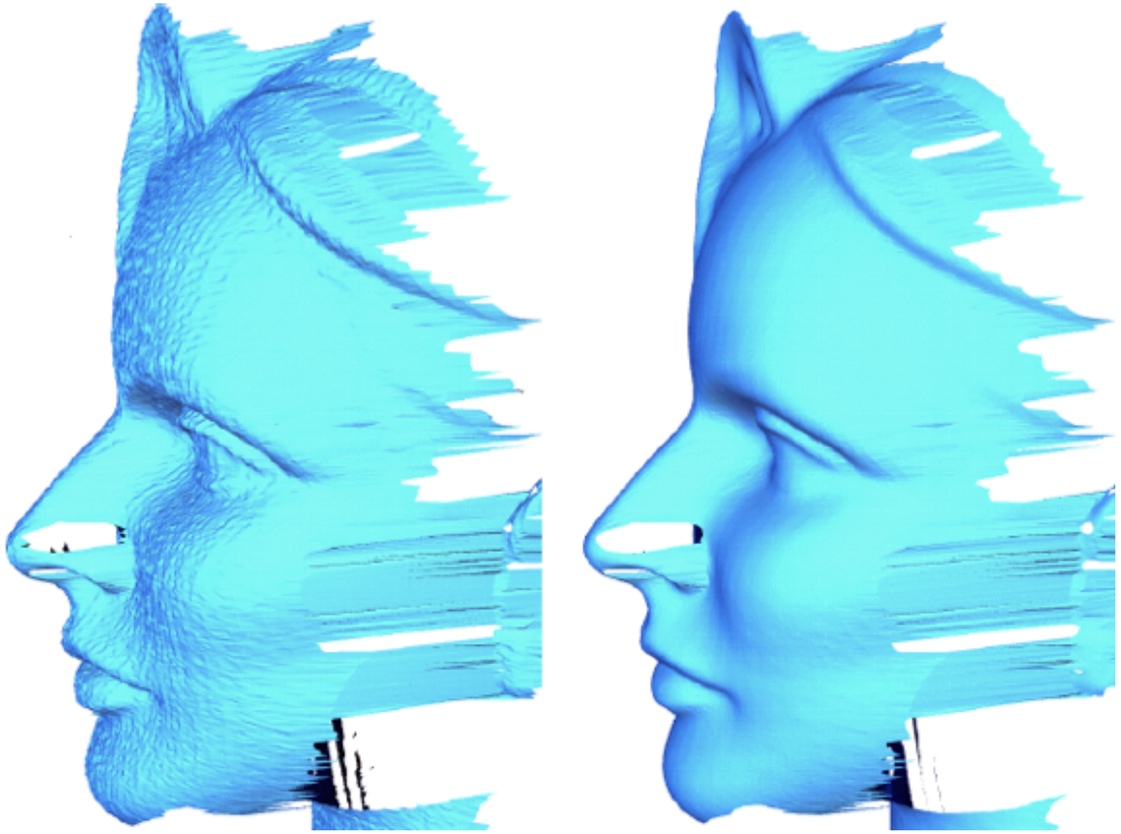
Smoothing

Acknowledgements: Olga Sorkine-Hornung and Mario Botsch

Motivation

- Scanned surfaces can be noisy

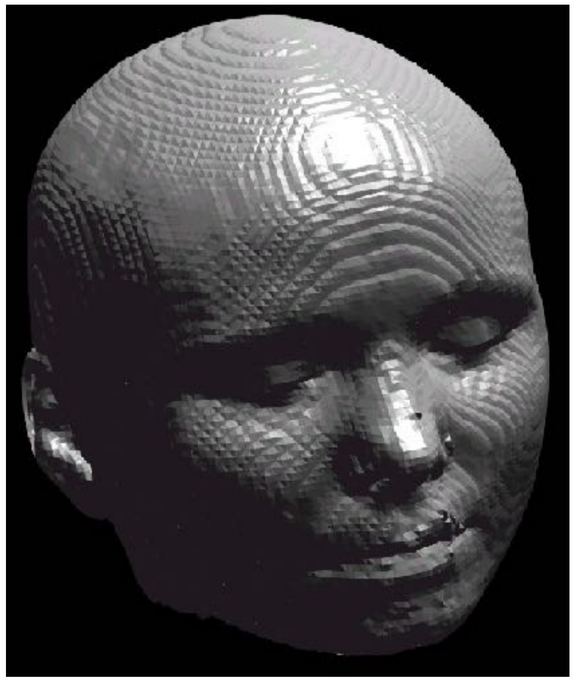


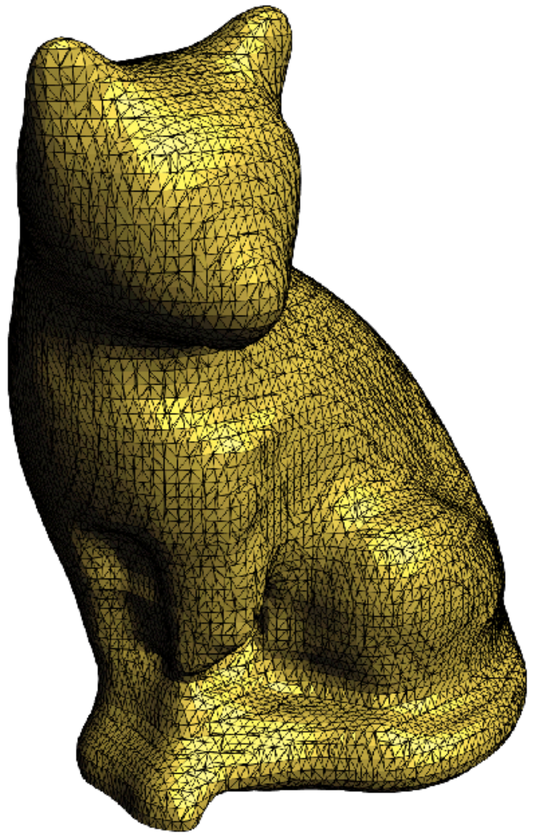
Motivation

- Scanned surfaces can be noisy

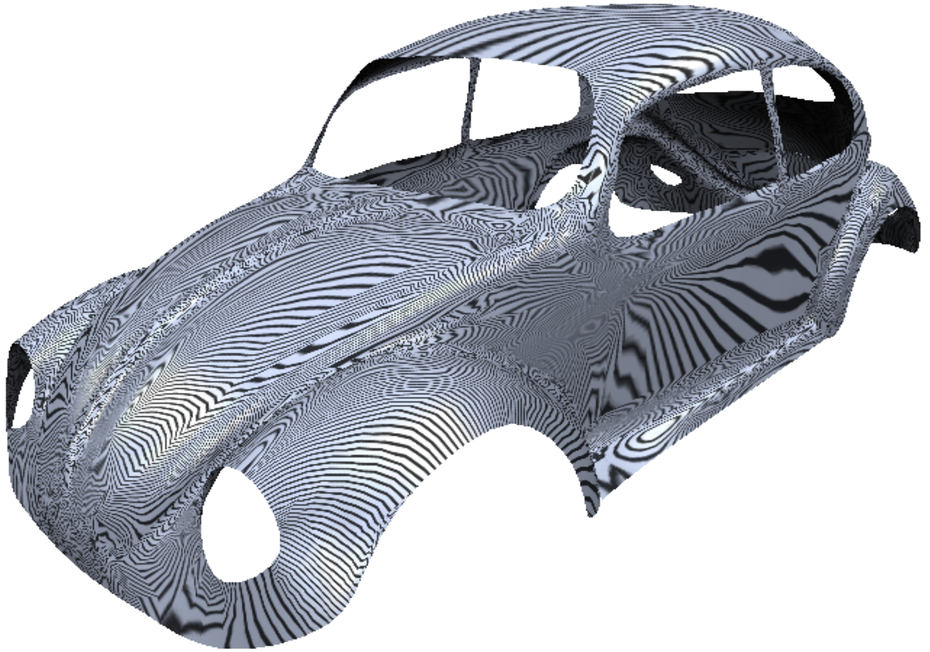
Motivation

- Marching Tetrahedra/Cubes meshes can be problematic



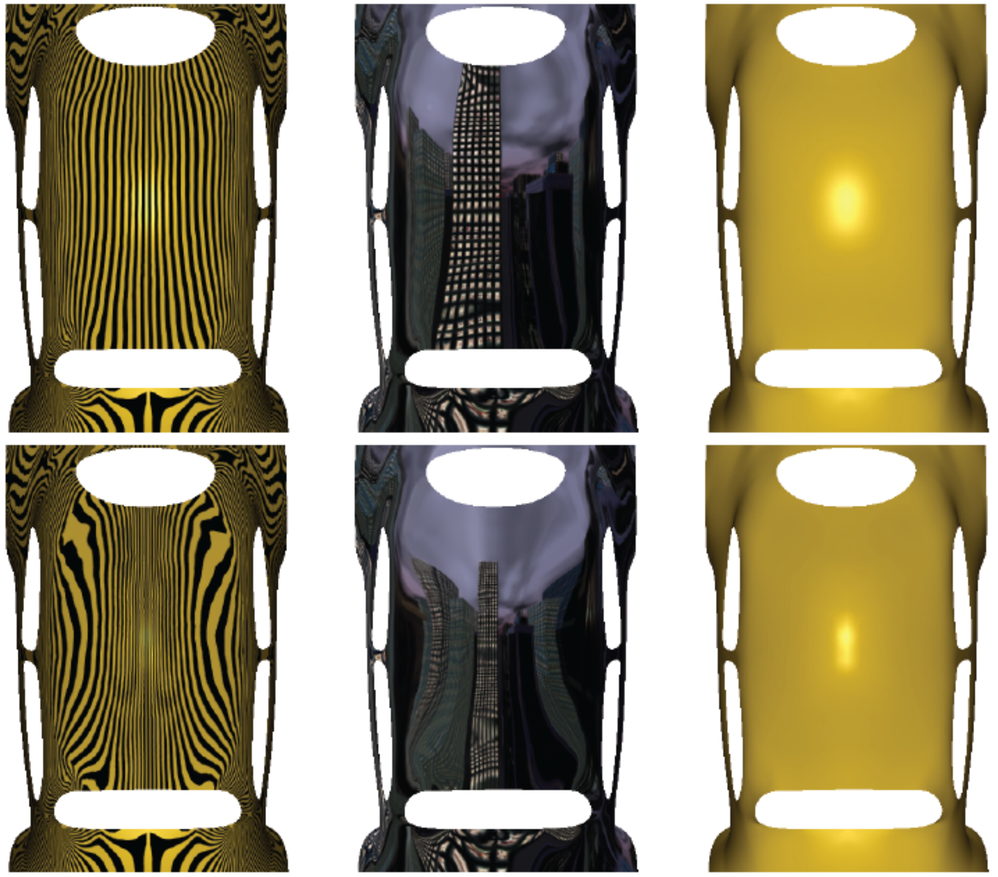


Visual Assessment of Surface Smoothness



Reflection Lines as an Inspection Tool

- **Shape optimization using reflection lines**
  E. Tosun, Y. I. Gingold, J. Reisman, D. Zorin
  Symposium on Geometry Processing 2007



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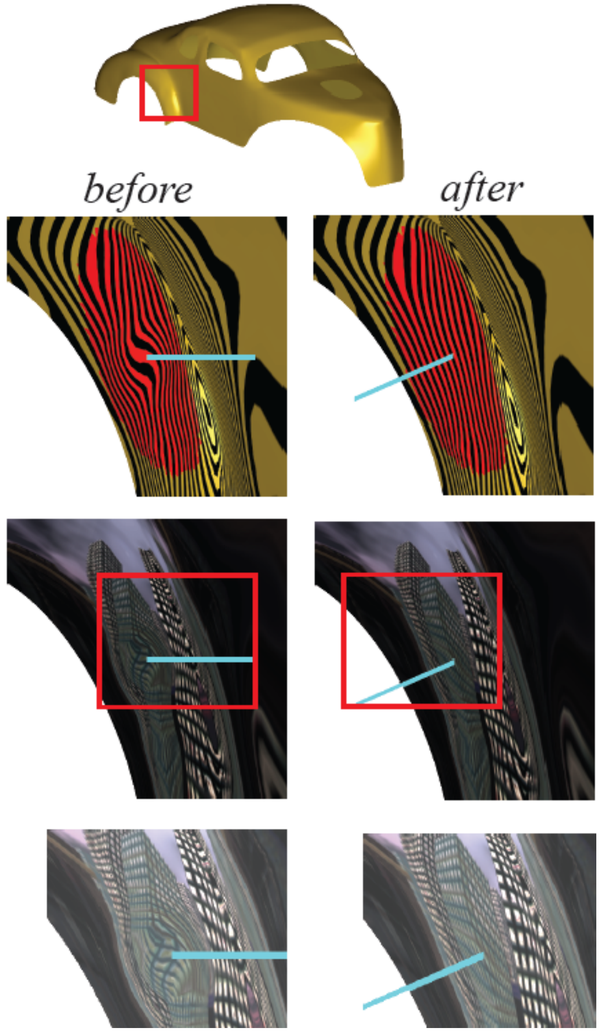
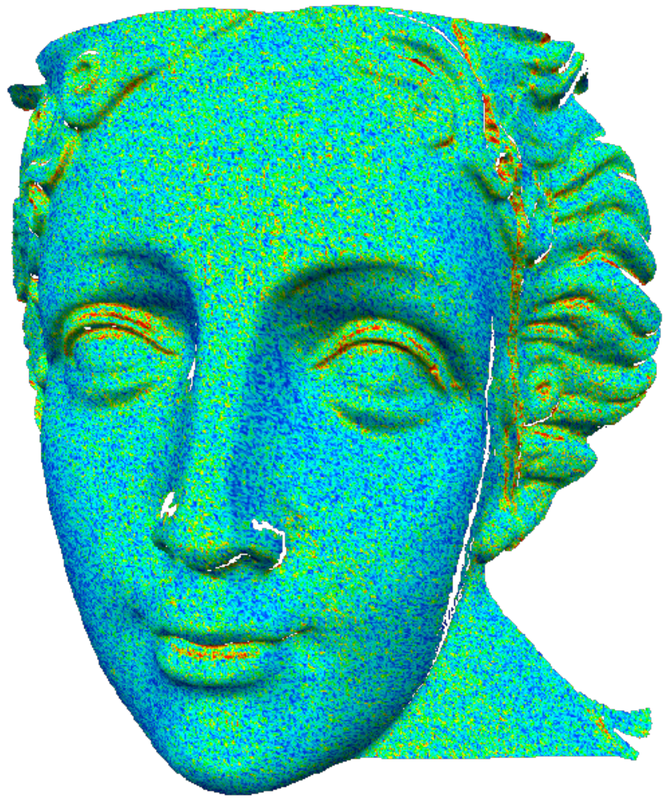


How to measure smoothness?

