Isomorphisms in $\mathcal{C}$

↳ Should formalize the idea that if there is an isomorphism between two objects, then the objects "share the same properties" inside the cat $\mathcal{C}$.

<u>Def</u> : $f : X \longrightarrow Y$ is an isomorphism if exists $g : Y \longrightarrow X$ so that

$$\begin{tikzcd} Y \arrow[d, "g"'] & X \arrow[l, "f"'] \arrow[dl, "id_X"] \\ X & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[d, "f"'] & Y \arrow[l, "g"'] \arrow[dl, "id_Y"] \\ Y & \end{tikzcd}$$

<u>Rmk</u> If g, g' are both such that $\begin{cases} g \circ f = id & g' f = id \\ f \circ g = id & f g' = id \end{cases}$ | Exercise

then $g = g'$

So even if formally "an isomorphism" comprises the tuple $(X, Y, f : X \to Y, g : Y \to X)$ so that $gf = 1$, $fg = 1$, g is unique when it exists

So g is <u>the</u> inverse morphism of f.

$f^{-1} =: g$

"Being isomorphic" is a relation on $\mathcal{C}_0$ objects of $\mathcal{C}$

$[X \cong Y]$ if $\exists\, f : X \to Y$ isomorphism.

Being iso— is an <u>equivalence relation</u> on $\mathcal{C}_0$

$[X \cong X]$ via the identity, if $X \cong Y \Rightarrow Y \cong X$

$id = id \circ id$
$id = id^{-1}$

if $\exists\, f : X \longrightarrow Y$ iso, then there is $f^{-1} : Y \longrightarrow X$

but f^{-1} is ITSELF an isomorphism with inverse $f = (f^{-1})^{-1}$

$$\begin{aligned} f \circ f^{-1} &= id \\ f^{-1} \circ f &= id \end{aligned} \Longrightarrow \text{ f is \underline{the} inverse for } f^{-1}.$$

<u>TRANSITIVITY</u> $\quad X \cong Y$ and $Y \cong Z \Longrightarrow X \cong Z$

$$\exists \begin{tikzcd} X \arrow[r, "f", "iso"'] & Y \arrow[l, bend left=60, "f^{-1}"] \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, "h", "iso"'] & Z \arrow[l, bend left=60, "h^{-1}"] \end{tikzcd}$$

$X \xrightarrow{f} Y \xrightarrow{h} Z$ is iso: its inverse is

$$\begin{tikzcd} Z \arrow[rr] \arrow[dr, "h^{-1}"'] & & X \\ & Y \arrow[ur, "f^{-1}"'] & \end{tikzcd}$$

$(h \circ f)^{-1} = f^{-1} \circ h^{-1}$

$$\begin{aligned} (h \circ f) \circ (f^{-1} \circ h^{-1}) &= (h \circ id) \circ h^{-1} \\ &= h \circ h^{-1} \\ &= id \end{aligned}$$

In the category of sets, $X \longrightarrow Y$ is iso

iff it is bijective

(iff it is injective & surjective)

An isomorphism in the category of vector spaces is a function that is linear and bijective

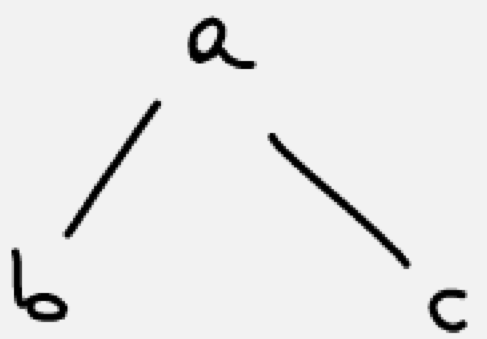
An isomorphism of monoids (or groups, or other algebraic structures) is a $\underbrace{\text{function preserving the operations}}_{\text{homomorphism}}$, and bijective

In other categories like topological spaces + continuous fun
partially ordered sets + monotone functions

might happen that an isomorphism

$(P, \leq) \longrightarrow (Q, \leq)$ is not just a function which is bijective and monotone

