Shape Deformation

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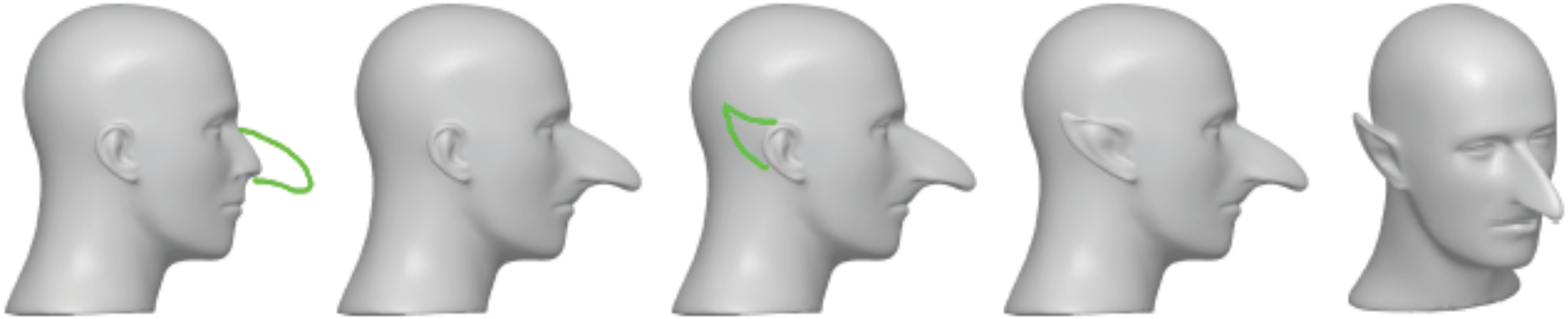
Why Shape Deformation?

Animation

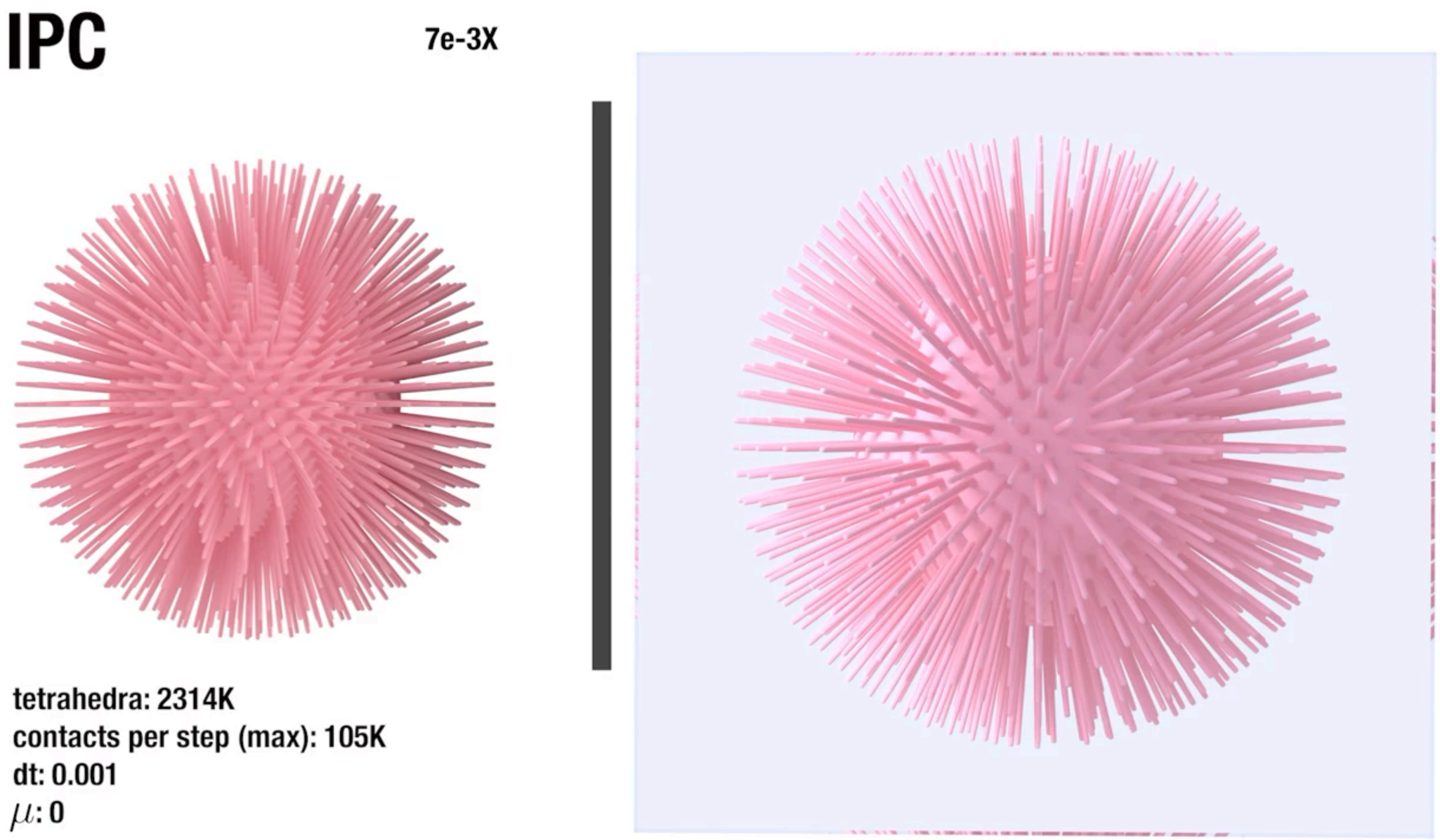
Why Shape Deformation?

Mesh Editing



Why Shape Deformation?

Simulation of Physical Phenomena



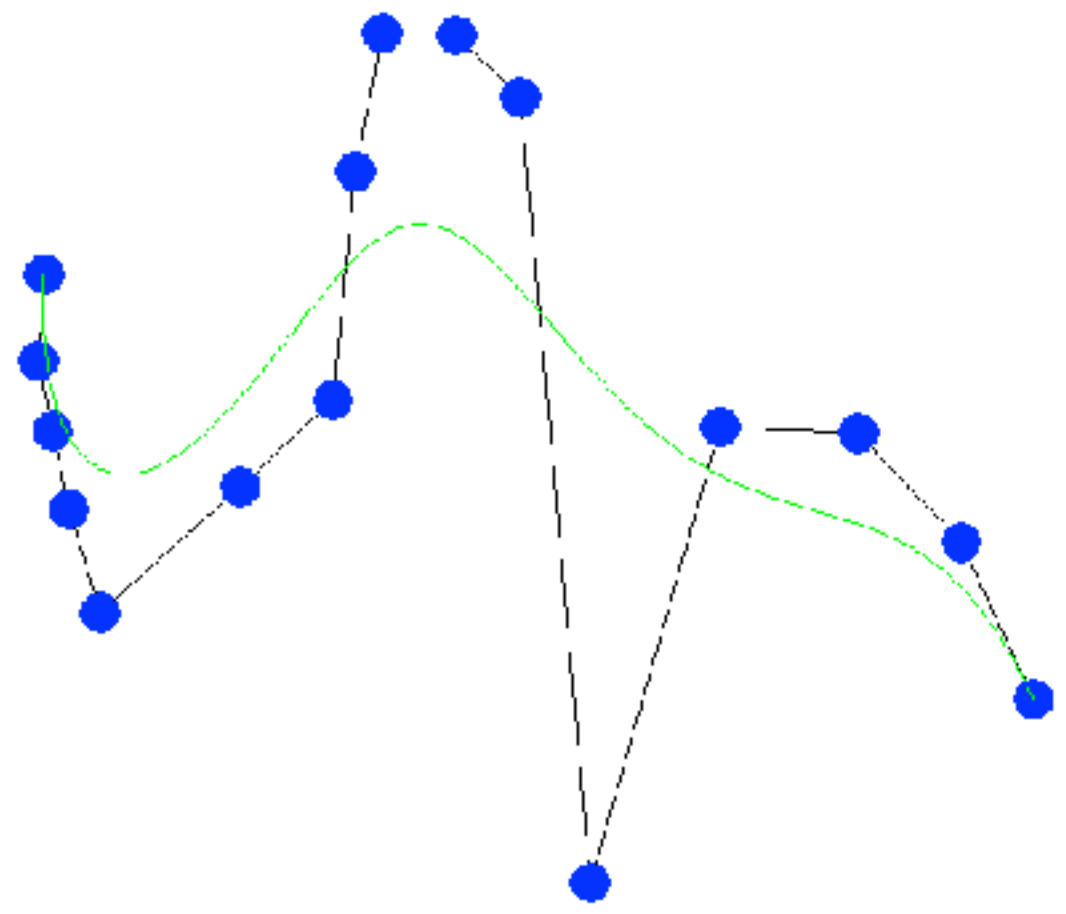
https://ipc-sim.github.io

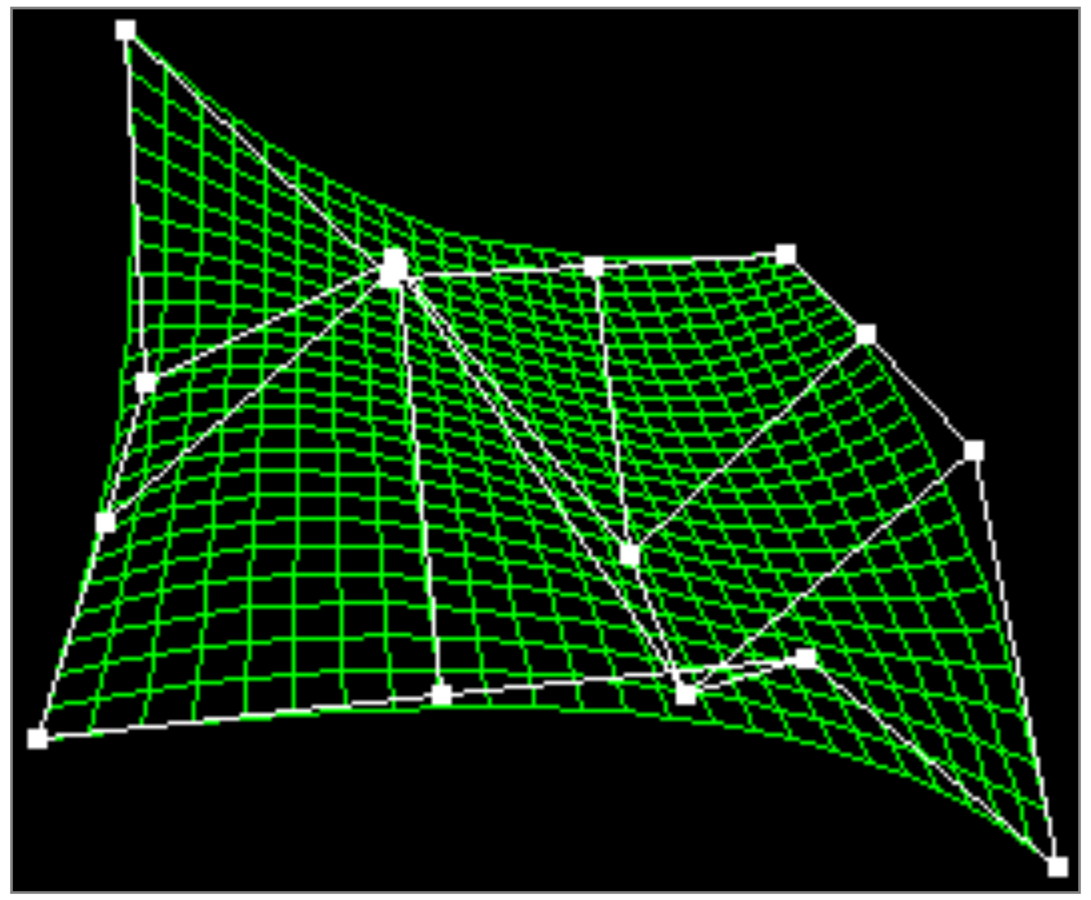
Parametric Curves and Surfaces

- Deformation by control point manipulation

- Built-in deformation mechanism

- Control structure is pre-set (you cannot pull on arbitrary points)





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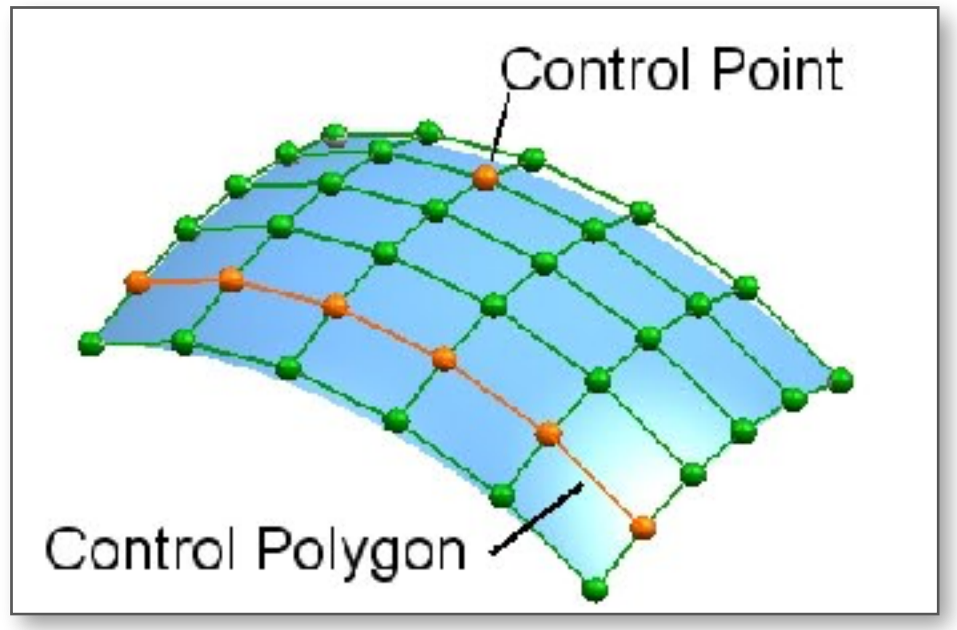
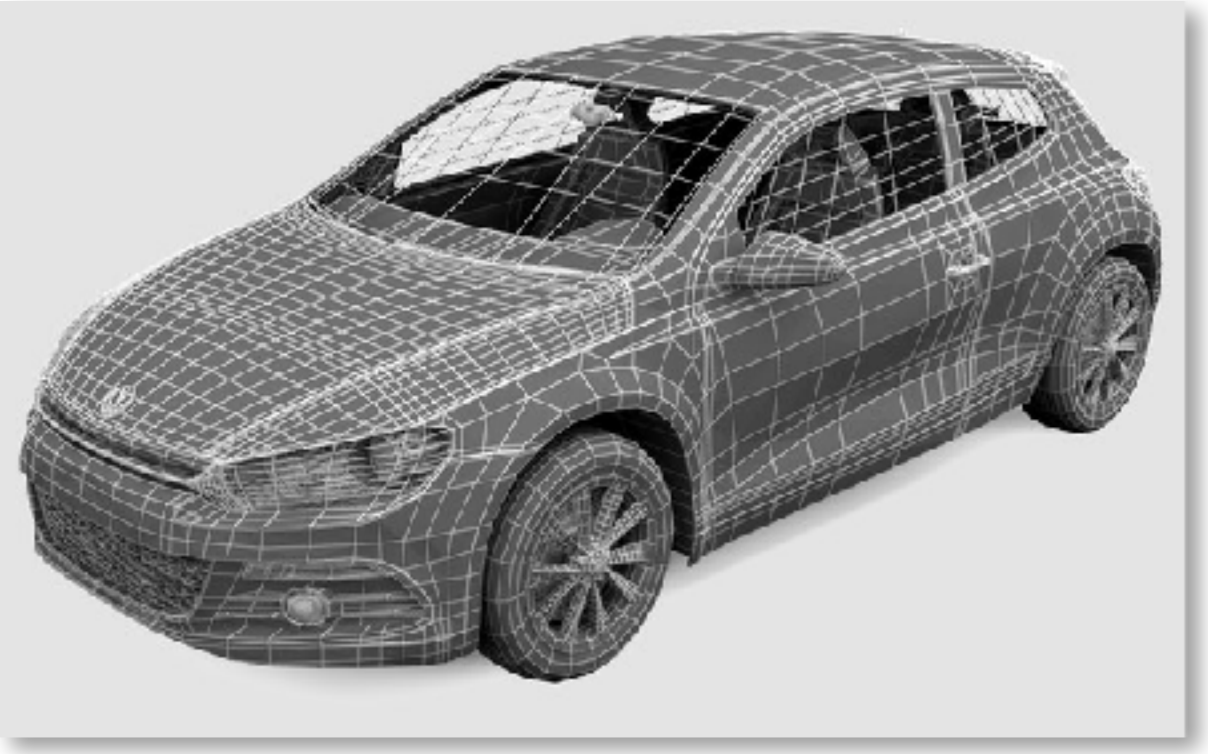
Traditional CAD vs Unstructured Meshes



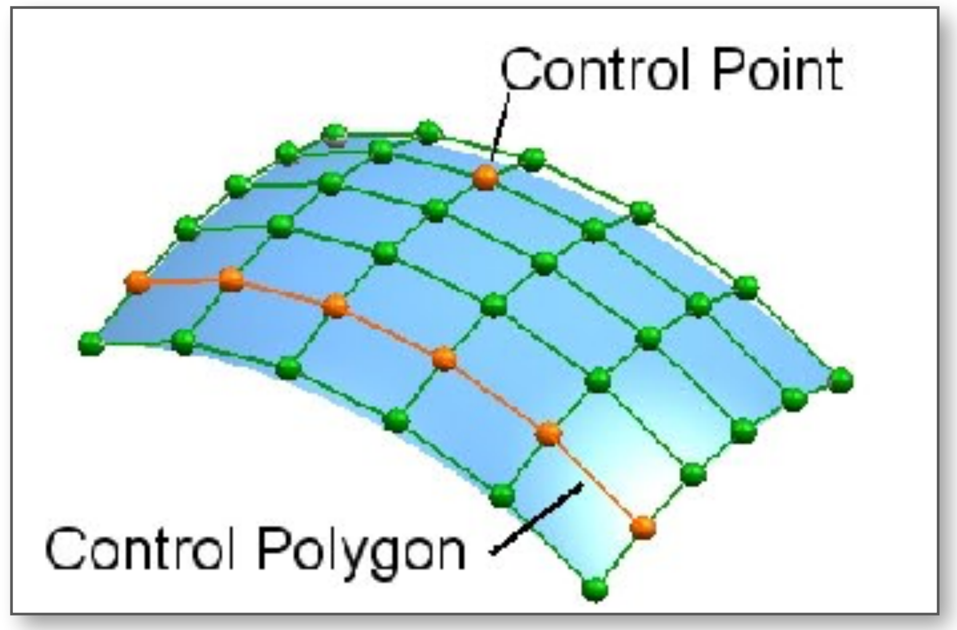
$$s(u,v) = \sum_{i,j} \boxed{\mathbf{p}_{i,j}} B_i(u) B_j(v)$$



$$\min_{\mathbf{x}'} \int_{\mathcal{S}} \|\Delta_{\mathcal{S}} \mathbf{x}' - \delta_0\|^2$$
$$s.t. \; \boxed{\mathbf{x}'|_{\mathcal{C}}} = \mathbf{x}_{\text{fixed}}$$

images from Jacobson et al., SGP 2010

Traditional CAD vs Unstructured Meshes



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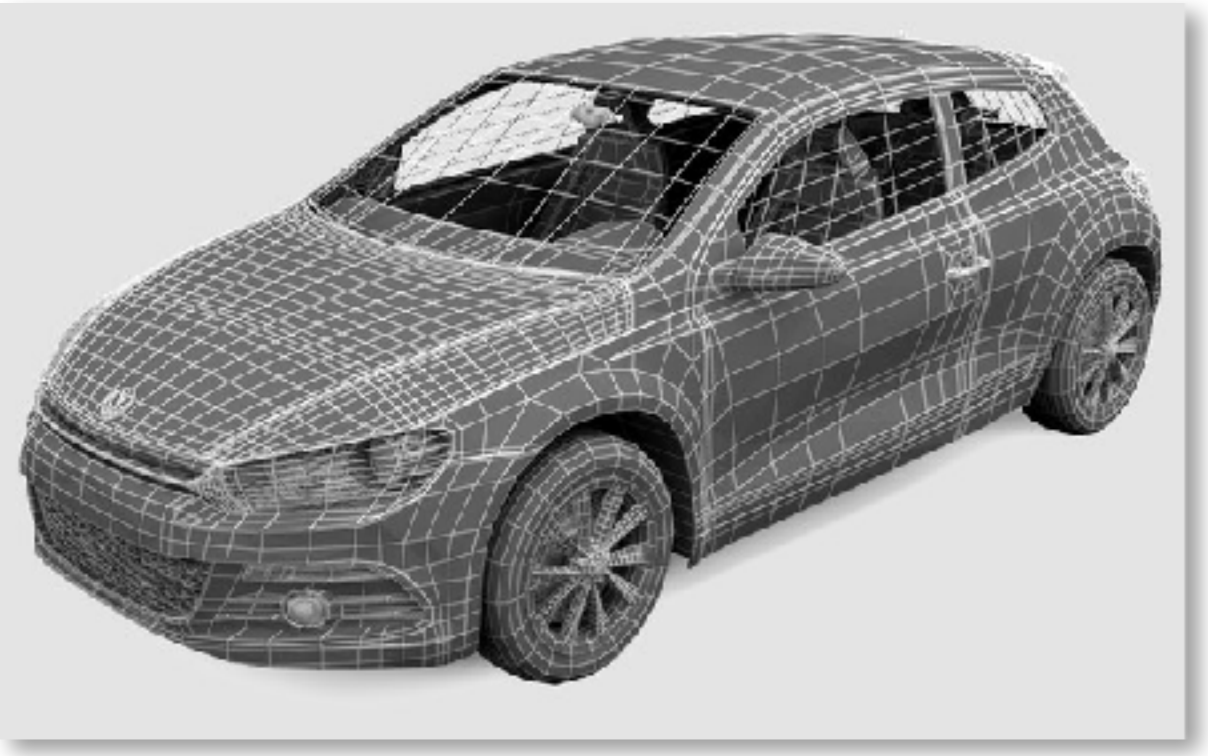
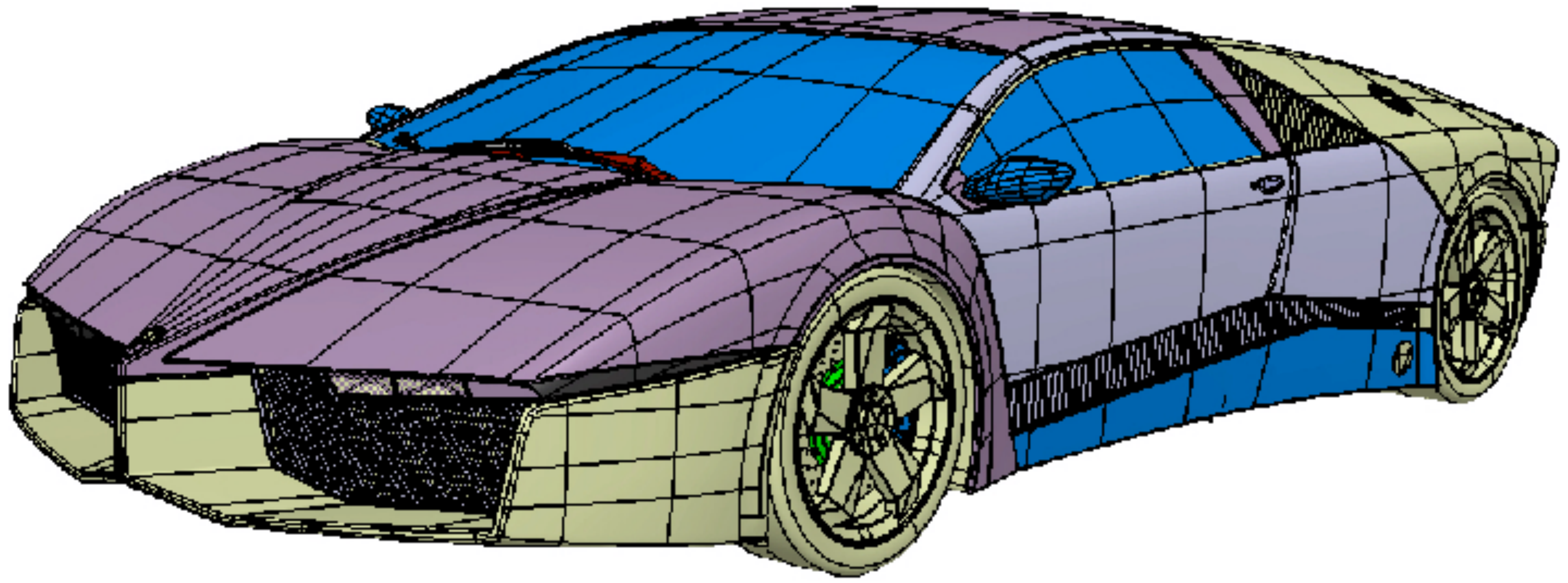
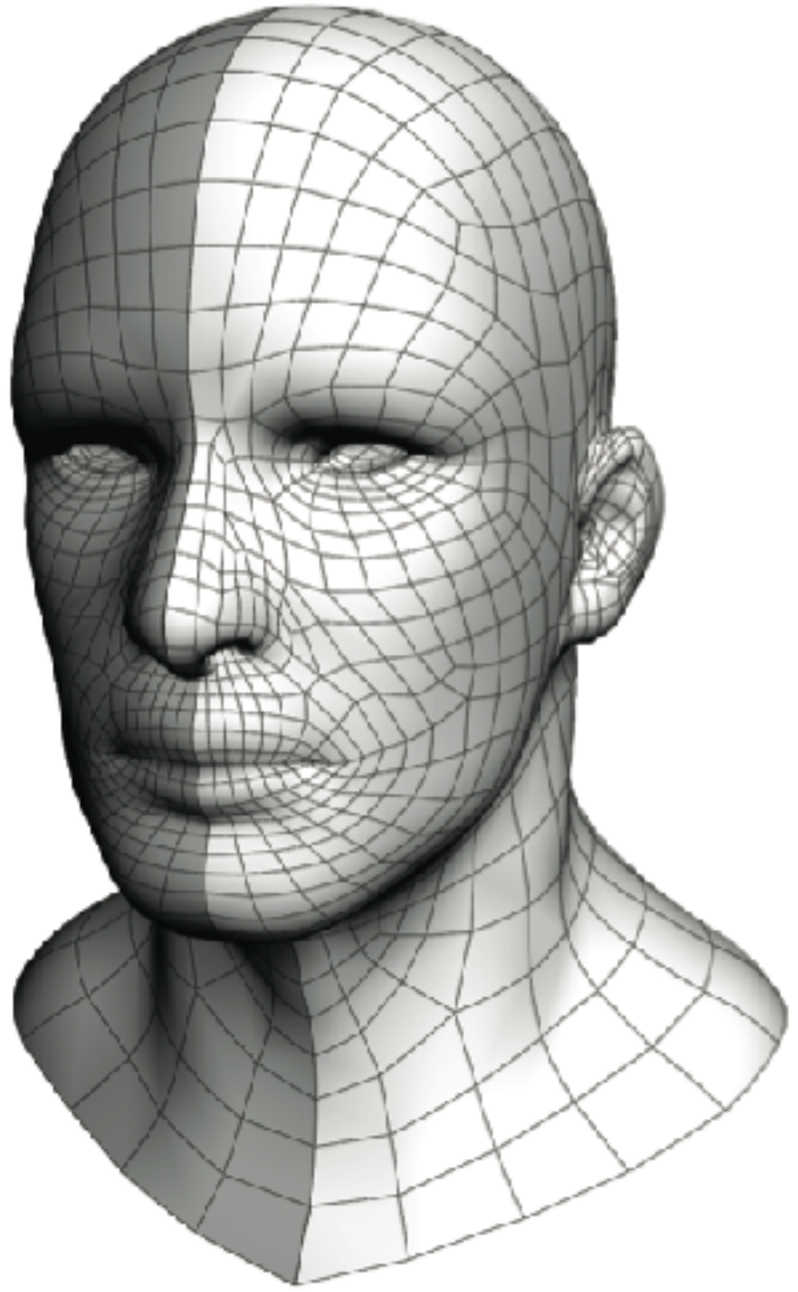


$$\min_{\mathbf{x}'} \int_{\mathcal{S}} \|\Delta_{\mathcal{S}} \mathbf{x}' - \delta_0\|^2$$
$$s.t. \ \boxed{\mathbf{x}'|_{\mathcal{C}}} = \mathbf{x}_{\text{fixed}}$$

Parametric Curves and Surfaces

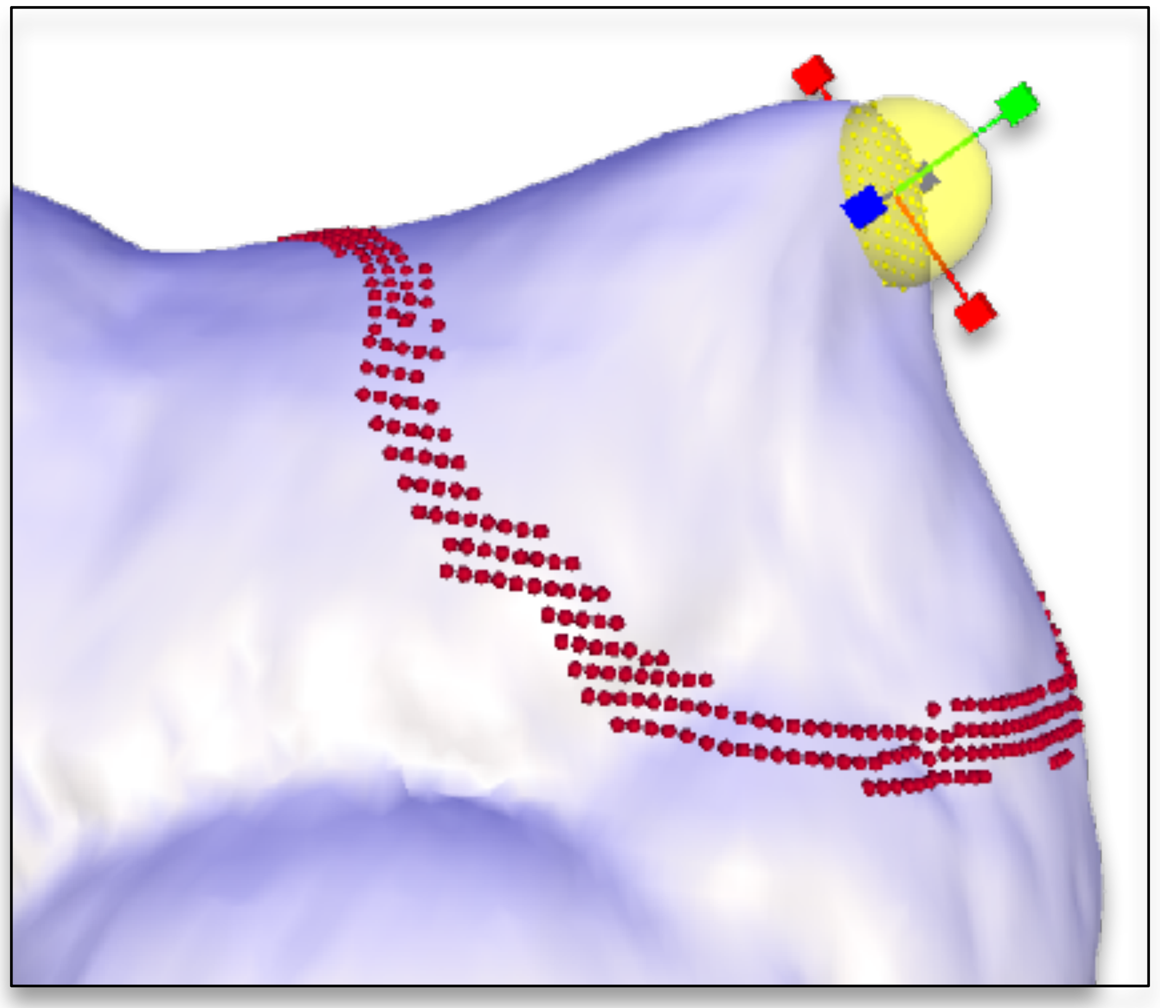
- Hard to change / adapt control structure to user needs
- Hard to experiment, need a precise idea of what will be modeled.





