A CATEGORICAL TRINITARISM

MOI

Ὡσαύτως καὶ αἱ τρεῖς ἡμέραι πρὸ τῶν φωστήρων γεγονυῖαι τύποι εἰσὶν τῆς τριάδος, τοῦ θεοῦ καὶ τοῦ λόγου αὐτοῦ καὶ τῆς σοφίας αὐτοῦ. τετάρτῳ δὲ τόπῳ ἐστὶν ἄνθρωπος ὁ προσδεὴς τοῦ φωτός, ἵνα ᾖ θεός, λόγος, σοφία, ἄνθρωπος. διὰ τοῦτο καὶ τετάρτῃ ἡμέρᾳ ἐγενήθησαν φωστῆρες.

Theophilus of Antioch, To Autolycus, II.XV

Every elementary categorical construction is subsumed by the notion of co/limit. This is not true any more for *enriched* categories, where a more general notion is needed to capture the intrisic 'two-dimensionality' of the theory.

The best we can do, in an enriched/higher setting, is to appeal the KWC *trinitarism* that defines Kan extension, weighted co/limits and co/ends. The plan of this first introductory lecture is to give a minimal amount of definitions in order to establish a language which will be useful later on.

These concepts (co/ends, especially) offer powerful computational tools. The daunting task of learning abstract homotopy theory will be relieved letting these machineries formally produce fairly involved results.

1. Kan extensions

Definition 1.1 : Let

(1)
$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathbf{D} \\ \mathbf{E} & \end{tikzcd}$$

be a diagram of categories and functors; a *left Kan extension* of F along G is a functor, denoted $\mathrm{Lan}_G F$, endowed with a natural transformation $\eta\colon F \Rightarrow \mathrm{Lan}_G F \circ G$, initial among these natural transformations.

This means that whenever another natural transformation $\alpha\colon F \Rightarrow KG$ is given, there is a unique natural transformation $\bar{\alpha}\colon \mathrm{Lan}_G F \Rightarrow K$ such that $\alpha = \eta \circ \bar{\alpha}_G$.

The situation is depicted in the following diagram:

$$\begin{tikzcd} \mathbf{C} \arrow[r, "F"] \arrow[d, "G"'] & \mathbf{D} \\ \mathbf{E} \arrow[ur, "\mathrm{Lan}_G F" description] \arrow[ur, bend right, "K"'] & \end{tikzcd} \tag{2}$$

We are mainly interested in a slightly less general definition, though. A *pointwise* left Kan extension is a left Kan extension $\mathrm{Lan}_G F$ that is preserved by every representable functor. A rather common situation where a pointwise left Kan extension exists is when the codomain $\mathbf{D}$ of F is cocomplete. In that case not only we find $\mathrm{Lan}_G F$ for *every* G, but we are also able to characterize it with a more intuitive universal property: the Kan extensions can be constructed, over each object (a point, in some sense) of the domain category, using a suitable colimit.

Proposition 1.2 : Let $G^*\colon [\mathbf{C}, \mathbf{D}] \to [\mathbf{E}, \mathbf{D}]$ be the functor induced by precomposition with G: then $\mathrm{Lan}_G F$ is the functor that acts, on objects and morphisms, as

$$e \mapsto \varinjlim_{(G\downarrow e)} Fc, \tag{3}$$

where $(G \downarrow e)$ is the *comma category* of arrows $Gc \to e$ and morphisms $c \to c'$ such that the obvious triangle commutes.

Moreover, the correspondence $\mathrm{Lan}_G(\_)\colon [\mathbf{C}, \mathbf{D}] \leftarrow [\mathbf{E}, \mathbf{D}]$ that sends $F \mapsto \mathrm{Lan}_G F$ is the left adjoint of G^* that sends H to HG.

Proof. A slightly more formal rephrasing of the above isomorphism is the following: consider the composition of functors

$$(G \downarrow e) \xrightarrow{\Sigma} \mathbf{C} \xrightarrow{F} \mathbf{D} \tag{4}$$

then the colimit

$$\varinjlim_{(c,\alpha:Gc\to e)\in(G\downarrow e)} F(\Sigma(c, \alpha)) \tag{5}$$

of this diagram on $\mathbf{D}$ is precisely the value of $\mathrm{Lan}_G F(e)$ which is then computed *pointwise*.

