P-Q-COENDS

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Contents

1. Motivation:

1.1. "symmetrize" tensors of higher arity. The so-called "Einstein summation convention" in linear algebra and differential geometry asserts that it is possible to suppress the summation symbol $\sum$ in every formula like

$$\sum_i c_i v_i$$

at the cost of writing "$c^i v_i$"; this means that contravariant tensors' indices are superscripts, while covariant tensors' indices are subscripts, and whenever homonymous indices appear in a string like $c^i v_i$, it means that we are

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summing over that index. So, for example, the first Bianchi identity:

$$\nabla_i R^i{}_j = \frac{1}{2}\nabla_j R$$

stands for $\sum_i \nabla_i R^i{}_j = \dots$, and the definition of R itself expands to a daunting

$$\begin{aligned}R_{ij} = &-\sum_{a,b}\frac{1}{2}\Big(\frac{\partial^2 g_{ij}}{\partial x^a\partial x^b} + \frac{\partial^2 g_{ab}}{\partial x^i\partial x^j} - \frac{\partial^2 g_{ib}}{\partial x^j\partial x^a} - \frac{\partial^2 g_{jb}}{\partial x^i\partial x^a}\Big)g^{ab}\\ &+\frac{1}{2}\sum_{a,b,c,d}\Big(\frac{1}{2}\frac{\partial g_{ac}}{\partial x^i}\frac{\partial g_{bd}}{\partial x^j} + \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jd}}{\partial x^b} - \frac{\partial g_{ic}}{\partial x^a}\frac{\partial g_{jb}}{\partial x^d}\Big)g^{ab}g^{cd}\\ &-\frac{1}{4}\sum_{a,b,c,d}\Big(\frac{\partial g_{jc}}{\partial x^i} + \frac{\partial g_{ic}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^c}\Big)\Big(2\frac{\partial g_{bd}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^d}\Big)g^{ab}g^{cd}.\end{aligned}$$

This convention does not allow for "unbalanced" expression to be summed over: the same number of subscript must be paired with the same number of superscripts.

In category theory, the analogue operation of "summing over repeated indices" is taking a **coend** of a functor

$$T : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{D}$$

intended as the quotient of $\coprod_C T(C,C)$ by the equivalence relation generated by the action of T on arrows; this analogy is not peregrine: if $S : \mathcal{A}^{\mathrm{op}} \times \mathcal{B} \to \mathsf{Set}$ and $T : \mathcal{B}^{\mathrm{op}} \times \mathcal{C} \to \mathsf{Set}$ are two "profunctors", their composition

$$ST(A,C) := \int^B S(A,B) \times T(B,C)$$

is akin to the matrix product of two matrices, seen as functions $S : [n] \times [m] \to K$, $T : [m] \times [r] \to K$.

(A perfect analogy is this: let A, B, C be discrete categories; then the profunctor composition of $S : A \times B \to \mathsf{Set}$ and $T : B \times C \to \mathsf{Set}$ is the matrix product of an $|A| \times |B|$ and a $|B| \times |C|$ matrix.)

1.2. **Question(s).**

- What if we want to sum/integrate/coend over an "unbalanced tensor" like
$$T : (\mathcal{C}^{\mathrm{op}})^p \times \mathcal{C}^q = \mathcal{C}^{(p,q)} \to \mathcal{D}$$
for $p, q \geq 1$?
- Is the resulting theory well-behaved as the classical one?
- No one would debate about the usefulness of "balanced" integrals; are the unbalanced ones good for something similar?

This work aims at answering all these questions in the positive:

- Yes, one can define a notion of **co/end** for "higher arity" functors $\mathcal{C}^{(p,q)} \to \mathcal{D}$;

- Yes and no; higher arity co/ends are particular instances of co/ends, where T has been "completely symmetrised" (see later for a definition); as such, they do not constitute a "new" object; instead, a specialisation of classical co/end calculus;
- Yes, the resulting theory is expressive enough to capture some new phenomena.

At this point, perhaps the most enlightening example is the following, appearing in a paper by Street and Dubuc:

Proposition 1. *Let $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be two functors; define the functor*

$$\mathrm{DNat}(F^{\uparrow}, G^{\downarrow}) : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Set}$$

sending (A, B) to $\mathcal{D}(F^B_A, G^A_B)$; then, the set of dinatural transformations $F \overset{\bullet\bullet}{\Longrightarrow} G$ is canonically isomorphic to the end of $DNat(F^{\uparrow}, G^{\downarrow})$, i.e. to the equaliser of the diagram

$$\prod_C \mathcal{D}(F^C_C, G^C_C) \underset{v}{\overset{u}{\rightrightarrows}} \prod_{A \to B} \mathcal{D}(F^B_A, G^A_B)$$

1.3. **generalised dinaturality recently introduced by A. Santamaria in his PhD thesis.** A. Santamaria and McGusker recently introduced in [MS] the notion of dinaturality we started from; yet, his notion is too general for our purposes for two reasons:

- they do not assume a "transformation" satisfies any condition globally, treating the

notion of naturality as a property of a single component (this reads as: a transformation between two functors of a certain "type" is di/natural at an index i, but it can be "unnatural" elsewhere).

- they do not assume that the type of the domain functor and the codomain functor are the same.

Our notational convention is also different: they take into account functors $\mathcal{C}^\alpha \to \mathcal{B}$, where α is a "binary multi-index", i.e. an element in the free monoid over the set $\{\oplus, \ominus\}$, and the convention is that $\mathcal{C}^{\varnothing} :=$, the terminal category, $\mathcal{C}^{\oplus} := \mathcal{C}$, $\mathcal{C}^{\ominus} := \mathcal{C}^{\mathrm{op}}$, and $\mathcal{C}^{\alpha \uplus \alpha'} := \mathcal{C}^\alpha \times \mathcal{C}^{\alpha'}$.

Here instead, we adopt a different convention: a generic power $\mathcal{C}^\alpha$ is always "reshuffled" in order for all its minus and plus signs to appear on the same side, respectively on the left and on the right. The categories $\mathcal{C}^\alpha$ and $\mathcal{C}^{(p,q)}$ so obtained are, of course, canonically isomorphic, and the tuple α is equivalent to the reshuffled tuple $(\ominus_1, \dots, \ominus_p, \oplus_1, \dots, \oplus_q)$.

Definition 1. *Let α, β be two multi-indices, and let $F : \mathcal{C}^\alpha \to \mathcal{D}$, $G : \mathcal{C}^\beta \to \mathcal{D}$ be functors. A* transformation *$\phi : F \to G$* of type *$|\alpha| \xrightarrow{\sigma} n \xleftarrow{\tau} |\beta|$ (with $n = |\boldsymbol{A}|$ a positive integer) is a family of morphisms in $\mathcal{D}$*

$$\phi_{A_1,\dots,A_n} : F(A_{\sigma 1}, \dots, A_{\sigma|\alpha|}) \to G(A_{\tau 1}, \dots, A_{\tau|\beta|}).$$

for each tuple of objects $A_1, \dots, A_n$ of $\mathcal{C}$.

Notice that α and β are *different* multi-indices in this definition, and σ, τ need not be injective or surjective, so we may have repeated or unused variables.

Definition 2. *Let $\phi = (\phi_{A_1,\dots,A_n}) : F \to G$ be a transformation. For $i \in \{1, \dots, n\}$, we say that ϕ is* dinatural in A_i *(or, more precisely,* dinatural in its i-th variable*) if and only if for all $A_1, \dots, A_{i-1}, A_{i+1}, \dots, A_n$ objects of $\mathcal{C}$ and for all $f : A \to B$ in $\mathcal{C}$ the following hexagon commutes:*

$$\begin{tikzcd}
 & F(\boldsymbol{A}[A/i]\sigma) \arrow[r, "\phi_{\boldsymbol{A}[A/i]}"] & G(\boldsymbol{A}[A/i]\tau) \arrow[dr, "G(\boldsymbol{A}[A,f/i]\tau)"] & \\
F(\boldsymbol{A}[B,A/i]\sigma) \arrow[ur, "F(\boldsymbol{A}[f,A/i]\sigma)"] \arrow[dr, "F(\boldsymbol{A}[B,f/i]\sigma)"'] & & & G(\boldsymbol{A}[A,B/i]\tau) \\
 & F(\boldsymbol{A}[B/i]\sigma) \arrow[r, "\phi_{\boldsymbol{A}[B/i]}"'] & G(\boldsymbol{A}[B/i]\tau) \arrow[ur, "G(\boldsymbol{A}[f,B/i]\tau)"'] &
\end{tikzcd}$$

where $\boldsymbol{A}$ is the n-tuple $(A_1, \dots, A_n)$ of the objects above with an additional (unused in this definition) object A_i of $\mathcal{C}$.

2. Higher arity co/wedges

N1) A generic tuple of objects,

$$\underline{A} := (A_1, \dots, A_n)$$

often split as the juxtaposition $\underline{A}'; \underline{A}''$ of two subtuples of length p, q,

$$\underline{A}' := (A_1, \dots, A_q), \qquad \underline{A}'' := (A_{p+1}, \dots, A_{p+q})$$

N2) As already said, the image of a split tuple $\underline{A}'; \underline{A}''$ under a functor of type $[\begin{smallmatrix} p \\ q \end{smallmatrix}]$, $F : \mathcal{C}^{(p,q)} \to \mathcal{D}$ is denoted $F^{\underline{A}'}_{\underline{A}''}$: the contravariant components come first, and the covariant component second. So: contravariant components are always *left* in the typing

$$F : \mathcal{C}^{(p,q)} \varnothing \mathcal{D}$$

of a functor, and *up* in its action on objects.

N3) Denoting a functor F of type $[\begin{smallmatrix} p \\ q \end{smallmatrix}]$ evaluated at a diagonal tuple: we write

$$F^{\boldsymbol{A}}_{\boldsymbol{A}} := F^{A,\dots,A}_{A,\dots,A},$$

where the superscript has p elements, and the subscript has q elements.

