

MA 225 Problem Set 5

facts and definitions You will need the following definitions and facts (at some point):

Definition 1. We call a set A *open* if, for each $p \in A$, there is a $\delta > 0$ so that the δ -neighborhood of p , $V(p, \delta)$, has $V(p, \delta) \subseteq A$. We call a set A *closed* if its complement is open.

Definition 2. Given sets A and B , we define

$$A \setminus B = \{x \mid x \in A \wedge x \notin B\}$$

and $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Fact. If $\alpha < \beta$, then the α - and β -neighborhoods of p are related as follows:

$$V(p, \alpha) \subseteq V(p, \beta)$$

exercises These problems don't require you to write proofs.

- Some of the following strings of symbols are ambiguous because they lack bracketing. Exhibit the ambiguity by supplying different possible bracketings, and explaining why the versions are different. Be careful – some may not be ambiguous at all!
 - $2^A \cap 2^B \setminus C \subseteq D \times 2^A$
Not ambiguous. While it can be written as $(2^A \cap 2^B) \setminus C \subseteq D \times 2^A$ and $2^A \cap (2^B \setminus C) \subseteq D \times 2^A$. In the equation, there exists a $p \in 2^A \cap 2^B$, but not in C . In the second equation, p is in 2^B but not C , and since p is in 2^A , and not C , p would only pull elements not in C from 2^A .
 - $A \cap B \cap C \subseteq D \cup A \cup B$. Not ambiguous, as cap and cup are commutative with themselves.
 - $A \cap B \cup C$.
 $(A \cap B) \cup C$ and $A \cap (B \cup C)$. A main difference is that in the first bracketing, all elements in C are used. In the second one, however, terms in C are only used if in A as well.
- Let A and B be sets. Consider the collection \mathcal{D} of all sets C with the property $C \subseteq A$ and $C \subseteq B$.
 - Write the definition of \mathcal{D} in set-builder notation.
 $\mathcal{D} = \{C \mid C \subseteq A \wedge C \subseteq B\}$
 - Between $A \in \mathcal{D}$ and $A \subseteq \mathcal{D}$, which is *sensible*? \in , as \subseteq would imply A was a set of sets. Furthermore, if A were to be a set of sets, as since it's said $C \subseteq A$, this would imply that the elements of C were sets of sets, meaning A was a set of set of sets, and this would continue.
 - Between $A \in \mathcal{D}$ and $A \subseteq \mathcal{D}$, is either *true*? As we discussed earlier, A can't be a subset of \mathcal{D} . It would work only if $A \subseteq B$, as $A \subseteq A$.
 - Among $A \cap B \in \mathcal{D}$, $A \cup B \in \mathcal{D}$, $A \cap B \subseteq \mathcal{D}$, $A \cup B \subseteq \mathcal{D}$, are any true? $A \cap B$, as the intersection of A and B is a subset of A , and the intersection of A and B are a subset of B .
- Let A and B be sets. Consider the collection \mathcal{E} of all sets F with the property: for any C with $C \subseteq A$ and $C \subseteq B$, $C \subseteq F$.
 - Write out the definition of \mathcal{E} in set-builder notation. $\mathcal{E} = \{F \mid C \subseteq A \wedge C \subseteq B \wedge C \subseteq F\}$

- (b) Between $A \in \mathcal{E}$ and $A \subseteq \mathcal{E}$, which is *sensible*? Same problem as number 2b. So $A \in \mathcal{E}$ makes more sense
 - (c) Between $A \in \mathcal{E}$ and $A \subseteq \mathcal{E}$, is either *true*? As stated before, $A \subseteq \mathcal{E}$, so this is false. $A \in \mathcal{E}$ is true only when $A \in B$ and when $A \in F$
 - (d) Among $A \cap B \in \mathcal{E}$, $A \cup B \in \mathcal{E}$, $A \cap B \subseteq \mathcal{E}$, $A \cup B \subseteq \mathcal{E}$, are any true?
4. In Definition 1, what kind of thing must δ be? What kind of a thing must a neighborhood be? δ must be a real number, as it is used in an inequality. A neighborhood must be a subset.
5. Write out what it means for the set J to be *not open*:
- (a) Do this in symbols.