To prove the statement,we build an explicit natural isomorphism

$$\mathsf{Nat}(F, HG) \underset{\Phi}{\overset{\Psi}{\leftrightarrows}} \mathsf{Nat}(\mathrm{Lan}_G F, H). \tag{6}$$

First of all we define a map Φ, sending $\eta\colon F \to HG$ to the cocone with components

$$\hat{\eta}_{c,\alpha}\colon Fc \xrightarrow{\eta_c} HGc \xrightarrow{H\alpha} Hc \tag{7}$$

This induces (via the universal property of the colimit) a unique map

$$\varinjlim_{(G\downarrow e)} Fc \xrightarrow{\Phi(\eta)_e} He \tag{8}$$

which is a natural transformation in e, since the square

$$\begin{tikzcd} \varinjlim_{(G\downarrow e)} Fc \arrow[r] \arrow[d] & He \arrow[d] \\ \varinjlim_{(G\downarrow e)} Fc \arrow[r] & He' \end{tikzcd} \tag{9}$$

commutes for $f\colon e \to e'$.

On the other hand we define Ψ taking the components $\beta_e\colon \beta\colon \varinjlim_{(G\downarrow e)} Fc \to He$ and considering the compositions

$$\begin{tikzcd} Fx \arrow[r, "\iota_{(x,1_{Gx})}"] \arrow[d] & \varinjlim_{(G\downarrow x)} Fc \arrow[r] \arrow[d] & HGx \arrow[d] \\ Fy \arrow[r, "\iota_{(y,1_{Gy})}"'] & \varinjlim_{(G\downarrow y)} Fc \arrow[r] & HGy \end{tikzcd} \tag{10}$$

The inner square commutes, and proves naturality in $x \to y$; the reason is that it splits into two sub-diagrams, each of which commutes. $\square$

Exercise 1.3 : Dualize; define right Kan extensions $\mathrm{Ran}_G F$ and their universal property (how is the direction of 1-cells affected by the dualization?); define pointwise right Kan extensions and show a similar adjunction formula

$$\mathsf{Nat}(HG, F) \cong \mathsf{Nat}(H, \mathrm{Ran}_G F). \tag{11}$$

1.1. **Examples.**

Example 1.4 : Let $\iota\colon H \leq G$ a group inclusion, regarded as a functor. The category of (left) G-sets is identified with the category of functors $[G, \mathbf{Vect}_k]$.

There exist then functors

$$[G, \mathbf{Vect}_k] \underset{\mathrm{Lan}_\iota}{\overset{\mathrm{Ran}_\iota}{\overset{\longleftarrow}{\underset{\longleftarrow}{\xrightarrow{\ \iota^*\ }}}}} [H, \mathbf{Vect}_k] \tag{12}$$

given by Kan extension. Their action on objects and morphisms is given by

$$V \mapsto k[G] \otimes_{k[H]} V$$
$$W \mapsto \hom_{k[H]}(k[G], W)$$

(the group algebra $k[G]$ is the free k-algebra on group elements of G, with multiplication $\sum_g a_g \cdot \sum_{g'} b_{g'} = \sum_h \sum_{gg'=h} a_g b_{g'}$, and similarly for H).

Exercise 1.5 : Prove the above statement, with the help of the following characterization for the two functors above:

- There is an equalizer

$$\hom_{k[H]}(k[G], W) \longrightarrow \hom_k(k[G], V) \underset{\hom(h,1)}{\overset{\hom(1,h)}{\rightrightarrows}} \hom_k(k[G], V) \tag{13}$$

- There is a coequalizer

$$\textstyle\bigoplus_{h\in H} k[G] \otimes_k V \underset{h\otimes 1}{\overset{1\otimes h}{\rightrightarrows}} k[G] \otimes_k V \longrightarrow k[G] \otimes_{k[H]} V. \tag{14}$$

The following is a standard statement that is often subsumed by the sentence ‘any functor F defined on presheaves on $\mathbf{C}$ is uniquely determined by its action on representables’.

Example 1.6 : Let $F : \mathbf{C} \to \mathbf{D}$ be a functor from a *small* to a *cocomplete* category; then there is a unique way to define a functor $\bar{F} : [\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}] \to \mathbf{D}$ that commutes with colimits, and that has a right adjoint N_F sending an object $d \in \mathbf{D}$ into the functor $c \mapsto \hom(Fc, d)$.

This left adjoint sends a presheaf P into the object $\mathrm{Lan}_y F(P)$ in $\mathbf{D}$.

Proof. It is enough to show that a natural transformation $\alpha\colon P \to N_F(d)$ determines a cocone for $(y \downarrow P) \to \mathbf{C} \xrightarrow{F} \mathbf{D}$: but from the commutativity of the square

(15)
$$\begin{tikzcd} c \arrow[d, "\varphi"'] & Pc \arrow[r] & \hom(Fc, d) \\ c' & Pc' \arrow[u, "P\varphi"] \arrow[r] & \hom(Fc', d) \arrow[u, "\hom(F\varphi, d)"'] \end{tikzcd}$$

it follows that the triangle

(16)
$$\begin{tikzcd} Fc \arrow[rr, "F\varphi"] \arrow[dr, "\alpha"'] & & Fc' \arrow[dl, "\alpha'"] \\ & d & \end{tikzcd}$$

commutes, and then sending $(c, x \in Pc)$ into $\alpha_c(x)\colon Fc \to d$ determines a cocone for FU. This concludes the proof, once we show that this is a bijection; of course it is. $\square$

Exercise 1.7 : Show that the above construction determines an equivalence of categories

(17)
$$\mathbf{Cat}(\mathbf{C}, \mathbf{D}) \xrightarrow{\text{Yan}} \mathbf{Adj}_{\mathrm{L}}([\mathbf{C}^{\mathrm{op}}, \mathbf{Sets}], \mathbf{D})$$

if morphisms on the right are natural transformations between lef adjoints (hint: simply show that the adjunction $\mathrm{Lan}_y \dashv y^*$ is an equivalence of categories).

