Practice Midterm Exam

1. Consider the system of linear equations:

$$\begin{array}{rcrcrcrcr} -3x_1 & - & 4x_2 & + & 4x_3 & + & 4x_4 & = & 2 \\ x_1 & + & 2x_2 & - & x_3 & - & 2x_4 & = & -2 \\ -3x_1 & - & 2x_2 & + & 5x_3 & + & 2x_4 & = & \alpha \end{array}$$

where α is any real number.

(a) Find the only value of α for which the system has solutions.

(b) For this value of α, find all solutions to the system of equations.

(c) Identify the reduced row echelon form of the coefficient matrix of this system of equations.

(a) Augmented matrix:

$$\left[\begin{array}{cccc|c} -3 & -4 & 4 & 4 & 2 \\ 1 & 2 & -1 & -2 & -2 \\ -3 & -2 & 5 & 2 & \alpha \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 1}]{3\text{Row 2} + \text{Row 1}} \left[\begin{array}{cccc|c} -3 & -4 & 4 & 4 & 2 \\ 0 & 2 & 1 & -2 & -4 \\ 0 & 2 & 1 & -2 & \alpha-2 \end{array}\right]$$

$$\xrightarrow{\text{Row 3} - \text{Row 2}} \left[\begin{array}{cccc|c} -3 & -4 & 4 & 4 & 2 \\ 0 & 2 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & \alpha+2 \end{array}\right]$$

For solutions to exist, need $\boxed{\alpha = -2}$

(b)

$\alpha = -2$

Row 1 + 2Row 2

$$\left[\begin{array}{cccc|c} -3 & 0 & 6 & 0 & -6 \\ 0 & 2 & 1 & -2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\frac{1}{2}\text{Row 2}]{-\frac{1}{3}\text{Row 1}} \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 2 \\ 0 & 1 & \frac{1}{2} & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(c) This is R

All solutions:

$x_1 - 2x_3 = 2$

$x_2 + \frac{1}{2}x_3 - x_4 = -2$

x_3, x_4 free

$$\longrightarrow \vec{x} = \begin{bmatrix} 2 + 2x_3 \\ -2 - \frac{1}{2}x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

2. (a) A matrix A is skew-symmetric if $A^T = -A$. Use the rules of transposes and the three properties of a subspace to show that the set S of all skew-symmetric $n \times n$ matrices is a subspace of the vector space $\mathbf{M}$ of $n \times n$ matrices.

(b) Find a spanning set for the subspace of skew-symmetric 2×2 matrices that has only one matrix in it. (That is, show that all skew-symmetric 2×2 matrices are multiples of one particular matrix.)

(a) 1. Is the zero matrix in S?

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \checkmark$$

(Works for any size, actually.)

2. If A, B are in S, what about $A+B$?

$$(A+B)^T = A^T + B^T = (-A) + (-B) = -(A+B)$$

So $A+B$ is in S. ✓

3. If A is in S, what about cA?

$$(cA)^T = c\,A^T = c(-A) = -(cA).$$

So cA is in S ✓

(b) If $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is skew-symmetric, then

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} = -\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow$$

$a = -a$, $c = -b$ $\longrightarrow$ $a = d = 0$, $c = -b$
$b = -c$, $d = -d$

$$\longrightarrow A = \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

This is a one-matrix spanning set.

3. (a) Find the inverse of

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}$$

(b) Use A^{-1} to solve the linear system of equations $A\mathbf{x} = (0, 1, 0, 0)$.

(a) Elimination method:

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 & 0 & 1 & 0 & 0 \\ 1 & 3 & 6 & 10 & 0 & 0 & 1 & 0 \\ 1 & 4 & 10 & 20 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row 3}-\text{Row 1, Row 4}-\text{Row 1}]{\text{Row 2}-\text{Row 1}} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & 9 & -1 & 0 & 1 & 0 \\ 0 & 3 & 9 & 19 & -1 & 0 & 0 & 1 \end{array}\right]$$

Row 3 - 2 Row 2
Row 4 - 3 Row 2

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & -2 & 1 & 0 \\ 0 & 0 & 3 & 10 & 2 & -3 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row 4}-3\,\text{Row 3}} \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right]$$

Row 1 - Row 4
Row 2 - 3 Row 4
Row 3 - 3 Row 4

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 2 & -3 & 3 & -1 \\ 0 & 1 & 2 & 0 & 2 & -8 & 9 & -3 \\ 0 & 0 & 1 & 0 & 4 & -11 & 10 & -3 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right] \xrightarrow[\text{Row 2}-2\,\text{Row 3}]{\text{Row 1}-\text{Row 3}} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & -2 & 8 & -7 & 2 \\ 0 & 1 & 0 & 0 & -6 & 14 & -11 & 3 \\ 0 & 0 & 1 & 0 & 4 & -11 & 10 & -3 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right]$$

Row 1 - Row 2

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & -6 & 4 & -1 \\ 0 & 1 & 0 & 0 & -6 & 14 & -11 & 3 \\ 0 & 0 & 1 & 0 & 4 & -11 & 10 & -3 \\ 0 & 0 & 0 & 1 & -1 & 3 & -3 & 1 \end{array}\right]$$

This is A^{-1}.

