MA 225 Problem Set 2: logic 2

facts and definitions The following fact is *true*, but we have not proved it so you **may not use it** anywhere in this homework set:

Forbidden Fruit Theorem. *Every integer is either even or odd.*

You **may** use the following fact:

Allowed Fact. *1 is not an even number.*

You will also need this definition:

Definition. We call the function f an *odd function* if

$$\forall x, f(-x) = -f(x)$$

We call the function f an *even function* if

$$\forall x, f(-x) = f(x)$$

exercises These problems don't require you to write proofs.

1. For proofs 1-5, write a symbolic version of each claim. Your symbolic version may only use logical symbols, variables, numbers, and the symbols $+, -, =$.
2. Let $H(t)$ represent *t is happy.* Let $S(x)$ represent *x is short.*
   (a) Compare the meanings of $\forall t, [H(t) \vee \sim H(t)]$ and $[\forall t, H(t)] \vee [\forall t, \sim H(t)]$. The first statement proposes that for all t, H(t) is happy or not happy, which is a tautology. The other states that for all t in H(t), all cases are happy, while for all cases of t this is not the case. This essentially means that none or all of t return happy for H(t)
   (b) Compare the meanings of $\forall t, [H(t) \wedge S(t)]$ and $[\forall t, H(t)] \wedge [\forall t, S(t)]$. They mean the same thing, for all of t, H(t) and S(t) are true
   (c) What's the general lesson about how $\vee$ and $\wedge$ interact with $\forall$?
   That $\forall$ distributes over $\wedge$ but not $\vee$
3. Let $H(t)$ represent *t is happy.* Let $S(x)$ represent *x is short.*
   (a) Compare the meanings of $\exists x : [H(x) \vee \sim H(x)]$ and $[\exists x : H(x)] \vee [\exists x : \sim H(x)]$. They both mean the same thingm that is that there exists a value of t which is true or false.
   (b) Compare the meanings of $\exists x : [H(x) \wedge S(x)]$ and $[\exists x : H(x)] \wedge [\exists x : S(x)]$.
   (c) What's the general lesson about how $\vee$ and $\wedge$ interact with $\exists$?
4. Consider the following line from William Shakespeare's *The Merchant of Venice*:

   All that glisters is not gold.

   (a) Use $G(t)$ to represent *t glisters* and $g(x)$ to represent *x is gold.* Write a symbolic sentence to represent **the literal meaning** of Shakespeare's line.
   (b) Write a symbolic sentence which represents **the meaning Shakespeare wants us to take** from the line. (*Hint.* If you find this request confusing, try revisiting your previous answer.)
   (c) Apply logical rules to simplify your symbolic answer to part (b). Then translate this simplified version into English.
5. The following joke by writer Joyce Carol Oates relies on an equivocation between two plausible interpretations of the original States United tweet. Distinguish between the two interpretations by expressing each as a symbolic sentence. Use $L(x, y, z)$ to represent *x shot y in z*, where the universe for x is all toddlers, the universe for y is all people, and the universe for z is all weeks.

muderoustoddler.jpg

6. Consider the two claims
   (1) *Every time I've been to Vegas, I was high on cocaine the whole time.*
   (2) *I have never done illegal narcotics.*

   Explain how it is that these claims could both be true (when spoken by the same person). What is the general lesson about the relationship between the claims $\forall x, P(x)$ and $\exists x : P(x)$?You have never been to Vegas. It means that even if a statement is true for all parts of t, it doesn't necesarily imply existance.

proofs Write a complete proof of each of the following statements.

1. The product of any integer with any even integer is even. <u>Claim:</u> Given any integer a, the product, p with an even integer b is even. **Proof:** Given b is an even integer, we can determine that it can be writen as b=2k.

$$\begin{aligned} p &= ba \\ &= 2ka && \text{(Replaces b with 2k)} \\ &= 2r && \text{(Uses r to represent ka)} \end{aligned}$$

   Therefore, as we can see, p=2r, and since this fits the formula 2j where j is an integer, ab where b is even, the product will be even.

2. The difference of two odd integers is always even. Claim: Given two odd numbers j and k, the difference D will always be even. Proof: We can represent j and k as 2l+1=n, where j and k are possible values of n where l is an integer. We can use this to see that:

$$\begin{aligned} D &= j - k \\ &= (2o + 1) - (2p + 1) && \text{(Replaced j with 2o+1 and k with 2p+1)} \\ &= 2o - 2p + 1 - 1 && \text{(distributed negative one)} \\ &= 2(o - p) && \text{(Subtracted ones and factored out 2.)} \\ D &= 2q && \text{(rewrote (o-p) as q)} \end{aligned}$$

   Therefore, D is in the form of 2l=n, which means that D is even.

3. If the sum of two integers is odd, their difference is odd.
4. ($\star\star$) No integer is both even and odd.
5. If m divides a and n divides b, then mn divides ab.
6. ($\star$) Given an integer t, if there are integers m and n so that $15m + 16n = t$, then there are integers r and s so that $3r + 8s = t$. Claim: Given an integer t, if there are integers m and n so that $15m + 16n = t$, then there are integers r and s so that $3r + 8s = t$. Proof: Since m and n must be integers, as well as r and s, we can set r=5m and s=2n, which satisfies the conditions of integers.

$$\begin{aligned} t &= 3r + 8s \\ &= 3(5m) + 8(2n) && \text{(plugged in values of r and s)} \\ t &= 15m + 16n && \text{(Solved for the values above)} \end{aligned}$$

   Since we know this to be true, there are integers which make 3r+8s=t when 15m+16n=t.
7. ($\star$) If there are integers m and n with $12m + 15n = 1$, then m and n are both positive.
8. ($\star$) The sum of two odd functions is [BLANK]. (Fill in the blank, and prove your answer.)
   Claim: The sum of two odd functions is odd. Proof: Let's consider two odd functions, f(x) and g(x) in which f(-x)=-f(x) and g(-x)=-g(x):

$$\begin{aligned} f(-x) + g(-x) &= -f(x) + -g(x) && \text{(addition of two functions)} \\ (f+g)(-x) &= -(f+g)(x) && \text{(factor our (-x) from left side and (-x) and -1 from right side)} \\ h(-x) &= -h(x) && \text{(Set f+g=h)} \end{aligned}$$

   Therefore, the sum of two odd functions is odd.
9. ($\star\star$) Show that there is a unique function which is both even and odd. Claim: There is a unique function that is both even and odd.
   Proof: Let us consider the class of even functions f(-x)=f(x) and odd functions f(-x)=-f(x). Let us also define a function as being an equation in which each input produces a unique output. Let us now define -f(x) graphically as the reflection of inputs across the x axis, as the distance between zero stays equidistant, however the direction in which these are is opposite. So, for a function to be both even and odd means for the outputs of f(-x) must equal both f(-x) and -f(-x). Because functions have unique outputs, -f(-x) must equal f(-x), or an equation in which the reflection and original converge. Because the only value which this holds true is 0, as this serves as the axis of reflection, and -0=0, the only function which would satisfy the conditions is f(x)=0.