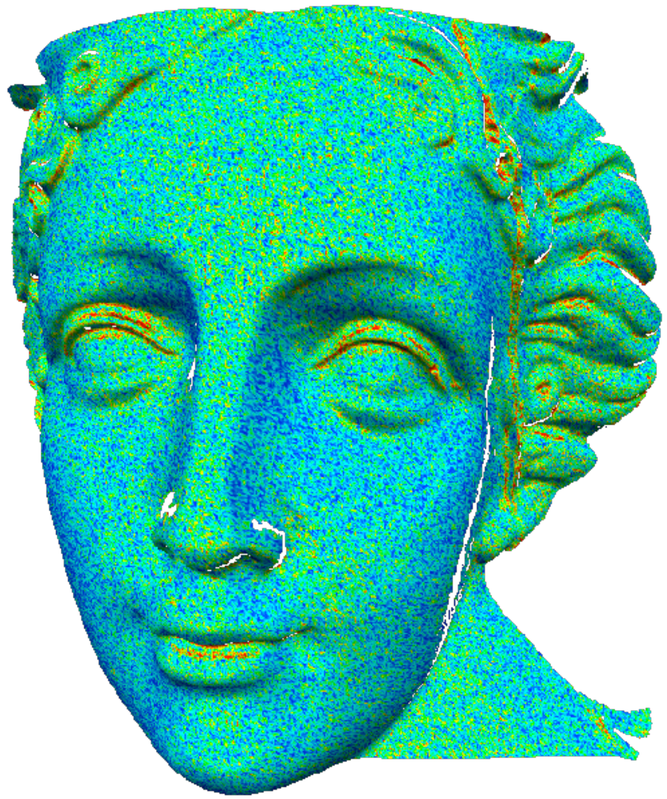
Curvature and Smoothness

Curvature and Smoothness

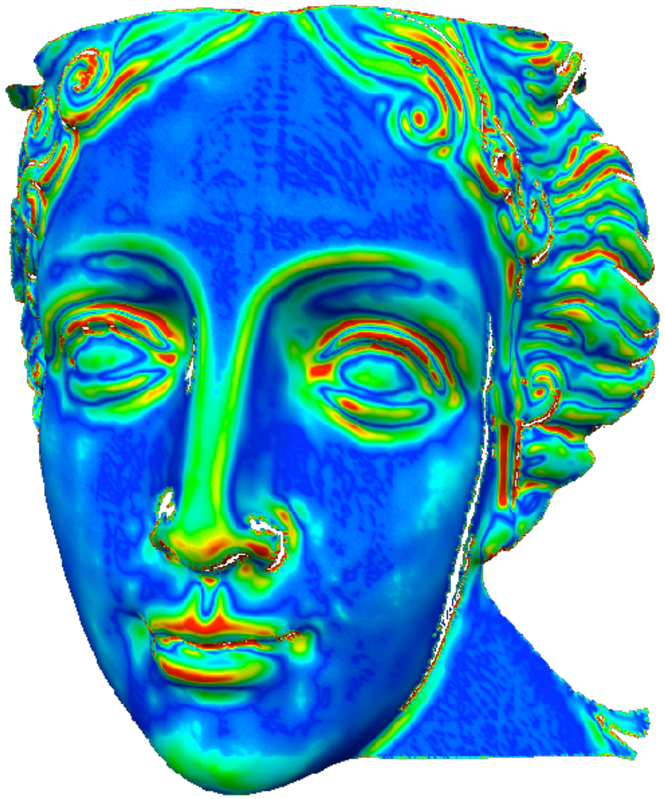


mean curvature plot

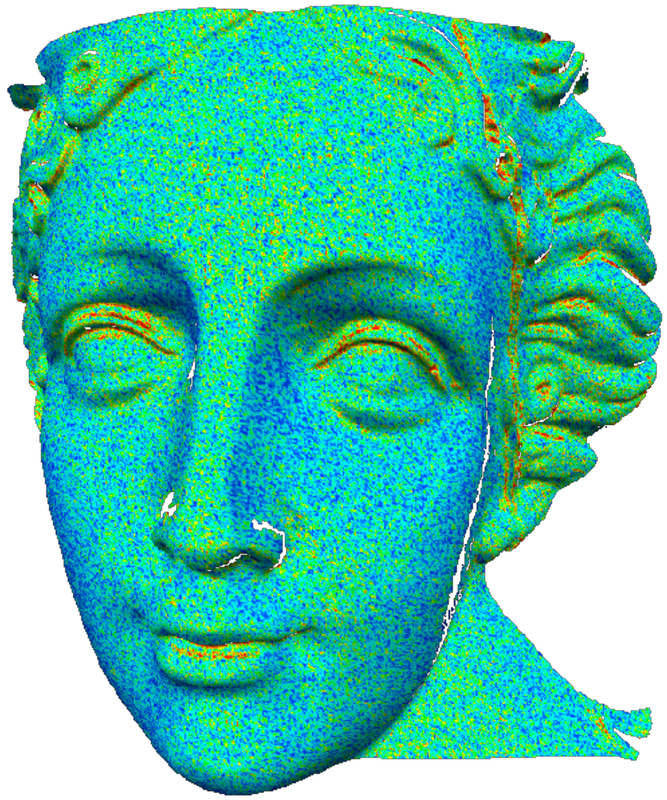
Curvature and Smoothness



mean curvature plot



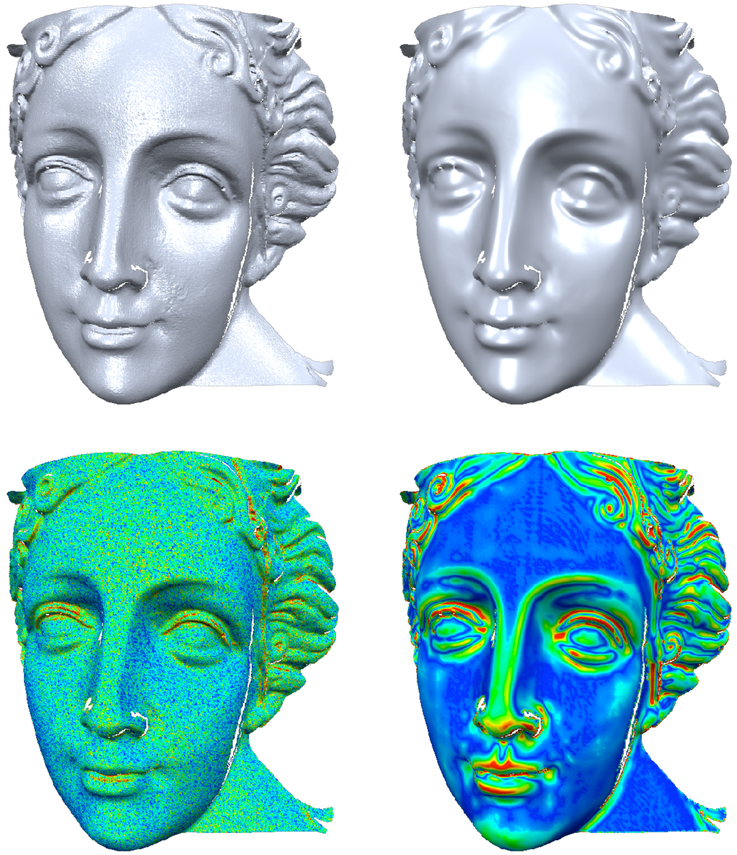
Curvature and Smoothness



mean curvature plot

Curvature and Smoothness

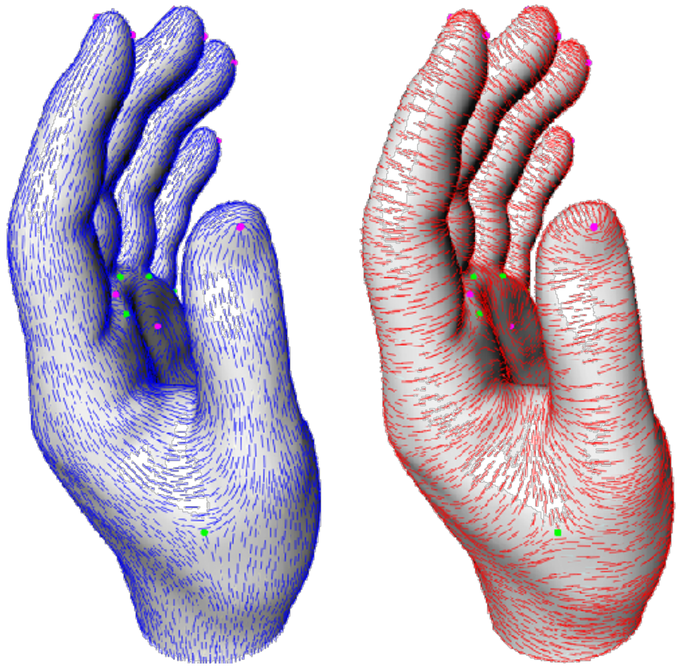
- Smoothing =
  reducing curvature?
- Smoothing =
  make curvature vary less?



Which curvature?

- Principal curvatures
  - Nonlinear and “discontinuous” operator in the definition (min, max)

$\kappa_{\min}, \kappa_{\max}$



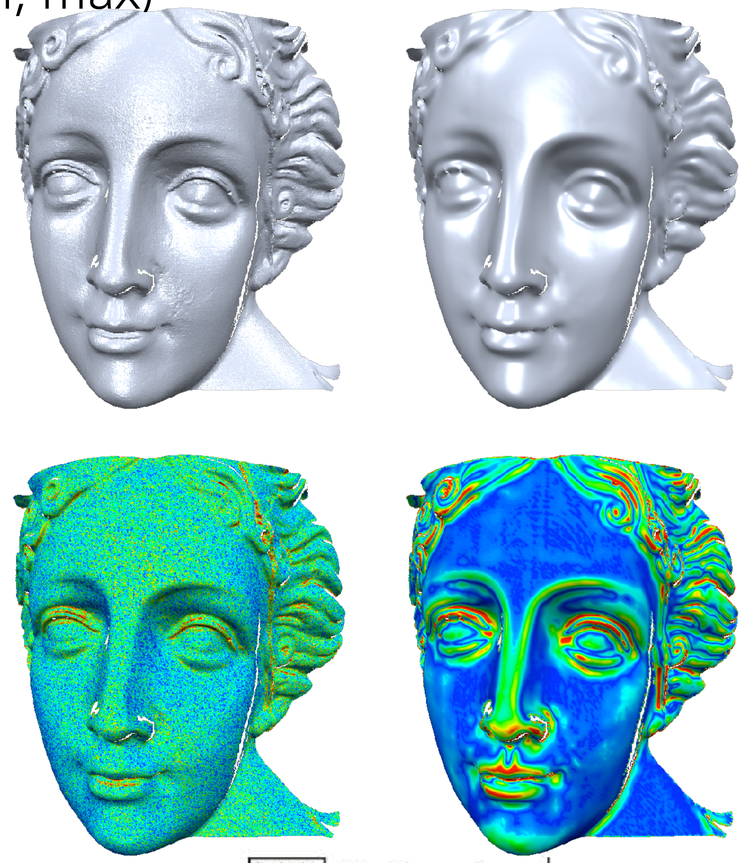
principal directions

Which curvature?

- Principal curvatures
  - Nonlinear and “discontinuous” operator in the definition (min, max)
- Gauss curvature K
  - Intrinsic-only, insensitive to embedding in

Which curvature?

