CGRA 151 Introduction to Computer Graphics
Mathematics Workbook

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This workbook is designed to help you get up to speed in the mathematics of vectors and matrices. Working through this sheet will help you to understand the mathematical underpinnings of various parts of the course. This material is assessed in the mathematics assignment and in the tests. If you are already familiar with all of this material then you can skip this workbook and go directly to doing the mathematics assignment.

1 Vectors

A *vector* is a way of packaging up several related *scalar variables* into a single object. A *scalar* is just a number, like 1 or 2.8182. A scalar is represented by an italic lowercase letter, such as a or x. A *vector* is represented by a boldface lowercase letter such as **a** or **x**. When writing vectors by hand, you cannot write in boldface so you can write vectors by putting a little arrow above them, e.g. $\vec{x}$.

Here are some simple vectors, in two and three dimensions:

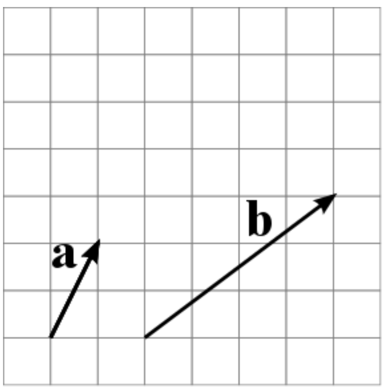
$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \quad \mathbf{d} = \begin{bmatrix} 8 \\ 9 \\ 0 \end{bmatrix}$$

Here are some more, using variables:

$$\mathbf{v_1} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \mathbf{v_2} = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \quad \mathbf{v_3} = \begin{bmatrix} x_3 \\ y_3 \\ w_3 \end{bmatrix} \quad \mathbf{v_4} = \begin{bmatrix} x_4 \\ y_4 \\ w_4 \end{bmatrix} \quad \mathbf{v_5} = \begin{bmatrix} x_5 \\ y_5 \\ w_5 \end{bmatrix}$$

2 Understanding vectors graphically

A vector can be thought of as a directed line segment. Below are graphical representations of two two-dimensional vectors, $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$



The first element of the vector is the distance that the vector travels in the x-direction, the second element is the distance that the vector travels in the y-direction. To visualise three-dimensional vectors requires three-dimensional space, so all of the graphical examples in this document will stick to two dimensions.

3 Vector addition

We use vectors because they allow us to write one equation instead of several. For example, this vector equation:

$$\mathbf{v_5} = \mathbf{v_3} + \mathbf{v_4}$$

is a fast way to write these three scalar equations:

$$\begin{aligned} x_5 &= x_3 + x_4 \\ y_5 &= y_3 + y_4 \\ w_5 &= w_3 + w_4 \end{aligned}$$

You can add vectors only if they have the same *dimension*. The *dimension* is the number of elements in the vector. For example, $\mathbf{a}, \mathbf{b}, \mathbf{v_1}, \mathbf{v_2}$ are all of dimension 2, while $\mathbf{c}, \mathbf{d}, \mathbf{v_3}, \mathbf{v_4}, \mathbf{v_5}$ are all of dimension 3.

To add vectors, you add the matching elements of each vector to each other Let us look at a couple of examples. We want to add togeher **a** and **b** to get a new vector, which we will call **e**.

$$\mathbf{e} = \mathbf{a} + \mathbf{b}$$

First we expand **a** and **b** to the full vectors:

$$\mathbf{e} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 3 \end{bmatrix}$$

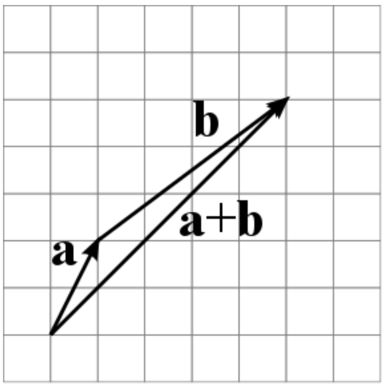
Then we add together the individual elements:

$$\mathbf{e} = \begin{bmatrix} 1+4 \\ 2+3 \end{bmatrix}$$

Which gives us the answer:

$$\mathbf{e} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}.$$

This can be interpreted graphically by putting the tail of the second vector at the head of the first, and the sum vector is a line from the tail of the first vector to the head of the second.



Doing this for vectors **c** and **d** gives us:

$$\begin{aligned} \mathbf{f} &= \mathbf{c} + \mathbf{d} \\ &= \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} + \begin{bmatrix} 8 \\ 9 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 5+8 \\ 6+9 \\ 7+0 \end{bmatrix} \\ &= \begin{bmatrix} 13 \\ 15 \\ 7 \end{bmatrix} \end{aligned}$$

Now it is your turn. Solve the following vector additions:

$$\begin{bmatrix} 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} 3.5 \\ 2.1 \end{bmatrix} + \begin{bmatrix} 1.2 \\ 4.4 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} -2.7 \\ 12 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.2 \\ -6.7 \\ 3.9 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

4 Multiplying a scalar by a vector

You cannot multiply two vectors together to create a new vector, because the operation does not make sense (there are however two operations that are similar to multiplication: vector dot product and vector cross product, you will meet dot product in Section 7). What you can do is multiply a vector by a scalar. For example, where k is a scalar and we use the vectors, $\mathbf{v_4}$ and $\mathbf{v_5}$, that we met earlier, then this equation:

$$\mathbf{v}_5 = k\mathbf{v}_4$$

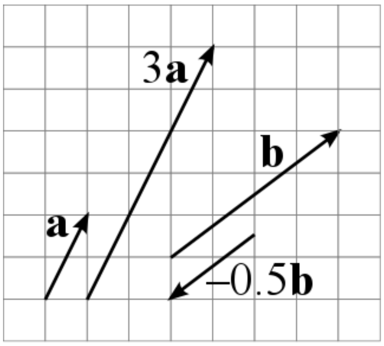
is a fast way to write these three scalar equations:

$$\begin{aligned} x_5 &= kx_4 \\ y_5 &= ky_4 \\ w_5 &= kw_4 \end{aligned}$$

For example, multiplying vector $\mathbf{c}$ by a factor of 3 gives us:

$$\begin{aligned} \mathbf{h} &= 3\mathbf{c} \\ &= 3 \times \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 3 \times 5 \\ 3 \times 6 \\ 3 \times 7 \end{bmatrix} \\ &= \begin{bmatrix} 15 \\ 18 \\ 21 \end{bmatrix}. \end{aligned}$$

