Meshes

Acknowledgements: Olga Sorkine-Hornung and Daniele Panozzo

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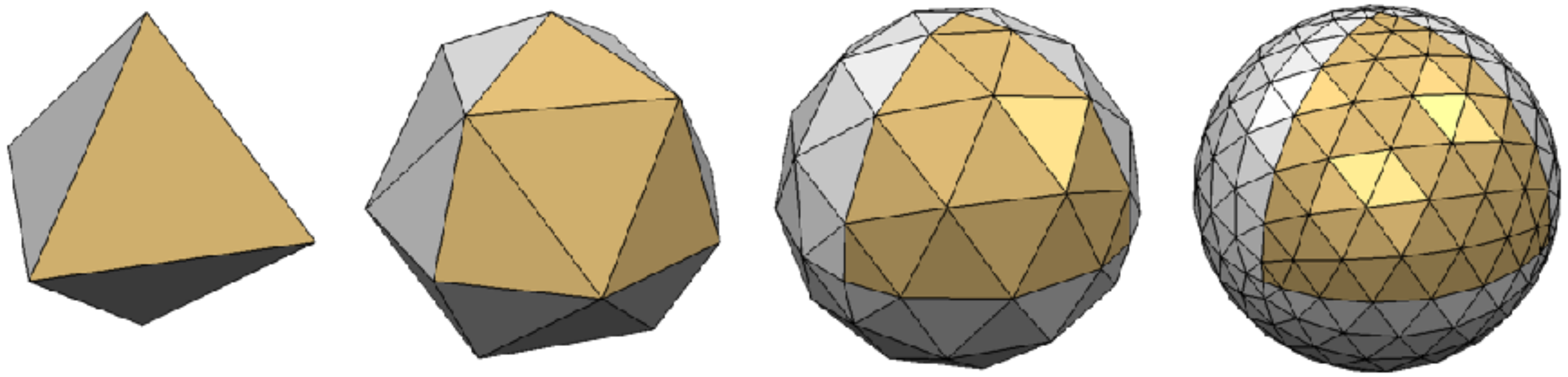
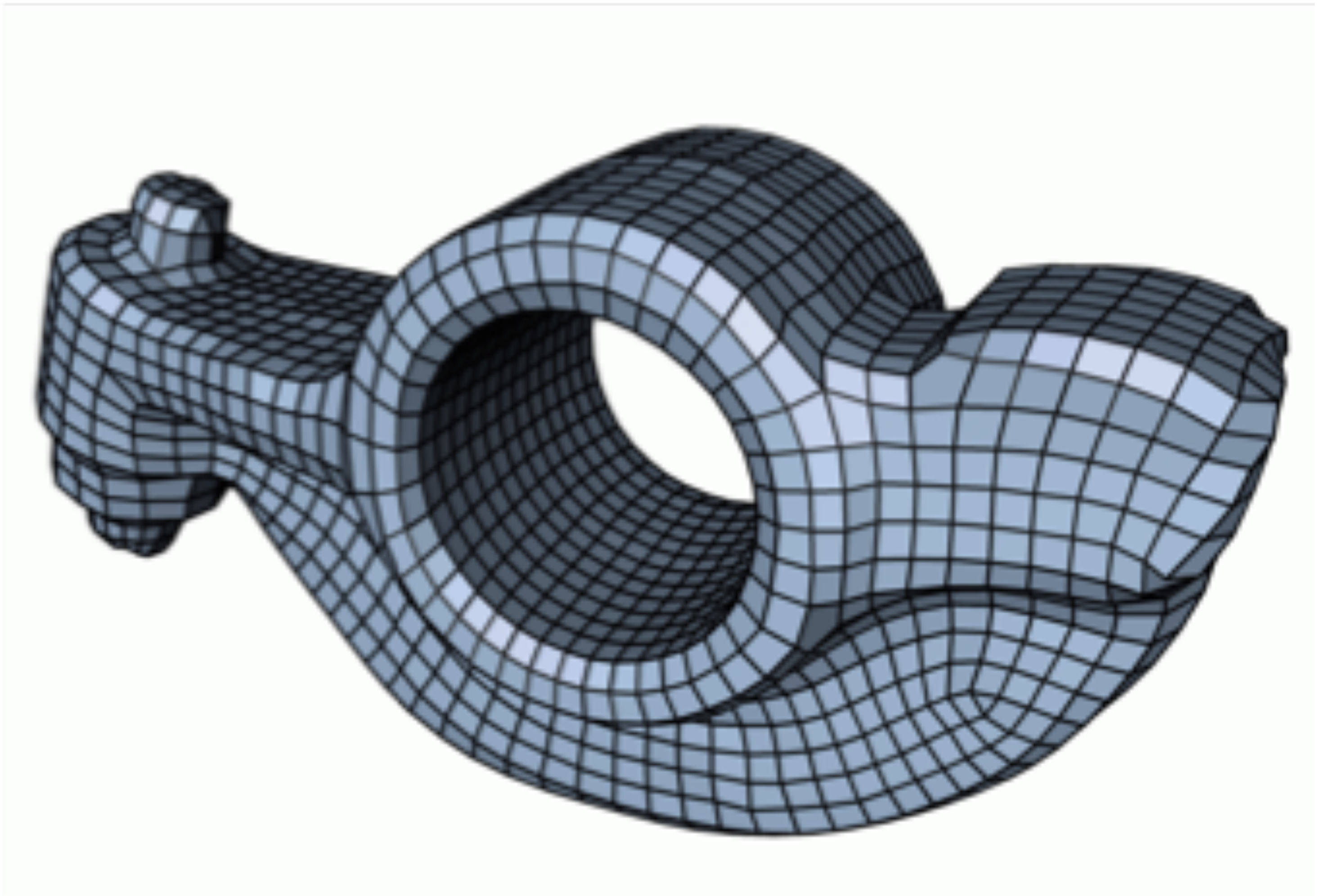
Libigl Tutorial

https://libigl.github.io/libigl-python-bindings/tut-chapter0/

Polygonal Meshes

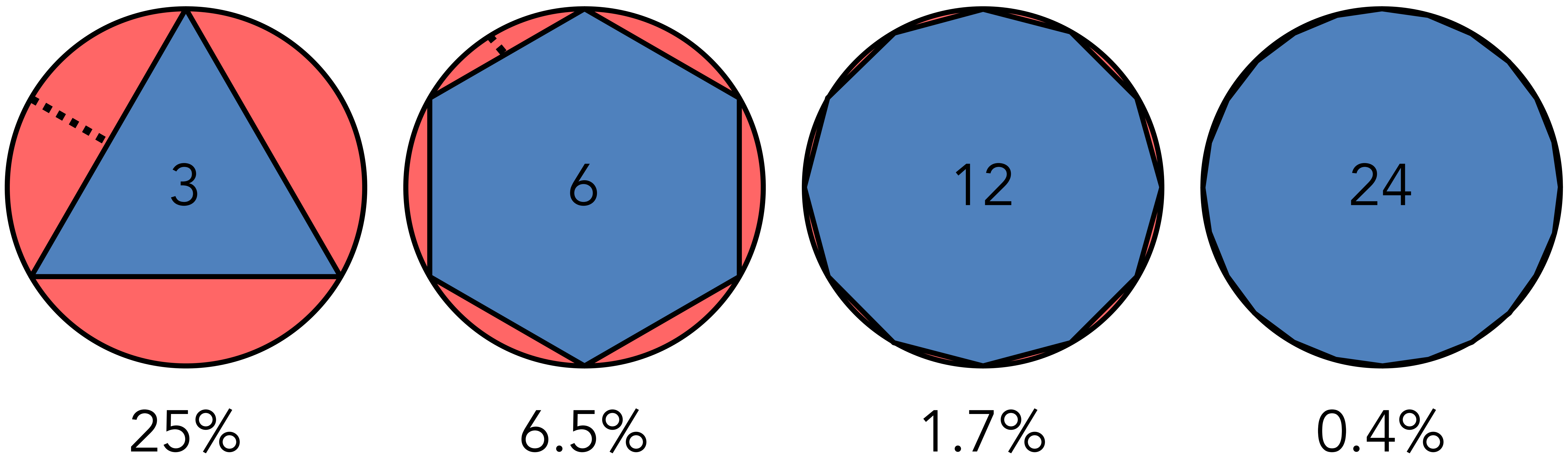
- Boundary representations of objects





Meshes as Approximations of Smooth Surfaces

- Piecewise linear approximation
  - Error is $O(h^2)$



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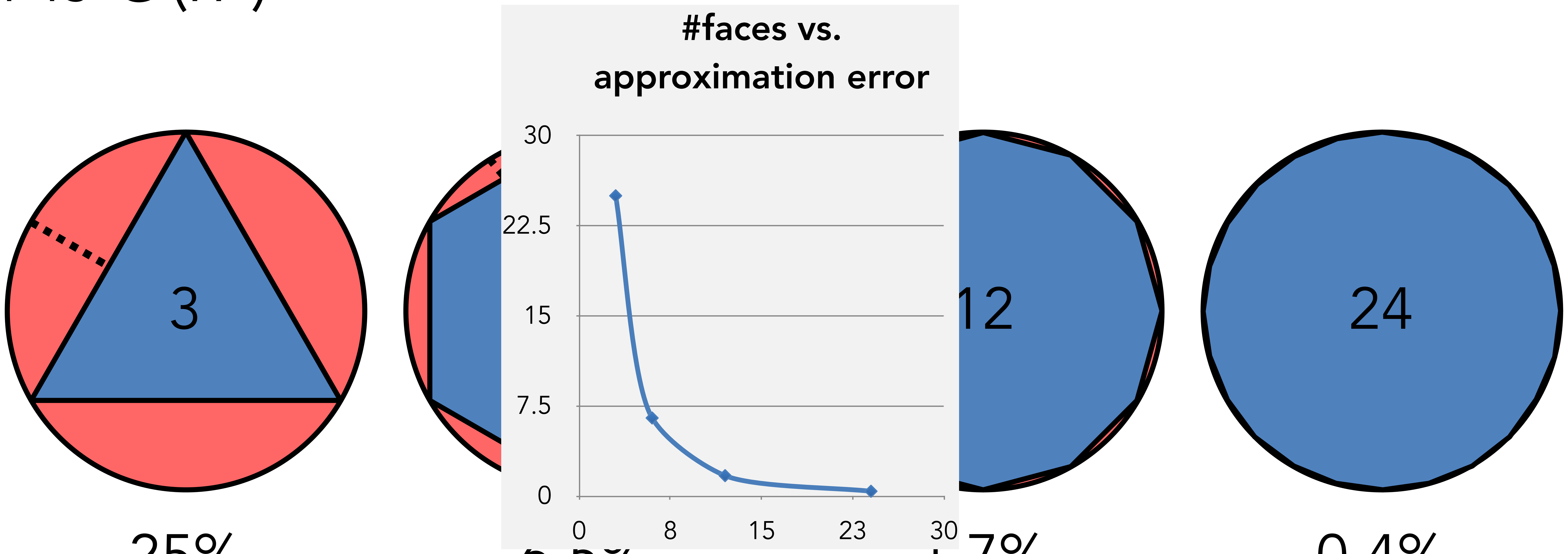

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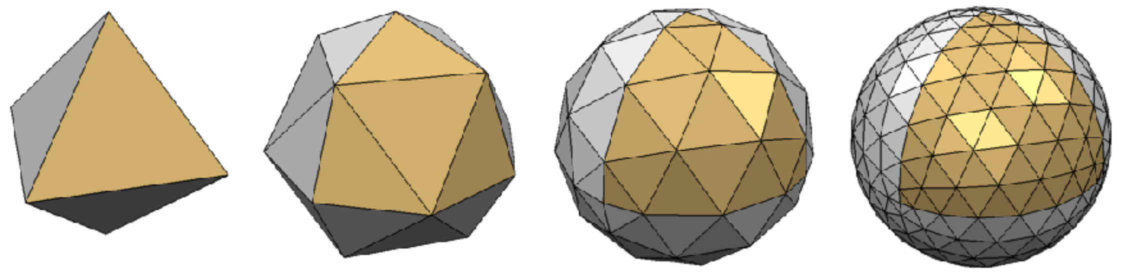
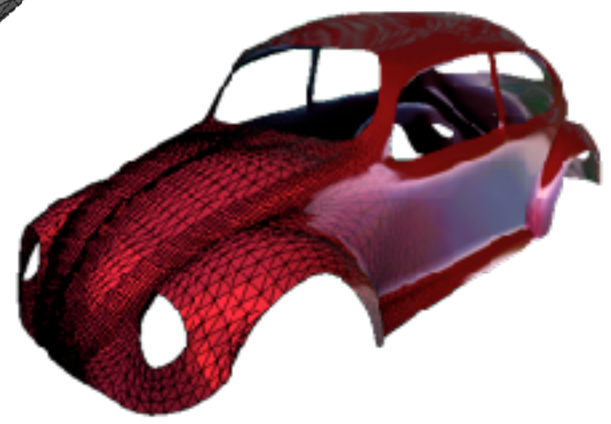
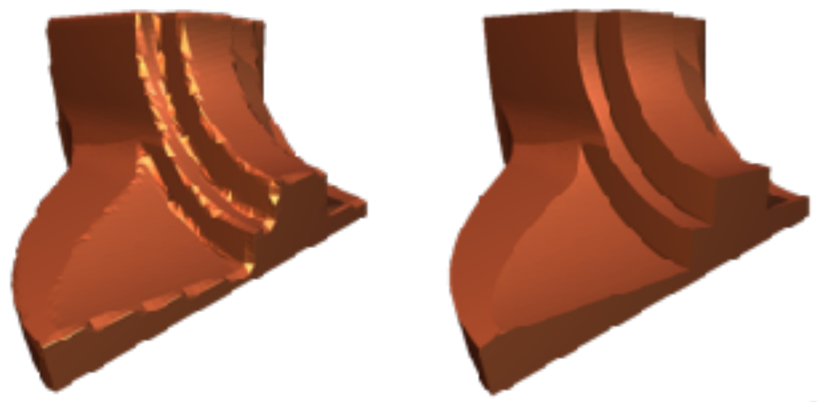
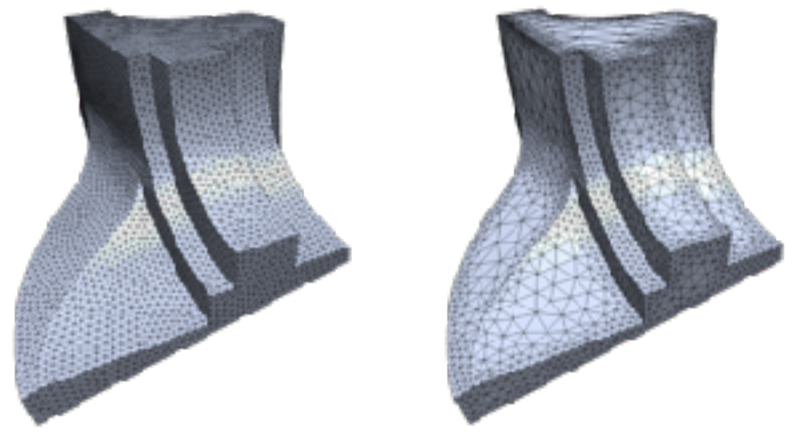
Polygonal Meshes

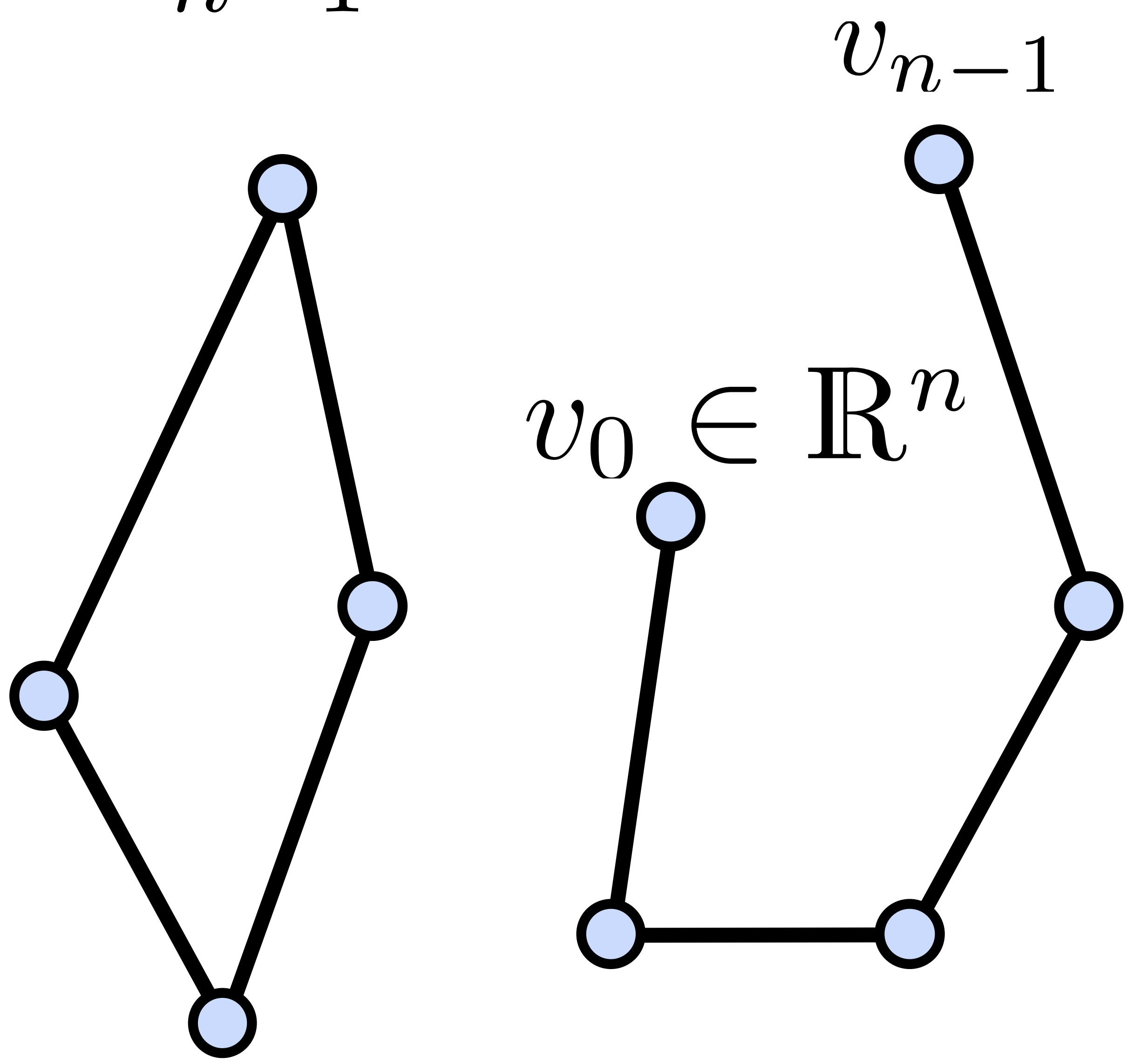
- Polygonal meshes are a good representation
  - approximation $O(h^2)$
  - arbitrary topology
  - piecewise smooth surfaces
  - adaptive refinement
  - efficient rendering


Polygon

- Vertices: $v_0, v_1, \ldots, v_{n-1}$
- Edges: $\{(v_0, v_1), \ldots, (v_{n-2}, v_{n-1})\}$

- Closed: $v_0 = v_{n-1}$
- Planar: all vertices on a plane
- Simple: not self-intersecting



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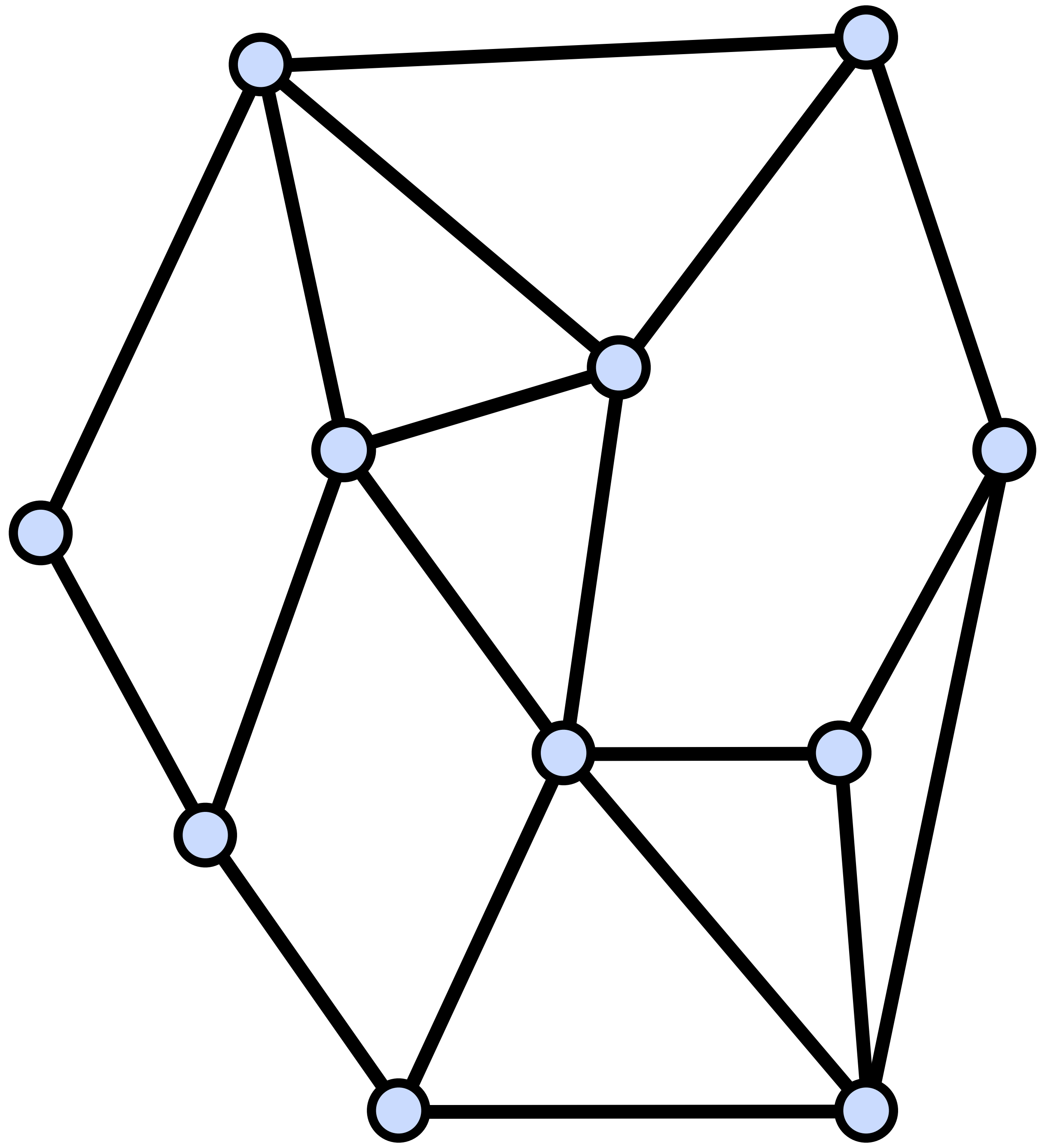
Polygonal Mesh



- A finite set M of closed, simple polygons Q_i is a polygonal mesh
- The intersection of two polygons in M is either empty, a vertex, or an edge

$$M = \langle \underbrace{V}_{\text{vertices}}, \underbrace{E}_{\text{edges}}, \underbrace{F}_{\text{faces}} \rangle$$

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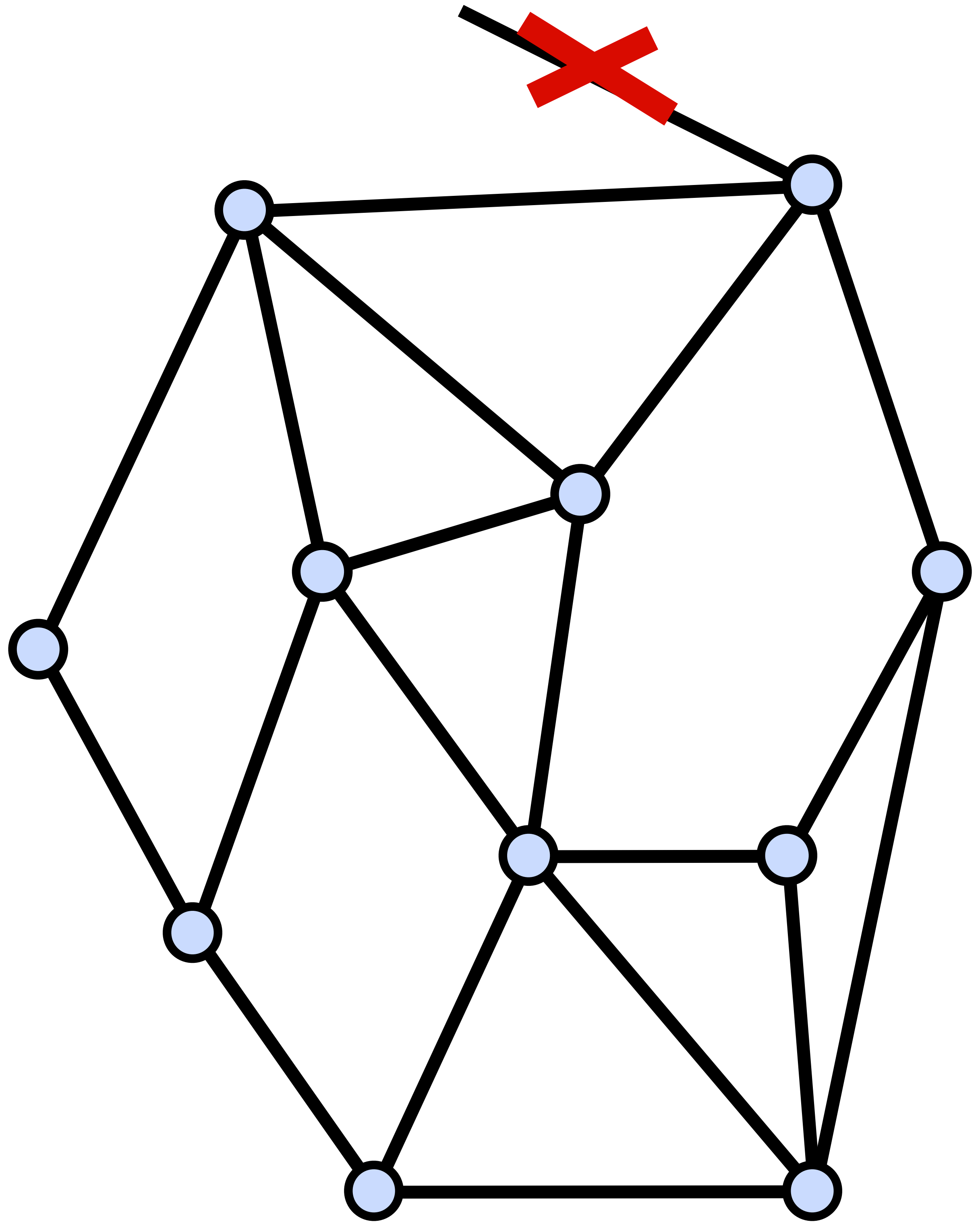
Polygonal Mesh



