清華大學

Homework 9 Solutions

4.1.6 $\left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 2 & 2 & 3 & b_2 \\ 3 & 4 & 5 & b_3 \end{array}\right] \xrightarrow[\text{Row3}-3\text{Row1}]{\text{Row2}-2\text{Row1}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -2 & -1 & b_2-2b_1 \\ 0 & -2 & -1 & b_3-3b_1 \end{array}\right] \xrightarrow{\text{Row3}-\text{Row2}} \left[\begin{array}{ccc|c} 1 & 2 & 2 & b_1 \\ 0 & -2 & -1 & b_2-2b_1 \\ 0 & 0 & 0 & -b_1-b_2+b_3 \end{array}\right]$

Look at $(-1)(\text{Eqn } 1) + (-1)\text{Eqn } 2 + (1)\text{ Eqn } 3 =$

$$\begin{aligned} -(x+2y+2z) &= -5 \\ -(2x+2y+3z) &= -5 \\ +(3x+4y+5z) &= 9 \\ \hline 0 &= -1 \end{aligned}$$

so $(1)\text{ Eqn } 1 + (1)\text{ Eqn } 2 + (-1)\text{ Eqn. } 3$ gives $0 = 1$.

$\boxed{y_1 = 1,\ y_2 = 1,\ y_3 = -1}$

$\vec{y} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ is in the <u>left</u> <u>nullspace</u> $N(A^T)$ since $\vec{y}^T A = \vec{0}$.

Recall: $C(A) \perp N(A^T)$, so if $\vec{b} = \begin{bmatrix} 5 \\ 5 \\ 9 \end{bmatrix}$ were in $C(A)$, it would be $\perp$ to $\vec{y}$. But since $\vec{y}^T \vec{b} = 1$, it is <u>not</u> in $C(A)$, meaning $A\vec{x} = \vec{b}$ has <u>no solution</u>.

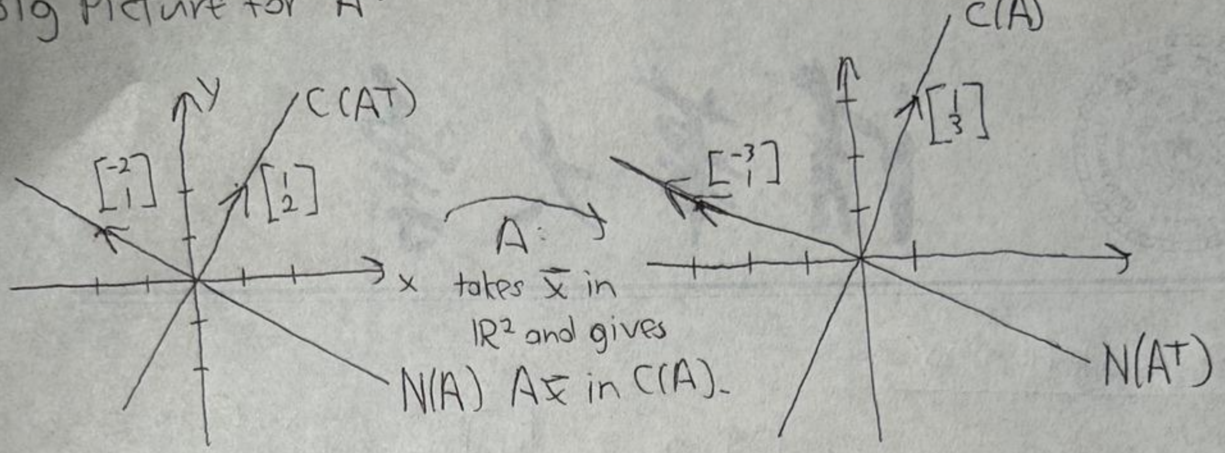
4.1.11 $A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $\quad C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right)$

$N(A)$: $x + 2y = 0 \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Row: $C(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$, $N(A^T)$: $-3\text{ Row } 1 + \text{Row } 2 = 0$

