FACOLTÀ DI SCIENZE MATEMATICHE FISICHE E NATURALI
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FUNCTORIAL FLAVORS OF BRIDGELAND'S SLICINGS

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The garboard strake leaks, the seams need caulking.
This form, this face, this life
Living to live in a world of time beyond me; let me
Resign my life for this life, my speech for that unspoken,

Thomas Eliot

Contents

1. Ouverture

1.1. **What is this thesis about.** Let us start with the question we shall try to answer in the following pages:

♫ *What is a cohomology theory?*

Classically speaking, what we know as cohomology is the functor

$$\mathscr{D}(\mathscr{A}) \xrightarrow{H^i} \mathscr{A}$$

from the derived category of some abelian category $\mathscr{A}$. Now, if we take a complex $C^\bullet \in \mathscr{D}(\mathscr{A})$, we can truncate it at some index i to get

$$\tau_i(C^\bullet) =$$

1.2. **Notation and conventions.** Some anti-Bourbaki conventions. We'll often consider objects of a category up to isomorphism. This means, for example, that when an isomorphism is obvious or canonical we'll just write equality and that all subcategories are assumed to be saturated (i.e. closed under isomorphisms). To keep things clean, we will avoid writing names for morphisms as much as possible and will denote 1 the identity of any object in any category.

2. Abstract homological algebra

2.1. Triangulated categories.

Definition 2.1. *Let $\mathscr{D}$ be an additive category, Σ an additive autoequivalence of $\mathscr{D}$. We denote for all $n \in \mathbb{Z}$, $*[n] = \Sigma^n(*)$. A **triangle** in $\mathscr{D}$ is a diagram of the form $X \longrightarrow Y \longrightarrow Z \longrightarrow X[1] = \Sigma(X)$, which we denote for simplicity:*

$$X \longrightarrow Y \longrightarrow Z \longrightarrow$$

The triangles of $\mathscr{D}$ form then an additive category, where the morphisms are triples a, b, c fitting in a commutative diagram:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, "c"] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

*A **triangulated category** is an additive category $\mathscr{D}$ equipped with an additive autoequivalence Σ, called **translation**, and a full (saturated) subcategory of its triangles $\triangle$, whose objects we call **distinguished triangles** of $\mathscr{D}$, so that:*

(Tr1) The triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow$$

*is distinguished. We call it the **trivial** triangle.*

*(Tr2) Any morphism $X \longrightarrow Y$ in $\mathscr{D}$ extends to a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow$. We call Z a **cone** of that morphism.*

(Tr3) The triangle $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow$ is distinguished if and only if the triangle

$$Y \longrightarrow Z \longrightarrow X[1] \xrightarrow{-f[1]}$$

*is distinguished. We call the second triangle the **rotation** of the first one.*

*(Tr4) If $X \xrightarrow{f} Y \xrightarrow{h} Z' \longrightarrow$, $Y \xrightarrow{g} Z \longrightarrow X' \longrightarrow$ and $X \xrightarrow{gf} Z \longrightarrow Y'$ are distinguished triangles, then there is a distinguished triangle $Z' \longrightarrow Y' \longrightarrow X' \longrightarrow$, called a **braid** generated by gf, so that the following commutes:*

$$\begin{tikzcd}
X \arrow[dr, "f"] \arrow[rr, bend left, "gf"] & & Z \arrow[dr] \arrow[rr, bend left] & & X' \arrow[dr] \arrow[rr, bend left] & & Z'[1] \\
& Y \arrow[ur, "g"] \arrow[dr, "h"] & & Y' \arrow[ur] \arrow[dr] & & Y[1] \arrow[ur, "h[1]"] & \\
& & Z' \arrow[ur] \arrow[rr, bend right] & & X[1] \arrow[ur, "f[1]"] & &
\end{tikzcd}$$

Let $\mathscr{D}$ be a triangulated category. Observe that the opposite category $\mathscr{D}^{\mathrm{op}}$ equipped with translation functor Σ^{-1} is clearly triangulated.
We put the sign in *(Tr3)* because, if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ is a distinguished triangle in $\mathscr{D}$, each three consecutive morphisms in the following long sequence form a distinguished triangle:

$$\cdots \xrightarrow{-g[-1]} Z[-1] \xrightarrow{-h[-1]} X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1] \xrightarrow{-f[1]} Y[1] \xrightarrow{-g[1]} \cdots$$

Example 2.2. *Let $\mathscr{A}$ be an abelian category. Recall that its derived category $\mathscr{D}(\mathscr{A})$ is the homotopy category of chain complexes in $\mathscr{A}$ localized to quasi-isomorphisms (i.e. morphisms which induce isomorphisms in homology). The translation functor of $\mathscr{D}(\mathscr{A})$ shifts the indices of a complex and changes the sign of the differential. Moreover, the cone* $\mathrm{cone}(f)$ *of a morphism f of complexes is the total complex of f (seen as a double complex). If we take as distinguished triangles in $\mathscr{D}(\mathscr{A})$ those isomorphic to triangles of the form*

$$X \xrightarrow{f} Y \longrightarrow \mathrm{cone}(f) \longrightarrow$$

the derived category becomes tiangulated. Moreover, the bounded derived category $\mathscr{D}^b(\mathscr{A})$ (consisting of complexes with homology vanishing for a cofinite set of indices) is a triangulated subcategory of $\mathscr{D}(\mathscr{A})$.

Proposition 2.3. (3 × 3 Lemma) *Suppose that $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$, $X' \xrightarrow{f'} Y' \longrightarrow Z' \longrightarrow$, $X \xrightarrow{a} X' \xrightarrow{a'} X'' \xrightarrow{a''}$, $Y \xrightarrow{b} Y' \longrightarrow Y'' \longrightarrow$ are distinguished triangles in $\mathscr{D}$. If the upper left square of the below diagram commutes, then there are distinguished triangles $Z \longrightarrow Z' \longrightarrow Z'' \longrightarrow$, $X'' \longrightarrow Y'' \longrightarrow Z'' \longrightarrow$ so that the following commutes:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r, "g"] \arrow[d, "b"] & Z \arrow[r, "h"] \arrow[d] & X[1] \arrow[d, "a[1]"] \\
X' \arrow[r, "f'"] \arrow[d, "a'"] & Y' \arrow[r] \arrow[d] & Z' \arrow[r] \arrow[d] & X'[1] \arrow[d, "a'[1]"] \\
X'' \arrow[r] \arrow[d, "a''"] & Y'' \arrow[r] \arrow[d] & Z'' \arrow[r] \arrow[d] & X''[1] \arrow[d, "-a''[1]"] \\
X[1] \arrow[r, "f[1]"] & Y[1] \arrow[r, "g[1]"] & Z[1] \arrow[r, "h[1]"] & X[2]
\end{tikzcd}$$

(caution: the bottom row may not be distinguished).

Proof. By taking a cone of $bf = f'a$, we have a distinguished triangle $X \xrightarrow{bf} Y' \longrightarrow V \longrightarrow$. By taking braids generated by bf and $f'a$ respectively, we have then two distinguished triangles:

$$Z \xrightarrow{s} V \xrightarrow{t} Y'' \longrightarrow$$

$$X'' \xrightarrow{s'} V \xrightarrow{t'} Z' \longrightarrow$$

By taking a cone of ts', we get a distinguished triangle

$$X'' \longrightarrow Y'' \longrightarrow Z'' \longrightarrow$$

which is one of the triangles of the thesis.
Now, by rotation we get a distinshed triangle $V \xrightarrow{t} Y'' \longrightarrow Z[1] \xrightarrow{-s[1]}$ and by taking a braid generated by ts' we have another distinguished triangle:

$$Z' \longrightarrow Z'' \longrightarrow Z[1] \longrightarrow$$

Rotating this last triangle, we have the other triangle of the thesis.
The commutativity follows by the way we applied *(Tr4)*. $\square$

Proposition 2.4. *Let $X \longrightarrow Y \longrightarrow Z \longrightarrow$ and $X' \longrightarrow Y' \longrightarrow Z' \longrightarrow$ distinguished triangles of $\mathscr{D}$, $Y \xrightarrow{b} Y'$ a morphism. Then the following are equivalent:*

1. *(1) there is a morphism $X \xrightarrow{a} X'$ so that the first square of the below diagram commutes*
2. *(2) there is a morphism $Z \xrightarrow{c} Z'$ so that the second square of the below diagram commutes*
3. *(3) b extends to a morphism between the two triangles:*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, "c"] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

Proof. Clearly, *(3)* implies the other ones. To show that *(1)* implies *(3)*, take cones of a and b and apply the 3×3 Lemma. But rotating the triangles, we see by the same argument that *(2)* implies *(3)*. □

The above Proposition means that we can start with two among three of the morphisms involved in a morphisms of triangles and obtain the third. We will refer to this opertation as **completing** the two morphisms to a morphism of triangles.

Proposition 2.5. *The composition of any two consecutive morphisms of a distinguished triangle is* 0*.*

Proof. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow$ be a distinguished triangle. Then complete f and the identity of X to a morphism of of triangles between that triangle and the trivial one:

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"] & X \arrow[r] \arrow[d, "f"] & 0 \arrow[r] \arrow[d] & X[1] \arrow[d, "1"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1] \end{tikzcd}$$

That shows $gf = 0$. For the other cases, just rotate the triangle. □

Definition 2.6. *Let $\mathscr{A}$ be an abelian category. A functor $\mathscr{D} \xrightarrow{H} \mathscr{A}$ is **cohomological** if for each distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow$ of $\mathscr{D}$ its image*

$$H(X) \longrightarrow H(Y) \longrightarrow H(Z)$$

is exact in $\mathscr{A}$.

Example 2.7. *Using the spectral sequence of a double complex one sees that the homology functor (in degree* 0*) $\mathscr{D}(\mathscr{A}) \xrightarrow{H^0} \mathscr{A}$ is cohomological. This will be anyway proved in a general context in the next section.*

Proposition 2.8. *For each object $E \in \mathscr{D}$, the functors $\mathrm{Hom}_{\mathscr{D}}(E, *)$ and $\mathrm{Hom}_{\mathscr{D}}(*, E)$ are cohomological (with values in $\mathrm{Mod}_{\mathbb{Z}}$ and $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{op}}$ respectively).*

Proof. We'll just show that the first functor is coomological, for the other one the argument is dual. Let $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow$ be a distinguished triangle, $E \xrightarrow{a} Y$ a morphism so that $ga = 0$. Complete a and $0 \longrightarrow Z$:

$$\begin{tikzcd} E \arrow[r, "1"] \arrow[d, "b"] & E \arrow[r] \arrow[d, "a"] & 0 \arrow[r] \arrow[d] & E[1] \arrow[d, "b[1]"] \\ X \arrow[r, "f"] & Y \arrow[r, "g"] & Z \arrow[r] & X[1] \end{tikzcd}$$

This shows that, for some b, $fb = a$, as desired. ☐

Definition 2.9. *A triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow$ in $\mathscr{D}$ (not necessarily distinguished) is **special** if for each $E \in \mathscr{D}$ the induced long sequence of abelian groups:*

$$\cdots \longrightarrow \mathrm{Hom}_{\mathscr{D}}(E,X) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(E,Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(E,Z) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(E,X[1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(E,Y[1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(E,Z[1]) \longrightarrow \cdots$$

is exact.

To be clear, saying that $\mathrm{Hom}_{\mathscr{D}}(E,*)$ is cohomological means taht distingusihed triangles are special. The converse is not true in general: if we change the sign of one of the morphisms in a distinguished triangle we obtain a special triangle which doesn't have to be distinguished.

Proposition 2.10. *Let*

$$\begin{tikzcd} X \arrow[r] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, "c"] & X[1] \arrow[d, "a[1]"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

be a morphism of special triangles. If two among a, b, c are isomorphisms, then so is the third.

Proof. As usual, up to rotation, we'll show the statement in the case a and c are isomorphisms. For each $E \in \mathscr{D}$, we have the diagram:

$$\begin{array}{ccccccccc}
\mathrm{Hom}_{\mathscr{D}}(E, Z[-1]) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(E, X) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(E, Y) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(E, Z) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(E, X[1]) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Hom}_{\mathscr{D}}(E, Z'[-1]) & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(E, X') & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(E, Y') & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(E, Z') & \longrightarrow & \mathrm{Hom}_{\mathscr{D}}(E, X'[1])
\end{array}$$

The rows are exact since the triangles are special, and the two left and two right vertical morphisms are isomorphisms of abelian groups by hypothesis. By the Five Lemma for abelian categories we conclude that the middle vertical morphism is an isomorphism, and since this is true for each $E \in \mathscr{D}$, by the Yoneda Lemma we conclude that b is an isomorphism. □

It follows immediately that the cone of a morphism f in $\mathscr{D}$ is unique up to (in general not unique) isomorphism, so we denote it $\mathrm{cone}(f)$. Unfortunately, it is not even possible to make a functorial choice for cones (indeed, when it's possible, then $\mathscr{D}$ is semisimple abelian, as shown in REFERENZAAAAA!!!!!) and thus we will always assume that an arbitrary choice of cones has been made in $\mathscr{D}$. We will establich a uniqueness property in the following theorem, which is a stronger version of **Proposition 2.4**.

Proposition 2.11. *Let* $X \xrightarrow{f} Y \longrightarrow Z \longrightarrow$ *and* $X' \longrightarrow Y' \xrightarrow{g} Z' \longrightarrow$ *distinguished triangles of* $\mathscr{D}$*,* $Y \xrightarrow{b} Y'$ *a morphism. Then* b *extends to a morphism of triangles:*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "a"] & Y \arrow[r] \arrow[d, "b"] & Z \arrow[r] \arrow[d, "c"] & X[1] \arrow[d, "{a[1]}"] \\
X' \arrow[r] & Y' \arrow[r, "g"] & Z' \arrow[r] & X'[1]
\end{tikzcd}$$

if and only if $gbf = 0$*. Moreover, if that's the case and* $\mathrm{Hom}_{\mathscr{D}}(X, Z'[-1]) = 0$*, then* a *and* c *are unique.*

Proof. The 'only if' part follows from **Proposition 2.5**. For the 'if' part, applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(X, *)$ to the second triangle, we get an exact sequence of abelian groups:

$$\mathrm{Hom}_{\mathscr{D}}(X, Z[-1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(X, X') \longrightarrow \mathrm{Hom}_{\mathscr{D}}(X, Y') \longrightarrow \mathrm{Hom}_{\mathscr{D}}(X, Z')$$

Since bf is in the kernel of the last map of the above sequence by hypothesis, it is in the image of the second map and we can then take

a in its preimage. If $\mathrm{Hom}_{\mathscr{D}}(X, Z'[-1]) = 0$, then a is unique since the second map is injective. To get c, just apply **Proposition 2.4**. $\square$

Proposition 2.12. *Let* $X \xrightarrow{f} Y$ *be a morphism in* $\mathscr{D}$. *Then* f *is an isomorphism if and only if* $\mathrm{cone}(f) = 0$.

Proof. Consider the diagram and the notation of **Proposition 2.5**. Since 1 is an automorphism of X, by **Proposition 2.10** $0 \longrightarrow Z = \mathrm{cone}(f)$ is an isomorphism if and only if f is an isomorphism. $\square$

Now we deal with direct sums. First of all, the following proposition shows that the cone commutes with direct sums.

Proposition 2.13. *Two triangles* $X \longrightarrow Y \longrightarrow Z \longrightarrow$ *and* $X' \longrightarrow Y' \longrightarrow Z' \longrightarrow$ *are both distinguished if and only if the triangle*

$$X \oplus X' \longrightarrow Y \oplus Y' \longrightarrow Z \oplus Z' \longrightarrow$$

is distinguished (the maps of the third triangle are the direct sums of the maps of the first two triangles).

Proof. Let's show the 'only if' part. Denote $Q = \mathrm{cone}(X \oplus X' \longrightarrow Y \oplus Y')$. By completing the inclusions of the summands in the direct sums, we get a morphisms $Z \longrightarrow Q$ and $Z' \longrightarrow Q$. Denote c the direct sum of these two morphisms. We have a morphism of triangles:

$$\begin{tikzcd}
X \oplus X' \arrow[r] \arrow[d, "1"] & Y \oplus Y' \arrow[r] \arrow[d, "1"] & Z \oplus Z' \arrow[r] \arrow[d, "c"] & X[1] \oplus X'[1] \arrow[d, "1"] \\
X \oplus X' \arrow[r] & Y \oplus Y' \arrow[r] & Q \arrow[r] & X[1] \oplus X'[1]
\end{tikzcd}$$

The upper triangle is special because it is direct sum of special triangles, and thus c is an isomorphism.
For the 'if' part, the argument is similar. Denote $Q = \mathrm{cone}(X \longrightarrow Y)$. Completing the projections to the summands of the direct sums, we have a morphism $Z \oplus Z' \longrightarrow Q$, and composing with the incluzion of Z, we get $Z \longrightarrow Q$. This is an isomorphism by the same argument as above, since direct summands of special triangles are special by an easy check. $\square$

In other words, the cone commutes with direct sums or, more precisely, $\triangle$ is an additive subcategory of triangles closed under direct summands. As a consequence, we have that for each $X, Y \in \mathscr{D}$ the triangle

$$X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow{0}$$

is distinguished (the maps are inclusion and projection), because it is the sum of a trivial and a rotated trivial triangle. Conversely, we have the following result.

Proposition 2.14. *Let* $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h}$ *be a distingusihde triangle of* $\mathscr{D}$*. If* $h = 0$*, then* g *has a right inverse. Moreover, for every right inverse* s *of* g*,*

$$X \oplus Z \xrightarrow{f \oplus s} Y$$

is an isomorphism.

Proof. Applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(Z, *)$ to the triangle, we see get an exact sequence of abelian groups:

$$\mathrm{Hom}_{\mathscr{D}}(Z, Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(Z, Z) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(Z, X[1])$$

The surjectivity of the first map is equivalent to g having right inverse and to $h = 0$. Thus the first statement of the theorem is proven. Now suppose g has right inverse, and thus $h = 0$. For each $W \in \mathscr{D}$, by applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(W, *)$ to the triangle we get a short exact sequence of abelian groups:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}}(W, X) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(W, Y) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(W, Z) \longrightarrow 0$$

By the splitting lemma $\mathrm{Hom}_{\mathscr{D}}(W, Y) = \mathrm{Hom}_{\mathscr{D}}(W, X \oplus Z)$ and by arbitrariness of W we get, using the Yoneda lemma, the desired result. □

Definition 2.15. *A* ***triangle functor*** *bewtween two triangulated categories* $\mathscr{D}$ *and* $\mathscr{D}'$ *with translation functors* Σ *and* Σ' *respectively is a functor* $\mathscr{D} \xrightarrow{F} \mathscr{D}'$ *equipped with an isomorphism of functors* $\{\xi_X\}_{X \in \mathscr{D}}$ *between* $F\Sigma$ *and* $\Sigma' F$ *so that for each distinguished triangle* $X \longrightarrow Y \longrightarrow Z \xrightarrow{h}$ *in* $\mathscr{D}$*, the triangle*

$$F(X) \longrightarrow F(Y) \longrightarrow F(Z) \xrightarrow{\xi_X F(h)}$$

is distinguished in $\mathscr{D}'$*.*
We denote $\mathrm{Aut}(\mathscr{D})$ *the group of triangle autoequivalences of* $\mathscr{D}$*.*

The class of triangulated categories has then a structure of a 2-category, whose 1-morphisms are triangle functors and a 2-morphism between two triangle functors F and F' (with attached natural isomorphisms ξ and ξ' respectively) is a natural transformation $F \xrightarrow{\alpha} F'$ so that the following commutes:

$$\begin{tikzcd}
F\Sigma \arrow[r, "\xi"] \arrow[d, "\alpha\Sigma"] & \Sigma' F \arrow[d, "\Sigma'\alpha"] \\
F'\Sigma \arrow[r, "\xi'"] & \Sigma' F'
\end{tikzcd}$$

Observe that we always have $\Sigma \in \mathrm{Aut}(\mathscr{D})$ (we equip Σ with -1, where 1 is the identity of Σ^2). When obvious, we will omit the ξ when speaking of triangle functors.
We do not require additivity in the definition of triangle and cohomological functors because it is automatic by the following proposition.

