Differential automata theory

Fosco Loregian

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Tallinn University of Technology

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10.4204/EPTCS.397.1 **2305.00272** **10.1007/978-3-031-66438-0_4**

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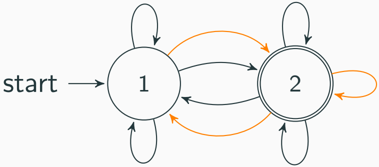

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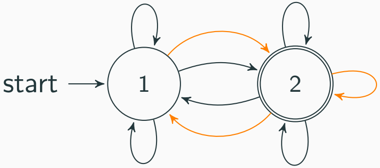
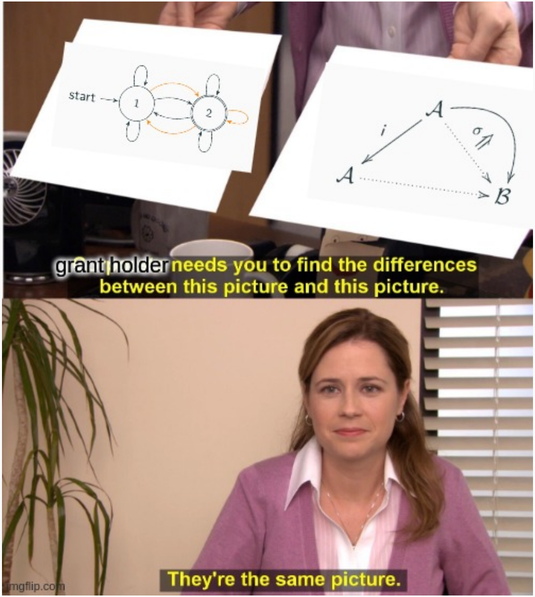


$$
\begin{tikzcd}
& \mathcal{A} \arrow[dl, "i"'] \arrow[dr, dotted] \arrow[dr, bend left=60] \arrow[dr, phantom, "\Uparrow\sigma", shift left=3] & \\
\mathcal{A} \arrow[rr, dotted] & & \mathcal{B}
\end{tikzcd}
$$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grant holder needs you to find the differences between this picture and this picture.
They're the same picture.
imgflip.com

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Techniques from representation theory, topos theory, homotopy theory, etc. are all useful and pop up constantly.

Formal category theory

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- homological algebra: find properties making a category $\mathcal{E}$ behave like $\mathbf{Ch}(\mathcal{A})$;
- homotopy theory: find properties making a category $\mathcal{E}$ behave like $\mathbf{Ho}(\mathbf{Top})$;
- ...

In stark contrast with topos theory, where a definitive answer to what are the intrinsic properties of a set-like category exists, FCT is an active field of study, with (yet) no unanimous consensus on what the right axioms are.

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A variety of different approaches to axiomatize the properties of **Cat** has been proposed, each with its own merits and drawbacks.

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I thought long and hard and…

9. arXiv:2303.03865 [pdf, other] math.CT cs.FL doi 10.4204/EPTCS.397.1

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Monads and limits in bicategories of circuits

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- fix objects $X, B \in \mathcal{K}_0$ and morphisms

$$\begin{tikzcd} B \arrow[r, "b"] & X \arrow[loop right, "f"] \end{tikzcd}$$

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We call such a (finite weighted) limit sketch an automata theory.

- construct step by step the following finite limits:

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\begin{tikzcd}
B \arrow[dr, "b"'] & & X \arrow[dl] \arrow[dr] & & X \arrow[dl, equal] \\
& X & & X &
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 & f/b \arrow[dl] \arrow[dr] & & & \\
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'Mealy' automata

'Moore' automata

Unraveling the definition if $\mathcal{K} = \mathbf{Cat}$, $B = 1$ (so $B : 1 \to \mathcal{X}$ picks an object of $\mathcal{X}$): given a diagram of categories and functors

$$1 \xrightarrow{B} \mathcal{X} \xleftarrow{F} \mathcal{X}$$

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- $\mathbf{Mly}(F, B)$ is the category $\overset{\textit{inserter}}{Alg(F)} \overset{\textit{pb}}{\times_{\mathcal{X}}} \overset{\textit{comma}}{(F/B)}$ of spans

$$X \longleftarrow FX \longrightarrow B$$

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- $\mathbf{Mre}(F, B)$ is the category $\overset{\textit{inserter}}{Alg(F)} \overset{\textit{pb}}{\times_{\mathcal{X}}} \overset{\textit{comma}}{(\mathcal{X}/B)}$ of disconnected spans

  $$X \longleftarrow FX, X \longrightarrow B$$

  and suitable morphisms.

