Chapter 3 Vector Spaces and Subspaces

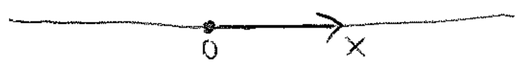
You can turn a set of numbers into a vector: $\{v_1, v_2, \dots, v_n\} \longrightarrow \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

Now, we want to turn a set of vectors into a vector space:

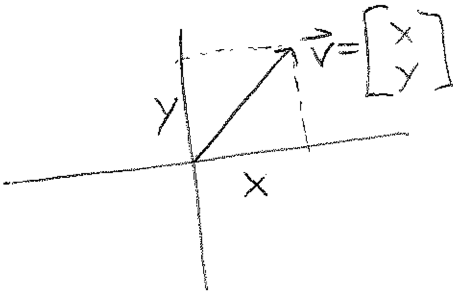
$\mathbb{R}^n$ = set of all vectors with n components, $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$.

That is, $\mathbb{R}^n$ = n-dimensional space.

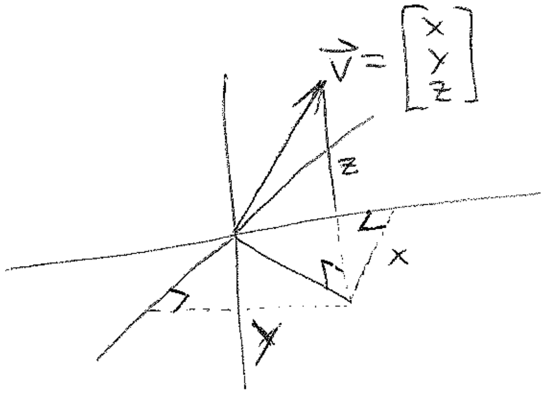
$\mathbb{R}^1$:



$\mathbb{R}^2$:

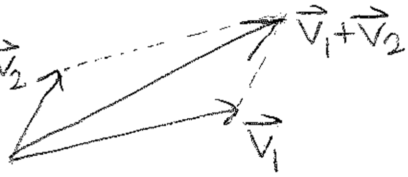


$\mathbb{R}^3$:



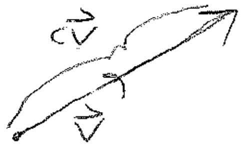
What can we do with vectors in $\mathbb{R}^n$?

We can add them:



$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$$

We can multiply by scalars:



$$c \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$$

Here are some properties of addition and scalar multiplication:

1. $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$
2. $\vec{v}_1 + (\vec{v}_2 + \vec{v}_3) = (\vec{v}_1 + \vec{v}_2) + \vec{v}_3$
3. There's a zero vector: $\vec{v} + \vec{0} = \vec{v}$
4. There are negative vectors: $\vec{v} + (-\vec{v}) = \vec{0}$
5. $1 \cdot \vec{v} = \vec{v}$
6. $(c_1 c_2) \cdot \vec{v} = c_1 \cdot (c_2 \cdot \vec{v})$

   real number mult. — Two scalar-vector multiplications — one scalar-vector multiplication
7. $c \cdot (\vec{v}_1 + \vec{v}_2) = c \cdot \vec{v}_1 + c \cdot \vec{v}_2$
8. $(c_1 + c_2) \cdot \vec{v} = c_1 \cdot \vec{v} + c_2 \cdot \vec{v}$

Now turn these properties of vectors in $\mathbb{R}^n$ into an abstract mathematical definition:

A vector space is a set of mathematical objects that we call "vectors." We call them vectors because they behave like vectors in $\mathbb{R}^n$:

① We have a way to "add" them: vector + vector = another vector

② We have a way to "multiply" them by scalars:
(real number) · vector = another vector

③ Addition of vectors and scalar multiplication obey the 8 rules I wrote down for vectors in $\mathbb{R}^n$.

Examples ① $\mathbb{R}^n$ is the basic example of a vector space.

② Matrices: you can add them and multiply by scalars:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}, \quad -2 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ -6 & -8 \end{bmatrix}$$

Have to be the same size to add.

If you fix m and n, then the set of all $m \times n$ matrices is a vector space (sometimes called $\mathbb{R}^{m \times n}$).

