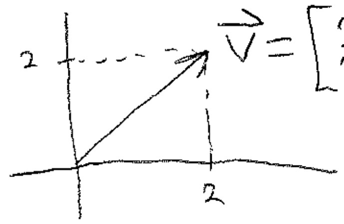
Last time: Mathematical objects in Linear Algebra,

vectors and scalars



$\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ (vector in the plane)

Operations: Vector addition

vector + vector = vector

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Scalar multiplication

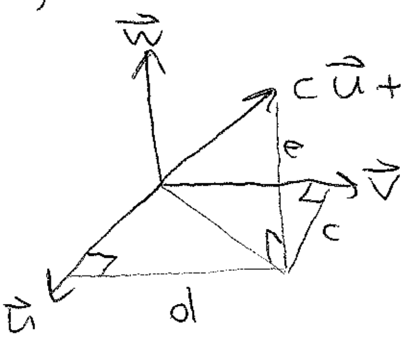
number × vector = vector

$$5 \cdot \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix}$$

Linear combinations of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ (in any dimension):

all vectors $c_1\vec{v}_1 + c_2\vec{v}_2 + \ldots + c_m\vec{v}_m$ where $c_1, c_2, \ldots, c_m$ are any scalars.

In 3 dimensions: usually, linear combinations of 3 vectors $\vec{u}, \vec{v}, \vec{w}$ fill up all of 3-dimensional space:

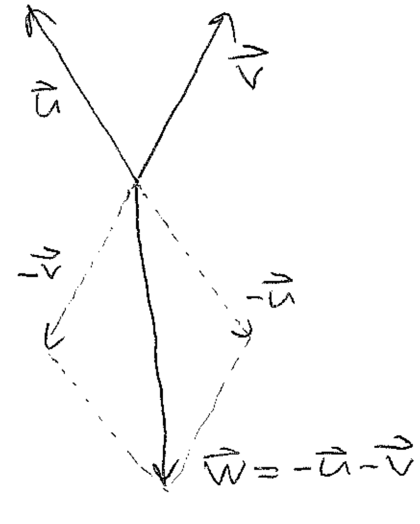


But not if one of $\vec{u}, \vec{v}, \vec{w}$ is already a linear combination of the other 2! Then set of linear combinations is probably a plane sitting inside 3-dim. space (or could be a line).

Problem 1.1.5 $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$

Let's compute $\vec{u} + \vec{v} + \vec{w} = \begin{bmatrix} 1-3+2 \\ 2+1-3 \\ 3-2-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ the zero vector

This tells us that $\vec{w} = -\vec{u} - \vec{v} = (-1)\vec{u} + (-1)\vec{v}$, a linear combination of $\vec{u}$ and $\vec{v}$.



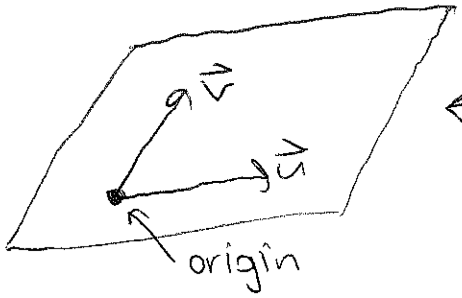
Any linear combination of $\vec{u}, \vec{v}, \vec{w}$ is just a linear combination of $\vec{u}, \vec{v}$:

$c\vec{u} + d\vec{v} + e\vec{w} =$

$c\vec{u} + d\vec{v} + e(-\vec{u} - \vec{v}) =$

$(c-e)\vec{u} + (d-e)\vec{v}$

So the set of linear combinations of $\vec{u}, \vec{v}, \vec{w}$, is just the plane determined by $\vec{u}$ and $\vec{v}$



plane through origin in 3-dimensional space

Here $\vec{u}, \vec{v}, \vec{w}$ are not "independent." Rather $\vec{w}$ is "dependent" on $\vec{u}$ and $\vec{v}$ (it's a linear combination).

Big question: How to tell when a set of vectors is "independent"? (We will study this question later in the course)

Section 1.2 We need one more operation to do geometry with vectors

measure: What are the basic things we measure in geometry?

The dot product of two vectors is a tool for measuring both length of a vector and angle between two vectors → Lengths and angles

Definition: The dot product of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is: 8

$$\vec{u} \cdot \vec{v} = \underbrace{u_1v_1 + u_2v_2 + \cdots + u_nv_n}$$

vector • vector = scalar (not another vector!)

Examples $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 1(3) + (-1)(1) + 2(2) = 6$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -3 \end{bmatrix} = 2(1) + 1(-3) = -1$

Some rules: ① $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (because $u_1v_1 = v_1u_1$, $u_2v_2 = v_2u_2$, ...)

② $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (Distributive law, because $u_1(v_1 + w_1) = u_1v_1 + u_1w_1$, ...)

