

Elementary Differential Geometry of Surfaces

Acknowledgements: Olga Sorkine-Hornung
CSC 486B/586B - Geometric Modeling - Teseo Schneider

Surfaces, Parametric Form

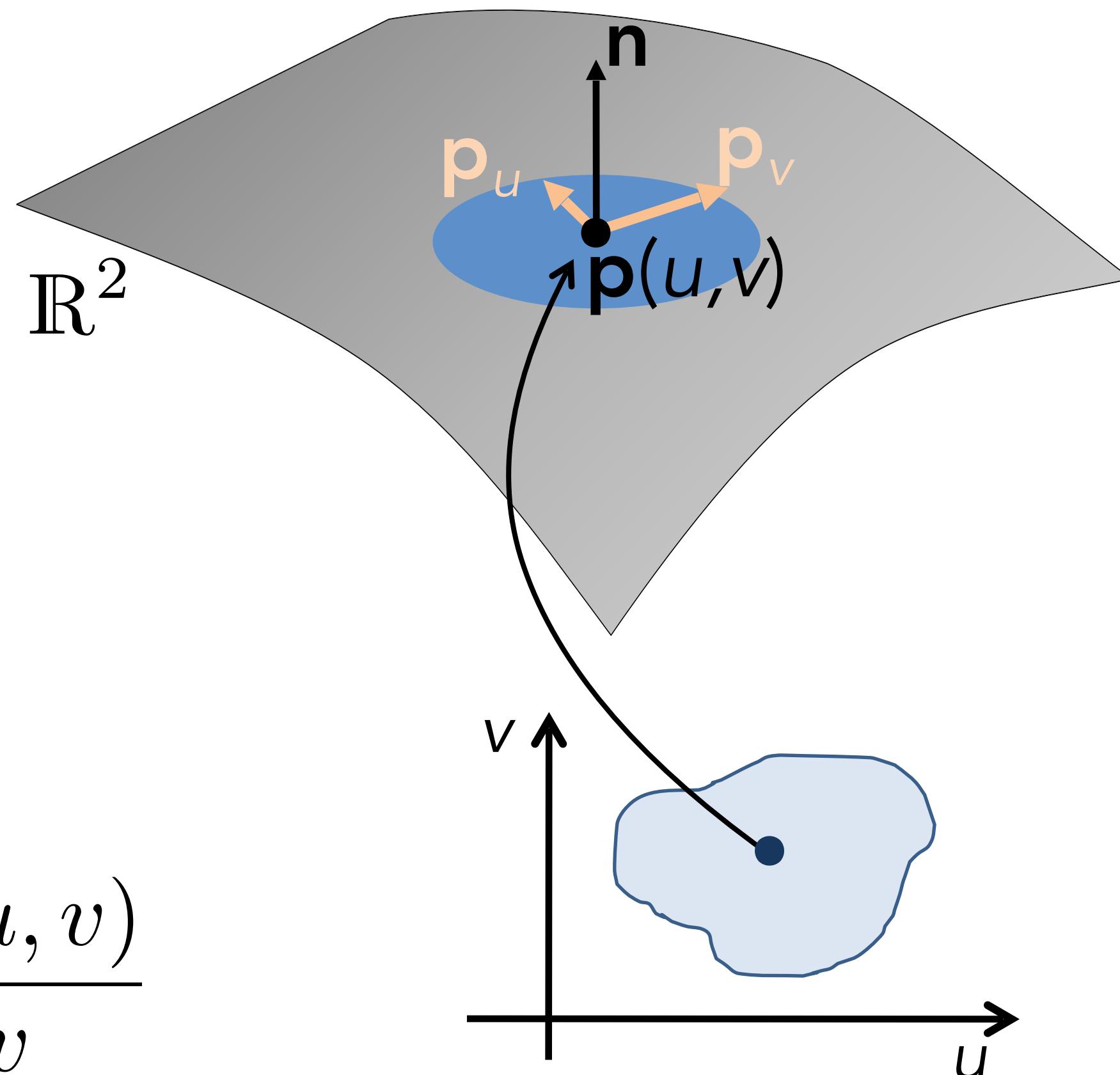
- Continuous surface

$$\mathbf{p}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}, \quad (u, v) \in \mathbb{R}^2$$

- Tangent plane at point $\mathbf{p}(u, v)$ is spanned by

$$\mathbf{p}_u = \frac{\partial \mathbf{p}(u, v)}{\partial u}, \quad \mathbf{p}_v = \frac{\partial \mathbf{p}(u, v)}{\partial v}$$

These vectors don't have to be orthogonal

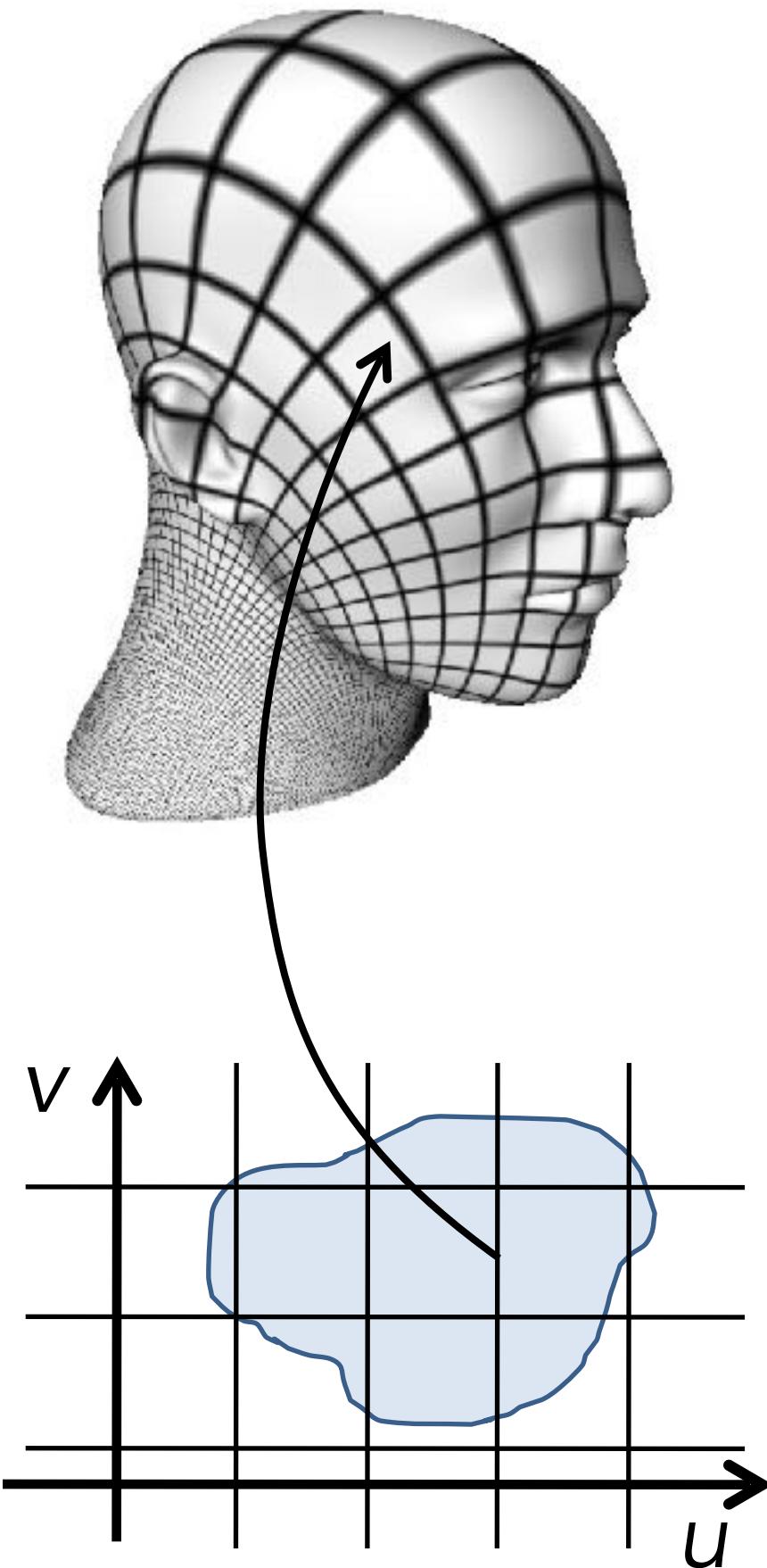


Isoparametric Lines

- Lines on the surface when keeping one parameter fixed

$$\gamma_{u_0}(v) = \mathbf{p}(u_0, v)$$

$$\gamma_{u_0}(u) = \mathbf{p}(u, v_0)$$



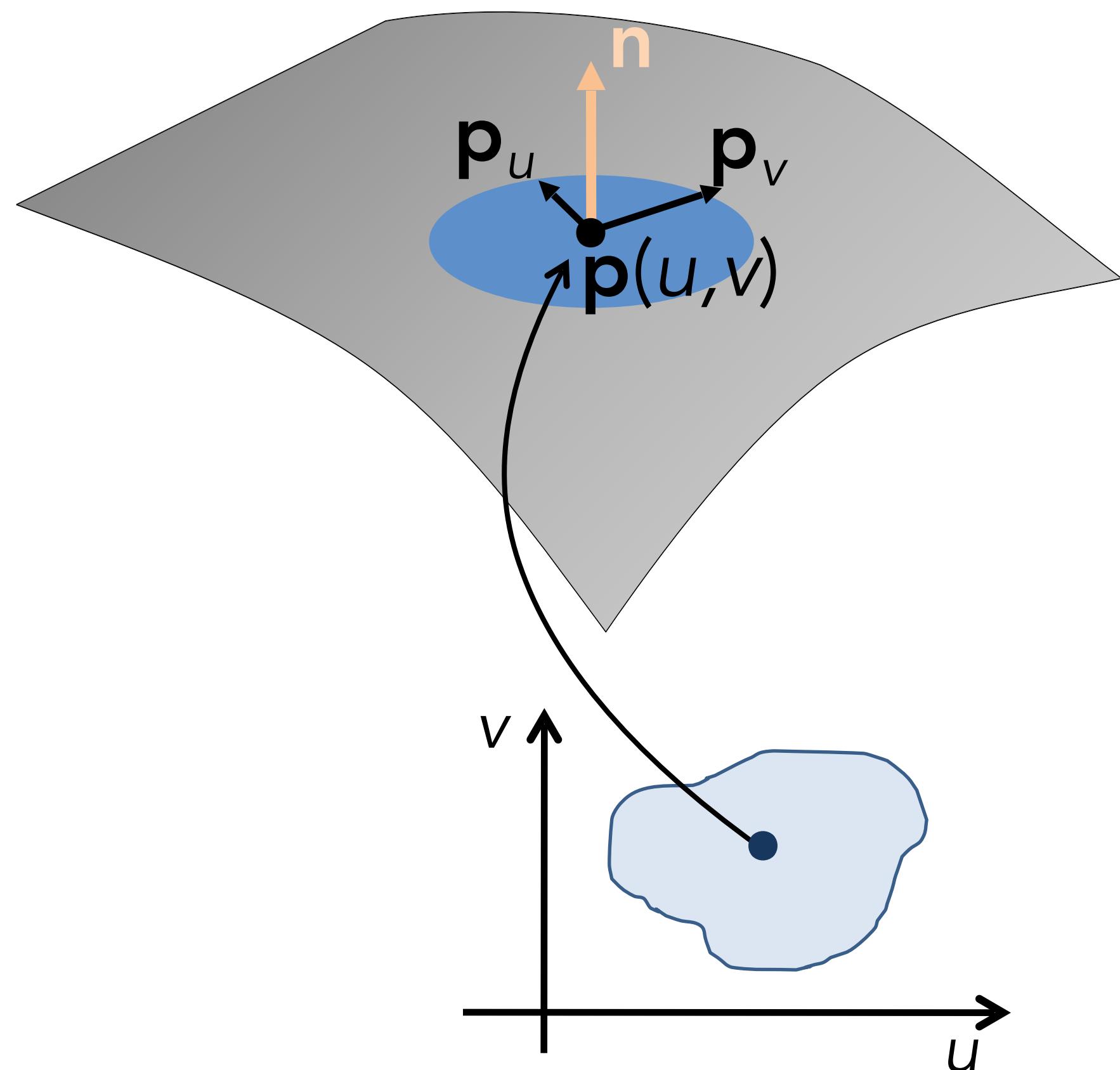
Surface Normals

- Surface normal:

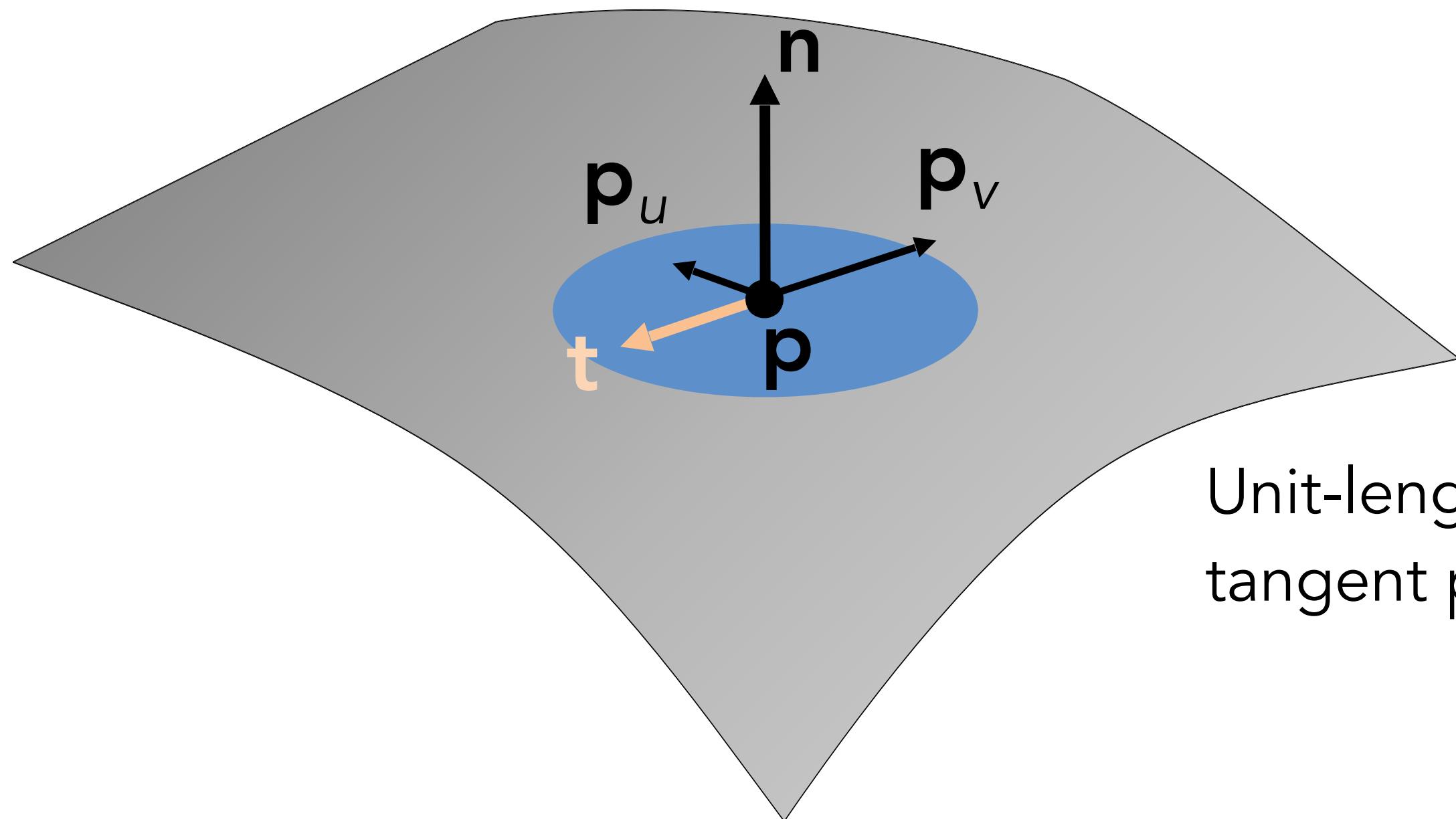
$$\mathbf{n}(u, v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

