# A 2-Categorical Analysis of the Tripos-to-Topos Construction

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**Abstract**

We characterize the tripos-to-topos construction of Hyland, John-stone and Pitts as a biadjunction in a bicategory enriched category of equipment-like structures. These abstract concepts are necessary to handle the presence of *oplax* constructs — the construction is only *oplax* functorial on certain classes of cartesian functors between triposes.

A by-product of our analysis is the decomposition of the tripos-to-topos construction into two steps, the intermediate step being a weakened version of quasitoposes.

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# 1  Introduction

Triposes were introduced by Hyland, Johnstone and Pitts [11] as a framework which enables to generalize the construction of the category of sheaves on a locale (complete Heyting algebra). Their motivating observations were that

- the alternative description of sheaves on a locale $A$ as '$A$-valued sets' which was independently introduced by Higgs [10], and by Fourman and Scott [6], really only depends on the fibered poset $\mathrm{fam}(A) : \mathrm{Fam}(A) \to \mathbf{Set}$ (the *family fibration* of $A$, see [12, Definition 1.2.1]), and

- Kleene's number realizability gives rise to a fibration of preorders on which Higgs, Fourman and Scott's construction is defined and yields a topos as well!

These observations lead to the question which properties of the fibration are really needed to allow the construction of toposes, and the definition of tripos gives sufficient conditions (but still stronger than necessary ones, as Pitts points out in [21]).

The topos that arises from the tripos associated to Kleene's number realizability is Hyland's *effective topos*, its introduction marks the starting point a whole new research field: *categorical realizability*[1].

The cross-fertilization between realizability and topos theory/category theory has proven fruitful to categorical logicians and topos theorists on the one hand, since it provides interesting examples of non-Grothendieck toposes, and to realizability on the other hand, since it brought new categorical tools, and a more 'semantic' way of thinking to a field which had traditionally been frightening due to its high amount of syntactical formalism. The new perspective on realizability lead to the discovery of new, 'global' connections between different notions of realizability, making use of geometrically motivated topos theoretic concepts such as *geometric morphism* and *subtopos* (associated to a Lawvere-Tierney topology). As examples, we mention

- Awodey, Birkedal and Scott's work [1], where a *local*[2] *geometric morphism*

$$\Delta \dashv \Gamma : \mathbf{RT}_o(\mathcal{A}, \mathcal{A}_\#) \to \mathbf{RT}(\mathcal{A}_\#) \dashv \nabla$$

  is exhibited between the *relative* realizability topos $\mathbf{RT}_o(\mathcal{A}, \mathcal{A}_\#)$ induced by an inclusion $\mathcal{A}_\# \subset \mathcal{A}$ of partial combinatory algebras and the realizability topos $\mathbf{RT}(\mathcal{A}_\#)$, and

- Birkedal and van Oosten's paper [2] which describes how the relative realizability topos $\mathbf{RT}_o(\mathcal{A}, \mathcal{A}_\#)$ and the *modified*[3] *relative* realizability topos $\mathbf{RT}_c(\mathcal{A}, \mathcal{A}_\#)$ can be viewed as open and closed complementary subtoposes

$$\mathbf{RT}_c(\mathcal{A}, \mathcal{A}_\#) \hookrightarrow \mathbf{RT}(\mathcal{A}, \mathcal{A}_\#) \hookleftarrow \mathbf{RT}_o(\mathcal{A}, \mathcal{A}_\#)$$

  of a larger topos $\mathbf{RT}(\mathcal{A}, \mathcal{A}_\#)$[4].

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[1]For an introduction to this field, we refer to Jaap van Oosten's recent textbook [28].

[2] A geometric morphism is called *local* if its unit is invertible and the direct image part has a further right adjoint.

[3]In light of [18], it is arguable whether modified (relative) realizability should really be viewed as a topos or if $\mathbf{RT}_o(\mathcal{A}, \mathcal{A}_\#)$ rather represents something else (since modified realizability has a typed notion of realizer), but this doesn't bother us here.

[4]See e.g. [14, A4.5] for the definition of open and closed subtoposes.

Abstractly, geometric morphisms and subtoposes are just adjunctions and idempotent monads in the 2-category of toposes and cartesian functors, and we have analogous concepts in an appropriate 2-category of triposes. Furthermore, the geometric morphisms and subtoposes in the previous examples are induced by analogous constructs between the corresponding triposes.

It turns out that it is much easier to make calculations on the level of triposes than on the level of toposes, to the extent that we would like to systematically reduce questions about functors between tripos-induced toposes to questions about morphisms between the corresponding triposes. But in order to do this, we need an abstract (i.e., universal) characterization of the construction which maps triposes to toposes and morphisms between triposes to functors between toposes. *This is the motivation and the objective of the present work.*

The question for a universal characterization of the tripos-to-topos construction is not a new one. Already in 2002, Pitts wrote [21]:

> The construction itself can be seen as the universal solution to the problem of realizing the predicates of a first order hyperdoctrine as subobjects in a logos with effective equivalence relations.

In a more recent, unpublished work [22], Rosolini and Maietti decompose the tripos-to-topos construction into a succession of fibrational completions.

These approaches answer the question for a universal characterization, but are not adequate as a framework for the above examples, since they (albeit implicitly) take place in the 2-categories of triposes and *regular* tripos morphisms (that is fibered functors that commute with $\wedge$ and $\exists$), and toposes and *regular* functors. In order to talk about arbitrary geometric morphisms and subtoposes/sub-triposes, this is too restrictive — we want to talk about functors and morphisms which only preserve finite limits and finite meets, respectively.

Already in [11], it was observed that it is possible to construct functors between toposes from tripos morphisms that merely commute with finite limits, but the abandonment of regularity leads to complications which require more sophisticated 2-dimensional techniques, as the following example demonstrates.

Let $\mathbb{B} = \{\mathsf{true}, \mathsf{false}\}$ be the locale of booleans, with $\mathsf{false} \leq \mathsf{true}$. Then $\mathrm{fam}(\mathbb{B})$ and $\mathrm{fam}(\mathbb{B} \times \mathbb{B})$ are triposes, and the induced toposes are equivalent to $\mathbf{Set}$ and $\mathbf{Set} \times \mathbf{Set}$, respectively. Between the locales we consider the meet-preserving maps

$$\delta = \langle \mathrm{id}, \mathrm{id} \rangle : \mathbb{B} \to \mathbb{B} \times \mathbb{B} \qquad \text{and} \qquad \wedge : \mathbb{B} \times \mathbb{B} \to \mathbb{B}$$
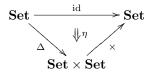
These maps give rise to tripos morphisms

$$\mathrm{fam}(\mathbb{B}) \xrightarrow{\mathrm{fam}(\delta)} \mathrm{fam}(\mathbb{B} \times \mathbb{B}) \xrightarrow{\mathrm{fam}(\wedge)} \mathrm{fam}(\mathbb{B}),$$

which in turn give rise to functors which happen to be the familiar

$$\mathbf{Set} \xrightarrow{\Delta = \langle \mathrm{id}, \mathrm{id} \rangle} \mathbf{Set} \times \mathbf{Set} \xrightarrow{(-\times-)} \mathbf{Set},$$

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Forming the composition of the maps, we get $\wedge \circ \delta = \mathrm{id}_{\mathbb{B}}$ and this gives rise to the identity functor. Therefore we obtain a *non-invertible* constraint cell

$$
\begin{array}{ccc}
\mathbf{Set} & \xrightarrow{\;\;\mathrm{id}\;\;} & \mathbf{Set} \\
& \Downarrow \eta & \\
\Delta \searrow & & \nearrow \times \\
& \mathbf{Set} \times \mathbf{Set} &
\end{array}
$$

where $\eta_I = \delta_I : I \to I \times I$ is the unit of the adjunction $\Delta \dashv (- \times -)$.

This means that the tripos-to-topos construction does not commute with composition of tripos morphisms (not even up to isomorphism), and hence it can not be a 2-functor or a pseudofunctor. The best that we can hope for is for it to be *oplax functorial*, which means that is commutes with identities and composition up to non-invertible 2-cell.

This turns out to be a major obstacle, since we would like to characterize the construction as a kind of left biadjoint to the construction that assigns its subobject fibration to a topos. Unfortunately, it is known that (op)lax functors are very badly behaved — horizontal compositions like $F\eta$ for example are simply not definable for (op)lax functors $F$ and transformations $\eta$ (see diagram (2.3) in Section 2.2). That means in particular that we can not transfer the algebraic definition of biadjunctions by unit, counit and modifications for the triangle equalities to the (op)lax world in a straightforward way.

To overcome these problems, we identify a class of oplax functors and transformations that compose well, more formally we define a three dimensional category of 2-categories with additional structure (so-called *pre-equipments*), and corresponding oplax functors and transformations in which there is an internal notion of biadjunction that fits our purposes. Ideas like these have come up at different places in the literature already, in particular in the context of double categories. The most general treatment can be found in Dominic Verity's thesis [29] on which we rely heavily.

The tripos-to-topos construction can then be described as a biadjunction between the pre-equipments **Trip** of triposes, and the pre-equipment **Top** of toposes. Trying to find a comprehensible description of the left adjoint, I observed that the construction naturally factors through a third pre-equipment — the *q-toposes* (suggestions for a better name are welcome), which are a generalization of quasitoposes where not all finite colimits are required. The q-toposes can be viewed as giving an official status to the so-called *weakly complete objects* that already occur in the original paper [11] by Hyland, Johnstone and Pitts.

## 1.1 Overview of the article

The article is divided in four main sections.

Section 2 provides the category theoretical background. We review Verity's notion of *bicategory enriched category*, and define the bicategory enriched category $\mathfrak{Spec}$ of pre-equipments, special functors, special transformations and modifications. Finally, we introduce *special biadjunctions* which are just biadjunctions in $\mathfrak{Spec}$ and which we use in the sequel for our characterization of the tripos-to-topos construction.

In Section 3 we define triposes, define their internal language, and explain how they form a pre-equipment.

In Section [4](#) we introduce q-toposes, define the pre-equipment **QTop** of q-toposes, explain how to interpret higher order intuitionistic logic in a q-topos and prove that the *coarse objects* in a q-topos form a topos.

Finally, in Section [5](#), we give a detailed exposition of the special adjunctions $F \dashv S : \mathbf{QTop} \to \mathbf{Trip}$ between triposes and q-toposes and $T \dashv U : \mathbf{Top} \to \mathbf{QTop}$ between q-toposes and toposes. These special biadjunctions form our characterization and decomposition of the tripos-to-topos construction.

## 1.2 Conventions, preliminaries

### Notation, terminology

In 2-categories, and in bicategory enriched categories as introduced in Section [2](#), we will write 'id' for identities in all dimensions, usually appropriately subscripted.

We will normally write $A \in \mathbb{C}$ instead of $A \in \mathrm{obj}(\mathbb{C})$ to mean that $A$ is an object of a category $\mathbb{C}$.

### Strict, strong, lax and oplax

We consider different kinds of functors and transformations between 2-categories, and we will use the adjectives *strict, strong, lax and oplax* to specify whether they have identity-, isomorphic or directed constraint cells. We also refer to strong functors as pseudofunctors.

For an oplax functor $F : \mathbf{A} \to \mathbf{B}$ and $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathbf{A}$, the direction of constraint cells is $F(gf) \to Fg\, Ff$ and $F\mathrm{id} \to \mathrm{id}$, and for an oplax transformation $\eta : F \to G : \mathbf{A} \to \mathbf{B}$, the direction of constrains is $\eta_B\, Ff \to Gf\, \eta_A$. For lax functors and transformation, the direction of constraints is the opposite (in parts of the literature, the meaning of lax and oplax is exchanged for transformations).

For definitions of strict, strong, and (op)lax functors, and transformations, see for example Leinster's [17] (Note that Leinster uses the traditional terms morphism and homomorphism for lax and strong functors).

### Size issues
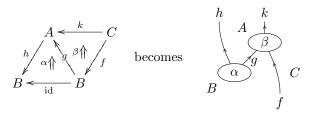
We will assemble possible large categories into 2-categories, and then assemble theses 2-categories into a three dimensional category. Formally, we need several Grothendieck universes to do this, but since we do not use concepts where relative sizes are important (such as local smallness) this does not pose problems (once we accept the existence of Grothendieck universes), and we will comment no more on that.

A related issue is that we will talk about presheaves of subobjects and representable presheaves, concepts which only make sense if the involved categories are well powered and locally small, respectively. While we want to avoid to appeal to local smallness and well-poweredness since they refer to relative sizes, we can always assume the existence of a universe which makes the involved categories *globally small* (and thus of course well powered).

### String diagrams

In addition to pasting diagrams, we will use string diagrams for 2-categorical reasoning since they are usually more concise and more importantly they make the structure of the calculations more apparent. Many string diagrammatic calculi have been developed and investigated for different kinds of monoidal

categories (see [23] for an overview), but we will only use the most basic variant, which exists in a version for monoidal category and a version for bicategories (which we will use). This basic version for bicategories is presented e.g. in [25], and since it is very easy, we explain it again here.

The basic idea of string diagrams is that they are a kind of dual graphs of pasting diagrams, as visualized in the following example for a composite of two 2-cells $\alpha, \beta$ in some generic 2-category $\mathbf{A}$.



In the pasting diagram on the left, objects are vertices, morphisms are edges and 2-cells can be viewed as faces. In the string diagram on the right, the 2-cells are nodes, the morphisms are edges again, but orthogonal to the edges in the pasting diagram, if we place the diagrams one upon the other; and the faces correspond to objects of the 2-category. Observe that the 2-cell $\alpha : \mathrm{id}_B \to hg$ is drawn as a node with zero inputs and 2 outputs; this is because in this context we think of $\mathrm{id}_B$ as the composite of the empty list of 1-cells. Normally we omit much of the typing information, i.e. the labels of lines and faces, because it clutters the diagrams and can be easily inferred from the context. Moreover, the orientation of our diagrams is always bottom-up and right-left, and we omit the redundant arrows on the edges as well.

We are not too much concerned about formal properties of the calculus of string diagrams itself, our use of them is heuristic rather than formalistic. In a sense, we view string diagrams as shorthands for more rigorous symbolic computations, which can always be reconstructed from them on demand (and for the size of the diagrams that we are using, this is not only a theoretical but a practical possibility — with a bit of practice you can even read the associated pasting diagram "between the lines").

**Existence of structures versus chosen structures**

By a finite limit category, do we mean a category such that for any finite diagram there exists a limiting cone, or a category equipped with a specific choice of such limiting cones? — Normally, one tends to be rather ambiguous about that, after all suitable choice principles always allow us to postulate an explicit family of limiting cones, even if we only assumed mere existence before. When we assemble our categories into 2-categories, however, we have to be more precise. We demonstrate this with a little example. Let $\mathbf{Fp}$ be the 2-category of finite product categories and $\mathbf{Cat}$ the 2-category of categories. We define two 2-functors $\boldsymbol{F}, \boldsymbol{G} : \mathbf{Fp} \to \mathbf{Cat}$, where

$$\boldsymbol{F}\mathbb{C} = \mathbb{C} \times \mathbb{C} \quad \text{and} \quad \boldsymbol{G}\mathbb{C} = \mathbb{C}$$

Now we want to define a transformation $\boldsymbol{\eta} : \boldsymbol{F} \to \boldsymbol{G}$ by

$$\boldsymbol{\eta}_{\mathbb{C}}(C, D) = C \times D.$$

This only makes sense if we have chosen products, otherwise the object part of $\boldsymbol{\eta}_{\mathbb{C}}$ is not well defined! On an informal level, one may be content to define such functors up to isomorphism, but at the latest when it comes to verifying coherence axioms of 2-categorical constructions, as we will have to do in Section 5, we really have to be precise what we are talking about.

Therefore, in the following whenever we talk about categories with certain limits or colimits, we implicitly require chosen such objects. This is equivalent to equipping the category with limit/colimit functors, because the morphism parts can be inferred by universality.

## 2 Pre-equipments

This section introduces the categorical backdrop to make sense of our analysis of the tripos-to-topos construction.

The overall aim of the article is to characterize the tripos-to-topos construction as a certain type of biadjunction. Since biadjunctions are naturally encountered in three dimensional categories (just like abstract (1-)adjunctions can be defined in arbitrary bicategories), we will explain how pre-equipments form such a three dimensional category. Before we can do this, however, we have to take yet another step back, and explain the notion of three dimensional category that we are going to use: Verity's bicategory enriched categories.

Almost everything in this section can be found in Verity's thesis [29], but for reasons of self containedness, because the thesis has not (yet) been published, and since the ideas are introduced there in much greater generality than we need, we repeat the necessary definitions and constructions here (mostly without proofs).

In the first subsection, we explain the concept of bicategory enriched category, which is closely related to — but more general than — the notion of Gray-category, and the abstract concept of biadjunction in bicategory enriched categories.

In the second subsection, we will then introduce *pre-equipments*, explain how they form a bicategory enriched category, and will have a closer look at the ensuing notion of biadjunction between pre-equipments.

### 2.1 Bicategory enriched categories

This section is the attempt to summarize the relevant parts of Section 1.3 of Verity's thesis [29].

As bicategory enriched categories are related to the more familiar notion of Gray categories, we begin by recalling the ideas behind the latter notion.

Informally, a Gray category is a three dimensional category whose hom-objects are 2-categories, where 1-cells induce strictly 2-functorial pre- and post-composition operations $(-)f$ and $f(-)$, and which does *not* have a primitive 'parallel' composition operation $\theta\eta$ for 2-cells

$$\mathbf{A} \underset{f'}{\overset{f}{\Rightarrow\eta}} \mathbf{B} \underset{g'}{\overset{g}{\Rightarrow\theta}} \mathbf{C} \, ,$$

but only specified coherent exchange isomorphisms of type

$$\theta f' \circ g\eta \xrightarrow{\cong} g'\eta \circ \theta f$$

$$\begin{array}{ccc} fg & \longrightarrow & f'g \\ \downarrow & \swarrow_{\cong} & \downarrow \\ fg' & \longrightarrow & f'g' \end{array}$$

between sequentializations of the parallel composition.

Formally, Gray categories are defined as categories enriched in the category of 2-categories equipped with a certain symmetric monoidal product, the *Gray product*. The Gray product is characterized by the natural bijection

$$\textbf{2-}\mathfrak{Cat}(\mathbf{A} \otimes_G \mathbf{B}, \mathbf{C}) \cong \textbf{2-}\mathfrak{Cat}(\mathbf{A}, [\mathbf{B}, \mathbf{C}]), \tag{2.1}$$

where $\textbf{2-}\mathfrak{Cat}(-, -)$ denotes the set of 2-functors between two 2-categories, and $[-, -]$ denotes the 2-category of 2-functors, pseudo-transformations and modifications between two 2-categories.

For bicategory enriched categories, the idea is the same, except that we replace 2-categories by bicategories and 2-functors by pseudofunctors[5]. However, we run into problems if we want to adapt the technique for Gray categories directly, since there is no tensor product on 2-categories/bicategories satisfying an equation like (2.1) if we replace 2-functors by pseudofunctors in the definition of $[-, -]$. The solution is to replace enrichment in monoidal categories by enrichment in multicategories — it turns out that there exists a multicategory structure on bicategories which behaves the way we want. The central definition is the following:

**Definition 2.1 ($n$-homomorphism)** Let $\mathbf{A}_1, \ldots, \mathbf{A}_n, \mathbf{B}$ be bicategories. An *n-homomorphism*

$$F : \mathbf{A}_1, \ldots, \mathbf{A}_n \to \mathbf{B}$$

is given by

- An object $F(A_1, \ldots, A_n) \in \mathbf{B}$ for each $n$-tuple $(A_1, \ldots, A_n)$ of objects with $A_i \in \mathbf{A}_i$.

- For each $1 \leq i \leq n$ and each $(n-1)$-tuple $(A_l)_{l \neq n}$ of objects with $A_i \in \mathbf{A}_i$ a *pseudofunctor*

$$F(A_1, \ldots, A_{i-1}, -, A_{i+1}, \ldots, A_n) : \mathbf{A}_i \to \mathbf{B}$$

  enriching the mapping on objects. We will often abbreviate this pseudofunctor by $F(-_i)$ omitting the constant objects.

- For all $1 \leq i < j \leq n$, all corresponding $(n-2)$-tuples of objects (suppressed in the notation), and all $f_i : A_i \to A'_i, f_j : A_j \to A'_j$ isomorphic

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[5] Note that the main complication does not arise from the replacement of 2-categories by bicategories, but rather from the replacement of 2-functors by pseudofunctors. In fact, we could have done all the definitions here using only 2-categories, as this is all we will need later, but I opted for bicategories, since it is closer to Verity's presentation, and the additional effort is negligible.

2-cells

$$F(A_i, A_j) \xrightarrow{\ F(A_i, f_j)\ } F(A_i, A'_j)$$

with vertical arrows $F(f_i, A_j)$ on the left, $F(f_i, A'_j)$ on the right, 2-cell $F(f_i, f_j)$ in the middle, and bottom arrow $F(A'_i, f_j): F(A'_i, A_j) \to F(A'_i, A'_j)$

such that

- The 1-cells $F(f_i)$ together with the 2-cells $F(f_i, f_j)$ give rise to pseudo-transformations of type

$$F(A_i, -_j) \to F(A'_i, -_j),$$

- The 1-cells $F(f_j)$ together with the 2-cells $F(f_i, f_j)$ give rise to pseudo-transformations of type

$$F(-_i, A_j) \to F(-_i, A'_j),$$

- For each triple $1 \le i < j < k \le n$, for all $f_i : A_i \to A'_i, f_j : A_j \to A'_j, f_k : A_k \to A'_k$ (and for all implicit $(n-3)$-tuples of objects), we have



$$\diamondsuit$$

Observe that a 0-homomorphism is just an object of $\mathbf{B}$, and a 1-homomorphism is a pseudofunctor.

The next step would be the definition of composition of $n$-homomorphisms, and the verification of the multicategory axioms. We won't give details here, since there are no surprises. The definition of composition is just 'what you would expect', and for the verification that the ensuing structure satisfies the axioms of a symmetric multicategory, we refer to Verity [29].

The following lemma is in analogy to (2.1).

