MA 225 Problem Set 5

facts and definitions You will need the following definitions and facts (at some point):

Definition 1. We call a set A *open* if, for each $p \in A$, there is a $\delta > 0$ so that the δ-neighborhood of p, $V(p, \delta)$, has $V(p, \delta) \subseteq A$. We call a set A *closed* if its complement is open.

Definition 2. Given sets A and B, we define

$$A \setminus B = \{x \mid x \in A \wedge x \notin B\}$$

and $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Fact. *If $\alpha < \beta$, then the α- and β-neighborhoods of p are related as follows:*

$$V(p, \alpha) \subseteq V(p, \beta)$$

exercises These problems don't require you to write proofs.

1. Some of the following strings of symbols are ambiguous because they lack bracketing. Exhibit the ambiguity by supplying different possible bracketings, and explaining why the versions are different. Be careful – some may not be ambiguous at all!
   (a) $2^A \cap 2^B \setminus C \subseteq D \times 2^A$
   Not ambiguous. While it can be written as $(2^A \cap 2^B) \setminus C \subseteq D \times 2^A$ and $2^A \cap (2^B \setminus C) \subseteq D \times 2^A$. In the equation, there exists a $p \in 2^A \cap 2^B$, but not in C. In the second equation, p is in 2^B but not C, and since p is in 2^A, and not C, p would only pull elements not in C from 2^A.
   (b) $A \cap B \cap C \subseteq D \cup A \cup B$. Not ambiguous, as cap and cup are communative with themself.
   (c) $A \cap B \cup C$.
   $(A \cap B) \cup C$ and $A \cap (B \cup C)$. A main difference is that in the first bracketing, all elements in C are used. In the second one, however, terms in C are only used if in A as well.
2. Let A and B be sets. Consider the collection $\mathcal{D}$ of all sets C with the property $C \subseteq A$ and $C \subseteq B$.
   (a) Write the definition of $\mathcal{D}$ in set-builder notation.
   $\mathcal{D} = \{C | C \subseteq A \wedge C \subseteq B\}$
   (b) Between $A \in \mathcal{D}$ and $A \subseteq \mathcal{D}$, which is *sensible*? $\in$, as $\subseteq$ would imply A was a set of sets. Furthermore, if A were to be a set of sets, as since it's said $C \subseteq A$, this would imply that the elements of C were sets of sets, meaning A was a set of set of sets, and this would continue.
   (c) Between $A \in \mathcal{D}$ and $A \subseteq \mathcal{D}$, is either *true*? As we discussed earlier, A can't be a subset of $\mathcal{D}$. In would work only if $A \subseteq B$, as $A \subseteq A$.
   (d) Among $A \cap B \in \mathcal{D}$, $A \cup B \in \mathcal{D}$, $A \cap B \subseteq \mathcal{D}$, $A \cup B \subseteq \mathcal{D}$, are any true? $A \cap B$, as the intersection of A and B is a subset of A, and the intersection of A and B are a subset of B.
3. Let A and B be sets. Consider the collection $\mathcal{E}$ of all sets F with the property: for any C with $C \subseteq A$ and $C \subseteq B$, $C \subseteq F$.
   (a) Write out the definition of $\mathcal{E}$ in set-builder notation. $\mathcal{E} = \{F | C \subseteq A \wedge C \subseteq B \wedge C \subseteq F\}$

(b) Between $A \in \mathcal{E}$ and $A \subseteq \mathcal{E}$, which is *sensible*? Same problem as number 2b. So $A \in \mathcal{E}$ makes more sense

(c) Between $A \in \mathcal{E}$ and $A \subseteq \mathcal{E}$, is either *true*?As stated before, $A \subseteq \mathcal{E}$, so this is false. $A \in \mathcal{E}$ is true only when $A \in B$ and when $A \in F$

(d) Among $A \cap B \in \mathcal{E}$, $A \cup B \in \mathcal{E}$, $A \cap B \subseteq \mathcal{E}$, $A \cup B \subseteq \mathcal{E}$, are any true?

