

OF LIMS AND SETS

FOSCO LOREGIAN

1. REDUCTION OF LIMITS TO PRODUCTS AND EQUALIZERS

Definition 1.1 (Some terminology). A (small) *diagram* in a category \mathcal{C} is a functor

$$\mathcal{J} \longrightarrow \mathcal{C} \quad (1.1)$$

whose domain is a small category. A *cone* for a diagram D consists of a pair (X, c) where X is an object of \mathcal{C} called *tip* of the cone, and c is a natural transformation $\Delta X \Rightarrow D$ from the constant functor at X ; so c consists of a family of arrows in \mathcal{C} ,

$$c_j : X \longrightarrow D_j \quad (1.2)$$

the *components* of the cone, such that for every morphism $f : i \rightarrow j$ in \mathcal{J} the triangle

$$\begin{array}{ccc} & X & \\ c_i \swarrow & & \searrow c_j \\ Di & \xrightarrow{Df} & Dj \end{array} \quad (1.3)$$

is commutative. The category of cones for D has

- objects the cones (X, c) for D ;
- arrows $h : (X, c) \rightarrow (Y, c')$ the arrows $h : X \rightarrow Y$ in \mathcal{C} , between the tips of the cones such that for every $j \in \mathcal{J}$ the triangle

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ c_j \searrow & & \swarrow c'_j \\ & Dj & \end{array} \quad (1.4)$$

The limit $(\lim D, p_j)$ of a diagram D consists of a terminal object in its category of cones. More than often one calls ‘limit of D ’ the tip of the terminal cone, leaving the maps of the cone implicit *this is almost always harmless but slightly incorrect: the limits is composed of **both** parts*.

If a diagram D has a limit $(\lim D, p_j)$ we say that \mathcal{C} *admits* the limit of D ; if for a fixed \mathcal{J} , every $D : \mathcal{J} \rightarrow \mathcal{C}$ has a limit, we say that \mathcal{C} has limits of shape \mathcal{J} or that it has \mathcal{J} -limits; if for every element \mathcal{J} of a subclass $\Phi \subseteq \mathbf{Cat}$ of categories, \mathcal{C} has limits of shape \mathcal{J} , we say that \mathcal{C} has limits of shape Φ or that it has Φ -limits. If \mathcal{C} has \mathbf{Cat} -limits, we say that \mathcal{C} is (small-)complete.

Definition 1.2. In particular, a category \mathcal{C} has all products if it has all limits of shape $S^\delta \rightarrow \mathcal{C}$ when S^δ is the discrete category over a set S , and \mathcal{C} has equalizers if it has limits over $\mathcal{J} = \{0 \rightrightarrows 1\}$: this is because

- a diagram of shape S^δ specifies precisely a family of objects X_s in \mathcal{C} , one for every $s \in S$; being the terminal cone in this case means that there exists a family of arrows $p_s : \prod_s X_s \rightarrow X_s$ indexed by S with the property that
- a diagram of shape $\{0 \rightrightarrows 1\}$ specifies precisely a pair of morphisms $f, g : D_0 \rightarrow D_1$ in \mathcal{C} ; being the terminal cone in this case means that there exists.

Theorem 1.3. The category **Set** of sets and functions has all products and all equalizers.

Proof.

- the product of a family of sets $\{X_s \mid s \in S\}$ is the usual Cartesian product $\prod_{s \in S} X_s$, constructed as the set of functions $S \rightarrow \bigcup_s X_s$ with the property that $f(s) \in X_s$. This allows to represent the elements of the set $\prod_{s \in S} X_s$ as S -indexed sequences $(x_s \mid s \in S, x_s \in X_s)$. Evidently, $\prod_{s \in S} X_s$ is equipped with projection maps $p_s : \prod_{t \in S} X_t \rightarrow X_s$ for every $s \in S$, picking the s th element of the S -sequence $(x_s \mid s \in S)$.

The universal property of the product $\prod_{s \in S} X_s$ is spelled as follows:

For every set Z and family of functions $z_s : Z \rightarrow X_s$, there exists a unique $\bar{z} : Z \rightarrow \prod_{s \in S} X_s$ such that $p_s \circ \bar{z} = z_s$.

Define \bar{z} to be the function sending $\zeta \in Z$ to the S -sequence $(z_s \zeta \mid s \in S)$. Clearly this is the only possible definition so that

$$\begin{array}{ccc} & \prod_{s \in S} X_s & \\ \nearrow \bar{z} & \downarrow p_s & \\ Z & \xrightarrow{z_s} & X_s \end{array} \quad (1.5)$$

is a commutative triangle for every $s \in S$.

