

ITI9200 — Category theory and Applications

Exercise Sheet 2 — Trimurti: Categories/Functors/Naturality Spring Semester

Basic category theory rests on three conceptual pillars: categories, functors, and natural transformations. The exercises below are meant to make you familiar with these notions and their basic properties, and extend a little bit the perspective on the definition of category.

Some preliminary definitions

Definition 1.1.

1. A *partial binary algebra* is a pair $(X, *)$ consisting of a class X and a partial binary operation $*$ on X ; i.e., a binary operation defined on a subclass of $X \times X$. (The value of $*(x, y)$ is denoted by $x * y$.)
2. If $(X, *)$ is a partial binary algebra, then an element u of X is called a *unit* of $(X, *)$ provided that

$$x * u = x \quad \text{whenever } x * u \text{ is defined,}$$

and

$$u * y = y \quad \text{whenever } u * y \text{ is defined.}$$

Definition 1.2. An *object-free category* is a partial binary algebra $\mathbf{C} = (M, \circ)$, where the members of M are called *morphisms*, that satisfies the following conditions:

1. *Matching Condition:* For morphisms f, g , and h , the following conditions are equivalent:
 - (a) $g \circ f$ and $h \circ g$ are defined,
 - (b) $h \circ (g \circ f)$ is defined, and
 - (c) $(h \circ g) \circ f$ is defined.
2. *Associativity Condition:* If morphisms f, g , and h satisfy the matching conditions, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

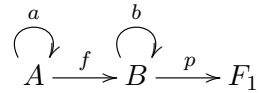
3. *Unit Existence Condition:* For every morphism f there exist units u_C and u_D of (M, \circ) such that $u_C \circ f$ and $f \circ u_D$ are defined.
4. *Smallness Condition:* For any pair of units (u_1, u_2) of (M, \circ) the class

$$\text{hom}(u_1, u_2) = \{ f \in M \mid f \circ u_1 \text{ and } u_2 \circ f \text{ are defined} \}$$

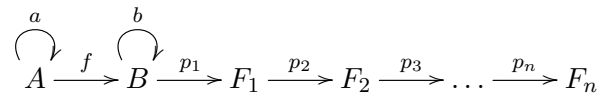
is a set.

Exercise 1:

Let \mathcal{Q} be the following directed graph:



- Determine the free category $F\langle\mathcal{Q}\rangle$ on \mathcal{Q} and prove that its set of morphisms determines a regular language in the alphabet $\Sigma = \{a, b, f, p\}$.
- Generalize as follows: if \mathcal{Q}_n is the directed graph



so that the previous \mathcal{Q} is \mathcal{Q}_1 , prove that the set of morphisms of $F\langle\mathcal{Q}_n\rangle$ determines a regular language in the alphabet $\Sigma = \{a, b, f, p_1, \dots, p_n\}$. Is this still true if $n \rightarrow \infty$?

Exercise 2:

Define the following categories associated to the functor $S_A : X \mapsto 1 + A \times X$.

- ∇S has objects the pairs (X, t) where $t \in S_A X$ is an element; a morphism $(X, t) \rightarrow (Y, v)$ in ∇S consists of a function $f : X \rightarrow Y$ such that the function Sf sends t to v :

$$Sf : 1 + A \times X \rightarrow 1 + A \times Y : t \mapsto v$$

- $\mathbf{coAlg}(S)$ has objects the pairs (X, ξ) where $\xi : X \rightarrow S_A X$ is a function; a morphism $(X, \xi) \rightarrow (Y, \theta)$ in $\mathbf{coAlg}(S)$ consists of a function $f : X \rightarrow Y$ such that

$$Sf \circ \xi = \theta \circ f.$$

Recall, or learn for the first time, that

- an initial objects in a category \mathcal{C} is an object I such that for every other object $X \in \mathcal{C}$, there exists a unique arrow $I \rightarrow X$;
- a terminal objects in a category \mathcal{C} is an object T such that for every other object $X \in \mathcal{C}$, there exists a unique arrow $X \rightarrow T$.

Prove that $\mathbf{coAlg}(S)$ has a terminal object; prove or disprove that ∇S has an initial object.

Exercise 3:

A *construct* consists of a pair (\mathcal{C}, U) where \mathcal{C} is a category and $U : \mathcal{C} \rightarrow \mathbf{Set}$ is a faithful functor. Two constructs (\mathcal{C}, U) and (\mathcal{D}, V) are (*strongly*) *equivalent* if there exist two functors F, G such that the two triangles

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow U & \swarrow V \\ & \mathbf{Set} & \end{array} \quad \begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} \\ & \searrow V & \swarrow U \\ & \mathbf{Set} & \end{array}$$

commute, and such that $F \circ G = \text{id}_{\mathcal{D}}$ and $G \circ F = \text{id}_{\mathcal{C}}$. When the construct functor U associated to a certain category is implicitly understood (for example, when \mathcal{C}, \mathcal{D} are categories of algebraic structures) we say that \mathcal{D}, \mathcal{C} are *concretely equivalent*.

Are the following pairs concretely equivalent?

- $\mathcal{C} = \mathbf{Cat}$ is the category of categories and functors, defined in the usual way, while U sends a category to its set of arrows; \mathcal{D} is the category of object-free categories, defined above, and V sends (M, \circ) to the set M .
- the category **Kop** of *Kuratowski spaces*, defined via an interior operator, and the category **Top** of topological spaces defined via a family of open subsets:
 - a *topological space* consists of a pair (X, τ) where $\tau \subseteq 2^X$ is a collection of subsets of X , the elements of which are called *open*, such that
 1. $\emptyset, X \in \tau$;
 2. if I is a set and $A_{\bullet} : I \rightarrow \tau$ an I -indexed family of elements $A_i \in \tau$, then $\bigcup_i A_i \in \tau$;
 3. if $A_1, A_2 \in \tau$, then $A_1 \cap A_2 \in \tau$.

An element of the form $X \setminus A$ for $A \in \tau$ is called a *closed subset*.

- A *Kuratowski space* consists of a set X equipped with a monotone function

$$j : 2^X \rightarrow 2^X$$

called the *interior operator* of X , satisfying the following properties:

1. $jX = X$;
2. for all $S \in 2^X$, $j(S) \subseteq S$;
3. for all $S \in 2^X$, $j(j(S)) = j(S)$;
4. for all $S, T \in 2^X$, $j(S) \cap j(T) \subseteq j(S \cap T)$.

Both U and V send a topological or Kuratowski space to its underlying set.