Categorical Ontology: Existence

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Abstract. The present paper approaches ontology and meta-ontology through Mathematics, and more precisely through the theory of *elementary toposes*; for us, an *ontology* is a mathematical object: it is a category $\mathcal{E}$, the universe of discourse in which our Mathematics (intended at large, as a theory of knowledge) can be deployed. The well-studied *internal language* of such categories is expressive enough to talk about existence, sometimes in a nuanced way that is akin to the way in which different philosophers talked; technically speaking, the presence of an object $\Omega_{\mathcal{E}}$ parametrizing the truth values of the internal propositional calculus prescribes the 'modes of existence' for the objects of a fixed ontology/category.

This approach resembles, but is more general than, the one leading to *fuzzy logics*, as most choices of $\mathcal{E}$ and thus of $\Omega_{\mathcal{E}}$ yield nonclassical, many-valued logics.

As both a test-bench for our theory, and a literary *divertissement*, we propose a possible category-theoretic solution of the famous Tlön's "nine copper coins" paradox, and of other seemingly paradoxical construction in Jorge Luis Borges' literary work.

We conclude with some vistas on the most promising applications of our future work.

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1. Preliminaries on variable set theory

अव्यक्तादीनि भूतानि व्यक्तमध्यानि भारत ।
अव्यक्तनिधनान्येव तत्र का परिदेवना ॥ २८॥

—O scion of Bharat, all created beings are unmanifest before birth, manifest in life, and again unmanifest on death. So why grieve?

Śrīmadbhagavadgītā II, 28

The present section sets up the notation for the subsequent discussion: we sketch the theory of *variable sets*. Shortly put, a variable set is a family of sets indexed by another set I, i.e. (more formally) a class function $I \to \mathsf{Set}$. A well-known categorical construction entails that a variable set over I is nothing more than a functor $I \to \mathsf{Set}$, regarding I as a discrete category.

Proposition 1.1. Let I be a set, regarded as a discrete category, and let Set^I be the category of functors $F : I \to \mathsf{Set}$; moreover, let Set/I the slice category. Then, there is an equivalence (actually, an isomorphism when a coherent choice of coproduct has been made: see [3, 1.5.1]) between Set^I and Set/I.

Proof. metti una ref

□

Notation 1.2. The present remark is meant to establish a bit of terminology: by virtue of Proposition 1.1 above, an object of the category of variable sets is equally denoted pair $(A, f : A \to I)$, as a function $h : I \to \mathsf{Set}$, or as the family of sets $\{h(i) \mid i : I\} = \{A_i \mid i : I\}$. We call the function f the *structure map* of the variable set A, and we call the function F_h the functor *associated*, or corresponding, to the variable set in study. Common parlance almost always blurs the distinction between these objects.

Remark 1.3. A more abstract look at this result establishes the equivalence $\mathsf{Set}/I \cong \mathsf{Set}^I$ as a particular instance of the *Grothendieck construction* (see [22, 1.1]): for every small category $\mathcal{C}$, the category of functors $\mathcal{C} \to \mathsf{Set}$ is equivalent to the category of *discrete fibrations* on $\mathcal{C}$ (see [22, 1.1]). In this case, the domain $\mathcal{C} = I$ is a discrete category, hence all functors $\mathcal{E} \to I$ are, trivially, discrete fibrations.

Remark 1.4. The next crucial step of our analysis is the observation that the category of variable sets is a *topos*: we break the result into the verification of the various axioms. Our proof relies on the fact that the category of sets is itself a topos: in particular, it is cartesian closed, and admits the set $\{\perp, \top\}$ as subobject classifier.[1]

[1] We choose to employ a classical model of set theory, as opposed to an intuitionistic model where the classifier Ω consists of a more general Heyting algebra H; a general procedure to obtain a Ω-many valued logic of set theory is to take the topos $\mathcal{E} = \mathrm{Sh}(H)$ of *sheaves* on a Heyting algebra H: then, there is an isomorphism $\Omega_{\mathcal{E}} \cong H$. The core of all our argument is

give refs once and for all for these

Proposition 1.5. The category of variable sets is Cartesian closed in the sense of [3, p.335].

Proposition 1.6. The category of variable sets has a subobject classifier.

Remark 1.7. A straightforward but important remark is now in order. The structure of subobject classifier of Ω_I, and in particular the shape of a characteristic function $\chi_S : A \to \Omega_I$ for a subobject $S \subseteq A$ in Set/I, is explicitly obtained using the structure map f of the variable set $f : A \to I$.

