Time to Criterion Latent Growth Models

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Abstract

Latent growth models, a special class of longitudinal models within the broader structural equation modeling (SEM) domain, provide researchers a framework for investigating questions about change over time; yet rarely is time itself modeled as a focal parameter of interest. In the current paper, rather than treating time purely as an index of measurement occasions, the proposed Time to Criterion (T2C) model draws from Preacher and Hancock's (2012) latent growth model reparameterization guidelines in order to model individual variability (i.e., to treat as a random effect) in one's time to achieve a criterion level of a given outcome. As such, the T2C model also allows researchers to model predictors and distal outcomes of time, as well as benefiting more generally from the flexibility afforded by being embedded within the broader SEM framework to accommodate such real-world data issues as missingness, complex error structures, nonnormality, and nested data. In this study we derive T2C from the linear latent growth model and discuss model assumptions and interpretation. By illustrating the model using real data, we demonstrate both its utility for applied research and its implementation in conventional SEM software. We also discuss and illustrate an extension of the model for nonlinear growth. Overall, the T2C model presents a novel and interpretable growth parameterization for further understanding processes of change.

KEY WORDS: latent growth curve models, longitudinal data, structural equation modeling, reparameterization, structured latent curve model

Time to Criterion Latent Growth Models

The time it takes to acquire new knowledge or skills is a question of interest across disciplines, including developmental science and psychology (Adi-Japha, Badir, Dorfberger, & Karni, 2014; Bloom, 1974; Megherbi, Elbro, Oakill, Segui, & New, 2018), health and physiology (Billat et al., 2000; Brown et al., 1998; Huang et al., 2012; Kim et al., 2015; Niles et al., 2009; Shane, Pettitt, Morgenthal, & Smith, 2008; Wayne et al., 2005; Weir et al., 2007), education (Anderson, 1976; Burke, 2016; Etkina, 2008; Fredrick & Walberg, 1980; Ishitani, 2006; Zampetakis, 2008), and disability research (Masera et al., 2015; Ruffino, Gori, Boccardi, Molteni, & Facoetti, 2014; Schipper, Fransen, den Broeder, & Van Riel, 2010). Naturally, many such studies utilize survival analysis (Allison, 2014; Kleinbaum & Klein, 2010; Singer & Willett, 2003) to investigate time-to-event processes. Although several of the events considered in these studies are inherently categorical (e.g., degree completion, mortality), others are continuous outcomes that must be dichotomized to conform to the survival analysis framework (e.g., language fluency, blood pressure levels). Beyond survival analysis approaches, other studies consider time elapsed as an outcome itself, modeling time as a continuous variable using such methods as analysis of variance or multiple regression. Research in applied literature, however, has recently placed a greater focus on modeling longitudinal outcomes from a random effects or latent variable growth framework (Felt, Depaoli, & Tiemensma, 2017; Grimm, Ram, & Hamagami, 2011; McCoach, Rambo, & Welsh, 2013; McNeish & Matta, 2017; Sterba, 2014; van de Schoot et al., 2018).

Growth models (McArdle, 1986; Meredith & Tisak, 1990; Muthén, 1991) take advantage of repeated measures data to investigate individual change processes over time, but in general do not provide inferential information regarding individual variability in time to reach specific goals. In recognition of this gap and to better address research questions regarding individual acquisition times, the current paper presents the Time to Criterion (T2C) latent growth model. The T2C model draws from foundational work by Preacher and Hancock (2012, 2015) on reparameterization and allows researchers to model time as a latent variable while simultaneously taking advantage of the benefits of latent growth modeling and broader structural equation modeling (SEM) framework. The purpose of this paper is to derive the T2C model for both linear and nonlinear latent growth models (LGMs), demonstrate the estimation of the model using conventional SEM software, and discuss applications of the model to real data.

LGMs and the Case for Reparameterization

LGMs use repeated measures data to estimate inter- and intra-individual change over time (for a technical discussion, see Bollen & Curran, 2006; Fitzmaurice, Laird, & Ware, 2012; Singer & Willett, 2003). The simple linear LGM, presented mathematically in Eq. (1) and graphically in Figure 1, estimates two primary aspects of growth: (1) the mean response on outcomes y_{ij} for individual j at a reference time point such as the common i = 0 (intercept), and (2) the expected trajectory of that response across measurement occasions for individual case j (slope):

$$y_{ij} = \alpha_j + \beta_j t_i + \varepsilon_{ij} \begin{cases} i = \text{time} \\ j = \text{case} \end{cases} . \tag{1}$$

We can similarly write out this model using matrix notation, where a vector of outcomes **y** for person j measured at I occasions (i = {0, 1, … , I – 1}) are modeled as a function of the y-intercepts **ν** (an I×1vector generally fixed to **0** to identify the means of the growth factors), an I×P matrix of factor loadings **Λ** (where P = number of growth factors; **Λ** is generally constrained such that the set of loadings for the growth intercept factor is fixed to **1** and the set of loadings for the slope factor is generally fixed to **t**, representing the I measurement occasions), a P×1 vector of weights $\boldsymbol{\eta}_j$ containing each of the person-specific latent growth factors (in this case α_j and β_j, with distribution $\boldsymbol{\eta}_j \sim \mathcal{N}\left(\boldsymbol{\mu}, \boldsymbol{\Sigma}\right)$), and an I×1 vector of occasion-specific errors $\boldsymbol{\varepsilon}_j$ with distribution $\boldsymbol{\varepsilon}_j \sim \mathcal{N}\left(\mathbf{0}, \boldsymbol{\Theta}\right)$. The I×I variance-covariance matrix $\boldsymbol{\Theta}$ is often structured to assume local independence of observations for person j across time points i, conditional on the values of the latent growth factors $\boldsymbol{\eta}_j$ as $\boldsymbol{\Theta} = \begin{bmatrix} \theta_0 & & & \\ 0 & \theta_1 & & \\ \vdots & \ddots & \ddots & \\ 0 & \cdots & 0 & \theta_{(I-1)} \end{bmatrix}$, which we impose throughout the current study (for further discussion of error covariance structures in the context of reparameterization, see Blozis, 2004; Preacher & Hancock, 2015). The model in matrix notation is thus given as follows:

$$\mathbf{y}_j = \boldsymbol{\nu} + \boldsymbol{\Lambda}\boldsymbol{\eta}_j + \boldsymbol{\varepsilon}_j \qquad (2)$$

We will utilize both sets of model notation throughout this paper.

Estimating the LGM from within the SEM framework allows for considerable flexibility in several aspects of the model, from the inclusion of time-varying and time-invariant covariates to the accommodating more complex error variance/covariance structures. One rigid aspect of the SEM framework, however, requires that estimated parameters enter models linearly, which may not sufficiently answer relevant research questions regarding growth and change processes. Extending previous work in structured latent curve models (Blozis, 2004; Browne, 1993), Preacher and Hancock (2012, 2015) demonstrated the reparameterization of growth curve models to extract parameters of interest, to allow for individual differences in those parameters, to linearize nonlinear model parameters (e.g., half-life), and to render parameters of interest estimable in conventional SEM software.

To start, consider the linear LGM. Although the model itself is relatively simple, the potential benefits of reparameterization are easy to see. Take, for example, one useful application of a reparameterized linear LGM, presented in Preacher and Hancock (2012). Starting with an unconditional linear growth model with random intercept and slope, the authors extracted the *aperture* (Hancock & Choi, 2006), the point at which individuals are most similar (i.e., where the model-implied variance is at its minimum), by taking advantage of the fact that this point occurs where the covariance between intercept and slope is equal to 0. In this way, the authors exchanged an otherwise uninformative parameter (the intercept) for one of greater research interest (aperture location). The T2C growth model, as derived in this paper, indeed sacrifices the random intercept factor in favor of a substantively more meaningful piece of information: time to achieve a specified criterion level on an outcome. In the latent variable framework, prior to reparameterization, time is commonly treated as an index of growth, which can be estimated or fixed to a specific value (see, e.g., Hancock, Harring, & Lawrence, 2013), yet is commonly assumed to be the same for all subjects measured at occasion *i* (although this assumption can be relaxed; see Aydin, Leite, & Algina, 2014; Mehta & West, 2000). These time indices are parameterized as factor loadings, which reflect a hypothesized growth trajectory, yet do not permit exploration of factors predictive of time to criterion or effects of time to criterion on distal outcomes. Bridging this gap, the T2C model explicitly integrates time to criterion into the growth model as a latent variable, providing opportunity to address a broad variety of research questions regarding individual variability in growth timing.

Time to Criterion vs. Time to Event

Time to criterion refers to the individual time to reach a pre-defined level on an outcome and is modeled in the reparameterized T2C growth model as a latent growth factor. Although it is possible to relax the following assumptions, the simplest case of the T2C growth model implies the time to criterion latent factor and the measured outcomes are continuous, multivariate normal variables, and the change process is linear. *Time to event* also describes individual time in reaching a pre-defined outcome status, yet it diverges from time to criterion in the way it is modeled and the research question it answers (for a discussion of joint estimation of survival and growth, see Anastasios & Davidian, 2004; Henderson, Diggle, & Dobson, 2000; Muthén & Masyn, 2005; Xu & Zeger, 2001). Within the survival analysis framework, time to event comprises one component of the dependent variable, while the other component is the *event status*; survival analysis thus models the probability that an individual will experience a well-defined discrete event state during a measured time period (Allison, 2014; Kleinbaum & Klein, 2010; Singer & Willet, 2003). Although the survival analysis family is represented by a wide variety of models, from parametric regression-type models to non-parametric tabulations, most survival models assume that there exists (1) a defined target event under observation, (2) a time point at which no one in the population has experienced the event, and (3) a scale for measuring time (Singer & Willett, 2003). T2C similarly requires a scale for measuring time (i.e., continuous or discrete) and a well-defined definition of the *criterion*; however, T2C does not presuppose that a time period exists where no one in the population has reached criterion.

In addition to the three assumptions stated above, another important difference between survival analytic methods and the T2C growth model pertains to the handling of missing data. Because time-to-event processes generally follow individuals over time, loss to follow up is an inevitable issue, either due to a competing risk (e.g., death) or due to a study design issue (e.g., attrition). In order to handle this specific type of missing data, survival models censor cases when an event time is unknown (see e.g., Allison, 2014). Censorship is necessary in survival analysis because such models, which traditionally track irreversible events like death, model probabilities of an event's occurrence at a particular time *given that it has not already been observed* (i.e., the hazard function). The basic survival model then necessarily assumes that measurement cannot or should not continue once the event (or a competing event) has been reached (though, once again, this assumption can be relaxed; see Allison, 2014; Austin, Lee, & Fine, 2016; Fine & Gray, 1999). The T2C model, on the other hand, is geared toward understanding time to criterion within the context of a growth process and is thus proposed for use with defined but not irreversible criteria, such as achieving a proficiency level on a standardized test. As such, data are not censored; rather, missing data can be handled in T2C using standard procedures available in the latent variable framework (e.g., full-information maximum likelihood, multiple imputation; see Enders, 2010, 2013).

Last, and most pertinent, survival models do not, in general, treat time as a model-estimated parameter, whereas the T2C growth model is parameterized expressly to do so. In the case of survival models, determination of individual survival times is often of great interest; however, predicted survival times are generally viewed as inaccurate and the process itself is topic of much debate (Collins, Reitsma, Altman, & Moons, 2015; Henderson & Keiding, 2005; Lau et al., 2007; Steyerberg et al., 2013). Although individual prediction may be of motivating interest for employing the T2C parameterization, the model retains utility for testing theory regarding whether and how the length and variability in time to achieve a criterion level is related to other time-varying or time-invariant variables of interest. The T2C model thus allows for not only the estimation of the mean and variance of the time parameter, which can be accomplished without reparameterization, but additionally it allows for easy implementation within SEM software, opening the door to more complex theory testing and data-model fit assessment within the latent variable framework.

The rest of this paper is organized as follows. First, we present the four steps of the linear T2C derivation (reparameterization, linearization, structured latent curve model expression, and parameter estimation). Second, we derive a nonlinear T2C growth model, following the same four steps discussed in the linear derivation. Third, we discuss the impact of the derivation process on interpretation of model parameters and compare the new interpretation with that of the original model specification. And finally, we provide guidelines for selection of meaningful values of the fixed criterion value for the T2C model.

Linear T2C Derivation

Preacher and Hancock (2012) provided a four-step process for extracting desired model parameters and respecifying LGMs for estimation in SEM software:

1. Reparameterize the model in terms of the parameter of interest;
2. Linearize the resulting equation using a first-order Taylor series expansion (if necessary);
3. Express the resulting function in the modified structured latent curve model (SLCM) specification to allow for estimation in SEM software;
4. Estimate model parameters.

Below we provide the derivation of the linear T2C growth model, following the first three steps above. The fourth step, estimation of parameters, will be contained within the real data examples that follow. Technical aspects related to parameter substitution, the accuracy of linearization as an approximation method, and the interpretation of linearized parameters are also discussed within this section.

