Интегрирование 12/02/24

тригонометрических функций.

① Универсальная триг. замена

$\int R(\sin x, \cos x)\,dx$

R - рациональна относительно

$\sin x$, $\cos x$

$\boxed{t = tg\frac{x}{2}}$ $\boxed{\cos^2 x = \frac{1}{1+tg^2 x}}$

$x = 2\,arctg\,t$ $dx = \frac{2\,dt}{t^2+1}$

$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,tg\left(\frac{x}{2}\right)\cos^2\frac{x}{2} =$

$= \frac{2tg\frac{x}{2}}{1+tg^2\frac{x}{2}} = \frac{2t}{t^2+1}$

$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}\left(1 - tg^2\frac{x}{2}\right) =$

$= \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}$

$\int R(\sin x, \cos x)\,dx = \underbrace{\int R\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right)\frac{2dt}{1+t^2}}_{\text{рациональна относительно } t}$

② $\int R(\sin 2x, \cos 2x, \operatorname{tg} x)\,dx$

$\cos^2 x, \sin^2 x$

$$\boxed{t = \operatorname{tg} x}$$

$$dx = \frac{dt}{t^2+1}$$

$$\sin 2x = 2\sin x \cos x = 2\operatorname{tg} x \cdot \cos^2 x = \frac{2\operatorname{tg} x}{1+\operatorname{tg}^2 x} = \frac{2t}{t^2+1}$$

$$\cos 2x = \cos^2 x - \sin^2 x = \cos^2 x(1-\operatorname{tg}^2 x) = \frac{1-\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = \frac{1-t^2}{1+t^2}$$

$$\cos^2 x = \frac{1}{1+\operatorname{tg}^2 x} = \frac{1}{1+t^2}$$

$$\sin^2 x = \operatorname{tg}^2 x \cdot \cos^2 x = \frac{\operatorname{tg}^2 x}{1+\operatorname{tg}^2 x} = \frac{t^2}{1+t^2}$$

① $$\int \frac{dx}{4\sin x + 3\cos x + 5} = \left| \begin{array}{ll} t = \operatorname{tg}\frac{x}{2} & \sin x = \frac{2t}{t^2+1} \\ dx = \frac{2dt}{t^2+1} & \cos x = \frac{1-t^2}{1+t^2} \end{array} \right| =$$

$t =$

$$= \int \frac{2dt}{(t^2+1)\left(\frac{8t}{t^2+1} + \frac{3(1-t^2)}{1+t^2} + 5\right)} =$$

$$= \int \frac{2dt}{8t + 3(1-t^2) + 5(t^2+1)} = \int \frac{2dt}{2t^2+8t+8} =$$

$$= \int \frac{dt}{(t+2)^2} = \int (t+2)^{-2}\,d(t+2) = \frac{(t+2)^{-1}}{-1} = -\frac{1}{t+2}$$

! Идея: модели в детале.

$$= -\frac{1}{t+2} + C = \frac{-1}{\operatorname{tg}\frac{x}{2}+2} + C$$

$$\int \frac{(1+\cos x)\,dx}{1+\cos x+\sin x} = \left| \begin{array}{ll} t = \operatorname{tg}\frac{x}{2} & \sin x = \frac{2t}{t^2+1} \\ dx = \frac{2dt}{t^2+1} & \cos x = \frac{1-t^2}{1+t^2} \end{array} \right| =$$

$$= \int \frac{2\left(1+\frac{1-t^2}{1+t^2}\right)\frac{dt}{1+t^2}}{\left(1+\frac{1-t^2}{1+t^2}+\frac{2t}{t^2+1}\right)} = \int \frac{\frac{4dt}{1+t^2}}{1+t^2+1-t^2+2t}$$

$$= \int \frac{4\,dt}{(1+t^2)(2+2t)} = \int \frac{2dt}{(1+t^2)(1+t)}$$

$$\frac{2}{(1+t)(t^2+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2+1}$$

$$2 = At^2 + A + Bt^2 + Bt + Ct + C$$

$$\begin{array}{l} t^2: \\ t^1: \\ t^0: \end{array} \begin{cases} A+B=0 \\ B+C=0 \\ A+C=2 \end{cases} \quad \begin{cases} -C+2-C=0 \\ B=-C \\ A=-C+2 \end{cases} \rightarrow \begin{cases} C=1 \\ B=-1 \\ A=1 \end{cases}$$

$$\int \frac{1}{t+1}dt + \int \frac{-t+1}{t^2+1}dt = \ln|t+1| + \int \frac{dt}{t^2+1} - \int \frac{t}{t^2+1}$$

$$= \ln|t+1| + \operatorname{arctg} t - \frac{1}{2}\ln|t^2+1| + C =$$

$$= \ln\left|\operatorname{tg}\frac{x}{2}+1\right| + \operatorname{arctg}\operatorname{tg}\frac{x}{2} - \frac{1}{2}\ln\left|\operatorname{tg}^2\frac{x}{2}+1\right| + C$$

$$\int \frac{tg\,x\,dx}{\sin^2 x - 5\cos^2 x + 4} = \left| \begin{array}{ll} t = tg\,x & \sin^2 x = \frac{t^2}{1+t^2} \\ x = arctg\,t & \cos^2 x = \frac{1}{1+t^2} \\ dx = \frac{dt}{t^2+1} & \end{array} \right| =$$

$$= \int \frac{t\,dt}{(t^2+1)\left(\frac{t^2}{1+t^2} - 5\frac{1}{1+t^2} + 4\right)} = \int \frac{t\,dt}{t^2 - 5 + 4t^2 + 4} =$$

