cavities

The following is a link to different modes in electromagnetism.

Waveguides

A waveguide is a structure that guides waves, such as electromagnetic waves or sound, with minimal loss of energy by restricting the transmission of energy to one direction. Without the physical constraint of a waveguide, wave intensities decrease according to the inverse square law as they expand into three dimensional space.

There are different types of waveguides for different types of waves. The original and most common meaning is a hollow conductive metal pipe used to carry high frequency radio waves, particularly microwaves. Dielectric waveguides are used at higher radio frequencies, and transparent dielectric waveguides and optical fibers serve as waveguides for light. In acoustics, air ducts and horns are used as waveguides for sound in musical instruments and loudspeakers, and specially-shaped metal rods conduct ultrasonic waves in ultrasonic machining.

Principle

Waves propagate in all directions in open space as spherical waves. The power of the wave falls with the distance R from the source as the square of the distance (inverse square law). A waveguide confines the wave to propagate in one dimension, so that, under ideal conditions, the wave loses no power while propagating. Due to total reflection at the walls, waves are confined to the interior of a waveguide.

Propagation mode and cutoff frequencies

A propagation mode in a waveguide is one solution of the wave equations, or, in other words, the form of the wave. Due to the constraints of the boundary conditions, there are only limited frequencies and forms for the wave function which can propagate in the waveguide. The lowest frequency in which a certain mode can propagate is the cutoff frequency of that mode. The mode with the lowest cutoff frequency is the fundamental mode of the waveguide, and its cutoff frequency is the waveguide cutoff frequency.

Interlude - Helmholtz equation
The Helmholtz equation is given by:

$$\nabla^2 f = -k^2 f$$
$$\text{where } k^2 \text{ is the eigenvalue and } f \text{ is the eigenfunction.}$$

In the spherical coordinates, the solution to the equation is given by the spherical Besel function and *spherical harmonics.*

Propagation modes are computed by solving the Helmholtz equation alongside a set of boundary conditions depending on the geometrical shape and materials bounding the region. The usual assumption for infinitely long uniform waveguides allows us to assume a propagating form for the wave, i.e. stating that every field component has a known dependency on the propagation direction (i.e. z). More specifically, the common approach is to first replace all unknown time-varying unknown fields $u(x, y, z, t)$ (assuming for simplicity to describe the fields in cartesian components) with their complex phasors representation $U(x, y, z)$, sufficient to fully describe any infinitely long single-tone signal at frequency f, (angular frequency $\omega = 2\pi f$, and rewrite the Helmholtz equation and boundary conditions accordingly. Then, every unknown field is forced to have a form like $U(x, y, z) = \hat{U}(x, y)e^{-\gamma z}$, where the γ term represents the propagation constant (still unknown) along the direction along which the waveguide extends to infinity. The Helmholtz equation can be rewritten to accommodate such form and the resulting equality needs to be solved for γ and $\hat{U}(x, y)$, yielding in the end an eigenvalue equation for γ and a corresponding eigenfunction $\hat{U}(x, y)_\gamma$ for each solution of the former.

The propagation constant γ of the guided wave is complex, in general. For a lossless case, the propagation constant might be found to take on either real or imaginary values, depending on the chosen solution of the eigenvalue equation and on the angular frequency ω. When γ is purely real, the mode is said to be "below cutoff", since the amplitude of the field phasors tends to exponentially decrease with propagation; an imaginary γ, instead, represents modes said to be "in propagation" or "above cutoff", as the complex amplitude of the phasors does not change with z.

Impedance matching

In circuit theory, the impedance is a generalization of electrical resistance in the case of alternating current, and is measured in ohms (Ω). A waveguide in circuit theory is described by a transmission line having a length and characteristic impedance. In other words, the impedance indicates the ratio of voltage to current of the circuit component (in this case a waveguide) during propagation of the wave. This description of the waveguide was originally intended for alternating current, but is also suitable for electromagnetic and sound waves, once the wave and material properties (such as pressure, density, dielectric constant) are properly converted into electrical terms (current and impedance for example).

Impedance matching is important when components of an electric circuit are connected (waveguide to antenna for example): The impedance ratio determines how much of the wave is transmitted forward and how much is reflected. In connecting a waveguide to an antenna a complete transmission is usually required, so an effort is made to match their impedances.

An impedance mismatch creates a reflected wave, which added to the incoming waves creates a standing wave. An impedance mismatch can be also quantified with the standing wave ratio (SWR or VSWR for voltage), which is connected to the impedance ratio and reflection coefficient by

Characteristic impedance

The characteristic impedance or surge impedance (usually written Z0) of a uniform transmission line is the ratio of the amplitudes of voltage and current of a single wave propagating along the line; that is, a wave travelling in one direction in the absence of reflections in the other direction. Alternatively, and equivalently, it can be defined as the input impedance of a transmission line when its length is infinite. Characteristic impedance is determined by the geometry and materials of the transmission line and, for a uniform line, is not dependent on its length.

Termination impedance

In electronics, electrical termination is the practice of ending a transmission line with a device that matches the characteristic impedance of the line. Termination prevents signals from reflecting off the end of the transmission line. Reflections at the ends of unterminated transmission lines cause distortion which can produce ambiguous digital signal levels and mis-operation of digital systems. Reflections in analog signal systems cause such effects as video ghosting, or power loss in radio transmitter transmission lines.

Shunt impedance

In accelerator physics, shunt impedance is a measure of the strength with which an eigenmode of a resonant radio frequency structure (e.g., in a microwave cavity) interacts with charged particles on a given straight line, typically along the axis of rotational symmetry. If not specified further, the term is likely to refer to longitudinal effective shunt impedance.

Q factor

In physics and engineering, the quality factor or Q factor is a dimensionless parameter that describes how underdamped an oscillator or resonator is. It is defined as the ratio of the initial energy stored in the resonator to the energy lost in one radian of the cycle of oscillation. Q factor is alternatively defined as the ratio of a resonator's centre frequency to its bandwidth when subject to an oscillating driving force. These two definitions give numerically similar, but not identical, results. Higher Q indicates a lower rate of energy loss and the oscillations die out more slowly. A pendulum suspended from a high-quality bearing, oscillating in air, has a high Q, while a pendulum immersed in oil has a low one. Resonators with high quality factors have low damping, so that they ring or vibrate longer.

Damping

Damping is an influence within or upon an oscillatory system that has the effect of reducing or preventing its oscillation. In physical systems, damping is produced by processes that dissipate the energy stored in the oscillation.

