MA 225 Problem Set 7

facts and definitions

Definition 1. We call a relation E on the set A *right-Euclidean* if for any $x, y, z \in A$, $x \mathrel{\mathrm{E}} y$ and $x \mathrel{\mathrm{E}} z$ together guarantee $y \mathrel{\mathrm{E}} z$.

Definition 2. We call a relation E on the set A *Euclidean* if for any $x, y, z \in A$, $x \mathrel{\mathrm{E}} y$ and $x \mathrel{\mathrm{E}} z$ together guarantee $y \mathrel{\mathrm{E}} z$, and $y \mathrel{\mathrm{E}} x$ and $z \mathrel{\mathrm{E}} x$ together guarantee $y \mathrel{\mathrm{E}} z$.

Definition 3. We call a relation E on the set A *antisymmetric* if for any $x, y \in A$, $x \mathrel{\mathrm{E}} y$ and $y \mathrel{\mathrm{E}} x$ together guarantee $x = y$.
We call a relation E on the set A *asymmetric* if for any $x, y \in A$, $x \mathrel{\mathrm{E}} y$ guarantees $y \not\mathrel{\mathrm{E}} x$.

exercises These problems don't require you to write proofs.

1. Write a blueprint for a proof of *If blah and yadda yadda, then E is right-Euclidean.*

   Claim 1. *If blah and yadda yadda, then E is right-Euclidean*

   Proof. Let $x, y, z \in A$ and xEy and xEz.
   ...
   So yEz is guarunteed. Therefore E is Right-Euclidian. □

2. Write a blueprint for a proof of *If blah and so on, then $\mathcal{P}$ is a partition of A.*

   Claim 2. *If blah and so on, then $\mathcal{P}$ is a partition of A.*

   Proof. Let $x \in \mathcal{P}$.
   ...
   Thereore $x \neq \varnothing$
   Let $m \in A$ ...
   Therefore $m \in \bigcup \mathcal{P}$. Let $J, Y \in \mathcal{P}$.
   ...
   So $J \cap Y = \varnothing$. So $\mathcal{P}$ is a partition of A. □

3. "R is antisymmetric" means something different from "R is not symmetric". Give an example to demonstrate and then explain in terms of logic. R is antisymmetric means that for any relation on set A with $x, y \in A$, xRy guaruntees yRx is false, while not symmetric means there exists an $x, y \in A$ that xRy doesn't guaruntee yRx. This means $(Risantisymmetric) \Rightarrow (Risnotsymmetric)$

4. "R is asymmetric" means something different from "R is not symmetric". Give an example to demonstrate and then explain in terms of logic. R is asymmetric means that for any relation on set A with $x, y \in A$, if xRy and yRx are true, then $x = y$, while not symmetric means there exists an $x, y \in A$ that xRy doesn't guaruntee yRx. This means $(Risasymmetric) \Rightarrow (Risnotsymmetric)$

proofs Write a complete proof for each of the following statements.

1. Let R be a relation on the set A.
   (a) ($\star$) If Domain$(R) = A$, and R is symmetric and transitive, then R is reflexive.

   Claim 3. *If* Domain$(R) = A$, *and R is symmetric and transitive, then R is reflexive.*

Proof. Let $x \in R$. So $x = (a, b)$ where $a, b \in A$ (R is a relation on set A and $a \in A$ is given.). Since R is symmetiric, $(b, a) \in A$. Since $x \in R$ and $(b, a) \in R$, and R is transitive, $(a, a) \in R$. Therefore R is reflexive. $\square$

(b) ($\star$) Explain, both by giving an example and in general, why the assumption Domain$(R) = A$ is necessary.
The assumption Domain $R = A$ is required because these properties will not hold when using an element for the Domain that is not in A as R is on the relationship A. For instance, if $g \in R$ and $g = (x, y)$ where $x, y \in \mathbb{R}$ $xy = yx$, this would not work if you were to use an element of a set of horses (what does it mean to multiply a horse?). Therefore Domain $R = A$ is required.

2. Let R and S be equivalence relations on a set A. Show that $S \cap R$ is an equivalence relation.

Claim 4. *$S \cap R$ is an equivalence relation on the set A if S and R are equivalence relationships.*

Proof. (Reflexive) Let $x \in S \cap R$. Then $x = (a, a)$ where $a \in A$. Therefore $x \in S$ and $x \in R$ (We know this because elements are related to themselves in equivalence classes). Therefore $S \cap R$ is Reflexive.
(Symmetric) Let $j \in S \cap R$. Then $j = (c, d)$ where $c, d \in A$. Since $j \in S$ and S is an equivallence relationship, $(d, c) \in S$. Since $j \in R$, and R is an equivallence relationship, $(d, c) \in R$. Therefore $S \cap R$ is symmetric.
(Transitive) Let $k, i \in S \cap R$. So $k = (e, f)$ and $i = (f, g)$ where $e, f, g \in A$. So $k, i \in S$ and $k, i \in R$. Since S is an equivallence class and is transitive, i and k imply (e, g). Since R is an equivallence class and is transitive, i and k imply (e, g). Therefore $S \cap R$ is transitive. So $S \cap R$ is an equivallence class. $\square$