- Assuming *regular* parameterization, i.e.,

$$\mathbf{p}_u \times \mathbf{p}_v \neq 0$$



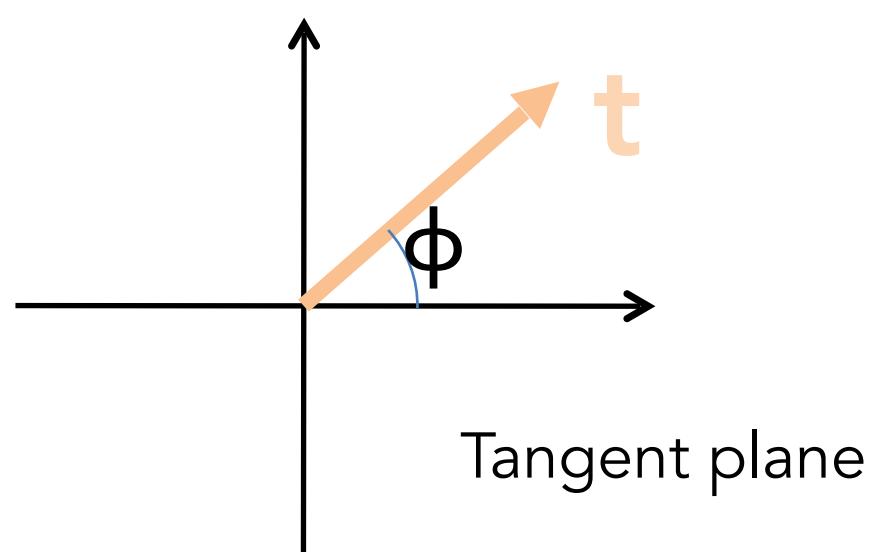
Normal Curvature



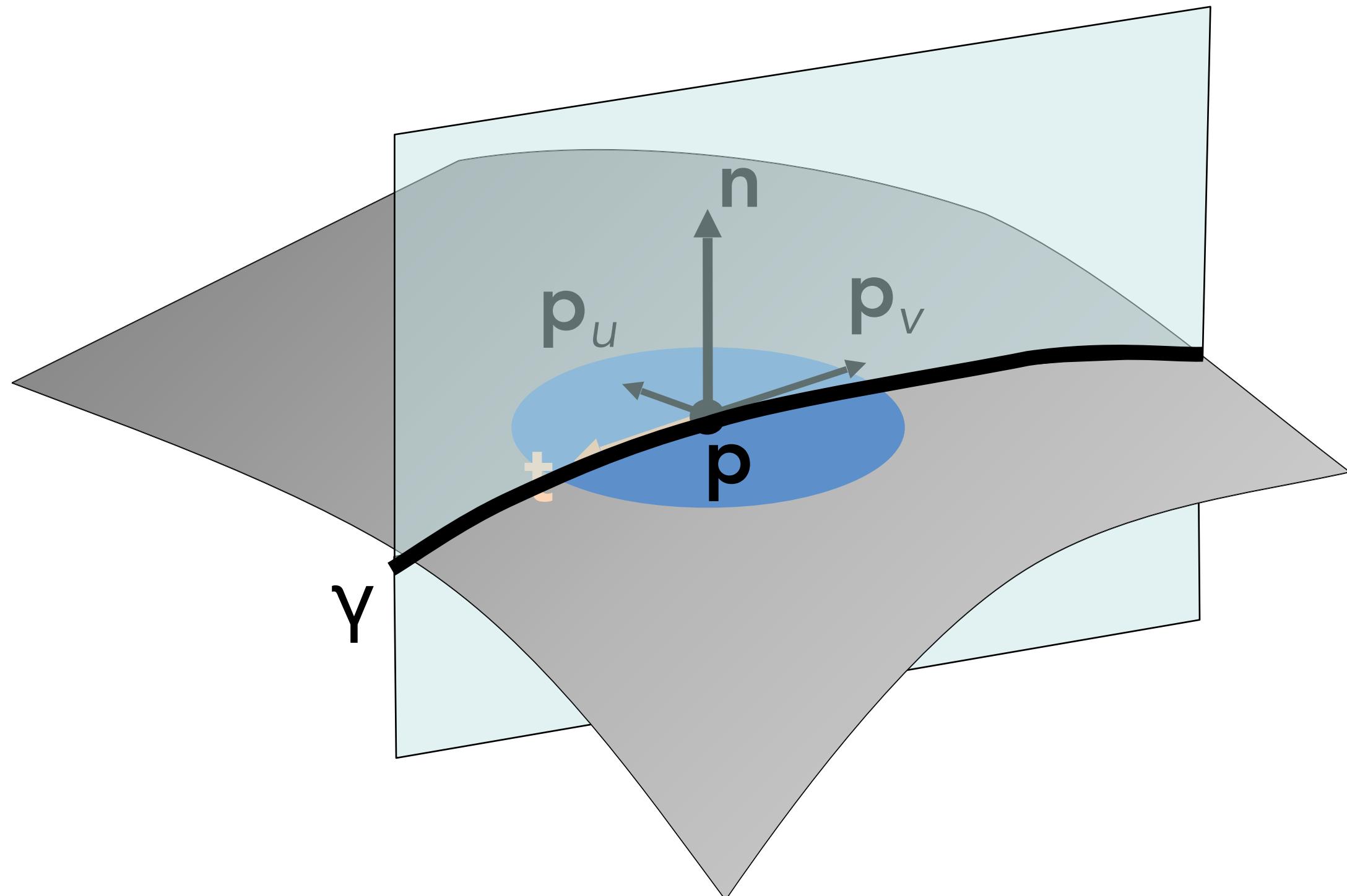
$$\mathbf{n}(u, v) = \frac{\mathbf{p}_u \times \mathbf{p}_v}{\|\mathbf{p}_u \times \mathbf{p}_v\|}$$

Unit-length direction \mathbf{t} in the tangent plane (if \mathbf{p}_u and \mathbf{p}_v are orthogonal):

$$\mathbf{t} = \cos \varphi \frac{\mathbf{p}_u}{\|\mathbf{p}_u\|} + \sin \varphi \frac{\mathbf{p}_v}{\|\mathbf{p}_v\|}$$



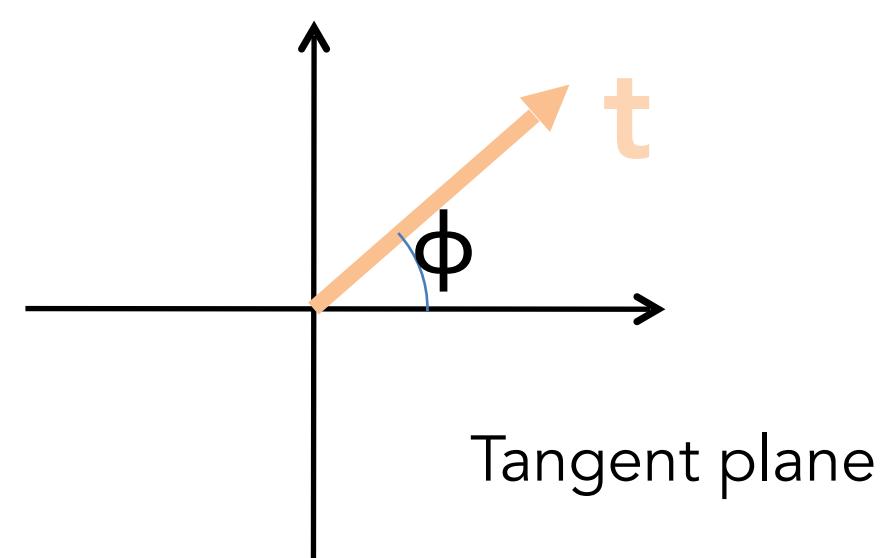
Normal Curvature



The curve γ is the intersection of the surface with the plane through \mathbf{n} and \mathbf{t} .

Normal curvature:

$$\kappa_n(\phi) = \kappa(\gamma(p))$$



Surface Curvatures

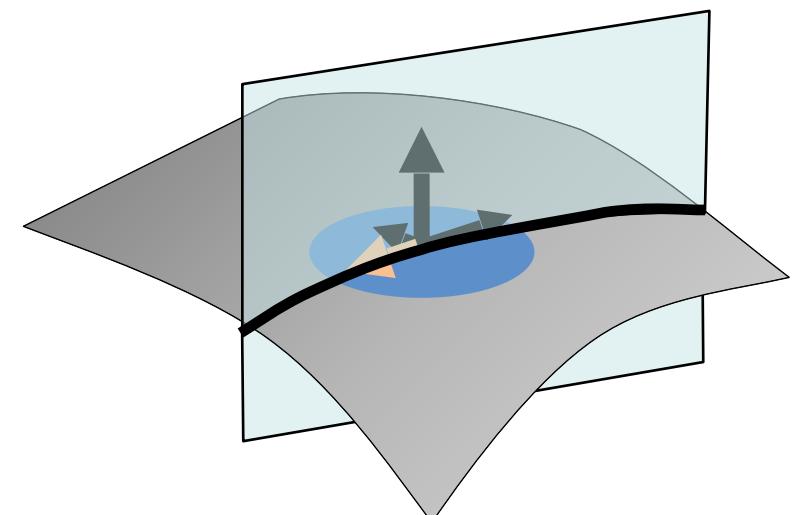
- Principal curvatures
 - Minimal curvature
 - Maximal curvature
- Mean curvature
- Gaussian curvature

$$\kappa_1 = \kappa_{\min} = \min_{\varphi} \kappa_n(\varphi)$$

$$\kappa_2 = \kappa_{\max} = \max_{\varphi} \kappa_n(\varphi)$$

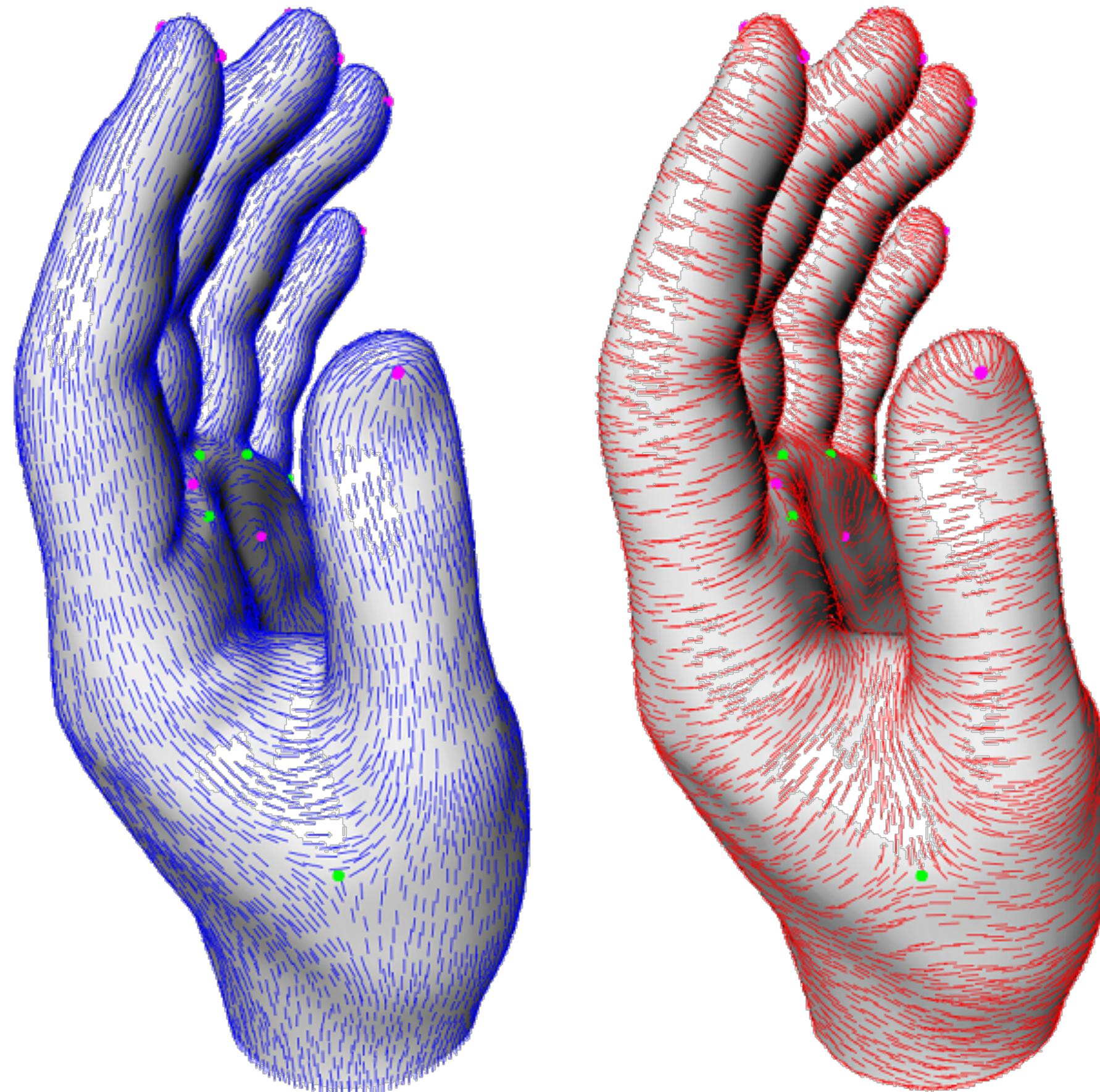
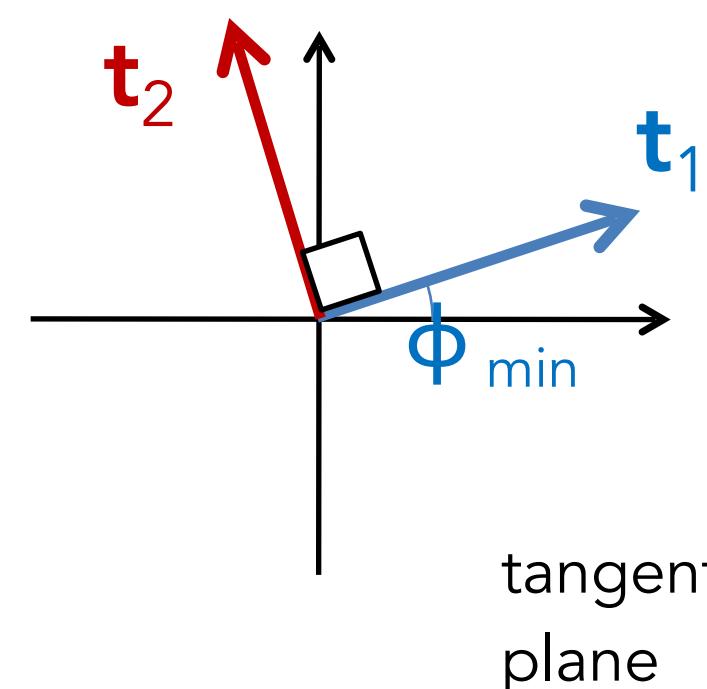
$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi) d\varphi$$

$$K = \kappa_1 \cdot \kappa_2$$



Principal Directions

- Principal directions:
tangent vectors
corresponding to
 ϕ_{\max} and ϕ_{\min}

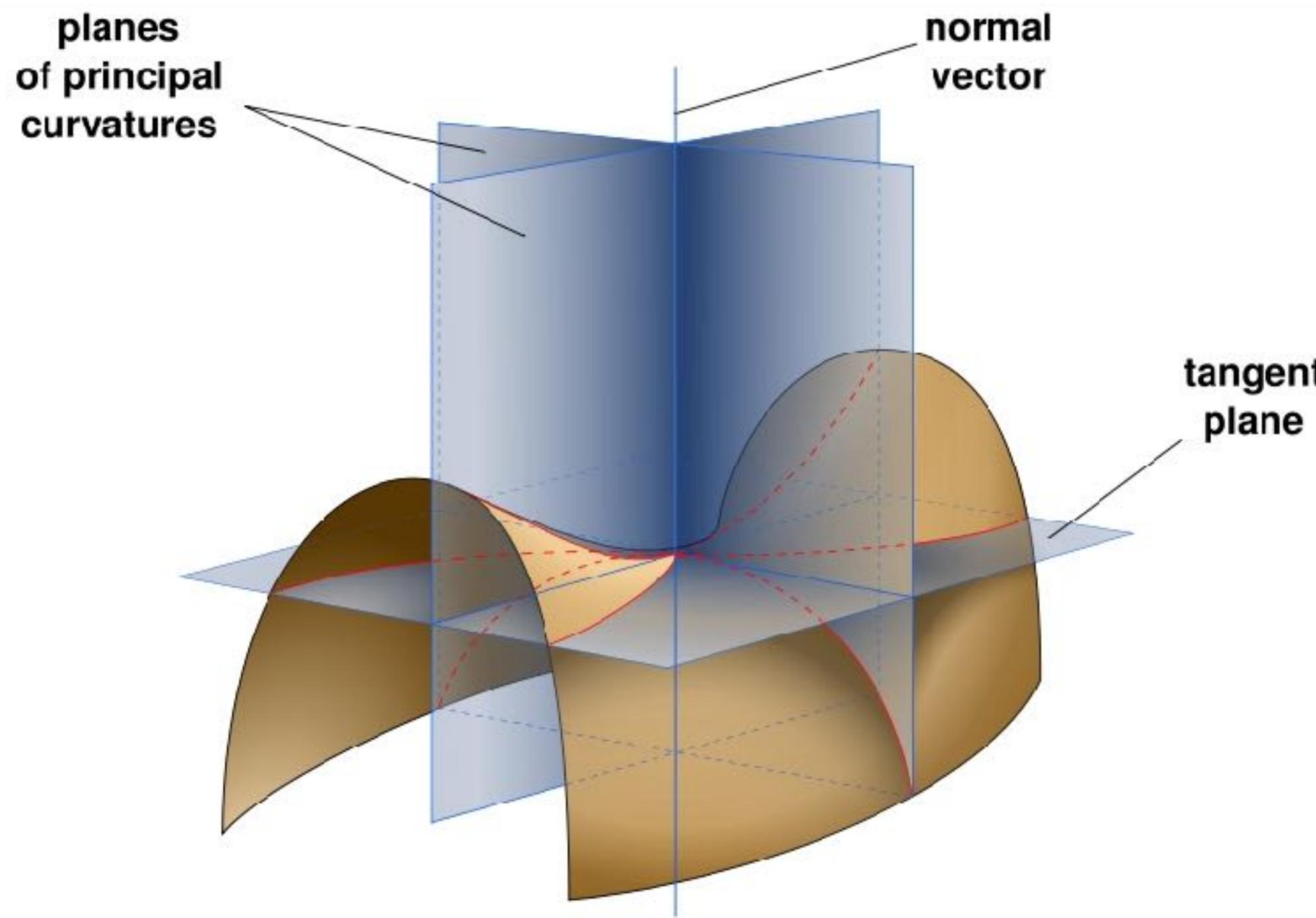


min curvature

max curvature

Pierre Alliez, David Cohen-Steiner, Olivier Devillers, Bruno Lévy, and Mathieu Desbrun. 2003. Anisotropic polygonal remeshing. *ACM Trans. Graph.* 22, 3 (July 2003), 485-493. DOI: <http://dx.doi.org/10.1145/882262.882296>

