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Research Statement

I consider myself a 'category theorist': an individual trained in basic mathematics, who gained some deeper understanding of its natural language, driven by the desire to reveal its fundamental patterns. Uninterested in muscular displays of computational prowess, a category theorist strives not to prove a theorem, but to 'explain why said theorem is obvious'.

This perspective shapes my attitude towards mathematics: I engaged with many different things, relishing them all, but without fixating on specialisation (the 'competent man' trope advocated by R. Heinlein is dear to my heart and I believe it applies to mathematics). I embody the spirit of a taxonomist, intrigued by the interplay of mathematical dialects converging into an intricate universal language. Through this lens, algebra, logic, geometry, topology, analysis, and mathematical physics appear strikingly similar, if not indistinguishable.

This does not imply a lack of interest or concern; quite the contrary, I am fairly excited to see how the same language can be used to describe phenomena and theories as different as number theory, topology/logic, fractals, combinatorics...

1. Overview of past research

1.1. Homotopy theory and algebraic topology.

I started my research as an algebraic topologist, since it seemed the most profound application of categorical language available at the time. Helped by my advisor D. Fiorenza, I was able to demonstrate that in the setting of stable ∞-categories the theory of *t-structures* introduced by Beilinson, Bernstein and Deligne in [BBD82] as a tool to study sheaves on stratified manifolds, is subordinated to a flexible and expressive calculus of *factorisation systems*.

- On one hand, factorisation systems are one of the oldest and most profitable assets of category theory: the first instance of such a structure was produced by Mac Lane in 1948 [ML48], and the notion later found countless applications in abstract algebra, logic, homotopy theory, higher-dimensional category theory.
- On the other hand, the theory of t-structures is a fundamental tool in modern algebraic geometry, and it is also a key ingredient in the theory of *mixed Hodge modules* [PS08], also having fundamental links with arithmetic geometry.

Linking these two apparently disconnected worlds has been the content of two papers [FL16, FLM19] co-authored with my advisor and later extended by one of his MSc students, G. Marchetti. A thorough analysis of t-structures on stable ∞-categories constituted my PhD thesis [Lor16].

1.2. Formal category theory.

For the subsequent five years, my research has been motivated by the desire to understand better how categorical language can unveil the deep behaviour of mathematical structures needed in algebraic topology.

The central theorem of my PhD thesis was later generalised to the setting of a generic triangulated category in joint work with S. Virili [LV20]; this result can only be completely understood in terms of the theory of *Grothendieck derivators*, an inherently 2-categorical approach to homotopy theory. From there, I turned my attention to *formal category theory* [SW78, Woo82, Web07, Gra74].

From that moment on, I was lured into what I consider the most elegant approach to category theory, providing deep insights on categorical logic and algebra [Gui82, Web16, Gui75].

This line of research led to [LL23, DLL19, ALL23], where we try to shed some light on fundamental aspects of category theory, e.g. the theory of accessible and presentable categories, generalised adjoint functor theorems, and the relation between profunctors and the Yoneda lemma. The first paper was published in Journal of Pure and Applied Algebra.

1.3. **Coend calculus.** For me, one of the most useful concepts in category theory is that of a *coend*. During my PhD I redacted a note on the subject that slowly gained popularity among students and eventually became a book [Lor21a] edited by Cambridge University Press in 2021.

The book is a collection of results on coends, and it is intended to put together coherently a wealth of material that has so far been confined to research papers or survey reports in a way accessible to anyone with basic knowledge of category theory. It is also a collection of examples of how coends can be used to conduct formal proofs in category theory (using the 'calculus of coends' in the title), and how they can be used to express categorical concepts concisely and elegantly.

For a very long time I had the impression that one can turn the act of 'conducting proofs in category theory using coend calculus' into a completely formal statement. Together with A. Laretto, the PhD student I am co-advising, we are in the process of making this precise, [LLV24].

