

Single Patch Parametrization

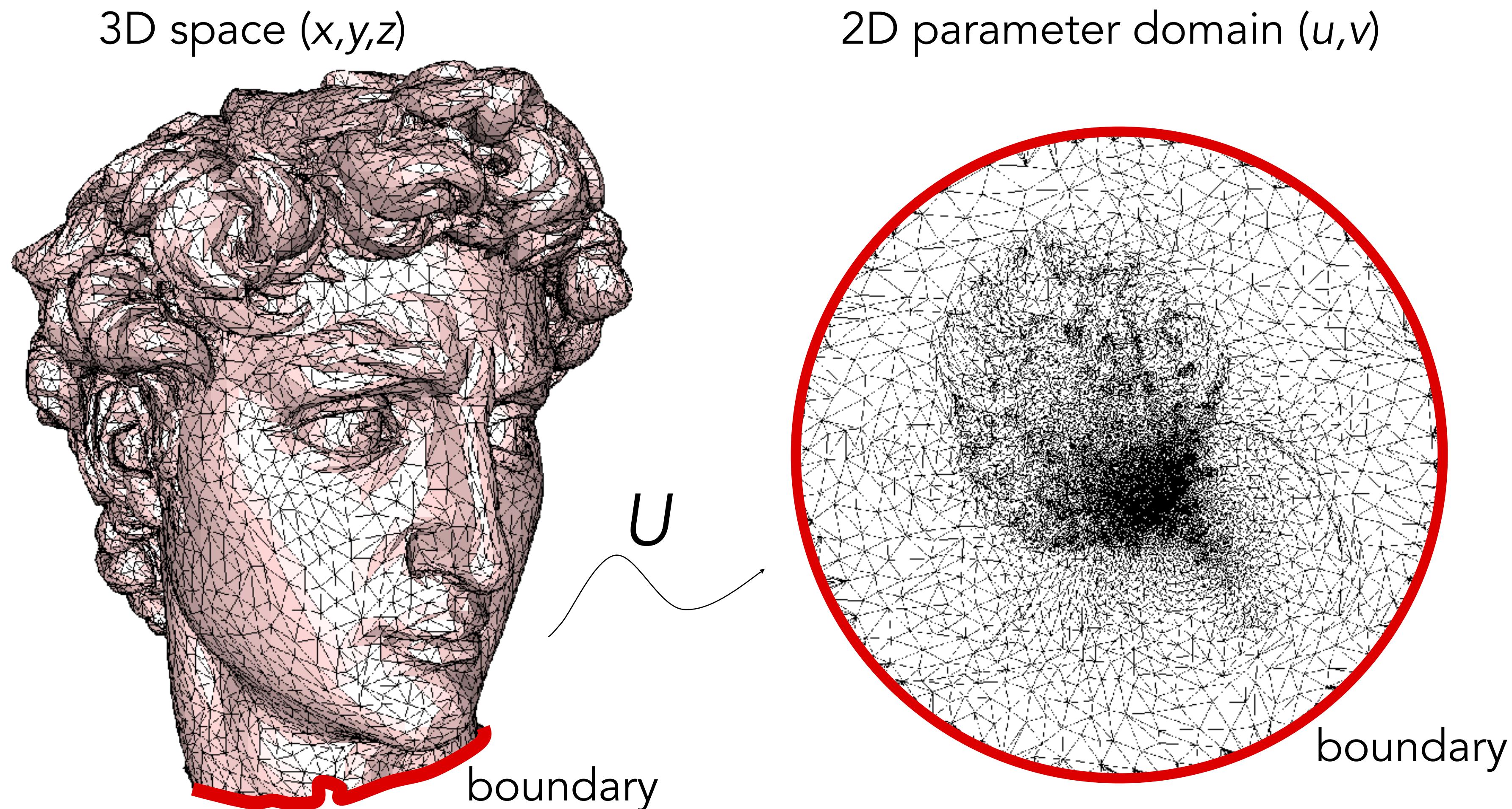
Acknowledgements: Olga Sorkine-Hornung
CSC 486B/586B - Geometric Modeling - Teseo Schneider



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Surface Parameterization



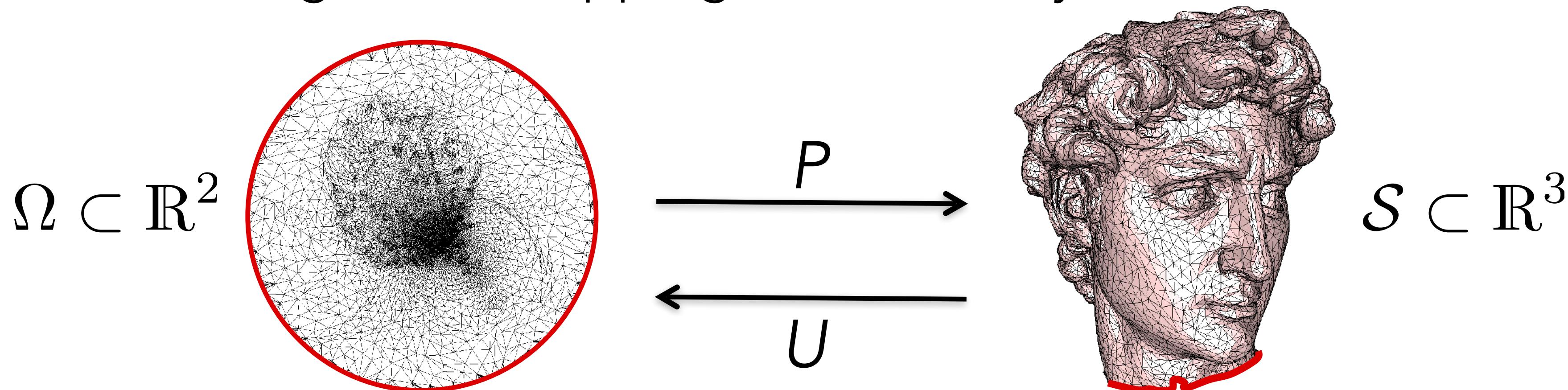
Parameterization – Definition

- Mapping P between a 2D domain Ω and the mesh S embedded in 3D (the inverse = flattening)

- Each mesh vertex has a corresponding 2D position:

$$U(\mathbf{v}_i) = (u_i, v_i)$$

- Inside each triangle, the mapping is affine (barycentric coordinates)

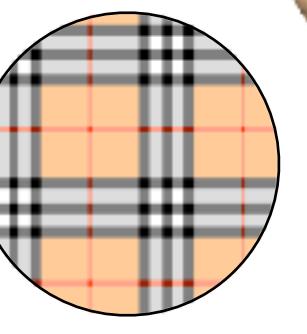
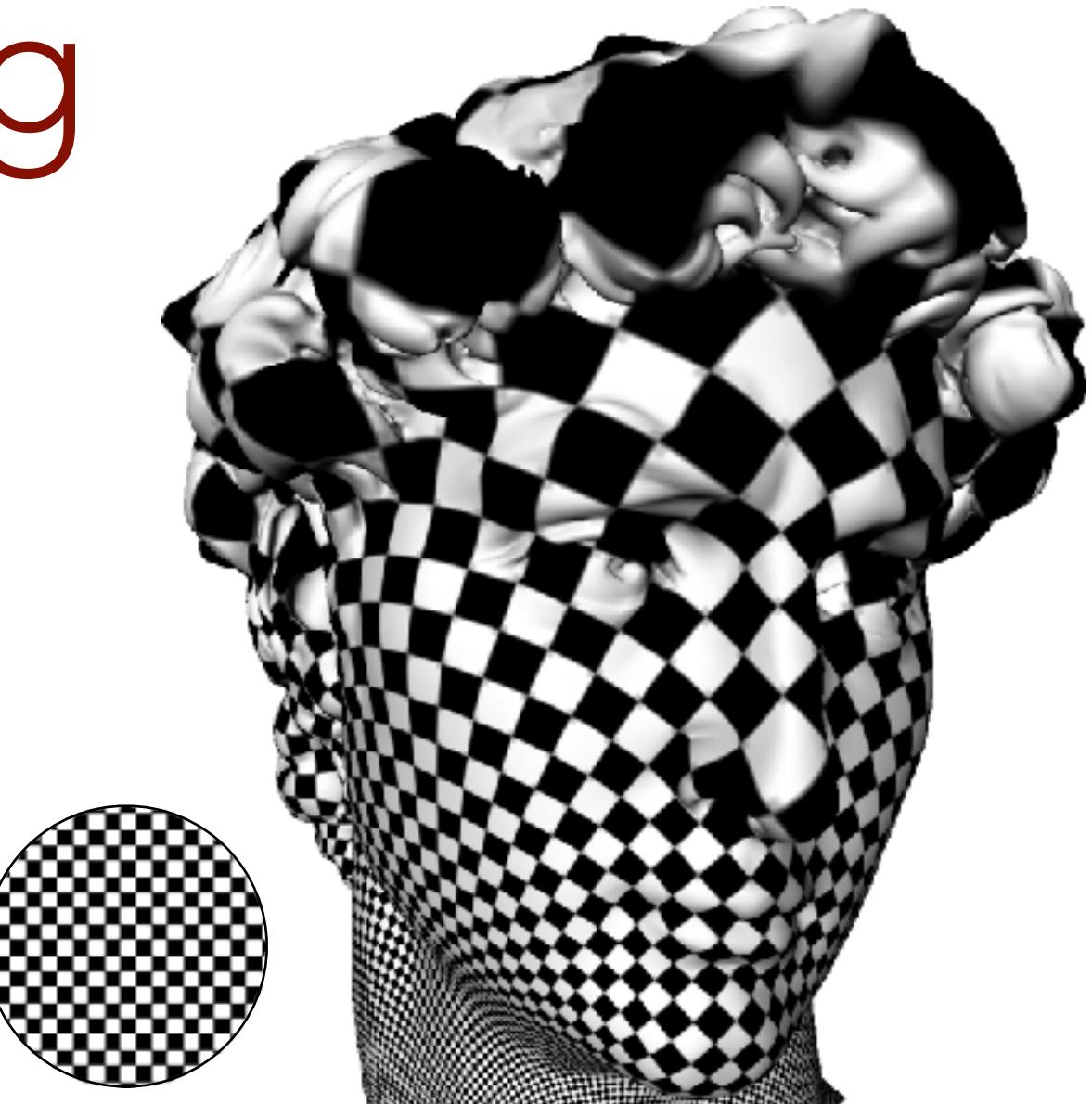
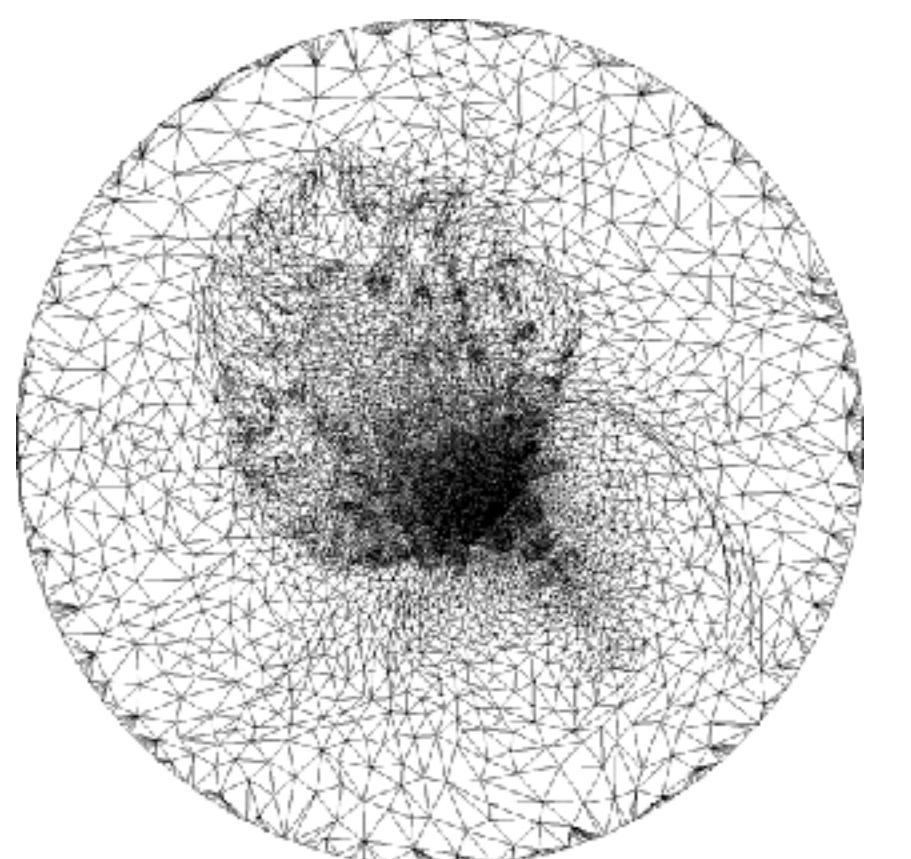


Why?

Why?

- Allows us to do processing in 2D and then map the results onto the 3D surface
- It is easier to operate in the 2D domain (but you have to create a parametrization first)

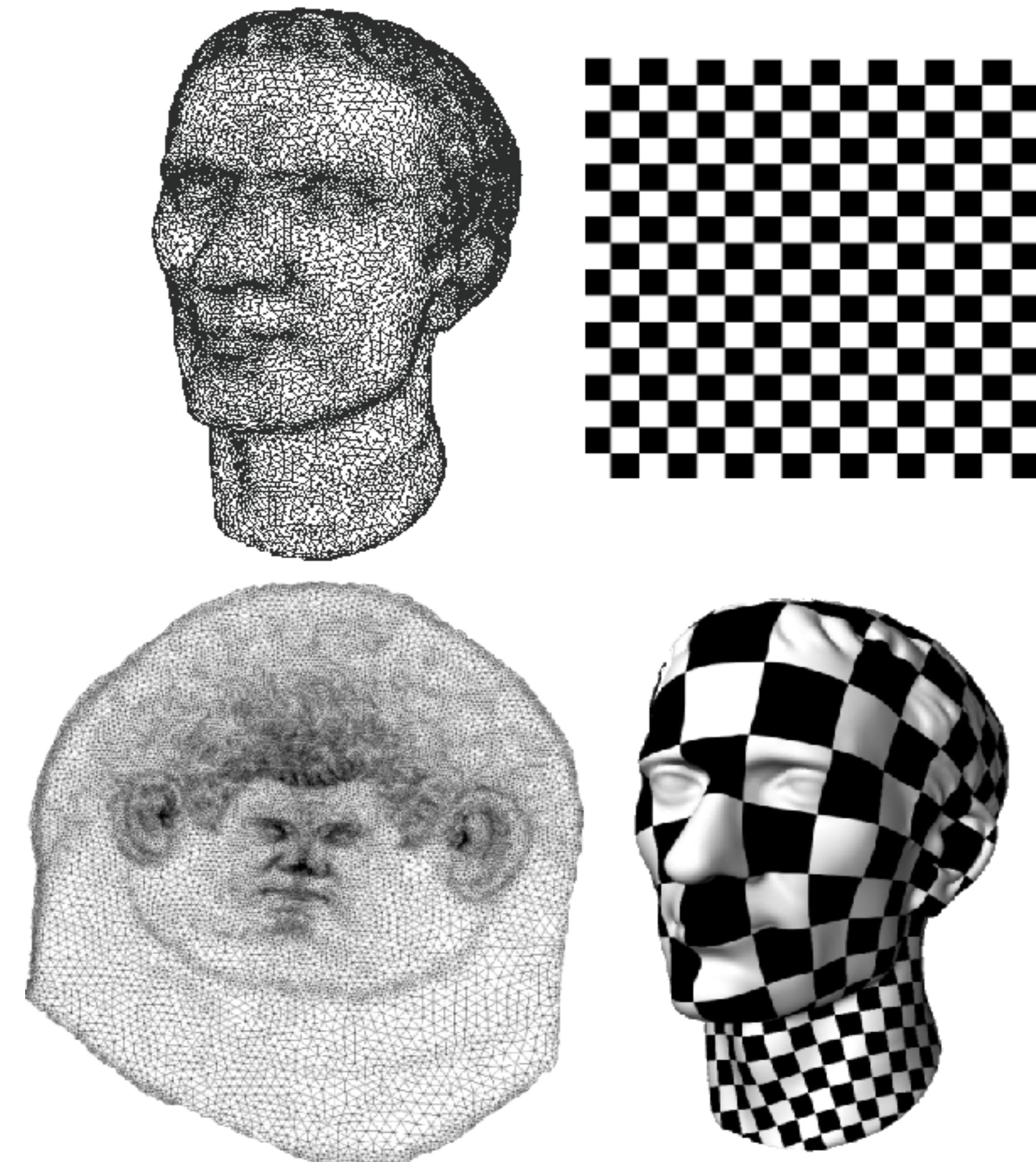
Texture Mapping



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Texture Mapping

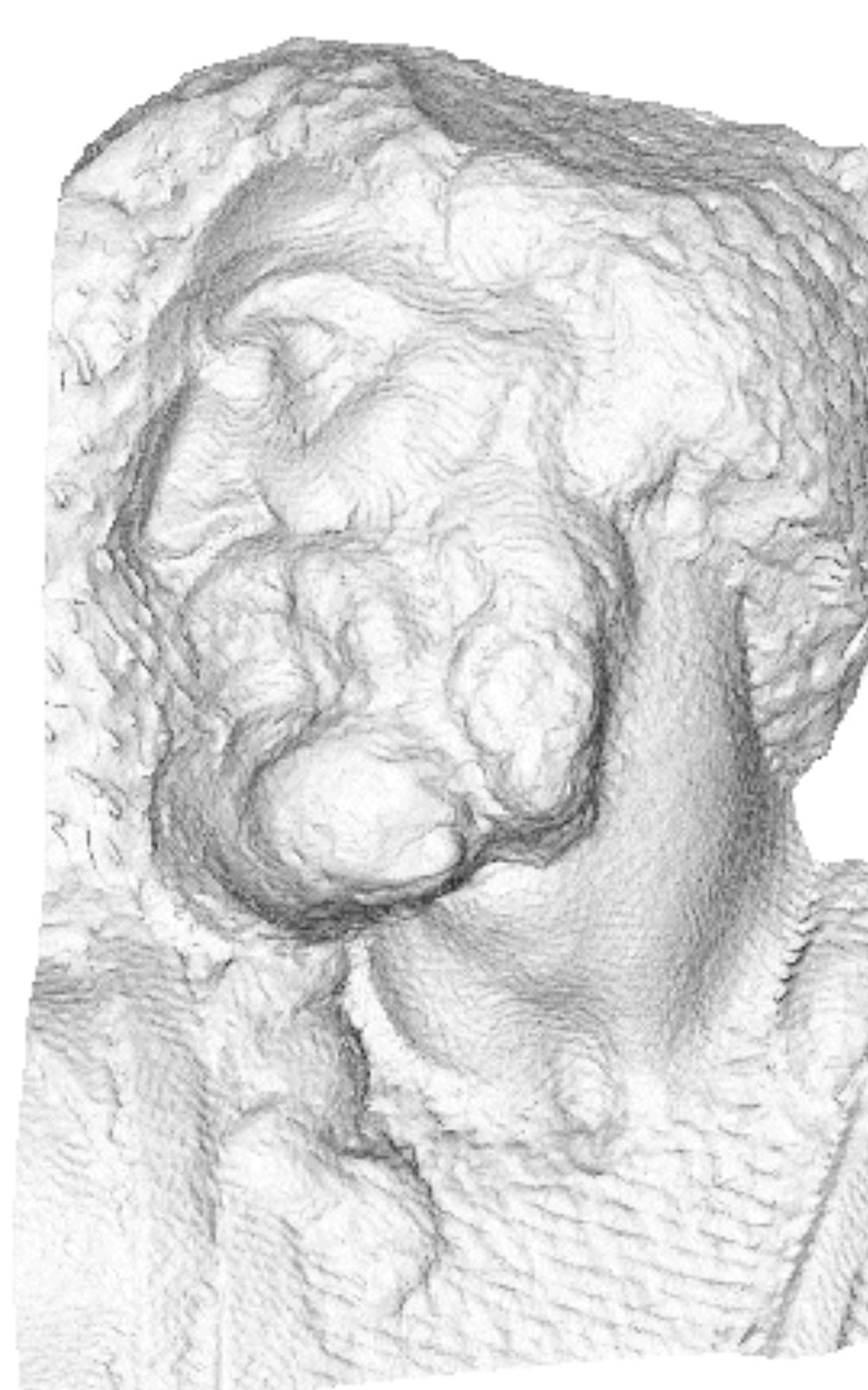


Texture Mapping

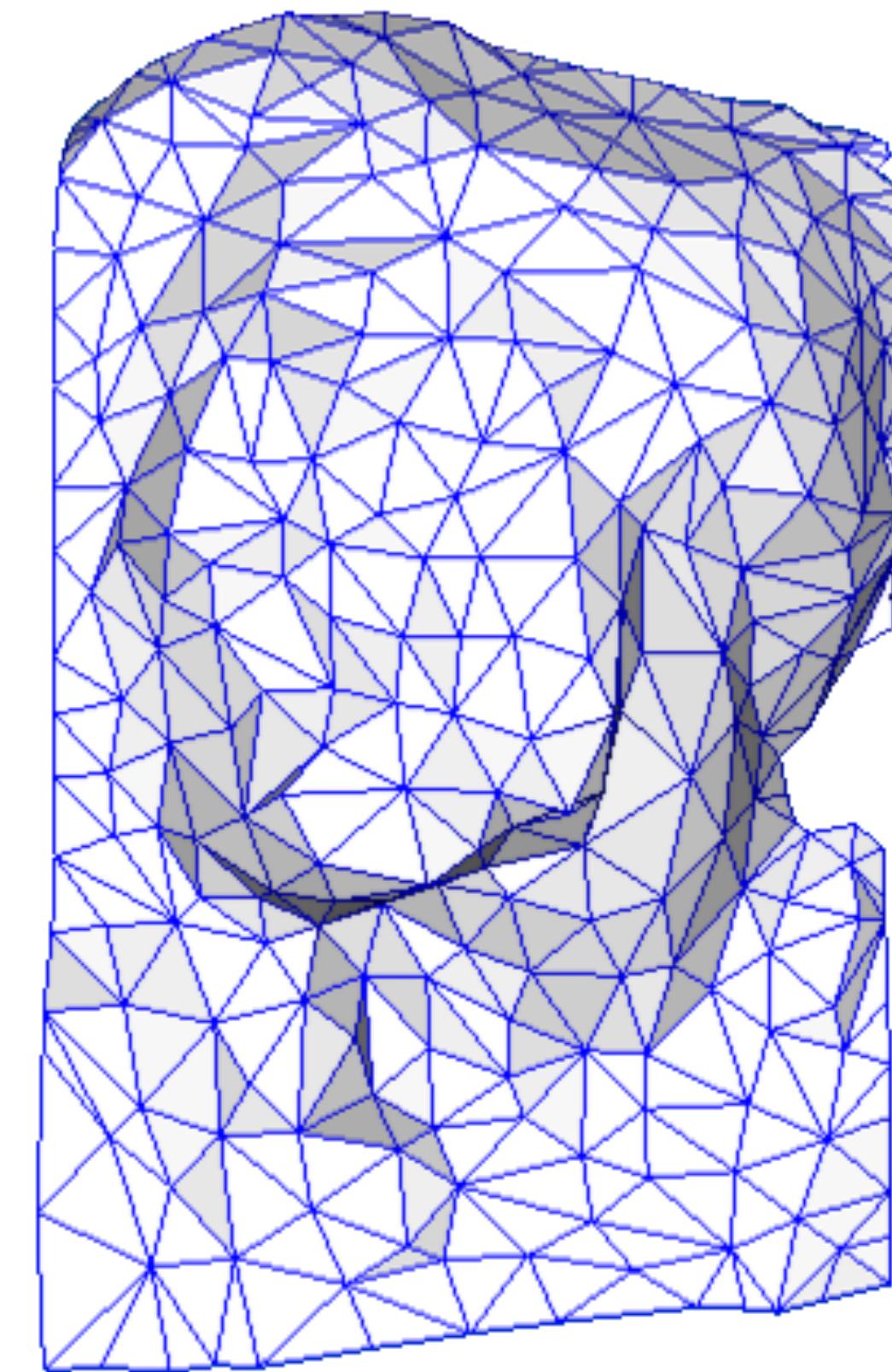


Image from Vallet and Levy, techreport INRIA

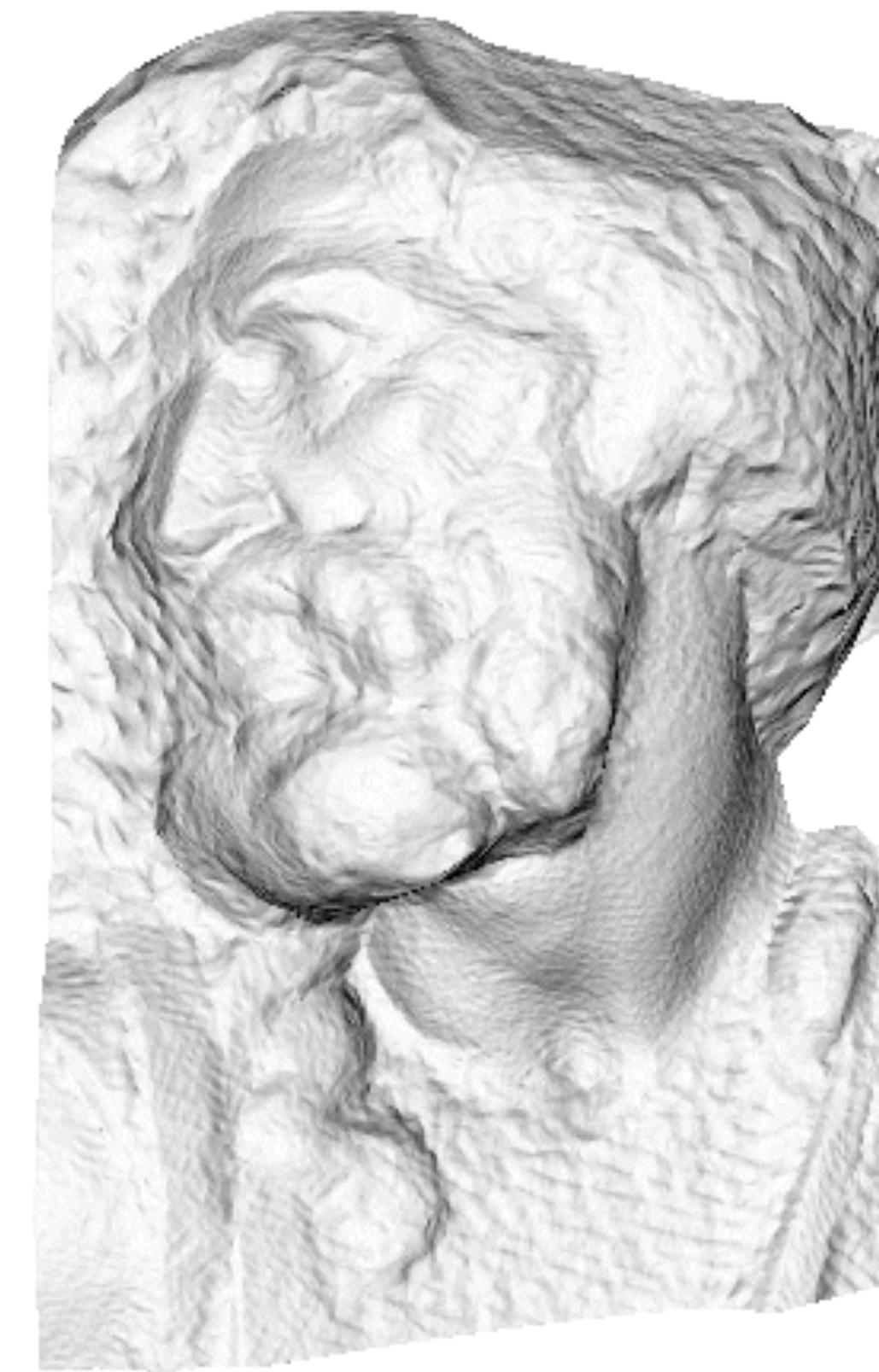
Normal/Bump Mapping



original mesh
4M triangles

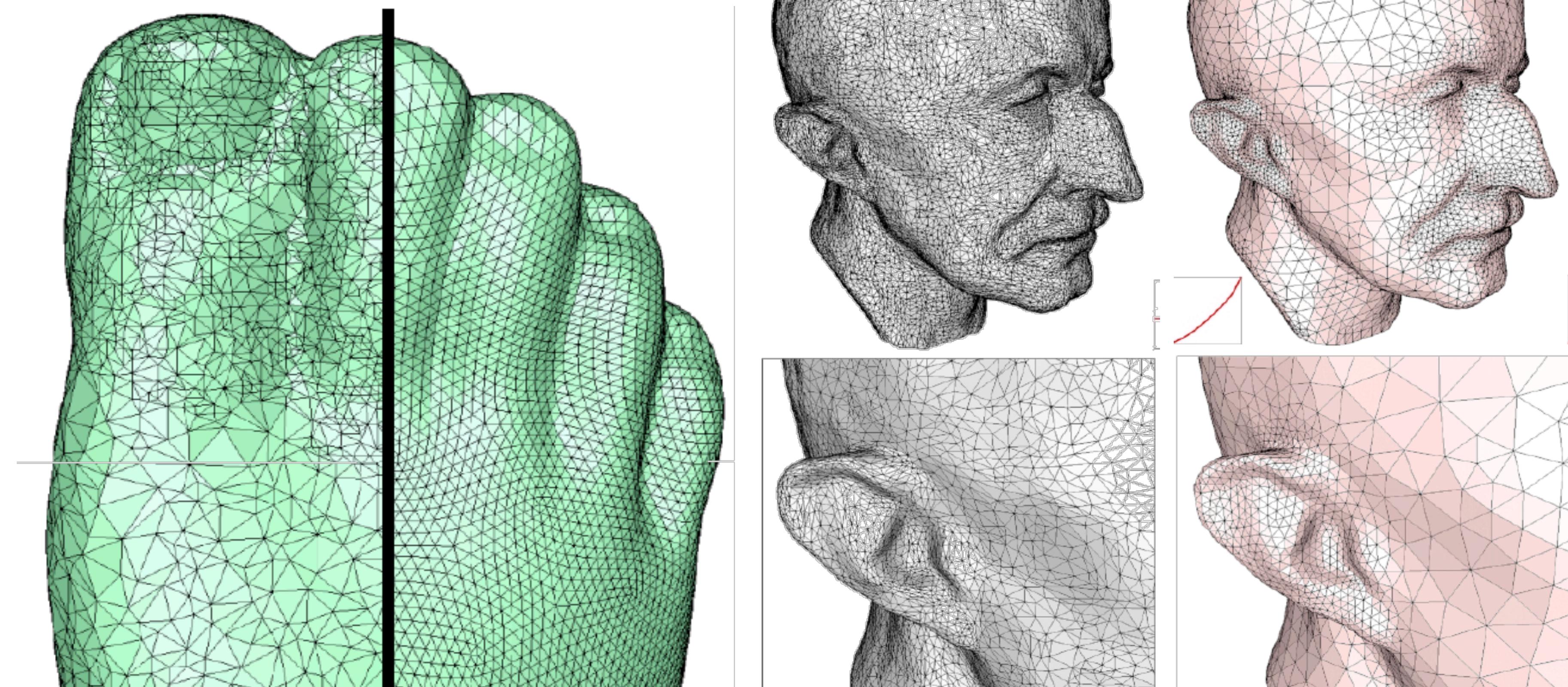


simplified mesh
500 triangles



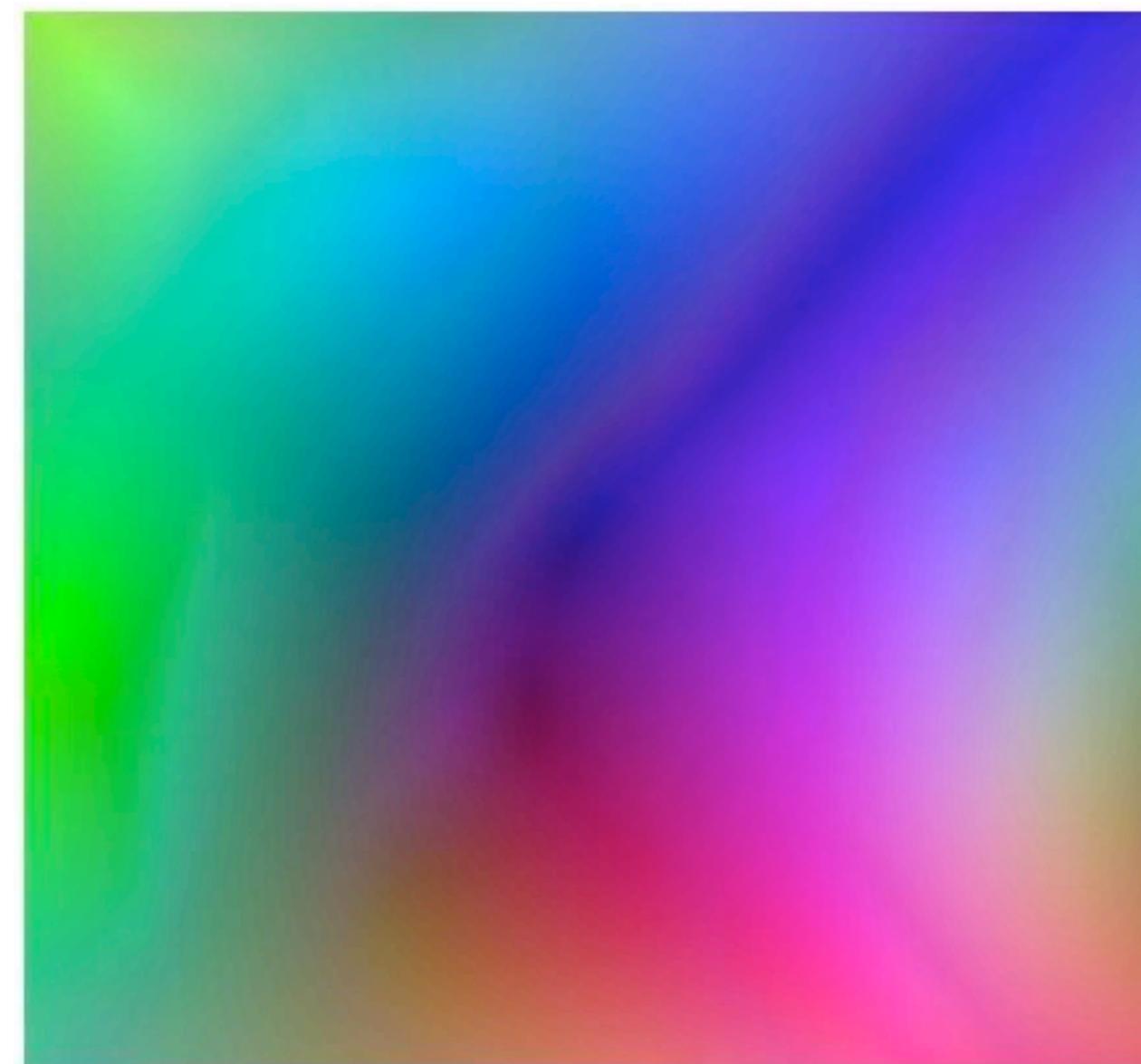
simplified mesh
and normal mapping
500 triangles

Remeshing

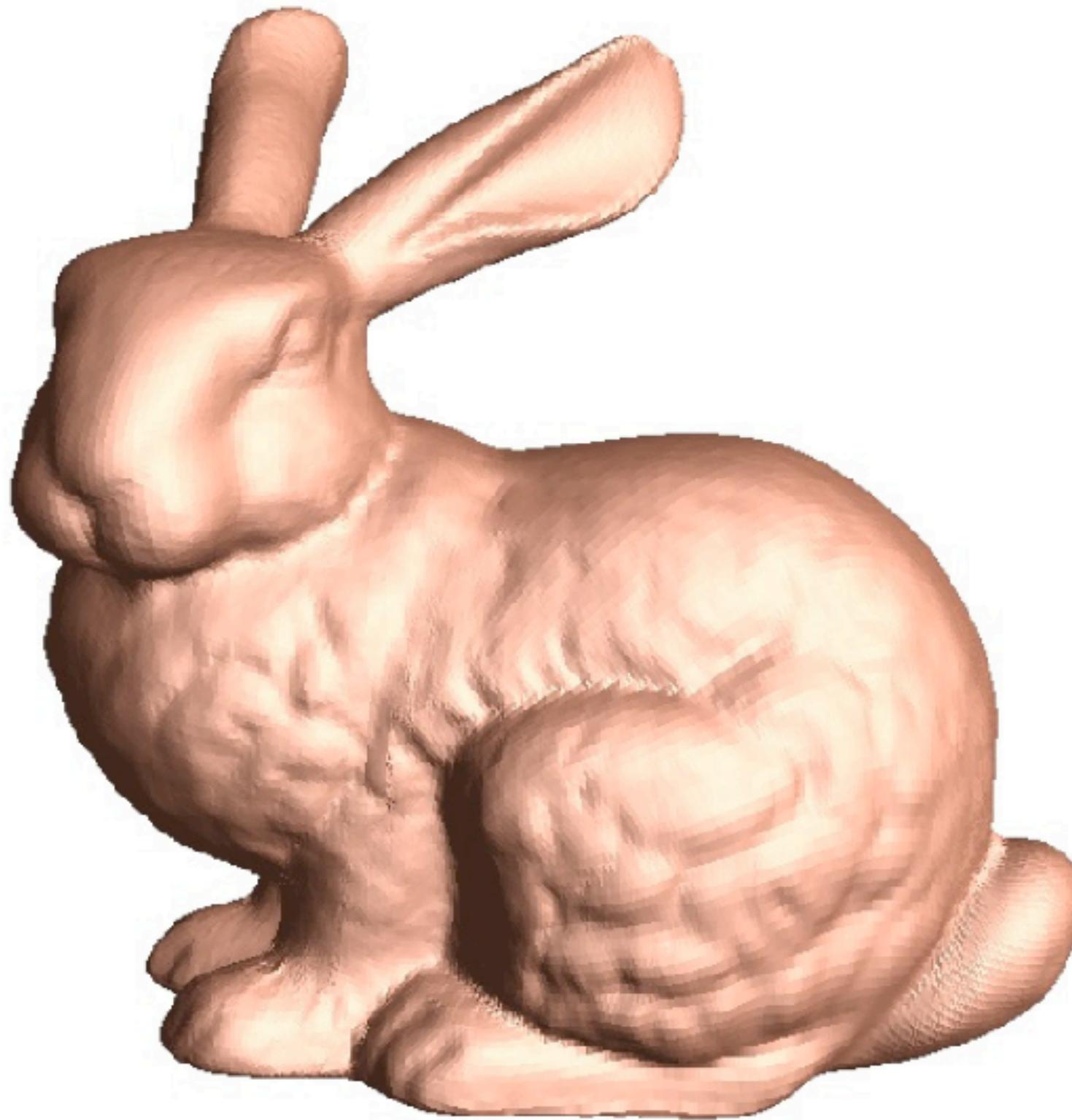


"Interactive Geometry Remeshing", Alliez et al., SIGGRAPH 2002

Compression

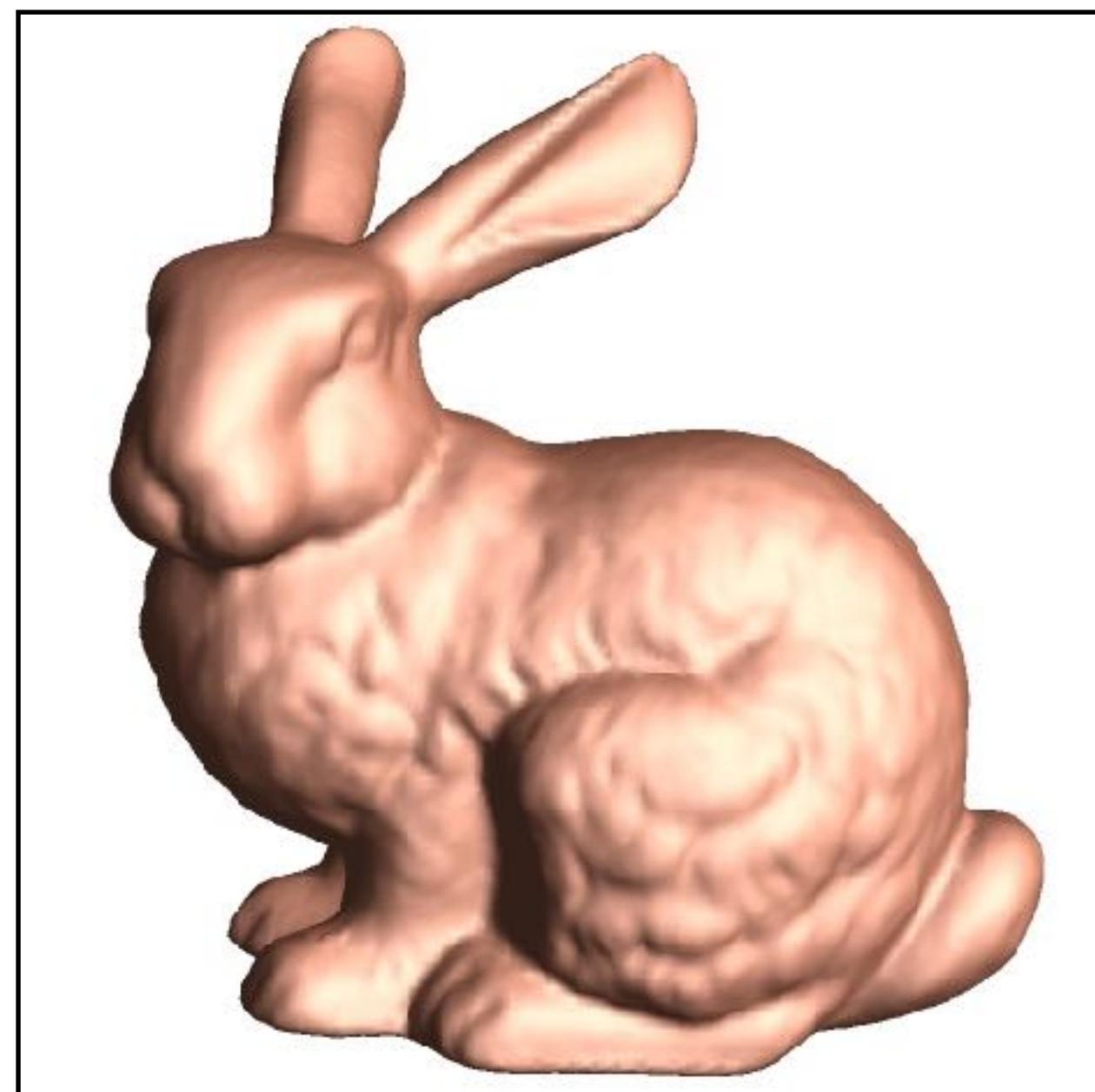


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"Geometry images", Gu et al., SIGGRAPH 2002

