Better towers

March 6, 2017

Let J^+ be (the nerve of) the category $(J^\lhd)^\rhd = (J^\rhd)^\lhd$.

Let $J \to \mathfrak{ts}(\mathbf{C})$ be a slicing considered with its canonical extension ([hearts, 1.15]) $J^+ \to \mathfrak{ts}(\mathbf{C})$.

We define

- A J(-Postnikov) tower to be a functor $(J^+)^{\mathrm{op}} \to \mathbf{C} : X_{-\infty} \leftarrow \cdots \leftarrow X_i \xrightarrow{i \le j} X_j \leftarrow \cdots \leftarrow X_{+\infty}$ such that
  1. $X_{-\infty} \cong 0$ and $X_{+\infty} \cong X$;
  2. ($X_j = R_j(X_{+\infty})$ for $j \in J$).
- A J(-Postnikov) pretower to be a functor $J^{\mathrm{op}} \to \mathbf{C}$ (such that $X_j \cong R_j(X_{j+_J 1})$)[1]

We denote $\varphi\colon \mathrm{Post}^+(\mathbf{C}) \hookrightarrow \mathrm{Post}(\mathbf{C})$ the inclusion of Postnikov towers into Postnikov pretowers. We say that *Postnikov towers are convergent* in $\mathbf{C}$ if φ is an equivalence, whose inverse is given by taking the limit $\varprojlim(X_0 \leftarrow X_1 \leftarrow \cdots)$.

Let $\mathbf{W} \subseteq \mathbf{C} \times J^+$ be the category of all pairs (C, j) where $C \in \mathbf{C}_{\le j}$ is a j-truncated object (in the obvious sense). Then (since $\mathbf{W}$ is the category of elements of $J \to \mathrm{Cat}_\infty : j \mapsto \mathbf{C}_{\le j}$) there exists a cocartesian fibration $p\colon \mathbf{W} \to J^+$ such that $p^\leftarrow(j) = \mathbf{C}_{\le j}$; this fibration classifies a tower of functors

$$\mathbf{C}_{\le -\infty} \leftarrow \cdots \leftarrow \mathbf{C}_{\le j} \leftarrow \cdots \leftarrow \mathbf{C}$$

Postnikov towers are then identified with cocartesian sections of p, and Postnikov pretowers with the cocartesian sections of $\tilde{p}$ in the pullback

$$\begin{tikzcd}
\mathbf{W} \times_{(J^+)^{\mathrm{op}}} J^{\mathrm{op}} \arrow[r] \arrow[d, "\tilde{p}"'] & \mathbf{W} \arrow[d, "p"] \\
J^{\mathrm{op}} \arrow[r] & (J^+)^{\mathrm{op}}
\end{tikzcd}$$

[1] J is a $\mathbb{Z}$-poset with action $+_J\colon J \times \mathbb{Z} \to J$; since $j \le j +_J 1$ (contravariant) functoriality gives $X_{j+_J 1} \to X_j$, and the request now is that $X_j \cong R_j(X_{j+_J 1})$. Equivalently –since the associated normal torsion theory $(\mathscr{E}_j, \mathscr{M}_j)$ is firmly $\mathscr{E}$-reflective– $X_{j+_J 1} \xrightarrow{e_j} X_j$ and $\tau_{\ge j}(X_j) \cong 0$.