

Cohesion in Rome

Fosco Loregian



December 10, 2019

Toposes

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vi en el Aleph la tierra, y en la tierra otra
vez el Aleph y en el Aleph la tierra, vi mi
cara y mis vísceras, vi tu cara, y sentí
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Definizione di fascio su uno spazio

Let (X, τ) be a topological space; a *sheaf on X* is a functor $F : \tau^{\text{op}} \rightarrow \underline{\text{Set}}$ such that for every $U \in \tau$ and every covering $\{U_i\}$ of U one has

- if $s, t \in FU$ are such that $s|_i = t|_i$ in FU_i for every $i \in I$, then $s = t$ in FU .

¹We denote $s|_i$ the image of $s \in FU$ under the nmeless map $FU \rightarrow FU_i$ induced by the inclusion $U_i \subseteq U$.

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- if $s_i \in FU_i$ is a family of elements such that $s_i|_{ij} = s_j|_{ij}$, then there exists a $s \in FU$ such that $s|_i = s_i$.¹

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- if $S \Rightarrow yX$ is a covering sieve and $f : Y \rightarrow X$ is a morphism of \mathcal{C} , then the morphism $f^*S \Rightarrow Y$ obtained in the pullback

$$\begin{array}{ccc} f^*S & \longrightarrow & S \\ \downarrow & \lrcorner & \downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

is again a covering sieve.

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- Let $S \Rightarrow yX$ be a covering sieve on X , and let T be any sieve on X . If for each object Y of \mathcal{C} and each arrow $f : Y \rightarrow X$ in SY the pullback sieve $f^* T$ is a covering sieve on Y , then T is a covering sieve on X .

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A Grothendieck site is a category with a Grothendieck topology, i.e. a function j that assigns to every object a family of covering sieves.

We denote a site as the pair (\mathcal{C}, j) .

Sheaves on a site

A *sheaf* on a small site \mathcal{C} is a functor $F : \mathcal{C}^{\text{op}} \rightarrow \underline{\text{Set}}$ such that for every covering sieve $R \rightarrow yU$ and every diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & F \\ m \downarrow & \nearrow & \\ yU & & \end{array}$$

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The full subcategory of sheaves on a site (\mathcal{C}, j) is denoted $\text{Sh}(\mathcal{C}, j)$.

Giraud Theorem

By general facts on locally presentable categories, the subcategory of sheaves on a site is reflective via a functor

$$r : \text{Cat}(\mathcal{C}^{\text{op}}, \underline{\text{Set}}) \rightarrow \text{Sh}(\mathcal{C}, j)$$

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Historical note

Grothendieck was the first to note that in every topos of sheaves the **internal language** is sufficiently expressive to concoct **higher-order logic** and he strived to advertise his intuitions to an audience of logicians.

But it wasn't until Lawvere devised the notion of **elementary topos** that the community agreed on the potential of this theory.

Elementary toposes

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- is cartesian closed;
- has a *subobject classifier*, i.e. an object $\Omega \in \mathcal{E}$ such that the functor $\text{Sub} : \mathcal{E}^{\text{op}} \rightarrow \underline{\text{Set}}$ sending A into the set of isomorphism classes of monomorphisms $\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}$ is representable by the object Ω .

Elementary toposes

The natural bijection $\mathcal{E}(A, \Omega) \cong \text{Sub}(A)$ is obtained pulling back a “characteristic arrow” $\chi_U : A \rightarrow \Omega$ along a *universal arrow* $t : 1 \rightarrow \Omega$ to obtain the monic U , as in the diagram

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- $\chi_- : \begin{bmatrix} U \\ \downarrow \\ A \end{bmatrix} \mapsto \chi_m$ and
- $- \times_{\Omega} t : \chi_U \mapsto \chi_U \times_{\Omega} t.$

Grothendieck \subset elementary

En los libros herméticos está escrito que lo que hay abajo es igual a lo que hay arriba, y lo que hay arriba, igual a lo que hay abajo; en el Zohar, que el mundo inferior es reflejo del superior.[†]

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[†]Microcosm principle: a topos, i.e. a place where subobjects are well-behaved, is but a well-behaved subobject in the 2-category of presheaf categories.

