# Differential automata theory

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This work arises from the synergy between three main interests:

- formal category theory
- differential algebra
- automata theory

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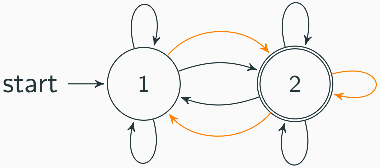
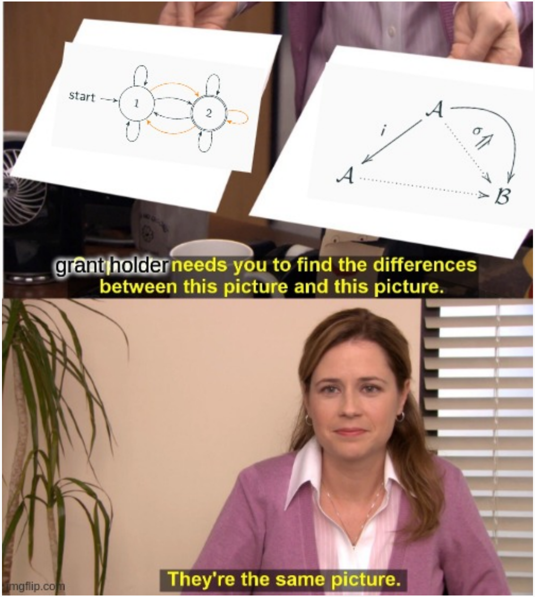
For the past two years, I have been convinced that

> *A fragment of formal category theory is the mathematical foundation for the theory of 'state machines'.*




$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "i"'] \arrow[dr, dotted] \arrow[dr, bend left=60] \arrow[dr, phantom, shift left=3, "\overset{\sigma}{\Nearrow}"] & \\ \mathcal{A} \arrow[rr, dotted] & & \mathcal{B} \end{tikzcd}$$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This means two things:

- if you are a category theorist, there is some interesting mathematics to unravel trying to understand the theory of state machines.
- if you are not, there is some interesting application for what has been considered, in the last 47 years (Street-Walters: 1978) the pinnacle of abstract nonsense.

Approach the subject from the direction you prefer, you will gain something.

Techniques from representation theory, topos theory, homotopy theory, etc. are all useful and pop up constantly.

# Formal category theory

In the broadest possible sense, formal category theory is the study of what properties make a 2-category ‘behave like **Cat**’ abstracting away from its ‘concreteness’.

Not a new idea:

- algebraic theories: find properties making a category $\mathcal{E}$ behave like $\mathbf{Mod}(T)$;
- topos theory: find properties making a category $\mathcal{E}$ behave like **Set**;
- homological algebra: find properties making a category $\mathcal{E}$ behave like $\mathbf{Ch}(\mathcal{A})$;
- homotopy theory: find properties making a category $\mathcal{E}$ behave like $\mathbf{Ho}(\mathbf{Top})$;
- ...

In stark contrast with topos theory, where a definitive answer to what are the intrinsic properties of a set-like category exists, FCT is an active field of study, with (yet) no unanimous consensus on what the right axioms are.

A variety of different approaches to axiomatize the properties of **Cat** has been proposed, each with its own merits and drawbacks.

- enriched category theory / categories are monoids and-or modules, take I
- internal category theory / categories are models of a theory
- category theory relative to a base topos / categories are generalized spaces, take I
- calculus of fibrations / categories are generalized spaces, take II
- double categories / categories are monoids and-or modules, take I

Squinting your eyes,

A formal theory of categories is supposed to give you

- 2-dimensional structures
- monads
- Kleisli constructions
- calculus of Kan exts
- calculus of bimodules
- universal properties
- free cocompletions

A theory of automata is about

- morphisms $\equiv$ processes
- recognition of languages
- behaviour
- minimization

I thought long and hard and. . .

