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Interpolation property for bicartesian closed categories

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Summary. We show that proofs in the intuitionistic propositional logic factor through interpolants – in this way we prove a stronger interpolation property than the usual one which gives only the existence of interpolants.

Translating that to categorical terms, we show that Pushouts (bipushouts) of bicartesian closed categories have the interpolation property (Theorem 3.2).

1 Introduction

A very successful approach to category theory is the one by Lambek and Lawvere in which they consider certain categories "coming from nature" as certain formal systems coming from logic. The whole approach one may call categorical logic. There is an important characteristic in Lambek's approach which is less emphasized in the approach of Lawvere, namely for Lambek these formal systems have not only formulas and the notion of provability, but also they have the equality among proofs – the notion which appeared in classical proof theory as well. For us, this is an essential feature, and we like to call this part of categorical logic categorical proof theory.

While for most of the proof theorists the notion of equality was just a byproduct of proof reduction which in turn was used "just" to investigate provability – the very notion of the equality of the proofs was also under consideration most explicitly by Prawitz. It turns out that the two equalities (of Prawitz and Lambek) are almost the same (for certain fragments of logic exactly the same). Therefore, Lambek's conclusion is that formulas in the formal systems are objects in the corresponding categories and that proofs (or rather their equivalence classes) are arrows in these categories. It was also noted that in that manner proofs become "real mathematical objects" – and perhaps some nonintended

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mathematical techniques could be applied to investigate them. We believe that even the fact that the proofs become more "real" is a step forward in the understanding what a "general" theory of proofs is [Göd65].

The formal systems investigated in this paper are the ones coming from intuitionistic propositional logic. In the presence of the equality of proofs that is the same as to investigate bicartesian (or cartesian) closed categories – these are categories "quite often appearing in nature" e.g. toposes are like that. The general goal is to formulate and to prove for these formal systems (in the presence of equality of the proofs) some of the well known properties which hold in the presence of provability only. Here we shall prove that interpolation property holds for the intuitionistic propositional logic considering not only provability but also equality of proofs. Let us be a bit more precise about this.

In logic, by interpolation we usually mean a statement as follows: suppose we have a proof of a statement C from hypothesis B (i.e. $B \to C$) where B is in a language L_1 and C is in a language L_2, then there exists a statement A in the language $L_1 \cap L_2$ so that we can prove $B \to A$ and $A \to C$. There are many proofs of statements of this type for different formal systems. Some of them are purely syntactic and they are obtained as corollaries to cut elimination or normalization.

In our setting (or better to say: in Lambek's approach) the statements of cut elimination and normalization are less elegant but interpolation remains (almost) as elegant as in the basic case. The above statement of interpolation in this setting has to have a form as follows: suppose again B is in the language L_1 and C in a language L_2 and suppose that there exists a proof $B \xrightarrow{t} C$ in the language $L_1 \cup L_2$, then there exists A in $L_1 \cap L_2$ and there are $B \xrightarrow{r} A$ in L_1 and $A \xrightarrow{s} C$ in L_2 such that $t = sr$. One can see that this kind of interpolation is a genuine improvement over the usual interpolation. We may also add that we allow the presence of axioms and even the presence of additional equations among proofs. An example involving additional constants the reader can find after Corollary 3.10.

We also want to obtain a categorical reformulation of the above statement (independent of such notions as language and theory); therefore we have to formulate the interpolation property in the appropriate form, i.e. as a statement about Pushouts (also called bipushouts). It turns out that this again generalizes even further the statement of interpolation. Let us just say that our main result says that Pushouts in the 2-category of bicartesian closed categories satisfy the naturally formulated interpolation property. The same holds for the 2-category of cartesian closed categories.

This is not the first time that interpolation is investigated from categorical viewpoint – perhaps the best known work is the one by Pitts [Pit83a, Pit83b, Pit87, Pit88, Pit92] there, as well as in almost all the other references, the interpolation "happens" in a poset (usually in the lattice of subobjects of an object) so gain we can say that these variants of interpolation concern the provability only – and not the equality of proofs. There are, however, exceptions: in [Pit86] it has been shown that the cocomma squares of the categories with finite limits satisfy the interpolation property in the same sense as we define it; also Pavlović in [Pav91] considers interpolation in a fibrational context and the fibrations do not have to be posetal – the results there are of a general nature and they do not resolve the question as to whether a particular doctrine e.g. of bicartesian closed categories has the interpolation property or not. Another categorical formulation of interpolation is given in [KP86] for the category of Banach spaces.

Let us now briefly describe the contents of the paper:

Following the Introduction is the second part called basics of bicartesian closed categories in which we give basic definitions and the relation between two versions of typed lambda calculus and corresponding categories (bicartesian closed and cartesian closed). Although the connections of this type are well known (cf. [LS86]), we give slightly different presentation; in particular our notion of internal language is different from the existing ones. Also, we think that we give the most explicit connection between bicartesian closed categories and the corresponding language. Let us just add that we adopt Curry-Howard isomorphism, so when we speak about lambda terms the reader can picture proofs in natural deduction (for a recent exposition of the isomorphism see [GLT89]).

In the third part we prove the interpolation result. In Sect. 3.1 we present the right set of reductions. Then in Sect. 3.2 we prove that the proofs in intuitionistic propositional logic enjoy a stronger interpolation property than required by the ordinary Craig interpolation. We use ideas from Prawitz' proof of the statement – the differences and similarities are explained at the beginning of the section. Section 3.3 contains the first restatement of the previous fact in categorical terms and then, it continues with the proof of our main result that bicartesian closed categories have the interpolation property. And finally, in Sect. 3.4 we give a couple of applications of the interpolation by showing that both of the main theorems on interpolation in Heyting algebras from [Pit83a] easily follow from our interpolation result. Since these results themselves generalize the Craig interpolation for intuitionistic propositional logic in that way we show that our result generalizes the statement even further.

2 Basics of bicartesian closed categories

In this section we shall give the definitions of bicartesian closed category (and cartesian closed category) and we shall explain the Lambek-type connection between these categories and the appropriate typed λ-calculi. There are many papers where various variants of typed lambda calculi with finite coproducts are dealt with, but we are not aware of the existence of the explicit comparison as done below; however, we have to admit, the comparison is direct.

The above categories are determined by the existence of certain adjoint functors. Briefly, we can say that bicartesian closed categories are the ones with finite products, finite coproducts and exponents; cartesian closed categories are the ones with finite products and exponents.

Let us now give precise definitions of these notions:

Definition 2.1. A category $\mathscr{B}$ is (strict) *bicartesian closed* if it has objects 1 and 0, and for every two objects A, $B \in \mathscr{B}$ there are objects – denoted $A \times B$, $A+B$ and A^B; let us write it in a tabular form as follows:

$\times$	1 $A \times B$
$+$	0 $A+B$
$\rightarrow$	A^B

The category also has to have the following arrows:

$\times$	$0_A \in \hom(A, 1)$ $\pi_{A,B} \in \hom(A \times B, A)$ $\pi'_{A,B} \in \hom(A \times B, B)$
$+$	$\square_A \in \hom(0, A)$ $\iota_{A,B} \in \hom(A, A+B)$ $\iota'_{A,B} \in \hom(B, A+B)$
$\rightarrow$	$\varepsilon_{A,B} \in \hom(A^B \times B, A)$

and the following operations on homsets:

$\times$	$\hom(C, A) \times \hom(C, B) \xrightarrow{\langle , \rangle} \hom(C, A \times B)$
$+$	$\hom(A, C) \times \hom(B, C) \xrightarrow{[,]} \hom(A+B, C)$
$\rightarrow$	$\hom(A \times B, C) \xrightarrow{*} \hom(A, C^B)$

(the operations should have indexes, but since they are uniquely determined by their arguments we omit then). These (constants and) operations have to satisfy the following equations:

$\times$	(T) $\quad f = 0_A$ (Pr_1) $\quad \pi_{A_1,A_2}\langle f_1, f_2\rangle = f_1$ (Pr_2) $\quad \pi'_{A_1,A_2}\langle f_1, f_2\rangle = f_2$ (SP) $\quad \langle \pi_{A,B}g, \pi'_{A,B}g\rangle = g$
$+$	(I) $\quad s = \square_A$ (In_1) $\quad [s_1, s_2]\iota_{A_1,A_2} = s_1$ (In_2) $\quad [s_1, s_2]\iota'_{A_1,A_2} = s_2$ (IC) $\quad [r\iota_{A,B}, r\iota'_{A,B}] = r$
$\rightarrow$	(B) $\quad \varepsilon_{A,B}\langle h^*\pi_{C,B}, \pi'_{C,B}\rangle = h$ (H) $\quad (\varepsilon_{A,B}\langle k\pi_{C,B}, \pi'_{C,B}\rangle)^* = k$

for every arrow $f \in \hom(A, 1)$, $f_i \in \hom(C, A_i)$, $g \in \hom(C, A \times B)$, $s \in \hom(0, A)$, $s_i \in \hom(A_i, C)$, $r \in \hom(A+B, C)$, $h \in \hom(C \times B, A)$ and $k \in \hom(C, A^B)$.

If a category has only finite products and exponents ("$\times, \rightarrow$" parts) we call it *cartesian closed*. And as we said earlier – a category with finite products, coproducts and exponents ("$\times, +, \rightarrow$" parts) is called a bicartesian closed category.

In the definition (as it stands) we allow nonuniqueness of objects 1, 0, $A \times B$, $A+B$ and A^B (for every A and B). When we want to stress this we call the category *nonstrict*. In the case that we choose only one object to represent the above constructs we call such a category *strict*.

We will use the following abbreviations: if $A_i \xrightarrow{f_i} A_i'$, $i=1, 2$ then $f_1 \times f_2 = \langle f_1\pi_1, f_2\pi_2 \rangle$: $A_1 \times A_2 \rightarrow A_1' \times A_2'$, $f+g = [\iota_1 f_1, \iota_2 f_2]$: $A_1 + A_2 \rightarrow A_1' + A_2'$. Also, $\mathbf{A} = A_1 \times \ldots \times A_n$ is used to denote products when the brackets are nested on the left; and if $\mathbf{B}$ is a subsequence of $\mathbf{A}$ then $\pi_{\mathbf{A}}^{\mathbf{B}}: \mathbf{A} \rightarrow \mathbf{B}$ denotes the canonical projection, and similarly for coproducts.

Definition 2.2. The 2-category $\mathscr{BCC}$ of bicartesian closed categories has as 0-cells (small) bicartesian closed categories, as 1-cells functors preserving bicartesian closed structure (bc-functors), and as 2-cells natural isomorphisms. We will also work in the 2-category – the "strict" version of the doctrine $\mathscr{BCC}$ – that is the 0-cells in $\mathscr{BCC}_s$ are strict bicartesian closed categories, 1-cells are strict bc-functors – that is functors which preserve the chosen structure "on the nose" e.g. $F(A \times B) = F(A) \times F(B)$. The 2-cells in $\mathscr{BCC}_s$ are natural isomorphisms. Similarly, $\mathscr{CCC}$ will denote the 2-category of the cartesian closed categories and $\mathscr{CCC}_s$ its strict variant. Often we refer to these 2-categories as (strict) *doctrines*. There is a nice adjointness between the strict and nonstrict doctrines as proved in [BKP89]; here we can just say that the obvious forgetful functor (forgetting the strict structure) $\|: \mathscr{BCC}_s \rightarrow \mathscr{BCC}$ has a left adjoint and the units are equivalences of categories.

Let us just add that the consideration of natural isomorphisms as 2-cells is not too strong a restriction. By now, it is part of the 2-categorical folklore that the doctrines with similar kind of closed structure require natural isomorphisms as 2-cells – otherwise they are not tripleable over the 2-category of categories. For a discussion see [BKP89]. We can also add that in our case these doctrines with all the natural transformations as 2-cells do not have Pushouts – the central object of study in our paper.

Definition 2.3 (Typed $\boldsymbol{\lambda\delta}$, $\boldsymbol{\lambda}$-calculi). A typed $\lambda\delta$-calculus is a formal system which consists of three classes: Types, Terms and Equations. They have to satisfy the following conditions:

Types. Types are freely generated from a set of basic types – sorts and the following rules: 1, $0 \in$ Types; if A, $B \in$ Types then $A \times B$, $A+B$, $A^B \in$ Types. Again using the tables we can write it as

$\times$	1 $A \times B$
$+$	0 $A+B$
$\rightarrow$	A^B

Terms. For each type A we have countably many variables of type A (we denote them as x_i^A or x_i: A) and they are terms, also if s: 0, s_i: C, r: $A+B$ a: $A_1 \times A_2$, a_i: $A_i (i=1, 2)$, f: A^B, b: B are terms then

$\times$	$*:1$ $\pi(a): A_1, \pi'(a): A_2$ $\langle a_1, a_2 \rangle: A_1 \times A_2$
$+$	$\varepsilon^C(s): C$ $\iota_{B,A}(b): B+A$ $\iota'_{A,B}(b): A+B$ $\delta x^A \cdot s_1^C, x^B \cdot s_2^C; r^{A+B}: C$
$\to$	$(f`b): A$ $\lambda x^A \cdot b: B^A$

are terms. (The notions of free and bounded variables in a term t are standard – let us just be explicit about the δ-form: $FV(\delta x^A \cdot s_1^C,\ x^B \cdot s_2^C;\ r^{A+B}: C) = (FV(s_1 - \{x^A\}) \cup (FV(s_2) - \{x^B\}) \cup FV(r)$ ($FV(t)$ denotes the set of the free variables in t.))