$$J = \{p | \forall \delta : \delta \leq 0 \vee V(p, \delta) \not\subseteq J\}$$
 - (b) Do this in English. For all δ in a J , if δ less than or equal to 0, or $V(p, \delta)$ is not a subset, then J is not open.
6. (a) Write a blueprint for the claim *If blah blah and such, then A is open*.
 If $\forall p \in A$, then there exists a δ which is greater than 0 and such that $V(p, \delta)$ is a subset of A , then A is open.
- (b) Write a blueprint for the claim *If blah blah and such, then A is closed*. If $\forall p \notin A$, then there exists a δ which is greater than 0 and such that $V(p, \delta)$ is disjoint from A , then A is closed.

proofs Prove the following claims.

1. For any $n \in \mathbb{N}$, $\sum_{k=0}^n B_{n,k} = 2^n$.
2. If $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$, then $A = B$ and $B = C$.

Claim 1. If $A \subseteq B$ and $B \subseteq C$ and $C \subseteq A$, then $A = B$ and $B = C$.

Proof. Let $s \in B$. So $s \in C$ according to $B \subseteq C$. So $s \in A$ according to $C \subseteq A$. So $B \subseteq A$, and $A \subseteq B$ according to definition, therefore $A = B$.

Let $p \in C$. So $p \in A$, according to $C \subseteq A$. So $p \in B$ according to $A \subseteq B$. So $B \subseteq C$, and $C \subseteq B$ according to definition, therefore $C = B$. \square

3. (\star) For any n , $\left(\bigcup_{j=1}^n A_j\right)^c = \bigcap_{j=1}^n A_j^c$. (*Hint.* n is a what?.)

Claim 2. For any n , $\left(\bigcup_{j=1}^n A_j\right)^c = \bigcap_{j=1}^n A_j^c$.

Proof. Assume $\left(\bigcup_{j=1}^n A_j\right)^c = \bigcap_{j=1}^n A_j^c$. Let $a \in A_j$. We can use proof by induction to prove this, as n is a natural number. We can see that this function holds true for $n = 1$, as $A_1^c = A_1^c$. We define k as $k=n$ and evaluate for $k+1$ to show that:

$$\begin{aligned}
 \bigcap_{j=1}^{k+1} A_j^c &= \bigcap_{j=1}^{k+1} A_j^c \\
 &= \bigcap_{j=1}^k A_j^c \cap A_{k+1}^c \\
 &= \left(\bigcup_{j=1}^k A_j\right)^c \cap A_{k+1}^c && \text{(Use inductive hypothesis)} \\
 &= \left(\bigcup_{j=1}^k A_j \cup A_{k+1}\right)^c && \text{(Used de Morgan's Law)} \\
 &= \left(\bigcup_{j=1}^{k+1} A_j\right)^c
 \end{aligned}$$

Therefore, through induction we can conclude that $\left(\bigcup_{j=1}^n A_j\right)^c = \bigcap_{j=1}^n A_j^c$ for any n . \square

4. Some facts about sets:

(a) $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$

Claim 3. $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$

Proof. Let $t \in (X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$. This means $t \in X$, but not in Y , and from this it is not in Z . This can be rewritten as $(X \setminus Z) \setminus (Y \setminus Z)$, as t is an element of X , but not of Z , and t is not in Y .

