STABLE ∞-CATEGORIES: THE WHATS, THE WHYS AND THE HOWS.

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Abstract. Lurification.

Introduction. These notes have an introductory purpose, having the aim to demonstrate how easy Homological Algebra can become when regarded from a higher categorical POV.

Half of this simplification is provided by the formalism of *Homotopical Algebra*, which finds in (co)homology theory its most fruitful application:

- "resolutions" of complexes serve as *fibrant/cofibrant approximations* for the objects of $\text{Ch}(\mathcal{A})$;
- the abelian category $\mathcal{A}$ "has enough injectives" if these approximations (*replacements*) can be performed for all objects $C_* \in \text{Ch}(\mathcal{A})$;
- homotopies of complexes are precisely what is needed to define homotopies between objects in model categorical sense.

Among other milestones in Homological Algebra, *triangulated categories* occupy a major role: the first attempt to axiomatize the phenomena giving rise to these structures was given by A. Dold and D. Puppe in their *Homologie nicht-additiver Funktoren*, and in A. Heller's *Stable homotopy theories*, where the author tried to axiomatize an additional structure carried by the *stable homotopy category* (the homotopy category of topological spectra).

Motivated by this result, Grothendieck and Verdier recognized a similar structure on the homotopy category of complexes, which further localized modding out objects in the kernel of the projection functor $Q\colon \text{ch}(\mathcal{A}) \to \text{k}(\mathcal{A})$ gives the *derived category* of complexes in the abelian category $\mathcal{A}$.

A rather cumbersome set of axioms was outlined by Verdier in his *thèse* [Ver67], aimed to capture the behaviour of a peculiar class of diagrams $A \to B \to C \to \Sigma A$, called *distinguished triangles*, acting like exact sequences and involving an additive autoequivalence $\Sigma\colon \mathbf{C} \to \mathbf{C}$.

But the systematization of the theory was far from being satisfactory at that point, since the origin of the axioms was obscure and really far

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from being canonical. As an example of this lack of universality, one can take the axiom embedding a map $f\colon X \to Y$ in a distinguished triangle $X \xrightarrow{f} Y \to Z \to$, a procedure which is not well-defined in any sensible way. And furthermore, as it is noted in [**MK07**], the derived category of an abelian category $\mathscr{A}$, taken as a triangulated category alone, has no universal property.

Another more practical problem is that localizing at quasi-isomorphisms is in some sense "too coarse" a procedure to retain interesting homotopical informations about the category of complexes. These informations are hidden in a higher structure the localization procedure outlined by Verdier is not able to preserve.

Despite this highly unsatisfactory situation, a great deal of refined mathematics stemmed out from the theory of triangulated categories:

- Verdier-Grothendieck primary task (to shed a light on the construction of derived functors) is easily achieved;
- In a suitable sense the derived category of sheaves on a good space contains the informations to rebuild the space from scratch (this is a result in reconstruction theory, mainly worked out in [**BO01**]);
- A great deal of informations about $\mathscr{A}$ can be desumed from the study of a peculiar kind of subcategories of $\mathbf{D}(\mathscr{A})$ (the adjacent classes of a "t-structure") on $\mathbf{D}(\mathscr{A})$) and of a generic triangulated category $\mathbf{D}$;
- There are countless applications in the representation theory of rings, modules and algebraic geometry.

On the other hand, a deeper analysis of the axioms giving the definition of triangulated category became more and more a priority. With the passing of time, it became evident that triangulated category behaved like "1-dimensional shadows" of a higher notion: they arise as *decategorification* of a structure taking place in the ∞-categorical world, and the axioms defining them are designed to keep track of the 1-categorical trace of this more refined notion.

A fruitful analogy is the following: shadows of objects retain no informations about their colours; in the same spirit, triangulated categories retain little informations about the higher structure generating them.

In some sense, whenever you give a definition in the language of categories and functors, you can't escape from referring to "simple" universal properties of maps: no matter how much a limit is complicated, it will always be the equalizer of some pair of maps.

Because of these reasons, it would be desirable to have at our disposal a more intrinsic notion of triangulated category, satisfying some reasonable requests of universality: whenever a higher category $\mathbf{C}$ enjoys a property which we will call "stability", then

- its homotopy category $\mathrm{Ho}(\mathbf{C})$ carries a triangulated structure in the classical sense;
- the axioms characterizing a triangulated structure are "easily verified and well-motivated consequences of evident universal arguments"; (see [**Lur11**, Remark **1.1.2.16**]);
- classical derived categories arising in Homological Algebra can be regarded as homotopy categories of stable ∞-categories functorially associated to an abelian $\mathscr{A}$ (see [**Lur11**, §**1.3.1**]).

From this point of view both the problem and the solution are clearly put in the correct frame: endowing the homotopy category with a triangulated structure is often sufficient in most practical purposes. However, as soon as one needs to remember the shape of homotopy co/limits that existed in the stable category, a triangulated structure is not enough.

Notations and conventions. Categories are denoted as boldface letters $\mathbf{C}, \mathbf{D}$ etc. *Functors* between categories are always denoted as capital Latin letters like F, G, H, K etc.; the category of functors $\mathbf{C} \to \mathbf{D}$ is denoted as $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$, $\mathbf{D}^{\mathbf{C}}$, $[\mathbf{C}, \mathbf{D}]$ and suchlike; morphisms in $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ (i.e. natural transformations) are written in Greek alphabet. The simplex category $\mathbf{\Delta}$ is the *topologist's delta*, having objects *nonempty* finite ordinals $\Delta[n] := \{0 < 1 \cdots < n\}$ regarded as categories in the obvious way. We adopt [**Lur09**] as a reference for the language of quasicategories and simplicial sets; in particular, we treat "quasicategory" and "∞-category" as synonymes.

1. Axioms become corollaries.

1.1. **Triangles, in several flavours.**

1.1.1. *SW-stabilization and Spectra.* Let $\mathbf{A}$ be a category. We want to cope with the following problem: given an endofunctor $\Sigma\colon \mathbf{A} \to \mathbf{A}$ we want to find the best embedding of $\mathbf{A}$ into a category $\mathrm{Sw}(\mathbf{A})$ (the *Spanier-Whitehead stabilization* of $\mathbf{A}$) in which Σ becomes an automorphism. We will give two answers; albeit perfectly (=canonically) equivalent, the first is so conceptual it can't provide almost any information about $\mathrm{Sw}(\mathbf{A})$.