Mesh Deformation

- Naïve method: dragging single vertices
- Smarter:
  - Introduce a small set of deformation handles
    - Makes deformation/editing easier
    - Introduces a trade-off between degrees of freedom and simplicity of the deformation task
  - Create a small set of control parameters
    - Affine transformations



Common Paradigms

- Space deformations
  - Deforms some 2D/3D space using a *cage*
  - Deformation propagation to all points in the space
  - Independent of shape representation

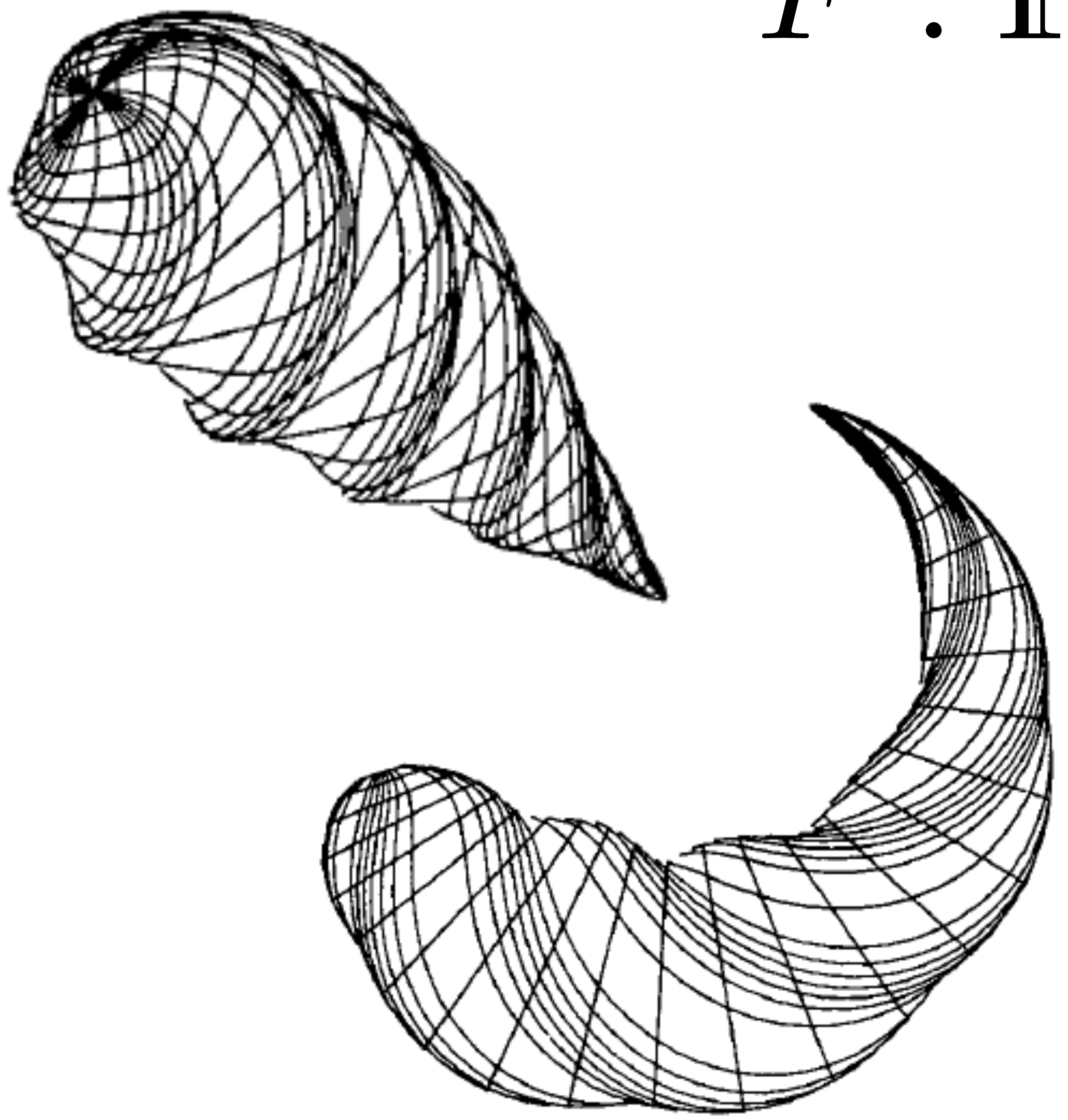
- Surface based deformations
  - Optimization on the surface
  - Physically motivated: variants of elastic energy minimization

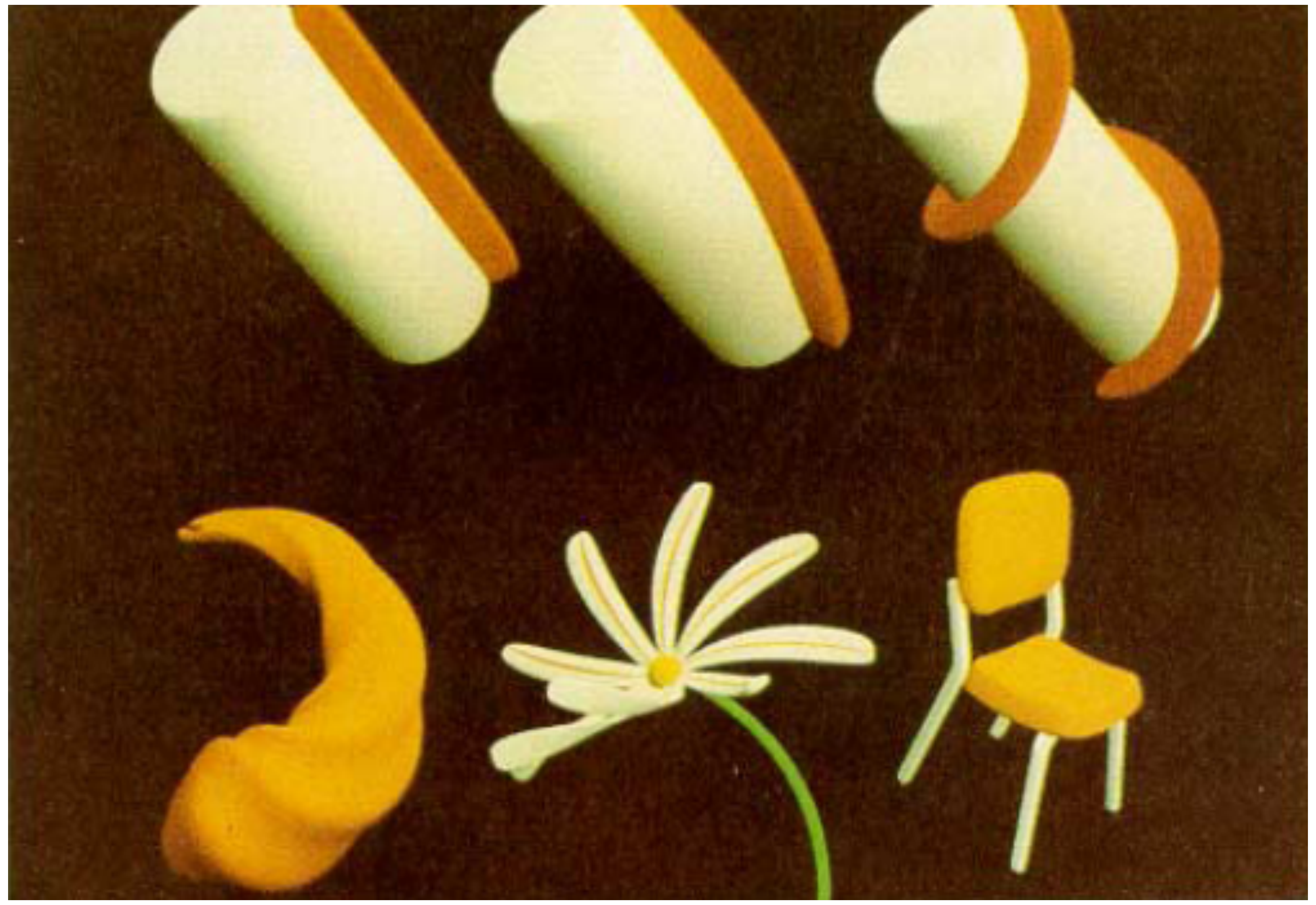
Space Deformations

Early seminal works in computer graphics

- Global and local deformation of solids [Barr 1984] http://dl.acm.org/citation.cfm?id=808573

$$F : \mathbb{R}^3 \to \mathbb{R}^3$$





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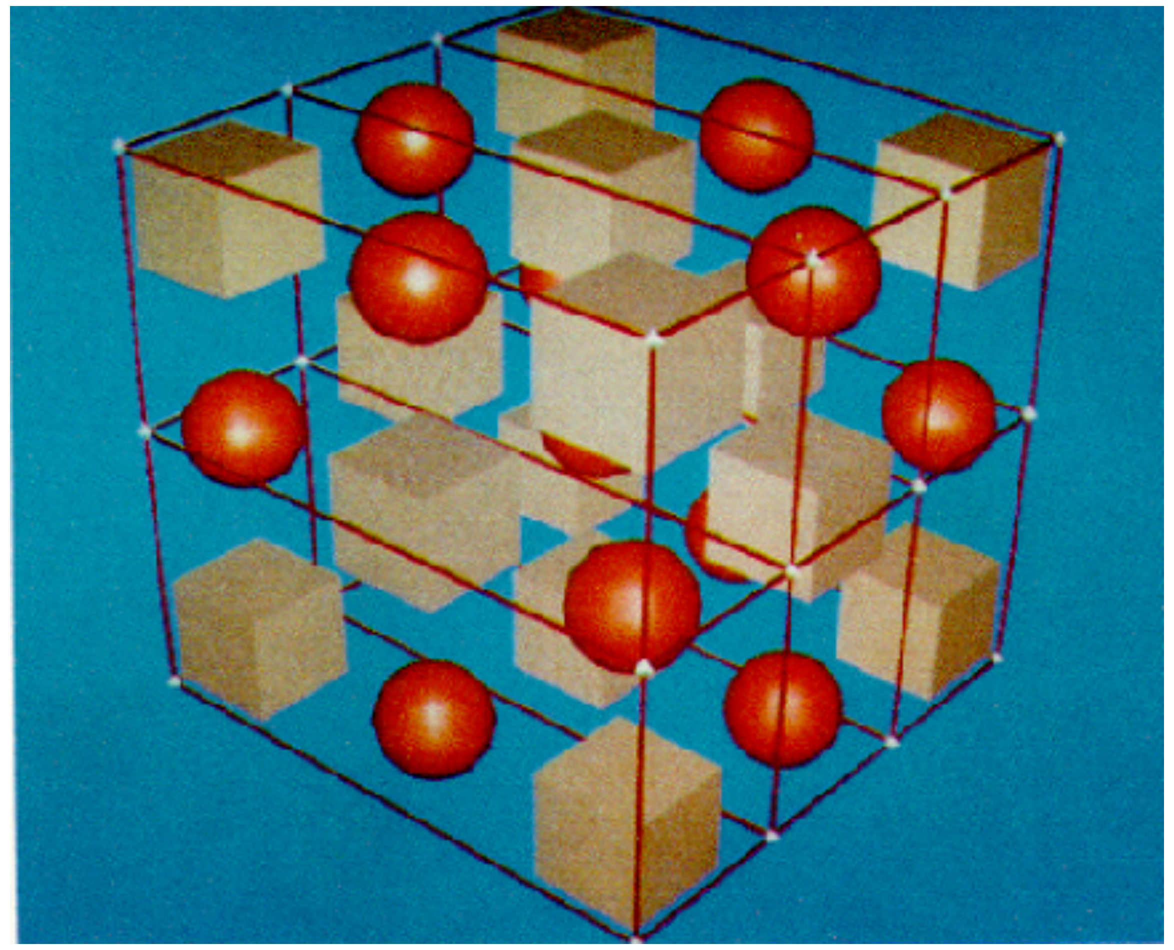
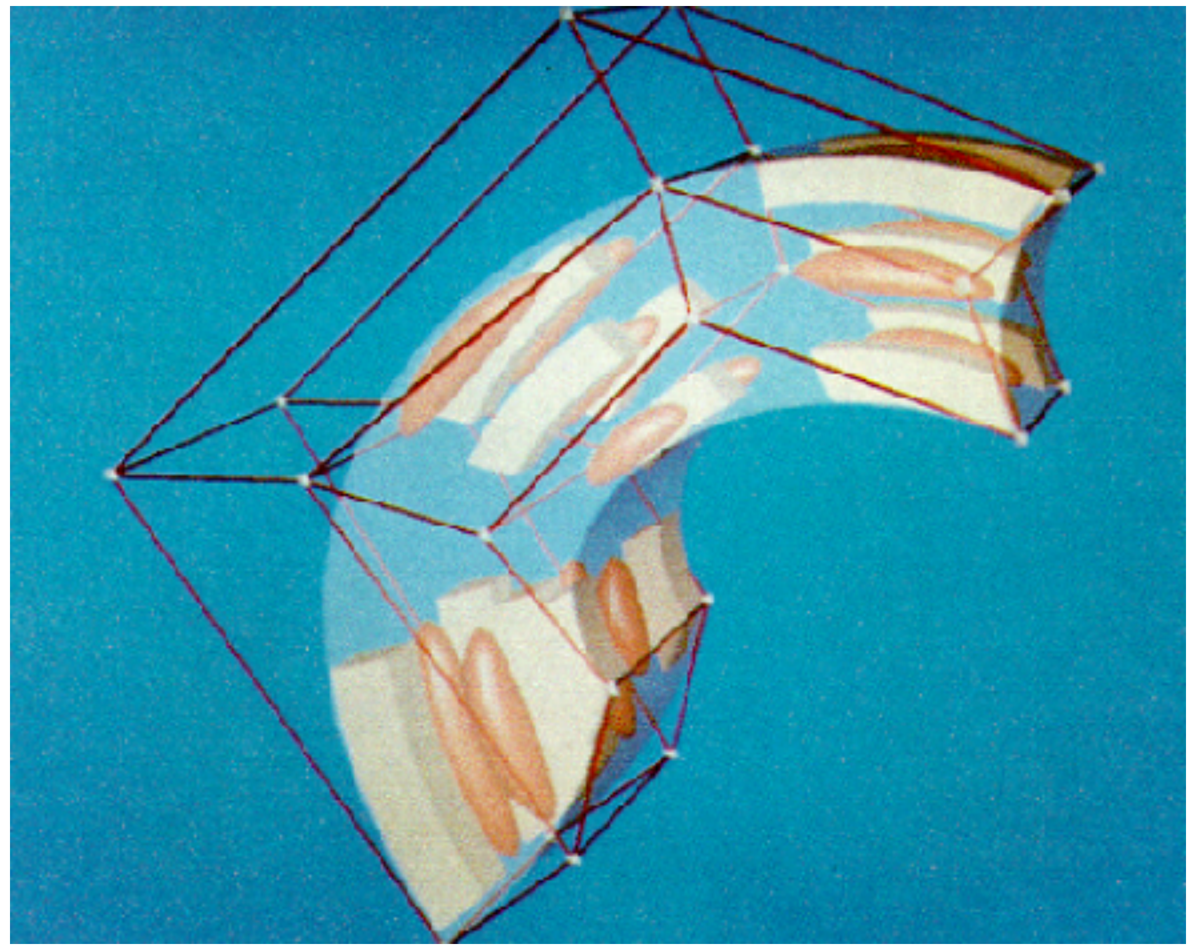
Early seminal works in computer graphics

$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

- Free form deformations
  [Sederberg and Parry 1986] http://dl.acm.org/citation.cfm?id=15903
  - Uses trivariate tensor product polynomial basis





Early seminal works in computer graphics

- Can be designed to be volume preserving

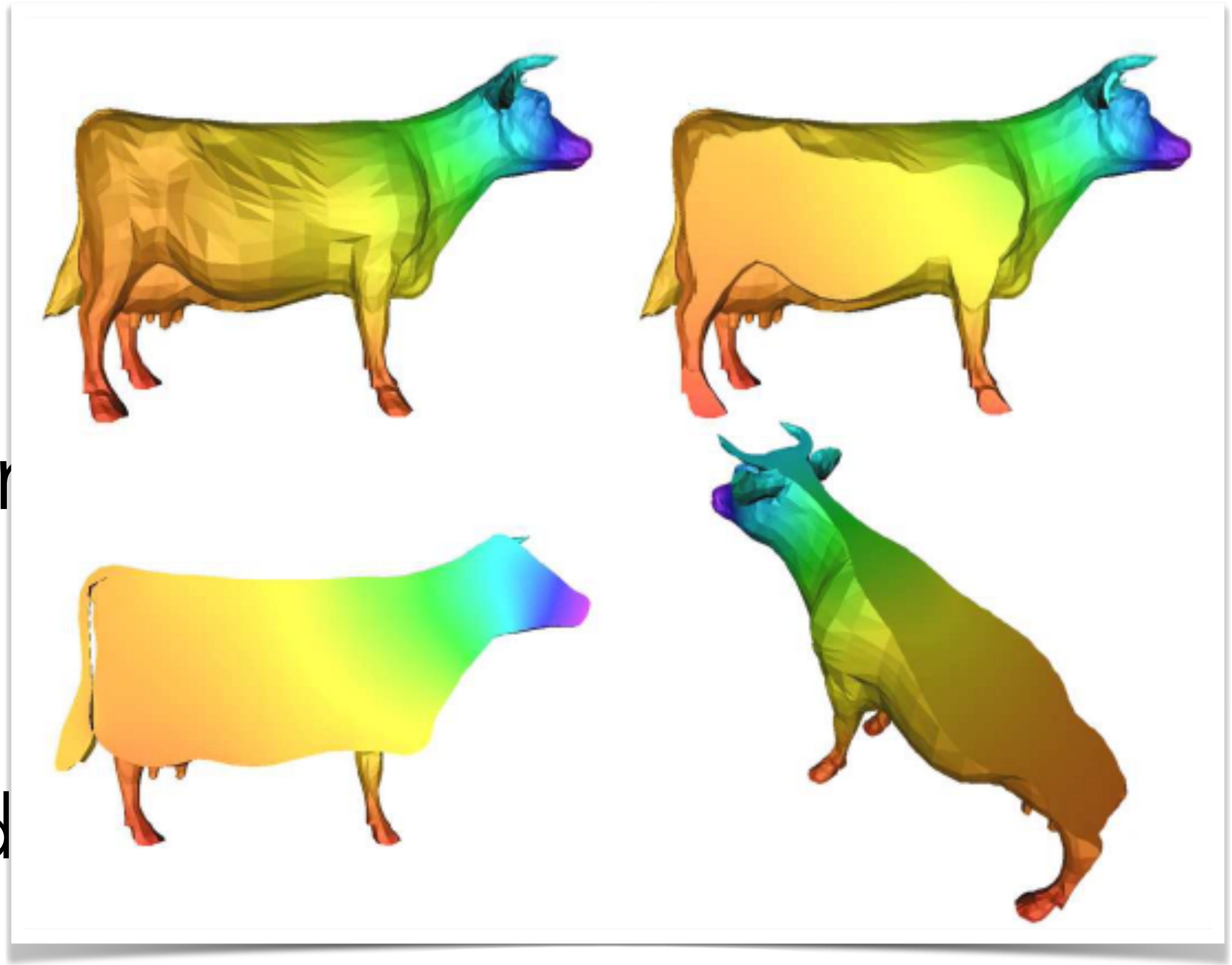
$$f : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\mathbf{F}(x,y,z) = (F(x,y,z), G(x,y,z), H(x,y,z))$$

then the Jacobian is the determinant

$$Jac(\mathbf{F}) = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial y} & \frac{\partial G}{\partial z} \\ \frac{\partial H}{\partial x} & \frac{\partial H}{\partial y} & \frac{\partial H}{\partial z} \end{vmatrix}$$

Basic Idea



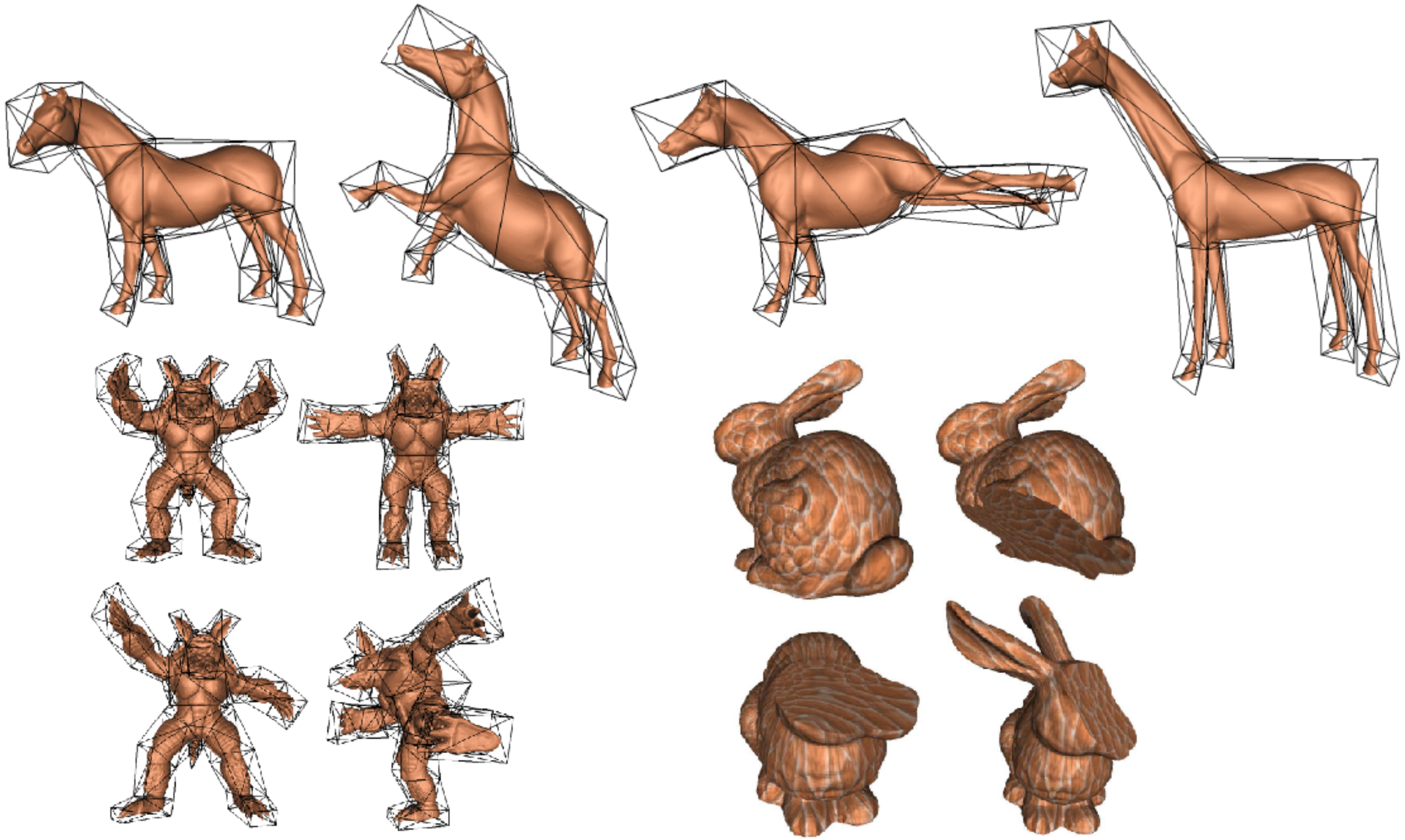
- Design a set of coordinates for all points i
w.r.t. the "cage" vertices
    - Each point **x** can be represented as a weighted

$$\mathbf{x} = \sum_{i=1}^{k} w_i(\mathbf{x})\, \mathbf{p}_i$$

    - When the cage changes, the coords stay the same, substitute the new cage geometry:

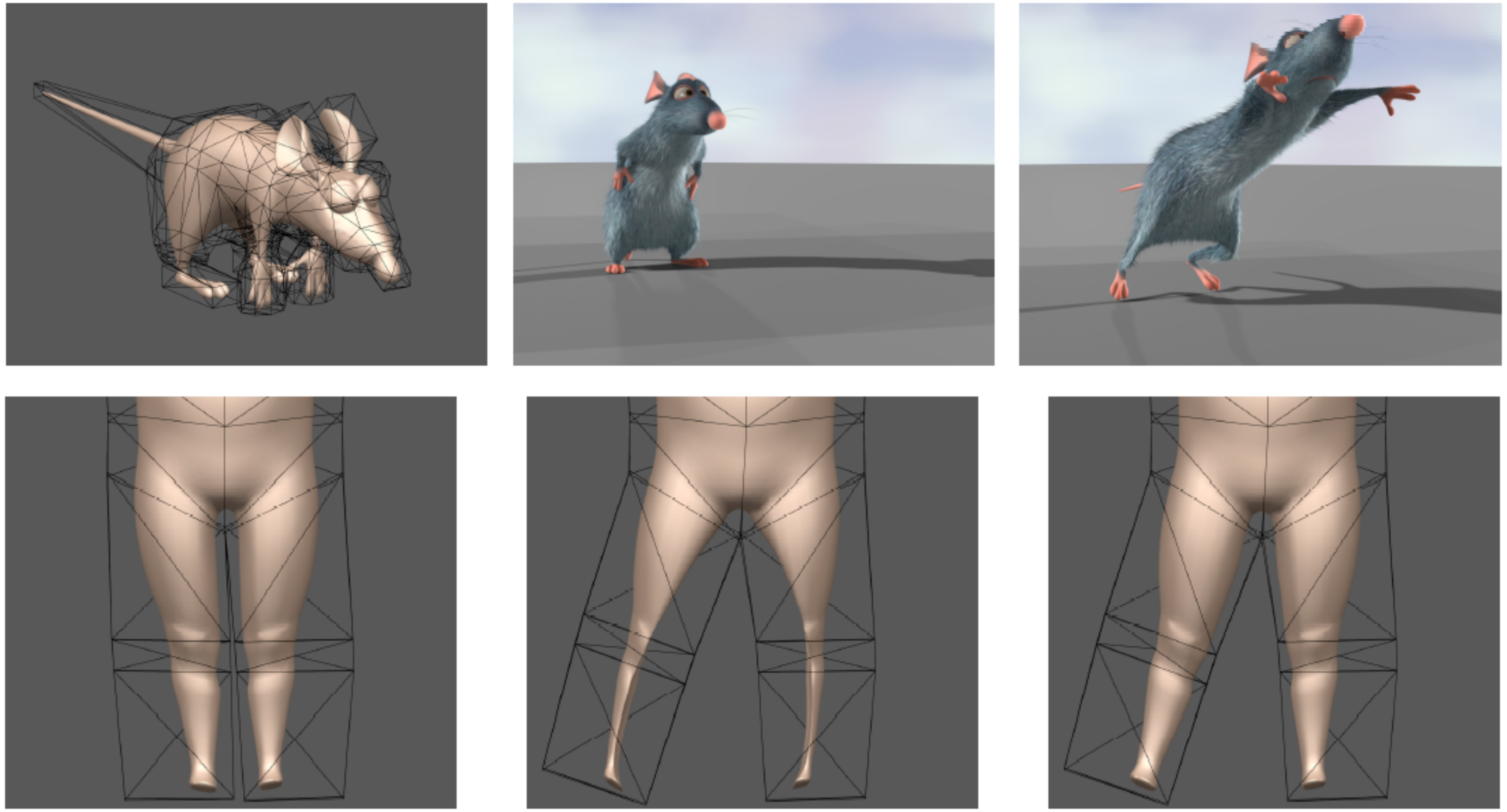
$$\mathbf{x}' = \sum_{i=1}^{k} w_i(\mathbf{x})\, \mathbf{p}'_i$$

