

# 1 Symmetric RW on finitely generated groups

[DOMA17]

The next two results are generalizations of the case of finite stochastic matrices in [DOMA17] and [DOMA17], respectively.

**Proposition 1.1.** *Let  $\Gamma$  be a countable discrete finitely generated group, and  $\mu$  be an admissible finitely supported symmetric probability measure on  $\Gamma$ . Then,  $\pi_z$  is an irreducible representation for each  $z \in \Gamma$ .*

*Proof.* For  $z \in \Gamma$ , we get that  $\mathbb{K}(\mathcal{H}_z) \triangleleft \pi_z(\mathcal{T}(\Gamma, \mu))$  from [DOAD21]. Since the image of  $\pi_z$  contains a copy of compact operators, it is indeed an irreducible subalgebra of  $\mathbb{B}(\mathcal{H}_z)$ .  $\square$

**Proposition 1.2.** *Let  $\Gamma$  be a countable discrete finitely generated group, and  $\mu$  be an admissible finitely supported symmetric probability measure on  $\Gamma$ . Then, the representations  $\pi_z|_{\mathcal{T}(\Gamma, \mu)}$  and  $\pi_{z'}|_{\mathcal{T}(\Gamma, \mu)}$  are not unitarily equivalent for  $z, z' \in \Gamma$  such that  $z \neq z'$ .*

*Proof.* We suppose that  $\pi_z$  and  $\pi_{z'}$  are unitarily equivalent for  $z, z' \in \Gamma$  such that  $z \neq z'$ , i.e. there exists a unitary  $U: \mathcal{H}_z \rightarrow \mathcal{H}_{z'}$  such that  $U\pi_z(T) = \pi_{z'}U(T)$  for all  $T \in \mathcal{T}(\Gamma, \mu)$ . Now, for  $m \in \mathbb{N}$ , let  $Q^{(m)}$  denote the orthogonal projection from  $\mathcal{H}$  onto  $\mathcal{H}^{(m)}$ , and for  $x \in \Gamma$ , define  $Q_z^{(0)} := S_{z,z}^{(0)}Q^{(0)} = Q^{(0)}S_{z,z}^{(0)}$ . Then, by [DOAD21], we get that  $Q_z^{(0)} \in \mathcal{T}(\Gamma, \mu)$ . Moreover, for  $e_{z,z}^{(0)} \in \mathcal{H}_z$ , we obtain

$$(\pi_{z'}(Q_z^{(0)})U)(e_{z,z}^{(0)}) = (U\pi_z(Q_z^{(0)}))(e_{z,z}^{(0)}) = U(e_{z,z}^{(0)}).$$

On the contrary, since  $(\pi_{z'}(Q_z^{(0)})U)(e_{z,z}^{(0)}) = c \cdot e_{z',z'}^{(0)}$  for some non-zero  $c \in \mathbb{C}$ , we have

$$(\pi_{z'}(Q_z^{(0)})U)(e_{z,z}^{(0)}) = c \cdot \pi_{z'}(S_{z,z}^{(0)})(e_{z',z'}^{(0)}) = c \cdot \delta_{z,z'} e_{z',z'}^{(0)}.$$

This clearly leads to a contradiction  $0 = c \cdot \delta_{z,z'} e_{z',z'}^{(0)} = U(e_{z,z}^{(0)}) \neq 0$ .  $\square$

*Notation.* For  $n, m \in \mathbb{N}$  and  $(x, y) \in \text{Gr}(P)$ , denote the operators

$$T_{x,y}^{(n)}: \mathcal{H}(\Gamma, \mu) \rightarrow \mathcal{H}(\Gamma, \mu),$$

$$e_{y',z}^{(m)} \mapsto \delta_{y,y'} \sqrt{\frac{P^m(y, z)}{P^{n+m}(x, z)}} e_{x,z}^{(n+m)},$$

where the adjoints are given by

$$(T_{x,y}^{(n)})^*(e_{x',z}^{(n+m)}) = \delta_{x,x'} \sqrt{\frac{P^m(y, z)}{P^{n+m}(x, z)}} e_{y,z}^{(m)}.$$

This construction allows for  $T_{x,y}^{(n)}(\mathcal{H}_z^{(m)}) \subset \mathcal{H}_z^{(n+m)}$ .

