Problem 3.2.1(a) Find R for $A=\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&3\end{bmatrix}$

$$\begin{bmatrix}1&2&2&4&6\\1&2&3&6&9\\0&0&1&2&3\end{bmatrix}\xrightarrow{\text{Row 2-Row 1}}\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&1&2&3\end{bmatrix}\xrightarrow{\text{Row 3-Row 2}}\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$$

leading 1's

This is U; lower-left variables are eliminated.

$$\xrightarrow{\text{Row 1-2 Row2}}\begin{bmatrix}1&2&0&0&0\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$$ This is R.

leading 1's in Cols 1 and 3 → these are the "pivot columns"
columns 2,4,5 give free variables.

We can easily solve $A\vec{x}=\vec{0}$ using R: $[A \mid \vec{0}] \longrightarrow [R \mid \vec{0}]$

$$\begin{cases}x_1+2x_2=0\\x_3+2x_4+3x_5=0\end{cases}\xrightarrow[\text{in terms of free variables}]{\text{solve for } x_1, x_3}\begin{aligned}x_1&=-2x_2\\x_3&=-2x_4-3x_5\end{aligned}$$

from R

All solutions: $$\vec{x}=\begin{bmatrix}-2x_2\\x_2\\-2x_4-3x_5\\x_4\\x_5\end{bmatrix}=\underbrace{x_2\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\-2\\1\\0\end{bmatrix}+x_5\begin{bmatrix}0\\0\\-3\\0\\1\end{bmatrix}}$$

Vectors in N(A): Three free variables, three special solutions

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For an $m\times n$ A, the number of leading 1's in R is important. It is called the rank of the matrix. (Use r for rand of A)
Since A has n total columns, the number of free variables is $n-r$ (also the number of special solutions in N(A)).

Interesting question: What matrices have $r=1$?
Answer: "Outer products" of vectors, $A=\vec{u}\,\vec{v}^T$

$m\times n \quad m\times 1 \quad 1\times n$

Example: $A = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1(3) & 1(1) & 1(4) \\ -1(3) & -1(1) & -1(4) \\ 2(3) & 2(1) & 2(4) \end{bmatrix}$

Every row is a multiple of the 1st row.

$$A \xrightarrow[\text{Row3}-2\text{Row1}]{\text{Row2}+\text{Row1}} \begin{bmatrix} 3 & 1 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{3}\text{Row 1}} \begin{bmatrix} 1 & 1/3 & 4/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

one pivot column, one leading 1

Fun fact: You can write any A as a linear combination of "outer products": $A = \vec{u}_1 \vec{v}_1^T + \vec{u}_2 \vec{v}_2^T + \cdots + \vec{u}_r \vec{v}_r^T$

The rank r is the minimum number of outer products required to add up to r.

Last time Looked at column and null spaces of $m \times n$ A:

$C(A)$ = linear combinations of the columns (subspace of $\mathbb{R}^m$)

$N(A)$ = all $\vec{x}$ such that $A\vec{x} = \vec{0}$ (subspace of $\mathbb{R}^n$)

Why do we care?

$C(A)$ tells us if $A\vec{x} = \vec{b}$ has any solution (it does if $\vec{b}$ is a vector in $C(A)$)

$N(A)$ tells us how many solutions $A\vec{x} = \vec{b}$ can have.

To see why, let's review the full elimination method for solving $A\vec{x} = \vec{b}$:

$$[A \mid \vec{b}] \longrightarrow [R \mid \vec{d}]$$

Augmented matrix — "reduced row echelon form" of A.

① Eliminate lower left variables in A first.

② Identify pivot columns (they contain the first non-zero entries for the rows.)

③ Turn first non-zero entries into leading 1's.

④ Eliminate all variables above the leading 1's.

This is easier if $\vec{b} = \vec{0}$ (solving $A\vec{x} = \vec{0}$, i.e., finding $N(A)$):

Example Find R for $A = \begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & -2 & 1 & 2 \\ 3 & 1 & 2 & -1 \end{bmatrix}$

Step ②: These are the pivot columns

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 2 & -2 & 1 & 2 \\ 3 & 1 & 2 & -1 \end{bmatrix} \xrightarrow[\text{Row 3} - 3\,\text{Row 1}]{\text{Step ①: Row 2} - 2\,\text{Row 1}} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -8 & -1 & 0 \\ 0 & -8 & -1 & -4 \end{bmatrix} \xrightarrow{\text{Row 3} - \text{Row 2}} \begin{array}{cccc} \downarrow & \downarrow & & \downarrow \\ \end{array} \begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & -8 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix} \xrightarrow[-\frac{1}{4}\,\text{Row 3}]{\text{Step ③: } -\frac{1}{8}\,\text{Row 2}}$$

$$\begin{bmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1/8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 1} - \text{Row 3}]{\text{Step ④: Eliminate these}} \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & 1/8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Row 1} - 3\,\text{Row 2}} \begin{bmatrix} 1 & 0 & 5/8 & 0 \\ 0 & 1 & 1/8 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = R.$$