Mean Value Coordinates



[Ju et al. 2005]
http://dl.acm.org/citation.cfm?id=1186822.1073229

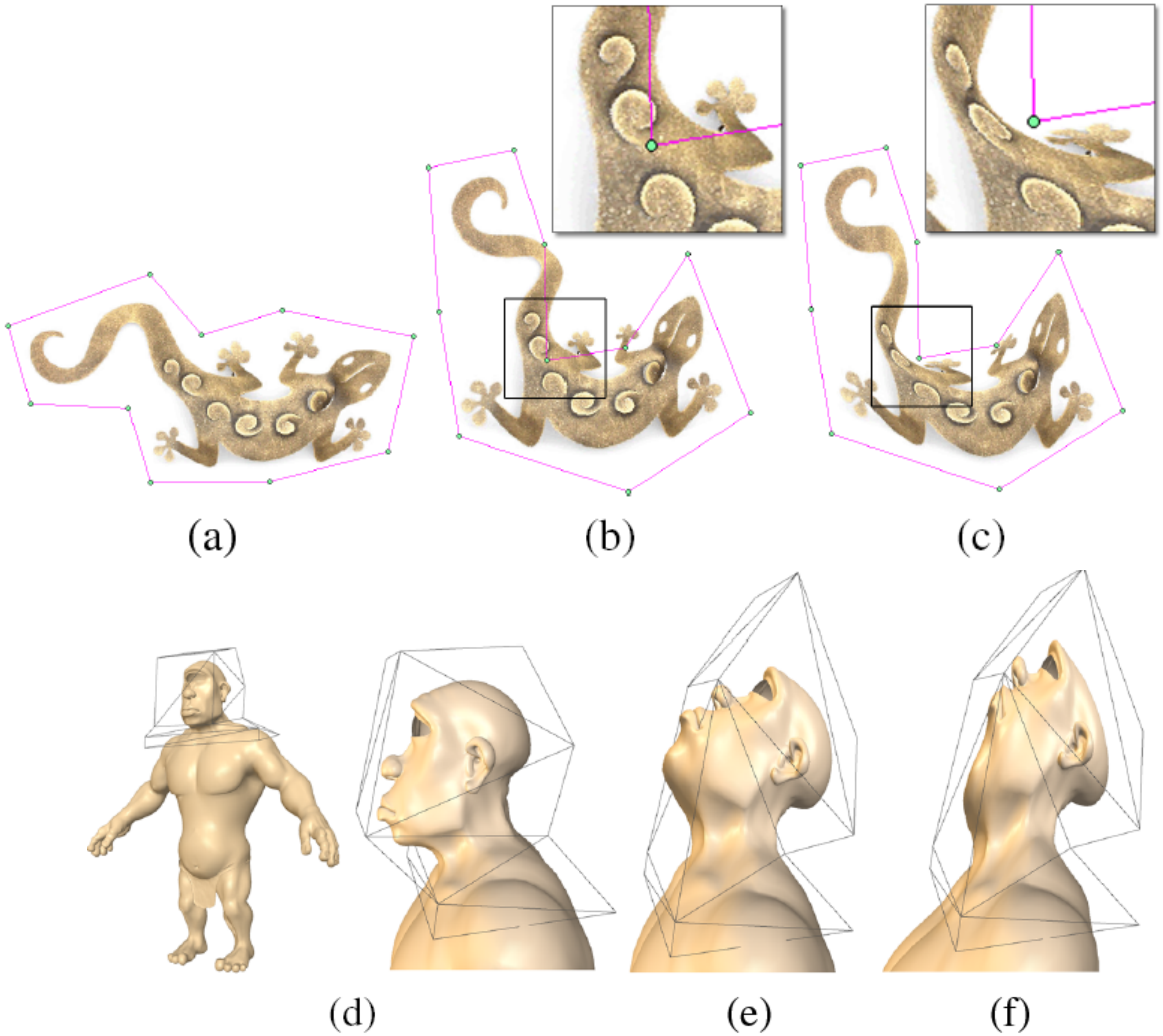
Harmonic Coordinates



[Joshi et al. 2007]

http://dl.acm.org/citation.cfm?id=1276466

Green Coordinates



http://dl.acm.org/citation.cfm?id=1360677

Space Deformations

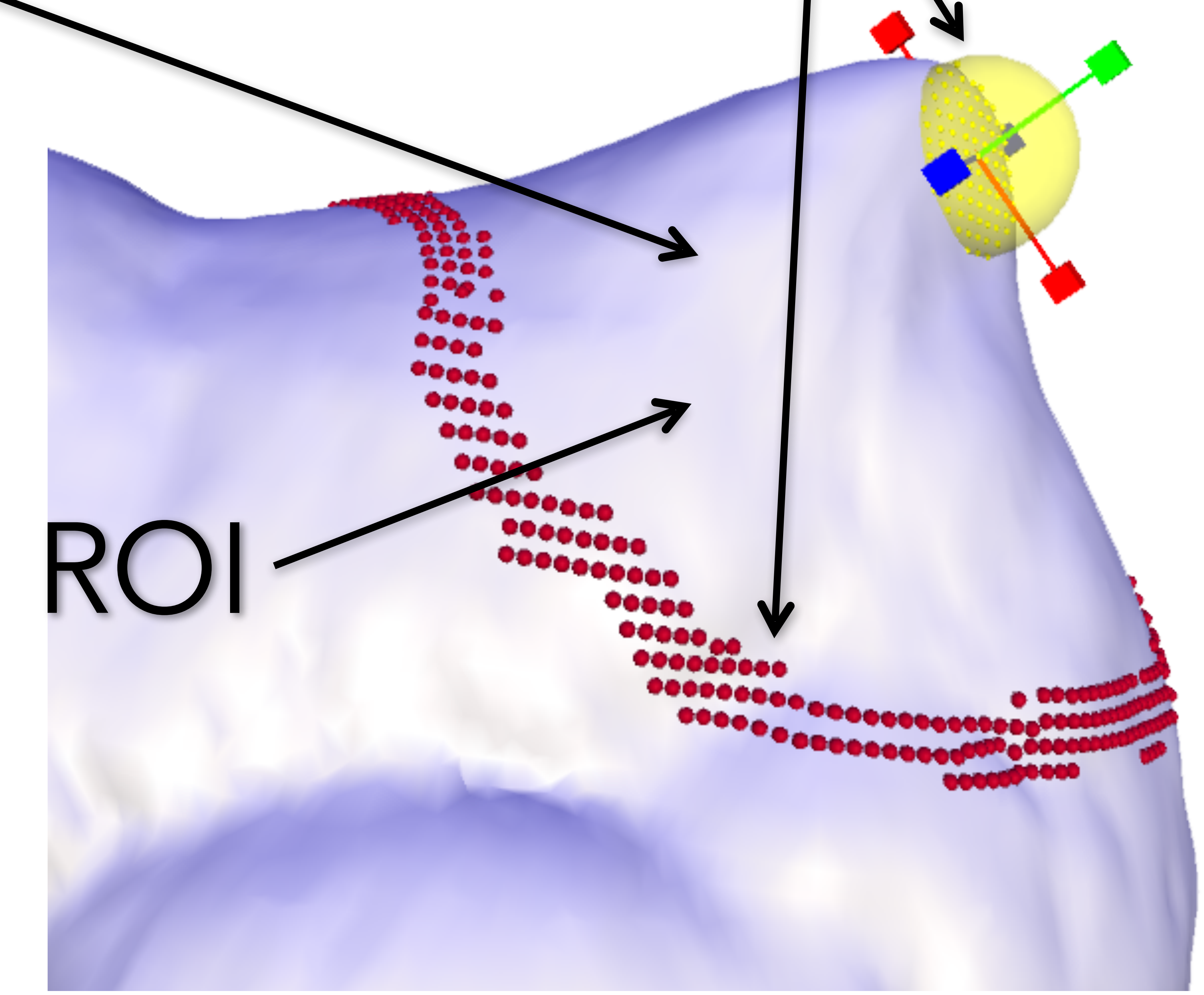
- Complexity depends mainly on the cage; linear in the number of mesh elements
  - Parallel execution with GPU accelerators
- Can handle disconnected components or even just point sets
- Harder to control the surface properties since the whole space is being warped

Surface-Based Deformations

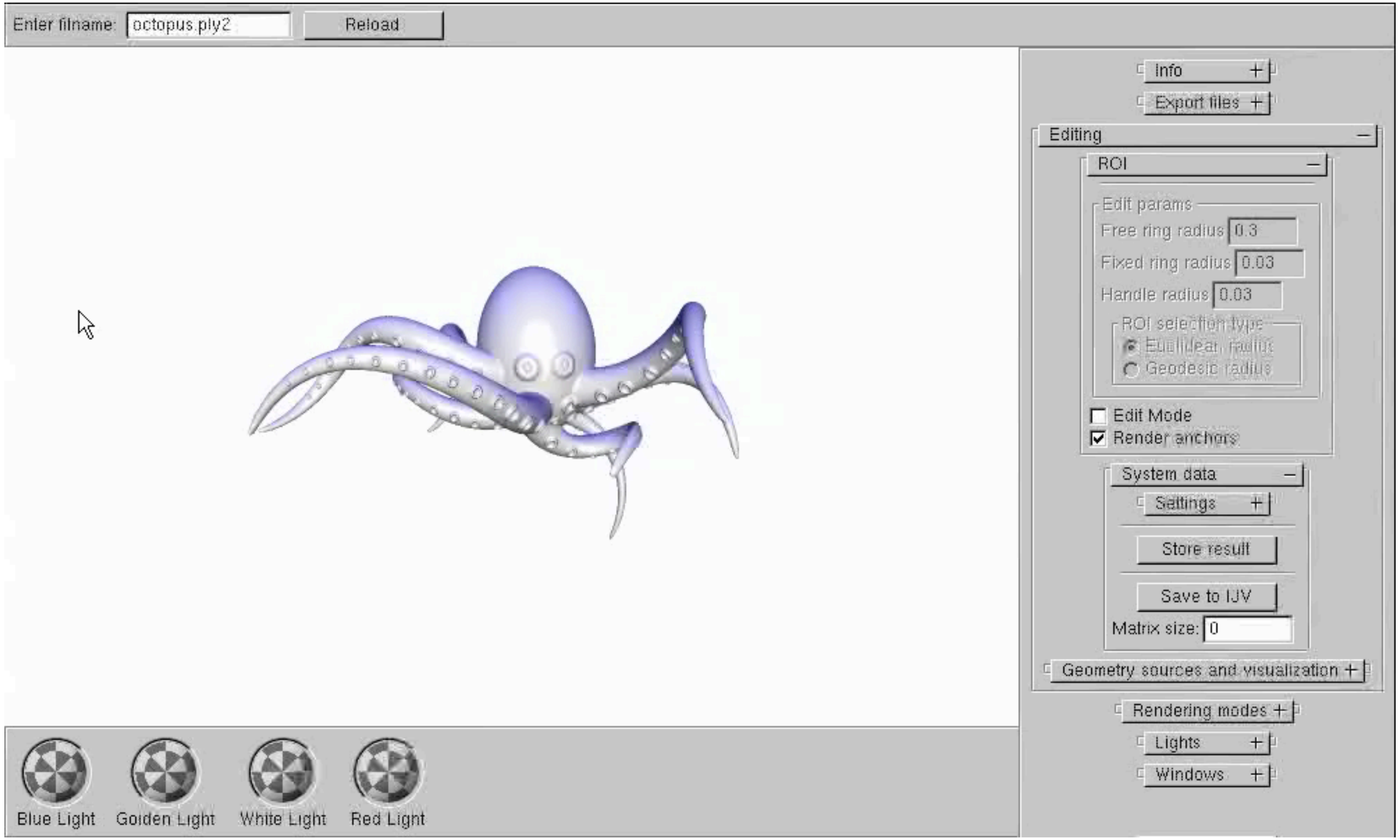
Surface-based Deformation: ROI-Handle Editing Metaphor

$$\mathbf{x}_{\text{def}} = \underset{\mathbf{x}'}{\operatorname{argmin}} \, E(\mathbf{x}') \quad s.t. \quad \mathbf{x}'_i = \mathbf{c}_i$$

- ROI is bounded by a belt (static anchors)
- Manipulation through handle(s) – affine transformations



Surface-based Deformation: ROI-Handle Editing Metaphor



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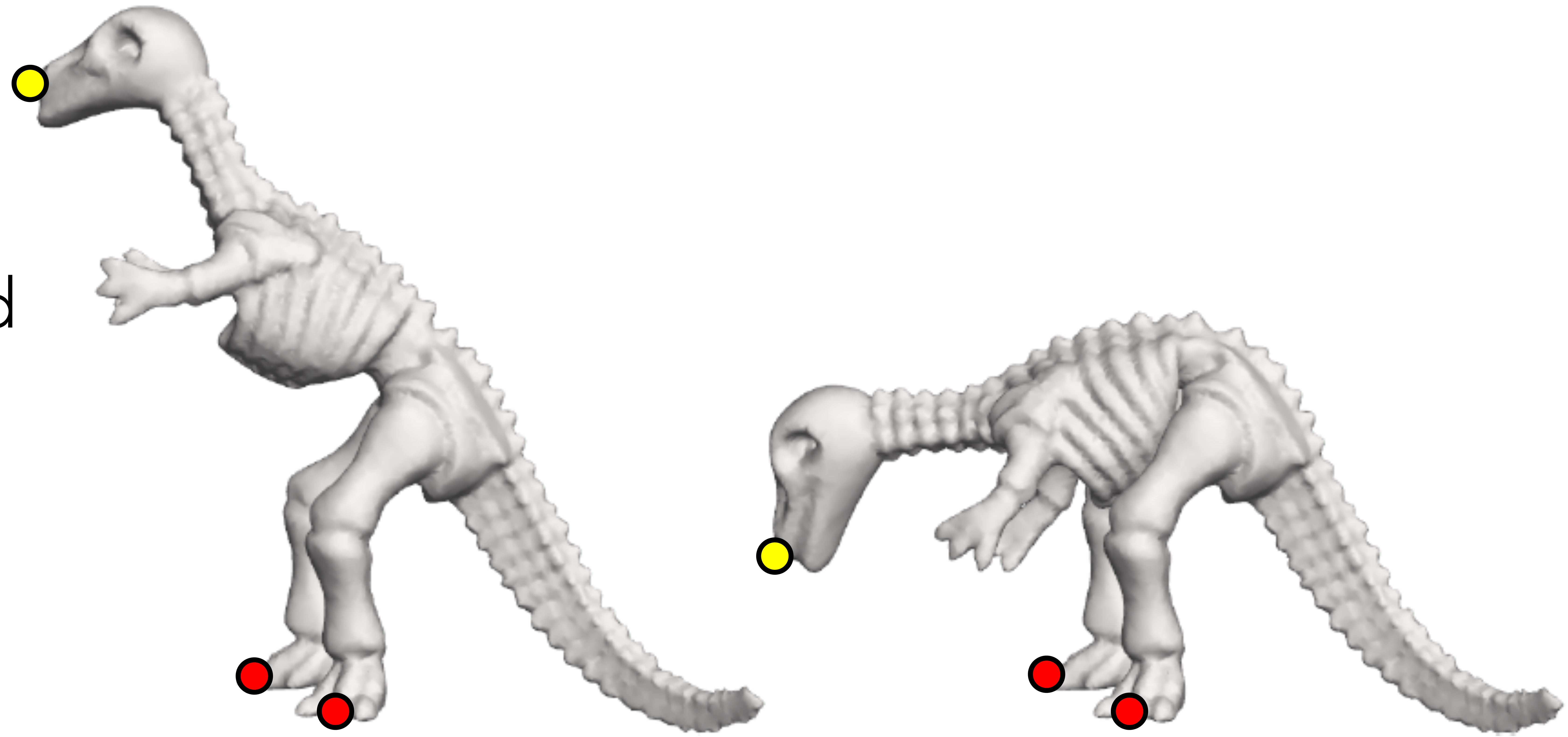
How to Define $E(\mathbf{x}')$?

- Intuitive deformations:
  - Smooth deformation on the global scale
  - Preserve local details (curvatures)
- Invariants: $E(\mathbf{x}')$ should be zero if $\mathbf{x}'$ is a rigid transformation of $\mathbf{x}$ original geometry



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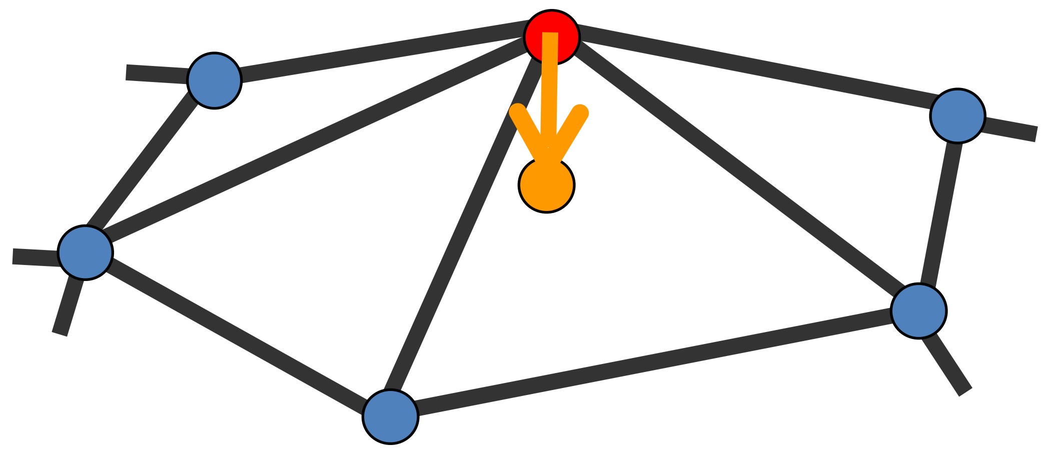
Recap: Differential Coordinates

- Detail = *smooth*(surface) – surface
- Smoothing = averaging



$$\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij} (\mathbf{x}_j - \mathbf{x}_i) \approx -2 H_i \mathbf{n}_i$$

Recap: Differential Coordinates

- Represent ***local detail*** at each surface point
  - More descriptive of the shape than just *xyz*
- Linear transition from *xyz* to **δ**
- Useful for operations on surfaces where surface details are important

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Simple Laplacian Editing

- Preserve mean curvature normal [≈differential coordinates] at every point in the ROI [≈ every vertex of the ROI]

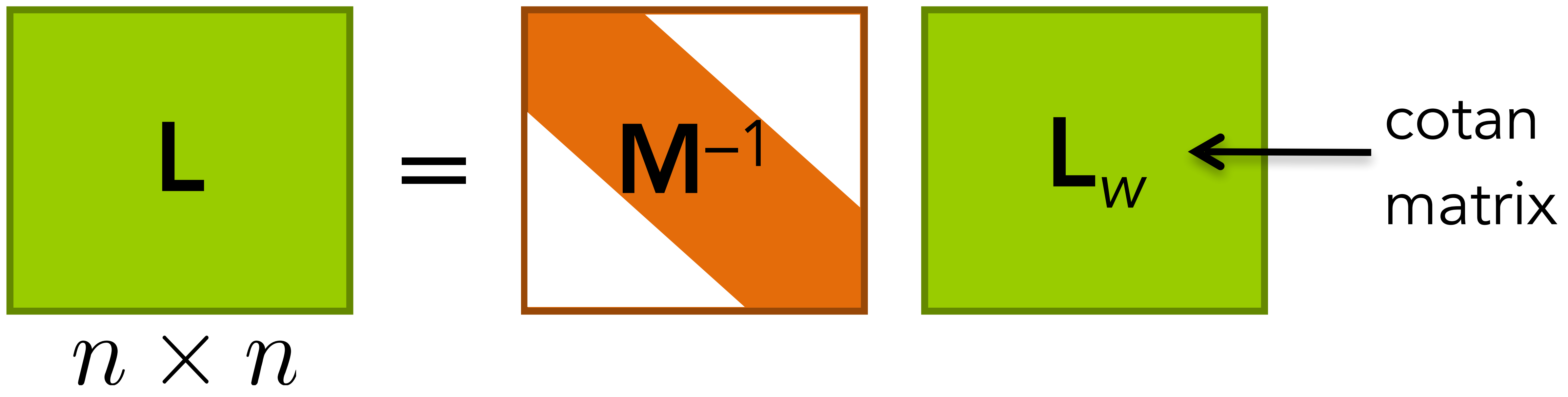
continuous: $$E(\mathcal{S}') = \int_{\mathcal{S}'} \|\Delta\mathbf{x}' - \delta\|^2 d\mathbf{x}'$$

discrete: $$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}_i') - \delta_i\|^2$$

Simplifying the Laplacian Energy

$$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}_i') - \delta_i\|^2 = \sum_{i=1}^{n} A_i \left(\Delta(\mathbf{x}_i')^T \Delta(\mathbf{x}_i') - 2\Delta(\mathbf{x}_i')^T \delta_i + \delta_i^T \delta_i\right) =$$

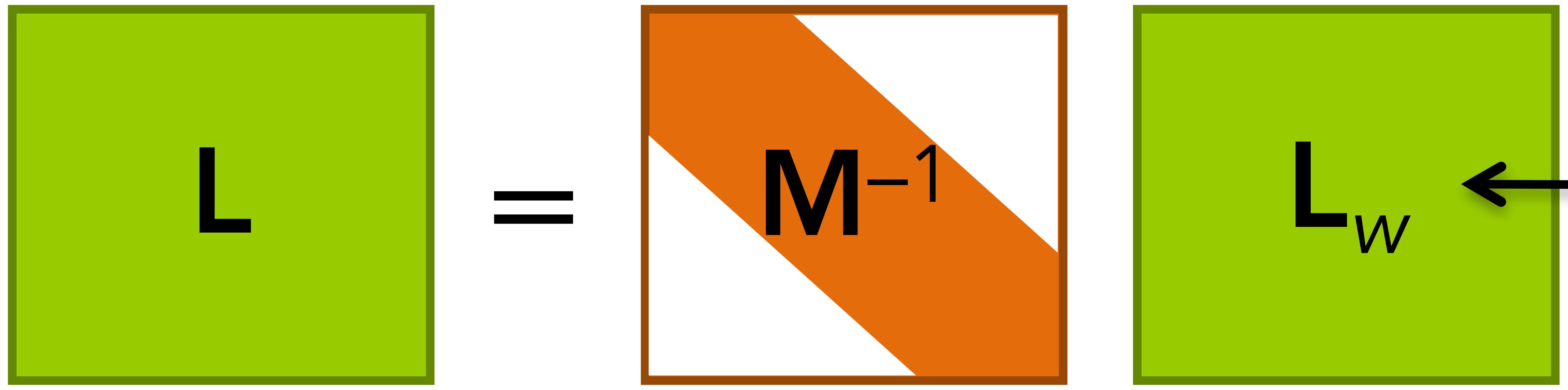
$$= \mathbf{x}'^T \underline{L^T M L}\mathbf{x}' - 2\mathbf{x}'^T \underline{L^T M}\delta + \text{const}$$



Simplifying the Laplacian Energy

$$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}'_i) - \delta_i\|^2 = \sum_{i=1}^{n} A_i \left(\Delta(\mathbf{x}'_i)^T \Delta(\mathbf{x}'_i) - 2\Delta(\mathbf{x}'_i)^T \delta_i + \delta_i^T \delta_i\right) =$$

$$= \mathbf{x}'^T \underline{L^T M} L\mathbf{x}' - 2\mathbf{x}'^T \underline{L^T M} \delta + \text{const}$$



$n \times n$

cotan matrix

$$L^T M L = (M^{-1} L_w)^T M (M^{-1} L_w) = L_w M^{-1} M M^{-1} L_w =$$

$$= L_w M^{-1} L_w$$

Symmetric sparse matrix!

Minimizing the Laplacian Energy

- To find the minimum, gradient = 0 and substitute the modeling constraints

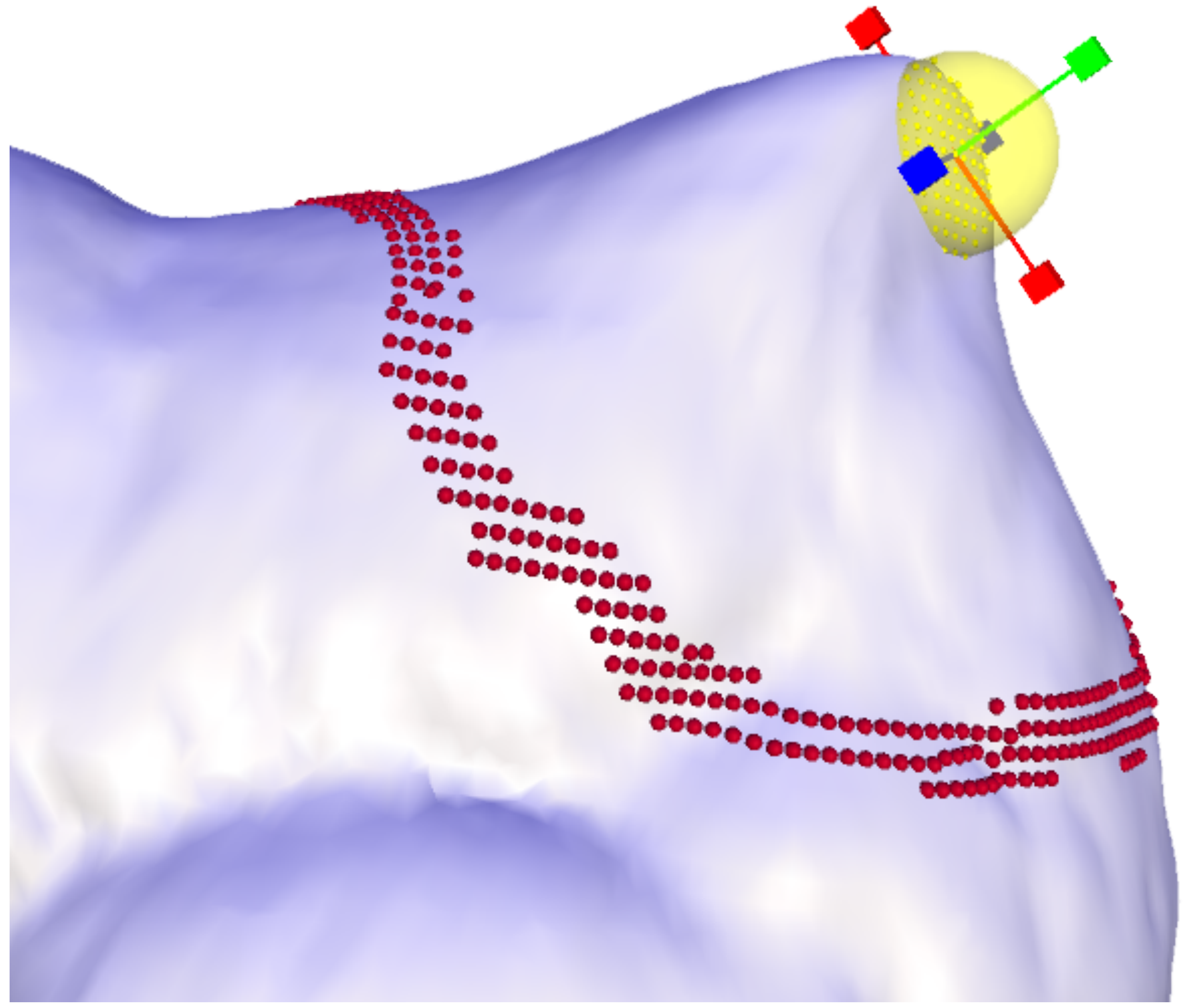
$$E(\mathbf{x}') = \mathbf{x}'^T L_w M^{-1} L_w \mathbf{x}' - 2\mathbf{x}'^T L_w \delta + \text{const}$$

$$\frac{\partial}{\partial \mathbf{x}'} E(\mathbf{x}') = 2 L_w M^{-1} L_w \mathbf{x}' - 2 L_w \delta$$

$$\mathbf{x}'_i = \mathbf{c}_i, \ i \in \mathcal{C}$$

Minimizing the Laplacian Energy

$$A\mathbf{x}' = \mathbf{b}$$



Matrix depends on the initial mesh and the indices of the constraints only. Matrix is fixed!

