# P-Q-COENDS

## THÉO DE OLIVEIRA SANTOS<sup>‡</sup> AND FOSCO LOREGIAN<sup>§</sup>

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## 1. MOTIVATION:

1.1. "symmetrize" tensors of higher arity. The so-called "Einstein summation convention" in linear algebra and differential geometry asserts that it is possible to suppress the summation symbol  $\sum$  in every formula like

 $c_i v_i$ 

at the cost of writing " $c^iv_i$ "; this means that contravariant tensors' indices are superscripts, while covariant tensors' indices are subscripts, and whenever homonymous indices appear in a string like  $c^iv_i$ , it means that we are

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summing over that index. So, for example, the first Bianchi identity:

$$\nabla_i R^i{}_j = \frac{1}{2} \nabla_j R$$

stands for  $\sum_{i} \nabla_{i} R^{i}{}_{j} = \dots$ , and the definition of R itself expands to a daunting

$$\begin{split} R_{ij} &= -\frac{1}{2} \Big( \frac{\partial^2 g_{ij}}{\partial x^a \partial x^b} + \frac{\partial^2 g_{ab}}{\partial x^i \partial x^j} - \frac{\partial^2 g_{ib}}{\partial x^j \partial x^a} - \frac{\partial^2 g_{jb}}{\partial x^i \partial x^a} \Big) g^{ab} \\ &+ \frac{1}{2} \Big( \frac{1}{2} \frac{\partial g_{ac}}{\partial x^i} \frac{\partial g_{bd}}{\partial x^j} + \frac{\partial g_{ic}}{\partial x^a} \frac{\partial g_{jd}}{\partial x^b} - \frac{\partial g_{ic}}{\partial x^a} \frac{\partial g_{jb}}{\partial x^d} \Big) g^{ab} g^{cd} \\ &- \frac{1}{4} \Big( \frac{\partial g_{jc}}{\partial x^i} + \frac{\partial g_{ic}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^c} \Big) \Big( 2 \frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d} \Big) g^{ab} g^{cd}. \end{split}$$

This convention does not allow for "unbalanced" expression to be summed over: the same number of subscript must be paired with the same number of superscripts.

In category theory, the analogue operation of "summing over repeated indices" is taking a **coend** of a functor

$$T: \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$$

intended as the quotient of  $\coprod_C T(C,C)$  by the equivalence relation generated by the action of T on arrows; this analogy is not peregrine: if  $S: \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Set}$  and  $T: \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$  are two "profunctors", their composition

$$ST(A,C) := \int_{-B}^{B} S(A,B) \times T(B,C)$$

is akin to the matrix product of two matrices, seen as functions  $S:[n]\times[m]\to K,\,T:[m]\times[r]\to K.$ 

(A perfect analogy is this: let \$A,B,C\$be discrete categories; then the profunctor composition of  $S:A\times B\to \mathsf{Set}$  and  $T:B\times C\to \mathsf{Set}$  is the matrix product of an  $|A|\times |B|$  and a  $|B|\times |C|$  matrix.)

1.2. **Question(s).** What if we want to sum/integrate/coend over an "unbalanced tensor" like

$$T: (\mathcal{C}^{\mathrm{op}})^p \times \mathcal{C}^q = \mathcal{C}^{(p,q)} \to \mathcal{D}$$

for  $p, q \geq 1$ ?

Is the resulting theory well-behaved as the classical one?

No one would debate about the usefulness of "balanced" integrals; are the unbalanced ones good for something similar?

This work aims at answering all these questions in the positive:

• Yes, one can define a notion of **co/end** for "higher arity" functors  $\mathcal{C}^{(p,q)} \to \mathcal{D}$ :

- Yes and no; higher arity co/ends are particular instances of co/ends, where T has been "completely symmetrised" (see later for a definition); as such, they do not constitute a "new" object; instead, a specialisation of classical co/end calculus;
- Yes, the resulting theory is expressive enough to capture some new phenomena.

At this point, perhaps the most enlightening example is the following, appearing in a paper by Street and Dubuc:

**Proposition 1.** Let  $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$  be two functors; define the functor

$$\mathrm{DNat}(F^{\uparrow},G^{\downarrow}):\mathcal{C}^{op}\times\mathcal{C}\to\mathcal{S}et$$

sending (A, B) to  $\mathcal{D}(F_A^B, G_B^A)$ ; then, the set of dinatural transformations  $F \stackrel{\bullet \bullet}{\Longrightarrow} G$  is canonically isomorphic to the end of  $DNat(F^{\uparrow}, G^{\downarrow})$ .

- 1.3. generalised dinaturality recently introduced by A. Santamaria in his PhD thesis. A. Santamaria and McGusker recently introduced in [MS] the notion of dinaturality we started from; yet, his notion is too general for our purposes for two reasons:
  - they do not assume a "transformation" satisfies any condition globally, treating the

notion of naturality as a property of a single component (this reads as: a transformation between two functors of a certain "type" is di/natural at an index i, but it can be "unnatural" elsewhere).

• they do not assume that the type of the domain functor and the codomain functor are the same.

Our notational convention is also different: they take into account functors  $\mathcal{C}^{\alpha} \to \mathcal{B}$ , where  $\alpha$  is a "binary multi-index", i.e. an element in the free monoid over the set  $\{\oplus,\ominus\}$ , and the convention is that  $\mathcal{C}^{\varnothing} :=$ , the terminal category,  $\mathcal{C}^{\oplus} := \mathcal{C}$ ,  $\mathcal{C}^{\ominus} := \mathcal{C}^{\text{op}}$ , and  $\mathcal{C}^{\alpha \uplus \alpha'} := \mathcal{C}^{\alpha} \times \mathcal{C}^{\alpha'}$ .

Here instead, we adopt a different convention: a generic power  $C^{\alpha}$  is always "reshuffled" in order for all its minus and plus signs to appear on the same side, respectively on the left and on the right. The categories  $C^{\alpha}$  and  $C^{(p,q)}$  so obtained are, of course, canonically isomorphic, and the tuple  $\alpha$  is equivalent to the reshuffled tuple  $(\ominus_1, \ldots, \ominus_p, \oplus_1, \ldots, \oplus_q)$ .

**Definition 1.** Let  $\alpha, \beta$  be two multi-indices, and let  $F : \mathcal{C}^{\alpha} \to \mathcal{D}, G : \mathcal{C}^{\beta} \to \mathcal{D}$  be functors. A transformation  $\phi : F \to G$  of type  $|\alpha| \xrightarrow{\sigma} n \xleftarrow{\tau} |\beta|$  (with n = |A| a positive integer) is a family of morphisms in  $\mathcal{D}$ 

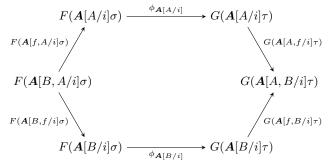
$$(\phi_{\mathbf{A}}: F(\mathbf{A}\sigma) \to G(\mathbf{A}\tau))_{\mathbf{A} \in \mathcal{C}^n}.$$

This translates into a family

$$\phi_{A_1,\ldots,A_n}: F(A_{\sigma 1},\ldots,A_{\sigma |\alpha|}) \to G(A_{\tau 1},\ldots,A_{\tau |\beta|}).$$

Notice that  $\alpha$  and  $\beta$  are different multi-indices in this definition, and  $\sigma, \tau$  need not be injective or surjective, so we may have repeated or unused variables.