. arXiv:2305.00272 [pdf, other] math.CT cs.FL

The semibicategory of Moore automata

**Authors:** Guido Boccali, Bojana Femić, Andrea Laretto, Fosco Loregian, Stefano Luneia

Completeness for Categories of Generalized Automata ((Co)algebraic pearls)

Authors Guido Boccali, Andrea Laretto, Fosco Loregian, Stefano Luneia

9. arXiv:2303.03865 [pdf, other] math.CT cs.FL doi 10.4204/EPTCS.397.1

Bicategories of Automata, Automata in Bicategories

**Authors:** Guido Boccali, Andrea Laretto, Fosco Loregian, Stefano Luneia

. arXiv:2501.01882 [pdf, other] math.CT cs.FL

Monads and limits in bicategories of circuits

**Authors:** Fosco Loregian

. arXiv:2401.04242 [pdf, ps, other] math.CT cs.FL doi 10.1007/978-3-031-66438-0_4

Automata and coalgebras in categories of species

**Authors:** Fosco Loregian

A category of automata **is just**™ the (2-)category of models of an enriched sketch (Borceux-Quinteiro 1998) constructed as follows:

- take a 2-category $\mathcal{K}$ with finite strict 2-limits;
- fix objects $X, B \in \mathcal{K}_0$ and morphisms

$$\begin{tikzcd} B \arrow[r, "b"] & X \arrow[loop right, "f"] \end{tikzcd}$$

We call such a (finite weighted) limit sketch an automata theory.

- construct step by step the following finite limits:

$$
\begin{tikzcd}[column sep=small]
& & \mathbf{Mly}(f,b) \arrow[dl] \arrow[dr] & & \\
& f/b \arrow[dl] \arrow[dr] \arrow[rr, phantom, "pb"] & & Ins(f,1) \arrow[dl] \arrow[dr] & \\
B \arrow[dr, "b"'] \arrow[rr, phantom, "\underset{comma}{\Leftarrow}"] & & X \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\underset{insert}{\Leftarrow}"] & & X \arrow[dl, equal] \\
& X & & X &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
& & \mathbf{Mre}(f,b) \arrow[dl] \arrow[dr] & & \\
& X/b \arrow[dl] \arrow[dr] \arrow[rr, phantom, "pb"] & & Ins(f,1) \arrow[dl] \arrow[dr] & \\
B \arrow[dr, "b"'] \arrow[rr, phantom, "\underset{comma}{\Leftarrow}"] & & X \arrow[dl, equal] \arrow[dr, equal] \arrow[rr, phantom, "\underset{insert}{\Leftarrow}"] & & X \arrow[dl, equal] \\
& X & & X &
\end{tikzcd}
$$

‘Mealy’ automata　　　　‘Moore’ automata

Unraveling the definition if $\mathcal{K} = \mathbf{Cat}$, $B = 1$ (so $B : 1 \to \mathcal{X}$ picks an object of $\mathcal{X}$): given a diagram of categories and functors

$$1 \xrightarrow{B} \mathcal{X} \xleftarrow{F} \mathcal{X}$$

- $\mathbf{Mly}(F, B)$ is the category $\overset{\textit{inserter}}{Alg(F)} \overset{\textit{pb}}{\times_{\mathcal{X}}} \overset{\textit{comma}}{(F/B)}$ of spans

  $$X \longleftarrow FX \longrightarrow B$$

  and suitable morphisms.
- $\mathbf{Mre}(F, B)$ is the category $\overset{\textit{inserter}}{Alg(F)} \overset{\textit{pb}}{\times_{\mathcal{X}}} \overset{\textit{comma}}{(\mathcal{X}/B)}$ of disconnected spans

  $$X \longleftarrow FX, X \longrightarrow B$$

  and suitable morphisms.

In particular when $\mathcal{X}$ is monoidal, and $F = A \otimes -$, one studies categories where **objects** and morphisms are as follows:

$$\begin{tikzcd} X & A \otimes X \arrow[l] \arrow[r] \arrow[d, "A\otimes f"] & B \arrow[d, equal] \\ Y & A \otimes Y \arrow[l] \arrow[r] & B \arrow[from=1-1, to=2-1, "f"] \end{tikzcd}$$

