$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 6 & 8 & 9 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \longrightarrow \left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2b_1+b_2 \\ 0 & 0 & -1 & -3 & -3b_1+b_3 \end{array}\right] \longrightarrow$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 4 & b_1 \\ 0 & 0 & 1 & 3 & -2b_1+b_2 \\ 0 & 0 & 0 & 0 & -5b_1+b_2+b_3 \end{array}\right]$$

Keeps track of which linear combinations of the rows we are creating.

Tells us that $-5\text{ Row }1 + \text{Row }2 + \text{Row }3 = \vec{0}^T$

So $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$ is in $N(A^T)$

Since $\dim N(A^T) = m - r = 3 - 2 = 1$, we just need this <u>one</u> non-zero vector in $N(A^T)$ to get a basis.

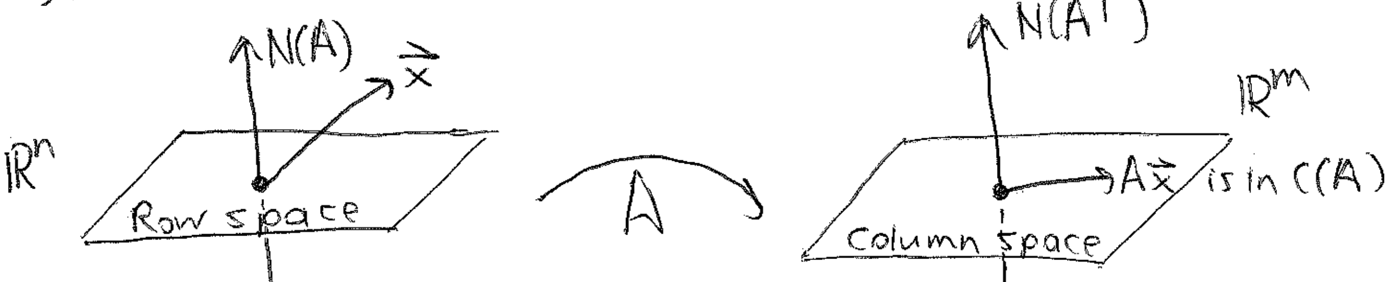
If all 3 rows of A had been independent, we would have gotten $N(A^T) = \{\vec{0}\}$.

<u>Chapter 4: Orthogonality</u>

"Big picture" from last week:

$$\mathbb{R}^n \xrightarrow{\ m\times n \text{ matrix } A\ } \mathbb{R}^m$$

$$\vec{x} \longrightarrow A\vec{x}$$



$\dim \text{Row}(A) + \dim N(A) = n$ $\qquad$ $\dim C(A) + \dim N(A^T) = m$

$\dim C(A)$ <u>and</u> $\dim \text{Row}(A)$ are rank of A, r.

I drew these subspaces perpendicular to each other.

Are they really? <u>Yes!</u>

Claim 1: Every vector in $N(A)$ is perpendicular (or orthogonal) to every vector in $R(A)$.

Why? $\vec{x}$ in $N(A)$ means $A\vec{x} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, so

$$\underbrace{(\text{row}1)\cdot\vec{x} = 0,\ (\text{row}2)\cdot\vec{x} = 0,\ \ldots,\ (\text{row}\ m)\cdot\vec{x} = \vec{0}}_{\text{Entries of } A\vec{x}}$$

So $\vec{x} \perp$ every row $\longrightarrow \vec{x} \perp$ every linear combination of the rows.

Claim 2: Same with column and left null spaces.

Why? Switch A with A^T in Claim 1: $N(A^T) \perp R(A^T)$

$\rightarrow N(A^T) \perp C(A)$.

Another proof: Vectors $\vec{y}$ in $N(A^T)$ satisfy $A^T\vec{y} = 0$

Vectors in $C(A)$ look like $A\vec{x}$.

