Cohesion in Rome

Fosco Loregian

TAL TECH

December 1, 2019

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Definizione di fascio su uno spazio

Let (X, τ) be a topological space; a *sheaf on X* is a functor $F : \tau^{\mathrm{op}} \to \underline{\mathrm{Set}}$ such that for every $U \in \tau$ and every covering $\{U_i\}$ of U one has

- if $s, t \in FU$ are such that $s|_i = t|_i$ in FU_i for every $i \in I$, then $s = t$ in FU.

[1] We denote $s|_i$ the image of $s \in FU$ under the nmeless map $FU \to FU_i$ induced by the inclusion $U_i \subseteq U$.

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- if $s_i \in FU_i$ is a family of elements such that $s_i|_{ij} = s_j|_{ij}$, then there exists a $s \in FU$ such that $s|_i = s_i$.[1]

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- if $S \Rightarrow yX$ is a covering sieve and $f : Y \to X$ is a morphism of $\mathcal{C}$, then the morphism $f^*S \Rightarrow Y$ obtained in the pullback

$$\begin{tikzcd} f^*S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

  is again a covering sieve.

Grothendieck topologies

- Let $S \Rightarrow yX$ be a covering sieve on X, and let T be any sieve on X. If for each object Y of $\mathcal{C}$ and each arrow $f : Y \to X$ in SY the pullback sieve f^*T is a covering sieve on Y, then T is a covering sieve on X.

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A Grothendieck site is a category with a Grothendieck topology, i.e. a function j that assigns to every object a family of covering sieves.

We denote a site as the pair $(\mathcal{C}, j)$.

Sheaves on a site

A *sheaf* on a small site $\mathcal{C}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \underline{\mathrm{Set}}$ such that for every covering sieve $R \to yU$ and every diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "m"'] & F \\ yU \arrow[ur, dotted] & \end{tikzcd}$$

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The full subcategory of sheaves on a site $(\mathcal{C}, j)$ is denoted $\mathrm{Sh}(\mathcal{C}, j)$.

Giraud Theorem

By general facts on locally presentable categories, the subcategory of sheaves on a site is reflective via a functor

$$r : \mathsf{Cat}(\mathcal{C}^{\mathsf{op}}, \underline{\mathsf{Set}}) \to \mathsf{Sh}(\mathcal{C}, j)$$

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Historical note

Grothendieck was the first to note that in every topos of sheaves the internal language is sufficiently expressive to concoct higher-order logic and he strived to advertise his intuitions to an audience of logicians.

But it wasn't until Lawvere devised the notion of elementary topos that the community agreed on the potential of this theory.

Elementary toposes

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- is cartesian closed;
- has a *subobject classifier*, i.e. an object $\Omega \in \mathcal{E}$ such that the functor $\mathrm{Sub} : \mathcal{E}^{\mathrm{op}} \to \underline{\mathrm{Set}}$ sending A into the set of isomorphism classes of monomorphisms $\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}$ is representable by the object Ω.

Elementary toposes

The natural bijection $\mathcal{E}(A, \Omega) \cong \mathrm{Sub}(A)$ is obtained pulling back a "characteristic arrow" $\chi_U : A \to \Omega$ along a *universal arrow* $t : 1 \to \Omega$ to obtain the monic U, as in the diagram

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so, the bijection is induced by the maps

- $\chi_- : \begin{bmatrix} U \\ \downarrow \\ A \end{bmatrix} \mapsto \chi_m$ and
- $- \times_\Omega t : \chi_U \mapsto \chi_U \times_\Omega t$.

Grothendieck $\subset$ elementary

> En los libros herméticos está escrito que lo que hay abajo es igual a lo que hay arriba, y lo que hay arriba, igual a lo que hay abajo; en el Zohar, que el mundo inferior es reflejo del superior.[†]
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†Microcosm principle: a topos, i.e. a place where subobjects are well-behaved, is but a well-behaved subobject in the 2-category of presheaf categories.

Logic of categories

What is the internal logic of a category?

This would have deserved a dedicated seminar but:

- Every category $\mathcal{C}$ is a universe in which we can interpret type theory;

This can be made precise in various ways:

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- every morphism $X \to A$ is a (generalised) term of type A, in a context X.