**Definition 2.** Let  $\phi = (\phi_{A_1,\dots,A_n}): F \to G$  be a transformation. For  $i \in \{1,\dots,n\}$ , we say that  $\phi$  is dinatural in  $A_i$  (or, more precisely, dinatural in its *i*-th variable) if and only if for all  $A_1,\dots,A_{i-1},A_{i+1},\dots,A_n$  objects of C and for all  $f: A \to B$  in C the following hexagon commutes:



where  $\mathbf{A}$  is the n-tuple  $(A_1, \ldots, A_n)$  of the objects above with an additional (unused in this definition) object  $A_i$  of C.

## 2. Higher arity co/wedges

N1) A generic tuple of objects,

$$A := (A_1, \ldots, A_n)$$

often split as the juxtaposition  $\underline{A}'; \underline{A}''$  of two subtuples of length p, q,

$$\underline{A}' := (A_1, \dots, A_q), \qquad \underline{A}'' := (A_{p+1}, \dots, A_{p+q})$$

N2) As already said, the image of a split tuple  $\underline{A}'; \underline{A}''$  under a functor of type  $\begin{bmatrix} p \\ q \end{bmatrix}$ ,  $F: \mathcal{C}^{(p,q)} \to \mathcal{D}$  is denoted  $F_{\underline{A}''}^{\underline{A}'}$ : the contravariant components come first, and the covariant component second. So: contravariant components are always left in the typing

$$F: \mathcal{C}^{(p,q)} \varnothing \mathcal{D}$$

of a functor, and *up* in its action on objects.

N3) Denoting a functor F of type  $\begin{bmatrix} p \\ q \end{bmatrix}$  evaluated at a diagonal tuple: we write

$$F_{\pmb{A}}^{\pmb{A}}:=F_{A,\dots,A}^{A,\dots,A},$$

where the superscript has p elements, and the subscript has q elements

N<sub>4</sub>) Substitution of an object at a prescribed index

$$A[X/i] := (A_1, \dots, A_{i-1}, X, A_{i+1}, \dots, A_n).$$

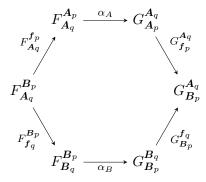
N<sub>5</sub>) Substitution of a tuple at a prescribed tuple of indices

$$\underline{A}[X_1,\ldots,X_r/i_1,\ldots,i_r] := ((\underline{A}[X_1/i_1])[X_2/i_2]\cdots)[X_r/i_r].$$

**Definition 3.** A(p,q)-dinatural transformation  $\alpha: F \stackrel{\bullet \bullet}{\Longrightarrow} G$  is a collection

$$\{\alpha_A: F_{A,\dots,A}^{\stackrel{p\ times}{A,\dots,A}} \longrightarrow G_{A,\dots,A}^{\stackrel{q\ times}{A,\dots,A}} \mid A \in \mathcal{C}_o\}$$

of morphisms of  $\mathcal{D}$  indexed by the objects of  $\mathcal{C}$  such that, for each morphism  $f: A \to B$  of  $\mathcal{C}$ , the diagram



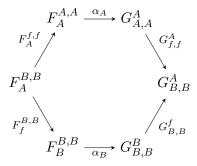
commutes.

A different name for this same notion: a (p,q)-to-(q,p)-dinatural transformation.

**Example 1.** For (p,q)=(2,1), a (2,1)-dinatural transformation is a collection

$$\left\{ \alpha_A : F_A^{A,A} \to G_{A,A}^A \mid A \in \mathcal{C}_o \right\}$$

of morphisms of  $\mathcal{D}$  such that, for each morphism  $f:A\to B$  of  $\mathcal{C}$ , the following hexagonal diagram commutes:



2.1. Why the weird "(p,q)-to-(q,p)" definition? We could have stick to Santamaria's definition of "(p,q)-to-(r,s)" dinaturality; we could have stick to the notion of (p,q)-to-(p,q) dinaturality. Our definition sits in the middle:

the type of domain and codomain of a "higher arity" dinatural transformation  $\alpha: F \xrightarrow{\bullet \bullet} G$  are different, but just

swapped: the contravariant length of F is the covariant length of G, and vice-versa.

It is important, even if straightforward, to note that as far as higher arity co/wedges (i.e. higher arity dinatural transformations from/to a constant functor) are concerned, the notions of (p,q)-dinaturality and (p,q)-to-(r,s)-dinaturality agree and yield the same theory of higher arity co/ends.

(Recall Mac Lane principle: what is the "right" level of generality?)

**Definition 4.** Let  $D: \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$  be a functor and let  $X \in \mathcal{D}_o$ .

- CW1) A (p,q)-wedge for D under X is a (p,q)-dinatural transformation  $\theta: X \stackrel{\bullet \bullet}{\Longrightarrow} D$  from the constant functor of type  $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$  with value X to D:
- CW2) A (p,q)-cowedge for D over X is a (p,q)-dinatural transformation  $\zeta: D \stackrel{\bullet \bullet}{\Longrightarrow} X$  from D to the constant functor of type  $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$  with value X.

#### Remark 1.

CWU1) A(p,q)-wedge  $\theta: X \stackrel{\bullet \bullet}{\Longrightarrow} D$  is a collection

$$\{\theta_A: X \to D_A^A : A \in \mathcal{C}_o\}$$

of morphisms of  $\mathcal C$  such that, for each morphism  $f:A\to B$  of  $\mathcal C$ , the diagram

$$X \xrightarrow{\theta_B} D_B^B$$

$$\theta_A \downarrow \qquad \qquad \downarrow D_B^f$$

$$D_A^A \xrightarrow{D_A^A} D_B^A$$

commutes.

CWU2) A(p,q)-cowedge  $\zeta: D \stackrel{\bullet \bullet}{\Longrightarrow} X$  is a collection

$$\{\zeta_A: D_{\boldsymbol{\Delta}}^{\boldsymbol{A}} \to X : A \in \mathcal{C}_o\}$$

of morphisms of  $\mathcal C$  such that, for each morphism  $f:A\to B$  of  $\mathcal C$ , the diagram

commutes.