The damping ratio is a dimensionless measure describing how oscillations in a system decay after a disturbance. Many systems exhibit oscillatory behavior when they are disturbed from their position of static equilibrium.
The damping ratio is a system parameter, denoted by ζ (zeta), that can vary from undamped ($\zeta = 0$), underdamped ($\zeta < 1$) through critically damped ($\zeta = 1$) to overdamped ($\zeta > 1$).

The general equation for an exponentially damped sinusoid may be represented as:

$$y(t) = A \cdot e^{-\lambda t} \cdot \cos(\omega t - \phi)$$

- Damping ratio: ζ is a non-dimensional charazterization of the decay rate relative to the frequency, approximately $\zeta = \lambda/\omega$, or exactly $\zeta = \lambda\sqrt{\lambda^2 + \omega^2} < 1$.
- Q factor: $Q = 1/(2\zeta) = \alpha/\omega_n$ (α is the exponential decay rate and ω_n is the natural frequency of the system) is another non-dimensional characterization of the amount of damping; high Q indicates slow damping relative to the oscillation.

Mode (electromagnetism)

The mode of electromagnetic radiation describes the field pattern of the propagating waves. Electromagnetic modes are analogous to the normal modes of vibration in other systems, such as mechanical systems.

- Free space modes
  Plane waves, waves in which the electric and magnetic fields are both orthogonal to the direction of travel of the wave. These are the waves that exist in free space far from any antenna.
- Modes in waveguides and transmission lines
  - Transverse modes, modes that have at least one of the electric field and magnetic field entirely in a transverse direction.
    - Transverse electromagnetic mode (TEM), as with a free space plane wave, both the electric field and magnetic field are entirely transverse.
    - Transverse electric (TE) modes, only the electric field is entirely transverse. Also notated as H modes to indicate there is a longitudinal magnetic component.
    - Transverse magnetic (TM) modes, only the magnetic field is entirely transverse. Also notated as E modes to indicate there is a longitudinal electric component.
  - Hybrid electromagnetic (HEM) modes, both the electric and magnetic fields have a component in the longitudinal direction. They can be analysed as a linear superposition of the corresponding TE and TM modes.
    - HE modes, hybrid modes in which the TE component dominates.
    - EH modes, hybrid modes in which the TM component dominates.

Evanescent field/Evanescent wave

In electromagnetics, an evanescent field, or evanescent wave, is an oscillating electric and/or magnetic field that does not propagate as an electromagnetic wave but whose energy is spatially concentrated in the vicinity of the source (oscillating charges and currents). Even when there is a propagating electromagnetic wave produced (e.g., by a transmitting antenna), one can still identify as an evanescent field the component of the electric or magnetic field that cannot be attributed to the propagating wave observed at a distance of many wavelengths (such as the far field of a transmitting antenna).

A hallmark of an evanescent field is that there is no net energy flow in that region. Since the net flow of electromagnetic energy is given by the average Poynting vector, this means that the Poynting vector in these regions, as averaged over a complete oscillation cycle, is zero.

Although all electromagnetic fields are classically governed according to Maxwell's equations, different technologies or problems have certain types of expected solutions, and when the primary solutions involve wave propagation the term "evanescent" is frequently applied to field components or solutions which do not share that property. For instance, the propagation constant of a hollow metal waveguide is a strong function of frequency (a so-called dispersion relation). Below a certain frequency (the cut-off frequency) the propagation constant becomes an imaginary number. A solution to the wave equation having an imaginary wavenumber does not propagate as a wave but falls off exponentially, so the field excited at that lower frequency is considered evanescent. It can also be simply said that propagation is "disallowed" for that frequency. The formal solution to the wave equation can describe modes having an identical form, but the change of the propagation constant from real to imaginary as the frequency drops below the cut-off frequency totally changes the physical nature of the result. The solution may be described as a "cut-off mode" or an "evanescent mode"; while a different author will just state that no such mode exists. Since the evanescent field corresponding to the mode was computed as a solution to the wave equation, it is often discussed as being an "evanescent wave" even though its properties (such as not carrying energy) are inconsistent with the definition of wave.

In summary then, the appropriate expressions for the TE^z_{mn} modes are, according to (8-1), (8-14a), and (8-14b),

$$\underline{\mathrm{TE}^{+z}_{mn}}$$

$$E_x^+ = A_{mn}\frac{\beta_y}{\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z} \tag{8-15a}$$

$$E_y^+ = -A_{mn}\frac{\beta_x}{\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z} \tag{8-15b}$$

$$E_z^+ = 0 \tag{8-15c}$$

$$H_x^+ = A_{mn}\frac{\beta_x\beta_z}{\omega\mu\varepsilon}\sin(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z} \tag{8-15d}$$

$$H_y^+ = A_{mn}\frac{\beta_y\beta_z}{\omega\mu\varepsilon}\cos(\beta_x x)\sin(\beta_y y)e^{-j\beta_z z} \tag{8-15e}$$

$$H_z^+ = -jA_{mn}\frac{\beta_c^2}{\omega\mu\varepsilon}\cos(\beta_x x)\cos(\beta_y y)e^{-j\beta_z z} \tag{8-15f}$$

where

$$\beta_c^2 \equiv \left(\frac{2\pi}{\lambda_c}\right)^2 = \beta^2 - \beta_z^2 = \beta_x^2 + \beta_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 \tag{8-15g}$$

$$\boxed{(f_c)_{mn} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}} \qquad \begin{array}{l} m = 0, 1, 2, \ldots \\ n = 0, 1, 2, \ldots \\ m \text{ and } n \text{ not zero simultaneously} \end{array} \tag{8-16}$$

where $(f_c)_{mn}$ represents the cutoff frequency of a given mn mode. Modes that have the same cutoff frequency are called *degenerate*.

$$(\beta_z)_{mn} = \begin{cases} \pm\sqrt{\beta^2 - \beta_c^2} = \pm\beta\sqrt{1 - \left(\frac{\beta_c}{\beta}\right)^2} \\ \quad = \pm\beta\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2} = \pm\beta\sqrt{1 - \left(\frac{f_c}{f}\right)^2} & \text{for } \beta > \beta_c,\ f > f_c \quad \text{(8-17a)} \\ 0 & \text{for } \beta = \beta_c,\ f = f_c \quad \text{(8-17b)} \\ \pm j\sqrt{\beta_c^2 - \beta^2} = \pm j\beta\sqrt{\left(\frac{\beta_c}{\beta}\right)^2 - 1} \\ \quad = \pm j\beta\sqrt{\left(\frac{\lambda}{\lambda_c}\right)^2 - 1} = \pm j\beta\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} & \text{for } \beta < \beta_c,\ f < f_c \quad \text{(8-17c)} \end{cases}$$

