1 Ideals and representations

1.1 Space-time Martin boundary

We now provide definitions and notation for the space-time Martin boundary as in [REF].

Definition 1.1 (Space-time graph)**.** Let Γ be a countable discrete group, and μ be an admissible probability measure on Γ. Then for $z \in \Gamma$, the set of vertices

$$\mathrm{ST}_z = \mathrm{ST}_z(\Gamma, \mu) := \{(y, m) \in \Gamma \times \mathbb{Z}_+ \,:\, P^m(y, z) > 0\},$$

where we add an edge between $(x, n), (y, m) \in \mathrm{ST}_z$ if $m = n+1$ and $P(x, y) > 0$, is called the *space-time graph* of the random walk (Γ, μ).

Definition 1.2 (Space-time Martin chain)**.** Let Γ be a countable discrete group, and μ be an admissible probability measure on Γ. Then the Markov chain $(Y_n)_{n\in\mathbb{N}}$ on ST_e with the transition probabilities given by

$$\mathbb{P}[Y_{n+1} = (y, m+1) \,|\, Y_n = (x, m)] = P(x, y),$$

is called the *space-time Martin chain.*

Remark. Note that for ST_z, the transition probabilities take the form

$$\mathbb{P}[Y_{n+1} = (y, m+1) \,|\, Y_n = (x, m)] =?,$$

Remark. Transience and reducibility

The following proposition serves as a definition for a space-time Martin kernel.

Proposition 1.1 (REF)**.** *Let Γ be a countable discrete group, and μ be an admissible probability measure on Γ. Then the corresponding space-time Martin kernel is given by*

$$\begin{aligned} K_{ST_e} \colon \mathrm{ST}_e \times \mathrm{ST}_e &\to (0, \infty), \\ ((x, m), (y, n)) &\mapsto \frac{P^{n-m}(x, y)}{P^n(e, y)}. \end{aligned}$$

Remark. Using the above remark, we see that for an arbitrary $z \in \Gamma$, the space-time Martin kernel is given by

$$\begin{aligned} K_{\mathrm{ST}_z} \colon \mathrm{ST}_z \times \mathrm{ST}_z &\to (0, \infty), \\ ((x, m), (y, n)) &\mapsto ?. \end{aligned}$$

Definition 1.3 (Space-time Martin compactification)**.** Let Γ be a countable discrete group, and μ be an admissible probability measure on Γ. Then the 1-Martin compactification of the space-time Markov chain associated with (Γ, μ) is called the *space-time Martin compactification* of Γ and is denoted by $\Delta_{\mathrm{ST}}\Gamma$. We denote the corresponding space-time Martin boundary by $\partial_{\mathrm{ST}}\Gamma$.

Proposition 1.2 (REF)**.** *Let Γ be a countable discrete group, μ be an admissible probability measure on Γ. Then a sequence $(y_k, n_k)_{k\in\mathbb{N}}$ in ST_e converges to a point $\xi \in \partial_{ST}\Gamma$ if for all $(x, m) \in \mathrm{ST}_e$,*

$$\frac{P^{n-m}(x, y)}{P^n(e, y)} \xrightarrow{k\to\infty} h_\xi(x, m)$$

for some function $h_\xi \colon \mathrm{ST}_e \to \mathbb{R}_+$.

1.2 Toeplitz algebra and representations

Definition 1.4 (Fock space)**.** Let Γ be a countable discrete group, and μ be an admissible probability measure on Γ. Then, the Hilbert space

$$\mathcal{H} = \mathcal{H}(\Gamma, \mu) := \bigoplus_{z\in\Gamma} \ell^2(\mathrm{ST}_z(\Gamma, \mu))$$

is called the *Fock space* of the random walk (Γ, μ). For each $z \in \Gamma$, we let $\{e^{(m)}_{x,z}\}_{(x,m)\in\mathrm{ST}_z}$ denote the orthonormal basis of the space

$$\mathcal{H}_z = \mathcal{H}_z(\Gamma, \mu) := \ell^2(\mathrm{ST}_z(\Gamma, \mu)).$$

Remark. Notice that trough the identification $\oplus_{m\in\mathbb{N}}\ell^2(\mathrm{ST}^{(m)}_z) = \ell^2(\sqcup_{m\in\mathbb{N}} \mathrm{ST}^{(m)}_z)$ we obtain the following natural decomposition $\oplus_{z\in\Gamma}\mathcal{H}_z = \oplus_{z\in\Gamma} \oplus_{m\in\mathbb{N}} \mathcal{H}^{(m)}_z$, where we denote $\mathcal{H}^{(m)}_z := \ell^2(\mathrm{ST}^{(m)}_z)$.

