A CATEGORICAL TRINITARISM

MOI

> Ὡσαύτως καὶ αἱ τρεῖς ἡμέραι πρὸ τῶν φωστήρων
> γεγονυῖαι τύποι εἰσὶν τῆς τριάδος, τοῦ θεοῦ καὶ
> τοῦ λόγου αὐτοῦ καὶ τῆς σοφίας αὐτοῦ. τετάρτῳ
> δὲ τόπῳ ἐστὶν ἄνθρωπος ὁ προσδεὴς τοῦ φωτός,
> ἵνα ᾖ θεός, λόγος, σοφία, ἄνθρωπος. διὰ τοῦτο
> καὶ τετάρτη ἡμέρᾳ ἐγενήθησαν φωστῆρες.

—————————————————————————

Theophilus of Antioch, To Autolycus, ɪɪ.xv

Every elementary categorical construction is subsumed by the notion of co/limit. This is not true any more for *enriched* categories, where a more general notion is needed to capture the intrisic 'two-dimensionality' of the theory.

The best we can do, in an enriched/higher setting, is to appeal the ᴋᴡᴄ *trinitarism* that defines Kan extension, weighted co/limits and co/ends. The plan of this first introductory lecture is to give a minimal amount of definitions in order to establish a language which will be useful later on.

These concepts (co/ends, especially) offer powerful computational tools. The daunting task of learning abstract homotopy theory will be relieved letting these machineries formally produce fairly involved results.

1. Kan extensions

Definition 1.1 : Let

$$(1) \qquad\qquad \mathbf{C} \xrightarrow{\ F\ } \mathbf{D} \atop {\scriptstyle G \downarrow} \atop \mathbf{E}$$

be a diagram of categories and functors; a *left Kan extension* of F along G is a functor, denoted $\mathrm{Lan}_G F$, endowed with a natural transformation $\eta\colon F \Rightarrow \mathrm{Lan}_G F \circ G$, initial among these natural transformations.

This means that whenever another natural transformation $\alpha\colon F \Rightarrow KG$ is given, there is a unique natural transformation $\bar{\alpha}\colon \operatorname{Lan}_G F \Rightarrow K$ such that $\alpha = \eta \circ \bar{\alpha}_G$.

The situation is depicted in the following diagram:

(2)

We are mainly interested in a slightly less general definition, though. A *pointwise* left Kan extension is a left Kan extension $\operatorname{Lan}_G F$ that is preserved by every representable functor. A rather common situation where a pointwise left Kan extension exists is when the codomain $\mathbf{D}$ of F is cocomplete. In that case not only we find $\operatorname{Lan}_G F$ for *every* G, but we are also able to characterize it with a more intuitive universal property: the Kan extensions can be constructed, over each object (a point, in some sense) of the domain category, using a suitable colimit.

Proposition 1.2 : Let $G^*\colon [\mathbf{C}, \mathbf{D}] \to [\mathbf{E}, \mathbf{D}]$ be the functor induced by precomposition with G: then $\operatorname{Lan}_G F$ is the functor that acts, on objects and morphisms, as

$$(3) \qquad e \mapsto \varinjlim_{(G \downarrow e)} Fc,$$

where $(G \downarrow e)$ is the *comma category* of arrows $Gc \to e$ and morphisms $c \to c'$ such that the obvious triangle commutes.

Moreover, the correspondence $\operatorname{Lan}_G(\_)\colon [\mathbf{C}, \mathbf{D}] \leftarrow [\mathbf{E}, \mathbf{D}]$ that sends $F \mapsto \operatorname{Lan}_G F$ is the left adjoint of G^* that sends H to HG.

Proof. A slightly more formal rephrasing of the above isomorphism is the following: consider the composition of functors

$$(4) \qquad (G \downarrow e) \xrightarrow{\Sigma} \mathbf{C} \xrightarrow{F} \mathbf{D}$$

then the colimit

$$(5) \qquad \varinjlim_{(c,\alpha\colon Gc \to e)\in(G \downarrow e)} F(\Sigma(c, \alpha))$$

of this diagram on $\mathbf{D}$ is precisely the value of $\operatorname{Lan}_G F(e)$ which is then computed *pointwise*.

To prove the statement,we build an explicit natural isomorphism

$$(6) \qquad \mathsf{Nat}(F, HG) \underset{\Phi}{\overset{\Psi}{\rightleftharpoons}} \mathsf{Nat}(\mathrm{Lan}_G F, H).$$

First of all we define a map Φ, sending $\eta\colon F \to HG$ to the cocone with components

$$(7) \qquad \hat{\eta}_{c,\alpha}\colon Fc \xrightarrow{\quad \eta_c \quad} HGc \xrightarrow{\quad H\alpha \quad} Hc$$

This induces (via the universal property of the colimit) a unique map

$$(8) \qquad \varinjlim_{(G\downarrow e)} Fc \xrightarrow{\quad \Phi(\eta)_e \quad} He$$

which is a natural transformation in e, since the square

$$(9) \qquad \begin{array}{ccc} \varinjlim_{(G\downarrow e)} Fc & \longrightarrow & He \\ \downarrow & & \downarrow \\ \varinjlim_{(G\downarrow e)} Fc & \longrightarrow & He' \end{array}$$

commutes for $f\colon e \to e'$.