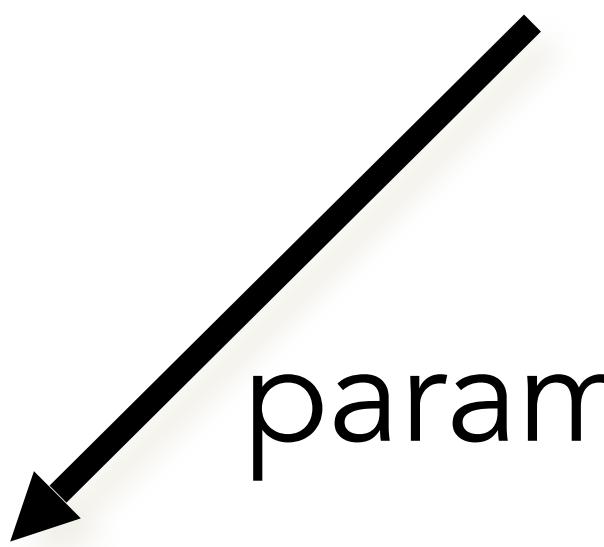
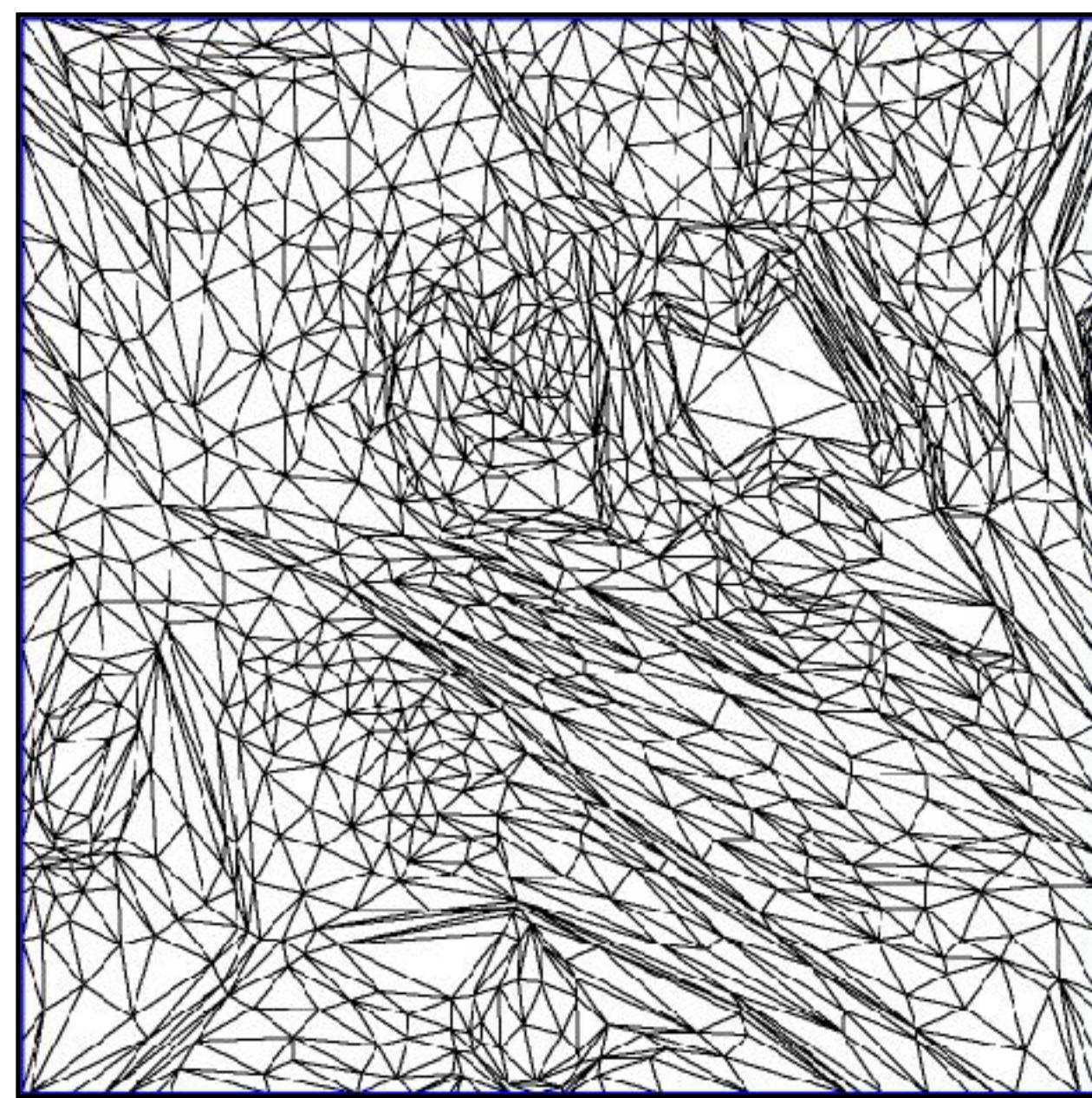
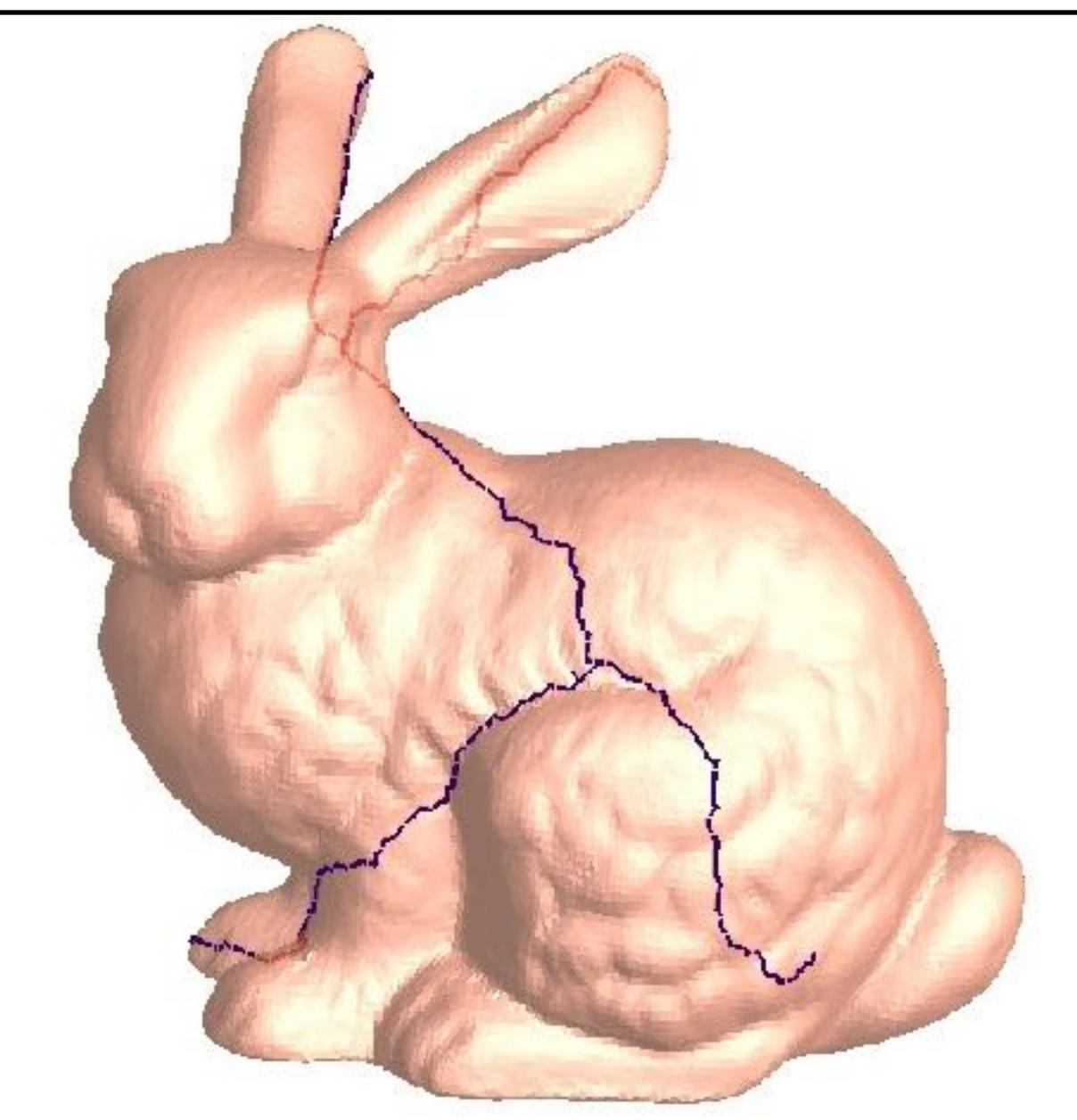
<http://research.microsoft.com/en-us/um/people/hoppe/proj/gim/>



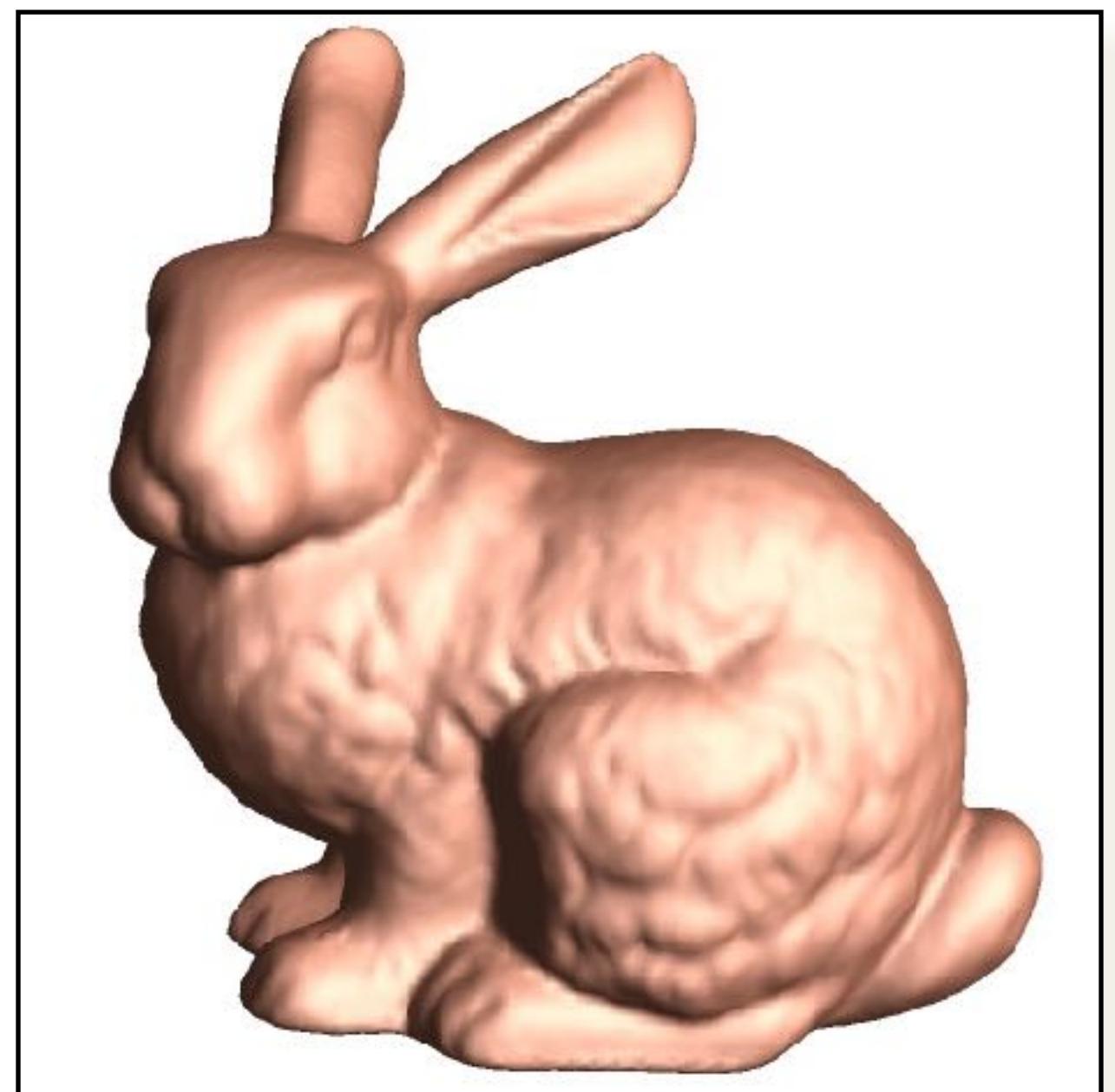
Geometry Images



cut



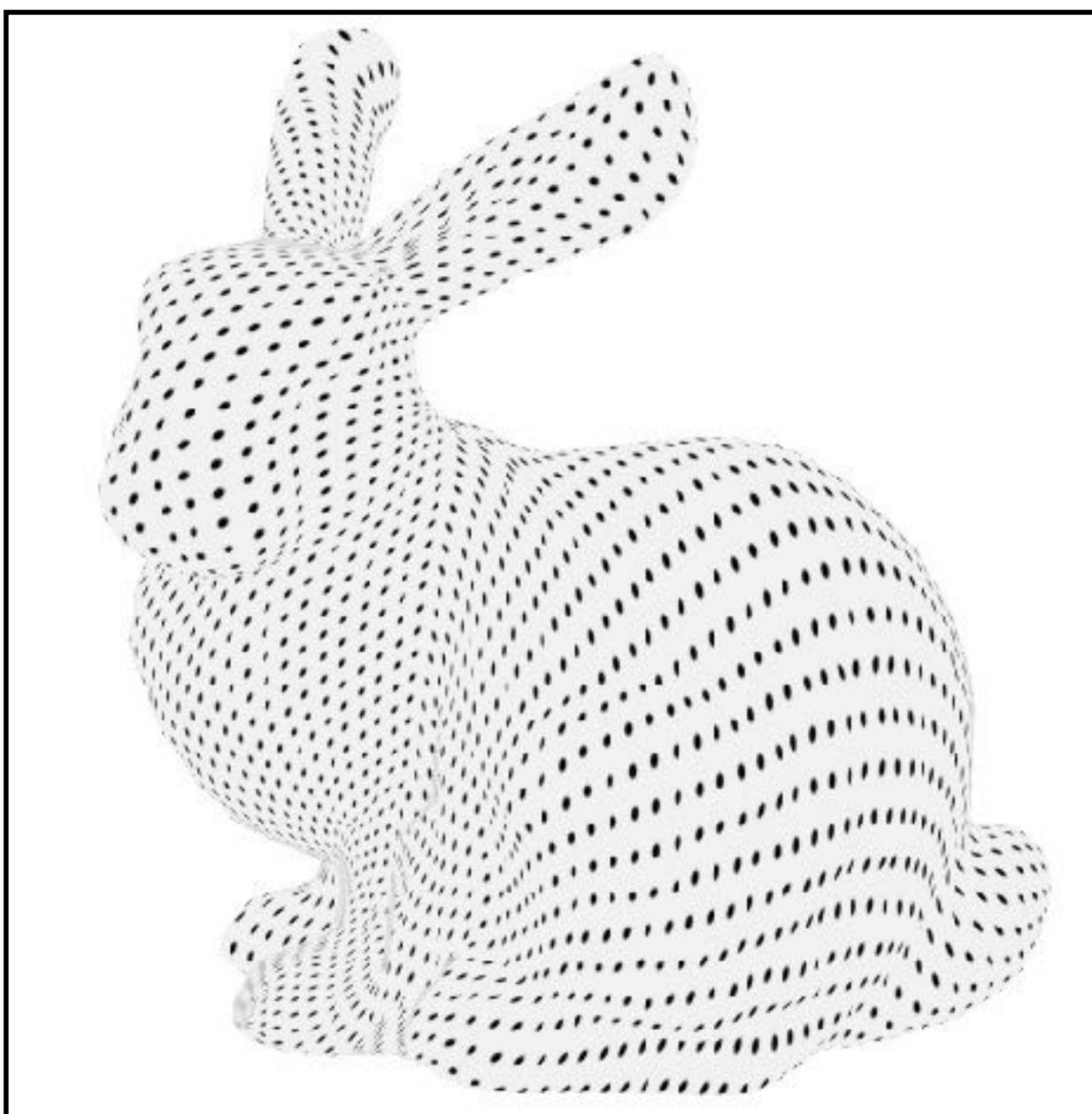
parametrize



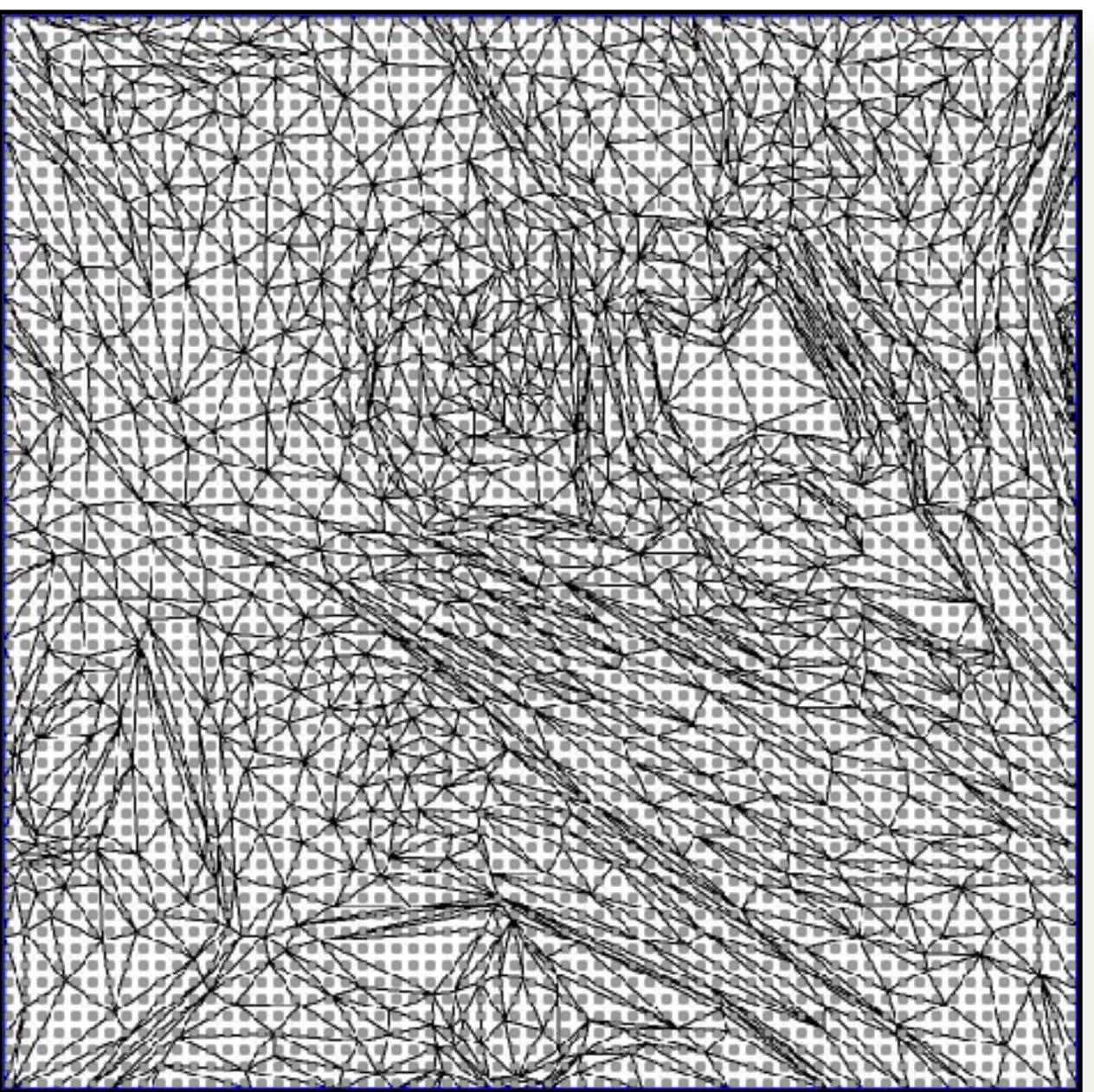
Geometry Images

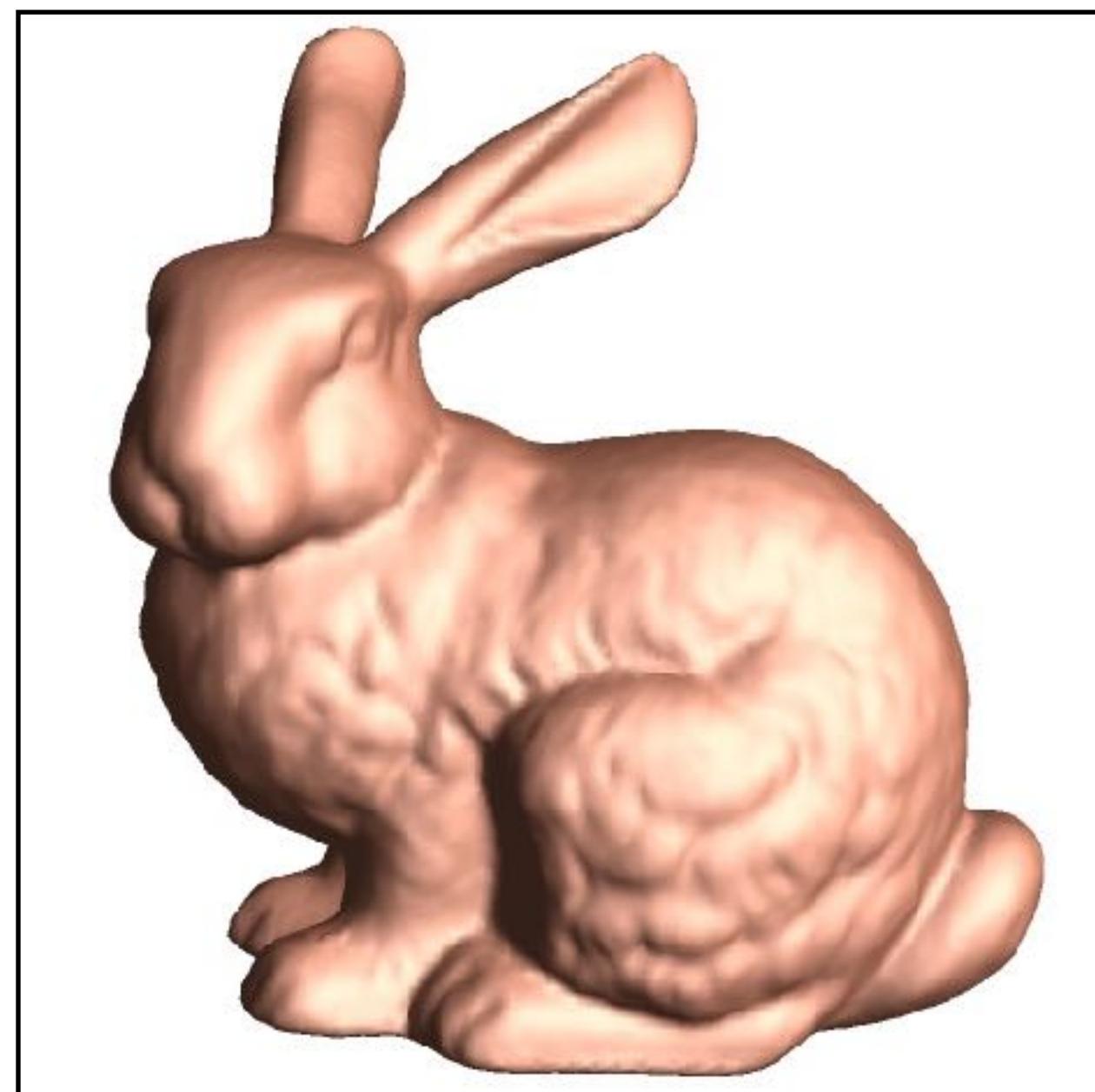


cut



sample

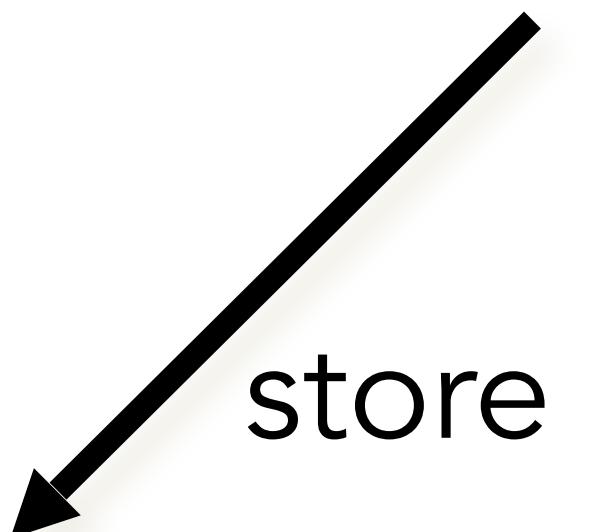
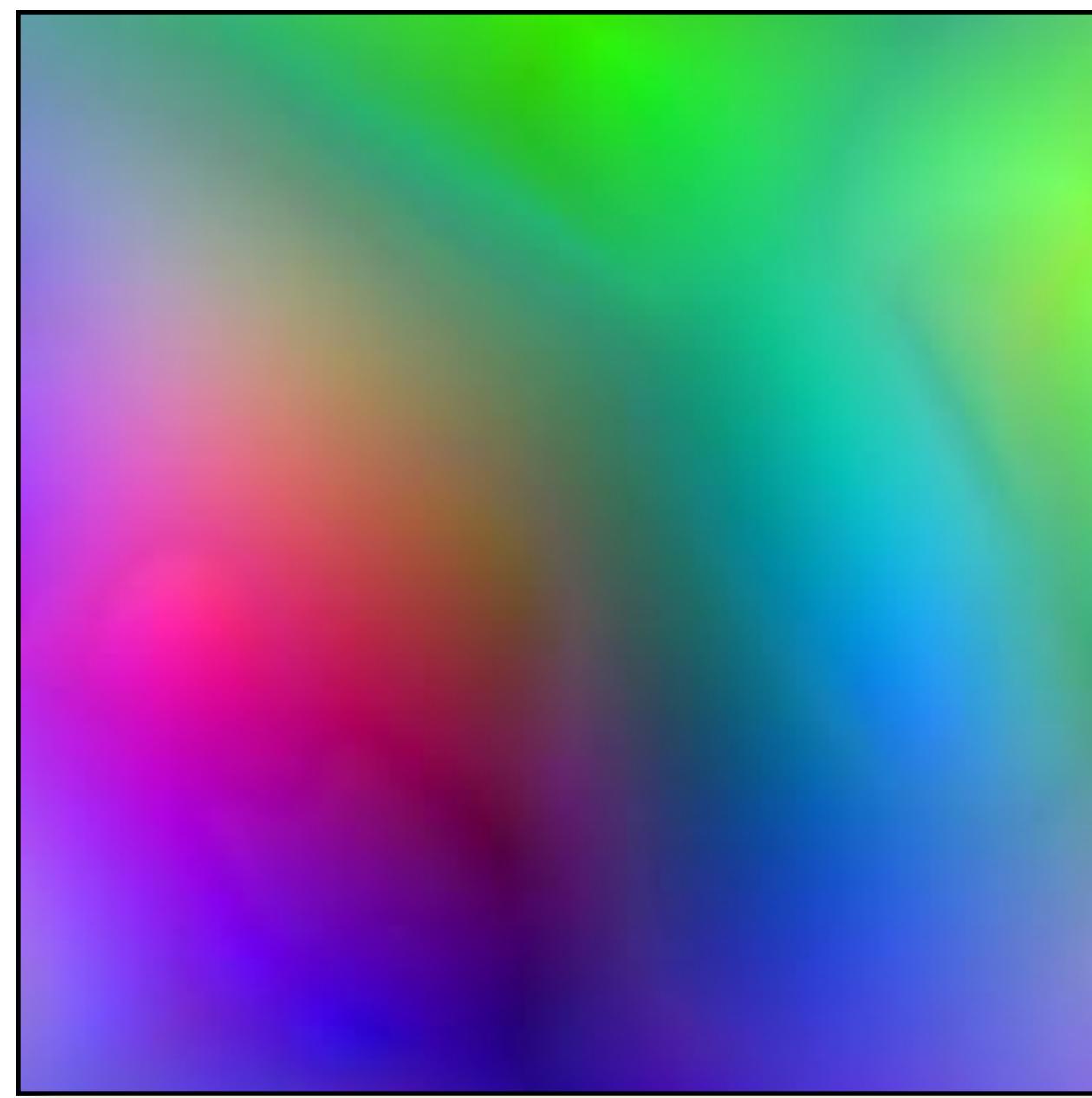
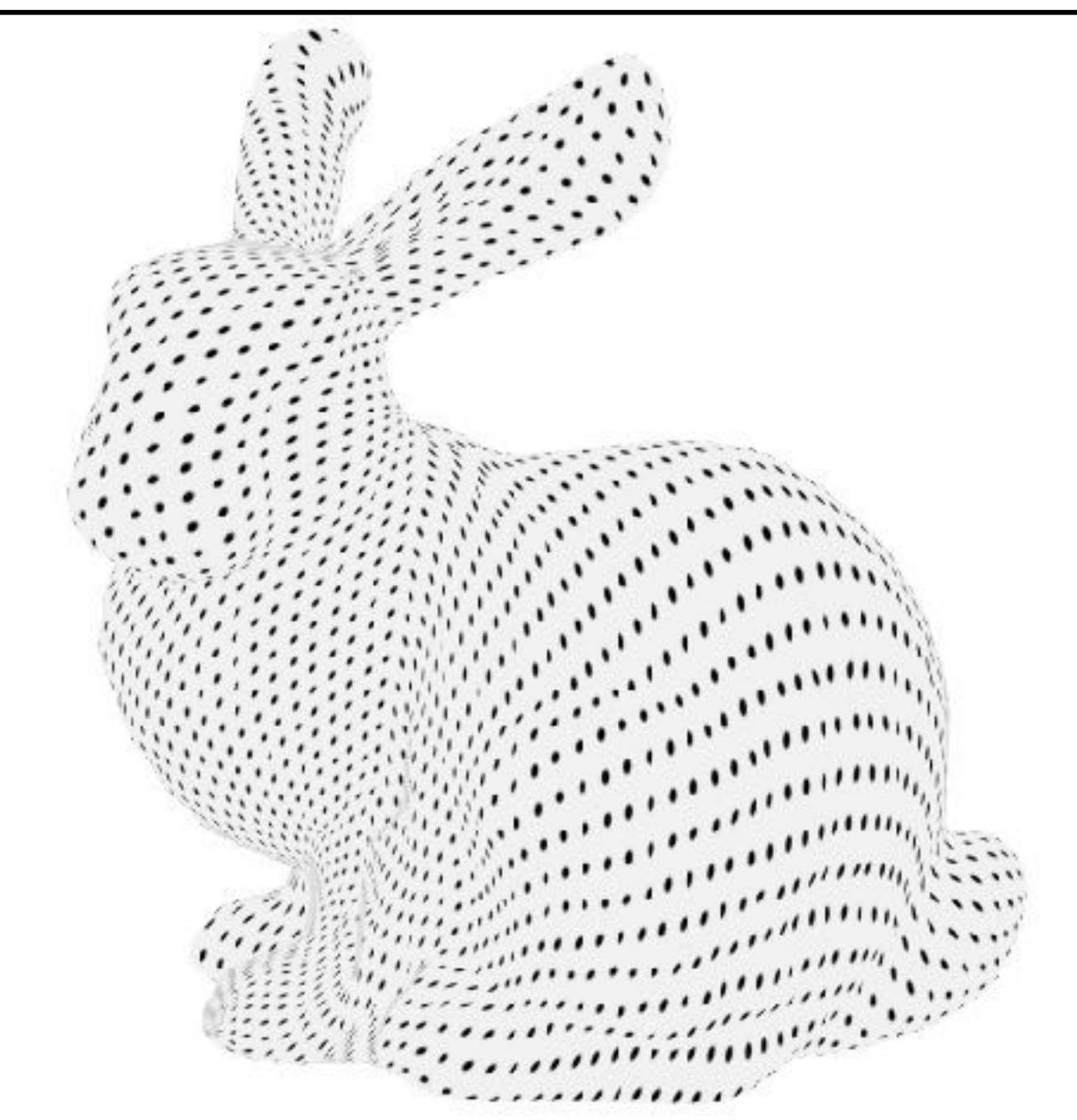




Geometry Images



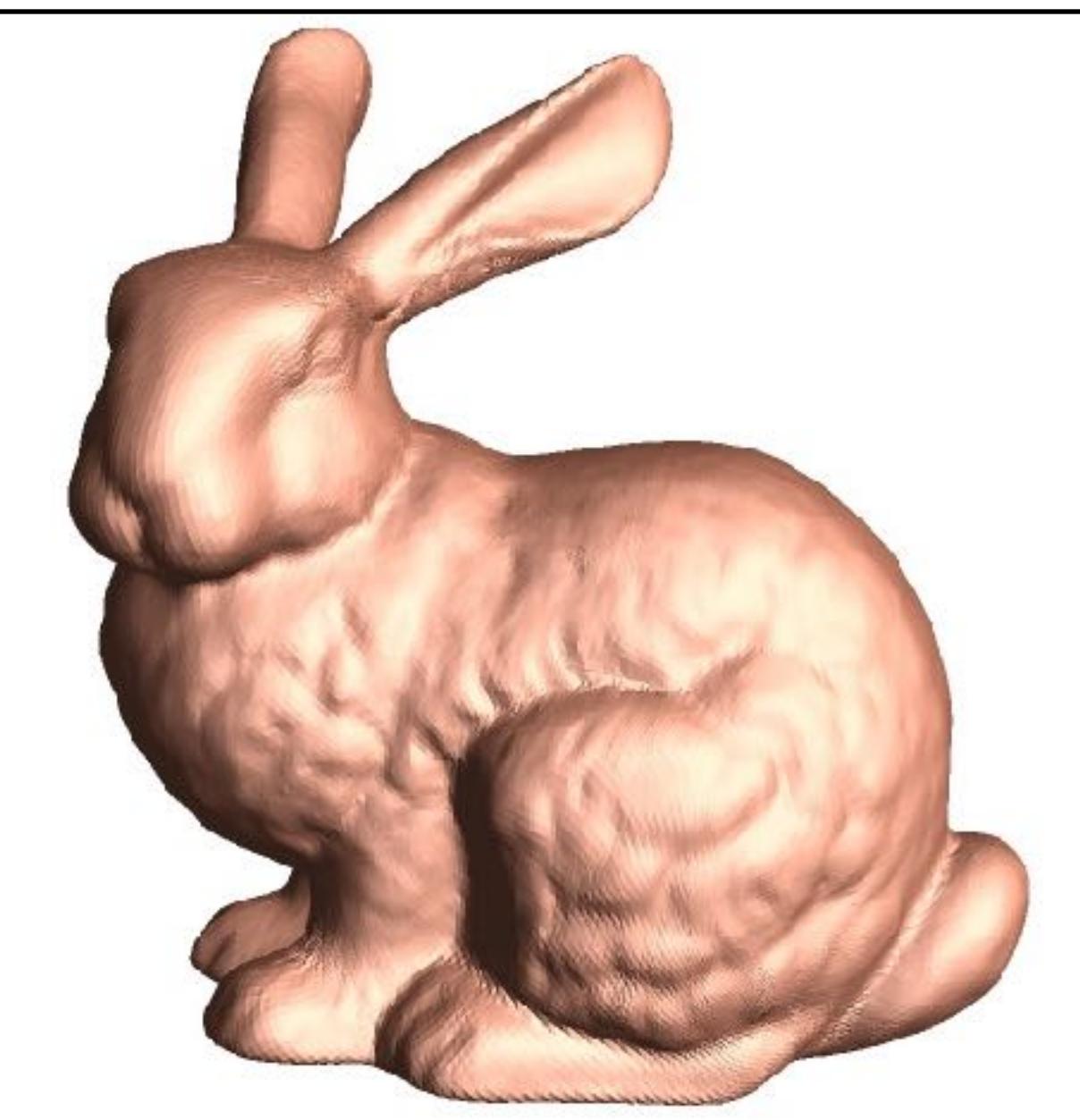
cut



store



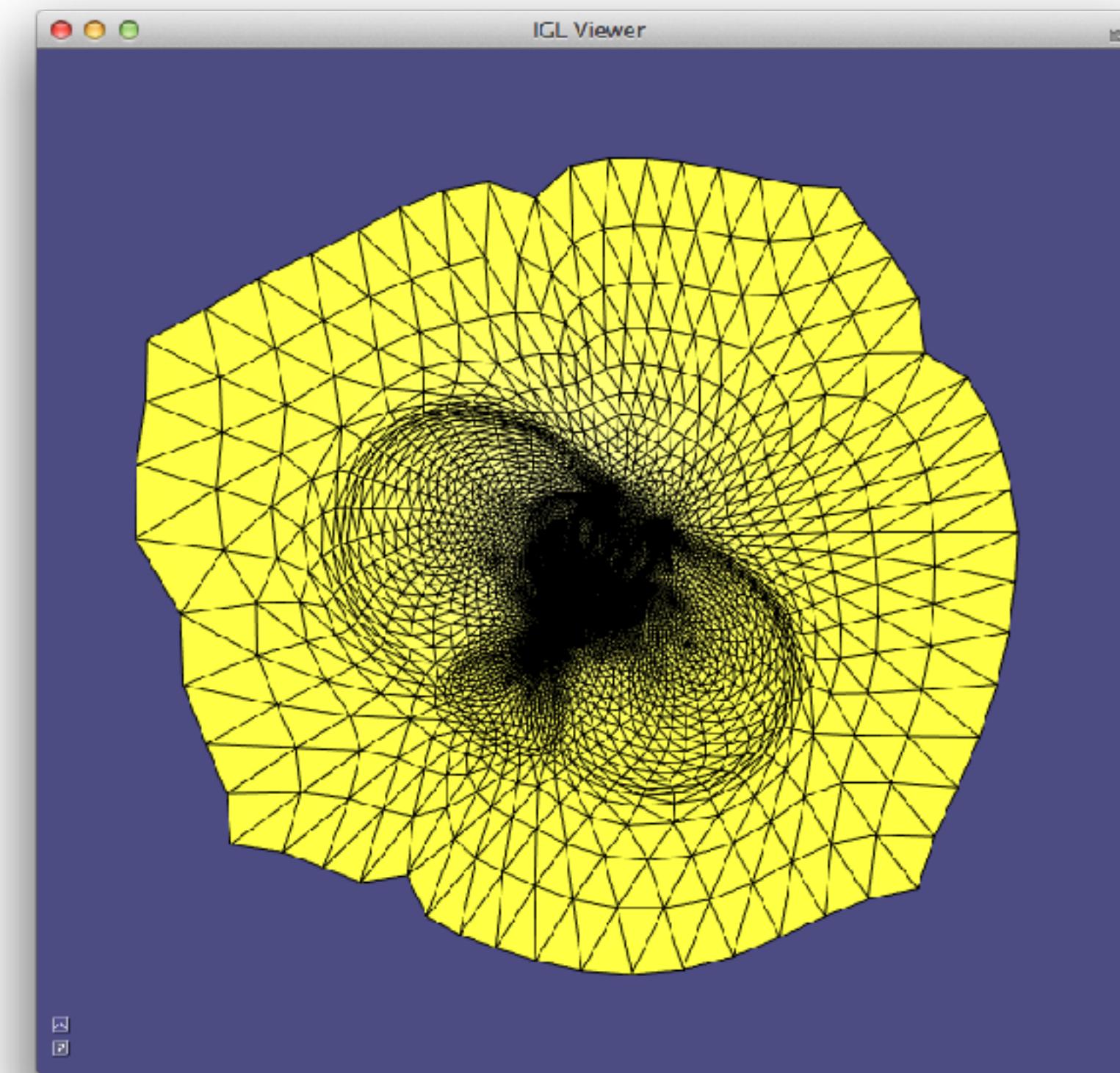
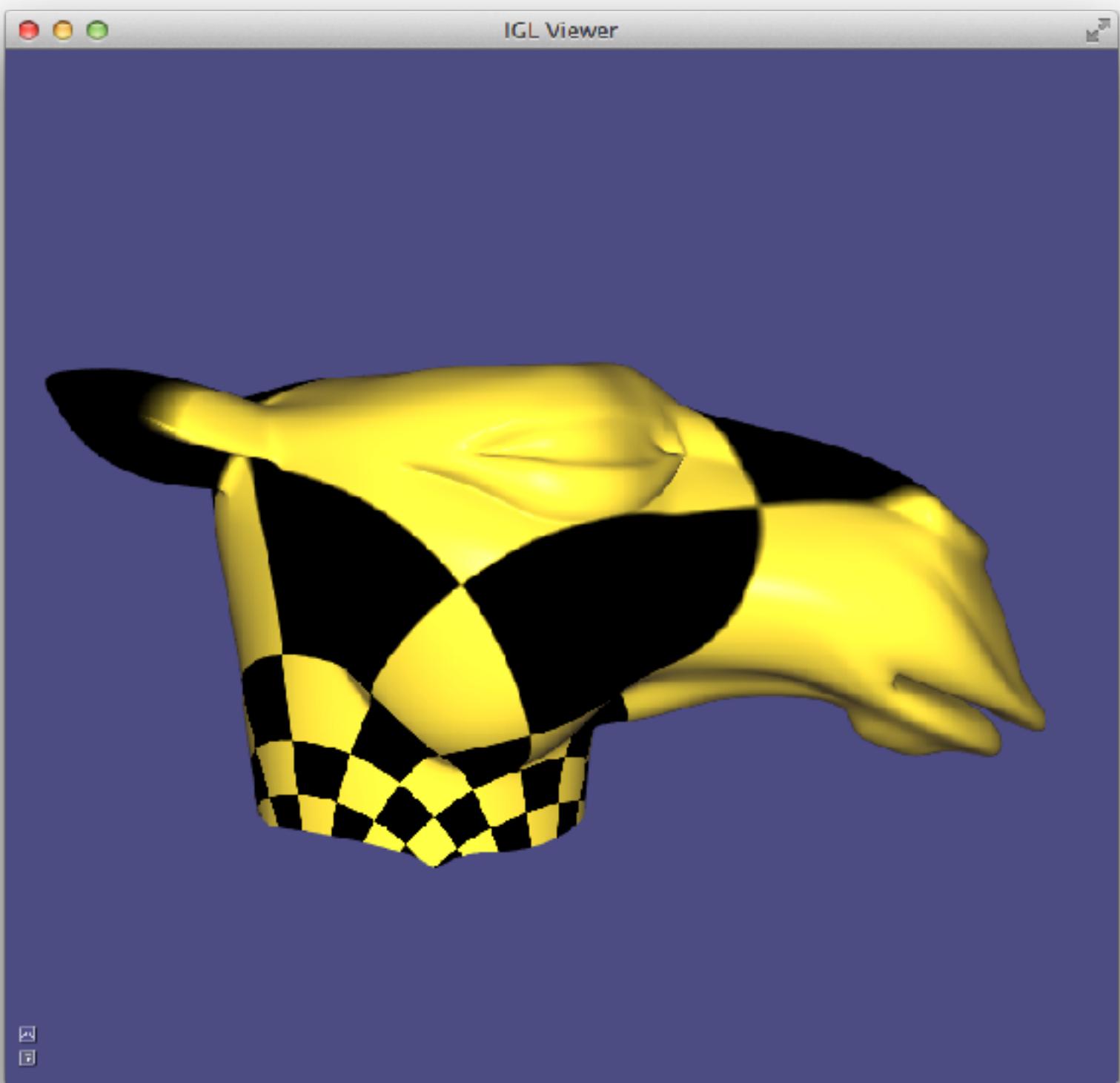
render



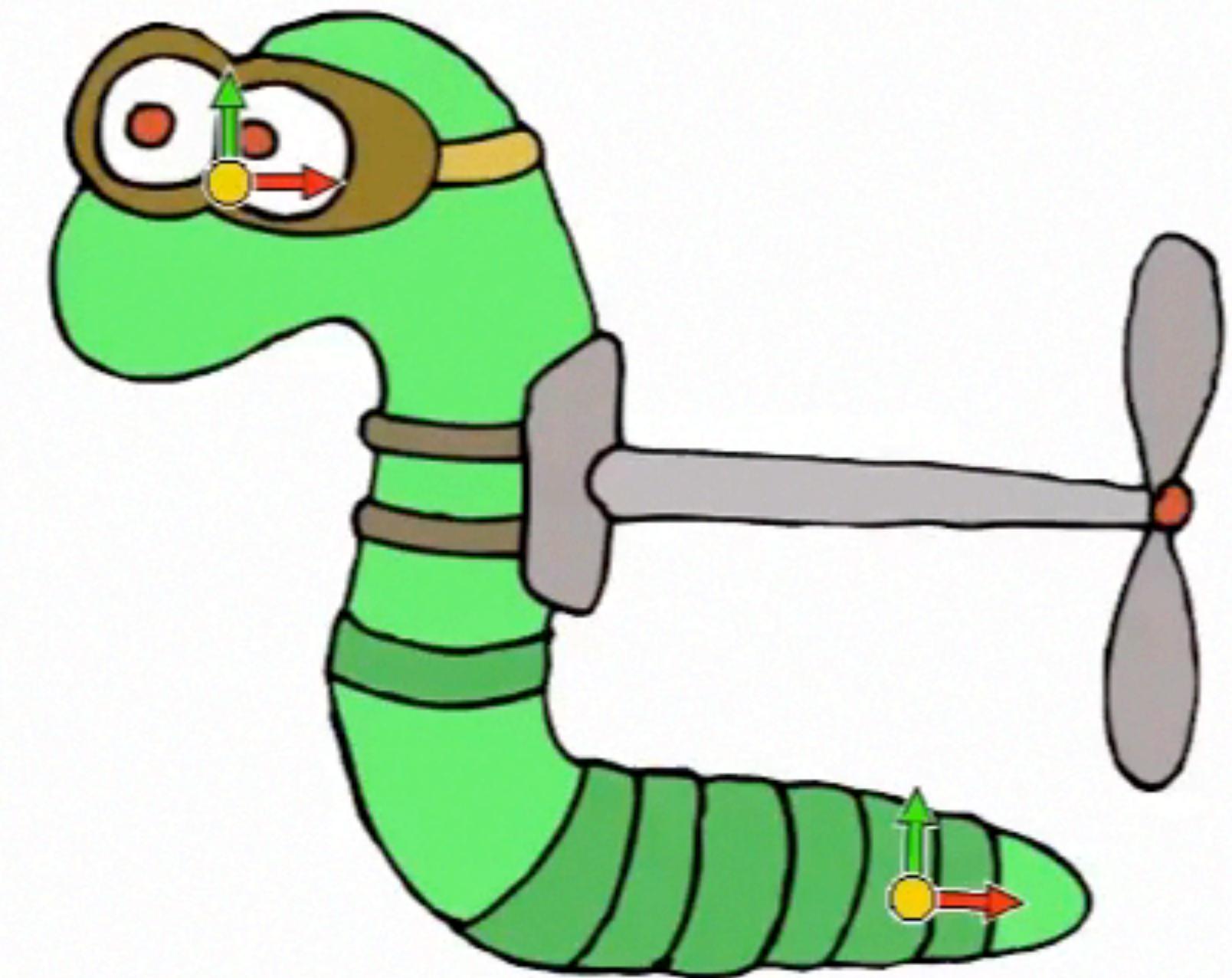
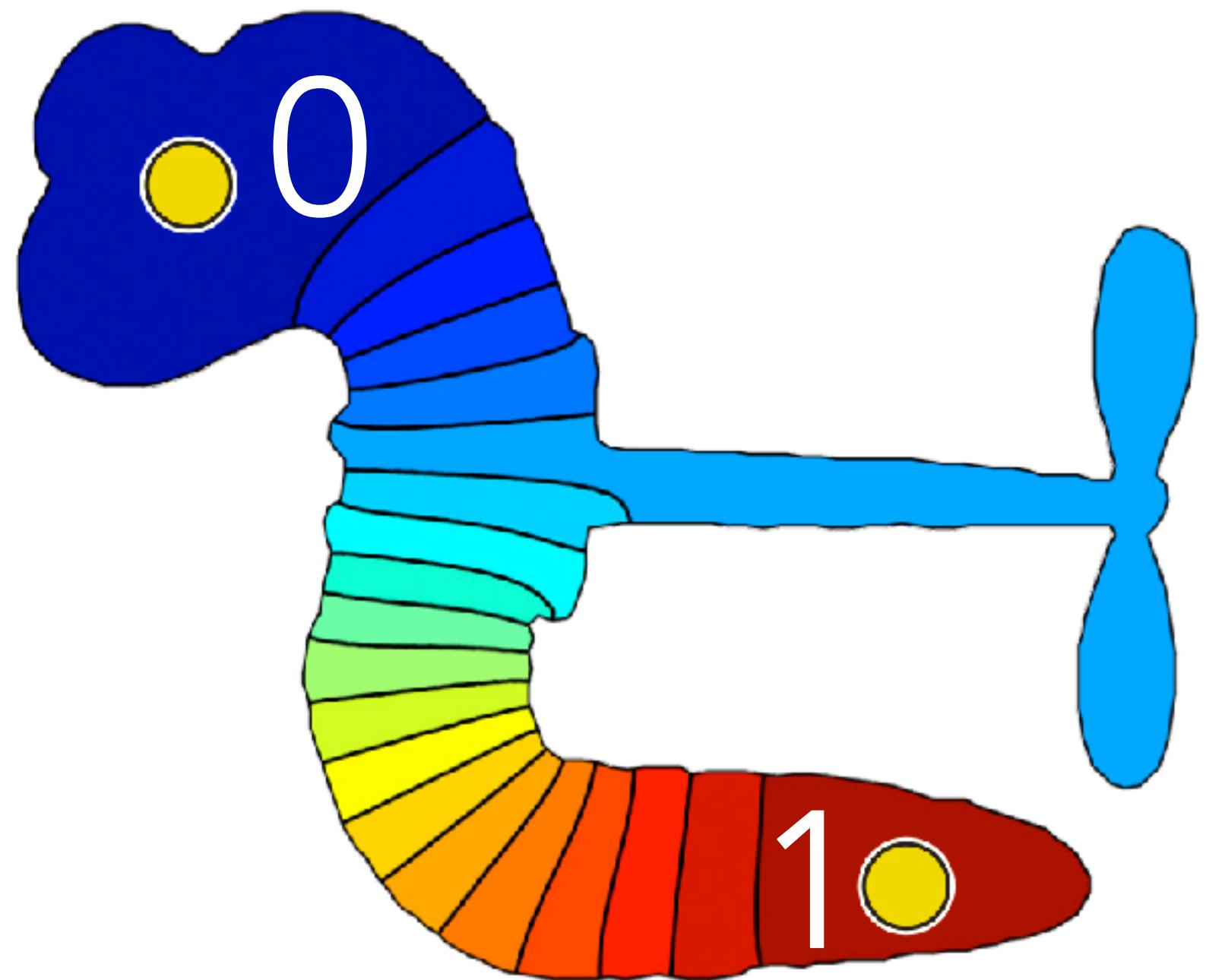
$$[r,g,b] = [x,y,z]$$

