Vector spaces : spaces of "vectors" that you can add and
multiply by scalars (+8 rules for addition and mult.)

Vector spaces can be pretty abstract. For example, the "vectors" could be functions (you can add functions and multiply them by real number scalars.)

But for now, the vector spaces we mainly want to look at are subspaces of $\mathbb{R}^n$ ← all vectors with n real number components.

Subspace = set of vectors in $\mathbb{R}^n$ such that:

① Includes zero vector $\vec{0}$ (so subspace is non-empty)

② Closed under addition: If $\vec{v}$ and $\vec{w}$ are in subspace, so is $\vec{v}+\vec{w}$

③ Closed under scalar multiplication: If $\vec{v}$ is in subspace, so is $c\vec{v}$ for any c.

} Closed under linear combinations: If $\vec{v}, \vec{w}$ are in subspace, so are all $c\vec{v}+d\vec{w}$.

First big example: $C(A)$ = column space of $m \times n$ matrix A.

$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}$

↖ ↑ ↖
columns of A

I.e., $C(A)$ = set of all linear combinations
$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

A vector $\vec{b}$ in $\mathbb{R}^m$ is in $C(A)$ precisely when the system of
↑
each column of A
has m components

equations $A\vec{x}=\vec{b}$ has at least one solution.

This is because "$\vec{b} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$" means the same thing as

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}.$$

Example: $A = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$ Take $(-2)\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 1\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ ← This is a vector in $C(A)$.

So the system $\begin{cases} x+4y=2 \\ 2x+5y=1 \\ 3x+6y=0 \end{cases}$ has a solution. It's $(x,y)=(-2,1)$.

What about $\begin{cases} x+4y=-3 \\ 2x+5y=-2 \\ 3x+6y=1 \end{cases}$?? Same question as: Is $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ in $C(A)$?

Check: $\begin{bmatrix} 1 & 4 & | & -3 \\ 2 & 5 & | & -2 \\ 3 & 6 & | & 1 \end{bmatrix} \xrightarrow[\text{Row 3-3 Row 1}]{\text{Row 2-2 Row 1}} \begin{bmatrix} 1 & 4 & | & -3 \\ 0 & -3 & | & 4 \\ 0 & -6 & | & 10 \end{bmatrix} \xrightarrow{\text{Row 3-2 Row 2}} \begin{bmatrix} 1 & 4 & | & -3 \\ 0 & -3 & | & 4 \\ 0 & 0 & | & 2 \end{bmatrix}$

$\rightsquigarrow \begin{cases} x+4y=-3 \\ -3y=4 \\ 0y=2 \end{cases}$ ⟵ Inconsistent equation, No solution ⟵ So $\begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$ is not in $C(A)$.

sticks out of the plane somehow ⟶
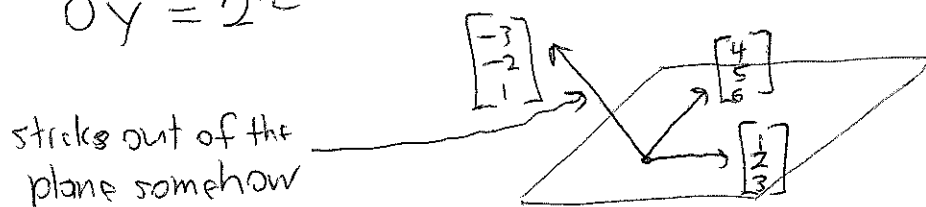
Comment: The set of all linear combinations of _any_ subset in $\mathbb{R}^n$ is a subspace, called the "span."

In our example, $C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$

Important general problem: If S is a subspace, can you find a set of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ _in S_ such that $S = \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m)$?

↑ have to be vectors in the subspace   called a "spanning set"

Example: $S = xy$-plane in $\mathbb{R}^3$, i.e. $S = $ all $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$

Since $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $S = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$ ⟵ different spanning sets, but same number of vectors in each.

