Linear Algebra – Fall 2020
Midterm Exam

NAME:

STUDENT ID:

Instructions:

- Show answers and arguments in the space provided. You may use the back of the pages also, but indicate clearly any such material that you want graded. Answers given without supporting work may receive zero credit.
- This is a closed book exam: no calculators, notes, or formula sheets.

QUESTION	POINTS	SCORE
1	14	
2	10	
3	12	
4	14	
5	14	
6	12	
7	14	
8	10	
TOTAL	100	

1. (a) (10 points) Find all solutions of the system of linear equations:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - 5x_4 + 6x_5 &= 1 \\ -x_1 - 2x_2 - x_3 + x_4 - x_5 &= 1 \\ 4x_1 + 8x_2 + 5x_3 - 8x_4 + 9x_5 &= -2 \end{aligned}$$

(b) (4 points) Find the reduced row echelon form R of the coefficient matrix of the system.

2. (a) (6 points) How long is the vector $\mathbf{v} = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

   (b) (4 points) Pick any numbers x, y, z such that $x + y + z = 0$. Find the angle between your vector $\mathbf{v} = (x, y, z)$ and the vector $\mathbf{w} = (z, x, y)$.

3. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 2 \\ 3 & 2 & 24 \end{bmatrix}.$$

(a) (6 points) Find the LU decomposition of A.

(b) (6 points) Use the LU decomposition to solve the linear system of equations $A\mathbf{x} = (1, 3, 0)$.

4. (a) (8 points) Show that the set of all vectors (b_1, b_2, b_3) such that $b_1 + b_2 + b_3 = 0$ is a subspace of $\mathbf{R}^3$. (Verify all three properties of a subspace.)

   (b) (6 points) Show that the set of all vectors (b_1, b_2, b_3) such that $b_1 \leq b_2 \leq b_3$ is *not* a subspace of $\mathbf{R}^3$. (Show that at least one property of a subspace fails.)

5. Consider the system of linear equations:

$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 1 \\ x_1 & + & 2x_2 & + & 3x_3 & = & 2 \\ x_1 & + & 4x_2 & + & 9x_3 & = & -2 \end{array}$$

(a) (10 points) Find the *inverse* of the coefficient matrix of the system of equations.

(b) (4 points) Use the inverse matrix to solve the system of linear equations.

6. (12 points) Determine whether the following vectors form a basis for $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 11 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

If the vectors are *not* linearly independent, show how to write one of the them as a linear combination of the others.

7. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & -1 \\ 3 & 4 & -6 & 8 \\ 0 & -1 & 3 & 4 \end{bmatrix}.$$

(a) (6 points) Find a linear relation on the components of $\mathbf{b} = (b_1, b_2, b_3, b_4)$ that guarantees that $\mathbf{b}$ is a vector in the column space $\mathbf{C}(A)$.

(b) (8 points) Find a spanning set (the special solutions) for the null space $\mathbf{N}(A)$.

8. (10 points) Find all 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$$A^T \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} A.$$

Show that every matrix A that satisfies this property is a scalar multiple of one particular 2×2 matrix.

This page left blank for any additional work.