So $A = X \Lambda X^{-1}$:

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$

Λ is a projection matrix, so $\Lambda^2 = \Lambda$ (also $\Lambda^n = \Lambda$). This means $A^n = A$ for any n as well. But A is <u>not</u> an <u>orthogonal</u> projection matrix because A isn't symmetric.

<u>Fun application of diagonalizing to Fibonacci numbers</u>

$F_0 = 0,\ F_1 = 1,\ F_2 = 1,\ F_3 = 2,\ F_4 = 3,\ 5, 8, 13, 21, 34, 55, 89, \ldots$

In general, $F_k = F_{k-1} + F_{k-2}$ (recursion formula)

Use eigenvalues to find a "closed-form" formula for F_k.

<u>Idea</u>: $\begin{bmatrix} F_1 \\ F_0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} F_2 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} F_1 + F_0 \\ F_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$

In general: $\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} F_k + F_{k-1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_k \\ F_{k-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} F_{k-1} + F_{k-2} \\ F_{k-1} \end{bmatrix}$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^2 \begin{bmatrix} F_{k-1} \\ F_{k-2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^3 \begin{bmatrix} F_{k-2} \\ F_{k-3} \end{bmatrix} = \cdots = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$$

$F_{k-2} + F_{k-3}$ same

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

We can use <u>eigenvalues</u> to compute matrix powers: $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

F_k is the <u>second</u> component of this vector.

<u>Eigenvalues</u>: $\begin{vmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 - \lambda - 1 = 0 \rightarrow \lambda = \dfrac{1 \pm \sqrt{(-1)^2 + 4(1)(-1)}}{2(1)}$

$$= \frac{1 \pm \sqrt{5}}{2}$$

"+" eigenvalue is called the "Golden Ratio". Typical notation is ϕ: $\phi = \frac{1+\sqrt{5}}{2}$

"−" eigenvalue is $1 - \left(\frac{1+\sqrt{5}}{2}\right) = 1 - \phi$

Eigenvectors: $\lambda = \phi$, Solve $\begin{bmatrix} 1-\phi & 1 \\ 1 & -\phi \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2nd equation tells us that $x_1 - \phi x_2 = 0$, so $\vec{x} = \begin{bmatrix} \phi \\ 1 \end{bmatrix}$ is an eigenvector.

For $\lambda = 1-\phi$, Solve $\begin{bmatrix} 1-(1-\phi) & 1 \\ 1 & -(1-\phi) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2nd equation says $x_1 - (1-\phi)x_2 = 0$, so $\vec{x} = \begin{bmatrix} 1-\phi \\ 1 \end{bmatrix}$ is an eigenvector.

So
$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} \phi & 1-\phi \\ 1 & 1 \end{bmatrix}}_{X}\underbrace{\begin{bmatrix} \phi & 0 \\ 0 & 1-\phi \end{bmatrix}}_{\Lambda}\underbrace{\frac{1}{\phi-(1-\phi)}\begin{bmatrix} 1 & -(1-\phi) \\ -1 & \phi \end{bmatrix}}_{X^{-1}}$$

Finally:
$$\begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^k \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \phi & 1-\phi \\ 1 & 1 \end{bmatrix}\begin{bmatrix} \phi^k & 0 \\ 0 & (1-\phi)^k \end{bmatrix}\frac{1}{2\phi-1}\begin{bmatrix} 1 & -(1-\phi) \\ -1 & \phi \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} \phi^{k+1} & (1-\phi)^{k+1} \\ \phi^k & (1-\phi)^k \end{bmatrix}\frac{1}{2\phi-1}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{2\phi-1}\begin{bmatrix} \phi^{k+1} - (1-\phi)^{k+1} \\ \phi^k - (1-\phi)^k \end{bmatrix}$$

Closed-form formula: F_k = 2nd component =

$$\frac{1}{2\left(\frac{1+\sqrt{5}}{2}\right)-1}\left(\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

It's a little remarkable that this expression is a positive integer, since there are so many $\sqrt{5}$'s in here. (But note: formula stays same when you change $\sqrt{5} \to -\sqrt{5}$, means $\sqrt{5}$'s have to cancel out.)

What does this formula tell us about F_k?

