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1. $A^TA = \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}$

$\begin{vmatrix} 20-\lambda & 10 \\ 10 & 5-\lambda \end{vmatrix} = \lambda^2 - 25\lambda = \lambda(\lambda-25) = 0 \rightarrow \lambda = 25, 0$

$\rightarrow \sigma = 5, 0$

$\lambda = 25: \begin{bmatrix} -5 & 10 \\ 10 & -20 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = 2x_2 \rightarrow \vec{x} = x_2\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Normalize: $\vec{v}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\lambda = 0: \begin{bmatrix} 20 & 10 \\ 10 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = -2x_1 \rightarrow \vec{x} = x_1\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

Normalize: $\vec{v}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{5}\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} = \frac{1}{5\sqrt{5}}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\vec{u}_2$ = unit basis for $N(A^T)$: $\begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = -2x_2$

$\rightarrow \vec{x} = x_2\begin{bmatrix} -2 \\ 1 \end{bmatrix} \rightarrow \vec{u}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

So $A = U\Sigma V^T = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$

2. $B^TB = \begin{bmatrix} 0 & 1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 16 \end{bmatrix} \rightarrow \lambda = 1, 16, \ \sigma = 4, 1$

$\lambda = 16: \begin{bmatrix} -15 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = 0 \rightarrow \vec{x} = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\lambda = 1: \begin{bmatrix} 0 & 0 \\ 0 & 15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_2 = 0 \rightarrow \vec{x} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{4}\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{\sigma_2}A\vec{v}_2 = \frac{1}{1}\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So $A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

3. $C^TC = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 3 & 5 \end{bmatrix}$

$\begin{vmatrix} 5-\lambda & 3 \\ 3 & 5-\lambda \end{vmatrix} = \lambda^2 - 10\lambda + 16 = (\lambda-2)(\lambda-8) = 0 \rightarrow \lambda = 2, 8$

$\sigma = \sqrt{8}, \sqrt{2}$

$\lambda = 8: \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = x_2 \rightarrow \vec{x} = x_2\begin{bmatrix} 1 \\ 1 \end{bmatrix} \rightarrow \vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda = 2: \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow x_1 = -x_2 \rightarrow \vec{x} = x_2\begin{bmatrix} -1 \\ 1 \end{bmatrix} \rightarrow \vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\vec{u}_1 = \frac{1}{\sigma_1}A\vec{v}_1 = \frac{1}{\sqrt{8}}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{u}_2 = \frac{1}{\cancel{\sigma_2}}A\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So $A = U\Sigma V^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$

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4. $D^TD = \begin{bmatrix}1&1\\1&0\end{bmatrix}\begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}2&1\\1&1\end{bmatrix}$

$$\begin{vmatrix}2-\lambda & 1\\ 1 & 1-\lambda\end{vmatrix} = \lambda^2-3\lambda+1=0 \to \lambda = \frac{3\pm\sqrt{9-4}}{2} = \frac{3}{2}\pm\frac{\sqrt{5}}{2}$$

$$\sigma = \sqrt{\frac{3}{2}\pm\frac{\sqrt{5}}{2}} = \frac{1}{2}\pm\frac{\sqrt{5}}{2}$$

These are just the eigenvalues of D (one is the Golden Ratio)

$$= \phi,\ 1-\phi$$

$$\lambda=\phi^2: \begin{bmatrix}2-\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right) & 1\\ 1 & 1-\left(\frac{3}{2}+\frac{\sqrt{5}}{2}\right)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to x_1 = \left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)x_2 \Rightarrow \vec{x} = x_2\begin{bmatrix}\phi\\1\end{bmatrix}$$

$$\lambda=(1-\phi)^2: \begin{bmatrix}2-\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right) & 1\\ 1 & 1-\left(\frac{3}{2}-\frac{\sqrt{5}}{2}\right)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} \to x_1 = \left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)x_2$$

$$\to \vec{x} = x_2\begin{bmatrix}1-\phi\\1\end{bmatrix} \to \vec{v}_2 = \frac{1}{\sqrt{(1-\phi)^2+1}}\begin{bmatrix}1-\phi\\1\end{bmatrix}$$

$$\left(\frac{1}{2}-\frac{\sqrt{5}}{2}\right)^2+1 = \frac{5}{2}-\frac{\sqrt{5}}{2}$$

$$\vec{u}_1 = \frac{1}{\phi}\begin{bmatrix}1&1\\1&0\end{bmatrix}\frac{1}{\sqrt{\phi^2+1}}\begin{bmatrix}\phi\\1\end{bmatrix} = \frac{1}{\phi\sqrt{\phi^2+1}}\begin{bmatrix}\overset{\phi^2}{1+\phi}\\ \phi\end{bmatrix} = \frac{1}{\sqrt{\phi^2+1}}\begin{bmatrix}\phi\\1\end{bmatrix} = \vec{v}_1$$

$$\vec{u}_2 = \frac{1}{1-\phi}\begin{bmatrix}1&1\\1&0\end{bmatrix}\frac{1}{\sqrt{(1-\phi)^2+1}}\begin{bmatrix}1-\phi\\1\end{bmatrix} = \frac{1}{(1-\phi)\sqrt{(1-\phi)^2+1}}\begin{bmatrix}\overset{(1-\phi)^2}{2-\phi}\\ 1-\phi\end{bmatrix} = \vec{v}_2$$

SVD decomposition is $D = Q\Delta Q^T$ (i.e., $U=V=Q$)

because D is already symmetric.