Midterm Exam Solutions

1. (a) (10 points) Find all solutions of the system of linear equations:

$$\begin{array}{rcrcrcrcr} x_1 & + & & & 2x_3 & + & 4x_4 & = & -8 \\ & & x_2 & - & 3x_3 & - & x_4 & = & 6 \\ 3x_1 & + & 4x_2 & - & 6x_3 & + & 8x_4 & = & 0 \\ & - & x_2 & + & 3x_3 & + & 4x_4 & = & -12 \end{array}$$

(b) (4 points) Identify the reduced row echelon form R of the coefficient matrix of the system.

Augmented matrix:

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 3 & 4 & -6 & 8 & 0 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \xrightarrow{\text{Row 3} - 3\,\text{Row 1}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 4 & -12 & -4 & 24 \\ 0 & -1 & 3 & 4 & -12 \end{array}\right] \xrightarrow[\text{Row 4} + \text{Row 2}]{\text{Row 3} - 4\,\text{Row 2}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & -6 \end{array}\right] \xrightarrow[\text{Then: Row 3} \leftrightarrow \text{Row 3}]{\frac{1}{3}\text{Row 4}} \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & -8 \\ 0 & 1 & -3 & -1 & 6 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[\text{Row 2} + \text{Row 3}]{\text{Row 1} - 4\,\text{Row 3}}$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & -3 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightsquigarrow \begin{array}{l} x_1 + 2x_3 = 0 \\ x_2 + 3x_3 = 4 \\ x_4 = -2 \\ x_3 \text{ free} \end{array}$$

(a) All solutions: $\vec{x} = \begin{bmatrix} -2x_3 \\ 4+3x_3 \\ x_3 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 0 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ +3 \\ 1 \\ 0 \end{bmatrix}$ for any x_3

(b) $R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

2. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}.$$

(a) (6 points) Find the LU decomposition of A.

(b) (6 points) Use the LU decomposition to solve the linear system of equations $A\mathbf{x} = (1, 0, 0)$.

(a) $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \xrightarrow[\text{Row 3} - \text{Row 1}]{\text{Row 2} - \text{Row 1}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{bmatrix} \xrightarrow{\text{Row 3} - \boxed{3}\,\text{Row 2}} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ U

$$\boxed{A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}}_{U}}$$

(b) Solve $L(\underbrace{U\vec{x}}_{\vec{y}}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$L\vec{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightsquigarrow$
$$\begin{aligned} y_1 &= 1 \\ y_1 + y_2 &= 0 \\ y_1 + 3y_2 + y_3 &= 0 \end{aligned}$$

$y_1 = 1$, $y_2 = -y_1 = -1$

$y_3 = -y_1 - 3y_2 = -1 + 3 = 2$

$\vec{y} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

Now solve $U\vec{x} = \vec{y}$:

$x_1 + x_2 + x_3 = 1 \to x_1 = 1 - (-3) - 1 = 3$

$x_2 + 2x_3 = -1 \to x_2 = -1 - 2 = -3$

$2x_3 = 2 \to x_3 = 1$

$$\boxed{\vec{x} = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}}$$

3. (a) (12 points) Find the inverse of the matrix:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

(b) (2 points) Use A^{-1} to solve the system of equations $A\mathbf{x} = (0, 1, 0, 0)$.

(a)
$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row4}-\text{Row1}]{\text{Row3}-\text{Row1}} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row 3}+\text{Row 2}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[\text{Row3}+\text{Row4}]{\text{Row 2}+2\text{Row 4}} \left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & -2 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row 2}+\text{Row 3}}$$

$$\left[\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -4 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{\text{Row 1}-\text{Row 2}} \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 5 & -2 & -1 & -3 \\ 0 & 1 & 0 & 0 & -4 & 2 & 1 & 3 \\ 0 & 0 & 1 & 0 & -2 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right]$$

(a) $$A^{-1} = \begin{bmatrix} 5 & -2 & -1 & -3 \\ -4 & 2 & 1 & 3 \\ -2 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}$$

(b) $$\vec{x} = A^{-1}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -2 & -1 & -3 \\ -4 & 2 & 1 & 3 \\ -2 & 1 & 1 & 1 \\ -1 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

4. (a) (5 points) How long is the vector $\mathbf{v} = (1, 1, \ldots, 1)$ in 9 dimensions? Find a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$ and a unit vector $\mathbf{w}$ that is perpendicular to $\mathbf{v}$.

(b) (5 points) Suppose x, y, and z are any non-zero numbers such that $x + y + z = 0$. Find the angle between $\mathbf{v} = (x, y, z)$ and the vector $\mathbf{w} = (z, x, y)$.

