



Homework 10 Solutions

4.3.1 Idea: find line $y = C + Dt$ which contains $(0,0), (1,8), (3,8), (4,20)$

(if possible): $0 = C + D(0)$
 $8 = C + D(1)$
 $8 = C + D(3)$
 $20 = C + D(4)$

$$\rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_A \begin{bmatrix} C \\ D \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}}_b \quad (\text{no solution})$$

Normal equations for best approximate solution:

$$A^T A \vec{x} = A^T \vec{b} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 4 & 8 \\ 8 & 26 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \end{bmatrix} \xrightarrow[\frac{1}{2} \text{Row 2}]{\frac{1}{4} \text{Row 1}} \begin{bmatrix} 1 & 2 & | & 9 \\ 4 & 13 & | & 56 \end{bmatrix} \xrightarrow{\text{Row 2} - 4 \text{Row 1}}$$

$$\begin{bmatrix} 1 & 2 & | & 9 \\ 0 & 5 & | & 20 \\ 1 & 4 \end{bmatrix} \xrightarrow{\text{Row 1} - 2 \text{Row 2}} \begin{bmatrix} 1 & 0 & | & 1 \\ 0 & 1 & | & 4 \end{bmatrix} \rightarrow \begin{matrix} C=1 \\ D=4 \end{matrix} \rightarrow y = 1 + 4t$$

The four heights p_i :

Four errors e_i :

Total error:

$$\begin{aligned} p_1 &= 1 + 4(0) = 1 \\ p_2 &= 1 + 4(1) = 5 \\ p_3 &= 1 + 4(3) = 13 \\ p_4 &= 1 + 4(4) = 17 \end{aligned}$$

$$\begin{aligned} e_1 &= 1 - 0 = 1 \\ e_2 &= 5 - 8 = -3 \\ e_3 &= 13 - 8 = 5 \\ e_4 &= 17 - 20 = -3 \end{aligned}$$

$$\begin{aligned} E &= 1^2 + (-3)^2 + 5^2 + (-3)^2 \\ &= 1 + 9 + 25 + 9 \\ &= 44 \end{aligned}$$

(2)

4.3.10 Want to solve:

$$0 = C + D(0) + E(0)^2 + F(0)^3$$

$$8 = C + D(1) + E(1)^2 + F(1)^3$$

$$8 = C + D(3) + E(3)^2 + F(3)^3$$

$$20 = C + D(4) + E(4)^2 + F(4)^3$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \\ F \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}$$

$$\begin{array}{l} \text{Row 2 - Row 1} \\ \text{Row 3 - Row 1} \\ \text{Row 4 - Row 1} \end{array} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 8 \\ 0 & 3 & 9 & 27 & | & 8 \\ 0 & 4 & 16 & 64 & | & 20 \end{bmatrix} \xrightarrow{\begin{array}{l} \text{Row 3 - 3Row 2} \\ \text{Row 4 - 4Row 2} \end{array}} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 8 \\ 0 & 0 & 6 & 24 & | & -16 \\ 0 & 0 & 12 & 60 & | & -12 \end{bmatrix}$$

$$\xrightarrow{\text{Row 4 - 2Row 3}} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 8 \\ 0 & 0 & 6 & 24 & | & -16 \\ 0 & 0 & 0 & 12 & | & 20 \end{bmatrix} \xrightarrow{\begin{array}{l} \frac{1}{6} \text{ Row 3} \\ \frac{1}{12} \text{ Row 4} \end{array}} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & 1 & | & 8 \\ 0 & 0 & 1 & 4 & | & -8/3 \\ 0 & 0 & 0 & 1 & | & 5/3 \end{bmatrix} \xrightarrow{\begin{array}{l} \text{Row 2 - Row 4} \\ \text{Row 3 - 4Row 4} \end{array}}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & 0 & | & 19/3 \\ 0 & 0 & 1 & 0 & | & -28/3 \\ 0 & 0 & 0 & 1 & | & 5/3 \end{bmatrix} \xrightarrow{\begin{array}{l} \text{Row 2} \\ -\text{Row 3} \end{array}} \begin{bmatrix} 1 & 0 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & 0 & | & 47/3 \\ 0 & 0 & 1 & 0 & | & -28/3 \\ 0 & 0 & 0 & 1 & | & 5/3 \end{bmatrix} \rightarrow \begin{array}{l} C = 0 \\ D = 47/3 \\ E = -28/3 \\ F = 5/3 \end{array}$$

