

## Practice Final Exam

1. (a) Find the unique value of  $c$  such that the system of equations has at least one solution:

$$x_1 + x_2 + x_3 + x_4 = 3$$

$$x_1 + 2x_2 + 3x_3 + 4x_4 = c$$

$$x_1 - x_2 - 3x_3 - 5x_4 = 1$$

- (b) For the value of  $c$  you found, find *all* solutions to the system of equations.

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 & c \\ 1 & -1 & -3 & -5 & 1 \end{array} \right] \xrightarrow[\text{Row 3 - Row 1}]{\text{Row 2 - Row 1}} \left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 3 & c-3 \\ 0 & -2 & -4 & -6 & -2 \end{array} \right] \xrightarrow[\text{Row 3 + 2 Row 2}]{\text{Row 1 - Row 2}}$$

$$\left[ \begin{array}{cccc|c} 1 & 0 & -1 & -2 & -c+6 \\ 0 & 1 & 2 & 3 & c-3 \\ 0 & 0 & 0 & 0 & 2c-8 \end{array} \right]$$

(a) To get solution(s), need  $2c - 8 = 0$ , or  $\boxed{c = 4}$

(b) When  $c = 4$ :

$$x_1 - x_3 - 2x_4 = 2$$

$$x_2 + 2x_3 + 3x_4 = 1$$

free  
variables

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 + 2 \\ -2x_3 - 3x_4 + 1 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

particular  
solution

null space  
vectors

2. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}.$$

(a) Find all eigenvalues of  $A$ .

(b) Find a basis of  $\mathbf{R}^3$  consisting of eigenvectors for  $A$ .

(c) Find the angles between the eigenvectors in the basis.

Solve  $\det(A - \lambda I) = 0$ :

$$(a) \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 3-\lambda & -2 \\ 1 & -2 & 3-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} - 1 \begin{vmatrix} 1 & -2 \\ 1 & 3-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 3-\lambda \\ 1 & -2 \end{vmatrix}$$

$$= (2-\lambda) \left[ (3-\lambda)(3-\lambda) - 4 \right] - \left[ (3-\lambda) + 2 \right] + \left[ -2 - (3-\lambda) \right]$$

$$= (2-\lambda) \left( \lambda^2 - 6\lambda + 5 \right) + 2(\lambda - 5)$$

$$(\lambda - 5)(\lambda - 1)$$

$$= (\lambda - 5) \left( \underbrace{(2-\lambda)(\lambda-1) + 2}_{-\lambda^2 + 3\lambda} \right) = -\lambda(\lambda-3)(\lambda-5) = 0$$

$$\boxed{\lambda = 0, 3, 5}$$

(b) Eigenvectors for  $\lambda = 0$ : Solve  $A\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix} \xrightarrow[\text{Row 3} - \frac{1}{2} \text{Row 1}]{\text{Row 2} - \frac{1}{2} \text{Row 1}} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 5/2 & -5/2 \\ 0 & -5/2 & 5/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1/2 & 1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Row 1} - \frac{1}{2} \text{Row 2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{matrix} x_1 + x_3 = 0 \\ x_1 - x_3 = 0 \end{matrix} \longrightarrow \vec{x} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda=3: \text{Solve } (A-3I)\vec{x}=\vec{0}$$

$$\begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 0 \end{bmatrix} \xrightarrow[\text{Row 3+Row 1}]{\text{Row 2+Row 1}} \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[\text{Row 3+Row 2}]{-\text{Row 1+Row 2}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{aligned} x_1 - 2x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned} \rightarrow \vec{x} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda=5: \text{Solve } (A-5I)\vec{x}=\vec{0}$$

$$\begin{bmatrix} -3 & 1 & 1 \\ 1 & -2 & -2 \\ 1 & -2 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -2 & -2 \\ -3 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 2+3Row 1}} \begin{bmatrix} 1 & -2 & -2 \\ 0 & -5 & -5 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{aligned} x_1 &= 0 \\ x_2 + x_3 &= 0 \end{aligned} \rightarrow \vec{x} = x_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\text{Basis of eigenvectors: } \left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$$