These two kinds of vectors are perpendicular:

$$(A\vec{x})\cdot\vec{y} \underset{\uparrow}{=} (A\vec{x})^T\vec{y} = (\vec{x}^T A^T)\vec{y} = \vec{x}^T(A^T\vec{y}) = \vec{x}^T\vec{0} = 0.$$

This formula is why we care about transposes.

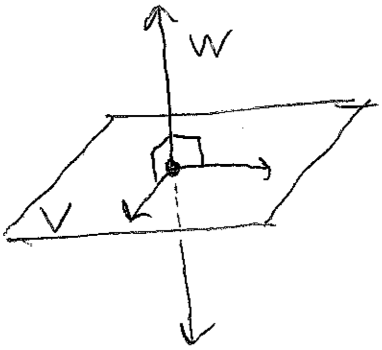
So $A\vec{x} \perp \vec{y}$ if y is in $N(A^T)$.

Definition: Two subspaces V and W in $\mathbb{R}^n$ are orthogonal if every vector in V is perpendicular to every vector in W.

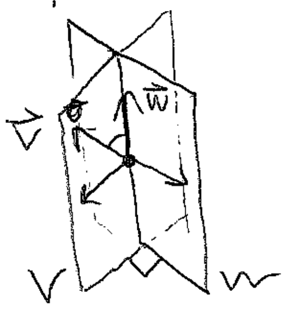
Notation: $V \perp W$

Examples: $N(A) \perp C(A^T)$ in $\mathbb{R}^n$, $N(A^T) \perp C(A)$ in $\mathbb{R}^m$.

Pictures: Orthogonal subspaces:



<u>Not</u> quite orthogonal:



<u>Some</u> vectors in $\vec{v}$ are $\perp$ to every vector in W, but <u>not</u> <u>all</u>

<u>Important Fact</u>: If $V \perp W$, then the <u>only</u> vector that is in <u>both</u> V and W is $\vec{0}$:

<u>Why?</u> If $\vec{v}$ is in both, it has to be $\perp$ to itself:

$$\vec{v}^T\vec{v} = 0, \text{ or } v_1^2 + v_2^2 + \dots + v_n^2 = 0$$

only works if <u>every</u> $v_1, v_2, \dots, v_n = 0$, i.e., $\vec{v} = \vec{0}$.

<u>Definition</u>: <u>Orthogonal</u> <u>complement</u> of a subspace V:

$V^\perp$ = set of all vectors in $\mathbb{R}^n$ that are $\perp$ to <u>all</u> vectors in V.

Is $V^\perp$ a subspace? Yes!

① $\vec{0}$ is in $V^\perp$: $\vec{0} \cdot (\text{every } \vec{v}) = 0$.

② If $\vec{x}, \vec{y}$ are in $V^\perp$, so is $\vec{x}+\vec{y}$: $(\vec{x}+\vec{y}) \cdot (\text{every } \vec{v}) =$

$$\vec{x}\cdot\vec{v} + \vec{y}\cdot\vec{v} = 0 + 0 = 0.$$

③ If $\vec{x}$ is in $V^\perp$, so is $c\vec{x}$: $(c\vec{x})\cdot(\text{every } \vec{v}) = c(\vec{x}\cdot\vec{v})$

$$= c(0) = 0.$$

<u>Example</u>: $V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 3 \\ 3 \end{bmatrix} \right\}$ in $\mathbb{R}^4$. Find a basis for $V^\perp$.

(This is a basis for V.)

Note: If $\vec{x} \cdot$ (every basis vector for V) $= 0$,

then $\vec{x} \cdot$ (every linear combination of basis vectors) $= 0$ also,

so $\vec{x} \cdot$ (every $\vec{v}$ in V) $= 0 \longrightarrow \vec{x}$ is in $V^{\perp}$.

So we just need to find all $\vec{x}$ such that:

$$\vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ 4 \end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix} 2 \\ 3 \\ 3 \\ 3 \end{bmatrix} = 0$$

Or, $\underbrace{\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 3 & 3 & 3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So, $V = \text{Row}(A)$ and $V^{\perp} = N(A)$!