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This can be made precise in various ways:

<https://ncatlab.org/nlab/show/relation+between+type+theory+and+category+theory>

Logic of toposes

flavor of type theory	equivalent to	flavor of category theory	
intuitionistic propositional logic/simply-typed lambda calculus		cartesian closed category	
multiplicative intuitionistic linear logic		symmetric closed monoidal category	(various authors since ~68)
first-order logic		hyperdoctrine	(Seely 1984a)
classical linear logic		star-autonomous category	(Seely 89)
extensional dependent type theory		locally cartesian closed category	(Seely 1984b)
homotopy type theory without univalence (intensional M-L dependent type theory)		locally cartesian closed $(\infty, 1)$-category	(Cisinski 12- (Shulman 12))
homotopy type theory with higher inductive types and univalence		elementary $(\infty, 1)$-topos	see here
dependent linear type theory		indexed monoidal category (with comprehension)	(Vákár 14)

Axiomatic Cohesion

What is cohesion

Cohesion is the mutual attraction of molecules sticking together to form *droplets*, caused by mild electrical attraction between them.

Figure 1: Droplets of mercury “exhibiting cohesion”

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Question

Which formal axioms describe the mathematics behind this intuition? What is *axiomatic cohesion*?

Axioms to answer this question have been devised by Lawvere [Law1] (worth reading, but quite mystical!).

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- Discretely and codiscretely cohesive spaces embed in \mathcal{H} , with fully faithful functors: in that

$$\mathcal{H}(\text{disc}(S), \text{disc}(T)) \cong \underline{\text{Set}}(S, T)$$

$$\mathcal{H}(\text{codisc}(S), \text{codisc}(T)) \cong \underline{\text{Set}}(S, T)$$

Axiomatic cohesion

An adjunction

$$\Pi \dashv \text{disc} \dashv \Gamma \dashv \text{codisc} : \mathcal{H}$$
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(Γ “forgets cohesion”: it sends a space to its underlying set of points)

Formal fact. Every quadruple of adjoints induces a triple of adjoints.

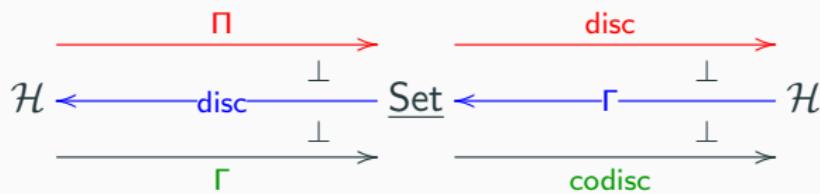
Formal fact. Every quadruple of adjoints induces a triple of adjoints.

- There is an adjoint triple of idempotent co/monads on \mathcal{H} , induced by the cohesion:

$$\begin{array}{ccccc} & \xrightarrow{\Pi} & & \xrightarrow{\text{disc}} & \\ \mathcal{H} & \xleftarrow{\text{disc}} & \perp & \perp & \mathcal{H} \\ & \xleftarrow{\perp} & \text{Set} & \xleftarrow{\Gamma} & \perp \\ & \xrightarrow{\Gamma} & & \xrightarrow{\perp} & \perp \\ & & \xrightarrow{\text{codisc}} & & \end{array}$$

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monad	comonad	monad
\int	$(-)^b$	$(-)^{\sharp}$
$\text{disc} \circ \Pi$	$\text{disc} \circ \Gamma$	$\text{codisc} \circ \Gamma$
pron.: <i>shape</i>	pron.: <i>flat</i>	pron.: <i>sharp</i>

Modalities, pieces

The triple of adjoints

$$\begin{array}{ccc} & \lrcorner & \\ \mathcal{H} & \xrightleftharpoons{\quad} & \mathcal{H} \\ & \sharp & \end{array}$$

is called the **shape, flat, sharp** string of “co/modalities”
(idempotent co/monads) for the cohesive topos \mathcal{H} .

