The Fibration of Algebras

Davide Castelnovo, Greta Coraglia, Fosco Loregian, Ülo Reimaa

December 7, 2023

Abstract

We study fibrations arising from indexed categories of the following form: fix a functor $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$, so that to each $F_A = F(A, -)$ one can associate a category of algebras $\mathsf{Alg}_{\mathcal{X}}(F_A)$ (or Eilenberg–Moore, or a Kleisli category if each F_A is a monad, coEilenberg–Moore if it's a comonad, etc.). The functor $\left[\begin{smallmatrix} & \int \mathsf{Alg}(F_\bullet) \\ p_F & \downarrow \\ & \mathcal{A} \end{smallmatrix}\right]$ over $\mathcal{A}$ having typical fiber over A the category $\mathsf{Alg}_{\mathcal{X}}(F_A)$ is the *fibration of algebras* of F.

We build on the particularly fruitful intuition that this construction presents the total category of such fibrations as categorified semidirect products $\mathcal{A} \ltimes_F \mathcal{X}$ of $\mathcal{A}$ acting on $\mathcal{X}$ through the endomorphisms F_A; examples of such construction arise in disparate areas of mathematics, such as the representation theory of Lie algebras, the category-theoretic study of computational effects in programming language design, topos theory and algebraic geometry, categorical logic.

Such a variety of examples calls for a general theory: outlining its essential features is the purpose of this work. Fibrations of algebras are all classified by pulling back along a universal one; our theory can be regarded as arising when a category is acted on by a free monoidal category –so as a categorified 'semiautomaton' in the sense of [EKKK74]– and, as a consequence, as a special kind of *actegory* ([Bén67] and [McC00], see [CG22] for a survey). This motivates the parallel with semidirect products, which is particularly fruitful here, considering the extensive relation that has been drawn between the semidirect product operation and the Grothendieck construction.

The theory we build is general enough to lend itself to a 'formal' interpretation: we first remark that a fibration $\left[\begin{smallmatrix} & \mathcal{E} \\ p & \downarrow \\ & \mathcal{A} \end{smallmatrix}\right]$ is a fibration of algebras for $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ if and only if p is monadic over the trivial fibration $\left[\begin{smallmatrix} & \mathcal{A} \times \mathcal{X} \\ \pi & \downarrow \\ & \mathcal{A} \end{smallmatrix}\right]$, and these monads can in turn be characterised locally, as monads in the slice 2-category $\mathsf{Cat}/\mathcal{A}$, or globally, as monads in a 2-category itself of the form $\mathsf{Cat} \ltimes \mathsf{Cat}$: this point of view allows for a swift generalisation of the theory to an abstract 2-category, with little effort.

Natural choices are pointed or additive categories, categories with a prescribed class of (co)limits. Under very mild assumptions, a fibration of algebras can be presented as a short sequence of the form

$$1 \longrightarrow \mathcal{X} \longrightarrow \mathcal{A} \ltimes \mathcal{X} \longrightarrow \mathcal{A} \longrightarrow 1$$

whose exactness is close to saying that $\mathcal{X}$ and $\mathcal{A}$ form a torsion theory on $\mathcal{A} \ltimes \mathcal{X}$.

Chapter 1

Introduction

Summary of chapter

We organise our exposition in chapters, each of which starts with a brief summary of its content. For this reason, the customary 'structure of the paper' in section 1.1 is less of a road-map, and more of a discussion on why we organise the exposition in this way.

After the introduction, in §1.2 we recall some basic properties of fibrations that we will need throughout the paper. A thorough presentation of the theory of fibrations is the scope of [Jac98, Ch. 1], [Joh02, B1], [Str23]; a survey swiftly getting to the heart of the theory is [LR20]: our aim here is to give a self-contained overview of the main concepts and ideas in the topic, building the terminology that we will adopt from that point on. The reader who is fluent in the matter might want to skip this section entirely.

1.1 Towards a general theory of parametricity

The purpose of the present paper is to give a uniform account of a phenomenon spanning from logic, to geometry, to abstract algebra and algebraic topology. The spirit of the enterprise is best appreciated enumerating a few telling examples.

Discussion 1.1.1 (The simple fibration)**.** Let $\mathcal{A}$ be a category with finite products. Each object $A \in \mathcal{A}$ induces a comonad $T_A = \_ \times A : \mathcal{A} \longrightarrow \mathcal{A}$, the so-called *coreader comonad* [AU19, p. 3]. The coKleisli category $\mathsf{coKl}(T_A)$ of this comonad consists of what is called the *simple slice category* $\mathcal{A}/\!\!/A$ over A, described in [Jac98, Exercise 1.3.4.(ii)] with type-theoretic applications in mind.

The objects of $\mathcal{A}/\!\!/A$ are the same of $\mathcal{A}$, and a coKleisli map is a morphism $f : X \times A \to Y$ in $\mathcal{A}$. In particular, each such morphism can be regarded as an family of maps $f_a : X \to Y$, 'abstractly parametrised' by A (this is precisely what happens if $\mathcal{A} = \mathsf{Set}$, because $f : X \times A \to Y$ transposes to a function $A \to Y^X$). Let it be noted that the full slice category $\mathcal{A}/A$, too, can be recovered in a similar fashion: it is the entire category of Eilenberg–Moore coalgebras $\mathsf{coEM}(T_A)$.

Our interest in this classical construction stems from the fact that we can collect all such simple slices $\mathcal{A}/\!\!/A$ in a single category, *fibred* over $\mathcal{A}$, and build a (cloven) fibration $\begin{bmatrix} \mathbf{s}(\mathcal{A}) \\ \downarrow \\ \mathcal{A} \end{bmatrix}$, cf. again [Jac98, Exercise 1.3.4.(ii)].

This 'simple slice' construction is important in categorical logic because of its role in finding categorical semantics for *simple type theory*, a framework introduced by Church in [Chu40]; see also [Far23, Hin97, nLa23].

Moving to a different example, recall that every monoid (say, in Set) gives rise to a monad $M \times \_$; the category of Eilenberg–Moore algebras of such a monad is the category of *representations* of M. It has been known since the early days of fibered category theory, by algebraic geometers, that we can package into a single functor the information about 'all actions of all monoids' at the same time:

Discussion 1.1.2 (The fibration of modules)**.** Let $\mathcal{A}$ be a category with finite products. Any internal monoid M in $\mathcal{A}$ defines a monad $M \times \_$ on $\mathcal{A}$, whose Eilenberg–Moore category is precisely the category $\mathcal{A}^M$ of objects carrying an action of M. For example, if $\mathcal{A} = \mathsf{Set}$ is the category of sets and functions, Set^M is the category of M-sets, or in other words, the category of functors $M \longrightarrow \mathsf{Set}$ from the one-object category M. If $\mathcal{A} = \mathsf{Ab}$ is the category of Abelian groups, and we let M vary over the entire category $\mathsf{Mon}(\mathcal{A})$ of internal monoids R in $\mathcal{A}$ (i.e., unital rings), we recover what algebraic geometer and algebraic topologists [Qui70] call the *fibration of modules* $\begin{bmatrix} \mathsf{Mod} \\ \downarrow \\ \mathsf{Mon}(\mathcal{A}) \end{bmatrix}$, the typical fiber of which is the category R-Mod of modules over R, as R varies over $\mathcal{A}$-monoids.

Of course Discussion 1.1.2 works the same for $\otimes$-modules over a more general monoidal base $(\mathcal{A}, \otimes, I)$.

Discussion 1.1.3 (The fibration of cocommutative Hopf algebras)**.** Consider, now, an algebraically closed field k of characteristic zero; an important theorem of Cartier, Gabriel and Kostant [Kos77], [EGNO15, 5.10.2] asserts that every cocommutative Hopf algebra H over k arises from a semidirect product of a group G (or rather, its group algebra $k[G]$) acting on a Lie algebra L over k. More formally, the correspondence $G \mapsto k[G]$-Lie_k sending a group to the category of Lie algebras with an action of $k[G]$ is a (contravariant) functor defining, in a similar fashion as Discussion 1.1.2, a fibred category collecting Eilenberg–Moore algebras of the monad $k[G] \otimes \_$. If we denote the total category of such fibration as $\mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Lie}_k$ and write the fibration associated to $G \mapsto k[G]$-Lie_k as a 'first projection' map

$$\mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Lie}_k \xrightarrow{p} \mathsf{Grp} \tag{1.1.1}$$

we can reformulate the CGK theorem as a fibered equivalence of categories: call $(\_)^{\mathrm{g}}$: $\mathsf{CCHopf} \longrightarrow \mathsf{Grp}$ the functor sending a cocommutative Hopf algebra H to its set H^{g} of group-like elements, then CGK states that $(\_)^{\mathrm{g}} \cong p$ in $\mathsf{Fib}(\mathsf{Grp})$.

Remark 1.1.4. The notation in (1.1.1) is motivated by the intuition that each fiber has something to do with semidirect products, as stated before, but this is only a suggestion for now. We will soon adopt such notation in section 2.1, fully motivating our choice only in chapter 4. The idea, however, can already be explained: we would like to treat Grp as a category *acting* on Lie_k, via the functor sending G to $k[G]$-Lie_k. The result of collating together the various categories $k[G]$-Lie_k with the reindexing functors induced by group homomorphisms shall be thought as the semidirect product of the categories Grp and Lie_k. It is certainly interesting that there is a micro-macrocosm principle at work here: a category where each object can be presented as a semidirect product *is itself* a semidirect product of simpler categories.

There is another instance of this micro-macrocosm principle, arising in categorical algebra: every category $\mathcal{B}$ acts on itself giving rise to its *fibration of points*.

Discussion 1.1.5 (The fibration of points)**.** For every object B of a pointed category $\mathcal{B}$ with finite coproducts, denote by $\mathsf{Pt}_B(\mathcal{B})$ the category of *points* of B, i.e. the category of split epimorphisms with codomain B (implicitly considered together with a prescribed right inverse). This is the typical fiber of a fibration called the *fibration of points*, $\begin{bmatrix} \mathsf{Pt}(\mathcal{B}) \\ \downarrow \\ \mathcal{B} \end{bmatrix}$; a specific branch of categorical algebra (the theory of *protomodular categories*) is occupied with finding reasonable conditions so that, in the category $\mathcal{B}$, an operation akin to the semidirect product in Grp can be performed.

Under suitable such assumptions, each fiber $\mathsf{Pt}_B(\mathcal{B})$ can be regarded as the category of Eilenberg–Moore algebras for a monad $B \flat \_$ depending on a *parameter object* B, so that there is an action $\_ \flat \_ : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B}$ such that $\mathsf{Pt}(\mathcal{B}) \cong \mathcal{B} \ltimes^{\mathsf{EM}}_{\flat} \mathcal{B}$.

The present paper aims at providing a common framework in which all these examples, and many others, fit naturally and can be studied in a uniform way, and where Remark 1.1.4 can be explained and formalised. Evidently, one major feature shared by all these constructions is that we have considered a 'bundle' (i.e., a fibration) over a category $\mathcal{A}$ whose fiber over an object A is the category of algebras of an endofunctor $F_A : \mathcal{X} \longrightarrow \mathcal{X}$ depending on the parameter A.

For this reason, we want to consider a general theory of *parametric endofunctors* $A \mapsto F_A$, arising from functors of type

$$F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}] \tag{1.1.2}$$

or, equivalently, and up to currying, of type $\mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$, and the way we associate to F_A a category of algebras. Of course, by the word we can mean algebras as broadly as the structure on F_A allows: if F_A is a co/monad, it will be natural to attach a co/Eilenberg–Moore (or Kleisli) category to it; if F_A is a mere pointed endofunctor, it will be natural to consider its pointed endofunctor algebras, and so on. The context will determine in basically every concrete case what notion of algebra is attached to a specific functor, but we have to establish a notation in order to avoid confusion.

Notation 1.1.6. When we write $\mathcal{A} \ltimes \mathcal{X}$ we are considering a category, equipped with a functor into $\mathcal{A}$, and obtained from an action of $\mathcal{A}$ on another category $\mathcal{X}$, embodied in a functor $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ (or, which is equivalent under currying, $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$). When we want to make the dependence from F explicit, we write $\mathcal{A} \ltimes_F \mathcal{X}$. The fiber over $A \in \mathcal{A}$ of $\mathcal{A} \ltimes \mathcal{X}$ is the category of endofunctor algebras $\mathsf{Alg}(F_A)$.

When the parametric functor is a *parametric monad*, by which we mean a functor such that teach $F_A = F(A, \_)$ is a monad, we usually write it as T, as it is customary, and we can build a category $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$, fibered over $\mathcal{A}$ and whose fiber over A is the Eilenberg–Moore category $\mathsf{EM}(T_A)$ (or $\mathcal{A} \ltimes^{\mathsf{EM}}_T \mathcal{X}$ when we want to make the dependence on T explicit).

If T is a parametric monad, it is then clear what we mean by $\mathcal{A} \ltimes^{\mathsf{Kl}} \mathcal{X}$, at least at an informal level: we are collecting all Kleisli categories of the various T_A as fibers of a fibration.

The dual constructions must also be taken into account, if we want to build a complete theory of these objects:[1]

- $\mathcal{A} \oplus_F \mathcal{X}$ collects the endofunctor *co*algebras of $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$;
- $\mathcal{A} \oplus^{\mathsf{EM}}_S \mathcal{X}$ collects the coEilenberg–Moore categories of a parametric comonad $S : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ and $\mathcal{A} \oplus^{\mathsf{Kl}}_S \mathcal{X}$ collects its coKleisli categories.

[1] If F is a parametric endofunctor the association $A \mapsto \mathsf{Alg}(F_A)$, as well as all the others mentioned above, is a pseudofunctor $\mathcal{A} \longrightarrow \mathsf{Cat}$, and as such it defines a fibration (if $\mathsf{Alg}(F_A)$ depends contravariantly on A) or an opfibration (if it depends covariantly) over $\mathcal{A}$, whose typical fiber is exactly the category of 'F_A-algebras' intended in one of the above senses.

So, the subject of our study can be summarised in a single sentence: we aim to taxonomise *fibrations and opfibrations associated with parametric endofunctors of sorts.*

Our claim is that all instances we have listed so far, the simple fibration, the fibration of points for a protomodular category, the CGK theorem,... are instances of a general phenomenon. We aim to build a general theory taxonomising the expressions of this overarching concept. To the untrained eye, it might seem that very little unites the representation theory of Lie algebras and categorical logic. Yet, they arise from a similar procedure, applied to different initial data.

Our scope here is to clarify this situation outline the common properties shared by a seemingly scattered variety of examples, and what structural theorems one can have available assuming just a bit more about $\mathcal{A}$, $\mathcal{X}$, or the subcategory of $[\mathcal{X}, \mathcal{X}]$ which F factors through (for example: the category of parameters $\mathcal{A}$, or the category $\mathcal{X}$, might be κ-accessible, or cocomplete, and each F_A an endofunctor commuting with κ-filtered colimits; or we can consider only additive functors $\mathcal{X} \longrightarrow \mathcal{X}$ of an abelian category $\mathcal{A}$; or strong monoidal ones, and so on).

Moreover, a number of structural results can quite easily be proved, now that we established the nomenclature to refer to the problem. Not only all the functors that we call *fibrations of algebras* are classified by pulling back along a universal one such fibration of algebras of the same type, but our theory can be regarded as arising when a category is acted on by a free monoidal category –so as a categorified 'semiautomaton' in the sense of [EKKK74]– and, as a consequence, as a special kind of *actegory* ([Bén67] and [McC00], see [CG22] for a survey). The parallel with semidirect products is particularly fruitful here, considering the extensive relation that has been drawn between the semidirect product operation and the Grothendieck construction [BW90, 12.2.4], [Wel80, Man22].

Such a general theory has a long history, and in fact the part of the present work where we struggled the least was finding examples. All along §3 we recollect instances of fibrations of algebras arising from a variety of fields: type theory (various classes of fibrations in categorical logic [Jac98] arise as fibrations of algebras, and polynomial functors [GK13] all give rise to a fibration of their endofunctor algebras), topos theory, representation theory (the aforementioned CGK theorem [Kos77]), algebraic topology [Qui70], categorical algebra (the fibration of points of [Bou17, BB04], see Example 3.3.17; Beck modules, in Example 3.3.7), linking it to computer science (with particular attention to *dinatural* parametricity explored in [Atk09b, Atk09a]; see section 3.5), and more.

We aim to provide additional evidence to a well-established operative principle allowing to manipulate variable categories: that parametricity as intended by French category theorists between the 50s and the 80s ('a family of categories continuously varying over a category of indices', a point of view championed, among others, by J. Bénabou), and studied by computer scientists (the 'nontrivial dependence of a functor from a set of states' whence different outcomes of a computation arise), can be reconciled easily when regarded as two manifestations of the same general principle.

Technicalities aside, our hope is to involve different communities in a fertile ground for improvement and generalisation: the growing boundary region between computer science, categorical logic, and pure category theory will hopefully fruitfully exploit the consequences of our investigation.

Structure of the paper.

As category theorists, we believe the most fruitful way to present this story is providing two different keys to the reader.

The *analytic* perspective focuses on the specifics of the fibrations of algebras, on concrete examples, and on structural theorems largely relying on fibred category theory, and occupies section 2.1 and chapter 3. All fibrations of algebras of a given type (we outline three: mere endofunctor algebras, pointed algebras, and Eilenberg-Moore –plus, obviously, their duals) arises pulling back along a universal one, and thus all properties of such a universal fibration of algebras defined in Definition 2.1.2 (and its dual in Definition 2.1.4) that are stable under pullback transfer to each fibration of algebra for a specific parametric endofunctor. The existence of limits and colimits in fibers, and in the total categories, can be analyzed, and the existence of adjoints to reindexings can be proved.

The *synthetic* perspective, on the other hand, adopts a more algebraic style, akin to formal category-theoretic methods [SW78], and is based on the micro/macro-cosmic intuition that a parametric endofunctor $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ can evidently be thought of as an algebra in its own right, for the endo-2-functor $\mathcal{A} \times \_ : \mathsf{Cat} \longrightarrow \mathsf{Cat}$ or, which is equivalent, as an Eilenberg–Moore algebra for the monad $\mathcal{A}^* \times \_$ induced by the free monoidal category on $\mathcal{A}$. This will be the intuition guiding the beginning of our chapter 4. As such, the operation of forming $\mathcal{A} \ltimes \mathcal{X}$ out of $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ *is itself* a functor of type

$$\_ \ltimes \_ : \mathsf{Cat} \ltimes \mathsf{Cat} \longrightarrow \mathsf{Cat} \tag{1.1.3}$$

(cf. Theorem 3.3.6, where this is phrased in terms of an example).

This perspective provides a clear connection with the theory of *graded monads* of [Smi08, MPS15, FKM16, MU22, OWE20]: the theory of fibrations of algebras can be thought of as the theory of graded monads over a 'free grading monoid' $\mathcal{A}^*$, in the same way a (set-based) semiautomaton is but an object with an action of the free monoid A^* on the input alphabet A. We explore this point of view to a certain depth, as well as its relations with the state of the art, in Remark 4.1.5.

Far from being the only formal consideration in this respect, this shift on perspective provides a useful further abstraction of our theory: a parametric endofunctor $F : \mathcal{A}\times\mathcal{X} \longrightarrow \mathcal{X}$ can be thought of as an abstract way in which $\mathcal{A}$ acts on $\mathcal{X}$, and thus the fibration of algebras arising from can be thought of as a categorified *semidirect product* $\mathcal{A} \ltimes_F \mathcal{X}$; we explore this point of view in section 4.3, with particular focus on the possibility of building a category $\mathsf{Ext}(\mathcal{A}, \mathcal{X})$ of 'extensions of $\mathcal{X}$ by $\mathcal{A}$', in which fibrations of Eilenberg–Moore algebras form a well-behaved subcategory. The heart of the matter is that a fibration $p : \mathcal{E} \longrightarrow \mathcal{A}$ gives rise to a span

$$\mathcal{X} \xleftarrow{V} \mathcal{E} \xrightarrow{p} \mathcal{A} \tag{1.1.4}$$

i.e. to a unique functor $\langle p, V\rangle : \mathcal{E} \longrightarrow \mathcal{A} \times \mathcal{X}$ into the product, and one can prove (cf. Theorem 4.2.1 and its dual) that p is a fibration of (co)Eilenberg–Moore algebras of a parametric (co)monad $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ if and only if $\langle p, V\rangle$ is (co)monadic.

It is to be noted that this perspective is very pliable and 'formal', as the main results of section 4.3 concerning 'exact sequences' of functors export without much effort to the fibrations in an abstract Cartesian 2-category $\mathcal{K}$, with some additional assumptions.

A diagram similar to (1.1.4) can be considered in such a 2-category, and a 'fibration of (co)algebras' for a parametric 1-cell $f : A \times X \longrightarrow X$ can be *defined* as a (co)monadic 1-cell $p : E \longrightarrow A \times X$.

Moreover, (1.1.4) can be extended to a diagram of adjoints

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "V"] \arrow[r, shift left, "p"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left] \end{tikzcd} \tag{1.1.5}$$

under mild assumptions on $\mathcal{A}, \mathcal{X}$, and this turns (1.1.5) into a short exact sequence to all intents and purposes.

Going even further, the same diagram can be interpreted in an abstract 2-category other than Cat; this is done at the end of chapter 4. The interest in developing the theory so formally resides, evidently, in the desire to restrict the domain in which adjunctions like (1.1.4) are considered: for example, $\mathcal{A}, \mathcal{X}$ could be pointed, Abelian, accessible, or monoidal categories, while p, V and their adjoints might suitably preserve these additional properties/structure.

Notation and conventions.

For most of our discussion, a category $\mathcal{A}$ of 'parameters' and categories $\mathcal{X}, \mathcal{Y}, \dots$ 'acted on by $\mathcal{A}$' will be fixed. This action is embodied in a 'parametric endofunctor', i.e. into a functor $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$. When the functor is a monad, we write $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ instead, and when it's a comonad, we write $S : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$. A monad structure will usually be denoted $\langle T_A, \mu^A, \eta^A \rangle$, and a comonad structure $\langle S_A, \delta^A, \epsilon^A \rangle$; clearly, $\mu^A, \eta^{\cdot} \delta^A, \epsilon^A$ are natural maps in A. We say that a parametric endofunctor $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ *admits parametric free algebras* if each forgetful functor $U_A : \mathsf{Alg}_{\mathcal{X}}(F_A) \longrightarrow \mathcal{X}$ has a left adjoint. Clearly, if the (parametric) endofunctor is a monad and we consider its Kleisli or Eilenberg–Moore categories it admits parametric free algebras. It is clear what we mean when we say that a parametric endofunctor admits (parametric) cofree coalgebras. In each of the case we study, the parametric endofunctors, monads, comonads $F, T, S, \dots$ embody the abstract action of $\mathcal{A}$ on $\mathcal{X}$; the total category of the fibration of algebras p they generate will be denoted as a semidirect product $\mathcal{A} \ltimes_F \mathcal{X}, \mathcal{A} \ltimes_T^{\mathsf{EM}} \mathcal{X}$, etc. to stress the intuition that a monoid $\mathcal{A}$ is acting on another monoid $\mathcal{X}$ through monoid homomorphisms. In this perspective, we occasionally use the notation $\{A\} \ltimes \mathcal{X}$ to refer to the fiber of the fibration p over A; clearly, $\{A\} \ltimes \mathcal{X}$ is just $\mathsf{Alg}_{\mathcal{X}}(F_A)$. It will be of particular interest to study how $\{\varnothing\} \ltimes \mathcal{X}, \{\mathbf{1}\} \ltimes \mathcal{X}$ interact with $\mathcal{X}$ and with the other algebras (for example, every T_A-algebra $(X, x)^A$ is also a $T_\varnothing$-algebra via reindexing along $\varnothing \to A$). Along the whole paper, we employ basic terminology from 2-dimensional category theory (such as: the definition of pseudofunctor, [Gra66, Gra74], see also [Lac10, JY21] for introductory texts) and in subsection 4.6.2 we freely employ terminology and results about 2-dimensional limits and colimits; the go-to reference for such constructions are is classical [Kel89] (the notation and nomenclature of which we adopt mostly in full), a nicely written overview is in John Bourke's PhD thesis [Bou11, Ch. 2], and a concrete description of some bilimits is in [MP89, Ch. 5] tied to the completeness theorem for the 2-category of accessible categories.

1.2 Fibrations and universal fibrations

The notion of *fibration* originated in the context of topology and is based on an old idea of Grothendieck [GAV72, Vis05], that the correspondence between sheaves and local homeomorphisms on a space X can be generalised to non-thin categories. In fact, in Grothendieck work shows that *every* small category $\mathcal{B}$ shall be thought of as some sort of 'generalised space' [JT91, HS01]; in this perspective, every functor $F : \mathcal{B}^{\mathrm{op}} \to \mathsf{Cat}$ – not only those defined over a category of open subsets – might be thought of as a certain generalised fibre bundle over $\mathcal{B}$ whose fibres are exactly the various categories FB.

A complementary take on fibrations stems from the work of F.W. Lawvere [Law63], where it was observed that such an object could quite appropriately represent key behaviour of logical systems, as one is usually inclined to consider formulas or types in a given context (an object Γ of a base category of *contexts*, usually freely generated), and to collect all of those over a given fixed context into a set, poset, space, or category (interpreting the judgment $\Gamma \vdash t$). In this perspective, the reader can consider $\mathcal{B}$ to be a category of *contexts and substitutions*, and the functor $F : \mathcal{B}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ to be this process of collection.

One of the fundamental results in the theory of fibrations is the *Grothendieck construction*, which substantiates the idea that a given (pseudo)functor taking values in Cat can be thought of a bundle-like structure, and ultimately the theory of *fibrations* provides an analogue for the notion of a space/category 'spreading' (*étalé*, cf. [Ten75]) over another and for the notion of local homeomorphism.

1.2.1 Main definitions

In differential geometry and algebraic topology, it is common to denote $p : E \to B$ a fiber bundle; the topological spaces E, B are respectively thought as the *total space* (E stands for *espace*) and the *base* of the bundle. We maintain such an intuition here.

Definition 1.2.1 (Cartesian morphism)**.** Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ a functor and $f : E' \to E$ a morphism in $\mathcal{E}$. We say that f is *p-Cartesian* or *Cartesian* over u if $p(f) = u$ and for any other $g : Z \to E$ and w such that $pg = u \circ w$ there is a unique $h : Z \to E'$ in $\mathcal{E}$ such that $p(h) = w$ and $f \circ h = g$. The situation is conveniently depicted in the following diagram:

$$\begin{tikzcd}
& Z \arrow[dl, dotted, "h"'] \arrow[dr, "g"] & & \\
E' \arrow[rr, "f"'] & & E & \mathcal{E} \\
& {} \arrow[d, mapsto] & & \\
& pZ \arrow[dl, "w"'] \arrow[dr, "pg"] & & \\
* \arrow[rr, "u=pf"'] & & * & \mathcal{B}
\end{tikzcd} \tag{1.2.1}$$

In such a situation, we say that f is a *(p-)Cartesian lifting* (also called a *prone* morphism in [Joh02]) of u.

Remark 1.2.2. As it is the case for every universal property, a p-Cartesian lifting for a given morphism is essentially unique: from the uniqueness of h above we deduce that two Cartesian liftings of a given $u : B \to pE$ are isomorphic in the slice $\mathcal{E}/E$.

Definition 1.2.3 (Fibration)**.** A functor $p : \mathcal{E} \to \mathcal{B}$ is a *fibration* if for all E in $\mathcal{E}$, each $u : B \to pE$ has a Cartesian lifting. We also say that $\mathcal{E}$ is *fibered over* $\mathcal{B}$ or that $\mathcal{E}$ is *over* $\mathcal{B}$.

Building on the topological intuition hinted above, oftentimes $\mathcal{B}$ is called the *base category* and $\mathcal{E}$ the *total category* of p.

Definition 1.2.4 (Vertical morphism, fibers)**.** A morphism $f : E' \to E$ such that $p(f)$ is an identity in the base is called a *vertical* morphism. For B in $\mathcal{B}$ we write $\mathcal{E}_C$ for the subcategory of $\mathcal{E}$ of objects and vertical maps over B: this is the *fiber over* B. We might also denote $\mathcal{E}_u$ the subcategory of $\mathcal{E}$ of objects and maps that are over u in $\mathcal{B}$.

Many properties that one usually lists of both Cartesian and vertical maps, such as that of being classes that are closed under composition, are contained in the following result (see [Jac98, Exercise 1.1.3.(i)]).

Proposition 1.2.5 (Vertical-Cartesian factorisation system)**.** Consider $p : \mathcal{E} \to \mathcal{B}$ a fibration. The classes of vertical and Cartesian morphisms form a orthogonal factorisation system on $\mathcal{E}$.

Proposition 1.2.5 entails at once a number of corollaries, that follow from the general calculus of factorisation systems [FK72, §2], [Bor94a, §5.5]:

C1) if g and gf are vertical, then so is f; dually, if gf and f are Cartesian, so is g;

C2) pullbacks of vertical maps along Cartesian ones exist and are vertical; dually, pushouts of Cartesian maps along vertical one exist and are Cartesian

C3) the classes of vertical arrows and Cartesian arrows determine each other under the relation of orthogonality.

Example 1.2.6 (The fundamental fibration)**.** One of the main examples of fibration is that related to the slice construction. Consider a category $\mathcal{B}$ and the codomain functor $\mathsf{cod} : \mathcal{B}^2 \to \mathcal{B}$: the universal property of Cartesian liftings becomes that of pullbacks, so that a square over a map in $\mathcal{B}$ is cod-Cartesian if and only if it is a pullback. In fact, the very term *Cartesian* is inspired by this example.

More generally, it is possible to show that

- cod is an opfibration (the dual of a fibration: see Definition 1.2.16), and
- cod is a fibration if and only if $\mathcal{B}$ has pullbacks.

Dual results hold for the domain functor dom. In fact, we can present the functors dom and cod together, in a span called a *two-sided fibration*

$$\mathcal{B} \xleftarrow{\mathsf{cod}} \mathcal{B}^2 \xrightarrow{\mathsf{dom}} \mathcal{B} \tag{1.2.2}$$

The slice category $\mathcal{B}/B$ is the fiber over B of the codomain functor cod; dually, the coslice $B/\mathcal{B}$ is the fiber of dom.

Example 1.2.7 (The simple slice, [Jac98, 1.3])**.** Another slice-like fibration is the so called *simple slice*, which is of much interest to the study of simple type theory, and will be thoroughly discussed in chapter 3.

Let $\mathcal{B}$ be a category with products, and consider the category $s(\mathcal{B})$ whose

- objects are pairs (I, X) of objects in $\mathcal{B}$;
- arrows $(J, Y) \to (I, X)$ are pairs (u, f) of maps in $\mathcal{B}$, with $u : J \to I$ and $f : J \times Y \to X$.

Composition can be defined using the universal property of products, meaning that the composite of maps

$$(K, Z) \xrightarrow{(v,g)} (J, Y) \xrightarrow{(u,f)} (I, X) \tag{1.2.3}$$

is defined as the pair $(u \circ v, f \circ \langle g, v \circ pr_1 \rangle)$, and identities are of the form (id, pr_2). The obvious projection functor $s(\mathcal{B}) \longrightarrow \mathcal{B}$ sending

$$(I, X) \mapsto I, \quad (u, f) \mapsto u \tag{1.2.4}$$

is a fibration. Its fibers are also denoted $s(\mathcal{B})_I$, they have objects X in $\mathcal{B}$ and maps $X' \to X$ are $f : I \times X' \to X$ in $\mathcal{B}$: one can think of these as I-indexed families $f_i : X' \to X$ with fixed domain and codomain.

Definition 1.2.8 (Cleavage)**.** A *cleavage* S for a fibration $p : \mathcal{E} \to \mathcal{B}$ is a choice for each E in $\mathcal{E}$ and $u : C' \to pE$ of a Cartesian lifting of u at E. We denote it $s_{E,u} : S(E,u) \to E$. A cleavage induces for each $u : B' \to B$ in $\mathcal{B}$ a functor $\mathcal{E}_B \to \mathcal{E}_{B'}$ by Cartesianity of $s_{E,u}$.

When no explicit definition of a cleavage is made, one can also write u^* for $S(-,u)$.

Although this process might seem strictly functorial, it is not always the case that for composable morphisms u and v one has

$$S(-, u \circ v) = S(-, v) \circ S(-, u). \tag{1.2.5}$$

However, one can show that the two are uniquely isomorphic. One can see such a behavior in Example 1.2.7 and in many other examples that occur naturally in the mathematical practice. Similarly, one does not necessarily have $S(-, \mathrm{id}) = \mathrm{id}$, but only a canonical isomorphism.

Definition 1.2.9 (Split fibration)**.** A cleavage S for p is called *split* or *splitting* if the following properties are satisfied:

- $s_{E,\mathrm{id}B} = \mathrm{id}E$;
- $s_{E,u\circ v} = s_{E,u} \circ s_{S(E,u),v}$.

A fibration is called *split* if it is endowed with a split cleavage.

Theorem 1.2.10 ([Gra66, Theorem 2.10])**.** The following conditions are equivalent for a functor $p : \mathcal{E} \longrightarrow \mathcal{B}$

1. p is a fibration;
2. for each E in $\mathcal{E}$, $p/E : \mathcal{E}/E \longrightarrow \mathcal{B}/pE$ has a right adjoint right inverse (*rari*).

