So $e^{tA} = X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots & e^{\lambda_n t} \end{bmatrix} X^{-1}$

So here are the solutions to $\vec{u}'(t) = A\vec{u}(t)$ when $A = X\Lambda X^{-1}$:

$$\vec{u}(t) = X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots & e^{\lambda_n t} \end{bmatrix} X^{-1}\vec{u}(0)$$

Let's check this with our previous example, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$:

$\lambda_1 = 1$, $\lambda_2 = 3$, $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $A = X\Lambda X^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$\frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

The solutions to $\vec{u}'(t) = A\,\vec{u}(t)$ are:

$$\vec{u}(t) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} -e^t & e^{3t} \\ e^t & e^{3t} \end{bmatrix} \begin{bmatrix} -\frac{1}{2}u_1(0) + \frac{1}{2}u_2(0) \\ \frac{1}{2}u_1(0) + \frac{1}{2}u_2(0) \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(u_1(0) - u_2(0))e^t + \frac{1}{2}(u_1(0) + u_2(0))e^{3t} \\ -\underbrace{\frac{1}{2}(u_1(0) - u_2(0))}e^t + \underbrace{\frac{1}{2}(u_1(0) + u_2(0))}e^{3t} \end{bmatrix}$$

These are the C and D from before.

Solve when $\vec{u}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$:

$u_1(t) = -\frac{1}{2}e^t + \frac{5}{2}e^{3t}$, $u_2(t) = \frac{1}{2}e^t + \frac{5}{2}e^{3t}$, like before.

Matrix exponential is still useful when eigenvalues don't behave well (repeated roots)

<u>Example</u> $y'' - 2y' + y = 0$

2nd-order equation for $y(t)$

Trick to turn into a <u>1st-order system</u>: write $u_1(t) = y(t)$, $u_2(t) = y'(t)$.

Then: $u_1'(t) = u_2(t)$ (by definition)

$$u_2'(t) = y''(t) = -y(t) + 2y'(t) = -u_1(t) + 2u_2(t)$$

That is: $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, or $\begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$

Eigenvalues: $\begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 = 0 \longrightarrow \lambda = 1, 1$

Eigenvectors: solve $\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow x_1 = x_2 \rightarrow \vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

only one independent eigenvector
(matrix is not diagonalizable)

One solution is $\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = Ce^t \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} Ce^t \\ Ce^t \end{bmatrix}$

But since we have a 2×2 system, we should have a second independent solution.

Good news: Matrix exponential does give us all solutions, if only we can calculate e^{tA}!

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = e^{tA}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} \overset{\text{Trick}}{=} e^{tI + t(A-I)}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

Warning: $e^{A+B} = e^A e^B$ only if $AB = BA$.

$$e^{tI} e^{t(A-I)}$$

$$e^{tI} = \begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}$$

$$e^{t(A-I)} = I + t(A-I) + \frac{t^2}{2}(A-I)^2 + \dots =$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{t^2}{2}\underbrace{\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}}_{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\,!!} + \dots =$$

$$= \begin{bmatrix} +1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} + \frac{t^2}{2}\, 0 + 0 + 0 + \cdots$$

<u>Conclusion</u>: $$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}}_{e^{It}} \underbrace{\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \right)}_{e^{t(A-I)}} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix}}_{e^{At}} \underbrace{\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}}_{\vec{u}(0)}$$

<u>Example</u>: Initial value problem with $y(0)=0$, $y'(0)=1$

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \underbrace{\begin{bmatrix} e^t & 0 \\ 0 & e^t \end{bmatrix}}_{e^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \underbrace{\begin{bmatrix} 1-t & t \\ -t & 1+t \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\begin{bmatrix} t \\ 1+t \end{bmatrix}} = \begin{bmatrix} t e^t \\ (1+t)e^t \end{bmatrix}$$

$\boxed{\text{So } y(t) = te^t}$

The 2nd basic solution to $y''-2y'+y=0$ (besides $y(t)=e^t$)

Matrix exponential is <u>also</u> useful when you have complex eigenvalues:

<u>Example</u> $y''=-y \rightarrow u_1(t)=y(t) \rightarrow u_1'=u_2$

$\searrow u_2(t)=y'(t) \qquad u_2'=y''=-y=-u_1$

So $$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

Eigenvalues: $\begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = 0 \rightarrow$

$\lambda^2+1=0 \rightarrow \lambda = \pm i$

So if you don't mind working with complex numbers, the solutions are: $\vec{u}(t) = Ce^{it}\vec{x}_1 + De^{-it}\vec{x}_2$

complex eigenvectors

If you <u>don't</u> like complex exponentials, you could try matrix exponential instead:

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \exp\left(t\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

$$\hookrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{I} + t\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{A} + \frac{t^2}{2}\underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{A^2} + \frac{t^3}{3!}\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A^3} +$$

$$\frac{t^4}{4!}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{A^4 = I\,!} + \frac{t^5}{5!}\underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{\text{Back to } A!} + \cdots$$

$$= \begin{bmatrix} 1 - t^2/2! + t^4/4! - \cdots & t - t^3/3! + t^5/5! - \cdots \\ -t + t^3/3! - t^5/5! + \cdots & 1 - t^2/2! + t^4/4! - \cdots \end{bmatrix}$$

Power series for $\sin t$ and $\cos t$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Conclusion: $$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} y(0)\cos t + y'(0)\sin t \\ -y(0)\sin t + y'(0)\cos t \end{bmatrix}$$

Section 6.4 Symmetric Matrices

We cannot diagonalize every $n \times n$ A: $A = X \Lambda X^{-1}$ because $\mathbb{R}^n$ might not have a basis of eigenvectors.

Helps with calculating A^N, for example; $A^N = X \Lambda^N X^{-1}$.

But if S is symmetric, $S = S^T$, we can always diagonalize.

We can say even more:

"Spectral Theorem"

1. Eigenvalues of symmetric S are all real numbers.
2. S can be diagonalized, even if there are repeated eigenvalues,
3. Eigenvectors with different eigenvalues are orthogonal.

From 2 and 3, $\mathbb{R}^n$ has an orthonormal basis of eigenvectors:

- If eigenvalues are all different, eigenvectors are already orthogonal, so just need to rescale to get unit vectors.
- If an eigenvalue is repeated, can use Gram-Schmidt process to get an orthonormal basis for each eigenspace.

→ This means $S = Q \Lambda Q^T$

For orthogonal matrix, inverse = transpose

diagonal matrix of real eigenvalues

matrix of orthonormal eigenvectors → orthogonal matrix

Example (Problem 6.4.7)

$$S = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix}$$

Eigenvalues: $\det(S-\lambda I) = \begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix} = (1-\lambda)\begin{vmatrix} -1-\lambda & -2 \\ -2 & -\lambda \end{vmatrix} + 2\begin{vmatrix} 0 & -1-\lambda \\ 2 & -2 \end{vmatrix}$

$= (1-\lambda)(\lambda^2+\lambda-4) + 4(1+\lambda) = \cancel{\lambda^2} + \lambda - \cancel{4} - \lambda^3 - \cancel{\lambda^2} + 4\lambda + \cancel{4} + 4\lambda$

$= 9\lambda - \lambda^3 = \lambda(9-\lambda^2) = \lambda(3-\lambda)(3+\lambda) = 0 \longrightarrow \lambda = 0, 3, -3$

Eigenvectors for $\lambda=0$: Solve $S\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \xrightarrow[-\text{Row }2]{\text{Row }3-2\text{Row }1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \xrightarrow{\text{Row }3+2\text{Row }2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} x_1 = -2x_3 \\ x_2 = -2x_3 \\ x_3 \text{ free} \end{array}$$

So $\vec{x} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}$

$\lambda=3$: Solve $(S-3I)\vec{x} = \vec{0}$

$$\begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow[2\text{Row }1]{\text{Row }3\,-} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & -2 & -1 \end{bmatrix} \xrightarrow[+2\text{Row }2]{\text{Row }3} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\begin{array}{l} x_1 = x_3 \\ x_2 = -\frac{1}{2}x_3 \\ x_3 \text{ free} \end{array} \longrightarrow \vec{x} = x_3 \begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix} = \left(\frac{1}{2}x_3\right)\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$

$\lambda=-3$: Solve $(S+3I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 2 & -2 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 2 & -2 & 3 \end{bmatrix} \xrightarrow{\text{Row }3-2\text{Row }1} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow[+2\text{Row }2]{\text{Row }3} \begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

So $\begin{array}{l} x_1 = -\frac{1}{2}x_3 \\ x_2 = x_3 \\ x_3 \text{ free} \end{array} \longrightarrow \vec{x} = x_3 \begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix} = \left(\frac{1}{2}x_3\right)\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$

Basis for $\mathbb{R}^3$ of eigenvectors: $\begin{bmatrix}-2\\-2\\1\end{bmatrix}, \begin{bmatrix}2\\-1\\2\end{bmatrix}, \begin{bmatrix}-1\\2\\2\end{bmatrix}$ orthogonal, but not orthonormal.

