On the unicity of the formal theory of categories

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- All I'm about to say is joint work with Ivan Di Liberti (Masaryk UNIversity, CZ).
- A few questions remain open, but we'll be on arXiv soon!
- Every comment or advice is welcome.

Plan of the talk

- Fast and painless intro to formal category theory
- Setting the stage: bits of 2-category theory I'll need along the discussion
- the current state of art: Yoneda lemma VS bimodules
- Main theorem: "Yoneda structures $\sim$ proarrow equipments."
- Examples:
  - Representable Yoneda structure
  - Two-sided Yoneda structure
  - Isbell duality and its generalizations
- An open and motivating problem: derivators.

Introduction

- Formal category theory is a branch of 2-category theory
- it serves to axiomatize the structural part of category theory

In the words of John Gray

1.

«The purpose of category theory is to try to describe certain general aspects of the structure of mathematics. Since category theory is also part of mathematics, this categorical type of description should apply to it as well as to other parts of mathematics.»

2.

«The basic idea is that the category of small categories, **Cat**, is a "2-category with properties"; one should attempt to identify those properties that enable one to do the "structural parts of category theory".»

1. reads as "category theory is a theory we can interpret in different contexts"; these contexts are 2-categories.
2. reads as "our task is unravel the properties enabling to treat an abstract 2-category $\mathcal{K}$ as if it were **Cat**".

Not a new idea:

- cf. topos theory: treat $\mathcal{E}$ as if it were **Set**;
- cf. categorical algebra: treat $\mathcal{A}$ as if it were $\mathsf{Alg}(\mathbb{T})$;
- cf. homological algebra: treat $\mathcal{C}$ as if it were $\mathsf{Ch}(\mathcal{A})$;

Perhaps surprisingly though, the bare structure of a 2-category does not suffice to embody "all" category theory in a formal way.

Sure you can do many things:

- adjunctions: pairs of 1-cells $f : X \leftrightarrows Y : g$ with co/unit
- monads: endo-1-cells $t : A \to A$ with multiplication and unit
- Kan extensions
- fibrations

but many other important things are missing:

- a "pointwise" formula to compute Kan extensions
- equivalent characterization of adjunctions: $Y(f,1) \cong X(1,g)$

Y) the Yoneda lemma ("category theory is the Yoneda lemma")

E) a calculus of modules ("category theory is the theory of multi-object monoids")

Idea

Take a 2-category $\mathcal{K}$, and take 1. and 2. very seriously.

(Y): Yoneda structures

- Category theory is the class of corollaries of Yoneda lemma.
- A *Yoneda structure* imposes on $\mathcal{K}$ enough structure so that every "small" object A has a "Yoneda embedding" $y_A : A \to PA$.
- Formally encode the Yoneda lemma in universal properties of certain diagrams

$$\begin{tikzcd} & A \arrow[dl, "y_A"'] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \chi^f"] & \\ PA & & B \arrow[ll, "B(f{,}1)"] \end{tikzcd}$$

representing f-nerves.

(E): proarrow equipments

- Category theory is the theory of multi-object monoids.
- And monoids are well-known for their irresistible tendency to act on objects!
- This leads to the notion of C-D-bimodule as a functor $C^\vee \times D \to \mathit{Set}$. All such bimodules live in a bicategory Mod, and there is an embedding

$$p : \mathit{Cat} \hookrightarrow \mathit{Mod}$$

  such that every $f \in \mathit{Cat}$ has a right adjoint when embedded in Mod.
- Every such $p : \mathcal{K} \to \overline{\mathcal{K}}$ equips $\mathcal{K}$ with proarrows.

The main theorem of this talk is an equivalence between a certain class of Yoneda structures and a certain class of proarrow equipments:

$$\begin{tikzcd} & \left\{\begin{array}{c}\text{yosegi}\\ \text{boxes}\end{array}\right\} \arrow[dr, Rightarrow] & \\ \left\{\begin{array}{c}\text{cocomplete}\\ \text{Yoneda}\\ \text{structures}\end{array}\right\} \arrow[ur, Rightarrow] & & \left\{\begin{array}{c}\text{Yoneda}\\ \text{equipments}\end{array}\right\} \arrow[ll, Rightarrow] \end{tikzcd}$$

What allows to pass from YS to E is the notion of a yosegi, a certain special kind of 2-monad.