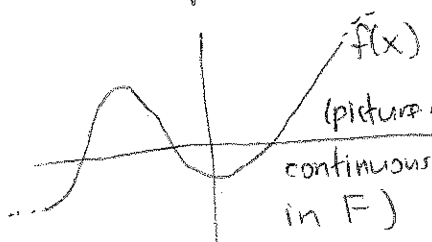
③ The zero vector space: Has one vector in it, $\vec{0}$

$\vec{0} + \vec{0} = \vec{0}$, $c \cdot \vec{0} = \vec{0}$ (c any real number), $-\vec{0} = \vec{0}$

You could call this vector space $\mathbb{R}^0$.

Here is maybe the most important example, besides $\mathbb{R}^n$:

④ Vector spaces of functions: F = set of all functions with domain $(-\infty, \infty)$



(picture of a continuous function in F)

We can add functions:
$(f+g)(x) = f(x) + g(x)$

We can multiply by scalars: $(c \cdot f)(x) = c\, f(x)$

Example (Problem 3.1.6) Linear combinations of $f(x) = x^2$, $g(x) = 5x$:

For example, $3 \cdot f(x) - 4 \cdot g(x) = 3x^2 - 4(5x) = 3x^2 - 20x$.

You can think of a function $f(x)$ as being like a vector with infinitely many components: one component, $f(x)$, for each real number x.

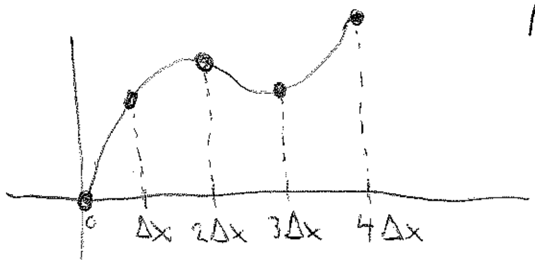
Thinking of functions as vectors in a vector space allows us to do calculus using linear algebra ideas!

In $\mathbb{R}^n$: $\vec{x} \xrightarrow{\text{matrix } A} A\vec{x}$ (another vector)

In F: $f(x) \xrightarrow[\text{derivative } d/dx]{} f'(x)$ (another function)

Calculus operations like derivatives behave like matrix-vector products ... but we won't do much of this now.

One example: "Discretizing" a function $f(x)$



Approximate $f(x)$ by its values at these points:

$$f(x) \approx \begin{bmatrix} f(0) \\ f(\Delta x) \\ f(2\Delta x) \\ f(3\Delta x) \\ f(4\Delta x) \end{bmatrix}$$

What about $f'(x)$?

$$f'(0) \approx \frac{f(\Delta x) - f(0)}{\Delta x}, \quad f'(\Delta x) \approx \frac{f(2\Delta x) - f(\Delta x)}{\Delta x}, \dots, f'(3\Delta x) \approx \frac{f(4\Delta x) - f(3\Delta x)}{\Delta x}$$

if Δx is small

(Let's not approximate $f'(4\Delta x)$ here)

$$\text{so} \begin{bmatrix} f'(0) \\ f'(\Delta x) \\ f'(2\Delta x) \\ f'(3\Delta x) \end{bmatrix} \approx \frac{1}{\Delta x} \begin{bmatrix} f(\Delta x) - f(0) \\ f(2\Delta x) - f(\Delta x) \\ f(3\Delta x) - f(2\Delta x) \\ f(4\Delta x) - f(3\Delta x) \end{bmatrix} = \frac{1}{\Delta x} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} f(0) \\ f(\Delta x) \\ f(2\Delta x) \\ f(3\Delta x) \\ f(4\Delta x) \end{bmatrix}$$

Textbook calls this a "backward difference" matrix

Approximate $f'(x)$ by a matrix-vector multiplication

For now, we mainly want to understand matrices and linear equations $A\vec{x} = \vec{b}$. So the vectors we care about live in $\mathbb{R}^n$. But they may live in special <u>subsets</u> of $\mathbb{R}^n$ that we call <u>subspaces</u>:

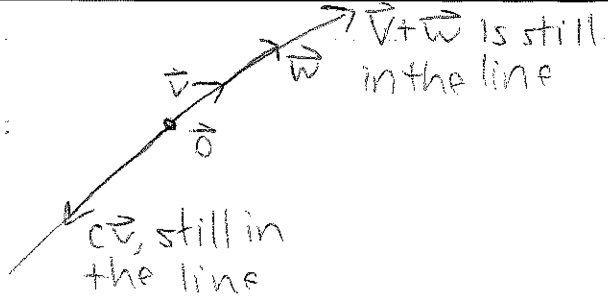
A <u>subspace</u> is a set of vectors in a vector space that satisfies:

1. The zero vector $\vec{0}$ is in the subspace.
2. If vectors $\vec{v}$ and $\vec{w}$ are in the subspace, so is $\vec{v} + \vec{w}$.
3. If $\vec{v}$ is in the subspace and c is a scalar, the $c \cdot \vec{v}$ is in the subspace.