③ $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v})$ (Associativity: you can move parentheses)

scalar multiplication

dot product

regular multiplication of real numbers

Proof $(c\vec{u}) \cdot \vec{v} = \begin{bmatrix} cu_1 \\ cu_2 \\ \vdots \\ cu_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = (cu_1)v_1 + (cu_2)v_2 + \cdots + (cu_n)v_n$

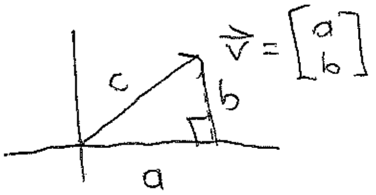
$= c(u_1v_1) + c(u_2v_2) + \cdots + c(u_nv_n) = c(u_1v_1 + u_2v_2 + \cdots + u_nv_n)$

$= c(\vec{u} \cdot \vec{v})$ ✓

These rules show that the dot product behaves like real number multiplication, so it makes sense to call it a "product."

Now what can we *do* with the dot product?

Lengths In 2D:



c = length of $\vec{v}$

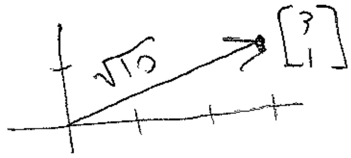
Pythagorean Theorem:

$$c^2 = \underbrace{a^2 + b^2}_{\text{This is } \vec{v} \cdot \vec{v}}$$

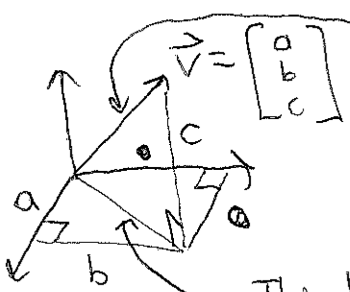
Notation: $\|\vec{v}\|$ = length of $\vec{v}$

Result: $\|\vec{v}\|^2 = \vec{v}\cdot\vec{v}$, or $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$

Example: $\left\|\begin{bmatrix}3\\1\end{bmatrix}\right\| = \sqrt{3^2+1^2} = \sqrt{10}$



In 3D:



This length $= \sqrt{a^2+b^2}$

$$\|\vec{v}\|^2 = \left(\sqrt{a^2+b^2}\right)^2 + c^2$$
$$= a^2+b^2+c^2 = \vec{v}\cdot\vec{v}$$
(again)

So $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$ in 3 dimensions as well.

Example: $\left\|\begin{bmatrix}3\\1\\-2\end{bmatrix}\right\| = \sqrt{3^2+1^2+(-2)^2} = \sqrt{14}$

What about vectors in n dimensions? Same formula:

$\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}}$, that is, $\sqrt{v_1^2+v_2^2+\dots+v_n^2}$

How does scalar multiplication affect lengths?

$$\|c\vec{v}\| = \sqrt{(c\vec{v})\cdot(c\vec{v})} = \sqrt{c^2(\vec{v}\cdot\vec{v})} = \sqrt{c^2}\cdot\sqrt{\vec{v}\cdot\vec{v}} = |c|\cdot\|\vec{v}\|$$

positive square root

$\|\vec{v}\|$

This is what I told you last time.

A <u>unit vector</u> has length $\|\vec{v}\| = 1$.

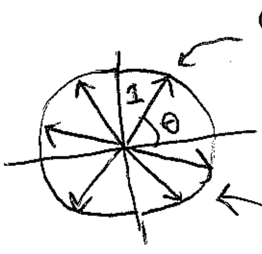
We can scale any <u>non-zero</u> vector to turn it into a unit vector.

<u>Example</u> $\vec{v} = \begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}$, $\|\vec{v}\| = \sqrt{1^2+(-1)^2+(-1)^2+1^2} = 2$ (<u>not</u> a unit vector)

But $\vec{u} = \frac{1}{2}\vec{v}$ <u>is</u> a unit vector: $\|\vec{u}\| = \frac{1}{2}\|\vec{v}\| = \frac{1}{2}(2) = 1$

($-\frac{1}{2}\vec{v}$ is <u>also</u> a unit vector)

<u>Example</u> Find <u>all</u> unit vectors in two dimensions.



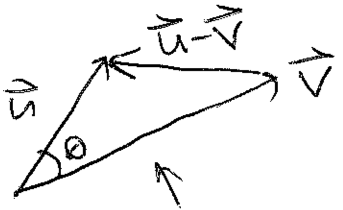
Coordinates are $(\cos\theta, \sin\theta)$

unit circle $x^2+y^2=1$

Every unit vector in 2D looks like $\vec{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ for some angle θ.

