1 Introduction

2 Theory

3 Waveguides

In order to understand cavities, we start off with the discussion on waveguides. An electromagnetic wave, when confined to the interior of a hollow pipe is called a waveguide. We shall closely follow the derivation from [].The generalized Maxwell equations in terms of E-field and H-field are given by:

$$\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t} \tag{1}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{2}$$

$$\nabla \times \mathbf{H} = \vec{j} + \frac{\partial D}{\partial t} \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4}$$

In vacuum, these equations become:

$$\nabla \times \mathbf{E} = \mu_0 \cdot \frac{\partial H}{\partial t} \tag{5}$$

$$\nabla \cdot \mathbf{E} = 0 \tag{6}$$

$$\nabla \times \mathbf{H} = \epsilon_0 \cdot \frac{\partial E}{\partial t} \tag{7}$$

$$\nabla \cdot \mathbf{H} = 0 \tag{8}$$

Upon solving these equations, we get plane wave solutions which looks like this:

$$\Delta \vec{E}\,(\vec{r}) + \frac{\omega}{c^2} \vec{E}\,(\vec{r}) = 0$$

$$\Delta \vec{B}\,(\vec{r}) + \frac{\omega}{c^2} \vec{B}\,(\vec{r}) = 0$$

Now, let us look at a waveguide which is aligned along the z-direction. Taking the ansatz $\vec{E} = \vec{E}(x, y).e^{i(\omega t - kz)}$ and the separation $\Delta = \Delta_\perp + \frac{\partial^2}{\partial z^2}$, for the longitudinal fields yields:

$$\Delta_\perp E_z + k_c^2 E_z = 0 \tag{9}$$

$$\Delta_\perp H_z + k_c^2 H_z = 0 \tag{10}$$

where

$$k_c^2 = \frac{\omega^2}{c^2} - k^2$$

The quantity k_c is called the critical wave number and is a characteristic property of the cavity, as we shall see. Solving these equations further, we see that it is sufficient to know the longitudinal fields, E_z and B_z because we can easily determine the transverse components from them.

$$ik_c^2 \vec{E}_\perp = k \vec{\nabla}_\perp E_z + \omega \mu_0 \vec{\nabla}_\perp H_z \times \hat{e}_z \tag{11}$$

$$ik_c^2 \vec{H}_\perp = k \vec{\nabla}_\perp H_z - \omega \epsilon_0 \vec{\nabla}_\perp E_z \times \hat{e}_z \tag{12}$$

Now, there are two waves to classify these waves:

1. $k_c^2 = 0$
   (a) $\vec{\Delta_\perp} E_z \neq 0$ and $\vec{\Delta}_\perp H_z \neq 0$: HE or EH hybrid waves.
   (b) $\vec{\Delta_\perp} E_z = 0$ and $\vec{\Delta}_\perp H_z = 0$: TEM waves.
2. $k_c^2 \neq 0$ In this case we do not get any propagation if $\omega \leq c \cdot k_c$. These waves are called evanescent waves or the cut off condition. The possible propagation modes are:
   (a) $E_z = 0$: TE (transversal electric) or H waves.
   (b) $H_z = 0$: TM (transversal magnetic) or E waves.

Corresponding to this critical wave, we have a critical frequency, which is $\omega_c = k_c.c$, below which there is no propagation. One interesting thing to note here is that for a hollow waveguide, only TE or TM modes are possible, TEM is not, because no wave would exist in this case []. But for a coaxial cable, which consists of straight wire surrounded by a conduction sheath, we can get TEM modes.

3.1 Cylindrical waveguides

We now consider a cylindrical waveguide with an inner radius a. This imposes the following boundary conditions to the walls of the waveguide:

- $E_\phi = 0$, $E_z = 0$ for $r = a$
- $H_r = 0$ for $r = a$

The field distrubution solution for a cylindrical waveguide can be seperated into angualar and radial parts, whose solutions are then given by the Bessel/Neumann functions. These can be then substituted in the eq.(11)-(12) and upon imposing the constraints from the boundary condition gives us the corresponding TE- and TM-modes.

3.2 Cylindrical waveguides resonator

The time has now come, to talk about cavities itself. If we now insert two conducting plates perpendicular to the z-axis, the incoming wave is reflected completely, giving us standing waves. Because of this, the z-dependence changes like:

$$a \cdot e^{ikz} \implies A \cdot \sin(kz + \phi_0)$$

The following condition is imposed so that the longitudinal boundary conditions are fulfilled: $k = p \cdot \pi/l$. The longitudinal field looks like:

$$\mathbf{TE_{mnp} - Modes} : H_z = H_{mn} \cdot J_m(k_c r) \cos(m\phi) \cdot \sin(p\pi/l \cdot z) \cdot e^{\omega_{mnp} t} \text{ where } k_c a = j'_{mn}$$
$$\mathbf{TM_{mnp} - Modes} : E_z = E_{mn} \cdot J'_m(k_c r) \cos(m\phi) \cdot \cos(p\pi/l \cdot z) \cdot e^{\omega_{mnp} t} \text{ where } k_c a = j_{mn}$$

For resonant frequency, we have:

$$\omega_{mnp} = c \cdot \sqrt{(j_{mn}/a)^2 + (p\pi/l)^2}$$

Here, J_m is the m-th Bessel function, $J_m^{'}$ is its derivative. And j_{mn} and $j_{mn}^{'}$ are the n-th zeropoints of the m-th Bessel function and its derivative.
The resonant modes can be written in the linear form as:

$$(d\nu)^2 = \left(\frac{c j_{mn}^{(')}}{\pi}\right)^2 + \left(\frac{c}{2}\right)^2 p^2 \left(\frac{d}{l}\right)^2 \tag{13}$$

where $d = 2 \cdot a$ is the diameter of the cavity.