**Lemma 2.2** *Let* $\mathbf{A}_1, \ldots, \mathbf{A}_{n+1}, \mathbf{B}$ *be bicategories. There are natural bijections*

$$\mathbf{hom}(\mathbf{A}_1, \ldots, \mathbf{A}_{n+1}; \mathbf{B}) \cong \mathbf{hom}(\mathbf{A}_1, \ldots, \mathbf{A}_n; [\![\mathbf{A}_{n+1}, \mathbf{B}]\!]),$$

*where* $\mathbf{hom}$ *denotes sets of $n$-homomorphisms, and* $[\![-, -]\!]$ *denotes the bicategory of pseudofunctors, pseudo-transformations and modifications.*

A bicategory enriched category is now just given by a set $\mathbf{X}_0$ of objects, for each pair $X, Y$ of objects a bicategory $\mathbf{X}(X, Y)$, identity 0-homomorphisms $\mathrm{id}_X$ (which are just objects of $\mathbf{X}(X, X)$), and composition 2-homomorphisms

$$\mathrm{comp}_{\mathrm{X,Y,Z}} : [\mathbf{X}(X, Y), \mathbf{X}(Y, Z)] \longrightarrow \mathbf{X}(X, Z),$$

subject to *strict* associativity and identity axioms. In bicategory enriched categories, we call the 0-, 1-, and 2-cells of the bicategories $\mathbf{X}(X, Y)$ *1-, 2-, and 3-cells* of the bicategory enriched category, respectively, and we denote horizontal composition of 1-, 2- and 3-cells by juxtaposition (i.e. $\mathrm{comp}_{X,Y,Z}(f, g) = gf$), vertical composition of 2- and 3-cells by $(- \circ -)$, and depth-wise composition of 3-cells by $(- \cdot -)$. For $\eta : f \to f'$ in $\mathbf{X}(X, Y)$ and $\theta : g \to g'$ in $\mathbf{X}(Y, Z)$ we denote the exchange isomorphism for horizontal composition by

$$\theta\eta : \theta f' \circ g\eta \xrightarrow{\cong} g'\eta \circ \theta f.$$

In pasting form this looks like

$$
\begin{array}{ccc}
gf & \xrightarrow{g\eta} & gf' \\
{\scriptstyle \theta f} \downarrow & \Swarrow_{\theta\eta} & \downarrow {\scriptstyle \theta f'} \\
g'f & \xrightarrow[g'\eta]{} & g'f'
\end{array} \quad ,
$$

and in string diagrams we denote exchange isomorphisms by braidings[6]

$$
\begin{array}{cc}
g'\eta & \theta f \\
& \\
\theta f' & g\eta
\end{array} \quad .
$$
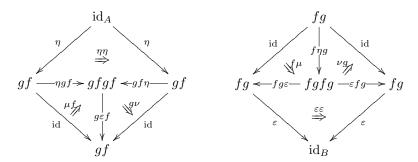
Now that we know what a bicategory enriched category is, we can finally introduce the desired abstract notion of biadjunction.

**Definition 2.3** Let $\mathbf{X}$ be a bicategory enriched category, and let $A, B$ be objects of $\mathbf{X}$. A biadjunction between $A$ and $B$ is given by

- 1-cells $\qquad\qquad f : A \to B \qquad\qquad g : B \to A,$
- 2-cells $\qquad\qquad \eta : \mathrm{id}_A \to gf \qquad\quad \varepsilon : fg \to \mathrm{id}_B$
- *invertible* 3-cells $\quad \mu : \mathrm{id}_g \xrightarrow{\cong} g\varepsilon \circ \eta g \qquad \nu : \varepsilon f \circ f\eta \xrightarrow{\cong} \mathrm{id}_f$

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[6]The notation as a braiding is motivated by thinking about bicategory enriched categories in a three dimensional way (the string diagrams that we use and that live 'locally' in a two dimensional section $\mathbf{X}(X, Y)$ of a bicategory enriched category can actually be viewed as projections of surface diagrams), but as mentioned earlier we don't want to talk too much about string diagrams themselves, so for us the notation as braiding is just a definition of a shorthand for a pasting diagram denoting a 2-cell in $\mathbf{X}(X, Z)$.

such that the diagrams



of isomorphic 3-cells compose to identities in $\mathbf{X}(A, A)$ and $\mathbf{X}(B, B)$, respectively. Note that strictly speaking, these diagrams are not well typed, as e.g. the domain of $\mu f$ is not $\mathrm{id}_{gf}$, but $\mathrm{id}_g f$, and horizontal composition is only pseudofunctorial. We omit the constraint isomorphisms since they are easy to fill in, and the diagrams are clearer and easier to memorize in this form. $\diamond$

For reference, here are the axioms for biadjoints in string diagrammatic notation:



$$(2.2)$$

Observe that they are rotated and reflected relative to the pasting diagrams to conform with our convention for the orientation of string diagrams. Furthermore, as for the pasting diagram version there are some hidden constraint isomorphisms, since e.g. the 3-cell $g\nu$ has type $g(\varepsilon f \circ f\eta) \to g\,\mathrm{id}_f$, but its environment in the diagram expects the type $g\varepsilon f \circ gf\eta \to \mathrm{id}_{gf}$.

We remark that Verity does not require the axioms in his definition of biadjunction. He calls a biadjunction that additionally satisfies the axioms a *locally adjoint biadjunction*.

Since we want to use biadjunctions to characterize things, we attach great value to the following lemma, which is a categorification of the fact that adjoints are unique up to isomorphism.
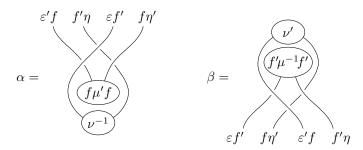
**Lemma 2.4** *Let* $\mathbf{X}$ *be a bicategory enriched category, and let*

$$(f \dashv g : B \to A, \eta, \varepsilon, \mu, \nu) \quad and \quad (f' \dashv g : B \to A, \eta', \varepsilon', \mu', \nu')$$

*be two biadjunctions sharing the same right adjoint* $g$. *Then* $f$ *and* $f'$ *are equivalent.*

*Proof.* The 2-cells between $f$ and $f'$ are given by $\varepsilon f' \circ f\eta' : f \to f'$ and $\varepsilon' f \circ f'\eta : f' \to f$. The fact that they are mutually inverse equivalences is witnessed by the isomorphic 3-cells $\alpha : \mathrm{id}_f \xrightarrow{\cong} \varepsilon' f \circ f'\eta \circ \varepsilon f' \circ f\eta'$ and $\beta : \varepsilon f' \circ f\eta' \circ \varepsilon' f \circ f'\eta \xrightarrow{\cong} \mathrm{id}'_f$

which are defined as

$$\alpha = \qquad \beta =$$

The interested reader is invited to prove that this equivalence is even an *adjoint* equivalence. ∎

## 2.2 Pre-equipments

We will now introduce the bicategory enriched category $\mathfrak{Spec}$ of pre-equipments and special functors and have a closer look on its biadjunctions, which we call *special* biadjunctions. This concept is the goal of our higher dimensional 'digressions' — we will later characterize the tripos-to-topos construction as a special biadjunction between triposes and toposes.

A pre-equipment is almost the same as what Verity calls a *weak proarrow equipment* (almost, since he doesn't have the closedness condition under vertical isomorphisms and furthermore he considers bicategories, not 2-categories), but the bicategory enriched categories that he considers are bigger, because his notions of morphisms and transformations (he studies several of them) are more general than the one we are interested in. We will elaborate on this after giving our definitions.

**Definition 2.5**  1. A *pre-equipment* is given by a 2-category $\mathbf{C}$ together with a designated subcategory $\mathbf{C}_r$ of the 1-cells which is closed under vertical isomorphisms.

   We think of the 1-cells in $\mathbf{C}_r$ as particularly 'nice' arrows and we call them *regular 1-cells*.

   We call a pre-equipment *geometric*[7] if all left adjoints in it are regular.

2. A *special functor* between pre-equipments $\mathbf{C}$ and $\mathbf{D}$ is an oplax functor $F : \mathbf{C} \to \mathbf{D}$ such that $Ff$ is a regular 1-cell whenever $f$ is a regular 1-cell, all identity constraints $F\mathrm{id}_A \to \mathrm{id}_{FA}$ are invertible, and the composition constraints $F(gf) \to Fg\,Ff$ are invertible whenever $g$ is a regular 1-cell.

3. A *special transformation* between special functors $F, G$ is an oplax (see Section 1.2) transformation $\eta : F \to G$ such that all $\eta_A$ are regular 1-cells and the naturality constraint $\eta_B\,Ff \to Gf\,\eta_A$ is invertible whenever $f$ is a regular 1-cell.

---

[7]The 'geometric' refers to *geometric morphism*. We view adjunctions in pre-equipments as geometric morphisms, and Lemma 2.9 says that these are preserved by special functors between geometric pre-equipments.

Every pre-equipment $\mathbf{C}$ gives rise to a double category $\widetilde{\mathbf{C}}$ where the vertical 1-cells are the 1-cells of $\mathbf{C}$, the horizontal arrows are the 1-cells of $\mathbf{C}_r$, and

$$
\text{2-cells} \quad
\begin{array}{ccc} A & \!\!\to\!\! & B \\ \downarrow & \!\!\searrow\!\! & \downarrow \\ C & \!\!\to\!\! & D \end{array}
\quad \text{in } \widetilde{\mathbf{C}} \text{ are 2-cells} \quad
\begin{array}{ccc} A & \!\!\to\!\! & B \\ \downarrow & \!\!\nearrow\!\! & \downarrow \\ C & \!\!\to\!\! & D \end{array}
\quad \text{in } \mathbf{C}.
$$

In Verity's bicategory enriched categories of equipments (see [29, Sections 1.4 and 1.5]), the 1-cells are certain double functors between these induced double categories, which are 'strong' in horizontal direction and lax or oplax in vertical direction. Special functors in the sense of the previous definition give rise to this kind of double functors, but not every double functor comes from a special functor. This is because a 1-cell in $\mathbf{C}_r$ appears in $\widetilde{\mathbf{C}}$ as a horizontal and a vertical cell, but these two need not to be mapped to the same or isomorphic 1-cells by a double functor $F : \widetilde{\mathbf{C}} \to \widetilde{\mathbf{D}}$ in the sense of Verity.

Now, we want to prove that pre-equipments, special functors, special transformations and modifications form a bicategory enriched category. If we wanted to minimize our effort in doing so, we could just prove that the morphisms and transformations that we consider are special cases of Verity's *comorphisms* and transformations, and are closed under composition.

However, to present a more closed flow of ideas, we prefer to describe the steps which are necessary to establish directly that the given definitions give rise to a bicategory enriched category. Since the proofs are for the most part straightforward once you know what to do, we do not prove every little detail, but only remark on subtleties and important points.

It is well known that oplax functors, oplax transformations, and modifications between 2-categories $\mathbf{A}, \mathbf{B}$ form a 2-category $\mathfrak{Oplax}(\mathbf{A}, \mathbf{B})$ (see e.g. [17, Section 2.0]). Moreover, it is easy to verify that special transformations are closed under composition, and thus for pre-equipments $\mathbf{C}, \mathbf{D}$, there is a locally full sub-2-category $\mathfrak{Spec}(\mathbf{C}, \mathbf{D})$ of $\mathfrak{Oplax}(\mathbf{C}, \mathbf{D})$ which consists of special functors, special transformations and modifications. For pre-equipments $\mathbf{C}, \mathbf{D}, \mathbf{E}$, we have to define composition 2-homomorphisms
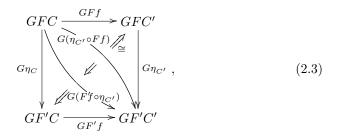
$$
\mathrm{comp}_{\mathbf{C}, \mathbf{D}, \mathbf{E}} : \mathfrak{Spec}(\mathbf{C}, \mathbf{D}), \mathfrak{Spec}(\mathbf{D}, \mathbf{E}) \longrightarrow \mathfrak{Spec}(\mathbf{C}, \mathbf{E}).
$$

For special functors $F, G$, $\mathrm{comp}(F, G)$ is just the composition of oplax functors (which is again special as is easily seen), and the definition of the pseudofunctors $\mathrm{comp}(F, -)$ is also straightforward. Postcomposition $\mathrm{comp}(-, G)$ is more interesting. Crucial here is the observation that in the world of pseudofunctors and pseudo-transformations, every $G \in \mathfrak{Pseudo}(\mathbf{D}, \mathbf{E})$ induces a pseudofunctor

$$
\mathrm{comp}(-, G) : \mathfrak{Pseudo}(\mathbf{C}, \mathbf{D}) \to \mathfrak{Pseudo}(\mathbf{C}, \mathbf{E}),
$$

but this does *not* generalize to oplax functors and transformations. The reason is that for a pseudo-transformation $\eta : F \to F' : \mathbf{C} \to \mathbf{D}$ and $f : C \to C'$, the

constraint 2-cell $G\eta_f$ is defined by the pasting diagram

$$
\begin{array}{ccc}
GFC & \xrightarrow{\ \ GFf\ \ } & GFC' \\
 & G(\overline{\eta_{C'}\circ Ff})\ \nearrow\!\!\!\!/\ {\cong} & \\
G\eta_C\ \Big\downarrow & \Swarrow & \Big\downarrow\ G\eta_{C'} \\
 & \nearrow\!\!\!\!/\ G(\overline{F'f\circ\eta_{C'}}) & \\
GF'C & \xrightarrow[\ \ GF'f\ \ ]{} & GF'C'
\end{array}
\qquad (2.3)
$$

but this only makes sense if the upper right composition constraint is invertible. Now this is not the case in general for oplax functors and transformations, but it *is* whenever $G$ and $\eta$ are special since then $\eta_{C'}$ is regular, and this implies the invertibility of the composition constraint by the definition of special functor. Postcomposition $G\alpha$ of modifications with functors is easy again, and in the end, $\mathrm{comp}(-,G)$ is a pseudofunctor for the same reasons as it is in the pseudo case.

To make $\mathrm{comp}(-,-)$ into a 2-homomorphism, we still have to define modifications $\theta\eta : G'\eta \circ \theta F \to \theta F' \circ G\eta$ for special transformations $\eta : F \to F'$ and $\theta : G \to G'$ and check that they have the desired properties. There is only one way to do define $\theta\eta$ — given $C \in \mathbf{C}$, we define the component of $\theta\eta$ at $C$ as the constraint cell of $\theta$ at $\eta_C$, i.e. $(\theta\eta)_C = \theta_{\eta_C}$. It follows from the fact that $\eta$ and $\theta$ are special that this is an isomorphism, and we leave the remaining verification that the such defined 2-cells give rise to pseudo-natural transformations

$$\mathrm{comp}(\eta,-) : \mathrm{comp}(F,-) \to \mathrm{comp}(F',-)$$

and

$$\mathrm{comp}(-,\theta) : \mathrm{comp}(-,G) \to \mathrm{comp}(-,G')$$

to the reader.

The identities of our bicategory enriched category are just given by identity-2-functors $\mathrm{id}_{\mathbf{C}} \in \mathfrak{Spec}(\mathbf{C},\mathbf{C})$, and the verifications of associativity and identity axioms do not bear any surprises either. We are thus able to state:

**Lemma 2.6** *Pre-equipments, special functors, special transformations and modifications together form a bicategory enriched category* $\mathfrak{Spec}$. ∎

As a first example, we define the pre-equipment of toposes. In Sections 3 and 4, we will furthermore introduce the pre-equipments **Trip** and **QTop** of triposes and q-toposes, respectively.

**Example 2.7** The pre-equipment **Top** of toposes has the 2-category of toposes, finite limit preserving functors and arbitrary natural transformations as underlying 2-category, and regular (i.e. epi preserving) functors as regular 1-cells. **Top** is a geometric pre-equipment, since epimorphisms are preserved by left adjoints. ◇

We call a biadjunction in $\mathfrak{Spec}$ a *special biadjunction* (we do not use the term biadjunction of pre-equipments since this expression should be reserved for the more general double categorical notion). Special biadjunctions enjoy the following interesting property.

**Lemma 2.8** *Let $F \dashv U : \mathbf{D} \to \mathbf{C}$ be a special biadjunction. Then $U$ is a strong functor.* ∎

A proof of a more general lemma (with double functors instead of special functors) appears in [29]. Similar results are also known for monoidal categories and double categories, the first systematic treatment of this phenomenon (for lax morphisms of pseudo-algebras) is [15, Theorem 1.5]. We do not prove the lemma here, because we don't really need it — for the biadjunctions that we consider we know that the right adjoints are pseudofunctors anyway. But we chose to mention the statement, since it somehow fits into the picture: The right adjoints in our case are forgetful functors — there is no reason for them to have non-invertible composition constraints. The left adjoints, however, are free constructions which naturally have more 'degrees of freedom', which in a sense justifies them being oplax.

Finally, some remarks about *geometric* pre-equipments. Wood's *proarrow equipments* have the property that all 1-cells of the designated subcategory have right adjoints, and this is the important property for abstract category theory, for which proarrow equipments were introduced, since it allows to give an abstract treatment of phenomena related to contravariance.

In the pre-equipments that we consider, the reverse inclusion holds, i.e. every left adjoint is in the designated subcategory (for example inverse image parts of geometric morphisms are regular), and we call a pre-equipment having this property *geometric*. The following lemma is an easy observation about special functors between geometric pre-equipments.

**Lemma 2.9** *Let $\mathbf{C}$ be a geometric pre-equipment, and let $F : \mathbf{C} \to \mathbf{D}$ be a special functor. If $(f \dashv u : B \to A, \eta, \varepsilon)$ is an adjunction in $\mathbf{C}$, then $(Ff \dashv Fu : FB \to FA, \phi_{u,f} \, F\eta \, \phi_{\mathrm{id}_A}^{-1}, \phi_{\mathrm{id}_B} \, F\varepsilon \, \phi_{f,u}^{-1})$ is an adjunction in $\mathbf{D}$ (By $\phi$ we denote the identity and composition constraints of $F$).*

*If $f \dashv u$ is a reflection (i.e., has isomorphic counit), then so is $Ff \dashv Fu$.* ∎

## 2.3   Equipments and related concepts in the literature

Proarrow equipments were first introduced by Wood in [31] as a framework for abstract category theory.

Lax functors between double categories were considered, besides by Verity, in the work of Grandis and Paré [9]. Their definitions are a bit different since they use pseudo-double categories instead of double bicategories, as Verity does. Shulman gives yet another — a bit more restrictive and therefore easier and shorter — variant of the definitions in his work on framed bicategories [24]. Shulman's bicategories are 'pseudocategories' in **Cat** (just like monoidal categories are pseudomonoids in **Cat**), thus they have strict composition vertically and bicategorical composition horizontally. Furthermore, his lax double functors commute with vertical composition on the nose.

The notion of special transformation appeared (without name) already in 1993 in Johnstone's [13, Lemma 1.1] where it is used to define a less general version of what we call special biadjunction, under the name *semi-oplax adjunction*. This article is notable since it was a starting point for the present work.

We adopted the adjective *special* from [5], where Day, McCrudden and Street define special functors for those pre-equipments in which the regular arrows coincide with the left adjoints.

**Vertical and horizontal**

When associating a double category to a pre-equipment, we have a choice to make, namely whether we want to view the regular 1-cells as horizontal or vertical 1-cells in the double category. In the context of general double-categories, this correpsonds to the question whether we want to view lax double functors as strong on the horizontal or on the vertical 2-category.

The first convention, where regular 1-cells are horizontal and lax double functors are horizontally strong, is used by Grandis, Paré and Verity, whereas for Shulman, regular 1-cells are horizontal, and lax double functors are vertically strong (and even strict).

We do not use double categories explicitly, but the fact that we usually draw the components of natural transformations vertically in naturality squares corresponds the convention used by Shulman (since components of special transformations are regular).

## 3  Triposes

In one sentence, triposes are fibrational models of non-extensional intuitionistic higher order logic. For a general introduction to fibrations in categorical logic and their internal language, we refer to Jacobs' book [12]. We will now give the definition of tripos; what it means for a tripos to be a model of higher order logic will be explained in Section 3.1.

**Definition 3.1** A *Heyting algebra* is a partial order that is bicartesian closed as a category. More explicitly, it is a poset with finite meets, finite joins and an operation $(- \Rightarrow -)$ universally characterized by

$$\varphi \wedge \psi \leq \gamma \quad \text{iff} \quad \varphi \leq \psi \Rightarrow \gamma. \qquad \qquad \diamondsuit$$

**Definition 3.2** Let $\mathbb{C}$ be a category with finite products. A *tripos over* $\mathbb{C}$ is a fibration
$$\mathcal{P} : \mathbb{X} \to \mathbb{C}$$
such that

1. All fibers of $\mathcal{P}$ are Heyting algebras.

2. Reindexing along morphisms in $\mathbb{C}$ preserves all structure of Heyting algebras.

3. For every $f : A \to B$ in $\mathbb{C}$, the reindexing map $f^* : \mathcal{P}_B \to \mathcal{P}_A$ has left and right adjoints
$$\exists_f \dashv f^* \dashv \forall_f,$$
   such that for every pair $f : A \to B, \quad g : X \to Y$ of morphisms in $\mathbb{C}$ and all $\varphi \in \mathcal{P}_{B \times X}$, we have

$$Q_{A \times g} \left( (f \times X)^* \varphi \right) = (f \times Y)^* (Q_{B \times g} \varphi), \tag{3.1}$$

17

where $Q$ is either $\forall$ or $\exists$.

4. $\mathcal{P}$ has *weak power objects*, i.e., for every $A \in \mathbb{C}$ there is an object $\mathfrak{P}A \in \mathbb{C}$ and a predicate $(\ni_A) \in \mathcal{P}_{\mathfrak{P}A \times A}$ such that for all predicates $\varphi \in \mathcal{P}_{C \times A}$ we are given a map[8] $\chi_A(\varphi) : C \to \mathfrak{P}A$ such that $\varphi = (\chi_A(\varphi) \times A)^*(\ni_A)$, which is written diagramatically[9] as

$$\varphi \rightsquigarrow \ni_A$$

$$C \times A \xrightarrow{\chi_A(\varphi) \times A} \mathfrak{P}(A) \times A \tag{3.2}$$

$\diamondsuit$

**Remark 3.3** The third clause of Definition 3.2 requires some clarifications. Condition (3.1) is the *Beck-Chevalley condition*, we will abbreviate it by (BC). It is usually stated in the form

"For every pullback square $\begin{array}{ccc} P & \xrightarrow{q} & B \\ p\downarrow & \ulcorner & \downarrow g \\ A & \xrightarrow{f} & C \end{array}$ in $\mathbb{C}$ and all $\varphi \in \mathcal{P}_A$, we have

$Q_q(p^*\varphi) \cong g^*(Q_f(\varphi))$ where $Q$ is either of $\forall, \exists$."

This definition is not appropriate in our setting, since we only assume $\mathbb{C}$ to have products (not arbitary finite limits), and in Section 3.3 we consider functors between the bases of triposes which only preserve finite products. The deep reason why we have to abandon general finite limits in favor of finite products will become apparent in Section 5.1.3 — the functors $D_{\mathcal{P}}$ defined there only preserve products.

It has long been observed that the full strength of the classical Beck-Chevalley condition is not necessary to ensure soundness of the interpration of logical systems. Jacobs [12] gives definitions using the pullback squares

$$\begin{array}{ccc} A \times X & \xrightarrow{A \times !} & A \\ f \times X \downarrow & & \downarrow f \\ B \times X & \xrightarrow{B \times !} & B \end{array} \quad \text{and} \quad \begin{array}{ccc} A \times X & \xrightarrow{A \times \delta} & A \times X \times X \\ f \times X \downarrow & & \downarrow f \times X \times X \\ B \times X & \xrightarrow{B \times \delta} & B \times X \times X \end{array} \tag{3.3}$$

which are definable from finite product structure [12, Definitions 1.9.1 and 3.4.1]. (3.1) corresponds to the classical condition restricted to squares of the form

$$\begin{array}{ccc} A \times X & \xrightarrow{A \times g} & A \times Y \\ f \times X \downarrow & & \downarrow f \times Y \\ B \times X & \xrightarrow{B \times g} & B \times Y \end{array} . \tag{3.4}$$

---

[8]This map is not supposed to be uniquely determined by the stated property, but we assume that the tripos is equipped with a choice of such maps.

