Квадратичные иррациональности 21/2/24

III

$$\int R(x, \sqrt{ax^2+bx+c})\,dx \longrightarrow$$

$$\longrightarrow \int R(x, \sqrt{\pm x^2 \pm m^2})\,dx$$

(выделение полного квадрата)

a) $\int R(x, \sqrt{m^2-x^2})\,dx = \left[\begin{array}{l} x = m\sin t \\ x = m\cos t \end{array}\right]$

б) $\int R(x, \sqrt{x^2-m^2})\,dx = \left[\begin{array}{l} x = \frac{m}{\cos t} \\ x = \frac{m}{\sin t} \end{array}\right]$

в) $\int R(x, \sqrt{x^2+m^2})\,dx = \left[\begin{array}{l} x = m\,tg\,t \\ x = m\,ctg\,t \end{array}\right]$

1) $\int \frac{x^3\,dx}{\sqrt{9-x^2}} = \left[\begin{array}{l} x = 3\sin t \\ dx = 3\cos t\,dt \end{array}\right] =$

$$= \int \frac{27\sin^3 t \cdot 3\cos t}{\sqrt{9-9\sin^2 t}}\,dt =$$

$$= 27\int \sin^3 t\,dt = 27\int (1-\cos^2 t)\sin t\,dt =$$

$$= -27\int (1-\cos^2 t)\,d(\cos t) =$$

$$= -27t + \frac{27\cdot\cancel{3}\cos^3 t}{3} + C = -27\cos t + 9\cos^3 t =$$

$=$

$\sin t = \frac{x}{3}$

$\cos t = \sqrt{1-\sin^2 t}$

$\cos t = \sqrt{1-\frac{x^2}{9}}$

$= -27\sqrt{1-\frac{x^2}{9}} + 9\sqrt[3]{1-\frac{x^2}{9}} + C$

② $\int \frac{\sqrt{x^2+16}\,dx}{x^2} = \int \left[\begin{array}{l} x = 4\,tg\,t \\ dx = \frac{4}{\cos^2 t}dt \end{array}\right]$

$= \int \frac{4\sqrt{16tg^2t+16}\,dt}{16\,tg^2t} = \int \frac{\sqrt{tg^2t+1}\,dt}{tg^2t\cdot\cos^2 t} =$

$= \int \frac{dt}{\cos t\cdot \sin^2 t} = \int \frac{\cos t\,dt}{\sin^2 t} + \int \frac{dt}{\cos t} =$

$= \int \frac{d(\sin t)}{\sin^2 t} + \ln\left|tg\left(\frac{t}{2}+\frac{\pi}{4}\right)\right| =$

$= -\frac{1}{\sin t} + \ln\left|tg\left(\frac{t}{2}+\frac{\pi}{4}\right)\right| + C =$

$= -\cancel{\sqrt{1+ctg^2t}} = -\sqrt{1+\frac{16}{x^2}} + \ln\left(tg\left(\frac{arctg\frac{x}{4}}{2}+\frac{\pi}{4}\right)\right)$

$+C$

③ $\int \frac{\sqrt{x^2-4}}{x^2}dx = \left|\begin{array}{l} x = \frac{2}{\sin t} \\ dx = 2(-\sin^{-2}t\cdot\cos t)dt \end{array}\right| =$

$\quad\quad \overset{\shortparallel}{-\frac{2\cos t}{\sin^2 t}dt}$

$$= -2\int \frac{\sqrt{\left(\frac{4}{\sin^2 t} - 4\right)}\frac{\cos t}{\sin^2 t}dt}{\frac{4}{\sin^2 t}} = -\int \sqrt{1-\sin^2 t}\cdot\cos t\, dt =$$

$$= -\int \sqrt{\frac{1-\sin^2 t}{\sin^2 t}}\cdot \cos t\, dt = -\int \frac{\cos^2 t\, dt}{\sin t} =$$

$$= -\int \frac{1-\sin^2 t}{\sin t} = -\int \frac{1}{\sin t}dt + \int \sin t\, dt =$$

$$= -\cos t - \ln\left|tg\frac{t}{2}\right| + C =$$

$$= \left| \begin{array}{l} \sin t = \frac{2}{x} \\ \cos t = \sqrt{1-\sin^2 t} = \sqrt{1-\frac{4}{x^2}} \end{array} \right| =$$

$$= -\sqrt{1-\frac{4}{x^2}} - \ln\left|tg\left(\frac{\arcsin\frac{2}{x}}{2}\right)\right| + C$$

IV Дифференциальный бином

$$\int x^m(ax^n+b)^p dx \;:\; a,b \in R;\ m,n,p \in Q$$

Выражается в элементарных функциях в трёх случаях.

1) $p \in Z$ $\quad x = t^s$; s = НОК – знаменат. m и n.