Proposition 2.16. *Triangle and cohomological functors are additive.*

Proof. Let $\mathscr{D} \xrightarrow{F} \mathscr{D}'$ be a triangle functor between triangulated categories (we omit the ξ). Considering the image of the very trivial triangle (the one with all zero vertices) of $\mathscr{D}$, we get a distinguished triangle in $\mathscr{D}'$:

$$F(0) \xrightarrow{1} F(0) \xrightarrow{1} F(0) \longrightarrow$$

By **Proposition 2.5**, $1 = 0$ and thus $F(0) = 0$, which also tells us that the image of any zero morphism is the zero morphism. For each $X, Y \in \mathscr{D}$, since $X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow{0}$ is a distinguished triangle in $\mathscr{D}$, we have that

$$F(X) \longrightarrow F(X \oplus Y) \longrightarrow F(Y) \xrightarrow{0}$$

is a distinguished triangle in $\mathscr{D}'$. Since the last morphism of the latter triangle is 0, we get by **Proposition 2.14** that $F(X \oplus Y) = F(X) \oplus F(Y)$ in $\mathscr{D}'$, as desired.
The proof for cohomological functors is very similar (use the Splitting Lemma for abelian categories and the right inverse from **Proposition 2.14**). $\square$

Proposition 2.17. *Adjoints (left or right) of triangle functors are triangle.*

Proof. Since $\mathscr{D}^{\mathrm{op}}$ is triangulated, it suffices to show the statement for, say, right adjoints. Now let $\mathscr{D} \xrightarrow{F} \mathscr{D}'$ be a triangle functor and $\mathscr{D} \xrightarrow{G} \mathscr{D}'$ its right adjoint. For each $X \in \mathscr{D}, Y \in \mathscr{D}'$, since Σ is a triangle equivalence and F is triangle, using adjunction property, we get:

$$\mathrm{Hom}_{\mathscr{D}}(X, G(Y[1])) = \mathrm{Hom}_{\mathscr{D}}(F(X[-1]), G(Y)) = \mathrm{Hom}_{\mathscr{D}}(X, G(Y)[1])$$

By the Yoneda Lemma and arbitrariness of X and Y, we get a natural isomorphism $\Sigma' G = G\Sigma$. Now, let $X \longrightarrow Y \longrightarrow Z \longrightarrow$ be a distinguished triangle in $\mathscr{D}'$. Taking the cone, we get a distinguished triangle $G(X) \longrightarrow G(Y) \longrightarrow W \longrightarrow$ in $\mathscr{D}$. Using adjunction property, we get isomorphisms $F(G(X)) \longrightarrow X$ and $F(G(Y)) \longrightarrow Y$ and we can complete them to a morphism of distinguished triangles:

$$\begin{array}{ccccccc}
F(G(X)) & \longrightarrow & F(G(Y)) & \longrightarrow & F(W) & \longrightarrow & F(G(X))[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & X[1]
\end{array}$$

The morphism $F(W) \longrightarrow Z$ is an isomorphism by **Proposition 2.10** and induces, by adjunction, an isomorphism $W \longrightarrow G(Z)$. This means that $G(X) \longrightarrow G(Y) \longrightarrow W = G(Z) \longrightarrow$ is a distinguished triangle in $\mathscr{D}$, and thus G is triangle. □

Definition 2.18. *An **extension-closed** subcategory of a triangulated category $\mathscr{D}$ is a full subcategory $\mathscr{C} \subseteq \mathscr{D}$ containing 0 so that for each distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow$ in $\mathscr{D}$, if $X, Z \in \mathscr{C}$, then $Y \in \mathscr{C}$.*
*We say that $\mathscr{C}$ is **thick** if it is an extension-closed triangulated subcategory.*
*If $S \subseteq \mathscr{D}$ is a subset, we denote $\langle S \rangle$ (resp. $\langle\langle S \rangle\rangle$) the smallest extension-closed (resp. thick) subcategory of $\mathscr{D}$ containing S, and call it the extension-closed (resp. thick) subcategory **generated** by S (we set $\langle \emptyset \rangle = 0$).*

Clearly, extension-closed subcategories are additive by the remark after **Proposition 2.13**.

Proposition 2.19. *Let $\mathscr{D} \xrightarrow{F} \mathscr{D}'$ be a full triangle functor, $S \subseteq \mathscr{D}$ a subset. Then $F(\langle S \rangle) = \langle F(S) \rangle$. A similar statement holds for full cohomological functors (where the extension-closeness on the right side is thought in the abelian sense).*

Proof. We have:

$$\langle S \rangle = \bigcup_{i \in \mathbb{Z}_{\geq 0}} S_i$$

where $S_0 = S$ and S_{i+1} is the set of objects $X \in \mathscr{D}$ so that there is a distinguished triangle $T \longrightarrow X \longrightarrow T' \longrightarrow$ with $T, T' \in S_i$. If $X \in S_{i+1}$, applying F to the latter triangle we get a distinguished triangle $F(T) \longrightarrow F(X) \longrightarrow F(T') \longrightarrow$ in $\mathscr{D}'$ and since $\langle F(S) \rangle$ is extension-closed, we conclude inductively that $F(\langle S \rangle) \subseteq \langle F(S) \rangle$.
To conclude, we have to show that $F(\langle S \rangle)$ is extension-closed. Pick a distinguished triangle

$$F(X) \longrightarrow T \longrightarrow F(Y) \xrightarrow{f}$$

in $\mathscr{D}'$ with $X, Y \in \langle S \rangle$. Since F is full, there is a morphism $Y[-1] \xrightarrow{g} X$ in $\mathscr{D}$ so that $F(g)[1] = f$. Since $\langle S \rangle$ is extension-closed, $\text{cone}(g) \in \langle S \rangle$, and since F is triangle $F(\text{cone}(g)) = \text{cone}(f)[-1] = T$, and thus $T \in F(\langle S \rangle)$, as desired.
The proof for cohomological functors is very similar. $\square$

Example 2.20. *The kernel (the full subcategory of objects sent to* 0*) of a cohomological functor is extension-closed. The kernel of a triangle functor is thick. Indeed, using Verdier localization (see REFERENZA-AAAAAAAAA) one shows that all the thick subcategories are obtained this way.*

2.2. **t-structures.** From now on, for set-theoretic purposes, we assume that $\mathscr{D}$ is (essentially) small.

Definition 2.21. *A **t-structure** on $\mathscr{D}$ is a full additive subcategory $\mathfrak{t} \subseteq \mathscr{D}$ so that:*

(1) $\mathfrak{t}[1] \subseteq \mathfrak{t}$

(2) For each $X \in \mathscr{D}$ there is a distinguished triangle

$$T \longrightarrow X \longrightarrow T' \longrightarrow$$

so that $T \in \mathfrak{t}$ and $T' \in \mathfrak{t}^{\perp}$

We denote $\mathfrak{ts}(\mathscr{D})$ the poset of t-structures on $\mathscr{D}$ (ordered by opposite inclusion).

Observe that $\mathfrak{ts}(\mathscr{D})$ is a bounded poset: the maximum is the zero subcategory, the minimum the whole $\mathscr{D}$.

Proposition 2.22. *Let $\mathfrak{t}$ be a t-structure on $\mathscr{D}$. Associating to $X \in \mathscr{D}$ the triangle of (2) defines a functor (unique up to isomorphism)*

$$\mathscr{D} \xrightarrow{\tau_{\mathfrak{t}}} \triangle$$

Moreover, composing $\tau_{\mathfrak{t}}$ with the projection to the left (resp. right) vertex defines a right (resp. left) adjoint to the inclusion $\mathfrak{t} \subseteq \mathscr{D}$ (resp. $\mathfrak{t}^{\perp} \subseteq \mathscr{D}$) which we denote $\tau_{\mathfrak{t}}^{\geq}$ (resp. $\tau_{\mathfrak{t}}^{<}$).

Proof. Let $X \longrightarrow Y$ be a morphism in $\mathscr{D}$. Using *(2)* we have distinguished triangles $T \longrightarrow X \longrightarrow T' \longrightarrow$ and $F \longrightarrow Y \longrightarrow F' \longrightarrow$. Since $T[1] \in \mathfrak{t}$ by *(1)* and $F' \in \mathfrak{t}^{\perp}$, we have $\mathrm{Hom}_{\mathscr{D}}(T[1], F') = \mathrm{Hom}_{\mathscr{D}}(T, F'[-1]) = 0$ and thus by **Proposition 2.11** we get a unique, functorial morphism between our two triangles. This shows the first part of the statement. Thus, we denote $\tau_{\mathfrak{t}}(X)$ the triangle of X, $\tau_{\mathfrak{t}}^{\geq}(X) = T$ and $\tau_{\mathfrak{t}}^{<}(X) = T'$. Now, pick $Z \in \mathfrak{t}$. Since again $\mathrm{Hom}_{\mathscr{D}}(Z, T') = \mathrm{Hom}_{\mathscr{D}}(Z, T'[-1]) = 0$, applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(Z, *)$ to the distinguished triangle

$$\tau_{\mathfrak{t}}^{\geq}(X) \longrightarrow X \longrightarrow \tau_{\mathfrak{t}}^{<}(X) \longrightarrow$$

we get

$$\mathrm{Hom}_{\mathscr{D}}(Z, \tau_{\mathfrak{t}}^{\geq}(X)) = \mathrm{Hom}_{\mathscr{D}}(Z, X)$$

which is the desired adjunction property. Similarly, picking $Z \in \mathfrak{t}^{\perp}$ and applying $\mathrm{Hom}_{\mathscr{D}}(*, Z)$ to the same triangle, we get

$$\mathrm{Hom}_{\mathscr{D}}(\tau_{\mathfrak{t}}^{<}(X), Z) = \mathrm{Hom}_{\mathscr{D}}(X, Z)$$

which concludes the proof. $\square$

Since adjoints of additive functors are additive, $\tau_{\mathfrak{t}}^{\geq}$ and $\tau_{\mathfrak{t}}^{<}$ are additive and thus $\tau_{\mathfrak{t}}$ is additive too (this also follows from **Proposition 2.13**). Now, besides $\mathrm{ts}(\mathscr{D})$ is not functorial in $\mathscr{D}$, it is still well-defined: taking the image $F(\mathfrak{t})$ of a t-structure $\mathfrak{t}$ under a triangle autoequivalense F defines an action of $\mathrm{Aut}(\mathscr{D})$ on $\mathrm{ts}(\mathscr{D})$, and we have:

$$\tau_{F(\mathfrak{t})} = F\tau_{\mathfrak{t}}F^{-1}$$

Proposition 2.23. *Associanting to a t-structure $\mathfrak{t}$ the functor $\tau_{\mathfrak{t}}$ defines a functor*

$$\mathfrak{ts}(\mathscr{D})^{\mathrm{op}} \xrightarrow{\tau_*} \triangle^{\mathscr{D}}$$

($\mathfrak{ts}(\mathscr{D})$ is thought as a posetal category)

Proof. Let $\mathfrak{t} \subseteq \mathfrak{q}$ be t-structures on $\mathscr{D}$, $X \in \mathscr{D}$. Since $\tau_{\mathfrak{t}}^{\geq}(X) \in \mathfrak{t}$ and $\tau_{\mathfrak{q}}^{<}(X) \in \mathfrak{q}^{\perp} \subseteq \mathfrak{t}^{\perp}$, we can again use **Proposition 2.11** to extend the identity of X to a morphism of triangles $\tau_{\mathfrak{t}}(X) \longrightarrow \tau_{\mathfrak{q}}(X)$. This is the desired functorial natural transformation. $\square$

Example 2.24. *Let $\mathscr{A}$ be an abelian category. Denote by $\mathfrak{t}$ the full subcategory of complexes with cohomology concentrated in strictly negative degree. Then $\mathfrak{t}$ is a t-structure on $\mathscr{D}(\mathscr{A})$ called standard t-structure, and the functors $\tau_{\mathfrak{t}}^{\geq}$ and $\tau_{\mathfrak{t}}^{<}$ correspond to the truncation of a complex on the right and the left of 0 respectively. Indeed, the the name 't-structure' stands for 'truncation structure'.*

Proposition 2.25. *Let $\mathfrak{t}$ be a t-structure on $\mathscr{D}$, $X \longrightarrow Y \longrightarrow Z \longrightarrow$ a distingusihed triangle in $\mathscr{D}$. If $Z \in \mathfrak{t}^{\perp}$, then $\tau_{\mathfrak{t}}^{\geq}(X) = \tau_{\mathfrak{t}}^{\geq}(Y)$. If $X \in \mathfrak{t}$, then $\tau_{\mathfrak{t}}^{<}(Y) = \tau_{\mathfrak{t}}^{<}(Z)$.*

Proof. Pick $W \in \mathfrak{t}$. Applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(W, *)$ to the triangle and using adjunction property, we see

$$\mathrm{Hom}_{\mathscr{D}}(W, \tau_{\mathfrak{t}}^{\geq}(X)) = \mathrm{Hom}_{\mathscr{D}}(W, \tau_{\mathfrak{t}}^{\geq}(Z))$$

By arbitrariness of W, using the Yoneda Lemma (applied to the category $\mathfrak{t}$), we get the result.
The proof of the second statement is very similar (just pick $W \in \mathfrak{t}^{\perp}$ and apply $\mathrm{Hom}_{\mathscr{D}}(*, W)$ instead). $\square$

Proposition 2.26. *Let $\mathfrak{t}$ be a t-structure on $\mathscr{D}$. Then $\mathfrak{t}$ and $\mathfrak{t}^{\perp}$ are extension-closed.*

Proof. Let $X \longrightarrow Y \longrightarrow Z \longrightarrow$ be a distinguished triangle with $X, Z \in \mathfrak{t}$. Then $\tau_{\mathfrak{t}}^{<}(Y) = \tau_{\mathfrak{t}}^{<}(Z) = 0$ by **Proposition 2.25**, which means that $Y \in \mathfrak{t}$, as desired. The proof for $\mathfrak{t}^{\perp}$ is very similar. $\square$

Proposition 2.27. *If $\mathfrak{t} \subseteq \mathfrak{q}$ are t-structures on $\mathscr{D}$, then:*

$$\tau_{\mathfrak{t}}^{\geq}\tau_{\mathfrak{q}}^{<} = \tau_{\mathfrak{q}}^{<}\tau_{\mathfrak{t}}^{\geq} = 0$$
$$\tau_{\mathfrak{t}}^{\geq}\tau_{\mathfrak{q}}^{\geq} = \tau_{\mathfrak{q}}^{\geq}\tau_{\mathfrak{t}}^{\geq} = \tau_{\mathfrak{t}}^{\geq}$$
$$\tau_{\mathfrak{q}}^{<}\tau_{\mathfrak{t}}^{<} = \tau_{\mathfrak{t}}^{<}\tau_{\mathfrak{q}}^{<} = \tau_{\mathfrak{q}}^{<}$$
$$\tau_{\mathfrak{t}}^{<}\tau_{\mathfrak{q}}^{\geq} = \tau_{\mathfrak{q}}^{\geq}\tau_{\mathfrak{t}}^{<}$$

Moreover, the last functor above coincides pointwise with the cone of the natural transformation $\tau_{\mathfrak{t}}^{\geq} \longrightarrow \tau_{\mathfrak{q}}^{\geq}$ induced by the inclusion $\mathfrak{t} \subseteq \mathfrak{q}$.

Proof. The first identity follows from the definitions. Now, pick $X \in \mathscr{D}$. By applying the 3×3 **Lemma** to the morphism of triangles $\tau_{\mathfrak{t}}(X) \longrightarrow \tau_{\mathfrak{q}}(X)$ induced by the inclusion $\mathfrak{t} \subseteq \mathfrak{q}$ (**Proposition 2.23**), we obtain a commutative diagram:

$$\begin{tikzcd}
\tau_{\mathfrak{t}}^{\geq}(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d, "1"] & \tau_{\mathfrak{t}}^{<}(X) \arrow[r] \arrow[d] & \tau_{\mathfrak{t}}^{\geq}(X)[1] \arrow[d] \\
\tau_{\mathfrak{q}}^{\geq}(X) \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & \tau_{\mathfrak{q}}^{<}(X) \arrow[r] \arrow[d] & \tau_{\mathfrak{q}}^{\geq}(X)[1] \arrow[d] \\
A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & A[1] \arrow[d] \\
\tau_{\mathfrak{t}}^{\geq}(X)[1] \arrow[r] & X[1] \arrow[r] & \tau_{\mathfrak{t}}^{<}(X)[1] \arrow[r] & \tau_{\mathfrak{t}}^{\geq}(X)[2]
\end{tikzcd}$$

Since $\tau_{\mathfrak{t}}^{\geq}(X) \in \mathfrak{t} \subseteq \mathfrak{q}$, by rotating the left column and applying extension closeness we get that $A[-1] \in \mathfrak{q}$ and thus, since $\mathfrak{q}[1] \subseteq \mathfrak{q}$, $A \in \mathfrak{q}$. Similarly, $B \in \mathfrak{t}^{\perp}$. Since $B = A[1]$ by **Proposition 2.12**, we conclude $A \in \mathfrak{t}^{\perp}$ and $B \in \mathfrak{q}$. But then the left column is just $\tau_{\mathfrak{t}}(\tau_{\mathfrak{q}}^{\geq}(X))$ and the third column is just the rotation of $\tau_{\mathfrak{q}}(\tau_{\mathfrak{t}}^{<}(X))$, as desired. □

Let $\mathfrak{t}$ be a t-structure on $\mathscr{D}$.