My experience with coend calculus was decisive in writing the joint work [CEG+20]. After an extremely long editorial process, the paper was accepted by Compositionality, and along the past four years became a go-to reference among users of category-theoretic approaches to modular data accessors in functional programming.

Together with D. Palombi and F. Genovese, we managed to apply coend calculus to the foundations of trading protocols in a trustless environment; our [GLP21b], where we describe how *escrows* can be mathematically described as a certain subclass of the same 'profunctor optics' of [CEG+20], was recently accepted by the Journal of Financial Technology.

2. Overview of present research

2.1. **Categorical semantics of Petri nets.** My collaboration with Genovese and Palombi did not stop there. Together, we published a series of papers on the categorical foundations of Petri nets.

- in [GLP21c] we define a new flavour of Petri nets where transitions can only fire a limited number of times; said transitions have a lifespan after which they are unable to fire again; the paper was published in ICGT 2021.
- In [GLP22] we provide a categorical semantics for when a given Petri net is *bounded*, meaning that, starting from a given marking, no place will hold more than a predetermined number of tokens throughout any possible firing; the paper was published in ACT 2021.
- In [GLP21a] we study a particular variety of hierarchical nets, where the firing of a transition in the parent net must correspond to an execution in some child net; in simple terms, Petri nets are *nested*; the paper was published in GCM 2021.

2.2. **Categorical automata theory.** This fruitful experience taught me that the theory of state machines is an interesting toy for a category theorist: the French school wrote extensively on the subject [Gui78, Bai75, Gui80, Gui74], and some subsequent work followed [BK81, KR90, KKR83], motivating key concepts as that of a quantaloid or of a category enriched over a bicategory.

Guitart and Bainbridge's style is, to put it mildly, lacking in clarity; the proofs are often omitted, or completely obscure when sketched; it's Bourbaki's spirit haunting applied mathematics. I took the challenge to revive this language, taking its fundamental implication (that a piece of formal category theory *coincides* with the mathematical foundation of 'abstract state machines', intended *à la* [EKKK, PA70, AM75b, AM75a]) to its natural consequences, trustful that this would have been a fruitful endeavour.

And fruitful it was: this project is the one that I am currently most involved in. Together with A. Laretto, the PhD student I am co-advising, and gradually attracting other people, we started a stream of research which is currently dealing with the following projects:

- in [BLLL23a] we study how deterministic (=Mealy) automata organise into a bicategory, building on previous work of Katis, Sabadini and Walters; we link their bicategory of 'processes' to a bicategory of Mealy machines constructed in 1974 by R. Guitart. Then, we define Mealy and Moore machines inside a bicategory $\mathbb{B}$, specialising to various choices of $\mathbb{B}$ like categories, relations, and profunctors. This paper has been accepted for ACT 2023.
- in [BLLL23b] we provide a slick proof of completeness and cocompleteness for categories of generalised automata in the sense of [AT90]; a pleasant selling point of our work is its certification: we mechanised some of our main results in the proof assistant Agda. This paper has been accepted for CALCO 2023.
- in [BFL$^+$23] we study the *semibicategory* of Moore automata: a structure that is like a bicategory but lacks identity 1-cells. We show that such a structure is far from pathological, providing a general way to build local adjunctions between said semibicategory and the genuine bicategory of Mealy automata studied in our previous [BLLL23a].
- in [Lor24] I study Mealy (and to a minor extent, Moore) monoidal automata where the base monoidal category is that of Joyal's *combinatorial species*. Species are my favourite category, and the theory of *differential 2-rigs* [LT21] intertwines automata theory at a deep level (which I hope to unveil in the next future).
- in [Lor25a] I study monads in a *double category* of Mealy automata, not (only) for the fun of it, but because one can characterize them as a gadget well-known to representation theorists (especially in the area of Hopf algebra theory).
- In [Lor25b] I have extended this idea that double categories are useful to study automata, and I studied a categorification of transducers [**?**, **?**] organized in a double category of *two-dimensional transducers*; there's plenty of reasons to study this gadget if you're a category theorist!