This will turn out to be very useful along our main section, where we shall note that a proposition in the internal language of Set/I amounts to a function $p : U \to \Omega_I$, having as domain a variable set $u : U \to I$, whose structure map uniquely determines the "strength" (see Remark 3.2) of the proposition p. In a nutshell, $p(x)$ is a truth value in Ω_I; the fact that p is a morphism *of variable sets* however forces this truth value to be $(u(x), \epsilon) : I \times \{\bot, \top\}$. We invite the baffled reader not to worry now; we will duly justify each of these conceptual steps along section 3 and 2.

Proposition 1.8. The category of variable sets is cocomplete and accessible.

Corollary 1.9. The category of variable sets is a Grothendieck topos.

2. The internal language of variable sets

> I am hard but I am fair; there is no racial bigotry here. [...] Here you are all equally worthless.
>
> GySgt Hartman

Definition 2.1. The internal language of a topos $\mathcal{E}$ is a formal language defined by *types* and *terms*; suitable terms form the class of variables. Other terms form the class of *formulæ*.

- *Types* are the objects of $\mathcal{E}$
- *Terms* of type X are morphisms of codomain X, usually denoted with Greek letters $\alpha, \beta, \sigma, \tau, \ldots : U \to X$.
  - Suitable terms are variables: the identity arrow of $X \in \mathcal{E}$ is the variable $x : X \to X$. For technical reasons we shall keep a countable number of variables of the same type distinguished:[1] $x, x', x'', \ldots : X \to X$ are all interpreted as 1_X.

very rarely affected by the choice to cut the complexity of our Ω to be the bare minimum; this is mainly due to expository reasons. The reader shall feel free to replace $\{\bot, \top\}$ with a more generic choice of Heyting algebra, and they are invited to adapt the arguments of section section 3 accordingly.

[1] These technical reasons lie on the evident necessity to be free to refer to the same free variable an unbounded number of times. This can be formalised in various ways: we refer the reader to [20] and [18].

- Generic terms may depend on multiple variables; the domain of a term of type X is the *domain of definition* of a term.

A number of inductive clauses define the other terms of the language:

- the identity arrow of an object $X \in \mathcal{E}$ is a term of type X;
- given terms $\sigma : U \to X$ and $\tau : V \to Y$ there exists a term $\langle \sigma, \tau \rangle$ of type $X \times Y$ obtained from the pullback

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, "\langle\sigma,\tau\rangle"] & X \times V \arrow[d] \\ U \times Y \arrow[r] & X \times Y \end{tikzcd} \tag{2.1}$$

- Given terms $\sigma : U \to X, \tau : V \to X$ of the same type X, there is a term $[\sigma = \tau] : W \xrightarrow{\langle\sigma,\tau\rangle} X \times X \xrightarrow{\delta_X} \Omega$, where $\delta_X : X \times X \to \Omega$ is defined as the classifying map of the mono $X \hookrightarrow X \times X$.
- Given a term $\sigma : U \to X$ and a term $f : X \to Y$, there is a term $f[\sigma] := f \circ \sigma : U \to Y$.
- Given terms $\theta : V \to Y^X$ and $\sigma : U \to X$, there is a term

$$W\langle\theta, \sigma\rangle \to Y^X \times X \xrightarrow{\mathrm{ev}} Y \tag{2.2}$$

- In the particular case $Y = \Omega$, the term above is denoted

$$[\sigma \in \theta] : W\langle\theta, \sigma\rangle \to \Omega \tag{2.3}$$

- If x is a variable of type X, and $\sigma : X \times U \to Z$, there is a term

$$\lambda x.\sigma : U \xrightarrow{\eta} (X \times U)^X \xrightarrow{\sigma^X} Z^X \tag{2.4}$$

  obtained as the mate of σ.