Let us just illustrate where ε and δ come from. For that recall the rules for elimination of the connectives $\perp$ and $\vee$ (in natural deduction):

$$\dfrac{\begin{matrix}\Gamma\\ \vdots\\ s\\ \vdots\\ \perp\end{matrix}}{C}\ \varepsilon^C(s) \qquad \dfrac{\begin{matrix}\Gamma_1\\ \vdots\\ r\\ \vdots\\ A \vee B\end{matrix} \quad \begin{matrix}\Gamma_2 \cancel{A}\\ \vdots\\ s_1\\ \vdots\\ C\end{matrix} \quad \begin{matrix}\Gamma_3 \cancel{B}\\ \vdots\\ s_2\\ \vdots\\ C\end{matrix}}{C}\ \delta x^A \cdot s_1^C, x^B \cdot s_2^C; r^{A+B}$$

As usually we allow cancellation of some (none or all) of the hypothesis A and B. In our notation these would be denoted by x^A respectively x^B.

Equations. They always have the following form $s =_X t$ where s, $t \in$Terms and X is a set of (typed) variables such that $FV(s) \cup FV(t) \subseteq X$.

Convention: when $FV(s) \cup FV(t) = X$ we often omit X in $s =_X t$. Also, typing is omitted whenever convenient.

The following expressions are equations (we call them axioms of $\lambda\delta$-calculus):

	(T)	$f^1 = *$	
$\times$	(Pr_i)	$\pi_i(\langle f_1, f_2 \rangle) = f_i$	$i=1,2$
	(SP)	$\langle \pi_1(g), \pi_2(g) \rangle = g$	
	(I)	$s^C = \varepsilon^C(x^0)$	$x^0 \in FV(s^C)$
$+$	(In_i)	$\delta x_1^{A_1} \cdot s_1, x_2^{A_2} \cdot s_2; \iota_i(r) = s_i(r/x_i)$	$i=1,2$
	(γ)	$\delta x^A \cdot v(\iota_1(x^A)/z^{A+B}), y^B \cdot v(\iota_2(y^B)/z^{A+B}); w = v(w/z)$	$x^A, y^B \notin FV(v)$
	(β)	$(\lambda x^A \cdot h`r) = h(r/x^A)$	
$\to$	(η)	$\lambda x^B \cdot (k`x^B) = k$	$x^B \notin FV(k)$

for every term $f{:}1$, $f_i{:}A_i$, $g{:}A\times B$, $s{:}C$, $s_i{:}C$, v^C, $w{:}A+B$, $h{:}B$, $r{:}A$ and $k{:}A^B$ such that s, v, k satisfy the conditions on the free variables as stated above (also, the notation $h(r/x)$ denotes the substitution of r instead of all free occurrences of x in h but first taking care of clashes of variables – so we are all the time working under α-congruence since it is possible to do that naively as in untyped λ-calculus and it is safe for our purposes).

Equations are obtained also by the following rules (we also say that proofs are formed from the axioms and the following rules):

$$(R)\frac{}{t=_X t}\quad (S)\frac{s=_X t}{t=_X s}\quad (Tran)\frac{r=_X s \quad s=_Y t}{r=_{X\cup Y} t}$$

$$(\xi)\frac{t=_{X\cup\{x\}} s}{\lambda x\cdot t=_X \lambda x\cdot s}\quad (Sub')\frac{a^B=_X b^B \quad s^{A^B}=_Y t^{A^B}}{(s`a)=_{X\cup Y}(t`b)}$$

$$(\xi d)\frac{s_1^C=_{X\cup\{x^A\}} t_1^C \quad s_2^C=_{Y\cup\{y^B\}} t_2^C \quad r^{A+B}=_Z u^{A+B}}{\delta x^A\cdot s_1, y^B\cdot s_2; r=_{X\cup Y\cup Z}\delta x^A\cdot t_1, y^B\cdot t_2; u}$$

The need for having indexed equations – contexts is well known – see [LS86] or [Čub94]. We can have some other basic types (sorts) and some other basic terms (constants). The part of the calculus denoted by "$\times, \rightarrow$" we shall call *λ-calculus*, and as we said earlier all the parts together we call *$\lambda\delta$-calculus.* All types and terms of a certain calculus we call the *language*; sometimes we are less precise and we call only the set of basic types and basic constants the language. A set of equations added to the above system we will call a theory of the calculus e.g. *$\lambda\delta$-theory* or just theory.

And one more piece of terminology: sometimes we will speak about *type-terms* i.e. when we want to be specific about the basic types used to build a complex type (using the operations as in the first table of the current definition) then $\mathcal{T}(X_1,\dots X_n)$ denotes a type built out of the basic types $X_1,\dots,X_n$. As usually done, we will overuse slightly the notation and we will write sometimes $\mathcal{T}(A_1,\dots,A_n)$ to denote the object in a strict bicartesian closed category build out of the objects $A_1,\dots,A_n$ and the operations on objects as in the first table of Definition 2.1.

In the presence of (Pr_i) and $(Tran)$ one can see that the reflexivity (rule (R)) is not needed. Also, it is a simple exercise to see that the following rules are derivable (the usual care about clashes of variables is needed for the second rule):

$$(W)\frac{t=_X s}{t=_{X\cup Y} s}\quad (Sub)\frac{a^B=_X b^B \quad s=_{Y\cup\{x^B\}} t}{s(a/x)=_{X\cup Y} t(b/x)}$$

Also, one can show that any two terms t^C, r^C are equal over a context which contain a variable of the type 0 (use $t^C=\pi_1(\langle t^C, x^0\rangle)=\varepsilon^C(x^0)$). The following lemma is going to be used:

Lemma 2.4. *For any term $F(Z^C)$ such that x_1, $x_2\notin FV(F)$ and any $u_1, u_2{:}C$ (and w of the appropriate type)*

$$F((\delta x_1\cdot u_1, x_2\cdot u_2; w)/Z)=\delta x_1\cdot F(u_1/Z), x_2\cdot F(u_2/Z); w.$$

(Hint: take $v\equiv F((\delta x_1\cdot u_1, x_2\cdot u_2; z^{A+B})/Z)$ and use (γ)).

Also, for a term t^0 and $F{:}D$ one can show that:

$$F(\varepsilon^C(t)/Z)=\varepsilon^D(t)$$

The following expression $(x_1^{A_1}, \dots, x_n^{A_n} \rhd t)$ called term with context is going to be often used, it denotes a term t and a sequence of variables such that $FV(t) \subseteq x_1^{A_1}, \dots, x_n^{A_n}$.

Definition 2.5. An *interpretation* M of a language L in a bicartesian closed category $\mathscr{B}$ is a function which assigns objects to basic types (sorts), and satisfies $M(A \# B) = M(A) \# M(B)$, where $\#$ is product, coproduct or exponent; also $M(\#) = \#$ where $\#$ is 1 or 0 (hence, $M(\mathscr{T}(X_1, \dots, X_n)) = \mathscr{T}(M(X_1), \dots, M(X_n))$ where $\mathscr{T}(X_1, \dots, X_n)$ is a type-term as at the end of Definition 2.3). If the language L has some basic constants it is assumed that the category $\mathscr{C}$ had them prescribed in advance, more precisely if $c : \mathscr{T}(X_1, \dots, X_n)$ is a basic constant in the language L we assume that there exists an arrow in hom $(1, \mathscr{T}(M(X_1), \dots, M(X_n)))$ – such an arrow we will often also denote by c). Then the interpretation assigns arrows to terms as follows (using induction on complexity of terms):

- $M(x_1^{A_1}, \dots, x_n^{A_n} \rhd x_i) = \pi_{\mathbf{A}}^{A_i}$.
- $M(\mathbf{x}:\mathbf{A} \rhd *) = 0_{\mathbf{A}}$. If the context were empty then we would have $M(\rhd *) = 1_1$.
- $M(\mathbf{x}:\mathbf{A} \rhd c) = c\,0_{\mathbf{A}}$ (here c is a constant). Also we could have empty context, then $M(\rhd c) = c$.
- $M(\mathbf{x}:\mathbf{A} \rhd \pi_i(t)) = \pi_i M(\mathbf{x}:\mathbf{A} \rhd t) i = 1, 2$.
- $M(\mathbf{x}:\mathbf{A} \rhd \langle t_1, t_2 \rangle) = \langle M(\mathbf{x}:\mathbf{A} \rhd t_1), M(\mathbf{x}:\mathbf{A} \rhd t_2) \rangle$.
- $M(\mathbf{x}:\mathbf{A} \rhd \varepsilon^C(t)) = \Box_C M(\mathbf{x}:\mathbf{A} \rhd t)$.
- $M(\mathbf{x}:\mathbf{A} \rhd \iota_i(t)) = \iota_i M(\mathbf{x}:\mathbf{A} \rhd t)$.
- $M(\mathbf{x}:\mathbf{A} \rhd \delta y_1^{B_1} \cdot u, y_2^{B_2} \cdot v; w)$
$= [M(\mathbf{x}:\mathbf{A}, y_1 : B_1 \rhd u), M(\mathbf{x}:\mathbf{A}, y_2 : B_2 \rhd v)]\, d \langle 1_{\mathbf{A}}, M(\mathbf{x}:\mathbf{A} \rhd w) \rangle$
- $M(\mathbf{x}:\mathbf{A} \rhd (t_1 ` t_2)) = \varepsilon \langle M(\mathbf{x}:\mathbf{A} \rhd t_1), M(\mathbf{x}:\mathbf{A} \rhd t_2) \rangle$.
- $M(\mathbf{x}:\mathbf{A} \rhd \lambda y^B \cdot t) = (M(\mathbf{x}:\mathbf{A},\ y : B \rhd t))^*$, if $\mathbf{x}:\mathbf{A}$ were not there we would have $M(\rhd \lambda y^B \cdot t) = (M(y^B \rhd t)\pi'_{1, M(B)})^*$.

The map $d : \mathbf{A} \times (B_1 + B_2) \to \mathbf{A} \times B_1 + \mathbf{A} \times B_2$ mentioned above is the canonical iso which exists in any bicartesian closed category.

Let I_1, I_2 be two interpretations of a theory T in a category $\mathscr{B}$. Then a *morphism* ψ from I_1 to I_2 is a family of arrows in $\mathscr{B}$ indexed by the set of types from T such that they commute with the basic symbols from the language i.e. $\pi_{I_2(A_1) \times I_2(A_2)}^{I_2(A_i)} \psi_{A_1 \times A_2} = \psi_{A_i} \pi_{I_1(A_1) \times I_1(A_2)}^{I_1(A_i)}$, $\psi_{A_1 + A_2} \iota_{I_1(A_i)}^{I_1(A_1) + I_1(A_2)} = \iota_{I_1(A_i)}^{I_1(A_1) + I_1(A_2)} \psi_{A_i}$, $\varepsilon_{I_2(A_1), I_2(A_2)}(\psi_{A_1^{A_2}} \times \psi_{A_2}) = \psi_{A_1} \varepsilon_{I_1(A_1), I_1(A_2)}$ and for every basic constant ξ^C, $\psi_C I_1(\rhd \xi^C) = I_2(\rhd \xi^C)$. It is interesting to notice that these conditions alone are enough to establish that for every term with context $(\mathbf{x}:\mathbf{A} \rhd t^C)$ the following holds:

$$I_2(\mathbf{x}:\mathbf{A} \rhd t^C)\psi_{\mathbf{A}} = \psi_C I_1(\mathbf{x}:\mathbf{A} \rhd t^C).$$

A *model* of a $\lambda\delta$-theory T is an interpretation such that all the equations from T are preserved. A morphism between two models we will call a *homomorphism*. A homomorphism $M_1 \overset{\psi}{\Rightarrow} M_2 \in Mod_T \mathscr{B}$ is an *isomorphism* iff all the components of the family are iso in $\mathscr{B}$.

For an interpretation $I : L \to \mathscr{B}$ (model $I : T \to \mathscr{B}$) and for a bicartesian closed functor $F : \mathscr{B} \to \mathscr{D}$ by $F \circ I$ we denote the interpretation $F \circ I : L \to \mathscr{D}$ (model $F \circ I : T \to \mathscr{D}$) defined as follows: on basic types $F \circ I(A) = F(I(A))$ and on basic constants $F \circ I(c) = F(I(c))$. Now it is easy to see that the first equation is actually true for all types and that the second equations generalize to the terms with contexts i.e. $F\circ I(x{:}A\rhd t)=F(I(x{:}A\rhd t))$. So indeed $F\circ I$ is an interpretation of L. That $F\circ I$ is also a model (if I is one) will follow from the soundness below.

Similarly, if we are given a (homo)morphism $\psi{:}I_1\Rightarrow I_2$ between two interpretations (models) of a language L (theory T) in a bc-category $\mathscr{B}$ and if $F{:}\mathscr{B}\to\mathscr{D}$ is a bc-functor then $F\circ\psi$ will denote the (homo)morphism between $F\circ I_1$ and $F\circ I_2$ defined as follows: $F\circ\psi_A=F(\psi_A)$; it is not hard to check that this is indeed a homomorphism. Also, of course, if ψ was an isomorphism $F\circ\psi$ remains one too.

And finally, if $\mathscr{A}\underset{G}{\overset{F}{\underset{\Downarrow\theta}{\rightrightarrows}}}\mathscr{B}$ is a natural isomorphism in $\mathscr{BCC}$ and if $T\xrightarrow{M}\mathscr{A}$ is a model, then $T\underset{G\circ M}{\overset{F\circ M}{\underset{\Downarrow\theta\circ M}{\rightrightarrows}}}\mathscr{B}$ is an isomorphism of models defined as expected i.e. $\theta\circ M_A=\theta_{M(A)}$.

