On the unicity of the formal theory of categories

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- Our original motivation was to investigate the formal category theory of derivators.

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  - Two-sided Yoneda structure
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- Examples:
  - Representable Yoneda structure
  - Two-sided Yoneda structure
  - Isbell duality and its generalizations
- An open and motivating problem: derivators.

Introduction

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«The category of small categories, **Cat**, is a "2-category with properties"; one should attempt to identify those properties that enable one to do the "structural parts of category theory".»

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Not a new idea:

- cf. topos theory: treat a category$\mathcal{E}$ as if it were **Set**;
- cf. categorical algebra: treat a category $\mathcal{A}$ as if it were $\mathsf{Alg}(\mathbb{T})$;
- cf. homological algebra: treat a category $\mathcal{C}$ as if it were $\mathsf{Ch}(\mathcal{A})$;

Perhaps surprisingly though, the bare structure of a 2-category does not suffice to embody "all" category theory in a formal way. (In short: there are features of 2-category theory induced by the fact that 2-**Cat** is a 3-category)

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- internal fibrations...

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- E) a calculus of modules (every bimodule $M : A^\vee \times B \to S$ can be straightened to a fibration over $A^\vee \times B$)

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Idea

Take a 2-category $\mathcal{K}$, and take Y) and E) very seriously.

Y): Yoneda structures

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- Category theory is the class of corollaries of Yoneda lemma.
- A *Yoneda structure* imposes on $\mathcal{K}$ enough structure so that every "small" object A has a "Yoneda embedding" $y_A : A \to PA$.
- Formally encode the Yoneda lemma in universal properties of certain diagrams

$$\begin{tikzcd} & A \arrow[dl, "y_A"'] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \chi^f"] & \\ PA & & B \arrow[ll, "B(f{,}1)"] \end{tikzcd}$$

representing f-nerves.

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- This leads to the notion of *C-D-bimodule* as a functor $C^{\mathrm{op}} \times D \to Set$. All such bimodules live in a bicategory Mod, and there is an embedding

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  such that every $f \in Cat$ has a right adjoint when embedded in Mod.
- Every such $p : \mathcal{K} \to \overline{\mathcal{K}}$ equips $\mathcal{K}$ with proarrows.

The main theorem of this talk is an equivalence between a certain class of Yoneda structures and a certain class of proarrow equipments:

$$\left\{\begin{matrix}\text{cocomplete}\\ \text{Yoneda}\\ \text{structures}\end{matrix}\right\} \Longleftarrow\!\!=\!=\!=\!=\!=\!=\!= \left\{\begin{matrix}\text{Yoneda}\\ \text{equipments}\end{matrix}\right\}$$

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What allows to pass from YS to E is the notion of a yosegi, a certain special kind of 2-monad.

Preliminaries

2-dimensional universality

The main 2-dimensional universal constructions we are interested in

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\swArrow\eta"] & B \\ C \arrow[ur, dotted, "\mathrm{Lan}_g f"'] & \end{tikzcd} \qquad \frac{\mathrm{Lan}_g f \to h}{f \to hg}$$

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(Exercise: properly dualize this statement)

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filled by a 2-cell $\chi : y_A \to B(f, f) =: B(f, 1) \cdot f$.

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  - The pair $\langle 1_{\mathbf{P}A}, 1_{y_A} \rangle$ exhibits the pointwise left extension $\mathrm{Lan}_{y_A} y_A$ (shortly, 'the Yoneda embedding is dense').

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  - Given a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, the pasting of 2-cells

$$\begin{tikzcd}
A \arrow[rr, "y_A"] \arrow[d, "f"'] & & \boldsymbol{P}A \\
B \arrow[r, "y_B"] \arrow[d, "g"'] \arrow[ur, phantom, "\Downarrow \chi^{y_B f}"] & \boldsymbol{P}B \arrow[ur, "\boldsymbol{P}f"'] & \\
C \arrow[ur, "C(g,1)"'] \arrow[u, phantom, shift left=4, "\Downarrow \chi^g"] & &
\end{tikzcd}$$

exhibits the pointwise extension $\mathrm{Lan}_{gf} y_A = C(gf,1)$ (shortly, $\boldsymbol{P}$ is a functor).