We can use R to find $N(A)$ easily: Solve $A\vec{x} = \vec{0}$

$$[A \mid \vec{0}] \xrightarrow{\text{elimination}} [R \mid \vec{0}] \longrightarrow \begin{cases} x_1 + \frac{5}{8}x_3 = 0 \\ x_2 + \frac{1}{8}x_3 = 0 \\ x_4 = 0 \end{cases}$$

"free variable"

pivot column variables

All solutions look like: $\vec{x} = \begin{bmatrix} -5/8\, x_3 \\ -1/8\, x_3 \\ x_3 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -5/8 \\ -1/8 \\ 1 \\ 0 \end{bmatrix}$ "special solution" Provides a spanning set for $N(A)$.

In general: All solutions to $A\vec{x}=\vec{0}$ is a linear combination of "special solutions," and there is one special solution for each free variable.

What about $A\vec{x}=\vec{b}$ with $\vec{b}\neq\vec{0}$? $[A \mid \vec{b}] \xrightarrow{\text{elimination}} [R \mid \vec{d}]$

not $\vec{0}$ any more

still easy to solve for pivot column variables in terms of free variables.

Example $\begin{bmatrix} 1 & 2 & 2 & 4 \\ 1 & 2 & 3 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightsquigarrow [A \mid \vec{b}] = \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 1 & 2 & 3 & 6 & 2 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right]$

$$\xrightarrow{\text{Row 2}-\text{Row 1}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow[-\text{Row 2}]{\text{Row 3}} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Row 1}-2\text{Row 2}} \begin{array}{c} R \qquad\qquad \vec{d} \\ \left[\begin{array}{cccc|c} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

Eliminate

pivot columns

consistent equation $0=0$, so solutions exist

Equations $R\vec{x}=\vec{d}$ look like: $\begin{cases} x_1 + 2x_2 = -1 \\ x_3 + 2x_4 = 1 \end{cases}$

pivot column variables ← free variables

We can get one particular solution by setting all free variables $=0$.

So $x_1=-1$, $x_2=0$, $x_3=0$, $x_4=0$ $\rightsquigarrow$ One solution is $\vec{x}_p = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

What if $\vec{y}$ is another solution, $A\vec{y}=\vec{b}$? Then

$A(\vec{y}-\vec{x}_p) = A\vec{y} - A\vec{x}_p = \vec{b}-\vec{b} = \vec{0}$

This is a solution to $A\vec{x}=\vec{0}$, so $\vec{y}-\vec{x}_p$ is in $N(A)$

So if $\vec{y}$ is any solution, we can write $\vec{y} = \vec{x}_p + \underbrace{(\vec{y}-\vec{x}_p)}_{\vec{x}_n}$

it's a sum of a particular solution and a null space vector.

On the other hand, any nullspace vector is a linear combination of "special solutions":

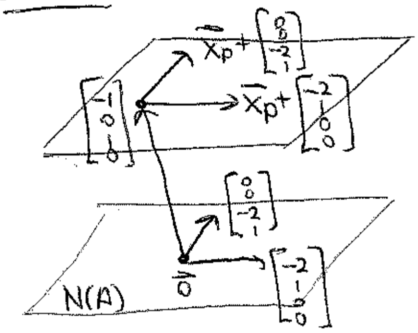
$$\begin{cases} x_1+2x_2=0 \\ x_3+2x_4=0 \end{cases} \rightsquigarrow \begin{matrix} x_1=-2x_2 \\ x_3=-2x_4 \end{matrix} \rightsquigarrow \vec{x}=\begin{bmatrix}-2x_2\\x_2\\-2x_4\\x_4\end{bmatrix}=x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\-2\\1\end{bmatrix}$$

So here is the <u>complete solution</u> to $A\vec{x}=\vec{b}$: All solutions look like

$$\vec{x}=\vec{x}_p+\vec{x}_n=\begin{bmatrix}-1\\0\\1\\0\end{bmatrix}+x_2\begin{bmatrix}-2\\1\\0\\0\end{bmatrix}+x_4\begin{bmatrix}0\\0\\-2\\1\end{bmatrix}$$

$\vec{x}_p\neq\vec{0}$, so plane of solutions <u>doesn't</u> contain $\vec{0}$ (<u>not</u> a subspace)

<u>two</u> free variables means solutions form a <u>plane</u> in $\mathbb{R}^4$.