**Proposition**

The assignment $(A, B) \mapsto \mathbf{Mly}_{\mathcal{X}}(A \otimes -, B) = \mathbf{Mly}_{\mathcal{X}}(A, B)$ defines an indexed category

$$\mathcal{X}^{\mathrm{op}} \times \mathcal{X} \longrightarrow \mathbf{Cat}$$

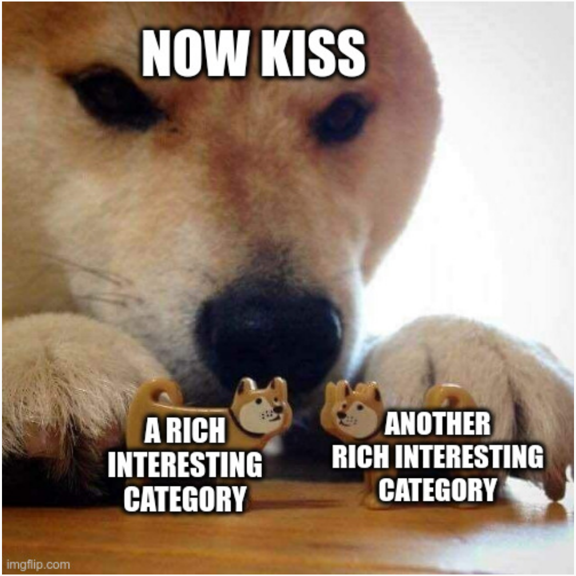
which when $\mathcal{X}$ is Cartesian forms the hom-category of the bicategory of Mealy automata.

In Automata and Coalgebras [...] species
10.4204/EPTCS.380.10 I studied a particular case of this construction, when

- $\mathcal{K}$ is the category of Joyal's combinatorial species;
- $F = \partial$ is the derivative of species.

This is a lot of structure:

> *Categories of automata (in fact, fibers of a monoidal fibration), with objects having a combinatorial meaning, equipped with a notion of derivative functor, in which to do categorified differential algebra / study non-reversible dynamical systems induced by a diff. op.*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# Differential 2-rigs

The pair $(\mathbf{Spc}, \partial)$ is an instance of a differential 2-rig (L/L-Trimble 2020), i.e. a category equipped with a 'linear and Leibniz endofunctor' $\partial$

**Theorem**

The category **Spc** has a universal property qua 2-rig and qua differential 2-rig.

- it is the free cocomplete 2-rig on one generator;
- the category of species in a countable set of 'colours' is the free differential 2-rig on one generator.

## Species

Let $S$ be a set. Let $\mathcal{V}$ be a symmetric monoidal closed, complete and cocomplete category (a 'cosmos').

Regard $S$ as a discrete category, let $\mathbf{P}[S]$ be the free symmetric monoidal category on $S$.

**Definition**

The category $(S, \mathcal{V})$-**Spc** of $(S, \mathcal{V})$-species is the category of functors $\mathbf{P}[S] \to \mathcal{V}$ and natural transformations.

For today, $S = \{*\}$ is a singleton, and $\mathcal{V} = \mathbf{Set}$. Other choices are possible (e.g., $\mathcal{V} = \mathbf{Vect}_k$ is probably the version algebraic topologists are more familiar with). Then

$$\boxed{\mathbf{P} := \mathbf{P}[1] \qquad \mathbf{Spc} := (1, \mathbf{Set})\text{-}\mathbf{Spc} = [\mathbf{P}, \mathbf{Set}]}$$

# Species

- **Spc** is the category of copresheaves on **P**, presented as groupoid of natural numbers: objects finite sets $[n]$, morphisms bijections (in partic. $\mathbf{P}([n],[m]) = \varnothing$ if $n \neq m$)
- Rich supply of monoidal structure(s) interacting with each other (esp. when instead of **Set**-presheaves one takes $k$-linear presheaves)
- **Spc** is equipped with $\partial : \mathbf{Spc} \to \mathbf{Spc}$ that 'shifts' a functor by 1, $F'[n] := F[n+1]$
- Leibniz rule $(F \otimes G)' \cong F' \otimes G + F \otimes G'$ (Day convolution)
- Chain rule $(F \circ G)' \cong (F' \circ G) \otimes G'$ (operadic or 'plethystic' composition)
- $L \dashv \partial \dashv R$ (this is important and nontrivial!)

Axiomatizing these properties leads to D2Rs. First, let's study species better expanding on the items of this list.

