OF LIMS AND SETS

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1. Reduction of limits to products and equalizers

Definition 1.1 (Some terminology)**.** A (small) *diagram* in a category $\mathcal{C}$ is a functor

$$\mathcal{J} \longrightarrow \mathcal{C} \tag{1.1}$$

whose domain is a small category. A *cone* for a diagram D consists of a pair (X, c) where X is an object of $\mathcal{C}$ called *tip* of the cone, and c is a natural transformation $\Delta X \Rightarrow D$ from the constant functor at X; so c consists of a family of arrows in $\mathcal{C}$,

$$c_j : X \longrightarrow Dj \tag{1.2}$$

the *components* of the cone, such that for every morphism $f : i \to j$ in $\mathcal{J}$ the triangle

$$\begin{tikzcd} & X \arrow[dl, "c_i"'] \arrow[dr, "c_j"] & \\ Di \arrow[rr, "Df"'] & & Dj \end{tikzcd} \tag{1.3}$$

is commutative. The category of cones for D has

- objects the cones (X, c) for D;
- arrows $h : (X, c) \to (Y, c')$ the arrows $h : X \to Y$ in $\mathcal{C}$, between the tips of the cones such that for every $j \in \mathcal{J}$ the triangle

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "c_j"'] & & Y \arrow[dl, "c'_j"] \\ & Dj & \end{tikzcd} \tag{1.4}$$

The limit $(\lim D, p_j)$ of a diagram D consists of a terminal object in its category of cones. More than often one calls 'limit of D' the tip of the terminal cone, leaving the maps of the cone implicit *this is almost always harmless but slightly incorrect: the limits is composed of* ***both*** *parts.*

If a diagram D has a limit $(\lim D, p_j)$ we say that $\mathcal{C}$ *admits* the limit of D; if for a fixed $\mathcal{J}$, every $D : \mathcal{J} \to \mathcal{C}$ has a limit, we say that $\mathcal{C}$ has limits of shape $\mathcal{J}$ or that it has $\mathcal{J}$-limits; if for every element $\mathcal{J}$ of a subclass $\Phi \subseteq \mathsf{Cat}$ of categories, $\mathcal{C}$ has limits of shape $\mathcal{J}$, we say that $\mathcal{C}$ has limits of shape Φ or that it has Φ-limits. If $\mathcal{C}$ has Cat-limits, we say that $\mathcal{C}$ is (small-)complete.

Definition 1.2. In particular, a category $\mathcal{C}$ has all products if it has all limits of shape $S^\delta \to \mathcal{C}$ when S^δ is the discrete category over a set S, and $\mathcal{C}$ has equalizers if it has limits over $\mathcal{J} = \{0 \rightrightarrows 1\}$: this is because

- a diagram of shape S^δ specifies precisely a family of objects X_s in $\mathcal{C}$, one for every $s \in S$; being the terminal cone in this case means that there exists a family of arrows $p_s : \prod_s X_s \to X_s$ indexed by S with the property that
- a diagram of shape $\{0 \rightrightarrows 1\}$ specifies precisely a pair of morphisms $f, g : D_0 \to D_1$ in $\mathcal{C}$; being the terminal cone in this case means that there exists.

Theorem 1.3. The category Set of sets and functions has all products and all equalizers.

Proof.

- the product of a family of sets $\{X_s \mid s \in S\}$ is the usual Cartesian product $\prod_{s\in S} X_s$, constructed as the set of functions $S \to \bigcup_s X_s$ with the property that $f(s) \in X_s$. This allows to represent the elements of the set $\prod_{s\in S} X_s$ as *S-indexed sequences* $(x_s \mid s \in S, x_s \in X_s)$. Evidently, $\prod_{s\in S} X_s$ is equipped with projection maps $p_s : \prod_{t\in S} X_t \to X_s$ for every $s \in S$, picking the sth element of the S-sequence $(x_s \mid s \in S)$.

  The universal property of the product $\prod_{s\in S} X_s$ is spelled as follows:

  > For every set Z and family of functions $z_s : Z \to X_s$, there exists a unique $\bar{z} : Z \to \prod_{s\in S} X_s$ such that $p_s \circ \bar{z} = z_s$.

