Last week: Eigenvalues and Eigenvectors

$A\vec{x} = \lambda \vec{x}$ ← non-zero eigenvector

A: n×n matrix

λ: eigenvalue ← Solutions of $\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial of } A} = 0$

General comment: When you find eigenvectors for a matrix in the homework problems, your answers might be different from the answer key's.

Why? Probably because your ~~answer~~ eigenvectors are multiples or linear combinations of the answer key's.

This is okay! General fact: Suppose λ is an eigenvalue of A. Then the set of all eigenvectors + zero vector is a subspace.

called the eigenspace for λ.

Closed under addition: $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \lambda\vec{x} + \lambda\vec{y} = \lambda(\vec{x}+\vec{y})$

Scalar multiplication: $A(c\vec{x}) = c(A\vec{x}) = c(\lambda\vec{x}) = \lambda(c\vec{x})$

Linear combinations of eigenvectors are still eigenvectors (if they are non-zero)

Note: Eigenspace for λ is just null space of $A-\lambda I$.

So if your homework answer is "$\vec{x}$" and answer key's is "$c\vec{x}$", both are correct!

Today: Solving differential equations with linear algebra.

1×1 system of ordinary differential equations

$$\begin{cases} \dfrac{du}{dt} = \lambda u \\ u(0) = u_0 \end{cases}$$

← General solution is $u(t) = Ce^{\lambda t}$, C a scalar,

because $\frac{d}{dt} Ce^{\lambda t} = C\lambda e^{\lambda t} = \lambda(Ce^{\lambda t})$

Section 6.2 Diagonalizing a matrix

Idea: If you want to understand $n\times n$ A, it is best to use a basis of $\mathbb{R}^n$ that is well-suited to A.

Maybe the best basis would be: a basis of eigenvectors for A.

Example $A=\begin{bmatrix}7 & 6\\ -8 & -7\end{bmatrix}$ Eigenvalues/vectors satisfy:

$$A\vec{x}=\lambda\vec{x}$$

λ: eigenvalue

$\vec{x}$: eigenvector, non-zero

$(A-\lambda I)\vec{x}=\vec{0}$ has non-zero null space $\to$ not invertible $\longrightarrow$ $\boxed{\det(A-\lambda I)=0}$

$$\begin{vmatrix}7-\lambda & 6\\ -8 & -7-\lambda\end{vmatrix}=(7-\lambda)(-7-\lambda)+48=\lambda^2-7\lambda+7\lambda-49+48$$

$$=\lambda^2-1=0 \longrightarrow \lambda=\pm 1$$

Eigenvectors for $\lambda=1$: Solve $(A-I)\vec{x}=\vec{0}$ $\longrightarrow$

$$\begin{bmatrix}6 & 6\\ -8 & -8\end{bmatrix}\longrightarrow\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}\longrightarrow x_1=-x_2\longrightarrow \vec{x}=x_2\begin{bmatrix}-1\\ 1\end{bmatrix}$$

every eigenvector for $\lambda=1$ is a non-zero multiple of this one

For $\lambda=-1$: Solve $(A+I)\vec{x}=\vec{0}$ $\longrightarrow$

$$\begin{bmatrix}8 & 6\\ -8 & -6\end{bmatrix}\longrightarrow\begin{bmatrix}4 & 3\\ 0 & 0\end{bmatrix}\longrightarrow 4x_1=-3x_2\longrightarrow \vec{x}=x_2\begin{bmatrix}-3/4\\ 1\end{bmatrix}$$

What can you do with this?

Put two special eigenvectors into a matrix: $X=\begin{bmatrix}-1 & -3/4\\ 1 & 1\end{bmatrix}$

Columns are eigenvectors, so AX is nice:

$$AX = \underset{A}{\begin{bmatrix} 7 & 6 \\ -8 & -7 \end{bmatrix}} \underset{X}{\begin{bmatrix} -1 & -3/4 \\ 1 & 1 \end{bmatrix}} = \begin{bmatrix} -1 & 3/4 \\ 1 & -1 \end{bmatrix} = \underset{X}{\begin{bmatrix} -1 & -3/4 \\ 1 & 1 \end{bmatrix}} \underset{\Lambda}{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}}$$

diagonal eigenvalue matrix

Multiply on right by X^{-1}: $(AX)X^{-1} = (X\Lambda)X^{-1} \longrightarrow$

$A = X \underset{\text{diagonal}}{\Lambda} X^{-1}$ We have "diagonalized" A.