When k gets large: $\phi > 1 \rightsquigarrow \phi^k$ <u>grows</u> exponentially

$-1+\phi<1 \rightsquigarrow (\phi-1)^k$ <u>decays</u> exponentially

So if k is large, $F_k \approx \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k$ ← Shows F_k grows approximately exponentially, with base $\phi \approx 1.61803398\ldots$

Ratio of consecutive Fibonacci numbers:

$$F_{k+1}/F_k \approx \frac{\frac{1}{\sqrt{5}}\phi^{k+1}}{\frac{1}{\sqrt{5}}\phi^k} = \phi \quad \text{(if } k \text{ is large)}$$

<u>Examples</u>: $\phi = 1.61803398\ldots$

$F_6/F_5 = 8/5 = 1.60000000$

$F_7/F_6 = 13/8 = 1.62500000$

$F_8/F_7 = 21/13 = 1.61538462\ldots$

$F_9/F_8 = 34/21 = 1.61904762\ldots$

$F_{10}/F_9 = 55/34 = 1.61764706\ldots$

$F_{11}/F_{10} = 89/55 = 1.61818182\ldots$

$F_{12}/F_{11} = 144/89 = 1.61797753\ldots$

Last week: Eigenvalues and Eigenvectors

$$A\vec{x} = \lambda\vec{x}$$ ← non-zero eigenvector

A: n×n matrix

λ: eigenvalue ← Solutions of $\underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial of } A} = 0$

General comment: When you find eigenvectors for a matrix in the homework problems, your answers might be different from the answer key's.

Why? Probably because your ~~answer~~ eigenvectors are multiples or linear combinations of the answer key's.

This is okay! General fact: Suppose λ is an eigenvalue of A. Then the set of all eigenvectors + zero vector is a subspace.

called the eigenspace for λ.

Closed under addition: $A(\vec{x}+\vec{y}) = A\vec{x}+A\vec{y}$
$= \lambda\vec{x}+\lambda\vec{y} = \lambda(\vec{x}+\vec{y})$

Scalar multiplication: $A(c\vec{x}) = c(A\vec{x})$
$= c(\lambda\vec{x}) = \lambda(c\vec{x})$

Linear combinations of eigenvectors are still eigenvectors (if they are non-zero)

Note: Eigenspace for λ is just null space of $A-\lambda I$.

So if your homework answer is "$\vec{x}$" and answer key's is "$c\vec{x}$", both are correct!

Today: Solving differential equations with linear algebra.

1×1 system of ordinary differential equations

$$\begin{cases} \dfrac{du}{dt} = \lambda u \\ u(0) = u_0 \end{cases}$$

← General solution is $u(t) = Ce^{\lambda t}$, C a scalar,

because $\frac{d}{dt} Ce^{\lambda t} = C\lambda e^{\lambda t} = \lambda(Ce^{\lambda t})$

Since also $u(0) = u_0$:

$u_0 = C e^{\lambda(0)} = C \longrightarrow u(t) = e^{\lambda t} u_0$

Change to nxn system: $\vec{u} = \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}$

$u_1(t)$ and $u_2(t)$ are "coupled"

$$\frac{d\vec{u}}{dt} = A\vec{u}(t)$$

nxn matrix

Example: $\begin{cases} u_1'(t) = 2u_1(t) + u_2(t) \\ u_2'(t) = u_1(t) + 2u_2(t) \end{cases}$

Or, $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Guess an exponential solution: $\vec{u}(t) = e^{\lambda t}\vec{x}$

constant vector

Then we need: $\vec{u}'(t) = \lambda \cancel{e^{\lambda t}} \vec{x}$

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \cancel{e^{\lambda t}} \vec{x}$$

$\longrightarrow \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \vec{x} = \lambda \vec{x}$

That is: We need λ to be an eigenvalue and $\vec{x}$ to be a corresponding eigenvetor (to get a non-zero solution)

Eigenvalues in this example: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3$$

$\longrightarrow (\lambda - 1)(\lambda - 3) = 0 \longrightarrow \lambda = 1, 3$

Eigenvetors for $\lambda = 1$: Solve $(A - I)\vec{x} = \vec{0} \rightarrow \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

So we need $x_1 = -x_2 \to \vec{x} = x_2\begin{bmatrix}-1\\1\end{bmatrix}$

Eigenvectors for $\lambda = 3$: Solve $(A - 3I)\vec{x} = \vec{0}$, or

$\begin{bmatrix}-1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. So $x_1 = x_2 \longrightarrow \vec{x} = \begin{bmatrix}1\\1\end{bmatrix}x_2$

Two eigenvalue/eigenvector pairs $\longrightarrow$ two different non-zero solutions of the differential equations:

$\vec{u}(t) = e^t\begin{bmatrix}-1\\1\end{bmatrix} = \begin{bmatrix}-e^t\\e^t\end{bmatrix}$ and $\vec{u}(t) = e^{3t}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}e^{3t}\\e^{3t}\end{bmatrix}$

General solution of the system of differential equations:

All linear combinations of the two basic solutions

because it's a 2×2 system

$$\vec{u}(t) = Ce^t\begin{bmatrix}-1\\1\end{bmatrix} + De^{3t}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-Ce^t + De^{3t}\\Ce^t + De^{3t}\end{bmatrix}\begin{array}{l}\leftarrow u_1(t)\\\leftarrow u_2(t)\end{array}$$

We can pick out one particular solution by choosing initial values for $u_1(t)$ and $u_2(t)$.

