Exercises ITI9200

FL

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1 Weeks 1-2

Exercise 1:
(*Light jumping jacks, but GySgt Hartman is behind you shouting "BOURBAKI!"*)

1. Prove or disprove that the following operations define monoid structures:
   - the set $\mathbf{R}^+ = \{x \in \mathbf{R} \mid x > 0\}$ of strictly positive real numbers, with respect to the operation of division, $(a, b) \mapsto a/b$;
   - the set of pairs of integers (m, n), with the operation $(p, q) \star (r, s) := (pr - qs, ps + qr)$.
2. Let S be a *finite* set, consider the monoid S^S of all functions $f : S \to S$, with respect to function composition. Prove that the following conditions are equivalent:
   - f is an injective function;
   - f is a surjective function;
   - f is a bijective function.

   This is *blatantly false* when S is infinite, say the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Build a counterexample.

Exercise 2:
(*Epimenides, Cantor, and Gödel enter a bar. . .* , [1])
An *applicative construct* (AC for short) $(A, \circ)$ consists of a nonempty set A with a binary operation $\circ : A \times A \to A$. If $(f, a) \in A \times A$, we denote $\circ(f, a)$ as $f \circ a$ and read 'f *applied to* a'.
If $(A, \circ)$ is an AC we say that

- $f \in A$ *has a fixpoint* $\mu_f \in A$ if $f \circ \mu_f = \mu_f$;
- $f \in A$ *has a diagonalizer* $\delta_f \in A$ if for every $a \in A$ the identity
$$\delta_f \circ a = f(a \circ a)$$
holds (brackets position is important).

Prove *Smullyan's mythological fixpoint theorem*:

If f has a diagonalizer δ_f, then it has a fixpoint μ_f.

Exercise 3:
(*'I know what a category is. . . ' —Show me.*)

- Can a category with 7 objects and 5 morphisms exist?
- Count how many categories with 3 objects and (exactly) 5 morphisms there are.

2 Weeks 3-4

Exercise 4:

(*You never put your hand in the same stream twice.* —Heraclitus, probably debugging a recursive function)

The category **Stream** has

- as objects the sets $A, B, C, \ldots$
- as arrows $f : A \rightsquigarrow B$ the functions of the form

$$f : \textstyle\sum_{n\geq 1} A^n \longrightarrow B$$

where the domain $A^+ = \sum_{n\geq 1} A^n$ is the set of *non-empty lists* of elements of A, i.e., the set

$$A + (A \times A) + (A \times A \times A) + \ldots$$

whose elements are ordered sequences of the form $(a_1, \ldots, a_n)$ for $n \geq 1$ and $a_i \in A$ for each $i = 1, \ldots, n$.[1]

You are invited to verify the category axioms for **Stream**:

- The identity morphisms are the functions $\sum_{n=1} A^n \to A$ defined by sending $(a_1, \ldots, a_n)$ to a_n;
- composition is given, if $f : A \rightsquigarrow B$ and $g : B \rightsquigarrow C$, by the rule $g \circ f : A \rightsquigarrow C$

$$(a_1, \ldots, a_n) \longmapsto g\big(f(a_1), f(a_1, a_2), \ldots f(a_1, \ldots, a_{n-1}), f(a_1, \ldots, a_n)\big).$$

In other words, the composition $g \circ f$ computes the output that the function g generates from the inputs $f(a_1), f(a_1, a_2), \ldots f(a_1, \ldots, a_{n-1}), f(a_1, \ldots, a_n)$.

The intuition to keep in mind is that an arrow $f \in \mathbf{Stream}(A, B)$ consists of an 'algorithm' that, given a non-empty list of inputs $(a_1, \ldots, a_n)$, computes an output $f(a_1, \ldots, a_n) \in B$ (which may clearly also depend on n), for every $n \geq 1$.

Exercise 5:

(*The endofunction $f : \boldsymbol{C} \to \boldsymbol{C}$ on the class $\boldsymbol{C}$ of mathematical objects sending 'X' to* 'category theory and X' *admits a fixpoint*)

Let's gather some definitions:

- A *ponoid* is a monoid $(M, \cdot, 1)$ equipped with a partial order $\leq$ such that

$$a \leq b \text{ and } x \leq y \implies ax \leq by$$

for all $a, b, x, y \in M$.
- An ordered set $(P, \leq)$ is called *directed complete* when all *directed* subsets (the nonempty subsets $S \subseteq P$ such that for every $x, y \in S$ there is an upper bound $u \in S$) admit a *least* upper bound.
- A monoid is said to have a *absorbing element* when there is an element z with $z = zx = xz$ for all $x \in M$.

Given this,

[1] More formally, A^+ is the *free semigroup* generated by the set A, where a 'semigroup' is a set equipped with an associative binary operation.

