Differential automata theory

Fosco Loregian

May 26, 2025

Tallinn University of Technology

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10.4204/EPTCS.397.1 **2305.00272** **10.1007/978-3-031-66438-0_4**

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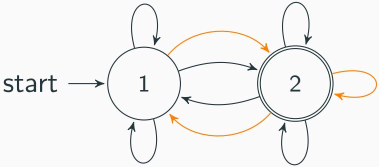

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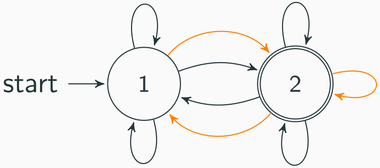
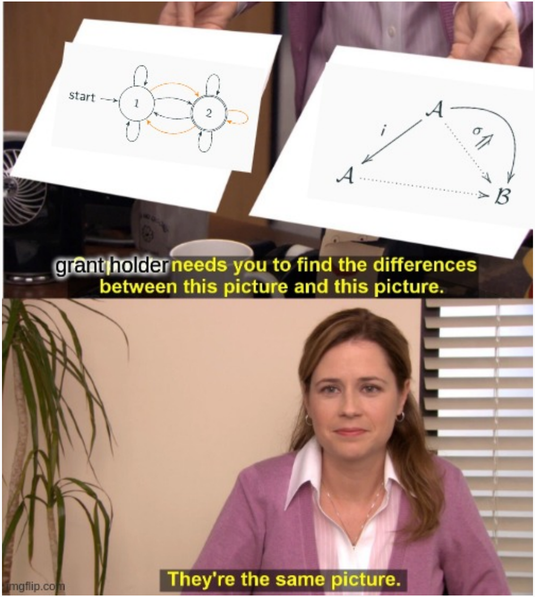


$$\begin{tikzcd}
& \mathcal{A} \arrow[dl, "i"'] \arrow[dr, dotted] \arrow[dr, bend left] \arrow[dr, phantom, "\sigma\Uparrow"] & \\
\mathcal{A} \arrow[rr, dotted] & & \mathcal{B}
\end{tikzcd}$$



start
1
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$\mathcal{A}$
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grant holder needs you to find the differences between this picture and this picture.
They're the same picture.
imgflip.com

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Techniques from representation theory, topos theory, homotopy theory, etc. are all useful and pop up constantly.

 Formal category theory

In the broadest possible sense, formal category theory is the study of what properties make a 2-category 'behave like **Cat**' abstracting away from its 'concreteness'.

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- homotopy theory: find properties making a category $\mathcal{E}$ behave like $\mathbf{Ho}(\mathbf{Top})$;
- ...

In stark contrast with topos theory, where a definitive answer to what are the intrinsic properties of a set-like category exists, FCT is an active field of study, with (yet) no unanimous consensus on what the right axioms are.

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A variety of different approaches to axiomatize the properties of **Cat** has been proposed, each with its own merits and drawbacks.

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I thought long and hard and...

9. arXiv:2303.03865 [pdf, other] math.CT cs.FL doi 10.4204/EPTCS.397.1

Bicategories of Automata, Automata in Bicategories

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Monads and limits in bicategories of circuits

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- take a 2-category $\mathcal{K}$ with finite strict 2-limits;
- fix objects $X, B \in \mathcal{K}_0$ and morphisms

$$\begin{tikzcd} B \arrow[r, "b"] & X \arrow[loop right, "f"] \end{tikzcd}$$

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We call such a (finite weighted) limit sketch an automata theory.

- construct step by step the following finite limits:

$$
\begin{tikzcd}
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‘Mealy’ automata

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‘Moore’ automata

Unraveling the definition if $\mathcal{K} = \mathbf{Cat}$, $B = 1$ (so $B : 1 \to \mathcal{X}$ picks an object of $\mathcal{X}$): given a diagram of categories and functors

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- $\mathbf{Mly}(F, B)$ is the category $\overset{\textit{inserter}}{Alg(F)} \overset{\textit{pb}}{\times_{\mathcal{X}}} \overset{\textit{comma}}{(F/B)}$ of spans

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- $\mathbf{Mre}(F, B)$ is the category $\overset{\textit{inserter}}{Alg(F)} \overset{\textit{pb}}{\times_{\mathcal{X}}} \overset{\textit{comma}}{(\mathcal{X}/B)}$ of disconnected spans

$$X \longleftarrow FX, X \longrightarrow B$$

  and suitable morphisms.