Note: The actual exam won't have this much calculation to find a matrix inverse.

(b) $A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow \vec{x} = A^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} 4 & -6 & 4 & -1 \\ -6 & 14 & -11 & 3 \\ 4 & -11 & 10 & -3 \\ -1 & 3 & -3 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -6 \\ 14 \\ -11 \\ 3 \end{bmatrix}$$

4. Suppose $A\mathbf{x} = \mathbf{b}$ is a linear system of n equations in n variables and $\mathbf{x}_1$, $\mathbf{x}_2$ are two solutions with $\mathbf{x}_1 \neq \mathbf{x}_2$.

   (a) Is the matrix A invertible? Explain.

   (b) Show that $\mathbf{x} = \mathbf{x}_1 + \alpha(\mathbf{x}_1 - \mathbf{x}_2)$ is also a solution to $A\mathbf{x} = \mathbf{b}$ for any scalar α. Which value of α gives the solution $\mathbf{x}_2$?

(a) Solution to $A\vec{x} = \vec{b}$ is not unique, so A can't be invertible.

(b) We know $A\vec{x}_1 = \vec{b}$ and $A\vec{x}_2 = \vec{b}$

Plug in $\vec{x} = \vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2)$:

$$
\begin{aligned}
A\vec{x} &= A\left(\vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2)\right) \\
&= A\vec{x}_1 + \alpha\left(A\vec{x}_1 - A\vec{x}_2\right) \\
&= \vec{b} + \alpha(\vec{b} - \vec{b}) \\
&= \vec{b}.
\end{aligned}
$$

So $\vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2)$ is also a solution to $A\vec{x} = \vec{b}$.

If $\vec{x}_1 + \alpha(\vec{x}_1 - \vec{x}_2) = \vec{x}_2$:

Then $(1+\alpha)\vec{x}_1 - (1+\alpha)\vec{x}_2 = \vec{0}$

This works if $\boxed{\alpha = -1}$

4

5. Set

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix}.$$

(a) Find the LU decomposition of A.

(b) Use the LU decomposition to solve the system of equations $A\mathbf{x} = (1,2,3)$ (that is, solve the two triangular systems $L\mathbf{y} = (1,2,3)$ and $U\mathbf{x} = \mathbf{y}$).

(a) $\begin{bmatrix} 2 & 1 & 1 \\ 6 & 4 & 5 \\ 4 & 1 & 3 \end{bmatrix} \xrightarrow[\text{Row 3} - \boxed{2}\text{Row 1}]{\text{Row 2} - \boxed{3}\text{Row 1}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{\text{Row 3} - \boxed{(-1)}\text{Row 2}} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}}_{U}$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$

$A\vec{x} = \vec{b}$

$L(\underbrace{U\vec{x}}_{\vec{y}}) = \vec{b}$

(b) First solve $L\vec{y} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then solve $U\vec{x} = \vec{y}$.

$$\begin{aligned} y_1 &= 1 \\ 3y_1 + y_2 &= 2 \\ 2y_1 - y_2 + y_3 &= 3 \end{aligned} \longrightarrow \begin{aligned} y_1 &= 1 \\ y_2 &= 2 - 3 = -1 \\ y_3 &= 3 - 2(1) + (-1) = 0 \end{aligned}$$

$$\begin{aligned} 2x_1 + x_2 + x_3 &= 1 \\ x_2 + 2x_3 &= -1 \\ 3x_3 &= 0 \end{aligned} \longrightarrow \begin{aligned} x_1 &= \tfrac{1}{2}(1 - (-1)) = 1 \\ x_2 &= -1 \\ x_3 &= 0 \end{aligned} \longrightarrow \boxed{\vec{x} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$$

6. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 & 3 & 1 \\ 2 & 4 & 5 & 5 & 4 \\ 3 & 6 & 7 & 8 & 5 \end{bmatrix}$$

(a) Find a spanning set (the special solutions) for the null space $\mathbf{N}(A)$.

(b) Find a linear relation on b_1, b_2, b_3 that guarantees that $\mathbf{b} = (b_1, b_2, b_3)$ is a vector in the column space $\mathbf{C}(A)$.

