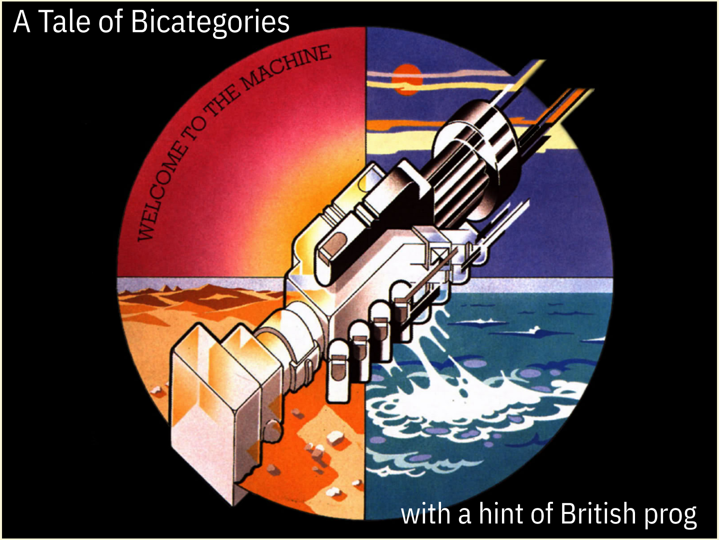
A Tale of Bicategories



with a hint of British prog

Prolegomena

j/w G. Boccali, A. Laretto, S. Luneia; arXiv:2303.03865.

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- Boccali, G., Femić, B., Laretto, A., ———, & Luneia, S. “The semibicategory of Moore automata.” arXiv:2305.00272.

A theory of abstract automata

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Let **K** be a strict 2-category with all finite weighted limits.

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I(f, 1_C) \arrow[r] & C
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{Mre}(f, b) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C/b \arrow[d] \\
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As such, **Mly** and **Mre** are parametric functors of type

$$\mathbf{K}(C, C)^{\mathrm{op}} \times \mathbf{K}/C \longrightarrow \mathbf{K}/C$$

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- the category of Mealy automata, where objects and morphisms are of the form

$$\begin{tikzcd} e \arrow[d, dotted, "\varphi"'] & Fe \arrow[l, "d"'] \arrow[r, "s"] \arrow[d, dotted, "F\varphi"] & b \arrow[d, equal] \\ e' & Fe \arrow[l, "d"'] \arrow[r, "s"] & b \end{tikzcd}$$

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- the category of Moore automata, where objects and morphisms are of the form

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In particular, if $F_A : \mathbf{K} \to \mathbf{K}$ is the functor depending on an object A (an ‘Alphabet’) Mealy and Moore automata are respectively diagrams of the form (E, d, s):

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- $d : A \otimes E \to E$ is an action of A on E (a dynamical system);
- s is an output function (think of $B = \{0, 1\}$ or $B = [0, 1]$, etc.)

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If **K** is monoidal closed, $F_A = A \otimes -$ is colimit-preserving and its algebras coincide with the coalgebras of its right adjoint $[A, -]$. This allows a number of deductions:

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- If **K** has countable sums, $d : A \otimes E \to E$ is an action of $A^* := \sum_n A^n$, and s extends similarly:

$$\begin{tikzcd}
& A \otimes E \arrow[dl, "d"'] \arrow[dr, "s"] \arrow[dd, "\eta_A \otimes E"'] & \\
E & & B \\
& A^* \otimes E \arrow[ul, dotted, "d^*"] \arrow[ur, dotted, "s^*"'] &
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This is called the canonical extension of (E, d, s).

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- If $(\mathbf{K}, \otimes)$ is monoidal and $T : \mathbf{K} \to \mathbf{K}$ is a commutative monad over it, we can lift the monoidal structure of $\mathbf{K}$ making the free functor $F : \mathbf{K} \to \mathbf{Kl}(T)$ strong monoidal.

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  Machines in $\mathbf{Kl}(T)$ are non-deterministic versions of the ones in **K**.

  Take T the powerset monad on **Set**, or a distribution/probability monad like the one of finite distributions –whose algebras are convex sets, and free algebras affine simplices).

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- If **K** is cocomplete (e.g., locally presentable), so are $\mathbf{Mly}(A, B), \mathbf{Mre}(A, B)$ for every A, B –with colimits created by $\mathbf{Mly}(A, B), \mathbf{Mre}(A, B) \to \mathcal{K}$ and connected limits created by the functor in the commæ.

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In particular, the terminal objects of $\mathbf{Mly}(A, B), \mathbf{Mre}(A, B)$ are respectively

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Observe that this can be deduced from the fact that when **K** is closed, we can characterize automata coalgebraically, see some work of Jacobs.

(Semi)bicategories of automata

A tale of bicategories

When **K** is Cartesian, **Mly**(A, B) is the hom-category of a bicategory.[1]

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Clearly, $C(\mathbf{K})$ exists for every monoidal category!

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- There exists a semibicategory **Mre** of Moore-type automata, a functor

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  and a right adjoint $\mathbf{Mly}(A, B) \to \mathbf{Mre}(A, B)$, altogether forming the components of a local adjunction.

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- the 1- and 2-full sub-bicategory of $\mathbf{Mac}^s$ spanned by monoids (=one-object categories);
- the 2-full sub-bicategory of **Mly** (over **Set**) whose 1-cells are Mealy automata between monoids such that the representation of A^* on E in $E \xleftarrow{d^*} A^* \otimes E \xrightarrow{s^*} B$ induces a functor $\Sigma : \mathcal{E}[d^*] \to B$, when B is a monoid.