Preliminaries

2-dimensional universality

The main 2-dimensional universal constructions we are interested in

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\swarrow\eta"] & B \\
C \arrow[ur, dotted, "\mathrm{Lan}_g f"'] &
\end{tikzcd}
\qquad
\frac{\mathrm{Lan}_g f \to h}{f \to hg}
$$

$$
\frac{\mathrm{Lift}_g f \to h}{f \to gh}
\qquad
\begin{tikzcd}
& C \arrow[d, "g"] \\
B \arrow[r, "f"'] \arrow[ur, dotted, "\mathrm{Lift}_g f"] \arrow[ur, phantom, "\nearrow\eta"', shift right=3] & A
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] \arrow[dr, phantom, "\nearrow\varepsilon"] & B \\
C \arrow[ur, dotted, "\mathrm{Ran}_g f"'] &
\end{tikzcd}
\qquad
\frac{hg \to f}{h \to \mathrm{Ran}_g f}
$$

$$
\frac{h \to \mathrm{Rift}_g f}{gh \to f}
\qquad
\begin{tikzcd}
& C \arrow[d, "g"] \\
B \arrow[r, "f"'] \arrow[ur, dotted, "\mathrm{Rift}_g f"] \arrow[ur, phantom, "\swarrow\varepsilon"', shift right=3] & A
\end{tikzcd}
$$

pointwise. . . . . . absolute

Proposition (formal description of adjoints)

The following conditions are equivalent, for a pair of 1-cells $f : A \leftrightarrows B : g$

- $f \dashv g$ with unit η and counit ϵ;
- The pair $\langle g, \eta \rangle$ exhibits the absolute Lan of 1 along f
- The pair $\langle g, \eta \rangle$ exhibits the Lan of 1 along f, and f preserves it.

(Exercise: properly dualize this statement)

A bit of coend calculus:

Proposition (ninja Yoneda lemma)

For every presheaf $F : \mathcal{C} \to \mathbf{Set}$, it holds

$$F(X) \cong \int_A FX^{\hom(A,X)} \qquad F(X) \cong \int^A FA \times \hom(X, A)$$

naturally in X.

Proposition (pointwise Kan extensions)

For a diagram $\mathcal{C} \xleftarrow{G} \mathcal{A} \xrightarrow{F} \mathcal{B}$, it holds

$$\mathrm{Lan}_G F(X) \cong \int^A \hom(GA, X) \times FA$$

naturally in X.

Yoneda structures

Let $\mathcal{K}$ be a 2-category; a Yoneda structure on $\mathcal{K}$ is made of

- Yoneda data
  - An ideal of admissible arrows $\mathfrak{J}$

    identity morphisms in the ideal determine admissible *objects*.
  - a family of morphisms $y_A : A \to \boldsymbol{P}A$ one for each admissible object A

    beware! $\boldsymbol{P}A$ is only syntactic sugar. It has no meaning whatsoever.
  - A family of triangles

$$\begin{tikzcd} & A \arrow[dl, "y_A"'] \arrow[dr, "f"] \arrow[d, phantom, "\Downarrow \chi^f"] & \\ PA & & B \arrow[ll, "B(f,1)"] \end{tikzcd}$$

filled by a 2-cell $\chi : y_A \to B(f, f) =: B(f, 1) \cdot f$.