Basis = special solutions of $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 2 & 3 & 3 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 3 & -1 & -5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -1/3 & -5/3 \end{bmatrix}$$

$x_1 = -2x_3 - 4x_4$

$x_2 = \frac{1}{3}x_3 + \frac{5}{3}x_4$

x_3, x_4 free

$$\longrightarrow \vec{x} = \begin{bmatrix} -2x_3 - 4x_4 \\ \frac{1}{3}x_3 + \frac{5}{3}x_3 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ 1/3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 5/3 \\ 0 \\ 1 \end{bmatrix}$$

These 2 vectors will form a <u>basis</u> for $V^{\perp}$.

Note that: $\dim V + \dim V^{\perp} = 2 + 2 = 4 = \dim$ of $\mathbb{R}^4$.

This always works (i.e. $\dim V + \dim V^{\perp} = n$ if V is in $\mathbb{R}^n$).

Also, this example illustrates the:

"<u>Fundamental Theorem of Linear Algebra, Part 2</u>":

For any $m \times n$ matrix A, $N(A) = C(A^T)^{\perp}$ (in $\mathbb{R}^n$)

rowspace

$N(A^T) = C(A)^{\perp}$ (in $\mathbb{R}^m$)

Nice consequence: If $\vec{b}$ is in $C(A)$, then there is a unique solution to $A\vec{x}_r = \vec{b}$ such that $\vec{x}_r$ comes from the row space.

Why? If we have two solutions: $A\vec{x}_r = \vec{b}$, $A\vec{x}_r' = \vec{b}$, where $\vec{x}_r, \vec{x}_r'$ are both in $C(A^T)$, then $A(\vec{x}_r - \vec{x}_r') = \vec{0}$

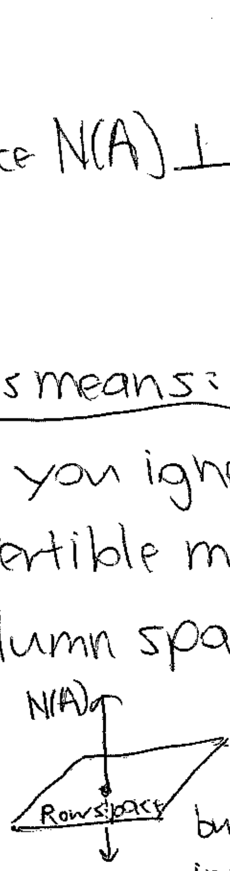
This vector is in both $C(A^T)$ and $N(A)$, so it's $\vec{0}$ by "important fact."

Since $N(A) \perp C(A^T)$, must have $\vec{x}_r - \vec{x}_r' = \vec{0}$, i.e., $\vec{x}_r = \vec{x}_r'$

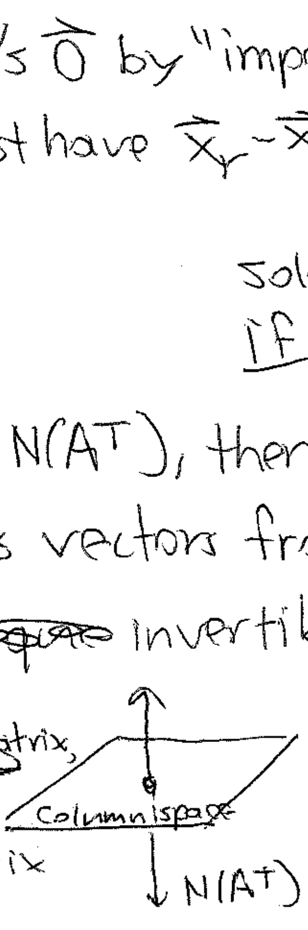
solution is actually unique, if $\vec{x}_r$ comes from $C(A^T)$.

