108-1. Adv-Algo. MidTerm 1

1. ① $\Theta(\phi^n)$ or $O(2^n)$. $\phi=\frac{1+\sqrt{5}}{2}$ (Lec03 P.20)

② Thm: $\begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$

Let $A=\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ By recursive squaring.

We can get A^n in $\Theta(\log_2 n)$ time.

F_n is in A^n. $A^n = \begin{cases} A^{n/2} \cdot A^{n/2} & \text{if } n \text{ even} \\ A^{\frac{n-1}{2}} \cdot A^{\frac{n-1}{2}} \cdot A & \text{if } n \text{ odd} \end{cases}$

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2. Let k be the num. of iterations executed. while-loop.

$k=1$. $3^{3^0} \le n < 3^{3^1}$

$k=2$ $3^{3^1} \le n < 3^{3^2}$

$k=3$ $3^{3^2} \le n < 3^{3^3}$, in general, $3^{3^{k-1}} \le n < 3^{3^k}$

$k-1 \le \log_3 \log_3 n < k$

$\log_3 \log_3 n < k \le \log_3 \log_3 n + 1$

$k = \lfloor \log_3 \log_3 n + 1 \rfloor = \lfloor \log_3 \log_3 n \rfloor + 1$

(a) $x = 5k+4 = 5\lfloor \log_3 \log_3 n \rfloor + 9$. $5\lfloor \log_3 \log_3 n \rfloor + 4$?

(b) $i = 3^{3^{\lfloor \log_3 \log_3 n \rfloor + 1}}$? $= 3^{3^k}$

3. (a) $R(1,n,k) = p_n R(1,n-1,k) + q_n p_{n-1} R(1,n-2,k)$
$+ q_n q_{n-1} p_{n-2} R(1,n-3,k) + \cdots$
$+ q_n q_{n-1} \cdots q_{n-k+2} p_{n-k+1} R(1,n-k,k)$.
$\forall i,\ q_i = 1 - p_i$.

(b) $R(1,n,k) = R(1,n-1,k) - q_n q_{n-1} \cdots q_{n-k+1} p_{n-k} R(1,n-k-1,k)$

4. Lec02, p.42-45. Master Thm.
$T(n) = a \cdot T(n/b) + f(n)$ $\quad n^{\log_b a} = \cancel{0.5} = \sqrt{n} = n^{0.5}$

(a) $f(n) = n = \Omega(n^{\log_b a + \varepsilon}) = \Omega(n^{0.5+0.2})$ Try Case 3.
$a \cdot f(\frac{n}{b}) = 3 \cdot \frac{n}{9} \leq cn$, if set $c = \frac{1}{2} < 1$.
$T(n) = \Theta(n)$.

(b) $f(n) = n^{0.5} \log_2 n = \Theta(n^{\log_b a} \log_2^k n)$ if set $k=1$.
Case 2 $\Rightarrow T(n) = \Theta(n^{\log_b a} \log_2^{k+1} n) = \Theta(n^{0.5} \log_2^2 n)$

(c) $f(n) = \log_2 n = O(n^{\log_b a - \varepsilon})$ If set $\varepsilon = \frac{1}{4}$.
Case 1 $\Rightarrow T(n) = \Theta(n^{\log_b a}) = \Theta(n^{0.5}) \ \sqrt{n}$

5 (a) Biased_Rand( ) / while (True)
{ { $a \leftarrow 4 \cdot$ Unbiased_Rand( )
$+ 2 \cdot$ Unbiased_Rand( )
$+$ Unbiased_Rand( ),

0~7 ⇒ 3个 bit.

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if (a≤3) return 1
else if (a=4) return 0 }
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5.67 ? output

}

(b) $Pr(\text{1st iter. of while outputs } 1) = \frac{4}{8}$

$Pr(\text{1st iter. of while outputs } 0) = \frac{1}{8}$

$Pr(\text{1st iter. of while has no output}) = \frac{3}{8}$

$Pr(\text{outputs } 1) = \frac{4}{8} + \frac{3}{8}\cdot\frac{4}{8} + \left(\frac{3}{8}\right)^2\cdot\frac{4}{8} + \cdots$

$= \frac{4/8}{1-3/8} = \frac{4/8}{5/8} = \frac{4}{5}$

$Pr(\text{Outputs } 0) = \frac{1}{8} + \frac{3}{8}\cdot\frac{1}{8} + \left(\frac{3}{8}\right)^2\frac{1}{8} + \cdots$

$= \frac{1/8}{1-3/8} = \frac{1}{5}.$

Let the time cost of one iter. of while is at most C (a constant).

The expected running time

$E(T) \le 1\cdot C + \frac{3}{8}C + \left(\frac{3}{8}\right)^2\cdot C + \cdots$

$= \frac{C}{1-\frac{3}{8}} = \frac{8C}{5} = \Theta(1).$

5. Let $\begin{cases}\text{the time cost be } T(n). \\ \text{the final value of count be } C(n).\end{cases}$

$$C(n)=\begin{cases}1 & \text{if } n=1,2\\ 2\cdot C(n-2)+C(n-1)+1, & \text{if } n>2.\end{cases}$$

$$T(n)=\begin{cases}K & \text{if } n=1,2\\ 2\cdot T(n-2)+T(n+1)+K, & \text{if } n>2.\end{cases}$$

an estimate of time cost

↓

a constant

Let's solve $C(n)$.

$$\begin{cases}x^2=2+x. & x=2,-1\\ y=2y+y+1, & y=\frac{-1}{2}\end{cases}$$

$C(n)=A\cdot 2^n+B(-1)^n+\frac{-1}{2}$.

$C(1)=2A-B+\frac{-1}{2}=1$

$C(2)=4A+B+\frac{-1}{2}=1.$

$6A=3.$

$A=\frac{1}{2}, B=\frac{-1}{2}$

Ans:

(b) finally count $=\frac{2^n-(-1)^n-1}{2}$

(a) $T(n)=\frac{2^n-(-1)^n-1}{2}\cdot K=\Theta(2^n)$

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7. (a) $(a_1 ? a_2)\ (a_3 ? a_4) \cdots (a_{63} ? a_{64})\ a_{65}$.

$\frac{64}{2}+(33-1)+(33-1)=96$

(b) $(65-1)+(\lceil \log_2 65\rceil-1)=64+5=69$

8. (a) $\Theta(n+2^b)$, (b) $\Theta(\frac{b}{r}(n+2^r))$. a. para

(c) If we use counting sort, $\theta(n+n^3) = \theta(n^3)$
not $O(n)$ ×

So we use Radix Sort, ~~let r = lo~~

Use n-ary number system,

$0 = \boxed{0}\boxed{0}\boxed{0}$

$1 = \boxed{0}\boxed{0}\boxed{1}$

$\vdots$

$n^3-1 = \boxed{n-1}\boxed{n-1}\boxed{n-1}$

$T(n) = \theta(3 \cdot (n+n)) = \theta(n) = O(n)$

9.

		A	B	C	A	B	D	A	B
	0	0	0	0	0	0	0	0	0
B	0	0	1	1	1	1	1	1	1
D	0	0	1	1	1	1	2	2	2
C	0	0	1	2	2	2	2	2	2
A	0	1	1	2	3	3	3	3	3
B	0	1	2	2	3	4	4	4	4
A	0	1	2	2	3	4	4	5	5

Length of LCS = 5

a LCS = BCABA.