ITI9200 — Category theory and Applications

Exercise Sheet 2 — Trimurti: Categories/Functors/Naturality
Spring Semester

Preliminary definitions

Definition 1.1.

1. A *partial binary algebra* is a pair $(X, *)$ consisting of a class X and a partial binary operation $*$ on X; i.e., a binary operation defined on a subclass of $X \times X$. (The value of $*(x, y)$ is denoted by $x * y$.)

2. If $(X, *)$ is a partial binary algebra, then an element u of X is called a *unit* of $(X, *)$ provided that
$$x * u = x \quad \text{whenever } x * u \text{ is defined,}$$
and
$$u * y = y \quad \text{whenever } u * y \text{ is defined.}$$

Definition 1.2. An *object-free category* is a partial binary algebra $\mathbf{C} = (M, \circ)$, where the members of M are called *morphisms*, that satisfies the following conditions:

1. *Matching Condition:* For morphisms f, g, and h, the following conditions are equivalent:
   (a) $g \circ f$ and $h \circ g$ are defined,
   (b) $h \circ (g \circ f)$ is defined, and
   (c) $(h \circ g) \circ f$ is defined.

2. *Associativity Condition:* If morphisms f, g, and h satisfy the matching conditions, then
$$h \circ (g \circ f) = (h \circ g) \circ f.$$

3. *Unit Existence Condition:* For every morphism f there exist units u_C and u_D of $(M, \circ)$ such that $u_C \circ f$ and $f \circ u_D$ are defined.

4. *Smallness Condition:* For any pair of units (u_1, u_2) of $(M, \circ)$ the class
$$\hom(u_1, u_2) = \{\, f \in M \mid f \circ u_1 \text{ and } u_2 \circ f \text{ are defined} \,\}$$
is a set.

Exercise 1:
Let $\mathcal{Q}$ be the following directed graph:

$$\begin{tikzcd} A \arrow[loop above, "a"] \arrow[r, "f"] & B \arrow[loop above, "b"] \arrow[r, "p"] & F_1 \end{tikzcd}$$

- Determine the free category $F\langle\mathcal{Q}\rangle$ on $\mathcal{Q}$ and prove that its set of morphisms determines a regular language in the alphabet $\Sigma = \{a, b, f, p\}$.
- Generalize as follows: if $\mathcal{Q}_n$ is the directed graph

  $$\begin{tikzcd} A \arrow[loop above, "a"] \arrow[r, "f"] & B \arrow[loop above, "b"] \arrow[r, "p_1"] & F_1 \arrow[r, "p_2"] & F_2 \arrow[r, "p_3"] & \dots \arrow[r, "p_n"] & F_n \end{tikzcd}$$

  so that the previous $\mathcal{Q}$ is $\mathcal{Q}_1$, prove that the set of morphisms of $F\langle\mathcal{Q}_n\rangle$ determines a regular language in the alphabet $\Sigma = \{a, b, f, p_1, \dots, p_n\}$. Is this still true if $n \to \infty$?

Exercise 2:

Define the following categories associated to the functor $S_A : X \mapsto 1 + A \times X$.

- ∇S has objects the pairs (X, t) where $t \in S_A X$ is an element; a morphism $(X, t) \to (Y, v)$ in S consists of a function $f : X \to Y$ such that the function Sf sends t to v:

$$Sf : 1 + A \times X \to 1 + A \times Y : t \mapsto v$$

- $\mathbf{coAlg}(S)$ has objects the pairs (X, ξ) where $\xi : X \to S_A X$ is a function; a morphism $(X, \xi) \to (Y, \theta)$ in $\mathbf{coAlg}(S)$ consists of a function $f : X \to Y$ such that

$$Sf \circ \xi = \theta \circ f.$$

Recall, or learn for the first time, that

- an initial objects in a category $\mathcal{C}$ is an object I such that for every other object $X \in \mathcal{C}$, there exists a unique arrow $I \to X$;
- a terminal objects in a category $\mathcal{C}$ is an object T such that for every other object $X \in \mathcal{C}$, there exists a unique arrow $X \to T$.

Prove that $\mathbf{coAlg}(S)$ has a terminal object; prove or disprove that ∇S has an initial object.

Exercise 3:

A *construct* consists of a pair $(\mathcal{C}, U)$ where $\mathcal{C}$ is a category and $U : \mathcal{C} \to \mathbf{Set}$ is a faithful functor. Two constructs $(\mathcal{C}, U)$ and $(\mathcal{D}, V)$ are *strongly equivalent* if there exist two functors

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "U"'] & & \mathcal{D} \arrow[dl, "V"] \\ & \mathbf{Set} & \end{tikzcd} \qquad \begin{tikzcd} \mathcal{D} \arrow[rr, "G"] \arrow[dr, "V"'] & & \mathcal{C} \arrow[dl, "U"] \\ & \mathbf{Set} & \end{tikzcd}$$

such that $F \circ G = \mathsf{id}_{\mathcal{D}}$ and $G \circ F = \mathsf{id}_{\mathcal{C}}$. The qualifier 'strong' is omitted when instead one only has invertible natural transformations $\epsilon : F \circ G \cong \mathsf{id}_{\mathcal{D}}$ and $\eta : \mathsf{id}_{\mathcal{C}} \cong G \circ F$, subject in addition to the *triangle identites*:

$$\begin{aligned} \epsilon_{FC} \circ F\eta_C &= 1_{FC} \\ G\epsilon_D \circ \eta_{GD} &= 1_{GD}. \end{aligned}$$

A (strong) equivalence of constructs is called a *concrete (strong) equivalence* when the construct functor associated to a certain category is implicitly understood.

Are the following pairs concretely equivalent, strongly or not?

- categories, defined in the usual way, and object-free categories, as defined above.
- Kuratowski spaces, defined via an interior operator, and topological spaces defined via a family of closed subsets:
  - a *topological space* consists of a pair (X, τ) where $\tau \subseteq 2^X$ is a collection of subsets of X, such that
    1. $\varnothing, X \in \tau$;
    2. if I is a set and $A_\bullet : I \to \tau$ an I-indexed family of elements $A_i \in \tau$, then $\bigcup_i A_i \in \tau$;
    3. if $A_1, A_2 \in \tau$, then $A_1 \cap A_2 \in \tau$.

    An element of τ is called an *open subset*; an element of the form $X \setminus A$ for $A \in \tau$ is called a *closed subset*.
  - A *Kuratowski space* consists of a set X equipped with a monotone function

    $$j : 2^X \to 2^X$$

    called the *interior operator* of X, satisfying the following properties:
    1. $jX = X$;
    2. for all $S \in 2^X$, $j(S) \subseteq S$;
    3. for all $S \in 2^X$, $j(j(S)) = j(S)$;
    4. for all $S, T \in 2^X$, $j(S) \cap j(T) \subseteq j(S \cap T)$.