This means:

If you ignore $N(A)$ and $N(A^T)$, then A behaves like an invertible matrix: takes vectors from Row space to Column space in an ~~unique~~ invertible way:



A is really a 3x3 non-invertible matrix, but it acts like an invertible 2x2 matrix



here's an invertible 2x2 matrix inside A.

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

(not invertible)

Now to understand an $m \times n$ matrix A: we need a good basis for $\mathbb{R}^n$ that is related to A:

Claim: We can get a basis for $\mathbb{R}^n$ by combining bases for null and column spaces:

$$\text{Basis} = \{ \underbrace{\text{basis vectors for } C(A^T)}_{r \text{ of them}},\ \underbrace{\text{basis vectors for } N(A)}_{n-r \text{ of them}} \}$$

get right number of vectors, n, but are they lin. ind?

Suppose (lin. comb. of row basis vectors) + (lin. comb. of N(A) basis) = $\vec{0}$.

call this $\vec{x}_r$ — call this $\vec{x}_n$

Then $\vec{x}_r + \vec{x}_n = \vec{0} \rightsquigarrow \vec{x}_r = -\vec{x}_n$

Shows $\vec{x}_r$ is in both $C(A^T)$ and $N(A)$

→ $\boxed{\vec{x}_r = \vec{0}}$ since $C(A^T) \perp N(A)$

→ $\boxed{\vec{x}_n = \vec{0}}$ as well.

The only way to get lin. comb = 0 is to set all coefficients = 0

→ linearly independent and a basis (since we have right number of vectors for a basis).

This proves the claim, which implies: — spanning set property

Every $\vec{x}$ in $\mathbb{R}^n$ = lin. comb. of basis vectors

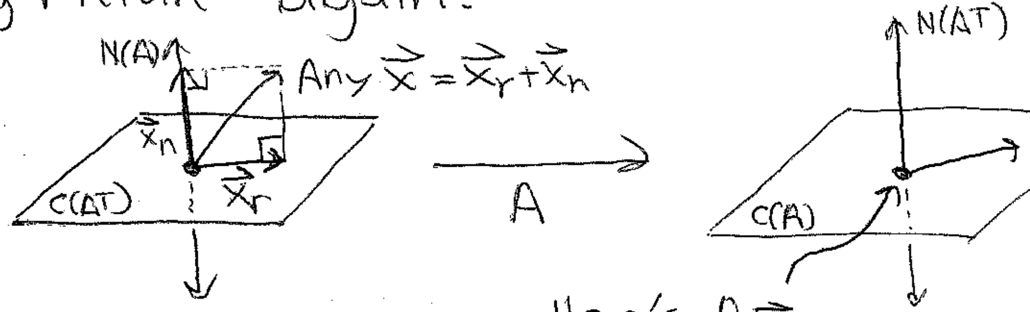
$$= \underbrace{\text{lin. comb. of row space vectors}}_{\vec{x}_r} + \underbrace{\text{lin comb. of null space vectors}}_{\vec{x}_n}$$

→ $\vec{x} = \vec{x}_r + \vec{x}_n$

$\vec{x}_n$: called "orthogonal projection" of $\vec{x}$ onto N(A).

$\vec{x}_r$: called "orthogonal projection" of $\vec{x}$ onto $C(A^T)$

"Big Picture" again:



$A\vec{x}$, also $A\vec{x}_r$:

$A\vec{x} = A(\vec{x}_r + \vec{x}_n)$
$= A\vec{x}_r + A\vec{x}_n$
$= A\vec{x}_r + \vec{0} = A\vec{x}_r$

Here's $A\vec{x}_n$. It's $\vec{0}$!

