ON THE CONCRETENESS OF CERTAIN CATEGORIES (*)

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In 1961 Kurosh defined a concrete category as one which has a faithful functor into the category of sets. Clearly, in order to be concrete a category must be locally-small, i.e. a *set* of maps between any two objects. The first locally-small category known to be non-concrete dates from 1961 and may be found in my book.

The *best* example of a non-concrete category is the category of spaces and homotopy classes of continuous functions, its non-concreteness being section two of this paper. In a more philosophic sense, homotopy has always been the best example of an abstract category — though its objects are spaces, the points of the spaces are irrelevant because the maps are not functions — best, because of all abstract categories it is the one most often lived in by real mathematicians. It is satisfying to know that its abstract nature is permanent, that there is no way of interpreting its objects as some sort of set and its maps as functions.

One interpretation of the non-concreteness of homotopy is that no functor, indeed no set of functors invariant on homotopy, from spaces into concrete categories can be used to distinguish non-homotopic maps. Of course, if we cut the class of spaces down to a set of spaces, the resulting homotopy category, being small, is concrete. The precise condition on the class of spaces we need for non-concreteness is that for some prime integer p the class contains all p-torsion Moore spaces in two adjacent dimensions (e.g. the class of all n-connected $(n+3)$-dimensional CW-complexes).

Hence the homotopy category of simply-connected spaces is not concrete and it makes no difference as far as concreteness goes whether we

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have base-points or not. Moreover, if we consider Moore spaces in dimension 4 and 5 we are considering a full-subcategory of the stable homotopy category, and hence stable homotopy is not concrete.

In the third section, we examine the concreteness of *Small* ($\mathfrak{A}$), $\mathfrak{A}$ a locally small category, *Small* ($\mathfrak{A}$) the category of covariant functors which are colimits of a small diagram of representables. *Small* ($\mathfrak{A}$) is always locally-small, virtually never concrete (e.g. for $\mathfrak{A}$ the category of abelian groups, commutative rings, sets with distinguished endormorphism, or any category in which these three examples may be fully embedded, covariantly or contravariantly).

If $\mathfrak{A}$ has finite coproducts we obtain a fairly simple necessary and sufficient condition that *Small* ($\mathfrak{A}$) be concrete. Examples of the latter are $\mathfrak{A}=$ the category of sets, vector spaces and their duals.

In section four we show first that the category of small categories and natural equivalence classes of functors is *not* concrete. On the other hand if we limit ourselves to small-categories with no more than K-objects, K a cardinal number, then it is concrete. Probably the only important case: $K=1$ in which case we obtain the category of monoids and conjugacy classes of homomorphisms.

Section five characterizes those categories for which there exists a set-valued functor which reflects isomorphism. Any locally-small category satisfies the characterization.

1. Two Lemmas.

At this writing all known proofs that a locally-small category is not concrete use the same idea, including the first example dating from 1961.

Given an object A define an equivalence relation on the maps into A as follows:

$$B \xrightarrow{f} A \equiv B' \xrightarrow{g} A \text{ if for all } \langle A \xrightarrow{k} C,\ A \xrightarrow{l} C \rangle (fk=fl) \Leftrightarrow (gk=gl).$$

In words, two maps into A are equivalent in the equalize the same pairs of maps leaving from A.

We define the *Generalized Regular Subobjects* of A to be the equivalence classes. Recall that a Regular Subobject of A is a subobject which appears as an equalizer. Any regular-subobject represents a unique generalized regular subobject.

LEMMA 1.1. *If $A \in \mathfrak{A}$ is such that it has more than a set of generalized regular subobjects then $\mathfrak{A}$ is not concrete.*

PROOF. Let $T : \mathfrak{A} \to Sets$ be any functor. We'll show that it is not an embedding. $T(A)$ has only a set of subsets, hence there must be $B \xrightarrow{f} A$, $B' \xrightarrow{g} A$ such that $f \not\equiv g$ (i.e. represent different generalized regular subobjects) but such that Image $(T(f)) = \text{Image}(T(g))$. Because $f \not\equiv g$ there exists a pair $\langle A \xrightarrow{k} C, A \xrightarrow{l} C \rangle$ such that $(fk = fl) \Leftrightarrow (gk = gl)$. Either $(fk = fl) \wedge (gk \neq gl)$ or $(fk \neq fl) \wedge (gk = gl)$. By interchanging f, g if necessary we may assume the latter.

Let

$$T(B') \xrightarrow{T(g)} T(A) \xrightarrow{T(k)} T(C) = T(B') \xrightarrow{T(g)} T(A) \xrightarrow{T(l)} T(C).$$

Because Im $(T(g)) =$ Im $(T(f))$ it must be the case that $T(g)$ and $T(f)$ equalize the same pairs. Hence

$$T(B') \xrightarrow{T(f)} T(A) \xrightarrow{T(k)} T(C) = T(B) \xrightarrow{T(f)} T(A) \xrightarrow{T(l)} T(C).$$

and T fails to distinguish fk and fl.

To show that a category is not concrete therefore we exhibit a proper-class of maps $\{B_i \to A\}$ which represent different generalized quotient objects.

In four different categories we do exactly this using the following lemma about abelian groups, which is an emmediate consequence of the theory of heights in p-torsion groups.

LEMMA 1.2. *For any prime integer p, there exists a transfinite sequence of p-torsion abelian groups $\{B_\alpha\}_{\alpha \in \text{ordinals}}$, each with a special element $x_\alpha \in B_\alpha$, $x_\alpha \neq 0$, $px_\alpha = 0$ such that for $\alpha < \beta$ and any function $f : B_\alpha \to B_\beta$ such that $f(px) = pf(x)$ all it is the case that $f(x_\alpha) = 0$.*

2. Homotopy is not concrete.

Given an abelian group G, the Moore Space for G in dimension $n \geqslant 1$, $M_n(G)$, is the connected CW-complex such that

$$H_n(M_n(G)) = G$$
$$H_j(M_n(G)) = 0 \qquad j \neq n$$

$$\pi_1(M_n(G)) = H_1(M_n(G)) \qquad \text{(i.e. for } n=1,\ \pi_1(M_1(G)) = G,\ n>1,\ \pi_1(M_n(G)) = 0).$$

We recall that we may construct $M_n(G)$ as follows: Let $0 \to \Sigma_I Z \to \Sigma_J Z \to G \to 0$ be exact. Define $V_I S^n \to V_J S^n$ (where V, the « wedge », is the coproduct in the category of spaces with base-point) by

$$S^n \xrightarrow{u_i} V_I S^n \to V_J S^n = \Sigma(Z \xrightarrow{u_i} \Sigma_I Z \to \Sigma_J Z \xrightarrow{p_j} Z)\cdot(u_j)$$

where the first Σ refers to summation in $\pi_n(V_J S^n)$, u_j the map $S^n \xrightarrow{u_j} V_I S^n$. Let

$$V_I S^n \to V_J S^n \to M_n(G) \to V_I S^{n+1} \to V_J S^{n+1}$$

be the mapping-cone sequence. Because

$$H_n(V_I S^n \to V_J S^n) = \Sigma_I S^n \to \Sigma_J S^n$$

and H carries mapping-cone sequences into exact sequences, we obtain the defining properties on $M_n(G)$.

Given another connected CW-complex M' such that

$$H_j(M') = H_j(M_n(G)), \qquad \pi_1(M') = \pi_1(M_n(G)),$$

we may choose an isomorphism $G \to H_n(M')$.

We define $V_J S^n \to M'$ so that $S^n \xrightarrow{u_i} V_J S^n \to M'$ to be a map such that

$$H_n(S^n \xrightarrow{u_i} V_J S^n \to M') = H_n(S^n) \to H_n(S^n)\, H_n(M') = Z \xrightarrow{u_i} \Sigma_J Z \to G\ H_n(M').$$

Then $V_I S^n \to V_J S^n \to M'$ is null-homotopic, and there exists

$$\begin{tikzcd} V_J S^n \arrow[r] \arrow[dr] & M_n(G) \arrow[d] \\ & M' \end{tikzcd}$$

which may be checked to be an H_n isomorphism, hence an H_*-isomorphism and a π_1 isomorphism. By Whitehead, it is a homotopy equivalence.