The **shape** of $X \in \mathcal{H}$ is the discrete object on the “fundamental groupoid” of X . The adjunction $\Pi \dashv \text{disc}$ has something to do with (topological) Galois theory.

Modalities, pieces

1. The **flat** functor corresponds to the **object of flat connections** on $X \in \mathcal{H}$: if G is a group,

$$\left\{ \begin{smallmatrix} \text{principal} \\ \text{bundles on } X \end{smallmatrix} \right\} \cong [X \rightarrow BG] \quad \left\{ \begin{smallmatrix} \text{flat con-} \\ \text{nections on } X \end{smallmatrix} \right\} \cong \left\{ \begin{array}{c} \nearrow \text{flat} \\ X \xrightarrow{\quad} BG \\ \downarrow \end{array} \right\}$$

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(keep in mind these equivalences: they will reappear later)

2. **sharp** of X , $\sharp X$, corresponds to the codiscrete object on the sets of **points** ΓX of X .
3. Co/discrete objects are precisely the objects for which $bX \cong X$, resp. $\sharp Y \cong Y$.

Every object fits in a “complex”:

Definition

There is a canonical natural transformation

$$\sharp X \xrightarrow{\epsilon_{(\text{disc} \dashv \Gamma), X}} X \xrightarrow{\eta_{(\Pi \dashv \text{disc}), X}} \int X$$

called the “points to pieces” map;

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It is a “comparison” between the action of Γ (send X into its “sections” or “set of points”) and Π (send X into its “pieces” or “components”).

- We say that **pieces have points** in the cohesive topos \mathcal{H} (or that “ \mathcal{H} satisfies *Nullstellensatz*”) if the points-to-pieces transformation $\alpha_X : \Gamma X \rightarrow \Pi X$ is surjective for all $X \in \mathcal{H}$.

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- We say that **discrete is concrete** in \mathcal{H} if natural transformation whose components are

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- ... and many others (see [Law]).

Proposition

- The adjunctions $\Pi \dashv \text{disc}$ and $\Gamma \dashv \text{codisc}$ exhibit the subcategories of discrete and codiscrete objects as reflective subcategories of \mathcal{H} ; these subcategories form **exponential ideals** in \mathcal{H} .
- If \mathcal{H} exhibits cohesion, then Set is equivalent to the full subcategory of \mathcal{H} whose objects are the X such that $\eta_{(\Gamma \dashv \text{codisc}), X} : X \rightarrow \text{codisc}(\Gamma(X))$ is an isomorphism.

Equivalently: every cohesive topos “contains the trivial cohesion of disconnected pieces ($=\underline{\text{Set}}$)”.

Non-trivial fact

Whenever a topos arises, there's an interaction between logic and geometry.

Definition

A monomorphism $\psi: S \rightarrow A$ in a cohesive topos \mathcal{H} is a proposition of type A in the internal logic of \mathcal{H} . We say that ψ is *discretely true* if the pullback $\psi^*(S) \rightarrow A$

$$\begin{array}{ccc} \psi^*(S) & \xrightarrow{\quad} & \flat S \\ \downarrow & \lrcorner & \downarrow \flat\psi \\ A & \xrightarrow[\eta]{} & \flat A \end{array}$$

is an isomorphism in \mathcal{H} , where $\eta: A \rightarrow \flat A$ is the \flat -unit of the flat monad.

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- Let $\mathcal{H} = \text{Sh}(\text{Cart}, J)$ be the topos of sheaves over cartesian spaces ($\text{hom}(m, n) = \text{smooth maps } \mathbb{R}^n \rightarrow \mathbb{R}^m$) is cohesive.