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & 3 & 1 & b_1 \\ 2 & 4 & 5 & 5 & 4 & b_2 \\ 3 & 6 & 7 & 8 & 5 & b_3 \end{array}\right] \xrightarrow[\text{Row } 3 - 3 \text{ Row } 1]{\text{Row } 2 - 2 \text{ Row } 1}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & 3 & 1 & b_1 \\ 0 & 0 & 1 & -1 & 2 & -2b_1 + b_2 \\ 0 & 0 & 1 & -1 & 2 & -3b_1 + b_3 \end{array}\right] \xrightarrow{\text{Row } 3 - \text{Row } 2}$$

$$\left[\begin{array}{ccccc|c} 1 & 2 & \boxed{2} & 3 & 1 & b_1 \\ 0 & 0 & 1 & -1 & 2 & -2b_1 + b_2 \\ 0 & 0 & 0 & 0 & 0 & -b_1 - b_2 + b_3 \end{array}\right]$$

(b) Solutions exist <u>only</u> when

$$-b_1 - b_2 + b_3 = 0$$

Row 1 − 2 Row 2

$$\begin{bmatrix} 1 & 2 & 0 & 5 & -3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Null space equations:

$$x_1 + 2x_2 + 5x_4 - 3x_5 = 0$$

$$x_3 - x_4 + 2x_5 = 0$$

x_2, x_4, x_5 free

<u>Spanning set</u>

(a) $\mathbf{N}(A) =$ all

$$\begin{bmatrix} -2x_2 - 5x_4 + 3x_5 \\ x_2 \\ x_4 - 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

7. Determine whether or not the following sets of vectors are bases for $\mathbb{R}^3$. In case the vectors are *not* linearly independent, find a way to write one vector as a linear combination of the other two.

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \right\}$

(a) Put into matrix:

$$A = \begin{bmatrix} 1 & 3 & 6 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 6 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$R \neq I$, so vectors are <u>not</u> a basis.

Dependence: $x_1 + 3x_3 = 0$
$x_2 + x_3 = 0$ $\longrightarrow N(A) = $ all $\begin{bmatrix} -3x_3 \\ -x_3 \\ x_3 \end{bmatrix}$

$$= x_3 \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix}$$

So $-3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1) \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = \vec{0}$

$$\longrightarrow \begin{bmatrix} 6 \\ 5 \\ 4 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

(b) $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 2 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Already 3 leading 1's

More elimination

$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

These vectors <u>are</u> a basis of $\mathbb{R}^3$.

8. Find numbers c that give dependent columns, so that a combination of the columns equals 0. For each value of c that you find, write one column of each matrix as a linear combination of the other two.

(a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & c \end{bmatrix}$

(b) $B = \begin{bmatrix} 1 & 0 & c \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$

(c) $C = \begin{bmatrix} c & c & c \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$

Can find c by inspection

(a) $\begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

So $c=3$ will work, and we have written the 3rd column as a linear combination of the first 2.

(b) $-\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$. So $c=-1$ will work.

(c) Here, $c=0$ should work. Find null space to get the linear combination.

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix} \xrightarrow[\text{Then: Row 1} \leftrightarrow \text{Row 3}]{\text{Row 3} - \text{Row 2}} \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 2} - 2\,\text{Row 1}}$$

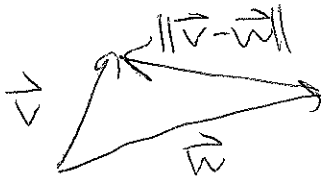
$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{Col 3} = 3\,\text{Col 1} - \text{Col 2}}$$

$$\begin{bmatrix} 0 \\ 5 \\ 6 \end{bmatrix} = 3\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

~~7. (12 points) Sketch the graph of the function $f(x) = \ldots$ Label all asymptotes, roots, maximum or minimum points, and inflection points.~~

Problem 3 $\|\vec{v}\| = 3$ and $\|\vec{w}\| = 5$

(a) Smallest and largest values of $\|\vec{v} - \vec{w}\|$:



Triangle inequality says

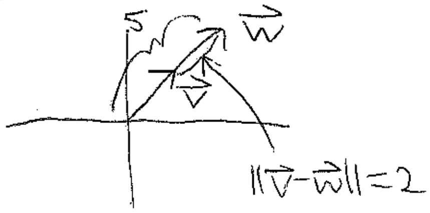
$\|\vec{v} - \vec{w}\| \leq \|\vec{v}\| + \|-\vec{w}\| = \|\vec{v}\| + \|\vec{w}\| = 3 + 5 = 8$

and $\|\vec{w}\| = \|(\vec{v} - \vec{w}) + (-\vec{v})\| \leq \|\vec{v} - \vec{w}\| + \|\vec{v}\|$

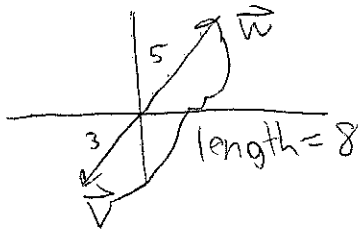
$\rightarrow \|\vec{v} - \vec{w}\| \geq \|\vec{w}\| - \|\vec{v}\| = 5 - 3 = 2$

So $2 \leq \|\vec{v} - \vec{w}\| \leq 8$

smallest value when:



largest value when:



(b) Schwarz inequality:

$|\vec{v} \cdot \vec{w}| \leq \|\vec{v}\| \|\vec{w}\| = 3 \cdot 5 = 15$

$\rightarrow -15 \leq \vec{v} \cdot \vec{w} \leq 15$

smallest value when they point in opposite directions:

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos 180°$

$= -15$

largest value when they point in same direction

$\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos 0°$

$= 15$