# Homework 4 Solutions



**2.3.9** $E_{21}$: Subtract Row 1 from Row 2 on identity matrix: $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(a)

$P_{23}$: Exchange Rows 2 and 3 of identity: $P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$M = P_{23}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

(b) $E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

If you switch Rows 2 and 3 first, then subtract Row 1 from Row 3, that is the same as subtracting Row 1 from the original Row 2, and then switching Rows 2 and 3. So you get the same $M$: $P_{23}E_{21} = E_{31}P_{23}$, even though $E$'s are different. ~~This shows~~

**2.3.12**

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}_{\begin{bmatrix} 7 & 8 & 9 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{bmatrix}}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 4 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$$

**2.3.17** $(x,y) = (1,4)$: $4 = a + b(1) + c(1)^2$

$= (2,8)$: $8 = a + b(2) + c(2)^2$

$= (3,14)$: $14 = a + b(3) + c(3)^2$

$$\longrightarrow \begin{cases} a + b + c = 4 \\ a + 2b + 4c = 8 \\ a + 3b + 9c = 14 \end{cases}$$

$$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \\ 14 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 14 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 2} - \text{Row 1}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right] \xrightarrow{\text{Row 3} - 2\text{Row 2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right] \xrightarrow{\frac{1}{2}\text{Row 3}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[\text{Row 2} - 3\text{Row 3}]{\text{Row 1} - \text{Row 3}}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{\text{Row 1} - \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{array}{l} a = 2 \\ b = 1 \\ c = 1 \end{array}$$

$\rightsquigarrow y = 2 + x + x^2$ is the correct parabola.

<u>2.3.28</u> If $AB = I$ and $BC = I$:

$$A = AI = A(BC) = \underbrace{(AB)C}_{\text{associativity}} = IC = C$$

<u>2.4.6</u> $$(A+B)^2 = \left(\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\right)^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}^2 = \begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix}$$

$$A^2 + 2AB + B^2 = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$$

Different

$$= \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} + 2\begin{bmatrix} 7 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 1+14+1 & 2 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$$

Correct rule: $(A+B)^2 = (A+B)(A+B) = A^2 + BA + AB + B^2$

Different, <u>not</u> $2AB$

<u>2.4.15</u> (a) If $A$ is $m \times n$, then $A^2$ is $(m \times n)(m \times n)$

have to be same, $m = n$

So <u>true</u>, $A$ is square ($m = n$) if $A^2$ is defined.

(b) $A$ $k \times \cancel{m}\,l$, $\rightsquigarrow$ $BA$: $(m \times n) \cdot (k \times l)$ $\quad$ $AB$: $(k \times l)(m \times n)$
$B$ $m \times n$

same, $n = k$ $\quad$ same, $l = m$

So $A$ has to be $n \times m$ and $B$ has to be $m \times n$ if $AB$, $BA$ are <u>both</u> defined. So <u>false</u>, $A$ and $B$ don't have to be square ($m \neq n$ is okay).

(3)

(c) We saw that if $AB$, $BA$ are both defined and $B$ is $m \times n$, then $A$ is $n \times m$.

$\rightsquigarrow$ $AB$ is $(n\times m)(m\times n) = n\times n$; square

$\rightsquigarrow$ $BA$ is $(m\times n)(n\times m) = m\times m$, square

So true, $AB$ and $BA$ are both square.

(d) False, $A$ doesn't need to be $I$ since $B$ might not be invertible.

For example, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then

$$AB = B \text{ if } \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$\rightsquigarrow$ $a=1$, $c=0$, no condition on $b, d$

So $\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix} B = B$, $\begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$ might not be $I$.

Note: If $B$ is invertible, then indeed $A$ would have to be $I$.

Proof: $AB = B \longrightarrow (AB)B^{-1} = BB^{-1} = I$

$$\begin{array}{c} (AB)B^{-1} \\ \| \\ A(BB^{-1}) = AI = A \end{array}$$

same if $B^{-1}$ exists

2.4.18 (a) $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $\quad a_{ij} = \text{min. of } i \text{ and } j \rightarrow$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

(b) $a_{ij} = (-1)^{i+j}$, $A = \begin{bmatrix} (-1)^{1+1} & (-1)^{1+2} & (-1)^{1+3} \\ (-1)^{2+1} & (-1)^{2+2} & (-1)^{2+3} \\ (-1)^{3+1} & (-1)^{3+2} & (-1)^{3+3} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

(c) $a_{ij} = \frac{i}{j}$, $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 2 & 1 & 2/3 \\ 3 & 3/2 & 1 \end{bmatrix}$

2.4.21 $\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So $A\vec{v} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}$, $A^2\vec{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}$, $A^3\vec{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$,

$$A^4\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

2.4.26 $\begin{bmatrix} 1 & 0 \\ 2 & 4 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 3 & 3 & 0 \\ 6 & 6 & 0 \\ 6 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 4 & 8 & 4 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 0 \\ 10 & 14 & 4 \\ 7 & 8 & 1 \end{bmatrix}$$

2.4.32 $AX = A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \vec{x}_3 \end{bmatrix} = \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & A\vec{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

So $AX = I$, the identity matrix.

## Graded Problem 1

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix}$

commute if $\begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix} = \begin{bmatrix} c & d \\ 0 & 0 \end{bmatrix} \rightsquigarrow a = d$ and $c = 0$

So all matrices that commute with $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ look like $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, $a, b$ any real numbers

$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} b & 0 \\ d & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \rightsquigarrow$ Need $\begin{cases} a = d \\ b = 0 \end{cases}$

All matrices commuting with $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ look like $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$, $a, c$ any real numbers

If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ commutes with both $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then it has to look like both $\begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ c & a \end{bmatrix}$. That is, $a = d$ and $b = c = 0$.

So $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$, $a$ any real number, are all such matrices.

## Graded Problem 2.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 & 2 \\ 0 & 3 & -3 & 0 \\ 0 & -3 & 3 & 0 \\ 2 & -1 & 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 16 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 1 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1-2+1-1 & -1+2+1+0 \\ 1-4+3-4 & -1+4+3+0 \\ 1-8+9-16 & -1+8+9+0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -4 & 6 \\ -14 & 16 \end{bmatrix}$$