A CONSTRUCTION WITH YONEDA STRUCTURES

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Abstract. In Cat, we call an object μ -presentable (for μ a regular cardinal) when hom $(a, _)$ commutes with μ -filtered colimits.

- A category C is μ-accessible when it has a small subcategory S ⊆ C whose objects are all μ-presentable and such that every object of C is a μ-filtered colimit of objects in S (such a S is called a generator for C).
- A category C is called *locally μ-presentable* if it is μ-accessible and cocomplete (equivalently, μ-accessible and complete).

I want to reproduce these two definitions inside a 2-category $\mathcal K$ with a Yoneda structure. To do it, I take the definitions in \mathbf{Cat} and try to rephrase them formally so that they become straighforwardly referred to the 0-,1- and 2-cells in $\mathcal K$.

Claim: There is a monad $T_{\mathcal{C},\mu}$ on $[\mathcal{C},\mathbf{Set}]$ such that $T_{\mathcal{C}}(F)\cong F$ iff F commutes with μ -filtered colimits.

The unit of this monad induces by restriction a 2-cell

$$\begin{array}{c}
\mathbb{C}^{\mathrm{op}} \xrightarrow{y} [\mathbb{C}, \mathbf{Set}] \\
y \downarrow & T_{\mathbb{C},\mu}
\end{array}$$

$$[\mathbb{C}, \mathbf{Set}]$$

Now if \mathcal{C} has a generator \mathcal{G} , composing with $j:\mathcal{G}\to\mathcal{C}$ one has $\alpha*j:Tyj(a)\cong yj(a)=\hom(a,_)$, since all objects \mathcal{G} are μ -presentable.

The fact that every object in \mathcal{C} is a colimit of objects in \mathcal{G} can be expressed by saying that j is dense: Lan_i $j \cong 1$. So, it seems that

Claim : \mathcal{C} is μ -accessible iff

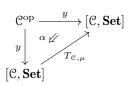
- it has a dense generator $j: \mathcal{G} \to \mathcal{C}$,
- such that if $\alpha := \eta * y$ is the restriction of the unit of $T_{\mathcal{C},\mu}$ to representables, the whiskering $\alpha * j$ is invertible.

Is it possible to find T? If yes, we're really near to define accessible and locally presentable objects in \mathcal{K} with a YS: in fact, the definition of locally presentable is way easier: since \mathcal{C} is locally presentable if it is a localization of a category of presheaves, it seems enough to say that \mathcal{C} is locally presentable if it has a generator j such that $\mathcal{G}(j,1)$ has a left adjoint and it is fully faithful.

I would like to know

- if this circle of ideas makes any sense at all; has anybody tried to do "the formal category theory of locally presentable and accessible objects"?
- if (and how) the existence of T can be verified, and when it exists in $\mathcal{K} \neq$ **Cat** what is its definition; I expect it to exist as a consequence of a specific property of \mathcal{K} (or maybe even from a new axiom?)

 \bullet if the necessity to work with the contravariant Yoneda in



poses a problem: are we dealing with a...right Yoneda structure (ie with right extensions and liftings)? If yes, can the theorems about these dualized Yoneda structure be dualized in the straightforward way?