

# ITI9200 — Category theory and Applications

## Exercise Sheet 2 — Trimurti: Categories/Functors/Naturality Spring Semester

Basic category theory rests on three conceptual pillars: categories, functors, and natural transformations. The exercises below are meant to make you familiar with these notions and their basic properties, and extend a little bit the perspective on the definition of category.

### Some preliminary definitions

#### Definition 1.1.

1. A *partial binary algebra* is a pair  $(X, *)$  consisting of a class  $X$  and a partial binary operation  $*$  on  $X$ ; i.e., a binary operation defined on a subclass of  $X \times X$ . (The value of  $*(x, y)$  is denoted by  $x * y$ .)
2. If  $(X, *)$  is a partial binary algebra, then an element  $u$  of  $X$  is called a *unit* of  $(X, *)$  provided that

$$x * u = x \quad \text{whenever } x * u \text{ is defined,}$$

and

$$u * y = y \quad \text{whenever } u * y \text{ is defined.}$$

**Definition 1.2.** An *object-free category* is a partial binary algebra  $\mathbf{C} = (M, \circ)$ , where the members of  $M$  are called *morphisms*, that satisfies the following conditions:

1. *Matching Condition:* For morphisms  $f, g$ , and  $h$ , the following conditions are equivalent:
  - (a)  $g \circ f$  and  $h \circ g$  are defined,
  - (b)  $h \circ (g \circ f)$  is defined, and
  - (c)  $(h \circ g) \circ f$  is defined.
2. *Associativity Condition:* If morphisms  $f, g$ , and  $h$  satisfy the matching conditions, then

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

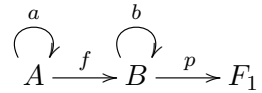
3. *Unit Existence Condition:* For every morphism  $f$  there exist units  $u_C$  and  $u_D$  of  $(M, \circ)$  such that  $u_C \circ f$  and  $f \circ u_D$  are defined.
4. *Smallness Condition:* For any pair of units  $(u_1, u_2)$  of  $(M, \circ)$  the class

$$\text{hom}(u_1, u_2) = \{ f \in M \mid f \circ u_1 \text{ and } u_2 \circ f \text{ are defined} \}$$

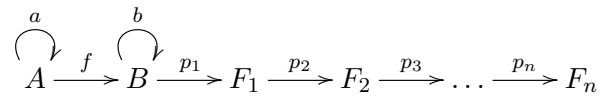
is a set.

**Exercise 1:**

Let  $\mathcal{Q}$  be the following directed graph:



- Determine the free category  $F\langle\mathcal{Q}\rangle$  on  $\mathcal{Q}$  and prove that its set of morphisms determines a regular language in the alphabet  $\Sigma = \{a, b, f, p\}$ .
- Generalize as follows: if  $\mathcal{Q}_n$  is the directed graph



so that the previous  $\mathcal{Q}$  is  $\mathcal{Q}_1$ , prove that the set of morphisms of  $F\langle\mathcal{Q}_n\rangle$  determines a regular language in the alphabet  $\Sigma = \{a, b, f, p_1, \dots, p_n\}$ . Is this still true if  $n \rightarrow \infty$ ?

**Exercise 2:**

Define the following categories associated to the functor  $S_A : X \mapsto 1 + A \times X$ .

- $\nabla S$  has objects the pairs  $(X, t)$  where  $t \in S_A X$  is an element; a morphism  $(X, t) \rightarrow (Y, v)$  in  $\nabla S$  consists of a function  $f : X \rightarrow Y$  such that the function  $Sf$  sends  $t$  to  $v$ :

$$Sf : 1 + A \times X \rightarrow 1 + A \times Y : t \mapsto v$$

- $\mathbf{coAlg}(S)$  has objects the pairs  $(X, \xi)$  where  $\xi : X \rightarrow S_A X$  is a function; a morphism  $(X, \xi) \rightarrow (Y, \theta)$  in  $\mathbf{coAlg}(S)$  consists of a function  $f : X \rightarrow Y$  such that

$$Sf \circ \xi = \theta \circ f.$$

Recall, or learn for the first time, that

- an initial objects in a category  $\mathcal{C}$  is an object  $I$  such that for every other object  $X \in \mathcal{C}$ , there exists a unique arrow  $I \rightarrow X$ ;
- a terminal objects in a category  $\mathcal{C}$  is an object  $T$  such that for every other object  $X \in \mathcal{C}$ , there exists a unique arrow  $X \rightarrow T$ .

Prove that  $\mathbf{coAlg}(S)$  has a terminal object; prove or disprove that  $\nabla S$  has an initial object.

**Exercise 3:**

A *construct* consists of a pair  $(\mathcal{C}, U)$  where  $\mathcal{C}$  is a category and  $U : \mathcal{C} \rightarrow \mathbf{Set}$  is a faithful functor. Two constructs  $(\mathcal{C}, U)$  and  $(\mathcal{D}, V)$  are (*strongly*) *equivalent* if there exist two functors  $F, G$  such that the two triangles

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{F} & \mathcal{D} \\ & \searrow U & \swarrow V \\ & \mathbf{Set} & \end{array} \quad \begin{array}{ccc} \mathcal{D} & \xrightarrow{G} & \mathcal{C} \\ & \searrow V & \swarrow U \\ & \mathbf{Set} & \end{array}$$

commute, and such that  $F \circ G = \text{id}_{\mathcal{D}}$  and  $G \circ F = \text{id}_{\mathcal{C}}$ . When the construct functor  $U$  associated to a certain category is implicitly understood (for example, when  $\mathcal{C}, \mathcal{D}$  are categories of algebraic structures) we say that  $\mathcal{D}, \mathcal{C}$  are *concretely equivalent*.

Are the following pairs concretely equivalent?

- $\mathcal{C} = \mathbf{Cat}$  is the category of categories and functors, defined in the usual way, while  $U$  sends a category to its set of arrows;  $\mathcal{D}$  is the category of object-free categories, defined above, and  $V$  sends  $(M, \circ)$  to the set  $M$ .
- the category **Kop** of *Kuratowski spaces*, defined via an interior operator, and the category **Top** of topological spaces defined via a family of open subsets: refer to the following definitions.
  - A *topological space* consists of a pair  $(X, \tau)$  where  $\tau \subseteq 2^X$  is a collection of subsets of  $X$ , the elements of which are called *open*, such that
    1.  $\emptyset, X \in \tau$ ;
    2. if  $I$  is a set and  $A_{\bullet} : I \rightarrow \tau$  an  $I$ -indexed family of elements  $A_i \in \tau$ , then  $\bigcup_i A_i \in \tau$ ;
    3. if  $A_1, A_2 \in \tau$ , then  $A_1 \cap A_2 \in \tau$ .
  - A *Kuratowski space* consists of a set  $X$  equipped with a monotone function

$$j : 2^X \rightarrow 2^X$$

called the *interior operator* of  $X$ , satisfying the following properties:

1.  $jX = X$ ;
2. for all  $S \in 2^X$ ,  $j(S) \subseteq S$ ;
3. for all  $S \in 2^X$ ,  $j(j(S)) = j(S)$ ;
4. for all  $S, T \in 2^X$ ,  $j(S) \cap j(T) \subseteq j(S \cap T)$ .

Both functors  $U : \mathbf{Top} \rightarrow \mathbf{Set}$  and  $V : \mathbf{KSp} \rightarrow \mathbf{Set}$  send a topological or Kuratowski space to its underlying set.