Right-hand side contains the coordinates of the constraints (handles)

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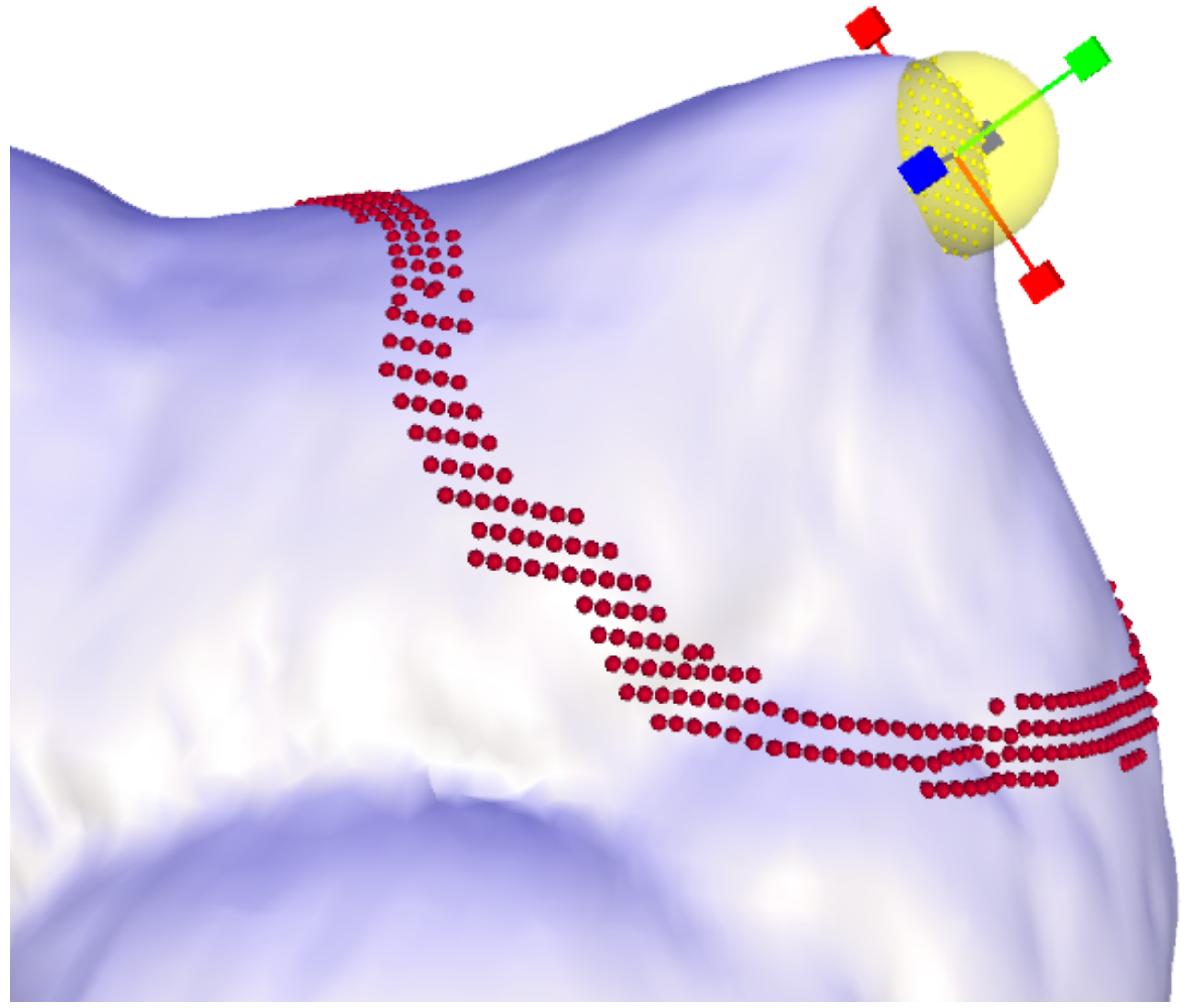
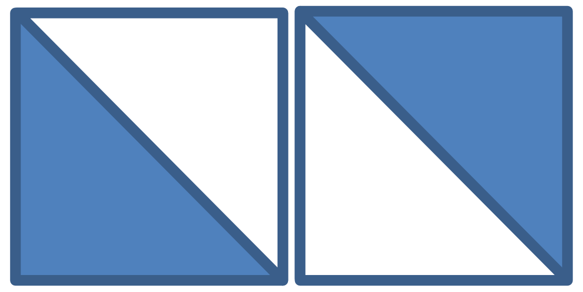
Minimizing the Laplacian Energy

$$A\mathbf{x}' = \mathbf{b}$$



Sparse Cholesky decomposition:

$$A = L_{\text{chol}} L_{\text{chol}}^T$$



At run-time: just back-substitution!

$$L_{\text{chol}} \mathbf{y} = \mathbf{b}$$

$$L_{\text{chol}}^T \mathbf{x}' = \mathbf{y}$$

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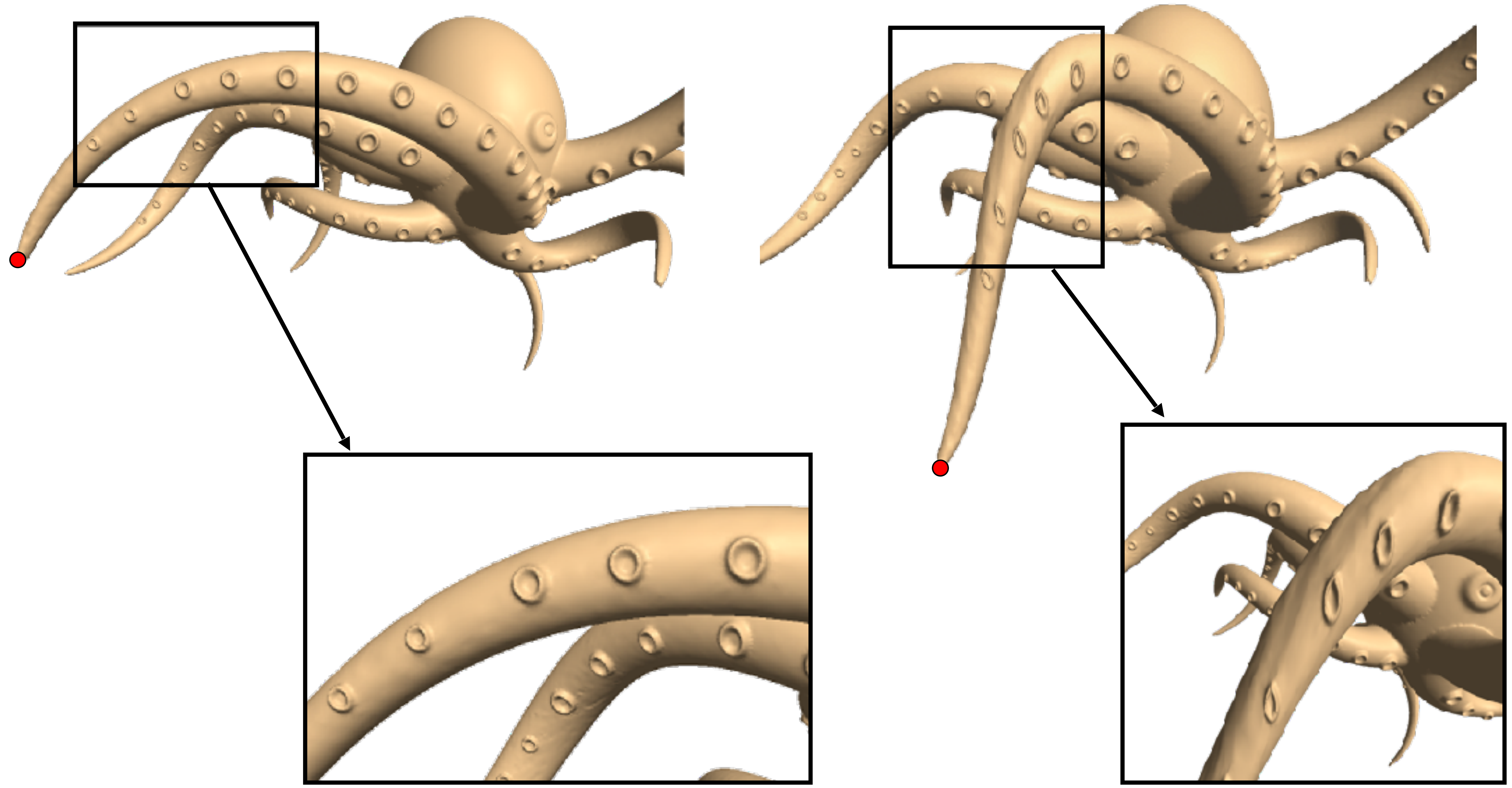
Fundamental Problem: Invariance to Transformations

- The basic Laplacian operator is ***translation***-invariant, but not **rotation-**invariant
- E(**x**') attempts to preserve the **original global** orientation of the details (the normal directions)



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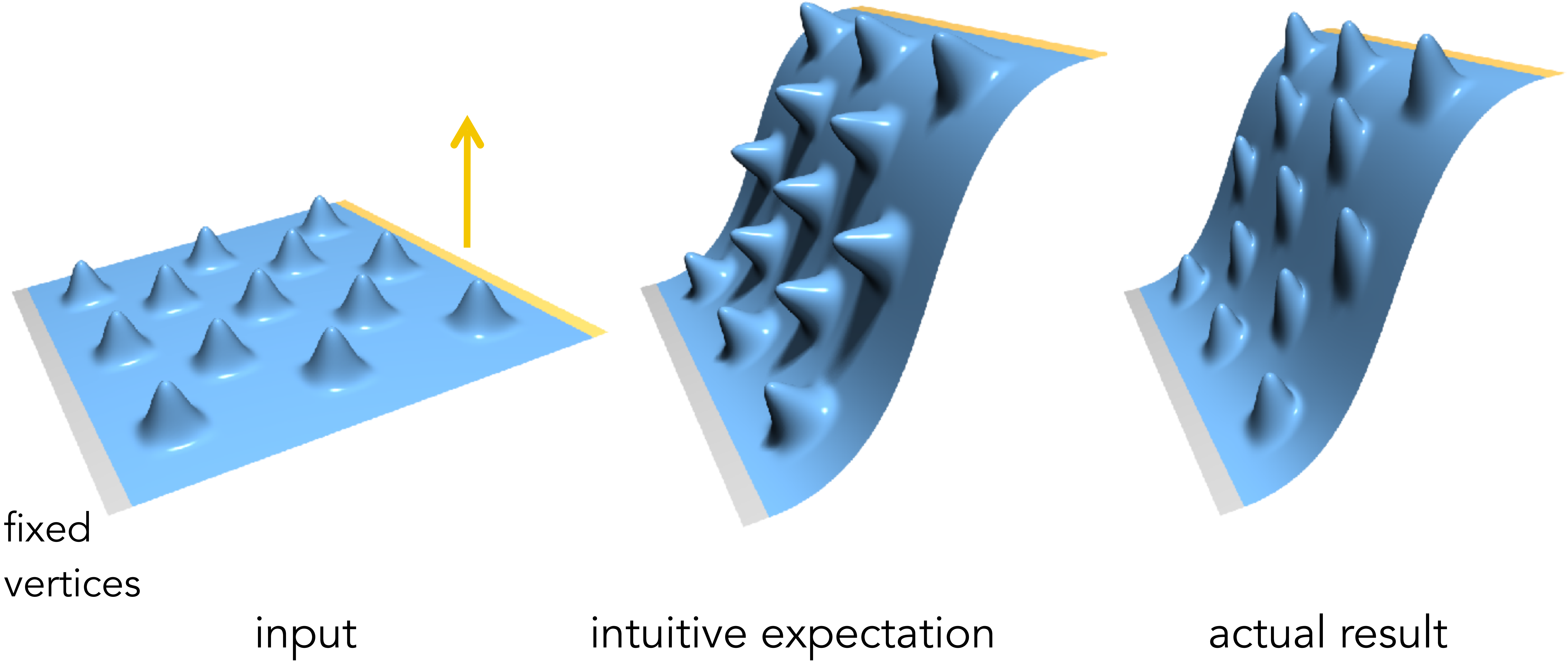

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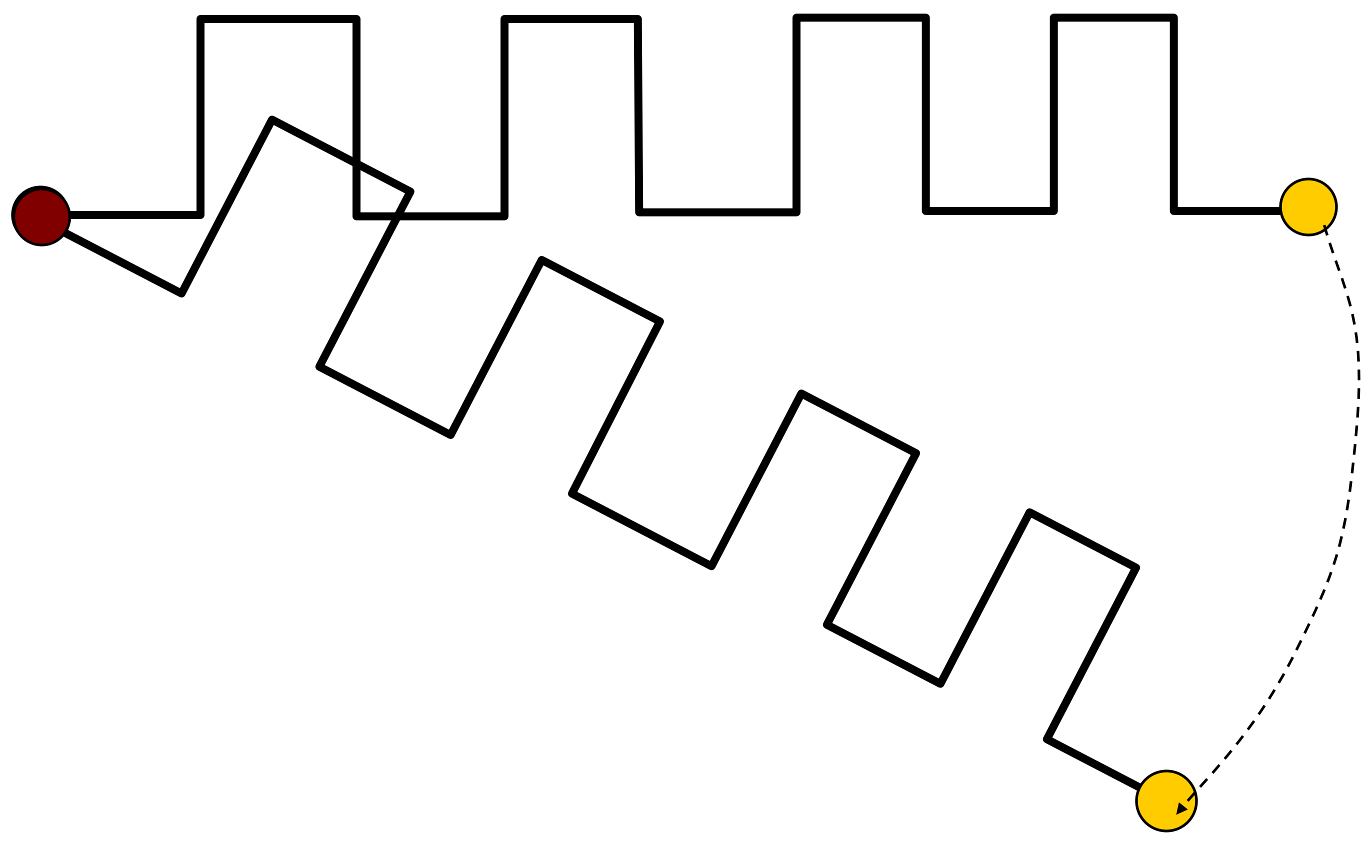

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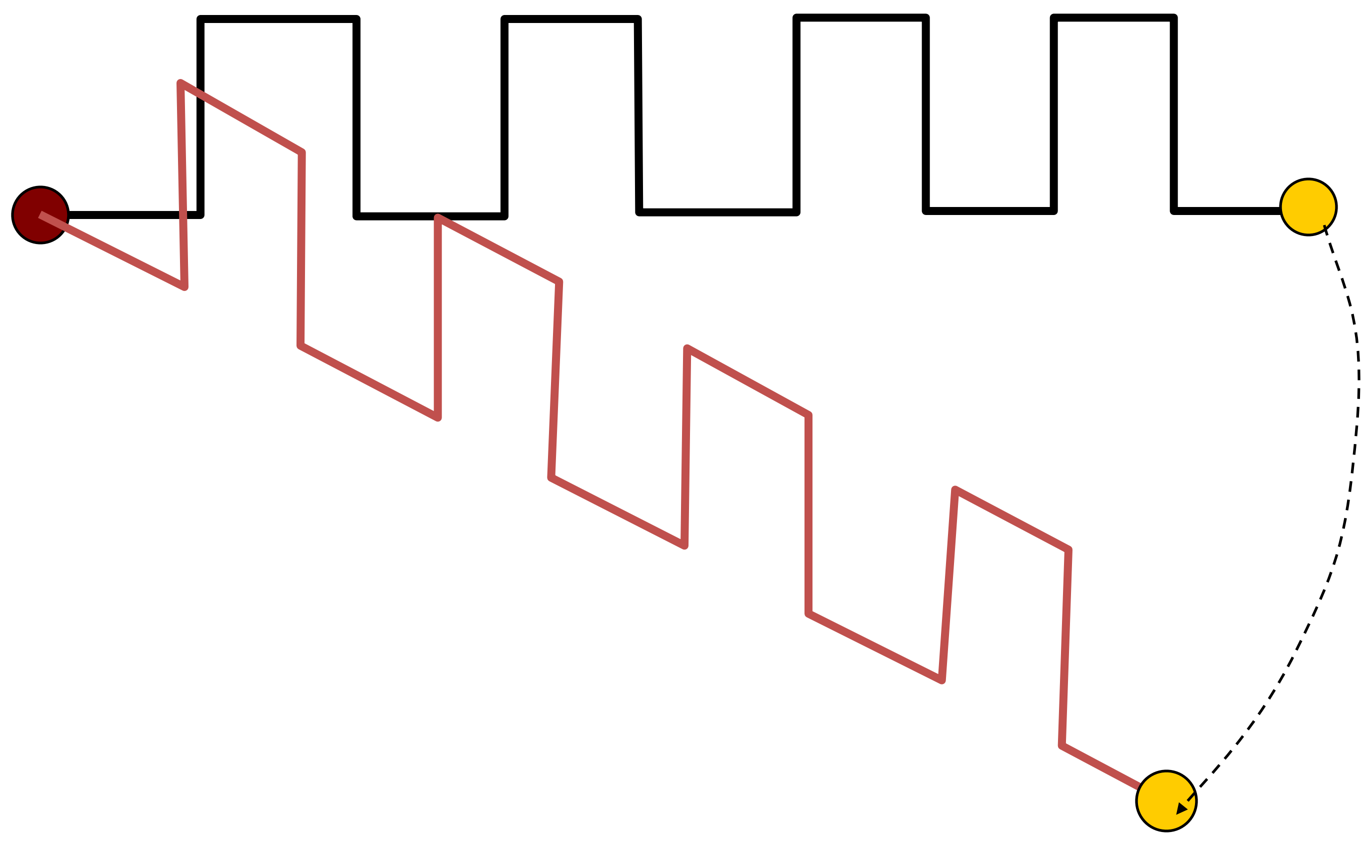


Energy Functional

- We need a rigid-invariant energy…

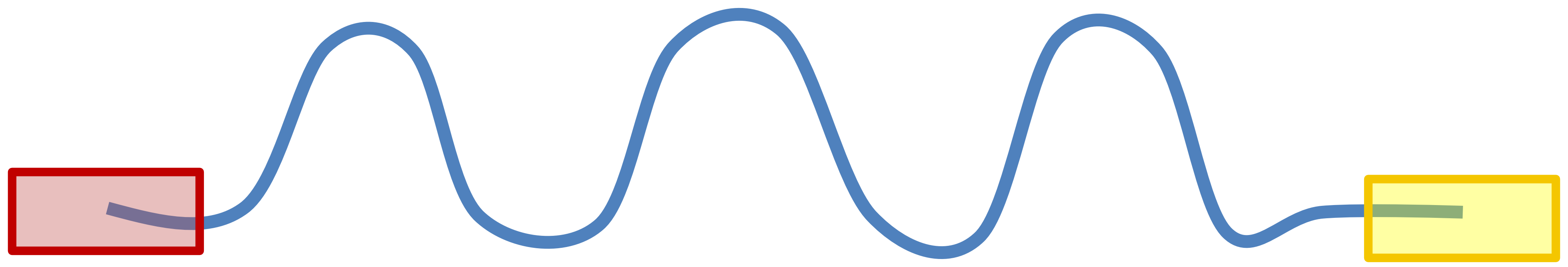
$$E(\mathbf{x}') = \sum_{i=1}^{n} A_i \|\Delta(\mathbf{x}'_i) - \delta_i\|^2$$

Need to locally
rotate the *target*
m.c. normals

Approach 1:
Multi-Resolution Mesh Editing

Multiresolution Approach

input



$\mathcal{S}$

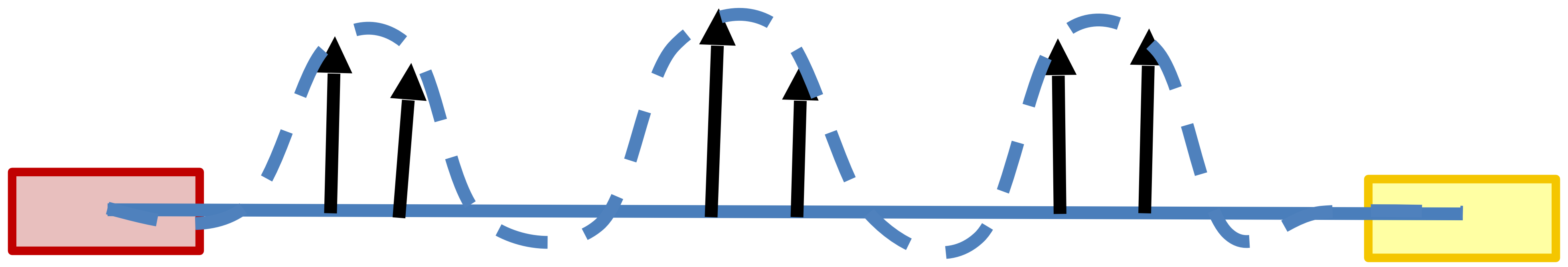
Multiresolution Approach

Smooth base surface

$\mathcal{B}$

Multiresolution Approach

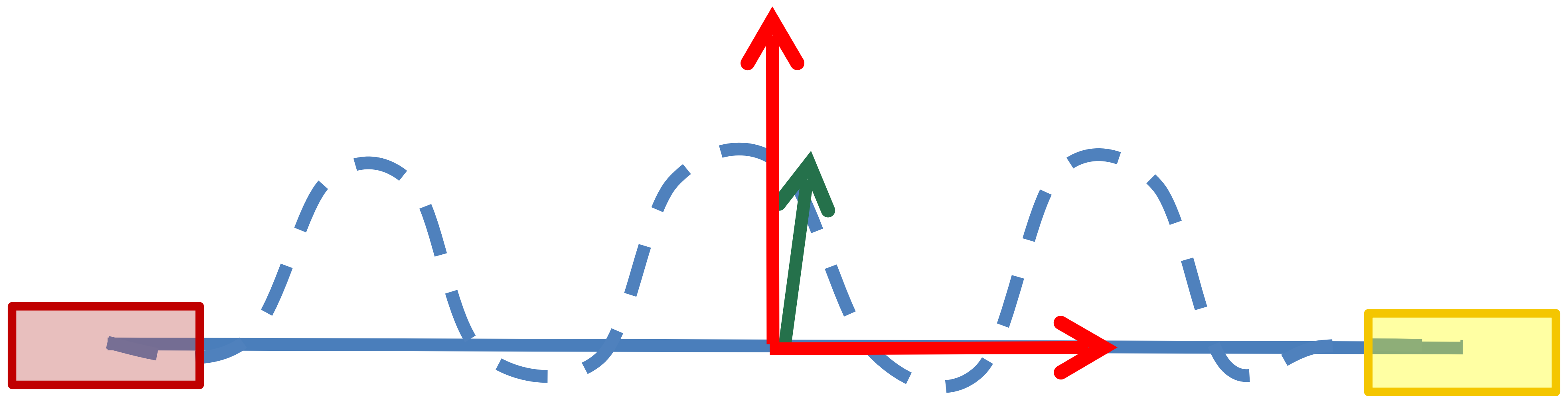
Details – displacement vectors



$$\mathcal{S} - \mathcal{B}$$

Multiresolution Approach

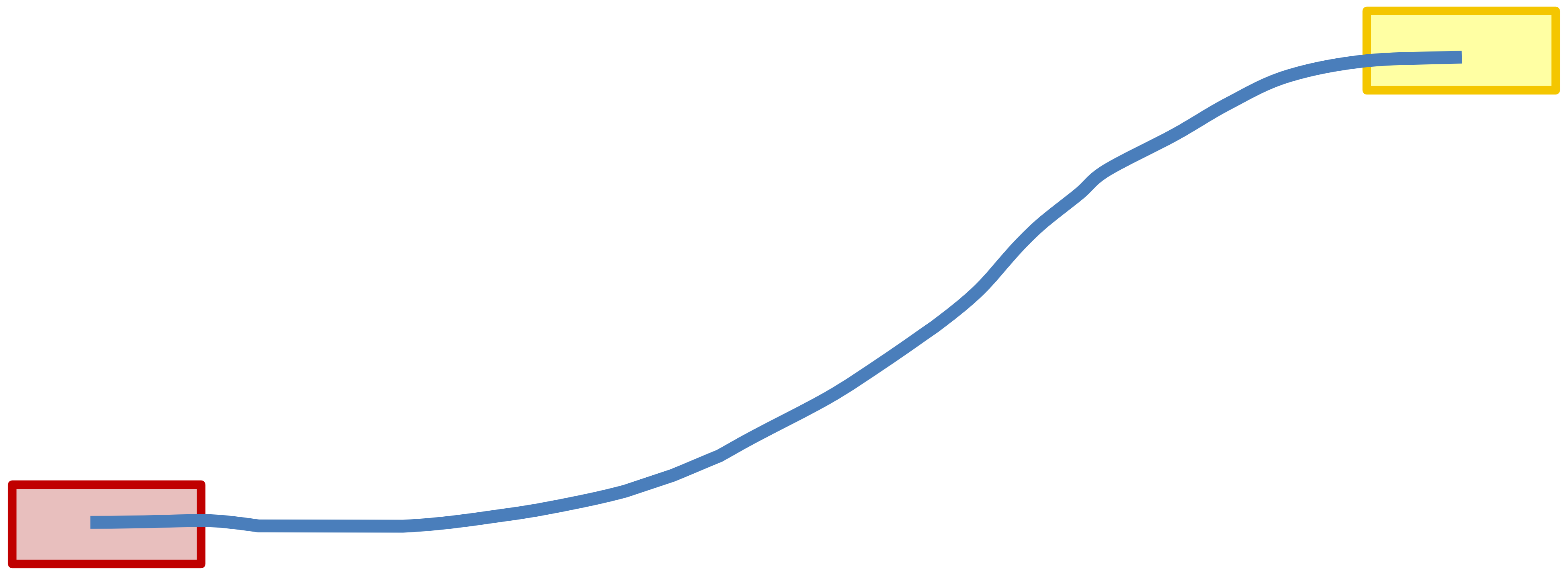
Encode details in the local frame of B



$$\mathbf{d}_i = a_1\mathbf{t}_i + a_2\mathbf{n}_i$$

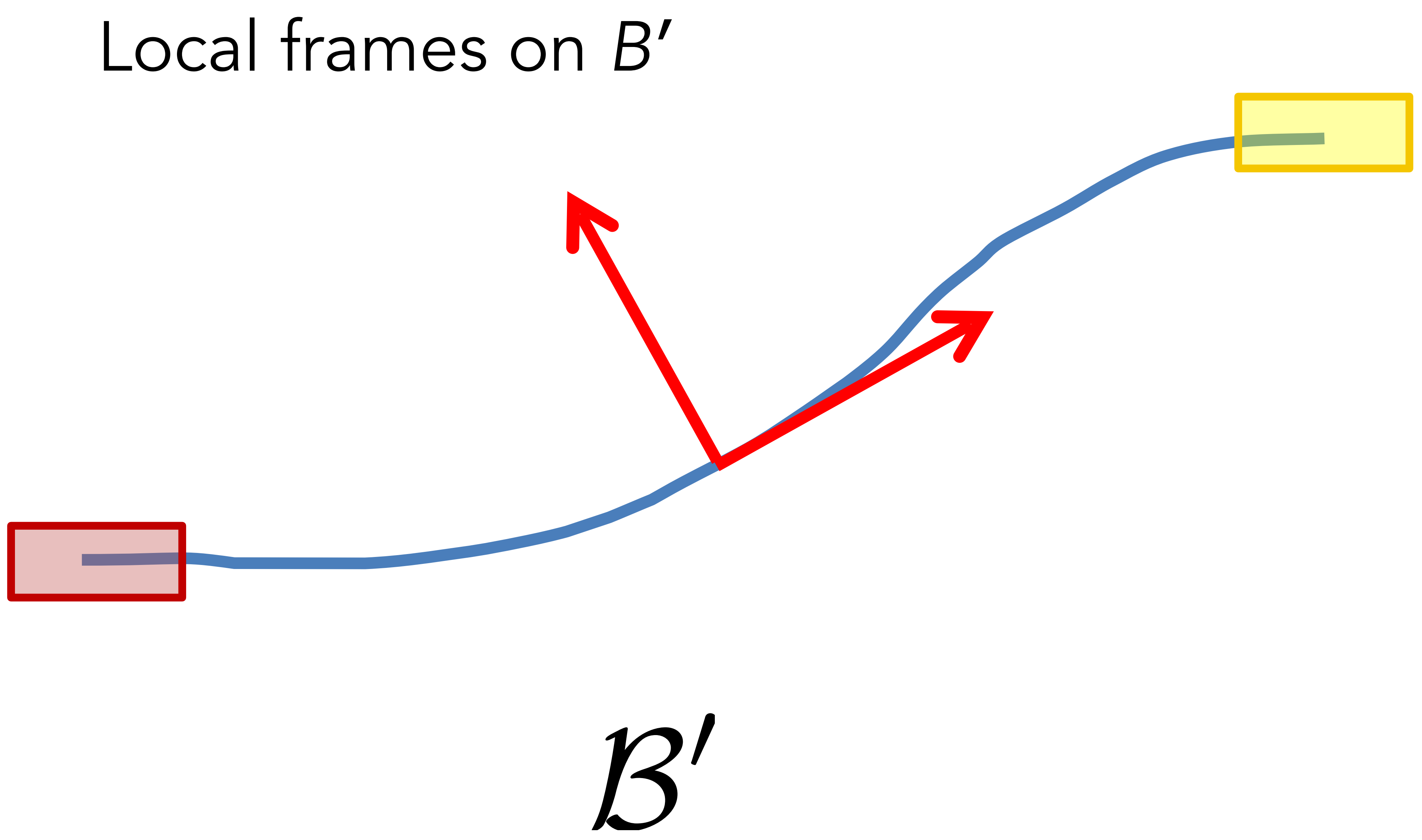
Multiresolution Approach

Deform smooth base surface



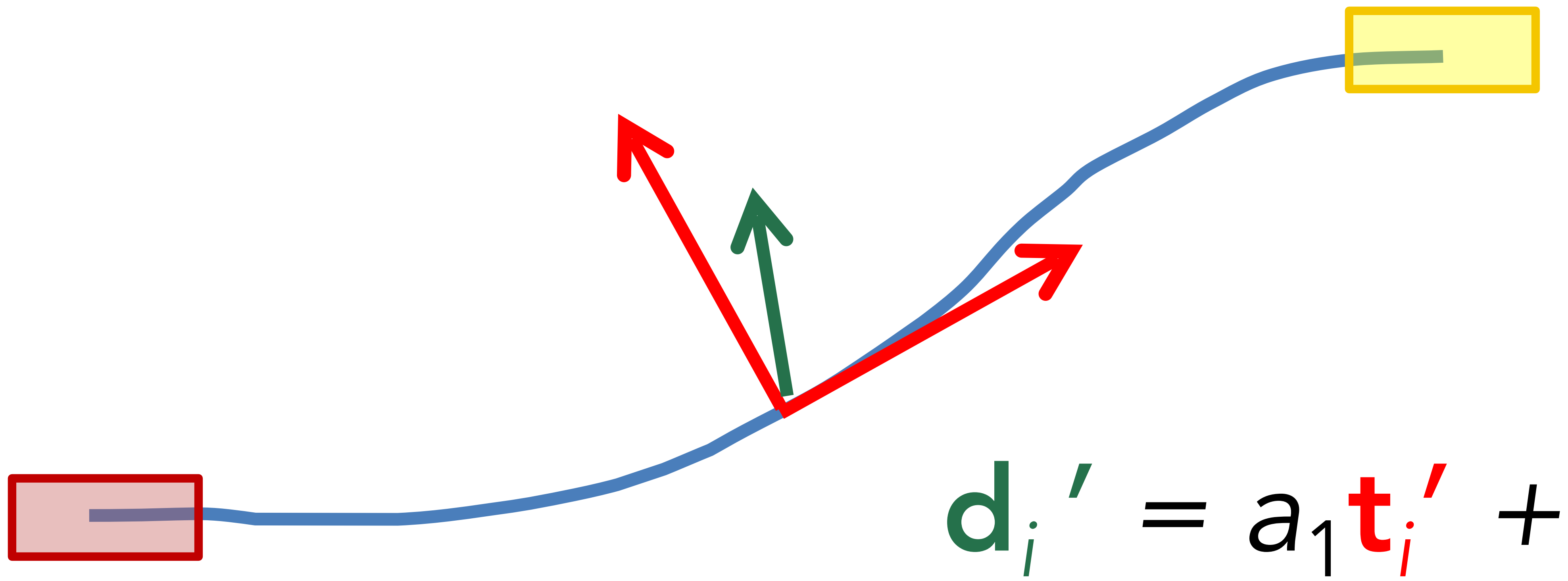
$\mathcal{B}'$

Multiresolution Approach



Multiresolution Approach

Add details back – in local frame!



