1. (a) Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}.$$

(b) Use the LU decomposition of A to final *all* solutions of the linear system of equations $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[R3-\boxed{7}R1]{R2-\boxed{4}R1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \xrightarrow{R3-\boxed{2}R2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix} = U$

$L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix}$ So $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 7 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$

(b) $L(\underbrace{U\vec{x}}_{\vec{y}}) = \vec{b}$ First solve $L\vec{y} = \vec{b}$:

$$\begin{aligned} y_1 &= -1 \\ 4y_1 + \cancel{0}y_2 &= 2 \\ 7y_1 + 2y_2 + y_3 &= 5 \end{aligned}$$

$\longrightarrow$ $y_2 = 2 - 4(-1) = 6$

$y_3 = 5 - 7(-1) - 2(6)$
$= 0$

Now solve $U\vec{x} = \vec{y} = \begin{bmatrix} -1 \\ 6 \\ 0 \end{bmatrix}$

$x_1 + 2x_2 + 3x_3 = -1 \rightarrow x_1 = -1 - 2(-2-2x_3) - 3x_3 = 3 + x_3$

$-3x_2 - 6x_3 = 6 \longrightarrow x_2 = -2 - 2x_3$

$0 = 0 \leftarrow$ solutions exist

All solutions: $\vec{x} = \begin{bmatrix} 3 + x_3 \\ -2 - x_3 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, x_3 = any real number.

2. (a) Use the determinant to find out for which values of the constant k the matrix

$$\begin{bmatrix} 1 & k & 1 \\ 1 & k+1 & k+2 \\ 1 & k+2 & 2k+4 \end{bmatrix}$$

is invertible.

(b) Find non-zero numbers a, b, c, d, e, f, g, h such that the matrix

$$\begin{bmatrix} a & b & c \\ d & k & e \\ f & g & h \end{bmatrix}$$

is invertible for *all* real numbers k, or explain why no such matrix exists.

(a) $$\begin{vmatrix} 1 & k & 1 \\ 1 & k+1 & k+2 \\ 1 & k+2 & 2k+4 \end{vmatrix} \underset{R3-R1}{\overset{R2-R1}{=}} \begin{vmatrix} 1 & k & 1 \\ 0 & 1 & k+1 \\ 0 & 2 & 2k+3 \end{vmatrix} \overset{R3-2R2}{=} \begin{vmatrix} 1 & k & 1 \\ 0 & 1 & k+1 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 1 \leftarrow$ non-zero for <u>all</u> k, so the matrix is invertible for all values of k.

(b) Need det $\neq 0$ for <u>all</u> k:

$$\begin{vmatrix} a & b & c \\ d & k & e \\ f & g & h \end{vmatrix} \overset{\text{2nd row}}{=} -d\begin{vmatrix} b & c \\ g & h \end{vmatrix} + k\begin{vmatrix} a & c \\ f & h \end{vmatrix} - e\begin{vmatrix} a & b \\ f & g \end{vmatrix}$$

This 2x2 det needs to be 0, because otherwise, we could solve for a value of k such that the 3x3 det is 0.

Let's choose $a=c=f=h=1$: $\det = k\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} - d\begin{vmatrix} b & 1 \\ g & 1 \end{vmatrix} - e\begin{vmatrix} 1 & b \\ 1 & g \end{vmatrix}$

$= -d(b-g) - e(g-b) = (b-g)(e-d)$

These need to be non-zero:

let's choose $b=e=2$, $d=g=1$

So the final matrix is:

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & k & 2 \\ 1 & 1 & 1 \end{bmatrix}, \quad \det \text{ is } 1\begin{vmatrix} k & 2 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & k \\ 1 & 1 \end{vmatrix}$$

$$= k-2-2(-1)+1-k = 1 \neq 0 \checkmark$$

3. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 0 & 1 \\ -1 & 2 & 1 & 1 \\ 2 & -3 & -1 & 0 \\ 5 & -7 & -2 & 1 \end{bmatrix}.$$

(a) Find the reduced row echelon form R of A.

(b) Find bases for the null space, row space, column space, and left null space of A.