Remark 2.6. Suppose that $N_1,\ N_2\in Mod_T\mathscr{B}$. It is easy to see that a family of isos $N_1(X)\xrightarrow{\psi_X}N_2(X)$ (X is a basic type (sort) from the language) extends in at most one way (if any) to an isomorphism $N_1\overset{\psi}{\Rightarrow}N_2$. (By induction on the complexity of types one can show that the isomorphisms ψ^C must satisfy the following: $\psi_1=1_1=\psi_1^{-1}$, $\psi_0=1_0=\psi_0^{-1}$, if X is a basic type ψ_X and ψ_X^{-1} are given above, $\psi_{A\times B}=\psi_A\times\psi_B$ and $\psi_{A\times B}^{-1}=\psi_A^{-1}\times\psi_B^{-1}$, $\psi_{A+B}=\psi_A+\psi_B$ and $\psi_{A+B}^{-1}=\psi_A^{-1}+\psi_B^{-1}$, and $\psi_{A^B}=(\psi_A\varepsilon(1\times\psi_B^{-1}))^*$ and $\psi_{A^B}^{-1}=(\psi_A^{-1}\varepsilon(1\times\psi_B))^*$.)

To guarantee also the existence of an isomorphism $N_1\overset{\psi}{\Rightarrow}N_2$ which extends the given family, the family has to satisfy the following: for every basic constant ξ^C from the language, $\psi_C N_1(\rhd\xi^C)=N_2(\rhd\xi^C)$. The isomorphisms ψ^C are defined as above.

Proposition 2.7 (Soundness). *Let T be a $\lambda\delta$-theory. Let M be a model of T in a bicartesian closed category. Then*

$$\textit{If } T\vdash f=_X g \textit{ then } M(X\rhd f)=M(X\rhd g).$$

Proof. As usually, this can be proved by induction on the complexity of proofs. □

Definition 2.8. To every bicartesian closed category $\mathscr{C}$ we can associate a $\lambda\delta$-language $L_{\mathscr{C}}$, called the *internal language*, as follows:
– The objects become the set of basic types. When we want to be precise, the basic type corresponding to an object A we will denote by X_A (this is required when we want to make distinction between types such as $X_A\times X_B$ and $X_{A\times B}$).
– The arrows from the specified terminal object 1 become the basic constants – but in several different ways! More precisely: the basic constants of type $\mathscr{T}(X_{A_1},\ldots,X_{A_n})$ are the arrows $\hom_{\mathscr{C}}(1,\mathscr{T}(A_1,\ldots,A_n))$. (Thus, we have (at least) two different constants $c_f{:}X_{A_1\times A_2}$ and $c_f{:}X_{A_1}\times X_{A_2}$ corresponding to the same $(1\xrightarrow{f}A_1\times A_2)\in\mathscr{C}$.)

The *standard interpretation* M is the interpretation which to every symbol of the internal language assigns the intended meaning: $X_A \mapsto A$ and $c_f : \mathcal{T}(X_{A_1}, \ldots, X_{A_n}) \mapsto f : 1 \to \mathcal{T}(A_1, \ldots, A_n)$.

The corresponding λ-theory $T_{\mathscr{C}}$ contains all equations satisfied by the standard interpretation: $t^A =_X s^A \in T_{\mathscr{C}}$ iff $M(X \rhd t) = M(X \rhd s)$.

The above notions make sense in case of a nonstrict ("ordinary") bicartesian closed category $\mathscr{B}$, not only in case of strict bcc, except that in the nonstrict case we first have to choose a strict structure on $\mathscr{B}$ and then the interpretation of the complex types e.g. $M(X \times Y)$ is the chosen product of $M(X)$ and $M(Y)$ in $\mathscr{B}$.

Proposition 2.9 (Completeness). *For a given $\lambda\delta$-theory T there exists a canonical model $M : T \to \mathscr{B}_T$ such that $M(X \rhd u) = M(X \rhd v)$ only if $T \vdash u =_X v$.*

Proof. This is a standard construction and it is given as follows.

Objects. Objects are types.

Arrows. They are classes of equivalent terms with contexts. To compare $(x_1 : A_1, \ldots, x_n : A_n \rhd f^D(x_1^{A_1}, \ldots, x_n^{A_n}))$ with $(y_1 : B_1, \ldots, y_m : B_m \rhd g^D(y_1^{B_1}, \ldots, y_m^{B_m}))$ we first have to have $(\ldots(A_1 \times A_2) \times \ldots) \times A_n \equiv (\ldots(B_1 \times B_2) \times \ldots) \times B_m$, call it C. (So, assuming $m \leqq n$ it says that $B_m \equiv A_n, \ldots, B_2 \equiv A_{n-m+2}$ and $B_1 \equiv (\ldots(A_1 \times A_2) \times \ldots) \times A_{n-m+1}$.) Then we say that they are equivalent iff

$$T \vdash f(\pi_1(z), \ldots, \pi_n(z)) =_z g(\pi_1(z), \ldots, \pi_m(z)).$$

The class above gives an arrow $C \to D$.

Composition. $(y^{\mathbf{B}} \rhd g)(x^{\mathbf{A}} \rhd f) = (x^{\mathbf{A}} \rhd g(\pi_1(f)/y_1, \ldots, \pi_m(f)/y_m))$. Here f is of the type $\mathbf{B}$.

Units. $1_A = (x : A \rhd x)$.

Cartesian structure. This is going to be defined on the representatives of arrows which have one free variable.
- $0_A = (x : A \rhd *)$.
- $\pi_{A,B} = (x : A \times B \rhd \pi(x))$.
- $\langle (x : A \rhd f(x)), (y : A \rhd g(y)) \rangle = (x : A \rhd \langle f(x), g(x) \rangle)$. (Sic!)

Closed structure.
- $\varepsilon_{A,B} = (x : A^B \times B \rhd (\pi_1(x)`\pi_2(x)))$.
- $(x : A \times B \rhd f(x))^* = (x_1 : A \rhd \lambda x_2 \cdot f(\langle x_1, x_2 \rangle))$.

Coproducts.
- $\Box_A = (x^0 \rhd \varepsilon^A(x^0))$
- $\iota_i = (x^{A_i} \rhd \iota_i(x^{A_i}))$
- $[(x^A \rhd f^C), (y^B \rhd g^C)] = (z^{A+B} \rhd \delta x \cdot f, y \cdot g; z)$

The equivalence classes which correspond to $(\rhd c)$, where c is a constant from the language we will denote also by c.

As usual the first thing to check is independence on representatives. But this is true because of the substitution rule (Sub) for typed $\lambda\delta$-calculus. It is also easy to see that the above construction gives a bicartesian closed category. The canonical interpretation which assigns types to the same-name-objects, constants to the same-name-arrows is obviously a model of T. The whole construction is such that 'by definition' completeness follows. $\Box$

Corollary 2.10. *The canonical model $M:T\to\mathscr{B}_T$ classifies all models of T in the following sense: the map $\mathscr{BCC}(\mathscr{B}_T,\mathscr{D})\xrightarrow{-\circ M}Mod_T\mathscr{D}$ is an isomorphism of categories.*

Let us be more explicit about the 2-dimensional property of the canonical model: suppose that N_1, $N_2\in Mod_T\mathscr{D}$ are two isomorphic models (i.e. for every type A in T there exists an isomorphism $N_1(A)\xrightarrow{\psi_A}N_2(A)$ in $\mathscr{D}$ such that for every term $(x^A\rhd t^B)\in T$, $\psi_B N_1(x^A\rhd t^B)=N_2(x^A\rhd t^B)\psi_A$) then there exists unique natural isomorphism $F_1\overset{\Psi}{\Rightarrow}F_2$ such that $\Psi\circ M=\psi$, in other words: for every type A $\Psi_A=\psi_A$.

Having in mind the Remark 2.6, we can require less in the above statement about the 2-dimensional property of $M:T\to\mathscr{B}_T$, that is, we could give the isomorphism between N_1, N_2 giving only a family of isomorphisms $N_1(X)\xrightarrow{\psi_X}N_2(X)$, now X is just a basic type, satisfying the following: for every basic constant ξ^C from the language, $\psi_C N_1(\rhd\xi^C)=N_2(\rhd\xi^C)$.

Proof. For the 1-dimensional part, let us just prove surjectivity of the above map $-\circ M$. Take a model $N:T\to\mathscr{D}$ we have to find a bc-functor $F:\mathscr{B}_T\to\mathscr{D}$ such that $N=F\circ M$. F on $Ob(\mathscr{B}_T)$ is easily defined since $Ob(\mathscr{B}_T)$ are types of T so $F(A)=N(A)$. Since the arrows of $\mathscr{B}_T$ are classes of equivalent terms with contexts we are going to define $F=N$ on arrows (also) (recall the definition of interpretation). Now we have to show that F does not depend on the choice of representatives and that F is indeed bc-functor. The first part follows from the completeness, and the second from the definition of the bc-structure on $\mathscr{B}_T$.

As for the 2-dimensional property, let us just say that naturality of $\Psi:F_1\Rightarrow F_2$ is "the same thing" as the homomorphism property of $\psi:N_1\Rightarrow N_2$. □

3 The interpolation result

In this part we formulate and prove our main result, that is the interpolation property of bicartesian closed categories as well as cartesian closed categories. Also, at the end we give couple of applications which show that our interpolation is indeed a strong generalization of the corresponding result for Heyting algebras.

Let us formulate more precisely what we are after:

Definition 3.1. A square consisting of categories, functors and a natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathscr{B} \arrow[d, "H"] \\ \mathscr{C} \arrow[r, "K"'] \arrow[ur, phantom, "\overset{\tau}{\swarrow}"] & \mathscr{D} \end{tikzcd}$$

has the interpolation property if for every two objects $C\in\mathscr{C}$ and $B\in\mathscr{B}$ and every arrow $H(B)\xrightarrow{d}K(C)$ in $\mathscr{D}$ there exist an object $A\in\mathscr{A}$ and arrows $B\xrightarrow{b}F(A)$ in $\mathscr{B}$ and $G(A)\xrightarrow{c}C$ in $\mathscr{C}$ such that $d=K(c)\tau_A H(b)$.

We will consider only those squares of the above type with τ a natural isomorphism.

Theorem 3.2. *In the 2-category $\mathscr{BCC}$ of bicartesian closed categories all Pushouts have the interpolation property.*

We use the term "Pushout" to specify that we have in mind appropriate version of the 2-categorical weighted bicolimit – the precise definition is coming later.

This result holds not only for bicartesian closed categories but also for cartesian closed categories.

The above theorem is proved in two steps. First we prove a stronger interpolation property for intuitionistic propositional logic than known in the literature – namely, as mentioned in the introduction, we not only obtain the interpolant but we also show that the new proofs (of the interpolant and from the interpolant) when composed are actually equal to the proof which we started with. Additional care is needed to handle the presence of axioms; the presence of additional equalities among proofs turns out to be no problem at all. Identifying proofs with terms we can precisely state this as follows.

Proposition 3.3. *Let L_1 and L_2 be two languages, and let T_1 and T_2 be two $\lambda\delta$-theories on the respective languages. Let T_0 be a theory on the language $L_1 \cap L_2$ such that $T_0 \subset T_1 \cap T_2$ (we may as well assume that the theories are deductively closed). Let $(x^B \rhd t^C)$ be a term in the language $L_1 \cup L_2$ such that the type B is in L_1 and the type C is in L_2. Then, there is a type A in $L_1 \cap L_2$ and terms $(x^B \rhd r^A)$ in L_1, and $(y^A \rhd s^C)$ in L_2 such that:*

$$T_1 \cup T_2 \vdash t =_{x^B} s(r/y).$$

The proof of this proposition is given at the end of Sect. 3.2. In the Sect. 3.3 we give the first reformulation of the above proposition in categorical terms and we obtain the interpolation property for "ordinary" 2-pushouts in the 2-category $\mathscr{BCC}_s$ (bicartesian closed categories with the chosen structure, strict be-functors and natural isomorphisms as 2-cells) – one can notice, though, that the functors associated to the above proposition (i.e. $\mathscr{B}_{T_0} \to \mathscr{B}_{T_i}$) have a particular property – they are inclusions on objects. This is so because L_0 is a subset of L_i – no collapsing of types has occurred. Although the main result – Theorem 3.2 does not require any assumption of this type – these somewhat unusual functors will play an important role in the proof of it.

Behind the scenes of the rest of the proof of Theorem 3.2 is the connection between the strict and nonstrict doctrines á la [BKP89] and the equivalence of pseudopushouts and 2-pushouts of functors which are injections on objects in the strict doctrine. The proof is presented in this manner in [Čub93]; here, however, we give only the minimal amount of these information which is needed for the proof.

3.1 Prawitz' permutative reductions

As was mentioned earlier, to obtain the interpolation property for bicartesian closed categories we are going to analyze a syntactic proof of the Craig interpolation property for intuitionistic propositional logic. There, the strategy is to show weak normalization of the appropriate system of proof-reductions, and then to study the normal form. It is hard to attack the whole set of equations as given in Definition 2.3, but luckily enough we don't have to do that. We can take a system of equations which is strictly weaker than the one in the definition and this will suffice! It is well known that adding disjunction to $\{\top, \wedge, \rightarrow\}$-fragment of intuitionistic logic brings difficulties of a new kind to the analysis of proofs. E.g. to prove a satisfactory form of normalization of proofs, i.e. one which will give the subformula property, it is no longer enough to consider just β-like reductions (denoted by R_2 below) but one has to add a substantial part of η-like reductions (or rather expansions) for disjunctions (and the connective "false") (denoted by C and E below). This is already done in [Pra65], for a "recent" discussion connecting this to linear logic see [GLT89]. We choose to work with Prawitz' original reductions, in the form given in [GLT89].

Notation: for two terms t and r we write $t \succ r$ if r is an immediate subterm of t. Reflexive and transitive closure of $\succ$ we shall denote $\succ^*$, so that $t \succ^* s$ means that s is a subterm of t.

