Interference Transform: Estimating the frequency and phase of low resolution samples

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Abstract

The Fourier Transform, naturally due to the context and time in which it was idealized, was formulated without regard to discrete waves or computational complexity. Some developments, such as the Discrete Fourier Transform and the Fast Fourier Transform algorithm updated the theory to the digital, discrete era. Some shortcomings were introduced with those methods, such as diminishing resolution in dealing with a small number of os samples. In this paper we propose a transform specifically designed for the discrete case, optimizing the inference of frequency and amplitude from whatever sampled points are available. The algorithm is based on the representation of the set of samples in a frequency versus phase space and the investigation of the interference pattern thus generated. Each sample defines a sinusoidal planar wave in this space, with a particular amplitude, direction and frequency. The determination of the point in which maximum constructive interference occurs translates to the determination of the most accurate sinusoidal description of the whole set of samples. TODO: improvement of the naïve algorithm. better accuracy TODO: an alternative to wavelet and short-time Fourier transform

Keywords: Fast Fourier Transform, Digital Signal Processing, Wavelet Transform

1 Introduction

2 Literature Review

3 Methodology

3.1 Theory

One can represent the discrete wave as an an array $W[i]$ of $n-1$ real numbers. Such wave is related, as illustrated in figure 1, to an hypothetical continuous wave $S(t)$ of which its components are sampled in uniform time intervals at a rate of fps frames per second.

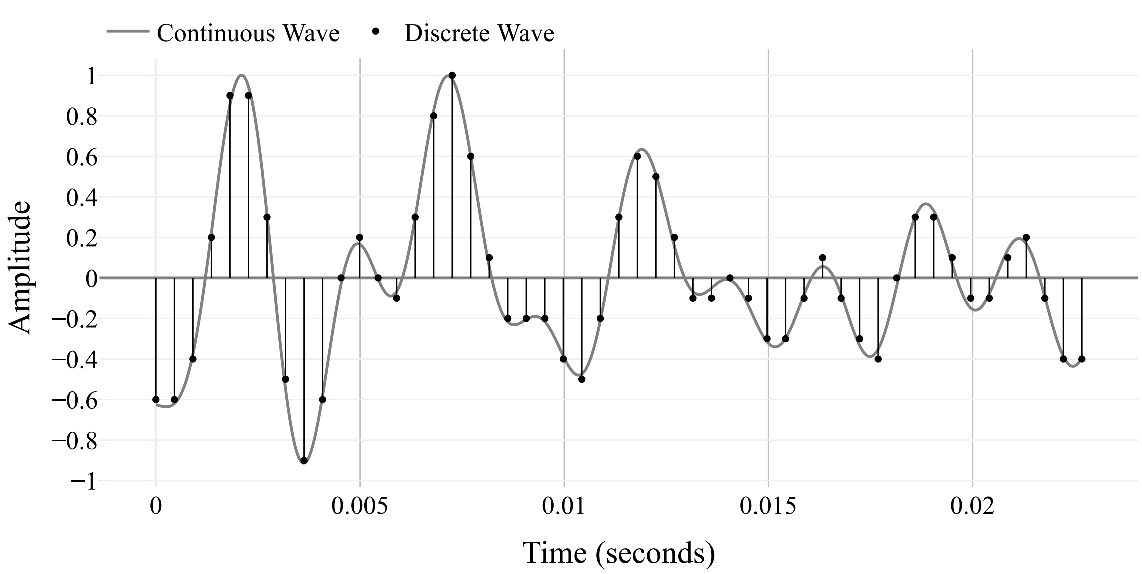

Figure 1: Discrete samples (black points) from a continuous wave (grey line). Note that the amplitude is also truncated

Thus, we have: $W[i] \in \mathbb{R}\ \forall\ i \in \mathbb{N}, 0 \le i < n$ $S(t) \in \mathbb{R}\ \forall\ t \in \mathbb{R}, 0 \le t < t_f$ where $t_f = \frac{n}{\text{fps}}$.

The relationship is better illustrated comparing two equivalent simple sinusoids; a continuous one and its discrete counterpart. Those can be described, respectively as

$$S(t) = a \cos\left(\phi + 2\pi \mathcal{F} t\right)$$

and

$$W[i] = a \cos\left(\phi + 2\pi f \frac{i}{n}\right)$$

where:
$\mathcal{F} \in \mathbb{R}^+$, absolute frequency of the continuous wave in Hz;
$f \in \mathbb{R}^+$, relative frequency of the discrete wave in cycles per n samples;
$f = \frac{n}{\text{fps}} \mathcal{F}$;
$i = t\ \text{fps}, i \in \mathbb{N}$, frame number;
$t \in \mathbb{R}$, time in seconds;
$n \in \mathbb{N}$, number os discrete samples;
$0 \le \frac{i}{n} < 1$;
$a \in \mathbb{R}^+$, amplitude;
$\phi \in \mathbb{R}^+, 0 \le \phi < 2\pi$, phase;

Figure 2 illustrates the relationship between the two. The values for the amplitude of both coincide, in contrast with the sampled example, as the case here is conceptually different: both sinusoids were constructed from identical parameters.