$$\mathbf{d}_i' = a_1\mathbf{t}_i' + a_2\mathbf{n}_i'$$

$$\mathcal{B}'$$

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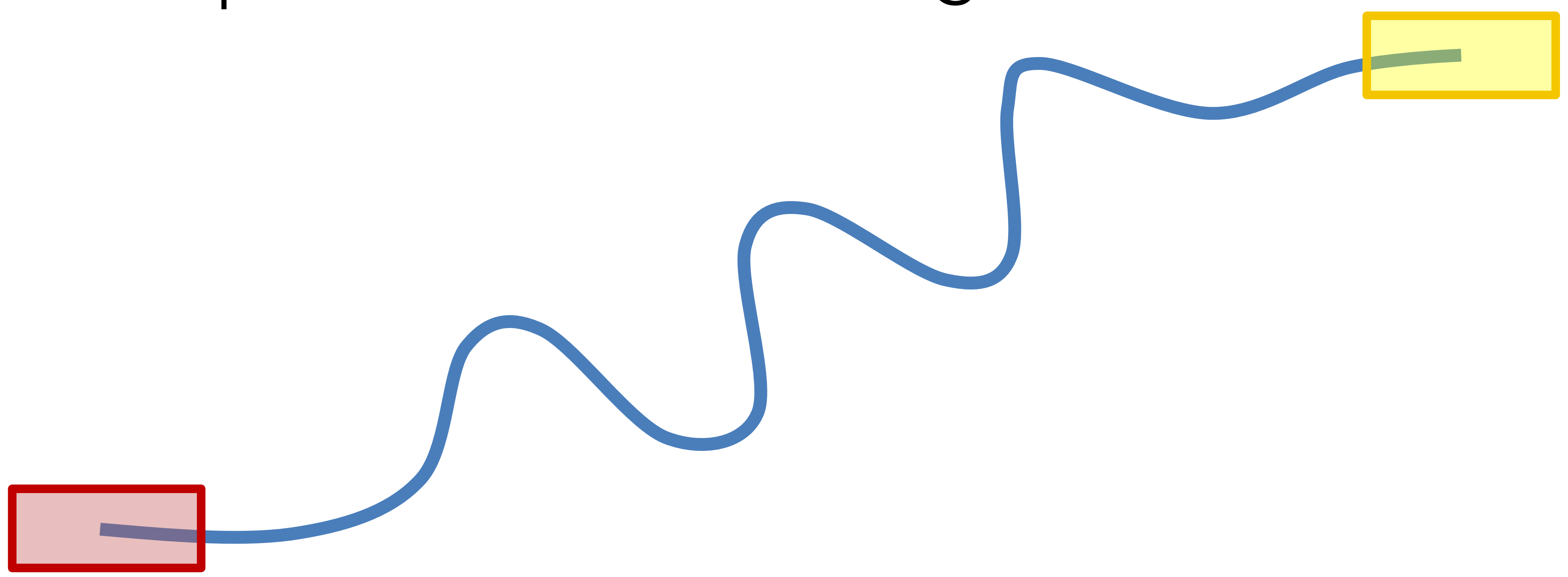
Multiresolution Approach

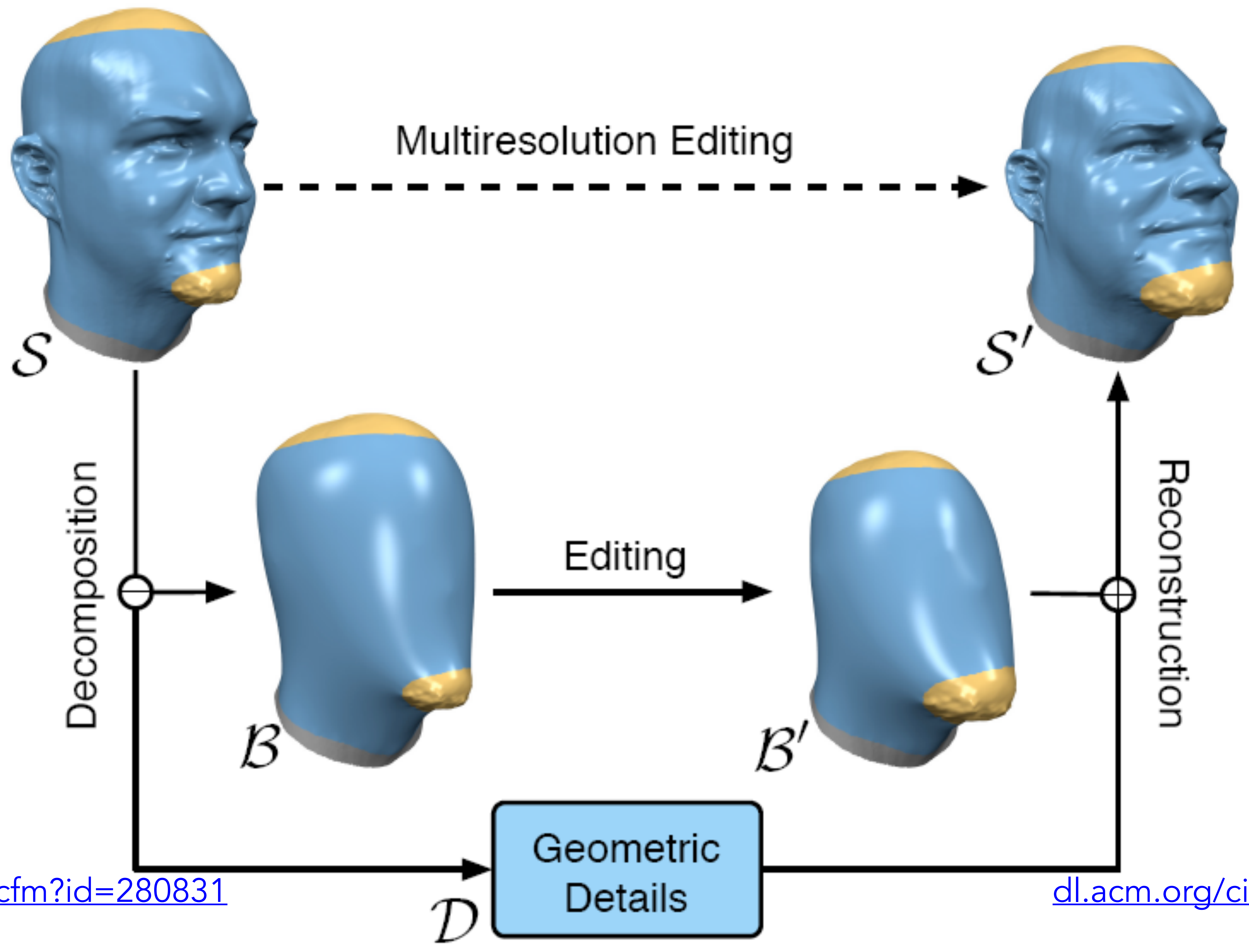
Displace the vertices to get the result



$\mathcal{S}'$

Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004



http://dl.acm.org/citation.cfm?id=280831

dl.acm.org/citation.cfm?id=1015772

Multiresolution Approach

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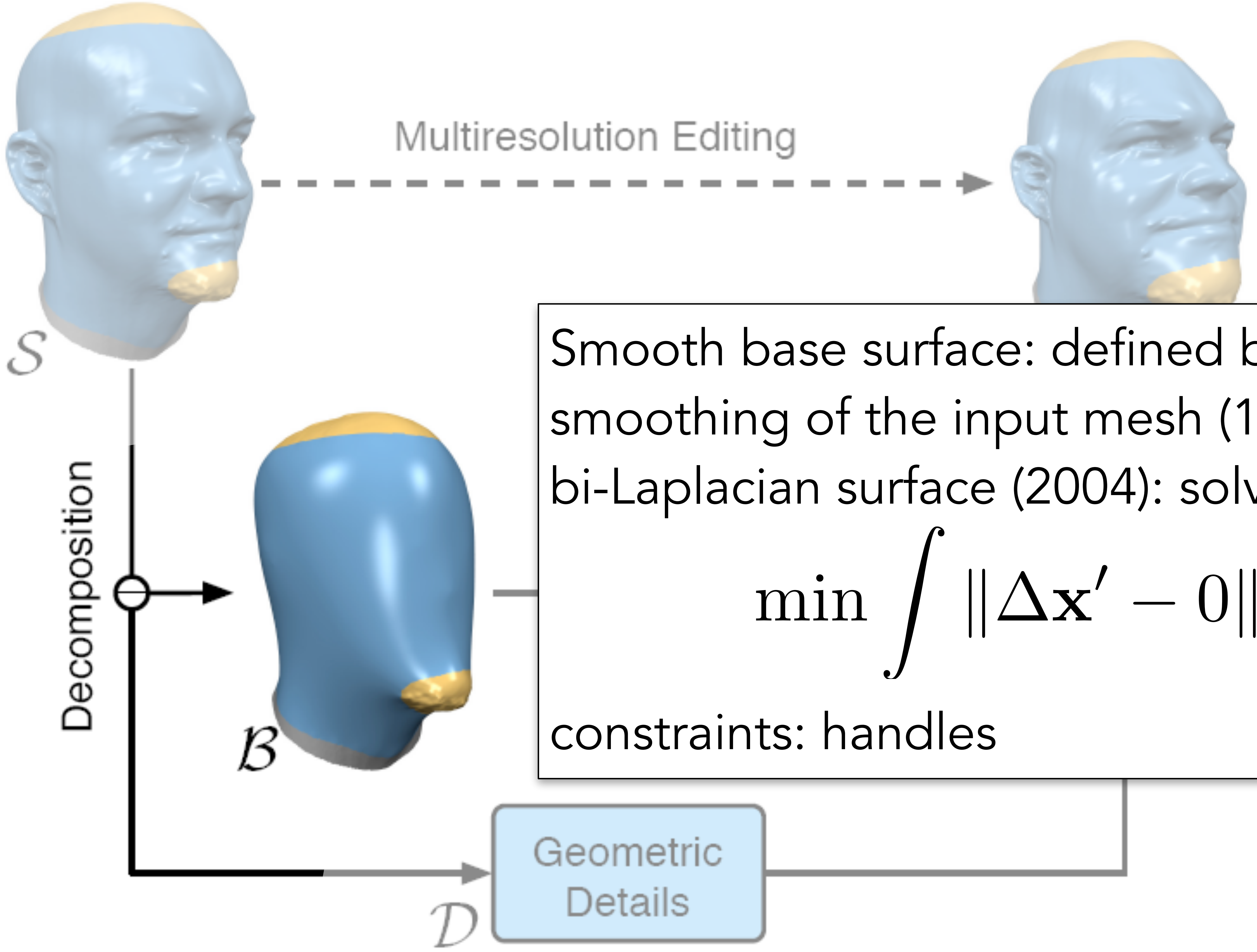


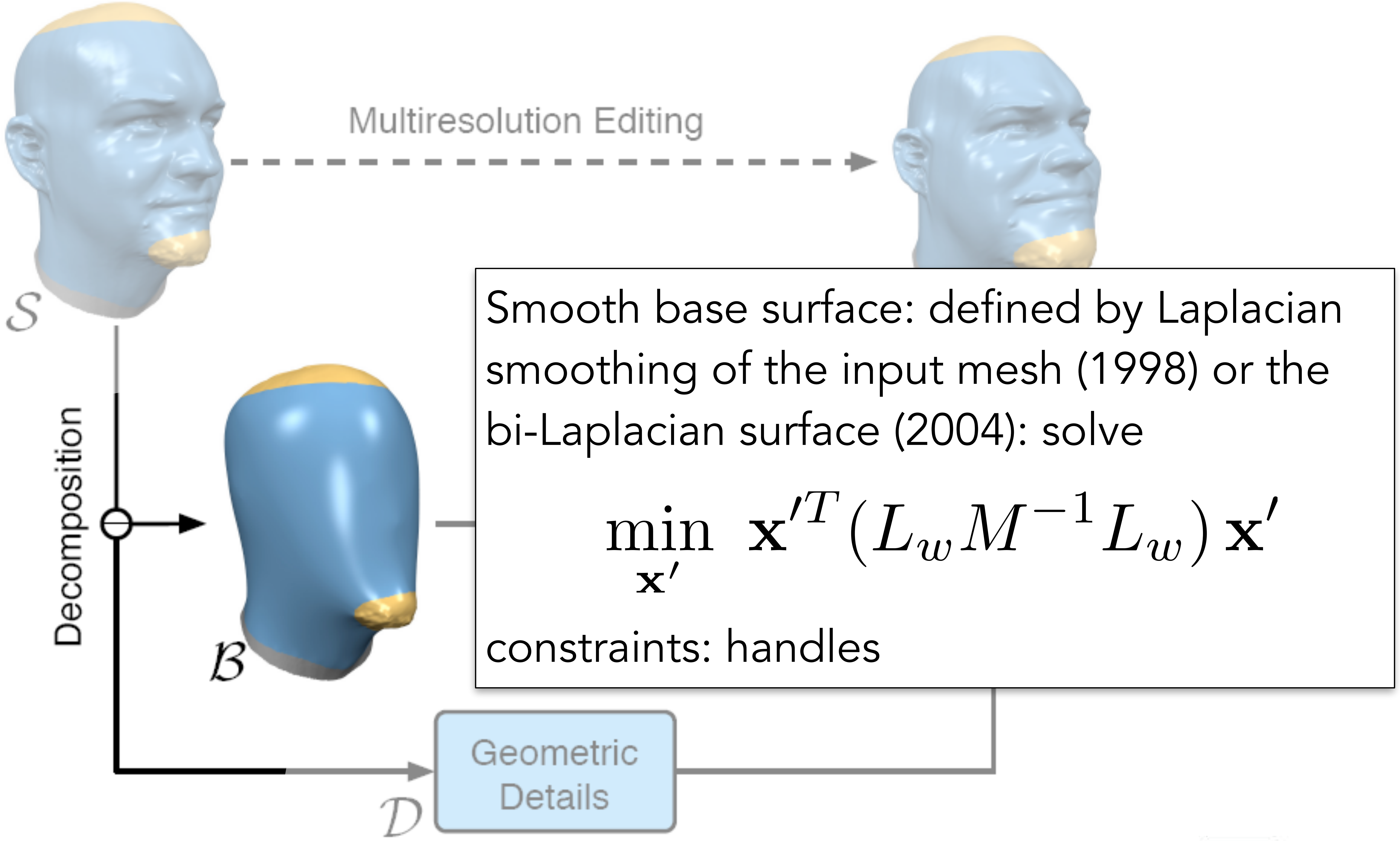
Smooth base surface: defined by Laplacian smoothing of the input mesh (1998) or the bi-Laplacian surface (2004): solve

$$\min \int \|\Delta \mathbf{x}' - 0\|^2 d\mathbf{x}'$$

constraints: handles

Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004



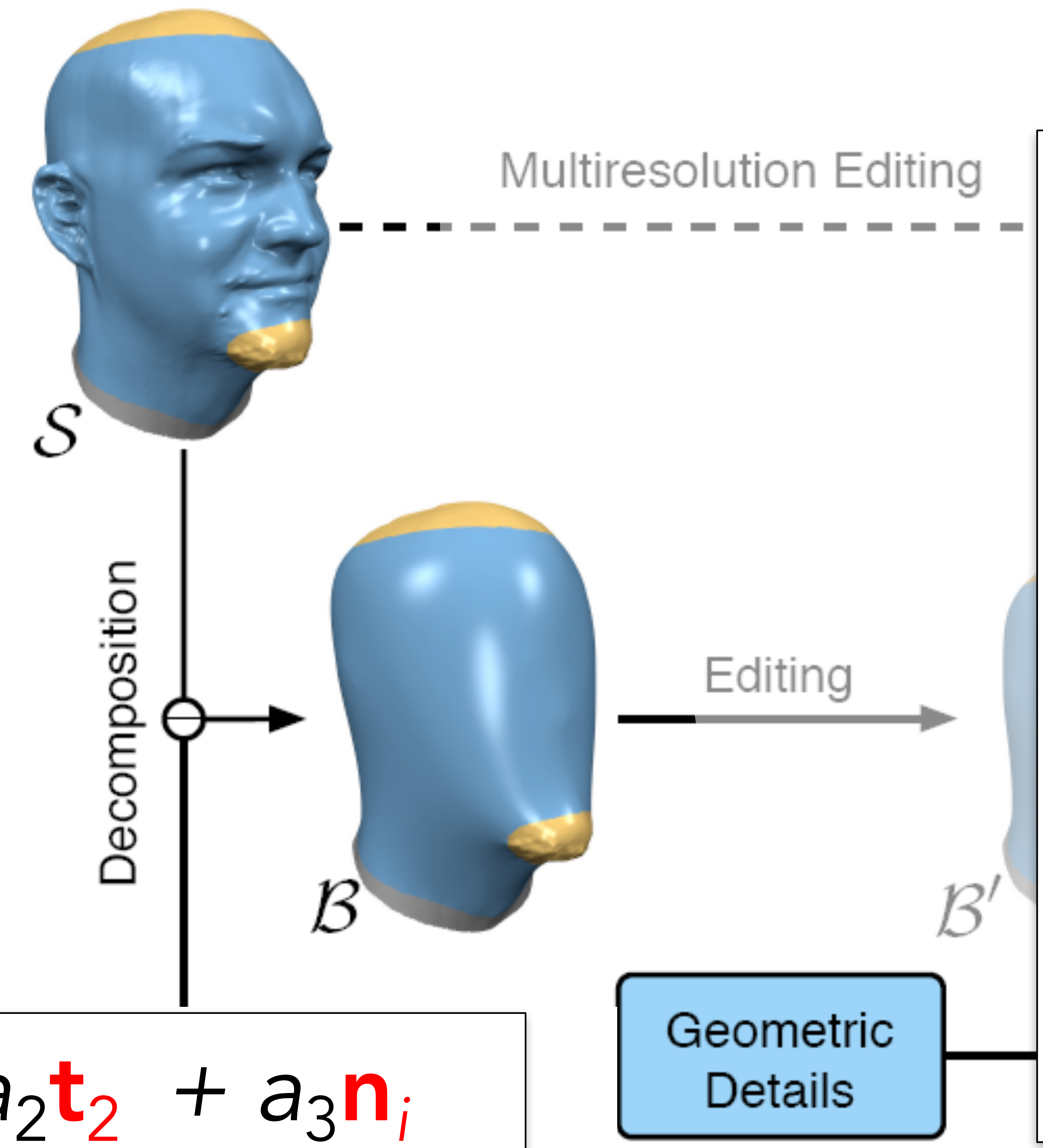
Smooth base surface: defined by Laplacian smoothing of the input mesh (1998) or the bi-Laplacian surface (2004): solve

$$\min_{\mathbf{x}'} \ \mathbf{x}'^T (L_w M^{-1} L_w)\, \mathbf{x}'$$

constraints: handles

Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004



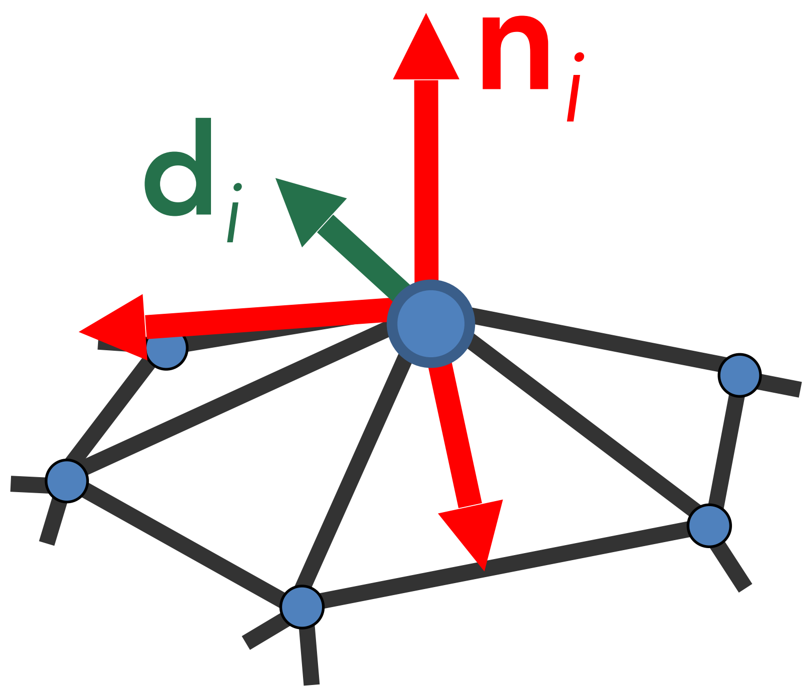
$$\mathbf{d}_i = a_1\mathbf{t}_1 + a_2\mathbf{t}_2 + a_3\mathbf{n}_i$$

Geometric details: simply the difference vectors between base surface B and input S

For each vertex, compute

$$\mathbf{d}_i = \mathbf{v}_i - \mathbf{b}_i$$

Represent $\mathbf{d}_i$ in the **local frame** of $\mathbf{b}_i$



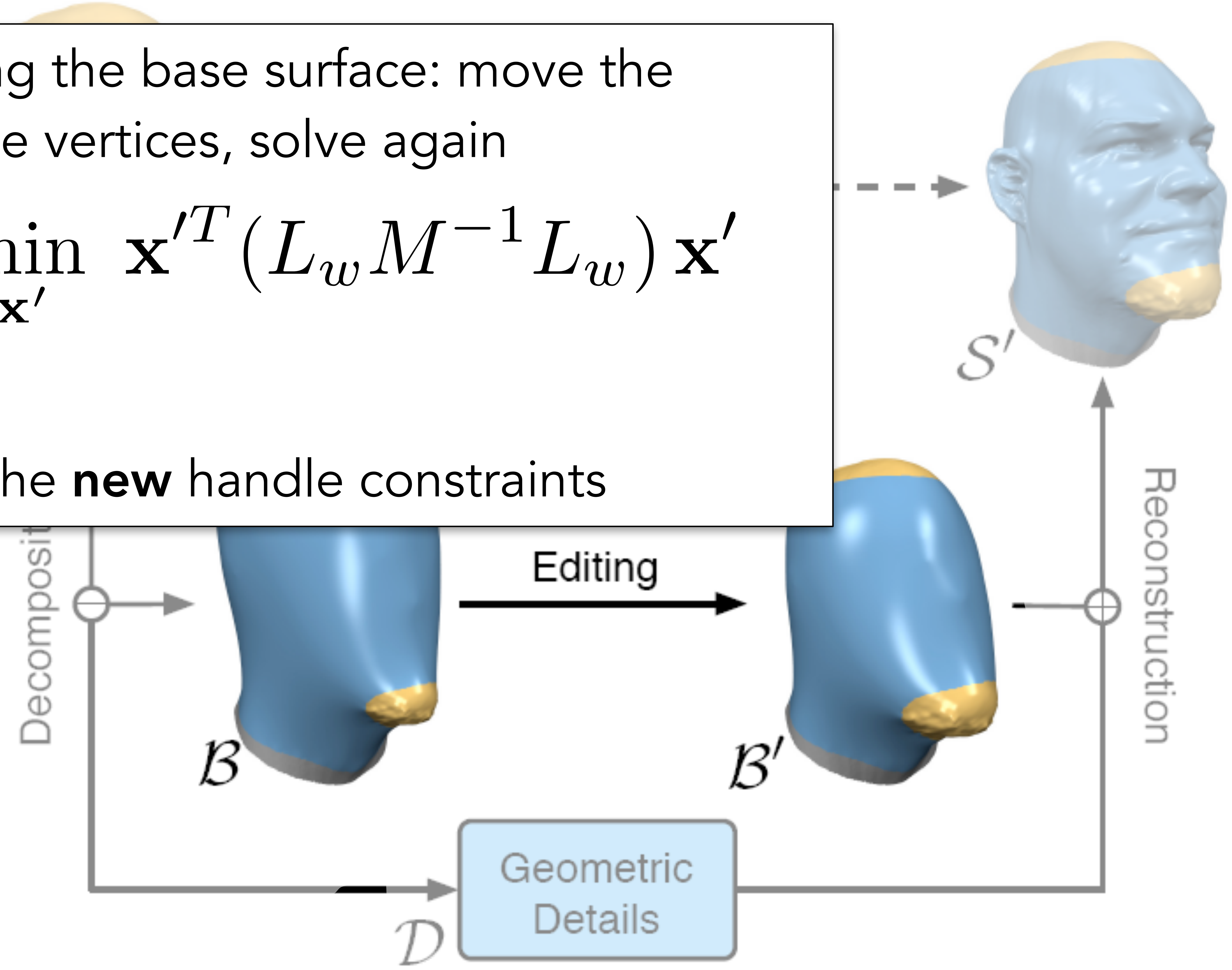
Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004

Editing the base surface: move the handle vertices, solve again

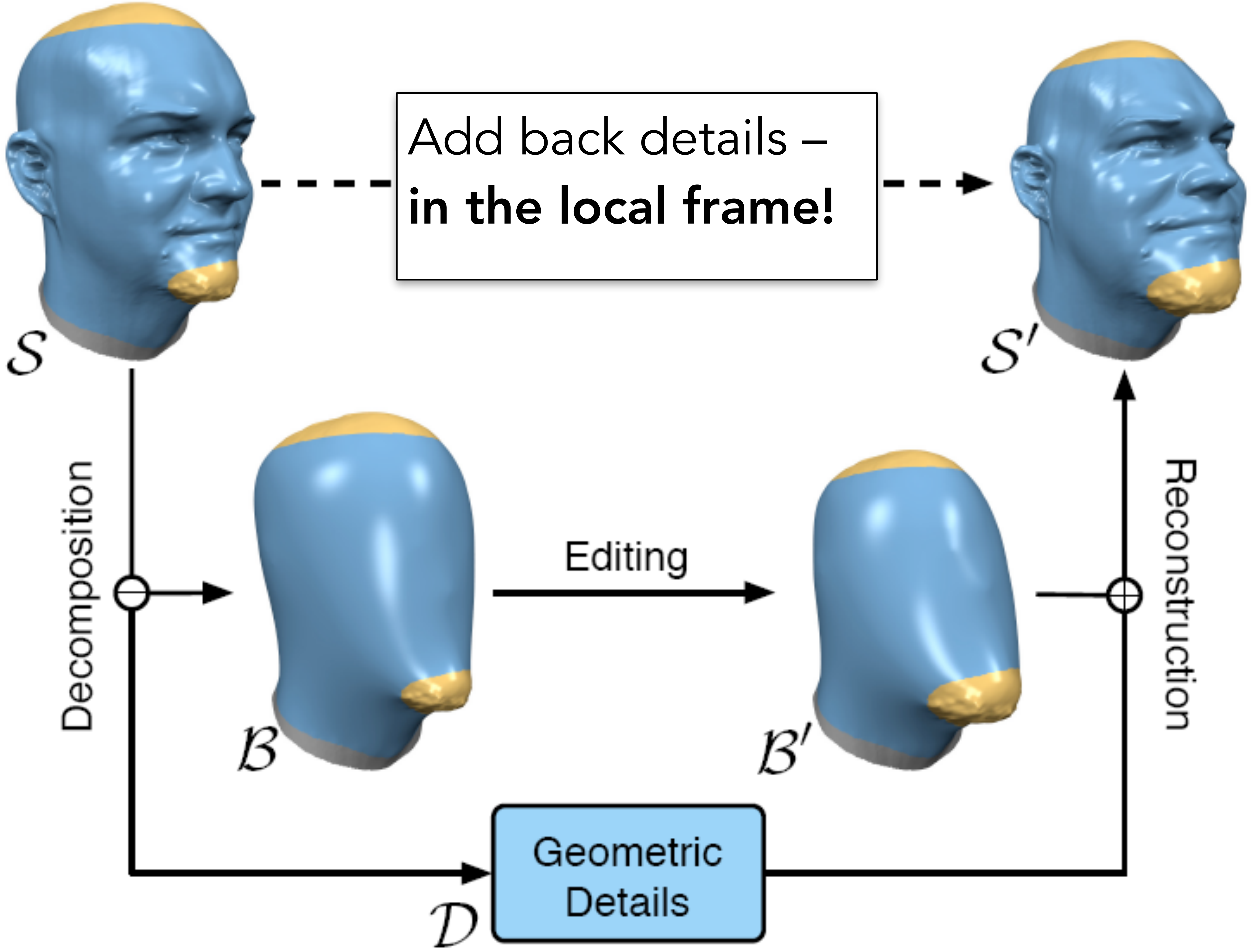
$$\min_{\mathbf{x}'} \; \mathbf{x}'^T \left(L_w M^{-1} L_w \right) \mathbf{x}'$$

with the **new** handle constraints



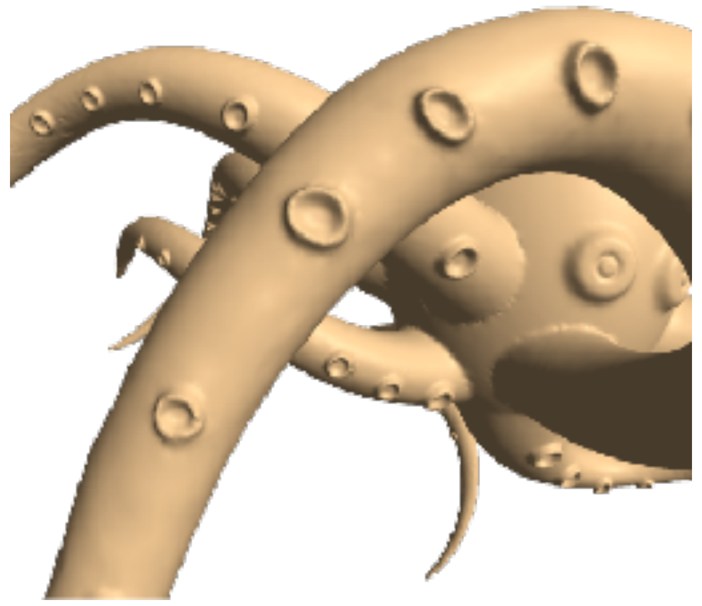
Multiresolution Approach

- Kobbelt et al. SIGGRAPH 98, Botsch and Kobbelt SIGGRAPH 2004

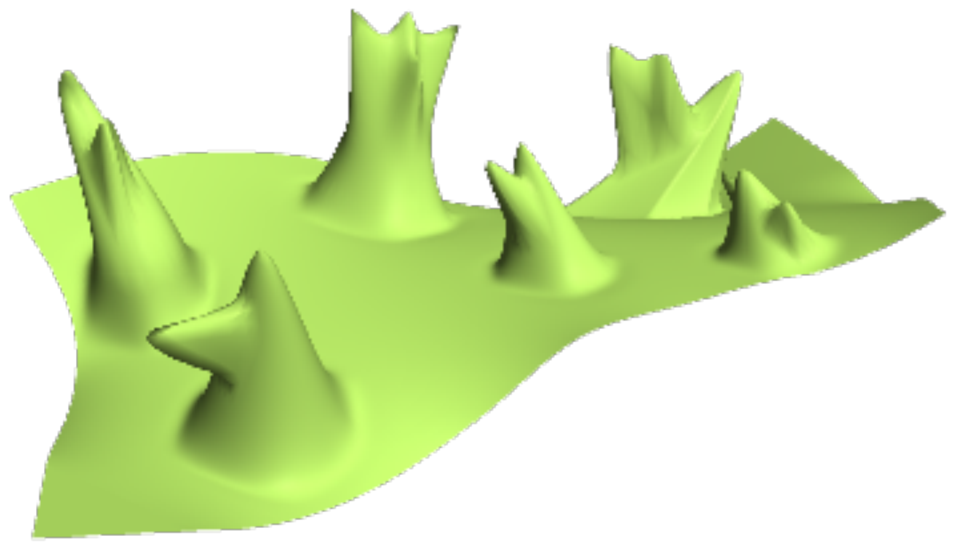


Discussion

- Pros:
  - Fast! Linear solve for the base surface deformation, and then add back displacements
  - Intuitive, easy to implement
- Cons:
  - works only for small height fields