In order for the waves to be traveling in the $+z$ direction, the expressions for β_z as given by (8-17a) through (8-17c) reduce to

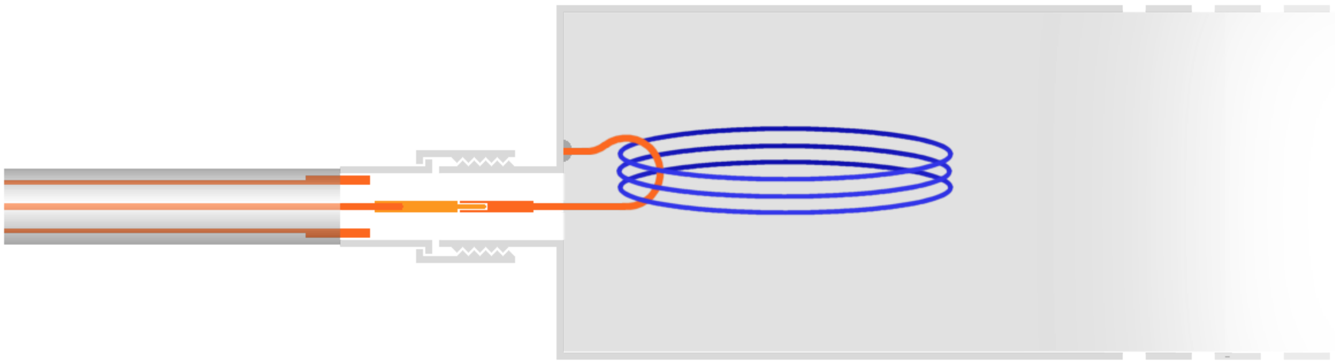
$$\boxed{(\beta_z)_{mn} = \begin{cases} \beta\sqrt{1 - \left(\frac{\lambda}{\lambda_c}\right)^2} = \beta\sqrt{1 - \left(\frac{f_c}{c}\right)^2} & \text{for } f > f_c \quad \text{(8-18a)} \\ 0 & \text{for } f = f_c \quad \text{(8-18b)} \\ -j\beta\sqrt{\left(\frac{\lambda}{\lambda_c}\right)^2 - 1} = -j\beta\sqrt{\left(\frac{f_c}{f}\right)^2 - 1} & \text{for } f < f_c \quad \text{(8-18c)} \end{cases}}$$

"Substituting the expressions for βz as given by (8-18a) through (8-18c) in the expressions for E and H as given by (8-15a) through (8-15f), it is evident that (8-18a) leads to propagating waves, (8-18b) to standing waves, and (8-18c) to evanescent (reactive) or nonpropagating waves. Evanescent fields are exponentially decaying fields that do not possess real power. Thus, (8-18b) serves as the boundary between propagating and nonpropagating waves, and it is usually referred to as the cutoff, which occurs when βz = 0. When the frequency of operation is selected to be higher than the value of $(f_c)_{mn}$ for a given mn mode, as given by (8-16), then the fields propagate unattenuated. If, however, f is selected to be smaller then $(f_c)_{mn}$, then the fields are attenuated. Thus, the waveguide serves as a high pass filter." - Advanced engineering electromagnetics, C. Balanis

Coupling

Waveguide coupling is a process in which the part of electromagnetic energy associated with one waveguide is shared with another waveguide.
The transmission, propagation, and reflection of guided waves happens in waveguides with or without excitation. Waveguide coupling is a process in which the part of electromagnetic energy associated with one waveguide is shared with another waveguide. Waveguides can be coupled to generators, excitation sources, or other waveguides to exchange energy in and out. When waveguides are coupled to power sources, various propagation modes are excited and there can be evanescent modes to store energy.

Loop coupling enables coupling to the magnetic field in the waveguide. In loop coupling, a conductor is inserted into the waveguide and bends into a loop. The center of the loop is at an equal distance from the top and bottom walls of the waveguide. When current flows through the loop, it generates a magnetic field component that couples with the waveguide fields. For high efficiency, the loop should be inserted at the point where the magnetic field is at its maximum strength.



Another way of injecting energy into a waveguide is by setting up an H field in the waveguide. This can be accomplished by inserting a small loop which carries a high current into the waveguide, as shown in the figure. A magnetic field builds up around the loop and expands to fit the waveguide. If the frequency of the current in the loop is within the bandwidth of the waveguide, energy will be transferred to the waveguide.
For the most efficient coupling to the waveguide, the loop is inserted at one of several points where the magnetic field will be of greatest strength. By loop coupling in a rectangular waveguide first a H field is produced which causes an E field.

Smith chart

The Smith chart, invented by Phillip H. Smith (1905–1987) and independently by Mizuhashi Tosaku, is a graphical calculator or nomogram designed for electrical and electronics engineers specializing in radio frequency (RF) engineering to assist in solving problems with transmission lines and matching circuits. The Smith chart can be used to simultaneously display multiple parameters including impedances, admittances, reflection coefficients, S_{nn}, scattering parameters, noise figure circles, constant gain contours and regions for unconditional stability, including mechanical vibrations analysis.

The Smith chart is most frequently used at or within the unity radius region. However, the remainder is still mathematically relevant, being used, for example, in oscillator design and stability analysis. While the use of paper Smith charts for solving the complex mathematics involved in matching problems has been largely replaced by software based methods, the Smith chart is still a very useful method of showing how RF parameters behave at one or more frequencies, an alternative to using tabular information. Thus most RF circuit analysis software includes a Smith chart option for the display of results and all but the simplest impedance measuring instruments can plot measured results on a Smith chart display.