Example (of an initial value problem): Suppose $u_1(0) = 2$, $u_2(0) = 3$, or $\vec{u}(0) = \begin{bmatrix}2\\3\end{bmatrix}$.

Then $\begin{cases}-Ce^0 + De^{3(0)} = 2\\Ce^0 + De^{3(0)} = 3\end{cases} \longrightarrow \begin{cases}-C + D = 2\\C + D = 3\end{cases}$ $\quad 2C = 1 \to C = \frac{1}{2}$, $\quad 2D = 5 \to D = \frac{5}{2}$

Get $\vec{u}(t) = \begin{bmatrix}-\frac{1}{2}e^t + \frac{5}{2}e^{3t}\\\frac{1}{2}e^t + \frac{5}{2}e^{3t}\end{bmatrix}$

Another perspective for solving differential equations:
"Matrix exponential" of an $n \times n$ matrix A.

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \frac{A^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

Technically, this is $\lim_{N\to\infty} \underbrace{\sum_{n=0}^{N} \frac{A^n}{n!}}_{\text{"partial sum" } n\times n \text{ matrix, } S_N}$, means <u>every entry</u> of S_N approaches the corresponding entry of e^A as $N \to \infty$.

Derivatives: $\frac{d}{dt} e^{tA} = \frac{d}{dt} \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n \overset{(?)}{=} \sum_{n=0}^{\infty} \frac{d}{dt} \frac{t^n}{n!} A^n$

$$= \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} A^n = A \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n = A e^{tA}.$$

Shows that solutions to $\vec{u}'(t) = A\vec{u}(t)$ <u>should</u> be

$\vec{u}(t) = e^{tA}\vec{u}(0)$ — the initial value, can be any constant vector

But can we actually <u>calculate</u> this matrix exponential?

Yes if A is diagonalized! $A = X \Lambda X^{-1}$ — eigenvalue matrix: $\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$

Because then $e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$

$$= \sum_{n=0}^{\infty} \frac{t^n}{n!} \underbrace{(X\Lambda X^{-1})^n}_{X\Lambda^n X^{-1}} = X \left(\underbrace{\sum_{m=0}^{\infty} \frac{t^m}{m!} \begin{bmatrix} \lambda_1^m & & & 0 \\ & \lambda_2^m & & \\ & & \ddots & \\ 0 & & & \lambda_n^m \end{bmatrix}}_{} \right) X^{-1}$$

Diagonal matrix with entries
$\sum_{m=0}^{\infty} \frac{t^m \lambda_1^m}{m!}, \sum_{m=0}^{\infty} \frac{t^m \lambda_2^m}{m!}, \ldots, \sum_{m=0}^{\infty} \frac{t^m}{m!} \lambda_n^m$

So $e^{tA} = X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots & e^{\lambda_n t} \end{bmatrix} X^{-1}$

So here are the solutions to $\vec{u}'(t) = A\vec{u}(t)$ when $A = X\Lambda X^{-1}$:

$$\vec{u}(t) = X \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & e^{\lambda_2 t} & \\ 0 & & \ddots & e^{\lambda_n t} \end{bmatrix} X^{-1}\vec{u}(0)$$

Let's check this with our previous example, $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$:

$\lambda_1 = 1$, $\lambda_2 = 3$, $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

So $A = X\Lambda X^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}^{-1}$

$\frac{1}{-2}\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

The solutions to $\vec{u}'(t) = A\vec{u}(t)$ are:

$$\vec{u}(t) = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{t} & 0 \\ 0 & e^{3t} \end{bmatrix}\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} u_1(0) \\ u_2(0) \end{bmatrix}$$

$$= \begin{bmatrix} -e^{t} & e^{3t} \\ e^{t} & e^{3t} \end{bmatrix}\begin{bmatrix} -\frac{1}{2}u_1(0) + \frac{1}{2}u_2(0) \\ \frac{1}{2}u_1(0) + \frac{1}{2}u_2(0) \end{bmatrix} = \begin{bmatrix} \underbrace{\tfrac{1}{2}(u_1(0) - u_2(0))}e^{t} + \underbrace{\tfrac{1}{2}(u_1(0)+u_2(0))}e^{3t} \\ -\tfrac{1}{2}(u_1(0) - u_2(0))e^{t} + \tfrac{1}{2}(u_1(0)+u_2(0))e^{3t} \end{bmatrix}$$

These are the C and D from before.

Solve when $\vec{u}(0) = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$:

$u_1(t) = -\frac{1}{2}e^{t} + \frac{5}{2}e^{3t}$, $u_2(t) = \frac{1}{2}e^{t} + \frac{5}{2}e^{3t}$, like before.

Matrix exponential is still useful when eigenvalues don't behave well (repeated roots)

<u>Example</u> $y'' - 2y' + y = 0$

2nd-order equation for $y(t)$

Trick to turn into a <u>1st-order system</u>: write $u_1(t) = y(t)$, $u_2(t) = y'(t)$.