1. (a) (10 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 & -1 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & -1 \\ -1 & -1 & -1 & 2 \end{bmatrix}.$$

(b) (2 points) Use A^{-1} to solve the linear system of equations $A\mathbf{x} = (1, -1, 1, -1)$.

(a)
$$\left[\begin{array}{cccc|cccc} 2 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ -1 & -1 & -1 & 2 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}(\text{Row }1+\text{Row }2+\text{Row }3)]{\text{Row }4+} \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{array}\right]$$

$$\xrightarrow[\text{Row }3+2\text{Row }4]{\text{Row }1+2\text{Row }4,\ \text{Row }2+2\text{Row }4} \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 2 & 1 & 1 & 2 \\ 0 & 2 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{array}\right] \xrightarrow[\frac{1}{2}\text{Row }3,\ 2\text{Row }4]{\frac{1}{2}\text{Row }1,\ \frac{1}{2}\text{Row }2}$$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1/2 & 1/2 & 1 \\ 0 & 1 & 0 & 0 & 1/2 & 1 & 1/2 & 1 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{array}\right] \longrightarrow \boxed{A^{-1} = \begin{bmatrix} 1 & 1/2 & 1/2 & 1 \\ 1/2 & 1 & 1/2 & 1 \\ 1/2 & 1/2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}}$$

(b)
$$\vec{x} = A^{-1}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 1/2 & 1 \\ 1/2 & 1 & 1/2 & 1 \\ 1/2 & 1/2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$$

$$\longrightarrow \boxed{\vec{x} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}}$$

2. (10 points) Find the determinants of the following matrices:

$$A = \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}$$

Also determine whether A is invertible or not, and find all values of t such that B is *not* invertible.

$$\det A = \begin{vmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{vmatrix} \overset{\text{Row 2-Row 1}}{\underset{\text{Row 3-Row 1}}{=}} \begin{vmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} \overset{\text{Row 3}}{\underset{-2\text{ Row 2}}{=}}$$

$$\begin{vmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = \boxed{0, \text{ \underline{not} invertible}}$$

$$\det B = \begin{vmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{vmatrix} = 1\begin{vmatrix} 1 & t \\ t & 1 \end{vmatrix} - t\begin{vmatrix} t & t \\ t^2 & 1 \end{vmatrix} + t^2\begin{vmatrix} t & 1 \\ t^2 & t \end{vmatrix}$$

$$= 1(1-t^2) - t(t-t^3) + t^2\cancel{(t^2-t^2)}$$

$$= (1-t^2)(1-t^2) = \boxed{(1-t^2)^2, \text{ not invertible if } t = \pm 1}$$

3. (14 points) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ -1 & 1 & 0 & 1 \\ 4 & -2 & 2 & -2 \end{bmatrix}.$$

(a) Find the reduced row echelon form R of A.

(b) Find bases for the null space, row space, column space, and left null space of A.

(a)
$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & b_1 \\ 1 & 1 & 2 & 1 & b_2 \\ -1 & 1 & 0 & 1 & b_3 \\ 4 & -2 & 2 & -2 & b_4 \end{array}\right] \xrightarrow[\text{Row } 2]{\text{Row } 1 \leftrightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ -1 & 1 & 0 & 1 & b_3 \\ 4 & -2 & 2 & -2 & b_4 \end{array}\right]$$

$$\xrightarrow[\text{Row}4-4\text{ Row}1]{\text{Row}3+\text{Row}1} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 0 & 2 & 2 & 2 & b_2+b_3 \\ 0 & -6 & -6 & -6 & -4b_2+b_4 \end{array}\right] \xrightarrow[\text{Row }4+6\text{ Row }2]{\text{Row}3-2\text{Row }2} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_3 \\ 0 & 0 & 0 & 0 & 6b_1-4b_2+b_4 \end{array}\right]$$

$$\xrightarrow{\text{Row }1-\text{Row }2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -b_1+b_2 \\ 0 & 1 & 1 & 1 & b_2 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_3 \\ 0 & 0 & 0 & 0 & 6b_1-4b_2+b_4 \end{array}\right]$$

This is R

(b) Null space: Solve $\begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 + x_4 = 0 \end{cases} \longrightarrow$

$$\vec{x} = \begin{bmatrix} -x_3 \\ -x_3 - x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for N(A)

Column space: Linearly independent pivot columns in A:

Basis of $C(A)$: $\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -2 \end{bmatrix} \right\}$

Row Space: Non-zero rows of R

Basis of $C(A^T)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Left null space: $\dim = 4 - \text{rank}(A) = 2$, $N(A^T) = C(A)^T$

If $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ is a column space vector, calculations for part

(a) show that $\begin{cases} -2b_1 + b_2 + b_3 = 0 \\ 6b_1 - 4b_2 + b_4 = 0 \end{cases}$ $\rightarrow$

$\begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ -4 \\ 0 \\ 1 \end{bmatrix}$ are two linearly independent vectors in $C(A)^T$

$= N(A^T)$, so they are a basis for $N(A^T)$.