3. Higher arity co/ends

**Definition 5.** Let  $D: \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$  be a functor.

PQ1) The (p,q)-end of D is, if it exists, the pair  $\left((p,q)\int_{A\in\mathcal{C}}D_{\underline{A}}^{\underline{A}},\omega\right)$  formed by an object

$$\int_{A\in\mathcal{C}} D_{\underline{A}}^{\underline{A}}$$

of  $\mathcal{D}$ , and a (p,q)-wedge

$$\omega: \int_{A\in\mathcal{C}} D_{\underline{A}}^{\underline{A}} \stackrel{\bullet \bullet}{\Longrightarrow} D$$

for  $(p,q)\int_{A\in\mathcal{C}}D_{\underline{A}}^{\underline{A}}$  over D, such that the (p,q)-wedge postcomposition natural transformation

$$\omega_*: \mathsf{h}\left(-, {}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}}\right) \Longrightarrow \mathsf{Wd}^{(p,q)}_{(-)}(D)$$

is a natural isomorphism.

PQ2) The (p,q)-coend of D is, if it exists, the pair  $\binom{(p,q)}{A} \in \mathcal{C}$   $D_{\underline{A}}^{\underline{A}}, \xi$  formed by an object

$$\int\limits_{-\infty}^{(p,\,q)} A \in \mathcal{C} \ D_{\underline{A}}^{\underline{A}}$$

of  $\mathcal{D}$ , and a (p,q)-cowedge

$$\xi: D \stackrel{\bullet \bullet}{\Longrightarrow} \int_{A}^{(p,q)} D_{\underline{A}}^{\underline{A}}$$

for  ${}^{(p,q)}\int^A D_{\underline{A}}^{\underline{A}}$  under D, such that the (p,q)-cowedge postcomposition natural transformation

is a natural isomorphism.

**Remark 2.** This means that the (p,q)-end of D is the terminal object of the category of wedges of D, whose morphisms  $h: (\alpha: \Delta_X \stackrel{\bullet \bullet}{\Longrightarrow} D) \to (\beta: \Delta_Y \stackrel{\bullet \bullet}{\Longrightarrow} D)$  are defined as the morphisms  $h: X \to Y$  of  $\mathcal{D}$  such that for every  $A \in \mathcal{C}_o$  one has  $\beta_A \circ h = \alpha_A$ :

$$X \xrightarrow{h} Y$$

$$\alpha_A \qquad \beta_A$$

$$D_{\underline{A}}^{\underline{A}}.$$

## 3.1. Basic properties.

**Proposition 2** (Properties of (p,q)-ends and (p,q)-coends). Let  $D: \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$  be a functor.

PE1) Functoriality. Let  $D: \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$  be a functor. The assignments  $D \mapsto_{(p,q)} \int_A D_A^{\underline{A}},^{(p,q)} \int_A^A D_A^{\underline{A}}$  define functors

with domain the category of functors from C of type  $\begin{bmatrix} p \\ q \end{bmatrix}$  to D and natural transformations between them.

PE2) (p,q)-Wedges and (p,q)-diagonals. For each  $X \in \mathcal{C}_o$  we have natural bijections

$$\begin{split} & \operatorname{Wd}_{(-)}^{(p,q)}(D) \cong \operatorname{Wd}_{(-)}\big(\Delta_*^{(p,q)}(D)\big), \\ & \operatorname{CWd}_{(-)}^{(p,q)}(D) \cong \operatorname{CWd}_{(-)}\big(\Delta_*^{(p,q)}(D)\big). \end{split}$$

where  $\Delta_{p,q}$  is the "twisted diagonal" functor

$$\Delta_{p,q} := \underbrace{\Delta^{op} \times \cdots \times \Delta^{op}}_{p \ times} \times \underbrace{\Delta \times \cdots \times \Delta}_{q \ times}.$$

PE3) (p,q)-Ends as ordinary ends. We have natural isomorphisms

$$(p,q) \int_{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \cong \int_{A \in \mathcal{C}} \Delta_*^{(p,q)}(D)_A^A,$$

$$(p,q) \int_{\underline{A}}^{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \cong \int_{A \in \mathcal{C}} \Delta_*^{(p,q)}(D)_A^A.$$

where  $\Delta_{p,q}$  is the twisted diagonal functor. In other words, the (p,q)end functor factors as a composition

$$\operatorname{Fun}(\mathcal{C}^{(p,q)}, \mathcal{D}) \xrightarrow{\Delta_*^{(p,q)}} \operatorname{Fun}(\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}) \xrightarrow{\int_A} \mathcal{D},$$

and similarly so do (p,q)-coends.

PE4) (p,q)-Ends as limits. The (p,q)-end and (p,q)-coend of D fit respectively into an equaliser and into a coequaliser diagram

$$(p,q) \int_{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \xrightarrow{\longrightarrow} \prod_{A \in \mathcal{C}_o} D_{\underline{A}}^{\underline{A}} \xrightarrow{\lambda} \prod_{A \to B} D_{\underline{B}}^{\underline{A}}$$

$$\prod_{A \to B} D_{\underline{B}}^{\underline{A}} \xrightarrow{\lambda'} \prod_{A \in \mathcal{C}_o} D_{\underline{A}}^{\underline{A}} \xrightarrow{(p,q)} \int_{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}}$$

for suitable maps  $\lambda, \rho, \lambda', \rho'$ , induced by the morphisms  $D_{\underline{u}}^{\underline{A}}, D_{\underline{B}}^{\underline{u}}$ . PE5) (p,q)-Ends as limits, again. We have natural isomorphisms

$$(p,q) \int_{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \cong \lim \left( \operatorname{Tw}(\mathcal{C}) \twoheadrightarrow \Sigma_{p,q} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D} \right),$$

$$(p,q) \int_{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \cong \operatorname{colim} \left( \operatorname{Tw}(\mathcal{C}) \twoheadrightarrow \Sigma_{p,q} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D} \right),$$

where  $\Sigma_{p,q} \colon \operatorname{Tw}(\mathcal{C}) \longrightarrow \mathcal{C}^{(p,q)}$  is the composition  $\Delta^{(p,q)} \circ \Sigma$ , with  $\Sigma$  the usual forgetful functor from  $\operatorname{Tw}(\mathcal{C})$  to  $\mathcal{C}^{op} \times \mathcal{C}$ . Explicitly,  $\Sigma^{(p,q)}$  is the functor

$$\operatorname{Tw}(\mathcal{C}) \longrightarrow \mathcal{C}^{(p,q)}$$

$$\begin{bmatrix} f_{\downarrow}^{A} \\ f_{\downarrow}^{+} \end{bmatrix} \longmapsto (\underline{A}, \underline{B})$$

$$\begin{bmatrix} \phi_{\uparrow}^{A} & \xrightarrow{f} & B \\ C & \xrightarrow{g} & D \end{bmatrix} \longmapsto (\underline{\phi}, \underline{\psi})$$

PE6) (p,q)-Ends as limits, yet again. There exists a category  $\mathsf{Tw}^{(p,q)}(\mathcal{C})$  together with a universal fibration

$$\Sigma \colon \mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \mathcal{C}^{(p,q)}$$

inducing natural isomorphisms

$$(p,q) \int_{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \cong \lim \Big( \mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \Sigma \mathcal{C}^{(p,q)} \stackrel{D}{\longrightarrow} \mathcal{D} \Big),$$

$$(p,q) \int_{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \cong \operatorname{colim} \Big( \mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \Sigma \mathcal{C}^{(p,q)} \stackrel{D}{\longrightarrow} \mathcal{D} \Big).$$

PE $\gamma$ ) (p,q)-Ends as (p+r,q+s)-ends. we have

$$(p,q) \int_{A \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \cong (p+r,q+s) \int_{A \in \mathcal{C}} \partial_s^r(D) \underline{\underline{A}},$$

$$(p,q) \int_{\underline{A} \in \mathcal{C}} D_{\underline{A}}^{\underline{A}} \cong (p+r,q+s) \int_{A \in \mathcal{C}} \partial_s^r(D) \underline{\underline{A}},$$

where  $\delta_s^r(-)$  is "(r,s)-dummyfication".