# Species

A species $F : \mathbf{P} \to \mathbf{Set}$ is a family of right $S_n$-sets $X_n$:

$$\mathbf{Cat}(\mathbf{P}, \mathbf{Set}) \cong \mathbf{Cat}\big(\textstyle\sum_{n\geq 0} S_n, \mathbf{Set}\big) \cong \textstyle\prod_{n\geq 0} \mathbf{Cat}(S_n, \mathbf{Set})$$

- The species $E$ of elements; constant at the singleton / $S_n$ action is always trivial
- The species $P$ of subsets; sends $[n]$ to $2^n = \{U \subseteq [n]\}$ / $S_n$ action is by permuting a subset
- The species $Sym$ of permutations; sends $[n]$ to $S_n$ / $S_n$ action is by multiplication
- The species $L$ of linear orders; sends $[n]$ to the set of linear orders $L_n$ on $[n]$ / $S_n$ action is by conjugation
- The species $Cyc$ of cyclic orders, def'd similarly.

# Species

- $[n] \oplus [m] := [n+m]$ defines a (symmetric) monoidal structure on $\mathbf{P}$;
- $\mathbf{Spc}$ inherits a Day convolution (symmetric, closed) monoidal structure
$$\mathbf{Spc}(F * G, H) \cong \mathbf{Spc}(F, \{G, H\})$$
- There is a functor $\partial : \mathbf{Spc} \to \mathbf{Spc}$ defined by 'shifting $F$ by 1'
$$\partial F[A] := F[A+1] \qquad \text{Try to prove the Leibniz rule!}$$

$$f(X) = \sum_{n \geq 0} \frac{a_n}{n!} X^n \qquad F[X] = \sum_{n \geq 0} \frac{F[n]}{\sim_{S_n}} X^n$$
$$\frac{d}{dX} f(X) = \sum_{n \geq 0} \frac{a_{n+1}}{(n+1)!} X^n \qquad \partial F[X] = \sum_{n \geq 0} \frac{F[n+1]}{\sim_{S_{n+1}}} X^n$$

- $E' \cong E \qquad Cyc' \cong L \qquad P' \cong E + E$ (*natural* isos of functors)
- $\partial$ has a left adjoint (easy to describe: $\partial = \{y[1], -\}$ hence $L = y[1] * -$), but also a right adjoint (because $y[1]$ is a tiny object)

# D2Rs

This motivates the definition of a differential 2-rig (D2R):

A 2-rig $(\mathcal{R}, \otimes)$ equipped with an endofunctor $\partial : \mathcal{R} \to \mathcal{R}$ such that

- $\partial(A + B) \cong \partial A + \partial B$
- $\partial(A \otimes B) \cong \partial A \otimes B + A \otimes \partial B$

Equivalently: $\partial$ is equipped with two tensorial strengths, forming a coproduct diagram

$$\partial A \otimes B \to \partial(A \otimes B) \leftarrow A \otimes \partial B$$

This realizes the Leibniz rule as a universal property.

 

# Differential 2-rigs

(and differential automata)

1. sketch the technology one can develop for D2Rs, categorifying differential algebra;
2. apply some of these ideas to a specific case for a category of automata.

Clearly, 2. is just a pretext for 1.

# Free objects and quotients

# Freeness results

**Spc** is the free (cocomplete) 2-rig $F[t]$ on a single generator $\{t\}$; it acquires a differential structure much like $k[x]$ does.

**Spc** is also initial among cocomplete 2-rigs.

The free differential 2-rig on a single generator is also a category of species:

$$F_\partial[Y, Y', Y'', \dots] \cong \mathbf{Set}^{\mathbf{P}[y_0, y_1, y_2, \dots]}$$

More generally one can define the free 2-rig on a category...

And extend scalars over $\mathcal{R}$ –defining the free $\mathcal{R}$-algebra on $S$: $F[S] \otimes \mathcal{R}$

$\mathcal{R}$ a 2-rig; $\mathcal{R}[t] = \mathcal{R} \otimes_{\mathbf{P}} F[t] = \mathcal{R} \otimes_{\mathbf{P}} \mathbf{Spc}$

# Geometry of D2Rs

$$\{\text{derivations on } R\} \cong \left\{ s : \begin{tikzcd} R[t]/t^2 \arrow[d, "ev_0"] \\ R \arrow[u, bend left, "s"] \end{tikzcd} \right\}$$

