

Homework 1 Solutions

①

1.1.3 $\vec{v} + \vec{w} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$$\vec{v} + \vec{w} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

+ $\vec{v} - \vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

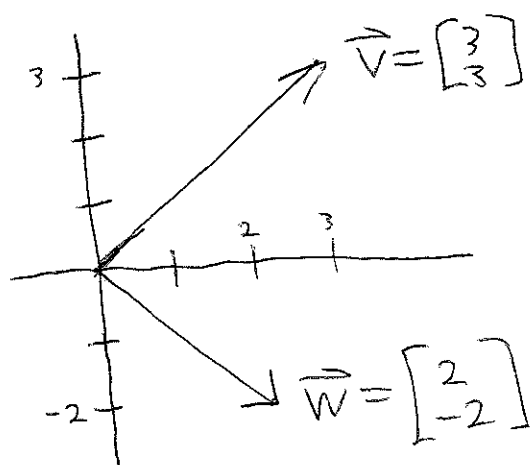
- $\vec{v} - \vec{w} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

$$2\vec{v} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$$

$$2\vec{w} = \begin{bmatrix} 4 \\ -4 \end{bmatrix}$$

$$\vec{v} = \frac{1}{2} \begin{bmatrix} 6 \\ 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\vec{w} = \frac{1}{2} \begin{bmatrix} 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$



1.1.6 Every linear combination looks like $c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} =$

$$= \begin{bmatrix} c \\ -2c+d \\ c-d \end{bmatrix}. \text{ components add up to } c + (-2c+d) + (c-d)$$

$$= (1-2+1)c + (1-1)d = \boxed{0}$$

Write $\begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix}$ as a linear combination: $\begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix} = c \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

$$\rightarrow \begin{cases} 1c + 0d = 3 \rightarrow c = 3 \end{cases}$$

$$\begin{cases} -2c + d = 3 \end{cases} \rightarrow -2(3) + d = 3 \rightarrow d = 9$$

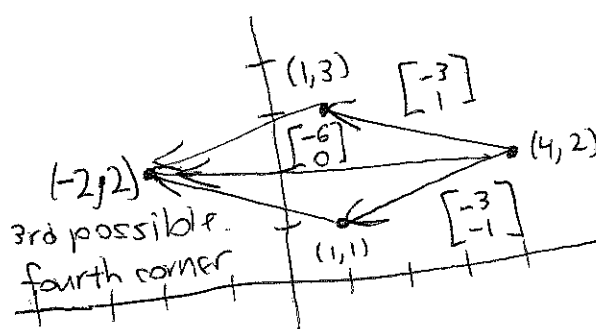
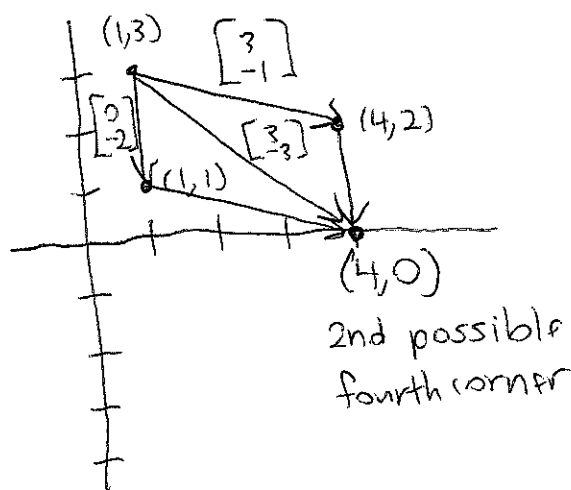
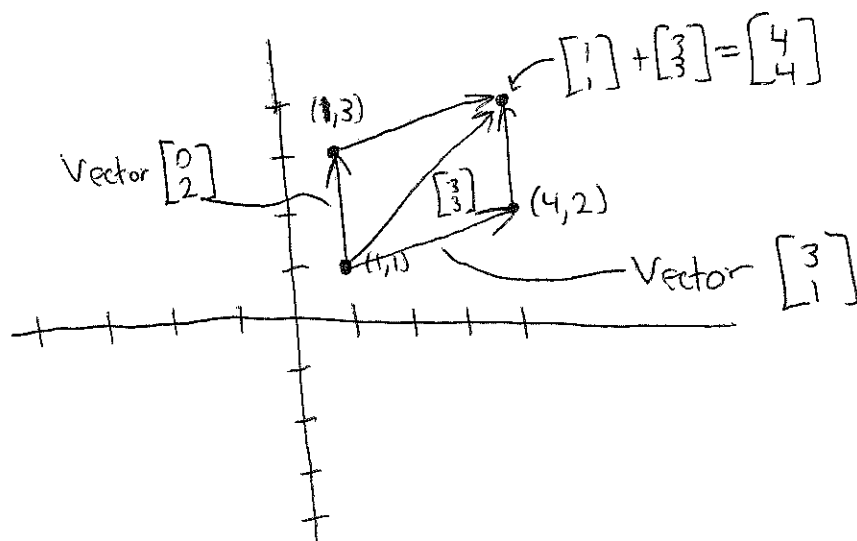
$$\begin{cases} c - d = -6 \end{cases}$$

