Higher-Order Effects with Implicitly Resolved Elaborations

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Higher-Order Effects with Implicitly Resolved Elaborations

THESIS

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Cover picture: Random maze.

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Abstract

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Preface

Preface here.

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Chapter 1

Introduction

In an idealized perspective on computation, programs and their subcomponents are *pure* functions. They take some input and for every set of inputs they return the same output, without interacting with other parts of the program or with the system. There is a certain elegance to this view: pure programs are relatively easy to reason about and analyse.

However, in practice, many programs are *impure*. These programs might interact with memory and the network, write to and read from the terminal. Additionally, programs might be clearer when they can be expressed with more complex control flow structures like exceptions and coroutines. We call these extensions on top of pure computation *effects* (Moggi 1989b).

Some programming languages, such as C and many of the languages it inspired, have opted to give the programmer unrestriced access to effectful operations. In particular, any part of the program can access memory, the filesystem and even allowed to use `goto` to jump to other parts outside of regular control flow. This unrestricted use has famously been criticized by Dijkstra (1968) and others. An open challenge in language design is then to design a language which allows for (limited) impure calculation, while retaining the attractive qualities of pure languages.

A promising aproach to striking this balance is the framework of algebraic effects by Plotkin and Power (2001) and extended by Plotkin and Pretnar (2009). Here, algebraic effects are only allowed within *handlers*, which have some, but still limited, control over the flow of the program via continuations. In recent years, some languages (e.g. Koka (Leijen 2017)) have been created with support for algebraic effects. However, there are some common patterns of effects that cannot be represented as algebraic effects, namely higher-order effects, that is, effects whose operations that take effectful computations as arguments.

To overcome this limitation, Bach Poulsen and Rest (2023) have extended algebraic effects with *hefty algebras*. They introduce *elaborations* in addition to handlers, which can encode higher-order effects.

In this thesis, we build on the work by Bach Poulsen and Rest (2023), applying the theory of hefty algebras as a first-class construct in a novel programming language called *Elaine*. Like Koka and other languages with support for algebraic effects, Elaine supports handlers. In addition, elaborations and higher-order effects are also supported.

We provide a novel reduction semantics and type system for a language with supporting elaborations. Furthermore, we introduce two transformations on Elaine programs. The first transformation is a type directed inference of elaborations, allowing the names of elaborations in use to be omitted. The second is a transformation from a program with higher-order effects and elaborations to a program with only algebraic effects. This transformation shows that elaborations can be added to existing languages and libraries for effects with relative ease.

In addition, we provide a full implementation of this language[1]. This implementation

[1]available at `https://github.com/tertsdiepraam/thesis/tree/main/elaine`

includes a parser, type checker, pretty printer, type checker and the two transformations described in this thesis.

Chapter 2

Background

While algebraic effects are a relatively new concept, the analysis of effectful computation and the representation of effects in programming language has a rich history. This history is relevant to this thesis because the various approaches to modelling effects proposed over time can be found in many popular programming languages. A comparison between these languages and languages with algebraic and higher-order effects hence requires comparison between theories. This chapter details this history.

Give definitions of effects, pure/impure, effectless/effectful

2.1 Motivation for Effects

First, let us pose a simple question: why study effects? As Moggi (1989b) notes, analyzing only pure computation leaves out many aspects of programs, such as side-effects, non-determinism and non-termination. Reasoning about the behaviour of a program then necessarily needs to include an analysis of effects.

Proving either the presence or absence of these properties is very valuable. For example, a deterministic, side-effect-less computation can be reliably cached. Without side-effects, computation can also be executed out of order without any observable difference. Hence, many sophisticated compilers rely on effect tracking to inform the optimizations they perform. Citation needed A theory of effectful computation can therefore help compiler engineers prove the correctness of their transformations. Citation needed Hence, a language with a strong formal model for effects is easier to compile or optimize than one without.

In addition, support for effects in the type system of a programming language could allow programmers to write stricter and safer APIs. In a language where the type system can reason about effects, the compiler can make certain guarantees beyond type checking only the input and output types of a function.

These guarantees fall into two categories: communicating presence and requiring absence. Communicating presence gives library authors the ability to tell the type system what effects their functions require. This informs users of libraries about what those functions can do (e.g. accessing the network or being able to run without terminating). This is often most useful when the effects are interaction with the environment,in which case effects can function as capabilities (Brachthäuser, Schuster, and Ostermann 2020).

A library author might also want to communicate that a function should not contain certain effects. For example, a hash function is generally understood to be deterministic and effectless, because it needs to be reproducible. However, this is often not guaranteed by the type system of mainstream programming languages. Citation needed

2.2 Monads and Monad Transformers

The study of effects starts right at the two foundational theories of computation: λ-calculus and Turing machines. Their respective treatment of effects could not be more different. The former is only concered with pure computation, while the latter consists solely of effectful operations.

In λ-calculus, effects are not modelled; every function is a function in the mathematical sense, that is, a pure computation (Moggi 1989b). Hence, many observable properties of programs are ignored, such as non-determinism and side-effects. In their seminal paper, Moggi (1989b) unified *monads* with computational effects, which they initially called notions of computation. Moggi identified that for any monad $T : C \to C$ and a type of values A, the type TA is the type of a computation of values of type A.

Since many programming languages have the ability to express monads from within the language, monads became a popular way to model effectful computation in functional programming languages. In particular, Peyton Jones and Wadler (1993) introduced a technique to model effects via monads in Haskell. This technique keeps the computation pure, while not requiring any extensions to the type system.

