1 Introduction

2 Theory

3 Waveguides

In order to understand cavities, we start off with the discussion on waveguides. An electromagnetic wave, when confined to the interior of a hollow pipe is called a waveguide. We shall closely follow the derivation from [].The generalized Maxwell equations in terms of E-field and H-field are given by:

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} \tag{3.1}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{3.2}$$

$$\nabla \times \mathbf{H} = \vec{j} + \frac{\partial D}{\partial t} \tag{3.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{3.4}$$

In vacuum, these equations become:

$$\nabla \times \mathbf{E} = \mu_0 \cdot \frac{\partial H}{\partial t} \tag{3.5}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{3.6}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \cdot \frac{\partial E}{\partial t} \tag{3.7}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{3.8}$$

Upon solving these equations, we get plane wave solutions which looks like this:

$$\Delta \vec{E}\,(\vec{r}) + \frac{\omega}{c^2}\vec{E}\,(\vec{r}) = 0$$

$$\Delta \vec{B}\,(\vec{r}) + \frac{\omega}{c^2}\vec{B}\,(\vec{r}) = 0$$

Now, let us look at a waveguide which is aligned along the z-direction. Taking the ansatz $\vec{E} = \vec{E}(x, y).e^{i(\omega t - kz)}$ and the separation $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$, for the longitudinal fields yields:

$$\Delta_\perp E_z + k_c^2 E_z = 0 \tag{3.9}$$

$$\Delta_\perp H_z + k_c^2 H_z = 0 \tag{3.10}$$

where

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

The quantity k_c is called the critical wave number and is a characteristic property of the cavity, as we shall see. Solving these equations further, we see that it is sufficient to know the longitudinal fields, E_z and B_z because we can easily determine the transverse components from them.

$$ik_c^2 \vec{E}_\perp = k\vec{\nabla}_\perp E_z + \omega\mu_0 \vec{\nabla}_\perp H_z \times \hat{e}_z \tag{3.11}$$

$$ik_c^2 \vec{H}_\perp = k\vec{\nabla}_\perp H_z - \omega\epsilon_0 \vec{\nabla}_\perp E_z \times \hat{e}_z \tag{3.12}$$

Now, there are two waves to classify these waves:

1. $k_c^2 = 0$
   (a) $\vec{\Delta_\perp} E_z \neq 0$ and $\vec{\Delta}_\perp H_z \neq 0$: HE or EH hybrid waves.
   (b) $\vec{\Delta_\perp} E_z = 0$ and $\vec{\Delta}_\perp H_z = 0$: TEM waves.
2. $k_c^2 \neq 0$ In this case we do not get any propagation if $\omega \leq c \cdot k_c$. These waves are called evanescent waves or the cut off condition. The possible propagation modes are:
   (a) $E_z = 0$: TE (transversal electric) or H waves.
   (b) $H_z = 0$: TM (transversal magnetic) or E waves.

Corresponding to this critical wave, we have a critical frequency, which is $\omega_c = k_c.c$, below which there is no propagation. One interesting thing to note here is that for a hollow waveguide, only TE or TM modes are possible, TEM is not, because no wave would exist in this case []. But for a coaxial cable, which consists of straight wire surrounded by a conduction sheath, we can get TEM modes.

3.1 Cylindrical waveguides

We now consider a cylindrical waveguide with an inner radius a. This imposes the following boundary conditions to the walls of the waveguide:

- $E_\phi = 0$, $E_z = 0$ for $r = a$
- $H_r = 0$ for $r = a$

The field distribution solution for a cylindrical waveguide can be separated into angular and radial parts, whose solutions are then given by the Bessel/Neumann functions. These can be then substituted in the eq.(3.11)-(3.12) and upon imposing the constraints from the boundary condition gives us the corresponding TE- and TM-modes.

3.2 Cylindrical waveguides resonator

The time has now come, to talk about cavities itself. If we now insert two conducting plates perpendicular to the z-axis, the incoming wave is reflected completely, giving us standing waves. Because of this, the z-dependence changes like:

$$a \cdot e^{ikz} \implies A \cdot \sin(kz + \phi_0)$$

The following condition is imposed so that the longitudinal boundary conditions are fulfilled: $k = p \cdot \pi/l$. The longitudinal field looks like:

$$\mathbf{TE_{mnp} - Modes} : H_z = H_{mn} \cdot J_m(k_c r) \cos(m\phi) \cdot \sin(p\pi/l \cdot z) \cdot e^{\omega_{mnp} t} \text{ where } k_c a = j'_{mn}$$
$$\mathbf{TM_{mnp} - Modes} : E_z = E_{mn} \cdot J'_m(k_c r) \cos(m\phi) \cdot \cos(p\pi/l \cdot z) \cdot e^{\omega_{mnp} t} \text{ where } k_c a = j_{mn}$$

For resonant frequency, we have:

$$\omega_{mnp} = c \cdot \sqrt{(j_{mn}/a)^2 + (p\pi/l)^2}$$

Here, J_m is the m-th Bessel function, $J_m^{'}$ is its derivative. And j_{mn} and $j_{mn}^{'}$ are the n-th zeropoints of the m-th Bessel function and its derivative.
The resonant modes can be written in the linear form as:

$$(d\nu)^2 = \left(\frac{c j_{mn}^{(')}}{\pi} \right)^2 + \left(\frac{c}{2} \right)^2 p^2 \left(\frac{d}{l} \right)^2 \tag{3.13}$$

here d is the diameter of the cavity. When we plot different modes on a graph, we get a mode map (Fig.3.1). The mode map allows one to read off the resonant frequencies for different diameters and length of the cavity.

