HEARTS AND TOWERS IN STABLE ∞-CATEGORIES

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1. Histoire d'$\mathcal{O}(J)$

Recall that a lower set in a poste J is a subset $L \subseteq J$ such that if $x \in L$ and $y \leq x$ then $y \in L$. Dually, one defines upper sets.

Definition 1.1 : Let J be a poset. A *slicing* of J is a pair (L, U), where L is a lower set in J, U is an upper set, $L \cap U = \emptyset$ and $L \cup U = J$. The collection of all slicings of J will be denoted by $\mathcal{O}(J)$.

Remark 1.2 : Since the complement of an upper set is a lower set and vice versa, the projection on the second factor is a bijection

$$\mathcal{O}(J) \xrightarrow{\sim} \{\text{upper sets of } J\}$$

This induces a natural partial order on $\mathcal{O}(J)$: we set $(L_1, U_1) \leq (L_2, U_2)$ if and only if $U_2 \subseteq U_1$. Notice that $\mathcal{O}(J)$ has a minimum given by the slicing $(\emptyset, J)$ and a maximum given by the slicing $(J, \emptyset)$.

Remark 1.3 : If J is a $\mathbb{Z}$-poset, then so is $\mathcal{O}(J)$. The natural $\mathbb{Z}$-action on $\mathcal{O}(J)$ is given by

$$(L, U) + n = (L + n, U + n),$$

where $L + n = \{x + n \mid x \in L\}$ and $U + n = \{x + n \mid x \in U\}$.

Remark 1.4 : Every element x in J determines two slicings of J: $((-\infty, x), [x, +\infty))$ and $((-\infty, x], (x, +\infty))$. Here $(-\infty, x)$ is the lower set $\{y \in J \mid y < x\}$, and similarly for $(-\infty, x]$, $(x, +\infty)$ and $[x, +\infty)$. This gives two natural morphisms of posets $J \longrightarrow \mathcal{O}(J)$. If J is a $\mathbb{Z}$-poset, then these morphisms are $\mathbb{Z}$-equivariant.

Example 1.5 : The morphism $n \mapsto [n, +\infty)$ induces an isomorphism of $\mathbb{Z}$-posets $\mathbb{Z} \cup \{\pm\infty\} \xrightarrow{\sim} \mathcal{O}(\mathbb{Z})$. The morphisms $x \mapsto (x, +\infty)$ and $x \mapsto [x, +\infty)$ together induce an isomorphism of posets $\mathbb{Z}$-posets

$$\mathbb{R} \times_{\text{lex}} \Delta^1 \cup \{\pm\infty\} \xrightarrow{\sim} \mathcal{O}(\mathbb{R}),$$

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where Δ^1 is the totally ordered set $\{0,1\}$ with $0<1$.

Before introducing the main definition of this section, let us recall that a *t-structure* on a stable ∞-category $\mathbf{C}$ consists of a pair $\mathfrak{t}=(\mathbf{L},\mathbf{U})$ of full sub-∞-categories satisfying the following properties:

(i) orthogonality: $\mathbf{C}(X,Y)$ is contractible for each $X\in\mathbf{U}$, $Y\in\mathbf{L}$;
(ii) one has $\mathbf{U}[1]\subseteq\mathbf{U}$ and $\mathbf{L}[-1]\subseteq\mathbf{L}$;
(iii) Any object $X\in\mathbf{C}$ fits into a (homotopy) fiber sequence $X_{\mathbf{U}}\longrightarrow X\longrightarrow X_{\mathbf{L}}$, with $X_{\mathbf{U}}$ in $\mathbf{U}$ and $X_{\mathbf{L}}$ in $\mathbf{L}$.

