Fosco Loregian

May 8, 2020

- Ph.D. at SISSA - Trieste 🇮🇹

  Stable homotopy theory, $\infty$-categories, derived AG

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- Tallinna Tehnikaülikooli

  2-categories; functorial semantics; categorical probability theory and its applications

# STABLE HOMOTOPY THEORY

$\infty$-categories: a thickening of the notion of category, suitable for homotopy-invariant mathematics (algebraic geometry, algebraic topology type theory).

Turns out some parts of Mathematics are easier if stated in these terms:

- homological algebra : the scary part of algebraic topology

  higher algebra: the linear algebra of $\infty$-categories
- 1-topos theory : a synthetic type theory,

  $\infty$-topos theory: a synthetic homotopy theory of homotopy types

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- with all finite limits and colimits
- such that a square is cartesian iff cocartesian

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Stable, rational, p-adic, ... homotopy theory become pieces of the commutative algebra of $\infty$-categories

A t-structure on a triangulated $\mathcal{D}$ is a pair of triangulated subcategories of D such that every object X lies in a sequence

$$X_{\leq} \to X \to X_{\geq} \to X_{\leq}[1]$$

[FL14] : On stable ∞-categories a t-structure is a factorization system (E,M)

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(Algebraic) geometry reduces to a piece of categorical algebra.

## Plan: redo $t$-structures

- [FL15] The set of t-structures has a natural choice of $\mathbb{Z}$-action ($\mathbb{Z}$ = the integers); so, study $\mathbb{Z}$-equivariant monotone maps from a poset J to TS(C).

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- [FL15b] Every stratified manifold $(X, \mathfrak{s})$ generates a pair of t-structure that can be glued together

  recollements, stratified schemes, representation of algebras

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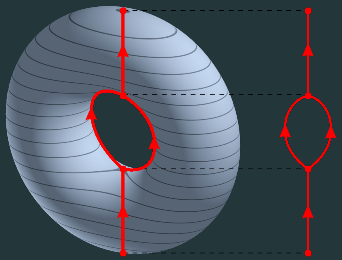
**Conjecture**
Blakers-Massey in positive characteristic is a theorem about factorization systems.

# Morse theory as a theory of FS



Fact: Bord(n) is the free (∞,n)-symmoncat on the point.

Monoidal functors $Z : Bord(n) \to Vect$ are completely classified.

Morse theory is the theory of suitable factorization systems on Bord(n).

critical points of a Morse function correspond to

critical values [L-PhD, Ch.7] of a certain slicing $J : \mathbb{R} \to FS(Bord(n))$.

# DERIVATORS

A derivator is a strict 2-functor

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satisfying stacky conditions. They form the 2-category Der.

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[Lor18] : reflective subderivators correspond to reflective factorization systems, and to algebras for idempotent monads (the "formal theory of monads" [*S80*] still holds in Der, a monad $T : D \to D$ is just defined objectwise)

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If this is true

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- a convincing form of adjoint functor theorem for derivators; existence of a six-operation calculus for mappings of derivators. 2-categorical account of Grothendieck duality complicated diagrams...
- profunctors between derivators; fibered derivators; operads in derivator theory; applications in representation theory of algebras, stable homotopy, ...

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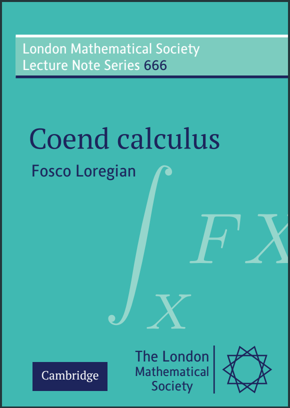
# COENDS AND DG-STUFF

I have written a book on coend calculus, soon to appear under Cambridge University Press LNMs:

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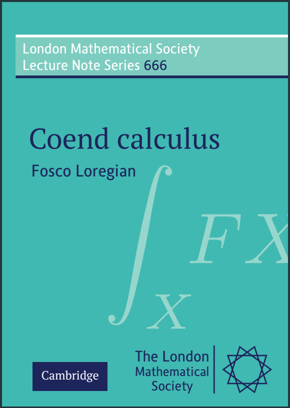


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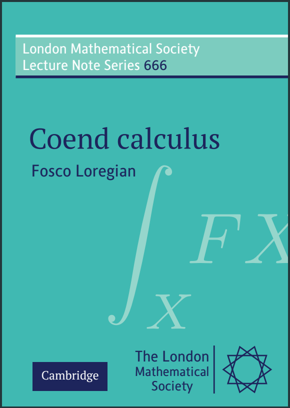


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- They find application in (monoidal) category theory, algebraic topology, algebraic geometry, categorical logic, representation theory (see ch.7 for an application to DG-categories), functional programming...
- The book is being extensively cited (45 citations on Scholar May 8, 2020)

In [L20, 7.2.2]

For example: if $\mathcal{A}$ is any dg-category its identity profunctor $\mathcal{A} \rightsquigarrow \mathcal{A}$ is a functor $\mathcal{A}^{\mathrm{op}} \boxtimes \mathcal{A} \to \mathrm{Ch}(\mathbb{Z})$, so that the coherent end

$$\oint_A \mathcal{A}(A, A) \tag{7.82}$$

i.e. the object of derived natural transformations of the identity functor $\mathrm{id}_{\mathcal{A}}$, recovers the *Hochschild complex* of $\mathcal{A}$. Then, if $\mathcal{A}$ is an associative algebra regarded as a one-object dg-category concentrated in degree zero, the object $H^n(\int_* A)$ is the *Hochschild cohomology* of $A$, understood in the classical sense of, say, [Pie82, Ch. 11].

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Applications to Kuznetsov-Lunts categorical resolutions of singularities: a smooth DG-category is a $\mathcal{D}$ such that its identity profunctor $h : \mathcal{D} \rightsquigarrow \mathcal{D}$ is a perfect object (read as: a variety is smooth if the diagonal map $\Delta : X \to X \times X$ is smooth)

# TEACHING AND ORGANIZATIONAL ACTIVITIES

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- Reviewer for zbMath and AMS.

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- more in detail, “2-semantics” of algebraic theories: profunctorial PROPs and theories, categorical algebra of cartesian bicategories…