$\rightarrow N(A^T) = \text{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$

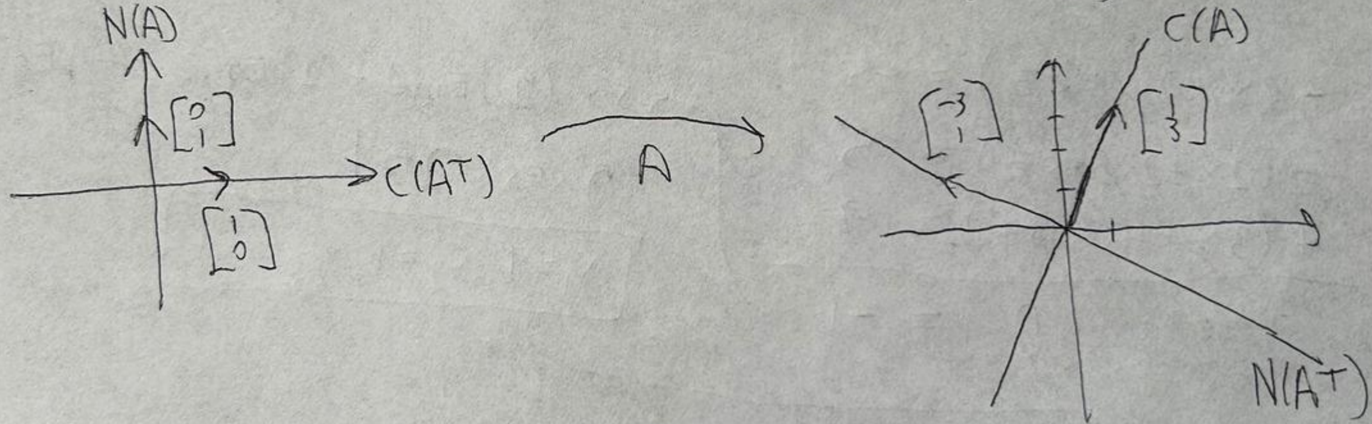
Big Picture for A:



$$B = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix} \xrightarrow[-3\text{Row }1]{\text{Row }2} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) \qquad N(A): x = 0 \rightsquigarrow \text{span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Row: $C(A^T) = \text{span}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$, $N(A^T)$: Row 3 -3Row 1 $= 0$

So $N(A^T) = \text{span}\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right)$



<u>4.1.17</u> If $S = \{\vec{0}\}$: $S^\perp$ = all $\vec{x}$ such that $\vec{x} \cdot \vec{0} = 0$

$= \underline{\text{all}}\ \vec{x}$ in $\mathbb{R}^3 \longrightarrow \boxed{S^\perp = \mathbb{R}^3}$

If $S = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)$: $S^\perp$ = all $\vec{x}$ such that $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \longrightarrow$

$x = -y - z \longrightarrow$ all $\begin{bmatrix} -y-z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Basis for $S^\perp$

So $S^\perp = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right)$

If $S = \text{span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\-1\end{bmatrix}\right)$: $S^\perp$ = all $\vec{x}$ such that $\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\vec{x} = 0$ <u>and</u>

$\begin{bmatrix}1\\1\\-1\end{bmatrix}\cdot\vec{x} = 0$. $\rightsquigarrow \begin{bmatrix}1 & 1 & 1\\1 & 1 & -1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \vec{0}$ Null space of $\begin{bmatrix}1 & 1 & 1\\1 & 1 & -1\end{bmatrix}$:

$$\begin{bmatrix}1 & 1 & 1\\1 & 1 & -1\end{bmatrix} \xrightarrow[\text{Row 2}]{\text{Row 1} -} \begin{bmatrix}1 & 1 & 1\\0 & 0 & -1\end{bmatrix} \xrightarrow[-\text{Row 2}]{\text{Row 1}+\text{Row 2}} \begin{bmatrix}1 & 1 & 0\\0 & 0 & 1\end{bmatrix} \longrightarrow \begin{matrix}x+y=0\\z=0\end{matrix}$$

So $S^\perp$ = all $\vec{x}$ $= \begin{bmatrix}-y\\y\\0\end{bmatrix}$ = all $y\begin{bmatrix}-1\\1\\0\end{bmatrix}$.