(a) $\|\vec{v}\| = \sqrt{\underbrace{1^2 + 1^2 + \cdots + 1^2}_{9\text{ times}}} = \sqrt{9} = \boxed{3}$

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}(1, 1, \ldots, 1) = \boxed{\left(\frac{1}{3}, \frac{1}{3}, \ldots, \frac{1}{3}\right)}$$

For $\vec{w}$: $(1, -1, 1, -1, \underbrace{0, \ldots, 0}_{5\text{ times}}) \perp \vec{v}$ because

$(1, 1, \ldots, 1) \cdot (1, -1, 1, -1, 0, \ldots, 0) = 1 - 1 + 1 - 1 = 0$. So we could take:

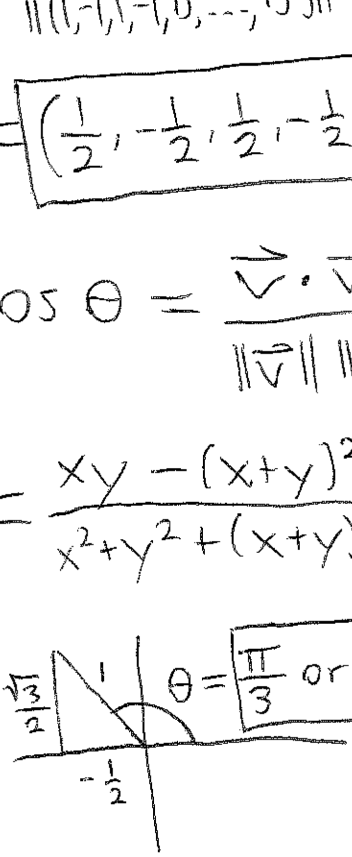
$$\vec{w} = \frac{1}{\|(1,-1,1,-1,0,\ldots,0)\|}(1,-1,1,-1,0,\ldots,0) = \frac{1}{2}(1,-1,1,-1,0,\ldots,0)$$

$$= \boxed{\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, 0, \ldots, 0\right)}$$

(b) $$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\,\|\vec{w}\|} = \frac{xz + yx + zy}{\sqrt{x^2+y^2+z^2}\sqrt{z^2+x^2+y^2}} = \frac{xy + (x+y)z}{x^2+y^2+z^2}$$

$$= \frac{xy - (x+y)^2}{x^2+y^2+(x+y)^2} = \frac{-x^2 - xy - y^2}{2x^2 + 2xy + 2y^2} = -\frac{1}{2}$$

$\theta = \boxed{\frac{\pi}{3} \text{ or } 120^\circ}$



5. Suppose that Q is any $n \times n$ matrix such that $Q^T = Q^{-1}$.

(a) (5 points) Show that the columns $\mathbf{q}_1, \mathbf{q}_2, \ldots, \mathbf{q}_n$ are all unit vectors, and show that if $\mathbf{q}_i$ and $\mathbf{q}_j$ are two different columns of Q, then $\mathbf{q}_i$ and $\mathbf{q}_j$ are perpendicular.

(b) (5 points) For any angle θ, find a 2×2 example $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ such that $Q^T = Q^{-1}$ and $q_{11} = \cos\theta$.

See Homework 6 Solutions, Problem 2.7.39

6. Let S be the set of all 3×3 matrices A such that

$$A^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A$$

(a) (7 points) Show that S is a subspace of the vector space $\mathbb{R}^{3\times 3}$ of all 3×3 matrices. (Verify all three properties of a subspace.)

(b) (7 points) Find a spanning set for S that contains three matrices.

Let's write $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, So S = all 3×3 A such that

$$A^T C = -CA$$

(a) 1. Is the zero matrix $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ in S?

Yes! $0^T C = 0 \cdot C = 0 = -C \cdot 0$

2. If A and B are in S, is $A+B$ also in S?

Yes! $(A+B)^T C = (A^T + B^T) C = A^T C + B^T C$

$$= -CA - CB = -C(A+B)$$

3. If A is in S, is cA in S for any scalar c?

Yes! $(cA)^T C = c(A^T C) = c(-CA) = -C(cA)$

(b) Suppose $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is in S.

$$A^T C = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{bmatrix}$$

$$-CA = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} -a_1 & -b_1 & -c_1 \\ -a_3 & -b_3 & -c_3 \\ -a_2 & -b_2 & -c_2 \end{bmatrix}$$

For these to be the same:

$$\begin{aligned} &a_1 = -a_1, \quad a_3 = -b_1, \quad a_2 = -c_1 \\ &b_1 = -a_3, \quad b_3 = -b_3, \quad b_2 = -c_3 \\ &c_1 = -a_2 \quad c_3 = -b_2, \quad c_2 = -c_2 \end{aligned} \longrightarrow \begin{aligned} &a_1 = b_3 = c_2 = 0 \\ &b_1 = -a_3, \quad c_1 = -a_2, \\ &c_3 = -b_2 \end{aligned}$$

$$A = \begin{bmatrix} 0 & -a_3 & -a_2 \\ a_2 & b_2 & 0 \\ a_3 & 0 & -b_2 \end{bmatrix} = a_2 \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} + b_2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(a_2, a_3, b_2 are free)

These three matrices are a spanning set for S.

