1. (a) (10 points) Find all solutions of the system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - 5x_4 + 6x_5 &= 1 \\ -x_1 - 2x_2 - x_3 + x_4 - x_5 &= 1 \\ 4x_1 + 8x_2 + 5x_3 - 8x_4 + 9x_5 &= -2 \end{aligned}$$

(b) (4 points) Find the reduced row echelon form R of the coefficient matrix of the system.

Augmented matrix:

$$\left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 1 \\ -1 & -2 & -1 & 1 & -1 & 1 \\ 4 & 8 & 5 & -8 & 9 & -2 \end{array}\right] \xrightarrow[\text{Row } 3 - 4\text{ Row } 1]{\text{Row } 2 + \text{Row } 1} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 1 \\ 0 & 0 & 1 & -4 & 5 & 2 \\ 0 & 0 & -3 & 12 & -15 & -6 \end{array}\right]$$

$$\xrightarrow{\text{Row } 3 + 3\text{ Row } 2} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & 1 \\ 0 & 0 & 1 & -4 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Row } 1 - 2\text{ Row } 2}$$

$$\underbrace{\left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & -3 \\ 0 & 0 & 1 & -4 & 5 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]}_{\text{This is } R.} \rightsquigarrow \begin{aligned} x_1 + 2x_2 + 3x_4 - 4x_5 &= -3 \\ x_3 - 4x_4 + 5x_5 &= 2 \end{aligned}$$

x_2, x_4, x_5 free

All solutions: $\vec{x} = \begin{bmatrix} -2x_2 - 3x_4 + 4x_5 - 3 \\ x_2 \\ 4x_4 - 5x_5 + 2 \\ x_4 \\ x_5 \end{bmatrix}$

$$= \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

2

2. (a) (6 points) How long is the vector $\mathbf{v} = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

(b) (4 points) Pick any numbers x, y, z such that $x + y + z = 0$. Find the angle between your vector $\mathbf{v} = (x, y, z)$ and the vector $\mathbf{w} = (z, x, y)$.

(a) $\|\vec{v}\| = \sqrt{\vec{v}\cdot\vec{v}} = \sqrt{\underbrace{1+1+\ldots+1}_{9 \text{ times}}} = \sqrt{9} = 3$

Can scale $\vec{v}$ by its length to get a unit vector:

$\vec{u} = \frac{1}{3}\vec{v} = \underbrace{\left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)}_{9 \text{ times}}$

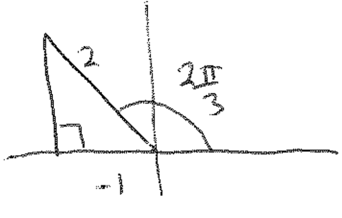
Perpendicular: $\vec{v}\cdot\vec{w} = \vec{0}$. $\vec{v}\cdot(1, -1, \underbrace{0, \ldots, 0}_{7 \text{ times}}) = 0$,

but need to scale by length:

$\vec{w} = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \underbrace{0, \ldots, 0}_{7 \text{ times}}\right)$ is a unit vector perpendicular to $\vec{v}$.

(b) Pick $x = 1$, $y = 4$, $z = -5$.

Angle: $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \dfrac{(1,4,-5)\cdot(-5,1,4)}{\sqrt{1+16+25}\,\sqrt{1+16+25}} = \dfrac{-21}{42} = -\dfrac{1}{2}$



$\cos\theta = -\frac{1}{2}^{3} \longrightarrow \boxed{\theta = \frac{2\pi}{3}}$

3.