Intuitively, the right adjoint performs the action of picking a lifting (the perceptive reader will have noticed that the choice involved in setting the rari for p is fixing a cleavage, and vice versa, a cleavage determines a rari for p).

1.2.2 2-categorical properties of fibrations

Fibrations, and morphisms of the slice $\mathsf{Cat}/\mathcal{B}$, organise into objects and morphisms of a 2-category.

Definition 1.2.11 (The 2-category of fibrations)**.** Call Fib the 2-category having

- for 0-cells fibrations;
- for 1-cells *strict fibration morphisms* $p \to p'$ i.e. pairs of functors (H, K) making the square

$$\begin{tikzcd} \mathcal{E} \arrow[r, "H"] \arrow[d, "p"'] & \mathcal{E}' \arrow[d, "p'"] \\ \mathcal{B} \arrow[r, "K"'] & \mathcal{B}' \end{tikzcd} \tag{1.2.6}$$

  commute and such that if a map s is p-Cartesian over u, then $H(s)$ is p'-Cartesian over $K(u)$;

- for 2-cells $(H_1, K_1) \to (H_2, K_2)$ pairs of natural transformations (ϕ, ψ) with $\phi : H_1 \Rightarrow H_2$ and $\psi : K_1 \Rightarrow K_2$ such that $p' * \phi = \psi * p$.

Most of the times one is particularly interested in categories over a fixed base $\mathcal{B}$, for instance in Theorem 1.2.13. We denote with $\mathsf{Fib}(\mathcal{B})$ the resulting 2-category.

One can quickly see that Fib is a 2-full subcategory of the arrow category $\mathsf{Cat}^{\to}$. Not only that, but the composition of such inclusion with $\mathsf{cod} : \mathsf{Cat}^{\to} \longrightarrow \mathsf{Cat}$ yields a fibration itself, with fibers precisely the categories $\mathsf{Fib}(\mathcal{B})$.

Proposition 1.2.12 (The fibration of fibrations, [Jac93, Prop. 2.6])**.** The functor that sends each fibration $p : \mathcal{E} \longrightarrow \mathcal{B}$ to its base $\mathcal{B}$ is itself a fibration

$$\mathsf{Fib} \longrightarrow \mathsf{Cat} \tag{1.2.7}$$

such that the fiber $\mathsf{Fib}_{\mathcal{B}}$ is $\mathsf{Fib}(\mathcal{B})$.

One could of course consider the 2-subcategories of all fibrations that are discrete or split – in this case we additionally ask that Cartesian functors preserve the cleavage.

1.2.3 The Grothendieck construction

Let us assume from now on that each fibration comes equipped with its own cleavage – this can be achieved either by adding strong choice axioms, or by slightly extending the definitions.

Theorem 1.2.13 (Grothendieck construction, [Gro71])**.** There exists a 2-equivalence

$$\mathsf{Fib}(\mathcal{B}) \cong \mathsf{Psd}[\mathcal{B}^{\mathrm{op}}, \mathsf{Cat}] \tag{1.2.8}$$

between the 2-category of fibrations (with base $\mathcal{B}$), Cartesian functors, and natural transformations, and that of contravariant pseudofunctors (from $\mathcal{B}$), pseudonatural transformations, and modifications. Here Cat stands for the 2-category of categories, functors and natural transformations.

The reader is invited to consult [JY21, Chapter 10] for a thorough immersion in the formalism of the Grothendieck construction, and *ibid.*, §4.1 for the definition of a pseudofunctor (as specialisation of the notion of a *lax 2-functor*).

Provided a fibration $\mathcal{E} \longrightarrow \mathcal{B}$, we can define a pseudofunctor $\mathcal{B}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ mapping each B in $\mathcal{B}$ to its fiber category, and each $B' \to B$ to the functor $\mathcal{E}_B \longrightarrow \mathcal{E}'_B$. The universal property of Cartesian liftings provides functoriality only up to isomorphism.

Conversely, a pseudofunctor $F : \mathcal{B}^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ gives rise to a fibration $p : \int F \longrightarrow \mathcal{B}$, where we denote $\int F$ (or $\int_{\mathcal{B}} F$ when the domain of F is not clear from the context) the *category of elements of F*, i.e. the category where

CE1) objects are pairs (B, X) with B in $\mathcal{B}$ and X in $F(B)$ in an object,

CE2) arrows $(B', Y) \to (B, X)$ are pairs (u, s) with $u : B' \to B$ in $\mathcal{B}$ and $s : Y \to F(u)(X)$ in $F(Y)$,

and p is the projection on the first component.

Cartesian functors are in a 1-to-1 correspondence with natural transformations between the relative pseudofunctors, since for a given context we have a functor between the respective fibers if and only if the functor between the total categories is Cartesian.

One can read a full proof in [Bor94b, Sec. 8.3], or learn more on the original work from Grothendieck in [Vis05, Section 3.1].

Remark 1.2.14 (Cartesian maps for a Grothendieck fibration)**.** With respect to $p : \int F \to \mathcal{B}$, Cartesian maps are precisely those (u, f) where f is an isomorphism.

Of course, additional properties of fibrations translate to properties of the corresponding pseudofunctors, and one could almost trivially restrict the 2-equivalence above to the respective 2-subcategories.

1.2.4 Opfibrations, bifibrations

A dual theory for *covariant* pseudofunctors $\mathcal{B} \longrightarrow \mathsf{Cat}$ gives rise to the notion of an *op*fibration. OpCartesian liftings are now initial as arrows with a given domain, and so on.

Definition 1.2.15 (OpCartesian morphism)**.** Let $p : \mathcal{E} \longrightarrow \mathcal{B}$ be a functor and $f : A' \to A$ a morphism in $\mathcal{E}$. We say that s is *p-opCartesian* or *opCartesian* over $u : C' \to C$ if it is Cartesian over u for the opposite functor $p^{\mathrm{op}} : \mathcal{E}^{\mathrm{op}} \longrightarrow \mathcal{B}^{\mathrm{op}}$.We say that f is a *(p-)opCartesian lifting* of u.

Dualising Theorem 1.2.10 one can say that a functor is an opfibration if and only if each A/p has a lali (left adjoint left inverse).

Definition 1.2.16 (Opfibration)**.** A functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is an *opfibration* if for all B in $\mathcal{E}$ we have that each $u : pB \to C$ has a opCartesian lifting.

Clearly, p is an opfibration if and only if p^{op} is a fibration.

Theorem 1.2.17 (Grothendieck construction)**.** There is an equivalence of categories

$$\mathsf{opFib}(\mathcal{B}) \cong \mathsf{Psd}[\mathcal{B}, \mathsf{Cat}]. \tag{1.2.9}$$

Remark 1.2.18. The category $\int F$, image of an $F : \mathcal{B} \longrightarrow \mathsf{Cat}$ through the isomorphism above, fits into a strict 2-pullback of categories and (pseudo)functors

$$\begin{tikzcd} \int F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{1}/\!\!/\mathsf{Cat} \arrow[d, "U"] \\ \mathcal{B} \arrow[r, "F"'] & \mathsf{Cat} \end{tikzcd} \tag{1.2.10}$$

where $\mathbf{1}/\!\!/\mathsf{Cat}$ is the *lax slice* [Kel74, §4] of 'laxly pointed categories', meaning of pairs $(\mathcal{A}, A)$ with $\mathcal{A}$ a category and A one of its objects, and of morphisms laxly preserving the points. The Grothendieck construction asserts that all opfibrations over $\mathcal{B}$ arise by pulling back along U.

A dual notion of this result holds for fibrations: they are all pullbacks along the opposite of the forgetful from the colax slice into Cat.

Functors that are *both* fibrations and opfibrations are particularly well-behaved:

Definition 1.2.19 (Bifibration)**.** A functor $p : \mathcal{E} \longrightarrow \mathcal{B}$ is an *bifibration* if it is both a fibration and an opfibration.

Lemma 1.2.20 (Characterizing bifibrations, [Jac98, 9.1.2])**.** A fibration is a bifibration if and only if each reindexing $S(-, u)$ has a left adjoint Σ_u.

Chapter 2

Fibrations of algebras

Summary of chapter

Once we fix a functor $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ so that $F_A : \mathcal{X} \longrightarrow \mathcal{X}$ for every A, naturally in A, the functor

$$\begin{aligned} \mathcal{A}^{\mathrm{op}} &\longrightarrow \mathsf{Cat} \\ A &\longmapsto \mathsf{Alg}(F_A) \end{aligned}$$

has an associated split fibration, which we study together with all other possible variations on this theme (if F is a monad, take its categories $\mathsf{EM}(F_A)$, or the Kleisli categories $\mathsf{Kl}(F_A)$, and dually if it is a comonad). We build a general theory of such construction and define a 'fibration of algebras' in Definition 2.1.1. All fibrations of algebras arise pulling back along a universal one, cf. Definition 2.1.2. In §2.2 we move to study the 'analytic' properties of such fibrations: preservation of limits by the reindexing functors $\alpha^* : \mathsf{Alg}_{\mathcal{X}}(F_{A'}) \longrightarrow \mathsf{Alg}_{\mathcal{X}}(F_A)$, and their failure to preserve colimits; existence of adjoints to reindexings.

2.1 Fibrations of co/algebras

2.1.1 Universal fibrations of co/algebras

Definition 2.1.1 (The category of endofunctor algebras)**.** Define the *category of (endofunctor) algebras* $\int \mathsf{Alg}_{\mathcal{X}}$ as follows:

- objects are pairs (F, x) where F is a functor $\mathcal{X} \longrightarrow \mathcal{X}$ and $x : FX \to X$ an F-algebra;
- an arrow $(F, x) \to (G, y)$ is a pair (α, f) given by a natural transformation $\alpha : F \Rightarrow G$ and an arrow $f : X \to Y$, making the following diagram in $\mathcal{X}$ commute.

$$\begin{tikzcd} FX \arrow[r, "Ff"] \arrow[d, "x"'] & FY \arrow[r, "\alpha_X"] & GY \arrow[d, "y"] \\ X \arrow[rr, "f"'] & & Y \end{tikzcd} \tag{2.1.1}$$

Composition is performed componentwise, while identities are $\mathrm{id}(F, x) = (\mathrm{id}F, \mathrm{id}X)$ for $x : FX \to X$.

One can see that a morphism in $\int \mathsf{Alg}_{\mathcal{X}}$ is just a homomorphism of F-algebras from $\left[\begin{smallmatrix} FX \\ x \downarrow \\ X \end{smallmatrix}\right]$ to $\alpha^* \left[\begin{smallmatrix} GY \\ y \downarrow \\ Y \end{smallmatrix}\right] = \left[\begin{smallmatrix} FY \\ y \circ \alpha_X \downarrow \\ Y \end{smallmatrix}\right]$.

Definition 2.1.2 (The universal fibration of endofunctor algebras)**.** The *fibration of algebras*

$$U : \textstyle\int \mathsf{Alg}_{\mathcal{X}} \longrightarrow [\mathcal{X}, \mathcal{X}] \tag{2.1.2}$$

is the functor which sends (X, x) and (α, f) to X and α respectively.

Notice that the fibration of algebras is precisely the functor obtained by applying the Grothendieck construction (1.2.3) to the pseudofunctor

$$\mathsf{Alg}_{\mathcal{X}}(\_) : [\mathcal{X}, \mathcal{X}]^{\mathrm{op}} \longrightarrow \mathsf{Cat} \tag{2.1.3}$$

sending F to $\mathsf{Alg}_{\mathcal{X}}(F)$ and $\alpha : F \Rightarrow G$ to $\alpha^* : \mathsf{Alg}_{\mathcal{X}}(G) \longrightarrow \mathsf{Alg}_{\mathcal{X}}(F) : \left[\begin{smallmatrix} GX \\ x \downarrow \\ X \end{smallmatrix}\right] \mapsto \left[\begin{smallmatrix} FX \\ x \circ \alpha_X \downarrow \\ X \end{smallmatrix}\right]$.

Remark 2.1.3. Since $\mathsf{Alg}_{\mathcal{X}} : [\mathcal{X}, \mathcal{X}]^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ is a functor, not only a pseudofunctor, $U : \int \mathsf{Alg}_{\mathcal{X}} \longrightarrow [\mathcal{X}, \mathcal{X}]$ is not only a fibration, but a *split* (1.2.9) one.

Of course we can dualise the previous definition as follows.

Definition 2.1.4 (The universal opfibration of endofunctor coalgebras)**.** Define the *category of (endofunctor) coalgebras* $\int \mathsf{coAlg}_{\mathcal{X}}$ as follows:

- objects are pairs (F, x) where F is a functor $\mathcal{X} \longrightarrow \mathcal{X}$ and $x : X \to FX$ an F-coalgebra;
- an arrow $(F, x) \to (G, y)$ is a pair (α, f) given by a natural transformation $\alpha : F \Rightarrow G$ and an arrow $f : X \to Y$, making the following diagram in $\mathcal{X}$ commute.

$$\begin{tikzcd} FX \arrow[r, "\alpha_X"] & GX \arrow[r, "Gf"] & GY \\ X \arrow[u, "x"] \arrow[rr, "f"'] & & Y \arrow[u, "y"'] \end{tikzcd} \tag{2.1.4}$$

The *opfibration of coalgebras* $\left[\begin{smallmatrix} \int \mathsf{coAlg}_{\mathcal{X}} \\ U \downarrow \\ [\mathcal{X}, \mathcal{X}] \end{smallmatrix}\right]$ is the functor which sends (F, x) and (α, f) to F and α respectively.

As it is the case for algebras, we can characterise the opfibration of coalgebras as the functor obtained applying the Grothendieck construction to the pseudofunctor sending $F \in [\mathcal{X}, \mathcal{X}]$ to $\mathsf{coAlg}_{\mathcal{X}}(F)$ and $\alpha : F \Rightarrow G$ to $\alpha_* : \mathsf{coAlg}_{\mathcal{X}}(F) \longrightarrow \mathsf{coAlg}_{\mathcal{X}}(G)$ sending (X, x) to $(X, \alpha_X \circ x)$.

2.1.2 Main definitions

Definition 2.1.5. A *fibration of endofunctor algebras* modeled on a parametric endofunctor $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ is a fibration resulting as the pullback of F along the universal fibration of algebras of Definition 2.1.2.

Therefore, a fibration of endofunctor algebras modeled on F is the left vertical leg of the strict pullback of categories below.

$$\begin{tikzcd} \mathcal{A} \ltimes_F \mathcal{X} \arrow[r] \arrow[d, "p_F"'] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{Alg}_{\mathcal{X}} \arrow[d, "U"] \\ \mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]} \end{tikzcd} \tag{2.1.5}$$

Remark 2.1.6. This entire work attempts to make a point that a good intuition for the category at the upper left corner of (2.1.5) is as a semidirect product of the category of parameters $\mathcal{A}$, acting on the category $\mathcal{X}$ via the parametric endofunctor $F : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]$. Whence our choice of a notation reminiscent of the semidirect product operation in group/-monoid theory. It will often be clear from the context which functor F will be considered, so we will drop the subscript F from $\mathcal{A} \ltimes_F \mathcal{X}$.

Remark 2.1.7. Unwinding the above definition, a fibration of algebras p_F has domain the category defined as follows:

- the objects are triples $(A; X, x)$ where $A \in \mathcal{A}$ is an object (intuitively the 'parameter') and $x^A : F_A X \to X$ is an algebra for $F_A = F(A, -)$; we will freely employ concise notations to denote an object of $\mathcal{A} \ltimes \mathcal{X}$, such as (X, x^A) or $(X, x)^A$;
- morphisms $(u, f) : (A; X, x) \to (A'; Y, y)$ consist of pairs $(u, f) \in \mathcal{A}(A, A') \times \mathcal{X}(X, Y)$ such that the following diagram commutes.

$$\begin{tikzcd} F_A X \arrow[r, "F_A f"] \arrow[d, "x^A"'] & F_A Y \arrow[r, "F_u Y"] & F_{A'} Y \arrow[d, "y^{A'}"] \\ X \arrow[rr, "f"'] & & Y \end{tikzcd} \tag{2.1.6}$$

Clearly, the reindexing functors $u^* : \mathsf{Alg}_{\mathcal{X}}(F_{A'}) \longrightarrow \mathsf{Alg}_{\mathcal{X}}(F_A)$ carry the A'-algebra $\left[\begin{smallmatrix} F_{A'}Y \\ \downarrow \\ Y \end{smallmatrix} \right]$ to the A-algebra $u^*(Y, y) = \left[y \circ F_u Y \begin{smallmatrix} F_A Y \\ \downarrow \\ Y \end{smallmatrix} \right]$, and the commutativity condition in (2.1.6), which is simply the same commutativity condition of (2.1.1) – just parametric in A – expresses the fact that $f : X \to Y$ is a morphism of algebras $(X, x) \to u^*(Y, y)$.

We can give a similar definition for a fibration of endofunctor coalgebras modeled on F; unwinding such a dualisation yields the following.

Definition 2.1.8. An opfibration of coalgebras q_F modeled on $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ has domain the category defined as follows:

- the objects are triples $(A; X, x^A)$ where $A \in \mathcal{A}$ is an object (the 'parameter') and $x^A : X \to F_A X$ is a coalgebra for $F_A = F(A, -)$;
- morphisms $(u, f) : (A; X, x) \to (A'; Y, y)$ consist of pairs $(u, f) \in \mathcal{A}(A, A') \times \mathcal{X}(X, Y)$ such that the following diagram commutes,

$$\begin{tikzcd} F_A X \arrow[r, "F_u X"] & F_{A'} X \arrow[r, "F_{A'} f"] & F_{A'} Y \\ X \arrow[u, "x"] \arrow[rr, "f"'] & & Y \arrow[u, "y"'] \end{tikzcd} \tag{2.1.7}$$

in such a way that $f : X \to Y$ is a morphism of coalgebras $u_*(X, x) \to (Y, y)$.

Using a similar approach, we can define other kinds of classifying fibrations: we replace the fibration of endofunctor algebras U of Definition Definition 2.1.2 with suitable subfibrations

$$\begin{tikzcd} \int \mathsf{Alg}_{\mathcal{X}*} \arrow[d, "U_\eta"] & \int \mathsf{EM}_{\mathcal{X}} \arrow[d, "U_\mu"] \\ {[\mathcal{X}, \mathcal{X}]_\eta} & {[\mathcal{X}, \mathcal{X}]_\mu} \end{tikzcd} \tag{2.1.8}$$

defined as follows:

- the subcategory $[\mathcal{X}, \mathcal{X}]_\eta \subseteq [\mathcal{X}, \mathcal{X}]$ to be the category of *pointed endofunctors* (F, η), i.e. those $F : \mathcal{X} \longrightarrow \mathcal{X}$ equipped with a natural transformation $\eta^F : \mathrm{id}\mathcal{X} \Rightarrow F$ from the identity functor, and U_η arises pulling back U;
- the subcategory $[\mathcal{X}, \mathcal{X}]_\mu \subseteq [\mathcal{X}, \mathcal{X}]_\eta$ to be the category of *monads* (T, μ, η), where the pointed endofunctor (T, η) is also equipped with an associative multiplication $\mu TT \Rightarrow T$ having η as unit; U_μ arises pulling back U_η.

Hence, we can arrange these classifying fibrations in a diagram of pullbacks

$$\begin{tikzcd} \int \mathsf{Alg}_{\mathcal{X}} \arrow[d] & \int \mathsf{Alg}_{\mathcal{X}*} \arrow[l] \arrow[d] & \int \mathsf{EM}_{\mathcal{X}} \arrow[l] \arrow[d] \\ {[\mathcal{X}, \mathcal{X}]} & {[\mathcal{X}, \mathcal{X}]_\eta} \arrow[l] & {[\mathcal{X}, \mathcal{X}]_\mu} \arrow[l] \end{tikzcd} \tag{2.1.9}$$

where the horizontal maps are the obvious inclusions.

Remark 2.1.9. The inclusions above are faithful, but clearly not full functors: we restrict a natural transformation to be compatible with the additional structure on endofunctors T, S, and in particular to be compatible

MH1) with the units, for pointed endofunctors; this means that $\alpha : T \Rightarrow S$ is such that $\alpha \circ \eta^T = \eta^S$;

MH2) with the multiplication and the unit, for monads; this means that $\alpha : T \Rightarrow S$ is such that $\alpha \circ \eta^T = \eta^S$ and in addition, $\alpha \circ \mu^T = \mu^S \circ (\alpha * \alpha)$.

Remark 2.1.10. The morphisms of monads that we consider are monad morphisms (intertwiners, or monad opfunctors in the terminology of [Str72]) over the identity functor (for the obvious forgetful functor $\mathsf{Mnd}(\mathsf{Cat}) \longrightarrow \mathsf{Cat} : (\mathcal{X}, T) \mapsto \mathcal{X}$); in simpler terms, in MH1 and MH2 we are considering monoid homomorphisms, if we recognise $\mathsf{Mnd}(\mathcal{X})$ as the category of internal monoids in $([\mathcal{X}, \mathcal{X}], \circ)$ in the well-known way.

Definition 2.1.11. Consider a fibration $\left[\begin{smallmatrix} \mathcal{E} \\ p\downarrow \\ \mathcal{A} \end{smallmatrix}\right]$ obtained as pullback

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & \bullet \arrow[d, "\Gamma"] \\ \mathcal{A} \arrow[r] & \bullet \end{tikzcd} \tag{2.1.10}$$

along another fibration Γ. We say that p is

- an *(endofunctor) algebra fibration* if $\Gamma = U$ in (2.1.5);
- a *pointed algebra fibration* if $\Gamma = U_\eta$ in (2.1.8);
- an *Eilenberg–Moore fibration* if $\Gamma = U_\mu$ in (2.1.8).

Similar refinements apply to the endofunctor coalgebra opfibration of Definition 2.1.8, and we define opfibrations of coalgebras, of copointed coalgebras, and of coEilenberg–Moore coalgebras as fitting in a diagram below.

$$\begin{tikzcd} \int \mathsf{coAlg}_{\mathcal{X}} \arrow[d, "U"] & \int \mathsf{coAlg}_{\mathcal{X}*} \arrow[l] \arrow[d, "U_\epsilon"] & \int \mathsf{coEM}_{\mathcal{X}} \arrow[l] \arrow[d, "U_\delta"] \\ {[\mathcal{X},\mathcal{X}]} & {[\mathcal{X},\mathcal{X}]_\epsilon} \arrow[l] & {[\mathcal{X},\mathcal{X}]_\delta.} \arrow[l] \end{tikzcd} \tag{2.1.11}$$

From this, we can define an opfibration of coalgebras, copointed coalgebras, or a coEilenberg–Moore opfibration as a functor fitting, respectively, in the following kinds of pullback:

$$\begin{tikzcd} \mathcal{A} \oplus \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{coAlg}_{\mathcal{X}} \arrow[d, "U"] \\ \mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \oplus^* \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{coAlg}_{\mathcal{X}*} \arrow[d, "U_\epsilon"] \\ \mathcal{A} \arrow[r, "F"'] & {[\mathcal{X},\mathcal{X}]_\epsilon} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{coEM}_{\mathcal{X}} \arrow[d, "U_\delta"] \\ \mathcal{A} \arrow[r, "T"'] & {[\mathcal{X},\mathcal{X}]_\delta} \end{tikzcd} \tag{2.1.12}$$

(Recall our informal convention that the letter F denotes a parametric –pointed– endofunctors and T a parametric monad.)

2.1.3 Kleisli's version

Let us follow the notation in (2.1.8) that considers the category $[\mathcal{X},\mathcal{X}]_\mu$ of monads $T : \mathcal{X} \longrightarrow \mathcal{X}$ and natural transformations compatible with the monad structure (=monoid homomorphisms). We then have the following.

Proposition 2.1.12. There exists a pseudofunctor $\mathsf{Kl}_{\mathcal{X}} : [\mathcal{X},\mathcal{X}]_\mu \to \mathsf{Cat}$ which sends a monad (T,η,μ) to its Kleisli category, defined on morphisms $\alpha : T \Rightarrow S$ as follows:

- each $\alpha_* : \mathsf{Kl}_{\mathcal{X}}(T) \longrightarrow \mathsf{Kl}_{\mathcal{X}}(S)$ is the identity on objects;
- on morphisms a Kleisli map $X \to TY$ goes to $X \to TY \to SY$.

Functoriality follows from the fact that α makes diagrams MH1 and MH2 commute: the fact that $\alpha_*(\eta^T_X) = \eta^S_X$ is precisely condition MH1, and composition is easily seen to be preserved.

Dualising this line of reasoning, we obtain the following result.

Proposition 2.1.13. There exists a functor $\mathsf{Kl}_{\mathcal{X}} : [\mathcal{X},\mathcal{X}]_\delta^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ which sends a comonad (S,ϵ,δ) to its Kleisli category.

Each reindexing α^* in this case is the identity on objects, and sends a coKleisli map $SX \xrightarrow{f} Y$ to $TX \xrightarrow{\alpha_X} SX \xrightarrow{f} Y$.

Note that the correspondence for free algebras is covariant, and the one for free coalgebras is contravariant, in contrast with the case of Eilenberg–Moore. Motivated by this definition, we can mimic what we have done in Definition 2.1.5, Definition 2.1.8.

Definition 2.1.14. The *universal opfibration of Kleisli categories* is the category having:

- as objects the pairs (T, X) where T is a monad and X an object of the category $\mathsf{Kl}(T)$;
- as morphisms $(T, X) \to (S, Y)$ the pairs $\alpha : T \Rightarrow S$, a monad homomorphism, and $f \in \mathsf{Kl}(S)(\alpha_* X, Y)$.

Dually,

Definition 2.1.15. The *universal fibration of coKleisli categories* is the category having:

- as objects the pairs (T, X) where T is a comonad and X an object of the category $\mathsf{coKl}(T)$;
- as morphisms $(T, X) \to (S, Y)$ the pairs $\alpha : T \Rightarrow S$, a comonad homomorphism, and $f \in \mathsf{coKl}(S)(X, \alpha^* Y)$.

Definition 2.1.16. Let $T : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]_\mu$ be a parametric monad. The *opfibration of free algebras* modeled on T is the category having

- as objects the pairs (A, X) where $A \in \mathcal{A}$ is an object and $X \in \mathsf{Kl}(T_A)$;
- as morphisms $(u, f) : (A, X) \to (A', Y)$ the pairs $u : A \to A'$ and $f : u_* X \to Y$ (a Kleisli morphism in $\mathsf{Kl}(T_{A'})$).

(The functor $u_* : \mathsf{Kl}(T_A) \longrightarrow \mathsf{Kl}(T_{A'})$ acts as $T_{u,*}$, in the notation of Proposition 2.1.12.)

Dually,

Definition 2.1.17. Let $T : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]_\mu$ be a parametric monad. The *fibration of cofree coalgebras* modeled on T is the category having

- as objects the pairs (A, X) where $A \in \mathcal{A}$ is an object and $X \in \mathsf{coKl}(T_A)$;
- as morphisms $(u, f) : (A, X) \to (A', Y)$ the pairs $u : A \to A'$ and $f : X \to u^* Y$ (a coKleisli morphism in $\mathsf{coKl}(T_A)$).

(The functor $u^* : \mathsf{coKl}(T_{A'}) \longrightarrow \mathsf{coKl}(T_A)$ is defined similarly.)

The previous definitions are all straightforward to give. However, the following argument shows a certain care is required when one wants to relate the (co)Kleisli opfibration of a parametric (co)monad with its (co)Eilenberg–Moore fibration. We will say more on this in §section 4.5.

Remark 2.1.18. It would be tempting to assert that, given a parametric monad $T : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]_\mu$, the comparison functors $K_T : \mathsf{Kl}_\mathcal{X}(T) \longrightarrow \mathsf{EM}_\mathcal{X}(T)$ are the components of an object-wise fully faithful pseudonatural transformation $\mathsf{Kl}_\mathcal{X} \Rightarrow \mathsf{EM}_\mathcal{X}$ of sorts, and maybe even of a morphism of fibrations

$$\begin{tikzcd} \int \mathsf{Kl}_\mathcal{X} \arrow[rr] \arrow[dr] & & \int \mathsf{EM}_\mathcal{X} \arrow[dl] \\ & {[\mathcal{X}, \mathcal{X}]_\mu} & \end{tikzcd} \tag{2.1.13}$$

between the Kleisli opfibration and the Eilenberg–Moore fibration, in the slice 2-category over $[\mathcal{X}, \mathcal{X}]_\mu$.

A moment of reflection is however enough to realise that the opposite variance of the pseudofunctors $\mathsf{Kl}_{\mathcal{X}}(\_)$ and $\mathsf{EM}_{\mathcal{X}}(\_)$ gets in the way: in fact, the diagram

$$\begin{tikzcd} \mathsf{Kl}_{\mathcal{X}}(T) \arrow[r, "K"] \arrow[d, "\alpha_*"'] & \mathsf{EM}_{\mathcal{X}}(T) \\ \mathsf{Kl}_{\mathcal{X}}(S) \arrow[r, "K'"'] & \mathsf{EM}_{\mathcal{X}}(S) \arrow[u, "\alpha^*"'] \end{tikzcd} \tag{2.1.14}$$

is not commutative; the best one can do is assess that it is *laxly* commutative, i.e. that it is filled by a 2-cell $\kappa : K \Rightarrow \alpha^* \circ K' \circ \alpha_*$ closing the square.

2.2 Analytic properties of fibrations of algebras

Given how fibrations of algebras arise by pulling back a universal one, we need to study the properties of such 'classifying' fibration $\Gamma_{\mathcal{X}}$. This is what we do in the present subsection.

Recall how $\Gamma_{\mathcal{X}}$ is defined.

Remark 2.2.1. The category $\int \mathsf{Alg}_{\mathcal{X}}$ arises through the Grothendieck construction when applied to the functor $\Gamma : [\mathcal{X}, \mathcal{X}]^{\mathrm{op}} \longrightarrow \mathsf{Cat}$ sending F to its category of algebras $\mathsf{Alg}_{\mathcal{X}}(F)$ and $\alpha : F \Rightarrow G$ to $\alpha^* : \mathsf{Alg}_{\mathcal{X}}(G) \longrightarrow \mathsf{Alg}_{\mathcal{X}}(F)$ defined as:

- $\alpha^*(y) := y \circ \alpha_Y$ for every $y : G(Y) \to Y$;
- $\alpha^*(f) = f$ for every morphism $f : y \to y'$ in $\mathsf{Alg}_{\mathcal{X}}(G)$.

This, in particular implies that $\left[\begin{smallmatrix} & \int \mathsf{Alg}_{\mathcal{X}} \\ U & \downarrow \\ & [\mathcal{X},\mathcal{X}] \end{smallmatrix}\right]$ is a Grothendieck fibration.

Remark 2.2.2. Note that, as it should be, given the shape of (2.1.1) the fibre of U over F consists precisely of the morphisms $f : X \to Y$ with the property that $y \circ Ff = f \circ x$, i.e. of F-algebra morphisms. So, each fibre of U corresponds to the category of algebras of F.

Each natural transformation $\alpha : F \Rightarrow G$ induces a functor $\alpha^* : \mathsf{Alg}_{\mathcal{X}}(G) \longrightarrow \mathsf{Alg}_{\mathcal{X}}(F)$ (by reindexing or looking at the action of Γ on arrows) sending $y : GY \to Y$ to $y \circ \alpha_Y : FY \to Y$, therefore we get the following explicit description of the fibrational structure of U.

The next step is to characterise the shape of Γ-Cartesian arrows in $\int \mathsf{Alg}_{\mathcal{X}}$, and the *vertical-Cartesian* factorisation system (1.2.5) therein.

Remark 2.2.3 (On vertical-Cartesian factorisations)**.** Any given fibration induces a vertical-Cartesian factorisation system on its total category. In the case of the fibration of algebras, it is easily seen to work as follows: let $(\alpha, f) : (F, x) \to (G, y)$ a map in $\int \mathsf{Alg}_{\mathcal{X}}$, then it factors through $(\mathrm{id}F, f)$, vertical, and $(\alpha, \mathrm{id}Y)$, Cartesian.

Similarly, one can characterise (op)Cartesian arrows and the (op)Cartesian-vertical factorisation in all (op) fibrations of algebras introduced so far.

2.2.1 Reindexing and limits

In this section we will look at the limit-preserving properties of the reindexing functors $\alpha^\star : \mathsf{Alg}_{\mathcal{X}}(F) \longrightarrow \mathsf{Alg}_{\mathcal{X}}(G)$; as a consequence of the fact that each forgetful functor $\mathsf{Alg}_{\mathcal{X}}(F)$ creates limits, we will get that each α^* is continuous.

Let us first start by providing a recipe to compute limits of algebras.

Proposition 2.2.4. Given $F : \mathcal{X} \longrightarrow \mathcal{X}$, the forgetful functor $V_F : \mathsf{Alg}_{\mathcal{X}}(F) \longrightarrow \mathcal{X}$ creates limits.

Now recall the following well-known result. In [Str23, Theorem 8.5] it is stated for $\mathcal{J}$ a finite category.

Theorem 2.2.5. Let $\left[\begin{smallmatrix} & \mathcal{E} \\ p & \downarrow \\ & \mathcal{C} \end{smallmatrix}\right]$ be a fibration over a finitely complete category $\mathcal{C}$, and let $\mathcal{J}$ be a small category. Then the following are equivalent:

1. all fibers of p have limits of shape $\mathcal{J}$ and they are preserved by all reindexing functors;
2. $\mathcal{E}$ has limits of shape $\mathcal{J}$ and p preserves them.