(a) $$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & b_1 \\ -1 & 2 & 1 & 1 & b_2 \\ 2 & -3 & -1 & 0 & b_3 \\ 5 & -7 & -2 & 1 & b_4 \end{array}\right] \xrightarrow[\substack{R3-2R1 \\ R4-5R1}]{R2+R1} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & 2 & b_1+b_2 \\ 0 & -1 & -1 & -2 & -2b_1+b_3 \\ 0 & -2 & -2 & -4 & -5b_1+b_4 \end{array}\right] \xrightarrow[R4+2R2]{R3+R2}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & b_1 \\ 0 & 1 & 1 & 2 & b_1+b_2 \\ 0 & 0 & 0 & 0 & -b_1+b_2+b_3 \\ 0 & 0 & 0 & 0 & -3b_1+2b_2+b_4 \end{array}\right] \xrightarrow{R1+R2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 3 & 2b_1+b_2 \\ 0 & 1 & 1 & 2 & b_1+b_2 \\ 0 & 0 & 0 & 0 & -b_1+b_2+b_3 \\ 0 & 0 & 0 & 0 & -3b_1+2b_2+b_4 \end{array}\right]$$

$$R = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) Null space: $b_1 = b_2 = b_3 + b_4 = 0$

$$\begin{aligned} x_1 + x_3 + 3x_4 &= 0 \\ x_2 + x_3 + 2x_4 &= 0 \end{aligned} \longrightarrow \begin{aligned} x_1 &= -x_3 - 3x_4 \\ x_2 &= -x_3 - 2x_4 \\ x_3, x_4 &\text{ free} \end{aligned} \longrightarrow \vec{x} = \begin{bmatrix} -x_3 - 3x_4 \\ -x_3 - 2x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

The two "special solutions" form a basis for $N(A)$

Column space: Basis = "pivot columns" in A, i.e., columns 1 and 2 $= \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -3 \\ -7 \end{bmatrix} \right\}$

Row space: basis = non-zero rows of R

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix} \right\}$$

Left null space: $N(A^T) = C(A)^\perp$, so vectors in $N(A^T)$ are $\perp$ to typical column space vector $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$.

We got $-b_1 + b_2 + b_3 = 0$ if $\vec{b}$ is in $C(A)$, so
$-3b_1 + 2b_2 + b_4 = 0$

$\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}$ are $\perp$ to all vectors in $C(A)$. They are also independent and are enough for a basis of $N(A^T)$, so they are a basis.

4. Choose a basis for $\mathbf{R}^4$ from among the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 2 \\ 6 \\ 2 \\ 4 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 2 \\ 6 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix} \quad \mathbf{v}_6 = \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}$$

Then show how to write the remaining vectors as linear combinations of your basis vectors.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 1 \\ 3 & 6 & 9 & 6 & 3 & 4 \\ 1 & 2 & 4 & 1 & 2 & 1 \\ 2 & 4 & 9 & 1 & 2 & 2 \end{bmatrix} \xrightarrow[\substack{R3-R1 \\ R4-2R1}]{R2-3R1} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 3 & -3 & 0 & 0 \end{bmatrix} \xrightarrow[R2 \leftrightarrow R4]{\frac{1}{3}R4,\text{ then}}$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R3-R2} \begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R1-3R2-R3-R4}$$

Pivot columns

$$\begin{bmatrix} 1 & 2 & 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\leftarrow$ vectors from pivot columns are independent, and there are 4 of them, so they are a basis of $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix} \right\}$$

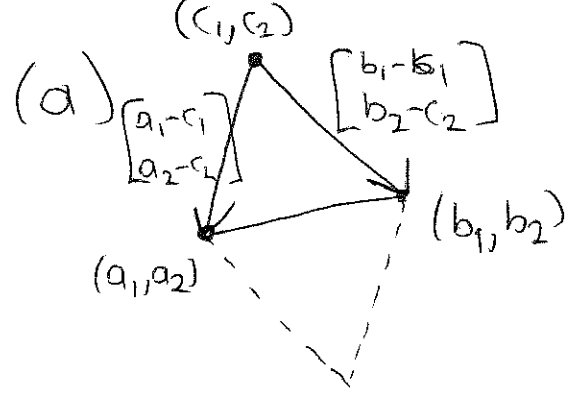
Entries show how to write $\vec{v}_2$ and $\vec{v}_4$ as linear combinations:

$\vec{v}_2 = 2\vec{v}_1$ $\qquad$ $\vec{v}_4 = 5\vec{v}_1 - \vec{v}_3$

5. (a) Find the area of the triangle in $\mathbb{R}^2$ with vertices $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$, and $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

(b) Suppose the volume of the box in $\mathbb{R}^3$ determined by the vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ is V, and suppose A is a 3×3 matrix. What is the volume of the box in $\mathbb{R}^3$ determined by $A\mathbf{x}_1$, $A\mathbf{x}_2$, and $A\mathbf{x}_3$?