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(a) LU decomposition of $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 24 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 24 \end{bmatrix} \xrightarrow[\text{Row } 3 - \boxed{3} \text{ Row } 1]{\text{Row } 2 - \boxed{2} \text{ Row } 1} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & -4 & 15 \end{bmatrix} \xrightarrow{\text{Row } 3 - \boxed{(-4)} \text{ Row } 2} \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}}_{U}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix}$$

So $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -4 \\ 0 & 0 & -1 \end{bmatrix}$

(b) Solve $A\vec{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \longrightarrow L(\underbrace{U\vec{x}}_{\vec{y}}) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$

Solve $L\vec{y} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ first, then solve $U\vec{x} = \vec{y}$.

$$\begin{aligned} y_1 \quad\quad\quad\quad &= 1 \\ 2y_1 + y_2 \quad\quad &= 3 \\ 3y_1 - 4y_2 + y_3 &= 0 \end{aligned} \longrightarrow \begin{aligned} y_1 &= 1 \\ y_2 &= 3 - 2(1) = 1 \\ y_3 &= -3(1) + 4(1) = 1 \end{aligned}$$

Now solve $U\vec{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ for $\vec{x}$:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ x_2 - 4x_3 &= 1 \\ -x_3 &= 1 \end{aligned} \longrightarrow \begin{aligned} x_1 &= 1 - 2(-3) - 3(-1) = 10 \\ x_2 &= 1 + 4(-1) = -3 \\ x_3 &= -1 \end{aligned}$$

So $\vec{x} = \begin{bmatrix} 10 \\ -3 \\ -1 \end{bmatrix}$

4. (a) (8 points) Show that the set of all vectors (b_1, b_2, b_3) such that $b_1 + b_2 + b_3 = 0$ is a subspace of $\mathbf{R}^3$. (Verify all three properties of a subspace.)

(b) (6 points) Show that the set of all vectors (b_1, b_2, b_3) such that $b_1 \le b_2 \le b_3$ is *not* a subspace of $\mathbf{R}^3$. (Show that at least one property of a subspace fails.)

(a) S = all $\vec{b}$ with $b_1 + b_2 + b_3 = 0$

1. Is $\vec{0}$ in S?

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow 0 + 0 + 0 = 0 \checkmark$$

2. If $\vec{b}$ and $\vec{c}$ are in S, what about $\vec{b} + \vec{c}$?

$$\vec{b} + \vec{c} = \begin{bmatrix} b_1 + c_1 \\ b_2 + c_2 \\ b_3 + c_3 \end{bmatrix} \rightsquigarrow (b_1 + c_1) + (b_2 + c_2) + (b_3 + c_3)$$

$$= (b_1 + b_2 + b_3) + (c_1 + c_2 + c_3)$$

$$= 0 + 0 = 0 \checkmark$$

3. If $\vec{b}$ is in S, what about $c\vec{b}$?

$$c\vec{b} = \begin{bmatrix} cb_1 \\ cb_2 \\ cb_3 \end{bmatrix} \rightsquigarrow cb_1 + cb_2 + cb_3$$

$$= c(b_1 + b_2 + b_3)$$

$$= c \cdot 0 = 0 \checkmark$$

(b) T = all $\vec{b}$ with $b_1 \le b_2 \le b_3$

T is <u>not</u> closed under scalar multiplication:

$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is in T, but $(-1)\begin{bmatrix} +1 \\ 2 \\ 3 \end{bmatrix}$ is <u>not</u> because $-1 > -2 > -3$.

5. Consider the system of linear equations:

$$\begin{array}{ccccccc} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & + & 4x_2 & + & 9x_3 & = & -2 \end{array}$$

(a) (10 points) Find the *inverse* of the coefficient matrix of the system of equations.

(b) (4 points) Use the inverse matrix to solve the system of linear equations.

(a) Elimination method:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}-\text{Row 1}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-3\text{Row 2}]{\text{Row 3}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right] \xrightarrow[\text{Row 2}-\text{Row 3}]{\text{Row 1}-\frac{1}{2}\text{Row 3}} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 3/2 & -1/2 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}\text{Row 3}]{\text{Row 1}-\text{Row 2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -5/2 & 1/2 \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -3/2 & 1/2 \end{array}\right]$$

This is the inverse.

(b) $A\vec{x} = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} \longrightarrow \vec{x} = A^{-1}\vec{b}$

$$= \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -3 \\ 7 \\ -3 \end{bmatrix}$$

6. (12 points) Determine whether the following vectors form a basis for $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 11 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

If the vectors are *not* linearly independent, show how to write one of the them as a linear combination of the others.