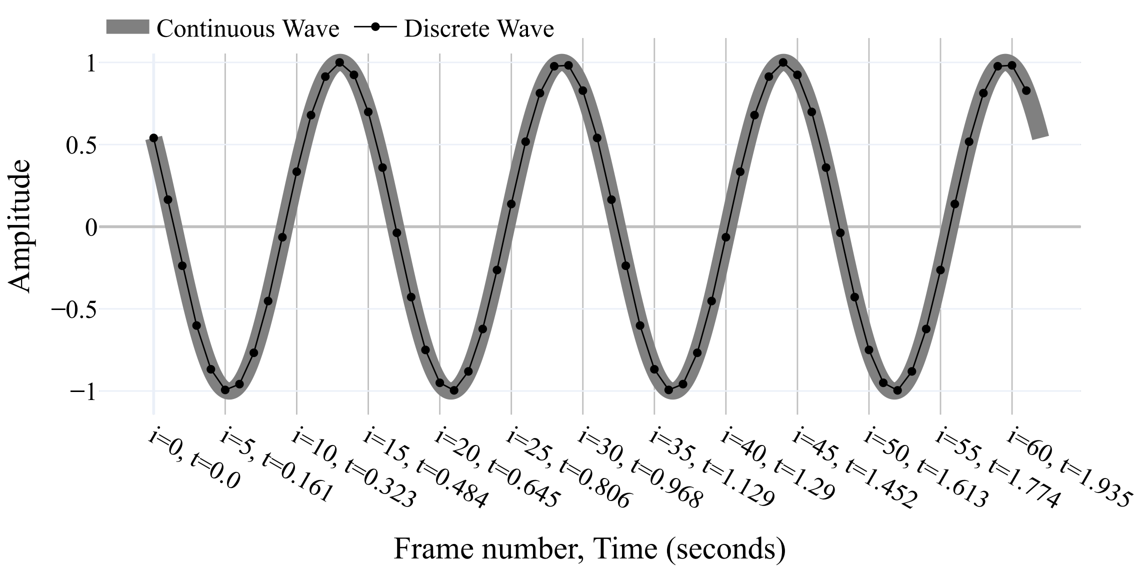

Figure 2: Continuous sinusoid (grey line) and the correspondent Discrete sinusoid (black points) illustrating the correspondence between frames i and time t

So it's easy to see that, for each frame of a given discrete wave $W[i]$, there exists an infinite family of continuous sinusoids that pass exactly through the point defined by i and $W[i]$. In fact, almost any sinusoid of the form $a \cos\left(\phi + 2\pi f \frac{i}{n}\right)$ can be forced to contain the point $(i, W[i])$, given a suitable choice of a, with the exception of the particular case where $W[i] \neq 0$ and $\cos\left(\phi + 2\pi \mathcal{F} \frac{i}{\text{fps}}\right)$. Of interest to this work are those sinusoids, as shown in Figure 3, that have one of their infinite extrema, either maximum or minimum, coinciding with the abscissa i or, setting $a = W[t]$, coincident with the point $(i, W[i])$.

One can define, for each frame, those waves mathematically as the family of continuous sinusoids that satisfy the equality $\cos\left(\phi + 2\pi \mathcal{F} t\right) = \text{sgn}(W[t])$ or, in terms of discrete sinusoids, $\cos\left(\phi + 2\pi f \frac{i}{n}\right) = \text{sgn}(W[t])$ where the sign function is defined as $\text{sgn}(x) = \{-1 \text{ if } x < 0,\ 0 \text{ if } x = 0,\ 1 \text{ if } x > 0\}$.

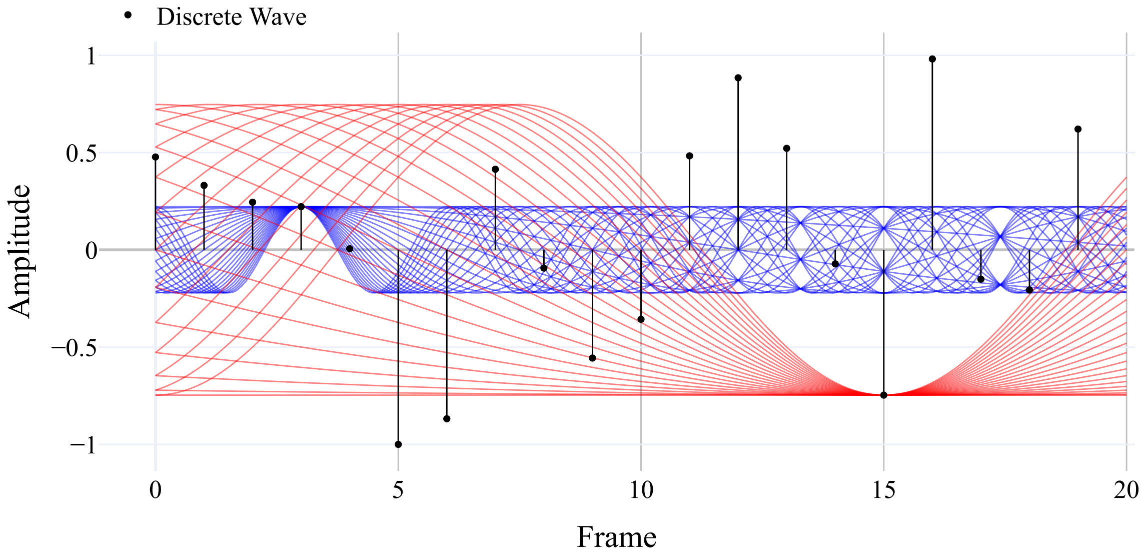

Figure 3: Sinusoids with maximum coinciding with sampled points i=3 and i=28

4 Results

5 Discussion + Conclusion

6 References