The following set of reduction we will denote by ρ; we find it convenient to partition ρ as follows:

$$R_2\begin{cases} (\lambda x \cdot t`s) \xrightarrow{\beta} t(s/x) & \\ \pi_i(\langle t_1, t_2\rangle) \xrightarrow{Pr_i} t_i & i=1,2 \\ \delta x_1 \cdot u_1, x_2 \cdot u_2 ; \iota_i(t) \xrightarrow{in_i} u_i(t/x_i) & i=1,2. \end{cases}$$

$$C\begin{cases} \pi_i(\delta x_1 \cdot u_1, x_2 \cdot u_2 ; t) \xrightarrow{1,2} \delta x_1 \cdot \pi_i(u_1), x_2 \cdot \pi_i(u_2); t \\ (\delta x_1 \cdot u_1, x_2 \cdot u_2 ; t)`r \xrightarrow{3} \delta x_1 \cdot u_1`r, x_2 \cdot u_2`r); t \\ \delta x_1 \cdot u_1, x_2 \cdot u_2 ; (\delta y_1 \cdot v_1, y_2 \cdot v_2 ; t) \\ \qquad \xrightarrow{4} \delta y_1 \cdot (\delta x_1 \cdot u_1 ; x_2 \cdot u_2 ; v_1), y_2 \cdot (\delta x_1 \cdot u_1, x_2 \cdot u_2 ; v_2); t \\ \varepsilon^A(\delta x \cdot u, y \cdot v; t) \xrightarrow{5} \delta x \cdot \varepsilon^A(u), y \cdot \varepsilon^A(v); t \end{cases}$$

$$E\begin{cases} \pi_i(\varepsilon^{A_1 \times A_2}(t)) \xrightarrow{1,2} \varepsilon^{A_i}(t) \\ \varepsilon^{A^B}(t)`r \xrightarrow{3} \varepsilon^A(t) \\ \delta x \cdot u^C, y \cdot v^C; \varepsilon^{A+B}(t) \xrightarrow{4} \varepsilon^C(t) \\ \varepsilon^A(\varepsilon^0(t)) \xrightarrow{5} \varepsilon^A(t) \end{cases}$$

The following "scheme" of reductions in natural deduction corresponds to the C-reductions:

$$\dfrac{\dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \begin{matrix}\cancel{A}\\ \vdots\\ C\end{matrix}\quad \begin{matrix}\cancel{B}\\ \vdots\\ C\end{matrix}}{C}\,\delta}{D}\,\tau \quad\rightarrow\quad \dfrac{\begin{matrix}\vdots\\ A\vee B\end{matrix}\quad \dfrac{\begin{matrix}\cancel{A}\\ \vdots\\ C\end{matrix}}{D}\,\tau \quad \dfrac{\begin{matrix}\cancel{B}\\ \vdots\\ C\end{matrix}}{D}\,\tau}{D}\,\delta$$

Here, τ stands for one of the five rules for elimination of connectives – so it can have one, two or three hypotheses; δ stands for the elimination of disjunction. Similarly one can represent the E-reductions.

To see that the system of equations generated by the above set of reductions ρ is weaker than the one in Definition 2.3 use Lemma 2.4. The following is not really needed for our purposes but let us notice that the equation $\delta x^A \cdot \iota_1(x^A), y^B \cdot \iota_2(y^B); w = w$ is not provable in the above system.

The above reduction system ρ is strongly normalizing and has the Church-Rosser property (stated and partially in [Pra71], [GLT89], [Gir71]); here we give just what we need and this is weak normalization (cf. [Pra65, p. 50]). A not original but complete proof of the theorem below one can find in [Čub93].

Theorem 3.4 (Prawitz' weak normalization). *Every term t in the $\lambda\delta$-calculus can be reduced to a ρ-normal form.*

3.2 *Interpolation in the λ-calculus setting*

Theorem 3.4 is used to prove the Craig interpolation theorem for intuitionistic propositional logic. In this section we give the proof which follows Prawitz' proof which is given as a hint in [Pra65]. There are several differences between our proof and the Prawitz proof. First the minor ones: he works in natural deduction and does not have the connective 1 ("true") in the language – we work with typed lambda calculus and we have all the propositional connectives. The most important difference is that we do more, i.e. we check that not only do we obtain an interpolant but also that the two proofs when composed are equal to the proof which we began with (actually we even get more: the two proofs when composed reduce to the normal form of the proof that we began with). Also we allow the presence of arbitrary axioms as well as additional equations of proofs and Prawitz does not consider this at all.

A general remark – whenever a variable appears "out of nowhere" it means that this is a brand new variable.

First we need the following lemma (cf. [Pra65, Cor. 3, p. 54]).

Lemma 3.5. *For a free $\lambda\delta$-calculus (no additional equations but with constants) the following holds: let t^C be a ρ-normal term which is one of: $\pi_i(s)$, $s`r$, $\delta x \cdot u, y \cdot v; s$, $\varepsilon^C(s)$ or an atomic term (i.e. a variable or a constant (or $*$)). Then there exists a chain $t \equiv t_0 \succ \ldots \succ t_n$ of successive subterms of t such that they are in the following relation: for every $0 \leq i < n$, t_i is one of $\pi_j(t_{i+1})$, $t_{i+1}`u$, $\delta x \cdot u, y \cdot v; t_{i+1}$ (for some u, v, x and y) or $t_i \equiv \varepsilon^B(t_{i+1})$, and moreover t_n is an atomic term (i.e. $t_n \equiv v^E$ where v is a variable or a constant (or $*$)).*

In particular if the term t is not $*$ then t_n is not $*$. So, if in addition the $\lambda\delta$-calculus doesn't have constants then t_n is a variable.

Furthermore, by C-normality we have: if $t_{n-1} \equiv dx \cdot u, y \cdot v; t_n$ then the above chain $t \equiv t_0 \succ \ldots \succ t_n$ is actually just $t \equiv \delta x \cdot u, y \cdot v; t_1$ and t_1 is either $x^{E_1+E_2}$ or a constant (for some ρ-normal terms u, v).

Similarly, by E-normality, if $t_{n-1} \equiv \varepsilon^B(t_n)$ then $t \equiv \varepsilon^B(t_1)$ and t_1 is a variable x^0 or a constant of type 0.

Proof. Induction on the complexity of t. If the complexity is zero this is no problem by the property of t_n. If $t \equiv \pi_i(s)$ then $s^{C_1 \times C_2} \equiv s_1`r_1$ for some ρ-normal terms r_1 and s_1, or $s \equiv \pi_j(s_1)$ for a ρ-normal term s_1, or s is of the zero complexity (s can't be a δ-form by C-normality, ε-form by E-normality nor $\langle u, v\rangle$ by R_2-normality, and the other cases don't type-match). All three cases are all right by the induction hypothesis. So to get the chain for t we just add t on the top of the chain for s. Similarly the case when $t \equiv s`r(s^{C_1^{C_2}} \equiv \pi_j(s_1)$ or $s \equiv s_1`r_1$ or s is atomic). Third case is when $t \equiv \delta x \cdot u, y \cdot v; s$. Then again s can be either $s_1`r$ or $\pi_j(s_1)$ or atomic; this is handled again by the induction hypothesis. Notice that t_n must have a complex type or type 0 unless $t \equiv *$. □

Definition 3.6. $A^+(A^-)$ is the set of atoms which occur positively (negatively) in A. For a context $\Gamma = x_1^{B_n}, \ldots, x_n^{B_n}$ we define $\Gamma^+ = \cup_i B_i^+ (\Gamma^- = \cup_i B_i^-)$. Also we define $1^+ = 1^- = 0^+ = 0^- = \phi$ (empty set).

Lemma 3.7. *For a free $\lambda\delta$-calculus without free constant terms we have: let $(\Gamma \triangleright t^C)$ be a ρ-normal term and let $\Gamma_1 \cup \Gamma_2 = \Gamma$ be a partition of the context. Then there are $(\Gamma_1 \triangleright r^A)$ and $(\Gamma_2, y^A \triangleright s^C)$ such that*

1. $t =_\Gamma s(r/y)$,
2. $A^+ \subseteq \Gamma_1^+ \cap (\Gamma_2^- \cup C^+)$,
3. $A^- \subseteq \Gamma_1^- \cap (\Gamma_2^+ \cup C^-)$.

Actually one proves in 1) that $s(r/y) \xrightarrow{\rho^*} t$.

Proof. First notice that in the case when $\Gamma_1 = \phi$ the result is obvious: just take $A \equiv 1$, $r \equiv *$ and $s \equiv t$. Now we proceed by induction on the complexity of t. We have the following cases:

Case 1: $t^C \equiv x^C$. We have two subcases: $x^C \in \Gamma_1$ then take $A \equiv C$, $r \equiv x^C$ and $s \equiv y^C$; and in the second subcase $x^C \in \Gamma_2$ then take $A \equiv 1$, $s \equiv x^C$ and $r \equiv *$.

Case 2: $t^C \equiv *$, so $C \equiv 1$. Just take $A \equiv 1$, $s \equiv *$ and $r \equiv *$.

Case 3: $t^C \equiv \langle t_1^{C_1}, t_2^{C_2}\rangle$, so $C \equiv C_1 \times C_2$ and $t_i^{C_i}$ are ρ-normal. By the induction hypothesis there are $(\Gamma_1 \triangleright r_i^{A_i})$ and $(\Gamma_2, y_i^{A_i} \triangleright s_i^{C_i})$ $(i = 1, 2)$ such that

$$t_i =_\Gamma s_i(r_i/y_i), \quad A_i^+ \subseteq \Gamma_1^+ \cap (\Gamma_2^- \cup C_i^+), \quad A_i^- \subseteq \Gamma_1^- \cap (\Gamma_2^+ \cup C_i^-).$$

Now take $A \equiv A_1 \times A_2$, $s \equiv \langle s_1(\pi_1(y^{A_1 \times A_2})/y_1), s_2(\pi_2(y^{A_1 \times A_2})/y_2)\rangle$ and $r \equiv \langle r_1, r_2\rangle$ and see that it satisfies the lemma.

Case 4: $t^C \equiv \iota(t_1^{C_1})$, so $C \equiv C_1 + C_2$ and $t_1^{C_1}$ is ρ-normal. By the induction hypothesis there are $(\Gamma_1 \triangleright r_1^{A_1})$ and $(\Gamma_2, y_1^{A_1} \triangleright s_1^{C_1})$ such that

$$t_1 \equiv_\Gamma s_1(r_1/y_1), \quad A_1^+ \subseteq \Gamma_1^+ \cap (\Gamma_2^- \cup C_1^+), \quad A_1^- \subseteq \Gamma_1^- \cap (\Gamma_2^+ \cup C_1^-).$$

Now take $A \equiv A_1$, $s \equiv \iota(s_1)$ and $r \equiv r_1$ and see that it satisfies the lemma.

Case 5: $t^C \equiv \lambda x^{C_2} \cdot t_1^{C_1}$, so $C \equiv C_1^{C_2}$, and $t_1^{C_1}$ is ρ-normal. Then by the induction hypothesis applied to the term $(\Gamma, x^{C_2} \triangleright t_1^{C_1})$ and the partition of its context as $\Gamma_1 \cup (\Gamma_2, x^{C_2})$ there are $(\Gamma_1 \triangleright r_1^{A_1})$ and $(\Gamma_2, x^{C_2}, y^{A_1} \triangleright s_1^{C_1})$ such that

$$t_1 =_{\Gamma, x} s_1(r_1/y_1), A_1^+ \subseteq \Gamma_1^+ \cap ((\Gamma_2^-, C_2^-) \cup C_1^+), A_1^- \subseteq \Gamma_1^- \cap ((\Gamma_2^+, C_2^+) \cup C_1^-).$$

Now take $A \equiv A_1$, $r \equiv r_1$ and $s \equiv \lambda x^{C_2} \cdot s_1$ and check that the lemma holds (use $(C_1^{C_2})^- = C_2^+ \cup C_1^-$).

Case 6: (In this case only, we use Lemma 3.5.) $t \equiv \pi_i(s)$ or $t \equiv s`r$ or $t \equiv \delta x \cdot u, y \cdot v; s$ or $t \equiv \varepsilon^C(s)$ where r, s, u and v are some ρ-normal terms. Then by Lemma 3.5 we have that there exists a chain of immediate subterms: $t \equiv t_0 \succ t_1 \succ \ldots \succ t_i \succ \ldots \succ t_n$ such that the following subcases can take place: $t^{E_i}_{n-1} \equiv \pi_i(x^{E_1 \times E_2})$ or $t^{E_1}_{n-1} \equiv x^{E_1^{E_2}}`u^{E_2}$ or $t_{n-1} \equiv \delta x \cdot u, y \cdot v; x^{E_1+E_2}$ or $t_{n-1} \equiv \varepsilon^B(x^0)$ for some ρ-normal terms u and v since in the calculus we don't have additional constants and $t \not\equiv *$.

Subcase 6.1.1: $t^{E_i}_{n-1} \equiv \pi_i(x^{E_1 \times E_2})$ and $x^{E_1 \times E_2} \in \Gamma_1$.

Let t' be like the term t expect that it has x^{E_i} instead of t_{n-1} (notice however, that t' can contain $x^{E_1 \times E_2}$) so $t \equiv_\Gamma t'(\pi_i(x^{E_1 \times E_2})/x^{E_i})$. The complexity of t' is lower then the complexity of t, so we can apply the induction hypothesis on $(\Gamma \cup \{x^{E_i}\} \triangleright t')$ and the partition $\Gamma, x^{E_i} = (\Gamma_1, x^{E_i}) \cup \Gamma_2$. Then by the induction hypothesis then exist $(\Gamma_1, x^{E_i} \triangleright R^A)$ and $(\Gamma_2, y^A \triangleright S^C)$ such that

$$t' =_{\Gamma, x^{E_i}} S(R/y), \quad A^+ \subseteq (\Gamma_1 \cup E_i)^+ \cap (\Gamma_2^- \cup C^+), \quad A^- \subseteq (\Gamma_1 \cup E_i)^- \cap (\Gamma_2^+ \cup C^-).$$

Recall that $x^{E_1 \times E_2}$ is in Γ_1. So we define $r = R(\pi_i(x^{E_1 \times E_2}))$, $s = S$ and A stays the same and we can check that the lemma is satisfied. First:

$$t =_\Gamma t'(\pi_i(x^{E_1 \times E_2})/x^{E_i}) =_\Gamma S((R(\pi_i(x^{E_1 \times E_2})/x^{E_i}))/y) \equiv s(r/y).$$

The second and third part of the conclusion are satisfied since $E_i^s \subset (E_1 \times E_2)^s$ where $s = +, -$. It is obvious that the stronger hypothesis gives the reduction instead of the equality.

