Appendix A. 2-dimensional supernaturality.

We recall (see §**1.2**; see [**EK66**, **ML98**]) that given categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and functors $P\colon \mathcal{A}\times\mathcal{B}^{\mathrm{op}}\times\mathcal{B} \to \mathcal{D}$, $Q\colon \mathcal{A}\times\mathcal{C}^{\mathrm{op}}\times\mathcal{C} \to \mathcal{D}$, an *extranatural transformation* $\alpha\colon P \stackrel{..}{\to} Q$ consist of a collection of arrows in $\mathcal{D}$

$$\big\{\alpha_{abc}\colon P(a,b,b) \longrightarrow Q(a,c,c)\big\}$$

indexed by triples of object in $\mathcal{A}\times\mathcal{B}\times\mathcal{C}$ such that the following hexagonal diagram commutes for every morphism $f\colon a \to a'$, $g\colon b \to b'$, $h\colon c \to c'$:

$$\begin{tikzcd}
P(a,b',b) \arrow[r, "{P(f,b',g)}"] \arrow[d, "{P(a,g,b)}"'] & P(a',b',b') \arrow[r, "\alpha_{a'b'c}"] & Q(a',c,c) \arrow[d, "{Q(a',c,h)}"] \\
P(a,b,b) \arrow[r, "\alpha_{abc'}"'] & Q(a,c',c') \arrow[r, "{Q(f,h,c')}"'] & Q(a',c,c');
\end{tikzcd}$$

Notice how this commutative hexagon can be equivalently described as the juxtaposition of three distinguished commutative squares, depicted in [**EK66**]: the three can be obtained letting respectively f and h, f and g, or g and h be identities in the former diagram, which collapses from time to time to

$$\begin{tikzcd}
P(a,b,b) \arrow[r, "{P(f,b,b)}"] \arrow[d, "\theta_{abc}"'] & P(a',b,b) \arrow[d, "\theta_{a'bc}"] \\
Q(a,c,c) \arrow[r, "{Q(f,c,c)}"'] & Q(a',c,c)
\end{tikzcd}
\qquad
\begin{tikzcd}
P(a,b',b) \arrow[r, "{P(a,b',g)}"] \arrow[d, "{P(a,g,b)}"'] & P(a,b',b') \arrow[d, "\theta_{ab'c}"] \\
P(a,b,b) \arrow[r, "\theta_{abc}"'] & Q(a,c,c)
\end{tikzcd}
\qquad
\begin{tikzcd}
P(a,b,b) \arrow[r, "\theta_{abc}"] \arrow[d, "\theta_{abc'}"'] & Q(a,c,c) \arrow[d, "{Q(a,c,h)}"] \\
Q(a,c',c') \arrow[r, "{Q(a,h,c')}"'] & Q(a,c,c')
\end{tikzcd}$$

Now we are interested in a 2-dimensional enhancement of this definition, for 2-categories $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ and 2-functors $P\colon \mathcal{A}\times\mathcal{B}^{\mathrm{op}}\times\mathcal{B} \to \mathcal{D}$ and $Q\colon \mathcal{A}\times\mathcal{C}^{\mathrm{op}}\times\mathcal{C} \to \mathcal{D}$.

A tentative definition of a 2-co/wedge was in fact given in [**Boz77**], and a particular case of the following definition, having P or Q a constant 2-functor, was given in []; we've been unable to find additional relevant literature on the topic.

Lemma A.1 [The $*_F$ notation]**:** ...

Definition A.2 : A *lax extranatural transformation* between *strict* 2-functors $P\colon \mathcal{A}\times\mathcal{B}^{\mathrm{op}}\times\mathcal{B} \to \mathcal{D}$ and $Q\colon \mathcal{A}\times\mathcal{C}^{\mathrm{op}}\times\mathcal{C} \to \mathcal{D}$ consists of a family of 1-cells $\theta_{abc}\colon P(a,b,b) \to Q(a,c,c)$ and 2-cells θ_{fgh}, one for each $f\colon a \to a'$, $g\colon b \to b'$, $h\colon c \to c'$, filling the diagram

$$\begin{tikzcd}
P(a,b',b) \arrow[r, "{P(f,b',g)}"] \arrow[d, "{P(a,g,b)}"'] & P(a',b',b') \arrow[r, "\theta_{a'b'c}"] \arrow[d, phantom, "\theta_{fgh}\swarrow"] & Q(a',c,c) \arrow[d, "{Q(a',c,h)}"] \\
P(a,b,b) \arrow[r, "\theta_{abc'}"'] & Q(a,c',c') \arrow[r, "{Q(f,h,c')}"'] & Q(a',c,c');
\end{tikzcd}$$

and such that the following coherence conditions are satisfied:

(1) The diagram of 2-cells

$$\begin{tikzcd}
P(a,b,b) \arrow[r, bend left, "{P(f,b,b)}"] \arrow[r, bend right, "{P(f',b,b)}"'] \arrow[r, phantom, "\Downarrow P_\alpha"] \arrow[d, "\theta_{abc}"'] \arrow[dr, phantom, "\swarrow\theta_{f'11}"] & P(a',b,b) \arrow[d, "\theta_{a'bc}"] \\
Q(a,c,c) \arrow[r, "{Q(f',c,c)}"'] & Q(a',c,c)
\end{tikzcd}
=
\begin{tikzcd}
P(a,b,b) \arrow[r, "{P(f,b,b)}"] \arrow[d, "\theta_{abc}"'] \arrow[dr, phantom, "\swarrow\theta_{f11}"] & P(a',b,b) \arrow[d, "\theta_{a'bc}"] \\
Q(a,c,c) \arrow[r, bend left, "{Q(f,c,c)}"] \arrow[r, bend right, "{Q(f',c,c)}"'] \arrow[r, phantom, "\Downarrow Q_\alpha"] & Q(a',c,c)
\end{tikzcd}$$

commutes for each 2-cell $\alpha\colon f \Rightarrow f'$;

(2) The diagram of 2-cells

$$
\begin{tikzcd}
P(a,b',b) \arrow[r, bend left, "{P(a,b,g)}"] \arrow[r, bend right, "{P(a,b,g')}"'] \arrow[r, phantom, "\Downarrow P_\beta"] \arrow[d, "{P(a,b',g')}"'] \arrow[dr, phantom, "\swarrow \theta_{1g'1}"] & P(a,b',b') \arrow[d, "\theta_{ab'c}"] \\
P(a,b,b) \arrow[r, "\theta_{abc}"'] & Q(a,c,c)
\end{tikzcd}
=
\begin{tikzcd}
P(a,b',b) \arrow[r, "{P(a,b',g)}"] \arrow[d, bend right, "{P(a,g',b)}"'] \arrow[d, bend left, "{P(a,g,b)}"] \arrow[d, phantom, "\Leftarrow P_\beta"] \arrow[dr, phantom, "\swarrow \theta_{1g1}"] & P(a,b',b') \arrow[d, "\theta_{ab'c}"] \\
P(a,b,b) \arrow[r, "\theta_{abc}"'] & Q(a,c,c)
\end{tikzcd}
$$

commutes for each 2-cell $\beta\colon g \Rightarrow g'$;

(3) The diagram of 2-cells

$$
\begin{tikzcd}
P(a,b,b) \arrow[r, "\theta_{abc}"] \arrow[d, "\theta_{abc'}"'] \arrow[dr, phantom, "\swarrow \theta_{11h}"] & Q(a,c,c) \arrow[d, "{Q(a,c,h)}"] \\
Q(a,c',c') \arrow[r, bend left, "{Q(a,h,c')}"] \arrow[r, bend right, "{Q(a,h',c')}"'] \arrow[r, phantom, "\Downarrow Q_\gamma"] & Q(a,c,c')
\end{tikzcd}
=
\begin{tikzcd}
P(a,b,b) \arrow[r, "\theta_{abc}"] \arrow[d, "\theta_{abc'}"'] \arrow[dr, phantom, "\swarrow \theta_{11h'}"] & Q(a,c,c) \arrow[d, bend right, "{Q(a,c,h')}"'] \arrow[d, bend left, "{Q(a,c,h)}"] \arrow[d, phantom, "\Leftarrow Q_\gamma"] \\
Q(a,c',c') \arrow[r, "{Q(a,h',c')}"'] & Q(a,c,c')
\end{tikzcd}
$$

commutes for each 2-cell $\gamma\colon h \Rightarrow h'$;

(4) The 2-cell $\theta_{\mathrm{id},\mathrm{id},\mathrm{id}}$ coincides with the identity 2-cell of $\theta_{abc}\colon P(a,b,b) \to Q(a,c,c)$;

(5) There are compatibilities with the composition...

F says : Open questions:

- Are these conditions sufficient (guess: no, we miss coherence for composition, identities)?
- Is it necessary to take into account also $A^{\mathrm{co}}, B^{\mathrm{co}}$, etc.?
- Is it possible to package these three conditions in a single commutative diagram of 2-cells (hint: maybe, using the $\beta *_P \alpha$ notation)?

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