MA 225 Problem Set 8

Recall the following definition:

1. Show that a relation R on the set A is total iff $A \times A = R \cup R^{-1}$.

   Claim 1. *A relation R on the set A is total iff $A \times A = R \cup R^{-1}$.*

   Proof. Let $A \times A = R \cup R^{-1}$ and $a \in A$. Since $a \times a$ $\square$

2. Formulate a set-theoretic criterion (similar to problem 1 above and problem 8 from Homework 6) for antisymmetry. Prove that your criterion is equivalent to the relation in question being antisymmetric.

   Claim 2. *A relation R on the set A is total iff $A \times A = R \cap R^{-1}$.*

   Proof. Let $A \times A = R \cap R^{-1}$. Consider the relation $(a, b) \in R$ where $a, b \in A$. So $(a, b) \in A \times A$ if $(a, b) \in R^{-1}$. Since the only way for that to occur is if $a = b$, R is antisymmetric.
   Let R be an antisymmetric relation. Consider $(a, b) \in R$ where $a, b \in A$. Let $(a, b) \in R$ and $(b, a) \in R$, so that $a = b$ (according to the definition of antisymmetric). So $(a, a) \in R$. Since $(a, a) \in R^{-1}$ (found inverse of (a, a)), and $(a, a) \in R$, $A \times A = R \cap R^{-1}$. $\square$

3. ($\star\star$) For each set A, there is a unique relation on A which is both a function and an equivalence relation. Find this relation and prove that it is unique.

   Claim 3. *Equals is the only relation which is both a function and equivalence relationship.*

   Proof. Let R be an equivalence relationship and function. Let $x, y \in R$ where $x = (a, b)$ and $y = (b, c)$ with $a, b, c \in A$. So $(a, c) \in R$. Since $(a, c), (a, b) \in R$, R is not injective and therefore not a function. $\square$

4. Show that the inverse of a partial order is a partial order.

   Claim 4. *The inverse of a partial order is a partial order.*

   Proof. Let R be a partial order and $(a, b), (b, c) \in R$. We'll show R^{-1} is a partial order
   Reflexive: $(a, a) \in R$ (R is reflexive). Therefore R^{-1} is reflexive.
   Transitive: $(a, c) \in R$ (Since R is transitive). So $(c, b), (b, a) \in R$ (inverses of (a, b) and (b, c)). Therefore if R is transitive, $(c, a) \in R$. Since $(c, a) \in R$ (inverse of (a, c)), R^{-1} is transitive.
   Antisymmetric: If $(a, b), (b, a) \in R$, $a = b$. Therefore if $(b, a), (a, b) \in R^{-1}$, $a = b$. So R^{-1} is symmetric.
   So R^{-1} is a partial order. $\square$

5. ($\star\star$) Let $\preceq$ be a partial order on the set A. Define a relation $\vdash$ on $A \times A$ by $(a, b) \vdash (c, d)$ if either $a \preceq c$ and $a \neq c$, or $a = c$ and $b \preceq d$. We call $\vdash$ the *lexicographic* or *dictionary* order on $A \times A$. Show that $\vdash$ is a partial order on $A \times A$. (*Hint 1.* It may help to consider why $\vdash$ is called the *dictionary* order. *Hint 2.* Both antisymmetry and transitivity involve assuming that two pairs of pairs are $\vdash$-related. Since there are two ways for two pairs to be $\vdash$-related, there are four cases to consider.)

6. Let $(A, \vdash)$ be a partially-ordered set (*i.e.*, $\vdash$ is a partial order on A). For each $a \in A$, define the *downset* of a to be
   $$D_a = \{x \in A | x \vdash a\}$$

   (a) No downset is empty.

   Claim 5. *No downset is empty.*

Proof. Let A be a set and D_a be a downset where $a \in A$ (So A is not empty). Since $a \vdash a$ (Reflexivity by definition of parital order), $a \in D_a$ (by definition of a downset). Therefore a downset on a set is not empty but contains a.

So a downset is never empty. □

(b) For any $a, b \in A$, $D_a \subseteq D_b$ iff $a \vdash b$.

(c) Consider $\text{Down}_\vdash = \{D_a | a \in A\}$, the set of all downsets with respect to $\vdash$. Explain why $(A, \vdash)$ and $(\text{Down}_\vdash, \subseteq)$ have essentially the same structure.

