

Example Approximate solution to

$$\begin{bmatrix} -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad A = QR = \begin{bmatrix} -2/\sqrt{5} & -3/\sqrt{70} \\ 1/\sqrt{5} & -6/\sqrt{70} \\ 0 & 5/\sqrt{70} \end{bmatrix} \begin{bmatrix} \sqrt{5} & 6/\sqrt{5} \\ 0 & \sqrt{14/5} \end{bmatrix}$$

$A \quad \vec{x} \quad \vec{b}$

Just need to solve  $R\hat{x} = Q^T \vec{b} = \begin{bmatrix} -2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -3/\sqrt{70} & -6/\sqrt{70} & 5/\sqrt{70} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 3/\sqrt{70} \end{bmatrix}$

So  $\sqrt{5}x + \frac{6}{\sqrt{5}}y = -\frac{2}{\sqrt{5}} \rightarrow x = \frac{1}{\sqrt{5}} \left( -\frac{2}{\sqrt{5}} - \frac{6}{\sqrt{5}} \cdot \frac{3}{5} \right) = -\frac{2}{5} - \frac{18}{25} = -\frac{28}{25}$

$\frac{\sqrt{14}}{5}y = -\frac{3}{\sqrt{70}} \rightarrow y = -\frac{3\sqrt{5}}{\sqrt{14} \cdot 70} = -\frac{3}{5}$

## Chapter 5 Determinants

The determinant is a function that takes a square matrix ( $n \times n$ ) as input, and outputs a real number:

$$\begin{matrix} A \\ n \times n \end{matrix} \xrightarrow{\det} \begin{matrix} \det(A), \text{ or } |A| \\ \text{in } \mathbb{R} \end{matrix}$$

It determines whether  $A$  is invertible:

$$\begin{cases} \det A \neq 0 \rightarrow A \text{ has an inverse} \\ \det A = 0 \rightarrow A \text{ has no inverse.} \end{cases}$$

But doesn't give us an efficient way of finding  $A^{-1}$ , so it's more useful as a theoretical tool.

Start with simplest examples:

1x1:  $A = [a] \rightarrow$  has inverse exactly when  $a \neq 0$ , because  $[a]^{-1} = [a^{-1}]$ .

So  $\det[a] = a$

2x2:  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  Formula for  $A^{-1}$ :  $\frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

So  $A^{-1}$  exists exactly when  $ad-bc \neq 0$ , and this is the determinant:  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

What about bigger matrices? There are several ways to define the determinant. To understand these formulas, it may be good first to look at what properties the determinant should have:

Properties: It turns out that det is the only possible function from  $n \times n$  matrices to  $\mathbb{R}$  that obeys 3 simple rules:

① The determinant of the identity matrix is 1:  $|I| = 1$ .

2x2 case:  $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = (1)(1) - (0)(0) = 1 \quad \checkmark$

② If you switch two rows in  $A$ , then det changes by  $-1$  factor:

2x2 case:  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[\text{rows}]{\text{switch}} \begin{vmatrix} c & d \\ a & b \end{vmatrix}$   
 $\downarrow \det \qquad \qquad \downarrow \det$   
 $ab-bc \xrightarrow[\text{by } -1]{\text{Multiply}} cb-da$

③ det is a "linear function" of each row separately. This means:

(a) If you multiply one row by a scalar, det also changes by that scalar factor

2x2 case:  $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[\pm \text{Row } 1]{\text{Row } 1 \rightarrow \pm \text{Row } 1}} \begin{vmatrix} \pm a & \pm b \\ c & d \end{vmatrix}$   
 $\downarrow \det \qquad \qquad \downarrow \det$   
 $ad-bc \xrightarrow[\text{by } \pm]{\text{Multiply}} \pm ad - \pm bc = \pm(ad-bc)$

(b) If you add one row of  $A$  to the same row of  $A'$ , and all the other rows of  $A$  and  $A'$  are the same, then determinants also add.

2x2 case:  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a' & b' \\ c & d \end{bmatrix}$   $\xrightarrow[\text{Keep Row 2 the same}]{\text{Add Row 1's}}$   $\begin{bmatrix} a+a' & b+b' \\ c & d \end{bmatrix}$

Two matrices with the same Row 2

$\downarrow$  Take both determinants

$\downarrow$  det

$\begin{matrix} ad-bc, a'd-b'c \\ \text{Add} \\ \text{determinants} \end{matrix}$   $\rightarrow$   $\begin{matrix} (a+a')d - (b+b')c = \\ (ad-bc) + (a'd-b'c) \end{matrix}$

Warning: Rule (3) does not mean:  $\begin{cases} \det(tA) = t \det A \\ \det(A+B) = \det A + \det B \end{cases}$

For  $tA$ , you multiply all n rows of  $A$  by  $t$ , so Rule (3) means you multiply  $\det A$  by  $t$   $n$  times:

