A CATEGORICAL TRINITARISM

MOI

> Ὡσαύτως καὶ αἱ τρεῖς ἡμέραι πρὸ τῶν φωστήρων
> γεγονυῖαι τύποι εἰσὶν τῆς τριάδος, τοῦ θεοῦ καὶ
> τοῦ λόγου αὐτοῦ καὶ τῆς σοφίας αὐτοῦ. τετάρτῳ
> δὲ τόπῳ ἐστὶν ἄνθρωπος ὁ προσδεὴς τοῦ φωτός,
> ἵνα ᾖ θεός, λόγος, σοφία, ἄνθρωπος. διὰ τοῦτο
> καὶ τετάρτη ἡμέρᾳ ἐγενήθησαν φωστῆρες.

<div align="right">Theophilus of Antioch, To Autolycus, II.XV</div>

Every elementary categorical construction is subsumed by the notion of co/limit. This is not true any more for *enriched* categories, where a more general notion is needed to capture the intrisic 'two-dimensionality' of the theory.

The best we can do, in an enriched/higher setting, is to appeal the KWC *trinitarism* that defines Kan extension, weighted co/limits and co/ends. The plan of this first introductory lecture is to give a minimal amount of definitions in order to establish a language which will be useful later on.

These concepts (co/ends, especially) offer powerful computational tools. The daunting task of learning abstract homotopy theory will be relieved letting these machineries formally produce fairly involved results.

1. Kan extensions

Definition 1.1 : Let

$$(1) \qquad \begin{array}{ccc} \mathbf{C} & \xrightarrow{\ F\ } & \mathbf{D} \\ {\scriptstyle G}\downarrow & & \\ \mathbf{E} & & \end{array}$$

be a diagram of categories and functors; a *left Kan extension* of F along G is a functor, denoted $\mathrm{Lan}_G F$, endowed with a natural transformation $\eta\colon F \Rightarrow \mathrm{Lan}_G F \circ G$, initial among these natural transformations.

This means that whenever another natural transformation $\alpha\colon F \Rightarrow KG$ is given, there is a unique natural transformation $\bar{\alpha}\colon \operatorname{Lan}_G F \Rightarrow K$ such that $\alpha = \eta \circ \bar{\alpha}_G$.

The situation is depicted in the following diagram:

(2)

We are mainly interested in a slightly less general definition, though. A *pointwise* left Kan extension is a left Kan extension $\operatorname{Lan}_G F$ that is preserved by every representable functor. A rather common situation where a pointwise left Kan extension exists is when the codomain $\mathbf{D}$ of F is cocomplete. In that case not only we find $\operatorname{Lan}_G F$ for *every* G, but we are also able to characterize it with a more intuitive universal property: the Kan extensions can be constructed, over each object (a point, in some sense) of the domain category, using a suitable colimit.

Proposition 1.2 : Let $G^*\colon [\mathbf{C}, \mathbf{D}] \to [\mathbf{E}, \mathbf{D}]$ be the functor induced by precomposition with G: then $\operatorname{Lan}_G F$ is the functor that acts, on objects and morphisms, as

(3)
$$e \mapsto \varinjlim_{(G \downarrow e)} Fc,$$

where $(G \downarrow e)$ is the *comma category* of arrows $Gc \to e$ and morphisms $c \to c'$ such that the obvious triangle commutes.

Moreover, the correspondence $\operatorname{Lan}_G(\_)\colon [\mathbf{C}, \mathbf{D}] \leftarrow [\mathbf{E}, \mathbf{D}]$ that sends $F \mapsto \operatorname{Lan}_G F$ is the left adjoint of G^* that sends H to HG.

Proof. A slightly more formal rephrasing of the above isomorphism is the following: consider the composition of functors

(4)
$$(G \downarrow e) \xrightarrow{\Sigma} \mathbf{C} \xrightarrow{F} \mathbf{D}$$

then the colimit

(5)
$$\varinjlim_{(c,\alpha\colon Gc \to e) \in (G \downarrow e)} F(\Sigma(c, \alpha))$$

of this diagram on $\mathbf{D}$ is precisely the value of $\operatorname{Lan}_G F(e)$ which is then computed *pointwise*.