PE8) Commutativity of (p,q)-ends with homs. We have natural isomorphisms

$$\mathcal{D}\left(-, {}_{(p,q)}\int_{A\in\mathcal{C}} D_{\underline{A}}^{\underline{A}}\right) \cong {}_{(p,q)}\int_{A\in\mathcal{C}} \mathcal{D}\left(-, D_{\underline{A}}^{\underline{A}}\right)$$
$$\mathcal{D}\left({}^{(p,q)}\int_{A\in\mathcal{C}} D_{\underline{A}}^{\underline{A}}, -\right) \cong {}_{(q,p)}\int_{A\in\mathcal{C}} \mathcal{D}\left(D_{\underline{A}}^{\underline{A}}, -\right).$$

## 4. Examples:

#### 4.1. Some of them are trivial.

**Example 2** (Some (p,q)-co/ends are trivial for trivial reasons). We show that the two (0,2)-ends and (0,2)-coends of the functor  $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$  that gives  $\mathcal{C}$  a monoidal structure are trivial under very mild assumptions on  $\mathcal{C}$ . This rules out a class of possibly interesting examples coming from multilinear algebra.

To fix ideas, let R be a commutative ring, and let's consider the category  $Mod_R$  (with additional care one can take left modules and right modules, of course). To show that

$$\lim_{(0,2) \to A} A \otimes A \cong \lim_{A,B \in Mod_R} A \otimes B \cong 0,$$

$$\lim_{(0,2) \to A} A \otimes A \cong \operatorname{colim}_{A,B \in Mod_R} A \otimes B \cong 0,$$

we just observe that  $Mod_R$  is a sifted category, because it admits finite coproducts. The fact that a category C is sifted if and only if C is non-empty and the diagonal functor  $\Delta_C \colon C \longrightarrow C \times C$  is final then yields the result.

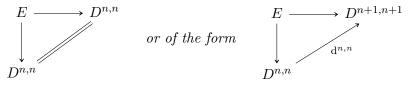
It's not difficult to see that if C is a sifted category, all diagonal functors  $\Delta: C \to C^n$  are final, because the product and composition of final functors is itself final. Thus the same result transports to higher coends of higher arity functors: for example,  $\bigwedge^k: Mod_R^n \to Mod_R$  sending M to  $\bigwedge^k M$ .

**Example 3** (More (p,q)-co/ends are trivial for similarly trivial reasons). Let R be a ring. The walking cochain complex ([?, Paragraph 35.1]) is the  $Mod_R$ -enriched category Ch whose set of objects is the set of integers, and where the hom-sets are R-modules freely generated by

$$\mathsf{C}([m],[n]) = \begin{cases} \{d,0\} & \text{if } m = n+1, \\ \{1,0\} & \text{if } m = n, \\ \{0\} & \text{otherwise.} \end{cases}$$

Now, a cochain complex is precisely a  $Mod_R$ -enriched functor from the Ch to  $Mod_R$ . Similarly, bicomplexes are  $Mod_R$ -enriched functors from  $Ch\boxtimes_{\mathsf{Mod}_R} Ch$  to  $\mathsf{Mod}_R$ .

Let  $D: Ch \boxtimes_{\mathsf{Mod}_R} Ch \longrightarrow \mathsf{Mod}_R$  be a bicomplex. We claim that its  $\mathsf{Mod}_R$ -enriched (2,0)-end E is just the zero module. Indeed, looking at (2,0)-wedges, we see that they are either of the form



Now, it follows from the second diagram that

$$E \cong \{(a_k)_{k \in \mathbb{Z}} \in \prod_{k \in \mathbb{Z}} D^{k,k} \mid a_{k+1} = d^{k,k}(a_k)\},$$

but differentials square to zero, so we must have  $a_k = 0$  for all  $k \in \mathbb{Z}$ , and E is indeed isomorphic to the zero module. A similar argument shows that  $(0,2)_{\Gamma}[k] \in Ch$   $D^{k,k} \cong 0$ .

<sup>&</sup>lt;sup>1</sup>This is due to [?]; see also [?, Proposition 5.3.2] or [?, Theorem 2.15] for reviews.

**Example 4** (Bisimplicial sets). Recall that a bisimplicial set ([?, Chapter IV], [?, §3.1.15]) is a functor  $X : \Delta^{op} \times \Delta \to Set$ ; moreover, the diagonalisation of a bisimplicial set  $X_{\bullet,\bullet} : \Delta^{op} \times \Delta^{op} \longrightarrow Set$  is the simplicial set  $d(X)_{\bullet} : \Delta^{op} \longrightarrow Set$  given by

$$d(X)_n := X_{n,n}$$
.

Joining the products and equalisers formula for (p,q)-coends we see that  ${}^{(2,0)}\int^{[n]\in\Delta^{op}}X_{n,n}$  is the coequaliser of he diagram

$$\coprod_{[n]\to[m]\in\Delta} X_{m,m}, \Longrightarrow \coprod_{[n]\in\Delta} X_{n,n}$$

giving

$$\int_{[n]\in\Delta}^{[n]\in\Delta}X_{n,n}\cong\pi_0(d(X)).$$

By a similar argument, we have

$$\int_{[0,0]} \int_{[n] \in \Delta} X_{n,n} \cong X_{0,0}.$$

## 4.2. Juicy examples:

4.2.1. A glance at weighted co/ends. We now introduce a natural factory of examples for higher arity co/ends. In a nutshell, weighted co/ends stand to co/ends in the same relation as weighted co/limits stand to limits.

**Definition 6** (Weighted co/end). Let C and D be V-enriched categories and  $D: C^{op} \otimes_{\mathcal{V}} C \longrightarrow D$  a V-functor, and  $W: C^{op} \times C \longrightarrow V$  a V-presheaf.

