

This shows x_3 is allowed to be any real number. But once we've picked x_3 , we must take $x_1 = -x_3$ and $x_2 = -x_3$.

So all solutions look like $\begin{bmatrix} -c \\ -c \\ c \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (c can be any real number)

For example, if we pick $c=1$: tells us that

$$(-1) \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note: Today we briefly looked at some of the major ideas in solving linear equations. We will look at these ideas in more detail later, so don't worry if you didn't completely understand everything today.

Chapter 2: Solving Linear Equations

Today, begin a systematic study of systems of linear equations.

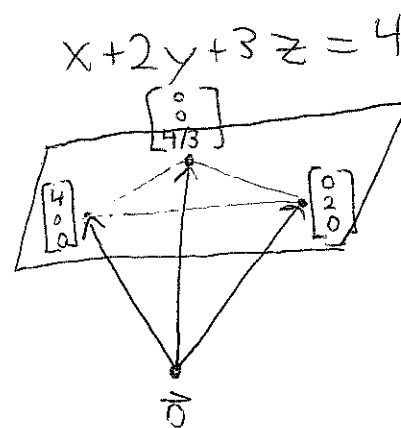
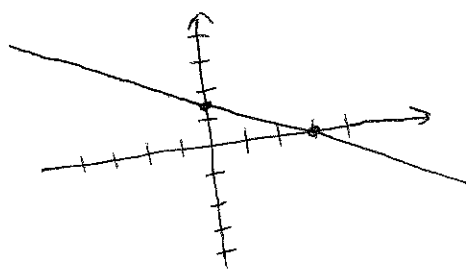
Algebraically: Linear equations involve 1st powers of the variables, as well as constants:

$x + 2y + 3z = 4$ is linear, $\sin x + xy + y^3 = 1$ is not

Geometrically: Linear equations describe lines in the plane, planes in space, "hyperplanes" in higher-dimensional space.

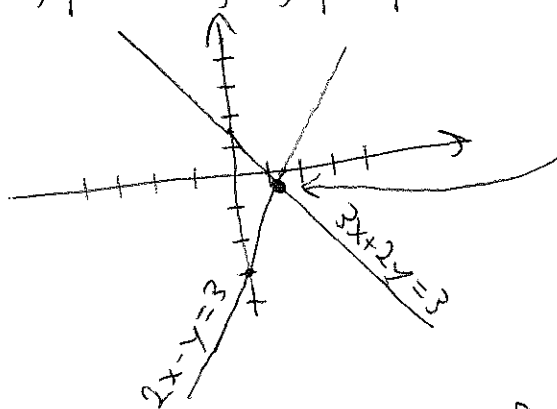
$$x + 2y = 3$$

$$(y = -\frac{1}{2}x + \frac{3}{2})$$



Linear systems can involve more than one linear equation. (19)
 They describe how lines, planes, hyperplanes intersect.

$$\begin{cases} 2x - y = 3 \\ 3x + 2y = 3 \end{cases}$$



This point has coordinates (x, y) that satisfy both equations.

What's the best way to find the point of intersection?

We will use elimination:

$$\begin{aligned} 2x - y &= 3 \\ 3x + 2y &= 3 \end{aligned}$$

Matrix form: $\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

The coefficient matrix

Optional first step \downarrow Divide Eqn. 1 by 2

$$x - \frac{1}{2}y = \frac{3}{2}$$

$$3x + 2y = 3$$

\downarrow Eliminate x from Eqn. 2:
 \downarrow Eqn. 2 \rightarrow Eqn. 2 - 3(Eqn. 1)

$$x - \frac{1}{2}y = \frac{3}{2}$$

$$0x + \frac{7}{2}y = -\frac{3}{2}$$

\downarrow Solve for y

$$x - \frac{1}{2}y = \frac{3}{2}$$

$$y = -\frac{3}{7}$$

\downarrow Eliminate y in Eqn. 1:
 \downarrow Eqn. 1 \rightarrow Eqn. 1 + $\frac{1}{2}$ (Eqn. 2)

$$\begin{aligned} x + 0y &= \frac{3}{2} + \frac{1}{2}\left(-\frac{3}{7}\right) = \frac{21-3}{14} = \frac{9}{7} \\ y &= -\frac{3}{7} \end{aligned}$$

\downarrow Divide Row 1 by 2

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ 3 \end{bmatrix}$$

\downarrow Row 2 \rightarrow
 Row 2 - 3(Row 1)

