MA 225 Worksheet: Useful Logical Facts

The following logical facts can be deduced by looking at appropriate truth tables, by formulating a tautologous propositional formula, or by just thinking it through.

	1	2	3	4	5
a	P	Q	$P \Rightarrow Q$	P $\wedge$ Q	P $\vee$ Q
b	True	True	True	True	True
c	True	False	False	False	True
d	False	True	True	False	True
e	False	False	True	False	False

(a) If $P \Rightarrow Q$ is true and P is true, then Q must be true.
Proof: Using the proof table above, we can use rows b and c which show when P is true. Then, looking on row b and c in columns 2 and 3, we can see that the only value which results in P $\Rightarrow$ Q being True is Q being True.

(b) If $P \Rightarrow Q$ is true and Q is false, then P must be false.
Proof: Looking at the proof table, we can see based off column 2 and 3, the only row which satisfies this is row e.

(c) If $P \wedge Q$ is true, then P is true; if P is true, then $P \vee Q$ is true.
Proof: We can prove the statement by looking at columns 1 2 and 4 of the truth table above to see that the only time P$\wedge$ Q is True is when P and Q are true, therefore P is True. Next, we can look at rows b and c where P are true and column 5 to see that when P is true, so is $P \vee Q$.

	1	2	3	4	5	6
	P	Q	R	P$\Rightarrow Q$	Q $\Rightarrow R$ R	$P \Rightarrow R$
a	True	True	True	True	True	True
b	True	True	False	True	False	False
c	True	False	True	False	True	True
d	True	False	False	False	True	False
e	False	True	True	True	True	True
f	False	True	False	True	False	True
g	False	False	True	True	True	True
h	False	False	False	True	True	True

(d) If $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true, then $P \Rightarrow R$ is true.
Proof: Looking at the proof directly above in rows a g e and h, which represent all combinations of P Q and R in which P $\Rightarrow$ Q and Q $\Rightarrow$ R are true, we can see that in column 6 that $P \Rightarrow R$ is always true.

(e) $P \Rightarrow (Q \vee R)$ is equivalent to $(P \wedge (\sim Q)) \Rightarrow R$ and $(P \wedge (\sim R)) \Rightarrow Q$.
Proof:

	1	2	3	4	5	6
	P	Q	R	$P \Rightarrow (Q \vee R)$	$(P \wedge (\sim Q)) \Rightarrow R$	$(P \wedge (\sim R)) \Rightarrow Q$
a	True	True	True	True	True	True
b	True	True	False	True	True	True
c	True	False	True	True	True	True
d	True	False	False	False	False	False
e	False	True	True	True	True	True
f	False	True	False	True	True	True
g	False	False	True	True	True	True
h	False	False	False	False	False	False

We can see from the Truth table directly above that for all values in columns 4 5 and 6, all values are equal, therefore the three statements are equivalent.

(f) $P \Rightarrow (Q \wedge R)$ is equivalent to $(P \Rightarrow Q) \wedge (P \Rightarrow R)$.
Proof:

	1	2	3	4	5
	P	Q	R	$P \Rightarrow (Q \wedge R)$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
a	True	True	True	True	True
b	True	True	False	False	False
c	True	False	True	False	False
d	True	False	False	False	False
e	False	True	True	True	True
f	False	True	False	True	True
g	False	False	True	True	True
h	False	False	False	False	False

Observing columns 4 and 5 of the truth table directly above, we can see that all the values are the same, therefore $P \Rightarrow (Q \wedge R)$ and $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ are equivalent.

(g) If $P \Rightarrow R$ is true and $Q \Rightarrow R$ is true, then $(P \vee Q) \Rightarrow R$ is true.
Proof: First, we must realize that in order for wither statement to be false, R would have to be false. For the first statement $P \Rightarrow R$, P would have to be True, and in order for the second statement $Q \Rightarrow R$ to be False, Q would have to be True. Therefore, the only way both could be True while R is False is if P and Q were false, which we can see that for (P $\vee Q) \Rightarrow R$, the only way the statement could be False is if P or Q were True, which was shown to set former equations to false.

(h) If $(P \vee Q) \Rightarrow R$ is true, then $P \Rightarrow R$ is true.
Proof: According to truth table 1, $P \vee Q$ is true whenever P is true. Therefore, since the R remianed constant in the two propositions, it is therefore shown that when (P $\vee Q) \Rightarrow Q$ is true, then so is P $\vee$ Q.

1. For statements (a)-(d), give a proof by just looking at the truth table for $\Rightarrow$, $\vee$, or $\wedge$.

2. For statements (e) and (f), give a proof by making truth tables for the claimed equivalences.

3. For statements (g) and (h), give a proof as follows:
   (a) Rewrite the $\Rightarrow$ in the formula $(P \vee Q) \Rightarrow R$ using $\vee$ and $\sim$, to get a formula that only involves $\vee$ and $\sim$.

   (b) Do the same for $P \Rightarrow R$ and $Q \Rightarrow R$.

   (c) After possibly doing some arithmetic, apply fact (c).

4. The logical facts above will each prove to be useful in writing proofs. Below are a list of names or shorthands which correspond to logical facts (a)-(h). Match the logical fact to the name or shorthand. For some of these you will need to consult an outside source.
   (I) $\Rightarrow$ *is transitive* d
   (II) *specialization* c
   (III) *prove each separately* f
   (IV) *modus tollens* a
   (V) *a fortiori* h
   (VI) *rule out the undesirable case* e
   (VII) *modus ponens / direct proof* b
   (VIII) *proof by cases* g