Final Exam Solutions

Problem 1

$$\left[\begin{array}{cccc|c} 0 & 1 & 1 & 1 & b_1 \\ 1 & 1 & 2 & 1 & b_2 \\ 1 & 2 & 2 & 0 & b_3 \\ 2 & 0 & 3 & 2 & b_4 \end{array}\right] \xrightarrow[\text{Row 2}]{\text{Row 1} \leftrightarrow} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 1 & 2 & 2 & 0 & b_3 \\ 2 & 0 & 3 & 2 & b_4 \end{array}\right] \xrightarrow[\text{Row 4}-2\text{Row 1}]{\text{Row 3}-\text{Row 1}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 0 & 1 & 0 & -1 & b_3-b_2 \\ 0 & -2 & -1 & 0 & b_4-2b_2 \end{array}\right] \xrightarrow[\text{Row 4}+2\text{Row 2}]{\text{Row 3}-\text{Row 2}} \left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 0 & 0 & -1 & -2 & b_3-b_2-b_1 \\ 0 & 0 & 1 & 2 & 2b_1-2b_2+b_4 \end{array}\right] \xrightarrow{\text{Row 4}+\text{Row 3}}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & b_2 \\ 0 & 1 & 1 & 1 & b_1 \\ 0 & 0 & 1 & 2 & b_1+b_2-b_3 \\ 0 & 0 & 0 & 0 & b_1-3b_2+b_3+b_4 \end{array}\right]$$

(a) Solutions exist if $\boxed{b_1-3b_2+b_3+b_4=0}$

(b) For $(b_1,b_2,b_3,b_4)=(1,1,1,1)$:

$$\left[\begin{array}{cccc|c} 1 & 1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{Row 2}-\text{Row 3}]{\text{Row 1}-2\text{Row 3}} \left[\begin{array}{cccc|c} 1 & 1 & 0 & -3 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{\text{Row 1}-\text{Row 2}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\rightarrow \begin{array}{l} x_1 = 2x_4 - 1 \\ x_2 = x_4 \\ x_3 = -2x_4 + 1 \\ x_4 \text{ free} \end{array} \rightarrow \boxed{\vec{x} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}}$$

Problem 2 (a)

$$\det A = \begin{vmatrix} a & r & r & r \\ a & b & s & s \\ a & b & c & t \\ a & b & c & d \end{vmatrix} \xrightarrow[\text{Row }-\text{Row 1}]{\text{Row 2}-\text{Row 1},\ \text{Row 3}-\text{Row 1}} \begin{vmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & b-r & c-r & t-r \\ 0 & b-r & c-r & d-r \end{vmatrix}$$

$$\xrightarrow[\text{Row 4}-\text{Row 2}]{\text{Row 3}-\text{Row 2}} \begin{vmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & c-s & d-s \end{vmatrix} \xrightarrow[\text{Row 3}]{\text{Row 4}-} \begin{vmatrix} a & r & r & r \\ 0 & b-r & s-r & s-r \\ 0 & 0 & c-s & t-s \\ 0 & 0 & 0 & d-t \end{vmatrix} =$$

$= a(b-r)(c-s)(d-t)$

A is invertible if $a \neq 0$, $b \neq r$, $c \neq s$, $d \neq t$.

(b) A is invertible, so $A^{-1} = \frac{1}{\det A} C^T \longrightarrow$

$$\det(A^{-1}) = \det\left(\frac{1}{\det A} C^T\right) \rightarrow \frac{1}{\det A} = \left(\frac{1}{\det A}\right)^n \det(C^T)$$

$$\rightarrow (\det A)^{n-1} = \det(C^T) = \det C. \quad \text{So } \boxed{\det C = (\det A)^{n-1}}$$

Problem 3

(a)

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 6 \\ 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \end{bmatrix} \xrightarrow[\substack{\text{Row 3}-3\text{Row 1} \\ \text{Row 4}-4\text{Row 1}}]{\text{Row 2}-2\text{Row 1}} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -2 & -3 & -4 \\ 0 & -2 & -4 & -6 & -8 \\ 0 & -3 & -6 & -9 & -12 \end{bmatrix} \xrightarrow[\substack{\text{Row 4}-3\text{Row 2} \\ \text{Row 2} \to -\text{Row 2}}]{\text{Row 3}-2\text{Row 2}}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1}-2\text{Row 2}} \underbrace{\begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}$$

This is the reduced row-echelon form R.