7. Let X be a set. For each $A \subseteq X$, define the *characteristic function* of A, $\chi_A : X \to \{0,1\}$ by

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

(a) If A and B are subsets of X, what does it mean about A and B that $\forall x \in X, \chi_A(x) \leq \chi_B(x)$? Prove your answer.

Claim 6. *If $\forall x \in X, \chi_A(x) \leq \chi_B(x)$, then $A \subseteq B$.*

Proof. Let $j \in A$. So $\chi_A(j) = 1$. Since $\chi_A(j) \leq \chi_B(j)$, $\chi_B(j) = 1$ (Since χ's highest possible value is 1). Since $j \in B$, $A \subseteq B$. □

(b) ($\star$) What does it mean about A and B that $\chi_A = \chi_B$? Formulate your answer as an "iff" statement and prove it.

Claim 7. $\chi_A = \chi_B$ *iff* $A = B$

Proof. Let $A = B$ and $x \in A$. So $x \in B$. Therefore $\chi_A(x) = 1$ and $\chi_B = 1$(According to the definition of χ). Consider now $x \notin A$. So $x \notin B$ $(A = B)$. Then $\chi_A = 0$ and $\chi_B = 0$ (By definition of χ). Now let $y \in B$. So $y \in B$. Therefore $\chi_B(x) = 1$ and $\chi_A = 1$(According to the definition of χ). So $\chi_A = \chi_B$ if $A = B$
Let $\chi_A = \chi_B$ and j be an element. Assume $A \neq B$. Then there is an element which $\chi_A = 0$ and $\chi_A = 1$ or $\chi_A = 1$ and $\chi_B = 0$.
Case $j \in A$ and $j \notin B$. Therefore $\chi_A(x) = 1$ and $\chi_B(x) = 0$. Since this implies $0 = 1$, $A \neq b$ cannot be true.
Case $j \in B$ and $j \notin A$. Therefore $\chi_B(x) = 1$ and $\chi_A(x) = 0$. Since this implies $0 = 1$, $A \neq b$ cannot be true.
Therefore, by contradiction, $A = B$. So $\chi_A = \chi_B$ iff $A = B$. □

8. For any $A, B \subseteq X$, we have

(a) $\chi_{A \cap B} = \chi_A \chi_B$

Claim 8. $\chi_{A \cap B} = \chi_A \chi_B$

Proof. There are four possibilities, $(x \in A \cap B)$, ($x \in A$ and $x \notin B$),($x \notin A$ and $x \in B$), and ($x \notin A$ and $x \notin B$).
Case $x \in A \cap B$. Since $x \in A$ and $x \in B$, $\chi_A = 1$ and $\chi_B = 1$, Therefore, plugging in $1 = (1)1$, if $x \in A \cap B$, $\chi_{A \cap B} = \chi_A \chi_B$.
Case $x \in A$ and $x \notin B$: So $\chi_A = 1$ and $\chi B = 0$. So $\chi_A \chi_B = 0$. Since $x \notin A \cap B$, $\chi_{A \cap B} = 0$. So $\chi_{A \cap B} = \chi_A \chi_B$. Case $x \notin A$ and $x \in B$: So $\chi_A = 0$ and $\chi B = 1$. So $\chi_A \chi_B = 0$. Since $x \notin A \cap B$, $\chi_{A \cap B} = 0$. So $\chi_{A \cap B} = \chi_A \chi_B$.
Case $x \notin A$ and $x \notin B$: So $\chi_A = 0$ and $\chi B = 0$. So $\chi_A \chi_B = 0$. Since $x \notin A \cap B$, $\chi_{A \cap B} = 0$. So $\chi_{A \cap B} = \chi_A \chi_B$.
Since $\chi_{A \cap B} = \chi_A \chi_B$ for all conditions, $\chi_{A \cap B} = \chi_A \chi_B$. □

