MA 225 Problem Set 3: induction 1

facts and definitions You will need the following definitions and facts (at some point):

Definition 1. We'll call strings of the symbols a and b *words.* Consider a class of words, the *legal words*, defined as follows. aba is a legal word. If W is a legal word, then so are abW, aWb, Wab, baW, bWa, and Wba. No words other than those obtained in this way are legal.

Definition 2. The *binomial coefficients* are a collection of natural numbers $B_{n,k}$, defined for a pair of nonnegative integers n, k with $0 \leq k \leq n$, as follows:

$$B_{n,k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ B_{n-1,k-1} + B_{n-1,k} & \text{if } 1 \leq k \leq n-1 \end{cases}$$

Definition 3. *Higher derivatives* are defined as follows: the zeroth derivative of a function f is f itself; we write

$$\frac{d^0}{dx^0} f(x) = f(x) = f^{(0)}(x)$$

For $n \in \mathbb{N}$, we define the n^{th} derivative as the derivative of the $(n-1)^{st}$ derivative:

$$\frac{d^n}{dx^n} f(x) = \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}} f(x)\right]$$

$$f^{(n)}(x) = \left(f^{(n-1)}\right)'(x)$$

Integration by Parts. *For any two differentiable functions f, g, we have*

$$\int f(x)g(x)\ dx = f(x)\left[\int g(x)\ dx\right] - \int f'(x)\left[\int g(x)\ dx\right]\ dx,$$

provided we adopt the convention that the constants of integration in both instances of $\int g(x)\ dx$ must be the same.

exercises These problems don't require you to write proofs.

1. Compute $B_{n,k}$ for $0 \leq n \leq 6$.

	k=0	k=1	k=2	k=3	k=4	k=5	k=6
n=0	1						
n=1	1	1					
n=2	1	2	1				
n=3	1	3	3	1			
n=4	1	4	6	4	1		
n=5	1	5	10	5	1		
n=6	1	6	15	20	15	6	1

2. Explain why the definition given for $B_{n,k}$ actually constitutes a definition; that is, why we can compute $B_{n,k}$ for any choice of n, k with $0 \leq k \leq n$. We can do this because one can work their way up from $B_{0,k}$, which is 1 and continue to get the rest of their solutions.

3. Identify, explain, and correct any correctable flaws in the following proofs: The problem with this proof is we don't see $n^2 + n$ is odd for n, instead it tells us that this is true, when in fact this isn't as 1+1=2 which is even. To make this proof work, we could change the claim to $n^2 + 1$ is always even.

Claim 1. *$n^2 + n$ is odd.*

Proof. $n = 1$ is odd.
Inductively, assume $n^2 + n$ is odd. Then

$$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = n^2 + n + 2(n+1)$$

so $(n+1)^2 + (n+1)$ is the sum of an odd number and an even number, hence itself odd. This completes the inductive step. □

In this proof, the writer didn't include even numbers, instead using the formula for odd numbers and finding all numbers in there.

Claim 2. *Every natural number is odd.*

Proof. $k = 1$ is clearly odd.
Inductively, assume k is odd. This means there is an integer p so that $k = 2p+1$. Consider $p+1$. Clearly $2(p+1)+1$ is odd. □

Didn't use a base case, instead only solving for $n+1$.

Claim 3. *Every natural number is both even and odd.*

Proof. Assume that k is both even and odd. Consider $k+1$.
Since k is even, there is p with $k = 2p$. So $k+1 = 2p+1$ is odd. Since k is odd, there is q with $k = 2q+1$. So

$$k+1 = (2q+1)+1 = 2(q+1)$$

is even. □

You didn't do the inductive step, you simply stated that there was a next term, not necessarily that this was divisible by 6.

Claim 4. *$n^3 - n$ is divisible by 6.*

Proof. For the base case: when $n = 1$, $n^3 - n = 0 = 6 \cdot 0$.
Now proceed inductively. Assume that for all k, $k^3 - k$ is divisible by 6. Then, since $n+1$ is one possible value of k, we have that $(n+1)^3 - (n+1)$ is divisible by 6. □

proofs Prove the following claims.

1. For any natural number p, 8 divides $5^{2p} - 1$.

   Claim 1. *For any natural number p, 8 divides $5^{2p} - 1$*

   Proof by Induction 1. We must first confirm that the base case, or $5^{2p} - 1$ is divisible by 8. We determine that the value of this function at 1 is 24, which is $8 * 3$, so this statement is valid. Next, we can see that the solution to this is $8m$, where m is a natural number. We can now use mathematical induction by setting p=n and plugging in p+1 to find:

$$8m = 5^{2(p+1)} - 1$$

(Solving for p+1, since we know that this is true when n is true when n=1)

$$= 5^2 * 5^{2p} - 1$$
$$= 25(8m+1) - 1$$
$$= 8(25m+3)$$

(Substituted 5^{2p} with $8m+1$, as $8m$ is what we are attempting to prove.)