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- Let $\psi: Z^p(U) \hookrightarrow \Omega^p(U)$ be the proposition in \mathcal{H} given by “the p -form ω is closed on a neighbourhood $V_x \subseteq U$ of a point $x \in U$ ”. Then ψ is discretely true (“every form is closed over a discrete space”).

Examples

The Sierpiński topos

Let $\mathcal{C} = \{0 \rightarrow 1\}$ be the interval category with a unique non-identity arrow.

The category of **presheaves** on \mathcal{C} forms a topos $\mathcal{H} = \underline{\text{Set}}^{\mathcal{C}}$ = arrows in Set, that exhibits cohesion:

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- the functor Γ sends an object $S \rightarrow I$ to its domain S ;
- the functor disc sends a set K into the identity $1: K \rightarrow K$;
- the functor codisc sends a set K into its terminal morphism $K \rightarrow *$.

The Sierpiński topos

Evidently these functors form an adjunction ($\Pi \dashv \text{disc} \dashv \Gamma \dashv \text{codisc}$) so that \mathcal{H} exhibits cohesion; this matches our intuition, in that

- The “points to pieces” transformation sends $f : S \rightarrow I$ into $S = \Gamma(f) \rightarrow \Pi(f) = I$;
- $\text{disc}(K)$ “keeps all the pieces of K maximally distinguished” and
- $\text{codisc}(K)$ “lumps all the pieces of K together”.

Pointed categories exhibit cohesion

Let \mathcal{C} small and with a terminal object. Then there exists a triple

$$\begin{array}{ccc} & \xrightarrow{\lim\limits_{\longrightarrow}} & \\ [\mathcal{C}^{\text{op}}, \underline{\text{Set}}] & \xleftarrow{\text{const}} & \underline{\text{Set}} \\ & \xrightarrow{\lim\limits_{\longleftarrow}} & \end{array}$$

that extends to $\lim\limits_{\longleftarrow} \dashv \mathcal{K}$:

$$S \xrightarrow{\mathcal{K}} \left(c \mapsto \underline{\text{Set}}(\mathcal{C}(*, c), S) \right)$$

(Dually, if \mathcal{C} has an initial object...)

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Proposition

If \mathcal{C} has both an initial and a terminal object (e.g. it is pointed) then $[\mathcal{C}^{\text{op}}, \underline{\text{Set}}]$ exhibits cohesion with

$$(\lim\limits_{\longrightarrow} \dashv \text{const} \dashv \lim\limits_{\longleftarrow} \dashv \aleph) : [\mathcal{C}^{\text{op}}, \underline{\text{Set}}] \xrightleftharpoons{\text{const}} \underline{\text{Set}}$$

Reflexive directed graphs

Consider the category \mathcal{C}

$$\begin{array}{ccc} & s & \\ 0 & \xrightleftharpoons[q]{\quad} & 1 \\ & t & \end{array}$$

such that $0 \rightrightarrows 1 \rightarrow 0$ is a reflexive coequalizer;

The category $[\mathcal{C}^{\text{op}}, \underline{\text{Set}}]$ is the category of **reflexive directed graphs** RDGph, and it exhibits cohesion, since the terminal geometric morphism

$$(\text{disc} \dashv \Gamma) : \text{RDGph} \leftrightarrows \underline{\text{Set}}$$

extends on the left with $\Pi : X \mapsto \text{coeq}(X_1 \rightrightarrows X_0)$ (simply the connected components of the graph).

Since this is a reflexive coequalizer, it preserves products.

(exercise: define disc, codisc)

Simplicial sets

Proposition

Let Δ be the simplex category having objects nonempty finite ordinals and morphisms monotone maps. The topos

$\mathcal{H} = [\Delta^{\text{op}}, \underline{\text{Set}}]$ exhibits cohesion, and in \mathcal{H} pieces have points.

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- disc sends a set S into the constant simplicial set in S having constant set of simplices and identities as faces and degeneracies.
- codisc sends a set S into the simplicial set whose n -simplices are $(n + 1)$ -tuples of elements of S (and faces and degeneracies forget and add elements accordingly).