4. (10 points) Choose a basis for $\mathbf{R}^4$ from among the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Then show how to write the remaining vector as a linear combinations of your basis vectors.

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow{\text{Row 2+Row 1}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[-\text{Row 3}]{\text{Row 4+Row 2}}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & +1 & -1 & 0 \\ 0 & 0 & 1 & -1 & 2 \end{bmatrix} \xrightarrow{\text{Row 4-Row 3}} \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 2-Row 4}]{\text{Row 1-Row 4}}$$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 2-Row 3}]{\text{Row 1-Row 3}} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5$ are independent

Basis for $\mathbb{R}^4$:

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_5\}$

$$\boxed{\vec{v}_4 = \vec{v}_1 + \vec{v}_2 - \vec{v}_3}$$

5. (a) (5 points) If all entries of A and A^{-1} are integers, prove that $\det A = 1$ or -1. Find a 2×2 example of such an A with no zero entries. *Hint:* What is $\det A$ times $\det A^{-1}$?

(b) (5 points) Suppose (x, y, z) and $(1, 1, 0)$ and $(1, 2, 1)$ lie on the same plane that goes through the origin. What determinant must be zero, and what equation does this give for the plane?

(a) $(\det A)(\det A^{-1}) = \det(AA^{-1}) = \det I = 1$, so $\det A^{-1} = \frac{1}{\det A}$. Also, $\det A$ ~~[illegible]~~ and $\det A^{-1}$ are both integers since A, A^{-1} have integer entries. The only integers with integer reciprocals are ± 1, so $\det A = +1$ or -1.

2x2 example: $A = \begin{bmatrix} 4 & 3 \\ 1 & 1 \end{bmatrix}$, $A^{-1} = \frac{1}{4(1)-3(1)} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$

(b) Since they are in the same plane, they are linearly dependent, so $\begin{bmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ is not invertible, and

$\begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} = 0$. This gives the equation:

$$x \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} - y \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} + z \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 0 \rightsquigarrow$$

$$\boxed{x - y + z = 0}$$

6. (a) (8 points) Find the best least squares line $C + Dt$ to fit the data points $(-1, 2)$, $(0, 2)$, $(1, -1)$, $(2, 0)$, and $(3, -2)$.

(b) (2 points) Sketch a graph of the data points and your least squares line.

(c) (2 points) Find the least squares error $\|\mathbf{e}\|$ of the best fit line.

(a) Try to solve:
$$\begin{aligned} C + (-1)D &= 2 \\ C + 0\,D &= 2 \\ C + 1\,D &= -1 \\ C + 2\,D &= 0 \\ C + 3\,D &= -2 \end{aligned} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \\ -2 \end{bmatrix}$$

No solution; instead solve normal equations:

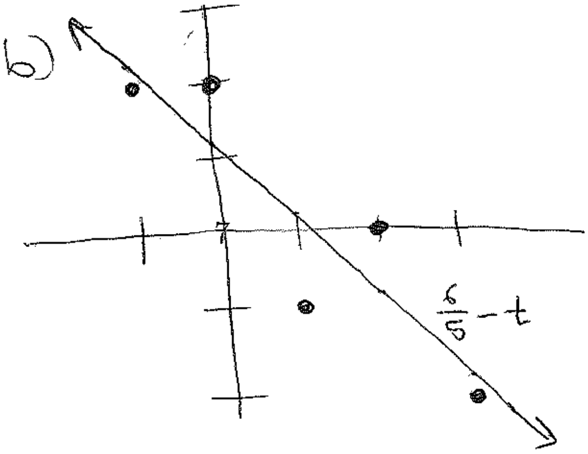
$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \\ -2 \end{bmatrix} \rightsquigarrow$$

$$\begin{bmatrix} 5 & 5 \\ 5 & 15 \end{bmatrix}\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ -9 \end{bmatrix} \longrightarrow \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{5(15)-5(5)}\begin{bmatrix} 15 & -5 \\ -5 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ -9 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -9 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 12 \\ -10 \end{bmatrix} = \begin{bmatrix} 6/5 \\ -1 \end{bmatrix}$$

$$\boxed{\text{Best line is } \frac{6}{5} - t}$$

(b)