Proposition 2.2.6. Let $\mathcal{J}$ a small category, and $\mathcal{X}$ a category which admits $\mathcal{J}$-limits; then $\alpha^* : \mathsf{Alg}_{\mathcal{X}}(G) \longrightarrow \mathsf{Alg}_{\mathcal{X}}(F)$ preserves limits of shape $\mathcal{J}$ for every $\alpha : F \Rightarrow G$.

Now Proposition 2.2.6 and Theorem 2.2.5, together with the fact that limits in $[\mathcal{X}, \mathcal{X}]$ are computed pointwise, entail the following corollary.

Corollary 2.2.7. If $\mathcal{X}$ is complete then $\int \mathsf{Alg}_{\mathcal{X}}$ is complete too and $\left[\begin{smallmatrix} & \int \mathsf{Alg}_{\mathcal{X}} \\ U & \downarrow \\ & [\mathcal{X},\mathcal{X}] \end{smallmatrix}\right]$ preserves all limits.

Remark 2.2.8. Note how this gives an explicit way to compute limits of shape $\mathcal{J}$ in the universal fibration of algebras: let $D : \mathcal{J} \longrightarrow \int \mathsf{Alg}_{\mathcal{X}}$ be a functor; its components are given by $(F_J; X_J, \left[\begin{smallmatrix} & F_J X \\ x_J & \downarrow \\ & X \end{smallmatrix}\right])$ and we can consider the limits $F = \lim_J F_J$ of all parameters, as well as the algebras $\alpha_J^*(X_J, x_J)$, each of which is an F-algebra by change of base; it is a routine exercise to check that the limit (X, x) of these objects in $\mathsf{Alg}_{\mathcal{X}}(F)$ yields the limit of D. A similar argument shows also how to compute limits in a fibration of algebras modelled on $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$: given a diagram $\mathcal{A} \ltimes \mathcal{X}$ with components $(A_J; X_J, x_J)$, compute the limit $A = \lim_J A_J$ of the parameters; reindex all F_{A_J}-algebras to make them F_A-algebras and compute the limit in that fiber.

Existence of colimits in a fibration of algebras is a more delicate issue that we will address in §2.2.3. In general this is not an easy problem to solve. In the case of parametric monads, we can rely on more powerful structural theorems, for which the language of chapter 4 will be essential to recognise $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ as Eilenberg–Moore categories. The question will be addressed in subsection 2.2.3.

2.2.2 Adjoints to reindexing

Fibrations whose reindexing functors have any adjoints are particularly well-behaved: from the point of view of the logic left/right adjoints to reindexing correspond to *dependent sums/products* (whence the notation we use for them below).

With some additional properties on the category $\mathcal{X}$ (=that it is κ-accessible) and restricting to the functors $F \in [\mathcal{X}, \mathcal{X}]$ (=the κ-accessible ones), it is possible to employ a specific form of the adjoint functor theorem to prove the existence of left adjoints for each reindexing.

Theorem 2.2.9. Let $\mathcal{X}$ be κ-presentable and assume that the fibration of algebras is restricted to just the κ-accessible functors $\mathcal{X} \longrightarrow \mathcal{X}$, then each reindexing α^* has a left adjoint $\sum_\alpha$.

The proof follows an argument that in early stages of this work has been dubbed by the authors a 'Freyd swindle';[1] we sketch the idea in the case $\kappa = \omega$, for bigger ordinals one argues similarly.

Proof. Let $\alpha^* : \mathsf{Alg}_{\mathcal{X}}(S) \longrightarrow \mathsf{Alg}_{\mathcal{X}}(T) : (X, x) \mapsto (X, \alpha_X \circ x)$ be the functor between algebras induced by a natural transformation $\alpha : T \Rightarrow S$, and consider a T-algebra $a : TA \to A$; consider the pushout

$$\begin{tikzcd} TA \arrow[r, "\alpha"] \arrow[d, "a"'] & SA \arrow[d] \\ A \arrow[r, "t_0"'] \arrow[ur, phantom, "\ulcorner"] & P_0 \end{tikzcd} \tag{2.2.1}$$

and define inductively the chain at the lower horizontal side of the diagram

$$\begin{tikzcd} TA \arrow[r, "\alpha"] \arrow[d, "a"'] & SA \arrow[r, "St_0"] \arrow[d] & SP_0 \arrow[r, "St_1"] \arrow[d] & SP_1 \arrow[r] \arrow[d] & \dots \arrow[r] & \boxed{?} \arrow[d] \\ A \arrow[r, "t_0"'] \arrow[ur, phantom, "\ulcorner"] & P_0 \arrow[r, "t_1"'] \arrow[ur, phantom, "\ulcorner"] & P_1 \arrow[r, "t_2"'] \arrow[ur, phantom, "\ulcorner"] & P_2 \arrow[r, "t_3"'] & \dots \arrow[r] \arrow[ur, phantom, "\ulcorner"] & P_\infty \end{tikzcd} \tag{2.2.2}$$

The accessibility assumption on S now implies that SP_∞ is the colimit of the upper horizontal chain, so $\boxed{?} = SP_\infty$. The pushout cocone then yields a canonical choice of a map $s_\infty : SP_\infty \to P_\infty$. ☐

Remark 2.2.10. Note (1.2.20) that each reindexing having a left adjoint is equivalent to the fact that p is a bifibration.

Remark 2.2.11. By contrast, universals, i.e. *right* adjoints $\prod_\alpha$ to reindexing functors α^* tend to be nonexistent. In fact, the existence of such a $\prod_\alpha$ requires a filling of the following horn

$$\begin{tikzcd} FY \arrow[rr, "Fh"] \arrow[dr, "\alpha_Y"] \arrow[dd, "\alpha^* y"'] & & FZ \arrow[dr, "\alpha_Z"] \arrow[dd, "z"'] & \\ & GY \arrow[rr, "Gh"] \arrow[dl, "y"] & & GZ \arrow[dl, dotted] \\ Y \arrow[rr, "h"'] & & Z & \end{tikzcd} \tag{2.2.3}$$

[1] The term 'swindle' entered some parts of mathematical practice as a non-derogatory way to refer to 'clever tricks, akin to sleights of hand, providing proofs of true statements based on illegal but evocative manipulations'. For example, the offhanded re-bracketing of an infinite sum (of real numbers, as in Euler's proofs of some analytic identities; of direct sums of R-modules, as in [Lam98, Corollary 2.7] where the term 'Eilenberg swindle' was apparently coined, or of connected sums of compact manifolds, [Tao09]).

From our category-theoretic standpoint, the sleight of hand is instead a demonstration of stubbornness: to build a certain universal object, you apply a certain universal construction, for example a pushout. The pushout will hardly compute the desired left adjoint $\sum_\alpha(A, a)$. But one insists, and applies the same construction again, to the new piece of data; the pushout at this second stage will hardly be the desired $\sum_\alpha(A, a)$. But you insist, ... and after a certain number of steps you get an answer in the form of a fix-point: in our case, $SP_\infty = S(\operatorname{colim} \dots) \cong \operatorname{colim} S \dots$ is the clever re-bracketing (but apart from a few little sins of omissions in what exactly the isomorphism means, the argument is rigorous). As for attributing the swindle to P.J. Freyd, more than one bright argument in his mathematical work is based on a similar 'do one thing for a transfinite number of steps' technique.

and that is in general not possible.[2]

Leveraging on the following result, we can prove that all fibrations obtained pulling back the universal fibration of algebras share its same properties in terms of limit-preserving reindexings:

Lemma 2.2.12. Let $\left[\begin{smallmatrix}\mathcal{E}\\ p\downarrow\\ \mathcal{C}\end{smallmatrix}\right]$ be a fibration such that each reindexing $u^* : \mathcal{E}_{C'} \longrightarrow \mathcal{E}_C$ preserves (co)limits; let $F : \mathcal{A} \longrightarrow \mathcal{C}$ be a functor and q the fibration obtained pulling back p along F. Then, each reindexing of q also preserves (co)limits.

2.2.3 The question of completeness and cocompleteness

From §2.2.1 we know that each fiber of a fibration of Eilenberg–Moore algebras is complete, with limits created by the forgetful functors; this will be a staple of our characterisation of $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ as an Eilenberg–Moore category in §4.2 (implying, incidentally, that limits in the whole category $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ are indeed created by a canonical monadic functor $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}$: this will put Remark 2.2.8 above in perspective, showing that there's essentially a unique way to compute limits of diagrams spanning different fibers). Colimits in categories of algebras, and limits in coalgebras, on the other hand, are notoriously way more difficult to compute –even when they exist, they tend to be complicated objects: think of initial endofunctor algebras.

We are then left with the problem of establishing how colimits in each fiber $\{A\} \ltimes \mathcal{X}$ of a fibration of algebras induce, if anything, *global* colimits in the whole $\mathcal{A} \ltimes_T \mathcal{X}$, provided some additional assumptions on T are made. The issue is somewhat subtle, as it is well-known that being 'internally' cocomplete as an object in Fib is a weaker property than having a cocomplete total category, [Str23, p. 87]. Are total categories of fibrations of algebras cocomplete in the weak sense, or in the strong sense? And how does this relate, if anything, to assumptions made on $\mathcal{A}, \mathcal{X}$ or $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$?

One preliminary remark is the well-known observation by Linton [Lin69a], an Eilenberg–Moore category over a cocomplete base has coproducts as long as it has certain specific reflexive coequalisers, and it is cocomplete if it has all reflexive coequalisers. Computing such coequalisers is, however, in general a difficult task. In the specific case of $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ regarded as an Eilenberg–Moore category, the computation of binary coproducts goes as follows: given algebras (X, ξ) and (Y, θ) their coproduct is the coequaliser of the reflexive pair

$$FUFU(X,\xi) * FUFU(Y,\theta) \underset{FU\epsilon_X * FU\epsilon_Y}{\overset{\epsilon FU(X,\xi)*\epsilon FU(Y,\theta)}{\rightrightarrows}} FU(X,\xi) * FU(Y,\theta) \tag{2.2.4}$$

where the coproduct of free algebras is simplified by the observation that $FUX * FUY \cong F(UX + UY)$ since F is a left adjoint. In the specific case of $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ the coequaliser that defines the coproduct $(X, \xi)^A * (Y, \theta)^B$ is then

$$\big(T_{A+B}(T_A X + T_B Y), \mu^{A+B}\big)^{A+B} \rightrightarrows \big(T_{A+B}(X + Y), \mu^{A+B}\big)^{A+B} \tag{2.2.5}$$

where it is not important what objects are involved, but rather that the coequaliser in question happens in a single fiber, that is over $A + B$ (coproduct in $\mathcal{A}$). This colimit can't in

[2] Note that for the fibrations of interest in this work, the request that u^* preserves colimits is highly restrictive: in natural examples, even preservation of initial objects can dramatically fail (consider, for example, the reindexings of the 'generation of list' endofunctor $(A, X) \mapsto 1 + A \times X$).

general be reduced further (although there are cases where it acquires a more explicit form, for example when A is the initial object of $\mathcal{A}$).

What will make the problem more tractable is restricting the class of functors we consider:

> In the following, the categories $\mathcal{A}, \mathcal{X}$ are both κ-accessible categories for some regular cardinal κ; furthermore, all the parametric functors T_A that we consider are κ-accessible.

Under these assumptions, the category of κ-accessible functors $[\mathcal{X}, \mathcal{X}]_{<\kappa}$ is itself accessible (although for a bigger cardinal $\lambda \gg \kappa$; if $\mathcal{X}$ is *κ-presentable* instead, the category $[\mathcal{X}, \mathcal{X}]_{<\kappa}$ is κ-presentable).

Now, a simple way to get accessibility of all the total categories in study, is to appeal the results of [MP89, Ch. 5]: the pseudofunctor $\mathsf{Alg}_{\mathcal{X}} : [\mathcal{X}, \mathcal{X}]^{\mathrm{op}}_{<\kappa} \longrightarrow \mathsf{Cat}$ is accessible in the sense of [MP89, 5.3.1], which means, in addition to the assumptions we made, that $\mathsf{Alg}_{\mathcal{X}}(\operatorname{colim} T_i) \cong \lim_i \mathsf{Alg}_{\mathcal{X}}(T_i)$ for every κ-filtered diagram of endofunctors –or, equivalently, for ordinals smaller than κ; this ensures that $\begin{bmatrix} \mathcal{A} \ltimes \mathcal{X} \\ \downarrow \\ \mathcal{A} \end{bmatrix}$ is accessible (which means, its total category is accessible, and its projection is an accessible functor).

Relying on this, we can cover in one fell swoop

- all cases in which reindexings are covariant: coEilenberg–Moore opfibrations, Kleisli opfibrations, endofunctor coalgebras. . . ; this can also be appreciated directly, relying on the completeness theorem for the 2-category of accessible categories, and presenting the opfibrations of (2.1.12) as pullbacks of accessible functors over $[\mathcal{X}, \mathcal{X}]_{<\kappa}$;
- the fibrations of endofunctor algebras over accessible categories, cf. Theorem 2.2.9;
- Eilenberg–Moore fibrations: notably, the proof of Theorem 2.2.9 can be carried over unchanged, and s_∞ will be an Eilenberg–Moore algebra assuming S, T are monads. But there is a slicker argument in that case: from [Bor94b, 4.3.2] we know that a category $\mathcal{C}^T$ of Eilenberg–Moore algebras for a monad $T : \mathcal{C} \longrightarrow \mathcal{C}$ has all colimits that T preserves. Then, it will immediately follow from Theorem 4.2.1 that if $\mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X} : (A, X) \mapsto (A, T_A X)$ preserves colimits of shape $\mathcal{J}$, then $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ has colimits of shape $\mathcal{J}$.

Chapter 3

Some motivating examples

Summary of chapter

Examples of fibrations of algebras are abundant; we try to classify them in the most systematic way possible. The paradigmatic example is that of a category acting on itself through the *regular representation*, Example 3.2.1, and this has many of the motivating examples in the introduction as special cases. This action can be by (co)monads, as in the case of the simple fibration or of the (bi)fibration of monoids (cf. Example 3.3.1) or by mere endofunctor, as it is the case for polynomials (cf. section 3.4) or when the simple fibration is 'twisted', Example 3.3.4 –a construction that is known under the name of *Artin gluing* if the base category is a topos–. The assignment $(\mathcal{A}, \mathcal{X}) \mapsto \mathcal{A} \ltimes \mathcal{X}$ (as well as $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$, etc.) is functorial from a suitably defined domain which is of the form $\mathsf{Cat} \ltimes \mathsf{Cat}$, with respect to an action that is the 2-dimensional analogue of the regular representation. The semidirect products of groups (Proposition 3.3.8) and monoids (Remark 3.3.9) can be characterised as functors $\mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Grp} \longrightarrow \mathsf{Grp}$ and $\mathsf{Mon} \ltimes^{\mathsf{EM}} \mathsf{Mon} \longrightarrow \mathsf{Mon}$. The case of dinatural dependence from a parameter is studied in §3.5, relating our Kleisli opfibration with Atkey's Kleisli category of a diparametric monad [Atk09a] (we call them 'diparametric' to avoid confusion: a diparametric monad is a monad parametric over $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$). In §3.5.1 we draw a connection between parametric endofunctors and *optics* [Tam06, Ver23] associating to every $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ a category of F-optics $\mathsf{Opt}_{\mathcal{X},F}$ which is, however, very difficult to characterise in practice.

3.1 Monoids and posets

Soon we will recover the motivating examples outlined in the introduction; but a fundamental design principle of category theory compels us to adopt 'negative thinking' any time we are introducing a new definition, and thus we are led to analyse what Definition 2.1.5, Definition 2.1.8 and all the rest become when $\mathcal{A}$ has only one object (i.e. when it is a monoid) or when it has at most one morphism between any two given objects (i.e. when it is a preorder).

Remark 3.1.1. When $\mathcal{A} = M$ is a monoid, a category fibered over M consists of the *translation category* associated to a representation of M in terms of endofunctors of a single category of algebras; that is, a parametric endofunctor $M \longrightarrow [\mathcal{X}, \mathcal{X}]$ corresponds to the choice of a single endofunctor $F : \mathcal{X} \longrightarrow \mathcal{X}$ and a representation of M through endofunctors of $\mathsf{Alg}(F)$.

In particular, when $M = A^*$ is a free monoid on a set A, i.e. the free category on the graph having a single vertex and one loop for every element $a \in A$, this amounts to a family $\alpha_a : \mathsf{Alg}(F) \Rightarrow \mathsf{Alg}(F)$ of natural transformations of the category $\mathsf{Alg}(F)$.

The case of poset gives rise to examples with a distinguished topological flavour, as a parametric endofunctor $P \longrightarrow [\mathcal{X}, \mathcal{X}]$ is essentially a presheaf of endofunctors (and a nice behaviour can be expected when this presheaf is in fact a sheaf).

Example 3.1.2 (The stack of subtopoi of a space)**.** If X is a topological space and $L = OX$ is its frame of open subsets regarded as a category, we can consider the association

$$\begin{aligned} L^{\text{op}} &\longrightarrow [\mathsf{Sh}(X), \mathsf{Sh}(X)] \\ U &\longmapsto T_U \end{aligned} \tag{3.1.1}$$

where, whenever U is an open subset of X, T_U is the monad associated to the open subtopos $\mathsf{Sh}(U)$:

$$\mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(U) \longrightarrow \mathsf{Sh}(X) \tag{3.1.2}$$

clearly, this monad is equivalently determined by the (reflective) subcategory $\mathsf{Sh}(U) \hookrightarrow \mathsf{Sh}(X)$, and we could have equivalently taken the category of sheaves over the closed set-theoretic complement $F = U^c$ of the open U, regarded with the subspace topology, and identifying a subcategory $\mathsf{Sh}(F) \hookrightarrow \mathsf{Sh}(X)$; the two subcategories determine each other as they fit in a ‘short exact sequence’ like

$$\mathsf{Sh}(F) \longrightarrow \mathsf{Sh}(X) \longrightarrow \mathsf{Sh}(U) \tag{3.1.3}$$

an arrangement of categories and functors that is called a (*unpointed*) *recollement* (we will study the relation between recollements and fibrations of algebras in subsection 3.3.1; Example 3.3.7 below is one strong point of connection between the two theories).

3.2 More monoids, algebras, transition systems

Example 3.2.1 (The regular representation of a Cartesian category)**.** Let $\mathcal{X}$ be a category having finite products; consider the transposed of the Cartesian product functor,

$$\_ \times \_ : \mathcal{X} \times \mathcal{X} \longrightarrow \mathcal{X} \tag{3.2.1}$$

sending an object A to the functor $\lambda X.A \times X$: in every Cartesian category, this endofunctor is a comonad S_A (mimicking what happens when a monoid M acts on itself on the left under the map $m \mapsto \lambda x.mx$, we call this the *regular representation* of $\mathcal{X}$ on itself). As such, we can consider

- its fibration of coKleisli categories; this is the category $\mathbf{s}(\mathcal{X})$ over $\mathcal{X}$, where each fiber $\mathbf{s}(\mathcal{X})_A$ over an object A has
  - the same objects of $\mathcal{X}$;
  - $\mathbf{s}(\mathcal{X})_A(X, Y) := \mathcal{X}(X \times A, Y)$.

CoKleisli composition, i.e. composition of intra-fiber arrows is defined as

$$X \times A \xrightarrow{X\times\Delta} X \times A \times A \xrightarrow{f\times A} Y \times A \xrightarrow{g} Z \tag{3.2.2}$$

The category $\mathbf{s}(\mathcal{C})_A$ is called the *simple slice* $\mathcal{C}/\!\!/A$ in [Jac98] and it is used in the categorical semantics of *simple type theory*, [Hin97, Far23].

- its opfibration of coEilenberg–Moore coalgebras $\mathsf{coAlg}_{\mathcal{X}}(S_\bullet)$; in this case the fiber over $A \in \mathcal{X}$ coincides with the whole slice category over A, so the fibration of coEilenberg–Moore algebras for $S_\bullet$ is simply the domain opfibration.

Notation 3.2.2. When the regular representation of $\mathcal{X}$ is considered as a parametric comonad, it usually falls under different names; one is to call $A \times \_$ the *coreader comonad* on $\mathcal{X}$ with parameter A. Considering that the sub-text of our entire discussion is that we are categorifying the notion of monoid action, we have a more representation-theoretic orientation, but readers should be aware that the notion is (largely) studied in categorical logic, with genuinely type-theoretic motivations in mind.

Remark 3.2.3 (Monoidal semiautomata)**.** There is interest in considering also mere endofunctor algebras for the regular representation $S : A \mapsto A \times \_$, and the same construction in fact makes sense in any monoidal category $(\mathcal{K}, \otimes)$: there, endofunctor algebras for $A \otimes \_$ consist of morphisms $d : A \otimes E \to E$, and algebra morphisms $f : (X, d) \to (Y, d')$ are the $f : X \to Y$ fitting in commutative squares

$$\begin{tikzcd} A \otimes X \arrow[r, "A\otimes f"] \arrow[d, "d"'] & A \otimes Y \arrow[d, "d'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{3.2.3}$$

so an $A \otimes \_$-algebra consists of a *semiautomaton* [KKM00] (also called a *Medvedev automaton*, cf. [EKKK74]) with *input alphabet* A. The associated fibration of endofunctor algebras is then what we call the *fibration of semiautomata* for $(\mathcal{K}, \otimes)$. Observe here that if $\mathcal{K}$ admits countable coproducts preserved by each $A \otimes \_$, i.e. if $(\mathcal{K}, \otimes)$ is, in the language of [LT23], a doctrine of $\boldsymbol{D}$-rigs for the KZ 2-monad of coproducts, then the fibration of semiautomata is precisely the fibration of $\mathcal{K}$-monoids (cf. Example 3.3.1 below) restricted to free monoids of the form $\sum_{n\geqslant 0} A^n$.

By virtue of this, we obtain a natural identification of the fibration of semiautomata and the fibration of Eilenberg–Moore algebras of Example 3.3.1 below, restricted to free monoids.

In short, whenever $\mathcal{X}$ is a Cartesian category with a free monoid construction (an adjunction $F : \mathcal{X} \rightleftarrows \mathsf{Mon}(\mathcal{X}) : U$), we can consider pullback diagrams

$$\begin{tikzcd} \mathcal{X} \ltimes \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{X} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{Alg}_{\mathcal{X}} \arrow[d] \\ \mathcal{X} \arrow[r, "F"'] & \mathsf{Mon}(\mathcal{X}) \arrow[r] & {[\mathcal{X}, \mathcal{X}]} \end{tikzcd} \tag{3.2.4}$$

Remark 3.2.4. Similar considerations can be made in the more general case of a category $\mathcal{C}$ (enriched and) tensored over a category $\mathcal{V}$; the tensor functor $- \odot - : \mathcal{V} \times \mathcal{C} \longrightarrow \mathcal{C}$ evidently gives rise to a parametric endofunctor, and an endofunctor algebra $V \odot C \to C$ is a 'family of endomorphisms $f_v : C \to C$' indexed by the 'elements' of V (this is precisely what happens when $\mathcal{C}$ is an ordinary category with small coproducts and $\mathcal{V} = \mathsf{Set}$).

This slight generalisation comes in handy when one wants to encode an action of an object V on C abstractly; for example, a monoid M acting on another monoid N through endomorphisms $a_m : N \to N$ can be represented as a map $UM \odot N \to N$, where $UM \odot N = \sum_{m \in M} N$ but the sum is performed in the category Mon of monoids.

Ultimately, we have a pullback

$$\begin{tikzcd} \mathcal{V} \ltimes \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{Alg}_{\mathcal{C}} \arrow[d] \\ \mathcal{V} \arrow[r, "\odot"'] & {[\mathcal{C}, \mathcal{C}]} \end{tikzcd} \tag{3.2.5}$$

Remark 3.2.5. Recall that if G is a group, one can form the semidirect product $\mathrm{Aut}(G) \ltimes G$ when $\mathrm{Aut}(G)$ acts on G by evaluating an automorphism f on an element g; this is called the *holomorph* of G; the same construction carries over to monoids, as $\mathrm{End}(M)$ acts on M with the same evaluation map. We can recapture this example, and generalise it to categories, via a suitable fibration of algebras.

Example 3.2.6 (The fibration of endofunctor algebras)**.** When $\mathcal{K} = [\mathcal{X}, \mathcal{X}]$ is the monoidal category of endofunctors of a fixed category $\mathcal{X}$, Example 3.2.1 provides a sort of universal paradigm for our construction, and an analogue of the notion of holomorph: indeed, any object $F : \mathcal{X} \longrightarrow \mathcal{X}$ has an associated category of $(F \circ \_)$-*algebras* (*(right) modules* in the sense of [Dub70, p. 61]) and if we let such F vary over all $[\mathcal{X}, \mathcal{X}]$ we obtain the fibration of endofunctor algebras of Definition 2.1.2, via the Grothendieck construction. Hence, $\int \mathsf{Alg}_{\mathcal{X}} \cong [\mathcal{X}, \mathcal{X}] \ltimes \mathcal{X}$ for the action so determined.

A module for $F : \mathcal{X} \longrightarrow \mathcal{X}$ is a functor $X : \mathcal{J} \longrightarrow \mathcal{X}$ equipped with a natural transformation $\alpha : F \circ X \Rightarrow X$ in $[\mathcal{J}, \mathcal{X}]$ in such a way that if $\mathcal{J}$ we recover the usual notion of an F-algebra (and if F carries additional structure, for example it is a monad, α is required to be compatible with it). But then, $\mathsf{Mod}(F)$ is also fibred over Cat, in such a way that its fiber over $\mathcal{J} = 1$ is the category $\mathsf{Alg}_{\mathcal{X}}(F)$ of F-algebras.

The theory of coalgebras provides a host of examples as well, a large family of which is given by labelled transition systems:

Example 3.2.7. Fix a set A and consider the (covariant in X and in A) functor $X \mapsto 2^{A \times X}$; its opfibration of endofunctor coalgebras has typical fiber the category of A-labelled transition systems, and one can collect it in the category

$$\mathsf{Set} \oplus_2 \mathsf{Set} \twoheadrightarrow \mathsf{Set} \tag{3.2.6}$$

Similarly, consider the *monad of finite distributions* [Jac18, Jac10] $D : \mathsf{Set} \longrightarrow \mathsf{Set}$ and the functor $D^\bullet : X \mapsto (1 + DX)^A$; its opfibration of endofunctor coalgebras is the category of *probabilistic* A-labelled transition systems [DEP02, LS91], and one can collect it in the category

$$\mathsf{Set} \oplus_D \mathsf{Set} \twoheadrightarrow \mathsf{Set} \tag{3.2.7}$$

For more examples, the reader is referred to the many places in which a parametric endofunctor appears, in [Jac16].

Endofunctor algebras, coupled with a slice construction, yield a family of parametric endofunctors of slice categories $\mathsf{Cat}/\mathcal{K}$; we address the reader looking for a concrete incarnation of the following construction to [Gui80, EKKK74, Gui74] and more recently [BLLL23b, BLLL23a].

Example 3.2.8 (Abstract Mealy and Moore automata)**.** The ingredients of the following construction are

- a possibly large category $\mathcal{K}$, and a small category $\mathcal{B}$;
- an endofunctor $F : \mathcal{K} \longrightarrow \mathcal{K}$, thought as a *generating input* of a dynamics;
- a functor $B : \mathcal{B} \longrightarrow \mathcal{K}$, thought of as a *generalised element of output* for the dynamics.

The categories $\mathsf{Mly}(F, B)$ and $\mathsf{Mre}(F, B)$ are then defined respectively as the following strict pullbacks in Cat:

$$\begin{tikzcd} \mathsf{Mly}(F,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & F/B \arrow[d] \\ \mathsf{Alg}(F) \arrow[r] & \mathcal{K} \end{tikzcd} \qquad \begin{tikzcd} \mathsf{Mre}(F,B) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{K}/B \arrow[d] \\ \mathsf{Alg}(F) \arrow[r] & \mathcal{K} \end{tikzcd} \tag{3.2.8}$$

where $\mathsf{Alg}(F)$ is the category of endofunctor algebras for F, F/B is the comma category of arrows $u : FX \to BY$, $\mathcal{K}/B$ the comma category of arrows $v : X \to BY$, and the functors into $\mathcal{K}$ are obtained from the comma construction defining F/B and $\mathcal{K}/B$.

As such, both $\mathsf{Mly}_{\mathcal{K}}, \mathsf{Mre}_{\mathcal{K}}$ are functors of type

$$\begin{tikzcd} {[\mathcal{K},\mathcal{K}]^{\mathrm{op}} \times \mathsf{Cat}/\mathcal{K}} \arrow[r] & \mathsf{Cat}/\mathcal{K} \\ (F,B) \arrow[r, mapsto] & \begin{bmatrix} \mathsf{Mly}_{\mathcal{K}}(F,B) \\ \downarrow \\ \mathcal{K} \end{bmatrix}, \begin{bmatrix} \mathsf{Mre}_{\mathcal{K}}(F,B) \\ \downarrow \\ \mathcal{K} \end{bmatrix} \end{tikzcd} \tag{3.2.9}$$

giving rise to a parametric endofunctor of $\mathsf{Cat}/\mathcal{K}$, with category of parameters $\mathcal{A} = [\mathcal{K}, \mathcal{K}]^{\mathrm{op}}$, and thus to a fibered category over $\mathcal{A}$ (both $\mathsf{Mly}_{\mathcal{K}}, \mathsf{Mre}_{\mathcal{K}}$ are considered categories over $\mathcal{K}$ via the diagonal in the pullback that defines them).

The notion of *Mealy* and *Moore automaton* of [Gui80, EKKK74] is obtained as particular instances of these constructions, when $\mathcal{K}$ is a Cartesian category, $F = A \times \_$ is the product for a fixed object A, the category $\mathcal{B}$ is terminal (and thus $B : \mathbb{1} \longrightarrow \mathcal{K}$ is an object of $\mathcal{K}$).

3.3 More monoids, algebras

Example 3.3.1 (Monoid objects acting on other objects)**.** Let $(\mathcal{K}, \otimes)$ be a monoidal category, and consider again the regular representation $T_M = M \otimes \_$ of Example 3.2.1, but restricted to the subcategory $\mathsf{Mon}(\mathcal{K})$ of monoids in $\mathcal{K}$; then the functor $T_M : \mathcal{K} \longrightarrow \mathcal{K}$ becomes a monad, and a T_M-algebra is precisely a (left) M-module, i.e. an object $A \in \mathcal{K}$ with a (left) action $a : M \otimes A \to A$; so, the category of $(M \otimes \_)$-algebras is the category of objects with an M-action, $\mathcal{K}^M$. A monoid homomorphism $f : M \to N$ now induces a functor that sends the left N-action $a : N \otimes X \to X$

$$M \otimes X \xrightarrow{f \otimes X} N \otimes X \xrightarrow{a} X. \tag{3.3.1}$$

This clearly sets up a functor $\mathsf{Mon}(\mathcal{K}) \longrightarrow \mathsf{Cat}$, and thus a fibration of Eilenberg–Moore algebras:

$$\mathsf{Mon}(\mathcal{K}) \ltimes^{\mathsf{EM}} \mathcal{K} \twoheadrightarrow \mathsf{Mon}(\mathcal{K}). \tag{3.3.2}$$

Example 3.3.2 (A sub-example: the fibration of modules over a ring)**.** As an example, when $\mathcal{K} = \mathsf{Ab}$ is the category of abelian groups, an internal monoid R is a ring, and the category Ab^R is the category of (left) R-modules. Collecting together all categories of modules over all rings we get a category Mod fibred over Ring.

Example 3.3.3 (A sub-example: the fibration of Eilenberg–Moore algebras)**.** When $\mathcal{K} = [\mathcal{X}, \mathcal{X}]$ is the monoidal category of endofunctors of a fixed category $\mathcal{X}$, this provides some sort of universal example for our construction: indeed, a $\mathcal{K}$-monoid is a monad, and for a fixed monad T on $\mathcal{K}$ a (right) T-module is exactly a T-algebra: the fibration of Eilenberg–Moore algebras then arises as the fibration of monads on $\mathcal{K}$.

From this, we obtain the identification between $\mathsf{Mnd}(\mathcal{X}) \ltimes^{\mathsf{EM}} [\mathcal{X}, \mathcal{X}]$ determined according to this action, and the category $\int \mathsf{EM}_{\mathcal{X}}$ of (2.1.8).

Example 3.3.4 (Artin Gluing and generalised gluing)**.** Given a left exact functor $F : \mathcal{E} \longrightarrow \mathcal{E}'$ between elementary toposes the *Artin gluing* of $\mathcal{E}, \mathcal{E}'$ along F is defined as the comma category $\mathcal{E}'/F$; with a similar reasoning as the one in Example 3.2.1, one can show that this arises as total category for the coEilenberg–Moore fibration of the parametric comonad $FE \times \_$.

Indeed, for a fixed $E \in \mathcal{E}$ we can define a comonad

$$(E, E') \longrightarrow (E, FE \times E') \tag{3.3.3}$$

and observe that such a correspondence boils down to a parametric functor $\mathcal{E} \longrightarrow [\mathcal{E}', \mathcal{E}']$ sending E to $FE \times \_$.