4. In Definition 1, what kind of thing must δ be? What kind of a thing must a neighborhood be? δ must be a real number, as it is used in an inequality. A neighborhood must be a subset.

5. Write out what it means for the set J to be *not open*:

   (a) Do this in symbols.
   $J = \{p | \forall \delta : \delta \leq 0 \vee V(p, \delta) \nsubseteq J\}$

   (b) Do this in English.For all δ in a J, if δ less than or equal to 0, or $V(p, \delta)$ is not a subset, then J is not open.

6. (a) Write a blueprint for the claim *If blah blah and such, then A is open.*
   If $\forall p \in A$, then there exists a δ which is greater than 0 and such that $V(p, \delta$, then A is open.

   (b) Write a blueprint for the claim *If blah blah and such, then A is closed.* If $\forall p \notin A$, then there exists a δ which is greater than 0 and such that $V(p, \delta$, then A is closed.

proofs Prove the following claims.

1. For any $n \in \mathbb{N}$, $\sum_{k=0}^{n} B_{n,k} = 2^n$.
2. If $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$, then $A = B$ and $B = C$.

   Claim 1. *If $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$, then $A = B$ and $B = C$.*

   Proof. Let $s \in B$. So $s \in C$ according to $B \subseteq C$. So $s \in A$ according to $C \subseteq A$. So $B \subseteq A$, and $A \subset B$ according to definition, therefore $A = B$.
   Let $p \in C$. So $p \in A$, according to $C \subseteq A$. So $p \in B$ according to $A \subseteq B$. So $B \subseteq C$, and $C \subseteq B$ according to definition, therefore $C = B$. □

3. $(\star)$ For any n, $\left(\bigcup_{j=1}^{n} A_j\right)^c = \bigcap_{j=1}^{n} A_j^c$. (*Hint.* n is a what?.)

   Claim 2. *For any n, $\left(\bigcup_{j=1}^{n} A_j\right)^c = \bigcap_{j=1}^{n} A_j^c$.*

   Proof. We can see that this function holds true for $n = 1$, as $\left(\bigcup_{j=1}^{1} A_j\right)^c = \bigcap_{j=1}^{1} A_j^c$ is equal to $A_1^c = A_1^c$, which is true, therefore the formula holds for the base case. Assume $\left(\bigcup_{j=1}^{n} A_j\right)^c = \bigcap_{j=1}^{n} A_j^c$. We can use proof by induction to prove this, as n is a natural number. We define k as k=n and evaluate for $k+1$ to show that:

$$\begin{aligned}
\bigcap_{j=1}^{k+1} A_j^c &= \bigcap_{j=1}^{k+1} A_j^c \\
&= \bigcap_{j=1}^{k} A_j^c \bigcap A_{k+1}^c \\
&= \left(\bigcup_{j=1}^{k} A_j\right)^c \bigcap A_{k+1}^c && \text{(Use inductive hypothesis)} \\
&= \left(\bigcup_{j=1}^{k} A_j \bigcup A_{k+1}\right)^c && \text{(Used de Morgan's Law)} \\
&= \left(\bigcup_{j=1}^{k+1} A_j\right)^c
\end{aligned}$$

   Therefore, through induction we can conclude that $\left(\bigcup_{j=1}^{n} A_j\right)^c = \bigcap_{j=1}^{n} A_j^c$ for any n. □

4. Some facts about sets:
   (a) $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$

Claim 3. $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$

Proof. Let $t \in (X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$. This means $t \in X$, but not in Y, and from this it is not in Z. This can be rewritten as $(X \setminus Z) \setminus (Y \setminus Z)$, as t is an element of X, but not of Z, and t is not in Y.
Let $n \in (X \setminus Z) \setminus (Y \setminus Z)$. Because n is never an element of $(X \setminus Z) \setminus (Y \setminus Z)$ by definition of \, but instead an element of X but not Y, this can be rewritten as $(X \setminus Y) \setminus Z$. Therefore $\subseteq (X \setminus Y) \setminus (X \setminus Z) \setminus (Y \setminus Z)$ Since the equation works for $\subseteq$ and $\supseteq$, $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$. □