$$b = \frac{47}{3}t - \frac{28}{3}t^2 + \frac{5}{3}t^3$$

$$p_1 = \frac{47}{3}(0) - \frac{28}{3}(0)^2 + \frac{5}{3}(0)^3 = 0 = b_1$$

$$p_2 = \frac{47}{3}(1) - \frac{28}{3}(1)^2 + \frac{5}{3}(1)^3 = 8 = b_2$$

$$p_3 = \frac{47}{3}(3) - \frac{28}{3}(3)^2 + \frac{5}{3}(3)^3 =$$

$$= 47 - 84 + 45 = 8 = b_3$$

$$p_4 = \frac{47}{3}(4) - \frac{28}{3}(4)^2 + \frac{5}{3}(4)^3 =$$

$$= \frac{188 - 448 + 320}{3} = 20 = b_4$$

Since each $p_i = b_i$,get error $e_i = 0$.

$$\text{I.e., } \vec{p} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix} = \vec{b}, \vec{e} = \vec{0}$$

Cubic goes through
all 4 points exactly.



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(3)

4.3.12 (a) $\vec{a}^T \vec{a} \hat{x} = \vec{b} \leadsto [1 \ 1 \ \dots \ 1] \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \hat{x} = [1 \ 1 \ \dots \ 1] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$

$\leadsto m \hat{x} = b_1 + b_2 + \dots + b_m \rightarrow \hat{x} = \frac{b_1 + b_2 + \dots + b_m}{m}$ (the mean)

(b) $\vec{e} = \vec{b} - \vec{a} \hat{x} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} - \frac{b_1 + b_2 + \dots + b_m}{m} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \frac{1}{m} \begin{bmatrix} (m-1)b_1 - b_2 - \dots - b_m \\ -b_1 + (m-1)b_2 - \dots - b_m \\ \vdots \\ -b_1 - b_2 - \dots + (m-1)b_m \end{bmatrix}$

so $\|\vec{e}\|^2 = \frac{1}{m^2} \left(((m-1)b_1 - b_2 - \dots - b_m)^2 + (-b_1 + (m-1)b_2 - \dots - b_m)^2 + \dots \right)$

For $i=1, 2, \dots, m$, we have an $(m-1)^2 b_i^2$ term and $m-1$ other b_i^2 terms

$\leadsto \|\vec{e}\|^2$ contains $((m-1)^2 + (m-1)) b_i^2 = m(m-1) b_i^2$

For $i, j=1, 2, \dots, m$, have $-2(m-1)b_i b_j$ coming from i th and j th terms, with $i \neq j$
also $2b_i b_j$ coming from other $m-2$ terms

$\leadsto \|\vec{e}\|^2$ contains $(2(m-2) - 4(m-1)) b_i b_j = -2m b_i b_j$

so $\|\vec{e}\|^2 = \frac{1}{m^2} \left(\sum_{i=1}^m m(m-1) b_i^2 - 2m \sum_{i \neq j} b_i b_j \right) = \frac{1}{m} \left(\sum_{i=1}^m (m-1) b_i^2 - 2 \sum_{i \neq j} b_i b_j \right)$

$\|\vec{e}\|$ is the square root of this

(c) If $\vec{b} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$, then $\hat{x} = \frac{1+2+6}{3} = 3$

(4)

So $\vec{e} = \vec{b} - \vec{a} \hat{x} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$

So $\vec{e} \perp \vec{p}$ because $[-2 \ -1 \ 3] \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} = -6 - 3 + 9 = 0$.

