1 Symmetric RW on finitely generated groups

The next two results are generalizations of the case of finite stochastic matrices in [DM17, Proposition 3.3] and [DM17, Proposition 3.4], respectively.

Proposition 1.1. *Let Γ be a countable discrete finitely generated group, and μ be an admissible finitely supported symmetric probability measure on Γ. Then, π_z is an irreducible representation for each $z \in \Gamma$.*

Proof. For $z \in \Gamma$, we get that $\mathbb{K}(\mathcal{H}_z) \lhd \pi_z(\mathcal{T}(\Gamma, \mu))$ from [Dor21, Proposition 4.4]. Since the image of π_z contains a copy of compact operators, it is indeed an irreducible subalgebra of $\mathbb{B}(\mathcal{H}_z)$. $\square$

Proposition 1.2. *Let Γ be a countable discrete finitely generated group, and μ be an admissible finitely supported symmetric probability measure on Γ. Then, the representations $\pi_z|_{\mathcal{T}(\Gamma,\mu)}$ and $\pi_{z'}|_{\mathcal{T}(\Gamma,\mu)}$ are not unitarily equivalent for $z, z' \in \Gamma$ such that $z \neq z'$.*

Proof. We suppose that π_z and $\pi_{z'}$ are unitarily equivalent for $z, z' \in \Gamma$ such that $z \neq z'$, i.e. there exists a unitary $U\colon \mathcal{H}_z \to \mathcal{H}_{z'}$ such that $U\pi_z(T) = \pi_{z'}U(T)$ for all $T \in \mathcal{T}(\Gamma, \mu)$. Now, for $m \in \mathbb{N}$, let $Q^{(m)}$ denote the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}^{(m)}$, and for $x \in \Gamma$, define $Q^{(0)}_z := S^{(0)}_{z,z}Q^{(0)} = Q^{(0)}S^{(0)}_{z,z}$. Then, by [Dor21, Proposition 4.4], we get that $Q^{(0)}_z \in \mathcal{T}(\Gamma, \mu)$. Moreover, for $e^{(0)}_{z,z} \in \mathcal{H}_z$, we obtain

$$(\pi_{z'}(Q^{(0)}_z)U)(e^{(0)}_{z,z}) = (U\pi_z(Q^{(0)}_z))(e^{(0)}_{z,z}) = U(e^{(0)}_{z,z}).$$

On the contrary, since $(\pi_{z'}(Q^{(0)})U)(e^{(0)}_{z,z}) = c \cdot e^{(0)}_{z',z'}$ for some non-zero $c \in \mathbb{C}$, we have

$$(\pi_{z'}(Q^{(0)}_z)U)(e^{(0)}_{z,z}) = c \cdot \pi_{z'}(S^{(0)}_{z,z})(e^{(0)}_{z',z'}) = c \cdot \delta_{z,z'}e^{(0)}_{z',z'}.$$

This clearly leads to a contradiction $0 = c \cdot \delta_{z,z'}e^{(0)}_{z',z'} = U(e^{(0)}_{z,z}) \neq 0$. $\square$

Notation. For $n, m \in \mathbb{N}$ and $(x, y) \in \mathrm{Gr}(P)$, denote the operators

$$\begin{aligned} T^{(n)}_{x,y}\colon \mathcal{H}(\Gamma, \mu) &\to \mathcal{H}(\Gamma, \mu), \\ e^{(m)}_{y',z} &\mapsto \delta_{y,y'}\sqrt{\frac{P^m(y, z)}{P^{n+m}(x, z)}}e^{(n+m)}_{x,z}, \end{aligned}$$

where the adjoints are given by

$$(T^{(n)}_{x,y})^*(e^{(n+m)}_{x',z}) = \delta_{x,x'}\sqrt{\frac{P^m(y, z)}{P^{n+m}(x, z)}}e^{(m)}_{y,z}.$$

This construction allows for $T^{(n)}_{x,y}(\mathcal{H}^{(m)}_z) \subset \mathcal{H}^{(n+m)}_z$.