A limitation of treating effects as monads is that monads do not compose well; the composition of two monads is not itself a monad. A solution to this are *monad transformers* (Moggi 1989a), which are functors over monads that add operations to a monad. A simple monad can then be obtained by applying a monad transformer to the `Identity` monad. The representation of a monad then becomes much like that of a list of monad transformers, with the `Identity` monad as `Nil` value. This "list" of transformers is ordered. For example, using the terminology from Haskell's `mtl` library, the monad `StateT a (ReaderT b Identity)` is distinct from `ReaderT b (StateT a Identity)`. The order of the monad transformers also determines the order in which they must be handled: the outermost monad transformer must be handled first.

In practice, this model has turned out to work quite well, especially in combination with `do`-notation, which allowed for easier sequential execution of effectful computations.

2.3 Algebraic Effects

Introduce algebraic effects, effect rows, and handlers.

2.3.1 Algebraic Theories

This section introduces algebraic theories. In particular, we will discuss algebraic theories with parametrized operations and general arities. This will form a foundation on which we can define algebraic effects. The definitions in this section follow Bauer (2018).

Definition 1 (Signature)**.** A *signature* $\Sigma = \{(op_i, P_i, A_i)\}$ is a collection of operation symbols op_i with corresponding parameter sets P_i and arity sets A_i. We will write operations symbols as follows:

$$op_i : P_i \rightsquigarrow A_i.$$

The arities are arbitrary sets. However, using the von Neumann ordinals, we can use natural numbers as arities. Hence we can call operation symbols with arities 1, 2 and 3 *unary*, *binary* and *ternary* respectively. An operation symbol with arity 0 is then called *constant* or *nullary*. Two other common arities are $\emptyset = \{\}$ and $\mathbf{1} = \{()\}$. We will refer to $()$ as the unit.

We can build terms with any given signature by composing the operations. Given some set X, we can build a set $\mathrm{Tree}_\Sigma(X)$ of *well-founded trees* over Σ generated by X. This set is defined inductively:

- for every $x \in X$ we have a tree $\mathsf{return}\, x$,
- and if $p \in P_i$ and $\kappa : A_i \to \mathrm{Tree}_\Sigma(X)$ then $op_i(p, k)$ is a tree, where op_i is the label for the root and the subtrees are given by κ.

A Σ-term is a pair of a context X and a tree $t \in \mathrm{Tree}_\Sigma(X)$. We write a Σ-term as

$$X \mid t.$$

While the notation for these trees is intentionally evocative of functions in many programming languages, it is important to note that the terms are only a representation of a tree and should be thought of as such.

Definition 2 (Σ-equation)**.** A Σ-equation is a pair of Σ-terms and a context X. We denote the equation

$$X \mid l = r.$$

Here, the $=$ symbol is just notation and its meaning is left unspecified. Note that we can only create equations with the $=$ symbol. We cannot, for instance, create an equation $X \mid l \neq r$.

When the relevant signature Σ is unambiguous, we will omit the Σ from the definitions above and simply speak of terms and equations. We can now build terms and equations with any signature we define. Hence, we can give a signature along with some associated laws that we intend to hold for that signature; this is the idea of an algebraic theory.

Definition 3 (Algebraic theory)**.** An *algebraic theory* (or *equational theory*) is a pair $\mathsf{T} = (\Sigma_\mathsf{T}, \mathcal{E}_\mathsf{T})$ consisting of a signature Σ_T and a collection $\mathcal{E}_\mathsf{T}$ of Σ_T-equations.

An algebraic theory is still hollow; it is only a specification, not an implementation. The implementation or meaning of the operations that we apply to the operation symbols needs to be given via an interpretation.

Definition 4 (Interpretation)**.** An *interpretation* I of a signature Σ is a

1. a *carrier set* $|I|$
2. and for each operation op_i a map

$$[\![op_i]\!]_I : P_i \times |I|_i^A \to |I|,$$

called an *operation.*

Additionally, we can define the interpretation of a tree, and by extension of a term, as a map

$$[\![t]\!]_I : |I|^X \to |I|.$$

This map is defined as

$$[\![\mathsf{return}\, x]\!]_I : \eta \mapsto \eta(x) \qquad [\![op_i(p, \kappa)]\!]_I : \eta \mapsto [\![op_i]\!]_I(p, \lambda a.[\![\kappa(a)]\!]_I(\eta)).$$

If the choice of I is obvious from the context, we will omit the subscript.

The *semantic bracket* $[\![]\!]_I$ is used to indicate that syntactic constructs are mapped to some (mathematical) interpretation of those symbols. In other words, an interpretation gives denotational semantics to a signature. this true?

Definition 5 (Model)**.** We say that an Σ-equation $X \mid l = r$ is *valid*, if the interpretations of l and r evaluate to the same map, that is,

$$[\![l]\!] = [\![r]\!].$$

A T-*model* M of an algebraic theory T is an interpretation of Σ_{T} for which all the equations in $\mathcal{E}_T$ valid.

We can relate models to each other with morhisms between their carrier sets. Given models L and M for T, we call such a morphism $\phi : |L| \to |M|$, a T-homomorphism if the following condition holds for all op_i in Σ_{T}:

$$\phi \circ [\![op_i]\!]_L(p, \kappa) = [\![op_i]\!]_M(k, \phi \circ k).$$

That is, if ϕ commutes with operations.