In particular when $\mathcal{X}$ is monoidal, and $F = A \otimes -$, one studies categories where **objects** and morphisms are as follows:

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$$
\begin{tikzcd}
X \arrow[d, "f"] & A \otimes X \arrow[l] \arrow[r] \arrow[d, "A\otimes f"] & B \arrow[d, equal] \\
Y & A \otimes Y \arrow[l] \arrow[r] & B
\end{tikzcd}
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$$\begin{tikzcd} X & A \otimes X \arrow[l] \arrow[r] \arrow[d, "A\otimes f"] & B \arrow[d, equal] \\ Y & A \otimes Y \arrow[l] \arrow[r] & B \arrow[from=1-1, to=2-1, "f"] \end{tikzcd}$$

Proposition

The assignment $(A, B) \mapsto \mathbf{Mly}_{\mathcal{X}}(A \otimes -, B) = \mathbf{Mly}_{\mathcal{X}}(A, B)$ defines an indexed category

$$\mathcal{X}^{\mathrm{op}} \times \mathcal{X} \longrightarrow \mathbf{Cat}$$

which when $\mathcal{X}$ is Cartesian forms the hom-category of the bicategory of Mealy automata.

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Categories of automata

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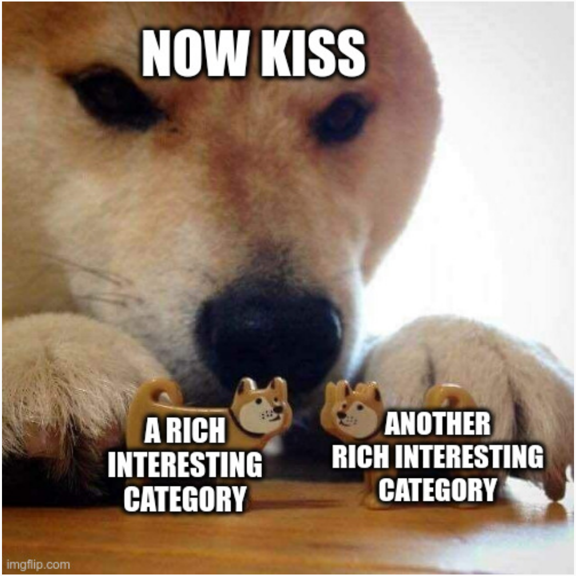
> *Categories of automata (in fact, fibers of a monoidal fibration), with objects having a combinatorial meaning, equipped with a notion of derivative functor, in which to do categorified differential algebra / study non-reversible dynamical systems induced by a diff. op.*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A RICH INTERESTING CATEGORY
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Differential 2-rigs

The pair $(\mathbf{Spc}, \partial)$ is an instance of a differential 2-rig (L/L-Trimble 2020), i.e. a category equipped with a 'linear and Leibniz endofunctor' ∂

Theorem

The category **Spc** has a universal property qua 2-rig and qua differential 2-rig.

- it is the free cocomplete 2-rig on one generator;
- the category of species in a countable set of 'colours' is the free differential 2-rig on one generator.

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$$\boxed{\mathbf{P} := \mathbf{P}[1] \qquad \mathbf{Spc} := (1, \mathbf{Set})\text{-}\mathbf{Spc} = [\mathbf{P}, \mathbf{Set}]}$$

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- $E' \cong E \qquad Cyc' \cong L \qquad P' \cong E + E$ (*natural* isos of functors)
- ∂ has a left adjoint (easy to describe: $\partial = \{y[1], -\}$ hence $L = y[1] * -$), but also a right adjoint (because $y[1]$ is a tiny object)

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This realizes the Leibniz rule as a universal property.

 Differential 2-rigs

(and differential automata)

1. sketch the technology one can develop for D2Rs, categorifying differential algebra;
2. apply some of these ideas to a specific case for a category of automata.

Clearly, 2. is just a pretext for 1.

Free objects and quotients

Freeness results

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More generally one can define the free 2-rig on a category...

And extend scalars over $\mathcal{R}$ –defining the free $\mathcal{R}$-algebra on S: $F[S] \otimes \mathcal{R}$

$\mathcal{R}$ a 2-rig; $\mathcal{R}[t] = \mathcal{R} \otimes_{\mathbf{P}} F[t] = \mathcal{R} \otimes_{\mathbf{P}} \mathbf{Spc}$

 Kähler differentials

Geometry of D2Rs

$$\{\text{derivations on } R\} \cong \left\{ s : \begin{tikzcd} R[t]/t^2 \arrow[d, "ev_0"] \\ R \arrow[u, bend left, "s"] \end{tikzcd} \right\}$$

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Given a 2-category $\mathcal{K}$ and a diagram

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the coinverter of f, g is a 1-cell $c : B \to Q$ such that $c * \alpha : cf \Rightarrow cg$ is invertible; (Q, c) is 1-initial and 2-initial among such pairs.