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Logic of toposes

flavor of type theory	equivalent to	flavor of category theory	
intuitionistic propositional logic/simply-typed lambda calculus		cartesian closed category	
multiplicative intuitionistic linear logic		symmetric closed monoidal category	(various authors since ~68)
first-order logic		hyperdoctrine	(Seely 1984a)
classical linear logic		star-autonomous category	(Seely 89)
extensional dependent type theory		locally cartesian closed category	(Seely 1984b)
homotopy type theory without univalence (intensional M-L dependent type theory)		locally cartesian closed (∞,1)-category	(Cisinski 12-(Shulman 12)
homotopy type theory with higher inductive types and univalence		elementary (∞,1)-topos	see here
dependent linear type theory		indexed monoidal category (with comprehension)	(Vákár 14)

Axiomatic Cohesion

What is cohesion

Cohesion is the mutual attraction of molecules sticking together to form *droplets*, caused by mild electrical attraction between them.

Figure 1: Droplets of mercury "exhibiting cohesion"

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Smooth manifolds can be probed via smooth open balls and every smooth space is a "coherent conglomerate" of *cohesive pieces*.

Question

Which formal axioms describe the mathematics behind this intuition? What is *axiomatic cohesion*?

Axioms to answer this question have been devised by Lawvere [Law1] (worth reading, but quite mystical!).

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- Discretely and codiscretely cohesive spaces embed in $\mathcal{H}$, with fully faithful functors: in that

$$\mathcal{H}(\mathrm{disc}(S), \mathrm{disc}(T)) \cong \underline{\mathrm{Set}}(S, T)$$

$$\mathcal{H}(\mathrm{codisc}(S), \mathrm{codisc}(T)) \cong \underline{\mathrm{Set}}(S, T)$$

Axiomatic cohesion

An adjunction

$$\Pi \dashv \mathrm{disc} \dashv \Gamma \dashv \mathrm{codisc} : \mathcal{H} \begin{tikzcd} {} \arrow[r, shift left=3, "\Pi"] & {} \arrow[l, shift left=1, "\mathrm{disc}"] \arrow[l, shift right=3, "\mathrm{codisc}"'] \end{tikzcd} \underline{\mathrm{Set}}$$

$$\mathcal{H} \begin{tikzcd}[column sep=huge] {} \arrow[r, shift left=4.5, "\Pi"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, phantom, shift left=0, "\perp"] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=1.5, "\Gamma"] & {} \arrow[l, shift right=-1.5, "\mathrm{disc}"] \arrow[l, shift left=4.5, "\mathrm{codisc}"] \end{tikzcd} \underline{\mathrm{Set}}$$

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exhibits the cohesion of $\mathcal{H}$ over <u>Set</u> if

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exhibits the cohesion of $\mathcal{H}$ over $\underline{\mathsf{Set}}$ if

- disc and codisc are fully faithful;
- the leftmost adjoint Π preserves finite products.

Axiomatic cohesion

An adjunction

$$\Pi \dashv \mathrm{disc} \dashv \Gamma \dashv \mathrm{codisc} : \mathcal{H} \begin{tikzcd} {} \arrow[r, shift left=3, "\Pi"] \arrow[r, leftarrow, shift left=1, "\mathrm{disc}"] \arrow[r, shift right=1, "\Gamma"] \arrow[r, leftarrow, shift right=3, "\mathrm{codisc}"'] & {} \end{tikzcd} \underline{\mathrm{Set}}$$

exhibits the cohesion of $\mathcal{H}$ over $\underline{\mathrm{Set}}$ if

- disc and codisc are fully faithful;
- the leftmost adjoint Π preserves finite products.

(Γ "forgets cohesion": it sends a space to its underlying set of points)

Formal fact. Every quadruple of adjoints induces a triple of adjoints.

