MA 225 Problem Set 9

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facts and definitions

Definition 1. Let $(A, \preceq)$ and $(B, \vdash)$ be partially-ordered sets (that is, $\preceq$ is a partial order on A and $\vdash$ is a partial order on B). We call $f : A \to B$ *monotone* if one of the following two conditions holds:

- $\forall x_1, x_2 \in A, x_1 \preceq x_2 \Rightarrow f(x_1) \vdash f(x_2)$ or
- $\forall x_1, x_2 \in A, x_1 \preceq x_2 \Rightarrow f(x_2) \vdash f(x_1)$

If $f : A \to B$ is monotone with respect to $\preceq$ and $\vdash$, we often write $f : (A, \preceq) \to (B, \vdash)$.

exercises These problems don't require you to write proofs.

1. Let $f : X \to Y$. Explain why $f_* : 2^X \to 2^Y$. Since the push forward is used to describe all of $x \in X$, and theyre relation to elements $y \in Y$, one can break up these relations into sets of subsets 2^X and relate them with their corrosponding elements in 2^Y.
2. In the definition of *monotone*, $\vdash$ does not have to be a total order. But let's assume it was. Explain two things: first, why someone might think *every* function to a totally ordered set is monotone; second, what mistake they would be making. One might think every totally ordered set is monotone since every term is comparable, given a $f(x_1), f(x_2)$, $f(x_1) \vdash f(x_2)$ or $f(x_2) \vdash f(x_1)$. However, only one can hold, so given leq, and given f maps to $\{1\}$, both would hold, and therefore this wouldn't be a monotone.

proofs Write a complete proof for each of the following statements.

1. Show that for any function $f : A \to B$, there is at most one function $g : B \to A$ so that $g \circ f = I_A$ and $f \circ g = I_B$.

   Claim 1. For any function $f : A \to B$, there is at most one function $g : B \to A$ so that $g \circ f = I_A$ and $f \circ g = I_B$.

   Proof. Let us assume there is another function, h which satisfies the conditions of g. For $h = g$, they must have the same domain and same rule. Since By definition $domain(h) = B$ and $domain(g) = B$, this is satisfied. Now let $f(a) = b$ where $a \in A$ and $b \in B$. So $h(f(a)) = a$ (Identity). $Soh(b) = a$. Now consider $g(f(a)) = a$. So $g(b) = a$ (Identity). So $h = g$, so g is unique. □

2. If $f : A \to B$ and there is a function $g : B \to A$ with the property that $g \circ f = I_A$ and $f \circ g = I_B$, then $f^{-1} : B \to A$.

   Claim 2. If $f : A \to B$ and there is a function $g : B \to A$ with the property that $g \circ f = I_A$ and $f \circ g = I_B$, then $f^{-1} : B \to A$.

   Proof. For $f^{-1} : B \to A$, the inverse must exist as a function. We'll show $g = f^{-1}$. Since $DOMAIN(g) = DOMAIN(f^{-1})$, this satisfies the first condition of functions being equal. Assume $f(a) = b$. Then $f^{-1}(b) = a$ (by definition of inverse). So $f^{-1} \circ f = I_A$ (composition), $f \circ f^{-1} = I_B$ (composition). Since these are the same rules as g, and f^{-1} shares a domain with g, $f^{-1} = g$, and therefore f^{-1} exists. □

3. Let $f : X \to Y$ be a function. Assume that $f^{-1} : Y \to X$.

   (a) Show that for any $Z \subseteq Y$, $(f^{-1})_*(Z) = f^*(Z)$.

Claim 3. For any $Z \subseteq Y$, $(f^{-1})_*(Z) = f^*(Z)$.

Proof. Functions are equal when they have the same rule and Domain. $Z = Domain(f_*^{-1}) = Domain(f^*(Z))$. Given $(f^*(z) = x$ for $z \in Z$ and $x \in X$. Therefor $f^{-1*}(x) = z$ (Inverse). Therefore the pullback can be defined as $f_*^{-1}(z) = x$. Since both functions have the same rule and domain, they are equal. □

(b) Explain why that makes it kind of okay that *SEStA* (and many others) use the notation $f^{-1}(C)$ where we use $f^*(C)$. Because $f^*(C)$ is equal to the inverse if f is bijective.

4. Let $h : X \to Y$ be a function and $C \subseteq Y$. Show that $h^*(C^c) = (h^*(C))^c$.

Claim 4. Let $h : X \to Y$ be a function and $C \subseteq Y$. Show that $h^*(C^c) = (h^*(C))^c$.

