MA 225 Problem Set 3: induction 1

facts and definitions You will need the following definitions and facts (at some point):

> **Definition 1.** We'll call strings of the symbols a and b *words.* Consider a class of words, the *legal words*, defined as follows. aba is a legal word. If W is a legal word, then so are abW, aWb, Wab, baW, bWa, and Wba. No words other than those obtained in this way are legal.
>
> **Definition 2.** The *binomial coefficients* are a collection of natural numbers $B_{n,k}$, defined for a pair of nonnegative integers n, k with $0 \leq k \leq n$, as follows:
>
> $$B_{n,k} = \begin{cases} 1 & \text{if } k = 0 \text{ or } k = n \\ B_{n-1,k-1} + B_{n-1,k} & \text{if } 1 \leq k \leq n-1 \end{cases}$$
>
> **Definition 3.** *Higher derivatives* are defined as follows: the zeroth derivative of a function f is f itself; we write
>
> $$\frac{d^0}{dx^0} f(x) = f(x) = f^{(0)}(x)$$
>
> For $n \in \mathbb{N}$, we define the n^{th} derivative as the derivative of the $(n-1)^{st}$ derivative:
>
> $$\frac{d^n}{dx^n} f(x) = \frac{d}{dx}\left[\frac{d^{n-1}}{dx^{n-1}} f(x)\right]$$
>
> $$f^{(n)}(x) = \left(f^{(n-1)}\right)'(x)$$
>
> **Integration by Parts.** *For any two differentiable functions f, g, we have*
>
> $$\int f(x)g(x)\ dx = f(x)\left[\int g(x)\ dx\right] - \int f'(x)\left[\int g(x)\ dx\right]\ dx,$$
>
> *provided we adopt the convention that the constants of integration in both instances of $\int g(x)\ dx$ must be the same.*

exercises These problems don't require you to write proofs.

1. Compute $B_{n,k}$ for $0 \leq n \leq 6$.
2. Explain why the definition given for $B_{n,k}$ actually constitutes a definition; that is, why we can compute $B_{n,k}$ for any choice of n, k with $0 \leq k \leq n$.
3. Identify, explain, and correct any correctable flaws in the following proofs:

   Claim 1. *$n^2 + n$ is odd.*

   Proof. $n = 1$ is odd.
   Inductively, assume $n^2 + n$ is odd. Then

   $$(n+1)^2 + (n+1) = n^2 + 2n + 1 + n + 1 = n^2 + n + 2(n+1)$$

   so $(n+1)^2 + (n+1)$ is the sum of an odd number and an even number, hence itself odd. This completes the inductive step. □

   Claim 2. *Every natural number is odd.*

   Proof. $k = 1$ is clearly odd.
   Inductively, assume k is odd. This means there is an integer p so that $k = 2p+1$. Consider $p+1$. Clearly $2(p+1)+1$ is odd. □

Claim 3. *Every natural number is both even and odd.*

Proof. Assume that k is both even and odd. Consider $k+1$.
Since k is even, there is p with $k = 2p$. So $k+1 = 2p+1$ is odd. Since k is odd, there is q with $k = 2q+1$. So

$$k + 1 = (2q + 1) + 1 = 2(q + 1)$$

is even. □

Claim 4. *$n^3 - n$ is divisible by 6.*

Proof. For the base case: when $n = 1$, $n^3 - n = 0 = 6 \cdot 0$.
Now proceed inductively. Assume that for all k, $k^3 - k$ is divisible by 6. Then, since $n+1$ is one possible value of k, we have that $(n+1)^3 - (n+1)$ is divisible by 6. □

proofs Prove the following claims.

1. For any natural number p, 8 divides $5^{2p} - 1$.

   Claim 5. *For any natural number p, 8 divides $5^{2p} - 1$*

   Proof by Induction: 1. We must first confirm that the base case, or $5^{2p} - 1$ is divisible by 8. We determine that the value of this function at 1 is 24, which is $8 * 3$, so this statement is valid. Next, we can see that the solution to this is $8m$, where m is a natural number. We can now use mathermatical induction and find:

$$\begin{aligned} 8m &= 5^{2(n+1} - 1 \\ &\text{(Solving for n+1, since we know that this is true when n is true when n=1)} \\ &= 5^2 * 5^{2n} - 1 \\ &= 25(8m + 1) - 1 \\ &= 8(25m + 3) \end{aligned}$$

   Therefore, we can see that by mathematical induction, $5^{2p} - 1$ will always be divisible by 8 as long as p is in the natural numbers.

2. For any natural number ℓ, $3^\ell \geq 1 + 2^\ell$.
3. Let $a_1, \ldots, a_n$ be real numbers. Then

$$2^{\left(\sum_{k=1}^{n} a_k\right)} = \prod_{k=1}^{n} 2^{a_k}$$

4. ($\star$) For any $k \in \mathbb{N}$, and any real numbers r, s, we have

$$r^k - s^k = (r - s) \sum_{\substack{p+q=k-1 \\ p,q \geq 0}} r^q s^p = (r - s)\left(r^{k-1} + r^{k-2}s + r^{k-3}s^2 + \cdots + r^2 s^{k-3} + r s^{k-2} + s^{k-1}\right)$$

5. Consider the possible results of flipping a fair coin n times. There are 2^n possible outcomes.
6. ($\star$) For any natural number q,

$$\sum_{i=1}^{2^q} \frac{1}{i} \geq 1 + \frac{q}{2}$$

7. Prove the power rule for derivatives: for any $n \in \mathbb{N}$, we have $\frac{d}{dx}[x^n] = nx^{n-1}$. You may **only** use the product rule for derivatives and the fact that $\frac{d}{dx}x = 1$.
8. ($\star$) The **power rule for integrals**: for any $n \in \mathbb{N}$, we have

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$

for some constant C.
You **may not** use the previous result. You may use **only** the following calculus facts: the linearity properties of the integral; $\int C dx = Cx + D$ for some constant D; $\frac{d}{dx}x = 1$; integration by parts.
9. $\frac{d^r}{dx^r}x^r = r!$
10. The **constant multiple rule for higher derivatives**: for any function f with at least n derivatives and any constant c, we have $\frac{d^n}{dx^n}[cf] = c\left[\frac{d^n}{dx^n}f\right]$. You may assume the constant multiple rule for derivatives.
11. The **sum rule for higher derivatives**: for any functions f, g with at least n derivatives, we have $\frac{d^n}{dx^n}[f+g] = \frac{d^n}{dx^n}f + \frac{d^n}{dx^n}g$. You may assume the sum rule for derivatives.
12. ($\star$) The **Binomial Theorem**: for any real numbers x, y, and any $n \in \mathbb{N}$,

$$(x+y)^n = B_{n,0}x^n + B_{n,1}x^{n-1}y + B_{n,2}x^{n-2}y^2 + \cdots + B_{n,n-2}x^2y^{n-2} + B_{n,n-1}xy^{n-1} + B_{n,n}y^n$$

(*Hint.* At some point you will need to "combine like terms".)
13. ($\star$) Any legal word has more as than bs.