

1. (15 points) Consider the system of linear equations

$$x_1 + 2x_2 + 3x_3 = b_1$$

$$2x_1 + 3x_2 + 4x_3 = b_2$$

$$3x_1 + 4x_2 + 5x_3 = b_3$$

(a) Find a linear relation involving b_1 , b_2 , and b_3 that guarantees the system has at least one solution.

(b) For $(b_1, b_2, b_3) = (1, 1, 1)$, find *all* solutions of the system of equations.

(a) Elimination:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 2 & 3 & 4 & b_2 \\ 3 & 4 & 5 & b_3 \end{array} \right] \xrightarrow[\text{Row 3} - 3\text{Row 1}]{\text{Row 2} - 2\text{Row 1}} \left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & -1 & -2 & -2b_1 + b_2 \\ 0 & -2 & -4 & -3b_1 + b_3 \end{array} \right] \xrightarrow[\text{Row 2} \rightarrow -\text{Row 2}]{\text{Row 3} - 2\text{Row 2}}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & b_1 \\ 0 & 1 & 2 & 2b_1 - b_2 \\ 0 & 0 & 0 & b_1 - 2b_2 + b_3 \end{array} \right] \xrightarrow{\text{Row 1} - 2\text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & -1 & -3b_1 + 2b_2 \\ 0 & 1 & 2 & 2b_1 - b_2 \\ 0 & 0 & 0 & b_1 - 2b_2 + b_3 \end{array} \right]$$

For solutions to exist, we need $b_1 - 2b_2 + b_3 = 0$

(b) For $(b_1, b_2, b_3) = (1, 1, 1)$, $1 - 2(1) + (1) = 0$, so solutions exist. Need to solve:

$$\begin{aligned} x_1 - x_3 &= -3b_1 + 2b_2 = -1 \\ x_2 + 2x_3 &= 2b_1 - b_2 = 1 \end{aligned} \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + x_3 \\ 1 - 2x_3 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

particular solution \rightarrow null space vectors

2. (12 points) Find the determinants of A and B :

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{bmatrix}, \quad B = \begin{bmatrix} a & 2 & b \\ 2 & 0 & 2 \\ b & 2 & a \end{bmatrix}.$$

What condition on a, b guarantees that B is *not* invertible?

For A , use row operations:

$$\begin{array}{c|c|c} \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \\ 1 & 4 & 16 & 64 \end{array} & \begin{array}{l} \text{Row 2-Row 1} \\ \text{Row 3-Row 1} \\ \text{Row 4-Row 1} \\ \text{(no change in det)} \end{array} & \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 2 & 8 & 26 \\ 0 & 3 & 15 & 63 \end{array} \end{array} \begin{array}{l} \text{Row 3-2Row 2} \\ \text{Row 4-3Row 2} \\ \text{(no change in det)} \end{array}$$

$$\begin{array}{c|c|c} \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 6 & 42 \end{array} & \begin{array}{l} \text{Row 4-3Row 3} \\ \text{(no change in det)} \end{array} & \begin{array}{cccc} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 2 & 12 \\ 0 & 0 & 0 & 6 \end{array} \end{array} = (1)(1)(2)(6) = 12$$

For B , expand across 2nd row:

$$\begin{vmatrix} a & 2 & b \\ 2 & 0 & 2 \\ b & 2 & a \end{vmatrix} = -2 \begin{vmatrix} 2 & b \\ 2 & a \end{vmatrix} - 2 \begin{vmatrix} a & 2 \\ b & 2 \end{vmatrix} = -2(2a-2b) - 2(2a-2b) \\ = -4(a-b) - 4(a-b) = -8(a-b)$$

B is not invertible when $a = b$ (then $\det B = 0$)

3. (16 points) Consider the matrix

$$A = \begin{bmatrix} 1 & -2 & -1 & 2 \\ 7 & -5 & 2 & 5 \\ -1 & -7 & -8 & 7 \end{bmatrix}$$

(a) Find the reduced row echelon form R of A .

