Linear Algebra – Fall 2022
Second Final Exam

NAME:

STUDENT ID:

Instructions:

- Show answers and arguments in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want graded. Answers given without supporting work may receive zero credit.
- This is a closed book exam: no calculators, notes, or formula sheets.

QUESTION	POINTS	SCORE
1	12	
2	10	
3	14	
4	10	
5	10	
6	12	
7	16	
8	16	
TOTAL	100	

1. (a) (10 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}.$$

(b) (2 points) Use A^{-1} to solve the linear system of equations $A\mathbf{x} = (1, -1, 1, -1)$.

2. (10 points) Find the determinants of the following matrices:

$$A = \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$$

Also determine whether A is invertible or not, and find all values of t such that B is *not* invertible.

3. (14 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ -1 & 1 & 0 & 1 \\ 4 & -2 & 2 & -2 \end{bmatrix}.$$

(a) Find the reduced row echelon form R of A.

(b) Find bases for the null space, row space, column space, and left null space of A.

4. (10 points) Choose a basis for $\mathbf{R}^4$ from among the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then show how to write the remaining vector as a linear combinations of your basis vectors.

5. (a) (5 points) If all entries of A and A^{-1} are integers, prove that $\det A = 1$ or -1. Find a 2×2 example of such an A with no zero entries. *Hint:* What is $\det A$ times $\det A^{-1}$?

   (b) (5 points) Suppose (x, y, z) and $(1, 1, 0)$ and $(1, 2, 1)$ lie on the same plane that goes through the origin. What determinant must be zero, and what equation does this give for the plane?

6. (a) (8 points) Find the best least squares line $C + Dt$ to fit the data points $(-1, 2)$, $(0, 2)$, $(1, -1)$, $(2, 0)$, and $(3, -2)$.

   (b) (2 points) Sketch a graph of the data points and your least squares line.

   (c) (2 points) Find the least squares error $\|\mathbf{e}\|$ of the best fit line.

7. Consider the symmetric matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) (10 points) Find all eigenvalues of A and an *orthonormal* basis of $\mathbf{R}^3$ consisting of eigenvectors for A.

(b) (2 points) Show how to write $A = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is diagonal.

(c) (4 points) Calculate $A^N\mathbf{x}$ for any vector $\mathbf{x} = (x, y, z)$ and any positive integer N.

8. Consider the matrix $A = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$.

(a) (5 points) Show that A is not diagonalizable.

(b) (9 points) Find the singular value decomposition $A = U\Sigma V^T$.

(c) (2 points) What is the maximum amount by which A stretches vectors in $\mathbb{R}^2$, and what is one vector that A stretches the most? That is, find the maximum value of $\|A\mathbf{x}\|/\|\mathbf{x}\|$, and find one vector $\mathbf{x}$ such that this ratio reaches the maximum value.

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