Tallinn University of Technology

Category Theory and its Applications – ITI9200

Exercise sheet 1

assigned: March 12, 2025
due: March 26, 2025

An intuitive representation of the slice category $\mathbf{Set}/A$ is the following: suppose that A is a finite set with r elements, a *colouring* of a set X consists of a function $c : X \to \{1, \dots, r\}$ (so that the elements of $X_i := c^{-1}(i)$ are the elements 'coloured' by colour $1 \le i \le r$); many problems in elementary enumerative combinatorics concern colourings of finite sets, and thus they can be stated through the slice category over $\{1, \dots, r\}$.

This exercises is meant to let you study *coloured graphs.* A (undirected, loop-free) graph $\mathcal{G}$ is made of a set of vertices V and a set of edges E, which can be thought as a set of 2-element subsets of V. If $\{v_1, v_2\} \in E$, we say that v_1, v_2 are *adjacent* in the graph $\mathcal{G}$.

Exercise 1:

Define an r*-colouring* of an undirected, loop-free graph $\mathcal{G} = (E, V)$ as a function $c : V \to \{1, \dots, r\}$, which assigns a colour $1 \le j \le r$ to each vertex V of $\mathcal{G}$, in such a way that two vertices v_1, v_2 connected by an edge are not of the same colour. Homomorphisms of coloured graphs *preserve the* r*-colouring* of domain and codomain, that is, they map vertices of one colour to vertices of the same colour.

Let $\mathcal{K}_r$ be the graph defined as follows (it's the *complete graph* on r vertices):

- the vertices of $\mathcal{K}_r$ are the elements of $\{1, \dots, r\}$;
- two vertices i, j are adjacent if and only if $i \neq j$.

► Prove that an r-colouring of a graph $\mathcal{G}$ corresponds precisely to the data of a graph homomorphism $\mathcal{G} \to \mathcal{K}_r$.

► Deduce that the category of coloured graphs is identified with the slice category $\mathbf{Gph}/\mathcal{K}_r$.

A binary operation of fundamental importance in the theory of categories is their *join*.

Exercise 2:

Given two categories $\mathcal{C}, \mathcal{D}$, the *join* of $\mathcal{C}$ and $\mathcal{D}$, denoted by $\mathcal{C} \star \mathcal{D}$, is the category defined as follows:

- the objects are the disjoint union of $\mathcal{C}_0$ and $\mathcal{D}_0$ (that is, the same objects as the sum $\mathcal{C} + \mathcal{D}$);
- the set of morphisms $(\mathcal{C} \star \mathcal{D})(X, Y)$ are specified by case splitting:
  - if $X = C, Y = C' \in \mathcal{C}_0$, then $(\mathcal{C} \star \mathcal{D})(X, Y) = \mathcal{C}(C, C')$;
  - if $X = D, Y = D' \in \mathcal{D}_0$, then $(\mathcal{C} \star \mathcal{D})(X, Y) = \mathcal{D}(D, D')$;
  - if $X = C \in \mathcal{C}_0$ and $Y = D \in \mathcal{D}_0$, there exists a *unique* arrow $u_{CD} : C \rightsquigarrow D$; such u_{CD} is called the *heteromorphism* connecting C and D;
  - if $X = D \in \mathcal{D}_0$ and $Y = C \in \mathcal{C}_0$, then $(\mathcal{C} \star \mathcal{D})(X, Y)$ is empty.

Composition is defined as in $\mathcal{C}$ (or as in $\mathcal{D}$) when both morphisms are in $\mathcal{C}$ (or in $\mathcal{D}$); composition with a heteromorphism results in a heteromorphism:

$$\begin{array}{ccc} C \longrightarrow C' \overset{u_{C'D}}{\rightsquigarrow} D & = & C \overset{u_{CD}}{\rightsquigarrow} D \\ C \underset{u_{CD}}{\rightsquigarrow} D \longrightarrow D' & = & C \underset{u_{CD'}}{\rightsquigarrow} D' \end{array}$$

► Show by induction that the composition of an n-tuple of morphisms in $\mathcal{C} \star \mathcal{D}$, of which at least one is a heteromorphism, is a heteromorphism (between the appropriate domain and codomain). Then prove that the join $\mathcal{C} \star \mathcal{D}$ is indeed a category.