[9]It has to be explained how to read diagram (3.2). Here, we are using a notation that is very common for fibrations — by drawing one object above another, e.g. $\varphi$ over $C \times A$, we assert that $\varphi$ is in the fiber over $C \times A$, i.e. $\mathcal{P}(\varphi) = C \times A$, and in the same way for morphisms. We use wavy arrows $\rightsquigarrow$ to denote cartesian morphisms. Thus, the diagram says that $\varphi$ is the cartesian lifting of $\ni_A$ along $\chi_A(\varphi) \times A$.

The class of squares of this form encompasses all squares used by Jacobs, in a slightly more concise way. Moreover, it expresses precisely the desired property, namely the commutation of substitution and (generalized) quantification.

As a side note, Jacobs' version of (BC) and (3.1) are equivalent, i.e., any fibration of Heyting algebras with $\exists$ satisfying (BC) for squares of the forms (3.3) already satisfies (BC) for all squares of the form (3.4). This can be seen by analyzing the proofs of the substitution lemma 3.7 and the soundness theorem 3.9 below. They only require (BC) for the squares (3.3) as hypothesis, but using the internal logic we can prove (BC) for all squares of the form (3.4). In fact, in our setting even Jacobs' set of squares is redundant — the condition for the right square in (3.3) can be derived using the equivalence $(\exists_\delta \varphi)(x, y) \dashv\vdash \varphi(x) \wedge x = y$. This equivalence is a consequence of the Frobenius law, which in turn follows from the existence of implication.

An advantage of the phrasing (3.1) of (BC) is that it does not rely on projections and diagonals, and thus is still meaningful in a monoidal setting. Indeed, Shulman proves the monoidal version of the condition for certain monoidal fibrations in [24, Corollary 16.4]. $\diamond$

## 3.1 Interpreting higher order logic in triposes

In this section, we explain how to interpret languages of higher order logic in triposes. This provides the basis for the *internal language* of a tripos, to be presented in the next section. Jacobs' book [12] gives a careful exposition of how to interpret different systems of predicate logic in fibrations, but for reasons of self-containedness, and because the internal language of a tripos will be a central tool in the following, we give a detailed and explicit description of the system that we use, how it can be interpreted in a tripos. Then, in the next section, we explain how the internal language — which is the language that we get for the 'maximal' choice of signature — can be used to reason and calculate in a tripos.

**Definition 3.4** A *signature* for a language of many sorted higher logic is given by a triple $\Sigma = (\mathbf{S}, \mathbf{F}, \mathbf{R})$ where

- $\mathbf{S}$ is a set of *base types*,

- $\mathcal{T}(\mathbf{S})$ is the set of *higher order types* generated by $\mathbf{S}$, that is the smallest set that contains all elements of $\mathbf{S}$ and is closed under the inductive clauses

  - $1 \in \mathcal{T}(\mathbf{S})$
  - $A, B \in \mathcal{T}(\mathbf{S}) \quad \Rightarrow \quad A \times B \in \mathcal{T}(\mathbf{S})$
  - $A \in \mathcal{T}(\mathbf{S}) \quad \Rightarrow \quad \mathbf{P}(A) \in \mathcal{T}(\mathbf{S})$,

- $\mathbf{F} = (\mathbf{F}_{\Delta,A} \ ; \ \Delta \in \mathcal{T}(\mathbf{S})^*, A \in \mathcal{T}(\mathbf{S}))$ is a family of sets of *function symbols* ($\mathcal{T}(\mathbf{S})^*$ is the set of lists of higher order types), where for $(A_1, \ldots, A_n) \equiv \Delta \in \mathcal{T}(\mathbf{S})^*$ and $A \in \mathcal{T}(\mathbf{S})$ we view an $f \in \mathbf{F}_{\Delta,A}$ as a function of type $f : A_1 \times \cdots \times A_n \to A$.

- $\mathbf{R} = (\mathbf{R}_\Delta \ ; \ \Delta \in \mathcal{T}(\mathbf{S})^*)$ is a family of sets of *relation symbols*, where for $\Delta \in \mathcal{T}(\mathbf{S})^*$, we view $R \in \mathbf{R}_\Delta$ as a relation of arity $\Delta$. $\diamond$

**Terms:**

$$\frac{}{\Delta \mid x_i : A_i}\qquad \frac{\Delta \mid t_i : B_i \quad (1 \le i \le n)}{\Delta \mid f(t_1,\ldots,t_n) : C}\; f \in \mathbf{F}_{(B_1,\ldots,B_n),C}$$

$$\frac{\Delta \mid t : B_1 \times B_2}{\Delta \mid \pi_i(t) : B_i}\; i = 1,2$$

**Formulas:**

$$\frac{}{\Delta \mid \top}\qquad\qquad \frac{}{\Delta \mid \bot}\qquad\qquad \frac{\Delta \mid \varphi \qquad \Delta \mid \psi}{\Delta \mid \varphi \wedge \psi}$$

$$\frac{\Delta \mid \varphi \qquad \Delta \mid \psi}{\Delta \mid \varphi \vee \psi}\qquad \frac{\Delta \mid \varphi \qquad \Delta \mid \psi}{\Delta \mid \varphi \Rightarrow \psi}\qquad \frac{\Delta \mid s : B \qquad \Delta \mid t : B}{\Delta \mid s = t}$$

$$\frac{\Delta, y{:}B \mid \varphi[y]}{\Delta \mid \exists y{:}B\,.\,\varphi[y]}\qquad \frac{\Delta, y{:}B \mid \varphi[y]}{\Delta \mid \forall y{:}B\,.\,\varphi[y]}\qquad \frac{\Delta \mid s : B \qquad \Delta \mid t : \mathbf{P}B}{\Delta \mid s \in t}$$

$$\frac{\Delta \mid t_i : B_i \quad (i = 1,\ldots,n)}{\Delta \mid R(t_1,\ldots,t_n)}\; \begin{array}{l} R \in \mathbf{R} \\ a_{\mathbf{R}}(R) = (B_1,\ldots,B_n) \end{array}$$

$\Delta \equiv x_1{:}A_1,\ldots,x_n{:}A_n$ denotes a context of typed variables.

Table 1: Terms and formulas in context over a signature $\Sigma = (\mathbf{S}, \mathbf{F}, \mathbf{R})$

From a signature $\Sigma$, we can inductively generate *terms* and *formulas*. To be able to define the semantics later, we have to keep track of free variables explicitly, using *contexts*. A context is a list $\Delta \equiv x_1 : A_1,\ldots,x_n : A_n$ of variable declarations, where $A_i \in \mathcal{T}(\mathbf{S})$. We will write terms and formulas in context as $(\Delta \mid t : B)$ and $(\Delta \mid \varphi)$ (the symbol $\vdash$ is reserved for the entailment relation between formulas). Table 1 gives the inductive clauses for terms and formulas. For reasons of conciseness and better readability, we will often supress contexts from the notation for terms and formulas, and types from the notation for formulas. We call the collections of terms and of formulas generated from $\Sigma$ together *the language generated by* $\Sigma$, and denote it by $\mathcal{L}(\Sigma)$.

**Definition 3.5 (Interpretation)** Given a signature $\Sigma = (\mathbf{S}, \mathbf{F}, \mathbf{R})$ and a tripos $\mathcal{P} : \mathbb{X} \to \mathbb{C}$, we may define an *interpretation* of $\mathcal{L}(\Sigma)$ in $\mathcal{P}$. This works as follows.

- To each base type symbol $S \in \mathbf{S}$, we associate an object $[\![S]\!] \in \mathbb{C}$ in the base of the tripos.

- We inductively extend this assignment to higher order types using the rules $[\![1]\!] = 1$, $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$ and $[\![\mathbf{P}A]\!] = \mathfrak{P}[\![A]\!]$.

- The interpretation of a list of types is given by

$$[\![A_1,\ldots,A_n]\!] = [\![A_1]\!] \times \cdots \times [\![A_n]\!]$$

The interpretation of a context $\Delta$ is the interpretation of the associated list of types (obtained by syntactically removing the variables).

- To each function symbol $f \in \mathbf{F}_{(A_1,\ldots,A_n),B}$ we associate a morphism $[\![f]\!] : [\![A_1,\ldots,A_n]\!] \to [\![B]\!]$ in $\mathbb{C}$.

- To each relation symbol $R \in \mathbf{R}_{(A_1,\ldots,A_n)}$ we associate a predicate $[\![R]\!] \in \mathcal{P}_{[\![A_1,\ldots,A_n]\!]}$.

- Now we can inductively define the semantics of terms by

$$[\![\Delta \mid x_i : A_i]\!] = \pi_i$$
$$[\![\Delta \mid f(t_1,\ldots,t_n) : B]\!] = [\![f]\!] \circ \langle [\![\Delta \mid t_1 : A_1]\!], \ldots, [\![\Delta \mid t_n : A_n]\!] \rangle$$

and of formulas by

$$[\![\Delta \mid \top]\!] = \top \in \mathcal{P}_{[\![\Delta]\!]}$$
$$[\![\Delta \mid \bot]\!] = \bot \in \mathcal{P}_{[\![\Delta]\!]}$$
$$[\![\Delta \mid \varphi \wedge \psi]\!] = [\![\Delta \mid \varphi]\!] \wedge [\![\Delta \mid \psi]\!]$$
$$[\![\Delta \mid \varphi \vee \psi]\!] = [\![\Delta \mid \varphi]\!] \vee [\![\Delta \mid \psi]\!]$$
$$[\![\Delta \mid \varphi \Rightarrow \psi]\!] = [\![\Delta \mid \varphi]\!] \Rightarrow [\![\Delta \mid \psi]\!]$$
$$[\![\Delta \mid s = t]\!] = \langle [\![\Delta \mid s : B]\!], [\![\Delta \mid t : B]\!] \rangle^*(\mathsf{eq}_{[\![B]\!]})$$
$$[\![\Delta \mid \exists y{:}\,B\,.\,\varphi[y]]\!] = \exists_{\pi^-} [\![\Delta, x{:}\,B \mid \varphi[x]]\!]$$
$$[\![\Delta \mid \forall y{:}\,B\,.\,\varphi[y]]\!] = \forall_{\pi^-} [\![\Delta, x{:}\,B \mid \varphi[x]]\!]$$
$$[\![\Delta \mid s \in t]\!] = \langle [\![\Delta \mid t : \mathbf{P}B]\!], [\![\Delta \mid s : B]\!] \rangle^*(\ni_{[\![B]\!]})$$
$$[\![\Delta \mid R(t_1,\ldots,t_n)]\!] = \langle [\![\Delta \mid t_1 : B_1]\!], \ldots, [\![\Delta \mid t_n : B_n]\!] \rangle^*([\![R]\!])$$

In the line for equality we use the notation $\mathsf{eq}_A = \exists_{\delta_A}(\top)$ where $\delta_A : A \to A \times A$ is the diagonal. In the clauses for existential and universal quantification, $\pi^-$ denotes the projection of type $[\![\Delta, B]\!] \to [\![\Delta]\!]$.

Observe that the interpretation of terms and formulas is compatible with types, i.e.,

$$[\![\Delta \mid t : A]\!] : [\![\Delta]\!] \to [\![A]\!] \quad \text{and} \quad [\![\Delta \mid \varphi]\!] \in \mathcal{P}_{[\![\Delta]\!]}. \qquad \diamondsuit$$

For the remainder of this section, $\Sigma = (\mathbf{S}, \mathbf{F}, \mathbf{R})$ is a fixed signature with a fixed interpretation $[\![-]\!]$ in a tripos $\mathcal{P} : \mathbb{X} \to \mathbb{E}$. Terms and formulas will always be terms and formulas generated from $\Sigma$.

**Convention 3.6** If $(\Delta \mid \psi)$ is a formula such that $[\![\Delta \mid \psi]\!] = \top$, then we say that $\psi$ *holds in* $\mathcal{P}$. More generally, if $[\![\varphi_1]\!] \wedge \cdots \wedge [\![\varphi_n]\!] \leq [\![\psi]\!]$ holds for formulas $\varphi_1,\ldots,\varphi_n,\psi$ in context $\Delta$, then we say that the judgment[10]

$$\Delta \mid \varphi_1,\ldots,\varphi_n \vdash \psi$$

holds in $\mathcal{P}$. $\qquad \diamondsuit$

The most important properties of $[\![-]\!]$ are the *substitution lemma* and the *soundness theorem*, stated now.

---

[10]As for terms and formulas, we will often suppress the context from the notation for judgments.

**Lemma 3.7 (Substitution lemma for triposes)** *Let $(\Delta \mid s_i : B_i), 1 \leq i \leq n$ and $(\Delta' \mid t[y_1, \ldots, y_n] : C)$ be terms and let $(\Delta' \mid \varphi[y_1, \ldots, y_n])$ be a formula, where $\Delta' = y_1 : B_1, \ldots, y_n : B_n$. Then we have*

1. $[\![\Delta \mid t[s_1, \ldots, s_n]]\!] = [\![\Delta' \mid t[y_1, \ldots, y_n]]\!] \circ \langle [\![\Delta \mid s_1]\!], \ldots, [\![\Delta \mid s_n]\!] \rangle$

2. $[\![\Delta \mid \varphi[s_1, \ldots, s_n]]\!] = \langle [\![\Delta \mid s_1]\!], \ldots, [\![\Delta \mid s_n]\!] \rangle^*([\![\Delta' \mid \varphi[y_1, \ldots, y_n]]\!])$ ∎

The soundness theorem says that the interpretation that we described is compatible with derivability of judgments in some logical system, thus we have to clarify which logical system we use before stating it.

**Definition 3.8 Non-extensional higher order intuitionistic logic** is intuitionistic predicate logic with explicit contexts of variables, formalized e.g. in natural deduction, with the additional axiom

$$\Delta, x : A, y : B \mid \Gamma \vdash \exists! z : A \times B \,.\, \pi_1(z) = x \wedge \pi_2(z) = y$$

for product types, and the additional *comprehension scheme*

$$\Delta \mid \Gamma \vdash \exists m : \mathbf{P}B \, \forall y \in B \,.\, y \in m \leftrightarrow \varphi[y]$$

for power types, where $(\Delta, y \in B \mid \varphi[y])$ is an arbitrary formula.

Since the explicit handling of variable contexts $\Delta$ is not contained in the standard presentation of intuitionistic logic, we give a complete natural deduction system in Appendix A. ◇

Now the soundness theorem can be phrased as follows.

**Theorem 3.9 (Soundness theorem for triposes)** *If the judgment*

$$\Delta \mid \varphi_1, \ldots, \varphi_n \vdash \psi$$

*is derivable in non-extensional higher order intuitionistic logic, then it holds in $\mathcal{P}$.* ∎

The substitution lemma and the soundness theorem are proved by induction on the structure of formulas/terms, and proofs respectively. This is fairly standard and straightforward, similar proofs can be found in [12].

A direct consequence of the soundness theorem is that if we have a theory over $\Sigma$ generated by a given set of axioms such that all the axioms hold in $\mathcal{P}$, then any statement that can be derived from the axioms does also hold in $\mathcal{P}$.

## 3.2 The internal language of a tripos

**Definition 3.10 (The internal language of a tripos)** Given a tripos $\mathcal{P} : \mathbb{X} \to \mathbb{C}$, we define the signature $\Sigma^{\mathcal{P}} = (\mathbf{S}^{\mathcal{P}}, \mathbf{F}^{\mathcal{P}}, \mathbf{R}^{\mathcal{P}})$, and at the same time an interpretation $[\![-]\!]$ of the language $\mathcal{L}(\mathcal{P})$ generated by $\Sigma^{\mathcal{P}}$ in $\mathcal{P}$, as follows:

- The set of base types is defined as

$$\mathbf{S}_{\mathcal{P}} = \mathbb{C} \quad \text{(the set of objects of } \mathbb{C}\text{)},$$

and $[\![C]\!] = C$ for $C \in \mathbf{S}^{\mathcal{P}}$.

- For $\Delta \in \mathcal{T}(\mathbf{S}^{\mathcal{P}})^*, A \in \mathcal{T}(\mathbf{S}^{\mathcal{P}})$, we define

$$\mathbf{F}^{\mathcal{P}}_{\Delta,A} = \mathbb{C}(\llbracket \Delta \rrbracket, \llbracket A \rrbracket)$$

and $\llbracket f \rrbracket = f$ for $f \in \mathbf{F}^{\mathcal{P}}_{\Delta,A}$.

- For $\Delta \in \mathcal{T}(\mathbf{S}^{\mathcal{P}})^*$, we define

$$\mathbf{R}^{\mathcal{P}}_{\Delta} = \mathcal{P}_{\llbracket \Delta \rrbracket}$$

and $\llbracket R \rrbracket = R$ for $R \in \mathbf{R}^{\mathcal{P}}_{\Delta}$.

The *internal language of* $\mathcal{P}$ is the language that is generated by $\Sigma^{\mathcal{P}}$. $\qquad \diamondsuit$

In the following, we will use the internal language freely and heavily when reasoning about triposes.

The power object of 1 has a special status since it is the type of propositions, therefore we introduce the notations

$$\begin{aligned}
\mathsf{Prop} &:= \mathfrak{P}1 \\
\mathsf{tr}(p) &:= \exists x \colon 1 \,.\, x \in p
\end{aligned}$$

for the power object of 1 and its element predicate.

## 3.3 Tripos morphisms

Fibrations form 2-categories in a natural way, the 1- and 2-cells being the fibered functors and fibered natural transformations (see [26, Definition 2.3]). For triposes, we only consider fibered functors that are compatible with a part of the logical structure.

**Definition 3.11** Let $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ and $\mathcal{Q} : \mathbb{Y} \to \mathbb{D}$ be two triposes.

- A *tripos morphism* is a pair of functors $(F, \Phi)$ with $F : \mathbb{C} \to \mathbb{D}$ and $\Phi : \mathbb{X} \to \mathbb{Y}$ with the following four properties.

  1. The square

  $$\begin{array}{ccc}
  \mathbb{X} & \xrightarrow{\Phi} & \mathbb{Y} \\
  {\scriptstyle \mathcal{P}}\downarrow & & \downarrow{\scriptstyle \mathcal{Q}} \\
  \mathbb{C} & \xrightarrow{F} & \mathbb{D}
  \end{array}$$

  commutes (on the nose).

  2. $\Phi$ maps cartesian arrows to cartesian arrows.

  3. $F$ preserves finite products.

  4. For each $A \in \mathbb{C}$, the restricted functor $\Phi_A : \mathcal{P}_A \to \mathcal{Q}_{FA}$ preserves finite meets.

- A tripos morphism $(F, \Phi)$ is called *regular* if it satisfies the following additional condition.

  5. $\Phi$ maps cocartesian arrows to cocartesian arrows. $\qquad \diamondsuit$

Conditions 1. and 2. in the definition of tripos morphism say that $\Phi$ is a fibered functor over $F$. The others are compatibility postulates. Their effect is best understood in terms of the internal language.

In order to express the interaction of tripos morphisms and the internal language, we need some more definitions.

**Definitions 3.12** Let $(F, \Phi) : \mathcal{P} \to \mathcal{Q}$ be a tripos morphism between triposes $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ and $\mathcal{Q} : \mathbb{Y} \to \mathbb{D}$.

1. We denote by $\mathcal{L}_0(\mathcal{P})$ the fragment of the internal language of $\mathcal{P}$ which is first order, i.e. without power types and the element predicate, but with product types, and we denote by $\mathcal{T}_0(\mathcal{P})$ the corresponding set of (first order) types.

2. For $A \in \mathcal{T}_0(\mathcal{P})$, we denote by $F\{A\} \in \mathcal{T}_0(\mathcal{Q})$ the type that is obtained by replacing all the occurring base types $C \in \mathbb{C}$ in $A$ by $FC$. For a list $\Delta = (C_1, \ldots, C_n)$, of objects of $\mathbb{C}$, we write $F\{\Delta\}$ for the list $(F\{C_1\}, \ldots, F\{C_n\})$, in the same way for contexts.

3. Since first order types are built up only from finite products which are preserved by $F$, there are obvious commutation isomorphisms which we name as follows:

$$\sigma_A : [\![F\{A\}]\!] \xrightarrow{\cong} F[\![A]\!]$$

$$\sigma_\Delta : [\![F\{\Delta\}]\!] \xrightarrow{\cong} F[\![\Delta]\!].$$

4. For a function symbol $f \in \mathbf{F}^{\mathcal{P}}_{\Delta, A}$, define $F_{\Delta, A}(f) \in \mathbf{F}^{\mathcal{Q}}_{F\{\Delta\}, A\{\Delta\}}$ as

$$F_{\Delta, A}(f) = \sigma_A^{-1} \circ Ff \circ \sigma_\Delta.$$

5. For a relation symbol $\varphi \in \mathbf{R}^{\mathcal{P}}_{\Delta}$, define $\Phi_\Delta(\varphi) \in \mathbf{R}^{\mathcal{Q}}_{F\{\Delta\}}$ by

$$\Phi_\Delta(\varphi) = \sigma_\Delta^*(\Phi\varphi).$$

6. For a term $(\Delta \mid t : A)$ of $\mathcal{L}_0(\mathcal{P})$, we denote by $(F\{\Delta\} \mid F\{t\} : F\{A\})$ the term of $\mathcal{L}_0(\mathcal{Q})$ that is obtained by replacing each of the occurring function symbols $g \in \mathbf{F}^{\mathcal{P}}_{\Delta', B}$ in $t$ by $F_{\Delta', B}(g)$.

7. For a formula $(\Delta \mid \varphi)$ of $\mathcal{L}_0(\mathcal{P})$, we denote by $(F\{\Delta\} \mid F\{\varphi\})$ the formula of $\mathcal{L}_0(\mathcal{Q})$ that is obtained by replacing each of the occurring function symbols $g \in \mathbf{F}^{\mathcal{P}}_{\Delta', B}$ in $t$ by $F_{\Delta', B}(g)$, and each relation symbol $\theta \in \mathbf{R}^{\mathcal{P}}_{\Delta}$ by $F_\Delta(\theta)$. $\diamondsuit$

The interaction of tripos morphisms and internal language is now expressed by the following lemma.

**Lemma 3.13** *Let $(F, \Phi) : \mathcal{P} \to \mathcal{Q}$ be a tripos morphism.*

- *Let $(\Delta \mid t : A)$ be a term in $\mathcal{L}_0(\mathcal{P})$. We have*

$$\sigma_A \circ [\![F\{t\}]\!] = F[\![t]\!] \circ \sigma_\Delta$$

- *Let $(\Delta \mid \varphi)$ be a formula in $\mathcal{L}_0(\mathcal{P})$. If $\varphi$ is built from atomic formulas (excluding equality) using only $\wedge$ and $\top$, then we have*

$$\llbracket \Phi\{\varphi\} \rrbracket = \sigma_\Delta^*(\Phi\llbracket \varphi \rrbracket).$$

  *If $(F, \Phi)$ is regular, then $\varphi$ may also contain $\exists$ and $=$.* ∎

This lemma looks very much like a categorical coherence theorem, but it is much easier to prove since we can do it by induction on the structure of terms and formula.

An important consequence of the lemma is that validity of sequents is preserved by tripos morphisms. Because we will heavily make use of this fact later, we spell it out explicitly.

**Corollary 3.14** *Let $(F, \Phi) : \mathcal{P} \to \mathcal{Q}$ be a tripos morphism, and let $\Delta \mid \Gamma \vdash \varphi$ be a valid judgment in $\mathcal{L}_0(\mathcal{P})$ containing no equality and only $\wedge, \top$ as logical symbols. Then $F\{\Delta\} \mid \Phi\{\Gamma\} \vdash \Phi\{\varphi\}$ is valid in $\mathcal{Q}$. If $(F, \Phi)$ is regular, then the statement also holds for judgments containing $\exists$ and $=$.* ∎

The next definition gives the 2-cells in the 2-category of triposes.

