\frac{-x+y}{\sqrt{2}}\right)^2=1 \qquad \frac{3}{2}x^2+2\left(-\frac{1}{2}\right)xy+\frac{3}{2}y^2=1$$

<u>positive</u> eigenvalues are required to make this an ellipse instead of a hyperbola.



Another useful test for positive definite matrices:

S is positive-definite $\longleftrightarrow$ all upper left subdeterminants of S are positive.

2×2: $\begin{bmatrix}a&b\\b&c\end{bmatrix}$, need a and $\begin{vmatrix}a&b\\b&c\end{vmatrix}>0$.

3×3: $\begin{bmatrix}a&b&c\\b&d&e\\c&e&f\end{bmatrix}$, need $a>0$, $\begin{vmatrix}a&b\\b&d\end{vmatrix}>0$, $\begin{vmatrix}a&b&c\\b&d&e\\c&e&f\end{vmatrix}>0$

Same for $n\times n$.

Show this for 2×2: How to guarantee all $\lambda > 0$?

$$\begin{vmatrix} a-\lambda & b \\ b & c-\lambda \end{vmatrix} = \lambda^2 - (a+c)\lambda + ac - b^2 = 0$$

$$\rightarrow \lambda = \frac{a+c \pm \sqrt{(a+c)^2 - (ac-b^2)}}{2}$$

For "−" eigenvalue to be >0, need $a+c > \sqrt{(a+c)^2 - (ac-b^2)}$

$$\rightarrow (a+c)^2 > (a+c)^2 - (ac-b^2) \longrightarrow \boxed{\underbrace{ac-b^2}_{\det S} > 0}$$

If $ac-b^2 > 0$ then $ac > b^2 \geq 0$, so a, c are both >0 or both <0

So we need $\boxed{a>0,}$ since otherwise ~~neg~~ $\frac{a+c-\sqrt{\cdots}}{2}$ would be negative.

Example: $S = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ is positive definite because

$2 > 0$, $\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3 > 0$, and $\begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 2\begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix}$

$$= 2(3) + (-2) = 4 > 0$$

Example What numbers d make $\begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$ positive definite?

$1 > 0$ ✓ $\begin{vmatrix} 1 & 2 \\ 2 & d \end{vmatrix} = d - 4 > 0 \longrightarrow \boxed{d > 4}$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} d & 4 \\ 4 & 5 \end{vmatrix} - 2\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} + 3\begin{vmatrix} 2 & d \\ 3 & 4 \end{vmatrix} = 5d - 16 - 2(-2) + 3(8 - 3d)$$

$$= -4d + 12 > 0 \longrightarrow \boxed{d < 3}$$

Inconsistent! No value of d works!

Topics to know for the final:

Chapter 4: Know what orthogonal vectors/subspaces are.
- Projections: Know how to project vectors onto a subspace and how to compute projection matrices.
- Least squares approximations (application of projection)
- Orthogonal matrices and orthonormal basis. Know how to turn any basis into orthonormal using Gram-Schmidt process.

Chapter 5: Determinants
- Know how to compute using row operations and cofactor exponsion.
- Applications to volumes, and cross products.

Chapter 6: Know how to find eigenvalues/eigenvectors. (Hint: on final, eigenvalues will never be more complicated than simple fractions.)
- Know how to diagonalize A (if there is a basis of eigenvectors), and use it to calculate A^N.
- Note: Differential equations will not be on the final exam.

Chapter 7: Know how to compute singular value decomposition of a 2×2 matrix, and know the geometric interpretation of singular values ($\sigma_1 = \max$ of $\|A\vec{x}\| / \|\vec{x}\|$)

Key topics to know from earlier in the course:

Matrix algebra:

- Matrix multiplication
- Inverses
- LU decomposition

$\left.\begin{array}{l}\text{- Inverses} \\ \text{- LU decomposition}\end{array}\right\}$ Know how to use these to solve linear equations.

Connections to geometry: Dot products, lengths, angles.

Chapter 3

- Definitions of subspace, null space, column space, row space, left null space.
- Finding all solutions to linear equations.
- Know what it means for vectors to be independent.
- Know how to find bases for subspaces, especially for the four subspaces associated to a matrix.