Principal Directions

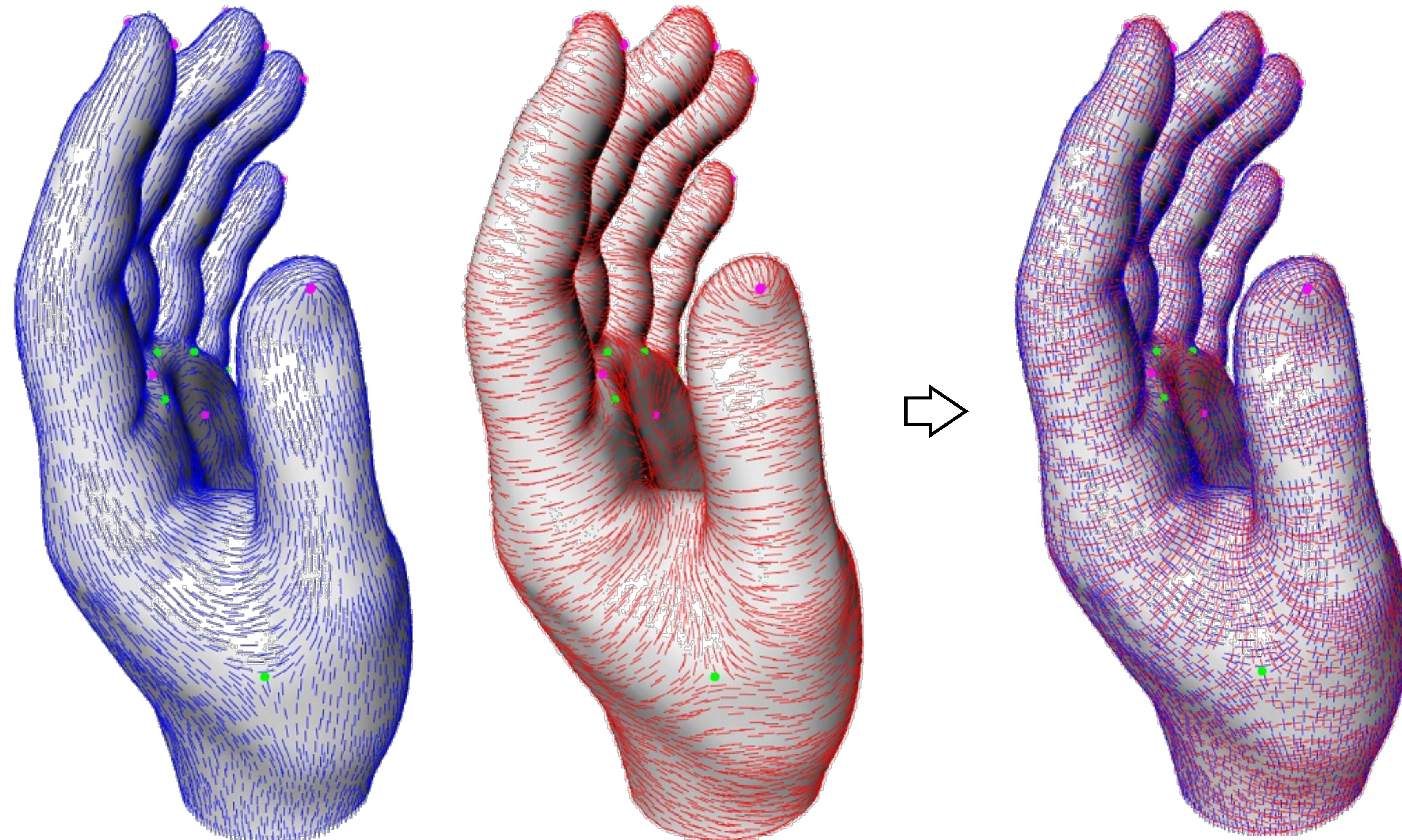


Euler's Theorem: Planes of principal curvature are **orthogonal** and independent of parameterization.

$$\kappa_n(\varphi) = \kappa_1 \cos^2 \varphi + \kappa_2 \sin^2 \varphi, \quad \varphi = \text{angle with } \mathbf{t}_1$$

By Eric Gaba (Sting) - Based upon a drawing in a book, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=1325452>

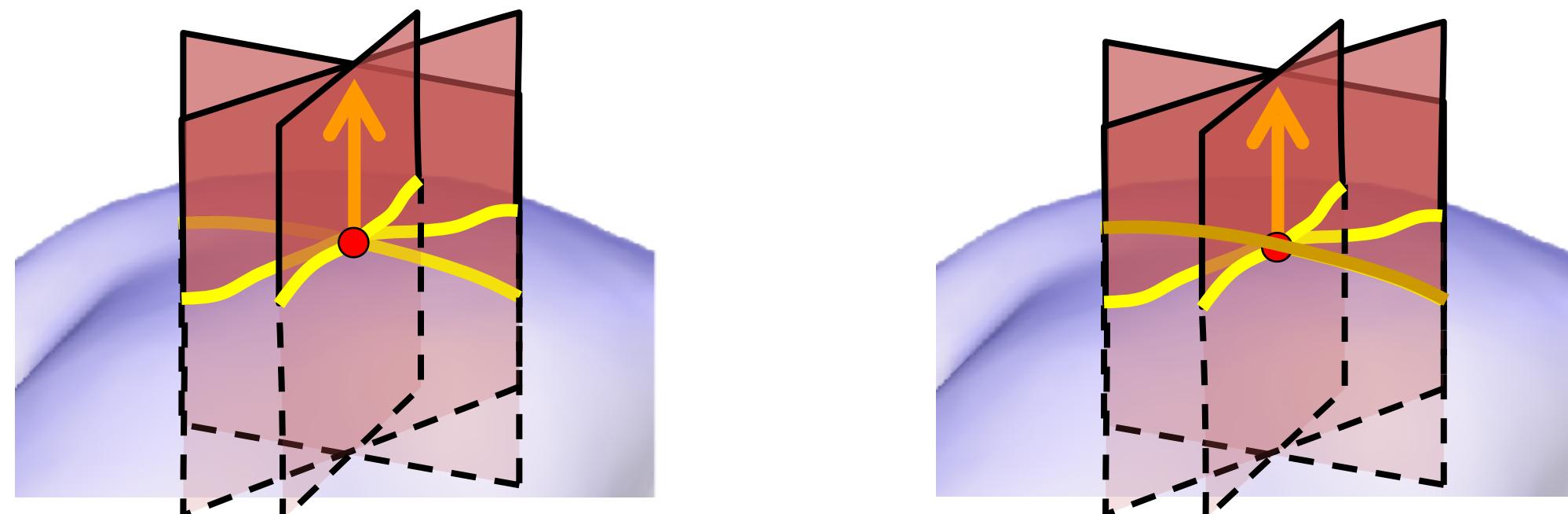
Principal Directions



Mean Curvature

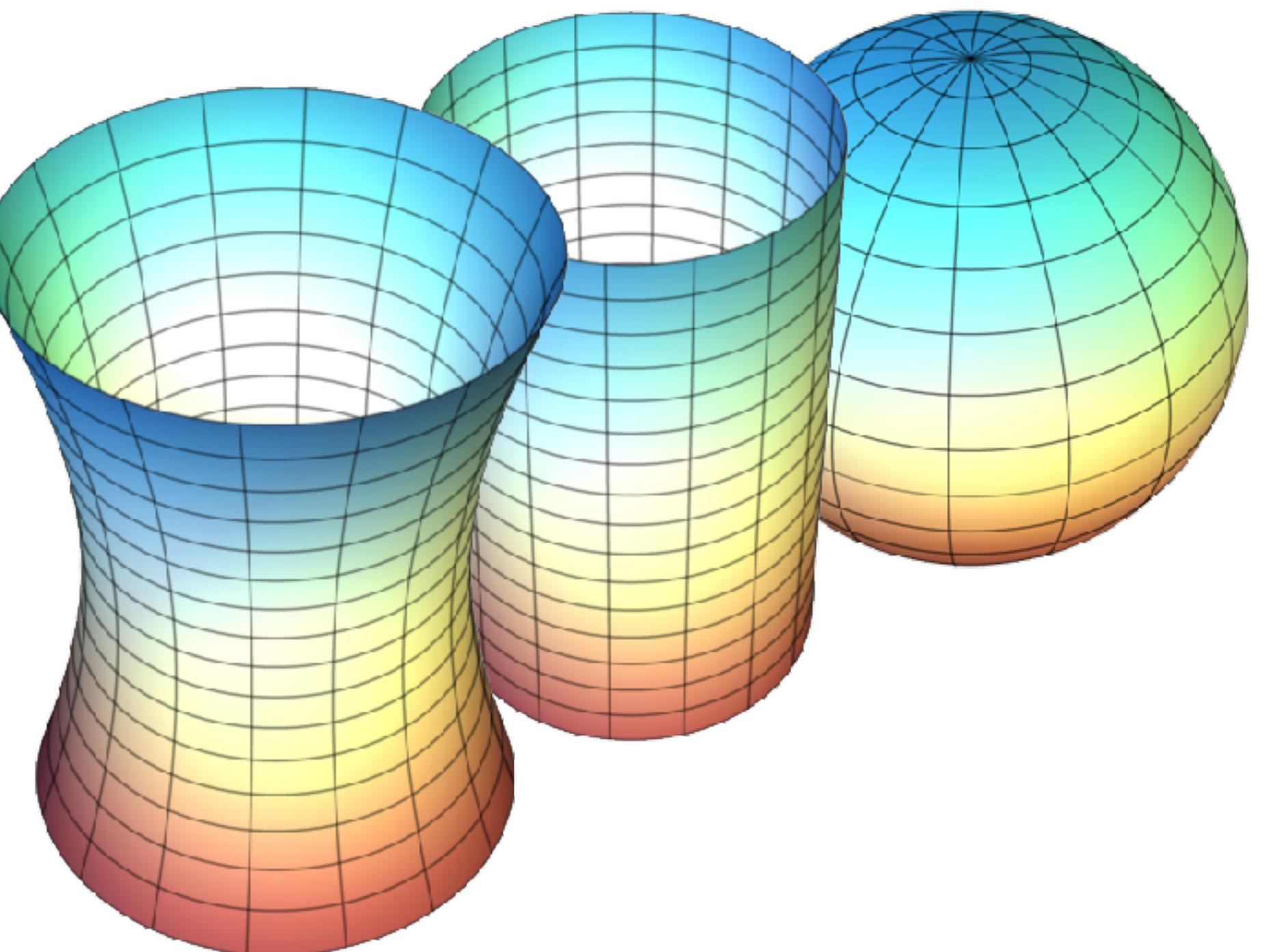
- Intuition for mean curvature

$$H = \frac{\kappa_1 + \kappa_2}{2} = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\varphi) d\varphi$$



Classification

- A point \mathbf{p} on the surface is called
 - Elliptic, if $K > 0$
 - Parabolic, if $K = 0$
 - Hyperbolic, if $K < 0$
- Developable surface iff $K = 0$

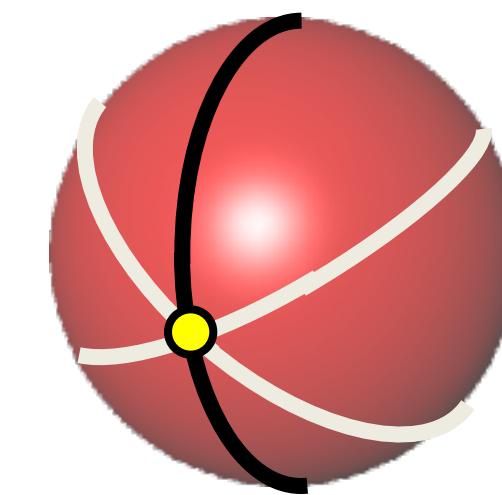


Local Surface Shape By Curvatures

Isotropic:

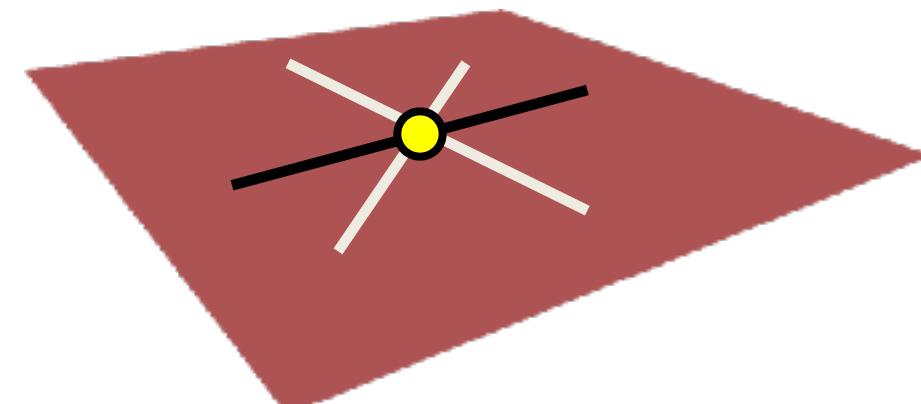
all directions are
principal directions

$$K > 0, \kappa_1 = \kappa_2$$



spherical (umbilical)

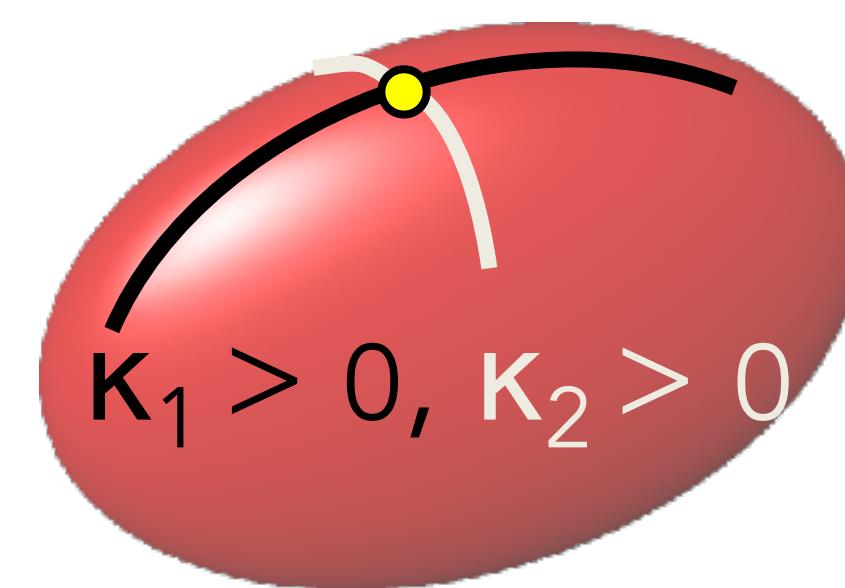
$$K = 0$$



planar

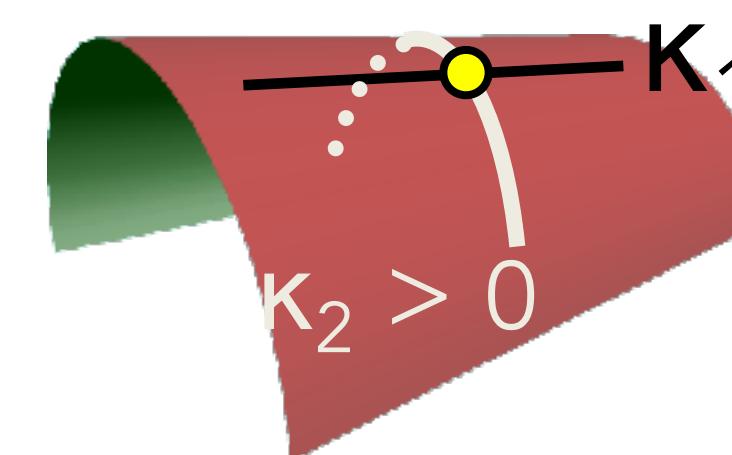
Anisotropic:
2 distinct
principal
directions