These rules can of course be also presented as the formation rules for a Gentzen-like deductive system: let us rewrite them in this formalism.

$$\frac{}{1_X : X \to X}\ \textsc{id} \qquad \frac{\sigma : U \to X \qquad \tau : V \to Y}{\langle\sigma, \tau\rangle : W\langle\theta, \sigma\rangle \to X \times Y}\ \Pi$$

$$\frac{\sigma : U \to X \qquad \tau : V \to X}{[\sigma = \tau] : W \to \Omega}\ \textsc{eq} \qquad \frac{\sigma : U \to X \qquad f : X \to Y}{f[\sigma] : U \to Y}\ \textsc{comp}$$

$$\frac{\theta : V \to Y^X \qquad \sigma : U \to X}{W\langle\theta, \sigma\rangle \to Y^X \times X \xrightarrow{\mathrm{ev}} Y}\ \textsc{ev} \qquad \frac{x : X \qquad \sigma : X \times U \to Z}{\lambda x.\sigma = \sigma^X \circ \eta : U \to (X \times U)^X \to Z^X}\ \lambda\text{-}\textsc{abs} \tag{2.5}$$

To formulas of the language of $\mathcal{E}$ we apply the usual operations and rules of first-order logic: logical connectives are induced by the structure of internal Heyting algebra of Ω: given formulas φ, ψ we define

- $\varphi \vee \psi$ is the formula $W\langle\varphi, \psi\rangle \to \Omega \times \Omega \xrightarrow{\vee} \Omega$;
- $\varphi \wedge \psi$ is the formula $W\langle\varphi, \psi\rangle \to \Omega \times \Omega \xrightarrow{\wedge} \Omega$;
- $\varphi \Rightarrow \psi$ is the formula $W\langle\varphi, \psi\rangle \to \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$;
- $\neg\varphi$ is the formula $U \to \Omega \xrightarrow{\neg} \Omega$.

Universal quantifiers admit an interpretation in the Mitchell-Bénabou language of $\mathcal{E}$: the following definition comes from [25, VI].

Definition 2.2. Let $\mathcal{E}$ be a topos, and let $\varphi : U \times V \to \Omega$ be a formula defined on a product type $U \times V$. The variable $u : U$ can be now quantified over yielding a new formula

$$\forall u.\varphi(u, v) : V \to \Omega \tag{2.6}$$

which no longer contains u as a free variable.

The term $\forall u.\varphi(u, v)$ is obtained by composition of $\lambda u.\varphi(u, v) : V \to V \to \Omega^U$ with the unique right adjoint $\forall_t : \Omega^U \to \Omega$ to the precomposition $t^* : \Omega \to \Omega^U$[2] in the diagram

$$V \xrightarrow{\lambda u.\varphi} \Omega^U \xrightarrow{\forall} \Omega \tag{2.7}$$

(see [25, IV.9]). Similarly, we obtain a term $\exists u.\varphi(u, v)$ from the composition

$$V \xrightarrow{\lambda u.\varphi} \Omega^U \xrightarrow{\exists} \Omega \tag{2.8}$$

with the left adjoint $\exists_t \dashv t$.

Each formula $\varphi : U \to \Omega$ defines a subobject $\{x \mid \varphi\} \subseteq U$ of its domain of definition; this is the subobject classified by φ, and must be thought as the subobject where "φ is true".

If $\varphi : U \to \Omega$ is a formula, we say that φ is *universally valid* if $\{x \mid \varphi\} \cong U$. If φ is universally valid in $\mathcal{E}$, we write "$\mathcal{E} \Vdash \varphi$" (read: "$\mathcal{E}$ believes in φ").

Examples of universally valid formulas:

- $\mathcal{E} \Vdash [x = x]$
- $\mathcal{E} \Vdash [(x \in_X \{x \mid \varphi\}) \iff \varphi]$
- $\mathcal{E} \Vdash \varphi$ if and only if $\mathcal{E} \Vdash \forall x.\varphi$
- $\mathcal{E} \Vdash [\varphi \Rightarrow \neg\neg\varphi]$

Let us now glance at the internal language of variable sets. This will turn out to be the cornerstone for the analysis in section 3.

Here we just unwind Definition 2.1 at its very surface; we invite the reader to endeavour in the instructive exercise to fill all details properly.

Definition 2.3. Types and terms of $\mathcal{L}(\mathsf{Set}/I)$ are respectively arrows $\left[\begin{smallmatrix} U \\ \downarrow \\ I \end{smallmatrix}\right]$ with codomain I, and commutative triangles

$$\begin{tikzcd} U \arrow[rr, "\sigma"] \arrow[dr, "u"'] & & X \arrow[dl, "x"] \\ & I & \end{tikzcd} \tag{2.9}$$

Given this, we define

- *product terms* as functions $\sigma : U_1 \times_I U_2 \times_I \cdots \times_I U_n \to X$;

[2] The arrow $t : U \to 1$ is the *t*erminal map, and $t^* : \Omega^1 \to \Omega^U$ is induced by precomposition; it sends a term $x : \Omega$ to the constant function $t^*x = \lambda u.x$.