  Define $\bar{z}$ to be the function sending $\zeta \in Z$ to the S-sequence $(z_s\zeta \mid s \in S)$. Clearly this is the only possible definition so that

$$\begin{tikzcd} & \prod_{s\in S} X_s \arrow[d, "p_s"] \\ Z \arrow[ur, "\bar{z}"] \arrow[r, "z_s"'] & X_s \end{tikzcd} \tag{1.5}$$

  is a commutative triangle for every $s \in S$.
- the equalizer of a pair of maps $f, g : X \to Y$ consists of the subset $E = \{x \in X \mid fx = gx\} \subseteq X$; it realizes the universal property

  > For every $u : Z \to X$ such that $f \circ u = g \circ u$, there exists a unique $\bar{u} : Z \to E$ such that u equals the composition $Z \to E \hookrightarrow X$.

  Since E is just a subset of X, the universal property of $\mathsf{eq}(f, g)$ can be rephrased as follows: every $u : Z \to X$ such that $f(u(z)) = g(u(z))$ for every $z \in Z$ takes values in the subset E, defined above. This is evident, as much as it is evident that E is chosen precisely in order to satisfy this property. □

Lemma 1.4.
If $E \xrightarrow{e} A$ is an equalizer of $A \underset{g}{\overset{f}{\rightrightarrows}} B$, then the following are equivalent:

(1) $f = g$,

(2) e is an epimorphism,

(3) e is an isomorphism,

(4) id_A is an equalizer of f and g. □

Theorem 1.5. The category Set of sets and functions has all limits.

Proof. Let $\mathcal{D}$ be a small category and $F : \mathcal{D} \to \mathsf{Set}$ a functor. For every arrow f in $\mathcal{D}$, we denote $\mathrm{s}(f)$ the source and $\mathrm{t}(f)$ the target of f; so, if $f : D \to D'$, $\mathrm{s}(f) = D$, $\mathrm{t}(f) = D'$.

We prove that the limit $\lim F$ of F is precisely the equalizer of the pair of maps

$$\prod_{D\in\mathcal{D}_0} FD \overset{\alpha^F}{\underset{\beta^F}{\rightrightarrows}} \prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) \tag{1.6}$$

where

- $\alpha^F((x_D \mid D \in \mathcal{D})) = (x_{\mathrm{t}(f)} \mid f \in \mathcal{D}_1)$;
- $\beta^F(((x_D \mid D \in \mathcal{D})) = (Ff(x_{\mathrm{s}(f)}) \mid f \in \mathcal{D}_1)$;

This means two things: if $\lim F$ exists, then it must be the equalizer of that pair; otoh, if that pair (α, β) has an equalizer, then such is the limit of F.

We have to prove that

(1) There exists a cone

$$\lim F \xrightarrow{\bar{p}} \prod_{D\in\mathcal{D}_0} FD \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) \tag{1.7}$$

where e equalizes the pair α, β;

(2) such cone is terminal; this will mean two things:
   - the universal property of $\lim F$ entails the universal property of $\mathsf{eq}(\alpha, \beta)$;
   - the universal property of $\mathsf{eq}(\alpha, \beta)$ entails the universal property of $\lim F$.

   Thus, there is a unique isomorphism $\mathsf{eq}(\alpha, \beta) \cong \lim F$.

Proving 1. is easy; if $(\lim F, p_D)$ exists, all projections $p_D : \lim F \to FD$ assemble into a unique map $\bar{p} : \lim F \to \prod_D FD$ (this is the universal property of $\prod_D FD$). Note in passing that by the lemma above if $\mathcal{D}$ is a discrete category, $\prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) = \prod_{D\in\mathcal{D}_0} FD$, α, β are invertible and thus $\lim F \cong \prod_{D\in\mathcal{D}_0} FD$, as it should be.

Now, $\bar{p} : \lim F \to \prod_D FD$ equalizes (α, β), because the components p_D form a cone: the triangle of sets and functions

$$\begin{tikzcd} & FD \arrow[dd, "Ff"] \\ \lim F \arrow[ur, "p_D"] \arrow[dr, "p_{D'}"'] & \\ & FD' \end{tikzcd} \tag{1.8}$$

is commutative, whence the fact that for all $\hat{x} \in \lim F$ and $f : D \to D'$ in $\mathcal{D}_1$ one has

$$\beta^F(\bar{p}(\hat{x})) = \big(Ff(p_{\mathrm{s}(f)}(\hat{x})) \mid f \in \mathcal{D}_1\big) = \big(p_{\mathrm{t}(f)}(\hat{x}) \mid f \in \mathcal{D}_1\big) = \alpha^F(\bar{p}(\hat{x})). \tag{1.9}$$

A similar argument for a general cone $(z : Z \to FD \mid D \in \mathcal{D}_0)$ proves that this is a cone for F if and only if it equalizes (α, β); thus, a cone for F must be a terminal cone wrt the property of equalizing (α, β); whence the conclusion. $\square$

More generally, for a category $\mathcal{C}$ to have all limits it is necessary and sufficient that it has all (small) products and equalizers.