Designing Scalar Functions

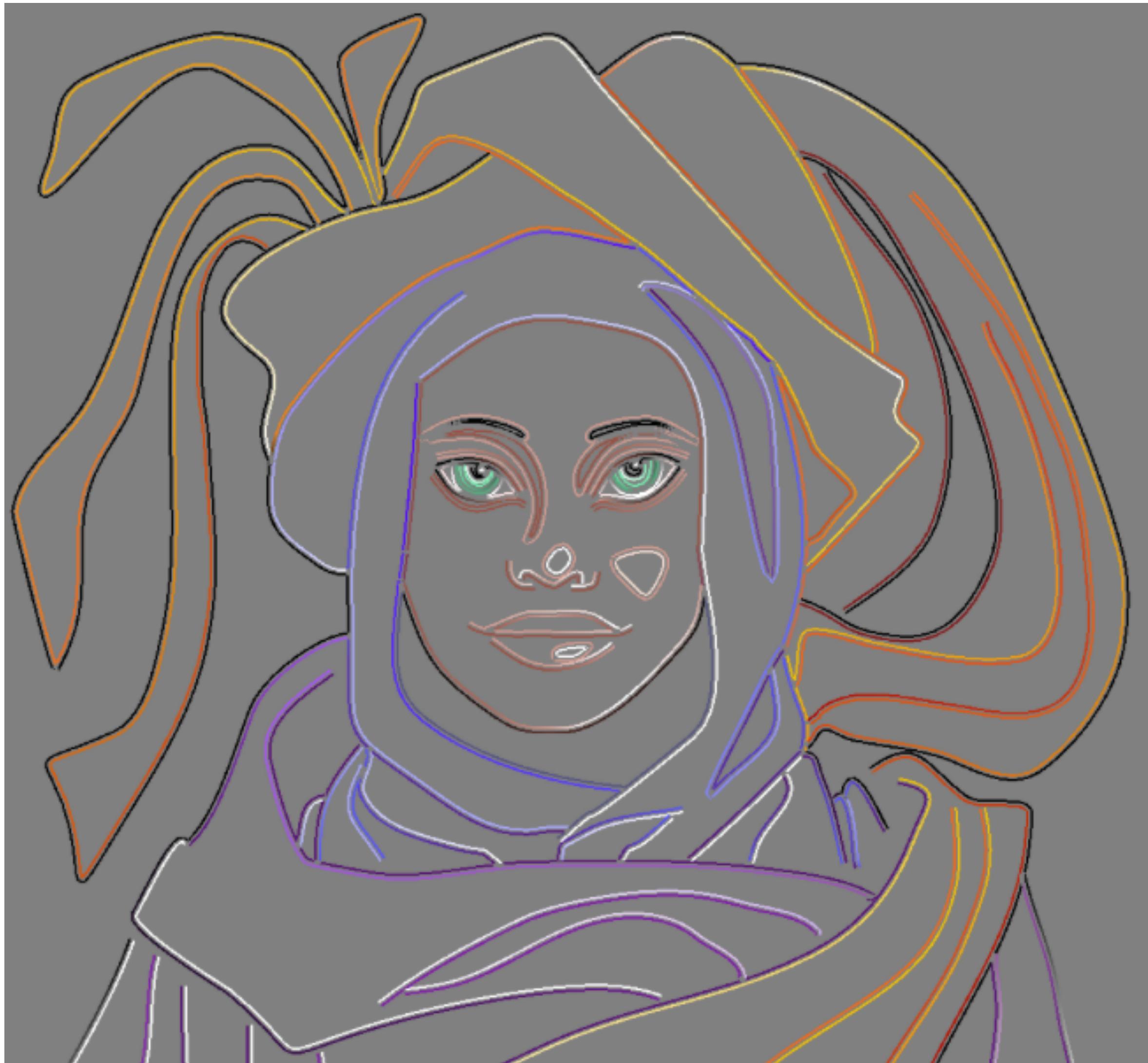
Parametrization = 2 Scalar Functions



Designing Scalar Functions

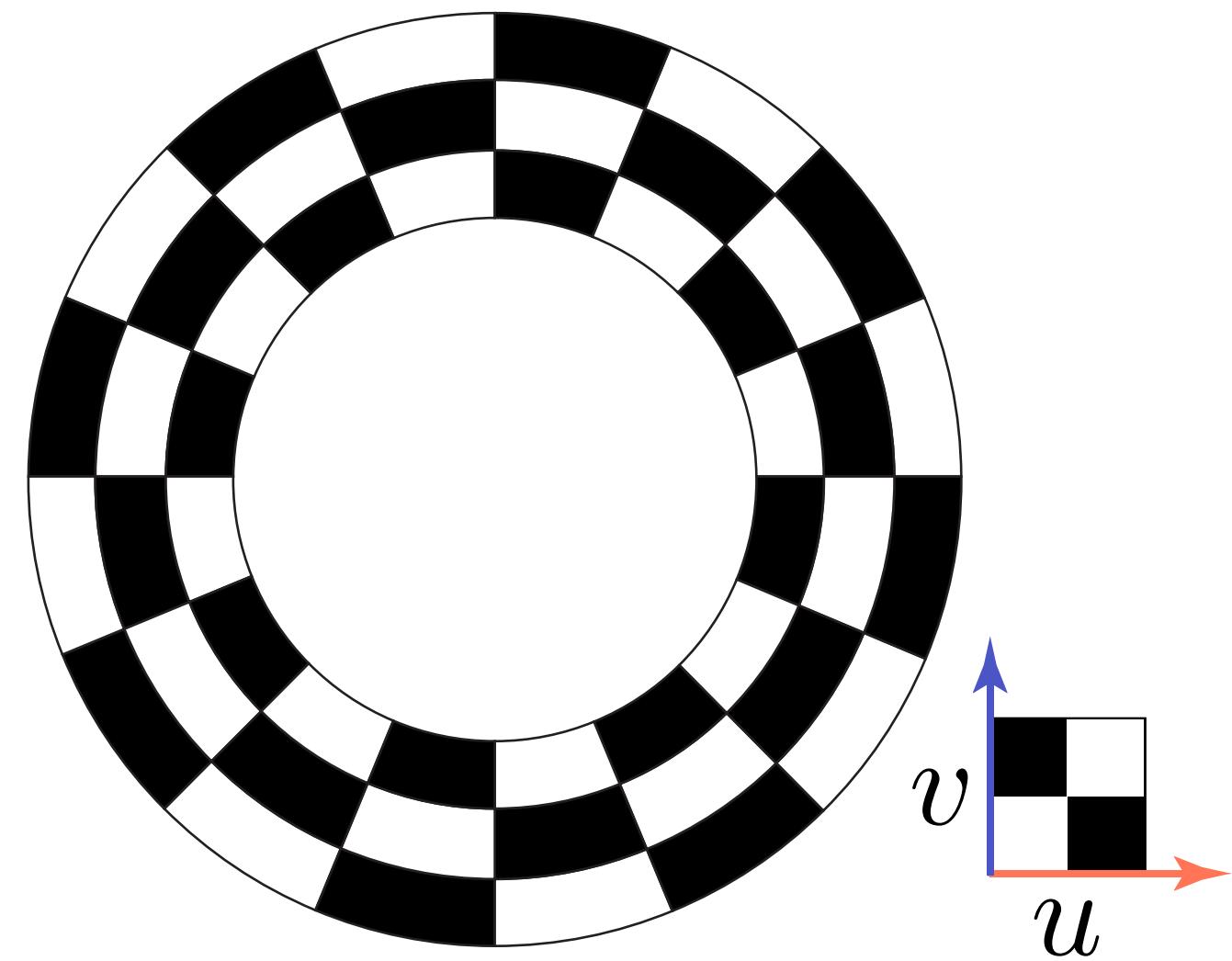


Designing Scalar Functions

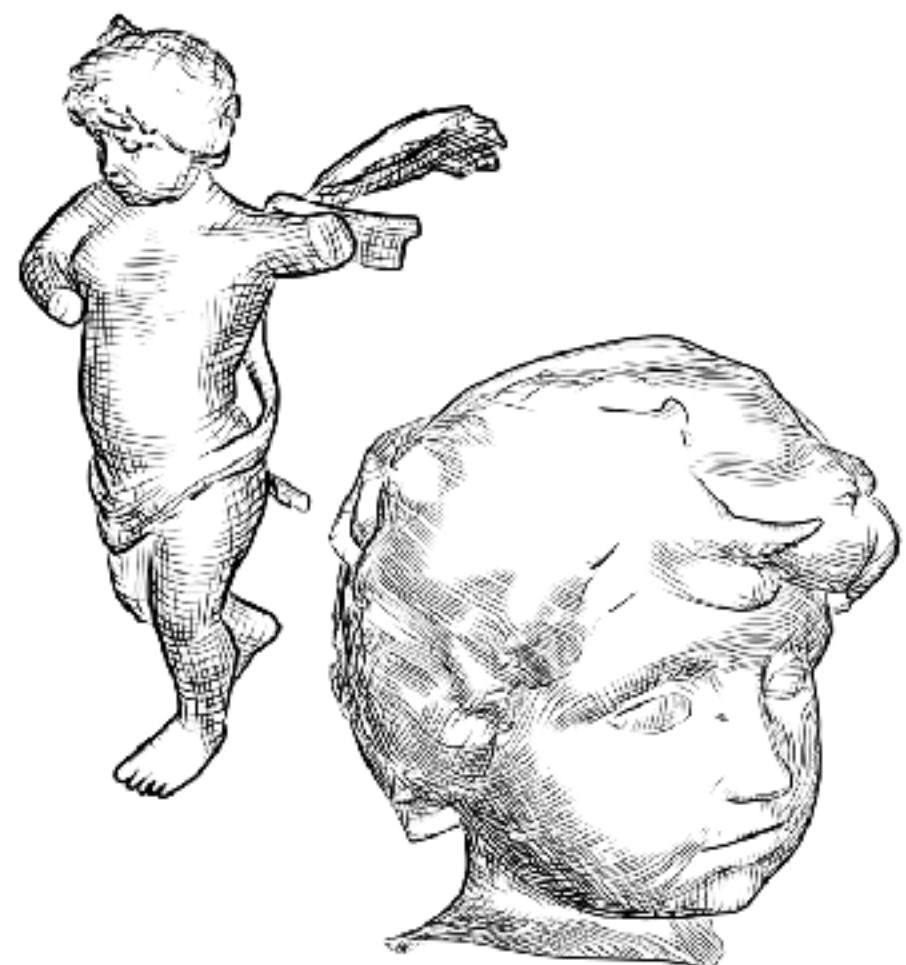


L.Rossignac©INRIA 2008

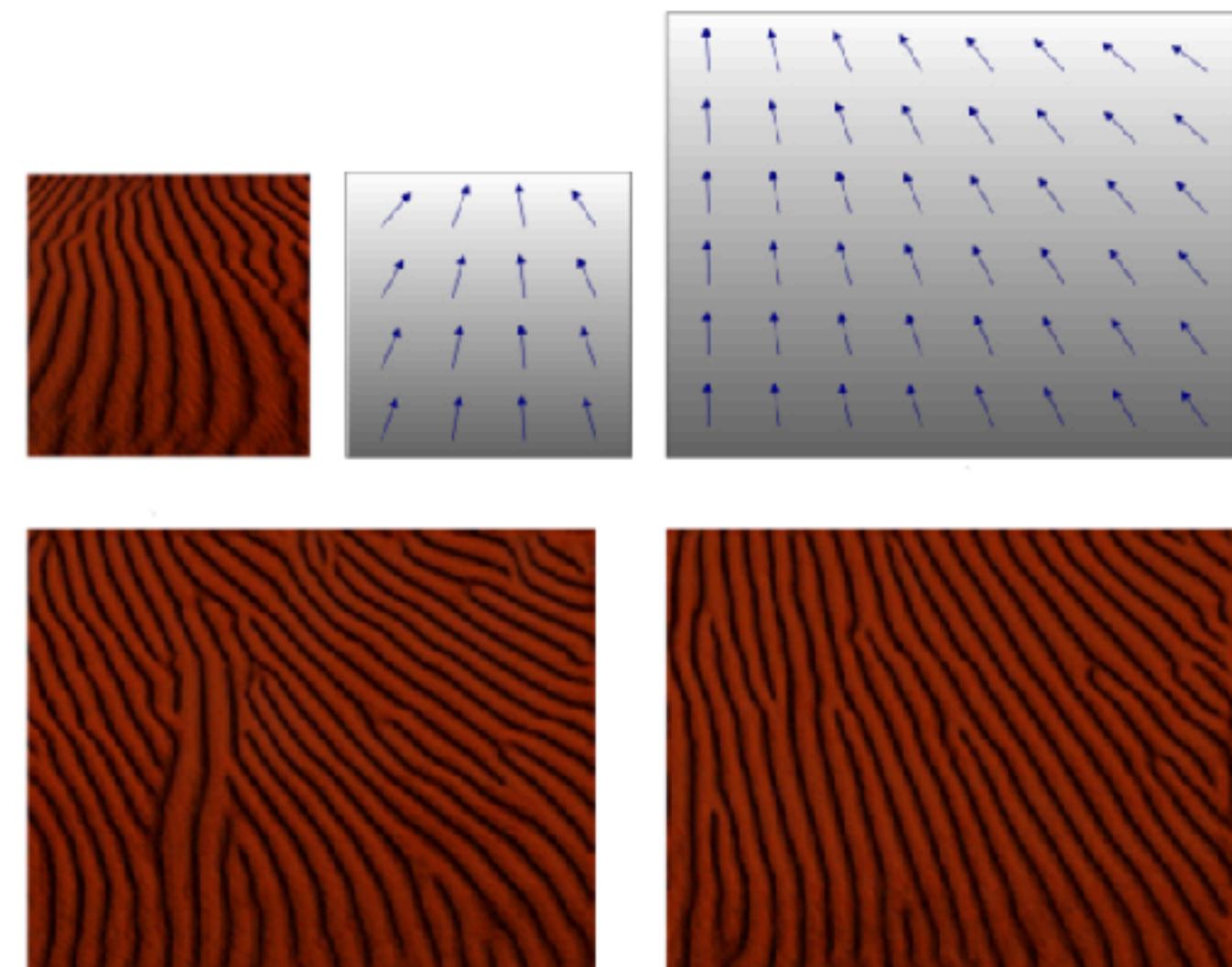
Two Scalar Functions



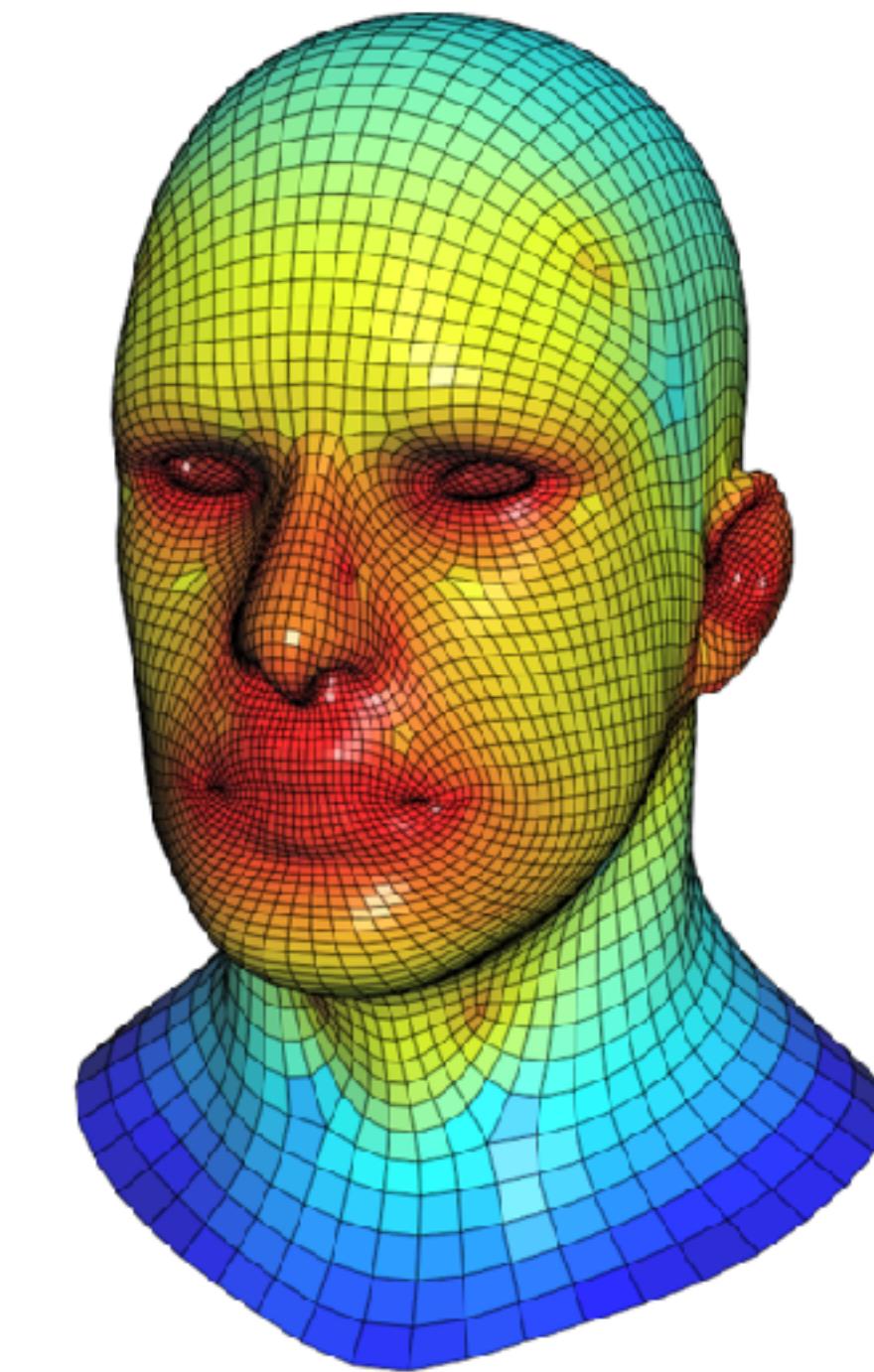
Designing Gradients



Hatching

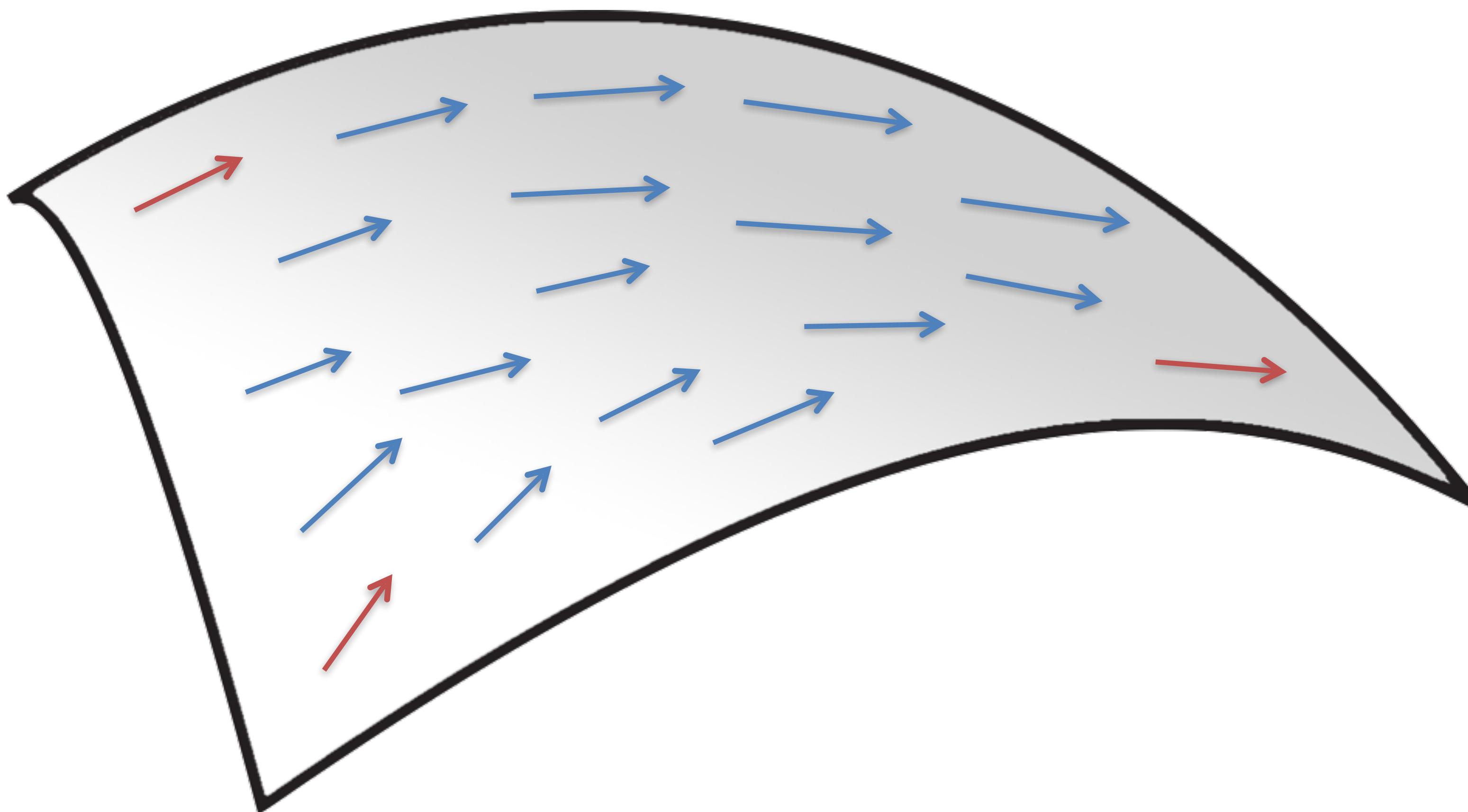


Texture Synthesis

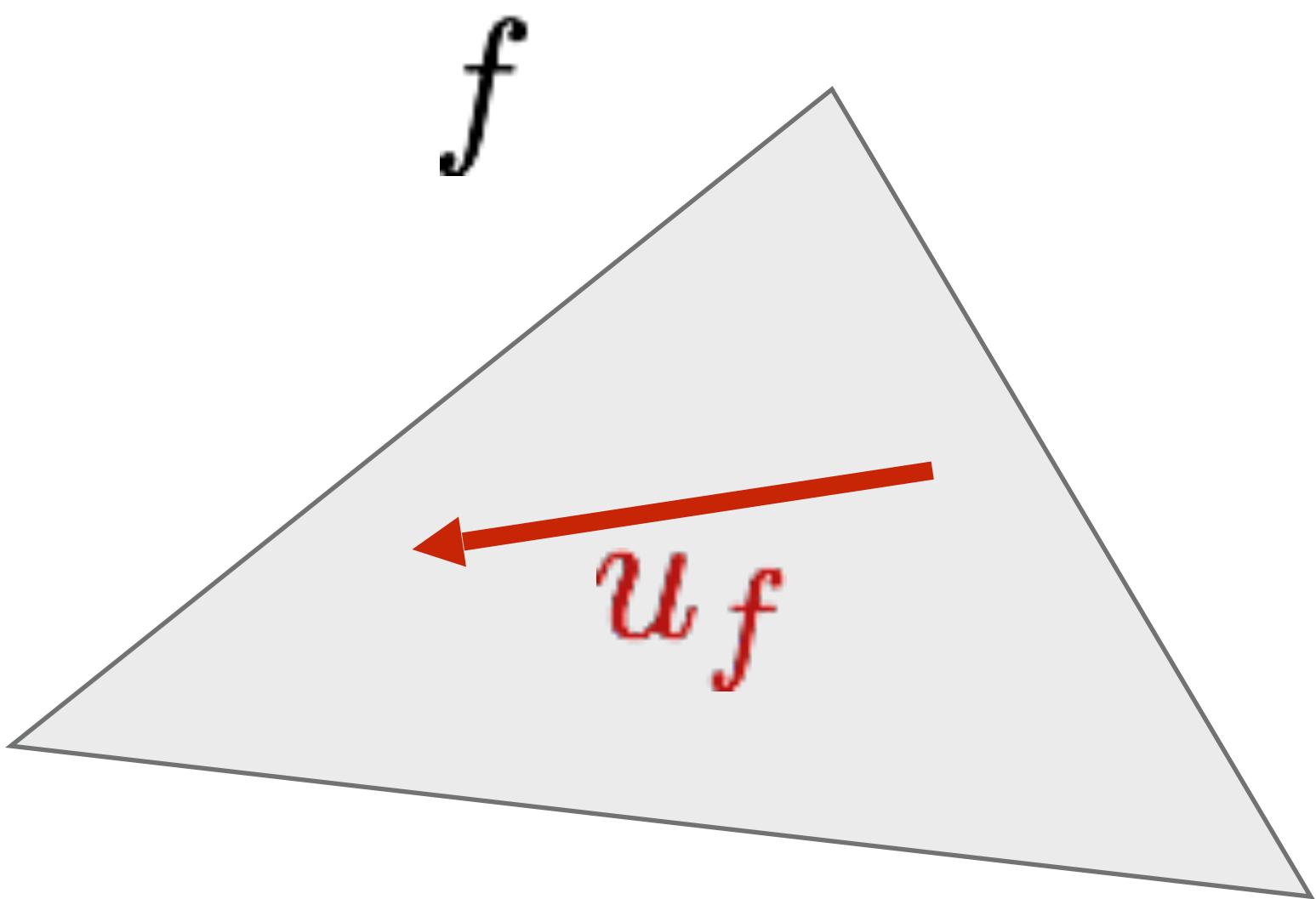


Meshting

Intepolating Vectors



Poisson Problem



Condition per triangle

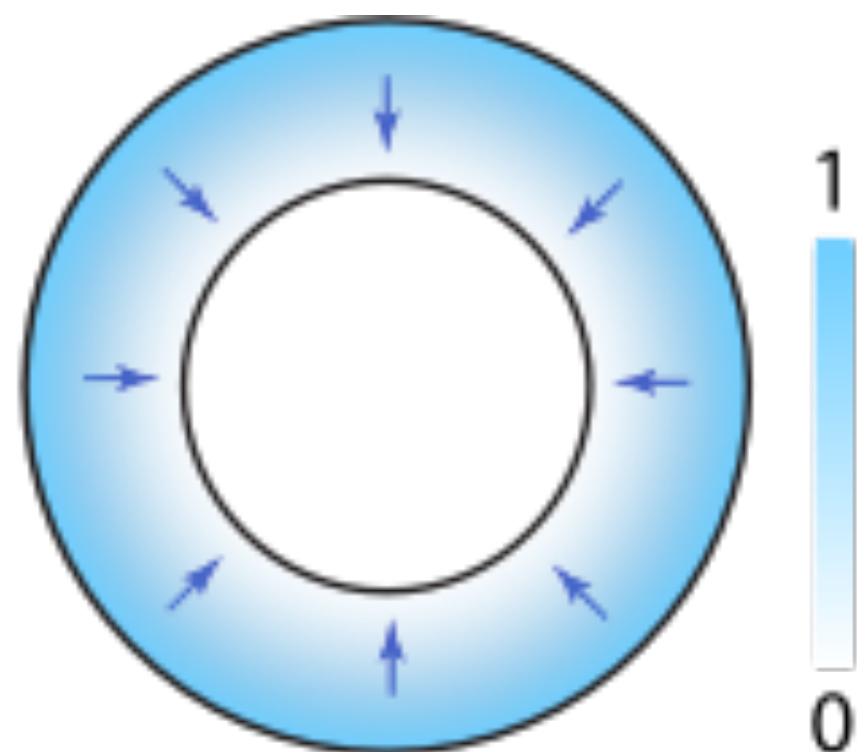
$$(\nabla s)_f = u_f$$

Error per triangle

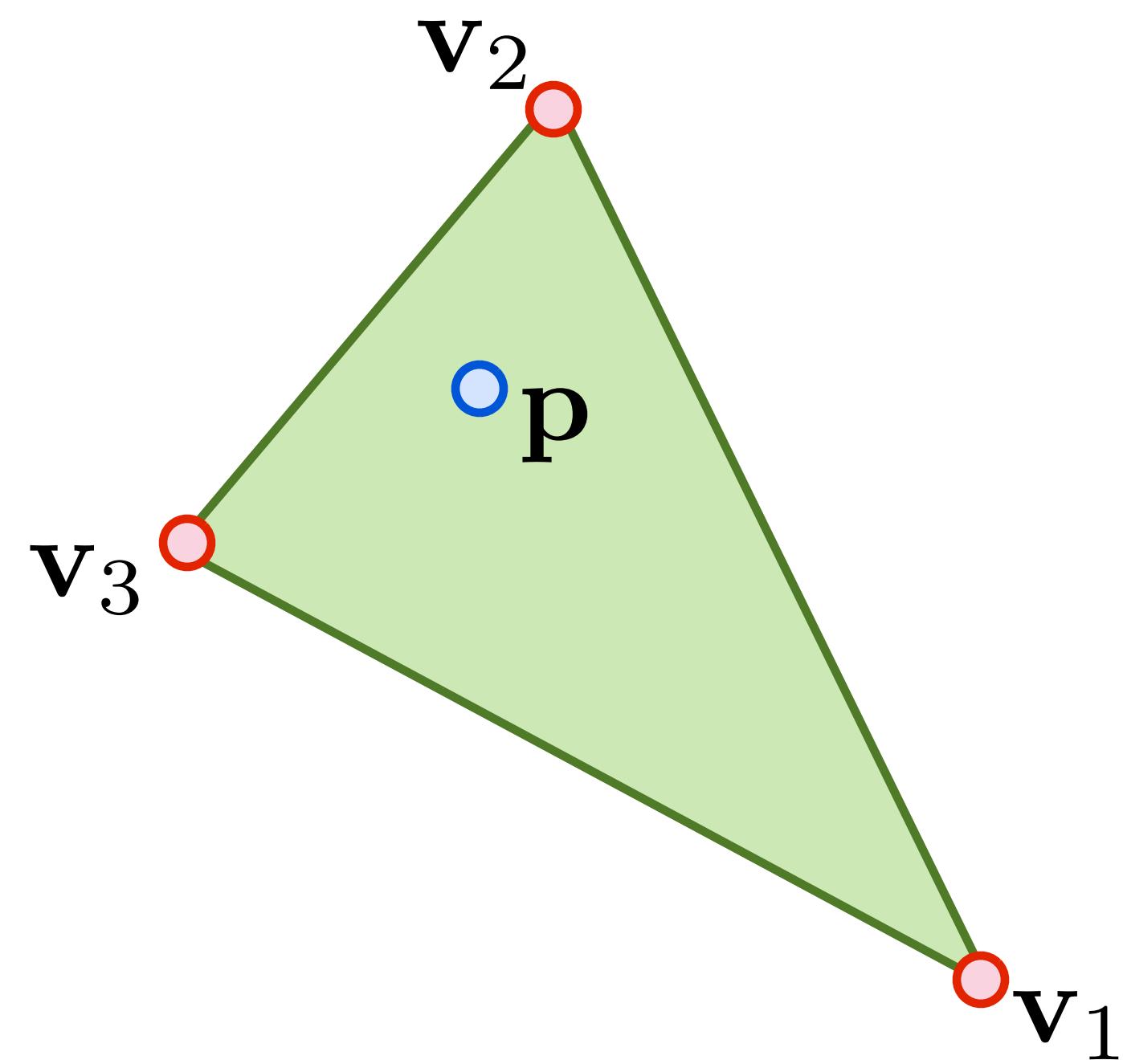
$$P_f = \|(\nabla s)_f - u_f\|_2$$

Error over the entire mesh

$$P = \sum_{f \in \mathcal{F}} P_f$$

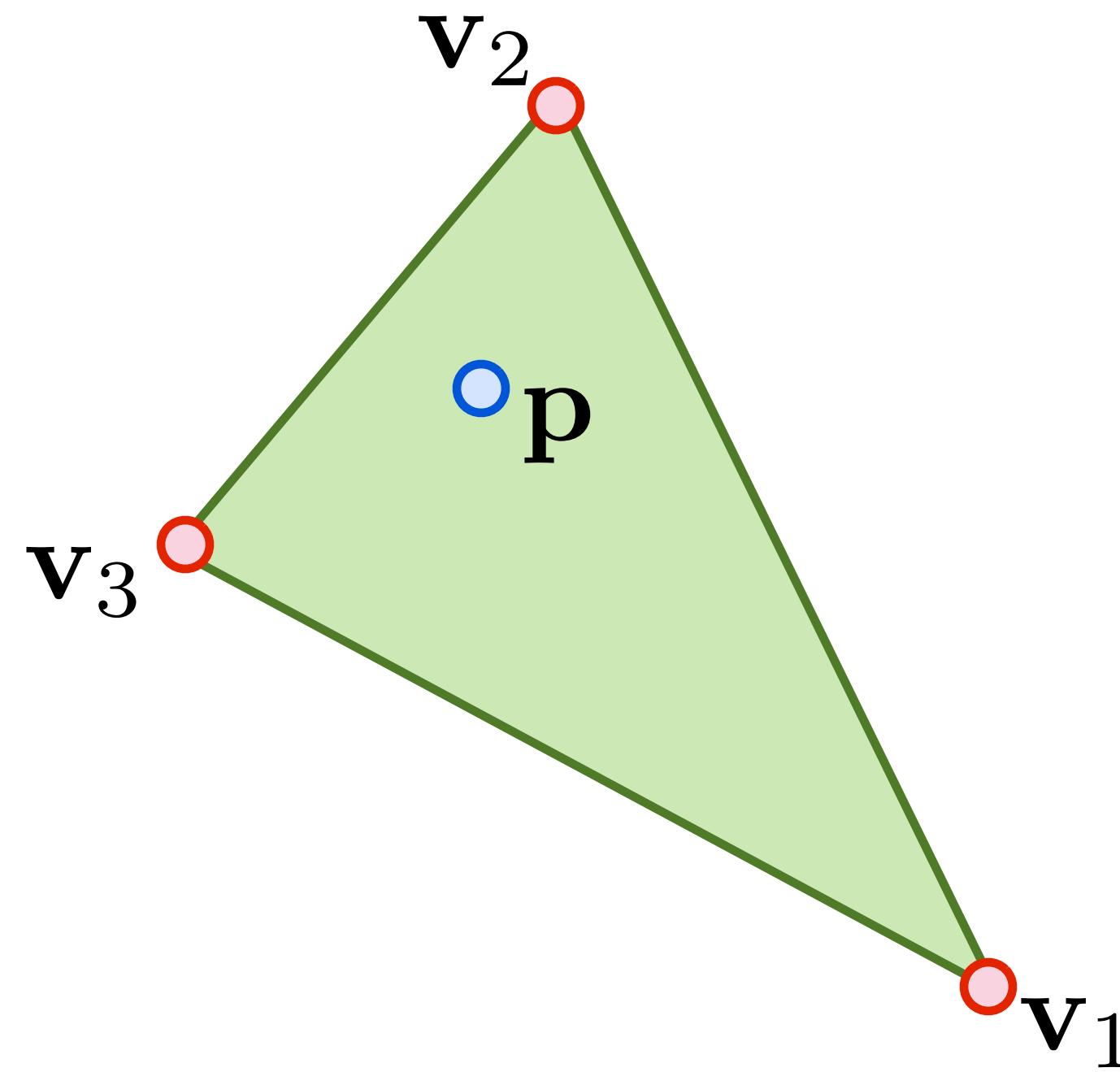


Barycentric Coordinates



$$\mathbf{p} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + w_3 \mathbf{v}_3$$

Barycentric Coordinates

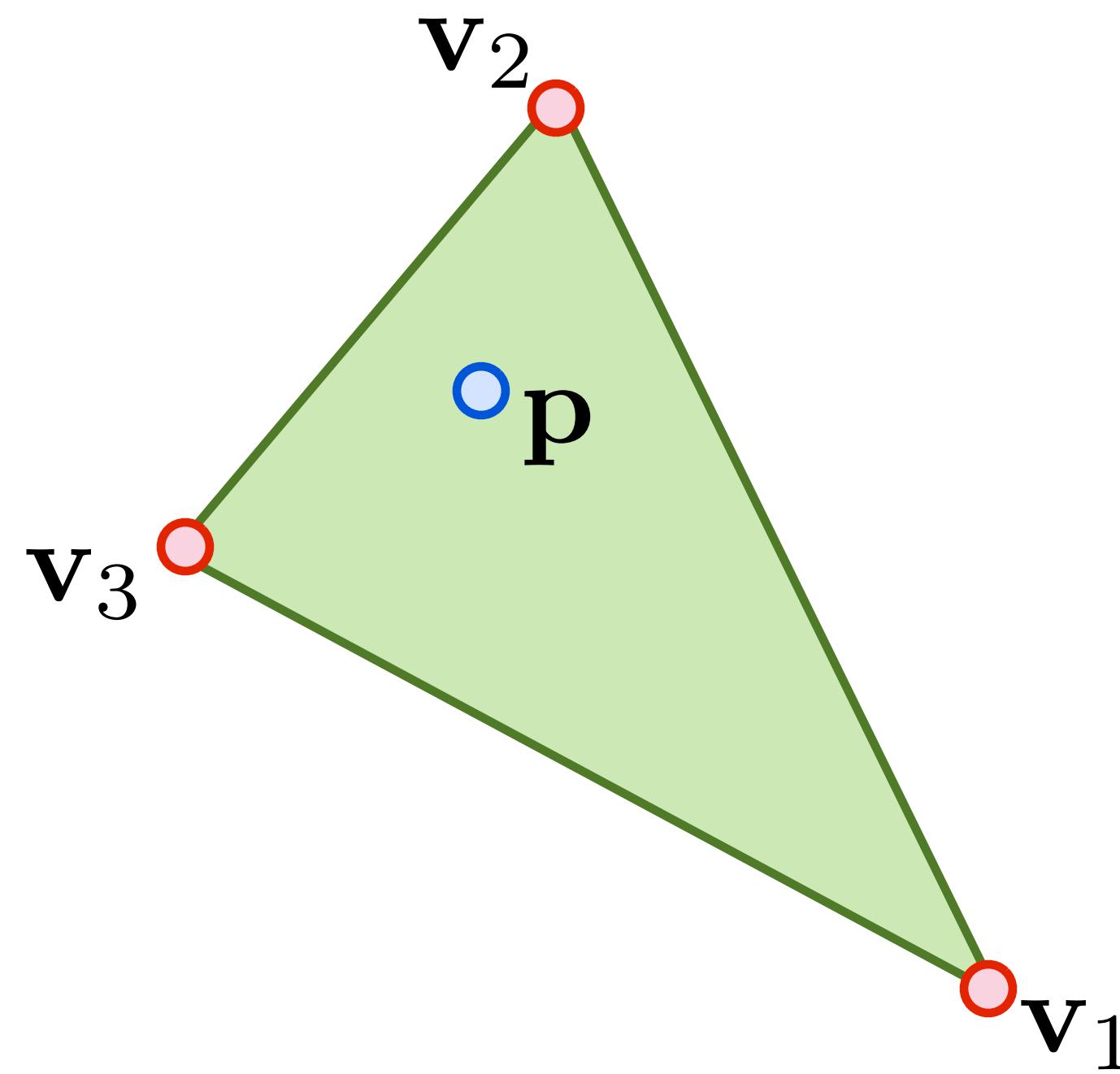


$$\mathbf{p} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + w_3 \mathbf{v}_3$$

Partition of unity: $w_1 + w_2 + w_3 = 1$

$$\mathbf{p} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + (1 - w_1 - w_2) \mathbf{v}_3$$

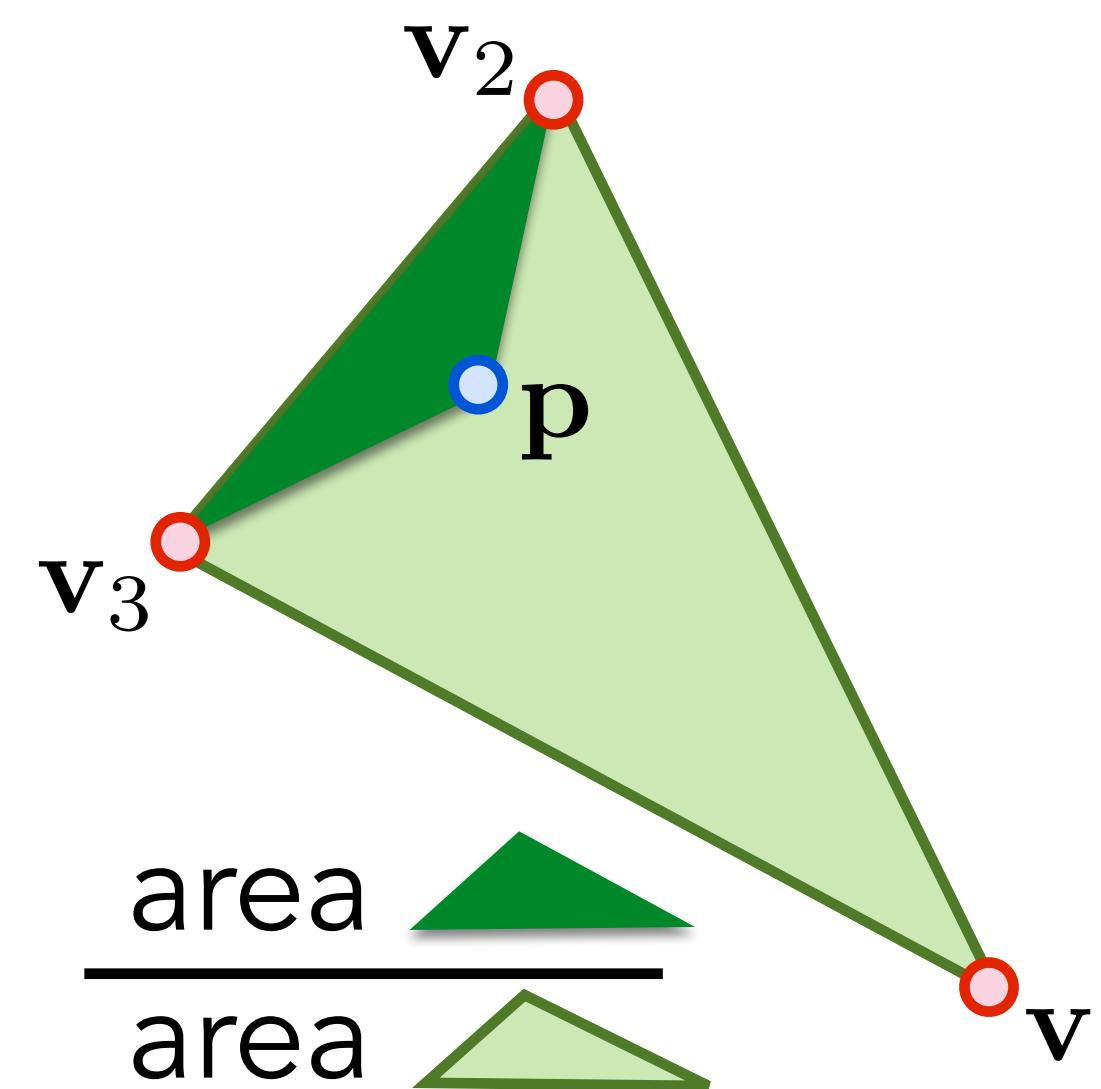
Barycentric Coordinates



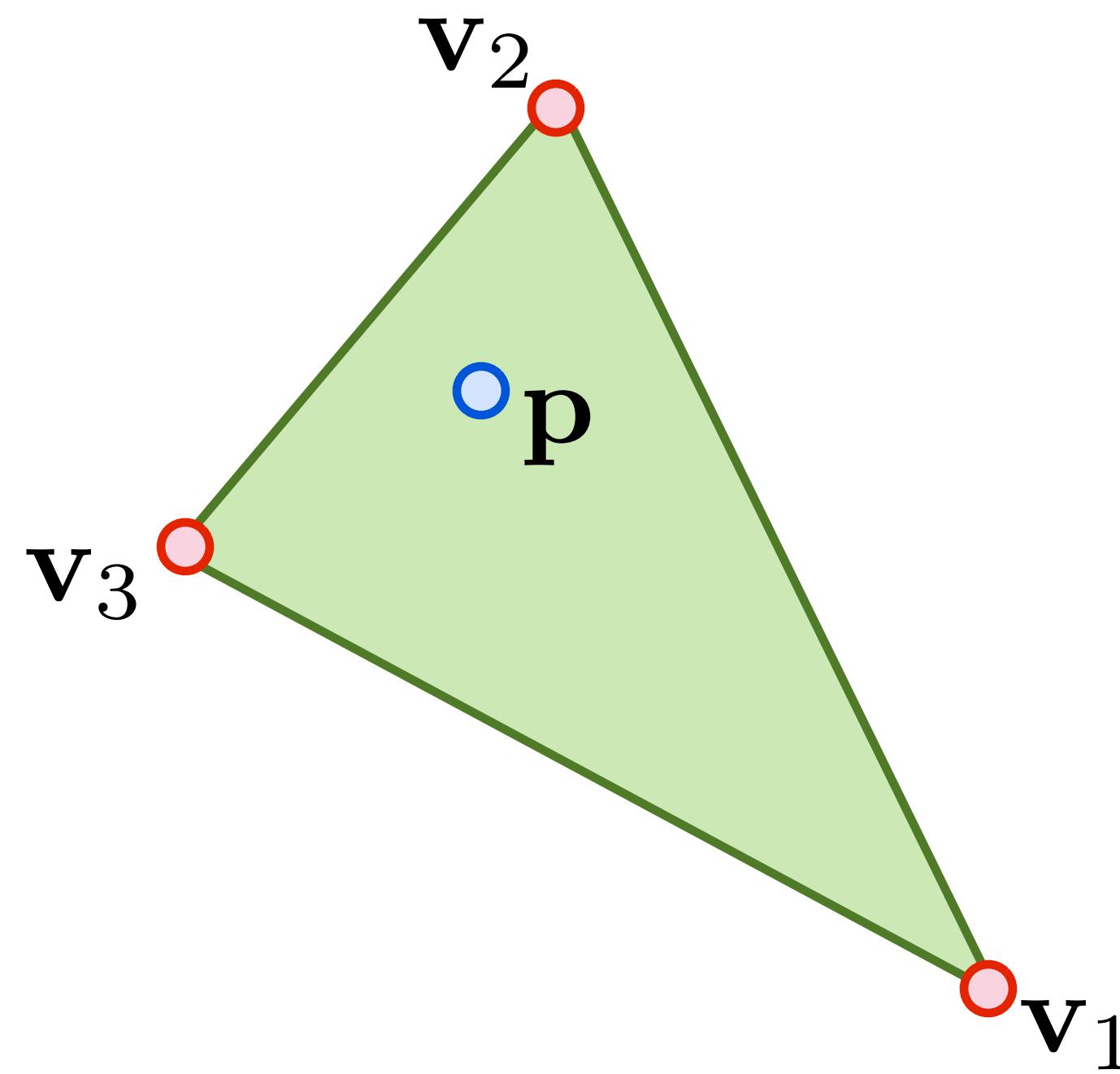
$$\mathbf{p} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + w_3 \mathbf{v}_3$$