$\theta = 0$: x-axis unit vector $\vec{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Notice: angle between $\vec{i}$ and $\vec{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ is θ, and $\underbrace{\vec{i} \cdot \vec{u} = \cos\theta}_{\text{relation between dot product and angle}}$

Angles: Dot product also measures angle between vectors



can be in any dimension. But since $\vec{u}, \vec{v}$ determine a plane (it's the plane of linear combinations $c_1\vec{u} + c_2\vec{v}$), we can measure angles in this plane

If $\theta = \frac{\pi}{2}$ (or $90°$), Pythagorean Thm. says $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

In general, Law of Cosines says:

$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\|\vec{v}\| \cdot \cos\theta$

$(\vec{u}-\vec{v}) \cdot (\vec{u}-\vec{v}) =$

$\vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v} =$

$\|\vec{u}\|^2 - 2(\vec{u}\cdot\vec{v}) + \|\vec{v}\|^2$

Simplify: $-2(\vec{u}\cdot\vec{v}) = -2\|\vec{u}\|\|\vec{v}\|\cos\theta \rightarrow \boxed{\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}}$

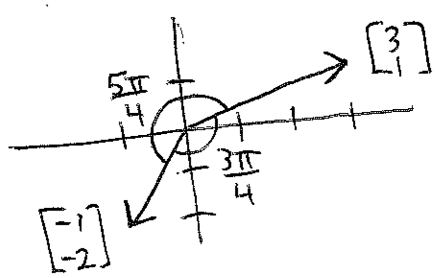
Example: Angle between $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ -2 \end{bmatrix}$

$$\cos\theta = \frac{\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ -2 \end{bmatrix}}{\left\|\begin{bmatrix} 3 \\ 1 \end{bmatrix}\right\| \left\|\begin{bmatrix} -1 \\ -2 \end{bmatrix}\right\|} = \frac{-3-2}{\sqrt{9+1}\sqrt{1+4}} = \frac{-5}{\sqrt{10}\sqrt{5}} = -\frac{1}{\sqrt{2}}$$

θ is an inverse cosine of $-\frac{1}{\sqrt{2}}$

Two options between 0 and 2π,

$\theta = \frac{3\pi}{4}$ or $\frac{5\pi}{4}$



Some applications of the dot product-angle formula:

Test for <u>perpendicular</u> or <u>orthogonal</u> vectors:

If $\vec{u}\cdot\vec{v}=0$, then $\cos\theta=0$, so $\theta=\frac{\pi}{2}$ (or $90°$).

<u>Schwarz</u> <u>Inequality</u>: $|\vec{v}\cdot\vec{w}|\le\|\vec{v}\|\,\|\vec{w}\|$

Why? Well, we know $\vec{v}\cdot\vec{w}=\|\vec{v}\|\,\|\vec{w}\|\underbrace{\cos\theta}_{\text{between } -1 \text{ and } 1}$

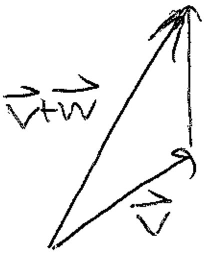
So $|\vec{v}\cdot\vec{w}|=\|\vec{v}\|\,\|\vec{w}\|\underbrace{|\cos\theta|}_{\le 1}\le\|\vec{v}\|\,\|\vec{w}\|$

Strang calls this "the most important inequality in mathematics"

Here's another important one:

<u>Triangle</u> <u>Inequality</u>: $\|\vec{v}+\vec{w}\|\le\|\vec{v}\|+\|\vec{w}\|$

or $\|\vec{v}-\vec{w}\|$ (replace $\vec{w}$ with $-\vec{w}$)



3rd side $\le$
1st side + 2nd side

To prove: compare $\|\vec{v}+\vec{w}\|^2$ with $(\|\vec{v}\|+\|\vec{w}\|)^2$

$(\vec{v}+\vec{w})\cdot(\vec{v}+\vec{w})=\|\vec{v}\|^2+2\,\vec{v}\cdot\vec{w}+\|\vec{w}\|^2$ $\qquad$ $\|\vec{v}\|^2+2\|\vec{v}\|\,\|\vec{w}\|+\|\vec{w}\|^2$

$$\le\|\vec{v}\|^2+2|\vec{v}\cdot\vec{w}|+\|\vec{w}\|^2$$

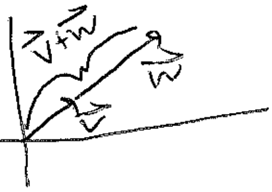
Schwarz $\longrightarrow$ $\le\|\vec{v}\|^2+2\|\vec{v}\|\,\|\vec{w}\|+\|\vec{w}\|^2$

$$=(\|\vec{v}\|+\|\vec{w}\|)^2$$

We see $\|\vec{v}+\vec{w}\|^2\le(\|\vec{v}\|+\|\vec{w}\|)^2$. Take $\sqrt{\ }$: $\|\vec{v}+\vec{w}\|\le\|\vec{v}\|+\|\vec{w}\|$ ✓

<u>Question</u>: Is it possible for $\|\vec{v}+\vec{w}\|$ to actually <u>equal</u> $\|\vec{v}\|+\|\vec{w}\|$?

Yes! <u>If</u> $\vec{v},\vec{w}$ are pointing in the same direction:



(Lengths add here)