Free models

Composing algebraic theories

2.3.2 Effects as Algebraic Theories

Plotkin and Power (2001) have shown that many effects can be represented as algebraic theories. For that to make sense, we first need to represent computation is in the context of an algebraic theory. A computation can either be pure or effectful. A pure computation only returns a value, while an effectful computation performs some operation and then continues. We then write

- $\mathsf{return}\, x$ for a pure computation,
- and $op(p, \kappa)$ for an effectful operation, where κ is the *continuation*.

As an example, take the `State` effect. To use the this effect, we need two operations: `put` and `get`. The computation

$$\mathtt{put}(a, \lambda\_.\mathtt{get}((), \lambda x.\, \mathsf{return}\, x)),$$

then first performs the `put` and then the `get`, returning the result of the `get`. Recall, that the $()$ symbol here represents the unit.

Naturally, this representation of computation matches the definition of trees given above. Hence, we can connect the dots. We can represent computations as terms, so what are the signatures and equations? We start with the signature for `State`:

$$\mathtt{put} : S \rightsquigarrow \mathbf{1} \quad \text{and} \quad \mathtt{get} : \mathbf{1} \rightsquigarrow S.$$

This signature indicates that `put` takes an S as parameter and resumes with $()$ and that `get` takes $()$ and resumes with S. Now we can define the equations that we want `State` to follow:

$$\begin{aligned}
\mathtt{get}((), \lambda s.\mathtt{get}((), \lambda t.\kappa(s,t))) &= \mathtt{get}((), \lambda s.\kappa(s,s)) \\
\mathtt{get}((), \lambda s.\mathtt{put}(s, \kappa)) &= \kappa() \\
\mathtt{put}(s, \lambda_.\mathtt{get}((), \kappa)) &= \mathtt{put}(s, \lambda_.\kappa(s)) \\
\mathtt{put}(s, \lambda_.\mathtt{put}(t, \kappa)) &= \mathtt{put}(t, \kappa)
\end{aligned}$$

This gives us an algebraic theory corresponding to the `State` monad. Plotkin and Power (2001) have shown that this theory gives rise to the canonical `State` monad. Many other effects can also be represented as algebraic theories, including but not limited to, non-determinism, non-termination, iteration, cooperative asynchronicity, traversal, input and output Citation needed. These effects are called *algebraic effects*.

Somewhere: algebraic theories to the free model to the free monad.

However, not all effects fit, for example, the `Exception` monad. The distinction has been described as the difference between effect *constructors* and effect *destructors* (Plotkin and Power 2003).

Why higher-order effects don't fit.

Continuing their work, Plotkin and Pretnar (2009) then introduced effect handlers, allowing the programmer to destruct effects by placing handlers around effectful expressions. This provides a way to treat exception handling using effects. However, the scope in which an effect is handled can only be changed by adding handlers. Effect operations cannot define their own scope. To support this, a system for higher-order effects is required, which are effects that take effectful operations as parameters.

2.4 Higher-Order Effects

Introduce higher-order effects, the composability problem, scoped effects, hefty algebras etc.

A solution to this were scoped effects (Wu, Schrijvers, and Hinze 2014). However, scoped effects require a significant increase in complexity and cannot express effects that are neither algebraic nor scoped, such as lambda abstractions (Berg et al. 2021). Latent effects (Berg et al. 2021) were subsequently introduced as an alternative that encapsulates a larger set of effects.

As an alternative approach, Bach Poulsen and Rest (2023) introduced hefty algebras. With hefty algebras, higher-order effects are treated separately from algebraic effects. Higher-order effects are not handled, but elaborated into algebraic effects, which can then be handled. The advantage is that the treatment of algebraic effects remains intact and that the process of elaboration is relatively simple.

Chapter 3

Related Work

- Koka
- Frank
- Eff
- etc.

In parallel with the work to define theoretical frameworks for effects, several libraries and languages have been designed that include effects as first-class concepts, allowing the programmer to define their own effects and handlers. For example, there are some libraries available for Haskell, like `fused-effects`[1] , `polysemy`[2] , `freer-simple`[3] and `eff`[4] , each encoding effects in a slightly different way.

Notable examples of languages with support for algebraic effects are Eff (Bauer and Pretnar 2015), Koka (Leijen 2017) and Frank (Lindley, McBride, and McLaughlin 2017). OCaml also gained support for effects (Sivaramakrishnan et al. 2021). By building algebraic effects into the language, instead of delegating to a library, is advantageous because the language can provide more convenient syntax. In these languages, some concepts that were traditionally only available with explicit language support, can be expressed by the programmer via algebraic effects. This includes exception handling and asynchronous programming.

Recognize advancements in algebraic effects beyond what is used here. E.g. named effects.

[1] `https://github.com/fused-effects/fused-effects`
[2] `https://github.com/polysemy-research/polysemy`
[3] `https://github.com/lexi-lambda/freer-simple`
[4] `https://github.com/hasura/eff`

Chapter 4

Elaine: Language Design

- Begin with examples
- Especially elaborations from HA paper
- Explain relation between Elaine and Hefty Algebras
- Spec
  - Syntax
  - Reduction semantics
  - Typing rules & row equivalence
  - Type checker: unification rules mostly

The language designed for this thesis is called “Elaine”. The distinguishing feature of this language is its support for higher-order effects via elaborations. It is not the first language with support for elaborations, with the first being Heft Citation needed. Put this somewhere

```
mod math {
   use std;
   pub let double = fn(x) {
      mul(2, x)
   };
   pub let abs = fn(x) {
      if lt(x, 0) { sub(0, x) } else { x }
   };
}

use math;
let main = {
   let x = -10;
   let y = double(x);
   abs(doubled)
};
```

A module is declared with **mod**, which takes a name and a block of declarations. Each declaration prefixed with **pub** will be imported to other modules that reference this module. Modules cannot be nested. The built-in `std` module defines a few basic functions for boolean, integer and string manipulation (e.g. `mul`, `lt` and `sub`). Functions are defined anonymously with **fn**, followed by a list of argument, a return type and a function body. Functions are called with parentheses.