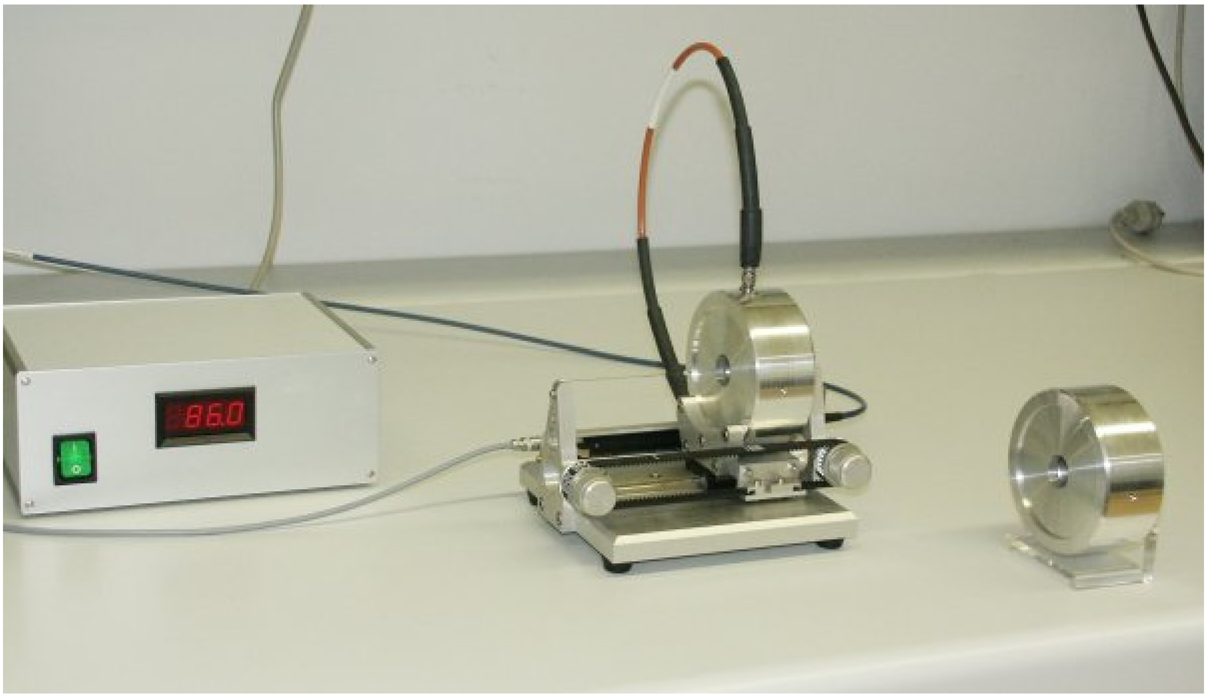
The experiment

Overview

This experiment of the master laboratory course introduces measurement methods and equipment for radio frequency (RF) applications, such as transmission lines and cavities. As cavities play a major role in various fields of physics and engineering, their properties are studied with emphasis on their usage in accelerator physics. For this, high-frequency electro-magnetic waves are coupled into a resonator whose response is monitored and fundamental operating parameters are measured and deduced.
The injected RF power is described by the reflection coefficient, stating the ratio of injected and reflected power. This important quantity can be measured with a vector network analyzer, which visualizes the absolute value of the complex reflection coefficient (scalar measurement) or represents it as vector in the complex plane (vectorial measurement). The frequency of the RF power lies in the range of some GHz (microwaves).
Properties resulting from of RF power transmission line compositions (especially coaxial cables) are examined in a short measurement at the beginning of the experiment.

Setup

Photographs of the experimental setup are shown below. First, the two microwave cavities with different coupling positions (top, side) are shown. The coupling mechanism consists of a wire loop extending into the resonator volume. One resonator is mounted onto a linear traveling stage whose position is electronically indicated. The second image shows the vector network analyzer performing a scalar reflection measurement on a microwave cavity.

