Calculus A(1): Homework 3

The total is 100 points. When we refer to a paragraph number (e.g. §2.5), we refer to the PDF of the textbook *Thomas Calculus* that you can find on the weblearn. The bonus exercises are optional and more difficult. We may (or may not) decide to grade one of your bonus exercises and use it to replace one assigned exercise (if it improves your total grade).

Routine exercises (do not hand-in)

1. §2.5, Exercises 22, 32, 36, 51
2. §2.6, Exercises 2, 5, 11, 16, 17, 36, 49, 52, 54, 55, 56
3. §2.7, Exercises 6, 7, 13, 18, 25

Assigned exercises (hand-in)

1. (10pts) Compute the following limits.

a. $\lim_{x\to 3^-} \frac{x^2-5x+6}{|x-3|}$ **b.** $\lim_{x\to 0^+} \frac{\tan x - \sin x}{x^3}$

2. (15pts) Compute the following limits (the limits can be infinite).

a. $\lim_{x\to 1^-} \left(\frac{1}{x^{\frac{1}{3}}} - \frac{1}{(x-1)^{\frac{4}{3}}}\right)$ **b.** $\lim_{x\to 0} \frac{\sqrt{1-\cos^2 x}}{1-\cos x}$ **c.** $\lim_{x\to 0} \frac{\sin(\sin x)}{x}$

3. (20pts) Find the asymptotes of the following functions.

a. $f(x) = \frac{x^2-6x+8}{3x-4}$ **b.** $f(x) = \sin\left(\frac{\pi}{x^2+1}\right)$

4. (20pts) Let $f : \mathbb{R} \to \mathbb{R}$ be such that for all $x \in \mathbb{R}$ we have $f(x) \neq x$. Assume that f is continuous. Prove that either $\forall x \in \mathbb{R}$ we have $f(x) > x$ or $\forall x \in \mathbb{R}$ we have $f(x) < x$.
5. (15pts) If a function f is continuous at x_0, explain why $|f(x)|$ and $f(x)^2$ are continuous at x_0. Is the converse true (either give a proof or a counter example)?
6. (20pts) Let $P(x)$ be a polynomial whose degree is an odd integer. Prove that P has a root in $\mathbb{R}$, i.e. there exists $\alpha \in \mathbb{R}$ such that $P(\alpha) = 0$.

Bonus exercises (optional)

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that for all $x, y \in \mathbb{R}$ we have $f(x + y) = f(x) + f(y)$. Prove that if f is continuous at $x_0 \in \mathbb{R}$, then $f(x) = f(1)x$ for all $x \in \mathbb{R}$. (Hint: consider first the case when $x \in \mathbb{Z}$, then $x \in \mathbb{Q}$ and finally $x \in \mathbb{R}$.)

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$ we have $f(f(x)) = x$. Prove that there exists $\eta \in \mathbb{R}$ such that $f(\eta) = \eta$. (Hint: Assume that for all η, $f(\eta) \neq \eta$. Then use Assigned exercise 4).