The categories $\mathbf{L}$ and $\mathbf{U}$ are called the lower sub-∞-category and the upper sub-∞-category of the t-structure $\mathfrak{t}$, respectivley. The collection $\mathrm{TS}(\mathbf{C})$ of all t-structures on a stable ∞-category $\mathbf{C}$ is a poset with respect to following order relation: given two t-structures $\mathfrak{t}_1=(\mathbf{L}_1,\mathbf{U}_1)$ and $\mathfrak{t}_2=(\mathbf{L}_2,\mathbf{U}_2)$, one has $\mathfrak{t}_1\leq\mathfrak{t}_2$ iff $\mathbf{U}_2\subseteq\mathbf{U}_1$. The ordered group $\mathbb{Z}$ acts on $\mathrm{TS}(\mathbf{C})$ in a way that is fixed by the action of the generator $+1$; this maps a t-structure $\mathfrak{t}=(\mathbf{L},\mathbf{U})$ to the *shifted* t-structure $\mathfrak{t}[1]=(\mathbf{L}[1],\mathbf{U}[1])$. Since $\mathfrak{t}\leq\mathfrak{t}[1]$ one sees that $\mathrm{TS}(\mathbf{C})$ is naturally a $\mathbb{Z}$-poset. Finally, the poset $\mathrm{TS}(\mathbf{C})$ has a minimum and a maximum given by $(\mathbf{C},\mathbf{0})$ and $(\mathbf{0},\mathbf{C})$, respectively. These are called the *trivial* t-structures.

Definition 1.6 : Let $(J,\leq)$ be a $\mathbb{Z}$-poset. A *J-slicing* of a stable ∞-category $\mathbf{C}$ is a $\mathbb{Z}$-equivariant morphism of posets $\mathfrak{t}\colon\mathcal{O}(J)\longrightarrow\mathrm{TS}(\mathbf{C})$ respecting minima and maxima on both sides.

More explicitly, a J-family is a family $\{\mathfrak{t}_{(L,U)}\}_{(L,U)\in\mathcal{O}(J)}$ of t-structures on $\mathbf{C}$ such that

(1) $\mathfrak{t}_{(L_1,U_1)}\leq\mathfrak{t}_{(L_2,U_2)}$ if $(L_1,U_1)\leq(L_2,U_2)$ in $\mathcal{O}(J)$;
(2) $\mathfrak{t}_{(L,U)+1}=\mathfrak{t}_{(L,U)}[1]$ for any $(L,U)\in\mathcal{O}(J)$.
(3) $\mathfrak{t}_{(J,\emptyset)}=(\mathbf{C},\mathbf{0})$ and $\mathfrak{t}_{(\emptyset,J)}=(\mathbf{0},\mathbf{C})$.

Notation 1.7 : We will denote the lower and the upper sub-∞-categories of the t-structure $\mathfrak{t}_{(L,U)}$ by $\mathbf{C}_L$ and $\mathbf{C}_U$, respectively, i.e., we write $\mathfrak{t}_{(L,U)}=(\mathbf{C}_L,\mathbf{C}_U)$. For $i\in J$, we will write $\mathbf{C}_{\geq i}$, $\mathbf{C}_{>i}$, $\mathbf{C}_{\leq i}$ and $\mathbf{C}_{<i}$ for $\mathbf{C}_{[i,+\infty)}$, $\mathbf{C}_{(i,+\infty)}$, $\mathbf{C}_{(-\infty,i]}$ and $\mathbf{C}_{(-\infty,i)}$, respectively. Note that, by $\mathbb{Z}$-equivariancy, we have $\mathbf{C}_{\geq i+1}=\mathbf{C}_{\geq i}[1]$, and similarly for the other cases.