1. Reparameterize the Target Function

The original function, given in Eq. (1) and restated in Eq. (2), is an unconditional linear LGM. In order to respecify the model in terms of the parameter of interest, time to criterion, the individual time t_{cj} to reach a fixed criterion level of interest c can be solved from other linear LGM parameters algebraically. It is important to emphasize here that c is a prespecified quantity and not an unknown or estimated model parameter (we discuss the appropriate identification of criterion values of c in more depth at the end of this paper). Eq. (3) shows the respecification:

$$\mathrm{c} = \alpha_j + \beta_j t_{\mathrm{c}j} \ , \tag{3}$$

where, as noted, c is the substantively-informed criterion value of y_{ij}, and t_{cj} is the value of time t at which person j reaches criterion c. The mean and variance of the time to criterion term, given $t_{\mathrm{c}j} = \frac{\mathrm{c} - \alpha_j}{\beta_j}$, are thus:

$$\begin{aligned} \operatorname{mean}\left(t_{\mathrm{c}j}\right) &= \mathrm{E}\left(\frac{\mathrm{c}-\alpha_j}{\beta_j}\right) \\ \operatorname{var}\left(t_{\mathrm{c}j}\right) &= \mathrm{E}\left[\left(\frac{\mathrm{c}-\alpha_j}{\beta_j}\right)^2\right] - \mathrm{E}\left[\left(\frac{\mathrm{c}-\alpha_j}{\beta_j}\right)\right]^2 \end{aligned} . \tag{4}$$

It is notable here that the transformation imposed implies that the t_{cj} term will likely not follow a normal distribution, and calculating the higher-order moments is left to the reader.[1]

Selecting a parameter from Eq. (3) to substitute for the new time to criterion parameter will involve a careful consideration of the substantive and theoretical trade-offs in sacrificing one parameter over another, and technical details related to different parameter substitutions are discussed in Step 2. We choose to solve for the individual intercept α_j, and substituting for it in the original equation yields the following:

[1] We thank an anonymous reviewer for this observation.

$$\alpha_j = \mathrm{c} - \beta_j t_{cj} \ , \tag{5}$$

$$y_{ij} = \left(\mathrm{c} - \beta_j t_{cj}\right) + \beta_j t_i + \varepsilon_{ij} \ . \tag{6}$$

Recognizing that the calculated value t_{cj} is now estimable as a model parameter, we can equivalently restate Eq. (6), replacing t_{cj} with Greek letter τ_j for clarity and consistency with the other model parameters:

$$y_{ij} = \left(\mathrm{c} - \beta_j \tau_j\right) + \beta_j t_i + \varepsilon_{ij} \ . \tag{7}$$

The resulting target function in Eq. (7) and the original function in Eq. (1) both describe the same growth process; however, in the new specification, the intercept (the expected value of the outcome for case j at time i = 0) has been substituted for a value of greater research interest, the time to criterion τ_j. The model given in Eq. (7) is thus a reparameterized linear growth curve model. In the next step, we demonstrate the linearization of the target function, a process that allows us to re-express the model parameters in terms that can be adapted to the structured latent curve framework for implementation in SEM software.

2. Linearization: First-Order Taylor series Expansion

Next, we apply a first-order Taylor series expansion to our reparameterized target function, Eq. (7), which has two effects: linearization, and preparation for the modified SLCM approach. In this case, the target function is already linear in the parameters, meaning that there is no model parameter for which that parameter's partial derivative contains itself (Bates & Watts, 1988), and thus, it is not strictly necessary to linearize the function. However, this case will not always be true when dealing with reparameterized functions, and it may not be readily clear when a parameter is linear or nonlinear. As in Preacher and Hancock (2012), we demonstrate the linearization step for the linear model to provide a clear foundation of the process for application to more complicated scenarios, such as the nonlinear T2C model presented in a later section.

Starting with the reparameterized function from Eq. (7), the first-order Taylor series expansion can be constructed as a linear combination of the expectation of y_{ij} evaluated at the mean of the growth parameters, β_j (slope) and τ_j (time to criterion), and each mean-deviated parameter coefficient multiplied by its first partial derivative:

$$\tilde{y}_{ij} = f\left(\boldsymbol{\mu}, t_i\right) + \left(\beta_j - \mu_\beta\right) \left.\frac{\partial y_{ij}}{\partial \beta_j}\right|_{\boldsymbol{\mu}_{\beta,\tau}} + \left(\tau_j - \mu_\tau\right) \left.\frac{\partial y_{ij}}{\partial \tau_j}\right|_{\boldsymbol{\mu}_{\beta,\tau}}, \tag{8}$$

where $f\left(\boldsymbol{\mu}, t_i\right) = \mathrm{E}\left[y_{ij}\right] = \left(\mathrm{c} - \mu_\beta \mu_\tau + \mu_\beta t_i\right)$, which is simply the target function evaluated at the population point. In Eq. (8), $\tilde{y}_{ij}$ is the first-order linearization of the target function, and τ_j is the new time to criterion parameter. The partial derivatives of the target function with respect to the slope and time to criterion parameters are given by the following:

$$\begin{aligned} \left.\frac{\partial y_{ij}}{\partial \beta_j}\right|_{\boldsymbol{\mu}_{\beta,\tau}} &= t_i - \mu_\tau \\ \left.\frac{\partial y_{ij}}{\partial \tau_j}\right|_{\boldsymbol{\mu}_{\beta,\tau}} &= -\mu_\beta \end{aligned} . \tag{9}$$

Note that neither partial derivative in Eq. (9) contains itself, which indicates that the parameters enter the model linearly after reparameterization. The final linearized solution, given Eqs. (8) and (9), is as follows:

$$\begin{aligned} \tilde{y} &= \left(\mathrm{c} - \mu_\beta \mu_\tau + \mu_\beta t_i\right) + \left(\beta_j - \mu_\beta\right)\left(t_i - \mu_\tau\right) + \left(\tau_j - \mu_\tau\right)\left(-\mu_\beta\right) \\ &= \left(\mathrm{c} + \mu_\beta \mu_\tau\right) + \beta_j\left(t_i - \mu_\tau\right) + \tau_j\left(-\mu_\beta\right) \end{aligned} . \tag{10}$$

Though not strictly necessary with a linear derivation, in general the Taylor series linearization step is important to construct the model parameters in terms that can be adapted to the modified SLCM method. Blozis and Harring (2016) demonstrated that the first-order Taylor series identically replicates the target function when all of the individually-varying parameters enter the model linearly, meaning that in Eq. (10) we have identically reproduced the target function of Eq. (7). When the target function is nonlinear in its individually-varying parameters, the Taylor series expansion will be an approximation, with its remainder comprised of higher-order polynomial terms. Given that SLCM approaches have used the first-order approximation exclusively (Preacher & Hancock, 2015), however, we also utilize the first-order Taylor polynomial here. How well the first-order Taylor series expansion will approximate a specific target function will depend on the parameter selected for substitution during the reparameterization step and the resulting linearity or nonlinearity of the individually-varying parameters in the reparameterized function.

To illustrate, recall that for the present reparameterization we substituted out the intercept factor α_j to allow for estimation of the time to criterion growth factor τ_j. Had there been a substantive or theoretical rationale for modeling the growth intercept α_j, we could have sacrificed the slope factor β_j instead, yielding the following alternatively reparameterized target function:

$$y_{ij} = \alpha_j + \frac{\left(\mathrm{c} - \alpha_j\right) t_i}{\tau_j} + \varepsilon_{ij} \quad . \tag{11}$$

The first partial derivative of Eq. (11) with respect to the time to criterion factor contains itself, evaluated at the population point, which indicates that the individually-varying time to criterion factor enters the model nonlinearly in this alternative target function (Bates & Watts, 1988):[2]

[2] The target function is required to be differentiable with respect to its parameters around the point of expansion to proceed with linearization and SLCM modification (Browne, 1993). The derivatives given in Eq. (12) with respect to α_j and τ_j are undefined when $\mu_\tau = 0$. When the average time to criterion value in the population is theorized to be 0, that is, if on average all participants initially fall at the criterion value, the quality of the linearization will be questionable for this particular parameterization and inference based on associated parameters will be suspect.

$$\begin{aligned} \left.\frac{\partial y_{ij}}{\partial \alpha_j}\right|_{\boldsymbol{\mu}_{\alpha,\tau}} &= 1 - \frac{t_i}{\mu_\tau} \\ \left.\frac{\partial y_{ij}}{\partial \tau_j}\right|_{\boldsymbol{\mu}_{\alpha,\tau}} &= \frac{(\mu_\alpha - \mathrm{c})t_i}{\mu_\tau^2} \end{aligned} . \tag{12}$$

Unlike the model in Eq. (10), the linearization of the model in Eq. (11) therefore will not identically reproduce its target function, despite that both models are derived from the same original linear function. In applying a modified SLCM specification (see Step 3 below; Preacher & Hancock, 2015) to the first-order linearization of Eq. (11), the mean trend of the population-level model will be constrained to be equal to the target function evaluated at the mean of its parameters $f(\boldsymbol{\mu}, t_i)$, but first-order linearization will produce an approximation to the individual-level response curves (Blozis & Harring, 2016, p. 916). Calculating the exact value of this approximation for a linearization taken with respect to multiple independent parameters is often intractable if not outright impossible (Courant & Fritz, 1989, pp. 68-69); as such, empirical estimation of the error of approximation may be a goal of future research. Further implications of the choice of substitution parameter may include model convergence or estimation time.

We close out this illustration by mentioning that the interpretation of the linearization of the model in Eq. (11) may differ from the interpretation of its target model, depending on the estimation method for the nonlinear model. We briefly discuss the differences in model interpretation at the end of Step 3 and in more detail in the section following the nonlinear derivation, but we note here that models that are linear in their individually-varying parameters, like the target reparameterization in Eq. (7) and its linearization in Eq. (10), have equivalent interpretations.

3. Modified Structured Latent Curve Specification

The standard SLCM approach uses the basis curves (the first partial derivatives of the target function with respect to each coefficient, evaluated at specific values of time t_i; Blozis, 2004) to construct a loading matrix $\mathbf{\Lambda}$ for each latent factor $\boldsymbol{\eta}_j$; $\mathbf{\Lambda}$ and the target function are then used to solve for the estimated elements of the factor mean vector. However, we employ a procedure based on a modified SLCM approach, which yields an equivalent model, has the benefit of being more intuitive, and permits the reparameterization of models with solely nonlinear parameters. This modified SLCM specification, discussed in Preacher and Hancock (2015), sets the factor mean vector to $\mathbf{0}$, although the average of each factor may be estimated as a byproduct of the imposition of other model constraints. To capture the trend of the target function, the modified SLCM approach then fixes the item intercept vector, $\boldsymbol{\nu}$, to take on the values of the target function, $f\left(\boldsymbol{\mu}, t_i\right)$. The procedure in the current paper differs from the approach in Preacher and Hancock (2015) in that, rather than fixing the mean vector to $\mathbf{0}$, additional terms created by multiplying out the factor means in Eq. (10) are distributed into the $\boldsymbol{\nu}$ intercept vector. The factor loading constraints remain equivalent between the two procedures—that is, in both procedures factor loadings will be constrained to their first partial derivatives, as given in Eq. (9)—but the item intercept constraints will differ (compare the item intercept vector elements for the fixed-factor mean approach, $\mathrm{c} - \mu_\beta \mu_\tau + \mu_\beta t_i$, with the item intercept vector elements for the estimated-factor mean approach, $\mathrm{c} + \mu_\beta \mu_\tau$).

Though slightly more complicated algebraically, following this revised approach not only allows the factor means to be directly estimated in the model, but it also carries an important interpretational consequence.[3] In setting the factor mean vector to $\mathbf{0}$, factors then behave as mean

[3] For example, in estimating the effect of gender on future math achievement conditional on time to criterion, the "centered" formulation of time to criterion would result in an interpretation of the effect of gender for time to

deviates, or as centered variables, in latent variable regression. Removing this constraint, as in the present procedure, results in essentially an uncentered interpretation of the factor in a regression path. Either formulation may be desirable. The "uncentered" form of the modified SLCM specification is shown below, and code for specifying both the "centered" and "uncentered" factor specifications is presented in the Online Supplemental Materials.