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- There is an adjoint triple of idempotent co/monads on $\mathcal{H}$, induced by the cohesion:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, shift left=8, "\Pi"] \arrow[r, shift right=8, "\Gamma"'] & \underline{\mathrm{Set}} \arrow[l, "\mathrm{disc}" description] \arrow[r, shift left=8, "\mathrm{disc}"] \arrow[r, shift right=8, "\mathrm{codisc}"'] & \mathcal{H} \arrow[l, "\Gamma" description]
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\begin{tikzcd}[column sep=huge]
\mathcal{H} \arrow[r, shift left=8pt, "\Pi"] \arrow[r, phantom, shift left=4pt, "\perp"] \arrow[r, phantom, shift right=4pt, "\perp"] \arrow[r, shift right=8pt, "\Gamma"'] & \underline{\mathrm{Set}} \arrow[l, "\mathrm{disc}"] \arrow[r, shift left=8pt, "\mathrm{disc}"] \arrow[r, phantom, shift left=4pt, "\perp"] \arrow[r, phantom, shift right=4pt, "\perp"] \arrow[r, shift right=8pt, "\mathrm{codisc}"'] & \mathcal{H} \arrow[l, "\Gamma"]
\end{tikzcd}
$$

monad	comonad	monad
$\int$	$(-)^\flat$	$(-)^\sharp$
$\mathrm{disc} \circ \Pi$	$\mathrm{disc} \circ \Gamma$	$\mathrm{codisc} \circ \Gamma$
pron.: *shape*	pron.: *flat*	pron.: *sharp*

Modalities, pieces

The triple of adjoints

$$\begin{tikzcd} \mathcal{H} \arrow[r, "\flat" description] & \mathcal{H} \arrow[l, bend right, "\int"'] \arrow[l, bend left, "\sharp"] \end{tikzcd}$$

is called the shape, flat, sharp string of "co/modalities" (idempotent co/monads) for the cohesive topos $\mathcal{H}$.

The shape of $X \in \mathcal{H}$ is the discrete object on the "fundamental groupoid" of X. The adjunction $\Pi \dashv \mathrm{disc}$ has something to do with (topological) Galois theory.

Modalities, pieces

1. The flat functor corresponds to the **object of flat connections** on $X \in \mathcal{H}$: if G is a group,

$$\left\{ \substack{\text{principal} \\ \text{bundles on } X} \right\} \cong \left[X \to BG \right] \qquad \left\{ \substack{\text{flat con-} \\ \text{nections on } X} \right\} \cong \left\{ \begin{tikzcd} & \flat BG \arrow[d] \\ X \arrow[r] \arrow[ur, dotted] & BG \end{tikzcd} \right\}$$

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2. sharp of X, $\sharp X$, corresponds to the codiscrete object on the sets of points ΓX of X.
3. Co/discrete objects are precisely the objects for which $\flat X \cong X$, resp. $\sharp Y \cong Y$.

Every object fits in a "complex":

Definition
There is a canonical natural trasformation

$$\sharp X \xrightarrow{\epsilon_{(\mathbf{disc} \dashv \Gamma), X}} X \xrightarrow{\eta_{(\Pi \dashv \mathbf{disc}), X}} \int X$$

called the "points to pieces" map;

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It is a "comparison" between the action of Γ (send X into its "sections" or "set of points") and Π (send X into its "pieces" or "components").

- We say that **pieces have points** in the cohesive topos $\mathcal{H}$ (or that "$\mathcal{H}$ satisfies *Nullstellensatz*") if the points-to-pieces transformation $\alpha_X \colon \Gamma X \to \Pi X$ is surjective for all $X \in \mathcal{H}$.

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- We say that **discrete is concrete** in $\mathcal{H}$ if natural transformation whose components are

$$\mathrm{disc}(S) \to \mathrm{codisc}(\Gamma(\mathrm{disc}(S))) \cong \mathrm{codisc}(S)$$

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- ...and many others (see [Law]).

Discrete and concrete

Proposition

- The adjunctions $\Pi \dashv \text{disc}$ and $\Gamma \dashv \text{codisc}$ exhibit the subcategories of discrete and codiscrete objects as reflective subcategories of $\mathcal{H}$; these subcategories form exponential ideals in $\mathcal{H}$.
- If $\mathcal{H}$ exhibits cohesion, then $\underline{\text{Set}}$ is equivalent to the full subcategory of $\mathcal{H}$ whose objects are the X such that $\eta_{(\Gamma \dashv \text{codisc}),X} : X \to \text{codisc}(\Gamma(X))$ is an isomorphism.