- Principal curvatures
  - Nonlinear and “discontinuous” operator in the definition (min, max)
- Gauss curvature K
  - Intrinsic-only, insensitive to embedding in
- Mean curvature H
  - Relatively simple to extract on meshes via Laplace-Beltrami:



$$\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}$$

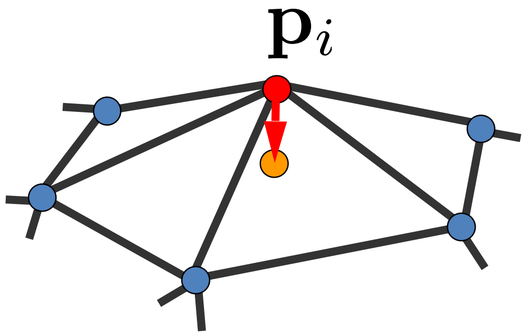
goal: $H = 0$ or $H = \text{const}$

Laplace as a Linear Operator

Recap: Laplace-Beltrami

$$\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}$$

- High-pass filter: extracts local surface detail
  - Detail = *smooth*(surface) – surface
  - Assumption: smoothing = averaging



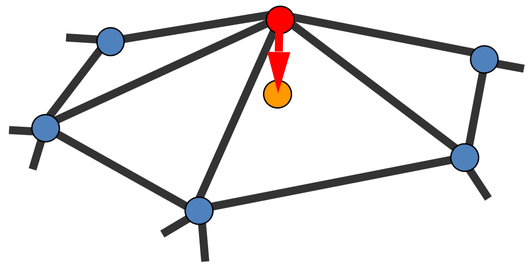
$$\Delta_{\mathcal{M}}(\mathbf{p}_i) = \delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{p}_j - \mathbf{p}_i)$$

Recap: Laplace-Beltrami

$$\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}$$

- The direction of $\boldsymbol{\delta}_i$ approximates the normal
- The size approximates the mean curvature

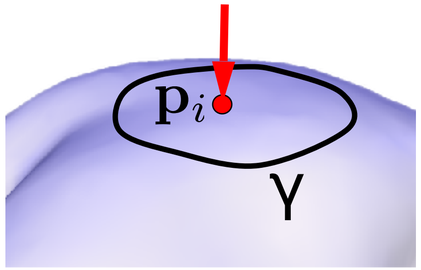
$$\mathbf{p}_i$$



$$\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{p}_j - \mathbf{p}_i)$$

L-B: Weighting Schemes

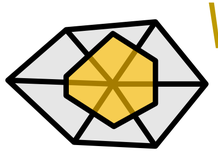
- Ignore geometry $\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{p}_j - \mathbf{p}_i)$



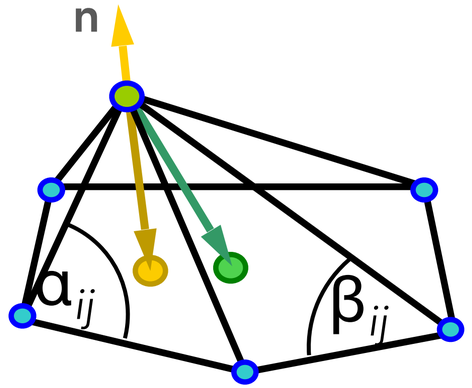
- Integrate over the Voronoi region of the vertex

δ_{uniform} : $W_i = 1$, $w_{ij} = 1/|N(i)|$

δ_{cotan} : $w_{ij} = 0.5(\cot \alpha_{ij} + \cot \beta_{ij})$



$W_i = A_i$



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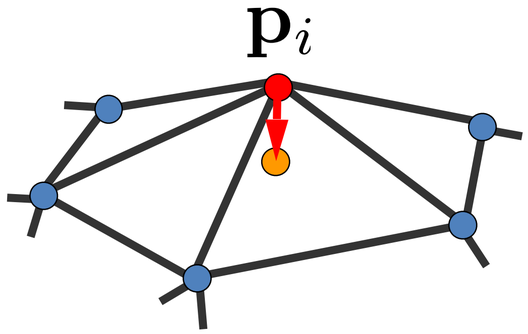
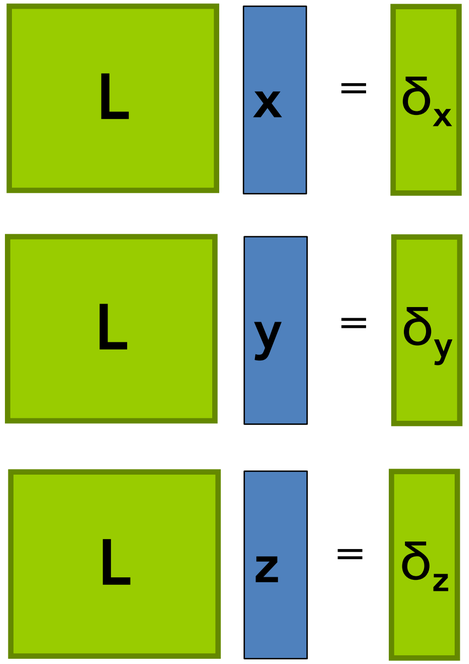
Laplacian Matrix

- The transition between *xyz* and **δ** is linear:



$$\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{p}_j - \mathbf{p}_i)$$



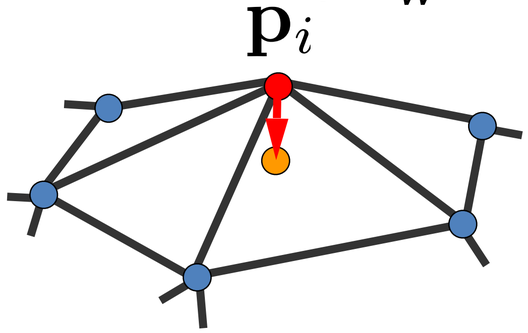
$$\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$$

$$\mathbf{L} \in \mathbb{R}^{n \times n}$$

$$\delta_\mathbf{x}, \delta_\mathbf{y}, \delta_\mathbf{z} \in \mathbb{R}^n$$

Laplacian Matrix

- Breaking down the Laplace matrix:
- **M** = mass matrix; $\mathbf{L}_w$ = stiffness matrix



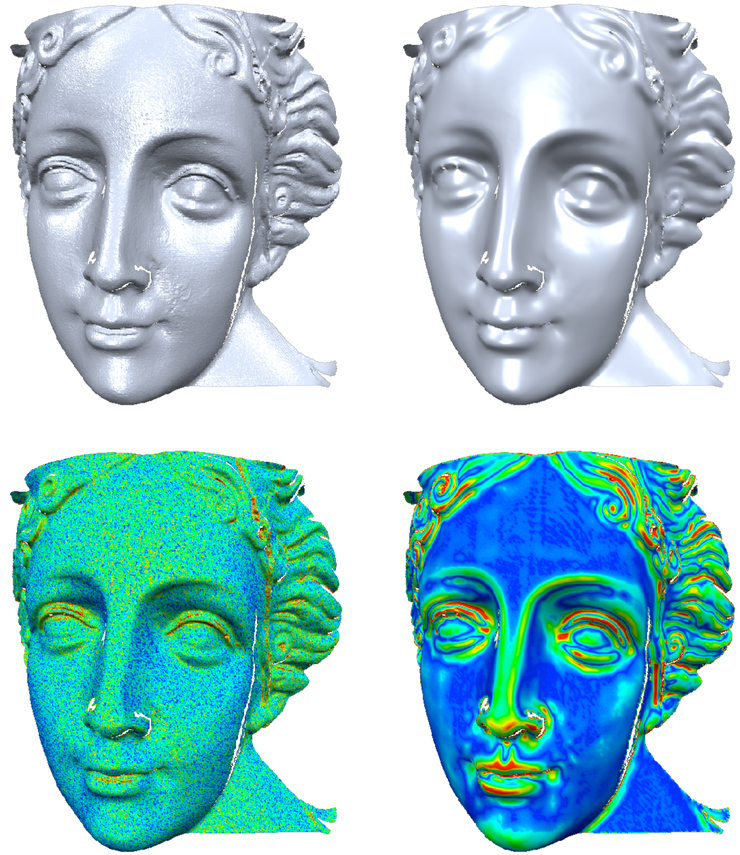
$$\delta_i = \frac{1}{W_i} \sum_{j \in \mathcal{N}(i)} w_{ij}(\mathbf{p}_j - \mathbf{p}_i)$$

How to do the smoothing?

$$\Delta_{\mathcal{M}}\mathbf{p} = -2H\mathbf{n}$$

goal: $H = 0$ or $H = \text{const}$

- Smooth *H*, obtain $\tilde{H}$
- Find a surface that has $\tilde{H}$ as mean curvature
  - *H* doesn't define the surface
  - **n** nonlinear in **p**

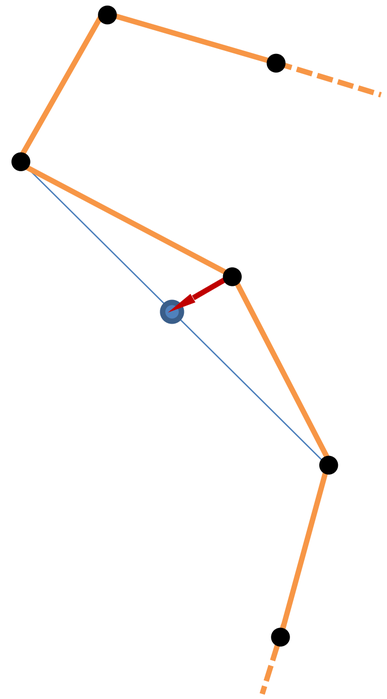


Method 1:
Smoothing by Flowing

Example – smoothing curves

- Laplace in 1D = second derivative:

$$L(\mathbf{p}_i) = \frac{1}{2}(\mathbf{p}_{i-1} - \mathbf{p}_i) + \frac{1}{2}(\mathbf{p}_{i+1} - \mathbf{p}_i)$$



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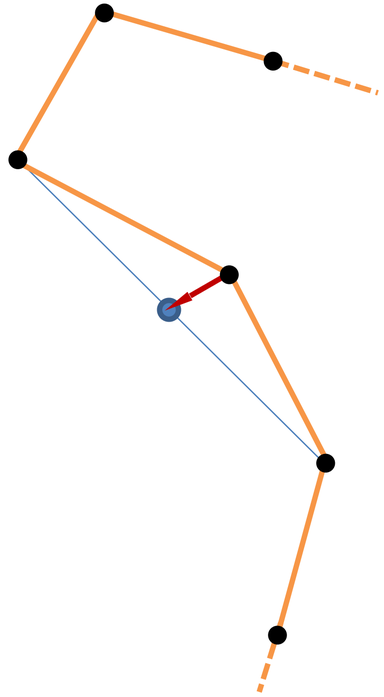

- In matrix-vector form for the whole curve

$$L\mathbf{p}$$

$$\mathbf{p} = [\mathbf{x}\ \mathbf{y}] \in \mathbb{R}^{n \times 2}$$

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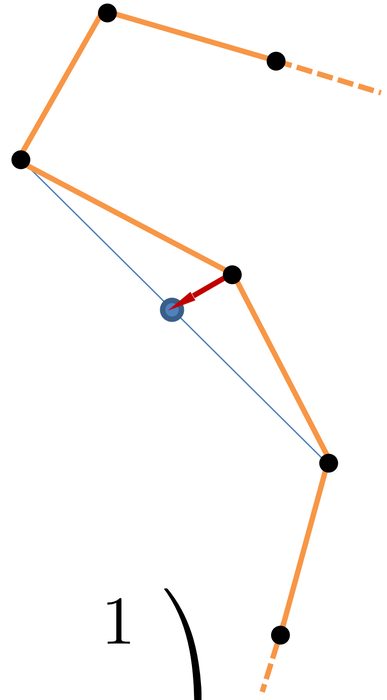


- In matrix-vector form for the whole curve

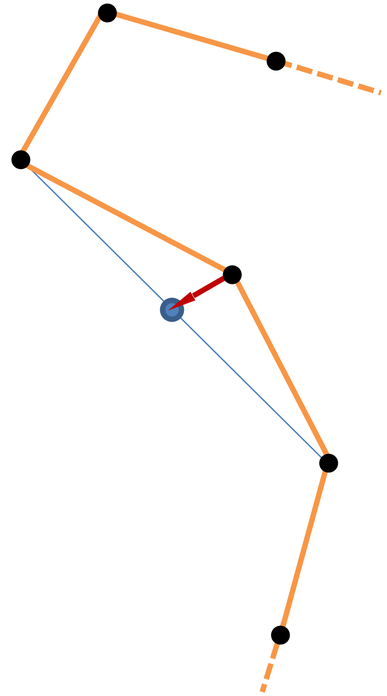
$$L\mathbf{p}$$

$$\mathbf{p} = [\mathbf{x}\ \mathbf{y}] \in \mathbb{R}^{n\times 2} \qquad L = \frac{1}{2}\begin{pmatrix} -2 & 1 & & & 1 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 1 & & & 1 & -2 \end{pmatrix}$$

Example – smoothing curves

- Flow to reduce curvature:

$$\tilde{\mathbf{p}}_i = \mathbf{p}_i + \lambda \frac{d^2}{ds^2}(\mathbf{p}_i)$$



- Scale factor $0 < \lambda < 1$
- Matrix-vector form:

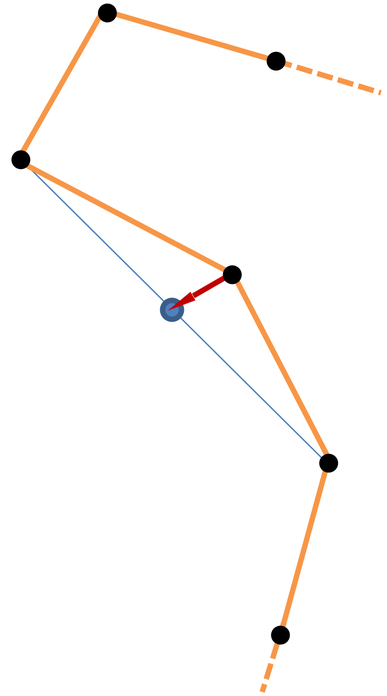
$$\tilde{\mathbf{p}} = \mathbf{p} + \lambda L \mathbf{p}, \quad \mathbf{p} \in \mathbb{R}^{n \times 2}$$

- Drawbacks?