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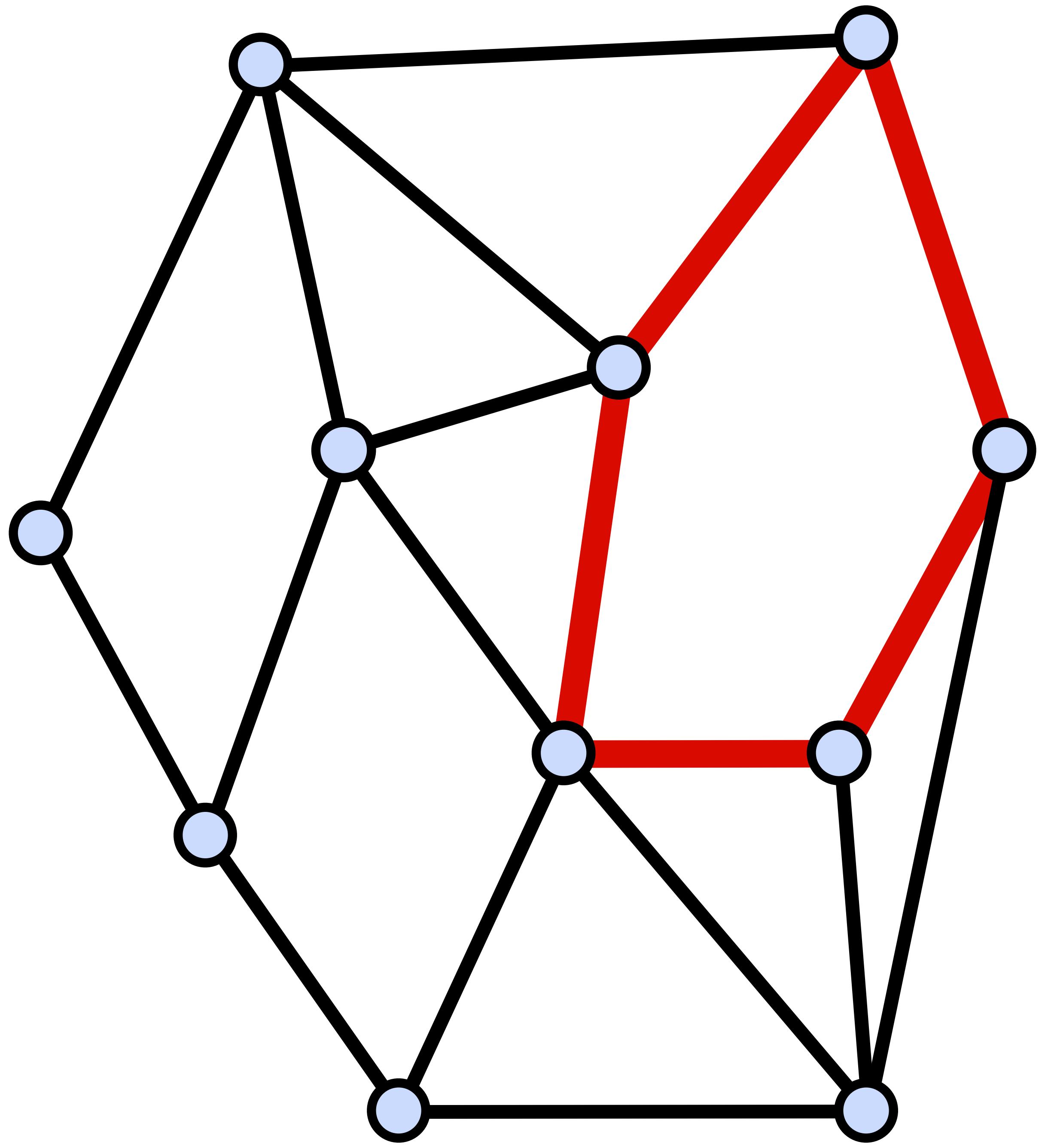

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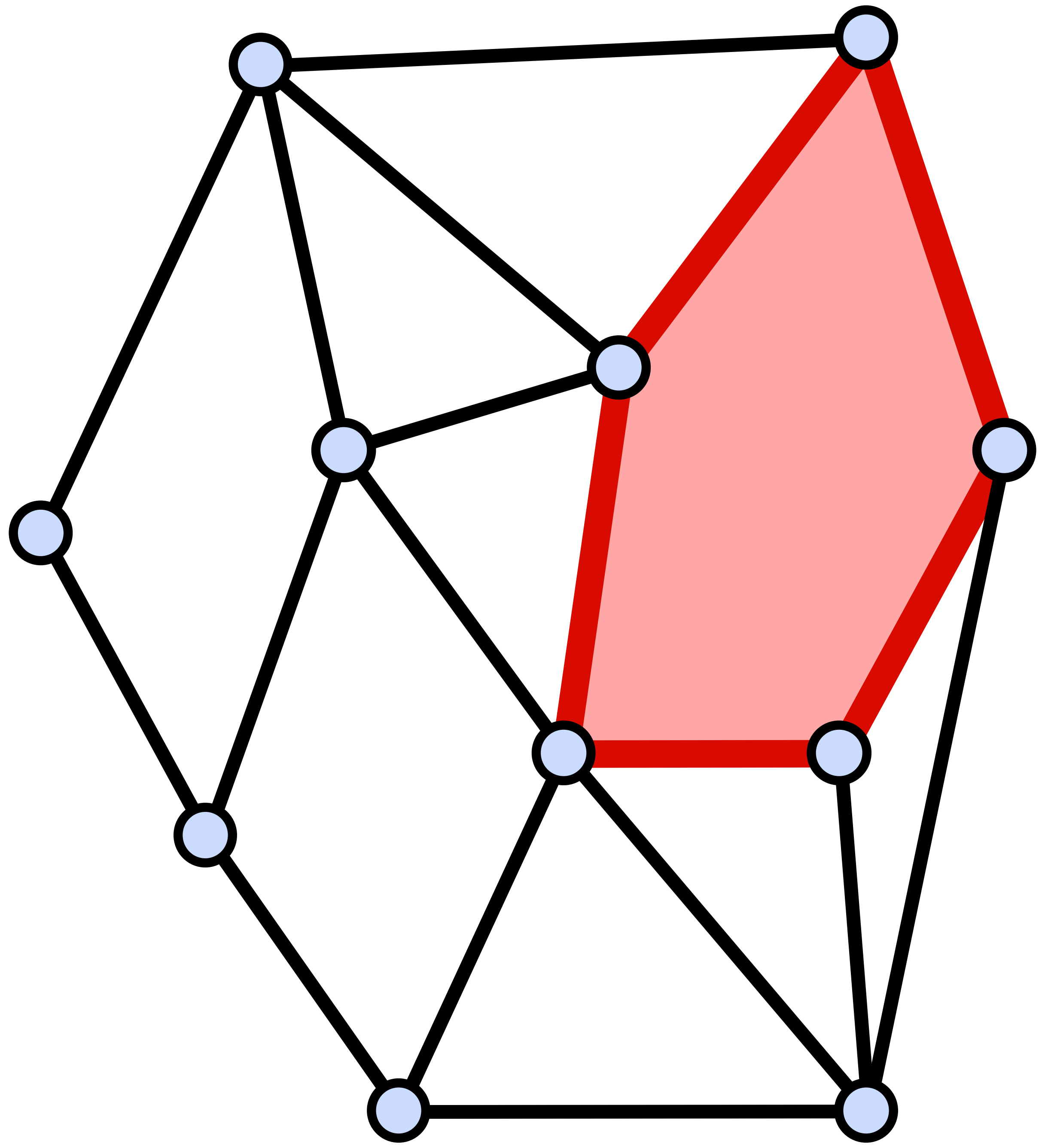

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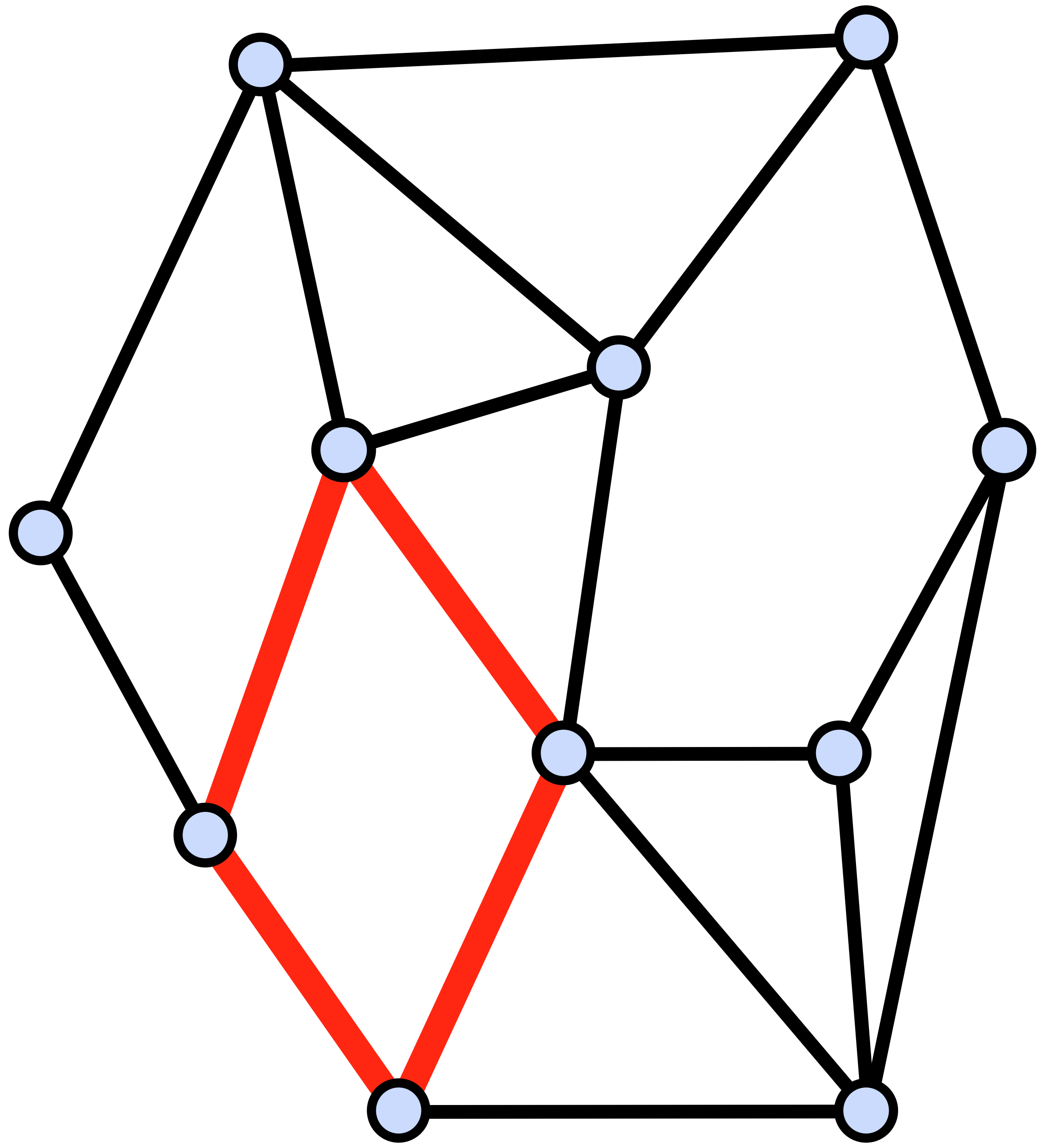

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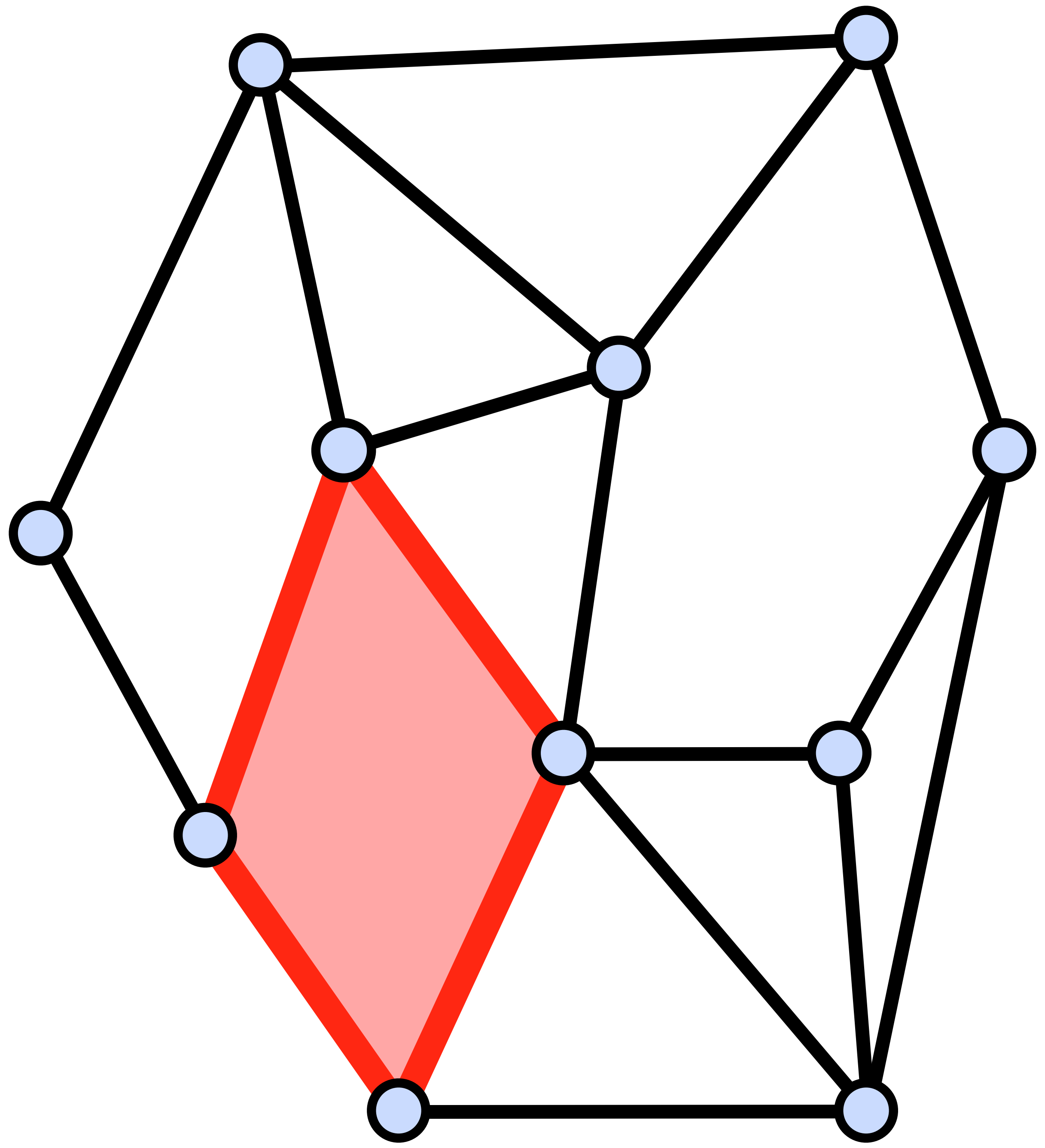

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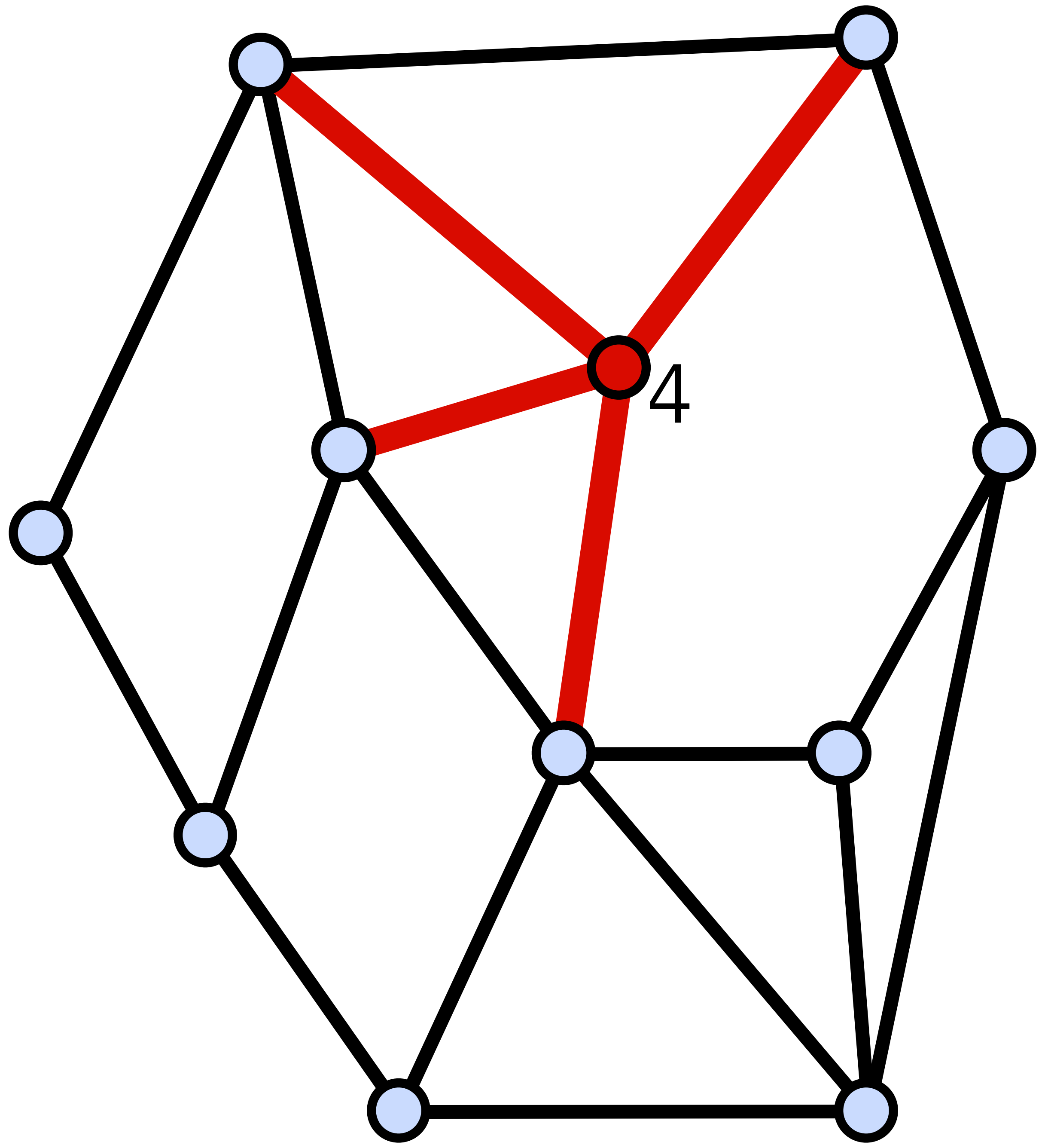

- Vertex **degree** or **valence** = number of incident edges

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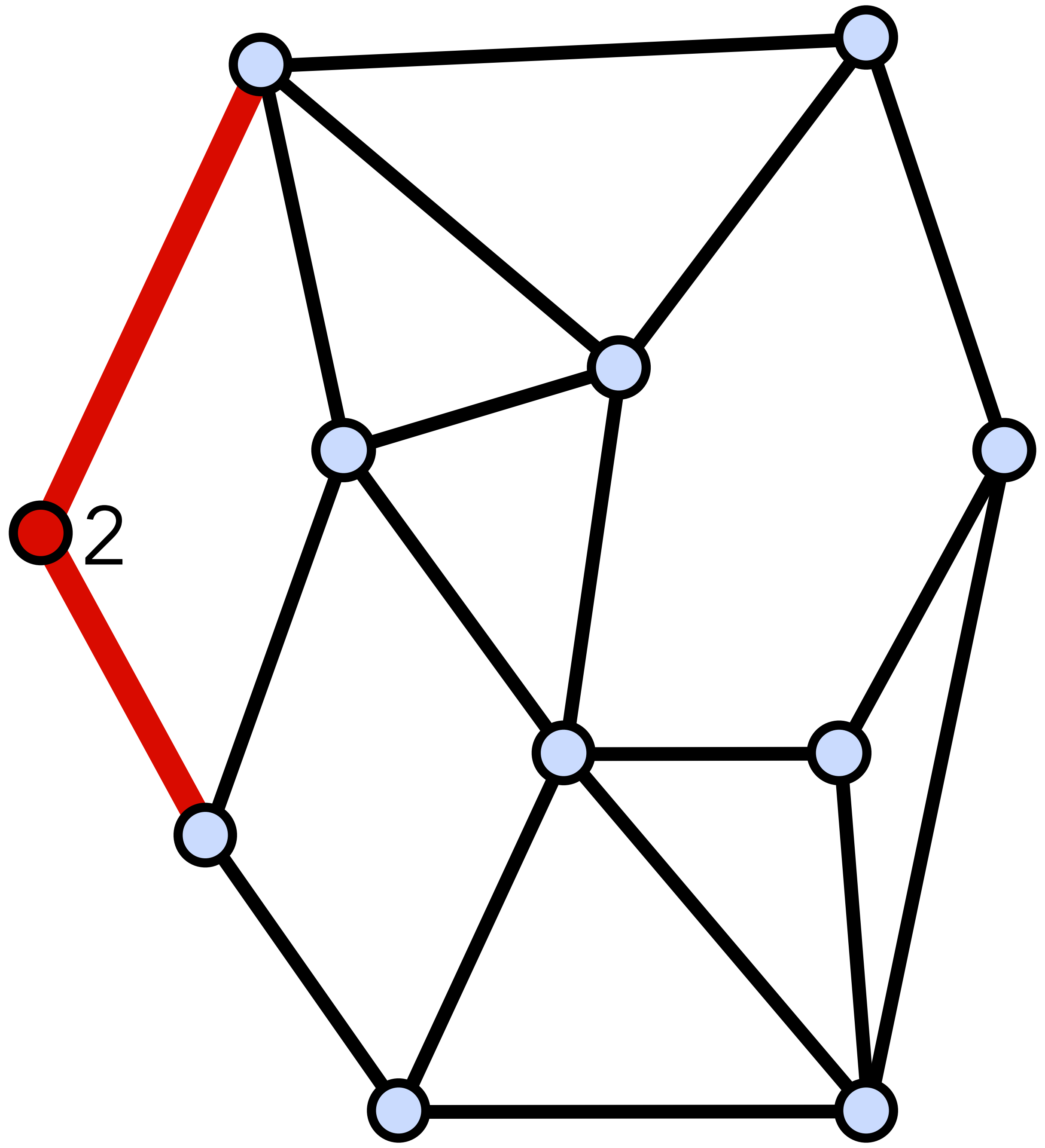
Polygonal Mesh