N4) Substitution of an object at a prescribed index

$$\underline{A}[X/i] := (A_1, \dots A_{i-1}, X, A_{i+1}, \dots, A_n).$$

N5) Substitution of a tuple at a prescribed tuple of indices

$$\underline{A}[X_1, \dots, X_r/i_1, \dots, i_r] := ((\underline{A}[X_1/i_1])[X_2/i_2] \cdots)[X_r/i_r].$$

Definition 3. *A* (p,q)-dinatural transformation $\alpha : F \overset{\bullet\bullet}{\Longrightarrow} G$ *is a collection*

$$\{\alpha_A : F^{\overbrace{A,\dots,A}^{p\ times}}_{\underbrace{A,\dots,A}_{q\ times}} \longrightarrow G^{\overbrace{A,\dots,A}^{q\ times}}_{\underbrace{A,\dots,A}_{p\ times}} \mid A \in \mathcal{C}_o\}$$

of morphisms of $\mathcal{D}$ *indexed by the objects of* $\mathcal{C}$ *such that, for each morphism* $f : A \to B$ *of* $\mathcal{C}$*, the diagram*

$$\begin{tikzcd}
& F^{\boldsymbol{A}_p}_{\boldsymbol{A}_q} \arrow[r, "\alpha_A"] & G^{\boldsymbol{A}_q}_{\boldsymbol{A}_p} \arrow[dr, "G^{\boldsymbol{A}_q}_{\boldsymbol{f}_p}"] & \\
F^{\boldsymbol{B}_p}_{\boldsymbol{A}_q} \arrow[ur, "F^{\boldsymbol{f}_p}_{\boldsymbol{A}_q}"] \arrow[dr, "F^{\boldsymbol{B}_p}_{\boldsymbol{f}_q}"'] & & & G^{\boldsymbol{A}_q}_{\boldsymbol{B}_p} \\
& F^{\boldsymbol{B}_p}_{\boldsymbol{B}_q} \arrow[r, "\alpha_B"'] & G^{\boldsymbol{B}_q}_{\boldsymbol{B}_p} \arrow[ur, "G^{\boldsymbol{f}_q}_{\boldsymbol{B}_p}"'] &
\end{tikzcd}$$

commutes.

A different name for this same notion: a (p,q)*-to-*(q,p)*-dinatural transformation.*

Example 1. *For* $(p,q)=(2,1)$*, a* $(2,1)$*-dinatural transformation is a collection*

$$\left\{\alpha_A : F^{A,A}_{A} \to G^{A}_{A,A} \;\middle|\; A \in \mathcal{C}_o\right\}$$

of morphisms of $\mathcal{D}$ *such that, for each morphism* $f : A \to B$ *of* $\mathcal{C}$*, the following hexagonal diagram commutes:*

$$\begin{tikzcd}
& F^{A,A}_{A} \arrow[r, "\alpha_A"] & G^{A}_{A,A} \arrow[dr, "G^{A}_{f,f}"] & \\
F^{B,B}_{A} \arrow[ur, "F^{f,f}_{A}"] \arrow[dr, "F^{B,B}_{f}"'] & & & G^{A}_{B,B} \\
& F^{B,B}_{B} \arrow[r, "\alpha_B"'] & G^{B}_{B,B} \arrow[ur, "G^{f}_{B,B}"'] &
\end{tikzcd}$$

2.1. **Why the weird "(p,q)-to-(q,p)" definition?** We could have stick to Santamaria's definition of "(p,q)-to-(r,s)" dinaturality; we could have stick to the notion of (p,q)-to-(p,q) dinaturality. Our definition sits in the middle:

> the type of domain and codomain of a "higher arity" dinatural transformation $\alpha : F \overset{\bullet\bullet}{\Longrightarrow} G$ are different, but just

swapped: the contravariant length of F is the covariant length of G, and vice-versa.

It is important, even if straightforward, to note that as far as higher arity co/wedges (i.e. higher arity dinatural transformations from/to a constant functor) are concerned, the notions of (p,q)-dinaturality and (p,q)-to-(r,s)-dinaturality agree and yield the same theory of higher arity co/ends.

(Recall Mac Lane principle: what is the "right" level of generality?)

Definition 4. *Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor and let $X \in \mathcal{D}_o$.*

CW1) *A* (p,q)-wedge for D under X *is a (p,q)-dinatural transformation $\theta : X \overset{\bullet\bullet}{\Longrightarrow} D$ from the constant functor of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ with value X to D;*

CW2) *A* (p,q)-cowedge for D over X *is a (p,q)-dinatural transformation $\zeta : D \overset{\bullet\bullet}{\Longrightarrow} X$ from D to the constant functor of type $\left[\begin{smallmatrix} q \\ p \end{smallmatrix}\right]$ with value X.*

Remark 1.

CWU1) *A (p,q)-wedge $\theta : X \overset{\bullet\bullet}{\Longrightarrow} D$ is a collection*

$$\left\{\theta_A : X \to D^{\boldsymbol{A}}_{\boldsymbol{A}} \;\;:\;\; A \in \mathcal{C}_o\right\}$$

of morphisms of $\mathcal{C}$ such that, for each morphism $f : A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} X \arrow[r, "\theta_B"] \arrow[d, "\theta_A"'] & D^{\boldsymbol{B}}_{\boldsymbol{B}} \arrow[d, "D^{\boldsymbol{f}}_{\boldsymbol{B}}"] \\ D^{\boldsymbol{A}}_{\boldsymbol{A}} \arrow[r, "D^{\boldsymbol{A}}_{\boldsymbol{f}}"'] & D^{\boldsymbol{A}}_{\boldsymbol{B}} \end{tikzcd}$$

commutes.

CWU2) *A (p,q)-cowedge $\zeta : D \overset{\bullet\bullet}{\Longrightarrow} X$ is a collection*

$$\left\{\zeta_A : D^{\boldsymbol{A}}_{\boldsymbol{A}} \to X \;\;:\;\; A \in \mathcal{C}_o\right\}$$

of morphisms of $\mathcal{C}$ such that, for each morphism $f : A \to B$ of $\mathcal{C}$, the diagram

$$\begin{tikzcd} X & D^{\boldsymbol{B}}_{\boldsymbol{B}} \arrow[l, "\zeta_B"'] \\ D^{\boldsymbol{A}}_{\boldsymbol{A}} \arrow[u, "\zeta_A"] & D^{\boldsymbol{A}}_{\boldsymbol{B}} \arrow[l, "D^{\boldsymbol{A}}_{\boldsymbol{f}}"] \arrow[u, "D^{\boldsymbol{f}}_{\boldsymbol{B}}"'] \end{tikzcd}$$

commutes.

3. Higher arity co/ends

Definition 5. *Let $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ be a functor.*

PQ1) *The* (p,q)-end *of* D *is, if it exists, the pair* $\left({}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}}, \omega\right)$ *formed by an object*

$$ {}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}} $$

of $\mathcal{D}$*, and a* (p,q)*-wedge*

$$ \omega : {}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}} \overset{\bullet\bullet}{\Longrightarrow} D $$

for ${}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}}$ *over* D*, such that the* (p,q)*-wedge postcomposition natural transformation*

$$ \omega_* : \mathsf{h}\left(-, {}_{(p,q)}\int_{A\in\mathcal{C}} D^{A}_{\underline{A}}\right) \Longrightarrow \mathsf{Wd}^{(p,q)}_{(-)}(D) $$

is a natural isomorphism.

PQ2) *The* (p,q)-coend *of* D *is, if it exists, the pair* $\left({}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}}, \xi\right)$ *formed by an object*

$$ {}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}} $$

of $\mathcal{D}$*, and a* (p,q)*-cowedge*

$$ \xi : D \overset{\bullet\bullet}{\Longrightarrow} {}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}} $$

for ${}^{(p,q)}\int^{A} D^{A}_{\underline{A}}$ *under* D*, such that the* (p,q)*-cowedge postcomposition natural transformation*

$$ \xi^* : \mathsf{h}\left({}^{(p,q)}\int^{A\in\mathcal{C}} D^{A}_{\underline{A}}, -\right) \Longrightarrow \mathsf{CWd}^{(p,q)}_{(-)}(D) $$

is a natural isomorphism.

Remark 2. *This means that the* (p,q)*-end of* D *is the terminal object of the category of wedges of* D*, whose morphisms* $h : (\alpha : \Delta_X \overset{\bullet\bullet}{\Longrightarrow} D) \to (\beta : \Delta_Y \overset{\bullet\bullet}{\Longrightarrow} D)$ *are defined as the morphisms* $h : X \to Y$ *of* $\mathcal{D}$ *such that for every* $A \in \mathcal{C}_o$ *one has* $\beta_A \circ h = \alpha_A$*:*

$$
\begin{tikzcd}
X \arrow[rr, "h"] \arrow[dr, "\alpha_A"'] & & Y \arrow[dl, "\beta_A"] \\
& D^{A}_{\underline{A}}. &
\end{tikzcd}
$$

3.1. Basic properties.

Proposition 2 (Properties of (p,q)-ends and (p,q)-coends)**.** *Let* $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ *be a functor.*

PE1) Functoriality. *Let* $D : \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ *be a functor. The assignments* $D \mapsto {}_{(p,q)}\!\int_A D^{\underline{A}}_{\underline{A}}, {}^{(p,q)}\!\int^A D^{\underline{A}}_{\underline{A}}$ *define functors*

$$
\begin{aligned}
{}_{(p,q)}\!\int_{A\in\mathcal{C}} &: \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \longrightarrow \mathcal{D},\\
{}^{(p,q)}\!\int^{A\in\mathcal{C}} &: \mathsf{Cat}\big(\mathcal{C}^{(p,q)}, \mathcal{D}\big) \longrightarrow \mathcal{D}
\end{aligned}
$$

with domain the category of functors from $\mathcal{C}$ *of type* $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$ *to* $\mathcal{D}$ *and natural transformations between them.*

PE2) (p,q)*-Wedges and* (p,q)*-diagonals. For each* $X \in \mathcal{C}_o$ *we have natural bijections*

$$
\begin{aligned}
\mathsf{Wd}^{(p,q)}_{(-)}(D) &\cong \mathsf{Wd}_{(-)}\big(\Delta^{(p,q)}_{*}(D)\big),\\
\mathsf{CWd}^{(p,q)}_{(-)}(D) &\cong \mathsf{CWd}_{(-)}\big(\Delta^{(p,q)}_{*}(D)\big).
\end{aligned}
$$

where $\Delta_{p,q}$ *is the "twisted diagonal" functor*

$$
\Delta_{p,q} := \underbrace{\Delta^{\mathrm{op}} \times \cdots \times \Delta^{\mathrm{op}}}_{p \text{ times}} \times \underbrace{\Delta \times \cdots \times \Delta}_{q \text{ times}}.
$$