 Kähler differentials

$$\{\text{derivations on } \mathcal{R}\} \cong \hom_{\textbf{2-Rig}}(\mathcal{R}, \mathcal{R}[t]/(t^2))$$

$$\mathcal{R}[t]/(t^2) \cong \text{coinverter}\Big( \begin{tikzcd} \mathcal{R}[t] \arrow[r, bend left, "\varnothing"] \arrow[r, bend right, "-\otimes t^2"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}[t] \end{tikzcd} \Big)$$

a certain kind of 2-dimensional colimit

# Intermezzo

## Definition

Given a 2-category $\mathcal{K}$ and a diagram

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\alpha"] & B \end{tikzcd}$$

the coinverter of $f, g$ is a 1-cell $c : B \to Q$ such that $c * \alpha : cf \Rightarrow cg$ is invertible; $(Q, c)$ is 1-initial and 2-initial among such pairs.

E.g. if $\mathcal{C}$ is a category, $W \subseteq \mathcal{C}^2$ a class of maps; the coinverter of

$$\begin{tikzcd} (W \subseteq \mathcal{C}^2) \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \end{tikzcd}$$

is the Gabriel-Zisman localization $\mathcal{C}[W^{-1}]$.

# Geometry of D2Rs

Let $\mathcal{R}$ be a cocomplete 2-rig.

Consider the unique 2-cell $\varnothing \Rightarrow (- \otimes t^2)$, where $- \otimes t^2$ 'multiplies by $t^2$'; the coinverter

$$\begin{tikzcd} \mathcal{R}[t] \arrow[r, bend left, "\varnothing"] \arrow[r, bend right, "- \otimes t^2"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}[t] \arrow[r, two heads, "q"] & C \end{tikzcd}$$

coincides with the procedure of killing off polynomials divisible by $t^2$, hence $C$ is the 'quotient 2-rig' by the ideal $(t^2)$.

$$(a + tb)(c + td) = ac + (ad + bc)t + \cancel{t^2 bd}$$

# Geometry of D2Rs

Now we would like to build the ‘space of sections’ of a canonical ‘evaluation at 0’ map

$$\begin{tikzcd}
\mathcal{R}[t] \arrow[r, bend left, "\varnothing"] \arrow[r, bend right, "-\otimes t^2"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}[t] \arrow[rr, two heads, "q"] \arrow[dr, "@_0"'] & & \mathcal{R}[\epsilon] \arrow[dl, dotted, "ev_0"] \\
& & \mathcal{R} &
\end{tikzcd}$$

$@_0 : \mathcal{R} \otimes_{\mathbf{P}} \mathbf{Spc} \to \mathcal{R} : (A, F) \mapsto \sum_{F[0]} A$ is induced by the universal property of coproducts.

**Theorem**

$$Der[\mathcal{R}] \cong \{\text{sections}/\mathcal{R} \text{ of } ev_0 : \mathcal{R}[\epsilon] \to \mathcal{R}\}$$

## Geometry of D2Rs

- similarly: quotient for a principal ideal, say $\mathfrak{J} = (p)$, is coinverter of

$$\begin{tikzcd} \mathcal{R}[t] \arrow[r, bend left, "\varnothing"] \arrow[r, bend right, "-\otimes p(t)"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}[t] \arrow[r, two heads, "q"] & \mathcal{R}[t]/(p) \end{tikzcd}$$

- Ideals are easy to define, but
  - Domains? $A \otimes B \cong \varnothing \Rightarrow A = B = \varnothing$?
  - quotient for a non-principal ideal $\mathfrak{J} = (p_i \mid i \in I)$ is a...?
  - What's a 2-PID?
- quotients like $\mathcal{R}[X, Y]/(Y^2 + 1 \cong X^2)$ (categorified hyperbola) acquire a differential structure, $\partial Y = X, \partial X = Y$; can be done more in general?

# Jet spaces

# Categorified jet spaces

Given a D2R $(\mathcal{R}, \otimes, \partial)$ let $\mathbf{Alg}(\partial)$ be the category of $\partial$-algebras.

- objects: $(X, \xi : \partial X \to X)$;
- morphisms: $f : X \to Y$ compatible with the structure map.

**Theorem**

$\mathbf{Alg}(\partial)$ is itself a 2-rig and $\partial$ lifts to a derivation $\partial'$ on $\mathbf{Alg}(\partial)$, compatible with the forgetful functor.