A more precise way to word this result is that there are two pasted pullbacks of categories

$$\begin{tikzcd} \mathcal{E}'/F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{s}(\mathcal{E}') \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Alg}_{\mathcal{E}'} \arrow[d] \\ \mathcal{E} \arrow[r, "F"'] & \mathcal{E}' \arrow[r] & [\mathcal{E}', \mathcal{E}'] \end{tikzcd} \tag{3.3.4}$$

(notation as in Example 3.2.1). We can further observe two things:

- first, the obvious fact that the assignment $E \mapsto FE \times \_$ is a strong monoidal functor $\mathcal{E} \longrightarrow [\mathcal{E}', \mathcal{E}']_\delta$ (notation as in Equation 2.1.8);
- second, that the assignment $F \mapsto \mathcal{E}'/F$ is a functor $\mathsf{Lex}(\mathcal{E}, \mathcal{E}') \longrightarrow \mathsf{Fib}/\mathcal{E}$ obtained taking F to the fibration of algebras for the comonad $FE \times \_$.

Tweaking the above example a bit, and taking advantage of the 2-dimensional structure of Cat, we can appeal to a 'macrocosm principle' and build the fibration (or rather, 2-fibration) of fibrations of algebras.

The construction we have in mind considers the regular representation $\mathsf{Cat} \longrightarrow [\mathsf{Cat}, \mathsf{Cat}]$ of Cat; its endofunctor algebras (more precisely, the endofunctor algebras of the underlying endofunctor of the comonad arising from Example 3.2.1) consist of functors $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ for some category $\mathcal{X}$ (clearly, parametric endofunctors with parameters $\mathcal{A}$ are precisely semiautomata on $\mathcal{X}$, with 'alphabet' $\mathcal{A}$), and the category of *oplax morphisms of algebras* has objects the pairs $(U, \delta) : (F, \mathcal{X}) \to (G, \mathcal{Y})$ where $U : \mathcal{X} \longrightarrow \mathcal{Y}$ is a functor and δ is a 2-cell filling the diagram

$$\begin{tikzcd} \mathcal{A} \times \mathcal{X} \arrow[r, "F"] \arrow[d, "\mathcal{A} \times U"'] \arrow[dr, phantom, "\swarrow_\delta"] & \mathcal{X} \arrow[d, "U"] \\ \mathcal{A} \times \mathcal{Y} \arrow[r, "G"'] & \mathcal{Y} \end{tikzcd} \tag{3.3.5}$$

Remark 3.3.5. If $\mathcal{A}$ is a monoidal category acting on $\mathcal{X}$, and U s the identity this definition recovers the notion of *distributive law for monoidal actions* exposed in [Sko04].

The importance of this construction is understood in terms of the following observation: sending $\mathcal{A}$ to the category $\mathsf{Alg}^\ell(\mathcal{A}\times \_)$ of its algebras and lax homomorphisms is a pseudo-functor $\mathsf{Cat} \longrightarrow \mathsf{Cat}$ of which we can consider once again the associated (very large) fibration obtaine as left vertical arrow in the pullback

$$\begin{tikzcd} \mathsf{Cat} \ltimes \mathsf{Cat} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Alg}^\ell_{\mathsf{Cat}} \arrow[d] \\ \mathsf{Cat} \arrow[r] & [\mathsf{Cat}, \mathsf{Cat}] \end{tikzcd} \tag{3.3.6}$$

providing a 'universe' for the fibration of algebras construction, in the sense that follows.

Theorem 3.3.6. There is a 2-functor

$$\_ \ltimes \_ : \mathsf{Cat} \ltimes \mathsf{Cat} \longrightarrow \mathsf{Cat} \tag{3.3.7}$$

sending the typical object $(\mathcal{A}; \mathcal{X}, F : \mathcal{A}\times\mathcal{X} \longrightarrow \mathcal{X})$ to the fibration of (endofunctor) algebras $\mathcal{A} \ltimes_F \mathcal{X}$ obtained as the pullback in Definition 2.1.5. Functoriality is over morphisms (U, δ) like in (3.3.5), and on 2-cells $\alpha : (U, \delta) \to (V, \sigma)$ compatible with δ, σ in the usual sense.

Example 3.3.7 (The fibration of Beck modules)**.** Let $\mathcal{X}$ be a category with finite limits. The fibration of *Beck modules* can be obtained considering the assignment

$$G \mapsto \mathsf{Ab}(\mathcal{X}/G) \tag{3.3.8}$$

where $\mathcal{X}$ is a category with finite limits, and $\mathsf{Ab}(\mathcal{X}/G)$ is the category of abelian groups in the slice $\mathcal{X}/G$; an object of $\mathsf{Ab}(\mathcal{X}/G)$ consists of a morphism

$$\begin{tikzcd} B \times_G B \arrow[rr, "m"] \arrow[dr, "b\times b"'] & & B \arrow[dl, "b"] \\ & G & \end{tikzcd} \tag{3.3.9}$$

and a section $u : G \to B$ for b such that in the slice $\mathcal{X}/G$ the axioms for an abelian group are satisfied (associativity, unitality and commutativity).

In case $\mathcal{X} = \mathsf{Grp}$ is the category of groups and group homomorphisms, this is an example of a fibration of algebras, but seeing why it is so is not entirely trivial:

- first of all, observe that the map $b : B \to G$ is a split epimorphism of groups (with inverse u); from this, one derives that the group G acts naturally on $\ker b$, which is Abelian.
- This sets up an equivalence between the category $\mathsf{Ab}(\mathcal{X}/G)$ and the category of actions of G on abelian groups, i.e. group homomorphisms

  $$\upsilon : G \longrightarrow \mathrm{Aut}(H) \tag{3.3.10}$$

  for some abelian group H;

- but then, the universal property of the group ring construction yields that v exhibits H as a $\mathbf{Z}[G]$-module, so that

$$\mathsf{Ab}(\mathsf{Grp}/G) \cong \mathbf{Z}[G]\text{-}\mathsf{Mod}. \tag{3.3.11}$$

All in all, a compact way to present this result is that there are pullback diagrams

$$\begin{tikzcd} \mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Ab} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Ring} \ltimes^{\mathsf{EM}} \mathsf{Ab} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \textstyle\int \mathsf{EM}_{\mathcal{X}} \arrow[d] \\ \mathsf{Grp} \arrow[r, "\mathbf{Z}[-]"'] & \mathsf{Ring} \arrow[r] & [\mathsf{Ab}, \mathsf{Ab}] \end{tikzcd} \tag{3.3.12}$$

where the rightmost square is just a particular instance of Example 3.3.1 when $\mathcal{K} = \mathsf{Ab}$, once we recognise that $\mathsf{Ring} = \mathsf{Mon}(\mathsf{Ab})$.

Observe that this presentation style allows to argue similarly

- when instead of $\mathbf{Z}$ as a base ring of coefficients one considers more generally a commutative ring k and a k-algebra R acting on k-modules; this yields a fibration of Eilenberg–Moore algebras in the same fashion (3.3.12) above does;
- considering general $\otimes$-monoid in a monoidal category $(\mathcal{K}, \otimes)$ as in Example 3.3.1, and the corresponding fibration of Eilenberg–Moore algebras, and pulling back from the category of Hopf monoids, provided one has available an analogue of the group ring functor, as left adjoint to the functor selecting grouplike elements of an Hopf monoid.

We conclude the chapter giving an application, in terms of an elegant description of the semidirect product operation as a left adjoint functor $\mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Grp} \longrightarrow \mathsf{Grp} : (G, H) \mapsto G \ltimes H$.

Let's go back at the fibration of monoids of Example 3.3.1.

Note that the typical object of the category $\mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Set}$ arising from Example 3.3.1 is a triple $(G; A, \alpha)$ where G is a group, A is a set, and $\alpha : G \times A \to A$ a (left) action of G on A; a morphism $(G; A, \alpha) \to (H; B, \beta)$ is a pair (u, v) of a group homomorphism $u : G \to H$ and a function $v : A \to B$ such that the square

$$\begin{tikzcd} G \times A \arrow[r, "\alpha"] \arrow[d, "u \times v"'] & A \arrow[d, "v"] \\ H \times B \arrow[r, "\beta"'] & B \end{tikzcd} \tag{3.3.13}$$

is commutative. Similarly, the typical object of $\mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Grp}$ is a *G-group*, i.e. a triple $(G; H, \alpha)$ where G acts on a *group* H under group automorphisms. Then,

Proposition 3.3.8. Let G be a group, and $\alpha : G \times G \to G$ be the conjugation action $(g, h) \mapsto g^{-1}hg$.

Then, there is a functor

$$r_\alpha : \mathsf{Grp} \longrightarrow \mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Grp} \tag{3.3.14}$$

sending G to the object $(G; G, \alpha) \in \mathsf{Grp} \ltimes^{\mathsf{EM}} \mathsf{Grp}$. The functor r_α has a left adjoint, sending a G-group (G, H, ψ) to the semidirect product $G \ltimes_\psi H$.

Remark 3.3.9. In a similar vein, the semidirect product *of monoids* [Jan03, 3.9], [BJ98, 3.2] can be characterised as a functor

$$\_ \ltimes \_ : \mathsf{Mon} \ltimes^{\mathsf{EM}} \mathsf{Mon} \longrightarrow \mathsf{Mon} \tag{3.3.15}$$

which is cocontinuous; by virtue of the reasoning in subsection 2.2.3, $\mathsf{Mon} \ltimes^{\mathsf{EM}} \mathsf{Mon}$ is locally presentable, and thus $\_ \ltimes \_$ is a left adjoint; it might however be difficult to establish its right adjoint, as the conjugation representation of Proposition 3.3.8 doesn't exist for monoids.

3.3.1 Recollements and fibrations of algebras

The above example lends itself to the following generalisation, for which we assume that the base category $\mathcal{X}$ is pointed (=it has a zero object denoted 0) and finitely complete and cocomplete.

Then, there is a sequence of functors

$$\mathsf{Ab}(\mathcal{X}) \xrightarrow{i_0} \textstyle\int_{\mathcal{X}} \mathsf{Ab}(\mathcal{X}/\_) \xrightarrow{U} \mathcal{X} \tag{3.3.16}$$

where U is the codomain functor, and i_0 the 'inclusion of the fiber at 0' functor $\mathsf{Ab}(\mathcal{X}/0) \cong \mathsf{Ab}(\mathcal{X}) \hookrightarrow \int_{\mathcal{X}} \mathsf{Ab}(\mathcal{X}/\_)$ in the category of elements of Theorem 1.2.13, in such a way that the square

$$\begin{tikzcd} \mathsf{Ab}(\mathcal{X}/0) \arrow[r, "i_0"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int_{\mathcal{X}} \mathsf{Ab}(\mathcal{X}/\_) \arrow[d, "U"] \\ 1 \arrow[r, "0"'] & \mathcal{X} \end{tikzcd} \tag{3.3.17}$$

is a pullback.

We can immediately observe that

Lemma 3.3.10. The functors U and i_0 both have a left and a right adjoint,

$$U_L \dashv U \dashv U_R \qquad\qquad i_L \dashv i_0 \dashv i_R \tag{3.3.18}$$

and moreover there is a natural isomorphism $U_L \cong U_R$.

The functors are defined as follows:

- $U_L = [1\_]$ sends an object A to its identity arrow $\begin{bmatrix} A \\ | \\ A \end{bmatrix}$;
- i_R sends an object $\left(A, {}_h\!\!\begin{smallmatrix} E \\ \downarrow \\ A \end{smallmatrix}\right) \in \mathsf{Ab}(\mathcal{X}/A)$ to its kernel $\ker h$ in $\mathcal{X}$;
- i_L can be proved to exist using the fact that U is a bifibration; however, can't be easily described explicitly.

It is easily seen that these indeed form adjunctions, and that i_0 and $[1\_]$ are fully faithful.

Remark 3.3.11. The functor $U_L = [1\_]$ is also a left adjoint to U, as in a commutative square of the form

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & E \arrow[d, "h"] \\ X \arrow[r, "v"'] & A \end{tikzcd} \tag{3.3.19}$$

the morphism $u : X \to E$ is uniquely determined by the compatibility $sv = u$ with the right inverse of h, an $s : A \to E$ such that $hs = 1_A$: if $hu_1 = hu_2 = v$ and $sv = u$ then

$$u_1 = shu_1 = sv = shu_2 = u_2. \tag{3.3.20}$$

This defines a diagram of adjoints

$$\begin{tikzcd} \mathsf{Ab}(\mathcal{X}/0) \arrow[r, "i_0" description] & \int_{\mathcal{X}} \mathsf{Ab}(\mathcal{X}/\_) \arrow[l, shift right=2, "i_L"'] \arrow[l, shift left=2, "i_R"] \arrow[r, "U" description] & \mathcal{X} \arrow[l, shift right=2, "U_L"'] \arrow[l, shift left=2, "U_L"] \end{tikzcd} \tag{3.3.21}$$

Now, recall from [BBD82] – and see [BR07, Jan65, Han14] and more recently [Lor16] the notion of a *recollement* (usually stated when the categories at play are Abelian or triangulated):

Definition 3.3.12. A *recollement* of additive (more often, Abelian) categories is an arrangement of categories and functors

$$\begin{tikzcd} \mathcal{L} \arrow[r, "i_0" description] & \mathcal{D} \arrow[l, shift right=2, "i_L"'] \arrow[l, shift left=2, "i_R"] \arrow[r, "q" description] & \mathcal{R} \arrow[l, shift right=2, "q_L"'] \arrow[l, shift left=2, "q_R"] \end{tikzcd} \tag{3.3.22}$$

subject to the following axioms:

R1) there are adjunctions $i_L \frac{\epsilon_L}{\eta_L}\Big| i_0 \frac{\epsilon_R}{\eta_R}\Big| i_R$ and $q_L \frac{\bar\epsilon_L}{\bar\eta_L}\Big| q \frac{\bar\epsilon_R}{\bar\eta_R}\Big| q_R$;

R2) the functors i_0, q_R, q_L are fully faithful;

R3) the essential image of i_0 coincides with the *kernel* of q, i.e. with the subcategory $\{X \in \mathcal{D} \mid qX \cong 0\}$;

R4) the squares

$$\begin{tikzcd} q_L q \arrow[r, "\bar\epsilon_L"] \arrow[d] & 1_{\mathcal{D}} \arrow[d, "\eta_L"] \\ 0 \arrow[r] & i_0 i_L \end{tikzcd} \qquad \begin{tikzcd} i_0 i_R \arrow[r, "\epsilon_R"] \arrow[d] & 1_{\mathcal{D}} \arrow[d, "\bar\eta_R"] \\ 0 \arrow[r] & q_R q \end{tikzcd} \tag{3.3.23}$$

obtained from units and counits of the adjunctions in item R1 are *exact*, which in this particular context means that they are both a pullback and a pushout (=pullout squares).

Now, a recollement is called *Frobenius* if $q_R \cong q_L$, and we have that

Theorem 3.3.13. The arrangement of categories and functors in (3.3.21) forms a Frobenius recollement.

Proof. We only need to show Axiom item R4, the other conditions having already been established. But this follows from explicitly writing the units and counits in question, as the two squares in item R4 are objectwise pullouts in $\mathcal{D} = \int_{\mathcal{X}} \mathsf{Ab}(\mathcal{X}/\_)$. □

Remark 3.3.14. In the unpointed case, the situation isn't as nice: for example, if $\mathcal{X} = \mathsf{Set}$ we can consider the category $\mathsf{Ab}(\mathsf{Set}/A) \cong \mathsf{Ab}(\mathsf{Set}^A) \cong \mathsf{Ab}^A = [A, \mathsf{Ab}]$ of functors $A \longrightarrow \mathsf{Ab}$

as typical fiber of a fibration $\int_A \mathsf{Ab}^A$ over Set, and the forgetful functor $U : \int_A \mathsf{Ab}^A \longrightarrow \mathsf{Set} : (A, G) \mapsto A$. In this case we have an arrangement

$$\begin{tikzcd} \mathsf{Ab} \arrow[r, shift right, "i_0"'] & \int_A \mathsf{Ab}^A \arrow[l, shift right, "i_L"'] \arrow[r, "U" description] & \mathsf{Set} \arrow[l, shift right=2, "U_L"'] \arrow[l, shift left=2, "U_L"] \end{tikzcd} \tag{3.3.24}$$

where i_0 lacks a right adjoint.

Remark 3.3.15. Every recollement as in (3.3.22) induces an adjunction between endofunctors of $\mathcal{D}$ obtained as follows:

- there is an adjunction

$$\begin{tikzcd} \mathcal{D} \arrow[r, shift left, "\langle q, i_L \rangle"] \arrow[r, phantom, "\perp"] & \mathcal{R} \times \mathcal{L} \arrow[l, shift left, "q_R \times i_0"] \end{tikzcd} \tag{3.3.25}$$

  where $q_R \times i_0$ is the functor $\mathcal{R} \times \mathcal{L} \xrightarrow{\langle q_R, i_0 \rangle} \mathcal{D} \times \mathcal{D} \xrightarrow{\times} \mathcal{D}$;
- similarly, there is an adjunction

$$\begin{tikzcd} \mathcal{R} \times \mathcal{L} \arrow[r, shift left, "q_L + i_0"] \arrow[r, phantom, "\perp"] & \mathcal{D} \arrow[l, shift left, "\langle q, i_R \rangle"] \end{tikzcd} \tag{3.3.26}$$

  where $q_L + i_0$ is the functor $\mathcal{R} \times \mathcal{L} \xrightarrow{\langle q_L, i_0 \rangle} \mathcal{D} \times \mathcal{D} \xrightarrow{+} \mathcal{D}$;
- now these adjunctions can be composed into an adjunction $F : \mathcal{D} \rightleftarrows \mathcal{D} : G$, that in case $\mathcal{D}$ is pointed and has biproducts, we can further specify the adjunction as follows: $FD = q_L q D \oplus i_0 i_L D$ and $GD = q_R q D \oplus i_0 i_R D$.

Remark 3.3.16. For the adjunction $F : \mathcal{D} \rightleftarrows \mathcal{D} : G$ we obtain two trivial exact sequences $0 \to q_L q D \to FD \to i_0 i_L D \to 0$, $0 \to i_0 i_R D \to GD \to q_R q D \to 0$.

In light of Theorem 4.2.1, it is an interesting problem to determine when the adjunction (3.3.26) of Remark 3.3.15 is (monadic and) fibered over $\mathcal{R}$, so that $\mathcal{R}$ is the category of parameters for a parametric endofunctor (monad) $\mathcal{R} \times \mathcal{L} \longrightarrow \mathcal{R} \times \mathcal{L}$.
This problem will be studied in full generality in a future work.

One of the motivating examples of a fibration of algebras is the category of actions, induced by the fibration of points, as considered in [BJK05, §3.2], [BJ98]. We reproduce the example here using our terminology, and we rely heavily on material that will only be presented and contextualised in the next chapter; but this example is important.

Example 3.3.17. The *freestanding split epi* is the category generated by

$$\begin{tikzcd} 0 \arrow[r, shift left, "r"] & 1 \arrow[l, shift left, "s"] \end{tikzcd} \tag{3.3.27}$$

subject to the relations requiring that $r \circ s = \mathrm{id}_1$ (and thus $s \circ r$ is an idempotent). A *point* in a category $\mathcal{B}$ is a functor P from the freestanding split epi to $\mathcal{B}$, and a *point at B* is, for every object B, a point such that $P1 = B$. If $\mathcal{B}$ has pullbacks and a zero object, for every B, the subcategory $\mathsf{Pt}_B(\mathcal{B})$ is the fiber at B of the *fibration of points* $\left[\begin{smallmatrix} & \mathsf{Pt}(\mathcal{B}) \\ p & \downarrow \\ & \mathcal{B} \end{smallmatrix}\right]$. Note that, by naturality, a morphism of points must commute with both the split epimorphism and its splitting.

Let us now assume that $\mathcal{B}$ has pullbacks, a zero object, and finite coproducts.

The functor $p\colon \mathsf{Pt}(\mathcal{B}) \to \mathcal{B}$ mapping a point to its codomain is a fibration known as the *fibration of points*, reindexing being given by pullback. The fiber $\mathsf{Pt}_B(\mathcal{B})$ over B has a zero object, the identity of B, split by the identity of B. This means that p has a fully faithful left and right adjoint s. We therefore have the diagram

$$\mathcal{B} \cong \mathsf{Pt}_0(\mathcal{B}) \underset{i_R}{\overset{i}{\rightleftarrows}} \mathsf{Pt}(\mathcal{B}) \; \begin{smallmatrix} \xleftarrow{s} \\ \xrightarrow{} \\ \xleftarrow[s]{} \end{smallmatrix} \; \mathcal{B}. \tag{3.3.28}$$

The category $\mathcal{B}$ is said to be *protomodular* when the functor $W = \langle p, i_R \rangle\colon \mathsf{Pt}(\mathcal{B}) \longrightarrow \mathcal{B} \times \mathcal{B}$ into the product category is conservative, and it is said to be a *category with semidirect products* when W is monadic. Note that since (by definition) $\pi_{\mathcal{B}} \circ W = p$, the functor W is a morphism from p to the trivial fibration $\pi_{\mathcal{B}}$ (the projection is on the first factor):

$$\begin{tikzcd} \mathsf{Pt}(\mathcal{B}) \arrow[rr, "W"] \arrow[dr, "p"'] & & \mathcal{B} \times \mathcal{B} \arrow[dl, "\pi_{\mathcal{B}}"] \\ & \mathcal{B} & \end{tikzcd} \tag{3.3.29}$$

Thus, we have a family of monadic functors $W_B : \mathsf{Pt}_B(\mathcal{B}) \longrightarrow \mathcal{B}$ given by reindexings, and Theorem 4.2.1 (more precisely, (4.2.1)) now yields a parametric monad

$$\_ \flat \_ : \mathcal{B} \times \mathcal{B} \longrightarrow \mathcal{B} \tag{3.3.30}$$

(the multiplication of the monad $B \flat \_$ is described at [BJK05, p. 246]; the unit is given in terms of a universal property that characterises $B \flat X \to B + X$ as an equaliser). Ultimately, there is a pullback diagram

$$\begin{tikzcd} \mathsf{Pt}(\mathcal{B}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{EM}_{\mathcal{B}} \arrow[d] \\ \mathcal{B} \arrow[r] & {[\mathcal{B}, \mathcal{B}]_\mu} \end{tikzcd} \tag{3.3.31}$$

presenting the fibration of points as a fibration of Eilenberg–Moore algebras.

3.4 Polynomials on (pre)toposes

Example 3.4.1 (Polynomials, first take, [MP00])**.** Given a locally Cartesian closed pretopos $\mathcal{E}$ and an object $f : X \to A$ of $\mathcal{E}/A$ we can define a polynomial endofunctor on $\mathcal{E}$ as the following composition

$$P_f : \; \mathcal{E} \xrightarrow{\pi} \mathcal{E}/A \xrightarrow{\langle f, - \rangle} \mathcal{E}/A \xrightarrow{d} \mathcal{E} \tag{3.4.1}$$

where

- π is the functor $X \mapsto \left[\pi_A \begin{smallmatrix} A \times X \\ \downarrow \\ A \end{smallmatrix} \right]$;
- $\langle -, - \rangle : (\mathcal{E}/A)^{\mathrm{op}} \times \mathcal{E}/A \longrightarrow \mathcal{E}/A$ is the internal hom of $\mathcal{E}/A$ (so that $\langle f, \_ \rangle$ is right adjoint to pulling back along f);

- d is the forgetful functor from the slice category (note that $d \dashv \pi$).

By very general facts a morphism $h : (X, f) \to (X', f')$ in $\mathcal{E}/A$ induces a natural transformation $h^* : \langle f', - \rangle \Rightarrow \langle f, - \rangle$ and thus a natural transformation $P_{f'} \Rightarrow P_f$ by whiskering on the left and on the right.

Thus, we obtain a functor

$$\begin{aligned} \mathcal{E}/A &\longrightarrow [\mathcal{E}, \mathcal{E}] \\ \left[\begin{smallmatrix} B \\ \downarrow \\ A \end{smallmatrix}\right] &\longmapsto P_f \end{aligned} \tag{3.4.2}$$

and from this we obtain a fibration having typical fiber $\mathsf{Alg}_{\mathcal{E}}(P_f)$.

Example 3.4.2 (Polynomials, second take, [GK13])**.** We can also define a category $/\mathsf{pol}/_I$ of polynomials having objects the diagrams

$$kf : \; I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} I \tag{3.4.3}$$

and morphisms $kf \to kf'$ diagrams

$$\begin{tikzcd} & B \arrow[r, "f"] \arrow[dl] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A \arrow[dr] \arrow[d] & \\ I & B' \arrow[l] \arrow[r, "f'"'] & A' \arrow[r] & I \end{tikzcd} \tag{3.4.4}$$

where the central square is a pullback. To each object $kf : I \xleftarrow{s} B \xrightarrow{f} A \xrightarrow{t} I$ of $/\mathsf{pol}/_I$ one can associate a polynomial endofunctor $P_k f$ of $\mathcal{E}/I$ defined thanks to the parametric adjunction $\Sigma_g \dashv \Delta_g \dashv \Pi_g$ (with parameter a morphism $g : U \to V$) as the composition

$$P_k f : \; \mathcal{E}/I \xrightarrow{\Delta_s} \mathcal{E}/B \xrightarrow{\Pi_f} \mathcal{E}/A \xrightarrow{\Sigma_t} \mathcal{E}/I \tag{3.4.5}$$

to the effect that a morphism like in (3.4.4) induces a natural transformation $P_k f \Rightarrow P_{kf'}$. So, we obtain a fibration having typical fiber $\mathsf{Alg}_{\mathcal{E}}(P_k f)$.

Note that indexing over $\mathcal{E}/A$ the construction in [MP00] gives a contravariant functor, whereas the particular case $I = 1$ in [GK13] still yields a covariant one. In order to obtain [MP00] as a particular instance of the polynomial functors in [GK13] one has to consider as indexing (what boils down to) the category of Cartesian morphisms for the codomain fibration $\left[\begin{smallmatrix} \mathcal{E}^{\to} \\ \downarrow \\ \mathcal{E} \end{smallmatrix}\right]$, i.e. the category of morphisms $f : X \to A$ and Cartesian commutative squares.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner"] & A \arrow[d, "v"] \\ X' \arrow[r, "f'"'] & A' \end{tikzcd} \tag{3.4.6}$$

In particular, [MP00] contains a description of the endofunctor algebras for such polynomials, with a particular focus on *initial* algebras, which model what in Martin-Löf type theory are W-types. There, their existence is preserved through (a certain) Artin glueing, i.e. our Example 3.3.4, so that [MP00, Theorem 6.3] can be formulated by means of an 'iterated' endofunctor algebras construction having fibered initial objects. We leave the details, and applications to polynomial monads as in [GK13], for future work, but some discussion on the preservation of initial objects through reindexing is already present in §4.4.

3.5 The case of dinatural parametricity

Besides a natural dependency on a parameter, another way in which an endofunctor $T_A : \mathcal{C} \longrightarrow \mathcal{C}$ can be 'indexed over A' is the following: let $L : \mathcal{A} \times \mathcal{C} \longrightarrow \mathcal{D}$ and $R : \mathcal{A}^{\mathrm{op}} \times \mathcal{D} \longrightarrow \mathcal{C}$ be two functors such that for each $A \in \mathcal{A}$ there exist adjunction isomorphisms

$$\mathcal{D}(L(A, X), Y) \cong \mathcal{C}(X, R(A, Y)) \tag{3.5.1}$$

natural in all variables A, X, Y; then, naturality in A is equivalent to the request that the counit and unit

$$L_A R_A \xRightarrow{\epsilon^A} 1 \qquad\qquad 1 \xRightarrow{\eta^A} R_A L_A \tag{3.5.2}$$

of the adjunction $L_A := L(A, -) \dashv R(A, -) =: R_A$ form, respectively, a cowedge and a wedge in the object A. The pervasive abundance of adjoint situations in Mathematics, an eminent example of which is given by the Cartesian closed adjunction of Set, where the functor

$$X \mapsto X^A \times A \tag{3.5.3}$$

depends on A in a dinatural fashion, and the evaluation $(f, a) \mapsto fa$, i.e. the counit $\epsilon : X \mapsto X^A \times A \to X$ is a cowedge in A, compels us to take into account *dinatural parametricity*.

The notion of a *diparametric monad* provides a useful notion for interpreting programming languages that record information about the effects performed by the typing information (see [Atk09b] where a variety of monads of interest are studied in their parametricity; cf. also [OWE20], albeit tangentially), but it doesn't seem that something similar to our 'fibration of algebras' point of view has ever been considered.

The problem in trying to fit this host of examples in our picture is, however, that if the dependence on the parameter is dinatural, there is no way to induce a natural transformation $T_A \Rightarrow T_{A'}$ o $T_{A'} \Rightarrow T_A$ given a morphism $u : A \to A'$ in $\mathcal{A}$.

We can, however, circumvent the problem considering the entire 'twisted product' $\mathcal{A}^{\mathrm{op}} \times \mathcal{A}$ as an indecomposable space of parameters, and then consider the family of endofunctors

$$\begin{aligned} \mathcal{A}^{\mathrm{op}} \times \mathcal{A} &\longrightarrow [\mathcal{D}, \mathcal{D}] \\ (A, A') &\longmapsto L_{A'} R_A \end{aligned} \tag{3.5.4}$$

of which we are free to consider the endofunctor algebras.

Definition 3.5.1. Adopt the notation $(\boldsymbol{L} \dashv \boldsymbol{R})_{\mathcal{A}} = \{L_A \dashv R_A \mid A \in \mathcal{A}\}$ for a parametric adjunction as in (3.5.1), and denote

$$S : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow [\mathcal{D}, \mathcal{D}] : (A, A') \mapsto L_{A'} R_A \qquad T : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow [\mathcal{C}, \mathcal{C}] : (A, A') \mapsto R_A L_{A'} \tag{3.5.5}$$

the parametric endofunctors obtained as in (3.5.2).

- The *fibration of algebras* (resp., of *twisted arrow algebras*) of the parametric adjunction $(\boldsymbol{L} \dashv \boldsymbol{R})_{\mathcal{A}}$ is defined as the central (resp., left) vertical leg of the pullback square

$$\begin{tikzcd} \mathsf{Alg}^{\tau}(\boldsymbol{L} \dashv \boldsymbol{R}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Alg}(\boldsymbol{L} \dashv \boldsymbol{R})_{\mathcal{A}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{Alg}_{\mathcal{C}} \arrow[d] \\ \mathsf{Tw}(\mathcal{A}) \arrow[r, "\Sigma"'] & \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \arrow[r, "T"'] & {[\mathcal{C}, \mathcal{C}]} \end{tikzcd} \tag{3.5.6}$$

- The *opfibration of coalgebras* (resp., of *twisted arrow coalgebras*) of the parametric adjunction $(\boldsymbol{L} \dashv \boldsymbol{R})_{\mathcal{A}}$ is defined as the central (resp., left) vertical leg of the pullback square

$$\begin{tikzcd} \mathsf{coAlg}^{\tau}(\boldsymbol{L} \dashv \boldsymbol{R}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Alg}(\boldsymbol{L} \dashv \boldsymbol{R})_{\mathcal{A}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{coAlg} \arrow[d] \\ \mathsf{Tw}(\mathcal{A}) \arrow[r, "\Sigma"'] & \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \arrow[r, "S"'] & [\mathcal{D}, \mathcal{D}] \end{tikzcd} \tag{3.5.7}$$

Remark 3.5.2. It is worth spelling out explicitly the structure of $\mathsf{Alg}(\boldsymbol{L} \dashv \boldsymbol{R})$ and $\mathsf{Alg}^{\tau}(\boldsymbol{L} \dashv \boldsymbol{R})$.

An object of $\mathsf{Alg}(\boldsymbol{L} \dashv \boldsymbol{R})$ consists of a triple $(A, A'; X)$ where X is a $R_A L_{A'}$-algebra (for short: an AA'-algebra with respect to $\boldsymbol{L} \dashv \boldsymbol{R}$), i.e. a map $\left[\begin{smallmatrix} & R_A L_{A'} X \\ x^{AA'} & \downarrow \\ & X \end{smallmatrix}\right]$. A morphism $(A, A'; X) \to (B, B', Y)$ consists of a triple $(u, v; f) \in \mathcal{A}(B, A) \times \mathcal{A}(A', B') \times \mathcal{X}(X, Y)$ such that f is a morphism of algebras $X \to (u, v)^* Y$, where Y becomes an AA'-algebra posing

$$(u, v)^* Y = R_A L_{A'} Y \xrightarrow{R_u * L_v} R_B L_{B'} Y \to Y \tag{3.5.8}$$

This means that the diagram

$$\begin{tikzcd} R_A L_{A'} X \arrow[r, "R_A L_{A'} f"] \arrow[d, "x^{AA'}"'] & R_A L_{A'} Y \arrow[d, "y^{BB'}"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{3.5.9}$$

is commutative.

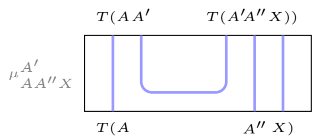
An object of $\mathsf{Alg}^{\tau}(\boldsymbol{L} \dashv \boldsymbol{R})$ admits a similar description, only that objects are *pairs* $(u : A \to A'; X)$ where X is a $R_A L_{A'}$-algebra, and a morphism $(a : A \to A'; X) \to (b; B \to B'; Y)$ consists, similarly, of a morphism $(u, v) \in \mathsf{Tw}(\mathcal{A})$ and a morphism $f : X \to (u, v)^* Y$.