- terms $\langle \sigma, \tau \rangle$ obtained as the (diagonal map in the) pullback

$$\begin{tikzcd} W\langle \sigma, \tau \rangle \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \times_I V \arrow[d] \arrow[ddr, bend left] & \\ U \times_I Y \arrow[r] \arrow[drr, bend right] & X \times_I Y \arrow[dr] & \\ & & I \times_I I \cong I \end{tikzcd} \tag{2.10}$$

- terms $[\sigma = \tau]$, obtained as compositions

$$W\langle \sigma, \tau \rangle \xrightarrow{\langle \sigma, \tau \rangle} X \times_I Y \xrightarrow{\delta} \Omega_I \tag{2.11}$$

where $\delta_X : X \times_I X \to \Omega$ classifies the mono $m : X \to X \times_I X$ obtained by the universal property of the pullback $X \times_I X$ as the kernel pair of $\left[\begin{smallmatrix} X \\ x \downarrow \\ I \end{smallmatrix} \right]$.

The cartesian closed structure of Set/I (cf. Proposition 1.5) yields terms (we denote $B^A := A \pitchfork B$ in the notation of Proposition 1.5)

- $[\sigma \in \theta] : W\langle \theta, \sigma \rangle \to Y^X \times_I X \to \Omega_I$ from suitable σ, θ;
- $U \to (X \times_I U)^X \xrightarrow{\sigma} Z^X$ (λ-abstraction on $x : X$) from a suitable $\sigma : X \times_I U \to Z$.

Quantifiers can be deduced in a similar way, starting from their general definition in Definition 2.2.

Remark 2.4. Like every other Grothendieck topos, the category Set/I has a *natural number object* (see [25, VI.1], [20, p.46]); here we shall outline its construction. It is a general fact that such a natural number object in the category of variable sets, consists of the constant functor on $\mathbb{N} : \mathsf{Set}$, when we realise variable sets as functors $I \to \mathsf{Set}$: thus, in fibered terms, the natural number object is just $\pi_I : \mathbb{N} \times I \to I$.

A natural number object provides the category $\mathcal{E}$ it lives in with a notion of *recursion* and with a notion of $\mathcal{E}$-induction principle: namely, we can interpret the sentence

$$\big(Q0 \wedge \textstyle\bigwedge_{i \le n} Qi \Rightarrow Q(i+1)\big) \Rightarrow \textstyle\bigwedge_{n : \mathbb{N}} Qn \tag{2.12}$$

for every $Q : \mathbb{N} \to \Omega_I$.

In the category of variable sets, the universal property of $\pi_I : \mathbb{N} \times I \to I$ amounts to the following fact: given any diagram of solid arrows

$$\begin{tikzcd} I \arrow[r, "0"] \arrow[d, equal] & \mathbb{N} \times I \arrow[r, "s \times I"] \arrow[d, dotted, "u"] & \mathbb{N} \times I \arrow[d, dotted, "u"] \\ I \arrow[r, "x"'] & X \arrow[r, "f"'] & X \end{tikzcd} \tag{2.13}$$

where every arrow carry a structure of morphism over I (and $0 : i \mapsto (0, i)$, $s \times I : (n, i) \mapsto (n + 1, i)$), there is a unique way to complete it with the dotted arrow, i.e. with a function $u : \mathbb{N} \times I \to X$ such that

$$u \circ (s \times I) = f \circ u. \tag{2.14}$$

Clearly, u must be defined by induction: if it exists, the commutativity of the left square amounts to the request that $u(0, i) = x(i)$ for every $i : I$. Given this, the inductive step is

$$u(s(n, i)) = u(n + 1, i) = f(u(n, i)). \tag{2.15}$$

This recursively defines a function with the desired properties; it is clear that these requests uniquely determine u.

Such a terse exposition does not exhaust such a vast topic as recursion theory conducted with category-theoretic tools. The interested reader shall consult [16] for a crystal-clear introductory account, and [11, 12] for more recent and modern development of recursion theory.