Example – smoothing curves

- Flow to reduce curvature:

$$\tilde{\mathbf{p}}_i = \mathbf{p}_i + \lambda \frac{d^2}{ds^2}(\mathbf{p}_i)$$

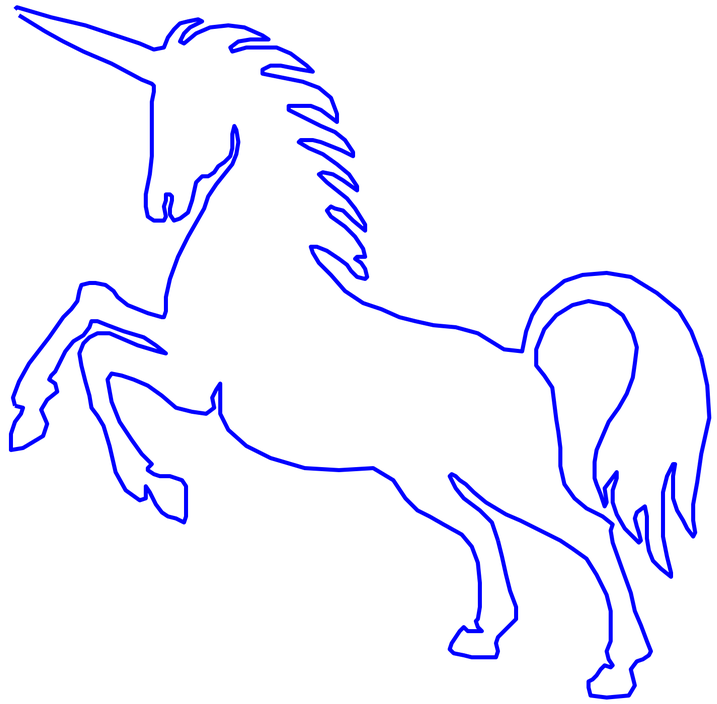


- Scale factor $0 < \lambda < 1$
- Matrix-vector form:

$$\tilde{\mathbf{p}} = \mathbf{p} + \lambda L \mathbf{p}, \quad \mathbf{p} \in \mathbb{R}^{n \times 2}$$

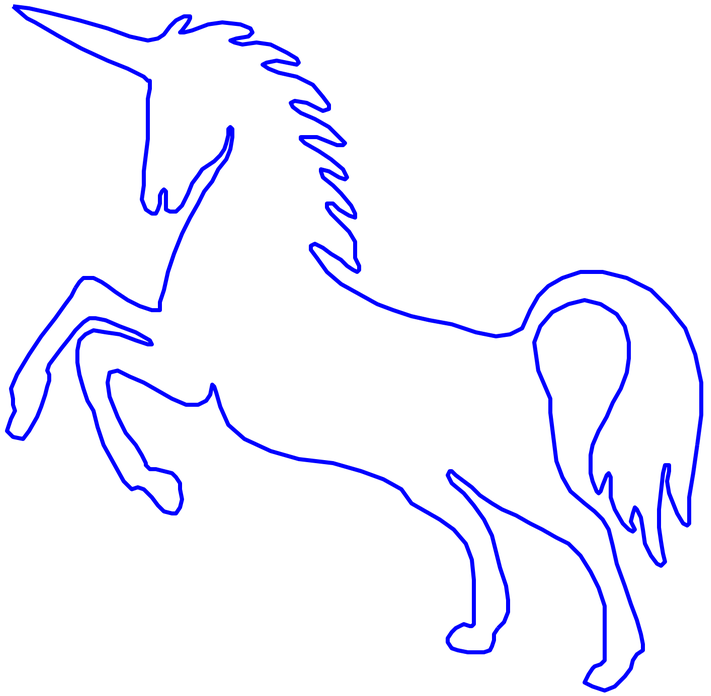
- May shrink the shape; can be slow

Filtering Curves



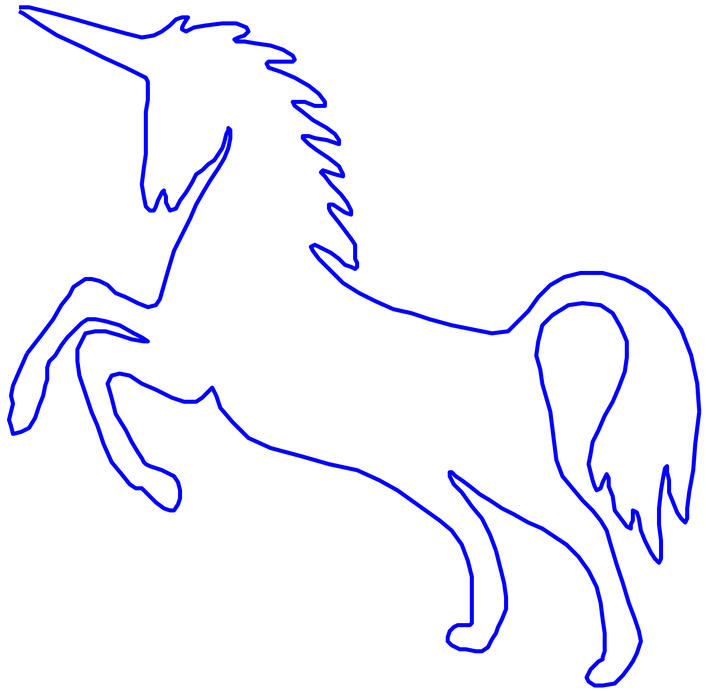
Original curve

Filtering Curves



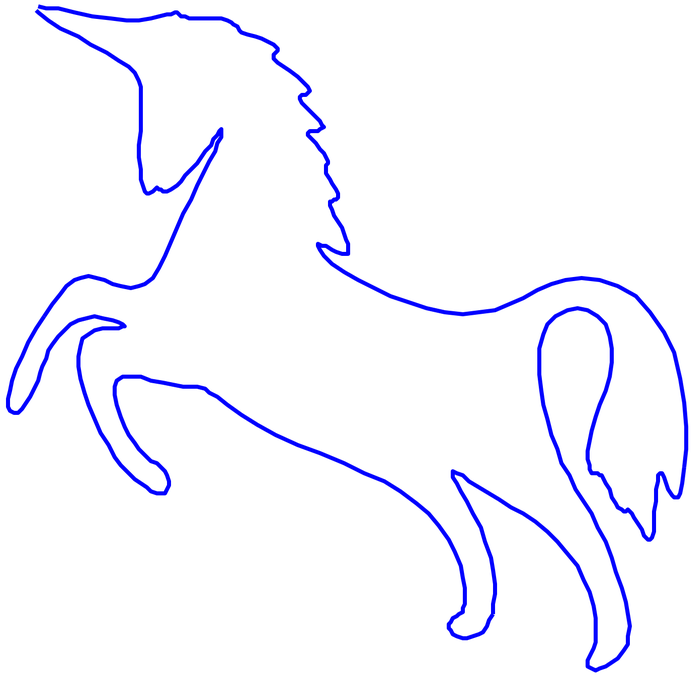
1st iteration; λ=0.5

Filtering Curves



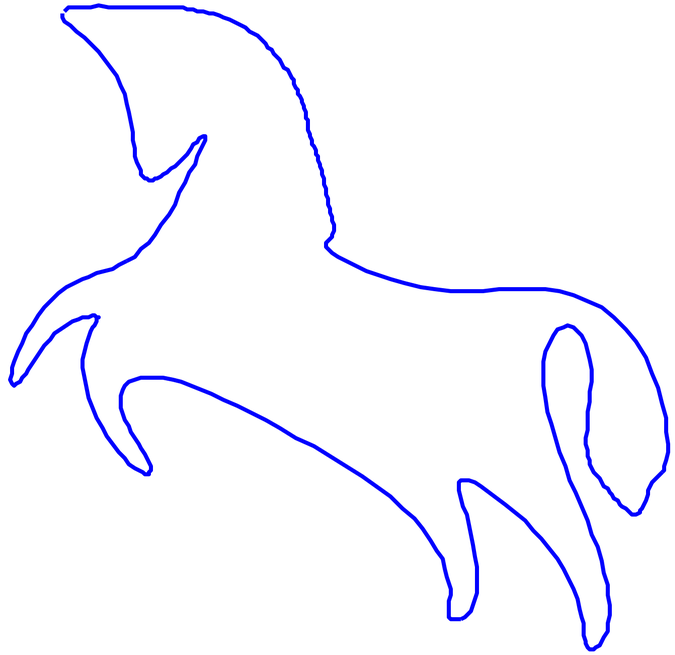
2nd iteration; λ=0.5

Filtering Curves



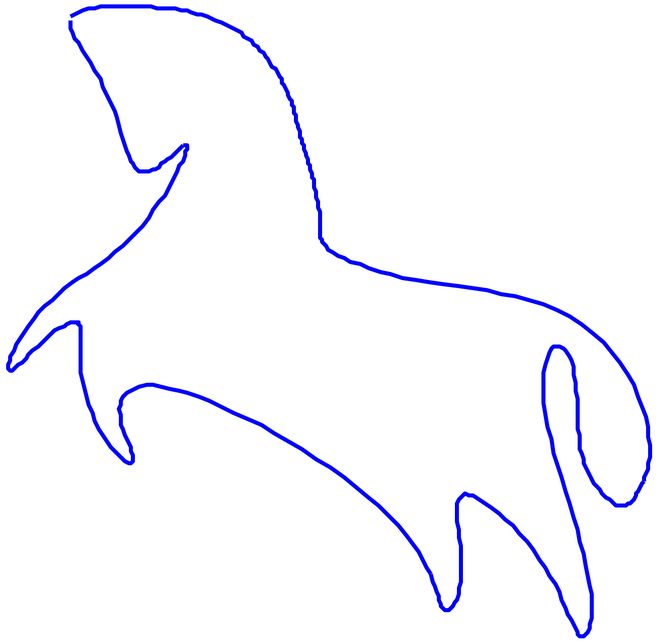
8th iteration; λ=0.5

Filtering Curves



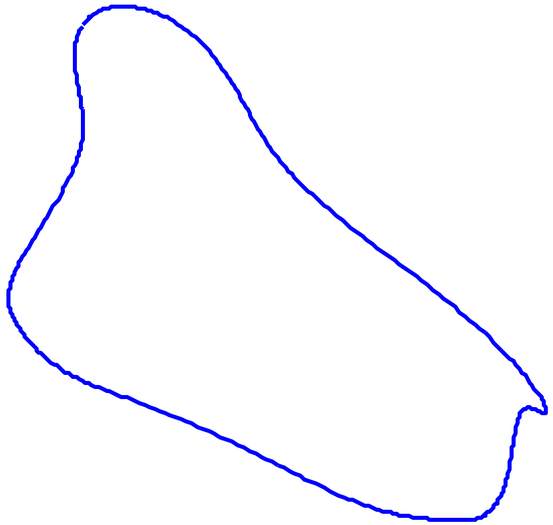
27th iteration; λ=0.5

Filtering Curves



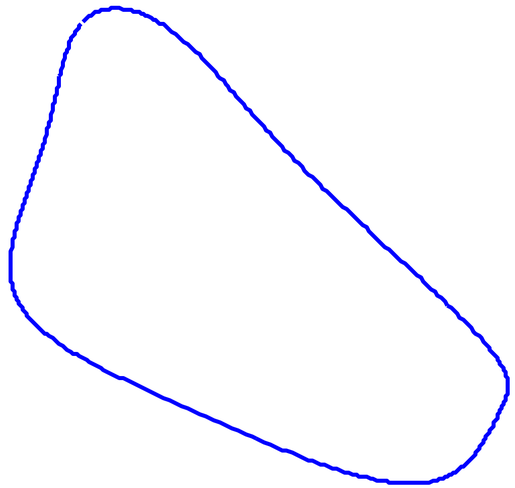
50th iteration; λ=0.5

Filtering Curves



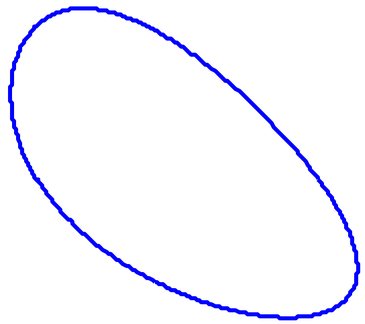
500th iteration; λ=0.5

Filtering Curves



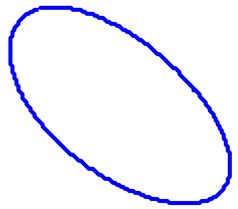
1000th iteration; λ=0.5

Filtering Curves



5000th iteration; λ=0.5

Filtering Curves



10000th iteration; λ=0.5

Filtering Curves

50000th iteration; λ=0.5

Smoothing as Mean Curvature Flow

- Model smoothing as a diffusion process

$$\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta \mathbf{p} = -2\lambda H \mathbf{n}$$

Smoothing as Mean Curvature Flow

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$$\boxed{\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta \mathbf{p}} = -2\lambda H \mathbf{n}$$

- Discretize in time, forward differences:

$$\frac{\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)}}{dt} = \lambda L \mathbf{p}^{(n)}$$

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$$\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)} = dt\, \lambda L \mathbf{p}^{(n)}$$

$$\boxed{\mathbf{p}^{(n+1)} = (I + dt\, \lambda L)\mathbf{p}^{(n)}}$$

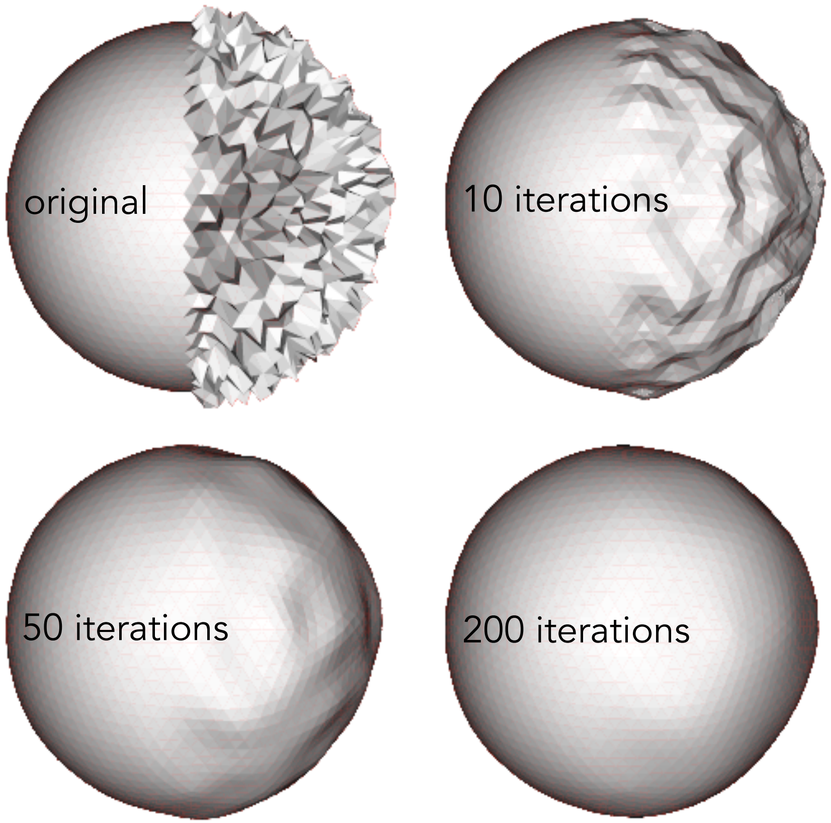
Explicit integration! Unstable unless time step *dt* is small

Taubin Smoothing: Explicit Steps

- Iterate:

$$\tilde{\mathbf{p}} = \mathbf{p} + \lambda L\mathbf{p} = (I + \lambda L)\mathbf{p}$$