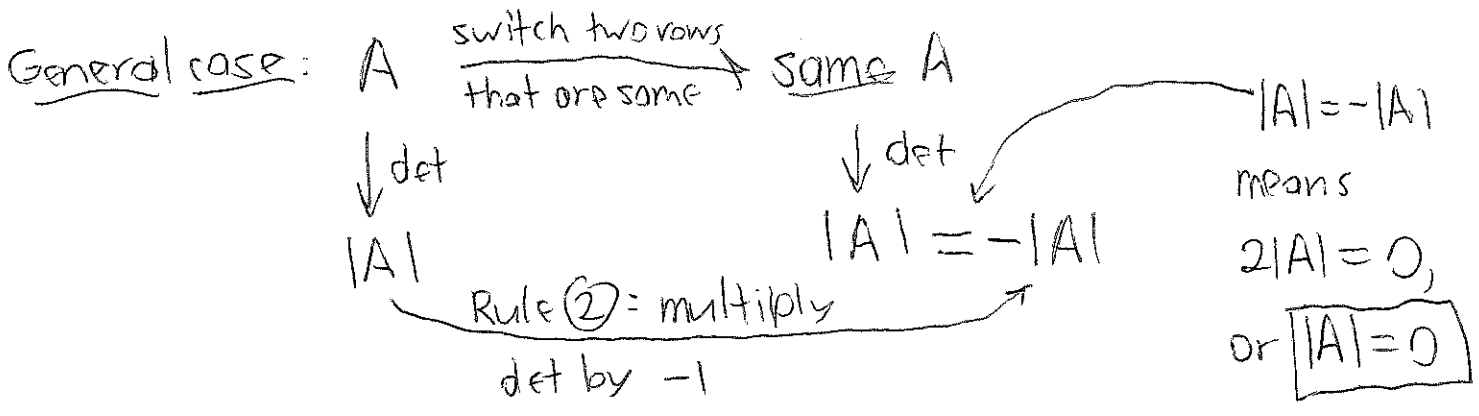
Correct relation:  $\det(tA) = t^n \det A$  if  $A$  is  $n \times n$ .

But there is no relation between  $\det(A+B)$  and  $\det A, \det B$  in general.

Using Rules 1-3, we can find many more properties of determinants and also find ways to calculate them for big matrices:

(4) If 2 rows of  $A$  are the same, then  $\det A = 0$ .

2x2 case:  $\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ba = 0 \quad \checkmark$



(5) The big row operation: If we add a multiple of one row to another row, the determinant doesn't change.

2x2 case:  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow[\text{Row 2} + t \text{ Row 1}]{\text{Row 2} \rightarrow} \begin{bmatrix} a & b \\ c+ta & d+tb \end{bmatrix}$

det  $\downarrow$

$\downarrow$  det

$$ad - bc = a(d+tb) - b(c+ta) = ad - bc + (atb - bta)$$

General case: det of new A =  $\begin{vmatrix} \text{Row 1} \\ \text{Row 2} + t \text{ Row 1} \\ \text{Row 3} \\ \vdots \\ \text{Row n} \end{vmatrix} \xrightarrow[\text{Rule (3b)}]{\text{Rule}} \begin{vmatrix} \text{Row 1} \\ \text{Row 2} \\ \vdots \\ \text{Row n} \end{vmatrix} + \begin{vmatrix} \text{Row 1} \\ t \text{ Row 1} \\ \vdots \\ \text{Row n} \end{vmatrix}$

old A

Rule (3a) =  $\det(\text{old A}) + t \begin{vmatrix} \text{Row 1} \\ \text{Row 1} \\ \vdots \\ \text{Row n} \end{vmatrix} \xleftarrow{\text{same rows}} \text{Rule (4)} = \det(\text{old A}) + t(0)$

So  $\det(\text{new A}) = \det(\text{old A})$

Rules (2), (3), and (5) tell us how row operations affect determinants. This means we can calculate determinants using elimination!

Example Find determinant of  $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix}$ .

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[\text{det changes by } (-1)]{\text{Row 1} \leftrightarrow \text{Row 2}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 2 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & -4 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

This shows:  $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{vmatrix} \xrightarrow{\text{Rule (3a)}} (-1)(-2) \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$

Rule (1) =  $(-1)(-2)(1) = \boxed{2}$

Since  $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 2 \end{vmatrix} \neq 0$ , this matrix should be invertible. (131)

But we don't need determinants to see that! We saw that the reduced row echelon form is  $I$ , which shows that  $A$  is invertible. In fact, the elimination method for determinants gives us a way to prove the:

Big Determinant Theorem:  $\begin{cases} \text{If } A \text{ is invertible, then } \det(A) \neq 0 \\ \text{If } A \text{ is not invertible, then } \det(A) = 0. \end{cases}$

Prove this:  $A \xrightarrow{\text{elimination}} R$ , reduced row echelon form

The three eliminations change  $|A|$  by:

- ① Factor of  $-1$  (row switches)
- ② Factor of  $\pm$  (multiply a row by non-zero  $\pm$ )
- ③ Nothing (factor of 1) (add a multiple of one row to another)

Whatever elimination operations we have to do, they change  $|A|$  by a non-zero scalar: So  $|R| = (\text{non-zero scalar}) |A|$ ,  
or  $|A| = (\text{non-zero})^{-1} |R|$

