P-Q-COENDS

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Contents

1. Motivation:

1.1. "symmetrize" tensors of higher arity. The so-called "Einstein summation convention" in linear algebra and differential geometry asserts that it is possible to suppress the summation symbol $\sum$ in every formula like

$$c_i v_i$$

at the cost of writing "$c^i v_i$"; this means that contravariant tensors' indices are superscripts, while covariant tensors' indices are subscripts, and whenever homonymous indices appear in a string like $c^i v_i$, it means that we are

Date: September 17, 2020.

summing over that index. So, for example, the first Bianchi identity:

$$\nabla_i R^i{}_j = \frac{1}{2}\nabla_j R$$

stands for $\sum_i \nabla_i R^i{}_j = \dots$, and the definition of R itself expands to a daunting

$$\begin{aligned} R_{ij} = -\frac{1}{2}\Big(&\frac{\partial^2 g_{ij}}{\partial x^a \partial x^b} + \frac{\partial^2 g_{ab}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ib}}{\partial x^j \partial x^a} - \frac{\partial^2 g_{jb}}{\partial x^i \partial x^a}\Big) g^{ab} \\ &+ \frac{1}{2}\Big(\frac{1}{2}\frac{\partial g_{ac}}{\partial x^i}\frac{\partial g_{bd}}{\partial x^j} + \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jd}}{\partial x^b} - \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jb}}{\partial x^d}\Big) g^{ab} g^{cd} \\ &- \frac{1}{4}\Big(\frac{\partial g_{jc}}{\partial x^i} + \frac{\partial g_{ic}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^c}\Big)\Big(2\frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d}\Big) g^{ab} g^{cd}. \end{aligned}$$

This convention does not allow for "unbalanced" expression to be summed over: the same number of subscript must be paired with the same number of superscripts.

In category theory, the analogue operation of "summing over repeated indices" is taking a **coend** of a functor

$$T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$$

intended as the quotient of $\coprod_C T(C, C)$ by the equivalence relation generated by the action of T on arrows; this analogy is not peregrine: if $S : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Set}$ and $T : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$ are two "profunctors", their composition

$$ST(A, C) := \int^B S(A, B) \times T(B, C)$$

is akin to the matrix product of two matrices, seen as functions $S : [n] \times [m] \to K$, $T : [m] \times [r] \to K$.

(A perfect analogy is this: let A,B,Cbe discrete categories; then the profunctor composition of $S : A \times B \to \mathsf{Set}$ and $T : B \times C \to \mathsf{Set}$ is the matrix product of an $|A| \times |B|$ and a $|B| \times |C|$ matrix.)

1.2. **Question(s).** What if we want to sum/integrate/coend over an "unbalanced tensor" like

$$T : (\mathcal{C}^{\mathrm{op}})^p \times \mathcal{C}^q = \mathcal{C}^{(p,q)} \to \mathcal{D}$$

for $p, q \geq 1$?

Is the resulting theory well-behaved as the classical one?

No one would debate about the usefulness of "balanced" integrals; are the unbalanced ones good for something similar?

This work aims at answering all these questions in the positive:

- Yes, one can define a notion of **co/end** for "higher arity" functors $\mathcal{C}^{(p,q)} \to \mathcal{D}$;

- Yes and no; higher arity co/ends are particular instances of co/ends, where T has been "completely symmetrised" (see later for a definition); as such, they do not constitute a "new" object; instead, a specialisation of classical co/end calculus;
- Yes, the resulting theory is expressive enough to capture some new phenomena.

At this point, perhaps the most enlightening example is the following, appearing in a paper by Street and Dubuc:

Proposition 1. *Let $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be two functors; define the functor*

$$\mathrm{DNat}(F^{\uparrow}, G^{\downarrow}) : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Set}$$

sending (A, B) to $\mathcal{D}(F^B_A, G^A_B)$; then, the set of dinatural transformations $F \overset{\bullet\bullet}{\Longrightarrow} G$ is canonically isomorphic to the end of DNat$(F^{\uparrow}, G^{\downarrow})$.

1.3. **generalised dinaturality recently introduced by A. Santamaria in his PhD thesis.** A. Santamaria and McGusker recently introduced in [MS] the notion of dinaturality we started from; yet, his notion is too general for our purposes for two reasons:

- they do not assume a "transformation" satisfies any condition globally, treating the

notion of naturality as a property of a single component (this reads as: a transformation between two functors of a certain "type" is di/natural at an index i, but it can be "unnatural" elsewhere).

- they do not assume that the type of the domain functor and the codomain functor are the same.

Our notational convention is also different: they take into account functors $\mathcal{C}^\alpha \to \mathcal{B}$, where α is a “binary multi-index”, i.e. an element in the free monoid over the set $\{\oplus, \ominus\}$, and the convention is that $\mathcal{C}^\varnothing :=$, the terminal category, $\mathcal{C}^\oplus := \mathcal{C}$, $\mathcal{C}^\ominus := \mathcal{C}^{\mathrm{op}}$, and $\mathcal{C}^{\alpha \uplus \alpha'} := \mathcal{C}^\alpha \times \mathcal{C}^{\alpha'}$.

Here instead, we adopt a different convention: a generic power $\mathcal{C}^\alpha$ is always “reshuffled” in order for all its minus and plus signs to appear on the same side, respectively on the left and on the right. The categories $\mathcal{C}^\alpha$ and $\mathcal{C}^{(p,q)}$ so obtained are, of course, canonically isomorphic, and the tuple α is equivalent to the reshuffled tuple $(\ominus_1, \dots, \ominus_p, \oplus_1, \dots, \oplus_q)$.

Definition 1. *Let α, β be two multi-indices, and let $F : \mathcal{C}^\alpha \to \mathcal{D}$, $G : \mathcal{C}^\beta \to \mathcal{D}$ be functors. A* transformation $\phi : F \to G$ of type $|\alpha| \xrightarrow{\sigma} n \xleftarrow{\tau} |\beta|$ *(with $n = |\boldsymbol{A}|$ a positive integer) is a family of morphisms in $\mathcal{D}$*

$$\left(\phi_{\boldsymbol{A}} : F(\boldsymbol{A}\sigma) \to G(\boldsymbol{A}\tau)\right)_{\boldsymbol{A} \in \mathcal{C}^n}.$$

This translates into a family

$$\phi_{A_1,\dots,A_n} : F(A_{\sigma 1}, \dots, A_{\sigma|\alpha|}) \to G(A_{\tau 1}, \dots, A_{\tau|\beta|}).$$

Notice that α and β are *different* multi-indices in this definition, and σ, τ need not be injective or surjective, so we may have repeated or unused variables.

Definition 2. *Let $\phi = (\phi_{A_1,\dots,A_n}) : F \to G$ be a transformation. For $i \in \{1, \dots, n\}$, we say that ϕ is* dinatural in A_i *(or, more precisely,* dinatural in its i-th variable*) if and only if for all $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ objects of $\mathcal{C}$ and for all $f : A \to B$ in $\mathcal{C}$ the following hexagon commutes:*

$$\begin{tikzcd}
 & F(\boldsymbol{A}[A/i]\sigma) \arrow[r, "\phi_{\boldsymbol{A}[A/i]}"] & G(\boldsymbol{A}[A/i]\tau) \arrow[dr, "G(\boldsymbol{A}[A,f/i]\tau)"] & \\
F(\boldsymbol{A}[B,A/i]\sigma) \arrow[ur, "F(\boldsymbol{A}[f,A/i]\sigma)"] \arrow[dr, "F(\boldsymbol{A}[B,f/i]\sigma)"'] & & & G(\boldsymbol{A}[A,B/i]\tau) \\
 & F(\boldsymbol{A}[B/i]\sigma) \arrow[r, "\phi_{\boldsymbol{A}[B/i]}"'] & G(\boldsymbol{A}[B/i]\tau) \arrow[ur, "G(\boldsymbol{A}[f,B/i]\tau)"'] &
\end{tikzcd}$$

where $\boldsymbol{A}$ is the n-tuple $(A_1, \dots, A_n)$ of the objects above with an additional (unused in this definition) object A_i of $\mathcal{C}$.