- $\lambda > 0$ to smooth
- $\lambda < 0$ to inflate
- Originally proposed with uniform Laplacian weights



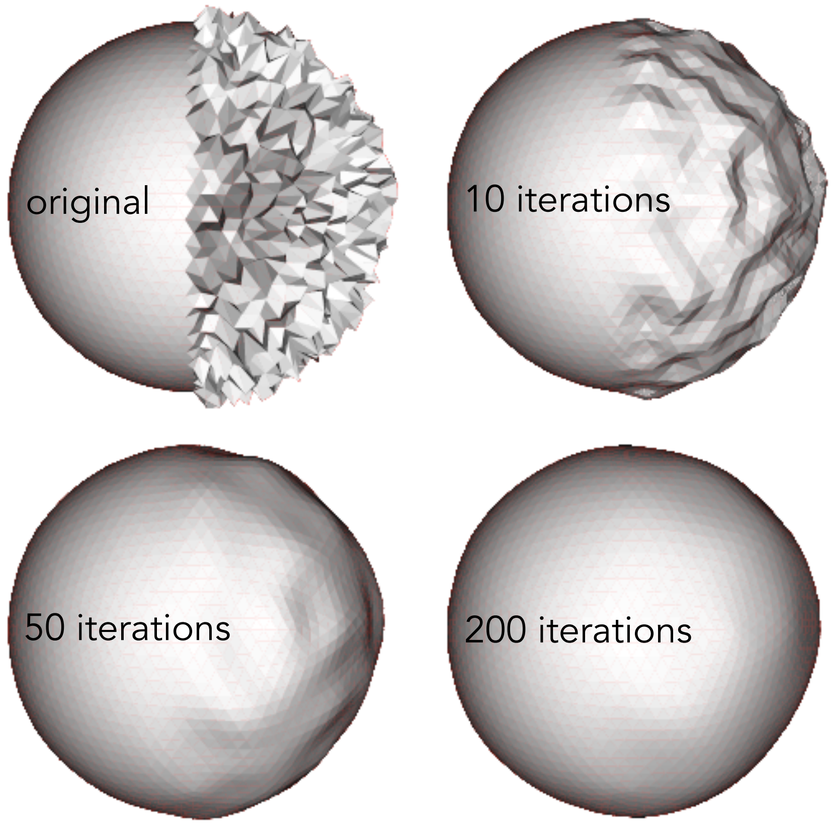
A Signal Processing Approach to Fair Surface Design
Gabriel Taubin, **ACM SIGGRAPH 95**

Taubin Smoothing: Explicit Steps

- Iterate:

$$\tilde{\mathbf{p}} = \mathbf{p} + \lambda L\mathbf{p} = (I + \lambda L)\mathbf{p}$$
$$\tilde{\mathbf{p}} = \mathbf{p} + \mu L\mathbf{p} = (I + \mu L)\mathbf{p}$$

- $\lambda > 0$ to smooth
- $u < 0$ to inflate
- Originally proposed with uniform Laplacian weights



A Signal Processing Approach to Fair Surface Design
Gabriel Taubin, **ACM SIGGRAPH 95**

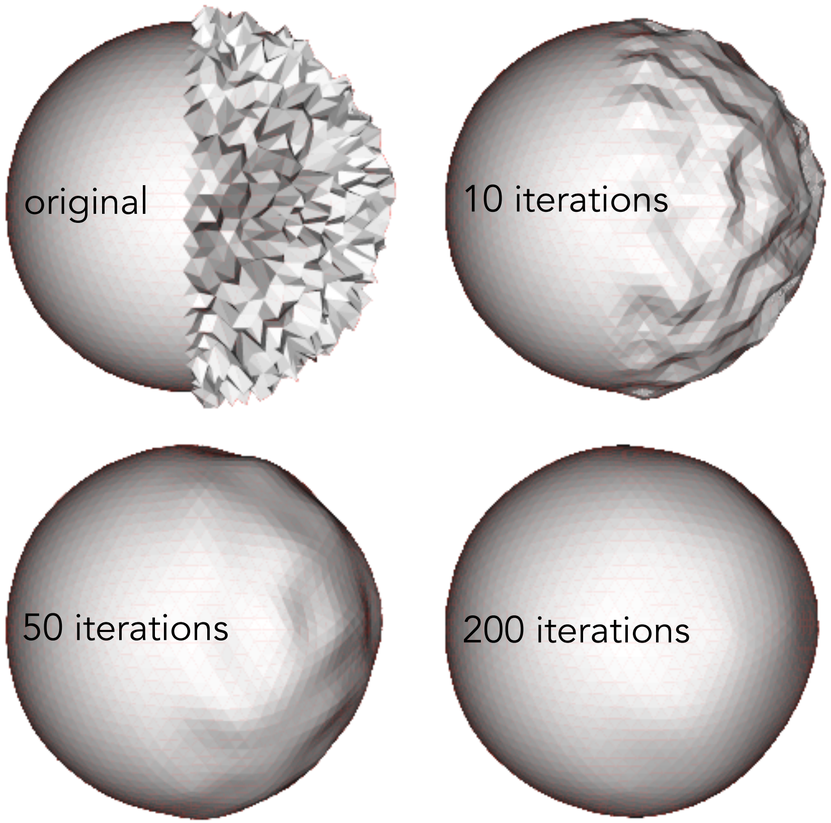
Taubin Smoothing: Explicit Steps

- Per-vertex iterations

$$\tilde{\mathbf{p}}_i = \mathbf{p}_i + \lambda L(\mathbf{p}_i)$$
$$\tilde{\mathbf{p}}_i = \mathbf{p}_i + \mu L(\mathbf{p}_i)$$

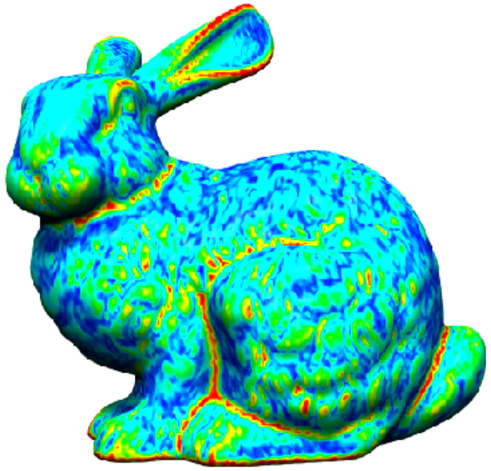
- Simple to implement

- Requires many iterations
- Need to tweak λ, u

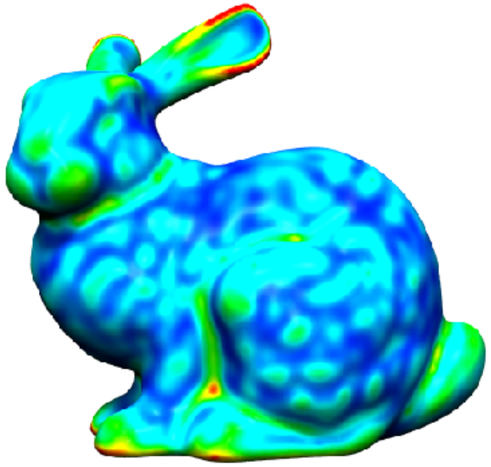


A Signal Processing Approach to Fair Surface Design
Gabriel Taubin, **ACM SIGGRAPH 95**

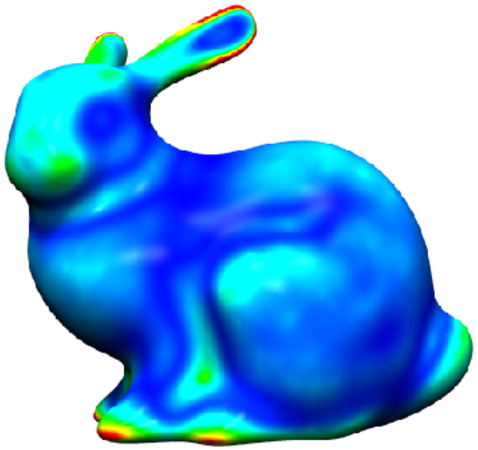
Example



0 iterations



10 iterations



100 iterations

Smoothing as (mean curvature) Flow

- Model smoothing as a diffusion process

$$\frac{\partial \mathbf{p}}{\partial t} = \lambda \Delta \mathbf{p} = -2\lambda H \mathbf{n}$$

- Backward Euler for unconditional stability

$$\frac{\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)}}{dt} = \lambda L \mathbf{p}^{(n+1)}$$

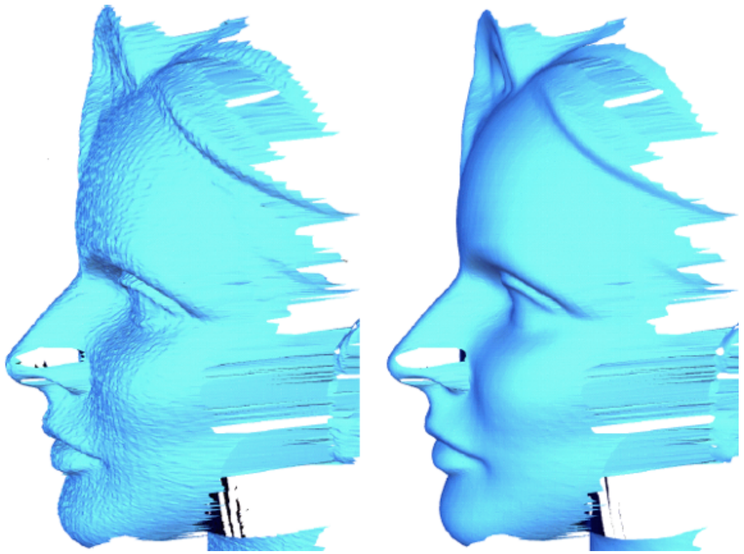
$$\mathbf{p}^{(n+1)} - \mathbf{p}^{(n)} = dt\, \lambda L \mathbf{p}^{(n+1)}$$

$$(I - dt\, \lambda L)\mathbf{p}^{(n+1)} = \mathbf{p}^{(n)}$$

Implicit Fairing: Implicit Euler Steps

- In each iteration, solve for the smoothed $\tilde{\mathbf{p}}$:

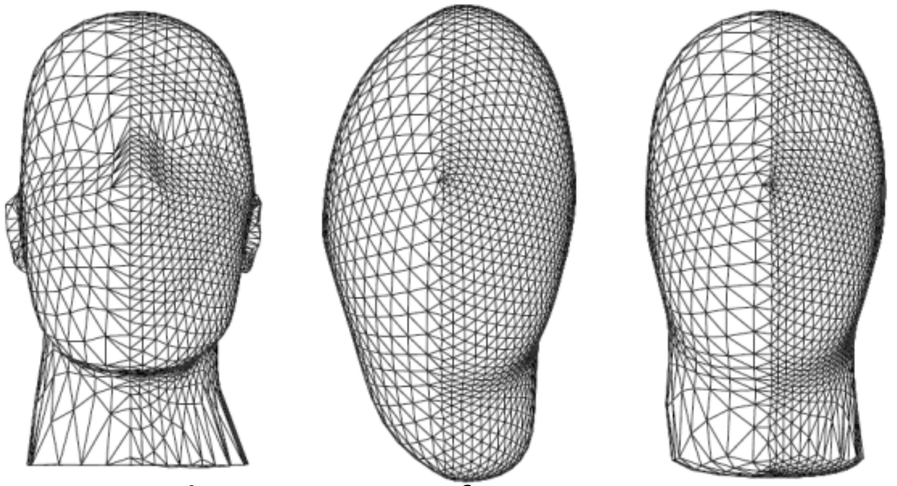
$$(I - \lambda L)\tilde{\mathbf{p}} = \mathbf{p}$$



Implicit fairing of irregular meshes using diffusion and curvature flow
M. Desbrun, M. Meyer, P. Schroeder, A. Barr, **ACM SIGGRAPH 99**

Mesh Independence

- Result of smoothing with uniform Laplacian depends on triangle density and shape
  - Why?
- Asymmetric results although underlying geometry is symmetric

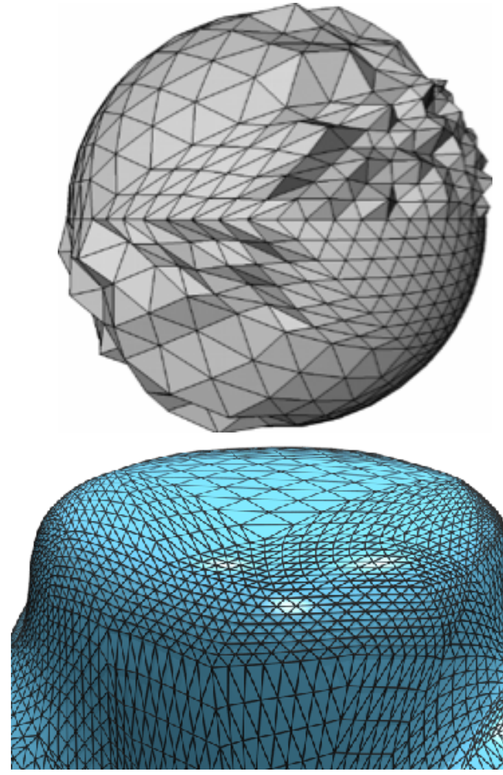


original uniform cotan

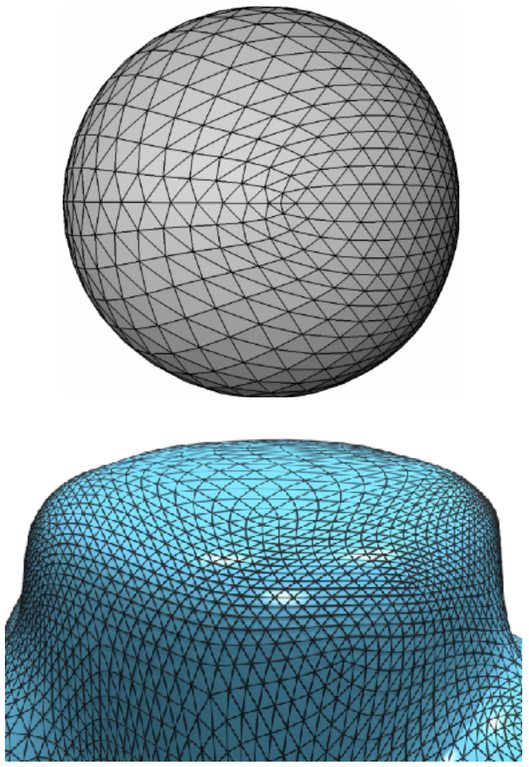
Comparison of the weights

- Explicit flow with different weights:

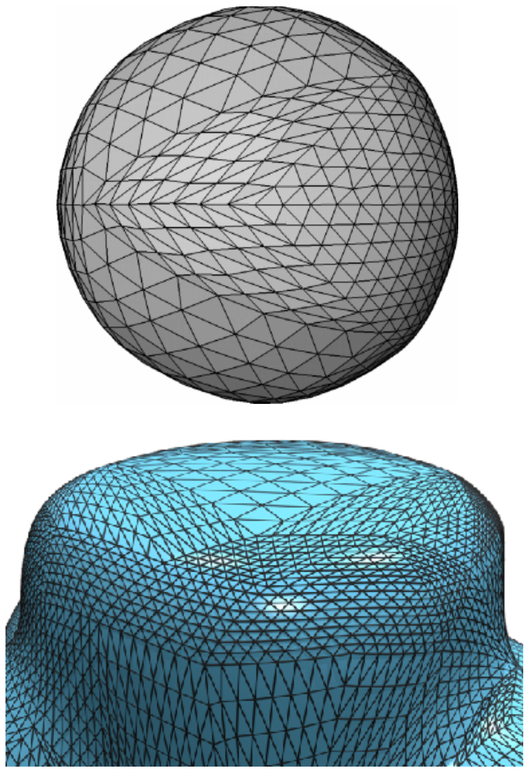
original



uniform

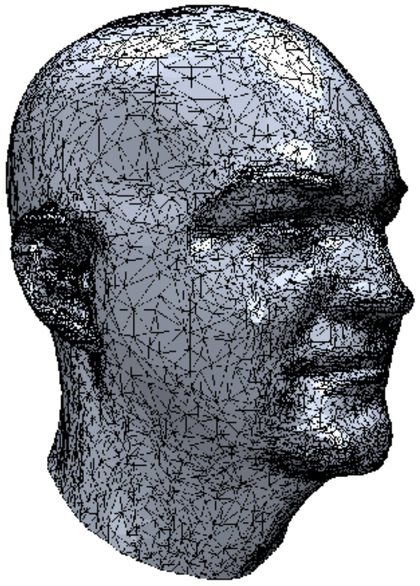


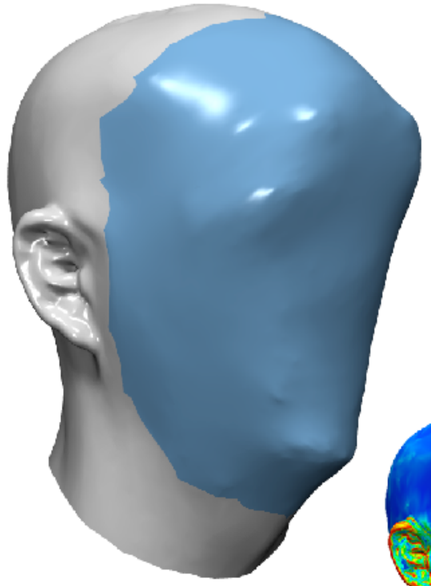
cotan



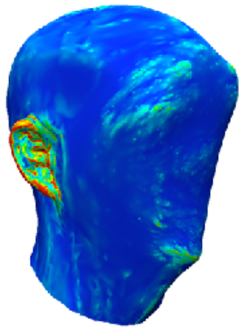
Implicit Fairing

- The importance of using the right weights

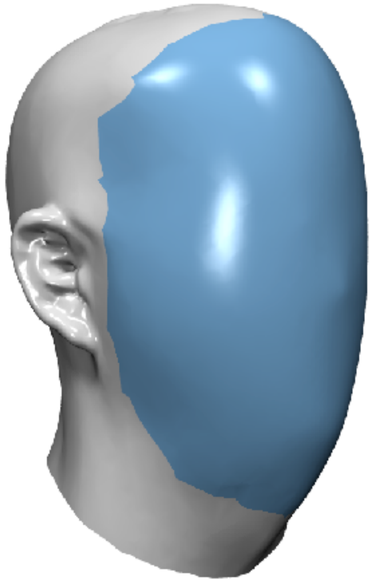




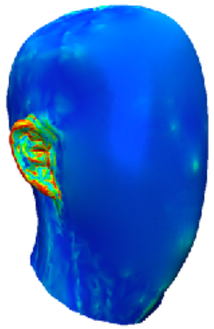
uniform



Mean curvature



cotan



Mean curvature

Method 2:
Smoothing as Optimization

Minimizing a smoothness energy

- Let's go for $H = 0$

$$\Delta_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n}$$

goal: $H = 0$ or $H = \text{const}$

- only trivial solution, no connection to initial surface p

$$\Delta_{\mathcal{M}} \tilde{\mathbf{p}} = 0$$

Minimizing a smoothness energy

- Let's go for $H = 0$

$$\Delta_{\mathcal{M}} \mathbf{p} = -2H\mathbf{n}$$

goal: $H = 0$ or $H = \text{const}$

- only trivial solution, no connection to initial surface p

- Let's regularize!

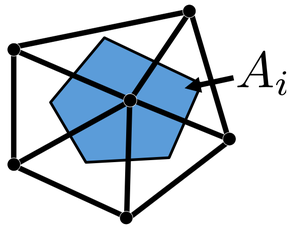
$$\Delta_{\mathcal{M}} \tilde{\mathbf{p}} = 0$$

$$\min_{\tilde{\mathbf{p}}} \int_{\mathcal{M}} \underbrace{\|\Delta_{\mathcal{M}} \tilde{\mathbf{p}}\|^2}_{\text{small } H} + \underbrace{w\|\tilde{\mathbf{p}} - \mathbf{p}\|^2}_{\text{stay close to original surface}}$$

weighting factor

Minimizing a smoothness energy

- Discretize: $$\min_{\tilde{\mathbf{p}}} \int_{\mathcal{M}} \|\Delta_{\mathcal{M}} \tilde{\mathbf{p}}\|^2 + w\|\tilde{\mathbf{p}} - \mathbf{p}\|^2$$



$$\min_{\tilde{\mathbf{p}}} \sum_{i=1}^{n} A_i \left(\|L\tilde{\mathbf{p}}_i\|^2 + w\|\tilde{\mathbf{p}}_i - \mathbf{p}_i\|^2 \right)$$

- Minimize!

$$\frac{\partial}{\partial \tilde{\mathbf{p}}} = 0 \qquad \text{remember:} \quad \frac{\partial}{\partial \mathbf{x}}(\mathbf{x}^T A \mathbf{x}) = (A + A^T)\mathbf{x}$$

tip: google “The Matrix Cookbook”

Minimizing a smoothness energy

$$\min_{\tilde{\mathbf{p}}} \sum_{i=1}^{n} A_i \left(\|L\tilde{\mathbf{p}}_i\|^2 + w\|\tilde{\mathbf{p}}_i - \mathbf{p}_i\|^2 \right)$$

$$\tilde{\mathbf{p}} = [\tilde{\mathbf{x}}\ \tilde{\mathbf{y}}\ \tilde{\mathbf{z}}] = \begin{pmatrix} \tilde{p}_{1x} & \tilde{p}_{1y} & \tilde{p}_{1z} \\ \tilde{p}_{2x} & \tilde{p}_{2y} & \tilde{p}_{2z} \\ \vdots & \vdots & \vdots \\ \tilde{p}_{nx} & \tilde{p}_{ny} & \tilde{p}_{nz} \end{pmatrix} \in \mathbb{R}^{n\times 3}$$

$$E(\tilde{\mathbf{p}}) = (L\tilde{\mathbf{p}})^T M(L\tilde{\mathbf{p}}) + w(\tilde{\mathbf{p}} - \mathbf{p})^T M(\tilde{\mathbf{p}} - \mathbf{p})$$

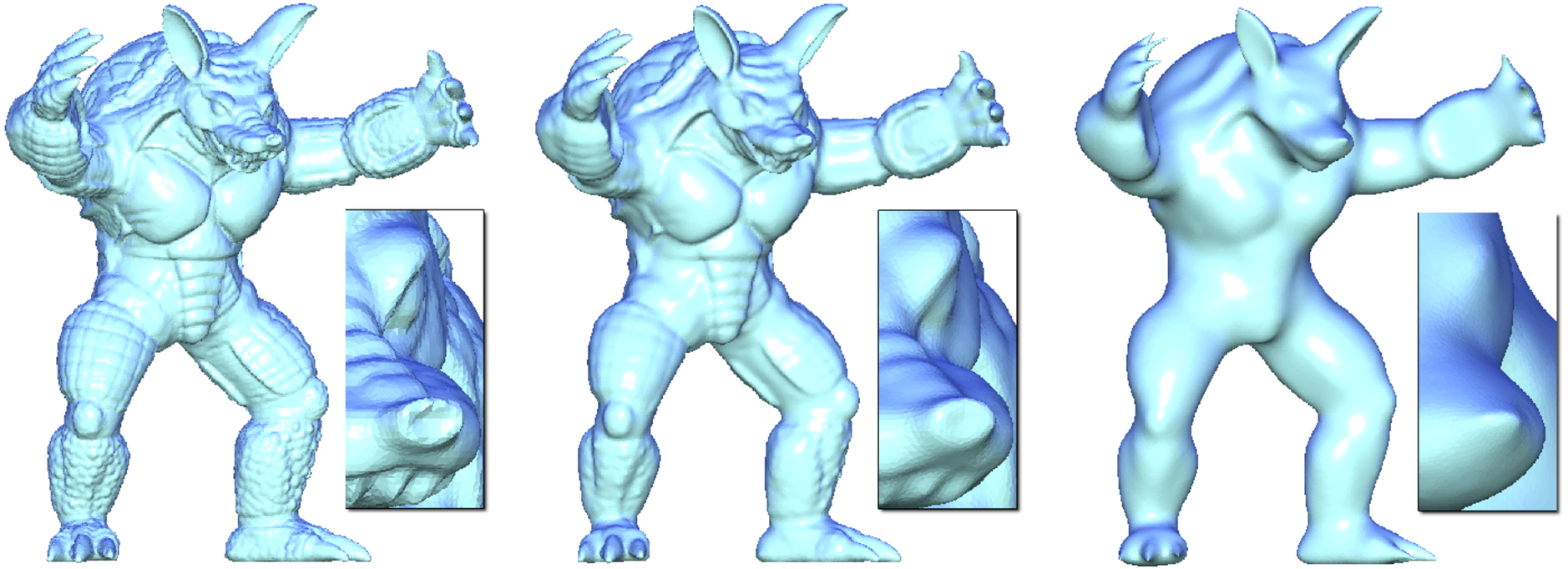
$$\frac{\partial E}{\partial \tilde{\mathbf{p}}} = 2L^T ML\tilde{\mathbf{p}} + 2wM(\tilde{\mathbf{p}} - \mathbf{p}) \stackrel{!}{=} 0$$

$$\Rightarrow \underline{\left(L^T ML + wM\right)\tilde{\mathbf{p}} = wM\mathbf{p}}$$ compare with implicit Euler!

$NB:$

$$L = M^{-1}L_w \ \Rightarrow\ L^T ML = L_w M^{-1} L_w$$

Results



original $w = 0.2$ $w = 0.02$

Customize the energy functional

$$\min_{\tilde{\mathbf{p}}} \sum_{i=1}^{n} A_i \left(\|L\tilde{\mathbf{p}}_i\|^2 + w\|\tilde{\mathbf{p}}_i - \mathbf{p}_i\|^2 \right)$$

add a varying weight here to control feature preservation

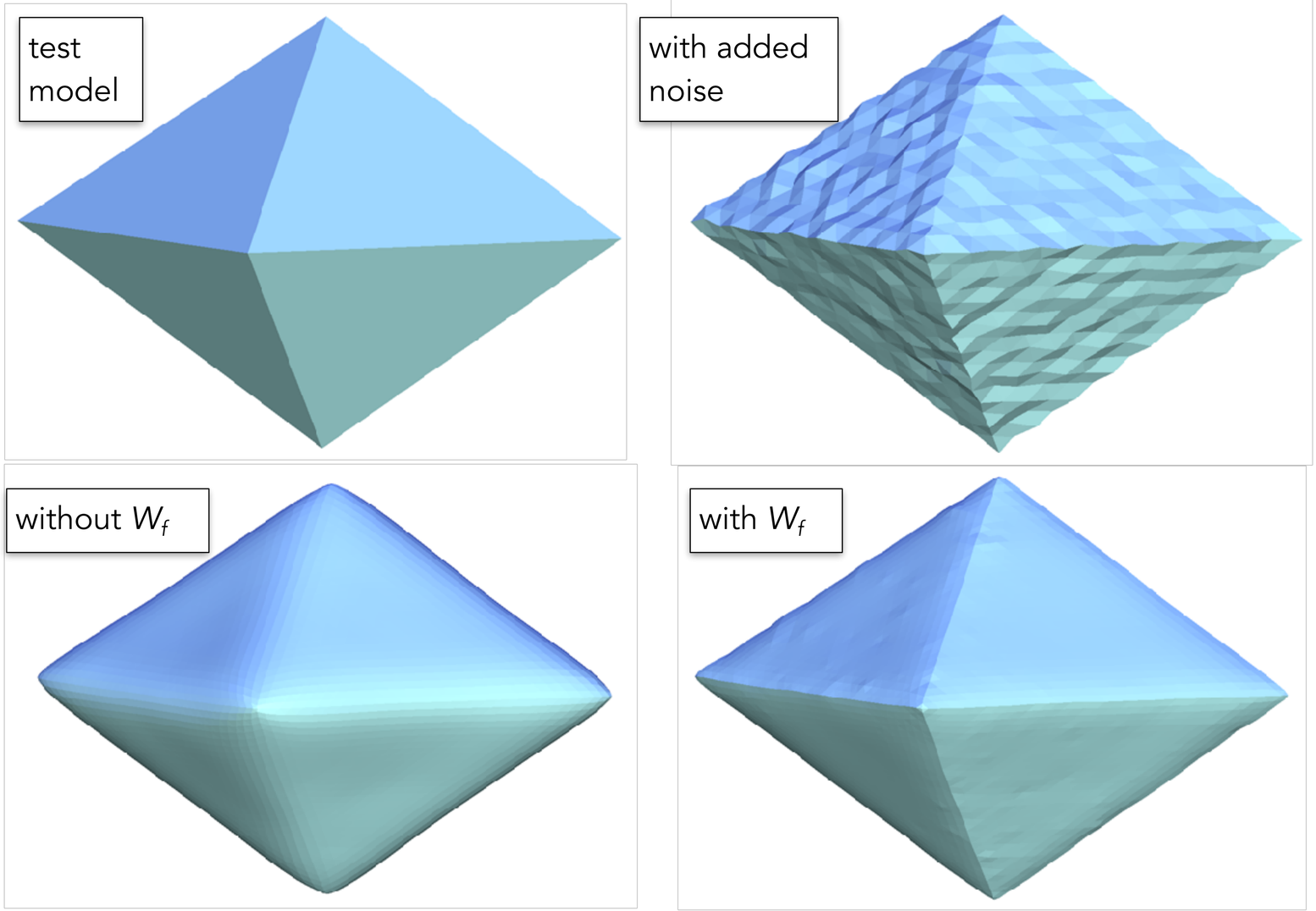
$$E(\tilde{\mathbf{p}}) = \tilde{\mathbf{p}}^T L^T (M^{0.5} W_f M^{0.5}) L\tilde{\mathbf{p}} + w(\tilde{\mathbf{p}} - \mathbf{p})^T M(\tilde{\mathbf{p}} - \mathbf{p})$$