Partition of unity: $w_1 + w_2 + w_3 = 1$

$$\mathbf{p} = \underline{w_1} \mathbf{v}_1 + w_2 \mathbf{v}_2 + (1 - w_1 - w_2) \mathbf{v}_3$$



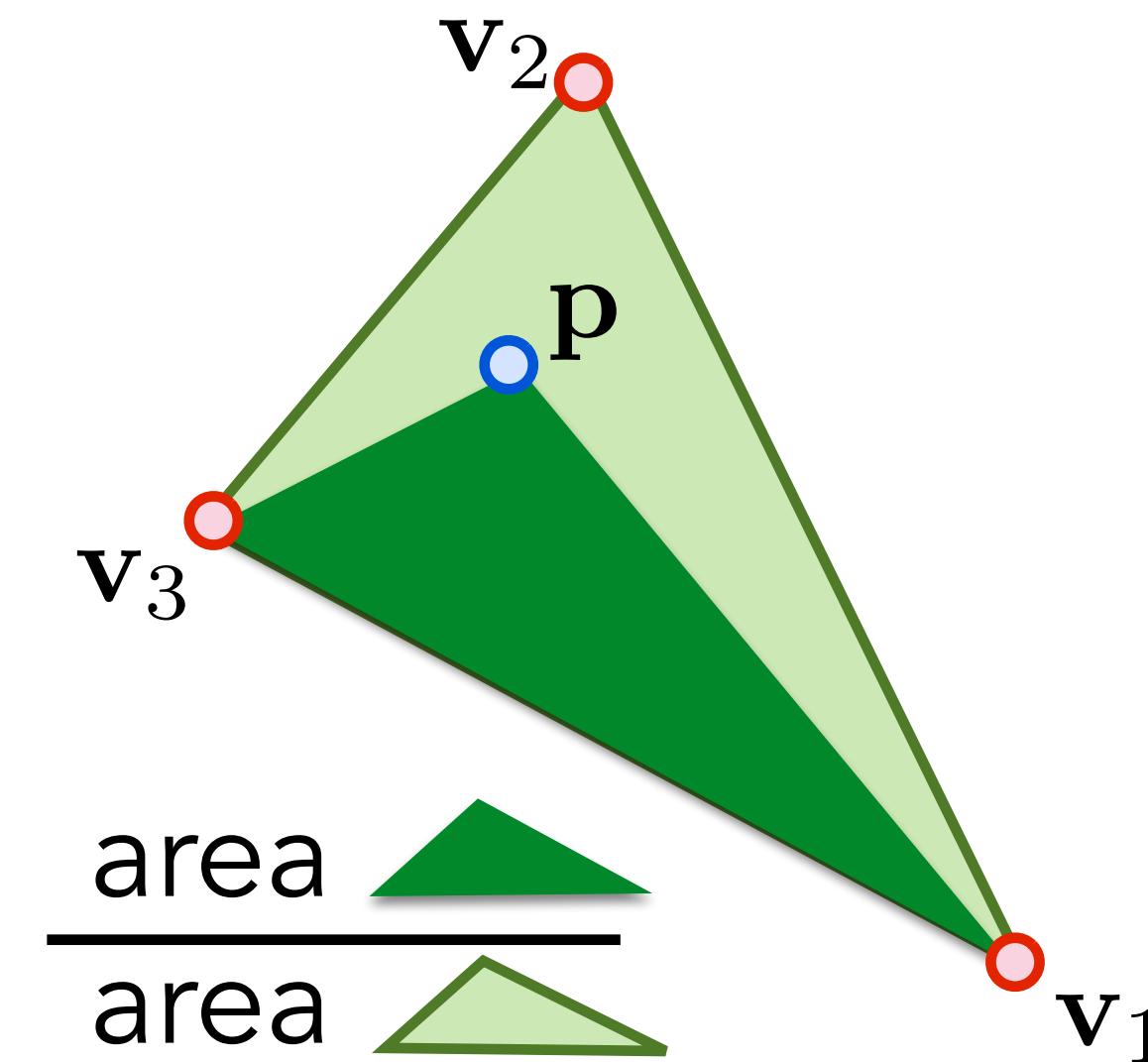
Barycentric Coordinates



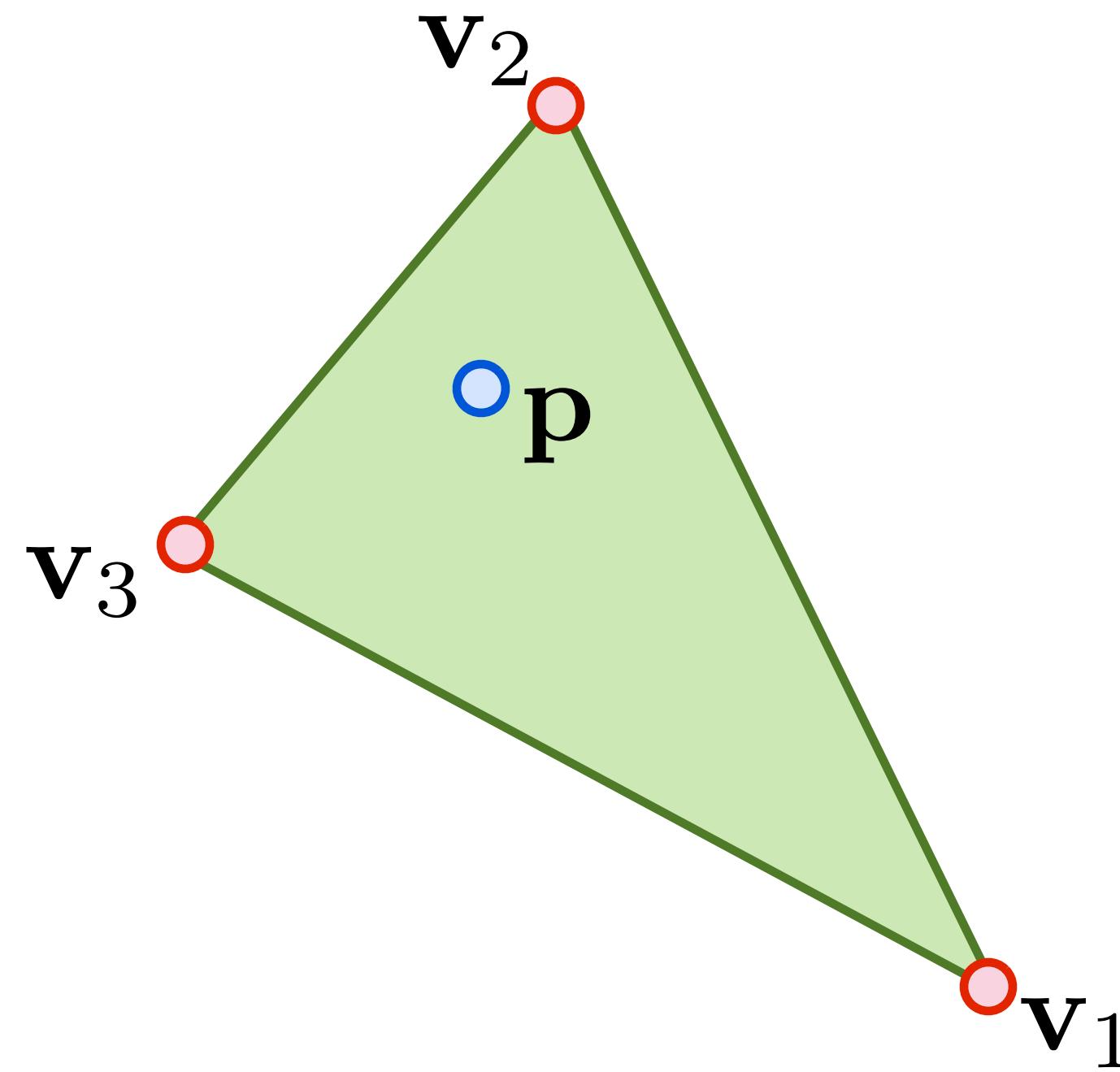
$$\mathbf{p} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + w_3 \mathbf{v}_3$$

Partition of unity: $w_1 + w_2 + w_3 = 1$

$$\mathbf{p} = w_1 \mathbf{v}_1 + \underline{w_2} \mathbf{v}_2 + (1 - w_1 - w_2) \mathbf{v}_3$$



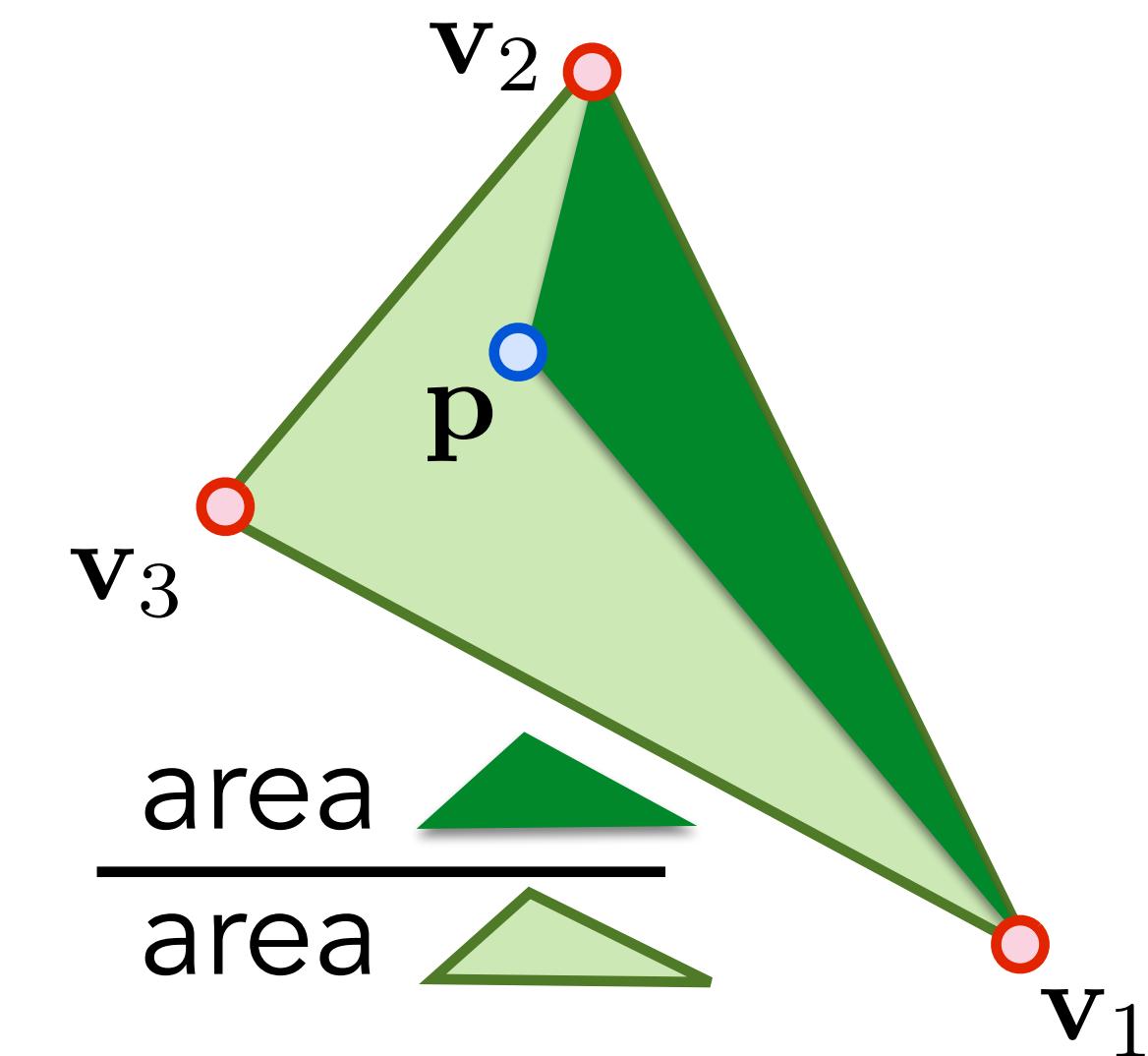
Barycentric Coordinates



$$\mathbf{p} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + w_3 \mathbf{v}_3$$

Partition of unity: $w_1 + w_2 + w_3 = 1$

$$\mathbf{p} = w_1 \mathbf{v}_1 + w_2 \mathbf{v}_2 + \underline{(1 - w_1 - w_2)} \mathbf{v}_3$$

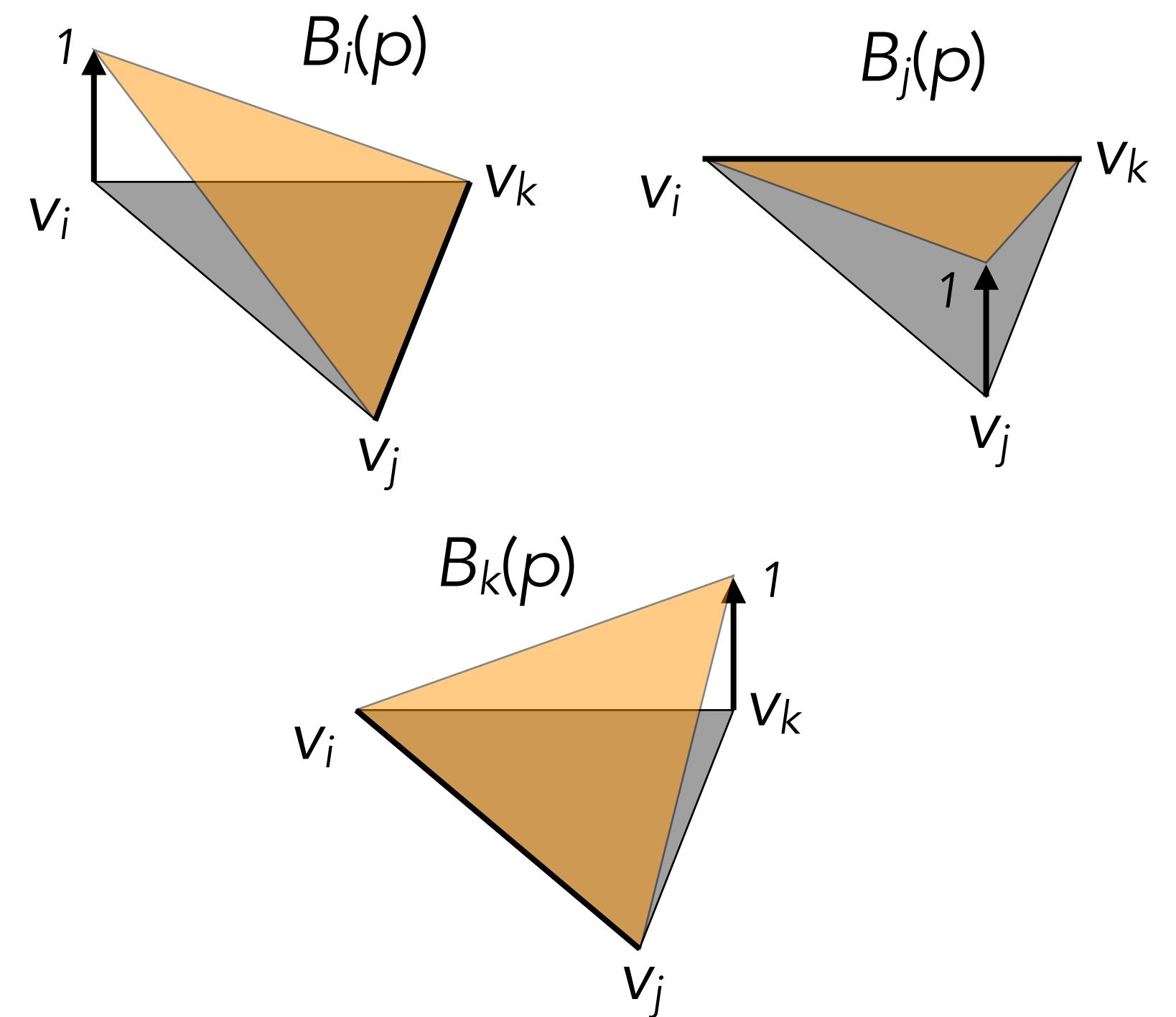


Piecewise linear functions on meshes

Hat functions and PL interpolation

$$f(\mathbf{p}) = B_i(\mathbf{p})f_i + B_j(\mathbf{p})f_j + B_k(\mathbf{p})f_k$$

$$B_i(\mathbf{p}) + B_j(\mathbf{p}) + B_k(\mathbf{p}) = 1$$



Piecewise linear functions on meshes

Hat functions and PL interpolation

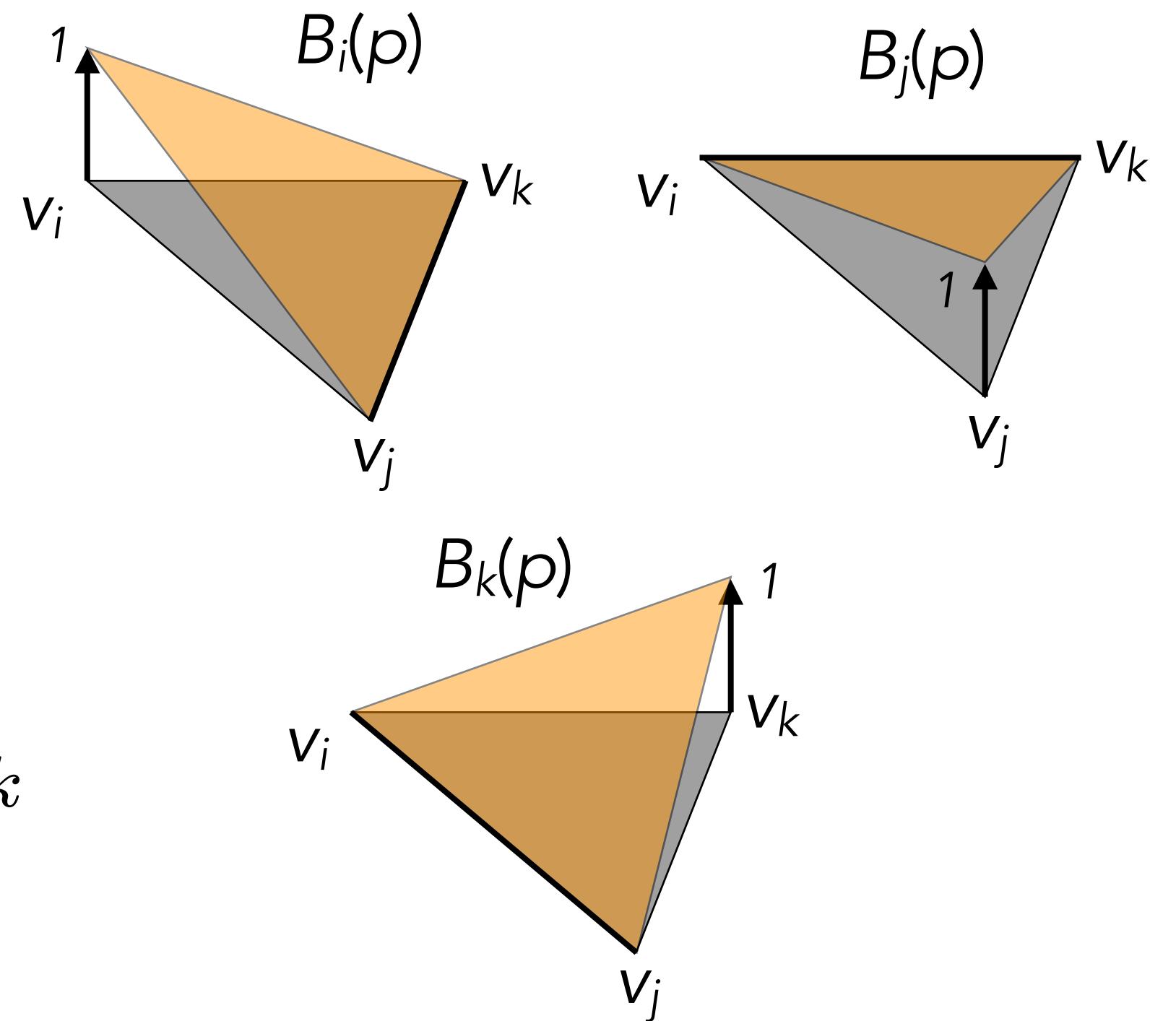
$$f(\mathbf{p}) = B_i(\mathbf{p})f_i + B_j(\mathbf{p})f_j + B_k(\mathbf{p})f_k$$

$$B_i(\mathbf{p}) + B_j(\mathbf{p}) + B_k(\mathbf{p}) = 1$$

Gradients

$$\nabla f(\mathbf{p}) = \nabla B_i(\mathbf{p})f_i + \nabla B_j(\mathbf{p})f_j + \nabla B_k(\mathbf{p})f_k$$

$$\nabla B_i(\mathbf{p}) + \nabla B_j(\mathbf{p}) + \nabla B_k(\mathbf{p}) = 0$$



Piecewise linear functions on meshes

Hat functions and PL interpolation

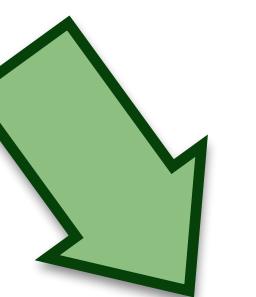
$$f(\mathbf{p}) = B_i(\mathbf{p})f_i + B_j(\mathbf{p})f_j + B_k(\mathbf{p})f_k$$

$$B_i(\mathbf{p}) + B_j(\mathbf{p}) + B_k(\mathbf{p}) = 1$$

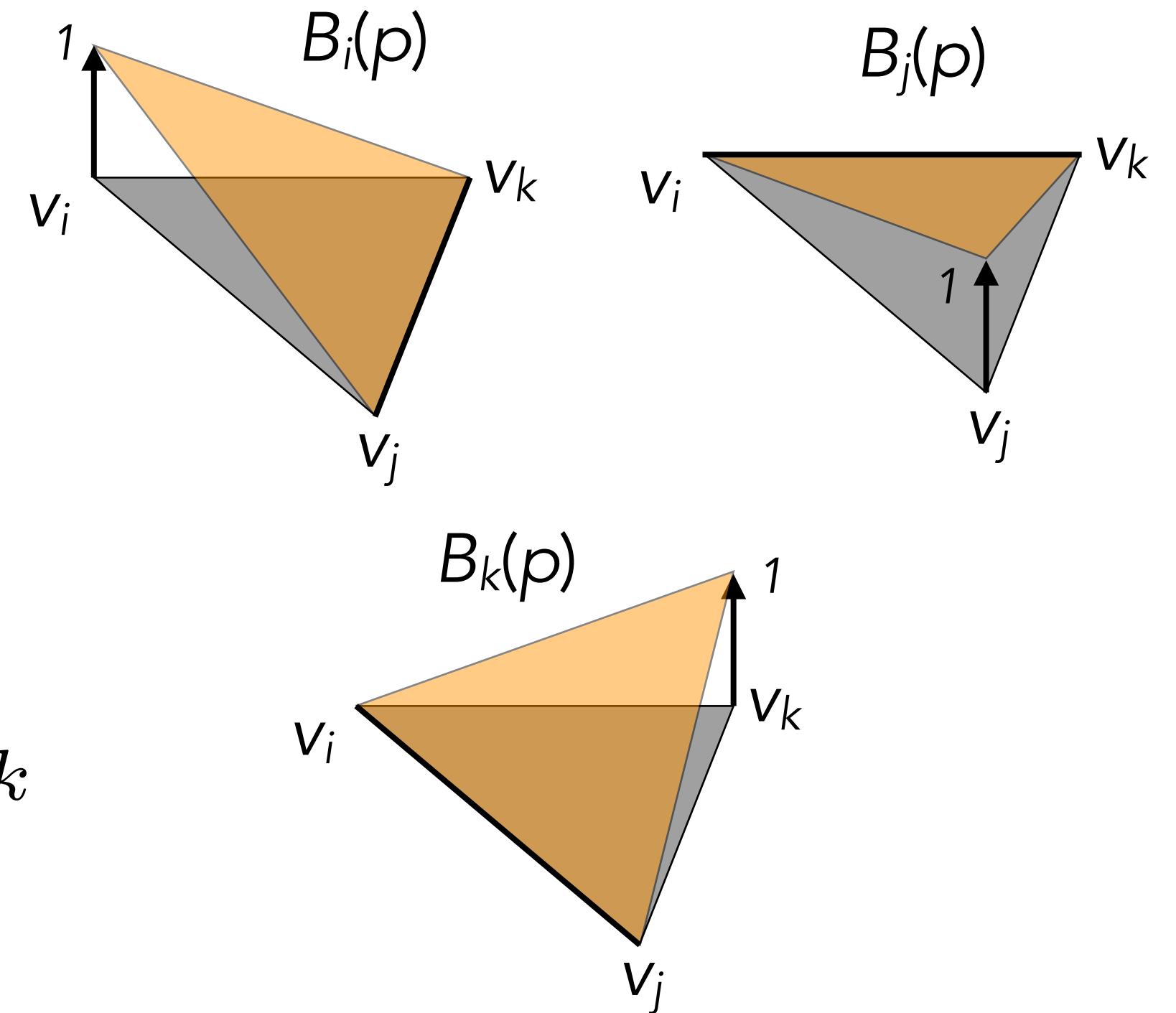
Gradients

$$\nabla f(\mathbf{p}) = \nabla B_i(\mathbf{p})f_i + \nabla B_j(\mathbf{p})f_j + \nabla B_k(\mathbf{p})f_k$$

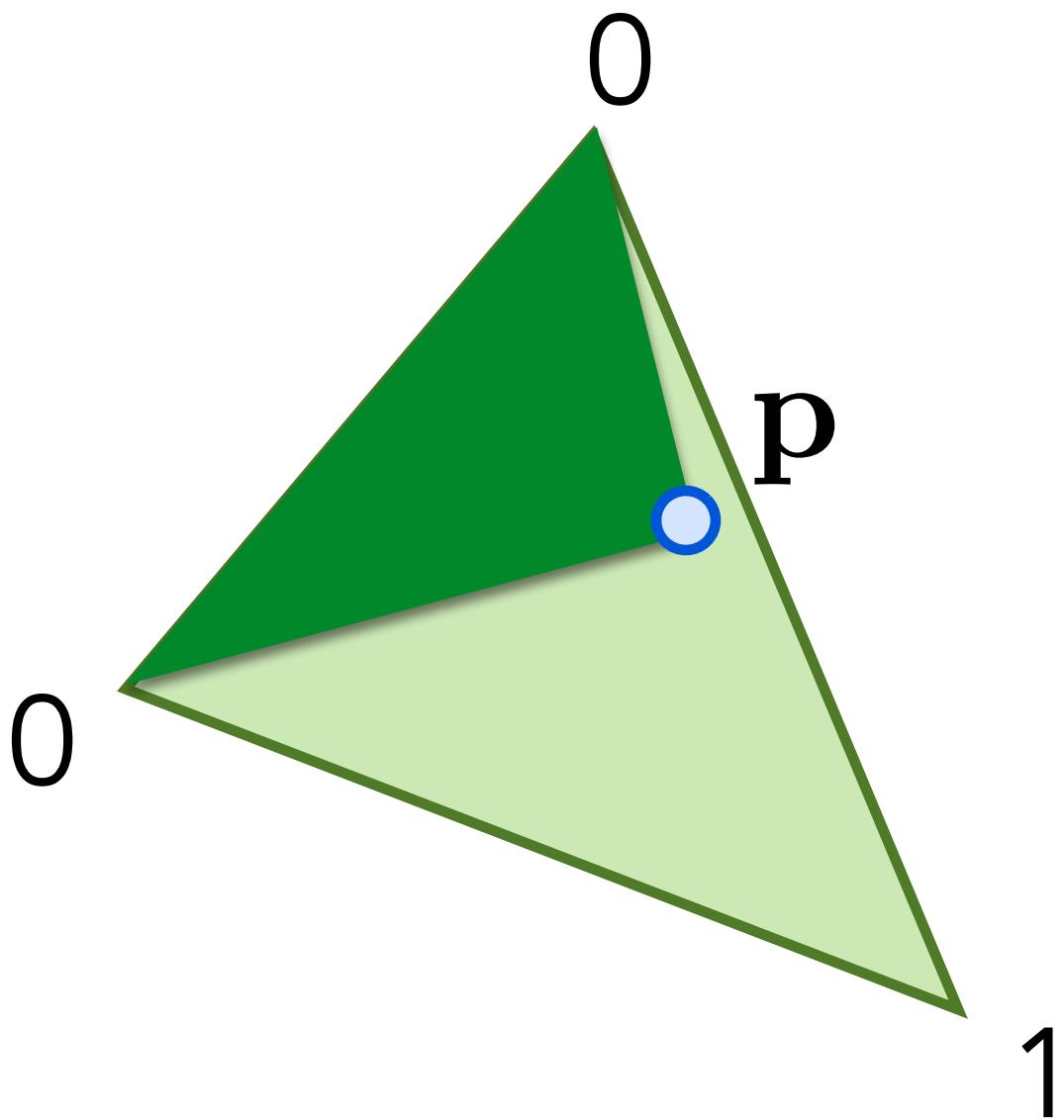
$$\nabla B_i(\mathbf{p}) + \nabla B_j(\mathbf{p}) + \nabla B_k(\mathbf{p}) = 0$$



$$\nabla f(\mathbf{p}) = (f_j - f_i)\nabla B_j(\mathbf{p}) + (f_k - f_i)\nabla B_k(\mathbf{p})$$

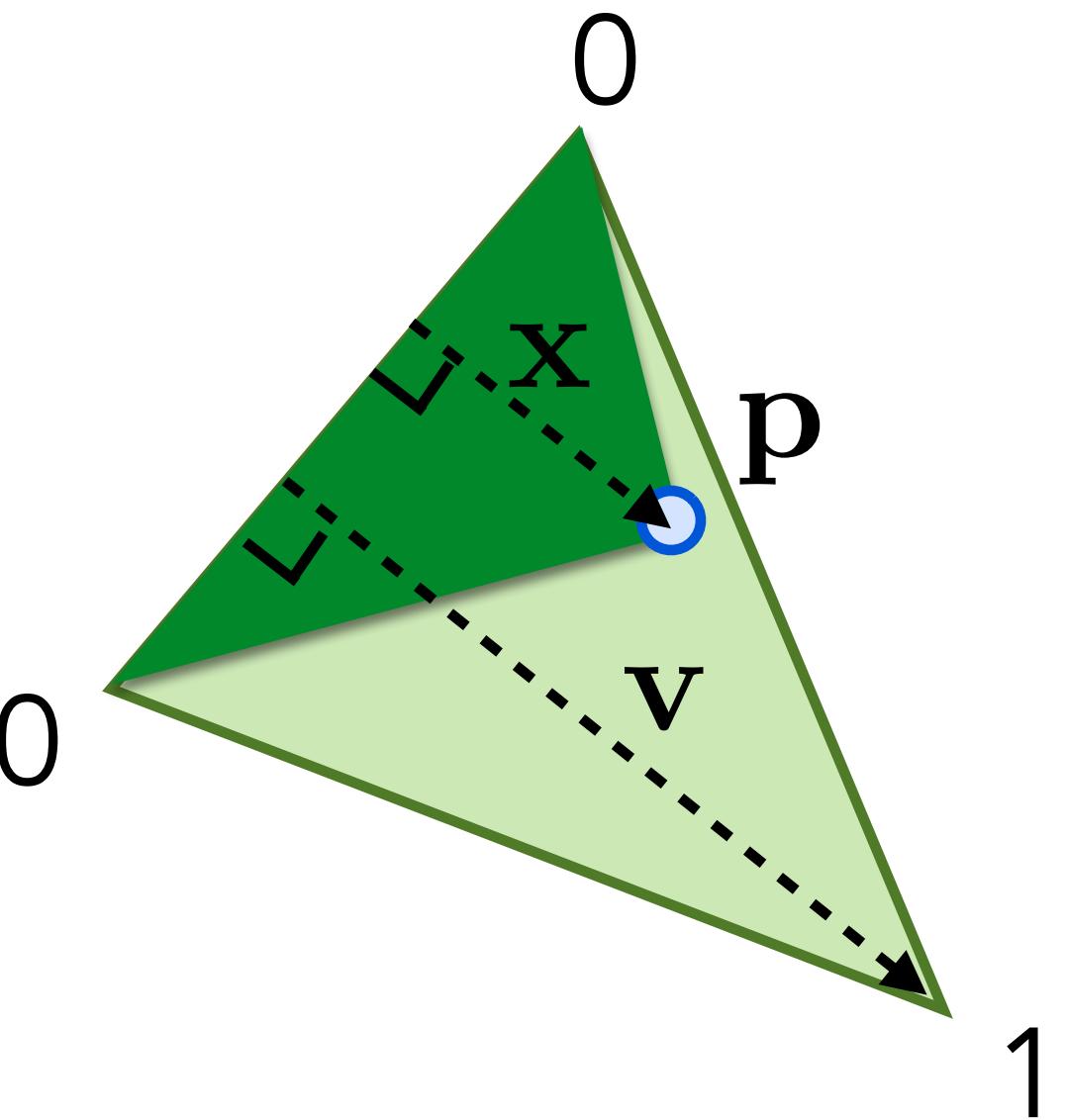


The hat function



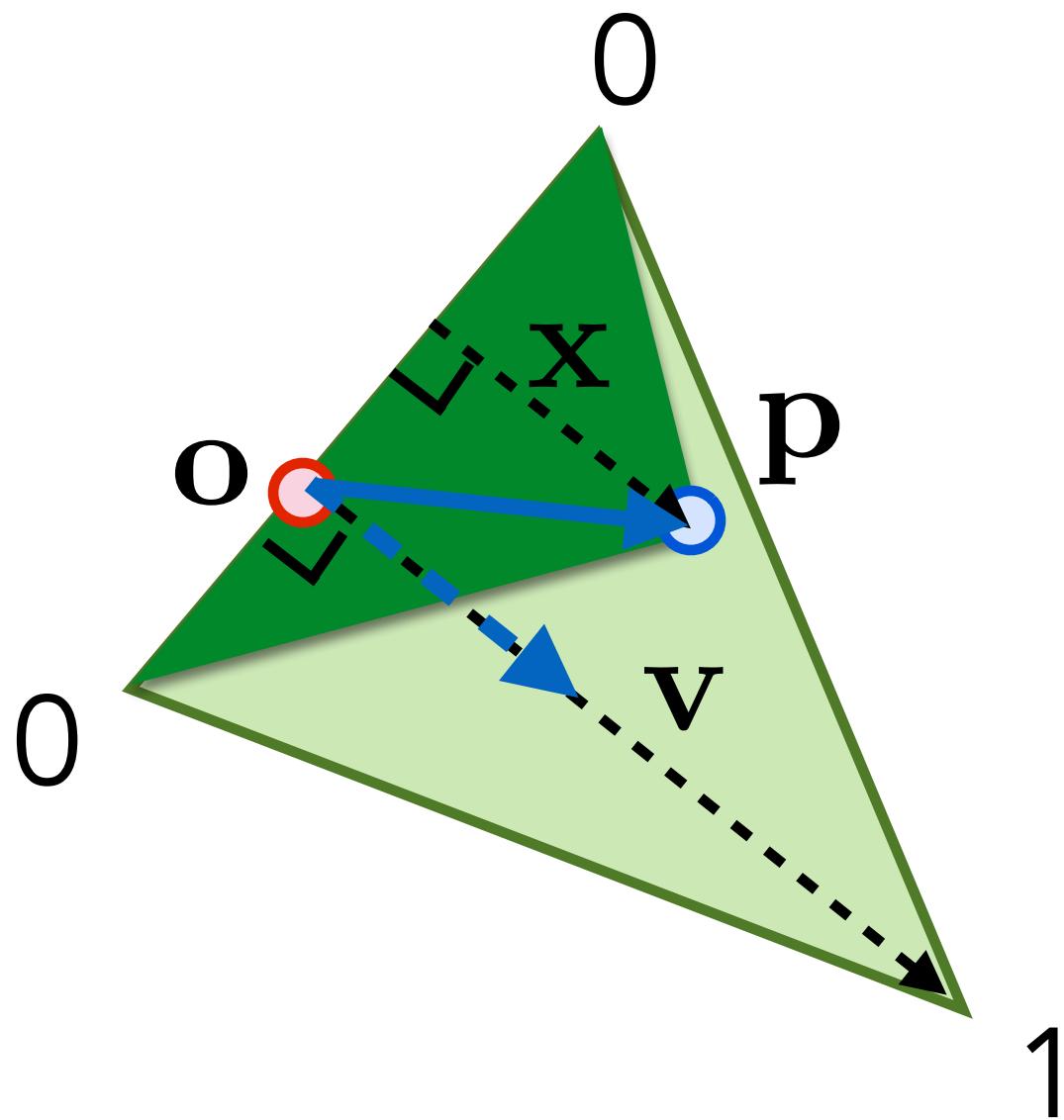
$$B(p) = \frac{\text{area } \triangle}{\text{area } \triangle}$$

The hat function



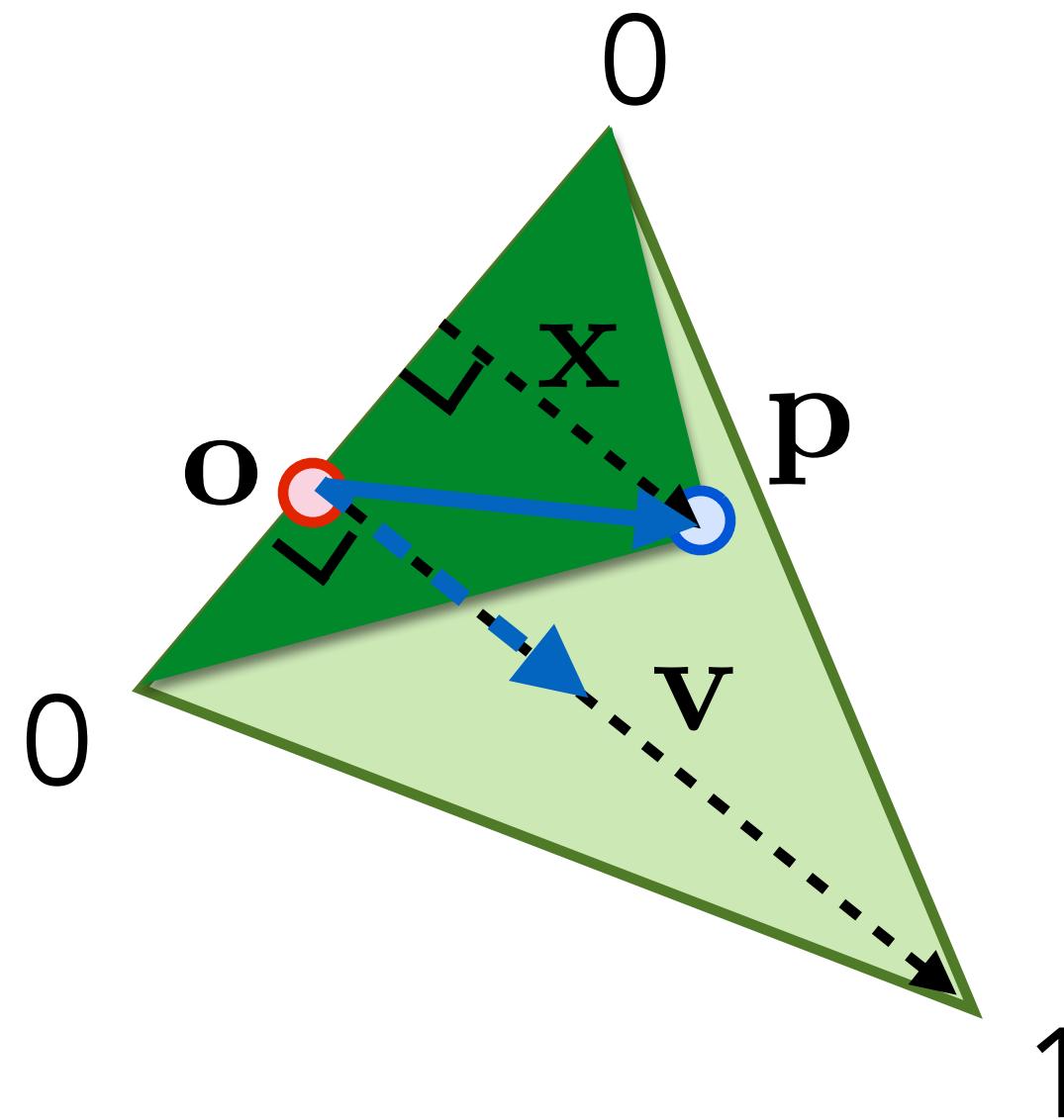
$$B(p) = \frac{\text{area } \triangle}{\text{area } \triangle} = \frac{\|x\|}{\|v\|}$$

The hat function



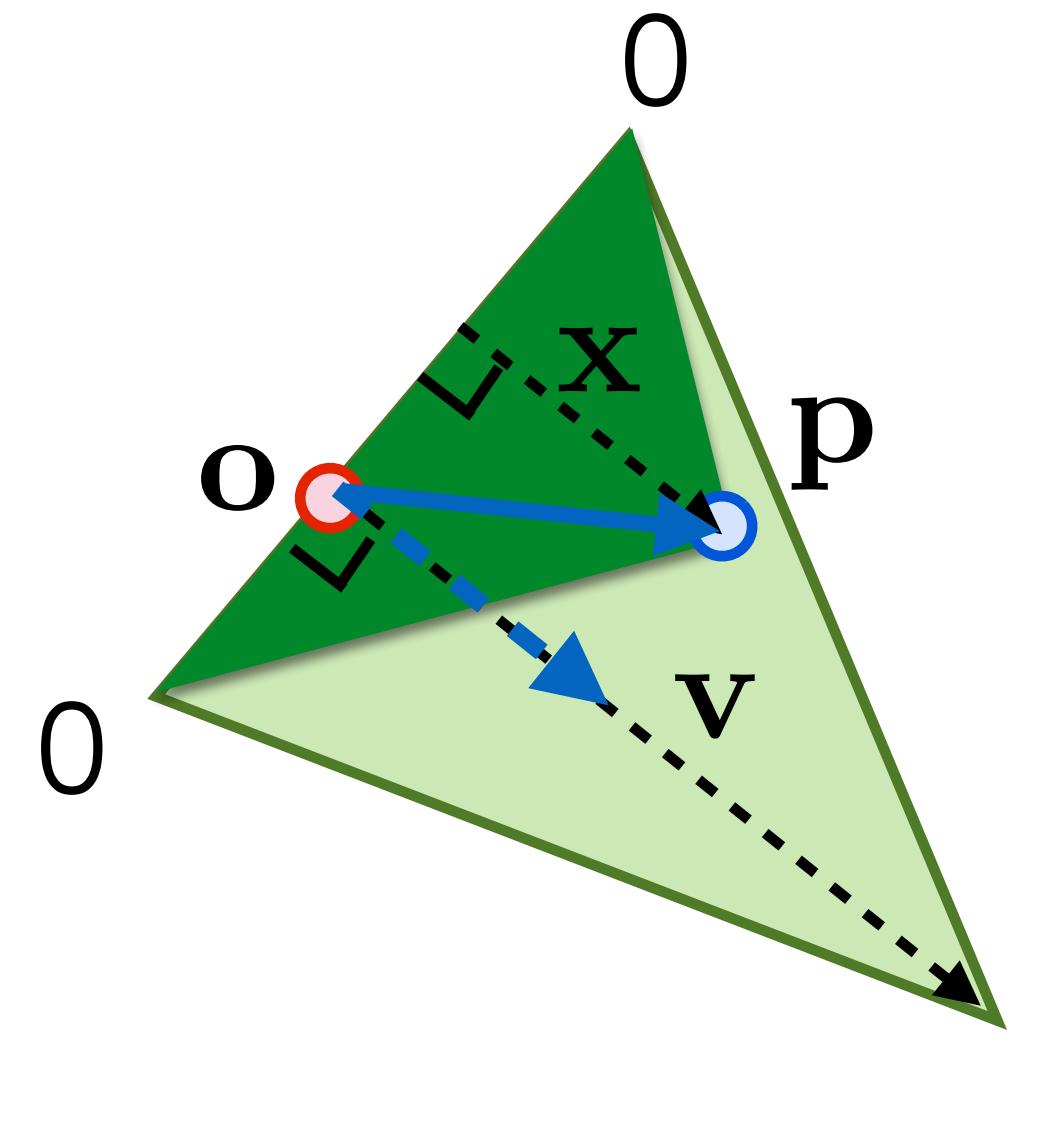
$$B(p) = \frac{\text{area } \triangle opv}{\text{area } \triangle ovp} = \frac{\|x\|}{\|v\|} = \frac{(p - o) \cdot \frac{v}{\|v\|}}{\|v\|}$$

The hat function



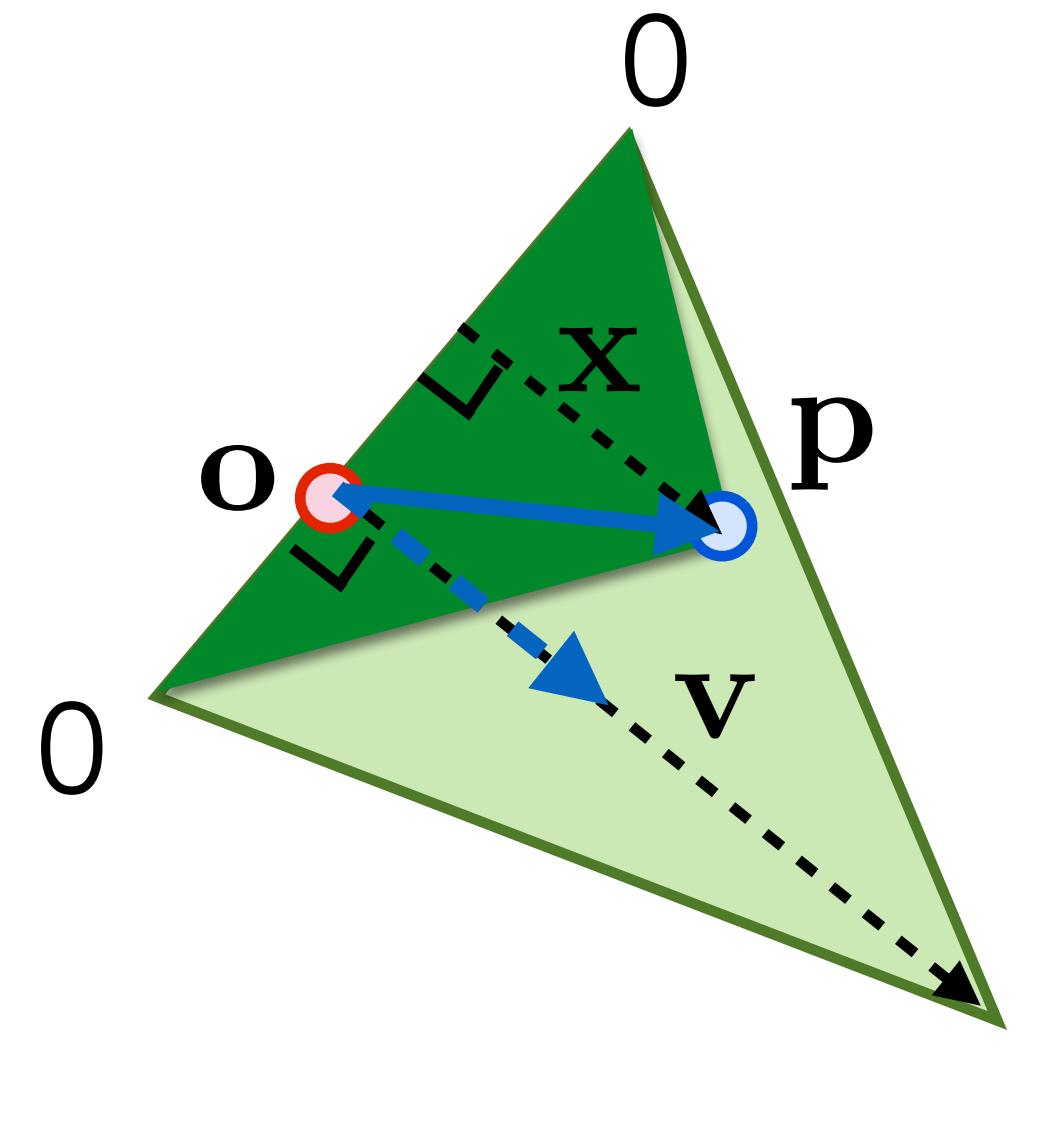
$$B(p) = \frac{\text{area } \triangle}{\text{area } \triangle} = \frac{\|x\|}{\|v\|} = \frac{(p - o) \cdot \frac{v}{\|v\|}}{\|v\|} = \frac{(p - o) \cdot v}{\|v\| \|v\|}$$

