Calculus A(1): Homework 4

The total is 100 points. When we refer to a paragraph number (e.g. §3.1), we refer to the PDF of the textbook *Thomas Calculus* that you can find on the weblearn. The bonus exercises are optional and more difficult. We may (or may not) decide to grade one of your bonus exercises and use it to replace one assigned exercise (if it improves your total grade).

Routine exercises (do not hand-in)

1. §3.1, Exercises 6, 12, 13, 24, 32, 38, 41, 44
2. §3.2, Exercises 41, 46, 49, 52
3. §3.4, Exercises 8, 10, 13, 44, 49, 54
4. §3.5, Exercises 6, 20, 35, 37, 52, 58, 60, 88, 114

Assigned exercises (hand-in)

1. (20pts) Find an equation for the tangent to the curve at the given point.

$$\textbf{a.}\ y = \frac{x}{x^2 - x + 1} \quad \text{at } (2, \frac{2}{3}) \qquad\qquad \textbf{b.}\ y = \frac{\tan x - x}{\cos x} \quad \text{at } (\pi, \pi)$$

2. (20pts) Find the derivatives of the following functions.

$$\textbf{a.}\ f(x) = \sin(x^2 - 2x + 3) \qquad\qquad \textbf{b.}\ f(x) = \frac{\tan x}{x}$$

$$\textbf{c.}\ f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}} \qquad\qquad \textbf{d.}\ f(x) = (\cos(x^2 - 1))^{\frac{1}{3}}$$

3. (20pts) Find an equation for the line tangent to the parametric curve at the point defined by the given value of t.

$$\textbf{a.}\ x = \cos^4 t, \quad y = \sin^4 t, \quad t = \frac{\pi}{4} \qquad\qquad \textbf{b.}\ x = \frac{t}{1+t}, \quad y = \frac{1-t}{1+t}, \quad t = \frac{1}{2}$$

4. (20pts) Suppose that $f(x)$ is differentiable on $\mathbb{R}$, prove that:

   (1) If $f(x)$ is an even function, then $f'(x)$ is an odd function, in particular $f'(0) = 0$.

   (2) If $f(x)$ is a periodic function, then so is $f'(x)$.

5. (20pts) Let

$$f(x) = \begin{cases} 0, & x = 0 \\ x^\alpha \sin \frac{1}{x}, & x \neq 0 \end{cases}$$

   Prove that $f(x)$ is differentiable at $x = 0$ when $\alpha > 1$, and it is not differentiable at $x = 0$ when $\alpha \leq 1$.

Bonus exercises (optional)

1. (Hard) Let $n \in \mathbb{N}$. Let f and g be two functions which are differentiable n times on an interval I. Prove the **Leibniz Formula**:

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(k)} g^{(n-k)}$$

   where $\binom{n}{k}$ is the binomial coefficient defined by $\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-m+1)}{k!}$

2. Let $f(x) = x^2 \sin x$ and $n \in \mathbb{N}$. Find the nth derivative of f. (You can use the Leibniz Formula without proof.)