- the equalizer of a pair of maps $f, g : X \rightarrow Y$ consists of the subset $E = \{x \in X \mid fx = gx\} \subseteq X$; it realizes the universal property

For every $u : Z \rightarrow X$ such that $f \circ u = g \circ u$, there exists a unique $\bar{u} : Z \rightarrow E$ such that u equals the composition $Z \rightarrow E \hookrightarrow X$.

Since E is just a subset of X , the universal property of $\text{eq}(f, g)$ can be rephrased as follows: every $u : Z \rightarrow X$ such that $f(u(z)) = g(u(z))$ for every $z \in Z$ takes values in the subset E , defined above. This is evident, as much as it is evident that E is chosen precisely in order to satisfy this property. \square

Lemma 1.4.

If $E \xrightarrow{e} A$ is an equalizer of $A \xrightleftharpoons[f]{g} B$, then the following are equivalent:

- (1) $f = g$,
- (2) e is an epimorphism,
- (3) e is an isomorphism,
- (4) id_A is an equalizer of f and g .

\square

Theorem 1.5. The category **Set** of sets and functions has all limits.

Proof. Let \mathcal{D} be a small category and $F : \mathcal{D} \rightarrow \mathbf{Set}$ a functor. For every arrow f in \mathcal{D} , we denote $s(f)$ the source and $t(f)$ the target of f ; so, if $f : D \rightarrow D'$, $s(f) = D$, $t(f) = D'$.

We prove that the limit $\lim F$ of F is precisely the equalizer of the pair of maps

$$\prod_{D \in \mathcal{D}_0} FD \xrightarrow[\beta^F]{\alpha^F} \prod_{f \in \mathcal{D}_1} F(t(f)) \quad (1.6)$$

where

- $\alpha^F((x_D \mid D \in \mathcal{D})) = (x_{t(f)} \mid f \in \mathcal{D}_1)$;
- $\beta^F((x_D \mid D \in \mathcal{D})) = (Ff(x_{s(f)}) \mid f \in \mathcal{D}_1)$;

This means two things: if $\lim F$ exists, then it must be the equalizer of that pair; otoh, if that pair (α, β) has an equalizer, then such is the limit of F .

We have to prove that

- (1) There exists a cone

$$\lim F \xrightarrow{\bar{p}} \prod_{D \in \mathcal{D}_0} FD \xrightarrow[\beta]{\alpha} \prod_{f \in \mathcal{D}_1} F(t(f)) \quad (1.7)$$

where e equalizes the pair α, β ;

- (2) such cone is terminal; this will mean two things:

- the universal property of $\lim F$ entails the universal property of $\text{eq}(\alpha, \beta)$;
- the universal property of $\text{eq}(\alpha, \beta)$ entails the universal property of $\lim F$.

Thus, there is a unique isomorphism $\text{eq}(\alpha, \beta) \cong \lim F$.

Proving 1. is easy; if $(\lim F, p_D)$ exists, all projections $p_D : \lim F \rightarrow FD$ assemble into a unique map $\bar{p} : \lim F \rightarrow \prod_D FD$ (this is the universal property of $\prod_D FD$). Note in passing that by the lemma above if \mathcal{D} is a discrete category, $\prod_{f \in \mathcal{D}_1} F(t(f)) = \prod_{D \in \mathcal{D}_0} FD$, α, β are invertible and thus $\lim F \cong \prod_{D \in \mathcal{D}_0} FD$, as it should be.

Now, $\bar{p} : \lim F \rightarrow \prod_D FD$ equalizes (α, β) , because the components p_D form a cone: the triangle of sets and functions

$$\begin{array}{ccc} & FD & \\ p_D \nearrow & \downarrow Ff & \\ \lim F & & FD' \\ p_{D'} \searrow & & \end{array} \quad (1.8)$$

is commutative, whence the fact that for all $\hat{x} \in \lim F$ and $f : D \rightarrow D'$ in \mathcal{D}_1 one has

$$\beta^F(\bar{p}(\hat{x})) = (Ff(p_{s(f)}(\hat{x})) \mid f \in \mathcal{D}_1) = (p_{t(f)}(\hat{x}) \mid f \in \mathcal{D}_1) = \alpha^F(\bar{p}(\hat{x})). \quad (1.9)$$

A similar argument for a general cone $(z : Z \rightarrow FD \mid D \in \mathcal{D}_0)$ proves that this is a cone for F if and only if it equalizes (α, β) ; thus, a cone for F must be a terminal cone wrt the property of equalizing (α, β) ; whence the conclusion. \square

More generally, for a category \mathcal{C} to have all limits it is necessary and sufficient that it has all (small) products and equalizers.