Using the approach for specifying the mean trend as discussed above, we can formulate the reparameterized LGM as in Eq. (2), where the individual components $\mathbf{v}$, $\mathbf{\Lambda}$, and $\boldsymbol{\eta}_j$, specified for I time points from time $i = \{0, 1, \ldots, I-1\}$, where $T_{(I-1)}$ represents the value of time at the Ith and final measurement occasion, are as follows:

$$\mathbf{v} = \begin{bmatrix} \mathrm{c} + \mu_\beta \mu_\tau \\ \mathrm{c} + \mu_\beta \mu_\tau \\ \vdots \\ \mathrm{c} + \mu_\beta \mu_\tau \end{bmatrix}$$
$$\mathbf{\Lambda} = \begin{bmatrix} 0 - \mu_\tau & -\mu_\beta \\ 1 - \mu_\tau & -\mu_\beta \\ \vdots & \vdots \\ T_{(I-1)} - \mu_\tau & -\mu_\beta \end{bmatrix}, \tag{13}$$
$$\boldsymbol{\eta}_j = \begin{bmatrix} \beta_j \\ \tau_j \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\mu} = \begin{bmatrix} \mu_\beta \\ \mu_\tau \end{bmatrix}$, and $\boldsymbol{\Sigma} = \begin{bmatrix} \psi_\beta & \\ \psi_{\tau\beta} & \psi_\tau \end{bmatrix}$. The resulting individual model equations are thus stated as:

criterion at its mean; the "uncentered" formulation would result in an interpretation of the effect of gender for time to criterion equal to 0.

$$\begin{bmatrix} y_{0j} \\ y_{1j} \\ \vdots \\ y_{(I-1)j} \end{bmatrix} = \begin{bmatrix} \mathrm{c} + \mu_\beta \mu_\tau \\ \mathrm{c} + \mu_\beta \mu_\tau \\ \vdots \\ \mathrm{c} + \mu_\beta \mu_\tau \end{bmatrix} + \begin{bmatrix} 0 - \mu_\tau & -\mu_\beta \\ 1 - \mu_\tau & -\mu_\beta \\ \vdots & \vdots \\ T_{(I-1)} - \mu_\tau & -\mu_\beta \end{bmatrix} \begin{bmatrix} \beta_j \\ \tau_j \end{bmatrix} + \begin{bmatrix} \varepsilon_{0j} \\ \varepsilon_{1j} \\ \vdots \\ \varepsilon_{(I-1)j} \end{bmatrix}. \tag{14}$$

The full path diagram of the reparameterized linear T2C model in Eq. (14) is presented in Figure 2. From the "uncentered" modified SLCM specification in Eq. (14), it is evident that the repeated measures y_{ij} now depend on the value of the fixed criterion c, the means of the intercept and time factors μ_β and μ_τ, individual weights β_j and τ_j, the value of time at the measurement occasions t_i, and a time-specific error ε_{ij}. If we look back at Eq. (7), we see that the y_{ij} from the target model are also dependent upon c, β_j, τ_j, t_i, and ε_{ij}. Because both models are linear in their individually-varying parameters, the expected values of β_j and τ_j, $\mathrm{E}\begin{bmatrix} \tau_j \\ \beta_j \end{bmatrix} = \begin{bmatrix} \tau \\ \beta \end{bmatrix}$, from the target model specification in Eq. (7) will have the same interpretation as the means of the growth parameters, μ_β and μ_τ, from the SLCM specification in Eq. (14). In the linear case, this interpretation is equivalently stated as "the expected growth factor for a subject drawn at random" or "the value of the growth factor that describes the population mean trajectory."

However, for models with individually-varying parameters that enter nonlinearly, as with our example from Eq. (11), the expected value of the growth parameters in the target model specification in general will not be equivalent to the SLCM means of those parameters, e.g., $\mathrm{E}\left[\tau_j\right] \neq \mu_\tau$ (Blozis & Harring, 2016). Depending on the estimation method, the expected value of the growth parameters in Eq. (11) will generally take on a *subject-specific* (SS) interpretation only, meaning that $\mathrm{E}\left[\tau_j\right]$ is the expected value of τ_j for a subject drawn at random, or the "typical" time to criterion value τ across subjects. Often this "typical value" is interpreted as the value of the growth parameter τ_j for a subject j whose individual deviation u_j from τ is equal to 0, i.e., for $u_j = 0$, $\tau_j = \tau + u_j = \tau$. Still, these “typical value” parameters do not fully describe the SS population mean trajectory. For SS models, the function that describes the population mean trajectory is dependent not only on the “typical value” of the growth parameters but also on the parameters governing the density of the individual deviations, e.g., the density of the distribution of u_j (for further explanation, see Davidian & Giltinan, 2003, sec. 2.4). On the other hand, regardless of estimation method, the SLCM specification for reparameterized nonlinear individually-varying parameters loses the “typical value” interpretation but retains the *population average* (PA) interpretation such that μ_τ in the alternative model represents the average time to criterion in the population and describes the population trajectory (Preacher & Hancock, 2015). In summary, for fully nonlinear models, the parameters of the target model specification will not carry the same interpretation as the parameters of the SLCM specification (Blozis & Harring, 2016). We now demonstrate the implementation and interpretation of the linear T2C growth model using real data from an education context.

4. Estimate Model Parameters

In order to exemplify some of the research questions that the linear T2C growth model can address, data were utilized from Project STAR (Achilles et al., 2008; Word et al., 1990), a longitudinal analysis of student achievement in classrooms of varying sizes. Project STAR (Achilles et al., 2008; Word et al., 1990) was initially a four-year longitudinal study that examined over 7000 students representing roughly 300 classrooms in 79 schools from kindergarten through 3rd grade. Students were randomized to classrooms of varying size to determine the impact of class size on learning outcomes. The second phase of Project STAR, the Lasting Benefits Study, followed up with the same youth from fourth grade through high school and beyond to explore potential lasting benefits of class size assignment. The full sample includes over 11,000 students who participated in either phase of the project for at least one year. The present demonstration analyzed math achievement scores measured at four time points (5^{th} – 8^{th} grades). For the purposes of this demonstration, non-independence of observations was accounted for by using design-based corrections that adjust model standard errors by the first school available for a given student, though further aspects of study design (e.g., stratification, weights) could be accounted for within this framework by standard SEM software (see Stapleton, 2013). Descriptive statistics of math scores by grade (Table 1) show the mean math achievement and variability across time points, both in aggregate and by gender, including mean and standard deviation of the distal high school outcome, ACT Math score. Math score variability appears relatively homogeneous across grade level. On average, the mean math scores of female students were slightly higher than the mean math scores for male students at each grade level; however, the average female student's performance on the math section of the ACT was lower than the average male student's performance.

Individual math achievement trajectories from 5^{th} through 8^{th} grades for a random subsample of students (*n*=45) are presented in Figure 3. The mean linear trend between grade levels assessed by Project STAR and math scores is overlaid with a dashed line. For this illustration a target criterion score has been prespecified at c_{math} = 762, which roughly corresponds to a passing score for the state's 8^{th} grade mathematics competency test (see Pate-Bain, Boyd-Zaharias, Cain, Word, & Binkley, 1997; more detail on the identification of a criterion c is provided at the end of this study). The overall mean in years to reach the c_{math} threshold could be calculated as in Eq. (4); however, a careful examination of the spaghetti plot reveals considerable variability in time to reach this particular criterion, with the earliest individual achieving a score of 762 prior to 5^{th} grade, while others have yet to reach criterion even after 8th grade. To reiterate, research questions to be addressed with these data include whether or not time to criterion is related to individual growth over time, whether or not female students achieve criterion faster on average than male students, and whether students' average time to criterion has any apparent effect on ACT math performance above and beyond gender.

To address these questions, an unconditional linear longitudinal growth model was first specified at four time points (Grades 5 – 8 from the Lasting Benefits Study follow-up to Project STAR) and fit with Mplus, Version 7 (Muthén & Muthén, 2012) to the math assessment data using maximum likelihood estimation with robust standard errors (MLR). Global fit was assessed and the model was found to be tenable ($LL_{\text{unconditional}} = -93879.41$; Scaling Correction for $\text{MLR}_{\text{unconditional}} = 3.948$; $\text{Npar}_{\text{unconditional}} = 9$; $RMSEA_{\text{unconditional}} = 0.030$, 90% CI = [0.021, 0.039]). The model was specified according to Eq. (14) with uncorrelated residuals and heterogeneous residual variances over time (Blozis, 2007; Preacher & Hancock, 2015). Given that our model provided adequate fit to the data, we did not test further error covariance structures, though misspecification in the individual-level covariance structure leads to bias in the growth parameters and should be investigated (Harring & Blozis, 2014).

Next, the reparameterized T2C model with criterion of $c_{\text{math}} = 762$ was fit to the data, yielding identical data-model fit as expected. Table 2 contains results for both the original unconditional linear LGM and linear T2C models. As a modification of the constraints specified in Eq. (13), factor loadings were fixed such that the first time point was set at 5 rather than 0, meaning that the reference time point t_0 was set to Grade 0 (i.e., Kindergarten) rather than Grade 5 (that is, five years prior to Grade 5).[4] This modification affects the interpretation of the mean of

[4] Our data do not provide individual-specific times of measurement for assessment at each grade level, but it is possible to specify the model with individually-varying measurement occasions. We direct the interested reader to the example Mplus code presented for the Preacher and Hancock (2015) inverse quadratic model, the first model discussed, retrievable from http://quantpsy.org/pubs/preacher_hancock_2015.supp.pdf.

the time factor, such that μ_τ, the expected time to reach criterion, now yields the expected grade level at which criterion will be reached ($\hat{\mu_\tau} = 6.728$, $p < 0.0001$). That is, on average, students are expected to reach a criterion score of 762 toward the end of their Grade 6 year or just before the beginning of Grade 7. Furthermore, there exists statistically significant variance in the population in terms of time to reach the criterion score (5.422, $p < 0.0001$), and the time factor is statistically significantly negative in its correlation with the slope factor (–0.442, $p < 0.0001$) indicating that, as would be expected, students who achieve criterion sooner (lower τ_j) grew faster (higher β_j). We can also interpret the loadings for the time to criterion and slope factors. Following Eq. (14), the loadings for τ_j have been constrained equal to the additive inverse of the slope factor mean μ_β (–16.532, $p < 0.0001$). Similarly, the loadings for β_j have been constrained equal to the value of time at measurement occasion i minus the mean of the time to criterion factor μ_τ, that is, $t_i - \mu_\tau$, where $i = \{5, 6, 7, 8\}$. These loadings are each statistically significant in the model (e.g., $t_5 - \mu_\tau = -1.728$, $p < 0.0001$).

Having specified the T2C model, assessed data-model fit, and examined key parameters in the unconditional model, predictors of growth and distal outcomes were then added. Table 2 also contains results for both the conditional linear LGM and linear T2C models, with growth depending upon a gender predictor (0 = Female, 1 = Male), and these models with the gender predictor as well as a distal outcome of ACT Math Score. Global fit for the conditional models was again assessed and found adequate ($LL_{\text{conditional}} = -93859.39$; Scaling Correction for $\text{MLR}_{\text{conditional}} = 3.411$; $\text{Npar}_{\text{conditional}} = 11$; $RMSEA_{\text{conditional}} = 0.028$, 90% CI = [0.020, 0.036]). Briefly, we found no evidence for an effect of gender on the slope factor β_j; however, there was a statistically significant effect of gender on the time to criterion factor τ_j (0.359, $p < 0.0001$), indicating that male students reach criterion on average over a third of a year later than female students. Because we specified the effect of gender on both growth factors, it may be advantageous to test the cumulative effect of this variable using a scaled likelihood ratio test (Satorra, 2000). We find that the cumulative effect of gender is significant in our model (χ^2_{scaled} = 40.342, df = 2). Finally, the conditional T2C model with predictor and distal outcome ($LL_{\text{conditional, distal}}$ = –101925.17; Scaling Correction for $\text{MLR}_{\text{conditional, distal}}$ = 3.129; $\text{Npar}_{\text{conditional, distal}}$ = 16; $RMSEA_{\text{conditional, distal}}$ = 0.026, 90% CI = [0.019, 0.033]) provides evidence for a direct effect on ACT Math scores of gender (0.623, $p < 0.0001$), reflecting males' generally higher math performance on the ACT; evidence for a direct effect on ACT Math of rate of change β_j controlling for gender (0.238, $p = 0.003$), indicating unsurprisingly that a quicker rate of growth relates to better subsequent math performance; and evidence for a direct effect on ACT Math of time to criterion τ_j controlling for gender (–1.510, $p < 0.0001$), indicating that a longer time to reach criterion is associated with poorer subsequent performance.

For this last model, we were unable to compute a scaled likelihood ratio test for the cumulative effect of the addition of ACT to the model due to a negative scaled chi-square value (Satorra, 2000), which, due to the complexity of the nonlinear constraints, is not resolved by implementing the strictly positive scaling method (Satorra & Bentler, 2010). This complexity leads us to mention that reparameterized models may require more precise starting values to achieve convergence than the default values provided by conventional SEM software. Starting values for unaltered parameters (e.g., residual variances, means and variances for the slope factor) can be based on the estimates obtained from the linear LGM specification; starting values for new parameters (e.g., means and variances for the time to criterion factor) can be calculated from Eq. (4); and starting values for constrained parameters (i.e., growth factor loadings and y-intercepts) can be calculated from the constraints in Eqs. (13) and (14). The relevant Mplus model code for specifying the linear T2C growth model and sample R code for calculating these starting values are provided in the Online Supplemental Materials.

Now that we have demonstrated the derivation and implementation of the linear T2C growth model and introduced the nuances of parameter interpretation for linear and, briefly, nonlinear SLCMs, we extend the T2C growth framework and present a T2C growth model based on a fully nonlinear target function.