Gradient of the hat function



$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

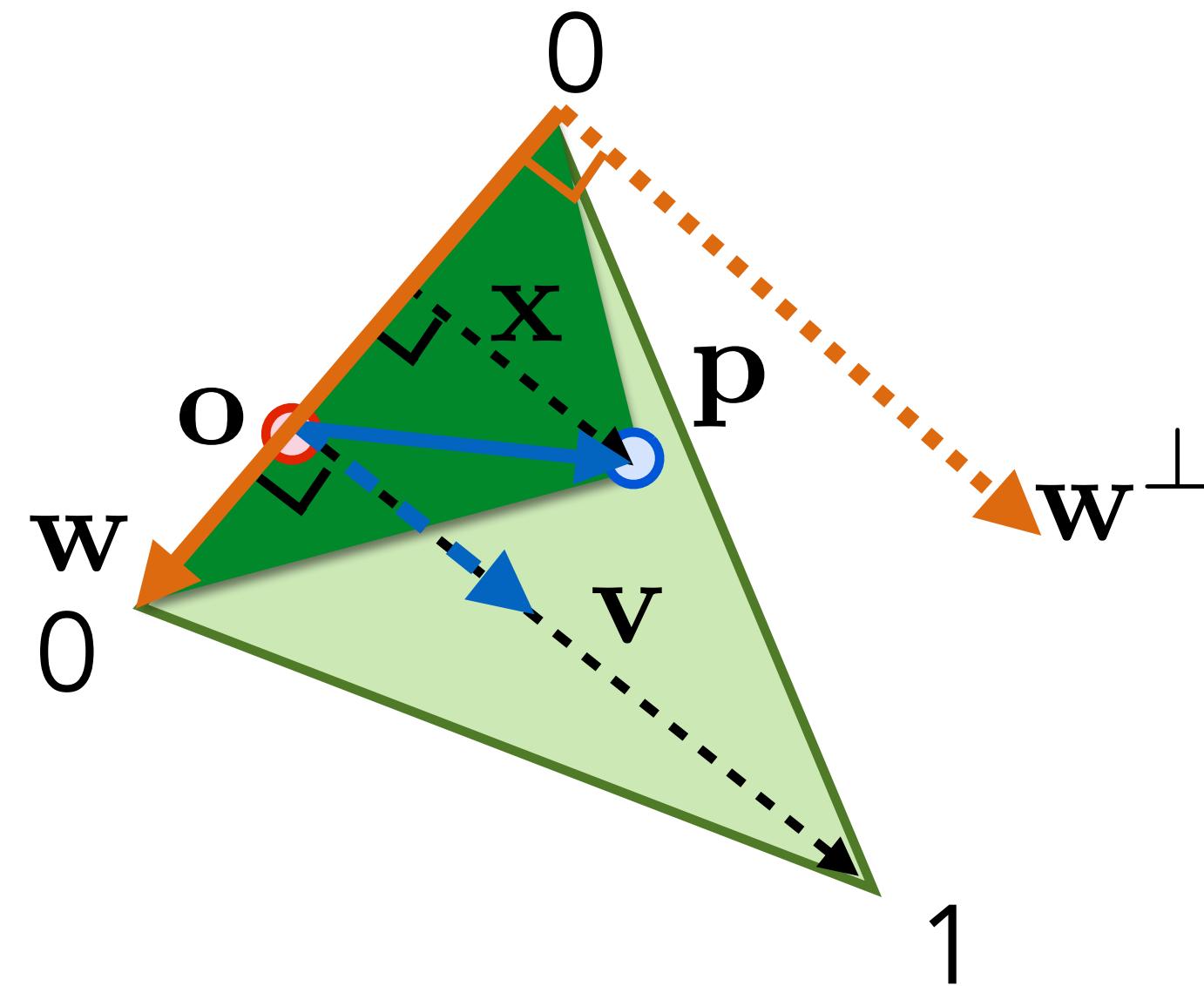
Gradient of the hat function



$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

Gradient of the hat function

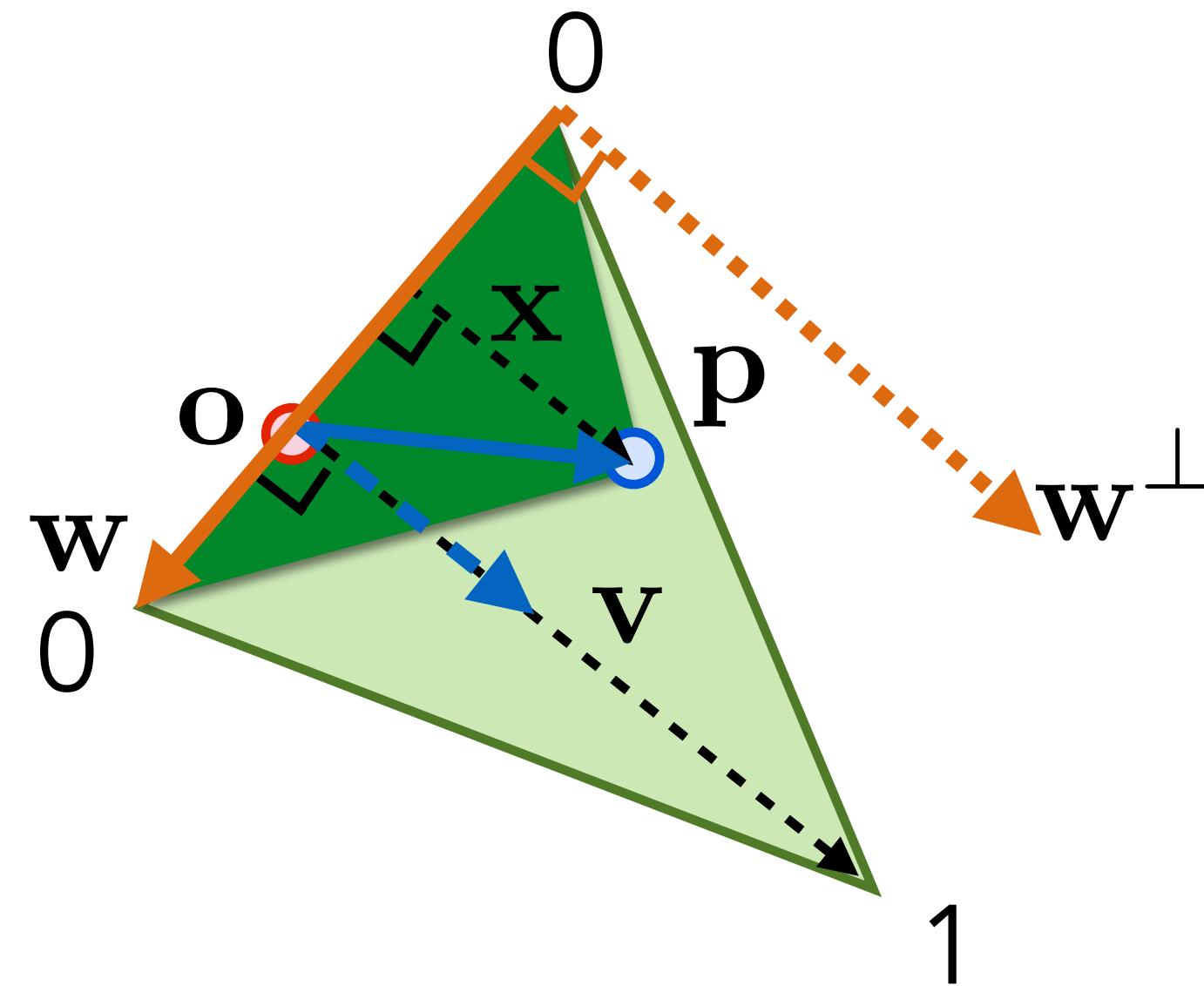


$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\|}$$

$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

Gradient of the hat function

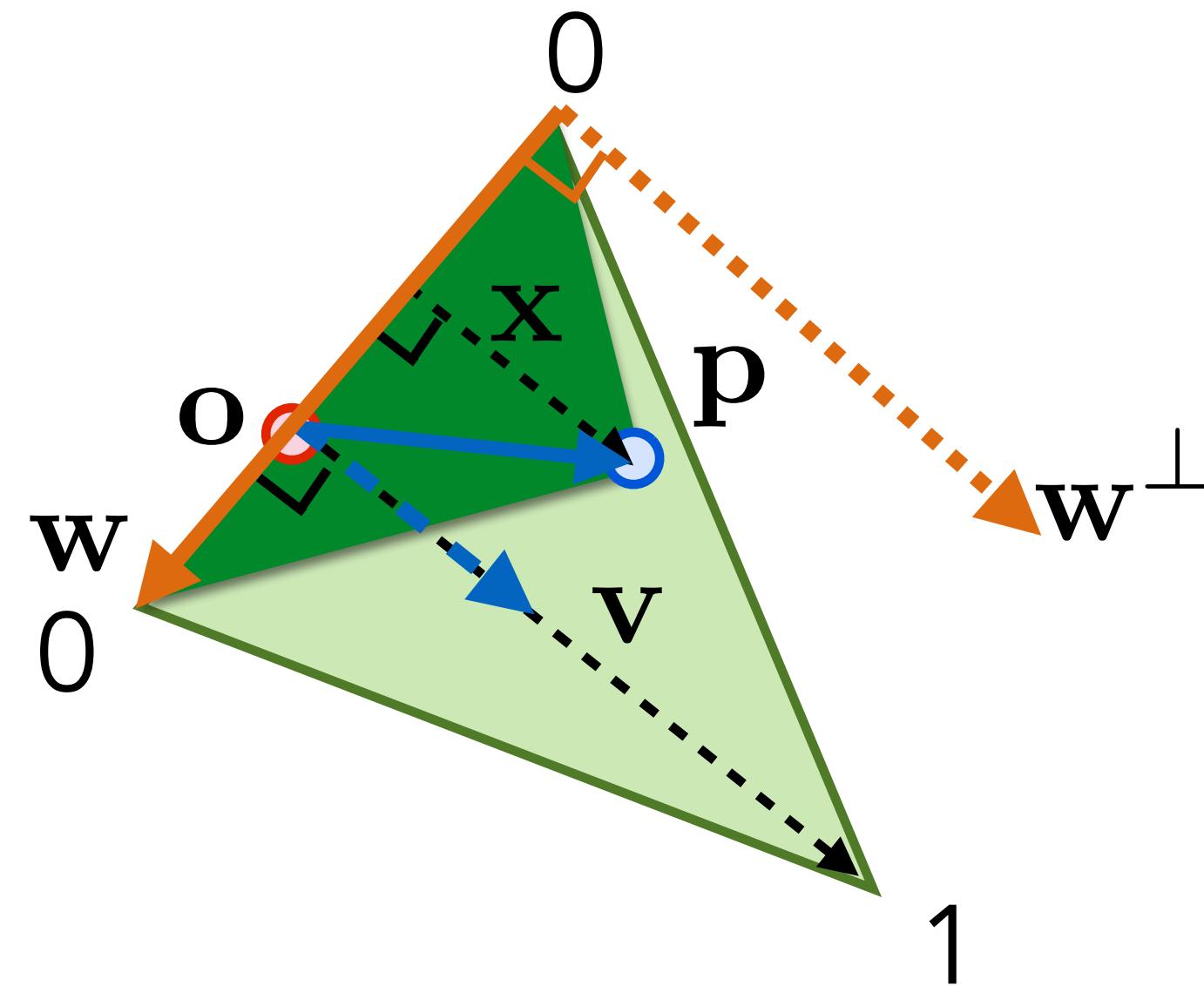


$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$
$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\| \|\mathbf{v}\|}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\|}$$

Gradient of the hat function



$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

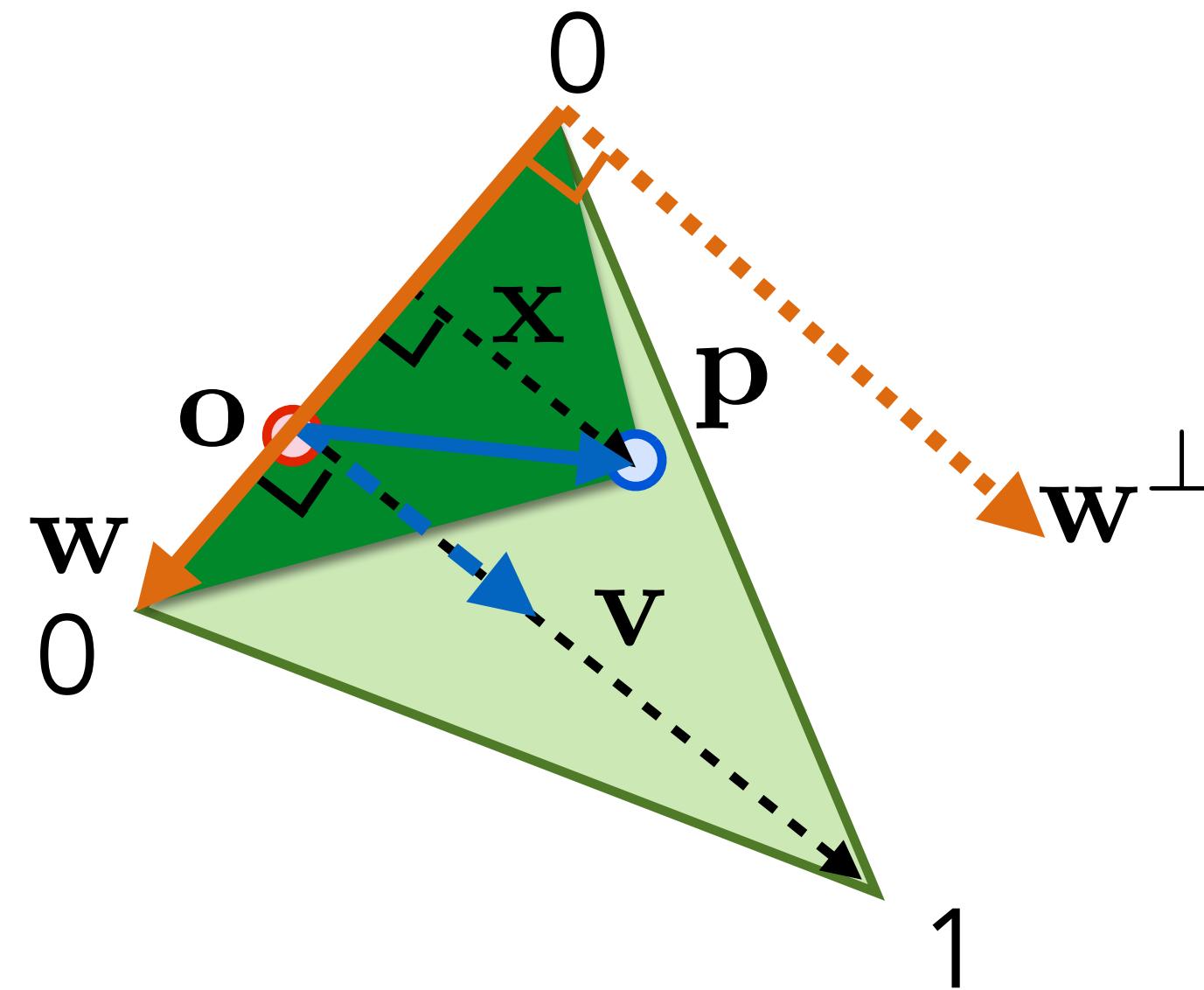
$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\| \|\mathbf{v}\|}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\|}$$

$$\|\mathbf{w}^\perp\| = \|\mathbf{w}\|$$

Gradient of the hat function



$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

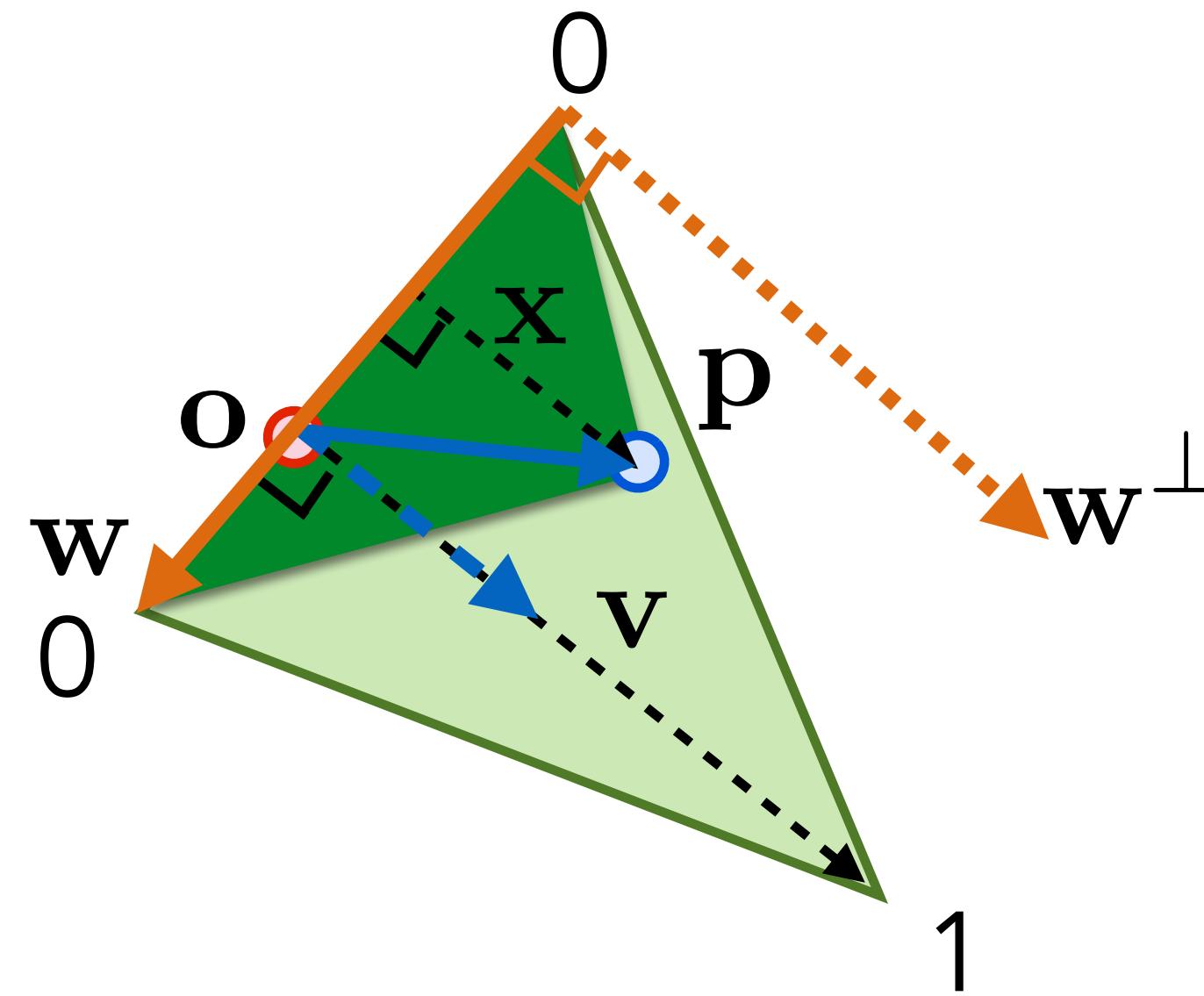
$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{\|\mathbf{w}\| \|\mathbf{v}\|}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\|}$$

$$\|\mathbf{w}^\perp\| = \|\mathbf{w}\|$$

Gradient of the hat function



$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\| \|\mathbf{v}\|}$$

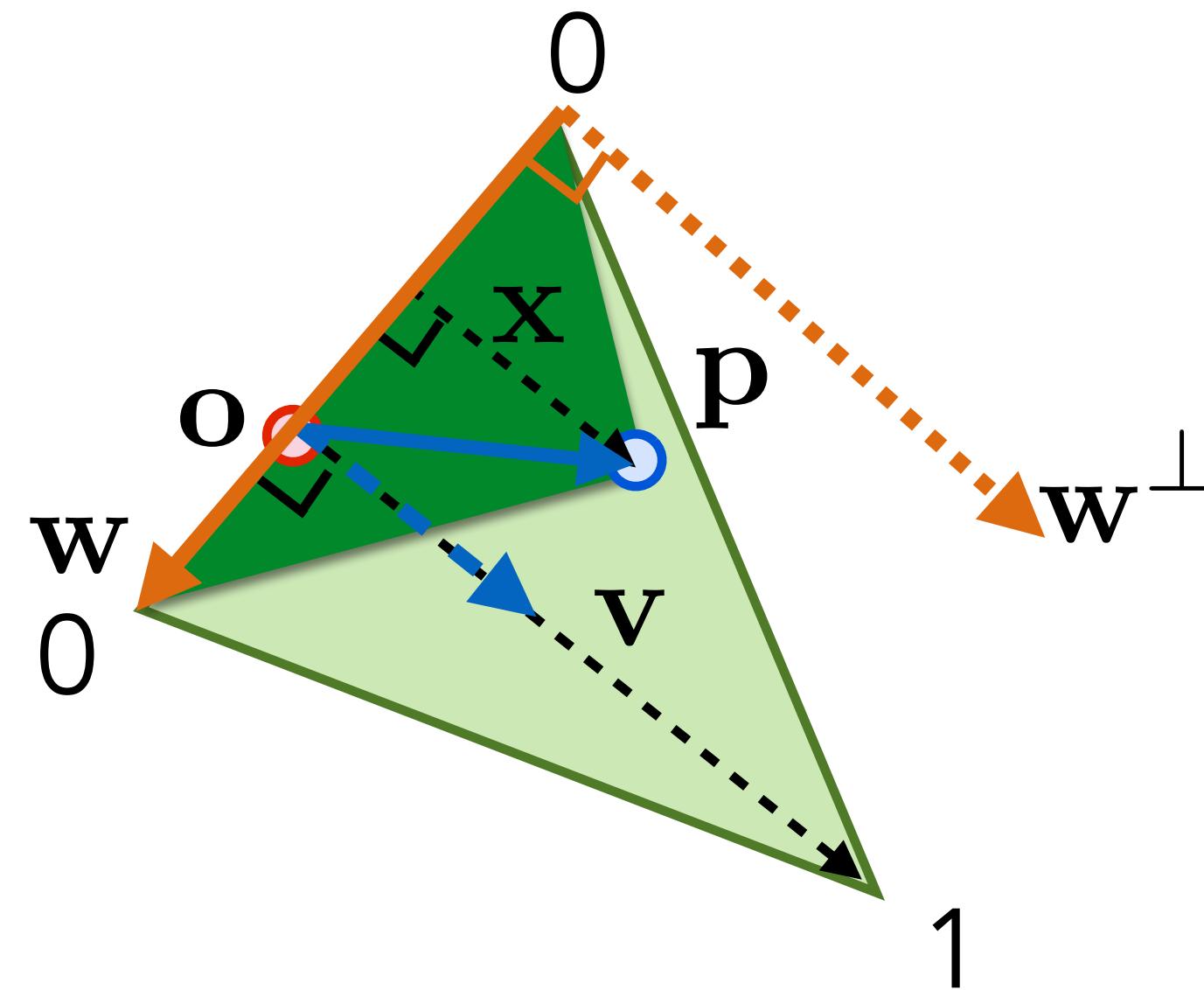
$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{\|\mathbf{w}\| \|\mathbf{v}\|}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\|}$$

$$\|\mathbf{w}^\perp\| = \|\mathbf{w}\|$$

$$A = \frac{\|\mathbf{v}\| \|\mathbf{w}\|}{2}$$

Gradient of the hat function



$$B(\mathbf{p}) = \frac{(\mathbf{p} - \mathbf{o}) \cdot \mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{v}}{\|\mathbf{v}\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\| \|\mathbf{v}\|}$$

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\mathbf{w}^\perp}{\|\mathbf{w}^\perp\|}$$

$$\|\mathbf{w}^\perp\| = \|\mathbf{w}\|$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{\|\mathbf{w}\| \|\mathbf{v}\|}$$

$$\nabla B(\mathbf{p}) = \frac{\mathbf{w}^\perp}{2A}$$

$$A = \frac{\|\mathbf{v}\| \|\mathbf{w}\|}{2}$$

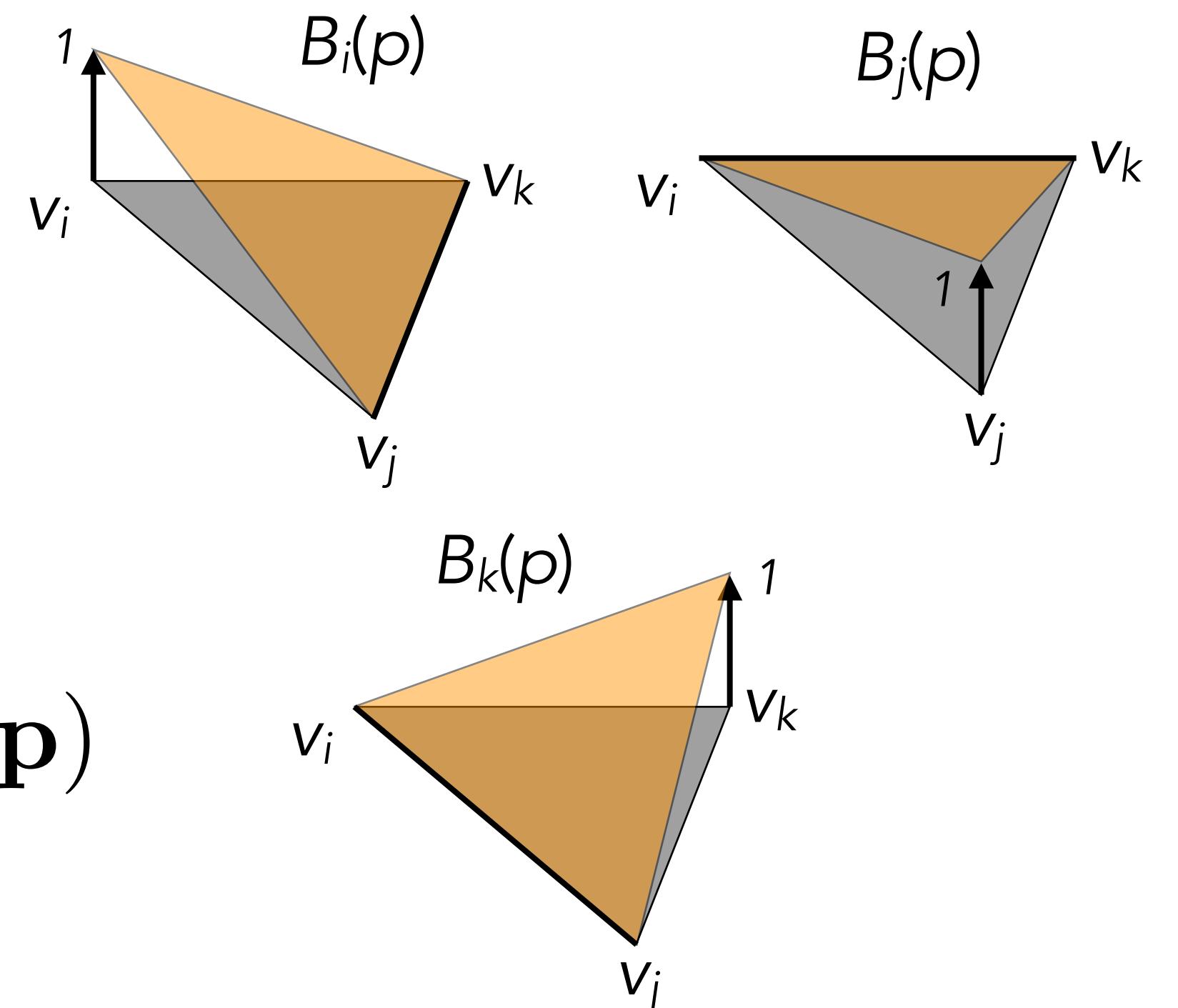
Piecewise linear functions on meshes

Hat functions and PL interpolation

$$f(\mathbf{p}) = B_i(\mathbf{p})f_i + B_j(\mathbf{p})f_j + B_k(\mathbf{p})f_k$$

Gradients

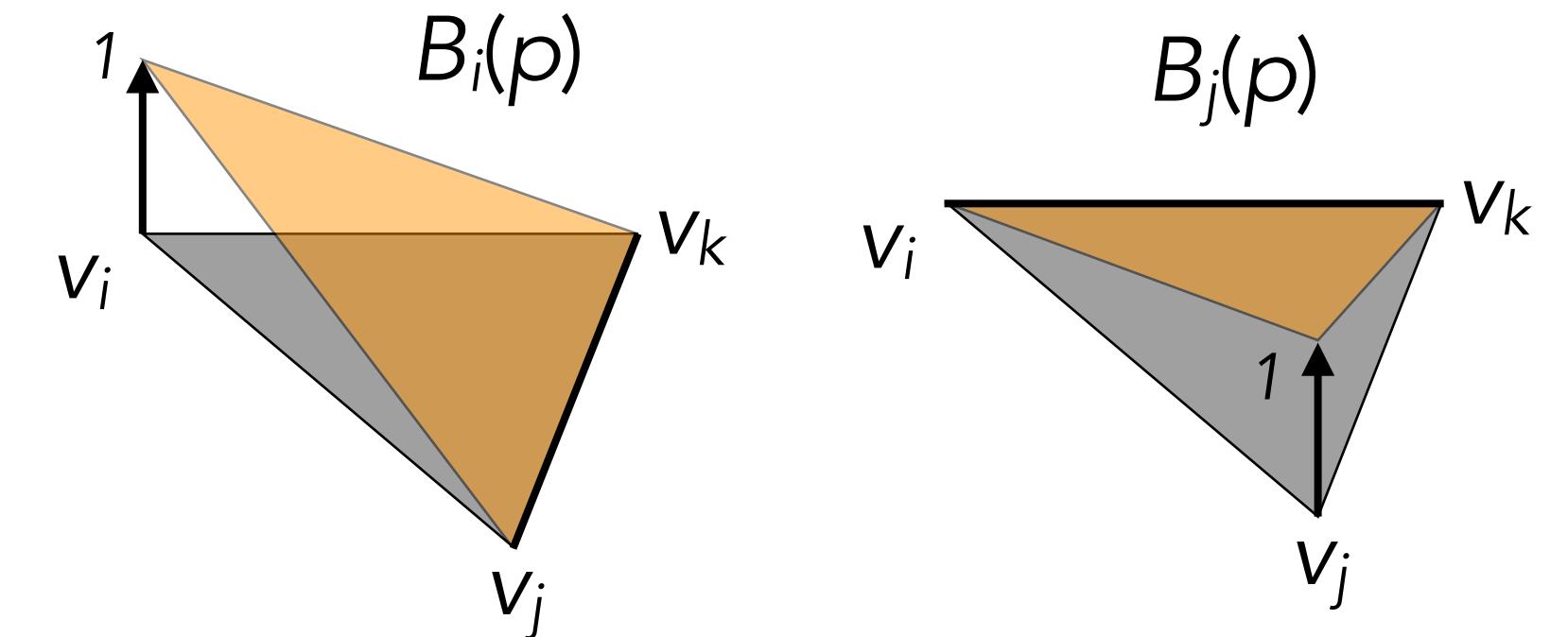
$$\nabla f(\mathbf{p}) = (f_j - f_i)\nabla B_j(\mathbf{p}) + (f_k - f_i)\nabla B_k(\mathbf{p})$$



Piecewise linear functions on meshes

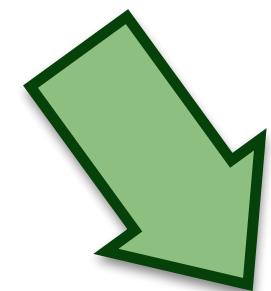
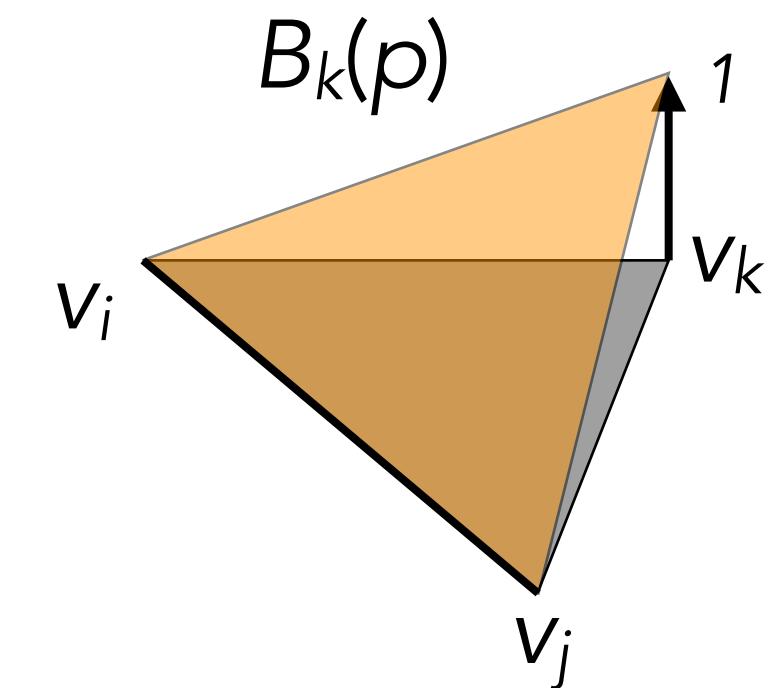
Hat functions and PL interpolation

$$f(\mathbf{p}) = B_i(\mathbf{p})f_i + B_j(\mathbf{p})f_j + B_k(\mathbf{p})f_k$$



Gradients

$$\nabla f(\mathbf{p}) = (f_j - f_i) \nabla B_j(\mathbf{p}) + (f_k - f_i) \nabla B_k(\mathbf{p})$$



$$\nabla f(\mathbf{p}) = (f_j - f_i) \frac{(\mathbf{v}_i - \mathbf{v}_k)^\perp}{2A} + (f_k - f_i) \frac{(\mathbf{v}_j - \mathbf{v}_i)^\perp}{2A}$$

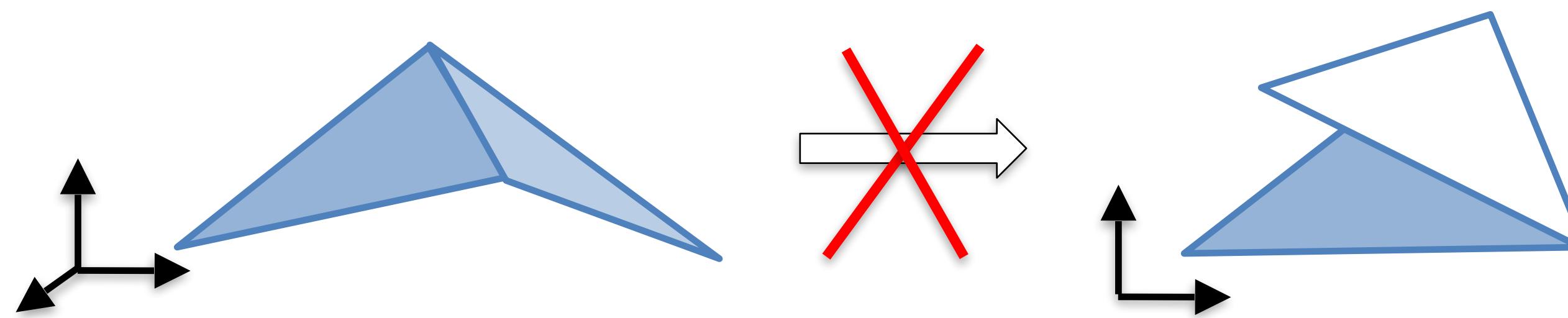
Properties

Desiderata?

- What are “good” parameterizations?
- How do we define “good”?

Bijection

- Locally bijective (1-1 and onto): No triangles fold over.



- Globally bijective:
locally bijective +
no “distant” areas
overlap

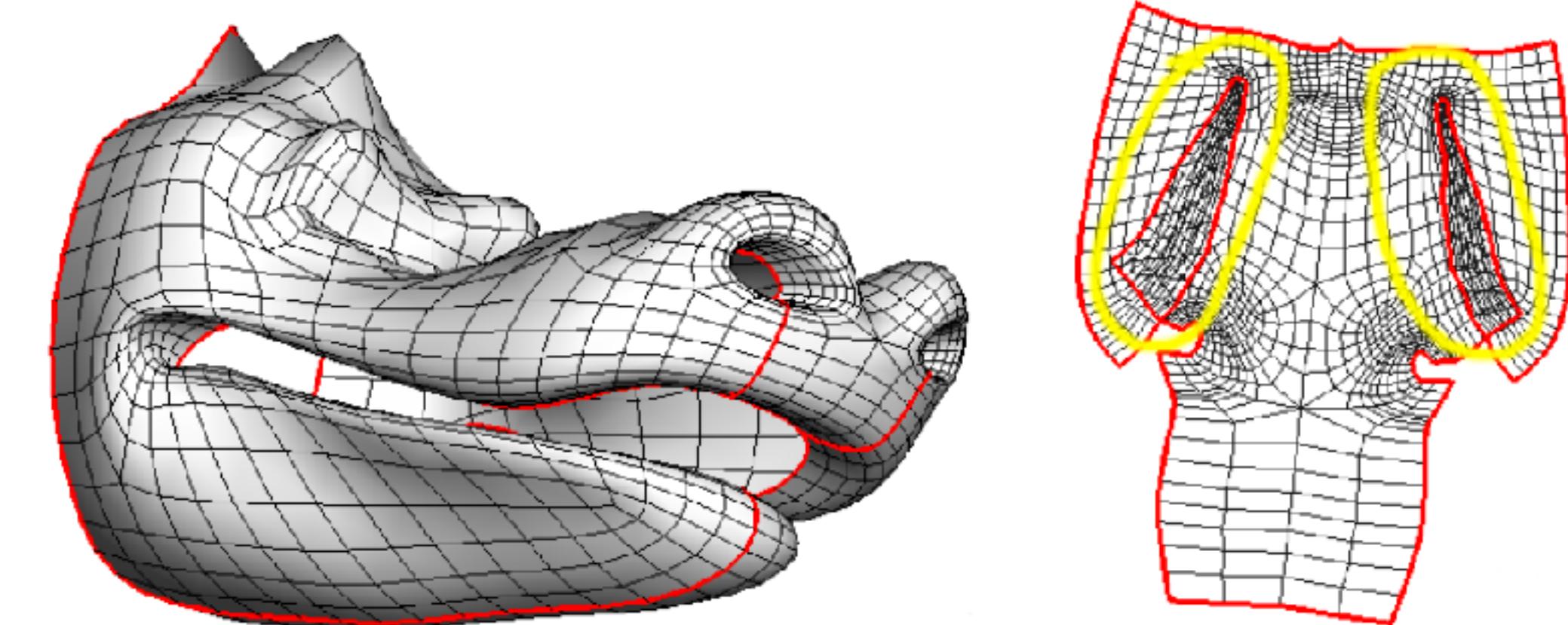
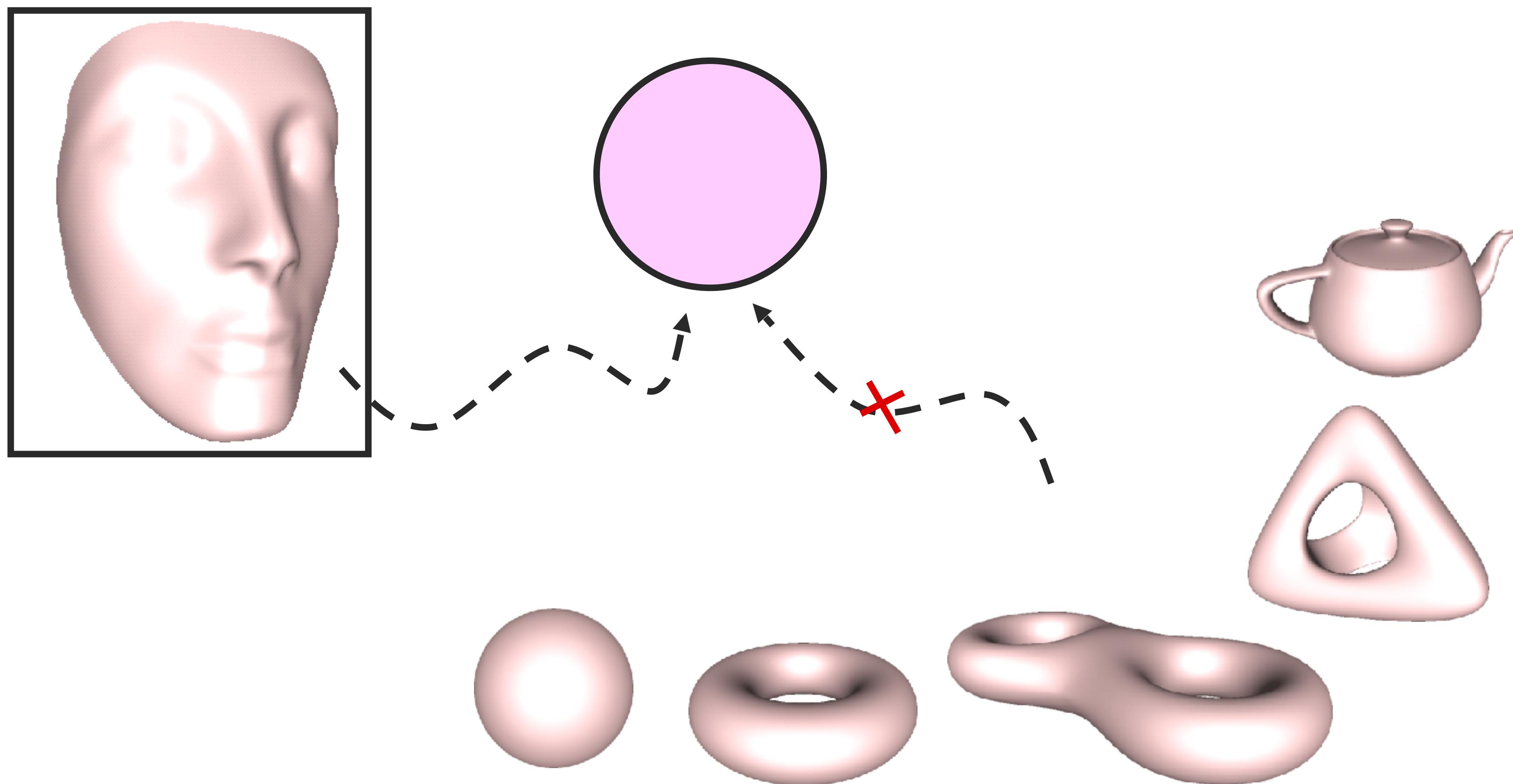
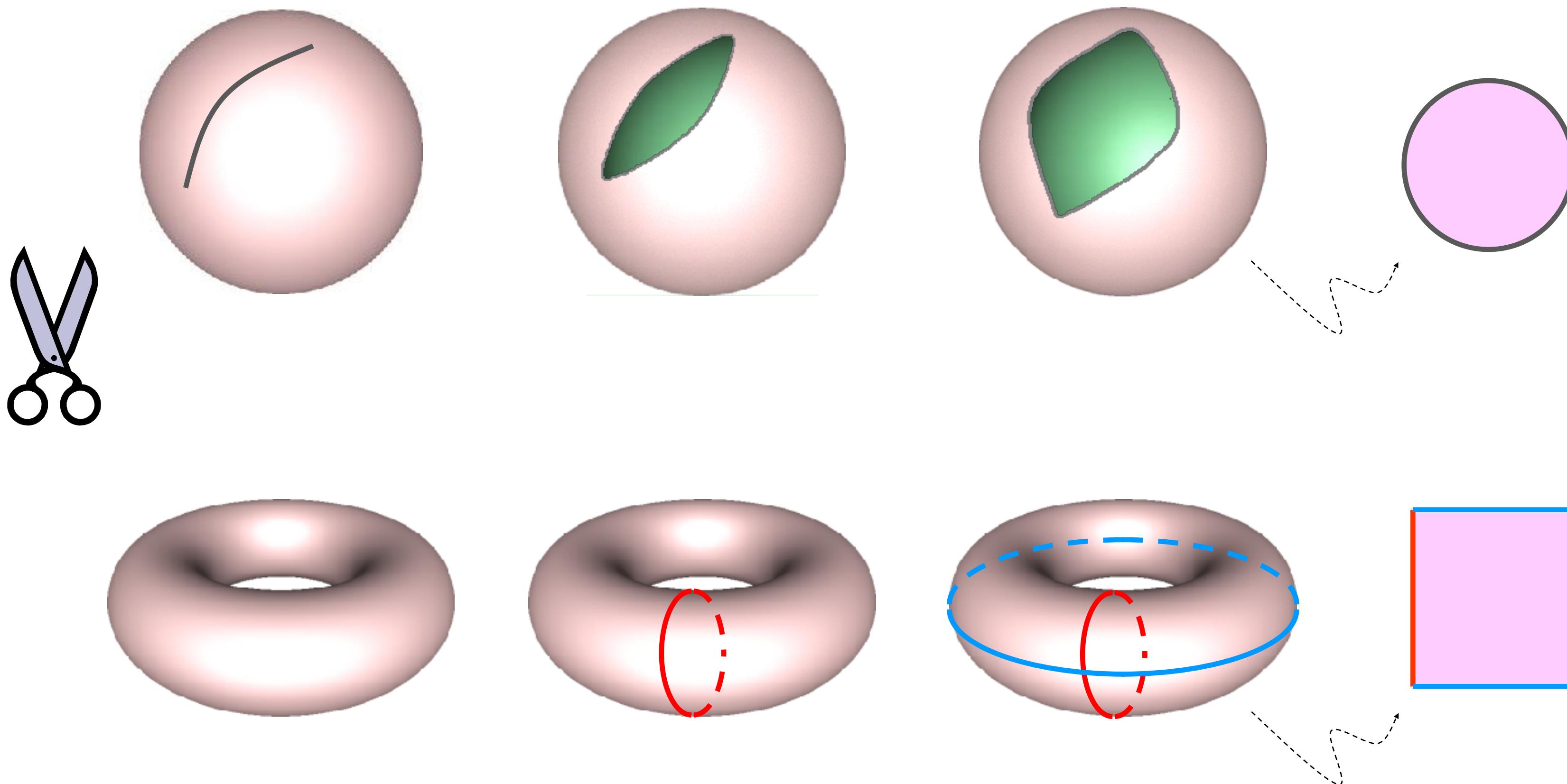


