# MA 225 Problem Set 4: induction 2

**facts and definitions** You will need the following definitions and facts (at some point):

**Definition 1.** The *Fibonacci numbers* are defined as follows:

$$f_k = \begin{cases} 1 & \text{if } k = 1 \text{ or } k = 2 \\ f_{k-1} + f_{k-2} & \text{if } k \geq 3 \end{cases}$$

**Definition 2.** $0! = 1$

**Principle of Mathematical Induction.** *Let $P(n)$ be an open sentence with universe the natural numbers. If $P(1)$ is true and $P(n)$ is inductive, then for any $n$, $P(n)$ is true.*

**Principle of Complete Induction.** *Let $P(n)$ be an open sentence with universe the natural numbers. If $P(1)$ is true and $P(n)$ is completely inductive, then for any $n$, $P(n)$ is true.*

**exercises** These problems don't require you to write proofs.

1. Explain why "$n$ is even" is completely inductive, but "$n$ is odd" is not completely inductive.
2. Is either of the above sentences inductive?
3. We showed that if $P(n)$ is inductive, then the set of values $n$ for which $P(n)$ is true must look like $\{n_0, n_0 + 1, n_0 + 2, \ldots\}$. Characterize what the set of values $n$ for which $Q(n)$ is true, assuming $Q(n)$ is completely inductive.

**proofs** Prove the following claims.

1. Let $P(n)$ be an inductive sentence. Then $P(n)$ is completely inductive.
2. ($\star$) In class we proved PCI, assuming PMI. Prove the converse: give a proof of the PMI that only assumes PCI.
3. The parity of the Fibonacci numbers follows the pattern: odd, odd, even, odd, odd, even, . . .
4. There are no common factors of $f_n$ and $f_{n+1}$, other than 1.
5. ($\star\star$) The *Tower of Hanoi* puzzle consists of $n$ disks of different radii, stacked in decreasing order of radius (so the largest disk is on the bottom) on one rod; two other rods are nearby. The goal of the game is to move the entire stack to another of the rods, in the same order. The rules are:
   - You may move the top disk of a stack onto another rod.
   - A disk may only be placed on top of a larger disk.
   - No other moves are allowed.

   If the Tower of Hanoi puzzle starts with $d$ disks, you can solve it in $2^d - 1$ moves.
6. $\displaystyle\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$.
7. ($\star\star$) The **product rule for higher derivatives**: given any $n \in \mathbb{N}$ and functions $f, g$ with at least $n$ derivatives, we have

$$\begin{aligned} \frac{d^n}{dx^n}[fg] =& B_{n,0} f^{(n)} g + B_{n,1} f^{(n-1)} g' + B_{n,2} f^{(n-2)} g'' + \cdots \\ & + B_{n,n-2} f'' g^{(n-2)} + B_{n,n-1} f' g^{(n-1)} + B_{n,n} f g^{(n)}. \end{aligned}$$

   (*Hint.* At some point you will need to "combine like terms".)
8. ($\star\star$) If $n \geq 3$, then the sum of the interior angles of a convex $n$-gon is $(n-2) \cdot 180°$.
9. Define the numbers $g_n$ as follows:

$$g_n = \begin{cases} 2 & \text{if } n = 1 \text{ or } n = 2 \\ g_{n-1} g_{n-2} & \text{if } n \geq 3 \end{cases}$$

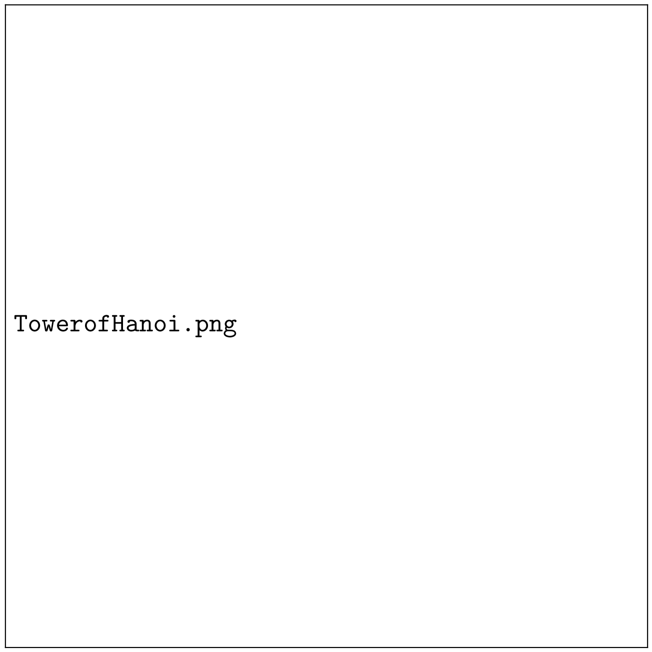


figure 1: A Tower of Hanoi puzzle in the midst of being solved ($d = 8$).

For all $n$, $g_n = 2^{f_n}$.

10. $f_k \leq 2^k$.
11. ($\star\star$) Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (These are the roots of the equation $x^2 - x - 1 = 0$.) Then

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

(*Hint.* You will need to use the fact that $\alpha$ and $\beta$ the solutions of the given equation.)

12. For any $n$ and any $0 \leq k \leq n$, $B_{n,k} = \frac{n!}{k!(n-k)!}$. (*Hint.* Induct on $n$.)
13. Prove two claims from Homework 3 using the Well-Ordering Principle.