1.2. **Construction via monads.**

Let $\mathbf{N}$ be the set of natural numbers considered as a category: $\mathbf{N}$ has a monoidal product, the sum of natural numbers (coproduct of finite sets), such that the empty set is the unit object.

It is a general fact that $T = (-) \times \mathbf{N}$ is a monad on $\mathbf{Cat}$, and the T-algebras $\mathbf{Cat}^{\mathbf{N}}$ can be described as pairs categories-endofunctor $(\mathbf{A}, \Sigma)$. Algebra morphisms are simply functor 'equivariant' with respect to the given endomorphisms. More explicitly, a T-algebra is a pair $(\mathbf{A}, \Sigma)$ where $\mathbf{A}$ is a category, and $\Sigma\colon \mathbf{A} \to \mathbf{A}$ is a functor such that the diagrams

$$\begin{tikzcd}
\mathbf{A}\times\mathbf{1} \arrow[r, "\mathbf{A}\times\eta"] \arrow[dr, "\sim"'] & \mathbf{A}\times\mathbf{N} \arrow[d, "\tilde\Sigma"] \\
& \mathbf{A}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{A}\times\mathbf{N}\times\mathbf{N} \arrow[r, "\tilde\Sigma\times\mathbf{N}"] \arrow[d, "\mathbf{A}\times\mu"'] & \mathbf{A}\times\mathbf{N} \arrow[d, "\tilde\Sigma"] \\
\mathbf{A}\times\mathbf{N} \arrow[r, "\tilde\Sigma"'] & \mathbf{A}
\end{tikzcd}$$

(where $\tilde\Sigma(A, n) = \Sigma^n A$ and η, μ are the monoid maps of $\mathbf{N}$) commute.

Notation 1.1 : In the following, T-algebras will be called *categories with endomorphism.*

Let now $\mathbf{N} \hookrightarrow \mathbf{Z}$ the obvious monoid homomorphism. When regarded as a category, it is a (strict) group in **Cat**, so $S = (-) \times \mathbf{Z}$ is a fortiori a triple. As above one sees that the category of S-algebras consists of pairs $(\mathbf{A}, \Sigma)$ where $\Sigma\colon \mathbf{A} \to \mathbf{A}$ is an *auto*morphism. Similar diagrams are requested to commute, so that if we consider the restriction $\Sigma_{(n)} = \Sigma|_{\mathbf{A}\times\{n\}}$ for any $n \in \mathbf{Z}$, and we identify $\mathbf{A} \times \{n\} \cong \mathbf{A}$, then we have that $\Sigma_{(1)} = \Sigma$, $\Sigma_{(-1)} = \Sigma^{-1}$ and so on.

Notation 1.2 : S-algebras are called *categories with automorphism.*

Remark 1.3 : The homomorphism $\iota\colon \mathbf{N} \hookrightarrow \mathbf{Z}$ induces a morphism of monads $T \to S$, which we call again ι; this in turns induces a forgetful functor

$$U\colon \mathbf{Cat}^{\mathbf{Z}} \hookrightarrow \mathbf{Cat}^{\mathbf{N}}$$

(the forgetful action of U is clear when its action is explicited: it simply forgets that an automorphism Σ of $\mathbf{A}$ has an inverse.

The general problem of this entire section is:

Does U possess a left adjoint $F\colon \mathbf{Cat}^{\mathbf{N}} \to \mathbf{Cat}^{\mathbf{Z}}$?

We can simply argue that the answer is yes, thanks to regularity hypoteses on the categories involved; but we want a more precise description of what's going on. To this end, given $(A, \Sigma) \in \mathbf{Cat}^{\mathbf{N}}$ let us consider the coequalizer diagram in **Cat**:

$$\mathbf{A}\times\mathbf{N}\times\mathbf{Z} \underset{(\mathbf{A}\times\mu)\circ(\iota_A\times\mathbf{Z})}{\overset{\Sigma\times\mathbf{Z}}{\rightrightarrows}} \mathbf{A}\times\mathbf{Z} \longrightarrow F(\mathbf{A}, \Sigma)$$

Now, all monads of the form $(-) \times M$, where M is a monoid in a monoidal(ly cocomplete) category $(\mathbf{A}, \times)$ preserve colimits, hence there is a unique S-algebra structure on $F(\mathbf{A}, \Sigma)$ such that

$$\bar\Sigma\colon F(\mathbf{A}, \Sigma) \times \mathbf{Z} \to F(\mathbf{A}, \Sigma)$$

is an automorphism of $F(\mathbf{A}, \Sigma)$ and the correspondence $\bar F\colon (\mathbf{A}, \Sigma) \mapsto (F(\mathbf{A}, \Sigma), \bar\Sigma)$ is the desired left adjoint. The category $F(\mathbf{A}, \Sigma)$ can be considered the *free category with automorphism* on the category with endomorphism $(\mathbf{A}, \Sigma)$.

This category satisfies the desired universal property: there exists a functor

$$\alpha\colon (\mathbf{A},\Sigma)\to \bar{F}(\mathbf{A},\Sigma)$$

(the *unit* of the adjunction we built) such that for any S-algebra morphism $H\colon (\mathbf{A},\Sigma)\to(\mathbf{B},\Theta)$ where $(\mathbf{B},\Theta)$ is a T-algebra, there is a unique T-algebra morphism $\bar{H}\colon \bar{F}(\mathbf{A},\Sigma)\to(\mathbf{B},\Theta)$ such that the following diagram commutes:

$$\begin{tikzcd}
(\mathbf{A},\Sigma) \arrow[rr, "\alpha", "(\mathbf{A},\Sigma)"'] \arrow[dr, "H"'] & & \bar{F}(\mathbf{A},\Sigma) \arrow[dl, dotted, "\bar{H}"] \\
& (\mathbf{B},\Theta). &
\end{tikzcd}$$

The category of *topological spectra* consists of the Spanier-Whitehead stabilization of the category of CW-complexes.