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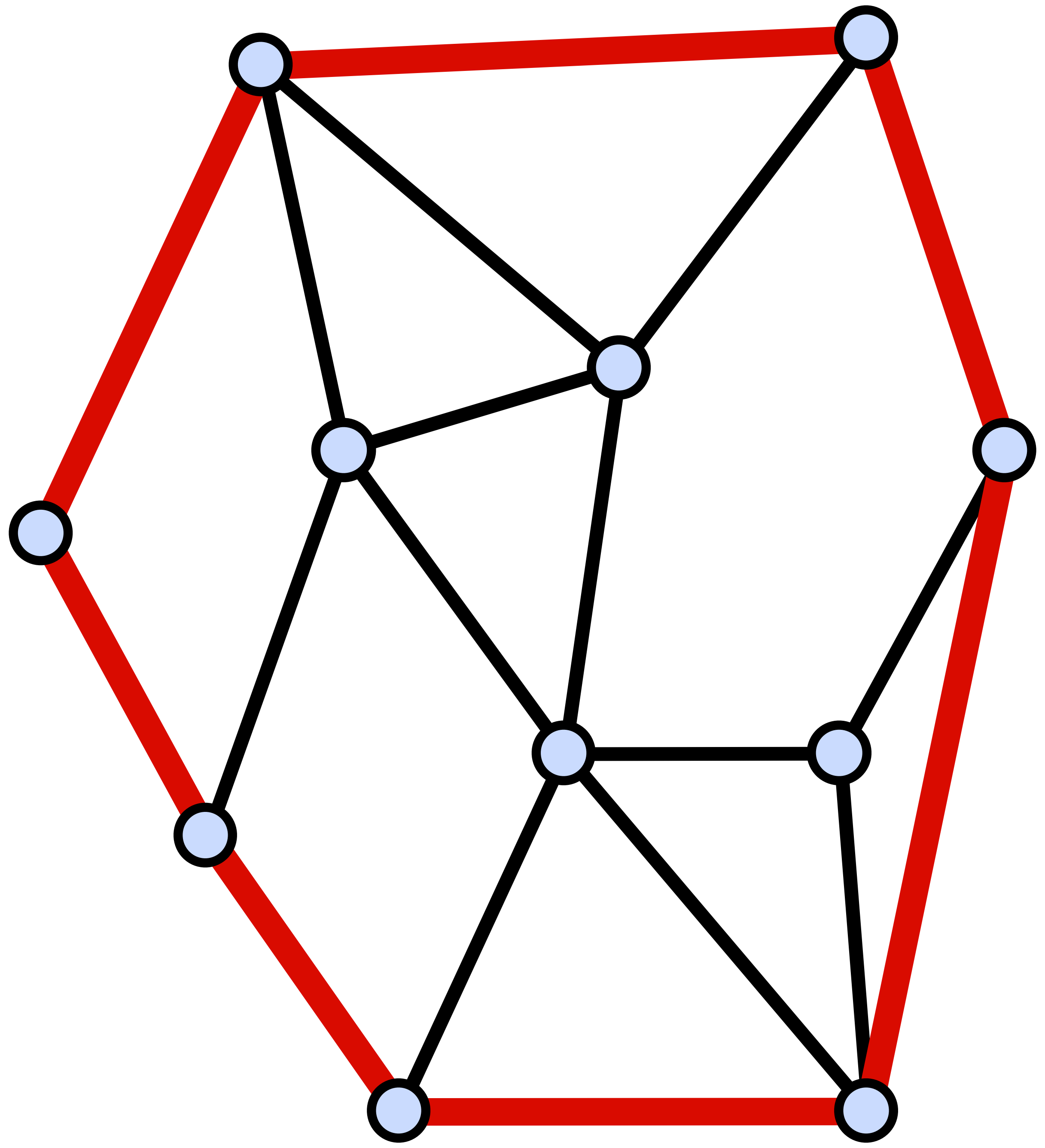
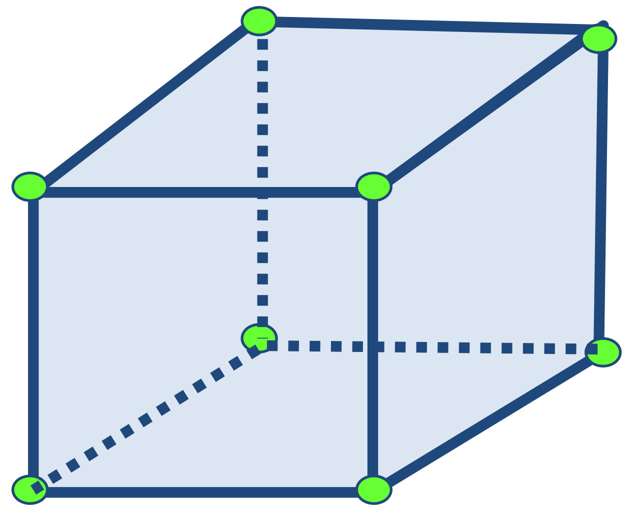
Polygonal Mesh



- **Boundary**: the set of all edges that belong to only one polygon
  - Either empty or forms closed loops
  - If empty, then the polygonal mesh is closed



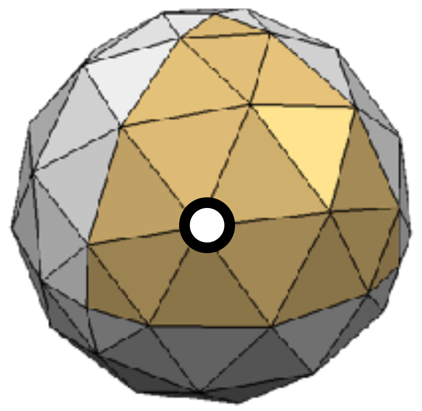
Triangle Meshes

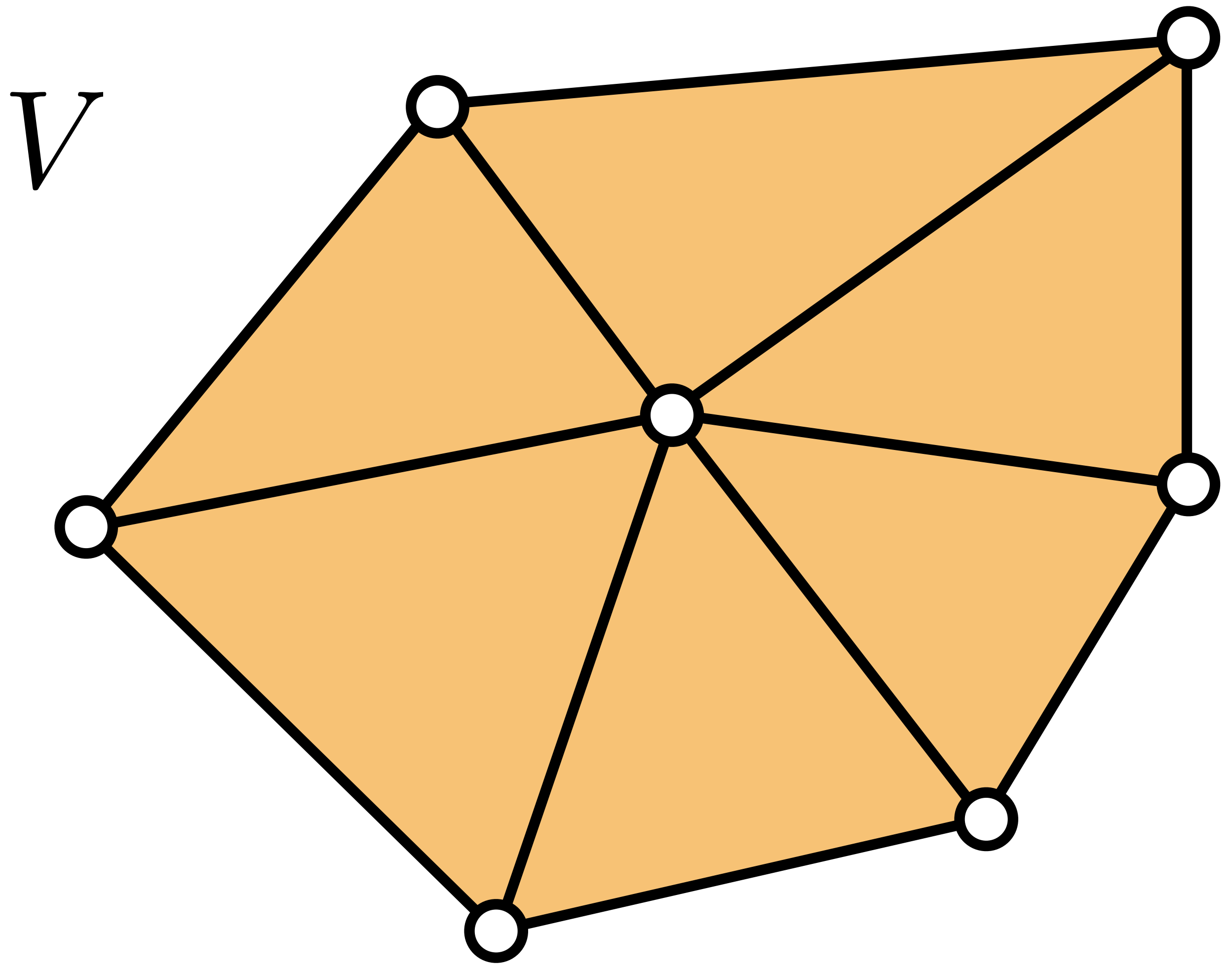
- Connectivity: vertices, edges, triangles

$$V = \{v_1, \ldots, v_n\}$$

$$E = \{e_1, \ldots, e_k\}, \quad e_i \in V \times V$$

$$F = \{f_1, \ldots, f_m\}, \quad f_i \in V \times V \times V$$



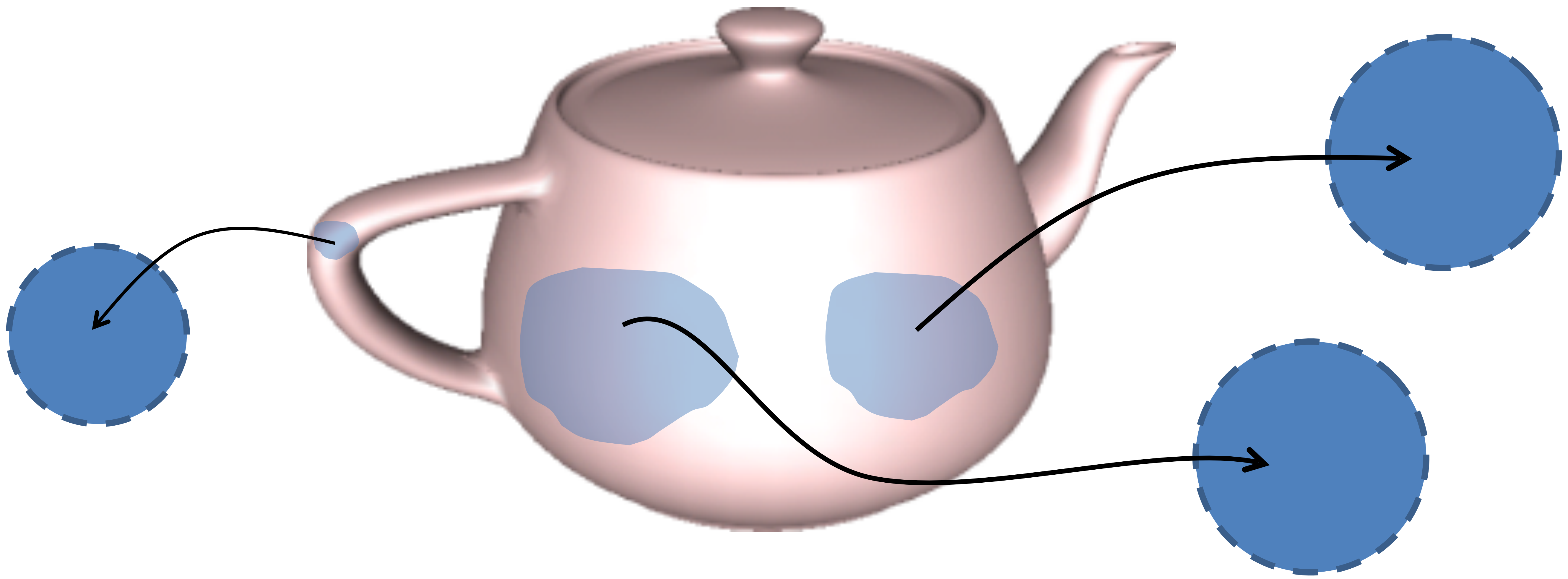


- Geometry: vertex positions

$$P = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\}, \quad \mathbf{p}_i \in \mathbb{R}^3$$

Manifolds

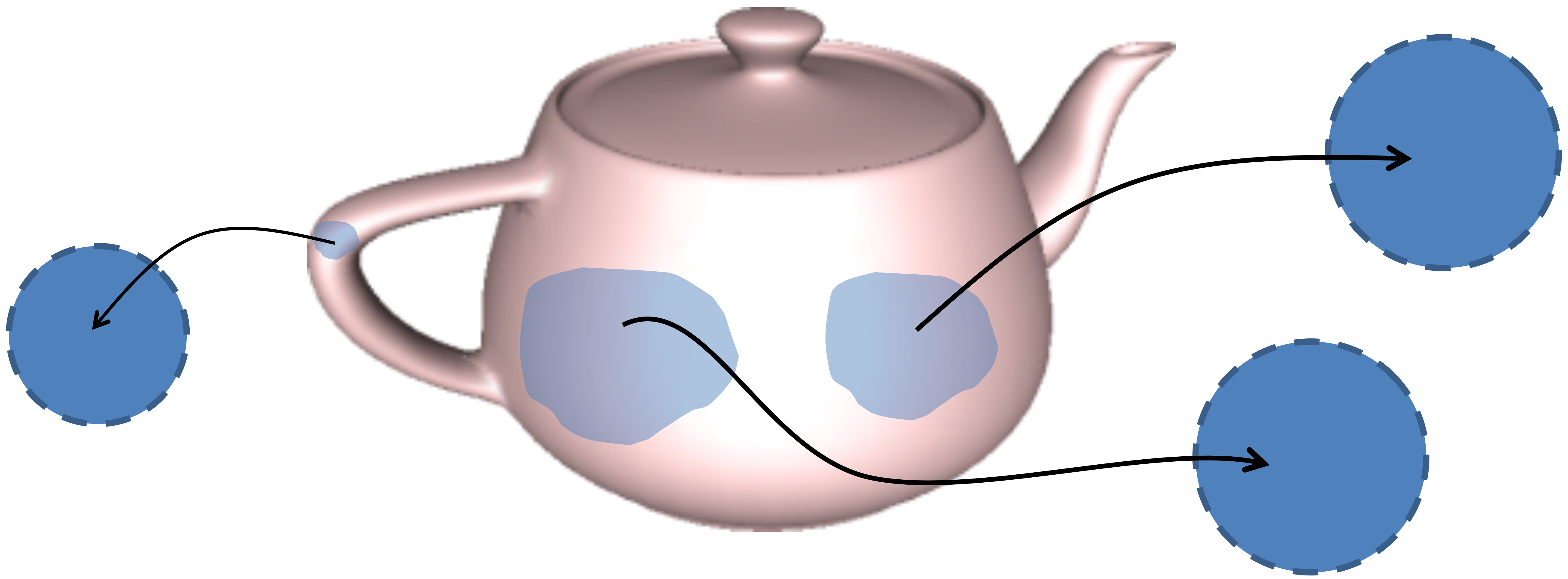
- A surface is a closed **2-manifold** if it is everywhere locally homeomorphic
  to a disk



Manifolds

- For every point **x** in M, there is an **open** ball $B_{\mathbf{x}}(r)$ of radius $r > 0$ centered at **x** such that $M \cap B_{\mathbf{x}}$ is homeomorphic to an open disk

$$B_{\mathbf{x}}(r) = \{\mathbf{y} \in \mathbb{R}^3 \ s.t. \ \|\mathbf{y} - \mathbf{x}\| < r\}$$



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Manifolds

- Manifold with boundary: a vicinity of each boundary point is homeomorphic to a half-disk

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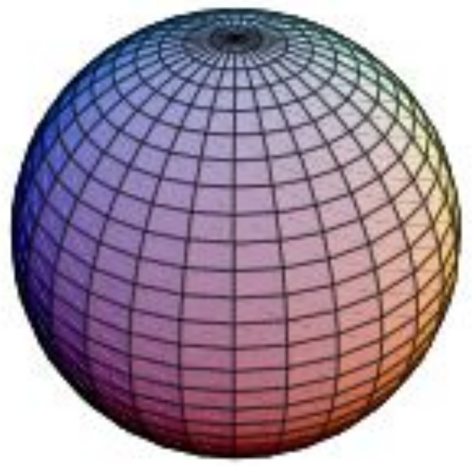
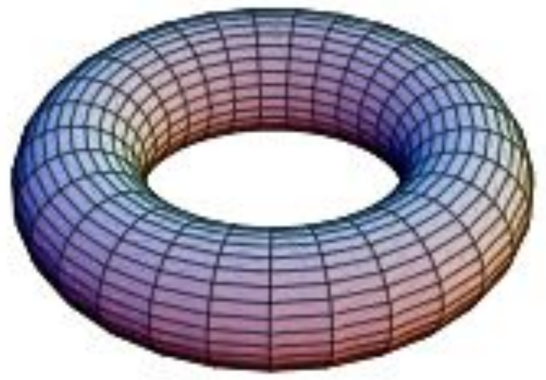
Examples

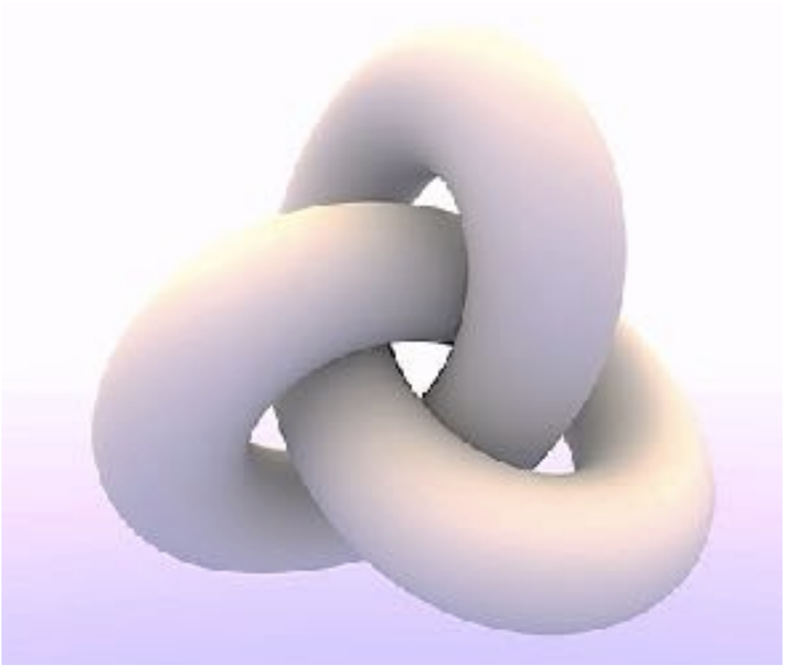
- For each case, decide if it is a 2-manifold (possibly with boundary) or not. If not, explain why not.



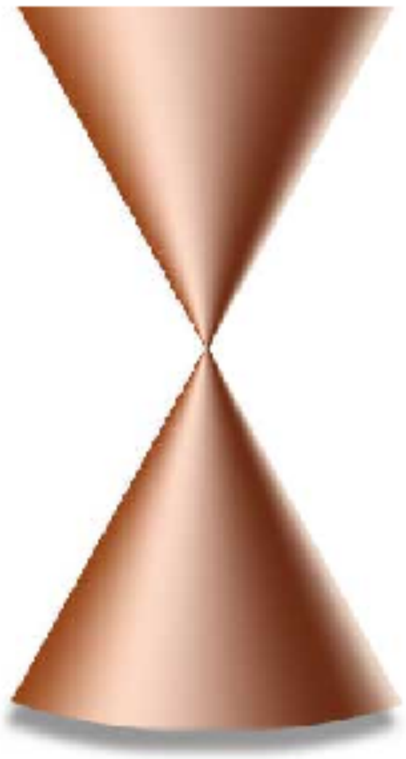
Case 1



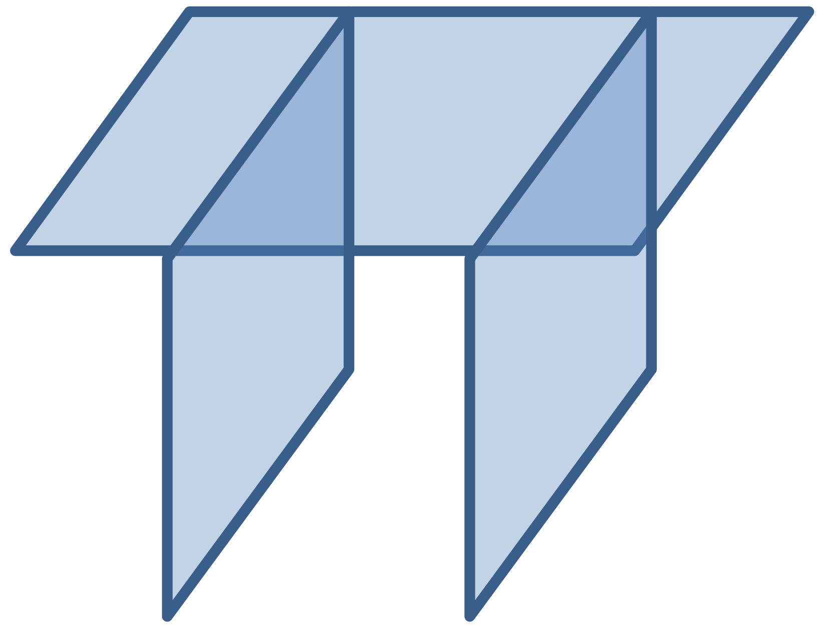
Case 2



Case 3



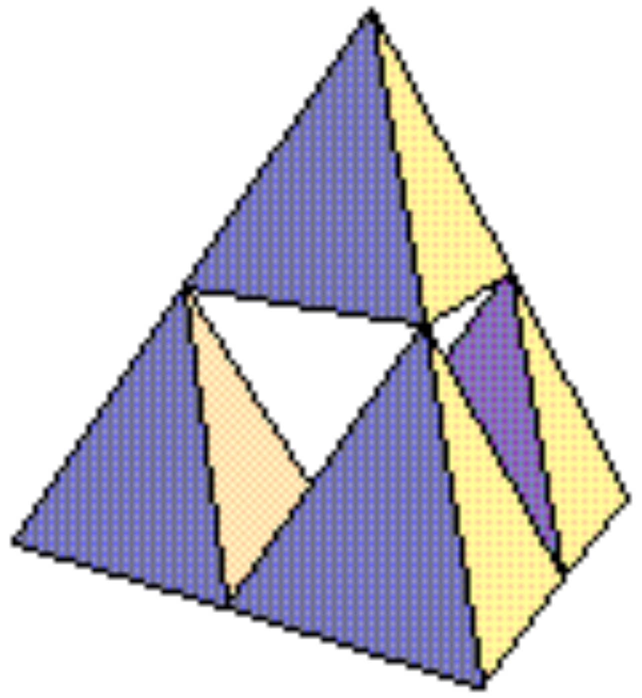
Case 4



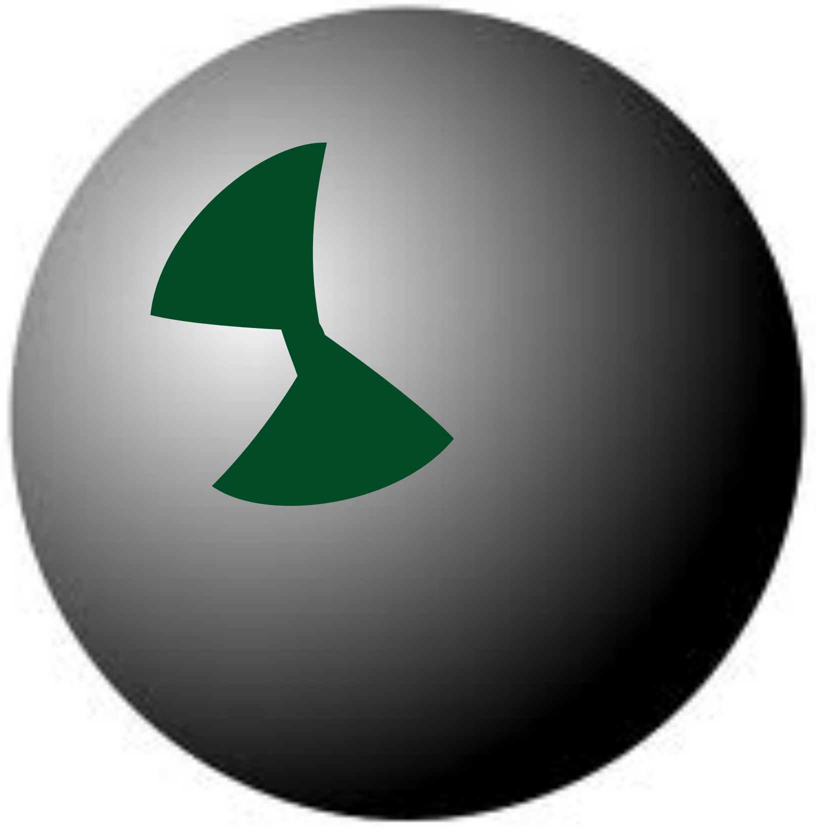
Case 5

Examples

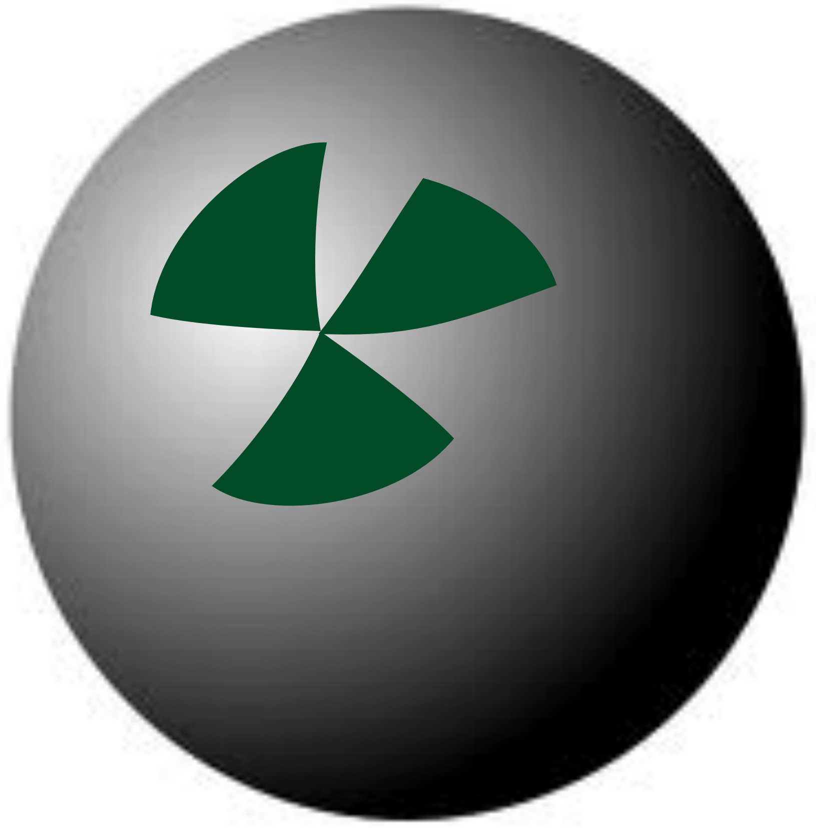
- Bonus cases



Case 6



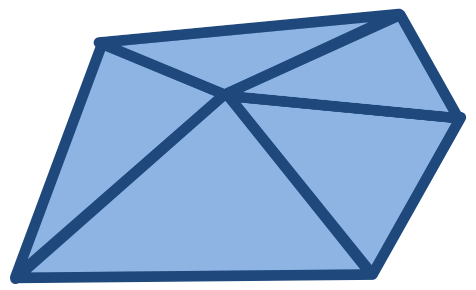
Case 7



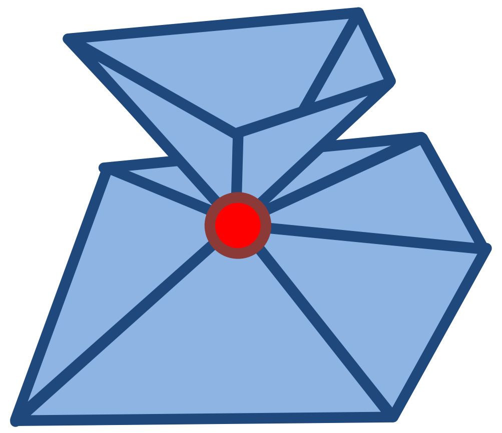
Case 8

Manifolds

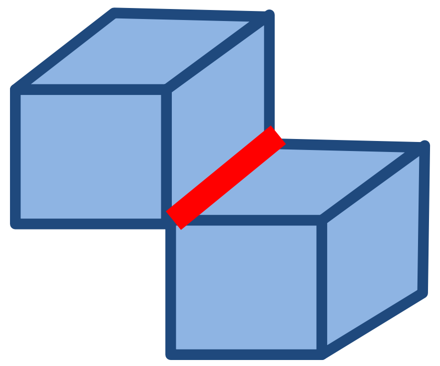
- In a manifold mesh, there are at most 2 faces sharing an edge
  - Boundary edges: have one incident face
  - Inner edges have two incident faces
- A manifold vertex has 1 connected ring of faces around it, or 1 connected half-ring (boundary)