Theorem 1.6. The following conditions are equivalent:

- \mathcal{C} has all small products and all equalizers;
- \mathcal{C} is small-complete.

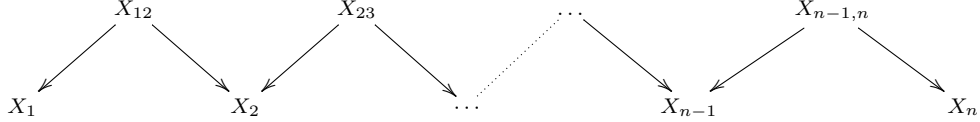
Proof. □

Exercise 1.7. The *triquazer* $\text{triq}(f, g, h)$ of functions $f, g, h : X \rightarrow Y$ is defined as the limit of the diagram

$$\mathcal{J} = \left\{ 0 \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \\ \xrightarrow{h} \end{array} 1 \right\} \rightarrow \mathbf{Set}. \quad (1.10)$$

- Spell out the universal property of $\text{triq}(f, g, h)$;
- by virtue of Theorem 1.6 above, $\text{triq}(f, g, h)$ must be expressible as an equalizer of two maps between products. How?

Exercise 1.8. An *Urizen compass* \mathbf{G}_n is a limit of a diagram of the following form:



write the universal property for \mathbf{G}_3 and \mathbf{G}_4 ; express \mathbf{G}_3 and \mathbf{G}_4 as equalizers of maps between products.

2. SYMMETRY AND ASSOCIATIVITY, INTERCHANGE OF LIMITS, YONEDA AND LIMITS

The universal property of a limit entails that

2.1. Functoriality of limits.

Theorem 2.1. A natural transformation $\alpha : D \Rightarrow D'$ in the category of functors $D : \mathcal{J} \rightarrow \mathbf{Set}$, induces a morphism $\lim \alpha : \lim D \rightarrow \lim D'$ between the limits. Moreover, \lim preserves identities and compositions, hence it is a functor

$$\lim : [\mathcal{J}, \mathbf{Set}] \longrightarrow \mathbf{Set} \quad (2.1)$$

The natural transformation α has components $Dj \rightarrow D'j$, and the composition $\lim D \xrightarrow{p_j} Dj \xrightarrow{\alpha_j} D'j$ is easily seen to be a cone for D' . Then, there is a unique $\bar{\alpha} : \lim D \rightarrow \lim D'$ such that each diagram

$$\begin{array}{ccc} \lim D & \xrightarrow{\bar{\alpha}} & \lim D' \\ p_j \downarrow & & \downarrow p'_j \\ Dj & \xrightarrow{\alpha_j} & D'j \end{array} \quad (2.2)$$

commutes. Uniqueness proves that $\overline{\beta \circ \alpha} = \bar{\beta} \circ \bar{\alpha}$ and $\overline{\text{id}_D} = \text{id}_{\lim D}$.

2.2. Yoneda and limits. One can use Yoneda lemma to express the universal property of limits; given a diagram $D : \mathcal{J} \rightarrow \mathcal{C}$, one can express the property

For all $A \in \mathcal{C}_0$, there is a (canonical, natural) isomorphism

$$\mathcal{C}(A, \lim_{j:\mathcal{J}} D_j) \cong \lim_{j:\mathcal{J}} \mathcal{C}(A, D_j) \quad (2.3)$$

(where on the right hand side we mean the limit of the composite

$$\begin{array}{ccc} \mathcal{J} & \longrightarrow & \mathcal{C} \longrightarrow \mathbf{Set} \\ j & \longmapsto & D_j \longmapsto \mathcal{C}(A, D_j) \end{array} \quad (2.4)$$

which we know exists in the category of sets) as the representability of the functor

$$\begin{array}{ccc} \lim \mathcal{C}(-, D_j) : \mathcal{C}^{\text{op}} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \lim_{j:\mathcal{J}} \mathcal{C}(A, D_j) \end{array} \quad (2.5)$$