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{7}{2} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$$

\downarrow Row 2 \rightarrow
 $\frac{2}{7}$ (Row 2)

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{7} \end{bmatrix}$$

\downarrow Row 1 \rightarrow
 \downarrow Row 1 + $\frac{1}{2}$ (Row 2)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{9}{7} \\ -\frac{3}{7} \end{bmatrix}$$

So the solution is $(x, y) = \left(\frac{9}{7}, -\frac{3}{7}\right)$

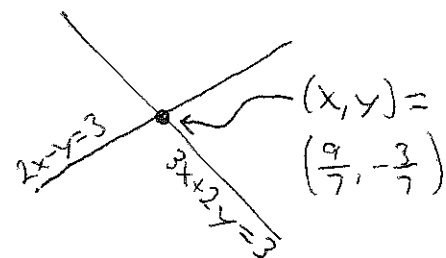
The final matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix. It has the property $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$ Doesn't change the vector, just like $1 \cdot x = x$ for scalars.

We have two different ways of thinking about linear equations:

"Row picture"
(lines / planes / etc.
intersecting in space)

$$\begin{cases} 2x - y = 3 \\ 3x + 2y = 3 \end{cases}$$

1st row of a matrix
2nd row of a matrix



$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

"Column picture"
(writing a vector
as a linear combination)

$$x \begin{bmatrix} 2 \\ 3 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

Columns of a matrix

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = \frac{9}{7} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + \left(-\frac{3}{7}\right) \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

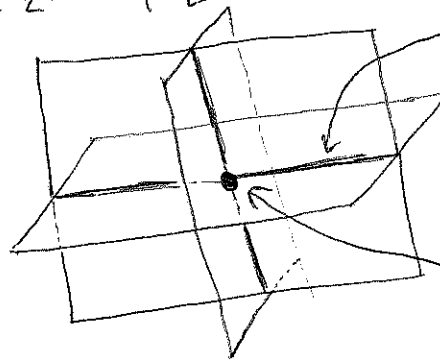
Row picture advantage: Geometrically natural
Column picture advantage: Easier to imagine
in higher dimensions.

In some sense, the matrix-vector product equation unifies these two pictures.

Now for a 3×3 example:

Row picture: $\begin{cases} 2x - y + z = 1 \\ -x + 2y - z = 1 \\ x - y + 2z = 1 \end{cases}$ ← These are equations of planes in 3-dim. space

Harder to visualize:



Two planes in 3-dim. space (usually) intersect in a line.

Three planes in 3-dim. space usually intersect in a single point (the solution to the system of linear equations.)

Matrix form of the equations:

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

↖ Coefficient matrix

Column picture: $\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ ← Can we write this as a linear combination of the other 3?

Elimination Procedure:

$$\begin{aligned} 2x - y + z &= 1 \\ -x + 2y - z &= 1 \\ x - y + 2z &= 1 \end{aligned}$$

Eliminate x from 2nd and 3rd eqns:
 2nd eqn \rightarrow 2nd $+$ $\frac{1}{2}$ (1st)
 3rd eqn \rightarrow 3rd $+$ $\frac{1}{2}$ (1st)

$$\begin{aligned} 2x - y + z &= 1 \\ 0x + \frac{3}{2}y - \frac{1}{2}z &= \frac{3}{2} \\ 0x - \frac{1}{2}y + \frac{3}{2}z &= \frac{1}{2} \end{aligned}$$

$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Row 2 \rightarrow Row 2 $+$ $\frac{1}{2}$ (Row 1)
 Row 3 \rightarrow Row 3 $+$ $\frac{1}{2}$ (Row 1)

$$\begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eliminate y from 3rd eqn
(use 2nd eqn, not 1st)
↓ 3rd eqn \rightarrow 3rd $+$ $\frac{1}{3}$ (2nd)

$$\begin{aligned} 2x - y + z &= 1 \\ 0x + \frac{3}{2}y - \frac{1}{2}z &= \frac{3}{2} \\ 0x + 0y + \frac{4}{3}z &= 1 \end{aligned}$$

↓ Solve for z

$$\begin{aligned} 2x - y + z &= 1 \\ \frac{3}{2}y - \frac{1}{2}z &= \frac{3}{2} \\ \boxed{z = \frac{3}{4}} \end{aligned}$$