Also, $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = \frac{x+y}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{x-y}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, so also $S = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}\right)$

But $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is _not_ a spanning set for S, even though ⟵ not vectors in S. $\begin{bmatrix} x \\ y \\ 0 \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} - (x+y)\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$$\vec{x} \xrightarrow{\quad A \quad} A\vec{x} \leftarrow \text{This vector}$$

$$\underset{\text{in } \mathbb{R}^n}{\uparrow} \qquad \underset{\text{in } \mathbb{R}^m}{\uparrow} \qquad \text{is in } C(A)$$

This vector is in $N(A)$ <u>if</u> $A\vec{x} = \vec{0}$. I.e., $N(A) = $ <u>all</u> vectors $\vec{x}$ in $\mathbb{R}^n$ such that $A\vec{x} = \vec{0}$.

<u>Example</u> $A = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}$ We want to find <u>all</u> solutions for $A\vec{x} = \vec{0}$

Put into augmented matrix:

$$\begin{bmatrix} 1 & 1 & 2 & 4 & | & 0 \\ 1 & 2 & 2 & 5 & | & 0 \\ 1 & 3 & 2 & 6 & | & 0 \end{bmatrix} \xrightarrow[\text{Row 3-Row 1}]{\text{Row 2-Row 1}} \begin{bmatrix} 1 & 1 & 2 & 4 & | & 0 \\ 0 & 1 & 0 & 1 & | & \\ 0 & 2 & 0 & 2 & | & \end{bmatrix} \xrightarrow{\text{Row 3-2 Row 2}} \begin{bmatrix} 1 & \textcircled{1} & 2 & 4 & | & 0 \\ 0 & 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

one more variable ← we can eliminate

$$\xrightarrow[\text{Row 1-Row 2}]{} \begin{bmatrix} 1 & 0 & 2 & 3 & | & 0 \\ 0 & 1 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \longrightarrow$$

System is reduced to <u>two</u> "independent" equations : $\begin{cases} x_1 + 2x_3 + 2x_4 = 0 \\ x_2 + x_4 = 0 \end{cases}$

Can solve for x_1 and x_2 in terms of x_3 and x_4 : $\begin{aligned} x_1 &= -2x_3 - 2x_4 \\ x_2 &= -x_4 \end{aligned}$

"free variables," they are free to take <u>any</u> real number value

So vectors in $N(A)$ look like: $\vec{x} = \begin{bmatrix} -2x_3 - 2x_4 \\ -x_4 \\ x_3 \\ x_4 \end{bmatrix} \leftarrow$ We have found <u>all</u> solutions.

Or: $\vec{x} = x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \rightsquigarrow$ So $N(A) = $ <u>all</u> linear combinations of the 2 "special solutions"

two "special solutions"

$$= \text{span}\left(\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

A <u>plane</u> in $\mathbb{R}^4$.

Now that we can find $N(A)$ for a matrix, let's check that it <u>really</u> Is a subpace of $\mathbb{R}^n$:

① Is $\vec{0}$ in $N(A)$? $A\vec{0} = \vec{0}$ ✓ Yes, it is.

② If $\vec{x}$ and $\vec{y}$ are in $N(A)$, what about $\vec{x} + \vec{y}$?

$$A(\vec{x} + \vec{y}) = \underbrace{A\vec{x}}_{\vec{0}} + \underbrace{A\vec{y}}_{\vec{0}} = \vec{0} + \vec{0} = \vec{0} \checkmark \quad \text{So } \vec{x} + \vec{y} \text{ is in } N(A)$$

③ If $\vec{x}$ is in $N(A)$, what about $c\vec{x}$?

$$A(c\vec{x}) = c\underbrace{(A\vec{x})}_{\vec{0}} = c\vec{0} = \vec{0} \checkmark \quad \text{So } c\vec{x} \text{ is in } N(A).$$

So $N(A)$ is a subspace!

Some more examples:

$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ Find all vectors in $N(A)$.