Given $G \to G'$, we may find $M_n(G) \to M_n(G')$ such that

$$H_n(M_n(G)) \to H_n(M_n(G')) = G \to G'.$$

Choose

$$\begin{array}{ccccccccc} 0 & \to & \Sigma_I Z & \to & \Sigma_J Z & \to & G & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \Sigma_{I'} Z & \to & \Sigma_{J'} Z & \to & G' & \to & 0 \end{array}$$

As for $\Sigma_I Z \to \Sigma_J Z$ we may find maps $V_I S^n \to V_{I'} S^n$, $V_J S^n \to V_{J'} S^n$ such that

$$\begin{array}{ccc} V_I S^n & \to & V_J S^n \\ \downarrow & & \downarrow \\ V_{I'} S^n & \to & V_{J'} S^n \end{array} \quad \text{commutes,}$$

returning the original square with free-group as vertices upon application by H_n. We may extend to the mapping-cones:

$$\begin{array}{ccc} V_J S^n & \to & M_n(G) \\ \downarrow & & \downarrow \\ V_{J'} S^n & \to & M_n(G'). \end{array}$$

Choose $n \geqslant 1$ and a prime integer p. Let $\{x_\alpha, B_\alpha\}$ be as described in 1.2. For each α let $M_\alpha = M_n(B_\alpha)$. Assuming, as we certainly can, that $B_0 = Z_p$, let $B_0 \to B_\alpha$ be a map with x_α in its image. For each α, let

$$M_0 \to M_\alpha \to C_\alpha \to \Sigma M_0 \to \Sigma_\alpha$$

be a mapping-cone sequence, Σ refering to suspension. Note that C_α is a Moore space, namely for Cok $(B_0 \to B_\alpha)$. ΣM_α is a Moore space in dimension $n+1$.

LEMMA 2.1. For $\alpha > \beta$, $C_\alpha \to \Sigma M_0 \to \Sigma M_\beta$ is not null-homotopic.

PROOF. Because $\Sigma M_0 \to \Sigma M_\alpha$ is the mapping-cone of $C_\alpha \to \Sigma M_0$, if $C_\alpha \to \Sigma M_0 \to \Sigma M_\beta = 0$ then there exists

$$\begin{tikzcd} \Sigma M_0 \arrow[r] \arrow[dr] & \Sigma M_\alpha \arrow[d] \\ & \Sigma M_\beta \end{tikzcd}$$

Applying H_{n+1} we obtain

$$\begin{tikzcd} & Z_\alpha \arrow[dd] \\ Z_p \arrow[ur] \arrow[dr] & \\ & Z_\beta \end{tikzcd}$$

and $x_\alpha \to 0$, a contradiction.

The maps $\{C_\alpha \to \Sigma M_0\}$ each represent a different generalized subobject and by 1.1 we are done.

THEOREM 2.2. *Let $\mathcal{H}$ be the homotopy-category obtained from p-torsion Moore-spaces for dimensions n and $n+1$, $T: \mathcal{H} \to$ Sets any functor. Then there exists $(X \to Y) \in \mathcal{H}$, $X \to Y \neq X \to Y$ such that $T(f) = T(0)$. $\mathcal{H}$ is not concrete.*

PROOF. Because $T(\Sigma M_0)$ has only a set of subsets, there must exist $\alpha > \beta$ such that Image $(T(C_\alpha \to \Sigma M_0)) =$ Im $(T(C_\beta \to \Sigma M_0))$. Since $C_\beta \to \Sigma M_0 \to \Sigma M_\beta = 0$, we have

$$T(C_\beta) \to T(\Sigma M_0) \to T(\Sigma M_\beta) = T(C_\beta) \to T(\Sigma M_0) \xrightarrow{T(0)} T(\Sigma M_\beta).$$

Because $T(C_\alpha \to \Sigma M_0)$ has the same image as $T(C_\beta \to \Sigma M_0)$ we have

$$T(C_\alpha) \to T(\Sigma M_0) \to T(\Sigma M_\beta) = T(C_\alpha) \to T(\Sigma M_0) \xrightarrow{T(0)} T(\Sigma M_\beta)$$

and for

$$X \xrightarrow{f} Y = C_\alpha \to \Sigma M_\beta \neq 0$$

we have $T(f) = T(0)$.

Given a cardinal number K we say that $f: X \to Y$ is a K-phantom map if f is not null-homotopic, but for every subcomplex $X' \subset X$ with less than K cells, $f \mid X'$ is null-homotopic. A $\aleph_0$-phantom map is what elsewhere has been called just a phantom map (it disappears on every observable piece).

COROLLARY 2.3. *For every cardinal K, there exist K-phantom maps.*

PROOF. Let $\mathcal{H}$ be as in 2.2, choose a representative from every homotopy-type of complexes with fewer than K cells and define T to be the product of the functors they represent. Let $X \xrightarrow{f} Y$ be such that $f \neq 0$, $T(f) = T(0)$. Then f is a K-phantom map.

3. Categories of small functors.

Given a locally-small category $\mathcal{A}$, let $(\mathcal{A}, \mathcal{S})$ be the category of set-valued functors. Because $\mathcal{A}$ is locally-small each representable functor is an object in $(\mathcal{A}, \mathcal{S})$. Given $A \in \mathcal{A}$, $H^A \in (\mathcal{A}, \mathcal{S})$ is the functor represented by A. $T \in (\mathcal{A}, \mathcal{S})$ is a PETTY-FUNCTOR if there exists a coproduct of representable ΣH^{A_i} and an epimorphism $\Sigma H^{A_i} \to T$. *Petty* $(\mathcal{A})$ is the full-subcategory of petty functors. It is locally-small: given any F, we obtain a monomorphism $(T, F) \to (\Sigma H^{A_i}, F)$ and $(\Sigma H^{A_i}, F) \simeq \Pi(H^{A_i}, F) \simeq \Pi F(A_i)$ a set. The last isomorphism is the Yoneda lemma.

$T \in (\mathcal{S}, \mathcal{A})$ is small if it is a colimit of representables. *Small* $(\mathcal{A})$ is the full subcategory of small functors. Clearly small implies petty and *Small* $(\mathcal{A}) \subset$ *Petty* $(\mathcal{A})$.

We recall that a category is well-powered if every object has at most a set of subobjects.

PROPOSITION 3.1. *The following are equivalent*:

a) *Petty* $(\mathcal{A})$ *is concrete;*

b) *Small* $(\mathcal{A})$ *is concrete;*

c) *For every* $A \in \mathcal{A}$, H^A *has at most a set of quotient objects;*

d) *Petty* $(\mathcal{A})$ *is well-powered and well-co-powered.*

PROOF.

a) $\Rightarrow$ b) E immediate;

b) $\Rightarrow$ c).

By the dual of lemma 1.1 the concreteness of *Small* $(\mathcal{A})$ implies that H^A has a most a set of generalized regular quotient objects in *Small* $(\mathcal{A})$. If $H^A \to Q$, $H^A \to Q'$ represent different quotient objects then they must represent different generalized quotient objects: Suppose there does *not* exist

$$\begin{tikzcd} & Q \arrow[dd] \\ H^A \arrow[ur] \arrow[dr] & \\ & Q' \end{tikzcd}$$

Let

$$\begin{tikzcd} C \arrow[r, "g"] \arrow[d, "f"'] & H^A \arrow[d] \\ H^A \arrow[r] & Q \end{tikzcd}, \qquad \begin{tikzcd} C' \arrow[r] \arrow[d] & H^A \arrow[d] \\ H^A \arrow[r] & Q' \end{tikzcd}$$

be pullbacks (in $(\mathcal{A}, \mathcal{S})$ since C, C' might not be petty). They are also pushouts. Hence

$$C \xrightarrow{f} H^A \to Q' \neq C \xrightarrow{g} H^A \to Q'.$$

Let $H^B \to C$ be such that

$$H^B \to C \xrightarrow{f} H^A \to Q' \neq H^B \to C \xrightarrow{g} H^A \to Q.$$

Then

$$H^B \to C \xrightarrow{f} H^A \to Q \neq H^B \to C \xrightarrow{g} H^A \to Q$$

and $H^A \to Q$, $H^A \to Q'$ represent different generalized quotient objects in *Small* $(\mathcal{A})$.