So $\begin{bmatrix}-1\\1\\0\end{bmatrix}$ is a basis for $S^\perp$ in this case.

<u>4.1.30</u> If $\vec{b}$ is a vector in $C(B)$, then $\vec{b} = B\vec{x}$ for some $\vec{x}$.

So $A\vec{b} = A(B\vec{x}) = (AB)\vec{x} = 0\vec{x} = \vec{0}$, since $AB = 0$, This means $\vec{b}$ is in $N(A)$.

Now: A is 3×4 $\quad \dim N(A) = 4 - \text{rank}(A)$

B is 4×5 $\quad \dim C(B) = \text{rank}(B)$

We also know $C(B)$ is inside $N(A)$, so

$\dim C(B) \leq \dim N(A)$

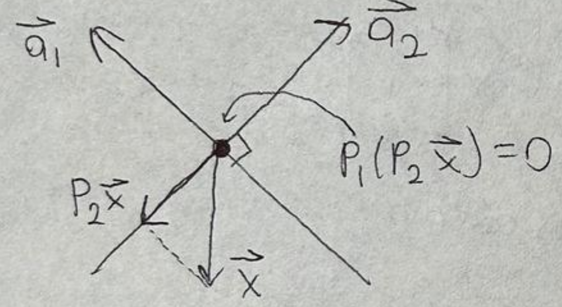
so $\text{rank } B \leq 4 - \text{rank}(A) \longrightarrow \boxed{\text{rank } A + \text{rank } B \leq 4}$

4.2.5 $\vec{a}_1 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, $P_1 = \dfrac{\vec{a}_1 \vec{a}_1^T}{\vec{a}_1^T \vec{a}_1} = \dfrac{\begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 2 \end{bmatrix}}{\begin{bmatrix} -1 & 2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}} = \dfrac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix}$

$\vec{a}_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $P_2 = \dfrac{\vec{a}_2 \vec{a}_2^T}{\vec{a}_2^T \vec{a}_2} = \dfrac{\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 2 & -1 \end{bmatrix}}{\begin{bmatrix} 2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}} = \dfrac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix}$

So $P_1 P_2 = \dfrac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \dfrac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} = \dfrac{1}{81} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

Why is $P_1 P_2 = 0$? Because $\vec{a}_1 \perp \vec{a}_2$:



$P_1 P_2$ means project to span$(\vec{a}_2)$ first, then project the result to span$(\vec{a}_1)$. But since $\vec{a}_1 \perp \vec{a}_2$, the result in span$(\vec{a}_2)$ is $\perp$ to $\vec{a}_1$, and gets projected to $\vec{0}$ by P_1. I.e., $P_1 P_2 \vec{x} = \vec{0}$ for all $\vec{x}$, which means $P_1 P_2 = 0$.

4.2.6 Project onto span$(\vec{a}_1)$: $P_1 \vec{b} = \dfrac{1}{9} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 4 & 4 \\ -2 & 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/9 \\ -2/9 \\ -2/9 \end{bmatrix} = \vec{p}_1$

Project onto span$(\vec{a}_2)$: $P_2 \vec{b} = \dfrac{1}{9} \begin{bmatrix} 4 & 4 & -2 \\ 4 & 4 & -2 \\ -2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/9 \\ 4/9 \\ -2/9 \end{bmatrix} = \vec{p}_2$

Project onto span$(\vec{a}_3)$: $\vec{p}_3 = \dfrac{\vec{a}_3 \vec{a}_3^T}{\vec{a}_3^T \vec{a}_3} \vec{b} = \underbrace{\dfrac{1}{9} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & -1 & 2 \end{bmatrix}} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 4/9 \\ -2/9 \\ 4/9 \end{bmatrix}$