Problem 4.1.12 Draw this picture for $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{x} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$C(A) = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\0\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$

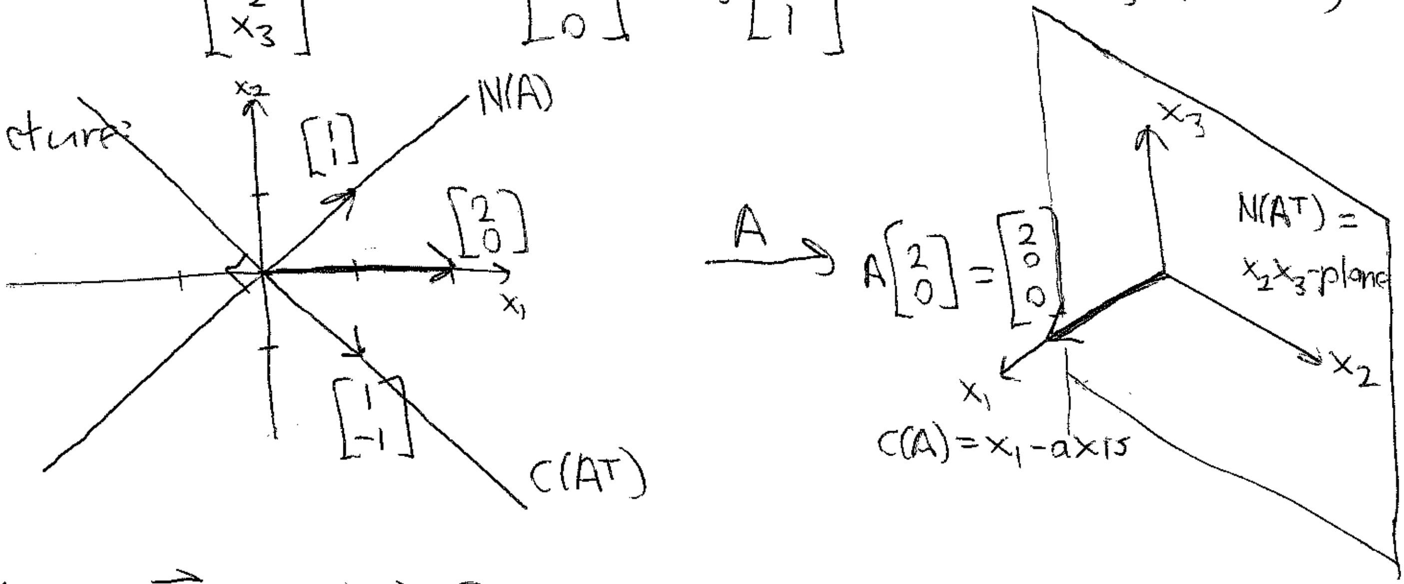
$N(A): \begin{bmatrix}1 & -1\\0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ means $x_1 = x_2 \rightsquigarrow N(A) = \text{all } x_2\begin{bmatrix}1\\1\end{bmatrix}$

$C(A^T) = \text{span}\left(\begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$ perpendicular!

$N(A^T): \begin{bmatrix}1 & 0 & 0\\-1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ means $x_1 = 0 \rightsquigarrow$

$N(A^T) = \text{all}\begin{bmatrix}0\\x_2\\x_3\end{bmatrix} = \text{all } x_2\begin{bmatrix}0\\1\\0\end{bmatrix} + x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$ $(= x_2x_3\text{-plane})$

The picture:



What are $\vec{x}_r$ and $\vec{x}_n$? Need to write

$\begin{bmatrix}2\\0\end{bmatrix} = \underbrace{c\begin{bmatrix}1\\-1\end{bmatrix}}_{\vec{x}_r} + \underbrace{d\begin{bmatrix}1\\1\end{bmatrix}}_{\vec{x}_n}$ Looks like $c = d = 1$ will work.

Section 4.2 Projections

Big linear algebra problem: Figure out how to write $\vec{x} = \vec{x}_r + \vec{x}_n$

To say another way: Figure out how to project $\vec{x}$ onto a subspace (such as $N(A)$).

This means: Write $\vec{x} = \vec{v} + \vec{e}$
$\vec{v}$ in V, $\vec{e}$ in $V^\perp$