Thus the concreteness of *Small* $(\mathcal{A})$ implies that H^A has at most a set of *small* quotient objects. Given any quotient $H^A \to Q$, Q small or not, consider the pullback

$$\begin{array}{ccc} C & \to & H^A \\ \downarrow & & \downarrow \\ H^A & \to & Q \end{array} .$$

Let $\{C_\alpha\}$ be an ascending chain through the ordinal numbers of petty subfunctors of C such that $\cup C_\alpha = C$.

Let

$$\begin{array}{ccc} C_\alpha & \to & H^A \\ \downarrow & & \downarrow \\ H^A & \to & Q_\alpha \end{array}$$

be a pushout. Then in the lattice of quotient objects of H^A, Q is the greatest lower bound of $\{Q_\alpha\}$. If we show that each Q_α is small then since there are only a set of small quotients, the sequence must stabilize and $Q = Q_\alpha$ large α. For each α choose $\Sigma H^B \to C_\alpha$ epi. Then

$$\begin{array}{ccc} \Sigma H^B & \to & H^A \\ \downarrow & & \downarrow \\ H^A & \to & Q_\alpha \end{array}$$

is still a pushout and Q_α is small. We note that we have, in fact, proved: H^A has at most a set of quotient objects $\Rightarrow$ All quotients of H^A are small.

$c) \Rightarrow d)$.

First we show that H^A has at most a set of subobjects. For each $S \subset H^A$ consider the pushout

$$\begin{array}{ccc} S & \to & H^A \\ \downarrow & & \downarrow \\ * & \to & Q \end{array}$$

where $*$ is the terminal functor (always a single point). Different subobjects yield in this manner different quotient object, hence H^A has at most a set of subobjects. Since every subobject of H^A is a union of petty subobjects we have shown: Every subobject of H^A is petty.

Petty $(\mathfrak{A})$ is well-powered as follows: given petty T choose $\Sigma H^{A_i} \to T$. For each subobject $T' \rightarrowtail T$ we obtain a unique subobject of ΣH^{A_i} with the pullback

$$\begin{array}{ccc} p' & \to & \Sigma H^{A_i} \\ \downarrow & & \downarrow \\ T' & \to & T \end{array}$$

It suffices to show that ΣH^A has at most a set of subobjects. Given $P \subset \Sigma H^{A_i}$ let

$$\begin{array}{ccc} P_i & \to & H^{A_i} \\ \downarrow & & \downarrow{\scriptstyle u_i} \\ P & \to & \Sigma H \end{array}$$

be a pullback. $P = \cup P_i$ (indeed $P = \Sigma P_i$). This is true because it is true in the category of sets. We have just shown there are only a set of possibilities for P_i, hence for P. We also have: Every subfunctor of a petty functor is petty.

For the well-co-powering of *Petty* $(\mathfrak{A})$ it clearly suffices to show that any coproduct ΣH^{A_i} has at most a set of quotient-objects. We will show that only a set of isomorphism types occurres among the quotient-objects, which together with local-smallness, suffices.

We first construct a diagram scheme: vertices will be I together with $I \times I$; arrows will be $\langle i, j\rangle \to i$, $\langle i, j\rangle \to j$. Consider diagrams $D : \mathfrak{D} \to$ *Petty* $(\mathfrak{A})$ such that D_i is a quotient of H^{A_i}, $D_{\langle i,j\rangle}$ is a subfunctor of D_i, $D_{\langle i,j\rangle} \to D_i$ and $D_{\langle i,j\rangle} \to D_j$ monomorphisms. Clearly only of a set of non-isomorphic such diagrams exist since there are only a set of possibilities for each D_i and only a set possibilities for each $D_{i,j}$ (because we have shown that $Petty(\mathfrak{C})$ is well-powered). Finally for every epic $\Sigma A^{A_i} \to Q$ there exists a diagram as stipulated whose colimit is Q, namely

$$D_i = \text{Image}\,(H^{A_i} \to \Sigma H^{A_i} \to Q),\ D_{\langle i,j\rangle} = D_i \cap D_j\,.$$

As in $b) \Rightarrow c)$ we can show that every quotient of H^{A_i} is small. Hence $Petty(\mathfrak{C}) = Small(\mathfrak{C})$ under hypothesis $c)$.

$d) \Rightarrow a)$.

We assume that $Petty(\mathfrak{C})$ is well-bipowered and show it to be concrete. We shall construct two set-valued functors which together distinguish maps. (Their product would be the embedding). The first functor is more easily described as a contravariant functor. ((Sets)^{0p} is concrete hence this suffices).

Define $Q : Petty(\mathfrak{C}) \to Sets$ to be the contravariant functor which assigns to each object its set of quotient objects. Given a map $T' \xrightarrow{f} T$ and $[T \to T'']\varepsilon Q(T)$ define $Q(f)[T \to T'']$ to be the coimage of $T' \to T \to T''$. The uniqueness of the factorization of maps is needed here.

Given $f, g : T' \to T$ if $Q(f) = Q(g)$ then

$$Q(f)[T \xrightarrow{1} T] = Q(g)[T \xrightarrow{1} T]$$

and Coimage (f) = Coimage (g). Thus there is an epic $T' \to A$, monomorphism $f, g' : A \to T$ such that $T' \to A \xrightarrow{f'} T = f$, $T' \to A \xrightarrow{f'} T = g$.

For each isomorphism type in $Petty(\mathfrak{C})$ pick a representative, A, and an epic $P_A \to A$, where P_A is a co-product of representables. We define a subfunctor $S_A \subset (P_A, -)$ as follows: $(P_A \xrightarrow{f} X)\varepsilon S(X)$ iff there does *not* exist a map $\text{Im}(f) \to A$ such that

$$\begin{tikzcd} P_A \arrow[r] \arrow[dr] & \text{Im}(f) \arrow[d] \\ & A \end{tikzcd}$$

Let

$$\begin{tikzcd} S_A \arrow[r] \arrow[d] & (P_A, -) \arrow[d] \\ * \arrow[r] & F_A \end{tikzcd}$$

be a pushout (in ($Petty(\mathcal{A})$, $\mathcal{S}$)).

Observe that for any X,

$$\begin{tikzcd} S_A(X) \arrow[r] \arrow[d] & (P_A, X) \arrow[d] \\ * \arrow[r] & F_A(X) \end{tikzcd}$$

still being a pushout, that $F_A(X) \neq *$ iff $S_A(X) \neq (P_A, X)$ iff there exists a maps $P_A \xrightarrow{f} A$ and $\mathrm{Im}(f) \rightarrow A$ such that

$$\begin{tikzcd} P_A \arrow[r] \arrow[dr] & \mathrm{Im}(f) \arrow[d] \\ & A \end{tikzcd}$$

In particular, A must appear as a quotient of a subobject of X. Given X there are only a set of possibilities for such A, that is, for all but a set of A's, $F_A(X) = *$. Hence ΠF_A exists in the category of set-valued functors from $Petty(\mathcal{A})$.

Given $f, g : T' \rightarrow T$ if $Q(f) = Q(g)$ and $\Pi F_A(f) = \Pi F_A(g)$ then $f = g$. As we have already shown, there exists epic $T' \rightarrow A$ and monics f', $g' : A \rightarrow T$ such that $T' \rightarrow A \xrightarrow{f'} T = f$, $T' \rightarrow A \xrightarrow{g'} T = g$. Pick $P_A \rightarrow T'$ such that

$$\begin{tikzcd} P_A \arrow[d, "h"'] \arrow[dr] & \\ T \arrow[r] & A . \end{tikzcd}$$

Using $F_A(f) = F_A(g)$ we have

$$F_A(T) \xrightarrow{F_A(f)} F_A(T') = F_A(T) \xrightarrow{F_A(g)} F_A(g).$$

Starting with the element $h \in F_A(T)$ we have

$$P_A \xrightarrow{h} T \xrightarrow{f} T' = P_A \xrightarrow{h} T \xrightarrow{g} T'$$

have

$$P_A \rightarrow A \xrightarrow{f'} T' = P_A \rightarrow A \xrightarrow{g'} T'$$

and $f' = g'$, $f = g$.