2. Higher arity co/wedges

N1) A generic tuple of objects,

$$\underline{A} := (A_1, \dots, A_n)$$

often split as the juxtaposition $\underline{A}'; \underline{A}''$ of two subtuples of length p, q,

$$\underline{A}' := (A_1, \dots, A_q), \qquad \underline{A}'' := (A_{p+1}, \dots, A_{p+q})$$

N2) As already said, the image of a split tuple $\underline{A}'; \underline{A}''$ under a functor of type $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$, $F : \mathcal{C}^{(p,q)} \to \mathcal{D}$ is denoted $F^{\underline{A}'}_{\underline{A}''}$: the contravariant components come first, and the covariant component second. So: contravariant components are always *left* in the typing

$$F : \mathcal{C}^{(p,q)} \varnothing \mathcal{D}$$

of a functor, and *up* in its action on objects.

N3) Denoting a functor F of type $\left[\begin{smallmatrix}p\\q\end{smallmatrix}\right]$ evaluated at a diagonal tuple: we write

$$F^{\boldsymbol{A}}_{\boldsymbol{A}} := F^{A,\dots,A}_{A,\dots,A},$$

where the superscript has p elements, and the subscript has q elements.

N4) Substitution of an object at a prescribed index

$$\underline{A}[X/i] := (A_1, \dots A_{i-1}, X, A_{i+1}, \dots, A_n).$$

N5) Substitution of a tuple at a prescribed tuple of indices

$$\underline{A}[X_1, \dots, X_r/i_1, \dots, i_r] := ((\underline{A}[X_1/i_1])[X_2/i_2] \cdots)[X_r/i_r].$$

Definition 3. *A* (p, q)-dinatural transformation $\alpha : F \overset{\bullet\bullet}{\Longrightarrow} G$ *is a collection*

$$\{\alpha_A : F^{\overbrace{A,\dots,A}^{p\ times}}_{\underbrace{A,\dots,A}_{q\ times}} \longrightarrow G^{\overbrace{A,\dots,A}^{q\ times}}_{\underbrace{A,\dots,A}_{p\ times}} \mid A \in \mathcal{C}_o\}$$

of morphisms of $\mathcal{D}$ *indexed by the objects of* $\mathcal{C}$ *such that, for each morphism* $f : A \to B$ *of* $\mathcal{C}$*, the diagram*

$$\begin{tikzcd}
& F^{\boldsymbol{A}_p}_{\boldsymbol{A}_q} \arrow[r, "\alpha_A"] & G^{\boldsymbol{A}_q}_{\boldsymbol{A}_p} \arrow[dr, "G^{\boldsymbol{A}_q}_{\boldsymbol{f}_p}"] & \\
F^{\boldsymbol{B}_p}_{\boldsymbol{A}_q} \arrow[ur, "F^{\boldsymbol{f}_p}_{\boldsymbol{A}_q}"] \arrow[dr, "F^{\boldsymbol{B}_p}_{\boldsymbol{f}_q}"'] & & & G^{\boldsymbol{A}_q}_{\boldsymbol{B}_p} \\
& F^{\boldsymbol{B}_p}_{\boldsymbol{B}_q} \arrow[r, "\alpha_B"'] & G^{\boldsymbol{B}_q}_{\boldsymbol{B}_p} \arrow[ur, "G^{\boldsymbol{f}_q}_{\boldsymbol{B}_p}"'] &
\end{tikzcd}$$

commutes.

A different name for this same notion: a (p, q)*-to-*(q, p)*-dinatural transformation.*

Example 1. *For* $(p, q)=(2, 1)$*, a* $(2, 1)$*-dinatural transformation is a collection*

$$\left\{\alpha_A : F^{A,A}_A \to G^A_{A,A} \;\middle|\; A \in \mathcal{C}_o\right\}$$

of morphisms of $\mathcal{D}$ *such that, for each morphism* $f : A \to B$ *of* $\mathcal{C}$*, the following hexagonal diagram commutes:*

$$\begin{tikzcd}
& F^{A,A}_A \arrow[r, "\alpha_A"] & G^A_{A,A} \arrow[dr, "G^A_{f,f}"] & \\
F^{B,B}_A \arrow[ur, "F^{f,f}_A"] \arrow[dr, "F^{B,B}_f"'] & & & G^A_{B,B} \\
& F^{B,B}_B \arrow[r, "\alpha_B"'] & G^B_{B,B} \arrow[ur, "G^f_{B,B}"'] &
\end{tikzcd}$$

2.1. **Why the weird "(p,q)-to-(q,p)" definition?** We could have stick to Santamaria's definition of "(p, q)-to-(r, s)" dinaturality; we could have stick to the notion of (p, q)-to-(p, q) dinaturality. Our definition sits in the middle:

> the type of domain and codomain of a "higher arity" dinatural transformation $\alpha : F \overset{\bullet\bullet}{\Longrightarrow} G$ are different, but just

swapped: the contravariant length of F is the covariant length of G, and vice-versa.

It is important, even if straightforward, to note that as far as higher arity co/wedges (i.e. higher arity dinatural transformations from/to a constant functor) are concerned, the notions of (p,q)-dinaturality and (p,q)-to-(r,s)-dinaturality agree and yield the same theory of higher arity co/ends.

(Recall Mac Lane principle: what is the "right" level of generality?)

Definition 4. *Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor and let $X \in \mathcal{D}_o$.*

- CW1) *A* (p,q)-wedge for D under X *is a (p,q)-dinatural transformation $\theta : X \overset{\bullet\bullet}{\Longrightarrow} D$ from the constant functor of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ with value X to D;*
- CW2) *A* (p,q)-cowedge for D over X *is a (p,q)-dinatural transformation $\zeta : D \overset{\bullet\bullet}{\Longrightarrow} X$ from D to the constant functor of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ with value X.*

Remark 1.

CWU1) *A (p,q)-wedge $\theta : X \overset{\bullet\bullet}{\Longrightarrow} D$ is a collection*

$$\left\{\theta_A : X \to D^{\boldsymbol{A}}_{\boldsymbol{A}} \ : \ A \in \mathcal{C}_o\right\}$$

of morphisms of $\mathcal{C}$ such that, for each morphism $f : A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
X \arrow[r, "\theta_B"] \arrow[d, "\theta_A"'] & D^{\boldsymbol{B}}_{\boldsymbol{B}} \arrow[d, "D^{\boldsymbol{f}}_{\boldsymbol{B}}"] \\
D^{\boldsymbol{A}}_{\boldsymbol{A}} \arrow[r, "D^{\boldsymbol{A}}_{\boldsymbol{f}}"'] & D^{\boldsymbol{A}}_{\boldsymbol{B}}
\end{tikzcd}$$

commutes.

CWU2) *A (p,q)-cowedge $\zeta : D \overset{\bullet\bullet}{\Longrightarrow} X$ is a collection*

$$\left\{\zeta_A : D^{\boldsymbol{A}}_{\boldsymbol{A}} \to X \ : \ A \in \mathcal{C}_o\right\}$$

of morphisms of $\mathcal{C}$ such that, for each morphism $f : A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd}
X & D^{\boldsymbol{B}}_{\boldsymbol{B}} \arrow[l, "\zeta_B"'] \\
D^{\boldsymbol{A}}_{\boldsymbol{A}} \arrow[u, "\zeta_A"] & D^{\boldsymbol{A}}_{\boldsymbol{B}} \arrow[l, "D^{\boldsymbol{A}}_{\boldsymbol{f}}"] \arrow[u, "D^{\boldsymbol{f}}_{\boldsymbol{B}}"']
\end{tikzcd}$$

commutes.