Equivalently: every cohesive topos "contains the trivial cohesion of disconnected pieces ($= \underline{\text{Set}}$)".

Non-trivial fact

Whenever a topos arises, there's an interaction between logic and geometry.

Definition
A monomorphism $\psi\colon S \to A$ in a cohesive topos $\mathcal{H}$ is a proposition of type A in the internal logic of $\mathcal{H}$. We say that ψ is *discretely true* if the pullback $\psi^*(S) \to A$

$$\begin{tikzcd} \psi^*(S) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \flat S \arrow[d, "\flat\psi"] \\ A \arrow[r, "\eta"'] & \flat A \end{tikzcd}$$

is an isomorphism in $\mathcal{H}$, where $\eta : A \to \flat A$ is the $\flat$-unit of the flat monad.

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- Let $\psi \colon Z^p(U) \hookrightarrow \Omega^p(U)$ be the proposition in $\mathcal{H}$ given by "the p-form ω is closed on a neighbourhood U_x of a point". Then ψ is discretely true ("every form is closed over a discrete space").

Examples

The Sierpiński topos

Let $\mathcal{C} = \{0 \to 1\}$ be the interval category with a unique non-identity arrow.

The category of presheaves on $\mathcal{C}$ forms a topos $\mathcal{H} = \underline{\text{Set}}^{\mathcal{C}} =$ arrows in $\underline{\text{Set}}$, that exhibits cohesion:

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- the functor disc sends a set K into the identity $1\colon K \to K$;
- the functor codisc sends a set K into its terminal morphism $K \to *$.

The Sierpiński topos

Evidently these functors form an adjunction ($\Pi \dashv \mathrm{disc} \dashv \Gamma \dashv \mathrm{codisc}$) so that $\mathcal{H}$ exhibits cohesion; this matches our intuition, in that

- The "points to pieces" transformation sends $f : S \to I$ into $S = \Gamma(f) \to \Pi(f) = I$;
- $\mathrm{disc}(K)$ "keeps all the pieces of K maximally distinguished" and
- $\mathrm{codisc}(K)$ "lumps all the pieces of K together".

Pointed categories exhibit cohesion

Let $\mathcal{C}$ small and with a terminal object. Then there exists a triple

$$\begin{tikzcd}[\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}] \arrow[r, shift left=6, "\varinjlim"] \arrow[r, shift right=6, "\varprojlim"'] & \underline{\mathrm{Set}} \arrow[l, "\mathrm{const}" description]\end{tikzcd}$$

that extends to $\varprojlim \dashv \text{よ}$:

$$S \overset{\text{よ}}{\mapsto} \Big(c \mapsto \underline{\mathrm{Set}}(\mathcal{C}(*, c), S) \Big)$$

(Dually, if $\mathcal{C}$ has an initial object. . .)

Pointed categories exhibit cohesion

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Proposition

If $\mathcal{C}$ has both an initial and a terminal object (e.g. it is pointed) then $[\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}]$ exhibits cohesion with

$$(\varinjlim \dashv \mathrm{const} \dashv \varprojlim \dashv \text{ユ}) \colon [\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}] \overset{\mathrm{const}}{\leftrightarrows} \underline{\mathrm{Set}}$$

Reflexive directed graphs

Consider the category $\mathcal{C}$

$$0 \underset{t}{\overset{s}{\underset{\longleftarrow q \longrightarrow}{\rightrightarrows}}} 1$$

such that $0 \rightrightarrows 1 \to 0$ is a reflexive coequalizer;

The category $[\mathcal{C}^{\mathrm{op}}, \underline{\mathrm{Set}}]$ is the category of reflexive directed graphs RDGph, and it exhibits cohesion, since the terminal geometric morphism

$$(\mathrm{disc} \dashv \Gamma) \colon \mathrm{RDGph} \leftrightarrows \underline{\mathrm{Set}}$$

extends on the left with $\Pi \colon X \mapsto \mathrm{coeq}\Big(X_1 \rightrightarrows X_0\Big)$ (simply the connected components of the graph).