The design of Elaine is similar to Koka, with syntactical elements inspired by Rust. Apart from the elaborations and handlers, none of the language constructs should be particularly surprising: it has let bindings, if-else expressions, first-class functions, booleans, integers and strings. Below is a small sample program that prints whether the square of 4 is even or odd. For the full specification, we refer to . add ref

For simplicity of analysis, the language does not support recursion or any other looping construct.

```
# The standard library contains basic functions for manipulation
# of integers, booleans and strings.
use std;

# Functions are created with `fn` and bound with `let`, just like
# other values. The last expression in a function is returned.
let square = fn(x: Int) Int {
    mul(x, x)
};

let is_even = fn(x: Int) Bool {
    eq(0, modulo(x, 2))
};

# Type annotations can be inferred:
let square_is_even = fn(x) {
    let result = is_even(square(x));
    if result {
        "even"
    } else {
        "odd"
    }
};

let give_answer = fn(f, x) {
    let text = concat(show_int(x), " is ");
    let answer = f(x);
    concat(text, answer)
};

let main = give_answer(square_is_even, 4);
```

4.1 Algebraic Effects in Elaine

The programs in the previous section are all pure and contain no effects and should be fairly standard. Following the lead of Koka Citation needed, Elaine additionally has first class support for effects and effect handlers.

An effect is declared with the **effect** keyword. An effect needs a name and a set of operations. Operations are the functions that are associated with the effect. They can have an arbitrary number of arguments and a return type. Only the signature of operations can be given in an effect declaration, the implementation must be provided via handlers (see Section 4.1.2)

We will be using the following effects throughout this section. Add corresponding monads?

```
# Defines a single operation to get an implicit variable
effect Val {
   val() Int
}

# Exits the current handle
effect Abort {
   abort() ()
}

# Allows for a mutable state via a get and a set operation
effect State {
   get() Int
   set(Int) ()
}
```

4.1.1 Effect Rows

Contextual vs parametric effect rows (see effects as capabilities paper). The paper fails to really connect the two: contextual is just parametric with implicit variables. However, it might be more convenient. The main difference is in the interpretation of purity (real vs contextual). In general, I'd like to have a full section on effect row semantics. In the capabilities paper effect rows are sets, which makes it possible to do stuff like (Leijen 2005).

In Elaine, each type has an *effect row*. In the previous examples, this effect row has been elided, but it is still inferred by the type checker. Effect rows specify the effects that need be handled to within the expression. For simple values, that effect row is empty, denoted `<>`. For example, an integer has type `<> Int`. Without row elision, the `square` function in the previous section could therefore have been written as

```
let square = fn(x: <> Int) <> Int {
   mul(x, x)
}
```

Simple effect rows consist of a list of effect names separated by commas. The return type of a function that returns an integer and uses effects "A" and "B" has type `<A,B> Int`. Important here is that this type is equivalent to `<B,A> Int`: the order of effects in effect rows is irrelevant. However, the multiplicity is important, that is, the effect rows `<A,A>` and `<A>` are not equivalent. To capture the equivalence between effect rows, we therefore model them as multisets.

Additionally, we can extend effect rows with other effect rows. In the syntax of the language, this is specified with the `|` at the end of the effect row: `<A,B|e>` means that the effect row contains `A`, `B` and some (possibly empty) other set of effects.

We can use extensions to ensure equivalence between effect rows without specifying the full rows (which might depend on context). For example, the following function uses the `Abort` effect if the called function returns false, while retaining the effects of the wrapped function.

```
let abort_on_false = fn(f: fn() <|e> Bool) <Abort|e> () {
   if f() { () } else { abort() }
}
```

Effect rows need special treatment in the unification algorithm of the type checker, which is detailed in Section 4.3.1.

4.1.2 Effect Handlers

To define the implementation of an effect, one has to create a handler for said effect. Handlers are first-class values in Elaine and can be created with the **handler** keyword. They can then be applied to an expression with the **handle** keyword. When **handle** expressions are nested with handlers for the same effect, the innermost **handle** applies.

For example, if we want to use an effect to provide an implicit value, we can make an effect `Val` and a corresponding handler, which `resume`s execution with some values. The `resume` function represents the continuation of the program after the operation. Since handlers are first-class values, we can return the handler from a function to simplify the code. This pattern is quite common to create dynamic handlers with small variations.

```
let hVal = fn(x) {
   handler Val {
      return(x) { x }
      val() { resume(x) }
   }
};

let main = {
   let a = handle[hVal(6)] add(val(), val());
   let b = handle[hVal(10)] add(val(), val());
   add(a, b)
};
```

The handlers we have introduced for `Val` all call the `resume` function, but that is not required. Conceptually, all effect operations are executed by the **handle**, hence, if we return from the operation, we return from the **handle**. A handler therefore has great control over control flow.