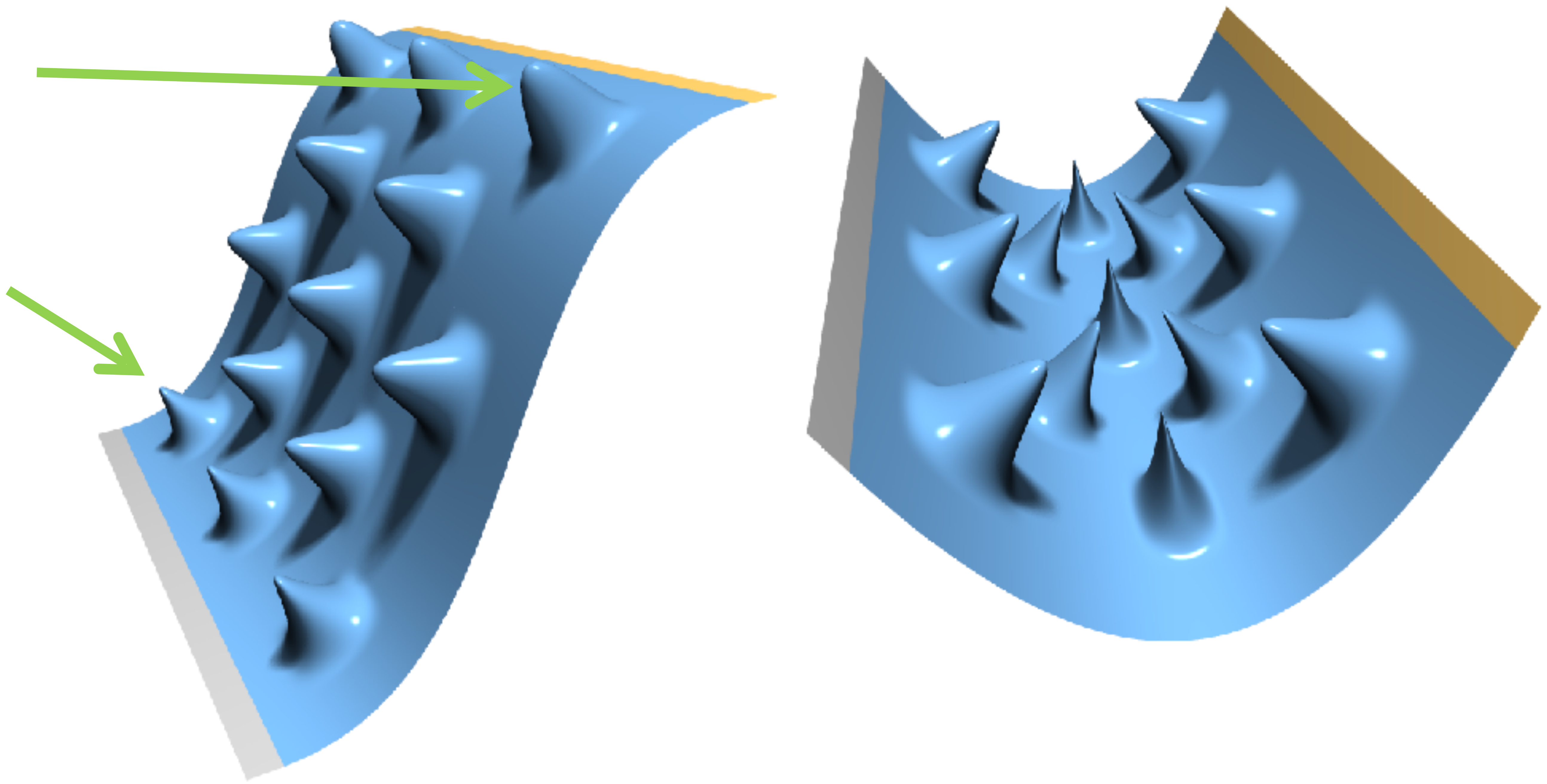
almost a height field



not a height field

Discussion

- If detail vectors are too big we get **overshooting** and **self-intersections**, especially in concave cases



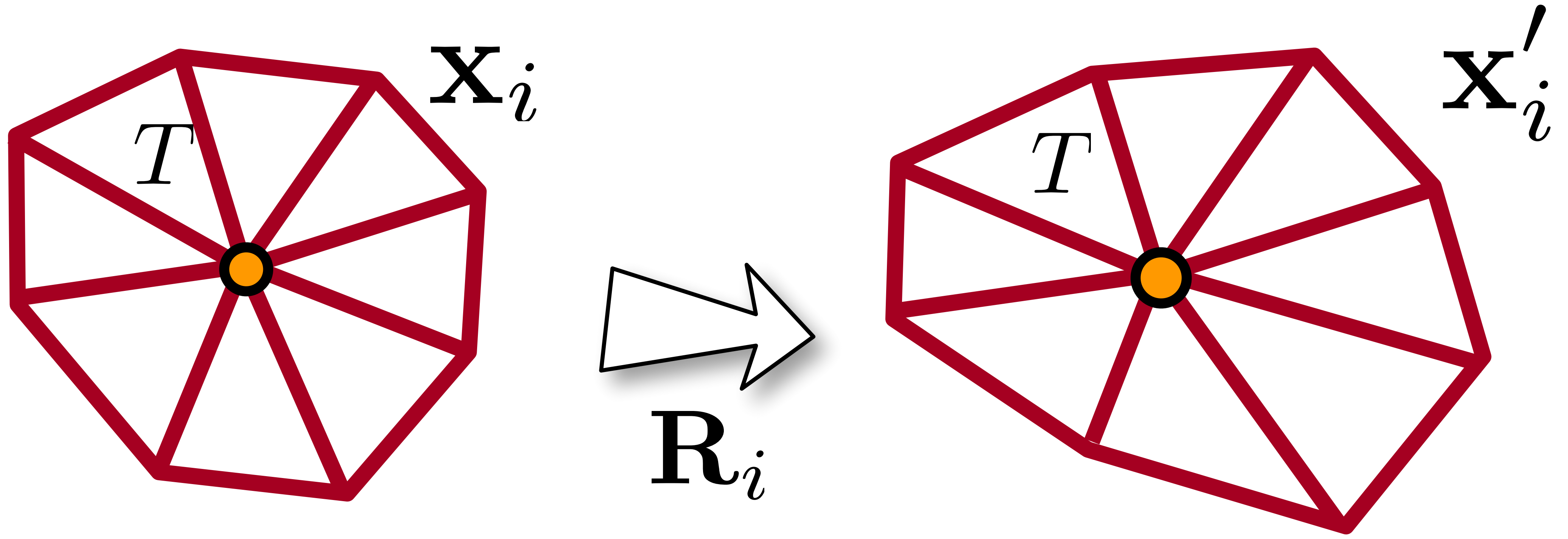
Approach 2:
As-Rigid-As-Possible Mesh Editing

Demo

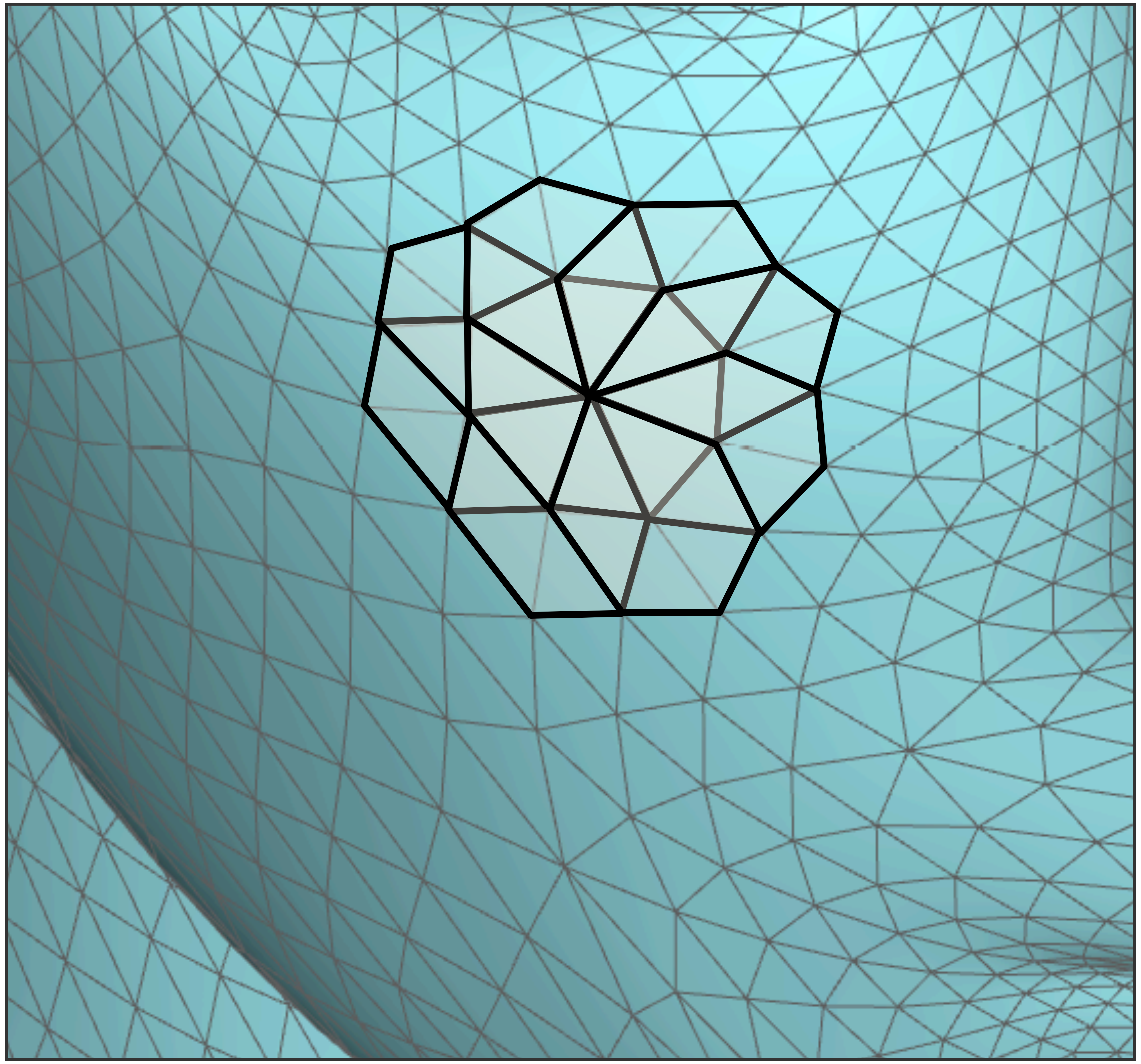
- https://libigl.github.io/libigl-python-bindings/tut-chapter3/

As-Rigid-As-Possible Deformation

- Preserve shape of cells covering the surface
- Ask each cell *i* to transform **rigidly** by best-fitting rotation $\mathbf{R}_i$

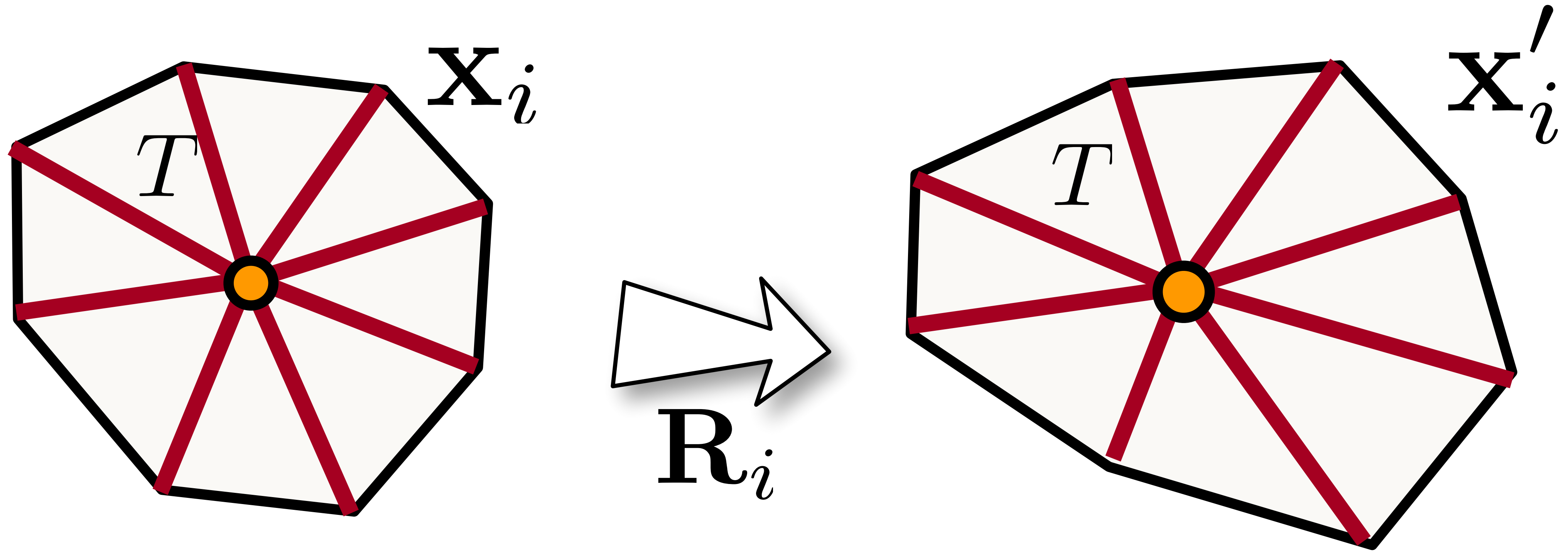


$$\min \sum_{T \in \mathrm{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$



As-Rigid-As-Possible Deformation

- Optimal $\mathbf{R}_i$ is uniquely defined by $\mathbf{x}_i\ \mathbf{x}'_i$

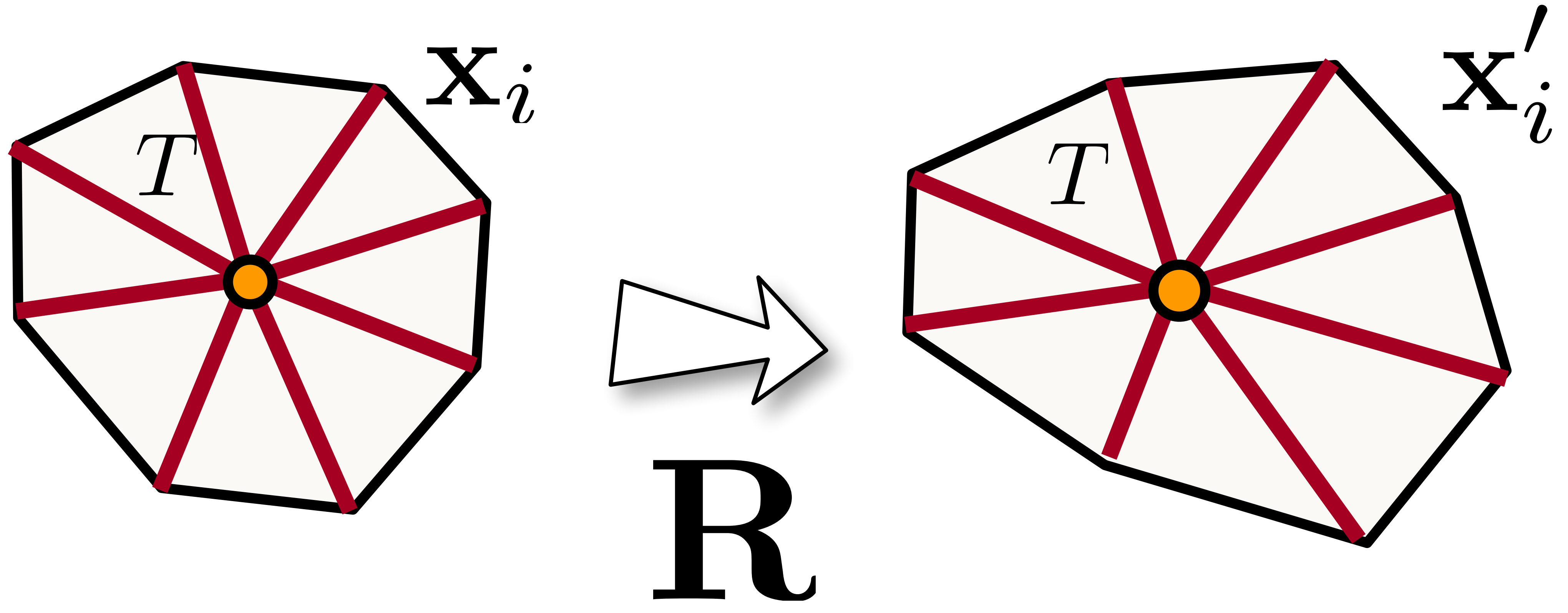


$$\min \sum_{T \in \mathrm{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

- so-called shape-matching problem, solved by a 3x3 SVD

R$_i$ is a nonlinear function of **x**

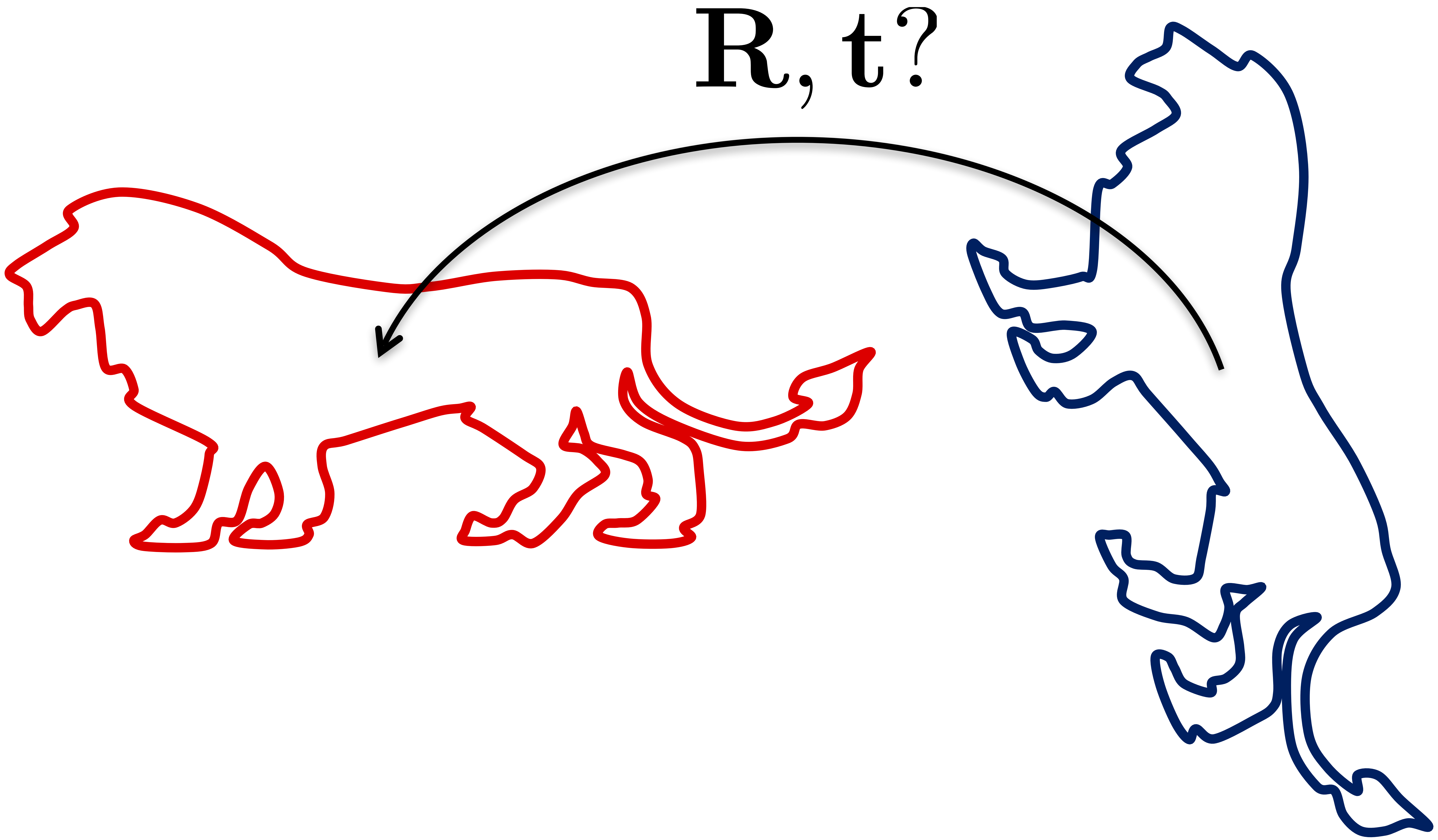
Optimal Rotation



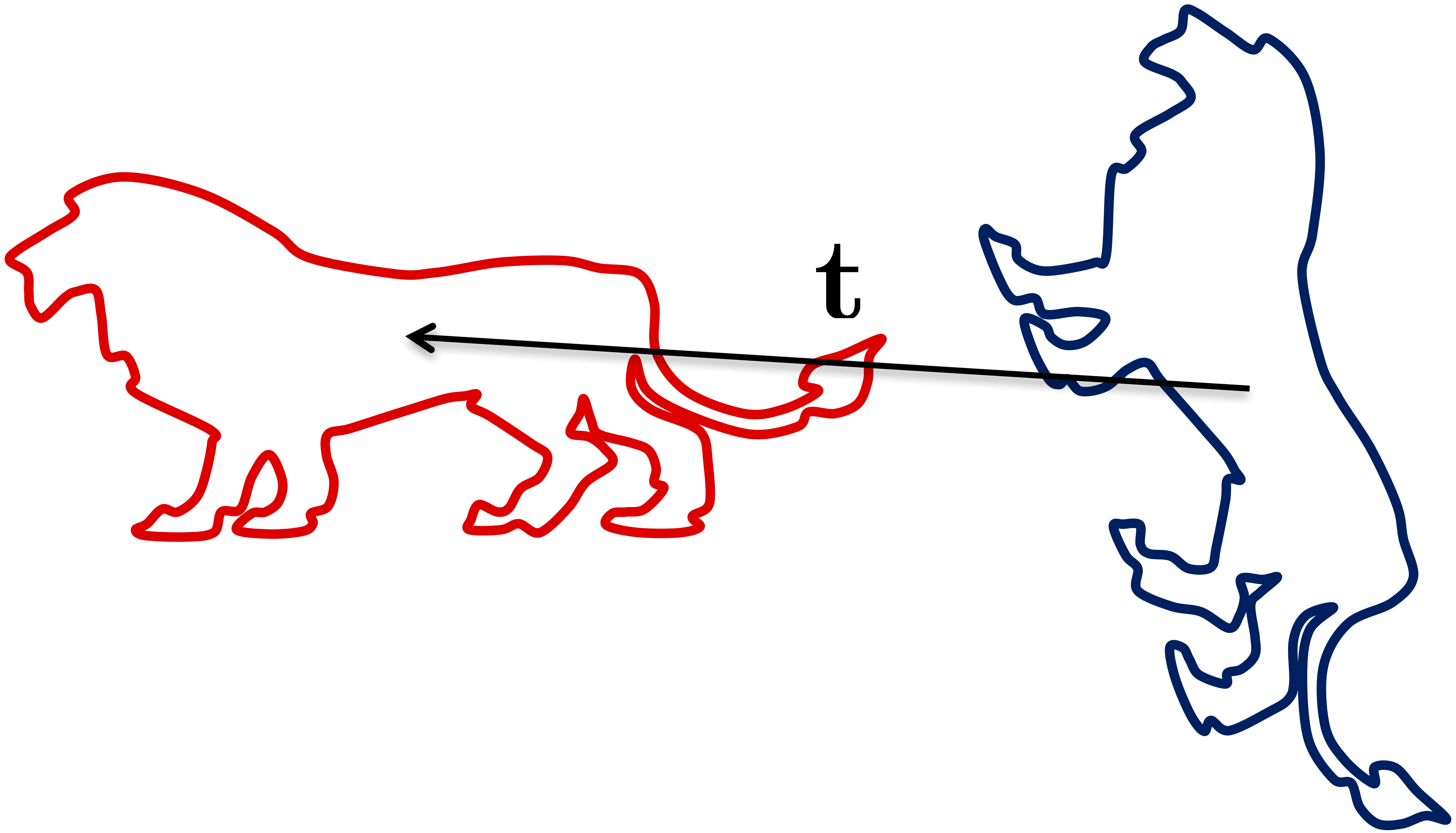
$$\min_{\mathbf{R}\in SO(3)} \sum_{T\in \text{Cell}_i} \sum_{(j,k)\in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

