Homework lec18

1. Consider the function
$$f(x,y) = (y-x^2)^2 + (1-x)^2$$
   (a) Plot this function using contour(). Try to get a feel for where the minimum is located. Use the window $-2 \le x \le 2$ and $-2 \le y \le 2$ at first then zoom in.

   (b) Solve the system
$$\frac{\partial f}{\partial x}(x,y) = 0$$
$$\frac{\partial f}{\partial y}(x,y) = 0$$
   with pen and paper in order to find the critical point(s). Note that this is a nonlinear system of equation! It is a pain to solve in general but this one is easy.

2. (a) Compute the gradient and Hessian matrix of the function
$$f(x_1,x_2) = (Ax)\cdot x = \left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\cdot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
   To do this first expand everything, then start computing the partial derivatives.

   (b) Compute the gradient of the function
$$f(x_1,x_2) = \Big(A(x-x^*)\Big)\cdot(x-x^*) = \left(\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}\right)\cdot\begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$
   Show that gradient can be written as:
$$\nabla f(x_1,x_2) = 2\Big(A(x-x^*)\Big) = 2\begin{bmatrix} a & b \\ b & c \end{bmatrix}\begin{bmatrix} x_1 - x_1^* \\ x_2 - x_2^* \end{bmatrix}$$

3. Compute the gradient and Hessian matrix of the function
$$f(x_1,x_2) = \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)\cdot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
   Note that in this problem the matrix is not symmetric. Show that gradient and Hessian can be written as:
$$\nabla f(x_1,x_2) = 2\begin{bmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{and} \qquad Hf(x_1,x_2) = 2\begin{bmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{bmatrix}$$

4. Compute the gradient and Hessian of
$$f(x_1,x_2) = v\cdot x = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\cdot\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$
   Explain why the Hessian is the way it is.

5. Consider the quadratic function
$$f(x_1,x_2) = \frac{1}{2}\left(0.8x_1^2 + 0.5x_1x_2 + 0.7x_2^2\right) + 1.55x_1 + 2.35x_2 + 0.5$$

(a) Use plot_wireframe to plot $f(x_1, x_2)$ on the window $-30 \leq x_1 \leq 30$ and $-30 \leq x_2 \leq 30$.

(b) Do the same plot, but this time use coutour(). Use the window $-5 \leq x_1 \leq 0$ and $-5 \leq x_2 \leq 0$. Use many contour lines. Try from the plot to identify roughly where the minimum is located.

(c) Solve the system

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = 0$$
$$\frac{\partial f}{\partial x_2}(x_1, x_2) = 0$$

in order to find the critical point. Note that this is a **linear system**. You can either solve it by hand or with linalg.solve(). Note that solving this system is equivalent to solving

$$\nabla f(x_1, x_2) = 0$$

6. Consider the quadratic function:

$$f(x_1, x_2) = \frac{1}{2}\left(\begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(a) "Expand" this function so that it is in the format

$$f(x_1, x_2) = ax_1^2 + bx_1x_2 + cx_2^2 + dx_1 + ex_2 + h$$

where a, b, c, d, e, h are numbers.

(b) Use plot_wireframe to plot $f(x_1, x_2)$ on the window $-30 \leq x_1 \leq 30$ and $-30 \leq x_2 \leq 30$. Use the "expanded" format as opposed to the matrix format to do this plot!

(c) Do the same plot, but this time use coutour instead of wireframe. Do it on the window $-3 \leq x_1 \leq 3$ and $-3 \leq x_2 \leq 3$. Try from the plot to identify roughly where is the minimum is located.

(d) Solve the system

$$\frac{\partial f}{\partial x_1}(x_1, x_2) = 0$$
$$\frac{\partial f}{\partial x_2}(x_1, x_2) = 0$$

in order to find the critical point. Note that this is a **linear system**. You can either solve it by hand or with linalg.solve().

7. Let's go back to the function from problem 1:

$$f(x, y) = (y - x^2)^2 + (1 - x)^2$$

You have already computed the location of the minimum with pen and paper. Now use Newton's algorithms to find the minimum. You should obviously get the same answer (or there is a bug in your code). Start with the initial guess $(x^{(0)}, y^{(0)}) = (-1, -1)$. Make sure that Newton's method recover the solution that you found with pen and paper in problem 1.

8. Here is an important problem. We are still working with the function

$$f(x, y) = (y - x^2)^2 + (1 - x)^2$$

from problem 1. But this time, instead of using Newton's method for minimization, let's use Newton's method to solve the nonlinear system of equations

$$\frac{\partial f}{\partial x}(x, y) = 0$$
$$\frac{\partial f}{\partial y}(x, y) = 0$$

Note that you solved this nonlinear system with pen and paper in problem 1. Here you are asked to solve it using Newton's method. Start with the initial guess $(x^{(0)}, y^{(0)}) = (-1, -1)$. How does this compare with what you did in the previous problem?

9. Consider the function

$$f(x, y) = \frac{1}{2}(x-1)^2 + \frac{1}{2}(y-1)^2 + ye^{-x}$$

   (a) Use contour() to plot the function on the window $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$. Use 100 contour lines.

   (b) As you see there is a local min in the top right corner of the plot. Use Newton's method to find the x and y coordinate of this local min with 5 digits accuracy after the decimal. What is the value of f at this local min.

   (c) What is the value of f at $x = -2$ and $y = -2$.

   (d) Do Newton's method with initial guess $x = -5$ and $y = -5$. Try to explain the behavior.