One more type of matrix where finding the inverse is easy:

Diagonal matrices: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$ Only non-zero entries go on the diagonal

To find inverse, just invert diagonal entries:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/3 & 0 \\ 0 & 0 & 0 & 1/4 \end{bmatrix}$$

How about $B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$?

No inverse because we can't invert 0.

Last time: An $n \times n$ matrix is invertible if it has an inverse matrix:

has to be square

$$AA^{-1} = I = A^{-1}A$$

$n \times n$ identity matrix: $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$ satisfies $IB = B = BI$ for all $n \times n$ B

What does invertibility say about solutions to linear equations?

If A has an inverse, then for any vector $\vec{b}$, the system $A\vec{x} = \vec{b}$ has a unique solution.

Why? Multiply on left by A^{-1}:

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b} \xrightarrow{\text{associativity}} (A^{-1}A)\vec{x} = A^{-1}\vec{b}$$

$$\rightsquigarrow I\vec{x} = A^{-1}\vec{b} \rightsquigarrow \vec{x} = A^{-1}\vec{b}$$

This is a vector. It's the only solution for $\vec{x}$.

In particular: Set $\vec{b} = \vec{0}$. If A has an inverse then $A\vec{x} = \vec{0}$ has only one solution. One solution is $\vec{x} = \vec{0}$, since $A\vec{0} = \vec{0}$. So $\vec{0}$ is the only solution to $A\vec{x} = \vec{0}$, if A is invertible.

Turn this logic around: If $A\vec{x} = \vec{0}$ has non-zero solutions, then A can't have an inverse.

Example Let's show $A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}$ doesn't have an inverse

by solving $A\vec{x}=\vec{0}$:

$$\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix} \longrightarrow \left[\begin{array}{ccc|c}1&2&3&0\\4&5&6&0\\7&8&9&0\end{array}\right] \xrightarrow[\text{Row 3}-7\text{Row 1}]{\text{Row 2}-4\text{Row 1}}$$

$$\left[\begin{array}{ccc|c}1&2&3&0\\0&-3&-6&0\\0&-6&-12&0\end{array}\right] \xrightarrow[\text{Row 2}\to -\frac{1}{3}\text{Row 2}]{\text{Row 3}-2\text{Row 2}} \left[\begin{array}{ccc|c}1&2&3&0\\0&1&2&0\\0&0&0&0\end{array}\right] \xrightarrow{\text{Row 1}-2\text{Row 2}} \left[\begin{array}{ccc|c}1&0&-1&0\\0&1&2&0\\0&0&0&0\end{array}\right]$$

We've reduced the system to: $x-z=0$
$y+2z=0$

We have no constraint on z, and then $x=z$, $y=-2z$

We have many solutions: $z=0$: $(x,y,z)=(0,0,0)$
$z=1$: $(x,y,z)=(1,-2,1)$
$z=-1$: $(x,y,z)=(-1,2,-1)$

So A^{-1} doesn't exist.

non-zero solutions

Some properties of inverse matrices:

① If A has an inverse, could it have more than one inverse?

No! To see why, what if B, C were both inverses of A?

Then $AB=I \xrightarrow[\text{C on left}]{\text{Multiply by}} C(AB)=CI=C \xrightarrow{\text{associativity}} \underbrace{(CA)}_{\text{This is also } I!}B=C$

$\longrightarrow IB=C \longrightarrow B=C$

So B and C have to be the same!

② If A and B both have inverses, what about AB?

Turns out, inverse of AB is not $A^{-1}B^{-1}$!

Actually, it's $B^{-1}A^{-1}$. Why?

$$(AB)(B^{-1}A^{-1})=A\underbrace{(BB^{-1})}_{I}A^{-1}=AA^{-1}=I$$

$$(B^{-1}A^{-1})(AB) = B^{-1}\underbrace{(A^{-1}A)}_{I}B = B^{-1}B = I. \checkmark$$

But, $(\underbrace{AB)(A^{-1}}B^{-1}) = ??$

They don't cancel, and we cant switch order.