Turn into unit vectors to get an orthonormal basis:

$$\vec{x}_1 = \frac{1}{3}\begin{bmatrix}-2\\-2\\1\end{bmatrix}, \quad \vec{x}_2 = \frac{1}{3}\begin{bmatrix}2\\-1\\2\end{bmatrix}, \quad \vec{x}_3 = \frac{1}{3}\begin{bmatrix}-1\\2\\2\end{bmatrix}$$

Now we can diagonalize S:

$$S = Q\Lambda Q^T = \underbrace{\begin{bmatrix}-2/3 & 2/3 & -1/3\\ -2/3 & -1/3 & 2/3\\ 1/3 & 2/3 & 2/3\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & -3\end{bmatrix}}\begin{bmatrix}-2/3 & -2/3 & 1/3\\ 2/3 & -1/3 & 2/3\\ -1/3 & 2/3 & 2/3\end{bmatrix}$$

check: $\begin{bmatrix}0 & 2 & 1\\ 0 & -1 & -2\\ 0 & 2 & -2\end{bmatrix}$ $\quad = \begin{bmatrix}1 & 0 & 2\\ 0 & -1 & -2\\ 2 & -2 & 0\end{bmatrix} = S$ ✓

Now let's prove some parts of the Spectral Theorem:

Orthogonal Eigenvectors: Say $\vec{x}_1$ and $\vec{x}_2$ are eigenvectors.

So $\begin{cases} S\vec{x}_1 = \lambda_1\vec{x}_1 \\ S\vec{x}_2 = \lambda_2\vec{x}_2 \end{cases}$ What is $\vec{x}_1 \cdot \vec{x}_2$?

$$(S\vec{x}_1)\cdot\vec{x}_2 = (\vec{x}_1^T S^T)\vec{x}_2 = \vec{x}_1^T(S\vec{x}_2) = \vec{x}_1\cdot(S\vec{x}_2)$$

$\uparrow$ S is symmetric

$$\lambda_1(\vec{x}_1\cdot\vec{x}_2) \qquad\qquad \lambda_2(\vec{x}_1\cdot\vec{x}_2)$$

$$(\lambda_1-\lambda_2)(\vec{x}_1\cdot\vec{x}_2) = 0$$

Only two possibilities: $\lambda_1 = \lambda_2$ or $\vec{x}_1\cdot\vec{x}_2 = 0$

So ~~if~~ $\vec{x}_1$ and $\vec{x}_2$ are eigenvectors for <u>different</u> eigenvalues ($\lambda_1 \neq \lambda_2$), then $\vec{x}_1 \perp \vec{x}_2$. ✓

<u>Real Eigenvalues</u> Say $S\vec{x} = \lambda\vec{x}$ Might also be complex...

Could λ be a <u>complex</u> number $\lambda = a+ib$? We want to show $b=0$.

Complex conjugate of λ: $\bar{\lambda} = a - ib$.

Note: To say $b=0$ is the same as to say $\bar{\lambda} = a - i0 = a + i0 = \lambda$.

So actually we need to show $\lambda = \bar{\lambda}$.

<u>Claim</u>: $\bar{\lambda}$ is <u>also</u> an eigenvalue of S. Why?

Start with $S(\vec{u} + i\vec{v}) = (a+ib)(\vec{u} + i\vec{v})$

Apply complex conjugation: $\bar{S}(\vec{u} - i\vec{v}) = (a - ib)(\vec{u} - i\vec{v})$

<u>same</u> as S, since S is <u>real</u>.

So $\bar{\lambda} = a - ib$ <u>is</u> an eigenvalue, has eigenvector $\vec{u} - i\vec{v}$.

Now let's look at another dot product:

$$(\vec{u} - i\vec{v})^T (S(\vec{u} + i\vec{v})) \underset{S \text{ is symmetric}}{=} ((\vec{u} - i\vec{v})^T S^T)(\vec{u} + i\vec{v})$$

$$= \lambda (\vec{u} - i\vec{v})^T (\vec{u} + i\vec{v})$$

$$= \lambda\left(\vec{u}^T\vec{u} - i\cancel{\vec{v}^T\vec{u}} + i\cancel{\vec{u}^T\vec{v}} - i^2\,\vec{v}^T\vec{v}\right)$$

$$= \lambda\left(\|\vec{u}\|^2 - (-1)\|\vec{v}\|^2\right)$$

$$= \lambda\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$$

$$((\vec{u} - i\vec{v})^T S^T)(\vec{u} + i\vec{v}) = (S(\vec{u} - i\vec{v}))^T (\vec{u} + i\vec{v})$$

$$= \bar{\lambda}(\vec{u} - i\vec{v})^T(\vec{u} + i\vec{v})$$

also equals

$$\bar{\lambda}\underbrace{\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)}_{\text{not } 0 \text{ since } \vec{u} \text{ and } \vec{v} \text{ can't both be } \vec{0}}$$

not 0 since $\vec{u}$ and $\vec{v}$ can't <u>both</u> be $\vec{0}$ (since $\vec{u} + i\vec{v}$ is a non-zero eigenvector)

$$\lambda\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right) = \bar{\lambda}\left(\|\vec{u}\|^2 + \|\vec{v}\|^2\right)$$

Cancel $\|\vec{u}\|^2 + \|\vec{v}\|^2$, get $\lambda = \bar{\lambda} \rightarrow \boxed{\lambda \text{ is real}}$