3. Higher arity co/ends

Definition 5. *Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor.*

PQ1) The (p,q)-end *of* D *is, if it exists, the pair* $\left({}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}}, \omega\right)$ *formed by an object*

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}} $$

of $\mathcal{D}$*, and a* (p,q)*-wedge*

$$ \omega : {}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}} \overset{\bullet\bullet}{\Longrightarrow} D $$

for ${}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}}$ *over* D*, such that the* (p,q)*-wedge postcomposition natural transformation*

$$ \omega_{*} : \mathsf{h}\left(-, {}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}}\right) \Longrightarrow \mathsf{Wd}^{(p,q)}_{(-)}(D) $$

is a natural isomorphism.

PQ2) The (p,q)-coend *of* D *is, if it exists, the pair* $\left({}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}}, \xi\right)$ *formed by an object*

$$ {}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}} $$

of $\mathcal{D}$*, and a* (p,q)*-cowedge*

$$ \xi : D \overset{\bullet\bullet}{\Longrightarrow} {}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}} $$

for ${}^{(p,q)}\int^{A} D^{A}_{\underline{A}}$ *under* D*, such that the* (p,q)*-cowedge postcomposition natural transformation*

$$ \xi^{*} : \mathsf{h}\left({}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}}, -\right) \Longrightarrow \mathsf{CWd}^{(p,q)}_{(-)}(D) $$

is a natural isomorphism.

Remark 2. *This means that the* (p,q)*-end of* D *is the terminal object of the category of wedges of* D*, whose morphisms* $h : (\alpha : \Delta_X \overset{\bullet\bullet}{\Longrightarrow} D) \to (\beta : \Delta_Y \overset{\bullet\bullet}{\Longrightarrow} D)$ *are defined as the morphisms* $h : X \to Y$ *of* $\mathcal{D}$ *such that for every* $A \in \mathcal{C}_o$ *one has* $\beta_A \circ h = \alpha_A$*:*

$$
\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\alpha_A"'] & & Y \arrow[dl, "\beta_A"] \\
& D^{A}_{\underline{A}}. &
\end{tikzcd}
$$

3.1. **Basic properties.**

Proposition 2 (Properties of (p,q)-ends and (p,q)-coends)**.** *Let* $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ *be a functor.*

PE1) Functoriality. *Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor. The assignments $D \mapsto {}_{(p,q)}\int_A D^{\underline{A}}_{\underline{A}}, {}^{(p,q)}\int^A D^{\underline{A}}_{\underline{A}}$ define functors*

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} : \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \longrightarrow \mathcal{D}, $$
$$ {}^{(p,q)}\int^{A\in\mathcal{C}} : \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \longrightarrow \mathcal{D} $$

with domain the category of functors from $\mathcal{C}$ of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ to $\mathcal{D}$ and natural transformations between them.

PE2) (p,q)-Wedges and (p,q)-diagonals. For each $X \in \mathcal{C}_o$ we have natural bijections

$$ \mathsf{Wd}^{(p,q)}_{(-)}(D) \cong \mathsf{Wd}_{(-)}\big(\Delta^{(p,q)}_*(D)\big), $$
$$ \mathsf{CWd}^{(p,q)}_{(-)}(D) \cong \mathsf{CWd}_{(-)}\big(\Delta^{(p,q)}_*(D)\big). $$

where $\Delta_{p,q}$ is the "twisted diagonal" functor

$$ \Delta_{p,q} := \underbrace{\Delta^{op} \times \cdots \times \Delta^{op}}_{p\ times} \times \underbrace{\Delta \times \cdots \times \Delta}_{q\ times}. $$

PE3) (p,q)-Ends as ordinary ends. We have natural isomorphisms

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \cong \int_{A\in\mathcal{C}} \Delta^{(p,q)}_*(D)^A_A, $$
$$ {}^{(p,q)}\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \cong \int^{A\in\mathcal{C}} \Delta^{(p,q)}_*(D)^A_A. $$

where $\Delta_{p,q}$ is the twisted diagonal functor.In other words, the (p,q)-end functor factors as a composition

$$ \mathrm{Fun}(\mathcal{C}^{(p,q)}, \mathcal{D}) \xrightarrow{\Delta^{(p,q)}_*} \mathrm{Fun}(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}) \xrightarrow{\quad \int_A \quad} \mathcal{D}, $$

and similarly so do (p,q)-coends.

PE4) (p,q)-Ends as limits. The (p,q)-end and (p,q)-coend of D fit respectively into an equaliser and into a coequaliser diagram

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \longrightarrow \prod_{A\in\mathcal{C}_o} D^{\underline{A}}_{\underline{A}} \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{A\to B} D^{\underline{A}}_{\underline{B}} $$
$$ \coprod_{A\to B} D^{\underline{A}}_{\underline{B}} \underset{\rho'}{\overset{\lambda'}{\rightrightarrows}} \coprod_{A\in\mathcal{C}_o} D^{\underline{A}}_{\underline{A}} \longrightarrow {}^{(p,q)}\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} $$

for suitable maps $\lambda, \rho, \lambda', \rho'$, induced by the morphisms $D^{\underline{A}}_{\underline{u}}, D^{\underline{u}}_{\underline{B}}$.

PE5) (p,q)-Ends as limits, again. We have natural isomorphisms

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \cong \lim\Big(\mathrm{Tw}(\mathcal{C}) \twoheadrightarrow \Sigma_{p,q}\mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big), $$
$$ {}^{(p,q)}\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \cong \mathrm{colim}\Big(\mathrm{Tw}(\mathcal{C}) \twoheadrightarrow \Sigma_{p,q}\mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big), $$

where $\Sigma_{p,q}\colon \mathrm{Tw}(\mathcal{C}) \longrightarrow \mathcal{C}^{(p,q)}$ *is the composition* $\Delta^{(p,q)} \circ \Sigma$*, with* Σ *the usual forgetful functor from* $\mathrm{Tw}(\mathcal{C})$ *to* $\mathcal{C}^{op} \times \mathcal{C}$*. Explicitly,* $\Sigma^{(p,q)}$ *is the functor*

$$\begin{array}{ccc}
\mathrm{Tw}(\mathcal{C}) & \longrightarrow & \mathcal{C}^{(p,q)} \\
\begin{bmatrix} A \\ f\downarrow \\ B \end{bmatrix} & \longmapsto & (\underline{A}, \underline{B}) \\
\begin{bmatrix} A & \xrightarrow{f} & B \\ \phi\uparrow & & \downarrow\psi \\ C & \xrightarrow[g]{} & D \end{bmatrix} & \longmapsto & (\underline{\phi}, \underline{\psi})
\end{array}$$

PE6) *(p,q)-Ends as limits, yet again. There exists a category* $\mathsf{Tw}^{(p,q)}(\mathcal{C})$ *together with a universal fibration*

$$\Sigma\colon \mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \mathcal{C}^{(p,q)}$$

inducing natural isomorphisms

$$\begin{aligned}
{}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \lim\Big(\mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \Sigma\mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big), \\
{}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \operatorname{colim}\Big(\mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \Sigma\mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big).
\end{aligned}$$

PE7) *(p,q)-Ends as $(p+r,q+s)$-ends. we have*

$$\begin{aligned}
{}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong {}_{(p+r,q+s)}\!\int_{A\in\mathcal{C}} \eth^r_s(D)^{\underline{A}}_{\underline{A}}, \\
{}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong {}^{(p+r,q+s)}\!\int^{A\in\mathcal{C}} \eth^r_s(D)^{\underline{A}}_{\underline{A}},
\end{aligned}$$

where $\eth^r_s(-)$ *is "(r,s)-dummyfication".*

PE8) *Commutativity of (p,q)-ends with homs. We have natural isomorphisms*

$$\begin{aligned}
\mathcal{D}\left(-, {}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}}\right) &\cong {}_{(p,q)}\!\int_{A\in\mathcal{C}} \mathcal{D}\Big(-, D^{\underline{A}}_{\underline{A}}\Big) \\
\mathcal{D}\left({}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}}, -\right) &\cong {}_{(q,p)}\!\int_{A\in\mathcal{C}} \mathcal{D}\Big(D^{\underline{A}}_{\underline{A}}, -\Big).
\end{aligned}$$

