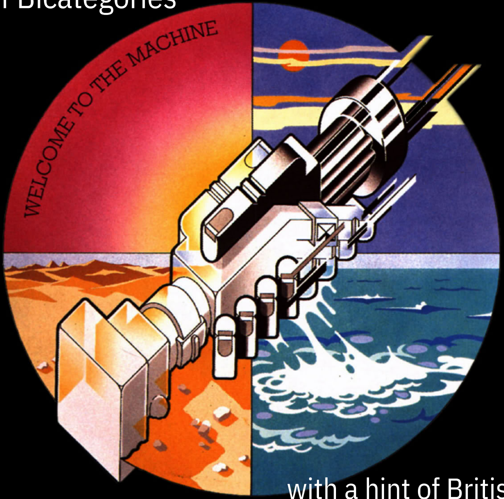
A Tale of Bicategories



with a hint of British prog

Prolegomena

j/w G. Boccali, A. Laretto, S. Luneia; EPTCS.397.1 .

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- Boccali, G., Femić, B., Laretto, A., ____, & Luneia, S. “The semibicategory of Moore automata.” arXiv:2305.00272 .

A theory of abstract automata

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and the object of (f, b)-Moore machines is the pullback on the right.

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and the object of (f, b)-Moore machines is the pullback on the right.

As such, **Mly** and **Mre** are parametric functors of type

$$\mathbf{K}(C, C)^{\mathrm{op}} \times \mathbf{K}/C \longrightarrow \mathbf{K}/C$$

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- the category of Mealy automata, where objects and morphisms are of the form

$$\begin{tikzcd} e \arrow[d, dotted, "\varphi"'] & Fe \arrow[l, "d"'] \arrow[r, "s"] \arrow[d, dotted, "F\varphi"] & b \arrow[d, equal] \\ e' & Fe \arrow[l, "d"'] \arrow[r, "s"] & b \end{tikzcd}$$

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- the category of Moore automata, where objects and morphisms are of the form

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In particular, if $F_A : \mathbf{K} \to \mathbf{K}$ is the functor depending on an object A (an 'Alphabet') Mealy and Moore automata are respectively diagrams of the form (E, d, s):

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- $d : A \otimes E \to E$ is an action of A on E (a dynamical system);
- s is an output function (think of $B = \{0, 1\}$ or $B = [0, 1]$, etc.)

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- If $\mathbf{K}$ has countable sums, $d : A \otimes E \to E$ is an action of $A^* := \sum_n A^n$, and s extends similarly:

$$\begin{tikzcd}
 & A \otimes E \arrow[dl, "d"'] \arrow[dr, "s"] \arrow[dd, "\eta_A \otimes E"'] & \\
E & & B \\
 & A^* \otimes E \arrow[ul, dotted, "d^*"] \arrow[ur, dotted, "s^*"'] &
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This is called the canonical extension of (E, d, s).

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- If $(\mathbf{K}, \otimes)$ is monoidal and $T : \mathbf{K} \to \mathbf{K}$ is a commutative monad over it, we can lift the monoidal structure of $\mathbf{K}$ making the free functor $F : \mathbf{K} \to \mathbf{Kl}(T)$ strong monoidal.

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  Machines in $\mathbf{Kl}(T)$ are non-deterministic versions of the ones in $\mathbf{K}$.

  Take T the powerset monad on **Set**, or a distribution/probability monad like the one of finite distributions –whose algebras are convex sets, and free algebras affine simplices).

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- If **K** is cocomplete (e.g., locally presentable), so are $\mathbf{Mly}(A, B)$, $\mathbf{Mre}(A, B)$ for every A, B –with colimits created by the forgetful into $\mathcal{K}$ and connected limits created by the functor in the commæ.

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In particular, the terminal objects of $\mathbf{Mly}(A, B), \mathbf{Mre}(A, B)$ are respectively

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($A^+ =$ free semigroup on A; $A^* =$ free monoid on A).

