MA 225 Problem Set 6

facts and definitions You will need the following definitions and facts (at some point):

> **Definition 1.** We call a relation R on a set A *symmetric* if for any $x, y \in A$, $x \mathrel{\mathrm{R}} y$ guarantees $y \mathrel{\mathrm{R}} x$. We call R *reflexive* if for any $x \in A$, $x \mathrel{\mathrm{R}} x$. We call R *transitive* if for any $x, y, z \in A$, $x \mathrel{\mathrm{R}} y$ and $y \mathrel{\mathrm{R}} z$ together guarantee $x \mathrel{\mathrm{R}} z$.

> **Definition 2.** We call a relation R on the set A *intransitive* if for any $x, y, z \in A$, $x \mathrel{\mathrm{R}} y$ and $y \mathrel{\mathrm{R}} z$ together guarantee $\sim (x \mathrel{\mathrm{R}} z)$.

exercises These problems don't require you to write proofs.

1. If A has exactly a elements and B has exactly b elements, how many relations are there from A to B? ab

2. Some sources call a relation R symmetric if for any $x, y \in A$, $x \mathrel{\mathrm{R}} y \Leftrightarrow y \mathrel{\mathrm{R}} x$. Explain two things: first, why this definition is different from ours, and second, why any relation which is symmetric according to this definition will be symmetric according to ours and vice versa. (You may be able to turn the second explanation into a formal $\star$ proof.) This definition is different from ours because our definition states if xRy guarantees yRx, then R is symmetric, or $xRy \Leftarrow yRx$.

   Claim 1. *If a relation R is symmetric under the definition for any $x, y \in A$, $x \mathrel{\mathrm{R}} y$ guarantees $y \mathrel{\mathrm{R}} x$ and $xRy \Leftrightarrow yRx$*

   Proof. For any $x, y \in A$ let, $x \mathrel{\mathrm{R}} y$ guarantee $y \mathrel{\mathrm{R}} x$. Then $xRy \Leftarrow yRx$ Since x and y are non-specific terms, they can be switched, therefore $yRx \Leftarrow xRy$. So if for any $x, y \in A$, $x \mathrel{\mathrm{R}} y$ guarantees $y \mathrel{\mathrm{R}} x$, $xRy \Leftrightarrow yRx$.
   Let $xRy \Leftrightarrow yRx$. Then $xRy \Rightarrow yRx$. So if for any $x, y \in A$, $x \mathrel{\mathrm{R}} y$ guarantees $y \mathrel{\mathrm{R}} x$. $\square$

3. For each of the following, explain whether or not it is reflexive, symmetric, transitive (in our universe)? Explain your answers

   (a) F: $(x, y) \in F$ if x is y's father. It is not reflexive, as that would be $F(x, x)$, or that x is his own father. It is not symetric, as $F(x, y)$ means x is y's father, as $F(y, x)$ means y is x's father, which is not the same. It is not transitive, as given x, y and z people, if x is y's father, and y is z's father, x is not z's father, but instead his grandfather. R_R represents people who are their own fathers, which is $\varnothing$ in our universe. S_R represents people who are their father's father, which does not exist in our universe. Finally, T_R represents people who are both the father and grandfather of a person, which is extremely rare in our universe.

   (b) $\neq$, as a relation on $\mathbb{R}$ $\neq$ is not reflexive, as given $a \in \mathbb{R}$, there is no a which $a \neq a$. It is symmetric, as if $a, b \in \mathbb{R}$, then if $a \neq b$, then $b \neq a$. It is not transitive either, an example of this is $4 \neq 3$ and $3 \neq 4$, however it is not true that $4 \neq 4$. R_R represents the set of real numbers that are not equal to themselves, which is none of them. S_R are numbers, x and y, that if $x \neq y$ and $y \neq x$, which is all real numbers. T_R are numbers which x y and z which are real numbers, if $x \neq y$ and $y \neq z$ then $x \neq z$, which includes numbers which x and z aren't equal.

