Section: Approximating functions - Lecture 1

Given a function with one spatial variable, e.g., $f(x) = \sin(x)$, we want to do the followings:

1. Find a way to "store" the function in computer
2. To evaluate the function values efficiently
3. To evaluate the derivative of the function

[1.1] How to "store" a function in computer

Very often we don't have a mathematical formulation for the function. We only have limited information about the function such as function values at several given points.

For example, suppose we have a function $f(x)$ defined on $[-1, 1]$ and we know its values at $x = -1, 0$ and 1. Then we store in computer $f(-1), f(0)$ and $f(1)$.

Of course if we know the function values at more points, we can store more in computer. BUT! There are infinitely many points on $[-1, 1]$ (and in fact uncountablly many) and there is no way to store all of them.

There is a theorem that might help us:

Weierstrass Approximation Theorem:

If f is continuous on $[a, b]$ and if $\epsilon > 0$, then there is a polynomial p satisfying $|f(x) - p(x)| \leq \epsilon$ on the interval $[a, b]$.

So, one way of thinking is to "represent" a continuous function by a finite degree polynomial, and we only store in computer the coefficients of the polynomial.

For example, if we have $f(-1) = 4, f(0) = 2$ and $f(1) = 2$. Then the polynomial that passes through these points is $p(x) = 2 - x + x^2$. So we store in computer three values, $2, -1$ and 1. If we want to evalute the function at a point x_0 we just evaluate $p(x_0)$.

Sounds good?! In principle if we have $n + 1$ points of a function, we can construct a nth-degree polynomial that passes through these $n + 1$ data points. We then "expect, hope, pray" that this new polynomial is a good representation of the function, i.e., $f(x) \approx p(x)$ for all x.

Remark 1:

There is a HUGE difference between storing the "discrete data points" and "discrete coefficients" of a function. The former one is a discrete description and we only have information on the given points, while the later one is actually a continuous description and we can get the value of every single point on the domain.

Remark 2:

One can imagine that there is a vector space consists of all the polynomials. A natural basis for this space is $\{1, x, x^2, x^3, \ldots\}$. So every polynomial, under this basis, have a coordinate. For example, the coordinate of $f(x) = (x-1)(x-2)$ is $[2, -3, 1, 0, \cdots]$.

Remark 3:

For periodic functions on $[0, 2\pi]$, the natural basis functions are $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x), \ldots\}$. A function being present in this basis is called "Fourier series":

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx).$$

The constants $a_0, a_1, b_1, a_2, b_2, \ldots$ are called Fourier coefficients.

[1.2] How to evaluate the function values efficiently

In general we represent a function by a "simpler" expression that is easier to evaluate. For example, if we want to evaluate $\sin(0.38)$, how to do that? One way is to use Taylor expansion as

$$\sin(0.38) \approx 0.38 - \frac{0.38^3}{6} + \frac{0.38^5}{120} + \cdots.$$

Note that the expression on the right hand side consists of adding, subtracting, multiplying and dividing values that can be performed easily.

In the cases where we don't have a close form Taylor series of a given function, we can represent the function by a polynomial, and then use the values of the polynomial to approximate the function values.

[1.3] How to evaluate the derivative of a function, how to integrate a function on a given domain

If we can represent a function by a polynomial $p(x)$, then we can use the derivative of $p(x)$ as an approximation for the derivative of the function.

Similarly, if we can represent a function by a polynomial $p(x)$, then we can integrate $p(x)$ and use this value as an approximation for the integration of the function.

Note that the above two results are based on the fact that we can differentiate and integrate a polynomial exactly.

Conclusion

To summarize, it "looks like" if we can do polynomial we can do everything. So the natural question is "how to represent nicely a function by a polynomial?" This will be the main topic of the next lecture.