Remark. Note that the map $\mathrm{ST}_z \to \mathrm{ST}_{g^{-1}z}, (y, m) \mapsto (g^{-1}y, m)$ induces a canonical action of the group $\Gamma \curvearrowright \mathcal{H}(\Gamma, \mu)$ via the unitaries

$$\begin{aligned} U_g \colon \mathcal{H}(\Gamma, \mu) &\to \mathcal{H}(\Gamma, \mu), \\ e^{(m)}_{x,z} &\mapsto e^{(m)}_{g^{-1}x,g^{-1}z}. \end{aligned}$$

These maps are well-defined since $U_g(\mathcal{H}_z) = \mathcal{H}_{g^{-1}z}$, and each of them sends an orthonormal basis to an orthonormal basis.

Remark. The space ... in [DM17] is isometrically isomorphic to the Fock space above. We change the notation to allude to the space-time Markov chain defined on the space-time graph.

Notation. For $n, m \in \mathbb{N}$ and $(x, y) \in \mathrm{Gr}(P)$, denote the bounded operators

$$\begin{aligned} S^{(n)}_{x,y} \colon \mathcal{H}(\Gamma, \mu) &\to \mathcal{H}(\Gamma, \mu), \\ e^{(m)}_{y',z} &\mapsto \delta_{y,y'} \sqrt{\frac{P^n(x, y)P^m(y, z)}{P^{n+m}(x, z)}} e^{(n+m)}_{x,z}, \end{aligned}$$

where the adjoints are given by

$$(S^{(n)}_{x,y})^*(e^{(n+m)}_{x',z}) = \delta_{x,x'} \sqrt{\frac{P^m(x, y)P^n(y, z)}{P^{m+n}(x, z)}} e^{(n)}_{y,z}.$$

Moreover, we have that $S^{(n)}_{x,y}(\mathcal{H}^{(m)}_z) \subset \mathcal{H}^{(n+m)}_z$.

Definition 1.5 (Toeplitz algebra, tensor algebra)**.** Let Γ be a countable discrete group, and μ be an admissible probability measure on Γ. Then,

(i) the C^*-algebra

$$\mathcal{T}(\Gamma, \mu) := C^*(S^{(n)}_{x,y} \,:\, (x, y) \in \mathrm{Gr}(P^n), n \in \mathbb{N}) \subset \mathbb{B}(\mathcal{H}(\Gamma, \mu)) \tag{1}$$

is called the *Toeplitz algebra* associated with the random walk (Γ, μ);

(ii) the operator algebra

$$\mathcal{T}^+(\Gamma,\mu) := \overline{\mathrm{Alg}}^{\|\cdot\|}(S^{(n)}_{x,y} \,:\, (x,y)\in \mathrm{Gr}(P^n), n\in\mathbb{N}) \subset \mathbb{B}(\mathcal{H}(\Gamma,\mu)) \tag{2}$$

is called the *tensor algebra* associated with the random walk (Γ,μ).

Definition 1.6 (Cuntz algebra)**.** Let Γ be a countable discrete group, and μ be an admissible probability measure on Γ. Then, the space

$$\mathcal{J} = \mathcal{J}(\Gamma,\mu) := \mathcal{T}(\Gamma,\mu) \cap \prod_{z\in\Gamma} \mathbb{K}(\mathcal{H}_z(\Gamma,\mu)) \tag{3}$$

is called the *Cuntz ideal* of $\mathcal{T}(\Gamma,\mu)$, and the quotient C^*-algebra

$$\mathcal{O} = \mathcal{O}(\Gamma,\mu) := \mathcal{T}(\Gamma,\mu)/\mathcal{J}(\Gamma,\mu) \tag{4}$$

is called the *Cuntz algebra* associated with the random walk (Γ,μ).

Definition 1.7 (Cuntz-Pimsner-Viselter algebra)**.** Let Γ be a countable discrete group, and μ be an admissible probability measure on Γ. Then, the space

$$\mathcal{I} = \mathcal{I}(\Gamma,\mu) := \bigoplus_{z\in\Gamma} \mathbb{K}(\mathcal{H}_z(\Gamma,\mu)) \tag{5}$$

is called the *Cuntz-Pimsner-Viselter ideal* of $\mathcal{T}(\Gamma,\mu)$, and the quotient C^*-algebra

$$\mathcal{V} = \mathcal{V}(\Gamma,\mu) := \mathcal{T}(\Gamma,\mu)/\mathcal{I}(\Gamma,\mu) \tag{6}$$

is called the *Cuntz-Pimsner-Viselter algebra* associated with the random walk (Γ,μ).