Theorem 1.6. The following conditions are equivalent:

- $\mathcal{C}$ has all small products and all equalizers;
- $\mathcal{C}$ is small-complete.

Proof. $\square$

Exercise 1.7. The *triqualizer* $\mathsf{triq}(f, g, h)$ of functions $f, g, h : X \to Y$ is defined as the limit of the diagram

$$\mathcal{J} = \left\{ \begin{tikzcd} 0 \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & 1 \end{tikzcd} \right\} \to \mathsf{Set}. \tag{1.10}$$

- Spell out the universal property of $\mathsf{triq}(f, g, h)$;
- by virtue of Theorem 1.6 above, $\mathsf{triq}(f, g, h)$ must be expressible as an equalizer of two maps between products. How?

Exercise 1.8. An *Urizen compass* G_n is a limit of a diagram of the following form:

$$\begin{tikzcd} & X_{12} \arrow[dl] \arrow[dr] & & X_{23} \arrow[dl] \arrow[dr] & & \dots \arrow[dl, dotted, no head] \arrow[dr] & & X_{n-1,n} \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 & & \dots & & X_{n-1} & & X_n \end{tikzcd}$$

write the universal property for G_3 and G_4; express G_3 and G_4 as equalizers of maps between products.

2. Symmetry and associativity, interchange of limits, Yoneda and limits

The universal property of a limit entails that

2.1. Functoriality of limits.

Theorem 2.1. A natural transformation $\alpha : D \Rightarrow D'$ in the category of functors $D : \mathcal{J} \to \mathsf{Set}$, induces a morphism $\lim \alpha : \lim D \to \lim D'$ between the limits. Moreover, lim preserves identities and compositions, hence it is a functor

$$\lim : [\mathcal{J}, \mathsf{Set}] \longrightarrow \mathsf{Set} \tag{2.1}$$

The natural transformation α has components $Dj \to D'j$, and the composition $\lim D \xrightarrow{p_j} Dj \xrightarrow{\alpha_j} D'j$ is easily seen to be a cone for D'. Then, there is a unique $\bar{\alpha} : \lim D \to \lim D'$ such that each diagram

$$\begin{tikzcd} \lim D \arrow[r, "\bar{\alpha}"] \arrow[d, "p_j"'] & \lim D' \arrow[d, "p'_j"] \\ Dj \arrow[r, "\alpha_j"'] & D'j \end{tikzcd} \tag{2.2}$$

commutes. Uniqueness proves that $\overline{\beta \circ \alpha} = \bar{\beta} \circ \bar{\alpha}$ and $\overline{\mathrm{id}_D} = \mathrm{id}_{\lim D}$.

2.2. Yoneda and limits. One can use Yoneda lemma to express the universal property of limits; given a diagram $D : \mathcal{J} \to \mathcal{C}$, one can express the property

For all $A \in \mathcal{C}_0$, there is a (canonical, natural) isomorphism

$$\mathcal{C}(A, \lim_{j:\mathcal{J}} D_j) \cong \lim_{j:\mathcal{J}} \mathcal{C}(A, Dj) \tag{2.3}$$

(where on the right hand side we mean the limit of the composite

$$\begin{array}{ccccc} \mathcal{J} & \longrightarrow & \mathcal{C} & \longrightarrow & \mathsf{Set} \\ j & \longmapsto & Dj & \longmapsto & \mathcal{C}(A, Dj) \end{array} \tag{2.4}$$

which we know exists in the category of sets) as the representability of the functor

$$\begin{array}{rcl} \lim \mathcal{C}(\_, Dj) : \mathcal{C}^{\mathrm{op}} & \longrightarrow & \mathsf{Set} \\ A & \longmapsto & \lim_{j:\mathcal{J}} \mathcal{C}(A, Dj) \end{array} \tag{2.5}$$