   (c) "lives within one mile of" This is reflexive, as all things in our universe are within a mile of themselves. It is also symmetric, as if a is within a mile of b, b is within a mile of a. It is not transitive, however, as if x is a mile from y, and y is a mile from z, then x can be 2 miles from z, which is not within a mile. R_R are people which live within a mile of themselves, which is everyone. S_R are places which if within a mile of another place, then the other place is within a mile of the original place. T_R includes places which for places x, y and z, if x is within a mile of y, then y is within a mile of z, which includes a portion of places within our universe.

(d) S: $x\,\mathrm{S}\,y$ if $x^2+y^2=1$ This is not reflexive, as given $2S2$, $8=1$, which is not true. It is symmetric however, as addition is communative, so if xSy is true, or $x^2+y^2=1$, then this can be changed to $y^2+x^2=1$ or ySx. It is not transitive, an example being $0S1$ is true, and $1S0$ is true, however $1S1$ implies $1=2$ which is false. R_R is the set of numbers which $x^2+x^2=1$ holds true. S_R is the set of ordered pairs which if x and y are switched, the statement still holds true (given they were true before switching), so in this case all numbers. T_R are the set of relations given xSy and ySz, xSz held true.

4. For each relation in exercise 3, explain what R_R, S_R, and T_R represent for that relation.

5. R_R is called the *reflexive closure of* R. Explain why someone might call it *the smallest reflexive relation which contains* R. R_R is the samallest set that is reflixive and contains R, as

6. (a) Make a blueprint for the claim *If blah blah and such, then* R *is reflexive.* Let $x \in R$
Good ol proofin steps...
Therefore $x=(a,a)$. So R is reflexive.

(b) Make a blueprint for the claim *If blah blah and such, then* S *is intransitive.* Let $x, y \in S$. So $x=(a,b)$ and $y=(b,c)$ Therefore if x, and y are in R, $(a,c) \notin R$

proofs Prove the following claims.

1. ($\star$) Give an example which shows that we could have

$$(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$$

(Prove that your example does what you say it does.)

Claim 2.
$$(A \times B) \cup (C \times D) \neq (A \cup C) \times (B \cup D)$$

Proof. Let $x \in (A \cup C) \times (B \cup D)$. Let $x = a \times d$, where $a \in A$, $d \in D$, but $a \notin C$ and $d \notin B$. So $a \in (A \cup C)$ and $d \in B \cup D$, which is true. Therefore $x \in (A \times B)$ or $x \in (C \cup D)$ if the statement is equal. So if $x \in (A \times B) \cup (C \times D)$, then $a \in A$ and $d \in B$ or $a \in C$ or $d \in D$. Since $a \notin C$ and $d \notin B$, the statements are not equivalent. □

2. Prove the clauses of Theorem 6.2.13: for any sets A, B, C, D and relations R from A to B, S from B to C, and T from C to D,

(a) $\left(R^{-1}\right)^{-1} = R$

Claim 3. $\left(R^{-1}\right)^{-1} = R$

Proof. Let $x \in (\left(R^{-1}\right)^{-1}$. Then $x = (m,n)$. So $(n,m) \in R^{-1}$. So $(m,n) \in R$. Then $m \in A$ and $n \in B$. Therefore $x \in R$, so $\left(R^{-1}\right)^{-1}$.
Let $p \in R$. Then $p = (q,r)$ where $q \in A$ and $r \in B$. So $(r,q) \in R^{-1}$. So $(q,r) \in \left(R^{-1}\right)^{-1}$. Then $p \in \left(R^{-1}\right)^{-1}$. Therefore $R \subseteq \left(R^{-1}\right)^{-1}$. So $\left(R^{-1}\right)^{-1} = R$. □

(b) $T \circ (S \circ R) = (T \circ S) \circ R$ (What's a good way to refer to this fact?)