► Determine:

- the join of two chains $\Delta[n] \star \Delta[m]$, if $\Delta[n]$ is the category $\{0 \to 1 \to \cdots \to n\}$;
- the join of two cubes $P[n] \star P[m]$, if $P[n]$ is the n-dimensional cube of subsets of $\{1, \dots, n\}$; more generally, can the join of two partially ordered sets P, Q be represented?
- the join of two discrete finite categories $A^\delta \star B^\delta$, and of two codiscrete categories $A^\chi \star B^\chi$: is the result still discrete, still codiscrete?
- the iterated join $\mathcal{B}(\mathbb{N}, +, 0) \star \mathcal{B}(\mathbb{N}, +, 0)$ of the additive monoid of natural numbers, and the iterated join $\mathcal{B}(\mathbb{N}, \max, 0) \star \mathcal{B}(\mathbb{N}, \max, 0)$ of the monoid under the maximum operation, $n \vee m := \max\{n, m\}$;[a]
- whether there is a relation between $(\mathcal{C} \star \mathcal{D})^{\mathrm{op}}$ and $\mathcal{C}^{\mathrm{op}} \star \mathcal{D}^{\mathrm{op}}$ (equal, opposite to each other, no relation...)?

[a] If M is a monoid, to avoid ambiguity, we denote by $\mathcal{B}M$ the category with a single object defined by M in the usual way.

Recall the construction of the right cone $S^{\triangleright}$ over a set S: it is the category whose objects are $S \cup \{\infty\}$ and whose arrows are the set $\{\lambda_s : s \to \infty \mid s \in S\}$. Similarly, $S^{\triangleleft}$ is the category whose objects are $S \cup \{-\infty\}$ and whose arrows are the set $\{\omega_s : -\infty \to s \mid s \in S\}$.

► Express $S^{\triangleright}$ and $S^{\triangleleft}$ as joins of two categories; describe $S^{\triangleleft} \star S^{\triangleright}$, $S^{\triangleleft} \star S^{\triangleleft}$, $S^{\triangleright} \star S^{\triangleright}$, $S^{\triangleright} \star S^{\triangleleft}$.

Combine the constructions of product, sum, slice, and coslice with the join construction:

► Can the join $(\mathcal{A} \times \mathcal{B}) \star \mathcal{C}$ be described in terms of $\mathcal{A} \star \mathcal{C}$, $\mathcal{B} \star \mathcal{C}$? How can $(\mathcal{C} + \mathcal{D})/X$ be described as X varies in $(\mathcal{C} + \mathcal{D})_0$? How can $(\mathcal{A}/A) \star (\mathcal{B}/B)$, $(A/\mathcal{A}) \star (B/\mathcal{B})$, $(\mathcal{A}/A) \star (B/\mathcal{B})$ be described? How can $(\mathcal{A} \star \mathcal{B}) + \mathcal{C}$ be described?

This exercise plays with three different representations of the category **Rel** of (finite) sets and relations.

Exercise 3:
Three versions of **Rel** (link).

Relations as spans. A relation between finite sets A, B consists of a subset R of the Cartesian product $A \times B$. As such, a relation $R \subseteq A \times B$ corresponds to a span

$$A \xleftarrow{a} R \xrightarrow{b} B$$

▶ What are $a : R \to A, b : R \to B$ if R is given as a subset of $A \times B$? Does a span of function having that shape always correspond to a relation, $R \to A \times B$, defined as the subset $(a(r), b(r)) \subseteq A \times B$?

Relations as cospans, or rather, as graphs. The cospan $c(R)$ associated with the relation R is defined as the category whose objects are the disjoint union of sets $A + B$, and where there is an arrow $s \to t$ if and only if $s \in A, t \in B$, and $(s,t) \in R$. This cospan

$$A \longrightarrow c(R) \longleftarrow B$$

is called the *graph* of R: to understand why,

▶ Draw the cospan $c(R)$ for simple relations between finite sets; fix a set X and characterize the properties of $c(R)$ for a relation $R \subseteq X \times X$ that is reflexive, symmetric, and transitive on X.

▶ What is the category $c(f)$ associated with a function, viewed as a total and single-valued relation?[a]

[a] Recall that a 'function' $f \subseteq A \times B$ is a relation f that satisfies the following property: for each $a \in A$, there exists a unique $b \in B$ such that $(a, b) \in f$.

▶ Given R as subset, what is *exactly* the relation between the span with tip R and the cospan with bottom $c(R)$? Are they interdefinable? Does one contain strictly more information than the other?

