Exercises ITI9200

February 8, 2025

1 Weeks 1-2

Exercise 1:
(*Light jumping jacks, but GySgt Hartman is behind you shouting "BOURBAKI!"*)

1. Prove or disprove that the following operations define monoid structures:
   - the set $\mathbf{R}^+ = \{x \in \mathbf{R} \mid x > 0\}$ of strictly positive real numbers, with respect to the operation of division, $(a, b) \mapsto a/b$;
   - the set of pairs of integers (m, n), with the operation $(p, q) \star (r, s) := (pr - qs, ps + qr)$.
2. Let S be a *finite* set, consider the monoid S^S of all functions $f : S \to S$, with respect to function composition. Prove that the following conditions are equivalent:
   - f is an injective function;
   - f is a surjective function;
   - f is a bijective function.

   This is *blatantly false* when S is infinite, say the set $\mathbb{N} = \{0, 1, 2, \dots\}$ of natural numbers. Build a counterexample.

Exercise 2:
(*Epimenides, Cantor, and Gödel enter a bar...*, [1])
An *applicative construct* (AC for short) $(A, \circ)$ consists of a nonempty set A with a binary operation $\circ : A \times A \to A$. If $(f, a) \in A \times A$, we denote $\circ(f, a)$ as $f \circ a$ and read 'f *applied to* a'.
If $(A, \circ)$ is an AC we say that

- $f \in A$ *has a fixpoint* $\mu_f \in A$ if $f \circ \mu_f = \mu_f$;
- $f \in A$ *has a diagonalizer* $\delta_f \in A$ if for every $a \in A$ the identity

  $$\delta_f \circ a = f(a \circ a)$$

  holds (brackets position is important).

Prove *Smullyan's mythological fixpoint theorem*:

If f has a diagonalizer δ_f, then it has a fixpoint μ_f.

Exercise 3:
(*'I know what a category is...' —Show me.*)

- Can a category with 7 objects and 5 morphisms exist?
- Count how many categories with 3 objects and (exactly) 5 morphisms there are.

References

[1] N. S. Yanofsky. A universal approach to self-referential paradoxes, incompleteness and fixed points. *Bulletin of Symbolic Logic*, 9(3):362–386, 2003.