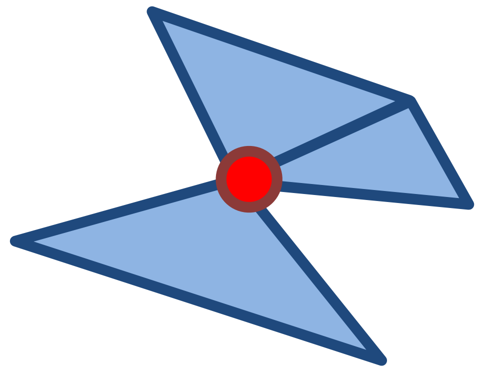
✓ manifold



✗ non-manifold vertex



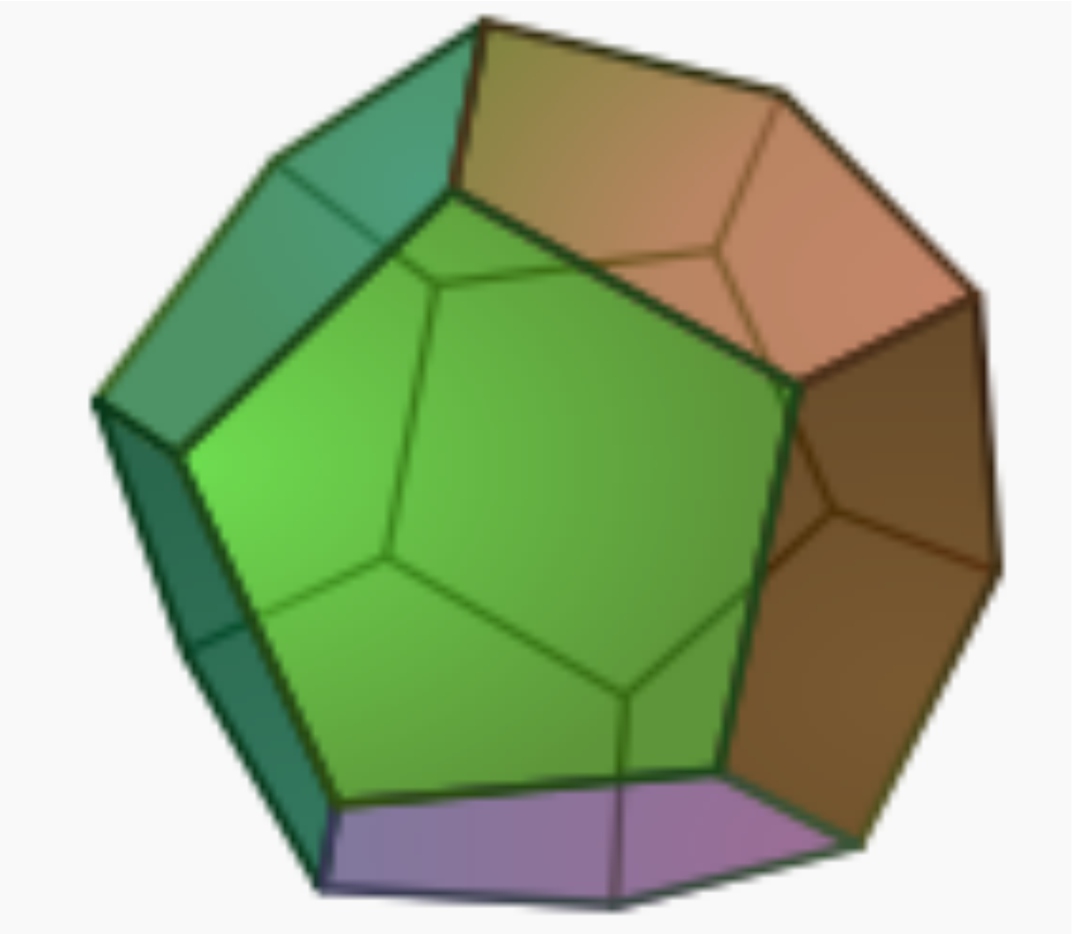
✗ non-manifold edge



✗ non-manifold vertex

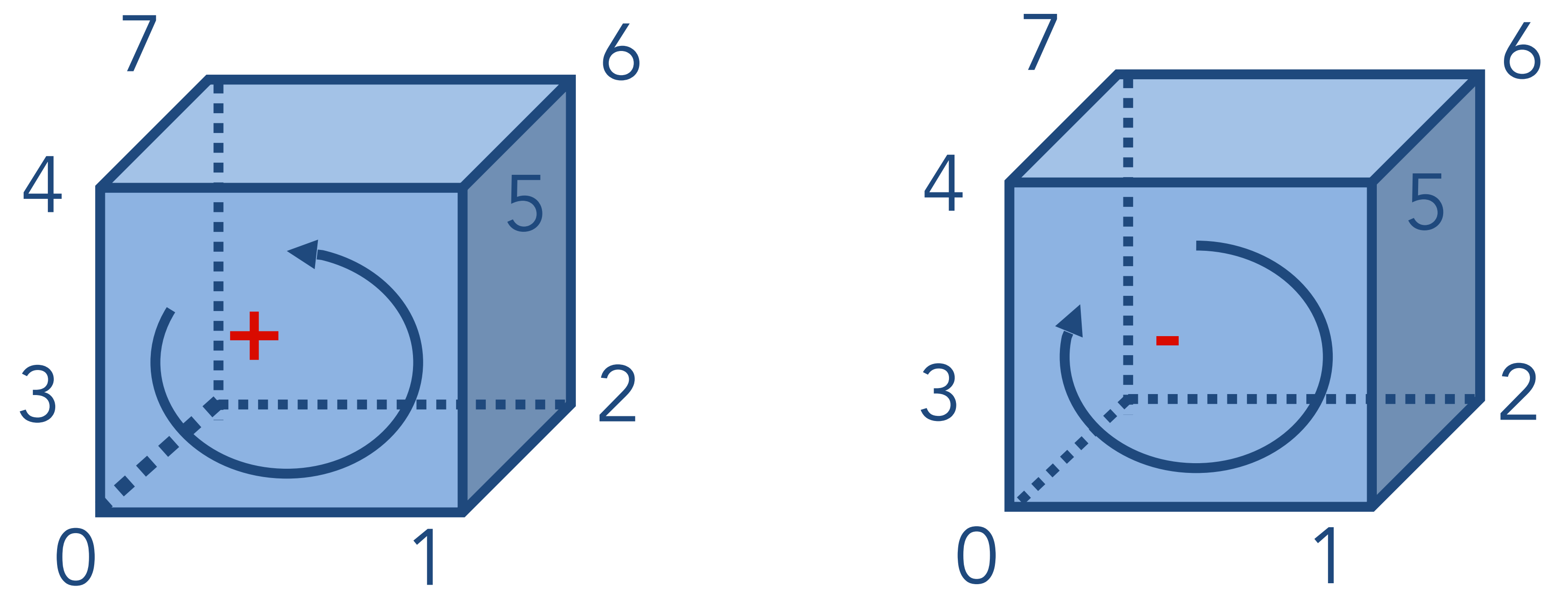
Manifolds

- If closed and not intersecting, a manifold divides the space into inside and outside
- A closed manifold polygonal mesh is called **polyhedron**



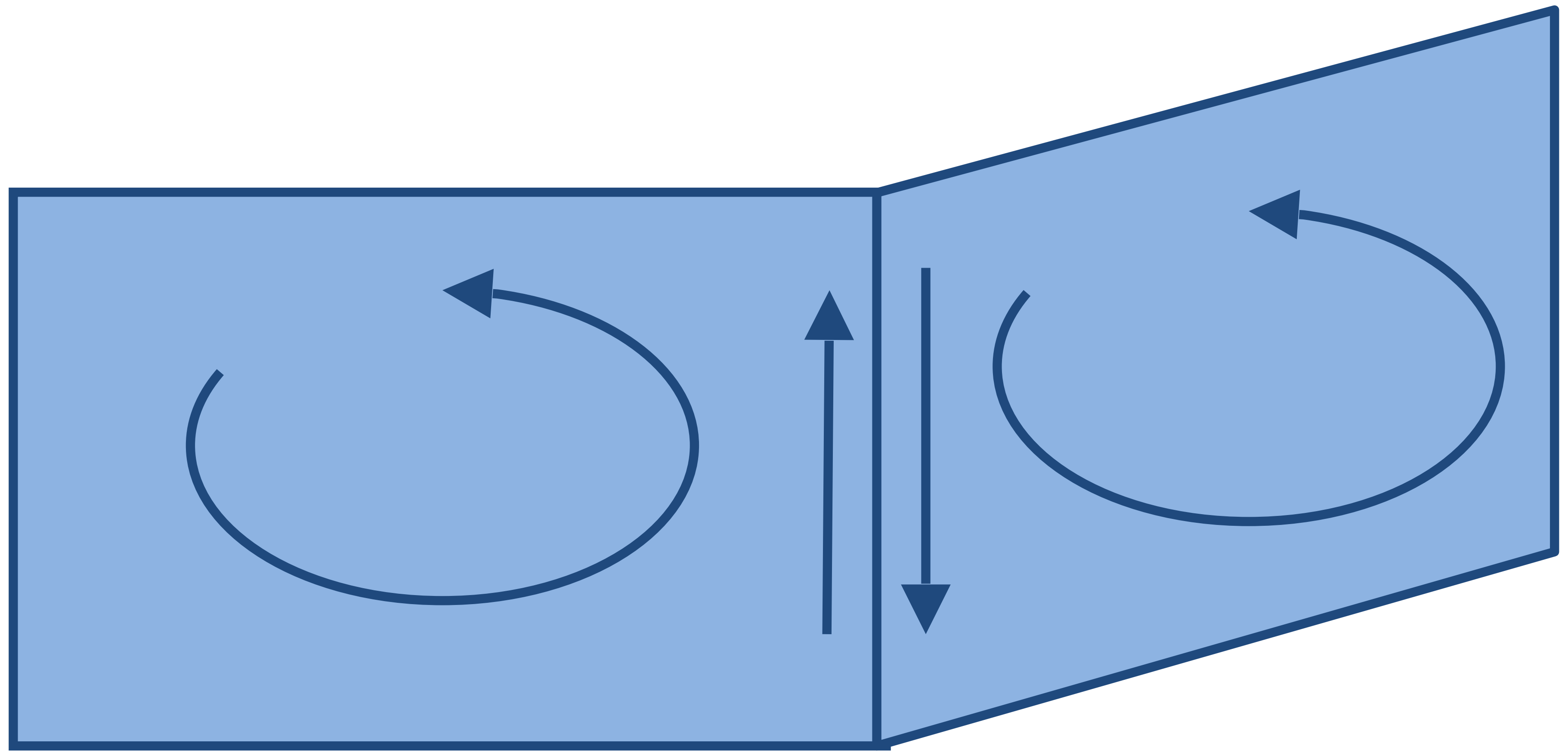
Orientation

- Every face of a polygonal mesh is orientable
  - Clockwise vs. counterclockwise order of face vertices
  - Defines sign/direction of the surface normal



Orientation

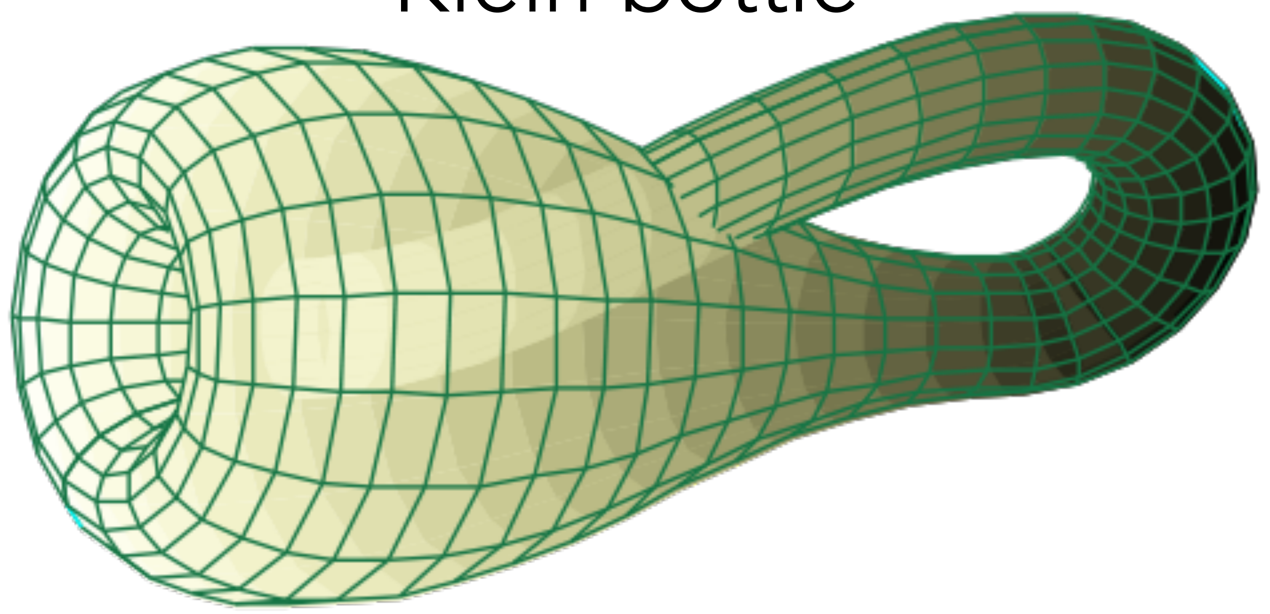
- Consistent orientation of neighboring faces:



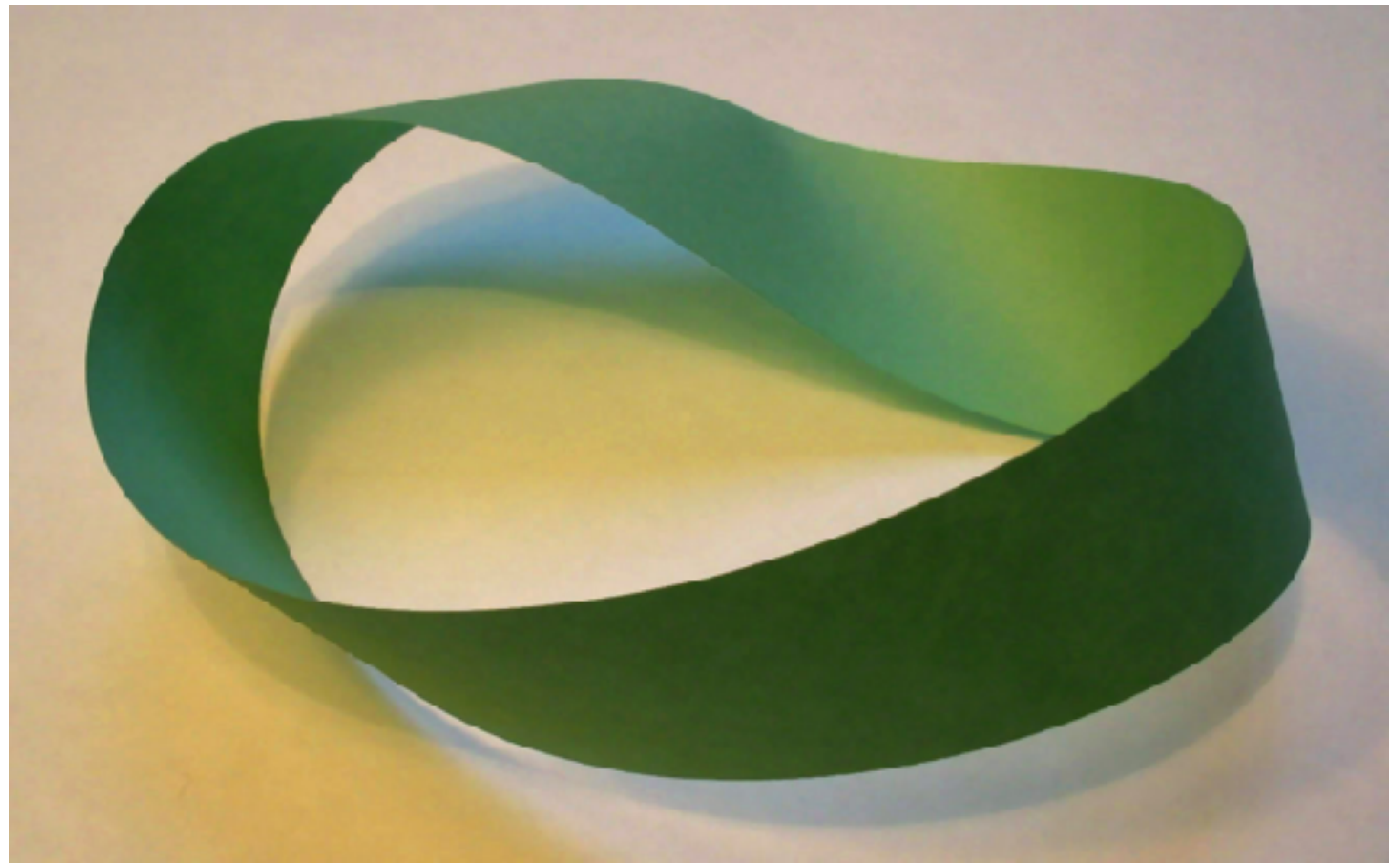
Orientability

- A polygonal mesh is orientable, if the incident faces to every edge can be consistently oriented
  - If the faces are consistently oriented for every edge, the mesh is oriented
- Notes
  - Every non-orientable closed mesh embedded in $\mathbb{R}^3$ intersects itself
  - The surface of a polyhedron is always orientable

Klein bottle



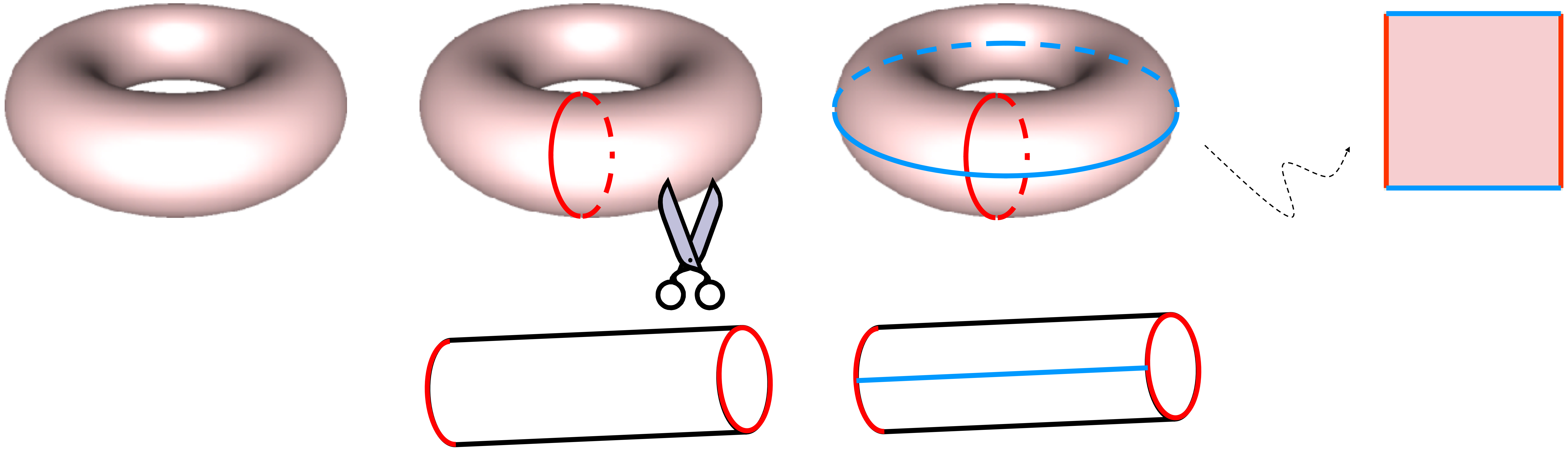
Möbius strip



By David Benbennick - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=50359
By Tttrung - Own work, CC BY-SA 3.0, https://commons.wikimedia.org/w/index.php?curid=960446

Global Topology of Meshes

- **Genus:** ½ × the maximal number of closed paths that do not disconnect the graph.
  - Informally, the number of handles ("donut holes").