In particular when $\mathcal{X}$ is monoidal, and $F = A \otimes -$, one studies categories where **objects** and morphisms are as follows:

$$(X, d, s) : \qquad X \xleftarrow{d} A \otimes X \xrightarrow{s} B$$

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$$(Y, d', s'): \qquad Y \longleftarrow A \otimes Y \longrightarrow B$$

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$$
\begin{tikzcd}
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\end{tikzcd}
$$

Proposition

The assignment $(A, B) \mapsto \mathbf{Mly}_{\mathcal{X}}(A \otimes -, B) = \mathbf{Mly}_{\mathcal{X}}(A, B)$ defines an indexed category

$$\mathcal{X}^{\mathrm{op}} \times \mathcal{X} \longrightarrow \mathbf{Cat}$$

which when $\mathcal{X}$ is Cartesian forms the hom-category of the bicategory of Mealy automata.

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> *Categories of automata (in fact, fibers of a monoidal fibration), with objects having a combinatorial meaning, equipped with a notion of derivative functor, in which to do categorified differential algebra*

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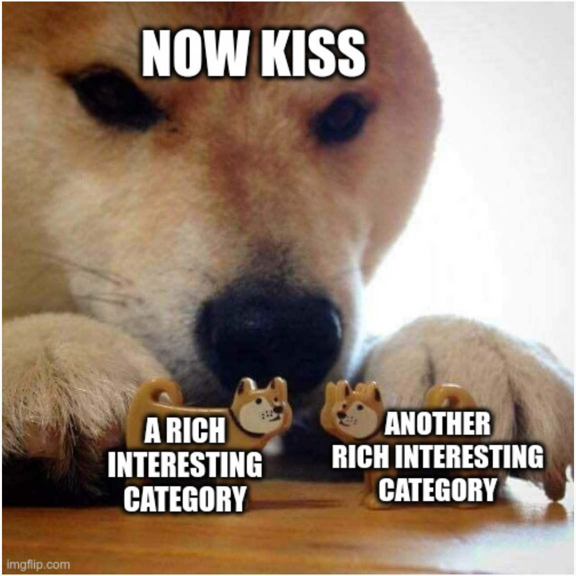
10.4204/EPTCS.380.10 I studied a particular case of this construction, when

- $\mathcal{K}$ is the category of Joyal's combinatorial species;
- $F = \partial$ is the derivative of species.

This is a lot of structure:

> *Categories of automata (in fact, fibers of a monoidal fibration), with objects having a combinatorial meaning, equipped with a notion of derivative functor, in which to do categorified differential algebra / study non-reversible dynamical systems induced by a diff. op.*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 Differential 2-rigs

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- it is the free cocomplete 2-rig on one generator;
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$$R[\![x]\!] : \mathbf{Ring} = \mathbf{Spc} : \mathbf{D2R}$$

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$$\boxed{\mathbf{P} := \mathbf{P}[1] \qquad \mathbf{Spc} := (1, \mathbf{Set})\text{-}\mathbf{Spc} = [\mathbf{P}, \mathbf{Set}]}$$

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Axiomatizing these properties leads to D2Rs. First, let's study species better expanding on the items of this list.

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- $E' \cong E \qquad Cyc' \cong L \qquad P' \cong E + E$ (*natural* isos of functors)
- ∂ has a left adjoint (easy to describe: $\partial = \{y[1], -\}$ hence $L = y[1] * -$), but also a right adjoint (because $y[1]$ is a tiny object)

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This realizes the Leibniz rule as a universal property.

Differential 2-rigs

(and differential automata)

1. sketch the technology one can develop for D2Rs, categorifying differential algebra;
2. apply some of these ideas to a specific case for a category of automata.

Clearly, 2. is just a pretext for 1.

Free objects and quotients

Freeness results

Spc is the free (cocomplete) 2-rig $F[t]$ on a single generator $\{t\}$; it acquires a differential structure much like $k[x]$ does.