Put into a matrix:

$$\begin{bmatrix} -1 & -1 & 3 & 1 \\ 1 & 3 & 1 & 2 \\ -1 & 3 & 11 & 3 \\ 1 & 1 & -3 & 4 \end{bmatrix} \xrightarrow[\substack{\text{Row 3}-\text{Row 1} \\ \text{Row 4}+\text{Row 1}}]{\text{Row 2}+\text{Row 1}} \begin{bmatrix} -1 & -1 & 3 & 1 \\ 0 & 2 & 4 & 3 \\ 0 & 4 & 8 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{5}\text{Row 4}]{\substack{-\text{Row 1} \\ \text{Row 3}-2\text{Row 2}}}$$

$$\begin{bmatrix} +1 & +1 & -3 & -1 \\ 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 3}+4\text{Row 4}]{\substack{\text{Row 1}+\text{Row 4} \\ \text{Row 2}-3\text{Row 4}}} \begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 3} \leftrightarrow \text{Row 4}]{\frac{1}{2}\text{Row 2}}$$

free variable, <u>not</u> independent

$$\begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1}-\text{Row 2}} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Linear relation: $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$

From null space vector

$$x_1 - 5x_3 = 0$$
$$x_2 + 2x_3 = 0$$
$$x_3 \text{ free}$$
$$x_4 = 0$$

$$\longrightarrow \vec{x} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

$$5\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} -1 \\ 3 \\ 3 \\ +1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 1 \\ 11 \\ -3 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \vec{0}$$

$$\begin{bmatrix} 3 \\ \cancel{0}1 \\ 11 \\ -3 \end{bmatrix} = -5\begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

7. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}.$$

(a) (6 points) Find a linear relation on b_1, b_2, b_3, b_4 that guarantees that $\mathbf{b} = (b_1, b_2, b_3, b_4)$ is a vector in the column space $\mathbf{C}(A)$.

(b) (8 points) Find a spanning set (the special solutions) for the null space $\mathbf{N}(A)$.

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & b_1 \\ 0 & 1 & -3 & -1 & b_2 \\ 3 & 4 & -6 & 8 & b_3 \\ 0 & -1 & 3 & 4 & b_4 \end{array}\right] \xrightarrow{\text{Row } 3 - 3\text{Row } 1} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & b_1 \\ 0 & 1 & -3 & -1 & b_2 \\ 0 & 4 & -12 & -4 & -3b_1 + b_3 \\ 0 & -1 & 3 & 4 & b_4 \end{array}\right]$$

$$\xrightarrow[\text{Row } 4 + \text{Row } 2]{\text{Row } 3 - 4\text{ Row } 2} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & b_1 \\ 0 & 1 & -3 & -1 & b_2 \\ 0 & 0 & 0 & 0 & -3b_1 - 4b_2 + b_3 \\ 0 & 0 & 0 & 3 & b_2 + b_4 \end{array}\right]$$

(a) Solutions exist only if

$$-3b_1 - 4b_2 + b_3 = 0$$

$\frac{1}{3}$ Row 4

Row 3 ↔ Row 4

(b)

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{Row } 2 + \text{Row } 3]{\text{Row } 1 - 4\text{Row } 3} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Null space: $x_1 + 2x_3 = 0$ $\quad x_3$ free

$x_2 - 3x_3 = 0$ $\quad x_4 = 0$

$$\vec{x} = \begin{bmatrix} -2x_3 \\ 3x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 3 \\ 1 \\ 0 \end{bmatrix}$$

One vector in the spanning set (one special solution)

8. (10 points) Find all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$A^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = - \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A.$$

Show that every matrix A that satisfies this property is a scalar multiple of one particular 2×2 matrix.

$$A^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} c & a \\ d & b \end{bmatrix}$$

$$-\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} -c & -d \\ -a & -b \end{bmatrix}$$

They are equal if $c = -c$, $a = -d$
$d = -a$, $b = -b$

So $d = -a$, $c = 0$, $b = 0$ $\longrightarrow$

$$A = \begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Every matrix A such that

$A^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A$ is a

multiple of this one.