Proof. Let $j \in h^*(C^c)$. So $j = (a, b)$ where $a \in C^c$ and $b \in X$. Since $h : X \to Y$, for all $x_1, x_2 \in X$, if $h(x_1) = h(x_2)$, then $x_1 = x_2$.
So $j \in (h^*(C))^c$ □

5. Let $f : X \to Y$ be a function, A, B subsets of X, and C, D subsets of Y.

(a) $f_*(A \cup B) = f_*(A) \cup f_*(B)$.

Claim 5. $f_*(A \cup B) = f_*(A) \cup f_*(B)$.

Proof. Let $j \in f_*(A \cup B)$. So $j = f(a)$ where $a \in A \cup B$. So $a \in A$ or $a \in B$. So either $j \in f_*(A)$ or $j \in f_*(B)$. Therefore $j \in f_*(A \cup B)$.

Let $k \in f_*(A) \cup f_*(B)$. So $k = f(b)$ where $b \in X$. So $k \in f_*(A)$ or $k \in f_*(B)$. So $b \in A \cup B$. Therefore $k \in f_*(A \cup B)$. So $f_*(A \cup B) = f_*(A) \cup f_*(B)$. □

(b) $f_*(A \cap B) \subseteq f_*(A) \cap f_*(B)$.

Claim 6. $f_*(A \cap B) \subseteq f_*(A) \cap f_*(B)$.

Proof. Let $j \in f_*(A \cap B)$. So $j = f(a)$ where $a \in A \cap B$. So $a \in A$ and $a \in B$. Since j maps to a unique output, and $a \in A \cap B$, $f_*(A) \cap f_*(B)$. Therefore $a \in f_*(A \cap B) \subseteq f_*(A) \cap f_*(B)$. □

(c) $f^*(C \cup D) = f^*(C) \cup f^*(D)$.

Claim 7. $f^*(C \cup D) = f^*(C) \cup f^*(D)$.

Proof. Let $x \in f^*(C \cup D)$. So $x \in X$. Since f is a function, elements in X can only be related to one element in Y. So $x \in f(C) \cup f(D)$ (since $x \in f^*(C \cup D)$ and $f^*(C \cup D)$ are linked to a unique element in $C \cup D$, they must be related to the element in $f^*(C)$ or $f^*(D)$). So $f^*(C \cup D) \subseteq f^*(C) \cup f^*(D)$.

Let $x \in f^*(C) \cup f^*(D)$. So $x \in X$. Since f is a function, elements in X can only be related to one element in Y. So $x \in f^*(C)$ or $x \in f^*(D)$. Since x is linked with a unique element in C or D (Definition of a function), $x \in f^*(C \cup D)$. So $f^*(C \cup D) = f^*(C) \cup f^*(D)$. □

(d) $f^*(C \cap D) = f^*(C) \cap f^*(D)$.

Claim 8. $f^*(C \cap D) = f^*(C) \cap f^*(D)$.

Proof. Let $x \in f^*(C \cap D)$. So $x \in X$. Since x can only be related to one term in $C \cap D$ (definition of a function), $x \in f^*(C) \cap f^*(D)$ (Since it can't be related to additional terms).

Let $x \in f^*(C) \cap f^*(D)$. So $x \in X$. So either $x \in f^*(C)$ or $x \in f^*(D)$. Since x can only be related to one term in $C \cap D$ (definition of a function), $x \in f^*(C \cap D)$ (Since it can't be related to additional terms). So $f^*(C \cap D) = f^*(C) \cap f^*(D)$. $\square$

6. Let $g : X \to Y$ be a function. Show that for any finite collection $A_1, \ldots, A_n$ of subsets of X, we have
$$g_* \left(\bigcup_{k=1}^{n} A_k \right) = \bigcup_{k=1}^{n} g_*(A_k)$$
(*Hint.* n is a natural number.)

Proof. (Proof by Mathematical Induction)
Base Case (n=1): $g_* A_1 = g_* A_k$

Inductive hypothesis $r = n$: $g_* \left(\bigcup_{k=1}^{r} A_k \right) = \bigcup_{k=1}^{r} g_*(A_k)$

Inductive Step:
$$\begin{aligned}
\bigcup_{k=1}^{r+1} g_*(A_k) &= \bigcup_{k=1}^{r+1} g_*(A_k) \\
&= \bigcup_{k=1}^{r} g_*(A_k) \cup g_*(A_{r+1}) \\
&= g_* \left(\bigcup_{k=1}^{r} A_k \right) \cup g_*(A_{r+1}) && \text{(Used inductive hypothesis)} \\
&= g_* \left(\bigcup_{k=1}^{r+1} A_k \right)
\end{aligned}$$
(Since input is either A_{r+1} or $g_* \left(\bigcup_{k=1}^{r} A_k \right)$, the input of g_* is $\bigcup_{k=1}^{r+1} A_k$)

