Homework 8 Solutions

3.3.18 $\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \xrightarrow[\text{Row 3}-\text{Row 1}]{\text{Row 2}-\text{Row 1}} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & q-1 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 3}-} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & q-2 \end{bmatrix}$

Is this 0 or not?

$\operatorname{rank}(A) = \begin{cases} 3 & \text{if } q \neq 2 \\ 2 & \text{if } q = 2 \end{cases}$, and A^T has the <u>same</u> rank, since

$$\operatorname{rank}(A^T) = \dim C(A^T) = \dim \operatorname{Row}(A) = \dim C(A) = \operatorname{rank}(A).$$

3.3.24 (a) We need A (${m \times n}$) to have no free variables (independent columns) but also a row of 0's at the bottom of R, so $m > n$.

Here's a simple example: $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}$

(b) R should have <u>no</u> row of 0's (so solutions always exist) <u>and</u> free variables. So $A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$ would work.

(c) R should have row of 0's <u>and</u> free variables. Here's an example:

$A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ (since $R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$)

(d) A should be invertible, for example $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

3.4.2 Put into a matrix and eliminate to find relations between columns:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 1}+} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 3}+} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & -1 & 0 & -1 & -1 \end{bmatrix}$$

$$\xrightarrow[+\text{Row 3}]{\text{Row 4}} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{Row 2}-\text{Row 3}]{\text{Row 1}-\text{Row 3}} \begin{bmatrix} 1 & 1 & 0 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[-\text{Row 2}]{\text{Row 1}} \begin{bmatrix} 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form shows largest number of independent vectors is 3: $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are independent, but then $\vec{v}_4 = -\vec{v}_1 + \vec{v}_2$, $\vec{v}_5 = -\vec{v}_1 + \vec{v}_3$, and $\vec{v}_6 = -\vec{v}_2 + \vec{v}_3$.

Should be $\vec{w}_2 + \vec{w}_3$, not $\vec{w}_1 + \vec{w}_2$.

<u>3.4.8</u> $c_1(\vec{v}_1) + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0} \longrightarrow$

$c_1(\vec{w}_1 + \vec{w}_2) + c_2(\vec{w}_1 + \vec{w}_3) + c_3(\vec{w}_2 + \vec{w}_3) = \vec{0} \longrightarrow$

$(c_1 + c_2)\vec{w}_1 + (c_1 + c_3)\vec{w}_2 + (c_2 + c_3)\vec{w}_3 = \vec{0}$

Since $\vec{w}_1, \vec{w}_2, \vec{w}_3$ are independent, this only happens if

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 + c_3 = 0 \\ c_2 + c_3 = 0 \end{cases} \rightsquigarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve by elimination:

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Can already see matrix is invertible, so <u>only</u> solution is $c_1 = 0, c_2 = 0, c_3 = 0$

This means $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are <u>independent</u>.

<u>3.4.11</u> (a) $V = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right) \rightsquigarrow$ <u>line</u>

since vectors are multiples of each other

(b) $V = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \text{span}\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) \rightsquigarrow$ <u>plane</u>

independent   doesn't contribute

(c) $V = \text{span}\left(\text{all } \begin{bmatrix} k \\ m \\ n \end{bmatrix} \text{ with } k, m, n = 0, 1, 2, 3, \ldots\right\} \rightsquigarrow$ <u>all of $\mathbb{R}^3$</u>

Includes a <u>basis</u> of $\mathbb{R}^3$, such as

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(d) $V = \text{span}\left(\text{all } \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ such that } x, y, z > 0\right) \rightsquigarrow$ all of $\mathbb{R}^3$

these vectors contain many bases for $\mathbb{R}^3$, such as $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \right\}$, so V contains

$\text{span(basis)} = \mathbb{R}^3$

3.4.20 plane $x - 2y + 3z = 0 \rightsquigarrow$ all $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $x = 2y - 3z$

$$= \text{all } \begin{bmatrix} 2y - 3z \\ y \\ z \end{bmatrix} = \text{all } y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$$

These two vectors form a basis for the plane

Intersection with xy-plane: all $y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$ such that $z = 0$

$z = 0$

$= \text{all } y \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ← This one vector is a basis for the line of intersection

Vectors $\perp$ to plane: All $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ such that $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

$\rightsquigarrow$ solutions to $\begin{bmatrix} 2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1/2 & 0 \\ -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 3/2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 1 & 2/3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \end{bmatrix}$$

