OF LIMS AND SETS

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1. Preliminaries

Definition 1.1 (Some terminology)**.** A (small) *diagram* in a category $\mathcal{C}$ is a functor

$$\mathcal{J} \longrightarrow \mathcal{C} \tag{1.1}$$

whose domain is a small category. A *cone* for a diagram D consists of a pair (X, c) where X is an object of $\mathcal{C}$ called *tip* of the cone, and c is a natural transformation $\Delta X \Rightarrow D$ from the constant functor at X; so c consists of a family of arrows in $\mathcal{C}$,

$$c_j : X \longrightarrow Dj \tag{1.2}$$

the *components* of the cone, such that for every morphism $f : i \to j$ in $\mathcal{J}$ the triangle

$$\begin{tikzcd} & X \arrow[dl, "c_i"'] \arrow[dr, "c_j"] & \\ Di \arrow[rr, "Df"'] & & Dj \end{tikzcd} \tag{1.3}$$

is commutative. The category of cones for D has

- objects the cones (X, c) for D;
- morphisms $h : (X, c) \to (Y, c')$ the morphisms $h : X \to Y$ in $\mathcal{C}$, between the tips of the cones such that for every $j \in \mathcal{J}$ the triangle

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "c_j"'] & & Y \arrow[dl, "c'_j"] \\ & Dj & \end{tikzcd} \tag{1.4}$$

The limit $(\lim D, p_j)$ of a diagram D consists of a terminal object in its category of cones. More than often one calls 'limit of D' the tip of the terminal cone, leaving the maps of the cone implicit *this is almost always harmless but slightly incorrect: the limits is composed of **both** parts.*

If a diagram D has a limit $(\lim D, p_j)$ we say that $\mathcal{C}$ *admits*, or contains, the limit of D. If for a fixed $\mathcal{J}$, every $D : \mathcal{J} \to \mathcal{C}$ has a limit, we say that $\mathcal{C}$ has limits of shape $\mathcal{J}$ or that it has $\mathcal{J}$-limits; if for every element $\mathcal{J}$ of a subclass $\Phi \subseteq \mathsf{Cat}$ of categories, $\mathcal{C}$ has limits of shape $\mathcal{J}$, we say that $\mathcal{C}$ has limits of shape Φ or that it has Φ-limits. If $\mathcal{C}$ has Cat-limits, we say that $\mathcal{C}$ is (small-)complete.

Definition 1.2. In particular, a category $\mathcal{C}$ has all products if it has all limits of shape $S^\delta \to \mathcal{C}$ when S^δ is the discrete category over a set S, and $\mathcal{C}$ has equalizers if it has limits over $\mathcal{J} = \{0 \rightrightarrows 1\}$.

Theorem 1.3. The category Set of sets and functions has all products and all equalizers.

Proof.

- the product of a family of sets $\{X_s \mid s \in S\}$ is the usual Cartesian product $\prod_{s\in S} X_s$, constructed as the set of functions $S \to \bigcup_s X_s$ with the property that $f(s) \in X_s$. This allows to represent the elements of the set $\prod_{s\in S} X_s$ as *S-indexed sequences* $(x_s \mid s \in S, x_s \in X_s)$. Evidently, $\prod_{s\in S} X_s$ is equipped with projection maps $p_s : \prod_{t\in S} X_t \to X_s$ for every $s \in S$, picking the sth element of the S-sequence $(x_s \mid s \in S)$.

  The universal property of the product $\prod_{s\in S} X_s$ is spelled as follows:

  > For every set Z and family of functions $z_s : Z \to X_s$, there exists a unique $\bar{z} : Z \to \prod_{s\in S} X_s$ such that $p_s \circ \bar{z} = z_s$.

  Define $\bar{z}$ to be the function sending $\zeta \in Z$ to the S-sequence $(z_s\zeta \mid s \in S)$. Clearly this is the only possible definition so that

$$\begin{tikzcd} & \prod_{s\in S} X_s \arrow[d, "p_s"] \\ Z \arrow[ur, "\bar{z}"] \arrow[r, "z_s"'] & X_s \end{tikzcd} \tag{1.5}$$

  is a commutative triangle for every $s \in S$.
- the equalizer of a pair of maps $f, g : X \to Y$ consists of the subset $E = \{x \in X \mid fx = gx\} \subseteq X$; it realizes the universal property

  > For every $u : Z \to X$ such that $f \circ u = g \circ u$, there exists a unique $\bar{u} : Z \to E$ such that u equals the composition $Z \to E \hookrightarrow X$.

  Since E is just a subset of X, the universal property of $\mathsf{eq}(f, g)$ can be rephrased as follows: every $u : Z \to X$ such that $f(u(z)) = g(u(z))$ for every $z \in Z$ takes values in the subset E, defined above. This is evident, as much as it is evident that E is chosen precisely in order to satisfy this property. □

Lemma 1.4.