WE1) The end of D weighted by W is, if it exists, the object  $\int_{A \in \mathcal{C}}^{W} D_A^A$  of  $\mathcal{D}$  with the property that

$$\hom_{\mathcal{D}}\left(-, \int_{A \in \mathcal{C}}^{W} D_{A}^{A}\right) \cong \operatorname{DiNat}_{\mathcal{V}}(W, \boldsymbol{hom}_{\mathcal{C}}(-, D)).$$

WE2) The coend of D weighted by W is, if it exists, the object  $\int_{W}^{A \in \mathcal{C}} D_{A}^{A}$  of  $\mathcal{D}$  with the property that

$$\hom_{\mathcal{D}}\left(\int_{W}^{A \in \mathcal{C}} D_{A}^{A}, -\right) \cong \operatorname{DiNat}_{\mathcal{V}}(W, \boldsymbol{hom}_{\mathcal{C}}(D, -)).$$

**Example 5** (Weighted co/ends are (2,2)-co/ends). A quick argument (to be discussed in future work [?]) gives (2,2)-co/end formulas for weighted co/ends:

$$\begin{split} &\int_{A\in\mathcal{C}}^{[W]} D_A^A \cong \int_{(2,2)} \int_{A\in\mathcal{C}} W_A^A \pitchfork D_A^A, \\ &\int_{[W]}^{A\in\mathcal{C}} D_A^A \cong \int^{(2,2)} \int^{A\in\mathcal{C}} W_A^A \odot D_A^A. \end{split}$$

**Example 6** (Weighting Increases Arity). Let  $F, G: \mathcal{C} \longrightarrow \mathcal{D}$  and  $W: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{V}$  be  $\mathcal{V}$ -functors. In analogy with

$$\mathbf{Nat}_{\mathcal{V}}(F,G) := \int_{A \in \mathcal{C}} oldsymbol{hom}_{\mathcal{D}}(F_A,G_A),$$

we define the object  $\mathsf{Nat}^{[W]}(F,G)$  of natural transformations from F to G weighted by W by

(1) 
$$\operatorname{Nat}^{[W]}(F,G) := \int_{A \in \mathcal{C}}^{[W]} \boldsymbol{hom}_{\mathcal{D}}(F_A, G_A).$$

Taking W to be mute in its contravariant variable, we can give a reformulation of the universal property of weighted limits:

$$\mathsf{h}\left(-, \lim\nolimits^W(D)\right) \cong \mathsf{Nat}^{[W]}\left(\Delta_{(-)}, D\right).$$

Defining DiNat<sub>V</sub><sup>[W]</sup>(F,G) by a similar formula, we also obtain the following isomorphism in the case of weighted ends:

$$\mathsf{h}\left(-,\int_{A\in\mathcal{C}}^{[W]}D_A^A
ight)\cong\mathsf{DiNat}_{\mathcal{V}}^{[W]}\left(\Delta_{(-)},D
ight).$$

This naturally suggests a definition of doubly-weighted ends:

$$\mathsf{h}\left(-,\int_{A\in\mathcal{C}}^{[W_1,W_2]}D_A^A\right)\cong\mathsf{DiNat}_{\mathcal{V}}^{[W_1]}(W_2,D).$$

Repeating this process give you ends weighted by a collection of n functors  $W_1, \ldots, W_n$ . These however, can be actually computed as (n+1, n+1)-ends ([?]):

$$\int_{A\in\mathcal{C}}^{[W_1,\ldots,W_n]} D_A^A \cong \int_{(n+1,n+1)} \int_{A\in\mathcal{C}} \left( (W_1)_A^A \times \cdots \times (W_n)_A^A \right) \odot D_A^A.$$

As such, we see that weighting an end increases its arity by (1,1).

4.2.2. Weighted Kan extensions. Another source of examples comes from "weighing" left and right Kan extensions. While the most general such weight is a profunctor, having type  $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ , weights of type  $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$  or  $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$  are specially interesting, as they give a more direct parallel with the classical theory of weighted co/limits.

Recall the definition of the object  $\mathsf{Nat}^{[W]}(F,G)$  of weighted natural transformations.

**Definition 7.** The left Kan extension of F along K weighted by W is, if it exists, the V-functor

$$\left( \mathsf{Lan}_K^{[W]} F \colon \mathcal{D} \longrightarrow \mathcal{E} \right) \colon \bigvee_{W \overset{\frown}{\longleftarrow} \mathcal{C}} \bigvee_{F} \bigvee_{\mathcal{E}} \mathsf{Lan}_K^{[W]} F$$

for which we have a V-natural isomorphism

(2) 
$$\mathbf{Nat}_{\mathcal{V}}\left(\mathsf{Lan}_{K}^{[W]}F,G\right) \cong \mathsf{Nat}_{\mathcal{V}}^{[W]}\left(F,G\circ K\right),$$

natural in G.

One defines weighted right Kan extensions in a dual manner:

**Definition 8.** The right Kan extension of F along K weighted by W is, if it exists, the V-functor

$$\left( \operatorname{Ran}_K^{[W]} F \colon \mathcal{D} \longrightarrow \mathcal{E} \right) \colon \int_{K}^{K} \operatorname{Ran}_K^{[W]} F$$
 
$$W \overset{\bullet}{\subset} \mathcal{C} \xrightarrow{F} \overset{\bullet}{\mathcal{E}}$$

for which we have a V-natural isomorphism

(3) 
$$\operatorname{Nat}_{\mathcal{V}}\left(G,\operatorname{Ran}_{K}^{[W]}F\right)\cong\operatorname{Nat}_{\mathcal{V}}^{[W]}\left(G\circ K,F\right),$$

natural in G.

**Example 7** (Weighted co/limits as weighted Kan extensions). Let  $D: \mathcal{C} \longrightarrow \mathcal{D}$  be a diagram on a category  $\mathcal{D}$ . Then we may canonically identify the left Kan extension of D along the terminal functor with its colimit:

Similarly, given a weight  $W: \mathcal{C}^{op} \longrightarrow \mathsf{Set}$ , we have

$$\mathsf{Lan}_!^{[W]}D \cong \lceil \mathrm{colim}^W(D) \rceil \qquad \qquad ! \nearrow \downarrow \lceil \mathrm{colim}^W(D) \rceil \\ W \iff \mathcal{C} \xrightarrow{D} \mathcal{D}$$

One can also prove that the following formulas hold ([?]):

(4)
$$\operatorname{Lan}_{K}^{[W]} F \cong \int_{[W]}^{A \in \mathcal{C}} \operatorname{\mathbf{hom}}_{\mathcal{C}}(K_{A}, -) \odot F_{A} \cong \int_{-\infty}^{(2, 2)} \operatorname{\mathbf{hom}}_{\mathcal{C}}(K_{A}, -) \odot F_{A},$$
(5)

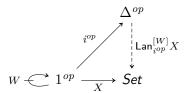
$$\mathsf{Ran}_{K}^{[W]}F\cong\int_{A\in\mathcal{C}}^{[W]}\mathbf{hom}_{\mathcal{C}}\left(-,K_{A}\right)\pitchfork F_{A}\cong\underset{(2,2)}{\int_{A\in\mathcal{C}}}\left(W_{A}^{A}\times\mathbf{hom}_{\mathcal{C}}(-,K_{A})\right)\pitchfork F_{A}.$$

Equipped with these, we now proceed to compute a few weighted Kan extensions.