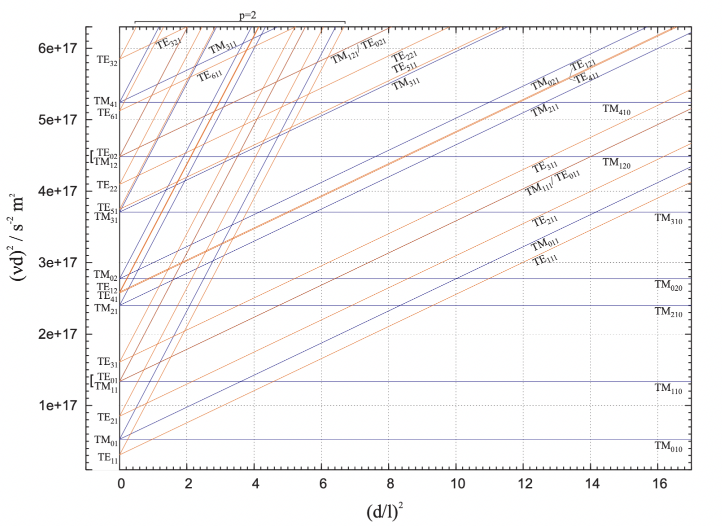

Figure 3.1: Mode map for $p \leq 2$ **??**

Since the derivative of the zeroth order Bessel function and the first order Bessel function, $j_{0n}^{'}$ and j_{1n}, are equal, the corresponding TE- and TM-modes have the same resonant frequencies. That is, TE_{0np}-modes and TM_{1np}-modes have the same resonant frequency.

4 Oscillating circuit

A cavity has a lot of characteristic quantities, which can be described by an equivalent circuit (Fig.4.1).
Coupling is a process through which electromagnetic waves can be coupled to a waveguide, or in this case, to a cavity. There are several ways to couple and in this experiment, we use loop coupling, which enables coupling to the magnetic field inside the cavity. In the figure 4.1, the LCR-circuit represents the cavity. The step-down transformer represents the loop coupling, Z_0 is the characteristic impedance and R_s is the Shunt impedance.

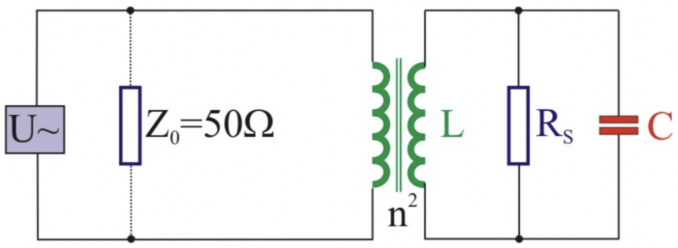

Figure 4.1: Equivalent circuit of a cavity with loop coupling

The characteristic quantities associated with the cavity are:

- Quality factor
- Coupling coefficient
- Reflection coefficient
- Shunt impedance

Let us look at these quantities in a bit more detail.

4.1 Quality factor

Quality factor is a dimensionless quantity which describes how underdamped an oscillator or resonator is. It is defined as:

$$Q_0 = \frac{2 \cdot \pi \text{stored energy}}{\text{losses per period}} = \frac{2\pi \cdot W}{T \cdot P} = \frac{\omega_0 \cdot W}{P} \tag{4.1}$$

where ω_0 is the angular resonant frequency. Looking at the case of driven oscillations, the unloaded quality factor can be determined from the Full Width Half Maximum (FWHM), $\Delta\omega_H$

$$Q_0 = \frac{\omega_0}{\Delta\omega_H} \tag{4.2}$$

4.2 Coupling coefficient

The coupling coefficient is defined as:

$$\kappa = \frac{Z_a}{Z_0} = \frac{R_s}{n^2 Z_0} = \frac{Q_0}{Q_{ext}} \tag{4.3}$$

where n is the transformer turn ratio, Q_0 is the unloaded quality factor and Q_{ext} is the external quality factor. If we know the coupling coefficient, the unloaded quality factor can be calculated using:

$$Q_0 = (1 + \kappa) \cdot Q \tag{4.4}$$

We also get 3 cases for the coupling coefficient, which are:

- $\kappa < 1$: undercritical coupling, $Q > Q_0/2$
- $\kappa = 1$: critical coupling, $Q = Q_0/2$ (no reflection)
- $\kappa > 1$: overcritical coupling, $Q < Q_0/2$

4.3 Reflection coefficient

In the conductor, we have an incoming wave $(\hat{U}_+, \hat{I}_+)$ and reflected wave $(\hat{U}_-, \hat{I}_-)$. Hence we define the complex reflection coefficient as:

$$\rho = \frac{\hat{U}_-}{\hat{U}_+} \tag{4.5}$$

The coupling coefficient and the reflection coefficient are related at resonance by:

$$\kappa = \left| \frac{1+\rho}{1-\rho} \right| \tag{4.6}$$

We shall discuss more about this in the next section.

4.4 Shunt impedance

Shunt impedance tells us how much energy is gained by a charged particle when it crosses the cavity. It is defined by:

$$R_s = \frac{U^2}{2P_V} = \frac{1}{P_V} \left| \int\limits_{L/2}^{-L/2} E_0\left(s\right) \cdot e^{i\omega_0 s/c} \cdot ds \right|^2 \tag{4.7}$$