Notation. For $z\in\Gamma$, we let π_z denote the representations of $\mathcal{T}(\Gamma,\mu)$ given by

$$\begin{aligned} \pi_z \colon \mathcal{T}(\Gamma,\mu) &\to \mathbb{B}(\mathcal{H}_z(\Gamma,\mu)), \\ T &\mapsto T|_{\mathcal{H}_z}, \end{aligned} \tag{7}$$

and for $n\in\mathbb{N}$, we let $\pi^{(n)}_z$ denote the representations of n-th amplification $M_n(\mathcal{T}(\Gamma,\mu))$ given by

$$\begin{aligned} \pi^{(n)}_z \colon M_n(\mathcal{T}(\Gamma,\mu)) &\to \mathbb{B}(\mathcal{H}_z(\Gamma,\mu)^{\oplus n}), \\ [T_{ij}]^n_{i,j=1} &\mapsto [T_{ij}|_{\mathcal{H}_z}]^n_{i,j=1}. \end{aligned} \tag{8}$$

Remark. We can extend an action $\Gamma \curvearrowright \mathcal{H}(\Gamma,\mu)$ via unitaries to a natural action $\Gamma \curvearrowright \mathcal{T}(\Gamma,\mu)$ via isometric $*$-isomorphisms

$$\begin{aligned} \mathrm{Ad}_g \colon \mathcal{T}(\Gamma,\mu) &\to \mathcal{T}(\Gamma,\mu), \\ S^{(n)}_{x,y} &\mapsto U_g S^{(n)}_{x,y} U_g^{-1}. \end{aligned}$$

These maps are well-defined since for $g\in\Gamma$, $n,m\in\mathbb{N}$, and $(x,y)\in\mathrm{Gr}(P)$, we get

$$\begin{aligned} (U_g S^{(n)}_{x,y} U_g^{-1})(e^{(m)}_{y',z}) &= U_g S^{(n)}_{x,y}(e^{(m)}_{gy',gz}) \\ &= U_g \left(\delta_{y,gy'} \sqrt{\frac{P^n(x,y)P^m(y,gz)}{P^{n+m}(x,gz)}} e^{(n+m)}_{x,gz} \right) \\ &= \delta_{y,gy'} \sqrt{\frac{P^n(x,y)P^m(y,gz)}{P^{n+m}(x,gz)}} e^{(n+m)}_{g^{-1}x,z} \\ &= \delta_{g^{-1}y,y'} \sqrt{\frac{P^n(g^{-1}x,g^{-1}y)P^m(g^{-1}y,z)}{P^{n+m}(g^{-1}x,z)}} e^{(n+m)}_{g^{-1}x,z} \\ &= S^{(n)}_{g^{-1}x,g^{-1}y}(e^{(m)}_{y',z}). \end{aligned}$$

This action leaves the tensor algebra invariant, and, moreover, is compatible with the representations above in the sense that $\pi_z \circ \mathrm{Ad}_g = \pi_{gz}$ as for $g \in \Gamma$, $n \in \mathbb{N}$ and $(x, y) \in \mathrm{Gr}(P)$, we obtain

$$\pi_z \circ \mathrm{Ad}_g(S^{(n)}_{x,y}) = S^{(n)}_{g^{-1}x,g^{-1}y}|_{\mathcal{H}_z} = S^{(n)}_{x,y}|_{\mathcal{H}_{gz}} = \pi_{gz}(S^{(n)}_{x,y}).$$

Note that the representations are not unitarily equivalent for different elements of the group, so Γ can not act via unitaries.

From [Dor21, Proposition 4.4], we see that $\mathcal{I}(\Gamma, \mu)$ is an ideal in the Toeplitz algebra $\mathcal{T}(\Gamma, \mu)$ whenever μ is finitely supported. Similar to [DM17, Section 3], using the following short exact sequence

$$0 \to \mathcal{I}(\Gamma, \mu) \to \mathcal{T}(\Gamma, \mu) \to \mathcal{V}(\Gamma, \mu) \to 0$$

and the discussion preceding [Arv76, Theorem 1.3.4], we see that given any representation ρ, it decomposes uniquely into a central direct sum $\rho = \rho_{\mathcal{I}} \oplus \rho_{\mathcal{V}}$ of representations, where $\rho_{\mathcal{I}}$ is the unique extension of $\rho|_{\mathcal{I}(\Gamma,\mu)}$ to $\mathcal{T}(\Gamma, \mu)$, and $\rho_{\mathcal{V}}$ annihilates $\mathcal{I}(\Gamma, \mu)$. Note that for an irreducible representation, one of the summands is trivial.