(b) Null space: $x_1 - x_3 - 2x_4 - 3x_5 = 0$
$x_2 + 2x_3 + 3x_4 + 4x_5 = 0$

$$\rightarrow \vec{x} = \begin{bmatrix} x_3 + 2x_4 + 3x_5 \\ -2x_3 - 3x_4 - 4x_5 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis for N(A)

Row Space: Basis = non-zero rows of R:

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ -2 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} \right\}$$

Column space: Basis = independent columns of A:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \right\}$$

Left Null space $= C(A)^T =$ all $\vec{x}$ such that ③

$$\begin{bmatrix}1\\2\\3\\4\end{bmatrix}\cdot\vec{x} = \begin{bmatrix}2\\3\\4\\5\end{bmatrix}\cdot\vec{x} = 0 \rightsquigarrow \text{Null space of } \begin{bmatrix}1&2&3&4\\2&3&4&5\end{bmatrix}$$

$$\xrightarrow{\text{Row 2}-2\text{Row 1}} \begin{bmatrix}1&2&3&4\\0&-1&-2&-3\end{bmatrix} \xrightarrow[-\text{Row 2}]{\text{Row 1}+2\text{Row 2}} \begin{bmatrix}1&0&-1&-2\\0&1&2&3\end{bmatrix} \longrightarrow \begin{matrix}x_1 = x_3 + 2x_4\\ x_2 = -2x_3 - 3x_4\end{matrix}$$

$$\vec{x} = x_3\begin{bmatrix}1\\-2\\1\\0\end{bmatrix} + x_4\begin{bmatrix}2\\-3\\0\\1\end{bmatrix}$$

Basis for $N(A^T)$

Problem 4

$$\begin{bmatrix}1&1&1&1&1\\1&2&0&3&4\\1&3&-1&6&10\end{bmatrix} \xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}-\text{Row 1}} \begin{bmatrix}1&1&1&1&1\\0&1&-1&2&3\\0&2&-2&5&9\end{bmatrix} \xrightarrow{\text{Row 3}-2\text{Row 2}}$$

$$\begin{bmatrix}1&1&1&1&1\\0&1&-1&2&3\\0&0&0&1&3\end{bmatrix} \xrightarrow[\text{Row 2}-2\text{Row 3}]{\text{Row 1}-\text{Row 3}} \begin{bmatrix}1&1&1&0&-2\\0&1&-1&0&-3\\0&0&0&1&3\end{bmatrix} \xrightarrow{\text{Row 1}-\text{Row 2}} \begin{bmatrix}1&0&2&0&1\\0&1&-1&0&-3\\0&0&0&1&3\end{bmatrix}$$

$\vec{v}_1, \vec{v}_2, \vec{v}_4$ are independent : Basis $\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}, \begin{bmatrix}1\\3\\6\end{bmatrix}\right\}$

$$\vec{v}_3 = 2\vec{v}_1 - \vec{v}_2 = 2\begin{bmatrix}1\\1\\1\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix}$$

$$\vec{v}_5 = \vec{v}_1 - 3\vec{v}_2 + 3\vec{v}_4 = \begin{bmatrix}1\\1\\1\end{bmatrix} - 3\begin{bmatrix}1\\2\\3\end{bmatrix} + 3\begin{bmatrix}1\\3\\6\end{bmatrix}$$

~~Problem 5 (a)~~ $\cancel{\begin{bmatrix}2&0&0&-1&1&0&0&0\\0&2&0&-1&0&1&0&0\\0&0&2&-1&0&0&1&0\\-1&-1&-1&2&0&0&0&1\end{bmatrix} \xrightarrow{\text{Row 4}+\frac{1}{2}(\text{Row 1}+\text{Row 2}+\text{Row 3})}}$

$$\left[\begin{array}{cccc|cccc} 2 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{array}\right] \xrightarrow[\text{Row 3}+2\text{ Row 4}]{\text{Row 1}+2\text{ Row 4},\ \text{Row 2}+2\text{ Row 4}} \left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 2 & 1 & 1 & 2 \\ 0 & 2 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 2 & 0 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 1/2 & 1/2 & 1/2 & 1/2 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{1}{2}\text{Row 3},\ 2\text{ Row 4}]{\frac{1}{2}\text{Row 1},\ \frac{1}{2}\text{Row 2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 1/2 & 1/2 & 1 \\ 0 & 1 & 0 & 0 & 1/2 & 1 & 1/2 & 1 \\ 0 & 0 & 1 & 0 & 1/2 & 1/2 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \end{array}\right] \rightarrow \boxed{A^{-1} = \begin{bmatrix} 1 & 1/2 & 1/2 & 1 \\ 1/2 & 1 & 1/2 & 1 \\ 1/2 & 1/2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}}$$