Example 1.8 : By Lemma **??** and Example 1.5, a $\mathbb{Z}$-slicing on $\mathbf{C}$ is equivalent to the datum of a t-structure $\mathfrak{t}_0=(\mathbf{C}_{<0},\mathbf{C}_{\geq 0})$. One has $\mathfrak{t}_n=(\mathbf{C}_{\geq n},\mathbf{C}_{<n})$ for any $n\in\mathbb{Z}$, consistently with the Notation **1.7**, $\mathfrak{t}_{-\infty}=(\mathbf{C},\mathbf{0})$ and $\mathfrak{t}_{+\infty}=(\mathbf{0},\mathbf{C})$. Notice that by our Remark **??**, as soon

as $\mathbf{C}_{\geq 1}$ is a proper subcategory of $\mathbf{C}_{\geq 0}$, then the inclusion $\mathbf{C}_{\geq n+1} \subseteq \mathbf{C}_{\geq n}$ is proper for all $n \in \mathbb{Z}$, i.e. the orbit $\mathfrak{t} + \mathbb{Z}$ is an infinite set.

Example 1.9 : By Example 1.5, $\mathbb{R}$-slicing on $\mathbf{C}$ is the datum of two t-structures $(\mathbf{C}_{<\lambda}, \mathbf{C}_{\geq\lambda})$ and $(\mathbf{C}_{\leq\lambda}, \mathbf{C}_{>\lambda})$ on $\mathbf{C}$ for any $\lambda \in \mathbb{R}$ in such a way that $\mathbf{C}_{\geq\lambda+1} = \mathbf{C}_{\geq\lambda}[1]$, etc., and with the inclusions $\mathbf{C}_{>\lambda} \subseteq \mathbf{C}_{\geq}\lambda$ for any $\lambda \in \mathbb{R}$ and

$$\mathbf{C}_{>\lambda_2} \subseteq \mathbf{C}_{\geq\lambda_2} \subseteq \mathbf{C}_{>\lambda_1} \subseteq \mathbf{C}_{\geq\lambda_1}$$

for any $\lambda_1 < \lambda_2$ in $\mathbb{R}$. $\mathbb{R}$-slicings have been introduced in [**?**], where they are called simply "slicings". Actually [**?**] imposes more restrictive conditions to ensure "compactness" of the factorization, we will come back to this later. Compare also [**?**].

Remark 1.10 : Since the subcategories $\mathbf{C}_L$ and $\mathbf{C}_U$ are the lower and the upper subcategories of a t-structure $\mathfrak{t}_{(L,U)}$ they are reflexive and coreflexive, respectively. In particular we have reflexion and coreflection functors

$$R_L\colon \mathbf{C} \longrightarrow \mathbf{C}_L; \qquad S_U\colon \mathbf{C} \longrightarrow \mathbf{C}_U.$$

For X an object in $\mathbf{C}$ we will usually write X_L for $R_L X$ and X_U for $S_U X$, and similarly for morphisms. Finally, by composing $R_L$and S_U with the inclusions of $\mathbf{C}_L$ and $\mathbf{C}_U$ in $\mathbf{C}$, we can look at R_L and S_U as endofunctors of $\mathbf{C}$.

Lemma 1.11 : (It is possible this lemma requires J to be totally ordered or some order relation between the slicing (L_1, U_1) and the slicing (L_2, U_2): check!) Let (L_1, U_1) and (L_2, U_2) two slicings of J and let $\mathfrak{t}$ be a J-slicing of $\mathbf{C}$. Then we have natural isomorphisms

$$\begin{aligned} R_{L_1} R_{L_2} &\cong R_{L_2} R_{L_1}; \qquad & S_{U_1} S_{U_2} &\cong S_{U_2} S_{U_1};\\ R_{L_1} S_{U_2} &\cong S_{U_2} R_{L_1}; \qquad & S_{U_1} R_{L_2} &\cong R_{L_2} S_{U_1} \end{aligned}$$

Moreover, if $(L_1, U_1) \leq (L_2, U_2)$ then we have natural isomorphisms

$$R_{L_1} R_{L_2} \cong R_{L_1}; \qquad S_{U_1} S_{U_2} \cong S_{U_2}$$

Aggiungere qualche riga sulle composizioni $R_{L_1} R_{L_2}$ e $S_{U_1} S_{U_2}$ con $(L_1, U_1) \leq (L_2, U_2)$.