$$K > 0$$



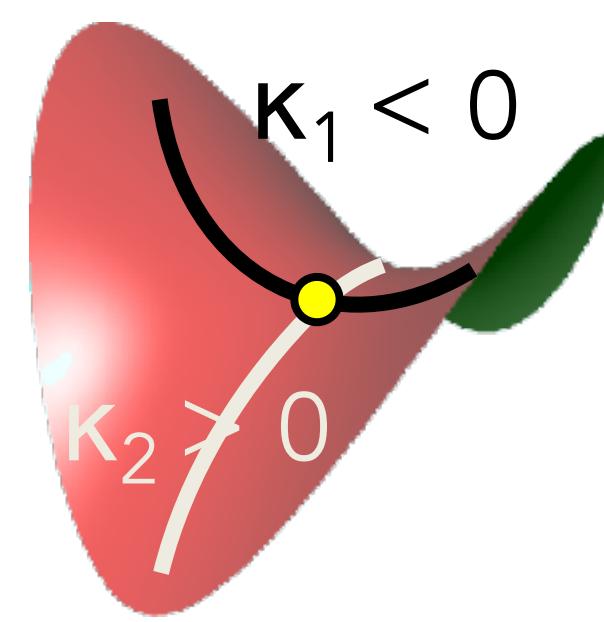
elliptic

$$K = 0$$



parabolic

$$K < 0$$



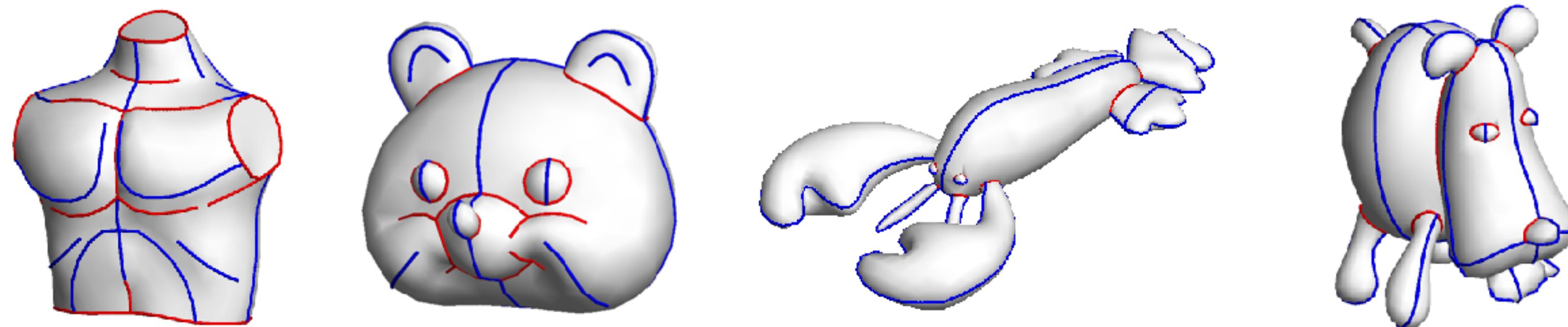
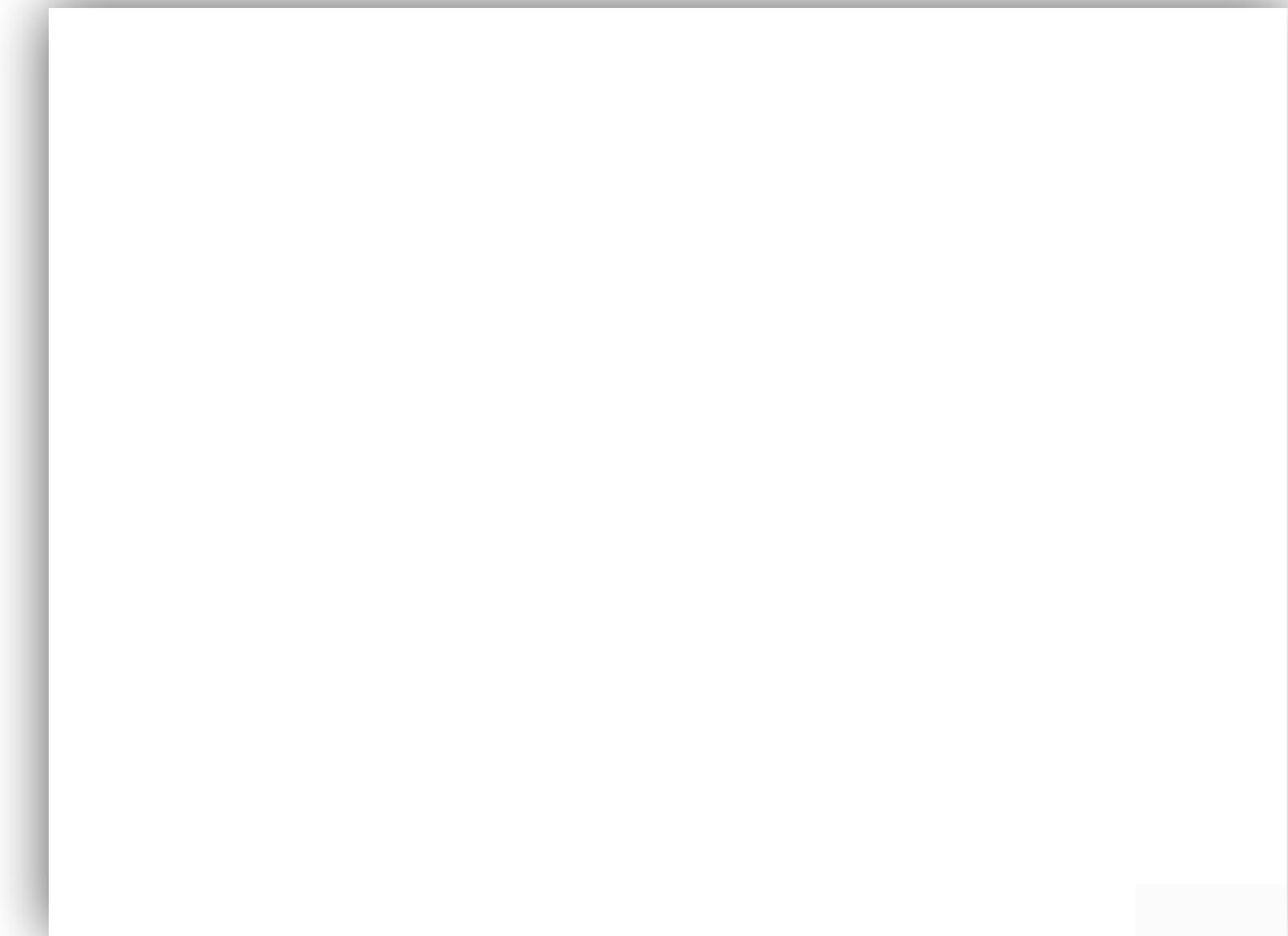
hyperbolic

Usage example: define fair surfaces

- FiberMesh

$$\min_{\mathcal{M}} \int_{\mathcal{M}} \left(\frac{d\kappa_n}{d\mathbf{t}_1} \right)^2 + \left(\frac{d\kappa_n}{d\mathbf{t}_2} \right)^2 dA$$

s.t. M interpolates the curves



Gauss-Bonnet Theorem

- For a closed surface M :

$$\int_{\mathcal{M}} K \, dA = 2\pi \chi(\mathcal{M})$$

$$\int K(\text{dolphin}) \, dA = \int K(\text{cow}) \, dA = \int K(\text{sphere}) \, dA = 4\pi$$

Gauss-Bonnet Theorem

- For a closed surface M :

$$\int_{\mathcal{M}} K \, dA = 2\pi \chi(\mathcal{M})$$

- Compare with planar curves:

$$\int_{\gamma} \kappa \, ds = 2\pi k$$

Fundamental Forms

- First fundamental form

$$\mathbf{I} = \begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \mathbf{p}_u^T \mathbf{p}_u & \mathbf{p}_u^T \mathbf{p}_v \\ \mathbf{p}_u^T \mathbf{p}_v & \mathbf{p}_v^T \mathbf{p}_v \end{pmatrix}$$

- Second fundamental form

$$\mathbf{II} = \begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} \mathbf{p}_{uu}^T \mathbf{n} & \mathbf{p}_{uv}^T \mathbf{n} \\ \mathbf{p}_{uv}^T \mathbf{n} & \mathbf{p}_{vv}^T \mathbf{n} \end{pmatrix}$$

- Together, they define a surface (if some compatibility conditions hold)

Fundamental Forms

- I allows to measure
 - length, angles, area, curvature
 - arc element

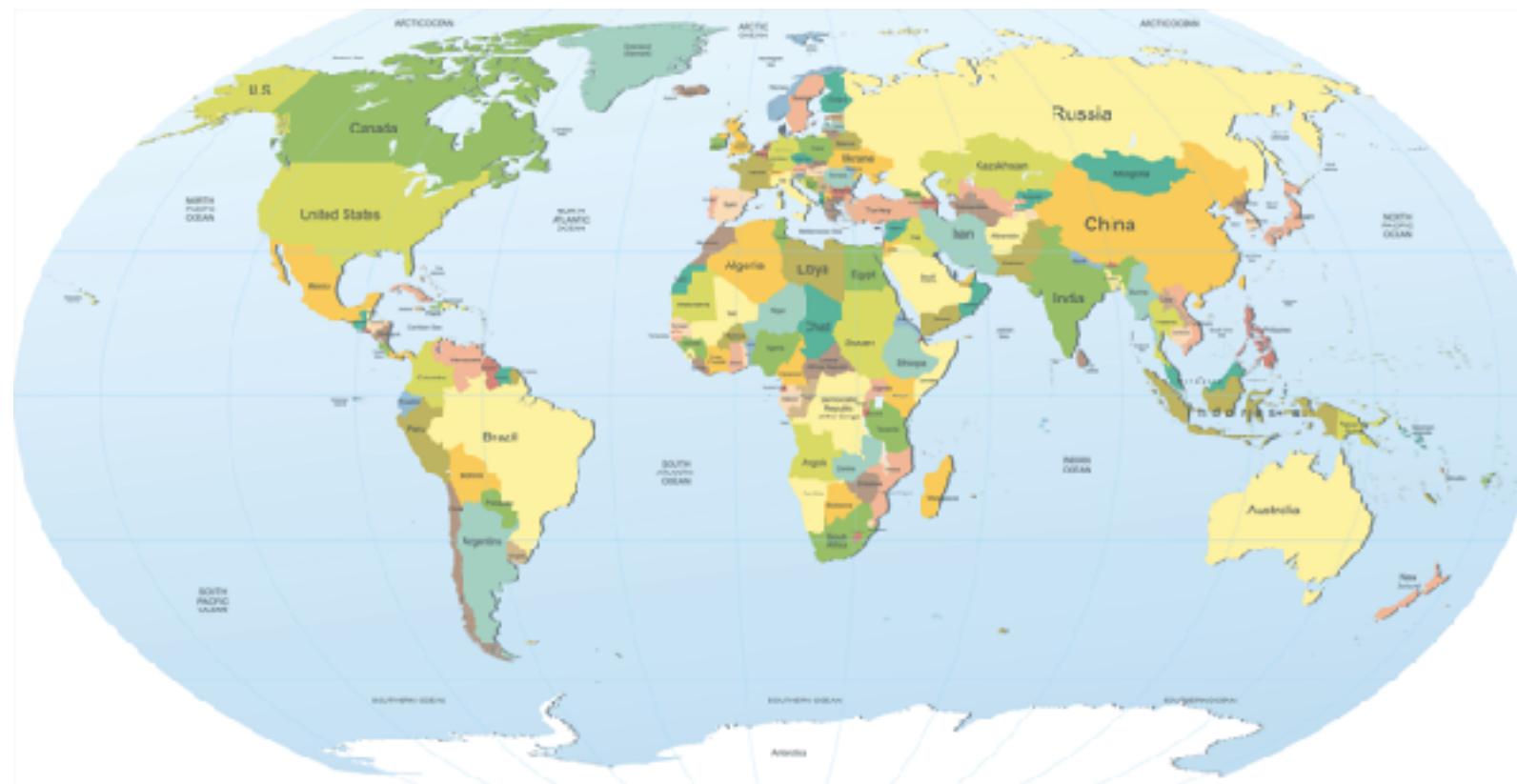
$$ds^2 = E du^2 + 2F dudv + G dv^2$$

- area element $dA = \sqrt{EG - F^2} dudv$



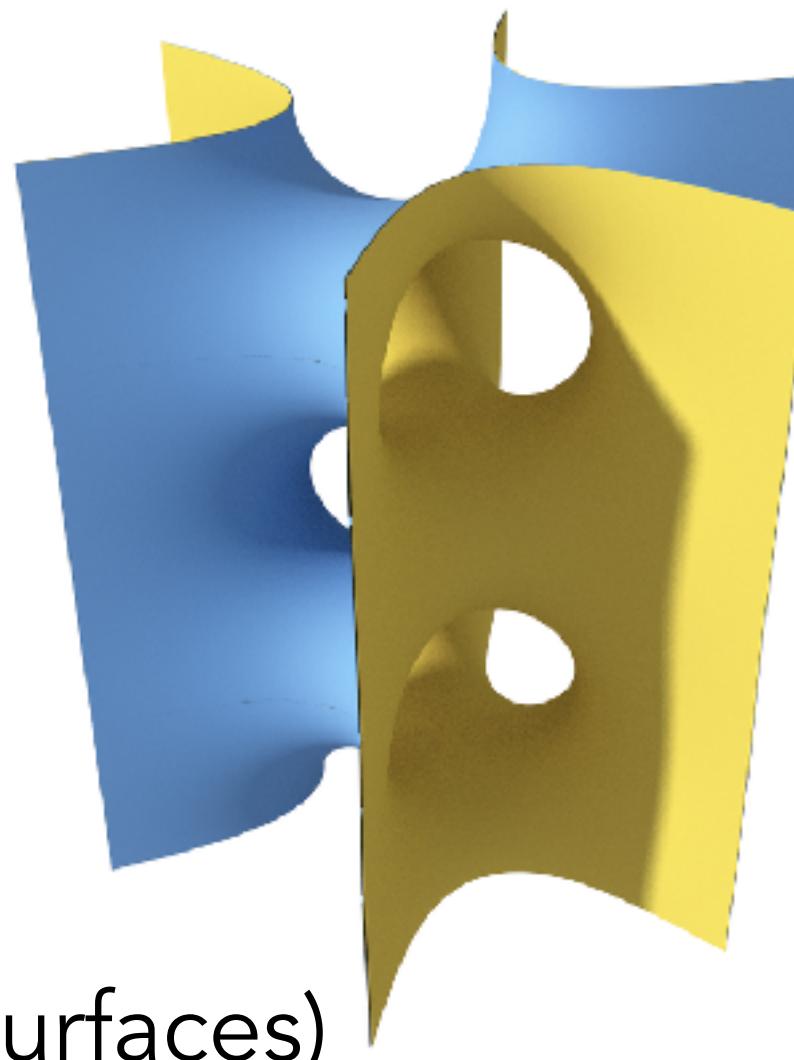
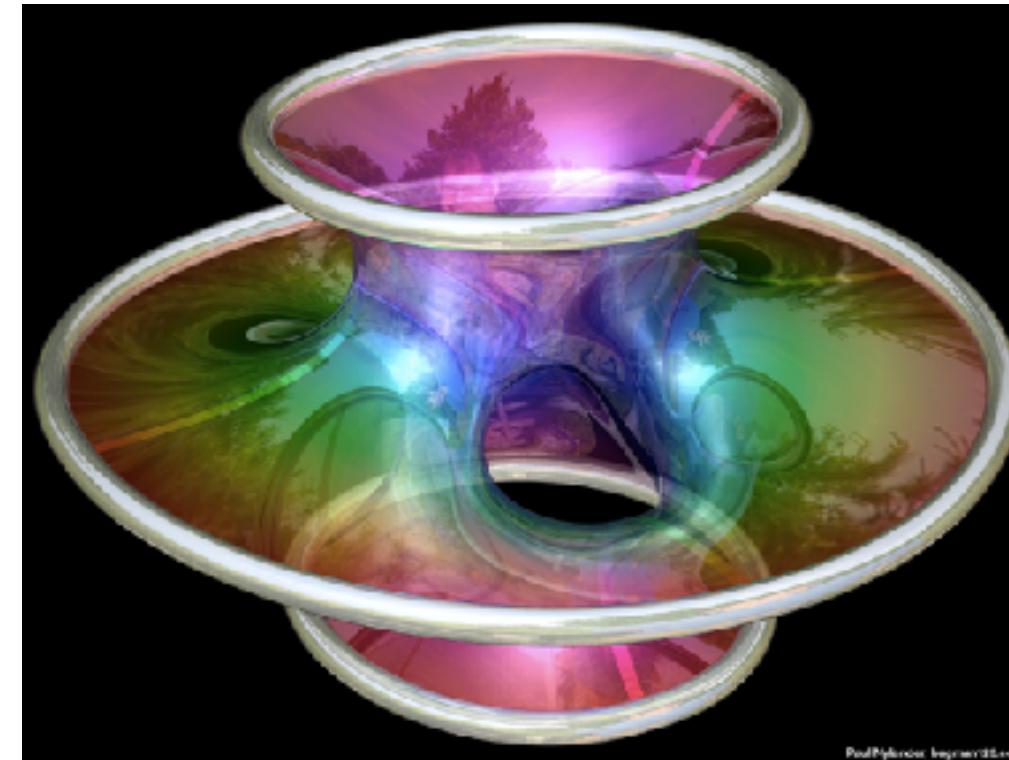
Intrinsic Geometry

- Properties of the surface that only depend on the first fundamental form
 - length
 - angles
 - Gaussian curvature (Theorema Egregium)



Interesting examples

- Hyperbolic geometry – spaces with constant **negative** Gauss curvature
- Constant mean curvature surfaces
- Minimal surfaces ($H = 0$)