(b) Find bases for the null space, row space, column space, and left null space of A .

$$(a) \begin{bmatrix} 1 & -2 & -1 & 2 \\ 7 & -5 & 2 & 5 \\ -1 & -7 & -8 & 7 \end{bmatrix} \xrightarrow[\text{Row 3 + Row 1}]{\text{Row 2 - 7 Row 1}} \begin{bmatrix} 1 & -2 & -1 & 2 \\ 0 & 9 & 9 & -9 \\ 0 & -9 & -9 & 9 \end{bmatrix} \xrightarrow[\frac{1}{9}\text{Row 2}]{\text{Row 3 + Row 2}}$$

$$\begin{bmatrix} 1 & -2 & -1 & 2 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1 + 2Row 2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$
 pivot columns free variables

$$(b) N(A) = \text{all solutions to } \begin{aligned} x_1 + x_3 &= 0 \\ x_2 + x_3 - x_4 &= 0 \end{aligned} \longrightarrow$$

$$N(A) = \text{all } \begin{bmatrix} -x_3 \\ -x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$\uparrow \quad \uparrow$
 special solutions, basis vectors for $N(A)$

Basis for row space, $C(A^T)$: non-zero rows in R :

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

Basis for $C(A)$: pivot columns in A: $\begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -5 \\ -7 \end{bmatrix}$

Left nullspace $N(A^T) = C(A)^T$ = all solutions to

$$\begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0, \begin{bmatrix} -2 \\ -5 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \longrightarrow$$

$$\begin{aligned} x_1 + 7x_2 + x_3 &= 0 \\ -2x_1 - 5x_2 - 7x_3 &= 0 \end{aligned} \longrightarrow \begin{bmatrix} 1 & 7 & -1 \\ -2 & -5 & -7 \end{bmatrix} \xrightarrow{\text{Row 2} + 2\text{Row 1}} \begin{bmatrix} 1 & 7 & -1 \\ 0 & 9 & -9 \end{bmatrix}$$

$\begin{matrix} 1 & -1 \end{matrix}$

$$\xrightarrow{\text{Row 1} - 7\text{Row 2}} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix} \longrightarrow \begin{aligned} x_1 &= -6x_3 \\ x_2 &= x_3 \\ x_3 &\text{ free} \end{aligned} \longrightarrow \vec{x} = x_3 \begin{bmatrix} -6 \\ 1 \\ 1 \end{bmatrix}$$

Basis for $N(A^T)$

Non-zero rows give another basis

$$\text{for } C(A) = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

4. (a) (7 points) If w_1, w_2 and w_3 are linearly independent vectors, show that the sums

$$v_1 = w_2 + w_3, \quad v_2 = w_1 + w_3, \quad v_3 = w_1 + w_2$$

are also independent. (Hint: Write $c_1 v_1 + c_2 v_2 + c_3 v_3 = \mathbf{0}$ in terms of the w 's. Find and solve equations for the c 's, to show they are zero.)

- (b) (5 points) If w_1, w_2 and w_3 are linearly independent vectors, show that the differences

$$v_1 = w_2 - w_3, \quad v_2 = w_1 - w_3, \quad v_3 = w_1 - w_2$$

are dependent. (Find a linear combination of the v 's that gives zero.)

(a) Need to show that the only solution to $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$

is $c_1 = c_2 = c_3 = 0$

$$c_1(\vec{w}_2 + \vec{w}_3) + c_2(\vec{w}_1 + \vec{w}_3) + c_3(\vec{w}_1 + \vec{w}_2) = \vec{0}$$

↓

$$(c_2 + c_3)\vec{w}_1 + (c_1 + c_3)\vec{w}_2 + (c_1 + c_2)\vec{w}_3 = \vec{0}$$

Because \vec{w} 's are independent, this only happens if

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 + c_3 = 0 \\ c_1 + c_2 = 0 \end{cases} \longrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 1} \leftrightarrow} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \xrightarrow{\text{Row 3} - \text{Row 1}}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 3} -} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\text{elimination}]{\text{more}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow$$

only solution is $c_1 = 0, c_2 = 0, c_3 = 0$, so \vec{v} 's are independent.

