problem 1 $f(x,y) = (y-x^2)^2 + (1-x)^2$

$$\frac{\partial f}{\partial x}(x,y) = 2(y-x^2)(-2x) + 2(1-x)(-1) = -4x(y-x^2) - 2(1-x)$$

$$\frac{\partial f}{\partial y}(x,y) = 2(y-x^2)$$

So we need to solve the nonlinear system $\begin{cases} -4x(y-x^2) - 2(1-x) = 0 & (1) \\ 2(y-x^2) = 0 & (2) \end{cases}$

From (2) we get $\boxed{y = x^2}$

and then from (1) we get $x = 1$. So there is a unique critical point at $(1,1)$

Problem 2

$$f(x_1, x_2) = \left(\begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix} \right) \cdot \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix} = \begin{bmatrix} a(x_1 - x_1^*) + b(x_2 - x_2^*) \\ b(x_1 - x_1^*) + c(x_2 - x_2^*) \end{bmatrix} \cdot \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

$$= a(x_1 - x_1^*)^2 + 2b(x_1 - x_1^*)(x_2 - x_2^*) + c(x_2 - x_2^*)^2$$

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = 2a(x_1 - x_1^*) + 2b(x_2 - x_2^*)$$

$$\frac{\partial f}{\partial x_2}(x_1, x_2) = 2b(x_1 - x_1^*) + 2c(x_2 - x_2^*)$$

$$\nabla f(x_1, x_2) = 2 \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

problem 3 $f(x_1,x_2) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} ax_1 + bx_2 \\ cx_1 + dx_2 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = ax_1^2 + bx_1x_2 + cx_1x_2 + dx_2^2$

$= ax_1^2 + (b+c)x_1x_2 + dx_2^2$

$$\frac{\partial f}{\partial x_1}(x_1,x_2) = 2ax_1 + (b+c)x_2$$

$$\frac{\partial f}{\partial x_2}(x_1,x_2) = (b+c)x_1 + 2dx_2$$

So $\nabla f(x_1,x_2) = 2\begin{bmatrix} a & \frac{b+c}{2} \\ \frac{b+c}{2} & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

For the hessian we have $\frac{\partial^2 f}{\partial x_1 \partial x_1} = 2a$ $\quad \frac{\partial^2 f}{\partial x_1 \partial x_2} = b+c$

$\frac{\partial^2 f}{\partial x_1 \partial x_2} = b+c$ $\quad \frac{\partial^2 f}{\partial x_2 \partial x_2} = 2d$

Problem 5

$$f(x_1,x_2) = \frac{1}{2}\left(0.8\,x_1^2 + 0.5\,x_1x_2 + 0.7\,x_2^2\right) + 1.55\,x_1 + 2.35\,x_2 + 0.5$$

$$\frac{\partial f}{\partial x_1}(x_1,x_2) = \frac{1}{2}\left(1.6\,x_1 + 0.5\,x_2\right) + 1.55$$

$$\frac{\partial f}{\partial x_2}(x_1,x_2) = \frac{1}{2}\left(0.5\,x_1 + 1.4\,x_2\right) + 2.35$$

So to solve $\begin{cases} \frac{\partial f}{\partial x_1}(x_1,x_2) = 0 \\ \frac{\partial f}{\partial x_2}(x_1,x_2) = 0 \end{cases}$ we need to solve $\begin{cases} 1.6\,x_1 + 0.5\,x_2 = -3.10 \\ 0.5\,x_1 + 1.4\,x_2 = -4.70 \end{cases}$

The solution is $x_1 = -1$ and $x_2 = -3$

Problem 6

We have $\boxed{f(x) = \frac{1}{2}(Ax)\cdot x + b\cdot x}$ where $A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$ and $b = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Since A is symmetric we have $\nabla f(x) = Ax + b$

So to solve $\nabla f(x) = 0$ we need to solve $Ax = -b$, that is

$$\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$ The solution is $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$

So the minimum of this function occurs at $(-2,-1)$

<u>Remark</u> "Expanding" this function gives

$$f(x_1,x_2) = \frac{1}{2}\left(3x_1^2 - 2x_1x_2 + 3x_2^2\right) + 5x_1 + x_2$$

Problem 7 we already computed the gradient in problem 1

$$\frac{\partial \ell}{\partial x}(x,y) = -4x\left(y-x^2\right) - 2(1-x) = -4xy + 4x^3 - 2 + 2x$$

$$\frac{\partial \ell}{\partial y}(x,y) = 2\left(y-x^2\right)$$

Let's compute the Hessian :

$$\frac{\partial^2 \ell}{\partial x^2} = -4y + 12x^2 + 2 \qquad \frac{\partial^2 \ell}{\partial x \partial y} = -4x$$

$$\frac{\partial^2 \ell}{\partial y \partial x} = -4x \qquad \frac{\partial^2 \ell}{\partial y^2} = 2$$

Problem 8

Recall that Newton's method to solve the system $\begin{cases} f_1(x,y)=0 \\ f_2(x,y)=0 \end{cases}$ consists

in solving $F(x^{(0)}) + DF(x^{(0)})(x - x^{(0)}) = 0$ where $F(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \end{bmatrix}$

initial guess

and $DF(x,y) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$

In our case we want to solve

$\begin{cases} \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases}$ so $f_1(x,y)$ is $\frac{\partial f}{\partial x}(x,y)$

and $f_2(x,y)$ is $\frac{\partial f}{\partial y}(x,y)$

this is the Hessian matrix!

and $DF(x,y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}$

So Newton's method becomes

$$\nabla f(x^{(0)}) + Hf(x^{(0)})(x - x_0) = 0$$

This is Newton's method for minimization

<u>Summary</u> Applying Newton's method to minimize $f(x,y)$ is the exact same thing than applying Newton's method to solve the system

$$\begin{cases} \dfrac{\partial f}{\partial x}(x,y) = 0 \\[2ex] \dfrac{\partial f}{\partial y}(x,y) = 0 \end{cases}$$

Problem 9

$$f(x,y) = \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1)^2 + y e^{-x}$$

$$\frac{\partial f}{\partial x}(x,y) = x-1 - y e^{-x} \qquad \frac{\partial f}{\partial y}(x,y) = y-1 + e^{-x}$$

Now let's compute the hessian:

$$\frac{\partial^2 f}{\partial x^2} = 1 + y e^{-x} \qquad \frac{\partial^2 f}{\partial x \partial y} = - e^{-x}$$

$$\frac{\partial^2 f}{\partial y \partial x} = - e^{-x} \qquad \frac{\partial^2 f}{\partial y^2} = 1$$