Definition 2.28. *We call* ***heart*** *of $\mathfrak{t}$ the full additive subcategory*

$$\heartsuit_{\mathfrak{t}} = \mathfrak{t} \cap \mathfrak{t}[1]^{\perp}$$

We call ***cohomology*** *in grade $n \in \mathbb{Z}$ induced by $\mathfrak{t}$ the additive functor*

$$H_{\mathfrak{t}}^{n} = \tau_{\mathfrak{t}[n]}^{\geq}\tau_{\mathfrak{t}[n+1]}^{<} = \tau_{\mathfrak{t}[n+1]}^{<}\tau_{\mathfrak{t}[n]}^{\geq}$$

which takes values in $\heartsuit_{\mathfrak{t}}[n]$.

Clearly, $X \in \heartsuit_{\mathfrak{t}}[n]$ if and only if $H^n_{\mathfrak{t}}(X) = X$.

Proposition 2.29. *$\heartsuit_{\mathfrak{t}}$ is an abelian category. Moreover, a short exact sequence $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ in $\heartsuit_{\mathfrak{t}}$ induces a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow$ in $\mathscr{D}$.*

Proof. Let $X \xrightarrow{f} Y$ be a morphism in $\heartsuit_{\mathfrak{t}}$. We define:

$$\ker(f) = H^0_{\mathfrak{t}}(\mathrm{cone}(f)[-1]) \xrightarrow{p} X$$

$$Y \xrightarrow{q} \mathrm{coker}(f) = H^0_{\mathfrak{t}}(\mathrm{cone}(f))$$

where the maps p, q are obtained by applying the functor $H^0_{\mathfrak{t}}$ to the triangle (and to its rotation) $X \longrightarrow Y \longrightarrow \mathrm{cone}(f) \longrightarrow$ and using $H^0_{\mathfrak{t}}(X) = X$ and $H^0_{\mathfrak{t}}(Y) = Y$.
Now, rotating the latter triangle and using extension-closeness, we see that $\mathrm{cone}(f) \in \mathfrak{t} \cap \mathfrak{t}[2]^{\perp}$, and thus

$$\ker(f) = \tau^{\geq}_{\mathfrak{t}[1]}(\mathrm{cone}(f))[-1]$$

$$\mathrm{coker}(f) = \tau^{<}_{\mathfrak{t}[1]}(\mathrm{cone}(f))$$

This means that $\tau_{\mathfrak{t}[1]}(\mathrm{cone}(f))$ is just

$$\ker(f)[1] \longrightarrow \mathrm{cone}(f) \longrightarrow \mathrm{coker}(f) \longrightarrow$$

Since q is the composition $Y \longrightarrow \mathrm{cone}(f) \longrightarrow \mathrm{coker}(f)$, by taking the generated braid we get a distinguished triangle

$$X[1] \longrightarrow \mathrm{cone}(q) \longrightarrow \ker(f)[2] \xrightarrow{p[2]}$$

We have shown that $\mathrm{cone}(q) = \mathrm{cone}(p)[1]$, and thus $\mathrm{im}(f) = \ker(q) = \mathrm{coker}(p) = \mathrm{coim}(f)$. Images and coimages coincide: this shows that $\heartsuit_{\mathfrak{t}}$ is abelian.
Now if $\ker(f) = 0$, by looking at $\tau_{\mathfrak{t}[1]}(\mathrm{cone}(f))$ above we see that $\mathrm{cone}(f) = \mathrm{coker}(f)$, which means that the cokernel of a monomorphisms in $\heartsuit_{\mathfrak{t}}$ is the cone of that same morphism in $\mathscr{D}$, which is a reformulation of the second part of the statement.

□

Observe that conversely, if $X \longrightarrow Y \longrightarrow Z \longrightarrow$ is a distinguished triangle in $\mathscr{D}$ with $X, Y, Z \in \heartsuit_{\mathfrak{t}}$, then $H^1_{\mathfrak{t}}(Z) = 0$ and $H^0_{\mathfrak{t}}(Z) = Z$, and thus $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is an exact sequence in $\heartsuit_{\mathfrak{t}}$. This also implies that $\heartsuit_{\mathfrak{t}}[n]$ is a heart for each $n \in \mathbb{Z}$, and Σ^n restricted to $\heartsuit_{\mathfrak{t}}$ is an exact equivalence of abelian categories.

Proposition 2.30. *For each $X, Y \in \heartsuit_{\mathfrak{t}}$ there is a (canonical) isomorphism of groups:*

$$\mathrm{Ext}^1_{\heartsuit_{\mathfrak{t}}}(X, Y) = \mathrm{Hom}_{\mathscr{D}}(X, Y[1])$$

Proof. If $0 \longrightarrow Y \longrightarrow Z \longrightarrow X \longrightarrow 0$ is an extension, by the second claim of **Proposition 2.29** we get a distinguished triangle $Y \longrightarrow Z \longrightarrow X \longrightarrow$ the last arrow of which is an element of $\mathrm{Hom}_{\mathscr{D}}(X, Y[1])$. Conversely, if we have a morphism $X \longrightarrow Y[1]$, by taking its cone Z we see by extension-closeness that $Z[-1] \in \heartsuit_{\mathfrak{t}}$ and is thus an extension by the above observation. □

Example 2.31. *If $\mathscr{A}$ is an abelian category with enough injectives, then something more is true by an easy check: for each $X, Y \in \mathscr{A}$, $n \geq 0$,*

$$\mathrm{Ext}^n_{\mathscr{A}}(X, Y) = \mathrm{Hom}_{\mathscr{D}(\mathscr{A})}(X, Y[n])$$

Proposition 2.32. *For each $n \in \mathbb{Z}$, the functor $\mathscr{D} \xrightarrow{H^n_{\mathfrak{t}}} \heartsuit_{\mathfrak{t}}[n]$ is cohomological.*

Proof. Since $H^n_{\mathfrak{t}} = H^0_{\mathfrak{t}[n]}$, we can assume $n = 0$. Pick a distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow$ in $\mathscr{D}$.
First, suppose that $X, Z \in \mathfrak{t}[1]^{\perp}$, and hence $Y \in \mathfrak{t}[1]^{\perp}$ too by extension-closeness. This means that $H^0_{\mathfrak{t}}$ and $\tau^{\geq}_{\mathfrak{t}}$ coincide on X, Y, Z. Now, for $W \in \heartsuit_{\mathfrak{t}}$, applying the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(W, *)$ to the triangle and using adjunction property we get an exact sequence of abelian groups:

$$\mathrm{Hom}_{\mathscr{D}}(W, H^0_{\mathfrak{t}}(X)) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(W, H^0_{\mathfrak{t}}(Y)) \longrightarrow \mathrm{Hom}_{\mathscr{D}}(W, H^0_{\mathfrak{t}}(Z))$$

By arbitrariness of W and since the Yoneda embedding of an abelian category reflects exactness, we get that

$$H^0_{\mathfrak{t}}(X) \longrightarrow H^0_{\mathfrak{t}}(Y) \longrightarrow H^0_{\mathfrak{t}}(Z)$$

is exact in $\heartsuit_{\mathfrak{t}}$.
Now, suppose just $Z \in \mathfrak{t}[1]^{\perp}$. By taking the cone of the composition $\tau^{\geq}_{[1]}(X) \longrightarrow X \longrightarrow Y$ we get a distinguished triangle

$$\tau^{\geq}_{[1]}(X) \longrightarrow Y \longrightarrow T \longrightarrow$$

and we have $\tau^{<}_{[1]}(Y) = \tau^{<}_{[1]}(T)$ by **Proposition 2.25**, and hence $H^0_{\mathfrak{t}}(Y) = H^0_{\mathfrak{t}}(T)$. By taking the braid generated by the composition $\tau^{\geq}_{\mathfrak{t}[1]}(X) \longrightarrow X \longrightarrow Y$, we get a distinguished triangle $\tau^{<}_{\mathfrak{t}[1]}(X) \longrightarrow T \longrightarrow Z \longrightarrow$, and applying $H^0_{\mathfrak{t}}$, since we are in the situation above and $H^0_{\mathfrak{t}}(\tau^{<}_{\mathfrak{t}[1]}(X)) = H^0_{\mathfrak{t}}(X)$, we get the result.

Now, in general, taking the braid generated by the composition $Y \longrightarrow Z \longrightarrow \tau^{<}_{\mathfrak{t}[1]}(Z)$ we get a distinguished triangle

$$X[1] \longrightarrow T' \longrightarrow \tau^{\geq}_{\mathfrak{t}[2]}(Z[1]) \longrightarrow$$

Again by **Proposition 2.25** $\tau^{<}_{\mathfrak{t}[2]}(X[1]) = \tau^{<}_{\mathfrak{t}[2]}(T')$ and hence $H^1_{\mathfrak{t}}(X[1]) = H^1_{\mathfrak{t}}(T')$. Applying then $H^0_{\mathfrak{t}}$ to the triangle $T'[-1] \longrightarrow Y \longrightarrow \tau^{<}_{\mathfrak{t}[1]}(Z) \longrightarrow$, since we are in the situation above and $H^0_{\mathfrak{t}}(\tau^{<}_{\mathfrak{t}[1]}(Z)) = H^0_{\mathfrak{t}}(Z)$, we get the result. □

Definition 2.33. *We say that* $\mathfrak{t}$ *is* ***bounded*** *if*

$$\mathscr{D} = \bigcup_{n,m\in\mathbb{Z}} \mathfrak{t}[n] \cap \mathfrak{t}[m]^{\perp}$$

We denote $\mathfrak{bts}(\mathscr{D}) \subseteq \mathfrak{ts}(\mathscr{D})$ *the set of bounded t-structures on* $\mathscr{D}$.

Example 2.34. *Let* $\mathscr{A}$ *be an abelian category and consider the t-structure* $\mathfrak{t}$ *on* $\mathscr{D}(\mathscr{A})$ *from* ***Example 2.24***. *Then* $\heartsuit_{\mathfrak{t}} = \mathscr{A}$, *with equivalence given by sending objects of* $\mathscr{A}$ *to the corresponding complexes concentrated in degree* 0. *Moreover, we have* $H^0 = H^0_{\mathfrak{t}}$. *While* $\mathfrak{t}$ *is not bounded in* $\mathscr{D}(\mathscr{A})$, *its intersection is bounded in* $\mathscr{D}^b(\mathscr{A})$.

Observe that if $\mathfrak{t}$ is bounded, then for each $X \in \mathscr{D}$ the set of indices $n \in \mathbb{Z}$ in which its cohomology doesn't vanish is finite.

Proposition 2.35. *Assume that* $\mathfrak{t}$ *is bounded. Then for each* $0 \neq X \in \mathscr{D}$ *there is a finite set* $\{k_1 > \cdots > k_n\} \subseteq \mathbb{Z}$ *and a factorization of the initial morphism, called* ***Postnikov tower*** *of* X *with respect to* $\mathfrak{t}$,

$$0 = X_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} X_n = X$$

so that $\operatorname{cone}(\alpha_i) = H^{k_i}_{\mathfrak{t}}(X)$ *for each* $1 \leq i \leq n$.

Proof. Take as the set the indices in which the cohomology of X doesn't vanish and put $X_i = \tau^{\geq}_{\mathfrak{t}[k_i]}(X)$. Now, since $k_i > k_{i+1}$, we have an induced natural transformation (valuated in X) $\tau^{\geq}_{\mathfrak{t}[k_i]}(X) \xrightarrow{\alpha_i} \tau^{\geq}_{\mathfrak{t}[k_{i+1}]}(X)$. Now, by the last claim of **Proposition 2.23** we have

$$\operatorname{cone}(\alpha_i) = \tau^{<}_{\mathfrak{t}[k_i]}(\tau^{\geq}_{\mathfrak{t}[k_{i+1}]}(X))$$

But for each $k_{i+1} < k < k_i$, since $H^k_{\mathfrak{t}}(X) = 0$ we see inductively by the relations in **Proposition 2.23** that $\tau^{\geq}_{\mathfrak{t}[k]}(X) = \tau^{\geq}_{\mathfrak{t}[k_i+1]}(X)$ and thus $\operatorname{cone}(\alpha_i) = H^{k_i}_{\mathfrak{t}}(X)$, as desired. □

We conclude saying that that $\mathfrak{ts}(\mathscr{D})$ can be very big, especially if $\mathscr{D}$ is not essentially small: for example, there is a proper class of t-structures on the derived categroy of abelian groups (see REFERENZAAAA!!!).

2.3. **The Grothendieck group.** ☕ Recall that the Grothendieck group of an abelian category $\mathscr{A}$ was introduced by A. Grothendieck in order to formulate his wonderful version of the Riemann-Roch theorem and is defined as the free abelian group on (isomorphism classes of) objects of $\mathscr{A}$ with a relation $B = A + C$ for each exact sequence $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathscr{A}$. We denote that group $K_0(\mathscr{A})$.

Definition 2.36. *The* ***Grothendieck group*** $K_0(\mathscr{D})$ *of* $\mathscr{D}$ *is the free abelian group on (isomorphism classes of) objects of* $\mathscr{D}$ *with a relation*

$$Y = X + Z$$

for each distinguished triangle $X \longrightarrow Y \longrightarrow Z \longrightarrow$ *in* $\mathscr{D}$.

STA ROBA SI TROVA IN THOMASON: THE CLASSIFICATION OF TRIANGULATED CATEGORIES. The Grothendieck group is functorial in the obvious way: a triangle functor $\mathscr{D} \longrightarrow \mathscr{D}'$ induces a homomorphism of groups between $K_0(\mathscr{D})$ and $K_0(\mathscr{D}')$, and thus the Grothendieck group is our first algebraic invariant for triangulated categories. The same kind of functoriality happens when considering cohomological functors instead of triangle.
Now, by considering the very trivial triangle with zero vertices, the identity element $0 \in K_0(\mathscr{D})$ is the zero object $0 \in \mathscr{D}$, so there is no clash of notation. Since $X \longrightarrow X \oplus Y \longrightarrow Y \xrightarrow{0}$ is distinguished, we have $X \oplus Y = X + Y$ for each $X, Y \in \mathscr{D}$. Moreover, by rotating multiple times the trivial triangle, we get

$$X[n] = (-1)^n X$$

for each $X \in \mathscr{D}$, $n \in \mathbb{Z}$. We thus have that every element in $K_0(\mathscr{D})$ is represented by an object of $\mathscr{D}$, which doesn't happen in an abelian category.

Example 2.37. *The Grothendieck group is not always interesting. As shown in REFERENZAAAA ????, if* R *is an artinian (commutative and unitary) ring then*

$$K_0(\mathscr{D}(\mathrm{Mod}_R^{\mathrm{fin}})) = 0$$

However, this doesn't happen for the bounded derived category (see ***Example 2.39****)!*

Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$. The existence of Postnikov towers shows that for each $X \in \mathscr{D}$

$$X = \sum_{n \in \mathbb{Z}} H^n_{\mathfrak{t}}(X)$$

in $K_0(\mathscr{D})$ and, since $H^n_{\mathfrak{t}}(X) \in \heartsuit_{\mathfrak{t}}[n]$, we get that X is an alternate sum of objects in $\heartsuit_{\mathfrak{t}}$.

Proposition 2.38. *Let $\mathfrak{t}$ be a bounded t-structure on $\mathscr{D}$. Then the inclusion $\heartsuit_{\mathfrak{t}} \subseteq \mathscr{D}$ induces an isomorphism of groups*

$$K_0(\heartsuit_{\mathfrak{t}}) = K_0(\mathscr{D})$$

Proof. Since exact sequences in $\heartsuit_{\mathfrak{t}}$ correspond to distinguished triangles in $\mathscr{D}$ with vertices in $\heartsuit_{\mathfrak{t}}$, the inclusion defines a morphism from $K_0(\heartsuit_{\mathfrak{t}})$ to $K_0(\mathscr{D})$. The inverse of this map is given by the alternate sum above. $\square$

Example 2.39. *Let k be a field. It is well-known that $K_0(\mathrm{Mod}_k^{\mathrm{fin}}) = \mathbb{Z}$ and thus the same equality holds for the bounded derived category of finite dimensional k-vector spaces. The natural map $\mathscr{D}^b(\mathrm{Mod}_k^{\mathrm{fin}}) \longrightarrow K_0(\mathrm{Mod}_k^{\mathrm{fin}}) = \mathbb{Z}$ is then the usual Euler characteristic.*

3. Slicing triangulated categories

Definition 3.1. *A $\mathbb{Z}$-poset is a poset J equipped with a group action by $\mathbb{Z}$ so that the action map*

$$\mathbb{Z} \times J \longrightarrow J$$

is a a morphism of posets (i.e. non decreasing), where the left member has the product order.

We write the action as $\phi + i$, with $\phi \in J$ and $i \in \mathbb{Z}$. The $\mathbb{Z}$-posets clearly define a category, denoted $\mathrm{Pos}_{\mathbb{Z}}$, where the morphisms are $\mathbb{Z}$-equivariant morphisms of posets. Just like the category of posets, this is a cartesian monoidal category enriched over herself. We further denote $\mathrm{tPos}_{\mathbb{Z}}$ the full subcategory of totally ordered $\mathbb{Z}$-posets.
Observe that the poset $\mathfrak{ts}(\mathscr{D})$ is a $\mathbb{Z}$-poset with $\mathbb{Z}$ acting by translation, and $\mathfrak{bts}(\mathscr{D}) \subseteq \mathfrak{ts}(\mathscr{D})$ is a sub-$\mathbb{Z}$-poset.
Let J be a $\mathbb{Z}$-poset.

Definition 3.2. *A J-**slicing** of $\mathscr{D}$ is a collection $\mathscr{P} = \{\mathscr{P}_\phi\}_{\phi \in J}$ of full additive subcategories $\mathscr{P}_\phi \subseteq \mathscr{D}$ so that:*

1. *$\mathscr{P}_{\phi+1} = \mathscr{P}_\phi[1]$ for all $\phi \in J$*
2. *$\mathscr{P}_\psi \subseteq \mathscr{P}_\phi^\perp$ if $\phi > \psi$*
3. *for each $0 \neq X \in \mathscr{D}$ there is a finite strictly decreasing sequence $\{\phi_1 > \cdots > \phi_n\} \subseteq J$ and a factorization of the initial morphism, called a **Postnikov tower** of X with respect to $\mathscr{P}$,*

$$0 = X_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} X_n = X$$

so that $0 \neq \mathrm{cone}(\alpha_i) \in \mathscr{P}_{\phi_i}$ for all $1 \leq i \leq n$

*The nonzero objects of $\mathscr{P}_\phi$ are called **semistable of phase** ϕ, and the simple ones are called **stable**.*
If $I \in 2^J$ is a subset, then $\mathscr{P}_I$ is the extension-closed subcategory generated by semistables with phase in I.
We denote $\mathbf{\Omega}_J(\mathscr{D})$ the set of J-slicings of $\mathscr{D}$.

Let $\mathscr{P}$ be a J-slicing on $\mathscr{D}$. We assume the notations above and state some trivial remarks. First of all, by *(2)*, $\mathscr{P}_\phi \cap \mathscr{P}_\psi = \{0\}$ if $\phi \neq \psi$, so the phase of a semistable object is well-defined. Moreover, since $\alpha_1 = 0$, we have $\mathrm{cone}(\alpha_1) = X_1$.