$\vec{x}_0 \quad \vec{x}_3 \quad \vec{x}_5$

$$\begin{aligned} \text{(c) Angles: } \vec{x}_0 \cdot \vec{x}_3 &= (-1)(2) + (1)(1) + (1)(1) = 0 \\ \vec{x}_0 \cdot \vec{x}_5 &= (-1)(0) + (1)(-1) + (1)(1) = 0 \\ \vec{x}_3 \cdot \vec{x}_5 &= (2)(0) + (1)(-1) + (1)(1) = 0 \end{aligned}$$



The three vectors are all at 90° angles to each other.

3. Determine whether the following sets of vectors are bases for  $\mathbb{R}^3$ . In case a set of vectors is *not* linearly independent, show how to write one of the vectors as a linear combination of the others:

(a)  $\left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} \right\}$       (b)  $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 25 \end{bmatrix} \right\}$

(a)  $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow[\text{Row 3} - 7 \text{ Row 1}]{\text{Row 2} - 4 \text{ Row 1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow{\text{Row 1} - 2 \text{ Row 2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = -2x_3 \end{matrix} \rightarrow \vec{x} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$R \neq I \rightarrow$  dependent, not a basis

Shows that  $1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix} + 1 \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = \vec{0}$ , or for example,

$$\begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ 5 \\ 8 \end{bmatrix}$$

(b)  $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 5 \\ 1 & 9 & 25 \end{bmatrix} \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 1} - \text{Row 2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 4 \\ 0 & 8 & 24 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{\text{Row 3} - \text{Row 2}}$

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{elimination}]{\text{more}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\uparrow$   
no free variables

$R = I \rightarrow$  independent, these vectors are a basis

4. Find bases for the null space, column space, row space, and left null space of the matrix:

$$A = \begin{bmatrix} -3 & 1 & 4 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 1 & 4 & 4 \\ 1 & 2 & -1 & -2 \\ -3 & 8 & 5 & 2 \end{bmatrix} \xrightarrow[\text{Row 3 - Row 1}]{3 \text{ Row 2} + \text{Row 4}} \begin{bmatrix} -3 & 1 & 4 & 4 \\ 0 & 5 & 1 & 0 \\ 0 & 5 & 1 & 0 \end{bmatrix} \xrightarrow{\text{Row 3 - Row 2}} \begin{bmatrix} -3 & 1 & 4 & 4 \\ 0 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1/3 & -4/3 & -4/3 \\ 0 & 1 & 1/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{3} \text{ Row 2}]{\text{Row 1} +} \begin{bmatrix} 1 & 0 & -9/5 & -10/5 \\ 0 & 1 & 1/5 & -2/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

N(A): Solve  $x_1 - \frac{9}{5}x_3 - \frac{10}{5}x_4 = 0$

$x_2 + \frac{1}{5}x_3 - \frac{2}{5}x_4 = 0$

two free variables,  
two basis vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 9/5 \\ -1/5 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 10/5 \\ 2/5 \\ 0 \\ 1 \end{bmatrix}$$

Basis for N(A).

C(A): Basis = pivot columns in A: cols. 1 and 2

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} \right\}$$

R(A) = C(A<sup>T</sup>): Basis = non-zero rows in R:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -9/5 \\ -10/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1/5 \\ -2/5 \end{bmatrix} \right\}$$

Since  $\dim C(A^T) = 2$ , you could also take any two independent rows of A to be your basis.

For N(A<sup>T</sup>), I'll show three different methods:

1. Solve  $A^T \vec{x} = 0$  by elimination:

$$\begin{bmatrix} -3 & 1 & -3 \\ 1 & 2 & 8 \\ 4 & -1 & 5 \\ 4 & -2 & 2 \end{bmatrix} \xrightarrow[\text{Row 3 - 4 Row 2}]{\text{Row 1} + 3 \text{ Row 2}} \begin{bmatrix} 0 & 7 & 21 \\ 1 & 2 & 8 \\ 0 & -9 & -27 \\ 0 & -10 & -30 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{-2 Row 2}]{\text{Row 1}} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

So  $x_1 + 2x_3 = 0$   
 $x_2 + 3x_3 = 0 \rightarrow \vec{x} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}$   
 Basis vector for  $N(AT)$ .