- subject to Yoneda axioms
  - The pair $\langle B(f,1), \chi^f \rangle$ exhibits the pointwise left extension $\mathrm{Lan}_f y_A$.
  - The pair $\langle f, \chi^f \rangle$ exhibits the absolute left lifting $\mathrm{LIFT}_{B(f,1)} y_A$ (shortly, $f \; {}_{y_A}\!\dashv B(f,1)$).
  - The pair $\langle 1_{\mathbf{P}A}, 1_{y_A} \rangle$ exhibits the pointwise left extension $\mathrm{Lan}_{y_A} y_A$ (shortly, 'the Yoneda embedding is dense').
  - Given a pair of composable 1-cells $A \xrightarrow{f} B \xrightarrow{g} C$, the pasting of 2-cells

$$\begin{tikzcd}
A \arrow[rr, "y_A"] \arrow[d, "f"'] \arrow[dr, phantom, shift left=3ex, "\Swarrow \chi^{y_B f}"] & & \mathbf{P}A \\
B \arrow[r, "y_B"] \arrow[d, "g"'] \arrow[d, phantom, shift left=4ex, "\Swarrow \chi^{g}"] & \mathbf{P}B \arrow[ur, "\mathbf{P}f"'] & \\
C \arrow[ur, "C(g,1)"'] & &
\end{tikzcd}$$

exhibits the pointwise extension $\mathrm{Lan}_{gf} y_A = C(gf,1)$ (shortly, $\mathbf{P}$ is a functor).

In **Cat**

- The ideal $\mathfrak{J}$ is made by functors $f : A \to B$ such that arrows $fa \to b$ form a set for every $a, b \in A, B$; admissible objects are made by locally small categories;
- y_A is of course the Yoneda embedding $A \to [A^{op}, \mathbf{Set}]$;
- χ^f is obtained from the action of f on arrows.

In **Cat**

Axiom 1

$B(f,1) = \lambda b.\lambda a.\, \hom_B(fa, b)$ exhibits with χ^f the pointwise left Kan extension of y_A along $f : A \to B$.

It is enough to check that

$$\begin{aligned}
[B, PA](N_f, G) &\cong \int_b PA(B(f,b), Gb) \\
&\cong \int_{ab} \mathbf{Set}(B(fa,b), G(b)(a)) \\
&\cong G(fa)(a) \\
[A, PA](y_A, G \cdot f) &\cong \int_a PA(y_A(a), G(fa)) \\
&\cong G(fa)(a).
\end{aligned}$$

In **Cat**

Axiom 2

The pair $\langle f, \chi^f \rangle$ exhibits a relative adjunction $f \;{}_{y_A}\!\dashv B(f, 1)$.

It is enough to check that

$$\begin{aligned}
[A, PA]\big(y_A, N_f \cdot g\big) &\cong \int_{a'} [A^{\mathrm{op}}, \mathbf{Set}]\big(y_A a', N_f \cdot g(a')\big) \\
&\cong \int_{a'} [A^{\mathrm{op}}, \mathbf{Set}]\big(y_A a', B(f-, ga')\big) \\
&\cong \int_{a'} B(fa', ga') \\
&\cong [A, B](f, g)
\end{aligned}$$

absoluteness can be checked by hand.

In Cat

Axiom 3-4

They admit a rephrasing as $\boldsymbol{P}$ is a functor, in that $\boldsymbol{P}(id) \cong id$ and $\boldsymbol{P}(gf) \cong \boldsymbol{P}f \cdot \boldsymbol{P}g$.

This is pretty obvious (although the isomorphisms

3) $[\boldsymbol{P}A, \boldsymbol{P}A](1, H) \cong [A, \boldsymbol{P}A](y_A, H \cdot y_A)$;
4) $C(gf, 1) = \mathrm{Lan}_{gf} y_A \cong \mathrm{Lan}_{y_B f} y_A \circ \mathrm{Lan}_g y_B = \boldsymbol{P}B(y_B f, 1) \cdot C(g, 1)$

can be checked directly).

All **Cat**-like 2-categories carry Yoneda structures in the obvious way:

- $\mathcal{V}$-**Cat** (enriched categories) with the enriched Yoneda embeddings;
- internal categories in a finitely complete category $\mathcal{A}$;
- the 2-category of pseudofunctors $\mathcal{A} \to \mathbf{Cat}$ for a small bicategory $\mathcal{A}$;
- ...