Proposition 3.3. *Let $\mathscr{P}$ be a J-slicing of $\mathscr{D}$, $A, B \subseteq J$ subsets. If $A > B$ (in the sense that $a > b$ for $a \in A$ and $b \in B$), then*

$$\mathscr{P}_B \subseteq \mathscr{P}_A^\perp$$

Proof. It simply follows from **Proposition 2.19** (considering the cohomological functor $\mathrm{Hom}_{\mathscr{D}}(*,*)$) and part *(2)* of defition of slicing. $\square$

Now, we denote $O(J) \subseteq 2^J$ the lattice of the upper sets of J: it is a $\mathbb{Z}$-poset when ordered by opposite inclusion and with the obvious action. Observe that $O(J)$ is totally ordered if and only if J is. If $J \xrightarrow{f} J'$ is a morphism of $\mathbb{Z}$-posets and $I' \in O(J')$, $f^{-1}(I') \in O(J)$, and this defines a 2-functor

$$\mathrm{Pos}_{\mathbb{Z}}^{\mathrm{op}} \xrightarrow{O(*)} \mathrm{Pos}_{\mathbb{Z}}$$

Proposition 3.4. *Let $\mathscr{P}$ be a J-slicing of $\mathscr{D}$ and suppose that J is totally ordered. If $I \in O(J)$ then $\mathscr{P}_I$ is a t-structure on $\mathscr{D}$ with $\mathscr{P}_I^{\perp} = \mathscr{P}_{J\setminus I}$. This defines a morphism of $\mathbb{Z}$-posets*

$$O(J) \xrightarrow{\mathscr{P}_*} \mathfrak{ts}(\mathscr{D})$$

which defines an injective map

$$\mathbf{\Omega}_J(\mathscr{D}) \longrightarrow \mathrm{Hom}_{\mathrm{Pos}_{\mathbb{Z}}}(O(J), \mathfrak{ts}(\mathscr{D}))$$

Proof. Let $0 \neq X \in \mathscr{D}$, $I \in O(J)$. In the usual notation, picking a Postnikov tower of X, denote $\underline{i} = \max_{\phi_i \in I}\{i\}$ (if it exists) and

$$T = \begin{cases} X_{\underline{i}} & X \neq 0 \text{ and } \phi_1 \in I \\ 0 & \text{otherwise} \end{cases}$$

(we set $T = 0$ if $X = 0$). Since I is an upper set, $T \in \mathscr{P}_I$ by looking at its Porstnikov tower. Now, we always have a morphism $T \longrightarrow X$, which is $\alpha_n \cdots \alpha_{\underline{i}+1}$ if $X \neq 0$ and $\phi_1 \in I$, and 0 otherwise. We define T' as the cone of that morphism. In the first situation, by braiding we get a distinguished triangle

$$\mathrm{cone}(\alpha_{n-1} \cdots \alpha_{\underline{i}+1}) \longrightarrow T' \longrightarrow \mathrm{cone}(\alpha_n)$$

and we inductively see $T' \in \mathscr{P}_{J\setminus I}$ by extension-closeness. But since I is an upper set, we have $I > J \setminus I$ in 2^J, and thus by **Proposition 3.3** $T' \in \mathscr{P}_{J\setminus I} \subseteq \mathscr{P}_I^{\perp}$, which shows that $\mathscr{P}_I$ is a t-structure, as desired.
The fact that indeed $\mathscr{P}_I^{\perp} = \mathscr{P}_{J\setminus I}$ now follows from the definitions: if $X \in \mathscr{P}_I^{\perp}$, then $T = 0$, which means that $\phi_1 \in J \setminus I$, and since $J \setminus I$ is a lower set, we have $X \in \mathscr{P}_{J\setminus I}$. $\square$

From now on, we suppose that J is totally ordered. Since $O(J)$ is a topology on J, called the upper Alexandrov topology, the above proposition can be charmingly interpreted as:

♫ *A J-slicing on $\mathscr{D}$ can be seen as a presheaf of t-structures on J with the upper Alexandrov topology*

If $\mathscr{P}$ is a J-slicing on $\mathscr{D}$, we denote for each $\phi \in J$:

$$\tau_{\mathscr{P}}^{\geq\phi} = \tau^{\geq}_{\mathscr{P}_{[\phi,+\infty[}} \qquad \tau_{\mathscr{P}}^{<\phi} = \tau^{<}_{\mathscr{P}_{[\phi,+\infty[}}$$
$$\tau_{\mathscr{P}}^{>\phi} = \tau^{\geq}_{\mathscr{P}_{]\phi,+\infty[}} \qquad \tau_{\mathscr{P}}^{\leq\phi} = \tau^{<}_{\mathscr{P}_{]\phi,+\infty[}}$$

Moreover the the image of $\mathbf{\Omega}_J(\mathscr{D})$ in $\mathrm{Hom}_{\mathrm{Pos}_{\mathbb{Z}}}(O(J)\mathfrak{ts}(\mathscr{D}))$ is an orbit of the $\mathbb{Z}$-action, and thus we can give $\mathbf{\Omega}_J(\mathscr{D})$ a structure of $\mathbb{Z}$-poset.

Proposition 3.5. *The Postnikov towers with respect to $\mathscr{P}$ are unique up to a unqiue isomorphism of diagrams.*

Proof. In the usual notation, we have $X_i = \tau_{\mathscr{P}}^{\geq\phi_i}(X)$ for each $0 \neq X \in \mathscr{D}$, and α_i is the natural transformation $\tau_{\mathscr{P}}^{\geq\phi_{i-1}}(X) \longrightarrow \tau_{\mathscr{P}}^{\geq\phi_i}(X)$ induced by the inequality $\phi_i > \phi_{i+1}$ (this follows from uniqueness in **Proposition 2.11**). This shows what we wanted. □

Proposition 3.6. *Let $\mathscr{P}$, $\mathscr{Q}$ be J-slicings on $\mathscr{D}$. If $\mathscr{P}_\phi \subseteq \mathscr{Q}_\phi$ for each $\phi \in J$, then $\mathscr{P} = \mathscr{Q}$.*

Proof. Pick $X \in \mathscr{Q}_\phi$. By hypothesis, the Postnikov tower of X with respect to $\mathscr{P}$ coincides with the Postnikov tower with respect to $\mathscr{Q}$. Since the latter is trivial (i.e. is the initial morphism), we get what we wanted. □

Now that we have uniqueness, we fix some notation:

$$H^\phi_{\mathscr{P}}(X) = \begin{cases} \mathrm{cone}(\alpha_i) & \phi = \phi_i \\ 0 & \text{otherwise} \end{cases}$$

$$\mathrm{supp}_{\mathscr{P}}(X) = \{\phi_1, \cdots, \phi_n\} = \{\phi \in J \mid H^\phi_{\mathscr{P}}(X) \neq 0\}$$
$$\phi^+_{\mathscr{P}}(X) = \phi_1 = \max \mathrm{supp}_{\mathscr{P}}(X)$$
$$\phi^-_{\mathscr{P}}(X) = \phi_n = \min \mathrm{supp}_{\mathscr{P}}(X)$$

This gives two morphisms of $\mathbb{Z}$-posets:

$$\mathbf{\Omega}_J(\mathscr{D}) \xrightarrow{\phi^{\pm}_{*}(X)} J$$

Proposition 3.7. *Let $\mathscr{P}$ be a J-slicing of $\mathscr{D}$, $0 \neq X \in \mathscr{D}$. For each $\phi \in J$:*

$$\tau_{\mathscr{P}}^{\leq\phi}\tau_{\mathscr{P}}^{\geq\phi} = \tau_{\mathscr{P}}^{\geq\phi}\tau_{\mathscr{P}}^{\leq\phi} = H^\phi_{\mathscr{P}}$$

$H^\phi_{\mathscr{P}}$ is thus an additive functor, called the ***cohomology*** *in grade ϕ induced by $\mathscr{P}$.*

Proof. The statement is obvious if $\phi \notin \mathrm{supp}_{\mathscr{P}}(X)$. Now, suppose $\phi = \phi_i$ for some i. By the last claim of **Proposition 2.27**, since α_i is the natural transformation $\tau_{\mathscr{P}}^{\geq \phi_{i-1}}(X) \longrightarrow \tau_{\mathscr{P}}^{\geq \phi_i}(X)$, $H_{\mathscr{P}}^{\phi_i}(X) = \tau_{\mathscr{P}}^{\geq \phi_i}(\tau_{\mathscr{P}}^{<\phi_{i-1}}(X)) = \tau_{\mathscr{P}}^{<\phi_{i-1}}(\tau_{\mathscr{P}}^{\geq \phi_i}(X))$. But by the proof of **Proposition 3.4**, $\tau_{\mathscr{P}}^{<\phi_{i-1}}(X) = \tau_{\mathscr{P}}^{\leq \phi_i}(X)$, and we can conclude. □

Proposition 3.8. *If $X \longrightarrow Y \longrightarrow Z \longrightarrow$ is a distingusihed triangle in $\mathscr{D}$ with nonzero vertices, then*

$$\min\{\phi_{\mathscr{P}}^-(X), \phi_{\mathscr{P}}^-(Z)\} \leq \phi_{\mathscr{P}}^-(Y) \leq \phi_{\mathscr{P}}^+(Y) \leq \max\{\phi_{\mathscr{P}}^+(X), \phi_{\mathscr{P}}^+(Z)\}$$

Proof. We'll just prove the right inequality, the other follows similarly. Denote $\phi = \phi_{\mathscr{P}}^+(Y)$. If $\phi > \phi_{\mathscr{P}}^+(Z)$, then looking at the Postnikov tower of Z one sees that $Z \in \mathscr{P}([\phi, +\infty[)^{\perp}$ and thus by **Proposition 2.25** $\tau_{\mathscr{P}}^{\geq \phi}(X) = \tau_{\mathscr{P}}^{\geq \phi}(Y) = Y \neq 0$. This means that $\phi_{\mathscr{P}}^+(X) \geq \phi$, which is what we wanted. □

Proposition 3.9. *Let $I \subseteq J$ an interval. Then $\mathscr{P}_I$ is the full subcategory of $\mathscr{D}$ consising of the zeroes of $\mathscr{D}$ and those $0 \neq X$ so that $\phi_{\mathscr{P}}^{\pm}(X) \in I$.*

Proof. Since I is an interval, the existence of Postnikov towers tells us that the $\mathscr{P}_I$ must contain the objects in the statement. But this already gives and extension-closed subcategory by **Proposition 3.8**, so we are done. □

Proposition 3.10. *For each $X, Y \in \mathscr{D}$,*

$$\mathrm{supp}_{\mathscr{P}}(X \oplus Y) = \mathrm{supp}_{\mathscr{P}}(X) \cup \mathrm{supp}_{\mathscr{P}}(Y)$$

Proof. For each $\phi \in J$, since $H_{\mathscr{P}}^{\phi}$ is additive we have

$$H_{\mathscr{P}}^{\phi}(X \oplus Y) = H_{\mathscr{P}}^{\phi}(X) \oplus H_{\mathscr{P}}^{\phi}(Y)$$

which can be nonzero if and only if either $H_{\mathscr{P}}^{\phi}(X)$ or $H_{\mathscr{P}}^{\phi}(Y)$ is nonzero. □

Proposition 3.11. *For each $\phi \in J$, $\mathscr{P}_\phi \subseteq \mathscr{D}$ is an extension-closed subcategory closed under direct summands.*

Proof. Since $\mathscr{P}_\phi = \mathscr{P}_{\{\phi\}}$ by **Proposition 3.9**, it is extension-closed. The closeness under direct summand follows from **Proposition 3.10**. □

Proposition 3.12. *Associating $\mathbf{\Omega}_J(\mathscr{D})$ to J defines a 2-functor*

$$\mathrm{tPos}_{\mathbb{Z}} \xrightarrow{\mathbf{\Omega}_*(\mathscr{D})} \mathrm{Pos}_{\mathbb{Z}}$$

We call it the ***slice functor****.*

Proof. Let $J \xrightarrow{f} J'$ be a morphism of (totally ordered) $\mathbb{Z}$-posets. Then for each J-slicing $\mathscr{P}$ of $\mathscr{D}$ we define

$$(f_{\underline{\Omega}}(\mathscr{P}))_\phi = \mathscr{P}_{f^{-1}(\phi)}$$

whith $\phi \in J'$. We will show that $f_{\underline{\Omega}}(\mathscr{P})$ is a J'-slicing on $\mathscr{D}$. Part *(1)* of the definition is trivial. If $\phi > \psi$ in J', since J is totally ordered, $f^{-1}(\phi) > f^{-1}(\psi)$ in 2^J and part *(2)* then follows from **Proposition 3.3**. Now, let $0 \neq X \in \mathscr{D}$ and

$$0 = X_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} X_n = X$$

its Postnikov tower with respect to $\mathscr{P}$. Starting from left, if we encounter an i so that $f(\phi_i) = f(\phi_{i+1}) = \phi$, taking the braid generated by $\alpha_{i+1}\alpha_i$ we get a distingusihed triangle

$$H^{\phi_i}_{\mathscr{P}}(X) \longrightarrow \mathrm{cone}(\alpha_{i+1}\alpha_i) \longrightarrow H^{\phi_{i+1}}_{\mathscr{P}}(X) \longrightarrow$$

By extensin-closeness, $\mathrm{cone}(\alpha_{i+1}\alpha_i) \in \mathscr{P}_{f^{-1}(\phi)} = (f_{\underline{\Omega}}(\mathscr{P}))_\phi$. We then replace the arrows

$$X_{i-1} \xrightarrow{\alpha_i} X_i \xrightarrow{\alpha_{i+1}} X_{i+1}$$

with the arrow $X_{i-1} \xrightarrow{\alpha_{i+1}\alpha_i} X_{i+1}$. Iterating this finite process, we get the desired Postnikov tower with respect to $f_{\underline{\Omega}}(\mathscr{P})$. $\square$

Now, since by definition $(f_{\underline{\Omega}}(\mathscr{P}))_I = \mathscr{P}_{f^{-1}(I)}$ for each interval $I \subseteq J$, one has a commutative diagram:

$$\begin{tikzcd}
\mathbf{\Omega}_J(\mathscr{D}) \arrow[r] \arrow[d, "f_{\Omega}"] & \mathrm{Hom}_{\mathrm{Pos}_{\mathbb{Z}}}(O(J), \mathfrak{ts}(\mathscr{D})) \arrow[d] \\
\mathbf{\Omega}_{J'}(\mathscr{D}) \arrow[r] & \mathrm{Hom}_{\mathrm{Pos}_{\mathbb{Z}}}(O(J'), \mathfrak{ts}(\mathscr{D}))
\end{tikzcd}$$

where the horizontal maps are given by **Proposition 3.4** and the right map is the precomposition by $O(J') \xrightarrow{f^{-1}} O(J)$. In other words, the slice functor is a sub-2-functor of $\mathrm{Hom}_{\mathscr{D}}(O(*), \mathfrak{ts}(\mathscr{D}))$, which we can restate as:

♫ *The slice functor is the direct image functor for presheaves of t-structures on the upper Alexandrov topologies*

Observe that by the proof of the above proposition, in the same notation, for each $X \in \mathscr{D}$ we have

$$\mathrm{supp}_{f_{\Omega}(\mathscr{P})}(X) = f(\mathrm{supp}_{\mathscr{P}}(X))$$

Now, we prove a reconstruction theorem for the slice functor.

Proposition 3.13. *Let $J \xrightarrow{f} J'$ be a morphism of (totally ordered) $\mathbb{Z}$-posets, $\mathscr{P}$ a J-slicing on $\mathscr{D}$. Then the Postnikov towers with respect to $\mathscr{P}$ can be recovered from the Postnikov towers with respect to $f_{\underline{\Omega}}(\mathscr{P})$ and the Postnikov towers with respect to $\mathscr{P}$ of semistables with respect to $f_{\underline{\Omega}}(\mathscr{P})$. Moreover for each $\phi \in J$*

$$H^{\phi}_{\mathscr{P}} = H^{\phi}_{\mathscr{P}} H^{f(\phi)}_{f_{\underline{\Omega}}(\mathscr{P})}$$

Proof. Pick $0 \neq X \in \mathscr{D}$. Consider its Postnikov tower with respect to $f_{\underline{\Omega}}(\mathscr{P})$

$$0 = X_0 \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_n} X_n = X$$

and consider the Postnikov tower of $H^{\phi_i}_{f_{\underline{\Omega}}(\mathscr{P})}(X)$ with respect to $\mathscr{P}$

$$0 = X_{i,0} \xrightarrow{\alpha_{i,1}} \cdots \xrightarrow{\alpha_{i,m_i}} X_{i,m_i} = H^{\phi_i}_{f_{\underline{\Omega}}(\mathscr{P})}(X)$$

Now, for $i > 1$ and $j > 0$, composing $\alpha_{i,m_i} \cdots \alpha_{i,j+1}[-1]$ with the composition $H^{\phi_i}_{f_{\underline{\Omega}}(\mathscr{P})}(X)[-1] \longrightarrow X_{i-1} \longrightarrow H^{\phi_{i-1}}_{f_{\underline{\Omega}}(\mathscr{P})}(X)$ we have a morphism $X_{i,j}[-1] \longrightarrow H^{\phi_{i-1}}_{f_{\underline{\Omega}}(\mathscr{P})}(X)$ and we denote $T_{i,j}$ its cone. We also set $T_{1,j} = X_{1,j}$. By the **3 × 3 Lemma** we get a commutative diagram:

$$\begin{tikzcd}
X_{i,j}[-1] \arrow[r] \arrow[d, "{\alpha_{i,j+1}[-1]}"] & H^{\phi_{i-1}}_{f_{\underline{\Omega}}(\mathscr{P})}(X) \arrow[r] \arrow[d, "1"] & T_{i,j} \arrow[r] \arrow[d] & X_{i,j} \arrow[d] \\
X_{i,j+1}[-1] \arrow[r] \arrow[d] & H^{\phi_{i-1}}_{f_{\underline{\Omega}}(\mathscr{P})}(X) \arrow[r] \arrow[d] & T_{i,j+1} \arrow[r] \arrow[d] & X_{i,j+1} \arrow[d] \\
H^{\phi_{i,j+1}}_{\mathscr{P}}(X)[-1] \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & H^{\phi_{i,j+1}}_{\mathscr{P}}(X) \arrow[d] \\
X_{i,j} \arrow[r] & H^{\phi_{i-1}}_{f_{\underline{\Omega}}(\mathscr{P})}(X)[1] \arrow[r] & T_{i,j}[1] \arrow[r] & X_{i,j}[1]
\end{tikzcd}$$

Thus we have constructed a morphsism $T_{i,j} \longrightarrow T_{i,j+1}$ whose cone is $H = H^{\phi_{i,j+1}}_{\mathscr{P}}(X)$, which means that the Postnikov tower of X with respect to $\mathscr{P}$ is

$$0 \longrightarrow T_{1,1} \longrightarrow T_{1,2} \longrightarrow \cdots \longrightarrow T_{n,m_n} = X$$

The second claim follows from **Proposition 2.12** applied to the third row of the above diagram. □

Proposition 3.14. *If $J \xrightarrow{f} J'$ is an isomorphism of (totally ordered) $\mathbb{Z}$-posets, then for each $\phi \in J$*

$$H^{\phi}_{\mathscr{P}} = H^{f(\phi)}_{f_{\Omega}(\mathscr{P})}$$

Proof. By **Proposition 3.13**

$$H^{\phi}_{\mathscr{P}} = H^{\phi}_{\mathscr{P}} H^{f(\phi)}_{f_{\Omega}(\mathscr{P})}$$

Since $f^{-1}(\phi) = \{\phi\}$ we have that $H^{f(\phi)}_{f_{\Omega}(\mathscr{P})} \in (f_{\Omega}(\mathscr{P}))_{f(\phi)} = \mathscr{P}_{\phi}$ and thus $H^{\phi}_{\mathscr{P}} H^{f(\phi)}_{f_{\Omega}(\mathscr{P})} = H^{f(\phi)}_{f_{\Omega}(\mathscr{P})}$, as desired. $\square$

Definition 3.15. *A J-slicing $\mathscr{P}$ on $\mathscr{D}$ is called **hereditary** if*

$$\mathscr{P}_{\psi} \subseteq \mathscr{P}_{\phi}^{\perp}$$

for $\phi + 1 < \psi$.