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Observe that this can be deduced from the fact that when **K** is closed, we can characterize automata coalgebraically, see some work of Jacobs.

(Semi)bicategories of automata

A tale of bicategories

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- define the bicategory $C(\mathbf{K})$ as the bicategory $\mathbf{Psd}(\mathbf{N}, \Sigma\mathbf{K})$ of pseudofunctors and lax natural transformations. Then, a 1-cell in $C(\mathbf{K})$ consists of a pair $(E, x) : E \otimes A \xrightarrow{x} B \otimes E$.

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Evidently, if $(\mathbf{K}, \otimes)$ is Cartesian, the universal property of products splits every x as $\langle s, d \rangle$ where (d, s) fit in the previous span.

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[1] This motivates the compact notation $(E, x) : A \rightarrow B$ to refer to a Mealy machine valued in $\mathbf{K}$.

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- A semibicategory is 'like a bicategory, but without identity 1-cells'
- There exists a semibicategory **Mre** of Moore-type automata, a functor

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and a right adjoint $\mathbf{Mly}(A, B) \to \mathbf{Mre}(A, B)$, altogether forming the components of a local adjunction.

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- the 1- and 2-full sub-bicategory of **Mac**^{s} spanned by monoids (=one-object categories);
- the 2-full sub-bicategory of **Mly** (over **Set**) whose 1-cells are Mealy automata between monoids such that the representation of A^* on E in $E \xleftarrow{d^*} A^* \otimes E \xrightarrow{s^*} B$ induces a functor $\Sigma : \mathcal{E}[d^*] \to B$, when B is a monoid.

Machines valued in a bicategory

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This idea is not *entirely* new; it resembles old (and obscure) work of Bainbridge, modeling the state space of abstract machines as a functor, of which one can take the left/right Kan extension along an 'input scheme'. See work of Petrişan et al.

A bimachine is a span in...

Definition

Let $\mathbb{B}$ be a bicategory; a bicategorical Moore (biMoore) machine in $\mathbb{B}$ is a diagram of 2-cells

$$e \Longleftarrow e \circ i, e \Longrightarrow o$$

between 1-cells e, i, o.[2]

[2] A 1-cell of states (états), of inputs, and of outputs.

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The fact that this span exists, *coherces the types* of i, o, e in such a way that i must be an endomorphism of an object A.

$$A \xrightarrow{i} A, \quad A \xrightarrow{i} A \xrightarrow{i} A, \quad A \xrightarrow{i} A \xrightarrow{i} A \xrightarrow{i} A, \ldots$$

all make sense.

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all make sense.

In the monoidal case, the fact that an input 1-cell stands on a different level from an output was completely obscured by the fact that every 1-cell is an endomorphism.

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Everything will be made a Kan extension

Recall that the terminal objects of $\mathbf{Mly}(A, B)$, $\mathbf{Mre}(A, B)$ are respectively $[A^+, B]$, $[A^*, B]$.

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$$
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> The terminal object of the category of biMoore machines[3] is the right extension of $o : A \to B$ along the free monad $i^\sharp : A \to A$.

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Examples

biMoore in Cat

Regarding **Cat** as a strict 2-category, a biMoore machine is a functor $E : \mathcal{C} \to \mathcal{D}$ closing a span $\mathcal{C} \xleftarrow{I} \mathcal{C} \xrightarrow{O} \mathcal{D}$ with suitable 2-cells.

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If $\mathcal{D} = \mathbf{Set}$, states and output are presheaves, and E is acted by an endofunctor; in this case, the behaviour of the terminal machine can be described as a known object: unpacking the end that defined $Ran_{I^\natural} O$ we obtain the functor

$$A \longmapsto [\mathcal{C}, \mathbf{Set}](\mathcal{C}(A, I^\natural \_), O)$$

sending an object A to the set of natural transformations $\alpha : \mathcal{C}(A, I^\natural \_) \Rightarrow O$; to each generalised A-element of $I^\natural C$ corresponds an element of the output space $\Upsilon_C(u) \in OC$.