4. Examples:

4.1. **Some of them are trivial.**

Example 2 (Some (p,q)-co/ends are trivial for trivial reasons)**.** *We show that the two $(0,2)$-ends and $(0,2)$-coends of the functor $\otimes : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ that gives $\mathcal{C}$ a monoidal structure are trivial under very mild assumptions on $\mathcal{C}$. This rules out a class of possibly interesting examples coming from multilinear algebra.*

To fix ideas, let R be a commutative ring, and let's consider the category Mod_R (with additional care one can take left modules and right modules, of course). To show that

$$
\begin{aligned}
{}^{(0,2)}\!\!\int_A A \otimes A &\cong \lim_{A,B\in Mod_R} A\otimes B \cong 0,\\
{}^{(0,2)}\!\!\int^A A \otimes A &\cong \operatorname{colim}_{A,B\in Mod_R} A\otimes B \cong 0,
\end{aligned}
$$

we just observe that Mod_R is a sifted category, because it admits finite coproducts. The fact that a category $\mathcal{C}$ is sifted if and only if $\mathcal{C}$ is non-empty and the diagonal functor $\Delta_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C}\times\mathcal{C}$ is final[1] *then yields the result.*

It's not difficult to see that if $\mathcal{C}$ is a sifted category, all diagonal functors $\Delta : \mathcal{C} \to \mathcal{C}^n$ are final, because the product and composition of final functors is itself final. Thus the same result transports to higher coends of higher arity functors: for example, $\bigwedge^k : Mod_R^n \to Mod_R$ sending M to $\bigwedge^k M$.

Example 3 (More (p,q)-co/ends are trivial for similarly trivial reasons)**.** *Let R be a ring. The* walking cochain complex *(*[**?**, Paragraph 35.1]*) is the Mod_R-enriched category Ch whose set of objects is the set of integers, and where the hom-sets are R-modules freely generated by*

$$
\mathsf{C}([m],[n]) = \begin{cases} \{d,0\} & \textit{if } m = n+1,\\ \{1,0\} & \textit{if } m = n,\\ \{0\} & \textit{otherwise.}\end{cases}
$$

Now, a cochain complex is precisely a Mod_R-enriched functor from the Ch to Mod_R. Similarly, bi*complexes are Mod_R-enriched functors from $\mathsf{Ch}\boxtimes_{\mathsf{Mod}_R}\mathsf{Ch}$ to Mod_R.*

Let $D\colon \mathsf{Ch}\boxtimes_{\mathsf{Mod}_R}\mathsf{Ch} \longrightarrow \mathsf{Mod}_R$ be a bicomplex. We claim that its Mod_R-enriched $(2,0)$-end E is just the zero module. Indeed, looking at $(2,0)$-wedges, we see that they are either of the form

$$
\begin{tikzcd}
E \arrow[r] \arrow[d] & D^{n,n} \\
D^{n,n} \arrow[ur, equal] &
\end{tikzcd}
\qquad \textit{or of the form} \qquad
\begin{tikzcd}
E \arrow[r] \arrow[d] & D^{n+1,n+1} \\
D^{n,n} \arrow[ur, "\mathrm{d}^{n,n}"'] &
\end{tikzcd}
$$

Now, it follows from the second diagram that

$$
E \cong \left\{ (a_k)_{k\in\mathbb{Z}} \in \textstyle\prod_{k\in\mathbb{Z}} D^{k,k} \;\;\middle|\;\; a_{k+1} = \mathrm{d}^{k,k}(a_k)\right\},
$$

but differentials square to zero, so we must have $a_k = 0$ for all $k\in\mathbb{Z}$, and E is indeed isomorphic to the zero module. A similar argument shows that ${}^{(0,2)}\!\!\int^{[k]\in\mathsf{Ch}} D^{k,k} \cong 0$.

[1] This is due to [**?**]; see also [**?**, Proposition 5.3.2] or [**?**, Theorem 2.15] for reviews.

Example 4 (Bisimplicial sets)**.** *Recall that a* bisimplicial set *(*[**?**, Chapter IV]*,* [**?**, §3.1.15]*) is a functor* $X : \Delta^{op} \times \Delta \to \mathsf{Set}$*; moreover, the* diagonalisation *of a bisimplicial set* $X_{\bullet,\bullet} \colon \Delta^{op} \times \Delta^{op} \longrightarrow \mathsf{Set}$ *is the simplicial set* $d(X)_\bullet \colon \Delta^{op} \longrightarrow \mathsf{Set}$ *given by*

$$d(X)_n := X_{n,n}.$$

Joining the products and equalisers formula for (p, q)*-coends we see that* ${}^{(2,0)}\!\int^{[n]\in\Delta^{op}} X_{n,n}$ *is the coequaliser of he diagram*

$$\coprod_{[n]\to[m]\in\Delta} X_{m,m}, \rightrightarrows \coprod_{[n]\in\Delta} X_{n,n}$$

giving

$${}^{(2,0)}\!\!\int^{[n]\in\Delta} X_{n,n} \cong \pi_0(d(X)).$$

By a similar argument, we have

$${}_{(2,0)}\!\!\int_{[n]\in\Delta} X_{n,n} \cong X_{0,0}.$$

4.2. **Juicy examples:**

4.2.1. *A glance at weighted co/ends.* We now introduce a natural factory of examples for higher arity co/ends. In a nutshell, weighted co/ends stand to co/ends in the same relation as weighted co/limits stand to limits.

Definition 6 (Weighted co/end)**.** *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be* $\mathcal{V}$*-enriched categories and* $D \colon \mathcal{C}^{op} \otimes_{\mathcal{V}} \mathcal{C} \longrightarrow \mathcal{D}$ *a* $\mathcal{V}$*-functor, and* $W : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$ *a* $\mathcal{V}$*-presheaf.*

- *WE1) The* end of D weighted by W *is, if it exists, the object* $\int^{W}_{A\in\mathcal{C}} D^A_A$ *of* $\mathcal{D}$ *with the property that*

$$\hom_{\mathcal{D}}\Big(-, \int^{W}_{A\in\mathcal{C}} D^A_A\Big) \cong \mathrm{DiNat}_{\mathcal{V}}(W, \boldsymbol{hom}_{\mathcal{C}}(-, D)).$$

- *WE2) The* coend of D weighted by W *is, if it exists, the object* $\int^{A\in\mathcal{C}}_{W} D^A_A$ *of* $\mathcal{D}$ *with the property that*

$$\hom_{\mathcal{D}}\Big(\int^{A\in\mathcal{C}}_{W} D^A_A, -\Big) \cong \mathrm{DiNat}_{\mathcal{V}}(W, \boldsymbol{hom}_{\mathcal{C}}(D, -)).$$

Example 5 (Weighted co/ends are $(2, 2)$-co/ends)**.** *A quick argument (to be discussed in future work* [**?**]*) gives* $(2, 2)$*-co/end formulas for weighted co/ends:*

$$\int^{[W]}_{A\in\mathcal{C}} D^A_A \cong {}_{(2,2)}\!\!\int_{A\in\mathcal{C}} W^A_A \pitchfork D^A_A,$$

$$\int^{A\in\mathcal{C}}_{[W]} D^A_A \cong {}^{(2,2)}\!\!\int^{A\in\mathcal{C}} W^A_A \odot D^A_A.$$