To prove the statement,we build an explicit natural isomorphism

$$(6) \qquad \mathsf{Nat}(F, HG) \overset{\Psi}{\underset{\Phi}{\leftrightarrows}} \mathsf{Nat}(\mathrm{Lan}_G F, H).$$

First of all we define a map Φ, sending $\eta\colon F \to HG$ to the cocone with components

$$(7) \qquad \hat{\eta}_{c,\alpha}\colon Fc \xrightarrow{\;\eta_c\;} HGc \xrightarrow{\;H\alpha\;} Hc$$

This induces (via the universal property of the colimit) a unique map

$$(8) \qquad \varinjlim_{(G\downarrow e)} Fc \xrightarrow{\;\;\Phi(\eta)_e\;\;} He$$

which is a natural transformation in e, since the square

$$(9) \qquad \begin{array}{ccc} \varinjlim\limits_{(G\downarrow e)} Fc & \longrightarrow & He \\ \downarrow & & \downarrow \\ \varinjlim\limits_{(G\downarrow e')} Fc & \longrightarrow & He' \end{array}$$

commutes for $f\colon e \to e'$.

On the other hand we define Ψ taking the components $\beta_e\colon \beta\colon \varinjlim_{(G\downarrow e)} Fc \to He$ and considering the compositions

$$(10) \qquad \begin{array}{ccccc} Fx & \xrightarrow{\iota_{(x,1_{Gx})}} & \varinjlim\limits_{(G\downarrow x)} Fc & \longrightarrow & HGx \\ \downarrow & & \downarrow & & \downarrow \\ Fy & \xrightarrow[\iota_{(y,1_{Gy})}]{} & \varinjlim\limits_{(G\downarrow y)} Fc & \longrightarrow & HGy \end{array}$$

The inner square commutes, and proves naturality in $x \to y$; the reason is that it splits into two sub-diagrams, each of which commutes. $\qquad\square$

Exercise 1.3 : Dualize; define right Kan extensions $\mathrm{Ran}_G F$ and their universal property (how is the direction of 1-cells affected by the dualization?); define pointwise right Kan extensions and show a similar adjunction formula

$$(11) \qquad \mathsf{Nat}(HG, F) \cong \mathsf{Nat}(H, \mathrm{Ran}_G F).$$

1.1. **Examples.**

Example 1.4 : Let $\iota\colon H \leq G$ a group inclusion, regarded as a functor. The category of (left) G-sets is identified with the category of functors $[G, \mathbf{Vect}_k]$.

There exist then functors

$$(12) \qquad [G, \mathbf{Vect}_k] \xrightleftharpoons[\mathrm{Lan}_\iota]{\overset{\mathrm{Ran}_\iota}{\longleftarrow}\;\iota^*\;\longrightarrow} [H, \mathbf{Vect}_k]$$

given by Kan extension. Their action on objects and morphisms is given by

$$V \mapsto k[G] \otimes_{k[H]} V$$
$$W \mapsto \mathrm{hom}_{k[H]}(k[G], W)$$

(the group algebra $k[G]$ is the free k-algebra on group elements of G, with multiplication $\sum_g a_g \cdot \sum_{g'} b_{g'} = \sum_h \sum_{gg'=h} a_g b_{g'}$, and similarly for H).

Exercise 1.5 : Prove the above statement, with the help of the following characterization for the two functors above:

- There is an equalizer

$$(13) \qquad \mathrm{hom}_{k[H]}(k[G], W) \longrightarrow \mathrm{hom}_k(k[G], V) \underset{\mathrm{hom}(h,1)}{\overset{\mathrm{hom}(1,h)}{\rightrightarrows}} \mathrm{hom}_k(k[G], V)$$

- There is a coequalizer

$$(14) \qquad \bigoplus_{h \in H} k[G] \otimes_k V \underset{h \otimes 1}{\overset{1 \otimes h}{\rightrightarrows}} k[G] \otimes_k V \longrightarrow k[G] \otimes_{k[H]} V.$$

The following is a standard statement that is often subsumed by the sentence 'any functor F defined on presheaves on $\mathbf{C}$ is uniquely determined by its action on representables'.