image from “Least Squares Conformal Maps”, Lévy et al., SIGGRAPH 2002

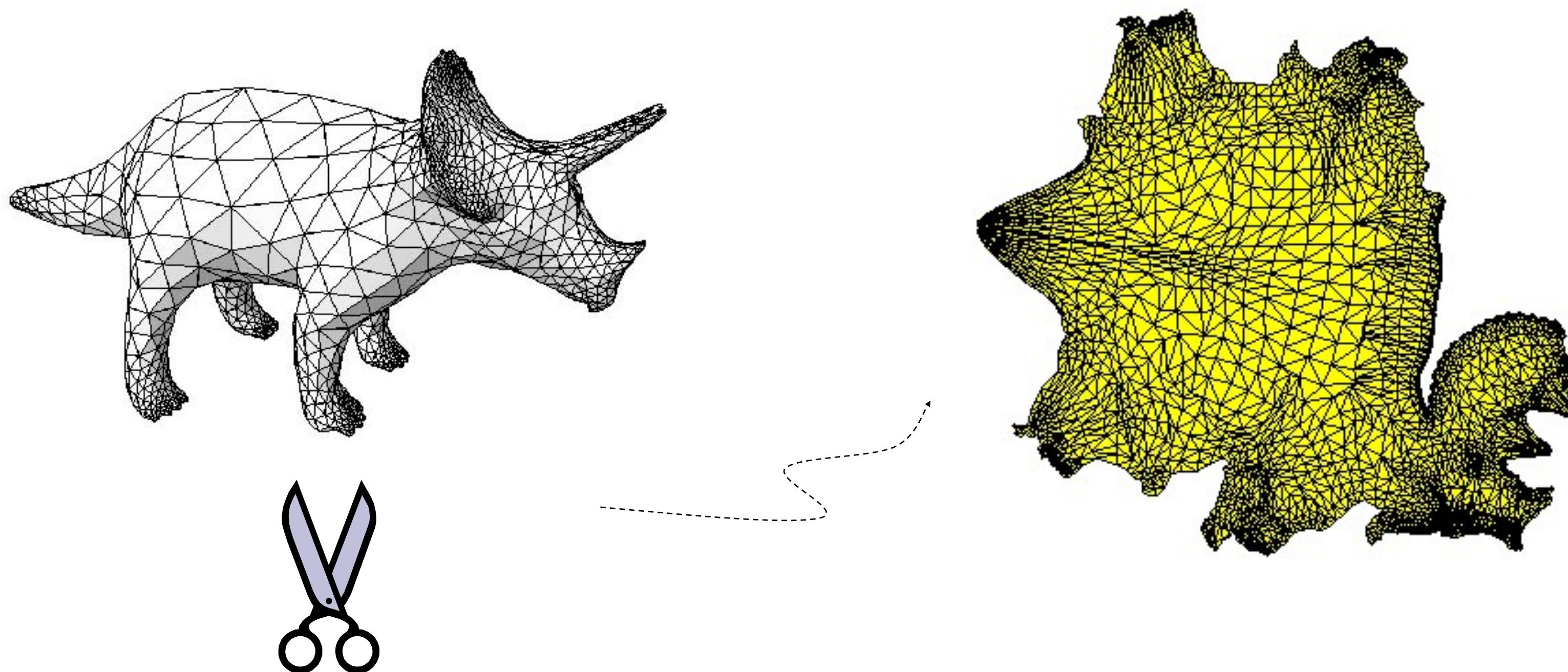
Bijectivity: Non-Disk Domains



Topological Cutting



Topological Cutting

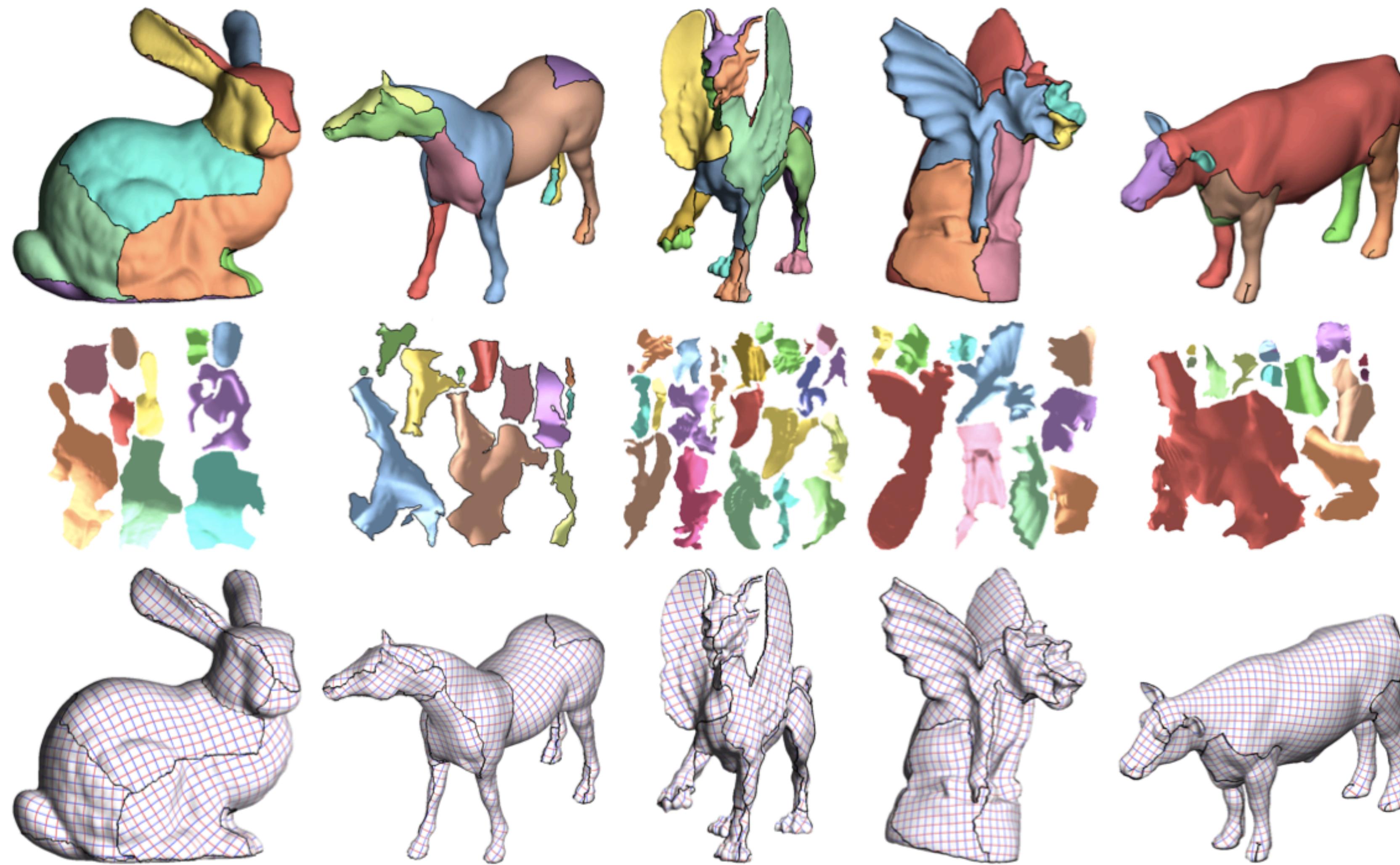


A. Sheffer, J. Hart:

Seamster: Inconspicuous Low-Distortion Texture Seam Layout, IEEE Vis 2002

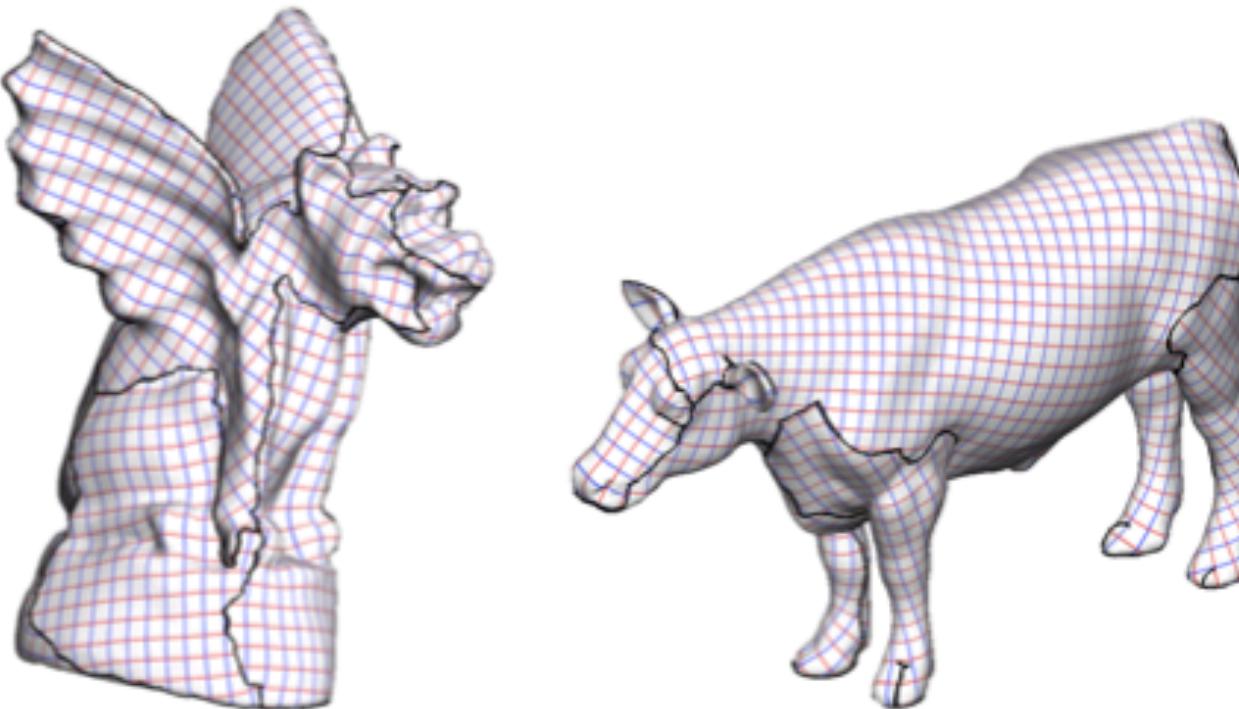
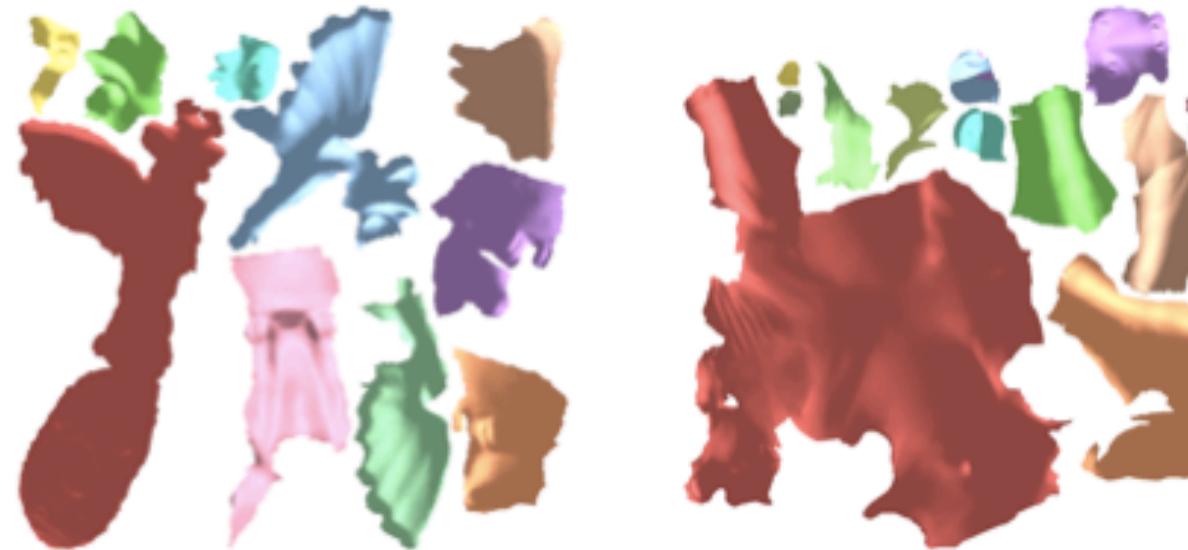
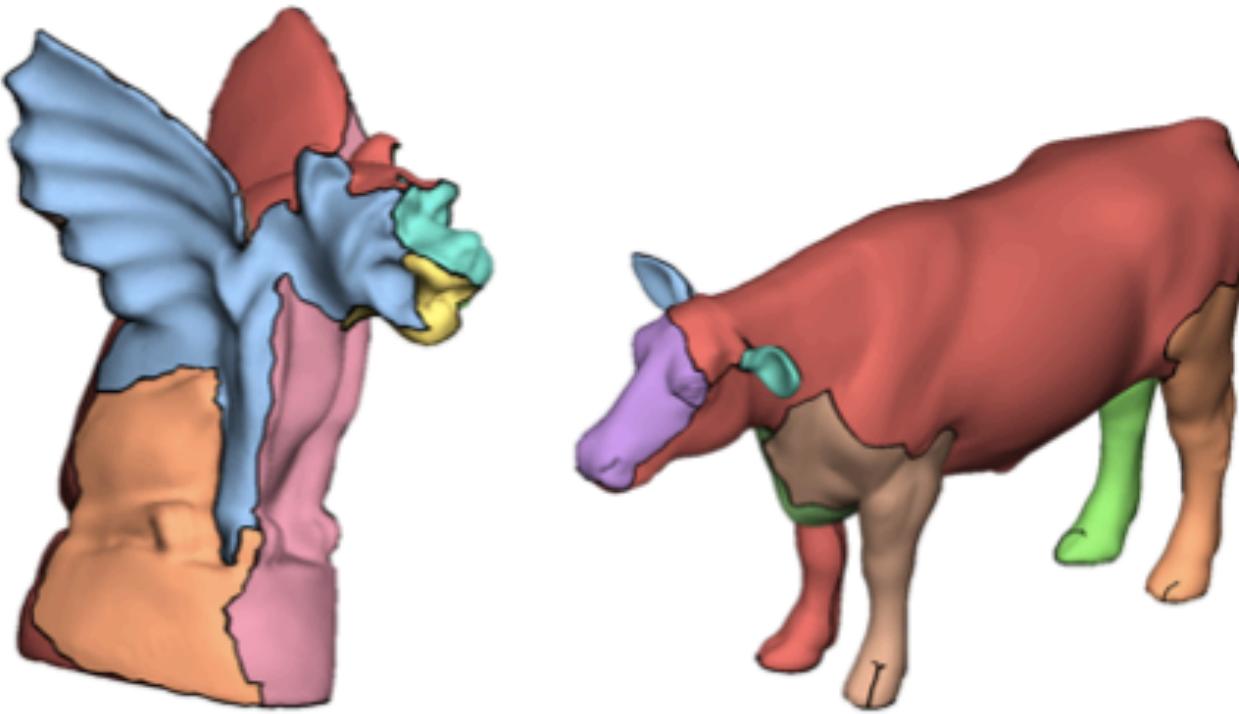
<http://www.cs.ubc.ca/~sheffa/papers/VIS02.pdf>

Segmentation



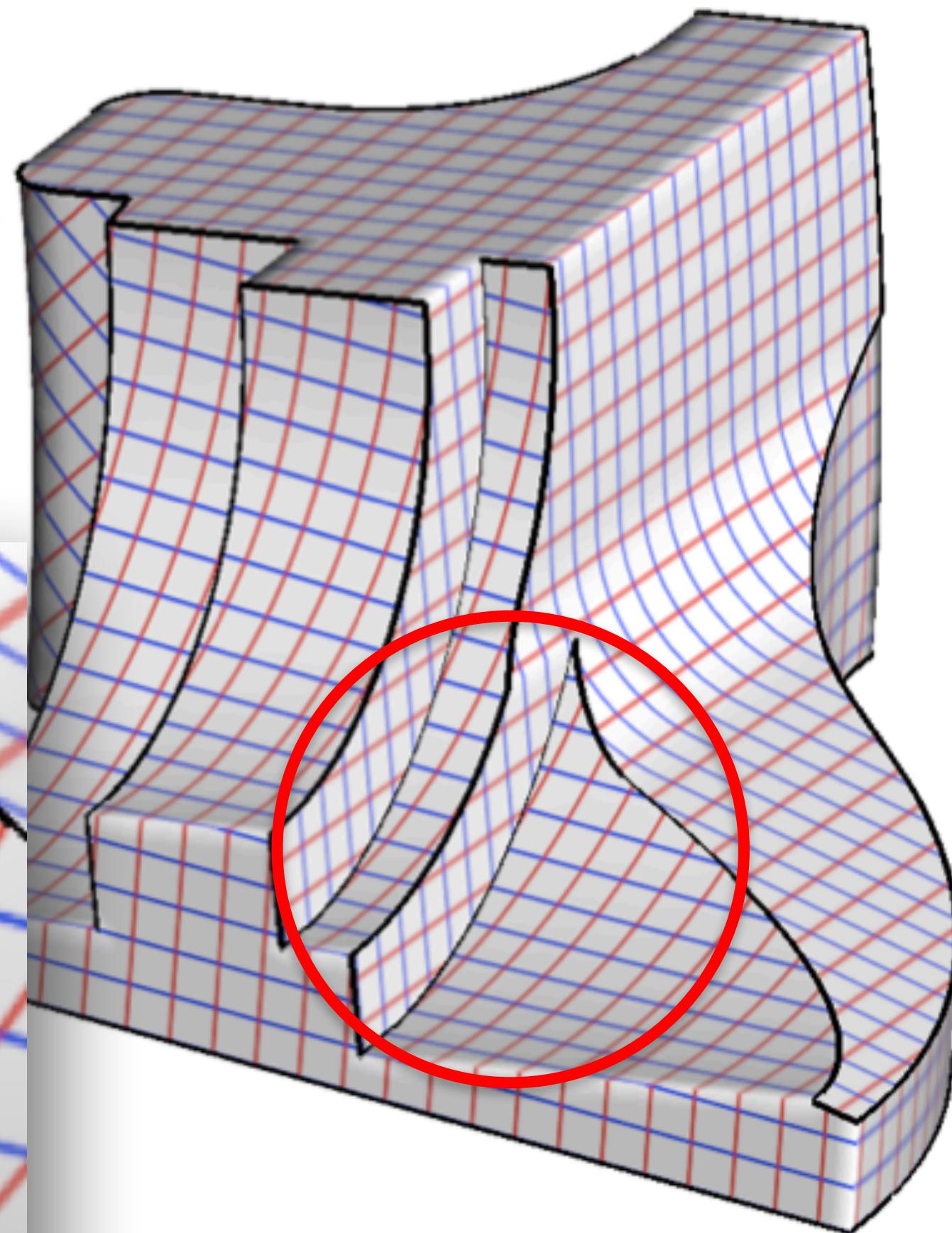
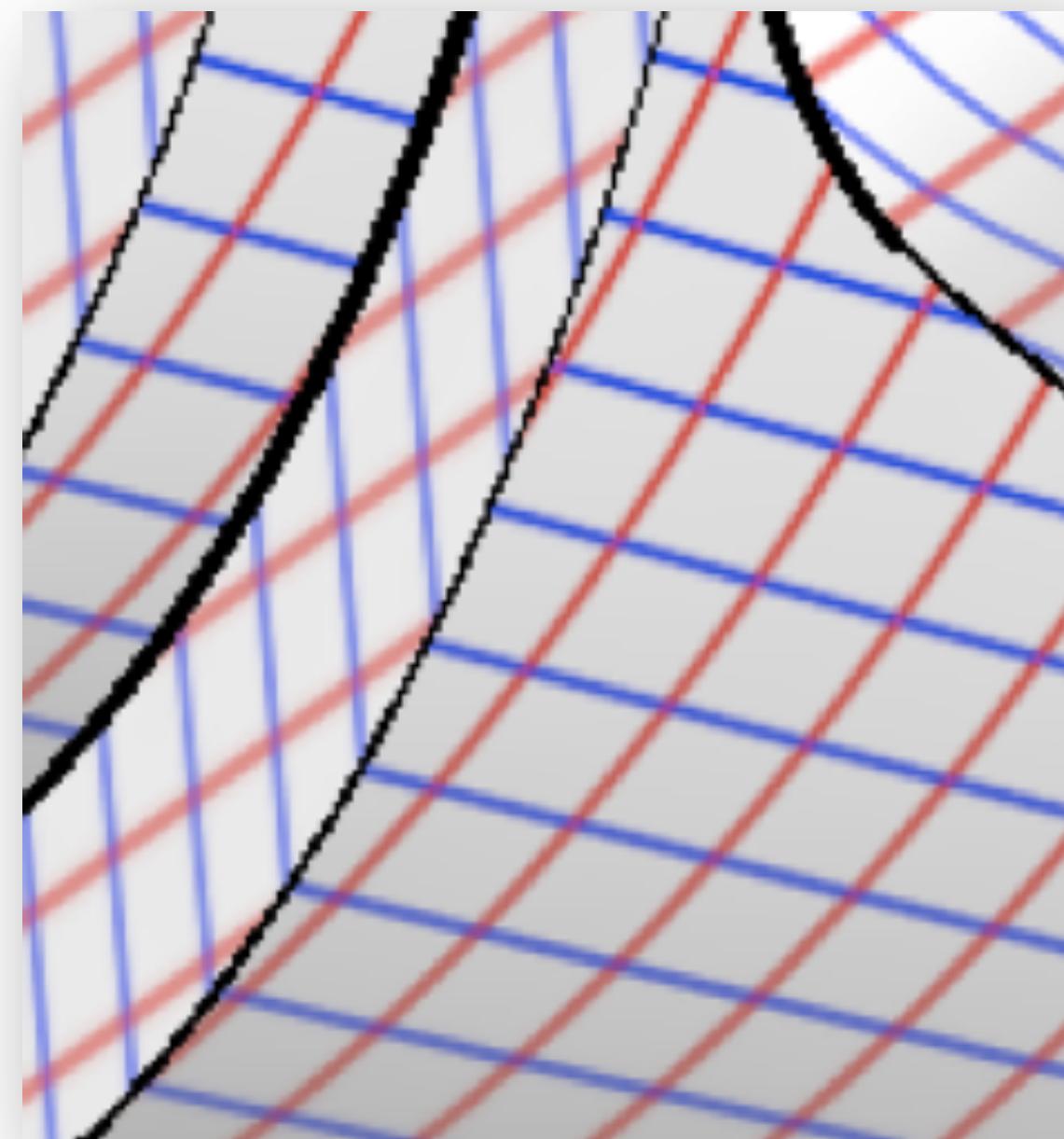
Segmentation

- D-Charts: Quasi-Developable Mesh Segmentation,
D. Julius, V. Kraevoy, A. Sheffer,
EUROGRAPHICS 2005
- Find patches that align to mesh features and are close to being developable surfaces



Good Cuts/Segmentations?

- Hide seams
- Small number/length of seams
- In quad meshing, we want the parameterization derivatives to be continuous across seams!

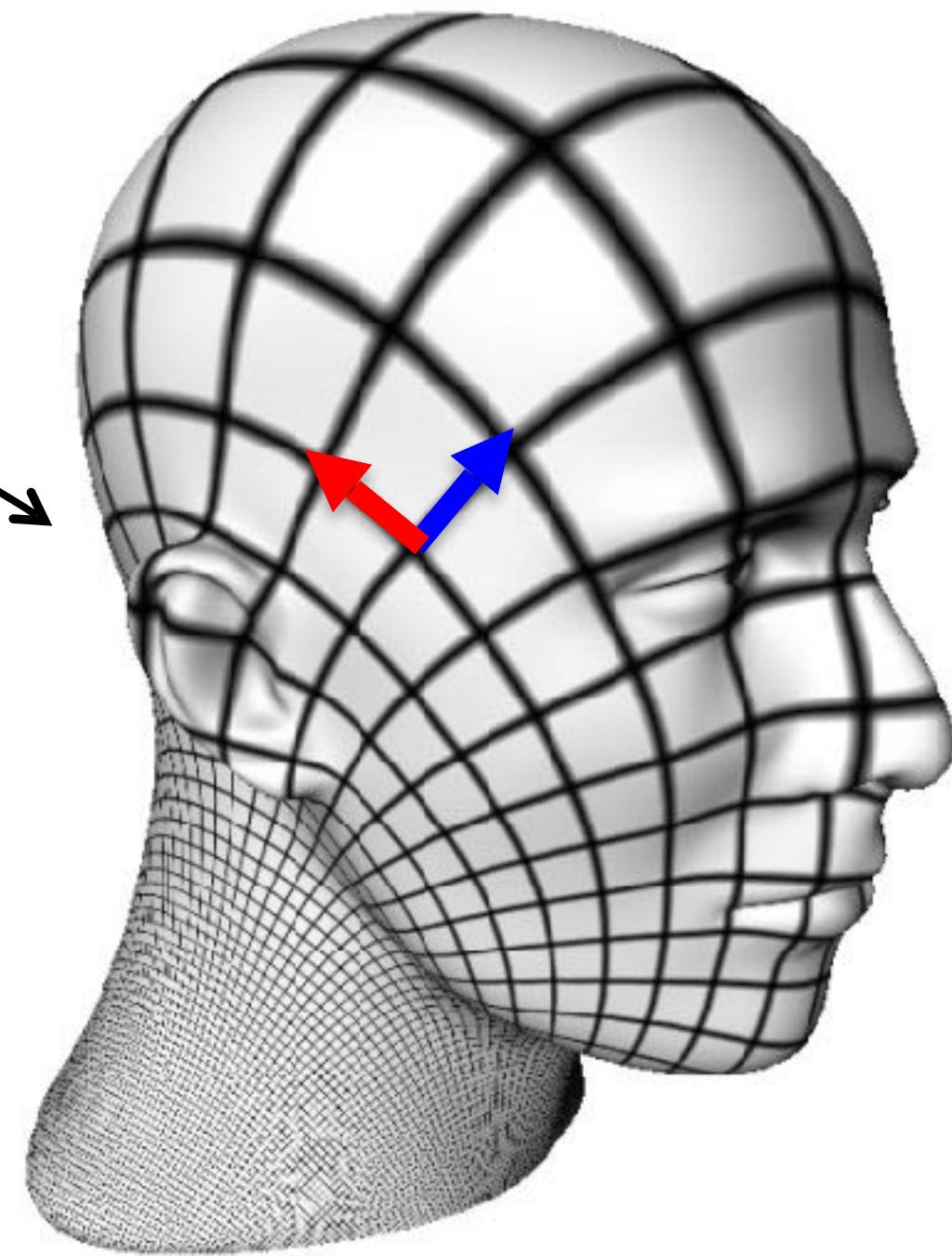
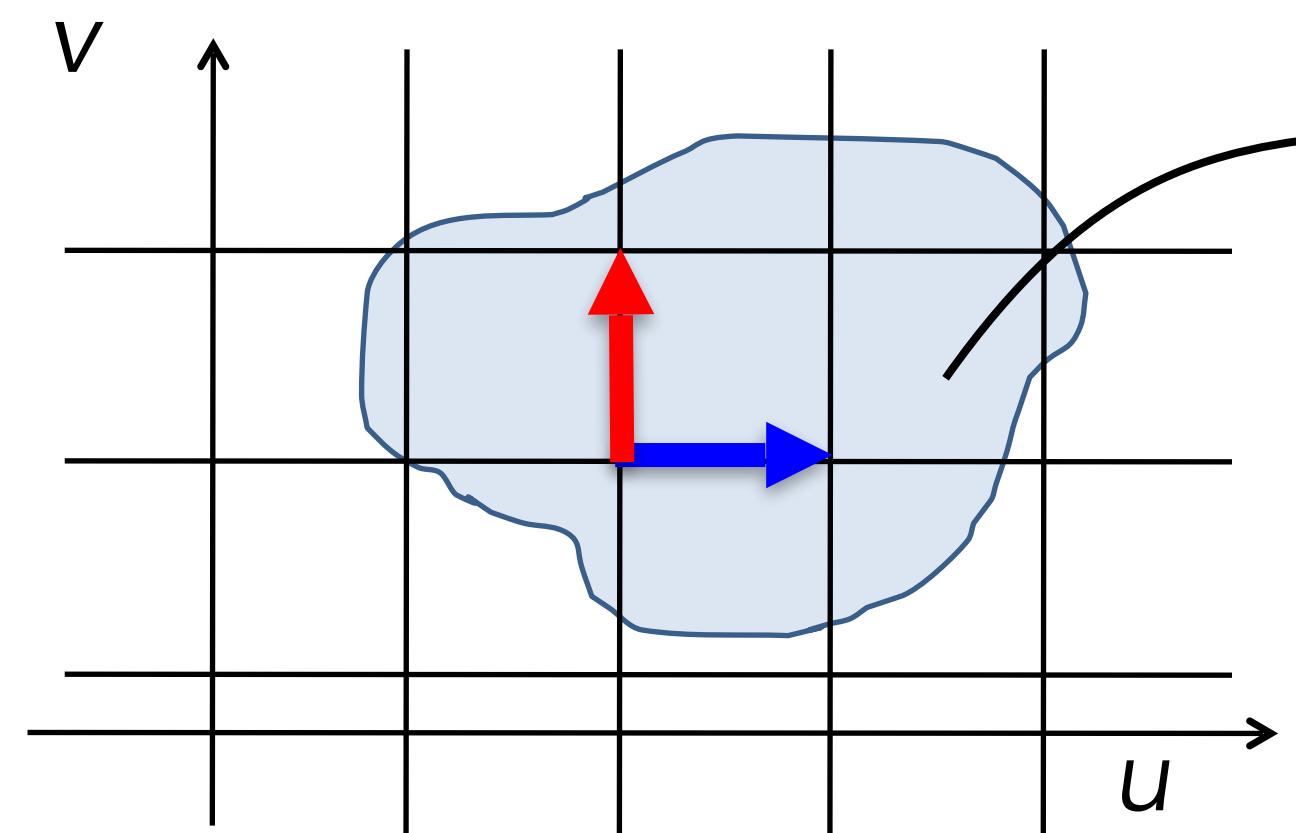


How to Measure Distortion?

Measures of Local Distortion

$$\mathbf{p}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2$$

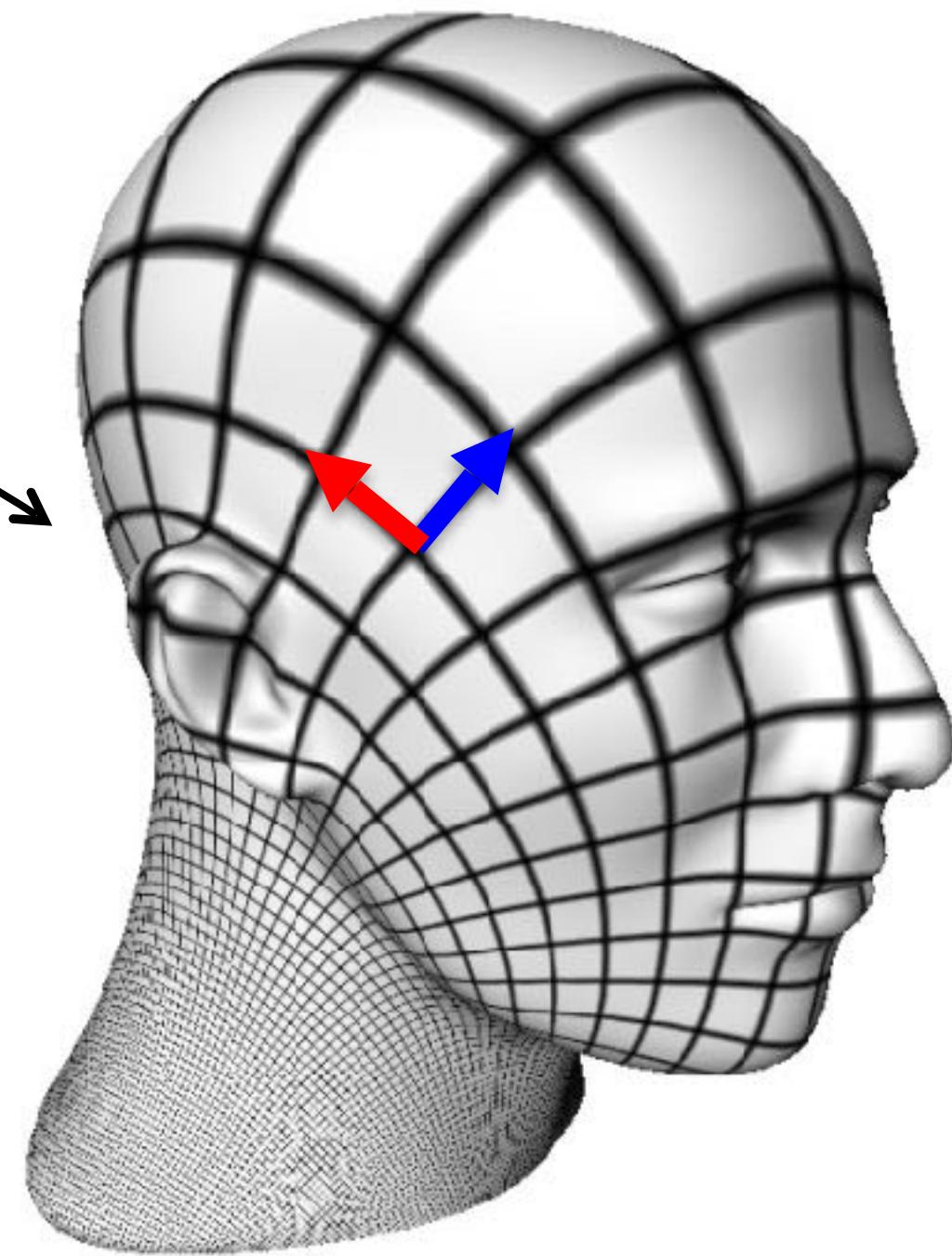
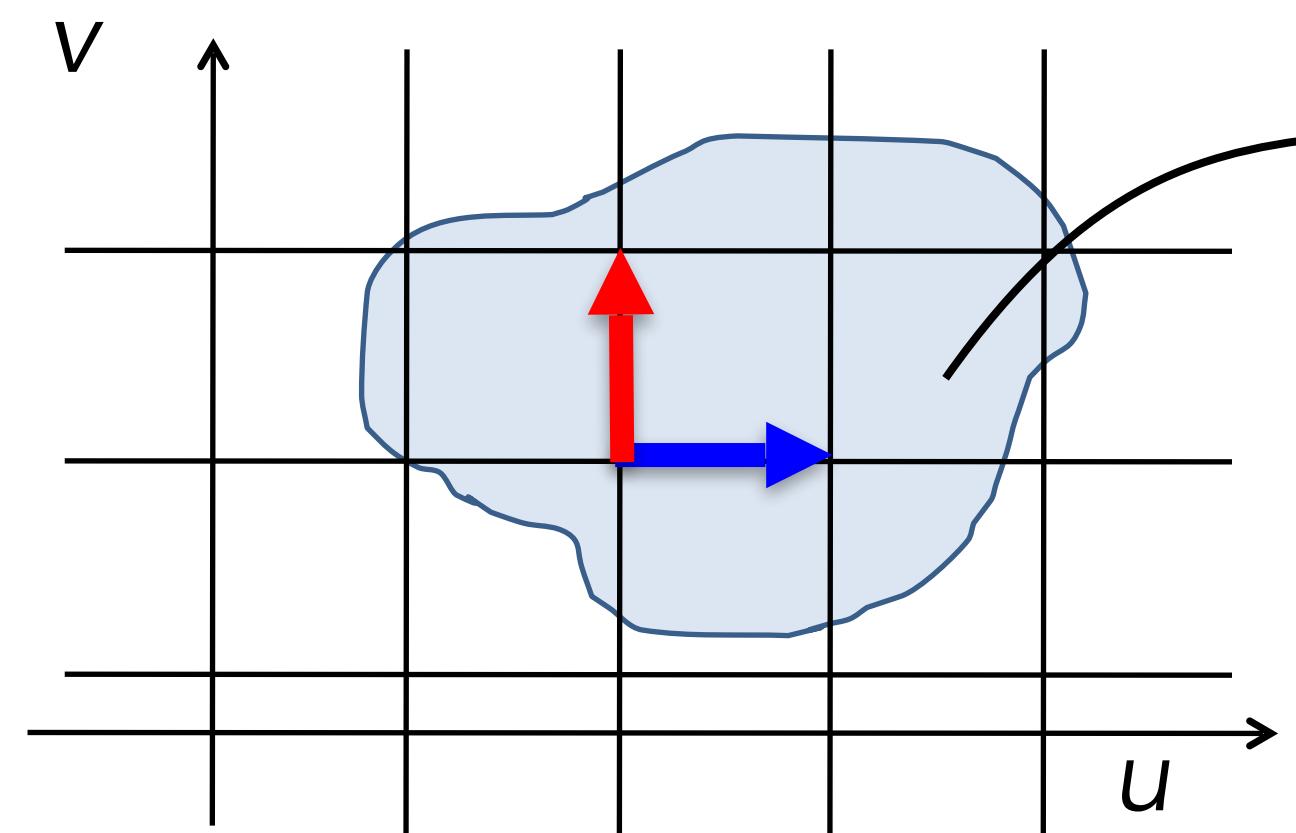
What happens
to tangent
vectors?



Measures of Local Distortion

$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u, v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u, v)}{\partial v}$$

What happens
to tangent
vectors?

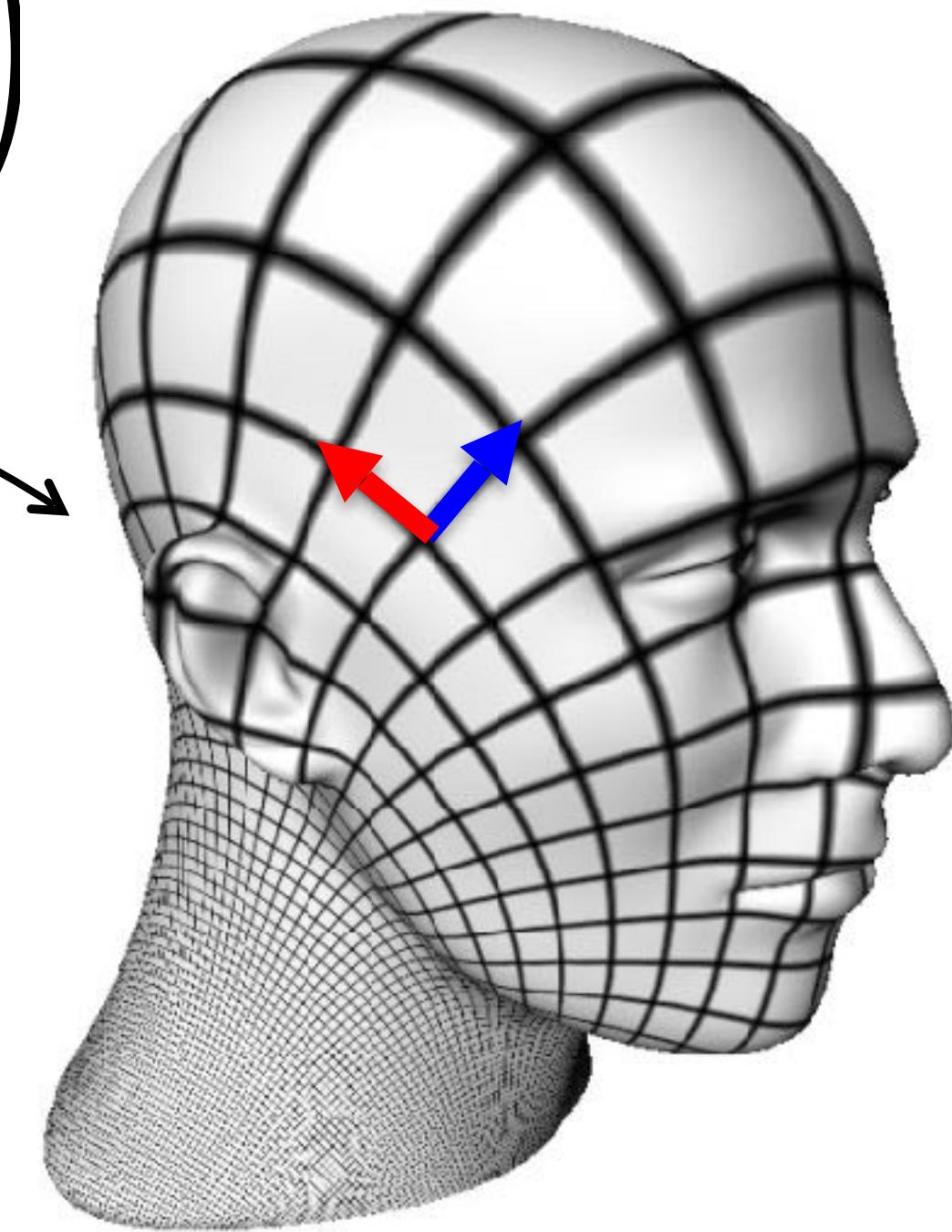
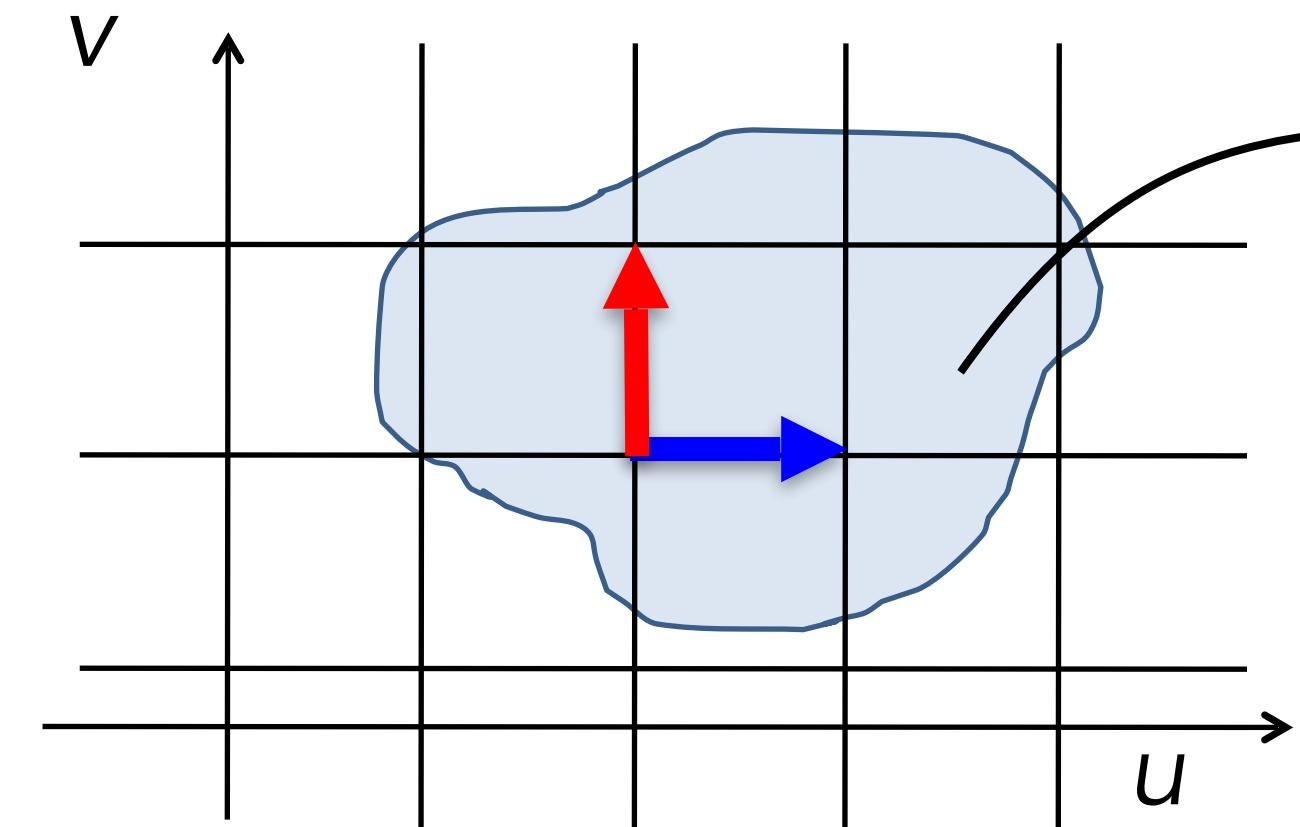


Measures of Local Distortion

- How do lengths and angles of tangents change?
 - First fundamental form!

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^T \mathbf{p}_u & \mathbf{p}_u^T \mathbf{p}_v \\ \mathbf{p}_u^T \mathbf{p}_v & \mathbf{p}_v^T \mathbf{p}_v \end{pmatrix}$$

What happens
to tangent
vectors?

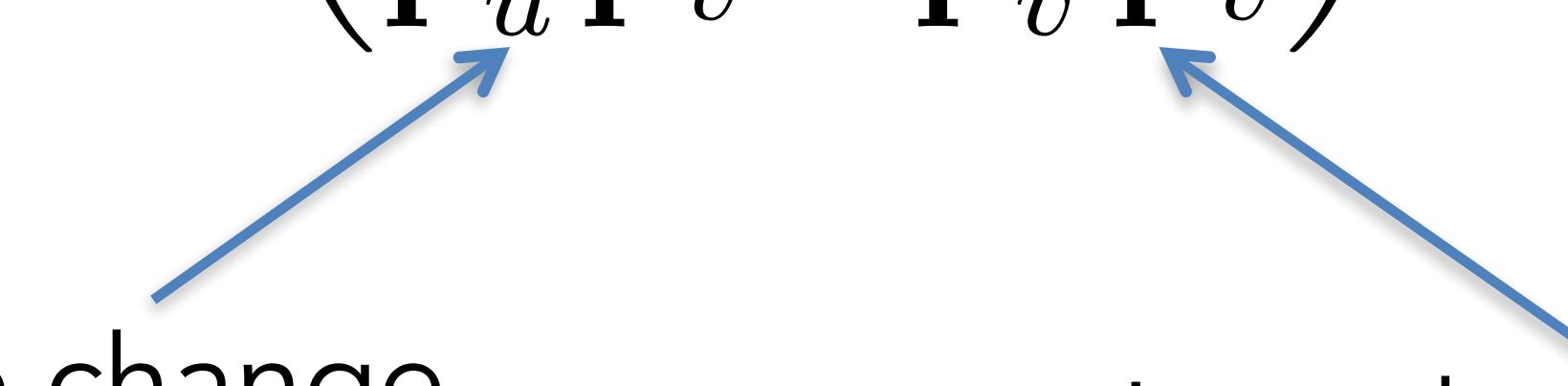


Measures of Local Distortion

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Angle change Length change



- Area distortion: area element:

$$dA = \sqrt{EG - F^2} dudv$$

Measures of Local Distortion

- How do lengths and angles of tangents change?
 - First fundamental form!