3x3 P: $P = \frac{\vec{a} \vec{a}^T}{\vec{a}^T \vec{a}} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

4.3.17 $\begin{cases} 7 = C + D(-1) \\ 7 = C + D(1) \\ 21 = C + D(2) \end{cases} \rightarrow \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{\vec{b}} = \underbrace{\begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}}_{\vec{b}}$

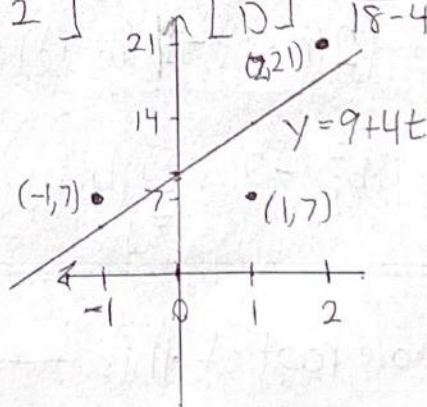
Least squares: $A^T A \hat{x} = A^T \vec{b}$

$\rightarrow \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 7 \\ 7 \\ 21 \end{bmatrix}$

$\rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 35 \\ 42 \end{bmatrix} \rightarrow \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{18-4} \begin{bmatrix} 6 & -2 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 35 \\ 42 \end{bmatrix}$

$= \frac{1}{14} \begin{bmatrix} 126 \\ 56 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \end{bmatrix}$

$y = 9 + 4t$





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4.4.2 $\left\| \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \right\| = \sqrt{2^2 + 2^2 + (-1)^2} = \sqrt{9} = 3$, $\left\| \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \right\| = 3$ also.

So $\vec{q}_1 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$, $\vec{q}_2 = \begin{bmatrix} -1/3 \\ 2/3 \\ 2/3 \end{bmatrix} \rightsquigarrow Q = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix}$

$$QQ^T = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 5/9 & 2/9 & -4/9 \\ 2/9 & 8/9 & 2/9 \\ -4/9 & 2/9 & 5/9 \end{bmatrix}$$

$$Q^T Q = \begin{bmatrix} 2/3 & 2/3 & -1/3 \\ -1/3 & 2/3 & 2/3 \end{bmatrix} \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ -1/3 & 2/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Projection matrix
for projection
onto $\text{span}(\vec{q}_1, \vec{q}_2)$

4.4.10 (a) Suppose $c_1 \vec{q}_1 + c_2 \vec{q}_2 + c_3 \vec{q}_3 = \vec{0}$

Dot with \vec{q}_1 : $c_1 (\underbrace{\vec{q}_1 \cdot \vec{q}_1}_1) + c_2 (\underbrace{\vec{q}_1 \cdot \vec{q}_2}_0) + c_3 (\underbrace{\vec{q}_1 \cdot \vec{q}_3}_0) = \underbrace{\vec{q}_1 \cdot \vec{0}}_0 \rightarrow c_1 = 0$

Similar: dot with $\vec{q}_2 \rightarrow c_2 = 0$, dot with $\vec{q}_3 \rightarrow c_3 = 0$.

Only solution is $c_1 = c_2 = c_3 = 0$, so $\{\vec{q}_1, \vec{q}_2, \vec{q}_3\}$ is independent.

(b) Write $Q = [\vec{q}_1 \ \vec{q}_2 \ \vec{q}_3]$, so $c_1 \vec{q}_1 + c_2 \vec{q}_2 + c_3 \vec{q}_3 = \vec{0}$ means $Q \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \vec{0}$

For orthogonal matrices, $Q^T Q = I$ (even if Q isn't square)

So $Q^T Q \vec{c} = Q^T \vec{0} \rightarrow I \vec{c} = \vec{0} \rightarrow \vec{c} = \vec{0} \rightarrow$ columns of Q are independent

4.4.18 $A = \frac{1}{\|\vec{a}\|} \vec{a} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$

(6)

\vec{B} = project \vec{b} onto $\text{span}(\vec{a})$, take difference \vec{e} :

$$\vec{e} = \vec{b} - \frac{\vec{a} \vec{a}^T \vec{b}}{\vec{a}^T \vec{a}} = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix}$$

Normalize: $\vec{B} = \frac{1}{\|\vec{e}\|} \vec{e} = \frac{1}{\sqrt{1/4 + 1/4 + 1 + 0}} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{3/2}} \begin{bmatrix} 1/2 \\ 1/2 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ -2 \\ 0 \end{bmatrix}$