Since this is a reflexive coequalizer, it preserves products.

(exercise: define $\mathrm{disc}, \mathrm{codisc}$)

Simplicial sets

Proposition
Let $\mathbf{\Delta}$ be the simplex category having objects nonempty finite ordinals and morphisms monotone maps. The topos $\mathcal{H} = [\mathbf{\Delta}^{\mathrm{op}}, \underline{\mathrm{Set}}]$ exhibits cohesion, and in $\mathcal{H}$ pieces have points.

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- disc sends a set S into the constant simplicial set in S having constant set of simplices and identities as faces and degeneracies.
- codisc sends a set S into the simplicial set whose n-simplices are $(n+1)$-tuples of elements of S (and faces and degeneracies forget and add elements accordingly).

Tangent cohesion

Consider the codomain fibration

$$\mathcal{C}^{\to} \xrightarrow{p} \mathcal{C}$$

of a finitely complete category $\mathcal{C}$, sending an arrow $f\colon X \to Y$ to its codomain. The fiber $p^{\leftarrow}(Y)$ is canonically isomorphic to the category $\mathcal{C}/Y$ of arrows over Y.

There exists a fibration $T\mathcal{C} \to \mathcal{C}$ having typical fiber the fiberwise abelianization of $\mathcal{C}/Y$, i.e. the category $\mathrm{Ab}(\mathcal{C}/Y)$ of abelian groups in $\mathcal{C}/Y$.

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Proposition
If $\mathcal{C}$ is locally presentable, then so is $T\mathcal{C}$; moreover, the projection $q \colon T\mathcal{C} \to \mathcal{C}$ creates co/limits.

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Proposition
If $\mathcal{C}$ is a topos over $\mathcal{S}$, then so is $T\mathcal{C}$; moreover, the projection $q\colon T\mathcal{C} \to \mathcal{C}$ creates co/limits.

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Proposition
There is a functor $\delta \colon T\mathcal{C} \to \mathcal{C}$ giving for each morphism in $T\mathcal{C}$ its domain. This functor is a right adjoint to a functor $\Omega \colon \mathcal{C} \to T\mathcal{C}$ that is also a *section* for q.

The object $\Omega(A)$ can be thought as the complex of differential forms on an internal abelian group $A \in \mathrm{Ab}(\mathcal{C}/X)$.

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Proposition
If $\mathcal{H}$ is a cohesive topos with cohesion $(\Pi \dashv \mathrm{disc} \dashv \Gamma \dashv \mathrm{codisc})$, then the tangent category is itself a cohesive topos.

Infinitesimal cohesion

Neighbourhoods of some spaces are “infintesimally extended around a single (global) point”. Cohesive structure can be refined to capture this phenomenon.

Infinitesimal cohesion

Let $\mathcal{H}$ be cohesive. An infinitesimal thickening of $\mathcal{H}$ is a new cohesive topos $\widetilde{\mathcal{H}}$ linked to the previous by a quadruple of adjoints

$$\begin{tikzcd} \mathcal{H} \arrow[d, hook, "(i_! \dashv i^* \dashv i_! \dashv i^!)"'] \\ \widetilde{\mathcal{H}} \end{tikzcd}$$

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such that $i_*, i_!$ are fully faithful and $i_!$ commutes with finite products.

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The functor i_* is fully faithful as well, and then we can consider $\mathcal{H}$ "sitting nicely" inside its thickening $\widetilde{\mathcal{H}}$.

Infinitesimal cohesion

- The cohesion exhibited by $\widetilde{\mathcal{H}}$ factors through that of $\mathcal{H}$, in that

$$(\Pi_{\widetilde{\mathcal{H}}} \dashv \mathrm{disc}_{\widetilde{\mathcal{H}}} \dashv \Gamma_{\widetilde{\mathcal{H}}}) \colon \; \begin{tikzcd} \widetilde{\mathcal{H}} \arrow[r, shift left=2, "i^*"] \arrow[r, shift right=2, "i^!"'] & \mathcal{H} \arrow[l, "i_*" description] \arrow[r, shift left=2, "\Pi"] \arrow[r, shift right=2, "\Gamma"'] & \underline{\mathrm{Set}} \arrow[l, "\mathrm{disc}" description] \end{tikzcd}$$

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- **Infinitesimal** cohesion describes formally infinitesimally extended neighbourhoods: if the functor i^* is interpreted as a contraction of a fat point onto its singleton, then $X \in \widetilde{\mathcal{H}}$ is infinitesimal if $i^*(X) \cong *$.