Graphically, scalar multiplication makes a vector longer or shorter, but it always still has the same direction (or completely the opposite direction if you multiply by a negative number). Here are graphical representations of $3 \times \mathbf{a}$ and $-0.5 \times \mathbf{b}$:



Try these examples:

$$4 \times \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$-2 \times \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$7.5 \times \begin{bmatrix} -2.7 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

$$3 \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \quad \end{bmatrix}$$

5 Vector subtraction

Vector subtraction works in the same way as vector addition, by getting you to subtract matching elements, This vector subtraction:

$$\mathbf{v}_5 = \mathbf{v}_3 - \mathbf{v}_4$$

is a fast way to write these three scalar subtractions:

$$\begin{aligned} x_5 &= x_3 - x_4 \\ y_5 &= y_3 - y_4 \\ w_5 &= w_3 - w_4 \end{aligned}$$

For example, subtracting vector $\mathbf{c}$ from vector $\mathbf{d}$ gives us:

$$\begin{aligned} \mathbf{g} &= \mathbf{d} - \mathbf{c} \\ &= \begin{bmatrix} 8 \\ 9 \\ 0 \end{bmatrix} - \begin{bmatrix} 5 \\ 6 \\ 7 \end{bmatrix} \\ &= \begin{bmatrix} 8-5 \\ 9-6 \\ 0-7 \end{bmatrix} \\ &= \begin{bmatrix} 3 \\ 3 \\ -7 \end{bmatrix} \end{aligned}$$

The other way to think of subtraction is as adding -1 times the vector, so:

$$\begin{aligned} \mathbf{v}_5 &= \mathbf{v}_3 - \mathbf{v}_4 \\ &= \mathbf{v}_3 + ((-1) \times \mathbf{v}_4) \end{aligned}$$

Try these subtractions:

$$\begin{bmatrix} 6 \\ 9 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

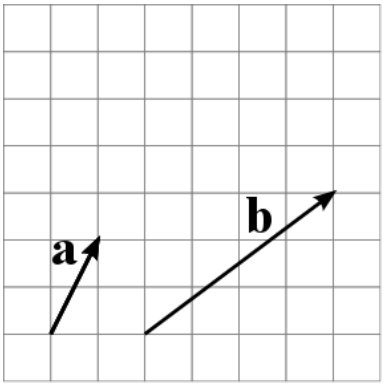
$$\begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \\ \quad \\ \quad \end{bmatrix}$$

6 Length of a vector

When we look at vectors graphically, we can see that they have a length. The length of a vector is denoted by vertical bars either side of the vector. The length of a two-dimensional vector is calculated using Pythagoras' theorem. So, for the vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, the length is calculated as:

$$|\mathbf{v}| = \sqrt{x^2 + y^2}$$

Consider a couple of examples, using the vectors we met earlier, $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$



We can calculate their lengths:

$$|\mathbf{a}| = \sqrt{1^2 + 2^2} = \sqrt{1+4} = \sqrt{5} \qquad |\mathbf{b}| = \sqrt{4^2 + 3^2} = \sqrt{16+9} = \sqrt{25} = 5$$

Pythagoras' theorem generalises to any number of dimensions.

For our three dimensional vector, $\mathbf{v_3} = \begin{bmatrix} x_3 \\ y_3 \\ w_3 \end{bmatrix}$, we get:

$$|\mathbf{v_3}| = \sqrt{x_3^2 + y_3^2 + w_3^2}$$

For the seven-dimensional vector, $\mathbf{v_7} = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix}$, we get:

$$|\mathbf{v_7}| = \sqrt{a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2}$$

Calculate the lengths of these five vectors:

$$\begin{bmatrix} 6 \\ 8 \end{bmatrix} \qquad \begin{bmatrix} -5 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} 3.5 \\ 2.1 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 8 \\ -7 \\ 6 \\ -5 \\ 4 \end{bmatrix}$$

7 Dot product

The vector dot product takes two vectors and returns a single scalar number. It turns out to be an extremely useful operator. To take the dot product of two vectors, first your two vectors must be of the same dimension (i.e., have the same number of elements). The calculation is to multiply together corresponding elements and then add up all the products to produce a single scalar number.

For example, let us take some of the vectors we met before:

$$\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix} \qquad \mathbf{v_3} = \begin{bmatrix} x_3 \\ y_3 \\ w_3 \end{bmatrix} \quad \mathbf{v_4} = \begin{bmatrix} x_4 \\ y_4 \\ w_4 \end{bmatrix}$$

$$\mathbf{a} \cdot \mathbf{b} = 1 \times 4 + 2 \times 3 = 10 \qquad \mathbf{v_3} \cdot \mathbf{v_4} = x_3 \times x_4 + y_3 \times y_4 + w_3 \times w_4$$

Try your hand at taking dot products:

$$\begin{bmatrix} 6 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} =$$

$$\begin{bmatrix} -2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} =$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \end{bmatrix} =$$

7.1 The dot product gives us the angle between two vectors

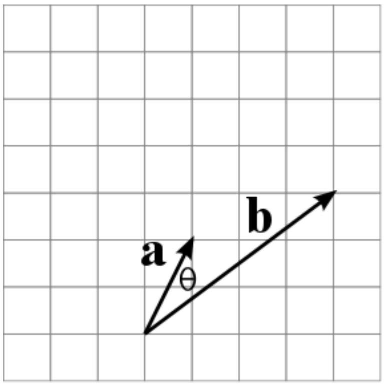
The dot product is useful because it can be proven to be equal to the product of the lengths of the two vectors times the cosine of the angle between the two vectors:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| \times |\mathbf{b}| \times \cos\theta$$

Proving that this is true is outside the scope of this course, but we can make good use of this piece of information. For example, if you want to know the cosine of the angle between two vectors you can rearrange the formula to get:

$$\cos\theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| \times |\mathbf{b}|}$$

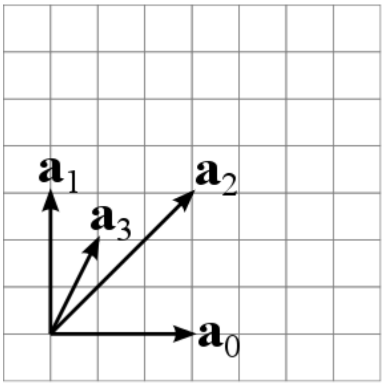
Take our old friends, $\mathbf{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$. If we view them graphically, with their tails put together, we might want to know the angle between them. Just looking at the picture, the angle looks to be somewhere between 20° and 30°.