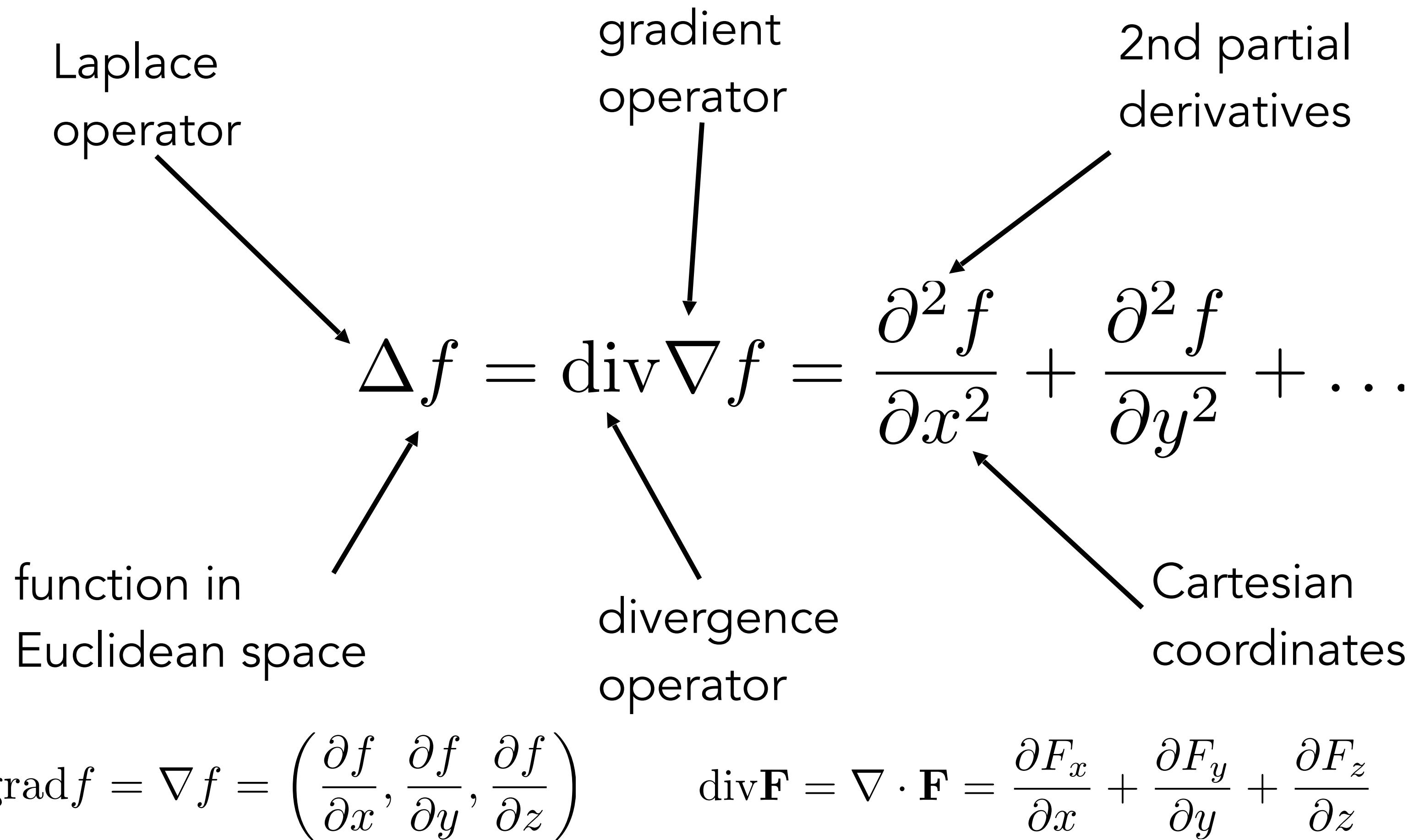


Soap surfaces (minimal surfaces)

By Anders Sandberg - Own work, CC BY-SA 3.0, <https://commons.wikimedia.org/w/index.php?curid=21537940>

Laplace Operator

$$f : \mathbb{R}^3 \rightarrow \mathbb{R} \quad \Delta f : \mathbb{R}^3 \rightarrow \mathbb{R}$$



Intuitive Explanation

The Laplacian $\Delta f(p)$ of a function f at a point p , up to a constant depending on the dimension, is the rate at which the average value of f over spheres centered at p deviates from $f(p)$ as the radius of the sphere grows.

Laplace-Beltrami Operator

- Extension of Laplace to functions on manifolds

$$f : \mathcal{M} \rightarrow \mathbb{R} \quad \Delta f : \mathcal{M} \rightarrow \mathbb{R}$$

$$\Delta_{\mathcal{M}} f = \operatorname{div}_{\mathcal{M}} \nabla_{\mathcal{M}} f$$

Diagram illustrating the components of the Laplace-Beltrami operator:

- Laplace-Beltrami (top left)
- function on surface \mathcal{M} (bottom left)
- gradient operator (top right)
- divergence operator (bottom right)

Arrows point from "Laplace-Beltrami" and "function on surface \mathcal{M} " to the left side of the equation. Arrows point from "gradient operator" and "divergence operator" to the right side of the equation.

Laplace-Beltrami Operator

- For coordinate functions: $f(x, y, z) = x$

$$\mathbf{p} = (x, y, z)$$

$$\Delta_{\mathcal{M}} \mathbf{p} = \operatorname{div}_{\mathcal{M}} \nabla_{\mathcal{M}} \mathbf{p} = -2H \mathbf{n} \in \mathbb{R}^3$$

Diagram illustrating the components of the Laplace-Beltrami operator:

- Laplace-Beltrami: $\Delta_{\mathcal{M}}$
- function on surface M : \mathbf{p}
- gradient operator: $\nabla_{\mathcal{M}}$
- divergence operator: $\operatorname{div}_{\mathcal{M}}$
- mean curvature: H
- unit surface normal: \mathbf{n}