Proarrow equipments

Definition (proarrow equipment)

A 2-functor $p : \mathcal{K} \to \overline{\mathcal{K}}$ equips $\mathcal{K}$ with proarrows if

- p is the identity on objects and locally fully faithful;
- p is such that for each 1-cell $f : A \to B$ in $\mathcal{K}$ $p(f)$ has a right adjoint in $\overline{\mathcal{K}}$.

Examples: the embedding $\mathbf{Cat} \hookrightarrow Mod$ into the category of bimodules/profunctors. The embedding of $\mathbf{Cat}_c$ (Cauchy complete categories) in geometric morphisms of presheaf toposes.

Very simple definition; a bit complicated to follow the literature. Recently framed into the more natural environment of (hyper)virtual double categories of (Shulman-Crutwell).

Question

How do these framework relate? It is reasonable to expect they do (they're both ways to encode a calculus of profunctors; $y_A : A \to \boldsymbol{P}A$ is the (mate of the)identity profunctor).

Strategy

1. in a Yoneda structure $\boldsymbol{P}$ is a functor by axiom 3-4; the correspondence $A \mapsto \boldsymbol{P}A$ is monad-like
2. the Kleisli category of $\boldsymbol{P}$ equips $\mathcal{K}$ with proarrows via the free part of its free-forget adjunction!
3. if only it was possible to go back to a Yoneda structure...

Turns out 1. is almost true; 2. is true; 3. is true if we restrict to so-called Yoneda equipments where p has additional properties.

The presheaf construction sending $A \mapsto [A^{\mathrm{op}}, \mathbf{Set}]$ and $f : A \to B$ to the inverse image $\boldsymbol{P}^* f : \boldsymbol{P}B \to \boldsymbol{P}A$ is such that

1. for each $f : A \to B$ there is an adjunction $\boldsymbol{P}_!(f) \dashv \boldsymbol{P}^*(f)$;
2. $\boldsymbol{P}_!$ is a relative monad (relative to the inclusion $\mathbf{Cat} \subset \mathbf{CAT}$);
3. the monad $\boldsymbol{P}_!$ is a KZ-doctrine.

Definition (yosegi box)

Let $j : \mathcal{A} \subset \mathcal{K}$ and $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ with the same properties, and such that moreover

1. the unit η_A of the monad $\boldsymbol{P}_!$ is fully faithful;
2. The cell $\boldsymbol{P}^* \eta_A$ exists (the codomain of η_A might be non-admissible!)

We call $\boldsymbol{P}$ a (j-relative) yosegi box.[a]

[a] *Yosegi-zaiku* (寄木細工) is a kind of marquetry featuring elaborate inlaid and mosaic designs; the defining properties of such a KZ-doctrine are tightly linked, rich of peculiar adornments.

The main theorem

Let $j : \mathcal{A} \subset \mathcal{K}$ the inclusion of a sub-2-category; the following conditions are equivalent.

1. $\mathcal{K}$ has a Yoneda structure with presheaf construction $\boldsymbol{P}$, and admissible objects those of $\mathcal{A}$, moreover, every $\boldsymbol{P}A$ is a cocomplete object;
2. the inclusion $j : \mathcal{A} \subset \mathcal{K}$ admits a *Yoneda equipment* $p \;{}_j\!\dashv u$;
3. The inclusion $j : \mathcal{A} \subset \mathcal{K}$ *extends to a yosegi box*.

- a Yoneda equipment is a triangle

$$\begin{tikzcd} & \mathcal{A} \arrow[dl, "j"'] \arrow[dr, "p"] \arrow[d, phantom, "\Rightarrow"] & \\ \mathcal{K} & & \mathcal{M} \arrow[ll, "u"] \end{tikzcd}$$

such that

  - p is a proarrow equipment à la Wood;
  - p is the j-relative left adjoint of $u : \mathcal{M} \to \mathcal{K}$;
  - the relative monad up generated by the relative adjunction, is lax idempotent with unit $\boldsymbol{\eta} : j \Rightarrow up$.
- the inclusion j extends to a yosegi if there is a relative lax-idempotent 2-monad $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ with relative unit $j \Rightarrow \boldsymbol{P}$ such that for each 1-cell $f : A \to B$, the 1-cell $\boldsymbol{P}(f)$ has a left adjoint $\boldsymbol{P}_! f$.