1.2.1. *Stable Model categories.*

1.2.2. *k-linear* DG*-categories.*

1.2.3. *Stable derivators.*

1.2.4. *Stable ∞-categories.*

$$\begin{tikzcd}
\mathscr{A} \arrow[r, mapsto, "\text{enrichment}/\mathrm{Ch}^+(\mathbf{Ab})"] & \mathscr{A}^\sim \arrow[r, mapsto, "\text{homwise Dold-Kan}"] & \mathscr{A}^\sim_{\mathbf{\Delta}} \arrow[r, mapsto, "N"] & \mathbf{D}_\infty(\mathscr{A}) \arrow[r, mapsto, "\mathrm{Ho}"] & \mathbf{D}(\mathscr{A}) \\
\mathbf{AbCat} \arrow[r, dotted, bend left] & \mathrm{Ch}^+(\mathbf{Ab})\text{-}\mathbf{Cat} \arrow[l, dotted, bend left] \arrow[r, dotted, bend left] & \mathbf{sAb}\text{-}\mathbf{Cat} \arrow[l, dotted, bend left] \arrow[r, dotted, bend left] & \infty\text{-}\mathbf{Cat}_{\mathrm{st}} \arrow[l, dotted, bend left] &
\end{tikzcd}$$

We start considering the category $\Delta[1]\times\Delta[1]=(\Lambda^2_2)^{\triangleleft}=(\Lambda^2_0)^{\triangleright}$ (see the diagrams besides; each of these descriptions will turn out to be useful) and denote it as $\square$ for short. In the same way, we denote pictorially the two horn-inclusions

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Lambda^2_0"] & (1,0) \\
(0,1) &
\end{tikzcd}$$

$$i_\ulcorner\colon {}^\ulcorner\!\to\square \quad \big(=\Lambda^2_0\to(\Lambda^2_0)^{\triangleright}\big)$$
$$i_\lrcorner\colon \lrcorner\to\square \quad \big(=\Lambda^2_2\to(\Lambda^2_2)^{\triangleleft}\big)$$

$$\begin{tikzcd}
& (1,0) \arrow[d] \\
(0,1) \arrow[r] \arrow[ur, phantom, "\Lambda^2_2"] & (1,1)
\end{tikzcd}$$

(see [**Lur09**, Notation **1.2.8.4**]) and the induced maps

$$i^*_\ulcorner\colon \mathrm{Map}(\square,\mathbf{C})\to\mathrm{Map}(\ulcorner,\mathbf{C}) \tag{1}$$
$$i^*_\lrcorner\colon \mathrm{Map}(\square,\mathbf{C})\to\mathrm{Map}(\lrcorner,\mathbf{C}) \tag{2}$$

from the category of commutative squares in $\mathbf{C}$, "restricting" a given diagram to its top or bottom part, respectively. These functors are part of a string of adjoints

$$(i_\ulcorner)_!\dashv \boxed{i^*_\ulcorner\dashv(i_\ulcorner)_*}\colon \mathrm{Map}(\square,\mathbf{C})\leftrightarrows\mathrm{Map}(\ulcorner,\mathbf{C}) \tag{3}$$
$$\boxed{(i_\lrcorner)_!\dashv i^*_\lrcorner}\dashv(i_\lrcorner)_*\colon \mathrm{Map}(\square,\mathbf{C})\leftrightarrows\mathrm{Map}(\lrcorner,\mathbf{C}) \tag{4}$$

where $(i_\ulcorner)_!$ and $(i_\lrcorner)_*$ are easily seen to be evaluations at the initial and terminal object of $\ulcorner$ and $\lrcorner$ respectively.

It's rather easy to see that, given $F \in \mathrm{Map}(\square, \mathbf{C})$ the canonical morphisms obtained from the boxed adjunctions,

$$\eta_{\ulcorner,F} \colon F \to (i_\ulcorner)_* i_\ulcorner^* F$$
$$\epsilon_{\lrcorner,F} \colon (i_\lrcorner)_! i_\lrcorner^* F \to F$$

give the canonical "comparison" arrow $F(1,1) \to \varinjlim i_\ulcorner^* F$ and $\varprojlim i_\lrcorner^* F \to F(0,0)$.

With these notations we can give the following definitions.

Proof. Let $\varnothing \to 1$ be the canonical arrow between the initial and terminal object of $\mathbf{C}$, which exist since $\mathbf{C}$ has finite limits.

$$\begin{tikzcd} \varnothing \arrow[r] & 1 \\ F \arrow[r] \arrow[u] & \varnothing \arrow[u] \end{tikzcd}$$

We can build the pushout and pullback squares besides and glue them together to get

$$\begin{tikzcd} F \arrow[r] \arrow[d] & \varnothing \arrow[r] \arrow[d] & 1 \arrow[d] \\ \varnothing \arrow[r] & 1 \arrow[r] & C \end{tikzcd}$$

$$\begin{tikzcd} 1 \arrow[r] & C \\ \varnothing \arrow[r] \arrow[u] & 1 \arrow[u] \end{tikzcd}$$

It's easy to see that $F \cong \varnothing$ and $C \cong 1 \amalg 1$ (and this is true independently from the pullout axiom, simply by definition); so now square ① besides must be a pullout, and $F \cong \varnothing \to \varnothing$ can only be the identity arrow, so that the arrow $1 \to C$ is an isomorphism too (since the class of isomorphisms is closed under cobase change). This entails that there is an isomorphism $1 \to \varnothing \cong 1 \amalg 1$, which allows to conclude: the square

$$\begin{tikzcd} \varnothing \arrow[r] \arrow[d] & 1 \arrow[d, equal] \\ 1 \arrow[r] & 1 \end{tikzcd}$$

is a pullout and isomorphisms are closed under base change too.

