A CONSTRUCTION WITH YONEDA STRUCTURES

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ABSTRACT. In **Cat**, we call an object *μ-presentable* (for μ a regular cardinal) when $\hom(a, \_)$ commutes with μ-filtered colimits.

- A category $\mathcal{C}$ is *μ-accessible* when it has a small subcategory $\mathcal{G} \subseteq \mathcal{C}$ whose objects are all μ-presentable and such that every object of $\mathcal{C}$ is a μ-filtered colimit of objects in $\mathcal{G}$ (such a $\mathcal{G}$ is called a *generator* for $\mathcal{C}$).
- A category $\mathcal{C}$ is called *locally μ-presentable* if it is μ-accessible and co-complete (equivalently, μ-accessible and complete).

I want to reproduce these two definitions inside a 2-category $\mathcal{K}$ with a Yoneda structure. To do it, I take the definitions in **Cat** and try to rephrase them formally so that they become straighforwardly referred to the 0-,1- and 2-cells in $\mathcal{K}$.

Claim : There is a monad $T_{\mathcal{C},\mu}$ on $[\mathcal{C}, \mathbf{Set}]$ such that $T_{\mathcal{C}}(F) \cong F$ iff F commutes with μ-filtered colimits.

The unit of this monad induces by restriction a 2-cell

$$\begin{tikzcd}
\mathcal{C}^{\mathrm{op}} \arrow[r, "y"] \arrow[d, "y"'] \arrow[dr, phantom, "\alpha \swarrow"] & {[\mathcal{C}, \mathbf{Set}]} \\
{[\mathcal{C}, \mathbf{Set}]} \arrow[ur, "T_{\mathcal{C},\mu}"'] &
\end{tikzcd}$$

Now if $\mathcal{C}$ has a generator $\mathcal{G}$, composing with $j : \mathcal{G} \to \mathcal{C}$ one has $\alpha * j : Tyj(a) \cong yj(a) = \hom(a, \_)$, since all objects $\mathcal{G}$ are μ-presentable.

The fact that every object in $\mathcal{C}$ is a colimit of objects in $\mathcal{G}$ can be expressed by saying that j is dense: $\mathrm{Lan}_j j \cong 1$. So, it seems that

Claim : $\mathcal{C}$ is μ-accessible iff

- it has a dense generator $j : \mathcal{G} \to \mathcal{C}$,
- such that if $\alpha := \eta * y$ is the restriction of the unit of $T_{\mathcal{C},\mu}$ to representables, the whiskering $\alpha * j$ is invertible.

Is it possible to find T? If yes, we're really near to define accessible and locally presentabile objects in $\mathcal{K}$ with a YS: in fact, the definition of locally presentable is way easier: since $\mathcal{C}$ is locally presentable if it is a localization of a category of presheaves, it seems enough to say that $\mathcal{C}$ is locally presentable if it has a generator j such that $\mathcal{G}(j, 1)$ has a left adjoint and it is fully faithful.

I would like to know

- if this circle of ideas makes any sense at all; has anybody tried to do "the formal category theory of locally presentable and accessible objects"?
- if (and how) the existence of T can be verified, and when it exists in $\mathcal{K} \neq$ **Cat** what is its definition; I expect it to exist as a consequence of a specific property of $\mathcal{K}$ (or maybe even from a new axiom?)

- if the necessity to work with the *contravariant* Yoneda in

$$\begin{tikzcd} \mathcal{C}^{\mathrm{op}} \arrow[r, "y"] \arrow[d, "y"'] \arrow[dr, phantom, "\alpha \Swarrow"] & {[\mathcal{C}, \mathbf{Set}]} \\ {[\mathcal{C}, \mathbf{Set}]} \arrow[ur, "T_{\mathcal{C},\mu}"'] & \end{tikzcd}$$

  poses a problem: are we dealing with a... right Yoneda structure (ie with right extensions and liftings)? If yes, can the theorems about these dualized Yoneda structure be dualized in the straightforward way?