

MA 225 Problem Set 9

facts and definitions

Definition 1. Let (A, \preceq) and (B, \vdash) be partially-ordered sets (that is, \preceq is a partial order on A and \vdash is a partial order on B). We call $f : A \rightarrow B$ *monotone* if one of the following two conditions holds:

- $\forall x_1, x_2 \in A, x_1 \preceq x_2 \Rightarrow f(x_1) \vdash f(x_2)$ or
- $\forall x_1, x_2 \in A, x_1 \preceq x_2 \Rightarrow f(x_2) \vdash f(x_1)$

If $f : A \rightarrow B$ is monotone with respect to \preceq and \vdash , we often write $f : (A, \preceq) \rightarrow (B, \vdash)$.

exercises These problems don't require you to write proofs.

1. Let $f : X \rightarrow Y$. Explain why $f_* : 2^X \rightarrow 2^Y$.
2. In the definition of *monotone*, \vdash does not have to be a total order. But let's assume it was. Explain two things: first, why someone might think *every* function to a totally ordered set is monotone; second, what mistake they would be making. One might think every totally ordered set is monotone since every term is comparable, given a $f(x_1), f(x_2)$, $f(x_1) \vdash f(x_2)$ or $f(x_2) \vdash f(x_1)$. However, only one can hold, so given *leq*, and given f maps to $\{1\}$, both would hold, and therefore this wouldn't be a monotone.

proofs Write a complete proof for each of the following statements.

1. Show that for any function $f : A \rightarrow B$, there is at most one function $g : B \rightarrow A$ so that $g \circ f = I_A$ and $f \circ g = I_B$.

Claim 1. For any function $f : A \rightarrow B$, there is at most one function $g : B \rightarrow A$ so that $g \circ f = I_A$ and $f \circ g = I_B$.

Proof. Let us assume there is another function, h which satisfies the conditions of g . For $h = g$, they must have the same domain and same rule. Since By definition $\text{domain}(h) = B$ and $\text{domain}(g) = B$, this is satisfied. Now let $f(a) = b$ where $a \in A$ and $b \in B$. So $h(f(a)) = a$ (Identity). So $h(b) = a$. Now consider $g(f(a)) = a$. So $g(b) = a$ (Identity). So $h = g$, so g is unique. \square

2. If $f : A \rightarrow B$ and there is a function $g : B \rightarrow A$ with the property that $g \circ f = I_A$ and $f \circ g = I_B$, then $f^{-1} : B \rightarrow A$.
3. Let $f : X \rightarrow Y$ be a function. Assume that $f^{-1} : Y \rightarrow X$.

(a) Show that for any $Z \subseteq Y$, $(f^{-1})_*(Z) = f^*(Z)$.

Claim 2. For any $Z \subseteq Y$, $(f^{-1})_*(Z) = f^*(Z)$.

Proof. Functions are equal when they have the same rule and Domain. $Z = \text{Domain}(f_*^{-1}) = \text{Domain}(f^*(Z))$. Given $(f^*(z) = x$ for $z \in Z$ and $x \in X$. Therefore $f^{-1*}(x) = z$ (Inverse). Therefore the pullback can be defined as $f_*^{-1}(z) = x$. Since both functions have the same rule and domain, they are equal. \square

- (b) Explain why that makes it kind of okay that *SEStA* (and many others) use the notation $f^{-1}(C)$ where we use $f^*(C)$. Because $f^*(C)$ is equal to the inverse if f is bijective.
4. Let $h : X \rightarrow Y$ be a function and $C \subseteq Y$. Show that $h^*(C^c) = (h^*(C))^c$.

Claim 3. Let $h : X \rightarrow Y$ be a function and $C \subseteq Y$. Show that $h^*(C^c) = (h^*(C))^c$.

Proof. Let $j \in h^*(C^c)$. So $j = (a, b)$ where $a \in C^c$ and $b \in X$. Since $h : X \rightarrow Y$, for all $x_1, x_2 \in X$, if $h(x_1) = h(x_2)$, then $x_1 = x_2$. So $j \in (h^*(C))^c$ □

5. Let $f : X \rightarrow Y$ be a function, A, B subsets of X , and C, D subsets of Y .

(a) $f_*(A \cup B) = f_*(A) \cup f_*(B)$.

