几个三角级数的表达式

令$z\in C, |z|<1, z=re^{i\theta},\ r\in[0,\,1), \theta\in[0,\ 2\pi)$. 由公式

$$\sum_{n=1}^{+\infty}\frac{z^n}{n}=-\ln(1-z),$$

可得

$$\sum_{n=1}^{+\infty}\frac{r^n\cos n\theta}{n}+i\sum_{n=1}^{+\infty}\frac{r^n\sin n\theta}{n}=-\ln(1-re^{i\theta}).$$

又因

$$\begin{aligned}\ln(1-re^{i\theta}) &= \ln(1-r\cos\theta-i\sin\theta)\\ &= \ln\sqrt{(1-r\cos\theta)^2+r^2\sin^2\theta}+i\arg(1-re^{i\theta})\\ &= \tfrac{1}{2}\ln(1-2r\cos\theta+r^2)+i\arctan\tfrac{-r\sin\theta}{1-r\cos\theta}\end{aligned}$$

可得, 当$r\in[0,\,1), \theta\in[0,\,2\pi)$时，

$$\begin{cases}\sum_{n=1}^{+\infty}\frac{r^n\cos n\theta}{n}=-\frac{1}{2}\ln(1-2r\cos\theta+r^2)\\ \sum_{n=1}^{+\infty}\frac{r^n\sin n\theta}{n}=\arctan\frac{r\sin\theta}{1-r\cos\theta}.\end{cases}\tag{0.1}$$

再令$\theta\in(0,\,2\pi), r\to 1^-$, 根据$Abel$第二定理，可得

$$\sum_{n=1}^{+\infty}\frac{\cos n\theta}{n}=-\frac{1}{2}\ln(2-2\cos\theta)=-\frac{1}{2}\ln 4\sin^2\frac{\theta}{2}=-\ln 2\sin\left(\frac{\theta}{2}\right),$$

$$\begin{aligned}\textstyle\sum_{n=1}^{+\infty}\frac{\sin n\theta}{n} &= \arctan\tfrac{\sin\theta}{1-\cos\theta}=\arctan\tfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}\\ &= \arctan(ctan(\tfrac{\theta}{2}))=\arctan\left(\tan\left(\tfrac{\pi-\theta}{2}\right)\right)=\tfrac{\pi-\theta}{2}.\end{aligned}$$

此即, 当$\theta\in(0,\ 2\pi)$时，

$$\begin{cases}\sum_{n=1}^{+\infty}\frac{\cos n\theta}{n}=-\ln\left(2\sin\frac{\theta}{2}\right)=-\ln 2-\ln(\sin\frac{\theta}{2}),\\ \sum_{n=1}^{+\infty}\frac{\sin n\theta}{n}=\frac{\pi-\theta}{2}.\end{cases}\tag{0.2}$$

更一般的有以下公式：当$\theta \in [0, 2\pi]$时, 有

1. $$\sum_{n=1}^{+\infty} \frac{\cos n\theta}{n^2} = \frac{\pi^2}{6} - \frac{\theta(2\pi - \theta)}{4} = \frac{(\pi - \theta)^2}{4} - \frac{\pi^2}{12}.$$
2. $$\sum_{n=1}^{+\infty} \frac{\sin n\theta}{n^3} = \frac{1}{12}\theta(\pi - \theta)(2\pi - \theta).$$
3. $$\sum_{n=1}^{+\infty} \frac{\cos n\theta}{n^4} = \frac{\pi^4}{90} - \frac{1}{12}\theta^2(\pi^2 - \pi\theta + \frac{\theta^2}{4}).$$
4. $$\sum_{n=1}^{+\infty} \frac{\sin n\theta}{n^5} = \frac{1}{720}\theta(\pi - \theta)(2\pi - \theta)(4\pi^2 + 6\pi\theta - 3\theta^3).$$

附录：Abel第二定理

若幂级数$f(z) = \sum_{n=0}^{+\infty} c_n z^n$在收敛的边界点$z^*$处收敛，则

$$\lim_{k \to +\infty} f(z_k) = f(z^*) = \sum_{n=0}^{+\infty} c_n (z^*)^n,$$

其中复数列$\{z_k\}$从$K_r = \{z \in C : |z| < r\}$内趋于点$z^*$.