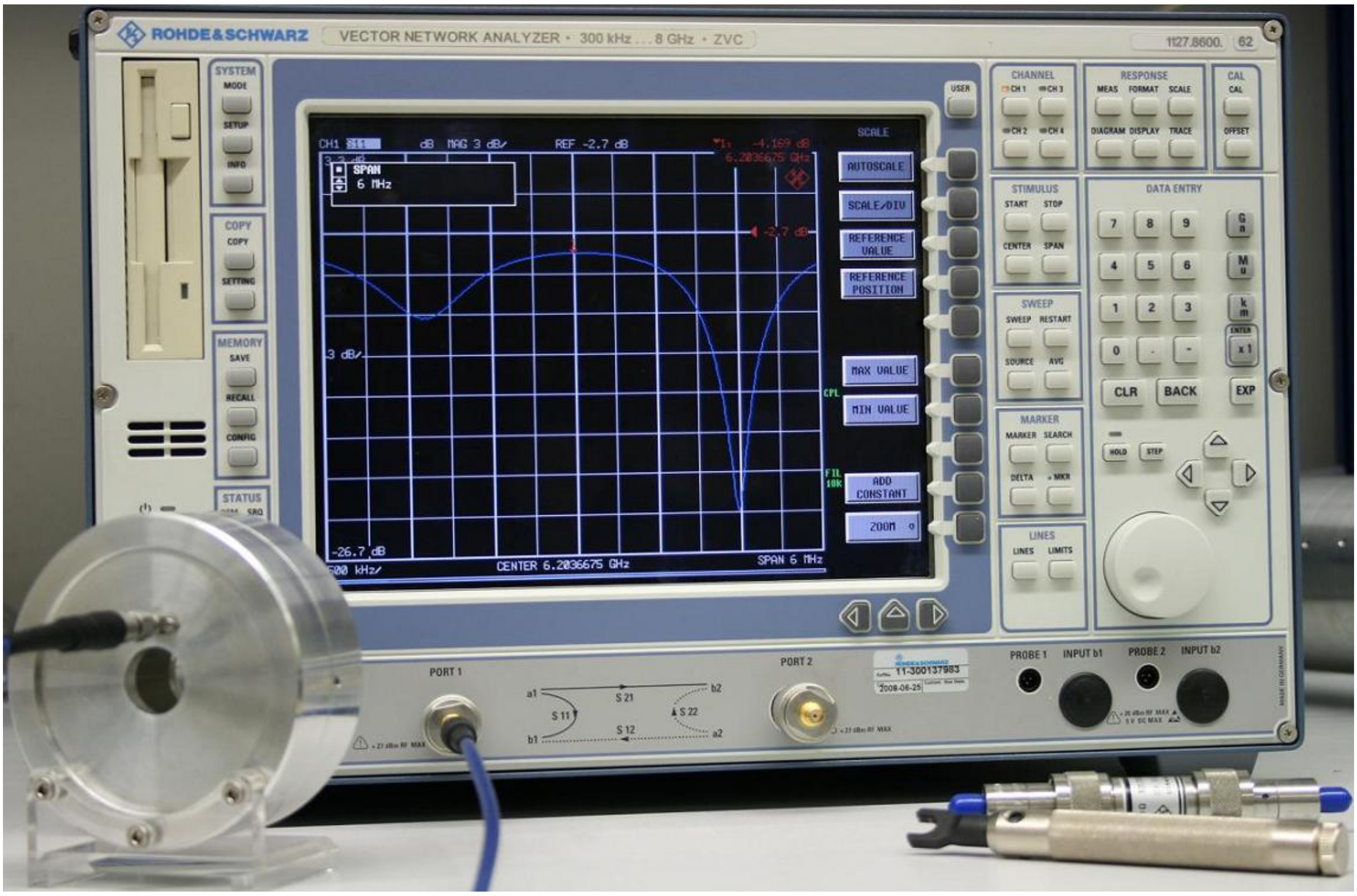


Resonant and non-resonant bead-pull measurement method

One of the two resonators is mounted onto a traveling platform which allows to perform a precise bead-pull measurement, for which a dielectric baffle (a teflon bead) is moved along the central axis of the resonator. The distortion of the resonance frequency allows the determination of the electric field distribution within the resonator. This allows to calculate an important figure in accelerator physics: the shunt impedance (the real cavity resistance in the case of resonance). The bead's position is measured through a sliding contact potentiometer, whose drop-off voltage is converted to millimeters.

Measurement of the resonator modes

To study the basic properties of a resonator, the frequencies of higher order resonator modes are measured (at certain frequencies the geometry of the cavity allows the formation of standing waves for which the RF injection is very efficient) and compared to theoretical calculations, which are based on the resonator's geometry. In addition, the shape of the resonance curve is analyzed. The network analyzer offers a scalar and vectorial measurement in the complex plane of the reflection coefficient.



What is dB?

The decibel (symbol: dB) is a relative unit of measurement equal to one tenth of a bel (B). It expresses the ratio of two values of a power or root-power quantity on a logarithmic scale. Two signals whose levels differ by one decibel have a power ratio of $10^{1/10}$ (approximately 1.26) or root-power ratio of $10^{1/20}$ (approximately 1.12).

A power quantity is a power or a quantity directly proportional to power, e.g., energy density, acoustic intensity, and luminous intensity. Energy quantities may also be labelled as power quantities in this context.

A root-power quantity is a quantity such as voltage, current, sound pressure, electric field strength, speed, or charge density, the square of which, in linear systems, is proportional to power. The term root-power quantity refers to the square root that relates these quantities to power.

The IEC Standard 60027-3:2002 defines the following quantities. The decibel (dB) is one-tenth of a bel: 1 dB = 0.1 B. The bel (B) is ½ ln(10) nepers: 1 B = ½ ln(10) Np.
Like the decibel, the neper is a unit in a logarithmic scale. While the bel uses the decadic (base-10) logarithm to compute ratios, the neper uses the natural logarithm, based on Euler's number (e ≈ 2.71828).

Power quantities

When referring to measurements of power quantities, a ratio can be expressed as a level in decibels by evaluating ten times the base-10 logarithm of the ratio of the measured quantity to reference value. Thus, the ratio of P (measured power) to P0 (reference power) is represented by LP, that ratio expressed in decibels, which is calculated using the formula:

$$L_p = \frac{1}{2}\ln(\frac{P}{P_o})\ Np = 10\ \log_{10}(\frac{P}{P_o})\ dB$$

In electronics, the decibel is often used to express power or amplitude ratios (as for gains) in preference to arithmetic ratios or percentages. One advantage is that the total decibel gain of a series of components (such as amplifiers and attenuators) can be calculated simply by summing the decibel gains of the individual components. Similarly, in telecommunications, decibels denote signal gain or loss from a transmitter to a receiver through some medium (free space, waveguide, coaxial cable, fiber optics, etc.) using a link budget.

The decibel unit can also be combined with a reference level, often indicated via a suffix, to create an absolute unit of electric power. For example, it can be combined with "m" for "milliwatt" to produce the "dBm". A power level of 0 dBm corresponds to one milliwatt, and 1 dBm is one decibel greater (about 1.259 mW).

In an optical link, if a known amount of optical power, in dBm (referenced to 1 mW), is launched into a fiber, and the losses, in dB (decibels), of each component (e.g., connectors, splices, and lengths of fiber) are known, the overall link loss may be quickly calculated by addition and subtraction of decibel quantities.

What is dBm?

dBm or dB_{mW} (decibel-milliwatts) is a unit of level used to indicate that a power level is expressed in decibels (dB) with reference to one milliwatt (mW). It is used in radio, microwave and fiber-optical communication networks as a convenient measure of absolute power because of its capability to express both very large and very small values in a short form. dBW is a similar unit, referenced to one watt (1000 mW).

The decibel (dB) is a dimensionless unit, used for quantifying the ratio between two values, such as signal-to-noise ratio. The dBm is also dimensionless, but since it compares to a fixed reference value, the dBm rating is an absolute one.

A power level of 0 dBm corresponds to a power of 1 milliwatt. A 10 dB increase in level is equivalent to a 10-fold increase in power. Therefore, a 20 dB increase in level is equivalent to a 100-fold increase in power. A 3 dB increase in level is approximately equivalent to doubling the power, which means that a level of 3 dBm corresponds roughly to a power of 2 mW. Similarly, for each 3 dB decrease in level, the power is reduced by about one half, making −3 dBm correspond to a power of about 0.5 mW.

To express an arbitrary power P in mW as x in dBm:

$$x = 10\log_{10}\frac{P}{1mW}$$

A brief summary

- We start with a waveguide and derive the resonant modes from the Maxwell's equations.
- First, we consider a propagation in cylindrical waveguides and see the different modes of propagation (modes of waveguide).
- Then, we transition from waveguide to cavity by introducing two conducting plates. This introduces additional longitudinal boundary conditions and causes formation of plane waves.