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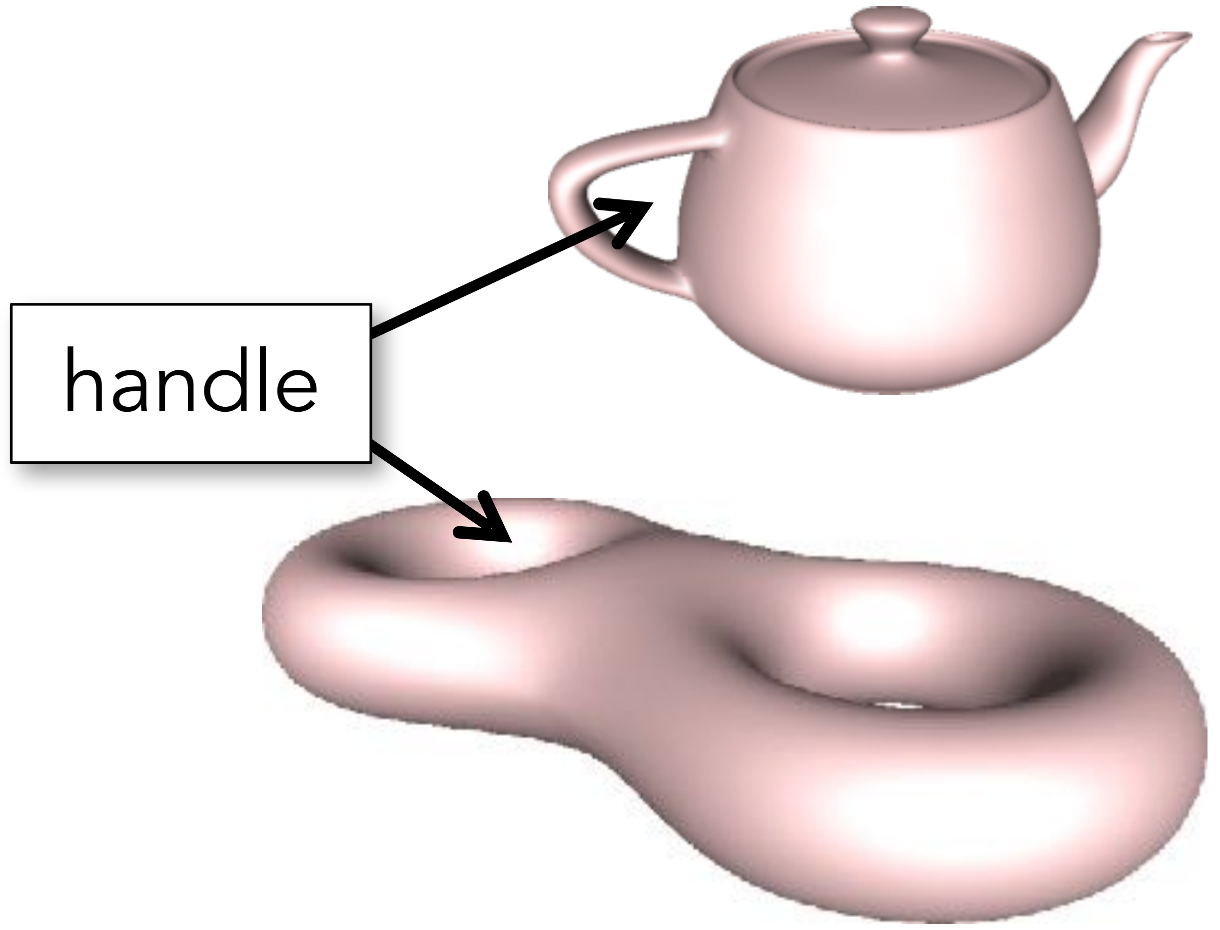

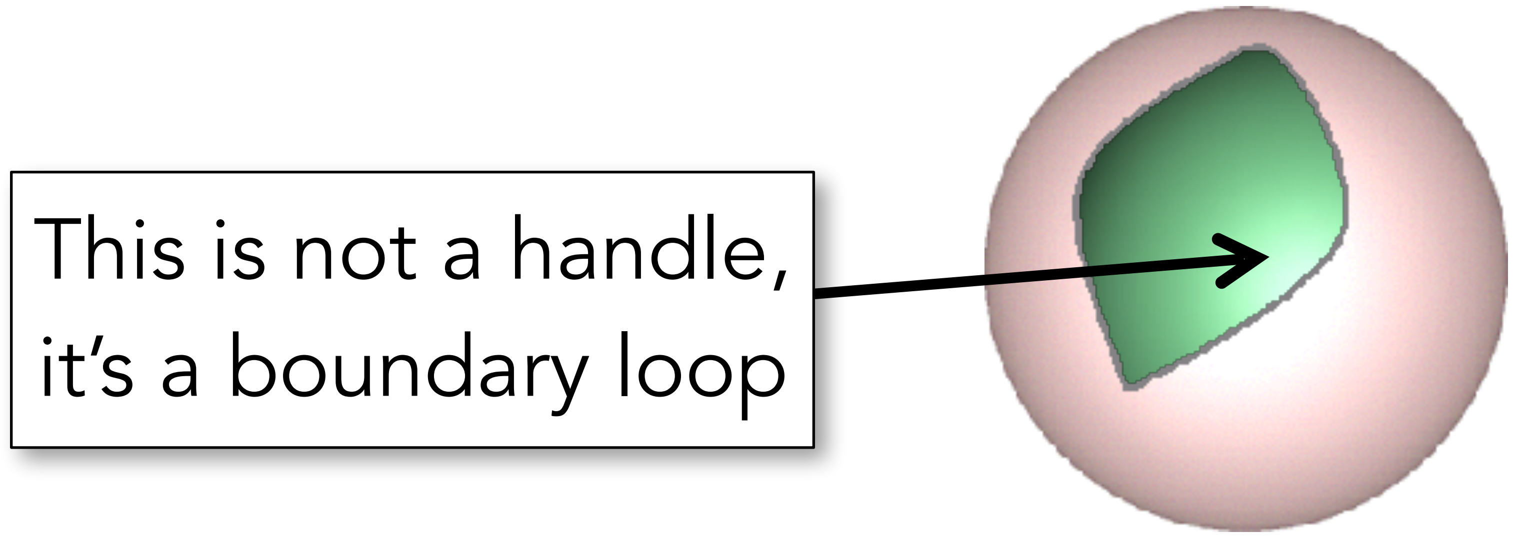

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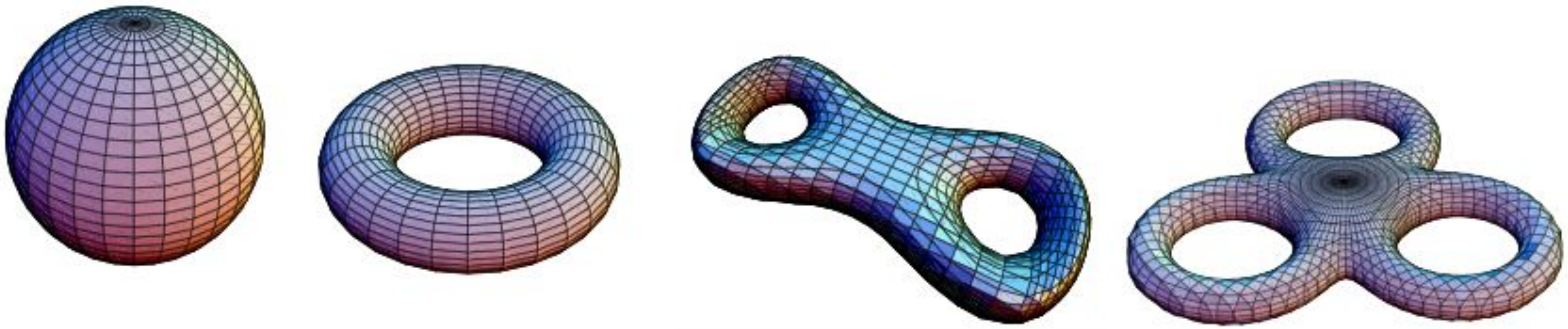


Genus 0 Genus 1 Genus 2 Genus 3

Euler-Poincaré Formula

- Theorem (Euler): The sum

$$\chi(M) = v - e + f$$

is **constant** for a given surface topology, no matter which (manifold) mesh we choose.

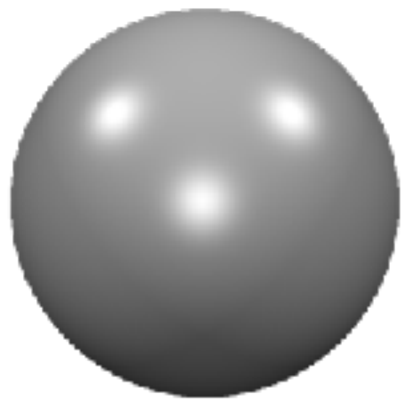
- *v* = number of vertices
- *e* = number of edges
- *f* = number of faces

Euler-Poincaré Formula

- For orientable meshes:

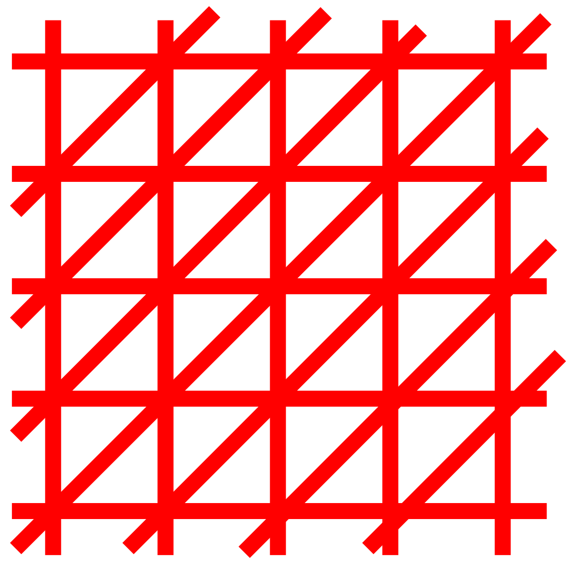
$$v - e + f = 2(c - g) - b = \chi(M)$$

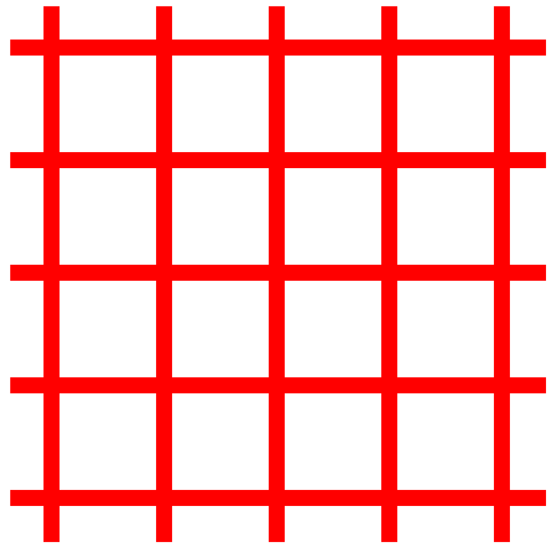
  - c = number of connected components
  - g = genus
  - b = number of boundary loops

$\chi($  $) = 2$ $\chi($  $) = 0$

Regularity

- Triangle mesh: average valence = 6
- Quad mesh: average valence = 4

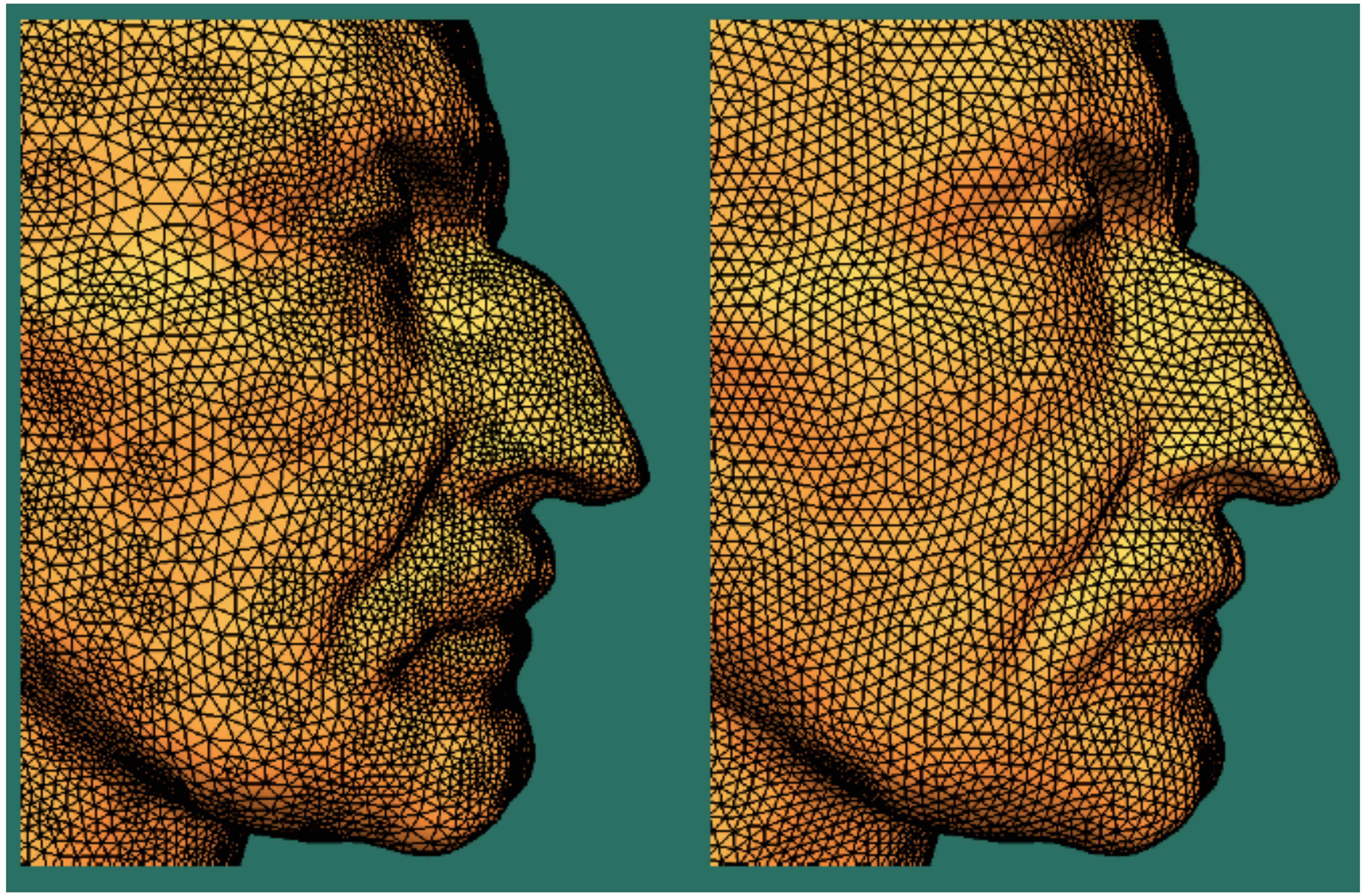




- **Regular mesh**: all faces have the same number of edges and all vertex degrees are equal
- **Quasi-regular mesh**:
  - a lot of vertices have degree 6 (4). Sometimes also refers to mostly equilateral faces.

Regularity

- Quasi-regular



Regularity

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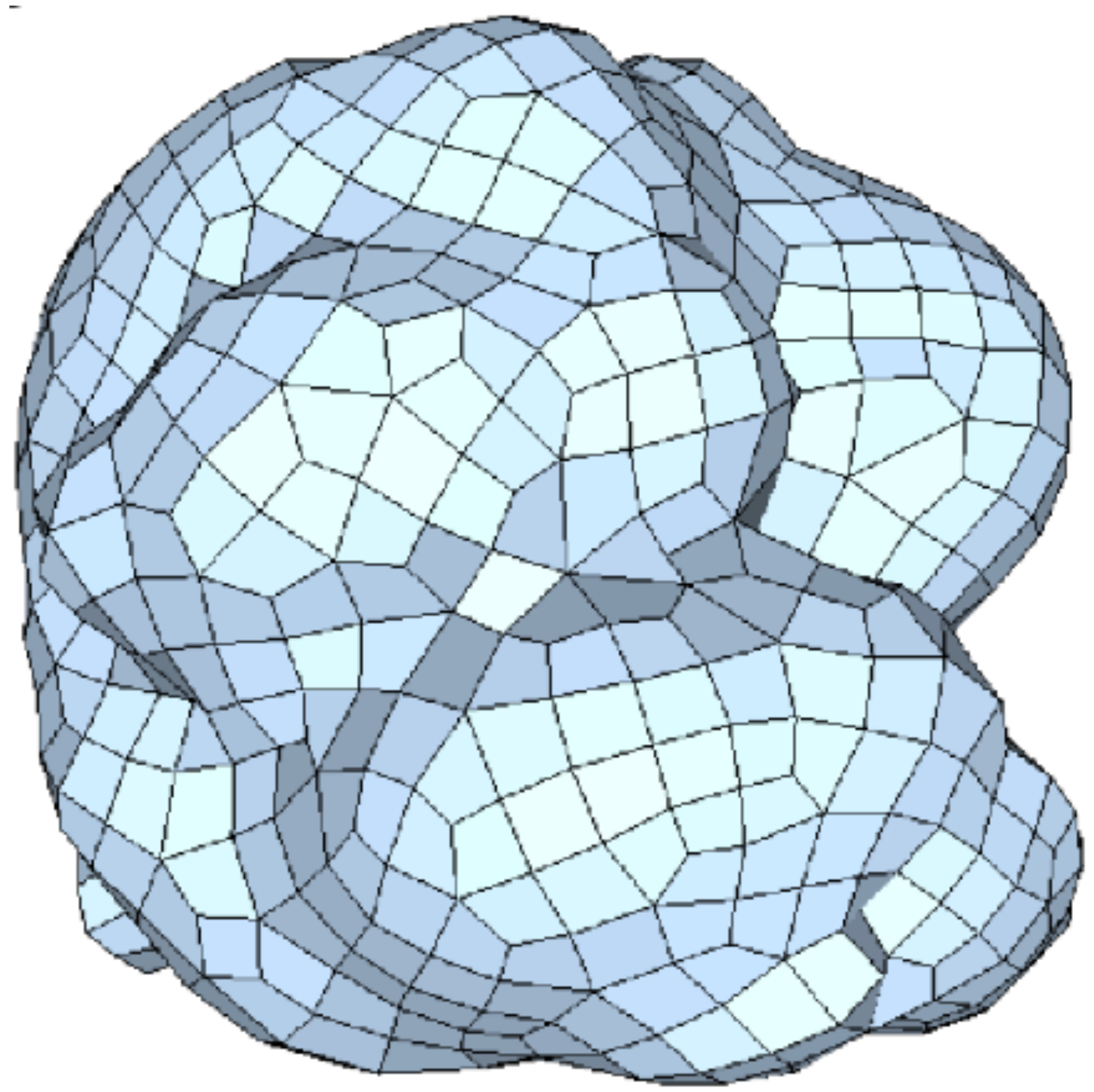
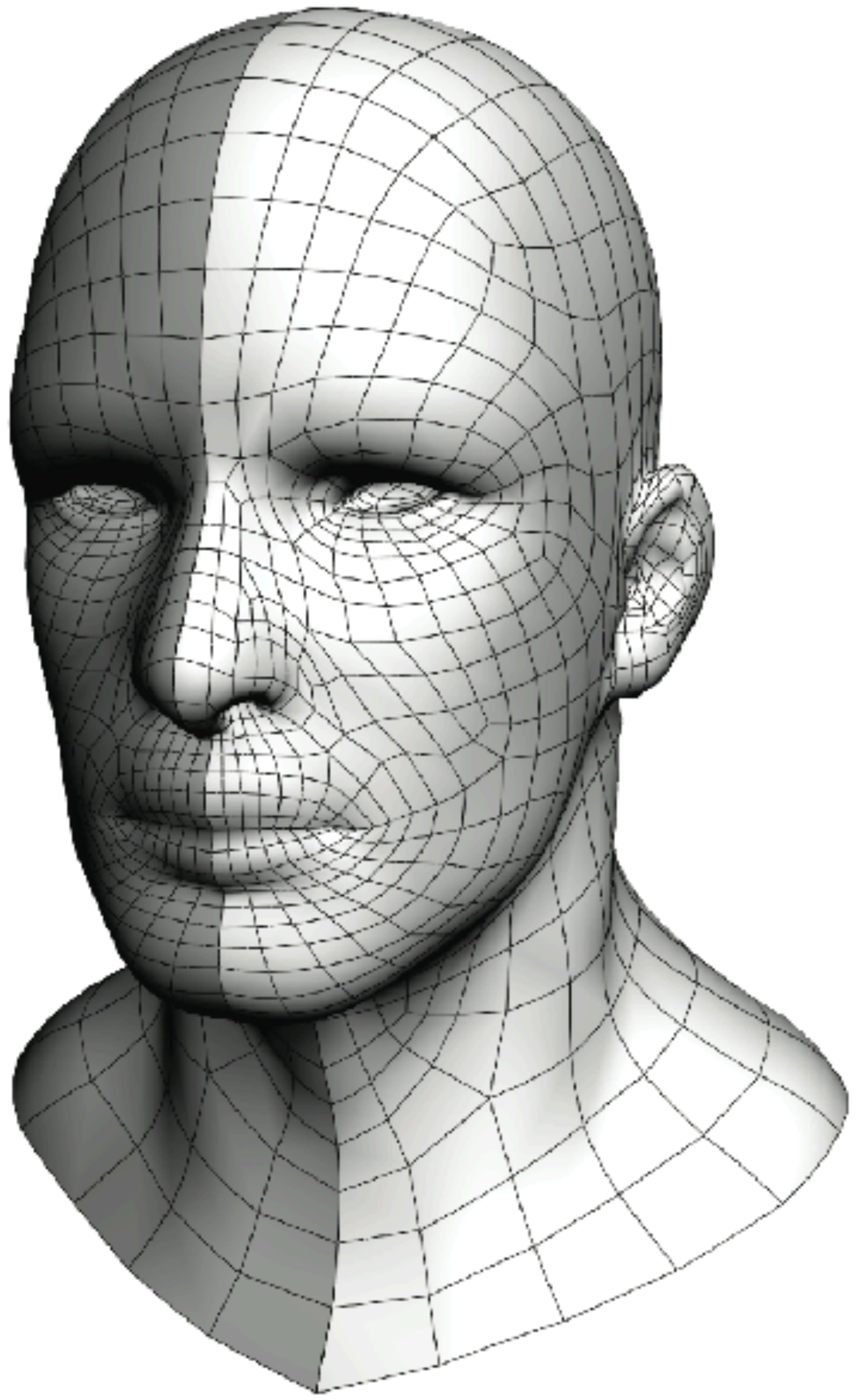
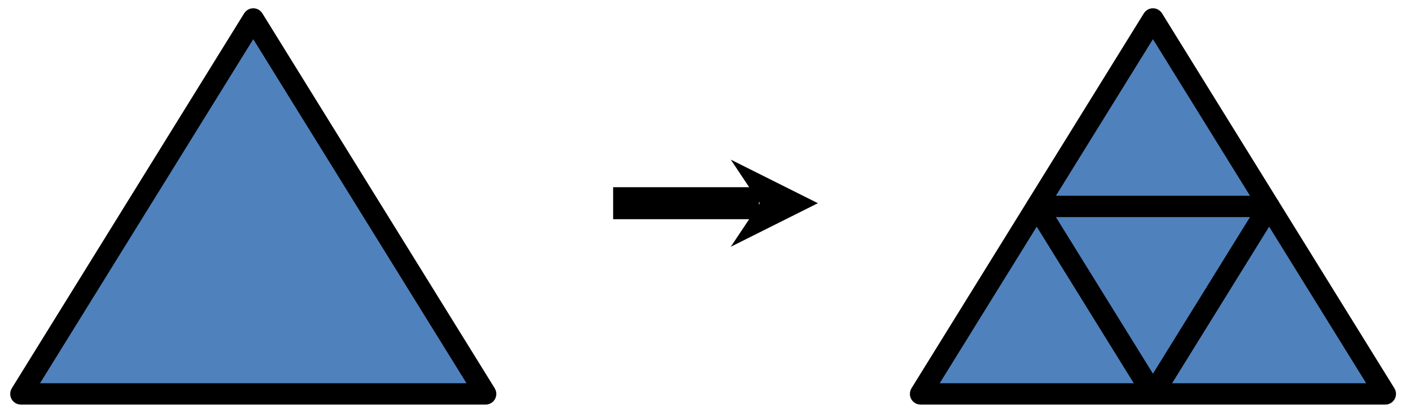
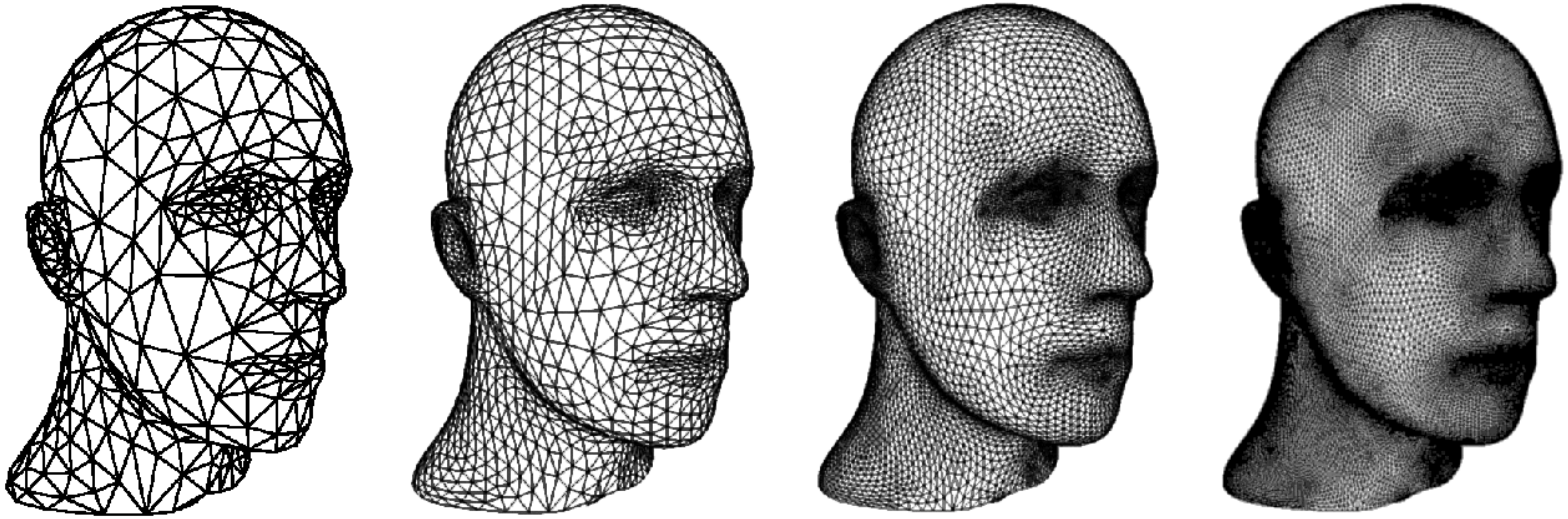