One thing we can do with this: Find <u>all</u> matrix powers A^n.

$$A^n = \underbrace{(X\Lambda X^{-1})(\cancel{X}\Lambda X^{-1}) \cdots (X\Lambda X^{-1})}_{n \text{ times}} = X \Lambda^n X^{-1}$$

cancel cancel

This is easy: $\begin{bmatrix} 1^n & 0 \\ 0 & (-1)^n \end{bmatrix}$

$$= \underset{X}{\begin{bmatrix} -1 & -3/4 \\ 1 & 1 \end{bmatrix}} \underset{\Lambda^n}{\begin{bmatrix} 1 & 0 \\ 0 & (-1)^n \end{bmatrix}} \underset{X^{-1}}{\frac{1}{-1/4} \begin{bmatrix} 1 & 3/4 \\ -1 & -1 \end{bmatrix}} = \begin{bmatrix} -1 & -\frac{3}{4}(-1)^n \\ 1 & (-1)^n \end{bmatrix} \begin{bmatrix} -4 & -3 \\ 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3(-1)^n & 3-3(-1)^n \\ -4+4(-1)^n & -3+4(-1)^n \end{bmatrix}$$

If n is odd, $(-1)^n = -1$ $\begin{bmatrix} 7 & 6 \\ -8 & -7 \end{bmatrix}$

If n is even, $(-1)^n = 1$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

So $A^n = \begin{cases} A & \text{if } n \text{ is odd} \\ I & \text{if } n \text{ is even} \end{cases}$

<u>In general</u>: A $n \times n$ matrix is "diagonalizable" if we can write $A = X\Lambda X^{-1}$ with Λ <u>diagonal</u>.

This works if $\mathbb{R}^n$ has a <u>basis</u> of eigenvectors for A.

Eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_n$

Eigenvectors: $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$

Then we create Λ with the eigenvalues: $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

X comes from the eigenvectors: $X = \begin{bmatrix} | & | & & | \\ \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \\ | & | & & | \end{bmatrix}$ ← invertible because $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are a basis for $\mathbb{R}^n$.

Let's check that indeed $A = X\Lambda X^{-1}$, or $AX = X\Lambda$:

$$A\cancel{X} = A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} = \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} \lambda_1\vec{x}_1 & \lambda_2\vec{x}_2 & \cdots & \lambda_n\vec{x}_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix} = X\Lambda \checkmark$$

↑

Multiplying a diagonal matrix on the right multiplies the columns of X by the scalar diagonal entries.

Notice: $A = X\Lambda X^{-1}$ does not mean $A = \Lambda$, because $X\Lambda \neq \Lambda X$

Also: to create X and Λ, you need to order the columns of X and entries of Λ consistently:

$\vec{x}_1$ is an eigenvector for λ_1, $\vec{x}_2$ is an eigenvector for λ_2, etc.

So you can choose a different order for the $\vec{x}$'s, but then you should adjust the order of λ's.

When does $\mathbb{R}^n$ have a basis of eigenvectors for A.

① What if all n eigenvalues are real and different? Then each of $\lambda_1, \lambda_2, \ldots, \lambda_n$ has an eigenvector: $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$. Eigenvectors for different eigenvalues are independent, so $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ is a basis → A is diagonalizable.

② What if A has complex number eigenvalues? Then eigenvectors will also be complex, so $\mathbb{R}^n$ won't have a basis of eigenvectors. But A might still be diagonalizable if you don't mind working with complex numbers.

③ What if A has a repeated eigenvalue? Each distinct eigenvalue will give you at least one eigenvector, but this might not be enough for a basis $\longrightarrow$ A might or might not be diagonalizable.