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~~8. (10 points) Consider a triangle with fixed perimeter P and such that two sides have equal length (an isosceles triangle). Prove that the triangle has maximum area when it is equilateral (that is, all three sides have the same length).~~

$$A = \left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & b_1 \\ -1 & -2 & -1 & 1 & -1 & b_2 \\ 4 & 8 & 5 & -8 & 9 & b_3 \end{array}\right] \xrightarrow[\text{Row 3} - 4\,\text{Row 1}]{\text{Row 2} + \text{Row 1}} \left[\begin{array}{ccccc|c} 1 & 2 & 2 & -5 & 6 & b_1 \\ 0 & 0 & 1 & -4 & 5 & b_1+b_2 \\ 0 & 0 & -3 & 12 & -15 & -4b_1+b_3 \end{array}\right]$$

$$\xrightarrow[\text{Row 1} - 2\,\text{Row 2}]{\text{Row 3} + 3\,\text{Row 2}} \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & -4 & -b_1-2b_2 \\ 0 & 0 & 1 & -4 & 5 & b_1+b_2 \\ 0 & 0 & 0 & 0 & 0 & -b_1+3b_2+b_3 \end{array}\right]$$

(a) $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is in the column space if $-b_1+3b_2+b_3=0$

(b) Null space equations $(b_1=b_2=b_3=0)$:

$x_1+2x_2+3x_4-4x_5=0$

$x_3-4x_4+5x_5=0$

x_2, x_4, x_5 free

$$\rightarrow \vec{x} = \begin{bmatrix} -2x_2-3x_4+4x_5 \\ x_2 \\ 4x_4-5x_5 \\ x_4 \\ x_5 \end{bmatrix} =$$

$$= x_2\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4\begin{bmatrix} -3 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} + x_5\begin{bmatrix} 4 \\ 0 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

the 3 special solutions

8. (a) (8 points) Determine whether the following vectors in $\mathbb{R}^4$ are linearly independent:

$$\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 11 \\ -3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}.$$

If the vectors are *not* linearly independent, show how to write one of the them as a linear combination of the others.

(b) (4 points) Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are non-zero vectors which are all perpendicular to each other, that is, $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set.

(a) $$\begin{bmatrix} -1 & -1 & 3 & 1 \\ 1 & 3 & 1 & 2 \\ -1 & 3 & 11 & 3 \\ 1 & 1 & -3 & 4 \end{bmatrix} \xrightarrow[\substack{\text{Row 3} - \text{Row 1} \\ \text{Row 4} + \text{Row 1}}]{\text{Row 2} + \text{Row 1}} \begin{bmatrix} -1 & -1 & 3 & 1 \\ 0 & 2 & 4 & 3 \\ 0 & 4 & 8 & 2 \\ 0 & 0 & 0 & 5 \end{bmatrix} \xrightarrow[-\text{Row 1}]{\text{Row 3} - 2\text{ Row 2}}$$

$$\begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 2 & 4 & 3 \\ 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 5 \end{bmatrix} \xrightarrow[\frac{1}{5}\text{ Row 4}]{\substack{\frac{1}{2}\text{ Row 2} \\ -\frac{1}{4}\text{ Row 3}}} \begin{bmatrix} 1 & 1 & -3 & -1 \\ 0 & 1 & 2 & 3/2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{Row 4} - \text{Row 3}]{\substack{\text{Row 1} + \text{Row 3} \\ \text{Row 2} - \frac{3}{2}\text{Row 3}}} \begin{bmatrix} 1 & 1 & -3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\text{Row 1} - \text{Row 2}} \begin{bmatrix} 1 & 0 & -5 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \leftarrow$$

Reduced row echelon for R is <u>not</u> I, so vectors are <u>dependent</u>

R shows that Col 3 = −5 Col 1 + 2 Col 2 :

$$\begin{bmatrix} 3 \\ 1 \\ 11 \\ -3 \end{bmatrix} = -5 \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} -1 \\ 3 \\ 3 \\ 1 \end{bmatrix}$$

(b) Suppose $c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = \vec{0}$. Need to show that $c_1 = c_2 = c_3 = 0$.

$$\vec{v}_1 \cdot (c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3) = \vec{v}_1 \cdot \vec{0} \longrightarrow c_1(\vec{v}_1 \cdot \vec{v}_1) + c_2(\vec{v}_1 \cdot \vec{v}_2) + c_3(\vec{v}_1 \cdot \vec{v}_3) = 0$$

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Since $\vec{v}_1 \cdot \vec{v}_2 = 0 = \vec{v}_1 \cdot \vec{v}_3 = 0$, we get $c_1 \|\vec{v}_1\|^2 = 0$

Since $\vec{v}_1 \neq 0$, $\|\vec{v}_1\|^2 \neq 0$ either, so $c_1 = 0$.

We can similarly dot both sides of equation

$c_1 \vec{v}_1 + c_2 \vec{v}_2 + c_3 \vec{v}_3 = \vec{0}$ with $\vec{v}_2$ and $\vec{v}_3$ to show that

$c_2 = 0$ and $c_3 = 0$ also.

So $c_1 = c_2 = c_3 = 0$, and $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.