Exercise 1.8 : Shape a dual statement regarding a universal property of the contravariant Yoneda embedding $\mathbf{C}^{\mathrm{op}} \xrightarrow{y} [\mathbf{C}, \mathbf{Sets}]^{\mathrm{op}}$. Is it really an independent theorem?

Exercise 1.9 : Show that $\mathrm{Lan}_1 F \cong F$ and $\mathrm{Ran}_1 F \cong F$, for every functor $F\colon \mathbf{A} \to \mathbf{B}$ (use the universal property).

Exercise 1.10 : Show that if $F\colon \mathbf{A} \to \mathbf{B}$ is a functor between small categories, then the following conditions are equivalent:

- F has a right adjoint;
- $\mathrm{Lan}_F 1$ exists and there is an isomorphism $\mathrm{Lan}_F F \cong F$;
- $\mathrm{Lan}_F 1$ exists and there is an isomorphism $\mathrm{Lan}_F L \cong L \circ \mathrm{Lan}_F 1$ for any functor $L\colon \mathbf{A} \to \mathbf{C}$.

Exercise 1.11 : Use the universal property to show that $G \mapsto \mathrm{Lan}_G$ is a pseudofunctor, by showing that for $\mathbf{A} \xrightarrow{F} \mathbf{B}, \mathbf{A} \xrightarrow{G} \mathbf{C} \xrightarrow{H} \mathbf{D}$ there is a uniquely determined laxity cell for composition

$$\mathrm{Lan}_H(\mathrm{Lan}_G F) \cong \mathrm{Lan}_{H\circ G} F. \tag{18}$$

Dualize for right Kan extensions.

2. Weighted co/limits

The notion of weighted colimit generalizes the notion of colimit to an enriched setting, and provides a fairly formal approach to the issues posed by abstract homotopy theory; it is noticeable that the notion of *homotopy colimit*[1] is captured, under mild assumptions on the model category, by a weighted colimit construction. Thus, the machinery produces a fairly explicit approach to the matter.

When looking for a classical (from now on, 'conical') colimit of a diagram $F\colon J \to \mathbf{C}$, we basically are looking for a representative for the functor $a \mapsto \mathrm{Cocones}(F, a) \cong \mathsf{Nat}(F, \Delta a)$, where $\Delta : \mathbf{C} \to \mathbf{C}^J$ is the 'constant diagram' functor.

The latter object, that uniquely determines the object $\varinjlim F$, can be rewritten as

$$\mathsf{Nat}(*, \hom(F, a)) \tag{19}$$

where now $\hom(F, a)$ is the functor $x \mapsto \hom(Fx, a)$ and $*$ is the terminal presheaf: then

$$\hom(\varinjlim F, a) \cong \mathsf{Nat}(*, \hom(F, a)). \tag{20}$$

A weighted colimit arises answering the following simple question: what happens if we replace the terminal presheaf $*$ with a more general $W\colon \mathbf{X}^{\mathrm{op}} \to \mathbf{Sets}$, and look for a representative of the functor $a \mapsto \mathsf{Nat}(W, \hom(F, a)$?

If we were able to solve this universal problem, we would obtain a natural isomorphism

$$\hom(\varinjlim{}^W F, a) \cong \mathsf{Nat}(W, \hom(F, a)). \tag{21}$$

for an object $\varinjlim{}^W F$ that we call the colimit of F weighted by W. A useful shorthand to denote the colimit of F weighted by W, alternative

[1] A homotopy colimit is a construction in a model category –not a real colimit– that corrects the fact that the functor $\varinjlim$ does not preserve weak equivalences.

to the ugly $\varinjlim^W F$, is $W \cdot F$, in such a way that its universal property is written

$$\hom(W \cdot F, a) \cong \mathsf{Nat}(W, \hom(F, a)). \tag{22}$$

2.1. **Examples.**

Example 2.1 : Let $W \colon * \to \mathbf{Sets}$ be the choice of a set X; let $F \colon * \to \mathbf{C}$ be the choice of an object c. A natural transformation $W \Rightarrow \hom(F, a)$ consists of a function of sets $X \to \hom(c, a)$, so that from the isomorphism

$$\hom(W \cdot F, a) \cong \mathbf{Sets}(X, \hom(c, a)) \tag{23}$$

and the uniqueness of adjoints we deduce that

$$W \cdot F = X \cdot c \cong \textstyle\coprod_{x \in X} c \tag{24}$$

Exercise 2.2 : Generalize to the case where $W \colon J \to \mathbf{Sets}$ is a parametric family of sets, and $F \colon J \to \mathbf{C}$ a parametric family of objects.