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^T \mathbf{p}_u & \mathbf{p}_u^T \mathbf{p}_v \\ \mathbf{p}_u^T \mathbf{p}_v & \mathbf{p}_v^T \mathbf{p}_v \end{pmatrix}$$

- The eigenvalues of \mathbf{I} tell us the maximal/minimal stretching of a tangent vector

Distortion on Triangle Meshes?

- Triangle in 3D is mapped to triangle in 2D
- Unique affine mapping

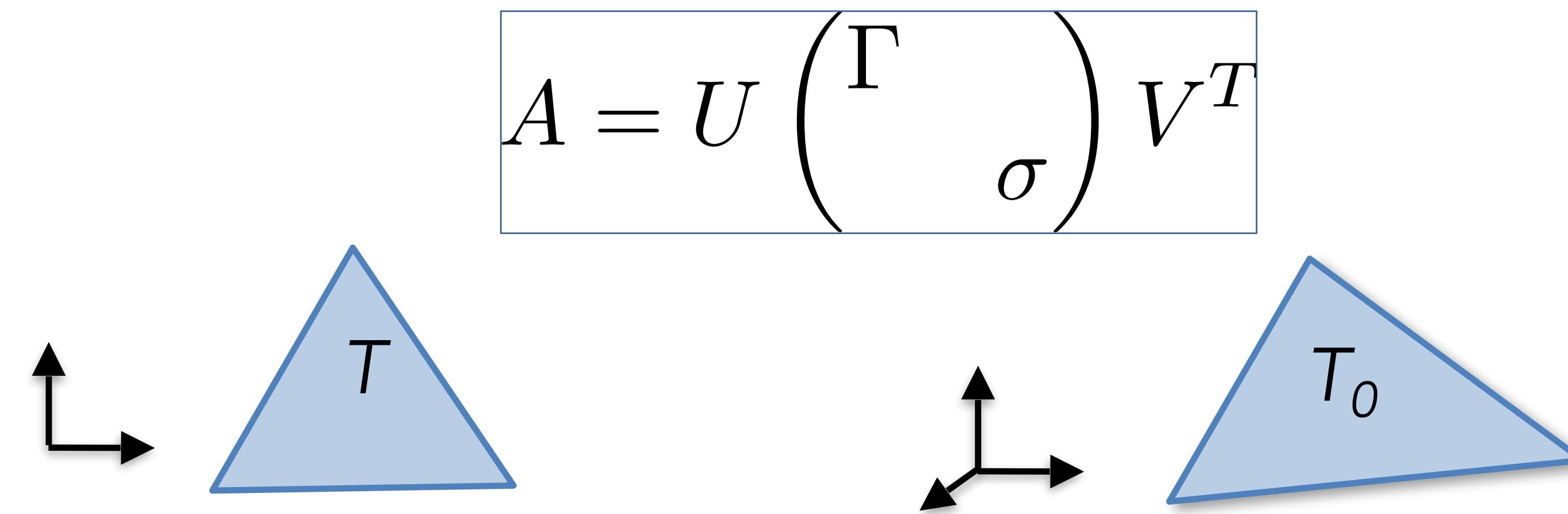


$$P : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad P(\mathbf{u}) = A\mathbf{u} + \mathbf{c}$$

$$[P_u \ P_v] = A \quad \xleftarrow{\text{Jacobian}}$$

Distortion on Triangle Meshes?

- SVD of the Jacobian reveals directions of extreme stretching



$$P : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad P(\mathbf{u}) = A\mathbf{u} + \mathbf{c}$$

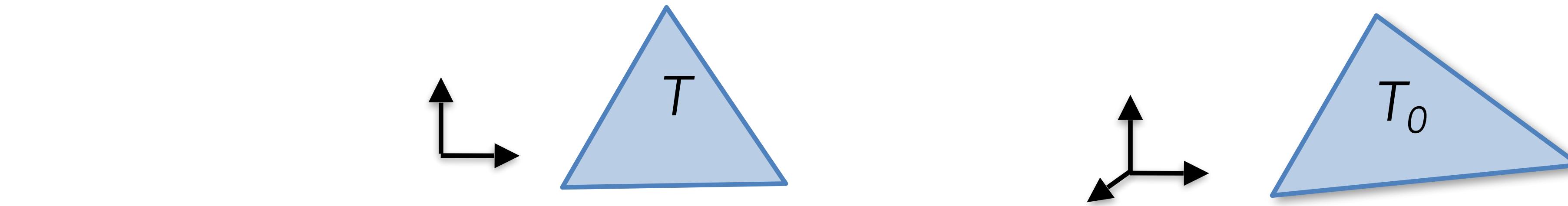
$$[P_u \ P_v] = A \xrightarrow{\text{SVD}} \mathbf{I} = A^T A$$

Distortion on Triangle Meshes?

- Possible distortion measures:

$$E(T) = \sqrt{\Gamma^2 + \sigma^2}$$

$$E(T) = \max \left\{ \Gamma, \frac{1}{\sigma} \right\}$$

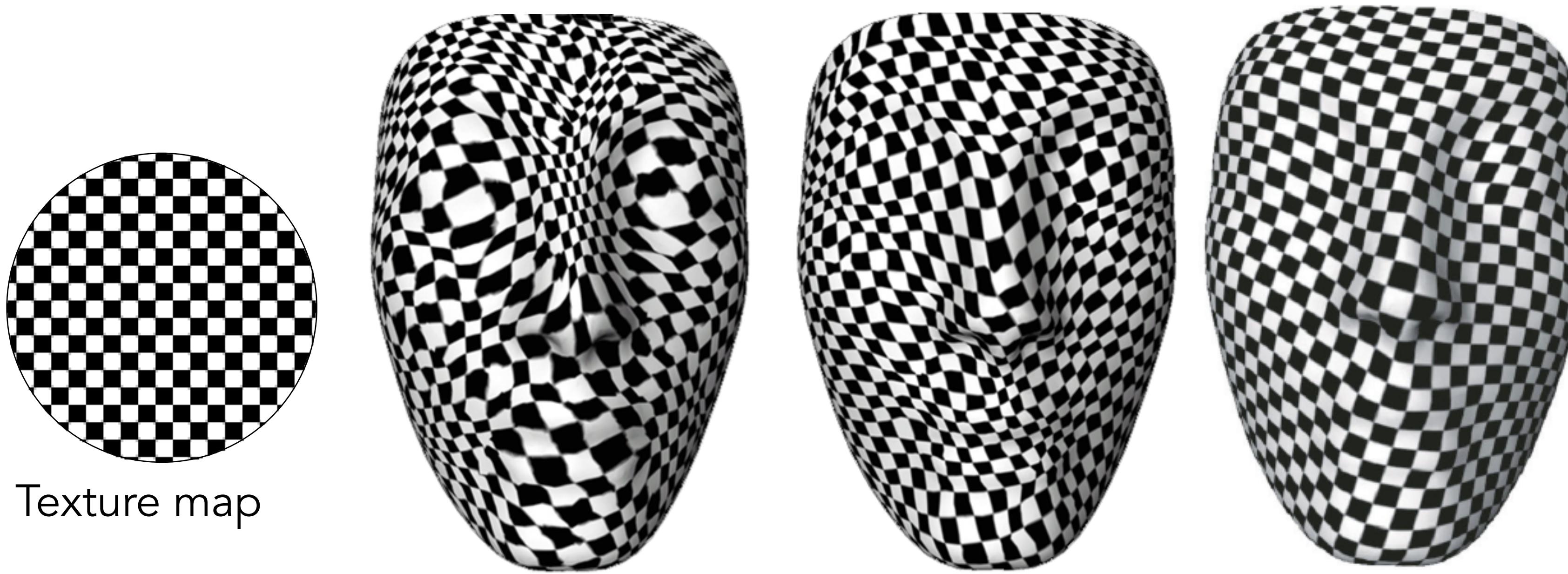


$$P : \mathbb{R}^2 \rightarrow \mathbb{R}^3, \quad P(\mathbf{u}) = A\mathbf{u} + \mathbf{c}$$

$$[P_u \ P_v] = A$$

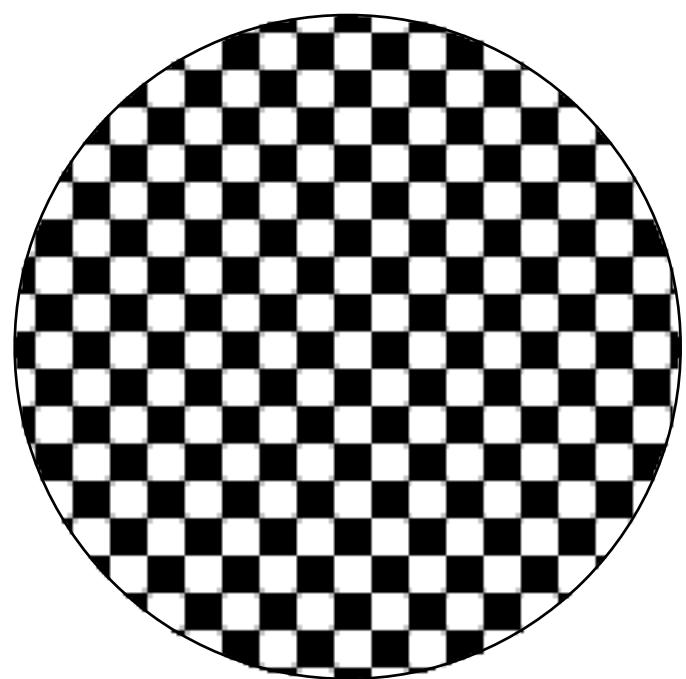
How To Compute Parameterizations?

Distortion Minimization

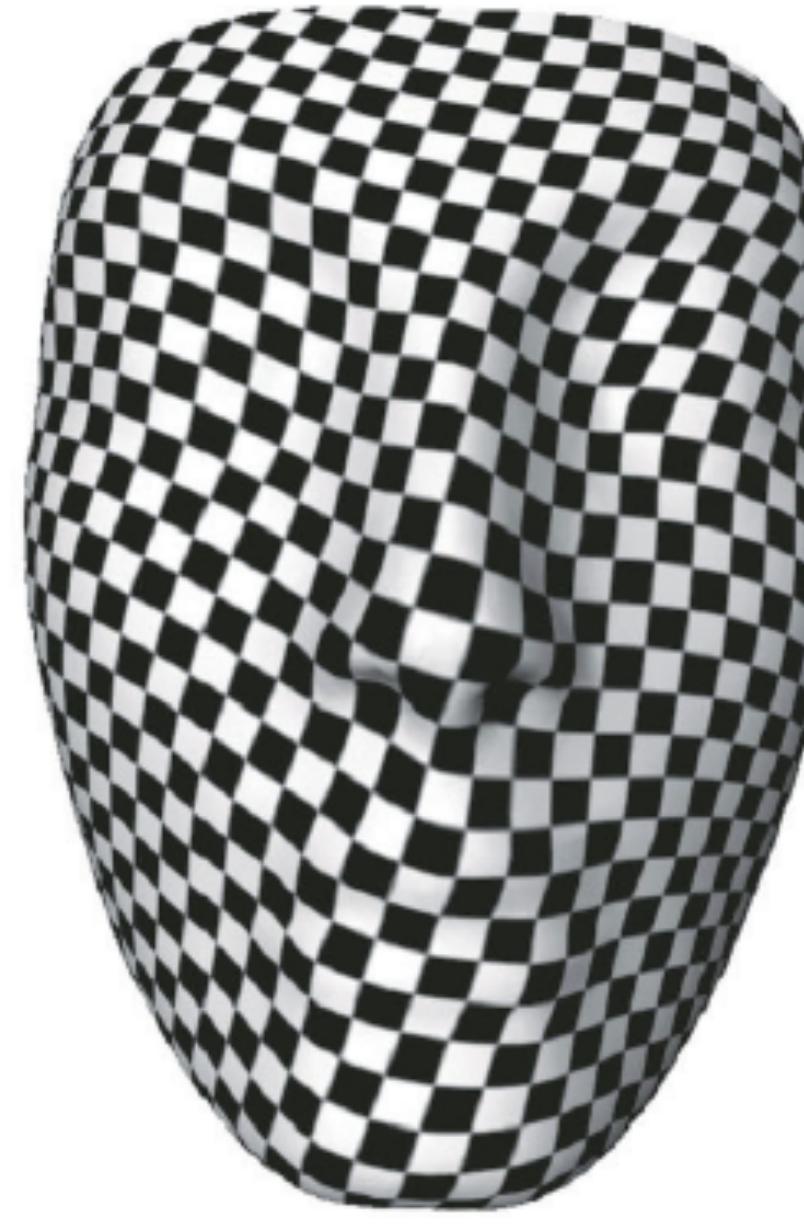
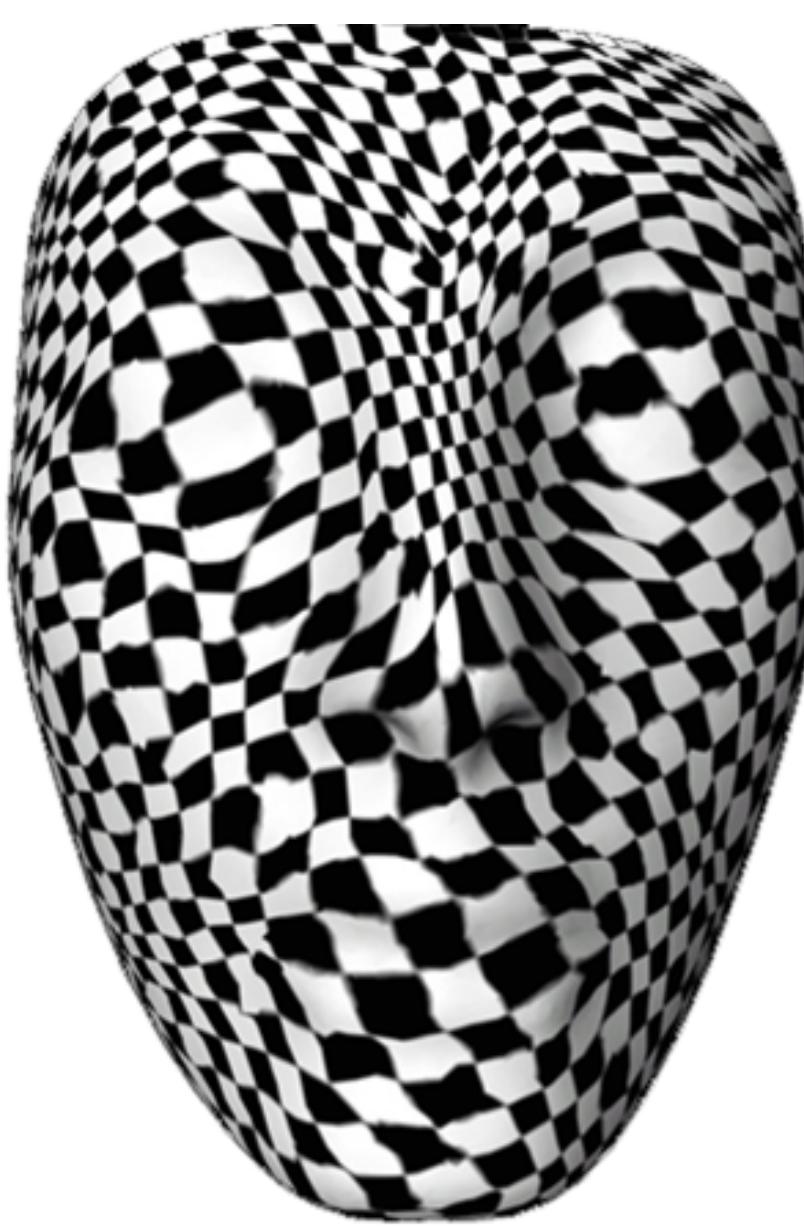


$$\operatorname{argmin}_{(u_1, v_1), \dots, (u_n, v_n)} \sum_T E(T)$$

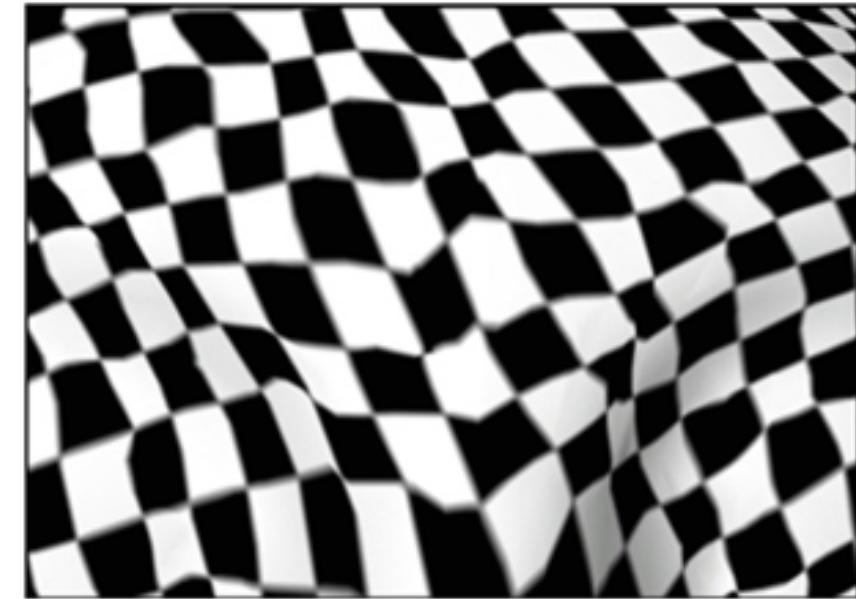
Distortion Minimization



Texture map



Kent et al '92



Floater 97



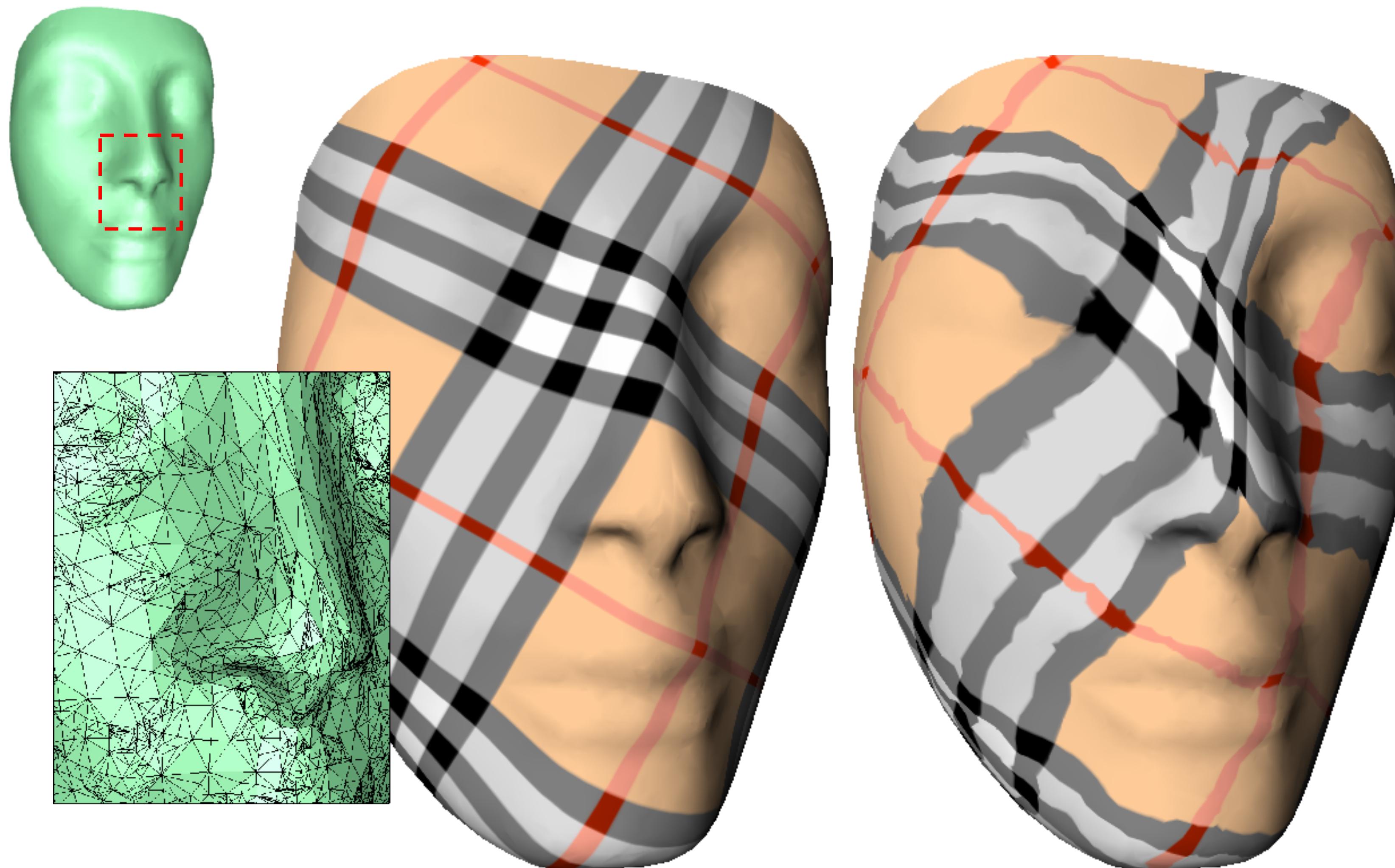
Sander et al '01

Literature for previous slide

- Kent et al. '92, "Shape transformation for polyhedral objects", Computer Graphics Vol. 26(2), 47-54
<http://ijcc.org/ojs/index.php/ijcc/article/viewFile/87/78>
- Floater '97, "Parametrization and smooth approximation of surface triangulations", Computer Aided Geometric Design Vol. 14 (1997), 231-250.
<http://www.mn.uio.no/math/english/people/aca/michaelf/papers/param.pdf>
- Sander et al. '01, "Texture mapping progressing meshes", SIGGRAPH 2001.
<http://research.microsoft.com/en-us/um/people/hoppe/proj/tmpm/>



Mesh Dependence

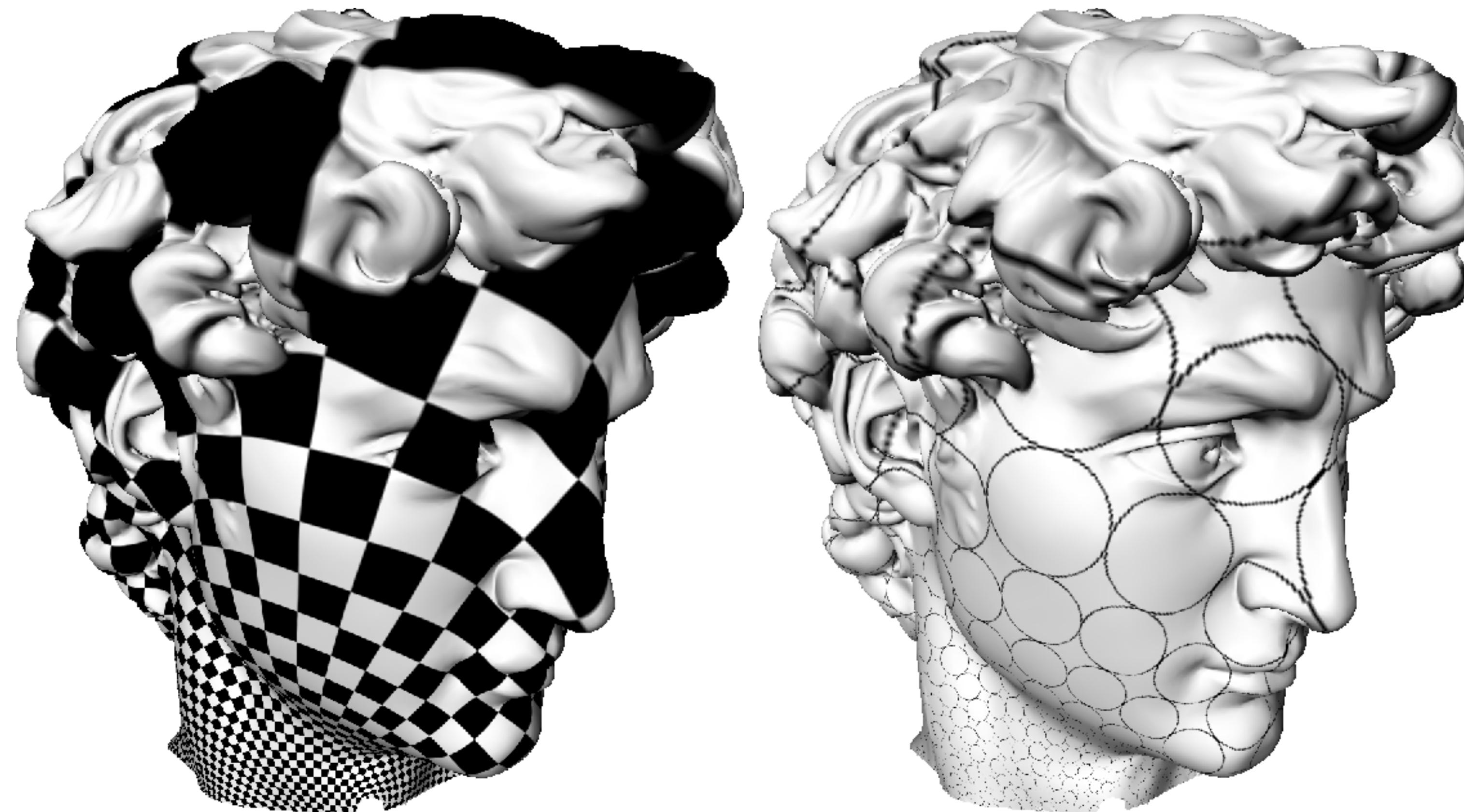


Alg. 1 – mesh-independent

Alg. 2 – ... less mesh-independent

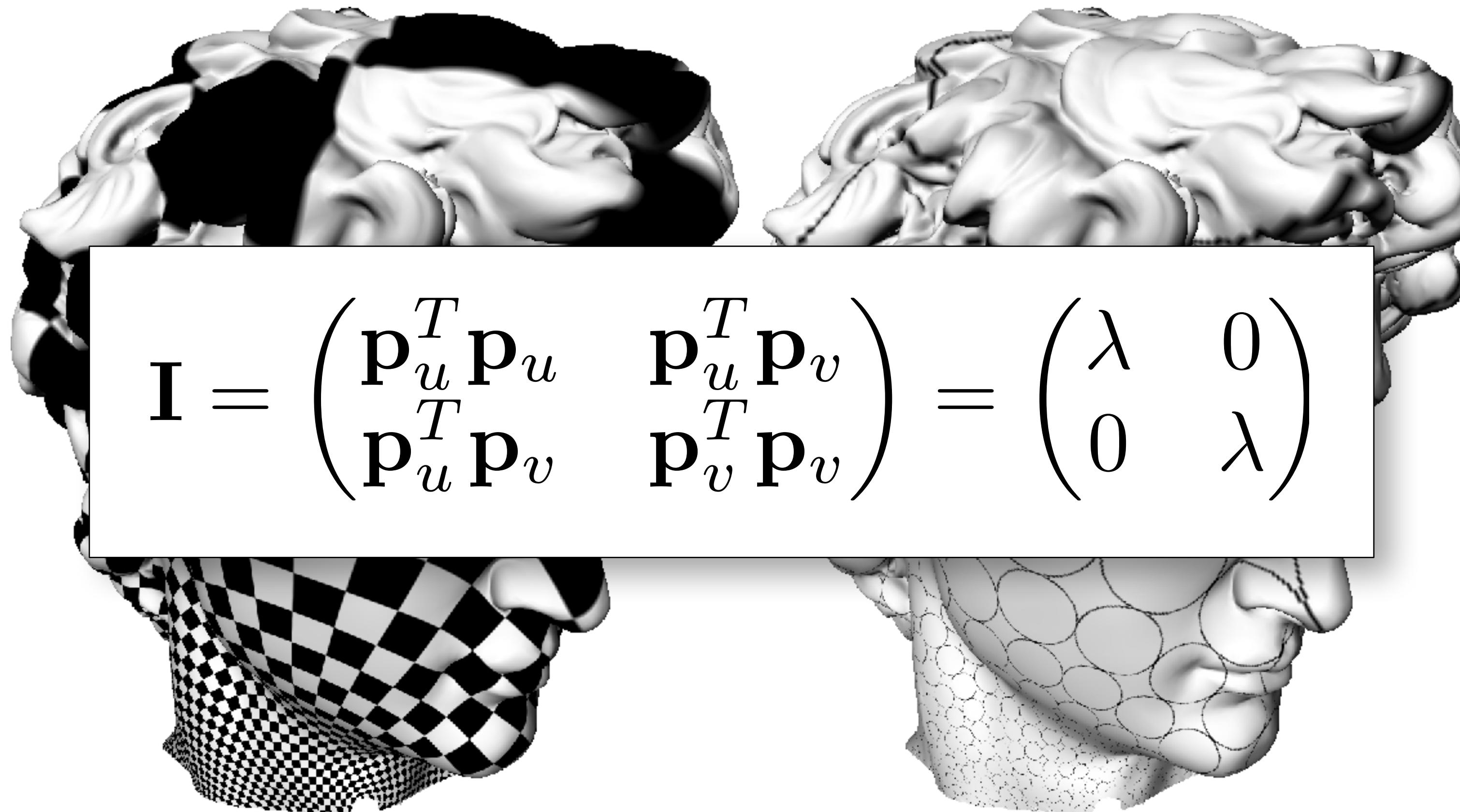
Conformal Parameterization

- Angle preservation; circles are mapped to circles



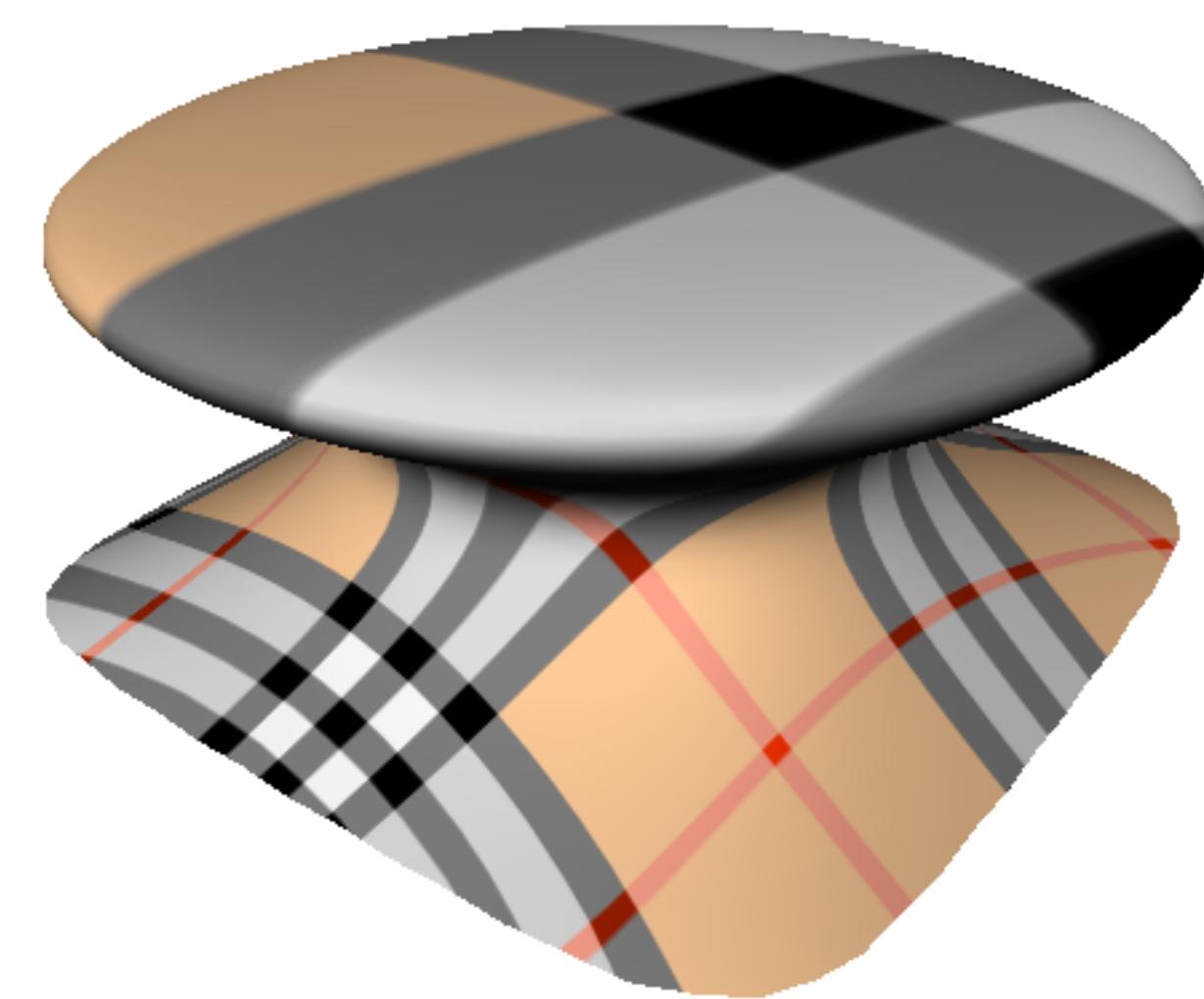
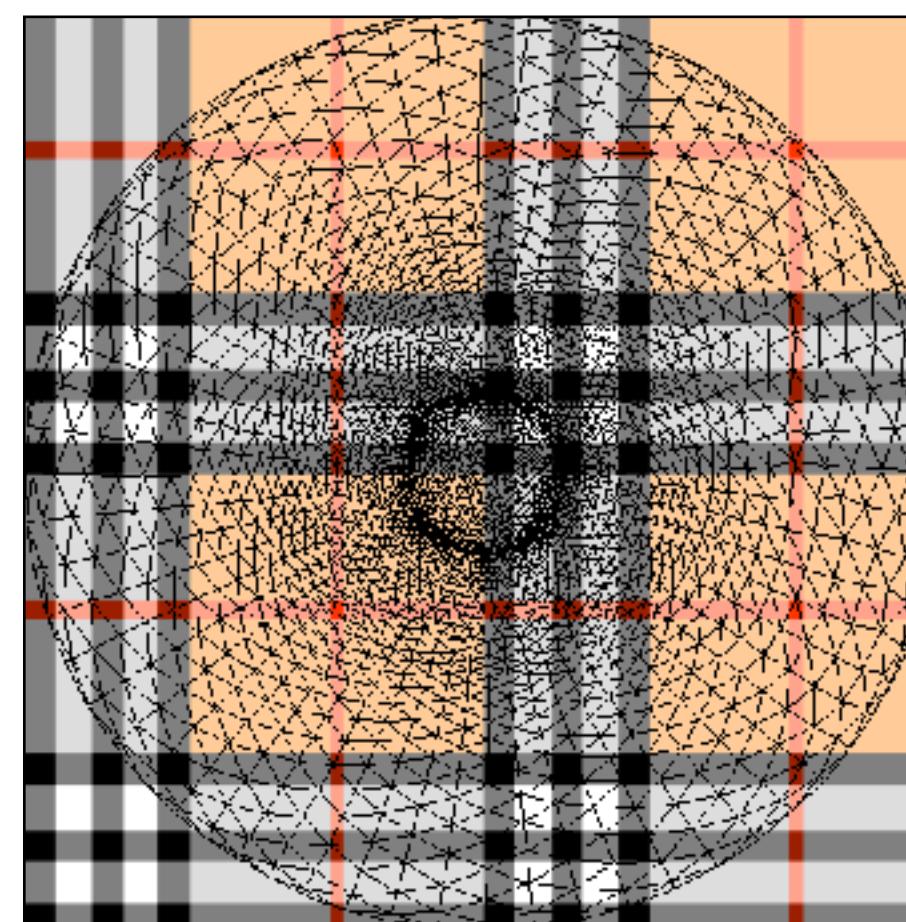
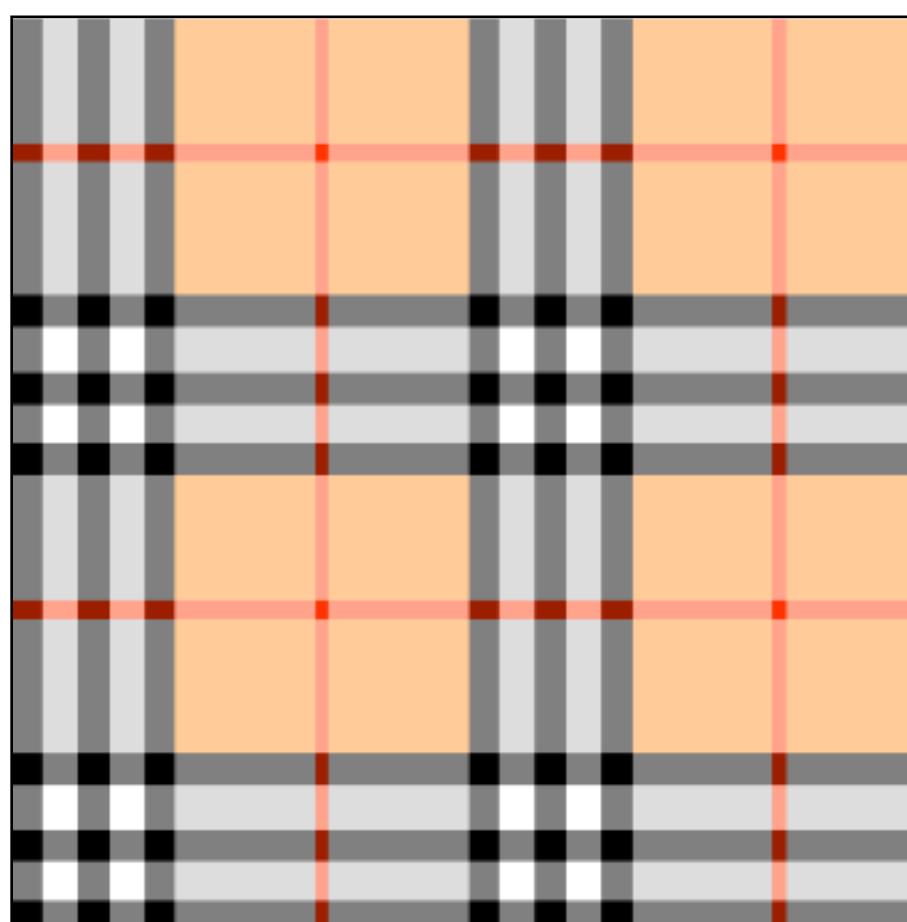
Conformal Parameterization

- Angle preservation; circles are mapped to circles



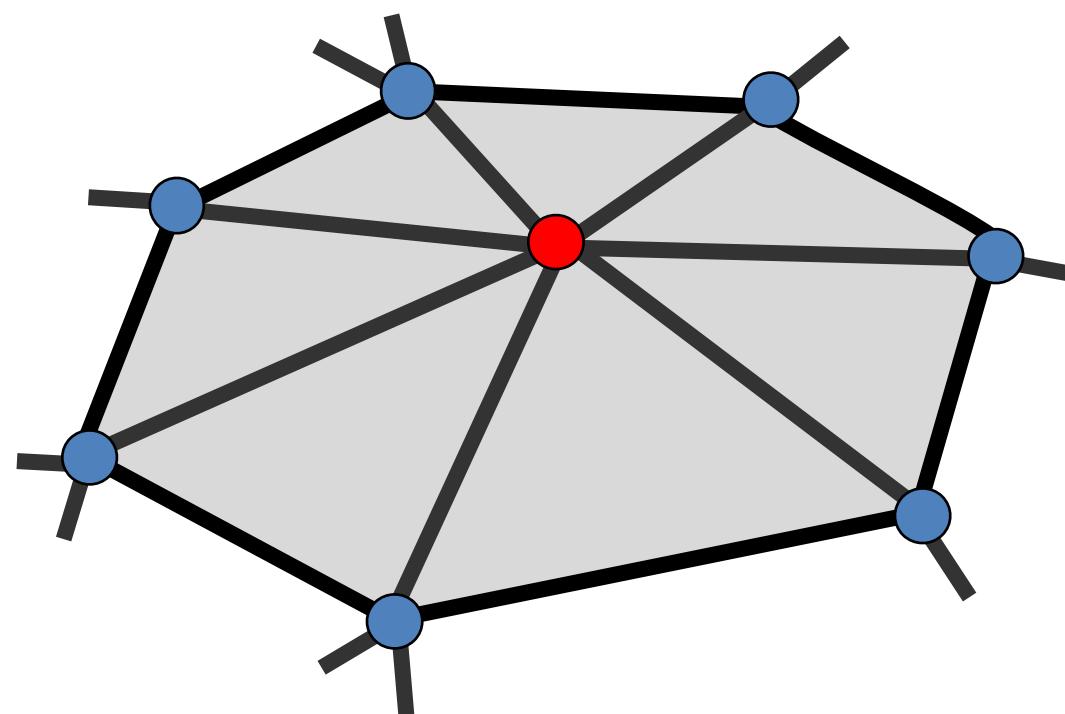
Area Distortion vs. Angle Distortion

- Is it possible to preserve both angles and areas at the same time?



Harmonic Mapping – Idea

- Want to flatten the mesh \rightarrow no curvature \rightarrow Laplace operator gives zero.