3. ($\star$) Consider the relation on $\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ given by

$$(m, n)\,\mathrm{CP}(r, s) \text{ means } ms = nr$$

Show that CP is an equivalence relation. **You may not use fractions anywhere in your proof!**

Claim 5. *CP is an equivallence class.*

Proof. (Reflexive) Let $x, y \in \mathbb{Z} \times (F \setminus \{0\})$. Then $x, y = (a, b)$ where $a, b \in \mathbb{Z}$. Since $ab = ba$ (Used definition of CP and communative property), xCPy is true and since $x = y$,xCPx is true and CP is reflexive.
(Symmetric) Let $j \in CP$. Then $j = ((a, b), (c, d))$ where $a, b, c, d \in \mathbb{Z}$. Therefore $ad = bc$. So $bc = ad$ (equals in symmetric. So $cb = da$ (multiplication is communative over multiplication). So $((c, d), (a, b)) \in CP$ therefore CP is symmetric.
(Transitive) Let $k, r \in CP$. Then $k = (g, o)CP(s, t)$, $r = (s, t)CP(e, h)$. So $gt = so$, and $sh = et$.Multiplying by the inverse of s on both sides gives us: $h = ets^{-1}$ and $o = gts^{-1}$. $he^{-1} = ts^{-1}$ and $og^{-1} = ts^{-1}$(Multiplied both sides by inverse). So $he^{-1} = og^{-1}$ (the equal sign is transitive). So $(g, o)CP(e, h) \in CP$. Therefore, since all of the relationships in CP are equivalence relationships, CP is an equivalence class. $\square$

4. We write $\frac{m}{n}$ for $[(m, n)]_{CP}$. Verify that $\frac{10}{5} = \frac{6}{3} = \frac{2}{1}$.

Claim 6. $\frac{10}{5} = \frac{6}{3} = \frac{2}{1}$.

Proof. Since $[(m, n)]_{CP} = \frac{m}{n}$ We can rewrite $\frac{10}{5} = \frac{6}{3}$ as $(10, 5)CP(6, 3)$. This means that $10 * 3 = 6 * 5$. Since this statement is true, $\frac{10}{5} = \frac{6}{3}$. We can now verify $\frac{6}{3} = \frac{2}{1}$ by using $6 * 1 = 3 * 2$(Used definition of CP), which is valid. So $\frac{10}{5} = \frac{6}{3} = \frac{2}{1}$. $\square$

5. Define $(m, n) \oplus (p, q) = (mq + pn, nq)$.
   (a) Show that if $(m, n)\,\mathrm{CP}(r, s)$ and $(p, q)\,\mathrm{CP}(t, s)$, then $(m, n) \oplus (p, q)\,\mathrm{CP}(r, s) \oplus (t, s)$.

   Claim 7. *If $(m, n)\,\mathrm{CP}(r, s)$ and $(p, q)\,\mathrm{CP}(t, s)$, then $(m, n) \oplus (p, q)\,\mathrm{CP}(r, s) \oplus (t, s)$.*

   Proof. $ms = nr$ and $ps = qt$ (According to the definition of CP). Furthermore, $(m, n) \oplus (p, q)\,\mathrm{CP}(r, s) \oplus (t, s)$ is equivalent to $(mq + pn, np)CP(rs + st, ts)$. We will show that if $ms = nr$ and $ps = qt$, then $(mq + pn)ts = (rs + st)np$.

   $$\begin{aligned} (mq + pn)ts &= (rs + st)np \\ mqst + pnst &= nprs + npts \\ ms(qt) + (ps)nt &= (nr)ps + (ps)nt \\ msps + qtnt &= msps + qtnt \qquad\qquad \text{(Used fact that } ms = nr \text{ and } ps = qt) \end{aligned}$$

   So if $(m, n)\,\mathrm{CP}(r, s)$ and $(p, q)\,\mathrm{CP}(t, s)$, then $(m, n) \oplus (p, q)\,\mathrm{CP}(r, s) \oplus (t, s)$. □

   (b) Rewrite the claim in part (a) using the fraction notation introduced in problem 4. If $\frac{m}{n} = \frac{r}{s}$, then and $\frac{p}{q} = \frac{t}{s}$ then $\frac{mq+np}{nq} = \frac{rs+st}{s^2}$

6. Let S and T be equivalence relations on a set A. Assume that $S \subseteq T$.
   (a) $S \subseteq T$ means that the condition $x\,\mathrm{S}\,y$ is easier/harder to satisfy than the condition $x\,\mathrm{T}\,y$. (Pick one and explain your answer.)