**Definition 3.15** Let $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ and $\mathcal{Q} : \mathbb{Y} \to \mathbb{D}$ be two triposes and consider two tripos morphisms $(F, \Phi), (G, \Gamma) : \mathcal{P} \to \mathcal{Q}$. A *transformation* from $(F, \Phi)$ to $(G, \Gamma)$ is a natural transformation $\eta : F \Rightarrow G$ with the property that for all $A \in \mathbb{C}$ and all $\psi \in \mathcal{P}_A$, we have

$$a \colon FA \mid \Phi\psi(a) \vdash \Gamma\psi(\eta_A(a)),$$

or diagrammatically

$$\psi \qquad\qquad \Phi\psi \longrightarrow \Gamma\psi$$

$$A \qquad\qquad FA \xrightarrow{\ \eta_A\ } GA \qquad\qquad \diamond$$

**Definition 3.16** We denote by **Trip** the pre-equipment consisting of triposes, tripos morphisms and tripos transformations, where the regular 1-cells are the regular tripos morphisms. $\diamond$

It is straightforward to check that this data does indeed constitute a pre-equipment. Furthermore, we have:

**Lemma 3.17** *The pre-equipment **Trip** is geometric (in the sense of Definition 2.5.1).*

*Proof.* Let $(F, \Phi) \dashv (G, \Gamma) : \mathcal{Q} \to \mathcal{P}$ be an adjunction between triposes $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ and $\mathcal{Q} : \mathbb{Y} \to \mathbb{D}$. The fact that we have an adjunction means that we have the bidirectional rule

$$\frac{\Phi\varphi \vdash f^*\psi}{\varphi \vdash g^*(\Gamma\psi)} \qquad \text{with special case} \qquad \frac{\Phi\varphi \vdash \psi}{\varphi \vdash \eta_C^*(\Gamma\psi)}$$

for the entailment relations in the fibers of $\mathcal{P}$ and $\mathcal{Q}$, where $f : FC \to D$ and $g : C \to GD$ are conjugate to each other via $F \dashv G$, $\varphi \in \mathcal{P}_C$, $\psi \in \mathcal{Q}_D$, and in the special case we have $f = \mathrm{id}_{FC}$.

Now the derivations

$$
\frac{\dfrac{\exists_f\varphi \vdash \exists_f\varphi}{\dfrac{\varphi \vdash f^*(\exists_f\varphi)}{\Phi\varphi \vdash \Phi(f^*(\exists_f\varphi)) \cong (Ff)^*(\Phi(\exists_f\varphi))}}}{\exists_{Ff}(\Phi\varphi) \vdash \Phi(\exists_f\varphi)}
$$

and

$$
\frac{\dfrac{\dfrac{\dfrac{\exists_{Ff}(\Phi\varphi) \vdash \exists_{Ff}(\Phi\varphi)}{\Phi\varphi \vdash (Ff)^*(\exists_{Ff}(\Phi\varphi))}}{\varphi \vdash \eta_A^*(\Gamma((Ff)^*(\exists_{Ff}(\Phi\varphi)))) \cong f^*(\eta_B^*(\Gamma(\exists_{Ff}(\Phi\varphi))))}}{\exists_f\varphi \vdash \eta_B^*(\Gamma(\exists_{Ff}(\Phi\varphi)))}}{\Phi(\exists_f\varphi) \vdash \exists_{Ff}(\Phi\varphi)}
$$

where $f : A \to B$ in $\mathbb{C}$ show that $(F, \Phi)$ commutes with existential quantification. ∎

# 4   Q-toposes

We introduce q-toposes as an intermediate step in our decomposition of the tripos-to-topos construction.

Q-toposes are similar to quasitoposes[11], in particular they have a classifier for strong monomorphisms, but they have less structure (that's why they have fewer letters in their name). Contrary to quasi-toposes, q-toposes are not required to be locally cartesian closed (not even cartesian closed), nor do they need to have all colimits.

Fortunately it turns out that in order to get a working higher order logic, neither of these features is needed, and it suffices to postulate the quasitopos version of powersets. In the end, we can not entirely do without colimits; we postulate pullback-stable effective quotients of *strong equivalence relations*, because we need them in a later proof.

The main difference between *toposes* and *quasitoposes* is that not every monomorphism in a quasitopos corresponds to a predicate in the internal logic. We rather restrict attention to a certain subclass of them, the *cocovers*. In the case of quasitoposes, there are two possible definitions of cocovers, one by orthogonality, corresponding to the concept of *strong monomorphism*, and one by a factorization property, corresponding to *extremal monomorphisms*. These two definitions coincide in the presence of pushouts, which we have in a quasitopos. For the weaker notion of q-topos, on the contrary, we do not require all pushouts, and we have to be careful which definition to choose. It turns out that the extremality definition is too weak, since the ensuing class of arrows might not even be stable under pullbacks. Therefore, we define cocovers (and at the same time *covers*, which we will need later) as follows.

**Definition 4.1** Let $\mathcal{C}$ be a category.

---

[11]Quasitoposes are due to Penon [19], my principal reference is the Elephant [14, A2.6].

- Let $f : A \to B$, $g : X \to Y$ in $\mathcal{C}$. We say that $f$ is *left orthogonal* to $g$ (or alternatively that $g$ is *right orthogonal* to $f$), if for any commuting square

$$
\begin{array}{ccc}
A & \longrightarrow & X \\
f \downarrow & {\scriptstyle h} \nearrow & \downarrow g \\
B & \longrightarrow & Y
\end{array}
$$

  there exists a *unique* $h : B \to X$ such that the two triangles commute.

- An epimorphism $e : A \to B$ is called a *cover*, or a *strong epimorphism*, if it is left orthogonal to all monomorphisms.

- A monomorphism $m : X \to Y$ is called a *cocover*, or a *strong monomorphism*, if it is right orthogonal to all epimorphisms. $\qquad \diamond$

In the presence of equalizers, we can drop the assumption that $e$ is a epimorphism from the definition of cover, since it follows from the orthogonality condition. Similarly, in the presence of kernel pairs and coequalizers of kernel pairs, we can drop the assumption that $m$ is a monomorphism from the definition of cocover. We will denote covers by the arrow '$\twoheadrightarrow$', and cocovers by the arrow '$\rightarrowtail$'.

It is easy to see that cocovers are stable under arbitrary pullbacks, and this allows us to construct a presheaf[12]

$$
\mathrm{sub}_c : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set},
$$

which assigns to each object $A$ the set of isomorphism classes of cocovers with common codomain $A$.

In order to be able to phrase the definition of q-topos in a concise manner, we send three more definitions ahead. For the first one, we assume that the reader is acquainted with the concept of an equivalence relation in a finite limit category as a monomorphism $\rho : R \rightarrowtail A \times A$ with certain properties. The precise definition can be found e.g. in [14, Definition 1.3.6].

**Definition 4.2** Let $\mathcal{C}$ be a category with finite limits.

- A *strong equivalence relation* is an equivalence relation $\rho : R \rightarrowtail A \times A$ which is a strong monomorphism.

- An *effective quotient* of an equivalence relation $\rho = \langle \rho_1, \rho_2 \rangle : R \rightarrowtail A \times A$ in $\mathcal{C}$ is a coequalizer $e : A \twoheadrightarrow Q$ of $\rho_1, \rho_2 : R \rightrightarrows A$ whose kernel pair is $\rho_1, \rho_2$.

  (Observe that equivalence relations which have effective quotients are necessarily strong.)

- An object $A$ of $\mathcal{C}$ is called *exponentiating* if the presheaf

$$
\mathcal{C}(- \times X, A)
$$

  is representable for all objects $X$. $\qquad \diamond$

---

[12]To avoid having to deal with size issues, we just assume that $\mathcal{C}$ is small with respect to some universe. See also Section 1.2.

Now the definition of q-topos is the following.

**Definition 4.3** A q-topos is a finite limit category $\mathcal{C}$ such that

- all presheaves $\mathrm{sub}_c(- \times A) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$ are representable,

- $\mathcal{C}$ has effective quotients of strong equivalence relations, and

- regular epimorphisms are stable under pullback. $\diamondsuit$

An immediate consequence of the definition is:

**Lemma 4.4** *Any q-topos is a* regular *category*. *This implies that the class of regular epimorphisms coincides with the class of covers.*

*Proof.* It is possible to define a regular category as a a category with finite limits and coequalizers of kernel pairs, where regular epimorphisms are stable under pullbacks (See e.g. [4, Definition 2.1.1], except that there not all finite limits are assumed). Kernel pairs are always strong equivalence relations, described as the pullback of $\delta_B : B \to B \times B$ along $f \times f$. hence the fact that every q-topos is regular follows directly from the definition. The fact that regular epimorphisms coincide with covers is proved e.g. in [4, Proposition 2.1.4]. ∎

The following lemma shows how to rephrase the first condition of Definition 4.3 in a way that is closer to the internal language which will be introduced in the next section.

**Lemma 4.5** *In a finite limit category $\mathcal{C}$, the presheaves*

$$\mathrm{sub}_c(- \times A) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$$

*are* all *representable if and only if the presheaf*

$$\mathrm{sub}_c(-) : \mathcal{C}^{\mathsf{op}} \to \mathbf{Set}$$

*can be represented by an exponentiating object.* ∎

We will always denote the object representing $\mathrm{sub}_c(-)$ by $\Omega$ and the element of $\mathrm{sub}_c(\Omega)$ which induces the natural isomorphism (generally known as 'generic predicate') by $\mathsf{t}$.

**Lemma 4.6** *1. The domain of $\mathsf{t} : U \rightarrowtail \Omega$ is terminal.*

*2. The class of cocovers coincides with the class of regular monomorphisms.*

*Proof. Ad 1.* The postcomposition map $\mathsf{t} \circ - : \mathcal{C}(A, U) \to \mathcal{C}(A, \Omega)$ induces a bijection between $\mathcal{C}(A, U)$ and arrows $f : A \to \Omega$ such that $f^*\mathsf{t}$ is an isomorphism. For each $A$, there is exactly one such arrow.

*Ad 2.* It is well known that in every category, regular monomorphisms are strong. Conversely, every cocover $m : U \rightarrowtail A$ is the equalizer of its classifying map $\chi_m : A \to \Omega$ and $\mathsf{t} \circ !_A$. ∎

## 4.1 The logic of q-toposes

The fibers of the presheaf $\mathrm{sub}_c$ have a natural ordering which is the inclusion of subobjects, i.e., for $m : U \rightarrowtail A, n : V \rightarrowtail A$

$$m \vdash_A n \quad \text{iff} \quad \exists h : U \to V \, . \, nh = m.$$

This ordering allows us to view $\mathrm{sub}_c$ (quotiented out by mutual inclusion) as a presheaf of posets, and the posetal fibration that we obtain from this presheaf via the Grothendieck construction is isomorphic to the *fibration of cocovers*

$$\boldsymbol{S}\mathcal{C} = \partial_1 : \mathrm{coc}(\mathcal{C}) \to \mathcal{C}, \tag{4.1}$$

i.e. the quotient by equivalence of the full subfibration on cocovers of the fundamental fibration $\partial_1 : \mathcal{C}{\downarrow}\mathcal{C} \to \mathcal{C}$.

The fibration of cocovers on a q-topos $\mathcal{C}$ is the home of the internal logic of $\mathcal{C}$. It is easy to see that it has fiberwise finite meets, which are just given by pullbacks. In this section we show that it is even a tripos, which will enable us later to define the forgetful functor from the pre-equipment of q-toposes to the pre-equipment of triposes. To do this we proceed in essentially the same way as Lambek and Scott do in [16] for the definition of the internal language of a topos[13].

To begin, we outline of the general strategy. First, we define a kind of minimal internal language of a q-topos $\mathcal{C}$, whose term constructors are projections, subset comprehension, element relation and equality. This internal language is called *type theory based on equality* in [16], we will refer to it as the *core calculus*. We view terms of type $\Omega$ of the core calculus as predicates, and give a intuitionistic sequent calculus style system of inference rules for them. The fact that predicates are only special terms reflects the higher order nature of the language.

Following Lambek and Scott, we will then show how to encode all propositional connectives and quantifiers in the core calculus in such a way that the usual rules of intuitionistic logic can be derived from the rules of the core calculus. The claim that the fibration of cocovers is a tripos then follows almost directly, because the terms of type $\Omega$ of the language correspond to the predicates in the fibration of cocovers. The only thing that is missing is full quantification (the language only gives quantification along projections), but using a standard encoding, quantification along arbitrary morphisms can also be obtained.

The core calculus of a q-topos $\mathcal{C}$ is given in Table 2.

The interpretation of the language in $\mathcal{C}$ is given as follows. Types are inductively interpreted by objects of $\mathcal{C}$, where $[\![X]\!] = X$, $[\![1]\!] = 1$, $[\![\Omega]\!] = \Omega$, $[\![PA]\!] = \Omega^{[\![A]\!]}$, and $[\![A \times B]\!] = [\![A]\!] \times [\![B]\!]$. Contexts are interpreted as cartesian products of their constituents, and terms are interpreted by suitably typed morphisms as follows:

$$
\begin{aligned}
[\![\Delta \mid x_i]\!] &= \pi_i \\
[\![\Delta \mid *]\!] &= !_{[\![\Delta]\!]} \\
[\![\Delta \mid \{x|\varphi[x]\}]\!] &= \Lambda([\![\Delta, x \mid \varphi[x]]\!])
\end{aligned}
$$

---

[13]Apparently this way of bootstrapping the internal logic of a topos is originally due to Boileau and Joyal [3].

**Types:**

$$A ::= X \mid 1 \mid \Omega \mid PA \mid A \times A \qquad X \in \mathcal{C}$$

**Terms:**

We use $\Delta$ to denote a context $x_1 \colon A_1, \ldots, x_n \colon A_n$ of typed variables.

$$\frac{}{\Delta \mid x_i : A_i} \; {}^{(i=1,\ldots,n)} \qquad\qquad \frac{}{\Delta \mid * : 1}$$

$$\frac{\Delta, x \colon A \mid \varphi[x] : \Omega}{\Delta \mid \{x \mid \varphi[x]\} : PA} \qquad\qquad \frac{\Delta \mid a : A \quad \Delta \mid b : B}{\Delta \mid (a,b) : A \times B}$$

$$\frac{\Delta \mid a : A \quad \Delta \mid M : PA}{\Delta \mid a \in M : \Omega} \qquad\qquad \frac{\Delta \mid a : A \quad \Delta \mid a' : A}{\Delta \mid a = a' : \Omega}$$

$$\frac{\Delta \mid a : X}{\Delta \mid f(a) : Y} \; f \in \mathcal{C}(X, Y)$$

**Deduction rules:**

Here, $p_1, \ldots, p_n, p, q$ denote terms of type $\Omega$, and $\Gamma$ denotes a sequence of such terms.

$$\frac{}{\Delta \mid p_1, \ldots, p_n \vdash p_i} \; \mathrm{Ax} \; {}_{(i=1,\ldots,n)} \qquad\qquad \frac{\Delta \mid \Gamma \vdash p \quad \Delta \mid \Gamma, p \vdash q}{\Delta \mid \Gamma \vdash q} \; \mathrm{Cut}$$

$$\frac{}{\Delta \mid \Gamma \vdash t = t} \; {=}\mathrm{R} \qquad\qquad \frac{\Delta, x \colon A \mid \Gamma \qquad \vdash \varphi[x, x]}{\Delta \mid \Gamma, s = t \vdash \varphi[s, t]} \; {=}\mathrm{L}$$

$$\frac{\Delta, x \colon A \mid \Gamma \vdash p[x] = (x \in M)}{\Delta \mid \Gamma \vdash \{x \mid p[x]\} = M} \; \mathrm{P}\text{-}\eta \qquad\qquad \frac{}{\Delta \mid \Gamma \vdash (a \in \{x \mid p[x]\}) = p[a]} \; \mathrm{P}\text{-}\beta$$

$$\frac{}{\Delta \mid \Gamma \vdash t = *} \; 1\text{-}\eta \qquad\qquad \frac{\Delta \mid \Gamma, p \vdash q \quad \Delta \mid \Gamma, q \vdash p}{\Delta \mid \Gamma \vdash p = q} \; \mathrm{Ext}$$

$$\frac{\Delta \mid \Gamma \vdash (a_1, a_2) = (a_1', a_2')}{\Delta \mid \Gamma \vdash a_i = a_i'} \; \times\text{-}\beta \; {}_{(i=1,2)} \qquad\qquad \frac{\Delta, x \colon A, y \colon B \mid \Gamma, t = (x, y) \vdash p[t]}{\Delta \mid \Gamma \qquad\qquad \vdash p[t]} \; \times\text{-}\eta$$

Table 2: The core calculus

$$\begin{aligned}
\llbracket \Delta \mid a \in M \rrbracket &= \varepsilon_{\llbracket A \rrbracket} \circ \langle \llbracket \Delta \mid M \rrbracket, \llbracket \Delta \mid a \rrbracket \rangle \\
\llbracket \Delta \mid (a,b) \rrbracket &= \langle \llbracket \Delta \mid a \rrbracket, \llbracket \Delta \mid b \rrbracket \rangle \\
\llbracket \Delta \mid s = t \rrbracket &= \chi_= \circ \langle \llbracket \Delta \mid s \rrbracket, \llbracket \Delta \mid t \rrbracket \rangle \\
\llbracket \Delta \mid f(a) \rrbracket &= f \circ \llbracket \Delta \mid a \rrbracket
\end{aligned}$$

Here $\pi_i$ denotes the suitable projection, $!_{\llbracket \Delta \rrbracket}$ denotes the terminal projection, $\Lambda : \mathcal{C}(\llbracket \Delta, x{:}\,A \rrbracket, \Omega) \to \mathcal{C}(\llbracket \Delta \rrbracket, \Omega^{\llbracket A \rrbracket})$ is the exponential transpose operation, $\varepsilon_{\llbracket A \rrbracket} : \Omega^{\llbracket A \rrbracket} \times \llbracket A \rrbracket \to \Omega$ is evaluation, and $\chi_= : A \times A \to \Omega$ is the classifying map of the diagonal $\delta_A : A \to A \times A$, i.e., the map that corresponds to $\delta_A \in \mathrm{sub}_c(A \times A)$ via the bijection $\mathrm{sub}_c(A \times A) \cong \mathcal{C}(A \times A, \Omega)$ ($\delta_A$ is indeed in $\mathrm{sub}_c(A \times A)$, since it is a split mono and therefore necessarily a cocover).

We make some comments on the deduction rules. The first two rules, (Ax) and (Cut), are just the usual structural rules known from sequent calculus style systems. (=L) and (=R) are equivalent to the usual rules for equality in first order logic. The rules ($P$-$\beta$) and ($P$-$\eta$) correspond to the $\beta$- and $\eta$-rules of lambda calculus if we identify subset comprehension and $\in$ with abstraction and application. Together with the rule (Ext), which says that two predicates which are equivalent are already equal, $P$-$\eta$ can be read as saying that *two sets are already equal if they have the same elements*, which is a more familiar phrasing of set-theoretic extensionality. Finally, there are a $\beta$- and an $\eta$ rule for pairing, and an $\eta$-rule for the unit type.

For the core calculus, we have a substitution lemma and a soundness theorem, which we state without proof because the proofs are standard and consist mainly of unfolding of definitions.

**Lemma 4.7 (Substitution lemma)** *For well formed terms* $\Delta \mid s_i : B_i, 1 \leq i \leq n$, *and* $x_1{:}\,B_1, \ldots, x_n{:}\,B_n \mid t : C$, *we have*

$$\llbracket \Delta \mid t[s_1/x_1, \ldots, s_n/x_n] \rrbracket = \llbracket x_1, \ldots, x_n \mid t \rrbracket \circ \langle \llbracket \Delta \mid s_1 \rrbracket, \ldots, \llbracket \Delta \mid s_n \rrbracket \rangle. \qquad \blacksquare$$

**Theorem 4.8 (Soundness theorem)** *If a sequent* $\Delta \mid p_1, \ldots, p_n \vdash q$ *is derivable in the core calculus, then we have*

$$\llbracket \Delta \mid p_1 \rrbracket^* \mathsf{t} \wedge \cdots \wedge \llbracket \Delta \mid p_n \rrbracket^* \mathsf{t} \vdash_{\llbracket \Delta \rrbracket} \llbracket \Delta \mid q \rrbracket^* \mathsf{t} \qquad (*)$$

*in* $\mathrm{sub}_c(\llbracket \Delta \rrbracket)$. $\qquad \blacksquare$

If the relation $(*)$ holds, we also say that *the judgment* $\Delta \mid p_1, \ldots, p_n \vdash q$ *holds in* $\mathcal{C}$.

The connectives of predicate logic can be encoded in the core calculus as follows:

$$\begin{aligned}
\top &\equiv & * = * \\
p \wedge q &\equiv & (p, q) = (\top, \top) \\
p \Rightarrow q &\equiv & p \wedge q = p \\
\forall x{:}\,A\,.\,p[x] &\equiv & \{x \mid p[x]\} = \{x \mid \top\} \\
\bot &\equiv & \forall z{:}\,\Omega\,.\,z \\
p \vee q &\equiv & \forall z{:}\,\Omega\,.\,(p \Rightarrow z) \wedge (q \Rightarrow z) \Rightarrow z \\
\exists x{:}\,A\,.\,p[x] &\equiv & \forall z{:}\,\Omega\,.\,(\forall x{:}\,A\,.\,p[x] \Rightarrow z) \Rightarrow z
\end{aligned}$$

It can now be derived from the rules of the core calculus that the such defined logical connectives validate the rules of intuitionistic predicate logic. For a proof of this, we refer to Lambek and Scott's book (alternatively, the reader may prove this herself as an instructive exercise).

Here are some basic principles that can be transferred directly from toposes to q-toposes:

**Lemma 4.9**      *1. Two parallel arrows $f, g : A \to B$ in $\mathcal{C}$ are equal if and only if $\big(x \mid\vdash f(x) = g(x)\big)$ holds in the internal logic.*

     *2. $f : A \to B$ is a monomorphism iff $f(x) = f(y) \vdash x = y$ holds in $\mathcal{C}$.*

     *3. $f : A \to B$ is an epimorphism iff $y{:}\,B \mid\vdash \exists x \,.\, f(x) = y$ holds in $\mathcal{C}$.*

*Proof.* *Ad 1.* Straightforward, since the interpretation of $\big(x \mid f(x) = g(x)\big)$ is the equalizer of $f$ and $g$.

*Ad 2.* The interpretation of $\big(x, y \mid f(x) = f(y)\big)$ is the kernel pair of $f$, and a morphism is monic iff its kernel pair is the diagonal.

*Ad 3.* Assume that $y{:}\,B \mid\vdash \exists x \,.\, f(x) = y$ holds. Let $h, k : B \to C$ be arbitrary and assume that $hf = kf$. Then the deduction

$$\dfrac{y \mid \,\vdash \exists x \,.\, fx = y \qquad \dfrac{\qquad\qquad x \mid \,\vdash h(f(x)) = k(f(x))}{x, y \mid \; fx = y \vdash h(y) = k(y)}}{y \mid \,\vdash h(y) = k(y)}$$

establishes the claim.

