

Section 5.2 Permutations and cofactors ①

Last time: $A \xrightarrow{\det} |A|$
 $n \times n$ matrix real number

$$\begin{cases} \det A \neq 0 \rightarrow A \text{ is invertible} \\ \det A = 0 \rightarrow A \text{ is not invertible} \end{cases}$$

We saw that determinants obey 3 basic properties:

① Identity matrix: $\det(I) = 1$

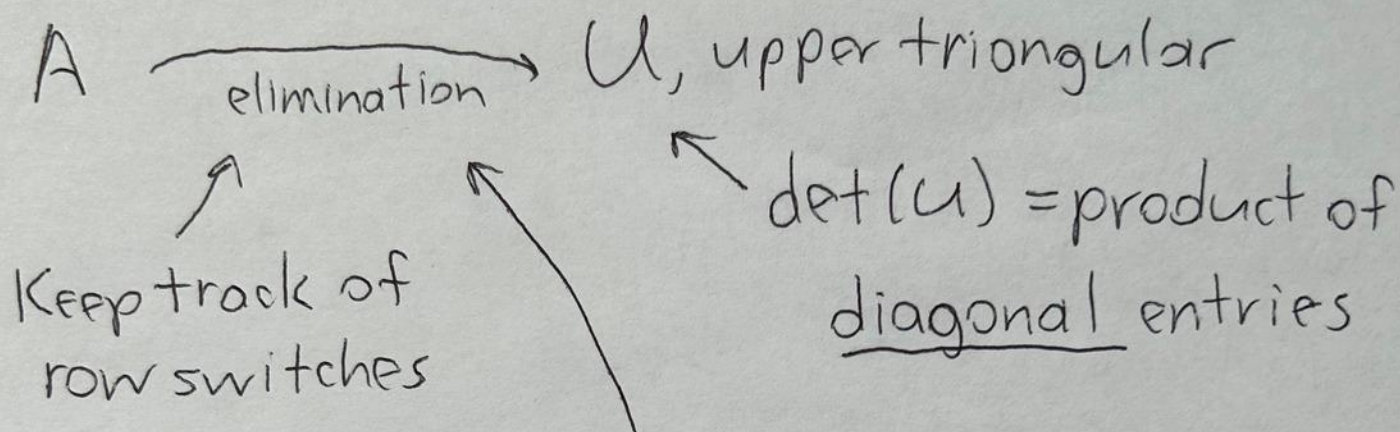
② Switch 2 rows \rightarrow multiplies determinant by -1 .

③ (a) Multiply one row by $t \rightarrow$ multiplies determinant by t .

(b) If one row is a sum of two rows \rightarrow addition rule.

$$\begin{vmatrix} \vec{a}_1 \\ \vec{a}_2 + \vec{a}_2' \\ \vdots \\ \vec{a}_n \end{vmatrix} = \begin{vmatrix} \vec{a}_1 \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix} + \begin{vmatrix} \vec{a}_1 \\ \vec{a}_2' \\ \vdots \\ \vec{a}_n \end{vmatrix}$$

Using only these 3 rules, we found many more ② properties of det, and a way to calculate them using row operations:



To keep things simple, don't do the row operation where you multiply a row by a non-zero scalar.

Then $\det(A) = \underbrace{(-1)(-1) \dots (-1)}_{\# \text{ of row switches}} \underbrace{(\det(U))}_{\text{product of pivots}}$

Example $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & -1 & -2 \end{vmatrix} =$

switch Row 2 and Row 3

from row switch

$- \begin{vmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{vmatrix} = -(1)(-1)(-1) = \boxed{-1}$

But: Is there a formula for $\det A$ that ③ obeys the 3 rules? (We have an algorithm, but a formula is sometimes more useful if we need to do calculations with symbols instead of real numbers.)

Yes! There is a formula. (Actually, more than one.) Let's derive it for 2×2 matrices using the 3 rules:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow{\text{Trick}} \begin{vmatrix} a+0 & 0+b \\ c & d \end{vmatrix} \xrightarrow{\text{Rule 3(b)}} \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix}$$

$$\xrightarrow{\text{Trick}} \begin{vmatrix} a & 0 \\ c+0 & 0+d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c+0 & 0+d \end{vmatrix} \xrightarrow{\text{Rule 3(b)}} \begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} + \begin{vmatrix} a & 0 \\ 0 & d \end{vmatrix}$$

$$+ \begin{vmatrix} 0 & b \\ c & 0 \end{vmatrix} + \begin{vmatrix} 0 & b \\ 0 & d \end{vmatrix} = ac \begin{vmatrix} 1 & 0 \\ 1 & 0 \end{vmatrix} + ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

Rule 3(a)

Rule 2 forces these to be 0.

$$+ pc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + bd \begin{vmatrix} 0 & 1 \\ 0 & 1 \end{vmatrix}$$

$$= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + bc \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

(4)

$$= ad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} - bc \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = \boxed{ad - bc}$$

Rule 2

Rule 1

We have just proved that this is the only formula for 2×2 matrices that obeys Rules 1, 2, 3.