Example 6 (Weighting Increases Arity)**.** *Let $F, G \colon \mathcal{C} \longrightarrow \mathcal{D}$ and $W \colon \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{V}$ be $\mathcal{V}$-functors. In analogy with*

$$\mathbf{Nat}_{\mathcal{V}}(F, G) := \int_{A \in \mathcal{C}} \boldsymbol{hom}_{\mathcal{D}}(F_A, G_A),$$

we define the object $\mathsf{Nat}^{[W]}(F, G)$ of natural transformations from F to G weighted by W *by*

$$\mathsf{Nat}^{[W]}(F, G) := \int_{A \in \mathcal{C}}^{[W]} \boldsymbol{hom}_{\mathcal{D}}(F_A, G_A). \tag{1}$$

Taking W to be mute in its contravariant variable, we can give a reformulation of the universal property of weighted limits:

$$\mathsf{h}\left(-, \lim^W(D)\right) \cong \mathsf{Nat}^{[W]}\left(\Delta_{(-)}, D\right).$$

Defining $\mathsf{DiNat}^{[W]}_{\mathcal{V}}(F, G)$ by a similar formula, we also obtain the following isomorphism in the case of weighted ends:

$$\mathsf{h}\left(-, \int_{A \in \mathcal{C}}^{[W]} D^A_A\right) \cong \mathsf{DiNat}^{[W]}_{\mathcal{V}}\left(\Delta_{(-)}, D\right).$$

This naturally suggests a definition of doubly-weighted ends*:*

$$\mathsf{h}\left(-, \int_{A \in \mathcal{C}}^{[W_1, W_2]} D^A_A\right) \cong \mathsf{DiNat}^{[W_1]}_{\mathcal{V}}(W_2, D).$$

Repeating this process give you ends weighted by a collection of n functors $W_1, \ldots, W_n$. These however, can be actually computed as $(n+1, n+1)$-ends ([**?**]*):*

$$\int_{A \in \mathcal{C}}^{[W_1, \ldots, W_n]} D^A_A \cong {}_{(n+1, n+1)}\int_{A \in \mathcal{C}} \left((W_1)^A_A \times \cdots \times (W_n)^A_A\right) \odot D^A_A.$$

As such, we see that weighting an end increases its arity by $(1, 1)$.

4.2.2. *Weighted Kan extensions.* Another source of examples comes from "weighing" left and right Kan extensions. While the most general such weight is a profunctor, having type $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, weights of type $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ are specially interesting, as they give a more direct parallel with the classical theory of weighted co/limits.

Recall the definition of the object $\mathsf{Nat}^{[W]}(F, G)$ of weighted natural transformations.

Definition 7. *The* left Kan extension of F along K weighted by W *is, if it exists, the $\mathcal{V}$-functor*

$$\left(\mathsf{Lan}^{[W]}_K F \colon \mathcal{D} \longrightarrow \mathcal{E}\right) : \qquad \begin{tikzcd} & & \mathcal{D} \arrow[d, dashed, "\mathsf{Lan}^{[W]}_K F"] \\ W \arrow[r, phantom, "\circlearrowleft"] & \mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathcal{E} \end{tikzcd}$$

for which we have a $\mathcal{V}$-natural isomorphism

$$\mathbf{Nat}_{\mathcal{V}}\left(\mathsf{Lan}_K^{[W]}F, G\right) \cong \mathsf{Nat}_{\mathcal{V}}^{[W]}\left(F, G \circ K\right), \tag{2}$$

natural in G.

One defines weighted right Kan extensions in a dual manner:

Definition 8. *The* right Kan extension of F along K weighted by W *is, if it exists, the $\mathcal{V}$-functor*

$$\left(\mathsf{Ran}_K^{[W]}F\colon \mathcal{D} \longrightarrow \mathcal{E}\right): \qquad \begin{tikzcd} & & \mathcal{D} \arrow[d, dashed, "\mathsf{Ran}_K^{[W]}F"] \\ W \arrow[r, hookleftarrow] & \mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=4, "\Swarrow"] & \mathcal{E} \end{tikzcd}$$

for which we have a $\mathcal{V}$-natural isomorphism

$$\mathbf{Nat}_{\mathcal{V}}\left(G, \mathsf{Ran}_K^{[W]}F\right) \cong \mathsf{Nat}_{\mathcal{V}}^{[W]}\left(G \circ K, F\right), \tag{3}$$

natural in G.

Example 7 (Weighted co/limits as weighted Kan extensions)**.** *Let $D\colon \mathcal{C} \longrightarrow \mathcal{D}$ be a diagram on a category $\mathcal{D}$. Then we may canonically identify the left Kan extension of D along the terminal functor with its colimit:*

$$\mathsf{Lan}_! D \cong \lceil \mathrm{colim}(D) \rceil \qquad \begin{tikzcd} & 1 \arrow[d, dashed, "\lceil \mathrm{colim}(D) \rceil"] \\ \mathcal{C} \arrow[ur, "!"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=4, "\Nearrow"] & \mathcal{D} \end{tikzcd}$$

Similarly, given a weight $W\colon \mathcal{C}^{op} \longrightarrow \mathsf{Set}$, we have

$$\mathsf{Lan}_!^{[W]} D \cong \lceil \mathrm{colim}^W(D) \rceil \qquad \begin{tikzcd} & & 1 \arrow[d, dashed, "\lceil \mathrm{colim}^W(D) \rceil"] \\ W \arrow[r, hookleftarrow] & \mathcal{C} \arrow[ur, "!"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=4, "\Nearrow"] & \mathcal{D} \end{tikzcd}$$

One can also prove that the following formulas hold ([**?**]):

(4)

$$\mathsf{Lan}_K^{[W]}F \cong \int_{[W]}^{A\in\mathcal{C}} \mathbf{hom}_{\mathcal{C}}\left(K_A, -\right) \odot F_A \cong {}^{(2,2)}\!\!\int^{A\in\mathcal{C}} \left(W_A^A \times \mathbf{hom}_{\mathcal{C}}(K_A, -)\right) \odot F_A,$$

(5)

$$\mathsf{Ran}_K^{[W]}F \cong \int_{A\in\mathcal{C}}^{[W]} \mathbf{hom}_{\mathcal{C}}\left(-, K_A\right) \pitchfork F_A \cong {}_{(2,2)}\!\!\int_{A\in\mathcal{C}} \left(W_A^A \times \mathbf{hom}_{\mathcal{C}}(-, K_A)\right) \pitchfork F_A.$$

Equipped with these, we now proceed to compute a few weighted Kan extensions.

Example 8. *Consider the functor $i^{op} : 1^{op} \to \Delta^{op}$; the left and right Kan extensions of a set $X_\bullet \colon 1 \longrightarrow \mathsf{Set}$ along i^{op} are given by*

$$\mathsf{Lan}_{i^{op}}(X) \cong \underline{X_\bullet}$$
$$\mathsf{Ran}_{i^{op}}(X) \cong \check{C}(X).$$

Now take a weight $W \colon 1^{op} \times 1 \longrightarrow \mathsf{Set}$:

$$\begin{tikzcd} & \Delta^{op} \arrow[d, dashed, "\mathsf{Lan}^{[W]}_{i^{op}} X"] \\ W \arrow[loop left] \; 1^{op} \arrow[ur, "i^{op}"] \arrow[r, "X"'] & \mathsf{Set} \end{tikzcd}$$

Then

$$\mathsf{Lan}^{[W]}_{i^{op}}(X) \cong \underline{W \times X_\bullet}$$
$$\mathsf{Ran}^{[W]}_{i^{op}}(X) \cong \check{C}(W \times X).$$

Example 9. *Now for the more interesting counterpart of the above:*

$$\begin{aligned} \pi^{op} \colon \Delta^{op} &\longrightarrow 1^{op} \\ [n] &\longmapsto \star \end{aligned}$$

The left and right Kan extensions of a simplicial set $X_\bullet \colon \Delta^{op} \longrightarrow \mathsf{Set}$ along π^{op} are given by

$$\mathsf{Lan}_{\pi^{op}}(X_\bullet) \cong \pi_0(X_\bullet)$$
$$\mathsf{Ran}_{\pi^{op}}(X_\bullet) \cong \mathsf{ev}_0(X_\bullet) := X_0.$$

Now, a weight $W^\bullet_\bullet \colon \Delta^{op} \times \Delta \longrightarrow \mathsf{Set}$ is wonderfully complicated: it is a cosimplicial space!

