



## MA 225 Problem Set 3: induction 1

**facts and definitions** You will need the following definitions and facts (at some point):

**Definition 1.** We'll call strings of the symbols  $a$  and  $b$  *words*. Consider a class of words, the *legal words*, defined as follows.  $aba$  is a legal word. If  $W$  is a legal word, then so are  $abW$ ,  $aWb$ ,  $Wab$ ,  $baW$ ,  $bWa$ , and  $Wba$ . No words other than those obtained in this way are legal.

**Definition 2.** The *binomial coefficients* are a collection of natural numbers  $B_{n,k}$ , defined for a pair of nonnegative integers  $n, k$  with  $0 \leq k \leq n$ , as follows:

$$B_{n,k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ B_{n-1,k-1} + B_{n-1,k} & \text{if } 1 \leq k \leq n-1 \end{cases}$$

**Definition 3.** *Higher derivatives* are defined as follows: the zeroth derivative of a function  $f$  is  $f$  itself; we write

$$\frac{d^0}{dx^0} f(x) = f(x) = f^{(0)}(x)$$

For  $n \in \mathbb{N}$ , we define the  $n^{\text{th}}$  derivative as the derivative of the  $(n-1)^{\text{st}}$  derivative:

$$\frac{d^n}{dx^n} f(x) = \frac{d}{dx} \left[ \frac{d^{n-1}}{dx^{n-1}} f(x) \right]$$

$$f^{(n)}(x) = \left( f^{(n-1)} \right)'(x)$$

**Integration by Parts.** For any two differentiable functions  $f, g$ , we have

$$\int f(x)g(x) \, dx = f(x) \left[ \int g(x) \, dx \right] - \int f'(x) \left[ \int g(x) \, dx \right] \, dx,$$

provided we adopt the convention that the constants of integration in both instances of  $\int g(x) \, dx$  must be the same.

**exercises** These problems don't require you to write proofs.

1. Compute  $B_{n,k}$  for  $0 \leq n \leq 6$ .
2. Explain why the definition given for  $B_{n,k}$  actually constitutes a definition; that is, why we can compute  $B_{n,k}$  for any choice of  $n, k$  with  $0 \leq k \leq n$ .
3. Identify, explain, and correct any correctable flaws in the following proofs:

**Claim 1.**  $n^2 + n$  is odd.

*Proof.*  $n = 1$  is odd.

Inductively, assume  $n^2 + n$  is odd. Then

$$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = n^2 + n + 2(n+1)$$

so  $(n+1)^2 + (n+1)$  is the sum of an odd number and an even number, hence itself odd. This completes the inductive step.  $\square$

**Claim 2.** Every natural number is odd.

*Proof.*  $k = 1$  is clearly odd.

Inductively, assume  $k$  is odd. This means there is an integer  $p$  so that  $k = 2p + 1$ . Consider  $p + 1$ . Clearly  $2(p+1) + 1$  is odd.  $\square$

**Claim 3.** *Every natural number is both even and odd.*

*Proof.* Assume that  $k$  is both even and odd. Consider  $k + 1$ .

Since  $k$  is even, there is  $p$  with  $k = 2p$ . So  $k + 1 = 2p + 1$  is odd. Since  $k$  is odd, there is  $q$  with  $k = 2q + 1$ . So

$$k + 1 = (2q + 1) + 1 = 2(q + 1)$$

is even. □

**Claim 4.**  $n^3 - n$  is divisible by 6.

*Proof.* For the base case: when  $n = 1$ ,  $n^3 - n = 0 = 6 \cdot 0$ .

Now proceed inductively. Assume that for all  $k$ ,  $k^3 - k$  is divisible by 6. Then, since  $n + 1$  is one possible value of  $k$ , we have that  $(n + 1)^3 - (n + 1)$  is divisible by 6. □

**proofs** Prove the following claims.

1. For any natural number  $p$ , 8 divides  $5^{2p} - 1$ .

**Claim 5.** *For any natural number  $p$ , 8 divides  $5^{2p} - 1$*

**Proof by Induction 1.** We must first confirm that the base case, or  $5^{2p} - 1$  is divisible by 8. We determine that the value of this function at 1 is 24, which is  $8 * 3$ , so this statement is valid. Next, we can see that the solution to this is  $8m$ , where  $m$  is a natural number. We can now use mathematical induction and find:

$$\begin{aligned} 8m &= 5^{2(n+1)} - 1 \\ (\text{Solving for } n+1, \text{ since we know that this is true when } n \text{ is true when } n=1) \\ &= 5^2 * 5^{2n} - 1 \\ &= 25(8m + 1) - 1 \\ &= 8(25m + 3) \end{aligned}$$

Therefore, we can see that by mathematical induction,  $5^{2p} - 1$  will always be divisible by 8 as long as  $p$  is in the natural numbers.