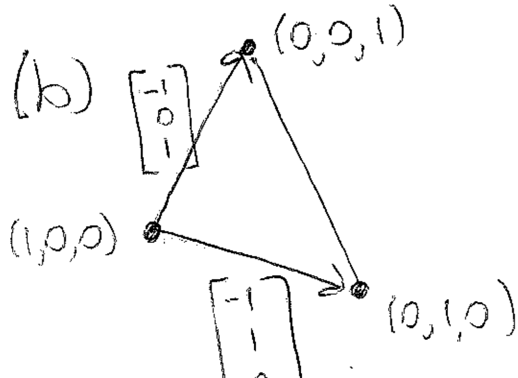
(b) $x = \begin{bmatrix} 1 & 1/2 & 1/2 & 1 \\ 1/2 & 1 & 1/2 & 1 \\ 1/2 & 1/2 & 1 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix}$

Problem 5 (a) The two boxes have volumes = absolute values of

$$\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{vmatrix} \quad \text{and} \quad \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 4 & 9 & 16 \end{vmatrix},$$

which must be the same because transpose matrices have equal determinants.

(b)



Area of triangle $= \frac{1}{2}$ Area of parallelogram

$$= \frac{1}{2} \left\| \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\|$$

$$= \frac{1}{2} \left\| \begin{vmatrix} i & j & k \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix} \right\|$$

$$= \frac{1}{2} \| -1i - 1j - 1k \| = \frac{1}{2}\sqrt{1^2+1^2+1^2} = \boxed{\frac{\sqrt{3}}{2}}$$

Problem ~~X~~ 6 (a) Try to solve.

$$\begin{aligned} C + D(0) &= 1 \\ C + D(1) &= 3 \\ C + D(2) &= 2 \\ C + D(3) &= 4 \\ C + D(4) &= 5 \end{aligned} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} C \\ D \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix}}_{\vec{b}}$$

No solution, instead solve normal equations:

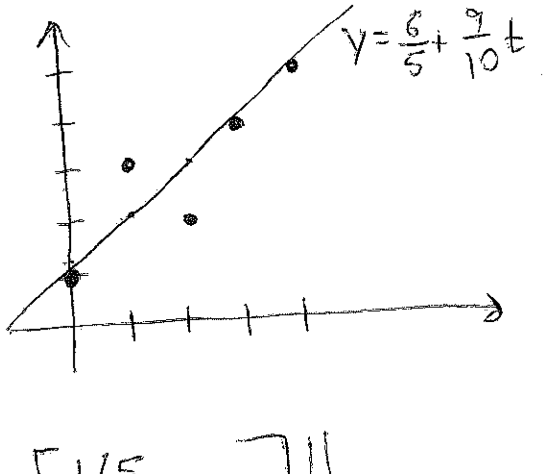
$$A^T A \hat{x} = A^T \vec{b} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 15 \\ 39 \end{bmatrix} \rightarrow \begin{bmatrix} C \\ D \end{bmatrix} = \frac{1}{5(30)-10^2} \begin{bmatrix} 30 & -10 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} 15 \\ 39 \end{bmatrix}$$

$$= \frac{1}{50} \begin{bmatrix} 30 & -10 \\ -10 & 5 \end{bmatrix} \begin{bmatrix} 15 \\ 39 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ +39 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 12 \\ 9 \end{bmatrix} = \begin{bmatrix} 6/5 \\ 9/10 \end{bmatrix}$$

Best fit line is $\frac{6}{5} + \frac{9}{10} t$

(b)



(c) $\text{Error} = \|A\hat{x} - \vec{b}\|$

$$= \left\| \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 6/5 \\ 9/10 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 5 \end{bmatrix} \right\|$$

$$= \left\| \begin{bmatrix} 1/5 \\ -9/10 \\ 1 \\ -1/10 \\ -1/5 \end{bmatrix} \right\| = \sqrt{\frac{4+81+100+1+4}{100}} = \sqrt{\frac{190}{100}} = \boxed{\sqrt{\frac{19}{10}}}$$

Problem #7 (a) $\begin{vmatrix} -\lambda & 1/2 & 1/2 \\ 1/2 & -\lambda & 1/2 \\ 1/2 & 1/2 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & 1/2 \\ 1/2 & -\lambda \end{vmatrix} - \frac{1}{2}\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -\lambda \end{vmatrix} + \frac{1}{2}\begin{vmatrix} 1/2 & -\lambda \\ 1/2 & 1/2 \end{vmatrix}$ 6

$$= -\lambda\left(\lambda^2 - \frac{1}{4}\right) - \frac{1}{2}\left(-\frac{1}{2}\lambda - \frac{1}{4}\right) + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\lambda\right)$$

$$= -\lambda\left(\lambda - \frac{1}{2}\right)\left(\lambda + \frac{1}{2}\right) + \frac{1}{4}\left(\lambda + \frac{1}{2}\right) + \frac{1}{4}\left(\lambda + \frac{1}{2}\right) = \left(\lambda + \frac{1}{2}\right)\left(-\lambda^2 + \frac{1}{2}\lambda + \frac{1}{2}\right)$$

$$= \left(\lambda + \frac{1}{2}\right)(1 - \lambda)\left(\lambda + \frac{1}{2}\right) \rightarrow \boxed{\lambda = 1, -\frac{1}{2}, -\frac{1}{2}}$$

