Linear Algebra – Homework 9

29 Nov 2023
Due: 7 Dec 2023

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 4.1.6. This system of equations $A\mathbf{x} = \mathbf{b}$ has *no solution* (they lead to $0 = 1$):

$$\begin{aligned} x + 2y + 2z &= 5 \\ 2x + 2y + 3z &= 5 \\ 3x + 4y + 5z &= 9 \end{aligned}$$

Find numbers y_1, y_2, y_3 to multiply these equations so they add to $0 = 1$. You have found a vector $\mathbf{y}$ in which subspace? Its dot product $\mathbf{y}^T\mathbf{b} = 1$, so no solution $\mathbf{x}$.

Problem 4.1.11. Draw Figure 4.2 (the "big picture") to show each subspace correctly for

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}.$$

Problem 4.1.17. If $\mathbf{S}$ is the subspace of $\mathbf{R}^3$ containing only the zero vector, what is $\mathbf{S}^\perp$? If $\mathbf{S}$ is spanned by $(1, 1, 1)$, what is $\mathbf{S}^\perp$? If $\mathbf{S}$ is spanned by $(1, 1, 1)$ and $(1, 1, -1)$, what is a basis for $\mathbf{S}^\perp$?

Problem 4.1.30. Suppose A is 3×4 and B is 4×5 and $AB = 0$. Show that every vector in $\mathbf{C}(B)$ is also a vector in $\mathbf{N}(A)$. Then prove from the dimensions of $\mathbf{N}(A)$ and $\mathbf{C}(B)$ that $\operatorname{rank}(A) + \operatorname{rank}(B) \leq 4$.

Problem 4.2.5. Compute the projection matrices $\mathbf{a}\mathbf{a}^T/\mathbf{a}^T\mathbf{a}$ onto the lines through $\mathbf{a}_1 = (-1, 2, 2)$ and $\mathbf{a}_2 = (2, 2, -1)$. Multiply these projection matrices and explain why their product P_1P_2 is what it is.

Problem 4.2.6. Project $\mathbf{b} = (1, 0, 0)$ onto the lines through $\mathbf{a}_1$ and $\mathbf{a}_2$ in Problem 4.2.5 and also onto $\mathbf{a}_3 = (2, -1, 2)$. Add up the three projections $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3$.

Problem 4.2.7. Continuing Problems 4.2.5 and 4.2.6, find the projection matrix P_3 onto $\mathbf{a}_3 = (2, -1, 2)$. Verify that $P_1 + P_2 + P_3 = I$. This is because the basis $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ is orthogonal!

Problem 4.2.19. To find the projection matrix onto the plane $x - y - 2z = 0$, choose two vectors in the plane and make them the columns of A. The plane will be the column space of A! Then compute $P = A(A^TA)^{-1}A^T$.

Problem 4.2.20. To find the projection matrix P onto the same plane $x - y - 2z = 0$, write down a vector $\mathbf{e}$ that is perpendicular to that plane. Compute the projection $Q = \mathbf{e}\mathbf{e}^T/\mathbf{e}^T\mathbf{e}$ and then $P = I - Q$.

Problem 4.2.25. Show that the projection matrix P onto an n-dimensional subspace S of $\mathbf{R}^m$ has rank $r = n$. (*Hint*: The projections $P\mathbf{b}$ fill the subspace S, so S is the _________ of P.)

Problem 4.2.34. Suppose P_1 and P_2 are projection matrices (that is, $P_i^2 = P_i = P_i^T$ for $i = 1, 2$). Prove that P_1P_2 is a projection matrix if and only if $P_1P_2 = P_2P_1$.

Graded Problems.

Problem 1. Suppose $\mathbf{V}$ is the subspace of $\mathbf{R}^5$ spanned by the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \\ 6 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

Find a basis for the orthogonal complement $\mathbf{V}^{\perp}$.

Problem 2. Find the projection matrix P onto the subspace of $\mathbf{R}^4$ spanned by $(1,0,1,0)$ and $(2,-1,2,-1)$. Use P to project the vector $\mathbf{x} = (1,2,3,4)$ onto this subspace. Also, find the length of the error vector, $\|\mathbf{e}\|$.