1.1. **A tale of intervals.** Although a few of the statements we are going to prove hold more generally for arbitrary $\mathbb{Z}$-posets, for the remainder of this section we will restrict our attention to $\mathbb{Z}$-posets which are totally ordered sets.

Definition 1.12 : Let J be a totally ordered set. An *interval* in J is a subset $I \subseteq J$ which can be written as the intersection of an upper set and a lower set.

Lemma 1.13 : In a totally ordered set, the upper set and the lower set intersecting in a nonempty interval I are uniquely determined by I.

Proof. Let $I \subseteq J$ be a interval and let

$$U_I = \bigcap_{U \supseteq I} U; \qquad L_I = \bigcap_{L \supseteq I} L,$$

with U and L ranging over the upper sets and the lower sets in J containing I, respectively. Then it is clear that $I \subseteq U_I \cap L_I$ and we want to show that actually $I = U_I \cap L_I$ and that if $I = U_0 \cap L_0$ then $U_0 = U_I$ and $L_0 = L_I$. By definition of interval there exist an upper set U_0 and a lower set L_0 such that $I = U_0 \cap L_0$. By definition of U_I and L_I we have $U_I \subseteq U_0$ and $L_I \subseteq L_0$. Therefore $I \subseteq L_I \cap U_I \subseteq L_0 \cap U_0 = I$ and so $I = U_I \cup L_I$. Now we want to show that $U_I = U_0$. Since $U_I \subseteq U_0$ we only need to show that $U_0 \subseteq U_I$. Let $x \in U_0$ and let $y \in I$. Since J is totally ordered, either $x \leq y$ or $x \geq y$. In the first case, since L_0 is a lower set, we have $x \in L_0$ and so $x \in L_0 \cup U_0 = I \subseteq U_I$. In the second case, since U_I is an upper set, we have directly $x \in U_I$. $\square$

By the above lemma, the following definition is well-posed.

Definition 1.14 : Let J be a totally ordered $\mathbb{Z}$-poset and let $\mathfrak{t}\colon \mathcal{O}(J) \longrightarrow \text{TS}(\mathbf{C})$ be a J-sliciang on a stable ∞-category $\mathbf{C}$. For every nonempty interval $I = L_I \cup U_I$ in J we set

$$\mathbf{C}_I = \mathbf{C}_{L_I} \cap \mathbf{C}_{U_I}.$$

We also set $\mathbf{C}_\emptyset = \{\mathbf{0}\}$.

Remark 1.15 : The whole of J is an interval, with $L_J = U_J = J$. From Definition 1.14 we obtaon $\mathbf{C}_J = \mathbf{C}$, as expected. Also, every upper set U is an interval, with $U_U = U$ and $L_U = J$. So from Definition 1.14 we find that the subcategory of $\mathbf{C}$ associated to U as an interval is precisely the subcategory $\mathbf{C}_U$ associated to U as an upper set. The same happens for lower sets. This shows that the notation introduced in Definition 1.14 is consistent with the notation for J-slicings.

Example 1.16 : For every i, j in J with $i \leq j$ one has the four intervals (i, j), $(i, j]$, $[i, j)$, $[i, j]$ and consequently the four subcategories $\mathbf{C}_{(i,j)}$, $\mathbf{C}_{(i,j]}$, $\mathbf{C}_{[i,j)}$ and $\mathbf{C}_{[i,j]}$of $\mathbf{C}$. In particular for every $i \in J$ we have the intervale $[i, i]$ consisting of the single element i. To avoid cumbersome notation, we will always write $\mathbf{C}_i$ for $\mathbf{C}_{[i,i]}$. The subcategories $\mathbf{C}_i$ with i ranging in J are called the *slices* of the J-slicing $\mathfrak{t}$.