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In the special case of $\{0, 1\}$-enriched profunctors (i.e., relations), the Kan extension of behaviour reduces to the maximal E such that $E \subseteq O$ and $E \circ I^\natural \subseteq E$ (here $\circ$ is the relational composition).

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$$(a, b) \in R \iff \forall a' \in A.((a', a) \in I^\natural \Rightarrow (a', b) \in O).$$

This relation expresses *reachability* of b from a:

$$a\, R\, b \iff \Big((a' = a) \vee (a' \xrightarrow{I} a_1 \xrightarrow{I} \dots \xrightarrow{I} a_n \xrightarrow{I} a) \Rightarrow a'\, O\, b\Big)$$

New maps

Intertwiners

Definition (Intertwiner between bicategorical machines)

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An *intertwiner* $(u, v) : (e, \delta, \sigma) \looparrowright (e', \delta', \sigma')$ consists of a pair of 1-cells $u : A \to A', v : B \to B'$ and a triple of 2-cells ι, ϵ, ω disposed as

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "i"'] \arrow[dr, phantom, "\swarrow\iota"] & A' \arrow[d, "i'"] & A \arrow[r, "u"] \arrow[d, "e"'] \arrow[dr, phantom, "\swarrow\epsilon"] & A' \arrow[d, "e'"] & A \arrow[r, "u"] \arrow[d, "o"'] \arrow[dr, phantom, "\swarrow\omega"] & A' \arrow[d, "o'"] \\
A \arrow[r, "u"'] & A' & B \arrow[r, "v"'] & B' & B \arrow[r, "v"'] & B'
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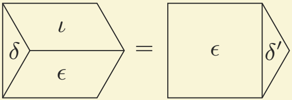
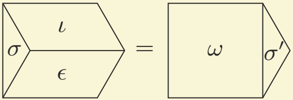


and



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1. there exist morphisms $\iota : I' \otimes U \to V \otimes I, \epsilon : E' \otimes U \to V \otimes E, \omega : O' \otimes U \to V \otimes O$;
2. the following two identities hold:

$$\epsilon \circ (d' \otimes U) = (V \otimes d) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$
$$\omega \circ (s' \otimes U) = (V \otimes s) \circ (\epsilon \otimes I) \circ (E' \otimes \iota)$$

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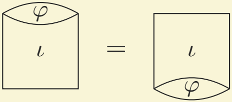

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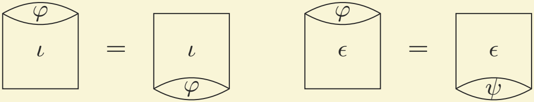

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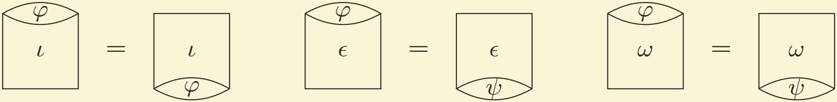

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Vistas

Monoidal topology and automata

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BiMoore and biMealy machines, when instantiated in $(T, \mathcal{V})$-**Prof**, a 2-categorical way to look at topological, (ultra)metric ways to study behaviour of a state machine The reachability relation becomes topological, (ultra)metric, probabilistic, sequential... according to suitable choices of T and $\mathcal{V}$.

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Conjecture

One can address nondeterministic biMoore automata in $\mathbb{B}$ as deterministic bicategorical automata in a proarrow equipment, porting all the paraphernalia (minimisation, behaviour, and bisimulation) into a bigger conceptual framework.

The En(i)d

The Enịd is a simphonic prog rock band from Southampthon;
suggested listening: *Ærie Færie Nonsense* and *Trippin the Light Fantastic*.