(b) $(\star)$ $2^{A \cap B} = 2^A \cap 2^B$
Claim 4. $2^{A \cap B} = 2^A \cap 2^B$

Proof. Let $r \in 2^{A \cap B}$, where r is a set. We must do this by first showing that $2^{A \cap B} \subseteq 2^A \cap 2^B$. Let $p \in A \cap B$. p can be defined as all elements which are in both A and B, and r is defined as all combinations of p. Because of the definition of the powerset, the subsets which contain p will be able to be produced by A and B, and by finding the intersection of these set of subsets, .
Therefore, since $2^{A \cap B} \subseteq 2^A \cap 2^B$ and $2^A \cap 2^B \subseteq 2^{A \cap B}$ are true, $2^{A \cap B} = 2^A \cap 2^B$ is also true. □

(c) $2^A \cup 2^B \subseteq 2^{A \cup B}$
Claim 5. $2^A \cup 2^B \subseteq 2^{A \cup B}$

Proof. Let $p \in 2^A \cup 2^B$. So either $p \in 2^A$ or 2^B. All of these elements are in $2^{A \cup B}$, as this consists of all subsets of the elements of $A \cup B$, as well as more. Therefore, $2^A \cup 2^B \subseteq 2^{A \cup B}$. □

(d) $A \Delta B = B \Delta A$
Claim 6. $A \Delta B = B \Delta A$

Proof. Let $p \in A \Delta B$. So $p \in (A \setminus B) \cup (B \setminus A)$ by definition. Since the operation $\cup$ is communative for sets (sets are not ordered and union just takes terms from both sets, therefore union is communative), this can be rewriten as $p \in (B \setminus A) \cup (A \setminus B)$, or $B \Delta A$ by definition. Therefore $A \Delta B = B \Delta A$.
Since $B \Delta A \subseteq B \Delta A$, and $A \Delta B \subseteq B \Delta A$, it can be concluded that $A \Delta B = B \Delta A$. □

(e) $A \Delta B = (A \cup B) \setminus (A \cap B)$
Claim 7. $A \Delta B = (A \cup B) \setminus (A \cap B)$

Proof. Let $p \in A \Delta B$. So by definition, $p \in (A \setminus B) \cup (B \setminus A)$. So $p \in (A \setminus B$ or $p \in B \setminus A$, or $p \in B$, and $p \notin A$.
Case 1 ($p \in A \setminus B$): So $p \in A$ and $p \notin B$. So $p \in A \cup B$ and $p \notin A \cap B$ therefore $p \in (A \cup B) \setminus (A \cap B)$.
Case 2 ($p \in B \setminus A$): So $p \in B$ and $p \notin A$. So $p \in A \cup B$ and $p \notin A \cap B$. Therefore $p \in (A \cup B) \setminus (A \cap B)$.
Therefore $A \Delta B \subseteq (A \cup B) \setminus (A \cap B)$.

Let $q \in (A \cup B) \setminus (A \cap B)$. So by definition of \, ($q \in A$ or $q \in B$), and $q \notin (A \cap B)$.
Case 1 ($q \in A$): Since $q \in A$, $q \notin B$, as $q \notin A \cap B$. So $q \in A \setminus B$, therefore $q \in (A \setminus B) \cup (B \setminus A)$ (Definition of $A \Delta B$). Therefore
Case 2 ($q \in B$): Since $q \in B$, $q \notin A$, as $q \notin A \cap B$. So $q \in B \setminus A$, therefore

$q \in (A \setminus B) \cup (B \setminus A)$ (Definition of $A\Delta B$). Therefore $q \in A\Delta B$.
Since $q \in A\Delta B$ for all cases, and $A\Delta B \subseteq (A\cup B)\setminus(A\cap B)$, $A\Delta B = (A\cup B)\setminus(A\cap B)$.
Since $A\Delta B \subseteq (A \cup B) \setminus (A \cap B)$, and $(A \cup B) \setminus (A \cap B) \subseteq A\Delta B$, this means that $A\Delta B = (A \cup B) \setminus (A \cap B)$. □

(f) If $A \cup B \subseteq C \cup D$ and $A \cap B = \varnothing$, and $C \subseteq A$, then $B \subseteq D$.

