Calculus A(1): Homework 8

The total is 100 points. When we refer to a paragraph number (e.g. §7.1), we refer to the PDF of the textbook *Thomas Calculus* that you can find on the weblearn. The bonus exercises are optional and more difficult. We may (or may not) decide to grade one of your bonus exercises and use it to replace one assigned exercise (if it improves your total grade).
We follow the convention taken in class and denote by log the logarithm in base e, which is the inverse of exp.

Routine exercises (do not hand-in)

1. §7.1 Exercises 4, 14, 17, 27, 39, 50
2. §7.2 Exercises 3, 9, 18, 33, 47, 68
3. §7.3 Exercises 4, 12, 23, 41, 49, 64
4. §7.4 Exercises 3, 8, 29, 36, 45, 61, 69, 84
5. §7.6 Exercises 5, 8
6. §7.7 Exercises 2, 24, 27, 64, 70, 83, 91, 109
7. §8.1 Exercises 2, 17, 42, 46, 51, 64
8. §8.2 Exercises 1, 13, 27, 28
9. §8.3 Exercises 7, 29, 36
10. §8.4 Exercises 1, 20, 23, 33
11. §8.5 Exercises 1, 18, 32, 33, 43

Assigned exercises (hand-in)

1. (10 points) Let I be an interval and $f : I \to \mathbb{R}$ be a one-to-one function. Prove that f is increasing if and only if f^{-1} is increasing (where f^{-1} denotes as usual the inverse function of f, whose domain is the range of f).

2. (20pts) Find the derivative of the following functions.

   a. $f(x) = \arcsin(\sqrt{\log(x)})$ **b.** $f(x) = x \arccos(x) - x^{\sqrt{x}}$

   c. $f(x) = (1+x^{\log(x)})e^{\arctan(x)}$ **d.** $f(x) = \left(\dfrac{(x+1)(x-5)}{(x+4)(x-3)}\right)^4$

3. (20 points) Find the following limits, if they exist.

a. $\lim_{x\to 0^+} x\log(x)$ **b.** $\lim_{x\to 0^+} x^x$

c. $\lim_{x\to +\infty} x^2 e^{-x}$ **d.** $\lim_{x\to +\infty} \dfrac{\exp((\log(x))^2)}{x}$

4. (10 points) Prove that for all $x \in \mathbb{R}$, we have $e^x \geq x+1$.

5. (40pts) Evaluate the following integrals.

a. $\displaystyle\int e^x \sec^2(e^x - 3)\ \mathrm{d}x$ **b.** $\displaystyle\int x(\log(x))^2\ \mathrm{d}x$

c. $\displaystyle\int \sqrt{4-x^2}\ \mathrm{d}x$ **d.** $\displaystyle\int \frac{1}{x^2\sqrt{1+x^2}}\ \mathrm{d}x$

e. $\displaystyle\int_0^{\pi} \frac{x\sin(x)}{1+\cos^2(x)}\ \mathrm{d}x$ **f.** $\displaystyle\int_1^3 (1+\log(x))x\log(x)\ \mathrm{d}x$

g. $\displaystyle\int_4^6 \frac{x^3+1}{x^3-5x^2+6x}\ \mathrm{d}x$ **h.** $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{2}}{2}} \frac{1}{\arcsin(x)\sqrt{1-x^2}}\ \mathrm{d}x$

Bonus exercises (optional)

1. Compute $\int \frac{1}{1+\sin(x)}\ \mathrm{d}x$. (Hint: Multiply both numerator and denominator by $1-\sin(x)$.)

2. **The geometric, logarithmic, and arithmetic mean inequality.**

   a. Show $f(x) = e^x$ is convex on $\mathbb{R}$.

   b. Show that if $0 < a < b$, we have

$$e^{(\log(a)+\log(b))/2} \cdot (\log(b) - \log(a)) < \int_{\log(a)}^{\log(b)} e^x \mathrm{d}x < \frac{a+b}{2} \cdot (\log(b) - \log(a)).$$

   (Hint: Consider the tangent line to the graph of e^x at $c = \frac{\log(a)+\log(b)}{2}$ and the secant line segment between $x = a$ and $x = b$. Use the properties of convex functions related to tangents and secants seen in class.)

   c. Use the inequality in part (b) to conclude that

$$\sqrt{ab} < \frac{b-a}{\log(b)-\log(a)} < \frac{a+b}{2}.$$