Subcase 6.1.2: $t^{E_i}_{n-1} \equiv \pi_i(x^{E_1 \times E_2})$ and $x^{E_1 \times E_2} \in \Gamma_2$. Again consider $(\Gamma, x^{E_i} \triangleright t')$ where t' is the same as t except that x^{E_i} appears instead of $\pi_i(x^{E_1 \times E_2})$, therefore $t'(\pi_i(x^{E_1 \times E_2})/x^{E_i}) \equiv t$. Since t' is less complex then t apply the induction hypothesis on the partition $\Gamma, x^{E_i} = \Gamma_1 \cup (\Gamma_2, x^{E_i})$. Then by the induction hypothesis then exist $(\Gamma_1 \triangleright R^A)$ and $(\Gamma_2, x^{E_i}, y^A \triangleright S^C)$ such that

$$t' =_{\Gamma, x^{E_i}} S(R/y), \quad A^+ \subseteq (\Gamma_1^+ \cap ((\Gamma_2 \cup E_i)^- \cup C^+), \quad A^- \subseteq \Gamma_1^- \cap ((\Gamma_2 \cup E_i)^+ \cup C^-).$$

Recall that $x^{E_1 \times E_2}$ is in Γ_2. So we define $r = R$, $s = S(\pi_i(x^{E_1 \times E_2}))$ and A stays the same and we can check that the lemma is satisfied. First, $t =_\Gamma t'(\pi_i(x^{E_1 \times E_2})/x^{E_i}) =_\Gamma S(\pi_i(x^{E_1 \times E_2})/x^{E_i})(R/y) \equiv s(r/y)$. The second and third part of the conclusion are satisfied since $E_i^s \subset (E_1 \times E_2)^s$ where $s = +, -$. As earlier, it is obvious that the stronger hypothesis gives the reduction instead of the equality.

Subcase 6.2.1: $t^{E_1}_{n-1} \equiv x^{E_1^{E_2}}`u^{E_2}$ and $x^{E_1^{E_2}} \in \Gamma_1$.

Then we apply the induction hypothesis on $(\Gamma \triangleright u^{E_2})$ and the "reverse" partition $\Gamma_2 \cup \Gamma_1 = \Gamma$ to get $(\Gamma_2 \triangleright R_1^{A_1})$ and $(\Gamma_1, y_1^{A_1} \triangleright S_1^{E_2})$ such that

$$u =_\Gamma S_1(R_1/y_1), \quad A_1^+ \subseteq \Gamma_2^+ \cap (\Gamma_1^- \cup E_2^+), \quad A_1^- \subseteq \Gamma_2^- \cap (\Gamma_1^+ \cup E_2^-).$$

Applying the induction hypothesis once more on $(\Gamma, z^{E_1} \triangleright w^C)$, where $w(t_{n-1}/z) \equiv t$, and the partition $(\Gamma_1, z^{E_1}) \cup \Gamma_2 = \Gamma, z^{E_1}$ to get $(\Gamma_1, z \triangleright R_2^{A_2})$ and $(\Gamma_2, y_2^{A_2} \triangleright S_2^C)$ such that

$$w =_{\Gamma, z} S_2(R_2/y_2), \quad A_2^+ \subseteq (\Gamma_1^+ \cup E_1^+) \cap (\Gamma_2^- \cup C^+), \quad A_2^- \subseteq (\Gamma_1^- \cup E_1^-) \cap (\Gamma_2^+ \cup C^-).$$

Now take $A=A_2^{A_1}$, $r\equiv\lambda y_1\cdot R_2((x`S_1)/z)$ and $s\equiv S_2((y^{A_2^{A_1}}`R_1)/y_2)$ and check that the lemma is satisfied. Indeed the first conclusion follows from: $s(r/y) \equiv S_2((\lambda y_1\cdot R_2((x`S_1)/z)`R_1)/y_2)\xrightarrow{\rho} S_2(R_2((x`S_1(R_1/y_1))/z))/y_2)$ ($y_1$ appears only in $S_1$)$\equiv S_2(R_2((x`u)/z))/y_2)\equiv S_2(R_2(t_{n-1}/z))/y_2)\equiv S_2(R_2/y_2)\ (t_{n-1}/z)$ (since z appears only in R_2)$=w(t_{n-1}/z)\equiv t$. To get $\xrightarrow{\rho}$ instead of $=$ in the last step use the stronger induction hypothesis: "actually one proves in 1) that $s(r/y)\xrightarrow{\rho} t$" not just the equality.

For the other two conclusions use that $x^{E_1^{E_2}}\in\Gamma_1$, so $E_1^s\subseteq\Gamma_1^s$ and $E_2^s\subseteq\Gamma_1^{-s}$ where $s=-,+$.

Subcase 6.2.2: $t_{n-1}^{E_1}\equiv x^{E_1^{E_2}}`u^{E_2}$ and $x^{E_1^{E_2}}\in\Gamma_2$.

Then we apply the induction hypothesis on $(\Gamma\rhd u^{E_2})$ and the partition $\Gamma_1\cup\Gamma_2=\Gamma$ to get $(\Gamma_1\rhd R_1^{A_1})$ and $(\Gamma_2, y_1^{A_1}\rhd S_1^{E_2})$ such that

$$u=_\Gamma S_1(R_1/y_1),\qquad A_1^+\subseteq\Gamma_1^+\cap(\Gamma_2^-\cup E_2^+),\qquad A_1^-\subseteq\Gamma_1^-\cap(\Gamma_2^+\cup E_2^-).$$

Applying the induction hypothesis once more on $(\Gamma, z^{E_1}\rhd w^C)$, where $w(t_{n-1}/z)\equiv t$, and the partition $\Gamma_1\cup(\Gamma_2, z^{E_1})=\Gamma, z^{E_1}$ to get $(\Gamma_1\rhd R_2^{A_2})$ and $(\Gamma_2, z^{E_1}, y_2^{A_2}\rhd S_2^C)$ such that

$$w=_{\Gamma,z}S_2(R_2/y_2),\qquad A_2^+\subseteq\Gamma_1^+\cap(\Gamma_2^-\cup E_1^-\cup C^+),\qquad A_2^-\subseteq\Gamma_1^-\cap(\Gamma_2^+\cup E_1^+\cup C^-).$$

Now take $A=A_1\times A_2$, $s\equiv S_2(\pi_2((y^{A_1\times A_2})/y_2))\ ((x`S_1(\pi_1(y^{A_1\times A_2})/y_1))/z)$ and $r\equiv\langle R_1, R_2\rangle$ and check that the lemma is satisfied. (We will check only the first conclusion. Indeed:

$$\begin{aligned}s(r/y)&\equiv S_2(\pi_2(\langle R_1, R_2\rangle)/y_2))\,((x`S_1(\pi_1(\langle R_1, R_2\rangle)/y_1))/z)\\&\xrightarrow{\rho} S_2(R_2/y_2)((x`S_1(R_1/y_1))/z)\\&\xrightarrow{\rho} S_2(R_2/y_2)((x`u)/z)\equiv S_2(R_2/y_2)(t_{n-1}/z)\\&\xrightarrow{\rho} w(t_{n-1}/z)\equiv t.\end{aligned}$$

The last two reductions where under the stronger induction hypothesis; otherwise we have equality.)

Subcase 6.3.1: $t_{n-1}\equiv\delta x_1^{E_1}\cdot t^1,\ x_2^{E_2}\cdot t^2;\ x^{E_1+E_2}$ and $x^{E_1+E_2}\in\Gamma_1$. First notice that by the end of the Lemma 3.5 $t\equiv\delta x_1\cdot t^1, x_2\cdot t^2; x^{E_1+E_2}$ for some ρ-normal terms t^1, t^2. Since the complexity of t^i is smaller the complexity of t we apply the induction hypothesis on $(\Gamma, x^{E_i}\rhd t^i)$, $(i=1,2)$ and the partition $\Gamma, x^{E_i}=(\Gamma_1, x^{E_i})\cup\Gamma_2$. So, we have that there are $(\Gamma_1, x^{E_i}\rhd r_i^{A_i})$ and $(\Gamma_2, y_i^{A_i}\rhd s_i^C)$ such that (for $i=1,2$):

$$t^i=_{\Gamma,x^{E_i}}s_i(r_i/y_i),\qquad A_i^+\subseteq(\Gamma_1, x^{E_i})^+\cap(\Gamma_2^-\cup C^+),\qquad A_i^-\subseteq(\Gamma_1, x^{E_i})^-\cap(\Gamma_2^+\cup C^-).$$

Now, let $A=A_1+A_2$, $r\equiv\delta x_1\cdot\iota_1(r_1), x_2\cdot\iota_2(r_2); x^{E_1+E_2}$ and $s\equiv\delta y_1\cdot s_1, y_2\cdot s_2; y^{A_1+A_2}$. Obviously s and r have right context; also the lemma is satisfied: first

$$\begin{aligned}s(r/y^{A_1+A_2})&\equiv\delta y_1\cdot s_1, y_2\cdot s_2;(\delta x_1\cdot\iota_1(r_1), x_2\cdot\iota_2(r_2); x^{E_1+E_2})\\&\xrightarrow{C_4}\delta x_1\cdot(\delta y_1\cdot s_1, y_2\cdot s_2;\iota_1(r_1)), x_2\cdot(\delta y_1\cdot s_1, y_2\cdot s_2;\iota_2(r_2)); x^{E_1+E_2}\\&\xrightarrow{\rho}\delta x_1\cdot s_1(r_1/y_1), x_2\cdot s_2(r_2/y_2); x^{E_1+E_2}\\&\xrightarrow{\text{ind. hyp.}}\delta x_1\cdot t^1, x_2\cdot t^2; x^{E_1+E_2}\equiv t.\end{aligned}$$

The second and the third part of the conclusion follow from $E_i^s \subset (E_1+E_2)^s$ where $s=+,-$.

Subcase 6.3.2: $t_{n-1} \equiv \delta x \cdot u, y \cdot v; x^{E_1+E_2}$ and $x^{E_1+E_2} \in \Gamma_2$. The same as the previous except that the partition of Γ is $\Gamma_1 \cup (\Gamma_2, x^{E_i})$.

Subcase 6.4: Finally assume $t_{n-1} \equiv \varepsilon(x^0)$. Again, by the end of the Lemma 3.5 $t \equiv \varepsilon^C(x^0)$. If $x^0 \in \Gamma_1$ then take $A=0$, $s \equiv \varepsilon^C(x^0))$ and $r \equiv x^0$. Obviously the lemma is satisfied. If $x^0 \in \Gamma_2$ then take $A=1$, $r \equiv *$ and take $s \equiv \varepsilon^C(x^0)$. Indeed $(\Gamma_1 \rhd *)$ and $(\Gamma_2, y^1 \rhd \varepsilon^C(x^0))$ satisfy the lemma. □

Now we have to prove a similar lemma but in the case when the $\lambda\delta$-calculus contains additional constant terms (but not additional equations). To see what kind of difficulty we have let's give an example: let $(x^B \rhd c^{CB} \text{`} x^B)$ then for $\Gamma_1 = x^B$ (and $\Gamma_2 = \phi$) the above lemma (as it is) would be false – A would have to be 1 but this wouldn't do. However the problem really doesn't exist, we just treat the additional constants as variables/additional hypothesis – what they actually are, and state the lemma carefully.

Lemma 3.8. *For a free $\lambda\delta$-calculus (with free constant terms) we have: let $(\Gamma \rhd t^C)$ be a ρ-normal term and let $\Gamma_1 \cup \Gamma_2 = \Gamma$ be a partition of the context; also let $\Sigma = \xi_1^{D_1}, \ldots, \Sigma_m^{D_m}$ be the set of the free constants which appear in t and let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a partition of that. Then there are $(\Gamma_1 \rhd r^A)$ and $(\Gamma_2, y^A \rhd s^C)$ such that*

1. $t =_\Gamma s(r/y)$,
2. $A^+ \subseteq (\Gamma_1^+ \cup \Sigma_1^+) \cap (\Gamma_2^- \cup \Sigma_2^- \cup C^+)$,
3. $A^- \subseteq (\Gamma_1^- \cup \Sigma_1^-) \cap (\Gamma_2^+ \cup \Sigma_2^+ \cup C^-)$,
4. *r^A contains only constants from Σ_1 and s^C contains only constants from Σ_2.*

Actually one proves in 1) that $s(r/y) \xrightarrow{\rho^*} t$.

Proof. We have to take care of two things: the first is that 1) is not stated with $\Gamma \cup \Sigma$ as context but just Γ, similarly the terms r and s are over the smaller contexts, and the second is the additional conclusion 4). We take the term T^C which is the same as t^C except that we put new variables $w_i^{D_i}$ instead of $\xi_i^{D_i}$. Then the above lemma gives $(\Gamma_1, \Sigma_1 \rhd R^A)$ and $(\Gamma_2, \Sigma_2, y^A \rhd S^C)$ such that $T =_{\Gamma \cup \Sigma} S(R/y)$ and the rest as in 2) and 3) and they don't have any constants (except maybe $*$) – we used the same name Σ for the set of the variables replacing ξ_i. Now just substitute the constants in the place of the appropriate variables and get the statement (r is the term R but with the constants instead of the variables from Σ_1, similarly s is the "new" S). □

In the previous example the lemma gives two solutions depending what the partition of the $\Sigma = w^{C^B}$ is (and with the given partition of context as $\Gamma_1 = x^B$). When $\Sigma_1 = C^B$ (and $\Sigma_2 = \phi$) then take $A = C$, $r \equiv t$ and $s^C \equiv y^C$. In the second case when $\Sigma_2 = C^B$ take $A = B$, $r \equiv x^B$ and $s^C \equiv (w^{C^B} \text{`} y^B)$.

