

# Homework 3 Solutions

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2.1.4 If  $z=2$ , then  $(x,y)$  satisfy  $\begin{cases} x+y+3(2)=6 \\ x-y+2=4 \end{cases}$

$\rightarrow \begin{cases} x+y=0 \\ x-y=2 \end{cases} \xrightarrow{\text{Add}} 2x=2 \rightarrow x=1$   
 $\xrightarrow{\text{Subtract}} 2y=-2 \rightarrow y=-1 \rightarrow \boxed{(x,y,z)=(1,-1,2)}$

If  $z=0$ , then  $(x,y)$  satisfy  $\begin{cases} x+y+3(0)=6 \\ x-y+0=4 \end{cases} \rightarrow \begin{cases} x+y=6 \\ x-y=4 \end{cases}$

$\xrightarrow{\text{Add}} 2x=10 \rightarrow x=5$   
 $\xrightarrow{\text{Subtract}} 2y=2 \rightarrow y=1 \rightarrow \boxed{(x,y,z)=(5,1,0)}$

Third point halfway in between should have average  $(x,y,z)$

coordinates:  $(x,y,z) = \frac{(1,-1,2) + (5,1,0)}{2} = \frac{(6,0,2)}{2} = \boxed{(3,0,1)}$

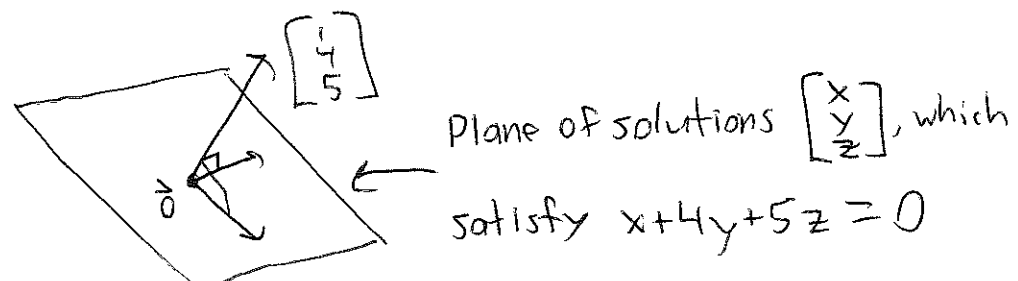
2.1.17  $\begin{bmatrix} y \\ z \\ x \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_P \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_Q \begin{bmatrix} y \\ z \\ x \end{bmatrix}$

(P and Q are inverses of each other)

2.1.22  $(1,4,5) \cdot (x,y,z) = 1x+4y+5z = \underbrace{\begin{bmatrix} 1 & 4 & 5 \end{bmatrix}}_A \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{\vec{x}}$

The solutions to  $A\vec{x} = \vec{0}$  lie on a plane perpendicular to the vector  $\begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

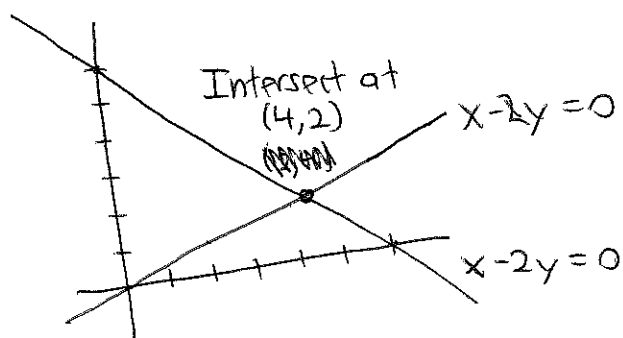
The columns of A are vectors in only 1-dimensional space.



2.1.26  $\begin{cases} x - 2y = 0 & (y = \frac{1}{2}x) \\ x + y = 6 & (y = 6 - x) \end{cases}$  subtract  $\rightarrow -3y = -6 \rightarrow y = 2$   
 $x = 2(2) = 4$

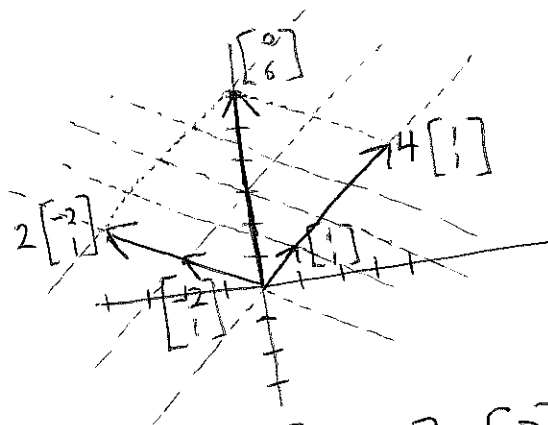
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Row Picture



