Last time: How do we find the dimension of a vectors space (like a subspace of $\mathbb{R}^n$)?

Answer: We find a basis: A set of vectors in the vector space that is linearly independent and a spanning set.

Then dimension = number of vectors in a basis

(Every vector space has a basis, and every basis has the same number of vectors, so this definition of dimension makes sense.)

Example: $\mathbb{R}^n$ has lots of different bases, but all of them have n vectors, so $\dim \mathbb{R}^n = n$.

How do you tell if a set of vectors in $\mathbb{R}^n$ is a basis?

$\left\{ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ in $\mathbb{R}^2$. Right number of vectors... but is it a basis?

Check independence: Does $x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ have non-zero solutions?

Check spanning: Does $x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ have a solution for all b_1, b_2?

Is every vector in $\mathbb{R}^2$ a linear combination of these two?

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 3 & 2 & b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & -1 & -3b_1+b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -2b_1+b_2 \\ 0 & -1 & -3b_1+b_2 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cc|c} 1 & 0 & -2b_1+b_2 \\ 0 & 1 & 3b_1-b_2 \end{array}\right] \longleftarrow$$ No row of 0's means solution always exists

Yes! Every $\vec{b}$ is a linear combination in a unique way:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = (-2b_1+b_2)\begin{bmatrix} 1 \\ 3 \end{bmatrix} + (3b_1-b_2)\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Only one choice for x, y (given b_1, b_2)

For $\vec{b}=\vec{0}$, only solution is $x=0$, $y=0$.

$\longrightarrow$ means vectors are also independent, so they are a basis.

General test for deciding if $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$:

$$\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\} \xrightarrow[\text{a matrix}]{\text{Put into}} A = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix} \xrightarrow{\text{Elimination}} R$$

If $R=I$:
- No free variables $\longrightarrow A\vec{x}=\vec{0}$ has unique solution
  $\longrightarrow$ vectors are independent
- No row of 0's $\longrightarrow A\vec{x}=\vec{b}$ always has a solution
  $\longrightarrow$ vectors span $\mathbb{R}^n$.

$\longrightarrow$ So $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis.

If R is not I: Vectors are not a basis (not independent, don't span)

Example: $\left\{ \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}2\\3\\4\end{bmatrix}, \begin{bmatrix}3\\4\\5\end{bmatrix} \right\}$ in $\mathbb{R}^3$

free variable

$$A = \begin{bmatrix}1&2&3\\2&3&4\\3&4&5\end{bmatrix} \longrightarrow \begin{bmatrix}1&2&3\\0&-1&-2\\0&-2&-4\end{bmatrix} \longrightarrow \begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix} \longrightarrow \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix} = R$$

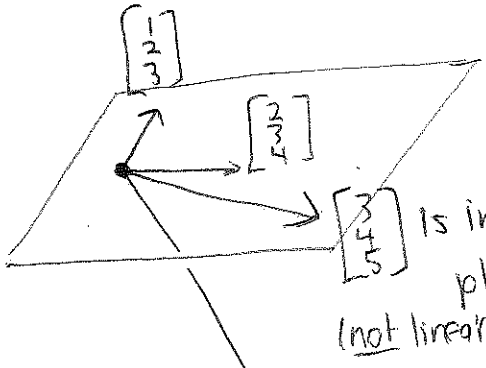
$R \neq I$, so vectors are dependent, not a basis

Null space: $x_1 - x_3 = 0$
$x_2 + 2x_3 = 0$

$$\rightarrow \vec{x} = \begin{bmatrix}x_3\\-2x_3\\x_3\end{bmatrix} = x_3\begin{bmatrix}1\\-2\\1\end{bmatrix}$$

$$\text{So } \begin{bmatrix}1\\2\\3\end{bmatrix} - 2\begin{bmatrix}2\\3\\4\end{bmatrix} + \begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$\text{or } \begin{bmatrix}3\\4\\5\end{bmatrix} = -\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}2\\3\\4\end{bmatrix}$$



is in the same plane (not linearly independent)

Here's a vector that isn't a lin. comb. of the three, so the three don't span $\mathbb{R}^3$.