Let $n \in (X \setminus Z) \setminus (Y \setminus Z)$. Because n is never an element of $(X \setminus Z) \setminus (Y \setminus Z)$ by definition of \setminus , but instead an element of X but not Y , this can be rewritten as $(X \setminus Y) \setminus Z$. Therefore $\subseteq (X \setminus Y) \setminus (X \setminus Z) \setminus (Y \setminus Z)$. Since the equation works for \subseteq and \supseteq , $(X \setminus Y) \setminus Z = (X \setminus Z) \setminus (Y \setminus Z)$. □

(b) $(\star) 2^{A \cap B} = 2^A \cap 2^B$

Claim 4. $2^{A \cap B} = 2^A \cap 2^B$

Proof. Let $r \in 2^{A \cap B}$, where r is a set. We must do this by first showing that $2^{A \cap B} \subseteq 2^A \cap 2^B$. Let $p \in A \cap B$. p can be defined as all elements which are in both A and B , and r is defined as all combinations of p . Because of the definition of the powerset, the subsets which contain p will be able to be produced by A and B , and by finding the intersection of these set of subsets, .

Therefore, since $2^{A \cap B} \subseteq 2^A \cap 2^B$ and $2^A \cap 2^B \subseteq 2^{A \cap B}$ are true, $2^{A \cap B} = 2^A \cap 2^B$ is also true. □

(c) $2^A \cup 2^B \subseteq 2^{A \cup B}$

Claim 5. $2^A \cup 2^B \subseteq 2^{A \cup B}$

Proof. Let $p \in 2^A \cup 2^B$. So either $p \in 2^A$ or 2^B . All of these elements are in $2^{A \cup B}$, as this consists of all subsets of the elements of $A \cup B$, as well as more. Therefore, $2^A \cup 2^B \subseteq 2^{A \cup B}$. □

(d) $A \Delta B = B \Delta A$

Claim 6. $A \Delta B = B \Delta A$

Proof. Let $p \in A \Delta B$. So $p \in (A \setminus B) \cup (B \setminus A)$ by definition. Since the operation \setminus is commutative (sets are not ordered and union just takes terms from both sets, therefore union is commutative), this can be rewritten as $p \in (B \setminus A) \cup (A \setminus B)$, or $B \Delta A$ by definition. Therefore $A \Delta B \subseteq B \Delta A$.

Let $q \in B \Delta A$. So $q \in (B \setminus A) \cup (A \setminus B)$ by definition. Since the operation \setminus is commutative (As explained earlier), this can be rewritten as $p \in (A \setminus B) \cup (B \setminus A)$, this can be rewritten as $p \in (A \setminus B) \cup (B \setminus A)$, or $A \Delta B$. Therefore $B \Delta A \subseteq A \Delta B$.

Since $B \Delta A \subseteq B \Delta A$, and $A \Delta B \subseteq B \Delta A$, it can be concluded that $A \Delta B = B \Delta A$. □

(e) $A \Delta B = (A \cup B) \setminus (A \cap B)$

Claim 7. $A \Delta B = (A \cup B) \setminus (A \cap B)$

Proof. Let $p \in A \Delta B$. So by definition, $p \in (A \setminus B) \cup (B \setminus A)$. This means that either $p \in A$ and $p \notin B$, or $p \in B$, and $p \notin A$, which means that $p \notin A \cap B$, however $p \in A \cup B$. We can use the \setminus operator to show to represent: $(A \cup B) \setminus (A \cap B)$. Therefore $A \Delta B \subseteq (A \cup B) \setminus (A \cap B)$.

Let $q \in (A \cup B) \setminus (A \cap B)$. So by definition of \setminus , $q \in (A \cup B)$, and $q \notin (A \cap B)$. We can rewrite this as $q \in (A \setminus B) \cup (B \setminus A)$, as this is saying q is in A but not B or B but not A , or that there is no intersections between A and B but it exists in A or B . $q \in (A \cup B) \setminus (A \cap B)$ by definition is equal to $q \in A \Delta B$. Therefore $A \Delta B = (A \cup B) \setminus (A \cap B) \subseteq A \Delta B$.