So $\vec{p}_1 + \vec{p}_2 + \vec{p}_3 =$

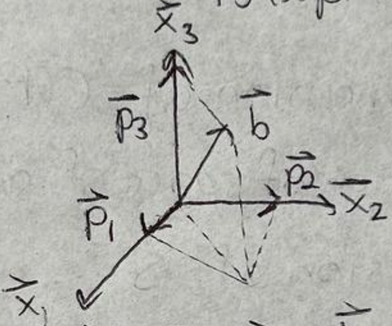
$\dfrac{1}{9} \left(\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ 4 \\ -2 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 4 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ For 4.2.7: $P_3 = \dfrac{1}{9} \begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$

4.2.7 $P_1+P_2+P_3=\frac{1}{9}\begin{bmatrix}1&-2&-2\\-2&4&4\\-2&4&4\end{bmatrix}+\frac{1}{9}\begin{bmatrix}4&4&-2\\4&4&-2\\-2&-2&1\end{bmatrix}+\frac{1}{9}\begin{bmatrix}4&-2&4\\-2&1&-2\\4&-2&4\end{bmatrix}$

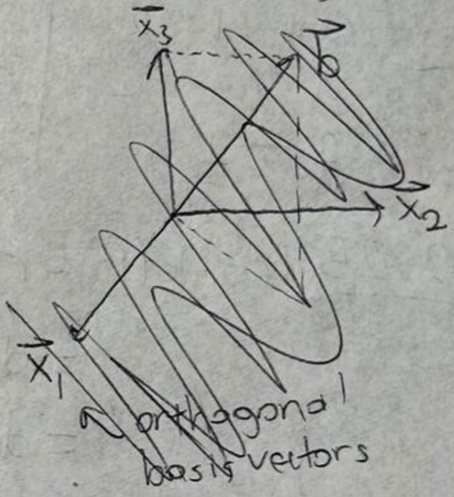
$=\frac{1}{9}\begin{bmatrix}9&0&0\\0&9&0\\0&0&9\end{bmatrix}=I$

General picture to explain:

Orthogonal basis



$\vec{p}_1+\vec{p}_2+\vec{p}_3=\vec{b}$



4.2.19 Plane $x-y-2z=0 \rightsquigarrow x=y+2z$

Plane $=$ all $\begin{bmatrix}y+2z\\y\\z\end{bmatrix}=$ all $y\begin{bmatrix}1\\1\\0\end{bmatrix}+z\begin{bmatrix}2\\0\\1\end{bmatrix}$, so $A=\begin{bmatrix}1&2\\1&0\\0&1\end{bmatrix}$, and

$P=A(A^TA)^{-1}A^T=\begin{bmatrix}1&2\\1&0\\0&1\end{bmatrix}\frac{1}{6}\begin{bmatrix}5&-2\\-2&2\end{bmatrix}\begin{bmatrix}1&1&0\\2&0&1\end{bmatrix}=\begin{bmatrix}1&2\\1&0\\0&1\end{bmatrix}\frac{1}{6}\begin{bmatrix}1&5&-2\\2&-2&2\end{bmatrix}$

$=\boxed{\frac{1}{6}\begin{bmatrix}5&1&2\\1&5&-2\\2&-2&2\end{bmatrix}}$

$\left(\begin{bmatrix}1&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&2\\1&0\\0&1\end{bmatrix}\right)^{-1}=\begin{bmatrix}2&2\\2&5\end{bmatrix}^{-1}=\frac{1}{6}\begin{bmatrix}5&-2\\-2&2\end{bmatrix}$

4.2.20 $\vec{e} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$ is $\perp$ to the plane $\rightsquigarrow$

$$Q = \frac{\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}\begin{bmatrix} 1 & -1 & -2 \end{bmatrix}}{\begin{bmatrix} 1 & -1 & -2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}} = \frac{1}{6}\begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ -2 & 2 & 4 \end{bmatrix}. \text{ Then } P = I - Q =$$

$$\frac{1}{6}\begin{bmatrix} 6-1 & 1 & 2 \\ 1 & 6-1 & -2 \\ 2 & -2 & 6-4 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 5 & 1 & 2 \\ 1 & 5 & -2 \\ 2 & -2 & 2 \end{bmatrix}$$ (same as before)

