Cohesion in Rome

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TAL TECH

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Outline

> [. . .] vi el Aleph, desde todos los puntos, vi en el Aleph la tierra, y en la tierra otra vez el Aleph y en el Aleph la tierra, vi mi cara y mis vísceras, vi tu cara, y sentí vértigo y lloré. . .
>
> JLB

Topos theory is a cornerstone of category theory linking together algebra, geometry and logic.

Simply said, in each topos it is possible to re-enact the totality of known Mathematics; today we focus on

- Logic (better said, a fragment of dependent type theory)
- Differential geometry
- Algebraic topology
- . . .

Topos theory

Definizione di fascio su uno spazio

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- if $s, t \in FU$ are such that $s|_i = t|_i$ in FU_i for every $i \in I$, then $s = t$ in FU.

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- if $s_i \in FU_i$ is a family of elements such that $s_i|_{ij} = s_j|_{ij}$, then there exists a $s \in FU$ such that $s|_i = s_i$.[1]

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Definizione di topa di Groto

A *sieve* on an object X of a category $\mathcal{C}$ is a subobject S of the hom functor $yX = \mathcal{C}(\_, X)$;

A *Grothendieck topology* on a category amounts to the choice of a family of *covering sieves* for every object $X \in \mathcal{C}$; this family of sieves is chosen in such a way that

- if $S \Rightarrow yX$ is a covering sieve and $f : Y \to X$ is a morphism of $\mathcal{C}$, then the morphism $f^*S \Rightarrow Y$ obtained in the pullback

$$\begin{tikzcd} f^*S \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d] \\ Y \arrow[r, "f"'] & X \end{tikzcd}$$

  is again a covering sieve.

- Let $S \Rightarrow yX$ be a covering sieve on X, and let T be any sieve on X. If for each object Y of $\mathcal{C}$ and each arrow $f : Y \to X$ in SY the pullback sieve f^*T is a covering sieve on Y, then T is a covering sieve on X.
- the identity $1 : yX \Rightarrow yX$ is a covering sieve.

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A Grothendieck site is a category with a Grothendieck topology, i.e. a function j that assigns to every object a family of covering sieves.

We denote a site as the pair $(\mathcal{C}, j)$.

Definizione di fascio su un sito

A *sheaf* on a small site $\mathcal{C}$ is a functor $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ such that for every covering sieve $R \to yU$ and every diagram

$$\begin{tikzcd} R \arrow[r, "f"] \arrow[d, "m"'] & F \\ yU \arrow[ur, dotted] & \end{tikzcd}$$

there is a unique dotted extension $yU \Rightarrow F$ (by the Yoneda lemma, this consists of a unique element $s \in FU$, **exercise**).

The full subcategory of sheaves on a site $(\mathcal{C}, j)$ is denoted $\mathrm{Sh}(\mathcal{C}, j)$.

Cat dei fasci è riflessiva, Giraud

By general facts on locally presentable categories, the subcategory of sheaves on a site is reflective via a functor

$$r : \mathbf{Cat}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set}) \to \mathrm{Sh}(\mathcal{C}, j)$$

called *sheafification* of a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$.

Historical note

Grothendieck was the first to note that in every topos of sheaves the internal language is sufficiently expressive to concoct higher-order logic and he strived to advertise his intuitions to an audience of logicians.

But it wasn't until Lawvere devised the notion of elementary topos that the community agreed on the potential of this theory.

Definizione di topos elementare

An *elementary topos* is a category $\mathcal{E}$ that

- it has finite limits;
- is cartesian closed;
- has a *subobject classifier*, i.e. an object $\Omega \in \mathcal{E}$ such that the functor $\mathrm{Sub} : \mathcal{E}^{\mathrm{op}} \to \mathbf{Set}$ sending A into the set of isomorphism classes of monomorphisms $\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}$ is representable by the object Ω.

The natural bijection $\mathcal{E}(A, \Omega) \cong \mathrm{Sub}(A)$ is obtained pulling back the monomorphism $U \subseteq A$ along a *universal arrow* $t : 1 \to \Omega$, as in the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d, "m"'] \arrow[dr, phantom, "\lrcorner"] & 1 \arrow[d, "t"] \\ A \arrow[r, "\chi_m"'] & \Omega \end{tikzcd}$$

so, the bijection is induced by the maps

- $\chi_- : \begin{bmatrix} U \\ \downarrow \\ A \end{bmatrix} \mapsto \chi_m$ and
- $- \times_\Omega t : \chi_U \mapsto \chi_U \times_\Omega t$.

Groto = elem + locpres (finito)

Cohesion

Intuizione sulla coesione

Definizione di coesione

Proprietà degli assiomi di coesione

"Momenti di opposizione"

Geometria continua, supergeometria, geometria rigida. . .

Un esempio workato out: de Rham in coesione