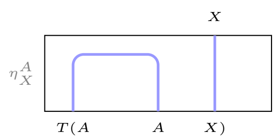
Note that Σ is a discrete fibration, and as a consequence so are $\left[\begin{smallmatrix} \mathsf{Alg}^{\tau}(\boldsymbol{L} \dashv \boldsymbol{R}) \\ \downarrow \\ \mathsf{Alg}(\boldsymbol{L} \dashv \boldsymbol{R}) \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} \mathsf{coAlg}^{\tau}(\boldsymbol{L} \dashv \boldsymbol{R}) \\ \downarrow \\ \mathsf{coAlg}(\boldsymbol{L} \dashv \boldsymbol{R}) \end{smallmatrix}\right]$.

Note how each $L_{A'} R_A$ is in general just a mere endofunctor, and a comonad if $A' = A$; similarly, for $R_A L_{A'}$. It seems like we can't build a fibration collecting the endofunctor algebras for $L_{A'} R_A$ if $A' \neq A$, and the coEilenberg–Moore algebras if $A' = A$, in the same total category (what would the reindexings of such a fibration be?).

The intuition on how to modify the definition of algebra to circumvent this problem is due to R. Atkey, and it is studied to a fairly large extent in [Atk09a] (where, for example, a Beck monadicity theorem is proved).

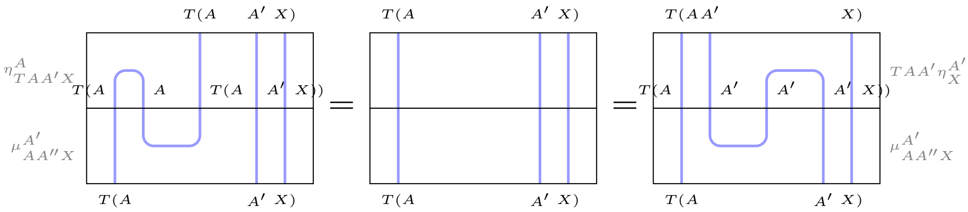
Definition 3.5.3. A *diparametric monad* consists of a functor $T : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ (or equivalently, of type $\mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$) equipped with extranatural transformations $\eta : \mathrm{id}\mathcal{X} \Rightarrow T$ and $\mu : T \circ T \Rightarrow T$ having components



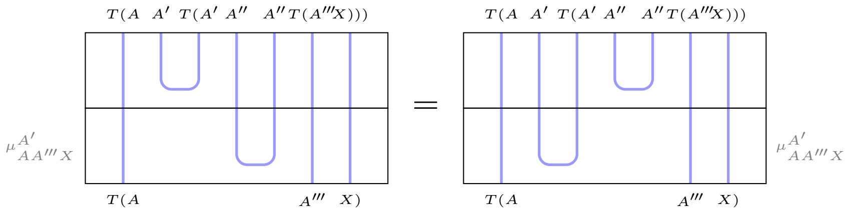


(this means that μ is natural in A, A'', X and dinatural in A', and η is natural in X and dinatural in A) such that the following axioms[1] are satisfied:

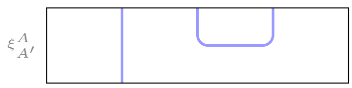
- left and right unitality:



- associativity:

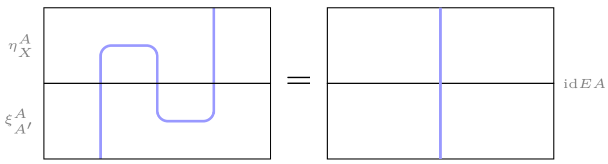


Definition 3.5.4. A *parametric algebra* for a diparametric monad as above consists of a parametric module; spelled out explicitly, this is a functor $E : \mathcal{A}^{\mathrm{op}} \longrightarrow \mathfrak{X}$ equipped with a family of morphisms $\xi^{A}_{A'} : T(A', A, EA) \to EA'$ subject to the following string-diagrammatic constraints: if the algebra map is described as

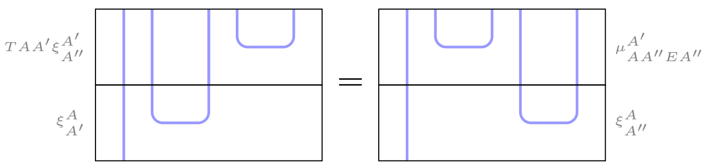


then we require the following compatibilities to hold:

- with the unit of the monad:



- with the multiplication of the monad:



[1] We express the axioms in string diagrammatic form, as the equational presentation in [Atk09a] is notationally quite daunting. The graphical presentation of axioms is not present in Atkey's paper, but it doesn't add anything new.

Remark 3.5.5. Note to what condition these string diagram reduce to, when $E : \mathcal{A}^{\mathrm{op}} \to \mathcal{X}$ is a constant functor at $E \in \mathcal{X}$: instead of an Eilenberg–Moore algebra for the monad $T(A, A, -)$, one has a map $T(A', A, E) \to E$ subject to certain compatibility conditions with the parametric unit and the parametric multiplication. The diparametric monad multiplication and unit are 'irreducible' to a monad structure on the various $T(A', A, -)$. This is compatible with the fact that for a given parametric adjunction, $R_A L_{A'}$ is not a monad if $A' \neq A$.

The obvious notion of morphism between parametric algebras $E, E' : \mathcal{A}^{\mathrm{op}} \to \mathcal{X}$ (a natural transformation $\gamma : E \Rightarrow E'$ compatible with the algebra maps in a suitable sense) gives the definition of the category of parametric algebras $\pi\mathsf{Alg}(T)$ for a diparametric monad as in Definition 3.5.3.

A similar line of reasoning defines *free* parametric algebras for a diparametric monad:

Definition 3.5.6. The category $\pi\mathsf{Kl}(T)$ of *parametric free algebras* for a diparametric monad (T, μ, η) has objects the pairs $(A, X) \in \mathcal{A} \times \mathcal{X}$, and morphisms $(A, X) \to (A', Y)$ the morphisms $X \to T(A, A', Y)$ in $\mathcal{X}$; the identity of (A, X) is given by $\eta_X : X \to T(A, A, X)$, and composition is defined using the multiplication maps $\mu^{A'}_{AA''X}$. Associativity and unitality then follows from the constraints of Definition 3.5.3.

Example of diparametric monads in the sense of Atkey come from elementary category theory and logic.

Example 3.5.7 (The state monad)**.** In every monoidal closed category $(\mathcal{C}, \otimes, I)$, the parametric adjunction

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, C^B) \tag{3.5.10}$$

gives rise to the *state monad* with unit and multiplication respectively given by the following.

$$\eta^A_X : X \longrightarrow (X \otimes A)^A \qquad \mu^B_{XAC} : ((X \otimes A)^B \otimes B)^C \longrightarrow (X \otimes A)^C \tag{3.5.11}$$

Example 3.5.8 (The continuation monad)**.** In every monoidal closed category $\mathcal{C}$, the parametric adjunction

$$\mathcal{C}(X, Y^A) \cong \mathcal{C}(Y, X^A) \tag{3.5.12}$$

gives rise to the *continuation monad* with unit and multiplication obtained as in Example 3.5.7.

Remark 3.5.9. Parametric monads are, much like graded monads, just monads in the sense of [Str72], in a 2-category other than Cat; a diparametric monad $T : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ is in fact just a monad in the bicategory of $[\mathcal{X}, \mathcal{X}]$-enriched profunctors, over the free $[\mathcal{X}, \mathcal{X}]$-enriched category generated by $\mathcal{A}$, and as such, a category enriched over $[\mathcal{X}, \mathcal{X}]$, with morphisms given by the morphisms in $\mathcal{A}$, plus the heteromorphisms prescribed by T. Indeed, the general monad laws boil down to the axioms of Definition 3.5.3 in the special case of $[\mathcal{X}, \mathcal{X}]$-enriched categories.

It is worth to note that however, Atkey provides every diparametric monad with the choice of a tensorial strength –an assumption that we do not include in the definition.

Theorem 3.5.10. There exists a comparison functor

$$K : \mathsf{Alg}(\boldsymbol{L} \dashv \boldsymbol{R}) \longrightarrow \pi\mathsf{Kl}(T) \tag{3.5.13}$$

where $T : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ is the diparametric monad $RL : (A, A') \mapsto R_A L_{A'}$.

On objects, K is defined as a projection $(A, A'; X) \mapsto (A', X)$; on morphisms, a triple $[u, v; f]$ as in Remark 3.5.2 goes to the composition

$$X \xrightarrow{\eta_X^{A'}} R_{A'}L_{A'}X \xrightarrow{R_{A'}L_{A'}f} R_{A'}L_{A'}Y \xrightarrow{R_{A'}*L_v} R_{A'}L_{B'}Y \tag{3.5.14}$$

which is a parametric Kleisli morphism $(X, A') \to (Y, B')$ according to Definition 3.5.6.

Note also that given a parametric adjunction $(\boldsymbol{L} \dashv \boldsymbol{R})$ we can build the fuctors T, S as in Equation 3.5.5 and we can consider the end and the coend

$$\int_A R^A L_A, \int^A L_A R^A : \mathcal{D} \longrightarrow \mathcal{D} \tag{3.5.15}$$

of T, S respectively. As (co)limits, and hence (co)ends, in functor categories are taken pointwise, $\int_A R^A L_A, \int^A L_A R^A$ are precisely the end and coend of a diagram of endofunctors.

Notice that from well-known facts on (co)monadicity of the category of (co)monads over $\mathcal{X}$ it follows that

- the functor $S^\flat = \int^A L_A R^A$ is a comonad on $\mathcal{D}$;
- the functor $T^\sharp = \int_A R^A L_A$ is a monad on $\mathcal{D}$.

Another instance of dinatural parametricity arises when, given a parametric endofunctor $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$, we want to consider its endofunctor algebras and coalgebras at the same time:

Definition 3.5.11. Define the pseudofunctor

$$\begin{aligned} \textsc{ca}_F : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} &\longrightarrow \mathsf{Cat} \\ (A, B) &\longmapsto \mathsf{Alg}_{\mathcal{X}}(F_A) \times \mathsf{coAlg}_{\mathcal{X}}(F_B) \end{aligned} \tag{3.5.16}$$

The fibration associated to $\textsc{ca}_F$ is the *fibration of coalgebras-algebras* obtained from F.

Similarly, given a diparametric monad $T : \mathcal{A} \longrightarrow \mathsf{Mnd}(\mathcal{X})$, its *Eilenberg–Moore-Kleisli fibration* is the fibration associated to the functor

$$\begin{aligned} \textsc{emk}_T : \mathcal{A}^{\mathrm{op}} \times \mathcal{A} &\longrightarrow \mathsf{Cat} \\ (A, B) &\longmapsto \mathsf{EM}(T_A) \times \mathsf{Kl}(T_B) \end{aligned} \tag{3.5.17}$$

Remark 3.5.12. From the fact that if $\mathcal{A}$ has an initial and a terminal object, its twisted arrow category has $\varnothing \to 1$ as a terminal object, we deduce that there are isomorphisms

$$\begin{aligned} \int^A \textsc{ca}_F(A, A) &= \int^A \mathsf{Alg}_{\mathcal{X}}(F_A) \times \mathsf{coAlg}_{\mathcal{X}}(F_B) \\ &\cong \mathsf{Alg}_{\mathcal{X}}(F_\varnothing) \times \mathsf{coAlg}_{\mathcal{X}}(F_1), \\ \int^A \textsc{emk}_T(A, A) &= \int^A \mathsf{EM}(T_A) \times \mathsf{Kl}(T_B) \\ &\cong \mathsf{EM}(T_\varnothing) \times \mathsf{Kl}(T_1). \end{aligned}$$

3.5.1 An optician's lament

In functional programming, a *lens* (and more generally, an *optic*) is a powerful abstraction that provides a composable and bidirectional data type to access and modify parts of other complex data structures. It draws its theoretical foundation from category theory (for the connection with lenses, particular optics, see [Hed19, Hed17]; see [Spi19] for a categorical overview and generalisation; [Ril18, PGW17, Boi18, CEG$^+$20]), specifically focusing on the notion of a lens as a certain type of morphism in categories of pairs of objets, representing abstract data structures.

Loosely speaking, a notion of optic arises every time a monoidal category $\mathcal{M}$ is acting on a category $\mathcal{X}$. To start our analysis, we recall how the construction of the category of optics goes: let $\mathcal{M}$ be a monoidal category, $\mathcal{X}$ a category equipped with two compatible left and right actions of $\mathcal{M}$, i.e. with functors $\leftharpoonup, \rightharpoonup$ such that $(M \rightharpoonup C) \leftharpoonup M' \cong M \rightharpoonup (C \leftharpoonup M')$.

Define a category $\mathsf{Opt}_{\mathcal{X}}$ having as objects the pairs $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ and hom-sets given by

$$\mathsf{Opt}_{\mathcal{X}}(\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right], \left[\begin{smallmatrix} C \\ D \end{smallmatrix}\right]) = \int^{M} \mathcal{X}(A, M \rightharpoonup C) \times \mathcal{X}(D \leftharpoonup M, B) \tag{3.5.18}$$

We can adapt this construction, first exposed in [Tam06, PS08] and used in [Ril18, CEG$^+$20] to describe modular data accessors, to attach a category of optics to a parametric endofunctor $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$.

Let $\mathcal{A}^*$ be the free monoidal category on $\mathcal{A}$ and consider the canonical extension $F^* : \mathcal{A}^* \longrightarrow [\mathcal{X}, \mathcal{X}]$ of F, given by the universal property of free monoidal categories; then one can define a category $\mathsf{Opt}_{\mathcal{X},F}$ having objects the pairs of objects of $\mathcal{X}$ and hom-sets

$$\mathsf{Opt}_{\mathcal{X}}(\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right], \left[\begin{smallmatrix} C \\ D \end{smallmatrix}\right]) = \sum_{n\in\mathbb{N}} \int^{\vec{X} = X_0,\dots,X_n} \mathcal{X}(A, \vec{X} \rightharpoonup C) \times \mathcal{X}(D \leftharpoonup \vec{X}, B) \tag{3.5.19}$$

where $\vec{X} \rightharpoonup C$, $B \leftharpoonup \vec{X}$ are shorthands for

$$\begin{aligned} \vec{X} \rightharpoonup C &:= F_{X_0}(\dots F_{X_{n-1}} F_{X_n} C) \\ D \leftharpoonup \vec{X} &:= F_{X_n}(\dots F_{X_1} F_{X_0} D) \end{aligned}$$

respectively.

Remark 3.5.13. Clearly, to determine (3.5.19) it is enough to compute the coends

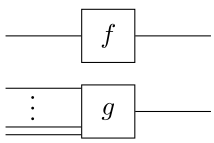
$$\int^{X} \mathcal{X}(A, X \rightharpoonup C) \times \mathcal{X}(D \leftharpoonup X, B), \quad \int^{XY} \mathcal{X}(A, (X,Y) \rightharpoonup C) \times \mathcal{X}(D \leftharpoonup (X,Y), B), \quad \dots \tag{3.5.20}$$

and sum all the results.

Example 3.5.14. If $\mathcal{A} = \mathcal{X}$ is a Cartesian category and F is the regular representation of Example 3.2.1, from (3.5.19) we have to compute the coends

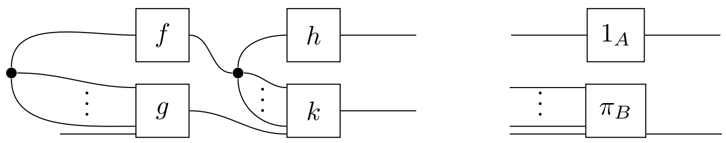
$$\int^{X_0,\dots,X_n} \mathcal{X}(A, X_0 \times \dots \times X_n \times C) \times \mathcal{X}(X_0 \times \dots \times X_n \times D, B) \tag{3.5.21}$$

whence we obtain the category having objects the pairs $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ and hom-sets given by the morphisms of type



(3.5.22)

and composition and identities respecively given by



(3.5.23)

This is usually called the category of *traversables.*

Note that there's very few other cases when the integral (3.5.19) can be explicitly evaluated.

In fact, even in the case where $\mathcal{A} = \mathcal{X}$ is a monoidal (nonCartesian) category and F is the regular representation, the integral (3.5.19) is not easy to compute: in fact, it is not even clear how to compute the coend at step $n = 2$, as the integral

$$\int^{XY} \mathcal{X}(A, Y \otimes X \otimes C) \times \mathcal{X}(D \otimes X \otimes Y, B) \tag{3.5.24}$$

doesn't seem to reduce to a known object.

Construction 3.5.15. A more interesting way to attach a category of optics to a parametric endofunctor comes from the theory of *dependent optics* exposed in [Ver23]; there, a category of optics is defined in terms of a pair of pseudofunctors $L, R : \mathcal{B}^{\text{op}} \longrightarrow \mathsf{Cat}$ from a bicategory $\mathcal{B}$; the objects of $\mathsf{Opt}_{L,R}$ are then 'dependent' pairs $(X, Y)^B := (B; X \in \mathcal{L}_B, Y \in \mathcal{R}_B)$, where $B \in \mathcal{B}$ is an object of the base bicategory, and X, Y objects of $\mathcal{L}_B = LB, \mathcal{R}_B = RB$ respectively. Hom-sets are defined as the coends

$$\mathsf{Opt}_{L,R}((X, Y)^B, (X', Y')^{B'}) = \int^{f \in \mathcal{B}(B,B')} \mathcal{L}_B(X, Lf(X')) \times \mathcal{R}_B(Rf(Y'), Y) \tag{3.5.25}$$

Specialising this construction to a pair of parametric endofunctors $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}], G : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ and the pseudofunctors associated to their fibration of algebras, we obtain the category with objects the dependent pairs $\left(\begin{smallmatrix} F_A X \\ \downarrow \\ X \end{smallmatrix}, \begin{smallmatrix} G_A Y \\ \downarrow \\ Y \end{smallmatrix}\right)$, and hom-sets the coends above.

We conclude this excursus in profunctor optics theory with a way to connect the category of (Set-based) lenses with the Kleisli opfibration of the parametric continuation monad of Example 3.5.8. The derivative functor was introduced in [Cap23] but this connection wasn't observed.

Construction 3.5.16. Let $\mathsf{Lens}(\mathsf{Set}) = \mathsf{Lens}$ be the category having objects the pairs $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ and morphisms

$$\mathsf{Lens}(\left[\begin{smallmatrix} X \\ S \end{smallmatrix}\right], \left[\begin{smallmatrix} Y \\ R \end{smallmatrix}\right]) := \mathsf{Set}(X, Y) \times \mathsf{Set}(X \times R, S) \tag{3.5.26}$$

(this is a particular case of Equation 3.5.18 when the action is the regular representation); now, define the endofunctor $\partial : \mathsf{Lens} \longrightarrow \mathsf{Lens}$ as follows:

- on objects $\partial \left[\begin{smallmatrix} X \\ S \end{smallmatrix}\right] = \left[\begin{smallmatrix} X \\ S^X \end{smallmatrix}\right]$;
- on morphisms, as a map

$$\mathsf{Lens}(\left[\begin{smallmatrix} X \\ S \end{smallmatrix}\right], \left[\begin{smallmatrix} Y \\ R \end{smallmatrix}\right]) \longrightarrow \mathsf{Lens}(\left[\begin{smallmatrix} X \\ S^X \end{smallmatrix}\right], \left[\begin{smallmatrix} Y \\ R^Y \end{smallmatrix}\right]) \tag{3.5.27}$$

so that a pair of functions $u : X \to Y$ and $f : X \times R \to S$ goes to the pair

$$(u, \partial_u f) \in \mathsf{Lens}(\left[\begin{smallmatrix} X \\ S^X \end{smallmatrix}\right], \left[\begin{smallmatrix} Y \\ R^Y \end{smallmatrix}\right]) = \mathsf{Set}(X, Y) \times \mathsf{Set}(X \times R^Y, S^X) \tag{3.5.28}$$

where $\partial_u f(x', t) = \lambda x.f(x, t(ux))$.

This is called the *reverse derivative* of a lens in [Cap23].

Lemma 3.5.17. Let M be the free monoid on two generators, $\mathbb{N}\langle\alpha, \beta\rangle$. Then, every ∂-coalgebra $(X, f) \in \mathsf{coAlg}(\partial)$ induces a representation of M on the Kleisli category of the parametric continuation monad of Example 3.5.8.

Theorem 3.5.18. The correspondence of the previous lemma is the action on objects of a functor

$$\Upsilon : \mathsf{coAlg}(\partial) \longrightarrow \pi\mathsf{Kl}(T) \tag{3.5.29}$$

where $\pi\mathsf{Kl}(T)$ is the Kleisli category in the sense of Atkey, of the parametric continuation monad $T : X \mapsto S^{(S^X)}$ on Set (cf. section 3.5).

The result is easily unraveled: an endofunctor coalgebra for ∂ consists of an element $(\bar{\alpha}, \beta)$ of $\mathsf{Set}(X, X) \times \mathsf{Set}(X \times S^X, S)$, i.e. of a pair of functions

$$X \xrightarrow{\bar{\alpha}} X \xrightarrow{\eta_X^S} S^{(S^X)} \qquad\qquad X \xrightarrow{\beta} S^{(S^X)} \tag{3.5.30}$$

(up to currying β and precomposing $\bar{\alpha}$ with the unit of the continuation monad. Clearly, this defines a representation of M sending $(\left[\begin{smallmatrix} X \\ S \end{smallmatrix}\right], (\bar{\alpha}, \beta))$ to the carrier X with the action induced by α, β.

The crux of the matter is to show that this correspondence is functorial. The generic morphism of coalgebras $\left[\begin{smallmatrix} \nu \\ \varphi \end{smallmatrix}\right] : \left[\begin{smallmatrix} X \\ S \end{smallmatrix}\right] \to \left[\begin{smallmatrix} Y \\ R \end{smallmatrix}\right]$ now induces a morphism $X \to S^{(R^Y)}$, defined as

$$x \mapsto \lambda r.\varphi(\nu, r(\nu x)). \tag{3.5.31}$$

Very boring routine calculations now show that this assignment is such that the identity in Lens goes to the unit η_X^S, and that composition of coalgebra maps in Lens goes to Kleisli composition in $\pi\mathsf{Kl}(T)$.

Chapter 4

Structure theorems

Summary of chapter

The present chapter is the heart of our work, where we expose a formal theory of fibrations of algebras; we elucidate the relation between fibrations of algebras and graded monads in Remark 4.1.5: the category of parametric algebras of a parametric monad $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ and of graded algebras for an associated graded monad $\hat{T}$ are equivalent. In this perspective, the theory of fibrations of algebras is a portion of the theory of graded monads.

Then, we move to characterise parametric monads intrinsically: Theorem 4.2.1 and Theorem 4.2.5 prove that a parametric monad is equivalently: a monad in $\mathsf{Fib}(\mathcal{A})$ over the trivial fibration $\left[\pi_{\mathcal{A}} \begin{smallmatrix} \mathcal{A}\times\mathcal{X} \\ \downarrow \\ \mathcal{A} \end{smallmatrix}\right]$; a monad in the coKleisli 2-category of the coreader comonad $\mathcal{A} \times \_$ over an object $\mathcal{X}$; a monad in the 2-category $\mathsf{Cat} \ltimes \mathsf{Cat}$ arising from Theorem 3.3.6.

In §4.3 we substantiate our view of fibrations of algebras as semidirect products, building the 'canonical null sequence' of a fibration of algebras

$$1 \longrightarrow \mathcal{X} \longrightarrow \mathcal{A} \ltimes \mathcal{X} \longrightarrow \mathcal{A} \longrightarrow 1$$

in Remark 4.3.6, and (4.3.4). We study the problem of the existence of adjoints to fibre inclusions in §4.4, which is naturally stated in terms of the canonical null sequence, and subsequently in §4.5 we move to relate the Eilenberg–Moore fibration $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ of a parametric monad T and the Kleisli fibration $\mathcal{A} \ltimes^{\mathsf{Kl}} \mathcal{X}$ via a *parametric Linton theorem*, Construction 4.5.2, built on the Yoneda structure on $\mathsf{Fib}(\mathcal{A})$ [SW78, Str81]. Lastly, in §4.6 we provide universal properties of the fibration of algebras $\mathcal{A} \ltimes \mathcal{X}$ and $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ as 2-dimensional limits (Proposition 4.6.5) and colimits (§5.1), and we draft the essentials of a theory of fibration of algebras in an abstract 2-category other than Cat. The fundamental result in this last part of the exposition is the well-known observation that the object of algebras of a endo-1-cell $f : X \to X$ is an inserter [Kel89].

4.1 On the relation with graded monads

Remark 4.1.1. In [BJK05] it is observed that to give a lax monoidal functor $(\mathcal{C}, +) \longrightarrow (\mathcal{K}, \otimes)$, where $\mathcal{K}$ is any monoidal category and $\mathcal{C}$ is coCartesian is the same as to give a functor $\mathcal{C} \longrightarrow \mathsf{Mon}(\mathcal{K})$, due to the universal property of the coCartesian monoidal structure.

We are particularly interested in this result when $\mathcal{K} = [\mathcal{X}, \mathcal{X}]$ is the category of endofunctors of $\mathcal{X}$ and $\otimes$ is functor composition.

In fact, in such a situation, thanks to the fact that every object of a coCartesian category $\mathcal{C}$ has a canonical monoid structure, we can prove that a functor $F : \mathcal{C} \longrightarrow \mathsf{Mon}(\mathcal{X})$ defines laxators

$$F_A \circ F_B \longrightarrow F_{A+B} \circ F_{A+B} \longrightarrow F_{A+B} \tag{4.1.1}$$

and vice versa, any lax monoidal functor $F : (\mathcal{C}, +) \longrightarrow ([\mathcal{X}, \mathcal{X}], \circ)$ is such that each F_A is a monad with multiplication

$$F_A \circ F_A \longrightarrow F_{A+A} \longrightarrow F_A \tag{4.1.2}$$

Remark 4.1.2. There is a rather intrinsic way to prove this result: given a parametric endofunctor $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$, a monad structure on each F_A amounts to the presence of 2-cells

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\Downarrow\mu"] & {[\mathcal{X}, \mathcal{X}]} \\ \mathcal{A} \times \mathcal{A} \arrow[r, "F \times F"'] & {[\mathcal{X}, \mathcal{X}] \times [\mathcal{X}, \mathcal{X}]} \arrow[u, "\circ"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[r, "t"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\eta"] & \mathbb{1} \arrow[d, "\lceil \mathrm{id}\mathcal{X} \rceil"] \\ \mathcal{A} \arrow[r, "F"'] & {[\mathcal{X}, \mathcal{X}]} \end{tikzcd} \tag{4.1.3}$$

while if $\mathcal{A}$ is coCartesian a lax monoidal structure on F amounts to the presence of 2-cells

$$\begin{tikzcd} \mathcal{A} \arrow[r, "F"] \arrow[dr, phantom, "\Nwarrow"] & {[\mathcal{X}, \mathcal{X}]} \\ \mathcal{A} \times \mathcal{A} \arrow[u, "+"] \arrow[r, "F \times F"'] & {[\mathcal{X}, \mathcal{X}] \times [\mathcal{X}, \mathcal{X}]} \arrow[u, "\circ"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{A} \arrow[d, equal] & \mathbb{1} \arrow[l, "\varnothing"'] \arrow[d, "\lceil \mathrm{id}\mathcal{X} \rceil"] \\ \mathcal{A} \arrow[r, "F"'] & {[\mathcal{X}, \mathcal{X}]} \end{tikzcd} \tag{4.1.4}$$

Then, if $\mathcal{A}$ is coCartesian, these two types of 2-cells correspond bijectively to each other under mating.[1]

It is an observation of J. Bénabou [Bén67] that a monad in a bicategory $\mathbb{B}$ consists exactly of a lax functor $\mathbb{1} \longrightarrow \mathbb{B}$, where $\mathbb{1}$ is the terminal bicategory. The notion of a *graded monad* [FKM16] arises to generalise this fact:

Definition 4.1.3 (Graded monad)**.** Let $\mathbb{B}$ be a bicategory; a *graded monad* in $\mathbb{B}$ consists of a lax functor $\Sigma\mathcal{K} \longrightarrow \mathbb{B}$, where $\Sigma\mathcal{K}$ is the one-object bicategory associated to a monoidal category $(\mathcal{K}, \otimes)$.

When there is no risk of misunderstanding, we will denote a graded monad simply as $T : \mathcal{K} \longrightarrow \mathbb{B}$, implicitly assuming that $\mathcal{K}$ is regarded as a one-object bicategory.

Unwinding this definition in the case $\mathbb{B} = \mathsf{Cat}$ is the strict 2-category of categories, functors and natural transformations, we get that a graded monad consists of the following data:

- a family of functors $T_M : \mathcal{X} \longrightarrow \mathcal{X}$;

[1] A *coCartesian object* in a Cartesian 2-category $\mathcal{K}$ can be defined as a 0-cell A such that the comonoid maps $A \to 1, A \to A \times A$ are right adjoints. CoCartesian objects in $(\mathsf{Cat}, \times)$ are precisely categories with (finite) coproducts.

- a natural transformation $\eta : \mathrm{id}\mathcal{X} \Rightarrow T_J$ where J is the monoidal unit of $\mathcal{K}$;
- a family of natural transformations $\mu^{MN} : T_M T_N \Rightarrow T_{M\otimes N}$,

satisfying the axioms of unitality and associativity:

- for $M, N \in \mathcal{K}$, the diagrams

$$\begin{tikzcd} T_N \arrow[r, "\eta T_N"] \arrow[dr, bend right] & T_J T_N \arrow[d, "\mu^{JN}"] \\ & T_{JN} \end{tikzcd} \qquad \begin{tikzcd} T_M T_J \arrow[d, "\mu^{MJ}"'] & T_M \arrow[l, "T_M \eta"'] \arrow[dl, bend left] \\ T_{MJ} & \end{tikzcd} \tag{4.1.5}$$

  are commutative, where the arrows $T_M \Rightarrow T_{MJ}$, $T_N \Rightarrow T_{JN}$ are induced by the unitors;
- for $L, M, N \in \mathcal{K}$, the diagram

$$\begin{tikzcd} T_L T_M T_N \arrow[r, "T_L \mu^{MN}"] \arrow[d, "\mu^{LM} T_N"'] & T_L T_{M\otimes N} \arrow[dd, "\mu^{L,MN}"] \\ T_{L\otimes M} T_N \arrow[d, "\mu^{LM,N}"'] & \\ T_{(L\otimes M)\otimes N} \arrow[r] & T_{L\otimes(M\otimes N)} \end{tikzcd} \tag{4.1.6}$$

  commutes, where the unnamed arrow comes from the associator.

This leads directly to the following definition, once recognised that the category of Eilenberg–Moore algebras for a monad $T : \mathcal{C} \longrightarrow \mathcal{C}$ is the lax limit of the lax functor $T : \mathbb{1} \longrightarrow \mathsf{Cat}$:

Definition 4.1.4 (Algebras for a graded monad)**.** The *category of algebras* for a graded monad $T : \mathcal{K} \longrightarrow \mathbb{B}$ is –provided it exists– the lax limit of the functor $T : \mathcal{K} \longrightarrow \mathbb{B}$.

Unwinding this definition in the case $\mathbb{B} = \mathsf{Cat}$ is the strict 2-category of categories, functors and natural transformations, we get that an algebra for a graded monad T consists of morphisms $T_M A_N \to A_{M\otimes N}$ satisfying suitable axioms (cf. [Fuj19, §4.1.1]).

This notion of graded algebra for a graded monad depends on all the parameters simultaneously; instead, we have been considering a *fiberwise* notion of parametric algebras. It is natural to wonder whether the two constructions are related meaningfully.

The relation between the two notions of algebras is made precise by the following remark: when we consider a parametric functor and its algebras in our sense, we are, in the end, just considering the algebras (in the graded sense) for the associated lax monoidal functor.

Remark 4.1.5 (On the relation with graded monads)**.** The category of parametric algebras for a parametric monad $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]_\mu$ and the category of graded algebras for the associated graded monad $\tilde{F} : (\mathcal{A}, +) \longrightarrow [\mathcal{X}, \mathcal{X}]$ are equivalent.

Proof. The proof fundamentally uses the equivalence of Remark 4.1.1. $\square$

4.2 EM fibrations as Eilenberg–Moore objects

A useful characterisation lemma for Eilenberg–Moore fibrations uses a monadicity criterion: it is known (cf. [Jac98, p. 79]) how to characterise monads in $\mathsf{Fib}(\mathcal{A})$ as monads on total categories having vertical unit and multiplication. This gives at once the following result, that turns out to be one of the most useful results of the present paper, although completely elementary.

Theorem 4.2.1. A fibration $\left[\begin{smallmatrix}\mathcal{E}\\ p\downarrow\\ \mathcal{A}\end{smallmatrix}\right]$ is an Eilenberg–Moore fibration if and only if there exists a morphism of fibrations $H : \left[\begin{smallmatrix}\mathcal{E}\\ p\downarrow\\ \mathcal{A}\end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix}\mathcal{A}\times\mathcal{X}\\ \pi_{\mathcal{A}}\downarrow\\ \mathcal{A}\end{smallmatrix}\right]$ which is monadic as a 1-cell in $\mathsf{Fib}(\mathcal{A})$.

This, in turn, is equivalent to the fact that H

- has a left adjoint L fibred over $\mathcal{A}$;
- the Eilenberg–Moore object for the monad HL induced by $L \dashv H$ is equivalent to p.