Now let's look at <u>basis</u> and <u>dimension</u> for <u>subspaces</u> of $\mathbb{R}^n$.

Important examples: $N(A)$ = all solutions to $A\vec{x}=\vec{0}$.

$C(A)$ = all $\vec{b}$ such that $A\vec{x}=\vec{b}$ has a solution.

<u>Two more</u> subspaces coming from $m\times n$ matrix A:

<u>Row space</u> = all linear combinations of the <u>rows</u> of A (a subspace of $\mathbb{R}^n$)

If $A=\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vdots \\ \vec{v}_m^T \end{bmatrix}$, then $R(A)=\text{span}(\vec{v}_1,\vec{v}_2,\dots,\vec{v}_m)$

= all linear combinations of the <u>columns</u> of A^T

$= C(A^T)$

<u>Left null space</u> $= N(A^T)$ (subspace of $\mathbb{R}^m$)

= all solutions to $A^T\vec{y}=\vec{0}$

= all solutions to $(A^T\vec{y})^T=\vec{0}^T$

= all solutions to $\boxed{\vec{y}^T A=\vec{0}^T}$

Let's look at these 4 subspaces for $A=\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix}$

Best way is to do elimination and find R:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix} \xrightarrow[\text{Row 3}-3\text{Row 1}]{\text{Row 2}-2\text{Row 1}} \begin{bmatrix} 1 & 2 & \boxed{3} & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & \boxed{-1} & -3 \end{bmatrix} \xrightarrow[\text{Row 3}+\text{Row 2}]{\text{Row 1}-3\text{Row 2}} \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

<u>Null space $N(A)$</u>

$x_1+2x_2-5x_4=0$

$x_3+3x_4=0$

$x_1=-2x_2+5x_4$

$x_3=-3x_4$

x_2, x_4 are free

$$N(A) = \text{all} \begin{bmatrix} -2x_2+5x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 5 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

<u>Two</u> special solutions, one for each free variable

The special solutions are a spanning set for $N(A)$, and they are also linearly independent. $\longrightarrow$ they are a basis for $N(A)$

$\longrightarrow \dim N(A) = 2$

General fact: $\dim N(A) = \#$ of special solutions

$= \#$ of free variables

$= \underset{\text{columns}}{n} - \underset{\text{rank}}{r}$

Column space $C(A)$

By definition, columns of A are a spanning set for $C(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 11 \\ 9 \end{bmatrix} \right\}$$

But they might not be independent. We have to get rid of dependent columns to get a basis. How to tell which columns are dependent?

Key fact: Row operations don't change relations between the columns. So columns of R are dependent exactly when the corresponding columns of A are dependent.

$$R = \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix}$$

R	A
Col 2 = 2 Col 1	Col 2 = 2 Col 1
Col 4 = (−5) Col 1 + 3 Col 3	Col 4 = (−5) Col 1 + 3 Col 3
{Col 1, Col 3} are independent	{Col 1, Col 3} are independent
	basis

So one basis for $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \\ 8 \end{bmatrix} \right\}$,

and $\dim C(A) = 2$.

Warning: $C(R)$ is not the same subspace as $C(A)$, though they have the same dimension.

Basis of $C(R) = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$. This is not a basis for $C(A)$.

General fact: Basis for $C(A)$ = pivot columns of A (the columns that will have a leading 1 in R).

So $\dim C(A) = \#$ leading 1's $= r$, rank of A.

Note: $\dim C(A) + \dim N(A) = r + (n-r) = \boxed{n}$.

Row space $R(A)$ One option: do elimination on A^T and find the pivot columns of A^T (since $R(A) = C(A^T)$).

Or: Use another Key Fact: Elimination on A doesn't change row space (so row space of A is the same as row space of R).

$$R = \begin{bmatrix} 1 & 2 & 0 & -5 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad A = \begin{bmatrix} - - - - - \end{bmatrix}$$

↳ Basis for Row(R) is just the two non-zero rows of R:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \right\} \leftarrow \text{This is also a basis for Row}(A)$$

Why does this work? Elimination involves taking reversible linear combinations of the rows, so it doesn't change the set of all linear combinations.