Claim 8. *Given $A \cup B \subseteq C \cup D$ and $A \cap B = \varnothing$, and $C \subseteq A$, then $B \subseteq D$.*

Proof. Let $r \in C$. Then $C \cup B \subseteq C \cup D$, as $r \in C \cup A$, and since $r \in C$. Furthermore, $C \cup B = \varnothing$, as C is a subset of A, and which $A \cup B = \varnothing$. So since $C \cup B \subseteq C \cup D$, and since C and B share no terms, $B \subseteq D$. □

5. ($\star$) The empty set is open. (*Hint.* Go to Vegas.)

Claim 9. *The empty set is open.*

Proof. The definition of an open set is that for all $s \in S$, where S is a set, there also exists a term with another property. By definition, the empty set does not contain any s, therefore all elements, in this case none, satisfy the conditions of being an open set. □ □

6. ($\star\star$) If A is open and B is open, then $A \cup B$ is open. (Yes, you can do this even though you have no idea what an open set is.)

Claim 10. *Given A and B are open, $A \cup B$ is open.*

Proof. Let $p \in A \cup B$. Since $\forall a \in A, \exists \delta : \delta > 0 \wedge V(a, \delta) \subseteq A$, and $\forall b \in B, \exists \delta : \delta > 0 \wedge V(b, \delta) \subseteq B$, $A \cup B$ consist of all elements in A as well as B, and the definition of an open set is still upheld by the union, $A \cup B$ is open. □

7. ($\star$) If $A_1, \ldots, A_p$ are open sets, then $\bigcup_{k=1}^{p} A_k$ is an open set. (*Hint.* p is a what?.)

Claim 11. *Given $A_1, \ldots, A_p$ are open sets, then $\bigcup_{k=1}^{p} A_k$ is an open set.*

Proof. Through proof by induction, we can determine $\bigcup_{k=1}^{p} A_k$ is open. We can first verify the base case, which results in $\bigcup_{k=1}^{1} A_k = A_1$, which is true based on the definition of A_1. Defining $j = p$, we will show that $\bigcup_{k=1}^{j} A_k$ is open for j+1:

$$\begin{aligned}
\bigcup_{k=1}^{j+1} A_k &= \bigcup_{k=1}^{j+1} A_k \\
&= \bigcup_{k=1}^{p} A_k \cup A_{j+1} \\
&= \bigcup_{k=1}^{p} A_k \cup A_{j+1}
\end{aligned}$$

According to our inductive assumption, $\bigcup_{k=1}^{j} A_k$ is an open set, as well as A_{j+1} by definition (they defined $A_1, A_2, ...A_p$, and the next term A_{p+1} as open). Since we proved in the previous proof that the union of two open sets is open, by proof of mathematical induction $\bigcup_{k=1}^{p} A_k$ is open. $\square$

8. ($\star$) If A is closed and B is closed, then $A \cap B$ is closed.

   Claim 12. *If A is closed and B is closed, then $A \cap B$ is closed.*

   Proof. Given both A and B are closed, A^c is open and B^c is open by definition. We can see from a previous proof that if A^c is opne and B^c is open, then $A^c \cup B^c$ is open as well. Using de Morgans law, we realize this is equal to $(A \cap B)^c$. Because this set is open, and it is the complement of $A \cap B$, $A \cap B$ is closed. $\square$

9. ($\star\star$) If A is open and B is open, then $A \cap B$ is open. (Again, you can do this without any knowledge of what a neighborhood is.)

   Claim 13. *If A is open and B is open, then $A \cap B$ is open. (Again, you can do this without any knowledge of what a neighborhood is.)*

   Proof. Let $p \in A \cap B$. By definition, we know that $p \in A$ and $p \in B$. We also know that because A is open, then for all $p \in A$, there exists a subset $V(p, \delta) \subseteq A$, and the same for B. Therefore, since for all p in $A \cap B$, there exists a subset $V(p, \delta)$, the set is open. $\square$