$x = \frac{1}{3} z$

$y = -\frac{2}{3} z$

Basis for vectors $\perp$ to plane (can also use $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$) $\quad z \begin{bmatrix} 1/3 \\ -2/3 \\ 1 \end{bmatrix}$

3.5.2 $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 5 & 8 \end{bmatrix} \xrightarrow[\text{2Row 1}]{\text{Row 2}-} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[\text{2Row 2}]{\text{Row 1}-} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}$

Column space $C(A)$: Cols 1 and 2 are independent: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\}$

Null space $N(A)$: $\begin{cases} x = -4z \\ y = 0 \end{cases} \to N(A) = \text{all } z \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow \text{Basis } \left\{ \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} \right\}$

Row space $C(AT)$: can use non-zero rows of R: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

Left null space $N(AT)$: $\dim N(AT) = \dim \mathbb{R}^2 - \text{rank}(\mathbb{B})$

$= 2 - 2 = 0$

So $N(AT) = \{\vec{0}\}$, basis = empty set

3.5.11 $A\vec{x} = \vec{b}$ has no solution for some $\vec{b}$ means reduced row echelon form of A has at least one row of 0's.

(a) Row of 0's $\to$ rank < # rows $\to$ ~~...~~ $r < m$

We also always have $r \leq n$.

We don't know what is the relation between m and n.

(b) According to the Fundamental Theorem of Linear Algebra,

$\dim N(AT) = m - r$, which is > 0.

So $N(AT)$ is not the 0 subspace: there is a non-zero $\vec{y}$ such that $AT\vec{y} = \vec{0}$.

3.5.18 $\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \xrightarrow[\text{Row 3}-7\text{Row 1}]{\text{Row 2}-4\text{Row 1}} \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & -6 & -12 & b_3 - 7b_1 \end{bmatrix} \xrightarrow{\text{Row 3}-2\text{Row 2}}$

$\begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$

Says that Row 3 − 2 Row 2 + Row 1 = 0

says that $N(AT)$ has all multiples of $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Null space of A: $\begin{bmatrix}1&2&3\\0&-3&-6\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&2&3\\0&1&2\\0&0&0\end{bmatrix} \to \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix}$

$\to \begin{array}{l} x = z \\ y = -2z \end{array} \to N(A) = \text{all} \begin{bmatrix}z\\-2z\\z\end{bmatrix} = z\begin{bmatrix}1\\-2\\1\end{bmatrix}$ (same null space as A^T, even though $A \neq A^T$!)

<u>3.5.24</u> $A^T\vec{y} = \vec{d}$ has solutions when $\vec{d}$ is in the column space of A^T, which is the row space of A. The solution is unique when the left null space $N(A^T)$ contains only the $\vec{0}$ vector.

<u>Graded Problem 1</u>: $\begin{bmatrix}2&3&-1&2\\4&6&2&2\\6&9&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}-1\\1\\-1\end{bmatrix}$ $\xrightarrow[\text{Row }3-3\text{Row }1]{\text{Row }2-2\text{Row }1}$

$\left[\begin{array}{cccc|c}2&3&-1&2&-1\\0&0&4&-2&3\\0&0&4&-4&2\end{array}\right] \xrightarrow{\text{Row }3-\text{Row }2} \left[\begin{array}{cccc|c}2&3&-1&2&-1\\0&0&4&-2&3\\0&0&0&-2&-1\end{array}\right] \to \left[\begin{array}{cccc|c}1&3/2&-1/2&1&-1/2\\0&0&1&-1/2&3/4\\0&0&0&1&1/2\end{array}\right]$

$\xrightarrow[\text{Row }2+\frac{1}{2}\text{Row }3]{\text{Row }1-\text{Row }3} \left[\begin{array}{cccc|c}1&3/2&-1/2&0&-1\\0&0&1&0&1\\0&0&0&1&1/2\end{array}\right] \xrightarrow{\text{Row }1+\frac{1}{2}\text{Row }2} \left[\begin{array}{cccc|c}1&3/2&0&0&-1/2\\0&0&1&0&1\\0&0&0&1&1/2\end{array}\right]$

$\to \begin{cases} x_1 + \frac{3}{2}x_2 = -\frac{1}{2} \\ x_3 = 1 \\ x_4 = 1/2 \end{cases} \to \vec{x} = \begin{bmatrix}-1/2-(3/2)x_2\\x_2\\1\\1/2\end{bmatrix} = \begin{bmatrix}-1/2\\0\\1\\1/2\end{bmatrix} + x_2\begin{bmatrix}-3/2\\1\\0\\0\end{bmatrix}$