$(W_f)_{ii}$ can be inverse-proportional to e.g. curvature at vertex *i*

(slight abuse of notation, the energy is actually the sum of the *xyz* components, like on slide #62)

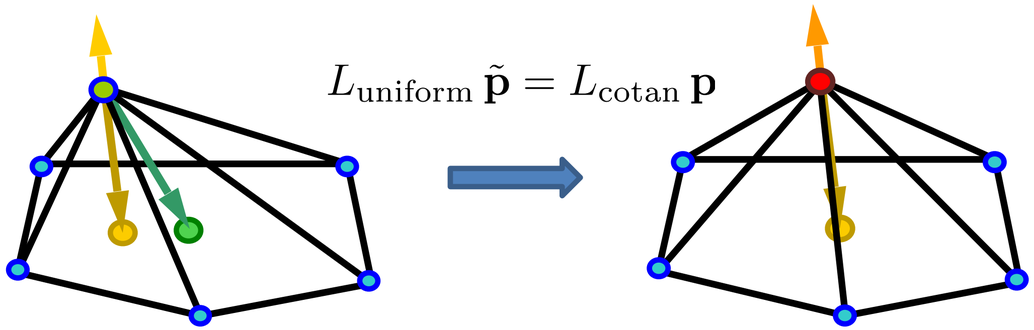
$$E(\tilde{\mathbf{p}}) = \sum_{\mathbf{v} \in \{\mathbf{x},\mathbf{y},\mathbf{z}\}} \tilde{\mathbf{v}}^T L^T \dots$$

Using W_f



Customize the energy functional

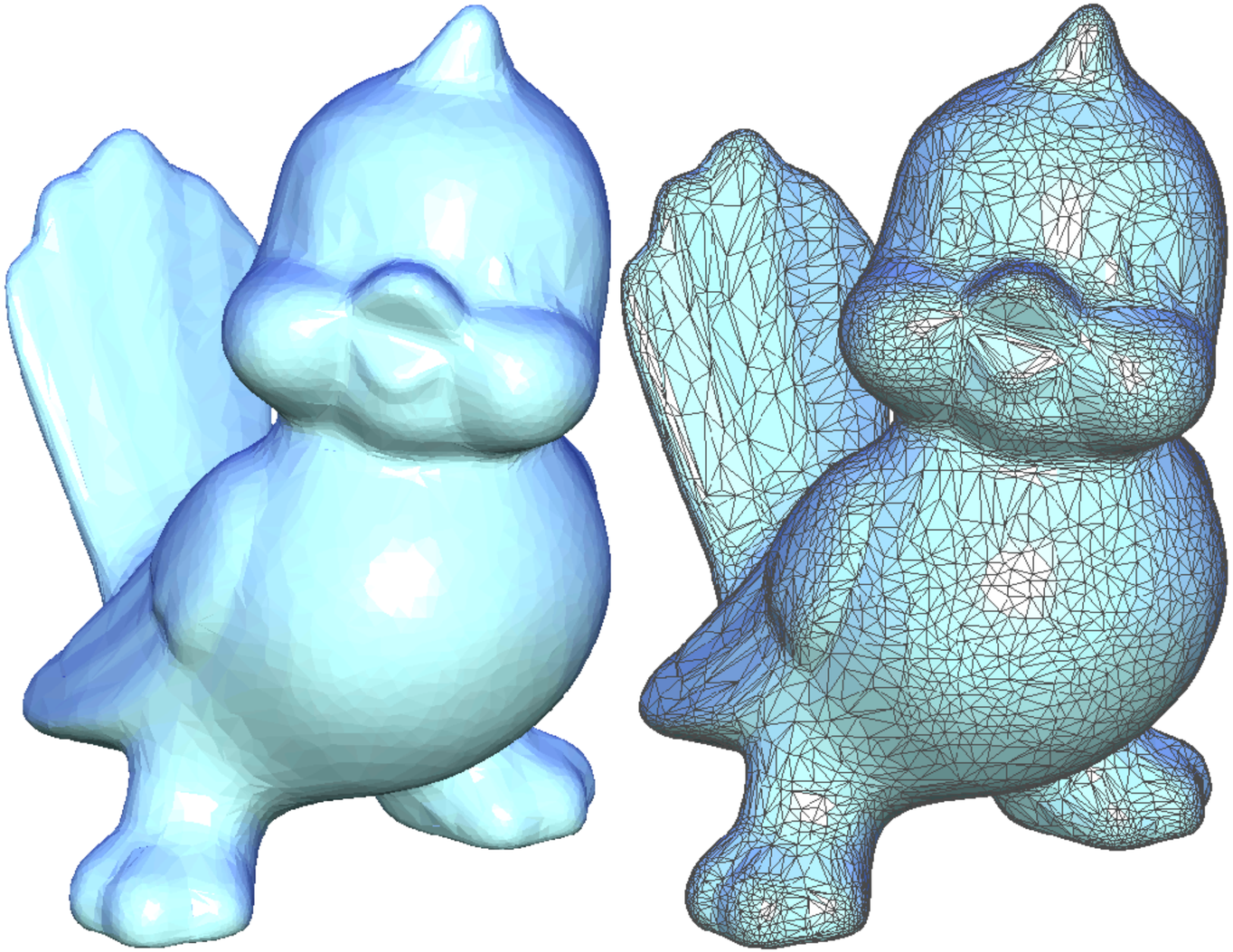
- Can do *tangential* smoothing!
  - Improve the shapes of mesh triangles without changing the surface shape



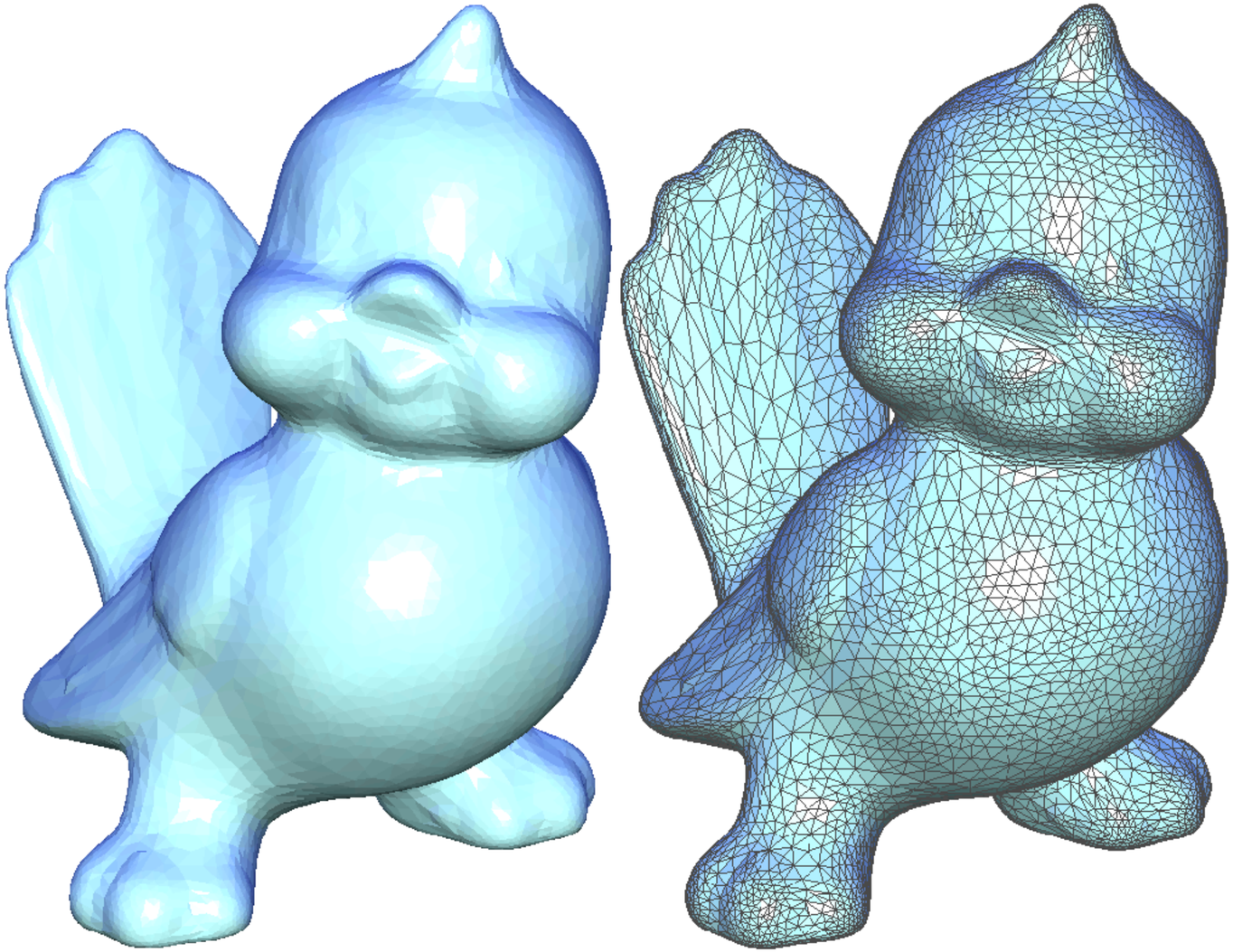
$$\min_{\tilde{\mathbf{p}}} \sum_{i=1}^{n} A_i \left(\| L_{\text{uniform}}\,\tilde{\mathbf{p}}_i - L_{\text{cotan}}\,\mathbf{p}_i \|^2 + w \| \tilde{\mathbf{p}}_i - \mathbf{p}_i \|^2 \right)$$

https://graphics.uni-bielefeld.de/downloads/publications/2004/sgp04.pdf

Original



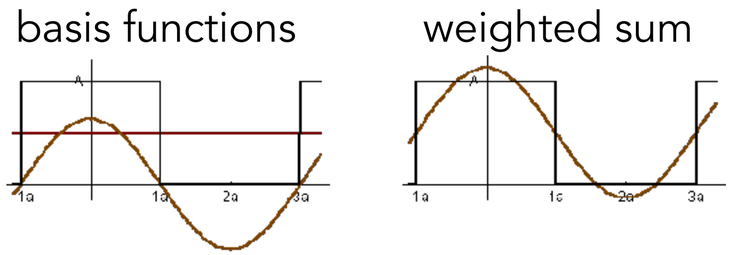
Triangle Shape Optimization



Method 3:
Smoothing as Filtering

Fourier analysis

- Represent a function as a weighted sum of sines and cosines (basis functions)

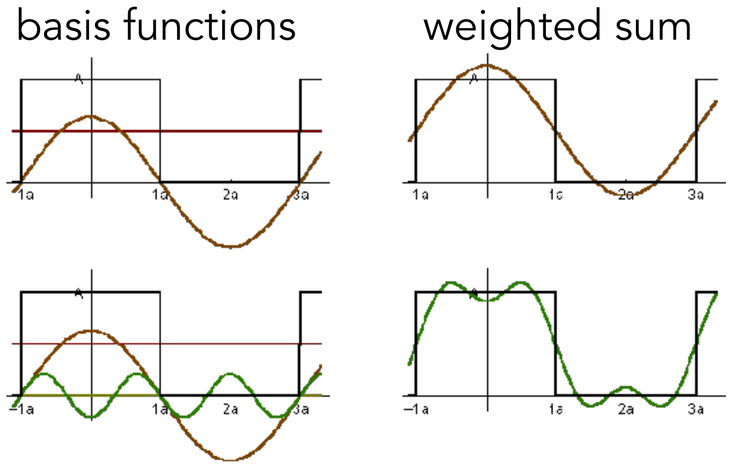


Joseph Fourier 1768 - 1830

$$f(x) = a_0 + a_1 \cos(x)$$

Fourier analysis

- Represent a function as a weighted sum of sines and cosines (basis functions)