(b) $\chi_{A\cup B} = \max\{\chi_A, \chi_B\}$

Claim 9. $\chi_{A\cup B} = \max\{\chi_A, \chi_B\}$

Proof. There are three possibilities, $x \in A$, $x \in B$ and $x \notin A \cup B$. Case $x \in A$: So $x \in \chi_{A\cup B}$. So $max\{\chi_A, \chi_B = 1$ (Since χ has a max value of 1 and $\chi_A = 1$). Therefore $\chi_{A\cup B} = \max\{\chi_A, \chi_B\}$. Case $x \in B$: So $x \in \chi_{A\cup B}$. So $max\{\chi_A, \chi_B\} = 1$ (Since χ has a max value of 1 and $\chi_B = 1$). Therefore $\chi_{A\cup B} = \max\{\chi_A, \chi_B\}$.
Case $x \notin A \cup B$: So $\chi_{A\cup B}(x) = 0$. So $max\{\chi_A, \chi_B\}(x) = 0$ (Since both $\chi_A(x)$ and $\chi_B(x) = 0$). Therefore $\chi_{A\cup B}(x) = \max\{\chi_A(x), \chi_B(x)\}$.
Since $\chi_{A\cup B}(x) = \max\{\chi_A(x), \chi_B(x)\}$ for all conditions, $\chi_{A\cup B}(x) = \max\{\chi_A(x), \chi_B(x)\}$. □

(c) $\chi_{A^c} = 1 - \chi_A$.

Claim 10. $\chi_{A^c} = 1 - \chi_A$

Proof. There are two possibilities, $x \in A$ or $x \notin A$ and therefore $x \in A^c$.
Case $x \in A$. So $\chi_A(x) = 1$ and $1 - \chi_A(x) = 0$. Since $x \notin A^c$, $\chi_A(x) = 0$. So $\chi_{A^c} = 1 - \chi_A$.
Case $x \in A^c$. So $x \notin A^c$, $\chi_{A^c}(x) = 1$. Furthermore $\chi_A(x) = 0$, therefore $1 - \chi_A(x) = 1$.So $\chi_{A^c} = 1 - \chi_A$.
Since $\chi_{A^c} = 1 - \chi_A$ for all conditions, $\chi_{A^c} = 1 - \chi_A$. □

9. ($\star$) Express $\chi_{A\setminus B}$ and $\chi_{A\Delta B}$ in terms of χ_A and χ_B. Prove your answer.

Claim 11. $\chi_{A\setminus B} = (1 - \chi_B)\chi_A$ *and* $\chi_{A\Delta B} = (1 - \chi_A)\chi_B + (1 - \chi_B)\chi_A$

Proof. ($\chi_{A\setminus B}$): There are four possibilities, ($x \in A \cap B$), ($x \in A$ and $x \notin B$),($x \notin A$ and $x \in B$), and ($x \notin A$ and $x \notin B$).
Case $x \in A \cap B$: So $\chi_A(x) = 1$ and $\chi_B(x) = 1$ (Since x is in both). So $(1 - \chi_B(x))\chi_A(x) = 0$. $x \notin A \setminus B$ (Since x is in B, according to the definition of setminus $x \notin A \setminus B$). So $\chi_{A\setminus B}(x) = 0$. Therefore $\chi_{A\setminus B} = (1 - \chi_B)\chi_A$.
Case $x \in A$ and $x \notin B$: Case $x \in A \cap B$: So $\chi_A(x) = 1$ and $\chi_B(x) = 0$ (Since by definition of χ). So $(1 - \chi_B(x))\chi_A(x) = 1$. $x \in A \setminus B$ (Since x is not in B but in A, according to the definition of setminus $x \in A \setminus B$). So $\chi_{A\setminus B}(x) = 1$. Therefore $\chi_{A\setminus B} = (1 - \chi_B)\chi_A$.
Case $x \notin A$ and $x \in B$: So $\chi_A(x) = 0$ and $\chi_B(x) = 1$ (According to the definition of χ). So $(1 - \chi_B(x))\chi_A(x) = 0$. $x \notin A \setminus B$ (Since x is in B, according to the definition of setminus $x \notin A \setminus B$). So $\chi_{A\setminus B}(x) = 0$. Therefore $\chi_{A\setminus B} = (1 - \chi_B)\chi_A$.
Case $x \notin A$ and $x \notin B$: So $\chi_A(x) = 0$ and $\chi_B(x) = 0$ (According to the definition of χ). So $(1 - \chi_B(x))\chi_A(x) = 0$. $x \notin A \setminus B$ (Since x is in B, according to the definition of setminus $x \notin A \setminus B$). So $\chi_{A\setminus B}(x) = 0$. Therefore $\chi_{A\setminus B} = (1 - \chi_B)\chi_A$. Since $\chi_{A\setminus B} = (1 - \chi_B)\chi_A$ for all conditions, $\chi_{A\setminus B} = (1 - \chi_B)\chi_A$.
($\chi_{A\Delta B} = (1 - \chi_A)\chi_B + (1 - \chi_B)\chi_A$):There are four possibilities, ($x \in A \cap B$), ($x \in A$ and $x \notin B$),($x \notin A$ and $x \in B$), and ($x \notin A$ and $x \notin B$).
Case $x \in A \cap B$: So $\chi_A = 1$ and $\chi_B = 1$ (According to the definition of χ). So $(1 - \chi_A(x))\chi_B(x) + (1 - \chi_B)\chi_A = 0$. Since $x \notin A\Delta B$ (x is in A and B, therefore according to the definition of Δ, $x \notin A\Delta B$). So $\chi_{A\Delta B} = 0$. Therefore $\chi_{A\Delta B} = (1 - \chi_A)\chi_B + (1 - \chi_B)\chi_A$.
Case $x \in A$ and $x \notin B$: So $\chi_A = 1$ and $\chi_B = 0$ (According to the definition of χ). So $(1 - \chi_A(x))\chi_B(x) + (1 - \chi_B)\chi_A = 1$. Since $x \notin A\Delta B$ (x is in A but not in B, therefore according to the definition of Δ, $x \in A\Delta B$). So $\chi_{A\Delta B} = 1$. Therefore $\chi_{A\Delta B} = (1 - \chi_A)\chi_B + (1 - \chi_B)\chi_A$.
Case $x \notin A$ $x \in B$: So $\chi_A = 0$ and $\chi_B = 1$ (According to the definition of χ). So $(1 - \chi_A(x))\chi_B(x) + (1 - \chi_B)\chi_A = 1$. Since $x \in A\Delta B$ (x is not in A but in B, therefore according to the definition of Δ, $x \in A\Delta B$). So $\chi_{A\Delta B} = 1$. Therefore $\chi_{A\Delta B} = (1 - \chi_A)\chi_B + (1 - \chi_B)\chi_A$.