Then

(1) *Taking $W = \Delta^\bullet$ almost gives the geometric realisation of $X_\bullet$:*

$$\mathsf{Lan}^{[\Delta^\bullet]}_{\pi^{op}}(X_\bullet) \cong \int^{[n]\in\Delta} \Delta^n \times X_n.$$

(2) *Dually, taking again $W = \Delta^\bullet$ but now a cosimplicial object $X^\bullet \colon \Delta \longrightarrow \mathsf{Set}$,*

$$\mathsf{Ran}^{[\Delta^\bullet]}_{\pi}(X^\bullet) = Tot(X_\bullet).$$

(3) *If ou take $W = \Delta^\bullet_\bullet = \hom_\Delta(-,-)$, then I think you get*

$$\begin{aligned} \mathsf{Lan}^{[\Delta^\bullet_\bullet]}_{\pi^{op}}(X_\bullet) &\cong {}^{(2,2)}\!\!\int^{[n]\in\Delta} \Delta^n_n \times X_n \\ &\cong {}^{(2,2)}\!\!\int^{[n]\in\Delta} \Delta^n_n \times X_n \end{aligned}$$

Example 10. *Using the fact that weighted left/right Kan extensions along the identity are adjoint to each other, we can study situations like*

$$\begin{tikzcd} & & \Delta^{op} \arrow[d, dashed, "?"] \\ W^{\bullet}_{\bullet} \arrow[r, hook] & \Delta^{op} \arrow[ur, equal, "1"] \arrow[r, "X_\bullet"'] \arrow[ur, phantom, shift left=4, "\Uparrow"] & \mathsf{Set} \end{tikzcd}$$

This gives rise to an adjunction $L : \mathsf{sSet} \leftrightarrows \mathsf{sSet} : R$ with

$$\begin{aligned} L(X_\bullet) &\cong {}_{(2,2)}\!\int_{[n]\in\Delta} W^n_n \odot X_n \cong {}_{(2,2)}\!\int_{[n]\in\Delta} W^n_n \times X_n, \\ R(X_\bullet) &\cong {}^{(2,2)}\!\int^{[n]\in\Delta} W^n_n \pitchfork X_n \cong {}^{(2,2)}\!\int^{[n]\in\Delta} [W^n_n, X_n]. \end{aligned}$$

Taking $W = \Delta^\bullet$ gives $L = R = 1$, so let's take something more complicated, like $\Delta^{\bullet}_{\bullet}$. Then

$$\begin{aligned} L(X_\bullet) &\cong {}_{(2,2)}\!\int_{[n]\in\Delta} \Delta^n[n] \times X_n \cong ? \\ R(X_\bullet) &\cong {}^{(2,2)}\!\int^{[n]\in\Delta} [\Delta^n[n], X_n] \cong ? \end{aligned}$$

Example 11 (Stalks of a sheaf ([**?**, Paragraph 6.8 and Section 7.1]))**.** *Let $i_p \colon \{p\} \hookrightarrow X$ be the inclusion of a point into a topological space X. We get an induced functor*

$$\begin{aligned} \mathcal{O}(i_p) \colon \mathcal{O}(X) &\longrightarrow \mathcal{O}(\{p\}) \\ U &\longmapsto i_p^{-1}(U) \end{aligned}$$

Considering now left Kan extensions along the opposite of $\mathcal{O}(i_p)$,

$$\begin{tikzcd} & \mathcal{O}(\{p\})^{op} \arrow[d, dashed, "\mathsf{Lan}_{\mathcal{O}(i_p)^{op}}\mathcal{F}"] \\ \mathcal{O}(X)^{op} \arrow[ur, "\mathcal{O}(i_p)^{op}"] \arrow[r, "\mathcal{F}"'] \arrow[ur, phantom, shift right=4, "\Uparrow"] & \mathsf{Set} \end{tikzcd}$$

we obtain a functor $\mathsf{Lan}_{\mathcal{O}(i_p)^{op}} \colon \mathsf{PSh}(X) \longrightarrow \mathsf{PSh}(\{p\})$, whose image at $\mathcal{F}$ is written $\lceil \mathcal{F}_p \rceil$ for simplicity. The restriction of this functor to $\mathsf{Sh}(X)$ can be identified with the stalk functor $(-)_p \colon \mathsf{Sh}(X) \longrightarrow \mathsf{Set}$: we have $\mathcal{O}(\{p\}) = \{\varnothing \hookrightarrow \{p\}\}$ and computing the images of $\varnothing$ and $\{p\}$ under $\lceil \mathcal{F}_p \rceil$ via the *usual colimit formula for left Kan extensions gives*

$$\begin{aligned}
\lceil \mathcal{F}_p \rceil(\{p\}) &\cong \operatorname{colim}\left(\left(\mathcal{O}(\lceil p \rceil) \downarrow \underline{\{p\}}\right)^{op} \xrightarrow{\pi^{op}} \mathcal{O}(X)^{op} \xrightarrow{\mathcal{F}} \mathsf{Set}\right),\\
&\cong \operatorname{colim}_{U \ni p}(\mathcal{F}(U)),\\
&\cong \mathcal{F}_p\\
\lceil \mathcal{F}_p \rceil(\varnothing) &\cong \operatorname{colim}\left((\mathcal{O}(\lceil p \rceil) \downarrow \underline{\varnothing})^{op} \xrightarrow{\pi^{op}} \mathcal{O}(X)^{op} \xrightarrow{\mathcal{F}} \mathsf{Set}\right),\\
&\cong \operatorname{colim}_{U \hookrightarrow \varnothing}(\mathcal{F}(U)),\\
&\cong \mathcal{F}(\varnothing).
\end{aligned}$$

(in case $\mathcal{F}$ is a sheaf, $\mathcal{F}(\varnothing)$ is the singleton set.) Consider the same situation, but now with a weight $W \colon \mathcal{O}(X) \times \mathcal{O}(X)^{op} \longrightarrow \mathsf{Set}$ (an "extradiagonal presheaf on X"):

$$\begin{tikzcd}
 & & \mathcal{O}(\{p\})^{op} \arrow[d, dashed, "\mathsf{Lan}^{[W]}_{\mathcal{O}(i_p)^{op}}\mathcal{F}"] \\
W \arrow[r, bend left, "\mid" marking] & \mathcal{O}(X)^{op} \arrow[ur, "\mathcal{O}(i_p)^{op}"] \arrow[r, "\mathcal{F}"'] \arrow[ur, phantom, shift right=4, "\Uparrow"] & \mathsf{Set}
\end{tikzcd}$$

We may compute $\mathsf{Lan}^{[W]}_{\mathcal{O}(i_p)^{op}}\mathcal{F} := \lceil \mathcal{F}^{[W]}_p \rceil$ as the weighted coend

$$\lceil \mathcal{F}^{[W]}_p \rceil := \int_{[W]}^{U \in \mathcal{O}(X)} \hom_{\mathcal{O}(X)^{op}}\left(\mathcal{O}(i_p^{op})(U), -\right)$$

$odotmathcalF(U)$

$$\cong \int^{U \in \mathcal{O}(X)} W_U^U \times \hom_{\mathcal{O}(X)}\left(\chi_p(U), -\right)$$

$timesmathcalF(U),$

where

$$\chi_p(U) = \begin{cases} \emptyset & \textit{if } p \notin U,\\ U & \textit{otherwise.}\end{cases}$$

For instance, taking W to be a sheaf $\mathcal{G}$ on X gives

$$\mathcal{F}_p^{[\mathcal{G}]} := \lceil \mathcal{F}_p^{[\mathcal{G}]} \rceil(\{p\}) \cong \left(\mathcal{F} \times \mathcal{G}\right)_p.$$

4.2.3. *A glance at extradiagonality.* "Extradiagonal" category theory arises when, instead of considering a natural transformation filling a higher-dimensional cell, we consider a *dinatural* one. Transformations that are more general than natural ones notoriously do not compose; yet, the category theory arising from this generalisation is interesting.