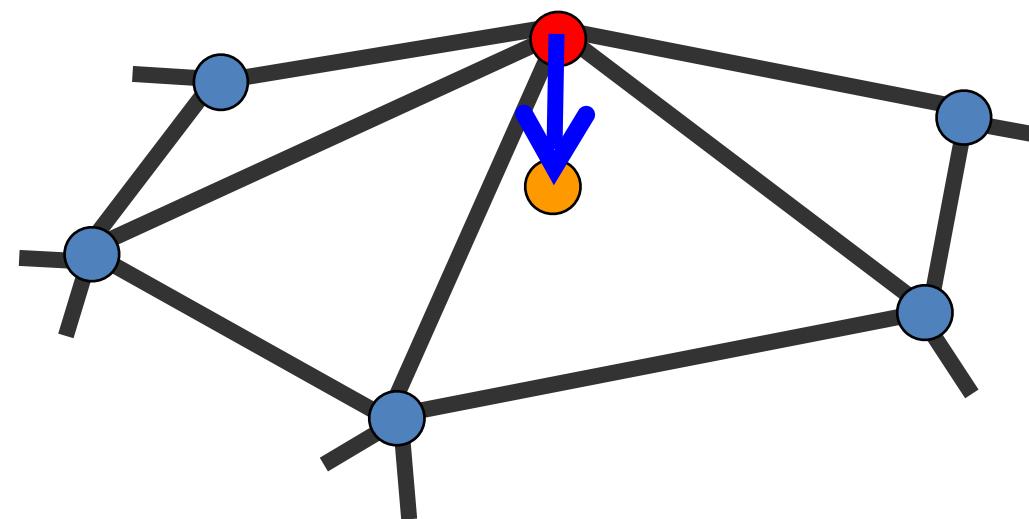
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graph TD; LB[\"Laplace-Beltrami\"] --> DeltaM["\u0394_M"]; fons["function on surface M"] --> DeltaM; grad["gradient operator"] --> gradOp["\u2207_M"]; div["divergence operator"] --> divOp["\u03c4div_M"]; H["mean curvature"] --> Hn["-2H\u2225"]; unit["unit surface normal"] --> Hn;
```

Differential Geometry on Meshes

- Assumption: meshes are piecewise linear approximations of smooth surfaces
- Can try fitting a smooth surface locally (say, a polynomial) and find differential quantities analytically
- But: it is often too slow for interactive settings

Discrete Differential Operators

- Approach: approximate differential properties at point \mathbf{v} as spatial average over local mesh neighborhood $N(\mathbf{v})$ where typically
 - \mathbf{v} = mesh vertex
 - $N_k(\mathbf{v})$ = k -ring neighborhood

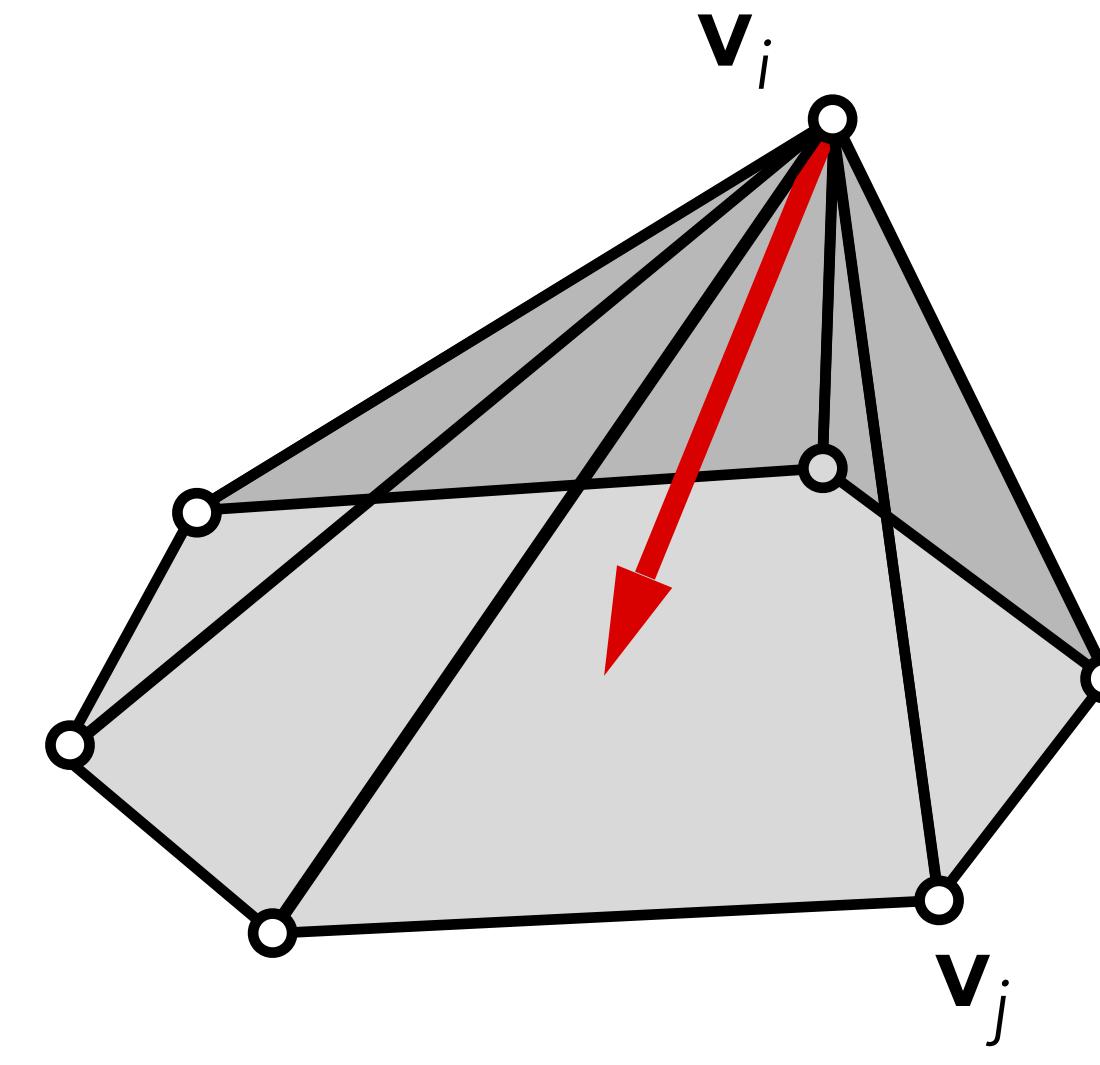


Discrete Laplace-Beltrami

$$\Delta_{\mathcal{M}} \mathbf{p} = -2H \mathbf{n}$$

- Uniform discretization:

$$L_u(\mathbf{v}_i) = \frac{1}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} (\mathbf{v}_j - \mathbf{v}_i) = \left(\frac{1}{|\mathcal{N}(i)|} \sum_{j \in \mathcal{N}(i)} \mathbf{v}_j \right) - \mathbf{v}_i$$

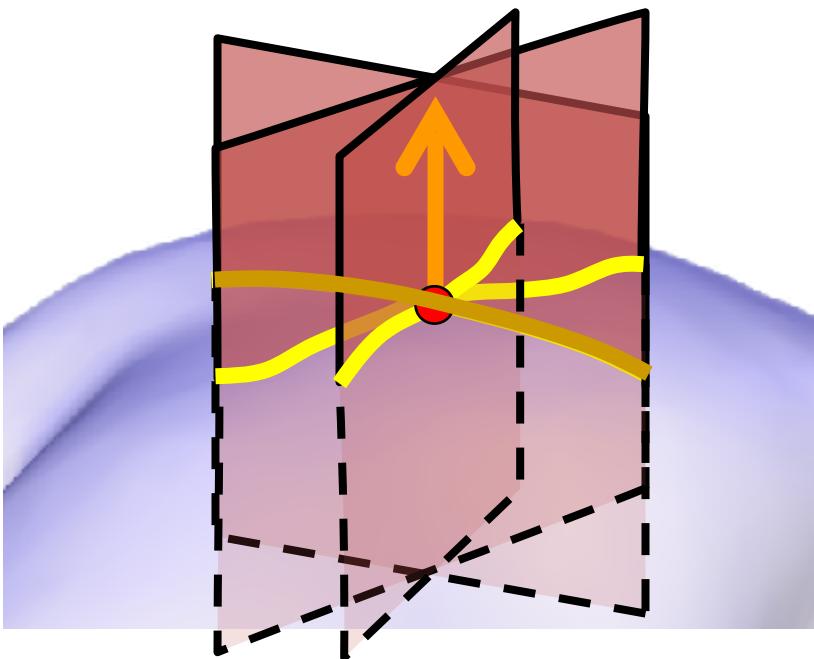


- Depends only on connectivity
= simple and efficient
- Bad approximation for
irregular triangulations

Discrete Laplace-Beltrami

$$\Delta_{\mathcal{M}} \mathbf{p} = -2H \mathbf{n}$$

- Intuition for uniform discretization



$$H = \frac{1}{2\pi} \int_0^{2\pi} \kappa(\varphi) d\varphi$$

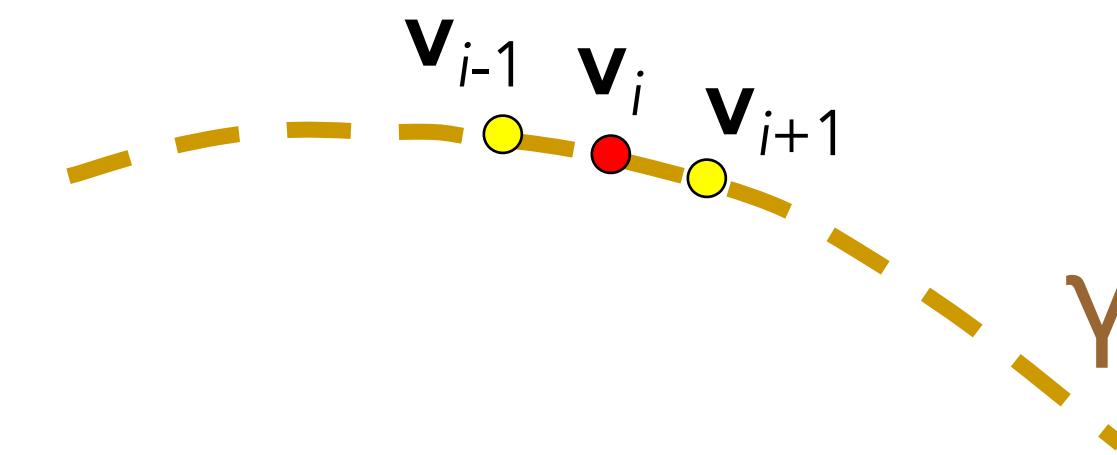
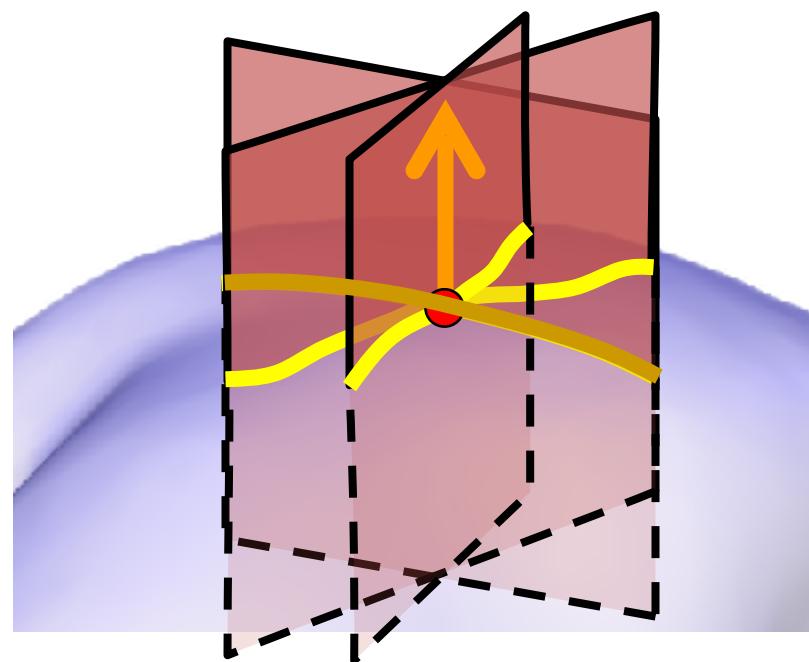
$$\kappa \mathbf{n} = \gamma''$$

$$-2H \mathbf{n} = -2 \left(\frac{1}{2\pi} \int_0^{2\pi} \kappa(\varphi) d\varphi \right) \mathbf{n} = -\frac{1}{\pi} \int_0^{2\pi} \kappa(\varphi) \mathbf{n} d\varphi = -\frac{1}{\pi} \int_0^{2\pi} \gamma'' d\varphi$$

Discrete Laplace-Beltrami

$$\Delta_{\mathcal{M}} \mathbf{p} = -2H \mathbf{n}$$

- Intuition for uniform discretization



$$H = \frac{1}{2\pi} \int_0^{2\pi} \kappa(\varphi) d\varphi$$