Condition c), that H^A has a most a set of quotient objects may be reformulated. We'll say that a quotient $H^A \to Q$ is *special* if it is a coequalizer of a pair of transformations $f,\ g : H^B \to H^A$. Every quotient of H^A is a colimit of special quotients hence it suffices to say that H^A has at most a set of special quotients. It is possible to translate into more primative condition, the statement that two pairs of maps $\langle f,\ g : A \to B \rangle$, $\langle f',\ g' : A \to B' \rangle$ are such that the coequalizer of $\langle H^f, H^g \rangle$ and $\langle H^{f'}, H^{g'} \rangle$ are the same. Doing so provides a necessary condition on $\mathfrak{A}$ that *Small* $(\mathfrak{A})$ be concrete.

A much easier condition results, however, when we may assume for every $A \in \mathfrak{A}$, that $H^A \times H^A$ is petty, e.g. if $A + A$ exists in $\mathfrak{A}$, or if a « weak-coproduct » or even a « pre-coproduct » of A with itself exists in $\mathfrak{A}$. The last condition is equivalent with the statement that $H^A \times H^A$ is petty. Namely: For locally-small categories $H^A \times H^A$ is petty iff there exists a set of objects $\{S_i\}_I$ such that for every pair of maps $f_1 f_2 : A \to B$ there exists $i \in I$ and maps $u_1 : A \twoheadrightarrow S_i$, $u_2 : A \twoheadrightarrow S_i$, $g,\ S_i \to B$ such that $u_1 g = f_1$, $u_2 g = f_2$.

Corollary 3.2. *Let $\mathfrak{A}$ be a locally-small category for which $H^A \times H^A$ is petty all $A \in \mathfrak{A}$. Then Small $(\mathfrak{A})$ is concrete iff for every $A \in \mathfrak{A}$ there exists a set of maps $\{A \twoheadrightarrow B_i\}_I$ such that for any $A \to X$ there exists*

$$\begin{tikzcd} & B_i \arrow[d] \\ A \arrow[ur] \arrow[r] & X \end{tikzcd} \qquad \begin{tikzcd} & X \arrow[d] \\ A \arrow[ur] \arrow[r] & B_i . \end{tikzcd}$$

Proof. We'll say that a subfunctor of H^A is principal if it is the image of a transformation $H^X \to H^A$. Given two maps $A \to X$, $A \to Y$, the resulting principal subfunctors of H^A are equal iff there exist

$$\begin{tikzcd} H^X \arrow[d] \arrow[dr] & \\ H^Y \arrow[r] & H^A \end{tikzcd}$$

and

$$\begin{tikzcd} H^Y \arrow[d] \arrow[dr] & \\ H^X \arrow[r] & H^A , \end{tikzcd}$$

hence iff there exist

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & X \arrow[dd] \\
A \arrow[ur] \arrow[dr] & \\
 & Y
\end{tikzcd}
\quad , \quad
\begin{tikzcd}[row sep=small, column sep=small]
 & X \arrow[dd] \\
A \arrow[ur] \arrow[dr] & \\
 & Y
\end{tikzcd}
.
$$

The set of maps $\{A \to B_i\}_I$ as described above exist iff H^A has at most a set of principal subfunctors. If *Small* $(\mathfrak{A})$ is concrete the last proposition said that H^A has at most a set of subfunctors, hence at most a set of principal subfunctors.

Conversely, if H^A has at most a set of principal subfunctors, it has at most a set of subfunctors, because every subfunctor is a union of principal subfunctors. As in the last proof, therefore, every petty functor has a most a set of subfunctors.

For each quotient $H^A \to Q$ we consider the pullback

$$
\begin{tikzcd}
P \arrow[r] \arrow[d] & H^A \arrow[d] \\
H^A \arrow[r] & Q
\end{tikzcd}
,
$$

and the induced monomorphisms $P \to H^A \times H^A$. Different quotients yield different subobjects of $H^A \times H^A$, hence under the hypothesis by the last proposition, Small $(\mathfrak{A})$ is concrete.

COROLLARY 3.3. *Let $\mathfrak{A}$ be a locally-small category for which $H^A \times H^A$ is petty all $A \in \mathfrak{A}$. Then Small $(\mathfrak{A})$ is concrete iff for every A there exists a set $\{B_i\}$ such that for every $A \to X$ there exists $i \in I$, $X \to B_i$, $B_i \to X$ such that*

$$
\begin{tikzcd}[row sep=small, column sep=small]
 & X \arrow[d] \\
A \arrow[ur] \arrow[dr] & B_i \arrow[d] \\
 & X
\end{tikzcd}
.
$$

In many categories, such as categories of algebras, spaces or their duals, we may restate the last condition of the corollary as follows:

Given A there exists a cardinal number K such that for every $A \to X$ there is an endomorphism $f\colon X \to X$ such that

$$\begin{tikzcd} & X \arrow[d, "f"] \\ A \arrow[ur] \arrow[r] & X \end{tikzcd}$$

and Image (f) is of cardinality less than K.

COROLLARY 3.4. *If $\mathfrak{A}$ a locally-small well-co-powered category for which $H^A \times H^A$ is petty all $A \in \mathfrak{A}$ and such that every map factors as an epimorphism followed by a splitting monomorphism (e.g. sets, vector-spaces or their duals) then Small $(\mathfrak{A})$ is concrete.*

To show that Small $(\mathfrak{A})$ is *not* concrete, we usually exhibit an object $A \in \mathfrak{A}$ and $\{A \to B_\alpha\}_{\alpha \in \mathrm{Ordinals}}$ such that for $\alpha > \beta$ or $\beta > \alpha$ there is no

$$\begin{tikzcd} & B_\alpha \arrow[d] \\ A \arrow[ur] \arrow[r] & B_\beta \end{tikzcd}$$

Clearly in such a case H^A has more than a set of principal subfunctors. Not that if $\mathfrak{A} \to \mathfrak{B}$ is a full embedding than the Kan extension *Small* $(\mathfrak{A}) \to$ *Small* $(\mathfrak{B})$ is an embedding. Hence if $\mathfrak{A}$ in such that Small $(\mathfrak{A})$ is not concrete then the same is true for $\mathfrak{B}$.

EXEMPLES:

Groups. Let A be the integers, B_α a simple-group of cardinality $\mathfrak{A}_\alpha$, $A \to B_\alpha$ any non trivial map.

(Semi groups)^{0p}. Let A be the terminal semi-group, $\{B_\alpha\}$ a class of semi-groups such that $(B_\alpha, B_\beta) = \emptyset$ all $\alpha \neq \beta$, the existence of which was demonstrated by Hedllin.

Fields. Let A be a prime field, B_α a field of characteristic equal to that of A and of cardinality $\aleph_\alpha$.

(Rings)^{0p}. Let A be the terminal ring $(0=1)$, B_α as above.

To see that *Small (Abelian Groups)* is not concrete we use 1.2, and the maps $Z \to B_\alpha$, $1 \mapsto x_\alpha$. For *Small (Abelian Groups)*^{0p} we use

1.2 again. For each α choose a map $B_\alpha \to Z_{p\infty}$ such that $x_\alpha \to \frac{1}{p}$. Then for $\alpha > \beta$ there does not exist

$$\begin{tikzcd} B_\alpha \arrow[d] \arrow[dr] & \\ & Z_{p\infty} . \\ B_p \arrow[ur] & \end{tikzcd}$$

Note that because we have full embeddings of (*Abelian Groups*) into (*Groups*), (*Monoids*), (*Semigroups*) this provides a proof that Small ($\mathfrak{A}$) is not concrete for any of these other three categories of their duals.