2) $\frac{m+1}{n} \in Z$; $p = \frac{k}{r}$; $t^r = ax^n + b$

3) $p + \frac{m+1}{n} \in Z \Rightarrow \frac{ax^n+b}{x^n} = t^r$

$$\int t\,\frac{24t\,dt}{(t^2+1)^2} = \left[\begin{array}{ll} u = 24t & du = 24dt \\ dv = \frac{t}{(t^2+1)^2}dt & v = \frac{1}{2}\int\frac{d(t^2)}{(t^2+1)^2} = \end{array}\right.$$

$$= \frac{1}{2}\cdot\frac{-1}{(t^2+1)}\Bigg] = \frac{-24t}{2(t^2+1)} + \frac{1}{2}\int\frac{1}{(t^2+1)}24dt =$$

$$= -\frac{12}{(t^2+1)} + 12\,\mathrm{arctg}\,t + C =$$

$$= \frac{12}{-\left(\sqrt{\frac{11-x}{x-23}}\right)} + 12\,\mathrm{arctg}\sqrt{\frac{11-x}{x-23}} + C$$

$$\int \cos\sqrt{\frac{1-x}{1+x}}\,\frac{dx}{(1+x)\sqrt{1-x^2}} = \left[\ t = \sqrt{\frac{1-x}{1+x}}\right.$$

$$t^2 = \sqrt{\frac{1+x}{1+x}} \qquad dx = \frac{-2t(t^2+1) - 2t(1-t^2)}{(t^2+1)^2}dt =$$

$$t^2 + xt^2 = 1-x \qquad = \frac{-\cancel{2t^3} - 2t - 2t + \cancel{2t^3}}{(t^2+1)^2} =$$

$$xt^2 + x = 1-t^2$$

$$x(t^2+1) = 1-t^2 \qquad = \frac{-4t}{(t^2+1)^2}dt$$

$$x = \frac{1-t^2}{t^2+1}$$

$$\int \cos t$$

2) 3) - Подстановка Чебышова

(4) $\int \frac{dx}{\sqrt[4]{x^4+1}} = \int (x^4+1)^{-\frac{1}{4}} dx \quad p = -\frac{1}{4} \notin \mathbb{Z}$

$m=0;\ n=4 \quad \frac{m+1}{n} = \frac{1}{4} \notin \mathbb{Z}$

$p + \frac{m+1}{n} = -\frac{1}{4} + \frac{1}{4} = 0 \in \mathbb{Z} \Rightarrow 3)$

$\boxed{\frac{x^4+1}{x^4} = t^4} \quad t^4 = 1 + \frac{1}{x^4} \Rightarrow x^4 = \frac{1}{t^4-1} \Rightarrow$

$\Rightarrow x = (t^4-1)^{-\frac{1}{4}}$

$dx = -\frac{1}{\cancel{4}}(t^4-1)^{-\frac{5}{4}} \cdot \cancel{4} t^3 dt = -\cancel{\frac{t^3 dt}{2}} =$

$= -(t^4-1)^{-\frac{5}{4}} \cdot t^3 dt$

$x^4+1 = \frac{1}{t^4-1} + 1 = \frac{t^4}{t^4-1}$

$\int -(t^4-1)^{-\frac{5}{4}} \cdot t^3 \cdot \left(\frac{t^4}{t^4-1}\right)^{-\frac{1}{4}} dt =$

$= \int -t^3 (t^4-1)^{-\frac{5}{4}} \left(\frac{t^4}{t^4-1}\right)^{-\frac{1}{4}} dt =$

$= \int -t^3 (t^4-1)^{-1} (t^4)^{-\frac{1}{4}} \cancel{dt} = -\int t^2 (t^4-1)^{-1} dt =$

$= -\int \frac{t^2 dt}{t^4-1} =$

$$= -\int \frac{t^2 dt}{(t^2+1)(t^2-1)} = -\frac{1}{2}\int \frac{(t^2-1)+(t^2+1)}{(t^2-1)(t^2+1)} dt$$

$$= -\frac{1}{2}\left(\int \frac{1}{t^2+1} dt + \int \frac{1}{t^2-1}\right) =$$

$$= -\frac{1}{2} \operatorname{arctg} t + \frac{1}{4} \ln\left|\frac{t-1}{t+1}\right| + C =$$

$$= -\frac{1}{2} \operatorname{arctg} \frac{\sqrt[4]{x^4+1}}{x} - \frac{1}{4} \ln\left|\frac{\frac{\sqrt[4]{x^4+1}}{x} - 1}{\frac{\sqrt[4]{x^4+1}}{x} + 1}\right| + C$$

$$\int \sqrt{\frac{11-x}{x-23}}\, dx = \left[t = \sqrt{\frac{11-x}{x-23}} \right.$$

$$t^2 = \frac{11-x}{x-23}$$

$$xt^2 - 23t^2 = 11 - x$$

$$xt^2 - 23t^2 - 11 + x = 0$$

$$xt^2 + x = 23t^2 + 11$$

$$x(t^2+1) = 23t^2 + 11$$

$$x = \frac{23t^2+11}{(t^2+1)}$$

$$dx = \frac{46t(t^2+1) - 2t(23t^2+11)}{(t^2+1)^2} dt =$$

$$= \frac{2t(23(t^2+1) - 23t^2 - 11)}{(t^2+1)^2} =$$

$$= \frac{2t(23t^2+23 - 23t^2 - 11)}{(t^2+1)^2} =$$

$$= \frac{24t\, dt}{(t^2+1)^2} dt$$

$$x+1=\frac{1-t^2}{1+t^2}+1=\frac{1-t^2+1+t^2}{1+t^2}=\frac{2}{1+t^2}$$

$$1-x^2=(1-x)(1+x)=\frac{4t^2}{(1+t^2)^2}$$

$$= \int \cos t\cdot\frac{-4t}{(1+t^2)^2}\cdot\frac{1+t^2}{2}\cdot\sqrt{\frac{(1+t^2)^2}{4t^2}}\,dt=$$

$$=-\int\cos t\,dt=-\sin t+C=-\sin\sqrt{\frac{1-x}{1+x}}+C$$