Remark 2.5. The object of natural numbers of Set/I is easily seen to match the definition of the *initial object* [3, 2.3.1] of the category Dyn/I so defined:

- the objects of Dyn/I are *dynamical systems* in Set/I, i.e. the triples (x, X, f), where $X : \mathsf{Set}/I$ (say, with structure map $\xi : X \to I$), $x : (I, \mathrm{id}_I) \to (X, \xi)$ and $f : X \to X$ is an endo-morphism of variable sets;
- given two dynamical systems (x, X, f) and (y, Y, g) a *morphism* between them is a function $u : X \to Y$ such that the diagram in (2.13) commutes in all its parts.

The reason why such a triple (x, X, f) identifies as a (discrete) dynamical system is easily seen: the function $x : I \to X$ works as initial seed for a recursive application of f, in such a way that every $f : X \to X$ defines a sequence

$$u_{n+1} := f(u_n) \tag{2.16}$$

of its iterates. The system now lends itself to all sorts of questions: is there a fixed point for $u_n(x)$? Does the limit of $u_n(x)$ belong to X (not obvious: consider $X = [0, 1[$ and $u_n(x) \equiv 1 - \frac{1}{n}$)? Is $u_n(x)$ continuous in x? Etc.

3. Nine copper coins, and other toposes

> Explicaron que una cosa es *igualdad*, y otra *identidad*, y formularon una especie de *reductio ad absurdum*, o sea el caso hipotético de nueve hombres que en nueve sucesivas noches padecen un vivo dolor. ¿No sería ridículo -interrogaron- pretender que ese dolor es el mismo?
>
> — *They explained that* equality *is one thing and* identity *another, and formulated a kind of* reductio ad absurdum*: the hypothetical case of nine men who on nine nights suffer a severe pain. Would it not be ridiculous -they questioned- to pretend that this pain is one and the same?*
>
> JLB —Tlön, Uqbar, Orbis Tertius

The main result of the present section is a roundup of examples showing that it is possible to concoct categories of variable sets Set/I where some seemingly paradoxical constructions coming from J.L. Borges' literary world have, instead, a perfectly 'classical' behaviour when looked in the internal logic of Set/I.

Each of the examples in our roundup Example 3.7, 3.13, 3.14 is organised as follows: we recall the statement in Borges' words. Then, we exhibit a topos in which the statement becomes admissible, when expressed in its internal language.

3.1. Choosing an internal logic

According to our description of the Mitchell-Bénabou language in the category of variable sets, *propositions* are morphisms of the form

$$p : U \to \Omega_I \tag{3.1}$$

where Ω_I is the subobject classifier of Set/I described in Proposition 1.6; now, recall that

- the object $\Omega_I = \Omega_0 \times I \to I$ becomes an object of Set/I when endowed with the projection $\pi_I : \Omega_I \to I$ on the second factor of its domain (Ω_0 is the subobject classifier of Set, having a top element $\top$ and a bottom element $\perp$);
- the universal monic $\mathsf{t} : I \to \Omega_I$ consists of a section of π_I, precisely the one that sends $i : I$ to the pair $(i, 1) : \Omega_I$;
- every subobject $U \hookrightarrow A$ of an object A results as a pullback (in Set/I) along t:

$$\begin{tikzcd}
U \arrow[rr, "u"] \arrow[dd, "m"'] \arrow[dr, "u"] & & I \arrow[dd, "\mathsf{t}"] \arrow[dl, equal] \\
& I & \\
A \arrow[rr, "\chi"'] \arrow[ur] & & \Omega_I \arrow[ul]
\end{tikzcd} \tag{3.2}$$

(see Proposition 1.6 for a complete proof).

The set I in this context acts as a *multiplier* of truth values, in that every proposition can have a pair (ϵ, t) as truth value: ϵ is the truth value, 'amplified' by $t : I$.

Notation 3.1. We introduce the following notation: a proposition $p : U \to \Omega_I$ is *true* (resp. *false*), in context $x : U$, with *strength* t, if $p(x) = (1, t)$ (resp., $p(x) = (0, t)$). We say, in short, that p is t-true in context $x : U$, and we denote these two judgments as

$$x : U \models p \qquad\qquad x : U \models \neg p. \tag{3.3}$$

This notation is extensible in the obvious way:

- $\vec{x} : U \models p$ means that $p(x_1, \dots, x_n) = (1, t)$ for $p : U = U_1 \times \cdots \times U_n \to \Omega_I$ and $\vec{x} = (x_1, \dots, x_n) : U$;
- $x : U \models p_1, \dots, p_n$ means that for every $1 \le i \le n$ one has $p_i(x) = (1, t)$;

every other combination is similarly defined.