Rotation group

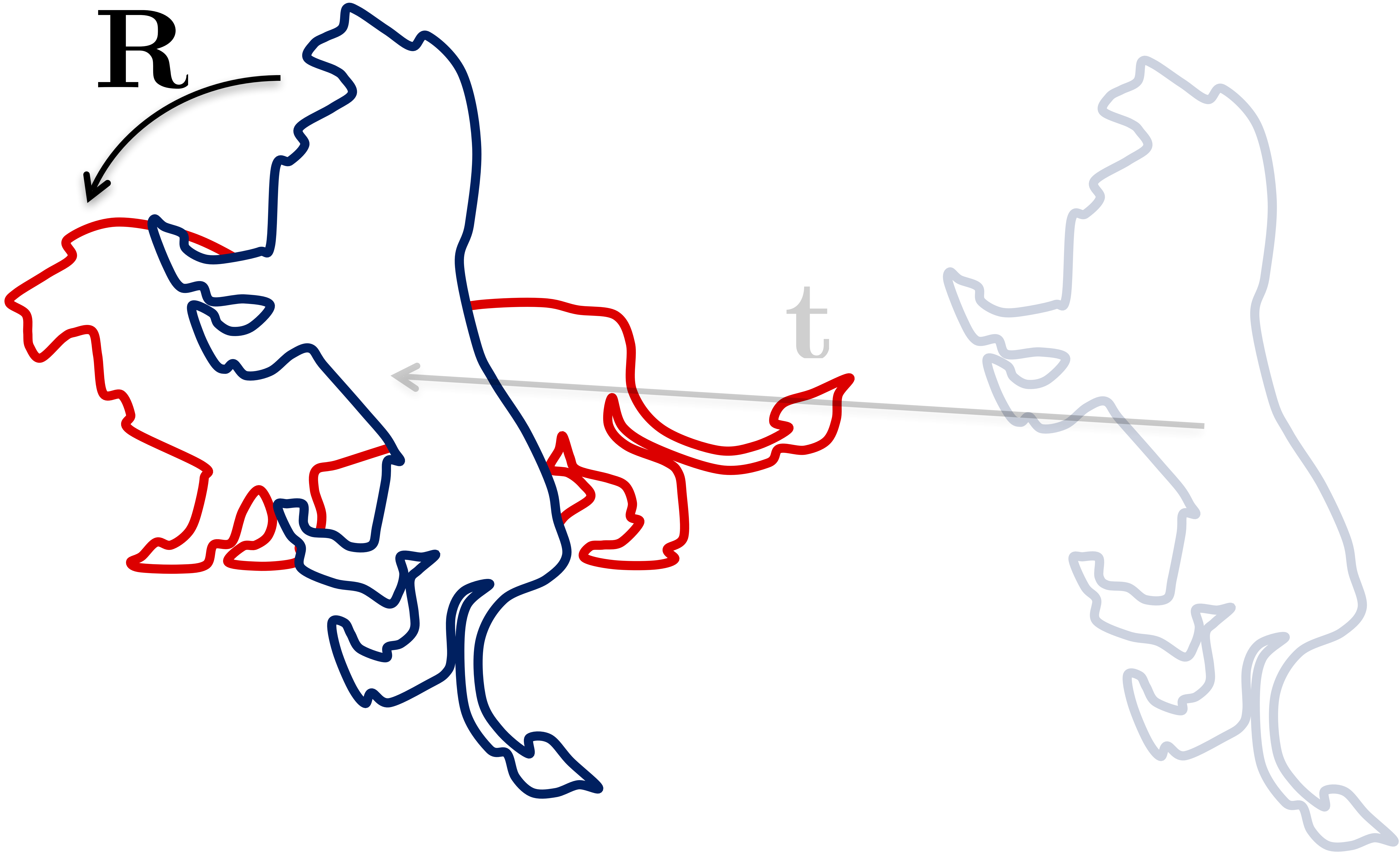
Shape Matching Problem



Shape Matching Problem



Shape Matching Problem



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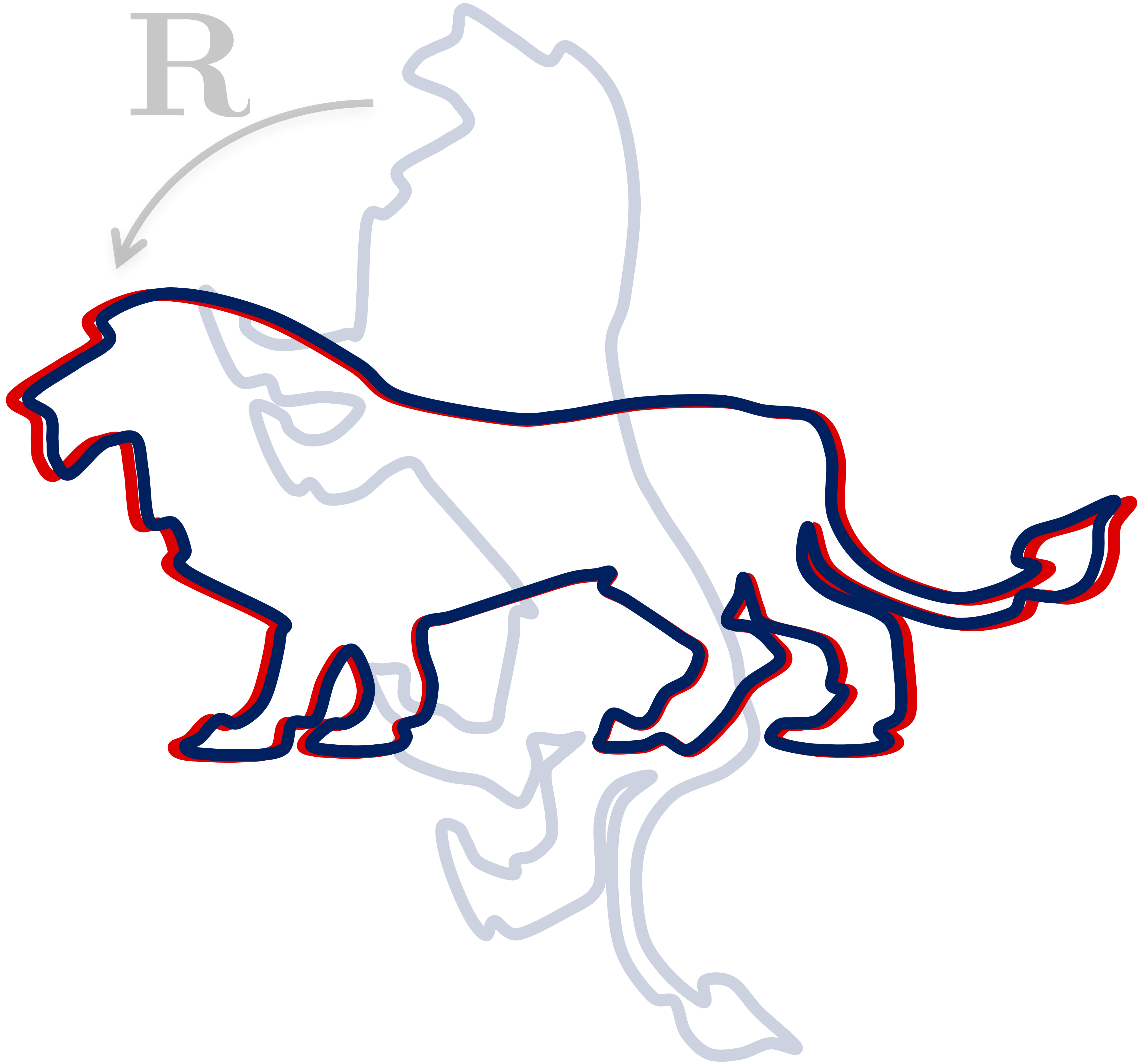
Shape Matching Problem



Shape Matching Problem

- Align two point sets

$$\mathcal{P} = \{\mathbf{p}_1, \ldots, \mathbf{p}_n\} \text{ and } \mathcal{Q} = \{\mathbf{q}_1, \ldots, \mathbf{q}_n\}$$

- Find a translation vector **t** and rotation matrix **R** so that

$$\sum_{i=1}^{n} \|(\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\|^2 \text{ is minimized}$$

Shape Matching – Solution

- Solve for translation first (w.r.t. **R**, **p**, and **q**)

$$\frac{\partial}{\partial \mathbf{t}} \sum_{i=1}^{n} \|(\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\|^2 = \sum_{i=1}^{n} 2\left((\mathbf{R}\mathbf{p}_i + \mathbf{t}) - \mathbf{q}_i\right) \stackrel{!}{=} 0$$

$$\mathbf{R}\sum_{i=1}^{n} \mathbf{p}_i + \sum_{i=1}^{n} \mathbf{t} - \sum_{i=1}^{n} \mathbf{q}_i = 0$$

$$\mathbf{t} = \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{q}_i\right)}_{\bar{\mathbf{q}}} - \mathbf{R}\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{p}_i\right)}_{\bar{\mathbf{p}}}$$

Point sets $\{\mathbf{q}_i\}$ and $\{\mathbf{R}\mathbf{p}_i\}$ have the same center of mass

Finding the Rotation $\mathbf{R}$

- To find the optimal **R**, we bring the centroids of both point sets to the origin

$$\mathbf{v}_i = \mathbf{p}_i - \bar{\mathbf{p}}, \qquad \mathbf{v}'_i = \mathbf{q}_i - \bar{\mathbf{q}}$$

- We want to find **R** that minimizes

$$\sum_{i=1}^{n} \|\mathbf{R}\mathbf{v}_i - \mathbf{v}'_i\|^2$$

Finding the Rotation **R**

$$\sum_{i=1}^{n} \|\mathbf{R}\mathbf{v}_i - \mathbf{v}_i'\|^2 = \sum_{i=1}^{n} (\mathbf{R}\mathbf{v}_i - \mathbf{v}_i')^T (\mathbf{R}\mathbf{v}_i - \mathbf{v}_i') =$$

$$= \sum_{i=1}^{n} \left(\mathbf{v}_i^T \underbrace{\mathbf{R}^T \mathbf{R}}_{\mathbf{I}} \mathbf{v}_i - \mathbf{v}_i'^T \mathbf{R}\mathbf{v}_i - \mathbf{v}_i^T \mathbf{R}^T \mathbf{v}_i' + \mathbf{v}_i'^T \mathbf{v}_i' \right)$$

These terms do not depend on **R**,
so we can ignore them in the minimization

Finding the Rotation $\mathbf{R}$

$$\underset{\mathbf{R}\in SO(3)}{\operatorname{argmin}} \sum_{i=1}^{n} \left(-\mathbf{v}_i'^T \mathbf{R} \mathbf{v}_i - \mathbf{v}_i^T \mathbf{R}^T \mathbf{v}_i' \right) = \underset{\mathbf{R}\in SO(3)}{\operatorname{argmax}} \sum_{i=1}^{n} \left(\mathbf{v}_i'^T \mathbf{R} \mathbf{v}_i + \underbrace{\mathbf{v}_i^T \mathbf{R}^T \mathbf{v}_i'} \right) =$$

$$= \boxed{\underset{\mathbf{R}\in SO(3)}{\operatorname{argmax}} \sum_{i=1}^{n} \mathbf{v}_i'^T \mathbf{R} \mathbf{v}_i}$$

$$\mathbf{v}_i^T \mathbf{R}^T \mathbf{v}_i' = \left(\mathbf{v}_i^T \mathbf{R}^T \mathbf{v}_i' \right)^T = \mathbf{v}_i'^T \mathbf{R} \mathbf{v}_i$$

Finding the Rotation $\mathbf{R}$

$$\sum_{i=1}^{n} \mathbf{v}_i'^T \mathbf{R} \mathbf{v}_i = tr\left(\mathbf{V}'^T \mathbf{R} \mathbf{V}\right)$$

$$\underbrace{\begin{bmatrix} \mathbf{v}_1'^T \\ \mathbf{v}_2'^T \\ \vdots \\ \mathbf{v}_n'^T \end{bmatrix}}_{\mathbf{V}'^T} \begin{bmatrix} \mathbf{R} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}}_{\mathbf{V}} = \begin{bmatrix} \mathbf{v}_1'^T \\ \mathbf{v}_2'^T \\ \vdots \\ \mathbf{v}_n'^T \end{bmatrix} \begin{bmatrix} \mathbf{R}\mathbf{v}_1 & \mathbf{R}\mathbf{v}_2 & \cdots & \mathbf{R}\mathbf{v}_n \end{bmatrix}$$

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Finding the Rotation $\mathbf{R}$

$$\sum_{i=1}^{n} \mathbf{v}_i'^T \mathbf{R} \mathbf{v}_i = tr\left(\mathbf{V}'^T \mathbf{R} \mathbf{V}\right)$$

$$\begin{bmatrix} \mathbf{v}_1'^T \\ \mathbf{v}_2'^T \\ \vdots \\ \mathbf{v}_n'^T \end{bmatrix} \begin{bmatrix} \mathbf{R}\mathbf{v}_1 & \mathbf{R}\mathbf{v}_2 & \cdots & \mathbf{R}\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \mathbf{v}_1'^T \mathbf{R}\mathbf{v}_1 & & & \\ & \mathbf{v}_2'^T \mathbf{R}\mathbf{v}_2 & & \\ & & \ddots & \\ & & & \mathbf{v}_n'^T \mathbf{R}\mathbf{v}_n \end{bmatrix}$$

Finding the Rotation $\mathbf{R}$

Take a look at the Matrix Cookbook!

- Find **R** that maximizes

$$tr\left(\mathbf{V'}^T\mathbf{R}\mathbf{V}\right) = tr\left(\mathbf{R}\mathbf{V}\mathbf{V'}^T\right)$$

- SVD: $\mathbf{V}\mathbf{V'}^T = \mathbf{U}\boldsymbol{\Sigma}\tilde{\mathbf{U}}^T$

$$tr\left(\mathbf{R}\mathbf{V}\mathbf{V'}^T\right) = tr\left(\underbrace{\mathbf{R}\,\mathbf{U}}\underbrace{\boldsymbol{\Sigma}\tilde{\mathbf{U}}^T}\right) = tr\left(\boldsymbol{\Sigma}\,\underbrace{\tilde{\mathbf{U}}^T\mathbf{R}\mathbf{U}}_{\text{orthonormal matrix}}\right)$$

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Finding the Rotation **R**

- We want to maximize

$$tr\left(\Sigma\,\mathbf{M}\right)$$

M: orthonormal matrix
all entries ≤ 1

$$\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \sigma_3 \end{bmatrix} \begin{bmatrix} m_{11} & \cdots & \\ \vdots & m_{22} & \vdots \\ & \cdots & m_{33} \end{bmatrix}$$

$$tr\left(\Sigma\,\mathbf{M}\right) = \sum_{i=1}^{3} \sigma_i\, m_{ii} \leq \sum_{i=1}^{3} \sigma_i$$

Finding the Rotation **R**

$$tr\left(\Sigma\,\mathbf{M}\right) = \sum_{i=1}^{3} \sigma_i\, m_{ii} \leq \sum_{i=1}^{3} \sigma_i$$

- Our best shot is m_{ii} = 1, i.e. to make **M = I**

$$\begin{aligned} \mathbf{M} = \tilde{\mathbf{U}}^T\mathbf{R}\mathbf{U} &\overset{!}{=} \mathbf{I} \\ \mathbf{R}\mathbf{U} &= \tilde{\mathbf{U}} \\ \boxed{\mathbf{R} = \tilde{\mathbf{U}}\mathbf{U}^T} \end{aligned}$$

Summary of Rigid Alignment

- Translate the input points to the centroids

$$\mathbf{v}_i = \mathbf{p}_i - \bar{\mathbf{p}}, \qquad \mathbf{v}'_i = \mathbf{q}_i - \bar{\mathbf{q}}$$

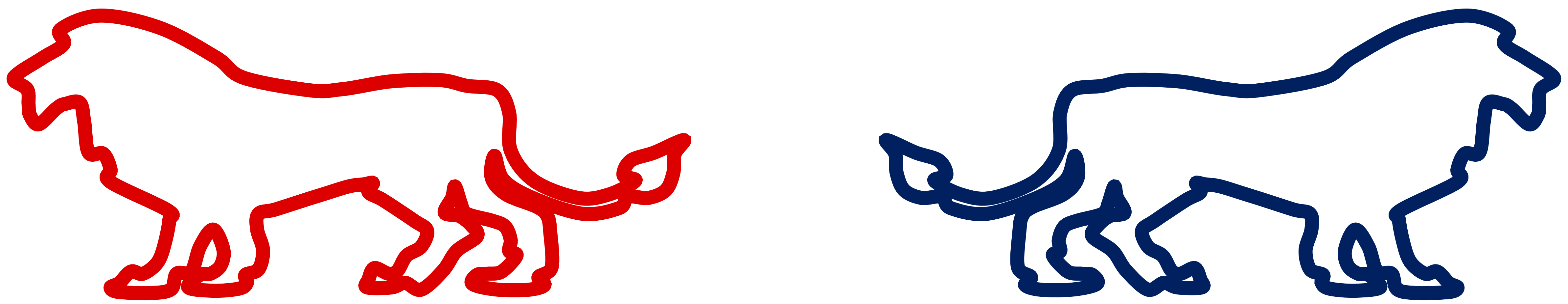
- Compute the "covariance matrix" $\mathbf{V}\mathbf{V}'^T$

- Compute its SVD: $\mathbf{V}\mathbf{V}'^T = \mathbf{U}\boldsymbol{\Sigma}\tilde{\mathbf{U}}^T$

- The optimal orthonormal **R** is $\mathbf{R} = \tilde{\mathbf{U}}\mathbf{U}^T$

Sign Correction

- It is possible that $\det(\tilde{\mathbf{U}}\mathbf{U}^T) = -1$: sometimes reflection is the best orthonormal transform



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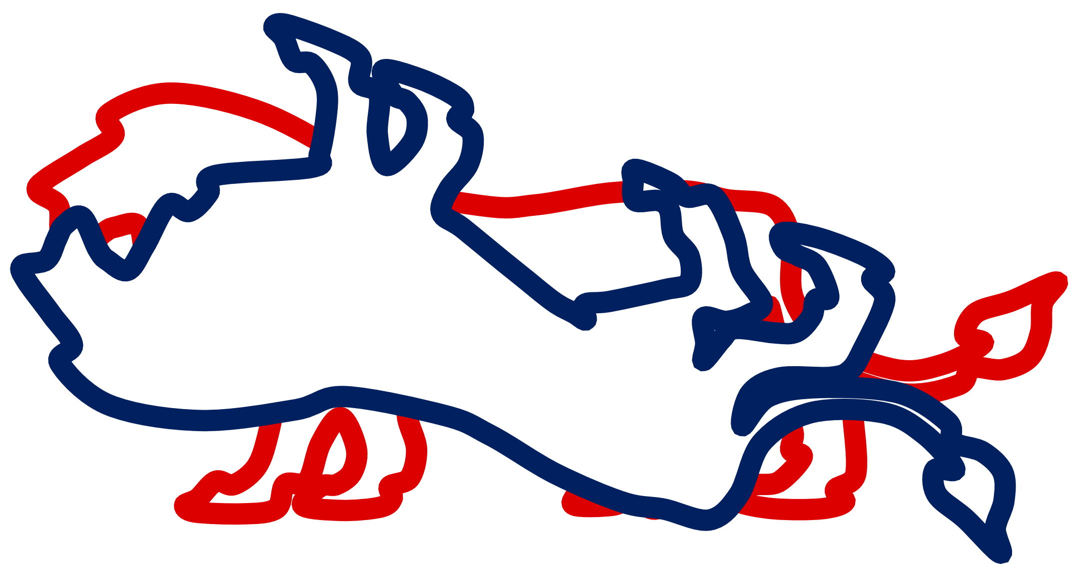
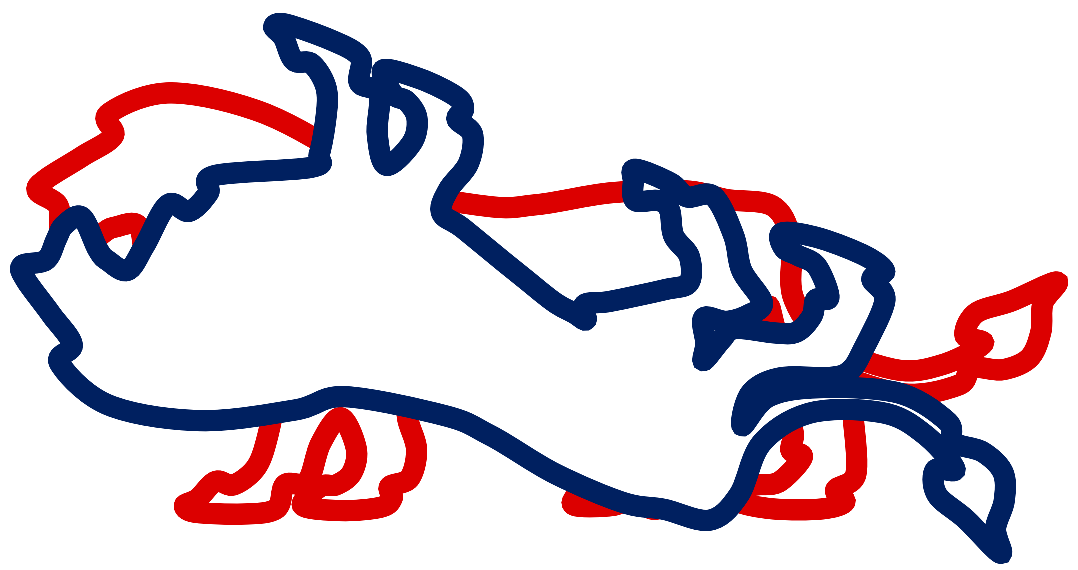


Sign Correction

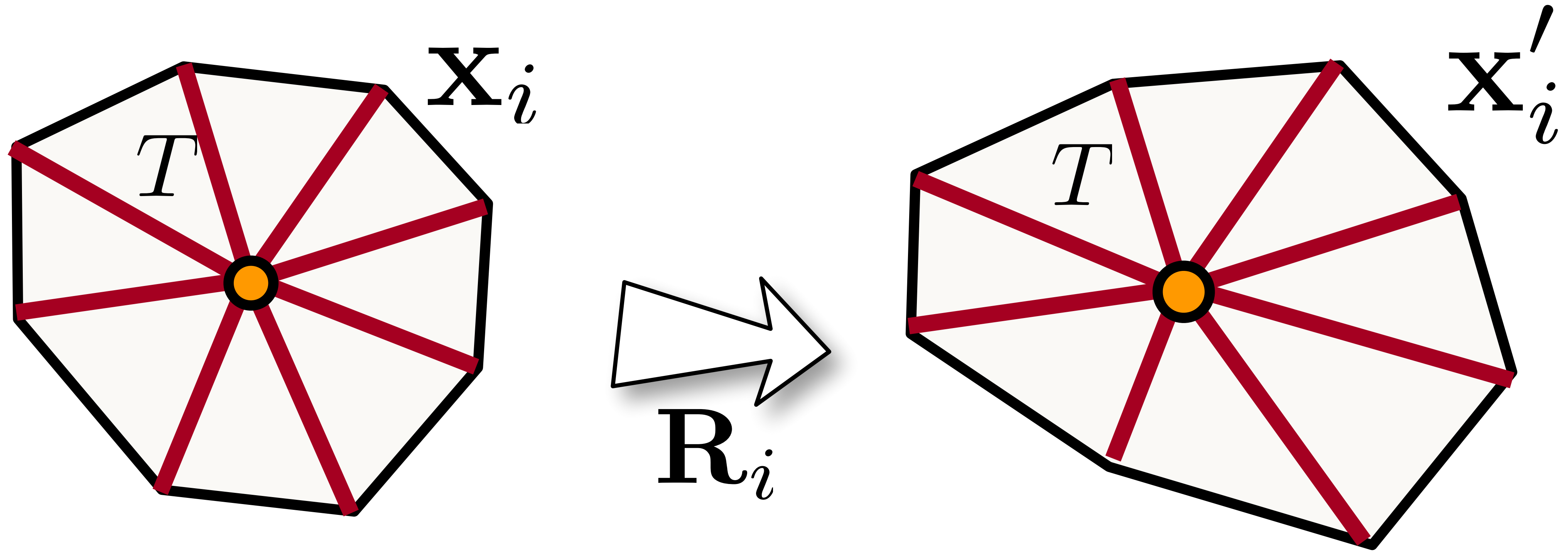
- To restrict ourselves to rotations only:
  take the last column of **U** (corresponding to the smallest singular value) and invert its sign.



- Why? See http://igl.ethz.ch/projects/ARAP/svd_rot.pdf

As-Rigid-As-Possible Deformation

- Optimal $\mathbf{R}_i$ is uniquely defined by $\mathbf{x}_i$ $\mathbf{x}'_i$



$$\min \sum_{T \in \text{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

- so-called shape-matching problem, solved by a 3x3 SVD

$\boxed{\mathbf{R}_i \text{ is a nonlinear function of } \mathbf{x}}$

As-Rigid-As-Possible Deformation

- Total ARAP energy: sum up for all the cells *i*

$$\sum_i \sum_{T \in \mathrm{Cell}_i} \sum_{(j,k) \in T} \|(\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i(\mathbf{x}_j - \mathbf{x}_k)\|^2$$

- Treat **x** and **R** as separate sets of variables
- Simple **local-global** iterative optimization process
  - Decreases the energy at each step

As-Rigid-As-Possible Deformation

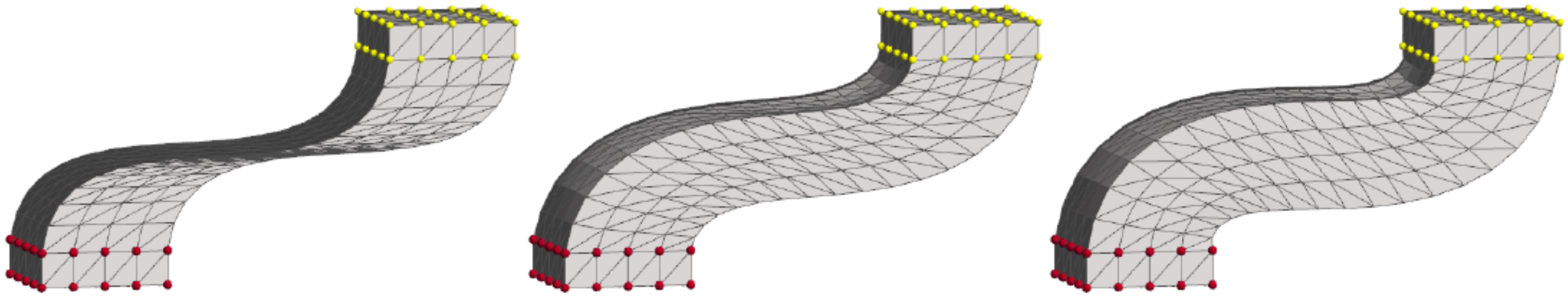
- Total ARAP energy: sum up for all the cells *i*

$$\sum_i \sum_{T \in \mathrm{Cell}_i} \sum_{(j,k) \in T} \left\| (\mathbf{x}'_j - \mathbf{x}'_k) - \mathbf{R}_i (\mathbf{x}_j - \mathbf{x}_k) \right\|^2$$

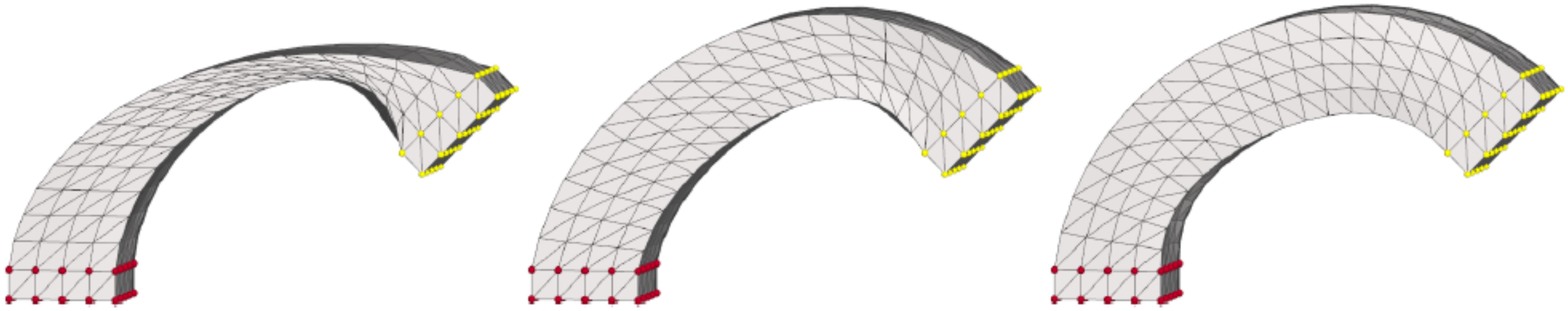
- Local step: keep **x'** fixed, find optimal $\mathbf{R}_i$ per cell *i*
- Global step: keep $\mathbf{R}_i$ fixed, solve for **x'** ➡ $\mathbf{L}\mathbf{x}' = \mathbf{b}$
quadratic minimization problem
  - The matrix **L** stays fixed, can pre-factorize