Hereditary slicings are remarkable for their splitting properties.

Proposition 3.16. *Let $\mathscr{P}$ be a hereditary J-slicing on $\mathscr{D}$. Then for each $X \in \mathscr{D}$*

$$X = \bigoplus_{\phi \in J} H^{\phi}_{\mathscr{P}}(X)$$

In particular, indecomposable objects of $\mathscr{D}$ are semistable with respect to $\mathscr{P}$.

Proof. Denote $\phi = \phi^{-}_{\mathscr{P}}(X)$, $H = H^{\phi}_{\mathscr{P}}(X)$. By looking at the last step of the Postnikov tower of X with respect to $\mathscr{P}$, we have a distinguished triangle

$$Y \longrightarrow X \longrightarrow H \longrightarrow$$

Now, since $\mathscr{P}$ is hereditary, $\mathrm{Hom}_{\mathscr{D}}(H, Y[1]) = 0$. This means that the last morphism of the above triangle is 0, and thus by **Proposition 2.14** $X = Y \oplus H$, and we iterate the argument on Y to get the desired result. $\square$

Example 3.17. *If R is an hereditary ring (i.e. $\mathrm{Ext}^2_R(*, *) = 0$) then the standard t-structure on $\mathscr{D}^b(\mathrm{Mod}_R)$, seen as a $\mathbb{Z}$-slicing, is hereditary by **Example 2.31**, which is the reason of our nomenclature. If R is semisimple, then we also have $\mathrm{Ext}^1_R(*, *) = 0$, and thus the splitting in **Proposition 3.16** defines an equivalence between $\mathscr{D}(\mathrm{Mod}_R)$ and the category of $\mathbb{Z}$-graded R-modules (with morphisms of degree 0).*

Definition 3.18. *Let $\mathscr{P}$ be a J-slicing, $\mathscr{Q}$ a J'-slicing. The ordered pair $(\mathscr{P}, \mathscr{Q})$ is said to be **consistent** if for each $\phi \in J$,$\psi \in J'$*

$$H^{\psi}_{\mathscr{Q}}(\mathscr{P}_{\phi}) \subseteq \mathscr{P}_{\phi}$$

Proposition 3.19. *Suppose that* $(\mathscr{P},\mathscr{Q})$ *is consistent and denote for each* $\phi \in J, \psi \in J'$

$$(\mathscr{P} \wedge \mathscr{Q})_{(\phi,\psi)} = \mathscr{P}_\phi \cap \mathscr{Q}_\psi$$

Then $\mathscr{P} \wedge \mathscr{Q}$ *is a* $J \times J'$*-slicing on* $\mathscr{D}$ *satisfying*

$$H^{(\phi,\psi)}_{\mathscr{P}\wedge\mathscr{Q}} = H^\psi_{\mathscr{Q}} H^\phi_{\mathscr{P}}$$

Proof. By hypothesis, the Postnikov towers with respect to $\mathscr{Q}$ of objects in $\mathscr{P}_\phi$ are still in $\mathscr{P}_\phi$. Combining them with the Postnikov towers with respect to $\mathscr{P}$ as in the proof of **Proposition 3.13** we get the desired result. □

Observe that if $(\mathscr{P},\mathscr{Q})$ is consistent, if we denote $J \times J \xrightarrow{\pi} J$ the projection, then

$$\pi_{\mathbf{\Omega}}(\mathscr{P} \wedge \mathscr{Q}) = \mathscr{P}$$

To conclude, we deal with group actions. First of all, by functoriality of slicings, $\mathrm{Aut}_{\mathrm{Pos}_{\mathbb{Z}}}(J)$ acts on $\mathbf{\Omega}_J(\mathscr{D})$ preserving hereditariness.

Proposition 3.20. $\mathrm{Aut}(\mathscr{D})$ *acts on* $\mathbf{\Omega}_J(\mathscr{D})$. *This action preserves hereditariness.*

Proof. For $F \in \mathrm{Aut}(\mathscr{D})$ and $\mathscr{P}$ a J-slicing on $\mathscr{D}$, define fot $\phi \in J$

$$(F.\mathscr{P})_\phi = F(\mathscr{P}_\phi)$$

This defines a J-slicing, since the Postnikov tower of X with respect to $F.\mathscr{P}$ is simply the image by F of the Postnikov tower of $F^{-1}(X)$ with respect to $\mathscr{P}$. □

The above proof also shows that for each $\phi \in J$

$$H^\phi_{F.\mathscr{P}} = F H^\phi_{\mathscr{P}} F^{-1}$$

Proposition 3.21. *The actions by* $\mathrm{Aut}(\mathscr{D})$ *and by* $\mathrm{Aut}_{\mathrm{Pos}_{\mathbb{Z}}}(J)$ *on* $\mathbf{\Omega}_J(\mathscr{D})$ *commute.*

Proof. Let F be a triangle autoequivalence of $\mathscr{D}$, f an automorphism of J as a $\mathbb{Z}$-poset, $\mathscr{P}$ a J-slicing on $\mathscr{D}$, $\phi \in J$. We have to show that

$$F(\mathscr{P}_{f^{-1}(\phi)}) = (F.\mathscr{P})_{f^{-1}(\phi)}$$

But this simply follows from **Proposition 2.19**. □

3.1. J **maps to $\mathbb{Z}$: abelian slicings.**

In some sense spinors describe the 'square root' of geometry and, just as understanding the square root of -1 took centuries, the same might be true of spinors.

Michael Atiyah

Proposition 3.22. *There is a (canonical) isomorphism of $\mathbb{Z}$-posets:*

$$\mathbf{\Omega}_{\mathbb{Z}}(\mathscr{D}) = \mathfrak{bts}(\mathscr{D})$$

which preserves the notions of Postnikov towers and cohomology.

Proof. Since $O(\mathbb{Z}) = \mathbb{Z}$ in the obvious way and $\mathrm{Hom}_{\mathrm{Pos}_{\mathbb{Z}}}(\mathbb{Z}, J) = J$ for each $\mathbb{Z}$-poset J, by **Proposition 3.4** we have a morphism of $\mathbb{Z}$-posets

$$\mathbf{\Omega}_{\mathbb{Z}}(\mathscr{D}) \longrightarrow \mathfrak{bts}(\mathscr{D})$$

The inverse of this maps is given by associating to $\mathfrak{t} \in \mathfrak{bts}(\mathscr{D})$ the collection $\{\heartsuit_{\mathfrak{t}}[n]\}_{n\in\mathbb{Z}}$, which is a $\mathbb{Z}$-slicing by **Proposition 2.35**. □

This means that some of the study of $\mathbb{Z}$-slicings has already been done. We now define the abelian counterpart of slicings, and prove a compatibiliy theorem.

Definition 3.23. *Let $\mathscr{A}$ be an abelian category, I a totally ordered set. An **abelian I-slicing** on $\mathscr{A}$ is a collection $\mathscr{P} = \{\mathscr{P}_\phi\}_{\phi\in I}$ of full extension-closed (in the abelian sense) additive subcategories $\mathscr{P}_\phi \subseteq \mathscr{A}$ so that:*

(1) $\mathscr{P}_\psi \subseteq \mathscr{P}_\phi^\perp$ if $\phi > \psi$

*(2) for each $0 \neq X \in \mathscr{A}$ there is a finite strictly decreasing sequence $\{\phi_1 > \cdots > \phi_n\} \subseteq I$ and a finite filtration by subobjects, called a **Harder-Narasimhan filtration** of X with respect to $\mathscr{P}$,*

$$0 = X_0 \subseteq \cdots \subseteq X_n = X$$

so that $0 \neq X_i/X_{i-1} \in \mathscr{P}_{\phi_i}$ for all $1 \leq i \leq n$

We denote $\mathbf{\Omega}_I(\mathscr{A})$ the set of abelian I-slicings on $\mathscr{A}$.

Proposition 3.24. *Let I be a totally-ordered set. Then there is a (canonical) identification*

$$\mathbf{\Omega}_{\mathbb{Z}\times I}(\mathscr{D}) = \coprod_{\mathfrak{t}\in\mathfrak{bts}(\mathscr{D})} \mathbf{\Omega}_I(\heartsuit_{\mathfrak{t}})$$

where the product on $\mathbb{Z} \times I$ is lexicographical and $\mathbb{Z}$ acts on the left factor.

Proof. If $\mathscr{P}$ is a $\mathbb{Z} \times I$-slicing on $\mathscr{D}$, we see applying the slice functor to the projection $\mathbb{Z} \times I \longrightarrow \mathbb{Z}$ that $\{\mathscr{P}_{\{n\}\times I}\}_{n\in\mathbb{Z}}$ defines a $\mathbb{Z}$-slicing on $\mathscr{D}$, and thus a t-structure whose heart is $\heartsuit = \mathscr{P}_{\{0\}\times I}$. But then $\{\mathscr{P}_{(0,\phi)}\}_{\phi\in I}$ is an abelian I-slicing on $\heartsuit$: the Harder-Narasimhan filtrations are simply the Postnikov towers with respect to $\mathscr{P}$, since distinguished triangles with vertices in $\heartsuit$ correspond to short exact sequences in $\heartsuit$.
Conversely, if $\mathfrak{t}$ is a t-structure and $\{\mathscr{P}_\phi\}_{\phi\in I}$ is an abelian I-slicing on $\heartsuit_{\mathfrak{t}}$, then $\mathscr{Q}_{(n,\phi)} = \mathscr{P}_\phi[n]$ defines a $\mathbb{Z} \times I$-slicing on $\mathscr{D}$ by (the proof of) **Proposition 3.13**. ☐

Since every abelian category is the heart of a bounded t-structure (for example of the standard one in its bounded derived category), all the properties (functoriality of cohomology, inequalities etc ...) from the previous paragraph hold similarly for abelian slicings, and we keep similar notations for the latter. Now, the proof of the above proposition also shows that if $\mathfrak{t}$ is a bounded t-structure on $\mathscr{D}$ and $\mathscr{P}$ an abelian I-slicing on $\heartsuit_{\mathfrak{t}}$, then associated $\mathbb{Z} \times I$-slicing $\mathscr{Q}$ on $\mathscr{D}$ satisfies:

$$H_{\mathscr{Q}}^{(n,\phi)} = \Sigma^n H_{\mathscr{P}}^{\phi} \Sigma^{-n} H_{\mathfrak{t}}^n$$

We now characterize the $\mathbb{Z}$-posets which are of the form required to apply **Proposition 3.24**.

Proposition 3.25. *A (totally ordered) $\mathbb{Z}$-poset J is isomorphic to the lexicographical product $\mathbb{Z} \times I$ for some totally ordered set I if and only if*

$$\mathrm{Hom}_{\mathrm{Pos}_{\mathbb{Z}}}(J, \mathbb{Z}) \neq \emptyset$$

Proof. First of all the 'only if' part is trivial (just consider the projection to $\mathbb{Z}$). For the 'if' part, pick a morphism $J \xrightarrow{f} \mathbb{Z}$ and set $I = f^{-1}(0)$. Then associating $\phi + n$ to $(n, \phi) \in \mathbb{Z} \times I$ defines an isomorphism $\mathbb{Z} \times I = J$. ☐

Now, denote $[n] = \{0, \cdots, n\}$ with usual order. An abelian $[n]$-slicing on $\mathscr{A}$ is called n-**torsion pair** in literature, and a 1-torsion pair is simply called torsion pair.

Example 3.26. *Let R be an integral domain and denote by $\mathscr{P}_1 \subseteq$* Mod_R *the full subcategory of torsion modules and by $\mathscr{P}_0$ the full subcategory of torsion-free modules. Then $\{\mathscr{P}_0, \mathscr{P}_1\}$ is a torsion pair on*

Mod_R, *which is the reason of our nomenclature. This can clearly be generalized to arbitrary integral schemes.*

Observe that giving a morphism of totally ordered $\mathbb{Z}$-posets $\mathbb{Z} \times [n] \longrightarrow J'$ is like giving a chain $\phi_0 \leq \phi_1 \leq \cdots \leq \phi_n \leq \phi_0 + 1$ in J'. Now, the following result generalizes both ????? ???? and **Proposition 3.22**.

Proposition 3.27. *There is a (canonical) isomorhpsim of $\mathbb{Z}$-posets*

$$\mathbf{\Omega}_{\mathbb{Z}\times[n]}(\mathscr{D}) = \mathrm{Hom}_{\mathrm{Pos}_{\mathbb{Z}}}(\mathbb{Z} \times [n], \mathfrak{bts}(\mathscr{D}))$$

Proof. First of all, there is an obvious isomorphism of $\mathbb{Z}$-posets $O(\mathbb{Z} \times [n]) = \mathbb{Z} \times [n]$, and thus we have a morphism of $\mathbb{Z}$ posets

$$\mathbf{\Omega}_{\mathbb{Z}\times[n]}(\mathscr{D}) \longrightarrow \mathrm{Hom}_{\mathrm{Pos}_{\mathbb{Z}}}(\mathbb{Z} \times [n], \mathfrak{bts}(\mathscr{D}))$$

To construct the inverse, take bounded t-structures $\mathfrak{t}_0 \supseteq \mathfrak{t}_1 \supseteq \cdots \supseteq \mathfrak{t}_n \supseteq \mathfrak{t}_0[1] = \mathfrak{t}_{n+1}$ and set for $0 \leq i \leq n$

$$\mathscr{P}_i = \mathfrak{t}_i \cap \mathfrak{t}_{i+1}^{\perp}$$

This inductively defines an abelian $[n]$-slicing on $\heartsuit_{\mathfrak{t}_0}$ and using **Proposition 2.35** we get a $\mathbb{Z} \times [n]$-slicing on $\mathscr{D}$. ☐

This, combined with **Proposition 3.24**, means that if $\mathfrak{t}$ is a bounded t-structure on $\mathscr{D}$ then giving a torsion pair on $\heartsuit_{\mathfrak{t}}$ is like giving a bounded t-structure $\mathfrak{q}$ with

$$\mathfrak{t}[1] \subseteq \mathfrak{q} \subseteq \mathfrak{t}$$

We call $\mathfrak{q}$ the **tilting** of $\mathfrak{t}$ with respect to the given torsion pair. Now, if I is a totally ordered set then giving a morphism of posets $I \longrightarrow [1]$ is like giving an upper set in $O(I)$ (the identification is given by seeing an upper set as its characteristic function). Thus, when $J = \mathbb{Z} \times I$, if we fix a bounded t-structure $\mathfrak{t}$, then all the t-structures of **Proposition 3.4** up to translation are obtained by tilting $\mathfrak{t}$ with respect to a torsion pair. Tilting is thus our main tool to construct new t-structures from old ones, and we give a characterization of the tilted heart in the following proposition.

Proposition 3.28. *Let $\mathfrak{t}$ a bounded t-structure on $\mathscr{D}$, $\{\mathscr{P}_0, \mathscr{P}_1\}$ a torsion pair on $\heartsuit_{\mathfrak{t}}$, $\mathfrak{q}$ the corresponding tilting. Then $\heartsuit_{\mathfrak{q}}$ consists of objects $X \in \mathscr{D}$ so that*

$$H^n_{\mathfrak{t}}(X) \in \mathscr{P}_n[n]$$

for $n = 0, 1$ and $H^n_{\mathfrak{t}}(X) = 0$ for $n \neq 0, 1$.

Proof. Denote by $\mathscr{Q}$ the corresponding $\mathbb{Z} \times [1]$-slicing obtained from **Proposition 3.24**. Then

$$\heartsuit_{\mathfrak{q}} = \mathscr{Q}_{\{(0,1),(1,0)\}}$$

By **Proposition 3.9**, $\heartsuit_{\mathfrak{q}}$ consists of objects X so that

$$H_{\mathscr{Q}}^{(n,\phi)}(X) = H_{\mathscr{P}}^{\phi}(H_{\mathfrak{t}}^{n}(X)[-n])[n] = 0$$

for $(n,\phi) \neq (0,1),(1,0)$, and this gives the desired result. $\square$

To conclude, we can also view the operation of tilting in a functorial way. If we consider the automorphism $\mathbb{Z} \times [1] \xrightarrow{f} \mathbb{Z} \times [1]$ given by

$$\begin{cases} f(n,0) = (n,1) \\ f(n,1) = (n+1,0) \end{cases}$$

$$\begin{tikzcd}
\cdots & (-1,0) \arrow[r, bend right, "f"'] & (-1,1) \arrow[r, bend right, "f"'] & (0,0) \arrow[r, bend right, "f"'] & (0,1) \arrow[r, bend right, "f"'] & (1,0) \arrow[r, bend right, "f"'] & (1,1) & \cdots
\end{tikzcd}$$

then $\mathfrak{q}$ is the bounded t-structure associated to $f_{\underline{\Omega}}(\mathscr{Q})$. Observe moreover that $f^2 = * + 1$, and thus:

♫ *Considering torsion pairs on hearts of bounded t-structures induces square roots of the translation.*

3.2. $\mathbb{Z}$ **acts trivially: semiorthogonal decompositions.** We now deal with the case of trivial action by $\mathbb{Z}$. First of all this immediately implies that all the t-structures of **Proposition 3.4** are fixed by translation, and thus have trivial (zero) heart and are not bounded. Moreover, saying that a J-slicing $\mathscr{P}$ is hereditary means that the subcategories $\mathscr{P}_\phi$ are (mutually) orthogonal.