**Example 8.** Consider the functor  $i^{op}: 1^{op} \to \Delta^{op}$ ; the left and right Kan extensions of a set  $X_{\bullet}: 1 \longrightarrow \mathsf{Set}$  along  $i^{op}$  are given by

$$\mathsf{Lan}_{i^{op}}(X) \cong \underline{X}_{ullet}$$
  
 $\mathsf{Ran}_{i^{op}}(X) \cong \check{C}(X).$ 

Now take a weight  $W: 1^{op} \times 1 \longrightarrow Set$ :



Then

$$\begin{split} &\mathsf{Lan}^{[W]}_{i^{op}}(X) \cong \underline{W \times X}_{\bullet} \\ &\mathsf{Ran}^{[W]}_{i^{op}}(X) \cong \check{C}(W \times X). \end{split}$$

**Example 9.** Now for the more interesting counterpart of the above:

$$\pi^{op} \colon \Delta^{op} \longrightarrow 1^{op}$$
$$[n] \longmapsto \star$$

The left and right Kan extensions of a simplicial set  $X_{\bullet} : \Delta^{op} \longrightarrow \mathsf{Set}$  along  $\pi^{op}$  are given by

$$\begin{split} \operatorname{Lan}_{\pi^{op}}(X_\bullet) &\cong \pi_0(X_\bullet) \\ \operatorname{Ran}_{\pi^{op}}(X_\bullet) &\cong \operatorname{ev}_0(X_\bullet) := X_0. \end{split}$$

Now, a weight  $W_{\bullet}^{\bullet} \colon \Delta^{op} \times \Delta \longrightarrow \mathit{Set}$  is wonderfully complicated: it is a cosimplicial space!

Then

(1) Taking  $W = \Delta^{\bullet}$  almost gives the geometric realisation of  $X_{\bullet}$ :

$$\mathrm{Lan}_{\pi^{op}}^{[\Delta^{\bullet}]}(X_{\bullet})\cong \int^{[n]\in\Delta}\Delta^{n}\times X_{n}.$$

(2) Dually, taking again  $W = \Delta^{\bullet}$  but now a cosimplicial object  $X^{\bullet} \colon \Delta \longrightarrow \operatorname{Set}$ .

$$\operatorname{Ran}_{\pi}^{[\Delta^{\bullet}]}(X^{\bullet}) = Tot(X_{\bullet}).$$

(3) If ou take  $W = \Delta^{\bullet}_{\bullet} = \hom_{\Delta}(-, -)$ , then I think you get

$$\begin{split} \operatorname{Lan}_{\pi^{op}}^{[\Delta_{\bullet}^{\bullet}]}(X_{\bullet}) &\cong \int\limits^{(2,\,2)} \int^{[n] \in \Delta} \Delta_{n}^{n} \times X_{n} \\ &\cong \int\limits^{(2,\,2)} \int^{[n] \in \Delta} \Delta_{n}^{n} \times X_{n} \end{split}$$

**Example 10.** Using the fact that weighted left/right Kan extensions along the identity are adjoint to each other, we can study situations like

$$W^{\bullet} \stackrel{\Delta^{op}}{\longleftarrow} \Delta^{op} \xrightarrow{X_{\bullet}} Set$$

This gives rise to an adjunction  $L : sSet \subseteq sSet : R$  with

$$L(X_{\bullet}) \cong \int_{[n] \in \Delta} W_n^n \odot X_n \cong \int_{[n] \in \Delta} W_n^n \times X_n,$$
  

$$R(X_{\bullet}) \cong \int_{[n] \in \Delta} W_n^n \cap X_n \cong \int_{[n] \in \Delta} W_n^n \times X_n.$$

Taking  $W = \Delta^{\bullet}$  gives L = R = 1, so let's take something more complicated, like  $\Delta^{\bullet}_{\bullet}$ . Then

$$L(X_{\bullet}) \cong \int_{[n] \in \Delta} \Delta^{n}[n] \times X_{n} \cong ?$$

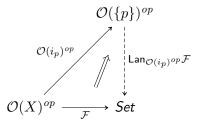
$$R(X_{\bullet}) \cong \int_{[n] \in \Delta} [\Delta^{n}[n], X_{n}] \cong ?$$

**Example 11** (Stalks of a sheaf ([?, Paragraph 6.8 and Section 7.1])). Let  $i_p: \{p\} \hookrightarrow X$  be the inclusion of a point into a topological space X. We get an induced functor

$$\mathcal{O}(i_p) \colon \mathcal{O}(X) \longrightarrow \mathcal{O}(\{p\})$$

$$U \longmapsto i_p^{-1}(U)$$

Considering now left Kan extensions along the opposite of  $\mathcal{O}(i_p)$ ,

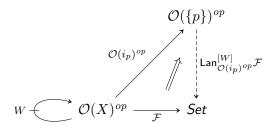


we obtain a functor  $\mathsf{Lan}_{\mathcal{O}(i_p)^{op}} \colon \mathsf{PSh}(X) \longrightarrow \mathsf{PSh}(\{p\})$ , whose image at  $\mathcal{F}$  is written  $\lceil \mathcal{F}_p \rceil$  for simplicity. The restriction of this functor to  $\mathsf{Sh}(X)$  can be identified with the stalk functor  $(-)_p \colon \mathsf{Sh}(X) \longrightarrow \mathsf{Set}$ : we have  $\mathcal{O}(\{p\}) = \{\varnothing \hookrightarrow \{p\}\}$  and computing the images of  $\varnothing$  and  $\{p\}$  under  $\lceil \mathcal{F}_p \rceil$  via the

usual colimit formula for left Kan extensions gives

$$\lceil \mathcal{F}_p \rceil (\{p\}) \cong \operatorname{colim} \left( \left( \mathcal{O}(\lceil p \rceil) \downarrow \underline{\{p\}} \right)^{op} \xrightarrow{\pi^{op}} \mathcal{O}(X)^{op} \xrightarrow{\mathcal{F}} \operatorname{Set} \right), \\
\cong \operatorname{colim}_{U \ni p} (\mathcal{F}(U)), \\
\cong \mathcal{F}_p \\
\lceil \mathcal{F}_p \rceil (\varnothing) \cong \operatorname{colim} \left( \left( \mathcal{O}(\lceil p \rceil) \downarrow \underline{\varnothing} \right)^{op} \xrightarrow{\pi^{op}} \mathcal{O}(X)^{op} \xrightarrow{\mathcal{F}} \operatorname{Set} \right), \\
\cong \operatorname{colim}_{U \hookrightarrow \varnothing} (\mathcal{F}(U)), \\
\cong \mathcal{F}(\varnothing).$$

(in case  $\mathcal{F}$  is a sheaf,  $\mathcal{F}(\varnothing)$  is the singleton set.) Consider the same situation, but now with a weight  $W: \mathcal{O}(X) \times \mathcal{O}(X)^{op} \longrightarrow \mathsf{Set}$  (an "extradiagonal presheaf on X"):



