

Section 5.3 Cramer's Rule, Inverses, Volumes (1)

Today look at some applications of determinants.
(The big application, eigenvalues, comes next chapter.)

First: Solving $n \times n$ linear systems

$$A\vec{x} = \vec{b} \xrightarrow{\text{arithmetic}} \text{Elimination algorithm}$$

\downarrow algebra

Cramer's Rule (for solving $\vec{x} = A^{-1}\vec{b}$)

Idea:
$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} x_1 & 0 & \dots & 0 \\ x_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \dots & 1 \end{bmatrix} = \begin{bmatrix} A\vec{x} & A \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} & \dots & A \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix} \end{bmatrix}$$

\swarrow should equal \vec{b}

$$= \begin{bmatrix} | & | & & | \\ \vec{b} & \vec{a}_2 & \dots & \vec{a}_n \\ | & | & & | \end{bmatrix} \leftarrow \text{Call this matrix } B_1 \text{ (you replace Col 1 of } A \text{ with } \vec{b}).$$

Now take det of both sides and use product rule:

$$(\det A) \begin{vmatrix} x_1 & 0 & \dots & 0 \\ x_2 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ x_n & 0 & \dots & 1 \end{vmatrix} = \det B_1$$
$$= x_1 (\det \text{ of } (n-1) \times (n-1) I) = x_1$$

So $x_1 = \frac{\det B_1}{\det A}$, if $\det A \neq 0$. ②

Same for x_2, \dots, x_n :

$$\begin{bmatrix} A \end{bmatrix} \begin{bmatrix} \begin{array}{c} 1 \\ \vdots \\ \vec{x} \\ \vdots \\ 0 \end{array} \begin{array}{c} 1 \\ \vdots \\ 1 \\ \vdots \\ -1 \end{array} \begin{array}{c} 0 \\ \vdots \\ \vdots \\ \vdots \\ 1 \end{array} \end{bmatrix} = \begin{bmatrix} \begin{array}{c} 1 \\ \vdots \\ \vec{a}_1 \\ \vdots \\ 1 \end{array} \begin{array}{c} 1 \\ \vdots \\ \vec{b} \\ \vdots \\ 1 \end{array} \begin{array}{c} 1 \\ \vdots \\ \vec{a}_n \\ \vdots \\ 1 \end{array} \end{bmatrix}$$

Put \vec{x} in Col i B_i (replace Col i of A with \vec{b}).

Expand across Row i :

$\det = x_i$

$$x_i = \frac{\det B_i}{\det A}$$

Example: Solve $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

We need to use 4 determinants:

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{vmatrix} = 2$$

$$\det B_1 = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 3 \\ 0 & 4 & 9 \end{vmatrix} \begin{matrix} \text{1st} \\ \text{col} \end{matrix} = - \begin{vmatrix} 1 & 1 \\ 4 & 9 \end{vmatrix} = -5$$

This is the C_{21} cofactor.

$$\det B_2 = \begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 3 \\ 1 & 0 & 9 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} = 8$$

This is the C_{22} cofactor

$$\det B_3 = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 1 & 4 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} = -3$$

This is the C_{23} cofactor

$$\text{Now: } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \frac{1}{\det A} \begin{bmatrix} \det B_1 \\ \det B_2 \\ \det B_3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -5 \\ 8 \\ -3 \end{bmatrix} = \begin{bmatrix} -5/2 \\ 4 \\ -3/2 \end{bmatrix}$$

What did we really do here? We found the 2nd column of A^{-1} !

$$\begin{bmatrix} A \\ A \end{bmatrix} \begin{bmatrix} A^{-1} \\ \text{col 1} \quad \text{col 2} \quad \dots \quad \text{col n} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

\uparrow \vec{x} $\quad \quad \quad \uparrow$ \vec{b}

The entries in Column j of A^{-1} come from the cofactors for Row j of A:

$$(A^{-1})_{ij} = \frac{C_{ji}}{\det A}$$

← Transpose!

cofactor matrix: $\begin{bmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{bmatrix}$

So we get an algebraic formula for A^{-1} : (4)

$$A^{-1} = \frac{1}{\det A} [C_{ij}]^T \leftarrow \text{important!} \quad (\text{if } \det A \neq 0).$$

↑

Remember: $C_{ij} = \pm \det$ of matrix you get after deleting Row i and column j from A .