T2C Modeling Extensions: Nonlinear Growth

The fit of the T2C model to the math achievement data across four grade levels, as presented in the previous section, was demonstrated to be adequate. It is well known, however, that developmental processes often follow nonlinear trajectories. We have several options when fitting nonlinear random effects models (Browne 1993; Cudeck & Harring, 2007; Grimm, Ram, & Hamagami, 2011). With nonlinear models, the score for person *j* at time *i* can be modeled as some functional form based on a set of parameters $\boldsymbol{\beta}_j$, and a set of variables $\mathbf{x}_{ij}$, plus an individual- and time-specific error term ε_{ij}. As the case for reparameterization proceeds from the argument that growth parameters should be useful and interpretable, the following section presents an illustrative model based on the common *Michaelis-Menten* (MM) function (for more on selecting an appropriate nonlinear model for repeated measures data, see Cudeck & Harring, 2010). The monotonic MM function utilizes three key growth parameters, all of which have clear and meaningful interpretation within a longitudinal context:

$$y_{ij} = f\left(\alpha_j, \delta_j, \varphi_j, t_i\right) + \varepsilon_{ij} = \alpha_j + \frac{\left(\delta_j - \alpha_j\right)t_i}{\varphi_j + t_i} + \varepsilon_{ij} \ , \tag{15}$$

where α_j is the intercept parameter, $f\left(t_i = 0\right) = \alpha_j$; δ_j is the upper asymptote parameter, $f\left(t_i = \infty\right) = \delta_j$; and φ_j is the time to half growth parameter, $f\left(t_i = \varphi_j\right) = \alpha_j + \frac{1}{2}\left(\delta_j - \alpha_j\right)$. This function, developed to explain kinetic processes in pharmacology, has been found to be useful for describing developmental processes as well (Cudeck & Harring, 2007; for a conditionally linear variant, see Harring, Kohli, Silverman, & Speece, 2012). Because many developmental processes proceed quickly at first and level off to some threshold, estimating a growth model based on the MM function may be very useful in identifying notable aspects of a growth process.

The MM function is notably nonlinear in one of its individually-varying parameters, φ_j. Thus, the base MM latent growth model (MM-LGM) must be linearized and respecified using the modified SLCM approach to be fit in the SEM framework following Steps 2 and 3 from the method presented above. This modification is not a reparameterization as such, though unlike the previous derivation, specifying the MM-LGM model in the SLCM framework does first require linearization. Note that preparing the MM-LGM model for estimation in SEM software is not necessary in order to proceed with the time to criterion reparameterization; however, the process is presented to provide a comparable PA interpretation between the base and reparameterized estimates of the MM model and is presented in its entirety for completeness in the Online Supplemental Materials. A path diagram of the resulting SLCM specification of the MM-LGM is presented in Figure 4.

Finally, before proceeding to Step 1, we would like to clarify one technical aspect of selecting the original nonlinear function. We noted previously that the MM function is monotonic. Applying the T2C reparameterization to a non-monotonic function without domain or range restrictions may result in multiple solutions of τ_j. One potential solution would specify time to criterion in terms of a series of piecewise growth functions, and this extension is a potential area for further investigation. The following section proceeds with the steps for reparameterization to specify the nonlinear T2C growth model based on the original MM function (MM-T2C).

1. Reparameterize the Target Function

Starting with the MM function from Eq. (15) and solving the equation for a criterion c yields:

$$\mathrm{c} = \alpha_j + \frac{(\delta_j - \alpha_j)t_{cj}}{\varphi_j + t_{cj}} \ , \tag{16}$$

such that t_{cj} is the value of t_i when y_{ij} is equal to c, as before. As in Eq. (4), we can again calculate the mean and variance of the time to criterion term, $t_{cj} = \frac{\varphi_j(\mathrm{c} - \alpha_j)}{\delta_j - \mathrm{c}}$:

$$\begin{aligned} \mathrm{mean}(t_{cj}) &= \mathrm{E}\left(\frac{\varphi_j(\mathrm{c}-\alpha_j)}{\delta_j - \mathrm{c}}\right) \\ \mathrm{var}(t_{cj}) &= \mathrm{E}\left[\left(\frac{\varphi_j(\mathrm{c}-\alpha_j)}{\delta_j - \mathrm{c}}\right)^2\right] - \mathrm{E}\left[\left(\frac{\varphi_j(\mathrm{c}-\alpha_j)}{\delta_j - \mathrm{c}}\right)\right]^2 \end{aligned} . \tag{17}$$

Once again, the transformation which generates the t_{cj} term will likely not result in a normally distributed variable, and calculation of higher-order moments are left to the reader. In general, simply knowing the moments of a variable's distribution does not allow us to incorporate time to criterion in the broader theory-testing framework available within structural equation modeling. To that end, we must first decide which parameter within the original model, Eq. (15), should be substituted to create a reparameterized time to criterion model.

The reparameterization of the linear T2C was conducted by sacrificing the information regarding the initial status intercept parameter in exchange for the time to criterion factor. In the case of the nonlinear MM-LGM, we have three candidate parameters to choose from in Eq. (16): intercept, upper asymptote, and time to half-growth. As previously discussed, the choice of parameter to substitute for time to criterion should be made by weighing the substantive and theoretical value of each existing parameter. For example, time to half-growth (i.e., the time to reach the value halfway between the starting point and the value of the upper asymptote) may not be a useful reference point in developmental processes, especially when a well-defined criterion of interest may exist in a practical context. At the same time, the information provided by the intercept and upper asymptote may be deemed valuable from a research perspective. The reparameterization thus proceeds with substitution of the time to half-growth factor in favor of the time to criterion factor. Solving Eq. (16) algebraically for φ_j yields

$$\varphi_j = \frac{\left(\delta_j - \mathrm{c}\right)t_{\mathrm{c}j}}{\mathrm{c} - \alpha_j} \quad , \tag{18}$$

and substitution results in the following specification:

$$y_{ij} = \alpha_j + \frac{\left(\delta_j - \alpha_j\right)t_i}{\left[\frac{\left(\delta_j - \mathrm{c}\right)t_{\mathrm{c}j}}{\mathrm{c} - \alpha_j}\right] + t_i} \quad . \tag{19}$$

Again we restate Eq. (19), replacing t_{cj} with Greek letter τ_j to indicate that it is now treated as a model-estimated parameter:

$$y_{ij} = \alpha_j + \frac{\left(\delta_j - \alpha_j\right)t_i}{\left[\frac{\left(\delta_j - \mathrm{c}\right)\tau_j}{\mathrm{c} - \alpha_j}\right] + t_i} \quad . \tag{20}$$

With that, the target function in Eq. (20) has been reparameterized to replace the time to half-growth parameter, a parameter which entered the original MM-LGM model nonlinearly, with the time to criterion parameter. Though it may not be evident from the reparameterized function, we will see below that not only does the new time to criterion parameter enter the MM-

T2C model nonlinearly but the other growth factors have become nonlinear as well, though other choices for parameter substitution would have different effects.

2. Linearization: First-Order Taylor series Expansion

While the first-order Taylor linearization was conducted primarily for didactic purposes in Step 2 of the linear T2C derivation, this step is necessary for specifying a model with individually-varying nonlinear parameters within the SLCM framework. The first-order linearization fits the reparameterized model to the population mean response of the data, while weights given by the set of growth parameters – α_j, δ_j, and τ_j – allow for the individual differences in growth (Blozis & Harring, 2016).

The first-order Taylor series expansion is given below. The mean-centered growth factors, α_j (intercept), δ_j (upper asymptote), and τ_j (time to criterion), are multiplied by their first partial derivatives and then added to the target function evaluated at the population point:

$$\tilde{y}_{ij} = f\left(\boldsymbol{\mu}, t_i\right) + \left(\alpha_j - \mu_\alpha\right)\left.\frac{\partial y_{ij}}{\partial \alpha_j}\right|_{\boldsymbol{\mu}_{\alpha,\delta,\tau}} + \left(\delta_j - \mu_\delta\right)\left.\frac{\partial y_{ij}}{\partial \delta_j}\right|_{\boldsymbol{\mu}_{\alpha,\delta,\tau}} + \left(\tau_j - \mu_\tau\right)\left.\frac{\partial y_{ij}}{\partial \tau_j}\right|_{\boldsymbol{\mu}_{\alpha,\delta,\tau}}, \tag{21}$$

where $f\left(\boldsymbol{\mu}, t_i\right) = \mathrm{E}\left[y_{ij}\right] = \mu_\alpha + \dfrac{\left(\mu_\delta - \mu_\alpha\right)t_i}{\left[\dfrac{\left(\mu_\delta - \mathrm{c}\right)}{\mathrm{c} - \mu_\alpha}\right] + t_i}$. The partial derivatives evaluated at the means are given below:

$$\begin{aligned}
\left.\frac{\partial y_{ij}}{\partial \alpha_j}\right|_{\boldsymbol{\mu}_{\alpha,\delta,\tau}} &= \frac{\mu_\tau\left(\mu_\tau - t_i\right)\left(\mathrm{c} - \mu_\delta\right)^2}{\left[-\mu_\delta\mu_\tau + \mathrm{c}\left(\mu_\tau - t_i\right) + \mu_\alpha t_i\right]^2} \\
\left.\frac{\partial y_{ij}}{\partial \delta_j}\right|_{\boldsymbol{\mu}_{\alpha,\delta,\tau}} &= \frac{-t_i\left(\mu_\tau - t_i\right)\left(\mathrm{c} - \mu_\alpha\right)^2}{\left[-\mu_\delta\mu_\tau + \mathrm{c}\left(\mu_\tau - t_i\right) + \mu_\alpha t_i\right]^2} \\
\left.\frac{\partial y_{ij}}{\partial \tau_j}\right|_{\boldsymbol{\mu}_{\alpha,\delta,\tau}} &= \frac{-t_i\left(\mu_\alpha - \mu_\delta\right)\left(\mathrm{c} - \mu_\alpha\right)\left(\mathrm{c} - \mu_\delta\right)}{\left[-\mu_\delta\mu_\tau + \mathrm{c}\left(\mu_\tau - t_i\right) + \mu_\alpha t_i\right]^2}
\end{aligned} \quad . \tag{22}$$

Note that the denominator for each partial derivative contains all three growth factors, evaluated at their population points, meaning that each growth factor is in fact nonlinear in the reparameterized target function, Eq. (20) (compare this model with the first partial derivatives for the MM-LGM function, shown in the Online Supplemental Materials). Therefore, as discussed in Step 2 of the linear T2C derivation, the linearization of the model in Eq. (20) will fit the mean trend of the population-level model, but first-order linearization will produce an approximation to the individual-level response curves (Blozis & Harring, 2016, p. 916). The linearized model will thus take on a PA interpretation, discussed further in Step 3.

It is also worth pointing out that the nonlinear derivation is much more cumbersome than the linear derivation. As such, many will find utilizing symbolic computation software such as Maple (Maple, 2018) or Mathematica (Wolfram, 2018) to be of assistance in deriving other nonlinear reparameterizations. We now proceed to demonstrate the specification of the linearized model as a modified SLCM.

3. SLCM Specification

As before, we utilize the results of the first-order Taylor expansion to generate the modified SLCM. Recall that in the linear T2C derivation, we discussed the implementation of the "uncentered" approach to the modified SLCM used by Preacher and Hancock (2015). Rather than fix the growth factor means to $\mathbf{0}$ as they had done, we estimate the factor means by relegating terms created by multiplying out the mean-deviated growth factors in Eq. (21) to the item intercept vector. Once again, this has the effect of directly estimating the factor means in the model as well as altering the interpretation of the growth factors in latent variable regression from a "centered" to an "uncentered" variable interpretation. Either or both interpretations may prove substantively useful, but we present the results of the "uncentered" approach below with code for both approaches in the Online Supplemental Materials.