We may compute  $\mathsf{Lan}_{\mathcal{O}(i_p)^{op}}^{[W]}\mathcal{F} := \lceil \mathcal{F}_p^{[W]} \rceil$  as the weighted coend

$$\begin{split} \lceil \mathcal{F}_p^{[W]} \rceil &:= \int_{[W]}^{U \in \mathcal{O}(X)} \hom_{\mathcal{O}(X)^{op}} \left( \mathcal{O} \big( i_p^{op} \big)(U), - \right) \\ odot math cal F(U) \\ &\cong \int^{U \in \mathcal{O}(X)} W_U^U \times \hom_{\mathcal{O}(X)} \left( \chi_p(U), - \right) \right) \end{split}$$

timesmathcal F(U),

where

$$\chi_p(U) = \begin{cases} \emptyset & \text{if } p \notin U, \\ U & \text{otherwise.} \end{cases}$$

For instance, taking W to be a sheaf G on X gives

$$\mathcal{F}_p^{[\mathcal{G}]} := \lceil \mathcal{F}_p^{[\mathcal{G}]} \rceil (\{p\}) \cong (\mathcal{F} \times \mathcal{G})_n.$$

4.2.3. A glance at extradiagonality. "Extradiagonal" category theory arises when, instead of considering a natural transformation filling a higher-dimensional cell, we consider a dinatural one. Transformations that are more general than natural ones notoriously do not compose; yet, the category theory arising from this generalisation is interesting.

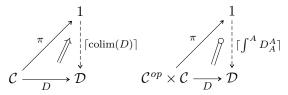
**Definition 9** (Diagonal left Kan extensions). The diagonal left Kan extension of a functor  $F: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{D}$  along a functor  $K: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{D}$  is, if it exists the functor  $\text{DiLan}_K F: \mathcal{D} \longrightarrow \mathcal{E}$  such that we have an isomorphism

$$\mathit{Nat}(\mathsf{DiLan}_K F, G) \cong \mathsf{DiNat}(F, G \circ K)$$

$$\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{F} \mathcal{E}$$

natural in G.

**Example 12.** Standard examples of diagonal left Kan extensions are ends: Generalising the fact that the left Kan extension of a functor  $D: \mathcal{C} \longrightarrow \mathcal{D}$  along the terminal functor  $\pi: \mathcal{C} \to 1$  can be identified with the colimit of  $\mathcal{D}$ , the diagonal left Kan extension of a functor  $D: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{D}$  along the terminal functor  $\pi: \mathcal{C}^{op} \times \mathcal{C} \to 1$  can be identified with the coend of  $\mathcal{D}$ .



Now, while ordinary Kan extensions can be computed via co/end formulas, diagonal Kan extensions admit (2,2)-co/end formulas ([?]):

(6) 
$$\operatorname{DiLan}_{K}F\cong \bigcap^{(2,2)}A\in\mathcal{C}}\mathcal{D}\left(K_{A}^{A},-\right)\odot F_{A}^{A},$$

(7) 
$$\operatorname{DiRan}_{K}F \cong \int_{(2,2)} \int_{A \in \mathcal{C}} \mathcal{D}\left(-, K_{A}^{A}\right) \pitchfork F_{A}^{A},$$

where the pairing is such that  $DiLan_K F$  is the coend of

$$(A,B) \mapsto \mathcal{D}\left(K_A^B,-\right) \odot F_B^A.$$

Alternatively, we may compute diagonal Kan extensions as hom-weighted Kan extensions ([?, ?]):

$$\begin{split} \operatorname{DiLan}_K & F \cong \int_{[\hom_{\mathcal{C}}(-,-)]}^{A,B \in \mathcal{C}} \mathcal{D}\left(K_A^B,-\right) \odot F_B^A, \\ \operatorname{DiRan}_K & F \cong \int_{A,B \in \mathcal{C}}^{[\hom_{\mathcal{C}}(-,-)]} \mathcal{D}\left(-,K_A^B\right) \pitchfork F_B^A. \end{split}$$

This is a generalisation of the fact that ends are hom-weighted limits. A forthcoming work [?] will address the topic of this remark in its entirety, studying the category theory arising from the notion of a weighted co/end.

**Example 13.** Let C be a closed monoidal category and  $D: C^{op} \times C \longrightarrow D$  be a diagram on D. What is  $DiLan_{[-,-]}D$  and  $DiRan_{[-,-]}D$ ?

$$\begin{array}{c|c} \mathcal{C} \\ \hline [-,-] & & \\ & & \\ & & \\ \mathcal{C}^{op} \times \mathcal{C} & \longrightarrow \mathcal{D} \end{array}$$

$$\mathsf{DiLan}_{[-,-]}D \cong \int^{A \in \mathcal{C}} \hom_{\mathcal{C}}([A,A],-) \odot D_A^A.$$

**Example 14.** Let  $D: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathcal{D}$  be a diagram on  $\mathcal{D}$ . What is  $\mathsf{DiLan}_y D$  and  $\mathsf{DiRan}_y D$ ?

$$\begin{array}{c|c}
\mathsf{PSh}(\mathcal{C}^{op} \times \mathcal{C}) \\
\downarrow \\
\mathcal{C}^{op} \times \mathcal{C} \xrightarrow{D} \mathcal{D}
\end{array}$$

In order to introduce the next example, we recall the following notation: we have an adjunction

$$(\pi \dashv i): 1 \xrightarrow[i]{\pi} \Delta,$$

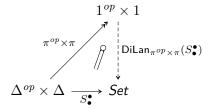
where

- i:  $1 \hookrightarrow \Delta$  is the functor choosing the terminal object;
- $\pi: \Delta \to 1$  is the terminal functor;

This induces a quadruple adjunction

$$\left(\pi_0\dashv\underline{(-)}_{\bullet}\dashv\operatorname{ev}_0\dashv\check{C}\right)\colon\operatorname{\mathsf{Set}} \Longrightarrow\operatorname{\mathsf{sSet}}_0$$

**Example 15.** Let  $S_{\bullet}^{\bullet} \colon \Delta^{op} \times \Delta \longrightarrow Set$  be a cosimplicial space. What is  $\mathsf{DiLan}_{\pi^{op} \times \pi}(S_{\bullet}^{\bullet})$ ?



