$$= \underbrace{\begin{bmatrix}1&0&0\\-1/2&1&0\\-1/2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&-1&1\end{bmatrix}} U$$

(Reverse ~~order~~ the row operations to get inverses)

$$\begin{bmatrix}1&0&0\\-1/2&1&0\\-1/2&-1&1\end{bmatrix} = L$$

$$\text{So } \underset{A}{\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}} = \underset{L}{\begin{bmatrix}1&0&0\\-1/2&1&0\\-1/2&-1&1\end{bmatrix}}\underset{U}{\begin{bmatrix}2&-1&-1\\0&3/2&-3/2\\0&0&0\end{bmatrix}}$$

(Check.)

Last time: Started LU decomposition

$$\text{n×n matrix} \rightarrow A = \underset{\text{lower triangular}}{L}\;\underset{\text{upper triangular}}{U}$$

How? $A \xrightarrow[\text{Same as: multiply A by lower triangular elimination matrices}]{\text{Eliminate lower left variables}} \overset{\text{upper triangular}}{U}$

$$A \rightarrow L_1 A \rightarrow L_2 L_1 A \rightarrow \cdots \rightarrow \underbrace{L_m \cdots L_2 L_1}_{\text{not L yet}} A = U$$

$$\text{Then: } A = \underbrace{L_1^{-1} L_2^{-1} \cdots L_m^{-1}}_{\text{This is L!}} U$$

4x4 Example:

$$A = \begin{bmatrix}1&1&1&1\\1&2&3&4\\1&3&6&10\\1&4&10&20\end{bmatrix} \xrightarrow[\text{Row 3 - Row 1, Row 4 - Row 1}]{\text{Eliminate lower left: Row 2 - Row 1}} \begin{bmatrix}1&1&1&1\\0&1&2&3\\0&2&5&9\\0&3&9&19\end{bmatrix}$$

Or: multiply by $\begin{bmatrix}1&0&0&0\\-1&1&0&0\\-1&0&1&0\\-1&0&0&1\end{bmatrix}$

$$\xrightarrow[\text{Row 4 - 3Row 1}]{\text{Row 3 - 2Row 1}} \begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&3&10\end{bmatrix} \xrightarrow{\text{Row - 3Row 3}} \begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}$$

or: Multiply by $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-2&1&0\\0&-3&0&1\end{bmatrix}$

Or: multiply by $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-3&1\end{bmatrix}$

Upper triangular; this is U

$$A \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} A \longrightarrow$$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}}_{\text{Not } L \text{ yet; this is } L^{-1}} A = U$$

Solve for A: $$A = \underbrace{\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \right)^{-1}} U$$

Switch order when you invert a product $\rightarrow$ $$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}^{-1}}$$

Reverse the row operations $\rightarrow$ $$\underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{bmatrix} \qquad = \qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \leftarrow \text{This is } L!$$

Conclusion: $$\underset{A}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}}$$

↑

LU decomposition works whenever you can do elimination $A \longrightarrow U$ <u>without</u> switching rows. How to find the entries of L more efficiently?

Diagonal entries of L: all 1's

Below-diagonal entries: Put ℓ_{ij} in row i, column j if you did the operation $\text{Row } i \to \text{Row } i - \ell_{ij} \text{ Row } j$

Example:
$$A = \begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix} \xrightarrow[\text{Row } 3 - \underset{\ell_{31}}{\boxed{1}} \text{ Row } 1]{\text{Row } 2 - \overset{\ell_{21}}{\boxed{1}} \text{ Row } 1} \begin{bmatrix}1&1&1\\0&1&2\\0&3&8\end{bmatrix} \xrightarrow{\text{Row } 3 - \underset{\ell_{32}}{\boxed{3}} \text{ Row } 2}$$

$$\begin{bmatrix}1&1&1\\0&1&2\\0&0&2\end{bmatrix} = U \qquad L = \begin{bmatrix}1&0&0\\1&1&0\\1&3&1\end{bmatrix}$$

Check that this is correct:
$$LU = \begin{bmatrix}1&0&0\\1&1&0\\1&3&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&2\\0&0&2\end{bmatrix} = \begin{bmatrix}1&1&1\\1&2&3\\1&4&9\end{bmatrix} = A \ \checkmark$$