   Claim 8. *The condition xSy is easier to satisfy than xTy*

   Proof. This is because for xTy to be true, xSy must also be true, while for xSy to be true, only xSy must be true. □

   (b) Let $a \in A$. What is the relationship between $[a]_S$ and $[a]_T$? Prove your answer.

   Claim 9. $[a]_S \subseteq [a]_T$

   Proof. Let $x \in [a]_S \subseteq [a]_T$. So $x \subseteq S$ (equivalence relations are made up of specific relations in another set of relation). So $x \in T$. Since x is an equivalence class an equivalence relation, $x \in [a]_S$. So $[a]_S \subseteq [a]_R$. □

   (c) ($\star$) What is the relationship between $A_{/S}$ and $A_{/T}$? Prove your answer.

7. Let A be a set with at least three elements.
   (a) If $\mathcal{P} = \{B_1, B_2\}$ is a partition of A, and $B_1 \neq B_2$, what can you say about B_1^c and B_2^c? Prove your answer.

   Claim 10. *If $\mathcal{P} = \{B_1, B_2\}$ is a partition of A, and $B_1 \neq B_2$, $B_1 \subseteq B_2^c$ and $B_2 \subseteq B_1^c$.*

   Proof. Let $x \in B_1$. So $x \notin B_2$ (By definition of a partition since $B_1 \neq B_2$). So $B_1 \subseteq B_2^c$. Let $x \in B_2$. So $x \notin B_1$ (By definition of a partition since $B_1 \neq B_2$). So $B_2 \subseteq B_1^c$. □

   (b) If $\mathcal{P} = \{B_1, B_2\}$ is a partition of A, $\mathcal{C}_1$ is a partition of B_1, and $\mathcal{C}_2$ is a partition of B_2, and $B_1 \neq B_2$, show that $\mathcal{C}_1 \cup \mathcal{C}_2$ is a partition of A.

   Claim 11. *$\mathcal{C}_1 \cup \mathcal{C}_2$ is a partition of A.*

   Proof. $B_1 \subseteq \bigcup \mathcal{C}_1$ and $B_2 \subseteq \bigcup \mathcal{C}_2$ (according to the definition of a partition). So $B_1 \cup B_2 \subseteq \bigcup(\mathcal{C}_1 \cup \mathcal{C}_2)$. Since $A \subseteq B_1 \cup B_2$, $A \subseteq \bigcup(\mathcal{C}_1 \cup \mathcal{C}_2)$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$ are partitions, $\varnothing \notin \mathcal{C}_1 \cup \mathcal{C}_2$. So $C_1 \cup C_2$ is a partition of A (By rules of partitions). □

(c) Why did we assume A has at least three elements? We assumed A had at least three elements because partitions B_1 and B_2 cannot be empty (by definition of partitions) and cannot overlap. Therefore there must be two elements which fill those partitions. Furthermore, since all sets contain the empty set and partitions can't contain this, we know that the empty set is in A, making a total of three elements.

8. Show that any asymmetric relation must be antisymmetric.

Claim 12. *Any asymmetric relation must be antisymmetric.*

Proof. □

9. ($\star$) Let S be a reflexive relation. Show that if S is right-Euclidean, then S is an equivalence relation.

Claim 13. *If S is right-Euclidean, then S is an equivalence relation.*

Proof. Let $x, y \in S$ where $x = (a, b)$ and $y = (a, c)$. So $(c, b) \in S$ (Right-Euclidian property). So $(c, a) \in S$ (Right-Euclidian property with (c, b) and x). Therefore S is symmetric. Then $(b, b) \in S$ (Right-Euclidian peoperty on (c, b) and (c, b)). Therefore S is reflexive. Then, since $(a, c) \in S$ and $(c, b) \in S$ gurantees (a, b), S is transitive. Since S is reflexive, transitive, and symmetric, S is an equivalence relation. □

10. ($\star$) Show that if E is transitive and Euclidean, then E is symmetric.

Claim 14. *Show that if E is transitive and Euclidean, then E is symmetric.*

Proof. Let $x, y \in E$ where $x = (a, b)$ $y = (b, c)$. So $(a, c) \in E$ (Transitive property). So $(b, a) \in E$ (left euclidian property on x and (a, c)). So $(c, a) \in E$ (Left euclidian property on y and (b, a)). So $(c, b) \in E$ (Right Euclidian property on (a, c) and x). Therefore E is symmetric. □