Specifically, $(-1)^{i+j}$

Let's check how this works for 2×2 :

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{bmatrix} = \begin{bmatrix} a_{11}C_{11} + a_{12}C_{12} & a_{11}C_{21} + a_{12}C_{22} \\ a_{21}C_{11} + a_{22}C_{12} & a_{21}C_{21} + a_{22}C_{22} \end{bmatrix}$$

Cofactor expansion for

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{21} & a_{22} \end{vmatrix} = 0$$

Cofactor expansions for $\det A$ (using Rows 1 and 2).

So matrix product is: $\begin{bmatrix} \det A & 0 \\ 0 & \det A \end{bmatrix}$

So just need to divide by $\det A$ to get I .

Note: Formula for A^{-1} is not too useful for calculating A^{-1} (elimination is usually faster)

But it can give some information about A^{-1} easily, ⁽⁵⁾ if we don't care exactly what A is.

Example: Show $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{bmatrix}^{-1}$ has integer entries (whole numbers, no fractions)

First find $\det A$ by elimination:

$$\begin{vmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 11 \\ 3 & 7 & 14 & 25 \\ 4 & 11 & 25 & 50 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 5 & 13 \\ 0 & 3 & 13 & 34 \end{vmatrix} \begin{array}{l} \text{Row 4} - 3 \text{ Row 3} \\ \text{Then: Row 2} \leftrightarrow \text{Row 3} \end{array}$$

$$= \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 13 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -2 & -5 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 13 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1$$

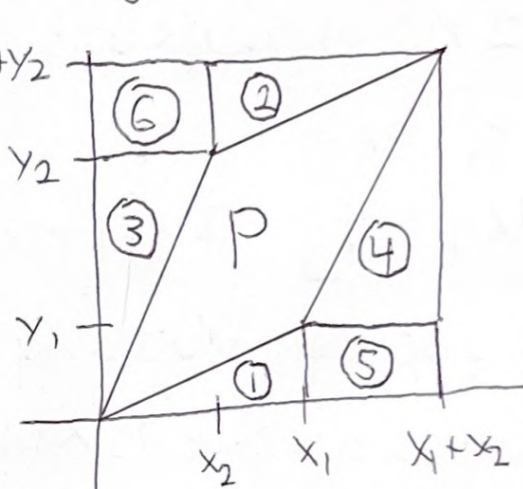
Now: entries of A^{-1} are $\frac{C_{ji}}{\det A} = -C_{ji}$

These are integers because $C_{ji} = \pm \det$ of a submatrix of $A = \text{integer}$ (since A has integer entries).

In general: If A has integer entries and $\det A = \pm 1$, then A^{-1} also has integer entries.

Areas and Volumes: Connect determinants to geometry (maybe this is the main reason determinants are natural.)

Let's find area of a parallelogram in the plane:



$$\text{Area of } P = (x_1 + x_2)(y_1 + y_2) - ((1) + (2) + (3) + (4) + (5) + (6))$$

$$\text{Area 1} = \text{Area 2} = \frac{1}{2} x_1 y_1 \quad \text{Area 5} = x_2 y_1$$

$$\text{Area 3} = \text{Area 4} = \frac{1}{2} x_2 y_2 \quad \text{Area 6} = x_2 y_1$$

$$\text{So Area of } P = (x_1 + x_2)(y_1 + y_2) - 2\left(\frac{1}{2} x_1 y_1 + \frac{1}{2} x_2 y_2 + x_2 y_1\right)$$

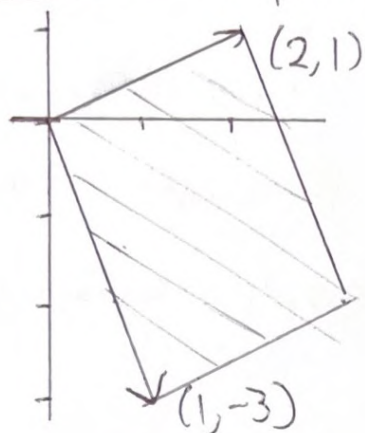
$$= x_1 y_1 + x_2 y_1 + x_1 y_2 + x_2 y_2 - x_1 y_1 - x_2 y_2 - 2 x_2 y_1$$

$$= x_1 y_2 - x_2 y_1 = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \quad \leftarrow \text{Determinant is an area!}$$

So if you know the coordinates of the vertices of a parallelogram, it's easy to find its area.

Example: $(x_1, y_1) = (2, 1)$, $(x_2, y_2) = (1, -3)$

⑦



$$\text{Area} = \begin{vmatrix} 2 & 1 \\ 1 & -3 \end{vmatrix} = -6 - 1 = -7$$

?? The minus sign comes from the order we chose for the two vectors. Actual area is the absolute value, 7.