The `Abort` effect uses this mechanism. It defines a single operation `abort`, which returns from the handler without resuming. To show the flexibility that the framework of algebraic effect handlers, provide we will demonstrate several possible handlers for `Abort`. The first ignores the result of the computation, but still halts execution.

```
let hAbort = handler Abort {
   return(x) { () }
   abort() { () }
};

let main = {
   handle[hAbort] {
      abort();
      f()
   };
   g()
};
```

In the program above, `f` will not get called because `hAbort` does not call the continuation, but `g` will be called, because it is used outside of the **handle**.

Alternatively, we can define a handler that defines a default value for failing expressions.

```
let hAbort = fn(default) {
   handler Abort {
      return(x) { x }
      abort() { default }
```

```
   }
};

let safe_div = fn(x: Int, y: Int) <Abort> Int {
   if eq(y, 0) {
      abort()
   } else {
      div(x, y)
   }
};

let main = handle[hAbort] safe_div(5, 0);
```

We can also map the `Abort` effect to the `Maybe` monad, which is the most common implementation.

Even for small handlers I need custom data types

```
let hAbort = handler Abort {
   return(x) { Just(x) }
   abort() { Nothing() }
};
```

Finally, we can ignore `abort` calls if we are writing an application in which we always want to try to continue execution no matter what errors occur.[1]

```
let hAbort = handler Abort {
   return(x) { x }
   abort() { resume(()) }
};
```

4.2 Higher-Order Effects in Elaine

4.3 Specification of Elaine

4.3.1 Type Checker

Unification of Effect Rows

Talk about (Leijen 2005).

During type checking effect rows are represented as a pair consisting of a multiset of effects and an optional extension variable. In this section we will use a more explicit notation than the syntax of the language by using the multiset representation directly. Hence, a row $\langle A_1, \dots, A_n | e_A \rangle$ is represented as the multiset $\{A_1, \dots, A_n\} + e_A$.

Like with regular Hindley-Milner type inference, two rows can be unified if we can find a substitution of effect row variables that make the rows equal. For effect rows, this yields 3 distinct cases.

If both rows are closed (i.e. have no extension variable) there are no variables to be substituted and we just employ multiset equality. That is, to unify rows A and B we check that $A = B$. If that is true, we do not need to unify further and unification has succeeded. Otherwise, we cannot make any substitutions to make them equal and unification has failed.

[1] With a never type, an alternative definition of `Abort` is possible where this handler is not permitted by the type system. The signature of `abort` would then be `abort() !`, where `!` is the never type and then `resume` could not be called.

If one of the rows is open, then the set of effects in that row need to be a subset of the effects in the other row. To unify the rows

$$A + e_A \quad \text{and} \quad B$$

we assert that $A \subseteq B$. If that is true, we can substitute e_n for the effects in $B - A$.

Finally, there is the case where both rows are open:

$$A + e_A \quad \text{and} \quad B + e_B.$$

In this case, unification is always possible, because both rows can be extended with the effects of the other. We create a fresh effect row variable e_C with the following substitutions:

$$\begin{aligned} e_A &\to (B - A) + e_C \\ e_B &\to (A - B) + e_C. \end{aligned}$$

In other words, A is extended with the effects that are in B but not in A and similarly, B is extended with the effects in A but not in A.

Chapter 5

Elaboration Resolution

- How it works
- Why it works

Chapter 6

Compiling Elaborations to Handlers

- How it works
- Why it works
- Preserves well-typedness
- Why syntactic substitution of elaborations with implementations does not work.
- Consequences for research on higher-order effects in general.

Chapter 7

Conclusion

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Appendix A

Elaine Specification

A.1 Syntax definition

$$
\begin{aligned}
\text{program } p &::= m \dots m \\
\text{module } m &::= \texttt{mod}\ x\ \{d \dots d\} \\
\\
\text{declaration } d &::= \texttt{pub}\ d' \ \mid\ d' \\
d' &::= \texttt{let}\ x = e; \\
&\mid \texttt{import}\ x; \\
&\mid \texttt{effect}\ \phi\ \{s, \dots, s\} \\
&\mid \texttt{type}\ x\ \{s, \dots, s\} \\
\text{expression } e &::= x \\
&\mid () \ \mid\ \texttt{true} \ \mid\ \texttt{false} \\
&\mid \texttt{fn}(x : T, \dots, x : T)\ T\ \{e\} \\
&\mid \texttt{if}\ e\ \{e\}\ \texttt{else}\ \{e\} \\
&\mid e(e, \dots, e) \\
&\mid x!(e, \dots, e) \\
&\mid \texttt{handler}\ \{return(x)\{e\}, o, \dots, o\} \\
&\mid \texttt{handle}[e]\ e \\
&\mid \texttt{elaboration}\ x! \to \Delta\ \{o, \dots, o\} \\
&\mid \texttt{elab}[e]\ e \\
&\mid \texttt{elab}\ e \\
&\mid \texttt{let}\ x = e;\ e \\
&\mid e;\ e \\
&\mid \{e\} \\
\\
\text{signature } s &::= x(T, \dots, T)\ T \\
\text{effect clause } o &::= x(x, \dots, x)\ \{e\} \\
\\
\text{type scheme } \sigma &::= T \ \mid\ \forall \alpha.\sigma \\
\text{type } T &::= \Delta\ \tau \\
\text{value type } \tau &::= x \ \mid\ () \ \mid\ \texttt{Bool} \\
&\mid (T, \dots, T) \to T
\end{aligned}
$$

$$
\begin{aligned}
&\mid \texttt{handler}\ x\ \tau\ \tau \\
&\mid \texttt{elaboration}\ x!\ \Delta \\
\text{effect row } \Delta &::= \langle\rangle \ \mid\ x \ \mid\ \langle \phi | \Delta \rangle \\
\text{effect } \phi &::= x \ \mid\ x!
\end{aligned}
$$