Regularity

- **Semi-regular mesh**: connectivity is a result of N>0 subdivision steps





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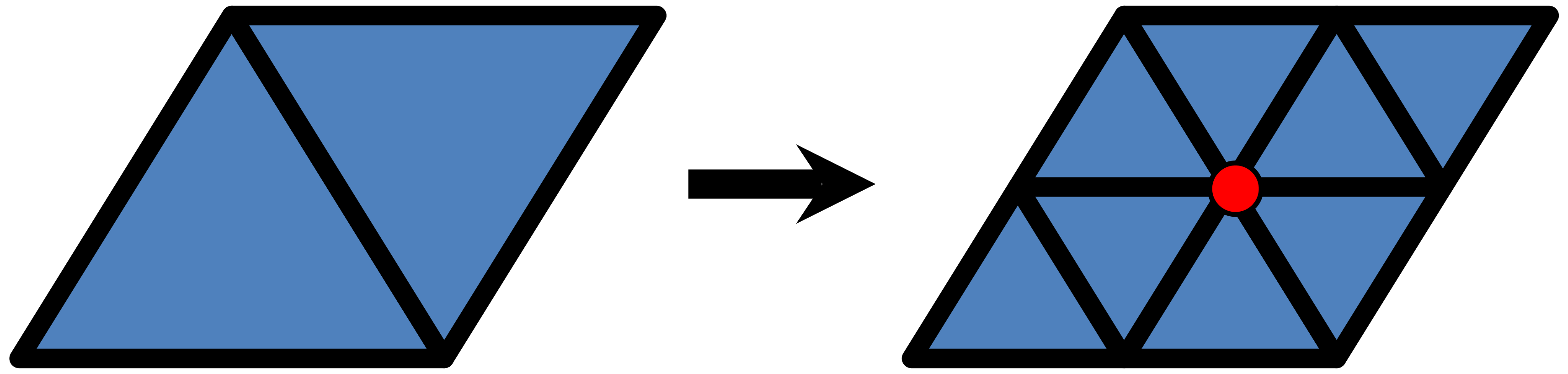

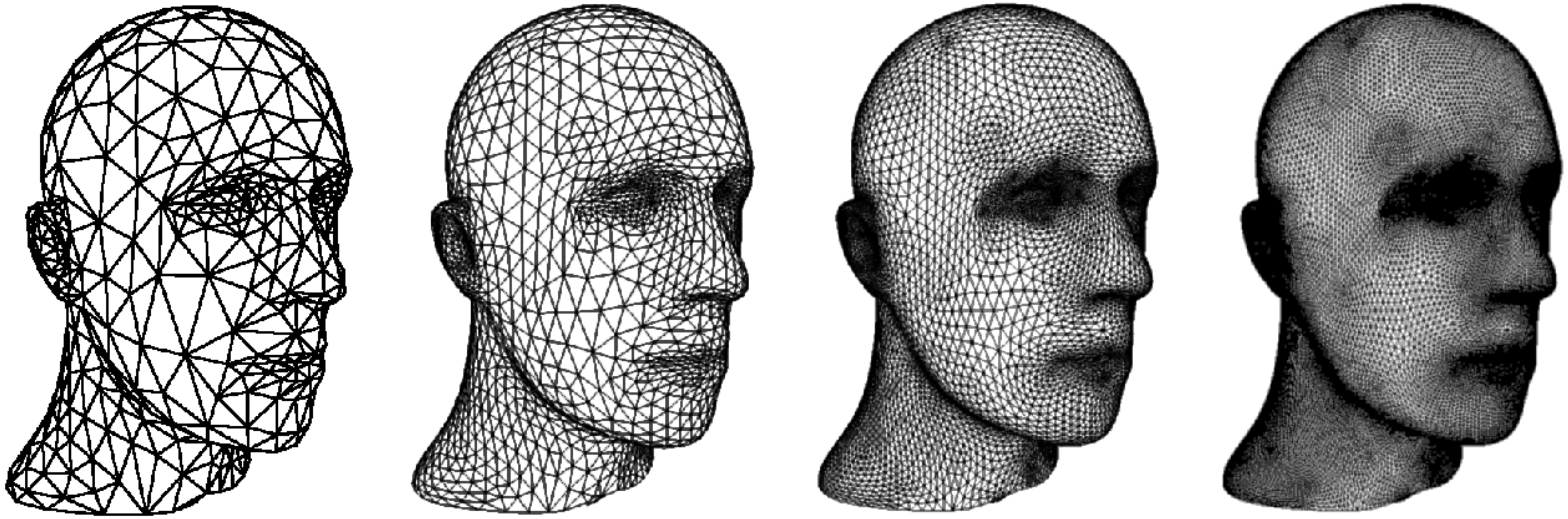

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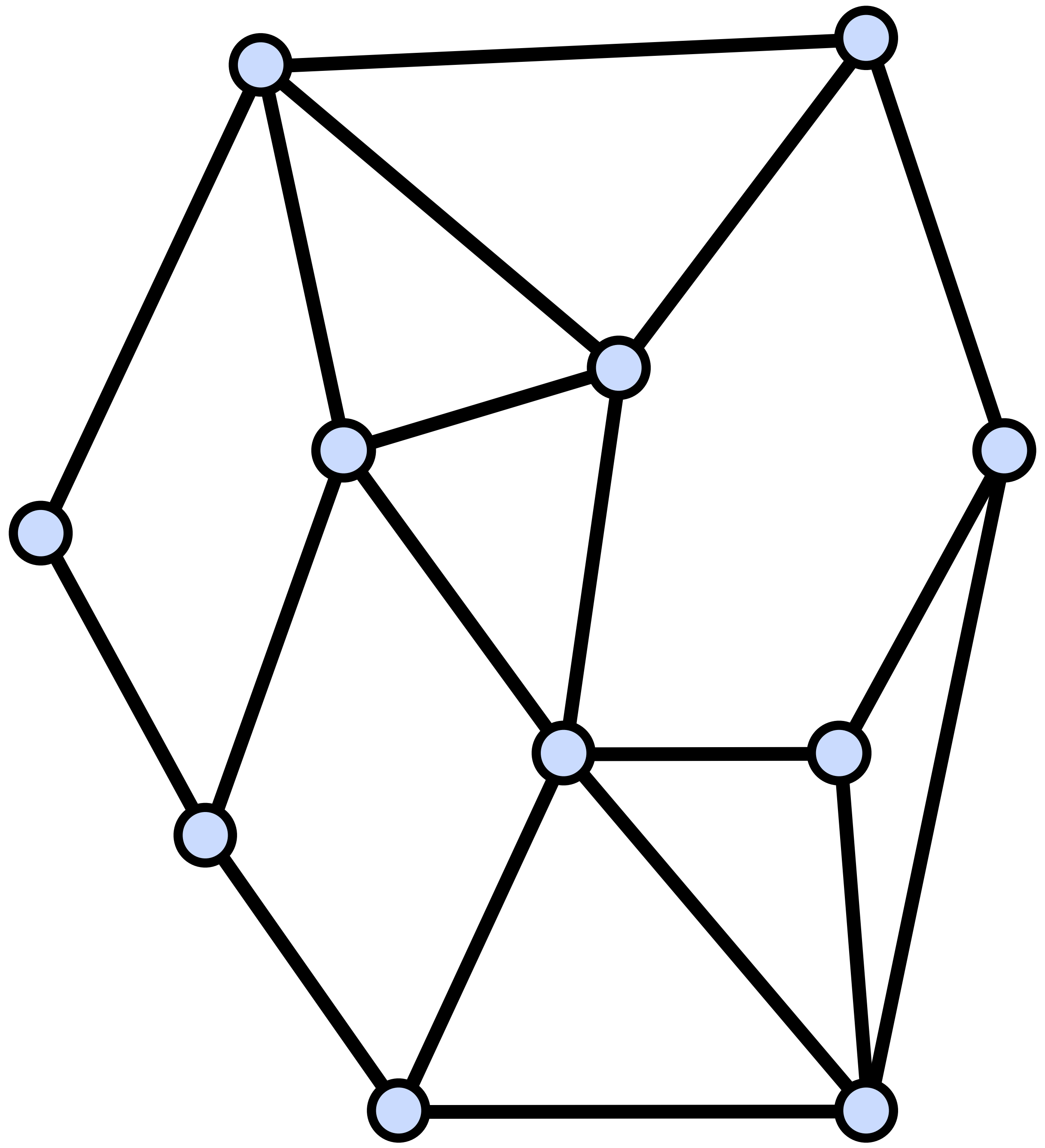
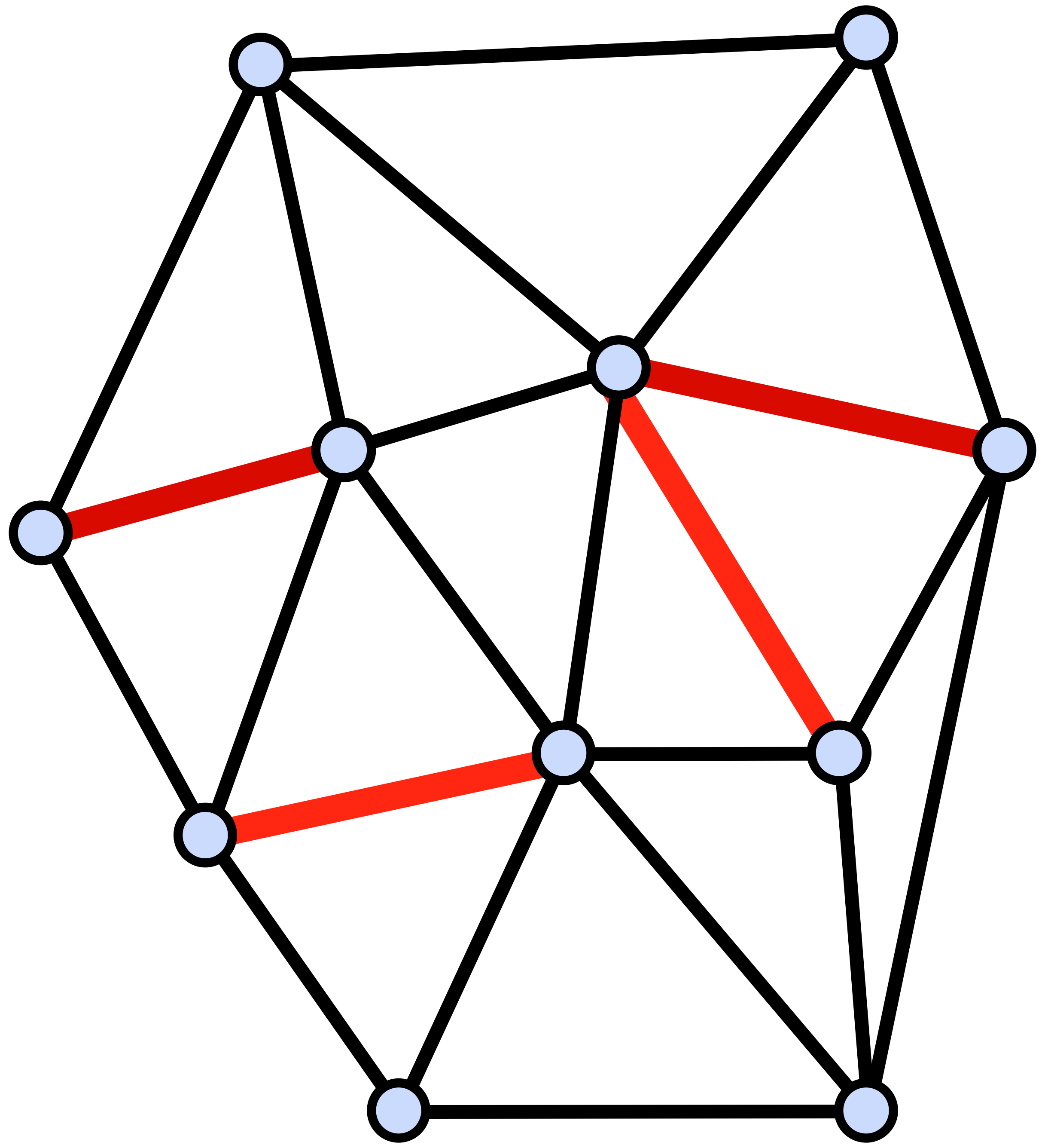
Triangulation



- Polygonal mesh where every face is a triangle

- Simplifies data structures
- Simplifies rendering
- Simplifies algorithms
- Each face planar and convex
- Any polygon can be triangulated

Triangulation



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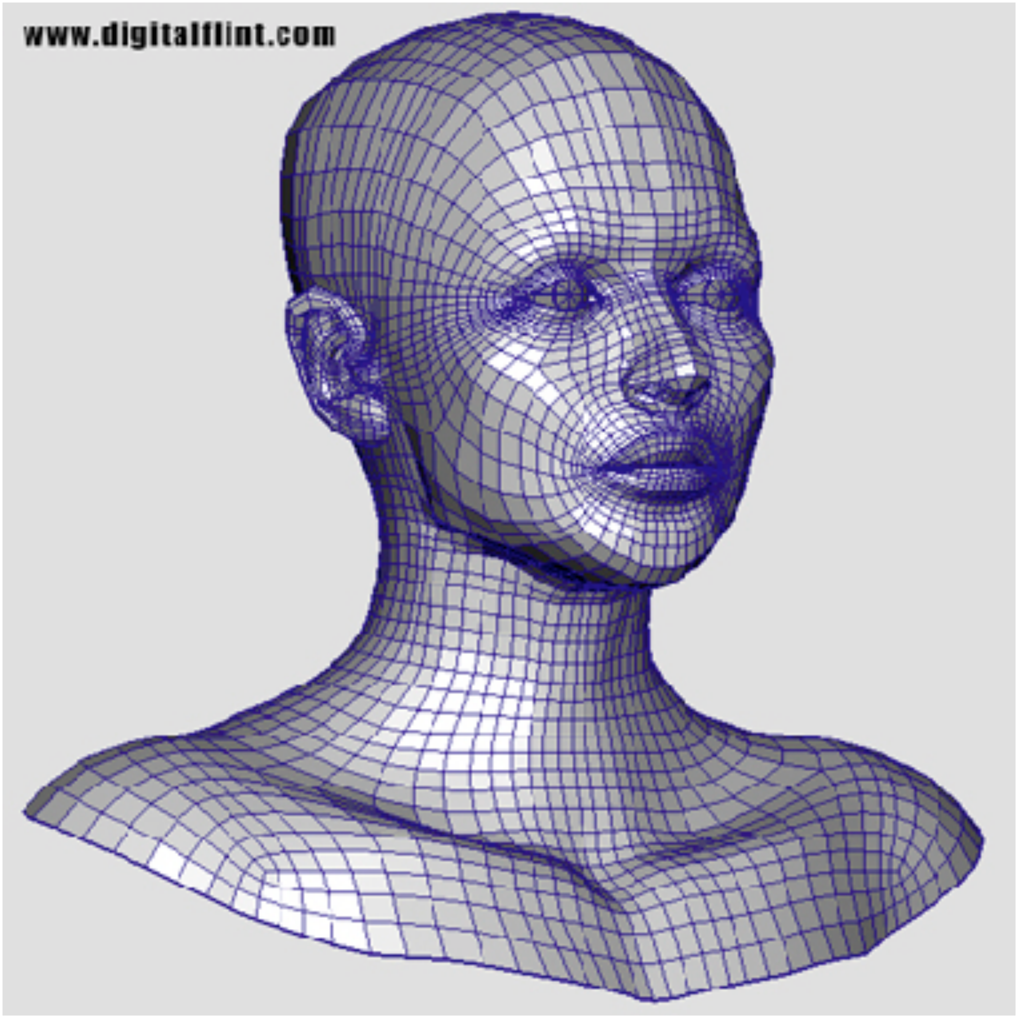
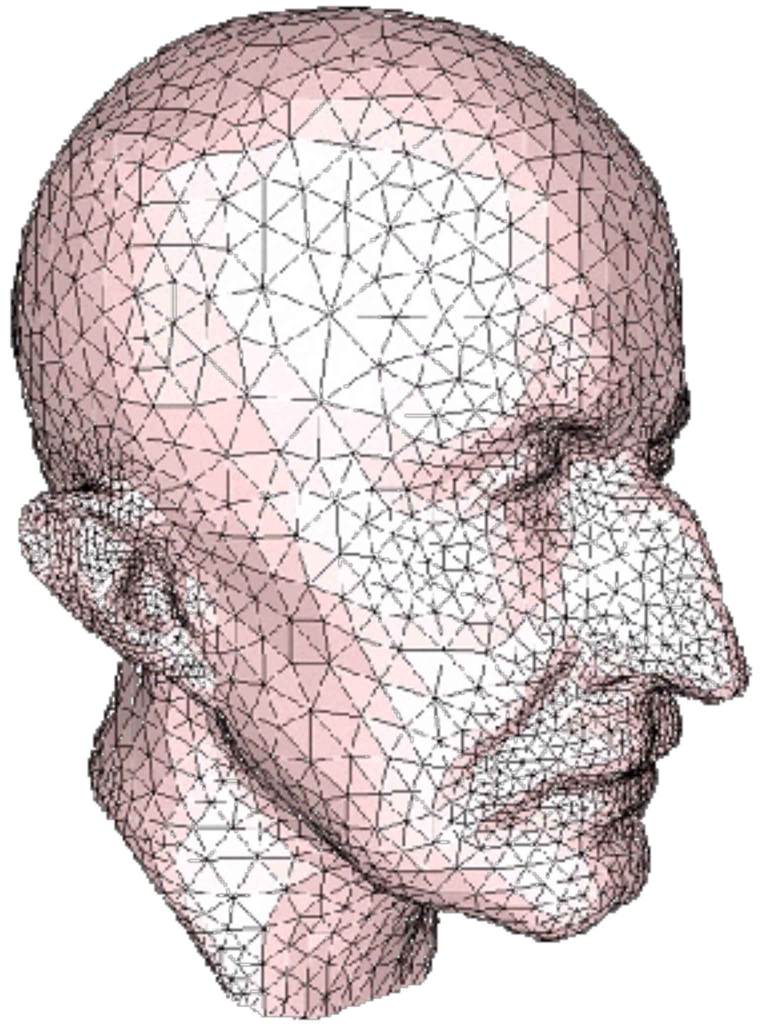
Polygonal vs. Triangle Meshes

- Triangles are flat and convex
  - Easy rasterization, normals
  - Uniformity (same # of vertices)
- 3-way symmetry is less natural

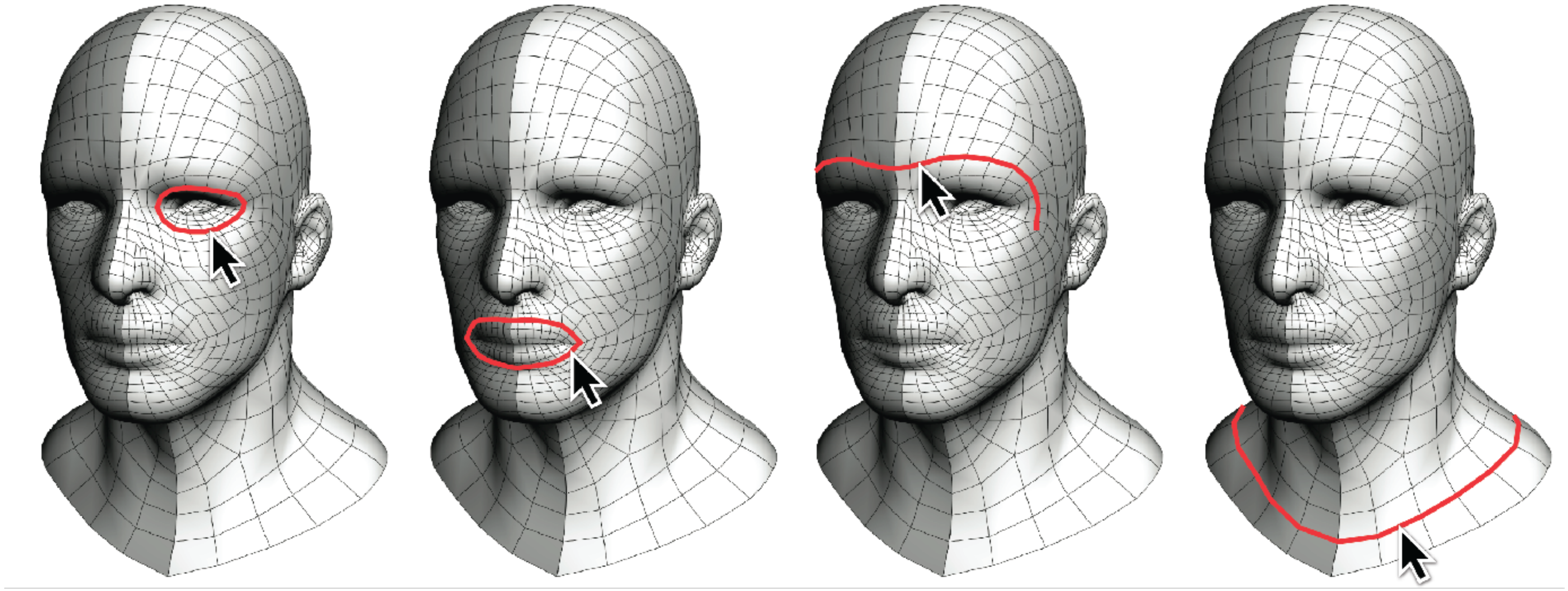


- General polygons are flexible
  - Quads have natural symmetry
- Can be non-planar, non-convex
  - Difficult for graphics hardware
- Varying number of vertices



Polygonal vs. Triangle Meshes

- Edge loops are ideal for editing



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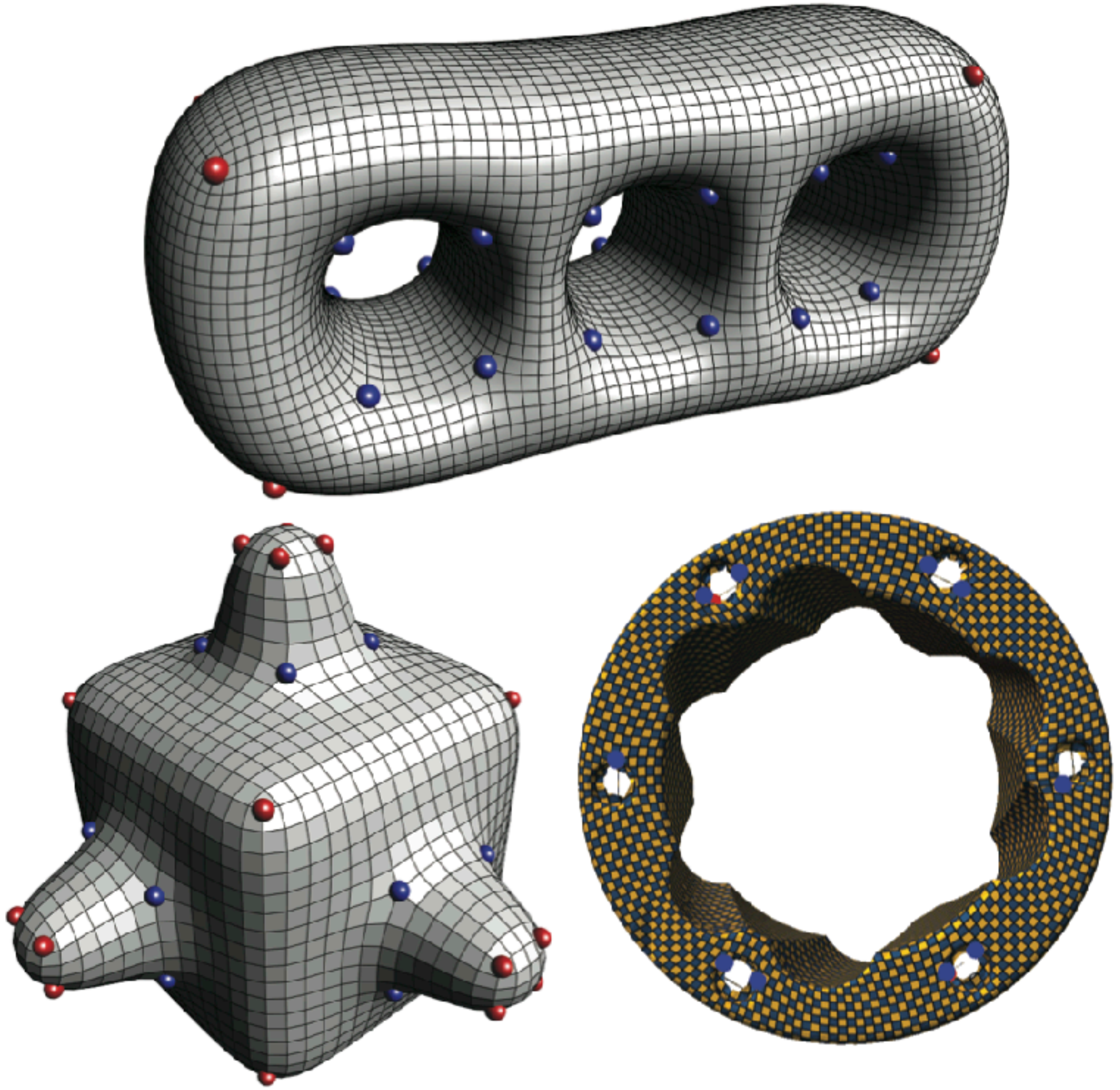
Polygonal vs. Triangle Meshes

- Quality of triangle meshes
  - Uniform Area
  - Angles close to 60
- Quality of quadrilateral meshes
  - Number of irregular vertices
  - Angles close to 90
  - Good edge flow