On the other hand we define Ψ taking the components $\beta_e\colon \beta\colon \varinjlim_{(G\downarrow e)} Fc \to He$ and considering the compositions

$$(10) \qquad \begin{array}{ccc} \varinjlim_{(G\downarrow x)} Fc & \longrightarrow & HGx \\ {\scriptstyle \iota_{(x,1_{Gx})}}\uparrow & & \\ Fx & & \\ & & HGy \\ \downarrow & & \uparrow \\ Fy & \xrightarrow{\scriptstyle \iota_{(y,1_{Gy})}} & \varinjlim_{(G\downarrow y)} Fc \end{array}$$

The inner square commutes, and proves naturality in $x \to y$; the reason is that it splits into two sub-diagrams, each of which commutes. $\qquad \square$

Exercise 1.3 : Dualize; define right Kan extensions $\mathrm{Ran}_G F$ and their universal property (how is the direction of 1-cells affected by the dualization?); define pointwise right Kan extensions and show a similar adjunction formula

$$\text{(11)} \qquad \mathsf{Nat}(HG, F) \cong \mathsf{Nat}(H, \mathrm{Ran}_G F).$$

1.1. **Examples.**

Example 1.4 : Let $H \leq G$ a group inclusion, regarded as a functor. The category of (left) G-sets is identified with the category of functors $[G, \mathbf{Sets}]$.

There exist then functors

$$\text{(12)} \qquad npoiapofa$$

given by Kan extension. Their action on objects and morphisms is given by

$$\text{(13)} \qquad fdsnpaia$$

The following is a standard statement that is often subsumed by the sentence 'any functor F defined on presheaves on $\mathbf{C}$ is uniquely determined by its action on representables'.

Example 1.5 : Let $F : \mathbf{C} \to \mathbf{D}$ be a functor from a *small* to a *cocomplete* category; then there is a unique way to define a functor $\bar{F} : [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}] \to \mathbf{D}$ that commutes with colimits, and that has a right adjoint N_F sending an object $d \in \mathbf{D}$ into the functor $c \mapsto \hom(Fc, d)$.

This left adjoint sends a presheaf P into the object $\mathrm{Lan}_y F(P)$ in $\mathbf{D}$.

Exercise 1.6 : Show that $\mathrm{Lan}_1 F \cong F$ and $\mathrm{Ran}_1 F \cong F$, for every functor $F \colon \mathbf{A} \to \mathbf{B}$ (use the universal property).

Exercise 1.7 : Show that if $F \colon \mathbf{A} \to \mathbf{B}$, $\mathbf{A}$ is small, F is cocontinuous and $\mathbf{B}$ is cocomplete, then $\mathrm{Lan}_F 1$ (where $1 \colon \mathbf{B} \to \mathbf{B}$ is the identity functor) is a left adjoint for F.

Exercise 1.8 : Use the universal property to show that $G \mapsto \mathrm{Lan}_G$ is a pseudofunctor, by showing that for $\mathbf{A} \xrightarrow{F} \mathbf{B}, \mathbf{A} \xrightarrow{G} \mathbf{C} \xrightarrow{H} \mathbf{D}$ there is a uniquely determined laxity cell for composition

$$\text{(14)} \qquad \mathrm{Lan}_H(\mathrm{Lan}_G F) \cong \mathrm{Lan}_{H \circ G} F.$$

Dualize for right Kan extensions.

2. Weighted co/limits

The notion of weighted colimit generalizes the notion of colimit to an enriched setting, and provides a fairly formal approach to the issues posed by abstract homotopy theory; it is noticeable that the notion of *homotopy colimit*[1] is captured, under mild assumptions on the model category, by a weighted colimit construction. Thus, the machinery produces a fairly explicit approach to the matter.

When looking for a classical (from now on, 'conical') colimit of a diagram $F \colon J \to \mathbf{C}$, we basically are looking for a representative for the functor $a \mapsto \mathrm{Cocones}(F, a) \cong \mathsf{Nat}(F, \Delta a)$, where $\Delta \colon \mathbf{C} \to \mathbf{C}^J$ is the 'constant diagram' functor.

The latter object, that uniquely determines the object $\varinjlim F$, can be rewritten as

$$(15) \qquad\qquad \mathsf{Nat}(*, \mathrm{hom}(F, a))$$

where now $\mathrm{hom}(F, a)$ is the functor $x \mapsto \mathrm{hom}(Fx, a)$ and $*$ is the terminal presheaf: then

$$(16) \qquad\qquad \mathrm{hom}(\varinjlim F, a) \cong \mathsf{Nat}(*, \mathrm{hom}(F, a)).$$

A weighted colimit arises answering the following simple question: what happens if we replace the terminal presheaf $*$ with a more general $W \colon \mathbf{X}^{\mathrm{op}} \to \mathbf{Sets}$, and look for a representative of the functor $a \mapsto \mathsf{Nat}(W, \mathrm{hom}(F, a))$?

If we were able to solve this universal problem, we would obtain a natural isomorphism

$$(17) \qquad\qquad \mathrm{hom}(\varinjlim{}^W F, a) \cong \mathsf{Nat}(W, \mathrm{hom}(F, a)).$$

for an object $\varinjlim{}^W F$ that we call the colimit of F weighted by W.

2.1. **Examples.**

3. Co/ends

---

[1]A homotopy colimit is a construction in a model category –not a real colimit– that corrects the fact that the functor $\varinjlim$ does not preserve weak equivalences.