The solution plane is <u>parallel</u> to N(A).

For this matrix A, R has a row of 0's, means $A\vec{x}=\vec{b}$ has <u>no</u> solution for most $\vec{b}$'s.

<u>In general</u>: If rank of A < #rows of A, then $A\vec{x}=\vec{b}$ usually has no solutions (i.e., most $\vec{b}$'s are <u>not</u> in C(A), so C(A) is <u>smaller</u> than $\mathbb{R}^m$).

<u>Problem 3.3.1</u> $A=\begin{bmatrix}2&4&6&4\\2&5&7&6\\2&3&5&2\end{bmatrix}$, $\vec{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$ Find a condition on b_1,b_2,b_3 for solutions of $A\vec{x}=\vec{b}$ to exist

$$A\vec{x}=\vec{b}\longrightarrow\left[\begin{array}{cccc|c}2&4&6&4&b_1\\2&5&7&6&b_2\\2&3&5&2&b_3\end{array}\right]\xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}-\text{Row 1}}\left[\begin{array}{cccc|c}2&4&6&4&b_1\\0&1&1&2&b_2-b_1\\0&-1&-1&-2&b_3-b_1\end{array}\right]\xrightarrow{\text{Row 3}+\text{Row 2}}$$

$$\left[\begin{array}{cccc|c}2&4&6&4&b_1\\0&1&1&2&b_2-b_1\\0&0&0&0&-2b_1+b_2+b_3\end{array}\right]\xrightarrow{\frac{1}{2}\text{Row 1}}\left[\begin{array}{cccc|c}1&2&3&2&\frac{1}{2}b_1\\0&1&1&2&b_2-b_1\\0&0&0&0&-2b_1+b_2+b_3\end{array}\right]\xrightarrow[-2\text{Row 2}]{\text{Row 1}}\left[\begin{array}{cccc|c}1&0&1&-2&\frac{5}{2}b_1-2b_2\\0&1&1&2&b_2-b_1\\0&0&0&0&-2b_1+b_2+b_3\end{array}\right]$$

Solutions exist <u>only</u> if $\boxed{-2b_1+b_2+b_3=0}$. This is the equation of a <u>plane</u> in $\mathbb{R}^3$, and this plane is exactly the column space C(A).

If $\vec{b}$ is in $C(A)$ (which means $-2b_1+b_2+b_3=0$), then there are many solutions.

Null space: Take $b_1=b_2=b_3=0$ (note $-2(0)+0+0=0$)

$$\begin{cases} x_1+x_3-2x_4=0 \\ x_2+x_3+2x_4=0 \end{cases} \longrightarrow \begin{array}{l} x_1=-x_3+2x_4 \\ x_2=-x_3-2x_4 \end{array}$$ (2 free variables, x_3 and x_4)

$N(A)$ = all vectors like $\begin{bmatrix} -x_3+2x_4 \\ -x_3-2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

special solutions

Now take $\vec{b}=\begin{bmatrix} 4 \\ 3 \\ 5 \end{bmatrix}$. Is $\vec{b}$ in $C(A)$? Check: $-2(4)+3+5=0$ ✓

In this case, get equations $\begin{cases} x_1+x_3-2x_4 = \frac{5}{2}(4)-2(3) = 4 \\ x_2+x_3+2x_4 = 3-4 = -1 \end{cases}$

One particular solution: set $x_3=x_4=0$, get $x_1=4$, $x_2=-1$

So $\vec{x}_p=\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. All solutions: $\vec{x}=\vec{x}_p+\vec{x}_n=\begin{bmatrix} 4 \\ -1 \\ 0 \\ 0 \end{bmatrix}+x_3\begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}+x_4\begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$

plane of solutions parallel to $N(A)$ again.

In these examples, $N(A)$ was a plane (non-zero) so we got a whole plane of solutions for $A\vec{x}=\vec{b}$ (or no solution if $\vec{b}$ is not in $C(A)$).
I.e., $N(A)$ tells you the max number of solutions $A\vec{x}=\vec{b}$ can have.

Question: When we do $[A \,|\, \vec{b}] \xrightarrow{\text{elimination}} [R \,|\, \vec{d}]$, what are the basic possibilities for R?

(1) R has a leading 1 in every row and every column.
(2) R has a leading 1 in every row, but not every column.
(3) R has a leading 1 in every column, but not every row.
(4) R has leading 1's missing from both rows and columns

We just saw examples of Case (4): For most $\vec{b}$, $A\vec{x}=\vec{b}$ has no solution, but if $\vec{b}$ is in $C(A)$, then $A\vec{x}=\vec{b}$ has infinitely many solutions,