(a)



Area $= \frac{1}{2}$(Area of parallelogram)

$$= \frac{1}{2} \left| \det \begin{bmatrix} a_1 - c_1 & b_1 - c_1 \\ a_2 - c_2 & b_2 - c_2 \end{bmatrix} \right|$$

$$= \frac{1}{2} \left| (a_1 - c_1)(b_2 - c_2) - (a_2 - c_2)(b_1 - c_1) \right|$$

$$= \frac{1}{2} \left| a_1 b_2 - c_1 b_2 - c_2 a_1 + \cancel{c_1 c_2} - a_2 b_1 + b_1 c_2 + a_2 c_1 - \cancel{c_1 c_2} \right|$$

$$= \frac{1}{2} \left| a_1 (b_2 - c_2) - a_2 (b_1 - c_1) + b_1 c_2 - c_1 b_2 \right|$$

$$= \frac{1}{2} \left| \det \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} - \det \begin{bmatrix} a_1 & c_1 \\ a_2 & c_2 \end{bmatrix} + \det \begin{bmatrix} b_1 & c_1 \\ b_2 & c_2 \end{bmatrix} \right|$$

$$= \frac{1}{2} \left| \det \begin{bmatrix} 1 & 1 & 1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{bmatrix} \right|$$

(b) We know $V = \left| \det \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} \right|$

So new volume is $\left| \det \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & A\vec{x}_3 \end{bmatrix} \right| = \left| \det \left(A \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} \right) \right|$

$$= \left| (\det A) \det \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} \right| = |\det A| \, V$$

6. (a) Find the best least squares line $C + Dt$ to fit the data points $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 2)$, and $(2, 1)$.

(b) Sketch a graph of the data points and your least squares line.

(c) Find the least squares error $\|\mathbf{e}\|$ of the best fit line.

(a) Try to solve:

$C + D(0) = 0$

$C + D(0) = 1$

$C + D(1) = 1$

$C + D(1) = 2$

$C + D(2) = 1$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}}_{\vec{b}}$$

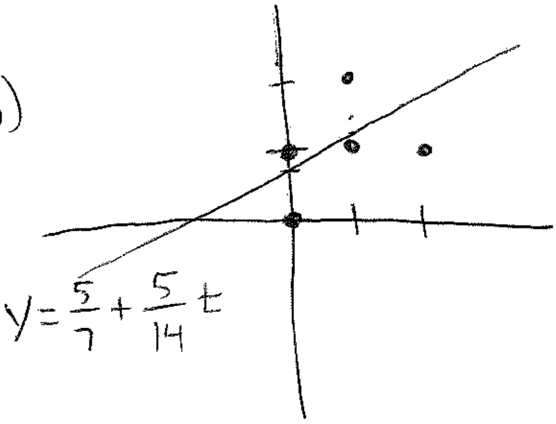
Probably no solution, so solve $A^TA\hat{x} = A^T\vec{b}$ instead.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 4 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 6 & -4 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \frac{1}{14} \begin{bmatrix} 10 \\ 5 \end{bmatrix} = \begin{bmatrix} 5/7 \\ 5/14 \end{bmatrix}$$

Best fit line is $y = \frac{5}{7} + \frac{5}{14}t$

(b)



(c) Error $= \|\underbrace{A\hat{x}}_{\text{y-values of best-fit line}} - \underbrace{\vec{b}}_{\text{actual y-values}}\| =$

$$\left\| \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 5/7 \\ 5/14 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 5/7 \\ -2/7 \\ 1/14 \\ -13/14 \\ 3/7 \end{bmatrix} \right\| = \sqrt{\frac{(10)^2 + (-4)^2 + 1^2 + (-13)^2 + 6^2}{14^2}} = \frac{1}{14}\sqrt{322}$$

7. Consider the symmetric matrix $A = \begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix}$.

(a) Find all eigenvalues of A and an *orthonormal* basis of $\mathbf{R}^3$ consisting of eigenvectors for A.