Proposition 3.29. *Suppose that the action of $\mathbb{Z}$ on J is trivial and let $\mathscr{P}$ be a J-slicing on $\mathscr{D}$. Then for each $\phi \in J$, $\mathscr{P}_\phi$ is thick, $H^\phi_{\mathscr{P}}$ is a triangle functor and*

$$K_0(\mathscr{D}) = \bigoplus_{\phi \in J} K_0(\mathscr{P}_\phi)$$

Proof. Since the action is trivial, $\mathscr{P}_\phi = \mathscr{P}_{\phi+1} = \mathscr{P}[1]$ and thus $\mathscr{P}$ is thick. By the same take, $\mathscr{P}_I$ is thick for each subset $I \in 2^J$. But then $\tau^{\geq \phi}_{\mathscr{P}}$, which is adjoint to the triangle inclusion $\mathscr{P}_{[\phi,+\infty[} \subseteq \mathscr{D}$, is triangle by **Proposition 2.17**, and similarly $\tau^{\leq \phi}_{\mathscr{P}}$ is triangle too. This implies that $H^\phi_{\mathscr{P}} = \tau^{\geq \phi}_{\mathscr{P}} \tau^{\leq \phi}_{\mathscr{P}}$ is triangle. Now, taking the sum of the morphisms of groups induced by the inclusions $\mathscr{P}_\phi \subseteq \mathscr{D}$ we have a morphism of groups

$$\bigoplus_{\phi \in J} K_0(\mathscr{P}_\phi) \longrightarrow K_0(\mathscr{D})$$

and its inverse is given by the morphism induced by $\oplus_{\phi \in J} H^\phi_{\mathscr{P}}$, as desired. $\square$

Observe that if $J \xrightarrow{f} J'$ be a morphism of (totally ordered) $\mathbb{Z}$-posets and the action of $\mathbb{Z}$ on J' is trivial, then each fiber of f is a sub-$\mathbb{Z}$-poset of J.

Proposition 3.30. *Let $J \xrightarrow{f} J'$ be a morphism of (totally ordered) $\mathbb{Z}$-posets and suppose that the action of $\mathbb{Z}$ on J' is trivial. Then there is a (canonical) identification:*

$$\mathbf{\Omega}_J(\mathscr{D}) = \coprod_{\mathscr{P} \in \mathbf{\Omega}_{J'}(\mathscr{D})} \prod_{\phi \in J'} \mathbf{\Omega}_{f^{-1}(\phi)}(\mathscr{P}_\phi)$$

Proof. It simply follows from the proof of **Proposition 3.13** with an argument very similar to the one used in **Proposition 3.24**. $\square$

We conclude the section by linking our theory to the classical theory of semiorthogonal decompositions.

Proposition 3.31. *If J is finite then $\mathbb{Z}$ acts trivially on J.*

Proof. Take $\phi \in \mathbb{Z}$. If $\phi + 1 \neq \phi$, then $\phi + 1 > \phi$ and thus we have an infinite chain $\cdots \phi + i + 1 > \phi + i > \cdots > \phi$, which is absurd since J is finite. Thus $\phi + 1 = \phi$, which means that the whole $\mathbb{Z}$ stabilizes ϕ and by arbitrariness of ϕ we get the result. □

As a consequence, if J is finite then it is isomorphic to $[n]$ (with trivial action), where $n + 1$ is the cardinality of J. Now, $[n]$-slicings are precisely the **semiorthogonal decompositions** which appear in literature. Clearly, a $[1]$-slicing is simply a t-structure which is fixed by translation. Moreover, if $J = \mathbb{Z}$ with trivial action, then a J-slicing is called **baric structure** in REFERENZAAAAA ????.

Definition 3.32. *A full subcategory $\mathscr{E} \subseteq \mathscr{D}$ is called **admissible** if $\{\mathscr{E}, {}^{\perp}\mathscr{E}\}$ and $\{\mathscr{E}^{\perp}, \mathscr{E}\}$ both define a $[1]$-slicing on $\mathscr{D}$.*

In other words, $\mathscr{E}$ is admissible if and only if it is a t-structure fixed by translation on both $\mathscr{D}$ and $\mathscr{D}^{\mathrm{op}}$.

Proposition 3.33. *Let $\mathscr{E}$ be an admissible subcategory of $\mathscr{D}$. Then the restriction*

$$ {}^{\perp}\mathscr{E} \xrightarrow{\tau_{\mathscr{E}}^{<}} \mathscr{E}^{\perp} $$

is a triangle equivalence.

Proof. □

Proposition 3.34. *There is a (totally ordered) $\mathbb{Z}$-poset $I(J)$ on which $\mathbb{Z}$ acts trivially and a morphism of $\mathbb{Z}$-posets $J \xrightarrow{\pi} I(J)$ whose fibers are singletons or (noncanonically) isomorphic to $\mathbb{Z} \times I$ for some totally ordered set I (depending on the fiber).*

Proof. Define the following equivalence relation on J: $\phi \sim \psi$ if and only if there are integers $a, b \in \mathbb{Z}$ so that $\phi + a \leq \psi \leq \phi + b$. This is clearly an equivalence relation. Define

$$ I(J) = J/\sim $$

One can easily check that if $\phi \sim \phi'$, $\psi \sim \psi'$, $\phi \not\sim \psi$ and $\phi > \psi$ then $\phi' > \psi'$, and thus the order of J descends to $I(J)$, on which we place the trivial $\mathbb{Z}$-action. Moreover $\phi + n \sim \phi$ for each $n \in \mathbb{Z}$, and thus the projection $J \xrightarrow{\pi} I(J)$ is a morphism of $\mathbb{Z}$-posets.
Now, let's show that the equivalence classes are as claimed. If the class of $\phi \in J$ is not a singleton (i.e. ϕ is not fixed by $\mathbb{Z}$), then the floor function

$$ \lfloor \psi \rfloor = \min_{\phi + n \geq \psi} \{n\} $$

for $\psi \sim \phi$ defines a morphism of $\mathbb{Z}$-posets $\pi(\phi) \xrightarrow{\lfloor * \rfloor} \mathbb{Z}$: since $\phi+n+1 > \phi+n$ by hypothesis, we must have $\lfloor \phi+1 \rfloor = \lfloor \phi \rfloor + 1$. Using **Proposition 3.25**, we get an isomorphism $\pi(\phi) = \mathbb{Z} \times [\phi, \phi+1[$, as desired. $\square$

Now, choose one element ϕ_α in each class which is not a singleton. Using **Proposition 3.24** and **Proposition 3.30** we have

$$\mathbf{\Omega}_J(\mathscr{D}) = \coprod_{\mathscr{P} \in \mathbf{\Omega}_{I(J)}(\mathscr{D})} \prod_{\alpha} \coprod_{\mathfrak{t} \in \mathfrak{bts}\left(\mathscr{P}_{\pi(\phi_\alpha)}\right)} \mathbf{\Omega}_{[\phi_\alpha, \phi_\alpha+1[}(\heartsuit_{\mathfrak{t}})$$

To conclude, observe that we have a fully faithful functor $\mathrm{tPos} \hookrightarrow \mathrm{tPos}_{\mathbb{Z}}$ which takes a poset to itself with trivial action. One easily checks that associating $I(J)$ to J defines a functor

$$\mathrm{tPos}_{\mathbb{Z}} \xrightarrow{I(*)} \mathrm{tPos}$$

which is right adjoint to the above one.

3.3. **The (he)art of gluing.** The slice functor hides greater potential. Indeed, even if we don't have a morphism of $\mathbb{Z}$-posets (but just a $\mathbb{Z}$-equivariant map), then the slice functor can still be applied, obtaining a partially defined map.

Proposition 3.35. *Let $J \xrightarrow{f} J'$ be a a $\mathbb{Z}$-equivariant map (not necessarily increasing) between (totally ordered) $\mathbb{Z}$-posets, $\mathscr{P}$ a J-slicing on $\mathscr{D}$. Suppose that $\mathscr{P}_\psi \subseteq \mathscr{P}_\phi^\perp$ whenever one of the following conditions holds:*

(a) $f(\phi) > f(\psi)$
(b) $f(\phi) + 1 > f(\psi)$ *and* $\phi + 1 < \psi$

Then

$$(f_\Omega(\mathscr{P}))_\phi = \mathscr{P}_{f^{-1}(\phi)}$$

defines a J'-slicing on $\mathscr{D}$.

Proof. Part *(1)* of definition of slicing follows from $\mathbb{Z}$-equivariance of f, while part *(2)* follows from condition *(a)* above. Pick $0 \neq X \in \mathscr{D}$ and consider its Postnikov tower with respect to $\mathscr{P}$:

$$0 = Y_0 \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} Y_n = X$$

with $\mathrm{cone}(\beta_i) = H_{\mathscr{P}}^{\phi_i}(X)$. Going from left to right, if we encounter an i so that $f(\phi_i) < f(\phi_{i+1})$ then by condition *(b)*

$$\mathrm{Hom}_{\mathscr{D}}(H_{\mathscr{P}}^{\phi_{i+1}}(X), H_{\mathscr{P}}^{\phi_i}(X)[1]) = 0$$

Using **Proposition 2.11** and the 3×3 **Lemma**, we can complete the identity of Y_i to get a diagram:

$$\begin{tikzcd}
H_{\mathscr{P}}^{\phi_{i+1}}(X)[-1] \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d, "1"] & Y_{i+1} \arrow[r] \arrow[d] & H_{\mathscr{P}}^{\phi_{i+1}}(X)[1] \arrow[d] \\
Y_{i-1} \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & H_{\mathscr{P}}^{\phi_i}(X) \arrow[r] \arrow[d] & Y_{i-1}[1] \arrow[d] \\
A \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & A[1] \arrow[r] \arrow[d] & A[1] \arrow[d] \\
H_{\mathscr{P}}^{\phi_{i+1}}(X) \arrow[r] & Y_i[1] \arrow[r] & Y_{i+1}[1] \arrow[r] & H_{\mathscr{P}}^{\phi_{i+1}}(X)[1]
\end{tikzcd}$$

We then replace β_i with $Y_{i-1} \longrightarrow A$ and β_{i+1} with $A \longrightarrow Y_{i+1}$. Iterating this process[1], we get a factorization

$$0 = Z_0 \xrightarrow{\gamma_1} \cdots \xrightarrow{\gamma_n} Z_n = X$$

with $\mathrm{cone}(\gamma_i) = H^{\phi_{k_i}}_{\mathscr{P}}(X)$ for some k_i and $f(\phi_{k_i}) \geq f(\phi_{k_{i+1}})$. We then conclude using the same argument as in **Proposition 3.12**. □

The above proof also shows that, the same hypotheses and notation, $(f_{\underline{\Omega}}(\mathscr{P}))_\phi$ consists of objects $X \in \mathscr{D}$ so that $H^{\psi}_{\mathscr{P}}(X) = 0$ for $f(\psi) \neq \phi$.

Now we review a particular case of the gluing construction for t-structures in ????? REFERENZA, but generalize it to any slicing. We start with two (totally ordered) $\mathbb{Z}$-posets J, J' and consider $J \times J'$ with lexicographical order and the product action.

Definition 3.36. *Let $\mathscr{P}$ be a $J \times J'$-slicing on $\mathscr{D}$. We call $\mathscr{P}$* ***gluable*** *if it satisties the assumptions of* ***Proposition 3.35*** *with respect to the map $J \times J' \xrightarrow{f} J' \times J$ that exchanges coordinates. In this case we denote*

$$\overline{\mathscr{P}} = f_{\underline{\Omega}}(\mathscr{P})$$

If $\mathscr{P}$ is a $J \times J'$-slicing, the gluability condition reads: $\mathscr{P}_{(\phi,\psi)} \subseteq \mathscr{P}^{\perp}_{(\phi',\psi')}$ if $\psi' > \psi$ or both $\psi' + 1 > \psi$ and $\phi' + 1 < \phi$. For example, the first condition is automatic when $\mathscr{P}$ is obtained using **Proposition 3.19** from a consistent pair of slicings or when $\mathbb{Z}$ acts trivially on J (in this case, it follows from the second one).

Proposition 3.37. *Let $\mathscr{P}$ be a gluable $J \times J'$-slicing on $\mathscr{D}$. For $i = 1, 2$, denote $\mathscr{P}^i = \pi^i_{\underline{\Omega}}(\mathscr{P})$, where $J \xleftarrow{\pi^1} J \times J' \xrightarrow{\pi^2} J'$ are the projections. Then for each $\psi \in J'$, $\mathscr{P}^2_\psi$ consists of objects $X \in \mathscr{D}$ so that*

$$H^{\phi}_{\mathscr{P}^1}(X) \in \mathscr{Q}_{(\phi,\psi)}$$

for each $\phi \in J$.

Proof. We have already seen that $X \in \mathscr{P}^2_\psi$ if and only if

$$H^{\lambda}_{\mathscr{P}}(X) = H^{\lambda}_{\mathscr{P}}(H^{\pi^1(\lambda)}_{\mathscr{P}^1}(X)) = 0$$

for $\pi^2(\lambda) \neq \psi$. By fixing $\pi^1(\lambda) = \phi$ and varying $\pi^2(\lambda)$, we get the desired result. □

[1]This is somehow reminiscent of the 'bubble sort' algorithm.

In particular, if $\mathbb{Z}$ acts trivially on J, $\mathscr{P}$ is a J-slicing on $\mathscr{D}$ and t_ϕ is a bounded t-structure on $\mathscr{P}_\phi$ for each $\phi \in J$, then using **Proposition 3.30** we get a $J \times \mathbb{Z}$-slicing $\mathscr{Q}$ which is gluable if and only if

$$\heartsuit_{\mathsf{t}_\psi}[n] \subseteq \heartsuit_{\mathsf{t}_\phi}^{\perp}$$

for $n \leq 0$ and $\phi < \psi$. In this case, the heart of $\pi^2_{\underline{\Omega}}(\mathscr{Q})$ (seen as a bounded t-structure) consists of objects $X \in \mathscr{D}$ so that

$$H^\phi_{\mathscr{P}}(X) \in \heartsuit_{\mathsf{t}_\phi}$$

for each $\phi \in J$. We can go even further: if we take a morphism of posets $J \xrightarrow{p} \mathbb{Z}$ we can define the $\mathbb{Z}$-equivariant map $J \times \mathbb{Z} \xrightarrow{f} \mathbb{Z}$ as

$$f(\phi, n) = n + p(\phi)$$

If $\mathscr{Q}$ is gluable, an easy check shows that it indeed satisfies the assumptions of **Proposition 3.35** with respect to f, and we can consider $f_{\underline{\Omega}}(\mathscr{Q})$.

4. Some applications

4.1. **Calabi-Yau categories.** Suppose now that $\mathscr{D}$ is k-linear over some field k (we also assume that Σ is k-linear).

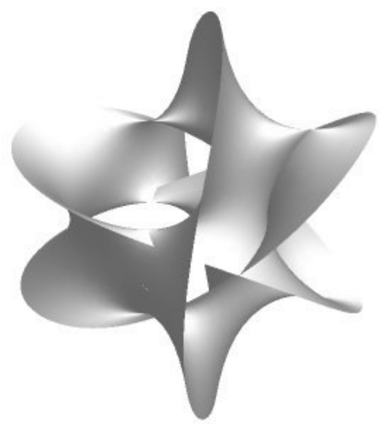


Figure 1. A depiction of a 3-dimensional real projection of a complex 2-dimensional section of the Fermat quintic in the 4-dimensional complex projective space.

Definition 4.1. *$\mathscr{D}$ is **Hom-finite** if for each $X, Y \in \mathscr{D}$ the graded k-vector space*

$$\mathrm{Hom}^\bullet_{\mathscr{D}}(X,Y) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{D}}(X, Y[i])$$

is finite dimensional.

Suppose moreover that $\mathscr{D}$ is Hom-finite.

Definition 4.2. *A **Serre functor** on $\mathscr{D}$ is a k-linear autoequivalence S so that there is a natural isomorphism of k-linear bifunctors from $\mathscr{D}^{\mathrm{op}} \times \mathscr{D}$ to Vect_k:*

$$\mathrm{Hom}_{\mathscr{D}}(X,Y) \xrightarrow{\varphi_{X,Y}} \mathrm{Hom}_{\mathscr{D}}(Y, S(X))^\vee$$

*If $n \in \mathbb{Z}$, we say that $\mathscr{D}$ is n-**Calabi-Yau** over k if Σ^n is a Serre functor on $\mathscr{D}$. We'll often refer to the above isomorphism as **Serre duality**.*

First of all, observe that if S is a Serre functor on $\mathscr{D}$, by bifunctoriality we have for each $X, Y \in \mathscr{D}$ a commutative diagram:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{D}}(X,Y) \arrow[r, "\varphi_{X,Y}"] \arrow[d, "S"] & \operatorname{Hom}_{\mathscr{D}}(Y,S(X))^{\vee} \\
\operatorname{Hom}_{\mathscr{D}}(S(X),S(Y)) \arrow[r, "\varphi_{S(X),S(Y)}"] & \operatorname{Hom}_{\mathscr{D}}(S(Y),S^2(X))^{\vee} \arrow[u, "S^{\vee}"']
\end{tikzcd}$$

which will legitimate us to write, as usual, equalities instead of isomorphisms.

