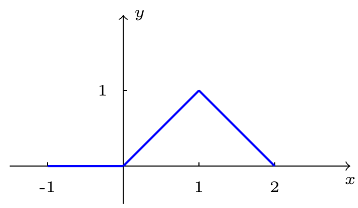
Solution of Midterm 2023

Part 1.

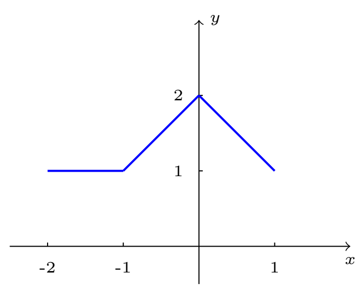
1. D
2. B
3. C
4. A

Part 2a.

1.



2.



Part 2b.

1. *Proof.*
$$\tan 2x = \frac{\sin 2x}{\cos 2x} = \frac{2\sin x \cos x}{\cos^2 x - \sin^2 x} = \frac{2\tan x}{1 - \tan^2 x}.$$
$\square$

2. *Solution.* Let $x = \tan\frac{\pi}{8}$, so we have $1 = \tan\frac{\pi}{4} = \frac{2x}{1-x^2}$, namely $x^2 + 2x - 1 = 0$. Then we get $x = \sqrt{2} - 1$ or $x = -\sqrt{2} - 1$. Finally, we get $\tan\frac{\pi}{8} = \sqrt{2} - 1$, since $\tan\frac{\pi}{8} > 0$. $\square$

Part 2c.

Solution. We have

$$f'(x) = \frac{2}{3}(1+x)^{-\frac{1}{3}},$$

so $f'(0) = \frac{2}{3}$ and $f(0) = 1$. And tangent line is $y = \frac{2}{3}x + 1$.

□

Part 2d.

Solution.

$$\lim_{x\to 0} \frac{(1+x+\sin x)^{\frac{1}{3}} - (1+\sin x)^{\frac{1}{3}}}{x}$$

$$\begin{aligned} &= \lim_{x\to 0} \frac{(1+x+\sin x) - (1+\sin x)}{x[(1+x+\sin x)^{\frac{2}{3}} + (1+x+\sin x)^{\frac{1}{3}}(1+\sin x)^{\frac{1}{3}} + (1+\sin x)^{\frac{2}{3}}]} \\ &= \frac{1}{3}. \end{aligned}$$

□

Remark. Some student use approximation $(1+x)^k \approx 1+kx$ when $x \to 0$ and give the answer:

$$\begin{aligned} \lim_{x\to 0} \frac{(1+x+\sin x)^{\frac{1}{3}} - (1+\sin x)^{\frac{1}{3}}}{x} &= \lim_{x\to 0} \frac{\frac{1}{3}(1+x+\sin x) - \frac{1}{3}(1+\sin x)}{x} \\ &= \frac{1}{3}. \end{aligned}$$

This solution is not strict, but we also accept it.
If you use any of the tools you haven't seen in class (such as L' Hospital rule) and you get the correct answer, you'll get half the marks (10pts).

Part 2e.

Proof. Let $g(x) = f(x) - x^2$, $g(x)$ is continuous on $[0, 2]$ since $f(x)$ is continuous on it. And $g(0) = f(0) > 0$, $g(2) = f(2) - 4 < 0$, since the Intermediate Value Property, there exists a point $c \in [0, 2]$, $g(c) = 0$, namely $f(c) = c^2$. □

Part 3.

Solution. Let

$$f(x) = \begin{cases} -2x + 2\sqrt{\pi} & x \le \sqrt{\pi} \\ \frac{\sin x^2}{x} & x > \sqrt{\pi} \end{cases}$$

It is easy to check $\lim\limits_{x\to+\infty} f(x) = 0$, then we show $f(x)$ is differentiable on $\mathbb{R}$. We will only show $f(x)$ is differentiable at $x = \sqrt{\pi}$, because it's easy to check $f(x)$ is differentiable on $\mathbb{R} \setminus \{\pi\}$.

We have

$$\lim_{x\to\sqrt{\pi}^-}\frac{f(x)-f(\sqrt{\pi})}{x-\sqrt{\pi}}=\lim_{x\to\sqrt{\pi}^-}\frac{-2(x-\sqrt{\pi})}{x-\sqrt{\pi}}=-2$$

and

$$\begin{aligned}\lim_{x\to\sqrt{\pi}^+}\frac{f(x)-f(\sqrt{\pi})}{x-\sqrt{\pi}}&=\lim_{x\to\sqrt{\pi}^+}\frac{\sin x^2}{x(x-\sqrt{\pi})}\\&=\lim_{x\to\sqrt{\pi}^+}-\frac{\sin(x^2-\pi)}{x(x-\sqrt{\pi})}\\&=\lim_{x\to\sqrt{\pi}^+}-\frac{(x^2-\pi)}{x(x-\sqrt{\pi})}\\&=-2\end{aligned}$$

So that $f(x)$ is differentiable on $\mathbb{R}$, but $f'(x)=-\frac{\sin x^2}{x^2}+2\cos x^2$, $\lim\limits_{x\to+\infty}f'(x)$ does not exist.

$\square$