(b) When is $c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$?

$$c_1(\vec{w}_2 - \vec{w}_3) + c_2(\vec{w}_1 - \vec{w}_3) + c_3(\vec{w}_1 - \vec{w}_2) = \vec{0}$$

$$(c_2 + c_3)\vec{w}_1 + (c_1 - c_3)\vec{w}_2 + (-c_1 - c_2)\vec{w}_3 = \vec{0}$$

Since \vec{w} 's are independent, this only happens if

$$\begin{cases} c_2 + c_3 = 0 \\ c_1 - c_3 = 0 \\ -c_1 - c_2 = 0 \end{cases} \rightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 1} \leftrightarrow} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix} \xrightarrow[\text{Row 4}]{\text{Row 3} +} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow{\text{Row 2} + \text{Row 3}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} c_1 = c_3 \\ c_2 = -c_3 \\ c_3 \text{ free} \end{matrix} \rightarrow \vec{c} = c_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

non-zero solutions, so
 \vec{v} 's are dependent.

Take $c_3 = 1$: Then $\vec{v}_1 - \vec{v}_2 + \vec{v}_3 = \vec{0}$, so we can write:

$$\vec{v}_1 = \vec{v}_2 - \vec{v}_3,$$

$$\vec{v}_2 = \vec{v}_1 + \vec{v}_3, \text{ and}$$

$$\vec{v}_3 = -\vec{v}_1 + \vec{v}_2$$

5. (a) (12 points) Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

- (b) (3 points) Use A^{-1} to solve the linear system of equations $Ax = (3, 6, 2)$.

(a) Use elimination:

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array} \right] \xrightarrow[\text{Row 3} - \frac{1}{2}\text{Row 1}]{\text{Row 2} - \frac{1}{2}\text{Row 1}} \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 1/2 & 1/2 & -1/2 & 0 & 1 \end{array} \right] \xrightarrow{\text{Row 2} - \frac{1}{3}\text{Row 2}}$$

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & 3/2 & 1/2 & -1/2 & 1 & 0 \\ 0 & 0 & 1/3 & -1/3 & -1/3 & 1 \end{array} \right] \xrightarrow[\frac{2}{3}\text{Row 2}]{\frac{1}{2}\text{Row 1}} \left[\begin{array}{ccc|ccc} 1 & 1/2 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/3 & -1/3 & 2/3 & 0 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array} \right] \xrightarrow[\text{Row 2} - \frac{1}{3}\text{Row 3}]{\text{Row 1} - \frac{1}{2}\text{Row 3}}$$

$$\left[\begin{array}{ccc|ccc} 1 & 1/2 & 0 & 1 & 1/2 & -3/2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array} \right] \xrightarrow{\text{Row 1} - \frac{1}{2}\text{Row 2}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 3 \end{array} \right]$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$(b) A\vec{x} = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} \rightarrow \vec{x} = A^{-1} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} +1 \\ 4 \\ -3 \end{bmatrix}$$

6. (a) (6 points) Find all eigenvalues of the symmetric matrix

$$A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}.$$

(b) (10 points) Find an *orthonormal* basis for \mathbf{R}^3 consisting of eigenvectors for A .

(a) Eigenvalues = solutions to $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ -1 & 2-\lambda & -1 \\ -1 & -1 & 2-\lambda \end{vmatrix} = (2-\lambda) \begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ -1 & 2-\lambda \end{vmatrix} + (-1) \begin{vmatrix} -1 & 2-\lambda \\ -1 & -1 \end{vmatrix}$$

$$= (2-\lambda) \left((2-\lambda)(2-\lambda) - 1 \right) + (-2 + \lambda - 1) - (1 + 2 - \lambda)$$

$$\lambda^2 - 4\lambda + 3 = (\lambda - 1)(\lambda - 3) \quad \lambda - 3$$

Factor out $\lambda - 3$:

$$= (\lambda - 3) \left((2-\lambda)(\lambda - 1) + 1 + 1 \right) = (\lambda - 3)(-\lambda^2 + 3\lambda) = -\lambda(\lambda - 3) = 0$$

$$\boxed{\lambda = 0, 3, 3}$$

(b) Eigenvectors for $\lambda = 0$: Solve $A\vec{x} = 0\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \xrightarrow[\text{Row 3} + \frac{1}{2}\text{Row 1}]{\text{Row 2} + \frac{1}{2}\text{Row 1}} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & -3/2 & 3/2 \end{bmatrix} \xrightarrow{\text{Row 3} + \text{Row 2}} \begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{2}{3}\text{Row 2}]{\frac{1}{2}\text{Row 1}} \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[\frac{1}{2}\text{Row 2}]{\text{Row 1} +} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{matrix} x_1 = x_3 \\ x_2 = x_3 \end{matrix} \rightarrow \vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Eigenvectors for $\lambda = 3$: Solve $A\vec{x} = 3\vec{x} \rightarrow (A - 3I)\vec{x} = 0$

$$\begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow x_1 + x_2 + x_3 = 0 \rightarrow \vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

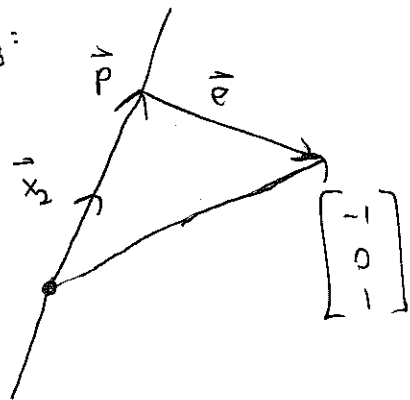
$$\rightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Here is a basis of eigenvectors = $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

But it's not orthonormal. Because A is symmetric, $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is already orthogonal to other two, but we need to normalize:

$$\vec{x}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \text{Then take } \vec{x}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

For \vec{x}_3 :



$$\vec{p} = P \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \frac{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}{\begin{bmatrix} -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{e} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$$

$$\text{Then } \vec{x}_3 = \frac{1}{\|\vec{e}\|} \vec{e} = \frac{1}{\sqrt{\frac{1}{4} + \frac{1}{4} + 1}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{2}{\sqrt{6}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Orthonormal basis of eigenvectors:

$$\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

7. (a) (4 points) What is the area of the parallelogram in \mathbf{R}^2 spanned by $\mathbf{x} = \begin{bmatrix} a \\ c \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} b \\ d \end{bmatrix}$?

(b) (8 points) Verify that for any 2×2 reflection matrix $Q = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$, the area of the parallelogram spanned by $Q\mathbf{x}$ and $Q\mathbf{y}$ is the same as your answer for part (a).

$$(a) \text{ Area} = \left| \det \begin{bmatrix} \vec{x} & \vec{y} \end{bmatrix} \right| = \left| \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right| = |ad - bc|$$

$$(b) \text{ Area} = \left| \det \begin{bmatrix} Q \begin{bmatrix} a \\ c \end{bmatrix} & Q \begin{bmatrix} b \\ d \end{bmatrix} \end{bmatrix} \right|$$

$$\begin{bmatrix} a \cos \theta + c \sin \theta \\ a \sin \theta - c \cos \theta \end{bmatrix} \quad \begin{bmatrix} b \cos \theta + d \sin \theta \\ b \sin \theta - d \cos \theta \end{bmatrix}$$

$$= \left| (a \cos \theta + c \sin \theta)(b \sin \theta - d \cos \theta) - (b \cos \theta + d \sin \theta)(a \sin \theta - c \cos \theta) \right|$$

$$= \left| \cancel{ab \cos \theta \sin \theta} + bc \sin^2 \theta - ad \cos^2 \theta - \cancel{cd \sin \theta \cos \theta} - \cancel{ba \cos \theta \sin \theta} - \cancel{da \sin^2 \theta} + bc \cos^2 \theta + \cancel{dc \sin \theta \cos \theta} \right|$$

$$= \left| bc(\sin^2 \theta + \cos^2 \theta) - ad(\cos^2 \theta + \sin^2 \theta) \right|$$

$$= |bc - ad| = |ad - bc|$$

8. (a) (10 points) Find the best least squares line $C + Dt$ to fit the data points $(-2, 4)$, $(-1, 2)$, $(0, -1)$, $(1, 0)$, and $(2, 0)$.
 (b) (2 points) Sketch a graph of the data points and your least squares line.
 (c) (3 points) Find the least squares error $\|e\|$ of the best fit line.