We now present a version of [DM17, Proposition 3.11] for symmetric random walks on finitely generated discrete groups.

Proposition 1.3. *Let Γ be a countable discrete finitely generated group which is not virtually $\mathbb{Z}$ or $\mathbb{Z}^2$, and μ be an admissible finitely supported symmetric probability measure on Γ. Then, π_z is a boundary representation for each $z \in \Gamma$.*

Proof. Similar to [DM17, Proposition 3.11], we will apply [Arv11, Theorem 7.2] to show that each π_z is completely strongly peaking in the sense of Definition (REF). Note using the above decomposition of representations for a given irreducible representation $\rho \neq \pi_z$, we end up with two possibilities. Either ρ is the unique extension of the irreducible representation of compact operators which is unitarily equivalent to a subrepresentation of the identity representation, and, thus, it is unitarily equivalent to some $\pi_{z'}$ for $z' \neq z$. Or ρ annihilates the ideal of compact operators, i.e. it can be expressed uniquely as a composition of another irreducible representation and a quotient map into the Cuntz-Pimsner-Viselter algebra. Hence, we reduce the problem to show that there exists $T \in \mathcal{T}^+(\Gamma, \mu)$ such that

$$\|\pi_z(T)\| > \max\{\sup_{z' \neq z} \|\pi_{z'}(T)\|, \|q(T)\|\}.$$

Using the action of Γ, we further reduce the problem for the first case to showing the inequality just for π_e since $\|\pi_z(T)\| = \|\pi_z \circ \mathrm{Ad}_{z^{-1}}(T)\| = \|\pi_e(T)\|$.
Now, let $z \in \Gamma$ with $z \neq e$. For $n \in \mathbb{N}$, $y \in \Gamma$, and $(x, m) \in \mathrm{ST}_y$, we get $T^{(n)}_{x,y} = \sqrt{\dfrac{1}{P^n(x,y)}} S^{(n)}_{x,y}$. Hence, these operators are bounded and $T^{(n)}_{x,y} \in \mathcal{T}^+(\Gamma, \mu)$. We choose $T := T^{(n)}_{e,e}$ and notice that each $\mathcal{H}^{(m)}_z$ and even the linear spans of each $e^m_{e,z}$ are reducing for both T^*T and TT^*. Thus, these operators are diagonalizable and for a fixed $n \in \mathbb{N}$, we get

$$\|\pi_e(T)\|^2 \geq \|\pi_e((T^{(n)}_{e,e})^*(T^{(n)}_{e,e}))(e^{(0)}_{e,e})\| = \left\| \frac{1}{P^n(e,e)} e^{(0)}_{e,e} \right\| = \frac{1}{P^n(e,e)}$$

and

$$\begin{aligned}
\|\pi_z(T)\|^2 &= \|\pi_z(TT^*)\| = \|TT^*|_{\mathcal{H}_z}\| \\
&= \sup_{m \in \mathbb{N}} \left\|(T^{(n)}_{e,e})(T^{(n)}_{e,e})^*(e^{(m)}_{e,z})\right\| = \sup_{m \in \mathbb{N}} \frac{P^{m-n}(e,z)}{P^m(e,z)} \left\|e^{(m)}_{e,z}\right\| \\
&= \sup_{m \in \mathbb{N}} K_{\mathrm{ST}}((e,n),(z,m)).
\end{aligned}$$

Hence, we have to show the following

$$\sup_{z \neq e} \|\pi_z(T)\|^2 = \sup_{\substack{m \in \mathbb{N} \\ z \neq e}} K_{\mathrm{ST}}((e,n),(z,m)) < \frac{1}{P^n(e,e)} \leq \|\pi_e(T)\|^2 .$$