The actual answer requires us to do some calculations:

$$\begin{aligned}
|\mathbf{a}| &= \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5} \approx 2.236 \\
|\mathbf{b}| &= \sqrt{4^2+3^2} = \sqrt{16+9} = \sqrt{25} = 5 \\
\mathbf{a}\cdot\mathbf{b} &= 1\times 4 + 2\times 3 = 10 \\
\cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\times|\mathbf{b}|} \approx \frac{10}{2.236\times 5} \approx 0.8944 \\
\theta &\approx \cos^{-1}(0.8944) \approx 26.57^\circ
\end{aligned}$$

Now it's your turn. Consider these four vectors:



$$\mathbf{a_0} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \mathbf{a_1} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad \mathbf{a_2} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad \mathbf{a_3} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

First calculate the lengths of the vectors:

$$|\mathbf{a_0}| = \qquad |\mathbf{a_1}| = \qquad |\mathbf{a_2}| = \qquad |\mathbf{a_3}| =$$

Let $\angle\mathbf{a_0a_2}$ be the angle between vectors $\mathbf{a_0}$ and $\mathbf{a_2}$. Calculate this angle:

$$\angle\mathbf{a_0a_2} = \cos^{-1}\left(\frac{\mathbf{a_0}\cdot\mathbf{a_2}}{|\mathbf{a_0}|\times|\mathbf{a_2}|}\right) =$$

Check your answer: does it look to be correct when you look at the diagram. The angle between $\mathbf{a_0}$ and $\mathbf{a_2}$, $\angle\mathbf{a_0a_2}$, is obviously 45°. Is that what you calculated?

Try calculating these angles:

$$\angle\mathbf{a_0a_1} =$$

$$\angle\mathbf{a_0a_3} =$$

$$\angle\mathbf{a_2a_3} =$$

7.2 The dot product tells us if two vectors are perpendicular

What you should have noticed in the previous section is that the dot product between two vectors at 90°, $\angle \mathbf{a_0 a_1}$, is zero. Two vectors at 90° to one another are said to be *perpendicular*. The dot product of two perpendicular vectors is *always* zero.

7.3 The dot product gives us the length of a vector

Given any vector, $\mathbf{v}$, we can calculate the length of the vector using the dot product:

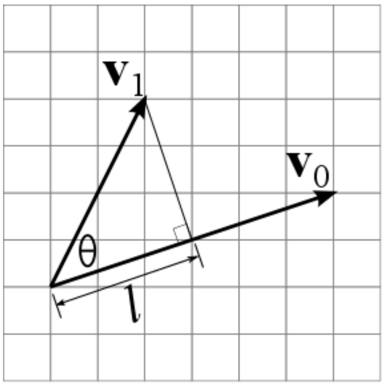
$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}}$$

Why is this true? The angle between a vector and itself is clearly zero and $\cos(0°) = 1$ so:

$$\begin{aligned} \mathbf{v} \cdot \mathbf{v} &= |\mathbf{v}| \times |\mathbf{v}| \times \cos(0°) \\ &= |\mathbf{v}| \times |\mathbf{v}| \times 1 \\ &= |\mathbf{v}|^2 \end{aligned}$$

7.4 The dot product gives us the projection of one vector onto another

We might want to project one vector onto another, this allows us to find how much of one vector points in the direction of another vector. Look at this diagram:



The length l is the length of the projection of vector $\mathbf{v_1}$ onto vector $\mathbf{v_0}$. We find the projection by connecting the starts of the two vectors, then dropping a perpendicular from the end of $\mathbf{v_1}$ onto vector $\mathbf{v_0}$. This makes a right angle.

From your high school mathematics you should know that the distance l is related to the cosine of the angle, θ, and the length of the hypothenuse, $|\mathbf{v_1}|$. That is, because it is a right-angled triangle:

$$l = |\mathbf{v_1}| \cos \theta$$

In Section 7.1 we saw that the dot product between $\mathbf{v_0}$ and $\mathbf{v_1}$ can be expressed by a similar formula:

$$\mathbf{v_0} \cdot \mathbf{v_1} = |\mathbf{v_0}| \times |\mathbf{v_1}| \times \cos \theta$$

Comparing the two formulas, you can then compute the distance l directly from the dot product:

$$l = \frac{\mathbf{v_0} \cdot \mathbf{v_1}}{|\mathbf{v_0}|}$$

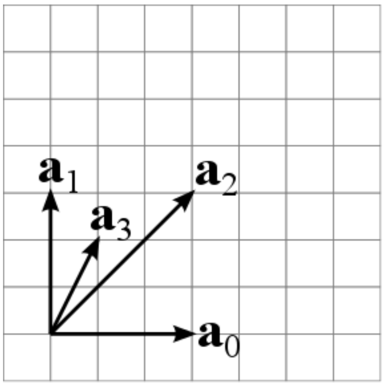
In the above example, we have these two vectors:

$$\mathbf{v_0} = \begin{bmatrix} 6 \\ 2 \end{bmatrix} \qquad \mathbf{v_1} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

