Linear Algebra – Homework 8

22 Nov 2023
Due: 30 Nov 2023

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 3.3.18. Use elimination to find the rank of A (it will depend on q). What is the rank of A^T?

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \quad \text{(rank depends on } q\text{)}$$

Problem 3.3.24 Give examples of matrices A for which the number of solutions to $A\mathbf{x} = \mathbf{b}$ is

(a) 0 or 1, depending on $\mathbf{b}$

(b) ∞, regardless of $\mathbf{b}$

(c) 0 or ∞, depending on $\mathbf{b}$

(d) 1, regardless of $\mathbf{b}$.

Problem 3.4.2. Find the largest possible number of independent vectors among

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

Problem 3.4.8. If $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{w}_3$ are independent vectors, show that the sums $\mathbf{v}_1 = \mathbf{w}_2 + \mathbf{w}_3$, $\mathbf{v}_2 = \mathbf{w}_1 + \mathbf{w}_3$, and $\mathbf{v}_3 = \mathbf{w}_1 + \mathbf{w}_2$ are also independent. (Write $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{0}$ in terms of the the $\mathbf{w}$'s. Find and solve equations for the c's, to show they are zero.)

Problem 3.4.11. Describe the subspace of $\mathbf{R}^3$ (is it a line or plane or $\mathbf{R}^3$?) spanned by

(a) the two vectors $(1, 1, -1)$ and $(-1, -1, 1)$.

(b) the three vectors $(0, 1, 1)$, $(1, 1, 0)$, and $(0, 0, 0)$.

(c) all vectors in $\mathbf{R}^3$ with whole number components

(d) all vectors with positive components.

Problem 3.4.20. Find a basis for the plane $x - 2y + 3z = 0$ in $\mathbf{R}^3$. Then find a basis for the intersection of that plane with the xy plane. Then find a basis for all vectors perpendicular to the plane.

Problem 3.5.2. Find bases and dimensions for the four subspaces associated to $B = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix}$.

Problem 3.5.11. A is an $m \times n$ matrix of rank r. Suppose there are right sides $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has *no* solution.

(a) What are all inequalities ($<$ or $\leq$) that *must* be true between m, n, and r?

(b) How do you know that $A^T\mathbf{y} = \mathbf{0}$ has solutions other than $\mathbf{y} = \mathbf{0}$?

Problem 3.5.18. Add the extra column $\mathbf{b}$ and reduce A to echelon form:

$$\begin{bmatrix} A & b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}.$$

A combination of rows of A has produced the zero row. What combination is it? (Look at $b_3 - 2b_2 + b_1$ on the right side.) Which vectors are in the nullspace of A^T and which vectors are in the nullspace of A?

Problem 3.5.24. $A^T\mathbf{y} = \mathbf{d}$ is solvable when $\mathbf{d}$ is in which of the four subspaces? The solution $\mathbf{y}$ is unique when the ___________ contains only the zero vector.

Graded Problems.

Problem 1. Find *all* solutions to the system of linear equations:

$$\begin{cases} 2\,x_1 + 3\,x_2 - x_3 + 2\,x_4 &= -1 \\ 4\,x_1 + 6\,x_2 + 2\,x_3 + 2\,x_4 &= 1 \\ 6\,x_1 + 9\,x_2 + x_3 + 2\,x_4 &= -1 \end{cases}$$

Problem 2. Determine whether these vectors form bases for $\mathbb{R}^4$. If they do not, show how to write one of them as a linear combination of the others.

(a) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix} \right\}$

Problem 3. Find bases for the column space $\mathbf{C}(A)$, null space $\mathbf{N}(A)$, row space $\mathbf{C}(A^T)$, and left null space $\mathbf{N}(A^T)$:

$$A = \begin{bmatrix} -1 & 2 & -3 & 4 \\ 3 & 4 & -1 & 0 \\ 2 & 1 & 1 & -2 \end{bmatrix}$$

What is the rank of A?