(b) Show how to write $A = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is diagonal.

(c) Show that $A^N = 9^{N-1}A$ for all positive integers N.

(a) $$\begin{vmatrix} 1-\lambda & -2 & 2 \\ -2 & 4-\lambda & -4 \\ 2 & -4 & 4-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} 4-\lambda & -4 \\ -4 & 4-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & \cancel{4-\lambda}\ -4 \\ 2 & +4-\lambda \end{vmatrix} + 2\begin{vmatrix} -2 & 4-\lambda \\ 2 & -4 \end{vmatrix}$$

$$= (\cancel{1}-\lambda)(\lambda^2-8\lambda)+2(2\lambda-8+8)+2(8-8+2\lambda)$$

$$= \lambda^2-\lambda^3-\cancel{8\lambda}+8\lambda^2+\cancel{8\lambda} = 9\lambda^2-\lambda^3 = \lambda^2(9-\lambda)$$

$\rightarrow \lambda = 0, 0, 9$ (eigenvalues)

$\underline{\lambda=9}$: $$\begin{bmatrix} -8 & -2 & 2 \\ -2 & -5 & -4 \\ 2 & -4 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/4 & -1/4 \\ -2 & -5 & -4 \\ 2 & -4 & -5 \end{bmatrix} \xrightarrow[R2-2R1]{R2+2R1}$$

$$\begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & -9/2 & -9/2 \\ 0 & -9/2 & -9/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/4 & -1/4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{R1-\frac{1}{4}R2} \begin{bmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow$$

$x_1 = \frac{1}{2}x_3$

$x_2 = -x_3$

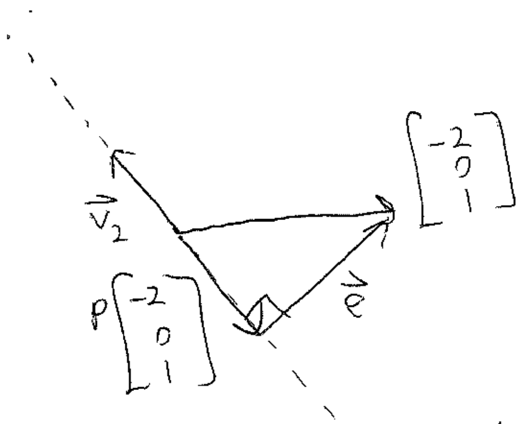
$$\rightarrow \vec{x} = x_3\begin{bmatrix} 1/2 \\ -1 \\ 1 \end{bmatrix} = \left(\tfrac{1}{2}x_3\right)\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Unit basis vector: $$\vec{v}_1 = \frac{1}{\sqrt{2^2+(-2)^2+1^2}}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$\underline{\lambda=0}$: $$\begin{bmatrix} 1 & -2 & 2 \\ -2 & 4 & -4 \\ 2 & -4 & 4 \end{bmatrix} \xrightarrow[R3-2R1]{R2+2R1} \begin{bmatrix} 1 & -2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow x_1 = 2x_2 - 2x_3$$

x_2, x_3 free

$$\vec{x} = x_2 \begin{bmatrix}2\\1\\0\end{bmatrix} + x_3 \begin{bmatrix}-2\\0\\1\end{bmatrix} \qquad \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix}2\\1\\0\end{bmatrix}$$



$$\vec{e} = \begin{bmatrix}-2\\0\\1\end{bmatrix} - \underbrace{p}_{\frac{\vec{v}_2 \vec{v}_2^T}{\vec{v}_2^T \vec{v}_2}} \begin{bmatrix}-2\\0\\1\end{bmatrix} = \begin{bmatrix}-2\\0\\1\end{bmatrix} - \frac{1}{5} \begin{bmatrix}2\\1\\0\end{bmatrix} \begin{bmatrix}2 & 1 & 0\end{bmatrix} \begin{bmatrix}-2\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}-2\\0\\1\end{bmatrix} + \frac{4}{5} \begin{bmatrix}2\\1\\0\end{bmatrix} = \frac{1}{5} \begin{bmatrix}-2\\4\\5\end{bmatrix}$$