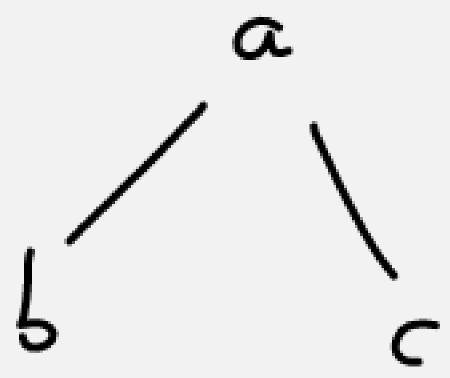
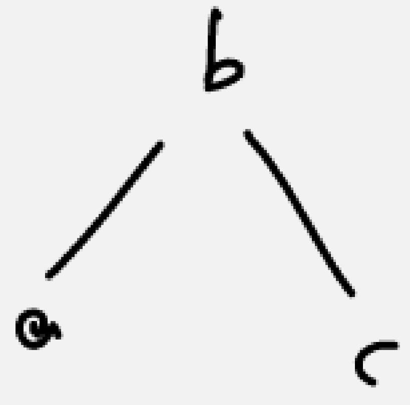
$\{a, b, c\} = P$



$b \leq a$
$c \leq a$

$P = Q = \{a, b, c\}$ $(a \leq a, b \leq b, c \leq c)$

but now $a \leq a$, $b \leq b$, $c \leq c$ discrete order

 $\cong$ 

the identity function $P_{disc} \xrightarrow{id} P_{\wedge}$ is bijective and monotone (evidently)

the only possible choice for its inverse is $id : P_{\wedge} \longrightarrow P_{disc}$.

BUT THIS IS NOT MONOTONE ! on $P_{\wedge}$ $b \leq a$, in P_{disc} not.

An iso in the category Pos is a function
- bijective
- monotone

[- its inverse is monotone as well]

the groupoid of natural numbers $\mathbb{B}ij$

Def (Groupoid) cat G is a groupoid if every arrow of G is iso

$0! = 1$

$n! = n \cdot (n-1)!$

Take the category of finite sets $\{1, \ldots, n\}$ and define a category taking only bijective functions

$\emptyset$ $(n = 0)$ $\quad \{1\}$ id $\quad$ id $\{1,2\}$ swap $\quad \{1,2,3\}$ $3!$ $\quad \ldots\ldots \quad$ $\{1 - n\}$ $n!$ of these

Def $\mathcal{C}$ category; the core of $\mathcal{C}$ $\left(\mathrm{core}(\mathcal{C})\right)$ is the category having same objects of $\mathcal{C}$ but taking only isos of $\mathcal{C}$ as arrow

$$\mathbb{B}\mathrm{ij} = \mathrm{core}(\text{Finite sets \& functions})$$

A <u>group</u> is precisely a groupoid with a single object

$$\begin{tikzcd}[column sep=huge, row sep=large]
\text{monoid} \arrow[d, "\substack{\text{is a}\\ \text{one object}\\ \dots}"] & \text{groups} \arrow[l, "\substack{\text{is a}\dots\\ \text{where all elements}}"', "\text{invertible}"] \arrow[d, "\substack{\text{a one}\\ \text{object}\\ \dots}"] \\
\text{categories} & \text{groupoids} \arrow[l, "\text{is a}\dots"', "\substack{\text{where all}\\ \text{arrows are}\\ \text{isomorphism.}}"]
\end{tikzcd}$$

There are other distinguished classes of arrows in a cat $\mathcal{C}$.
that deserve to be studied:

<u>Def</u> A MONOMORPHISM in $\mathcal{C}$ is an arrow $f : X \longrightarrow Y$ such that
if $u, v : A \longrightarrow X$ are two arrows such that
$f \circ u = f \circ v$ then $u = v$

<u>Def</u> An EPIMORPHISM in $\mathcal{C}$ is an arrow $f : X \longrightarrow Y$ such that
whenever $u \circ f = v \circ f$ then $u = v$

Being mono is a cancellation property on one side
epi on the other side

In the category $\underline{Set}$

1. A mono is an injective function

2. An epi is a surjective function

1) An injective function assumes every value precisely once ("mono")
A surjective function "covers" the codomain ("epi").
lies on the codomain

Any category $\mathcal{C}$: if f is an iso then f is both mono and epi.

In Set: f mono + epi $\Longrightarrow$ f iso

Cats where iso $\Longleftrightarrow$ mono & epi deserves a special name: BALANCED CATEGORIES

In Set epi $\equiv$ surjective

$E \Leftarrow S$

$e : X \longrightarrow A$ is a surjective function.