3. Plans for future research

Together with my co-authors, we are currently actively working on writing an unabridged version of [BFL$^+$23], and in joint work with D. Castelnovo, we plan to provide a proof of the celebrated Krohn-Rhodes theorem derived from first principles of the theory of 2-categories and formal enough to be easily extended to a larger class of bicategories of automata.

The experience of B. Femić with higher categories has been decisive in formulating the exciting conjecture that bicategories of Mealy automata enjoy properties similar to a free completion (more specifically, completions under Eilenberg-Moore objects); this is in line with, and a potent generalisation of, prior work by B. Paré [Par10]. We are currently working on a paper that will provide an extensive description and applications of this result.

An exciting open problem from [BLLL23a] is the following: given a monad T on Set and a quantale $\mathcal{V}$ [EGHK18, Ch. 2] we can define the locally thin bicategory of $(T, \mathcal{V})$-profunctors as in [HST14, Ch. III]; as the pair $(T, \mathcal{V})$ varies, one can recover a plethora of bicategories, yielding the categories of topological spaces, approach spaces [Low97], metric and ultrametric, closure spaces. . . as the $(T, \mathcal{V})$-*categories* of [HST14, §III.1.6]. When instantiated in $(T, \mathcal{V})$-profunctors, the general theory of [BLLL23a, §3] a 2-categorical way to look at topological, metric and loosely speaking 'fuzzy' approaches to automata theory.

Together with T. Trimble, I co-authored a paper on the theory of *differential 2-rigs* [LT21]. In simple terms, a differential 2-rig is a monoidal category $\mathcal{R}$ with coproducts, such that all functors $A \otimes \_, \_ \otimes B$ preserve sums and equipped with a *differential endofunctor* $\partial : \mathcal{R} \to \mathcal{R}$ preserving sums and satisfying the Leibniz property $\partial(A \otimes B) \cong \partial A \otimes B + A \otimes \partial B$.

I started studying this new concept in complete solitude and attracted Todd when a preliminary version of my work was circulating inside the community. This line of research has immense potential to generalise some parts of Joyal's theory of combinatorial species, and it might provide an unexpected connection between the theory of polynomial functors, differential algebra, monoidal categories, and combinatorial differential equations.

We are currently working on a paper that will provide a foundation for the theory of differential 2-rigs. An open problem in this direction that also intersects categorical automata theory is the following: let $(\mathcal{R}, \otimes, \partial)$ be a differential 2-rig. We can now define the category of ∂-automata following [AT90] and study its features (for example, [BLLL23b] yields at once that the categories of ∂-Mealy and ∂-Moore automata on the category of species are complete and cocomplete –actually, locally finitely presentable), with particular attention to objects that 'solve' differential equations in $\mathcal{R}$: call an object $A \in \mathcal{R}$ a *lax Napier object* if it is a ∂-coalgebra $a : A \to \partial A$, and a *pseudo-Napier object* if it is equipped with an isomorphism $A \cong \partial A$ (so in particular, the terminal ∂-coalgebra is pseudo-Napier, provided it exists).

4. Miscellaneous plans

A few of my plans for the future are:

- I'd like to publish all my old preprints on stable homotopy theory [Lor18, FL15]. I think they contain some ideas worth publishing, but algebraic topology does not pertain to me any more, and at this point in my life, I take this as a side project.
- I have a decent experience in the theory of factorisation systems; together with coends I think it's a piece of category theory that deserves a dedicated monograph, and never had one despite its usefulness. I have a few ideas on how to write such a book.
- I am interested in 'peripheral' ways to apply category theory; for example, in the foundations of living system theory [Lor21b] or in philosophy. Both

communities could use an experienced mathematician, and especially the former line of research widely intersects categorical automata theory: cf. [Bai73, War82, FB96, CLG+10, MLS09].

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