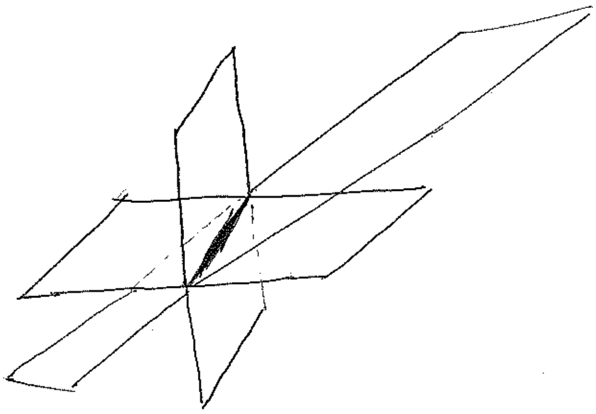
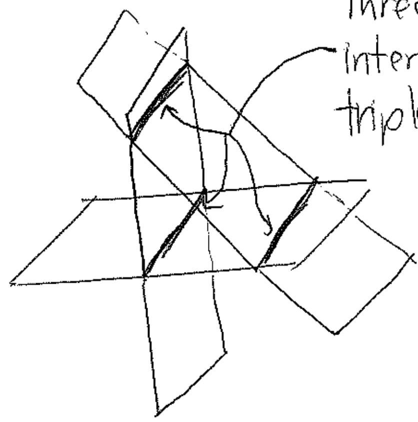
Important Warning: 3 planes in 3-dim. don't always intersect in a single point.

They might intersect in a line:

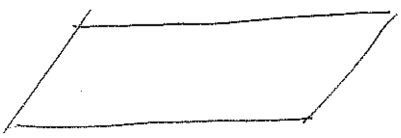


They might not have any common intersection:



Three pairwise intersections, but no triple intersection.

They might even intersect in a plane, if all three are really the same plane in disguise:



$$\begin{cases} x+2y+3z=4 \\ 2x+4y+6z=8 \\ -x-2y-3z=-4 \end{cases}$$

This means there are three possibilities when we solve a system of linear equations: No solution, exactly one solution, or infinitely many solutions (a whole line, plane, etc.)

(It will never have just 2 or 3 solutions!)

Last time

Pictures in 2 dimensions:

Row picture

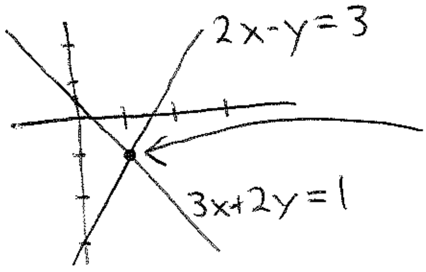
$$\begin{cases} 2x-y=3 \\ 3x+2y=1 \end{cases}$$

Matrix-vector equation

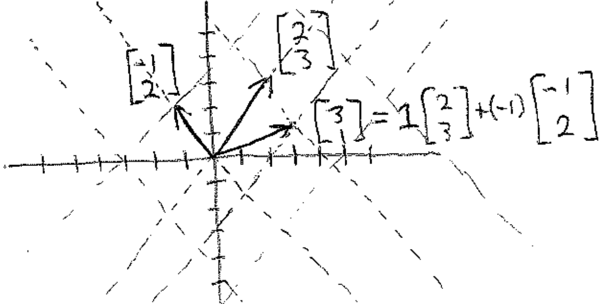
$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Column Picture

$$x\begin{bmatrix} 2 \\ 3 \end{bmatrix}+y\begin{bmatrix} -1 \\ 2 \end{bmatrix}=\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



one intersection, $(x,y)=(1,-1)$



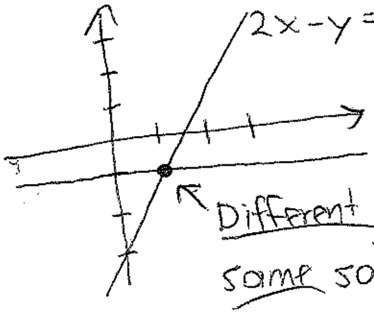
Solution using elimination:

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \xrightarrow[\text{Row 2} - \frac{3}{2}(\text{Row 1})]{\text{Row 2} \to} \underbrace{\begin{bmatrix} 2 & -1 \\ 0 & 7/2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -7/2 \end{bmatrix}}$$

First "pivot," the 1st non-zero entry in Row 1

a "multiplier"

New row picture:



Different equation, same solution

$\text{Row 2} \to \frac{2}{7}(\text{Row 2})$

$$\begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

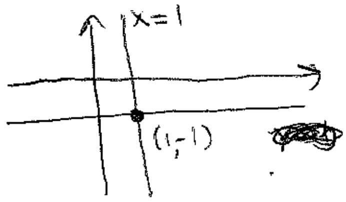
2nd "pivot," the 1st non-zero entry in the new Row 2

$\text{Row 1} \to \text{Row 1} + \text{Row 2}$

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \xrightarrow[\frac{1}{2}(\text{Row 1})]{\text{Row 1} \to} \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$$

we find the solution by doing a little arithmetic with the matrix-vector entries.

Final row picture:

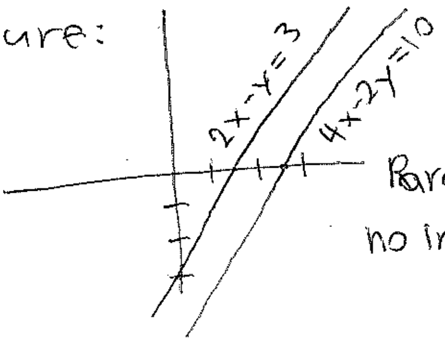


Row and column pictures with no solutions:

$$\begin{cases} 2x-y=3 \\ 4x-2y=10 \end{cases} \xrightarrow[\text{Eqn. 2} - 2(\text{Eqn 1})]{\text{Eqn. 2} \to} \begin{cases} 2x-y=3 \\ \underbrace{0x+0y=4} \end{cases}$$

$0=4$ ✗ no solution

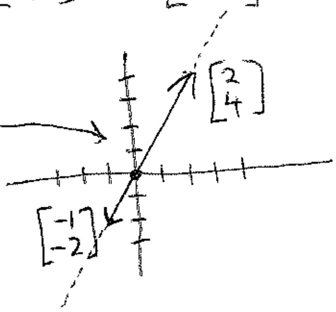
Row picture:



Parallel lines, no intersection

Column picture:

$$x\begin{bmatrix} 2 \\ 4 \end{bmatrix} + y\begin{bmatrix} -1 \\ -2 \end{bmatrix} \overset{??}{=} \begin{bmatrix} 3 \\ 10 \end{bmatrix}$$

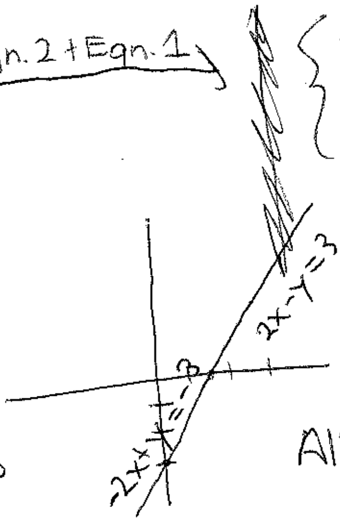


Parallel vectors, linear combinations only fill up a line, $\begin{bmatrix} 3 \\ 10 \end{bmatrix}$ is not on this line

Infinitely many solutions:

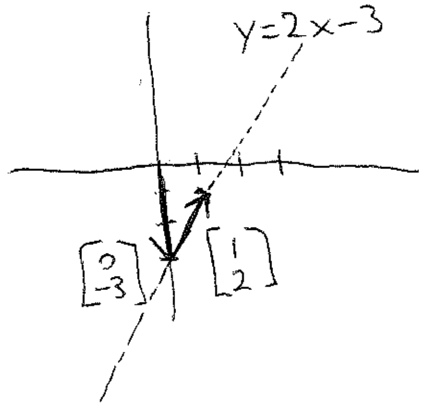
$$\begin{cases} 2x - y = 3 \\ -2x + y = -3 \end{cases} \xrightarrow{\text{Eqn. 2 + Eqn. 1}} \begin{cases} 2x - y = 3 \\ \underbrace{0x + 0y = 0}_{0=0,\ \text{okay}} \end{cases}$$

Same line: $y = 2x - 3$



→ All solutions are on the line $y = 2x - 3$. x can be anything, say $x = c$. Then $y = 2c - 3$

All solutions: $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ 2c-3 \end{bmatrix} = c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$



Direction of the line $y = 2x - 3$

$c\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \end{bmatrix}$

Point on the line

Now let's return to 3×3 systems in 3 dimensions

Row picture: Planes intersecting in space.

Column picture: Writing $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ as a linear combination of matrix columns.

We saw an example with a unique solution last time.

Today: $\begin{cases} x + y + z = b_1 \\ x + 2y + z = b_2 \\ 2x + y + 2z = b_3 \end{cases}$ Determine which values of b_1, b_2, b_3 give no solutions, and which give infinitely many.

1st "pivot": the first non-zero entry in 1st row

$$\begin{aligned} 1x + y + z &= b_1 \\ 1x + 2y + z &= b_2 \\ 2x + y + 2z &= b_3 \end{aligned}$$

Step 1: Use 1st "pivot" to eliminate x from 2nd and 3rd equations

2nd eqn → 2nd − 1st
3rd eqn → 3rd − 2(1st)

$$\begin{aligned} x + y + z &= b_1 \\ 1y &= b_2 - b_1 \\ -y &= b_3 - 2b_1 \end{aligned}$$

2nd "pivot"

use 2nd "pivot" to eliminate y from 3rd equation

3rd → 3rd + 2nd

$$\begin{aligned} x + y + z &= b_1 \\ y &= b_2 - b_1 \\ 0 &= -3b_1 + b_2 + b_3 \end{aligned}$$

From last equation: We get solutions only if $-3b_1 + b_2 + b_3 = 0$,
or $b_3 = 3b_1 - b_2$

If $b_3 = 3b_1 - b_2$, we get infinitely many solutions:

$$\begin{cases} x + y + z = b_1 \\ \quad\; y \quad\;\; = b_2 - b_1 \\ \quad 0 \quad\;\;\; = 0 \checkmark \end{cases} \xrightarrow[\text{1st eqn.} \to \text{1st} - \text{2nd}]{\text{Eliminate } y \text{ from 1st equation}} \begin{cases} x + z = 2b_1 - b_2 \\ \quad y = b_2 - b_1 \end{cases}$$

(no more simplification is possible)

We can choose z to be anything, let's say $z = c$

Then: $x = 2b_1 - b_2 - c$
$y = b_2 - b_1$
$z = c$

Infinitely many choices for c, infinitely many solutions

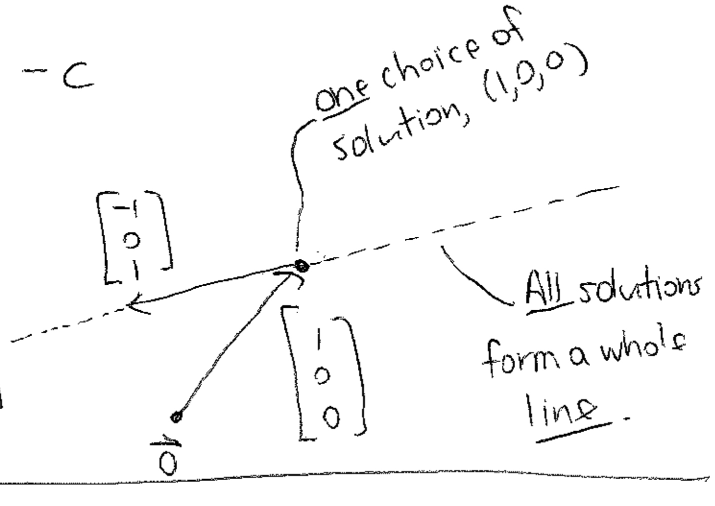
Specific examples: $b_1 = 1$, $b_2 = 1$, $b_3 = 1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{array}{c} \begin{matrix} b_1 \; & b_2 \; & b_3 \end{matrix} \\ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{array}$$ has no solutions because $3(1) - 1 \neq 1$