Therefore, through proof by mathematical induction, $g_* \left(\bigcup_{k=1}^{n} A_k \right) = \bigcup_{k=1}^{n} g_*(A_k)$. $\square$

7. Give an example which shows that we might not have equality in 5(b).

Proof. Let $f(x) = x^2$, $A = \{-3, -2, -1, 0\}$ and $b = \{0, 1, 2, 3\}$. So $f_*(A) \cap f_*(B)$, $f_*(A) = \{4, 1, 0, 9\}$ and $y f_*(B) = \{0, 1, 4, 9\}$, so $f_*(A) \cap f_*(B) = \{1, 4, 0, 9\}$. $f_*(A \cap B) = \{0\}$, so $f_*(A \cap B) \neq f_*(A) \cap f_*(B)$. $\square$

8. ($\star$) Let $f : X \to Y$. Assume that f is onto. Let $\mathcal{P}$ be a partition of Y. Define
$$f^*\mathcal{P} = \{f^*(A) | A \in \mathcal{P}\}$$
Show that $f^*\mathcal{P}$ is a partition of X.

Claim 9. $f^*\mathcal{P}$ is a partition of X.

Proof. Since $\mathcal{P}$ is a partition on Y, $Y = \bigcup_{x \in \mathcal{P}} P$. Since Y is surjective on X, $f^*\mathcal{P} \subseteq X$, and according to the definition of a function $f^*\mathcal{P} = X$ ($f^*\mathcal{P} \subseteq X$ since Y is surjective and partitioned by $\mathcal{P}$ and $X \subseteq f^*\mathcal{P}$ as by definition of a function, all elements in X are related to an element in Y). Furthermore, since $\forall x_1, x_2 \in X$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$, this implies that given an element, $y \in Y$, $f^*(y)$ will be unique so $\bigcup_{Y \in \mathcal{P}} f^*(Y)$. Since this satisfies all conditions of a partition, $f^*\mathcal{P}$ is a partition on X. $\square$

9. ($\star\star$) Assume that $h : X \to Y$ is onto. Consider an equivalence relation R on Y. Give an explicit expression for the equivalence classes of h^*R (defined in HW 8) in terms of the other objects in play. (*Hint.* Recall that equivalence classes of h^*R are subsets of X. Consider another problem on this sheet.)

10. ($\star$) Let $f : X \to Y$. Assume f is injective. If $\preceq$ is a partial order on Y, show that the relation $\vdash$ given by

$$\alpha \vdash \beta \text{ iff } f(\alpha) \preceq f(\beta)$$

is a partial order on X.

Claim 10. Let $f : X \to Y$. Assume f is injective. If $\preceq$ is a partial order on Y, show that the relation $\vdash$ given by

$$\alpha \vdash \beta \text{ iff } f(\alpha) \preceq f(\beta)$$

is a partial order on X.

Proof. (Reflexivity) Let $x \in X$. Since $f(x) \preceq f(x)$ (by definition of partial order, and since x maps to a unique value), $x \vdash x$ (Since $f(x)$ goes to a unique value due to injectivity). So $\vdash$ is reflexive on X.

(Transitivity) Let $x, y \in (\vdash, X)$ where $x = (a, b)$ and $y = (b, c)$. By the definition of $\preceq$, $f(a) \preceq f(b)$ and $f(b) \preceq f(c)$. Since $\preceq$ is a parital order, $f(a) \preceq f(c)$. By definition of $\vdash$ and a function, $a \vdash c$. So $\vdash$ is transitive on X.

(Antisymmetry) Let $x, y \in (\vdash, X)$ where $x = (a, b)$ and $y = (b, a)$. By the definition of $\preceq$, $f(a) \preceq f(b)$ and $f(b) \preceq f(a)$. Since $\preceq$ is a parital order, $a = b$. So since if $x, y \in (\vdash, X)$ $a = b$, $\vdash$ is antisymmetric. So $\vdash$ is a partial order on X. $\square$

11. ($\star$) Let $(Z, \preceq)$ and $(W, \vdash)$ be partially ordered sets. Assume $\preceq$ is a total order. Assume $g : (Z, \preceq) \to (W, \vdash)$ is monotone. Show that g is injective.