

Homework 6 Solutions

①

$$\underline{2.7.5} : \vec{x}^T A \vec{y} = [0, 1] \underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}_{A \vec{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = [0, 1] \begin{bmatrix} 2 \\ 5 \end{bmatrix} = 5$$

(a)

$$A \vec{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$(b) \vec{x}^T A = [4 \ 5 \ 6]$$

$$(c) A \vec{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\underline{2.7.11} \quad A = \begin{pmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \end{pmatrix} \begin{array}{l} \text{Row 1} \rightarrow \text{Row 3} \\ \text{Row 3} \rightarrow \text{Row 2} \\ \text{Row 2} \rightarrow \text{Row 1} \end{array} \rightsquigarrow P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

Multiplying A on the right by P_2 will exchange the columns of A .

$$\begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} \begin{array}{l} \text{Row 2} \\ \text{Row 3} \end{array} \begin{array}{l} \text{Row 2} \\ \text{Row 3} \end{array} \begin{bmatrix} 0 & 0 & 6 \\ 0 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[\text{col 3}]{\text{col 1} \leftrightarrow} \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\text{So } \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P_1} \underbrace{\begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{P_2}$$

(switch Rows 2 and 3 of I)

(switch cols. 1 and 3 of I)

$$\underline{2.7.16} \quad (a) \quad (A^2 - B^2)^T = (AA)^T - (BB)^T = A^T A^T - B^T B^T \\ = A^2 - B^2 \rightsquigarrow \text{certainly symmetric}$$

$$(b) [(A+B)(A-B)]^T = (A-B)^T(A+B)^T = (A^T - B^T)(A^T + B^T) \quad (2)$$

$$= (A-B)(A+B)$$

Note: $(A+B)(A-B) = A^2 + BA - AB + B^2$
 $(A-B)(A+B) = A^2 - BA + AB + B^2$ } not the same unless $AB = BA$

→ not necessarily symmetric

(c) $(ABA)^T = A^T B^T A^T = ABA$ → certainly symmetric

(d) $(ABAB)^T = B^T A^T B^T A^T = BABA$ → not necessarily symmetric.

For example: $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (both symmetric)

~~$ABAB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$~~

$$ABAB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$BABA = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

2.7.22 $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{\text{Row 1} \leftrightarrow \text{Row 2}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 3 & 4 \end{bmatrix} \xrightarrow{\text{Row 3} - 2 \times \text{Row 1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$

This is PA, with $P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 2 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

$P \quad A \quad L \quad U$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} \xrightarrow[\text{Row 3 - Row 1}]{\text{Row 2 - 2 Row 1}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix} \leftarrow \text{Need to switch Rows 2 and 3}$$

(3)

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 2 & 4 & 1 \end{bmatrix} \xrightarrow[\text{Row 3 - 2 Row 1}]{\text{Row 2 - 1 Row 1}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = U$$

$P \quad A$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$L \quad U$

2.7.39 $Q^T Q = I \rightsquigarrow$

$$\begin{bmatrix} \vec{q}_1^T \\ \vec{q}_2^T \\ \vdots \\ \vec{q}_n^T \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \dots & \vec{q}_n \end{bmatrix} = \begin{bmatrix} \vec{q}_1^T \vec{q}_1 & \vec{q}_1^T \vec{q}_2 & \dots & \vec{q}_1^T \vec{q}_n \\ \vec{q}_2^T \vec{q}_1 & \vec{q}_2^T \vec{q}_2 & \dots & \vec{q}_2^T \vec{q}_n \\ \vdots & \vdots & \ddots & \vdots \\ \vec{q}_n^T \vec{q}_1 & \vec{q}_n^T \vec{q}_2 & \dots & \vec{q}_n^T \vec{q}_n \end{bmatrix}$$

$Q^T \quad Q = I$

(a) I has 1's on diagonal: $\vec{q}_1^T \vec{q}_1 = 1, \vec{q}_2^T \vec{q}_2 = 1, \dots, \vec{q}_n^T \vec{q}_n = 1$
 $\rightsquigarrow \|\vec{q}_i\|^2 = 1$ for $i=1, 2, \dots, n$

(b) I has 0's off diagonal: $\vec{q}_i^T \vec{q}_j = 0$ if $i \neq j$.

(c) $Q = \begin{bmatrix} \cos \theta & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ $\cos^2 \theta + q_{21}^2 = 1 \rightsquigarrow q_{21} = \pm \sqrt{1 - \cos^2 \theta} = \pm \sin \theta$

~~Then~~ Then $q_{12} \cos \theta \pm q_{22} \sin \theta = 0$

can take $q_{12} = \sin \theta, q_{22} = \mp \cos \theta$

One example: $Q = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$ Another: $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$

3.1.4 zero vector: $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(4)

$\frac{1}{2}A = \frac{1}{2} \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}, -A = \begin{bmatrix} -2 & 2 \\ -2 & 2 \end{bmatrix}$

smallest subspace = $\text{span} \left(\begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} \right)$ = set of all $c \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix} = \begin{bmatrix} 2c & -2c \\ 2c & -2c \end{bmatrix}$

3.1.10 (a) Subspace: contains $\vec{0} = (0, 0, 0)$ (since $0=0$)

closed under addition: $(b_1, b_1, b_3) + (c_1, c_1, c_3) = (b_1 + c_1, b_1 + c_1, b_3 + c_3)$
 $\swarrow \quad \searrow$
 still equal

closed under scalar multiplication:

$c(b_1, b_1, b_3) = (cb_1, cb_1, cb_3)$
 $\swarrow \quad \searrow$
 still equal

(b) Not a subspace: $\vec{b} = (1, 0, 0)$ is in the set, but
 $2 \cdot (1, 0, 0) = (2, 0, 0)$ is not. (not closed under scalar multiplication, for example)

(c) Not a subspace: $(1, 0, 1)$ and $(0, 1, 0)$ are in the set, but
 $(1, 0, 1) + (0, 1, 0) = (1, 1, 1)$ is not (not closed under addition)

(d) Subspace: This is span of \vec{v}, \vec{w} , which is always a subspace.

(e) Subspace: We could check the 3 conditions, or we could

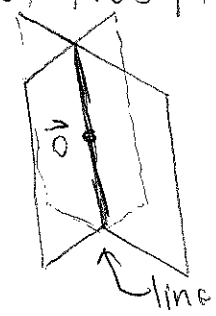
notice this set = all $\begin{bmatrix} b_1 \\ b_2 \\ -b_1 - b_2 \end{bmatrix} = \text{all } b_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + b_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} =$

$= \text{span} \left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right)$, a subspace.

(f) Not a subspace: $(1, 2, 3)$ is in the set, but $-1 \cdot (1, 2, 3) = (-1, -2, -3)$ is not (not closed under scalar multiplication)

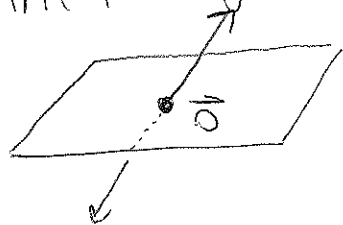
3.1.15 (a) Intersection of two planes through $(0,0,0)$ is

probably a line:

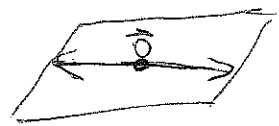


but, it could be a plane (if the two planes are really the same)

(b) The intersection of a plane and line through $(0,0,0)$ is probably a point (just $(0,0,0)$):



but it could be a line:



(c) (1) $S \cap T$ contains $\vec{0}$ since both S and T do.

(2) If \vec{x}, \vec{y} in $S \cap T$, what about $\vec{x} + \vec{y}$?

\vec{x}, \vec{y} both in $S \rightarrow \vec{x} + \vec{y}$ in S because S is a subspace.
 \vec{x}, \vec{y} both in $T \rightarrow \vec{x} + \vec{y}$ in T also

shows $\vec{x} + \vec{y}$ in $S \cap T$ ✓

(3) If \vec{x} in $S \cap T$, what about $c\vec{x}$?

\vec{x} in $S \rightarrow c\vec{x}$ is in S

\vec{x} in T also $\rightarrow c\vec{x}$ is in T also

$\rightarrow c\vec{x}$ is in $S \cap T$ ✓

3.1.20 (a)
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array} \right] \xrightarrow[\text{Row} + \text{Row } 1]{\text{Row } 2 - 2\text{Row } 1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_1 + b_3 \end{array} \right]$$

So $A\vec{x} = \vec{b}$ has a solution if ^{both} $b_2 = 2b_1$ and $b_1 + b_3 = 0$
 (two conditions are necessary)

(b)
$$\left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 2 & 8 & 4 & b_2 \\ -1 & -4 & -2 & b_3 \end{array} \right] \xrightarrow[\text{Row } 3 + \text{Row } 1]{\text{Row } 2 - 2\text{Row } 1} \left[\begin{array}{ccc|c} 1 & 4 & 2 & b_1 \\ 0 & 0 & 0 & b_2 - 2b_1 \\ 0 & 0 & 0 & b_3 + b_1 \end{array} \right]$$

Solution exists if $b_3 + b_1 = 0$.

3.1.25 $\vec{z} = \vec{x} + \vec{y}$ since $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y}$
 $= \vec{b} + \vec{b}^*$

(6)

Graded Problem 1: $S =$ all $n \times n$ A such that $A^T = A$

(1) Does S contain 0 matrix? $0^T = 0$ ✓

(2) Closed under addition: If A, B are symmetric, then
 $(A+B)^T = A^T + B^T = A+B$, so $A+B$ is symmetric also. ✓

(3) If A is in S , then $(cA)^T = cA^T = cA$, so cA is in S as well \rightarrow closed under scalar multiplication.

Graded Problem 2: (1) P_n contains the zero function:

$$0 = 0 + 0x + \dots + 0x^n \quad \checkmark$$

(2) Addition: $p(x) = a_0 + a_1x + \dots + a_nx^n$
 $+ q(x) = b_0 + b_1x + \dots + b_nx^n$

$$(p+q)(x) = (a_0+b_0) + (a_1+b_1)x + \dots + (a_n+b_n)x^n$$

\rightarrow still a polynomial with degree $\leq n$. ✓

(3) Scalar multiplication: $c p(x) = c(a_0 + a_1x + \dots + a_nx^n)$
 $= (ca_0) + (ca_1)x + \dots + (ca_n)x^n$

\rightarrow still a polynomial with degree $\leq n$ ✓