MA 225 Problem Set 11

exercises 1. Write three blueprints for a proof that *If blah blah and so on, then A is finite.*: each blue print should use a different approach. Consider $\phi : A \to \mathbb{N}_k$

...

Since if $\phi(a) = \phi(b)$ means $a = b$, ϕ is injective.

...

Since $\forall g \in \mathbb{N}_k, \exists a \in A : \phi(a) = g$, A is surjective. So A is finite.

Consider $\phi : A \to \mathbb{N}_k$.

...

So there exists a $\phi^{-1} : \mathbb{N}_k \to A$. So there exists a bijection between A and N_k, A is finite.

Consider the set A with the rule blah blah blah

...

So $A = \varnothing$, so A is finite.

2. Write two blueprints for a proof that *If blah blah and so on, then A is infinite.*.
Let A be finite.
Blah blah blah...
So $x \in A$. By the original definition, $x \notin A$. So A is infinite by contradiction.

   Consider $x \in B$ where B is an infinite set.
   Since $x \in A$, $B \subseteq A$. So A is infinite.

3. Why does problem 11 make the assumption that S has at least two elements? Because if it has less than 2, then there are no numbers, c which $a < c < b$ as there is no b.

proofs Write a complete proof for each of the following statements.

1. Long ago, we gave a bit of a handwavey argument that the product of two finite sets was finite. Let A be a finite set (say, with cardinality n) and B be a finite set (say, with cardinality m). Give a concrete proof that $A \times B$ is finite by explicitly constructing a bijection between $A \times B$ and some $\mathbb{N}_k$.

   Claim 1. *$A \times B$ is finite if A and B are finite.*

   Proof. $A \approx \mathbb{N}_n$ and $B \approx \mathbb{N}_m$. By proof 3, $A \times B \approx N_n \times N_k$. Since there are $n * k$ objects in $\mathbb{N}_n \times \mathbb{N}_k$, $\mathbb{N}_n \times \mathbb{N}_k \approx \mathbb{N}_j$. So $A \times B \approx \mathbb{N}_j$ (Transitivity) and therefore is finite. □

2. If $A_1, \ldots, A_p$ are each finite sets, then $\bigtimes_{k=1}^{p} A_k = \{(x_1, \ldots, x_p) | x_i \in A_i\}$ is finite.

   Claim 2. *If $A_1, \ldots, A_p$ are each finite sets, then $\bigtimes_{k=1}^{p} A_k = \{(x_1, \ldots, x_p) | x_i \in A_i\}$ is finite.*

   Proof. Base Case($p = 1$): A_1 is finite (by def)

   Inductive hypothesis($p = r$): $\bigtimes_{k=1}^{p} A_k$ is finite.

Inductive step$(r+1)$: We'll show $\bigtimes_{k=1}^{r+1} A_k$ is finite:

$$\bigtimes_{k=1}^{r+1} A_k$$
$$\bigtimes_{k=1}^{r} A_k \times A_{r+1}$$

Since A_{r+1} and $\bigtimes_{k=1}^{r} A_k$ are finite, $\bigtimes_{k=1}^{r} A_k \times A_{r+1}$ is finite (proof 1). So by PMI, $\bigtimes_{k=1}^{p} A_k = \{(x_1, \ldots, x_p) | x_i \in A_i\}$ is finite. □

3. Write out in detail the proof that if $A \approx X$ and $B \approx Y$, then $A \times B \approx X \times Y$.

   Claim 3. *If $A \approx X$ and $B \approx Y$, then $A \times B \approx X \times Y$.*

   Proof. Consider the bijections $\psi : A \to X$ and $\phi : B \to Y$.
   Let $\psi \times \phi : A \times B \to X \times Y$ which $(a, b) \mapsto (\psi(a), \phi(b))$ where $a \in A$ and $b \in B$.
   Injectivity: Let $\psi \times \phi((a_1, b_1)) = \psi \times \phi((a_2, b_2))$. In particular, $\psi(a_1) = \psi(a_2)$, so $a_1 = a_2$ (injectivity of ψ). Also $\phi(b_1) = \phi(b_2)$, so $b_1 = b_2$. Therefore $\psi \times \phi$ is injective.

   Surjectivity: Let $(x, y) \in X \times Y$ where $x \in X$ and $y \in Y$. So $\exists a \in A$ such that $\psi(a) = x$ and $\exists b \in B$ such that $\phi(b) = y$. So $\psi \times \phi((x, y)) = (a, b)$, so $\psi \times \phi$ is bijective. Therefore $A \times B \approx X \times Y$. □

4. ($\star\star$) It's a fact of arithmetic that for any natural numbers b, m, n, $b^{m+n} = b^m b^n$. State, and prove, the set equivalence which reduces to this fact if the three sets involved happen to be finite.