(a) Try to solve:

$$C + (-2)D = 4$$

$$C + (-1)D = 2$$

$$C + (0)D = -1$$

$$C + (1)D = 0$$

$$C + (2)D = 0$$

$$\begin{matrix} \rightarrow & \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \\ & A \quad \hat{x} \quad \vec{b} \end{matrix}$$

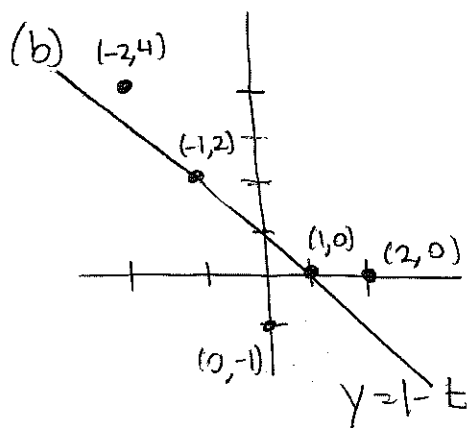
No solution!

Solve "normal equations" instead: multiply both sides by A^T :

$$A^T A \hat{x} = A^T \vec{b}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix} \rightarrow C = 1, D = -1 \rightarrow \text{Best fit line is } \boxed{y = 1 - t}$$



(c) Error $\|\vec{e}\| = \|A\hat{x} - \vec{b}\| =$

$$\left\| \begin{bmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ -1 \end{bmatrix} \right\|$$

$$= \sqrt{1 + 2^2 + 1} = \sqrt{6}$$

9. Consider 2×2 matrices

$$A = \begin{bmatrix} \frac{5}{2} & -\frac{1}{2} \\ 6 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}.$$

- (a) (8 points) Find an invertible matrix X and a diagonal matrix Λ such that $A = X\Lambda X^{-1}$.
 (b) (6 points) Use part (a) to calculate the matrix A^N for any positive integer N . What limit matrix does A^N approach as $N \rightarrow \infty$?
 (c) (6 points) Show that B is *not* diagonalizable.

(a) Λ comes from eigenvalues, X from eigenvectors.

Eigenvalues: Solve $\det(A - \lambda I) = 0$: $\begin{vmatrix} \frac{5}{2} - \lambda & -\frac{1}{2} \\ 6 & -1 - \lambda \end{vmatrix} = (\frac{5}{2} - \lambda)(-1 - \lambda) + 3$
 $= \lambda^2 - \frac{3}{2}\lambda - \frac{5}{2} + 3 = \lambda^2 - \frac{3}{2}\lambda + \frac{1}{2} = (\lambda - 1)(\lambda - \frac{1}{2}) = 0 \rightarrow \boxed{\lambda = 1, \frac{1}{2}}$

Eigenvectors for $\lambda = 1$: Solve $A\vec{x} = \vec{x} \rightarrow (A - I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 3/2 & -1/2 \\ 6 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/3 \\ 1 & -1/3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x} = x_2 \begin{bmatrix} 1/3 \\ 1 \end{bmatrix}$$

$\lambda = \frac{1}{2}$: Solve $(A - \frac{1}{2}I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 2 & -1/2 \\ 6 & -3/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/4 \\ 1 & -1/4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1/4 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x} = x_2 \begin{bmatrix} 1/4 \\ 1 \end{bmatrix}$$