2. Try to solve  $A\vec{x} = \vec{b}$  by elimination:

$$\left[ \begin{array}{cccc|c} -3 & 1 & 4 & 4 & b_1 \\ 1 & 2 & -1 & -2 & b_2 \\ -3 & 8 & 5 & 2 & b_3 \end{array} \right] \xrightarrow[\text{Row 3 - Row 1}]{3 \text{ Row 2} + \text{Row 1}} \left[ \begin{array}{cccc|c} -3 & 1 & 4 & 4 & b_1 \\ 0 & 7 & 1 & -2 & b_1 + 3b_2 \\ 0 & 7 & 1 & -2 & -b_1 + b_3 \end{array} \right]$$

$$\xrightarrow{\text{Row 3 - Row 2}} \left[ \begin{array}{cccc|c} -3 & 1 & 4 & 4 & b_1 \\ 0 & 7 & 1 & -2 & b_1 + 3b_2 \\ 0 & 0 & 0 & 0 & -2b_1 - 3b_2 + b_3 \end{array} \right]$$

Shows that  $-2(\text{Row 1}) - 3(\text{Row 2}) + (\text{Row 3}) = \vec{0} \rightarrow$

$$\begin{bmatrix} -2 & -3 & 1 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \vec{0} \rightarrow \begin{bmatrix} A^T \end{bmatrix} \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} = \vec{0}$$

Basis vector for  $N(AT)$ ; it's enough for a basis because  $\dim N(AT) = 3 - \dim C(A) = 1$

3. Remember that  $N(AT) = C(A)^\perp =$  all  $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$  such that

$$\begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \text{ and } \begin{bmatrix} -3 \\ 1 \\ -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 = \text{all solutions to } \begin{cases} x_1 + 2x_2 + 8x_3 = 0 \\ -3x_1 + x_2 - 3x_3 = 0 \end{cases}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 8 \\ -3 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 8 \\ 0 & 7 & -17 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = -3x_3 \end{cases}$$

$$\rightarrow \vec{x} = x_3 \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix} \text{ Basis vector}$$

5. (a) Diagonalize the matrix  $A$ : write  $A = X\Lambda X^{-1}$  where  $X$  is invertible and  $\Lambda$  is diagonal.

$$A = \begin{bmatrix} 10 & 12 \\ -6 & -7 \end{bmatrix}$$

- (b) Show that the matrix  $B$  is *not* diagonalizable:

$$B = \begin{bmatrix} 8 & 9 \\ -4 & -4 \end{bmatrix}$$

(a)  $\Lambda \rightarrow$  eigenvalues  
 $X \rightarrow$  eigenvectors

$$\begin{vmatrix} 10-\lambda & 12 \\ -6 & -7-\lambda \end{vmatrix} = (10-\lambda)(-7-\lambda) + 72$$

$$= \lambda^2 - 3\lambda + 2 = (\lambda-1)(\lambda-2) = 0 \rightarrow \lambda = 1, 2$$

For  $\lambda=1$ : Solve  $(A-I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 9 & 12 \\ -6 & -8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4/3 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x} = x_2 \begin{bmatrix} -4/3 \\ 1 \end{bmatrix} \quad \text{If } x_2=3: \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

For  $\lambda=2$ : Solve  $(A-2I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 8 & 12 \\ -6 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x} = x_2 \begin{bmatrix} -3/2 \\ 1 \end{bmatrix} \quad \text{If } x_2=2: \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$\text{So } A = \underbrace{\begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}}_X \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_\Lambda \underbrace{\begin{bmatrix} -4 & -3 \\ 3 & 2 \end{bmatrix}^{-1}}_{X^{-1}}$$