   Proof. □

5. ($\star\star$) It's a fact of arithmetic that for any natural numbers b, m, n, $b^{mn} = (b^m)^n$. State, and prove, the set equivalence which reduces to this fact if the three sets involved happen to be finite.

6. Let A be an infinite set and R an equivalence relation on A with the property that each equivalence class $[a]_R$ is finite. Show that $A_{/R}$ is infinite.

   Claim 4. *If A is an infinite set and R an equivalence relation on A with the property that each equivalence class $[a]_R$ is finite, then $A_{/R}$ is finite.*

   Proof. Assume $A_{/R}$ is finite. So $\bigcup_{x \in A_{/R}} x = A$. Since $\bigcup_{x \in A_{/R}} x$ is finite (Proof 9 Problem set 10) and A is not. So by contradiction, $A_{/R}$ cannot be finite, and therefore must be infinite. □

7. Show that the union of any collection of finitely many denumerable sets is denumerable; that is, for any $n \in \mathbb{N}$, if $A_1, \ldots, A_n$ are each denumerable, then $\bigcup_{k=1}^{n} A_k$ is denumerable. (*Hint.* n is a natural number.)

   Claim 5. *The union of any collection of finitely many denumerable sets is denumerable; that is, for any $n \in \mathbb{N}$, if $A_1, \ldots, A_n$ are each denumerable, then $\bigcup_{k=1}^{n} A_k$ is denumerable.*

Proof. Base case(n=1): $\bigcup_{k=1}^{n} A_k = A_1$ (denumerable by definition)

Inductive Hypothesis($r = n$): $\bigcup_{k=1}^{r} A_k$ is denumerable.

Inductive Step:

$$\bigcup_{k=1}^{r+1} A_k \qquad\qquad \bigcup_{k=1}^{r} A_k \cup A_r$$

So $\bigcup_{k=1}^{r} A_k \approx \mathbb{N}$ (inductive hypothesis). Since adding a denumerable number keeps a set denumerable, $\bigcup_{k=1}^{r} A_k \cup A_r$ is denumerable. By PMI, $\bigcup_{k=1}^{n} A_k$ is denumerable. □

8. Let A be a countable set and R an equivalence relation on A. Show that $A_{/R}$ is countable.

   Claim 6. *$A_{/R}$ is countable.*

   Proof. Assume $A_{/R}$ is not countable and A is countable. So $\bigcup_{a \in A} [a]_R = A$. However we earlier established A is countable, and $\bigcup_{a \in A} [a]_R$ isn't, this is a contradiction and so $A_{/R}$ is countable. □

9. Show that the set of all lines in the plane with rational slope and rational intercept is denumerable. (*Hint.* Write each line in slope-intercept form.)

10. ($\star$) No powerset is denumerable.

    Claim 7. *No powerset is denumerable.*

    Proof. Let A be a set.
    Case A is finite: Then $card(2^A) = 2^{card(A)}$. Since this is finite, 2^A is not denumerable.

    Case A is denumerable: Assume 2^A is denumerable. So $2^A = A_1, A_2, \cdots$ for infinite sets. Let $B = b_1, b_2, ...$ be a subset of A. Consider A_n term. Change the nth element to k in A_n and define $b_n = k$. Since $b_n \notin 2^A$ (since it differs by definition from every element in 2^A), this is a contradiction, as by definition $b_n \in 2^A$. So 2^A isn't denumerable in this case.

    Case A is uncountable: Since 2^A is not smaller than A, 2^A is not denumerable.
    So no powerset is denumerable. □

    b

11. ($\star\star$) Let $S \subseteq \mathbb{R}$ be a set with the property: given any $a, b \in S$ with $a < b$, there is some $c \in S$ with $a < c < b$. Assume S has at least two elements.

    (a) Show that S is infinite.

    Claim 8. *S is infinite.*

    Proof. Assume S is finite. Then $S = s_1, ...s_n$ where $n \in \mathbb{N}$ and $s_1 < s_2 < ... < s_n$. Consider $c = \frac{s_1+s_2}{2}$.Since $s_1, s_2 \in S$ and $s_1 < c < s_2$. So $c \in S$. However we claimed to have a complete list, and since there were no elements between s_1 and s_2, this is a contradiction. So S is infinite. □

    (b) Show that between any two distinct elements of S, there are infinitely many elements of S.

Claim 9. *Between any two distinct elements of S, there are infinitely many elements of S.*

Proof. Let $a, b \in A$. Consider the set C where $a, b \in C$ where $C = a, c_2...b$ and $a < c_2 < ... < b$. So C is infinite (previous proof). Since all elements of C are $< a$ and $> b$, there exists an infinite amount of elements between any two distinct points in A. $\square$