Conversely, assume that $f$ is an epimorphism. $f$ obviously equalizes the classifying maps of the predicates $\big(y{:}\,B \mid \top\big)$ and $\big(y{:}\,B \mid \exists x \,.\, fx = y\big)$, and since it is epic they are both equal, whence the second is valid in $\mathcal{C}$. ∎

Note that statement 3 stands in contrast to regular categories, where the judgment $y \mid\vdash \exists x \,.\, fx = y$ characterizes the *regular* epimorphisms. This discrepancy is due to the fact that the internal logic of a regular category has *all* monomorphisms as predicates, whereas we only consider the strong ones in q-toposes.

In [14], Johnstone shows that every quasitopos is coregular, i.e., its opposite category is regular. For q-toposes, this is too much to hope for, since we do not even require all colimits. However, we can still prove one of the most important consequences of coregularity — the existence of a epi/cocover factorization system.

**Lemma 4.10** *Let $f : A \to B$ in $\mathcal{C}$. The cocover $m : U \rightarrowtail A$ that is classified by the predicate $\big(y \mid \exists x \,.\, f(x) = y\big)$ is the minimal cocover through which $f$ factors, and in the corresponding factorization $f = me$, $e$ is an epimorphism. In particular, the class of epimorphisms and the class of cocovers together form a factorization system (see [8, 30]) on $\mathcal{C}$.*

*Proof.* The minimality condition is just a rephrasing of the universal property of existential quantification. If $e : A \to U$ were not an epimorphism, it would factor through the monomorphism $m' : U' \to U$ classified by $\big(y \mid \exists x \,.\, e(x) = y\big)$, which would be non-maximal by the previous lemma. Then $f$ would factor through $mm'$, contradicting the minimality of $m$.

To establish that the epimorphisms together with the cocovers constitute a factorization system, it now remains to show that the maps that are left orthogonal to all cocovers are precisely the epimorphisms. This follows from the *retract argument*: Let $g : C \to D$ be a morphism that is left orthogonal to all cocovers, and let $f = kh$ be its factorization into an epimorphism followed by a cocover. We obtain a square

$$
\begin{array}{ccc}
C & \xrightarrow{\;h\;} & X \\
g \downarrow & \nearrow{\scriptstyle l} & \downarrow k \\
D & \xrightarrow[\;\mathrm{id}\;]{} & D
\end{array}
\;,
$$

which has a lifting $l$ because $g$ is left orthogonal to all cocovers. It is now easy to see that $g$ is a retract of $h$ and this implies that $g$ is an epimorphism, because epimorphisms are closed under retracts. ∎

It is remarkable that to prove the existence of the factorization system, we need quite heavy machinery, in particular the higher order internal logic with its polymorphically defined existential quantification.

**Lemma 4.11** *1. If the square* (†) *is a pullback in* $\mathcal{C}$, *then the judgments* *(i)–(iii) hold in the internal logic. (The converse is not true in general.)*

$$
(\dagger)\quad
\begin{array}{ccc}
P & \xrightarrow{\;q\;} & B \\
p \downarrow & & \downarrow g \\
A & \xrightarrow[\;f\;]{} & C
\end{array}
\qquad
\begin{array}{llll}
(i) & z & | & \vdash f(pz) = g(qz) \\
(ii) & x,y & |\; fx = gy & \vdash \exists z \,.\, (pz, qz) = (x, y) \\
(iii) & z, z' & |\; (pz, qz) = (pz', qz') & \vdash z = z'
\end{array}
$$

*2. Epimorphisms are stable under pullback in a q-topos.*

*Proof. Ad 1.* Assume that the square is a pullback. $(i)$ just states that the square commutes. $(ii)$ holds, because $\big(x, y \mid fx = gy\big)$ as well as $\big(x, y \mid \exists z \,.\, (pz, qz) = (x, y)\big)$ classify the cocover $P \rightarrowtail A \times B$. $(iii)$ expresses the fact that $p$ and $q$ are jointly monic.

*Ad 2.* This follows from 1 and the characterization of epimorphisms in Lemma <span style="color:red">4.9.3</span>. ∎

**Lemma 4.12** *The fibration* $\mathbf{S}\mathcal{C}$ *of cocovers on a q-topos* $\mathcal{C}$ *is a tripos.*

*Proof.* The internal language gives us the propositional structure and quantification along projections. Quantification along arbitrary morphisms can be encoded as follows. For $f : A \to B$ and $\varphi : A \to \Omega$, we set

$$
\begin{array}{lcll}
\forall_f(\varphi)(y) & \equiv & \forall x\!:\! A \,.\, f(x) = y \Rightarrow \varphi(x) & \text{and} \\
\exists_f(\varphi)(y) & \equiv & \exists x\!:\! A \,.\, f(x) = y \wedge \varphi(x). &
\end{array}
$$

The Beck-Chevalley condition for pullback squares of the form

$$
\begin{array}{ccc}
A \times C & \longrightarrow & A \\
f \times C \downarrow & & \downarrow f \\
B \times C & \longrightarrow & B
\end{array}
$$

(where we quantify along the projections) follows directly from the Substitution Lemma 4.7. For general pullback squares, it follows from Lemma 4.11 and some calculations in the internal logic. Finally, it is easy to see that the power objects of the q-topos give us power objects in the fibration in the tripos sense. ∎

## 4.2 The subtopos of coarse objects

In Lemma 4.9, we saw that the internal logic of a q-topos is powerful enough to detect equality of arrows. However, the internal logic lacks another important feature: it is not capable to distinguish isomorphisms from maps which are monomorphisms and epimorphisms at the same time. This follows from the following lemma.

**Lemma 4.13** *Let $f : A \rightarrowtail\!\!\!\twoheadrightarrow B$ be monic as well as epic. Then the induced map $f^* : (\boldsymbol{SC})_B \to (\boldsymbol{SC})_A$ we do is an isomorphism of posets.*

*Proof.* Using the internal language, the map $\mathrm{sub}_c(f)$ can be written as

$$(\boldsymbol{SC})_B \ni \varphi \quad \mapsto \quad \big(a \mid \varphi(fa)\big).$$

A map in the converse direction is given by existential quantification:

$$(\boldsymbol{SC})_B \ni \psi \quad \mapsto \quad \big(b \mid \exists a . fa = b \wedge \psi(a)\big)$$

Using the characterizations of monomorphisms and epimorphisms of Lemma 4.9, it is easy to verify that these two maps are inverse to each other. ∎

So in a sense the arrows which are monic and epic at the same time disclose a mismatch between the category and the internal logic. This can be seen as a motivation for the following definition of *coarse objects*, which are just as blind as the internal logic, so that the correspondence between category and internal logic is restored if we restrict to the full subcategory on the coarse objects.

Coarse objects are also considered for quasitoposes, and the treatment here is a variation of the presentation in [14] for quasitoposes.

**Definition 4.14** An object $C$ of a q-topos $\mathcal{C}$ is called coarse, if for each morphism $f : A \rightarrowtail\!\!\!\twoheadrightarrow B$ which is monic and epic at once and all morphisms $g : A \to C$, there exists a morphism $h : B \to C$ such that $hf = g$. ◇

Because the arrow $f$ in the previous definition is an epimorphism, the mediating arrow $h$ is automatically unique.

**Lemma 4.15 (Properties of coarse objects)**    *1. If $C$ is coarse and $f : C \rightarrowtail\!\!\!\twoheadrightarrow A$ is monic as well as epic, then it is already an isomorphism.*

2. *If $C$ is coarse and $m : C \rightarrowtail A$ is a monomorphism, then $m$ is already a cocover.*

3. *If $C$ is coarse and $m : U \rightarrowtail C$ is a cocover, then $U$ is coarse.*

4. *Finite products of coarse objects are coarse.*

5. *For every object $A$ of $\mathcal{C}$, its power object $PA$ is coarse.*

6. *The full subcategory $\boldsymbol{TC}$ of $\mathcal{C}$ on the coarse objects is reflective.*

   *(In the following we denote this reflection by $J_{\mathcal{C}} \dashv I_{\mathcal{C}} : \boldsymbol{TC} \to \mathcal{C}$.)*

7. *The reflection functor $J_{\mathcal{C}} : \mathcal{C} \to \boldsymbol{TC}$ preserves finite limits and epimorphisms.*

8. $\boldsymbol{TC}$ *is a topos.*

9. *The embedding functor $I_{\mathcal{C}} : \boldsymbol{TC} \to \mathcal{C}$ maps epimorphisms to epimorphisms.*

*Proof. Ad 1.* By coarseness of $C$, there exists an arrow $g : A \to C$ such that $gf = \mathrm{id}_C$. Because $f$ is an epimorphism, it follows that $fg = \mathrm{id}_A$.

*Ad 2.* Let $\tilde{m}e = m$ be the factorization of $e$ into an epimorphism and a cocover. Then $e$ is a monomorphism as well as an epimorphism, and by *1.* it is already an isomorphism.

*Ad 3.* Let $f : A \to B$ be monic and epic, and $g : A \to U$. Then by coarseness of $C$, there exists a map $h : B \to C$ with $hf = mg$, and by orthogonality, there exists a map $k : B \to U$ such that $kf = g$ and $mk = h$.

$$
\begin{array}{ccc}
A & \xrightarrow{\;g\;} & U \\
{\scriptstyle f}\downarrow & \nearrow^{\;k} & \downarrow{\scriptstyle m} \\
B & \underset{h}{\dashrightarrow} & C
\end{array}
$$

*Ad 4.* To extend an arrow $\langle g_1, \ldots, g_n \rangle : A \to C_1 \times \cdots \times C_n$ along a monic-epic arrow $f : A \rightarrowtail\!\!\!\to B$, simply extend the components $g_i$ individually.

*Ad 5.* Let $f : A \rightarrowtail\!\!\!\to B$ be monic-epic and let $g : A \to PD$. The lifting of $g$ along $f$ can be elegantly expressed as $\big(b \mid \{d|\exists a . f(a) = b \wedge d \in g(a)\}\big)$. To see that this validates the required equation, simply substitute and simplify.

*Ad 6.* We have to give for each $A$ a coarse object $\overline{A} = J_{\mathcal{C}}(A)$ (we use the notations $\overline{A}$ and $J_{\mathcal{C}}A$ interchangeably) and a morphism $\eta_A : A \to \overline{A}$ such that for all coarse $C$ and all $f : A \to C$ there exists a unique $g : \overline{A} \to C$ with $g\eta_A = f$.

Consider the morphism $\big(x \mid \{y|y = x\}\big) : A \to PA$. Using Lemma 4.9.2, we deduce that it is a monomorphism, and construct its epi/cocover factorization $A \xrightarrow{\eta_A} \overline{A} \rightarrowtail PA$. It follows from 3 and 5 that $\overline{A}$ is coarse, and the required universal property of $\eta_A$ follows directly from the fact that it is a mono-epi.

*Ad 7.* Epimorphisms are preserved because they can be characterized in terms of colimits, and $J$ is a left adjoint.
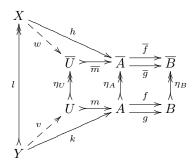
For the finite limits, we show that $J$ preserves the terminal object, binary products and equalizers.

We have already seen that the terminal object is coarse, hence $\overline{1} = 1$, i.e., the terminal object is preserved.

For products, we know from 4 that $\overline{A} \times \overline{B}$ is coarse and thus a product of $\overline{A}$ and $\overline{B}$ in $\boldsymbol{TC}$. Hence we have to show that $\overline{A \times B} \cong \overline{A} \times \overline{B}$. To show this, it is sufficient to find a monic-epic arrow $\eta'_{A \times B} : A \times B \rightarrowtail\!\!\!\to \overline{A} \times \overline{B}$, since this makes $\overline{A} \times \overline{B}$ into a coarse hull of $A \times B$ having the universal property that was described in 6. The obvious candidate arrow is the product $\eta_A \times \eta_B$ of the universal arrows for $A$ and $B$. It can be decomposed as $\eta_A \times \eta_B = \eta_A \times \mathrm{id}_B \circ \mathrm{id}_A \times \eta_B$, and

$\eta_A \times \mathrm{id}_B$ and $\mathrm{id}_A \times \eta_B$ are monic and epic since they are pullbacks of $\eta_A$ and $\eta_B$, respectively, and epimorphisms are pullback stable by Lemma 4.11.2.

Finally, consider a pair $f, g : A \to B$ of parallel arrows, with equalizer $m : U \rightarrowtail A$. We want to show that $\overline{m}$ equalizes $\overline{f}$ and $\overline{g}$. Let $h : X \to A$ such that $\overline{g}h = \overline{f}h$, form the pullback $k$ of $h$ along $\eta_A$, and consider the diagram



where $l$ is the pullback of $\eta_A$ along $h$ (and thus monic and epic). We can derive in the internal logic that $\overline{m}$ is a monomorphism from the facts that $\overline{m}\eta_U$ is a monomorphism and $\eta_U$ is an epimorphism. Furthermore we have $\eta_B f k = \overline{f} h l = \overline{g} h l = \eta_B g k$, and because $\eta_B$ is a monomorphism, this implies $gk = fk$. This induces a mediating arrow $v : Y \to U$, and by coarseness we can lift $\eta_U v$ along $l$ to obtain $w$. Doing some arrow chasing we conclude $\overline{m}w = h$, and $w$ is the unique such because $\overline{m}$ is a monomorphism. Hence, $\overline{m}$ is indeed an equalizer of $\overline{f}$ and $\overline{g}$.

*Ad 8.* It follows from 3 and 4 that $\boldsymbol{TC}$ has finite limits, because equalizers are cocovers. The powersets are coarse by 5, and from 2 and 3 it follows that the maps that are classified by arrows of type $A \to PB$ are precisely the monomorphisms $m : U \rightarrowtail A \times B$ with coarse $U$.

*Ad 9.* Let $e : A \to B$ be an epimorphism in $\boldsymbol{TC}$. We take its epi-cocover factorization $A \twoheadrightarrow D \rightarrowtail C$ in $\mathcal{C}$. Being a regular subobject of a coarse object, $D$ is also coarse, and thus the factorization is also an epi-mono factorization in $\boldsymbol{TC}$. Since $e$ is an epimorphism and $\boldsymbol{TC}$ is balanced (as a topos), the map $D \rightarrowtail C$ is an isomorphism. Thus, $e$ is an epimorphism in $\mathcal{C}$. $\blacksquare$

To conclude the section, we explain in which way we want to view q-toposes as a pre-equipment.

**Definition 4.16 (The pre-equipment QTop)** The underlying 2-category of **QTop** consists of q-toposes, finite limit preserving functors and arbitrary transformations.

The regular 1-cells are the functors that preserve epimorphisms and regular epimorphisms. $\Diamond$

The pre-equipment **QTop** is clearly geometric, since epimorphisms as well as regular epimorphisms may be characterized in terms of colimits, and these are preserved by left adjoints.

The attentive reader may have noticed that the definition of regular 1-cells does not mention cocovers. Should we not demand that they are also preserved? — It turns out that this comes for free, because we proved in 4.6 that the cocovers coincide with the regular monomorphisms, and those are preserved by any finite limit preserving functor.

# 5 The tripos-to-topos construction

To each topos, we can associate a tripos — its *subobject fibration* — in a 2-functorial manner. The present section is dedicated to proving that the such defined 2-functor has a *special left biadjoint*. In [7], I gave a very technical direct proof of this statement. Later I found a substantial simplification, which consists in decomposing the forgetful functor into two steps, the intermediate stage being q-toposes. We will now see how this allows a simple description of the special left biadjoints.

Sections 5.1 and 5.2 are devoted to the description of the special biadjunctions between triposes and q-toposes, and between q-toposes and toposes, respectively.

To simplify notation, we will in the following use the variables $\eta$ and $\varepsilon$ in an ambiguous sense, denoting unit and counit of whatever adjunction is at hand. We will use the convention that boldface $\boldsymbol{\eta}, \boldsymbol{\epsilon}$ denote unit and counit of 2-dimensional adjunctions, i.e., the special biadjunctions in our case, whereas we use normal $\eta, \varepsilon$ for 1-dimensional adjunctions.

## 5.1 The biadjunction $\boldsymbol{F} \dashv \boldsymbol{S}$ between triposes and q-toposes

In this section, we define a special biadjunction

$$(\boldsymbol{F} \dashv \boldsymbol{S} : \mathbf{QTop} \to \mathbf{Trip}, \boldsymbol{\eta}, \boldsymbol{\epsilon}, \nu, \mu)$$

between the pre-equipments of triposes and q-toposes.

The definitions and verifications of well-definedness of the constituents are a bit long-winded, therefore we devote a individual subsection to each of them.

### 5.1.1 The special right biadjoint functor $\boldsymbol{S}$

The forgetful functor

$$\boldsymbol{S} : \mathbf{QTop} \to \mathbf{Trip}$$

assigns to each q-topos $\mathcal{C}$ its fibration $\boldsymbol{S}\mathcal{C} = \partial_1 : \mathbf{coc}(\mathcal{C}) \to \mathcal{C}$ of cocovers. We defined this fibration in (4.1) and proved in Lemma 4.12 that it is indeed a tripos.

To a finite limit preserving functor $F : \mathcal{C} \to \mathcal{D}$, $\boldsymbol{S}$ assigns the tripos transformation

$$
\begin{array}{ccc}
\mathbf{coc}(\mathcal{C}) & \xrightarrow{\mathbf{coc}(F)} & \mathbf{coc}(\mathcal{D}) \\
\partial_1 \downarrow & & \downarrow \partial_1 \\
\mathcal{C} & \xrightarrow{\quad F \quad} & \mathcal{D}
\end{array} \quad .
$$

Here, $\mathbf{coc}(F)$ denotes the functor that maps the cocover $m : U \rightarrowtail A$ to $Fm : FU \rightarrowtail FA$, which is again a cocover by the remark after Definition 4.16.

To verify that this indeed defines a tripos morphism, we have to check the four conditions in Definition 3.11. Clearly the square commutes, and cartesian arrows are preserved by $\mathbf{coc}(F)$, because they are just pullback squares. Finite limits are preserved by $F$ by assumption, and fiberwise finite meets are preserved again because they are given by pullbacks.

The action of $S$ on 2-cells is easy again, a natural transformation $\eta : F \to G : \mathcal{C} \to \mathcal{D}$ is just mapped to itself and we leave it to the reader to verify that $\eta$ constitutes a tripos transformation from $(F, \mathbf{coc}(F))$ to $(G, \mathbf{coc}(G))$.

It is straightforward to verify that the previously described constructions all commute with compositions and identities on the nose and thus $F$ is a strict functor. We want it to be a special functor, and thus we have to check that it maps regular 1-cells in $\mathbf{QTop}$ to regular 1-cells in $\mathbf{Trip}$, i.e., that $\mathbf{coc}(F)$ preserves cocartesian arrows whenever $F$ preserves epimorphisms and regular epimorphisms. But cocartesian arrows in $\mathbf{coc}(C)$ and $\mathbf{coc}(D)$ are just squares

$$
\begin{array}{ccc}
U & \xrightarrow{\;h\;} & V \\
{\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\
A & \xrightarrow[\;f\;]{} & B
\end{array}
$$

where $m, n$ are cocovers and $h$ is an epimorphism, which are all preserved by $F$.

This completes the description of $S$.

### 5.1.2 The special left biadjoint functor $F$

The construction of the left biadjoint functor

$$F : \mathbf{Trip} \to \mathbf{QTop}$$

from triposes to q-toposes is more involved than that of $S$, which is to be expected because it is a kind of 'free' construction.

The object part of $F$ is close to — but not quite[14] — what is traditionally known as the 'tripos-to-topos construction' (i.e. the construction of the category $\mathbb{C}[\mathcal{P}]$ from a tripos $\mathcal{P}$, as described in [11, 20]). Concretely, the category $F\mathcal{P}$ for a tripos $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ is given as follows.

**Definition 5.1 (The category $F\mathcal{P}$)** Let $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ be a tripos.

- The objects of $F\mathcal{P}$ are pairs $(C, \rho)$, where $C \in \mathbb{C}$ and $\rho \in \mathcal{P}_{C \times C}$ is a *partial equivalence relation* (i.e., symmetric and transitive) in the logic of $\mathcal{P}$.

- Morphisms $f : (C, \rho) \to (D, \sigma)$ in $F\mathcal{P}$ are given by morphisms $f : C \to D$ in $\mathbb{C}$ which satisfy
$$x, y \mid \rho(x, y) \vdash \sigma(fx, fy)$$
in the logic of $\mathcal{P}$. Two such morphisms $f, g : C \to D$ are identified as morphisms from $(C, \rho)$ to $(D, \sigma)$ if
$$x \mid \rho(x, x) \vdash \sigma(fx, gx)$$
holds. (More concisely, we can define $F\mathcal{P}((C, \rho), (D, \sigma)) = \mathbb{C}(C, D)/\sim$, where $\sim$ is an (external) partial equivalence relation defined by $f \sim g$ iff $\rho(x, y) \vdash \sigma(fx, gy)$.)

- Composition and identities are inherited from $\mathbb{C}$. $\qquad\qquad\diamondsuit$

---

[14]The construction here has a more restrictive notion of morphism than the classical tripos-to-topos construction, and in particular does not produce toposes but only q-toposes.

It is straightforward to see that $\boldsymbol{F}\mathcal{P}$ is a well defined category. In Lemma 5.3, we will prove that it is even always a q-topos, but before we introduce some terminology.

**Definition 5.2** Let $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ be a tripos.

1. Let $f : C \to D$ and let $\rho \in \mathcal{P}_{C \times C}, \sigma \in \mathcal{P}_{D \times D}$ be partial equivalence relations. If the judgment

$$x, y \mid \rho(x, y) \vdash \sigma(fx, fy)$$

holds (i.e., if $f$ represents a morphism from $(C, \rho)$ to $(D, \sigma)$ in $\boldsymbol{T}\mathcal{P}$), we say that $f$ is *well defined with respect to $\rho$ and $\sigma$*.