This works in n dimensions too: Volume of n -dim box determined by $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$

$$= \left| \det \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix} \right|$$

↑
absolute value, not det!

or put $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ as the rows, get same det
since $\det A = \det A^T$.

How would you prove this volume formula? Show that the function Vol:

$$\begin{bmatrix} -\vec{a}_1- \\ -\vec{a}_2- \\ \vdots \\ -\vec{a}_n- \end{bmatrix}$$

Vol

"Signed" volume of box determined by $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n$ obeys the 3 rules of determinants.

Since \det is the only function that obeys all 3 (8) rules, the "signed" volume function has to be the same as \det .

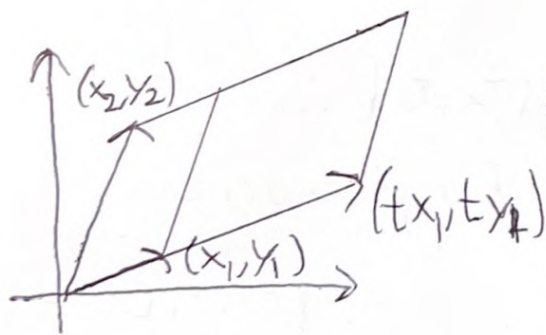
check these rules:

Rule 1: $\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \xrightarrow{\text{Vol}} 1 ??$

↪ This gives the unit "hypercube" in n dimensions. Volume = $\underbrace{(1)(1)\dots(1)}_{n \text{ times}} = 1$ ✓

Rule 2: How does volume change if we switch two rows? Volume doesn't change if we rearrange the vectors since the box they span is the same. But "signed" volume changes by a sign, like \det .

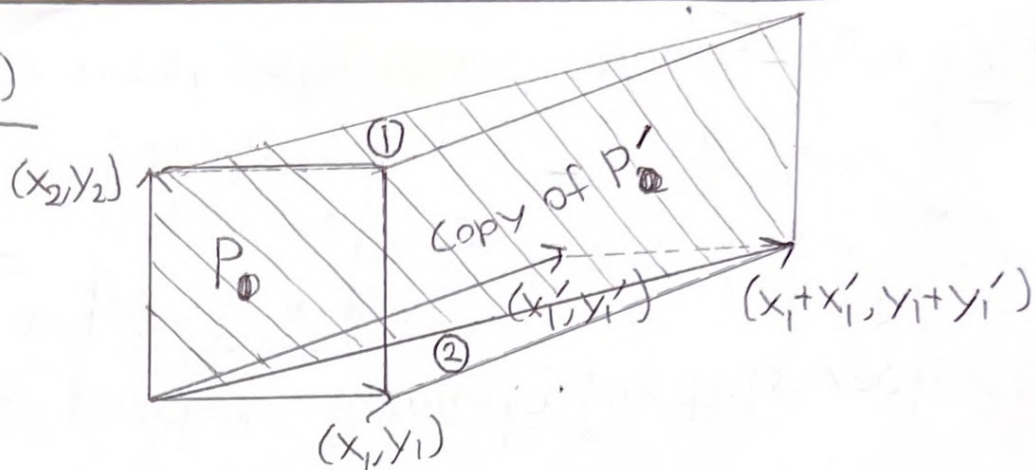
Rule 3(a) Multiply one row by t : $\begin{bmatrix} tx_1 & tx_1 \\ x_2 & x_2 \end{bmatrix}$



One side stretches by t while other dimensions of box are the same \rightarrow Volume is multiplied by t (just like \det).

Rule 3(b)

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$$\text{Area} \left(\begin{bmatrix} x_1+x_1' & y_1+y_1' \\ x_2 & y_2 \end{bmatrix} \right) = \text{shaded area} =$$

$$= \text{Area } P + \text{Area } P' + \text{Area (1)} - \text{Area (2)}$$

same, both = area of triangle
spanned by (x_1, y_1) and
 (x_1+x_1', y_1+y_1')

$$= \text{Area} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix} \right) + \text{Area} \left(\begin{bmatrix} x_1' & y_1' \\ x_2 & y_2 \end{bmatrix} \right)$$

Areas add, so Area function obeys Rule 3(b). ✓

Note: For n -dimensional space, you might wonder, Do we really know what " n -dimensional volume" really means? One way to fix this problem is to use determinants to define volume.