Definition 1.17 : Let $\mathfrak{t}$ be a J-slicing on $\mathbf{C}$. We say that $\mathbf{C}$ is *J-bounded* if

$$\mathbf{C} = \bigcup_{i,j \in J} \mathbf{C}_{[i,j]}.$$

Similarly, we say that $\mathbf{C}$ is *J-left-bounded* if $\mathbf{C} = \bigcup_{i \in J} \mathbf{C}_{[i,+\infty)}$ and *J-right-bounded* if $\mathbf{C} = \bigcup_{i \in J} \mathbf{C}_{(-\infty,i]}$. This notion is well known in the classical as well as in the quasicategorical setting: see [**?, ?**].

Remark 1.18 : Since $\mathbf{C}_{[i,j]} = \mathbf{C}_{[i,+\infty)} \cap \mathbf{C}_{[-\infty,j]}$ one immediately sees that $\mathbf{C}$ is J-bounded if and only if $\mathbf{C}$ is both J-left- and J-right-bounded.

Lemma 1.19 : Let L and U be a lower and an upper set in J, respectively, let $I = L \cap U$, and let $\mathbf{C}_I$ be the corresponding subcategory of $\mathbf{C}$, for a given J-slicing. Then the restriction of S_U to $\mathbf{C}_L$ and the restriction of R_L to $\mathbf{C}_U$ both take values in $\mathbf{C}_I$.

Proof. Let U_L be the upperset given by the complement of L in J, so that the t-structure with lower category $\mathbf{C}_L$ is $(\mathbf{C}_L, \mathbf{C}_{U_L})$. We split the proof in two cases. If $I = \emptyset$ then $U \subseteq U_L$ and so for any X in $\mathbf{C}_L$ we have $X_U \cong (X_{U_L})_U \cong \mathbf{0}_U = \mathbf{0}$. So $S_U|_{\mathbf{C}_L}$ does take its values in $\mathbf{C}_I = \mathbf{C}_\emptyset = \{\mathbf{0}\}$ in this case. If $I \neq \emptyset$, let $x \in U_L$ and $y \in I$. If $x \leq y$ then, since L is a lower set, we have $x \in L$ and so $x \in L \cap U_L = \emptyset$, which is clearly impossible. Therefore, since J is totally ordered, we have $x \geq y$ and so $U_L \subseteq [y, +\infty) \subseteq U$. Since S_U takes values in $\mathbf{C}_U$, we only need to show that it maps $\mathbf{C}_L$ into itself. In other words we want to show that if $X \in \mathbf{C}_L$ then $X_U \xrightarrow{\sim} (X_U)_L$. From the fiber sequence (non riesco a fare le frecce, kodi mi dà errore)

$$\begin{array}{cc} (X_U)_{U_L} & X_U \\ & \\ 0 & (X_U)_L \end{array}$$

we see we are reduced to showing that $(X_U)_{U_L} \cong \mathbf{0}$. Since $U_L \subseteq U$, we have $(X_U)_{U_L} \cong X_{U_L}$. But, since $X \in \mathbf{C}_L$ we have $X_{U_L} \cong \mathbf{0}$. This concludes the proof in the case $I \neq \emptyset$. The proof for R_L is completely analogous. $\square$

By the above lemma we can give the following

Definition 1.20 : Let $I = L \cap U \subseteq J$ be an interval, and let $\mathfrak{t}\colon \mathcal{O}(J) \longrightarrow \mathrm{TS}(\mathbf{C})$ be a J-sliciang on a stable ∞-category $\mathbf{C}$. The functor

$$\mathcal{H}^I\colon \mathbf{C} \longrightarrow \mathbf{C}_I$$

is defined as the composition $\mathcal{H}^I = R_U S_L = S_L R_U$.