2. Let $(C, \rho)$ be an object in $\boldsymbol{F}\mathcal{P}$. We call a predicate $\varphi \in \mathcal{P}_C$ *compatible with $\rho$*, if the judgments

$$x \mid \varphi(x) \vdash \rho(x, x) \quad \text{and} \quad x, y \mid \varphi(x), \rho(x, y) \vdash \varphi(y)$$

hold in $\mathcal{P}$ (intuitively, this means that $\varphi$ is a union of equivalence classes of $\rho$). In Pitts' [20] terminology, a predicate on $C$ compatible with $\rho$ is precisely a strict unary relation on $(C, \rho)$. $\diamondsuit$

**Lemma 5.3** *Let $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ be a tripos.*

1. *The category $\boldsymbol{F}\mathcal{P}$ has finite limits.*

2. *The formula*

$$\mathsf{eqv}(p, q) \quad := \quad \mathsf{tr}(p) \leftrightarrow \mathsf{tr}(q)$$

*defines a partial equivalence relation on $\mathsf{Prop} \in \mathbb{C}$.*

3. $e : (C, \rho) \to (D, \sigma)$ *is an epimorphism in $\boldsymbol{F}\mathcal{P}$ iff*

$$y \mid \sigma(y, y) \vdash \exists x . \rho(x, x) \wedge \sigma(ex, y)$$

*holds in $\mathcal{P}$.*

4. *If $\varphi \in \mathcal{P}_D$ is a predicate which is compatible with $\sigma$, then*

$$\sigma|_\varphi(x, y) \quad := \quad \sigma(x, y) \wedge \varphi(x)$$

*is a partial equivalence relation on $D$, and*

$$\mathrm{id} : (D, \sigma|_\varphi) \rightarrowtail (D, \sigma)$$

*is a cocover in $\boldsymbol{F}\mathcal{P}$.*

5. *For any morphism $f : (C, \rho) \to (D, \sigma)$ in $\boldsymbol{F}\mathcal{P}$, the predicate*

$$\psi(y) \quad := \quad \exists x . \rho(x, x) \wedge \sigma(fx, y)$$

*on $D$ is compatible with $\sigma$. Furthermore, $f : (C, \rho) \to (D, \sigma)$ factors through $(D, \sigma|_\psi) \rightarrowtail (D, \sigma)$, giving rise to a epi-cocover factorization.*

6. The assignments

$$
\begin{aligned}
\varphi &\mapsto & \big((C, \rho|_\varphi) \rightarrowtail (C, \rho)\big) \\
\big((U, \nu) \rightarrowtail (C, \rho)\big) &\mapsto & \big(y \mid \exists x . \nu(x, x) \wedge \rho(mx, y)\big)
\end{aligned}
$$

constitute a bijection between the isomorphism classes of cocovers with codomain $(C, \rho)$ and predicates on $C$ compatible with $\rho$.

7. The bijection established in the previous item provides a more convenient characterization of the fibration of cocovers on $\boldsymbol{F}\mathcal{P}$. In this representation, the reindexing of a predicate $\varphi \in \mathcal{P}_D$ that is compatible with $\sigma$ along $f : (C, \rho) \to (D, \sigma)$ is given by

$$
x \mid \rho(x, x) \wedge \varphi(fx) \ .
$$

8. $\boldsymbol{F}\mathcal{P}$ has effective quotients of strong equivalence relations, and up to post-composition by isomorphism the regular epimorphisms are precisely the morphisms of the form $\mathrm{id} : (D, \sigma) \to (D, \tau)$ with $\tau(x, x) \vdash \sigma(x, x)$.

9. Regular epimorphisms in $\boldsymbol{F}\mathcal{P}$ are stable under pullback.

10. The presheaves

$$
\mathrm{sub}_c(- \times (C, \rho)) : (\boldsymbol{F}\mathcal{P})^{\mathsf{op}} \to \mathbf{Set}
$$

are representable.

11. $\boldsymbol{F}\mathcal{P}$ is a q-topos.

*Proof. Ad 1.* Binary products of $(C, \rho)$ and $(D, \sigma)$ are given by $(C \times D, \rho \bowtie \sigma)$, where $\rho \bowtie \sigma$ is defined as

$$
(\rho \bowtie \sigma)(c, d, c', d') \quad := \quad \rho(c, c') \wedge \sigma(d, d') .
$$

$(1, \top)$ is a terminal object ($\top$ denotes the greatest predicate in the fiber over $1 \times 1 \cong 1$).

An equalizer of $f, g : (C, \rho) \to (D, \sigma)$ is given by $\mathrm{id} : (C, \tau) \to (C, \rho)$, where $\tau$ is defined as

$$
\tau(x, y) \quad := \quad \rho(x, y) \wedge \sigma(fx, gx) .
$$

*Ad 2.* This follows from the transitivity and symmetry of logical equivalence.

*Ad 3.* We give the proof of this statement in detail to give the reader an idea of how to carry out this kind of argument in the internal language. Assume that $e : (C, \rho) \to (D, \sigma)$ in $\boldsymbol{F}\mathcal{P}$ such that $\sigma(y, y) \vdash \exists x . \rho(x, x) \wedge \sigma(ex, y)$ holds in $\mathcal{P}$, and that $g, h : (D, \sigma) \to (E, \eta)$ such that $ge = he$ as morphisms of $\boldsymbol{F}\mathcal{P}$, i.e., $\rho(x, x) \vdash \eta(g(ex), h(ex))$ in $\mathcal{P}$. We infer

$$
\dfrac{\dfrac{\sigma(y, z) \vdash \eta(gy, gz)}{\sigma(y, ex) \vdash \eta(gy, g(ex))} \quad \rho(x, x) \vdash \eta(g(ex), h(ex)) \quad \dfrac{\sigma(z, y) \vdash \eta(hz, hy)}{\sigma(ex, y) \vdash \eta(h(ex), hy)}}{\sigma(ex, y), \rho(x, x) \vdash \eta(gy, hy)} .
$$

Here we use as hypotheses that $g$ and $h$ are well defined with respect to $\sigma, \eta$, and that $ge = he$; then we substitute and reason by transitivity and symmetry of the relations.

We proceed by

$$\dfrac{\sigma(y,y) \;\vdash\; \exists x\,.\,\sigma(fx,y)\wedge\rho(x,x) \qquad \dfrac{\sigma(fx,y),\rho(x,x) \;\vdash\; \eta(gy,hy)}{\exists x\,.\,\sigma(fx,y)\wedge\rho(x,x) \;\vdash\; \eta(gy,hy)}}{\sigma(y,y) \;\vdash\; \eta(gy,hy)}\;.$$

Here we use the remaining assumption and the conclusion of the previous derivation. The conclusion says that $g = h$ as morphisms of $\boldsymbol{F}\mathcal{P}$, as desired.

Conversely assume that $e : (C,\rho) \to (D,\sigma)$ is an epimorphism. By the (semi-)universal property of $\mathsf{Prop}$, there exist morphisms $g,h : D \to \mathsf{Prop}$ in $\mathbb{C}$ such that

$$\top \dashv\vdash \mathsf{tr}(gy) \quad\text{and}\quad \exists x\,.\,\rho(x,x)\wedge\sigma(ex,y) \dashv\vdash \mathsf{tr}(hy).$$

It is easy to see that these morphisms are well defined with respect to $\sigma$ and $\mathsf{eqv}$ and thus give rise to morphisms $g,h : (D,\sigma) \to (\mathsf{Prop},\mathsf{eqv})$ in $\boldsymbol{F}\mathcal{P}$. Moreover, these morphisms are equalized by $e$, i.e., $ge = he : (C,\rho) \to (\mathsf{Prop},\mathsf{eqv})$, as a calculation in the logic of $\mathcal{P}$ shows. Now $e : (C,\rho) \to (D,\sigma)$ is an epimorphism, and thus we have $g = h : (D,\sigma) \to (\mathsf{Prop},\mathsf{eqv})$, which is equivalent to the validity of the judgment $\sigma(y,y) \vdash \mathsf{eqv}(gy,hy)$. By unfolding the definition of $\mathsf{eqv}$ and making use of the characterizations of $g,h$, we thus deduce $\sigma(y,y) \vdash \exists x\,.\,\rho(x,x)\wedge\sigma(ex,y)$, as desired.

*Ad 4.* It is straightforward to verify that $\sigma|_\varphi$ is a partial equivalence relation, and that $\mathrm{id}_D$ is well defined with respect to $\sigma|_\varphi$ and $\sigma$.

To see that $\mathrm{id} : (D,\sigma|_\varphi) \to (D,\sigma)$ is a monomorphism, assume that $f,g : (C,\rho) \to (D,\sigma|_\varphi)$ such that the compositions with $\mathrm{id} : (D,\sigma|_\varphi) \to (D,\sigma)$ are equal in $\boldsymbol{F}\mathcal{P}$, i.e., $\rho(x,x) \vdash \sigma(fx,gx)$ in $\mathcal{P}$. Because $f : (C,\rho) \to (D,\sigma|_\varphi)$, we also have $\rho(x,x) \vdash \varphi(fx)$, and the conjunction $\rho(x,x) \vdash \sigma(fx,gx)\wedge\varphi(fx)$ of the two statements is equivalent to $f = g : (C,\rho) \to (D,\sigma|_\varphi)$.

It remains to show that the map is even a cocover. To see this, consider the square

$$\begin{array}{ccc}
(D,\sigma) & \xrightarrow{\ f\ } & (C,\rho|_\varphi) \\
{\scriptstyle e}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{id}} \\
(E,\eta) & \xrightarrow[\ g\ ]{} & (C,\rho)
\end{array}$$

with $e : (D,\sigma) \to (E,\eta)$ an epimorphism. We have to verify that $g$ is well defined with respect to $\eta$ and $\rho|_\sigma$, which amounts to establishing $\eta(y,y) \vdash \varphi(gx)$. This can be derived by applying the characterization of epimorphism of the previous item to $e$. Finally it is easy to see that the two induced triangles commute.

*Ad 5.* This is completely straightforward using the previously established facts.

*Ad 6.* Also straightforward.

*Ad 7.* Also straightforward.

*Ad 8.* Via the bijection established in 6, the strong equivalence relations on $(C,\rho)$ correspond to predicates $\tau \in \mathcal{P}_{C\times C}$ which are partial equivalence relations, compatible with $\rho \bowtie \rho$, and furthermore total with respect to $\rho$ in the sense that $\rho(x,x) \vdash \tau(x,x)$. It turns out that for partial equivalence relations, the conjunction of totality and compatibility can be expressed in a slightly

simplified manner: A partial equivalence relation $\tau$ is compatible with respect to $\rho \bowtie \rho$ and total (with respect to $\rho$), iff the judgments

$$\tau(x,x) \vdash \rho(x,x) \quad \text{and} \quad \rho(x,y) \vdash \tau(x,y) \tag{5.1}$$

hold in $\mathcal{P}$. Thus, summing up, the strong equivalence relations on $(C,\rho)$ correspond to the predicates on $C$ which are partial equivalence relations *and* satisfy the judgments (5.1).

Given such a representative $\tau$ of a strong equivalence relation, the obvious candidate for the quotient map is

$$\text{id} : (C,\rho) \to (C,\tau).$$

We have to verify that

$$(C \times C, (\rho \bowtie \rho)|_\tau) \rightrightarrows (C,\rho) \to (C,\tau)$$

is indeed a coequalizer diagram. This is straightforward and left to the reader (Hint: Show first that the map is an epimorphism using 3).

To verify that the quotient is effective, note that the kernel pair of a map $f : A \to B$ in a finite limit category can be computed as the pullback of the diagonal $\delta : B \to B \times B$ along $f \times f$, and then use the representation of the fibration of cocovers given in 7.

The second part of the claim follows because every regular epimorphism is the quotient of its kernel pair, which is a strong equivalence relation.

*Ad 9.* If we pull a regular epimorphism in 'normalized presentation' id : $(D,\sigma) \to (D,\tau)$ with $\tau(x,x) \vdash \sigma(x,x)$ back along a morphism $f : (C,\rho) \to (D,\tau)$, we get a square of the form

$$
\begin{array}{ccc}
(C \times D, \theta) & \xrightarrow{\pi_1} & (D,\sigma) \\
{\scriptstyle \pi_0}\downarrow & & \downarrow{\scriptstyle \text{id}} \\
(C,\rho) & \xrightarrow{\quad f \quad} & (D,\tau)
\end{array} \quad,
$$

where $\theta(c,d,c',d') := \rho(c,c') \wedge \sigma(d,d') \wedge \tau(fc,d)$. To see that $(C \times D, \theta) \to (C,\rho)$ is a regular epimorphism, observe that it is isomorphic to $(C \times D, \theta) \to (C \times D, \xi)$ with $\xi(c,d,c',d') := \rho(c,c') \wedge \tau(d,d') \wedge \tau(fc,d)$ via the isomorphism $\langle \text{id}, f \rangle : (C,\rho) \rightleftarrows (C \times D, \xi) : \pi_0$, and $(C \times D, \theta) \to (C \times D, \xi)$ is of the canonical form for regular epimorphisms.

*Ad 10.* For a given object $(C,\rho)$ of $\boldsymbol{F}\mathcal{P}$, we define its power object as $(\mathfrak{P}C, \mathfrak{P}\rho)$ with

$$
\begin{aligned}
(\mathfrak{P}\rho)(m,n) \quad &:= \quad (\forall x \,.\, x \in m \Rightarrow \rho(x,x)) \\
&\wedge (\forall x,y \,.\, x \in m \wedge \rho(x,y) \Rightarrow y \in m) \\
&\wedge (\forall x \,.\, x \in m \Leftrightarrow x \in n)
\end{aligned}
$$

The cocover corresponding to the element predicate is represented by

$$x \in_\rho m \quad := \quad x \in_C m \wedge (\mathfrak{P}\rho)(m,m)$$

The verification that these definitions make sense and give rise to a representation of $\mathrm{sub}_c(-\times(C,\rho))$ are tedious, but straightforward (again, the use of 7 is essential).

*Ad 11.* This follows from items 1, 8 and 10. ∎

**Remark 5.4** In Section 1.2, we stated that we always want to work with *chosen* limits and colimits whenever we speak about such structures. In light of this statement, if we asserted in the previous lemma that $\boldsymbol{F}\mathcal{P}$ always *has* finite limits and quotients of strong equivalence relations, from now on we think of it as equipped with the choices of limits and quotients whose constructions were sketched in the proof. On the contrary, we are happy with the mere *existence* of power objects. ◇

After having described the object part of $\boldsymbol{F}$, we come to its action on 1-cells. Assume that $(F,\Phi):\mathcal{P}\to\mathcal{Q}$ is a tripos morphism. The functor $\boldsymbol{F}(F,\Phi):\boldsymbol{F}\mathcal{P}\to\boldsymbol{F}\mathcal{Q}$ is given by

$$
\begin{array}{ccc}
(C,\rho) & \mapsto & (FC,\Phi_{C,C}\rho) \\
\downarrow{\scriptstyle f} & \mapsto & \downarrow{\scriptstyle Ff} \\
(D,\sigma) & \mapsto & (FD,\Phi_{D,D}\sigma)
\end{array}\quad.
$$

See Definition 3.12.5 for the notation $\Phi_{C,C}\rho$. To see that this construction is well defined, we have to show that $\Phi_{C,C}\rho,\Phi_{D,D}\sigma$ are partial equivalence relations and that $f\mapsto Ff$ is compatible with the partial equivalence relations on $\mathbb{C}(C,D)$ and $\mathbb{D}(FC,FD)$. This is a consequence of Corollary 3.14. Functoriality is clear, and furthermore we have:

**Lemma 5.5** *For every tripos morphism* $(F,\Phi):\mathcal{P}\to\mathcal{Q}$, $\boldsymbol{F}(F,\Phi)$ *preserves finite limits (thus in particular cocovers), and covers.*

*If* $\Phi$ *commutes with existential quantification* along projections, *then* $\boldsymbol{F}(F,\Phi)$ *also maps epimorphisms to epimorphisms.*

*Proof.* These claims follow from Corollary 3.14, using the construction of finite limits in the proof of Lemma 5.3.1, the characterization of regular epimorphisms in 5.3.8, and the characterization of epimorphisms given in 5.3.3. ∎

Now we come to the action of $\boldsymbol{F}$ on 2-cells. Let $\eta:(F,\Phi)\to(G,\Gamma):\mathcal{P}\to\mathcal{Q}$ be a tripos transformation. Then we can define a natural transformation $\boldsymbol{F}\eta:\boldsymbol{F}\mathcal{P}\to\boldsymbol{F}\mathcal{Q}$ whose component at $(C,\rho)$ is

$$
\eta_C:(FC,\Phi\rho)\to(GC,\Gamma\rho)
$$

The verification that this make sense is straightforward and left to the reader.

Now the description of $\boldsymbol{F}$ is complete, and we can show:

**Lemma 5.6** *The previous constructions establish a* strict special functor

$$
\boldsymbol{F}:\mathbf{Trip}\to\mathbf{QTop}.
$$

*Proof.* Well definedness follows from Lemmas 5.3.11 and 5.5. By strict, we mean that the construction is 2-functorial. This is straightforward to verify; note that here it is important that the fibers of the triposes are posets and not mere preorders. Finally, $\boldsymbol{F}$ is special because of Lemma 5.5. ∎

### 5.1.3 The unit $\eta$

The unit of $\boldsymbol{F} \dashv \boldsymbol{S}$ is a special transformation $\boldsymbol{\eta} : \mathrm{id}_{\mathbf{Trip}} \to \boldsymbol{SF}$. Its component at $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ is the tripos transformation

$$\boldsymbol{\eta}_{\mathcal{P}} = (D_{\mathcal{P}}, \Delta_{\mathcal{P}}) : \mathcal{P} \to \boldsymbol{SF}\mathcal{P},$$

which we describe now.

$D_{\mathcal{P}} : \mathbb{C} \to \boldsymbol{F}\mathcal{P}$ is defined by

$$
\begin{array}{ccc}
C & \mapsto & (C, =) \\
\downarrow f & \mapsto & \downarrow f \\
D & \mapsto & (D, =)
\end{array}
\quad .
$$

Using the same techniques as in the previous section, it is quite easy to see that $D_{\mathcal{P}}$ is a finite product preserving functor. *(It does, however, in general not preserve equalizers!)*

$\Delta_{\mathcal{P}} : \mathbb{X} \to \mathbf{coc}(\boldsymbol{T}\mathcal{P})$ maps predicates $\varphi \in \mathcal{P}_C$ to subobjects

$$(C, =|_{\varphi}) \rightarrowtail (C, =).$$

It is not difficult to verify that this defines a fibered functor over $D_{\mathcal{P}}$, and that $(D_{\mathcal{P}}, \Delta_{\mathcal{P}})$ is a regular tripos morphism.

For a tripos transformation $(F, \Phi) : \mathcal{P} \to \mathcal{Q}$, the transformation constraint



has components

$$\eta_{(F,\Phi),C} = \mathrm{id}_{FC} : (FC, =) \to (FC, \Phi=). \tag{5.2}$$

The verifications that $\eta_{(F,\Phi)}$ is a natural transformation that gives rise to a tripos transformation, and that $\eta$ is a lax transformation are left to the reader. To verify that $\eta$ is even a special transformation, we have to show that $\eta_{(F,\Phi)}$ is invertible whenever $(F, \Phi)$ commutes with existential quantification. This is apparent from the presentation (5.2) of the components of $\eta_{(F,\Phi)}$, because the equality predicate is defined in terms of existential quantification, and thus $(\Phi=) = (=)$.

### 5.1.4 The counit $\epsilon$

First of all a linguistic remark. In this section, we consider the category $\boldsymbol{FS}\mathcal{C}$, which is formally obtained by applying a tripos theoretic construction, defined

in the internal language of triposes, to the fibration of cocovers on a q-topos. This leads us to reasoning in $\boldsymbol{SC}$ using not the *core calculus* of Section 4 but the higher order logic of Section 3. This doesn't matter too much since higher order intuitionistic can be embedded into the core calculus anyway, but has the formal consequence that predicates will not be $\Omega$-valued morphisms as in Section 4, but equivalence classes of cocovers.

The components of the counit $\boldsymbol{\epsilon} : \boldsymbol{FS} \to \mathrm{id}_{\mathbf{QTop}}$ of $\boldsymbol{F} \dashv \boldsymbol{S}$ are functors

$$\epsilon_{\mathcal{C}} : \boldsymbol{FS}\mathcal{C} \to \mathcal{C},$$

which are defined as follows. An object of $\boldsymbol{FS}\mathcal{C}$ is a pair $(C, [r])$, where $C$ is an object of $\mathcal{C}$ and $[r]$ is an equivalence class of cocovers $r : R \rightarrowtail C \times C$ which is a partial equivalence relation in $\boldsymbol{SC}$. Let $m : C_0 \rightarrowtail C$ be a representative[15] of the predicate $(x \mid r(x, x))$. Then the predicate $[r_0](x, y) := [r](mx, my)$ is an equivalence relation on $C_0$, and we define $C/r$ to be its quotient in $\mathcal{C}$. We thus have a subquotient

$$C/r \xleftarrow{e} C_0 \xrightarrowtail{m} C.$$

Given another object $(D, [s])$ and a morphism $f : (C, [r]) \to (D, [s])$ in $\boldsymbol{FS}\mathcal{C}$, we claim that there is a unique pair of mediators for the diagram

$$
\begin{array}{ccccc}
C/r & \xleftarrow{\ e\ } & C_0 & \xrightarrow{\ m\ } & C \\
\ \vert & & \vert & & \vert \\
h\ \vert & & \vert\ g & & \Big\vert\ f \\
\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
D/s & \xleftarrow[\ e'\ ]{} & D_0 & \xrightarrow[\ m'\ ]{} & D
\end{array}
\qquad (5.3)
$$

Uniqueness is clear, since $m'$ is mono and $e$ is epic, and the existence of $g$ and $h$ follows from the validity of $[r](x, x) \vdash [s](fx, fx)$ and $[r_0](x, y) \vdash [s_0](gx, gy)$, respectively.

We define

$$\epsilon_{\mathcal{C}}(C, [r]) = C/r, \quad \epsilon_{\mathcal{C}}(D, [s]) = D/s, \quad \text{and} \quad \epsilon_{\mathcal{C}}(f) = h.$$

Functoriality of $\epsilon_{\mathcal{C}}$ follows from universal properties. Furthermore, we have the following:

**Lemma 5.7** *The functor $\epsilon_{\mathcal{C}} : \boldsymbol{FS}\mathcal{C} \to \mathcal{C}$ preserves finite limits, epimorphisms and regular epimorphisms. This means that it is a regular 1-cell in the pre-equipment* **QTop***.*

*Proof.* The terminal object is clearly preserved, since it is given by $(1, [\mathrm{id}])$ in $\boldsymbol{FS}\mathcal{C}$.

For products, remember that a product of $(C, [r])$ and $(D, [s])$ is given by $(C \times D, [r] \bowtie [s])$. We form the subquotient spans $Q \xleftarrow{e} C_0 \xrightarrowtail{m} C$ and $R \xleftarrow{p} D_0 \xrightarrowtail{n} D$ corresponding to the strong equivalence relations $[r]$ and $[s]$. We want

---

[15]The constructions here rely on the presence of chosen limits and colimits, and depend on the choice or representatives of predicates, for example for the construction of certain pullbacks. Thus, it seems that our decision to quotient out the subobject fibrations forces us here to use choice. This is not true, however, since we can always obtain a canonical representative of a subobject as $\chi^* \mathsf{t}$, where $\chi$ is the characteristic function of the subobject.

to show that $Q \times R \overset{e \times p}{\leftarrow} C_0 \times D_0 \overset{m \times n}{\rightarrowtail} C \times D$ is a subquotient span for $[r] \bowtie [s]$. First of all, we note that the classes of cocovers and regular epimorphisms are both closed under pullback and composition, and thus under products[16]. This implies that the legs of the product span are again a regular epi and a cover.

To see that $[m \times n](x, y) = [r](x, x) \wedge [s](y, y)$, observe that every predicate $[m]$ is equivalent to $\big(y \mid \exists x . [m]x = y\big)$. Thus $[m \times n]$ coincides with $\big(x, y \mid \exists x_0, y_0 . mx_0 = x \wedge ny_0 = y\big)$, which is equivalent to $\big(x, y \mid (\exists x_0 . mx_0 = x) \wedge (\exists y_0 . ny_0 = y)\big)$ and to $\big(x, y \mid r(x, x) \wedge s(y, y)\big)$.