<u>Graded Problem 2</u> (a) $\begin{bmatrix}2&-1&0&0\\-1&2&-1&-1\\0&-1&2&0\\0&-1&0&2\end{bmatrix} \xrightarrow{\text{Row }2+\frac{1}{2}\text{Row }1} \begin{bmatrix}2&-1&0&0\\0&3/2&-1&-1\\0&-1&2&0\\0&-1&0&2\end{bmatrix} \xrightarrow[\text{Row }4+\frac{2}{3}\text{Row }2]{\text{Row }3+\frac{2}{3}\text{Row }2}$

$\begin{bmatrix}2&-1&0&0\\0&3/2&-1&-1\\0&0&4/3&-2/3\\0&0&-2/3&4/3\end{bmatrix} \xrightarrow{\text{Row }4+\frac{1}{2}\text{Row }3} \begin{bmatrix}2&-1&0&0\\0&3/2&-1&-1\\0&0&4/3&-2/3\\0&0&0&1\end{bmatrix} \leftarrow$ We can already see that R will not have free variables, so the vectors are <u>independent</u>.

(b) $\begin{bmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{Row}4+\frac{1}{2}\text{Row}1]{\text{Row}2+\frac{1}{2}\text{Row}1} \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & -1 & 2 & -1 \\ 0 & -1/2 & -1 & 3/2 \end{bmatrix} \xrightarrow[\text{Row}4+\frac{1}{3}\text{Row}2]{\text{Row}3+\frac{2}{3}\text{Row}2}$

$\begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & 0 & 4/3 & -4/3 \\ 0 & 0 & -4/3 & 4/3 \end{bmatrix} \xrightarrow[\text{Then: } \frac{3}{4}\text{Row}3]{\text{Row}4+\text{Row}3} \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 3/2 & -1 & -1/2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\xrightarrow[\text{Then: } \frac{2}{3}\text{Row}2]{\text{Row}2+\text{Row}3} \begin{bmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow[\text{Then: } \frac{1}{2}\text{Row}1]{\text{Row}1+\text{Row}2} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Vectors are <u>dependent</u>: $\begin{bmatrix} -1 \\ 0 \\ -1 \\ 2 \end{bmatrix} = -\begin{bmatrix} 2 \\ -1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ -1 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 2 \\ -1 \end{bmatrix}$

<u>Graded Problem 3</u>: $\left[\begin{array}{cccc|c} -1 & 2 & -3 & 4 & b_1 \\ 3 & 4 & -1 & 0 & b_2 \\ 2 & 1 & 1 & -2 & b_3 \end{array}\right] \xrightarrow[\text{Row}3+2\text{Row}1]{\text{Row}2+3\text{Row}1}$ Then: $-$Row1

$\left[\begin{array}{cccc|c} 1 & -2 & 3 & -4 & -b_1 \\ 0 & 10 & -10 & 12 & 3b_1+b_2 \\ 0 & 5 & -5 & 6 & 2b_1+b_3 \end{array}\right] \xrightarrow[\text{Then } \frac{1}{10}\text{Row}2]{\text{Row}3-\frac{1}{2}\text{Row}2} \left[\begin{array}{cccc|c} 1 & -2 & 3 & -4 & -b_1 \\ 0 & 1 & -1 & 6/5 & \frac{3}{10}b_1+\frac{1}{10}b_2 \\ 0 & 0 & 0 & 0 & \boxed{\frac{1}{2}b_1-\frac{1}{2}b_2+b_3} \end{array}\right]$

$\xrightarrow{\text{Row}1+2\text{Row}2} \begin{bmatrix} 1 & 0 & 1 & -8/5 \\ 0 & 1 & -1 & 6/5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow$ <u>Nullspace</u>: $x_1 = -x_3 + \frac{8}{5}x_4$

$x_2 = x_3 - \frac{6}{5}x_4$

$\boxed{\text{Rank of } A = 2}$ $\vec{x} = x_3 \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 8/5 \\ -6/5 \\ 0 \\ 1 \end{bmatrix}$

Basis for $N(A)$

<u>Column space</u>: Cols 1 and 2 of A are independent:

Basis $\left\{ \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}$

<u>Row space</u>: Basis = non-zero rows of R

$= \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ -8/5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 6/5 \end{bmatrix} \right\}$

<u>Left nullspace</u>: $\dim \mathbb{R}^3 - \text{rank}(A) = 1$.

Basis: $\left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ 1 \end{bmatrix} \right\}$