Small (*Abelian Groups*) is more than non-concrete. Besides the fact that the representables have more than a set of subfunctors, they have subfunctors not even small. The second statement is not a consequence of the first. We showed in [ ], that for $\mathfrak{B}$ the category of sets with distinguished endomorphism that every subfunctor of a petty functor is petty (and hence petty = small). But *Small* ($\mathfrak{B}$) is not concrete.

The identity functor on abelian groups has a non-petty subfunctor: $S(X) = \{x \in X \mid \text{ for all } X \to Z,\ x \to 0\}$. The proof is Nunke's, though he used the proof to show a different subfunctor not petty.

The functor

$$(\textit{Abelian Group}) \xrightarrow{(-,Z_{p\infty})} (\textit{Abelian Group})$$

has a non-petty subfunctor, namely $F \subset (-, Z_{p\infty})$,

$$F(A) = \{f : A \to Z_{p\infty} \mid \text{Ker}(f) \supset \text{divisible part of } A\}.$$

4. The category of small categories and natural equivalence classes of functors.

The category of the above section title is called Coscanecof.

THEOREM 4.1. *Coscanecof is not concrete.*

PROOF. We shall write all operators on the right.

Let $\boldsymbol{A}$ be the free monoid on two generators named $\tilde{p}$, $\tilde{a}$. We shall view $\boldsymbol{A}$ as a category with one object.

Choose a prime integer p, and let $\langle x_\alpha, B_\alpha \rangle$ be as described in 1.2.

For each $y \in B_\alpha$, let $\tilde{y}$ denote the function from B_α to B_α obtained by translating by $y \cdot x\tilde{y} = x + y$. Let $\tilde{p}$ denote the function from B_α to B_α obtained by multipling by $p \cdot x\tilde{p} = xp$. Let $\boldsymbol{B}_\alpha$ be the monoid generated $\{\tilde{p}\} \cup \{\tilde{y} \mid y \in B_\alpha\}$. Note that $\tilde{y}\tilde{p} = \tilde{p}(\widetilde{yp})$. That is

$$x(\tilde{y}\tilde{p}) = (x+y)\tilde{p} = xp + yp = (xp)(\widetilde{yp}) = x(\tilde{p}(\widetilde{yp})).$$

Moreover $\tilde{y}\tilde{z} = (\widetilde{y+z})$ and every element of $\boldsymbol{B}_\alpha$ is of the form $\tilde{p}^n \cdot \tilde{y}$. The only units of $\boldsymbol{B}_\alpha$ are the translations.

Let $G_\alpha : \boldsymbol{A} \to \boldsymbol{B}_\alpha$ be the functor (i.e. Homomorphism) such that $\tilde{p}G = \tilde{p}$, $\tilde{\alpha}G = \tilde{x}_\alpha$.

We shall show for $\alpha \neq \beta$ that G_α and G_β represent different generalized regular quotients of $\boldsymbol{A}$, hence by the dual of lemma 1.1., coscanecof is not concrete.

For $\alpha > \beta$ we shall construct a category $\boldsymbol{C}_\alpha$, a pair of functors $F, F' : \boldsymbol{C}_\alpha \to \boldsymbol{A}$ such that $FG_\alpha \simeq F'G_\alpha$ but $FG_\beta \simeq F'G_\beta$, thus showing that G_α and G_β equalize different pairs.

The objects of $\boldsymbol{C}_\alpha$ is the set $\{[y] \mid y \in B_\alpha\}$. The maps are freely generate by the following

$$\text{for every } [y] \text{ a map } \tilde{p}_y : [y] \to [yp]$$

$$\text{and a map } \tilde{\alpha}_y : [y] \to [y + x_\alpha].$$

(A map of $\boldsymbol{C}_\alpha$ is, therefore, a sequence of $\tilde{p}$'s and $\tilde{\alpha}$'s which compose. No relations).

$F : \boldsymbol{C}_\alpha \to \boldsymbol{A}$ sends, as it must, each object of $\boldsymbol{C}_\alpha$ to the unique object of $\boldsymbol{A}$. For maps, $\tilde{p}_y F = \tilde{p}$, $\tilde{\alpha}_y F = \tilde{\alpha}$.

$F' : \boldsymbol{C}_\alpha \to \boldsymbol{A}$ is defined by $\tilde{p}_y F' = \tilde{p}$, $\tilde{\alpha}_y F' = 1$.

Consider, for the moment, an arbitrary monoid $\boldsymbol{M}$ and pair of functors $H, H' : \boldsymbol{C}_\alpha \to \boldsymbol{M}$. A natural equivalence $H' \xrightarrow{\eta} H$ is described a function $\eta : B_\alpha \to (\text{Units } \boldsymbol{M})$ such that

$$\begin{tikzcd}
{[y]H'} \arrow[r, "y\eta"] \arrow[d, "\tilde{p}_y H'"'] & {[y] \cdot H} \arrow[d, "\tilde{p}_y H"] \\
{[yp] \cdot H'} \arrow[r, "(yp)\eta"'] & {[yp] \cdot H}
\end{tikzcd}$$

and

$$
\begin{array}{ccc}
[y]H' & \xrightarrow{y\eta} & [y]H \\
{\scriptstyle \tilde{\alpha}_y H'}\big\downarrow & & \big\downarrow{\scriptstyle \tilde{\alpha}_y H} \\
[y+\alpha]\cdot H' & \xrightarrow[(y+x_\alpha)\eta]{} & [y+x_\alpha]\cdot H.
\end{array}
$$

Remembering that all objects in $\boldsymbol{M}$ are one, the condition on η becomes:

$$(y\eta)(\tilde{p}H)=(p_yH')((yp)\eta)$$

$$(y)(\tilde{\alpha}_yH)=(\tilde{\alpha}_yH')((y+x_\alpha)\eta).$$

Given

$$\gamma\,,\ G_\gamma : \boldsymbol{A}\to\boldsymbol{B}_\gamma\,, \qquad H=FG_\gamma\,, \qquad H'=F'G_\gamma$$

$FG_\gamma \simeq F'G_\gamma$ iff there exists a function $\eta : B_\alpha \to (\text{units } \boldsymbol{B}_\gamma)$ such that

$$(\widetilde{y\eta})\tilde{p}=\tilde{p}((\widetilde{yp})\eta)$$

$$(\widetilde{y\eta})_\gamma=\widetilde{(y+x_\alpha)\eta}.$$

Identifying the units of $\boldsymbol{B}_\gamma$ with the group B_γ, we may view η as a function from B_α to B_γ such that

$$(y\eta)p=(yp)\eta$$

$$y\eta+\chi_\gamma=(y+x_\alpha)\eta.$$

For $\gamma=\alpha$ we may take η to be the identity function obtaining $FG_\alpha=F'G_\alpha$. For $\gamma=\beta<\alpha$ we reach a contradiction by assuming the existence of such η. Note first that for $\gamma=0$ we have $(0\eta)p=(0\cdot p)\eta$, hence $0=0\eta$.

Letting $\gamma=0$ in the second equation, we have $\chi_\beta=\chi_\alpha\eta$, which by 1.2 is impossible in light of the first equation.

Consider the category whose objects are groups and whose maps are « conjugacy classes » of homomorphisms: f, $g : A\to B$ are conjugate if there is $\theta\in B$ such that $f=\theta^{-1}g\theta$. This category appears as a full subcategory og both coscanecof and the homotopy category. It is, however, concrete.

More generally, let K be a cardinal number, $\mathcal{C}_K$ the full subcategory of coscanecof of those small categories with no more than K objects. Note that the category of groups-and-conjugacy-classes-of-homomorphisms is a full subcategory of $\mathcal{C}_1$, the category of monoids-and-conjugacy-classses-of-homomorphisms.

THEOREM 4.2. *For every cardinal number K, $\mathcal{C}_K$ is concrete.*

PROOF. We shall construct a pair of functors from coscanecof to sets which together distinguish maps in $\mathcal{C}_K$. As for part $(d \Rightarrow a)$ of 3.1, the first functor is more easily described as a contravariant functor.