Since $A \Delta B \subseteq (A \cup B) \setminus (A \cap B)$, and $(A \cup B) \setminus (A \cap B) \subseteq A \Delta B$, this means that $A \Delta B = (A \cup B) \setminus (A \cap B)$. \square

(f) If $A \cup B \subseteq C \cup D$ and $A \cap B = \emptyset$, and $C \subseteq A$, then $B \subseteq D$.

Claim 8. *Given $A \cup B \subseteq C \cup D$ and $A \cap B = \emptyset$, and $C \subseteq A$, then $B \subseteq D$.*

Proof. Let $r \in C$. Then $C \cup B \subseteq C \cup D$, as $r \in C \cup A$, and since $r \in C$. Furthermore, $C \cup B = \emptyset$, as C is a subset of A , and which $A \cup B = \emptyset$. So since $C \cup B \subseteq C \cup D$, and since C and B share no terms, $B \subseteq D$. \square

5. (\star) The empty set is open. (*Hint.* Go to Vegas.)

Claim 9. *The empty set is open.*

Proof. The definition of an open set is that for all $s \in S$, where S is a set, there also exists a term with another property. By definition, the empty set does not contain any s , therefore all elements, in this case none, satisfy the conditions of being an open set. \square \square

6. $(\star\star)$ If A is open and B is open, then $A \cup B$ is open. (Yes, you can do this even though you have no idea what an open set is.)

Claim 10. *Given A and B are open, $A \cup B$ is open.*

Proof. Let $p \in A \cup B$. Since $\forall a \in A, \exists \delta : \delta > 0 \wedge V(a, \delta) \subseteq A$, and $\forall b \in B, \exists \delta : \delta > 0 \wedge V(b, \delta) \subseteq B$, $A \cup B$ consist of all elements in A as well as B , and the definition of an open set is still upheld by the union, $A \cup B$ is open. \square

7. (\star) If A_1, \dots, A_p are open sets, then $\bigcup_{k=1}^p A_k$ is an open set. (*Hint.* p is a what?.)

Claim 11. *Given A_1, \dots, A_p are open sets, then $\bigcup_{k=1}^p A_k$ is an open set.*

Proof. Through proof by induction, we can determine $\bigcup_{k=1}^p A_k$ is open. We can first verify

the base case, which results in $\bigcup_{k=1}^1 A_k = A_1$, which is true based on the definition of A_1 .

Defining $j = p$, we will show that $\bigcup_{k=1}^j A_k$ is open for $j+1$:

$$\begin{aligned} \bigcup_{k=1}^{j+1} A_k &= \bigcup_{k=1}^{j+1} A_k \\ &= \bigcup_{k=1}^p A_k \cup a_{j+1} \end{aligned}$$

According to our inductive assumption, $\bigcup_{k=1}^j A_k$ is an open set, as well as a_{j+1} by definition.

Since we proved in the previous proof that the union of two open sets is open, by proof of mathematical induction $\bigcup_{k=1}^p A_k$ is open. \square

8. (★) If A is closed and B is closed, then $A \cap B$ is closed.

Claim 12. *If A is closed and B is closed, then $A \cap B$ is closed.*

Proof. Given both A and B are closed, A^c is open and B^c is open by definition. We can see from a previous proof that if A^c is open and B^c is open, then $A^c \cup B^c$ is open as well. Using de Morgans law, we realize this is equal to $(A \cap B)^c$. Because this set is open, and it is the complement of $A \cap B$, $A \cap B$ is closed. \square

9. (★★) If A is open and B is open, then $A \cap B$ is open. (Again, you can do this without any knowledge of what a neighborhood is.)

Claim 13. *If A is open and B is open, then $A \cap B$ is open. (Again, you can do this without any knowledge of what a neighborhood is.)*

Proof. Let $p \in A \cap B$. By definition, we know that $p \in A$ and $p \in B$. We also know that because A is open, then for all $p \in A$, there exists a subset $V(p, \delta) \subseteq A$, and the same for B . Therefore, since for all p in $A \cap B$, there exists a subset $V(p, \delta)$, the set is open. \square