Example of Case ② $A = \begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}$ Eigenvalues: Solve $\det(A - \lambda I) = 0$

$$\begin{vmatrix} -5-\lambda & 13 \\ -2 & 5-\lambda \end{vmatrix} = (-5-\lambda)(5-\lambda) - 13(-2) = \lambda^2 + 5\lambda - 5\lambda - 25 + 26 = \lambda^2 + 1 = 0$$

Eigenvalues are complex numbers: $\lambda = \pm\sqrt{-1}$ (or, $\pm i$)

But since they are different, there are two independent complex eigenvectors, so $A = X \Lambda X^{-1}$

complex eigenvector matrix $\qquad \Lambda = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

This still gives us information about A^n. For example, A^4.

$$A^4 = (X\Lambda \cancel{X^{-1}})(\cancel{X}\Lambda X^{-1})(X\Lambda X^{-1})(X\Lambda X^{-1}) = X\Lambda^4 X^{-1} = XIX^{-1} = I$$

$$\begin{bmatrix} i^4 & 0 \\ 0 & (-i)^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So $A^4 = I$ (or $A^{-1} = A^3$), not obvious from A itself, but we can check:

$$\underbrace{\begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}}_{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}\underbrace{\begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}\begin{bmatrix} -5 & 13 \\ -2 & 5 \end{bmatrix}}_{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$$

Example of <u>non-diagonalizable</u> A with repeated eigenvalues: (11B)
See end of notes from last time. (12)

Even if A has repeated eigenvalues, it <u>might</u> still be diagonalizable =

<u>Example</u>: $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$ ← not invertible, non-zero null space, so <u>one</u> eigenvalue will be 0.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & -1-\lambda & 1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & -1-\lambda \\ 1 & -1 \end{vmatrix}$$

$$= (1-\lambda)\Big(\underbrace{(-1-\lambda)(1-\lambda)+1}_{\lambda^2-\lambda+\lambda-1+1}\Big) + \cancel{(1-\lambda-1) + (-1+1+\lambda)} = (1-\lambda)\lambda^2 = 0$$

Eigenvalues are: $\lambda = 1, 0$ ← repeated root

<u>Eigenvectors for $\lambda=1$:</u> Solve $(A-I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 0 & -1 & 1 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row 3}]{\text{Row 1} \leftrightarrow} \begin{bmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\text{-Row 1}]{\text{Row 2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\text{Then: -Row 2}]{\text{Row 3-Row 2}} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Row 1+Row 2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} x_1 - x_3 = 0 \\ x_2 - x_3 = 0 \end{matrix} \longrightarrow \vec{x} = x_3\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

<u>one</u> basis (?) eigenvector

<u>For $\lambda=0$:</u> Solve $A\vec{x} = \vec{0}$ (null space) $\longrightarrow x_1 - x_2 + x_3 = 0$

<u>two</u> free variables

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

<u>Two</u> linearly independent eigenvectors for the <u>same</u> eigenvalue, 0.

Only two different eigenvalues, but one of them has two linearly independent eigenvectors, so we still get a basis.

$$\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \vec{x_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{x_3} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

So $A = X\Lambda X^{-1}$:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

Λ is a projection matrix, so $\Lambda^2 = \Lambda$ (also $\Lambda^n = \Lambda$). This means $A^n = A$ for any n as well. But A is <u>not</u> an <u>orthogonal</u> projection matrix because A isn't symmetric.

<u>Fun application of diagonalizing to Fibonacci numbers</u>

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ 5, 8, 13, 21, 34, 55, 89, \ldots$

In general, $F_k = F_{k-1} + F_{k-2}$ (recursion formula)

Use~~d~~ eigenvalues to find a "closed-form" formula for F_k:

<u>Idea</u>: $\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_1 + F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$

In general: $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_k + F_{k-1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} + F_{k-2} \\ F_{k-1} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 \begin{bmatrix} F_{k-2} \\ F_{k-3} \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$F_{k-2} + F_{k-3}$ same $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We can use <u>eigenvalues</u> to compute matrix powers: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

F_k is the <u>second</u> component of this vector.

<u>Eigenvalues</u>: $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - \lambda - 1 = 0 \rightarrow \lambda = \dfrac{1 \pm \sqrt{(-1)^2 + 4(1)(-1)}}{2(1)}$

$$= \frac{1 \pm \sqrt{5}}{2}$$