Proposition 4.3. *$\mathscr{D}$ admits a Serre functor if and only if for each $X \in \mathscr{D}$ the functor $\operatorname{Hom}_{\mathscr{D}}(X,*)^{\vee}$ from $\mathscr{D}^{\mathrm{op}}$ to Mod_k is representable. Moreover, the Serre functor, if it exists, is unique up to natural isomorphism of k-linear functors.*

Proof. Clearly, if S if a Serre functor on $\mathscr{D}$, $S(X)$ represents $\operatorname{Hom}_{\mathscr{D}}(X,*)^{\vee}$. Conversely, for each $X \in \mathscr{D}$, choose a representative $S(X)$ for the functor $\operatorname{Hom}_{\mathscr{D}}(X,*)^{\vee}$ and a natural isomorphism of k-linear functors

$$\operatorname{Hom}_{\mathscr{D}}(*,S(X)) \xrightarrow{\varphi_X} \operatorname{Hom}_{\mathscr{D}}(X,*)^{\vee}$$

and this is the data that gives the desired Serre functor.
For the second part of the claim, if S and S' are Serre functors, then for each $X \in \mathscr{D}$, $S(X)$ and $S'(X)$ both represent the functor $\operatorname{Hom}_{\mathscr{D}}(X,*)^{\vee}$ and thus $S = S'$ by the Yoneda lemma. $\square$

Proposition 4.4. *Let S be a Serre functor on $\mathscr{D}$. Then:*

1. *S commutes (up to natural isomorphism) with all the autoequivalences of $\mathscr{D}$*
2. *S is a triangle functor*

Proof. Pick an autoequivalence F of $\mathscr{D}$. For each $X, Y \in \mathscr{D}$, using Serre duality we get:

$$\begin{aligned}
\operatorname{Hom}_{\mathscr{D}}(X, F(S(Y))) &= \operatorname{Hom}_{\mathscr{D}}(F^{-1}(X), S(Y)) \\
&= \operatorname{Hom}_{\mathscr{D}}(Y, F^{-1}(X))^{\vee} \\
&= \operatorname{Hom}_{\mathscr{D}}(F(Y), X)^{\vee} \\
&= \operatorname{Hom}_{\mathscr{D}}(X, S(F(Y)))
\end{aligned}$$

Using the Yoneda lemma we conclude $FS(Y) = SF(Y)$ naturally, as desired.
Now we prove *(2)*. First of all, using *(1)*, there is an isomorphism $S\Sigma = \Sigma S$. Pick a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow$ in $\mathscr{D}$. Now, since for each $E \in \mathscr{D}$

$$\mathrm{Hom}_{\mathscr{D}}(E, *) = \mathrm{Hom}_{\mathscr{D}}(*, S(E))^{\vee}$$

and $\mathrm{Hom}_{\mathscr{D}}(*, S(E))^{\vee}$ is cohomological, we have that the triangle $S(X) \longrightarrow S(Y) \longrightarrow S(Z) \longrightarrow$ is special. Putting $C = \mathrm{cone}(S(f))$, we get a distinguished triangle $S(X) \longrightarrow S(Y) \longrightarrow C \xrightarrow{h}$. By applying the Hom-functors to these two triangles and using Serre duality we get a commutative diagram with exact rows and columns:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{Hom}_{\mathscr{D}}(X[1], Y) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{D}}(C, S(Y)) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{D}}(Y, Y) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathscr{D}}(X[1], Z) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathscr{D}}(C, S(Z)) \arrow[r, "d"] \arrow[d, "\delta"] & \mathrm{Hom}_{\mathscr{D}}(Y, Z) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathrm{Hom}_{\mathscr{D}}(X, X) \arrow[r] & \mathrm{Hom}_{\mathscr{D}}(C, S(X)[1]) \arrow[r] & \mathrm{Hom}_{\mathscr{D}}(Y, X[1]) \arrow[r] & \cdots
\end{tikzcd}$$

By some diagram chasing involving **Proposition 2.4**, we get a morphism $C \xrightarrow{a} S(Z)$ so that $d(a) = g$ and $\delta(a) = h$. This means that the following is a morphism of special triangles:

$$\begin{tikzcd}
S(X) \arrow[r] \arrow[d, "1"] & S(Y) \arrow[r] \arrow[d, "1"] & C \arrow[r] \arrow[d, "a"] & S(X)[1] \arrow[d, "1"] \\
S(X) \arrow[r] & S(Y) \arrow[r] & S(Z) \arrow[r] & S(X)[1]
\end{tikzcd}$$

Applying **Proposition 2.10** we get $C = S(Z)$, as desired. $\square$

Example 4.5. *If M is an n-dimensional complex smooth projective variety, then the classical Serre duality in geometry tells us that the twist by $\omega_M[n]$ (where ω_M is the canonical bundle) is a Serre functor on $\mathscr{D}^b(\mathrm{Coh}(M))$. If M is moreover a Calabi-Yau variety (i.e. the canonical bundle $\omega_M = \mathcal{O}_M$ is trivial) then we see that $\mathscr{D}^b(\mathrm{Coh}(M))$ is n-Calabi-Yau over $\mathbb{C}$, which is the reason of our nomenclature.*

Observe that if $A \subseteq \mathscr{D}$ is a full subcategory and S is a Serre functor, then $S({}^{\perp}A) = A^{\perp}$.

Proposition 4.6. *Suppose that $\mathscr{D}$ has a Serre functor S and that J acts trivially on $\mathbb{Z}$. Let $\mathscr{P}$ be a J-slicing on $\mathscr{D}$. Then $\mathscr{P}_{\phi}$ is admissible and has a Serre functor for each $\phi \in J$.*

Proof. Since $\mathscr{P}_{\phi}$ is a t-structure on $\mathscr{P}_{]-\infty,\phi]}$ fixed by translation and the same holds for $\mathscr{P}_{]-\infty,\phi]}$ in $\mathscr{D}$, by ??? we can suppose $J = [2]$. Pick a t-structure $\mathfrak{t}$ on $\mathscr{D}$ fixed by translation and consider the [2]-slicing $\mathscr{P} = \{\mathfrak{t}^{\perp}, \mathfrak{t}\}$. By the above observation, acting by S on $\mathscr{P}$ we obtain a [2]-slicing

$$S.\mathscr{P} = \{S(\mathfrak{t}$$

□

Now, observe that we allow 'negative dimension' ($n \leq 0$) in the definition of n-Calabi-Yau categories. Such categories exist (see REFERENZA) and provide a counterexample to the existence of bounded t-structures, as shown below.

Proposition 4.7. *Suppose that $\mathscr{D}$ is n-Calabi-Yau with $n < 0$. Then any t-structure on $\mathscr{D}$ is fixed by translation. In particular, $\mathfrak{bts}(\mathscr{D}) = \emptyset$.*

Proof. Let $\mathfrak{t}$ be a t-structure on $\mathscr{D}$. For each $X \in \heartsuit_{\mathfrak{t}}$, using Serre duality

$$\mathrm{End}_{\mathscr{D}}(X) = \mathrm{Hom}_{\mathscr{D}}(X[-n], X)^{\vee}$$

Since $n < 0$, the latter is zero and thus $X = 0$. By arbitrariness of X, $\heartsuit_{\mathfrak{t}} = 0$, as desired.
The second claim follows from the fact that if a t-structure is fixed by translation then it has zero heart, and thus can't be bounded. □

Proposition 4.8. *Suppose that $\mathscr{D}$ is n-Calabi-Yau. Suppose moreover that one of the following conditions holds:*

(1) $n \leq 1$
(2) the action of $\mathbb{Z}$ on J is trivial

Then any J-slicing on $\mathscr{D}$ is hereditary.

Proof. Pick $X \in \mathscr{P}_{\phi}$, $Y \in \mathscr{P}_{\psi}$ and suppose $\phi + 1 < \psi$ in J. Then using Serre duality we get

$$\mathrm{Hom}_{\mathscr{D}}(X, Y) = \mathrm{Hom}_{\mathscr{D}}(Y, X[n])^{\vee}$$

Now, $X[n]$ is semistable of phase $\phi + n$ with respect to $\mathscr{P}$. If condition *(1)* holds, then $\phi + n \leq \phi + 1 < \psi$. It condition *(2)* holds, then $\phi + n = \phi < \psi$. In either case, the latter is zero, as desired. □

Proposition 4.9. *Let M be an n-dimensional Calabi-Yau variety and suppose that the action of $\mathbb{Z}$ on J is trivial. Then any J-slicing on $\mathscr{D}^b(\mathrm{Coh}(M))$ is trivial*[2].

Proof. Denote $\mathscr{D} = \mathscr{D}^b(\mathrm{Coh}(M))$ and let $\mathscr{P}$ be a J-slicing on $\mathscr{D}$. Now the structure sheaf $\omega_M = \mathcal{O}_M$ and the skyscraper sheaves of points $\mathcal{O}_x$ (with $x \in M$) are indecomposable in $\mathrm{Coh}(M)$ and, since the latter is closed under direct summands in $\mathscr{D}$ by **Proposition 3.11**, they are indecomposable in $\mathscr{D}$ as well. But then they are semistable with respect to $\mathscr{P}$ by **Proposition 4.8**. Now, using Serre duality, for each $x \in M$

$$\mathrm{Hom}_{\mathscr{D}}(\mathcal{O}_x, \mathcal{O}_M[1]) = \mathrm{Hom}_{\mathscr{D}}(\mathcal{O}_M, \mathcal{O}_x)^\vee \neq 0$$

and thus all those sheaves must have the same phase ϕ. This means that if $\psi \neq \phi$, then the subcategory $\mathscr{P}_\psi$ is orthogonal (either on right or left) to all the skyscraper sheaves and their translations, and thus $\mathscr{P}_\psi = 0$. In other words, $\mathscr{P}_\phi = \mathscr{D}$, as desired. □

Proposition 4.10. *Let M be an n-dimensional Calabi-Yau variety. If there is a nontrivial J-slicing on $\mathscr{D}^b(\mathrm{Coh}(M))$ then $J = \mathbb{Z} \times I$ for some totally ordered set I.*

Proof. If $\mathscr{P}$ is a nontrivial J-slicing, applying the slice functor to the morphism $J \longrightarrow I(J)$ we get a trivial $I(J)$-slicing by **Proposition 4.9** and we conclude using **Proposition 3.34**. □

Now we deal with elliptic curves.

Example 4.11. *Since the canonical bundle of a compact Riemann surface of genus g has degree $2g-2$, the latter can be 0 only if $g = 1$. This means that the one-dimensional Calabi-Yau varieties are exactly the complex elliptic curves.*

Let M be a complex elliptic curve and denote $\mathscr{D} = \mathscr{D}^b(\mathrm{Coh}(M))$. Denote moreover $\overline{\mathbb{Q}} = \mathbb{Q} \sqcup \{\infty\}$ with the usual order. If $q = \frac{a}{b} \in \overline{\mathbb{Q}} = \mathbb{Q} \sqcup \{\infty\}$ with $a \neq 0$ and $b > 0$ (or $a = 0$, $b = 1$, or $a = 1$, $b = 0$) then we define E_q as the set of indecomposable coherent sheaves on M of degree a and rank b. If $X \neq Y \in E_q$ then

$$\mathrm{Hom}_{\mathscr{D}}(X, Y) = \mathrm{Hom}_{\mathscr{D}}(Y, X) = 0$$

and if $p > q$ then for each $X \in E_p$, $Y \in E_q$

$$\mathrm{Hom}_{\mathscr{D}}(X, Y) = 0 \neq \mathrm{Hom}_{\mathscr{D}}(Y, X)$$

[2] A J-slicing $\mathscr{P}$ on $\mathscr{D}$ is called trivial if $\mathscr{P}_\phi = \mathscr{D}$ for some $\phi \in J$.

We denote the set of all indecomposable coherent sheaves on M by

$$E = \coprod_{q \in \overline{\mathbb{Q}}} E_q$$

Now, $\mathrm{Coh}(M)$ is a Krull-Schmidt category, and thus every coherent sheaf is direct sum of indecomposable ones. This means that if we choose a total order on each set E_q in an arbitrary way and take the lexicographical coproduct order on E, then

$$\mathscr{M}^0_X = \langle X \rangle$$

with $X \in E$ defines an abelian E-slicing $\mathscr{M}^0$ on $\mathrm{Coh}(M)$ which by **Proposition 3.24** induces a $\mathbb{Z} \times E$-slicing $\mathscr{M}$ on $\mathscr{D}$. We call $\mathscr{M}$ the **Mumford-Takemoto** slicing (beware that it depends on the choices made for E). We now show that it is indeed the 'finest' one.

Proposition 4.12. *Let $\mathscr{P}$ be a nontrivial J-slicing on $\mathscr{D}$. Then there is a choice for E and a morphism of $\mathbb{Z}$-posets $\mathbb{Z} \times E \xrightarrow{f} J$ so that*

$$\mathscr{P} = f_{\Omega}(\mathscr{M})$$

Proof. Each $X \in E$ is indecomposable and thus, by **Proposition 4.8** ($n = 1$), is semistable with respect to $\mathscr{P}$ of some phase $f(X) \in J$. Pulling back the ordering on J we get a partial order on E which we can extend, using the Zorn's lemma, to a total order. By the hom-vanishing properties above we have a morphism of posets $E \xrightarrow{f} J$ which extends to a morphism of $\mathbb{Z}$-posets $\mathbb{Z} \times E \xrightarrow{f} J$. By definition, for each $\phi \in J$ we have $(f_{\Omega}(\mathscr{M}))_\phi \subseteq \mathscr{P}_\phi$ and we conclude using **Proposition 3.6**. □

In particular, using **Proposition 3.4** we get all the bounded t-structures on $\mathscr{D}$, which are all tiltings of the standard one!

4.2. **Exceptional collections.** Now, let k be a field and suppose that $\mathscr{D}$ is k-linear and Hom-finite. Denote $\mathscr{D}(k) = \mathscr{D}^b(\mathrm{Mod}_k^{\mathrm{fin}})$ the bounded derived category of finite-dimensional k-vector spaces. First of all, by Example 3.17 $\mathscr{D}(k)$ is equivalent to the category of finite-dimensional $\mathbb{Z}$-graded k-vector spaces, and we'll implicitly use this identification for the rest of the section.
Now, we have a bifunctor $\mathscr{D}(k) \times \mathscr{D} \longrightarrow \mathscr{D}$ which sends $V = \oplus_{i\in\mathbb{Z}} V_i$ and $X \in \mathscr{D}$ to

$$V \otimes X = \bigoplus_{i\in\mathbb{Z}} X[-i]^{\oplus \dim_k(V_i)}$$

Moreover, we have by an easy check an isomorphism of functors

$$\mathrm{Hom}^\bullet_{\mathscr{D}}(Y, V \otimes X) = V \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(Y, X)$$

where the second tensor product is the one in the category of graded vector spaces. Thus, for $X, Y \in \mathscr{D}$ we have

$$\begin{aligned} \mathrm{End}_{\mathscr{D}(k)}(\mathrm{Hom}^\bullet_{\mathscr{D}}(X,Y)) &= \mathrm{Hom}^\bullet_{\mathscr{D}}(X,Y)^\vee \otimes \mathrm{Hom}^\bullet_{\mathscr{D}}(X,Y) \\ &= \mathrm{Hom}^\bullet_{\mathscr{D}}(X, \mathrm{Hom}^\bullet_{\mathscr{D}}(X,Y)^\vee \otimes Y) \\ &= \mathrm{Hom}^\bullet_{\mathscr{D}}(\mathrm{Hom}^\bullet_{\mathscr{D}}(X,Y) \otimes X, Y) \end{aligned}$$

The identity in the first member induces morphisms in $\mathscr{D}$:

$$X \longrightarrow \mathrm{Hom}^\bullet_{\mathscr{D}}(X,Y)^\vee \otimes Y$$

$$\mathrm{Hom}^\bullet_{\mathscr{D}}(X,Y) \otimes X[-1] \longrightarrow Y[-1]$$

We denote $R_Y X$ (resp. $L_X Y$) the cone of the first (resp. second) morphism and call it the **right** (resp. **left**) **mutation** of X induced by Y (resp. of Y induced by X).

Definition 4.13. *An object $E \in \mathscr{D}$ is called **exceptional** if the functor $* \otimes E$ is fully faithful, or, in other words, if*

$$\dim_k \mathrm{Hom}_{\mathscr{D}}(E, E[i]) = \begin{cases} 1 & i = 0 \\ 0 & \text{otherwise} \end{cases}$$

*An **exceptional collection** is a finite sequence of exceptional objects $\{E_0, \cdots, E_n\}$ so that*

$$\mathrm{Hom}^\bullet_{\mathscr{D}}(E_i, E_j) = 0 \ for\ i > j$$

We call an exceptional collection

*(a) **full** if $\mathscr{D}$ is the thick subcategory generated by the collection*
*(b) **ext** if $\mathrm{Hom}_{\mathscr{D}}(E_i, E_j[n]) = 0$ for $i < j$ and $n \leq 0$*
*(c) **strong** if $\mathrm{Hom}_{\mathscr{D}}(E_i, E_j[n]) = 0$ for $i < j$ and $n \neq 0$*

If E is an exceptional object, then by **Proposition 2.14** $\langle E\rangle$ consists of direct sums of copies of E and is equivalent to the category of finite dimensional k-vector spaces.

Proposition 4.14. *Let $\mathfrak{e} = \{E_0, \cdots, E_n\}$ an exceptional collection on $\mathscr{D}$. Then:*

(1) $\mathscr{R}^{\mathfrak{e}} = \{\langle\langle E_0\rangle\rangle, \cdots, \langle\langle E_n\rangle\rangle, {}^{\perp}\langle\langle \mathfrak{e}\rangle\rangle\}$ is an $[n+1]$-slicing on $\mathscr{D}$ with

$$\tau^{\geq i}_{\mathscr{R}^{\mathfrak{e}}}(X) = R_{E_i} R_{E_{i-1}} \cdots R_{E_0} X[i]$$

(2) $\mathscr{L}^{\mathfrak{e}} = \{\langle\langle E_0\rangle\rangle, \cdots, \langle\langle E_n\rangle\rangle, \langle\langle \mathfrak{e}\rangle\rangle^{\perp}\}$ is an $[n+1]^{\mathrm{op}}$-slicing on $\mathscr{D}^{\mathrm{op}}$ with

$$\tau^{\geq i}_{\mathscr{L}^{\mathfrak{e}}}(X) = L_{E_i} L_{E_{i+1}} \cdots L_{E_n} X$$

Proof. We'll just show *(1)*, the other claim is proved similarly. Pick $0 \neq X \in \mathscr{D}$ and denote

$$X_i = R_{E_{n-i}} R_{E_{n-i-1}} \cdots R_{E_0} X[i]$$

(we set $X_{n+1} = X$). Then by definition we have a distinguished triangle for each $0 \leq i \leq n$

$$X_i \longrightarrow X_{i+1} \longrightarrow \mathrm{Hom}^{\bullet}_{\mathscr{D}}(X_{i+1}, E_{n-i})^{\vee} \otimes E_{n-i}[i+1] \longrightarrow$$

Applying the cohomological functor $\mathrm{Hom}^{\bullet}_{\mathscr{D}}(*, E_{n-i})$ to the above triangle we see that $\mathrm{Hom}^{\bullet}_{\mathscr{D}}(X_i, E_{n-i}) = 0$. But then, applying $\mathrm{Hom}^{\bullet}_{\mathscr{D}}(*, E_j)$ with $j < n-i$ we see inductively that $\mathrm{Hom}^{\bullet}_{\mathscr{D}}(X_i, E_j) = 0$. In particular, $\mathrm{Hom}^{\bullet}_{\mathscr{D}}(X_0, E_i) = 0$ for all i which means $X_0 \in {}^{\perp}\langle\langle \mathfrak{e}\rangle\rangle$, and this concludes the construction of the desired Postnikov tower. ☐

Proposition 4.15. *Let $\mathfrak{e} = \{E_0, \cdots, E_n\}$ a full exceptional collection on $\mathscr{D}$. Then*

$$K_0(\mathscr{D}) = \mathbb{Z}^{n+1}$$

with $\mathfrak{e}$ as generators. In particular, all full exceptional collections have the same length.

Proof. We have $K_0(\mathrm{Mod}^{\mathrm{fin}}_k) = K_0(\mathscr{D}(k)) = \mathbb{Z}$ and using **Proposition 3.29** on the semiorthogonal decomposition from **Proposition 4.14** we get the result. ☐

If $\mathfrak{e} = \{E_0, \cdots, E_n\}$ is a full exceptional collection, considering the standard t-structure on $\mathscr{D}(k)$ and applying **Proposition 3.30** we see that

$$\mathscr{Q}_{(i,k)} = \langle E_i[k]\rangle$$

defines a $[n] \times \mathbb{Z}$-slicing $\mathscr{Q}$ on $\mathscr{D}$. The collection is ext if and only if $\mathscr{Q}$ is gluable, and considering the projection $\mathbb{Z} \times [n] \longrightarrow \mathbb{Z}$ we obtain a bounded t-structure whose heart is $\langle \mathfrak{e} \rangle$.