Hence the chain

$$\begin{tikzcd}
\dots \arrow[r] & \mathbf{Alg}(\partial') \arrow[r] \arrow[d, "\partial''"] & \mathbf{Alg}(\partial) \arrow[r] \arrow[d, "\partial'"] & \mathcal{R} \arrow[d, "\partial"] \\
\dots \arrow[r] & \mathbf{Alg}(\partial') \arrow[r] & \mathbf{Alg}(\partial) \arrow[r] & \mathcal{R}
\end{tikzcd}$$

# Categorified jet spaces

Define by mutual induction:

- $\mathcal{R}^{(0)} := \mathcal{R}$ and $\mathcal{R}^{(n+1)} := \mathbf{Alg}(\partial^{(n)}, \mathcal{R}^{(n)})$;
- $\partial^{(1)} := \partial$ and $\partial^{(n+1)} := \mathcal{R}^{(n+1)} \to \mathcal{R}^{(n+1)}$ defined lifting $\partial^{(n)}$.

**Definition**

From the chain of forgetful functors

$$\mathcal{R} \longleftarrow \mathbf{Alg}(\partial) \longleftarrow \mathbf{Alg}(\partial') \longleftarrow \mathbf{Alg}(\partial'') \longleftarrow \cdots$$

$$\mathbf{Jet}[\mathcal{R}, \partial] := \lim\left(\mathcal{R} \xleftarrow{U} \mathcal{R}^{(1)} \xleftarrow{U^{(1)}} \mathcal{R}^{(2)} \xleftarrow{U^{(2)}} \cdots\right).$$

# Categorified jet spaces

The typical object in $\mathbf{Jet}[\mathcal{R}, \partial]$ consists of a countable sequence

$$\vec{X} = \big(X, (X; \xi : \partial X \to X), ((X;\xi); \xi' : \partial'(X;\xi) \to (X;\xi)), \dots\big)$$

the $n^{\text{th}}$ element of which equips the $(n-1)^{\text{th}}$ with an algebra structure for $\partial^{(n)}$.

$$X \xleftarrow{\xi} \partial X \xleftarrow{\xi'} \partial\partial X \xleftarrow{\xi''} \partial\partial\partial X \leftarrow \dots$$

# Categorified jet spaces

Define the *k-jet* $J^k(\vec{X})$ of an object $\vec{X} \in \mathbf{Jet}[\mathcal{R}, \partial]$ as the image of $\vec{X}$ under the functor $J^k$ obtained from the limit projections $\pi_k : \mathbf{Jet}[\mathcal{R}, \partial] \to \mathcal{R}^{(k)}$ as

$$J^k := \langle \pi_0, \dots, \pi_k \rangle : \mathbf{Jet}[\mathcal{R}, \partial] \longrightarrow \prod_{i=0}^{k} \mathcal{R}^{(i)}$$

cf. differential geometry, where the $k$-jet of a real valued function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$(J^k_{x_0} f)(z) = \sum_{\ell=0}^{k} \frac{f^{(\ell)}(x_0)}{\ell!} z^\ell = f(x_0) + f'(x_0)z + \cdots + \frac{f^{(k)}(x_0)}{k!} z^k$$

# An application to automata

# Differential automata

Let $\mathcal{R}$ be a D2R; the assignment $(A, B) \mapsto \mathbf{Mly}(A \otimes -, B)$ defines a two-sided fibration via the Grothendieck construction

$$\underset{Mly:(A,B)\mapsto\mathbf{Mly}(A\otimes -,B)}{\mathbf{Psd}(\mathcal{R}^{\mathrm{op}} \times \mathcal{R}, \mathbf{Cat})} \underset{\sim}{\rightleftarrows} \underset{(V:\mathbf{Mly}_{\mathcal{R}}\twoheadrightarrow\mathcal{R}^{\mathrm{op}}\times\mathcal{R})}{\mathbf{Fib}/(\mathcal{R}^{\mathrm{op}} \times \mathcal{R})}$$

which is a D2R morphism with respect to a canonical differential structure on the domain $\mathbf{Mly}_{\mathcal{R}}$.