Now: If  $A$  is invertible: then  $R = I$ , so

$$|A| = (\text{non-zero scalar}) |I| = (\text{non-zero})(1) \neq 0 \quad \checkmark$$

$\nwarrow$  Rule ①

But if  $A$  is not invertible: Then  $R$  has a row of  $0$ 's.

$$|A| = (\text{non-zero scalar}) \begin{vmatrix} \text{stuff} \\ 0 & 0 & \dots & 0 \end{vmatrix} \xrightarrow[\text{Rule ③a}]{} (\text{non-zero})(0) \begin{vmatrix} \text{stuff} \\ 1 & 1 & \dots & 1 \end{vmatrix} = 0 \quad \checkmark$$

$\nwarrow$  These don't matter.

Another big determinant theorem:

$$\det(AB) = (\det A)(\det B), \text{ or } |AB| = |A||B|.$$

Remember this kind of rule doesn't work for addition!

Check  $2 \times 2$ :  $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} ap+br & aq+bs \\ cp+dr & cq+ds \end{bmatrix}$

$$\begin{array}{ccc} (\det A)(\det B) & \downarrow & \det(AB) \\ (ad-bc)(ps-qr) & \stackrel{??}{=} & (ap+br)(cq+ds) - (aq+bs)(cp+dr) \end{array}$$

Diagram illustrating the expansion of the determinant of the product matrix. The left side shows the product of two  $2 \times 2$  determinants. The right side shows the determinant of the product matrix, expanded as  $(ap+br)(cq+ds) - (aq+bs)(cp+dr)$ . Arrows and numbers (1, 2, 3, 4) indicate the terms that survive after cancellation. The terms  $(ap+br)(cq+ds)$  and  $(aq+bs)(cp+dr)$  are shown with arrows indicating that other terms cancel out.

General case: First possibility:  $A$  is not invertible. Then  $AB$  is not invertible either (if  $(AB)^{-1}$  existed, then we'd get  $A^{-1} = B(AB)^{-1}$ ). So  $\det(AB)$  and  $(\det A)(\det B)$  are both 0.

$\hookrightarrow 0$

2nd possibility:  $A$  is invertible, so  $R = I$

$$A \xrightarrow{\text{elimination}} I \quad \text{actually, this is } |A|.$$

$$|A| \longrightarrow (\text{non-zero scalar}) |I|$$

Now do the same elimination operations to  $AB$ . This is the same as multiplying by elimination matrices on the left, so:

$$AB \xrightarrow[\text{elimination}]{\text{same}} IB = B, \quad \text{Determinants: } |AB| \longrightarrow (\text{same non-zero}) |B|$$

$$\text{So } |AB| = |A||B| \quad \checkmark$$

Some more properties:

• If  $A$  is invertible, then  $|A^{-1}| = |A|^{-1}$ . Non-zero, ~~so~~ so  $|A|^{-1}$  exists!

Why?  $|A||A^{-1}| = |AA^{-1}| = |I| = 1 \longrightarrow |A^{-1}| = \frac{1}{|A|}$ .

↑  
product rule

- Det of triangular matrix is easy: product of diagonal entries (133)

$$\begin{vmatrix} a & \text{stuff} \\ 0 & b \\ 0 & 0 & c \end{vmatrix} \xrightarrow[\text{no change in det}]{\text{elimination}} \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} \xrightarrow[\text{Rule 3a}]{(a)(b)(c)} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = abc$$

↖ If one of  $a, b, c$  is 0, matrix isn't invertible, so  $\det = 0$ , still the same as  $abc$ .

Same for lower triangular matrices.

Example Find det of  $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

Just need to get to triangular form.

$$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{in det}]{\text{No change}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{change}]{\text{No}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \xrightarrow[\text{change}]{\text{No}} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 0 \\ 0 & 0 & 4/3 & 0 \\ 0 & 0 & 0 & 5/4 \end{bmatrix}$$

Determinant is  $2(\frac{3}{2})(\frac{4}{3})(\frac{5}{4}) = \boxed{5}$

- Transposes:  $\det(A^T) = \det(A)$  (no difference)

Why? Use LU decomposition:  $PA = LU \rightarrow |A| = |L||U|/|P|$   
 permutation matrix in case of row switches from product rule

Also:  $(PA)^T = (LU)^T \rightarrow A^T P^T = U^T L^T \rightarrow |A^T| = |U^T||L^T|/|P^T|$

Now:  $|U^T| = |U|$  (both are triangular, one upper and one lower, with same diagonal entries)

↖  $|L^T| = |L|$  (same reason)

$|P^T| = |P|$  (You can row-switch  $P$  to get  $I$ , so  $|P| = \pm 1$ . Also,  $P$  is orthogonal:  $P^T P = I$ , so  $|P||P^T| = 1 \rightarrow$  ~~both are~~ either both are  $+1$  or both are  $-1$ )

so  $|A^T| = |L||U|/|P|$   
 $\boxed{= |A|}$