Claim 4. $T \circ (S \circ R) = (T \circ S) \circ R$

Proof. Let $x \in T \circ (S \circ R)$. Then $x = (m,n)$.So $m = Domain(S \circ R)$. Since $Domain(S \circ R) \in A$, $m \in A$. Furthermore, $n \in D$. Since $Range(T \circ S) \in D$, and $Domain(R) \in A$,

$x \in (T \circ S) \circ R$. So $T \circ (S \circ R) \subseteq (T \circ S) \circ R$ Let $j \in (T \circ S) \circ R$. Then $x = (o, p)$. So $o = Range(T \circ S)$. So $o \in D$. Furthermore, $p \in A$. Since $Domain(T \circ S) \in D$ and $Domain(R \circ S) \in A$ and $Range(T) \in D$, $j \in (T \circ (S \circ R)$. Since $j \in T \circ (S \circ R)$, $(T \circ S) \circ R \subseteq T \circ (S \circ R)$. So $T \circ (S \circ R) = (T \circ S) \circ R$. □

(c) $I_B \circ R = R$ and $R \circ I_A = R$

Claim 5. *$I_B \circ R = R$ and $R \circ I_A = R$*

Proof. Let $x \in I_B \circ R$. So $x = (a, b)$ where $a \in A$ and $b \in B$. Let $j \in R \circ I_A$. So $x = (m, n)$ where $m \in A$ and $n \in B$. □

(d) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Claim 6. $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

Proof. Let $x \in (S \circ R)^{-1}$. Then $x = (m, n)$. So $(n, m) \in (S \circ R)$. So $n \in A$ and $m \in C$. Therefore $(m, n) \in R^{-1} \circ S^{-1}$ through inverses. So $x \in R^{-1} \circ S^{-1}$.
Let $h \in R^{-1} \circ S^{-1}$. So □

3. ($\star$) Suppose A is a set on which $\varnothing$ is a reflexive relation. This says something very strong about A.

Claim 7. *If $\varnothing$ is reflexive on set A, then A is the null set.*

Proof. Let $a \in A$. Since $\varnothing$ implies that for all of elements in A, there is are not relatiionships. Since $a \in A$, $a \varnothing a$ is false unless there is no a. Therefore, the only way $\varnothing$ is reflexive on set A is if A is the empty set. □

4. The complement of a symmetric relation is symmetric.

Claim 8. *The complement of a symmetric relation is symmetric.*

Proof. Let S be a symmetric relation and let xSy be true. Since S is symmetric, ySx. Since xSy and ySx, the complement of this is $x \not S y$ and $y \not S x$, which is symmetric by definition. Therefore the complement of S is symmetric. □

5. Let R be a relation on the set A.

(a) Show that $R_R = I_A \cup R$ is reflexive.

Claim 9. *$R_R = I_A \cup R$ is reflexive.*

Proof. Let $J \in R_R$, where J is a relationship. So J is I_A or R.
Case 1 ($J \in I_A$): So $J = (a, a)$, where $a \in A$. Therefore R_R is reflexive if $J \in I_A$.
Case 2 ($J \in R$): So $J = (a, b)$, where $a, b \in A$. Then let $a = b$ so $J = (a, a)$. Therefore R_R is reflexive if $J \in R$
Since R_R is reflexive under all conditions, it is reflexive. □

(b) Show that if S is a reflexive relation on A with $R \subseteq S$, then $R_R \subseteq S$.

Claim 10. *If S is a reflexive relation on A with $R \subseteq S$, then $R_R \subseteq S$.*

Proof. Let $x \in R_R$. Then $x \in R$ or $x \in I_A$ by definition.
Case 1($x \in R$): Since $R \subseteq S$, $x \in S$. Therefore $R_R \subseteq S$.
Case 2($x \in I_A$): So $x = (a, a)$ where $a \in A$. Since S is reflexive, $x \in S$ and $R_R \subseteq S$.
Since $R_R \subseteq S$ for all conditions, $R_R \subseteq S$. □

6. Let R be a relation on the set A.

(a) Show that $S_R = R \cup R^{-1}$ is a symmetric relation.