Yoneda says that it is enough to know how limits are computed in Set (and in a category of functors into Set) to define them uniquely (=up to unique iso) in any category $\mathcal{C}$, via the notion of representability; the 'limit' $(\lim D, p_j)$ of a diagram $D : \mathcal{J} \to \mathcal{C}$ consists of a terminal object in the category of elements of $\lim \mathcal{C}(\_, Dj)$ (which is *precisely* the category of cones for D ;-)), while the cone

$$p_j : \lim D \longrightarrow Dj \tag{2.6}$$

is the universal element: it corresponds to the identity of $\lim D$ in the bijection

$$\mathcal{C}(\lim D, \lim D) \cong \lim \mathcal{C}(\lim D, Dj) \tag{2.7}$$

for $A = \lim D$.

Reading (2.3) as the isomorphism

$$y(\lim D)(A) \cong \big(\lim yD\big)(A) \tag{2.8}$$

one can also argue that y (the Yoneda embedding) *preserves* all limits:

Definition 2.2. Let $D : \mathcal{J} \to \mathcal{C}$ be a diagram; let $F : \mathcal{C} \to \mathcal{X}$ be a functor; let $\lim D$ denote the limit of D and $\lim(FD)$ the limit of the composite functor $F \circ D : \mathcal{J} \to \mathcal{X}$; then, the functoriality of F defines a cone

$$\gamma_{F,j} : F(\lim D) \longrightarrow FDj \tag{2.9}$$

whence a unique morphism, in $\mathcal{X}$, $\gamma_F : F(\lim D) \to \lim(FD)$. We say that F *preserves* $\lim D$, or that it *commutes* with $\lim D$ if γ_F is an isomorphism in $\mathcal{X}$.

Proposition 2.3. Limits in the category $[\mathcal{C}^{\mathrm{op}}, \mathsf{Set}]$ are computed objectwise in Set: this means that given a diagram

$$D : \mathcal{J} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}] \tag{2.10}$$

its limit $\lim D$ is a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$ defined as $(\lim_j D_j)(C) = \lim_j \big(D_j C\big)$.

Remark 2.4. The yoneda embedding

$$y : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathsf{Set}] \tag{2.11}$$

preserves all limits.

Evidently, F preserves all limits if and only if it preserves all products and all equalizers; it's easy to show that y preserves products and equalizers directly from the definition.

2.3. **Limits commute with limits.** The following argument will work for any category $\mathcal{C}$ be a category admitting products and equalizers, but we will spell it out just for sets; given a S-indexed family of parallel arrows

$$X_s \overset{f_s}{\underset{g_s}{\rightrightarrows}} Y_s \tag{2.12}$$

one can consider the parallel maps induced between the products using Theorem 2.1,

$$\prod_s X_s \overset{\left(\prod_s f_s\right)}{\underset{\left(\prod_s g_s\right)}{\rightrightarrows}} \prod_s Y_s. \tag{2.13}$$

The map $\left(\prod_s f_s\right)$ is defined as the unique with the property that for every $s' \in S$, the square

$$\begin{tikzcd} \prod_s X_s \arrow[r, "\left(\prod_s f_s\right)"] \arrow[d, "p^X_{s'}"'] & \prod_s Y_s \arrow[d, "p^Y_{s'}"] \\ X_{s'} \arrow[r, "f_{s'}"'] & Y_{s'} \end{tikzcd} \tag{2.14}$$

commutes. Similarly, for $\left(\prod_s g_s\right)$.

Theorem 2.5. The equalizer of (2.13) above is the product $\prod_s E_s$ of the equalizers $(E_s \xrightarrow{h_s} X_s \rightrightarrows Y_s \mid s \in S)$. In other words,

$$\lim_{0 \rightrightarrows 1}(\lim_{S^\delta} D) \cong \lim_{S^\delta}(\lim_{0 \rightrightarrows 1} D). \tag{2.15}$$