We now present a version of [DOMA17] for symmetric random walks on finitely generated discrete groups.

**Proposition 1.3.** *Let  $\Gamma$  be a countable discrete finitely generated group which is not virtually  $\mathbb{Z}$  or  $\mathbb{Z}^2$ , and  $\mu$  be an admissible finitely supported symmetric probability measure on  $\Gamma$ . Then,  $\pi_z$  is a boundary representation for each  $z \in \Gamma$ .*

*Proof.* Similar to [DOMA17], we will apply [ARWI11] to show that each  $\pi_z$  is completely strongly peaking in the sense of Definition (REF). Note using the above decomposition of representations for a given irreducible representation  $\rho \neq \pi_z$ , we end up with two possibilities. Either  $\rho$  is the unique extension of the irreducible representation of compact operators which is unitarily equivalent to a subrepresentation of the identity representation, and, thus, it is unitarily equivalent to some  $\pi_{z'}$  for  $z' \neq z$ . Or  $\rho$  annihilates the ideal of compact operators, i.e. it can be expressed uniquely as a composition of another irreducible representation and a quotient map into the Cuntz-Pimsner-Viselter algebra. Hence, we reduce the problem to show that there exists  $T \in \mathcal{T}^+(\Gamma, \mu)$  such that

$$\|\pi_z(T)\| > \max\{\sup_{z' \neq z} \|\pi_{z'}(T)\|, \|q(T)\|\}.$$

Using the action of  $\Gamma$ , we further reduce the problem for the first case to showing the inequality just for  $\pi_e$  since  $\|\pi_z(T)\| = \|\pi_z \circ \text{Ad}_{z^{-1}}(T)\| = \|\pi_e(T)\|$ .

Now, let  $z \in \Gamma$  with  $z \neq e$ . For  $n \in \mathbb{N}$ ,  $y \in \Gamma$ , and  $(x, m) \in \text{ST}_y$ , we get  $T_{x,y}^{(n)} = \sqrt{\frac{1}{P^n(x, y)}} S_{x,y}^{(n)}$ . Hence, these operators are bounded and  $T_{x,y}^{(n)} \in \mathcal{T}^+(\Gamma, \mu)$ . We choose  $T := T_{e,e}^{(n)}$  and notice that each  $\mathcal{H}_z^{(m)}$  and even the linear spans of each  $e_{e,z}^m$  are reducing for both  $T^*T$  and  $TT^*$ . Thus, these operators are diagonalizable and for a fixed  $n \in \mathbb{N}$ , we get

$$\|\pi_e(T)\|^2 \geq \|\pi_e((T_{e,e}^{(n)})^*(T_{e,e}^{(n)}))(e_{e,e}^{(0)})\| = \left\| \frac{1}{P^n(e, e)} e_{e,e}^{(0)} \right\| = \frac{1}{P^n(e, e)}$$

and

$$\begin{aligned} \|\pi_z(T)\|^2 &= \|\pi_z(TT^*)\| = \|TT^*|_{\mathcal{H}_z}\| \\ &= \sup_{m \in \mathbb{N}} \|(T_{e,e}^{(n)})(T_{e,e}^{(n)})^*(e_{e,z}^{(m)})\| = \sup_{m \in \mathbb{N}} \frac{P^{m-n}(e, z)}{P^m(e, z)} \|e_{e,z}^{(m)}\| \\ &= \sup_{m \in \mathbb{N}} K_{\text{ST}}((e, n), (z, m)). \end{aligned}$$

Hence, we have to show the following

$$\sup_{\substack{z \neq e \\ m \in \mathbb{N} \\ z \neq e}} \|\pi_z(T)\|^2 = \sup_{\substack{m \in \mathbb{N} \\ z \neq e}} K_{\text{ST}}((e, n), (z, m)) < \frac{1}{P^n(e, e)} \leq \|\pi_e(T)\|^2.$$

Note that the space-time Markov kernel vanishes unless  $(z, m) \in \text{ST}_e$ . Viewing the graph  $\text{ST}_e$  as a dense subset of the space-time Martin compactification  $\Delta_{\text{ST}}\Gamma$  yields that the value of the supremum can be attained on either  $\text{ST}_e$  itself or on the space-time Martin boundary  $\partial_{\text{ST}}\Gamma$ .