$$= \int \frac{t\,dt}{5t^2 - 1} = \frac{1}{10}\int (5t^2-1)^{-1} d(5t^2-1)$$

$$= \frac{1}{10}\ln|5t^2 - 1| + C = \frac{1}{10}\ln|5tg^2 x - 1| + C$$

(4) $$\int \frac{(tg\,x + 4)}{3\cos^2 x + 2\sin 2x + 1} = \left| \begin{array}{ll} t = tg\,x & \sin 2x = \frac{2t}{t^2+1} \\ \cos^2 x = \frac{1}{1+t^2} & \\ dx = \frac{dt}{t^2+1} & \end{array} \right| =$$

$$= \int \frac{(t+4)\,dt}{(t^2+1)\left(\frac{3}{1+t^2} + \frac{4t}{t^2+1} + 1\right)} =$$

$$= \int \frac{(t+4)\,dt}{3 + 4t + t^2 + 1} = \int \frac{(t+4)\,dt}{(t+2)^2} =$$

$$= \int \frac{(t+2)+5}{(t+2)^2}dt = \int \frac{dt}{t+2} + 5\int \frac{dt}{(t+2)^2} =$$

$$= \ln|t+2| - \frac{5}{t+2} + C = \ln|tg\,x + 2| - \frac{5}{tg\,x+2} + C$$

③ $\int R(\cos x)\cdot \sin x\,dx = -\int R(\cos x)\,d(\cos x)$

$= [t=\cos x] = \ldots \int R(\sin x)\cos x\,dx =$

$= \int R(\sin x)\,d(\sin x) = [t=\sin x] = \ldots$

$\int \cos^n x\,\sin^{2n+1}x\,dx = \int \cos^n (\sin^2 x)^n \sin x\,dx$

$= -\int \cos^n x(1-\cos^2 x)^n d(\cos x) =$

$= -\int \sin^m x\,\cos^{2n+1}x\,dx = \int \sin^m x\,(\cos^2 x)^n \cos x\,dx =$

$= \int \sin^n x(1-\sin^2 x)^n d(\sin x) = [t=\sin x]$

⑤ $\int \sin^4 x\cdot\cos^5 x\,dx = \int \sin^4 x\,\cos^4 x\,\cos x\,dx =$

$= \int \sin^4 x(1-\sin^2 x)^2 d(\sin x) = \left| t=\sin x \right| =$

$= \int t^4(1-t^2)^2 dt = \int t^4(1-2t^2+t^4)\,dt =$

$= \int (t^4-2t^6+t^8)\,dt = \int t^4 dt - 2\int t^6 dt + \int t^8 dt =$

$= \frac{t^5}{5} - \frac{2t^7}{7} + \frac{t^9}{9} + C =$

$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9} + C$

④ $\int \sin^m x \cos^n x \, dx$ $\quad m+n<0$

$\quad m+n$ – чётное

$\boxed{t = tg x}$

$\int \frac{dx}{\cos x \sin^3 x} = \left[t = tg x \right] = \int \frac{\sin x \, dx}{\cos x \cdot \sin^4 x} =$

$= \int \frac{tg x \, dt}{(\sin^2 x)^2} = \left| \begin{array}{l} t = tg x \\ \sin^2 x = \frac{t^2}{t^2+1} \\ dx = \frac{dt}{t^2+1} \end{array} \right| = \int \frac{t \, dt}{\left(\frac{t^2}{t^2+1}\right)(t^2+1)} =$

$= \int \frac{(t^2+1)\,dt}{t^3} = \int \frac{1}{t} dt + \int \frac{dt}{t^3} = \ln|t| - \frac{2}{t^2} + C =$

$= \ln|tg x| - \frac{2}{tg^2 x} + C$

⑤ $\int \sin^n x \cos^m x \, dx$ $\quad n, m \geq 0$, n, m – чётные

$\sin x \cos x = \frac{\sin 2x}{2}$; $\cos^2 x = \frac{1+\cos 2x}{2}$

$\sin^2 x = \frac{1-\cos 2x}{2}$

№4 $\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 dx =$

$= \int \left(\frac{\sin 2x}{2}\right)^2 dx = \frac{1}{4} \int \sin^2 2x = \frac{1}{4} \int \frac{1-\cos 4x}{2} dx$

$= \frac{1}{8} \int (1-\cos 4x) dx = \frac{x}{8} - \frac{\sin 4x}{32} + C$

№8 $\int \cos^6 x\,dx = \int (\cos^2 x)^3 dx = \int \left(\frac{1+\cos 2x}{2}\right)^3 dx =$

$= \frac{1}{8}\int (1+\cos 2x)^3 dx = \frac{1}{8}\int (1+3\cos 2x+3\cos^2 2x+\cos^3 2x)\,dx$

$= \frac{1}{8}\left(\int dx + 3\int \cos 2x\,dx + 3\int \cos^2 2x\,dx + \int \cos^3 2x\,dx\right)$

$\int \cos^2 2x\,dx = \int \frac{1+\cos 4x}{2}\,dx = \frac{1}{2}\int (1+\cos 4x)\,dx =$

$= \frac{x}{2} + \frac{\sin 4x}{4} + C$

$\int \cos^3 2x\,dx = \int \cos^2 2x \cos 2x\,dx = \frac{1}{2}\int (1-\sin^2 2x)\,d(\sin 2x) =$

$= \frac{1}{2}\int (1-t^2)\,dt = \frac{1}{2}\sin 2x - \frac{\sin^3 2x}{3} + C$

$= \frac{x}{8} + \frac{3\sin 2x}{16} + \frac{3}{8}\left(\frac{x}{2} + \frac{\sin 4x}{4}\right) + \frac{1}{8}\left(\frac{1}{2}\sin 2x - \frac{\sin^3 2x}{3}\right)$