Propagation of waves and boundary condition

Wave equation:

$$\Delta\vec{E}(\vec{r}) + \frac{\omega^2}{c^2}\vec{E}(\vec{r}) = 0$$

$$\Delta\vec{H}(\vec{r}) + \frac{\omega^2}{c^2}\vec{H}(\vec{r}) = 0$$

Waveguides

Begin with a general waveguide aligned in the z-direction. For longitudinal fields:

$$\Delta_\perp E_z + k_c^2 E_z = 0 \qquad \text{mit} \qquad \boxed{\frac{\omega^2}{c^2} - k^2 = k_c^2}$$

$$\Delta_\perp H_z + k_c^2 H_z = 0$$

(dispersion relation of the waveguide)

k_c is the critical wavenumber is a characteristic of the cavity.

$$ik_c^2\vec{E}_\perp = k\vec{\nabla}_\perp E_z + \omega\mu_0\vec{\nabla}_\perp H_z \times \hat{e}_z$$

$$ik_c^2\vec{H}_\perp = k\vec{\nabla}_\perp H_z - \omega\varepsilon_0\vec{\nabla}_\perp E_z \times \hat{e}_z$$

We can classify different possible waves as:

1. $k_c^2 = 0$:

- Gives HE or EH hybrid waves (recall that HE = TE + TM; TE>TM & EH = TE + TM; TE<TM)
- Or TEM waves

2. $k_c^2 \neq 0$
   No propagation for $\omega \leq c \cdot k_c$: evanescent waves $\longleftrightarrow$ cut-off

- TE waves or
- TM waves

Below critical (wavenumber) frequency ($\omega_c = c \cdot k_c$), no propagation of waves in the waveguide

For cylindrical waveguide (with inner radius a), transverse fields:

$$ik_c^2\vec{E}_\perp = k\left\{\frac{\partial E_z}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial E_z}{\partial \varphi}\hat{e}_\varphi\right\} - \omega\mu_0\left\{\frac{\partial H_z}{\partial r}\hat{e}_\varphi - \frac{1}{r}\frac{\partial H_z}{\partial \varphi}\hat{e}_r\right\}$$

$$ik_c^2\vec{H}_\perp = k\left\{\frac{\partial H_z}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial H_z}{\partial \varphi}\hat{e}_\varphi\right\} + \omega\varepsilon_0\left\{\frac{\partial E_z}{\partial r}\hat{e}_\varphi - \frac{1}{r}\frac{\partial E_z}{\partial \varphi}\hat{e}_r\right\}$$

[insert page number 8 and 9 here]

Eigenmodes of cylindrical resonator

Insert conducting plates, perpendicular the z-axis. Incoming wave reflected off completely, giving standing waves.

Inserting eigenmodes of the waveguide, longitudinal fields become:

TE_{mnp}-Modes: $H_z = H_{mn} \cdot J_m(k_c r) \cdot \cos(m\varphi) \cdot \sin(p\pi/l \cdot z) \cdot e^{i\omega_{mnp}t}$

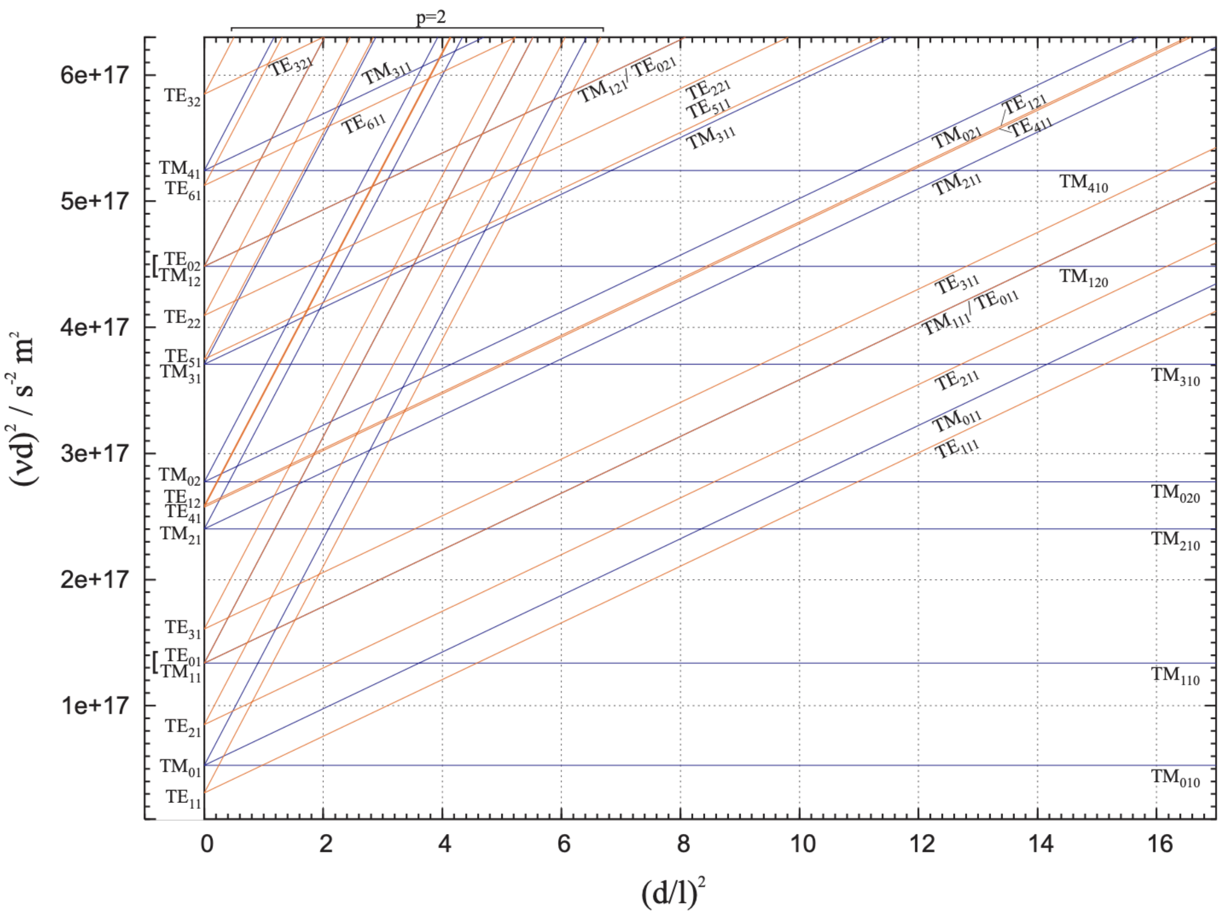
TM_{mnp}-Modes: $E_z = E_{mn} \cdot J_m(k_c r) \cdot \cos(m\varphi) \cdot \cos(p\pi/l \cdot z) \cdot e^{i\omega_{mnp}t}$

For the resonant frequencies one has: $\omega_{mnp} = c \cdot \sqrt{(j_{mn}/a)^2 + (p\pi/l)^2}$