Proof is technical, not enlightening, but amazingly all the pieces fit together as in a perfectly built carpentry work.

The key assumption here is that $\mathcal{A}$ 'embeds well' into $\mathcal{K}$ via j (as soon as one of the conditions above is true, j is very near to be dense).

- the Kleisli category of a yosegi $\boldsymbol{P} : \mathcal{A} \to \mathcal{K}$ equips its domain $\mathcal{A}$ with proarrows;
- every cocomplete Yoneda structure has a yosegi as presheaf construction;
- Yoneda equipments contain enough information to recover a Yoneda structure.

Applications

Examples

- Often, $\boldsymbol{P}$ is representable, because $\mathcal{K}$ is cartesian closed: $\boldsymbol{P} : A \mapsto [A^{\mathsf{op}}, \Omega]$; this is the case for many $\mathcal{K}$'s having a duality involution $(\_)^{\mathsf{op}}$, and entails that $\boldsymbol{P}$ has an adjoint

$$\boldsymbol{P}^\sharp \dashv \boldsymbol{P}$$

In **Cat**, $\boldsymbol{P}^\sharp$ is the "contravariant presheaf construction", sending A to $[A, \mathbf{Set}]^{\mathsf{op}}$.

- This self-duality of $\boldsymbol{P}$ is tightly linked to the possibility to instantiate the formal version of *Isbell duality*: there is an adjunction

$$\begin{tikzcd} & A \arrow[dl, "y_A"'] \arrow[dr, "y_A^\sharp"] & \\ \boldsymbol{P}A \arrow[rr, shift left, "\mathcal{O}"] & & \boldsymbol{P}^\sharp A \arrow[ll, shift left, "\mathrm{Spec}"] \end{tikzcd}$$

where $\mathcal{O} = \mathrm{Lan}_{y_A}(y_A^\sharp)$ and $\mathrm{Spec} = \mathrm{Lan}_{y_A^\sharp}(y_A)$.

- The pair $(\boldsymbol{P}^\sharp, \boldsymbol{P})$ forms a two-sided Yoneda structure/yosegi: we have copresheaf constructions working as free completion: $\boldsymbol{P}^\sharp : A \mapsto [A, \mathbf{Set}]^{\mathrm{op}}$ does precisely this in **Cat**; the axioms of a (left) Yoneda structure hold replacing left extensions and lifts with the corresponding right versions. These two structures are compatible.

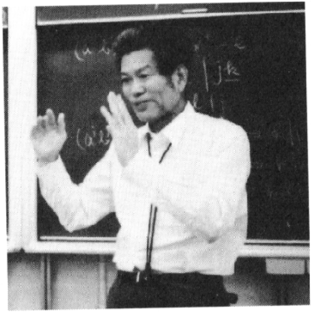
An enticing conjecture is that

On the 2-category of derivators there is a Yoneda structure that keeps track of the homotopy theory of derivators.

The yosegi it generates is represented by the derivator of spaces, which has the universal property of the "free cocompletion of the point". A sketch of the precise construction: consider the following diagram

$$
\begin{tikzcd}
\mathbf{cat} \arrow[r, "\nu"] \arrow[d, "y"'] & \mathbf{Cat}_\infty \arrow[r] & \mathrm{Der} \\
{[\mathbf{cat}^{\mathrm{op}}, \mathbf{cat}]} \arrow[urr, dotted] & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, mapsto] \arrow[d, mapsto] & \mathbf{sSet}^{NA} \arrow[r, mapsto] & \mathrm{Ho}(\mathbf{sSet}^{NA}) \\
\boldsymbol{y}(A) & &
\end{tikzcd}
$$

The arrow y is the Yoneda embedding of small categories into small prederivators; taking the Yoneda extension of the functor $\nu : A \mapsto \mathrm{sSet}^{NA}$ now we get a functor from pDer (lowercase p = small prederivators) to (presentable) ∞-categories, whose homotopy categories are thus derivators.



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Thanks!