I.e., define $\text{Vol}(\text{n-dim box})$ to equal

$$\left| \det \begin{bmatrix} \vec{a_1} \\ \vec{a_2} \\ \vdots \\ \vec{a_n} \end{bmatrix} \right| . \text{ But we would still want}$$

this function to behave like volume. For example:

Problem 5.3.20 Find volume of 4-dim
"hypercube" spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

means the vectors are
orthogonal

Since this is a hypercube, volume should equal
product of side lengths:

$$\left\| \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2} = 2 \rightarrow \text{same for other 3 vectors}$$

$$\rightarrow \text{Volume} \stackrel{??}{=} (2)(2)(2)(2) = 16$$

But let's check this using the determinant
volume formula.

Rule 3(b) (x_2, y_2)

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$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{vmatrix} \longrightarrow \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & -2 \\ 0 & -2 & -2 & 0 \\ 0 & -2 & 0 & -2 \end{vmatrix} \begin{matrix} \text{1st} \\ \text{col} \end{matrix} = 1 \begin{vmatrix} 0 & -2 & -2 \\ -2 & -2 & 0 \\ -2 & 0 & -2 \end{vmatrix}$$

$$= -(-2) \begin{vmatrix} -2 & -2 \\ 0 & -2 \end{vmatrix} + (-2) \begin{vmatrix} -2 & -2 \\ -2 & 0 \end{vmatrix} = 2(4) + (-2)(-4)$$

$$= \boxed{16} \checkmark \quad \text{Determinant produces correct volume}$$

Cross Product in \mathbb{R}^3 Given \vec{u}, \vec{v} in \mathbb{R}^3 , find a vector \perp to both of them (i.e., a vector in $\text{span}(\vec{u}, \vec{v})^\perp$).

The cross product formula for this vector:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2 v_3 - u_3 v_2) \vec{i} - (u_1 v_3 - u_3 v_1) \vec{j} + (u_1 v_2 - u_2 v_1) \vec{k}$$

(not a real determinant)

Here: $\vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. Components

(12)

of $\vec{u} \times \vec{v}$ are C_{11}, C_{12}, C_{13} cofactors.

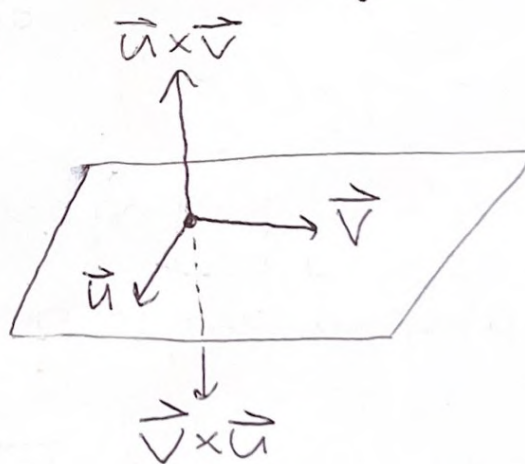
Why is $\vec{u} \times \vec{v} \perp \vec{u}, \vec{v}$?

$$\vec{u} \cdot (\vec{u} \times \vec{v}) = u_1 C_{11} + u_2 C_{12} + u_3 C_{13}$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = 0. \checkmark \quad \text{Same for } \vec{v} \cdot (\vec{u} \times \vec{v}).$$

From Rule 2 for determinants: If you switch \vec{u} and \vec{v} , cross product changes sign.

$$\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$$



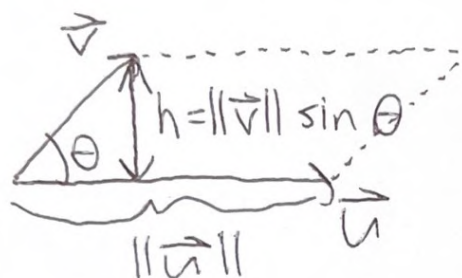
How long is $\vec{u} \times \vec{v}$? Check when \vec{u}, \vec{v} are in xy-plane:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & 0 \end{vmatrix} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So ~~10~~ $\|\vec{u} \times \vec{v}\| = \text{absolute value of } \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$ (13)

= Area of parallelogram spanned by \vec{u} and \vec{v} .

Can express this area using angle between \vec{u}, \vec{v} :



$$\begin{aligned} \text{So } \|\vec{u} \times \vec{v}\| &= \text{area} \\ &= \|\vec{u}\| \|\vec{v}\| |\sin \theta| \end{aligned}$$

Similar to dot product:

$$|\vec{u} \cdot \vec{v}| = \|\vec{u}\| \|\vec{v}\| |\cos \theta|$$

Triple Product Formula for volume of box in \mathbb{R}^3 :

$$\text{Vol of box spanned by } \vec{u}, \vec{v}, \vec{w} = \begin{vmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$= w_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - w_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + w_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \vec{w} \cdot (\vec{u} \times \vec{v}) \quad (\text{Take absolute value to get actual volume})$$