Next we want to show that $e \times p$ is a coequalizer of the components of $[r_0] \bowtie [s_0] = \big(x, y, x', y' \mid [r](mx, mx') \wedge [s](ny, ny')\big)$. Because $e \times p$ is a regular epimorphism, it suffices to show that its kernel pair is equivalent to $[r_0] \bowtie [s_0]$. This follows immediately, because the kernel pair can be expressed in the internal language as $\big(x, y, x', y' \mid ex = ex' \wedge py = py'\big)$.

A similar argument shows the preservation of equalizers.

Now let $f : (C, [r]) \to (D, [s])$ be an epimorphism in $\boldsymbol{FSC}$. To show that its image under $\epsilon_{\mathcal{C}}$ — i.e. the arrow $h$ in diagram (5.3) is again an epimorphism, we have to show that $\big(v \mid \vdash \exists u . fu = v\big)$ holds, but this can be derived from the valid judgments $\big(x \mid [r](x, x) \vdash \exists x_0 . mx_0 = x\big)$, $\big(u \mid \exists x_0 . ex_0 = u\big)$, $\big(y_0 \mid \vdash [s](ny_0, ny_0)\big)$ and $\big(y \mid [s](y, y) \vdash \exists x . [r](x, x) \wedge fx = y\big)$.

Finally, we have to show that $\epsilon_{\mathcal{C}}$ preserves regular epimorphisms. For a regular epimorphism in canonical representation, i.e. $\mathrm{id} : (C, [r]) \to (C, [s])$ with $[s](x, x) \vdash [r](x, x)$, the diagram of subquotient spans looks as follows

$$
\begin{array}{ccccc}
C/r & \overset{e}{\leftarrowtail} & C_0 & \rightarrowtail & C \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} \\
C/s & \underset{e'}{\leftarrowtail} & C_0 & \rightarrowtail & C
\end{array} ,
$$

and the claim follows from the facts that in a composition the second arrow is a cover whenever the composition is a cover (follows from orthogonality), and covers coincide with regular epimorphisms (Lemma 4.4). ∎

Next we define the transformation constraint

$$
\begin{array}{ccc}
\boldsymbol{FSC} & \overset{\boldsymbol{FSF}}{\longrightarrow} & \boldsymbol{FSD} \\
\downarrow{\scriptstyle \epsilon_{\mathcal{C}}} & \overset{\epsilon_F}{\Swarrow} & \downarrow{\scriptstyle \epsilon_{\mathcal{D}}} \\
\mathcal{C} & \underset{F}{\longrightarrow} & \mathcal{D}
\end{array}
$$

The subquotient span $C/r \overset{e}{\leftarrow} C_0 \overset{m}{\rightarrowtail} C$ associated to the image of an object $(C, [r]) \in \boldsymbol{FSC}$ under $\epsilon_{\mathcal{C}}$ gets mapped to

$$F(C/r) \overset{Fe}{\leftarrow} FC_0 \overset{Fm}{\rightarrowtail} FC \tag{5.4}$$

---

[16] The regular epimorphisms are closed under composition because they coincide with the covers.

by $F$. $Fe$ is not necessarily an epimorphism any more, but because $F$ preserves finite limits, $F([r]|_{[m]})$ is still its kernel.

On the other side of the diagram, $(C, [r])$ gets mapped first to $(FC, [Fr])$ by $\boldsymbol{FSF}$, and then to the object $C'_0/Fr$ in the subquotient span

$$FC/Fr \xleftarrow{m'} C'_0 \xrightarrowtail{e'} FC \tag{5.5}$$

by $\epsilon_{\mathcal{D}}$. Now the support $C'_0$ of $Fr$ is isomorphic to the image $FC_0$ of the support of $m$ under $F$, because the support is defined as a pullback, and those are preserved by $F$. If we combine (5.4) and (5.5) into a diagram

$$\begin{array}{ccc}
FC/Fr \xleftarrow{\quad e' \quad} C'_0 \xrightarrowtail{\quad m' \quad} FC \\
\downdownarrows \qquad\qquad \cong \Big\downarrow \quad\nearrow_{Fm} \\
F(C/r) \xleftarrow[Fe]{\quad} FC'_0
\end{array} \tag{5.6}$$

we see that the universal property of the quotient map $e'$ allows us to construct a mediating arrow from $FC/Fr$ to $F(C/r)$ (which is moreover monic since $e'$ and $Fe$ have isomorphic kernel pairs). This map is the component $\epsilon_{F,(C,r)}$ of the transformation constraint $\epsilon_F$. To prove that $\epsilon$ is a well defined oplax transformation, we have to verify naturality of $\epsilon_F$ and the transformation axioms for $\epsilon$, which is straightforward; all the commutations follow from universal properties.

To verify that $\epsilon$ is moreover a special transformation, we have to verify that $\epsilon_F$ is invertible whenever $F$ is regular. This becomes clear when looking at diagram (5.6). When $F$ is regular then $Fe$ is a regular epimorphism, because those are preserved by regular functors between q-toposes by definition. We then have two regular epimorphisms $e'$, $Fe$ with the same kernel pair and thus the mediating arrow is an isomorphism.

### 5.1.5 The modification $\nu$

The components of the modification

$$\nu : \boldsymbol{\epsilon_F} \circ \boldsymbol{F\eta} \to \mathrm{id}_{\boldsymbol{F}} : \boldsymbol{F} \to \boldsymbol{F} : \mathbf{Trip} \to \mathbf{QTop}$$

are natural transformations

$$\nu_{\mathcal{P}} : \boldsymbol{\epsilon_{F\mathcal{P}}} \circ \boldsymbol{F\eta_{\mathcal{P}}} \to \mathrm{id}_{\boldsymbol{F\mathcal{P}}} : \boldsymbol{F}\mathcal{P} \to \boldsymbol{F}\mathcal{P},$$

whose components are in turn morphisms of type

$$\nu_{\mathcal{P},(C,\rho)} : \boldsymbol{\epsilon_{F\mathcal{P}}}(\boldsymbol{F\eta_{\mathcal{P}}}(C,\rho)) \to (C,\rho) : \boldsymbol{F}\mathcal{P}.$$

Now $\boldsymbol{F\eta_{\mathcal{P}}}(C,\rho)$ is the object $((C,=),[\tilde{\rho}])$ in $\boldsymbol{FSF}\mathcal{P}$, where

$$\tilde{\rho} : (C \times C, (= \bowtie =)|_{\rho}) \rightarrowtail (C \times C, (= \bowtie =))$$

is the subobject induced by $\rho$.

General objects of $\boldsymbol{FSF}\mathcal{P}$ are of the form $((C,\rho),[r])$ where $r : U \rightarrowtail (C \times C, \rho \bowtie \rho)$ is a partial equivalence relation on $(C,\rho)$ which can be represented by a predicate $\tau \in \mathcal{P}_{C \times C}$.

We know from the proof of Lemma 5.3.8 and from Section 5.1.4 how to construct a subquotient span for the subquotient of $(C, \rho)$ by $r$. It is given by

$$(C, \tau) \leftarrow (C, \rho|_{\mathsf{supp}(\tau)}) \rightarrowtail (C, \rho).$$

We pointed out earlier that we have to be careful not to mix up chosen limits and colimits with arbitrary limits and colimits, so at this point we remark that $(C, \tau)$ is not precisely the image of $((C, \rho), [r])$ under $\boldsymbol{\epsilon}_{\boldsymbol{F}\mathcal{P}}$, because the object in the middle of the subquotient span (the support of the partial equivalence relation) is defined by a pullback in $\boldsymbol{F}\mathcal{P}$ and if we carry out the canonical pullback construction as an equalizer of a product, and use the choices of products and equalizers described in the proof of Lemma 5.3, we end up with a partial equivalence relation on $C \times C \times C$. Thus, all we can say at this point is that there is a canonical isomorphism $\boldsymbol{\epsilon}_{\boldsymbol{F}\mathcal{P}}((C, \rho), r) \cong (C, \tau)$ which we obtain by comparing the subquotient span above with the one arising from the canonical constructions.

Instantiating with $\boldsymbol{F}\boldsymbol{\eta}_{\mathcal{P}}(C, \rho) = ((C, =), [\tilde{\rho}])$, we obtain the desired isomorphism

$$\nu_{\mathcal{P},(C,\rho)} : \boldsymbol{\epsilon}_{\boldsymbol{F}\mathcal{P}}(\boldsymbol{\eta}_{\mathcal{P}}(C, \rho)) \xrightarrow{\cong} (C, \rho).$$

The reader is invited to verify that $\nu_{\mathcal{P}}$ is indeed a natural transformation and that $\nu$ is a modification.

### 5.1.6 The modification $\mu$

The modification $\mu$ has the type

$$\mu \ : \ \mathrm{id}_{\boldsymbol{S}} \to \boldsymbol{S}\boldsymbol{\epsilon} \circ \boldsymbol{\eta}_{\boldsymbol{S}} \ : \ \boldsymbol{S} \to \boldsymbol{S} \ : \ \mathbf{QTop} \to \mathbf{Trip}.$$

Its components are tripos transformations

$$\mu_{\mathcal{C}} \ : \ \mathrm{id}_{\boldsymbol{S}\mathcal{C}} \to \boldsymbol{S}\boldsymbol{\epsilon}_{\mathcal{C}} \circ \boldsymbol{\eta}_{\boldsymbol{S}\mathcal{C}} \ : \ \boldsymbol{S}\mathcal{C} \to \boldsymbol{S}\mathcal{C}$$

whose components are morphisms

$$\mu_{\mathcal{C},C} \ : \ C \to \epsilon_{\mathcal{C}}(D_{\boldsymbol{S}\mathcal{C}}C)$$

in $\mathcal{C}$. Applying the definition in the previous sections, we see that $\epsilon_{\mathcal{C}}(D_{\boldsymbol{S}\mathcal{C}}C) = C/\delta_C$ in the (degenerate) subquotient span

$$C/\delta_C \xleftarrow{\cong} \bullet \xrightarrow{\cong} C,$$

i.e. the (sub)quotient of $C$ by the diagonal predicate. Of course a possible subquotient span of $C$ by $\delta_C$ would be the span consisting of two identities, but we can not use this since we pledged earlier always to use instances of limits and (co)limits that are given by some construction of chosen (co)limits which is given to us together with the definition of $\mathcal{C}$. We now define $\mu_{\mathcal{C},C}$ in the most straightforward way — as composition of the two isomorphisms in the subquotient span.

It remains to check that this defines a natural transformation $\mu_{\mathcal{C}} : \mathrm{id}_{\mathcal{C}} \to \epsilon_{\mathcal{C}} \circ D_{\boldsymbol{S}\mathcal{C}}$ which induces a tripos transformation $\mu_{\mathcal{C}} : \mathrm{id}_{\boldsymbol{S}\mathcal{C}} \to \boldsymbol{S}\boldsymbol{\epsilon}_{\mathcal{C}} \circ \boldsymbol{\eta}_{\boldsymbol{S}\mathcal{C}}$, and that these tripos transformations give rise to a modification $\mu : \mathrm{id}_{\boldsymbol{S}} \to \boldsymbol{S}\boldsymbol{\epsilon} \circ \boldsymbol{\eta}_{\boldsymbol{S}}$.

The naturality of $\mu_{\mathcal{C}}$ follows directly from the definition of $\boldsymbol{\epsilon}_{\mathcal{C}}$, and the fact that $\mu_{\mathcal{C}}$ is a tripos transformation is also easy to see. The verification of the modification axiom boils down to showing that for any finite limit preserving functor $F : \mathcal{C} \to \mathcal{D}$ between q-toposes and any $C \in \mathcal{C}$, the square

$$
\begin{array}{ccc}
FC/\delta_C & \xrightarrow{\boldsymbol{\epsilon}_{\mathcal{D}}(\boldsymbol{\eta}_{SF,C})} & FC/F\delta_C \\
\mu_{\mathcal{D},FC} \Big\uparrow & & \Big\downarrow \boldsymbol{\epsilon}_{F,(C,[\delta])} \\
FC & \xrightarrow[F\mu_{\mathcal{C},C}]{} & F(C/\delta_C)
\end{array}
$$

commutes. A careful inspection of the constructions shows that all the arrows arise as mediating arrows between spans which are anchored at $FC$ on one side (and furthermore have isomorphic legs), and from this commutation is evident.

### 5.1.7  The axioms

To check that the data $F, S, \boldsymbol{\eta}, \boldsymbol{\epsilon}, \mu, \nu$ form a special biadjunction, we have to check the equalities of modifications stated in (2.2). Equality of modifications is componentwise equality, and for the components, which are tripos transformations for the first axiom, equality is in turn componentwise equality of the underlying natural transformations. By evaluating at $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ and $C \in \mathbb{C}$, we see that we have to check the commutativity of the following square.

$$
\begin{array}{ccc}
\boldsymbol{\epsilon}_{\boldsymbol{F}\mathcal{P}}(D_{\boldsymbol{SF}\mathcal{P}}(D_{\mathcal{P}}C)) & \xrightarrow{\boldsymbol{\epsilon}_{\boldsymbol{F}\mathcal{P}}(\boldsymbol{\eta}_{\boldsymbol{\eta}_{\mathcal{P}},C})} & \boldsymbol{\epsilon}_{\boldsymbol{F}\mathcal{P}}(\boldsymbol{F}\boldsymbol{\eta}_{\mathcal{P}}(D_{\mathcal{P}}C)) \\
\mu_{\boldsymbol{F}\mathcal{P},D_{\mathcal{P}}C} \Big\uparrow & & \Big\downarrow \nu_{\mathcal{P},D_{\mathcal{P}}C} \\
D_{\mathcal{P}}C & \xrightarrow[\mathrm{id}]{} & D_{\mathcal{P}}C
\end{array}
$$

Instantiating the constructions, we obtain

$$
\begin{array}{ccc}
(C,=)/= & \xrightarrow{\boldsymbol{\epsilon}_{\boldsymbol{F}\mathcal{P}}(\boldsymbol{\eta}_{\boldsymbol{\eta}_{\mathcal{P}},C})} & (C,=)/\Delta= \\
\mu_{\boldsymbol{F}\mathcal{P},(C,=)} \Big\uparrow & & \Big\downarrow \nu_{\mathcal{P},(C,=)} \\
(C,=) & \xrightarrow[\mathrm{id}]{} & (C,=)
\end{array} \quad .
$$

To see that this commutes, observe that the underlying arrows in $\mathbb{C}$ are all equal to $\mathrm{id}_C$, regardless of the partial equivalence relations.

The verification of the second axiom amounts to checking that for every q-topos $\mathcal{C}$ and every $(C, [r]) \in \boldsymbol{FS}\mathcal{C}$, the square

$$
\begin{array}{ccc}
\boldsymbol{\epsilon}_{\mathcal{C}}(\boldsymbol{FS}\boldsymbol{\epsilon}_{\mathcal{C}}(\boldsymbol{F}\boldsymbol{\eta}_{\boldsymbol{S}\mathcal{C}}(C,[r]))) & \xrightarrow{\boldsymbol{\epsilon}_{\boldsymbol{\epsilon}_{\mathcal{C}}}(\boldsymbol{F}\boldsymbol{\eta}_{\boldsymbol{S}\mathcal{C}}(C,[r]))} & \boldsymbol{\epsilon}_{\mathcal{C}}(\boldsymbol{\epsilon}_{\boldsymbol{FS}\mathcal{C}}(\boldsymbol{F}\boldsymbol{\eta}_{\boldsymbol{S}\mathcal{C}}(C,[r]))) \\
\boldsymbol{\epsilon}_{\mathcal{C}}(\boldsymbol{F}\mu_{\mathcal{C}}(C,[r])) \Big\uparrow & & \Big\downarrow \boldsymbol{\epsilon}_{\mathcal{C}}(\nu_{\boldsymbol{S}\mathcal{C}}\boldsymbol{S}\mathcal{C}(C,[r])) \\
\boldsymbol{\epsilon}_{\mathcal{C}}(C,[r]) & \xrightarrow[\mathrm{id}]{} & \boldsymbol{\epsilon}_{\mathcal{C}}(C,[r])
\end{array}
$$

commutes in $\mathcal{C}$. The verification of this is cumbersome, but in the end it boils down to observing that different mediating arrows between spans anchored at $C$ are the same, which is a similar argument to the one we sketched to establish that $\mu$ is a modification.

## 5.2 The biadjunction $T \dashv U$ between q-toposes and toposes

Now we come to the special biadjunction $(T \dashv U, \eta, \epsilon, \nu, \mu)$ between the pre-equipments of q-toposes and toposes Fortunately, this is a lot easier and less technical than the biadjunction $F \dashv S$ of the previous section.

### 5.2.1 The special functors $T$ and $U$

The functor

$$U : \mathbf{Top} \to \mathbf{QTop}$$

is just the inclusion. It is well defined because every topos is a q-topos and in a topos epimorphisms and regular epimorphisms coincide and therefore any functor that preserves the former automatically preserves the latter.
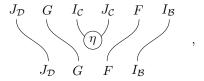
The object part of the functor

$$T : \mathbf{QTop} \to \mathbf{Top}$$

was already given in Lemma 4.15; $TC$ is the full subcategory of $\mathcal{C}$ on the coarse objects.

The action of $T$ on functors and natural transformations is given by

$$
\begin{aligned}
TF &= J_{\mathcal{D}} \ F \ I_{\mathcal{C}} & \text{where} \quad & F : \mathcal{C} \to \mathcal{D} \\
T\theta &= J_{\mathcal{D}} \ \theta \ I_{\mathcal{C}} & \text{where} \quad & \theta : F \to G : \mathcal{C} \to \mathcal{D}.
\end{aligned}
$$

It follows from Lemma 4.15.7 that $TF$ preserves finite limits, and from 4.15.9 that $TF$ is regular whenever $F$ is regular.

For $F : \mathcal{B} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$, the composition constraint $T(GF) \to TG\,TF$ is given by
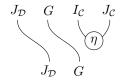


and the identity constraint $T\mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ is given by



The axioms for oplax functors follow formally from 2-categorical reasoning in **Cat**, using only the axioms for the adjunctions $J_{\mathcal{C}} \dashv I_{\mathcal{C}}$.

To show that $T$ is in fact a special functor, it remains to show that $T(GF) \to TG\,TF$ is invertible whenever $G$ is regular. This is a direct consequence of the following lemma.

**Lemma 5.8** *The natural transformation*



*is invertible whenever $G$ regular.*

*Proof.* Let $C \in \mathcal{C}$. The component at $C$ of the transformation in question is the mediating arrow in the diagram



Now if $G$ is regular, then $G\eta_C$ is also an epimorphism, and by coarseness of $\overline{GC}$, there exists a mediating arrow in the opposite direction, which turns out to be inverse (this follows because the other arrows are epimorphisms). ∎

### 5.2.2 The special transformations $\eta$ and $\epsilon$

The unit $\boldsymbol{\eta} : \mathrm{id}_{\mathbf{QTop}} \to \boldsymbol{U}\boldsymbol{T}$ of $\boldsymbol{T} \dashv \boldsymbol{U}$ is quite easy to define. Its components are just the reflection functors $J_{\mathcal{C}}$. The definition of special transformation requires them to be regular, and indeed they are, by Lemma 4.15.7.

The transformation constraint of $\boldsymbol{\eta}$ at $F : \mathcal{C} \to \mathcal{D}$ is given by



where $\eta$ is the unit of $J_{\mathcal{C}} \dashv I_{\mathcal{C}}$. It follows from general abstract nonsense that this does indeed define an oplax transformation, which is furthermore special as follows from Lemma 5.8.

Now we come to the counit. First of all, a little stylistic/philosophical remark to explain a design decision. Previously, we assumed that for every q-topos $\mathcal{C}$ we have a chosen reflection $J_{\mathcal{C}} \dashv I_{\mathcal{C}} : \boldsymbol{T}\mathcal{C} \to \mathcal{C}$ into the subtopos of coarse objects. Now if $\mathcal{C}$ is already a topos, then $\boldsymbol{T}\mathcal{C} = \mathcal{C}$ and we could choose $I_{\mathcal{C}} = J_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}$. However, such a choice depending on whether a q-topos is already a topos is horribly nonconstructive and not really necessary. Thus, we are content with the fact that whenever the q-topos is already a topos, we have an *adjoint auto-equivalence* $J_{\mathcal{C}} \dashv I_{\mathcal{C}}$.

We define the component of the counit of $\boldsymbol{T} \dashv \boldsymbol{U}$ at a topos $\mathcal{E}$ to be

$$d\epsilon_{\mathcal{E}} = I_{\boldsymbol{U}\mathcal{E}} : \boldsymbol{T}\boldsymbol{U}\mathcal{E} \to \mathcal{E},$$

which is regular because it is an equivalence.

The transformation constraint of $\epsilon$ at $F : \mathcal{E} \to \mathcal{F}$ is given by

$$\epsilon_F = \qquad\qquad = \qquad\qquad$$

$$\begin{array}{cc} F \quad \epsilon_{\mathcal{E}} & F \quad I_{U\mathcal{E}} \\ \epsilon_{\mathcal{F}} \;\; \boldsymbol{TU}F & I_{U\mathcal{F}} \;\; J_{U\mathcal{F}} \;\; F \;\; I_{U\mathcal{E}} \end{array}$$

where $\eta$ is the unit of $J_{\mathcal{F}} \dashv I_{\mathcal{F}}$. The reader may verify that this gives a strong transformation.

### 5.2.3  The modifications $\nu$ and $\mu$

The component $\nu_{\mathcal{C}}$ of the modification

$$\nu \quad : \quad \boldsymbol{\epsilon T} \circ \boldsymbol{T\eta} \quad \to \quad \mathrm{id}_{\boldsymbol{T}}$$

is a isomorphic 2-cell in the pre-equipment **QTop** and is given by
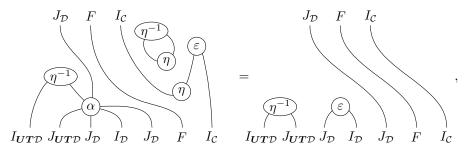
$$\begin{array}{cc} \boxed{\nu_{\mathcal{C}}} \;=\; \boxed{\eta^{-1}} \quad \boxed{\varepsilon} \\ \boldsymbol{\epsilon}_{\boldsymbol{TC}} \;\; \boldsymbol{T\eta}_{\mathcal{C}} \qquad I_{\boldsymbol{UTC}} \, J_{\boldsymbol{UTC}} \;\; J_{\mathcal{C}} \;\; I_{\mathcal{C}} \end{array} ,$$

where $\eta$ is invertible because $J_{\boldsymbol{UTC}} \dashv I_{\boldsymbol{UTC}} \; \varepsilon$ is an equivalence and is invertible because $J_{\mathcal{C}} \dashv I_{\mathcal{C}}$ is a reflection.

