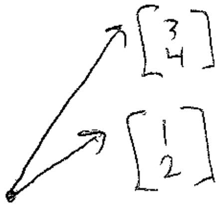
Section 1.3 Matrices

<u>So far</u>: We have introduced vectors and some algebraic operations:

$$\begin{bmatrix}1\\2\end{bmatrix}+\begin{bmatrix}3\\4\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}, \quad 3\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}3\\6\end{bmatrix}, \quad \underbrace{\begin{bmatrix}1\\2\end{bmatrix}\cdot\begin{bmatrix}3\\4\end{bmatrix}=(1)(3)+(2)(4) = 11}_{}$$



For geometry with vectors: lengths and angles

<u>Today</u>: Introduce <u>matrices</u> and linear equations

$$\begin{bmatrix}1 & 2 & 3\\4 & 5 & 6\end{bmatrix}$$

Here is a 2×3 matrix: It is a rectangular array of numbers. This one has 2 rows and 3 columns.

Here is a 3×2 matrix: $\begin{bmatrix}1 & 4\\2 & 5\\3 & 6\end{bmatrix}$ $\qquad 2\times 2$: $\begin{bmatrix}1 & 4\\2 & 5\end{bmatrix}$

1×4: $\begin{bmatrix}1 & 2 & 3 & 4\end{bmatrix}$

In general, an $m\times n$ matrix has m rows and n columns

Typical abstract notation: $A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\a_{21} & a_{22} & \cdots & a_{2n}\\\vdots & & \ddots & \vdots\\a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}$

<u>Notice</u>: A <u>column vector</u> in n dimensions is an $n\times 1$ matrix:

$$\vec{v}=\begin{bmatrix}v_1\\v_2\\\vdots\\v_n\end{bmatrix}$$

You can think of a matrix as a way to package information: the information of $m\times n$ real numbers (scalars)

Why would we want to package information in rectangular form?

One reason: We have some algebraic operations with matrices that let us manipulate the information in a matrix.

First operation: Matrix-vector multiplication

(m×n matrix)(n-dim vector) = (m-dim. vector)

$\hookrightarrow$ n×1 matrix

$$\underbrace{\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}}_{2\times 3 \text{ matrix}} \underbrace{\overset{\vec{v}}{\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}}_{3\times 1 \text{ column vector}} = \begin{bmatrix} 1(-1)+2(-1)+3(1) \\ 4(-1)+5(-1)+6(1) \end{bmatrix} = \underbrace{\overset{A\vec{v}}{\begin{bmatrix} 0 \\ -3 \end{bmatrix}}}_{2\text{-dim. column vector}}$$

"Row picture" of this operation:

The 1st component of $A\vec{v}$ = (1st row of A) $\cdot \vec{v}$ Dot product

2nd component of $A\vec{v}$ = (2nd row of A) $\cdot \vec{v}$

3rd ... } However many
4th ... } rows you have

"Column picture" of this operation:

Think of the 3 columns of $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ as 3 vectors in 2-dim. space: $\vec{u} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ (i.e., A contains the <u>information</u> of 3 vectors)

Now: the matrix-vector product $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ calculates a <u>linear combination</u>: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 4 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ 5 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 6 \end{bmatrix}$

$$= \begin{bmatrix} -1-2+3 \\ -4-5+6 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$$ <u>Same result</u>

Another example:

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + (-1)\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 8 & -5 \\ 12 & -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 6 \end{bmatrix}$$

2 column vectors in 3-dim. space, $\vec{u}, \vec{v}$

This vector is a linear combination of $\vec{u}$ and $\vec{v}$.

We now have two ways of thinking about linear combinations:

1. Numbers multiplying vectors: $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$
2. Matrix multiplying a vector:

$$\begin{array}{c} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \\ \begin{array}{ccc} \text{1st column} & \text{2nd column} & \end{array} \end{array} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

Problem 1.3.1 $\vec{s}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \vec{s}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \vec{s}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Use matrix-vector multiplication to calculate:

$$\underset{x_1}{3}\vec{s}_1 + \underset{x_2}{4}\vec{s}_2 + \underset{x_3}{5}\vec{s}_3 = \underset{\vec{s}_1 \ \vec{s}_2 \ \vec{s}_3}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix}} = \begin{bmatrix} 1(3)+0(4)+0(5) \\ 1(3)+1(4)+0(5) \\ 1(3)+1(4)+1(5) \end{bmatrix} = \underset{\vec{b}}{\begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}}$$

compute using dot products ("row picture")

Using matrix notation:

$$S\vec{x} = \vec{b}$$

Here, we knew S and $\vec{x}$, so we could calculate $\vec{b} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}$

Reverse problem: What if we knew S and $\vec{b}$? Could we find $\vec{x}$?

Problem 1.3.2: Solve $S\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\vec{x}$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longleftrightarrow \begin{bmatrix} x_1 + 0x_2 + 0x_3 \\ x_1 + x_2 + 0x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Or, $\begin{cases} x_1 = 1 \\ x_1 + x_2 = 1 \\ x_1 + x_2 + x_3 = 1 \end{cases}$ A system of linear equations!

We can solve it: $x_1 = 1$ ✓

$x_2 = 1 - x_1 = 1 - 1 = 0$ ✓

$x_3 = 1 - x_1 - x_2 = 1 - 1 - 0 = 0$ ✓

So $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and indeed $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Or: $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

This is a linear combination of the 3 vectors (actually, it <u>is</u> one of the 3 vectors)

<u>Harder question</u>: Is <u>every</u> vector in 3-dim. space a linear combination of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$?

<u>Solution</u>: Every vector in 3-dim space looks like this: $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

(b_1, b_2, b_3 can be any real numbers)

The question is: can we find x_1, x_2, x_3 so that

They would depend on b_1, b_2, b_3.