So: $(AB)^{-1} = B^{-1}A^{-1}$ (if A, B are both invertible).

Now the big question: Can we actually calculate inverse matrices in a practical way?

We need to solve the matrix equation $AX = I$ for the unknown X.

$$A\begin{bmatrix} \vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n \end{bmatrix} = I \longrightarrow \begin{bmatrix} A\vec{x}_1 & A\vec{x}_2 & \cdots & A\vec{x}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

We need to solve n systems of linear equations at the same time:

$$A\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad A\vec{x}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \cdots, \quad A\vec{x}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

same coefficient matrix for each system, can write down a big augmented matrix to solve all systems by elimination at the same time:

$$\left[\begin{array}{c|cccc} & 1 & 0 & \cdots & 0 \\ A & 0 & 1 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ & 0 & 0 & \cdots & 1 \end{array}\right] \xrightarrow[\text{elimination}]{\text{Apply}} \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & A^{-1} \\ \vdots & & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \end{array}\right]$$

solutions for $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ on the right side will be the columns of the inverse matrix.

Example Find inverse of $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

Solution: Apply elimination on a big augmented matrix.

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ -1 & 2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row } 2 + \frac{1}{2}\text{Row } 1} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -2 & 1/2 & 1 & 0 \\ 0 & -1 & 2 & 0 & 0 & 1 \end{array}\right]$$

Put identity matrix on the right.

$$\xrightarrow{\text{Row } 3 + \frac{2}{3}\text{Row } 2} \left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & -2 & 1/2 & 1 & 0 \\ 0 & 0 & 2/3 & 1/3 & 2/3 & 1 \end{array}\right] \xrightarrow{\text{Row } 2 + 3\text{Row } 3}$$

$$\left[\begin{array}{ccc|ccc} 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & 3/2 & 3 & 3 \\ 0 & 0 & 2/3 & 1/3 & 2/3 & 1 \end{array}\right] \xrightarrow{\text{Row } 1 + \frac{2}{3}\text{Row } 2} \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 2 & 2 & 2 \\ 0 & 3/2 & 0 & 3/2 & 3 & 3 \\ 0 & 0 & 2/3 & 1/3 & 2/3 & 1 \end{array}\right]$$

$$\xrightarrow[\frac{3}{2}\text{Row } 3]{\frac{1}{2}\text{Row } 1,\ \frac{2}{3}\text{Row } 2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1/2 & 1 & 3/2 \end{array}\right]$$

This is A^{-1}

Good to check: $\underbrace{\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1/2 & 1 & 3/2 \end{bmatrix}}_{A^{-1}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I}$

We have to divide by 2 in some entries; suggests that the "determinant" of A is the <u>non-zero</u> number 2.

Another example: Upper triangular

$$A = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{\text{Do elimination on:}} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & -4 & 1 & 0 & 0 & 0 \\ 0 & -1 & -2 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[\text{Row } 3 - 2\text{Row } 4]{\text{Row } 1 + 4\text{Row } 4,\ \text{Row } 2 - 3\text{Row } 4} \left[\begin{array}{cccc|cccc} 1 & 2 & 3 & 0 & 1 & 0 & 0 & 4 \\ 0 & -1 & -2 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row } 2 - 2\text{Row } 3]{\text{Row } 1 + 3\text{Row } 3}$$

$$\left[\begin{array}{cccc|cccc} 1 & 2 & 0 & 0 & 1 & 0 & +3 & -2 \\ 0 & -1 & 0 & 0 & 0 & 1 & -2 & +1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Then: }-\text{Row 2},\ -\text{Row 3}]{\text{Row1}+2\text{Row 2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

This is A^{-1}. It's still upper triangular.

Good to check:

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{A^{-1}} = I$$

Section 2.6 LU Decomposition

Factorizing a matrix as a product of simpler factors.