PE3) (p,q)*-Ends as ordinary ends. We have natural isomorphisms*

$$
\begin{aligned}
{}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \int_{A\in\mathcal{C}} \Delta^{(p,q)}_{*}(D)^{A}_{A},\\
{}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \int^{A\in\mathcal{C}} \Delta^{(p,q)}_{*}(D)^{A}_{A}.
\end{aligned}
$$

where $\Delta_{p,q}$ *is the twisted diagonal functor.In other words, the* (p,q)*-end functor factors as a composition*

$$
\mathrm{Fun}(\mathcal{C}^{(p,q)}, \mathcal{D}) \xrightarrow{\Delta^{(p,q)}_{*}} \mathrm{Fun}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}, \mathcal{D}) \xrightarrow{\quad \int_A \quad} \mathcal{D},
$$

and similarly so do (p,q)*-coends.*

PE4) (p,q)*-Ends as limits. The* (p,q)*-end and* (p,q)*-coend of* D *fit respectively into an equaliser and into a coequaliser diagram*

$$
{}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} \longrightarrow \prod_{A\in\mathcal{C}_o} D^{\underline{A}}_{\underline{A}} \underset{\rho}{\overset{\lambda}{\rightrightarrows}} \prod_{A\to B} D^{\underline{A}}_{\underline{B}}
$$

$$
\coprod_{A\to B} D^{\underline{A}}_{\underline{B}} \underset{\rho'}{\overset{\lambda'}{\rightrightarrows}} \coprod_{A\in\mathcal{C}_o} D^{\underline{A}}_{\underline{A}} \longrightarrow {}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}}
$$

for suitable maps $\lambda, \rho, \lambda', \rho'$*, induced by the morphisms* $D^{\underline{A}}_{\underline{u}}, D^{\underline{u}}_{\underline{B}}$*.*

PE5) (p,q)*-Ends as limits, again. We have natural isomorphisms*

$$
\begin{aligned}
{}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \lim\Big(\mathrm{Tw}(\mathcal{C}) \twoheadrightarrow \Sigma_{p,q}\mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big),\\
{}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \mathrm{colim}\Big(\mathrm{Tw}(\mathcal{C}) \twoheadrightarrow \Sigma_{p,q}\mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big),
\end{aligned}
$$

where $\Sigma_{p,q}\colon \mathrm{Tw}(\mathcal{C}) \longrightarrow \mathcal{C}^{(p,q)}$ *is the composition* $\Delta^{(p,q)} \circ \Sigma$*, with* Σ *the usual Sigma functor from* $\mathrm{Tw}(\mathcal{C})$ *to* $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$*. Explicitly,* $\Sigma^{(p,q)}$ *is the functor*

$$\begin{aligned} \mathrm{Tw}(\mathcal{C}) &\longrightarrow \mathcal{C}^{(p,q)} \\ \begin{bmatrix} A \\ f\downarrow \\ B \end{bmatrix} &\longmapsto (\underline{A}, \underline{B}) \\ \begin{bmatrix} A & \xrightarrow{f} & B \\ \phi\uparrow & & \downarrow\psi \\ C & \xrightarrow[g]{} & D \end{bmatrix} &\longmapsto (\underline{\phi}, \underline{\psi}) \end{aligned}$$

PE6) *(p,q)-Ends as limits, yet again. There exists a category* $\mathsf{Tw}^{(p,q)}(\mathcal{C})$ *together with a universal fibration*

$$\Sigma\colon \mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \mathcal{C}^{(p,q)}$$

inducing natural isomorphisms

$$\begin{aligned} {}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \lim\Big(\mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \Sigma\mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big), \\ {}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \operatorname{colim}\Big(\mathsf{Tw}^{(p,q)}(\mathcal{C}) \twoheadrightarrow \Sigma\mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big). \end{aligned}$$

PE7) *(p,q)-Ends as $(p+r,q+s)$-ends. we have*

$$\begin{aligned} {}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong {}_{(p+r,q+s)}\!\int_{A\in\mathcal{C}} \eth^r_s(D)^{\underline{A}}_{\underline{A}}, \\ {}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong {}^{(p+r,q+s)}\!\int^{A\in\mathcal{C}} \eth^r_s(D)^{\underline{A}}_{\underline{A}}, \end{aligned}$$

where $\eth^r_s(-)$ *is "(r,s)-dummyfication".*

PE8) *Commutativity of (p,q)-ends with homs. We have natural isomorphisms*

$$\begin{aligned} \mathcal{D}\left(-, {}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}}\right) &\cong {}_{(p,q)}\!\int_{A\in\mathcal{C}} \mathcal{D}\Big(-, D^{\underline{A}}_{\underline{A}}\Big) \\ \mathcal{D}\left({}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}}, -\right) &\cong {}_{(q,p)}\!\int_{A\in\mathcal{C}} \mathcal{D}\Big(D^{\underline{A}}_{\underline{A}}, -\Big). \end{aligned}$$

Theorem 3.1 (The Fubini Rule)**.** *Let* $D : \mathcal{A}^{(p,q)} \times \mathcal{B}^{(r,s)} \longrightarrow \mathcal{D}$ *be a functor. Then*

(1) $${}_{(p+r,q+s)}\!\int_{(A,B)} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}} \cong {}_{(p,q)}\!\int_{A}\,{}_{(r,s)}\!\int_{B} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}} \cong {}_{(r,s)}\!\int_{B}\,{}_{(p,q)}\!\int_{A} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}},$$

(2) $${}^{(p+r,q+s)}\!\int^{(A,B)} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}} \cong {}^{(p,q)}\!\int^{A}\,{}^{(r,s)}\!\int^{B} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}} \cong {}^{(r,s)}\!\int^{B}\,{}^{(p,q)}\!\int^{A} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}}$$

as objects of $\mathcal{D}$*, meaning that any of these expressions exist if and only if the others do, and, if so, they are are all canonically isomorphic.*

Remark 3 (Fubini does not reduce arity)**.** *Note that p, q, r, s can't be broken further: given a functor G of type $\left[\begin{smallmatrix} p \\ q \end{smallmatrix}\right]$, its (p,q)-end isn't in general expressible in terms of $(p-r, q-s)$-ends for suitable $r, s \geq 1$. This confirms the fact that iterated ends* are not *higher arity ends. Instead, higher arity ends are particular ends.*

That is, the Fubini rule does not allow us to reduce the arity of a higher arity co/end when $\mathcal{A} = \mathcal{B}$:

$$ {}_{(p,q)}\int_{A} {}_{(r,s)}\int_{B} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}} \cong {}_{(p+r,q+s)}\int_{(A,B)\in\mathcal{A}\times\mathcal{A}} D^{\boldsymbol{A},\boldsymbol{B}}_{\boldsymbol{A},\boldsymbol{B}} \ncong {}_{(p+r,q+s)}\int_{A\in\mathcal{C}} D^{\boldsymbol{A}}_{\boldsymbol{A}}. $$

This is already apparent from the classical Fubini rule, where, given a functor $T\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \times \mathcal{E}^{\mathrm{op}} \times \mathcal{E} \longrightarrow \mathcal{D}$ with $\mathcal{C} = \mathcal{E}$, we have once again

$$ \int_{(A,B)\in\mathcal{C}\times\mathcal{C}} T((A,B),(A,B)) \ncong \int_{A\in\mathcal{C}} T(A,A,A,A). $$

The main point in both cases is that we are integrating over a pair (A,B), and not over a single variable A.

From the point of view of adjoints, we have in (e.g.) the $(p,q) = (1,1)$ case

$$ (-) \odot \left(\mathsf{h}^{-_1}_{-_3} \times \mathsf{h}^{-_1}_{-_4} \times \mathsf{h}^{-_2}_{-_3} \times \mathsf{h}^{-_2}_{-_4}\right) \dashv {}^{(2,2)}\int^{A\in\mathcal{C}} D^{A,A}_{A,A} $$

$$ (-) \odot \underbrace{\mathsf{h}^{(-_1,-_2)}_{(-_3,-_4)}}_{\mathsf{h}^{-_1}_{-_3}\times\mathsf{h}^{-_2}_{-_4}} \dashv \int^{(A,B)\in\mathcal{C}\times\mathcal{C}} D^{(A,B)}_{(A,B)}, $$

and of course

$$ \mathsf{h}^{-_1}_{-_3} \times \mathsf{h}^{-_1}_{-_4} \times \mathsf{h}^{-_2}_{-_3} \times \mathsf{h}^{-_2}_{-_4} \neq \mathsf{h}^{(-_1,-_2)}_{(-_3,-_4)} = \mathsf{h}^{-_1}_{-_3} \times \mathsf{h}^{-_2}_{-_4}, $$

so $\int^{A\in\mathcal{C}} D^{A,A}_{A,A}$ and $\int^{(A,B)\in\mathcal{C}\times\mathcal{C}} D^{(A,B)}_{(A,B)}$ are different as well.

4. Examples:

4.1. **Some of them are trivial.**

Example 2 (Some (p,q)-co/ends are trivial for trivial reasons)**.**

- *The $(0,2)$-ends and $(0,2)$-coends of the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ that gives $\mathcal{C}$ a monoidal structure are trivial under very mild assumptions on $\mathcal{C}$. This rules out a class of possibly interesting examples coming from multilinear algebra.*

  Consider the category Mod_R (with additional care one can take left modules and right modules, of course). To show that

$$ {}_{(0,2)}\int_{A} A \otimes A \cong \lim_{A,B\in\mathsf{Mod}_R} A \otimes B \cong 0, $$

$$ {}^{(0,2)}\int^{A} A \otimes A \cong \operatorname{colim}_{A,B\in\mathsf{Mod}_R} A \otimes B \cong 0, $$

we just observe that Mod_R *is a sifted category, because it admits finite coproducts. The fact that a category* $\mathcal{C}$ *is sifted if and only if* $\mathcal{C}$ *is non-empty and the diagonal functor* $\Delta_{\mathcal{C}}\colon \mathcal{C} \longrightarrow \mathcal{C}\times\mathcal{C}$ *is final*[1] *then yields the result.*

- *If* $\mathcal{C}$ *is a sifted category, all diagonal functors* $\Delta : \mathcal{C} \to \mathcal{C}^n$ *are final, because the product and composition of final functors is itself final. Thus the same result transports to higher coends of higher arity functors: for example,* $\bigwedge^k : \mathsf{Mod}_R^n \to \mathsf{Mod}_R$ *sending* M *to* $\bigwedge^k M$*.*
- *Let* R *be a ring. The* walking cochain complex *(*[**?**, Paragraph 35.1]*) is the* Mod_R*-enriched category* Ch *whose set of objects is the set of integers, and where the hom-sets are* R*-modules freely generated by*

$$\mathsf{C}([m],[n]) = \begin{cases} \{d,0\} & \textit{if } m = n+1,\\ \{1,0\} & \textit{if } m = n,\\ \{0\} & \textit{otherwise.}\end{cases}$$