Tangent cohesion

Consider the codomain fibration

$$\mathcal{C}^{\rightarrow} \xrightarrow{p} \mathcal{C}$$

of a finitely complete category \mathcal{C} , sending an arrow $f: X \rightarrow Y$ to its codomain. The fiber $p^{-1}(Y)$ is canonically isomorphic to the category \mathcal{C}/Y of arrows over Y .

There exists a fibration $T\mathcal{C} \rightarrow \mathcal{C}$ having typical fiber the fiberwise abelianization of \mathcal{C}/Y , i.e. the category $\text{Ab}(\mathcal{C}/Y)$ of abelian groups in \mathcal{C}/Y .

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Proposition

If \mathcal{C} is locally presentable, then so is $T\mathcal{C}$; moreover, the projection $q: T\mathcal{C} \rightarrow \mathcal{C}$ creates co/limits.

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Proposition

If \mathcal{C} is a **topos** over \mathcal{S} , then so is $T\mathcal{C}$; moreover, the projection $q: T\mathcal{C} \rightarrow \mathcal{C}$ creates co/limits.

Tangent cohesion

Proposition

There is a functor $\delta: T\mathcal{C} \rightarrow \mathcal{C}$ giving for each morphism in $T\mathcal{C}$ its domain. This functor is a right adjoint to a functor $\Omega: \mathcal{C} \rightarrow T\mathcal{C}$ that is also a *section* for q .

The object $\Omega(A)$ can be thought as the complex of **differential forms** on an internal abelian group $A \in \text{Ab}(\mathcal{C}/X)$.

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Proposition

If \mathcal{H} is a cohesive topos with cohesion $(\Pi \dashv \text{disc} \dashv \Gamma \dashv \text{codisc})$, then the tangent category is itself a cohesive topos.

Infinitesimal cohesion

Neighbourhoods of some spaces are “infintesimally extended around a single (global) point”. Cohesive structure can be refined to capture this phenomenon.

Infinitesimal cohesion

Let \mathcal{H} be cohesive. An **infinitesimal thickening** of \mathcal{H} is a new cohesive topos $\tilde{\mathcal{H}}$ linked to the previous by a quadruple of adjoints

$$\begin{array}{ccc} & \mathcal{H} & \\ (i_! \dashv i^* \dashv i_! \dashv i^!) & \downarrow & \\ & \tilde{\mathcal{H}} & \end{array}$$

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such that i_* , $i_!$ are fully faithful and $i_!$ commutes with finite products.

If such a structure exists, \mathcal{H} “exhibits **infinitesimal cohesion**”.

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If such a structure exists, \mathcal{H} “exhibits **infinitesimal cohesion**”.

The functor i_* is fully faithful as well, and then we can consider \mathcal{H} “sitting nicely” inside its thickening $\tilde{\mathcal{H}}$.

Infinitesimal cohesion

- The cohesion exhibited by $\tilde{\mathcal{H}}$ factors through that of \mathcal{H} , in that

$$(\Pi_{\tilde{\mathcal{H}}} \dashv \text{disc}_{\tilde{\mathcal{H}}} \dashv \Gamma_{\tilde{\mathcal{H}}}) : \quad \begin{array}{ccccc} \tilde{\mathcal{H}} & \xrightleftharpoons[i^*]{\hspace{-1cm}} & \mathcal{H} & \xrightleftharpoons[\hspace{-1cm}]{} & \underline{\text{Set}} \\ & \xleftarrow[i_*]{\hspace{-1cm}} & & \xleftarrow[\text{disc}]{\hspace{-1cm}} & \\ & \xrightarrow[i^!]{\hspace{-1cm}} & & \xrightarrow[\Gamma]{\hspace{-1cm}} & \end{array}$$

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- Infinitesimal cohesion describes formally infinitesimally extended neighbourhoods: if the functor i^* is interpreted as a contraction of a fat point onto its singleton, then $X \in \tilde{\mathcal{H}}$ is infinitesimal if $i^*(X) \cong *$. This motivates the fact that

$$\tilde{\mathcal{H}}(*, X) \cong \tilde{\mathcal{H}}(i_!(*), X) \cong \mathcal{H}(*, i^*(X)) \cong \mathcal{H}(*, *) \cong *$$

so that \mathcal{H} sees X as a “small neighbourhood concentrated around a single point $*_X$ ”.