$A = LU$

- A: nxn matrix
- L: lower triangular
- U: upper triangular

This comes from linear systems / elimination.

Start with a system of linear equations $\xrightarrow[\text{left variables}]{\text{Eliminate lower}}$ Upper triangular system

$\updownarrow$

Coefficient matrix A $\xrightarrow[\text{matrices (lower-triangular)}]{\text{Multiply by elimination}}$ Upper triangular coefficient matrix U

Solve by back-substitution or more elimination

Example

$$\begin{aligned} 2x - 2y &= 2 \\ -2x - 3y &= 3 \end{aligned} \quad \xrightarrow{\text{Eqn. 2 + Eqn. 1}} \quad \begin{aligned} 2x - 2y &= 2 \\ -5y &= 5 \end{aligned}$$

$\updownarrow$

$$\underbrace{\begin{bmatrix} 2 & -2 \\ -2 & -3 \end{bmatrix}}_{\text{This is } A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \xrightarrow[\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}]{\text{Multiply both sides by}} \begin{bmatrix} 2 & -2 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

That is, $\begin{bmatrix}1&0\\1&1\end{bmatrix}A = U \xrightarrow[\text{for } A]{\text{Solve}} A = \begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1} U$

Elimination matrix $\begin{bmatrix}1&0\\1&1\end{bmatrix}$ does the "Row 2 $\to$ Row 2 + Row 1"

operation, so $\begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1}$ does the inverse row operation that undoes

it: "Row 2 $\to$ Row 2 − Row 1" $\rightsquigarrow \begin{bmatrix}1&0\\1&1\end{bmatrix}^{-1} = \begin{bmatrix}1&0\\-1&1\end{bmatrix} \leftarrow$ This is L.

So the LU decomposition (or factorization) of A is:

$$\underset{A}{\begin{bmatrix}2&-2\\-2&3\end{bmatrix}} = \underset{L}{\begin{bmatrix}1&0\\-1&1\end{bmatrix}} \underset{U}{\begin{bmatrix}2&-2\\0&-5\end{bmatrix}}$$

Basic idea: Elimination from A to an upper triangular system is the same as factoring A ~~into~~ as a product of lower and upper triangular matrices.

3×3 example: Let's eliminate $A = \begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix}$ to upper triangular

$$\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix} \xrightarrow[\text{Row 3}+\frac{1}{2}\text{Row 1}]{\text{Row 2}+\frac{1}{2}\text{Row 1}} \begin{bmatrix}2&-1&-1\\0&3/2&-3/2\\0&-3/2&3/2\end{bmatrix} \xrightarrow{\text{Row 3}+\text{Row 2}} \begin{bmatrix}2&-1&-1\\0&3/2&-3/2\\0&0&0\end{bmatrix}$$

Or: multiply by $\begin{bmatrix}1&0&0\\1/2&1&0\\1/2&0&1\end{bmatrix}$ Or: multiply by $\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}$

Upper triangular now; this is U

$$A \longrightarrow \begin{bmatrix}1&0&0\\1/2&1&0\\1/2&0&1\end{bmatrix} A \longrightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\1/2&1&0\\1/2&0&1\end{bmatrix} A = U$$

$$\text{So } A = \left(\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\1/2&1&0\\1/2&0&1\end{bmatrix}\right)^{-1} U$$

This is L.

$$= \begin{bmatrix}1&0&0\\1/2&1&0\\1/2&0&1\end{bmatrix}^{-1}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}^{-1} U$$

(switch order when you invert a product of matrices)

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} U$$

(Reverse ~~order~~ the row operations to get inverses)

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -1 & 1 \end{bmatrix} = L$$

So
$$\underset{A}{\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ -1/2 & -1 & 1 \end{bmatrix}}\underset{U}{\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3/2 & -3/2 \\ 0 & 0 & 0 \end{bmatrix}}$$

(Check.)