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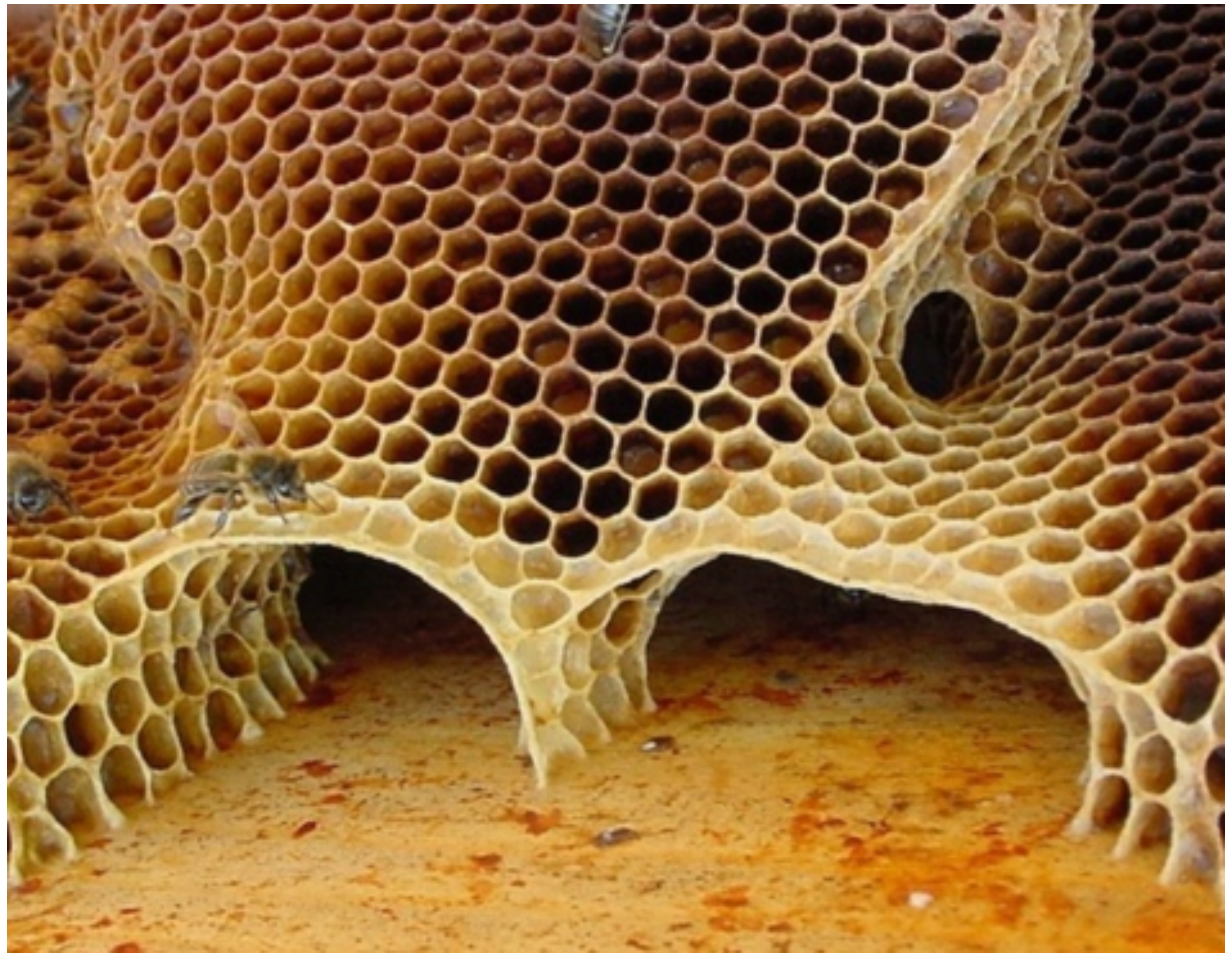
Polygonal vs. Triangle Meshes

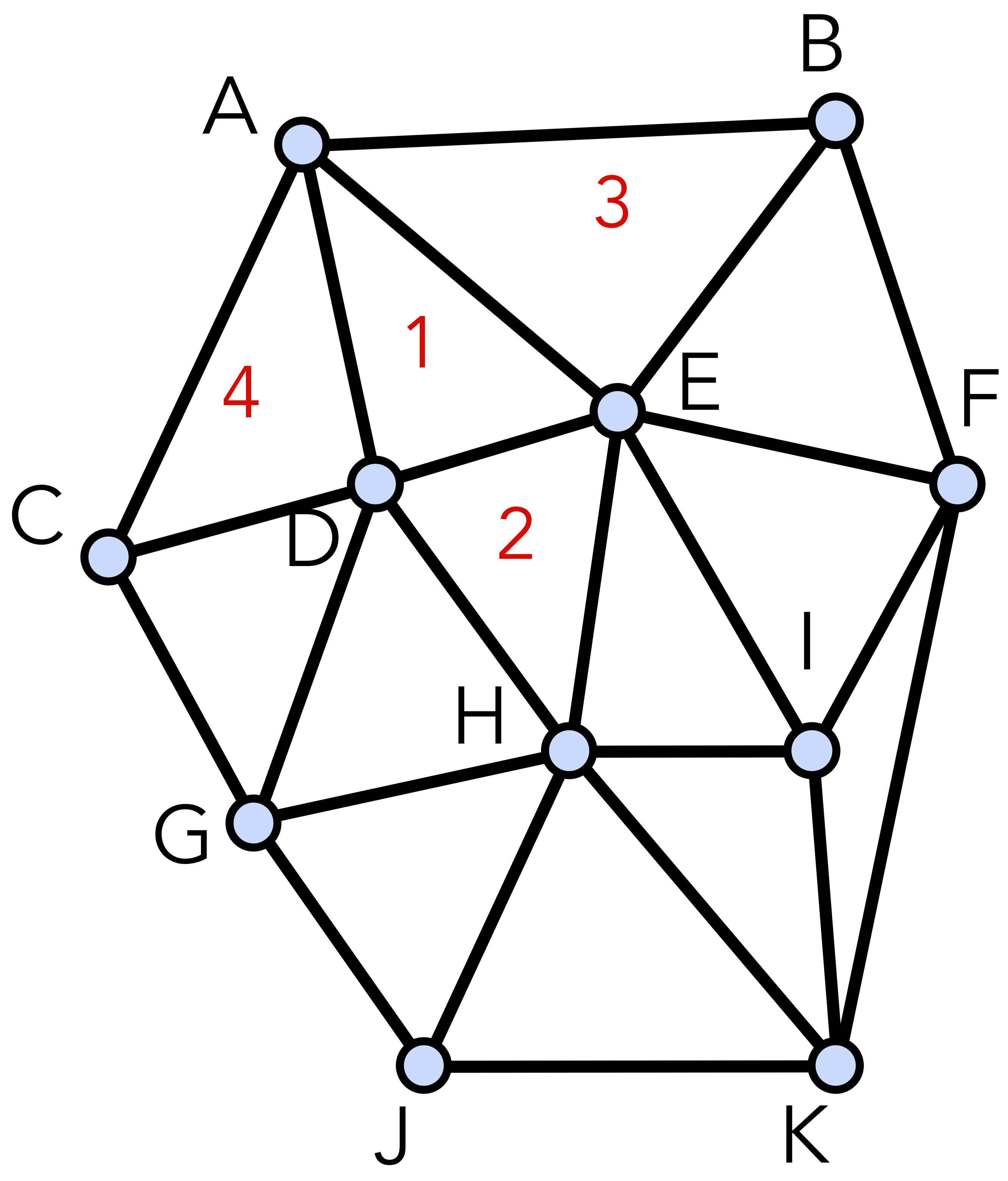


E. Van Egeraat

Data Structures

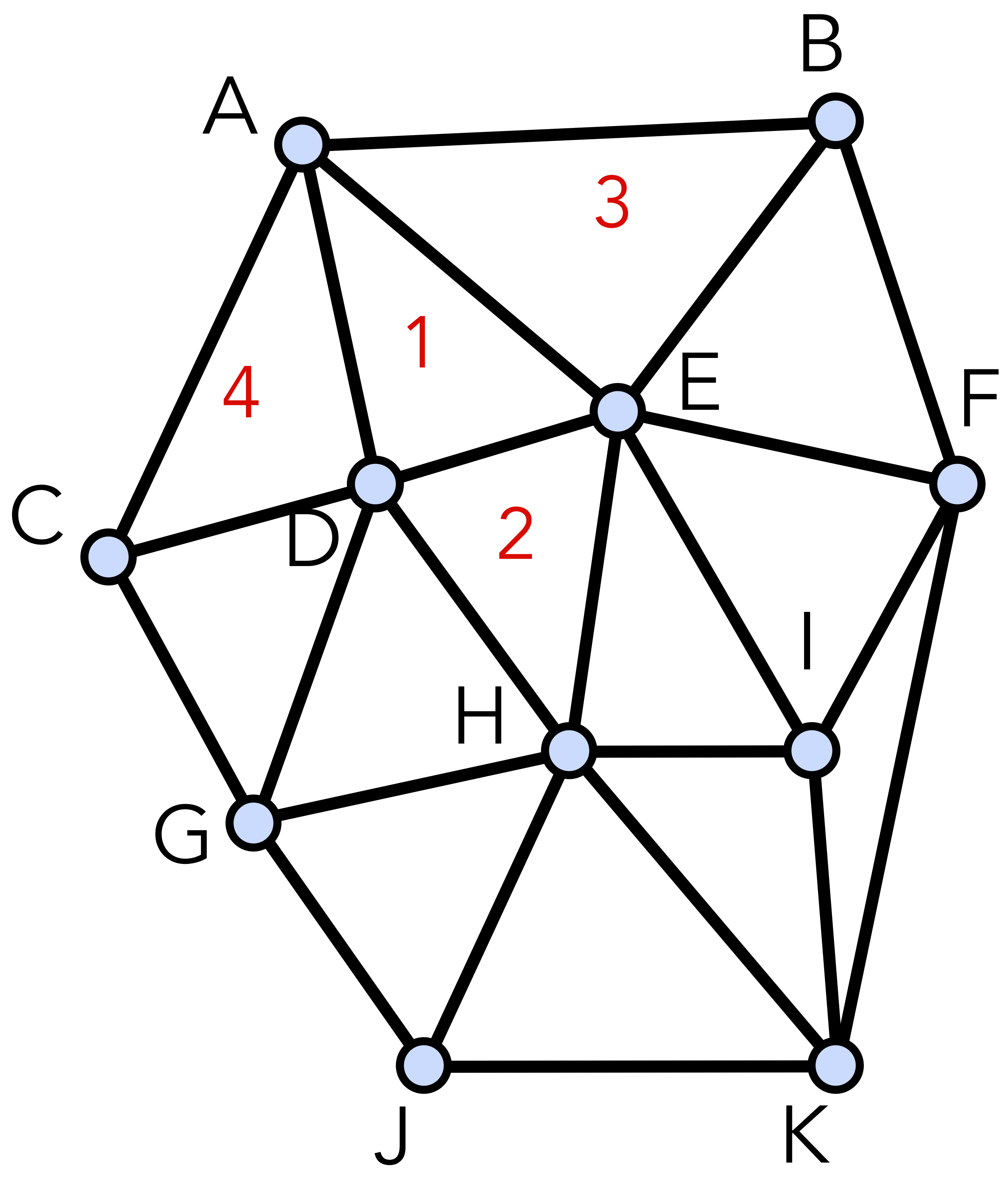
- What should be stored?
  - Geometry: 3D coordinates
  - Connectivity
    - Adjacency relationships
  - Attributes
    - Normal, color, texture coordinates
    - Per vertex, face, edge

Data Structures



- What should be supported?
  - Rendering
  - Geometry queries
    - What are the vertices of face #2?
    - Is vertex *A* adjacent to vertex *H*?
    - Which faces are adjacent to face #1?
  - Modifications
    - Remove/add a vertex/face
    - Vertex split, edge collapse

Data Structures



- How good is a data structure?
  - Time to construct
  - Time to answer a query
  - Time to perform an operation
  - Space complexity
  - Redundancy
- Criteria for design
  - Expected number of vertices
  - Available memory
  - Required operations
  - Distribution of operations

Triangle List

- STL format (used in CAD)
- Storage
  - Face: 3 positions
  - 4 bytes per coordinate
  - 36 bytes per face
    - Euler: $f = 2v$
    - $72 * v$ bytes for a mesh with v vertices
- No connectivity information

Triangles			
0	x0	y0	z0
1	x1	x1	z1
2	x2	y2	z2
3	x3	y3	z3
4	x4	y4	z4
5	x5	y5	z5
6	x6	y6	z6
…	…	…	…

Indexed Face Set

- Used in formats

  OBJ, OFF, WRL

- Storage
  - Vertex: position
  - Face: vertex indices
  - 12 bytes per vertex
  - 12 bytes per face
  - 36*v bytes for the mesh

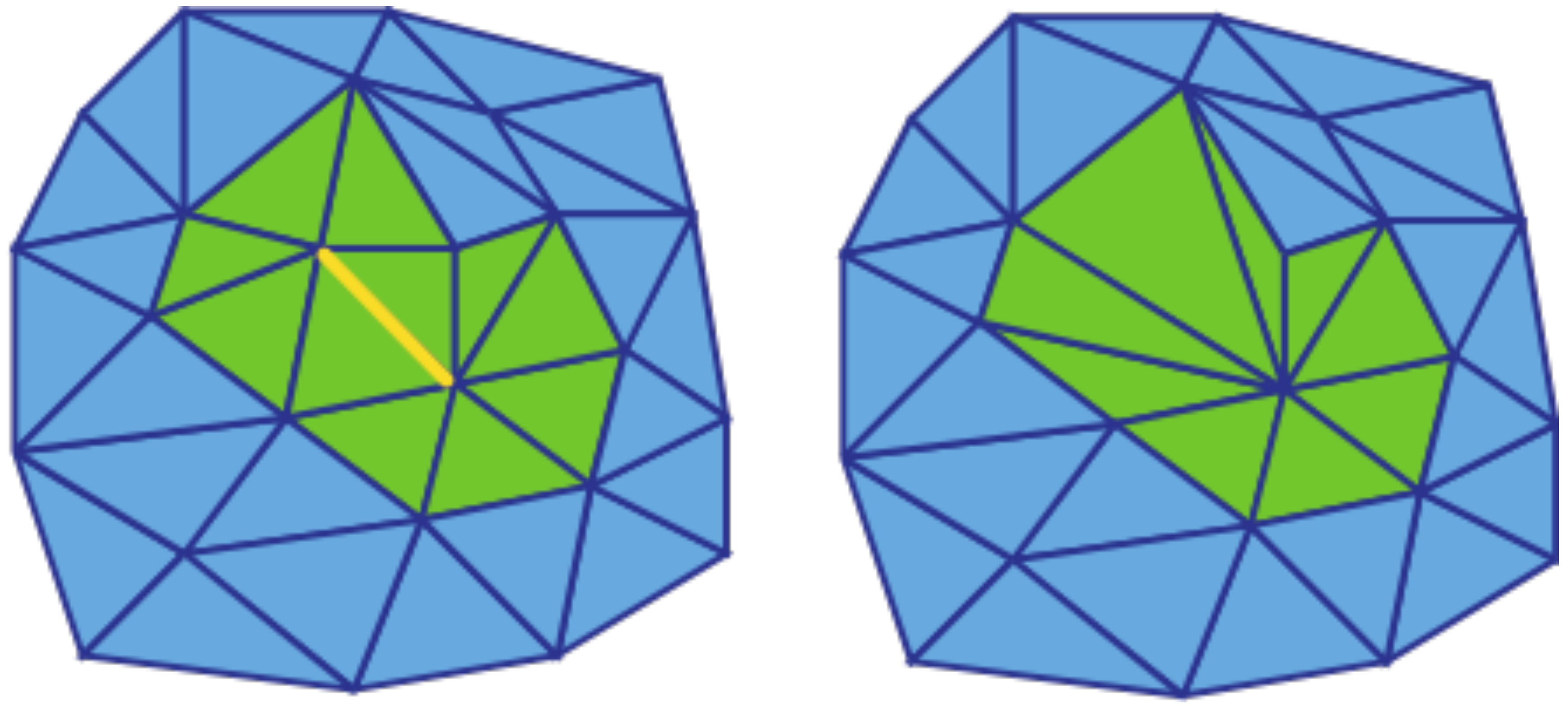
- No *explicit* neighborhood info

Vertices			
v0	x0	y0	z0
v1	x1	x1	z1
v2	x2	y2	z2
v3	x3	y3	z3
v4	x4	y4	z4
v5	x5	y5	z5
v6	x6	y6	z6
…	…	…	…

Triangles			
t0	v0	v1	v2
t1	v0	v1	v3
t2	v2	v4	v3
t3	v5	v2	v6
…	…	…	…

Indexed Face Set: Problems

- Information about neighbors is not explicit
  - Finding neighboring vertices/edges/faces costs $O(\#V)$ time!
  - Local mesh modifications cost $O(V)$

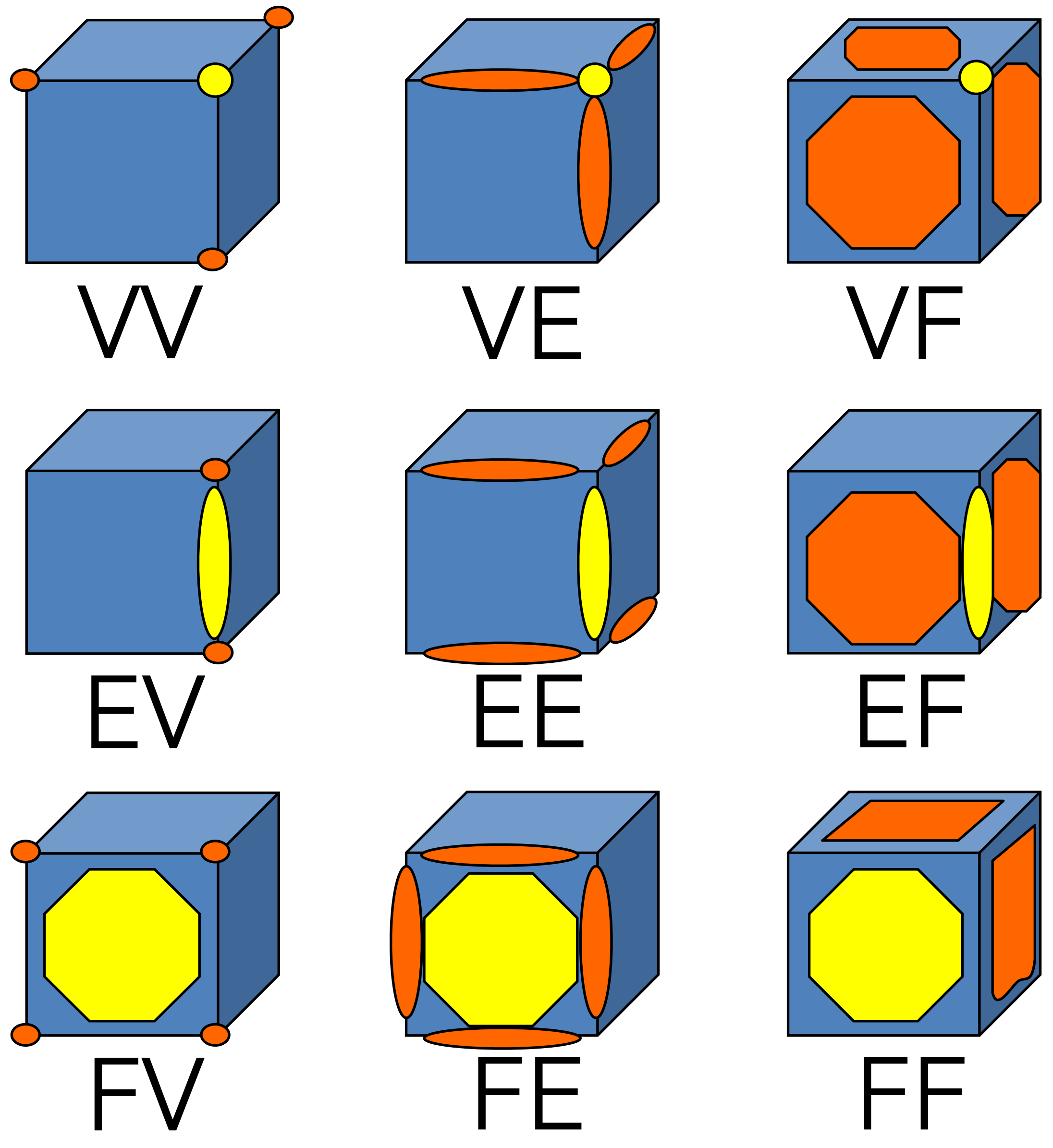


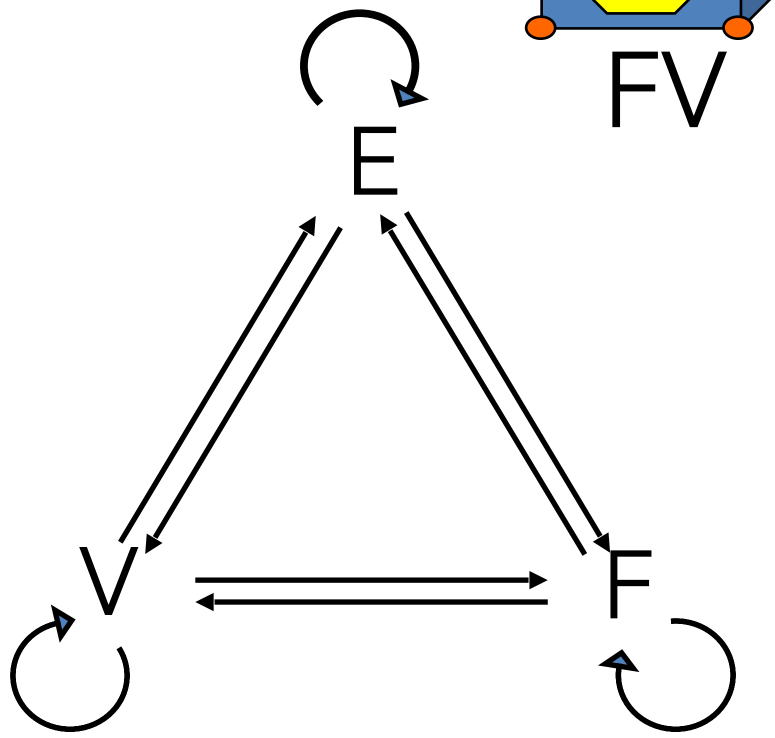
  - Breadth-first search costs $O(k*\#V)$
    where k = # found vertices

Neighborhood Relations

- All possible neighborhood relationships:

1. Vertex – Vertex VV
2. Vertex – Edge VE
3. Vertex – Face VF
4. Edge – Vertex EV
5. Edge – Edge EE
6. Edge – Face EF
7. Face – Vertex FV
8. Face – Edge FE
9. Face – Face FF





We'd like $O(1)$ time for queries and local updates of these relationships

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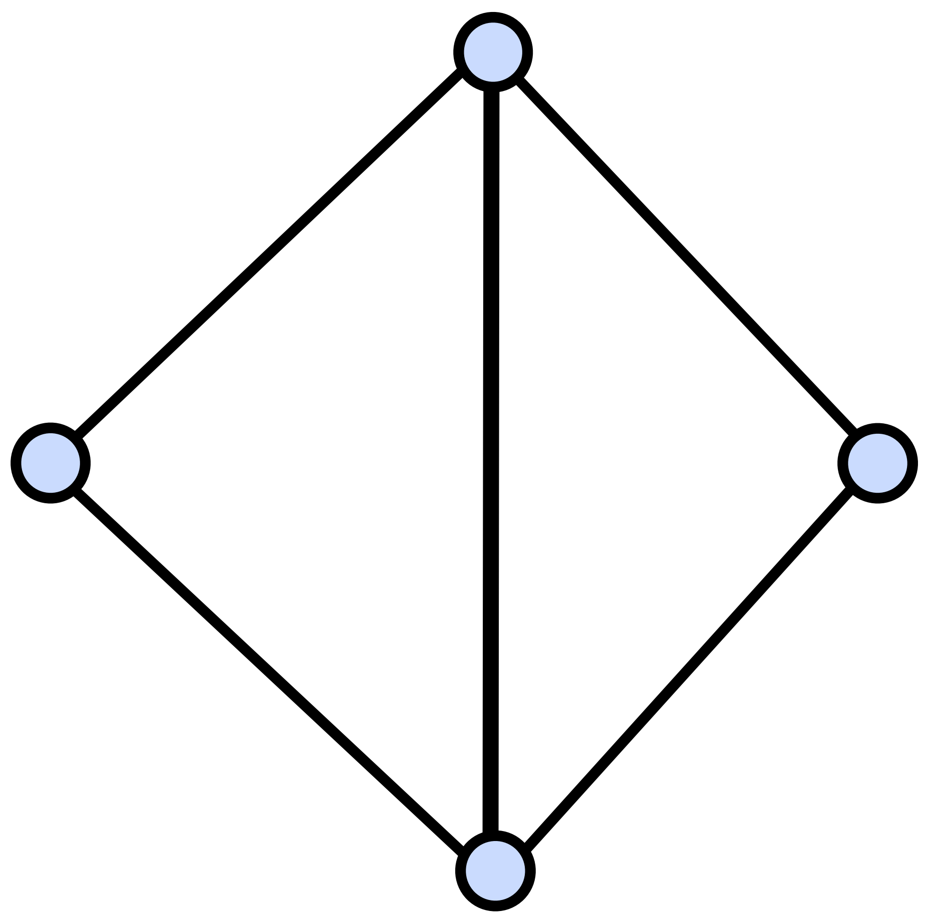
Halfedge data structure