Claim 4. $f_*(A \cup B) = f_*(A) \cup f_*(B)$.

Proof. Let $j \in f_*(A \cup B)$. So $j = f(a)$ where $a \in A \cup B$. So $a \in A$ or $a \in B$. So either $j \in f_*(A)$ or $j \in f_*(B)$. Therefore $j \in f_*(A \cup B)$.

Let $k \in f_*(A) \cup f_*(B)$. So $k = f(b)$ where $b \in X$. So $k \in f_*(A)$ or $k \in f_*(B)$. So $b \in A \cup B$. Therefore $k \in f_*(A \cup B)$. So $f_*(A \cup B) = f_*(A) \cup f_*(B)$. □

(b) $f_*(A \cap B) \subseteq f_*(A) \cap f_*(B)$.

Claim 5. $f_*(A \cap B) \subseteq f_*(A) \cap f_*(B)$.

Proof. Let $j \in f_*(A \cap B)$. So $j = f(a)$ where $a \in A \cap B$. So $a \in A$ and $a \in B$. Since j maps to a unique output, and $a \in A \cap B$, $f_*(A) \cap f_*(B)$. Therefore $a \in f_*(A \cap B) \subseteq f_*(A) \cap f_*(B)$. □

(c) $f^*(C \cup D) = f^*(C) \cup f^*(D)$.

Claim 6. $f^*(C \cup D) = f^*(C) \cup f^*(D)$.

Proof. Let $x \in f^*(C \cup D)$. So $x \in X$. Since f is a function, elements in X can only be related to one element in Y . So $x \in f^*(C) \cup f^*(D)$ (since $x \in f^*(C \cup D)$ and $f^*(C \cup D)$ are linked to a unique element in $C \cup D$, they must be related to the element in $f^*(C)$ or $f^*(D)$). So $f^*(C \cup D) \subseteq f^*(C) \cup f^*(D)$.

Let $x \in f^*(C) \cup f^*(D)$. So $x \in X$. Since f is a function, elements in X can only be related to one element in Y . So $x \in f^*(C)$ or $x \in f^*(D)$. Since x is linked with a unique element in C or D (Definition of a function), $x \in f^*(C \cup D)$. So $f^*(C \cup D) = f^*(C) \cup f^*(D)$. □

(d) $f^*(C \cap D) = f^*(C) \cap f^*(D)$.

6. Let $g : X \rightarrow Y$ be a function. Show that for any finite collection A_1, \dots, A_n of subsets of X , we have

$$g_* \left(\bigcup_{k=1}^n A_k \right) = \bigcup_{k=1}^n g_*(A_k)$$

(Hint. n is a natural number.)

Proof. (Proof by Mathematical Induction)

Base Case ($n=1$): $g_* A_1 = g_* A_k$

Inductive hypothesis $r = n$: $g_* (\bigcup_{k=1}^r A_k) = \bigcup_{k=1}^r g_*(A_k)$

Inductive Step:

$$\begin{aligned}
\bigcup_{k=1}^{r+1} g_*(A_k) &= \bigcup_{k=1}^{r+1} g_*(A_k) \\
&= \bigcup_{k=1}^r g_*(A_k) \cup g_*(A_{r+1}) \\
&= g_*\left(\bigcup_{k=1}^r A_k\right) \cup g_*(A_{r+1}) && \text{(Used inductive hypothesis)} \\
&= g_*\left(\bigcup_{k=1}^{r+1} A_k\right) \\
&\quad \text{(Since input is either } A_{r+1} \text{ or } g_*\left(\bigcup_{k=1}^r A_k\right), \text{ the input of } g_* \text{ is } \bigcup_{k=1}^{r+1} A_k)
\end{aligned}$$