Infinitesimal cohesion

- The cohesion exhibited by $\widetilde{\mathcal{H}}$ factors through that of $\mathcal{H}$, in that

$$(\Pi_{\widetilde{\mathcal{H}}} \dashv \mathrm{disc}_{\widetilde{\mathcal{H}}} \dashv \Gamma_{\widetilde{\mathcal{H}}}) \colon \begin{tikzcd} \widetilde{\mathcal{H}} \arrow[r, shift left=4, "i^*"] \arrow[r, shift right=4, "i^!"'] & \mathcal{H} \arrow[l, "i_*" description] \arrow[r, shift left=4, "\Pi"] \arrow[r, shift right=4, "\Gamma"'] & \underline{\mathrm{Set}} \arrow[l, "\mathrm{disc}" description] \end{tikzcd}$$

- **Infinitesimal** cohesion describes formally infinitesimally extended neighbourhoods: if the functor i^* is interpreted as a contraction of a fat point onto its singleton, then $X \in \widetilde{\mathcal{H}}$ is infinitesimal if $i^*(X) \cong *$. This motivates the fact that

$$\widetilde{\mathcal{H}}(*, X) \cong \widetilde{\mathcal{H}}(i_!(*), X) \cong \mathcal{H}(*, i^*(X)) \cong \mathcal{H}(*, *) \cong *$$

so that $\mathcal{H}$ sees X as a "small neighbourhood concentrated around a single point $*_X$".

Higher order cohesion: jet spaces

Most examples of infinitesimal cohesions come equipped with an infinite chain of thickening approximations.

Consider the infinitesimal shape modality $\Im := i_* i^*$
(it comes equipped with other two adjoints, $\Re \dashv \Im \dashv \&$)[2]

[2]This is the same general fact inducing $\int \dashv \flat \dashv \sharp$ adjunction.

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In several cases (like smooth manifolds) we have a **chain** of infinitesimal thickenings

$$
\begin{tikzcd}
\widetilde{\mathcal{H}}_0 \arrow[r, hook, shift left=3] \arrow[r, hook, shift right=1, "i_{*,(0)}"'] & \widetilde{\mathcal{H}}_1 \arrow[l, shift right=1, "i^*_{(0)}"'] \arrow[l, shift left=3] \arrow[r, hook, shift left=3] \arrow[r, hook, shift right=1, "i_{*,(1)}"'] & \widetilde{\mathcal{H}}_2 \arrow[l, shift right=1, "i^*_{(1)}"'] \arrow[l, shift left=3] \arrow[r, hook, shift left=3] \arrow[r, hook, shift right=1, "i_{*,(2)}"'] & \cdots \arrow[l, shift right=1, "i^*_{(2)}"'] \arrow[l, shift left=3] \arrow[r, hook, shift left=3] \arrow[r, hook, shift right=1, "i_{*,\#}"'] & \widetilde{\mathcal{H}}_\infty \arrow[l, shift right=1, "i^*_{\#}"'] \arrow[l, shift left=3] \arrow[r, hook, shift left=3] \arrow[r, hook, shift right=1, "i_{*,(\infty)}"'] & \mathcal{H} \arrow[l, shift right=1, "i^*_{(\infty)}"'] \arrow[l, shift left=3]
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\end{tikzcd}$$

here we speak of a sequence of orders of differential structures.

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Higher order cohesion: jet spaces

Each of these approximations comes equipped with an *order k infinitesimal shape* modality $\Im^{(k)}X$ in a sequence

$$X \to \Im X = \Im^{(0)}X \to \Im^{(1)}X \to \Im^{(2)}X \to \cdots$$

Example: Every cohesive topos exhibits infinitesimal cohesion via its **tangent** cohesive topos. This cohesion extends to any order of differential structure ("cohesive jet spaces").