Yoneda says that it is enough to know how limits are computed in \mathbf{Set} (and in a category of functors into \mathbf{Set}) to define them uniquely (=up to unique iso) in any category \mathcal{C} , via the notion of representability; the ‘limit’ $(\lim D, p_j)$ of a diagram $D : \mathcal{J} \rightarrow \mathcal{C}$ consists of a terminal object in the category of elements of $\lim \mathcal{C}(-, D_j)$ (which is *precisely* the category of cones for D ;-), while the cone

$$p_j : \lim D \longrightarrow D_j \quad (2.6)$$

is the universal element: it corresponds to the identity of $\lim D$ in the bijection

$$\mathcal{C}(\lim D, \lim D) \cong \lim \mathcal{C}(\lim D, D_j) \quad (2.7)$$

for $A = \lim D$.

Reading (2.3) as the isomorphism

$$y(\lim D)(A) \cong (\lim yD)(A) \quad (2.8)$$

one can also argue that y (the Yoneda embedding) *preserves* all limits:

Definition 2.2. Let $D : \mathcal{J} \rightarrow \mathcal{C}$ be a diagram; let $F : \mathcal{C} \rightarrow \mathcal{X}$ be a functor; let $\lim D$ denote the limit of D and $\lim(FD)$ the limit of the composite functor $F \circ D : \mathcal{J} \rightarrow \mathcal{X}$; then, the functoriality of F defines a cone

$$\gamma_{F,j} : F(\lim D) \longrightarrow FD_j \quad (2.9)$$

whence a unique morphism, in \mathcal{X} , $\gamma_F : F(\lim D) \rightarrow \lim(FD)$. We say that F *preserves* $\lim D$, or that it *commutes* with $\lim D$ if γ_F is an isomorphism in \mathcal{X} .

Proposition 2.3. Limits in the category $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ are computed objectwise in \mathbf{Set} : this means that given a diagram

$$D : \mathcal{J} \longrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}] \quad (2.10)$$

its limit $\lim D$ is a functor $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ defined as $(\lim_j D_j)(C) = \lim_j (D_j C)$.

Remark 2.4. The yoneda embedding

$$y : \mathcal{C} \longrightarrow [\mathcal{C}^{\text{op}}, \mathbf{Set}] \quad (2.11)$$

preserves all limits.

Evidently, F preserves all limits if and only if it preserves all products and all equalizers; it’s easy to show that y preserves products and equalizers directly from the definition.

2.3. Limits commute with limits. The following argument will work for any category \mathcal{C} be a category admitting products and equalizers, but we will spell it out just for sets; given a S -indexed family of parallel arrows

$$X_s \begin{array}{c} \xrightarrow{f_s} \\ \xrightarrow{g_s} \end{array} Y_s \quad (2.12)$$

one can consider the parallel maps induced between the products using Theorem 2.1,

$$\prod_s X_s \begin{array}{c} \xrightarrow{(\prod_s f_s)} \\ \xrightarrow{(\prod_s g_s)} \end{array} \prod_s Y_s. \quad (2.13)$$

The map $(\prod_s f_s)$ is defined as the unique with the property that for every $s' \in S$, the square

$$\begin{array}{ccc} \prod_s X_s & \xrightarrow{(\prod_s f_s)} & \prod_s Y_s \\ p_{s'}^X \downarrow & & \downarrow p_{s'}^Y \\ X_{s'} & \xrightarrow{f_{s'}} & Y_{s'} \end{array} \quad (2.14)$$

commutes. Similarly, for $(\prod_s g_s)$.

Theorem 2.5. The equalizer of (2.13) above is the product $\prod_s h_s : \prod_s E_s \rightarrow \prod_s X_s$ of the equalizers $(E_s \xrightarrow{h_s} X_s \rightrightarrows Y_s \mid s \in S)$. In other words,

$$\lim_{0 \rightrightarrows 1} (\lim_{S^\delta} D) \cong \lim_{S^\delta} (\lim_{0 \rightrightarrows 1} D). \quad (2.15)$$

A single function $v : Z \rightarrow \prod_s X_s$ corresponds to a family of functions $(v_s : Z \rightarrow X_s \mid s \in S)$, and the fact that $(\prod_s f_s) \circ v = (\prod_s g_s) \circ v$ is exactly equivalent to the fact that putting $v_s = p_s \circ v$

$$\forall s \in S. (f_s \circ v_s = g_s \circ v_s) \quad (2.16)$$

Indeed,

- if $(\prod_s f_s) \circ v = (\prod_s g_s) \circ v$, then for all $s \in S$

$$p_s^Y \circ (\prod_s f_s) \circ v = p_s^Y \circ (\prod_s g_s) \circ v \quad (2.17)$$

but the LHS of this equation is $f_s \circ p_s^X \circ v$, and RHS is $g_s \circ p_s^X \circ v$.