\vec{C} = project \vec{c} onto $\text{span}(\vec{A}, \vec{B})$, take difference \vec{e} :

$$\vec{e} = \vec{c} - \underbrace{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \\ 0 & 0 \end{bmatrix}}_Q \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{\vec{c}}$$

Projection matrix

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{6} \\ 0 & -2/\sqrt{6} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} - \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix}$$

Normalize: $\vec{C} = \frac{1}{\sqrt{1/9 + 1/9 + 1/9 + 1}} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix} = \frac{\sqrt{3}}{2} \begin{bmatrix} 1/3 \\ 1/3 \\ 1/3 \\ -1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/6 \\ \sqrt{3}/6 \\ \sqrt{3}/6 \\ -\sqrt{3}/2 \end{bmatrix}$

Note $\vec{A}, \vec{B}, \vec{C}$ are all \perp to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, just like $\vec{a}, \vec{b}, \vec{c}$



4.4.22: $\vec{A} = \frac{1}{\|\vec{a}\|} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

\vec{B} : project \vec{b} onto $\text{span}(\vec{a})$, take error \vec{e} :

$$\vec{e} = \vec{b} - \underbrace{\frac{\vec{a}\vec{a}^T}{\vec{a}^T\vec{a}} \vec{b}}_{\text{projection onto span}(\vec{a})} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - \frac{1}{6} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \underbrace{[1 \ 1 \ 2]}_{0, \vec{a} \perp \vec{b} \text{ already}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

So $\vec{b} = \frac{1}{\|\vec{e}\|} \vec{e} = \frac{1}{\|\vec{b}\|} \vec{b} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

\vec{C} : project \vec{c} onto $\text{span}(\vec{A}, \vec{B})$, take error \vec{e} :

$$\vec{e} = \vec{c} - \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{2} \\ 2/\sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} \\ 1/\sqrt{6} & -1/\sqrt{2} \\ 2/\sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} 9/\sqrt{6} \\ 1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \vec{C} = \frac{1}{\|\vec{e}\|} \vec{e} = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

4.4.3) (a) All columns of Q have length $\sqrt{1+1+1+1} = 2$, so $\boxed{C = \frac{1}{2}}$

(b) $\vec{p} = P \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} [1 \ -1 \ -1 \ -1]}{\begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

Projection for plane of 1st two columns: can use

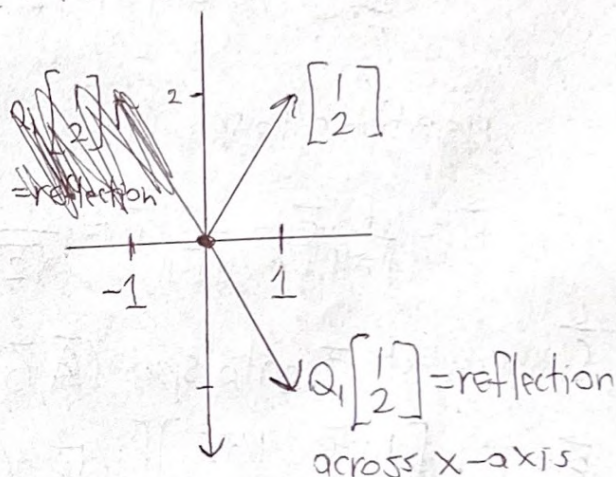
orthogonal columns: $p = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \\ -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1/2 & -1/2 & -1/2 & -1/2 \\ -1/2 & 1/2 & -1/2 & -1/2 \end{bmatrix}$

$$= \begin{bmatrix} 1/2 & -1/2 & 0 & 0 \\ -1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{bmatrix}$$

Multiply by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$: get $\vec{p} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