Therefore, we can see that by mathematical induction, $5^{2p} - 1$ will always be divisible by 8 as long as p is in the natural numbers.

2. For any natural number ℓ, $3^\ell \geq 1 + 2^\ell$.
3. Let $a_1, \ldots, a_n$ be real numbers. Then

$$2^{\left(\sum_{k=1}^n a_k\right)} = \prod_{k=1}^n 2^{a_k}$$

Claim 2. *Given $a_1, \ldots, a_n$ are real numbers then $2^{\left(\sum_{k=1}^n a_k\right)} = \prod_{k=1}^n 2^{a_k}$*

Proof by Induction 2. We can first test our base case by plugging in n=1, which gives us $2^{a_1} = 2^{a_1}$, so the base case is correct. We can now set k=r+1, to give us:

$$\begin{aligned}
&= 2^{\left(\sum_{r=1}^n a_r\right)+a_{r+1}} && \text{(replaces k with r+1)} \\
&= 2^{a_r+1} * 2^{\left(\sum_{r=1}^n a_r\right)} \\
&= 2^{a_{r+1}} * 2^{\left(\sum_{r=1}^n a_r\right)} \\
&= 2^{a_{r+1}} * \prod_{r=1}^n 2^{a_r} \\
&= \prod_{r=1}^{n+1} 2^{a_r}
\end{aligned}$$

As we can see, these two are equal at n=1 and all subsequent values, so this holds true. This was achieved with mathematical induction.

4. ($\star$) For any $k \in \mathbb{N}$, and any real numbers r, s, we have

$$r^k - s^k = (r-s) \sum_{\substack{p+q=k-1 \\ p,q \geq 0}} r^q s^p = (r-s)\left(r^{k-1} + r^{k-2}s + r^{k-3}s^2 + \cdots + r^2 s^{k-3} + rs^{k-2} + s^{k-1}\right)$$

5. Consider the possible results of flipping a fair coin n times. There are 2^n possible outcomes.

Claim 3. *Given n coin tosses, there are 2^n outcomes.*

Proof by Induction 3. We can first determine that 2^n does satisfy the base case of one coin flip, which we know has two possibilities. We must then solve for 2^1, which is 2, therefore verifying the base case. We can now use k with $k = n$ and solve for $k+1$, or2^{k+1}:

$$2^{n+1}$$
$$2(2^n)$$

Since the amount of flips will be twice the previous amount, as you can do all of the previous flip possibilities on heads and again on tails, this confirms that 2^n accurately describes the probability of heads and tails through Mathemartical Induction, we must conclude that given n coin tosses, there are a total of 2^n possibilities. $\square$

6. ($\star$) For any natural number q,

$$\sum_{i=1}^{2^q} \frac{1}{i} \geq 1 + \frac{q}{2}$$

7. Prove the power rule for derivatives: for any $n \in \mathbb{N}$, we have $\frac{d}{dx}[x^n] = nx^{n-1}$. You may **only** use the product rule for derivatives and the fact that $\frac{d}{dx}x = 1$.

   Claim 4. *The derivative of a function in the form of* $x^n = nx^{n-1}$.

   Proof by Induction 4. We can confirm the base case by confirming that $\frac{d}{dx}(x^1) = \frac{d}{dx}(x) = 1$. Now that we have proven the base case, we can use the inductive step in which $n = k+1$.

$$\begin{aligned}
nx^{n-1} &= \frac{d}{dx}x^{k+1} \\
&= \frac{d}{dx}x(x^k) \\
&= x^k + x\frac{d}{dx}x^k && \text{(Applied product rule)} \\
&= x^k + x(kx^{k-1}) && \text{(Substituted with formula } \tfrac{d}{dx}x^n \text{ given in claim)} \\
&= x^k + kx^k \\
&= (k+1)x^{(k+1)-1}
\end{aligned}$$

   As we can see, the input of k+1 in $\frac{d}{dx}x^n$ resulted in the same values being replaced in the second part. Therefore, through proof by mathematical induction, we can confirm that $x^n = nx^{n-1}$. $\square$

8. ($\star$) The **power rule for integrals**: for any $n \in \mathbb{N}$, we have

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

   for some constant C.
   You **may not** use the previous result. You may use **only** the following calculus facts: the linearity properties of the integral; $\int C dx = Cx + D$ for some constant D; $\frac{d}{dx}x = 1$; integration by parts.