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More generally one can define the free 2-rig on a category…

And extend scalars over $\mathcal{R}$ –defining the free $\mathcal{R}$-algebra on S: $F[S] \otimes \mathcal{R}$

$\mathcal{R}$ a 2-rig; $\mathcal{R}[t] = \mathcal{R} \otimes_{\mathbf{P}} F[t] = \mathcal{R} \otimes_{\mathbf{P}} \mathbf{Spc}$

Geometry of D2Rs

 Kähler differentials

Geometry of D2Rs

$$\{\text{derivations on } R\} \cong \left\{ s : \begin{tikzcd} R[t]/t^2 \arrow[d, "ev_0"] \\ R \arrow[u, bend left, "s", red] \end{tikzcd} \right\}$$

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Intermezzo

Definition

Given a 2-category $\mathcal{K}$ and a diagram

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E.g. if $\mathcal{C}$ is a category, $W \subseteq \mathcal{C}^2$ a class of maps; the coinverter of

$$\begin{tikzcd} (W \subseteq \mathcal{C}^2) \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \end{tikzcd}$$

is the Gabriel-Zisman localization $\mathcal{C}[W^{-1}]$.

Geometry of D2Rs

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Theorem

$$Der[\mathcal{R}] \cong \{\text{sections}/\mathcal{R} \text{ of } ev_0 : \mathcal{R}[\epsilon] \to \mathcal{R}\}$$

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- quotients like $\mathcal{R}[X, Y]/(Y^2 + 1 \cong X^2)$ (categorified hyperbola) acquire a differential structure, $\partial Y = X, \partial X = Y$; can be done more in general?

Jet spaces

Categorified jet spaces

Given a D2R $(\mathcal{R}, \otimes, \partial)$ let $\mathbf{Alg}(\partial)$ be the category of ∂-algebras.

- objects: $(X, \xi : \partial X \to X)$;
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Hence the chain

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Categorified jet spaces

Define by mutual induction:

- $\mathcal{R}^{(0)} := \mathcal{R}$ and $\mathcal{R}^{(n+1)} := \mathbf{Alg}(\partial^{(n)}, \mathcal{R}^{(n)})$;
- $\partial^{(1)} := \partial$ and $\partial^{(n+1)} := \mathcal{R}^{(n+1)} \to \mathcal{R}^{(n+1)}$ defined lifting $\partial^{(n)}$.

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Definition

From the chain of forgetful functors

$$\mathcal{R} \longleftarrow \mathbf{Alg}(\partial) \longleftarrow \mathbf{Alg}(\partial') \longleftarrow \mathbf{Alg}(\partial'') \longleftarrow \cdots$$

$$\mathbf{Jet}[\mathcal{R}, \partial] := \lim\left(\mathcal{R} \xleftarrow{U} \mathcal{R}^{(1)} \xleftarrow{U^{(1)}} \mathcal{R}^{(2)} \xleftarrow{U^{(2)}} \cdots\right).$$

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the n^{th} element of which equips the $(n-1)^{\text{th}}$ with an algebra structure for $\partial^{(n)}$.

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$$X \xleftarrow{\xi} \partial X \xleftarrow{\xi'} \partial\partial X \xleftarrow{\xi''} \partial\partial\partial X \leftarrow \dots$$

Categorified jet spaces

Define the *k*-jet $J^k(\vec{X})$ of an object $\vec{X} \in \mathbf{Jet}[\mathcal{R}, \partial]$ as the image of $\vec{X}$ under the functor J^k obtained from the limit projections $\pi_k : \mathbf{Jet}[\mathcal{R}, \partial] \to \mathcal{R}^{(k)}$ as

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cf. differential geometry, where the k-jet of a real valued function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$(J^k_{x_0} f)(z) = \sum_{\ell=0}^{k} \frac{f^{(\ell)}(x_0)}{\ell!} z^\ell = f(x_0) + f'(x_0)z + \cdots + \frac{f^{(k)}(x_0)}{k!} z^k$$

An application to automata

Differential automata

Let $\mathcal{R}$ be a D2R; the assignment $(A, B) \mapsto \mathbf{Mly}(A \otimes -, B)$ defines a two-sided fibration via the Grothendieck construction

$$\underset{Mly:(A,B)\mapsto\mathbf{Mly}(A\otimes-,B)}{\mathbf{Psd}(\mathcal{R}^{\mathsf{op}} \times \mathcal{R}, \mathbf{Cat})} \underset{\sim}{\overset{}{\rightleftarrows}} \underset{(V:\mathbf{Mly}_{\mathcal{R}}\twoheadrightarrow\mathcal{R}^{\mathsf{op}}\times\mathcal{R})}{\mathbf{Fib}/(\mathcal{R}^{\mathsf{op}} \times \mathcal{R})}$$

which is a D2R morphism with respect to a canonical differential structure on the domain $\mathbf{Mly}_{\mathcal{R}}$.