If $E \xrightarrow{e} A$ is an equalizer of $A \overset{f}{\underset{g}{\rightrightarrows}} B$, then the following are equivalent:

(1) $f = g$,

(2) e is an epimorphism,

(3) e is an isomorphism,

(4) id_A is an equalizer of f and g. □

Theorem 1.5. The category Set of sets and functions has all limits.

Proof. Let $\mathcal{D}$ be a small category and $F : \mathcal{D} \to \mathsf{Set}$ a functor. For every arrow f in $\mathcal{D}$, we denote $\mathrm{s}(f)$ the source and $\mathrm{t}(f)$ the target of f; so, if $f : D \to D'$, $\mathrm{s}(f) = D$, $\mathrm{t}(f) = D'$.

We prove that the limit $\lim F$ of F is precisely the equalizer of the pair of maps

$$\prod_{D\in\mathcal{D}_0} FD \underset{\beta^F}{\overset{\alpha^F}{\rightrightarrows}} \prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) \tag{1.6}$$

where

- $\alpha^F((x_D \mid D \in \mathcal{D})) = (x_{\mathrm{t}(f)} \mid f \in \mathcal{D}_1)$;
- $\beta^F(((x_D \mid D \in \mathcal{D})) = (Ff(x_{\mathrm{s}(f)}) \mid f \in \mathcal{D}_1)$;

This means two things: if $\lim F$ exists, then it must be the equalizer of that pair; otoh, if that pair (α, β) has an equalizer, then such is the limit of F.

We have to prove that

(1) There exists a cone

$$\lim F \xrightarrow{\bar{p}} \prod_{D\in\mathcal{D}_0} FD \underset{\beta}{\overset{\alpha}{\rightrightarrows}} \prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) \tag{1.7}$$

where e equalizes the pair α, β;

(2) such cone is terminal; this will mean two things:
- the universal property of $\lim F$ entails the universal property of $\mathsf{eq}(\alpha, \beta)$;
- the universal property of $\mathsf{eq}(\alpha, \beta)$ entails the universal property of $\lim F$.

Thus, there is a unique isomorphism $\mathsf{eq}(\alpha, \beta) \cong \lim F$.

Proving 1. is easy; if $(\lim F, p_D)$ exists, all projections $p_D : \lim F \to FD$ assemble into a unique map $\bar{p} : \lim F \to \prod_D FD$ (this is the universal property of $\prod_D FD$). Note in passing that by the lemma above if $\mathcal{D}$ is a discrete category, $\prod_{f\in\mathcal{D}_1} F(\mathrm{t}(f)) = \prod_{D\in\mathcal{D}_0} FD$, α, β are invertible and thus $\lim F \cong \prod_{D\in\mathcal{D}_0} FD$, as it should be.

Now, $\bar{p} : \lim F \to \prod_D FD$ equalizes (α, β), because the components p_D form a cone: the triangle of sets and functions

$$\begin{tikzcd} & FD \arrow[dd, "Ff"] \\ \lim F \arrow[ur, "p_D"] \arrow[dr, "p_{D'}"'] & \\ & FD' \end{tikzcd} \tag{1.8}$$

is commutative, whence the fact that for all $\hat{x} \in \lim F$ and $f : D \to D'$ in $\mathcal{D}_1$ one has

$$\beta^F(\bar{p}(\hat{x})) = \big(Ff(p_{\mathrm{s}(f)}(\hat{x})) \mid f \in \mathcal{D}_1\big) = \big(p_{\mathrm{t}(f)}(\hat{x}) \mid f \in \mathcal{D}_1\big) = \alpha^F(\bar{p}(\hat{x})). \tag{1.9}$$

A similar argument for a general cone $(z : Z \to FD \mid D \in \mathcal{D}_0)$ proves that this is a cone for F if and only if it equalizes (α, β); thus, a cone for F must be a terminal cone wrt the property of equalizing (α, β); whence the conclusion. $\square$

More generally, for a category $\mathcal{C}$ to have all limits it is necessary and sufficient that it has all (small) products and equalizers.

Theorem 1.6. The following conditions are equivalent:

- $\mathcal{C}$ has all small products and all equalizers;
- $\mathcal{C}$ is small-complete.

Proof. □

Exercise 1.7. The *triqualizer* $\mathsf{triq}(f,g,h)$ of functions $f,g,h : X \to Y$ is defined as the limit of the diagram

$$\mathcal{J} = \left\{ \begin{tikzcd} 0 \arrow[r, shift left=2, "f"] \arrow[r, "g" description] \arrow[r, shift right=2, "h"'] & 1 \end{tikzcd} \right\} \to \mathsf{Set}. \tag{1.10}$$

- Spell out the universal property of $\mathsf{triq}(f,g,h)$;
- by virtue of Theorem 1.6 above, $\mathsf{triq}(f,g,h)$ must be expressible as an equalizer of two maps between products. How?

References