9. $\frac{d^r}{dx^r}x^r = r!$

   Claim 5. $\frac{d^r}{dx^r}x^r = r!$

   Proof by Induction 5. We must first confirm this is through with the base case n=1. $\frac{d^1}{dx^1}x^1 = 1! = 1$. Now that we know the base case holds true, we can use the inductive step by having $r = k$ and substituting n for $r + 1$.

$$\begin{aligned}
r! &= \frac{d^{k+1}}{dx^{k+1}} x^{k+1} \\
&= \frac{d^k}{dx^k} \frac{d}{dx} x^{k+1} \\
&= (k+1) \frac{d^k}{dx^k} x^k && \text{(Uses Claim 4 as well as the constant multiple rule of derivatives.)} \\
&= (k+1)k! && \text{(Substitutes equation given in the claim.)} \\
&= (k+1)! && \text{(Uses definition of factorial)}
\end{aligned}$$

Therefore, through mathematical induction, $\frac{d^r}{dx^r} x^r = r!$ is true. □

10. The **constant multiple rule for higher derivatives**: for any function f with at least n derivatives and any constant c, we have $\frac{d^n}{dx^n}[cf] = c\left[\frac{d^n}{dx^n} f\right]$. You may assume the constant multiple rule for derivatives.

    Claim 6. *For any function f with at least n derivatives and any constant c, we have $\frac{}{d^n} dx^n [cf] = c\left[\frac{d^n}{dx^n} f\right]$.*

    Proof by Induction 6. We can first prove the base case by seeing that $\frac{d^1}{dx^1}[cf] = c\frac{d}{dx} f$ due to the constant multiple rule for derivatives. Next, we can use $k = n$ and substitute $k + 1$ to perform the inductive step:

$$\begin{aligned}
c\frac{d^n}{dx^n}[f] &= \frac{d^{k+1}}{dx^{k+1}}[cf] \\
&= \frac{d}{dx}\left(\frac{d^k}{dx^k}[cf]\right) \\
&= \frac{d}{dx}\left(c\frac{d^k}{dx^k}[f]\right) && \text{(Substituted using equation given in claim)} \\
&= c\frac{d}{dx}\left(\frac{d^k}{dx^k}[f]\right) && \text{(Used constant multiple rule for derivatives)} \\
&= c\frac{d^{k+1}}{d^{k+1}}[f]
\end{aligned}$$

    Therefore, we can see this works for n as well as n+1, therefore through mathematical induction $\frac{d^n}{dx^n}[cf] = c\left[\frac{d^n}{dx^n} f\right]$. □

11. The **sum rule for higher derivatives**: for any functions f, g with at least n derivatives, we have $\frac{d^n}{dx^n}[f+g] = \frac{d^n}{dx^n} f + \frac{d^n}{dx^n} g$. You may assume the sum rule for derivatives.

    Claim 7. *for any functions f, g with at least n derivatives, we have $\frac{d^n}{dx^n}[f+g] = \frac{d^n}{dx^n} f + \frac{d^n}{dx^n} g$.*

    Proof by Induction 7. We must first prove the base case, or with n=1. We can see that $\frac{d}{dx}[f+g] = \frac{d}{dx} f + \frac{d}{dx} g$ is true due to the sum rule for derivatives. Now that we know this is true, we can use k=n and solve for k+1.

$$
\begin{aligned}
\frac{d^{k+1}}{dx^{k+1}}f + \frac{d^{k+1}}{dx^{k+1}}g &= \frac{d^{k+1}}{dx^{k+1}}[f+g] \\
&= \frac{d}{dx}\frac{d^k}{dx^k}[f+g] \\
&= \frac{d}{dx}\frac{d^k}{dx^k}f + \frac{d}{dx^k}g && \text{(Substituted with equation given in claim)} \\
&= \frac{d^{k+1}}{dx^{k+1}}f + \frac{d^{k+1}}{dx^{k+1}}g
\end{aligned}
$$

Therefore, since $\frac{d^n}{dx^n}[f+g] = \frac{d^n}{dx^n}f + \frac{d^n}{dx^n}g$ is true in the base case of n=1 and for n+1, through mathematical induction this is true. □

12. (⋆) The **Binomial Theorem**: for any real numbers x, y, and any $n \in \mathbb{N}$,

$$(x+y)^n = B_{n,0}x^n + B_{n,1}x^{n-1}y + B_{n,2}x^{n-2}y^2 + \cdots + B_{n,n-2}x^2y^{n-2} + B_{n,n-1}xy^{n-1} + B_{n,n}y^n$$

(*Hint.* At some point you will need to "combine like terms".)

13. (⋆) Any legal word has more as than bs.

Claim 8. *Any legal word has more as than bs.*

Proof by Induction 8. We can see through the base case, aba, that there are more as than bs. Next, we must realize that all of the transformations (legal alterations of an existing legal word which provide another legal word) are the same in terms of the amount of as and bs added, so if can be proven for any of the transformations that there are more as and bs, then it is true for all of them. These transformations offer an inductive step, and since all of them offer an equal amount of as and bs, and the base case has an equal amount of them, there will always be more as than bs by mathematical induction. □