It is just the end of  $S_{\bullet}^{\bullet}$  (btw do you know what this is?):

$$\begin{split} \mathsf{DiLan}_{\pi^{op}\times\pi}(S_{\bullet}^{\bullet}) &\cong \int^{[n]\in\Delta} \hom_1(\star,\star) \odot S_n^n, \\ &\cong \int^{[n]\in\Delta} S_n^n. \end{split}$$

Similarly, given a set  $X \colon 1^{op} \times 1 \longrightarrow \mathsf{Set}$ , we have

$$\begin{array}{c|c} \Delta^{op} \times \Delta \\ & \downarrow \\ i^{op} \times i \end{array} \begin{array}{c} \Delta^{op} \times \Delta \\ & \downarrow \\ \operatorname{DiLan}_{i^{op} \times i}(X) \end{array}$$
 
$$1^{op} \times 1 \xrightarrow{X} \operatorname{Set}$$

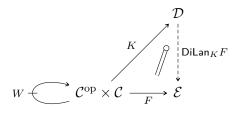
$$\begin{split} \mathsf{DiLan}_{i^{op}\times i}(X) &\cong \int^{\star\in 1} \hom_{\Delta^{op}\times \Delta}(([0],[0]),(-_1,-_2)) \odot X, \\ &\cong \hom_{\Delta^{op}\times \Delta}(([0],[0]),(-_1,-_2)) \odot X \\ &\cong \hom_{\Delta}([0],-_2) \odot X \\ &\cong \Delta^{-_2}[0] \odot X. \end{split}$$

Similarly, let  $X_{\bullet}^{\bullet} : \Delta^{op} \times \Delta \longrightarrow \mathit{Set}$  be a cosimplicial space again. What is  $\mathsf{DiLan}_{\Delta}(X_{\bullet}^{\bullet})$ ?

$$\begin{array}{c|c} & Set \\ & & \\ \Delta^{-2}[-_1] & & \\ & & \\ & & \\ \Delta^{op} \times \Delta & \xrightarrow{X^{\bullet}_{\bullet}} Set \end{array}$$

$$\mathsf{DiLan}_{\Delta}(X_{\bullet}^{\bullet}) \cong \int^{[n] \in \Delta} \mathit{Set}(\Delta^n[n], -) \odot X_n^n.$$

4.2.4. Weighted diagonal Kan extensions. In the same spirit, one can define weighted diagonal Kan extensions, mixing the two perspectives and considering now the diagram



just to discover that these are actually computed as (4,4)-co/ends:

$$\begin{split} \operatorname{DiLan}_{K}^{[W]} F &\cong \int^{(4,4)} \int^{A \in \mathcal{C}} \left( W_{A,A}^{A,A} \times \operatorname{\mathbf{hom}}_{\mathcal{C}} \left( K_{A}^{A}, - \right) \right) \odot F_{A}^{A}, \\ \operatorname{DiRan}_{K}^{[W]} F &\cong \int_{(4,4)} \int_{A \in \mathcal{C}} \left( W_{A,A}^{A,A} \times \operatorname{\mathbf{hom}}_{\mathcal{C}} \left( -, K_{A}^{A} \right) \right) \pitchfork F_{A}^{A}. \end{split}$$

At this point, the reader shall be convinced that the list of examples is virtually endless. We defer a thorough study of the topic to separate works [?, ?].

4.2.5. Daydreaming About Operads. Day convolution was introduced by B. Day in [?, ?], in order to classify monoidal structures on the category  $\mathsf{PSh}(\mathcal{C})$  of presheaves on  $\mathcal{C}$ . Day proved that  $\mathsf{PSh}(\mathcal{C})$  can be turned into a monoidal category in as many ways as  $\mathcal{C}$  can be turned into a pseudomonoid in the bicategory of profunctors.

We now propose a generalisation of this framework based on higher arity coends: let  $(\mathcal{C}, \otimes, I)$  be a monoidal category, and let  $\mathcal{K} := \mathsf{PSh}(\mathcal{C})$ . Higher arity Day convolution is defined as a family of functors  $\circledast_n : \mathcal{K}^n \to \mathcal{K}$ :

**Definition 10.** The **Day** (n,n)-convolution of an n-tuple of presheaves  $\mathcal{F}_1, \ldots, \mathcal{F}_n$  is the presheaf

$$\circledast_n(\mathcal{F}_1,\ldots,\mathcal{F}_n)\colon \mathcal{C}^{op}\longrightarrow Set$$

defined at  $A \in \mathcal{C}_o$  as the (n, n)-coend

$$\circledast_n(\mathcal{F}_1,\ldots,\mathcal{F}_n):=A\mapsto \bigcap^{(n,n)}A\in\mathcal{C}}\mathcal{F}_1(A)\times\cdots\times\mathcal{F}_n(A)\times\mathcal{C}\left(-,A^{\otimes n}\right),$$

where  $A^{\otimes n}$  is shorthand for the n-fold tensor product of A with itself.

**Example 16** (Day convolution operad). The **Day convolution operad** associated to  $(C, \otimes, I)$  is the free symmetric operad Day whose set of generating operations (see [?, Section 1.2.5]) is given by  $\{1, \circledast_2, \circledast_3, \ldots, \circledast_n, \ldots\}$ .

**Remark 3.** We spell out in detail the first four sets of n-ary operations of Day:

$$Day_1 = \{1\}$$

$$\mathsf{Day}_2 = \{ \circledast_2(-,-) \}$$

$$\mathsf{Day}_3 = \{ \circledast_3(-,-,-), \circledast_2(\circledast_2(-,-),-), \circledast_2(-,\circledast_2(-,-)) \}$$

$$\begin{split} \mathsf{Day}_4 &= \{ \circledast_4(-,-,-,-), \circledast_2(\circledast_3(-,-,-),-), \circledast_2(-,\circledast_3(-,-,-)), \circledast_2(\circledast_2(-,-),\circledast_2(-,-)), \\ & \circledast_3(-,\circledast_2(-,-),-), \circledast_3(\circledast_2(-,-),-,-), \circledast_3(-,-,\circledast_2(-,-)) \} \end{split}$$

All in all, the set  $\mathsf{Day}_n$  can be succinctly described as

$$\mathsf{Day}_n = \{\circledast_n\} \cup \sum_{p+q=n} \mathsf{Day}_p \times \mathsf{Day}_q$$

The operadic composition of Day is now defined via 'grafting' in the usual way:

$$\begin{split} \mathsf{Day}_n \times \mathsf{Day}_{k_1} \times \dots \times \mathsf{Day}_{k_n} & \longrightarrow \mathsf{Day}_{\sum k_i} \\ (\theta; \theta_1, \dots, \theta_k) & \longmapsto \theta(\theta_1(-_1, \dots, -_{k_1}), \dots, \theta_k(-_1, \dots, -_{k_n})) \end{split}$$

## 5. Kusarigamas

- 5.1. Reducing dinaturality to naturality.
- 5.2. higher arity twisted arrow categories.

- 6.1. weighing co/ends: the full story.
- 6.2. kusarigamas are a toy example of "extradiagonal" (for lack of a better name) transformation.
- 6.3. A graphical language for higher arity co/ends. <sup>‡</sup>Universidade de São Paulo, Instituto de Ciências Matemáticas e de Computação, Av. Trab. São Carlense, 400, 13566-590 São Carlos, Brasil theo.de.oliveira.santos@usp.br

§TALLINN UNIVERSITY OF TECHNOLOGY, INSTITUTE OF CYBERNETICS, AKADEEMIA TEE 15/2, 12618 TALLINN, ESTONIA FOSCO.LOREGIAN@TALTECH.EE