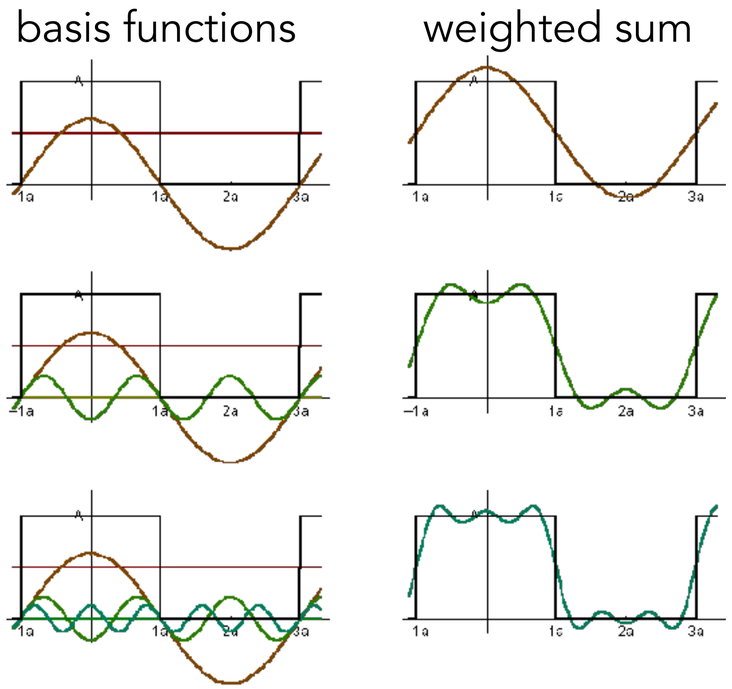
Joseph Fourier 1768 - 1830

$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x)$$

Fourier analysis

- Represent a function as a weighted sum of sines and cosines (basis functions)

Joseph Fourier 1768 - 1830

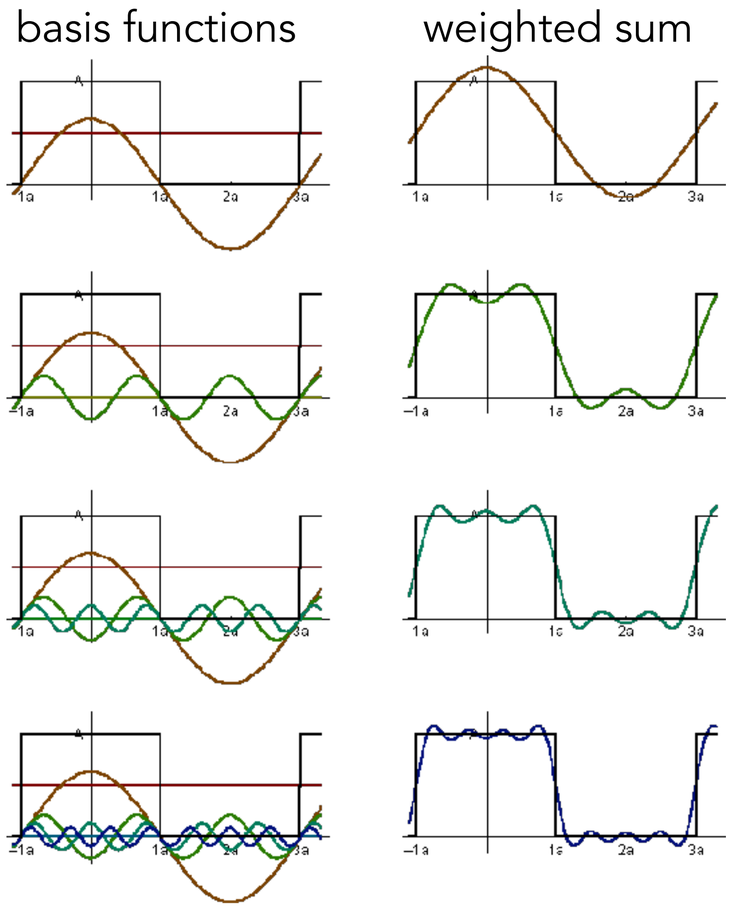


$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x)$$

Fourier analysis

- Represent a function as a weighted sum of sines and cosines (basis functions)

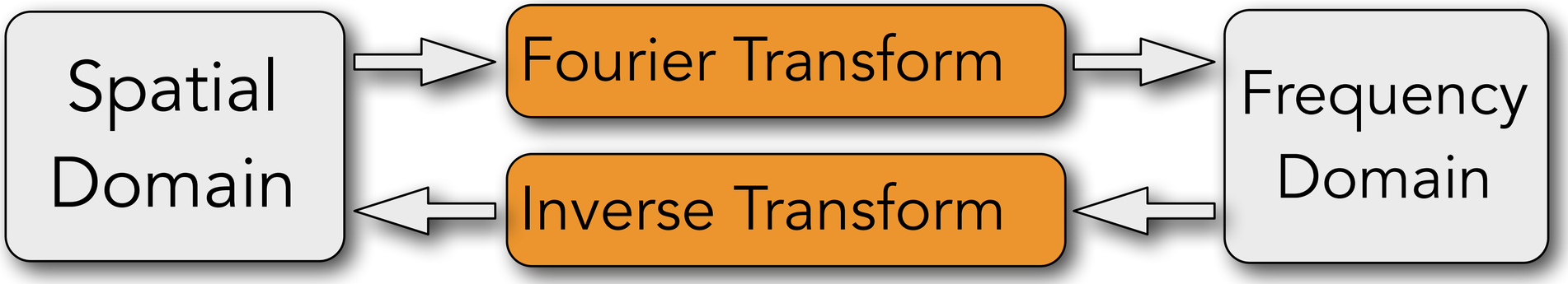
Joseph Fourier 1768 - 1830



$$f(x) = a_0 + a_1 \cos(x) + a_2 \cos(3x) + a_3 \cos(5x) + a_4 \cos(7x) + \dots$$

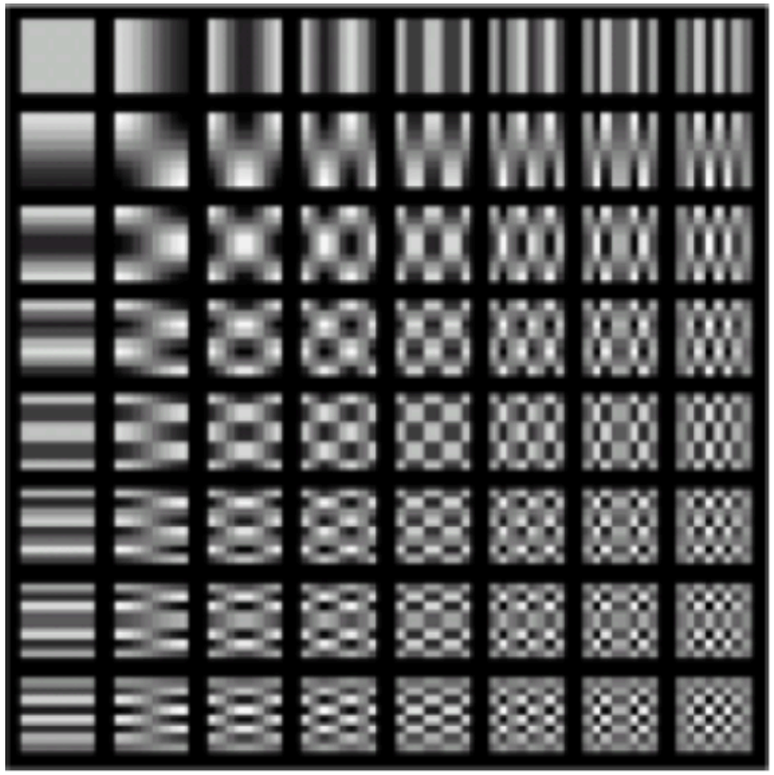
Fourier analysis

$$F(\omega) = \int_{-\infty}^{\infty} f(x) \mathrm{e}^{-2\pi i \omega x} \, dx$$



$$f(x) = \int_{-\infty}^{\infty} F(\omega) \mathrm{e}^{2\pi i \omega x} \, d\omega$$

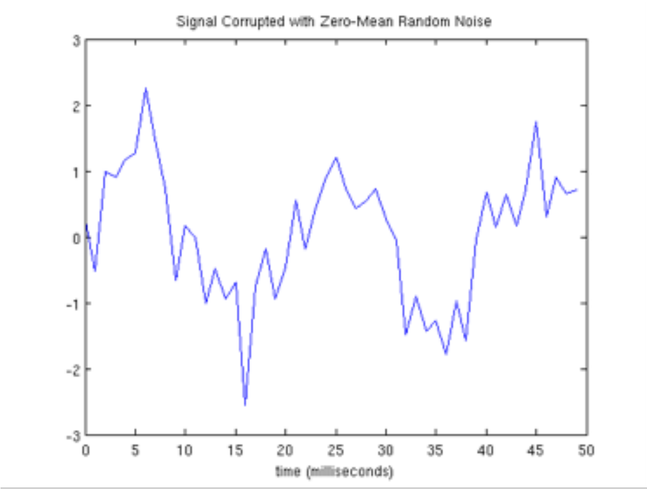
Also works on rectangular 2D domains

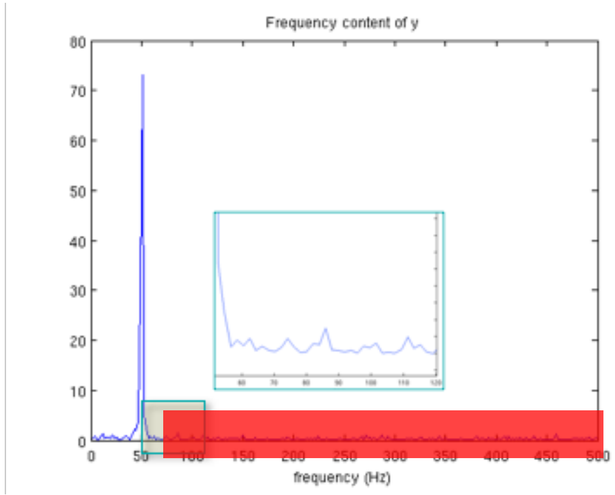


Fourier (DCT) basis functions for 8x8 grayscale images

$$\cos(2\pi\omega_h)\cos(2\pi\omega_v)$$

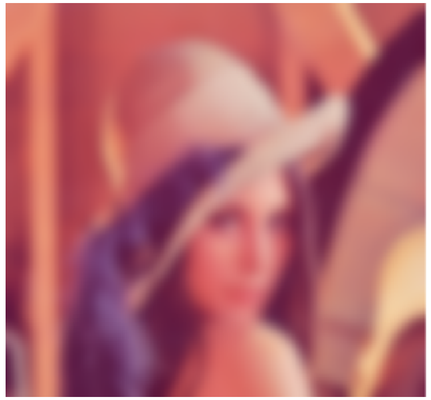
Smoothing = filtering high frequencies out





spatial domain

frequency domain



Extend Fourier to meshes?

- Basis functions??
- Fourier basis functions are eigenfunctions of the (standard) Laplace operator:

$$\Delta\left(e^{2\pi i\omega x}\right) = \frac{\partial^2}{\partial x^2} e^{2\pi i\omega x} = -(2\pi\omega)^2 e^{2\pi i\omega x}$$

- On meshes: take the eigenvectors of the Laplace-Beltrami matrix!

Spectral analysis on meshes

- Take your favorite L-B matrix *L*
- Compute eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_k$ with the *k* smallest eigenvalues
- Reconstruct the mesh geometry from these eigenvectors:

$$\mathbf{x} = [x_1, \ldots, x_n]^T \qquad \mathbf{y} = [y_1, \ldots, y_n]^T \qquad \mathbf{z} = [z_1, \ldots, z_n]^T$$

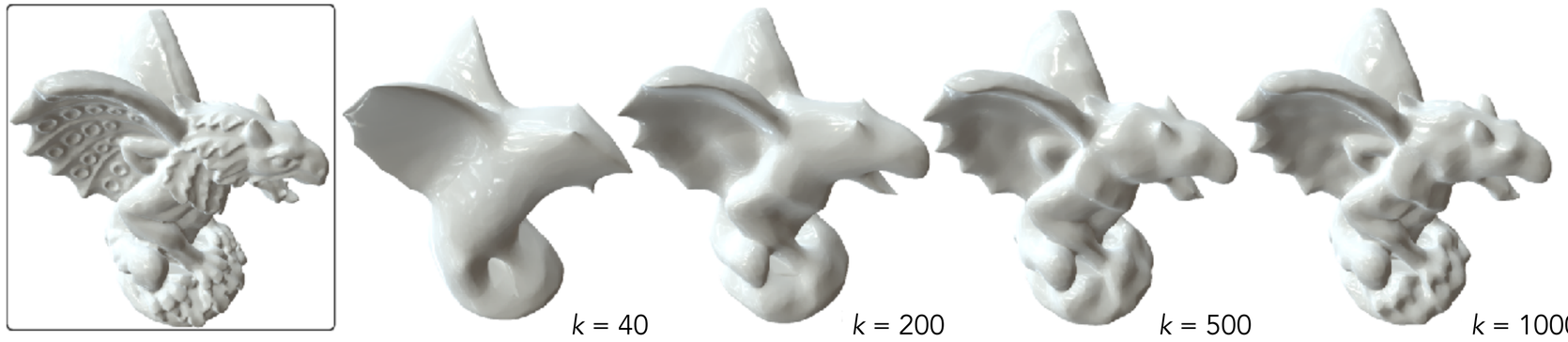
$$\tilde{\mathbf{x}} = \sum_{i=1}^{k} \left(\mathbf{x}^T \mathbf{e}_i\right) \mathbf{e}_i \qquad \tilde{\mathbf{y}} = \sum_{i=1}^{k} \left(\mathbf{y}^T \mathbf{e}_i\right) \mathbf{e}_i \qquad \tilde{\mathbf{z}} = \sum_{i=1}^{k} \left(\mathbf{z}^T \mathbf{e}_i\right) \mathbf{e}_i$$

$$\tilde{\mathbf{p}} = [\tilde{\mathbf{x}}\ \tilde{\mathbf{y}}\ \tilde{\mathbf{z}}] \in \mathbb{R}^{n \times 3}$$

Spectral analysis on meshes

too expensive for large meshes

- Take your favorite L-B matrix L
- Compute eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k$ with the k smallest eigenvalues
- Reconstruct the mesh geometry from these eigenvectors:



References

- Polygon Mesh Processing, Chapter 4