Now, a cochain complex is precisely a Mod_R*-enriched functor from the* Ch *to* Mod_R*. Similarly,* bi*complexes are* Mod_R*-enriched functors from* $\mathsf{Ch}\boxtimes_{\mathsf{Mod}_R}\mathsf{Ch}$ *to* Mod_R*.*

Let $D\colon \mathsf{Ch}\boxtimes_{\mathsf{Mod}_R}\mathsf{Ch} \longrightarrow \mathsf{Mod}_R$ *be a bicomplex. We claim that its* Mod_R*-enriched* $(2,0)$*-end* E *is just the zero module. Indeed, looking at* $(2,0)$*-wedges, we see that they are either of the form*

$$\begin{tikzcd} E \arrow[r] \arrow[d] & D^{n,n} \\ D^{n,n} \arrow[ur, equal] & \end{tikzcd} \qquad \textit{or of the form} \qquad \begin{tikzcd} E \arrow[r] \arrow[d] & D^{n+1,n+1} \\ D^{n,n} \arrow[ur, "\mathrm{d}^{n,n}"'] & \end{tikzcd}$$

Now, it follows from the second diagram that

$$E \cong \big\{(a_k)_{k\in\mathbb{Z}} \in \textstyle\prod_{k\in\mathbb{Z}} D^{k,k} \;\;|\;\; a_{k+1} = \mathrm{d}^{k,k}(a_k)\big\},$$

but differentials square to zero, so we must have $a_k = 0$ *for all* $k\in\mathbb{Z}$*, and* E *is indeed isomorphic to the zero module. A similar argument shows that* ${}^{(0,2)}\!\!\int^{[k]\in\mathsf{Ch}} D^{k,k} \cong 0$*.*

Example 3 (Bisimplicial sets)**.** *Recall that a* bisimplicial set *(*[**?**, Chapter IV], [**?**, §3.1.15]*) is a functor* $X : \Delta^{\mathrm{op}}\times\Delta \to \mathsf{Set}$*; moreover, the* diagonalisation *of a bisimplicial set* $X_{\bullet,\bullet}\colon \Delta^{\mathrm{op}}\times\Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ *is the simplicial set* $d(X)_\bullet\colon \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ *given by*

$$d(X)_n := X_{n,n}.$$

Joining the products and equalisers formula for (p,q)*-coends we see that* ${}^{(2,0)}\!\!\int^{[n]\in\Delta^{\mathrm{op}}} X_{n,n}$ *is the coequaliser of he diagram*

$$\coprod_{[n]\to[m]\in\Delta} X_{m,m}, \rightrightarrows \coprod_{[n]\in\Delta} X_{n,n}$$

[1] This is due to [**?**]; see also [**?**, Proposition 5.3.2] or [**?**, Theorem 2.15] for reviews.

giving

$$ {}^{(2,0)}\!\!\int^{[n]\in\Delta} X_{n,n} \cong \pi_0(d(X)). $$

By a similar argument, we have

$$ {}_{(2,0)}\!\!\int_{[n]\in\Delta} X_{n,n} \cong X_{0,0}. $$

4.2. **Juicy examples:**

4.2.1. *A glance at weighted co/ends.* Weighted co/ends stand to co/ends in the same relation as weighted co/limits stand to limits.

Definition 6 (Weighted co/end)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be $\mathcal{V}$-enriched categories and $D\colon \mathcal{C}^{\mathrm{op}} \otimes_{\mathcal{V}} \mathcal{C} \longrightarrow \mathcal{D}$ a $\mathcal{V}$-functor, and $W : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{V}$ a $\mathcal{V}$-presheaf.*

- WE1) *The* end of D weighted by W *is, if it exists, the object $\int^{W}_{A\in\mathcal{C}} D^A_A$ of $\mathcal{D}$ with the property that*

$$ \hom_{\mathcal{D}}\Big(X, \int^{W}_{A\in\mathcal{C}} D^A_A\Big) \cong \mathrm{DiNat}_{\mathcal{V}}(W, \mathbf{hom}_{\mathcal{C}}(X, D)) $$

naturally in $X \in \mathcal{D}$.

- WE2) *The* coend of D weighted by W *is, if it exists, the object $\int^{A\in\mathcal{C}}_{W} D^A_A$ of $\mathcal{D}$ with the property that*

$$ \hom_{\mathcal{D}}\Big(\int^{A\in\mathcal{C}}_{W} D^A_A, Y\Big) \cong \mathrm{DiNat}_{\mathcal{V}}(W, \mathbf{hom}_{\mathcal{C}}(D, Y)) $$

naturally in $Y \in \mathcal{D}$.

Example 4 (Weighted co/ends are $(2,2)$-co/ends)**.** *A quick argument (to be discussed in future work [**?**]) gives $(2,2)$-co/end formulas for weighted co/ends:*

$$ \int^{[W]}_{A\in\mathcal{C}} D^A_A \cong {}_{(2,2)}\!\!\int_{A\in\mathcal{C}} W^A_A \pitchfork D^A_A, $$
$$ \int^{A\in\mathcal{C}}_{[W]} D^A_A \cong {}^{(2,2)}\!\!\int^{A\in\mathcal{C}} W^A_A \odot D^A_A. $$

Example 5 (Weighting Increases Arity)**.** *Let $F, G\colon \mathcal{C} \longrightarrow \mathcal{D}$ and $W : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{V}$ be $\mathcal{V}$-functors. In analogy with*

$$ \mathbf{Nat}_{\mathcal{V}}(F, G) := \int_{A\in\mathcal{C}} \mathbf{hom}_{\mathcal{D}}(F_A, G_A), $$

we define the object $\mathsf{Nat}^{[W]}(F, G)$ of natural transformations from F to G weighted by W *by*

$$ \mathsf{Nat}^{[W]}(F, G) := \int^{[W]}_{A\in\mathcal{C}} \mathbf{hom}_{\mathcal{D}}(F_A, G_A). \tag{3} $$

Taking W to be mute in its contravariant variable, we can give a reformulation of the universal property of weighted limits:

$$\mathsf{h}\left(-, \lim^{W}(D)\right) \cong \mathsf{Nat}^{[W]}\left(\Delta_{(-)}, D\right).$$

Defining $\mathsf{DiNat}_{\mathcal{V}}^{[W]}(F, G)$ *by a similar formula, we also obtain the following isomorphism in the case of weighted ends:*

$$\mathsf{h}\left(-, \int_{A\in\mathcal{C}}^{[W]} D_A^A\right) \cong \mathsf{DiNat}_{\mathcal{V}}^{[W]}\left(\Delta_{(-)}, D\right).$$

This naturally suggests a definition of doubly-weighted ends*:*

$$\mathsf{h}\left(-, \int_{A\in\mathcal{C}}^{[W_1,W_2]} D_A^A\right) \cong \mathsf{DiNat}_{\mathcal{V}}^{[W_1]}(W_2, D).$$

*Repeating this process give you ends weighted by a collection of n functors $W_1, \ldots, W_n$. These however, can be actually computed as $(n+1, n+1)$-ends ([**?**]):*

$$\int_{A\in\mathcal{C}}^{[W_1,\ldots,W_n]} D_A^A \cong {}_{(n+1,n+1)}\!\int_{A\in\mathcal{C}} \left((W_1)_A^A \times \cdots \times (W_n)_A^A\right) \odot D_A^A.$$

As such, we see that weighting an end increases its arity by $(1, 1)$.

4.2.2. *Weighted Kan extensions.* Another source of examples comes from "weighing" left and right Kan extensions. While the most general such weight is a profunctor, having type $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, weights of type $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ or $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ are specially interesting, as they give a more direct parallel with the classical theory of weighted co/limits.

Recall the definition of the object $\mathsf{Nat}^{[W]}(F, G)$ of weighted natural transformations.

Definition 7. *The* left Kan extension of F along K weighted by W *is, if it exists, the $\mathcal{V}$-functor*

$$\left(\mathsf{Lan}_K^{[W]} F \colon \mathcal{D} \longrightarrow \mathcal{E}\right) : \qquad \begin{tikzcd} & & \mathcal{D} \arrow[d, dashed, "\mathsf{Lan}_K^{[W]} F"] \\ W \arrow[r, hook'] & \mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathcal{E} \end{tikzcd}$$

for which we have a $\mathcal{V}$-natural isomorphism

$$\mathbf{Nat}_{\mathcal{V}}\left(\mathsf{Lan}_K^{[W]} F, G\right) \cong \mathsf{Nat}_{\mathcal{V}}^{[W]}\left(F, G \circ K\right), \tag{4}$$

natural in G.

One defines weighted right Kan extensions in a dual manner:

Definition 8. *The* right Kan extension of F along K weighted by W *is, if it exists, the* $\mathcal{V}$*-functor*

$$\left(\mathsf{Ran}_K^{[W]}F\colon \mathcal{D}\longrightarrow\mathcal{E}\right):\qquad \begin{tikzcd} & & \mathcal{D} \arrow[d, dashed, "\mathsf{Ran}_K^{[W]}F"] \\ W \arrow[r, hook'] & \mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] \arrow[ur, phantom, shift left=3, "\Downarrow"] & \mathcal{E} \end{tikzcd}$$

for which we have a $\mathcal{V}$*-natural isomorphism*

$$\mathbf{Nat}_{\mathcal{V}}\left(G, \mathsf{Ran}_K^{[W]}F\right)\cong \mathsf{Nat}_{\mathcal{V}}^{[W]}\left(G\circ K, F\right), \tag{5}$$

natural in G*.*

Example 6 (Weighted co/limits as weighted Kan extensions)**.** *Let* $D\colon\mathcal{C}\longrightarrow\mathcal{D}$ *be a diagram on a category* $\mathcal{D}$*. Then we may canonically identify the left Kan extension of* D *along the terminal functor with its colimit:*

$$\mathsf{Lan}_!D\cong\lceil\mathrm{colim}(D)\rceil \qquad \begin{tikzcd} & 1 \arrow[d, dashed, "\lceil\mathrm{colim}(D)\rceil"] \\ \mathcal{C} \arrow[ur, "!"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathcal{D} \end{tikzcd}$$

Similarly, given a weight $W\colon\mathcal{C}^{\mathrm{op}}\longrightarrow\mathsf{Set}$*, we have*

$$\mathsf{Lan}_!^{[W]}D\cong\lceil\mathrm{colim}^W(D)\rceil \qquad \begin{tikzcd} & & 1 \arrow[d, dashed, "\lceil\mathrm{colim}^W(D)\rceil"] \\ W \arrow[r, hook'] & \mathcal{C} \arrow[ur, "!"] \arrow[r, "D"'] \arrow[ur, phantom, shift right=3, "\Uparrow"] & \mathcal{D} \end{tikzcd}$$

One can also prove that the following formulas hold ([**?**]):

$$\mathsf{Lan}_K^{[W]}F\cong\int_{[W]}^{A\in\mathcal{C}}\mathbf{hom}_{\mathcal{C}}\left(K_A,-\right)\odot F_A\cong {}^{(2,2)}\!\!\int^{A\in\mathcal{C}}\left(W_A^A\times\mathbf{hom}_{\mathcal{C}}(K_A,-)\right)\odot F_A, \tag{6}$$

$$\mathsf{Ran}_K^{[W]}F\cong\int^{[W]}_{A\in\mathcal{C}}\mathbf{hom}_{\mathcal{C}}\left(-,K_A\right)\pitchfork F_A\cong {}_{(2,2)}\!\!\int_{A\in\mathcal{C}}\left(W_A^A\times\mathbf{hom}_{\mathcal{C}}(-,K_A)\right)\pitchfork F_A. \tag{7}$$

Equipped with these, we now proceed to compute a few weighted Kan extensions.