E.g. if $\mathcal{C}$ is a category, $W \subseteq \mathcal{C}^2$ a class of maps; the coinverter of

$$\begin{tikzcd} (W \subseteq \mathcal{C}^2) \arrow[r, bend left, "s"] \arrow[r, bend right, "t"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \end{tikzcd}$$

is the Gabriel-Zisman localization $\mathcal{C}[W^{-1}]$.

Geometry of D2Rs

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Theorem

$$Der[\mathcal{R}] \cong \{\text{sections}/\mathcal{R} \text{ of } ev_0 : \mathcal{R}[\epsilon] \to \mathcal{R}\}$$

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- similarly: quotient for a principal ideal, say $\mathfrak{J} = (p)$, is coinverter of

$$\begin{tikzcd} \mathcal{R}[t] \arrow[r, bend left, "\varnothing"] \arrow[r, bend right, "-\otimes p(t)"'] \arrow[r, phantom, "\Downarrow"] & \mathcal{R}[t] \arrow[r, two heads, "q"] & \mathcal{R}[t]/(p) \end{tikzcd}$$

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$$\begin{tikzcd} \mathcal{R}[t] \arrow[r, bend left, "\varnothing"{name=U}] \arrow[r, bend right, "-\otimes p(t)"'{name=D}] \arrow[phantom, from=U, to=D, "\Downarrow"] & \mathcal{R}[t] \arrow[r, two heads, "q"] & \mathcal{R}[t]/(p) \end{tikzcd}$$

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- quotients like $\mathcal{R}[X, Y]/(Y^2 + 1 \cong X^2)$ (categorified hyperbola) acquire a differential structure, $\partial Y = X, \partial X = Y$; can be done more in general?

Jet spaces

Categorified jet spaces

Given a D2R $(\mathcal{R}, \otimes, \partial)$ let $\mathbf{Alg}(\partial)$ be the category of ∂-algebras.

- objects: $(X, \xi : \partial X \to X)$;
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\mathbf{Alg}(\partial) \arrow[r] \arrow[d, "\partial'"] & \mathcal{R} \arrow[d, "\partial"] \\
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Categorified jet spaces

Define by mutual induction:

- $\mathcal{R}^{(0)} := \mathcal{R}$ and $\mathcal{R}^{(n+1)} := \mathbf{Alg}(\partial^{(n)}, \mathcal{R}^{(n)})$;
- $\partial^{(1)} := \partial$ and $\partial^{(n+1)} := \mathcal{R}^{(n+1)} \to \mathcal{R}^{(n+1)}$ defined lifting $\partial^{(n)}$.

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Definition

From the chain of forgetful functors

$$\mathcal{R} \longleftarrow \mathbf{Alg}(\partial) \longleftarrow \mathbf{Alg}(\partial') \longleftarrow \mathbf{Alg}(\partial'') \longleftarrow \cdots$$

$$\mathbf{Jet}[\mathcal{R}, \partial] := \lim\left(\mathcal{R} \xleftarrow{U} \mathcal{R}^{(1)} \xleftarrow{U^{(1)}} \mathcal{R}^{(2)} \xleftarrow{U^{(2)}} \cdots \right).$$

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$$\vec{X} = \big(X, (X; \xi : \partial X \to X), ((X; \xi); \xi' : \partial'(X; \xi) \to (X; \xi)), \dots\big)$$

the n^{th} element of which equips the $(n-1)^{\mathrm{th}}$ with an algebra structure for $\partial^{(n)}$.

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$$X \xleftarrow{\xi} \partial X \xleftarrow{\xi'} \partial\partial X \xleftarrow{\xi''} \partial\partial\partial X \leftarrow \dots$$

Categorified jet spaces

Define the *k*-jet $J^k(\vec{X})$ of an object $\vec{X} \in \mathbf{Jet}[\mathcal{R}, \partial]$ as the image of $\vec{X}$ under the functor J^k obtained from the limit projections $\pi_k : \mathbf{Jet}[\mathcal{R}, \partial] \to \mathcal{R}^{(k)}$ as

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cf. differential geometry, where the k-jet of a real valued function $f : \mathbb{R} \to \mathbb{R}$ is defined as

$$(J^k_{x_0} f)(z) = \sum_{\ell=0}^{k} \frac{f^{(\ell)}(x_0)}{\ell!} z^\ell = f(x_0) + f'(x_0) z + \cdots + \frac{f^{(k)}(x_0)}{k!} z^k$$

An application to automata

Differential automata

Let $\mathcal{R}$ be a D2R; the assignment $(A, B) \mapsto \mathbf{Mly}(A \otimes -, B)$ defines a two-sided fibration via the Grothendieck construction

$$\underset{Mly:(A,B)\mapsto\mathbf{Mly}(A\otimes -,B)}{\mathbf{Psd}(\mathcal{R}^{\mathsf{op}} \times \mathcal{R}, \mathbf{Cat})} \underset{\sim}{\overset{}{\rightleftarrows}} \underset{(V:\mathbf{Mly}_{\mathcal{R}}\twoheadrightarrow\mathcal{R}^{\mathsf{op}}\times\mathcal{R})}{\mathbf{Fib}/(\mathcal{R}^{\mathsf{op}} \times \mathcal{R})}$$

which is a D2R morphism with respect to a canonical differential structure on the domain $\mathbf{Mly}_{\mathcal{R}}$.

