MA 225 Problem Set 6

facts and definitions You will need the following definitions and facts (at some point):

> **Definition 1.** We call a relation R on a set A *symmetric* if for any $x, y \in A$, $x \, \mathrm{R} \, y$ guarantees $y \, \mathrm{R} \, x$. We call R *reflexive* if for any $x \in A$, $x \, \mathrm{R} \, x$. We call R *transitive* if for any $x, y, z \in A$, $x \, \mathrm{R} \, y$ and $y \, \mathrm{R} \, z$ together guarantee $x \, \mathrm{R} \, z$.
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> **Definition 2.** We call a relation R on the set A *intransitive* if for any $x, y, z \in A$, $x \, \mathrm{R} \, y$ and $y \, \mathrm{R} \, z$ together guarantee $\sim (x \, \mathrm{R} \, z)$.

exercises These problems don't require you to write proofs.

1. If A has exactly a elements and B has exactly b elements, how many relations are there from A to B? ab
2. Some sources call a relation R symmetric if for any $x, y \in A$, $x \, \mathrm{R} \, y \Leftrightarrow y \, \mathrm{R} \, x$. Explain two things: first, why this definition is different from ours, and second, why any relation which is symmetric according to this definition will be symmetric according to ours and vice versa. (You may be able to turn the second explanation into a formal $\star$ proof.)
3. For each of the following, explain whether or not it is reflexive, symmetric, transitive (in our universe)? Explain your answers
   (a) F: $(x, y) \in F$ if x is y's father. It is not reflexive, as that would be $F(x, x)$, or that x is his own father. It is not symetric, as $F(x, y)$ means x is y's father, as $F(y, x)$ means y is x's father, which is not the same.
   (b) $\neq$, as a relation on $\mathbb{R}$ No, as given $a \in \mathbb{R}$, there is no a which $a \neq a$.
   (c) "lives within one mile of"
   (d) S: $x \, \mathrm{S} \, y$ if $x^2 + y^2 = 1$
4. For each relation in exercise 3, explain what R_R, S_R, and T_R represent for that relation.
5. R_R is called the *reflexive closure of* R. Explain why someone might call it *the smallest reflexive relation which contains* R.
6. (a) Make a blueprint for the claim *If blah blah and such, then R is reflexive.*
   (b) Make a blueprint for the claim *If blah blah and such, then S is intransitive.*

proofs Prove the following claims.

1. ($\star$) Give an example which shows that we could have

$$(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$$

   (Prove that your example does what you say it does.)
2. Prove the clauses of Theorem 6.2.13: for any sets A, B, C, D and relations R from A to B, S from B to C, and T from C to D,
   (a) $\left(R^{-1}\right)^{-1} = R$

   Claim 1. $\left(R^{-1}\right)^{-1} = R$

   Proof. Let $x \in (\left(R^{-1}\right)^{-1}$ and $x = (a, b)$, where $a \in A$ and $b \in B$. So $(y, x) \in R^{-1}$. So $(x, y) \in R$. Therefore $x \in R$, so $\left(R^{-1}\right)^{-1}$.
   Let $m \in R$ and $x = (g, h)$ where $g \in A$ and $h \in B$. □

   (b) $T \circ (S \circ R) = (T \circ S) \circ R$ (What's a good way to refer to this fact?)

   (c) $I_B \circ R = R$ and $R \circ I_A = R$
   (d) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

3. ($\star$) Suppose A is a set on which $\varnothing$ is a reflexive relation. This says something very strong about A.
4. The complement of a symmetric relation is symmetric.
5. Let R be a relation on the set A.
   (a) Show that $R_R = I_A \cup R$ is reflexive.
   (b) Show that if S is a reflexive relation on A with $R \subseteq S$, then $R_R \subseteq S$.
6. Let R be a relation on the set A.
   (a) Show that $S_R = R \cup R^{-1}$ is a symmetric relation.
   (b) Show that if S is any symmetric relation on A and $R \subseteq S$, then $S_R \subseteq S$.
7. Let R be a relation on the set A.
   (a) Show that $S_R = R \cup R^{-1}$ is a symmetric relation.
   (b) Show that if S is any symmetric relation on A and $R \subseteq S$, then $S_R \subseteq S$.
8. Let R be a relation on the set A.
   (a) Show that R is symmetric if and only if $R^{-1} = R$.
   (b) Show that R is transitive if and only if $R \circ R \subseteq R$.
9. Let R be a relation on the set A. Let G be a relation on A with the property that: *if S is any symmetric relation on A and $R \subseteq S$, then $G \subseteq S$.* Show that such G is unique.
10. Let R be a relation on the set A. Define T_R, a relation on A, by $x \, \mathrm{T_R} \, y$ if there are $a_0, a_1, \ldots, a_k \in A$ with $x = a_0$, $y = a_k$, and $a_i \, \mathrm{R} \, a_{i+1}$ for each $i = 0, 1, \ldots, k-1$.
    (a) ($\star$) Show that for any relation R, T_R is transitive.
    (b) ($\star\star$) If S is a transitive relation on A with $R \subseteq S$, show that $T_R \subseteq S$.