We can see that l appears to be a little bit bigger than 3. We can compute its exact length:

$$\begin{aligned}
\mathbf{v_0} \cdot \mathbf{v_1} &= 6 \times 2 + 2 \times 4 \\
&= 20 \\
|\mathbf{v_0}| &= \sqrt{6 \times 6 + 2 \times 2} \\
&= \sqrt{36 + 4} \\
&= \sqrt{40} \\
l &= \frac{\mathbf{v_0} \cdot \mathbf{v_1}}{|\mathbf{v_0}|} \\
&= \frac{20}{\sqrt{40}} \\
&= \sqrt{10} \approx 3.162
\end{aligned}$$

You should now try some examples yourself:

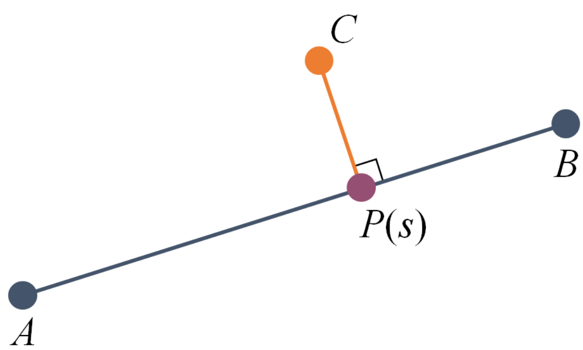


$$\mathbf{a_0} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \mathbf{a_1} = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \qquad \mathbf{a_2} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \qquad \mathbf{a_3} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Try doing projections of each of $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ onto $\mathbf{a_0}$. It is obvious from the diagram that the answers should be 0, 3, and 1 respectively. See if that is what the formula gives you. Then try projecting $\mathbf{a_3}$ onto each of $\mathbf{a_1}$ and $\mathbf{a_2}$.

7.5 Closest point on a line

The fact that the dot product gives the projection of one vector onto another (Section 7.4) proves to be a useful way to find the closest point on a straight line to any other point. You will recall that this is used in a couple of the algorithms in the course. We have this situation:



We have a straight line between two points, A and B. We have another point, C. We have dropped a perpendicular from point C to the line $\overline{AB}$. We denote the point where the perpendicular meets $\overline{AB}$ as $P(s)$. Given that we know A, B and C, we want to find the position of point $P(s)$, which is the closest point to C that lies on the line $\overline{AB}$. Comparing this to the diagram at the start of Section 7.4, we can see that, in that diagram, we would have:

$$\begin{aligned}
\mathbf{v_0} &= B - A \\
\mathbf{v_1} &= C - A
\end{aligned}$$

which immediately tells us that the distance, l, from A to $P(s)$ is:

$$\begin{aligned}
l &= \frac{\mathbf{v_0} \cdot \mathbf{v_1}}{|\mathbf{v_0}|} \\
&= \frac{(B - A) \cdot (C - A)}{|B - A|}
\end{aligned}$$

How does this help us to find the actual position of point $P(s)$? Recall that the s in the definition of $P(s)$ is a parameter that tells us how far between A and B the point $P(s)$ is. The formula for $P(s)$ is:

$$P(s) = (1 - s)A + sB$$

If $s = 0$ then we are at point A: $P(0) = A$. If $s = 1$ then we are at point B: $P(1) = B$. If $s = 0.5$ then we are half-way between: $P(0.5) = 0.5 \times A + 0.5 \times B$. The value l gives us the actual distance between A and $P(s)$. How do we convert this to the parameter s? Recall that, if $s = 0$, then $P(s) = A$ and $l = 0$. If $s = 1$, then $P(s) = B$ and $l = |B - A|$, so to convert l to s, we need simply to divide by the length of the line from A to B, which is $|B - A|$:

$$\begin{aligned}
s &= \frac{l}{|B - A|} \\
&= \frac{(B - A) \cdot (C - A)}{|B - A| \times |B - A|} \\
&= \frac{(B - A) \cdot (C - A)}{|B - A|^2}
\end{aligned}$$

8 Matrices

A matrix is a two-dimensional array of numbers. As with vectors, we use matrices to package up multiple equations into a single equation. A matrix is normally represented by a bold uppercase character. For example, we can package up these two equations as a single matrix equation.

$$\begin{aligned}
2x + 3y &= 5 \\
6x + 4y &= 8
\end{aligned}$$

This can be represented as

$$\begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$$

If we write $\mathbf{M} = \begin{bmatrix} 2 & 3 \\ 6 & 4 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{p} = \begin{bmatrix} 5 \\ 8 \end{bmatrix}$, then the equation becomes the very compact:

$$\mathbf{Mv} = \mathbf{p}$$

Notice how the x and y in the original equations have been packaged up as a column vector $\begin{bmatrix} x \\ y \end{bmatrix}$.

8.1 Multiplying a matrix and a vector

You can think of multiplication of a matrix and a vector as each *row* of the matrix diving into the column vector, with each element of the row then being multiplied by its matching element in the column vector and then finally all those products being added together to give a single new entry in the new vector on the right.

Let's try this with some numbers:

$$\begin{aligned}
\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 7 \\ 8 \end{bmatrix} &= \begin{bmatrix} 2 \times 7 + 3 \times 8 \\ 4 \times 7 + 5 \times 8 \end{bmatrix} \\
&= \begin{bmatrix} 14 + 24 \\ 28 + 40 \end{bmatrix} \\
&= \begin{bmatrix} 38 \\ 68 \end{bmatrix}
\end{aligned}$$

Looking at those calculations, you should see that each of the multiplications is a dot product. We are taking the dot product of each *row* of the matrix with the column vector to produce a single entry in the new vector.

Now it's your turn. As in the first example in this section, express the following three equations as a 3×3 matrix multiplied by a vector with three elements to produce another vector with three elements:

$$\begin{aligned}
3a + 4b + 5c &= 21 \\
7a + 12b + 13c &= 37 \\
9a + 17b + 25c &= 54
\end{aligned}$$

Try these matrix multiplications, which each produces a two-dimensional vector as a result:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \end{bmatrix}$$

Now attempt these 3×3 matrices multiplied by a three-dimensional vector to give another three-dimensional vector:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

$$\begin{bmatrix} 0.87 & -0.50 & 0 \\ 0.50 & 0.87 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \quad \\ \quad \end{bmatrix}$$

Recall from the course that we can use 3×3 matrices to represent transformations in two dimensions. The last two matrices above represent transformations: the second represents a scale by a factor of two in the y-direction combined with a translation of $(-3, -4)$. The third represents (approximately) a rotation by $30°$ because $\cos 30° \approx 0.87$ and $\sin 30° = 0.5$.

