Measure Theory Homework

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1.3. g_1 is not an algebra. Take $A = (-\infty, -3) \cup (2, +\infty)$ which is in g_1 but $A^c = [-3, 2]$ which is not in g_1. g_2 is an algebra and g_3 is σ-algebra.

1.7. By definition, $\emptyset$ has to be in the algebra generated by a set X. And by the requirement of closed under complementary, X then has to be too. So the smallest algebra must be $\{\emptyset, X\}$. The largest algebra should be the one contains all possible subsets, which is the power set.

1.10. First of all, $\emptyset \in S_\alpha$ for all α because S_α is a σ-algebra. Suppose $A \in \sigma(g)$ then $A \in S_\alpha$ for all α, then $A^c \in S_\alpha$ for all α by definition. Thus $A^c \in \sigma(g)$. Similarly we could prove it is closed under countable unions.

1.22. Take $C = A^c \cap B$ then as $A \subset B$, $|C| \geq 0$. And we have $\mu(C \cup A) = \mu(B)$ which is $\mu(C) + \mu(A) = \mu(B)$, so $\mu(B) - \mu(A) \geq 0$.

Define $F_n = \cup_{i=1}^n A_i$. It is easy to see that $F_n \subset F_{n+1}$. Therefore F_n is increasing. Also we know that $F_n \uparrow \cup_{i=1}^{+\infty} A_i$. Hence we have

$$\begin{aligned}
\mu(\cup_{i=1}^{+\infty} A_i) &= \lim_{n \to +\infty} \mu(F_n) \\
&= \lim_{n \to +\infty} \mu(A_1 \cup A_2 \cup \dots\dots \cup A_n) \\
&\leq \lim_{n \to +\infty} \sum_{i=1}^{n} \mu(A_i) = \sum_{i=1}^{+\infty} \mu(A_i).
\end{aligned}$$

1.23. Let $A = \emptyset$, then $A \cap B = \emptyset$, so $\lambda(A) = \mu(A \cap B) = 0$. Take $A_1, A_2, \dots\dots$ to be mutually exclusive sets, then $\lambda(\cup_i A_i) = \mu((\cup_i A_i) \cap B)$ which is $\mu(\cup_i (A_i \cap$

$B)) = \sum_i \mu(A_i \cap B) = \sum_i \lambda(A_i)$. So λ is a measure.

1.26. Note we have $A_i \downarrow \cap_i A_i = X$. We could write $A_{n_0} = X + \sum_{n=n_0}^{+\infty}(A_n - A_{n+1})$. So we have $\mu(A_{n_0}) = \mu(X) + \sum_{n=n_0}^{+\infty} \mu(A_n - A_{n+1})$ which completes the proof.

2.10. Because from the countable subadditivity of the outer measure we know that $\geq$ holds. As $\leq$ holds, the equal sign should also hold.

2.14. We know $o(\mathcal{O}) \subset \mathcal{M}$ from Caratheodory and construction of lebesgue measure as an infinite collection of the form $(a, b]$ and $(-\infty, a]$ we know that $\sigma(\mathcal{A}) \subset \mathcal{M}$. If $o \in \sigma(\mathcal{O})$ then $o \in \sigma(\mathcal{A})$ therefore $o \in \mathcal{M}$.

3.1. Let a_i be a sequence of a countable set A then $A_i = (a_i - 2^{-i-1}\epsilon, a_i + 2^{-i-1}\epsilon)$. So $A \subset \cup A_i$ and $\mu(\cup A_i) \leq 2^{-i}\epsilon = \epsilon$ which completes the proof.

3.7. Because they are complementary to each other so the condition could be replaced.

3.10. Let $F(f(x), g(x)) = f(x) + g(x)$ is countable. Then F is countable. Given condition 4 we know that $f + g$ is measurable. Similarly by taking $F(f(x), g(x)) = f(x)g(x)$ we could prove that fg is measurable. As f and g are measurable, then $\{x \in X : \max(f, g) < a\}$=$\{x \in X : g < a\} \cap \{x \in X : g < a\}$. M is closed under intersection. Similarly we have $\{x \in X : \min(f, g) > a\}$=$\{x \in X : g > a\} \cup \{x \in X : g > a\}$. And finally as $\{x \in X : |f|> a\}$=$\{x \in X : f > a\} \cup \{x \in X : f < -a\}$ we could prove that $|f|$ is measurable.

4.13. We know that $||f||= f^+ + f^-$. As $|f|< M$ we have $f^+ \leq M$ and/or $-M < f^-$. Therefore $\int_E f^+ d\mu < \infty$ and $\int_E f^- d\mu < \infty$ which completes the proof.

4.14. Suppose there exists a subset $A \subset E$ that f is not finite so $f = \infty$ on A. Therefore $\int_E f d\mu > \int_A f d\mu + \infty$ which contradicts.

4.15. Define $s = g - f$ so we have $s \geq 0$ and $\int_E s d\mu \geq 0$. Therefore we have $\int_E (g - f) d\mu \geq 0$ which completes the proof.

4.16. From the question we know that $\int_E f^+ d\mu < \infty$ and $\int_E f^- d\mu < \infty$. As $A \subset E$, we have $\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$. Therefore $f \in L^1(\mu, A)$.

4.21. Without loss, we could write $C = A \cap B^c$. So $\int_A f d\mu = \int_B f d\mu + \int_C f d\mu = \int_B f d\mu$ since $\mu(C) = 0$.