Initial Guess

- Can use naïve Laplacian editing



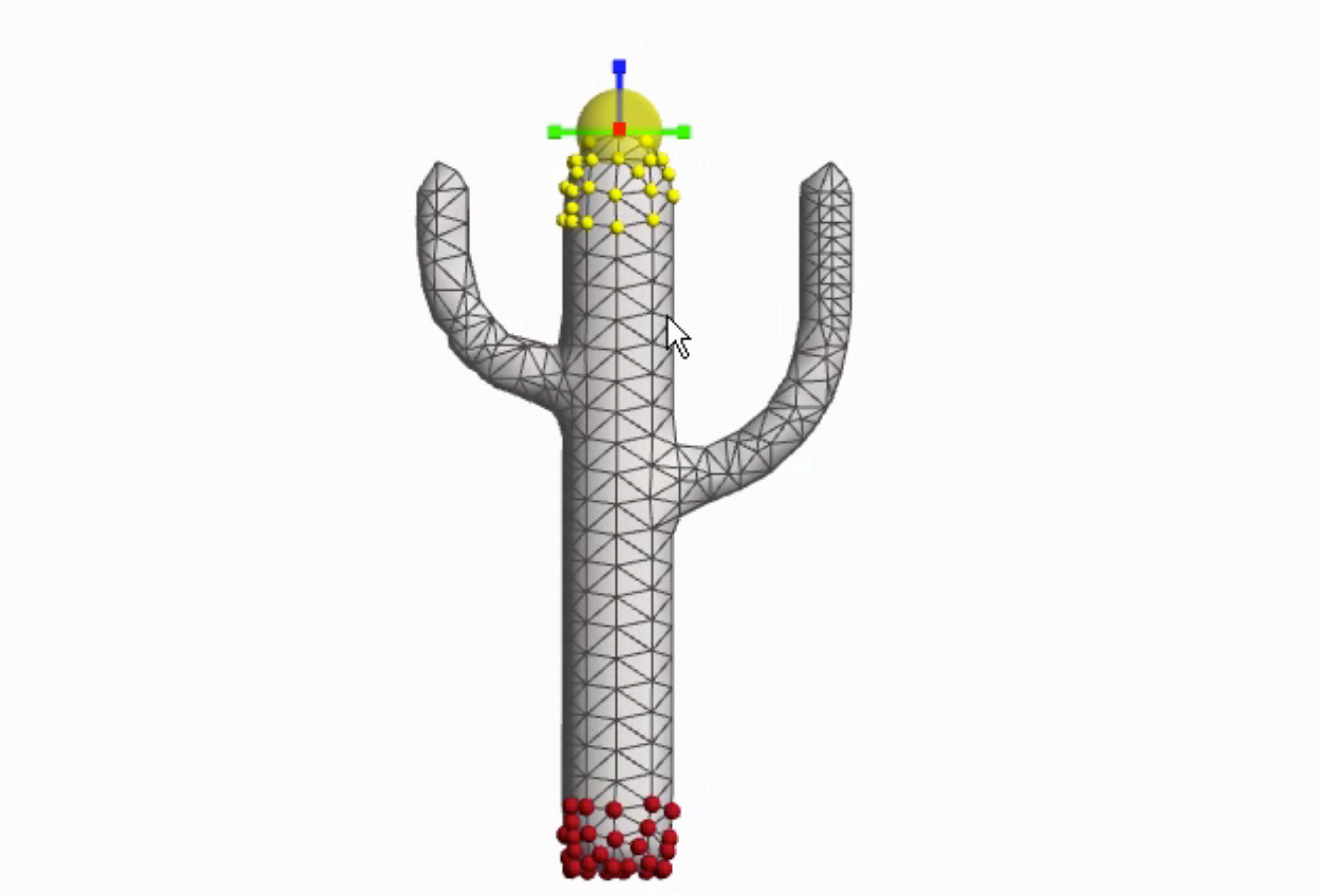
initial guess | 1 iteration | 2 iterations



initial guess | 1 iterations | 4 iterations

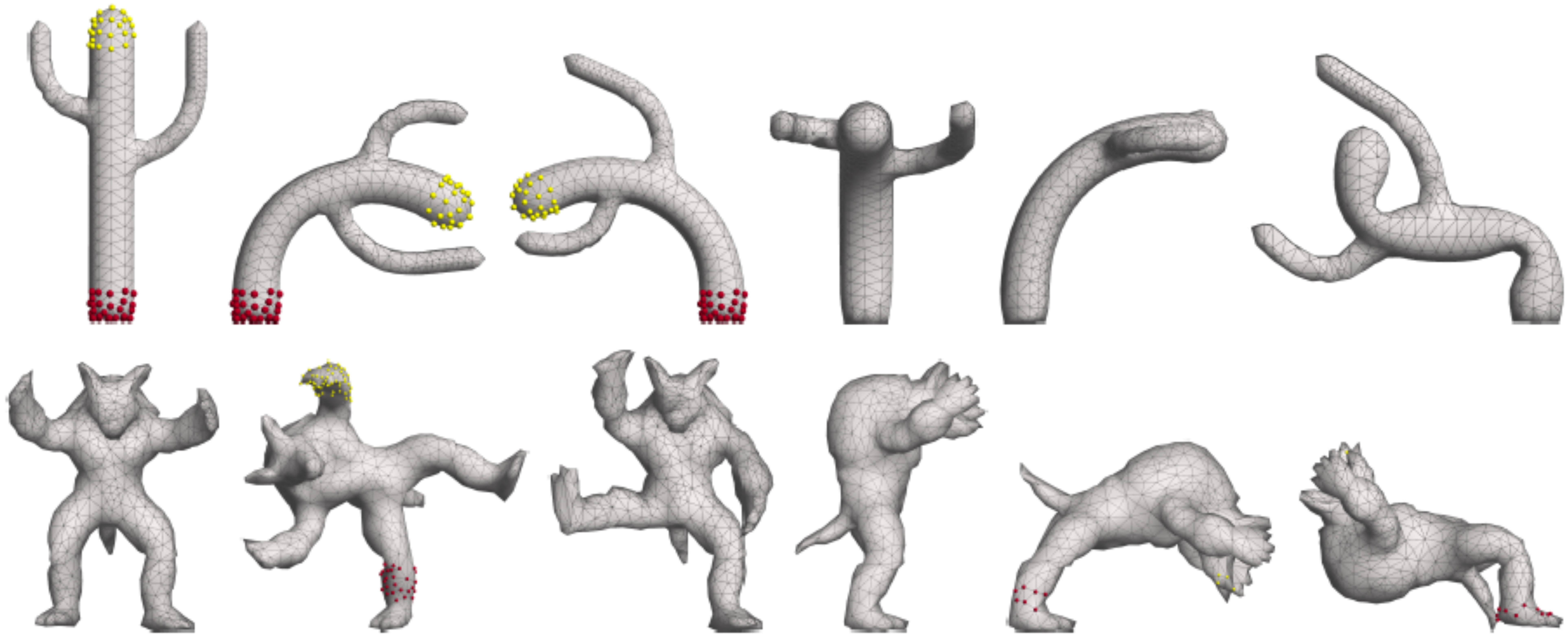
Initial Guess

- Can also use the previous frame
- Replace all handle vertex positions by the currently prescribed ones
- Fast convergence

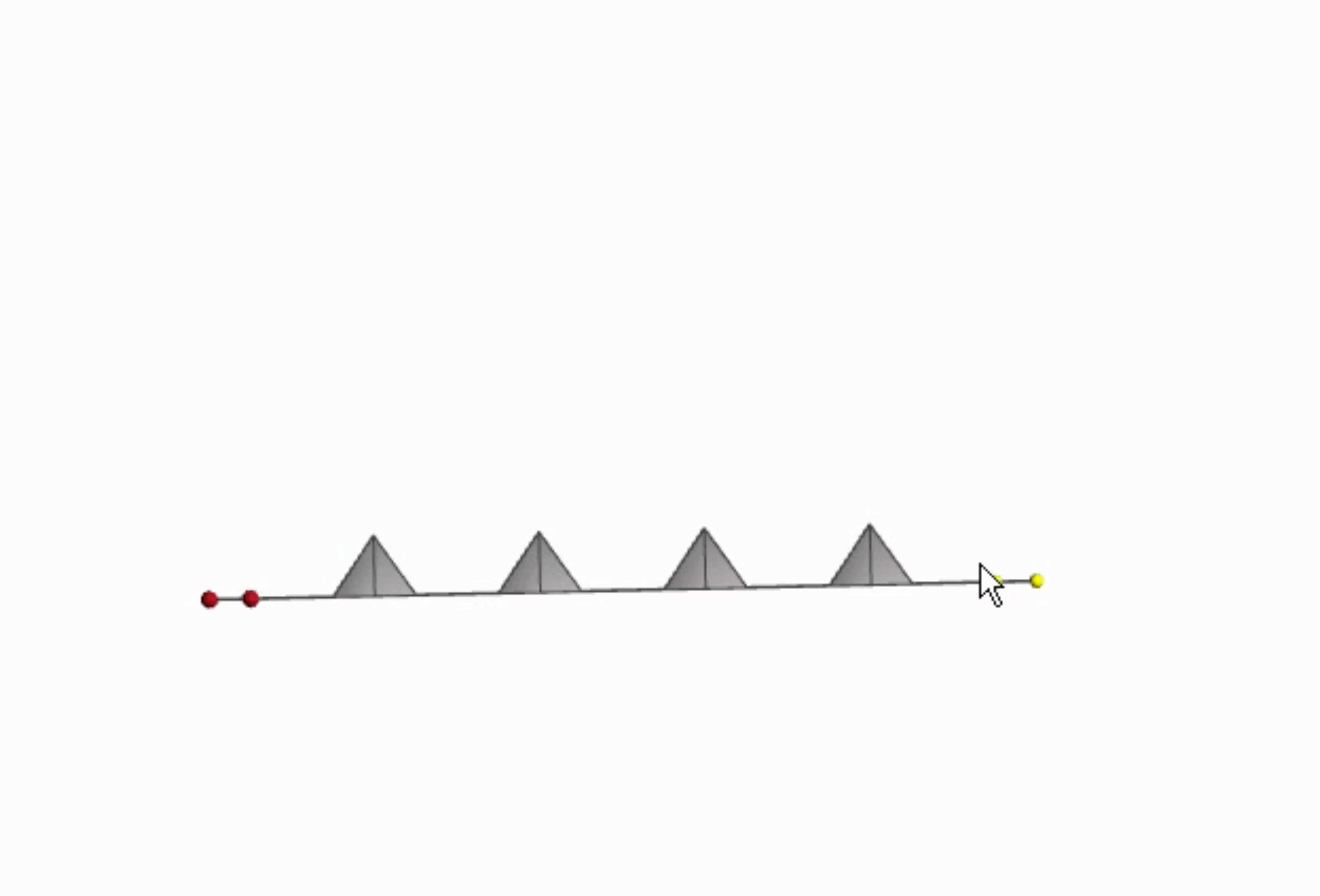


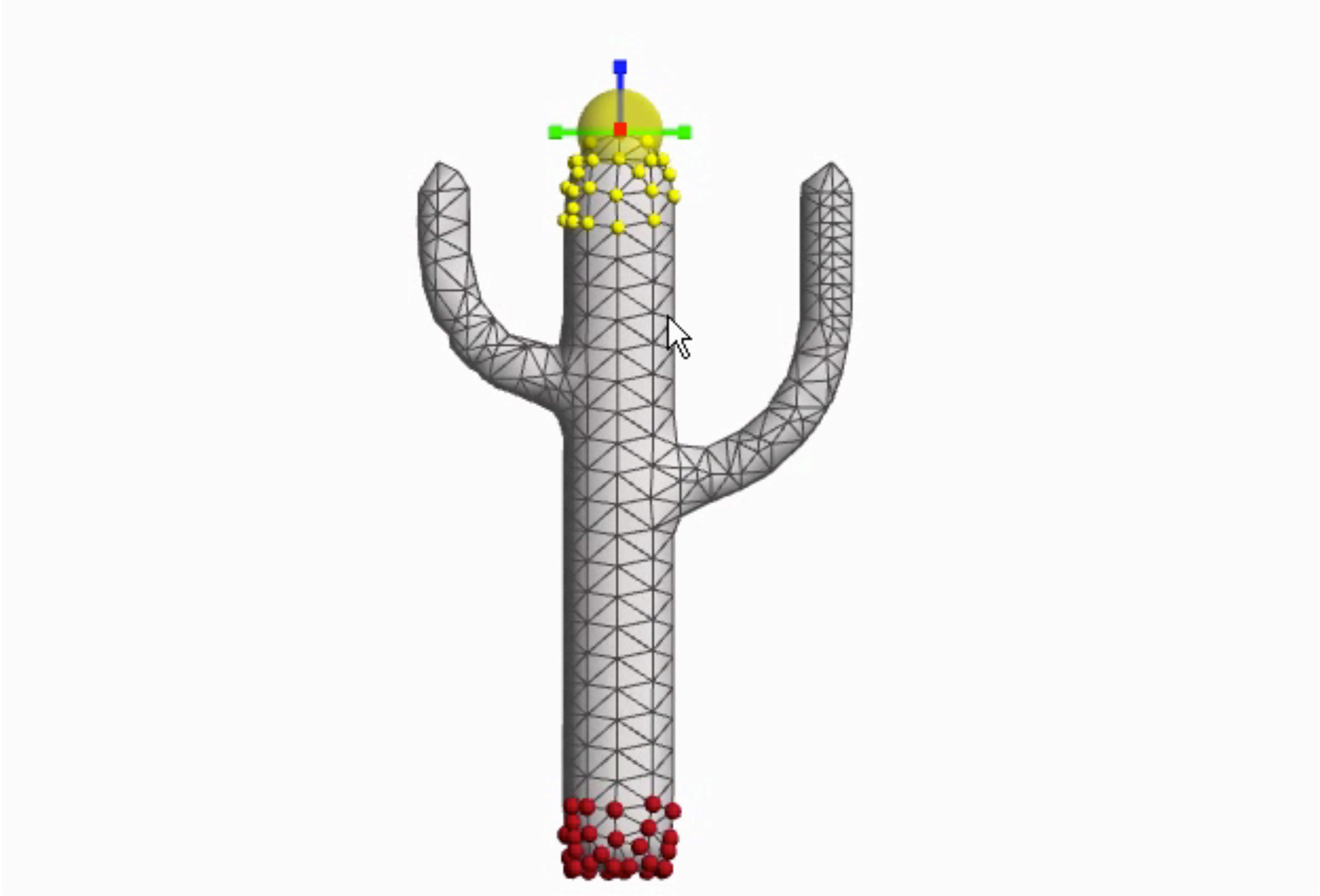
Large Rotations

- Use previous frame as the initial guess



Examples





Discussion

- Nonlinear deformation that models a elastic shell
- Simple to implement, no parameters to tune except number of iterations
- Each step is guaranteed to not increase the energy
- Each iteration is relatively cheap, no matrix re-factorization necessary

Discussion

- Works fine on small meshes
- On larger meshes: slow convergence
  - Each iteration is more expensive
  - Need more iterations because the conditioning of the system becomes worse as the matrix grows
- Material stiffness depends on the cell size
  - lots of wrinkles for fine meshes when using 1-rings as cells

Acceleration using Subspace Techniques

- Subspace created by influence weight functions for each handle
- Drastically reduces the number of degrees of freedom in the optimization

Alec Jacobson, Ilya Baran, Ladislav Kavan, Jovan Popović, and Olga Sorkine. "Fast Automatic Skinning Transformations," 2012.

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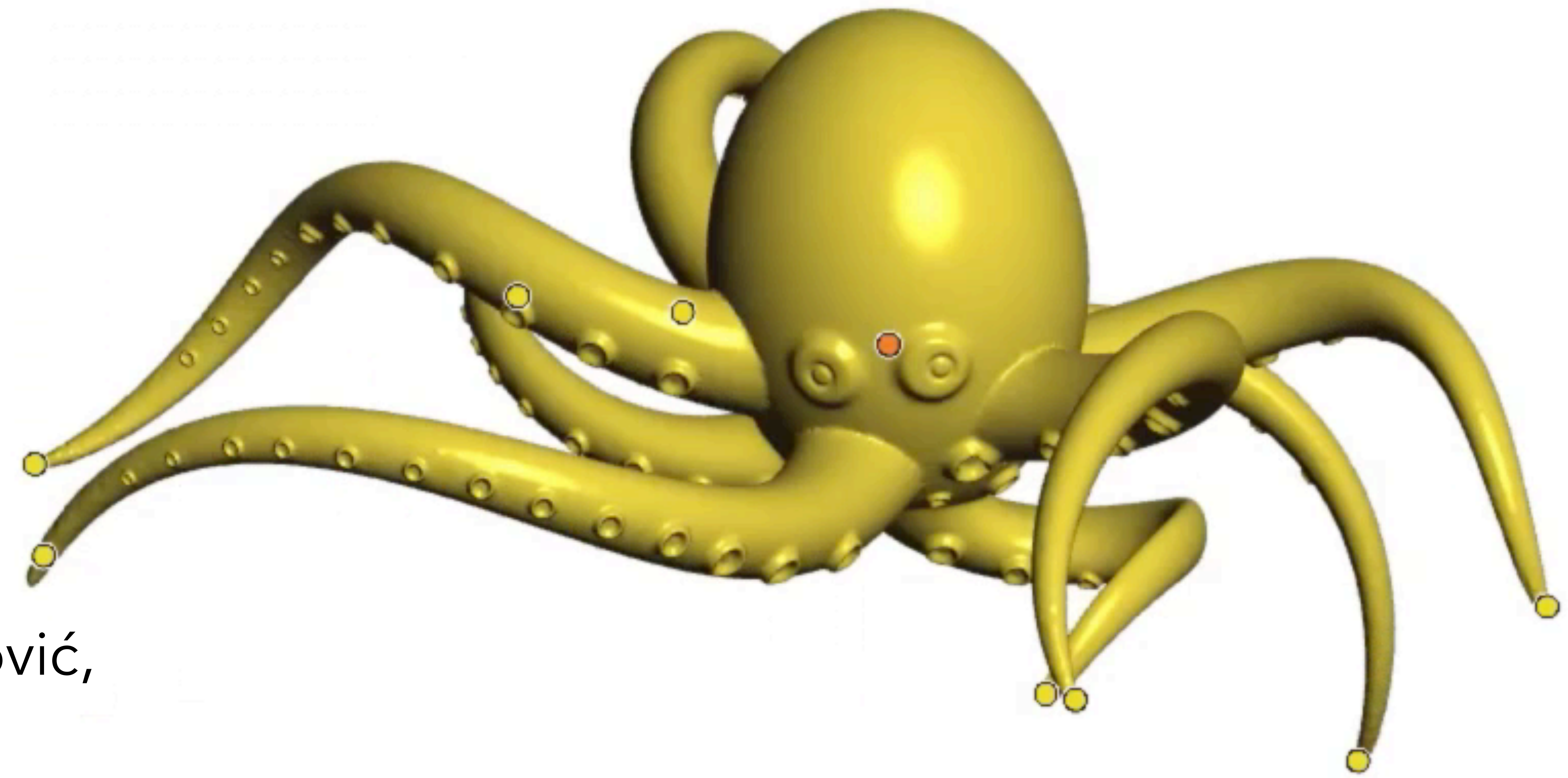

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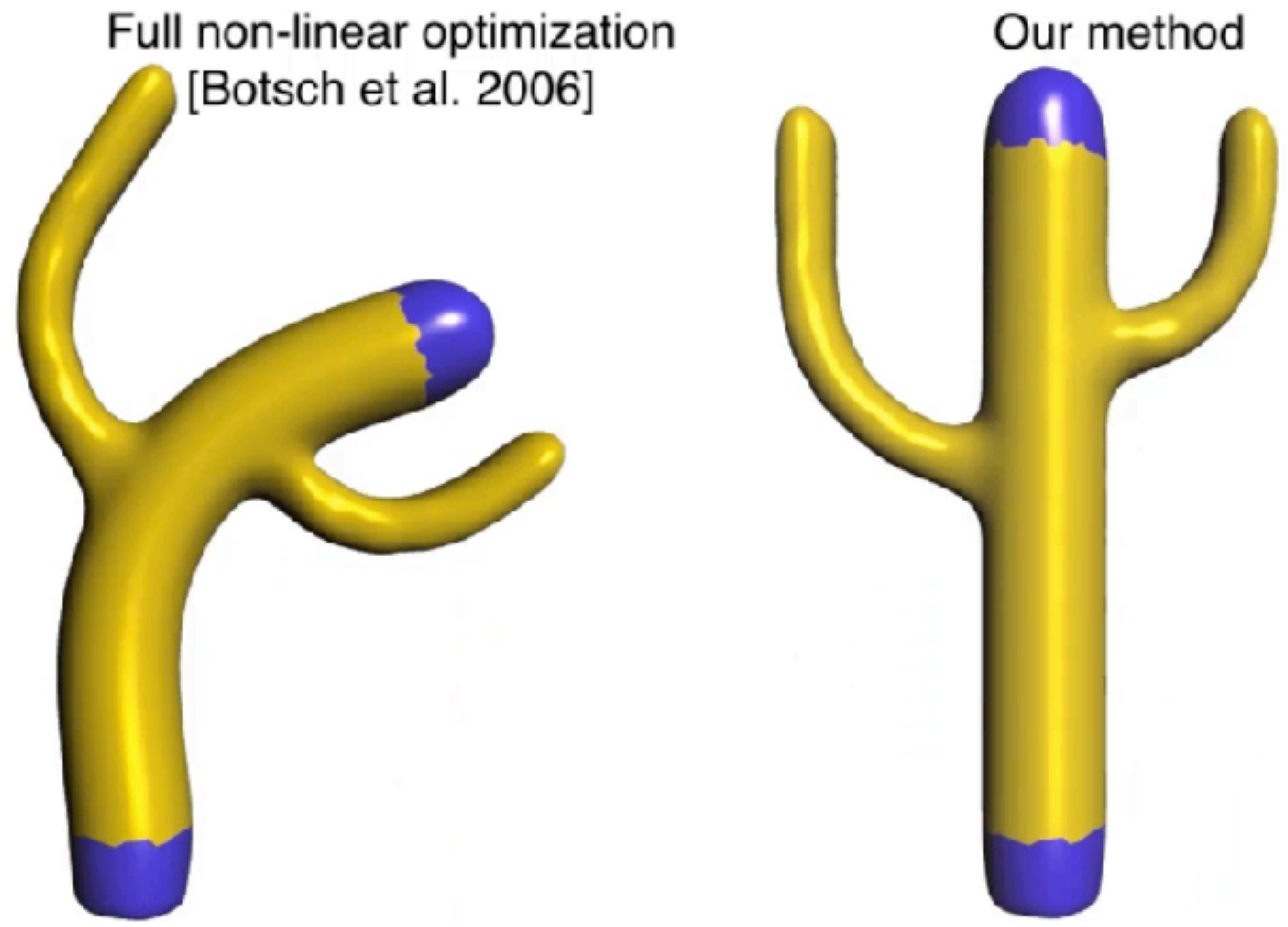

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Summary on Surface-Based Deformation

Summary on Surface-Based Deformation

- Find a mesh that optimizes some objective functional and satisfies modeling constraints

$$\mathbf{x}_{\text{def}} = \underset{\mathbf{x}'}{\operatorname{argmin}}\, E(\mathbf{x}') \quad s.t. \quad \mathbf{x}'_i = \mathbf{c}_i$$

Linear Methods

- Triangle gradient methods (2004-2005)

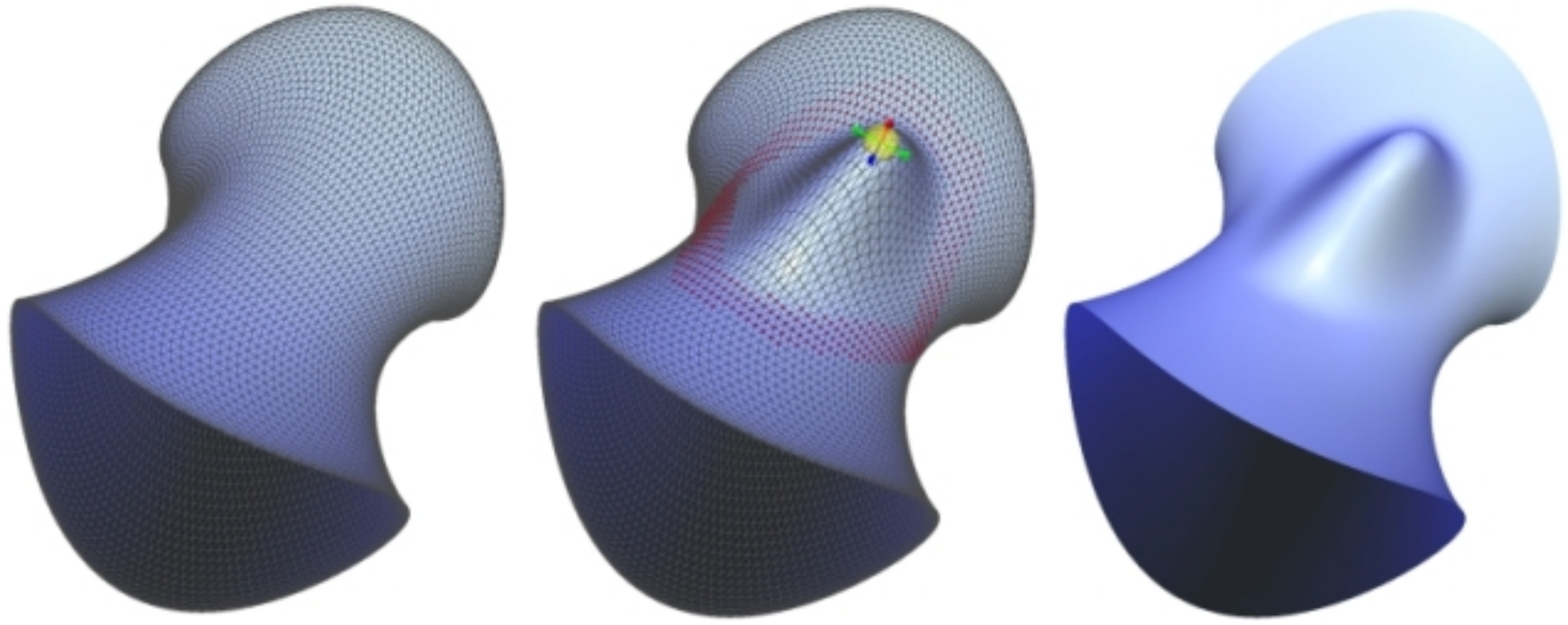
  http://dl.acm.org/citation.cfm?id=1015774

  http://www-ui.is.s.u-tokyo.ac.jp/~takeo/research/rigid/

- Laplacian surface editing (2004-2005)

  http://dl.acm.org/citation.cfm?id=1015772

  http://igl.ethz.ch/projects/Laplacian-mesh-processing/Laplacian-mesh-editing/



Nonlinear Methods

- As Rigid as Possible surface modeling

  http://igl.ethz.ch/projects/ARAP/

  http://dl.acm.org/citation.cfm?id=1778775

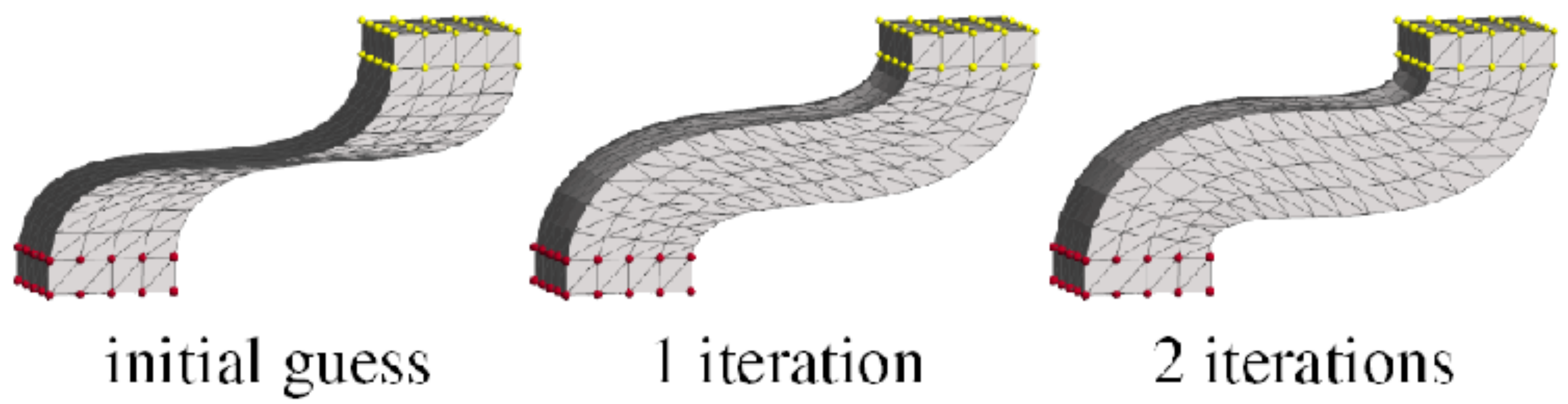
  http://igl.ethz.ch/projects/fast/

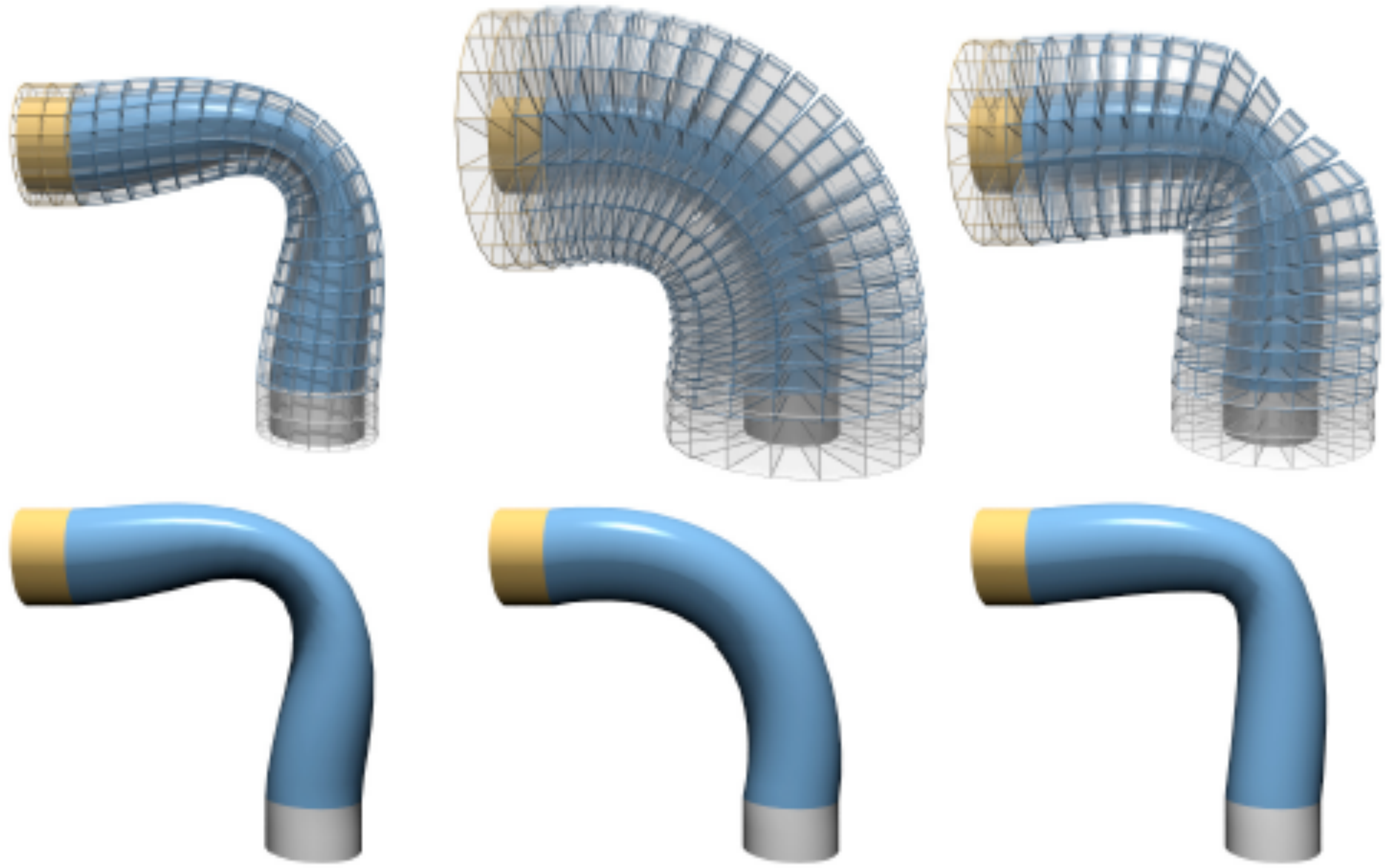
- PriMo

  http://dl.acm.org/citation.cfm?id=1281959

- Mesh Puppetry

  http://dl.acm.org/citation.cfm?id=1275808.1276479





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Surface-based Deformations

- Objective functional expressed in the mesh elements (vertices)
- Complexity depends on the mesh resolution
- Linear methods:
  - Solve a global linear system on the mesh
  - Usually suffer from some artifacts
- Nonlinear methods
  - Fewer artifacts but slower, and harder to implement

References

- Polygon Mesh Processing, Chapter 9