$V$ is a fibration of trajectories for discrete dynamical systems of endpoints $A, B$; each category of trajectories $\mathbf{Mly}(A, B)$ has a limit (=terminal) object

$$\textstyle\prod_{n\geq 1}[A^{\otimes n}, B]$$

(Analogy: the limit set of a dynamical system $\overline{A}^f := \bigcap_{n\geq 1} \overline{f^n(A)}$, where $f$ is en endomap of a metric space $A$.)

# Differential automata

If $\mathcal{R}$ is monoidal closed, $\mathbf{Mly}_{\mathcal{R}}$ is a category of coalgebras for a certain endofunctor $R : \mathcal{R}^{\mathsf{op}} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}^{\mathsf{op}} \times \mathcal{R} \times \mathcal{R}$, fibred over the projection $\pi_{12}$

$$(A, B, X) \longmapsto (A, B, [A, B \times X])$$

There is a distributive law $\delta : (1 \times \partial)R \Rightarrow R(1 \times \partial)$

$$\begin{tikzcd}
\mathbf{Mly}_{\mathcal{R}} \arrow[r, "\bar{\partial}"] \arrow[d] & \mathbf{Mly}_{\mathcal{R}} \arrow[d] \\
\mathcal{R}^{\mathsf{op}} \times \mathcal{R} \arrow[r, "1\times\partial"'] & \mathcal{R}^{\mathsf{op}} \times \mathcal{R}
\end{tikzcd}$$

lifting $\partial$ to a derivative functor $\bar{\partial}$ on $\mathbf{Mly}_{\mathcal{R}}$; the category of differential automata is the category of coalgebras for such $\bar{\partial}$.

# Differential automata

Let $\mathcal{R}$ be the D2R of species; observe that

- the species $L$ of linear orders is the free monoid on the monoidal unit (plays the role of an NNO in **Spc**);
- thus there are four equivalent descriptions for the category of $\mathbf{Spc}^L$ of $L$-algebras, building block for $\mathbf{Mly}([1], B)$:
  - the category of algebras for the functor $[1] * -$;
  - the category of EM algebras for the monad $L \otimes -$;
  - the category of coalgebras for the functor $\partial$;
  - the category of coEM algebras for the comonad $\{L, -\}$.

## Differential automata

Similar reasoning applies to scopic D2R, where $\partial$ has both a left and aright adjoint. There are plenty of variations on the theme of categories of species which are scopic D2Rs, e.g.:

- The category of $S$-species, i.e. functors $\mathbf{P}[S] \to \mathbf{Set}$ for an arbitrary set $S$; this supports partial derivatives, $\{\partial_s \mid s \in S\}$;
- $k$-vector ($S$-)species (Aguiar-Mahajan I,II,III,IV), i.e. functors $\mathbf{P}[S] \to \mathbf{Vect}_k$;

# Differential automata

- linear species, i.e. families of functors of the form $X_n : [S_n/S_n] \to \mathbf{Set}$, where $[S_n/S_n]$ is the action groupoid of the regular representation of $S_n$ on itself; (widely studied because differential equations admit unique solution here);
- Möbius species, where functors out of $\mathbf{P}[S]$ are valued in a category of posets with top and bottom (Möbius inversion formula has a category-theoretic proof);
- nominal sets, i.e. representations of the filtered colimit $S_1 \subset S_2 \subset S_3 \subset \dots$ of finite symmetric groups on the set of finite sets; (this is only a left scopic D2R; widely used in TCS).

# Differential automata

There are examples

- of species having no $\partial$-coalgebra structures, but acquire many when linearized (i.e. considered as $k$-vector species instead of **Set**-species);
- of species having a finite number of $\partial$-coalgebra structures (precisely four);
- of species having uncountably many $\partial$-coalgebra structures.

(The fact that a coalgebra map must be $S_n$-equivariant is often a strong restriction on the structure of the coalgebra!)

# Wrapping up

So:

- D2Rs $\cap$ automata $\approx$ differential discrete dynamics
- there's a ring theory to write for 2-rigs
- these objects are highly structured ($\partial I \neq 0$, self-similarity,$\dots$)
- it's 'difficult' for a category to be a diff-2-rig ($Der(\mathcal{R})$ knows about a 'dimension' of $\mathcal{R}$)
- yet, differential algebra is quite interesting (differential equations?)