Higher order cohesion: jet spaces

Most examples of infinitesimal cohesion come equipped with an infinite chain of thickening approximations.

Consider the **infinitesimal shape modality** $\Im := i_* i^*$
(it comes equipped with other two adjoints, $\Re \dashv \Im \dashv \&$)²

²This is the same general fact inducing $\int \dashv b \dashv \sharp$ adjunction.

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(it comes equipped with other two adjoints, $\Re \dashv \Im \dashv \&$)²

In several cases (like **smooth manifolds**) we have a **chain** of infinitesimal thickenings

$$\begin{array}{ccccccc} \widetilde{\mathcal{H}}_0 & \xrightarrow{i^{*(0)}} & \widetilde{\mathcal{H}}_1 & \xrightarrow{i^{*(1)}} & \widetilde{\mathcal{H}}_2 & \xrightarrow{i^{*(2)}} & \dots \\ \xleftarrow{i_{*,(0)}} & & \xleftarrow{i_{*,(1)}} & & \xleftarrow{i_{*,(2)}} & & \\ & & & & & & \\ \widetilde{\mathcal{H}}_\infty & \xrightarrow{i^{*(\infty)}} & \mathcal{H} & & \mathcal{H} & & \end{array}$$

²This is the same general fact inducing $\int \dashv \flat \dashv \sharp$ adjunction.

Higher order cohesion: jet spaces

Most examples of infinitesimal cohensions come equipped with an infinite chain of thickening approximations.

Consider the **infinitesimal shape modality** $\Im := i_* i^*$
(it comes equipped with other two adjoints, $\Re \dashv \Im \dashv \&$)²

In several cases (like **smooth manifolds**) we have a **chain** of infinitesimal thickenings

$$\begin{array}{ccccccc} \widetilde{\mathcal{H}}_0 & \xrightarrow{i^{*(0)}} & \widetilde{\mathcal{H}}_1 & \xrightarrow{i^{*(1)}} & \widetilde{\mathcal{H}}_2 & \xrightarrow{i^{*(2)}} & \dots \\ \xleftarrow{i_{*,(0)}} & & \xleftarrow{i_{*,(1)}} & & \xleftarrow{i_{*,(2)}} & & \\ & & & & & & \\ \widetilde{\mathcal{H}}_\infty & \xrightarrow{i^{*(\infty)}} & \mathcal{H} & & \mathcal{H} & & \end{array}$$

here we speak of a **sequence of orders of differential structures**.

²This is the same general fact inducing $\int \dashv \flat \dashv \sharp$ adjunction.

Higher order cohesion: jet spaces

Each of these approximations comes equipped with an *order k infinitesimal shape modality* $\Im^{(k)} X$ in a sequence

$$X \rightarrow \Im X = \Im^{(0)} X \rightarrow \Im^{(1)} X \rightarrow \Im^{(2)} X \rightarrow \dots$$

Example: Every cohesive topos exhibits infinitesimal cohesion via its **tangent** cohesive topos. This cohesion extends to any order of differential structure (“cohesive jet spaces”).

One can go **way** further, but the terminology becomes pretty dire:

Remark 2.2.13. The perspective of def. 2.2.12 has been highlighted in [Law91], where it is proposed (p. 7) that adjunctions of this form usefully formalize “many instances of the *Unity and Identity of Opposites*” that control Hegelian metaphysics [He1841].

[DCCT170811], 1040 pages of Hegel-ish mathematics

uses axiomatic cohesion of ∞ -toposes to axiomatise string theory.

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With Aufhebung.