We again present the reparameterized MM-T2C using the matrix model formulation following the model in Eq. (2). After combining algebraic terms within the intercept vector **ν** to achieve the "uncentered" SLCM modification, the individual model components **ν**, **Λ**, and $\boldsymbol{\eta}_j$—again specified for I time points from time $i = \{0, 1, \ldots, I - 1\}$, where $T_{(I-1)}$ represents the Ith and final measurement occasion—are presented below (recall that the elements of the **Λ** loading matrix will not be altered by using the centered or uncentered approaches):

$$\mathbf{v} = \begin{bmatrix} \dfrac{(\mu_\alpha - \mu_\delta)(\mathrm{c}^2 - 2\mathrm{c}\mu_\delta + \mu_\alpha\mu_\delta)\mu_\tau \cdot 0}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - 0) + \mu_\alpha \cdot 0\right]^2} \\ \dfrac{(\mu_\alpha - \mu_\delta)(\mathrm{c}^2 - 2\mathrm{c}\mu_\delta + \mu_\alpha\mu_\delta)\mu_\tau \cdot 1}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - 1) + \mu_\alpha \cdot 1\right]^2} \\ \vdots \\ \dfrac{(\mu_\alpha - \mu_\delta)(\mathrm{c}^2 - 2\mathrm{c}\mu_\delta + \mu_\alpha\mu_\delta)\mu_\tau T_{(I-1)}}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - T_{(I-1)}) + \mu_\alpha T_{(I-1)}\right]^2} \end{bmatrix}$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \dfrac{\mu_\tau(\mu_\tau - 0)(\mathrm{c} - \mu_\delta)^2}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - 0) + \mu_\alpha \cdot 0\right]^2} & \dfrac{-0 \cdot (\mu_\tau - 0)(\mathrm{c} - \mu_\alpha)^2}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - 0) + \mu_\alpha \cdot 0\right]^2} & \dfrac{-0 \cdot (\mu_\alpha - \mu_\delta)(\mathrm{c} - \mu_\alpha)(\mathrm{c} - \mu_\delta)}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - 0) + \mu_\alpha \cdot 0\right]^2} \\ \dfrac{\mu_\tau(\mu_\tau - 1)(\mathrm{c} - \mu_\delta)^2}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - 1) + \mu_\alpha \cdot 1\right]^2} & \dfrac{-1 \cdot (\mu_\tau - 1)(\mathrm{c} - \mu_\alpha)^2}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - 1) + \mu_\alpha \cdot 1\right]^2} & \dfrac{-1 \cdot (\mu_\alpha - \mu_\delta)(\mathrm{c} - \mu_\alpha)(\mathrm{c} - \mu_\delta)}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - 1) + \mu_\alpha \cdot 1\right]^2} \\ \vdots & \vdots & \vdots \\ \dfrac{\mu_\tau(\mu_\tau - T_{(I-1)})(\mathrm{c} - \mu_\delta)^2}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - T_{(I-1)}) + \mu_\alpha T_{(I-1)}\right]^2} & \dfrac{-T_{(I-1)}(\mu_\tau - T_{(I-1)})(\mathrm{c} - \mu_\alpha)^2}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - T_{(I-1)}) + \mu_\alpha T_{(I-1)}\right]^2} & \dfrac{-T_{(I-1)}(\mu_\alpha - \mu_\delta)(\mathrm{c} - \mu_\alpha)(\mathrm{c} - \mu_\delta)}{\left[-\mu_\delta\mu_\tau + \mathrm{c}(\mu_\tau - T_{(I-1)}) + \mu_\alpha T_{(I-1)}\right]^2} \end{bmatrix}$$

$$\boldsymbol{\eta}_j = \begin{bmatrix} \alpha_j \\ \delta_j \\ \tau_j \end{bmatrix} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

$$, \tag{23}$$

where $\boldsymbol{\mu} = \begin{bmatrix} \mu_\alpha \\ \mu_\delta \\ \mu_\tau \end{bmatrix}$, and $\boldsymbol{\Sigma} = \begin{bmatrix} \psi_\alpha & & \\ \psi_{\delta\alpha} & \psi_\delta & \\ \psi_{\tau\alpha} & \psi_{\tau\delta} & \psi_\tau \end{bmatrix}$. A path diagram of the reparameterized MM-T2C model is presented in Figure 5.

Before moving on to the demonstration of the MM-T2C growth model with real data, take a moment to compare the SLCM specification of the MM-T2C in Eq. (23) with the reparameterized target function given in Eq. (20). The outcomes $\mathbf{y}_j$ in the SLCM specification now depend on c, growth factor means μ_α, μ_β, and μ_τ, a set of individual weights α_j, δ_j, and τ_j, measurement occasion t_i, and an occasion-specific error term ε_{ij}. As discussed in Step 3 of the linear T2C derivation, because the models are nonlinear in the individually-varying parameters, the expected value of the growth parameters for Eq. (20) in general is not equal to the mean of the growth parameters in the SLCM specification given in Eq. (23). Depending on the estimation method used for the target function, the expectation of the growth parameters in Eq. (20), $\mathrm{E}\begin{bmatrix} \alpha_j \\ \delta_j \\ \tau_j \end{bmatrix} = \begin{bmatrix} \alpha \\ \delta \\ \tau \end{bmatrix}$, represents the “typical values” of the growth factors (Davidian & Giltinan, 2003), whereas the mean of the growth parameters in the SLCM specification, μ_α, μ_δ, and μ_τ, represents the values that describe the change process in the population longitudinal trend (see Blozis & Harring, 2016; Harring & Blozis, 2016). Given these nuances, we stress that the interpretation of the SLCM parameters should be approached with careful attention, demonstrated in the following data example.

4. Estimate Model Parameters

Data were utilized from the Kanfer-Ackerman air-traffic controller task (ATC; Kanfer & Ackerman, 1989), a multi-trial task developed to understand the influence of motivation on cognitive resources in the context of cognitively challenging learning and skill acquisition. Participants were required to move through several actions, including landing planes on the appropriate runway while managing a holding pattern and ensuring that the planes in the holding pattern did not run out of fuel. In the original study, three separate experiments were conducted with over 1000 trainees in the U.S. Air Force. The current example utilizes data from Experiment 2, in which participants completed one 10-minute test trial followed by nine 10-minute task trials and in which they received a performance goal prior to Trial 5. Descriptive statistics for the number of planes successfully landed across the first five trials are presented in Table 3. Individual skill learning trajectories for a random subsample (n=10) of the number of planes landed across the first five trials are shown in Figure 6, with the mean trend across trials represented with a dot-dashed line. The descriptive statistics presented in Table 3 indicate that the variability decreases over time and that the median trend closely aligns with the mean trend; however, the spaghetti plot in Figure 6 depicts a considerable amount of variability in individual trajectories. For example, some participants exhibit growth that increases quickly and levels off; some participants never quite get the hang of the task; and some participants perform well at first but then begin performing poorly afterward. One advantage of using the SLCM approach to fit the model is that the subject-specific trajectories are not required to follow the population trajectory as long as the sum of subject trajectories is equal to the overall mean (Blozis & Harring, 2016; McNeish & Matta, 2017). It is partially this flexibility in modeling change that makes Taylor expansion and SLCM approaches to growth modeling so useful. A specified criterion of landing 30 planes (c_{skill} = 30) is indicated in the figure, and the population average time to reach that criterion is also indicated at slightly more than two trials. Based on both the decreasing increases in means and medians shown in Table 3 and the overall shape of the trajectories shown in Figure 6, a nonlinear growth model likely will best fit the data.

The Kanfer-Ackerman ATC data were analyzed using the reparameterized nonlinear T2C (MM-T2C). Models were fit using MLR in Mplus 7 (Muthén & Muthén, 2012), as in the previous example. Parameter estimates for both the MM-LGM and the MM-T2C are presented in Table 4. The unconditional MM-LGM model had adequate fit to the data ($LL_{\text{unconditional}} = -2105.02$, $\text{Npar}_{\text{unconditional}} = 14$, $RMSEA_{\text{unconditional}} = 0.000$, 90% CI: [0.000, 0.074]). Because these results were suitable, an unconditional MM-T2C was specified with a pre-defined criterion of $c_{\text{skill}} = 30$, yielding data-model fit indices that were equivalent, as expected. Once the model was correctly specified and its fit deemed acceptable, model parameters were assessed. In this model, the mean of the time factor yields the expected number of trials in the overall population to achieve the criterion score of 30 planes landed (2.27, $p < 0.0001$). Evidence was also substantial that the number of trials to reach criterion varied in the population (9.14, $p < 0.0001$). Last, the time factor was negatively and statistically significantly correlated with the intercept factor (–0.80, $p < 0.0001$), meaning that, in the population, a longer time to reach criterion (higher τ_j) corresponded with a smaller number of planes landed in the first trial (lower α_j). This relation among the intercept and time factors could provide useful information about how to quickly identify participants who may not reach criterion by a specified time point.

Of secondary interest is that there was no evidence that the upper asymptote factor δ_j (in both MM-LGM and MM-T2C) nor the time-to-half-growth factor φ_j (in the MM-LGM model) varied significantly in the population. One potential reason for this finding is lack of statistical power with a small sample size (n=140), but given that hypothesis testing procedures for variance components tend to be too conservative (Fitzmaurice, Laird, & Ware, 2012) and that, as mentioned in the linear derivation section above, the complexity of the nonlinear constraints may prevent the calculation of the strictly positive scaled likelihood ratio test (Satorra & Bentler, 2010), it may be advisable to utilize bootstrapped confidence intervals for hypothesis testing (Yung & Bentler, 1996). Because no nonlinear constraints were imposed upon any variance components, as recommended by Savalei and Kolenikov (2008), the regularity condition is not violated, and confidence intervals may include positive, zero, and negative values (for more on testing variance components with inequality constraints, see Kolenikov & Bollen, 2012). Bias-corrected bootstrap confidence intervals were generated for these parameters, which provided evidence that these factors were indeed variable in the population.

Last, as was noted previously in the linear T2C section, identifying reasonable starting values will likely be crucial to the speedy and accurate estimation of the reparameterized nonlinear model. To generate these initial values, once again, unaltered parameters (e.g., residual variances, means and variances for the slope factor) can be based on the estimates obtained from the linear LGM specification, while the starting values for new parameters (e.g., means and variances for the time to criterion factor) can be calculated from Eq. (17); and starting values for constrained parameters (i.e., growth factor loadings and *y*-intercepts) can be calculated from the constraints in Eq. (23). Relevant Mplus code for specifying the MM-LGM and the MM-T2C as well as sample R code for calculating start values are provided in the Online Supplementary Materials.

Interpretation and Estimation of T2C Growth Models

An important concept discussed throughout this paper relates to the differences in interpretation between models with nonlinear individually-varying parameters and those same models following linearization. Whether a reparameterized target function and the resulting modified SLCM will be nonlinear in its individually-varying parameters will depend on the original function itself as well as the parameter selected for substitution for the time to criterion factor. We demonstrated in Eq. (11) that even starting with a linear growth model can result in a reparameterized model with one or more nonlinear growth parameters despite that other choices for parameter substitution may not (e.g. the linear T2C). Taylor series linearization and subsequent SLCM implementation fit the population-level model to the population means of the nonlinear reparameterized target function, while the individual-level curves are allowed to differ in trajectory from the curve of population means and will only approximate the target function (Blozis & Harring, 2016; Harring & Blozis, 2016; McNeish & Matta, 2017). Models linearized in this way, such as our nonlinear MM-T2C model, thus take on a PA interpretation.

In PA models, the marginal expectation of the outcome is of interest to the researcher, making these models suitable for use with experimental data when trying to answer questions regarding which treatment group or subpopulation responded best (Harring & Blozis, 2016). On the other hand, prior to first-order linearization our reparameterized target functions carry an SS interpretation, meaning that models represent the individual change functions of each participant. In SS models, the means of the individually-varying parameters represent the “typical values” for those parameters and in general are not equivalent to the values that describe the mean population response (Cudeck & Harring, 2007; Davidian & Giltinan, 2003; Harring & Blozis, 2016). SS models are useful when the effects of treatment on the change process of an individual participant are of interest to the researcher. However, if the reparameterized model is linear in its parameters, both the PA and SS interpretations are equivalent and valid (Harring & Blozis, 2016).

The last important point to make within the context of the current study relates to model estimation. The estimation procedure for the nonlinear mixed-effects model is complicated by the fact that the individually-varying parameters, or random effects (Serroyen, Molenberghs, Verbeke, & Davidian, 2009), enter the model in a nonlinear way, which prevents an analytic solution to the model's estimation (Blozis & Harring, 2016). Approximations are therefore required for the estimation of nonlinear mixed-effects models, one of which is the first-order Taylor series expansion (note: when the first-order Taylor series is used to estimate a nonlinear mixed-effects model, the resulting parameter estimates takes on a PA rather than SS interpretation; see Blozis & Harring, 2016). When estimating SLCMs, however, the random weights (e.g., the individual growth factors $\boldsymbol{\eta}_i$) enter the linearized model linearly in the SLCM. As such, the estimation of the SLCM then can be carried out using maximum likelihood methods (Blozis & Harring, 2016); however, regardless of estimation method used, a PA interpretation will always be valid for the SLCM (Blozis & Harring, 2016). In this paper, we used maximum likelihood with robust standard errors (Yuan & Bentler, 2000) to estimate each of our real-data demonstration models. This technique was developed to be robust to moderate departures from normality with respect to both observed and unobserved variables in the presence of missing data and has been shown to be superior to other estimation methods in recovering relations among latent parameters (Li, 2016; Yuan & Bentler, 2000).

The interpretation of nonlinear models with individually-varying parameters requires careful consideration. As described above, the interpretation of nonlinear mixed-effects models will depend on the estimation method used, whereas the interpretation of linearized SLCMs will be consistent across methods of estimation. Throughout this study, we provided a basic discussion of the differences in interpretation between nonlinear mixed-effects models (e.g., our reparameterized target functions) and linearized SLCMs, both in the derivation of those models and in the context of their application to real data. For a more technical review of these issues, we refer the reader Blozis and Harring (2016), Cudeck and Harring (2007), Harring and Blozis (2016), and Preacher and Hancock (2015).

