

MA 225 Worksheet: Useful Logical Facts

The following logical facts can be deduced by looking at appropriate truth tables, by formulating a tautologous propositional formula, or by just thinking it through.

	1	2	3	4	5
a	P	Q	$P \Rightarrow Q$	$P \wedge Q$	$P \vee Q$
b	True	True	True	True	True
c	True	False	False	False	True
d	False	True	True	False	True
e	False	False	True	False	False

- (a) If $P \Rightarrow Q$ is true and P is true, then Q must be true.

Proof: Using the proof table above, we can use rows b and c which show when P is true. Then, looking on row b and c in columns 2 and 3, we can see that the only value which results in $P \Rightarrow Q$ being True is Q being True.

- (b) If $P \Rightarrow Q$ is true and Q is false, then P must be false.

Proof: Looking at the proof table, we can see based off column 2 and 3, the only row which satisfies this is row e.

- (c) If $P \wedge Q$ is true, then P is true; if P is true, then $P \vee Q$ is true.

Proof: We can prove the statement by looking at columns 1 2 and 4 to see that the only time $P \wedge Q$ is True is when P and Q are true, therefore P is True. Next, we can look at rows b and c where P are true and column 5 to see that when P is true, so is $P \vee Q$.

- (d) If $P \Rightarrow Q$ and $Q \Rightarrow R$ are both true, then $P \Rightarrow R$ is true.

Proof: Looking at the proof table the only way which $P \Rightarrow R$ can be false is if P is True and R is false. We can then see that

- (e) $P \Rightarrow (Q \vee R)$ is equivalent to $(P \wedge (\sim Q)) \Rightarrow R$ and $(P \wedge (\sim R)) \Rightarrow Q$. Proof:

	1	2	3	4	5	6
	P	Q	R	$P \Rightarrow (Q \vee R)$	$(P \wedge (\sim Q)) \Rightarrow R$	$(P \wedge (\sim R)) \Rightarrow Q$
a	True	True	True	True	True	True
b	True	True	False	True	True	True
c	True	False	True	True	True	True
d	True	False	False	False	False	False
e	False	True	True	True	True	True
f	False	True	False	True	True	True
g	False	False	True	True	True	True
h	False	False	False	False	False	False

We can see from the proof table that for all values in columns 4 5 and 6, all values are equal, therefore the three statements are equivalent.

- (f) $P \Rightarrow (Q \wedge R)$ is equivalent to $(P \Rightarrow Q) \wedge (P \Rightarrow R)$. Proof:

	1	2	3	4	5
	P	Q	R	$P \Rightarrow (Q \wedge R)$	$(P \Rightarrow Q) \wedge (P \Rightarrow R)$
a	True	True	True	True	True
b	True	True	False	False	False
c	True	False	True	False	False
d	True	False	False	False	False
e	False	True	True	True	True
f	False	True	False	True	True
g	False	False	True	True	True
h	False	False	False	False	False

Observing columns 4 and 5, we can see that all the values are the same, therefore $P \Rightarrow (Q \wedge R)$ and $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ are equivalent.

- (g) If $P \Rightarrow R$ is true and $Q \Rightarrow R$ is true, then $(P \vee Q) \Rightarrow R$ is true. Proof:
(h) If $(P \vee Q) \Rightarrow R$ is true, then $P \Rightarrow R$ is true.

1. For statements (a)-(d), give a proof by just looking at the truth table for \Rightarrow , \vee , or \wedge .

2. For statements (e) and (f), give a proof by making truth tables for the claimed equivalences.

3. For statements (g) and (h), give a proof as follows:

(a) Rewrite the \Rightarrow in the formula $(P \vee Q) \Rightarrow R$ using \vee and \sim , to get a formula that only involves \vee and \sim .

(b) Do the same for $P \Rightarrow R$ and $Q \Rightarrow R$.

(c) After possibly doing some arithmetic, apply fact (c).

4. The logical facts above will each prove to be useful in writing proofs. Below are a list of names or shorthands which correspond to logical facts (a)-(h). Match the logical fact to the name or shorthand. For some of these you will need to consult an outside source.

(I) \Rightarrow *is transitive* d

(II) *specialization* c

(III) *prove each separately* f

(IV) *modus tollens* a

(V) *a fortiori* h

(VI) *rule out the undesirable case* e

(VII) *modus ponens / direct proof* b

(VIII) *proof by cases* g