- Prove that an absorbing element in a monoid is unique when it exists;
- prove that if $(M, \cdot, 1)$ is a directed complete ponoid with $1 \leq x$ for all $x \in M$, then M has an absorbing element.

Use the latter result to prove that every monotone function $f : P \to P$ over a directed complete order has a smallest fixpoint (this means: there exists an $x \in P$ such that $f(x) = x$, and $x \leq p$ for every p such that $f(p) = p$).

Exercise 6:
(*The most merciful thing in the world, I think, is the inability of the human mind to correlate all its contents.* —HPL)

Recall the definition of the category of discrete dynamical systems (dds for short):

- objects are triples $(X, x_0; f)$ where (X, x_0) is a pointed set, and $f : X \to X$ an endofunction.
- morphisms $u : (X, x_0; f) \to (Y, y_0; g)$ are basepoint-preserving functions $u : (X, x_0) \to (Y, y_0)$ such that $g(ux) = u(fx)$ for every $x \in X$.

To every discrete dynamical $\boldsymbol{X} = (X, x_0; f)$ system one can associate the *exploded-view* category $\mathbf{EW}(\boldsymbol{X})$ having

- objects the elements $x, y, z, \dots \in X$;
- there is an arrow $\langle n \rangle : x \to f^n(x)$ for every $x \in X$ and $n \in \mathbb{N}$.

The identity i_x is $\langle 0 \rangle : x \to f^0 x = x$. Composition of morphisms is defined as $\langle n + m \rangle$, if $\langle n \rangle : x \to f^n x$ and $\langle m \rangle : f^n x \to f^m(f^n x) = f^{n+m} x$.

- prove the category axioms for $\mathbf{EW}(\boldsymbol{X})$;
- consider a subset $S \subseteq X$ in a dds $\boldsymbol{X} = (X, x_0; f)$ and define, inductively,

$$S^{(0)} := S \qquad S^{(k+1)} := \{f(s) \mid s \in S^{(k)}\}.$$

  The *flow* $\Phi_{\boldsymbol{X}}(S)$ of S is defined as $\bigcup_{k \geq 0} S^{(k)}$.

Let $\boldsymbol{N} = (\mathbb{N}, 0, c)$ be the dynamical system defined by $c : \mathbb{N} \to \mathbb{N}$,

$$c(2k) = k \qquad c(2k + 1) = 3k + 2$$

- Compute the flow $\Phi_{\boldsymbol{N}}(S)$ of $S = \{3, 9, 15, 39, 43\}$ (draw a picture); do you see a pattern?
- (very hard, do *not* attempt. Seriously, stay away from this problem.) can you find $S, T \subseteq \boldsymbol{N}$ such that $\Phi_{\boldsymbol{N}}(S) \cap \Phi_{\boldsymbol{N}}(T) = \varnothing$?

3 Weeks 5-6

Exercise 7:
Let $\mathcal{C}$ be a category. Define a relation $\preceq$ on its class of objects:

$$A \preceq B \iff \hom_{\mathcal{C}}(A, B) \neq \varnothing$$

Show that

- (ref) for all $A \in \mathcal{C}_0$, $A \preceq A$;
- (tns) for every $A, B, C \in \mathcal{C}_0$, if $A \preceq B$ and $B \preceq C$, then $A \preceq C$.

In other words, $(\mathcal{C}_0, \preceq)$ is an order relation on the class $\mathcal{C}_0$, called the *order reflection* of the category $\mathcal{C}$. Find a category $\mathcal{C}$ such that $(\mathcal{C}_0, \preceq)$ does not satisfy the *antisymmetric* property:

$$(\mathsf{asy}) \quad A \preceq B, B \preceq A \Rightarrow A = B.$$

Explain what property $\mathcal{C}$ has, *qua* category, when $(\mathcal{C}_0, \preceq)$ has a $\preceq$-minimum element, i.e. when

$$\text{there exists } A_\perp \in \mathcal{C}_0 \text{ such that } A_\perp \preceq X \text{ for all } X \in \mathcal{C}_0.$$

Exercise 8:
Prove or disprove the following statement: for every category $\mathcal{C}$ there is a functor

$$\mathcal{C} \longrightarrow \mathcal{C}^{\mathrm{op}}$$

sending an object X to itself, and a morphism $f : X \to Y$ to $f^{\mathrm{op}} : Y \to X$.

Exercise 9:
Call *thin* a category where there exists at most one arrow between any two objects.
Show that a category $\mathcal{C}$ is thin if and only if every functor with domain $\mathcal{C}$ is faithful.
Show that a category $\mathcal{C}$ is empty if and only if every functor with domain $\mathcal{C}$ is full.

References

[1] N. S. Yanofsky. A universal approach to self-referential paradoxes, incompleteness and fixed points. *Bulletin of Symbolic Logic*, 9(3):362–386, 2003.