Example 7. *Consider the functor* $\imath^{\mathrm{op}}:1^{\mathrm{op}}\to\Delta^{\mathrm{op}}$*; the left and right Kan extensions of a set* $X_\bullet\colon 1\longrightarrow\mathsf{Set}$ *along* $\imath^{\mathrm{op}}$ *are given by*

$$\mathsf{Lan}_{\imath^{\mathrm{op}}}(X)\cong\underline{X}_\bullet$$
$$\mathsf{Ran}_{\imath^{\mathrm{op}}}(X)\cong\check{C}(X).$$

Now take a weight $W\colon 1^{\mathrm{op}} \times 1 \longrightarrow \mathsf{Set}$*:*

$$\begin{tikzcd}
 & & \Delta^{\mathrm{op}} \arrow[d, dashed, "\mathsf{Lan}^{[W]}_{\imath^{\mathrm{op}}} X"] \\
W \curvearrowright & 1^{\mathrm{op}} \arrow[ur, "\imath^{\mathrm{op}}"] \arrow[r, "X"'] & \mathsf{Set}
\end{tikzcd}$$

Then

$$\begin{aligned}
\mathsf{Lan}^{[W]}_{\imath^{\mathrm{op}}}(X) &\cong \underline{W \times X}_{\bullet} \\
\mathsf{Ran}^{[W]}_{\imath^{\mathrm{op}}}(X) &\cong \check{C}(W \times X).
\end{aligned}$$

Example 8. *Now for the more interesting counterpart of the above:*

$$\begin{aligned}
\pi^{\mathrm{op}}\colon \Delta^{\mathrm{op}} &\longrightarrow 1^{\mathrm{op}} \\
[n] &\longmapsto \star
\end{aligned}$$

The left and right Kan extensions of a simplicial set $X_\bullet\colon \Delta^{\mathrm{op}} \longrightarrow \mathsf{Set}$ *along* π^{op} *are given by*

$$\begin{aligned}
\mathsf{Lan}_{\pi^{\mathrm{op}}}(X_\bullet) &\cong \pi_0(X_\bullet) \\
\mathsf{Ran}_{\pi^{\mathrm{op}}}(X_\bullet) &\cong \mathsf{ev}_0(X_\bullet) := X_0.
\end{aligned}$$

Now, a weight $W^\bullet_\bullet\colon \Delta^{\mathrm{op}} \times \Delta \longrightarrow \mathsf{Set}$ *is wonderfully complicated: it is a cosimplicial space!*

Then

1. *Taking* $W = \Delta^\bullet$ *almost gives the geometric realisation of* $X_\bullet$*:*

$$\mathsf{Lan}^{[\Delta^\bullet]}_{\pi^{\mathrm{op}}}(X_\bullet) \cong \int^{[n]\in\Delta} \Delta^n \times X_n.$$

2. *Dually, taking again* $W = \Delta^\bullet$ *but now a cosimplicial object* $X^\bullet\colon \Delta \longrightarrow \mathsf{Set}$*,*

$$\mathsf{Ran}^{[\Delta^\bullet]}_{\pi}(X^\bullet) = Tot(X_\bullet).$$

3. *If ou take* $W = \Delta^\bullet_\bullet = \hom_\Delta(-,-)$*, then I think you get*

$$\begin{aligned}
\mathsf{Lan}^{[\Delta^\bullet_\bullet]}_{\pi^{\mathrm{op}}}(X_\bullet) &\cong {}^{(2,2)}\!\!\int^{[n]\in\Delta} \Delta^n_n \times X_n \\
&\cong {}^{(2,2)}\!\!\int^{[n]\in\Delta} \Delta^n_n \times X_n
\end{aligned}$$

Example 9. *Using the fact that weighted left/right Kan extensions along the identity are adjoint to each other, we can study situations like*

$$\begin{tikzcd} & & \Delta^{\mathrm{op}} \arrow[d, dashed, "?"] \\ W^\bullet_\bullet \arrow[r, hook] & \Delta^{\mathrm{op}} \arrow[ur, equal, "1"] \arrow[r, "X_\bullet"'] & \mathsf{Set} \arrow[ul, phantom, "\Uparrow"] \end{tikzcd}$$

This gives rise to an adjunction $L : \mathit{sSet} \leftrightarrows \mathit{sSet} : R$ with

$$\begin{aligned} L(X_\bullet) &\cong {}_{(2,2)}\!\int_{[n]\in\Delta} W^n_n \odot X_n \cong {}_{(2,2)}\!\int_{[n]\in\Delta} W^n_n \times X_n, \\ R(X_\bullet) &\cong {}^{(2,2)}\!\int^{[n]\in\Delta} W^n_n \pitchfork X_n \cong {}^{(2,2)}\!\int^{[n]\in\Delta} [W^n_n, X_n]. \end{aligned}$$

Taking $W = \Delta^\bullet$ gives $L = R = 1$, so let's take something more complicated, like $\Delta^\bullet_\bullet$. Then

$$\begin{aligned} L(X_\bullet) &\cong {}_{(2,2)}\!\int_{[n]\in\Delta} \Delta^n[n] \times X_n \cong ? \\ R(X_\bullet) &\cong {}^{(2,2)}\!\int^{[n]\in\Delta} [\Delta^n[n], X_n] \cong ? \end{aligned}$$

Example 10 (Weighing the stalks of a sheaf)**.** *Let $i_p \colon \{p\} \hookrightarrow X$ be the inclusion of a point into a topological space X. We get an induced functor*

$$\begin{aligned} \mathcal{O}(i_p) \colon \mathcal{O}(X) &\longrightarrow \mathcal{O}(\{p\}) \\ U &\longmapsto i_p^{-1}(U) \end{aligned}$$

Considering now left Kan extensions along the opposite of $\mathcal{O}(i_p)$,

$$\begin{tikzcd} & \mathcal{O}(\{p\})^{\mathrm{op}} \arrow[d, dashed, "\mathsf{Lan}_{\mathcal{O}(i_p)^{\mathrm{op}}}\mathcal{F}"] \\ \mathcal{O}(X)^{\mathrm{op}} \arrow[ur, "\mathcal{O}(i_p)^{\mathrm{op}}"] \arrow[r, "\mathcal{F}"'] & \mathsf{Set} \arrow[ul, phantom, "\Uparrow"] \end{tikzcd}$$

we obtain a functor $\mathsf{Lan}_{\mathcal{O}(i_p)^{\mathrm{op}}} \colon \mathsf{PSh}(X) \longrightarrow \mathsf{PSh}(\{p\})$, whose image at $\mathcal{F}$ is written $\lceil \mathcal{F}_p \rceil$ for simplicity. The restriction of this functor to $\mathsf{Sh}(X)$ can be identified with the stalk functor $(-)_p \colon \mathsf{Sh}(X) \longrightarrow \mathsf{Set}$: we have $\mathcal{O}(\{p\}) = \{\varnothing \hookrightarrow \{p\}\}$ and computing the images of $\varnothing$ and $\{p\}$ under $\lceil \mathcal{F}_p \rceil$ via the *usual colimit formula for left Kan extensions gives*

$$\begin{aligned}
\lceil \mathcal{F}_p \rceil(\{p\}) &\cong \operatorname{colim}\left(\left(\mathcal{O}(\lceil p \rceil) \downarrow \underline{\{p\}}\right)^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}} \mathcal{O}(X)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathsf{Set}\right), \\
&\cong \operatorname{colim}_{U \ni p}(\mathcal{F}(U)), \\
&\cong \mathcal{F}_p \\
\lceil \mathcal{F}_p \rceil(\varnothing) &\cong \operatorname{colim}\left((\mathcal{O}(\lceil p \rceil) \downarrow \underline{\varnothing})^{\mathrm{op}} \xrightarrow{\pi^{\mathrm{op}}} \mathcal{O}(X)^{\mathrm{op}} \xrightarrow{\mathcal{F}} \mathsf{Set}\right), \\
&\cong \operatorname{colim}_{U \hookrightarrow \varnothing}(\mathcal{F}(U)), \\
&\cong \mathcal{F}(\varnothing).
\end{aligned}$$

(in case $\mathcal{F}$ is a sheaf, $\mathcal{F}(\varnothing)$ is the singleton set.) Consider the same situation, but now with a weight $W \colon \mathcal{O}(X) \times \mathcal{O}(X)^{\mathrm{op}} \longrightarrow \mathsf{Set}$ (an "extradiagonal presheaf on X"):

$$\begin{tikzcd}
 & & \mathcal{O}(\{p\})^{\mathrm{op}} \arrow[d, dashed, "\mathsf{Lan}^{[W]}_{\mathcal{O}(i_p)^{\mathrm{op}}}\mathcal{F}"] \\
W \arrow[r, bend left] & \mathcal{O}(X)^{\mathrm{op}} \arrow[ur, "\mathcal{O}(i_p)^{\mathrm{op}}"] \arrow[r, "\mathcal{F}"'] & \mathsf{Set} \arrow[u, phantom, shift right=4ex, "\Nearrow"]
\end{tikzcd}$$

We may compute $\mathsf{Lan}^{[W]}_{\mathcal{O}(i_p)^{\mathrm{op}}}\mathcal{F} := \lceil \mathcal{F}_p^{[W]} \rceil$ as the weighted coend

$$\lceil \mathcal{F}_p^{[W]} \rceil := \int_{[W]}^{U \in \mathcal{O}(X)} \hom_{\mathcal{O}(X)^{\mathrm{op}}}\left(\mathcal{O}(i_p^{\mathrm{op}})(U), -\right)$$

$odotmathcalF(U)$

$$\cong \int^{U \in \mathcal{O}(X)} W_U^U \times \hom_{\mathcal{O}(X)}\left(\chi_p(U), -\right)$$

$timesmathcalF(U),$

where

$$\chi_p(U) = \begin{cases} \emptyset & \textit{if } p \notin U, \\ U & \textit{otherwise.} \end{cases}$$

For instance, taking W to be a sheaf $\mathcal{G}$ on X gives

$$\mathcal{F}_p^{[\mathcal{G}]} := \lceil \mathcal{F}_p^{[\mathcal{G}]} \rceil(\{p\}) \cong (\mathcal{F} \times \mathcal{G})_p.$$

4.2.3. *A glance at extradiagonality.* "Extradiagonal" category theory arises when, instead of considering a natural transformation filling a higher-dimensional cell, we consider a *dinatural* one. Transformations that are more general than natural ones notoriously do not compose; yet, the category theory arising from this generalisation is interesting.