What about 3×3 matrices? Apply the "trick" to the first row:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a+0+0 & 0+b+0 & 0+0+c \\ d & e & f \\ g & h & i \end{vmatrix}$$

$$\stackrel{\text{Rule 3(b)}}{=} \begin{vmatrix} a & 0 & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & e & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & e & f \\ g & h & i \end{vmatrix}$$

Now apply
← trick and
Rule 3(b) to
2nd row;

$$= \begin{vmatrix} a & 0 & 0 \\ 0 & e & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & 0 & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & b & 0 \\ d & 0 & f \\ g & h & i \end{vmatrix}$$

$$+ \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & f \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ d & 0 & 0 \\ g & h & i \end{vmatrix} + \begin{vmatrix} 0 & 0 & c \\ 0 & e & 0 \\ g & h & i \end{vmatrix}$$

some terms
have to be
0 by Rule 2.

$$= aei \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + bdi \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + bfg \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \quad (5)$$

3rd row
 now,
 and Rule
 (2)

$$+ afh \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} + cdh \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} + ceg \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}$$

These are all 6 3×3 permutation matrices

Det of permutation matrix = $(-1)(-1) \dots (-1)$
 Rule 2 \nearrow # of row switches to get I.

So $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} \quad \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix}$$

1st 2nd 1st 2nd

What is the pattern here?

- det A has 6 terms, one for each 3×3 permutation matrix
- Each term has a product of 3 matrix entries, one from each row of A, and one from each

column. (which column it is for each row depends ⑥ on the permutation).

— Each term has a ± 1 sign, which is $\det P$. This is called the sign of the permutation: $(-1)^{\# \text{row switches}}$

The same trick and the same Rules 1, 2, 3 give us the same "Big Formula" for bigger $n \times n$ matrices:

$$\det A = \sum_{\substack{\uparrow \\ \pm 1}} (\det P) a_{1,j_1} a_{2,j_2} \cdots a_{n,j_n}$$

sum over all
 $n \times n$ permutation
matrices P

This is the product of matrix entries for:

$$P = \begin{bmatrix} 0 & \cdots & 1 & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & 1 & \cdots & 0 \\ \vdots & & & & & \\ 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \end{bmatrix} \begin{matrix} \leftarrow \text{Row 1} \\ \leftarrow \text{Row 2} \\ \vdots \\ \leftarrow \text{Row } n \end{matrix}$$

$\uparrow \quad \uparrow \quad \uparrow$
 $\text{Col } j_n \quad \text{Col } j_1 \quad \text{Col } j_2$

How many?

Row 1: n choices of column
for the non-zero entry

Row 2: $n-1$ choices left

Row 3: $n-2$ choices left

\vdots

Row n : Only 1 choice remaining

So # of P 's =

$$\begin{aligned} & \longrightarrow n(n-1)(n-2) \cdots 1 \\ & = n! \end{aligned}$$

Note: This formula is an elegant mathematical formula, but it's not efficient for calculating because it has $n!$ terms.

↖ very large number!

Unless, a lot of the terms will be 0.

Example

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & 1 \end{vmatrix} = ??$$

↖ $4(3)(2)(1) = 24 \text{ terms}$

Terms in the formula will be 0, unless the permutation chooses Col 2 for Row 2:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & \textcircled{2} & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & \textcircled{1} \end{vmatrix}$$

Only $3(2)(1) = 6$ terms choose Col 2 for Row 2. Even most of these will be 0, unless the permutation picks Col 4 for Row 4.

So $\det A = (1)(2)(-1)(1) \underbrace{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{+1} + (3)(-1)(2) \underbrace{\begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}}_{-1}$

$$= -2 - (-6) = \boxed{4}$$

(8)

Cofactor Expansion: Let's revisit the 3×3 determinant formula.