$$3 - 9 = -6 \checkmark$$

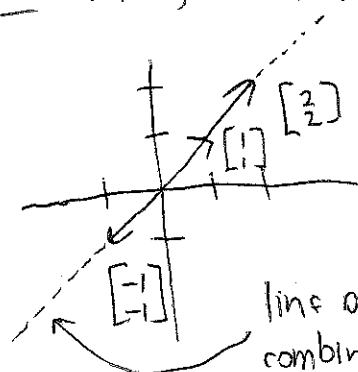
$$\rightarrow \text{so } \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + 9 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$\begin{bmatrix} 3 \\ 3 \\ 6 \end{bmatrix}$ is not a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ because its components don't add up to 0: $3+3+6=12 \neq 0$. (2)

1.1.9



1.1.25 First, can take $\vec{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ in 2-dim. space:



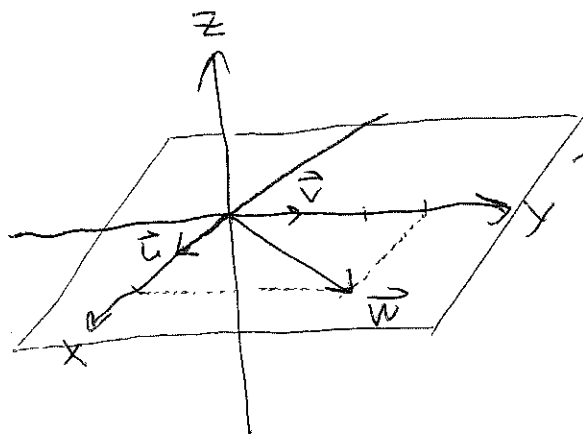
$$c\vec{u} + d\vec{v} + e\vec{w} = c\begin{bmatrix} 1 \\ 1 \end{bmatrix} + d\begin{bmatrix} 2 \\ 2 \end{bmatrix} + e\begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} c+2d-e \\ c+2d-e \end{bmatrix}$$

x and y coordinates always have to be equal

Second: can take $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and \vec{w} = some linear combination of \vec{u} and \vec{v} , for example $\vec{w} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = 2\vec{u} + 3\vec{v}$. Then all linear combinations

look like $c \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + e \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} c+2e \\ d+3e \\ 0 \end{bmatrix}$ \swarrow z-coordinate is always 0, so linear combinations fill up the xy-plane (3)



plane of linear combinations
= xy-plane (all vectors
with z-coordinate = 0)

1.1.26 $c \begin{bmatrix} 1 \\ 2 \end{bmatrix} + d \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 14 \\ 8 \end{bmatrix}$

\parallel
 $\begin{bmatrix} c+3d \\ 2c+d \end{bmatrix}$

$\begin{cases} c+3d=14 \\ 2c+d=8 \end{cases}$

Solve: Eqn 2 - 2 Eqn 1:

$-5d = -20,$

$d = 4$

so $c = 14 - 3(4) = 2$

$\begin{bmatrix} 14 \\ 8 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + 4 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

1.1.29 We need $c \begin{bmatrix} 1 \\ 3 \end{bmatrix} + d \begin{bmatrix} 2 \\ 7 \end{bmatrix} + e \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, or

$\begin{cases} c+2d+e=0 \\ 3c+7d+5e=1 \end{cases} \xrightarrow{\text{Eqn 2} - 3 \text{Eqn 1}} \begin{cases} c+2d+e=0 \\ d+2e=1 \end{cases}$

$\xrightarrow{\text{Eqn 1} - 2 \text{Eqn 2}} \begin{cases} c-3e=-2 \\ d+2e=1 \end{cases} \rightarrow \begin{cases} c=3e-2 \\ d=1-2e \\ e=\text{anything} \end{cases}$

For example, we get one solution if we pick $e=1$: $c=1, d=-1, e=1$

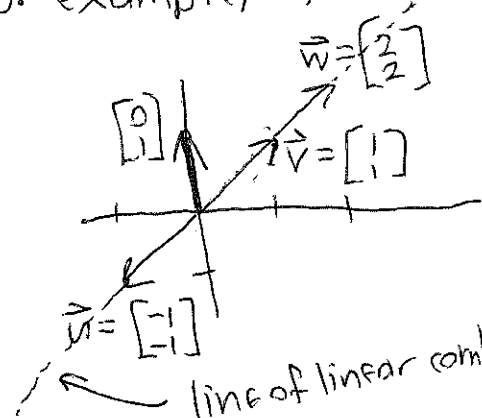
so $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + (-1) \begin{bmatrix} 2 \\ 7 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