Definition 9 (Diagonal left Kan extensions)**.** *The* diagonal left Kan extension *of a functor $F \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ along a functor $K \colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ is, if* *it exists the functor* $\mathsf{DiLan}_K F\colon \mathcal{D} \longrightarrow \mathcal{E}$ *such that we have an isomorphism*

$$\mathit{Nat}(\mathsf{DiLan}_K F, G) \cong \mathrm{DiNat}(F, G \circ K) \qquad \begin{tikzcd} & \mathcal{D} \arrow[d, dashed, "\mathsf{DiLan}_K F"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] & \mathcal{E} \end{tikzcd}$$

natural in G.

Example 11. *Standard examples of diagonal left Kan extensions are ends: Generalising the fact that the left Kan extension of a functor* $D\colon \mathcal{C} \longrightarrow \mathcal{D}$ *along the terminal functor* $\pi\colon \mathcal{C} \twoheadrightarrow 1$ *can be identified with the colimit of* $\mathcal{D}$*, the diagonal left Kan extension of a functor* $D\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ *along the terminal functor* $\pi\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \twoheadrightarrow 1$ *can be identified with the coend of* $\mathcal{D}$*.*

$$\begin{tikzcd} & 1 \arrow[d, dashed, "\ulcorner \mathrm{colim}(D) \urcorner"] \\ \mathcal{C} \arrow[ur, "\pi"] \arrow[r, "D"'] & \mathcal{D} \end{tikzcd} \qquad \begin{tikzcd} & 1 \arrow[d, dashed, "\ulcorner \int^A D^A_A \urcorner"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, "\pi"] \arrow[r, "D"'] & \mathcal{D} \end{tikzcd}$$

Now, while ordinary Kan extensions can be computed via co/end formulas, diagonal Kan extensions admit $(2,2)$-co/end formulas ([**?**]):

$$\mathsf{DiLan}_K F \cong {}^{(2,2)}\!\!\int^{A\in\mathcal{C}} \mathcal{D}\left(K^A_A, -\right) \odot F^A_A, \tag{8}$$

$$\mathsf{DiRan}_K F \cong {}_{(2,2)}\!\!\int_{A\in\mathcal{C}} \mathcal{D}\left(-, K^A_A\right) \pitchfork F^A_A, \tag{9}$$

where the pairing is such that $\mathsf{DiLan}_K F$ is the coend of

$$(A, B) \mapsto \mathcal{D}\left(K^B_A, -\right) \odot F^A_B.$$

Alternatively, we may compute diagonal Kan extensions as hom-weighted Kan extensions ([**?**, **?**]):

$$\mathsf{DiLan}_K F \cong \int^{A,B\in\mathcal{C}}_{[\hom_{\mathcal{C}}(-,-)]} \mathcal{D}\left(K^B_A, -\right) \odot F^A_B,$$
$$\mathsf{DiRan}_K F \cong \int^{[\hom_{\mathcal{C}}(-,-)]}_{A,B\in\mathcal{C}} \mathcal{D}\left(-, K^B_A\right) \pitchfork F^A_B.$$

This is a generalisation of the fact that ends are hom-weighted limits. A forthcoming work [**?**] will address the topic of this remark in its entirety, studying the category theory arising from the notion of a weighted co/end.

Example 12. *Let $\mathcal{C}$ be a closed monoidal category and $D\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ be a diagram on $\mathcal{D}$. What is $\mathsf{DiLan}_{[-,-]}D$ and $\mathsf{DiRan}_{[-,-]}D$?*

$$\begin{tikzcd} & \mathcal{C} \arrow[d, dashed, "\mathsf{DiLan}_{[-,-]}D"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, "{[-,-]}"] \arrow[r, "D"'] & \mathcal{D} \end{tikzcd}$$

$$\mathsf{DiLan}_{[-,-]}D \cong \int^{A \in \mathcal{C}} \hom_{\mathcal{C}}([A, A], -) \odot D^A_A.$$

Example 13. *Let $D\colon \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \longrightarrow \mathcal{D}$ be a diagram on $\mathcal{D}$. What is $\mathsf{DiLan}_y D$ and $\mathsf{DiRan}_y D$?*

$$\begin{tikzcd} & \mathsf{PSh}(\mathcal{C}^{\mathrm{op}} \times \mathcal{C}) \arrow[d, dashed, "\mathsf{DiLan}_y D"] \\ \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, hook, "y"] \arrow[r, "D"'] & \mathcal{D} \end{tikzcd}$$

$$\begin{aligned} \mathsf{DiLan}_y D &\cong \int_{\hom_{\mathcal{C}}(-,-)}^{A,B \in \mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C})} \left(y^B_A, -\right) \odot F^A_B, \\ &\cong \int^{A \in \mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C})} \left(y^A_A, -\right) \odot F^A_A, \\ &\cong {}^{(2,2)}\!\!\int^{A \in \mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C})} \left(y^A_A, -\right) \odot F^A_A, \\ &:= {}^{(2,2)}\!\!\int^{A \in \mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C})} \left(\hom_{\mathcal{C}^{\mathrm{op}}\times\mathcal{C}}(-, (A, A)), -\right) \odot F^A_A, \\ &:= {}^{(2,2)}\!\!\int^{A \in \mathcal{C}} \hom_{\mathsf{PSh}(\mathcal{C}^{\mathrm{op}}\times\mathcal{C})} \left(\mathsf{h}^A \times \mathsf{h}_A, -\right) \odot F^A_A. \end{aligned}$$

In order to introduce the next example, we recall the following notation: we have an adjunction

$$(\pi \dashv \imath)\colon\ 1 \underset{\imath}{\overset{\pi}{\rightleftarrows}} \Delta,$$

where

- $\imath\colon 1 \hookrightarrow \Delta$ is the functor choosing the terminal object;
- $\pi\colon \Delta \twoheadrightarrow 1$ is the terminal functor;

This induces a quadruple adjunction

$$\left(\pi_0 \dashv \underline{(-)}_{\bullet} \dashv \mathsf{ev}_0 \dashv \check{C}\right):\ \mathsf{Set} \overset{\pi_0}{\rightleftarrows} \mathsf{sSet}$$

Example 14. *Let* $S^\bullet_\bullet \colon \Delta^{\mathrm{op}} \times \Delta \longrightarrow \mathsf{Set}$ *be a cosimplicial space. What is* $\mathsf{DiLan}_{\pi^{\mathrm{op}}\times\pi}(S^\bullet_\bullet)$*?*

$$\begin{tikzcd}
& 1^{\mathrm{op}} \times 1 \arrow[d, dashed, "\mathsf{DiLan}_{\pi^{\mathrm{op}}\times\pi}(S^\bullet_\bullet)"] \\
\Delta^{\mathrm{op}} \times \Delta \arrow[ur, "\pi^{\mathrm{op}}\times\pi"] \arrow[r, "S^\bullet_\bullet"'] & \mathsf{Set}
\end{tikzcd}$$

It is just the end of $S^\bullet_\bullet$ *(btw do you know what this is?):*

$$\begin{aligned}
\mathsf{DiLan}_{\pi^{\mathrm{op}}\times\pi}(S^\bullet_\bullet) &\cong \int^{[n]\in\Delta} \hom_1(\star,\star) \odot S^n_n, \\
&\cong \int^{[n]\in\Delta} S^n_n.
\end{aligned}$$

Similarly, given a set $X \colon 1^{\mathrm{op}} \times 1 \longrightarrow \mathsf{Set}$*, we have*

$$\begin{tikzcd}
& \Delta^{\mathrm{op}} \times \Delta \arrow[d, dashed, "\mathsf{DiLan}_{\imath^{\mathrm{op}}\times\imath}(X)"] \\
1^{\mathrm{op}} \times 1 \arrow[ur, hook, "\imath^{\mathrm{op}}\times\imath"] \arrow[r, "X"'] & \mathsf{Set}
\end{tikzcd}$$

$$\begin{aligned}
\mathsf{DiLan}_{\imath^{\mathrm{op}}\times\imath}(X) &\cong \int^{\star\in 1} \hom_{\Delta^{\mathrm{op}}\times\Delta}(([0],[0]),(-_1,-_2)) \odot X, \\
&\cong \hom_{\Delta^{\mathrm{op}}\times\Delta}(([0],[0]),(-_1,-_2)) \odot X \\
&\cong \hom_{\Delta}([0],-_2) \odot X \\
&\cong \Delta^{-_2}[0] \odot X.
\end{aligned}$$

Similarly, let $X^\bullet_\bullet \colon \Delta^{\mathrm{op}} \times \Delta \longrightarrow \mathsf{Set}$ *be a cosimplicial space again. What is* $\mathsf{DiLan}_{\Delta}(X^\bullet_\bullet)$*?*

$$\begin{tikzcd}
& \mathsf{Set} \arrow[d, dashed, "\mathsf{DiLan}_{\Delta}(X^\bullet_\bullet)"] \\
\Delta^{\mathrm{op}} \times \Delta \arrow[ur, "{\Delta^{-_2}[-_1]}"] \arrow[r, "X^\bullet_\bullet"'] & \mathsf{Set}
\end{tikzcd}$$

$$\mathsf{DiLan}_{\Delta}(X^\bullet_\bullet) \cong \int^{[n]\in\Delta} \mathsf{Set}(\Delta^n[n],-) \odot X^n_n.$$

4.2.4. *Weighted diagonal Kan extensions.* In the same spirit, one can define weighted diagonal Kan extensions, mixing the two perspectives and considering now the diagram

$$\begin{tikzcd} & & \mathcal{D} \arrow[d, dashed, "\mathsf{DiLan}_K F"] \\ W & \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \arrow[ur, "K"] \arrow[r, "F"'] & \mathcal{E} \end{tikzcd}$$

just to discover that these are actually computed as $(4,4)$-co/ends:

$$\mathsf{DiLan}_K^{[W]} F \cong {}^{(4,4)}\!\!\int^{A \in \mathcal{C}} \left(W_{A,A}^{A,A} \times \mathbf{hom}_{\mathcal{C}} \left(K_A^A, - \right) \right) \odot F_A^A,$$
$$\mathsf{DiRan}_K^{[W]} F \cong {}_{(4,4)}\!\!\int_{A \in \mathcal{C}} \left(W_{A,A}^{A,A} \times \mathbf{hom}_{\mathcal{C}} \left(-, K_A^A \right) \right) \pitchfork F_A^A.$$

At this point, the reader shall be convinced that the list of examples is virtually endless. We defer a thorough study of the topic to separate works [**?**, **?**].

