MA 225 Problem Set 4: induction 2

facts and definitions You will need the following definitions and facts (at some point):

Definition 1. The *Fibonacci numbers* are defined as follows:

$$f_k = \begin{cases} 1 & \text{if } k = 1 \text{ or } k = 2 \\ f_{k-1} + f_{k-2} & \text{if } k \geq 3 \end{cases}$$

Definition 2. $0! = 1$

Principle of Mathematical Induction. *Let $P(n)$ be an open sentence with universe the natural numbers. If $P(1)$ is true and $P(n)$ is inductive, then for any n, $P(n)$ is true.*

Principle of Complete Induction. *Let $P(n)$ be an open sentence with universe the natural numbers. If $P(1)$ is true and $P(n)$ is completely inductive, then for any n, $P(n)$ is true.*

exercises These problems don't require you to write proofs.

1. Explain why "n is even" is completely inductive, but "n is odd" is not completely inductive. n is odd is inductive because when adding two even numbers, we get another even number, however when we do this for odd numbers, this does not apply for n or for all integers less than n.
2. Is either of the above sentences inductive? Yes. As seen later in proof 1 as well as 2, we prove that $PCI \Leftrightarrow PMI$. Therefore, since n is completely inductive, we can show that this is a solution. then
3. We showed that if $P(n)$ is inductive, then the set of values n for which $P(n)$ is true must look like $\{n_0, n_0 + 1, n_0 + 2, \ldots\}$. Characterize what the set of values n for which $Q(n)$ is true, assuming $Q(n)$ is completely inductive. The values which n is true for $Q(n)$ are $\{n, n-1, n-2, n-3 \ldots n_0\}$

proofs Prove the following claims.

1. Let $P(n)$ be an inductive sentence. Then $P(n)$ is completely inductive.

   Claim 1. *Let $P(n)$ be an inductive sentence. Then $P(n)$ is completely inductive.*

   Proof by Complete Induction 1. *We can show that $P(n)$ is completely inductive by looking at the definition of both. We know that in standard induction, we use $P(n_0), P(n_0 + 1)...P(n)$ in order to prove that $P(n+1)$ is true. For complete induction, however, we assume that $P(n)$ is true as well as $P(n-1), P(n-2)...P(n_0)$ where n_0 is the base case. Because of this, we can reverse the order of the terms if standard induction to get complete induction. Therefore, if $P(n)$ is inductive, then it is completely inductive as well.* □

2. ($\star$) In class we proved PCI, assuming PMI. Prove the converse: give a proof of the PMI that only assumes PCI.

   Claim 2. *Given $R(n)$ is completely inductive, then $R(t)$ is mathematically inductive.*

   Proof by Complete Induction 2. *We can first see that complete induction assumes that by assuming $R(n)$ holds true for n as well as values less than n, then we can gather $R(n + 1)$. Since regular induction requires a base case as well as for $R(n)$ to hold true for n, since complete induction implies both of these, we can see that if $R(n)$ is completely inductive, then it is mathematically inductive.* □

3. The parity of the Fibonacci numbers follows the pattern: odd, odd, even, odd, odd, even, . . .

   Claim 3. *The parity of the Fibonacci numbers is odd, odd, even, odd, odd, even...*

   Proof by Complete Induction 3. *We can first test the base cases, given as n=1 and n=2, by seeing that $f_1 = 1$ is odd, $f_2 = 1$, which is odd, and using the equation $f_n = f_{n-1} + f_{n-2}$, $1 + 1 = 2$, which is even. We can show by setting $n = k$ and solving for f_{k+1}. Let's assume that f_k is We can now see that $f_{k+1} = f_k + f_{k-1}$.*
   Because of the principal of complete induction, we can see that this pattern holds for all $n + 1$.

4. There are no common factors of f_n and f_{n+1}, other than 1.

   Claim 4.

