On the unicity of the formal theory of categories

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- Our original motivation was to investigate the formal category theory of derivators.

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  - Two-sided Yoneda structure
  - Isbell duality and its generalizations

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- Examples:
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  - Two-sided Yoneda structure
  - Isbell duality and its generalizations
- An open and motivating problem: derivators.

Introduction

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«The purpose of category theory is to try to describe certain general aspects of the structure of mathematics. Since category theory is also part of mathematics, this categorical type of description should apply to it as well as to other parts of mathematics.»

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«The category of small categories, **Cat**, is a "2-category with properties"; one should attempt to identify those properties that enable one to do the "structural parts of category theory".»

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Not a new idea:

- cf. topos theory: treat a category$\mathcal{E}$ as if it were **Set**;
- cf. categorical algebra: treat a category $\mathcal{A}$ as if it were $\mathsf{Alg}(\mathbb{T})$;
- cf. homological algebra: treat a category $\mathcal{C}$ as if it were $\mathsf{Ch}(\mathcal{A})$;

Perhaps surprisingly though, the bare structure of a 2-category does not suffice to embody "all" category theory in a formal way. (In short: there are features of 2-category theory induced by the fact that 2-**Cat** is a 3-category)

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- Kan extensions;
- internal fibrations. . .

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Idea

Take a 2-category $\mathcal{K}$, and take Y) and E) very seriously.

Y): Yoneda structures

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- Category theory is the class of corollaries of Yoneda lemma.
- A *Yoneda structure* imposes on $\mathcal{K}$ enough structure so that every "small" object A has a "Yoneda embedding" $y_A : A \to PA$.
- Formally encode the Yoneda lemma in universal properties of certain diagrams

$$\begin{tikzcd} & A \arrow[dl, "y_A"'] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \chi^f"] & \\ PA & & B \arrow[ll, "B(f{,}1)"] \end{tikzcd}$$

representing f-nerves.

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- This leads to the notion of C-D-bimodule as a functor $C^{\mathrm{op}} \times D \to \mathit{Set}$. All such bimodules live in a bicategory Mod, and there is an embedding

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- This leads to the notion of C-D-bimodule as a functor $C^{\mathrm{op}} \times D \to Set$. All such bimodules live in a bicategory Mod, and there is an embedding
$$p : Cat \hookrightarrow Mod$$
such that every $f \in Cat$ has a right adjoint when embedded in Mod.
- Every such $p : \mathcal{K} \to \overline{\mathcal{K}}$ equips $\mathcal{K}$ with proarrows.

The main theorem of this talk is an equivalence between a certain class of Yoneda structures and a certain class of proarrow equipments:

$$\left\{\begin{array}{c}\text{cocomplete}\\ \text{Yoneda}\\ \text{structures}\end{array}\right\} \Longleftarrow\!=\!=\!=\!=\!=\!=\!=\!= \left\{\begin{array}{c}\text{Yoneda}\\ \text{equipments}\end{array}\right\}$$

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\end{tikzcd}$$

What allows to pass from YS to E is the notion of a yosegi, a certain special kind of 2-monad.

Preliminaries

2-dimensional universality

The main 2-dimensional universal constructions we are interested in

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow\eta"] & B \\ C \arrow[ur, dotted, "\mathrm{Lan}_g f"'] & \end{tikzcd} \qquad \frac{\mathrm{Lan}_g f \to h}{f \to hg}$$

$$\frac{\mathrm{Lift}_g f \to h}{f \to gh} \qquad \begin{tikzcd} & C \arrow[d, "g"] \\ B \arrow[ur, dotted, "\mathrm{Lift}_g f"] \arrow[r, "f"'] \arrow[ur, phantom, "\nearrow\eta"'] & A \end{tikzcd}$$

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(Exercise: properly dualize this statement)

A bit of coend calculus:

Proposition (ninja Yoneda lemma)

For every presheaf $F : \mathcal{C} \to \mathbf{Set}$, it holds

$$F(X) \cong \int_A FX^{\hom(A,X)} \qquad F(X) \cong \int^A FA \times \hom(X, A)$$

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Proposition (pointwise Kan extensions)

For a diagram $\mathcal{C} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{B}$, it holds

$$\mathrm{Lan}_G F(X) \cong \int^A \hom(GA, X) \times FA$$

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  - A family of triangles

$$\begin{tikzcd} & A \arrow[dl, "y_A"'] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \chi^f"] & \\ PA & & B \arrow[ll, "B(f,1)"] \end{tikzcd}$$

filled by a 2-cell $\chi : y_A \to B(f,f) =: B(f,1) \cdot f$.

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  - The pair $\langle 1_{\boldsymbol{P}A}, 1_{y_A} \rangle$ exhibits the pointwise left extension $\mathrm{Lan}_{y_A} y_A$ (shortly, 'the Yoneda embedding is dense').