Machines valued in a bicategory

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This idea is not *entirely* new; it resembles old (and obscure) work of Bainbridge, modeling the state space of abstract machines as a functor, of which one can take the left/right Kan extension along an ‘input scheme’. See work of Petrişan et al.

A bimachine is a span in...

Definition

Let $\mathbb{B}$ be a bicategory; a bicategorical Moore (biMoore) machine in $\mathbb{B}$ is a diagram of 2-cells

$$e \Longleftarrow e \circ i, e \Longrightarrow o$$

between 1-cells e, i, o.[2]

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The fact that this span exists, *coherces the types* of i, o, e in such a way that i must be an endomorphism of an object A.

$$A \xrightarrow{i} A, \quad A \xrightarrow{i} A \xrightarrow{i} A, \quad A \xrightarrow{i} A \xrightarrow{i} A \xrightarrow{i} A, \dots$$

all make sense.

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all make sense.

In the monoidal case, the fact that an input 1-cell stands on a different level from an output was completely obscured by the fact that every 1-cell is an endomorphism.

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Everything will be made a Kan extension

Recall that the terminal objects of $\mathbf{Mly}(A,B)$, $\mathbf{Mre}(A,B)$ are respectively $[A^+,B]$, $[A^*,B]$.

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Analogously, given that a biMoore of fixed input and output i, o consists of a way of filling the dotted arrows in

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& A \arrow[dl, "i"'] \arrow[dr, dotted, "e"] \arrow[dr, bend left=60, "o"] \arrow[d, phantom, "\overset{\delta}{\Rightarrow}"] \arrow[dr, phantom, shift left=3, "\overset{\sigma}{\nearrow\!\!\nearrow}"] & \\
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The terminal object of the category of biMoore machines[3] is the right extension of $o : A \to B$ along the free monad $i^\sharp : A \to A$.

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Examples

biMoore in Cat

Regarding **Cat** as a strict 2-category, a biMoore machine is a functor $E : \mathcal{C} \to \mathcal{D}$ closing a span $\mathcal{C} \xleftarrow{I} \mathcal{C} \xrightarrow{O} \mathcal{D}$ with suitable 2-cells.

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If $\mathcal{D} = \mathbf{Set}$, states and output are presheaves, and E is acted by an endofunctor; in this case, the behaviour of the terminal machine can be described as a known object: unpacking the end that defined $Ran_{I^\natural} O$ we obtain the functor

$$A \longmapsto [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, I^\natural \_), O)$$

sending an object A to the set of natural transformations $\alpha : \mathcal{C}(A, I^\natural \_) \Rightarrow O$; to each generalised A-element of $I^\natural C$ corresponds an element of the output space $\Upsilon_C(u) \in OC$.

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In the bicategory **Prof** of profunctors, a biMoore machine $E : I \mathrel{\circ\!\!\rightarrow} O$ consists of a digraph I of inputs, and parallel profunctors E, O of states and output.

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In the special case of $\{0, 1\}$-enriched profunctors (i.e., relations), the Kan extension of behaviour reduces to the maximal E such that $E \subseteq O$ and $E \circ I^\natural \subseteq E$ (here $\circ$ is the relational composition).

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$$(a, b) \in R \iff \forall a' \in A.((a', a) \in I^\natural \Rightarrow (a', b) \in O).$$

This relation expresses *reachability* of b from a:

$$a\,R\,b \iff \Big((a' = a) \vee (a' \xrightarrow{I} a_1 \xrightarrow{I} \dots \xrightarrow{I} a_n \xrightarrow{I} a) \Rightarrow a'\,O\,b\Big)$$

New maps

Intertwiners

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A \arrow[r, "u"] \arrow[d, "i"'] & A' \arrow[d, "i'"] \arrow[dl, Rightarrow, "\iota"] \\
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\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "e"'] & A' \arrow[d, "e'"] \arrow[dl, Rightarrow, "\epsilon"] \\
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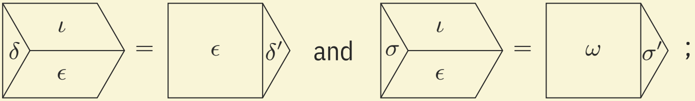

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2. the following two identities hold:

$$\epsilon \circ (d' \otimes U) = (V \otimes d) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$
$$\omega \circ (s' \otimes U) = (V \otimes s) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$

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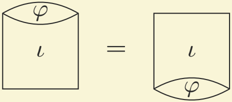

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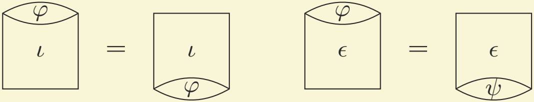

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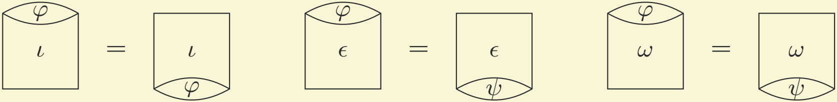

Vistas

Monoidal topology and automata

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BiMoore and biMealy machines, when instantiated in $(T, \mathcal{V})$-**Prof**, a 2-categorical way to look at topological, (ultra)metric ways to study behaviour of a state machine –the reachability relation becomes topological, (ultra)metric, probabilistic, sequential... according to suitable choices of T and $\mathcal{V}$.

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Conjecture

One can address nondeterministic biMoore automata in $\mathbb{B}$ as deterministic bicategorical automata in a proarrow equipment, porting all the paraphernalia (minimisation, behaviour, and bisimulation) into a bigger conceptual framework.

The En(i)d

The En¡d is a simphonic prog rock band from Southampthon;
suggested listening: *Ærie Færie Nonsense* and *Trippin the Light Fantastic*.