Note that the space-time Markov kernel vanishes unless $(z, m) \in \mathrm{ST}_e$. Viewing the graph ST_e as a dense subset of the space-time Martin compactification $\Delta_{\mathrm{ST}}\Gamma$ yields that the value of the supremum can be attained on either ST_e itself or on the space-time Martin boundary $\partial_{\mathrm{ST}}\Gamma$.
First, suppose that there exists $(z_0, m_0) \in \mathrm{ST}_e$ for which the value of the supremum is attained. Then, we find $n \in \mathbb{N}$ such that $0 < P^n(e,e) < 1$ and

$$P^n(e,e)P^{m_0}(e,z_0) < P^{n+m_0}(e,z_0).$$

Finally, we obtain

$$K_{\mathrm{ST}}((e,n),(z_0,m_0)) = \frac{P^{m_0-n}(e,z_0)}{P^{m_0}(e,z_0)} < \frac{1}{P^n(e,e)}.$$

Second, for $(z_0, m_0) \in \partial_{\mathrm{ST}}\Gamma$, it follows by the Poisson-Martin representation theorem that

$$K_{\mathrm{ST}}((e,n),(z_0,m_0)) = \int_{\partial^m_{\mathrm{ST}}\Gamma} K_{\mathrm{ST}}((e,n),(r,\zeta))d\nu((r,\zeta)) \leq \sup_{(\lambda,\xi)\in\partial^m_{\mathrm{ST}}\Gamma} K_{\mathrm{ST}}((e,n),(\lambda,\xi))$$

for some probability measure ν on $\partial^m_{\mathrm{ST}}\Gamma$. Now, using Theorem 4.2 (REF!!!), we get

$$K_{\mathrm{ST}}((e,n),(z_0,m_0)) \leq \sup_{(\lambda,\xi)\in\sqcup_{\lambda\in[0,R]}\partial^m_{M,\lambda}\Gamma} K_{\mathrm{ST}}((e,n),(\lambda,\xi)).$$

Now, either $\lambda = 0$ and $\xi \in \partial^m_{M,0}\Gamma$, implying

$$K_{\mathrm{ST}}((e,n),(\lambda,\xi)) = K_0(e,\xi)\chi_{|e|=n} \equiv 0,$$

or $\lambda \in (0, R]$ and $\xi \in \partial^m_{M,\lambda}\Gamma$, so that

$$K_{\mathrm{ST}}((e,n),(\lambda,\xi)) = \lambda^n K_\lambda(e,\xi) \leq R^n.$$

This trivially reduces to the inequality

$$R^n P^n(e,e) < 1,$$

which follows using [Woe00, Theorem 7.8].
Now, note that for $\|\pi_e(T)\| > \|q(T)\|$, it is enough to show

$$\|q(T)\|^2 \leq \|TT^* - Q_e^{(n)}TT^*\| < \frac{1}{P^n(e,e)} \leq \|\pi_e(T)\|^2,$$

where $Q_e^{(n)}TT^*$ is compact as a finite-rank operator. First, we see that

$$\|T\|^2 = \sup_{z\in\Gamma} \|TT^*|_{\mathcal{H}_z}\| = \sup_{z\in\Gamma}\sup_{m\in\mathbb{N}} \frac{P^{m-n}(e,z)}{P^m(e,z)}.$$

Since for any $(z,m) \in \mathrm{ST}_e$, clearly $P^n(e,e)P^{m-n}(e,z) \leq P^m(e,z)$ for $(z,m) \in \mathrm{ST}_e$ and for $(e,n) \in \mathrm{ST}_e$, the equality holds, we have that $\|T\|^2 = \dfrac{1}{P^n(e,e)}$. On the other hand, we get

$$\|TT^* - Q_e^{(n)}TT^*\| = \sup_{z\in\Gamma}\sup_{m>n} \frac{P^{m-n}(e,z)}{P^m(e,z)},$$

which we have shown above to satisfy the strict inequality. Hence, this concludes the proof. □