Summarize LU decomposition algorithm:

$$A \xrightarrow[\text{elimination}]{\text{lower left}} U \rightsquigarrow A = \underset{\uparrow \text{ lower triangular}}{L}\,\overset{\leftarrow \text{ upper triangular}}{U}$$

$$L = \begin{bmatrix}1&0&\cdots&0\\ &1&\cdots&0\\ \boxed{\ell_{ij}}&&\ddots&\vdots\\ &&&1\end{bmatrix}$$

Comes from operation
$\text{Row } i \to \text{Row } i - \ell_{ij} \text{ Row } j$
note the minus sign

Warning 1: Formula for L only works if you do the elimination operations in the right order:

Eliminate Column 1 variables using Row 1 first,
Eliminate Column 2 variables using Row 2 next,
⋮

Warning 2: LU decomposition only works if you can go from A to U without switching rows.

Now: What is the use of LU decomposition?

It turns one hard system of equations $A\vec{x}=\vec{b}$ into two easy triangular systems:

$$A\vec{x}=\vec{b} \rightsquigarrow L\underbrace{U\vec{x}}_{\text{call this vector }\vec{y}}=\vec{b}$$

First solve $L\vec{y}=\vec{b}$ for $\vec{y}$, then solve $U\vec{x}=\vec{y}$ for $\vec{x}$

same $\vec{y}$

Check that this works: Does $A\vec{x}$ really equal $\vec{b}$?

$$A\vec{x}=(LU)\vec{x}=L(\underbrace{U\vec{x}}_{\text{This is }\vec{y}})=\underbrace{L\vec{y}}_{\text{This is }\vec{b}}=\vec{b}\ \checkmark$$

Example solve
$$\begin{cases} x_1+x_2+x_3+x_4=1\\ x_1+2x_2+3x_3+4x_4=0\\ x_1+3x_2+6x_3+10x_4=-1\\ x_1+4x_2+10x_3+20x_4=0\end{cases}$$

Already know:
$$A=\underbrace{\begin{bmatrix}1&0&0&0\\1&1&0&0\\1&2&1&0\\1&3&3&1\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}1&1&1&1\\0&1&2&3\\0&0&1&3\\0&0&0&1\end{bmatrix}}_{U}$$

First solve $L\vec{y}=\vec{b}$:
$$\begin{aligned} y_1 &= 1\\ y_1+y_2 &= 0 \rightarrow y_2=-1\\ y_1+2y_2+y_3 &= -1 \rightarrow y_3=-1-1-2(-1)=0\\ y_1+3y_2+3y_3+y_4 &= 0 \rightarrow y_4=-1-3(-1)=2\end{aligned}$$

Now solve $U\vec{x}=\vec{y}$:
$$\begin{aligned} x_1+x_2+x_3+x_4 &= 1 \rightarrow 1-5-(-6)-2=0\\ x_2+2x_3+3x_4 &= -1 \rightarrow x_2=-1-2(-6)-3(2)=5\\ x_3+3x_4 &= 0 \rightarrow x_3=-3(2)=-6\\ x_4 &= 2\end{aligned}$$

so $\vec{x}=\begin{bmatrix}0\\5\\-6\\2\end{bmatrix}$

Note: The two triangular systems are easy to solve, but we still have to work hard to find L and U.

Solving systems using LU works best when you have a lot of systems with the same A:

$$A\vec{x}_1=\vec{b}_1,\ A\vec{x}_2=\vec{b}_2,\ A\vec{x}_3=\vec{b}_3, ---, A\vec{x}_{1000}=\vec{b}_{1000}$$

Even though you have 1000 systems, you only need to find LU once.