How about $b_1 = 1$, $b_2 = 1$, $b_3 = 2$? $\quad 3(1) - 1 = 2 \checkmark$

Solutions are: $x = 2b_1 - b_2 - c = 1 - c$
$y = b_2 - b_1 = 0$
$z = c$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 - c \\ 0 \\ c \end{bmatrix} \underset{\text{Rewrite}}{=} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$



Now let's do a 4×4 example:

$$\begin{cases} 2x_1 - x_2 = 2 \\ -x_1 + 2x_2 - x_3 - x_4 = -2 \\ -x_2 + 2x_3 = -2 \\ -x_2 + 2x_4 = 2 \end{cases}$$

Equations for 4 "hyperplanes" intersecting in 4-dimensional space

Systematic elimination: Eliminate lower left variables first, from left to right. Then eliminate upper right variables from right to left.

← 1st "pivot"

$$\begin{cases} 2x_1 - x_2 = 2 \\ -x_1 + 2x_2 - x_3 - x_4 = -2 \\ -x_2 + 2x_3 = -2 \\ -x_2 + 2x_4 = 2 \end{cases}$$

Eliminate first

Eqn. 2 → Eqn 2 + $\frac{1}{2}$(Eqn. 1)

$$\begin{cases} 2x_1 - x_2 = 2 \\ \frac{3}{2}x_2 - x_3 - x_4 = -1 \\ -x_2 + 2x_3 = -2 \\ -x_2 + 2x_4 = 2 \end{cases}$$

Eliminate next, use 2nd "pivot", 3/2

Row 3 → Row 3 + $\frac{2}{3}$(Row 2)

Row 4 → Row + $\frac{2}{3}$(Row 2)

$$\begin{aligned} 2x_1 - x_2 + 0x_3 + 0x_4 &= 2 \\ 0x_1 + \tfrac{3}{2}x_2 - x_3 - x_4 &= -1 \\ 0x_1 + 0x_2 + \tfrac{4}{3}x_3 - \tfrac{2}{3}x_4 &= -\tfrac{8}{3} \\ 0x_1 + 0x_2 - \tfrac{2}{3}x_3 + \tfrac{4}{3}x_4 &= \tfrac{4}{3} \end{aligned}$$

3rd "pivot"

Eliminate using Eqn. 3

Eqn. 4 → Eqn. 4 + $\frac{1}{2}$ Eqn. 3

$$\begin{aligned} 2x_1 - x_2 &= 2 \\ \tfrac{3}{2}x_2 - x_3 - x_4 &= -1 \\ \tfrac{4}{3}x_3 - \tfrac{2}{3}x_4 &= -\tfrac{8}{3} \\ x_4 &= 0 \end{aligned}$$

Coefficient matrix

Augmented matrix

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 2 \\ -1 & 2 & -1 & -1 & -2 \\ 0 & -1 & 2 & 0 & -2 \\ 0 & -1 & 0 & 2 & 2 \end{array}\right]$$

Row 2 → Row 2 + $\frac{1}{2}$ Row 1

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 2 \\ 0 & 3/2 & -1 & -1 & -1 \\ 0 & -1 & 2 & 0 & -2 \\ 0 & -1 & 0 & 2 & 2 \end{array}\right]$$

Row 3 → Row 3 + $\frac{2}{3}$(Row 2)

Row 4 → Row 4 + $\frac{2}{3}$(Row 2)