Definition 9 (Diagonal left Kan extensions)**.** *The* diagonal left Kan extension *of a functor $F\colon \mathcal{C}^{op}\times\mathcal{C}\longrightarrow\mathcal{D}$ along a functor $K\colon \mathcal{C}^{op}\times\mathcal{C}\longrightarrow\mathcal{D}$ is, if it exists the functor $\mathsf{DiLan}_K F\colon \mathcal{D}\longrightarrow\mathcal{E}$ such that we have an isomorphism*

$$\mathit{Nat}(\mathsf{DiLan}_K F, G)\cong \mathrm{DiNat}(F, G\circ K) \qquad \begin{tikzcd} & \mathcal{D} \arrow[d, dashed, "\mathsf{DiLan}_K F"] \\ \mathcal{C}^{op}\times\mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\Rrightarrow"] & \mathcal{E} \end{tikzcd}$$

natural in G.

Example 12. *Standard examples of diagonal left Kan extensions are ends: Generalising the fact that the left Kan extension of a functor $D\colon\mathcal{C}\longrightarrow\mathcal{D}$ along the terminal functor $\pi\colon\mathcal{C}\twoheadrightarrow 1$ can be identified with the colimit of $\mathcal{D}$, the diagonal left Kan extension of a functor $D\colon\mathcal{C}^{op}\times\mathcal{C}\longrightarrow\mathcal{D}$ along the terminal functor $\pi\colon\mathcal{C}^{op}\times\mathcal{C}\twoheadrightarrow 1$ can be identified with the coend of $\mathcal{D}$.*

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\lceil \mathrm{colim}(D)\rceil"] \\ \mathcal{C} \arrow[ur, "\pi"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=3, "\Rrightarrow"] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[d, dashed, "\lceil \int^A D^A_A\rceil"] \\ \mathcal{C}^{op}\times\mathcal{C} \arrow[ur, "\pi"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=3, "\Rrightarrow"] & \mathcal{D} \end{tikzcd}$$

Now, while ordinary Kan extensions can be computed via co/end formulas, diagonal Kan extensions admit $(2,2)$-co/end formulas ([**?**]):

$$\mathsf{DiLan}_K F \cong {}^{(2,2)}\!\!\int^{A\in\mathcal{C}} \mathcal{D}\left(K^A_A, -\right)\odot F^A_A, \tag{6}$$

$$\mathsf{DiRan}_K F \cong {}_{(2,2)}\!\!\int_{A\in\mathcal{C}} \mathcal{D}\left(-, K^A_A\right)\pitchfork F^A_A, \tag{7}$$

where the pairing is such that $\mathsf{DiLan}_K F$ is the coend of

$$(A,B)\mapsto \mathcal{D}\left(K^B_A, -\right)\odot F^A_B.$$

Alternatively, we may compute diagonal Kan extensions as hom-weighted Kan extensions ([**?**, **?**]):

$$\mathsf{DiLan}_K F \cong \int^{A,B\in\mathcal{C}}_{[\hom_{\mathcal{C}}(-,-)]} \mathcal{D}\left(K^B_A, -\right)\odot F^A_B,$$

$$\mathsf{DiRan}_K F \cong \int^{[\hom_{\mathcal{C}}(-,-)]}_{A,B\in\mathcal{C}} \mathcal{D}\left(-, K^B_A\right)\pitchfork F^A_B.$$

This is a generalisation of the fact that ends are hom-weighted limits. A forthcoming work [**?**] will address the topic of this remark in its entirety, studying the category theory arising from the notion of a weighted co/end.

Example 13. *Let $\mathcal{C}$ be a closed monoidal category and $D\colon \mathcal{C}^{op}\times\mathcal{C}\longrightarrow\mathcal{D}$ be a diagram on $\mathcal{D}$. What is $\mathsf{DiLan}_{[-,-]}D$ and $\mathsf{DiRan}_{[-,-]}D$?*

$$\begin{tikzcd}
 & \mathcal{C} \arrow[d, dashed, "\mathsf{DiLan}_{[-,-]}D"] \\
\mathcal{C}^{op}\times\mathcal{C} \arrow[ur, "{[-,-]}"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow"] & \mathcal{D}
\end{tikzcd}$$

$$\mathsf{DiLan}_{[-,-]}D \cong \int^{A\in\mathcal{C}} \hom_{\mathcal{C}}([A,A],-)\odot D^A_A.$$

Example 14. *Let $D\colon \mathcal{C}^{op}\times\mathcal{C}\longrightarrow\mathcal{D}$ be a diagram on $\mathcal{D}$. What is $\mathsf{DiLan}_y D$ and $\mathsf{DiRan}_y D$?*

$$\begin{tikzcd}
 & \mathsf{PSh}(\mathcal{C}^{op}\times\mathcal{C}) \arrow[d, dashed, "\mathsf{DiLan}_y D"] \\
\mathcal{C}^{op}\times\mathcal{C} \arrow[ur, hook, "y"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=3ex, "\Uparrow"] & \mathcal{D}
\end{tikzcd}$$

$$\begin{aligned}
\mathsf{DiLan}_y D &\cong \int_{\hom_{\mathcal{C}}(-,-)}^{A,B\in\mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{op}\times\mathcal{C})}\left(y^B_A,-\right)\odot F^A_B,\\
&\cong \int^{A\in\mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{op}\times\mathcal{C})}\left(y^A_A,-\right)\odot F^A_A,\\
&\cong {}^{(2,2)}\!\!\int^{A\in\mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{op}\times\mathcal{C})}\left(y^A_A,-\right)\odot F^A_A,\\
&:= {}^{(2,2)}\!\!\int^{A\in\mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{op}\times\mathcal{C})}\left(\hom_{\mathcal{C}^{op}\times\mathcal{C}}(-,(A,A)),-\right)\odot F^A_A,\\
&:= {}^{(2,2)}\!\!\int^{A\in\mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{op}\times\mathcal{C})}\left(\mathsf{h}^A\times\mathsf{h}_A,-\right)\odot F^A_A.
\end{aligned}$$

In order to introduce the next example, we recall the following notation: we have an adjunction

$$(\pi\dashv \mathrm{i})\colon\ \begin{tikzcd} 1 \arrow[r, hook, shift right, "\mathrm{i}"'] \arrow[r, phantom, "\perp"] & \Delta, \arrow[l, two heads, shift right, "\pi"'] \end{tikzcd}$$

where

- $\mathrm{i}\colon 1\hookrightarrow\Delta$ is the functor choosing the terminal object;
- $\pi\colon\Delta\twoheadrightarrow 1$ is the terminal functor;

This induces a quadruple adjunction

$$\left(\pi_0\dashv \underline{(-)}_{\bullet}\dashv \mathsf{ev}_0\dashv \check{C}\right)\colon\ \mathsf{Set} \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} \mathsf{sSet}\ \pi_0$$