First, suppose that there exists  $(z_0, m_0) \in \text{ST}_e$  for which the value of the supremum is attained. Then, we find  $n \in \mathbb{N}$  such that  $0 < P^n(e, e) < 1$  and

$$P^n(e, e)P^{m_0}(e, z_0) < P^{n+m_0}(e, z_0).$$

Finally, we obtain

$$K_{\text{ST}}((e, n), (z_0, m_0)) = \frac{P^{m_0-n}(e, z_0)}{P^{m_0}(e, z_0)} < \frac{1}{P^n(e, e)}.$$

Second, for  $(z_0, m_0) \in \partial_{\text{ST}}\Gamma$ , it follows by the Poisson-Martin representation theorem that

$$K_{\text{ST}}((e, n), (z_0, m_0)) = \int_{\partial_{\text{ST}}^m \Gamma} K_{\text{ST}}((e, n), (r, \zeta)) d\nu((r, \zeta)) \leq \sup_{(\lambda, \xi) \in \partial_{\text{ST}}^m \Gamma} K_{\text{ST}}((e, n), (\lambda, \xi))$$

for some probability measure  $\nu$  on  $\partial_{\text{ST}}^m \Gamma$ . Now, using Theorem 4.2 (REF!!!), we get

$$K_{\text{ST}}((e, n), (z_0, m_0)) \leq \sup_{(\lambda, \xi) \in \sqcup_{\lambda \in [0, R]} \partial_{M, \lambda}^m \Gamma} K_{\text{ST}}((e, n), (\lambda, \xi)).$$

Now, either  $\lambda = 0$  and  $\xi \in \partial_{M, 0}^m \Gamma$ , implying

$$K_{\text{ST}}((e, n), (\lambda, \xi)) = K_0(e, \xi) \chi_{|e|=n} \equiv 0,$$

or  $\lambda \in (0, R]$  and  $\xi \in \partial_{M, \lambda}^m \Gamma$ , so that

$$K_{\text{ST}}((e, n), (\lambda, \xi)) = \lambda^n K_\lambda(e, \xi) \leq R^n.$$

This trivially reduces to the inequality

$$R^n P^n(e, e) < 1,$$

which follows using [WOWO00].

Now, note that for  $\|\pi_e(T)\| > \|q(T)\|$ , it is enough to show

$$\|q(T)\|^2 \leq \|TT^* - Q_e^{(n)}TT^*\| < \frac{1}{P^n(e, e)} \leq \|\pi_e(T)\|^2,$$

where  $Q_e^{(n)}TT^*$  is compact as a finite-rank operator. First, we see that

$$\|T\|^2 = \sup_{z \in \Gamma} \|TT^*|_{\mathcal{H}_z}\| = \sup_{z \in \Gamma} \sup_{m \in \mathbb{N}} \frac{P^{m-n}(e, z)}{P^m(e, z)}.$$

Since for any  $(z, m) \in \text{ST}_e$ , clearly  $P^n(e, e)P^{m-n}(e, z) \leq P^m(e, z)$  for  $(z, m) \in \text{ST}_e$  and for  $(e, n) \in \text{ST}_e$ , the equality holds, we have that  $\|T\|^2 = \frac{1}{P^n(e, e)}$ . On the other hand, we get

$$\|TT^* - Q_e^{(n)}TT^*\| = \sup_{z \in \Gamma} \sup_{m > n} \frac{P^{m-n}(e, z)}{P^m(e, z)},$$

which we have shown above to satisfy the strict inequality. Hence, this concludes the proof.  $\square$