Need to solve $A\vec{x} = \vec{0}$: $\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 4 & 5 & 6 & | & 0 \\ 7 & 8 & 9 & | & 0 \end{bmatrix}$ $\xrightarrow[\text{Row 3 - 7 Row 1}]{\text{Row 2 - 4 Row 1}}$ $\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & -3 & -6 & | & 0 \\ 0 & -6 & -12 & | & 0 \end{bmatrix}$ $\xrightarrow[-2\,\text{Row 2}]{\text{Row 3}}$

"free variable" column

$\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & -3 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

↑ ↑
"pivot columns"

We can now solve for variables in the "pivot columns" in terms of the variable in the "free" column.
But first let's do a little more elimination:

$-\frac{1}{3}$ Row 2 $\begin{bmatrix} 1 & 2 & 3 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$ $\xrightarrow{\text{Row 1 - 2 Row 2}}$ $\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$

This matrix is called a "reduced row echelon form." It is the simplest form for reading off solutions.
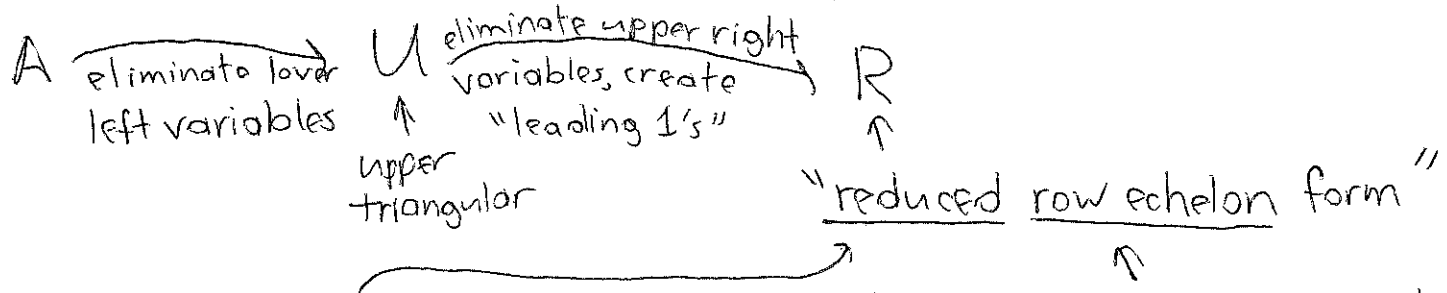
$x_1 - x_3 = 0$

$x_2 + 2x_3 = 0$

$\hookrightarrow x_1 = x_3$ $x_3 = $ free variable, can

$x_2 = -2x_3$ take any value

All solutions: $\vec{x} = \begin{bmatrix} x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ ← one "special solution"

$N(A) = $ all multiples (or, linear combinations) of the "special solution"

$= \text{span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right) = $ a line in $\mathbb{R}^3$.

Let's review the full elimination procedure:

$A \xrightarrow[\text{left variables}]{\text{eliminate lower}} U \xrightarrow[\text{variables, create}]{\text{eliminate upper right}} R$
$\qquad\qquad\qquad\qquad$ "leading 1's"

↑ upper triangular

"reduced row echelon form"

Means = all variables above leading 1's are eliminated

Means: non-zero entries in each row begin with a "leading 1", and leading 1's go down from upper left corner towards lower right.
(If a row has all 0's put it at the bottom.)

Example: What is R for

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$??

$\xrightarrow[\text{Row 3 - Row 1}]{\text{Row 2 - Row 1}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \xrightarrow{\text{Row 3 - 3 Row 2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

← This is U

two "leading 1's" already

$\xrightarrow[\text{to get a "leading 1" in Row 3}]{\frac{1}{2}\text{Row 3}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 2 - 2 Row 3}]{\text{Row 1 - Row 3}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Row 1 - Row 2}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

This is R; leading 1's go down diagonal, all variables above leading 1's are eliminated.

In this case, $R = I$! We can now easily determine $N(A)$:

Solve $A\vec{x} = \vec{0}$: $[A \mid \vec{0}] \xrightarrow{\text{elimination}} [R \mid \vec{0}] = [I \mid \vec{0}]$

↑ Elimination operations don't change $\vec{0}$.

↓ $x_1 = 0, x_2 = 0, x_3 = 0$

There's only one solution: $\vec{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, So $N(A) =$ the zero vector subspace.

This always works if $R = I$: $A \xrightarrow{\text{elimination}} I$ means $N(A) =$ zero subspace. Another way of saying this: the columns of A are "independent."