COHESION

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1. Introduction

Definition 1.1.

Definition 1.2. Let (X, τ) be a topological space; a *sheaf on* X is a functor $F : \tau^{\mathrm{op}} \to \mathsf{Set}$ such that for every $u \in \tau$ and every covering $\{U_i\}$ of U one has

Definition 1.3. A *sieve* on an object X of a category $\mathcal{C}$ is a subobject of the hom functor $yX = \mathcal{C}(\_, X)$; a *Grothendieck topology* on a category amounts to the choice of a family of *covering sieve* for every object $X \in \mathcal{C}$; this family is chosen in such a way that

- if $S \Rightarrow yX$ is a covering sieve and $fY \to X$ is a morphism of $\mathcal{C}$, then the morphism $f^S \Rightarrow Y$ obtained in

$$\begin{tikzcd} f^*S \arrow[dr, phantom, "\lrcorner", very near start] & RS \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

  is again a covering sieve.
- Let $S \Rightarrow yX$ be a covering sieve on X, and let T be any sieve on X. If for each object Y of $\mathcal{C}$ and each arrow $f : Y \to X$ in SY the pullback sieve f^*T is a covering sieve on Y. Then T is a covering sieve on X.
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Definition 1.4. A sheaf on a small site $\mathcal{C}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ such that for every covering sieve $R \to y(U)$ and every diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "m"'] & F \\ y(U) \arrow[ur, dotted] & \end{tikzcd}$$

there is a unique dotted extension $y(U) \Rightarrow F$ (by the Yoneda lemma, this consists of a unique element $s \in FU$). The full subcategory of sheaves on a site $(\mathcal{C}, j)$ is denoted $\mathrm{Sh}(\mathcal{C}, j)$.

By general facts on locally presentable categories, the subcategory of sheaves on a site is reflective via a functor

$$r : \mathsf{Cat}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set}) \to \mathrm{Sh}(\mathcal{C}, j)$$

called *sheafification* of a presheaf $F : \mathcal{C}^{\text{op}} \to \mathsf{Set}$.

Grothendieck was the first to note that in every topos of sheaves the internal language is sufficiently expressive to concoct higher-order logic and he strived to advertise his intuitions to an audience of logicians. But it wasn't until Lawvere devised the notion of *elementary topos* that the community agreed on the potential of this theory.

Definition 1.5. An *elementary topos* is a category $\mathcal{E}$ that

- is finitely complete (i.e. it admits finite products and equalizers, or a terminal object and pullbacks, or all limits of diagrams $D : \mathcal{J} \to \mathcal{E}$ where $\mathcal{J}$ is a finite category);
- is cartesian closed, i.e. the functor $A \times \_$ has a right adjoint $[A, \_]$ for every object $A \in \mathcal{E}$
- has a *subobject classifier*, i.e. an object $\Omega \in \mathcal{E}$ such that the functor $\mathrm{Sub} : \mathcal{E}^{\text{op}} \to \mathsf{Set}$ sending A into the set of isomorphism classes of monomorphisms $\left[\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}\right]$ is representable by the object Ω.

The natural bijection $\mathcal{E}(A, \Omega) \cong \mathrm{Sub}(A)$ is obtained pulling back the monomorphism $U \subseteq A$ along a *universal arrow* $t : 1 \to \Omega$, as in the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "t"] \\ A \arrow[r, "\chi_U"'] & \Omega \end{tikzcd}$$

so, the bijection is induced by the map $m \mapsto \chi_U$.

Definition 1.6. copincolla da nLab

2. Cohesion

intuition for Cohesion
cohesive topos
properties and thms
classes of cohesive toposes
"moments of opposition"
cohesion in smooth homotopy
de Rham in a cohesive topos