2. For any natural number  $\ell$ ,  $3^\ell \geq 1 + 2^\ell$ .
3. Let  $a_1, \dots, a_n$  be real numbers. Then

$$2^{(\sum_{k=1}^n a_k)} = \prod_{k=1}^n 2^{a_k}$$

**Claim 6.** *Given  $a_1, \dots, a_n$  are real numbers then*

$$2^{(\sum_{k=1}^n a_k)} = \prod_{k=1}^n 2^{a_k}$$

**Proof by Induction 2.** We can first test our base case by plugging in  $n=1$ , which gives us  $2^{a_1} = 2^{a_1}$ , so the base case is correct. We can now set  $k=r+1$ , to give us:

$$\begin{aligned}
 &= 2^{(\sum_{r=1}^n a_r + a_{r+1})} && \text{(replaces k with } r+1 \text{)} \\
 &= 2^{r+1} * 2^{(\sum_{r=1}^n a_r)} \\
 &= 2^{r+1} * \prod_{r=1}^n 2^{a_r} && \text{(Replaced sum formula with it's equal (As stated in class))} \\
 &= \prod_{r=1}^n 2^{a_r+1}
 \end{aligned}$$

As we can see, these two are equal at  $n=1$  and all subsequent values, so this holds true. This was achieved with mathematical induction.

4. (★) For any  $k \in \mathbb{N}$ , and any real numbers  $r, s$ , we have

$$r^k - s^k = (r - s) \sum_{\substack{p+q=k-1 \\ p,q \geq 0}} r^q s^p = (r - s) \left( r^{k-1} + r^{k-2}s + r^{k-3}s^2 + \dots + r^2s^{k-3} + rs^{k-2} + s^{k-1} \right)$$

5. Consider the possible results of flipping a fair coin  $n$  times. There are  $2^n$  possible outcomes.  
 6. (★) For any natural number  $q$ ,

$$\sum_{i=1}^{2^q} \frac{1}{i} \geq 1 + \frac{q}{2}$$

7. Prove the power rule for derivatives: for any  $n \in \mathbb{N}$ , we have  $\frac{d}{dx}[x^n] = nx^{n-1}$ . You may **only** use the product rule for derivatives and the fact that  $\frac{d}{dx}x = 1$ .  
 8. (★) The **power rule for integrals**: for any  $n \in \mathbb{N}$ , we have

$$\int x^n dx = \frac{1}{n+1} x^{n+1} + C$$

for some constant  $C$ .

You **may not** use the previous result. You may use **only** the following calculus facts: the linearity properties of the integral;  $\int C dx = Cx + D$  for some constant  $D$ ;  $\frac{d}{dx}x = 1$ ; integration by parts.

9.  $\frac{d^r}{dx^r} x^r = r!$   
 10. The **constant multiple rule for higher derivatives**: for any function  $f$  with at least  $n$  derivatives and any constant  $c$ , we have  $\frac{d^n}{dx^n} [cf] = c \left[ \frac{d^n}{dx^n} f \right]$ . You may assume the constant multiple rule for derivatives.  
 11. The **sum rule for higher derivatives**: for any functions  $f, g$  with at least  $n$  derivatives, we have  $\frac{d^n}{dx^n} [f + g] = \frac{d^n}{dx^n} f + \frac{d^n}{dx^n} g$ . You may assume the sum rule for derivatives.

12. (★) The **Binomial Theorem**: for any real numbers  $x, y$ , and any  $n \in \mathbb{N}$ ,

$$(x + y)^n = B_{n,0}x^n + B_{n,1}x^{n-1}y + B_{n,2}x^{n-2}y^2 + \cdots + B_{n,n-2}x^2y^{n-2} + B_{n,n-1}xy^{n-1} + B_{n,n}y^n$$

(*Hint.* At some point you will need to “combine like terms”.)

13. (★) Any legal word has more *as* than *bs*.