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ ??}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

we need to solve a matrix-vector equation for x_1, x_2, x_3

$$\begin{bmatrix} x_1 + 0x_2 + 0x_3 \\ x_1 + x_2 + 0x_3 \\ x_1 + x_2 + x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \longrightarrow \begin{cases} x_1 = b_1 \\ x_1 + x_2 = b_2 \\ x_1 + x_2 + x_3 = b_3 \end{cases}$$

We can solve for x_1, x_2, x_3 <u>in terms of</u> the general parameters b_1, b_2, b_3:

$x_1 = b_1$ ✓

$x_2 = b_2 - x_1 = b_2 - b_1$ ✓

$x_3 = b_3 - x_1 - x_2$

$= b_3 - b_1 - (b_2 - b_1) = b_3 - b_2$ ✓

This tells us:

$$b_1\begin{bmatrix}1\\1\\1\end{bmatrix}+(b_2-b_1)\begin{bmatrix}0\\1\\1\end{bmatrix}+(b_3-b_2)\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

No matter what I give you for $\vec{b}$, you can tell me $\vec{x}$:

If $\vec{b}=\begin{bmatrix}1\\4\\9\end{bmatrix}$, then $\vec{x}=\begin{bmatrix}b_1\\b_2-b_1\\b_3-b_2\end{bmatrix}=\begin{bmatrix}1\\4-1\\9-4\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}$.

Let's look at this formula more carefully:

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}b_1+0b_2+0b_3\\-b_1+b_2+0b_3\\0b_1-b_2+b_3\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

The original problem was: Solve $S\vec{x}=\vec{b}$ for $\vec{x}$.

The answer turns out to be: $\vec{x}=\underbrace{\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}}\begin{bmatrix}\vec{b}\end{bmatrix}$

$S\vec{x}=\vec{b}\longrightarrow\vec{x}=S^{-1}\vec{b}$

If you can find an "inverse" matrix, you can use it to solve linear equations

This matrix is called the <u>inverse</u> of S, or S^{-1}.

<u>Easy question</u>: What's the inverse of a 1×1 matrix?

$A=[a]$ (really just a number)

$A\vec{x}=\vec{b}$ is really just $ax=b \rightsquigarrow x=\underbrace{a^{-1}}b$

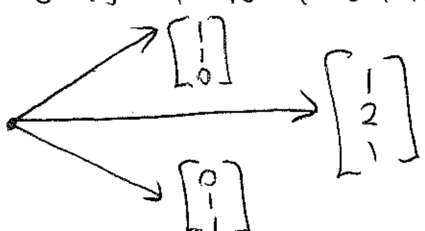
This is the inverse.

So $[a]^{-1}=[a^{-1}]$

<u>But this only works if $a\neq 0$!!</u>

It's possible that a matrix doesn't have an inverse.

<u>Example</u>: 3 vectors that lie in a plane



(the vector sum)

Can we write $\begin{bmatrix}1\\0\\0\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}$?

Try to solve: $\begin{bmatrix}1&0&1\\1&1&2\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\0\\0\end{bmatrix} \rightsquigarrow$

$$\begin{cases} x_1+x_3=1 \\ x_1+x_2+2x_3=0 \\ x_2+x_3=0 \end{cases}$$

$x_2+x_3=0 \longrightarrow x_3=-x_2$

$x_1+x_3=1 \longrightarrow x_1-x_2=1$

$x_1+x_2+2x_3=0 \longrightarrow x_1-x_2=0$

These equations are inconsistent!

There is <u>no solution</u> to this system of equations.

The problem is that $\begin{bmatrix}1\\2\\1\end{bmatrix}$ is a linear combination of $\begin{bmatrix}1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\end{bmatrix}$.

It is "dependent."

This means the matrix $\begin{bmatrix}1&0&1\\1&1&2\\0&1&1\end{bmatrix}$ has <u>no inverse</u> (just like 0^{-1} doesn't exist)

When the columns of the matrix are dependent (no inverse exists) we might also get <u>infinitely many</u> solutions to linear equations.

<u>Example</u> How many ways can we write the zero vector $\begin{bmatrix}0\\0\\0\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\1\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\1\end{bmatrix}$, and $\begin{bmatrix}1\\2\\1\end{bmatrix}$.

One way: $\begin{bmatrix}0\\0\\0\end{bmatrix}=0\begin{bmatrix}1\\1\\0\end{bmatrix}+0\begin{bmatrix}0\\1\\1\end{bmatrix}+0\begin{bmatrix}1\\2\\1\end{bmatrix}$

Other ways: Solve $\begin{bmatrix}1&0&1\\1&1&2\\0&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \longleftrightarrow \begin{cases} x_1+x_3=0 \\ x_1+x_2+2x_3=0 \\ x_2+x_3=0 \end{cases}$

$$\longrightarrow \begin{cases} x_1=-x_3 \\ x_2=-x_3 \end{cases}$$

$x_1+x_2+2x_3=-x_3-x_3+2x_3=0$ ✓ (No problem. Also, no new information about x_1, x_2, x_3)

This shows x_3 is allowed to be <u>any</u> real number. But once we've picked x_3, we must take $x_1 = -x_3$ and $x_2 = -x_3$.

So all solutions look like $\begin{bmatrix} -c \\ -c \\ c \end{bmatrix} = c \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$ (c can be any real number)

For example, if we pick $c=1$: tells us that

$$(-1)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + (-1)\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

<u>Note</u>: Today we <u>briefly</u> looked at some of the major ideas in solving linear equations. We will look at these ideas in more detail later, so don't worry if you didn't completely understand everything today.