We could continue eliminating z from 2nd eqn, and z and y from 1st eqn. Or, we can solve for y and x using substitution:

$$\frac{3}{2}y = \frac{3}{2} + \frac{1}{2}z = \frac{3}{2} + \frac{1}{2}\left(\frac{3}{4}\right) = \frac{15}{8}$$

$$\rightarrow \boxed{y = \frac{2}{3}\left(\frac{15}{8}\right) = \frac{5}{4}}$$

$$\rightarrow 2x = 1 + y - z = 1 + \frac{5}{4} - \frac{3}{4} = \frac{3}{2} \rightarrow \boxed{x = \frac{3}{4}}$$

So the 3 planes intersect at the single point $\boxed{(x, y, z) = \left(\frac{3}{4}, \frac{5}{4}, \frac{3}{4}\right)}$

Check: $2\left(\frac{3}{4}\right) - \frac{5}{4} + \frac{3}{4} = \frac{3}{2} - \frac{1}{2} = 1 \quad \checkmark$
 $-\frac{3}{4} + 2\left(\frac{5}{4}\right) - \frac{3}{4} = \frac{10-6}{4} = 1 \quad \checkmark$

$$\begin{aligned} \frac{3}{4} - \frac{5}{4} + 2\left(\frac{3}{4}\right) &= \\ -\frac{1}{2} + \frac{3}{2} &= 1 \quad \checkmark \end{aligned}$$

Column interpretation: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a linear combination of

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}: \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{3}{4} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + \frac{5}{4} \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix} + \frac{3}{4} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

(22)

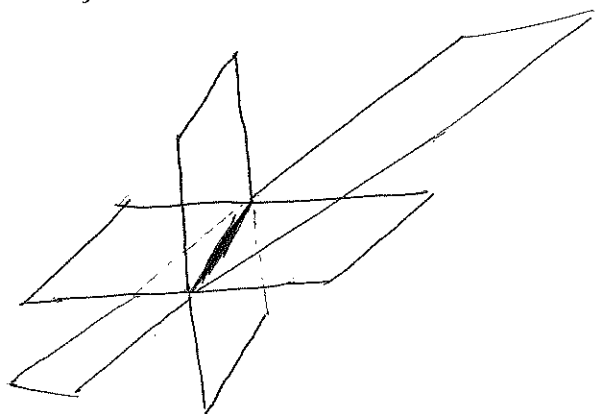
↓ Row 3 \rightarrow Row 3 $+$ $\frac{1}{3}$ (Row 2)

$$\begin{bmatrix} 2 & 1 & 1 \\ 0 & 3/2 & -1/2 \\ 0 & 0 & 4/3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 3/2 \\ 1 \end{bmatrix}$$

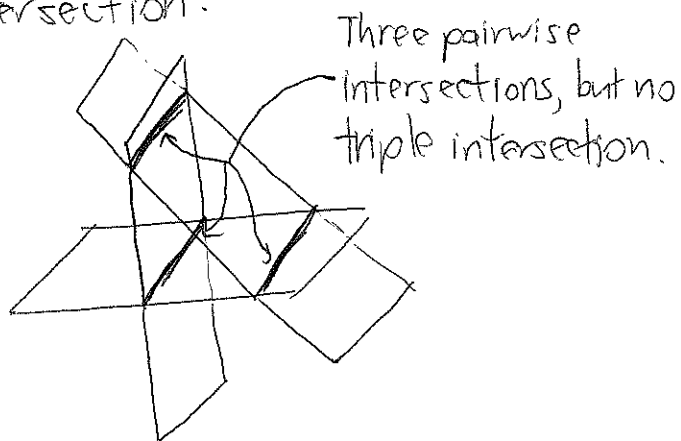
↖ An upper triangular matrix.

Important Warning: 3 planes in 3-dim. don't always intersect in a single point.

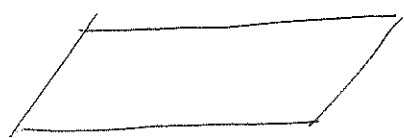
They might intersect in a line:



They might not have any common intersection:



They might even intersect in a plane, if all three are really the same plane in disguise:



$$\begin{cases} x + 2y + 3z = 4 \\ 2x + 4y + 6z = 8 \\ -x - 2y - 3z = -4 \end{cases}$$

This means there are three possibilities when we solve a system of linear equations: No solution, exactly one solution, or infinitely many solutions (a whole, line, plane, etc.)

(It will never have just 2 or 3 solutions!)