Example 1.6 : Let $F : \mathbf{C} \to \mathbf{D}$ be a functor from a *small* to a *cocomplete* category; then there is a unique way to define a functor $\bar{F} : [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}] \to \mathbf{D}$ that commutes with colimits, and that has a right adjoint N_F sending an object $d \in \mathbf{D}$ into the functor $c \mapsto \mathrm{hom}(Fc, d)$.

This left adjoint sends a presheaf P into the object $\mathrm{Lan}_y F(P)$ in $\mathbf{D}$.

Proof. It is enough to show that a natural transformation $\alpha\colon P \to N_F(d)$ determines a cocone for $(y \downarrow P) \to \mathbf{C} \xrightarrow{F} \mathbf{D}$: but from the commutativity of the square

(15)
$$
\begin{array}{ccc}
c & Pc \longrightarrow \mathrm{hom}(Fc, d) \\
\varphi\downarrow \quad P\varphi\uparrow \qquad\qquad \uparrow{\scriptstyle\mathrm{hom}(F\varphi,d)} \\
c' & Pc' \longrightarrow \mathrm{hom}(Fc', d)
\end{array}
$$

it follows that the triangle

(16)
$$
\begin{array}{c}
Fc \xrightarrow{\quad F\varphi \quad} Fc' \\
{\scriptstyle\alpha}\searrow \qquad \swarrow{\scriptstyle\alpha'} \\
d
\end{array}
$$

commutes, and then sending $(c, x \in Pc)$ into $\alpha_c(x)\colon Fc \to d$ determines a cocone for FU. This concludes the proof, once we show that this is a bijection; of course it is. $\qquad\square$

Exercise 1.7 : Show that the above construction determines an equivalence of categories

(17)
$$
\mathbf{Cat}(\mathbf{C}, \mathbf{D}) \xrightarrow{\qquad \mathrm{Yan} \qquad} \mathbf{Adj}_{\mathrm{L}}([\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}], \mathbf{D})
$$

if morphisms on the right are natural transformations between lef adjoints (hint: simply show that the adjunction $\mathrm{Lan}_y \dashv y^*$ is an equivalence of categories).

Exercise 1.8 : Shape a dual statement regarding a universal property of the contravariant Yoneda embedding $\mathbf{C}^{\mathrm{op}} \xrightarrow{y} [\mathbf{C}, \mathbf{Sets}]^{\mathrm{op}}$. Is it really an independent theorem?

Exercise 1.9 : Show that $\mathrm{Lan}_1 F \cong F$ and $\mathrm{Ran}_1 F \cong F$, for every functor $F\colon \mathbf{A} \to \mathbf{B}$ (use the universal property).

Exercise 1.10 : Show that if $F\colon \mathbf{A} \to \mathbf{B}$ is a functor between small categories, then the following conditions are equivalent:
- F has a right adjoint;
- $\mathrm{Lan}_F 1$ exists and there is an isomorphism $\mathrm{Lan}_F F \cong F$;
- $\mathrm{Lan}_F 1$ exists and there is an isomorphism $\mathrm{Lan}_F L \cong L \circ \mathrm{Lan}_F 1$ for any functor $L\colon \mathbf{A} \to \mathbf{C}$.

Exercise 1.11 : Use the universal property to show that $G \mapsto \mathrm{Lan}_G$ is a pseudofunctor, by showing that for $\mathbf{A} \xrightarrow{F} \mathbf{B}, \mathbf{A} \xrightarrow{G} \mathbf{C} \xrightarrow{H} \mathbf{D}$ there is a uniquely determined laxity cell for composition

$$(18) \qquad \mathrm{Lan}_H(\mathrm{Lan}_G F) \cong \mathrm{Lan}_{H \circ G} F.$$

Dualize for right Kan extensions.

2. Weighted co/limits

The notion of weighted colimit generalizes the notion of colimit to an enriched setting, and provides a fairly formal approach to the issues posed by abstract homotopy theory; it is noticeable that the notion of *homotopy colimit*[1] is captured, under mild assumptions on the model category, by a weighted colimit construction. Thus, the machinery produces a fairly explicit approach to the matter.

When looking for a classical (from now on, 'conical') colimit of a diagram $F \colon J \to \mathbf{C}$, we basically are looking for a representative for the functor $a \mapsto \mathsf{Cocones}(F, a) \cong \mathsf{Nat}(F, \Delta a)$, where $\Delta \colon \mathbf{C} \to \mathbf{C}^J$ is the 'constant diagram' functor.