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  - The pair $\langle 1_{\mathbf{P}A}, 1_{y_A} \rangle$ exhibits the pointwise left extension $\mathrm{Lan}_{y_A} y_A$ (shortly, 'the Yoneda embedding is dense').
  - Given a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, the pasting of 2-cells

$$\begin{tikzcd}
A \arrow[rr, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, "\Swarrow \chi^{y_B f}"] & & \mathbf{P}A \\
B \arrow[r, "y_B"] \arrow[d, "g"'] \arrow[dr, phantom, "\Swarrow \chi^{g}", near start] & \mathbf{P}B \arrow[ur, "\mathbf{P}f"'] & \\
C \arrow[ur, "C(g,1)"'] & &
\end{tikzcd}$$

exhibits the pointwise extension $\mathrm{Lan}_{gf} y_A = C(gf,1)$ (shortly, $\mathbf{P}$ is a functor).

In **Cat**

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- χ^f is obtained from the action of f on arrows.

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Axiom 1

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It is enough to check that

$$\begin{aligned}
[B, PA](N_f, G) &\cong \int_b PA(B(f,b), Gb) \\
&\cong \int_{ab} \mathbf{Set}(B(fa,b), G(b)(a)) \\
&\cong G(fa)(a) \\
[A, PA](y_A, G \cdot f) &\cong \int_a PA(y_A(a), G(fa)) \\
&\cong G(fa)(a).
\end{aligned}$$

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Axiom 2

The pair $\langle f, \chi^f \rangle$ exhibits a relative adjunction $f \;{}_{y_A}\!\dashv B(f,1)$.

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[A, PA]\big(y_A, N_f \cdot g\big) &\cong \int_{a'} [A^{\mathrm{op}}, \mathbf{Set}]\big(y_A a', N_f \cdot g(a')\big) \\
&\cong \int_{a'} [A^{\mathrm{op}}, \mathbf{Set}]\big(y_A a', B(f-, ga')\big) \\
&\cong \int_{a'} B(fa', ga') \\
&\cong [A, B](f, g)
\end{aligned}$$

absoluteness can be checked by hand.

In **Cat**

Axiom 3-4

They admit a rephrasing as $\boldsymbol{P}$ is a functor, in that $\boldsymbol{P}(id) \cong id$ and $\boldsymbol{P}(gf) \cong \boldsymbol{P}f \cdot \boldsymbol{P}g$.

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This is pretty obvious (although the isomorphisms

3) $[\boldsymbol{P}A, \boldsymbol{P}A](1, H) \cong [A, \boldsymbol{P}A](y_A, H \cdot y_A)$;
4) $C(gf, 1) = \mathrm{Lan}_{gf}\, y_A \cong \mathrm{Lan}_{y_B f}\, y_A \circ \mathrm{Lan}_g\, y_B = \boldsymbol{P}B(y_B f, 1) \cdot C(g, 1)$

can be checked directly).

All **Cat**-like 2-categories carry Yoneda structures in the obvious way:

- $\mathcal{V}$-**Cat** (enriched categories) with the enriched Yoneda embeddings;
- internal categories in a finitely complete category $\mathcal{A}$;
- the 2-category of pseudofunctors $\mathcal{A} \to$ **Cat** for a small bicategory $\mathcal{A}$;
- ...

Proarrow equipments

Definition (proarrow equipment)

A 2-functor $p : \mathcal{K} \to \overline{\mathcal{K}}$ equips $\mathcal{K}$ with proarrows if

- p is the identity on objects and locally fully faithful;
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Examples: the embedding **Cat** $\hookrightarrow$ *Mod* into the category of bimodules/profunctors. The embedding of $\mathbf{Cat}_c$ (Cauchy complete categories) in geometric morphisms of presheaf toposes.

Very simple definition; a bit complicated to follow the literature. Recently framed into the more natural environment of (hyper)virtual double categories of (Shulman-Crutwell).

Question

How do these framework relate? It is reasonable to expect they do (they're both ways to encode a calculus of profunctors; $y_A : A \to \boldsymbol{P}A$ is the (mate of the)identity profunctor).

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3. if only it was possible to go back to a Yoneda structure...

Turns out 1. is almost true; 2. is true; 3. is true if we restrict to so-called Yoneda equipments where p has additional properties.

The presheaf construction sending $A \mapsto [A^{\text{op}}, \mathbf{Set}]$ and $f : A \to B$ to the inverse image $\boldsymbol{P}^* f : \boldsymbol{P}B \to \boldsymbol{P}A$ is such that

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3. the monad $\boldsymbol{P}_!$ is a KZ-doctrine.