Relations as matrices. A relation $R \subseteq X \times Y$ can also be represented as a matrix whose entries are Boolean values: if $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$, then R consists of the matrix (r_{ij}) with entries 0 or 1, so that $r_{ij} = 1$ if and only if $(x_i, y_j) \in R$, and zero otherwise.

For the sake of concreteness, from now on let's just consider the special finite sets $[n] := \{1, \dots, n\}$ as objects.

For example, if $n = 4, m = 7$, the matrix

$$R = \begin{pmatrix} 0&1&1&0&1&1&0 \\ 1&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0 \\ 0&0&0&1&1&1&1 \end{pmatrix}$$

represents a relation $R : \{1, \dots, 7\} \to \{1, \dots, 4\}$: write it as a subset of $\{1, \dots, 7\} \times \{1, \dots, 4\} = \{(i, j) \mid 1 \le i \le 7, 1 \le j \le 4\}$; then depict its graph $c(R)$.

▶ Verify that the product (in the usual sense of linear algebra) of Boolean matrices corresponds to **composition** of relations; in other words, the matrix associated to the composition $S \circ R : \{1, \dots, m\} \xrightarrow{R} \{1, \dots, n\} \xrightarrow{S} \{1, \dots, p\}$ is precisely the matrix having entries

$$(S \circ R)_{ij} = \bigvee_{k=1}^{n} r_{ik} \wedge s_{kj}.$$ [a]

[a] Recall that $\wedge$ and $\vee$ are the usual Boolean operations of logical AND and OR:

$\wedge$	0	1
0	0	0
1	0	1

$\vee$	0	1
0	0	1
1	1	1

Pointwise operations between Boolean matrices are induced by said operations of logical AND, OR and NOT between Booleans; they are defined as follows:

- matrix OR: given $R, S : A \to B$ the matrix $R \vee S$ is defined as $(R \vee S)_{ij} = R_{ij} \vee S_{ij}$;
- matrix AND: given $R, S : A \to B$ the matrix $R \wedge S$ is defined as $(R \wedge S)_{ij} = R_{ij} \wedge S_{ij}$;
- matrix NOT: given $R : A \to B$, the matrix $\neg R$ is defined as $(\neg R)_{ij} = 0$ if and only if $R_{ij} = 1$, $(\neg R)_{ij} = 1$ if and only if $R_{ij} = 0$.

▶ Define all these operations as operations on relations-as-spans and relations-as-graphs.

▶ Are the following properties true for all Boolean matrices R, S, T (of course, taking them of the appropriate size when needed)? (Be mindful of what is true in linear algebra and what is false!)

- $(R \vee S) \wedge T = (R \wedge T) \vee (S \wedge T)$, and $R \vee (S \wedge T) = (R \vee S) \wedge (R \vee T)$.
- $(R \wedge S) \circ T = (R \circ T) \wedge (S \circ T)$, and $R \circ (S \vee T) = (R \circ S) \vee (R \circ T)$.
- $\neg(R \wedge S) = (\neg R) \vee (\neg S)$ and $\neg(R \vee S) = (\neg R) \wedge (\neg S)$.
- $(\neg R) \circ S = R \circ (\neg S)$.

A form of *Kronecker product* of Boolean matrices is also defined: given $R : A \to B$ and $S : X \to Y$ two Boolean matrices (note that the sizes of domain and codomain of

$R = (r_{ij}), S = (s_{pq})$ are possibly not related in any way) the Boolean matrix $R \otimes S : A \times X \to B \times Y$ is defined as follows:

$$R \otimes S = \begin{pmatrix} r_{11}S & \cdots & r_{1n}S \\ \vdots & \ddots & \vdots \\ r_{m1}S & \cdots & r_{mn}S \end{pmatrix} = \begin{pmatrix} r_{11}s_{11} & r_{11}s_{12} & \cdots & r_{11}s_{1q} & \cdots & \cdots & r_{1n}s_{11} & r_{1n}s_{12} & \cdots & r_{1n}s_{1q} \\ r_{11}s_{21} & r_{11}s_{22} & \cdots & r_{11}s_{2q} & \cdots & \cdots & r_{1n}s_{21} & r_{1n}s_{22} & \cdots & r_{1n}s_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ r_{11}s_{p1} & r_{11}s_{p2} & \cdots & r_{11}s_{pq} & \cdots & \cdots & r_{1n}s_{p1} & r_{1n}s_{p2} & \cdots & r_{1n}s_{pq} \\ \vdots & \vdots & & \vdots & \ddots & & \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots & & \ddots & \vdots & \vdots & & \vdots \\ r_{m1}s_{11} & r_{m1}s_{12} & \cdots & r_{m1}s_{1q} & \cdots & \cdots & r_{mn}s_{11} & r_{mn}s_{12} & \cdots & r_{mn}s_{1q} \\ r_{m1}s_{21} & r_{m1}s_{22} & \cdots & r_{m1}s_{2q} & \cdots & \cdots & r_{mn}s_{21} & r_{mn}s_{22} & \cdots & r_{mn}s_{2q} \\ \vdots & \vdots & \ddots & \vdots & & & \vdots & \vdots & \ddots & \vdots \\ r_{m1}s_{p1} & r_{m1}s_{p2} & \cdots & r_{m1}s_{pq} & \cdots & \cdots & r_{mn}s_{p1} & r_{mn}s_{p2} & \cdots & r_{mn}s_{pq} \end{pmatrix}$$

▶ Define the analogue of Kronecker product on relations-as-spans and relations-as-graphs.

▶ Are the following identities true for all Boolean matrices R, S, T (once again, taking them of the appropriate size when needed)?

- $(R \wedge S) \otimes T = (R \otimes T) \wedge (S \otimes T)$, and $(R \vee S) \otimes T = (R \otimes T) \vee (S \otimes T)$.
- $(R \circ S) \otimes (U \circ T) = (R \otimes U) \circ (S \otimes T)$, and $R \circ (S \otimes T) = (R \circ S) \otimes (R \circ T)$.
- $\neg(R \otimes S) = (\neg R) \otimes (\neg S)$.

Consider the following matrices:

$$B = \begin{pmatrix} 0&1&1&1&0&1&0 \\ 0&1&0&1&0&1&0 \\ 1&0&0&0&0&0&0 \\ 0&1&0&0&1&1&0 \end{pmatrix} \qquad P = \begin{pmatrix} 0&1&1&1&1 \\ 0&0&1&1&0 \\ 0&0&0&0&1 \\ 1&1&0&0&1 \\ 1&0&0&0&1 \end{pmatrix} \qquad Q = \begin{pmatrix} 1&1&1&1&0 \\ 1&0&0&1&0 \\ 1&0&0&1&0 \end{pmatrix}$$

Compute $A \wedge B$, $A \vee B$, $(\neg A)$; compute the powers $P \circ P, P \circ P \circ P, \dots, P^{\circ n}$ of P; compute $Q \circ P$; compute $B \otimes Q$ and compare it with $Q \otimes B$; draw their associated relations-as-graphs.

Given all this, how distant is the category of relations-as-Boolean-matrices **MRel** from the category $\mathbb{Z}/2\mathbb{Z}$-**Mat** having objects the natural numbers (i.e. the finite sets $[n] := \{1, \dots, n\}$) and morphisms $[n] \to [m]$ the $m \times n$ matrices with entries in the field with two elements? Can one define functors

$$\mathbf{MRel} \longrightarrow \mathbb{Z}/2\mathbb{Z}\text{-}\mathbf{Mat} \qquad \mathbb{Z}/2\mathbb{Z}\text{-}\mathbf{Mat} \longrightarrow \mathbf{MRel}$$

inverse to each other? Is every R 'linear'? How does one interpret the 'determinant' of a Boolean matrix $R : n \to n$? Etc.

Exercise 4:

On the distant planet Kobaïa, centuries ago, the inhabitants faced a challenge: how to effectively navigate ambiguity of language? After an era of profound intellectual curiosity, followed by three consecutive disastrous planetary wars, they turned to the only possible source of objectivity and peace —category theory.