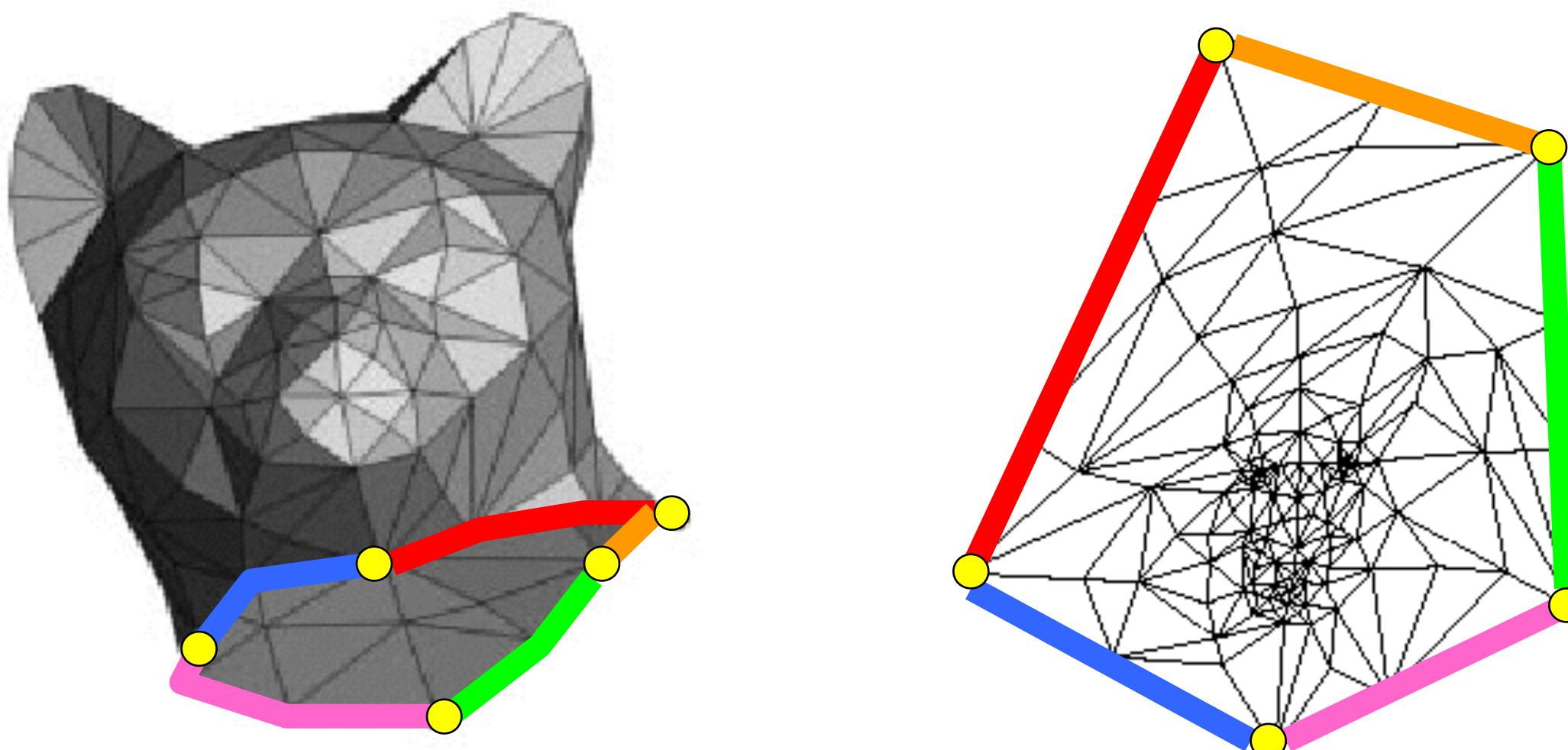
$\mathbf{u} = (u, v)$ domain

$$\Delta(\mathbf{u}) = 0$$

need boundary constraints
to prevent trivial solution;

Convex Mapping (Tutte, Floater)

- Boundary vertices are fixed
- Convex weights



$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ – inner vertices

$\mathbf{v}_{n+1}, \dots, \mathbf{v}_N$ – boundary vertices

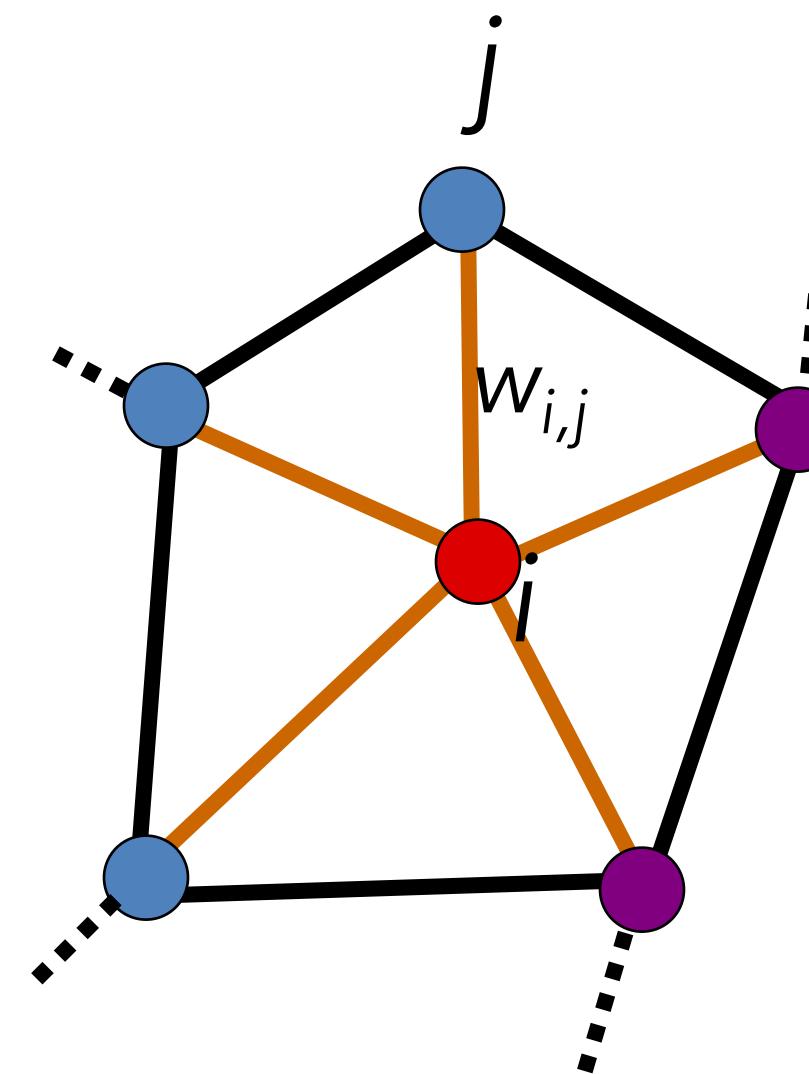
Convex Mapping (Tutte, Floater)

- Boundary vertices are fixed
- Convex weights

$$\Delta(\mathbf{u}_i) = 0, \quad i = 1, \dots, n$$

$$L(\mathbf{u}_i) = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij} (\mathbf{u}_j - \mathbf{u}_i) = 0, \quad i = 1, \dots, n$$

$$w_{ij} > 0$$



Convex Mapping (Tutte, Floater)

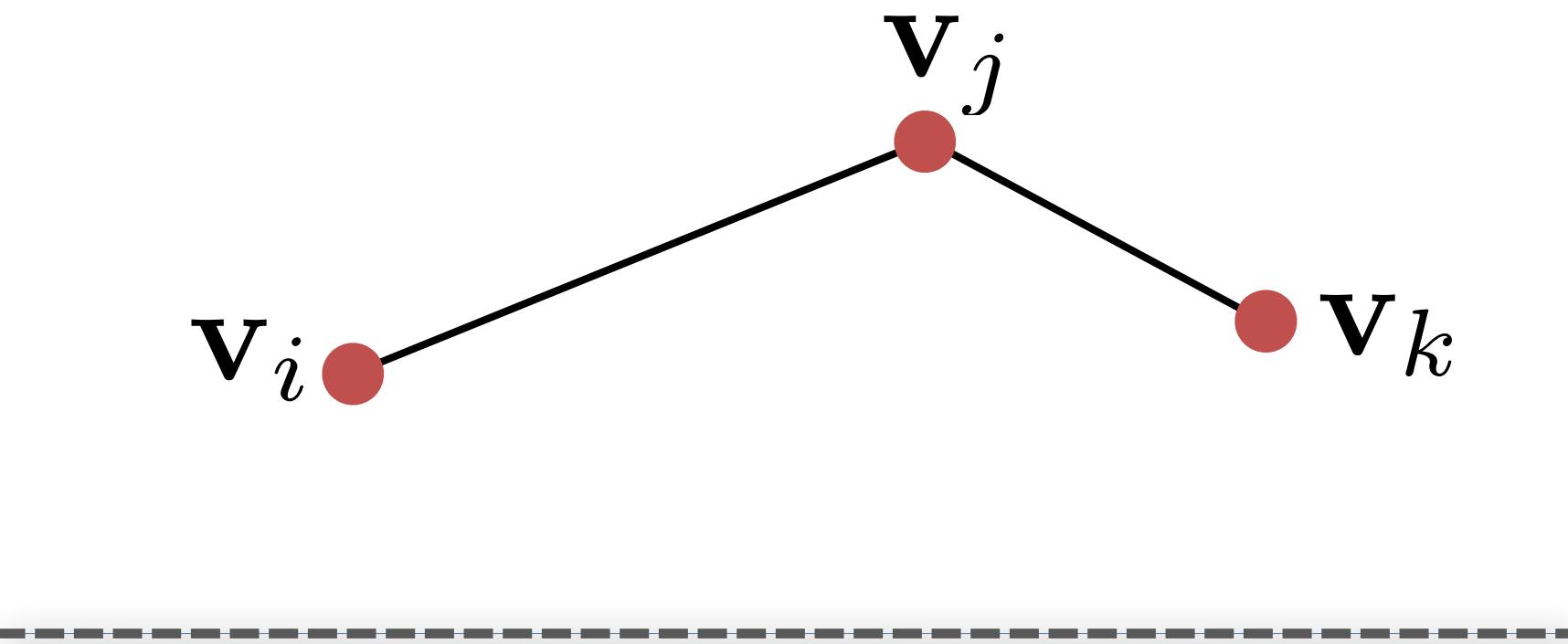
- Solve the linear system

$$L\mathbf{u} = 0 \quad \mathbf{u} \in \mathbb{R}^{n \times 2}$$

- The values of the boundary vertices are known and thus substituted (transfer to right-hand side)

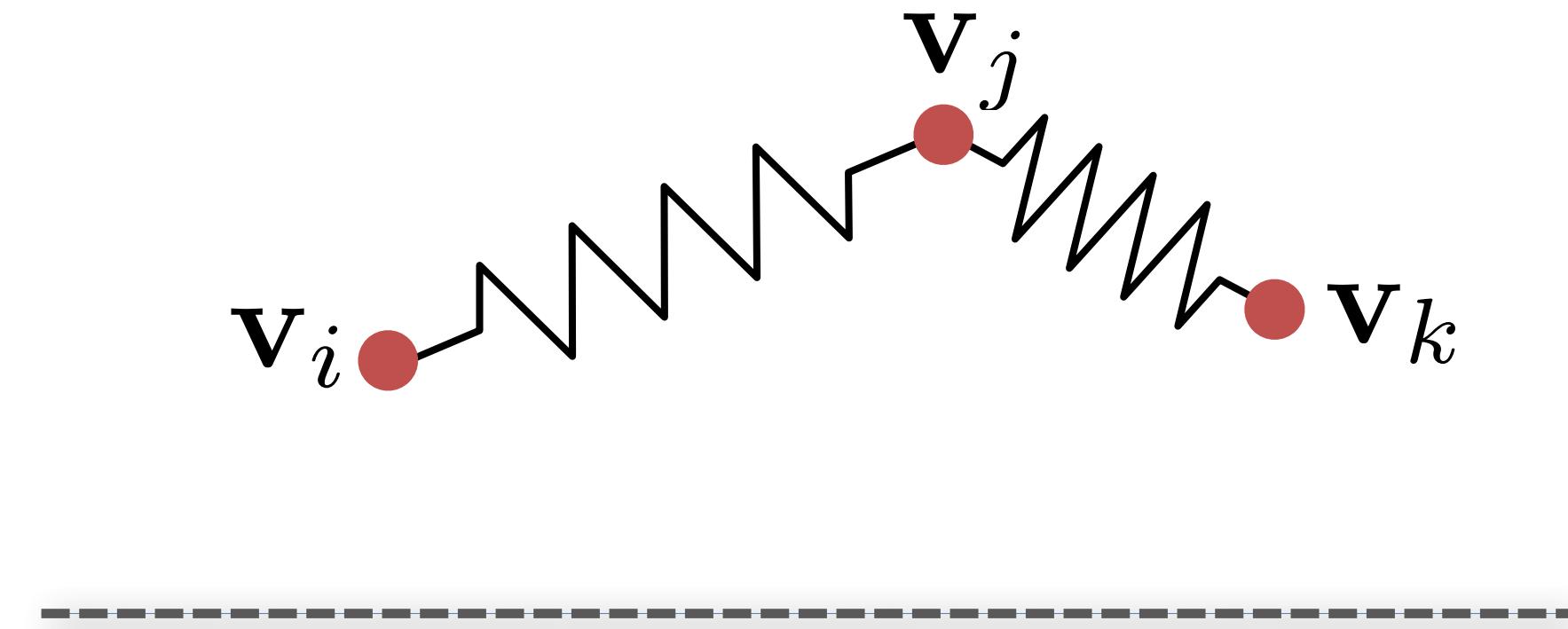
Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



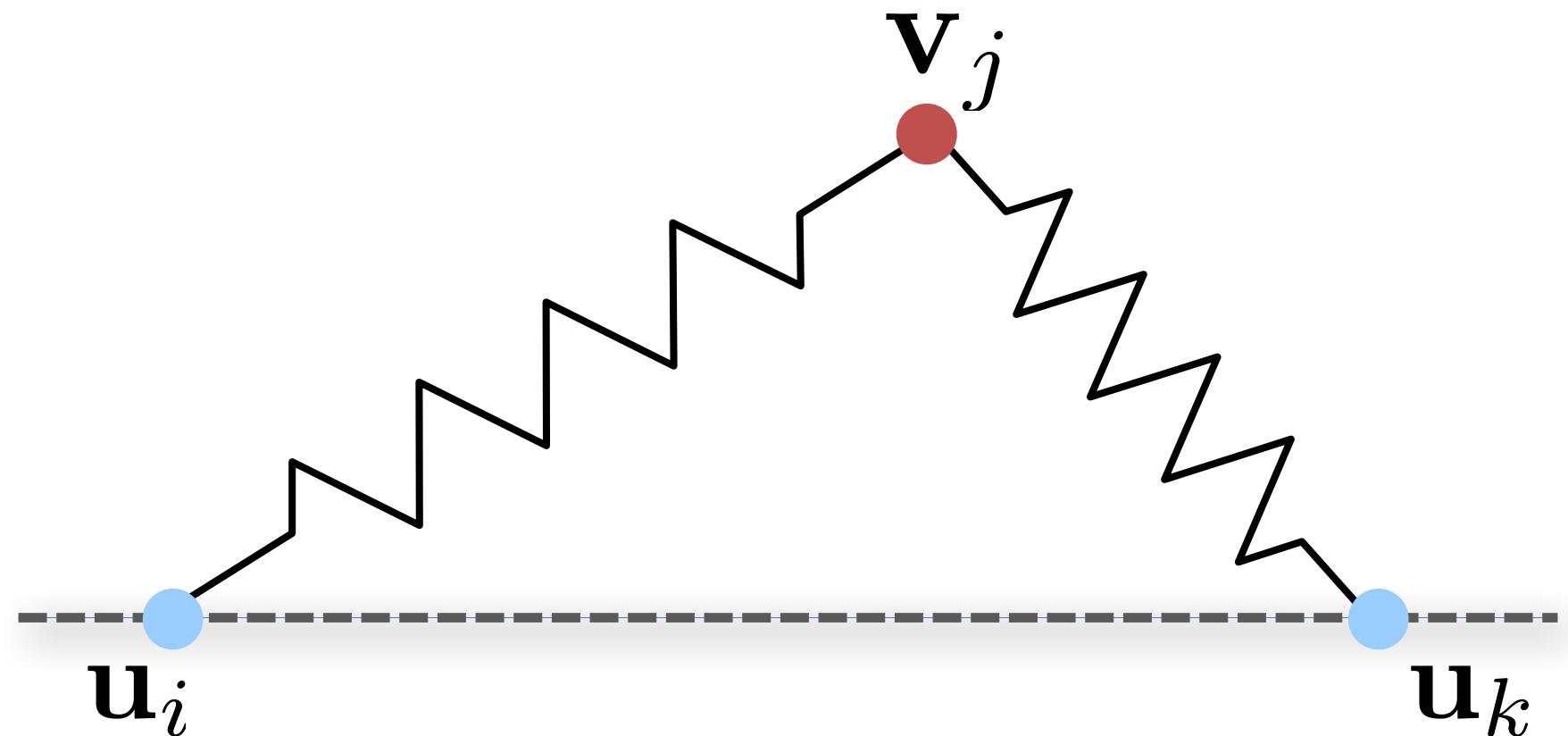
Harmonic Mapping

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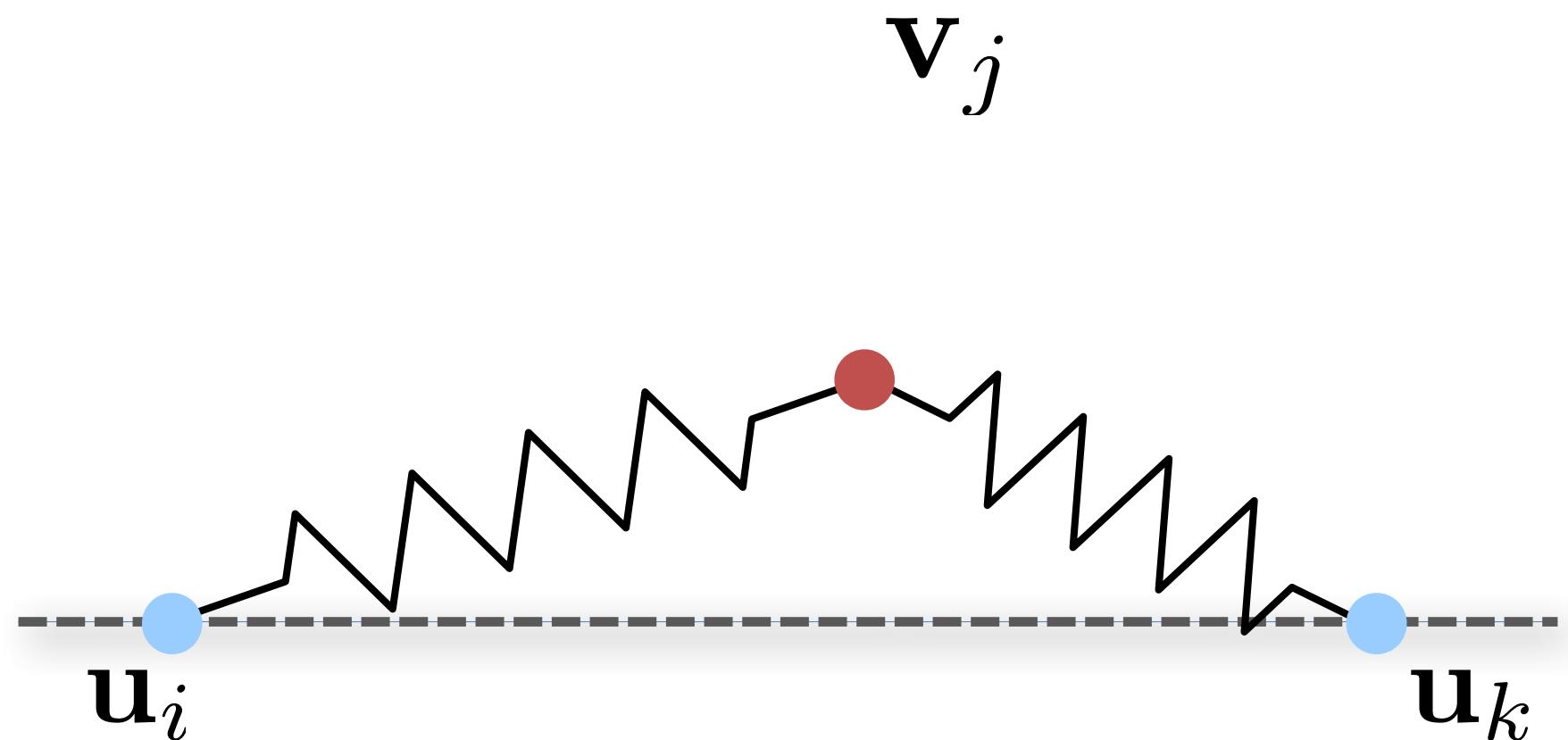
Harmonic Mapping

- Inner mesh edges as springs
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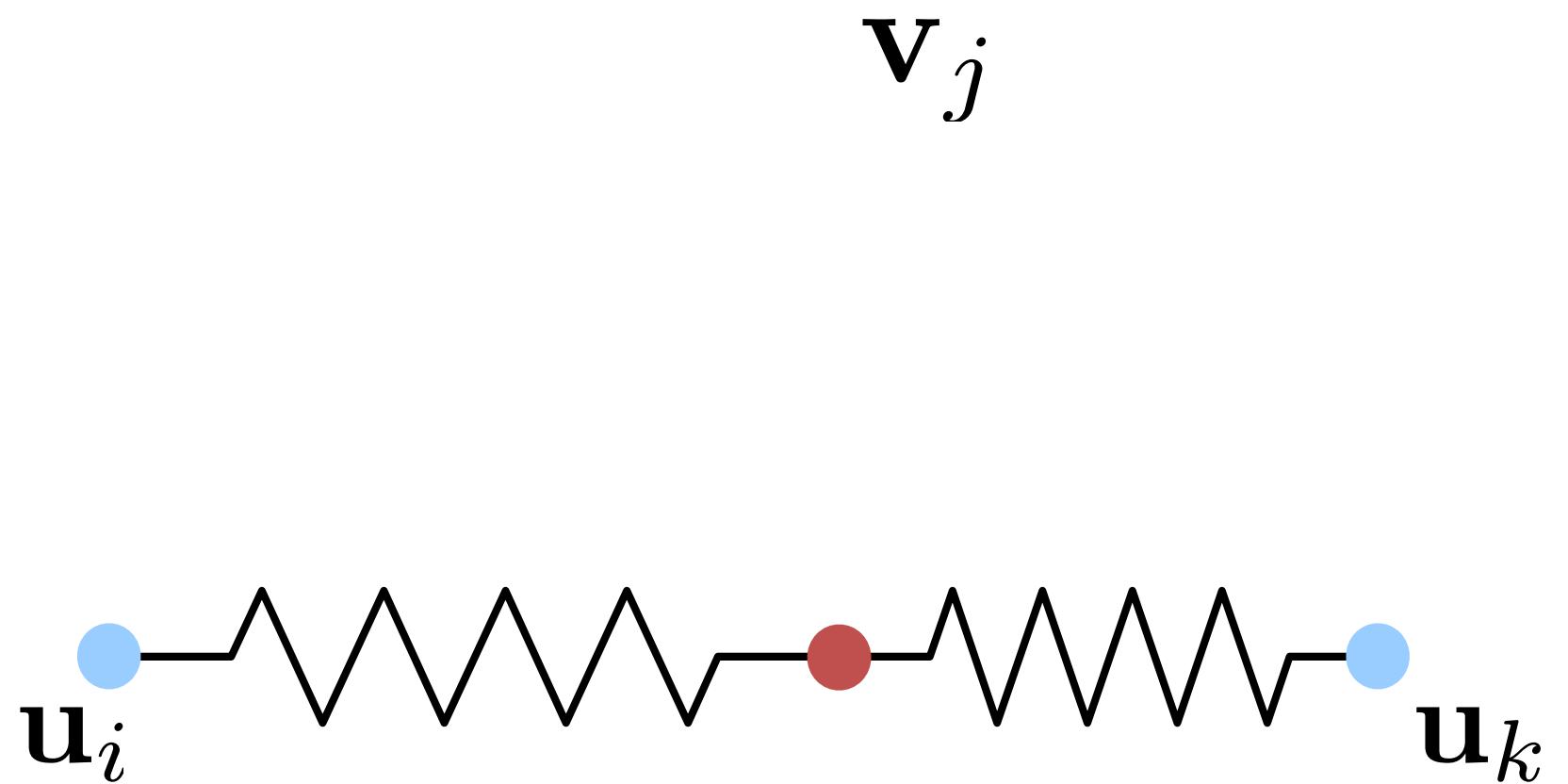
Harmonic Mapping

- Inner mesh edges as springs
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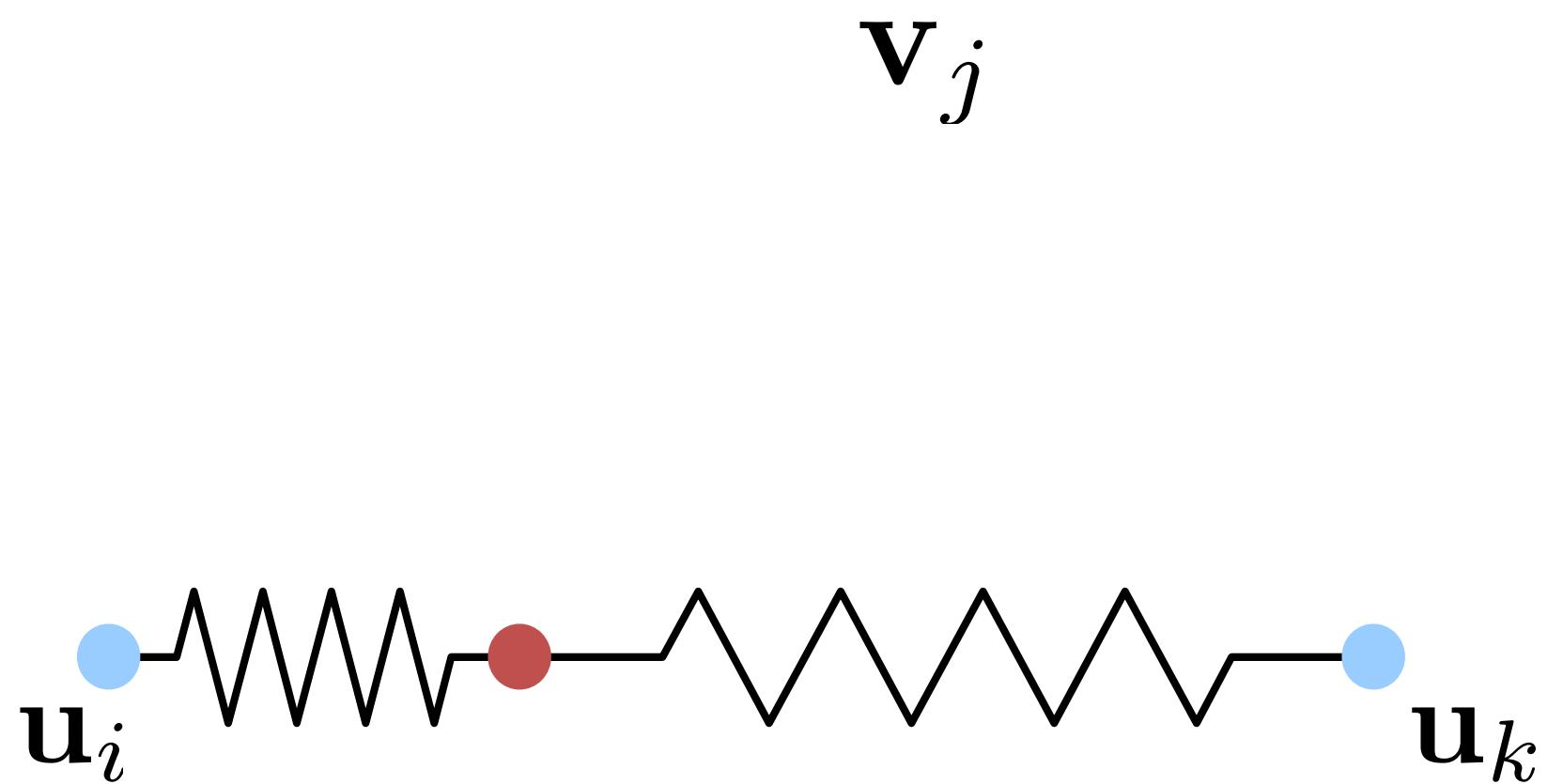
Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



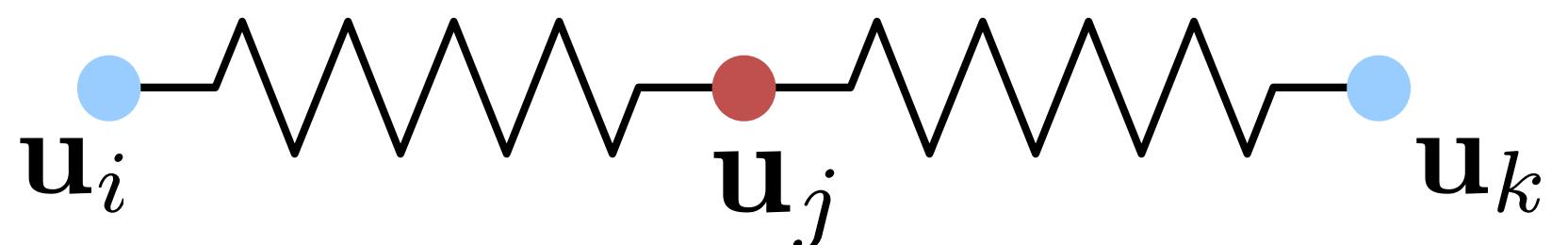
Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



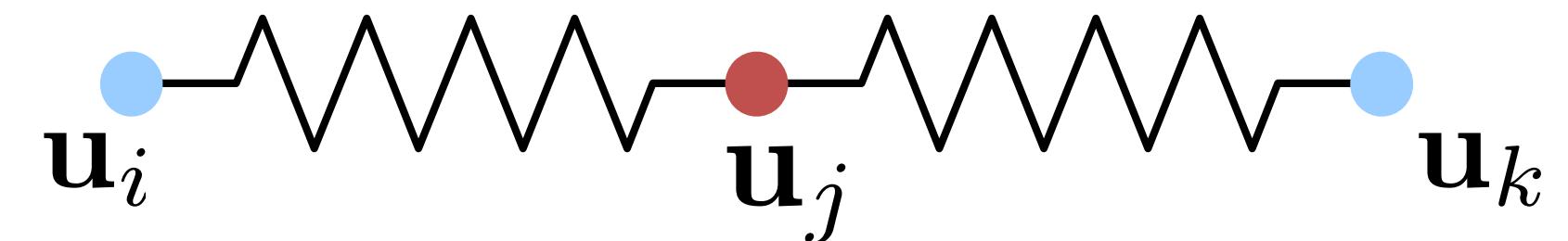
Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane



Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane
- Spring energy:

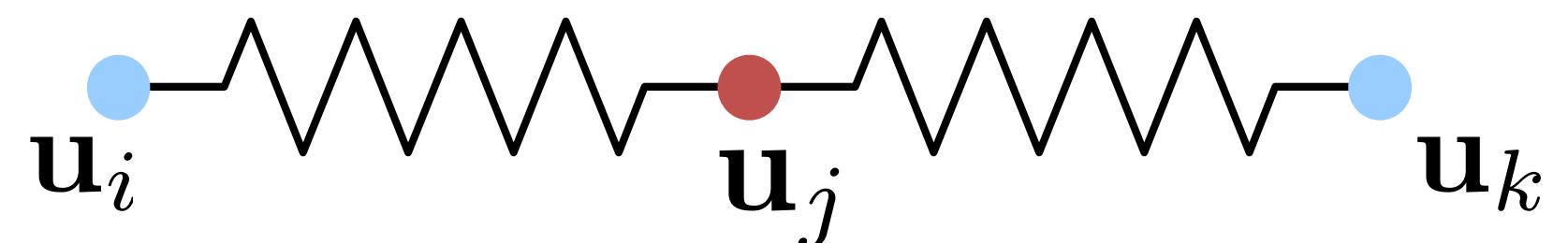


$$\frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$
$$\mathbf{u}_i, \mathbf{u}_j \in \mathbb{R}^2$$

Harmonic Mapping

- Inner mesh edges as springs
- Find minimum-energy state where all vertices lie in the 2D plane
- Total spring energy of the flattened mesh:

$$E(\mathbf{u}_1, \dots, \mathbf{u}_n) = \sum_{(i,j) \in \mathcal{E}} \frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$



Demo

- <https://libigl.github.io/libigl-python-bindings/tut-chapter4/>

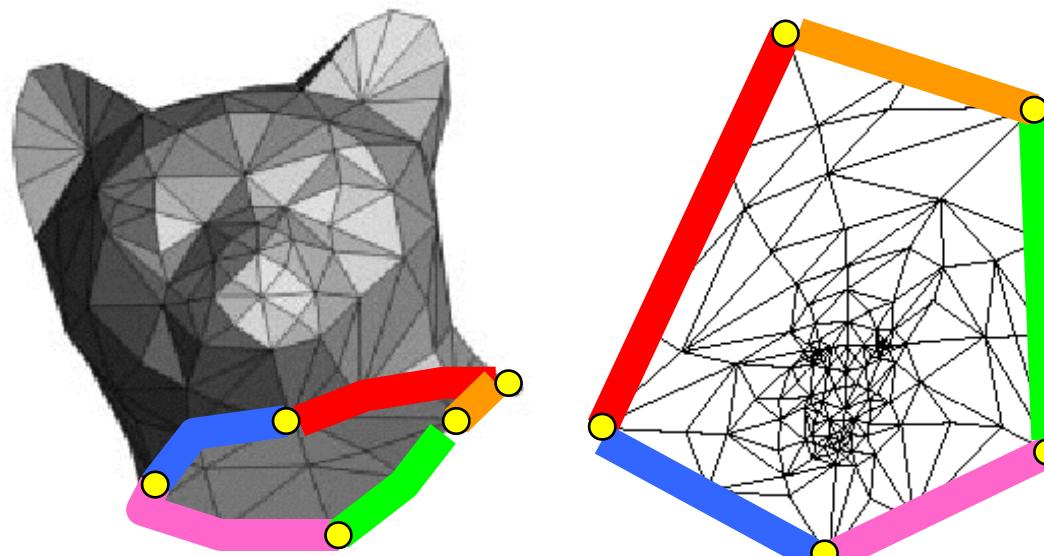
Minimizing Spring Energy

$$E(\mathbf{u}_1, \dots, \mathbf{u}_n) = \sum_{(i,j) \in \mathcal{E}} \frac{1}{2} k_{i,j} \|\mathbf{u}_i - \mathbf{u}_j\|^2$$

$$\frac{\partial E(\mathbf{u}_1, \dots, \mathbf{u}_n)}{\partial \mathbf{u}_i} = \sum_{j \in \mathcal{N}(i)} k_{i,j} (\mathbf{u}_i - \mathbf{u}_j) = 0$$

$$\sum_{j \in \mathcal{N}(i) \cap \mathcal{B}} k_{i,j} \mathbf{u}_i + \sum_{j \in \mathcal{N}(i) \setminus \mathcal{B}} k_{i,j} (\mathbf{u}_i - \mathbf{u}_j) = \sum_{j \in \mathcal{N}(i) \cap \mathcal{B}} k_{i,j} \mathbf{u}_j$$

unknown
flat vertex
positions



$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ – inner vertices
 $\mathbf{v}_{n+1}, \dots, \mathbf{v}_N$ – boundary vertices

known fixed
boundary
positions

Minimizing Spring Energy

- Sparse linear system of n equations to solve!

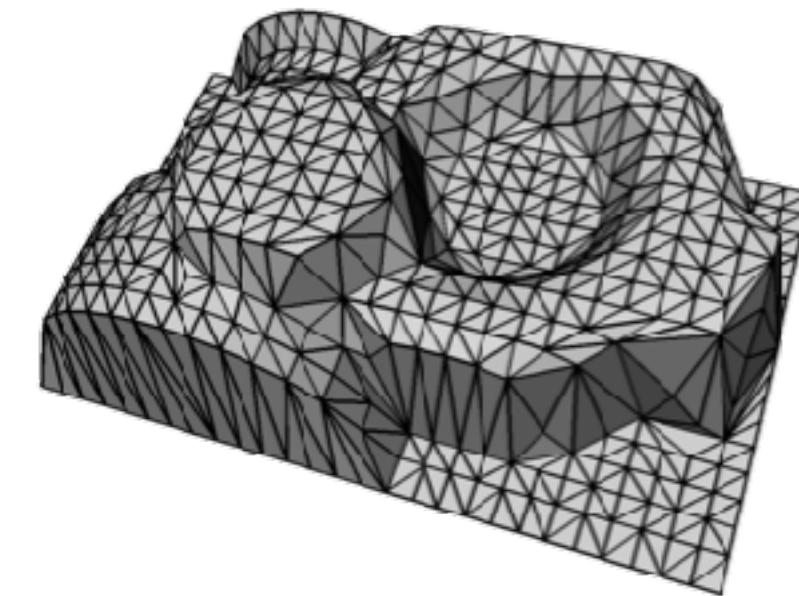
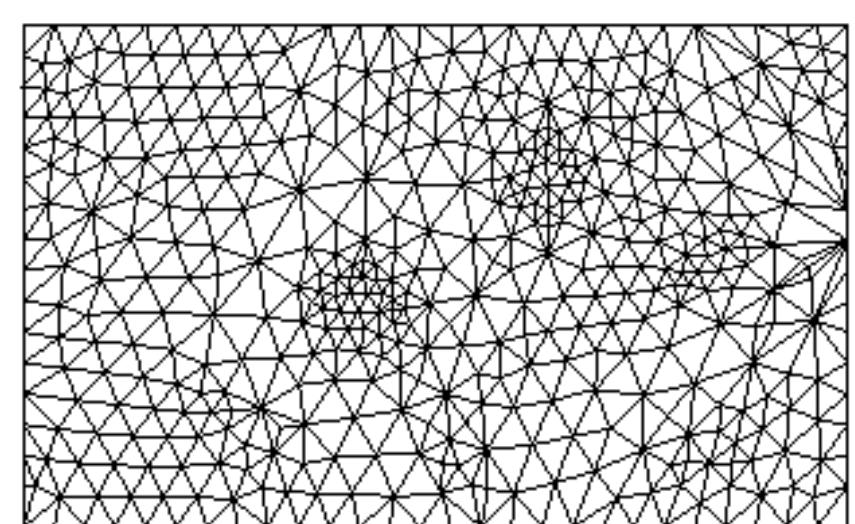
$$\sum_{j \in \mathcal{N}(i) \cap \mathcal{B}} k_{i,j} \mathbf{u}_i + \sum_{j \in \mathcal{N}(i) \setminus \mathcal{B}} k_{i,j} (\mathbf{u}_i - \mathbf{u}_j) = \sum_{j \in \mathcal{N}(i) \cap \mathcal{B}} k_{i,j} \mathbf{u}_j$$

$$\begin{pmatrix} \sum_j k_{i,j} & * & \cdots & -k_{i,j} \\ * & \sum_j k_{i,j} & * & \vdots \\ \vdots & * & \ddots & * \\ -k_{j,i} & \cdots & * & \sum_j k_{i,j} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \vdots \\ \mathbf{u}_n \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{u}}_1 \\ \bar{\mathbf{u}}_2 \\ \vdots \\ \bar{\mathbf{u}}_n \end{pmatrix}$$

Choice of spring constants $k_{i,j}$

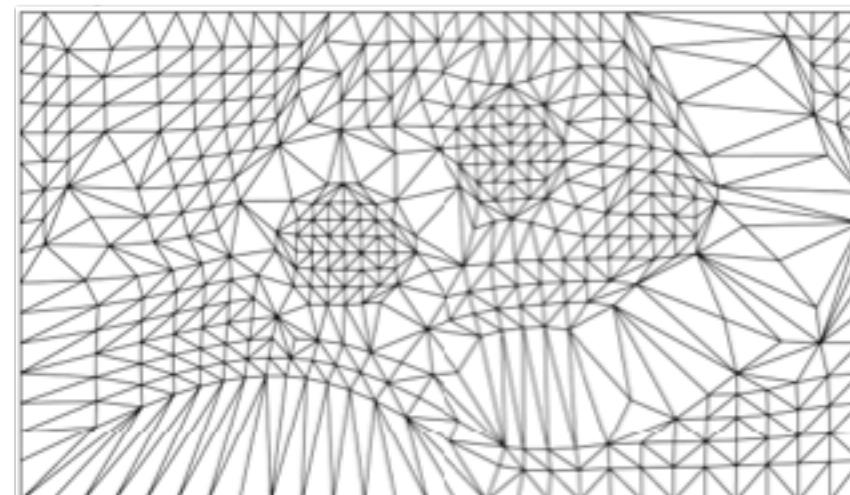
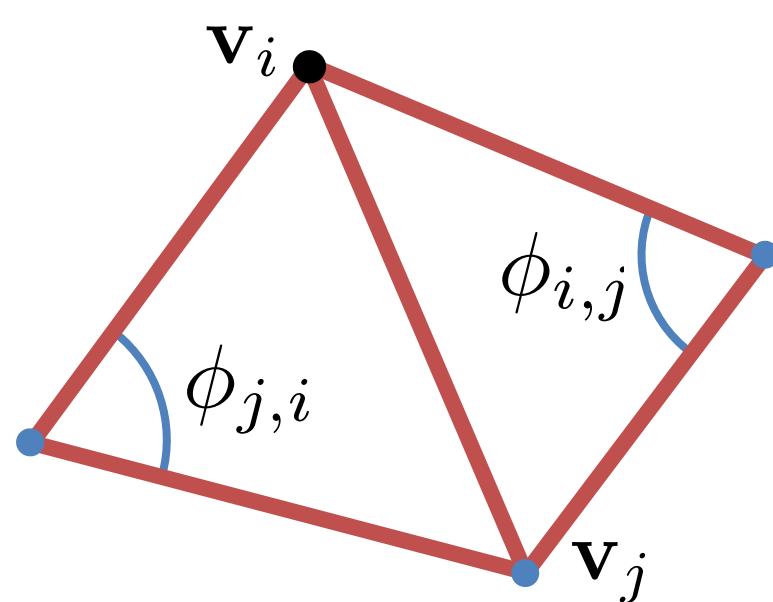
- Uniform

$$k_{i,j} = 1$$



- Cotan

$$k_{i,j} = \cot \phi_{i,j} + \cot \phi_{j,i}$$

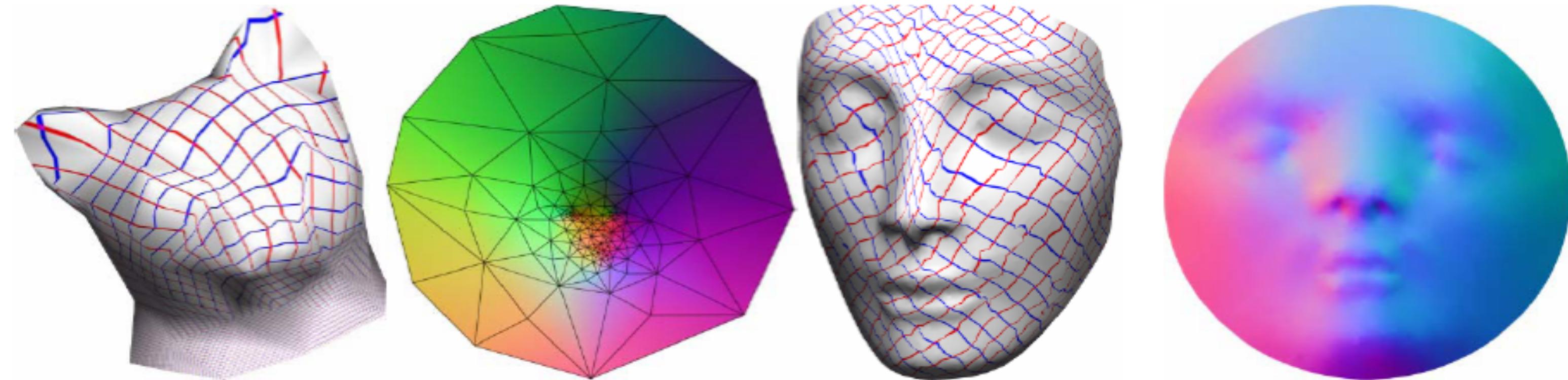


Tutte's Theorem

- If the weights are **nonnegative**, and the boundary is fixed to a **convex** polygon, the parameterization is **bijective**
- Tutte'63 proved for uniform weights
- Floater'97 extended to arbitrary nonnegative weights
 - W.T. Tutte. "How to draw a graph". Proceedings of the London Mathematical Society, 13(3):743-768, 1963.
 - Michael S. Floater. 1997. Parametrization and smooth approximation of surface triangulations. Comput. Aided Geom. Des. 14, 3 (April 1997), 231–250. DOI:[https://doi.org/10.1016/S0167-8396\(96\)00031-3](https://doi.org/10.1016/S0167-8396(96)00031-3)

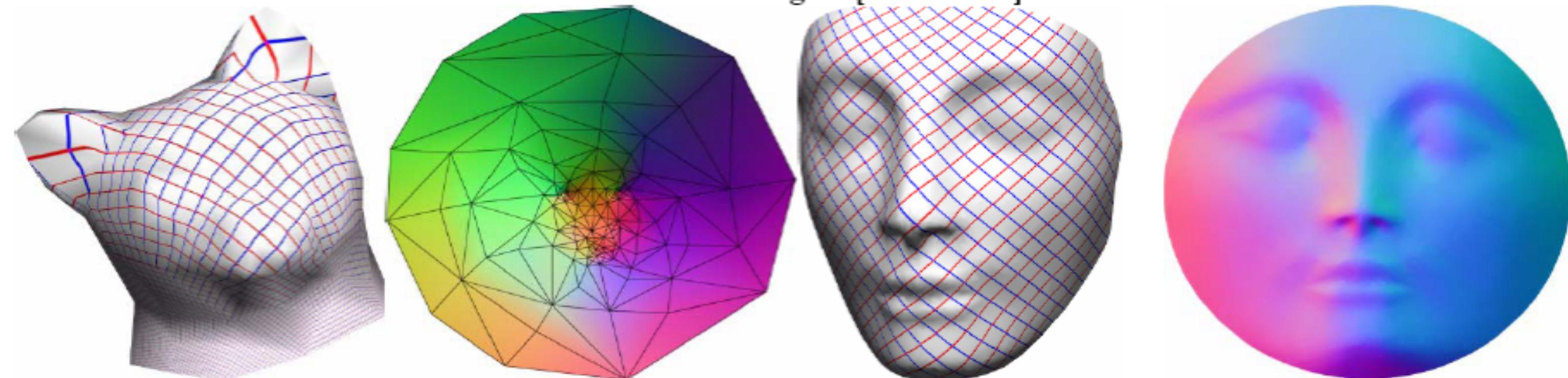
Comparison of Weights

uniform
weights



Parameterization with uniform weights [Tutte 1963] on a circular domain.

cotan
weights



Parameterization with harmonic weights [Eck et al. 1995] on a circular domain.

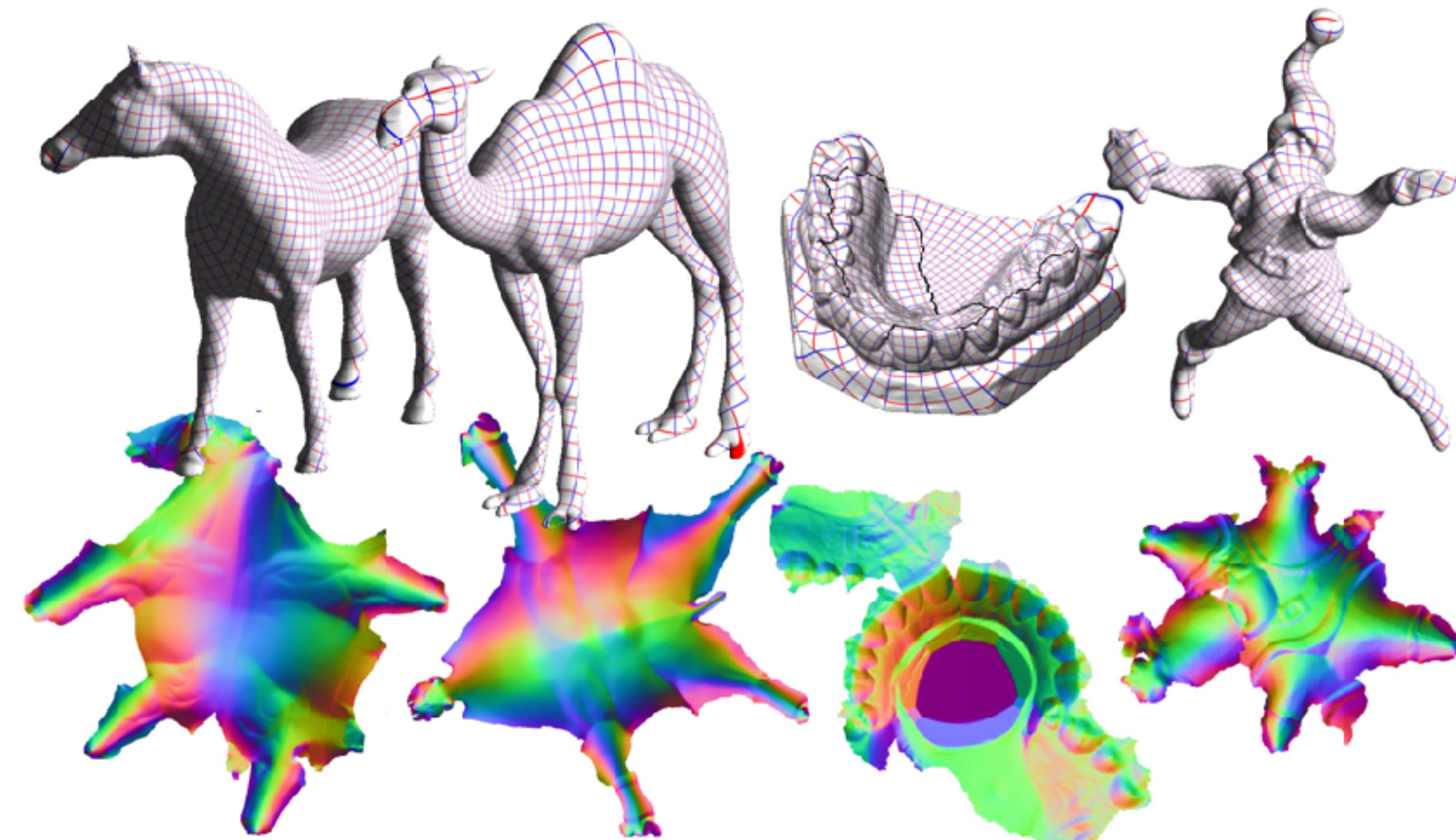
Eck et al. 1995, "Multiresolution analysis of arbitrary meshes", SIGGRAPH 1995

Discussion

- The results of cotan-weights mapping are better than those of uniform convex mapping (local area and angles preservation).
- But: the mapping is not always legal (the cotan weights can be negative for badly-shaped triangles...)
- In any case: sparse system to solve, so robust and efficient numerical solvers exist

Discussion

- Both mappings have the problem of **fixed boundary** – it constrains the minimization and causes **distortion**.
- More advanced methods do not require boundary conditions.



ABF++ method,
Sheffer et al. 2005
<http://www.cs.ubc.ca/~sheffa/ABF++/abf.htm>

Summary

- Parametrization have many uses in computer graphics: they allow to use 2D algorithms and data on surfaces
- Designing a parametrization is equivalent to designing two scalar functions
- Scalar functions can be designed interpolating values directly, or interpolating vectors and solving a Poisson problem

References

- Surface Parameterization: a Tutorial and Survey: <http://graphics.stanford.edu/courses/cs468-05-fall/Papers/param-survey.pdf>
- Polygon Mesh Processing: Chapter 5