The following lemma (cf. [Pra65, Cor. 5, p. 46]) is an immediate consequence of the previous lemma when $\Gamma_2 = \phi$ and the fact that every term has a ρ-normal form.

Lemma 3.9. *For every term $(\Gamma \rhd t^C)$ and for every partition $\Sigma = \Sigma_1 \cup \Sigma_2$ of the (free) constants from t, there exist $(\Gamma \rhd r^A)$ and $(y^A \rhd s^C)$ such that*

1. $t =_\Gamma s(r/y)$,
2. $A^+ \subseteq (\Gamma^+ \cup \Sigma_1^+) \cap (\Sigma_2^- \cup C^+)$,
3. $A^- \subseteq (\Gamma^- \cup \Sigma_1^-) \cap (\Sigma_2^+ \cup C^-)$,
4. r^A *contains only constants corresponding to* Σ_1 *and* s^C *contains only constants corresponding to* Σ_2.

And now an important corollary which we find interesting:

Corollary 3.10. *In a free $\lambda\delta$-calculus on a language $L = L_1 \cup L_2$ (with free constants) for every term $(x^B \triangleright t^C)$ such that $B \in L_1$ and $C \in L_2$ there exist $(x^B \triangleright r^A)$ and $(y^A \triangleright s^C)$ such that:*
1. $t =_{x^B} s(r/y)$,
2. $(x^B \triangleright r^A) \in L_1$,
3. $(y^A \triangleright s^C) \in L_2$.

(Notice that this implies that $A \in L_1 \cap L_2$.)

Before we prove the corollary let us give an example in categorical terminology: suppose that L_1 consists of three free objects/types X, Y, Z and suppose that it has only one free arrow/constant $a: Y \to Z$. Suppose also that L_2 consists of the same types and the only free arrow is $b: X \to Y$. Now suppose that we want to interpolate $ab: X \to Z$. For a moment it may look a bit surprising that there is any "useful" arrow in L_1 from X. But there is! It is quite easy to see that the interpolation is obtained from the following two arrows:

$$(\langle (a\pi')^*, 1_X \rangle : X \to Z^Y \times X) \in L_1 \quad \text{and} \quad (\varepsilon \langle \pi, b\pi' \rangle : Z^Y \times X \to Z) \in L_2.$$

Now we go back to

Proof of the previous corollary. Given $L_1 \cup L_2$ make Σ_1 to be the set of (free) constants from L_1, and Σ_2 to be the set of (free) constants from $L_2 - L_1$. Also notice that for a term u if all the variables and constants which appear are from a language L then the term u is on the language L i.e. all the types which appear in u are made out of the basic types which appear in the typing of the variables and constants. Now apply the previous lemma to obtain that all the constants from r^A are from L_1 and since x^B was in L_1 we have that $(x^B \triangleright r^A) \in L_1$. To show that $(y^A \triangleright s^C) \in L_2$ we reason similarly and in addition we check that $y^A \in L_2$ – from the previous lemma parts 2) and 3) we have that $A \subset \Sigma_2 \cup C$ and this gives $A \in L_2$ by the definition of Σ_2 and the assumption $C \in L_2$. The first conclusions in the corollary and in the previous lemma are the same. □

Finally we can notice that the above proof actually gives the proof of a stronger result which allows arbitrary $\lambda\delta$-theories, not only free ones – this was already mentioned as Proposition 3.3 which we can now prove:

Proof of Proposition 3.3. In the previous corollary we proved the statement without referring to theories, i.e. $\vdash t =_{x^B} s(r/y)$. From that, of course, follows $T_1 \cup T_2 \vdash t =_{x^B} s(r/y)$. □

It is interesting to notice that not every interpolant in the usual sense is one in our sense e.g. $X \times X \vdash X \times X$ has as an interpolant X but in our case X can't be an interpolant if the above proof is just $1_{X \times X}$ and X is atomic (since there is only one arrow in $\mathrm{hom}(X, X)$ – this itself can be proved by the subformula property – it would mean that X is isomorphic to $X \times X$ for every bicartesian closed category).

3.3 Interpolation in the categorical setting

Now, we want to see a categorical rewording of the previous proposition.

Proposition 3.11. *Let L_1 and L_2 be two languages, and let T_1 and T_2 be two $\lambda\delta$-theories in the respective languages. Let T_0 be a theory in the language $L_1 \cap L_2$ such that $T_0 \subset T_1 \cap T_2$ (again we may assume that the theories are deductively closed). To this situation we can associate the following diagram in $\mathscr{BCC}_s$:*

$$\begin{tikzcd} \mathscr{B}_{T_0} \arrow[r, "F_1"] \arrow[d, "F_2"'] & \mathscr{B}_{T_1} \\ \mathscr{B}_{T_2} & \end{tikzcd}$$

where $\mathscr{B}_{T_0} \xrightarrow{F_i} \mathscr{B}_{T_i}$ $i=1,2$ are obtained from the respective interpretations $T_0 \xrightarrow{M_i|L_0} \mathscr{B}_{T_i}$ where $T_i \xrightarrow{M_i} \mathscr{B}_{T_i}$ is the canonical model; see Corollary 2.10 ($M|_L$ means the reduct of the model M in the language L).

Now, we can form the following commutative square:

$$\begin{tikzcd} \mathscr{B}_{T_0} \arrow[r, "F_1"] \arrow[d, "F_2"'] & \mathscr{B}_{T_1} \arrow[d, "H"] \\ \mathscr{B}_{T_2} \arrow[r, "K"'] & \mathscr{B}_{T_1 \cup T_2} \end{tikzcd}$$

where H is the unique functor from Corollary 2.10 such that $H \circ M_1 = M_{1\cup 2|L_1}$, similarly $K \circ M_2 = M_{1\cup 2|L_2}$; here, $M_{1\cup 2}$ is the canonical model $T_1 \cup T_2 \to \mathscr{B}_{T_1 \cup T_2}$.

The above square has the interpolation property (where the 2-cell is the identity since the square commutes).

Proof. Take $B \in \mathscr{B}_{T_1}$ and $C \in \mathscr{B}_{T_2}$ and assume that there exists an arrow $H(B) \xrightarrow{d} K(C) \in \mathscr{B}_{T_1 \cup T_2}$. By the definition of a category associated to a theory there exists a term $(x^B \rhd t^C)$ in the language $L_1 \cup L_2$ such that $d = [x^B \rhd t^C]$. Now apply Proposition 3.3. □

It would be quite easy to see that the above square is a 2-pushout in the strict doctrine $\mathscr{BCC}_s$, but we are going to prove a more surprising fact: this square is also a Pushout in $\mathscr{BCC}$.

Let us just mention that the usual notion of pushout is not invariant under equivalence of categories. Also, one feels that in the doctrines as $\mathscr{BCC}$ pushouts "ought" to exist and yet they don't. All that is achieved by the right notion, that is the notion of Pushout (also known as bipushout) which is defined similarly to the ordinary pushouts except that one requires equivalences instead of isomorphisms.

Here is a precise definition of a Pushout.

Suppose we are given a diagram $\mathscr{A} \xrightarrow{F} \mathscr{B}$, $\mathscr{A} \xrightarrow{G} \mathscr{C}$ *in* $\mathscr{BCC}$. We want to construct a Pushout. That is, we want a diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathscr{B} \arrow[d, "H"] \arrow[dl, phantom, "\tau\,\swarrow"] \\ \mathscr{C} \arrow[r, "K"'] & \mathscr{D} \end{tikzcd}$$

where $\tau : HF \Rightarrow KG$ is a natural isomorphism which satisfies the following: for every two functors $\mathscr{B} \xrightarrow{P} \mathscr{E}$ and $\mathscr{C} \xrightarrow{Q} \mathscr{E}$ and a natural isomorphism $\theta : PF \Rightarrow QG$ there exists a functor $\mathscr{D} \xrightarrow{T} \mathscr{E}$ and there exist two natural isomorphisms $\theta_1 : P \Rightarrow TH$ and $\theta_2 : TK \Rightarrow Q$

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[d, "H"] \arrow[dr, "P"] \arrow[dl, phantom, "\swarrow \tau"] & \\
\mathcal{C} \arrow[r, "K"'] \arrow[drr, "Q"'] & \mathcal{D} \arrow[dr, "T"] \arrow[r, phantom, "\Downarrow \theta_1"] \arrow[d, phantom, "\Leftarrow \theta_2"] & {} \\
& {} & \mathcal{E} \arrow[loop right, "\theta"]
\end{tikzcd}$$

such that $(\theta_2 G)(T\tau)(\theta_1 F) = \theta$ and such that for every $\mathscr{D} \xrightarrow{T'} \mathscr{E}$ and natural isomorphisms $\phi_1 : TH \Rightarrow T'H$, $\phi_2 : TK \to T'K$ which satisfy $T'\tau \circ \phi_1 F = \phi_2 G \circ T\tau$ there exists unique natural isomorphism $\psi : T \Rightarrow T'$ such that $\phi_1 = \psi H$ and $\phi_2 = \psi K$.

First, we can notice that the interpolation property is invariant for Pushouts, i.e. if one Pushout over $\mathscr{A} \xrightarrow{F} \mathscr{B}$, $\mathscr{A} \xrightarrow{G} \mathscr{C}$ has the interpolation property than all the other Pushouts over the same diagram have this property. Even more is true, the interpolation property is indeed a 2-categorical (we even may say bicategorical) notion in the following sense.

Lemma 3.12. *Suppose we have the following diagram*

$$\begin{tikzcd}
\mathcal{A}' \arrow[rrr, "F'"] \arrow[ddd, "G'"'] & & & \mathcal{B}' \arrow[ddd, "H'"] \\
& \mathcal{A} \arrow[ul, "t_{\mathcal{A}}"] \arrow[r, "F"] \arrow[d, "G"'] & \mathcal{B} \arrow[ur, "t_{\mathcal{B}}"'] \arrow[d, "H"] \arrow[dl, phantom, "\swarrow \tau"] \arrow[ur, phantom, "\nwarrow t_F", shift left=4] & \\
& \mathcal{C} \arrow[dl, "t_{\mathcal{C}}"'] \arrow[r, "K"'] \arrow[dl, phantom, "\nwarrow t_G", shift left=4] & \mathcal{D} \arrow[dr, "t_{\mathcal{D}}"] \arrow[dr, phantom, "\nearrow t_H", shift left=4] \arrow[dr, phantom, "\swarrow t_K", shift right=4] & \\
\mathcal{C}' \arrow[rrr, "K'"'] & & & \mathcal{D}' \arrow[loop right, "\tau'"]
\end{tikzcd}$$

where $t_{\mathcal{A}}$, $t_{\mathcal{B}}$, $t_{\mathcal{C}}$ and $t_{\mathcal{D}}$ are equivalences of categories and $t_F: t_{\mathcal{B}}F \Rightarrow F't_{\mathcal{A}}$, $t_G: t_{\mathcal{C}}G \Rightarrow G't_{\mathcal{A}}$, $t_H: t_{\mathcal{D}}H \Rightarrow H't_{\mathcal{B}}$, $t_K: t_{\mathcal{D}}K \Rightarrow K't_{\mathcal{C}}$, $\tau: HF \Rightarrow KG$ and $\tau': H'F' \Rightarrow K'G'$ are natural isomorphisms such that

$$(1) \qquad K't_G \circ t_K G \circ t_{\mathcal{D}}\tau = \tau' t_{\mathcal{A}} \circ H't_F \circ t_H F : t_{\mathcal{D}}HF \Rightarrow K'G't_{\mathcal{A}}.$$

(This is essentially a strong transformation $\mathcal{I}^- \underset{\Phi^{-\prime}}{\overset{\Phi^-}{\Downarrow t^-}} \mathcal{BCC}$ *where $\mathcal{I}^-$ is a commutative square, and t is an equivalence in* $\mathrm{Hom}(\mathcal{I}^-, \mathcal{BCC})$*, see [MP89, Prop. 4.1.3].) Then:*

1. If one of the squares has the interpolation property the other one has it too.

2. Also, if one of the squares is a Pushout the other one is too.

Proof. The proof is easy – the first statement is fully proved in [Čub93] and the proof of the second statement is a part of the bicategorical folklore. □

Lemma 3.13. *Suppose $I = \{2 \xleftarrow{20} 0 \xrightarrow{01} 1\}$ is a 2-category and suppose* $\mathcal{I} \underset{\Phi'}{\overset{\Phi}{\Downarrow t}} \mathcal{BCC}$ *where t is an equivalence in* $\mathrm{Hom}(\mathcal{I}, \mathcal{BCC})$*. Suppose also $I: \mathcal{I} \hookrightarrow \mathcal{I}^-$ is an inclusion of the 2-categories (recall that $\mathcal{I}^-$ is just a commutative square), and suppose that there is a homomorphism $\mathcal{I}^- \xrightarrow{\Phi^-} \mathcal{BCC}$ such that $\Phi^- I = \Phi$.*

Then there exists a homomorphism $\mathcal{I}^- \xrightarrow{\Phi^{-\prime}} \mathcal{BCC}$ and an equivalence $\mathcal{I}^- \underset{\Phi^{-\prime}}{\overset{\Phi^-}{\Downarrow t^-}} \mathcal{BCC}$ *such that $\Phi^{-\prime} I = \Phi'$ and $t^- I = t$.*

Proof. It is easy to construct what is needed. □

To construct the Pushouts in $\mathcal{BCC}$ we need to examine more closely the relation between a bc-category $\mathcal{B}$ and the category $\mathcal{B}_{T_{\mathcal{B}}}$ (we will denote $\mathcal{B}_{T_{\mathcal{B}}}$ by $\mathcal{B}_s$). First recall that in the case of an "ordinary" (nonstrict) category to talk about the internal language $L_{\mathcal{B}}$ and the theory $T_{\mathcal{B}}$ we have to choose a bc-structure Σ on $\mathcal{B}$ and then these notions are defined as in Definition 2.8; however we will use just $L_{\mathcal{B}}$ for $L_{(\mathcal{B},\Sigma)}$, and $T_{\mathcal{B}}$ for $T_{(\mathcal{B},\Sigma)}$ when Σ is understood. Notice that although $\mathcal{B} \in \mathcal{BCC}$ was nonstrict $\mathcal{B}_{T_{\mathcal{B}}} \in \mathcal{BCC}_s$ is strict.