(b) Eigenvalues:  $\begin{vmatrix} 8-\lambda & 9 \\ -4 & -4-\lambda \end{vmatrix} = (8-\lambda)(-4-\lambda) + 36 = \lambda^2 - 4\lambda + 4$   
 $= (\lambda-2)^2 = 0 \rightarrow \lambda = 2, 2$

Eigenvectors: Solve  $(B-2I)\vec{x} = \vec{0}$   $\begin{bmatrix} 6 & 9 \\ -4 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3/2 \\ 0 & 0 \end{bmatrix}$

$\rightarrow \vec{x} = x_2 \begin{bmatrix} -3/2 \\ 1 \end{bmatrix}$   $\leftarrow$  Only one independent eigenvector, not enough for a basis of  $\mathbb{R}^2$ . So  $B$  is not diagonalizable.

6. Consider the  $(x, y)$  data points  $(-1, 0)$ ,  $(0, 1)$ ,  $(1, 3)$ , and  $(2, 9)$ .

- Find the best least squares fit by a linear function  $y = mx + b$  to the data.
- Plot your linear function from part (a) along the data on a coordinate system.
- Find the error of the least squares approximation.

(a) Try to solve ~~the~~  $\begin{cases} 1b + (-1)m = 0 \\ 1b + 0m = 1 \\ 1b + 1m = 3 \\ 1b + 2m = 9 \end{cases} \rightarrow \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_A \underbrace{\begin{bmatrix} b \\ m \end{bmatrix}}_{\vec{b}} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix}}_{\vec{b}}$

Inconsistent, no solution.

Instead, solve the "normal equations":

$$\underbrace{A^T A}_{\begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}} \hat{x} = A^T \underbrace{\vec{b}}_{\begin{bmatrix} 13 \\ 21 \end{bmatrix}}$$

$$\hat{x} = \begin{bmatrix} 4 & 2 \\ 2 & 8 \end{bmatrix}^{-1} \begin{bmatrix} 13 \\ 21 \end{bmatrix} = \frac{1}{28} \begin{bmatrix} 8 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 13 \\ 21 \end{bmatrix} = \begin{bmatrix} 31/14 \\ 29/14 \end{bmatrix} \quad \begin{matrix} m \\ b \end{matrix}$$

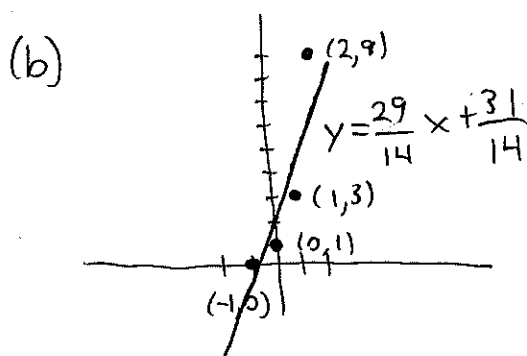
$$\boxed{y = \frac{29}{14}x + \frac{31}{14}}$$

(c) Error =  $\|\vec{e}\| = \|A\hat{x} - \vec{b}\|$

$$= \left\| \begin{bmatrix} 1/7 \\ 31/14 \\ 30/7 \\ 89/14 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 3 \\ 9 \end{bmatrix} \right\|$$

$$= \sqrt{\left(\frac{1}{7}\right)^2 + \left(\frac{17}{14}\right)^2 + \left(\frac{9}{7}\right)^2 + \left(-\frac{37}{14}\right)^2}$$

(Probably can't simplify any further without a calculator.)





7. (a) Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 7 & 14 \end{bmatrix}$$

(b) Use  $A^{-1}$  to solve the linear system of equations  $Ax = (1, 2, 1)$ .