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$$

$$= a(ei - fh) - b(di - fg) + c(dh - eg) \leftarrow \text{Separate out 1st-row terms}$$

$$= -d(bi - ch) + e(ai - cg) - f(ah - bg) \leftarrow \text{2nd row}$$

$$= g(bf - ce) - h(af - cd) + i(ae - bd) \leftarrow \text{3rd row}$$

These expressions Terms in parentheses are 2×2 determinants:

show that $\det A$

is indeed a linear function of the rows

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} + b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

"Expanding across" the 1st row

$$+ c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

These $\pm 2 \times 2$ determinants are called cofactors.

You can also find $\det A$ by "expanding across" Rows 2 and 3, but you have to be careful about \pm :

$$\text{Row 2: } \det A = -d \begin{vmatrix} b & c \\ h & i \end{vmatrix} + e \begin{vmatrix} a & c \\ g & i \end{vmatrix} - f \begin{vmatrix} a & b \\ g & h \end{vmatrix}$$

General formula for $n \times n$ A

(9)

The "cofactors": they are $\pm [(n-1) \times (n-1) \det]$ you get by deleting Row 1 and Col j

Expanding

across 1st row: $\det A = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

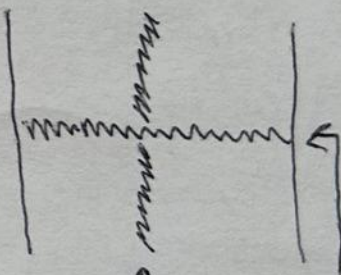
permutations that pick Col 2 for Row 2.

This collects all $(n-1)!$ terms in the "Big Formula" that pick Col 1 for Row 1

Expanding

across Row i: $\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$

$$C_{ij} \text{ cofactor} = (-1)^{i+j}$$



Delete Col j Delete Row i

Take det of what is left.

This is ± 1 . Gives alternating

sign pattern:

Row i \rightarrow

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Sign for C_{ij} is -1 here.

col j

The term $a_{ij}C_{ij}$ collects all $(n-1)!$ in the "Big Formula" that come from permutations which pick Col j for Row i.

(10)

Previous
Example

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & 1 \end{vmatrix} \xrightarrow[\text{row}]{\text{1st}} 1 \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix}$$

$$-2 \begin{vmatrix} 0 & 0 & 0 \\ -1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} + 3 \begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & 2 \\ 0 & 4 & 1 \end{vmatrix} - 4 \begin{vmatrix} 0 & 2 & 0 \\ -1 & 1 & -1 \\ 0 & 4 & 0 \end{vmatrix}$$

will be 0
(Row 1 = 0)

will be 0
(Row 3 = 2 Row 1)

$$= 1 \left(2 \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} + 0 \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} \right)$$

$$+ 3 \left(0 \begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} + 0 \begin{vmatrix} -1 & 1 \\ 0 & 4 \end{vmatrix} \right)$$

$$= 1 \left(2((-1)(1) - 2(0)) \right) + 3 \left(-2((-1)(1) - 2(0)) \right)$$

$$= -2 + 6 = 4 \text{ again}$$

It would be more efficient to take advantage of the 0's:

same sign pattern

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & 1 \end{vmatrix} \xrightarrow[\text{row}]{\text{2nd}} -0 \cdots +2 \begin{vmatrix} 1 & 3 & 4 \\ -1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} - 0 \cdots +0 \cdots$$

$$= 2 \begin{vmatrix} 1 & 3 & 4 \\ -1 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} \begin{matrix} \text{3rd} \\ \text{row} \end{matrix} = 2 \left(+1 \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} \right) \begin{vmatrix} 1 & 3 \\ -1 & -1 \end{vmatrix} \quad (11)$$

$$= 2 \left(\begin{matrix} \cancel{1(-1)} & -\cancel{1(-1)} \\ 1(-1) & 3(-1) \end{matrix} \right) = 4 \quad \checkmark$$

We can also do cofactor expansion down columns:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 0 & 0 \\ -1 & 1 & -1 & 2 \\ 0 & 4 & 0 & 1 \end{vmatrix} \begin{matrix} \text{1st} \\ \text{col} \end{matrix} = 1 \begin{vmatrix} 2 & 0 & 0 \\ 1 & -1 & 2 \\ 4 & 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} \dots \end{vmatrix}$$

$$+ (-1) \begin{vmatrix} 2 & 3 & 4 \\ 2 & 0 & 0 \\ 4 & 0 & 1 \end{vmatrix} - 0 \begin{vmatrix} \dots \end{vmatrix}$$

same cofactors,

same sign pattern

$$\begin{matrix} \text{2nd} \\ \text{col} \end{matrix} = 1 \left(-1 \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} \right) + (-1) \left(-3 \begin{vmatrix} 2 & 0 \\ 4 & 1 \end{vmatrix} \right)$$

$$= 1(-1)(2)(1) - 1(-3)(2)(1) = 4$$

That is:
$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix} = a_{1j} C_{1j} + a_{2j} C_{2j} + \dots + a_{nj} C_{nj}$$

collects all terms for permutations that choose Row n for Col j .