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V is a fibration of trajectories for discrete dynamical systems of endpoints A, B; each category of trajectories $\mathbf{Mly}(A, B)$ has a limit (=terminal) object

$$\textstyle\prod_{n\geq 1}[A^{\otimes n}, B]$$

(Analogy: the limit set of a dynamical system $\overline{A}^f := \bigcap_{n\geq 1} \overline{f^n(A)}$, where f is en endomap of a metric space A.)

Differential automata

If $\mathcal{R}$ is monoidal closed, $\mathbf{Mly}_{\mathcal{R}}$ is a category of coalgebras for a certain endofunctor $R : \mathcal{R}^{\mathsf{op}} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}^{\mathsf{op}} \times \mathcal{R} \times \mathcal{R}$, fibred over the projection π_{12}

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There is a distributive law $\delta : (1 \times \partial)R \Rightarrow R(1 \times \partial)$

$$\begin{tikzcd}
\mathbf{Mly}_{\mathcal{R}} \arrow[r, "\bar{\partial}"] \arrow[d] & \mathbf{Mly}_{\mathcal{R}} \arrow[d] \\
\mathcal{R}^{\mathsf{op}} \times \mathcal{R} \arrow[r, "1\times\partial"'] & \mathcal{R}^{\mathsf{op}} \times \mathcal{R}
\end{tikzcd}$$

lifting ∂ to a derivative functor $\bar{\partial}$ on $\mathbf{Mly}_{\mathcal{R}}$; the category of differential automata is the category of coalgebras for such $\bar{\partial}$.

Differential automata

Let $\mathcal{R}$ be the D2R of species; observe that

- the species L of linear orders is the free monoid on the monoidal unit (plays the role of an NNO in **Spc**);
- thus there are four equivalent descriptions for the category of $\mathbf{Spc}^L$ of L-algebras, building block for $\mathbf{Mly}([1], B)$:
  - the category of algebras for the functor $[1] * -$;
  - the category of EM algebras for the monad $L \otimes -$;
  - the category of coalgebras for the functor ∂;
  - the category of coEM algebras for the comonad $\{L, -\}$.

Differential automata

Similar reasoning applies to scopic D2R, where ∂ has both a left and aright adjoint. There are plenty of variations on the theme of categories of species which are scopic D2Rs, e.g.:

- The category of S-species, i.e. functors $\mathbf{P}[S] \to \mathbf{Set}$ for an arbitrary set S; this supports partial derivatives, $\{\partial_s \mid s \in S\}$;
- k-vector (S-)species (Aguiar-Mahajan I,II,III,IV), i.e. functors $\mathbf{P}[S] \to \mathbf{Vect}_k$;

Differential automata

- linear species, i.e. families of functors of the form $X_n : [S_n/S_n] \to \mathbf{Set}$, where $[S_n/S_n]$ is the action groupoid of the regular representation of S_n on itself; (widely studied because differential equations admit unique solution here);
- Möbius species, where functors out of $\mathbf{P}[S]$ are valued in a category of posets with top and bottom (Möbius inversion formula has a category-theoretic proof);
- nominal sets, i.e. representations of the filtered colimit $S_1 \subset S_2 \subset S_3 \subset \dots$ of finite symmetric groups on the set of finite sets; (this is only a left scopic D2R; widely used in TCS).

Differential automata

There are examples

- of species having no ∂-coalgebra structures, but acquire many when linearized (i.e. considered as k-vector species instead of **Set**-species);
- of species having a finite number of ∂-coalgebra structures (precisely four);
- of species having uncountably many ∂-coalgebra structures.

(The fact that a coalgebra map must be S_n-equivariant is often a strong restriction on the structure of the coalgebra!)

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- these objects are highly structured ($\partial I \neq 0$, self-similarity,$\dots$)
- it's 'difficult' for a category to be a diff-2-rig ($Der(\mathcal{R})$ knows about a 'dimension' of $\mathcal{R}$)
- yet, differential algebra is quite interesting (differential equations?)