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "①"] & \varnothing \arrow[d] \\ 1 \arrow[r] & C \end{tikzcd}$$

We now use this fact to show that products and coproducts coincide everywhere: build the outer square in the following diagram, out of the smaller square (which are precisely the pullbacks/pushouts needed to define products and coproducts):

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & X \times Y \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \arrow[d] \\ Y \arrow[r] & X \amalg Y \arrow[r] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & X \times Y \arrow[r] \arrow[d] & X \arrow[d] \\ 0 \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ X \arrow[r] & X \amalg Y \arrow[r] & X \end{tikzcd}$$

These two diagrams imply the presence of biproducts, in the form of a result shown by Freyd (see [Fre64]): there exists an object S (the *biproduct* of X, Y, hence unique up to unique isomorphism) such that

- There are arrows $Y \leftrightarrows S \leftrightarrows X$;
- The arrow $Y \to S \to Y$ compose to the identity of Y, and the arrow $X \to S \to X$ compose to the identity of X;
- There are "exact sequences" (in the sense of a pointed, finitely bicomplete category) $0 \to Y \to S \to X \to 0$ and $0 \to X \to S \to Y \to 0$.

It is evident that the diagrams above contain all these informations. $\square$

A pleasant consequence of Freyd characterization is that in any additive category the enrichment over the category of abelian groups is *canonical*; in fact, exploiting the isomorphism $Y \times Y \cong Y \amalg Y$ one is able to define the *sum* of $f, g \colon X \rightrightarrows Y$ as

$$f + g \colon X \xrightarrow{\begin{bmatrix} [[]1 \\ 1 \end{bmatrix} []]} X \times X \xrightarrow{(f,g)} Y \times Y \cong Y \amalg Y \xrightarrow{[\,[[]1\ 1\,][]]} Y$$

In fact, this result can be retrieved in the setting of stable quasicategories (see [Lur11, Lemma **1.1.2.9**]); we do not want to reproduce the whole argument: instead we want to investigate the construction of the *loop* and *suspension* functors in a pointed category.

Definition 1.4 : Let $\mathbf{C}$ be a finitely cocomplete, pointed quasicategory; denote by $\mathbf{C}^{\square}_{\text{cocart}}$ the full subcategory of $\text{Map}(\square, \mathbf{C})$ spanned by the cocartesian squares of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ 0' & \longrightarrow & Y \end{array}$$

where $0, 0'$ are (possibly different, but equivalent) zero objects of $\mathbf{C}$; dually, we can define $\mathbf{C}^{\square}_{\text{cart}}$ the full subcategory of $\text{Map}(\square, \mathbf{C})$ spanned by the cartesian squares of the form

$$\begin{array}{ccc} X & \longrightarrow & 0 \\ \big\downarrow & & \big\downarrow \\ 0' & \longrightarrow & Y \end{array}$$

where $0, 0'$ are (possibly different, but equivalent) zero objects of $\mathbf{C}$.

Proposition 1.5 [[Gro10, Prop. **5.3**]]**:** The canonical evaluations

$$\mathrm{e}_{(0,0)} \colon \mathbf{C}^{\square}_{\text{cocart}} \to \mathbf{C} \qquad \mathrm{e}_{(1,1)} \colon \mathbf{C}^{\square}_{\text{cart}} \to \mathbf{C}$$

(where (i, j) denotes the vertex $(i, j) \in \square_0$) are acyclic Kan fibrations.

From this it follows that we can choose sections (unique up to homotopy) s_Σ and s_Ω for $\mathrm{e}_{(0,0)}$ and $\mathrm{e}_{(1,1)}$ respectively. This leads to the following

Definition 1.6 [LOOP AND SUSPENSION FUNCTORS]**:** Let $\mathbf{C}$ be a finitely bicomplete, pointed quasicategory; we define the *suspension* of an object $X \in \mathbf{C}$ as the composition

$$\Sigma\colon \mathbf{C} \xrightarrow{s_\Sigma} \mathbf{C}^{\square}_{\text{cocart}} \xrightarrow{\mathrm{e}_{(1,1)}} \mathbf{C}$$

and (dually) the *looping* of X as the composition

$$\Omega\colon \mathbf{C} \xrightarrow{s_\Omega} \mathbf{C}^{\square}_{\text{cart}} \xrightarrow{\mathrm{e}_{(0,0)}} \mathbf{C}$$

More explicitly, ΣX and ΩX are uniquely determined from the homotopy cocartesian and homotopy cartesian squares...

Definition 1.7 [TRIANGULATED CATEGORY]**:** An additive category $\mathbf{C}$ is called *suspended* if it is endowed with an additive endofunctor $\Sigma\colon \mathbf{C} \to \mathbf{C}$; a category with suspension $(\mathbf{C}, \Sigma)$ is said to be *triangulated* if the following axioms are satisfied:

- PT(-1)) The suspension endofunctor is an equivalence of categories;
- PT0) There exists a class of diagrams in $\mathbf{C}$, called *distinguished triangles* of the form $X \to Y \to Z \to \Sigma X$ (often denoted $X \to Y \to Z \to^{+}$ for short) which is closed under isomorphism and contains every sequence of the form $X \xrightarrow{\mathrm{id}_X} X \to 0 \to \Sigma X$;
- PT1) Any arrow $f\colon \Delta[1] \to \mathbf{C}$ fits into at least one distinguished triangle $X \xrightarrow{f} Y \to Z \to \Sigma X$;
- PT2) (rotation) The diagram $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$ is distinguished if and only if the "rotated diagram" $Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$ is distinguished;
- PT3) (completion) In any diagram of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"'] & Z \arrow[r] & \Sigma X \arrow[d, "\Sigma f"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & \Sigma X' \end{tikzcd}$$

where the rows are distinguished triangles, there exists a morphism $h\colon Z \to Z'$ making the whole diagram a morphism of triangles (which, once regarded triangles as suitable functors $J \to \mathbf{C}$ are simply natural transformations between two such functors).
- TR) Given *three* distinguished triangles

$$X \xrightarrow{f} Y \to Y/X \to^{+} \qquad Y \xrightarrow{g} Z \to Z/Y \to^{+} \qquad X \xrightarrow{gf} Z \to Z/X \to^{+}$$

arranged in a *braid* diagram

$$
\begin{tikzcd}
X \arrow[rr, bend left, "gf"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left] \arrow[dr, "p"] & & Z/Y \arrow[rr, bend left] \arrow[dr] & \circlearrowleft & Y/X[1] \\
& Y \arrow[ur, "g"] \arrow[dr] & & Z/X \arrow[ur, dotted, "t"] \arrow[dr, "q"] & & Y[1] \arrow[ur] & \\
& & Y/X \arrow[ur, dotted, "s"] \arrow[rr, bend right] & & X[1] \arrow[ur] & &
\end{tikzcd}
$$

then there is a (non-unique) way to complete it with the arrows s, t indicated.