The formula for the resonant frequencies can be written as a linear equation as follows:

$$(d\nu)^2 = \left(\frac{c j^{(')}_{mn}}{\pi}\right)^2 + \left(\frac{c}{2}\right)^2 p^2 \left(\frac{d}{l}\right)^2$$

$d = 2a$ is the diameter, $j^{(')}_{mn}$ is the zero point of the Bessel function or its derivative. Plotting the lines of the different modes in a diagram one gets the so called mode map (here for p ≤ 2):



Zeroes of $J_m(x)$:

n	j_{0n}	j_{1n}	j_{2n}	j_{3n}	j_{4n}	j_{5n}
	-	0	0	0	0	0
1	2,40482	3,83171	5,13562	6,38016	7,58834	8,77148
2	5,52007	7,01559	8,41724	9,76102	11,06471	12,33860
3	8,65372	10,17347	11,61984	13,01520	14,37254	15,70017
4	11,79153	13,32369	14,79595	16,22347	17,61597	18,98013
5	14,93091	16,47063	17,95982	19,40942	20,82693	22,21780

Zeroes of $J'_m(x)$:

n	j'_{0n}	j'_{1n}	j'_{2n}	j'_{3n}	j'_{4n}	j'_{5n}
	0	-	0	0	0	0
1	3,83170	1,84118	3,05424	4,20119	5,31755	6,41562
2	7,01558	5,33144	6,70613	8,01524	9,28240	10,51986
3	10,17346	8,53632	9,96947	11,34592	12,68191	13,98719
4	13,32369	11,70600	13,17037	14,58585	15,96411	17,31284
5	16,47063	14,86359	16,34752	17,78875	19,19603	20,57551

Therefore the corresponding TE- and TM-Modes have the same resonant frequencies:

$$TE_{0np} = TM_{1np} \text{ for arbitrary values of n and p}$$

Cavity as an oscillating circuit

Characteristic quantities in the unloaded case

[page number 13]
Bunch of formulas for:

- voltages
- current
- time constant
- angular resonant frequency
- quality factor

And the case for weakly damped oscillation.

Driven oscillations

Use an external current as a driving force on the oscillation.
$I_{ext} = \hat{I} \cdot e^{i\omega t}$ for external current. Using $U = \hat{U} \cdot e^{i\omega t}$, the solution for the differential equation becomes:

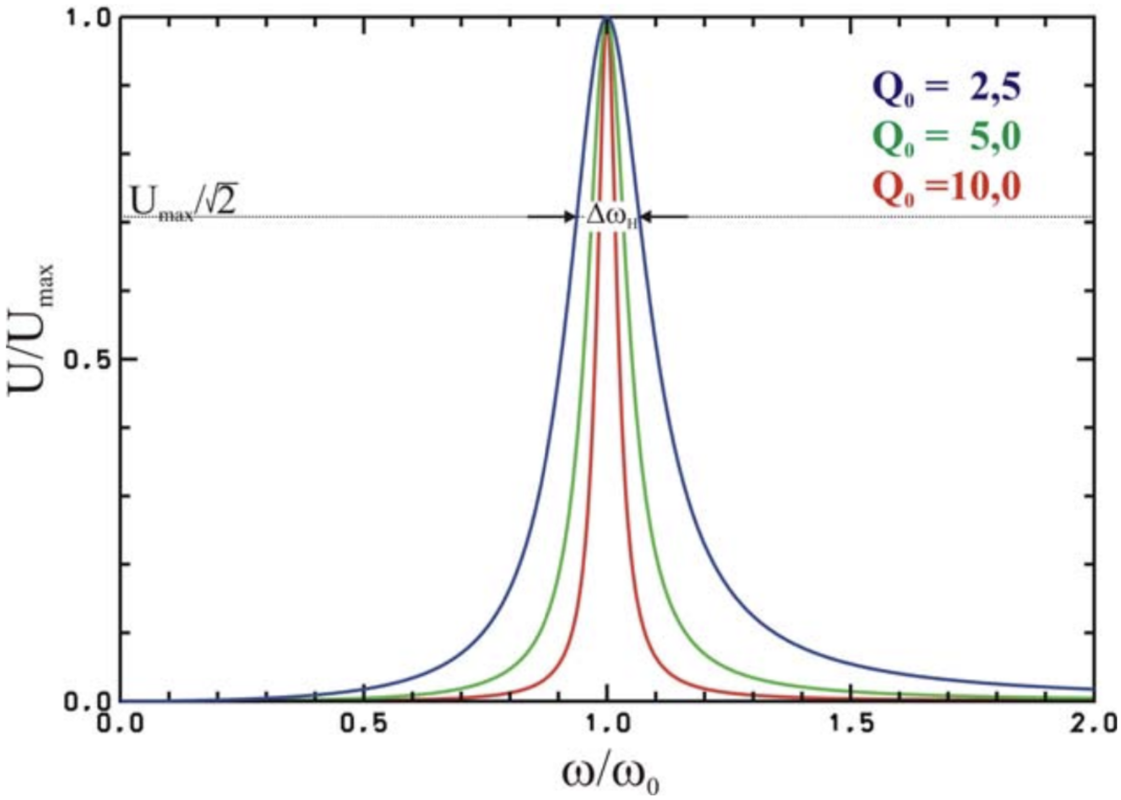
$$\hat{U} = \frac{i\omega \hat{I}_{ext} / C}{\omega_0^{\ 2} - \omega^2 + \frac{i\omega\omega_0}{Q_0}}$$

After substituting the value of Q factor,

$$\hat{U} = \frac{R \cdot \hat{I}_{ext}}{1 + iQ_0\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)} \overset{\Delta\omega \ll \omega_0}{\approx} \frac{R \cdot \hat{I}_{ext}}{1 + 2iQ_0 \frac{\Delta\omega}{\omega}}$$

$$\left|\hat{U}\right| = \frac{R \cdot \hat{I}_{ext}}{\sqrt{1 + Q_0^{\ 2}\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)^2}} \overset{\Delta\omega \ll \omega_0}{\approx} \frac{R \cdot \hat{I}_{ext}}{\sqrt{1 + 4Q_0^{\ 2}\left(\frac{\Delta\omega}{\omega}\right)^2}}$$

$$\tan\varphi = Q_0\left(\frac{\omega_0}{\omega} - \frac{\omega}{\omega_0}\right) \overset{\Delta\omega \ll \omega_0}{\approx} -2Q_0 \frac{\Delta\omega}{\omega}$$