Proposition 4.16. *Let $\mathfrak{e} = \{E_0, \cdots, E_n\}$ be a full ext exceptional collection on $\mathscr{D}$. Then $\langle \mathfrak{e} \rangle$ is an abelian category of finite length (i.e. both artinian and noetherian) whose simple objects (up to isomorphism) are $E_0, \cdots, E_n$.*

Proof. First of all, $\langle \mathfrak{e} \rangle$ is abelian since it is the heart of a t-structure. Since we obtained $\langle \mathfrak{e} \rangle$ from a $\mathbb{Z} \times [n]$-slicing, we have that $\{\langle E_0 \rangle, \cdots, \langle E_n \rangle\}$ is an n-torsion pair on $\langle \mathfrak{e} \rangle$. Pick $0 \neq X \in \langle \mathfrak{e} \rangle$ and consider its Harder-Narasimhan filtration

$$0 = X_0 \subseteq \cdots \subseteq X_m = X$$

Now for each i, $X_i / X_{i-1} = E_{k_i}^{\oplus t_i}$ for some k_i, t_i. If $t_i > 1$, then there is an $X_{i-1} \subseteq Y \subseteq X_i$ so that $Y / X_{i-1} = E_{k_i}$ and $X_i / Y = E_{k_i}^{t_i - 1}$. Iterating this argument, we get a filtration of X whose factors are all in $\mathfrak{e}$. If we show that the E_i's are simple, this is the desired Jordan-Hölder filtration. Let's show that the objects in $\mathfrak{e}$ are simple. Pick a subobject $0 \neq A \subseteq E_i$. By looking at the first step of the Harder-Narasimhan filtration, there is a j and an inclusion $E_j \subseteq A$. But by definition of ext exceptional collection, we have $i = j$ and thus $A = E_i$, as desired. Since we have already constructed Jordan-Hölder filtrations, we get the the finite-length property. $\square$

Example 4.17. *By* ***Proposition 4.14*** *and* ***Proposition 4.9*** *we see that Calabi-Yau varieties cannot have full exceptional collections on the bounded derived category of their coherent sheaves. However, using the Euler exact sequences, one sees that*

$$\{\mathcal{O}_{\mathbb{CP}^n}, \mathcal{O}_{\mathbb{CP}^n}(1)[-1], \cdots, \mathcal{O}_{\mathbb{CP}^n}(n)[-n]\}$$

is an ext full exceptional collection on $\mathscr{D}^b(\mathrm{Coh}(\mathbb{CP}^n))$ (REFERENZA-AAA). In this case, the abelian category obtained from ***Proposition 4.16*** *is equivalent to the bounded derived category of finite-dimensional modules over the complex path algebra of the quiver*

$$\begin{tikzcd} \bullet \arrow[r, bend left, "\alpha_0^0"] \arrow[r, bend right, "\alpha_0^n"'] \arrow[r, phantom, "\vdots"] & \bullet & \cdots & \bullet \arrow[r, bend left, "\alpha_{n-1}^0"] \arrow[r, bend right, "\alpha_{n-1}^n"'] \arrow[r, phantom, "\vdots"] & \bullet \end{tikzcd}$$

with relations $\alpha_{i+1}^j \alpha_i^k = \alpha_{i+1}^k \alpha_i^j$.

Definition 4.18. *Let $\mathfrak{e} = \{E_0, \cdots, E_n\}$ be an exceptional collection on $\mathscr{D}$, $0 \le i < n$. The i-**th right mutation** of that exceptional collection is the exceptional collection*

$$R_i(\mathfrak{e}) = \{E_0, \cdots, E_{i-1}, E_{i+1}, R_{E_{i+1}}(E_i), E_{i+2}, \cdots, E_n\}$$

Recall that the braid group B_{n+1} is defined as the free group on generators $\sigma_0, \cdots, \sigma_{n-1}$ with relations

$$\begin{aligned} \sigma_i\sigma_j &= \sigma_j\sigma_i & &for\ |i-j| > 1 \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} & &for\ 0 \le i < n-1 \end{aligned}$$

Proposition 4.19. *The braid group B_{n+1} acts on the set of exceptional collections on $\mathscr{D}$. This action preserves fullness.*

Proof. We define the action on the generators as

$$\sigma_i.\mathfrak{e} = R_i(\mathfrak{e})$$

and we have to check that it is well-defined. Clearly, the first relation of the braid group is satistied. For the second relation, it suffices to show that for an exceptional collection $\{E_0, E_1, E_2\}$ we have

$$R_{R_{E_2}(E_1)}(R_{E_2}(E_0)) = R_{E_2}(R_{E_1}(E_0))$$

By definition we have a distinguished triangle

$$E_0 \longrightarrow \mathrm{Hom}^\bullet_{\mathscr{D}}(E_0, E_1)^\vee \otimes E_1 \longrightarrow R_{E_1}(E_0) \longrightarrow$$

Now, since $\mathrm{Hom}^\bullet_{\mathscr{D}}(E_0, E_1) = \mathrm{Hom}^\bullet_{\mathscr{D}}(R_{E_2}(E_0), R_{E_2}(E_1))$, applying the triangle functor $R_{E_2}(*)$ to the above triangle we get distinguished triangle:

$$R_{E_2}(E_0) \longrightarrow \mathrm{Hom}^\bullet_{\mathscr{D}}(R_{E_2}(E_0), R_{E_2}(E_1))^\vee \otimes R_{E_2}(E_1) \longrightarrow R_{E_2}(R_{E_1}(E_0)) \longrightarrow$$

which is what we wanted. $\square$

5. $J = \mathbb{R}$: Bridgeland stability

5.1. **Stability functions.** Let $\mathscr{A}$ be an abelian category. We denote

$$\mathbb{H} = \mathbb{R}_{>0} e^{\sqrt{-1}]0,\pi]} \subseteq \mathbb{C}$$

the semi-closed upper half-plane.

Definition 5.1. *A **stability function** on $\mathscr{A}$ is a group homomorphism $K_0(\mathscr{A}) \xrightarrow{z} \mathbb{C}$ so that for each $0 \neq X \in \mathscr{A}$ we have*

$$z(X) \in \mathbb{H}$$

*The **phase** with respect to z of an object $0 \neq X \in \mathscr{A}$ is*

$$\phi_z(X) = \frac{1}{\pi} \arg(z(X)) \in]0,1]$$

*An object $0 \neq X \in \mathscr{A}$ is called z-**semistable** if for all nonzero subobjects $Y \subseteq X$ we have $\phi_z(Y) \leq \phi_z(X)$.*
*We say that z satisfies the **Harder-Narasimhan property** if*

$$\mathscr{P}^z_\phi = \{0 \neq X \in \mathscr{A} \mid X \text{ is } z\text{-semistable of phase } \phi\} \cup \{0\}$$

defines an abelian $]0,1]$-slicing on $\mathscr{A}$.
Moreover, for each $0 \neq X \in \mathscr{A}$, we define $\mathrm{HN}_z(X)$ as the convex hull of $z(Y)$ with $Y \subseteq X$ and $\phi_z(Y) \geq \phi_z(X)$.

The following is a very easy result which we prove using a picture.

Proposition 5.2. (See-saw property) *Let z be a stability function on $\mathscr{A}$. If $0 \longrightarrow X \longrightarrow Y \longrightarrow Z \longrightarrow 0$ is a short exact sequence of nonzero objects in $\mathscr{A}$, then either X, Y, Z have the same phase or*

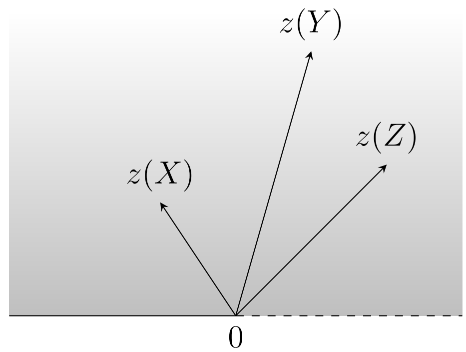
$$\min\{\phi_z(X), \phi_z(Z)\} < \phi_z(Y) < \max\{\phi_z(X), \phi_z(Z)\}$$

Proof. We have $z(Y) = z(X) + z(Z)$, and the result follows from Euclidean properties of $\mathbb{H}$. $\square$

The following lemma shows that in order to check the Harder-Narasimhan property we only care about the existence of Harder-Narasimhan filtrations.

Proposition 5.3. *Let z be a stability function on $\mathscr{A}$, $\phi, \psi \in]0,1]$. If $\phi > \psi$, then*

$$\mathscr{P}^z_\psi \subseteq (\mathscr{P}^z_\phi)^\perp$$



Proof. Let $X \xrightarrow{f} Y$ be a nonzero morphism in $\mathscr{A}$ with X z-semistable of phase ϕ and Y z-semistable of phase ψ. Since $\operatorname{im}(f)$ is a subobject of Y and a quotient of X, by semistability and see-saw property we get:

$$\phi \leq \phi_z(\operatorname{im}(f)) \leq \psi$$

which is absurd by hypothesis. $\square$

Since $\mathbb{Z}\times]0,1] = \mathbb{R}$ as a $\mathbb{Z}$-poset, if we have a bounded t-structure $\mathfrak{t}$ on $\mathscr{D}$ and a stability function on $\heartsuit_{\mathfrak{t}}$ with the Harder-Narasimhan property we get an $\mathbb{R}$-slicing on $\mathscr{D}$ by **Proposition 3.24**.

Proposition 5.4. *Let z be a stability function on $\mathscr{A}$ with the Harder-Narasimhan property. For each $\phi \in]0,1]$, $\mathscr{P}^z_\phi$ is an Abelian subcategory of $\mathscr{A}$.*

Proof. Let $X \xrightarrow{f} Y$ be a nonzero morphism in $\mathscr{P}^z_\phi$. $\square$

If $0 \neq Y \subseteq X$ is a subobject, since $X = Y + X/Y$ in $K_0(\mathscr{A})$ and $z(X/Y), z(X), z(Y) \in \mathbb{H}$, we have $\Im(z(X)) \geq \Im(z(Y))$. In particular, this means that $\mathrm{HN}_z(X)$ is contained below the horizontal line through $z(X)$. Moreover, if $z(Y) = z(X)$, then $z(X/Y) = 0$ and thus $X/Y = 0$, which means $X = Y$.

Proposition 5.5. *Let z be a stability function on $\mathscr{A}$, $0 \neq X \in \mathscr{A}$, $Y, Z \subseteq X$. If $z(Y)$ and $z(Z)$ are extremal points of $\mathrm{HN}_z(X)$ and $\phi_z(Z) > \phi_z(Y)$, then $Z \subseteq Y$.*

Proof. Consider the intersection $A = Y \cap Z$ and the span $B = Y + Z \subseteq X$ (there is a clash of notation: here the sum doesn't denote the operation of the Grothendieck group). Since there is an exact sequence

$$0 \longrightarrow A \longrightarrow Y \oplus Z \longrightarrow B \longrightarrow 0$$

we have $z(Y) + z(Z) = z(A) + z(B)$. But by convexity and the above observation ($A \subseteq Z$ and $Y \subseteq B$), $z(A)$ and $z(B)$ must lie on the right of the line through $z(Y)$ and $z(Z)$, so we have that $z(A)$, $z(B)$, $z(X)$, $z(Y)$ must be aligned and $\Im(z(B)) \geq \Im(z(Y)) \geq \Im(z(Z)) \geq \Im(z(A))$. By extremality, we get that $z(A) = z(Z)$ and $z(B) = z(Y)$. Since again $A \subseteq Z$, we have $A = Z$. ☐

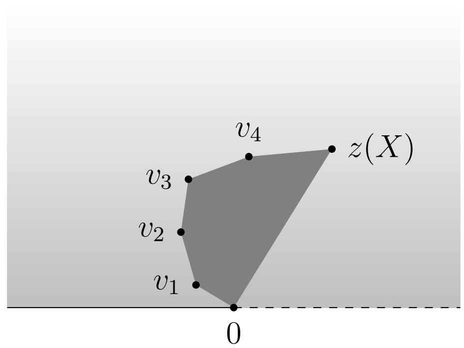


FIGURE 2. A depiction in the complex plane of $\mathrm{HN}_z(X)$ when the Harder-Narasimhan property holds.

Proposition 5.6. *Let z be a stability function on $\mathscr{A}$. Then z has the Harder-Narasimhan property if and only if for each $0 \neq X \in \mathscr{A}$, $\mathrm{HN}_z(X)$ is a polygon.*

Proof. Suppose that $\mathrm{HN}_z(X)$ is a polygon. Then it has a finite number of extremal points (vertices) $v_0 = 0, \cdots, v_n = z(X)$ (ordered by decreasing argument), and for each $1 \leq i < n$ there is a subobject $X_i \subseteq X$ so that $z(X_i) = v_i$. Now we claim that these subobjects form the filtration of the Harder-Narasimhan property. First of all, by Proposition ??????, $X_i \subseteq X_{i+1}$ for all i. Now we show that X_{i+1}/X_i is semistable. If not, there is a subobject $Y \subseteq X_{i+1}$ containing X_i so that $\phi_z(Y/X_i) > \phi_z(X_{i+1}/X_i)$. This tells us that $z(Y)$ lies in the open half plane on the left of the line through v_i and v_{i+1}, which is absurd by convexity. Now, $\phi_z(X_i/X_{i-1}) > \phi_z(X_{i+1}/X_i)$ again by convexity, so we have the first implication of the statement.

Conversely, suppose that z has the Harder-Narasimhan property. For each $0 \neq X \in \mathscr{D}$, we have a filtration by subobjects $0 = X_0 \subseteq \cdots \subseteq X_n = X$ as in the definition. Take a subobject $Y \subseteq X$. We show, by induction on n, that $z(Y)$ lies on the right of the line through $z(X_i)$ and $z(X_{i+1})$ for all i. By induction hypothesis, $z(Y \cap X_{n-1})$ lies on the right of the line through $z(X_i)$ and $z(X_{i+1})$ for $i < n$. Since $Y/(Y \cap X_{n-1})$

is a subobject of the semistable X/X_{n-1} and

$$z(Y) = z(Y \cap X_{n-1}) + z(Y/(Y \cap X_{n-1}))$$

we get what we wanted. □

By polygon we also mean a degenerate one (a segment). This happens if and only if X is semistable.

Proposition 5.7. *If $\mathscr{A}$ is a finite length (both Artinian and Noetherian) category, then every stability function on $\mathscr{A}$ has the Harder-Narasimhan property.*

Proof. Take a stability function z on $\mathscr{A}$. Suppose that for some object $0 \neq X \in \mathscr{A}$, $\mathrm{HN}_z(X)$ is not a polygon. Then it has an infinite set of extremal point. Then we can extract a sequence of increasing or decreasing phase, which by Proposition ???? corresponds to a strict sequence of subobjects, which is absurd by hypothesis. □

5.2. **The space of real slicings.**

Definition 5.8. *If $\mathscr{P}$, $\mathscr{Q}$ are $\mathbb{R}$-slicings on $\mathscr{D}$, we define their (possibly infinite) distance as*

$$d(\mathscr{P},\mathscr{Q}) = \sup_{0\neq X\in\mathscr{D}} \{|\phi^{\pm}_{\mathscr{P}}(X) - \phi^{\pm}_{\mathscr{Q}}(X)|\}$$

Proposition 5.9. *d defines a (possibly infinite) metric on $\mathbf{\Omega}_{\mathbb{R}}(\mathscr{D})$.*

Proof. Simmetry and triangle inequality are obvious. If $d(\mathscr{P},\mathscr{Q}) = 0$, then $\operatorname{supp}_{\mathscr{P}}(X) = \{\phi\}$ for some $\phi \in \mathbb{R}$ if and only if $\operatorname{supp}_{\mathscr{Q}}(X) = \{\phi\}$, which means $\mathscr{P} = \mathscr{Q}$. $\square$

The metric d induces thus a topology on $\mathbf{\Omega}_{\mathbb{R}}(\mathscr{D})$ so that each connected component is a metric space. Now, tilting gives a way to characterize the bounded t-structures obtained from real slicings in the same connected component.

Proposition 5.10. *Suppose $d(\mathscr{P},\mathscr{Q}) < \frac{1}{2}$. Then $\mathscr{P}_{]\frac{1}{2},+\infty[}$ is a tilting of both $\mathscr{P}_{]0,+\infty[}$ and $\mathscr{Q}_{]0,+\infty[}$.*

Proof. By hypothesis, we have a diagram of inclusions of bounded t-structures:

$$\begin{array}{ccccc} & & \mathscr{P}_{]1,+\infty[} & & \\ & & \cap\!\!\mid & & \\ \mathscr{Q}_{]1,+\infty[} & \subseteq & \mathscr{P}_{]\frac{1}{2},+\infty[} & \subseteq & \mathscr{Q}_{]0,+\infty[} \\ & & \cap\!\!\mid & & \\ & & \mathscr{P}_{]0,+\infty[} & & \end{array}$$

and we conclude using Proposition 3.27. $\square$

This topology behaves well with respect to the operations we constructed on slicings. First of all, $\operatorname{Aut}(\mathscr{D})$ acts clearly by isometries on $\mathbf{\Omega}_{\mathbb{R}}(\mathscr{D})$. Moreover, since $\operatorname{supp}_{f_{\Omega}(\mathscr{P})}(X) = f(\operatorname{supp}_{\mathscr{P}}(X))$ for $f \in \operatorname{Aut}_{\mathrm{Pos}_{\mathbb{Z}}}$ and $\mathscr{P} \in \mathbf{\Omega}_{\mathbb{R}}(\mathscr{D})$, the subgroup of continuous automorphisms of $\mathbb{R}$ as a $\mathbb{Z}$-poset acts by homeomorphisms on the real slicings.
Now, consider a semiorthogonal decomposition $\mathscr{P} = \{\mathscr{P}_0, \cdots, \mathscr{P}_n\}$ on $\mathscr{D}$. Denote by

$$S(\mathscr{P}) \subseteq \prod_{0\leq i\leq n} \mathbf{\Omega}_{\mathbb{R}}(\mathscr{P}_i)$$

the set of elements so that the obtained $[n]\times\mathbb{R}$-slicing is gluable. By composing gluing with the projection to the left factor $\mathbb{R}\times[n] \longrightarrow \mathbb{R}$ and applying the slice functor we obtain a map $S(\mathscr{P}) \xrightarrow{\gamma} \mathbf{\Omega}_{\mathbb{R}}(\mathscr{D})$.

Proposition 5.11. *The map*

$$S(\mathscr{P}) \xrightarrow{\gamma} \underline{\mathbf{\Omega}}_{\mathbb{R}}(\mathscr{D})$$

is continuous, where $S(\mathscr{P})$ has the product topology.

Proof. Pick $(\mathscr{Q}_i)_i \in S(\mathscr{P})$, and denote $\mathscr{Q}$ its image by γ. For $0 \neq X \in \mathscr{D}$ we easily see that

$$\phi^+_{\mathscr{Q}}(X) = \max_{i \in \operatorname{supp}_{\mathscr{P}}(X)} \phi^+_{\mathscr{Q}_i}(X) \qquad \phi^-_{\mathscr{Q}}(X) = \min_{i \in \operatorname{supp}_{\mathscr{P}}(X)} \phi^-_{\mathscr{Q}_i}(X)$$

which gives the desired continuity. $\square$