Fino a qui

Remark 1.21 : As it is natural to expect, if $i \geq j$, then $\mathbf{C}_{[i,j)}$ is contractible. Namely, since $j \leq i$ one has $\mathbf{C}_{<j} \subseteq \mathbf{C}_{<i}$ and so

$$\mathbf{C}_{[i,j)} = \mathbf{C}_{\geq i} \cap \mathbf{C}_{<j} \subseteq \mathbf{C}_{\geq i} \cap \mathbf{C}_{<i} = \mathbf{C}_{\geq_i 0} \cap \mathbf{C}_{<_i 0}$$

which corresponds to the contractible subcategory of zero objects in $\mathbf{C}$ (this is immediate, in view of the definition of the two classes).

Remark 1.22 : Let $\mathfrak{t}$ be a $\mathbb{Z}$-family of t-structures on $\mathbf{C}$. Then $\mathbf{C}$ is $\mathbb{Z}$-bounded (resp., $\mathbb{Z}$-left-bounded, $\mathbb{Z}$-right-bounded) if and only if $\mathbf{C}$ is bounded (resp., left-bounded, right-bounded)with respect to the t-structure $\mathfrak{t}_0$, agreeing with the classical definition of boundedness as given, e.g., in [**?**].

Remark 1.23 : For any i, j, h, k in J with $j \leq h$ one has

$$\mathbf{C}_{[i,j)} \subseteq \mathbf{C}_{[h,k)}^{\boxslash},$$

i.e., $\mathbf{C}(X, Y)$ is contractible whenever $X \in \mathbf{C}_{[h,k)}$ and $Y \in \mathbf{C}_{[i,j)}$ (one says that $\mathbf{C}_{[i,j)}$ is *right-orthogonal* to $\mathbf{C}_{[h,k)}$.Indeed, since $\mathbf{C}_{<j} = \mathbf{C}_{<_j 0} = \mathbf{C}_{\geq_j 0}^{\boxslash} = \mathbf{C}_{\geq j}^{\boxslash}$, and passing to the orthogonal reverses the inclusions, we have

$$\mathbf{C}_{[i,j)} \subseteq \mathbf{C}_{<j} = \mathbf{C}_{\geq j}^{\boxslash} \subseteq \mathbf{C}_{\geq h}^{\boxslash} \subseteq \mathbf{C}_{[h,k)}^{\boxslash}.$$

Definition 1.24 : Let $(\mathbf{C}, \mathfrak{t})$ be a stable ∞-category endowed with a t-structure, arising from the normal torsion theory $\mathbb{F} = (\mathcal{E}, \mathcal{M})$. For each $n \in \mathbb{Z}$, let $\mathbf{C}_{\geq n}$ and $\mathbf{C}_{<n}$ be the reflective and coreflective subcategories of $\mathbf{C}$ determined by the t-structure $\mathfrak{t}$.

Then $\mathfrak{t}$ is said to be

- *bounded* if $\bigcup \mathbf{C}_{\geq n} = \mathbf{C}$;

- *limited* if every $f\colon X \longrightarrow Y$ fits into a fiber sequence

$$\begin{tikzcd} F \arrow[r] \arrow[d, "{m[a]}"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d, "{e[b]}"] \\ 0 \arrow[r] & Y \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & C \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

  where $F = \mathrm{fib}(f), C = \mathrm{cofib}(f)$, and $m[a] \in \mathcal{M}[a], e[b] \in \mathcal{E}[b]$ for suitable integers $a, b \in \mathbb{Z}$;
- *narrow* if $\mathbf{C} = \bigcup_{a \le b} \mathbf{C}_{[a,b)}$, where $\mathbf{C}_{[a,b)} = \mathbf{C}_{\ge a} \cap \mathbf{C}_{<b}$.

Proposition 1.25 : Let $(\mathbf{C}, \mathfrak{t})$ be a stable ∞-category endowed with a t-structure. Then $\mathfrak{t}$ is narrow if and only if it is bounded, if and only if it is limited.

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