If $f, g : A \longrightarrow B$ are such that $f \circ e = g \circ e$

we have to prove that $f = g$

If for every $x \in X$ $f(e(x)) = g(e(x))$ then $f(a) = g(a)$

But since e is surjective any $a \in A$ is of the form $e(x)$ for some x

if $a \in A$ is $e(x_a)$ for some chosen $x_a \in X$ then

$a = e(x_a)$

$f(a) = f(e(x_a))$

$= g(e(x_a))$ by

$= g(a)$

Epi $\Rightarrow$ Surj

if $\begin{cases} e : X \longrightarrow A \text{ is an epi} \\ \text{whenever } f \circ e = g \circ e \text{ then } f = g \end{cases}$

then $\{ e$ is surjective : given $a \in A$ $\exists$ x_a so that $e(x_a) = a$

Assume e is not surjective, we can prove that e is not epi

$(P \Rightarrow Q$ iff $\neg Q \Rightarrow \neg P)$

$\rightarrow$ exists $a \in A$ so that for no x $e(x) = a$ $(\underline{a \notin im(e)})$

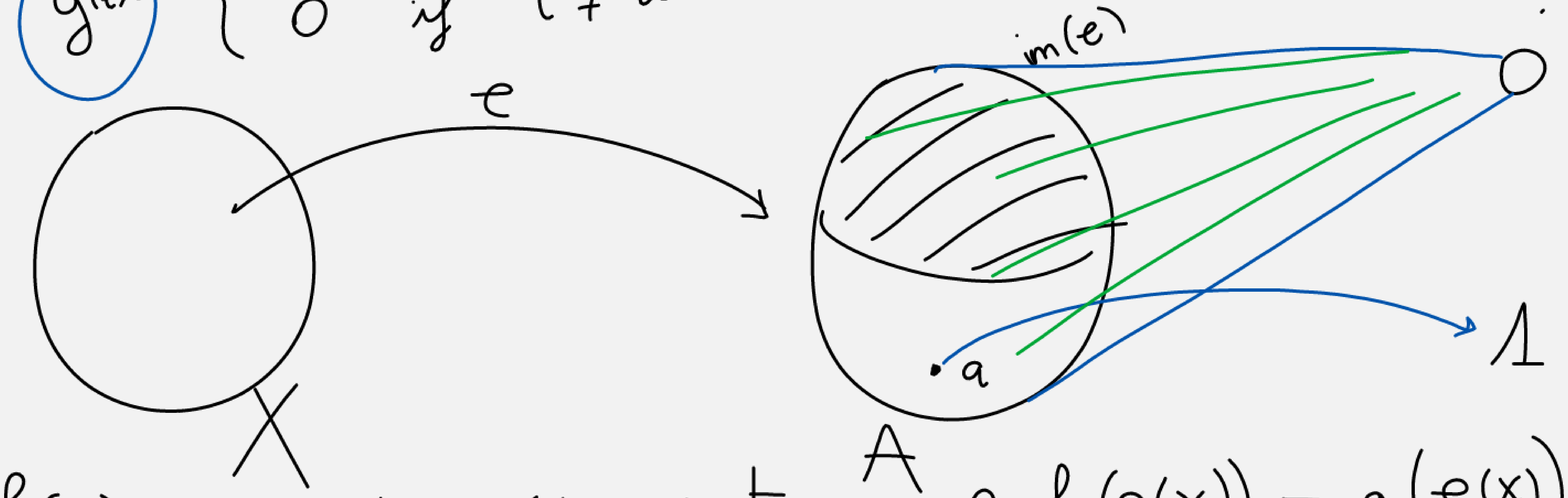
Build $\underline{\text{two}}$ functions $f, g : A \longrightarrow \#$, distinct $f \neq g$

having the property $f \circ e = g \circ e$

$f, g : A \longrightarrow \{0, 1\}$

$f =$ constant at 0 $\quad f(t) = 0$ for every $t \in A$

$$g(t) = \begin{cases} 1 & \text{if } t = a \\ 0 & \text{if } t \neq a \end{cases}$$



$f(a) = 0 \neq 1 = g(a)$ yet $0 = f(e(x)) = g(e(x)) = 0$

precisely bc of the assumption that $a \notin \mathrm{im}(e)$

It would have been easier to prove this by saying:

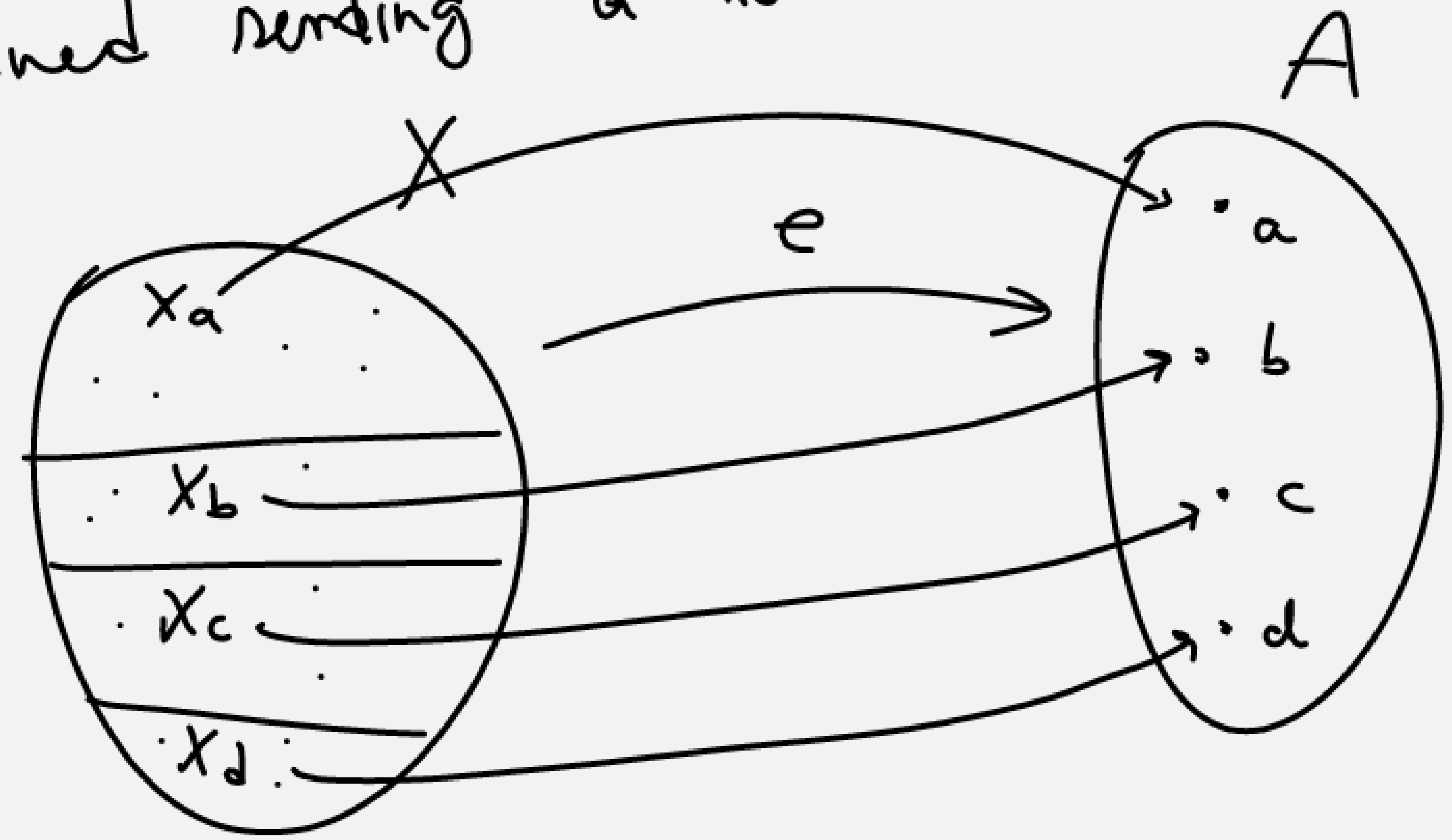
e a surjective $X \longrightarrow A$ $\Longrightarrow$ e has a <u>right inverse</u> : $\exists\ h : A \to X$: $e \circ h = id_A$

$\Downarrow$

$\forall\, a \in A,\ \exists\, x$ such that $e(x_a) = a$

h is defined sending a to such an x_a

$A \longrightarrow X$



Axiom of Choice

Given a family X_a of nonempty sets (e.g. $\emptyset \neq X_a = e^{\leftarrow}(a)$ e surjective)

I can choose $x_a \in X_a$ for every $a \in A$