   Proof by Complete Induction 4. *We can first define the base cases to be 1 and 2, and as we can see, and using $f_n = f_{n-1} - f_{n-2} = f_3 = 1 + 1 = 2$, which shares no common multiples. We can first define k=n and assume that there are no common multiples for f_k and $fk - 1$ and for all natural numbers less than k but more than 2. We will show that f_n shares a factor, c withf_{k+1}, then c must equal 1. We can solve for f_{k+1} by adding one to k to produce $f_{k+1} = f_k + f_{k-1}$. We can see that in order for c to be a common multiple of f_k and f_{k+1}, then it must be factored out and therefore f_{k-1}. However, according to complete induction, this holds true for k and values less than k, therefore k must share a factor with f_{k-1}. This will continue until we reach the base case, which only shares the factor 1. Therefore, according to the principle of complete induction, there is not common factors of $f(n)$ and $f(n+1)$.* ☐

5. ($\star\star$) The *Tower of Hanoi* puzzle consists of n disks of different radii, stacked in decreasing order of radius (so the largest disk is on the bottom) on one rod; two other rods are nearby. The goal of the game is to move the entire stack to another of the rods, in the same order. The rules are:

   - You may move the top disk of a stack onto another rod.
   - A disk may only be placed on top of a larger disk.
   - No other moves are allowed.

   If the Tower of Hanoi puzzle starts with d disks, you can solve it in $2^d - 1$ moves.

   Claim 5. *Given a Towe of Hanoi with d disks, it can be solved in $2^d - 1$ moves.*

   Proof by Complete Induction 5. *We must first find a way to represent the number of moves, h, that can be taken given $d - 1$ moves, or h_{d-1}. It is noted that in order to move the bottom piece, all pieces above it must be shifted to another rod. This causes us to move h_{d-1} disks, move the bottom disk, which adds an additional move, and finally move the rod with $d - 1$ disks to the bottom disk. This can yet again be described as taking h_{d-1} moves, therefore the total moves which can be taken for d disks is $h_d = 2h_{d-1} + 1$ (Formula 2). We can now show that $2h_d + 1 = 2^d - 1$(Equation 1). We must first prove the base case, which is d=1. Plugging in for one, we get $2^1 - 1$, or 1 move, which is correct, as you only need to move the one piece. We can define k as $k = d$ and solve for $k+1$:*

$$
\begin{aligned}
2h_{k+1} + 1 &= 2^{k+1} - 1 \\
&= 2 * 2^k - 1 \\
&= 2 * (2^{h_k} + 2) + 1 && \text{(Replaced using formuala 1)} \\
&= 4h_k + 3 \\
&= 2(2h_k) + 3 \\
&= 2h_{k+1} + 1
\end{aligned}
$$

   (Replaced d with k+1 in formula 2 through complete induction and solved for h with k disks)

   Therefore, since this is true for all $k + 1$, through complete mathematical induction we can show that given a Tower of Hanoi with d disks, it can be solved in $2^d - 1$ moves. ☐

6. $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$.

   Claim 6. $\sum_{k=1}^{n} f_k^2 = f_n f_{n+1}$

   Proof by Complete Induction 6. *We can start by determining the base cases. In this instance, the base cases are n=1 and n=2. We can see by plugging in these values $\sum_{k=1}^{1} f_k^2 = f_1 f_2 = 1$ and n=2:*

$\sum_{k=1}^{2} f_k^2 = f_2 f_3 = 2$. *We can define* $r = n$*, where we will assume that* k *satisfies the equation as well as values less than* r*. We will use this to show that the equation* $\sum_{k=1}^{2} f_k^2 = f_2 f_3 = 2$ *is satisfied under* $k+1$*:*

$$\begin{aligned} f_{r+2} f_{r+1} &= \sum_{k=1}^{r+1} n f_k^2 \\ &= \sum_{k=1}^{r+1} f_k^2 + f_{r+1}^2 \\ &= f_r f_{r+1} + f_{r+1}^2 && \text{(Substituted using induction hypothesis)} \\ &= f_{r+1}(f_r + f_{r+1} \\ &= f_{r+1} f_{r+2} \end{aligned}$$

(Use definition of fibbanachi sequence along with complete induction by adding 1 to r in definition and substitute.)