A.2 Typing judgments

The context $\Gamma = (\Gamma_M, \Gamma_V, \Gamma_E, \Gamma_\Phi)$ consists of the following parts:

$$
\begin{aligned}
&\Gamma_M : x \to (\Gamma_V, \Gamma_E, \Gamma_\Phi) && \text{module to context} \\
&\Gamma_V : x \to \sigma && \text{variable to type scheme} \\
&\Gamma_E : x \to (\Delta, \{f_1, \ldots, f_n\}) && \text{higher-order effect to elaboration type} \\
&\Gamma_\Phi : x \to \{s_1, \ldots, s_n\} && \text{effect to operation signatures}
\end{aligned}
$$

A Γ_T for data types might be added.

Whenever one of these is extended, the others are implicitly passed on too, but when declared separately, they not implicitly passed. For example, Γ'' is empty except for the single $x : T$, whereas Γ' implicitly contains Γ_M, Γ_E & Γ_Φ.

$$\Gamma'_V = \Gamma_V, x : T \qquad \Gamma''_V = x : T$$

If the following invariants are violated there should be a type error:

- The operations of all effects in scope must be disjoint.
- Module names are unique in every scope.
- Effect names are unique in every scope.

A.2.1 Effect row semantics

We treat effect rows as multisets. That means that the row $\langle A, B, B, C\rangle$ is simply the multiset $\{A, B, B, C\}$. The $|$ symbol signifies extension of the effect row with another (possibly arbitrary) effect row. The order of the effects is insignificant, though the multiplicity is. We define the operation set as follows:

$$
\begin{aligned}
\text{set}(\varepsilon) = \text{set}(\langle\rangle) &= \emptyset \\
\text{set}(\langle A_1, \ldots, A_n \rangle) &= \{A_1, \ldots, A_n\} \\
\text{set}(\langle A_1, \ldots, A_n | R \rangle) &= \text{set}(\langle A_1, \ldots, A_n \rangle) + \text{set}(R).
\end{aligned}
$$

Note that the extension uses the sum, not the union of the two sets. This means that $\text{set}(\langle A | \langle A \rangle \rangle)$ should yield $\{A, A\}$ instead of $\{A\}$.

Then we get the following equality relation between effect rows A and B:

$$A \cong B \iff \text{set}(A) = \text{set}(B).$$

In typing judgments, the effect row is an overapproximation of the effects that actually used by the expression. We freely use set operations in the typing judgments, implicitly calling the the set function on the operands where required. An omitted effect row is treated as an empty effect row ($\langle\rangle$).

Any effect prefixed with a ! is a higher-order effect, which must elaborated instead of handled. Due to this distinction, we define the operations $H(R)$ and $A(R)$ representing the higher-order and first-order subsets of the effect rows, respectively. The same operators are applied as predicates on individual effects, so the operations on rows are defined as:

$$H(\Delta) = \{\phi \in \Delta \ \mid\ H(\phi)\} \qquad \text{and} \qquad A(\Delta) = \{\phi \in \Delta \ \mid\ A(\phi)\}.$$

A.2.2 Type inference

We have the usual generalize and instantiate rules. But, the generalize rule requires an empty effect row.

Koka requires an empty effect row. Why?

$$\frac{\Gamma \vdash e : \sigma \qquad \alpha \notin \text{ftv}(\Gamma)}{\Gamma \vdash e : \forall \alpha . \sigma} \qquad \frac{\Gamma \vdash e : \forall \alpha . \sigma}{\Gamma \vdash e : \sigma[\alpha \mapsto T']}$$

Where ftv refers to the free type variables in the context.

A.2.3 Expressions

We freely write τ to mean that a type has an empty effect row. That is, we use τ and a shorthand for $\langle\rangle\ \tau$. The Δ stands for an arbitrary effect row. We start with everything but the handlers and elaborations and put them in a separate section.

$$\frac{\Gamma_V(x) = \Delta\ \tau}{\Gamma \vdash x : \Delta\ \tau} \qquad \frac{\Gamma \vdash e : \Delta\ \tau}{\Gamma \vdash \{e\} : \Delta\ \tau} \qquad \frac{\Gamma \vdash e_1 : \Delta\ \tau \qquad \Gamma_V, x : \tau \vdash e_2 : \Delta\ \tau'}{\Gamma \vdash \texttt{let}\ x = e_1; e_2 : \Delta\ \tau'}$$

$$\frac{}{\Gamma \vdash () : \Delta\ ()} \qquad \frac{}{\Gamma \vdash \texttt{true} : \Delta\ \texttt{Bool}} \qquad \frac{}{\Gamma \vdash \texttt{false} : \Delta\ \texttt{Bool}}$$

$$\frac{\Gamma_V, x_1 : T_1, \ldots, x_n : T_n \vdash c : T \qquad T_i = \langle\rangle \tau_i}{\Gamma \vdash \texttt{fn}(x_1 : T_1, \ldots, x_n : T_n)\ T\ \{e\} : \Delta\ (T_1, \ldots, T_n) \to T}$$

$$\frac{\Gamma \vdash e_1 : \Delta\ \texttt{Bool} \qquad \Gamma \vdash e_2 : \Delta\ \tau \qquad \Gamma \vdash e_3 : \Delta\ \tau}{\Gamma \vdash \texttt{if}\ e_1\ \{e_2\}\ \texttt{else}\ \{e_3\}\ : \Delta\ \tau}$$

$$\frac{\Gamma \vdash e : (\tau_1, \ldots, \tau_n) \to \Delta\ \tau \qquad \Gamma \vdash e_i : \Delta\ \tau_i}{\Gamma \vdash e(e_1, \ldots, e_n) : \Delta\ \tau}$$