4.2.25 $\text{rank}(P) = r =$ dimension of $C(P)$

$=$ dimension of space of all $P\vec{b}$

$=$ dimension of S

$= n$

Why are these the same? Well, $P\vec{b}$ is in S for any $\vec{b}$, and on the other hand, if $\vec{b}$ is already in S, then $P\vec{b} = \vec{b}$, so S is also contained in the set of all vectors $P\vec{b}$.

4.2.34 P is a projection matrix if and only if $P = P^T$ and $P^2 = P$.

What about P_1P_2? First, suppose $P_1P_2 = P_2P_1$. Then

$$(P_1P_2)^T = P_2^T P_1^T = P_2P_1 = P_1P_2 \checkmark$$

P_1 and P_2 are projections

and $(P_1P_2)^2 = P_1P_2P_1P_2 = P_1P_1P_2P_2 = P_1P_2 \checkmark$

So P_1P_2 is a projection.

on the other hand, suppose P_1P_2 is a projection matrix. Then

$(P_1P_2) = (P_1P_2)^T = P_2^T P_1^T = P_2P_1$. So P_1 and P_2 commute. $\checkmark$

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<u>Graded Problem 1</u> If $V = \text{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, then $V^\perp$ = all $\vec{x}$ such that

$$\begin{cases} \vec{v}_1^T\vec{x} = 0 \\ \vec{v}_2^T\vec{x} = 0 \\ \vec{v}_3^T\vec{x} = 0 \end{cases} = \text{all } \vec{x} \text{ such that } \begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix}\vec{x} = \vec{0} = \text{null space of}$$

$$\begin{bmatrix} \vec{v}_1^T \\ \vec{v}_2^T \\ \vec{v}_3^T \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 6 \\ 1 & 2 & 1 & 2 & 3 \end{bmatrix} \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 2} - \text{Row 1}} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 & -2 \end{bmatrix} \xrightarrow{\text{Row 3} + 2\,\text{Row 2}}$$

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{\text{Row 1} - 3\,\text{Row 2}} \begin{bmatrix} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} x_1 + 2x_2 + x_4 + 2x_5 = 0 \\ x_3 + x_4 + x_5 = 0, \\ x_2, x_4, x_5 \text{ free} \end{array}$$

$$V^\perp = \text{all} \begin{bmatrix} -2x_2 - x_4 - 2x_5 \\ x_2 \\ -x_4 - x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Basis for $V^\perp$

<u>Graded Problem 2</u> $P = A(A^TA)^{-1}A^T$ where $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}$

$$\left(\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} 2 & 4 \\ 4 & 10 \end{bmatrix}^{-1} = \frac{1}{4}\begin{bmatrix} 10 & -4 \\ -4 & 2 \end{bmatrix} = \begin{bmatrix} 5/2 & -1 \\ -1 & 1/2 \end{bmatrix}$$

$$\text{So } P = \begin{bmatrix} 1 & 2 \\ 0 & -1 \\ 1 & 2 \\ 0 & -1 \end{bmatrix} \underbrace{\begin{bmatrix} 5/2 & -1 \\ -1 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & 2 & -1 \end{bmatrix}}_{\begin{bmatrix} 1/2 & 1 & 1/2 & 1 \\ 0 & -1/2 & 0 & -1/2 \end{bmatrix}} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$$

$$\text{Project } \vec{x}: \quad P\vec{x} = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} = \vec{p}$$

$$\text{Error: } \vec{e} = \vec{x} - \vec{p} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix},$$

$$\|\vec{e}\| = \sqrt{(-1)^2 + (-1)^2 + 1^2 + 1^2} = 2$$