Some statistics from the textbook: (if A is nxn)

To find LU, it usually takes about $\frac{2}{3}n^3$ arithmetic operations

↑ multiplication and addition

To solve a triangular system usually takes about $2n^2$ arithmetic operations

So what if you solve 1000 systems individually?

$$\underbrace{A \xrightarrow[\frac{2}{3}n^3]{} U \xrightarrow[2n^2]{\text{solve triangular}} \text{solution}}_{1000\text{ times}} \longrightarrow 2000\left(\frac{n^3}{3}+n^2\right) \text{ operations}$$

Using LU: $A \to U$ once $\rightsquigarrow \frac{2}{3}n^3$

$$\underbrace{\text{Two triangular systems} \rightsquigarrow 4n^2}_{1000\text{ times}} \to \frac{2}{3}n^3+4000n^2 \text{ operations}$$

If n is big, then n^2 is much less than n^3, so solving by LU is almost 1000 times faster:

$$\frac{2000\left(\frac{n^3}{3}+n^2\right)}{\frac{2}{3}n^3+4000n^2} = \frac{\frac{2000n^3}{3}\left(1+\frac{3}{n}\right)}{\frac{2}{3}n^3\left(1+\frac{6000}{n}\right)} = 1000\cdot\underbrace{\frac{1+3/n}{1+6000/n}}_{\substack{\text{Goes to 1}\\ \text{as } n\to\infty}}$$

Transposes and Permutation

One final matrix operation: $\underset{\substack{\uparrow \\ \text{switch rows and columns of a matrix}}}{\underline{\text{transpose}}}$

$$m\times n \text{ matrix } A \xrightarrow[\text{into columns}]{\text{turn rows}} n\times m \text{ matrix } A^T$$

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \longrightarrow A^T=\begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

(i,j)-entry of A^T $=(j,i)$-entry of A

$$\begin{array}{c}\text{jth} \\ \text{row}\end{array}\rightarrow \underset{\substack{\uparrow \\ \text{ith column}}}{\begin{bmatrix} & | & \\ — & A_{ji} & — \\ & | & \end{bmatrix}} \longrightarrow A^T = \underset{\text{jth } \underline{\text{column}} \text{ now}}{\begin{bmatrix} & | & \\ — & (A^T)_{ij} & — \\ & | & \end{bmatrix}} \leftarrow \begin{array}{c}\text{ith } \underline{\text{row}} \\ \text{now}\end{array}$$

Another viewpoint: "Flip A over the diagonal" to get A^T:

$$A=\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \longrightarrow A^T=\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

<u>Rules</u> $(A+B)^T = A^T + B^T$ ← $(A+B)^T_{ij} = (A+B)_{ji}$

$(AB)^T = ??$ $\qquad\qquad\qquad\qquad = A_{ji} + B_{ji}$

$(A^{-1})^T = ??$ $\qquad\qquad\qquad = A^T_{ij} + B^T_{ij} = (A^T+B^T)_{ij}$

(i,j)-entry of $(AB)^T$ $= (j,i)$-entry of AB

$= (j\text{th row of } A)\cdot(i\text{th column of } B)$

$= (j\text{th } \underline{\text{column}} \text{ of } A^T)\cdot(i\text{th } \underline{\text{row}} \text{ of } B^T)$

$= (i\text{th row of } B^T)\cdot(j\text{th column of } A^T)$

$= (i,j)$-entry of B^TA^T.

so $\boxed{(AB)^T = B^TA^T}$ (switch order, just like for inverse)

Now we can find $(A^{-1})^T$:

$$AA^{-1} = I \longrightarrow (AA^{-1})^T = I^T \longrightarrow (A^{-1})^T A^T = I$$

$$A^{-1}A = I \longrightarrow (A^{-1}A)^T = I^T \longrightarrow \underbrace{A^T (A^{-1})^T = I}$$

$(A^{-1})^T$ lookslike inverse of A^T.

So : $\boxed{(A^{-1})^T = (A^T)^{-1}}$ This means that A^T is invertible if A is, and A^T isn't invertible if A isn't.

Example

$$\begin{tikzcd}
A = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix} \arrow[r, "\text{Inverse}"] \arrow[d, "\text{Transpose}"] & A^{-1} = \frac{1}{\underbrace{(1)(4)-(1)(3)}_{1}} \begin{bmatrix} 4 & -3 \\ -1 & 1 \end{bmatrix} \arrow[d, "\text{Transpose}"] \\
A^T = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \arrow[r, "\text{Inverse}"] & (A^{-1})^T = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} = (A^{-1})^T
\end{tikzcd}$$