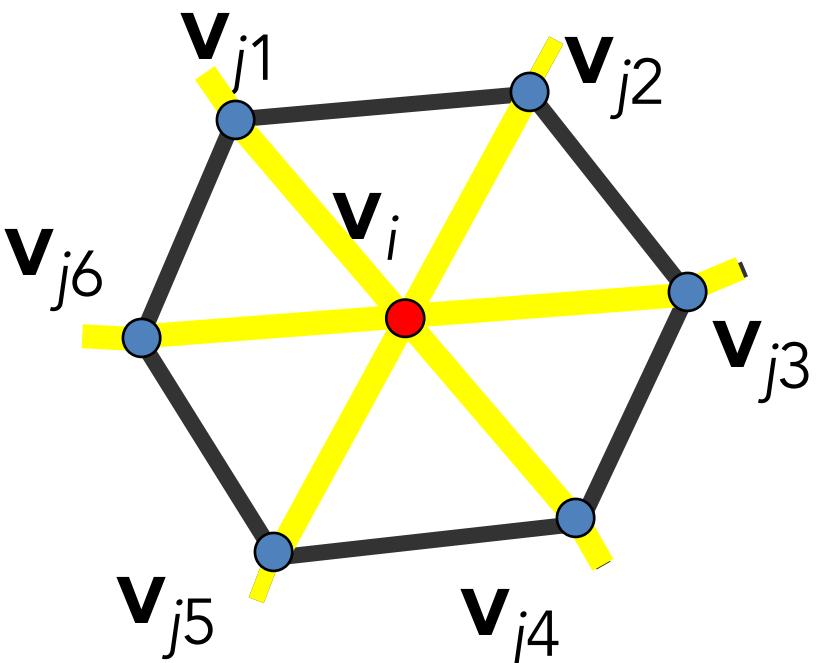
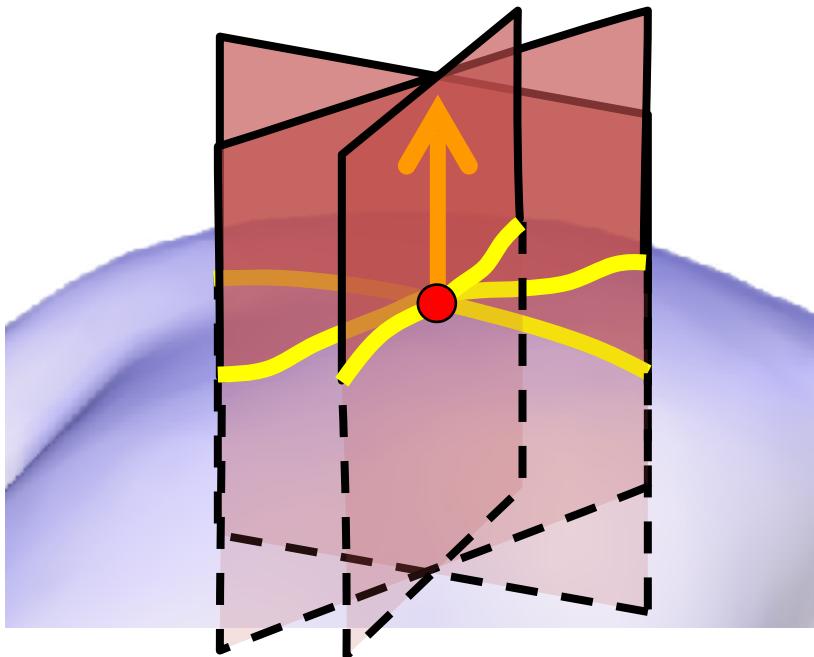
$$\kappa \mathbf{n} = \gamma''$$

$$\gamma'' \approx \frac{1}{h} \left(\frac{\mathbf{v}_{i+1} - \mathbf{v}_i}{h} - \frac{\mathbf{v}_i - \mathbf{v}_{i-1}}{h} \right) = -\frac{2}{h^2} \left(\frac{1}{2} (\mathbf{v}_{i-1} + \mathbf{v}_{i+1}) - \mathbf{v}_i \right)$$

Discrete Laplace-Beltrami

$$\Delta_{\mathcal{M}} \mathbf{p} = -2H \mathbf{n}$$

$$\gamma'' \approx \frac{1}{h} \left(\frac{\mathbf{v}_{i+1} - \mathbf{v}_i}{h} - \frac{\mathbf{v}_i - \mathbf{v}_{i-1}}{h} \right) = -\frac{2}{h^2} \left(\frac{1}{2}(\mathbf{v}_{i-1} + \mathbf{v}_{i+1}) - \mathbf{v}_i \right)$$



$$H = \frac{1}{2\pi} \int_0^{2\pi} \kappa(\varphi) d\varphi$$

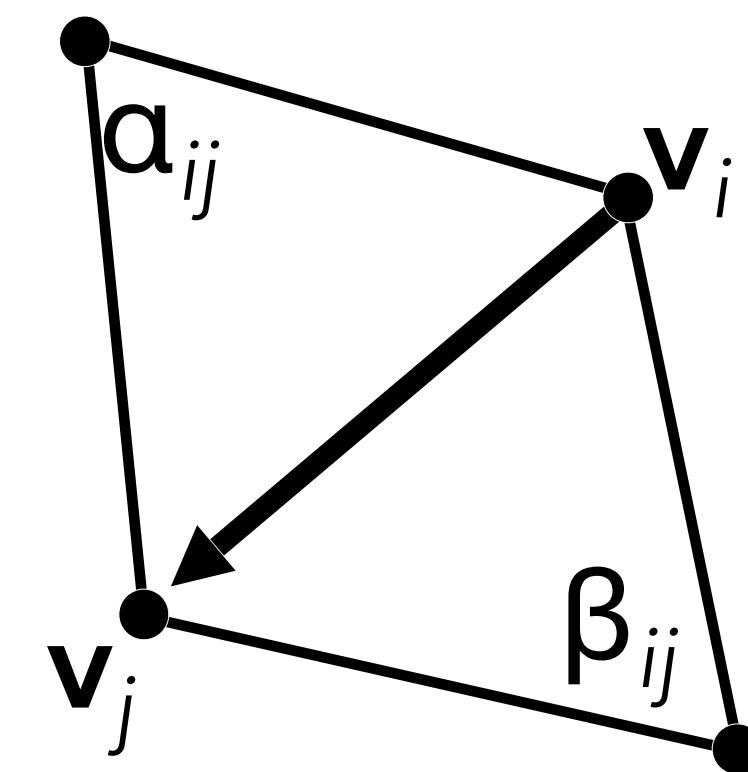
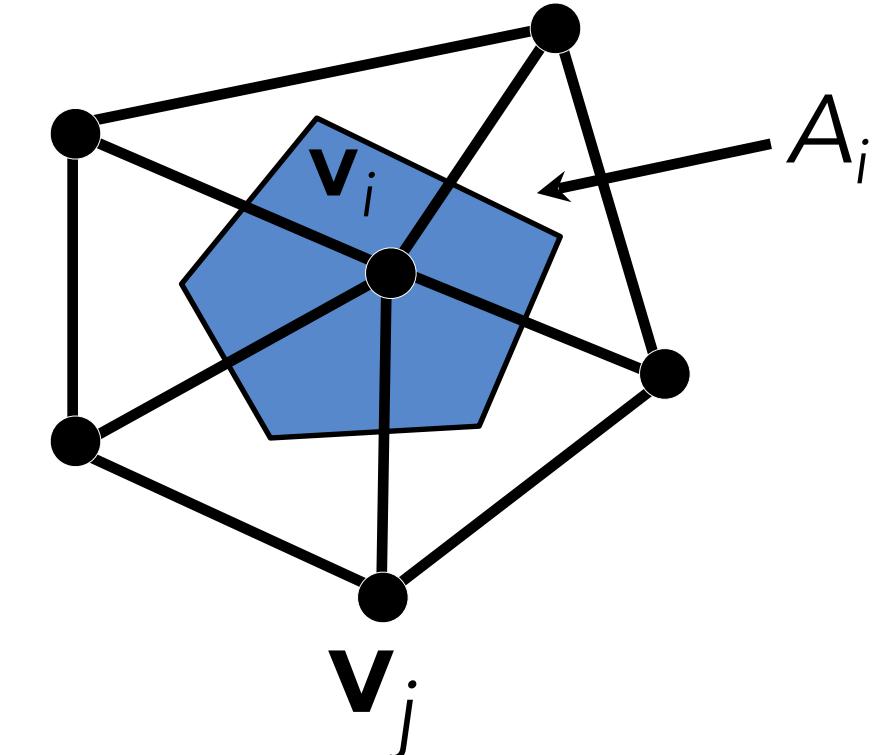
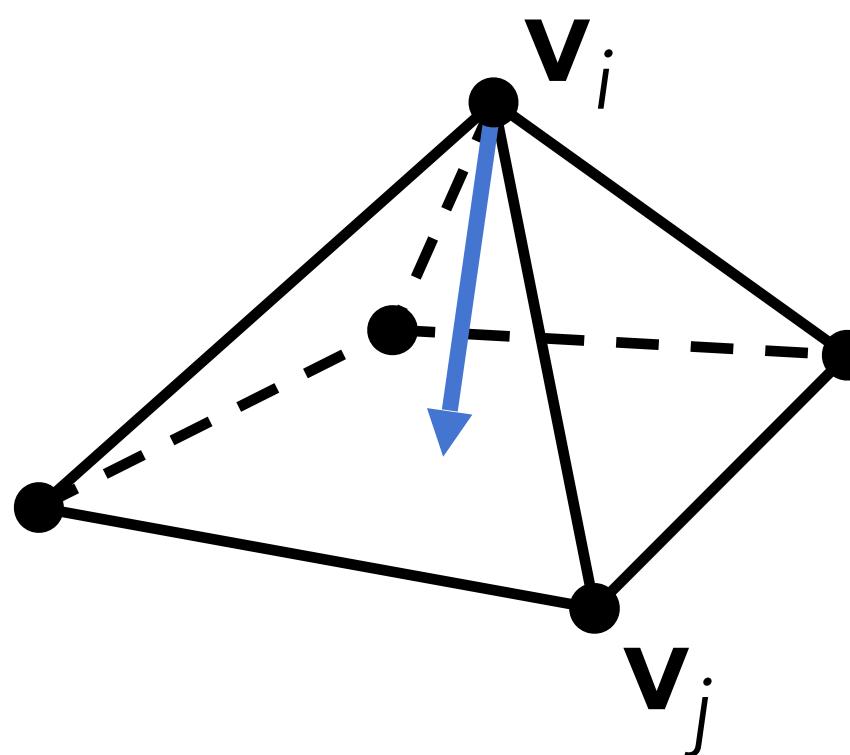
$$\begin{aligned} & \frac{1}{2}(\mathbf{v}_{j_1} + \mathbf{v}_{j_4}) - \mathbf{v}_i \quad + \\ & \frac{1}{2}(\mathbf{v}_{j_2} + \mathbf{v}_{j_5}) - \mathbf{v}_i \quad + \\ & \frac{1}{2}(\mathbf{v}_{j_3} + \mathbf{v}_{j_6}) - \mathbf{v}_i = \frac{1}{2} \sum_{j \in \mathcal{N}(i)} \mathbf{v}_j - 3\mathbf{v}_i = 3 \left(\frac{1}{6} \sum_{j \in \mathcal{N}(i)} \mathbf{v}_j - \mathbf{v}_i \right) \end{aligned}$$

$L_u(\mathbf{v}_i)$

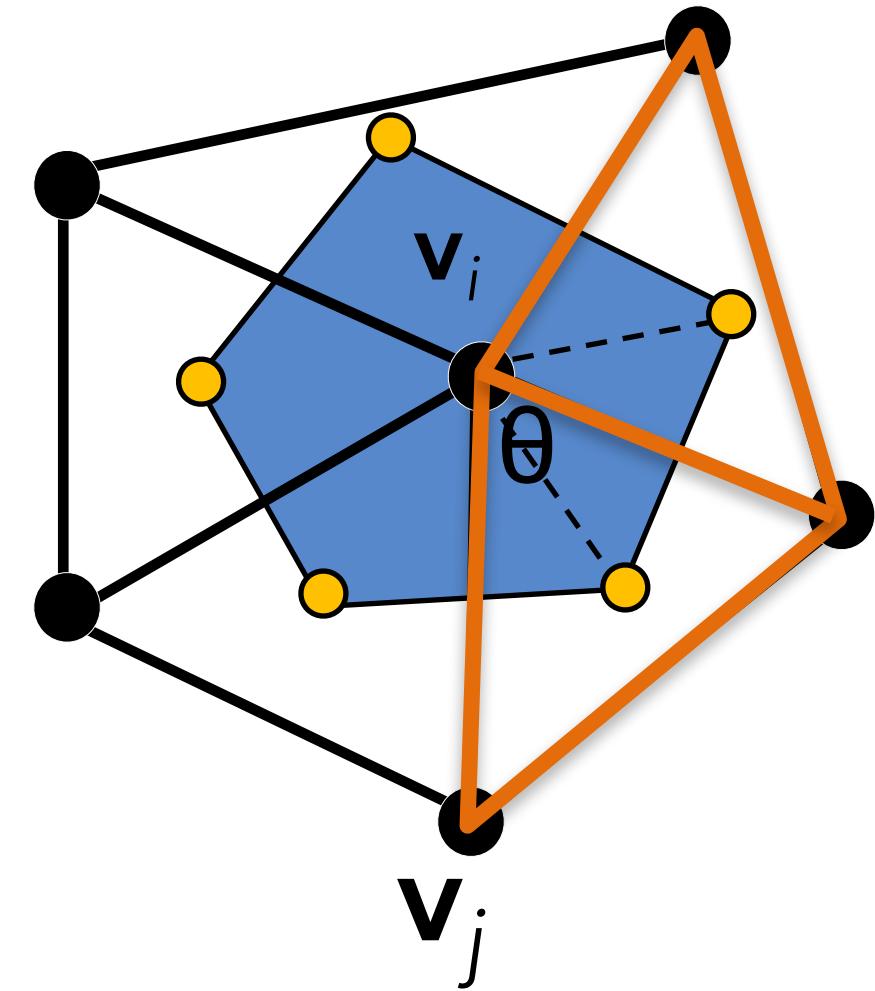
Discrete Laplace-Beltrami

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) (\mathbf{v}_j - \mathbf{v}_i)$$

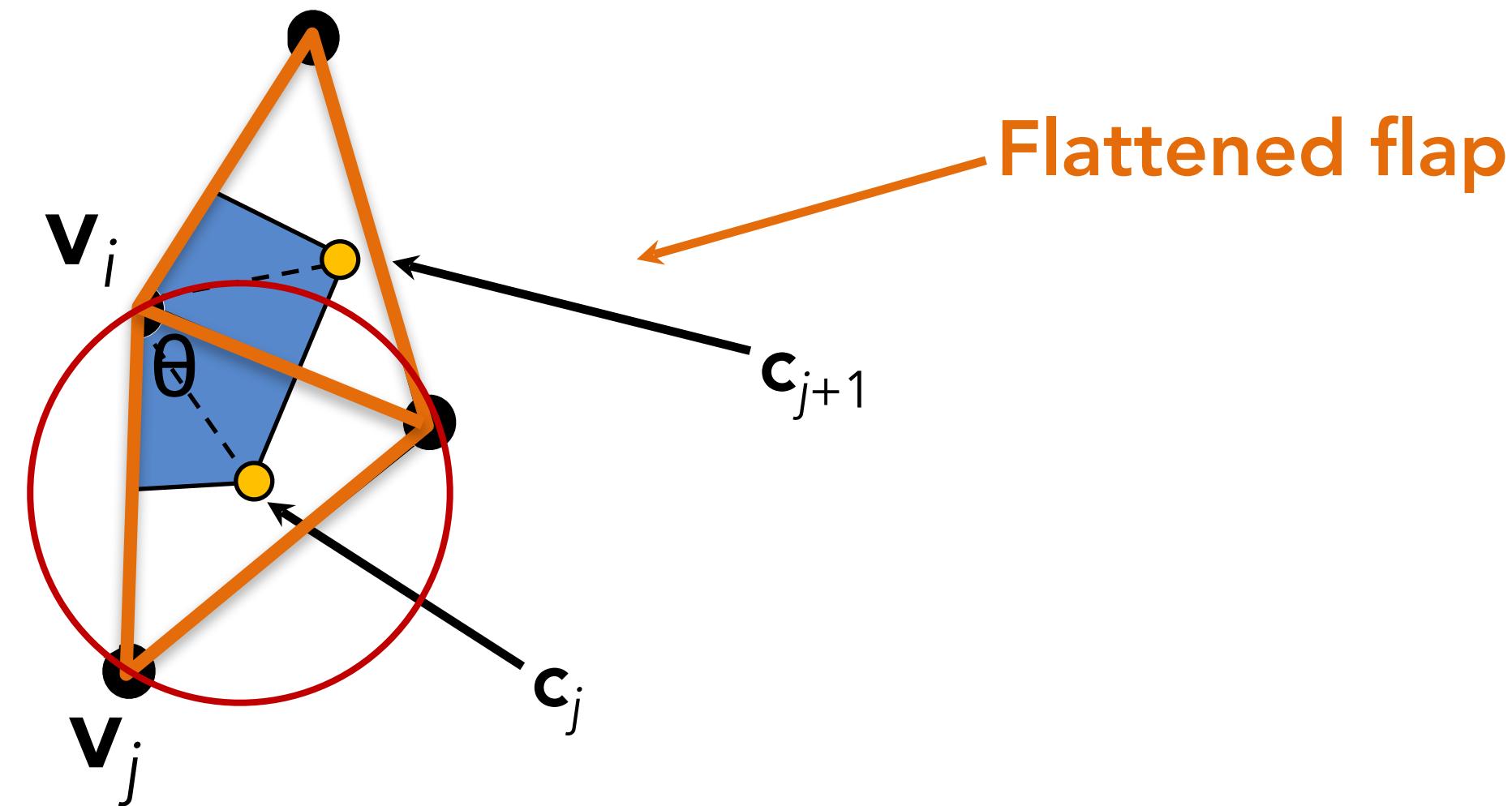


Voronoi Vertex Area



- Unfold the triangle flap onto the plane (without distortion)

Voronoi Vertex Area



$$\mathbf{c}_j = \begin{cases} \text{circumcenter of } \triangle(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_{j+1}) & \text{if } \theta < \pi/2 \\ \text{midpoint of edge } (\mathbf{v}_j, \mathbf{v}_{j+1}) & \text{if } \theta \geq \pi/2 \end{cases}$$

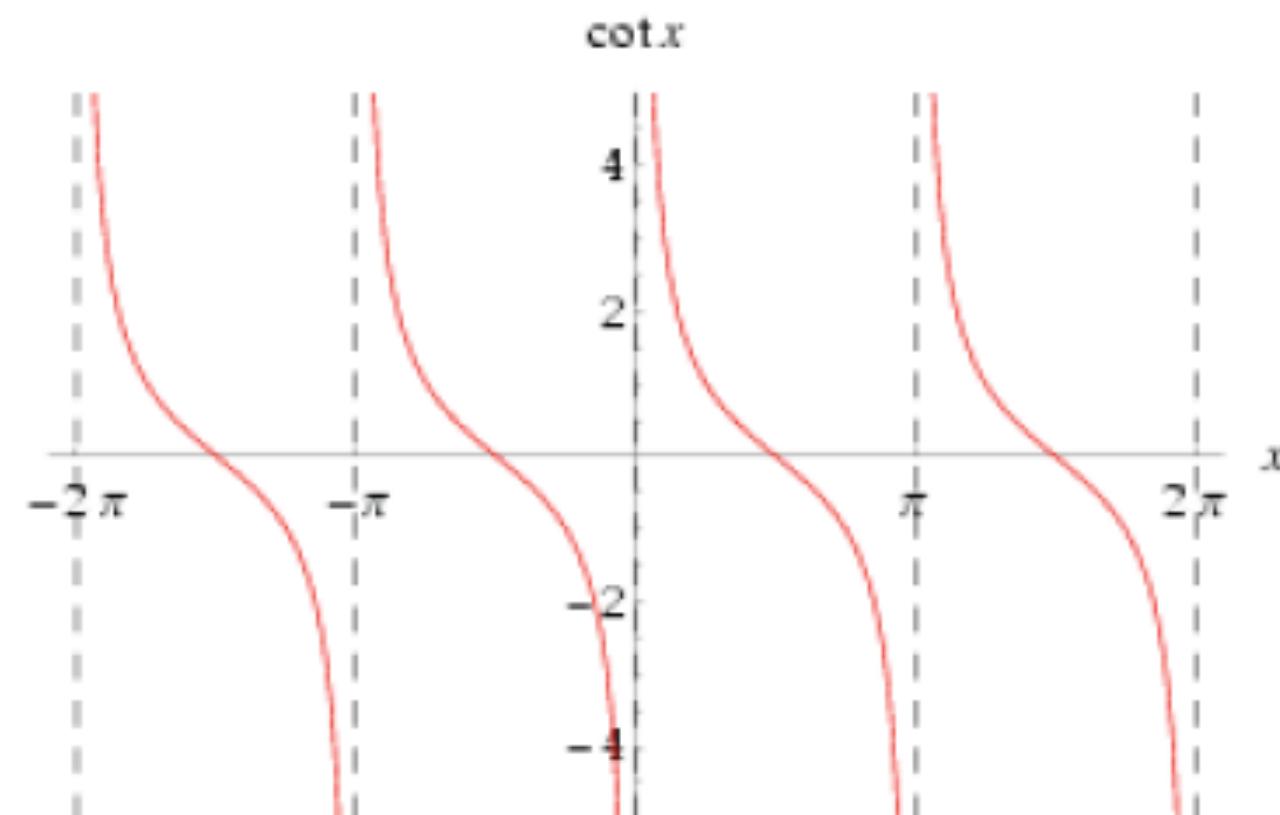
$$A_i = \sum_j \text{Area}(\triangle(\mathbf{v}_i, \mathbf{c}_j, \mathbf{c}_{j+1}))$$

Discrete Laplace-Beltrami

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) (\mathbf{v}_j - \mathbf{v}_i)$$

- Accounts for mesh geometry