We can flesh out the statement in more explicit terms. An Eilenberg–Moore fibration is equivalent to giving a functor

$$\mathcal{A} \longrightarrow (\mathsf{Cat}/\mathcal{X})^{\mathrm{op}}_m \tag{4.2.1}$$

where the codomain is the subcategory of functors $U : \mathcal{C} \to \mathcal{X}$ that are monadic.

The proof Theorem 4.2.1 can be given directly, but we prefer to proceed swiftly to more interesting results, and thus we provide a formal argument suited to a more general setting (that will be extensively adopted in §4.7).

Proof. Define two correspondences, in opposite directions, between fibrations p_T obtained as pullbacks

$$\begin{tikzcd} \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \int \mathsf{EM}_{\mathcal{X}} \arrow[d, "U_\mu"] \\ \mathcal{A} \arrow[r, "T"'] & [\mathcal{X},\mathcal{X}]_\mu \end{tikzcd} \tag{4.2.2}$$

and monads on $\mathcal{A} \times \mathcal{X}$, fibered over $\mathcal{A}$:

- on one side, given T (and thus the fibration p_T), we compose it with the forgetful functor $V : \int \mathsf{EM}_{\mathcal{X}} \longrightarrow \mathcal{X}$ sending $(A, E \in \mathsf{EM}(T_A)) \mapsto E$ of (4.3.4); this yields

  $$\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \xrightarrow{\langle p_T, V\rangle} \mathcal{A} \times \mathcal{X} \tag{4.2.3}$$

  and one can prove directly that this amounts to collating together all (monadic) functors $U_A : \mathsf{EM}(T_A) \longrightarrow \mathcal{X}$ and their left adjoints into a monadic functor;
- on the other side, let $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}$ be a monad, split as the composition $UF : \mathcal{A} \times \mathcal{X} \xrightarrow{F} \mathsf{EM}(T) \xrightarrow{U} \mathcal{A} \times \mathcal{X}$; then, we can consider the composition $\pi_{\mathcal{X}} \circ UF$, i.e.

  $$\mathcal{A} \times \mathcal{X} \xrightarrow{T} \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}\,. \tag{4.2.4}$$

Now, we have to prove that these two correspondences are mutually inverse; the heart of the matter is proving that the diagram

$$\begin{tikzcd}
\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[rrr] \arrow[d] & & & \int \mathsf{EM}_{\mathcal{X}} \arrow[d] \\
\mathcal{A} \arrow[r, "\eta"] & {[\mathcal{X}, \mathcal{A} \times \mathcal{X}]} \arrow[r, "{[\mathcal{X}, UF]}"] & {[\mathcal{X}, \mathcal{A} \times \mathcal{X}]} \arrow[r, "{[\mathcal{X}, \pi_{\mathcal{X}}]}"] & {[\mathcal{X}, \mathcal{X}]}
\end{tikzcd} \tag{4.2.5}$$

is a pullback; this is however a direct consequence of the pasting lemma for pullbacks:

$$\begin{tikzcd}
\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner", very near start] & & {[\mathcal{X}, \mathcal{A}] \times \int \mathsf{EM}_{\mathcal{X}}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \int \mathsf{EM}_{\mathcal{X}} \arrow[d] \\
\mathcal{A} \arrow[r, "\eta"] & {[\mathcal{X}, \mathcal{A} \times \mathcal{X}]} \arrow[r, "{[\mathcal{X}, UF]}"] & {[\mathcal{X}, \mathcal{A} \times \mathcal{X}]} \arrow[r, "{[\mathcal{X}, \pi_{\mathcal{X}}]}"] & {[\mathcal{X}, \mathcal{X}]}
\end{tikzcd} \tag{4.2.6}$$

and this concludes the argument. $\square$

Consider the functor

$$\langle p_T, V \rangle : \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}. \tag{4.2.7}$$

It is a general fact that its left adjoint L is the objectwise coproduct of the left adjoints to p_T and V, and coupled with the remarks of subsection 2.2.3 a direct computation shows that such lef adjoint acts sending (A, X) to the free A-algebra $(T_A X, \mu_X^A)^A$.

From here, it is clear how to construct a (tautological) equivalence of categories between the category of algebras for the induced monad $\langle p_T, V \rangle \circ L$ and $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$.

Remark 4.2.2. It's clear how to dualise the Theorem 4.2.1 to characterise $\mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X}$ as a coEilenberg-Moore object; we will postpone a precise statement to Theorem 4.2.6 below.

The above discussion yields an application to the aforementioned Cartier–Gabriel–Kostant theorem.

Example 4.2.3 (The Cartier–Gabriel–Kostant setting)**.** Let k be an algebraically closed field of characteristic zero, let Lie_k denote the category of Lie algebras over k and let CCHopf denote the category of cocommutative Hopf algebras over k. We then have the adjunctions

$$\begin{tikzcd}
\mathsf{Lie}_k \arrow[r, shift left, "U"] \arrow[r, phantom, "\perp"] & \mathsf{CCHopf} \arrow[l, shift left, "P"] \arrow[r, "\mathcal{G}" description] & \mathsf{Grp} \arrow[l, shift right, "{k[-]}"'] \arrow[l, shift left=2, "{k[-]}"]
\end{tikzcd} \tag{4.2.8}$$

where

- the functor $\mathcal{G}$ maps a Hopf algebra to the set of its grouplike elements (elements satisfying $\Delta(x) = x \otimes x$), which becomes a group if we restrict the multiplication of the algebra to it,
- the functor P maps a lie algebra to its primitive elements (elements satisfying $\Delta(x) = x \otimes 1 + 1 \otimes x$), which becomes a Lie algebra under the commutator operation $[x, y] = xy - yx$,
- the functor $k[-]$ maps a group G to its group algebra, which becomes a Hopf algebra if we define comultiplication on the generators $g \in G$ by $\Delta(g) = g \otimes g$ and the antipode on the generators $g \in G$ by $S(g) = g^{-1}$ (note that $k[\_] \dashv \mathcal{G} \dashv k[\_]$), and

- the functor U maps a Lie algebra L to its universal enveloping Hopf algebra, which is constructed by taking the free algebra on the vector space L, which is given by $\bigoplus_{n=0}^{\infty} L^{\otimes n}$, and taking the largest quotient algebra that kills the elements $[x, y] - (x \otimes y - y \otimes x)$.

The CGK theorem [Kos77], [EGNO15, 5.10.2] now can be rephrased into the following assertion, thus giving rise to an example of a fibration of algebras:

Theorem 4.2.4 (Cartier–Gabriel–Konstant theorem)**.** The functor $\langle \mathcal{G}, P \rangle : \mathsf{CCHopf} \longrightarrow \mathsf{Grp} \times \mathsf{Lie}_k$ is monadic over the trivial fibration $\left[\begin{smallmatrix} & \mathsf{Grp}\times\mathsf{Lie}_k \\ \pi_{\mathsf{Grp}} & \downarrow \\ & \mathsf{Grp} \end{smallmatrix}\right]$.

4.2.1 More characterisation results

Theorem 4.2.1 evidently has a dual analogue for fibration of Eilenberg–Moore coalgebras; in fact, we can appreciate best the nature of Theorem 4.2.1 above if we rephrase it as follows:

Theorem 4.2.5. The following pieces of data are equivalent:

- a parametric monad $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$, i.e. (upon currying) a functor $\mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]_\mu$;
- a monad $T : \left[\begin{smallmatrix} & \mathcal{A}\times\mathcal{X} \\ \pi_{\mathcal{A}} & \downarrow \\ & \mathcal{X} \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} & \mathcal{A}\times\mathcal{X} \\ \pi_{\mathcal{A}} & \downarrow \\ & \mathcal{X} \end{smallmatrix}\right]$ in the 2-category $\mathsf{Fib}(\mathcal{A})$;
- a monad $T : \mathcal{A} \times \mathcal{X} \rightsquigarrow \mathcal{X}$ in the 2-coKleisli category of the 2-comonad $\mathcal{A} \times -$, over the object $\mathcal{X}$;
- a monad in the (domain of the) 2-fibration $\mathsf{Cat} \ltimes \mathsf{Cat}$ obtained in Theorem 3.3.6.

Proof. It is enough to observe that the unit and multiplication of a parametric monad $T : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]_\mu$ can be seen as 2-cells $\eta : \pi_{\mathcal{X}} \Rightarrow T$ and $\mu : T \bullet T \Rightarrow T$ where $\pi_{\mathcal{X}} : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ is the projection functor and

$$T \bullet T : \mathcal{A} \times \mathcal{X} \xrightarrow{\Delta \times \mathcal{X}} \mathcal{A} \times \mathcal{A} \times \mathcal{X} \xrightarrow{\mathcal{A} \times T} \mathcal{A} \times \mathcal{X} \xrightarrow{T} \mathcal{X} \tag{4.2.9}$$

is a coKleisli composition. □

This statement is more prone to be dualised straightforwardly:

Theorem 4.2.6. The following pieces of data are equivalent:

- a parametric comonad $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$, i.e. (upon currying) a functor $\mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]_\delta$;
- a comonad $T : \left[\begin{smallmatrix} & \mathcal{A}\times\mathcal{X} \\ \pi_{\mathcal{A}} & \downarrow \\ & \mathcal{X} \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} & \mathcal{A}\times\mathcal{X} \\ \pi_{\mathcal{A}} & \downarrow \\ & \mathcal{X} \end{smallmatrix}\right]$ in the 2-category $\mathsf{Fib}(\mathcal{A})$;
- a comonad $T : \mathcal{A} \times \mathcal{X} \rightsquigarrow \mathcal{X}$ in the 2-coKleisli category of the 2-comonad $\mathcal{A} \times -$, over the object $\mathcal{X}$;
- a comonad in the (domain of the) 2-fibration $\mathsf{Cat} \ltimes \mathsf{Cat}$ obtained in Theorem 3.3.6.

Remark 4.2.7. A similar statement holds for coKleisli and Kleisli (op)fibrations: a functor $\left[\begin{smallmatrix} & \mathcal{E} \\ p & \downarrow \\ & \mathcal{A} \end{smallmatrix}\right]$ is equivalent to $\mathcal{A} \ltimes^{\mathsf{Kl}} \mathcal{X}$ (resp., $\mathcal{A} \oplus^{\mathsf{Kl}} \mathcal{X}$) if and only if it is the (co)Kleisli object of a monad fibered over the projection; Kleisli objects in the 2-category Fib are the subject of [Her93, 5.4.1, 5.4.2].

Remark 4.2.8. We turn to an application of this characterisation: Theorem 4.2.1, and 4.2.5 even more, are quite useful in pinpointing the correct notion of *distributive law* between parametric endofunctors $S, T : \mathcal{X} \longrightarrow \mathcal{X}$. Recall that the classical notion of distributive law consists of a 'lax intertwiner' between functors $\lambda : ST \Rightarrow TS$ to which we require to be compatible with the structure of S, T (monad and monad; comonad and monad; monad and comonad, etc.; cf. [AM20, Appendix C.1] for a thorough discussion on the matter of intertwining p monads and q comonads).

Now Theorem 4.2.5 allows describing distributive laws between parametric moands as distributive laws between a monad $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}$ on $\pi_{\mathcal{A}}$, and a monad $S : \mathcal{B} \times \mathcal{Y} \longrightarrow \mathcal{B} \times \mathcal{Y}$ on $\pi_{\mathcal{B}}$. (Unwinding what this latter characterisation boils down to is just a matter of bookkeeping the notation.)

Another application of this point of view is to relate endofunctor algebra fibrations and Eilenberg–Moore fibrations together.

Remark 4.2.9. Assume $\mathcal{X}$ is such that the forgetful functor

$$U : [\mathcal{X}, \mathcal{X}]_\mu \longrightarrow [\mathcal{X}, \mathcal{X}] \tag{4.2.10}$$

has a left adjoint L, we can compare the fibration of endofunctor algebras for a parametric functor $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ and the fibration of Eilenberg–Moore algebras for the associated free parametric monad $\tilde{F} : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}] \xrightarrow{L} [\mathcal{X}, \mathcal{X}]_\mu$.

In fair generality, these fibrations are even equivalent, as it can be seen observing that the fiberwise equivalences between $\mathsf{Alg}_{\mathcal{X}}(F_A)$ and $\mathsf{EM}(\tilde{F}_A)$ glue together to an equivalence of fibrations.

4.3 Fibrations of algebras as short exact sequences

The purpose of this section is to provide a more formal outlook on fibrations of algebras, inspired by the fact that under minimal assumptions on $\mathcal{A}, \mathcal{X}$ we can build 'short exact sequences of categories' out of a parametric endofunctor $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$.

Assumption 4.3.1. For the entire section, we will assume that the category $\mathcal{A}$ of parameters has an initial object and the category $\mathcal{X}$, the domain of the parametric endofunctors we consider, has a terminal object. Sometimes, we will mention the possibility that $\mathcal{A}$ is pointed or has both an initial and a terminal object, and likewise for $\mathcal{X}$.

Moreover, we assume that all parametric free algebras exist: more formally, each functor

$$U_A : \{A\} \ltimes \mathcal{X} \longrightarrow \mathcal{X} \tag{4.3.1}$$

has a left adjoint.

Remark 4.3.2. In case F is a parametric monad, this last assumption is always satisfied for obvious reasons when considering its fibration of EM-algebras and its opfibration of Kleisli categories.

Now that this is set up, we have to establish some notation and terminology:

Definition 4.3.3 (Null adjunctions, null sequences)**.** In the above assumption that $\mathcal{A}$ has an initial object and $\mathcal{X}$ a terminal one, we call the pair of adjoint functors

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "1"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left, "\varnothing"] \end{tikzcd} \tag{4.3.2}$$

obtained from the constant at the initial and terminal objects the *null adjunction*.

Equivalently, we call null adjunction any adjoint pair that factors through the discrete category on one object. Note that the essential uniqueness of adjoints makes the above one in (4.3.2) the unique null adjunction, up to isomorphism of functors.

Consider, now, a sequence of adjoints

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "u"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "u_R"] \arrow[r, shift left, "v"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "v_R"] \end{tikzcd} \tag{4.3.3}$$

that composes to the null adjunction; in such a situation, we call (4.3.3) a *null sequence of adjunctions.*

Under such assumptions,

Lemma 4.3.4. The category $\mathcal{A} \ltimes \mathcal{X}$ induced by a parametric endofunctor $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ fits in a null sequence of adjunctions

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "\Phi"] \arrow[r, phantom, "\perp"] & \mathcal{A} \ltimes \mathcal{X} \arrow[l, shift left, "V"] \arrow[r, shift left, "p_F"] \arrow[r, phantom, "\perp"] & \mathcal{A}. \arrow[l, shift left, "s"] \end{tikzcd} \tag{4.3.4}$$

The right adjoint s is determined by choosing the terminal object of each fibre. The functor V is obtained from the forgetful functors $U_A : \mathsf{Alg}_{\mathcal{X}}(F_A) \longrightarrow \mathcal{X}$, and its left adjoint from the free algebra at the initial parameter: $X \in \mathcal{X}$ goes to the free $F_\varnothing$-algebra on X.

Proposition 4.3.5. Given a null sequence

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left, "u"] \arrow[r, phantom, "\perp"] & \mathcal{B} \arrow[l, shift left, "u_R"] \arrow[r, shift left, "v"] \arrow[r, phantom, "\perp"] & \mathcal{C} \arrow[l, shift left, "v_R"] \end{tikzcd} \tag{4.3.5}$$

of adjunctions $u \underset{\eta^u}{\overset{\epsilon^u}{\dashv}} u^R$ and $v \underset{\eta^v}{\overset{\epsilon^v}{\dashv}} v^R$, there is precisely one natural transformation $uu^R \Rightarrow vv^R$, necessarily equal to the composition

$$uu^R \overset{\varepsilon^u}{\Longrightarrow} 1_{\mathcal{B}} \overset{\eta^v}{\Longrightarrow} vv^R \tag{4.3.6}$$

in the category $\mathcal{B}$.

From the previous discussion, we deduce the following:

Remark 4.3.6 (The canonical null sequence of a fibration of algebras)**.** There is a diagram of adjunctions

$$\begin{tikzcd} \mathcal{X} \arrow[rr, shift left, "\Phi"] \arrow[rr, phantom, "\perp"] \arrow[dr, shift left] & & \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[ll, shift left, "V"] \arrow[r, shift left, "p_F"] \arrow[r, phantom, "\perp"] \arrow[dl, shift left, "i_R"] & \mathcal{A} \arrow[l, shift left, "s"] \\ & \{\varnothing\} \ltimes \mathcal{X} \arrow[ul, shift left, "U_\varnothing"] \arrow[ur, shift left, "i_\varnothing"] & & \end{tikzcd} \tag{4.3.7}$$

where the upper line of Lemma 4.3.4 forms a null sequence and the left triangle of adjoints commutes, in the sense that the adjunction $(\Phi \dashv V)$ is the composite of the two adjunctions $\mathcal{X} \rightleftarrows \{\varnothing\} \ltimes \mathcal{X} \rightleftarrows \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$.

Remark 4.3.7. The diagram of functors

$$\mathcal{X} \xleftarrow{V} \mathcal{A} \ltimes \mathcal{X} \xrightarrow{p_F} \mathcal{A} \xrightarrow{!} 1 \tag{4.3.8}$$

defines a polynomial with $J = 1$ in the notation of [GK13], so a 'monomial functor on $\mathcal{X}$ many variables'. Recall that we introduced the topic in the context of fibrations of algebras in Example 3.4.2, but only in the case of endofunctors – meaning when, in the same notation, $J = I$.

Note that since p_F is a Grothendieck fibration, it is always a Conduché functor [Con72], so that the adjunction $p_F^* \dashv \Pi_{p_F}$ always exists and we can compose

$$\mathsf{Cat}/\mathcal{X} \xrightarrow{V^*} \mathsf{Cat}/\mathcal{A} \ltimes \mathcal{X} \xrightarrow{\Pi_{p_F}} \mathsf{Cat}/\mathcal{A} \xrightarrow{!_*} \mathsf{Cat}/1 = \mathsf{Cat}\,. \tag{4.3.9}$$

Definition 4.3.8. The polynomial canonically associated to the parametric endofunctor $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ is the composition $P[F] : \mathsf{Cat}/\mathcal{X} \longrightarrow \mathsf{Cat}$ obtained as in (4.3.9).

This leaves open how to compute $P[F]$ in motivating examples, as it is generally quite intricate to compute a dependent product functor Π_f.

Remark 4.3.9 (Adjoint split extensions)**.** The presence of a sequence of functors like (4.3.4) and its properties call for a general theory, which will be outlined in a forthcoming paper that we plan to organise as follows: consider the 2-category $\mathsf{LAdj}_{0,1}$ whose objects are categories having an initial and terminal object. $\mathsf{LAdj}_{0,1}$ is 2-pointed, hence we can consider sequences

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "i"] \arrow[r, phantom, "\perp"] & \mathcal{E} \arrow[l, shift left, "i_R"] \arrow[r, shift left, "p"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left, "p_R"] \end{tikzcd} \tag{4.3.10}$$

which are pairs of adjunctions composing to the zero adjunction $0 \dashv 1$ between constant functors. If the functor p additionally has a left adjoint p_L, then we talk of an *ace* (short for *Adjoint Split EXtension*), and if this latter adjunction is Frobenius (i.e. $p_L \cong p_R$) we talk of a *pointed* ace.

For particularly nice choices of sub-2-categories of $\mathsf{LAdj}_{0,1}$ such as semi-abelian categories or the opposites of toposes, as considered in [FM20], the central term $\mathcal{E}$ of an ace can be seen as the total category of a fibration of algebras; a nice example of this is the torsion theory, considered in [GKV16], arising from CGK theorem Example 4.2.3.

4.4 On adjoints to inclusions of the fibers

Lifting an adjunction along a fibration via pullback is a well-known problem that can be dated back to C. Hermida's PhD thesis [Her93], see for example his 3.2.3 and 3.2.5; the content of the present section hence is not new, but might be considered a specialisation to the case when there is an adjunction between the category $\mathcal{A}$ of parameters of a fibration of algebras, and another category $\mathcal{K}$; particularly interesting choice of $\mathcal{K}$ are the fibre over the initial or terminal object of $\mathcal{A}$ (so we determine an adjunction between the fibres $\{\varnothing\} \ltimes \mathcal{X}$ or $\{1\} \ltimes \mathcal{X}$ and $\mathcal{A} \ltimes \mathcal{X}$), which reduce to an adjunction $\mathcal{X} \leftrightarrows \mathcal{A} \ltimes \mathcal{X}$ under the relatively mild assumption that $T_\varnothing$ or $T_{\mathbf{1}}$ is the identity functor.

The terminology and notation of null sequences exposed in §4.3 here is particularly handy.

Proposition 4.4.1. Let $p : \mathcal{E} \longrightarrow \mathcal{A}$ be a fibration, and let $i : \mathcal{K} \hookrightarrow \mathcal{A}$ be (the inclusion functor of) a coreflective subcategory with coreflection $r : \mathcal{A} \longrightarrow \mathcal{K}$. Consider the pullback of p along i,

$$\begin{tikzcd} \mathcal{E}_{\mathcal{K}} \arrow[r, shift left, "j"] \arrow[r, phantom, "\perp"] \arrow[d, "p'"] & \mathcal{E} \arrow[l, shift left] \arrow[d, "p"] \\ \mathcal{K} \arrow[r, shift left, "i"] \arrow[r, phantom, "\perp"] & \mathcal{A} \arrow[l, shift left] \end{tikzcd} \tag{4.4.1}$$

i.e. the inclusion of the full subcategory $\mathcal{E}_{\mathcal{K}}$ over the objects such that pE lies in $\mathcal{K}$. Then, j has a right adjoint r'.

Note that a more concise way to rephrase Proposition 4.4.1 is that p' is a coreflective subfibration of p as an object of the 2-category Fib of fibrations (on variable bases).

Corollary 4.4.2. Consider a parametric endofunctor $T : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$; then, if $\mathcal{A}$ has an initial object $\varnothing$, the inclusion of the fiber over $\varnothing$, i.e. the functor $i : \{\varnothing\} \ltimes \mathcal{X} := \mathsf{Alg}_{\mathcal{X}}(T_\varnothing)$ has a right adjoint $i_R : \mathcal{A} \ltimes \mathcal{X} \longrightarrow \{\varnothing\} \ltimes \mathcal{X}$.

Proof. The inclusion $\{\varnothing\} \rightleftarrows \mathcal{A}$ is a coreflection. $\square$

Dually, if $i : \mathcal{H} \hookrightarrow \mathcal{A}$ is a reflective subcategory and $p : \mathcal{E} \longrightarrow \mathcal{A}$ is an *op*fibration, then there exists a left adjoint to $j : \mathcal{E}_{\mathcal{H}} \hookrightarrow \mathcal{E}$, obtained in a pullback similar to (4.4.1):

$$\begin{tikzcd} \mathcal{E}_{\mathcal{H}} \arrow[r, shift left, "j"] \arrow[r, phantom, "\top"] \arrow[d] & \mathcal{E} \arrow[l, shift left] \arrow[d, "p"] \\ \mathcal{H} \arrow[r, shift left, "i"] \arrow[r, phantom, "\top"] & \mathcal{A} \arrow[l, shift left] \end{tikzcd} \tag{4.4.2}$$

Corollary 4.4.3. Consider a parametric endofunctor $T : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$; then, if $\mathcal{A}$ has a terminal object $\mathbf{1}$, the inclusion of the fiber over $\mathbf{1}$, i.e. the functor $i : \{\mathbf{1}\} \ltimes \mathcal{X} := \mathsf{coAlg}_{\mathcal{X}}(T_{\mathbf{1}})$ has a left adjoint $i_L : \mathcal{A} \ltimes \mathcal{X} \longrightarrow \{\varnothing\} \ltimes \mathcal{X}$.

Proof. The inclusion $\{\mathbf{1}\} \rightleftarrows \mathcal{A}$ is a reflection. $\square$

Notice that by Lemma 1.2.20 the fibration p is a bifibration if and only if each reindexing functor u^* has a left adjoint $u_!$ (cf. Theorem 2.2.9 for sufficient conditions under which this is true for fibrations of endofunctor / Eilenberg–Moore algebras). Thus, we can consider the diagram

$$\begin{tikzcd} \{\mathbf{1}\} \ltimes \mathcal{X} \arrow[r, shift left] \arrow[r, phantom, "\top"] \arrow[d, shift left] \arrow[d, phantom, "\dashv"] & \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[l, shift left] \\ \{\varnothing\} \ltimes \mathcal{X} \arrow[r, shift left] \arrow[r, phantom, "\perp"] \arrow[u, shift left] & \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[l, shift left] \end{tikzcd} \tag{4.4.3}$$

where $u : \varnothing \to \mathbf{1}$ is the unique morphism, and try to outline conditions under which said diagram can be ‘completed’ with an adjunction of $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ compatible with the given one.

Observe also that the composite adjunction

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \{\varnothing\} \ltimes \mathcal{X} \arrow[l, shift left] \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathcal{A} \ltimes \mathcal{X} \arrow[l, shift left] \end{tikzcd} \tag{4.4.4}$$

coincides with the adjunction $(\Phi \dashv V)$ of Lemma 4.3.4; then, by uniqueness of adjoints, if $p : \mathcal{A} \ltimes \mathcal{X} \longrightarrow \mathcal{A}$ has a left adjoint l, $\langle p, V \rangle$ also has a left adjoint obtained as the pointwise coproduct $l + \Phi : (A, X) \mapsto lA + \Phi X$. Last, observe that such an adjoint is characterised similarly to Remark 3.3.15.

To conclude this analysis, we shall study the case when $\mathcal{A}$ has both an initial and a terminal object, which might not coincide. Then, if the initial object $\varnothing$ is strict, there is a fully faithful functor $\Delta[1] = \{0 \leqslant 1\} \longrightarrow \mathcal{A}$ choosing the unique arrow $u : \varnothing \to \mathbf{1}$, and this admits a left adjoint. Leveraging on Corollary 4.4.3, from any opfibration of coalgebras $\begin{bmatrix} \mathcal{A} \oplus \mathcal{X} \\ q \downarrow \\ \mathcal{X} \end{bmatrix}$ we obtain a reflection

$$\mathcal{H}' \underset{\perp}{\rightleftarrows} \mathcal{A} \oplus \mathcal{X} \tag{4.4.5}$$

where $\mathcal{H}'$ is the subcategory over the fibers on $\varnothing$ and $\mathbf{1}$, connected by the reindexing $u^* : \mathsf{EM}(T_{\mathbf{1}}) \longrightarrow \mathsf{EM}(T_{\varnothing})$.

Remark 4.4.4. The situation in which the existence of such adjoints to inclusions of fibres becomes interesting is the following general problem (we carry on the discussion in the case of parametric monads, with little adjustments –like assuming the existence of free endofunctor algebras for $F : \mathcal{A} \to [\mathcal{X}, \mathcal{X}]$– a similar analysis can be made for $\mathcal{A} \ltimes \mathcal{X}$, and of course dualised to $\mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X}$ and $\mathcal{A} \oplus \mathcal{X}$). Consider an arrangement of categories and functors

$$\begin{tikzcd} \{\varnothing\} \ltimes \mathcal{X} \arrow[r, shift left, "i=i_\varnothing"] \arrow[r, phantom, "\perp"] & \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[l, shift left, "i_R"] \arrow[r, "p_R"'] & \mathcal{A} \arrow[l, shift right, "p_L"'] \arrow[l, shift left=2] \end{tikzcd} \tag{4.4.6}$$

where p_L is the functor sending $A \in \mathcal{A}$ to the free T_A-algebra on the initial object of $\mathcal{X}$, and p_R as usual chooses the terminal object in the fiber over $A \in \mathcal{A}$; then we can form the commutative square of natural transformations

$$\begin{tikzcd} i i_R p_L p \arrow[r, "i i_R * \epsilon"] \arrow[d, "\epsilon' * p_L p"'] & i i_R \arrow[d, "\epsilon'"] \\ p_L p \arrow[r, "\epsilon"'] & 1 \end{tikzcd} \tag{4.4.7}$$

and address the following issues: the square is generally quite far from being a pushout, but the real pushout k of $i i_R * \epsilon$ and $\epsilon' * p_L p$ acquires thanks to (4.4.7) a canonical choice of copoint $\kappa : k \Rightarrow 1$.

This naturally begs the question: when is k a comonad? Given that both $H = i i_R$ and $K = p_L p$ are comonads, a natural additional assumption on the above diagram is that the composition HK is itself a comonad (so that $i i_R * \epsilon$ and $\epsilon' * p_L p$ become comonad homomorphisms). As it is well-known, this is equivalent to the presence of a comonad-comonad distributive law $HK \Rightarrow KH$. Then, the pushout of $K \Leftarrow HK \Rightarrow H$ in the category of comonads over $\mathcal{E} = \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ is created by the forgetful functor into endofunctors $\mathsf{Mnd}(\mathcal{E}) \longrightarrow [\mathcal{E}, \mathcal{E}]$.

In our specific situation, however, no nontrivial distributive law can exist: the composition $KH = p_L p i i_R$ is the constant functor at the initial object (because such is pi), and then a distributive law would have components $HK(A; X, x) \to \varnothing$.

Remark 4.4.5. One can ask a similar question when, instead of the fibre over $\varnothing$, we consider the whole $\mathcal{X}$ and the adjunction $(\Phi \dashv V)$ of Lemma 4.3.4: then we have a diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "i=\Phi"] \arrow[r, phantom, "\perp"] & \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[l, shift left, "i_R=V"] \arrow[r, "p_R"'] & \mathcal{A} \arrow[l, shift right, "p_L"'] \arrow[l, shift left=2] \end{tikzcd} \tag{4.4.8}$$

a commutative square like (4.4.7) also exists in this case, and one can wonder under which conditions it is a pushout or induces a coreflection of sorts. Note that in this case $i = \Phi$ is not full; note also that the two problems just outlined coincide when $T_\varnothing$ is the identity functor (it happens, for example, for the 'exception' fibration of endofunctor algebras $T_A X = A + X$).

Given the above considerations, in general one can only compute the pushout (4.4.7) explicitly; unwinding the definitions of the various functors one is left with the pushout of the following span: fix an object $(X, x)^A \in \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$, and consider

$$\begin{tikzcd}
(\varnothing; \begin{bmatrix} T_\varnothing T_A \varnothing \\ \downarrow \\ T_A\varnothing \end{bmatrix}) \arrow[r, "(!_A, \mathrm{id} T_A\varnothing)"] \arrow[d, "(\mathrm{id}\varnothing, \bar{f})"'] & (A; \begin{bmatrix} T_A T_A \varnothing \\ \mu \downarrow \\ T_A\varnothing \end{bmatrix}) \\
(\varnothing; \begin{bmatrix} T_\varnothing X \\ x \circ T_! X \downarrow \\ X \end{bmatrix})
\end{tikzcd} \tag{4.4.9}$$

where $\bar{f} : T_A\varnothing \to X$ is induced by $\varnothing \to X$ given the A-algebra structure on X. Note that the vertical leg of the span in question is just the image of $(T_A\varnothing, \mu_A^\varnothing)^A \to (X, x)^A$ under the reindexing along the initial map $\varnothing \to A$ (note also that the $\varnothing$ as subscript of T is the initial object of $\mathcal{A}$, while the $\varnothing$ in its argument is the initial object of $\mathcal{X}$); then, given that $p : \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \longrightarrow \mathcal{A}$ must preserve this pushout, the above diagram can be completed as

$$\begin{tikzcd}
(\varnothing; \begin{bmatrix} T_\varnothing T_A \varnothing \\ \downarrow \\ T_A\varnothing \end{bmatrix}) \arrow[r, "(!_A, \mathrm{id} T_A\varnothing)"] \arrow[d, "(\mathrm{id}\varnothing, \bar{f})"'] & (A; \begin{bmatrix} T_A T_A \varnothing \\ \mu \downarrow \\ T_A\varnothing \end{bmatrix}) \arrow[d] \\
(\varnothing; \begin{bmatrix} T_\varnothing X \\ x \circ T_! X \downarrow \\ X \end{bmatrix}) \arrow[r] \arrow[ur, phantom, "\ulcorner"] & (A; Q)
\end{tikzcd} \tag{4.4.10}$$

for an object Q that arises as a colimit in $\{A\} \ltimes \mathcal{X}$. Very little can be said more in general, but in a specific example, a specific parametric monad can make this computation simpler.

Example 4.4.6. In the fibration of monoids of Example 3.3.1, the pushout in (4.4.10) defines the functor $k : \mathsf{Mon} \ltimes^{\mathsf{EM}} \mathsf{Set} \to \mathsf{Mon} \ltimes^{\mathsf{EM}} \mathsf{Set}$ sending an M-set $(X, a : M \times X \to X)$ to the set X itself, equipped with the trivial action. Note that this functor is copointed with a natural transformation $\epsilon : k \Rightarrow \mathrm{id}$, having components the zero maps $(0_M, \mathrm{id}X) : (X, \pi_X)^M \to (X, a)^M$;

Note also that the functor k is idempotent (up to isomorphism: $k \circ k \cong k$) and well-copointed, but not an idempotent comonad (because $\epsilon * k = k * \epsilon$ is not an isomorphism).

