

# LINEAR ALGEBRA — HOMEWORK 10

6 Dec, 2023

Due: 14 Dec, 2023

**Textbook Problems.** These problems will not be graded, but you must submit solutions to receive full credit for the homework.

**Problem 4.3.1.** With  $b = 0, 8, 8, 20$  at  $t = 0, 1, 3, 4$ , set up and solve the normal equations  $A^T A \hat{\mathbf{x}} = A^T \mathbf{b}$ . For the best straight line, find the four heights  $p_i$  (the  $y$ -values for the line at  $t = 0, 1, 3, 4$ ), and the errors  $e_i$  (the differences between the  $y$ -values of the line and the  $y$ -values of the data points). What is the minimum value  $E = e_1^2 + e_2^2 + e_3^2 + e_4^2$ ?

**Problem 4.3.10.** For the closest cubic  $b = C + Dt + Et^2 + Ft^3$  to the same four points in Problem 4.3.1, write down the four equations  $A\mathbf{x} = \mathbf{b}$ . Solve them by elimination. This cubic now goes exactly through the points. What are  $\mathbf{p}$  and  $\mathbf{e}$ ?

**Problem 4.3.12.** This problem projects  $\mathbf{b} = (b_1, b_2, \dots, b_m)$  onto the line through  $\mathbf{a} = (1, 1, \dots, 1)$ . We solve  $m$  equations  $\mathbf{a}\mathbf{x} = \mathbf{b}$  in 1 unknown (by least squares).

- (a) Solve  $\mathbf{a}^T \mathbf{a} \hat{x} = \mathbf{a}^T \mathbf{b}$  to show that  $\hat{x}$  is the *mean* (the average) of the  $b$ 's.
- (b) Find  $\mathbf{e} = \mathbf{b} - \mathbf{a}\hat{x}$  and the *variance*  $\|\mathbf{e}\|^2$  and the *standard deviation*  $\|\mathbf{e}\|$ .
- (c) The horizontal line  $\hat{x} = 3$  is closest to  $\mathbf{b} = (1, 2, 6)$ , since 3 is the average of 1, 2, and 6. Check that  $\mathbf{p} = (3, 3, 3)$  is perpendicular to  $\mathbf{e}$  and find the  $3 \times 3$  projection matrix  $P$ .

**Problem 4.3.17.** Write down three equations for the line  $b = C + Dt$  to go through  $b = 7$  at  $t = -1$ ,  $b = 7$  at  $t = 1$ , and  $b = 21$  at  $t = 2$ . Find the least squares solution to  $\hat{\mathbf{x}} = (C, D)$  and draw the closest line.

**Problem 4.4.2.** The vectors  $(2, 2, -1)$  and  $(-1, 2, 2)$  are orthogonal. Divide them by their lengths to find orthonormal vectors  $\mathbf{q}_1$  and  $\mathbf{q}_2$ . Put those into the columns of  $Q$  and multiply  $Q^T Q$  and  $Q Q^T$ .

**Problem 4.4.10.** Orthonormal vectors are automatically linearly independent:

- (a) Vector proof: When  $c_1 \mathbf{q}_1 + c_2 \mathbf{q}_2 + c_3 \mathbf{q}_3 = \mathbf{0}$ , what dot product leads to  $c_1 = 0$ ? Similarly,  $c_2 = 0$  and  $c_3 = 0$ . Thus the  $\mathbf{q}$ 's are independent.
- (b) Matrix proof: Show that  $Q\mathbf{x} = \mathbf{0}$  leads to  $\mathbf{x} = \mathbf{0}$ . Since  $Q$  might not be square, you can use  $Q^T$  but not  $Q^{-1}$ .

**Problem 4.4.18.** Find orthogonal vectors  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  by Gram-Schmidt from  $\mathbf{a}, \mathbf{b}, \mathbf{c}$ :

$$\mathbf{a} = (1, -1, 0, 0), \quad \mathbf{b} = (0, 1, -1, 0), \quad \mathbf{c} = (0, 0, 1, -1).$$

$\mathbf{A}, \mathbf{B}, \mathbf{C}$  and  $\mathbf{a}, \mathbf{b}, \mathbf{c}$  are both bases for the vectors perpendicular to  $\mathbf{d} = (1, 1, 1, 1)$ .

**Problem 4.4.22.** Find orthogonal vectors  $\mathbf{A}, \mathbf{B}, \mathbf{C}$  by Gram-Schmidt from

$$\mathbf{a} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}.$$

**Problem 4.4.31.**

- (a) Choose  $c$  so that  $Q$  is an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}.$$

(b) Project  $\mathbf{b} = (1, 1, 1, 1)$  onto the first column. Then project  $\mathbf{b}$  onto the plane of the first two columns.

**Problem 4.4.32.** If  $\mathbf{u}$  is a unit vector, then  $Q = I - 2\mathbf{u}\mathbf{u}^T$  is a reflection matrix. Find  $Q_1$  from  $\mathbf{u} = (0, 1)$  and  $Q_2$  from  $\mathbf{u} = (0, \sqrt{2}/2, \sqrt{2}/2)$ . Draw the reflections when  $Q_1$  and  $Q_2$  multiply the vectors  $(1, 2)$  and  $(1, 1, 1)$ .

### Graded Problems.

**Problem 1.** Physics tells us that near the Earth's surface, the height  $h(t)$  of an object dropped at time  $t = 0$  from an initial height of  $h_0$  obeys the equation  $h(t) = h_0 - \frac{1}{2}gt^2$ , where  $g$  is the acceleration due to gravity. In an experiment, a ball is dropped from an initial height  $h_0 = 50$ , and its distances above the ground are measured to be 50, 44, 32, 6 at times  $t = 0, 1, 2, 3$ . Use the least squares method to find the parabola  $C + Dt + Et^2$  that best fits these data points, and use your value for  $E$  to estimate the acceleration  $g$ .

*Note:* You do not need to worry about units in this problem. Also, the numbers in this exercise will get a bit large. You will not have to deal with such large numbers on the final exam.

**Problem 2.** Use the Gram-Schmidt process as necessary to find orthonormal bases for both the row space and null space of

$$A = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}.$$

Combine these two orthonormal bases to get an orthonormal basis for  $\mathbf{R}^3$ .