- Introduce orientation into data structure
  - Oriented edges



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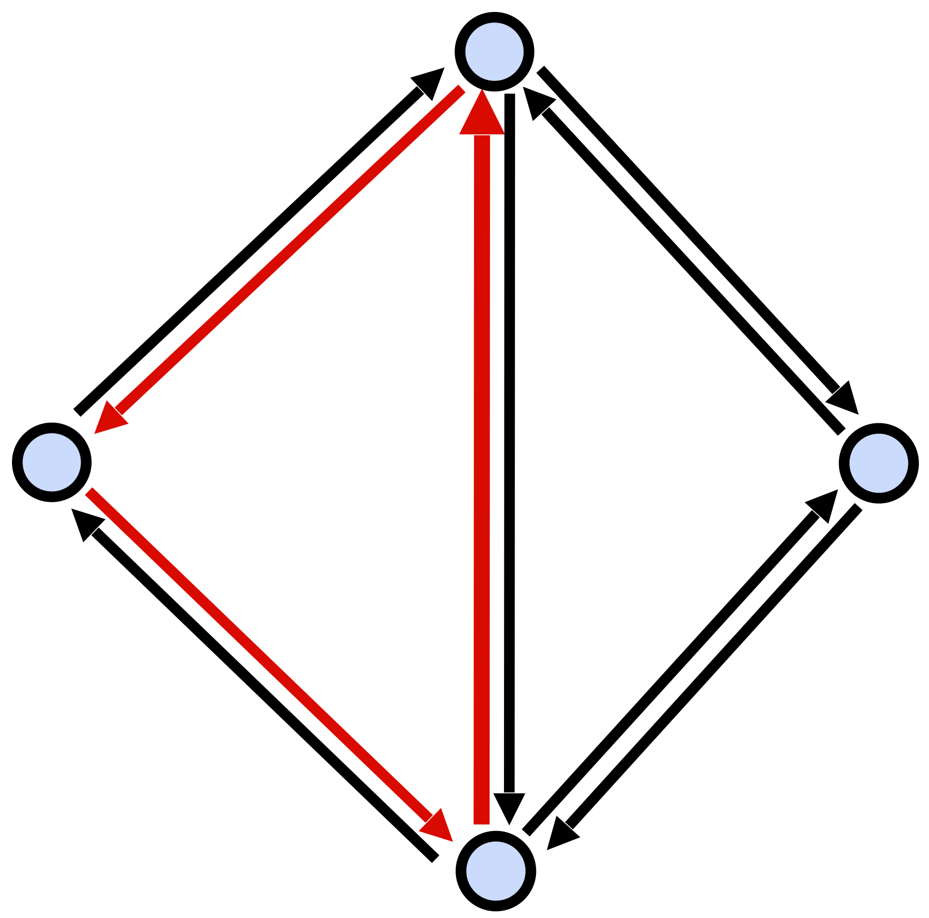
Halfedge data structure

- Introduce orientation into data structure
  - Oriented edges



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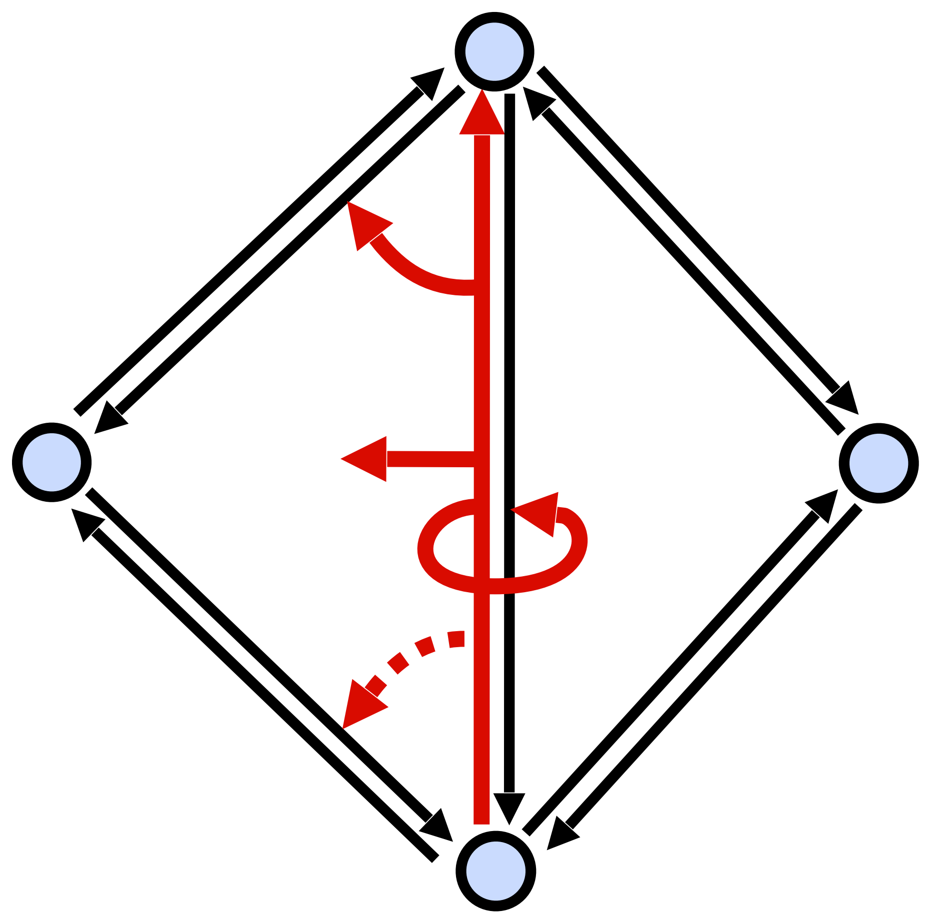
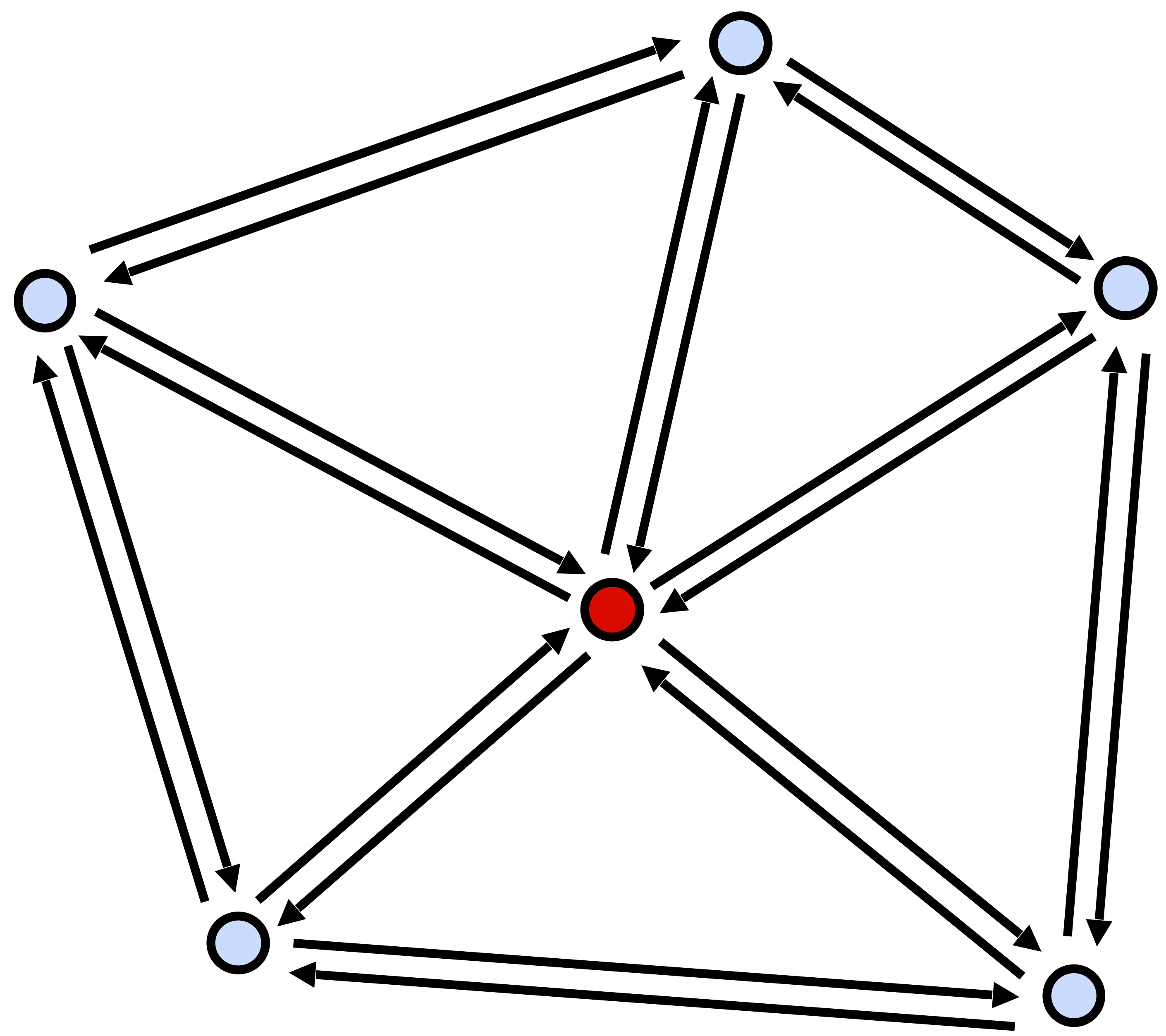
Halfedge data structure

- Introduce orientation into data structure
  - Oriented edges
- Vertex
  - Position
  - 1 outgoing halfedge index
- Halfedge
  - 1 origin vertex index
  - 1 incident face index
  - 3 next, prev, twin halfedge indices
- Face
  - 1 adjacent halfedge index
- Easy traversal, full connectivity



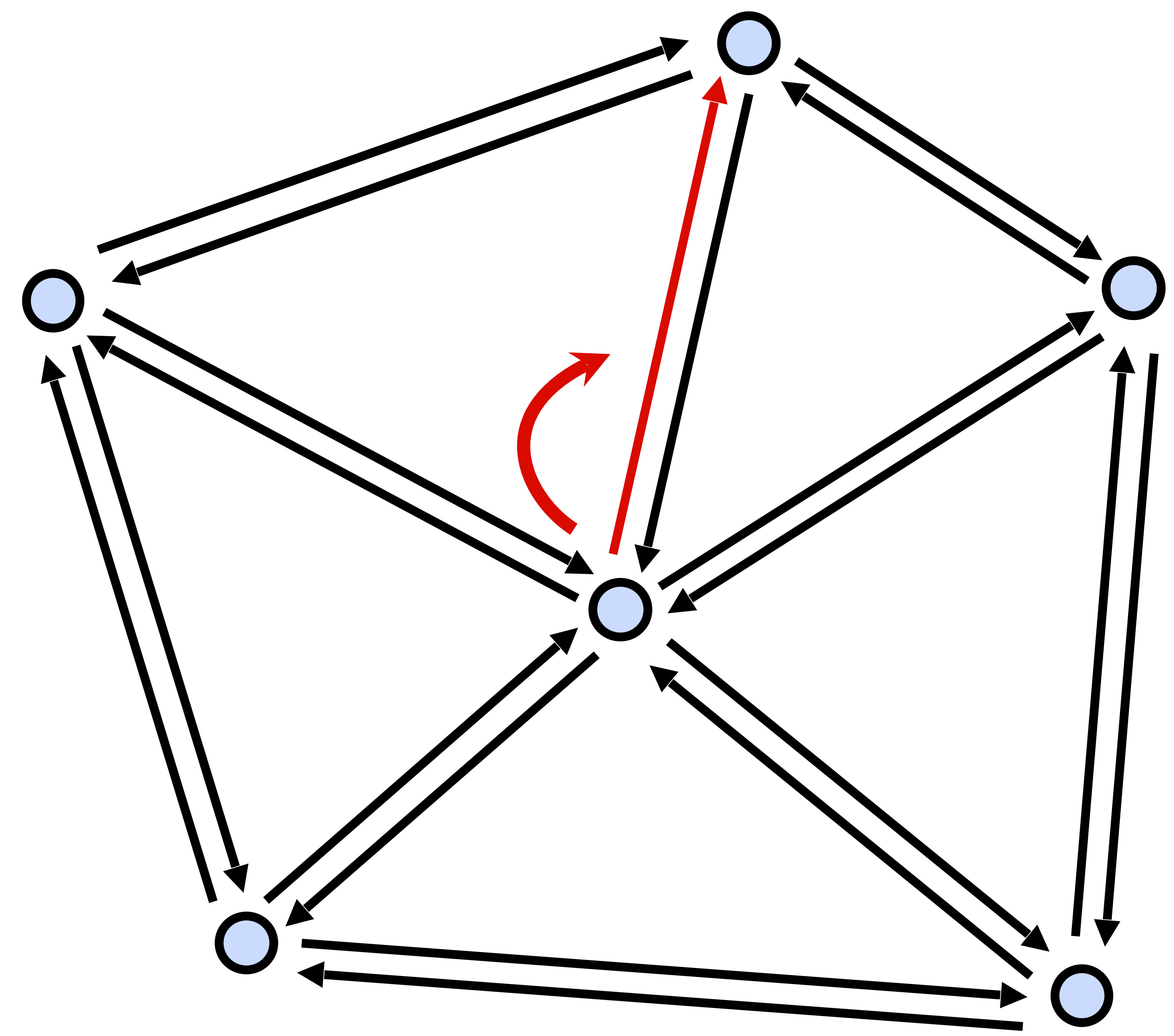
Halfedge data structure

- One-ring traversal
  - Start at vertex



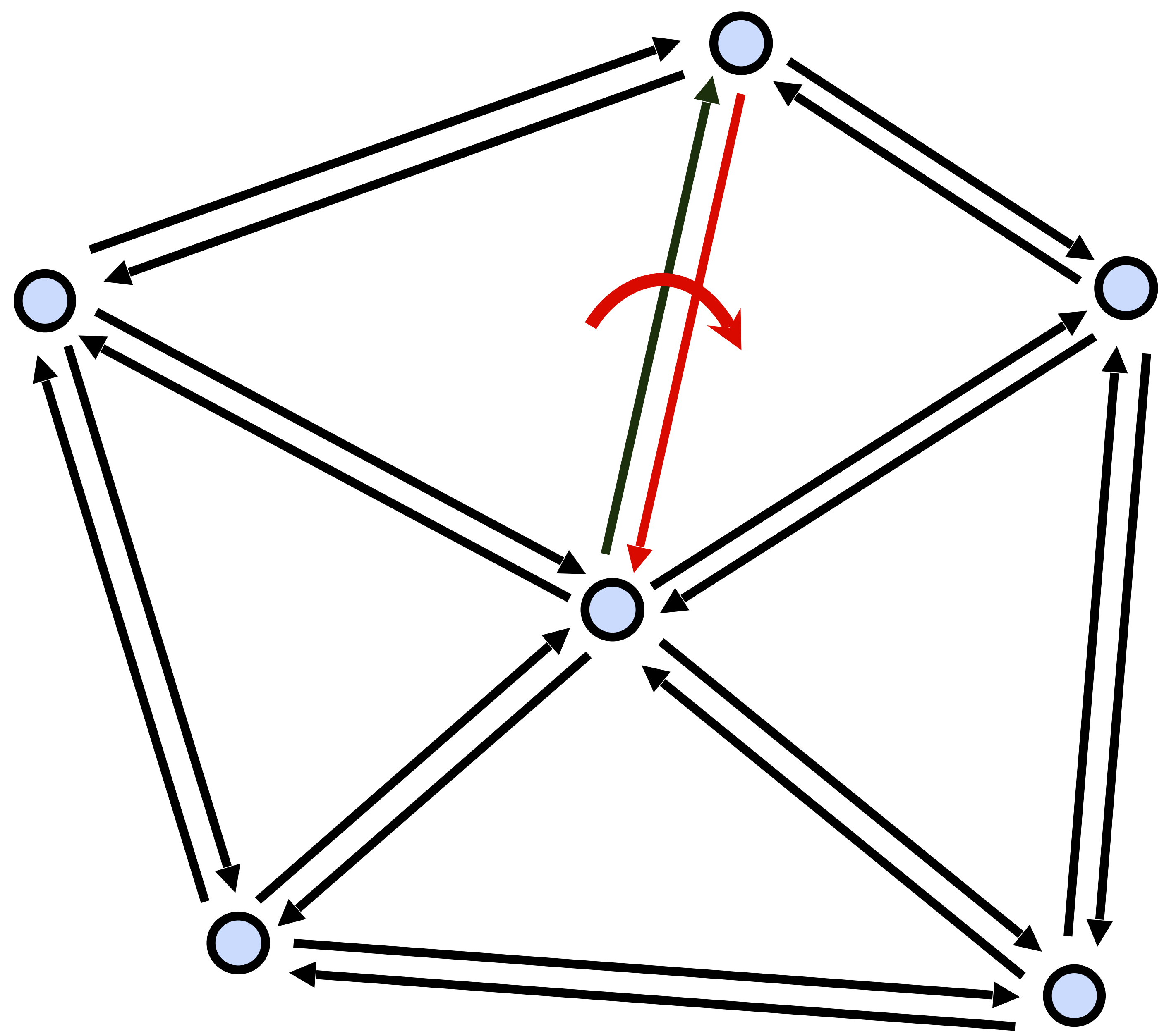
Halfedge data structure

- One-ring traversal
  - Start at vertex
  - Outgoing halfedge



Halfedge data structure

- One-ring traversal
  - Start at vertex
  - Outgoing halfedge
  - Twin halfedge



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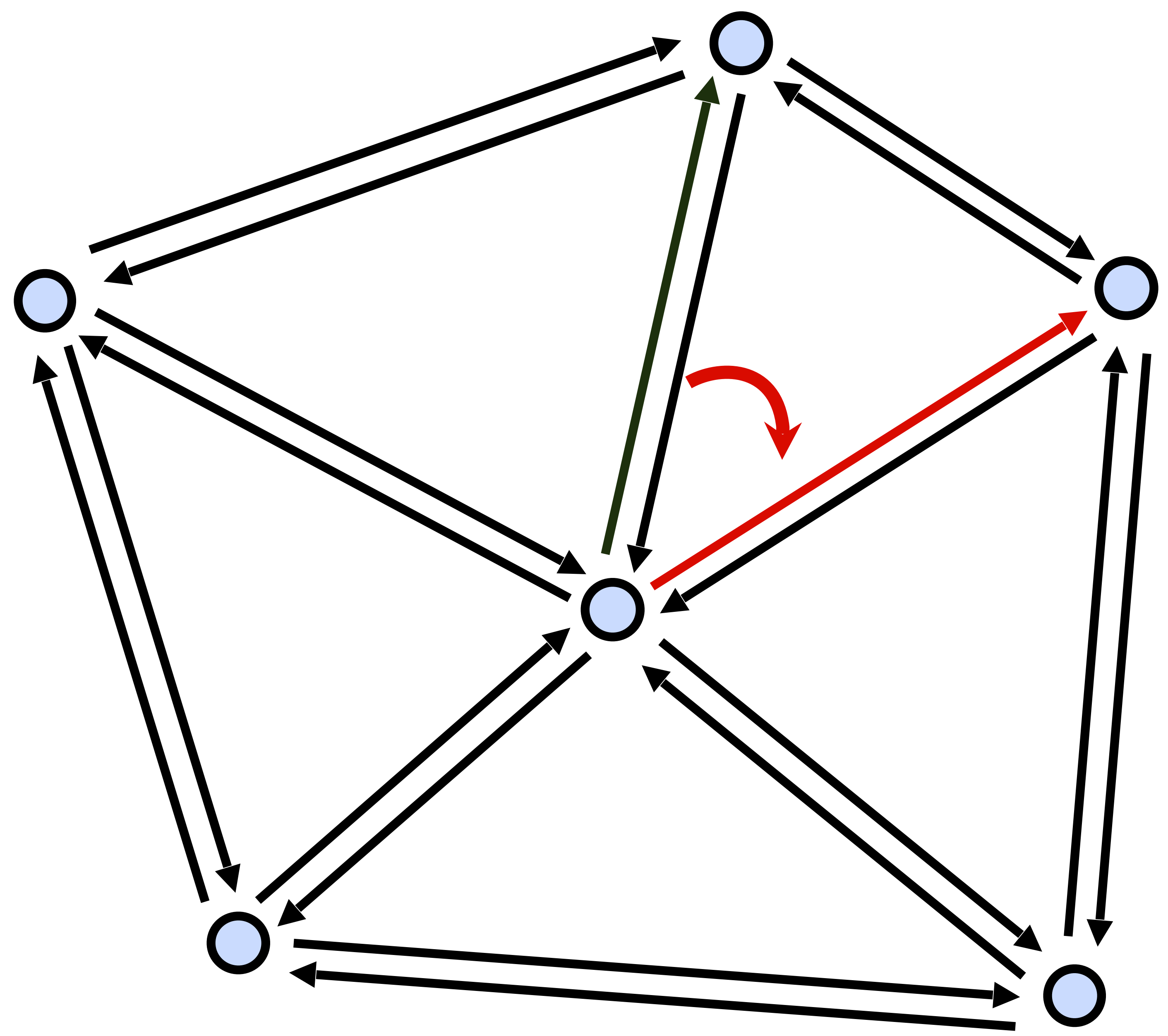
Halfedge data structure

- One-ring traversal
  - Start at vertex
  - Outgoing halfedge
  - Twin halfedge
  - Next halfedge



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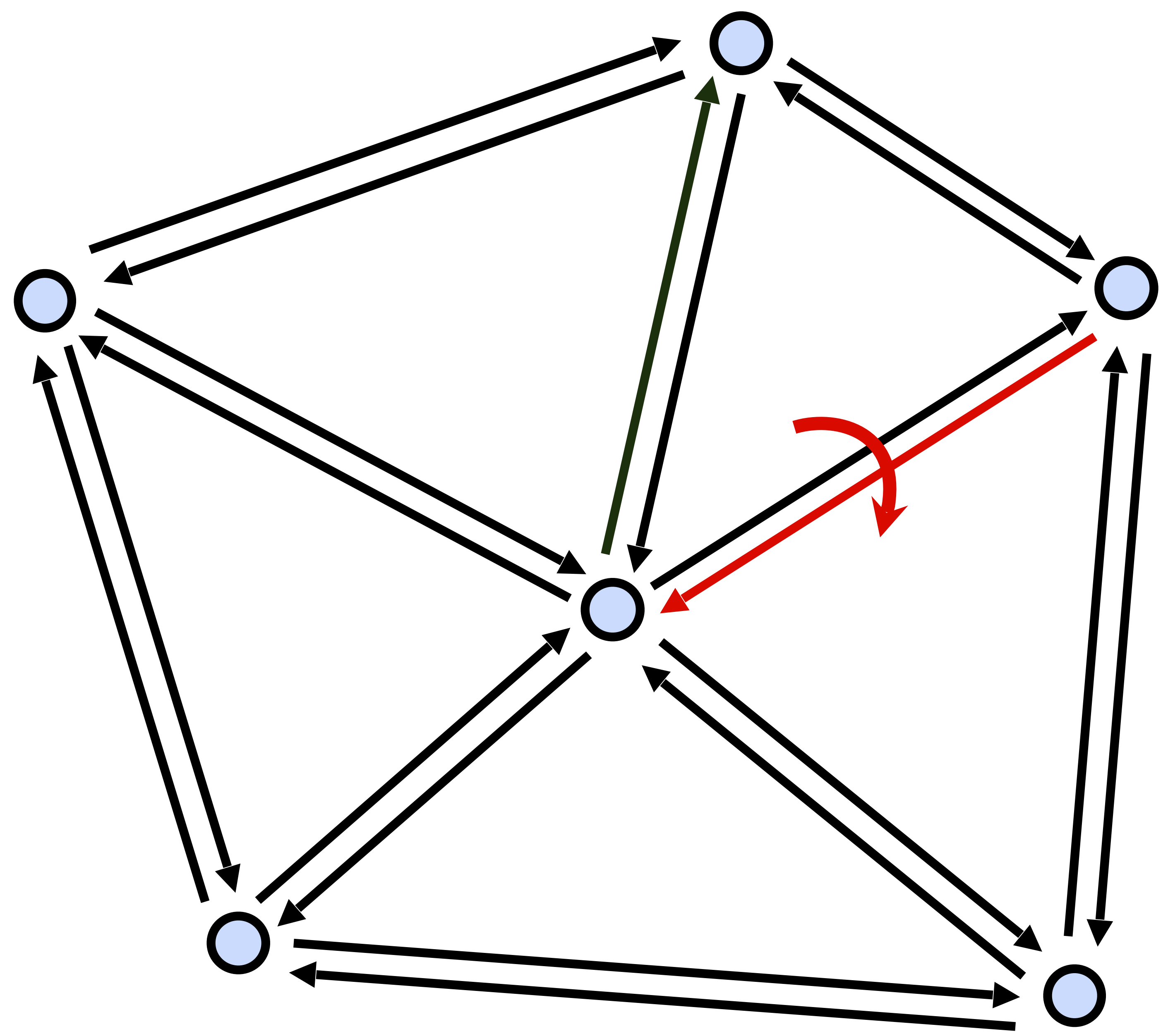
Halfedge data structure

- One-ring traversal
  - Start at vertex
  - Outgoing halfedge
  - Twin halfedge
  - Next halfedge
  - Twin …



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Halfedge data structure

- Pros: *(assuming bounded vertex valence)*
  - $O(1)$ time for neighborhood relationship queries
  - $O(1)$ time and space for local modifications (edge collapse, vertex insertion…)
- Cons:
  - Heavy – requires storing and managing extra pointers
  - Not as trivial as Indexed Face Set for rendering with OpenGL/DirectX

Halfedge Libraries

- CGAL
  - www.cgal.org
  - Computational geometry
- OpenMesh
  - www.openmesh.org
  - Mesh processing

- We will not implement a half-edge data structure in the class. Instead we will work with Indexed Face Set and augment it to have fast queries.

References

- Polygon Mesh Processing Book, Chapter 2