PROPERTIES OF CONVEX FUNCTIONS

Definition. Let I be an interval and $f : I \to \mathbb{R}$ be a differentiable function on I. We say that f is *convex* on I if f' is increasing on I, *i.e.* if for all $a, b \in I$, if $b > a$ then $f'(b) > f'(a)$. We say that f is *concave* on I if f' is decreasing on I.

Remark. Some references (including our book Thomas' Calculus) call convex functions *concave up* and concave functions *concave down*. Some references sometimes also call our convex functions *strictly convex* functions, but we follow the convention of the book Thomas' Calculus and do not use the term "strictly convex". Note that according to our definition, a linear function $f(x) = mx + b$ (for some $m, b \in \mathbb{R}$) is neither convex nor concave.

Remark. f is concave if and only if $-f$ is convex.

Proposition. Let I be an interval and $f : I \to \mathbb{R}$ be a twice differentiable function on I. If for all $x \in I$ we have $f''(x) > 0$ then f is convex.

Indeed, if f'' is positive on I, then f' is increasing.

There is a characterisation of convex functions in terms of tangents and secants to the graph of f.

Theorem. Let I be an interval and $f : I \to \mathbb{R}$ be a differentiable function on I. The following assertions are equivalent.

(i) f is convex on I.

(ii) The graph of f is above its tangent at every point of I, *i.e.* for all $a \in I$, for all $x \in I$ with $x \neq a$, we have
$$f(x) > f(a) + f'(a)(x - a) \ .$$

(iii) The graph of f is below its secant on each closed interval $[a, b]$ of I, *i.e.* for all $a, b \in I$ with $a \neq b$, for all $\lambda \in (0, 1)$ we have
$$f(\lambda a + (1 - \lambda)b) < \lambda f(a) + (1 - \lambda)f(b) \ .$$