Therefore, through proof by mathematical induction, $g_*\left(\bigcup_{k=1}^n A_k\right) = \bigcup_{k=1}^n g_*(A_k)$. □

7. Give an example which shows that we might not have equality in 5(b).

Proof. Let $f(x) = x^2$, $A = \{-3, -2, -1, 0\}$ and $b = \{0, 1, 2, 3\}$. So $f_*(A) \cap f_*(B)$, $f_*(A) = \{4, 1, 0, 9\}$ and $f_*(B) = \{0, 1, 4, 9\}$, so $f_*(A) \cap f_*(B) = \{1, 4, 0, 9\}$. $f_*(A \cap B) = \{0\}$, so $f_*(A \cap B) \neq f_*(A) \cap f_*(B)$. □

8. (★) Let $f : X \rightarrow Y$. Assume that f is onto. Let \mathcal{P} be a partition of Y . Define

$$f^*\mathcal{P} = \{f^*(A) \mid A \in \mathcal{P}\}$$

Show that $f^*\mathcal{P}$ is a partition of X .

Claim 7. $f^*\mathcal{P}$ is a partition of X .

Proof. Since \mathcal{P} is a partition on Y , $Y = \bigcup_{A \in \mathcal{P}} A$. Since f is surjective on X , $f^*\mathcal{P} \subseteq X$, and according to the definition of a function $f^*\mathcal{P} = X$ ($f^*\mathcal{P} \subseteq X$ since f is surjective and partitioned by \mathcal{P} and $X \subseteq f^*\mathcal{P}$ as by definition of a function, all elements in X are related to an element in Y). Furthermore, since $\forall x_1, x_2 \in X$ if $f(x_1) = f(x_2)$ then $x_1 = x_2$, this implies that given an element, $y \in Y$, $f^*(y)$ will be unique so $\bigcup_{A \in \mathcal{P}} f^*(A)$. Since this satisfies all conditions of a partition, $f^*\mathcal{P}$ is a partition on X . □

9. (★★) Assume that $h : X \rightarrow Y$ is onto. Consider an equivalence relation R on Y . Give an explicit expression for the equivalence classes of h^*R (defined in HW 8) in terms of the other objects in play. (*Hint.* Recall that equivalence classes of h^*R are subsets of X . Consider another problem on this sheet.)
10. (★) Let $f : X \rightarrow Y$. Assume f is injective. If \preceq is a partial order on Y , show that the relation \vdash given by

$$\alpha \vdash \beta \text{ iff } f(\alpha) \preceq f(\beta)$$

is a partial order on X .

Claim 8. Let $f : X \rightarrow Y$. Assume f is injective. If \preceq is a partial order on Y , show that the relation \vdash given by

$$\alpha \vdash \beta \text{ iff } f(\alpha) \preceq f(\beta)$$

is a partial order on X .

Proof. (Reflexivity) Let $x \in X$. Since $f(x) \preceq f(x)$ (by definition of partial order, and since x maps to a unique value), $x \vdash x$ (Since $f(x)$ goes to a unique value due to injectivity). So \vdash is reflexive on X .

(Transitivity) Let $x, y \in (\vdash, X)$ where $x = (a, b)$ and $y = (b, c)$. By the definition of \preceq , $f(a) \preceq f(b)$ and $f(b) \preceq f(c)$. Since \preceq is a partial order, $f(a) \preceq f(c)$. By definition of \vdash and a function, $a \vdash c$. So \vdash is transitive on X .

(Antisymmetry) Let $x, y \in (\vdash, X)$ where $x = (a, b)$ and $y = (b, a)$. By the definition of \preceq , $f(a) \preceq f(b)$ and $f(b) \preceq f(a)$. Since \preceq is a partial order, $a = b$. So since if $x, y \in (\vdash, X)$ $a = b$, \vdash is antisymmetric. So \vdash is a partial order on X . \square

11. (\star) Let (Z, \preceq) and (W, \vdash) be partially ordered sets. Assume \preceq is a total order. Assume $g : (Z, \preceq) \rightarrow (W, \vdash)$ is monotone. Show that g is injective.