Linear Algebra – Homework 13

27 Dec 2023
Due: 4 Jan 2024

Textbook Problems. These problems will not be graded, but you must submit solutions to receive full credit for the homework.

Problem 6.2.7. Write down all 2×2 matrices that have eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Problem 6.2.9. Suppose G_{k+2} is the average of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_{k+1} \end{aligned} \qquad \longrightarrow \qquad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = X\Lambda^n X^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$, show that $\lim_{k\to\infty} G_k = \frac{2}{3}$.

Problem 6.2.15. $A^k = X\Lambda^k X^{-1}$ approaches the 0 matrix as $k \to \infty$ if and only if every λ has absolute value less than ________. Which of these matrices has $A^k \to 0$?

$$A_1 = \begin{bmatrix} 0.6 & 0.9 \\ 0.4 & 0.1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} 0.6 & 0.9 \\ 0.1 & 0.6 \end{bmatrix}.$$

Problem 6.2.16. Find Λ and X to diagonalize A_1 in Problem 6.2.15. What is the limit of Λ^k as $k \to \infty$? What is the limit of $X\Lambda^k X^{-1}$? In the columns of this limiting matrix you see the ________.

Problem 6.2.30. The "Cayley-Hamilton Theorem" states that if $p(\lambda)$ is the characteristic polynomial of an $n \times n$ matrix A, then the $n \times n$ matrix $p(A)$ is the zero matrix.

(a) If $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, then the determinant of $A - \lambda I$ is $(\lambda - a)(\lambda - d)$. Check that $(A - aI)(A - dI) = \textit{zero matrix}$, as predicted by the Cayley-Hamilton Theorem.

(b) Test the Cayley-Hamilton Theorem on the Fibonacci matrix $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$. The theorem predicts that $A^2 - A - I = 0$, since the polynomial $\det(A - \lambda I)$ is $\lambda^2 - \lambda - 1$.

Problem 6.3.4. A door is opened between rooms that hold $v(0) = 30$ people and $w(0) = 10$ people. The movement between rooms is proportional to the difference $v - w$:

$$\frac{dv}{dt} = w - v \qquad \text{and} \qquad \frac{dw}{dt} = v - w.$$

Show that the total $v(t)+w(t)$ is constant (40 people). Find the matrix in $d\mathbf{u}/dt = A\mathbf{u}$ and its eigenvalues and eigenvectors. What are v and w at $t = 1$ and $t = \infty$?

Problem 6.3.21. Write $A = \begin{bmatrix} 1 & 4 \\ 0 & 0 \end{bmatrix}$ in the form $X\Lambda X^{-1}$. Find e^{At} from $Xe^{\Lambda t}X^{-1}$.

Problem 6.4.8. Find all orthogonal matrices that diagonalize $S = \begin{bmatrix} 9 & 12 \\ 12 & 16 \end{bmatrix}$.

Problem 6.4.21. Find the eigenvector matrices Q for S and X for B. Show that X is still invertible at $d = 1$, even though $\lambda = 1$ is repeated. Are those eigenvectors perpendicular?

$$S = \begin{bmatrix} 0 & d & 0 \\ d & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -d & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & d \end{bmatrix} \qquad \text{have} \qquad \lambda = 1, d, -d.$$

Graded Problem.

Find an orthonormal basis of $\mathbf{R}^3$ consisting of eigenvectors for the symmetric matrix:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}.$$

Then compute the matrix power A^N for any positive integer N.