Why does this work? Because $\det A = \det A^T$! (12)

So expanding down Col j = expand across Row j of $A^T \rightarrow$ gives same determinant.

We now have two ways to calculate determinants:
Row operations and cofactor expansion.

Usually, row operations are more efficient, but cofactor expansion is good when A has many 0's.

Problem
5.2.16

$$F_n = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & -1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \left. \vphantom{\begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 1 & -1 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}} \right\} n \times n$$

Row operations: need to n of them (lot of work)

cofactor expan. using 1st row: reveals a pattern.

1st row
 \downarrow

$$\det F_n = 1 \begin{vmatrix} 1 & -1 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \\ 0 & 0 & \cdots & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{vmatrix}$$

This is $\det F_{n-1}$ not F_{n-1} !

(13)

$$= \det F_{n-1} + 1 \cdot 1$$

1st column

$$\begin{vmatrix} 1 & -1 & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & 1 & 1 \end{vmatrix}$$

This is $\det F_{n-2}$

$$\text{So } \det F_n = \det F_{n-1} + \det F_{n-2}.$$

$$\text{Also: } \det F_1 = |1| = 1$$

$$\det F_2 = \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} = 2$$

These
determinants
are
Fibonacci
numbers.

$$\checkmark F_1 \checkmark F_2 \checkmark F_3 \checkmark F_4 \checkmark F_5 \checkmark F_6 \checkmark F_7 \checkmark F_8$$

$$1, 2, 3, 5, 8, 13, 21, 34, \dots$$

Let's do F_4 explicitly:

$$\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = 1 \begin{vmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix} - (-1) \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$= 2 + 1 + 2 = 5. \checkmark$$

With row operations:

(14)

$$\begin{vmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3/2 & -1 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 3/2 & -1 \\ 0 & 0 & 0 & 5/3 \end{vmatrix} \xrightarrow{\text{Row 3} - \frac{1}{2} \text{Row 2}} \xrightarrow{\text{Row 4} - \frac{2}{3} \text{Row 3}} 1(2)\left(\frac{3}{2}\right)\left(\frac{5}{3}\right) = 5 \checkmark$$

A similar example:

$$A_n = \begin{bmatrix} 2 & -1 & \dots & 0 & 0 \\ -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & \dots & -1 & 2 \end{bmatrix} \left. \vphantom{\begin{bmatrix} 2 & -1 & \dots & 0 & 0 \\ -1 & 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 2 & -1 \\ 0 & 0 & \dots & -1 & 2 \end{bmatrix}} \right\} n \times n \text{ matrix}$$

$$\det A_1 = |2| = 2$$

$$\det A_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

In general:

(15)

$$\begin{vmatrix} 2 & -1 & & 0 \\ -1 & 2 & & \\ & & \ddots & \\ 0 & & & 2 & -1 \\ & & & -1 & 2 \end{vmatrix} = 2 \begin{vmatrix} 2 & -1 & & 0 \\ -1 & & & \\ & & \ddots & \\ 0 & & & 2 & -1 \\ & & & -1 & 2 \end{vmatrix}$$

 $\det A_{n-1}$

$$= -(-1) \begin{vmatrix} -1 & -1 & \dots & 0 & 0 \\ 0 & 2 & -1 & \dots & 0 & 0 \\ 0 & -1 & & \ddots & & \\ \vdots & & & \ddots & & 1 \\ 0 & & & & 2 & -1 \\ & & & & -1 & 2 \end{vmatrix}$$

$$= 2 \det A_{n-1} + 1 \left(-1 \det A_{n-2} \right) \quad \uparrow \quad A_{n-2}$$

$$= 2 \det A_{n-1} - \det A_{n-2} \quad \leftarrow \text{"Recursive formula"}$$

$$\det A_3 = 2 \det A_2 - \det A_1 = 2(3) - 2 = 4$$

$$\det A_4 = 2 \det A_3 - \det A_2 = 2(4) - 3 = 5$$

 \vdots

$$\det A_n = n+1 \text{ in general}$$

$$\det A_{n+1} = 2 \det A_n - \det A_{n-1}$$

$$= 2(n+1) - (n) = n+2 = (n+1) + 1 \quad \checkmark$$