To proceed further we need additional notation: Let $(f: A \to B) \in \mathcal{B}$ (and $\mathcal{B}$ is a strict bc-category). Then, as earlier, $\hat{f}: 1 \to B^A$ denotes the unique transpose of f, i.e. $\hat{f} = (f\pi'_{1\times A})^*$. Let $\mathbf{A}_i$ be a finite sequence of objects from $\mathcal{B}$ and let $\mathbf{X}_{\mathbf{A}_i}$ be the corresponding sequence of basic types from $L_{\mathcal{B}}$. Let $\mathcal{T}_i$, $i = 1, 2$ be two type-terms satisfying the following: $\mathcal{T}_1(\mathbf{A}_i) = A$ and $\mathcal{T}_2(\mathbf{A}_i) = B$ (as one may recall from Definition 2.3 the type-term simply means a type built out of the basic types which are specified in the parenthesis – since the same operations exist on objects of a (strict) bcc we can use this notation for the objects as well). For a particular kind of arrow in $\mathscr{B}_s$ we use the following notation:

$$\mathscr{T}_1(\mathbf{X}_{\mathbf{A}_i}) \xrightarrow{c_f} \mathscr{T}_2(\mathbf{X}_{\mathbf{A}_i}) = [x : \mathscr{T}_1(\mathbf{X}_{\mathbf{A}_i}) \triangleright c_f^{\mathscr{T}_2(\mathbf{X}_{\mathbf{Ai}})^{\mathscr{T}_1(\mathbf{X}_{\mathbf{Ai}})}} {}^{\backprime} x^{\mathscr{T}_1(\mathbf{X}_{\mathbf{Ai}})}].$$

Often we will talk about a special kind of the above arrow – those which have basic types for domain and codomain e.g. $X_A \xrightarrow{c_f} X_B$; we shall call them elementary arrows in $\mathscr{B}_s$.

Also recall the fact mentioned earlier that there exists a forgetful 2-functor

$$\mathscr{BCC}_s \xrightarrow{||} \mathscr{BCC}$$

which on objects (0-cells) just forgets the chosen structure (on 1- and 2-cells doesn't do anything).

Lemma 3.14. *For every $\mathscr{B} \in \mathscr{BCC}$ there is an equivalence*

$$\mathscr{B} \xrightarrow{\eta_{\mathscr{B}}} |\mathscr{B}_s|$$

defined as follows: $\eta_{\mathscr{B}}(A \xrightarrow{f} B) = X_A \xrightarrow{c_f} X_B$.

Proof. Consider the following commutative diagram:

$$\begin{tikzcd} & T_{(\mathscr{B},\Sigma)} \arrow[dl, "M_{\mathscr{B}}"'] \arrow[dr, "M"] & \\ \mathscr{B}_{T_{(\mathscr{B},\Sigma)}} \arrow[rr, "\mu_{(\mathscr{B},\Sigma)}"'] & & (\mathscr{B},\Sigma) \end{tikzcd}$$

where M and $M_{\mathscr{B}}$ are the canonical models and $\mu_{\mathscr{B}}$ is defined to be the functor induced by Collorary 2.10. Explicitly, $\mu_{\mathscr{B}}(X_A) = A$, X_A a basic type of $L_{\mathscr{B}}$ (corresponding to an object $A \in \mathscr{B}$) and $\mu_{\mathscr{B}}(1 \xrightarrow{c_f} \mathscr{T}(\mathbf{X}_{\mathbf{A}_i})) = 1 \xrightarrow{f} \mathscr{T}(\mathbf{A}_i)$. It is easy to see that $\mu_{\mathscr{B}}$ is an equivalence of categories (it is onto on objects – therefore essentially surjective, also it is full since "c_f is mapped on f" and it is faithful by the definition of $T_{(\mathscr{B},\Sigma)}$ and by the completeness of $T_{(\mathscr{B},\Sigma)}$ with respect to $M_{\mathscr{B}}$ since $u =_x v \in T_{(\mathscr{B},\Sigma)}$ iff $M(x \triangleright u) = M(x \triangleright v)$).

Now notice that $|\mu_{\mathscr{B}}| \eta_{\mathscr{B}} = 1_{\mathscr{B}}$ by definition of $\eta_{\mathscr{B}}$ (and $\mu_{\mathscr{B}}$). Obviously then, $\eta_{\mathscr{B}}$ is an equivalence of categories because $|\mu_{\mathscr{B}}|$ has a pseudo-inverse. □

Now we want to see what is a natural strict functor between $\mathscr{A}_s$ and $\mathscr{B}_s$ corresponding to a nonstrict, bc-functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$ which is an inclusion on objects. (It will be shown later that this is not an essential restriction and yet things get simpler in our setting.)

Lemma 3.15. *Let $\mathscr{A} \xrightarrow{F} \mathscr{B} \in \mathscr{BCC}$ be an inclusion on objects. Let Σ_1 be an arbitrary strict structure on $\mathscr{A}$. Then, there exists a strict structure Σ_2 on $\mathscr{B}$ such that $(\mathscr{A}, \Sigma_1) \xrightarrow{F} (\mathscr{B}, \Sigma_2) \in \mathscr{BCC}_s$.*

Proof. Let $\{B_\alpha\}_{\alpha<\kappa}$ be the set of objects of $\mathscr{B}$. Let us just define strict products. $B_\alpha \times B_\beta = F(A \times A')$ if $B_\alpha = F(A)$ and $B_\beta = F(A')$; otherwise choose any product of B_α and B_β to be $B_\alpha \times B_\beta$ (the arrow part of the definition is equally simple). For the terminal object in Σ_2 choose $F(1)$ where 1 is the terminal object in Σ_1. $\square$

One can easily see that some condition as inclusion on object is needed for the above lemma to hold. As an example consider the functor $\mathscr{A} \xrightarrow{F} \mathscr{B} \in \mathscr{BCC}$ such that $\mathscr{A}$ is the 4 element Boolean algebra, $\mathscr{B}$ the category with just two isomorphic (but different) objects and F functor which maps the three non-bottom elements to one of the objects and the bottom to the other object. It is easy to see that there is no strict structure on the codomain category which would make F strict.

Lemma 3.16. *Suppose that $F:(\mathscr{A},\Sigma_1) \to (\mathscr{B},\Sigma_2)$ is a strict bc-functor which is an inclusion on objects. Then one can define a bc-functor $F_s : \mathscr{A}_s \to \mathscr{B}_s$ in the following way: on basic types $X_A \mapsto Y_{F(A)}$ and on basic arrows $c_f \mapsto c_{F(f)}$ (the rest is determined since F is strict bc-functor). Up to the renaming of symbols we can assume that we have inclusion of the languages $L_{\mathscr{A}} \subseteq L_{\mathscr{B}}$ and of the theories $T_{\mathscr{A}} \subseteq T_{\mathscr{B}}$. Then F_s so constructed is like the functors F_i in Proposition* 3.11. *Moreover, one can show that the following diagram commutes:*

(2)
$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "\eta_{\mathscr{A}}"'] & \mathscr{B} \arrow[d, "\eta_{\mathscr{B}}"] \\ \mathscr{A}_s \arrow[r, "F_s"'] & \mathscr{B}_s \end{tikzcd}$$

Recall, also, that the components of η are equivalences of categories.

Now we can start constructing Pushouts in the doctrine $\mathscr{BCC}$. By Lemmas 3.12 and 3.13 to show that every Pushout over a diagram $\Phi: I \to \mathscr{BCC}$ has the interpolation property it is enough to show that for a Pushout over one equivalent diagram $\Phi': I \to \mathscr{BCC}$.

We can restrict our attention on slightly more special diagrams and for that we need the following easy lemma.

Lemma 3.17. *Every functor $\mathscr{A} \xrightarrow{F} \mathscr{B}$ can be factored (in the doctrine $\mathscr{BCC}$) as*

$$\begin{tikzcd} \mathscr{A} \arrow[rr, "F"] \arrow[dr, "F'"'] & & \mathscr{B} \arrow[dl, "F''"] \\ & \mathscr{B}' & \end{tikzcd}$$

where F' is an inclusion on objects and F'' is an equivalence of categories.

Proof. Let us first define the category $\mathscr{B}'$ as follows: the objects of $\mathscr{B}'$ are objects of $\mathscr{B}$ and also pairs $(F(A), A)$ where A is an object of $\mathscr{A}$. Arrows in

$\mathscr{B}'$ are only arrows between first coordinates – more explicitly $\hom_{\mathscr{B}'}((F(A), A), B) = \hom_{\mathscr{B}}(F(A), B)$, the other cases are similar. The composition and identities in $\mathscr{B}'$ are the ones from $\mathscr{B}$. This is indeed in the doctrine because the properties of the doctrine are defined up to a (unique coherent) iso anyway. The definition of F' is the obvious one: $(A_1 \xrightarrow{f} A_2) \mapsto ((F(A_1), A) \xrightarrow{F(f)} (F(A_2), A_2))$. This functor is in the doctrine, basically for the same reason that $\mathscr{B}'$ is.

To construct F'' we do the following: $((F(A), A) \xrightarrow{g} B) \mapsto (F(A) \xrightarrow{g} B)$, similarly for the other cases. The pseudo-inverse of F'' is just the inclusion of $\mathscr{B}$ in $\mathscr{B}'$. □

Since $\mathscr{C} \xrightarrow{G} \mathscr{A} \xrightarrow{F} \mathscr{B}$ is an equivalent diagram to $\mathscr{C}' \xrightarrow{G'} \mathscr{A} \xrightarrow{F'} \mathscr{B}'$, by the above lemmas we can assume that the functors in the diagram for which we want to construct Pushout are inclusions on objects.

We are now all set for the proof of our main result.

Proof of Theorem 3.2. Start from a diagram

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathscr{B} \\ \mathscr{C} & \end{tikzcd}$$

in the doctrine $\mathscr{BCC}$ where both functors are inclusions on objects.

To construct what is needed first construct an equivalent diagram

(3)
$$\begin{tikzcd} \mathscr{A}_s \arrow[r, "F_s"] \arrow[d, "G_s"'] & \mathscr{B}_s \\ \mathscr{C}_s & \end{tikzcd}$$

The functors F_s and G_s are given as in Lemma 3.16 and the lemma also shows that the two diagrams are equivalent. The same lemma says that this diagram is of the same type as the first one in Proposition 3.11.

All we have to show is that the following diagram is a Pushout in $\mathscr{BCC}$.

(4)
$$\begin{tikzcd} \mathscr{A}_s \arrow[r, "F_s"] \arrow[d, "G_s"'] & \mathscr{B}_s \arrow[d, "U"] \\ \mathscr{C}_s \arrow[r, "V"'] & \mathscr{D} \end{tikzcd}$$

Notation: $\mathscr{D}$ is a short name for $\mathscr{B}_{T_{\mathscr{B}} \cup T_{\mathscr{C}}}$; as earlier, the union is not disjoint, i.e. the symbols coming from $L_{\mathscr{A}}$ are identified, and functors U and V are induced by the inclusions of the languages.

This diagram is the same as the second one given in Proposition 3.11 (since $T_{\mathscr{A}} \subseteq T_{\mathscr{B}} \cap T_{\mathscr{C}}$ up to the renaming of symbols). This square has the interpolation property as shown in the same proposition.

First, let's take care of the 1-dimensional part of the definition of a Pushout. So, suppose we are given $P:\mathscr{B}_s \to \mathscr{E}$, $Q:\mathscr{C}_s \to \mathscr{E}$ and a natural isomorphism $\theta : PF_s \Rightarrow QG_s$. We have to construct a bc-functor $R:\mathscr{D} \to \mathscr{E}$ and two natural isomorphisms $\theta_1 : P \Rightarrow RU$ and $\theta_2 : RV \Rightarrow Q$ such that $(\theta_2 G_s)(\theta_1 F_s) = \theta$.

Using Corollary 2.10 one can give the above R as a model of $T_{\mathscr{B}} \cup T_{\mathscr{C}}$ in $\mathscr{E}$ in the following manner. On types:

$$R(D) = \begin{cases} P(D) & \text{if } D \in L_{\mathscr{B}} - L_{\mathscr{A}} \\ Q(D) & \text{if } D \in L_{\mathscr{A}} \end{cases}$$

Notice that R is not given only on the basic types but not on all the types either. This is not a problem, on the rest of types we chose: e.g. $R(X_B \times X_C)$ we chose to be a product of $R(X_B)$ and $R(X_C)$ in $\mathscr{E}$. Notice that in the above definition we essentially used the inclusion of languages, that is the fact that F is an inclusion on objects.