As we can see above, through proof by complete induction we have shown that $\sum_{k=1}^{2} f_k^2 = f_2 f_3 = 2$ *is true for all* k*.* □

7. ($\star\star$) The **product rule for higher derivatives**: given any $n \in \mathbb{N}$ and functions f, g with at least n derivatives, we have

$$\begin{aligned} \frac{d^n}{dx^n}[fg] =& B_{n,0} f^{(n)} g + B_{n,1} f^{(n-1)} g' + B_{n,2} f^{(n-2)} g'' + \cdots \\ &+ B_{n,n-2} f'' g^{(n-2)} + B_{n,n-1} f' g^{(n-1)} + B_{n,n} f g^{(n)}. \end{aligned}$$

(*Hint.* At some point you will need to "combine like terms".)

8. ($\star\star$) If $n \geq 3$, then the sum of the interior angles of a convex n-gon is $(n-2) \cdot 180°$.

Claim 7. *If* $n \geq 3$*, then the sum of the interior angles of a convex* n*-gon is* $(n-2) \cdot 180°$ *(Formula 1).*

Proof by Complete Induction 7. *We can first prove the base case by using a 3-gon, which is* $180°$*. Through plugging in, we can see that* $(3-2)180° = 180°$*, which matches up with our expected value. We can mathematically represent the change in angles by defining adding a vertex as creating a triangle and placing two of its verticies on existing verticies. Because of this definition, we can represent the angles present in a shape as* a_n*, where* $a_n = a_{n-1} + 180°$ *(Formula 2). We can now define* $k = n$*, where we will assume the open statement* $(k-2) \cdot 180°$*. We can now use complete induction to show that this is valid for* $k+1$*:*

$$\begin{aligned} a_n + 180° &= (k-1)180° \\ &= 180° k - 180\circ \\ &= a_{n-1} + 360° && \text{(Substitute using inductive hypothesis)} \\ &= a_n + 180 && \text{(Substituted using Formula 2)} \end{aligned}$$

Therefore, as seen above, it can be seen through complete induction that $(n-2) \cdot 180°$ *is valid for convex* n*-gons.* □

9. Define the numbers g_n as follows:

$$g_n = \begin{cases} 2 & \text{if } n = 1 \text{ or } n = 2 \\ g_{n-1} g_{n-2} & \text{if } n \geq 3 \end{cases}$$

For all n, $g_n = 2^{f_n}$.

10. $f_k \leq 2^k$.

Claim 8. *$f_k \leq 2^k$ is true for all $k \in \mathbb{N}$.*

Proof by Complete Induction 8. *We will show that $f_k \leq 2^k$ is true for all k through the Principle of Complete Induction. First, we must prove a base case, in this case we will use n=1, or $g_1 = 2^{f_1}$, or $2 = 2$ and the same for 2, as this results in the same value. (since both result in 2. Next we set k=n to show that $g_k = 2^{f_k}$ by assuming this works for all natural numbers before k and k to show that it works for n+1:*

$$\begin{aligned}
g_{k+1} &= 2^{f_{k+1}} \\
&= 2^{f_k + f_{k-1}} \\
&= 2^{f_k} 2^{f_{k-1}} \\
&= g_n 2^{f_{k-1}} && \text{(Used inductive hypothesis for substitution)} \\
&= g_{k+1} 2^{f_k} \\
g_{k+2} &= g_{k+1} g_k && \text{(Added 1 to k in definition)} \\
g_{k+1} &= g_k g_{k+1} && \text{(Used definition above in order to substitute)}
\end{aligned}$$

As we can see, this results in the desired results of showing that $g_k = 2^{f_k}$ by proof of complete mathematical induction. □

11. (⋆⋆) Let $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ (These are the roots of the equation $x^2 - x - 1 = 0$.) Then

$$f_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$$

(*Hint.* You will need to use the fact that α and β the solutions of the given equation.)

12. For any n and any $0 \leq k \leq n$, $B_{n,k} = \frac{n!}{k!(n-k)!}$. (*Hint.* Induct on n.)

13. Prove two claims from Homework 3 using the Well-Ordering Principle.