We get another solution if we pick $e=-1$: $c=-5, d=3, e=-1$

so $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = (-5) \begin{bmatrix} 1 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 7 \end{bmatrix} + (-1) \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

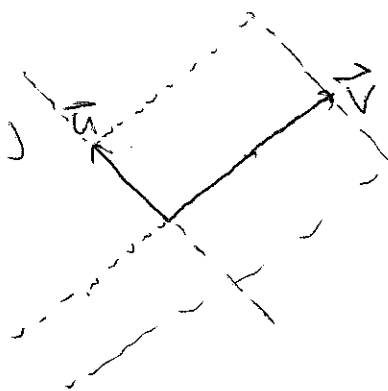
If $\vec{u}, \vec{v}, \vec{w}$ are only 3 vectors in 2-dim. space, we might not be able ⁽⁴⁾ to write $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{u}, \vec{v}, \vec{w}$ in two different ways.

For example, $\vec{u}, \vec{v}, \vec{w}$ might all be on a single line that doesn't include $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.



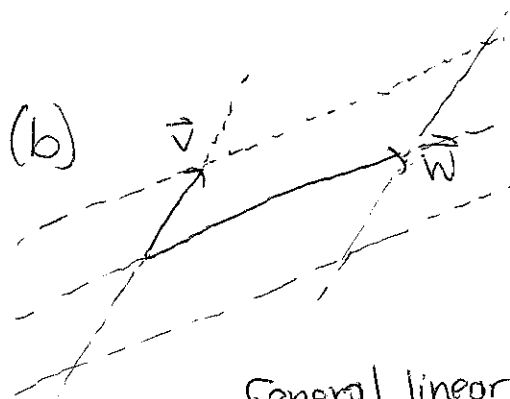
← No way to write $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as a linear combination of $\vec{u}, \vec{v}, \vec{w}$.

Graded problem (a)



\vec{u} and \vec{v} are not multiples of each other (they are "independent") so because there are two vectors, their linear combinations fill up a plane.

General linear combination: $c\vec{u} + d\vec{v} = c\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2c \\ 2d \\ 2d \end{bmatrix}$



\vec{v} and \vec{w} are not multiples of each other, so their linear combinations also fill up a plane.

General linear combination: $e\vec{v} + f\vec{w} = e\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix} + f\begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2f \\ 2e+2f \\ 2e+3f \end{bmatrix}$

(e) Question: What vectors can we write as both $\begin{bmatrix} 2c \\ 2d \\ 2d \end{bmatrix}$ and $\begin{bmatrix} 2f \\ 2e+2f \\ 2e+3f \end{bmatrix}$?

Need $\begin{cases} 2c = 2f \\ 2d = 2e + 2f \\ 2d = 2e + 3f \end{cases} \rightarrow \begin{cases} c = f \\ d = e + f \\ d = e + \frac{3}{2}f \end{cases} \rightarrow f = d - e = \frac{3}{2}f$

$c = 0$ also

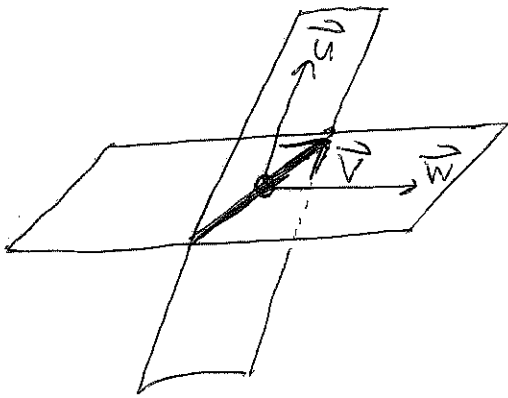
only works if $f = 0$

Now $d = e + f \xrightarrow{f=0} d = e$. So these vectors look like $\begin{bmatrix} 2(0) \\ 2(d) \\ 2(d) \end{bmatrix}$, or $\begin{bmatrix} 2(0) \\ 2(d) + 2(0) \\ 2(d) + 3(0) \end{bmatrix}$ (same)

→ d $\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$ (all multiples of \vec{v})

(5)

So if a vector is a linear comb. of \vec{u}, \vec{v} and a linear comb. of \vec{v}, \vec{w} , then it is a multiple of \vec{v} . These multiples fill up a line:



Intersection of these two planes
is a line.