4.2.5. *Daydreaming About Operads.* Day convolution was introduced by B. Day in [**?**, **?**], in order to classify monoidal structures on the category $\mathsf{PSh}(\mathcal{C})$ of presheaves on $\mathcal{C}$. Day proved that $\mathsf{PSh}(\mathcal{C})$ can be turned into a monoidal category in as many ways as $\mathcal{C}$ can be turned into a pseudomonoid in the bicategory of profunctors.

We now propose a generalisation of this framework based on higher arity coends: let $(\mathcal{C}, \otimes, I)$ be a monoidal category, and let $\mathcal{K} := \mathsf{PSh}(\mathcal{C})$. Higher arity Day convolution is defined as a family of functors $\circledast_n : \mathcal{K}^n \to \mathcal{K}$:

Definition 10. *The* ***Day*** (n,n)***-convolution*** *of an n-tuple of presheaves $\mathcal{F}_1, \dots, \mathcal{F}_n$ is the presheaf*

$$\circledast_n(\mathcal{F}_1, \dots, \mathcal{F}_n) \colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathsf{Set}$$

defined at $A \in \mathcal{C}_o$ as the (n,n)-coend

$$\circledast_n(\mathcal{F}_1, \dots, \mathcal{F}_n) := A \mapsto {}^{(n,n)}\!\!\int^{A \in \mathcal{C}} \mathcal{F}_1(A) \times \cdots \times \mathcal{F}_n(A) \times \mathcal{C}\left(-, A^{\otimes n}\right),$$

where $A^{\otimes n}$ is shorthand for the n-fold tensor product of A with itself.

Example 15 (Day convolution operad)**.** *The* ***Day convolution operad associated to*** $(\mathcal{C}, \otimes, I)$ *is the free symmetric operad* Day *whose set of generating operations (see* [**?**, Section 1.2.5]*) is given by* $\{1, \circledast_2, \circledast_3, \dots, \circledast_n, \dots\}$.

Remark 4. *We spell out in detail the first four sets of n-ary operations of* Day*:*

$$
\begin{aligned}
&\mathsf{Day}_1 = \{1\}\\
&\mathsf{Day}_2 = \{\circledast_2(-,-)\}\\
&\mathsf{Day}_3 = \{\circledast_3(-,-,-), \circledast_2(\circledast_2(-,-),-), \circledast_2(-,\circledast_2(-,-))\}\\
&\mathsf{Day}_4 = \{\circledast_4(-,-,-,-), \circledast_2(\circledast_3(-,-,-),-), \circledast_2(-,\circledast_3(-,-,-)), \circledast_2(\circledast_2(-,-),\circledast_2(-,-)),\\
&\qquad \circledast_3\,(-,\circledast_2(-,-),-), \circledast_3(\circledast_2(-,-),-,-), \circledast_3(-,-,\circledast_2(-,-))\}
\end{aligned}
$$

All in all, the set Day_n *can be succinctly described as*

$$\mathsf{Day}_n = \{\circledast_n\} \cup \sum_{p+q=n} \mathsf{Day}_p \times \mathsf{Day}_q$$

The operadic composition of Day *is now defined via 'grafting' in the usual way:*

$$
\begin{aligned}
\mathsf{Day}_n \times \mathsf{Day}_{k_1} \times \cdots \times \mathsf{Day}_{k_n} &\longrightarrow \mathsf{Day}_{\sum k_i}\\
(\theta;\theta_1,\dots,\theta_k) &\longmapsto \theta(\theta_1(-_1,\dots,-_{k_1}),\dots,\theta_k(-_1,\dots,-_{k_n}))
\end{aligned}
$$

5. Kusarigamas

Kusarigamas are functors of type $[\begin{smallmatrix} q\\ p\end{smallmatrix}]$ attached to a functor G of type $[\begin{smallmatrix} p\\ q\end{smallmatrix}]$, and enjoying a universal property among these. They are functors

$$
\begin{aligned}
\daleth^{p,q} &: \mathsf{Cat}(\mathcal{C}^{(p,q)},\mathcal{D}) \longrightarrow \mathsf{Cat}(\mathcal{C}^{(q,p)},\mathcal{D}),\\
\Gamma^{p,q} &: \mathsf{Cat}(\mathcal{C}^{(p,q)},\mathcal{D}) \longrightarrow \mathsf{Cat}(\mathcal{C}^{(q,p)},\mathcal{D}),
\end{aligned}
$$

that can be regarded as

- Universal objects among (p,q)-dinatural transformations, through which all other (p,q)-dinaturals factor:

$$\mathrm{DiNat}^{(p,q)}\,(F,G) \cong \mathsf{Nat}\,(F,\Gamma^{p,q}(G)) \cong \mathsf{Nat}\,(\daleth^{p,q}(F),G)\,;$$

- Functors that can be inductively defined through suitable Kan extensionsstarting from the case $[\begin{smallmatrix} 1\\ 1\end{smallmatrix}]$:

$$\Gamma^{p,q}(G) \cong \mathsf{Ran}_{\Delta^*_{p,q}G}\Gamma^{1,1}(G); \qquad \daleth^{p,q}(F) \cong \mathsf{Lan}_{\Delta^*_{p,q}F}\daleth^{1,1}(F).$$

The paramount property of the co/kusarigama functors is that

> given a category $\mathcal{C}$, the category of elements of $\daleth^{p,q}(1)$, where $1 : \mathcal{C}^{(p,q)} \to \mathsf{Set}$ is the terminal presheaf, is the universal fibration needed to build a higher-arity version of the *twisted arrow category* (i.e., the category of elements of $\hom_{\mathcal{C}}$).

This makes it possible to express the (p,q)-co/end of $G : \mathcal{C}^{(p,q)} \to \mathcal{D}$ as a co/limit over the (p,q)-twisted arrow category of $\mathcal{C}$:

$$
\begin{aligned}
{}_{(p,q)}\!\int_{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \lim\Big(\mathsf{Tw}^{(p,q)}(\mathcal{C}) \xrightarrow{\Sigma_{(p,q)}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big),\\
{}^{(p,q)}\!\int^{A\in\mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \operatorname{colim}\Big(\mathsf{Tw}^{(p,q)}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\Sigma_{(p,q)}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big).
\end{aligned}
$$

5.1. **Reducing dinaturality to naturality, and other properties.**

Construction 1 (Constructing co/kusarigamas)**.**

- *Suppose that $\mathcal{D}$ is cocomplete. Then*

$$
{}^{(p,q)}\!\int^{A\in\mathcal{C}} \left(\mathsf{h}^{-}_{\boldsymbol{A}_q} \times \mathsf{h}^{\boldsymbol{A}_p}_{-}\right) \odot F^{\boldsymbol{A}_p}_{\boldsymbol{A}_q}
$$

  meaning the (p,q)-coend of

$$
\begin{aligned}
\mathcal{C}^{(p,q)} &\longrightarrow \mathsf{Cat}(\mathcal{C}^{(q,p)},\mathcal{D})\\
(\underline{A},\underline{B}) &\longmapsto \hom_{\mathcal{C}^{(q,p)}}\left((\underline{B},\underline{A});(-,-)\right) \odot F^{\underline{A}}_{\underline{B}},
\end{aligned}
$$

  is the cokusarigama of F.

- *Suppose that $\mathcal{D}$ is complete. Then*

$$
{}_{(q,p)}\!\int_{A\in\mathcal{C}} \left(\mathsf{h}^{\boldsymbol{A}_p}_{-} \times \mathsf{h}^{-}_{\boldsymbol{A}_q}\right) \pitchfork G^{\boldsymbol{A}_q}_{\boldsymbol{A}_p},
$$

  meaning the (q,p)-coend of

$$
\begin{aligned}
\mathcal{C}^{(q,p)} &\longrightarrow \mathsf{Cat}(\mathcal{C}^{(p,q)},\mathcal{D})\\
(\underline{A},\underline{B}) &\longmapsto \hom_{\mathcal{C}^{(q,p)}}\left((\underline{A},\underline{B});(-,-)\right) \pitchfork G^{\underline{A}}_{\underline{B}},
\end{aligned}
$$

  is the kusarigama of G.