Identifying an Appropriate Criterion

As a final note in this section, the identification of an appropriate criterion c is of utmost importance in building a theoretically defensible T2C growth model and is crucial to obtaining useful and interpretable estimates of the mean (μ_τ) and variance (ψ_τ) of τ_j. In time to event models, event states are often defined via cutpoints, which may result in discretization of a continuous variable and result in a loss of information; furthermore, identifying meaningful definitions for both the discretized event status and the time of occurrence may pose a challenge. In T2C growth models, using cutpoints to define criteria poses less of a threat to information loss because the individual growth curves and continuous outcomes are preserved. The issue of identifying a useful definition for the event status remains, however. Because the values of μ_τ and ψ_τ will depend on the chosen value of c, we strongly discourage the use of sample statistics for determining c, as this implementation has a very recursive interpretation. For example, the expected time to reach the sample average value $\mathrm{c} = \bar{y}_i$ of the measurement taken at time t_i will be approximately equal to t_i, which may not provide much useful information. That said, we argue that appropriate, pre-defined criteria exist in many real-world settings, and selecting an effective criterion score will be key for conducting a strong analysis.

One example of published criteria comes from large scale educational assessments. Large education data sources, such as the National Assessment of Educational Progress (NAEP; US DOE) which provides comparisons to data used in longitudinal education studies, publish cut scores used in demarcating between those students who have achieved some level of mastery in a given skill. These cut scores are determined using rigorous item response theory and large scale assessment methods and are generated from information gathered from large numbers of test takers across the country. Criteria developed in this manner provide external metrics for achievement that may be meaningfully applied to T2C growth models.

However, not all useful criteria are generated from analyses calibrated over thousands of individuals. Substantive criteria set by external agencies may be of interest in research as well. For example, for those interested in factors impacting effective new media entrepreneurship, several social media metrics may be used to describe the success of a given product, such as the number of unique views as well as the number of repeat views from other well-known social media users (Gustafsson & Khan, 2017). One very clear criterion exists within a subset of the new media sphere: YouTube for Creators, an ad-sharing and resource-granting benefits program eligible to creators who upload original videos on YouTube.com and who satisfy a set of requirements, such as the total number of “watch time” hours clocked in a given year (Google, 2018). The factors related to the time it takes for YouTube entrepreneurs to reach this criterion level of watch time hours could easily be modeled with a T2C growth model, potentially providing useful information for benefactor agencies, in this case YouTube, as well as for the individual entrepreneurs themselves.

Last, should several criteria be available, as is often the case in cross-cultural mental health research that employs cut scores to screen psychiatric conditions (Wang & Gorenstein, 2013), expert review may be necessary to determine an appropriate criterion c for use with the T2C growth model. Further, though beyond the scope of the present study, it is not strictly necessary that all populations share the same criterion. Multiple group analysis, for example, would allow for the specification of a unique criterion for each analyzed subpopulation. The impact of this variable criterion specification on the testable assumptions of the multiple group model is a potential area for future inquiry.

Summary and Conclusions

Drawing from prior LGM reparameterization work (Preacher & Hancock, 2012, 2015), the current study derived latent growth models specifically for evaluating variability in subjects' time to reach a criterion level of interest on an outcome variable. A linear model, as well as an example of a nonlinear parameterization, were presented within the SEM framework, which further accommodates such issues as missing data, time-varying and time-invariant covariates, and complex error covariance structures. This study also illustrated the implementation of these models in SEM software using real data.

One issue discussed throughout the current study concerns two formulations of the nonlinear model: the *population average* (PA) model versus the *subject specific* (SS) model. Preacher and Hancock (2015) mentioned this issue in the context of reparameterization in latent growth, while Cudeck and Harring (2007) and Harring and Blozis (2016) discussed this issue in detail from a nonlinear mixed-effects modeling framework, and Blozis and Harring (2016) discussed the issue at length from both a nonlinear mixed-effects and an SLCM perspective. To summarize, PA models follow the curve of the overall mean trend across the population; SS models, on the other hand, model individual trajectories, but may not necessarily follow the curve of the overall means. In a linear (or conditionally linear) model, the typical values based on the PA and the SS models both follow the same curve; however, when parameters enter the model nonlinearly, the SS model no longer follows the curve of means. This discrepancy can, in turn, require special attention for interpretation when using a nonlinear T2C variant, as with other nonlinear growth models. Following linearization, the T2C models presented in this paper now follow the PA trajectory. The reader is referred to Blozis and Harring (2016), Cudeck and Harring (2007), Harring and Blozis (2016), and Preacher and Hancock (2015) for further details.

Finally, this study has considered a key component of the T2C model: identification of the criterion. As discussed in the context of time-to-event status definitions, in general, events in survival analysis are discrete, non-overlapping states with a single and simultaneous start point. However, obtaining precise, meaningful definitions for both the event status and the time of occurrence may pose a challenge. Event states are often defined via cutpoints, a practice evident from the literature, thus constituting discretized continuous outcomes. Although potentially useful for addressing a given research question, discretized outcomes come at the cost of a loss of potentially valuable information. T2C models may impose a practical cutpoint to indicate the criterion, but the specification of this cutpoint as criterion does not result in a loss of information or destruction of data because the continuous outcomes and growth trajectories are preserved. As such, the incorporation of the criterion point serves to highlight a key growth milestone without altering the data. Selecting the "correct" criterion is therefore crucial to the extent that the criterion is well-defined and relevant to the stated research question. Often, a practical criterion is accepted as an explicit or implicit cutpoint delineating progress, success, achievement, as so forth, in a given field, especially in clinical fields (e.g., the clinical threshold for PTSD; van de Schoot et al., 2018). We recommend utilizing substantively meaningful and potentially externally reviewed criteria whenever possible.

The applicability of, and potential for, the T2C growth model go well beyond the LGM specifications and contexts presented here. Modeling time to criterion may be useful, for example, in biomarker tracking, business skills acquisition, entrepreneurship, and any field where outcomes can be repeatedly assessed over time. Future methodological work with this model may extend to variations on the nonlinear model, including specifying splines at the criterion point, examining population heterogeneity in time to criterion by expanding the model into a growth mixture modeling framework, or simultaneously examining time to criterion and time to event by specifying a joint estimation of growth and survival models. Such extensions and applications within the flexible LGM/SEM framework represent exciting possibilities that are eagerly awaited.

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Table 1.

Means and Standard Deviations of Math Standard Scores by Grade Level and Gender for Students Enrolled in Tennessee's Project STAR (n_{analytic} = 6737)

	Full Sample (100%)	Males (48.7%)	Females (51.3%)
Math Score	Mean (SD)	Mean (SD)	Mean (SD)
Grade 5	734.59 (42.07)	731.85 (45.44)	737.33 (38.20)
Grade 6	751.93 (43.96)	748.66 (47.80)	754.98 (39.80)
Grade 7	767.87 (42.00)	765.81 (45.70)	769.67 (38.41)
Grade 8	785.33 (46.68)	783.46 (49.42)	786.98 (44.06)
ACTMath	18.23 (4.34)	18.89 (4.70)	17.77 (4.02)

Table 2.

Parameter Estimates for Linear LGM and Linear T2C Model, Project STAR Grades 5 – 8 ($n_{analytic}$ = 6737)

	Linear LGM			Linear T2C (c_{math} = 762)		
Parameters	U	C	CDO	U	C	CDO
Factor Means (**μ**)						
Intercept Factor (α)	650.77* (3.14)	655.68* (3.15)	655.24* (3.14)	—	—	—
Slope Factor (β)	16.53* (0.47)	16.22* (0.48)	16.29* (0.48)	16.53* (0.47)	16.22* (0.48)	16.29* (0.48)
Time to Criterion (τ)	—	—	—	6.73* (0.08)	6.56* (0.08)	6.55* (0.08)
Factor Variances (**Ψ**)						
Intercept Factor (α)	1495.1* (389.8)	1478.4* (388.7)	1403.4* (363.0)	—	—	—
Slope Factor (β)	26.17* (8.35)	26.11* (8.35)	24.46* (7.69)	25.58* (6.68)	26.11* (8.35)	24.46* (7.69)
Time to Criterion (τ)	—	—	—	6.46* (0.21)	5.49* (0.37)	5.43* (0.36)
Cov(α, β)	–89.00 (55.03)	–88.14 (54.94)	–77.23 (50.90)	—	—	—
Cov(α, τ)	—	—	—	—	—	—
Cov(β, τ)	—	—	—	–5.27* (0.77)	–5.12* (0.76)	–5.10* (0.76)
Factor Loadings (**Λ**)						

Intercept Factor (α)						
y_5	1	1	1	—	—	—
y_6	1	1	1	—	—	—
y_7	1	1	1	—	—	—
y_8	1	1	1	—	—	—
Slope Factor (β)						
y_5	5	5	5	–1.73* (0.08)	–1.56* (0.08)	–1.55* (0.08)
y_6	6	6	6	–0.73* (0.08)	–0.56* (0.08)	–0.55* (0.08)
y_7	7	7	7	0.27* (0.08)	–0.44* (0.08)	0.45* (0.08)
y_8	8	8	8	1.27* (0.08)	1.44* (0.08)	1.45* (0.08)
Time to Criterion (τ)						
y_5	—	—	—	–16.53* (0.47)	–16.22* (0.48)	–16.29* (0.48)
y_6	—	—	—	–16.53* (0.47)	–16.22* (0.48)	–16.29* (0.48)
y_7	—	—	—	–16.53* (0.47)	–16.22* (0.48)	–16.29* (0.48)
y_8	—	—	—	–16.53* (0.47)	–16.22* (0.48)	–16.29* (0.48)
Predictors of Growth (**π**)						

α on Male	—	–10.02* (2.35)	–10.42* (2.35)	—	—	—
β on Male	—	0.64 (0.34)	0.71* (0.34)	—	0.64 (0.34)	0.71* (0.34)
τ on Male	—	—	—	—	0.36* (0.06)	0.36* (0.06)
Distal Outcomes ($\boldsymbol{\gamma}$)						
ACTMath on α	—	—	0.09* (0.01)	—	—	—
ACTMath on β	—	—	0.85* (0.05)	—	—	0.24* (0.08)
ACTMath on τ	—	—	—	—	—	–1.51* (0.11)
ACTMath on Male	—	—	0.62* (0.14)	—	—	0.62* (0.14)
Item Intercepts ($\boldsymbol{\nu}$)						
y_5	0	0	0	873.23* (3.14)	868.32* (3.15)	868.76* (3.14)
y_6	0	0	0	873.23* (3.14)	868.32* (3.15)	868.76* (3.14)
y_7	0	0	0	873.23* (3.14)	868.32* (3.15)	868.76* (3.14)
y_8	0	0	0	873.23* (3.14)	868.32* (3.15)	868.76* (3.14)
ACTMath	—	—	–58.12* (3.47)	—	—	22.43* (1.89)
Residual Variances ($\boldsymbol{\Theta}$)						
y_5	506.79* (54.23)	506.40* (54.00)	510.11* (52.30)	506.74* (54.23)	506.40* (54.00)	510.10* (52.31)

y_6	546.22* (27.67)	545.15* (27.52)	548.40* (27.21)	546.22* (27.67)	545.16* (27.52)	548.40* (27.21)
y_7	361.38* (24.31)	361.89* (24.26)	356.17* (22.26)	361.39* (24.31)	361.89* (24.26)	356.17* (22.26)
y_8	498.71* (29.49)	499.02* (29.44)	512.91* (25.95)	498.68* (29.49)	499.02* (29.44)	512.91* (25.95)
ACTMath	—	—	5.13* (0.52)	—	—	5.13* (0.52)

Note: U=unconditional; C=conditional; CDO=conditional with distal outcome; Reported values=Estimate (Std. Error); Estimates without standard errors have been constrained; Asterisk (*)=p<0.05; α=intercept factor; β=slope factor; τ=time to criterion factor.

Table 3.

Measures of Central Tendency Describing Number of Planes Landed in the Kanfer-Ackerman Air-Traffic Controller Task (n_{analytic} = 140)

Statistic	Trial 1	Trial 2	Trial 3	Trial 4	Trial 5
Mean	20.12	25.52	29.32	32.40	34.19
(SD)	(9.68)	(9.67)	(9.78)	(9.28)	(8.58)
Median	20.50	26.00	30.00	33.00	34.00
(IQR)	(14.00)	(11.00)	(12.25)	(10.25)	(10.00)

Table 4.