8.2 Adding two matrices

Adding two matrices is the same as adding two vectors. You can only do the addition if the matrices have the same dimensions and you do the addition by adding together corresponding elements. For example:

$$\begin{aligned}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1+1 & 2+0 & 3+-3 \\ 4+0 & 5+2 & 6+-4 \\ 7+0 & 8+0 & 9+1 \end{bmatrix} \\ &= \begin{bmatrix} 2 & 2 & 0 \\ 4 & 7 & 2 \\ 7 & 8 & 10 \end{bmatrix}\end{aligned}$$

8.3 Multiplying two matrices

You multiply two matrices by taking the dot products of *row vectors* in the first matrix with *column vectors* in the second matrix.

$$\begin{bmatrix} \boxed{1 \quad 2 \quad 3} \\ 4 \quad 5 \quad 6 \\ 7 \quad 8 \quad 9 \end{bmatrix} \times \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 0 \\ 0 \end{matrix}} & \begin{matrix} 0 \\ 2 \\ 0 \end{matrix} & \begin{matrix} -3 \\ -4 \\ 1 \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{1} & 4 & -8 \\ 4 & 10 & -20 \\ 7 & 16 & -44 \end{bmatrix}$$

In the above example, we have put a box around one of the row vectors and a box around one of these column vectors, the dot product of these two gives one of the numbers in the product matrix, again shown in a box:

$$1 \times 1 + 2 \times 0 + 3 \times 0 = 1$$

As another example, the bottom right product, -44, is produced by taking the dot product the bottom row in the first matrix by the right-most column in the second matrix:

$$7 \times -3 + 8 \times -4 + 9 \times 1 = -44$$

Do the dot products for the other seven combinations of a row from the first matrix and a column from the second matrix. See if your answers agree with the ones given.

When calculating matrix products by hand, it is best to work left-to-right, top-to-bottom. So you take the top row of the first matrix and multiply it by each of the columns of the second matrix, in turn, to produce the top row of the product matrix. Then you move on to do the same with the second row, then the third, and so on. Now try these two matrix multiplications:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{3}/2 & -0.5 & 0 \\ 0.5 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

The first product has an identity matrix (all ones on the main diagonal, zeros everywhere else) times a matrix. This should have a product equal to the second matrix. Multiplying any matrix by an identity matrix produces the same matrix as the product.

The second product is multiplying a rotation matrix (rotation by 30°) by a translation matrix (translation by $(2, 3)$). From your experience with Processing you will know that if you rotate first then translate, the translation happens in the rotated universe. The product of the two matrices should have exactly the same rotation matrix in the top-left 2×2 block and with a rotated translation in right-hand column.

Now try doing the translation first and the rotation second. In this case the translation (third column) is not affected by the rotation:

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \sqrt{3}/2 & -0.5 & 0 \\ 0.5 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \quad & \quad & \quad \\ & & \\ & & \end{bmatrix}$$

8.4 Matrices representing transformations

In the course you are taught that we represent two-dimensional transformations by using 3×3 matrices. We have three basic transformations:

scale by factors of s_x and s_y $\quad \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}$

rotate by angle θ $\quad \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

translate a distance (t_x, t_y) $\quad \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}$

To transform a point, (x, y), we first convert it to a three-dimensional vector by adding the extra homogeneous coordinate, which we set to 1, then multiply by the transformation matrix, then convert back to two-dimensions. For example, to rotate point $(3, 4)$ by 30°:

$$\begin{aligned} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} &= \begin{bmatrix} 3\cos 30^\circ - 4\sin 30^\circ \\ 3\sin 30^\circ + 4\cos 30^\circ \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 \times \sqrt{3}/2 - 4 \times 0.5 \\ 3 \times 0.5 + 4 \times \sqrt{3}/2 \\ 1 \end{bmatrix} \\ &\approx \begin{bmatrix} 0.598 \\ 4.964 \\ 1 \end{bmatrix} \end{aligned}$$

So point $(3, 4)$ rotates about the origin to point $(0.598, 4.964)$.

Try scaling point $(3, 4)$ by a factor of 2 in the x-direction and a factor of 0.5 in the y-direction:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \end{bmatrix}$$

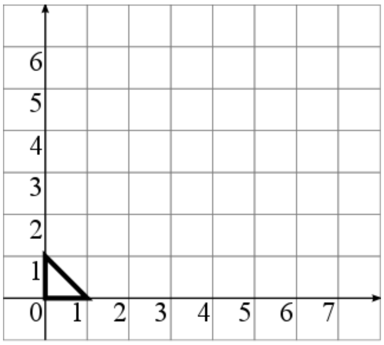
Try translating point $(3, 4)$ by a distance $(5, -2)$:

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \quad \\ \\ \end{bmatrix}$$

The big advantage of using matrices to represent transformations is that we concatenate multiple transformations together by multiplying matrices, so that we only ever have to multiply each point by a single matrix, which combines all the transformations in one place.

8.5 Example transformations

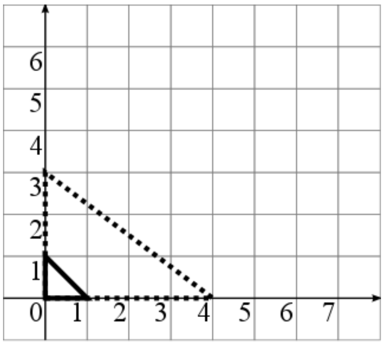
When we are trying to get to grips with a transformation matrix, one good idea is to see what happens to three special points: $(0,0)$, $(1,0)$, and $(0,1)$. In a scale or rotation, the origin, $(0,0)$, will not move. The three points are easy to compute with because they comprise only zeroes and ones. Those three points together give you a good idea of what sort of transformation is going on; they make a small triangle, as shown below.