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 2 \\ 0 & 3/2 & -1 & -1 & -1 \\ 0 & 0 & 4/3 & -2/3 & -8/3 \\ 0 & 0 & -2/3 & 4/3 & 4/3 \end{array}\right]$$

Row 4 → Row 4 + $\frac{1}{2}$ (Row 3)

Eliminate using Row 4

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 2 \\ 0 & 3/2 & -1 & -1 & -1 \\ 0 & 0 & 4/3 & -2/3 & -8/3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Row 2 → Row 2 + Row 4

Row 3 → Row 3 + $\frac{3}{2}$ Row 4

Eliminate using Row 3

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 2 \\ 0 & 3/2 & -1 & 0 & -1 \\ 0 & 0 & 4/3 & 0 & -8/3 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Row 2 → Row 2 + $\frac{3}{4}$(Row 3)

(Also Row 3 → $\frac{1}{2}$ Row 3)

Eliminate

$$\left[\begin{array}{cccc|c} 2 & -1 & 0 & 0 & 2 \\ 0 & 3/2 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Row 1 → Row 1 + $\frac{2}{3}$(Row 2)

(Row 2 → $\frac{2}{3}$ Row 2)

Row 1 → $\frac{1}{2}$ Row 1

$$\left[\begin{array}{cccc|c} \cancel{2}\,1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Solution = final column of final augmented matrix:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -2 \\ 0 \end{bmatrix}$$

column picture: $\begin{bmatrix} 2 \\ -2 \\ -2 \\ 2 \end{bmatrix} = (-2)\begin{bmatrix} \text{2nd} \\ \text{col} \end{bmatrix} + (-2)\begin{bmatrix} \text{3rd} \\ \text{col} \end{bmatrix}$

~~right~~ last column of first augmented matrix

columns of coefficient matrix

$$= (-2)\begin{bmatrix} -1 \\ 2 \\ -1 \\ -1 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ -1 \\ 2 \\ 0 \end{bmatrix} \checkmark$$

So far: We have applied two operations for solving linear systems by elimination:

1. Add a multiple of one equation ~~to~~ another.
2. Multiply both sides of an equation by a non-zero scalar.

These change the equations, but not the solutions.

Why? Because these operations are reversible:

- If I add 2(Eqn. 1) to Eqn. 2, I can reverse this by subtracting 2(Eqn. 1) back off.
- If I multiply Eqn. 3 by c, I can reverse this by dividing Eqn. 3 by c (as long as $c \neq 0$!)

There's one more reversible operation we can perform, though it might seem a bit silly:

3. Switch the order of two equations.

Why would we want to switch the order of two equations?

Might need to if we want to follow our systematic elimination method (i.e., eliminate lower left variables first)

Example

$$\begin{cases} x_2 + x_3 = 4 \\ x_1 + x_2 + x_3 = 1 \\ x_1 + 2x_2 + 3x_3 = 2 \end{cases} \longleftrightarrow \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 1 & 4 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \end{array}\right]$$

First step: Eliminate these using Eqn. 1. But we can't!

Instead: Switch Rows 1 and 2 first.

$$\xrightarrow{\text{Row 1} \leftrightarrow \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 1 & 2 & 3 & 2 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 3} \to} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 1 & 2 & 1 \end{array}\right] \xrightarrow[\text{Row 3} - \text{Row 2}]{\text{Row 3} \to}$$

Now eliminate this

Eliminate this

Eliminate

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 4 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow[\text{Row 2} - \text{Row 3}]{\text{Row 1} - \text{Row 3}} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right] \xrightarrow{\text{Row 1} - \text{Row 2}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

Eliminate

or use back-substitution

Solution vector, $x_1 = -3,\ x_2 = 7,\ x_3 = -3$

$$x_1 + x_2 + x_3 = 1$$

$$x_2 + x_3 = 4 \longrightarrow x_2 = 4 - x_3 = 4 - (-3) = \boxed{7}$$

$$\boxed{x_3 = -3}$$

$$x_1 = 1 - x_2 - x_3 = 1 - 7 - (-3) = \boxed{-3}$$