Parameter Estimates for Nonlinear MM-LGM and Nonlinear MM-T2C, Kanfer-Ackerman ATC, Times 0 – 4 (n_{analytic} = 140)

	Nonlinear MM-LGM	Nonlinear MM-T2C (c_{skill} = 30)
Parameters*	Unconditional	Unconditional
Factor Means (**μ**)		
Intercept Factor (α)	20.07* (0.83)	20.07* (0.83)
Upper Asymptote (δ)	52.45* (4.66)	52.45* (4.66)
Time to ½ Growth (φ)	5.14* (1.32)	—
Time to Criterion (τ)	—	2.27* (0.28)
Factor Variances (**Ψ**)		
Intercept Factor (α)	87.36* (11.25)	87.36* (11.25)
Upper Asymptote (δ)	1818.26 (1622.11)	1818.27 (1621.27)
Time to ½ Growth (φ)	142.71 (124.23)	—
Time to Criterion (τ)	—	9.14* (2.17)
Cov(α, δ)	–58.88 (89.73)	–58.88 (89.72)
Cov(α, φ)	–19.41 (24.08)	—
Cov(δ, φ)	482.24 (439.00)	—
Cov(α, τ)	—	–22.61* (3.86)
Cov(δ, τ)	—	42.80 (34.59)
Cov(φ, τ)	—	—
Factor Loadings (**Λ**)		

Intercept Factor (α)		
y_0	1	1
y_1	0.84* (0.04)	0.39* (0.06)
y_2	0.72* (0.05)	0.06 (0.06)
y_3	0.63* (0.06)	–0.13* (0.06)
y_4	0.56* (0.06)	–0.24* (0.06)
Upper Asymptote (δ)		
y_0	0	0
y_1	0.16* (0.04)	–0.03* (0.01)
y_2	0.28* (0.05)	–0.01 (0.01)
y_3	0.37* (0.06)	0.03 (0.02)
y_4	0.44* (0.06)	0.08* (0.03)
Time to ½ Growth (φ)		
y_0	0	—
y_1	–0.86* (0.25)	—
y_2	–1.27* (0.29)	—
y_3	–1.47* (0.27)	—
y_4	–1.55* (0.24)	—
Time to Criterion (τ)		
y_0	—	0
y_1	—	–1.95* (0.30)
y_2	—	–2.88* (0.35)
y_3	—	–3.32* (0.38)

y_4	—	–3.51* (0.42)
Item Intercepts (**ν**)		
y_0	0	0
y_1	4.42* (0.24)	23.66* (2.23)
y_2	6.53* (0.35)	34.99* (1.53)
y_3	7.54* (0.66)	40.38* (1.52)
y_4	7.97* (0.96)	42.70* (2.67)
Residual Variances (**Θ**)		
y_0	5.58 (6.62)	5.58 (6.62)
y_1	4.31* (1.72)	4.31* (1.72)
y_2	6.19* (1.48)	6.19* (1.48)
y_3	8.11* (1.91)	8.11* (1.91)
y_4	5.61 (3.54)	5.61 (3.54)

Note: Reported values=Estimate (Std. Error); Estimates without standard errors have been constrained; Asterisk (*)=p<0.05; α=intercept factor; δ=upper asymptote factor; φ=time to half growth factor; τ=time to criterion factor

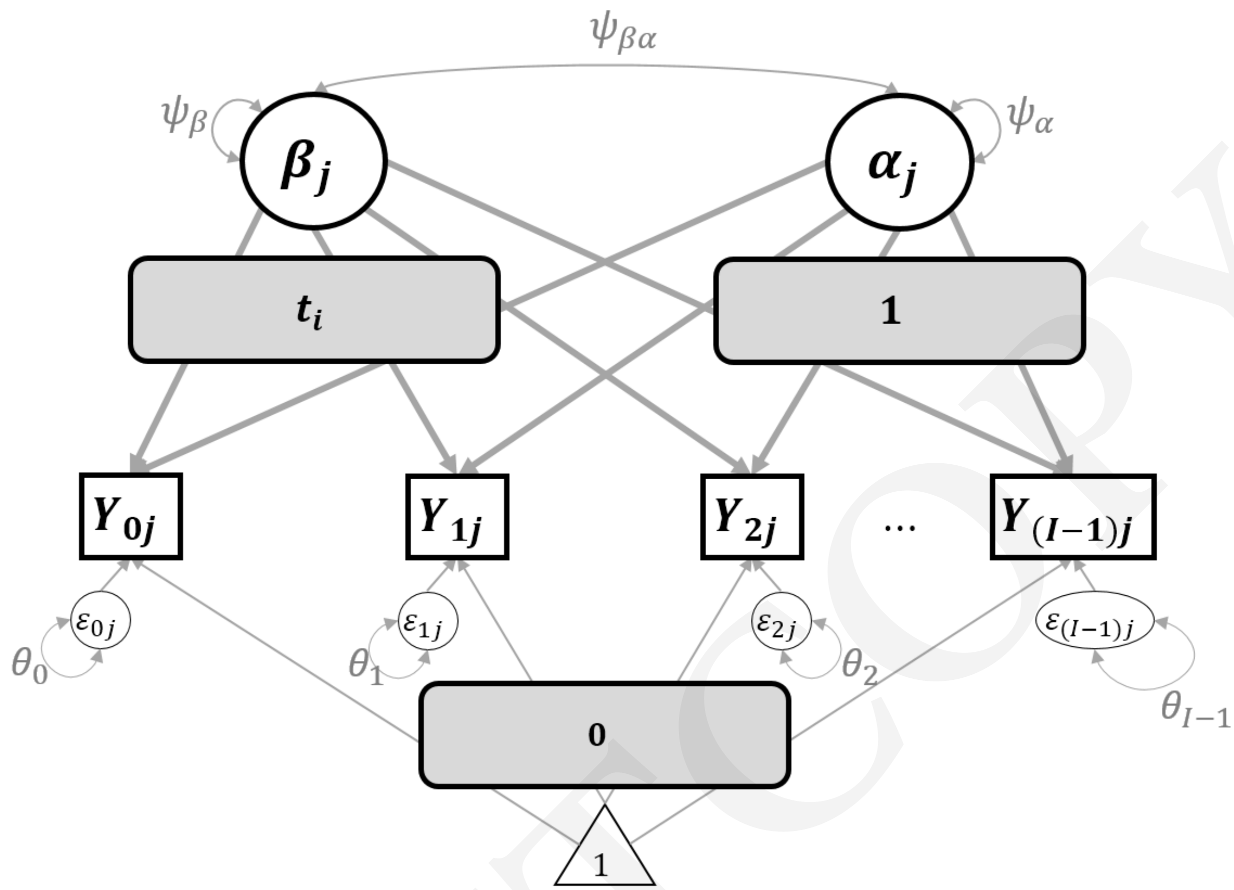

Figure 1: A linear latent growth model with y_{ij} measured at I time points, a random slope factor (β_j) and a random intercept factor (α_j). Slope factor loadings are fixed at t_i, and intercept factor loadings are fixed at 1. The mean vector of the y_{ij} measured variables has been fixed to **0** for identification of the latent growth factor means. For clarity, the mean structure of the latent growth factors, $\boldsymbol{\mu}_{\beta,\alpha} = \begin{bmatrix} \mu_\beta \\ \mu_\alpha \end{bmatrix}$, is not depicted.

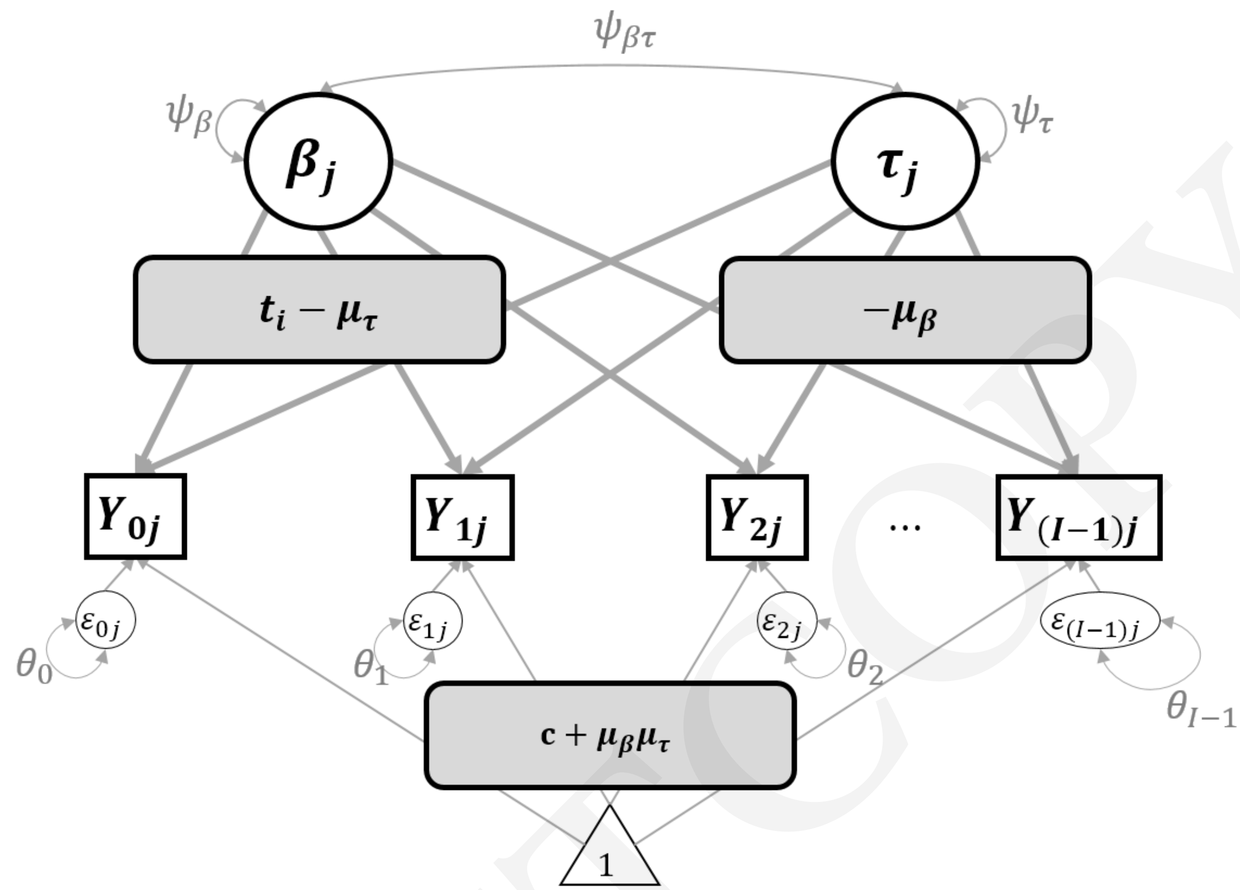

Figure 2: A reparameterized linear latent growth model with y_{ij} measured at *I* time points, a random slope (β_j), a random Time to Criterion factor (τ_j), and a criterion fixed at c. For clarity, the mean structure of the latent growth factors, $\boldsymbol{\mu}_{\beta,\tau} = \begin{bmatrix} \mu_\beta \\ \mu_\tau \end{bmatrix}$, is not depicted.

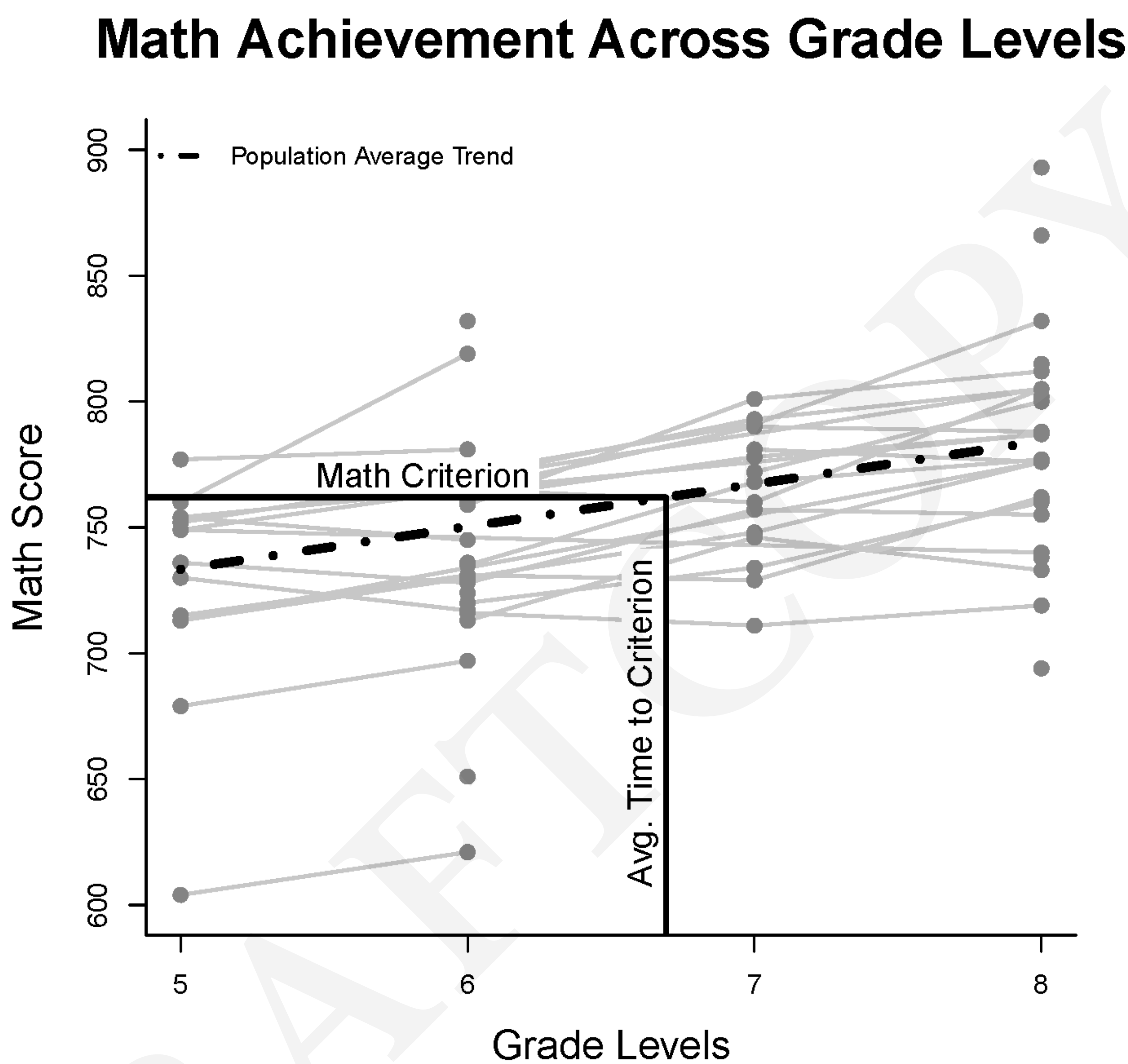

Figure 3: Individual math trajectories and Average Time (t_c) to Math Criterion (c_{math} = 762) for a random subsample of students enrolled in Project STAR, grades 5-8 ($n_{subsample}$ = 45).