(c) Error $\|\hat{e}\| = \Bigg\| \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 6/5 \\ -1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ -1 \\ 0 \\ -2 \end{bmatrix}}_{\begin{bmatrix} 11/5 - 2 \\ 6/5 - 2 \\ 1/5 + 1 \\ -4/5 - 0 \\ -9/5 + 2 \end{bmatrix}} \Bigg\| = \Bigg\| \begin{bmatrix} 1/5 \\ -4/5 \\ 6/5 \\ -4/5 \\ -1/5 \end{bmatrix} \Bigg\|$

$$= \frac{1}{5}\sqrt{1^2 + (-4)^2 + 6^2 + (-4)^2 + (-1)^2}$$

$$= \boxed{\frac{1}{5}\sqrt{70}}$$

7. Consider the symmetric matrix $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

(a) (10 points) Find all eigenvalues of A and an *orthonormal* basis of $\mathbf{R}^3$ consisting of eigenvectors for A.

(b) (2 points) Show how to write $A = Q\Lambda Q^T$ where Q is an orthogonal matrix and Λ is diagonal.

(c) (4 points) Calculate $A^N\mathbf{x}$ for any vector $\mathbf{x} = (x, y, z)$ and any positive integer N.

(a) $\begin{vmatrix} 1-\lambda & -1 & 1 \\ -1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix} = (1-\lambda)\left((1-\lambda)^2-1\right)-(-1)\left(-1+\lambda+1\right)+1\left(1-1+\lambda\right)$

$= (1-\lambda)(\lambda^2-2\lambda)+2\lambda = -\lambda\left((\lambda-1)(\lambda-2)-2\right)$

$= -\lambda(\lambda^2-3\lambda) = 0 \longrightarrow \lambda = 0, 0, 3$

$\underline{\lambda=3}$: $\begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\xrightarrow[\text{Row3}+\frac{1}{2}\text{Row1}]{\text{Row2}-\frac{1}{2}\text{Row1}}$ $\begin{bmatrix} -2 & -1 & 1 \\ 0 & -3/2 & -3/2 \\ 0 & -3/2 & -3/2 \end{bmatrix} \longrightarrow$

$\begin{bmatrix} 1 & +1/2 & -1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{2}\text{Row2}]{\text{Row1}-} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} x_1 = x_3 \\ x_2 = -x_3 \end{matrix} \xrightarrow{x_3=1} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

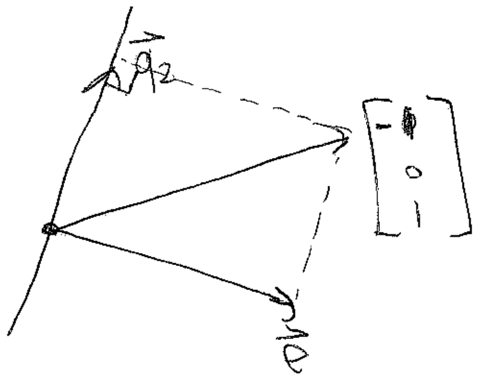
$\underline{\lambda=0}$: $\begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow[\text{Row3}-\text{Row1}]{\text{Row2}+\text{Row1}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow x_1 = x_2 - x_3$

$\xrightarrow{x_2=1,\ x_3=0} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$\xrightarrow{x_2=0,\ x_3=1} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Basis of eigenvectors, <u>not</u> orthonormal

Basis: $\vec{q}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}$, $\vec{q}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\vec{q}_3$: Gram-Schmidt



$$\vec{e} = \begin{bmatrix}-1\\0\\1\end{bmatrix} - (\text{Projection})\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$= \begin{bmatrix}-1\\0\\1\end{bmatrix} - \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix}\underbrace{\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{-1}$$

$$= \begin{bmatrix}-1\\0\\1\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1/2\\1/2\\1\end{bmatrix} \xrightarrow[\text{vector}]{\text{unit}} \vec{q}_3 = \frac{1}{\sqrt{\frac{1}{4}+\frac{1}{4}+1}}\begin{bmatrix}-1/2\\1/2\\1\end{bmatrix}$$

$$= \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

<u>Basis:</u> $\left\{ \frac{1}{\sqrt{3}}\begin{bmatrix}1\\-1\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\\0\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-1\\1\\2\end{bmatrix} \right\}$

(b) $$A = Q\Lambda Q^T = \begin{bmatrix}1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6}\\ -1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6}\\ 1/\sqrt{3} & 0 & 2/\sqrt{6}\end{bmatrix}\begin{bmatrix}3 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{3} & -1/\sqrt{3} & 1/\sqrt{3}\\ 1/\sqrt{2} & 1/\sqrt{2} & 0\\ -1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6}\end{bmatrix}$$