A congruence on a category $\boldsymbol{A}$ is an equivalence relation on the maps of $\boldsymbol{A}$ such that

$$x \equiv y \Rightarrow zx \equiv zy$$

$$x \equiv y \Rightarrow xz \equiv yz,$$

where it is understood that (« $zx \equiv zy$ » and « $xz \equiv yz$ » mean that if either side is defined then so is the other. In particular, $1_A \equiv 1_B \Rightarrow A = B$. We may define $\boldsymbol{A}/\equiv$ to be the category with the same objects as $\boldsymbol{A}$ and with $\equiv$ classes as maps. The obvious functor $\boldsymbol{A} \to \boldsymbol{A}/\equiv$ is full and one-to-one onto on objects.

Given any functor $T : \boldsymbol{A} \to \boldsymbol{B}$ we define on $\boldsymbol{A}$ the congruence induced by T by $x \equiv y \Leftrightarrow T(x) = T(y)$ and Domain $(x) =$ Domain (y), Range $(x) =$ Range (y).

There exists unique

$$\begin{tikzcd} \boldsymbol{A} \arrow[r] \arrow[dr] & \boldsymbol{A}/\equiv \arrow[d] \\ & \boldsymbol{B} \end{tikzcd} \quad \text{and } \boldsymbol{A}/\equiv \;\to \boldsymbol{B}$$

is faithfull, (though not necessarily one-to-one on objects). If T is equivalent to T' they induce the same congruence.

We define $C(\boldsymbol{A})$ to be the set of congruences on $\boldsymbol{A}$. C becomes a contravariant functor as follows: given $T : \boldsymbol{A} \to \boldsymbol{B}$ and a congruence $\equiv$ on $\boldsymbol{B}$, $C(T)$ of $\equiv$ is the congruence on $\boldsymbol{A}$ induced by

$$\boldsymbol{A} \xrightarrow{T} \boldsymbol{B} \to \boldsymbol{B}/\equiv.$$

C is a functor from coscanecof to sets.

If $C(T)=C(T')$ for T, $T' : \boldsymbol{A} \to \boldsymbol{B}$, then T and T' induce the same congruence on $\boldsymbol{A}$, as may be seen by starting with the identity congruence on $\boldsymbol{B}$.

Let R be a class of small categories of no more than K objects, exactly one representative from each isomorphism-type (not isomorphism as defined in coscanecof, but as defined by one-to-one onto functors). Given $\boldsymbol{C} \in R$ we'll say that a congruence $\equiv$ on $\boldsymbol{C}$ is small, if

$$|\boldsymbol{C}| + \aleph_0 = |\boldsymbol{C}/\equiv| + \aleph_0$$

where the absolute value signs denote the cardinately of the set of maps.

Note that if $\boldsymbol{C} \to \boldsymbol{B}$ induces a congruences *not* small then for any $\boldsymbol{B} \to \boldsymbol{D}$, $\boldsymbol{C} \to \boldsymbol{B} \to \boldsymbol{D}$ induces a congruence not small. Hence we may define the subfunctor $N \subset (\boldsymbol{C}, -)$ of functors which induce congruences *not* small. Let

$$\begin{array}{ccc} N & \to & (\boldsymbol{C}, -) \\ \downarrow & & \downarrow \\ * & \longrightarrow & S_{\boldsymbol{C}} \end{array}$$

be a pushout in the category of set-valued functors, from coscanecof where $*$ is the terminal-functor, that which is always equal to a single-point set. For any $\boldsymbol{B}$,

$$\begin{array}{ccc} N(\boldsymbol{B}) & \to & (\boldsymbol{C}, \boldsymbol{B}) \\ \downarrow & & \downarrow \\ * & \longrightarrow & S_{\boldsymbol{C}}(\boldsymbol{B}) \end{array}$$

is a pushout in the category of sets, and $S_{\boldsymbol{C}}(\boldsymbol{B})$ may be constructed as $[(\boldsymbol{C}, \boldsymbol{B}) - N(\boldsymbol{B})] \cup \{*\}$, the set of natural equivalence classes of functors from $\boldsymbol{C}$ to $\boldsymbol{B}$ which induce small congruences plus a « base point ». Hence $S_{\boldsymbol{C}}(\boldsymbol{B}) \neq \{*\}$ iff there exists $\boldsymbol{C} \to \boldsymbol{B}$ inducing a small congruence.

We note that if $\boldsymbol{A} \to \boldsymbol{B}$ is faithfull, $\boldsymbol{A} \in \mathfrak{C}_K$ then $|\boldsymbol{A}| \leqslant K \cdot |\boldsymbol{B}|$. if $S_{\boldsymbol{C}}(\boldsymbol{B}) \neq \{*\}$ then there exists $\boldsymbol{C} \to \boldsymbol{B}$ which induces a small congruence $\equiv$, hence there exists a faithfull functor $\boldsymbol{C}/\equiv \to \boldsymbol{B}$, and

$$|\boldsymbol{C}| + \aleph_0 = |\boldsymbol{C}/\equiv| + \aleph_0 \leqslant K \cdot |\boldsymbol{B}| + \aleph_0 .$$

Fixing $\boldsymbol{B}$, we thus obtain an upper bound $K \cdot |\boldsymbol{B}| + \aleph_0$, on $|\boldsymbol{C}|$ for $\boldsymbol{C}$ such that $S_{\boldsymbol{C}}(\boldsymbol{B}) \neq \{*\}$. There are only a set of such $\boldsymbol{C}$'s in R, hence

$S_C(\boldsymbol{B})=\{*\}$ for all but a set of $\boldsymbol{C}$'s in R. The product $\Pi_R S_C$ thus exists in the category of set-valued functors. Because for each $\boldsymbol{C}\in\mathcal{R}$, there is a transformation $*\to S_C$ and hence for $\boldsymbol{C}$, $\boldsymbol{C}'\in R$ a transformation $S_C\to *\to S_C\to \Pi_R S_C$ we obtain a transformation S_C such that

$$\mathbb{S}_C\to\Pi_R S_C\to\mathbb{S}_C=1_{S_C}.$$

Suppose T, T': $\boldsymbol{A}\to\boldsymbol{B}$, $\boldsymbol{A}$, $\boldsymbol{B}\in\mathcal{C}_K$ are such that $C(T)=C(T')$ and $\Pi_R S_C(T)=\Pi_R S_C(T')$. We shall show that $T=T'$ in coscanecof. As already observed, $C(T)=C(T')$ implies that T, T' induce the same congruence $\equiv$. There exist faithfull functors F, F': $\boldsymbol{A}/\equiv\to\boldsymbol{B}$ such that

$$\boldsymbol{A}\to\boldsymbol{A}/\equiv\xrightarrow{F}\boldsymbol{B}=T,$$

$$\boldsymbol{A}\to\boldsymbol{A}/\equiv\xrightarrow{F'}\boldsymbol{B}=T',$$

For each map in $\boldsymbol{A}/\equiv$ choose an ancestor in $\boldsymbol{A}$, and let $\boldsymbol{A}'$ be the subcategory of $\boldsymbol{A}$ generated by the choices. $|\boldsymbol{A}'|+\aleph_0=|\boldsymbol{A}/\equiv|+\aleph_0$. Let $\boldsymbol{C}\in R$ be such that there exists an isomorphism $\boldsymbol{C}\to\boldsymbol{A}'$. The congruence induced by $\boldsymbol{C}\to\boldsymbol{A}'\to\boldsymbol{A}\to\boldsymbol{A}/\equiv$ is small. Because F, F' are faithfull, $\boldsymbol{C}\to\boldsymbol{A}/\equiv\xrightarrow{F}\boldsymbol{B}$ and $\boldsymbol{C}\to\boldsymbol{A}/\equiv\xrightarrow{F'}\boldsymbol{B}$ each induce the same congruence as $\boldsymbol{C}\to/\equiv$. $\Pi_R S_{\boldsymbol{C}}(F)=\Pi_R S_{\boldsymbol{C}}(F')$, $\mathbb{S}_{\boldsymbol{C}}$ is a retract of $\Pi_R S_{\boldsymbol{C}}(F)$, hence $\mathbb{S}_{\boldsymbol{C}}(F)=\mathbb{S}_{\boldsymbol{C}}(F')$. Starting with $\boldsymbol{C}\to\boldsymbol{A}/\equiv$ as an element of $\mathbb{S}_{\boldsymbol{C}}(\boldsymbol{A}/\equiv)$ and applying $\mathbb{S}_{\boldsymbol{C}}(F)$, $\mathbb{S}_{\boldsymbol{C}}(F')$ we obtain

$$\boldsymbol{C}\to\boldsymbol{A}/\equiv\xrightarrow{F}\boldsymbol{B}\simeq\boldsymbol{C}\to\boldsymbol{A}/\equiv\xrightarrow{F'}\boldsymbol{B}$$