$\underline{\lambda = 1}$: $\begin{bmatrix} -1 & 1/2 & 1/2 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & -3/4 & 3/4 \\ 0 & 3/4 & -3/4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1/2 & -1/2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$

$$\longrightarrow \vec{x} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \xrightarrow{\text{normalize}} \vec{q}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\underline{\lambda = -\frac{1}{2}}$: $\begin{bmatrix} +1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow x_1 = -x_2 - x_3$

$$\longrightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Project onto span $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and take difference:

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \underbrace{\frac{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}}{\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}}_{\text{Projection matrix}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Normalize:

$$\vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}(1)\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} \xrightarrow{\text{Normalize:}} \vec{q}_3 = \frac{1}{\sqrt{1 + \frac{1}{4} + \frac{1}{4}}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$$

Orthonormal Basis: $\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix} \right\}$

(b) $A = Q\Lambda Q^T = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$

(c) $\lim\limits_{N\to\infty} A^N = \lim\limits_{N\to\infty} Q\Lambda^N Q^T = Q \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} Q^T$

$= \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}$

<u>Problem #8</u> (a) $\begin{vmatrix} \frac{1}{2}-\lambda & -3/2 \\ 3/2 & \frac{7}{2}-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + \frac{7}{4} + \frac{9}{4} = \lambda^2 - 4\lambda + 4 = 0$

$\rightarrow (\lambda-2)^2 = 0 \rightarrow \lambda = 2, 2$

Eigenvectors: $\begin{bmatrix} -\frac{3}{2} & -\frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \rightarrow x_1 = -x_2 \rightarrow \vec{x} = x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Only one linearly independent eigenvector, <u>not</u> diagonalizable.

(b) $A^T A = \begin{bmatrix} 1/2 & 3/2 \\ -3/2 & 7/2 \end{bmatrix} \begin{bmatrix} 1/2 & -3/2 \\ 3/2 & 7/2 \end{bmatrix} = \begin{bmatrix} 5/2 & 9/2 \\ 9/2 & 29/2 \end{bmatrix}$

$\begin{vmatrix} 5/2-\lambda & 9/2 \\ 9/2 & 29/2-\lambda \end{vmatrix} = \lambda^2 - 17\lambda + \frac{145}{4} - \frac{81}{4} = \lambda^2 - 17\lambda + 16 = 0$

$\rightarrow (\lambda-1)(\lambda-16) = 0 \rightarrow \lambda = 1, 16 \rightarrow \boxed{\sigma = 1, 4}$

<u>$\lambda=1$</u>: $\begin{bmatrix} 3/2 & 9/2 \\ 9/2 & 27/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = -3x_2 \rightarrow \vec{x} = x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix} \rightarrow \vec{v}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

<u>$\lambda=16$</u>: $\begin{bmatrix} -27/2 & 9/2 \\ 9/2 & -3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = 3x_1 \rightarrow \vec{x} = x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow \vec{v}_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{u}_1 = \frac{1}{\sigma_1} A\vec{v}_1 = \frac{1}{4}\begin{bmatrix} 1/2 & -3/2 \\ 3/2 & 7/2 \end{bmatrix}\begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix} = \frac{1}{4\sqrt{10}}\begin{bmatrix} -8/2 \\ 24/2 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 3 \end{bmatrix}$$ (8)

$$\vec{u}_2 = \frac{1}{\sigma_2} A\vec{v}_2 = \frac{1}{1}\begin{bmatrix} 1/2 & -3/2 \\ 3/2 & 7/2 \end{bmatrix}\begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} = \frac{1}{2\sqrt{10}}\begin{bmatrix} -6 \\ -2 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$A = U\Sigma V^T = \cancel{[1/\sqrt{10} \;\; 3/\sqrt{10}]}$$

$$= \begin{bmatrix} -1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & -1/\sqrt{10} \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{10} & 3/\sqrt{10} \\ -3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$

(c) Maximum amount = largest singular ~~vector~~ value $\sigma_1 = 4$.

Vectors stretched by 4 are in $\text{span}(\vec{v}_1) = \text{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$.

So could take $\vec{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Check: $\left\|A\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\| = \left\|\begin{bmatrix} -4 \\ 12 \end{bmatrix}\right\| = 4\left\|\begin{bmatrix} -1 \\ 3 \end{bmatrix}\right\| = 4\left\|\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right\|$