With the exception of Axiom TR, which is somehow characteristic, and by no means the less natural among triangulated category axioms, one can easily see that all of them are easy consequences of universal properties of (homotopy) limits: distinguished triangles are precisely those diagrams of the form

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Z \arrow[r] & \Sigma X
\end{tikzcd}
$$

where both squares are pullout. Every arrow $f\colon \Delta[1] \to \mathbf{C}$ fits into a distinguished triangle, since every arrow admits a cofiber $Y \to C$ given by the homotopy colimit of the diagram $0 \leftarrow X \to fY$; this can be completed on its own right taking the cofiber of the arrow $Y \to C$; the 2-for-3 property applied to the diagram

$$X$$

implies now that $W \simeq \Sigma X$; the completion axiom is an immediate consequence of the universal property of homotopy pushouts.

Hence we concentrate on a detailed proof of Axiom TR: in the classical, 1-categorical theory, it can be justified in at least two ways:

- The *freshman algebraist's theorem* holds in triangulated categories: $\frac{Z/X}{Y/X} \cong Z/Y$
- Given the braid diagram above, not only the triangle ending with $Z/Y \to (Y/X)[1]$ can be completed, but the choice can be made in a *coeherent* manner.

In fact once translated the braid diagram in a diagram in a stable quasicategory $\mathbf{C}$ we are in the following situation:

$$
\begin{tikzcd}
X \arrow[r, red, "f"] \arrow[d] \arrow[rr, green, bend left, "gf"] & Y \arrow[r, blue, "g"] \arrow[d, red] & Z \arrow[r] \arrow[d, green] \arrow[dd, blue, bend right] & 0 \arrow[d] & \\
0 \arrow[r] & Y/X \arrow[d] \arrow[rr, red, bend right] & Z/X \arrow[r, green] & X[1] \arrow[r] \arrow[d, red] & 0 \arrow[d] \\
& 0 \arrow[r] & Z/Y \arrow[r, blue] & Y[1] \arrow[r, red] & (Y/X)[1]
\end{tikzcd}
$$

where different colours denote different fiber sequences (i.e., triangles in the homotopy category). Axiom TR says that we can find arrows $Y/X \to Z/X \to Z/Y$ such that the triangle $Y/X \to Z/X \to Z/Y \to (Y/X)[1]$ is distinguished.

The completion axiom (which we already know to hold) now implies that the diagram

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Y/X \arrow[r] & X[1] \arrow[d, "f[1]"] \\
Y \arrow[r, "g"'] & Z \arrow[r] & Z/X \arrow[r] & Y[1]
\end{tikzcd}
$$

can be completed with an arrow $Y/X \xrightarrow{\varphi} Z/X$ completing the square

$$
\begin{tikzcd}
Y \arrow[r] \arrow[d] & Z \arrow[d] \\
Y/X \arrow[r, "\varphi"'] & Z/X.
\end{tikzcd}
$$

Now consider the ojects $V = \mathrm{hocolim}\Big(Y/X \leftarrow Y \xrightarrow{g} Z\Big)$ and $W = \mathrm{hocolim}\Big(0 \leftarrow Y/X \xrightarrow{\varphi} Z/X\Big)$;

2-out-of-3 now implies that the outer rectangle is a pushout, hence $W \cong Z/Y$. It remains to prove that $V \cong Z/X$; this follows from the 2-out-of-3 property applied to the two joined pullout diagrams

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\
0 \arrow[r] & Y/X \arrow[r] & V.
\end{tikzcd}
$$

2. First examples.

About TR4 axiom. The octahedral axiom is kind of the *deus ex machina* of triangulated categories: it says that given three distinguished triangles

$$X \xrightarrow{f} Y \to Y/X \to X[1]$$
$$Y \xrightarrow{g} Z \to Z/Y \to Y[1]$$
$$X \xrightarrow{gf} Z \to Z/X \to X[1]$$

arranged in a *braid* diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "gf"] \arrow[dr, "f"'] & & Z \arrow[rr, bend left] \arrow[dr, "p"] & & Z/Y \arrow[rr, bend left] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & Y/X[1] \\
& Y \arrow[ur, "g"'] \arrow[dr] & & Z/X \arrow[ur, dotted, "t"] \arrow[dr, "q"] & & Y[1] \arrow[ur] & \\
& & Y/X \arrow[ur, dotted, "s"] \arrow[rr, bend right] & & X[1] \arrow[ur] & &
\end{tikzcd}$$

then there is a (non-unique) way to complete it with the arrows s, t indicated. This can be justified in at least two ways:

- The *freshman algebraist's theorem* holds in triangulated categories: $\frac{Z/X}{Y/X} \cong Z/Y$
- Given the braid diagram above, not only the triangle ending with $Z/Y \to (Y/X)[1]$ can be completed, but the choice can be made in a *coeherent* manner.

In fact once translated the braid diagram in a diagram in a stable ∞-category $\mathbf{C}$ we are in the following situation:

$$\begin{tikzcd}
X \arrow[r, red, "f"] \arrow[rr, green, bend left, "gf"] \arrow[d] & Y \arrow[r, blue, "g"] \arrow[d, red] & Z \arrow[r] \arrow[d, green] \arrow[d, blue, bend right] & 0 \arrow[d] & \\
0 \arrow[r] & Y/X \arrow[d] \arrow[rr, red, bend right] & Z/X \arrow[r, green] & X[1] \arrow[r] \arrow[d, red] & 0 \arrow[d] \\
& 0 \arrow[r] & Z/Y \arrow[r, blue] & Y[1] \arrow[r, red] & (Y/X)[1]
\end{tikzcd}$$

where different colours denote different fiber sequences (i.e., triangles in the homotopy category). Axiom TR4 says that we can find arrows $Y/X \to$

$Z/X \to Z/Y$ such that the triangle $Y/X \to Z/X \to Z/Y \to (Y/X)[1]$ is distinguished.