A.2.4 Declarations and Modules

The modules are gathered into Γ_M and the variables that are in scope are gathered in Γ_V. Each module has a the type of its public declarations. Note that these are not accumulative; they only contain the bindings generated by that declaration. Each declaration has the type of both private and public bindings. Without modifier, the public declarations are empty, but with the `pub` keyword, the private bindings are copied into the public declarations.

$$\frac{\Gamma_{i-1} \vdash m_i : \Gamma_{m_i} \qquad \Gamma_{M,i} = \Gamma_{M,i-1}, \Gamma_{m_i}}{\Gamma_0 \vdash m_1 \ldots m_n : ()}$$

$$\frac{\Gamma_{i-1} \vdash d_i : (\Gamma'_i; \Gamma'_{\text{pub},i}) \qquad \Gamma_i = \Gamma_{i-1}, \Gamma'_i \qquad \Gamma \vdash \Gamma'_{\text{pub},1}, \ldots, \Gamma'_{\text{pub},n}}{\Gamma_0 \vdash \texttt{mod}\ x\ \{d_1 \ldots d_n\} : (x : \Gamma)}$$

$$\frac{\Gamma \vdash d : \Gamma'}{\Gamma \vdash d : (\Gamma'; \varepsilon)} \qquad \frac{\Gamma \vdash d : \Gamma'}{\Gamma \vdash \texttt{pub}\ d : (\Gamma'; \Gamma')} \qquad \frac{}{\Gamma \vdash \texttt{import}\ x : \Gamma_M(x)}$$

$$\frac{\begin{array}{c} f_i = \forall \alpha.(\tau_{i,1}, \ldots, \tau_{i,n_i}) \to \alpha\ x \\ \Gamma'_V = x_1 : f_1, \ldots, x_m : f_m \end{array}}{\Gamma \vdash \texttt{type}\ x\ \{x_1(\tau_{1,1}, \ldots, \tau_{1,n_1}), \ldots, x_m(\tau_{m,1}, \ldots, \tau_{m,n_m})\} : \Gamma'}$$

$$\frac{\Gamma \vdash e : T}{\Gamma \vdash \texttt{let}\ x = e : (x : T)}$$

A.2.5 First-Order Effects and Handlers

Effects are declared with the `effect` keyword. The signatures of the operations are stored in Γ_Φ. The types of the arguments and resumption must all have no effects.

A handler must have operations of the same signatures as one of the effects in the context. The names must match up, as well as the number of arguments and the return type of the expression, given the types of the arguments and the resumption. The handler type then includes the handled effect ϕ, an "input" type τ and an "output" type τ'. In most cases, these will be at least partially generic.

The handle expression will simply add the handled effect to the effect row of the inner expression and use the the input and output type.

$$\frac{s_i = op_i(\tau_{i,1}, \ldots, \tau_{i,n_i}) : \tau_i \qquad \Gamma'_\Phi(x) = \{s_1, \ldots, s_n\}}{\Gamma \vdash \texttt{effect}\ x\ \{s_1, \ldots, s_n\} : \Gamma'}$$

$$\frac{\Gamma \vdash e_h : \texttt{handler}\ \phi\ \tau\ \tau' \qquad \Gamma \vdash e_c : \langle \phi | \Delta \rangle\ \tau}{\Gamma \vdash \texttt{handle}\ e_h\ e_c : \Delta\ \tau'}$$

$$\frac{\begin{array}{c} A(\phi) \qquad \Gamma_\Phi(\phi) = \{s_1, \ldots, s_n\} \qquad \Gamma, x : \tau \vdash e_{\text{ret}} : \tau' \\ \begin{bmatrix} s_i = x_i(\tau_{i,1}, \ldots, \tau_{i,m_i}) \to \tau_i \qquad o_i = x_i(x_{i,1}, \ldots, x_{i,m_i})\ \{e_i\} \\ \Gamma_V, resume : (\tau_i) \to \tau', x_{i,1} : \tau_{i,1}, \ldots, x_{i,i_m} : \tau_{i,i_m} \vdash e_i : \tau' \end{bmatrix}_{1 \le i \le n} \end{array}}{\Gamma \vdash \texttt{handler}\ \{\texttt{return}(x)\{e_{\text{ret}}\}, o_1, \ldots, o_n\} : \texttt{handler}\ \phi\ \tau\ \tau'}$$

A.2.6 Higher-Order Effects and Elaborations

The declaration of higher-order effects is similar to first-order effects, but with exclamation marks after the effect name and all operations. This will help distinguish them from first-order effects.

Elaborations are of course similar to handlers, but we explicitly state the higher-order effect x! they elaborate and which first-order effects Δ they elaborate into. The operations do not get a continuation, so the type checking is a bit different there. As arguments they take the effectless types they specified along with the effect row Δ. Elaborations are not added to the value context, but to a special elaboration context mapping the effect identifier to the row of effects to elaborate into.