General fact: Basis for $\underset{\text{Row}(A)}{C(A^T)}$ = Non-zero rows of R

$\dim C(A^T) = \#$ non-zero rows in R
$= \#$ leading 1's
$= r$, rank of A.

$C(A)$ and $C(A^T)$ have same dimension, but they are usually different subspaces.

More bases for $C(A^T)$: Once you know that $\dim C(A^T) = r$, you can just pick r linearly independent rows of A itself (but you have to check that they are linearly independent.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix}, \quad r = 2 \rightsquigarrow 3 \text{ possible bases: } \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \\ 11 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \\ 9 \end{bmatrix} \right\}, \left\{ \begin{bmatrix} 2 \\ 4 \\ 7 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 8 \\ 9 \end{bmatrix} \right\}$$

Left null space $N(A^T)$ – we know its dimension:

Remember: $\dim N(A) + \dim C(A) = \#$ columns in A

Same for A^T: $\dim N(A^T) + \underbrace{\dim C(A^T)}_{r} = \underbrace{\# \text{columns in } A^T}_{m}$

So $\dim N(A^T) = m - r$

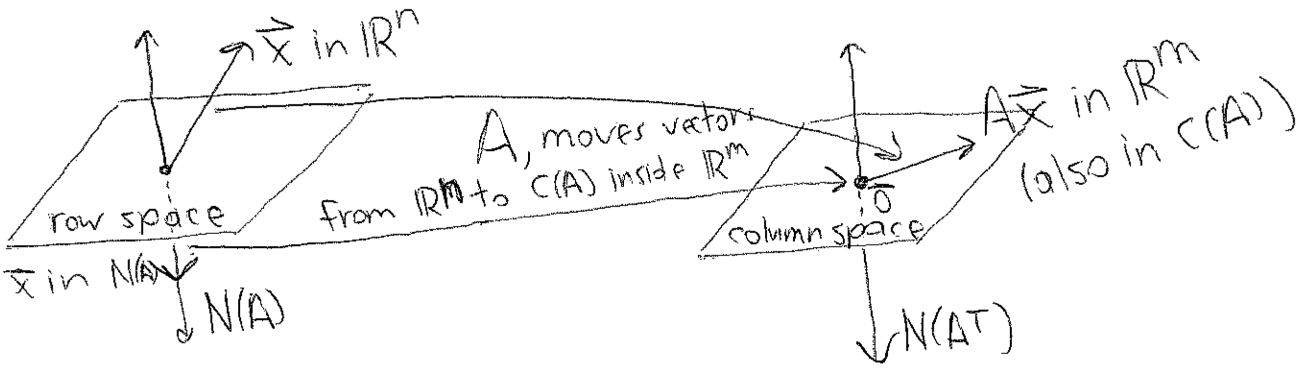
Summary: "Fundamental Theorem of Linear Algebra, Part 1":

There are 3 important numbers for A: m, n, and r.

– r = dim of column space and dim of row space

– $N(A)$, $N(A^T)$ have dimensions $n-r$, $m-r$.

The "big picture" when $m = n = 3$, $r = 2$



Now: can we actually find a basis for $N(A^T)$.

One option: Do elimination on A^T and find the null space as usual.

Different approach: Left null space = all $\vec{y}$ such that $\underbrace{\vec{y}^T A}$ $= \vec{0}^T$

Linear combination of the rows of A.

We can find $N(A^T)$ by finding which linear combinations of the rows of A give $\vec{0}$:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2b_1+b_2 \\ 0 & 0 & -1 & -3 & -3b_1+b_3 \end{array}\right] \longrightarrow$$

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$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2b_1+b_2 \\ 0 & 0 & 0 & 0 & -5b_1+b_2+b_3 \end{array}\right]$$

Keeps track of which linear combinations of the rows we are creating.

Tells us that $-5 \text{ Row } 1 + \text{Row } 2 + \text{Row } 3 = \vec{0}^T$

So $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$ is in $N(A^T)$

Since $\dim N(A^T) = m - r = 3 - 2 = 1$, we just need this <u>one</u> non-zero vector in $N(A^T)$ to get a basis.

If all 3 rows of A had been independent, we would have gotten $N(A^T) = \{\vec{0}\}$.