Column picture:

$$4 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



2.1.29  $\vec{u}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ ,  $\vec{u}_1 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} .8 \\ .2 \end{bmatrix}$ ,  $\vec{u}_2 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .8 \\ .2 \end{bmatrix} = \begin{bmatrix} .7 \\ .3 \end{bmatrix}$

$$\vec{u}_3 = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} .7 \\ .3 \end{bmatrix} = \begin{bmatrix} .65 \\ .35 \end{bmatrix}$$

For all 4 vectors, sum of x and y coordinates = 1

(We can prove this will always work: suppose  $x + y = 1$ . Then

$$\begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .8x + .3y \\ .2x + .7y \end{bmatrix}, \text{ and we still get } (.8x + .3y) + (.2x + .7y)$$

$$= (.8 + .2)x + (.3 + .7)y = 1x + 1y = 1.)$$

2.2.6  $\begin{cases} 2x + by = 16 \\ 4x + 8y = 9 \end{cases} \rightarrow \left[ \begin{array}{cc|c} 2 & b & 16 \\ 4 & 8 & 9 \end{array} \right] \xrightarrow{\text{Row 2} - 2\text{Row 1}} \left[ \begin{array}{cc|c} 2 & b & 16 \\ 0 & 8-2b & g-32 \end{array} \right]$

Singular if  $8 - 2b = 0$ , or  $b = 4$ .

If system is singular, need  $g = 32$  to get solutions.

If  $b = 4$ ,  $g = 32$ , all solutions satisfy  $2x + 4y = 16$ , or  $x = 8 - 2y$

one solution:  $y = 1$ ,  $x = 8 - 2(1) = 6$ ; another:  $y = 2$ ,  $x = 8 - 2(2) = 4$

$$2.2.13 \left[ \begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 4 & -5 & 1 & 7 \\ 2 & -1 & -3 & 5 \end{array} \right] \xrightarrow[\text{from Row 3}]{\substack{\text{Subtract } 2 \cdot \text{Row 1} \\ \text{from Row 2.} \\ \text{Subtract } 1 \cdot \text{Row 1}}} \left[ \begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 2 & -3 & 2 \end{array} \right] \xrightarrow[\text{from Row 3}]{\text{Subtract } 2 \cdot \text{Row 2}} \textcircled{3}$$

$$\left[ \begin{array}{ccc|c} 2 & -3 & 0 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -5 & 0 \end{array} \right] \longrightarrow \begin{array}{l} 2x - 3y = 3 \rightarrow x = \frac{1}{2}(3+3y) = \frac{1}{2}(3+3) = \boxed{3} \\ y + z = 1 \rightarrow y = 1 - z = 1 - 0 = \boxed{1} \\ -5z = 0 \rightarrow z = \boxed{0} \end{array}$$

2.2.18 (This problem has more than one solution.)

~~ANNOYING~~ Rows 2 and 3 should be multiples of Row 1. For example:

$$\begin{array}{rcl} x + 2y + 3z = b_1 & & x + 2y + 3z = b_1 \\ -x - 2y - 3z = b_2 & \xrightarrow{\text{Eqn. 2} + \text{Eqn. 1}} & 0 = b_1 + b_2 \\ 4x + 8y + 12z = b_3 & \xrightarrow{\text{Eqn 3} - 4 \text{ Eqn. 1}} & 0 = -4b_1 + b_3 \end{array}$$

This system has infinitely many solutions if  $b_1 + b_2 = 0$  and  $-4b_1 + b_3 = 0$ .  
It has no solutions if  $b_1 + b_2 \neq 0$  or  $-4b_1 + b_3 \neq 0$ .

So to get solutions, we need  $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = b_1 \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 8 \\ 8 \\ 0 \end{bmatrix}$  satisfies this, but

$\begin{bmatrix} 1 \\ 1 \\ 100 \end{bmatrix}$  does not.

$$2.2.21 \left[ \begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array} \right] \xrightarrow{\text{Row 2} - \frac{1}{2} \text{Row 1}} \left[ \begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array} \right] \xrightarrow{\text{Row 3} - \frac{2}{3} \text{Row 2}}$$

$$\left[ \begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 1 & 0 \\ 0 & 0 & 1 & 2 & 5 \end{array} \right] \xrightarrow{\text{Row 4} - \frac{3}{4} \text{Row 3}} \left[ \begin{array}{cccc|c} 2 & 1 & 0 & 0 & 0 \\ 0 & 3/2 & 1 & 0 & 0 \\ 0 & 0 & 4/3 & 1 & 0 \\ 0 & 0 & 0 & 5/4 & 5 \end{array} \right]$$