(because neither is the base point) of $S_{\boldsymbol{C}}(\boldsymbol{B})$. $\boldsymbol{C}\to\boldsymbol{A}/\equiv$ is an epimorphism in coscanecof, because it is one-to-one, onto on objects and full — a function from the objects of $\boldsymbol{C}$ to isomorphism in $\boldsymbol{B}$ emmediatly gives rise to a function from the objects of $\boldsymbol{A}/\equiv$ to isomorphism in $\boldsymbol{B}$, naturality is preserved because $\boldsymbol{C}\to\boldsymbol{A}/\equiv$ is full. Hence F and F' are naturally equivalent. Hence $\boldsymbol{A}\to\boldsymbol{A}/\equiv\xrightarrow{F}\boldsymbol{B}=T$ and $\boldsymbol{A}\to\boldsymbol{A}/\equiv\xrightarrow{F'}\boldsymbol{B}=$ $=T'$ are naturally equivalent.

For the case $K=1$ the proof may be slicked-up.

Let T': $(Monoids)\to Sets$ be defined as follows:

$$T'(A)=\{\langle\alpha, f\rangle\,|\,\alpha \text{ an ordinal}, f:\alpha\to A \text{ a one-to-one function}\}$$

where we use the von-Neumann convention that $\alpha = \{\beta \mid \beta < \alpha\}$. $T'(A)$ has a « zero », namely $\langle 0, \emptyset \rangle$.

Given $A \xrightarrow{g} B$ we define $T'(g)$ by

$$(T'(g))\langle \alpha, f \rangle = \begin{cases} \langle \alpha, fg \rangle & \text{if } fg \text{ is one-to-one} \\ \langle 0, \emptyset \rangle & \text{otherwise.} \end{cases}$$

We define $T : \mathcal{C}_1 \to Sets$ as follows: Given a monoid A, define an equivalence relation $\equiv$ on $T'(A)$ by $\langle \alpha, f \rangle \equiv \langle \alpha, f' \rangle$ if there exists $u, u^{-1} \in A$ such that $f' = ufu^{-1}$. Define $T(A)$ to be the set of $\equiv$-classes. For $g, g' : A \to B$, $g' = \nu g \nu^{-1}$ we note that fg is one-to-one iff fg' is one-to-one and T becomes a functor in the natural way. It is an embedding: if $T(g) = T(g')$ we may first show that g and g' induce the same congruence $\equiv$ just by examining the action of $T(g) = T(g')$ on the elements of $T(A)$ of the form $\langle 2, f \rangle$. Let $\langle \alpha, f \rangle$ be a function which chooses a set of representation of $\equiv$. Since $(T(g))\langle \alpha, f \rangle = (T(g'))\langle \alpha, f \rangle$ and fg is one-to-one, we have $\nu \in B$ such that $fg' = \nu fg \nu^{-1}$. Then $g' = \nu g \nu^{-1}$ and $g = g'$ in $\mathcal{C}_1$.

5. Isomorphism reflecting functors.

Le $\mathcal{A}$ be an arbitrary category, locally small or not. Given $A \in \mathcal{A}$ we define an equivalence relation $\equiv$ on the maps with A as range:

$$X' \to A \equiv X \to A \text{ if for all } A \to Y$$

$[X \to A \to Y$ is an isomorphism$] \Leftrightarrow [X \to A \to Y$ is an isomorphism$]$.

Let $T(A)$ be the family of $\equiv$-classes. For some suitably large notion of « class », T becomes a class valued functor in a natural way; given the class of $(X \xrightarrow{f} A)$ in $T(A)$ and a map $g : A \to B$

$$(T(g))[f] = [fg].$$

THEOREM 5.1. *Let $\mathcal{S}$ be a category of sets (or classes) closed under the formation of power-sets and subsets. Then the following are equivalent.*

a) *There exists an $\mathcal{S}$-valued functor on $\mathcal{A}$ which reflects the existence of left-inverses;*

b) *There exists and* $\mathcal{S}$*-valued functor on* $\mathcal{A}$ *which reflects the existence of right-inverses;*

c) *There exists an* $\mathcal{S}$*-valued functor on* $\mathcal{A}$ *which reflects isomorphisms;*

d) T *is* $\mathcal{S}$*-valued.*

PROOF. $c) \Rightarrow d)$.

Given $A \in \mathcal{A}$ we first note that

LEMMA 5.2. *If* $(X \to A) \equiv (X' \to A)$ *then*

$(X \to A$ *has a right-inverse*$) \Leftrightarrow (X' \to A$ *has a right-inverse*$)$.

PROOF OF LEMMA. If $X \to A$ has a right-inverse, $A \to Y$, then because $(X \to A) \equiv (X' \to A)$, $X' \to A \to Y$ is an isomorphism.

Let $Y \to X'$ be its inverse. Then $A \to Y \to X'$ is a right-inverse of $X' \to A$.

LEMMA 5.12. *If* $X \to A$ *and* $X' \to A$ *each have no right-inverse then* $(X \to A) \equiv (X' \to A)$.

PROOF OF LEMMA. Clearly $X \to A \to Y$ can never be an isomorphism since such delivers a right-inverse for $X \to A$. Similarly $X' \to A \to Y$ is never an isomorphism and thus for all $A \to Y$,

$$(X \to A \to Y \text{ is isomorphic}) \Leftrightarrow (X' \to A \to Y \text{ is isomorphic}).$$

End of Lemma-Proof.

If there exists $X \to A$ without a right-inverse we obtain a « trivial element » of $T(A)$ which we'll call $*$. All other elements of $T(A)$ are classes of splitting monomorphisms.

If $X \to A$, $X' \to A$ are both splitting monomorphisms and if $F : \mathcal{A} \to \mathcal{S}$ is an isomorphism-reflecting functor and if

$$\text{Image}\,(F(X) \to F(A)) = \text{Image}\,(F(X') \to F(A))$$

then $(X \to A) \equiv (X' \to A)$.

Hence we obtain a one-to-one function $T(A) \to 2^{F(A)}$ by sending the trivial class of $T(A)$ (if it exists) to $\emptyset \in 2^{F(A)}$, and sending every other class to $\text{Im}\,(F(f))$ for some representative f of the class.

$b) \Rightarrow d)$.

LEMMA 5.13. $(X \to A) \equiv (X \to A)$ *iff for all* $A \to Y$

$$(X \to A \to Y) \text{ has a right-inverse} \Leftrightarrow (X' \to A \to Y) \text{ has a right-inverse.}$$

PROOF OF LEMMA. Suppose $(X \to A) \equiv (X' \to A)$ and that $X \to A \to Y$ has a right-inverse, $Y \to X$. Then $X \to A \to Y \to X$ is an isomorphism, hence $X' \to A \to Y \to X$ is an isomorphism. Let $X \to X'$ be its inverse. Then $Y \to X \to X'$ is a right-inverse for $X' \to Y$.