Example 2.3 : Let $W \colon \{0 \to 1\} \to \mathbf{Sets}$ be the choice of a function of sets $f : X \leftarrow Y$, and $F \colon \{0 \to 1\} \to \mathbf{C}$ be the choice of a morphism $u : c \to c'$ in $\mathbf{C}$. Then a natural transformation $W \Rightarrow \hom(F, a)$ consists of a pair of functions α_ϵ $(\epsilon = 0, 1)$ such that

$$\begin{tikzcd} X \arrow[r, "\alpha_0"] & \hom(c, a) \\ Y \arrow[u] \arrow[r, "\alpha_1"'] & \hom(c', a) \arrow[u] \end{tikzcd} \tag{25}$$

Show that $W \cdot F$ fits into a pushout diagram

$$\begin{tikzcd} Y \cdot c \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Y \cdot c' \arrow[d] \\ X \cdot c \arrow[r] & W \cdot F. \end{tikzcd} \tag{26}$$

Definition 2.4 : Let $W \colon \mathbf{C} \to \mathbf{Sets}$ be a functor; the *category of elements* $\mathbf{C} \int W$ of W is the category having objects the pairs $(c \in \mathbf{C}, u \in Wc)$, and morphisms $(c, u) \to (c', v)$ those $f \in \mathbf{C}(c, c')$ such that $W(f)(u) = v$.

Notation 2.5 : The exotic notation "$\mathbf{C} \int W$" for the category of elements of W comes from the wondrous paper [**?**].

Proposition 2.6 : The category $\mathbf{C}\int W$ can be equivalently characterized as

- The category which results from the pullback

$$\begin{tikzcd} \mathbf{C}\int W \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{Sets}_* \arrow[d, "U"] \\ \mathbf{C} \arrow[r, "W"'] & \mathbf{Sets} \end{tikzcd} \tag{27}$$

  where $U\colon \mathbf{Sets}_* \to \mathbf{Sets}$ is the forgetful functor which sends a pointed set to its underlying set;
- The comma category of the cospan $\{*\} \to \mathbf{Sets} \xleftarrow{W} \mathbf{C}$, where $\{*\} \to \mathbf{Sets}$ chooses the terminal object of $\mathbf{Sets}$;
- The opposite of the comma category $(y \downarrow \ulcorner W \urcorner)$, where $\ulcorner W \urcorner\colon \{*\} \to [\mathbf{C}, \mathbf{Sets}]$ is the *name* of the functor W, i.e. the unique functor choosing the presheaf $W \in [\mathbf{C}, \mathbf{Sets}]$.

Proposition 2.7 : The category of elements $\mathbf{C}\int W$ of a functor $W\colon \mathbf{C} \to \mathbf{Sets}$ comes equipped with a canonical "Grothendieck fibration" to the domain of W, which we denote $\Sigma\colon \mathbf{C}\int W \to \mathbf{C}$, defined forgetting the distinguished element $u \in Wc$.

Remark 2.8 [☉☉, The Grothendieck construction trivializes weights]**:** The definition of weighted co/limit can be extended in the case $F\colon \mathbf{C} \to \mathbf{A}$ is a $\mathscr{V}$-enriched functor between $\mathscr{V}$-categories, and $W\colon \mathbf{C} \to \mathscr{V}$ is a $\mathscr{V}$-co/presheaf; this is the setting where the notion acquires a supremacy over the "classical" one (where the weight is the terminal presheaf).

When $\mathscr{V} = \mathbf{Sets}$, indeed, the construction sending a (co)presheaf into its category of elements turns out to trivialize the theory of weighted limits: as the following discussion shows, in such a situation every weighted limit can be expressed as a 'classical' (we prefer to call them *conical*, due to the shape of the weight) limit.

Proposition 2.9 [**Sets**-weighted limits are limits]**:** As shown in Prop. **2.7**, the category $\mathbf{C}\int W$ comes equipped with a fibration $\Sigma\colon \mathbf{C}\int W \to \mathbf{C}$; this is such that for any functor $F\colon \mathbf{C} \to \mathbf{A}$ one has

$$\varprojlim^W F \cong \varprojlim_{(c,x)\in\mathbf{C}\int W} F \circ \Sigma. \tag{28}$$

Proof. We show that the two objects share the same universal property. □

3. Co/ends