Example 15. *Let* $S^\bullet_\bullet\colon \Delta^{op}\times\Delta \longrightarrow \mathsf{Set}$ *be a cosimplicial space. What is* $\mathsf{DiLan}_{\pi^{op}\times\pi}(S^\bullet_\bullet)$*?*

$$\begin{tikzcd}
 & 1^{op}\times 1 \arrow[d, dashed, "\mathsf{DiLan}_{\pi^{op}\times\pi}(S^\bullet_\bullet)"] \\
\Delta^{op}\times\Delta \arrow[ur, "\pi^{op}\times\pi"] \arrow[r, "S^\bullet_\bullet"'] \arrow[ur, phantom, shift right=4ex, "\Uparrow"] & \mathsf{Set}
\end{tikzcd}$$

It is just the end of $S^\bullet_\bullet$ *(btw do you know what this is?):*

$$\begin{aligned}
\mathsf{DiLan}_{\pi^{op}\times\pi}(S^\bullet_\bullet) &\cong \int^{[n]\in\Delta} \hom_1(\star,\star)\odot S^n_n,\\
&\cong \int^{[n]\in\Delta} S^n_n.
\end{aligned}$$

Similarly, given a set $X\colon 1^{op}\times 1\longrightarrow \mathsf{Set}$*, we have*

$$\begin{tikzcd}
 & \Delta^{op}\times\Delta \arrow[d, dashed, "\mathsf{DiLan}_{i^{op}\times i}(X)"] \\
1^{op}\times 1 \arrow[ur, hook, "i^{op}\times i"] \arrow[r, "X"'] \arrow[ur, phantom, shift right=4ex, "\Uparrow"] & \mathsf{Set}
\end{tikzcd}$$

$$\begin{aligned}
\mathsf{DiLan}_{i^{op}\times i}(X) &\cong \int^{\star\in 1} \hom_{\Delta^{op}\times\Delta}(([0],[0]),(-_1,-_2))\odot X,\\
&\cong \hom_{\Delta^{op}\times\Delta}(([0],[0]),(-_1,-_2))\odot X\\
&\cong \hom_\Delta([0],-_2)\odot X\\
&\cong \Delta^{-_2}[0]\odot X.
\end{aligned}$$

Similarly, let $X^\bullet_\bullet\colon \Delta^{op}\times\Delta\longrightarrow \mathsf{Set}$ *be a cosimplicial space again. What is* $\mathsf{DiLan}_\Delta(X^\bullet_\bullet)$*?*

$$\begin{tikzcd}
 & \mathsf{Set} \arrow[d, dashed, "\mathsf{DiLan}_\Delta(X^\bullet_\bullet)"] \\
\Delta^{op}\times\Delta \arrow[ur, "\Delta^{-_2}[-_1]"] \arrow[r, "X^\bullet_\bullet"'] \arrow[ur, phantom, shift right=4ex, "\Uparrow"] & \mathsf{Set}
\end{tikzcd}$$

$$\mathsf{DiLan}_\Delta(X^\bullet_\bullet)\cong \int^{[n]\in\Delta} \mathsf{Set}(\Delta^n[n],-)\odot X^n_n.$$

4.2.4. *Weighted diagonal Kan extensions.* In the same spirit, one can define weighted diagonal Kan extensions, mixing the two perspectives and considering now the diagram

$$
\begin{tikzcd}
 & & \mathcal{D} \arrow[d, dashed, "\mathsf{DiLan}_K F"] \\
W \arrow[r, bend left] & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] & \mathcal{E}
\end{tikzcd}
$$

just to discover that these are actually computed as $(4,4)$-co/ends:

$$
\begin{aligned}
\mathsf{DiLan}_K^{[W]} F &\cong {}^{(4,4)}\!\!\int^{A\in\mathcal{C}} \left(W^{A,A}_{A,A} \times \mathbf{hom}_{\mathcal{C}}\left(K^A_A, -\right)\right) \odot F^A_A, \\
\mathsf{DiRan}_K^{[W]} F &\cong {}_{(4,4)}\!\!\int_{A\in\mathcal{C}} \left(W^{A,A}_{A,A} \times \mathbf{hom}_{\mathcal{C}}\left(-, K^A_A\right)\right) \pitchfork F^A_A.
\end{aligned}
$$

At this point, the reader shall be convinced that the list of examples is virtually endless. We defer a thorough study of the topic to separate works [**?**, **?**].

4.2.5. *Daydreaming About Operads.* Day convolution was introduced by B. Day in [**?**, **?**], in order to classify monoidal structures on the category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$. Day proved that $\mathsf{PSh}(\mathcal{C})$ can be turned into a monoidal category in as many ways as $\mathcal{C}$ can be turned into a pseudomonoid in the bicategory of profunctors.

We now propose a generalisation of this framework based on higher arity coends: let $(\mathcal{C}, \otimes, I)$ be a monoidal category, and let $\mathcal{K} := \mathsf{PSh}(\mathcal{C})$. Higher arity Day convolution is defined as a family of functors $\circledast_n : \mathcal{K}^n \to \mathcal{K}$:

Definition 10. *The* ***Day*** (n,n)***-convolution*** *of an n-tuple of presheaves $\mathcal{F}_1, \dots, \mathcal{F}_n$ is the presheaf*

$$\circledast_n(\mathcal{F}_1, \dots, \mathcal{F}_n) \colon \mathcal{C}^{op} \longrightarrow \mathit{Set}$$

defined at $A \in \mathcal{C}_o$ as the (n,n)-coend

$$\circledast_n(\mathcal{F}_1, \dots, \mathcal{F}_n) := A \mapsto {}^{(n,n)}\!\!\int^{A\in\mathcal{C}} \mathcal{F}_1(A) \times \cdots \times \mathcal{F}_n(A) \times \mathcal{C}\left(-, A^{\otimes n}\right),$$

where $A^{\otimes n}$ is shorthand for the n-fold tensor product of A with itself.

Example 16 (Day convolution operad)**.** *The* ***Day convolution operad associated to*** $(\mathcal{C}, \otimes, I)$ *is the free symmetric operad* Day *whose set of generating operations (see* [**?**, Section 1.2.5]*) is given by* $\{1, \circledast_2, \circledast_3, \dots, \circledast_n, \dots\}$.

Remark 3. *We spell out in detail the first four sets of n-ary operations of* Day*:*

$\mathsf{Day}_1 = \{1\}$

$\mathsf{Day}_2 = \{\circledast_2(-,-)\}$

$\mathsf{Day}_3 = \{\circledast_3(-,-,-), \circledast_2(\circledast_2(-,-),-), \circledast_2(-,\circledast_2(-,-))\}$

$$\begin{aligned}\mathsf{Day}_4 = \{&\circledast_4(-,-,-,-), \circledast_2(\circledast_3(-,-,-),-), \circledast_2(-,\circledast_3(-,-,-)), \circledast_2(\circledast_2(-,-),\circledast_2(-,-)),\\ &\circledast_3(-,\circledast_2(-,-),-), \circledast_3(\circledast_2(-,-),-,-), \circledast_3(-,-,\circledast_2(-,-))\}\end{aligned}$$

All in all, the set Day_n *can be succinctly described as*

$$\mathsf{Day}_n = \{\circledast_n\} \cup \sum_{p+q=n} \mathsf{Day}_p \times \mathsf{Day}_q$$

The operadic composition of Day *is now defined via 'grafting' in the usual way:*

$$\begin{aligned}\mathsf{Day}_n \times \mathsf{Day}_{k_1} \times \cdots \times \mathsf{Day}_{k_n} &\longrightarrow \mathsf{Day}_{\sum k_i}\\ (\theta;\theta_1,\dots,\theta_k) &\longmapsto \theta(\theta_1(-_1,\dots,-_{k_1}),\dots,\theta_k(-_1,\dots,-_{k_n}))\end{aligned}$$

5. Kusarigamas

5.1. Reducing dinaturality to naturality.

5.2. higher arity twisted arrow categories.

6. Future work (?)

6.1. weighing co/ends: the full story.

6.2. kusarigamas are a toy example of "extradiagonal" (for lack of a better name) transformation.

6.3. A graphical language for higher arity co/ends.

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