Explicitly,

$$
\begin{aligned}
\Gamma^{p,q}(G)(\underline{X},\underline{Y}) &\cong {}_{(q,p)}\!\int_{A\in\mathcal{C}} \left(\mathsf{h}^{A}_{X_1} \times \cdots \times \mathsf{h}^{A}_{X_p} \times \mathsf{h}^{Y_1}_{A} \times \cdots \times \mathsf{h}^{Y_q}_{A}\right) \pitchfork G^{A,\dots,A}_{A,\dots,A}\\
\daleth^{p,q}(F)(\underline{X},\underline{Y}) &\cong {}^{(q,p)}\!\int^{A\in\mathcal{C}} \left(\mathsf{h}^{A}_{X_1} \times \cdots \times \mathsf{h}^{A}_{X_p} \times \mathsf{h}^{Y_1}_{A} \times \cdots \times \mathsf{h}^{Y_q}_{A}\right) \odot F^{A,\dots,A}_{A,\dots,A}
\end{aligned}
$$

Proposition 3 (Properties of Co/kusarigamas)**.** *Let $D, F, G : \mathcal{C}^{(p,q)} \rightrightarrows \mathcal{D}$ be functors, where $\mathcal{D}$ is a bicomplete category.*

PK1) Adjointness. We have an adjunction

$$
\mathsf{Cat}(\mathcal{C}^{(p,q)},\mathcal{D}) \underset{\Gamma^{q,p}}{\overset{\daleth^{p,q}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} \mathsf{Cat}\left(\mathcal{C}^{(q,p)},\mathcal{D}\right).
$$

PK2) *Commutativity with homs. Let $F : \mathcal{C}^{(p,q)} \to \mathcal{D}$ be a functor, and let us consider the functors*

$$\mathcal{D}(F,1) : \mathcal{D} \to \mathsf{Cat}(\mathcal{C}^{(q,p)}, \mathit{Set}), D \mapsto \left((\underline{A},\underline{B}) \mapsto \mathcal{D}\left(F\tfrac{\underline{A}}{\underline{B}}, D\right)\right),$$

$$\mathcal{D}(1,F) : \mathcal{D}^{\mathrm{op}} \to \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathit{Set}), D \mapsto \left((\underline{A},\underline{B}) \mapsto \mathcal{D}\left(D, F\tfrac{\underline{A}}{\underline{B}}\right)\right),$$

then the diagrams

$$\begin{tikzcd} & \mathcal{D} \arrow[dl, "\mathcal{D}(\daleth^{p,q}(F),1)"'] \arrow[dr, "\mathcal{D}(F,1)"] & \\ \mathsf{Cat}(\mathcal{C}^{(q,p)}, \mathit{Set}) \arrow[rr, "\Gamma^{q,p}"'] & & \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathit{Set}) \end{tikzcd} \qquad \begin{tikzcd} & \mathcal{D} \arrow[dl, "\mathcal{D}(1,\Gamma^{p,q}(F))"'] \arrow[dr, "\mathcal{D}(1,F)"] & \\ \mathsf{Cat}(\mathcal{C}^{(p,q)}, \mathit{Set}) \arrow[rr, "\Gamma^{p,q}"'] & & \mathsf{Cat}(\mathcal{C}^{(q,p)}, \mathit{Set}) \end{tikzcd}$$

commute:

$$\mathcal{D}(\daleth^{p,q}(F),1) \cong \Gamma^{q,p}(\mathcal{D}(F,1)) \qquad \mathcal{D}(1,\Gamma^{p,q}(D)) \cong \Gamma^{p,q}(\mathcal{D}(1,D)).$$

PK3) *Limits of kusarigamas. We have functorial isomorphisms*

$$\int_{(p,q)\, \underline{A}\in\mathcal{C}} D\tfrac{\underline{A}}{\underline{A}} \cong \lim\left(\Gamma^{p,q}(D)\right), \qquad \int^{(p,q)\, \underline{A}\in\mathcal{C}} D\tfrac{\underline{A}}{\underline{A}} \cong \operatorname{colim}\left(\daleth^{q,p}(D)\right).$$

PK4) *Higher arity co/kusarigamas from $(1,1)$-co/kusarigamas. The cokusarigama*

$$\daleth^{p,q}(F) \colon \mathcal{C}^{(q,p)} \longrightarrow \mathcal{D}$$

of a functor $F \colon \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$ is the left Kan extension of the $(1,1)$-cokusarigama of $\Delta^*_{p,q}(F)$ along $\Delta_{q,p}$:

$$\daleth^{p,q}(F) = \mathsf{Lan}_{\Delta_{q,p}}\left(\daleth\big(\Delta^*_{p,q}(F)\big)\right) \qquad \begin{tikzcd} & \mathcal{C}^{(q,p)} \arrow[d, dashed, "\daleth^{p,q}(F)"] \\ \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \arrow[ur, "\Delta_{q,p}"] \arrow[r, "\daleth\left(\Delta^*_{p,q}(F)\right)"'] \arrow[ur, phantom, shift right=4, "\Uparrow"] & \mathcal{D}. \end{tikzcd}$$

Dually, the kusarigama

$$\Gamma^{q,p}(G) \colon \mathcal{C}^{(p,q)} \longrightarrow \mathcal{D}$$

of $G \colon \mathcal{C}^{(q,p)} \longrightarrow \mathcal{D}$ is the right Kan extension of the $(1,1)$-kusarigama of $\Delta^*_{q,p}(G)$ along $\Delta_{p,q}$:

$$\Gamma^{q,p}(G) = \mathsf{Ran}_{\Delta_{p,q}}\left(\Gamma\big(\Delta^*_{q,p}(G)\big)\right) \qquad \begin{tikzcd} & \mathcal{C}^{(p,q)} \arrow[d, dashed, "\Gamma^{q,p}(G)"] \\ \mathcal{C}^{\mathrm{op}}\times\mathcal{C} \arrow[ur, "\Delta_{p,q}"] \arrow[r, "\Gamma\left(\Delta^*_{q,p}(G)\right)"'] \arrow[ur, phantom, shift right=4, "\Downarrow"] & \mathcal{D}. \end{tikzcd}$$

5.2. **Higher arity twisted arrow categories.**

Definition 11. *The* (p,q)-twisted arrow category *is the category* $\mathsf{Tw}^{(p,q)}(\mathcal{C})$ *defined as the category of elements* $\mathcal{C}^{(q,p)} \int ⅃^{p,q}(1)$ *of* $⅃^{p,q}(1)$*:*

$$\begin{tikzcd} \mathsf{Tw}^{(p,q)}(\mathcal{C}) \arrow[r, two heads, "\Sigma_{(p,q)}"] \arrow[d, two heads] \arrow[dr, phantom, "\Nearrow"] & \mathcal{C}^{(p,q)} \arrow[d, "⅃^{p,q}(1)"] \\ 1 \arrow[r, hook, "1"'] & \mathit{Set}. \end{tikzcd}$$

It is well-known that this comma object (as all categories of elements) admits an equivalent description as the comma object

$$\begin{tikzcd} \mathsf{Tw}^{(p,q)}(\mathcal{C}) \arrow[r, two heads, "\Sigma_{(p,q)}"] \arrow[d, two heads] \arrow[dr, phantom, "\Nearrow"] & \mathcal{C}^{(p,q)} \arrow[d, "y_{\mathcal{C}^{(p,q)}}"] \\ 1 \arrow[r, hook, "\ulcorner ⅃^{p,q}(1) \urcorner"'] & \mathsf{PSh}(\mathcal{C}^{(p,q)}), \end{tikzcd}$$

where $y_{\mathcal{C}^{(p,q)}}$ is the Yoneda embedding and $\lceil ⅃^{p,q}(1) \rceil$ is the "name" of the functor $⅃^{p,q}(1)$ picking out the object $⅃^{p,q}(1)$ of $\mathsf{PSh}(\mathcal{C}^{(p,q)})$. All in all, this means that the (p,q)-twisted arrow category of $\mathcal{C}$ admits the following equivalent descriptions:

Proposition 4. $\mathsf{Tw}^{(p,q)}(\mathcal{C})$ *can be equivalently characterised as:*

PQT1) *The full-subcategory of* $\mathsf{PSh}(\mathcal{C})_{/⅃^{p,q}(1)}$ *spanned by representable presheaves, i.e. the category whose*
- *Objects are natural transformations of the form* $\mathsf{h}_{\boldsymbol{A}} \longrightarrow ⅃^{p,q}(1)$ *with* $\boldsymbol{A} \in \left(\mathcal{C}^{(p,q)}\right)_o$*;*
- *Morphisms are natural transformations* $\mathsf{h}_{\boldsymbol{f}} \colon \mathsf{h}_{\boldsymbol{A}} \longrightarrow \mathsf{h}_{\boldsymbol{B}}$ *such that the diagram*

$$\begin{tikzcd} \mathsf{h}_{\boldsymbol{A}} \arrow[rr, "\mathsf{h}_{\boldsymbol{f}}"] \arrow[dr] & & \mathsf{h}_{\boldsymbol{B}} \arrow[dl] \\ & ⅃^{p,q}(1) & \end{tikzcd}$$

commutes.

PQT2) *The category whose*
- *Objects are triples* $(\boldsymbol{X}, \boldsymbol{Y}, t)$ *where* $(\boldsymbol{X}, \boldsymbol{Y})$ *is an object of* $\mathcal{C}^{(p,q)}$*, and* t *is an element of* $⅃^{p,q}(1)^{\boldsymbol{X}}_{\boldsymbol{Y}}$*;*
- *Morphisms are "basepoint preserving" morphisms* $(\boldsymbol{X}, \boldsymbol{Y}) \to (\boldsymbol{X}', \boldsymbol{Y}')$*.*

PQT3) *The category whose*
- *Objects are collections* $\{f_{ij} \colon A_i \longrightarrow B_j\}$ *of morphisms of* $\mathcal{D}$ *with* $0 \leq i \leq p$ *and* $0 \leq j \leq q$*;*

- *Morphisms are collections of factorisations of the codomain through the domain, of the form*

$$\begin{tikzcd} A_i \arrow[r, "f"] & B_j \arrow[d, "\psi_j"] \\ A'_i \arrow[u, "\phi_i"] \arrow[r, "g"'] & B'_j, \end{tikzcd}$$

one for each $0 \leq i \leq p$ *and each* $0 \leq j \leq q$.

If $\mathcal{C}$ *has finite products and coproducts, we gain an additional equivalent description of* $\mathsf{Tw}^{(p,q)}(\mathcal{C})$*:*

TWD1) *The category whose*
- *Objects are morphisms* $A_1 \coprod \cdots \coprod A_p \longrightarrow B_1 \times \cdots \times B_q$*;*
- *Morphisms are factorisations of the codomain through the domain, of the form*

$$\begin{tikzcd} A_1 \coprod \cdots \coprod A_p \arrow[r, "f"] & B_1 \times \cdots \times B_q \arrow[d, "\psi_1 \times \cdots \times \psi_q"] \\ A'_1 \coprod \cdots \coprod A'_p \arrow[u, "\phi_1 \coprod \cdots \coprod \phi_p"] \arrow[r, "g"'] & B'_1 \times \cdots \times B'_q. \end{tikzcd}$$

From this,

$$\begin{aligned} {}^{(p,q)}\!\!\int_{A \in \mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \lim\Big(\mathsf{Tw}^{(p,q)}(\mathcal{C}) \xrightarrow{\Sigma_{(p,q)}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big), \\ {}^{(p,q)}\!\!\int^{A \in \mathcal{C}} D^{\underline{A}}_{\underline{A}} &\cong \operatorname{colim}\Big(\mathsf{Tw}^{(p,q)}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\Sigma_{(p,q)}} \mathcal{C}^{(p,q)} \xrightarrow{D} \mathcal{D}\Big). \end{aligned}$$

6. Future work (?)

6.1. weighing co/ends: the full story. This and that

6.2. kusarigamas are a toy example of "extradiagonal" (for lack of a better name) transformation. This and that

6.3. A graphical language for higher arity co/ends. This and that

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