We want to verify that this does indeed define a modification. If we instantiate the modification axiom

$$\begin{array}{ccc} \boldsymbol{T}F \quad \boxed{\nu_{\mathcal{C}}} & & \boldsymbol{T}F \\ & = & \boxed{\nu_{\mathcal{D}}} \\ \boldsymbol{\epsilon}_{\boldsymbol{TD}} \;\; \boldsymbol{T\eta}_{\mathcal{D}} \;\; \boldsymbol{T}F & & \boldsymbol{\epsilon}_{\boldsymbol{TD}} \;\; \boldsymbol{T\eta}_{\mathcal{D}} \;\; \boldsymbol{T}F \end{array} \qquad (5.7)$$

with the definitions, we obtain[17]

$$\begin{array}{ccc} J_{\mathcal{D}} \;\; F \;\; I_{\mathcal{C}} & & J_{\mathcal{D}} \;\; F \;\; I_{\mathcal{C}} \\ \boxed{\eta^{-1}} \; \boxed{\eta} \; \boxed{\varepsilon} & & \\ \boxed{\eta^{-1}} \;\; \boxed{\alpha} \;\; \boxed{\eta} & = & \\ I_{\boldsymbol{UTD}} \; J_{\boldsymbol{UTD}} \; J_{\mathcal{D}} \; I_{\mathcal{D}} \; J_{\mathcal{D}} \; F \; I_{\mathcal{C}} & & I_{\boldsymbol{UTD}} \; J_{\boldsymbol{UTD}} \; J_{\mathcal{D}} \; I_{\mathcal{D}} \; J_{\mathcal{D}} \; F \; I_{\mathcal{C}} \end{array} ,$$

----

[17]The subtle point is how to unfold the right crossing in the left diagram of (5.7). This is the constraint cell $\boldsymbol{T\eta}_F$, which is obtained by applying $\boldsymbol{T}$ to the constraint cell $\boldsymbol{\eta}_F$ and then pre- and postcomposing with composition constraints of $\boldsymbol{T}$.

where $\alpha$ is the inverse of

$$J_{\boldsymbol{UT}\mathcal{D}} \quad J_{\mathcal{D}} \quad I_{\mathcal{D}} \quad J_{\mathcal{D}}$$
$$\overset{\eta}{\phantom{x}}$$
$$J_{\boldsymbol{UT}\mathcal{D}} \quad J_{\mathcal{D}} \quad ,$$

which is invertible by Lemma 5.8. The equality now follows formally from the triangle equalities for the adjunctions $J \dashv I$.

The component of the modification

$$\mu \quad : \quad \mathrm{id}_U \quad \to \quad \boldsymbol{U\epsilon} \circ \boldsymbol{\eta U}$$

at $\mathcal{E} \in \mathbf{Top}$ is given by

$$\boldsymbol{U\epsilon}_{\mathcal{E}} \; \boldsymbol{\eta}_{\boldsymbol{U}\mathcal{E}} \qquad I_{\boldsymbol{U}\mathcal{E}} \; J_{\boldsymbol{U}\mathcal{E}}$$
$$\boxed{\mu_{\mathcal{E}}} \quad = \quad \eta \qquad .$$

The modification axiom for $\mu$ is verified as follows (the first and third equalities just instantiate definitions):

$$\boldsymbol{UF} \; \boldsymbol{U\epsilon}_{\mathcal{E}} \; \eta_{\boldsymbol{U}\mathcal{E}} \qquad F \; I_{\boldsymbol{U}\mathcal{E}} \; J_{\boldsymbol{U}\mathcal{E}} \qquad F \; I_{\boldsymbol{U}\mathcal{E}} \; J_{\boldsymbol{U}\mathcal{E}} \qquad \boldsymbol{UF} \; \boldsymbol{U\epsilon}_{\mathcal{E}} \; \eta_{\boldsymbol{U}\mathcal{E}}$$

### 5.2.4 The axioms

Finally we have to verify the biadjunction axioms. The calculation for the first axiom looks as follows:

$$\boxed{\boldsymbol{U\nu}_{\mathcal{C}}} \; \boldsymbol{\eta}_{\mathcal{C}} \qquad = \qquad \cdots \qquad J_{\mathcal{C}} \qquad J_{\mathcal{C}} \; \boldsymbol{\eta}_{\mathcal{C}}$$

And here is the calculation for the second axiom:

$$\boldsymbol{\epsilon}_{\mathcal{E}} \qquad I_{\boldsymbol{U}\mathcal{E}} \qquad I_{\boldsymbol{U}\mathcal{E}} \qquad \boldsymbol{\epsilon}_{\mathcal{E}}$$

This finishes the proof that we have a special biadjunction $\boldsymbol{T} \dashv \boldsymbol{U}$. We can compose with the special biadjunction $\boldsymbol{F} \dashv \boldsymbol{S}$ of Section 5.1 to obtain the tripos-to-topos construction, i.e., a left biadjoint $\boldsymbol{T} \circ \boldsymbol{F}$ of the forgetful functor $\boldsymbol{S} \circ \boldsymbol{U}$ from toposes to triposes.

## 5.3 Summary, examples

After a lot of in-detail constructions and verifications, we try to give a kind of big picture. We have related three geometric pre-equipments by a pair of special biadjunctions.

$$\mathbf{Trip} \underset{\boldsymbol{F}}{\overset{\boldsymbol{S}}{\rightleftarrows}} \mathbf{QTop} \underset{\boldsymbol{T}}{\overset{\boldsymbol{U}}{\rightleftarrows}} \mathbf{Top}$$

As right special biadjoints, $\boldsymbol{S}$ and $\boldsymbol{T}$ are necessarily strong by Lemma 2.8, and from Lemma 5.6 we know that $F$ is strong as well (in fact all three functors are even strict). The counit of $\boldsymbol{T} \dashv \boldsymbol{U}$ is also strong, but the remaining constructs are genuinely oplax, as we will exemplify in the following.

That $\boldsymbol{T}$ is necessarily oplax becomes apparent already from the example in the introduction (if $\boldsymbol{T}$ were strong as well as $\boldsymbol{F}$, then so would be their composite).

To show that the counit of $\boldsymbol{F} \dashv \boldsymbol{S}$ is not a strong transformation, we have to give a finite limit preserving functor $F : \mathcal{C} \to \mathcal{D}$ between q-toposes such that the transformation constraint $\epsilon_F$ is not invertible.

Consider the poset $D = \{\bot, l, r\}$ with $\bot \leq l$ and $\bot \leq r$, and the global sections functor $\Gamma : \hat{D} \to \mathbf{Set}$. We define a contravariant presheaf $A$ on $D$ by $A(\bot) = \{0, 1\}$, $A(l) = \{0\}$, and $A(r) = \{1\}$ where the restriction maps are just the inclusions, and define $R$ to be the maximal equivalence relation $R \rightarrowtail A \times A$ on $A$. Then $(A, R)$ is an object of $\boldsymbol{F}\boldsymbol{S}\hat{D}$, and we follow it along the two sides of the square

$$
\begin{array}{ccc}
\boldsymbol{F}\boldsymbol{S}(\hat{D}) & \xrightarrow{\boldsymbol{F}\boldsymbol{S}(\Gamma)} & \boldsymbol{F}\boldsymbol{S}(\mathbf{Set}) \\
{\scriptstyle \epsilon_{\hat{D}}} \downarrow & & \downarrow {\scriptstyle \epsilon_{\mathbf{Set}}} \\
\hat{D} & \xrightarrow{\Gamma} & \mathbf{Set}
\end{array}
$$

We have $\epsilon_{\mathbf{Set}}(\boldsymbol{F}\boldsymbol{S}(\Gamma)(A, R)) = \varnothing$, since $A$ doesn't have any global elements, but $\Gamma(\epsilon_{\hat{D}}(A, R)) = 1$, since $A$ has global support, thus $\epsilon_{\Gamma}$ is not a natural isomorphism.

Finally, we will have a closer look at the units of $\boldsymbol{F} \dashv \boldsymbol{S}$ and $\boldsymbol{T} \dashv \boldsymbol{U}$, as a demonstration how our two-step decomposition gives additional information in form of an epi-mono factorization.

Consider a triposes $\mathcal{P} : \mathbb{X} \to \mathbb{C}$ and $\mathcal{Q} : \mathbb{Y} \to \mathbb{D}$ and a tripos transformation $(F, \Phi) : \mathcal{P} \to \mathcal{Q}$. the unit constraint cell of the composite biadjunction $\boldsymbol{T}\boldsymbol{F} \dashv \boldsymbol{S}\boldsymbol{U}$ is a natural transformation which is given by the pasting

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\quad F \quad} & \mathbb{D} \\
{\scriptstyle D_{\mathcal{P}}} \downarrow & {\scriptstyle \boldsymbol{\eta}_{(F,\Phi)}\Downarrow} & \downarrow {\scriptstyle D_{\mathcal{Q}}} \\
\boldsymbol{F}\mathcal{P} & \xrightarrow{\boldsymbol{F}(F,\Phi)} & \boldsymbol{F}\mathcal{Q} \\
{\scriptstyle J_{\boldsymbol{F}\mathcal{P}}} \downarrow & {\scriptstyle \boldsymbol{\eta}_{\boldsymbol{F}(F,\Phi)}\Downarrow} & \downarrow {\scriptstyle J_{\boldsymbol{F}\mathcal{Q}}} \\
\boldsymbol{T}\boldsymbol{F}\mathcal{P} & \xrightarrow[\boldsymbol{T}\boldsymbol{F}(F,\Phi)]{} & \boldsymbol{T}\boldsymbol{F}\mathcal{Q}
\end{array}
\qquad , \qquad (5.8)
$$

and we have the following lemma.

**Lemma 5.9** *1. The components of $\boldsymbol{\eta}_{(F,\Phi)}$ are regular epimorphisms. $\boldsymbol{\eta}_{(F,\Phi)}$ is a natural isomorphism whenever $\Phi$ commutes with $\exists$ along diagonals.*

*2. The components of $\boldsymbol{\eta}_{\boldsymbol{F}(F,\Phi)}$ are monomorphisms. $\boldsymbol{\eta}_{\boldsymbol{F}(F,\Phi)}$ is a natural isomorphism whenever $\Phi$ commutes with existential quantification along projections.*

*3. For $C \in \mathbb{C}$,*

$$J_{\boldsymbol{F}\mathbb{Q}}D_{\mathbb{Q}}FC \xrightarrow{J_{\boldsymbol{F}\mathbb{Q}}\boldsymbol{\eta}_{(F,\Phi),C}} J_{\boldsymbol{F}\mathbb{Q}}\boldsymbol{F}(F,\Phi)D_{\mathbb{P}}C \xrightarrow{\boldsymbol{\eta}_{\boldsymbol{F}(F,\Phi),D_{\mathbb{P}}C}} \boldsymbol{T}\boldsymbol{F}(F,\Phi)J_{\boldsymbol{F}\mathbb{P}}D_{\mathbb{P}}C$$

*is an epi-mono factorization of the unit constraint of $\boldsymbol{T}\boldsymbol{F} \dashv \boldsymbol{S}\boldsymbol{U}$ at $F$ and $C$.*

*Proof. Ad 1.* This follows directly from the definition of $\boldsymbol{\eta}_{(F,\Phi)}$ in (5.2) and the characterization of regular epimorphisms in $\boldsymbol{F}\mathbb{Q}$ in Lemma 5.3.8.

*Ad 2.* The component of $\boldsymbol{\eta}_{\boldsymbol{F}(F,\Phi)}$ at $A \in \boldsymbol{F}\mathbb{P}$ is given by $J_{\boldsymbol{F}\mathbb{Q}}\boldsymbol{F}(F,\Phi)\eta_A$, where $\eta_A : A \rightarrowtail \overline{A}$ is the unit of the reflection of $\boldsymbol{F}\mathbb{P}$ onto the coarse objects. It is a monomorphism since $\eta_A$ is monic and $\boldsymbol{F}(F,\Phi)$ and $J_{\boldsymbol{F}\mathbb{Q}}$ preserve finite limits. Since $\boldsymbol{T}\boldsymbol{F}\mathbb{Q}$ is balanced and $J_{\boldsymbol{F}\mathbb{Q}}$ preserves epis as a left adjoint, $\boldsymbol{\eta}_{\boldsymbol{F}(F,\Phi),A}$ will be an isomorphism as soon as $\boldsymbol{F}(F,\Phi)$ preserves epimorphisms. But epimorphisms $f : (C,\rho) \to (D,\sigma)$ in $\boldsymbol{F}\mathbb{P}$ are characterized by the judgement $\sigma(d,d) \vdash \exists c.\rho(c,c) \wedge \sigma(fc,d)$ and since this judgement only contains conjunction and existential quantification along projections, epimorphisms will be preserved by $\boldsymbol{F}(F,\Phi)$ if $\Phi$ preserves those.

*Ad 3.* This follows from 1 and 2, since $J_{\boldsymbol{F}\mathbb{Q}}$ preserves epimorphisms. ∎

The tripos transformation $\mathrm{fam}(\wedge) : \mathrm{fam}(\mathbb{B} \times \mathbb{B}) \to \mathrm{fam}(\mathbb{B})$ is an example which commutes with $\exists$ along diagonals, but not along projections. The double negation topology on the modified realizability tripos [27] is an example that commutes with $\exists$ along projections, but not along diagonals.

# 6 Epilogue: Conclusion, observations and first applications

Now that we finally know that the tripos-to-topos construction is part of a special biadjunction and in particular a special functor, we want to see whether this helps us in any way to understand what happens in the realizability constructions that we mentioned in the beginning. Of special interest here are geometric morphisms and subtoposes.

Geometric morphisms of toposes arise naturally in our setting, because they are just adjunctions in the 2-category **Top**. Adjunctions in **Trip** are a generalization of what is usually known as geometric morphism of triposes (normally [11], they are only considered on a fixed base category). Now we know from Lemma 2.9 that special functors preserve adjunctions between geometric pre-equipments, and this gives a conceptual explanation why the tripos-to-topos construction transforms geometric morphisms of triposes into geometric morphisms of toposes.

Subtoposes coincide with idempotent monads in the 2-category **Top**, and subtriposes coincide (almost, there is again the question of fixed or varying

base for the triposes) with idempotent monads in **Trip**. Monads are not in general preserved by oplax functors (nor by special functors), but nevertheless (as already described in [11]) we can construct a local operator on a topos from a local operator on the corresponding tripos. So how can we understand this? — From an abstract point of view, this works because every idempotent monad in **Trip** can be decomposed into an adjunction with isomorphic counit, so we can apply the tripos-to-topos construction to the decomposition and reassemble the local operator in **Top** (idempotency is preserved by the second statement of Lemma 2.9). For the case of and idempotent monad $(T, \Theta) : \mathcal{P} \to \mathcal{P}$ on a tripos $\mathcal{P} : \mathbb{X} \to \mathbb{C}$, the second tripos of the factorization has as base the full subcategory of $\mathbb{C}$ on the objects $C$ where $\eta_C : C \to TC$ is an isomorphism, and the fibers are the subpreorders of $\mathcal{P}_C$ on the $\Theta$-stable predicates.

We may also ask what happens to comonads. These are preserved by special functors simply because they can be defined as oplax functors having as domain, and oplax functors can be composed. However, it is possible that an idempotent comonad on a tripos gives rise to a non-idempotent comonad on a topos. An example of this is the comonad induced by the adjunction $\delta \dashv \wedge : \mathbf{Set}(-, \mathbb{B} \times \mathbb{B}) \to \mathbf{Set}(-, \mathbb{B})$ from the introduction. The induced comonad of triposes, which is given by the fiberwise operation $(a, b) \mapsto (a \wedge b, a \wedge b)$ on $\mathbb{B}$ is idempotent, but its image

$$\mathbf{Set} \times \mathbf{Set} \to \mathbf{Set} \times \mathbf{Set}, \qquad (A, B) \mapsto (A \times B, A \times B)$$

under $\boldsymbol{T} \circ \boldsymbol{S}$ is not.

# A    Non-extensional higher order intuitionistic logic

In this appendix, we present the logical system mentioned in Section 3.1. The derivation rules are given in Table 3. Observe that the variable conditions for quantification and equality, that are normally perceived as somewhat disturbing, arise naturally in the presence of explicit variable contexts. The equality rules are equivalent to the traditional ones, but are meant to resemble the rules for existential quantification since from the categorical point of view, equality is existential quantification along diagonals

This system is called non-extensional since there is no axiom or rule that states that two sets (= individuals of power type) are equal whenever they have the same elements.

$$\frac{}{\Delta \mid \Gamma \vdash \top}$$

$$\frac{\Delta \mid \Gamma \vdash \varphi \quad \Delta \mid \Gamma \vdash \psi}{\Delta \mid \Gamma \vdash \varphi \wedge \psi}$$

$$\frac{\Delta \mid \Gamma \vdash \varphi_i}{\Delta \mid \Gamma \vdash \varphi_1 \vee \varphi_2} \ i = 1, 2$$

$$\frac{\Delta \mid \Gamma, \varphi \vdash \psi}{\Delta \mid \Gamma \vdash \varphi \Rightarrow \psi}$$

$$\frac{\Delta, x{:}A \mid \Gamma \vdash \xi[x]}{\Delta \mid \Gamma \vdash \forall x \,.\, \xi[x]}$$

$$\frac{\Delta \mid \Gamma \vdash \xi[t]}{\Delta \mid \Gamma \vdash \exists x \,.\, \xi[x]}$$

$$\frac{}{\Delta \mid \Gamma \vdash t = t}$$

$$\frac{\Delta \mid \Gamma \vdash \bot}{\Delta \mid \Gamma \vdash \varphi}$$

$$\frac{\Delta \mid \Gamma \vdash \varphi_1 \wedge \varphi_2}{\Delta \mid \Gamma \vdash \varphi_i} \ i = 1, 2$$

$$\frac{\Delta \mid \Gamma \vdash \varphi \vee \psi \quad \Delta \mid \Gamma, \varphi \vdash \gamma \quad \Delta \mid \Gamma, \psi \vdash \gamma}{\Delta \mid \Gamma \vdash \gamma}$$

$$\frac{\Delta \mid \Gamma \vdash \varphi \Rightarrow \psi \quad \Delta \mid \Gamma \vdash \varphi}{\Delta \mid \Gamma \vdash \psi}$$

$$\frac{\Delta \mid \Gamma \vdash \forall x \,.\, \xi[x]}{\Delta \mid \Gamma \vdash \xi[t]}$$

$$\frac{\Delta \mid \Gamma \vdash \exists x \,.\, \xi[x] \quad \Delta, x{:}A \mid \Gamma, \xi[x] \vdash \psi}{\Delta \mid \Gamma \vdash \psi}$$

$$\frac{\Delta \mid \Theta[s,t] \vdash s = t \quad \Delta, x{:}A \mid \Theta[x,x] \vdash \rho[x,x]}{\Delta \mid \Theta[s,t] \vdash \rho[s,t]}$$

$$\frac{}{\Delta \mid \Gamma \vdash \exists m : \mathbf{P}A \ \forall x \in A \,.\, x \in m \leftrightarrow \xi[x]}$$

$$\frac{}{\Delta, x{:}A, y{:}B \mid \Gamma \vdash \exists! z{:}A \times B \,.\, \pi_1(z) = x \wedge \pi_2(z) = y}$$

$\Delta = x_1{:}A_1, \ldots, x_n{:}A_n$ is a context of variables,
$\Gamma$ is a list of formulas in context $\Delta$,
$\varphi, \varphi_1, \varphi_2, \psi, \gamma$ are formulas in context $\Delta$,
$\xi[x]$ is a formula in context $(\Delta, x{:}A)$,
$\Theta[x, y]$ is a list of formulas in context $(\Delta, x{:}A, y{:}A)$,
$\rho[x, y]$ is a formula in context $(\Delta, x{:}A, y{:}A)$, and
$s, t : A$ are terms in context $\Delta$.

Table 3: Deduction rules of non-extensional higher order intuitionistic logic.

# References

[1] S. Awodey, L. Birkedal, and D.S. Scott. Local realizability toposes and a modal logic for computability. *Mathematical Structures in Computer Science*, 12(03):319–334, 2002.

[2] Lars Birkedal and Jaap van Oosten. Relative and modified relative realizability. *Ann. Pure Appl. Logic*, 118(1-2):115–132, 2002.

[3] A. Boileau and A. Joyal. La logique des topos. *The Journal of Symbolic Logic*, 46(1):6–16, 1981.

[4] Francis Borceux. *Handbook of categorical algebra. 2*, volume 51 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1994. Categories and structures.

[5] Brian Day, Paddy McCrudden, and Ross Street. Dualizations and antipodes. *Appl. Categ. Structures*, 11(3):229–260, 2003.

[6] M. Fourman and D. Scott. Sheaves and logic. *Applications of sheaves*, pages 302–401, 1979.

[7] Jonas Frey. A universal characterisation of the tripos-to-topos construction. Master thesis, available at `http://www.pps.jussieu.fr/~frey/main.pdf`.

[8] P. J. Freyd and G. M. Kelly. Categories of continuous functors. I. *J. Pure Appl. Algebra*, 2:169–191, 1972.

[9] M. Grandis and R. Paré. Limits in double categories. *Cahiers de Topologie et Géométrie Différentielle Catégoriques*, 40:162–220, 1999.

[10] D. Higgs. A category approach to Boolean-valued set theory. *preprint, University of Waterloo*, 1973.

[11] J. M. E. Hyland, P. T. Johnstone, and A. M. Pitts. Tripos theory. *Math. Proc. Cambridge Philos. Soc.*, 88(2):205–231, 1980.

[12] B. Jacobs. *Categorical logic and type theory*. Elsevier Science Ltd, 2001.

[13] P. T. Johnstone. Fibrations and partial products in a 2-category. *Appl. Categ. Structures*, 1(2):141–179, 1993.

[14] Peter T. Johnstone. *Sketches of an elephant: a topos theory compendium. Vol. 1*, volume 43 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 2002.

[15] G. Kelly. Doctrinal adjunction. In *Category Seminar*, pages 257–280. Springer, 1974.

[16] J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*, volume 7 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986.

[17] T. Leinster. Basic bicategories. *Arxiv preprint math.CT/9810017*, 1998.

[18] P. Lietz and T. Streicher. Impredicativity entails untypedness. *Mathematical Structures in Computer Science*, 12(03):335–347, 2002.

[19] Jacques Penon. Quasi-topos. *C. R. Acad. Sci. Paris Sér. A-B*, 276:A237–A240, 1973.

[20] A. M. Pitts. *The theory of triposes*. PhD thesis, Cambridge Univ., 1981.

[21] Andrew M. Pitts. Tripos theory in retrospect. *Math. Structures Comput. Sci.*, 12(3):265–279, 2002.

[22] G. Rosolini and M. Maietti. Quotient completion for preordered fibrations. Conference talk at "Category Theory 2008" in Calais.

[23] P. Selinger. A survey of graphical languages for monoidal categories. *New Structures for Physics*, pages 275–337, 2009.

[24] M. Shulman. Framed bicategories and monoidal fibrations. *Theory and Applications of Categories*, 20(18):650–738, 2008.

[25] R. Street and D. Verity. Low-dimensional topology and higher-order categories. *Proceedings of CT95, Halifax*, 1995. Available electronically at http://www.maths.mq.edu.au/~street/LowDTop.pdf.

[26] Thomas Streicher. Fibred Categories à la Jean Bénabou. Available at http://www.mathematik.tu-darmstadt.de/~streicher.

[27] J. Van Oosten. The modified realizability topos. *Journal of pure and applied algebra*, 116(1-3):273–289, 1997.

[28] J. Van Oosten. *Realizability: an introduction to its categorical side*. Elsevier Science Ltd, 2008.

[29] D. Verity. Enriched categories, internal categories and change of base (PhD thesis, University of Cambridge, 1992). *Repr. Theory Appl. Categ.*, 2011. to appear.

[30] Wikipedia. Factorization system. http://en.wikipedia.org/wiki/Factorization_system#Equivalent_definition (Version of 10-January-2010).

[31] R. J. Wood. Abstract proarrows. I. *Cahiers Topologie Géom. Différentielle*, 23(3):279–290, 1982.