- Conversely, if $\forall s \in S. (f_s \circ v_s = g_s \circ v_s)$, then one uses uniqueness: for all $s \in S$,

$$p_s^Y \circ (\prod_s f_s) \circ v = f_s \circ p_s^X \circ v = g_s \circ p_s^X \circ v = p_s^Y \circ (\prod_s g_s) \circ v \quad (2.18)$$

but there exists a *unique* $w : Z \rightarrow \prod_s Y_s$ such that for all $s \in S$ one has that $p_s^Y \circ w$ equals the common value $f_s \circ p_s^X \circ v = g_s \circ p_s^X \circ v$, hence $w = (\prod_s f_s) \circ v = (\prod_s g_s) \circ v$.

Now, given this, each v_s equalizes (f_s, g_s) , and thus factors through the equalizer $Z \xrightarrow{\bar{v}_s} E_s \xrightarrow{h_s} X_s$ so that $v_s = h_s \circ \bar{v}_s$; by the universal property of $\prod_s E_s$, there is

a unique map $\bar{v} : Z \rightarrow \prod_s E_s$, which must be so that $\prod_s h_s \circ \bar{v} = v$:

$$\begin{array}{ccc} \prod E_s & \xrightarrow{\prod h_s} & \prod X_s \xrightleftharpoons[(\prod g_s)]{(\prod f_s)} \prod Y_s \\ & \nwarrow \bar{v} & \uparrow v \\ & & Z \end{array} \quad (2.19)$$

2.4. Unitality and associativity of products. One can use the universal property of products to exhibit natural isomorphisms

$$A \times 1 \cong A \cong 1 \times A \quad (A \times B) \times C \cong A \times (B \times C) \quad A \times B \cong B \times A \quad (2.20)$$

Indeed, we just have to show that A has the same universal property of $A \times 1$, that $B \times A$ has the same universal property of $A \times B$, etc.

- A comes equipped with two projections

$$1 \longleftarrow A \Longrightarrow A \quad (2.21)$$

which satisfy the universal property $1 \times A$ is required to have.

- Consider the universal problem

$$\begin{array}{ccccc} & & B \times A & & \\ & p_A \swarrow & \vdots & \searrow p_B & \\ A & \xleftarrow{p_A} & A \times B & \xrightarrow{p_B} & B \end{array} \quad (2.22)$$

It must have a unique solution $\sigma = \langle p_B, p_A \rangle$. Uniqueness of $\text{id}_{A \times B}$ solving the universal problem

$$\begin{array}{ccccc} & & A \times B & & \\ & p_B \swarrow & \vdots & \searrow p_A & \\ A & \xleftarrow{p_A} & A \times B & \xrightarrow{p_B} & B \end{array} \quad (2.23)$$

implies that $\sigma \circ \sigma = \text{id}_{A \times B}$.

- Associativity uses diagram chasing at its full potential:

$$\begin{array}{ccccc} & & (A \times B) \times C & & \\ & p_{A \times B} \swarrow & \searrow \langle p_B p_{A \times B}, p_C \rangle & & \\ & A \times B & & B \times C & \\ p_A \swarrow & & \searrow p_B & \nearrow p_{B \times C} & \searrow \\ A & \xleftarrow{\langle p_A p_{A \times B}, \langle p_B p_{A \times B}, p_C \rangle \rangle} & A \times (B \times C) & \xrightarrow{p_B} & B \\ & \nwarrow p'_A & & & \searrow \\ & & & & C \end{array} \quad (2.24)$$

the blue arrow $\langle p_B p_{A \times B}, p_C \rangle$ is induced by its arguments $p_B p_{A \times B}, p_C$; similarly for the red arrow $\langle p_A p_{A \times B}, \langle p_B p_{A \times B}, p_C \rangle \rangle$. In a similar fashion one can induce an arrow $A \times (B \times C) \rightarrow (A \times B) \times C$; these two arrows are mutually inverse to each other, and unique.

Exercise 2.6. Let $D : \mathcal{J} \rightarrow \mathcal{C}$ be a diagram; show that if \mathcal{J} has an initial object 0 , then $\lim D = D0$.