Later, we could add more precise syntax for which effects need to be present in the arguments of the elaboration operations.

The elab expression then checks that a elaboration for all higher-order effects in the inner expression are in scope and that all effects they elaborate into are handled.

It is not possible to elab-

$$\frac{s_i = op_i!(\tau_{i,1}, \ldots, \tau_{i,n_i}) : \tau_i \qquad \Gamma'_\Phi(x!) = \{s_1, \ldots, s_n\}}{\Gamma \vdash \texttt{effect}\ x!\ \{s_1, \ldots, s_n\} : \Gamma'}$$

$$\frac{\begin{gathered}\Gamma_\Phi(x!) = \{s_1, \ldots, s_n\} \qquad \Gamma'_E(x!) = \Delta \\ \begin{bmatrix} s_i = x_i!(\tau_{i,1}, \ldots, \tau_{i,m_i})\ \tau_i \qquad o_i = x_i!(x_{i,1}, \ldots, x_{i,m_i})\{e_i\} \\ \Gamma, x_{i,1} : \Delta\ \tau_{i,1}, \ldots, x_{i,n_i} : \Delta\ \tau_{i,n_i} \vdash e_i : \Delta\ \tau_i \end{bmatrix}_{1 \le i \le n}\end{gathered}}{\Gamma \vdash \texttt{elaboration}\ x! \to \Delta\ \{o_1, \ldots, o_n\} : \Gamma'}$$

$$\frac{\left[\Gamma_E(\phi) \subseteq \Delta\right]_{\phi \in H(\Delta')} \qquad \Gamma \vdash e : \Delta'\ \tau \qquad \Delta = A(\Delta')}{\Gamma \vdash \texttt{elab}\ e : \Delta\ \tau}$$

A.3 Desugaring

Fold over the syntax tree with the following operation:

$$\begin{aligned} D(\texttt{fn}(x_1 : T_1, \ldots, x_n : T_n)\ T\ \{e\}) &= \lambda x_1, \ldots, x_n.e \\ D(\texttt{let}\ x = e_1;\ e_2) &= (\lambda x.e_2)(e_1) \\ D(e_1; e_2) &= (\lambda\_.e_2)(e_1) \\ D(\{e\}) &= e \\ D(e) &= e \end{aligned}$$

A.4 Elaboration resolution

A.5 Semantics

A.5.1 Reduction contexts

$$\begin{aligned} E ::= {} & [] \mid E(e_1, \ldots, e_n) \mid v(v_1, \ldots, v_n, E, e_1, \ldots, e_m) \\ & \mid \texttt{if}\ E\ \{e\}\ \texttt{else}\ \{e\} \\ & \mid \texttt{let}\ x = E;\ e \mid E;\ e \\ & \mid \texttt{handle}[E]\ e \mid \texttt{handle}[v]\ E \\ & \mid \texttt{elab}[E]\ e \mid \texttt{elab}[v]\ E \end{aligned}$$

$$\begin{aligned} X_{op} ::= {} & [] \mid X_{op}(e_1, \ldots, e_n) \mid v(v_1, \ldots, v_n, X_{op}, e_1, \ldots, e_m) \\ & \mid \texttt{if}\ X_{op}\ \{e_1\}\ \texttt{else}\ \{e_2\} \\ & \mid \texttt{let}\ x = X_{op};\ e \mid X_{op};\ e \\ & \mid \texttt{handle}[X_{op}]\ e \mid \texttt{handle}[h]\ X_{op}\ \text{if}\ op \notin h \\ & \mid \texttt{elab}[X_{op}]\ e \mid \texttt{elab}[\epsilon]\ X_{op}\ \text{if}\ op! \notin e \end{aligned}$$

A.5.2 Reduction rules

$$
\begin{array}{rcl}
c(v_1, \ldots, v_n) & \longrightarrow & \delta(c, v_1, \ldots, v_n) \\
 & & \quad \text{if } \delta(c, v_1, \ldots, v_n) \text{ defined} \\
(\lambda x_1, \ldots, x_n.e)(v_1, \ldots, v_n) & \longrightarrow & e[x_1 \mapsto v_1, \ldots, x_n \mapsto v_n] \\
\texttt{if true } \{e_1\} \texttt{ else } \{e_2\} & \longrightarrow & e_1 \\
\texttt{if false } \{e_1\} \texttt{ else } \{e_2\} & \longrightarrow & e_2 \\
 & & \\
\texttt{handle}[h]\ v & \longrightarrow & e[x \mapsto v] \\
 & & \quad \text{where } \mathsf{return}(x)\{e\} \in H \\
\texttt{handle}[h]\ X_{op}[op(v_1, \ldots, v_n)] & \longrightarrow & e[x_1 \mapsto v_1, \ldots, x_n \mapsto v_n, resume \mapsto k] \\
 & & \quad \text{where } op(x_1, \ldots, x_n)\{e\} \in h \\
 & & \qquad\quad k = \lambda y\ .\ \texttt{handle}[h]\ X_{op}[y] \\
\texttt{elab}[\epsilon]\ v & \longrightarrow & v \\
\texttt{elab}[\epsilon]\ X_{op!}[op!(e_1, \ldots, e_n)] & \longrightarrow & \texttt{elab}[\epsilon]\ X_{op!}[e[x_1 \mapsto e_1, \ldots, x_n \mapsto e_n]] \\
 & & \quad \text{where } op!(x_1, \ldots, x_n)\{e\} \in \epsilon
\end{array}
$$