(c) $$A^N\vec{x} = Q\Lambda^N Q^T\vec{x} = \begin{bmatrix}\vec{q}_1 & \vec{q}_2 & \vec{q}_3\end{bmatrix}\begin{bmatrix}3^N & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\vec{q}_1^T\vec{x}\\ \vec{q}_2^T\vec{x}\\ \vec{q}_3^T\vec{x}\end{bmatrix}$$

$$= \begin{bmatrix}\vec{q}_1 & \vec{q}_2 & \vec{q}_3\end{bmatrix}\begin{bmatrix}3^N\vec{q}_1^T\vec{x}\\ 0\\ 0\end{bmatrix} = 3^N(\vec{q}_1^T\vec{x})\,\vec{q}_1$$

$$= \frac{3^N}{\sqrt{3}}(x-y+z)\begin{bmatrix}1/\sqrt{3}\\ -1/\sqrt{3}\\ 1/\sqrt{3}\end{bmatrix} = \boxed{3^{N-1}(x-y+z)\begin{bmatrix}1\\-1\\1\end{bmatrix}}$$

8. Consider the matrix $A = \begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}$.

(a) (5 points) Show that A is not diagonalizable.

(b) (9 points) Find the singular value decomposition $A = U\Sigma V^T$.

(c) (2 points) What is the maximum amount by which A stretches vectors in $\mathbb{R}^2$, and what is one vector that A stretches the most? That is, find the maximum value of $\|A\mathbf{x}\|/\|\mathbf{x}\|$, and find one vector $\mathbf{x}$ such that this ratio reaches the maximum value.

(a) Eigenvalues: $\begin{vmatrix} -2-\lambda & 4 \\ -1 & 2-\lambda \end{vmatrix} = -(\lambda+2)(2-\lambda) - (-4) = \lambda^2 = 0$

$\rightarrow \lambda = 0, 0$

Eigenvectors: $\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x_1 = 2x_2 \rightsquigarrow x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Only one independent eigenvector, not enough for a basis of $\mathbb{R}^2$, so A is <u>not</u> diagonalizable.

(b) Singular values $= \sqrt{\ }$ of eigenvalues of A^TA

$A^TA = \begin{bmatrix} -2 & -1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} -2 & 4 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix} \rightsquigarrow \begin{vmatrix} 5-\lambda & -10 \\ -10 & 20-\lambda \end{vmatrix} =$

$(5-\lambda)(20-\lambda) - 100 = \lambda^2 - 25\lambda = 0 \rightarrow \lambda = 0, 25$

$\sigma = 0, 5$

$\sigma_1 = 5$, $\vec{v}_1 =$ eigenvector for $\lambda = 25$:

$\begin{bmatrix} -20 & -10 \\ -10 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow 2x_1 + x_2 = 0 \rightsquigarrow \vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\sigma_2 = 0$, $\vec{v}_2 =$ eigenvector for $\lambda = 0$: 9

$\begin{bmatrix} 5 & -10 \\ -10 & 20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow x_1 - 2x_2 = 0 \rightsquigarrow \vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{5}\begin{bmatrix}-2 & 4\\ -1 & 2\end{bmatrix}\frac{1}{\sqrt{5}}\begin{bmatrix}1\\ -2\end{bmatrix} = \frac{1}{5\sqrt{5}}\begin{bmatrix}-10\\ -5\end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix}-2\\ -1\end{bmatrix}$

$\vec{u}_2$ = basis for $N(A^T)$: $\begin{bmatrix}-2 & -1\\ 4 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix} \rightsquigarrow 2x_1 + x_2 = 0$

$\to \vec{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\ -2\end{bmatrix}$

So $A = \begin{bmatrix}\vec{u}_1 & \vec{u}_2\end{bmatrix}\begin{bmatrix}\sigma_1 & 0\\ 0 & \sigma_2\end{bmatrix}\begin{bmatrix}\vec{v}_1^T\\ \vec{v}_2^T\end{bmatrix} = \begin{bmatrix}-2/\sqrt{5} & 1/\sqrt{5}\\ -1/\sqrt{5} & -2/\sqrt{5}\end{bmatrix}\begin{bmatrix}5 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1/\sqrt{5} & -2/\sqrt{5}\\ 2/\sqrt{5} & 1/\sqrt{5}\end{bmatrix}$

(c) Max stretch factor = max singular value = 5

Vectors that are stretched the most: span$(\vec{v}_1)$, so

$\vec{x} = \begin{bmatrix}1\\ -2\end{bmatrix}$ would be an example