To define R on the basic constants we have to introduce a family of isomorphisms θ'_D in $\mathscr{E}$ where D is an arbitrary type in $L_{\mathscr{B}} \cup L_{\mathscr{C}}$. This family is defined inductively as in Remark 2.6, here we will give just the basis of the induction:

$$\theta'_{X_D} = \begin{cases} \theta_D & \text{if } X_D \in L_{\mathscr{A}} \\ 1_{P(X_D)} & \text{if } X_D \in L_{\mathscr{B}} - L_{\mathscr{A}} \\ 1_{Q(X_D)} & \text{if } X_D \in L_{\mathscr{C}} - L_{\mathscr{A}}. \end{cases}$$

Now we can define R on basic arrows as follows:

$$R(\rhd d_f : D) = \begin{cases} \theta'_D P([\rhd d_f]) & \text{if } d_f \in L_{\mathscr{B}} \\ Q([\rhd d_f]) & \text{otherwise.} \end{cases}$$

Notice the in case that f is in $\mathscr{A}$ then (using naturality of θ) we get $R(d_f) = \theta_D P([\rhd d_f]) = Q([\rhd d_f])$. The rest of R is defined by induction on the complexity of terms as in the definition of interpretation, see 2.5.

Now we have to check that R is a model of $T_{\mathscr{B}} \cup T_{\mathscr{C}}$. Since every equation is in $T_{\mathscr{B}}$ or in $T_{\mathscr{C}}$ it will follow from the following

Claim. *For every term* $(x_1 : B_1, \dots, x_n : B_n \rhd t : B) \in L_{\mathscr{B}}$ *the following holds:*

$$R(x_1 : B_1, \dots, x_n : B_n \rhd t : B) = \theta'_B P([x_1 : B_1, \dots, x_n : B_n \rhd t : B]) \theta'^{-1}_{B_1 \times \dots \times B_n}.$$

Also, for every term $(x_1 : C_1, \dots, x_n : C_n \rhd t : C) \in L_{\mathscr{C}}$ *the following holds:*

$$R(x_1 : C_1, \dots, x_n : C_n \rhd t : C) = Q([x_1 : C_1, \dots, x_n : C_n \rhd t : C]).$$

Both parts are easily proved by induction on the complexity of t. (Notice again that the case when $t \in L_{\mathscr{A}}$ is not a problem since

$$\theta'_A P([x_1 : A_1, \dots, x_n : A_n \rhd t : A]) \theta'^{-1}_{A_1 \times \dots \times A_n} = Q([x_1 : C_1, \dots, x_n : C_n \rhd t : C])$$

by the naturality of θ.

Let us now give the two natural isomorphisms $\theta_1 : P \Rightarrow RU$ and $\theta_2 : RV \Rightarrow Q$ such that $(\theta_2 G_s)(\theta_1 F_s) = \theta$. Since $RV = Q$ we take $\theta_2 = 1_Q$ and we define θ_1 as the restriction of θ' on $L_{\mathscr{B}}$ that is $\theta_1 = \theta'|_{L_{\mathscr{B}}}$ (indeed θ_1 is a natural isomorphism between P and RV by the first part of the above claim using the fact that V is induced by the inclusion $L_{\mathscr{B}} \subseteq L_{\mathscr{B}} \cup L_{\mathscr{C}}$. Indeed, $(\theta_2 G_s)(\theta_1 F_s) = \theta$ since $\theta'|_{L_{\mathscr{A}}} = \theta$.

To show the 2-dimensional universal property of the Pushout suppose that we are given two bc-functors $R, R' : \mathscr{D} \to \mathscr{E}$ and two natural isomorphisms $\phi_1 : RU \Rightarrow R'U$, $\phi_2 : RV \Rightarrow R'V$ which satisfy $\phi_1 F_s = \phi_2 G_s$. We want a unique natural isomorphism $\psi : R \Rightarrow R'$ such that $\phi_1 = \psi U$ and $\phi_2 = \psi V$. To give this natural isomorphism is the same as to give an isomorphism of models of $T_{\mathscr{B}} \cup T_{\mathscr{C}}$ determined by R and R' which will satisfy the given equations. Therefore, by Remark 2.6, if the isomorphism exists it must be given on basic types as follows:

$$\psi_{X_D} = \begin{cases} \phi_1(X_D) & \text{if } X_D \in L_{\mathscr{B}} \\ \phi_2(X_D) & \text{if } X_D \in L_{\mathscr{C}} \end{cases}.$$

Notice that ψ_{X_D}, $X_D \in L_{\mathscr{A}}$, is well defined since $\phi_1 F_s = \phi_2 G_s$ and F_s, G_s are induced by the inclusions on the languages. The rest of ψ is given by the induction on the complexity of types as in Remark 2.6. Obviously, such ψ (if it exists) satisfy $\phi_1 = \psi U$ and $\phi_2 = \psi V$. To see that this ψ exists (that is that this is an isomorphism of models) it is enough to show that for every basic constant ξ^D from the language $L_{\mathscr{B}} \cup L_{\mathscr{C}}$, $\psi_D R(\rhd \xi^D) = R'(\rhd \xi^D)$. Again, since ξ^D is in $L_{\mathscr{B}}$ or $L_{\mathscr{C}}$; the equation will follow, in the first case, from the naturality of ϕ_1 (since U is induced by the inclusion $L_{\mathscr{B}} \subseteq \mathscr{L}_{\mathscr{B}} \cup L_{\mathscr{C}}$ and $\psi|_{L_{\mathscr{B}}} = \phi_1$), and in the second case from naturality of ϕ_2 (since V is induced by the inclusion $L_{\mathscr{C}} \subseteq L_{\mathscr{B}} \cup L_{\mathscr{C}}$ and $\psi|_{L_{\mathscr{C}}} = \phi_2$). This finishes the proof of Theorem 3.2. □

3.4 Some consequences

In this section we show that our interpolation is a genuine generalization of the well known property of Heyting algebras. The theorem is first proved in the important work by Maksimova [Mak77]. The theorem as stated in [Pit83a, Thm. B.] is the following:

Theorem 3.18. *Every pushout square in the category* **Ha** *of Heyting algebras (and structure preserving morphisms) has the interpolation property.*

Proof. Every Heyting algebra is a bicartesian closed category and homomorphisms of these algebras are bc-functors. We can view **Ha** as a 2-category (2-cells being identities). Therefore there is an inclusion $I : \mathbf{Ha} \to \mathscr{BCC}$. Also, it is easy to show that a left adjoint to this functor is "posetal collapse" $P : \mathscr{BCC} \to \mathbf{Ha}$. To construct a pushout of $C \xleftarrow{g} A \xrightarrow{f} B$ in **Ha** we can do the following: include the diagram in $\mathscr{BCC}$ and construct a Pushout there. Then apply the functor P, in this way we obtain the square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "P(H)"] \\ C \arrow[r, "P(K)"'] & P(\mathscr{D}) \end{tikzcd}$$

Since as a left adjoint P preserves Colimits, this square is a pushout. Also a posetal collapse of a square which has the interpolation property is again a square with the interpolation property. From that the theorem follows. □

We shall come back to Heyting algebras in a moment but before that let us establish another interesting fact.

Proposition 3.19. *The full functors are stable under Pushouts in $\mathscr{BCC}$ and $\mathscr{CCC}$ doctrines.*

Proof. We want to show that in a Pushout square as below if F is a full functor then K must be.

$$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] \arrow[d, "G"'] & \mathscr{B} \arrow[d, "\mathscr{H}"] \arrow[dl, phantom, "\tau \Swarrow"] \\ \mathscr{C} \arrow[r, "K"'] & \mathscr{D} \end{tikzcd}$$

Suppose that $x:K(C_1)\to K(C_2)$ is an arrow in $\mathscr{D}$. Since $H(F(1))$ is a terminal object and $K(C_2^{C_1})$ is an exponent of $K(C_2)$ by $K(C_1)$ (both in $\mathscr{D}$) then (as in any ccc) there exists unique arrow $\hat{x}:H(F(1))\to K(C_2^{C_1})$ in $\mathscr{D}$ such that $K(\varepsilon)u = x$ where $u:K(C_1)\to K(C_2^{C_1}\times C_1)$ is the unique arrow such that $K(\pi)u=\hat{x}0_{K(C_1)}$ and $K(\pi')u=1_{K(C_1)}$ (and $0_{K(C_1)}:K(C_1)\to H(F(1))$ is a unique arrow).

Now we can apply our interpolation theorem to $\hat{x}:H(F(1))\to K(C_2^{C_1})$ and we get $A\in\mathscr{A}$, $(b:F(1)\to(F(A))\in\mathscr{B}$ and $(c:G(A)\to C_2^{C_1})\in\mathscr{C}$ such that $\hat{x} = K(c)\tau_A H(b)$. Since F is full by the assumption there exists an arrow $a:1\to A$ in $\mathscr{A}$ such that $F(a)=b$. By naturality of τ we obtain that $K(G(a))\tau_1=\tau_A H(F(a))$ and by the way a was chosen we have that $\hat{x}=K(cG(a))\tau_1$. Now we can check that $K(\langle cG(a)0_{C_1}, 1_{C_1}\rangle):K(C_1)\to K(C_1^{C_2}\times C_1)$ satisfies the equations defining the above u and then by the uniqueness of u we have that $u=K(\langle cG(a)0_{C_1}, 1_{C_1}\rangle)$ (here, 0_{C_1} is the unique arrow $C_1\to G(1)$). So finally, $x=K(\varepsilon\langle cG(a)0_{C_1}, 1_{C_1}\rangle)$, i.e. K is indeed full. □

We think that full and faithful functors are not stable under Pushouts in these doctrines, but the example which we have requires further justification. Coming back to Heyting algebras, let us prove the other main theorem from [Pit83a] – Thm A.

Theorem 3.20. *Monomorphisms are stable under pushout in* **Ha**.

Suppose

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "k"] \\ C \arrow[r, "h"'] & D \end{tikzcd}$$

is a pushout in **Ha** and f is mono, we want to show that this h is mono. By the proof of the previous theorem we know that the above square is posetal collapse of a Pushout square from $\mathscr{BCC}$ (i.e. $k=P(K)$ and $h=P(H)$). Since the monomorphism g as a functor is full – it follows by the previous proposition that H is full. Also, posetal collapse of a full functor is a monomorphism i.e. h is a monomorphism. □

Since our interpolation result was valid for cartesian closed categories as well we can say that the same statements hold for their posetal collapse. These are known in the literature as Brouwerian semilattices so we can just conclude that in the category of Brouwerian semilattices pushouts have the interpolation property and that monomorphisms are stable under pushouts as well.

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References

[BKP89] Blackwell, R., Kelly, G.M., Power, J.: Two-dimensional monad theory. J. Pure Appl. Algebra **59**(59), 1–41 (1989)

[Čub93] Čubrić, Dj.: Results in Categorical Proof Theory. PhD thesis, McGill University, Montréal, 1993

[Čub94] Čubrić, Dj.: Embedding of a free cartesian closed category into the category of sets. J. Pure Appl. Algebra, 1994. To appear

[Gir71] Girard, J.-Y.: Interprétation fonctionelle et élimination des coupures de l'arithmétique d'ordre supérieur. PhD thesis, Universite Paris VII, 1971

[GLT89] Girard, J.-Y., Lafont, Y., Taylor, P.: Proofs and Types, volume 7 of Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, 1989

[Göd65] Gödel, K.: Remarks before the Princeton bicentennial conference in mathematics. In: Davis, M. (ed.) The Undecidable. Raven Press, 1965

[KP86] Kaijser, S., Pelletier, J.W.: Interpolation Functors and Duality, volume 1208 of Lect. Notes Math. Springer-Verlag, 1986

[LS86] Lambek, J., Scott, P.J.: Introduction to Higher Order Categorical Logic. Cambridge University Press, 1986

[Mak77] Maksimova, L.L.: Craig's interpolation theorem and amalgamable varieties. Soviet. Math. Dokl. **18**, 1550–1553 (1977)

[MP89] Makkai, M., Paré, R.: Accessible categories: the Foundations of Categorical Model Theory, volume 104 of Contemp. math. American Mathematical Society, 1989

[Pav91] Pavlović, D.: Categorical interpolation: descent and the Beck-Chevalley condition without direct images. In: Carboni, A. et al. (eds.) Category Theory, number 1488 in Lect. Notes Math., p. 306–326, 1991

[Pit83a] Pitts, A.M.: Amalgamation and interpolation in the category of Heyting algebras. J. Pure Appl. Algebra **29**, 155–165 (1983)

[Pit83b] Pitts, A.M.: An application of open maps to categorical logic. J. Pure Appl. Algebra **29**, 313–326 (1983)

[Pit86] Pitts, A.M.: Lax descent for essential surjections, 1986. Talk given at a Cambridge category theory conference

[Pit87] Pitts, A.M.: Interpolation and conceptual completeness for pretoposes via category theory. In: Kueker, D.W., Lopez-Escobar, E.G.K., Smith, C.H. (eds.) Mathematical logic and theoretical computer science, volume 106 of Lect. Notes Pure Appl Math., p. 301–327. Marcel Dekker, 1987

[Pit88] Pitts, A.M.: Applications of sup-lattice enriched category theory to sheaf theory. Proc. London Math. Soc. **57**(3), 433–480 (1988)

[Pit92] Pitts, A.M.: On an interpretation of second order quantification in first order intuitionistic propositional logic. J. Symb. Logic **57**, 33–52 (1992)

[Pra65] Prawitz, D.: Natural Deduction, A Proof-Theoretic Study, volume 3 of Stockholm Studies in Philosophy. Almqvist & Wiskell, Uppsala, 1965

[Pra71] Prawitz, D.: Ideas and results in proof theory. In: Fenstad, J.E. (ed.) Proc. of the Second Scandinavian Logic Symposium, p. 235–307. North-Holland, 1971