Now can use back substitution:

$$\frac{5}{4}t = 5 \rightarrow \boxed{t = 4}$$

$$\frac{4}{3}z + t = 0 \rightarrow z = \frac{3}{4}(-4) = \boxed{-3}, \quad \frac{3}{2}y + z = 0 \rightarrow y = \frac{2}{3}(+3) = \boxed{2},$$

$$2x + y = 0 \rightarrow x = \frac{1}{2}(-2) = \boxed{-1}, \quad \boxed{(x, y, z, t) = (-1, 2, -3, 4)}$$

$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 0 \\ -1 & 2 & -1 & 0 & | & 0 \\ 0 & -1 & 2 & -1 & | & 0 \\ 0 & 0 & -1 & 2 & | & 5 \end{bmatrix} \xrightarrow{\text{Row 2} + \frac{1}{2} \text{Row 1}} \begin{bmatrix} 2 & -1 & 0 & 0 & | & 0 \\ 0 & 3/2 & -1 & 0 & | & 0 \\ 0 & -1 & 2 & -1 & | & 0 \\ 0 & 0 & -1 & 2 & | & 5 \end{bmatrix} \xrightarrow{\text{Row 3} + \frac{2}{3} \text{Row 2}}$$

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$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 0 \\ 0 & 3/2 & -1 & 0 & | & 0 \\ 0 & 0 & 4/3 & -1 & | & 0 \\ 0 & 0 & -1 & 2 & | & 5 \end{bmatrix} \xrightarrow{\text{Row 4} + \frac{3}{4} \text{Row 3}} \begin{bmatrix} 2 & -1 & 0 & 0 & | & 0 \\ 0 & 3/2 & -1 & 0 & | & 0 \\ 0 & 0 & 4/3 & -1 & | & 0 \\ 0 & 0 & 0 & 5/4 & | & 5 \end{bmatrix} \xrightarrow{\text{Row 3} + \frac{4}{5} \text{Row 4}}$$

Continue elimination this time:

$$\begin{bmatrix} 2 & -1 & 0 & 0 & | & 0 \\ 0 & 3/2 & -1 & 0 & | & 0 \\ 0 & 0 & 4/3 & 0 & | & 4 \\ 0 & 0 & 0 & 5/4 & | & 5 \end{bmatrix} \xrightarrow{\text{Row 2} + \frac{3}{4} \text{Row 3}} \begin{bmatrix} 2 & -1 & 0 & 0 & | & 0 \\ 0 & 3/2 & 0 & 0 & | & 3 \\ 0 & 0 & 4/3 & 0 & | & 4 \\ 0 & 0 & 0 & 5/4 & | & 5 \end{bmatrix} \xrightarrow{\text{Row 1} + \frac{2}{3} \text{Row 2}} \begin{bmatrix} 2 & 0 & 0 & 0 & | & 2 \\ 0 & 3/2 & 0 & 0 & | & 3 \\ 0 & 0 & 4/3 & 0 & | & 4 \\ 0 & 0 & 0 & 5/4 & | & 5 \end{bmatrix}$$

$$\begin{array}{l} \frac{1}{2} \text{Row 4} \\ \frac{2}{3} \text{Row 2} \\ \frac{3}{4} \text{Row 3} \\ \frac{4}{5} \text{Row 4} \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & 0 & | & 3 \\ 0 & 0 & 0 & 1 & | & 4 \end{bmatrix}$$

$$(x, y, z, t) = (1, 2, 3, 4)$$

### Graded Problem

(a) Matrix-vector equation:

$$\begin{bmatrix} 2 & -2 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Vector equation:

$$x \begin{bmatrix} 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$(b) \begin{bmatrix} 2 & -2 & | & 1 & 0 \\ -1 & 2 & | & 0 & 1 \end{bmatrix} \xrightarrow{\text{Row 1} + \frac{1}{2} \text{Row 2}} \begin{bmatrix} 2 & -2 & | & 1 & 0 \\ 0 & 1 & | & \frac{1}{2} & 1 \end{bmatrix} \xrightarrow{\text{Row 1} + 2 \text{Row 2}} \begin{bmatrix} 2 & 0 & | & 2 & 2 \\ 0 & 1 & | & \frac{1}{2} & 1 \end{bmatrix}$$

We can solve both cases for  $\vec{b}$  at the same time this way

$\downarrow (1/2) \text{Row 1}$

$$\begin{bmatrix} 1 & 0 & | & 1 & 1 \\ 0 & 1 & | & \frac{1}{2} & 1 \end{bmatrix}$$

$$\text{If } \vec{b} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \text{ then } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

$$\text{If } \vec{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \text{ then } \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \text{ so } \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$