Let's first look at transforming these using a scale matrix, scaling by a factor of 4 in the x-direction and 3 in the y-direction. The matrix to do this is:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

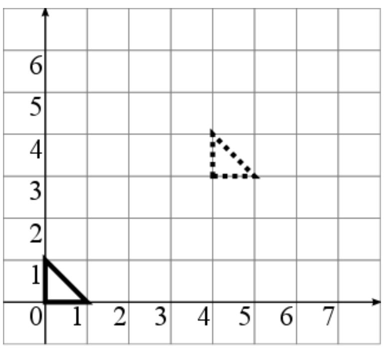
Apply this matrix to our three special points, $(0,0)$, $(1,0)$, and $(0,1)$. This tells you where the points will be drawn in the original coordinate system after they have been transformed. Your answers should match the vertices of the dotted triangle in the diagram below.



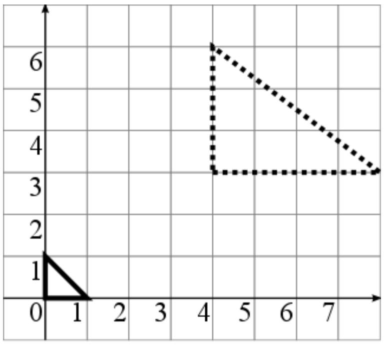
Now try transforming using a translation matrix, moving the universe by a distance of $(4,3)$. The matrix to do this is:

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply this matrix to our three special points, $(0,0)$, $(1,0)$, and $(0,1)$. Your answers should match the dotted triangle in the diagram below. Remember that what we are doing is calculating where these points will move to, in the original coordinate system, after they have been transformed.



Now, how do we combine these two transformations to produce the dotted triangle below?



This is a scaled and translated version of the triangle. In Processing we would want to translate the universe first, to put the origin at $(4, 3)$, then scale about that new origin by the appropriate factors. So the matrix we want is the product of the translation and the scale:

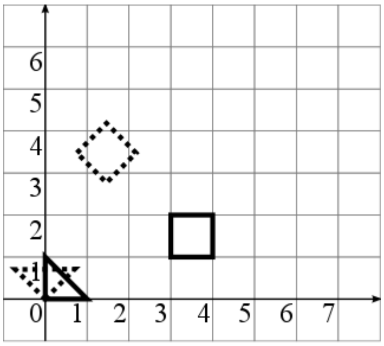
$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiply these two matrices together and then apply the result to the three special points, $(0, 0)$, $(1, 0)$, and $(0, 1)$. Your answers should match the dotted triangle in the diagram.

What do you get if you apply the transformations in the opposite order?

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's now look at rotation. Here is a rotation by 45°.



The matrix for this rotation is:

$$\begin{bmatrix} 0.7071 & -0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $\cos(45°) = \sin(45°) = \sqrt{2}/2 \approx 0.7071$. Apply this matrix to our three special points and also to the four vertices of the square. Notice that the triangle and the square are both rotating about the origin. What would you need to do to get the square to rotate about its own centre?

Answers

Section 3 (Vector addition):

$$\begin{bmatrix} 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6+4 \\ 9+2 \end{bmatrix} = \begin{bmatrix} 10 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 7 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -2+3 \\ 7+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 3.5 \\ 2.1 \end{bmatrix} + \begin{bmatrix} 1.2 \\ 4.4 \end{bmatrix} = \begin{bmatrix} 3.5+1.2 \\ 2.1+4.4 \end{bmatrix} = \begin{bmatrix} 4.7 \\ 6.5 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1-1 \\ 2+2 \\ 3+4 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} -2.7 \\ 12 \\ 0 \end{bmatrix} + \begin{bmatrix} 1.2 \\ -6.7 \\ 3.9 \end{bmatrix} = \begin{bmatrix} -2.7+1.2 \\ 12-6.7 \\ 0+3.9 \end{bmatrix} = \begin{bmatrix} -1.5 \\ 5.3 \\ 3.9 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \end{bmatrix} = \begin{bmatrix} 1+8 \\ 2+7 \\ 3+6 \\ 4+5 \\ 5+4 \end{bmatrix} = \begin{bmatrix} 9 \\ 9 \\ 9 \\ 9 \\ 9 \end{bmatrix}$$

Section 4 (Multiplying a scalar by a vector):

$$4 \times \begin{bmatrix} 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 4 \times 6 \\ 4 \times 9 \end{bmatrix} = \begin{bmatrix} 24 \\ 36 \end{bmatrix}$$

$$-2 \times \begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -2 \times -2 \\ -2 \times 7 \end{bmatrix} = \begin{bmatrix} 4 \\ -14 \end{bmatrix}$$

$$7.5 \times \begin{bmatrix} -2.7 \\ 12 \\ 0 \end{bmatrix} = \begin{bmatrix} 7.5 \times -2.7 \\ 7.5 \times 12 \\ 7.5 \times 0 \end{bmatrix} = \begin{bmatrix} 20.25 \\ 90 \\ 0 \end{bmatrix}$$

$$3 \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \times 1 \\ 3 \times 2 \\ 3 \times 3 \\ 3 \times 4 \\ 3 \times 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \\ 15 \end{bmatrix}$$

Section 5 (Vector subtraction):

$$\begin{bmatrix} 6 \\ 9 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 6-4 \\ 9-2 \end{bmatrix} = \begin{bmatrix} 2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 8-1 \\ 7-2 \\ 6-3 \\ 5-4 \\ 4-5 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 3 \\ 1 \\ -1 \end{bmatrix}$$