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- χ^f is obtained from the action of f on arrows.

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It is enough to check that

$$[B, PA](N_f, G) \cong [A, PA](y_A, G \cdot f)$$

This can be proved directly.

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Axiom 2

The pair $\langle f, \chi^f \rangle$ exhibits a relative adjunction $f \;{}_{y_A}\!\dashv B(f, 1)$.

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and this can be checked diretly; absoluteness can be checked by hand too.

In **Cat**

Axiom 3-4

They admit a rephrasing as $\boldsymbol{P}$ is a functor, in that $\boldsymbol{P}(id) \cong id$ and $\boldsymbol{P}(gf) \cong \boldsymbol{P}f \cdot \boldsymbol{P}g$.

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This is pretty obvious (although the isomorphisms

3) $[\boldsymbol{P}A, \boldsymbol{P}A](1, H) \cong [A, \boldsymbol{P}A](y_A, H \cdot y_A)$;
4) $C(gf, 1) = \mathrm{Lan}_{gf} y_A \cong \mathrm{Lan}_{y_B f} y_A \circ \mathrm{Lan}_g y_B = \boldsymbol{P}B(y_B f, 1) \cdot C(g, 1)$

can be checked directly).

All **Cat**-like 2-categories carry Yoneda structures in the obvious way:

- $\mathcal{V}$-**Cat** (enriched categories) with the enriched Yoneda embeddings;
- internal categories in a finitely complete category $\mathcal{A}$;
- the 2-category of pseudofunctors $\mathcal{A} \to \mathbf{Cat}$ for a small bicategory $\mathcal{A}$;
- ... and maybe the 2-category of (pre)derivators

This would give an account of "how much category theory" is expressible in **PDer** and **Der**.

Proarrow equipments

Definition (proarrow equipment)

A 2-functor $p : \mathcal{K} \to \overline{\mathcal{K}}$ equips $\mathcal{K}$ with proarrows if

- p is the identity on objects and locally fully faithful;
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Very simple definition; a bit complicated to follow the literature. Recently framed into the more natural environment of (hyper)virtual double categories of (Shulman-Crutwell).

Question

How do these framework relate? It is reasonable to expect they do (they're both ways to encode a calculus of profunctors; $y_A : A \to \boldsymbol{P}A$ is the (mate of the)identity profunctor).

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Turns out 1. is almost true; 2. is true; 3. is true if we restrict to so-called Yoneda equipments where p has additional properties.

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1. for each $f : A \to B$ there is an adjunction $\boldsymbol{P}_!(f) \dashv \boldsymbol{P}^*(f)$;
2. $\boldsymbol{P}_!$ is a relative monad (relative to the inclusion $\mathbf{Cat} \subset \mathbf{CAT}$);

The presheaf construction sending $A \mapsto [A^{\mathrm{op}}, \mathbf{Set}]$ and $f : A \to B$ to the inverse image $\boldsymbol{P}^* f : \boldsymbol{P}B \to \boldsymbol{P}A$ is such that

1. for each $f : A \to B$ there is an adjunction $\boldsymbol{P}_!(f) \dashv \boldsymbol{P}^*(f)$;
2. $\boldsymbol{P}_!$ is a relative monad (relative to the inclusion $\mathbf{Cat} \subset \mathbf{CAT}$);
3. the monad $\boldsymbol{P}_!$ is a KZ-doctrine.

Definition (yosegi box)

Let $j : \mathcal{A} \subset \mathcal{K}$ and $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ with the same properties, and such that moreover

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[a] *Yosegi-zaiku* (寄木細工) is a kind of marquetry featuring elaborate inlaid and mosaic designs; the defining properties of such a KZ-doctrine are tightly linked, rich of peculiar adornments.

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3. The inclusion $j : \mathcal{A} \subset \mathcal{K}$ *extends to a yosegi box*.