A single function $v : Z \to \prod_s X_s$ corresponds to a family of functions $(v_s : Z \to X_s \mid s \in S)$, and the fact that $\left(\prod_s f_s\right) \circ v = \left(\prod_s g_s\right) \circ v$ is exactly equivalent to the fact that putting $v_s = p_s \circ v$

$$\forall s \in S.(f_s \circ v_s = g_s \circ v_s) \tag{2.16}$$

Indeed,

- if $\left(\prod_s f_s\right) \circ v = \left(\prod_s g_s\right) \circ v$, then for all $s \in S$

$$p^Y_s \circ \left(\prod_s f_s\right) \circ v = p^Y_s \circ \left(\prod_s g_s\right) \circ v \tag{2.17}$$

  but the LHS of this equation is $f_s \circ p^X_s \circ v$, and RHS is $g_s \circ p^X_s \circ v$.
- Conversely, if $\forall s \in S.(f_s \circ v_s = g_s \circ v_s)$, then one uses uniqueness: for all $s \in S$,

$$p^Y_s \circ \left(\prod_s f_s\right) \circ v = f_s \circ p^X_s \circ v_s = g_s \circ p^X_s \circ v_s = p^Y_s \circ \left(\prod_s g_s\right) \circ v \tag{2.18}$$

  but there exists a *unique* $w : Z \to \prod_s Y_s$ such that for all $s \in S$ one has that $p^Y_s \circ w$ equals the common value $f_s \circ p^X_s \circ v_s = g_s \circ p^X_s \circ v_s$, hence $w = \left(\prod_s f_s\right) \circ v = \left(\prod_s g_s\right) \circ v$.

Now, given this, each v_s equalizes (f_s, g_s), and thus factors through the equalizer $Z \xrightarrow{\bar{v}_s} E_s \subseteq X_s$; there is a unique map $\bar{v} : Z \to \prod_s E_s$ such that $\prod_s h_s \circ \bar{v} = v$:

$$
\begin{tikzcd}
\prod E_s \arrow[r, "\prod_s h_s"] & \prod_s X_s \arrow[r, shift left, "(\prod_s f_s)"] \arrow[r, shift right, "(\prod_s g_s)"'] & \prod_s Y_s \\
& Z \arrow[ul, "\bar{v}"] \arrow[u, "v"'] &
\end{tikzcd}
\tag{2.19}
$$

2.4. **Unitality and associativity of products.** One can use the universal property of products to exhibit natural isomorphisms

$$A \times 1 \cong A \cong 1 \times A \qquad (A \times B) \times C \cong A \times (B \times C) \qquad A \times B \cong B \times A \tag{2.20}$$

Indeed, we just have to show that A has the same universal property of $A \times 1$, that $B \times A$ has the same universal property of $A \times B$, etc.

- A comes equipped with two projections

$$
\begin{tikzcd}
1 & A \arrow[l] \arrow[r, equal] & A
\end{tikzcd}
\tag{2.21}
$$

  which satisfy the universal property $1 \times A$ is required to have.
- Consider the universal problem

$$
\begin{tikzcd}
& B \times A \arrow[dl, "p_A"'] \arrow[dr, "p_B"] \arrow[d, dotted] & \\
A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B
\end{tikzcd}
\tag{2.22}
$$

  It must have a unique solution $\sigma = \langle p_B, p_A \rangle$. Uniqueness of $\mathrm{id}_{A \times B}$ solving the universal problem

$$
\begin{tikzcd}
& A \times B \arrow[dl, "p_B"'] \arrow[dr, "p_A"] \arrow[d, dotted] & \\
A & A \times B \arrow[l, "p_A"] \arrow[r, "p_B"'] & B
\end{tikzcd}
\tag{2.23}
$$

  implies that $\sigma \circ \sigma = \mathrm{id}_{A \times B}$.
- Associativity uses diagram chasing at its full potential:

$$
\begin{tikzcd}
& & (A \times B) \times C \arrow[dl, "p_{A \times B}"'] \arrow[drr, "\langle p_B p_{A \times B}, p_C \rangle"] \arrow[ddd, bend right, "{\langle p_a p_{A \times B}, \langle p_B p_{A \times B}, p_C \rangle \rangle}"'] \arrow[ddddrrrr, bend left] & & & & \\
& A \times B \arrow[dl, "p_A"'] \arrow[ddrrrr, "p_B"] & & & B \times C \arrow[ddr] \arrow[ddrr] & & \\
A & & & & & & \\
& & A \times (B \times C) \arrow[ull, "p'_A"] \arrow[uurr, "p_{B \times C}"'] & & & B & C
\end{tikzcd}
\tag{2.24}
$$