This notation is chosen in order not to make explicit reference to the set of truth values we take for our background logic in Set; all the results that we state independent from the assumption that the subobject classifier Ω_0 of Set is the usual two-element Boolean algebra $\{0 < 1\}$. (This will be, however, our natural choice.)

Some of the usual introduction and elimination rules apply to the $\models$ judgment, for example

$$\frac{a : A \models p(\_, b) \qquad b \models p(a, \_)}{(a, b) : A \times B \models p}\ \Pi\text{-}\textsc{intro} \tag{3.4}$$

This is not accidental; however, we will not say more on the structure of the deductive system so generated, as it would derange us from our main topic of discussion.

Remark 3.2. A proposition in the internal language of variable sets is a morphism of the following kind: a function $p : U \to \Omega_I$, defined on a certain domain, and such that

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "u"'] & \Omega_0 \times I \arrow[d, "\pi"] \\ I \arrow[r, equal] & I \end{tikzcd} \tag{3.5}$$

(it must be a morphism of variable sets!) This means that $\pi p(x : U) = u(x : U)$, so that $p(x) = (\epsilon_x, u(x))$ for $\epsilon_x = 0, 1$ and u is uniquely determined by the 'variable domain' U. We can succintly denote this fact in the above notation, writing

$$\frac{}{x : (U, u) \models \epsilon_x \cdot p} \tag{3.6}$$

where $\epsilon_x \cdot p$ is p if $\epsilon_x = 1$, and $\neg p$ otherwise.

This is an important observation: the strength with which p is true/false is completely determined by the structure of its domain, in the form of the function $u : U \to I$ that renders the pair (U, u) an object of Set/I.

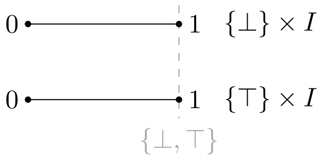
Remark 3.3. To get a concrete grip on the different classes of propositions we can build in an internal language, it is now convenient to restrain the structure of I in order to satisfy our intuition that it is a space of *strengths*: it is for example possible to equip I with an order structure, or a natural topology.

Among all such different choices of truth multiplier, yielding different categories of variable sets, and different kinds of internal logic therein, we will concentrate our study on I's that are dense, linear orders with LUP; thus not really far from being a closed, bounded subset of the real line.[1]

Even if this assumption is never strictly necessary, a natural choice for I is a *continuum* (=a dense total order with LUP –see [26]), a basic model of which is the closed unit interval $[0, 1]$ of the real line.

An alternative choice drops the density assumption: in that case the (unique) finite total order $\Delta[n] = \{0 < 1 < \cdots < n\}$, or the countable total order $I = \omega = \{0, 1, 2, \ldots, n, \ldots\}$ are all pretty natural choices for I (although it is way more natural for I to have a minimum *and* a maximum element).[2]

Remark 3.4. In each of these cases 'classical' logic is recovered as a projection: propositions p can be true or false with strength 1,[3] the maximum element of I:



[1]There is another more philosophical reason for this assumption, that will appear clear throughout the section: density of I is meant to allow 'intersubjective concordance' between two interlocutors X and Y. Engaging in a debate on existence of the nine coins, X, Y might disagree about how strongly some of them exist; the assumption that I is a dense order makes it possible to always find an intermediate strength between the one p_X, assigned by X, and the one $p_Y > p_X$ assigned by Y to the same set of coins, as there must be a third truth value $z : p_X < z < p_Y$ lying between the two.
[2]We're only interested in the notion of an abstract interval here: a continuum X endowed with an operation $X \to X \vee X$ of 'zooming', uniquely defined by this property. In a famous paper Freyd characterises 'the interval' as the terminal interval coalgebra: see [13, §1]; for our purposes, note that $[0, 1]$ is a natural choice: it is a frame, thus a Heyting algebra $\mathfrak{H} = ([0, 1], \wedge, \vee, \Rightarrow)$ with respect to the pseudo-complement operation given by $(x \Rightarrow z) := \bigvee_{x \wedge y \leq z} y$ (it is immediate that $x \wedge a \leq b$ if and only if $a \leq x \Rightarrow b$ for every $a, b : [0, 1]$).
[3]Here I is represented as an interval whose minimal and maximal element are respectively 0 and 1; of course these are just placeholders, but it is harmless for the reader to visualise I as the interval $[0, 1]$.