One can go way further, but the terminology becomes pretty dire:

Remark 2.2.13. The perspective of def. 2.2.12 has been highlighted in [Law91], where it is proposed (p. 7) that adjunctions of this form usefully formalize "many instances of the *Unity and Identity of Opposites*" that control Hegelian metaphysics [He1841].

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Supergeometry: rheonomy

We can speak of supergeometry and show that certain categories of supersmooth manifolds exhibit cohesion (but not over Set. . .):

$$\begin{tikzcd}
\mathsf{SuperSmoothS} \arrow[d, shift right=4.5, "\Pi"'] \arrow[d, shift left=1.5, "\Gamma"'] \\
\mathsf{SuperS} \arrow[u, shift right=1.5, "d"'] \arrow[u, shift right=-4.5, "c"']
\end{tikzcd}$$

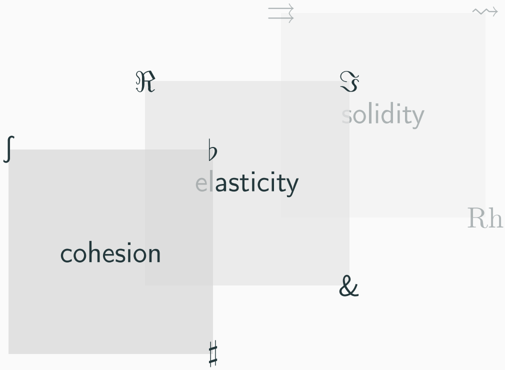
The quadruple of adjoints generates the triple

$$\rightrightarrows \quad \dashv \quad \rightsquigarrow \quad \dashv \quad \mathrm{Rh}$$

(in some sense “fermions” $\dashv$ “bosons”)

Supergeometry: rheonomy

There is a "quadruple-to-triple" pattern here:



de Rham cohomology in cohesion

- Let $\mathcal{H}$ be a cohesive topos, and $0 \to A$ a pointed object (e.g. an internal abelian group); then, A fits into a pullback square

$$\begin{tikzcd} \flat_{\mathrm{dR}} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \flat A \arrow[d] \\ 0 \arrow[r] & A \end{tikzcd}$$

  where $\flat_{\mathrm{dR}} A$ is the object of coefficients for de Rham cohomology.

- Let $X \in \mathcal{H}$ any object; we define $\int_{\mathrm{dR}} X$ to be the pushout

$$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \arrow[d] \\ \int X \arrow[r] & \int_{\mathrm{dR}} X \end{tikzcd}$$

  where $\int_{\mathrm{dR}} X$ is the de Rham object associated to X.

de Rham cohomology in cohesion

There is an adjunction

$$\begin{tikzcd} */\mathcal{H} \arrow[r, shift left, "\flat_{\mathsf{dR}}"] & \mathcal{H} \arrow[l, shift left, "\int_{\mathsf{dR}}"] \end{tikzcd}$$

The mapping space $*/\mathcal{H}(\int_{\mathsf{dR}} X, A) \cong \mathcal{H}(X, \flat_{\mathsf{dR}} A)$ is called the de Rham space of X with coefficients in A and denoted $\mathbf{H}^0_{\mathsf{dR}}(X, A)$.

de Rham cohomology in cohesion

Consider the pullback defining $\flat_{\mathrm{dR}} A$ and apply the limit-preserving functor $\mathcal{H}(X, -)$: the square

$$\begin{tikzcd} \mathcal{H}(X, \flat_{\mathrm{dR}} A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{H}(X, \flat A) \arrow[d] \\ 0 \arrow[r] & \mathcal{H}(X, A) \end{tikzcd}$$

remains a pullback and the object $\mathcal{H}(X, \flat A)$ identifies to A-valued differential forms, and the maps $X \to \flat_{\mathrm{dR}} A$ are the flat ones: under the $\int \dashv \flat$ adjunction, a map $X \to \flat A$ mates to a smooth map $\int X \to A$, corresponding to a square $\begin{smallmatrix} X & \to & 0 \\ \downarrow & & \downarrow \\ \int X & \to & A \end{smallmatrix}$

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