$$\text{so } \vec{v}_3 = \frac{1}{\|e\|} \vec{e} = \frac{1}{\sqrt{25+16+4}} \begin{bmatrix}-2\\4\\5\end{bmatrix} = \frac{1}{\sqrt{45}} \begin{bmatrix}-2\\4\\5\end{bmatrix} = \frac{1}{3\sqrt{5}} \begin{bmatrix}-2\\4\\5\end{bmatrix}$$

$$\text{orthonormal basis} = \left\{ \begin{bmatrix}1/3\\-2/3\\2/3\end{bmatrix}, \begin{bmatrix}2/\sqrt{5}\\1/\sqrt{5}\\0\end{bmatrix}, \begin{bmatrix}-2/3\sqrt{5}\\4/3\sqrt{5}\\5/3\sqrt{5}\end{bmatrix} \right\}$$

$$\text{(b) } A = \underbrace{\begin{bmatrix}1/3 & 2/\sqrt{5} & -2/3\sqrt{5}\\ -2/3 & 1/\sqrt{5} & 4/3\sqrt{5}\\ 2/3 & 0 & 5/3\sqrt{5}\end{bmatrix}}_{Q} \underbrace{\begin{bmatrix}9 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix}1/3 & -2/3 & 2/3\\ 2/\sqrt{5} & 1/\sqrt{5} & 0\\ -2/3\sqrt{5} & 4/3\sqrt{5} & 5/3\sqrt{5}\end{bmatrix}}_{Q^T}$$

$$\text{(c) } A^N = Q \Lambda^N Q^T = 9^N Q \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} Q^T$$

$$= 9^N \begin{bmatrix}1/3 & 0 & 0\\ -2/3 & 0 & 0\\ 2/3 & 0 & 0\end{bmatrix} \begin{bmatrix}1/3 & -2/3 & 2/3\\ \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots\end{bmatrix} = 9^N \begin{bmatrix}1/9 & -2/9 & 2/9\\ -2/9 & 4/9 & -4/9\\ 2/9 & -4/9 & 4/9\end{bmatrix}$$

$$= 9^{N-1} A$$

8. Consider the matrix $A = \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) (5 points) Show that A is not diagonalizable.

(b) (8 points) Find the singular value decomposition $A = U\Sigma V^T$.

(c) (3 points) What is the maximum amount by which A stretches vectors in $\mathbb{R}^2$, and what is one vector that A stretches the most? That is, find the maximum value of $\|A\mathbf{x}\|/\|\mathbf{x}\|$, and find one vector $\mathbf{x}$ such that this ratio reaches the maximum value.

(a) $\begin{vmatrix} -1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (-1-\lambda)(1-\lambda)+1 = \lambda^2-\lambda+\lambda-1+1 = \lambda^2 \to \lambda = 0,0$

Eigenvectors: $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x=y \to \vec{x} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Not enough for a basis of eigenvectors, so A is not diagonalizable.

(b) $A^TA = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$\begin{vmatrix} 2-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = \lambda^2-2\lambda-2\lambda+4-4 = \lambda(\lambda-4) \to \lambda = 0, 4$

$\sigma = 0, 2$

$\underline{\lambda=4}: \begin{bmatrix} -2 & -2 \\ -2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x=-y \to \vec{x} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\underline{\lambda=0}: \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x=y \to \vec{x} = x\begin{bmatrix} 1 \\ 1 \end{bmatrix} \to \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$, $\Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

Then $\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\vec{u}_2$ = basis vector for $N(A^T)$ (unit)

$A^T \vec{x} = \vec{0}$: $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \to x = -y \to \vec{x} = x\begin{bmatrix} 1 \\ -1 \end{bmatrix} \to \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Finally, get $A = \underbrace{\begin{bmatrix} -1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}}_{\Sigma} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

(b) Maximum amount A stretches vectors is largest singular value ~~vector~~, $\sigma_1 = 2$. One vector $\vec{x}$ that gets stretched the most is $\vec{v}_1$, or any non-zero multiple. So for example, $\vec{x}$ could be $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$