After the reform operated by Emperor Horžtavak, standard Kobaïan contains three different registers of communication: *poetic*, *structural*, and *imperial.*

- **Poetic** Kobaïan contains nouns, adjectives, adverbs and verbs. It is willingly kept as morphologically simple as possible: verbs have no conjugations, there is marker to distinguish subject and object, there is no distinction of gender, no plural... In short, poetic Kobaïan is (approximated by what on planet Earth we call) a *closed, isolating language.*

- **Structural** Kobaïan opts for an intermediate solution, as it is used for objective, everyday communication; there is a difference between an animate and inanimate object, a prefix (*wö-*) transforms nouns into adjectives[1] and a prefix *â-* to turn adjectives into adverbs,[2] there is a system of pronouns[3]

- **Imperial** Kobaïan is reserved for communication with the *Lusztzess* (the Emperor) and the aristocracy of Kobaïa. It is a highly refined and complex language, designed to minimize ambiguity through the maximal use of morphological richness. There are 74 cases divided in six families, the family of **transrelative** cases contains, alone, the cases *oblique, absolutive, dative, ergative, effectuative, inducive, affective, instrumental, activative, derivative*, and *situative.* There are 7 tones (strictly ascending, mid-ascending, ascending, void, descending, mid-descending, strictly descending); the part of speech which corresponds to nouns and verbs in Earthling languages is called a formative: the generation of a formative follows the scheme

  $$(((\mathrm{Cv}\ +)\ \mathrm{VL}\ +)\ \mathrm{Cg/Cs}\ +)\ \mathrm{Vr} + (\mathrm{Cx/Cv} + \mathrm{Vp/VL}\ +)\ \mathrm{Cr} + \mathrm{Vc}\ (+\ \mathrm{Ci} + \mathrm{Vi}) + \\ + \mathrm{Ca}\ (+\ \mathrm{VxC}\ )\ \ [+\ \mathrm{tone}]\ [+\ \mathrm{stress}]$$

  where a central root **Cr** can be declined into a case **Vc** and other auxiliary or optional morphological markers can be added to mark

  - **Vr**: pattern (the **static** designation assigns *immutable* names, as opposed to the **contextual** pattern and the **temporary** pattern) and function (determines the function of the noun being specified: the four functions are *stative, dynamic, manifestive*, and *descriptive*).

[1] *Wötâ flöv zï wökoëhl.*: my hovercraft (*flöv*) is full of eels (lit.: *the wideboat of-I is eel-y*)
[2] *Hurt zanka föhr âföhr*: sunrise glows brightly (lit.: *first sun shines in a shiny manner*).
[3] *Tâ, nê, zî* correspond to *I,you,he/she/it*, and plural was formed by repeating the singular, *tâtâ, nênê, zîzî*, although there are attested shortened forms *tât, nên, zîz.*

– **Ca**: specifies one among nine possible **configurations** of the object defined (a single contextual unit embodying the stem concept; an aggregation of incoherent objects of diverse kind; a grouping or set of the basic stem units...), four possible **affiliations** (consolidative, associative, variative, and coalescent), four possible **perspectives** (monadic, unbounded, nomic, and abstract), six possible **extensions** (delimitive, proximal, inceptive, terminative, depletive, and graduative) and two possible **essences** (normal and representative).
– tone and stress: one of the seven tones and a stress (accent can fall on the ultimate, penultimate, or antepenultimate vowel).

Provide evidence that each dialect of Kobaïan behaves like a category $\mathbf{PKob}, \mathbf{SKob}$, and $\mathbf{IKob}$, with terms of the language as objects and grammatical transformations as morphisms. The usual way to do this (but there might be others, explore them) is to take a subset $L \subseteq A^\star$ of 'grammatically correct' words in an alphabet A (call it, evocatively, a *language* obtained from A) and then consider a relation $R \subseteq L \times L$ of 'production' of a word w into a word w'. *Qua* relation, R gives rise to a category taking its (reflexive and) transitive closure $R^\star := \Delta + R + R \circ R + \dots R \circ \dots \circ R + \dots$. For example, if N (Noun), D (aDjective) and V (Verb) are the names of three parts of speech, one can introduce a relation R through the generators

$$\begin{aligned} N &\to ND, & N &\to DN \\ N_1 &\to N_1VN_2, & N_2 &\to N_1VN_2 \end{aligned}$$

saying that from a token N (*flöv*, hovercraft) one obtains a token ND (*flöv wökoëhl*, hovercraft eel-y), and from nouns N_1, N_2 one obtains sentences N_1VN_2 by attaching N_1V- or -VN_2 to either the left or right.

Is it possible to build functors of 'poietization'

$$P : \mathbf{IKob} \longrightarrow \mathbf{PKob}$$

working as dictionary from imperial to poetic Kobaïan? Is it possible to define a functor that *creates* morphological features, i.e. an 'imperialization' procedure

$$I : \mathbf{PKob} \longrightarrow \mathbf{IKob}$$

to make artists and aristocrat communicate effectively?