Definition (yosegi box)

Let $j : \mathcal{A} \subset \mathcal{K}$ and $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ with the same properties, and such that moreover

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[a] *Yosegi-zaiku* (寄木細工) is a kind of marquetry featuring elaborate inlaid and mosaic designs; the defining properties of such a KZ-doctrine are tightly linked, rich of peculiar adornments.

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3. The inclusion $j : \mathcal{A} \subset \mathcal{K}$ *extends to a yosegi box*.

- a Yoneda equipment is a triangle

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "j"'] \arrow[dr, "p"] \arrow[d, phantom, "\Rightarrow"] & \\ \mathcal{K} & & \mathcal{M} \arrow[ll, "u"] \end{tikzcd}$$

such that

  - p is a proarrow equipment à la Wood;
  - p is the j-relative left adjoint of $u : \mathcal{M} \to \mathcal{K}$;
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  - the relative monad up generated by the relative adjunction, is lax idempotent with unit $\eta : j \Rightarrow up$.
- the inclusion j extends to a yosegi if there is a relative lax-idempotent 2-monad $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ with relative unit $j \Rightarrow \boldsymbol{P}$ such that for each 1-cell $f : A \to B$, the 1-cell $\boldsymbol{P}(f)$ has a left adjoint $\boldsymbol{P}_! f$.

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- Yoneda equipments contain enough information to recover a Yoneda structure.

Applications

Examples

- Often, $\boldsymbol{P}$ is representable, because $\mathcal{K}$ is cartesian closed: $\boldsymbol{P} : A \mapsto [A^{\mathsf{op}}, \Omega]$; this is the case for many $\mathcal{K}$'s having a duality involution $(\_)^{\mathsf{op}}$, and entails that $\boldsymbol{P}$ has an adjoint

$$\boldsymbol{P}^\sharp \dashv \boldsymbol{P}$$

  In **Cat**, $\boldsymbol{P}^\sharp$ is the "contravariant presheaf construction", sending A to $[A, \mathbf{Set}]^{\mathsf{op}}$.

- This self-duality of $\boldsymbol{P}$ is tightly linked to the possibility to instantiate the formal version of *Isbell duality*: there is an adjunction

$$\begin{tikzcd} & A \arrow[dl, "y_A"'] \arrow[dr, "y_A^\sharp"] & \\ \boldsymbol{P}A \arrow[rr, shift left, "\mathcal{O}"] & & \boldsymbol{P}^\sharp A \arrow[ll, shift left, "\mathrm{Spec}"] \end{tikzcd}$$

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- The pair $(\boldsymbol{P}^\sharp, \boldsymbol{P})$ forms a two-sided Yoneda structure/yosegi: we have copresheaf constructions working as free completion: $\boldsymbol{P}^\sharp : A \mapsto [A, \mathbf{Set}]^{\mathrm{op}}$ does precisely this in **Cat**; the axioms of a (left) Yoneda structure hold replacing left extensions and lifts with the corresponding right versions. These two structures are compatible.

Formal ∞-category theory

Let $\mathbf{QCat}_2$ be the homotopy 2-category of the **Kan**-enriched category $\mathbf{QCat}$ of ∞-categories; there is an "∞-Yoneda structure" on $\mathbf{QCat}$ that descends to a Yoneda structure on $\mathbf{QCat}_2$.

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The Yoneda structure is representable by an object $\mathbb{S}$ that *classifies opfibrations*: the ∞-category of spaces.

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Question

Can do the same for other examples of ∞-cosmoi (also those nonequivalent to $\mathbf{QCat}$)?

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$$\begin{tikzcd}
\mathbf{cat} \arrow[r, "\nu"] \arrow[d, "y"'] & \mathbf{Cat}_\infty \arrow[r] & \mathrm{Der} \\
{[\mathbf{cat}^{\mathrm{op}}, \mathbf{cat}]} \arrow[urr, dotted] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, mapsto] & \mathbf{sSet}^{NA} \arrow[r, mapsto] & \mathrm{Ho}(\mathbf{sSet}^{NA}) \\
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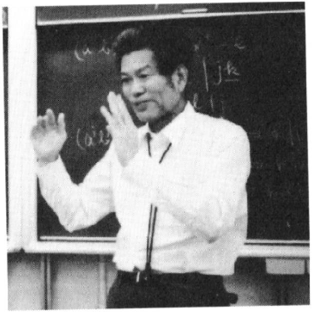
The arrow y is the Yoneda embedding of small categories into small prederivators; taking the Yoneda extension of the functor $\nu : A \mapsto \mathrm{sSet}^{NA}$ now we get a functor from pDer (lowercase p = small prederivators) to (presentable) ∞-categories, whose homotopy categories are thus derivators.



米田信夫

Thanks!