Section 6 (Length of a vector):

$$\left\| \begin{bmatrix} 6 \\ 8 \end{bmatrix} \right\| = \sqrt{6^2 + 8^2} = \sqrt{36 + 64} = \sqrt{100} = 10$$

$$
\begin{aligned}
\left\|\begin{bmatrix} -5 \\ 12 \end{bmatrix}\right\| &= \sqrt{-5^2+12^2} = \sqrt{25+144} = \sqrt{169} = 13 \\
\left\|\begin{bmatrix} 3.5 \\ 2.1 \end{bmatrix}\right\| &= \sqrt{3.5^2+2.1^2} = \sqrt{12.25+4.41} = \sqrt{16.66} \approx 4.082 \\
\left\|\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right\| &= \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14} \approx 3.742 \\
\left\|\begin{bmatrix} 8 \\ -7 \\ 6 \\ -5 \\ 4 \end{bmatrix}\right\| &= \sqrt{8^2+-7^2+6^2+-5^2+4^2} = \sqrt{64+49+36+25+16} = \sqrt{190} \approx 13.784
\end{aligned}
$$

Section 7 (Dot product):

$$
\begin{aligned}
\begin{bmatrix} 6 \\ 9 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} &= 6 \times 4 + 9 \times 2 = 24 + 18 = 42 \\
\begin{bmatrix} -2 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= -2 \times 3 + 7 \times 1 = -6 + 7 = 1 \\
\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 2 \\ 4 \end{bmatrix} &= 1 \times -1 + 2 \times 2 + 3 \times 4 = -1 + 4 + 12 = 15 \\
\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 7 \\ 6 \\ 5 \\ 4 \end{bmatrix} &= 1 \times 8 + 2 \times 7 + 3 \times 6 + 4 \times 5 + 5 \times 4 = 8 + 14 + 18 + 20 + 20 = 80
\end{aligned}
$$

Section 7.1 (The dot product gives us the angle between two vectors):

$$
\begin{aligned}
|\mathbf{a_0}| &= \sqrt{3^2+0^2} = \sqrt{9+0} = \sqrt{9} = 3 \\
|\mathbf{a_1}| &= \sqrt{0^2+3^2} = \sqrt{0+9} = \sqrt{9} = 3 \\
|\mathbf{a_2}| &= \sqrt{3^2+3^2} = \sqrt{9+9} = \sqrt{18} \approx 4.243 \\
|\mathbf{a_3}| &= \sqrt{1^2+2^2} = \sqrt{1+4} = \sqrt{5} \approx 2.236
\end{aligned}
$$

$$
\begin{aligned}
\angle \mathbf{a_0 a_2} &= \cos^{-1}\left(\frac{\mathbf{a_0} \cdot \mathbf{a_2}}{|\mathbf{a_0}| \times |\mathbf{a_2}|}\right) \\
&\approx \cos^{-1}\left(\frac{3 \times 3 + 0 \times 3}{3 \times 4.243}\right) \\
&\approx \cos^{-1}(0.707) \\
&\approx 45^\circ
\end{aligned}
$$

$$
\begin{aligned}
\angle \mathbf{a_0 a_1} &= \cos^{-1}\left(\frac{\mathbf{a_0} \cdot \mathbf{a_1}}{|\mathbf{a_0}| \times |\mathbf{a_1}|}\right) \\
&= \cos^{-1}\left(\frac{3 \times 0 + 0 \times 3}{3 \times 3}\right) \\
&= \cos^{-1}(0) \\
&= 90^\circ
\end{aligned}
$$

$$
\begin{aligned}
\angle \mathbf{a_0 a_3} &= \cos^{-1}\left(\frac{\mathbf{a_0}\cdot\mathbf{a_3}}{|\mathbf{a_0}|\times|\mathbf{a_3}|}\right) \\
&\approx \cos^{-1}\left(\frac{3\times 1 + 0\times 2}{3\times 2.236}\right) \\
&\approx \cos^{-1}(0.447) \\
&\approx 63.435^\circ
\end{aligned}
$$

$$
\begin{aligned}
\angle \mathbf{a_2 a_3} &= \cos^{-1}\left(\frac{\mathbf{a_2}\cdot\mathbf{a_3}}{|\mathbf{a_2}|\times|\mathbf{a_3}|}\right) \\
&\approx \cos^{-1}\left(\frac{3\times 1 + 3\times 2}{4.243\times 2.236}\right) \\
&\approx \cos^{-1}(0.947) \\
&\approx 18.435^\circ
\end{aligned}
$$

Section 7.4 (The dot product gives us the projection of one vector onto another)

$$
\begin{aligned}
\text{Project } \mathbf{a_1} \text{ onto } \mathbf{a_0}\text{: } l &= \frac{\mathbf{a_0}\cdot\mathbf{a_1}}{|\mathbf{a_0}|} \\
&= \frac{0}{3} \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\text{Project } \mathbf{a_2} \text{ onto } \mathbf{a_0}\text{: } l &= \frac{\mathbf{a_0}\cdot\mathbf{a_2}}{|\mathbf{a_0}|} \\
&= \frac{9}{3} \\
&= 3
\end{aligned}
$$

$$
\begin{aligned}
\text{Project } \mathbf{a_3} \text{ onto } \mathbf{a_0}\text{: } l &= \frac{\mathbf{a_0}\cdot\mathbf{a_3}}{|\mathbf{a_0}|} \\
&= \frac{3}{3} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\text{Project } \mathbf{a_3} \text{ onto } \mathbf{a_0}\text{: } l &= \frac{\mathbf{a_0}\cdot\mathbf{a_3}}{|\mathbf{a_0}|} \\
&= \frac{3}{3} \\
&= 1
\end{aligned}
$$

$$
\begin{aligned}
\text{Project } \mathbf{a_3} \text{ onto } \mathbf{a_1}\text{: } l &= \frac{\mathbf{a_1}\cdot\mathbf{a_3}}{|\mathbf{a_1}|} \\
&= \frac{0\times 1 + 3\times 2}{3} \\
&= \frac{6}{3} \\
&= 2
\end{aligned}
$$

$$
\begin{aligned}
\text{Project } \mathbf{a_3} \text{ onto } \mathbf{a_2}\text{: } l &= \frac{\mathbf{a_2}\cdot\mathbf{a_3}}{|\mathbf{a_2}|} \\
&\approx \frac{9}{4.243} \\
&\approx 2.121
\end{aligned}
$$