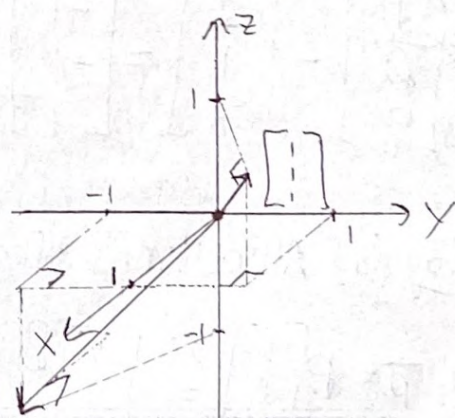
4.4.32 $Q_1 = I - 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$Q_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$



$$Q_2 = I - 2 \begin{bmatrix} 0 \\ \sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} 0 & \sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}, \quad Q_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$



Reflection in plane \perp to $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ (spanned by x-axis and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$)



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Graded Problem 1

$$t = 0, 1, 2, 3 \rightarrow \text{Try to solve}$$

$$h = 50, 44, 32, 6$$

$$\begin{aligned} C+D(0)+E(0)^2 &= 50 \\ C+D(1)+E(1)^2 &= 44 \\ C+D(2)+E(2)^2 &= 32 \\ C+D(3)+E(3)^2 &= 6 \end{aligned}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 50 \\ 44 \\ 32 \\ 6 \end{bmatrix}$$

$A \qquad \qquad \qquad \vec{b}$

No solution, solve
normal equations $A^T A \hat{x} = A^T \vec{b}$
instead.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 50 \\ 44 \\ 32 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 6 & 14 \\ 6 & 14 & 36 \\ 14 & 36 & 98 \end{bmatrix}$$

$$\begin{bmatrix} 132 \\ 126 \\ 226 \end{bmatrix}$$

Best fit parabola:

$$h(t) = \frac{248}{5} + \frac{3}{5}t - 5t^2$$

$$\rightarrow g \approx 2(5) = 10$$

Divide all
rows by 2

$$\begin{bmatrix} 2 & 3 & 7 & 66 \\ 3 & 7 & 18 & 63 \\ 7 & 18 & 49 & 113 \end{bmatrix} \xrightarrow{\text{Row 2} - \frac{3}{2}\text{Row 1}} \begin{bmatrix} 2 & 3 & 7 & 66 \\ 0 & 5/2 & 15/2 & -36 \\ 7 & 18 & 49 & 113 \end{bmatrix} \xrightarrow{\text{Row 3} - \frac{7}{2}\text{Row 1}} \begin{bmatrix} 2 & 3 & 7 & 66 \\ 0 & 5/2 & 15/2 & -36 \\ 0 & 15/2 & 49/2 & -118 \end{bmatrix}$$

$$\begin{array}{l} \text{Row 3} - \\ 3 \text{ Row 2} \end{array} \rightarrow \begin{bmatrix} 2 & 3 & 7 & 66 \\ 0 & 5/2 & 15/2 & -36 \\ 0 & 0 & -10 & -10 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3/2 & 7/2 & 33 \\ 0 & 1 & 3 & -72/5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{Row 1} - \frac{3}{2}\text{Row 2}} \begin{bmatrix} 1 & 0 & 1/2 & 101/2 \\ 0 & 1 & 3 & -72/5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{\text{Row 2} - 3\text{Row 3}} \begin{bmatrix} 1 & 0 & 1/2 & 101/2 \\ 0 & 1 & 0 & -10 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\text{Row 1} - \frac{1}{2}\text{Row 3} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 248/5 \\ 0 & 1 & 0 & 3/5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Warning: there

Graded Problem 2

(10)

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 2} - \frac{1}{2}\text{Row 1}} \begin{bmatrix} 2 & 1 & 2 \\ 0 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[2\text{Row 2}]{\frac{1}{2}\text{Row 1}} \begin{bmatrix} 1 & 1/2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{Row 1}]{-\frac{1}{2}\text{Row 2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \text{Row space} = \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right)$$

$$\text{Null space: } \begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0 \end{aligned} \longrightarrow \vec{x} = \begin{bmatrix} -x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$\longrightarrow \text{basis vector} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Orthonormal basis for Rowspace:
basis vectors are already \perp , just
need to rescale first one:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Orthonormal basis
for null space:

$$\frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

These 3 vectors together
form an orthonormal basis
for \mathbb{R}^3 .