Remark 4.4.7. We now sketch how to dualise the above analysis:

D1) provided that parametric free algebras exist, we can carry on a similar analysis for endofunctor algebras in a diagram analogue to (4.4.6); in addition,

D2) we can apply the above line of reasoning to the reflection $\{\mathbf{1}\} \ltimes \mathcal{X} \rightleftarrows \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ instead; moreover,

D3) we can study *coreflections* of fibers of a fibration of *co*algebras, and coreflections $\mathcal{H}' \leftrightarrows \mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X}$ and $\mathcal{A} \oplus \mathcal{X}$ as in (4.4.5).

More explicitly, one can study diagrams like

$$\begin{tikzcd} \{A\} \oplus^{\mathsf{EM}} \mathcal{X} \arrow[r, shift left, "i_A"] \arrow[r, phantom, "\top"] & \mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X} \arrow[l, shift left, "i_L"] \arrow[r, "p"] & \mathcal{A} \arrow[l, shift right, "p_L"'] \arrow[l, shift left, "p_R"] \end{tikzcd} \tag{4.4.11}$$

for the inclusion of a fiber $\{A\} \oplus \mathcal{X} \hookrightarrow \mathcal{A} \oplus \mathcal{X}$ (or $\mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X}$), or for the forgetful-cofree adjunction $V' : \mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X} \leftrightarrows \mathcal{X} : \Phi'$. In the latter case, the diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, shift left, "\Phi'"] \arrow[r, phantom, "\top"] & \mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X} \arrow[l, shift left, "V'"] \arrow[r, "p"] & \mathcal{A} \arrow[l, shift right, "p_L"'] \arrow[l, shift left, "p_R"] \end{tikzcd} \tag{4.4.12}$$

is composed by functors Φ', sending X to the cofree coalgebra $S_1 X$ on the terminal parameter with carrier X, V', sending a S_A-coalgebra to its carrier, p_L sending A to the initial object in the fiber $\{A\} \oplus^{\mathsf{EM}} \mathcal{X}$, and p_R, sending A to the cofree S_A-coalgebra on the carrier $\mathbf{1}$ (the terminal object of $\mathcal{X}$).

For the coEilenberg–Moore opfibration above, this construction yields a pullback of natural transformations

$$\begin{tikzcd} h \arrow[r, Rightarrow] \arrow[d, Rightarrow] \arrow[dr, phantom, "\lrcorner"] & p_R p \arrow[d, Rightarrow] \\ \Phi' V' \arrow[r, Rightarrow] & \Phi' V' p_R p \end{tikzcd} \tag{4.4.13}$$

rendering h a pointed endofunctor (the reasoning is entirely similar, but dual, to the one for (4.4.7)); in the specific case of the coreader comonad $A \times \_$ in a Cartesian category (cf. Example 3.2.1), the pullback in question is easily computed, and $h : (A, \left[\begin{smallmatrix} X \\ x\downarrow \\ A \end{smallmatrix}\right]) \mapsto (1, \left[\begin{smallmatrix} A\times X \\ \downarrow \\ 1 \end{smallmatrix}\right])$ is an idempotent monad on $\mathcal{A} \oplus^{\mathsf{EM}} \mathcal{X}$.

4.5 A parametric Linton theorem

The scope of the present section is to transport the following classical result (cf. [Lin69b, Lin74]) to the realm of parametric monads: relying on the presheaf construction in the 2-category Fib [Str81], we can relate together the opfibration of Kleisli of §2.1.3 and the fibration of Eilenberg–Moore of Definition 2.1.11.

Theorem 4.5.1 (Linton's characterisation of algebras through free algebras)**.** Let $T : \mathcal{C} \longrightarrow \mathcal{C}$ be a monad; then the Eilenberg–Moore category $\mathsf{EM}(T)$ fits in a pullback

$$\begin{tikzcd} \mathsf{EM}(T) \arrow[r, "\hat{K}"] \arrow[d, "U"'] \arrow[dr, phantom, "\lrcorner"] & {[\mathsf{Kl}(T)^{\mathrm{op}}, \mathsf{Set}]} \arrow[d, "\_ \circ F_T"] \\ \mathcal{C} \arrow[r, "y_{\mathcal{C}}"'] & {[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]} \end{tikzcd} \tag{4.5.1}$$

where $y_{\mathcal{C}}$ is the Yoneda embedding, $F_T^* = \_ \circ F_T$ precomposes with the (opposite of) the free functor $F_T : \mathcal{C} \longrightarrow \mathsf{Kl}(T)$ and $\hat{K}$ is isomorphic to the 'nerve' of the comparison functor $K : \mathsf{Kl}(T) \longrightarrow \mathsf{EM}(T)$, $\hat{K} : (A, \alpha) \mapsto \mathsf{EM}(T)(K\_, (A, \alpha))$.

Here we want a similar result for a parametric monad $T : \mathcal{A} \longrightarrow \mathsf{Mnd}(\mathcal{X})$; we will obtain it as a consequence of the following general fact, which can be seen as modelled on [Str81, 6.1].

Construction 4.5.2. Let $\mathcal{A}$ be a category and $\left[\begin{smallmatrix} \mathcal{F} \\ b\downarrow \\ \mathcal{A} \end{smallmatrix}\right]$ be a bifibration; then there is a functor

$$\varpi : b/\mathsf{OpFib}(\mathcal{A}) \longrightarrow \mathsf{Fib}(\mathcal{A})/b \tag{4.5.2}$$

defined on objects $h : b \to q$ as follows, if $\left[\begin{smallmatrix} \mathcal{E} \\ q\downarrow \\ \mathcal{A} \end{smallmatrix}\right]$ is an opfibration over $\mathcal{A}$: consider the pseudofunctor $F_q : \mathcal{A} \longrightarrow \mathsf{Cat}$ associated to q under the Grothendieck construction, and the composition

$$\begin{array}{rcl} \mathcal{A}^{\mathrm{op}} = \mathcal{A}^{\mathrm{coop}} & \longrightarrow \mathsf{Cat}^{\mathrm{coop}} \longrightarrow & \mathsf{Cat} \\ A \longmapsto & F_q(A) = q^{-1}A \longmapsto & [F_q(A)^{\mathrm{op}}, \mathsf{Set}] \end{array} \tag{4.5.3}$$

and its associated fibration $\left[\begin{smallmatrix} \bar{\mathcal{E}} \\ \boldsymbol{P}q\downarrow \\ \mathcal{A} \end{smallmatrix}\right]$; consider now the pullback of fibrations

$$\begin{tikzcd} \varpi q \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner"] & b \arrow[d, "y_b"] \\ \boldsymbol{P}q \arrow[r, "\boldsymbol{P}h"'] & \boldsymbol{P}b \end{tikzcd} \tag{4.5.4}$$

or more explicitly, the pullback of categories $\mathcal{Z} = \bar{\mathcal{E}} \times_{\bar{\mathcal{F}}} \mathcal{F}$:

$$\begin{tikzcd} & \mathcal{Z} \arrow[rr, "v"] \arrow[dl, "u"'] \arrow[ddr, "\varpi q"] & & \mathcal{F} \arrow[dl, "y_b"'] \arrow[ddl, bend left, "b"] \\ \bar{\mathcal{E}} \arrow[rr, "\boldsymbol{P}h"] \arrow[drr, "\boldsymbol{P}q"'] & & \bar{\mathcal{F}} \arrow[d, "\boldsymbol{P}b"] & \\ & & \mathcal{A} & \end{tikzcd} \tag{4.5.5}$$

Then, we obtain a fibration $\left[\begin{smallmatrix} \mathcal{Z} \\ \varpi q\downarrow \\ \mathcal{A} \end{smallmatrix}\right]$ over $\mathcal{A}$ with a morphism $v : \varpi q \to b$.

The trivial fibration is certainly a bifibration, and thus from the Kleisli opfibration of $T : \mathcal{A} \longrightarrow \mathsf{Mnd}(\mathcal{X})$ as in (2.1.16) we obtain a morphism of opfibrations $f : \left[\begin{smallmatrix} \mathcal{A}\times\mathcal{X} \\ \pi\ \downarrow \\ \mathcal{A} \end{smallmatrix}\right] \longrightarrow \left[\begin{smallmatrix} \mathcal{A}\ltimes^{\mathsf{Kl}}\mathcal{X} \\ q^T\ \downarrow \\ \mathcal{A} \end{smallmatrix}\right]$ collating together all the free functors; from this we can form the pullback

$$\begin{tikzcd} \mathcal{Z}_A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \boldsymbol{P}(\mathsf{Kl}(T_A)) \arrow[d] \\ \mathcal{X} \arrow[r] & \boldsymbol{P}\mathcal{X} \end{tikzcd} \tag{4.5.6}$$

where $\boldsymbol{P}\mathcal{C} = [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$, which is nothing but the pullback exhibiting, fiberwise, the pullback in $\mathsf{Fib}(\mathcal{A})$

$$\begin{tikzcd}
 & \mathcal{Z} \arrow[rr] \arrow[dl] \arrow[ddr] & & \mathcal{A}\times\mathcal{X} \arrow[dl] \arrow[ddl, bend left] \\
 \boldsymbol{P}(\mathcal{A}\ltimes^{\mathsf{Kl}}\mathcal{X}) \arrow[rr] \arrow[drr] & & \mathcal{A}\times\boldsymbol{P}\mathcal{X} \arrow[d] & \\
 & & \mathcal{A} &
\end{tikzcd} \tag{4.5.7}$$

Linton theorem now ensures that $\mathcal{Z}_A \cong \mathsf{EM}(T_A)$, but the construction that we provided is 'global', not just an objectwise version on the result in [Lin74]: we get an isomorphism *of fibred categories* $\mathcal{Z} \cong \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$. It is worth noting that this is a particular instance of a result true in a sufficiently good Yoneda structure on a 2-category with pullbacks, induced by a presheaf construction $\boldsymbol{P} : \mathcal{K}^{\mathrm{coop}} \longrightarrow \mathcal{K}$ admitting a left 2-adjoint (cf. [SW78, Prop. 22] or [Ark22, 5.5.2] for a sharper statement): in the Yoneda structure on fibrations, Linton's theorem becomes the following statement.

Proposition 4.5.3. Let C be an admissible object, with an admissible monad $t : C \to C$, such that the free algebra object C_t exists and it is admissible; then the pullback square

$$\begin{tikzcd}
Q \arrow[r, "v"] \arrow[d, "u"'] \arrow[dr, phantom, "\lrcorner", very near start] & \boldsymbol{P}C_t \arrow[d] \\
C \arrow[r] & \boldsymbol{P}C
\end{tikzcd} \tag{4.5.8}$$

exhibits Q as the algebra object C^t, so that $u : Q \to C$ becomes isomorphic in the slice $\mathcal{K}/C$ to the forgetful, and v to the nerve of the comparison $C^t(k, 1) : C^t \to P(C_t)$.

All in all, the result boils down to the following observation: for every admissible object B, the presheaf construction of the Yoneda structure $\boldsymbol{P} : \mathcal{K}^{\mathrm{coop}} \longrightarrow \mathcal{K}$ induces a 2-functor

$$B/\mathcal{K} = (\mathcal{K}^{\mathrm{op}}/B)^{\mathrm{op}} \xrightarrow{\text{apply } \boldsymbol{P}} \mathcal{K}^{\mathrm{op}}/\boldsymbol{P}B \xrightarrow{\text{pb along } y_B^*} \mathcal{K}/B. \tag{4.5.9}$$

In Construction 4.5.2 we are using the Yoneda structure on $\mathsf{Fib}(\mathcal{A}) \cong \mathsf{Psd}(\mathcal{A}^{\mathrm{op}}, \mathsf{Cat})$ described in [Str81].

4.6 Universal properties of fibrations of algebras

4.6.1 Fixpoints and orbits

When studying the representation theory of a monoid or a group, it is common to fix an action $a : G \times A \to A$ and define

- the space of orbits of A/G, as the pushout

$$\begin{tikzcd}
G \cdot A \arrow[r, "\nabla"] \arrow[d, "a'"'] & A \arrow[d] \\
A \arrow[r] \arrow[ur, phantom, "\ulcorner", very near end] & A/_a\, G
\end{tikzcd} \tag{4.6.1}$$

  where $a' : G \cdot A \to A$ is induced by the action, if $G \cdot H := \sum_{g \in G} H$ is the copower of $|G|$ copies of H, and ∇ is the fold map (if one wants, induced by the trivial action).

- The subspace of fixpoints of the action, as the equaliser[2]

$$\begin{tikzcd} A^G \arrow[r] & A \arrow[r, shift left, "a''"] \arrow[r, shift right, "d"'] & A^G \end{tikzcd} \tag{4.6.2}$$

  where a'' is induced transposing the action (and A^G is the $|G|$-fold product of copies of A), and d is the diagonal map.

We can easily reproduce both these constructions starting from a parametric endofunctor $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ and define

- the analogue for the space of orbits, as the iso-coinserter

$$\begin{tikzcd} \mathcal{A} \times \mathcal{X} \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Swarrow\alpha"] & \mathcal{X} \arrow[d, "q"] \\ \mathcal{X} \arrow[r, "q"'] & \mathcal{X}/_F \mathcal{A} \end{tikzcd} \tag{4.6.3}$$

- the analogue for the subspace of fixpoints, as the iso-inserter of the same pair:

$$\begin{tikzcd} \mathcal{X}^{(\mathcal{A})} \arrow[r, "p"] \arrow[d, "p"'] \arrow[dr, phantom, "\Swarrow\omega"] & \mathcal{A} \times \mathcal{X} \arrow[d, "F"] \\ \mathcal{A} \times \mathcal{X} \arrow[r, "\pi"'] & \mathcal{X} \end{tikzcd} \tag{4.6.4}$$

Now, what are the mutual relations between $\mathcal{X}/_F \mathcal{A}$, $\mathcal{X}^{(\mathcal{A})}$ and $\mathcal{A} \ltimes \mathcal{X}$?

- If $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ is a parametric monad such that $T_\varnothing$ is the identity functor, then multiplication yields an invertible 2-cell

$$\begin{tikzcd} \mathcal{A} \times \mathcal{X} \arrow[r, "F"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\Swarrow\mu"] & \mathcal{X} \arrow[d, "\Phi"] \\ \mathcal{X} \arrow[r, "\Phi"'] & \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \end{tikzcd} \tag{4.6.5}$$

  (recall from Lemma 4.3.4 that Φ selects the free algebra over the initial parameter) filling the square and thus giving a unique functor $R : \mathcal{X}/_F \mathcal{A} \longrightarrow \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ such that $R \circ q = \Phi$ and $R * \alpha = \mu$.
- Dually, one can define a functor

$$J : \mathcal{X}^{(\mathcal{A})} \longrightarrow \mathcal{A} \ltimes \mathcal{X} \tag{4.6.6}$$

  sending every object $(A; X, \xi : F_A X \cong X)$ to $(X, \xi)^A \in \mathcal{A} \ltimes \mathcal{X}$ regarded as an endofunctor algebra with invertible structure map. This functor is fully faithful, and in case F is accessible, J is an accessible functor of accessible (in fact, locally presentable) categories, and thus $\mathcal{X}^{(\mathcal{A})}$ is complete and cocomplete; with the adjoint functor theorem we can ensure an adjoint for J, but there seems to be no explicit, straightforward procedure to describe it.

[2] In the (fairly common) applications to algebraic topology, the space of fixpoints can be replaced with the space of *homotopy* fixpoints, by replacing the above equaliser with an appropriately defined homotopy limit. Inspired by a similar procedure, we also define homotopy fixpoints, as isomorphisms $FX \cong X$.

4.6.2 Fibrations of algebras as limits

The fact that the category of endofunctor algebras of $F : \mathcal{X} \longrightarrow \mathcal{X}$ can be seen as an inserter (cf. [AR94, p. 121]) is the special case of the following characterisation when $\mathcal{A} = 1$ is the terminal category.

Proposition 4.6.1. Let $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$ be a parametric endofunctor; then there is an inserter diagram

$$\begin{tikzcd} \mathcal{A} \ltimes \mathcal{X} \arrow[r, "U"] \arrow[d, "U"'] \arrow[dr, phantom, "\Swarrow_\theta"] & \mathcal{A} \times \mathcal{X} \arrow[d, "F"] \\ \mathcal{A} \times \mathcal{X} \arrow[r, "\pi_\mathcal{X}"'] & \mathcal{X} \end{tikzcd} \tag{4.6.7}$$

Now, it is still well-known but rarely spelt out in detail (see [AR94, §2.78]) that the Eilenberg–Moore category of a monad $T : \mathcal{X} \to \mathcal{X}$ admits a similar description as a joint equifier; more precisely, consider the inserter $\mathrm{Ins}(T, 1)$ (i.e. the category of endofunctor algebras of the underlying endofunctor of T) and a pair of parallel 2-cells

$$\begin{tikzcd} \mathrm{Ins}(T,1) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta\circ\eta U \quad \Downarrow\mathrm{id}U"] & \mathcal{X} \end{tikzcd} \tag{4.6.8}$$

of which we take the equifier $k_1 : \mathcal{E}_1 \longrightarrow \mathrm{Ins}(T, 1)$. Consider now the pair of 2-cells

$$\begin{tikzcd} \mathrm{Ins}(T,1) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta\circ T\theta \quad \Downarrow\theta\circ\mu U"] & \mathcal{X} \end{tikzcd} \tag{4.6.9}$$

The equifier of the whiskered 2-cells $(\theta \circ T\theta) * k_1, (\theta \circ \mu U) * k_1$ is the category of Eilenberg–Moore algebras for T. We can invoke an analogous result for the monad on $\mathcal{A} \times \mathcal{X}$ that corresponds to the parametric monad $T : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]_\mu$ under Theorem 4.2.5, but we can also cook up an equifier formula that employs only $T : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$.

Recall Theorem 4.2.5. From that statement, we can consider T as a monad in $\mathsf{coKl}(\mathcal{A}\times-)$, and take the inserter of T and $\pi_\mathcal{X}$, the 2-cell $\theta \circ \eta U$,

$$\begin{tikzcd} \mathcal{A} \ltimes \mathcal{X} \arrow[r, "U"] \arrow[d, "U"'] \arrow[dr, phantom, "\Swarrow_\theta"] & \mathcal{A} \times \mathcal{X} \arrow[d, "F"'] \arrow[d, bend left=60, "\pi_\mathcal{X}"] \arrow[d, phantom, shift left=3, "\overset{\Leftarrow}{\eta}"] \\ \mathcal{A} \times \mathcal{X} \arrow[r, "\pi_\mathcal{X}"'] & \mathcal{X} \end{tikzcd} \tag{4.6.10}$$

and the equifier $k_1 : \mathcal{E}_1 \longrightarrow \mathcal{A} \ltimes \mathcal{X}$ of $\theta \circ \eta U$ and of the identity 2-cell of $\pi_\mathcal{X} \circ U$.

Proposition 4.6.2. There is an equifier diagram

$$\begin{tikzcd} \mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \arrow[r] & \mathcal{E}_1 \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow \; \Downarrow"] & \mathcal{X} \end{tikzcd} \tag{4.6.11}$$

where the 2-cells equified are obtained from the compositions

$$(T \bullet T) \circ U \overset{\mu U}{\Rightarrow} T \circ U \overset{\theta}{\Rightarrow} \pi_\mathcal{X} U \quad \text{and} \quad (T \bullet T) \circ U \overset{T\theta}{\Rightarrow} (T \bullet \pi_\mathcal{X}) \circ U = T \circ U \overset{\theta}{\Rightarrow} \pi_\mathcal{X} U \tag{4.6.12}$$

whiskered with k_1.

The diagrams in Proposition 4.6.1 and Proposition 4.6.2 can be interpreted in any Cartesian 2-category $\mathcal{K}$ and provide definitions for the object of algebras of $f : A \times X \to X$ and of Eilenberg–Moore algebras of $t : A \times X \to X$:

Proposition 4.6.3. The object of algebras of a parametric endo-1-cell $f : A \times X \to X$ of $\mathcal{K}$ is the inserter of f, π_X:

$$\begin{tikzcd} \mathrm{Ins}(f,\pi_X) \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "\not\Downarrow_\theta"] & A \times X \arrow[d, "f"] \\ A \times X \arrow[r, "\pi_X"'] & X \end{tikzcd} \tag{4.6.13}$$

What is a parametric monad in $\mathcal{K}$, i.e. a 1-cell $t : A \times X \to X$ such that 'each $t_A : X \to X$ is a monad? Leveraging on Theorem 4.2.5 we can define the 'simple 2-slice' $\mathcal{K}/\!\!/A$ and consequently a parametric monad as a monad over one of its objects X:

Definition 4.6.4. Let $\mathcal{K}$ a Cartesian 2-category, and A one of its objects; the *simple 2-slice* is the 2-category where

- objects are the same of $\mathcal{K}$;
- the hom-category of 1- and 2-cells $X \to Y$ is just $\mathcal{K}(A \times X, Y)$.

With this at hand, a *parametric monad* on X, with object of parameters A, is just a monad in $\mathcal{K}/\!\!/A$, i.e. a 1-cell $t : A \times X \to X$ equipped with 2-cells

$$\begin{tikzcd} A \times X \arrow[r, bend left, "\pi_X"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\eta"] & X \end{tikzcd} \qquad \begin{tikzcd} A \times X \arrow[r, bend left, "t \bullet t"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\mu"] & X \end{tikzcd} \tag{4.6.14}$$

subject to appropriate unit and associativity conditions ($t \bullet t$ is coKleisli composition in $\mathcal{K}/\!\!/A$, which gives an explanation for why $T \bullet T$ appeared in (4.2.9)).

Proposition 4.6.5. The object of Eilenberg–Moore algebras of a parametric monad $t : A \times X \to X$ in $\mathcal{K}$ as above is the following iterated equifier construction:

- first, one takes the equifier of $\theta \circ (\eta * u)$ obtained as pasting

$$\begin{tikzcd} \mathrm{Ins}(t,\pi_X) \arrow[r, "u"] \arrow[d, "u"'] \arrow[dr, phantom, "\not\Downarrow_\theta"] & A \times X \arrow[d, "F"'] \arrow[d, bend left=60, "\pi_X"] \arrow[d, phantom, shift left=3, "\Leftarrow_\eta"] \\ A \times X \arrow[r, "\pi_X"'] & X \end{tikzcd} \tag{4.6.15}$$

  and of the identity 2-cell $\pi_X \circ u \Rightarrow \pi_X \circ u$; this results in a diagram

$$\begin{tikzcd} E_1 \arrow[r, "k_1"] & \mathrm{Ins}(t,\pi_X) \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\theta\circ\eta u \ \Downarrow 1"] & X \end{tikzcd} \tag{4.6.16}$$

- then, one takes the equifier of

$$(t \bullet t) \circ u \Rightarrow t \circ u \Rightarrow \pi_X u \quad \text{and} \quad (t \bullet t) \circ u \Rightarrow (t \bullet \pi_X) \circ u = t \circ u \Rightarrow \pi_X u \tag{4.6.17}$$

  similarly to (4.6.12).

4.7 Formal theory of fibrations of algebras

A trained eye will have noticed that the discussion carried up to this point is inspired by the desire to interpret the constructions attached to a fibration of algebras within an abstract 2-category endowed with sufficient additional structure (for example, finite limits and colimits). The purpose of this section is to make this precise in two different ways.

4.7.1 Fibration of algebras in a 2-category

This section is intended to formalise the results of subsection 4.6.2.

Our blanket assumption for the entire section is that $\mathcal{K}$ is a 2-category with all (strict) finite 2-limits. Then, we can leverage on the results of §4.6.2 and give the following definitions.

Definition 4.7.1. Let A be an object of $\mathcal{K}$. A *A-parametric endocell* (resp., an *A-parametric monad*, *A-parametric comonad*) is an endo-1-cell (resp., monad, comonad) $X \to X$ in the coKleisli category $\mathsf{coKl}(A \times -)$; equivalently, it's a 1-cell $f : A \times X \to X$ (resp., satisfying monad, comonad laws in $\mathsf{coKl}(A \times -)$).

Remark 4.7.2. We can build the analogue construction of Theorem 3.3.6 and define the *oplax regular representation* fibration as the 2-fibration arising as the left vertical side of the strict 2-pullback

$$\begin{tikzcd} \mathcal{K} \ltimes \mathcal{K} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Alg}^{\ell} \arrow[d] \\ \mathcal{K} \arrow[r] & {[\mathcal{K}, \mathcal{K}]} \end{tikzcd} \tag{4.7.1}$$

and a parametric endocell (or monad, comonad) can be defined as an endo-1-cell in $\mathcal{K} \ltimes \mathcal{K}$ in complete analogy with (3.3.6).

Definition 4.7.3. Let $f : A \times X \to X$ be a parametric endocell in $\mathcal{K}$, regarded as a coKleisli map; its *object of algebras* $\mathsf{Alg}(f)$ is the inserter $I(f, \pi_X)$ between $f : X \rightsquigarrow X$ and the identity $X \rightsquigarrow X$, i.e. the inserter in $\mathcal{K}$

$$\begin{tikzcd} I(f, \pi_X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow_{\theta}"] & A \times X \arrow[d, "f"] \\ A \times X \arrow[r, "\pi_X"'] & X \end{tikzcd} \tag{4.7.2}$$

The object of coalgebras $\mathsf{coAlg}(f)$ is, dually, the inserter $I(\pi_X, f)$.

Definition 4.7.4.

- The *object of EM-algebras* of a parametric monad is the iterated equifier obtained from $\mathsf{Alg}(f)$ in the same way of Proposition 4.6.5.
- The *object of coEM-coalgebras* is, dually, obtained from similar equifiers from $\mathsf{coAlg}(f)$.

Remarkably, the definition we gave is a fibration in $\mathcal{K}$, and in particular a Street fibration: to see this, note that it's enough to prove the result in Cat, i.e. to prove that every corepresentable functor $\mathcal{K}(E, -)$ sends the composition

$$\mathsf{Alg}(f) \longrightarrow A \times X \xrightarrow{\pi_A} A \tag{4.7.3}$$

to a fibration in Cat, and this latter statement follows from the fact that $\mathcal{A} \ltimes \mathcal{X}$'s natura of inserter has been demonstrated in Proposition 4.6.1, and $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ has been characterised as an iterated equifier in Proposition 4.6.2. This way of reasoning 'representably' is well-known and based on the fact that one can consider, for a monad (E, t) in $\mathcal{K}$, and a generic object E, the category $\mathcal{K}(E, X)$, and the induced monad

$$t_* : \mathcal{K}(E, X) \longrightarrow \mathcal{K}(E, X) \tag{4.7.4}$$

given by post-composition, whose algebras are to be intended as 'generalised t-algebras with domain X' (cf. [LS02]). This produces an algebra-like 2-functor

$$\mathsf{Alg}(t) : \mathcal{K}^{\mathrm{op}} \longrightarrow \mathsf{Cat}, \quad E \mapsto \mathsf{Alg}_{\mathcal{K}(E,X)}(t_*) \,. \tag{4.7.5}$$

When $\mathsf{Alg}(t)$ is representable, the representing object is the *Eilenberg–Moore object* of the monad t.

Theorem 4.7.5. [Str72] There exists a right 2-adjoint to $\mathcal{K} \longrightarrow \mathsf{Mnd}(\mathcal{K})$ if and only if $\mathcal{K}$ has Eilenberg–Moore objects for all its monads.

The correspondence above arises from a bifunctor

Lemma 4.7.6. For a 2-category $\mathcal{K}$ and C one of its objects, there is a 2-functor

$$\begin{aligned} \mathsf{Alg} : [X,X]_\mu^{\mathrm{coop}} \times \mathcal{K}^{\mathrm{op}} &\longrightarrow \mathsf{Cat} \\ (t,E) &\longmapsto \mathsf{Alg}_{\mathcal{K}(E,X)}(t_*) \end{aligned}$$

where $[C,C]_\mu$ is the 2-category obtained as in Remark 2.1.10, with trivial 2-cells.

4.7.2 A 2-category of extensions

This section is intended to formalise the ideas of Remark 4.3.9, of the results in Lemma 4.3.4, Remark 4.3.6, and provides a birdseye view of our forthcoming worl.

We fix a 2-category $\mathcal{K}$ with all finite weighted limits (in particular, it has finite products) and coproducts.

The terminology in the following definition is either standard or straightforward.

Definition 4.7.7. An object C of $\mathcal{K}$ *has a terminal* (resp., *initial*) *object* when the unique 1-cell $C \to 1$ has a right (resp., left) adjoint $1 \to C$; the object C is *pointed* when the left and right adjoints to $C \to 1$ (exist and) coincide. Clearly, if two objects $A, X \in \mathcal{K}$ are pointed, it is meaningful to speak of the 'zero adjunction' between them, i.e. of the composition of adjoints

$$\begin{tikzcd} X \arrow[r, shift left, "!"] \arrow[r, phantom, "\perp"] & 1 \arrow[l, shift left, "0"] \arrow[r, shift left, "0"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "!"] \end{tikzcd} \tag{4.7.6}$$

(obviously: in Cat, an object is pointed if and only if it is a category with an initial and a terminal object, that are isomorphic; and the zero adjunction is just the 'constant at zero object' pair of functors, $0 \dashv 0$).

Definition 4.7.8 (Slices and coslices of adjunctions)**.** Let Adj be the 2-category obtained from $\mathcal{K}$, having objects adjoint pairs $f : C \rightleftarrows D : u$ in $\mathcal{K}$ and morphisms the pairs h, k such that the following square commutes choosing both the left and the right adjoints:

$$\begin{tikzcd} C \arrow[r, shift left, "f"] \arrow[d, "h"'] & D \arrow[l, shift left, "u"] \arrow[d, "k"] \\ C' \arrow[r, shift left, "f'"] & D' \arrow[l, shift left, "u'"] \end{tikzcd} \qquad \begin{aligned} k \circ f &= f' \circ h \\ \\ u' \circ k &= h \circ u \end{aligned} \tag{4.7.7}$$

If A is an object of $\mathcal{K}$ we define the 2-category of adjunctions over A as the comma object

$$\begin{tikzcd} \mathsf{Adj}/A \arrow[r] \arrow[d] \arrow[dr, phantom, "\Nearrow"] & \{A\} \arrow[d] \\ \mathsf{Adj} \arrow[r, equal] & \mathsf{Adj} \end{tikzcd} \tag{4.7.8}$$

or equivalently, as the morphisms of Adj into the identity adjunction of A and dually, if X is an object of $\mathcal{K}$ we define the 2-category of adjunctions under X as the comma object

$$\begin{tikzcd} X/\mathsf{Adj} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Swarrow"] & \{X\} \arrow[d] \\ \mathsf{Adj} \arrow[r, equal] & \mathsf{Adj} \end{tikzcd} \tag{4.7.9}$$

or equivalently, as the morphisms of Adj out of the identity adjunction of X.

Unwinding the above definitions, the typical morphisms $h : (f, u) \to (f', u')$ of Adj/A and X/Adj are the following triangle on the left and on the right respectively:

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, shift left, "f"] & & F \arrow[dl, shift left, "f'"] \\ & A \arrow[ul, shift left, "u"] \arrow[ur, shift left, "u'"] & \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[dl, shift left] \arrow[dr, shift left] & \\ E \arrow[ur, shift left] \arrow[rr, "h"'] & & F \arrow[ul, shift left] \end{tikzcd} \tag{4.7.10}$$

where the triangle commutes both choosing left adjoints, and choosing right adjoints:

$$f' \circ h = f \text{ and } h \circ u = u' \qquad\qquad h \circ f = f' \text{ and } u' \circ h = u.$$

From now on, we assume that the objects A, X that we consider are pointed, in the sense of Definition 4.7.7 above.

Definition 4.7.9 (The category of sequences)**.** The 2-category $\mathsf{Seq}(A, X)$ is defined by the following strict pullback:

$$\begin{tikzcd} \mathsf{Seq}(A,X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathsf{Adj}/A \arrow[d, "d"] \\ X/\mathsf{Adj} \arrow[r, "c"'] & \mathcal{K} \end{tikzcd} \tag{4.7.11}$$

where the 2-functors $X/\mathsf{Adj} \xrightarrow{c} \mathcal{K} \xleftarrow{d} \mathsf{Adj}/A$ are defined sending $f : X \rightleftarrows E : u$ and $f : E \rightleftarrows A : u$ to E.

Unwinding the above definition, the typical object of $\mathsf{Seq}(A, X)$ is a sequence of adjoints

$$\mathbb{E} = \Big\langle \begin{tikzcd} X \arrow[r, shift left, "f_X"] \arrow[r, phantom, "\perp"] & E \arrow[l, shift left, "u_X"] \arrow[r, shift left, "f_A"] \arrow[r, phantom, "\perp"] & A \arrow[l, shift left, "u_A"] \end{tikzcd} \Big\rangle \tag{4.7.12}$$

and the typical morphism $\mathbb{E} \to \mathbb{F}$ is a 1-cell $h : E \to F$ in $\mathcal{K}$ between the central terms of $\mathbb{E}, \mathbb{F}$:

$$\begin{tikzcd} & E \arrow[dl, shift left] \arrow[dr, shift left] \arrow[dd, "h"] & \\ X \arrow[ur, shift left] \arrow[dr, shift left] & & A \arrow[ul, shift left] \arrow[dl, shift left] \\ & F \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd} \tag{4.7.13}$$

The definition explains unambiguously what it means that this diagram 'commutes'.