Section 8.1 (Multiplying a matrix and a vector)

$$\begin{aligned} 3a + 4b + 5c &= 21 \\ 7a + 12b + 13c &= 37 \\ 9a + 17b + 25c &= 54 \end{aligned}$$

converts to the matrix equation:

$$\begin{bmatrix} 3 & 4 & 5 \\ 7 & 12 & 13 \\ 9 & 17 & 25 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 21 \\ 37 \\ 54 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 5 + 2 \times 6 \\ 3 \times 5 + 4 \times 6 \end{bmatrix} = \begin{bmatrix} 5 + 12 \\ 15 + 24 \end{bmatrix} = \begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \times 5 - 1 \times 6 \\ 2 \times 5 + 0 \times 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 10 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \times 1 + 2 \times 2 + 3 \times -1 \\ 4 \times 1 + 5 \times 2 + 6 \times -1 \\ 7 \times 1 + 8 \times 2 + 9 \times -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \times 3 + 0 \times 2 - 3 \times 1 \\ 0 \times 3 + 2 \times 2 - 4 \times 1 \\ 0 \times 0 + 0 \times 0 + 1 \times 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0.87 & -0.50 & 0 \\ 0.50 & 0.87 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.87 \\ 0.5 \\ 1 \end{bmatrix}$$

Section 8.3 (Multiplying two matrices)

$$\begin{aligned} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \\ = & \begin{bmatrix} 1 \times 1 + 0 \times 2 + 0 \times 3 & 1 \times 4 + 0 \times 5 + 0 \times 6 & 1 \times 7 + 0 \times 8 + 0 \times 9 \\ 0 \times 1 + 1 \times 2 + 0 \times 3 & 0 \times 4 + 1 \times 5 + 0 \times 6 & 0 \times 7 + 1 \times 8 + 0 \times 9 \\ 0 \times 1 + 0 \times 2 + 1 \times 3 & 0 \times 4 + 0 \times 5 + 1 \times 6 & 0 \times 7 + 0 \times 8 + 1 \times 9 \end{bmatrix} \\ = & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} & \begin{bmatrix} \sqrt{3}/2 & -0.5 & 0 \\ 0.5 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \\ \approx & \begin{bmatrix} 0.866 & -0.5 & 0 \\ 0.5 & 0.866 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \\ = & \begin{bmatrix} 0.866 \times 1 - 0.5 \times 0 + 0 \times 0 & 0.866 \times 0 - 0.5 \times 1 + 0 \times 0 & 0.866 \times 2 - 0.5 \times 3 + 0 \times 1 \\ 0.5 \times 1 + 0.866 \times 0 + 0 \times 0 & 0.5 \times 0 + 0.866 \times 1 + 0 \times 0 & 0.5 \times 2 + 0.866 \times 3 + 0 \times 1 \\ 0 \times 1 + 0 \times 0 + 1 \times 0 & 0 \times 0 + 0 \times 1 + 1 \times 0 & 0 \times 2 + 0 \times 3 + 1 \times 1 \end{bmatrix} \\ \approx & \begin{bmatrix} 0.866 & -0.5 & 0.232 \\ 0.5 & 0.866 & 3.598 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \sqrt{3}/2 & -0.5 & 0 \\ 0.5 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} \sqrt{3}/2 & -0.5 & 2 \\ 0.5 & \sqrt{3}/2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Section 8.4 (Matrices representing transformations):

$$\begin{aligned} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} &= \begin{bmatrix} 2 \times 3 + 0 \times 4 + 0 \times 1 \\ 0 \times 3 + 0.5 \times 4 + 0 \times 1 \\ 0 \times 3 + 0 \times 4 + 1 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 \times 3 + 0 \times 4 + 5 \times 1 \\ 0 \times 3 + 1 \times 4 - 2 \times 1 \\ 0 \times 3 + 0 \times 4 + 1 \times 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

Section 8.5 (Example transformations)
Scaling:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

So $(0,0) \to (0,0)$; $(1,0) \to (4,0)$; $(0,1) \to (0,3)$.

Translation:

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix}$$

So $(0,0) \to (4,3)$; $(1,0) \to (5,3)$; $(0,1) \to (4,4)$.

Translation followed by scaling:

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 4 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$$

So $(0,0) \to (4,3)$; $(1,0) \to (8,3)$; $(0,1) \to (4,6)$.

Scaling followed by translation:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 16 \\ 0 & 3 & 9 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice how the translation has happened in the scaled universe, so that the points get moved in the original coordinate system by a distance of $(4 \times 4, 3 \times 3) = (16, 9)$.

Rotation by $45°$:
$(0,0) \to (0,0)$; $(1,0) \to (0.7071, 0.7071)$; $(0,1) \to (-0.7071, 0.7071)$; $(3,1) \to (1.4142, 2.8284)$; $(4,1) \to (2.1213, 3.5355)$; $(3,2) \to (0.7071, 3.5355)$; $(4,2) \to (2.1213, 4.2426)$.

You can check if your answers are right by looking at the diagram and seeing whether the points you produce are giving coordinates that match the drawn locations of the points.

Getting the square to rotate about its own centre:
Translate the universe to put the origin at the centre of the square, rotate the universe, translate the universe to put the origin back at the origin. So multiply these three matrices:

$$\begin{bmatrix} 1 & 0 & 3.5 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 0.7071 & -0.7071 & 0 \\ 0.7071 & 0.7071 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & -3.5 \\ 0 & 1 & -1.5 \\ 0 & 0 & 1 \end{bmatrix}$$

I find it easier to think of this set of transformations from a point's point of view. From the point's point of view you read this list of matrices backwards because the point is multiplied on the right, so it gets to see the rightmost matrix first. Let's work out what happens to a point: it first gets translated towards the origin by $(-3.5, -1.5)$ then gets rotated about the real origin by $45°$, then gets moved back by $(3.5, 1.5)$. Prove that this works by multiplying the three matrices together and then transforming the four vertices of the square. If you get it right, you should get the square rotated by $45°$ about the centre of the square, so: $(3,1) \to (3.5, 0.7929)$; $(4,1) \to (4.2071, 1.5)$; $(3,2) \to (2.7929, 1.5)$; $(4,2) \to (3.5, 2.2.071)$.