The latter object, that uniquely determines the object $\varinjlim F$, can be rewritten as

$$(19) \qquad \mathsf{Nat}(*, \hom(F, a))$$

where now $\hom(F, a)$ is the functor $x \mapsto \hom(Fx, a)$ and $*$ is the terminal presheaf: then

$$(20) \qquad \hom(\varinjlim F, a) \cong \mathsf{Nat}(*, \hom(F, a)).$$

A weighted colimit arises answering the following simple question: what happens if we replace the terminal presheaf $*$ with a more general $W \colon \mathbf{X}^{\mathrm{op}} \to \mathbf{Sets}$, and look for a representative of the functor $a \mapsto \mathsf{Nat}(W, \hom(F, a))$?

If we were able to solve this universal problem, we would obtain a natural isomorphism

$$(21) \qquad \hom(\varinjlim{}^W F, a) \cong \mathsf{Nat}(W, \hom(F, a)).$$

for an object $\varinjlim{}^W F$ that we call the colimit of F weighted by W. A useful shorthand to denote the colimit of F weighted by W, alternative

---

[1] A homotopy colimit is a construction in a model category –not a real colimit– that corrects the fact that the functor $\varinjlim$ does not preserve weak equivalences.

to the ugly $\varinjlim^W F$, is $W \cdot F$, in such a way that its universal property is written

$$(22) \qquad \hom(W \cdot F, a) \cong \mathsf{Nat}(W, \hom(F, a)).$$

2.1. **Examples.**

Example 2.1 : Let $W \colon * \to \mathbf{Sets}$ be the choice of a set X; let $F \colon * \to \mathbf{C}$ be the choice of an object c. A natural transformation $W \Rightarrow \hom(F, a)$ consists of a function of sets $X \to \hom(c, a)$, so that from the isomorphism

$$(23) \qquad \hom(W \cdot F, a) \cong \mathbf{Sets}(X, \hom(c, a))$$

and the uniqueness of adjoints we deduce that

$$(24) \qquad W \cdot F = X \cdot c \cong \coprod_{x \in X} c$$

Exercise 2.2 : Generalize to the case where $W \colon J \to \mathbf{Sets}$ is a parametric family of sets, and $F \colon J \to \mathbf{C}$ a parametric family of objects.

Example 2.3 : Let $W \colon \{0 \to 1\} \to \mathbf{Sets}$ be the choice of a function of sets $f : X \leftarrow Y$, and $F \colon \{0 \to 1\} \to \mathbf{C}$ be the choice of a morphism $u : c \to c'$ in $\mathbf{C}$. Then a natural transformation $W \Rightarrow \hom(F, a)$ consists of a pair of functions α_ϵ ($\epsilon = 0, 1$) such that

$$(25) \qquad \begin{array}{ccc} X & \xrightarrow{\ \alpha_0\ } & \hom(c, a) \\ \uparrow & & \uparrow \\ Y & \xrightarrow[\ \alpha_1\ ]{} & \hom(c', a) \end{array}$$

Show that $W \cdot F$ fits into a pushout diagram

$$(26) \qquad \begin{array}{ccc} Y \cdot c & \longrightarrow & Y \cdot c' \\ \downarrow & \ulcorner & \downarrow \\ X \cdot c & \longrightarrow & W \cdot F. \end{array}$$

Definition 2.4 : Let $W \colon \mathbf{C} \to \mathbf{Sets}$ be a functor; the *category of elements* $\mathbf{C} \!\int\! W$ of W is the category having objects the pairs $(c \in \mathbf{C}, u \in Wc)$, and morphisms $(c, u) \to (c', v)$ those $f \in \mathbf{C}(c, c')$ such that $W(f)(u) = v$.

Notation 2.5 : The exotic notation "$\mathbf{C} \!\int\! W$" for the category of elements of W comes from the wondrous paper [**?**].