The completion axiom (which we already know to hold) now implies that the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[r] \arrow[d, "g"] & Y/X \arrow[r] & X[1] \arrow[d, "f[1]"] \\ Y \arrow[r, "g"'] & Z \arrow[r] & Z/X \arrow[r] & Y[1] \end{tikzcd}$$

can be completed with an arrow $Y/X \xrightarrow{\varphi} Z/X$ completing the square

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Z \arrow[d] \\ Y/X \arrow[r, "\varphi"'] & Z/X. \end{tikzcd}$$

Now consider the ojects $V = \text{hocolim}\Big(Y/X \leftarrow Y \xrightarrow{g} Z\Big)$ e $W = \text{hocolim}\Big(0 \leftarrow Y/X \xrightarrow{\varphi} Z/X\Big)$;

$$\begin{tikzcd} Y \arrow[rr] \arrow[dd] & & Z \arrow[dl] \arrow[dd] \\ & V \arrow[dr] & \\ Y/X \arrow[ur] \arrow[rr, "\varphi"'] \arrow[d] & & Z/X \arrow[d] \\ 0 \arrow[rr] & & W \end{tikzcd}$$

2-out-of-3 now implies that the outer rectangle is a pushout, hence $W \cong Z/Y$. It remains to prove that $V \cong Z/X$; this follows from the 2-out-of-3 property applied to

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[d] \\ 0 \arrow[r] & Y/X \arrow[r] & V. \end{tikzcd}$$

About the completion axiom. The completion axiom says that any diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[r, "h"] & X[1] \arrow[d, "p[1]"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & X'[1] \end{tikzcd}$$

where the rows are distinguished triangles can be completed to a morphism of triangles $(p, q, r\colon Z \to Z')$.

In the ∞-categorical setting this can be derived extremely easily starting from universal properties of (homotopy) pushouts: since in the cubic diagram

$$\begin{tikzcd}
 & X' \arrow[rr] \arrow[dl] \arrow[dd] & & Y' \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[dl] \arrow[dd] \\
X \arrow[rr] \arrow[dd] & & Y \arrow[rr] \arrow[dd] & & 0 \arrow[dd] & \\
 & 0 \arrow[rr] \arrow[dl] & & Z' \arrow[rr] & & X'[1] \arrow[dl] \\
0 \arrow[rr] & & Z \arrow[rr] & & X[1] &
\end{tikzcd}$$

the objects Z, Z' are respectively $\mathrm{cofib}(f)$ and $\mathrm{cofib}(f')$ we get the desired arrow ($p[1]$ is obtained via a universal property too: the commutativity of the diagram besides induces a –unique up to homotopy– dotted arrow).

$$\begin{tikzcd}
X' \arrow[drrr, bend left] \arrow[dddr, bend right] & & & \\
 & X \arrow[rr] \arrow[dd] & & 0 \arrow[dl] \arrow[dd] \\
 & & X'[1] \arrow[dr, dotted] & \\
 & 0 \arrow[ur] \arrow[rr] & & X[1]
\end{tikzcd}$$

2.1. **Other results.**

About the triangulated 5 lemma. The triangulated 5-lemma says that in a morphism of triangles

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "p"'] & Y \arrow[r, "g"] \arrow[d, "q"'] & Z \arrow[r, "h"] \arrow[d, "r"'] & X[1] \arrow[d, "p[1]"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & X'[1]
\end{tikzcd}$$

where p, q are isomorphisms, then also r is an iso.

The proof simply exploits the fact that the functor hom is "decent" (its codomain has product, and it respects them: see [**Nee01**], Def. **1.1.12**) and the Yoneda lemma. More precisely:

Lemma 2.1 : (notations as before) If $H\colon \mathbf{C} \to \mathbf{Ab}$ is (decent and) homological, and $H(p[n]), H(q[n])$ are isomorphisms for any $n \in \mathbb{Z}$, then also $H(r[n])$ is an isomorphism.

Lemma 2.2 : (notations as before) If $H\colon \mathbf{C} \to \mathbf{Ab}$ is (decent and) homological, and p, q are isomorphisms, then $H(r)$ is an isomorphism too.

Now, being $\hom(X, -)$ (decent and) homological for any $X \in \mathbf{C}$, we get that

$$\hom(X, r)\colon\ \hom(X, Z) \to \hom(X, Z')$$

is an isomorphism for any $X \in \mathbf{C}$, or in other words that $\hom(-, r)$ is a natural isomorphism. Being the Yoneda embedding fully faithful, r must be an isomorphism.

In the ∞-categorical setting, we must appeal a similar version of the Yoneda lemma. First of all, the condition "p, q isomorphisms" in the diagram before translates into the condition "p, q homotopy equivalences" in the morphism of fiber sequences

$$
\begin{tikzcd}
& X' \arrow[rr] \arrow[dl] \arrow[dd] & & Y' \arrow[rr] \arrow[dl] \arrow[dd] & & 0 \arrow[dl] \arrow[dd] \\
X \arrow[rr] \arrow[dd] & & Y \arrow[rr] \arrow[dd] & & 0 \arrow[dd] & \\
& 0 \arrow[rr] \arrow[dl] & & Z' \arrow[rr] \arrow[dl] & & X'[1] \arrow[dl] \\
0 \arrow[rr] & & Z \arrow[rr] & & X[1] &
\end{tikzcd}
$$

This entails that there is a completion

$$
\begin{tikzcd}
X' \arrow[r, "f"] \arrow[d, "p"'] & Y' \arrow[r, "g"] \arrow[d, "q"'] & Z' \arrow[r, "h"] \arrow[d, "r"] & X'[1] \arrow[d, "{p[1]}"] \\
X \arrow[r, "f'"] \arrow[d, dotted] & Y \arrow[r, "g'"] \arrow[d, dotted] & Z \arrow[r, "h'"] \arrow[d, dotted] & X[1] \arrow[d, dotted] \\
0 \arrow[r, dotted] \arrow[d, dotted] & 0 \arrow[r, dotted] \arrow[d, dotted] & Z'' \arrow[r, dotted] \arrow[d, dotted] & 0 \arrow[d, dotted] \\
X'[1] \arrow[r] & Y'[1] \arrow[r] & Z'[1] \arrow[r] & X'[2]
\end{tikzcd}
$$

(embed p, q in distinguished triangles: since they are isomorphisms, triangles can be completed with zero maps; then use May's Lemma).