Claim 11. *$S_R = R \cup R^{-1}$ is a symmetric relation.*

Proof. Let $x \in S_R$. Then x=(a,b) where $a, b \in A$. So $x \in R$ or $x \in R^{-1}$.
Case 1 ($x \in R$): So $(b, a) \in R^{-1}$. Therefore if $x \in R$, S_R is symmetric.
Case 2 ($x \in R^{-1}$): So $(b, a) \in R$. Therefore if $x \in R$, S_R is symmetric.
Since S_R is symmetric for all cases, S_R is symmetric. □

(b) Show that if S is any symmetric relation on A and $R \subseteq S$, then $S_R \subseteq S$.

Claim 12. *If S is any symmetric relation on A and $R \subseteq S$, then $S_R \subseteq S$.*

Proof. Let $x \in S_R$. Therefore $x = (a, b)$, where $a, b \in A$. So $x \in R$ or $x \in R^{-1}$.
Case 1 ($x \in R$): Since $x \in R$ and $R \subseteq S$, $x \in S$.
Case 2 ($x \in R^{-1}$): Since $x \in R^{-1}$. Therefore $(b, a) \in R$. Since S is symmetric, $(b, a) \in R$, and $R \subseteq S$ then $x \in S$.
Since $x \in S$ for all of S_R, then $S_R \subseteq S$.

□

7. Let R be a relation on the set A.

(a) Show that R is symmetric if and only if $R^{-1} = R$.

Claim 13. *R is symmetric if and only if $R^{-1} = R$.*

Proof. Let $x \in R$. Then $x = (a, b)$ where $a, b \in A$. So, according to $R = R^{-1}$, (a, b) implies (b, a). Therefore whenever $R^{-1} = R$, R is symmetric.
Let xRy imply yRx, where $x, y \in A$(Definition of symmetric). So $R \subseteq R^{-1}$. Since x and y are arbitrary elements in A, yRx implies xRy. So $R^{-1} \subseteq R$. Therefore, $R = R^{-1}$.
So, R is symmetric if and only if $R^{-1} = R$. □

(b) Show that R is transitive if and only if $R \circ R \subseteq R$.

Claim 14. *R is transitive if and only if $R \circ R \subseteq R$.*

Proof. Let $x \in R \circ R$. Then $x = (a, b)$, where $a, b \in A.R \circ R \subseteq R$ can be rewritten as $aRc \circ cRb \subseteq aRb$ where $c \in A$. Therefore, since aRc and cRb implies x according to $R \circ R \subseteq R$, x is transitive.
Let xRy and yRz imply xRz (let R be transitive), where $x, y, z \in A$. So $xRy \circ yRz \subseteq xRz$, as the $Range(xRy) = Domain(yRz)$ and this results in xRz. Therefore R is transitive if and only if $R \circ R \subseteq R$. □

8. Let R be a relation on the set A. Let G be a relation on A with the property that: *if S is any symmetric relation on A and $R \subseteq S$, then $G \subseteq S$.* Show that such G is unique.

9. Let R be a relation on the set A. Define T_R, a relation on A, by $x \,\mathrm{T_R}\, y$ if there are $a_0, a_1, \ldots, a_k \in A$ with $x = a_0$, $y = a_k$, and $a_i \,\mathrm{R}\, a_{i+1}$ for each $i = 0, 1, \ldots, k - 1$.

(a) ($\star$) Show that for any relation R, T_R is transitive.

Claim 15. *For any relation R, T_R is transitive.*

Proof. Let $j, w \in T_R$. Then $j = (x, y)$ and $w = (y, z)$, where $x, y, z \in A$. Then j can be described as $x = a_0$ and $y = a_k$ $xRa_1 ... a_{k-1}Ry$. w can be described as $yRa_{i+1} ... a_{m-1}Rz$ where m is the amount of elements in A between x and z inclusively. Since this holds true for all terms in between x and z, xT_Rz holds for any j and w in T_R, therefore T_R is transitive. □

(b) ($\star\star$) If S is a transitive relation on A with $R \subseteq S$, show that $T_R \subseteq S$.