Remark 4.7.10. The intuition behind the terminology is clear: an object of $\mathsf{Seq}(A, X)$ can be thought of as a sequence of the form

$$1 \longrightarrow X \xrightarrow{f_X} E \xrightarrow{f_A} A \longrightarrow 1 \tag{4.7.14}$$

where each arrow is a left adjoint.

Among short sequences, the *exact* ones play a fundamental role: this motivates us to single out a subcategory of $\mathsf{Seq}(A, X)$ of sequences where the composition $f_A \circ f_X$ is zero (or equivalently, the composition $u_X \circ u_A$ is).

Definition 4.7.11. The category of exact sequences $\mathsf{Ext}(A, X)$ is defined as the full subcategory of $\mathsf{Seq}(A, X)$ spanned by all objects as in (4.7.12), where the square

$$\begin{tikzcd} X \arrow[r, "f_X"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d, "f_A"] \\ 1 \arrow[r, "0"'] & A \end{tikzcd} \tag{4.7.15}$$

is a pullback in $\mathcal{K}$; this characterises X as the *fiber* of $f_A : E \to A$, and by uniqueness of adjoints the composition $u_X \circ u_A$ is also 'exactly zero'.

The next result shall be compared with the fact that in homological algebra, the set of extensions $\mathrm{Ext}(M, N)$ between two modules is an Abelian group.

Theorem 4.7.12. $\mathsf{Ext}(A, X)$ is a symmetric monoidal category.

Chapter 5

Where do we go from here?

Summary of chapter

We end the paper describing what remains an open problem or an enticing direction in which to develop the theory further.

5.1 Fibrations of algebras as colimits

Given that $\mathcal{A} \ltimes \mathcal{X}$ and $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ have been characterised as having limit properties, it is an interesting question whether they also have *colimit* properties, modelled on the intuition that the semidirect product of groups and monoids can be presented in a colimit form.

Consider the following colimit characterisations of the semidirect product of groups and monoids (cf. [BJK05, §4.3]):

Fact 5.1.1 (Semidirect products of groups as a pushout)**.** Let G be a group and H a G-group, i.e. another group over which G acts by automorphisms; then there is a pushout diagram

$$\begin{tikzcd} G \cdot H \arrow[r, "\gamma"] \arrow[d, "a"'] & G * H \arrow[d] \\ H \arrow[r] \arrow[ur, phantom, "\ulcorner"] & G \ltimes H \end{tikzcd} \tag{5.1.1}$$

where $G \cdot H := \sum_{g \in G} H$ is the copower of $|G|$ copies of H, a arises from the action in the obvious way, $G * H$ is the coproduct of groups, and $\gamma(g,h) := h^g = g^{-1}hg$, and $G \ltimes H$ is the semidirect product of groups.

Fact 5.1.2 (Semidirect product of monoids as a coequaliser)**.** Let M, N be monoids, and let $a : M \times N \to N$ be an action of M on N under monoid homomorphisms. Then there is a coequaliser diagram

$$\begin{tikzcd} M \times N \arrow[r, shift left] \arrow[r, shift right] & M * N \arrow[r] & M \ltimes N \end{tikzcd} \tag{5.1.2}$$

where the upper map is defined as $(m,n) \mapsto [m, a(m,n)]$ (seen as equivalence class of a word in $M * N$) and the lower map as $(m,n) \mapsto [n,m]$. (If the monoids are groups, this boils down to identifying the action $a(m,n)$ with the conjugation action.)

This leads to the question of whether it is possible to equip $\mathcal{A} \ltimes \mathcal{X}$ and $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ with colimit-like properties. Although there seems to be evidence that this conjecture can be answered positively, the question remains open at the moment of writing. We limit ourselves to drafting some remarks, and we reserve to the near future the opportunity for further investigation.

For categories, the copower operation $\mathcal{A} \cdot \mathcal{X}$ amounts to the Cartesian product $\mathcal{A} \times \mathcal{X}$, and a legitimate candidate for a is, then, just our parametric endofunctor $F : \mathcal{A} \times \mathcal{X} \longrightarrow \mathcal{X}$. We are then left with the question of what interesting 2-dimensional colimits (coinserters, coequifiers, cocomma objects) arise combining F and the functors of Remark 4.3.6.

Recall the following concrete presentation for cocomma objects in Cat:

Lemma 5.1.3. Given a span of functors,

$$\mathcal{B} \xleftarrow{G} \mathcal{E} \xrightarrow{F} \mathcal{C} \tag{5.1.3}$$

the cocomma category $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right]$ can be obtained as follows: its objects are those of the disjoint union $\mathcal{B} + \mathcal{C}$, and morphisms are specified so that

- $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right](B, B') = \mathcal{B}(B, B')$;
- $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right](C, C') = \mathcal{C}(C, C')$;
- the hom-set $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right](C, B)$ of *heteromorphisms* $C \rightsquigarrow B$ is determined as the coend

$$\int^{E} \mathcal{C}(C, FE) \times \mathcal{B}(GE, B) \tag{5.1.4}$$

All hom-sets $\left[\begin{smallmatrix} F \\ G \end{smallmatrix}\right](B, C)$ are empty, and the composition of a $\mathcal{B}$-morphism and a $\mathcal{C}$-morphism with an heteromorphism is performed using elementary coend manipulations.

Remark 5.1.4. Note that this is just the cograph of the profunctor $F^* \diamond G_* : \mathcal{C} \rightsquigarrow \mathcal{B}$.

Now, let $\mathcal{X} \,\S_T\, (\mathcal{A} \times \mathcal{X})$ denote the cocomma category

$$\begin{tikzcd} \mathcal{A} \times \mathcal{X} \arrow[r, equal] \arrow[d, "T"'] \arrow[dr, phantom, "\Nearrow"] & \mathcal{A} \times \mathcal{X} \arrow[d] \\ \mathcal{X} \arrow[r] & \mathcal{X} \,\S_T\, (\mathcal{A} \times \mathcal{X}) \end{tikzcd} \tag{5.1.5}$$

the typical object of which is

$$\mathsf{unpack}\, C\, \mathsf{as}\, [X \in \mathcal{X} \mid (A', X') \in \mathcal{A} \times \mathcal{X}] \tag{5.1.6}$$

(or $C = [X \in \mathcal{X} \mid (A', X') \in \mathcal{A} \times \mathcal{X}]$ for short.)

The coend formula above shows that the set of heteromorphisms in $\mathcal{X} \,\S_T\, (\mathcal{A} \times \mathcal{X})$ can be computed as

$$\hom(X'', (A', X')) \cong \mathsf{Kl}(T_{A'})(X'', X') \tag{5.1.7}$$

Remark 5.1.5. There exists a natural transformation $\alpha : \Phi \circ T \Rightarrow F_T$ (where F_T is the free functor of the Kleisli adjunction $F_T : \mathcal{A} \times \mathcal{X} \rightleftarrows \mathcal{A} \ltimes^{\mathsf{Kl}} \mathcal{X} : U_T$ that can be deduced from Theorem 4.2.1), from which, using the universal property of $\mathcal{X} \,\S_T\, (\mathcal{A} \times \mathcal{X})$ we deduce the existence of a unique functor $\Gamma : \mathcal{X} \,\S_T\, (\mathcal{A} \times \mathcal{X}) \longrightarrow \mathcal{A} \ltimes^{\mathsf{Kl}} \mathcal{X}$, defined using F_T and Φ.

The following theorem is easily proved, showing the adjunction isomorphism

$$\hom_{\mathcal{X} \S_T (\mathcal{A} \times \mathcal{X})} \big(\Gamma C, (A; T_A X, \mu^A_X)\big) \cong \hom_{\mathcal{A} \ltimes^{\mathsf{Kl}} \mathcal{X}} \big(C, W(A; T_A X, \mu^A_X)\big) \tag{5.1.8}$$

directly.

Theorem 5.1.6. The functor obtained in Remark 5.1.5 has a right adjoint W, defined as

$$\begin{aligned} W : \mathcal{A} \ltimes^{\mathsf{Kl}} \mathcal{X} &\longrightarrow \mathcal{X} \S_T (\mathcal{A} \times \mathcal{X}) \\ (A; T_A X, \mu^A_X) &\longmapsto (A, T_A X) \end{aligned} \tag{5.1.9}$$

5.2 Applications to torsion theories

Let $\mathcal{C}$ be a category with a terminal object. From [RT07] we know that there is a tight connection between torsion theories on $\mathcal{C}$ and fibrations with $\mathcal{C}$ as total category: more precisely,

Theorem 5.2.1. A class $\mathcal{M}$ of arrows in $\mathcal{C}$ is the right class of a torsion theory $\mathfrak{T} = (\mathcal{E}, \mathcal{M})$ if and only if there is a prefibration[1] such that the following conditions are satisfied:

- P preserves the terminal object, i.e. $P1 \cong 1$;
- $\mathcal{M}$ coincides with the class of P-cartesian morphisms in $\mathcal{C}$;
- $\mathcal{M}$ has the 3-for-2 property.

The important part in the proof of this theorem is the fact that the reflector $R : \mathcal{C} \longrightarrow \mathcal{F}$ of a torsion theory onto the class of torsion-free objects is always a prefibration. The other half of the proof consists in proving that the vertical-Cartesian factorisation of a prefibration is a torsion theory.

Leveraging on this result, we can pull back a torsion theory on $\mathcal{A} \ltimes \mathcal{X}$ given a torsion theory on $\mathcal{A}$:

Theorem 5.2.2. Let $F : \mathcal{A} \longrightarrow [\mathcal{X}, \mathcal{X}]$ be a parametric endofunctor. Let $\mathfrak{T} = (\mathcal{E}, \mathcal{M})$ be a torsion theory on the category of parameters $\mathcal{A}$ of F, and let p_F be the fibration of algebras associated to F. Then, there is a torsion theory $p_F^*\mathfrak{T}$ on $\mathcal{A} \ltimes \mathcal{X}$ whose right class is p-induced, i.e. a morphism $(u, f) : (X, \xi)^A \to (Y, \theta)^B$ is in $p^*\mathcal{M}$ if and only if $u : A \to B$ is in $\mathcal{M}$.

Proof. First of all, note that there is a composite fibration

$$q : \mathcal{A} \ltimes \mathcal{X} \xrightarrow{p} \mathcal{A} \xrightarrow{R} \mathcal{F}_A \tag{5.2.1}$$

the claim that there is an induced torsion theory on $\mathcal{A} \ltimes \mathcal{X}$ now follows from the fact that q-vertical maps, i.e. the class $q^{-1}(\mathsf{Iso})$, coincide with Rp-vertical maps, i.e. with $p^{-1}R^{-1}(\mathsf{Iso}) = p^{-1}(\mathcal{M}_A)$. This determines uniquely the left class of q-cartesian maps, i.e. with $({}^{\perp}p^{-1}(\mathcal{M}_A))^{\perp}$.

The same line of reasoning builds a torsion theory on $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X}$ and all the other fibrations of algebras; properly dualising Theorem 5.2.1 and Theorem 5.2.2 we can instead lift left classes and induce torsion theories on opfibrations of coalgebras. □

This is however a rather coarse way to induce a torsion theory on $\mathcal{A} \ltimes \mathcal{X}$ that does not take into account any structure of $\mathcal{X}$; fortunately, we can do slightly better: if we have torsion theories $\mathfrak{A}$ on $\mathcal{A}$ *and* $\mathfrak{X}$ *on* $\mathcal{X}$, we can induce one $\mathfrak{A} \times \mathfrak{X}$ on the product category $\mathcal{A} \times \mathcal{X}$, and lift it from $\mathcal{A} \times \mathcal{X}$ to $\mathcal{A} \ltimes \mathcal{X}$ using the following lemma:

[1] A *prefibration* $P : \mathcal{C} \longrightarrow \mathcal{B}$ is a functor such that each slice functor $P_{/C} : \mathcal{C}/C \longrightarrow \mathcal{B}/PC$ has a rari. There is a dual notion of a pre-opfibration, recognising *left* classes of torsion theories.

Lemma 5.2.3. Let $\mathfrak{T} = (\mathcal{E}, \mathcal{M})$ be a factorisation system on a category $\mathcal{C}$; let T be a monad on $\mathcal{C}$ that preserves the left class, i.e. such that $T\mathcal{E} \subseteq \mathcal{E}$; then there is a factorisation system $U^*\mathfrak{T} = (\mathcal{E}', \mathcal{M}') = U^{-1}(\mathfrak{F})$ on the category of T-algebras defined by preimage, i.e. $\mathcal{E}' := U^{-1}(\mathcal{E}), \mathcal{M}' := U^{-1}(\mathcal{M})$.

Apply the above theorem to the monadic functor $\mathcal{A} \ltimes^{\mathsf{EM}} \mathcal{X} \longrightarrow \mathcal{A} \times \mathcal{X}$ to get a torsion theory $U^*(\mathfrak{A} \times \mathfrak{X})$; now, it is an interesting question to study how does $q^*\mathfrak{T}$ and $U^*(\mathfrak{A} \times \mathfrak{X})$ compare. (Also, the first construction is but the second, when the torsion theory on $\mathcal{X}$ is trivial.)

Acknowledgements

Like many others before, this project started small. The plan was merely to understand the true nature of an exercise, specifically [Jac98, Exercise 1.3.4.(ii)], and knowing whether by any chance a general theory of ‘parametric algebras’ was treated somewhere, explaining *why* the result was true (or better, what it was a particular instance of).

Firmly believing that something so obvious should have been observed somewhere by someone else, FL posted a completely innocent question on MathOverflow [Lor] in March 2022 and promptly forgot about it. It is no understatement to say that the present article merely consists of a solution to [Jac98, *ibi*] that a particularly stubborn category theorist might give. To this day, the question on MathOverflow remains unanswered and almost entirely ignored.

In May 2022, GC and DC were simultaneously in Tallinn for a short time, in the final stage of their PhD; after scanning the web once more, now aided by their expertise in categorical logic, FL scribbled a list of examples that are fibrations of algebras, some covariant, some contravariant. The list grew bigger and bigger and bigger. Then it doubled in size and again. We noticed that we could package the property of being a fibration of algebras in what appears now as Definition 2.1.5; this motivated us to write the backbone of what is now chapter 3.

Shortly after FL travelled to Brno, a city dear to his heart, to attend the 106$^{\text{th}}$ PSSL in honour of Jiří Rosický’s 75 years. What now appears as Theorem 4.2.1 was keenly suggested to FL in *Punkt.*, a kavárna in Bayerova 7, by Ivan Di Liberti, a friend and clever logician dear to his heart.

There *must* have been more to this story, between the lines of such a beautiful, concise characterisation. In Brno, FL asked Jiří Adámek whether he knew the true face of the theorem we were looking at. He did not. This was the final endorsement we needed to fully engage with the problem, trustful that it would have been a fruitful endeavour. For a very long time, however, we were stuck in the mud of our question: examples were abundant, even too many, and no convincing narrative could tie them together. Months passed, and frustration mounted at our inability to see past the illusory multitude of the hylic forms; FL presented some scattered results first to the research group in Tallinn and then to the 3rd ItaCa meeting in Pisa in December 2022. After the talk, Beppe Metere suggested that Beck modules could have been another source of fibrations of algebras. As Example 3.3.7 shows, he was right. The ItaCa talk [CCL22] is essentially a cry for help: what is this story *really* about?

Help came as a polite, unprompted email that ÜR sent to FL in March 2023 from the University of Tartu. ÜR asked whether we knew about a construction for which ‘there was not much to find about in the literature, although it seemed to appear quite often in practice’. It was a fibration of algebras, in the context of protomodular categories and representation

theory. *Első számú fasiszta*, it seems, is not without a sense of irony if He left to His children yet another slew of examples (and another category theorist) *in Estonia*, just for them to discover after a year of pilgrimages.

The story goes that Xiangyan Zhixian spent several years studying the sūtras, making very little headway. One day, his master Guishan asked him what his face was before he was born –a question to which Zhixian could not respond, to his shame. After more and more pointless study, enraged, he burned the sūtras and left the monastery, built a hut nearby and stayed there alone. One day, while he was weeding, a piece of rock which Zhixian had dislodged struck a bamboo tree. At that sound, *tock!*, in the clear silence of the morning, Zhixian burst out laughing; suddenly, his mind was open.

Similarly, most of this article emerged instantaneously during a single meeting on a single morning in the first week of April 2023 that FL spent in Tartu.

Without the Universities and cities of Brno, Tartu, and Tallinn, without the patience of FL's coauthors, without A. and M., without Ülo's email, and without ItaCa, probably none of these theorems would have been recorded. To each of the people mentioned above and many others, goes our warmest thank you.

Bibliography

[AM20] M. Aguiar and S. Mahajan, *Bimonoids for hyperplane arrangements*, Cambridge University Press, February 2020.

[AR94] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994.

[Ark22] N. Arkor, *Monadic and higher-order structure*, Ph.D. thesis, University of Cambridge, 2022, available at `https://www.repository.cam.ac.uk/handle/1810/338940`.

[Atk09a] R. Atkey, *Algebras for parameterised monads*, Algebra and Coalgebra in Computer Science, Springer Berlin Heidelberg, 2009, pp. 3–17.

[Atk09b] ———, *Parameterised notions of computation*, Journal of Functional Programming **19** (2009), no. 3-4, 335–376.

[AU19] D. Ahman and T. Uustalu, *Decomposing Comonad Morphisms*, CALCO 2019 (Dagstuhl, Germany), Leibniz International Proceedings in Informatics (LIPIcs), vol. 139, 2019, pp. 14:1–14:19.

[BB04] F. Borceux and D. Bourn, *Mal'cev, Protomodular, Homological and Semi-abelian Categories*, Springer Netherlands, 2004.

[BBD82] A. Beĭlinson, J. Bernstein, and P. Deligne, *Faisceaux pervers*, Analysis and topology on singular spaces, I (Luminy, 1981), Astérisque, vol. 100, Soc. Math. France, Paris, 1982, pp. 5–171.

[Bén67] J. Bénabou, *Introduction to bicategories*, Reports of the Midwest Category Seminar, Springer, Berlin, 1967, pp. 1–77.

[BJ98] D. Bourn and G. Janelidze, *Protomodularity, descent, and semidirect products.*, Theory and Applications of Categories **4** (1998), 37–46 (eng).

[BJK05] F. Borceux, G. Janelidze, and G.M. Kelly, *Internal object actions*, Comment. Math. Univ. Carolin. **46** (2005), no. 2, 235–255.

[BLLL23a] G. Boccali, A. Laretto, F. Loregian, and S. Luneia, *Bicategories of automata, automata in bicategories*, `arXiv:2303.03865` preprint, 2023.

[BLLL23b] ———, *Completeness for Categories of Generalized Automata*, 10th Conference on Algebra and Coalgebra in Computer Science (CALCO 2023) (Dagstuhl, Germany), vol. 270, 2023, pp. 20:1–20:14.

[Boi18] G. Boisseau, *Understanding profunctor optics: A representation theorem (extended abstract)*, Companion Proceedings of the 2nd International Conference on the Art, Science, and Engineering of Programming (New York, NY, USA), Programming '18, Association for Computing Machinery, 2018, p. 30–32.

[Bor94a] F. Borceux, *Handbook of categorical algebra. 1*, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994, Basic category theory.

[Bor94b] ———, *Handbook of categorical algebra 2*, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, Cambridge, 1994, Categories and structures.

[Bou11] J. Bourke, *Codescent objects in 2-dimensional universal algebra*, Ph.D. thesis, University of Sydney, 2011.

[Bou17] D. Bourn, *From groups to categorial algebra*, Compact Textbooks in Mathematics, Springer International Publishing, 2017.

[BR07] A. Beligiannis and I. Reiten, *Homological and homotopical aspects of torsion theories*, Mem. Amer. Math. Soc. **188** (2007), no. 883, viii+207.

[BW90] M. Barr and C. Wells, *Category theory for computing science*, vol. 1, Prentice Hall New York, 1990, available online at `https://www.math.mcgill.ca/triples/Barr-Wells-ctcs.pdf`.

[Cap23] M. Capucci, *Diegetic representation of feedback in open games*, Electronic Proceedings in Theoretical Computer Science **380** (2023), 145–158.

[CCL22] D. Castelnovo, G. Coraglia, and F. Loregian, *A few remarks on the fibration of algebras*, `https://www.youtube.com/watch?v=nlT2T_cvl-U`, December 2022.

[CEG+20] B. Clarke, D. Elkins, J. Gibbons, F. Loregian, B. Milewski, E. Pillmore, and M. Román, *Profunctor optics, a categorical update*, 2020.

[CG22] M. Capucci and B. Gavranović, *Actegories for the working amthematician*, 2022, `arXiv:2203.16351` preprint.

[Chu40] A. Church, *A formulation of the simple theory of types*, Journal of Symbolic Logic **5** (1940), no. 2, 56–68.

[Con72] François Conduché, *Au sujet de l'existence d'adjointsa droite aux foncteurs "image réciproque" dans la catégorie des catégories*, CR Acad. Sci. Paris **275** (1972), A891–894.

[DEP02] J. Desharnais, A. Edalat, and P. Panangaden, *Bisimulation for labelled Markov processes*, Information and Computation **179** (2002), no. 2, 163–193.

[Dub70] E.J. Dubuc, *Kan extensions in enriched category theory*, Lecture Notes in Mathematics, Vol. 145, Springer-Verlag, Berlin-New York, 1970. MR 0280560

[EGNO15] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik, *Tensor categories*, Mathematical Surveys and Monographs, vol. 205, American Mathematical Society, Providence, RI, 2015. MR 3242743

[EKKK74] H. Ehrig, K.-D. Kiermeier, H.-J. Kreowski, and W. Kühnel, *Universal theory of automata. A categorical approach*, Teubner Studienbücher. Informatik. Stuttgart: B. G. Teubner., 1974 (English).

[Far23] W.M. Farmer, *Simple type theory*, Springer International Publishing, 2023.

[FK72] P.J. Freyd and G.M. Kelly, *Categories of continuous functors. I*, J. Pure Appl. Algebra **2** (1972), 169–191.

[FKM16] S. Fujii, S. Katsumata, and P.-A. Melliès, *Towards a formal theory of graded monads*, FoSSaCS. Foundations of Software Science and Computation Structures (B. Jacobs and C. Löding, eds.), 2016, pp. 513–530.

[FM20] Peter F. Faul and Graham R. Manuell, *Artin glueings of frames as semidirect products*, Journal of Pure and Applied Algebra **224** (2020), no. 8, 106334.

[Fuj19] S. Fujii, *A 2-categorical study of graded and indexed monads*, 2019, `arXiv:1904.08083` preprint.

[GAV72] A. Grothendieck, M. Artin, and J. L. Verdier, *SGA4. théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos*, Lecture Notes in Mathematics, Vol. 269, Springer-Verlag, Berlin-New York, 1972.

[GK13] N. Gambino and J. Kock, *Polynomial functors and polynomial monads*, Mathematical Proceedings of the Cambridge Philosophical Society **154** (2013), no. 1, 153–192.

[GKV16] M. Gran, G. Kadjo, and J. Vercruysse, *A torsion theory in the category of cocommutative Hopf algebras*, Appl. Categ. Structures **24** (2016), no. 3, 269–282.

[Gra66] J.W. Gray, *Fibred and cofibred categories*, Proceedings of the Conference on Categorical Algebra (Berlin, Heidelberg), Springer Berlin Heidelberg, 1966, pp. 21–83.

[Gra74] ———, *Formal category theory: Adjointness for 2-categories*, Springer Berlin Heidelberg, 1974.

[Gro71] A. Grothendieck, *Revêtements étales et groupe fondamental*, vol. 224, LNM, no. 1, SV, 1971.

[Gui74] R. Guitart, *Remarques sur les machines et les structures*, Cahiers de Topologie et Géométrie Différentielle Catégoriques **15** (1974), 113–144.

[Gui80] R. Guitart, *Tenseurs et machines*, Cahiers de topologie et géométrie différentielle **21** (1980), no. 1, 5–62 (fr). MR 569117

[Han14] Y. Han, *Recollements and Hochschild theory*, Journal of Algebra **397** (2014), 535–547.

[Hed17] J. Hedges, *Coherence for lenses and open games*, 2017, `arXiv:1704.02230` preprint.

[Hed19] J. Hedges, *Limits of bimorphic lenses*, 2019, `arXiv:1808.05545` preprint.

[Her93] C.A. Hermida, *Fibrations, logical predicates and indeterminates*, Ph.D. thesis, The University of Edinburgh, 1993.

[Hin97] J.R. Hindley, *Basic simple type theory*, Cambridge University Press, July 1997.

[HS01] A. Hirschowitz and C. Simpson, *Descente pour les n-champs (descent for n-stacks)*, 2001, `arXiv:math/9807049` preprint.

[Jac93] B. Jacobs, *Comprehension categories and the semantics of type dependency*, Theoretical Computer Science **107** (1993), no. 2, 169–207.

[Jac98] ———, *Categorical logic and type theory: Volume 141*, Studies in Logic and the Foundations of Mathematics, Elsevier Science, London, England, December 1998.

[Jac10] ———, *Convexity, duality and effects*, Theoretical Computer Science (Cristian S. Calude and Vladimiro Sassone, eds.), vol. 323, Springer Berlin Heidelberg, 2010, pp. 1–19.

[Jac16] ———, *Introduction to coalgebra*, Cambridge University Press, October 2016.

[Jac18] ———, *From probability monads to commutative effectuses*, Journal of Logical and Algebraic Methods in Programming **94** (2018), 200–237.

[Jan65] J.P. Jans, *Some aspects of torsion*, Pacific J. Math. **15** (1965), no. 4, 1249–1259.

[Jan03] G. Janelidze, *Internal crossed modules*, Georgian Mathematical Journal **10** (2003), no. 1, 99–114.

[Joh02] P.T. Johnstone, *Sketches of an Elephant: A topos theory compendium*, vol. I, Oxford University Press, 2002.

[JT91] A. Joyal and M. Tierney, *Strong stacks and classifying spaces*, Category theory (Como, 1990), Lecture Notes in Math., vol. 1488, Springer, Berlin, 1991, pp. 213–236.

[JY21] N. Johnson and D. Yau, *2-dimensional categories*, Oxford University Press, January 2021.

[Kel74] G. M. Kelly, *On clubs and doctrines*, Category Seminar, Springer Berlin Heidelberg, 1974, pp. 181–256.

[Kel89] G.M. Kelly, *Elementary observations on 2-categorical limits*, Bulletin of the Australian Mathematical Society **39** (1989), no. 2, 301–317.

[KKM00] M. Kilp, U. Knauer, and A.V. Mikhalev, *Monoids, acts and categories*, De Gruyter, December 2000.

[Kos77] B. Kostant, *Graded manifolds, graded lie theory, and prequantization*, Lecture Notes in Mathematics, Springer Berlin Heidelberg, 1977, pp. 177–306.

[Lac10] S. Lack, *A 2-categories companion*, Towards higher categories, IMA Vol. Math. Appl., vol. 152, Springer, New York, 2010, pp. 105–191.

[Lam98] T.Y. Lam, *Lectures on modules and rings*, 1999 ed., Graduate Texts in Mathematics, Springer, New York, NY, October 1998 (en).

[Law63] F.W. Lawvere, *Functorial semantics of algebraic theories*, Proceedings of the National Academy of Sciences of the United States of America **50** (1963), no. 5, 869.

[Lin69a] F.E.J. Linton, *Coequalizers in categories of algebras*, Sem. on Triples and Categorical Homology Theory, Springer, Berlin, 1969, pp. 75–90. Lecture Notes in Math., Vol. 80.

[Lin69b] ———, *An outline of functorial semantics*, p. 7–52, Springer Berlin Heidelberg, 1969.

[Lin74] ———, *Relative functorial semantics, III: triples vs. theories*, mimeographed notes available at `https://ncatlab.org/nlab/files/Linton_RelativeFunctorialSemanticsIII.pdf` (December 7, 2023), 1974.

[Lor] F. Loregian, *Fibrations from parametric monads*, MathOverflow question, `https://mathoverflow.net/q/418434` (version: 2022-03-18); [username: fosco].

[Lor16] ———, *t-structures in stable* $(\infty, 1)$*-categories*, Ph.D. thesis, SISSA, 2016, `web archive:1963/35202`.

[LR20] F. Loregian and E. Riehl, *Categorical notions of fibration*, Expositiones Mathematicae **38** (2020), no. 4, 496–514.

[LS91] K.G. Larsen and A. Skou, *Bisimulation through probabilistic testing*, Information and Computation **94** (1991), no. 1, 1–28.

[LS02] S. Lack and R. Street, *The formal theory of monads II*, Journal of Pure and Applied Algebra **175** (2002), no. 1, 243–265, Special Volume celebrating the 70th birthday of Professor Max Kelly.

[LT23] F. Loregian and T. Trimble, *Differential 2-rigs*, Electronic Proceedings in Theoretical Computer Science **380** (2023), 159–182.

[Man22] G. Manuell, *Monoid extensions and the grothendieck construction*, Semigroup Forum **105** (2022), no. 2, 488–507.

[McC00] P. McCrudden, *Categories of representations of coalgebroids*, Advances in Mathematics (2000), 299–332 (English).

[MP89] M. Makkai and R. Paré, *Accessible categories: The foundations of categorical model theory*, Contemporary Mathematics, vol. 104, American Mathematical Society, Providence, Rhode Island, 1989.

[MP00] I. Moerdijk and E. Palmgren, *Wellfounded trees in categories*, Annals of Pure and Applied Logic **104** (2000), no. 1, 189–218.

[MPS15] S. Milius, D. Pattinson, and L. Schröder, *Generic Trace Semantics and Graded Monads*, 6th Conference on Algebra and Coalgebra in Computer Science (CALCO2015), LIPIcs 2015, 2015, pp. 253–269.

[MU22] D. McDermott and T. Uustalu, *Flexibly graded monads and graded algebras*, Mathematics of Program Construction (Cham), Springer International Publishing, 2022, pp. 102–128.

[nLa23] nLab authors, *Categorical semantics of dependent type theory, §3*, `https://ncatlab.org/nlab/show/categorical+semantics+of+dependent+type+theory#simple_fibrations`, August 2023, Revision 71.

[OWE20] D. Orchard, P. Wadler, and H. Eades, *Unifying graded and parameterised monads*, Electronic Proceedings in Theoretical Computer Science **317** (2020), 18–38.

[PGW17] M. Pickering, J. Gibbons, and N. Wu, *Profunctor optics: Modular data accessors*, The Art, Science, and Engineering of Programming **1** (2017), no. 2, 51.

[PS08] C. Pastro and R. Street, *Doubles for monoidal categories*, Theory and Applications of Categories **21** (2008), 61–75 (eng).

[Qui70] D. Quillen, *On the (co-)homology of commutative rings*, Applications of Categorical Algebra (Proc. Sympos. Pure Math., Vol. XVII, New York, 1968), American Mathematical Society, 1970, pp. 65–87. MR 257068

[Ril18] M. Riley, *Categories of optics*, 2018, `arXiv:1809.00738` preprint.

[RT07] J. Rosický and W. Tholen, *Factorization, fibration and torsion*, J. Homotopy Relat. Struct. **2** (2007), no. 2, 295–314. MR 2369170

[Sko04] Z. Skoda, *Distributive laws for actions of monoidal categories*, 2004, `arXiv:math/0406310` preprint.

[Smi08] A. L. Smirnov, *Graded monads and rings of polynomials*, Journal of Mathematical Sciences **151** (2008), no. 3, 3032–3051.

[Spi19] D.I. Spivak, *Generalized lens categories via functors* $\mathcal{C}^{\mathrm{op}} \to \mathsf{Cat}$, 2019, `arXiv:1908.02202` preprint.

[Str72] R. Street, *The formal theory of monads*, J. Pure Appl. Algebra **2** (1972), no. 2, 149–168.

[Str81] ———, *Conspectus of variable categories*, Journal of Pure and Applied Algebra **21** (1981), no. 3, 307–338.

[Str23] T. Streicher, *Fibered categories* à la *Jean Benabou*, 2023, `arXiv:1801.02927` preprint.

[SW78] R. Street and R. Walters, *Yoneda structures on 2-categories*, J. Algebra **50** (1978), no. 2, 350–379.

[Tam06] D. Tambara, *Distributors on a tensor category*, Hokkaido Mathematical Journal **35** (2006), no. 2, 379–425.

[Tao09] T. Tao, *Mazur's swindle*, Oct 2009, `https://terrytao.wordpress.com/2009/10/05/mazurs-swindle/`.

[Ten75] B. R. Tennison, *Sheaf theory*, Cambridge University Press, December 1975.

[Ver23] P. Vertechi, *Dependent optics*, Electronic Proceedings in Theoretical Computer Science **380** (2023), 128–144.

[Vis05] A. Vistoli, *Grothendieck topologies, fibered categories and descent theory*, Fundamental algebraic geometry, Math. Surveys Monogr., vol. 123, Amer. Math. Soc., Providence, RI, 2005, pp. 1–104.

[Wel80] C. Wells, *A Krohn-Rhodes theorem for categories*, Journal of Algebra **64** (1980), no. 1, 37–45.

Davide Castelnovo: Università degli studi di Padova
E-mail: `davide.castelnovo@math.unipd.it`

Greta Coraglia: Dipartimento di Filosofia, Università degli Studi di Milano
E-mail: `greta.coraglia@unimi.it`

Fosco Loregian: Department of Software Science, Tallinn University of Technology
E-mail: `fosco.loregian@gmail.com`

Ülo Reimaa: Institute of Mathematics and Statistics, University of Tartu
E-mail: `ulo.reimaa@ut.ee`