Can take $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$, $X = \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix}$, so

$$A = \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix}^{-1}$$

(b) $A^N = (X\Lambda X^{-1})^N = X\Lambda^N X^{-1}$

$$\begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1^N & 0 \\ 0 & (\frac{1}{2})^N \end{bmatrix} \begin{bmatrix} 1 & -1/4 \\ -1 & 1/3 \end{bmatrix}$$

$\begin{bmatrix} 12 & -3 \\ -12 & 4 \end{bmatrix}$
 $//$

$\frac{1}{\frac{1}{3} - \frac{1}{4}} \begin{bmatrix} 1 & -1/4 \\ -1 & 1/3 \end{bmatrix}$
 $= \frac{1}{1/12} = 12$

$$A^N = \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (\frac{1}{2})^N \end{bmatrix} \begin{bmatrix} 12 & -3 \\ -12 & 4 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 12 & -3 \\ -12/2^N & 4/2^N \end{bmatrix}$$

$$= \begin{bmatrix} 4 - 3(\frac{1}{2})^N & -1 + (\frac{1}{2})^N \\ 12 - 12(\frac{1}{2})^N & -3 + 4(\frac{1}{2})^N \end{bmatrix}$$

As $N \rightarrow \infty$, $(\frac{1}{2})^N \rightarrow 0$. So $\lim_{N \rightarrow \infty} A^N = \begin{bmatrix} 4 & -1 \\ 12 & -3 \end{bmatrix}$

(c) Eigenvalues for B: $\begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2 = 0$

$\rightarrow \lambda = 1, 1$

Eigenvectors: Solve $(B - I)\vec{x} = \vec{0}$

$$\begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{x} = x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

only one linearly independent eigenvector, not enough for a basis of \mathbb{R}^2 . So B isn't diagonalizable.

10. (a) (6 points) Find a basis for the subspace S of \mathbb{R}^4 spanned by all solutions of

$$x_1 + x_2 + x_3 - x_4 = 0.$$

(b) (5 points) Find a basis for the orthogonal complement S^\perp .

(c) (6 points) Find \mathbf{b}_1 in S and \mathbf{b}_2 in S^\perp so that $\mathbf{b}_1 + \mathbf{b}_2 = (1, 1, 1, 1)$.

(a) $x_1 + x_2 + x_3 - x_4 = 0$ means

$$x_1 = -x_2 - x_3 + x_4, \text{ or } \vec{x} = \begin{bmatrix} -x_2 - x_3 + x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Every vector in S is a linear combination of these three, and they are independent. So they are a basis of S .

(b) Since $\dim S = 3$, $\dim S^\perp = 4 - 3 = 1$, so any one non-zero vector in S^\perp is a basis. One vector in S^\perp is $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, since the definition of S is all \vec{x} such that $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \cdot \vec{x} = 0$.

So $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$ is a basis for S^\perp .

(c) \vec{b}_1 is projection of $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ onto S , and \vec{b}_2 is the projection onto S^\perp . Projection matrix P for S^\perp is easier:

$$P = \frac{\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix}}{\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$$

$$\text{Then } \vec{b}_2 = p \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 2 \\ 2 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$\text{Finally, } \vec{b}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$$

Another solution: write $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ as a linear combination of vectors in

a basis for \mathbb{R}^4 : $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} +1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right\}$:

$$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_1 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \rightarrow \begin{bmatrix} -1 & -1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix}$$

$\underbrace{\hspace{10em}}_{\vec{b}_1} \qquad \underbrace{\hspace{10em}}_{\vec{b}_2}$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \begin{array}{l} \text{Row 4} + \text{Row 1} \\ \leftarrow \\ + \text{Row 2} - \\ \text{Row 3} \end{array} \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 & 1 \end{bmatrix}$$

1 1/2

$$\begin{array}{l} \text{Row 1} - \text{Row 4} \\ \text{Row 2} - \text{Row 4} \\ \text{Row 3} + \text{Row 4} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 3/2 \\ 0 & 0 & 0 & 1 & 1/2 \end{bmatrix} \rightarrow \vec{b}_1 = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 3/2 \end{bmatrix}$$

$$\vec{b}_2 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ -1/2 \end{bmatrix}$$