(a) Use elimination:

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 4 & 7 & 0 & 1 & 0 \\ 3 & 7 & 14 & 0 & 0 & 1 \end{array} \right] \xrightarrow[\text{Row 3} - 3 \text{ Row 1}]{\text{Row 2} - 2 \text{ Row 1}} \left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 5 & -3 & 0 & 1 \end{array} \right] \xrightarrow[\text{Row 3}]{\text{Row 2} \leftrightarrow \text{Row 3}}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 5 & -3 & 0 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array} \right] \xrightarrow[\text{Row 2} - 5 \text{ Row 3}]{\text{Row 1} - 3 \text{ Row 3}} \left[ \begin{array}{ccc|ccc} 1 & 2 & 0 & 7 & -3 & 0 \\ 0 & 1 & 0 & 7 & -5 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array} \right] \xrightarrow[\text{-2 Row 2}]{\text{Row 1}}$$

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & -7 & 7 & -2 \\ 0 & 1 & 0 & 7 & -5 & 1 \\ 0 & 0 & 1 & -2 & 1 & 0 \end{array} \right]$$

$A^{-1}$

$$(b) \bar{x} = A^{-1} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 & 7 & -2 \\ 7 & -5 & 1 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 0 \end{bmatrix}$$

8. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 2 \end{bmatrix}.$$

(a) Find the  $LU$  decomposition of  $A$ .

(b) Find the volume of the box in  $\mathbf{R}^3$  that is spanned by the columns of  $A$ .

(a)  $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & -2 & 2 \end{bmatrix} \xrightarrow[\text{Row 3} - \boxed{1} \text{ Row 1}]{\text{Row 2} - \boxed{1} \text{ Row 1}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow[-\boxed{\frac{1}{2}} \text{ Row 2}]{\text{Row 3}} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = U$

So  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$L \qquad U$

(b) Volume =  $|\det A| = |(1)(2)(1)| = 2$

9. (a) Find the determinants of the matrices

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -3 & 0 & 3 \end{bmatrix}$$

(b) Are the matrices  $A$  and  $B$  invertible?

$$(a) \det A = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 4 \\ 1 & -1 & 2 & -2 \\ 1 & -1 & 8 & -8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 1 & -3 \\ 0 & -2 & 7 & -9 \end{vmatrix} \xrightarrow{R_4 - R_3} = - \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -3 \\ 0 & 0 & 3 & 3 \\ 0 & -2 & 7 & -9 \end{vmatrix}$$

$$\begin{array}{l} \text{Row 4 - Row 2} \\ \hline \end{array} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 6 & -6 \end{vmatrix} \begin{array}{l} \text{Row 4 -} \\ \hline \text{2Row 3} \end{array} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & 3 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & -12 \end{vmatrix} = -(1)(-2)(3)(-12)$$

$$= \boxed{-72}$$

$$\det B = 2 \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} = 2(3-3) = \boxed{0}$$

(b)  $A$  is invertible,  $B$  is not.

10. (a) Find an orthonormal basis for the subspace  $V$  of  $\mathbb{R}^4$  spanned by  $(1, 1, 1, 1)$  and  $(3, 2, 2, 1)$ .  
 (b) Find the projection matrix  $P$  for the orthogonal projection onto  $V$ , and compute the projection of  $(0, 0, 1, 1)$  onto  $V$ .

(a) First basis vector:  $\vec{x}_1 = \frac{1}{\text{length}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{1^2+1^2+1^2+1^2}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Second:

$\vec{x}_2 = \frac{1}{\|\vec{e}\|} \vec{e} \leftarrow \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} - \vec{p}$

$\vec{p} = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}} = \frac{1}{4} (3+2+2+1) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}$

Projection matrix

Orthonormal basis:

$\vec{x}_2 = \frac{1}{\|\vec{e}\|} \left( \begin{bmatrix} 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right) = \frac{1}{\text{length}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$

$\left\{ \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

(b)  $P = A(A^T A)^{-1} A^T = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \left( \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

columns = basis for  $V$

Can use this basis, which is orthogonal, though not quite orthonormal

$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/2 & 0 & 0 & -1/2 \end{bmatrix} = \begin{bmatrix} 3/4 & 1/4 & 1/4 & -1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ -1/4 & 1/4 & 1/4 & 3/4 \end{bmatrix}$

$P = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/2 \end{bmatrix}$

$P \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix}$