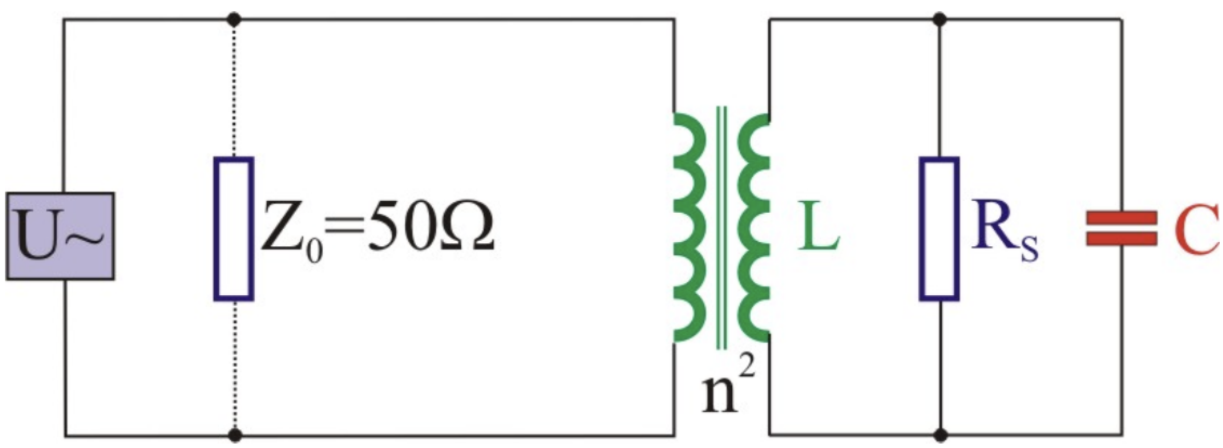


Unloaded quality factor can be determined by measuring FWHM (Full Width at Half Minimum) $\Delta\omega_H$

$$Q_0 = \frac{\omega_0}{\Delta\omega_H}, \qquad \Delta\omega_H = \underline{\textbf{full}}\ \text{width at half maximum at}\ \frac{U_{max}}{\sqrt{2}}$$

Loaded case by coupling to high frequency

In case of loop coupling, following equivalent circuit:



The purpose of the coupling is to carry the microwaves coming out of the generator to the resonator as complete as possible (without reflections). To achieve this, the transmission line from the generator to the resonator needs to be terminated by its characteristic wave impedance (typically 50Ω). The impedance of the resonator is a complex quantity and only real in the case of resonance. It is then called Shunt-impedance R_S.

The order of magnitude is typically MΩ! Therefore it is transformed down to $Z_a = R_s/n^2$ via loop coupling; in the equivalent circuit this corresponds to a transformer with the turn ratio n. The relevant quantity for the reflection is the ratio between termination impedance and characteristic impedance. We therefore define the coupling coefficient:

$$\boxed{\kappa = \frac{Z_a}{Z_0} = \frac{R_S}{n^2 \cdot Z_0}}$$

The resonator is additionally loaded by the external transmission line:

$$\frac{1}{R} = \frac{1}{R_S} + \frac{1}{n^2 \cdot Z_0} \quad \Rightarrow \quad \frac{1}{Q} = \frac{1}{Q_0} + \frac{1}{Q_{\mathrm{ext}}}.$$

The unloaded quality factor is reduced to Q because of the appearance of an additional external quality factor Q_{ext} . Formally taking the external power dissipation P_{ext} into account, we get the relations:

$$Q_{\mathrm{ext}} = \frac{\omega_0 \cdot W}{P_{\mathrm{ext}}} \quad \Rightarrow \quad Q = \frac{\omega_0 \cdot W}{P + P_{\mathrm{ext}}}$$

$$\kappa = \frac{Q_0}{Q_{ext}} = \frac{P_{ext}}{P} = \frac{R_S}{n^2 \cdot Z_0}$$

We get 3 cases:

1. $\kappa < 1$: under critical coupling, $Q > Q_o/2$
2. $\kappa = 1$: critical coupling, $Q = Q_o/2$, no reflection.

3. $\kappa < 1$: over critical coupling, $Q < Q_o/2$

If the coupling coefficient is known, we can determine the unloaded quality factor Q_o from the loaded quality factor Q

$$Q_0 = (1+\kappa)\cdot Q$$

The complex reflection coefficient

Dependence on termination impedance

Incoming wave: $\hat{U}_+, \hat{I}_+$, Reflected wave: $\hat{U}_-, \hat{I}_-$, in a conductor.
Complex reflection coefficient:

$$\rho = \frac{\hat{U}_-}{\hat{U}_+}$$

Termination impedance Z_a and characteristic impedance Z_o

$$Z_a = \frac{\hat{U}}{\hat{I}} = \frac{\hat{U}_+ + \hat{U}_-}{\hat{I}_+ + \hat{I}_-}, \qquad Z_0 = \frac{\hat{U}_+}{\hat{I}_+} = \frac{\hat{U}_-}{-\hat{I}_-}$$

$$Z_a = \frac{1+\rho_0}{1-\rho_0}\cdot Z_0 \qquad \Leftrightarrow \qquad \rho_0 = \frac{Z_a - Z_0}{Z_a + Z_0} = \frac{(Z_a/Z_0)-1}{(Z_a/Z_0)+1}$$

Reflection close to an "insulated resonance"(?)

Case of non-overlapping resonances, complex reflection coefficient is:

$$\rho_0(\Delta\omega) = \frac{\kappa - \left(1 + 2iQ_0 \,{}^{\Delta\omega}\!/\!_{\omega}\right)}{\kappa + \left(1 + 2iQ_0 \,{}^{\Delta\omega}\!/\!_{\omega}\right)}.$$

If the reflection coefficient is measured at a distance l to the coupling (this is due to the presence of a transmission line between the position where the measurement takes place and the coupling), twice the delay factor of the wave in the line is added.

$$\left| \rho(\Delta\omega) = \rho_0(\Delta\omega)\cdot e^{-2ikl} = \frac{\kappa - \left(1 + 2iQ_0 \,{}^{\Delta\omega}\!/\!_{\omega}\right)}{\kappa + \left(1 + 2iQ_0 \,{}^{\Delta\omega}\!/\!_{\omega}\right)} \cdot e^{-2ikl} \right|$$

Measurement of $|\rho|$

Resonance curve