This in turn implies that $Z' \cong 0$, since the third row can be chosen so as to make $0 \to 0' \to Z'' \to 0$ distinguished. Then the Yoneda lemma entails that ir is sufficient to show that the induced map $\mathrm{Map}_{\mathbf{C}}(Z', X) \to \mathrm{Map}_{\mathbf{C}}(Z, X)$ is a weak homotopy equivalence for any $X \in \mathbf{C}$; again (see [Lur11], Notation **1.1.2.16**) this is implied by the fact that the abelian groups

in one of the forms suited for ∞-categories

$$\mathrm{Ext}^k_{\mathbf{C}}(Z', X), \mathrm{Ext}^k_{\mathbf{C}}(Z, X)$$

are isomorphic for each $k \leq 0$ via the induced map. Now this is obvious, since $\mathrm{Ext}^*(X, -) = \pi_*(\hom_{\mathbf{C}}(X, -))$ is homological.

The heart of a t-structure is abelian. It means: the homotopy category of the heart of an ∞-stable t-structure is abelian.

Lemma 2.3 : Let $X \to Y \to Z \to X[1]$ be a fiber sequence. Then if $X, Z \in \mathbf{D}_{\geq}/\mathbf{D}_{\leq}$ so is Y.

This is obvious until a t-structure on $\mathbf{C}$ is defined as a t structure on $\mathrm{Ho}(\mathbf{C})$, since the objects are the same on both sides.

Lemma 2.4 : The ∞-Yoneda lemma gives that $\mathrm{Map}(-,=)$ reflects homotopy exact sequences.

Let now $f\colon \Delta[1] \to \mathbf{C}$, embedded in a fiber sequence

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & Z \arrow[r] & X[1]; \end{tikzcd}$$

from the first lemma follows that $Z \in \mathbf{D}^{\leq} \cap \mathbf{D}^{\geq -1}$; now

- (The homotopy image of) f has a kernel and a cokernel, since for any $W \in \mathbf{C}^{\heartsuit}$ we have two exact sequences of abelian groups
  $$\cancel{\mathrm{Map}(X[1], W)} \to \mathrm{Map}(Z, W) \to \mathrm{Map}(Y, W) \to \mathrm{Map}(X, W)$$
  $$\cancel{\mathrm{Map}(W, Y[-1])} \to \mathrm{Map}(W, Z[-1]) \to \mathrm{Map}(W, X) \to \mathrm{Map}(W, Y).$$
  Now (by adjunction) $\mathrm{Map}(Z, W) \cong \mathrm{Map}(\tau^{\geq} Z, W)$ and $\mathrm{Map}(W, Z[-1]) \cong \mathrm{Map}(W, \tau^{\leq}(Z[-1]))$, so in the end we are left with exact sequences
  $$0 \to \mathrm{Map}(\tau^{\geq} Z, W) \to \mathrm{Map}(Y, W) \to \mathrm{Map}(X, W)$$
  $$0 \to \mathrm{Map}(W, \tau^{\leq}(Z[-1])) \to \mathrm{Map}(W, X) \to \mathrm{Map}(W, Y).$$
  This is a translation of the universal properties of kernels and cokernels. So we showed that
  $$\begin{cases} \ker f \cong Z[-1]^{\leq} \\ \operatorname{coker} f \cong Z^{\geq} \end{cases}$$
- It remains to check that
  $$\mathrm{coim} f = \operatorname{coker} \ker f \cong \ker \operatorname{coker} f = \operatorname{im} f.$$
  To this end, consider the braid
  $$\begin{tikzcd}[column sep=small] Z[-2]^{\leq} \arrow[rr, bend left] \arrow[dr] & & X \arrow[rr, bend left] \arrow[dr, dotted] & & Y \arrow[rr, bend left] \arrow[dr] \arrow[rr, phantom, "\circlearrowleft"] & & Z^{\geq} \\ & Z[-1] \arrow[ur] \arrow[dr] & & I \arrow[ur] \arrow[dr, dotted] & & Z \arrow[ur] & \\ & & Z[-1]^{\geq} \arrow[rr] \arrow[ur] & & Z[-1]^{\leq} \arrow[ur] & & \end{tikzcd}$$

and the resulting triangle $X \to I \to Z[-1]^{\leq} \to X[1]$, from which it follows that $I \in \mathbf{C}^{\heartsuit}$. On the one side, from the triangle $\ker f \to X \to I \to \ker f[1]$ we get $\mathrm{coim} f \cong \mathrm{coker}(\ker f \to I) \cong I^{\geq} \cong I$; on the other side from the triangle $I \to Y \to Z^{\geq} \to I[1]$ we get $\operatorname{im} f \cong \ker(Y \to Z^{\geq}) \cong I[1][-1]^{\leq} \cong I$. □

3. An exercise from May

Proposition 3.1 : Suppose given the following diagram

$$(3 \times 3) \qquad \begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "i"'] & Y \arrow[r, "g"] \arrow[d, "j"'] & Z \arrow[r, "h"] \arrow[d, dotted, "k"] & X[1] \arrow[d, "{i[1]}"] \\
X' \arrow[r, "f'"] \arrow[d, "i'"'] & Y' \arrow[r, "g'"] \arrow[d, "j'"'] & Z' \arrow[r, "h'"] \arrow[d, dotted, "k'"] & X'[1] \arrow[d, "{i'[1]}"] \\
X'' \arrow[r, dotted, "f''"] \arrow[d, "i''"'] & Y'' \arrow[r, dotted, "g''"] \arrow[d, "j''"'] & Z'' \arrow[r, dotted, "h''"] \arrow[d, dotted, "k''"] & X''[1] \arrow[d, "{-i''[1]}"] \\
X[1] \arrow[r, "{f[1]}"'] & Y[1] \arrow[r, "{g[1]}"'] & Z[1] \arrow[r, "{-h[1]}"'] & X[2]
\end{tikzcd}$$

where the upper left square is commutative, the two rows are distinguished, and where Z'' and the dotted arrows do not exist a priori. Then one can find such a Z'' and the arrows completing the diagram; the bottom right square commutes up to a sign, and all four rows are distinguished.