- a Yoneda equipment is a triangle

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "j"'] \arrow[dr, "p"] \arrow[d, phantom, "\Rightarrow"] & \\ \mathcal{K} & & \mathcal{M} \arrow[ll, "u"] \end{tikzcd}$$

such that
  - p is a proarrow equipment à la Wood;
  - p is the j-relative left adjoint of $u : \mathcal{M} \to \mathcal{K}$;
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  - the relative monad up generated by the relative adjunction, is lax idempotent with unit $\boldsymbol{\eta} : j \Rightarrow up$.
- the inclusion j extends to a yosegi if there is a relative lax-idempotent 2-monad $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ with relative unit $j \Rightarrow \boldsymbol{P}$ such that for each 1-cell $f : A \to B$, the 1-cell $\boldsymbol{P}(f)$ has a left adjoint $\boldsymbol{P}_! f$.

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- every cocomplete Yoneda structure has a yosegi as presheaf construction;
- Yoneda equipments contain enough information to recover a Yoneda structure.

Applications

Examples

- Often, $\boldsymbol{P}$ is representable, because $\mathcal{K}$ is cartesian closed: $\boldsymbol{P} : A \mapsto [A^{\mathrm{op}}, \Omega]$; this is the case for many $\mathcal{K}$'s having a duality involution $(\_)^{\mathrm{op}}$, and entails that $\boldsymbol{P}$ has an adjoint

$$\boldsymbol{P}^{\sharp} \dashv \boldsymbol{P}$$

In **Cat**, $\boldsymbol{P}^{\sharp}$ is the "contravariant presheaf construction", sending A to $[A, \mathbf{Set}]^{\mathrm{op}}$.

- This self-duality of $\boldsymbol{P}$ is tightly linked to the possibility to instantiate the formal version of *Isbell duality*: there is an adjunction

$$\begin{tikzcd} & A \arrow[dl, "y_A"'] \arrow[dr, "y_A^\sharp"] & \\ \boldsymbol{P}A \arrow[rr, shift left, "\mathcal{O}"] & & \boldsymbol{P}^\sharp A \arrow[ll, shift left, "\mathrm{Spec}"] \end{tikzcd}$$

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- The pair $(\boldsymbol{P}^\sharp, \boldsymbol{P})$ forms a two-sided Yoneda structure/yosegi: we have copresheaf constructions working as free completion: $\boldsymbol{P}^\sharp : A \mapsto [A, \mathbf{Set}]^{\mathrm{op}}$ does precisely this in **Cat**; the axioms of a (left) Yoneda structure hold replacing left extensions and lifts with the corresponding right versions. These two structures are compatible.

Formal ∞-category theory

Let $\mathbf{QCat}_2$ be the homotopy 2-category of the $\mathbf{Kan}$-enriched category $\mathbf{QCat}$ of ∞-categories; there is an "∞-Yoneda structure" on $\mathbf{QCat}$ that descends to a Yoneda structure on $\mathbf{QCat}_2$.

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The Yoneda structure is representable by an object $\mathbb{S}$ (the ∞-category of spaces) that *classifies opfibrations*: the ∞-category of spaces.

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Question

Can do the same for other examples of ∞-cosmoi (also those nonequivalent to **QCat**)?

Formal CT of derivators

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$$\begin{tikzcd}
\mathbf{cat} \arrow[r, "\nu"] \arrow[d, "y"'] & {[\mathbf{cat}^{op}, \mathbf{Cat}_\infty]} \arrow[r] & \mathrm{Der} \\
{[\mathbf{cat}^{op}, \mathbf{cat}]} \arrow[urr, dotted] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, mapsto] & \mathbf{sSet}^{A\times -} \arrow[r, mapsto] & \mathrm{Ho}(\mathbf{sSet}^{A\times -}) \\
\boldsymbol{y}(A) & &
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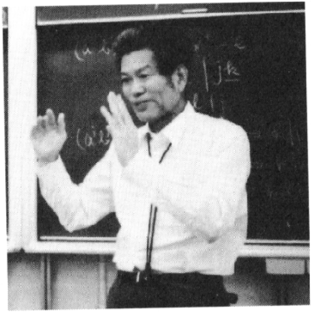

米田信夫

Thanks!