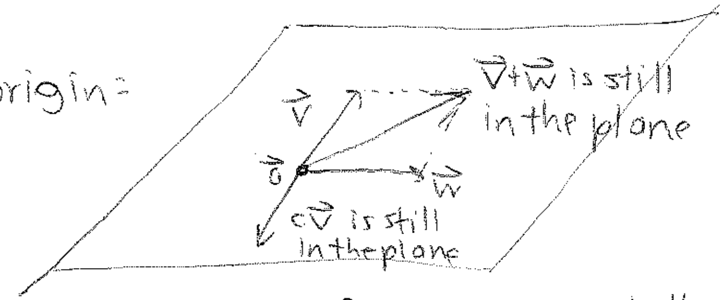
So a subspace is non-empty (it contains $\vec{0}$) and is "closed" under vector addition and scalar multiplication.

Subspaces in $\mathbb{R}^3$:

Lines through the origin:



Planes through the origin:

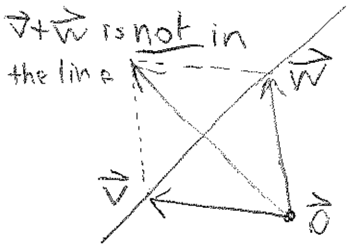


Two more subspaces that are kind of silly, but still work:

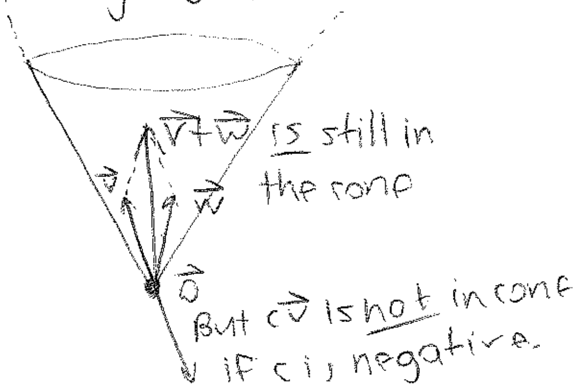
- All of $\mathbb{R}^3$ is a subspace of $\mathbb{R}^3$
- The zero vector $\vec{0}$ by itself is a subspace.

Non-subspaces of $\mathbb{R}^3$

- A line that doesn't go through $\vec{0}$:



- A plane that doesn't go through $\vec{0}$ is also not a subspace.
- A cone of vectors:



Important comment: Subspaces of $\mathbb{R}^n$ are themselves vector spaces. They are sets of vectors that you can add and multiply by scalars, and the result is always another vector in the subspace.

This is the main reason we care about vector spaces right now: we need to understand subspaces that are related to a matrix A.

Important rule for subspaces: If $\vec{v}$ and $\vec{w}$ are both in a subspace, then so are all linear combinations of $\vec{v}$ and $\vec{w}$.

$\hookleftarrow c\vec{v}+d\vec{w}$

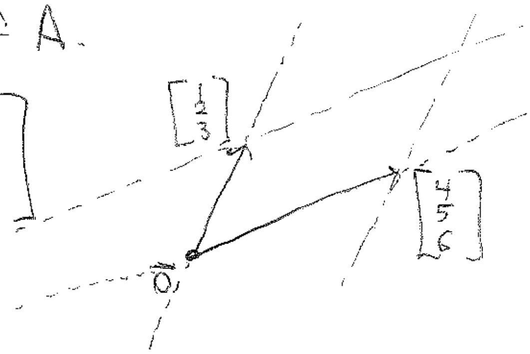
$$\begin{array}{ccccc} \vec{v} & \xrightarrow{} & c\vec{v} & & \\ \text{and} & \text{Rule } ③ & \text{and} & \text{Rule } ② & c\vec{v}+d\vec{w} \ \underline{\text{also}} \text{ in subspace.} \\ \vec{w} & \xrightarrow{} & d\vec{w} & & \\ \text{in subspace} & & \text{also in subspace} & & \end{array}$$

We can actually use linear combinations to <u>create</u> <u>subspaces</u>

Here's an important example: <u>Column space</u> of a matrix.