Proof. First of all notice that the bottom row is distinguished (since it is a triangle isomorphic to $(-f[1], -g[1], -h[1])$). Then, embed the arrow $a = jf = f'i$ in a distinguished triangle

$$X \xrightarrow{jf} Y' \xrightarrow{p} V \xrightarrow{q} X[1]$$

Now notice that we are in the position to apply the octahedral axiom to the following situation:

$$\begin{tikzcd}
X \arrow[rr, bend left, "jf"] \arrow[dr, "f"'] & & Y' \arrow[rr, bend left, "j'"] \arrow[dr, "p"'] & & Y'' \arrow[rr, bend left] \arrow[dr, "j''"'] & \circlearrowleft & Z[1] \\
& Y \arrow[ur, "j"'] \arrow[dr] & & V \arrow[ur, dotted, "t"] \arrow[dr, "q"'] & & Y[1] \arrow[ur, "g[1]"'] & \\
& & Z \arrow[ur, dotted, "s"] \arrow[rr, bend right, "h"'] & & X[1] \arrow[ur, "f[1]"'] & &
\end{tikzcd}$$

Hence we obtain a completion (s, t) such that the whole braid is commutative and

$$pj = sg, \quad tp = j', \quad qs = h, \quad j''t = f[1]q$$

The same procedure applied to $f'i$, with different triangles, produces a diagram

$$\begin{tikzcd}
X \arrow[rr, bend left, "f'i"] \arrow[dr, "i"'] & & Y' \arrow[rr, bend left, "j'"] \arrow[dr, "p"'] & & Z' \arrow[rr, bend left] \arrow[dr, "h'"'] & \circlearrowleft & X''[1] \\
& X' \arrow[ur, "f'"'] \arrow[dr, "i'"'] & & V \arrow[ur, dotted, "t'"] \arrow[dr, "q"'] & & X'[1] \arrow[ur, "i'[1]"'] & \\
& & X'' \arrow[ur, dotted, "s'"] \arrow[rr, bend right, "i''"'] & & X[1] \arrow[ur, "i[1]"'] & &
\end{tikzcd}$$

These give the following relations:

$$pf' = s'i', \quad t'p = g', \quad qs' = i'', \quad h't' = i[1]q$$

Now, define $k = t's$: then $kg = t'sg = t'pj = g'j$ and $h'k = h't's = i[1]qs = i[1]h$. Define $f'' = ts'$ (so that $f''i' = ts'i' = tpf' = j'f'$), and embed f'' in a distinguished triangle

$$X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \xrightarrow{h''} X''[1].$$

Apply again the octahedral axiom we get

$$\begin{tikzcd}
\bullet \arrow[rr, bend left, "f''"] \arrow[dr, "s'"'] & & \bullet \arrow[rr, bend left, "{g[1]j''}"] \arrow[dr, "g''"'] & \mathbf{b} & Z \arrow[rr, bend left] \arrow[dr, "{-s[1]}"'] & \circlearrowleft & \bullet \\
& \bullet \arrow[ur, "t"'] \arrow[dr, "t'"'] & \mathbf{l} & Z'' \arrow[ur, dotted, "k''"] \arrow[dr, "h''"'] & \mathbf{r} & \bullet \arrow[ur, "{t'[1]}"'] & \\
& & Z' \arrow[ur, dotted, "k'"] \arrow[rr, bend right, "{i'[1]h'}"'] & \mathbf{a} & \bullet \arrow[ur, "{s'[1]}"'] & &
\end{tikzcd}$$

Notice that $-t'[1]s[1] = -k[1]$. Hence we obtain a distinguished triangle $Z' \xrightarrow{k'} Z'' \xrightarrow{k''} Z \xrightarrow{-k[1]} Z[1]$, such that all the square up to the central commute (using $\mathbf{a}, \mathbf{b}$). Now the commutativity of squares $\mathbf{l}, \mathbf{r}$ concludes. $\square$

∞-*Proof.* We will try to mimic the previous proof in the ∞-categorical setting. Let $\mathbf{C}$ be a stable ∞-category, and suppose given the same arrows of diagram (3×3).

First of all, the embedding $jf = f'i$ into a d.t. amounts to build the diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & V \arrow[r] & W \cong X[1]
\end{tikzcd}$$

where $V \cong \operatorname{hocolim}\Big(0 \leftarrow X \to Y'\Big)$ and where $W \cong X[1]$ by the 2-out-of-3 property of pulation squares in a stable ∞-category $\mathbf{C}$; it follows from the axioms of triangulated category that the composition $V \to X[1] \to Y[1]$ is homotopic to zero, hence a fortiori $V \to Y[1] \xrightarrow{j[1]} Y'[1]$ is $\cong 0$. Hence the square

$$\begin{tikzcd}
V \arrow[r] \arrow[d] & Y[1] \arrow[d] \\
0 \arrow[r] & Y'[1]
\end{tikzcd}$$

factors through the homotopy fiber of $Y[1] \xrightarrow{j[1]} Y'[1]$, i.e. through Y''. $\square$

4. More elaborate constructions.

4.1. Verdier (and others) Localization.

4.2. Universal property of $\mathbf{T} \to \widehat{\mathbf{T}}$.

4.3. Cetera.

References

[BO01] A.I. Bondal and D.O. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, Compositio Mathematica **125** (2001), no. 03, 327–344.

[Fre64] P. Freyd, *Abelian categories. An introduction to the theory of functors*, Harper's Series in Modern Mathematics, Harper & Row, Publishers, New York, 1964. MR 0166240 (29 #3517)

[Gro10] M. Groth, *A short course on ∞-categories*, arXiv preprint arXiv:1007.2925 (2010), 77.

[Lur09] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659 (2010j:18001)

[Lur11] ______, *Higher algebra*, online version May 18, 2011.

[MK07] G. Maltsiniotis and B. Keller, *La K-theorie d'un derivateur triangule.*, Contemporary Mathematics **431** (2007), 341.

[Nee01] A. Neeman, *Triangulated categories*, Ann. of Math. Studies, vol. 148, Princeton University Press, 2001.

[Ver67] J.L. Verdier, *Des catégories dérivées des catégories abéliennes*, Ph.D. thesis, Université de Paris, 1967.

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