Case $x \notin A$ and $x \notin B$: So $\chi_A = 0$ and $\chi_B = 0$ (According to the definition of χ). So $(1-\chi_A(x))\chi_B(x) + (1-\chi_B)\chi_A = 0$. Since $x \notin A\Delta B$ (x not in A or B, therefore according to the definition of Δ, $x \notin A\Delta B$). So $\chi_{A\Delta B} = 0$. Therefore $\chi_{A\Delta B} = (1-\chi_A)\chi_B + (1-\chi_B)\chi_A$.
Since $\chi_{A\Delta B} = (1-\chi_A)\chi_B + (1-\chi_B)\chi_A$ for all conditions, $\chi_{A\Delta B} = (1-\chi_A)\chi_B + (1-\chi_B)\chi_A$. □

10. ($\star$) Let $h : X \to Y$. Let R be an equivalence relation on Y. Define a relation h^*R on X by:

$$x_1 \, \mathrm{h}^* \mathrm{R} \, x_2 \text{ if and only if } h(x_1) \, \mathrm{R} \, h(x_2)$$

Show that h^*R is an equivalence relation on X.

Claim 12. *h^*R is an equivalence relation on X.*

Proof. (Reflexive) Let $a \in X$. So $h(a) \in Y$ (By definition of a map). So $h(a)Rh(a)$ (Since R is an equivalence relation and therefore Reflexive). So ah^*Ra (Definition of h^*). So h^* is reflexive.
(Symmetric) Let $(a,b) \in h^*R$. So $(h(a),h(b)) \in Y$ (By definition of h^*R). So $(h(b),h(a)) \in R$ (Since R is an equivalence relation therefore symmetric). By definition of h^*R, $(b,a) \in h^*R$. Since $(a,b),(b,a) \in h^*R$, h^*R is symmetric.
(Transitive) Let $(a,b),(b,c) \in h^*R$. So $(h(a),h(b)),(h(b),h(c))) \in Y$ (By definition of h^*R). So $(h(a),h(c)) \in R$ (Since R is an equivalence relation therefore transitive). By definition of h^*R, $(a,c) \in h^*R$. Since $(a,b),(b,c),(a,c) \in h^*R$, h^*R is transitive.
Since R is reflexive, transitive, and symmetric, R is an equivallence relation on X. □