Conversely suppose that for all $A \to Y$,

$$(X \to A \to Y) \text{ has a right-inverse} \Leftrightarrow X' \to A \to Y \text{ has a right-inverse.}$$

Let $A \to Y$ be such that $X \to A \to Y$ is an isomorphism. We wish to show that $X' \to A \to Y$ is an isomorphism. Since $X \to A \to Y$ has a right-inverse, $X' \to A \to Y$ must have a right-inverse, $Y \to X$.

But again $X \to A \to Y \to X'$ must have a right-inverse, hence must be monomorphic. Since $X \to A \to Y$ is an isomorphism, $Y \to X'$ must be monomorphic. A monomorphic right-inverse must be an isomorphism because

$$Y \to X' \to Y \to X' = Y' \to Y \to X' \text{ and } Y \to X' \to Y = 1.$$

Hence $X' \to A \to Y$ is an isomorphism.

End of Lemma-Proof.

The same argument as for isomorphism-reflecting functors works: if $F : \mathcal{A} \to \mathcal{S}$ reflects the existence of right-inverses, and if $X \to A$, $X' \to A$ both have right-inverses and $\text{Image}\,(X \to A) = \text{Im}\,(X' \to A)$, then $(X \to A) \equiv (X' \to A)$ and we obtain one-to-one $T(A) \to 2^{F(A)}$.

$d) \Rightarrow b)$.

We define a functor $T_* : \mathcal{A} \to \mathcal{S}$ as follows

$$T_*(A) = \begin{cases} T(A) \text{ if } T(A) \text{ has a trivial class} \\ T(A) \cup \{*\} \text{ otherwise.} \end{cases}$$

For all A, $* \in T_*(A)$. It is understood that $(T(f))(*) = *$ all f.

T_* reflects the existence of right-inverses. Suppose $g : A \to B$ is such that $T_*(g)$ is one-to-one. Let $[1] \in T(A) \subset T_*(A)$ be the class of the identity map. $(T(g)[1] \neq *$, hence $A \to B$ is not in the trivial class of $T_*(B)$, hence by 5.11 has a right-inverse.

$d) \Rightarrow c)$.

The same functor, T_* reflects the existence of left-inverses (as does T). Suppose $g : A \to B$ is such that $T_*(g)$ is onto. Let $[1] \in T_*(B) \subset T(B)$ be the class of the identity map and choose $f : X \to A$ such that $(T_*(g))[f] = [\mathrm{E}]$, i.e.

$$(X \to A \to B) \equiv (B \xrightarrow{1} B),$$

and since $B \xrightarrow{1} B \xrightarrow{1} B$ is an isomorphism $X \to A \to B \xrightarrow{1} B$ is an isomorphism. Let $B \to X$ be its inverse. $B \to X \to A$ is a left-inverse for $A \to B$.

Thus T_* reflects the existence of both left and right inverses. A map is an isomorphism iff it has both a left and a right inverse.

We have shown $c \Rightarrow d$, $b \Rightarrow d$, $d \Rightarrow b$, $d \Rightarrow c$ hence $c \Leftrightarrow d \Leftrightarrow b$. That $a \Leftrightarrow c$ is dual to $b \Leftrightarrow c$.

A splitting subobject is a subobject represented by monomorphisms with right-inverses.

Corollary 5.2. *If $\mathcal{A}$ is such that for every $A \in \mathcal{A}$ the class of splitting-subobjects is in $\mathcal{S}$ then there is an $\mathcal{S}$-valued functor which reflects isomorphism.*

Proof. If $X \to A$, $X' \to A$ represent the same splitting subobject, i.e. if there exist

$$\begin{tikzcd} X \arrow[dr] \arrow[dd] & \\ & A, \\ X' \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[dr] \arrow[dd] & \\ & A \\ X \arrow[ur] & \end{tikzcd}$$

then $X \to A$, $X' \to A$ represent the same class in T.

Corollary 5.3. *If $\mathcal{A}$ is such that for every $A \in \mathcal{A}$ the class of idempotents, of A is in $\mathcal{S}$ then there is an $\mathcal{S}$-valued functor which reflects isomorphism.*

Proof. For each splitting subobject of A choose a representative $X \to A$ and a right-inverse $A \to X$ and define

$$(\text{Splitting subobjects of } A) \to (\text{Idempotents of } A)$$

by $[X \to A] \mapsto [A \to X \to A]$.

The function is one-to-one: If $X \to A$, $X' \to A$ are such that there exist right-inverse $A \to X$, $A \to X'$ with

$$A \to X \to A = A \to X' \to A$$

then

$$\begin{tikzcd} X \arrow[d] \arrow[dr] & \\ A \arrow[d] & A \\ X \arrow[ur] & \end{tikzcd}$$

and

$$\begin{tikzcd} X' \arrow[d] \arrow[dr] & \\ A \arrow[d] & A. \\ x \arrow[ur] & \end{tikzcd}$$

Corollary 5.4. *If $\mathcal{A}$ is locally-small there exists a set-valued functor which reflects isomorphisms.*

We call a functor $F : \mathcal{A} \to \mathcal{S}$ proper if for every $\mathcal{S}' \subset \mathcal{S}$ equivalent to a small category, $F^{-1}(\mathcal{S}')$ is equivalent to a small category. (Proper maps between topological spaces, it may be recalled, are those such that inverse images of compacts are compact).

For locally-small $\mathcal{A}$, F is proper iff for every cardinal K only a set of isomorphism classes are represented by $\{A \mid F(A)$ has cardinality $K\}$.

Theorem. *There exists a proper set-valued functor from $\mathcal{A}$ iff $\mathcal{A}$ is locally-small and its isomorphism classes may be indexed by the cardinals.*

Proof. Note that the last condition may stated as follows: there exists a proper functor $\wp(\mathcal{A}) \to \mathcal{S}$, where $\wp(\mathcal{A})$ is the subcategory of isomorphisms.

$\Rightarrow$ If $F : \mathcal{A} \to \mathcal{S}$ is proper then for $A, B \in \mathcal{A}$ let $\mathcal{S}'$ be the full subcategory of $F(A)$, $F(B)$.

Since $F^{-1}(\mathcal{S}')$ must have small skeleton, (A, B) must be small. The last condition is clear in light of our remarks above.

$\Leftarrow$ It suffices to construct a proper functor from any skeleton of $\mathcal{A}$, hence we assume that $\mathcal{A}$ is skeletal. Let K : (Objects $\mathcal{A}$) $\to$ Cardinals be a one-to-one function.

For each $A \in \mathcal{A}$ define $S^A \subset (A, -)$ by

$$(A \to B) \in S(B) \text{ iff } A \to B \text{ has no right-inverse.}$$

Let

$$\begin{tikzcd} S^A \arrow[r] \arrow[d] & (A, -) \arrow[d] \\ * \arrow[r] & T^A \end{tikzcd}$$

be a pushout, where $*$ is the terminal functor. For each $B \in \mathcal{A}$,

$$\begin{tikzcd} S^A(B) \arrow[r] \arrow[d] & (A, B) \arrow[d] \\ * \arrow[r] & T^A(B) \end{tikzcd}$$

is a pushout in the category of sets. $T^A(B)$ may be constructed as $((A, B) - S^A(B)) \cup \{*\}$, that is, the set of maps from $A \to B$ which do have a right-inverse together with $\{*\}$. $T^A(B) \neq \{*\}$ iff there exists $A \to B$ with a right inverse.

For fixed B, $T^A(B) = \{*\}$ for all but a set of A. The argument is in 5.3: there are only a set of splitting subobjects of B, hence a set of isomorphisms types which appear as splitting subobjects.

Define

$$F = \prod_{A \in \mathcal{A}} (T^A)^{K(A)}.$$

For each B, $(T^A(B))^{K(A)} = \{*\}^{K(A)}$, for all but a set of A's, and the product exists as a set valued functor.

If $F(B)$ has cardinality L, then $K(B) < L$ because $(T^B(B))^{K(B)} \subset F$, $T^B(B)$ has at least 2 elements, hence L must be at least as large as $2^{K(B)}$. Thus $\{B \mid F(B)$ has cardinality $L\} \subset \{B \mid K(B) < L\}$ a set.

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