If A is a $m \times n$ matrix, its column space $C(A)$ = set of <u>all</u> linear combinations of the columns of A.

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}, \quad C(A) = \underline{\text{all}}\ c\begin{bmatrix}1\\2\\3\end{bmatrix} + d\begin{bmatrix}4\\5\\6\end{bmatrix}$$



$C(A)$ = <u>plane</u> sitting inside $\mathbb{R}^3$ ← # rows of A

↖ <u>2 independent</u> columns

Why is this a subspace?

① Contains $\vec{0}$: $\vec{0} = 0\begin{bmatrix}1\\2\\3\end{bmatrix} + 0\begin{bmatrix}4\\5\\6\end{bmatrix}$

② Closed under addition: $\left(c_1\begin{bmatrix}1\\2\\3\end{bmatrix} + d_1\begin{bmatrix}4\\5\\6\end{bmatrix}\right) + \left(c_2\begin{bmatrix}1\\2\\3\end{bmatrix} + d_2\begin{bmatrix}4\\5\\6\end{bmatrix}\right)$

$$= (c_1+c_2)\begin{bmatrix}1\\2\\3\end{bmatrix} + (d_1+d_2)\begin{bmatrix}4\\5\\6\end{bmatrix}$$ ← <u>still</u> a linear combination

③ Closed under scalar multiplication:

$$e\left(c\begin{bmatrix}1\\2\\3\end{bmatrix} + d\begin{bmatrix}4\\5\\6\end{bmatrix}\right) = (ec)\begin{bmatrix}1\\2\\3\end{bmatrix} + (ed)\begin{bmatrix}4\\5\\6\end{bmatrix}$$ ← <u>still</u> a linear combination.

Why is $C(A)$ important? It tells us when we can find solutions to linear equations $A\vec{x} = \vec{b}$

A: $m \times n$ matrix; $\vec{x}$: in $\mathbb{R}^n$; $\vec{b}$: in $\mathbb{R}^m$

Vector form of the equation:

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

columns of A

$$\downarrow$$

$$\underbrace{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n}_{\text{linear combination, in } C(A)} = \vec{b}$$

If there's a solution, the $\vec{b}$ has to be a vector in $C(A)$. If there's no solution, then $\vec{b}$ isn't in $C(A)$:

So $A\vec{x} = \vec{b}$ has a solution $\Leftrightarrow$ $\vec{b}$ is a vector in $C(A)$, the column space.

Example $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ again. $(-2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ← This is a vector in $C(A)$

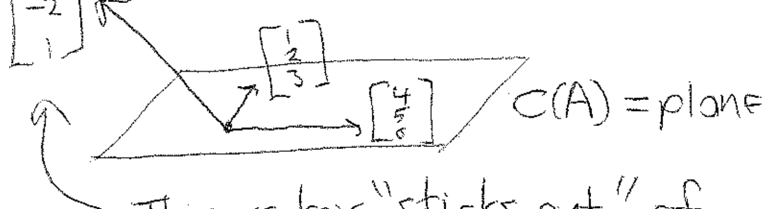
So the system $\begin{cases} x + 4y = 2 \\ 2x + 5y = 1 \\ 3x + 6y = 0 \end{cases}$ has a solution. It's $(x, y) = (-2, 1)$

What about $\begin{cases} x + 4y = -3 \\ 2x + 5y = -2 \\ 3x + 6y = 1 \end{cases}$??

$$\left[\begin{array}{cc|c} 1 & 4 & -3 \\ 2 & 5 & -2 \\ 3 & 6 & 1 \end{array}\right] \xrightarrow[\text{Row 3} - 3\,\text{Row 1}]{\text{Row 2} - 2\,\text{Row 1}}$$

$$\left[\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & -3 & 4 \\ 0 & -6 & 10 \end{array}\right] \xrightarrow{\text{Row 3} - 2\,\text{Row 2}} \left[\begin{array}{cc|c} 1 & 4 & -3 \\ 0 & -3 & 4 \\ 0 & 0 & 2 \end{array}\right] \rightsquigarrow \begin{cases} x + 4y = -3 \\ -3y = 4 \\ \underbrace{0y = 2}_{\text{inconsistent equation}} \end{cases}$$

No solution $\longrightarrow$ $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ is not a vector in $C(A)$



This vector "sticks out" of the plane somehow.