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V is a fibration of trajectories for discrete dynamical systems of endpoints A, B; each category of trajectories $\mathbf{Mly}(A, B)$ has a limit (=terminal) object

$$\prod_{n\geq 1}[A^{\otimes n}, B]$$

(Analogy: the limit set of a dynamical system $\overline{A}^f := \bigcap_{n\geq 1} \overline{f^n(A)}$, where f is en endomap of a metric space A.)

Differential automata

If $\mathcal{R}$ is monoidal closed, $\mathbf{Mly}_{\mathcal{R}}$ is a category of coalgebras for a certain endofunctor $R : \mathcal{R}^{\mathsf{op}} \times \mathcal{R} \times \mathcal{R} \to \mathcal{R}^{\mathsf{op}} \times \mathcal{R} \times \mathcal{R}$, fibred over the projection π_{12}

$$(A, B, X) \longmapsto (A, B, [A, B \times X])$$

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There is a distributive law $\delta : (1 \times \partial)R \Rightarrow R(1 \times \partial)$

$$\begin{tikzcd}
\mathbf{Mly}_{\mathcal{R}} \arrow[r, "\bar\partial"] \arrow[d] & \mathbf{Mly}_{\mathcal{R}} \arrow[d] \\
\mathcal{R}^{\mathsf{op}} \times \mathcal{R} \arrow[r, "1\times\partial"'] & \mathcal{R}^{\mathsf{op}} \times \mathcal{R}
\end{tikzcd}$$

lifting ∂ to a derivative functor $\bar\partial$ on $\mathbf{Mly}_{\mathcal{R}}$; the category of differential automata is the category of coalgebras for such $\bar\partial$.

Differential automata

Let $\mathcal{R}$ be the D2R of species; observe that

- the species L of linear orders is the free monoid on the monoidal unit (plays the role of an NNO in **Spc**);
- thus there are four equivalent descriptions for the category of $\mathbf{Spc}^L$ of L-algebras, building block for $\mathbf{Mly}([1], B)$:
  - the category of algebras for the functor $[1] * -$;
  - the category of EM algebras for the monad $L \otimes -$;
  - the category of coalgebras for the functor ∂;
  - the category of coEM algebras for the comonad $\{L, -\}$.

Differential automata

Similar reasoning applies to scopic D2R, where ∂ has both a left and aright adjoint. There are plenty of variations on the theme of categories of species which are scopic D2Rs, e.g.:

- The category of S-species, i.e. functors $\mathbf{P}[S] \to \mathbf{Set}$ for an arbitrary set S; this supports partial derivatives, $\{\partial_s \mid s \in S\}$;
- k-vector (S-)species (Aguiar-Mahajan I,II,III,IV), i.e. functors $\mathbf{P}[S] \to \mathbf{Vect}_k$;

Differential automata

- linear species, i.e. families of functors of the form $X_n : [S_n/S_n] \to \mathbf{Set}$, where $[S_n/S_n]$ is the action groupoid of the regular representation of S_n on itself; (widely studied because differential equations admit unique solution here);
- Möbius species, where functors out of $\mathbf{P}[S]$ are valued in a category of posets with top and bottom (Möbius inversion formula has a category-theoretic proof);
- nominal sets, i.e. representations of the filtered colimit $S_1 \subset S_2 \subset S_3 \subset \dots$ of finite symmetric groups on the set of finite sets; (this is only a left scopic D2R; widely used in TCS).

Differential automata

There are examples

- of species having no ∂-coalgebra structures, but acquire many when linearized (i.e. considered as k-vector species instead of **Set**-species);
- of species having a finite number of ∂-coalgebra structures (precisely four);
- of species having uncountably many ∂-coalgebra structures.

(The fact that a coalgebra map must be S_n-equivariant is often a strong restriction on the structure of the coalgebra!)

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- these objects are highly structured ($\partial I \neq 0$, self-similarity,$\dots$)
- it's 'difficult' for a category to be a diff-2-rig ($Der(\mathcal{R})$ knows about a 'dimension' of $\mathcal{R}$)
- yet, differential algebra is quite interesting (differential equations?)