CALCULUS A(1) TSINGHUA UNIVERSITY BLANK MIDTERM EXAM (Fall 2023)

NAME :

STUDENT ID:

NOTE: There are 3 Parts to this BLANK MIDTERM EXAM (total of 7 pages). For Part 1 (multiple choice) be sure to indicate your answer clearly as no partial credit will be awarded. Each question has a unique right answer.

Part 3 consists of a bonus exercise, which is not compulsory, and can give you some extra points. It is advised to finish all the compulsory questions before attempting the bonus ones. If your total is > 100 points, your assigned grade will be 100.

In order to receive full credit for Parts 2 and 3, you must show work to explain your reasoning. If you require additional sheets for Parts 2 or 3, be sure to put your name and ID to each additional page that you turn in with this exam. Use of calculators will NOT be permitted. You have 90 minutes complete this test.

Part 1 (20pts)

1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$, we have $g(x) = f(x-4) - 2$. The graph of g is obtained from the graph of f by

   (A) shifting from 4 units on the left and 2 units down.
   (B) shifting from 4 units on the right and 2 units down.
   (C) shifting from 4 units on the left and 2 units up.
   (D) shifting from 4 units on the right and 2 units up.
   (E) None of the above

2. What is the value of the following limit:

$$\lim_{x \to -\infty} \sqrt{x^4 + 3} - x^2 \text{ ?}$$

   (A) 3
   (B) $\sqrt{3}$
   (C) -3
   (D) 0
   (E) None of the above.

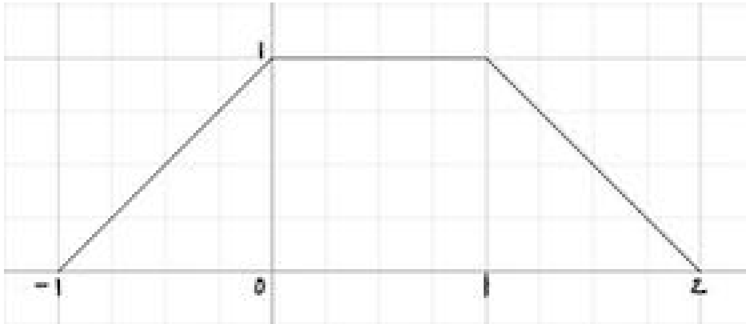
3. Which of the assertions below is equivalent to "$\lim_{x \to x_0} f(x) \neq +\infty$" (for a function $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$)?

   (A) $\exists M > 0$ such that $\forall \delta > 0$ there exists $x \in \mathbb{R}$ such that $0 <\mid x - x_0 \mid< \delta$ and $f(x) \leq M$
   (B) $\forall M > 0$ $\exists \delta > 0$ such that if $0 <\mid x - x_0 \mid< \delta$ then we have $f(x) \leq M$.
   (C) $\exists M > 0$ such that $\forall \delta > 0$, if $0 <\mid x - x_0 \mid< \delta$ then we have $f(x) \leq M$.
   (D) $\forall M > 0$ $\forall \delta > 0$, if $0 <\mid x - x_0 \mid< \delta$ then we have $f(x) \leq M$.
   (E) None of the above

4. What is the (smallest) period of the function $f(x) = (\sin(3x))^2$?

   (A) $\frac{2\pi}{3}$
   (B) $\frac{\pi}{2}$
   (C) $\frac{\pi}{3}$
   (D) 2π
   (E) None of the above.

Part 2a. (20 pts) Consider the function f whose graph is represented below.



Draw the graph of each function (you do not need to justify):

1. $y = -f(-x)$.

2. $y = f(2x) - 1$.

3. $y = -2f(x + 1)$.

Part 2b. (10pts) Show that the equation $x^2+y^2+2y-6x+1=0$ is the equation of a circle, and determine its center and radius.

Part 2c. (10pts) Find the tangent line to the graph of $f(x) = \tan(x)$ at $x = \frac{\pi}{4}$. (You need to justify your answer).

Part 2d. (20pts) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 \cos(\frac{1}{x})$ for $x \neq 0$ and $f(0) = 0$. Prove that f is differentiable at $x = 0$, and compute $f'(0)$.

Part 2e. (20pts) Let $f : [0,1] \to \mathbb{R}$ be a **continuous** function such that for all $x \in [0,1]$, we have $0 \leq f(x) \leq 1$. Show that the graph of f intersects the diagonal $y = x$, i.e. you need to prove that there exists $c \in [0,1]$ such that $f(c) = c$.

Part 3 (BONUS QUESTION). (10pts)
Find the real numbers a and b such that the function

$$f : x \mapsto \begin{cases} \frac{1}{2}x & \text{if } x \in [0,1] \\ ax^2 + bx + 1 & \text{if } x \in (1, +\infty) \end{cases}$$

is differentiable at $x = 1$.