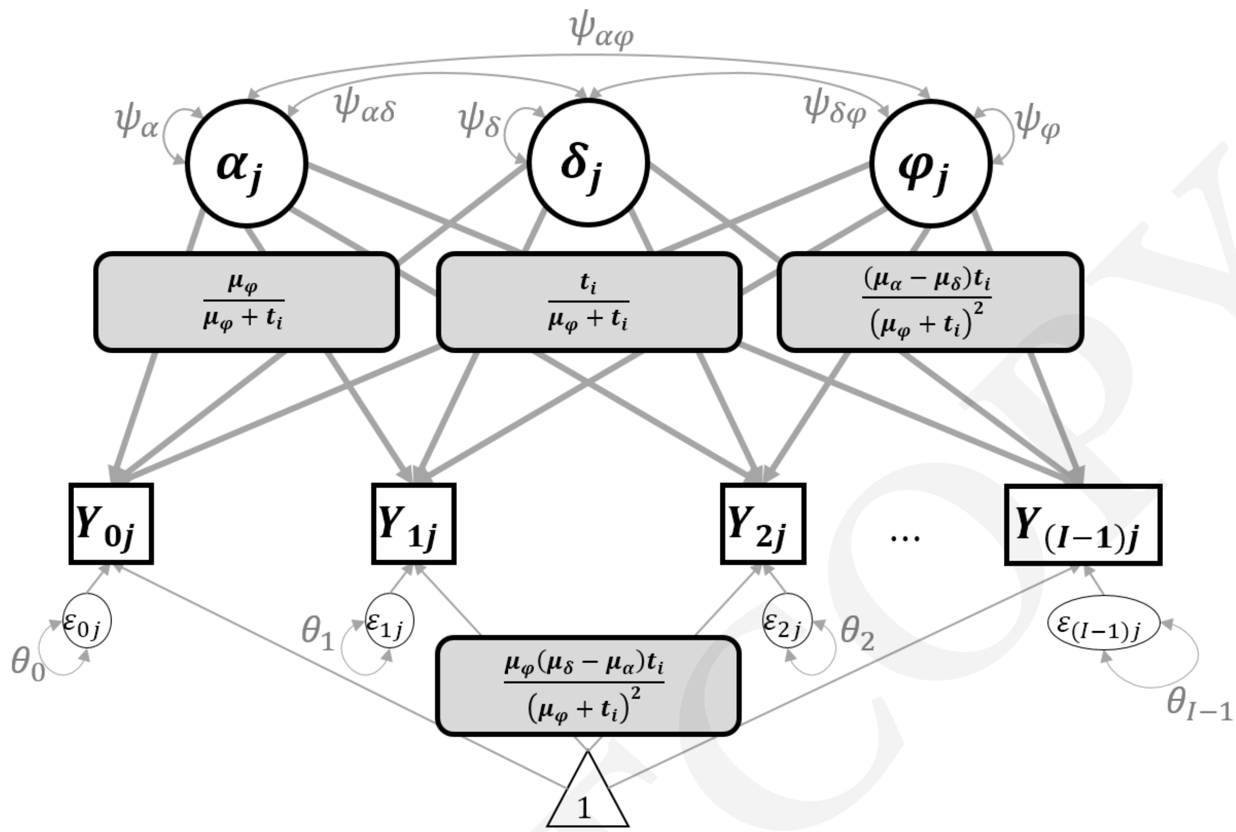

Figure 4. A nonlinear latent growth model (*Michaelis-Menten*) with y_{ij} measured at I time points, a random intercept (α_j), a random upper asymptote (δ_j), and a random time to half-growth parameter (φ_j). For clarity, the mean structure of the latent growth factors, $\boldsymbol{\mu}_{\alpha,\delta,\varphi} = \begin{bmatrix} \mu_\alpha \\ \mu_\delta \\ \mu_\varphi \end{bmatrix}$, is not depicted.

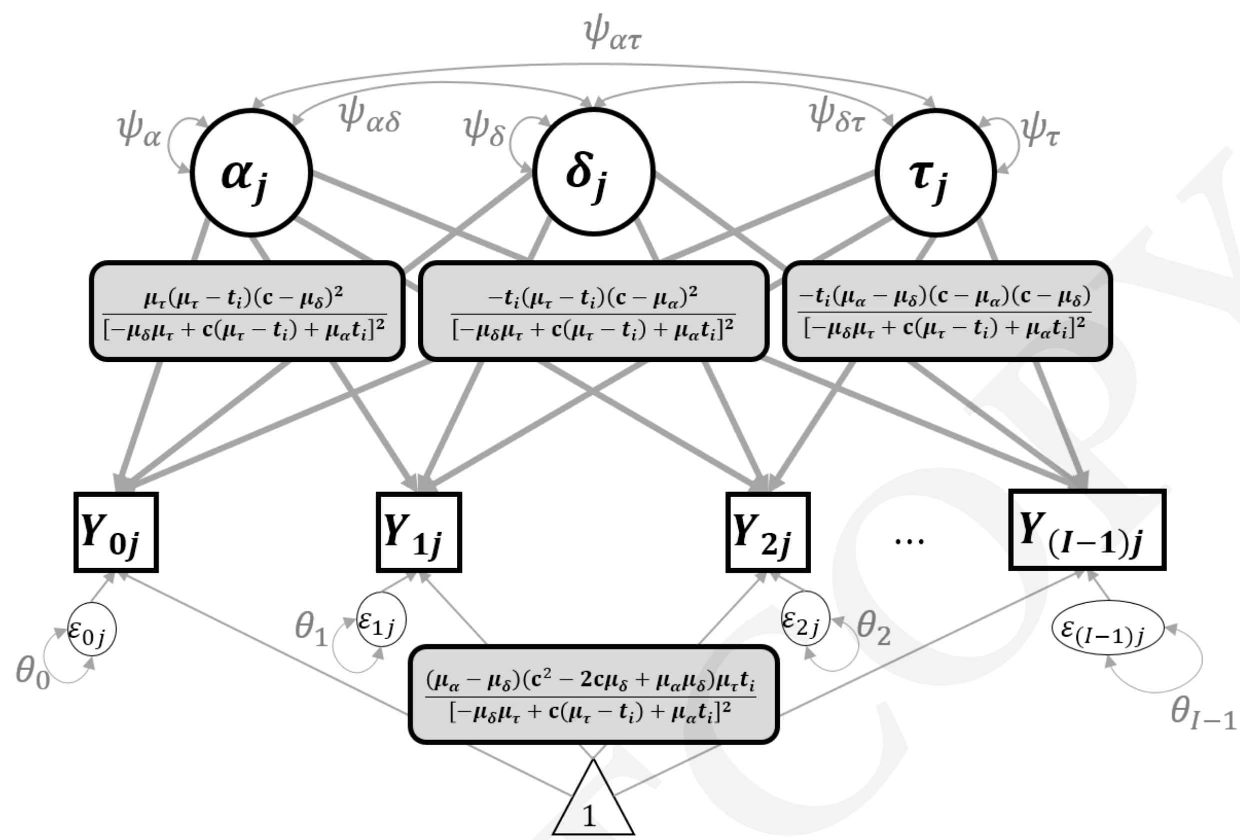

Figure 5. A reparameterized form of the model presented in Figure 4, such that time to half-growth (φ_j) is sacrificed to yield Time to Criterion (τ_j) and criterion is fixed at c. For clarity, the mean structure of the latent growth factors, $\boldsymbol{\mu}_{\alpha,\delta,\tau} = \begin{bmatrix} \mu_\alpha \\ \mu_\delta \\ \mu_\tau \end{bmatrix}$, is not depicted.

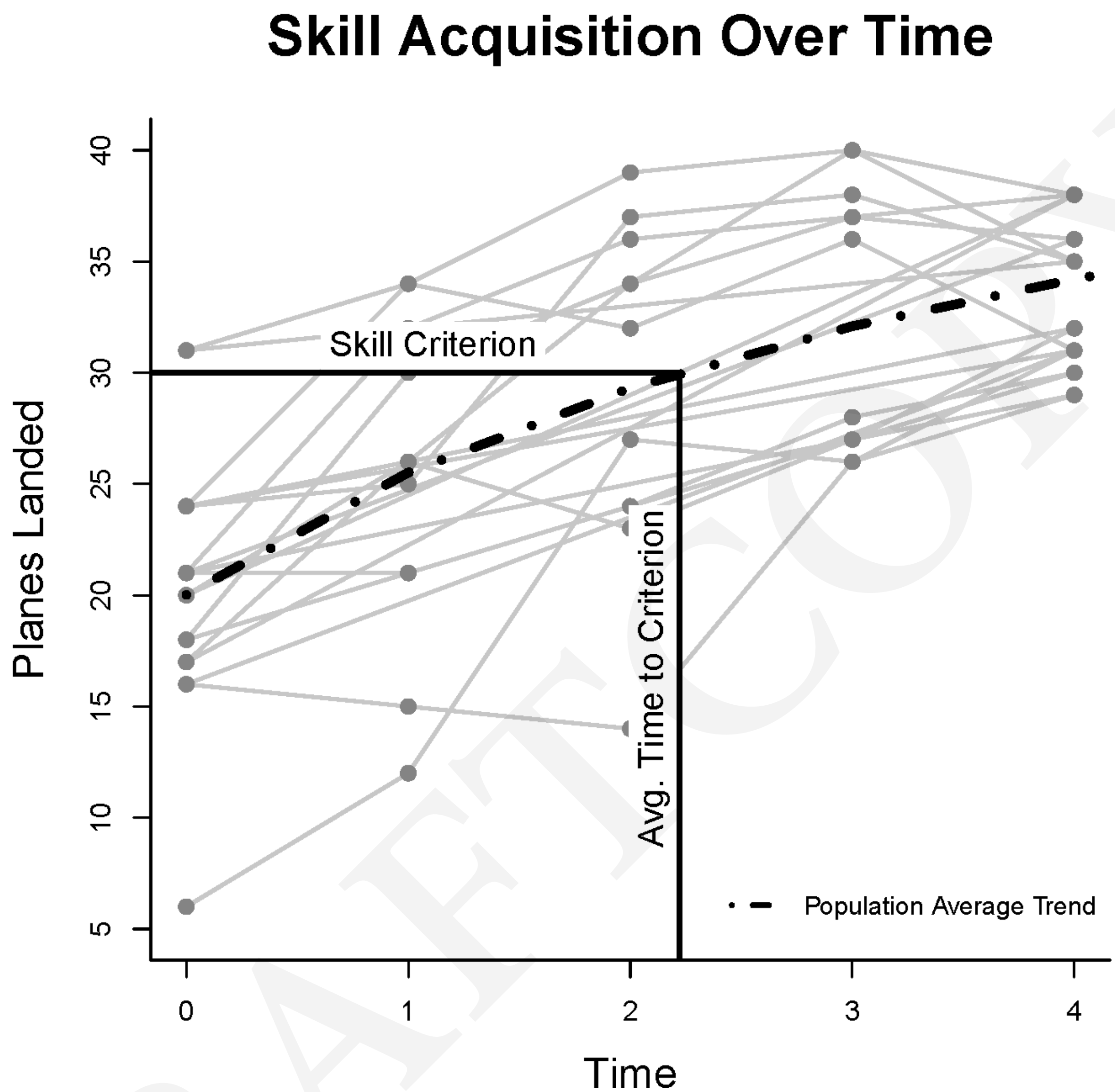

Figure 6: Individual skill acquisition trajectories and Average Time (t_c) to Skill Criterion (c_{skill} = 30) for a random subsample of participants in the Kanfer-Ackerman Air Traffic Controller Task (ATC) across 5 time points ($n_{subsample}$ = 10).