Supergroupoids: rheonomy

We can speak of **supergroupoids** and show that certain categories of supersmooth manifolds exhibit cohesion (but not over Set...):

$$\begin{array}{c} \text{SuperSmoothS} \\ \uparrow d \quad \uparrow \Gamma \quad \uparrow c \\ \Pi \downarrow \quad \downarrow \quad \downarrow \\ \text{SuperS} \end{array}$$

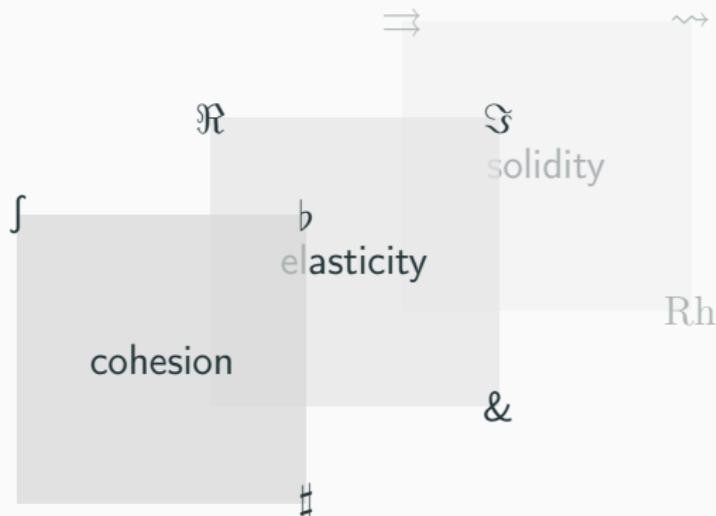
The quadruple of adjoints generates the triple

$$\Rightarrow \dashv \rightsquigarrow \dashv \text{Rh}$$

(in some sense “fermions” \dashv “bosons”)

Supergroup: rheonomy

There is a “quadruple-to-triple” pattern here:



de Rham cohomology in cohesion

- Let \mathcal{H} be a cohesive topos, and $0 \rightarrow A$ a pointed object (e.g. an internal abelian group); then, A fits into a pullback square

$$\begin{array}{ccc} b_{dR}A & \longrightarrow & bA \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & A \end{array}$$

where $b_{dR}A$ is the **object of coefficients** for de Rham cohomology.

- Let $X \in \mathcal{H}$ any object; we define $\int_{dR} X$ to be the pushout

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \int X & \longrightarrow & \int_{dR} X \end{array}$$

where $\int_{dR} X$ is the **de Rham object** associated to X .

de Rham cohomology in cohesion

There is an adjunction

$$\begin{array}{ccc} */\mathcal{H} & \xrightarrow{\flat_{dR}} & \mathcal{H} \\ & \xleftarrow{\sharp_{dR}} & \end{array}$$

The mapping space $*/\mathcal{H}(\sharp_{dR} X, A) \cong \mathcal{H}(X, \flat_{dR} A)$ is called the de Rham space of X with coefficients in A and denoted $\mathbf{H}_{dR}^0(X, A)$.

de Rham cohomology in cohesion

Consider the pullback defining $\flat_{dR} A$ and apply the limit-preserving functor $\mathcal{H}(X, -)$: the square

$$\begin{array}{ccc} \mathcal{H}(X, \flat_{dR} A) & \longrightarrow & \mathcal{H}(X, \flat A) \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & \mathcal{H}(X, A) \end{array}$$

remains a pullback and the object $\mathcal{H}(X, \flat A)$ identifies to **A-valued differential forms**, and the maps $X \rightarrow \flat_{dR} A$ are the **flat** ones: under the $\int \dashv \flat$ adjunction, a map $X \rightarrow \flat A$ mates to a smooth map

$$\int X \rightarrow A, \text{ corresponding to a square } \begin{array}{ccc} X & \xrightarrow{\quad} & 0 \\ \downarrow & & \downarrow \\ \int X & \rightarrow & A \end{array}$$

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