Proposition 2.6 : The category $\mathbf{C}\!\int\! W$ can be equivalently characterized as

- The category which results from the pullback

(27)
$$
\begin{array}{ccc}
\mathbf{C}\!\int\! W & \longrightarrow & \mathbf{Sets}_* \\
\downarrow & \lrcorner & \downarrow U \\
\mathbf{C} & \xrightarrow{\ W\ } & \mathbf{Sets}
\end{array}
$$

where $U\colon \mathbf{Sets}_* \to \mathbf{Sets}$ is the forgetful functor which sends a pointed set to its underlying set;
- The comma category of the cospan $\{*\} \to \mathbf{Sets} \xleftarrow{\ W\ } \mathbf{C}$, where $\{*\} \to \mathbf{Sets}$ chooses the terminal object of $\mathbf{Sets}$;
- The opposite of the comma category $(y \downarrow \lceil W \rceil)$, where $\lceil W \rceil\colon \{*\} \to [\mathbf{C}, \mathbf{Sets}]$ is the *name* of the functor W, i.e. the unique functor choosing the presheaf $W \in [\mathbf{C}, \mathbf{Sets}]$.

Proposition 2.7 : The category of elements $\mathbf{C}\!\int\! W$ of a functor $W\colon \mathbf{C} \to \mathbf{Sets}$ comes equipped with a canonical "Grothendieck fibration" to the domain of W, which we denote $\Sigma\colon \mathbf{C}\!\int\! W \to \mathbf{C}$, defined forgetting the distinguished element $u \in Wc$.

Remark 2.8 [◉◉, The Grothendieck construction trivializes weights]: The definition of weighted co/limit can be extended in the case $F\colon \mathbf{C} \to \mathbf{A}$ is a $\mathcal{V}$-enriched functor between $\mathcal{V}$-categories, and $W\colon \mathbf{C} \to \mathcal{V}$ is a $\mathcal{V}$-co/presheaf; this is the setting where the notion acquires a supremacy over the "classical" one (where the weight is the terminal presheaf).

When $\mathcal{V} = \mathbf{Sets}$, indeed, the construction sending a (co)presheaf into its category of elements turns out to trivialize the theory of weighted limits: as the following discussion shows, in such a situation every weighted limit can be expressed as a 'classical' (we prefer to call them *conical*, due to the shape of the weight) limit.

Proposition 2.9 [$\mathbf{Sets}$-weighted limits are limits]: As shown in Prop. **2.7**, the category $\mathbf{C}\!\int\! W$ comes equipped with a fibration $\Sigma\colon \mathbf{C}\!\int\! W \to \mathbf{C}$; this is such that for any functor $F\colon \mathbf{C} \to \mathbf{A}$ one has

(28)
$$
W \cdot F \cong \varinjlim_{(c,x) \in (\mathbf{C}\!\int\! W)^{\mathrm{op}}} F \circ \Sigma^{\mathrm{op}}.
$$

Proof. We show that the two objects share the same universal property: the proof is based on the fact that a natural transformation $\eta\colon\ \Rightarrow$

hom(F, d) corresponds to a cocone $F \circ \Sigma^{\mathrm{op}} \to d$. In fact the naturality of such an η, expressed in the commutative square

$$(29) \qquad \begin{array}{ccc} Wc & \longrightarrow & \hom(Fc, d) \\ \uparrow & & \uparrow \\ Wc' & \longrightarrow & \hom(Fc', d) \end{array}$$

translates into the commutativity of the triangle

$$(30) \qquad\qquad fapifna$$

testifying that $\eta_-(\_)$ are the components of a cocone

$$(31) \qquad \left\{ \eta_c(x) \colon Fc = F \circ \Sigma(c, x) \to d \mid (c, x) \in \mathbf{C}\textstyle\int W \right\}.$$

Conversely, a cocone for $F \circ \Sigma^{\mathrm{op}}$ does constitute a natural transformation $\Rightarrow \hom(F, d)$, defined reading the same equivalence (between squares and triangles) backward. $\qquad \square$

3. Co/ends

The language of coends is designed to subsume the whole discussion above in the same conceptual framework.

Definition 3.1 [DINATURAL TRANSFORMATION]**:**

Definition 3.2 [WEDGES AND COWEDGES]**:**

Definition 3.3 [CO/ENDS]**:**

Proposition 3.4 [CO/ENDS AS COLIMITS]**:**

Remark 3.5 [FUBINI]**:**

3.1. **Examples.**

Example 3.6 :

Example 3.7 :

Example 3.8 :

Example 3.9 :

Example 3.10 :

Example 3.11 :