- Potentially negative/infinite weights



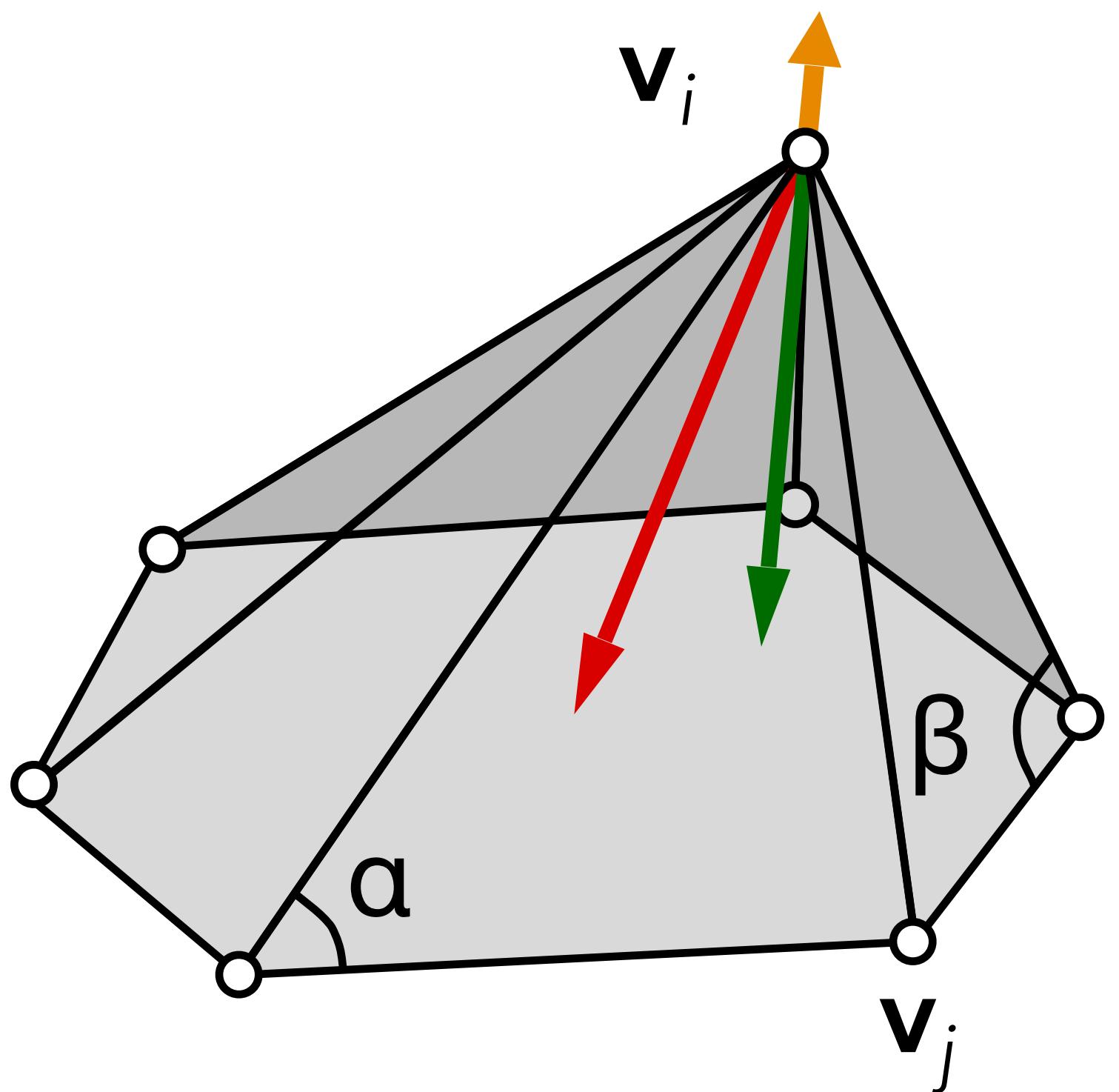
Discrete Laplace-Beltrami

- Cotangent formula

$$L_c(\mathbf{v}_i) = \frac{1}{A_i} \sum_{j \in \mathcal{N}(i)} \frac{1}{2} (\cot \alpha_{ij} + \cot \beta_{ij}) (\mathbf{v}_j - \mathbf{v}_i)$$

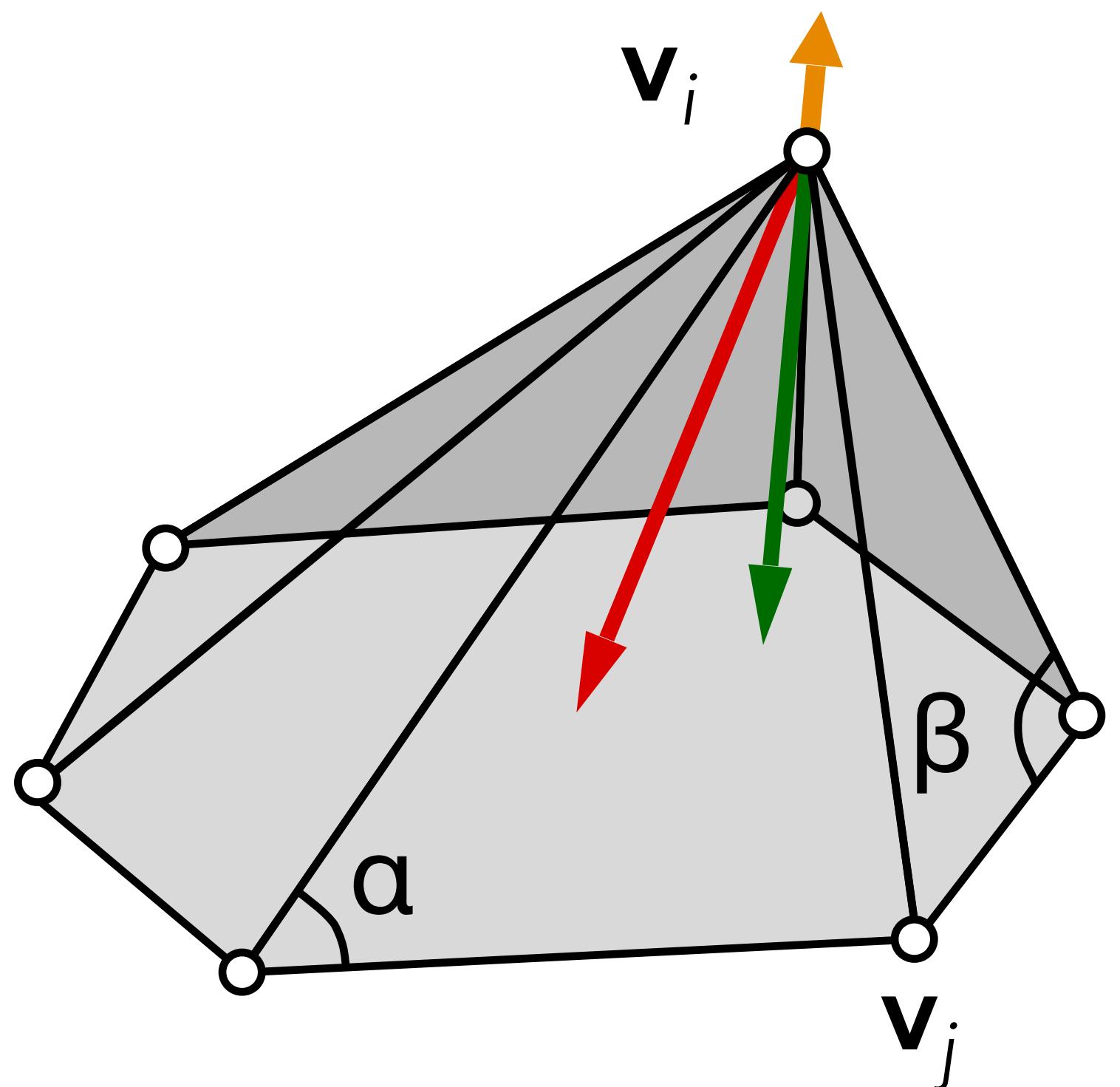
- Can be derived using linear finite elements
- Nice property: gives zero for planar 1-rings!

Discrete Laplace-Beltrami



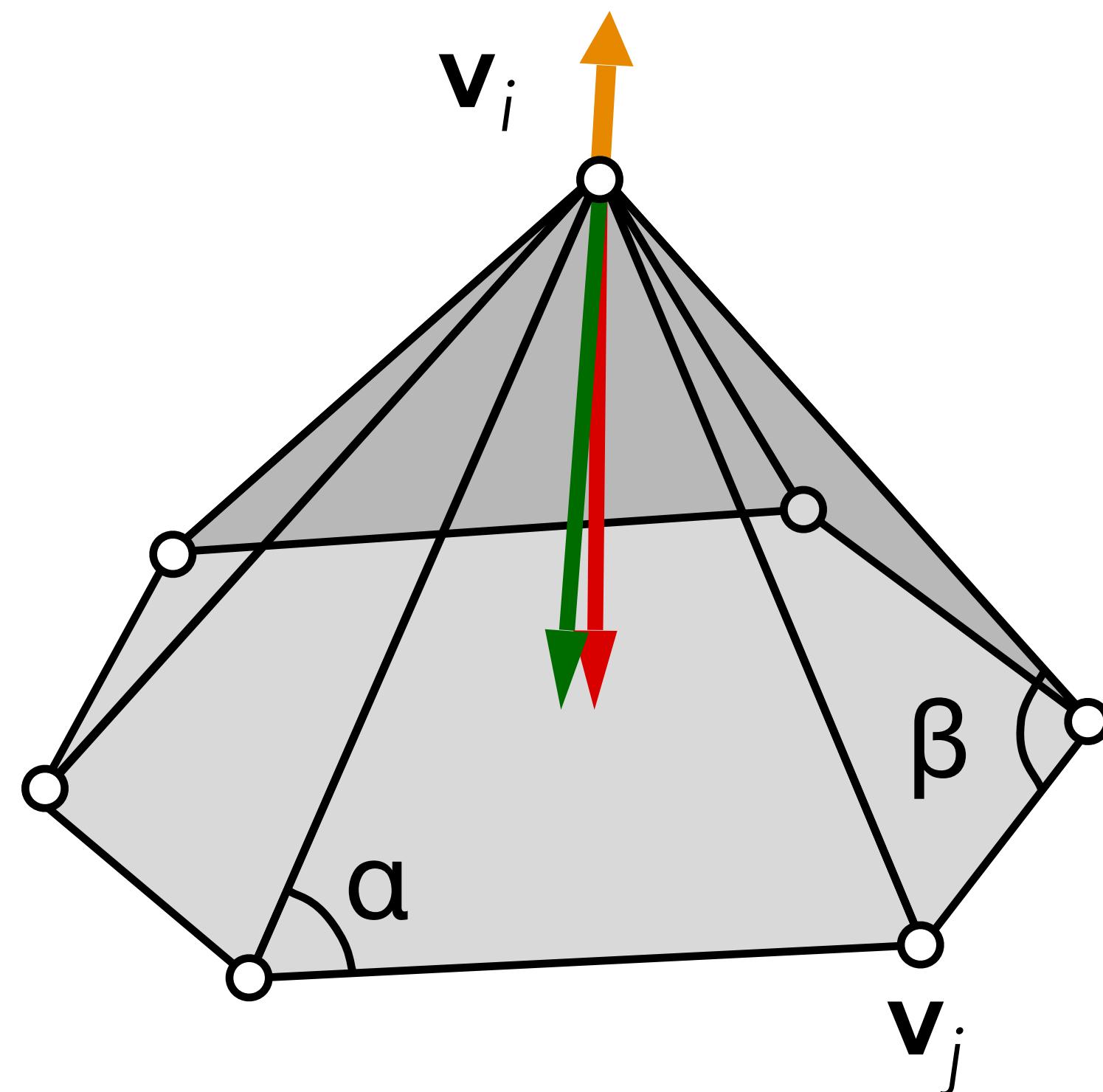
- Uniform Laplacian $L_u(v_i)$
- Cotangent Laplacian $L_c(v_i)$
- Normal

Discrete Laplace-Beltrami



- Uniform Laplacian $L_u(v_i)$
- Cotangent Laplacian $L_c(v_i)$
- Normal
- For nearly equal edge lengths
Uniform \approx Cotangent

Discrete Laplace-Beltrami



- Uniform Laplacian $L_u(v_i)$
- Cotangent Laplacian $L_c(v_i)$
- Normal
- For nearly equal edge lengths
Uniform \approx Cotangent

Cotan Laplacian allows computing discrete normal

Discrete Curvatures

- Mean curvature (sign defined according to normal)

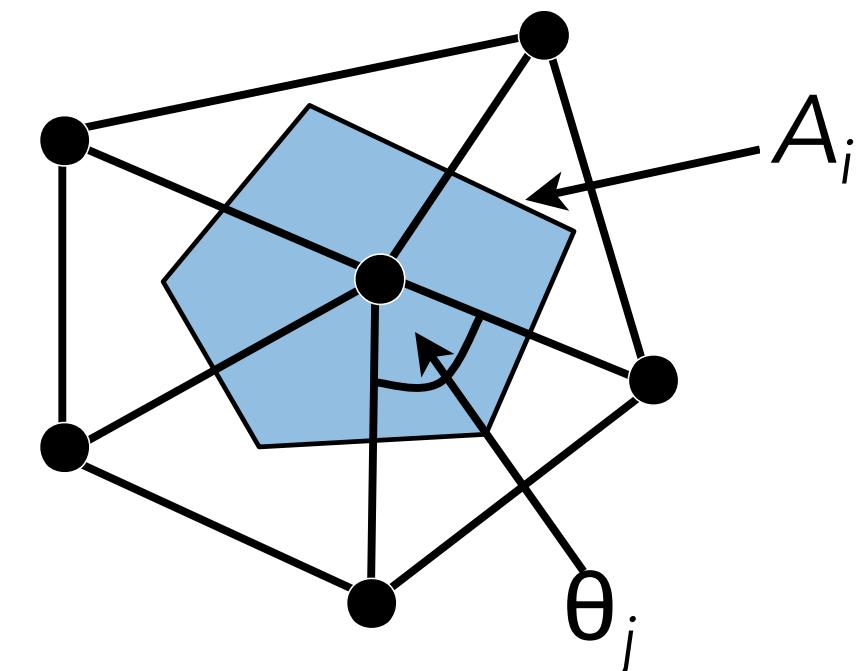
$$|H(\mathbf{v}_i)| = \|L_c(\mathbf{v}_i)\|/2$$

- Gaussian curvature

$$K(\mathbf{v}_i) = \frac{1}{A_i} (2\pi - \sum_j \theta_j)$$

- Principal curvatures

$$\kappa_1 = H - \sqrt{H^2 - K} \quad \kappa_2 = H + \sqrt{H^2 - K}$$

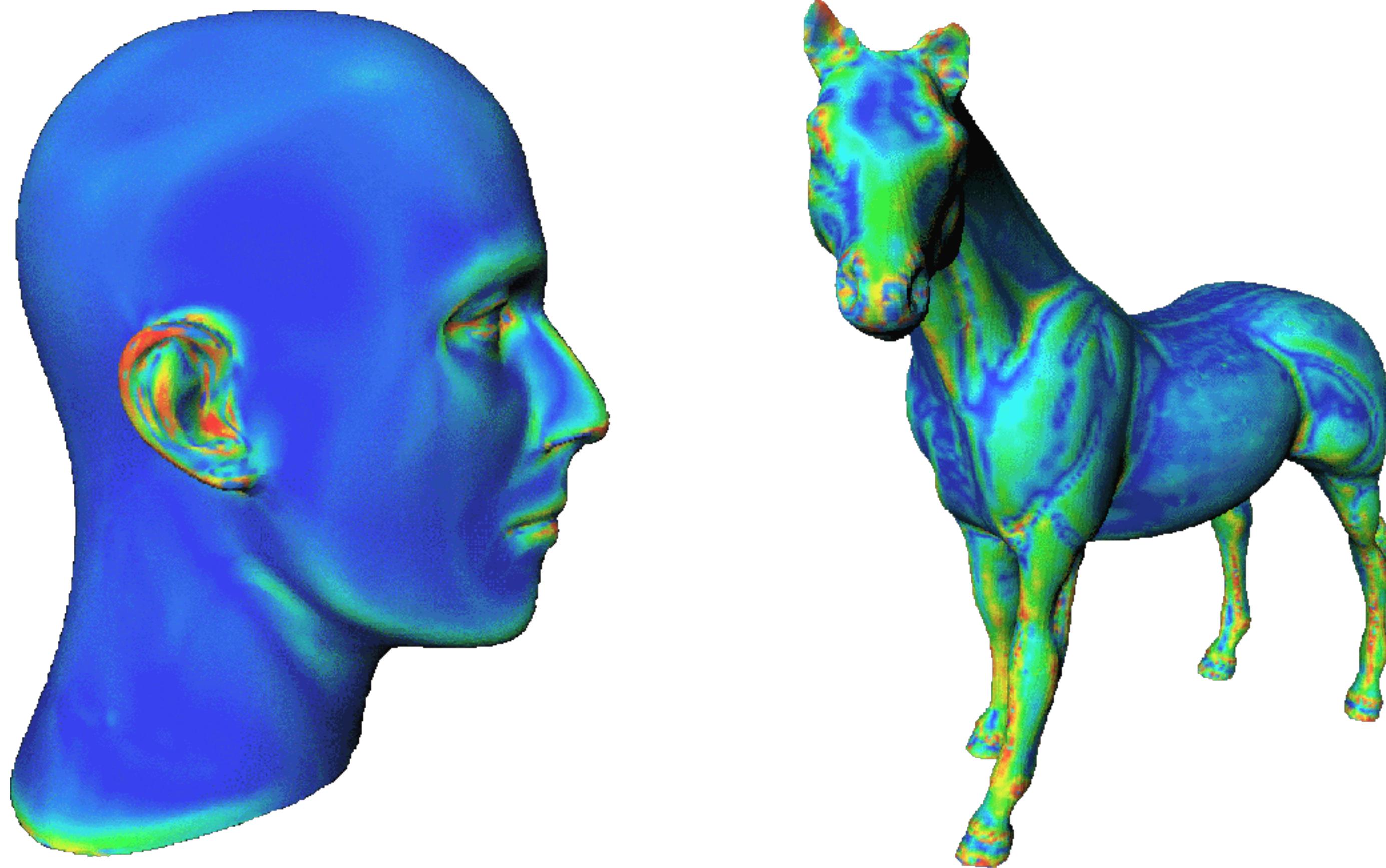


Discrete Gauss-Bonnet Theorem

- Total Gaussian curvature is fixed for a given topology

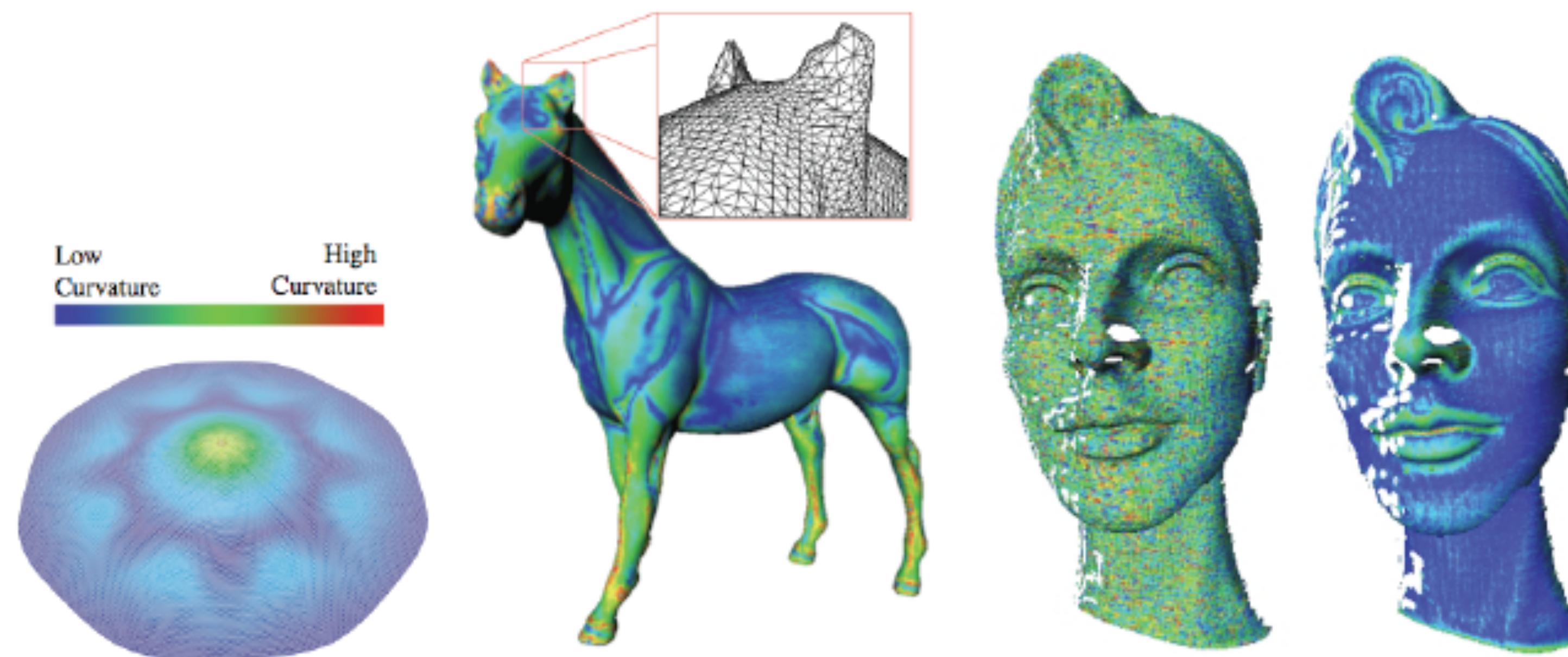
$$\int_{\mathcal{M}} K dA = \sum_i A_i K(\mathbf{v}_i) = \sum_i \left[2\pi - \sum_{j \in \mathcal{N}(i)} \theta_j \right] == 2\pi\chi(\mathcal{M})$$

Example: Discrete Mean Curvature



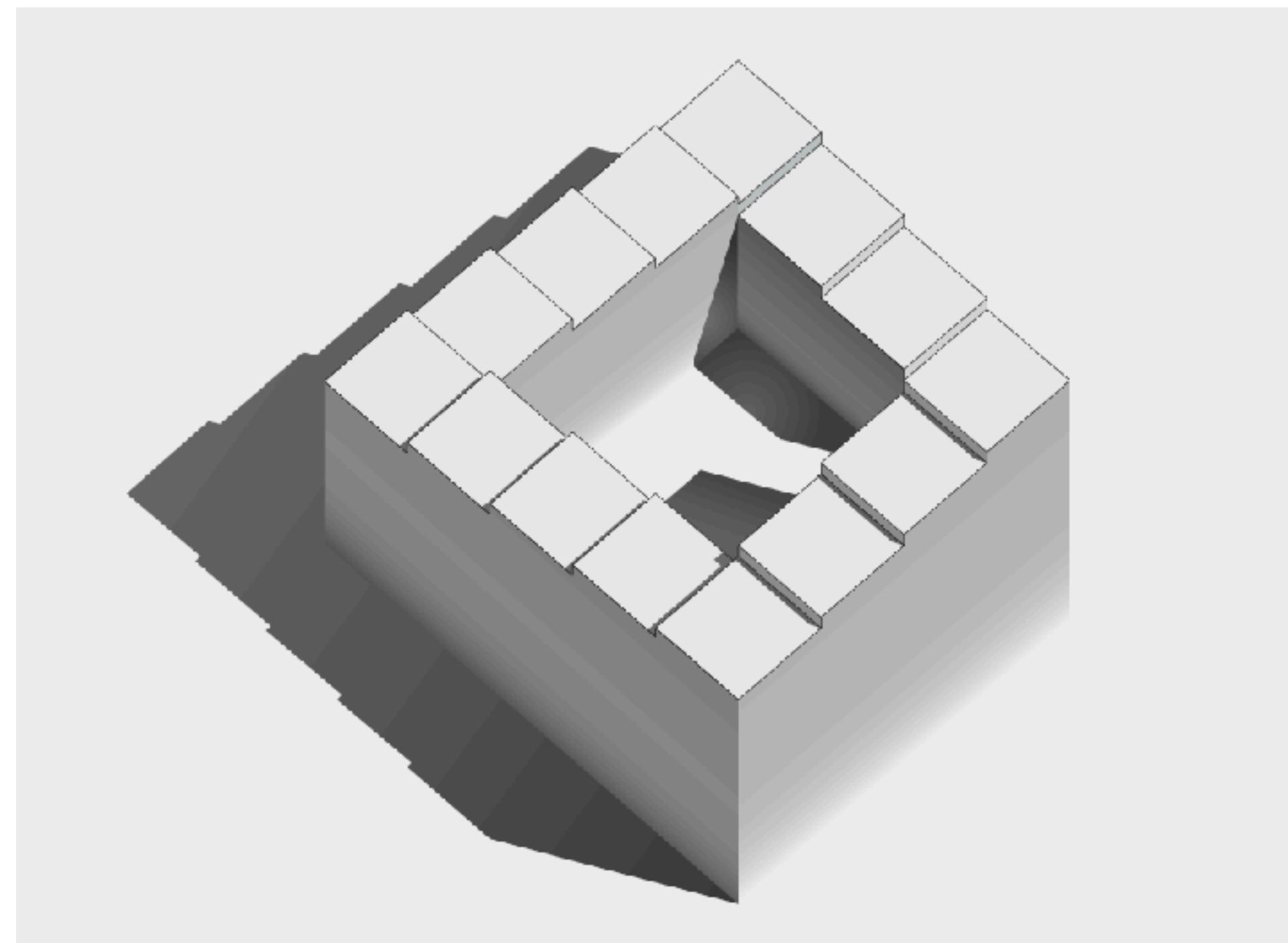
References

- M. Meyer, M. Desbrun, P. Schroeder, A. Barr
Discrete Differential-Geometry Operators for Triangulated 2-Manifolds, VisMath, 2002



References

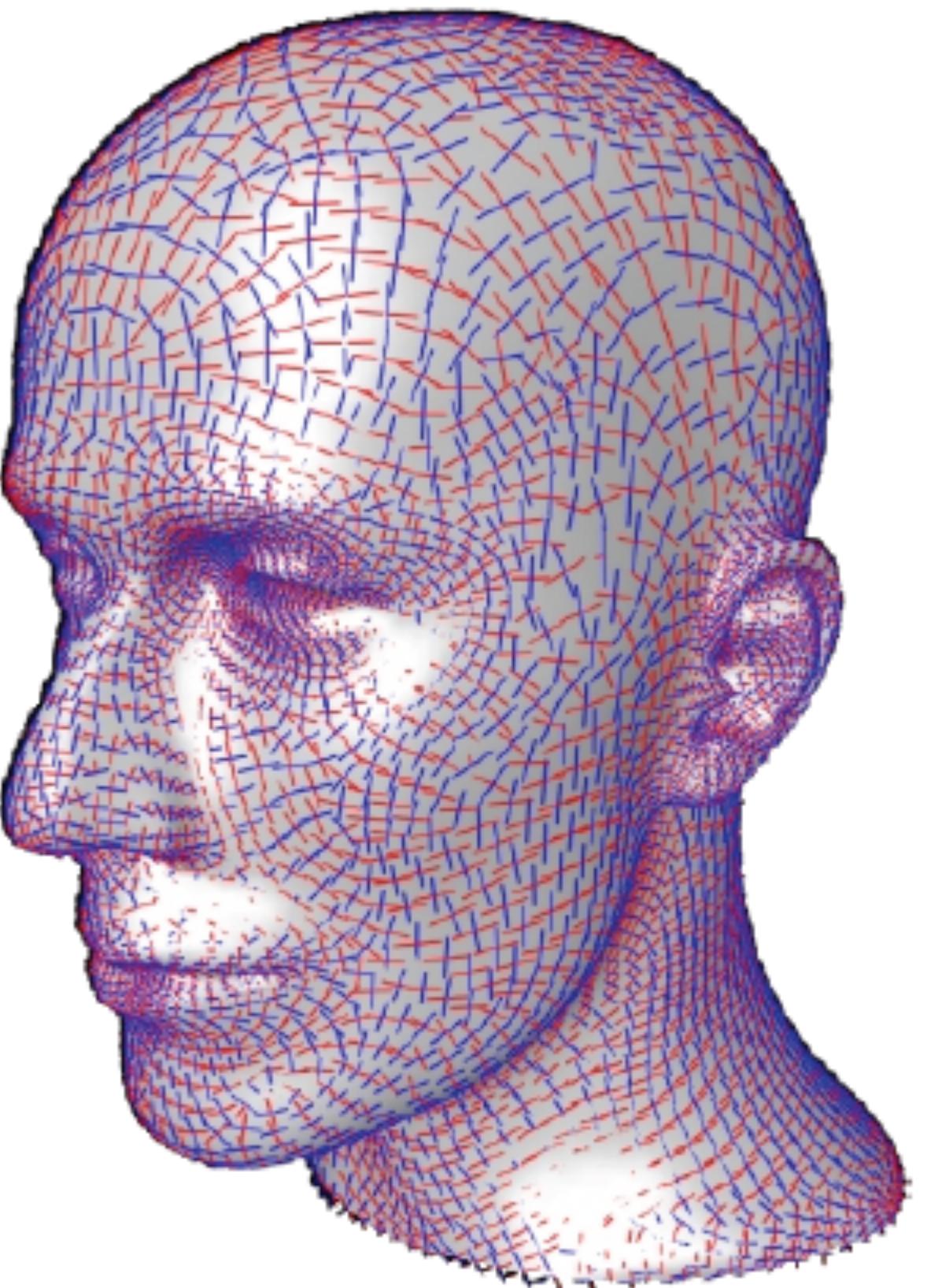
Discrete Differential Geometry: An Applied Introduction
Keenan Crane



<https://www.cs.cmu.edu/~kmcrane/Projects/DGPDEC/>

References

- libigl implements many discrete differential operators
- <https://libigl.github.io/libigl-python-bindings/tut-chapter1/>



Principal Directions