In order to aid the reader understand the explicit way in which I 'multiplies' truth values, we spell out explicitly the structure of the subobject classifier in $\mathsf{Set}/\Delta[2]$. In order to keep calling the minimum and maximum of I respectively 0 and 1 we call $\frac{1}{2}$ the intermediate point of $\Delta[2]$.

Remark 3.5. The subobject classifier of $\mathsf{Set}/\Delta[2]$ consists of the partially ordered set $\Delta[1] \times \Delta[2]$ that we can represent pictorially as a rectangle

$$\begin{tikzcd}
(0,1) \arrow[r, dash] \arrow[d, dash] & (\ ,1) \arrow[r, dash] \arrow[d, dash] & (0,1) \arrow[d, dash] \\
(0,0) \arrow[r, dash] & (\ ,0) \arrow[r, dash] & (0,0)
\end{tikzcd} \tag{3.7}$$

endowed with the product order. The resulting poset is partially ordered, and in fact a Heyting algebra, because it results as the product of two Heyting algebras: the Boole algebra $\{0 < 1\}$ and the frame of open subsets of the Sierpiński space $\acute{S} = (\{a, b\}, \tau_S)$ (the topology is $\tau_S = \{\varnothing, \{a\}, \{a, b\}\}$).

Remark 3.6. Given that $I = [0, 1]$ endowed with its usual Euclidean topology is one of our most natural choices, we explicitly define some interesting sets obtained out of Ω_I or out of a given proposition $p : U \to \Omega_I$ in the Mitchell-Bénabou language of $\mathsf{Set}/[0, 1]$:

- the two sets
$$A^\top = \big\{x : U \models p,\ t_x > 0\big\} \subseteq U$$
$$A^\perp = \big\{x : U \models \neg p,\ t_x > 0\big\} \subseteq U$$
the last set is canonically identifies to the subset of structure functions $\left[\begin{smallmatrix} U \\ u\downarrow \\ I \end{smallmatrix}\right]$ such that $u^\leftarrow 0 = \varnothing$.
- the two sets
$$B^\top = \big\{x : U \models p\big\} \subseteq U$$
$$B^\perp = \big\{x : U \models \neg p\big\} \subseteq U$$
these are obtained as fibers over the maximal element of I. A useful shorthand for the judgment $\models$ is just $\models$.
- the two sets
$$E_t^\top = \{x : U \models p\} \subseteq U$$
$$E_t^\perp = \{x : U \models \neg p\} \subseteq U$$
Clearly, $B^\top = \coprod_{t:I} E_t^\top$ and $B^\perp = \coprod_{t:I} E_t^\perp$.

A crucial decision at some point will be about the regularity with which the strength of p depends with respect to the variables on its domain of definition. Without a continuous dependence, small changes in context $x : U$ might drastically change the truth value $p(x)$.[4]

[4]There is no a priori reason to maintain that p is a continuous proposition; one might argue that discontinuous changes in truth value of p happen all the time in 'real life'; see

3.2. The unimaginable topos theory hidden in Borges' library

Jorge Luis Borges' literary work is well-known to host paradoxical worlds; oftentimes, seemingly absurd consequences follow by stretching ideas from logic and Mathematics to their limits: time, infinity, self-referentiality, duplication, recursion, the relativity of time, the illusory nature of our perceptions, eternity as a curse, the limits of language, and its capacity to generate worlds.

In the present section, we choose *Fictions*, Borges' famous collection of novels, as a source of inspiration to put to the test possible and impossible worlds together with their ontology.

In a few words, Borges' work generates 'impossible worlds': the term as it is is used in various ways in paraconsistent semantics since the classic work of Rescher and Brandom [30], where worlds are obtained starting from 'possible' ones, through recursive operations on standard Kripkean worlds.

Here we claim that interesting consequences originate from reversing the above perspective: instead of removing from the realm of possibility those worlds that do not comply with sensory experience tagging them as 'impossible', we accept their existence, for bizarre that it may seem, and we try to deduce ex post, *from their very existence*, a kind of logic that can consistently generate them.

The results of our analysis are surprising:

- we unravel how a mathematically deep universe Borges has inadvertently created: of the many compromises we had to take in order to reconcile literature and the underlying Mathematics,[5] we believe no one is particularly far-fetched;
- we unravel how ontological assumptions are context-dependent; they are not given: using category theory ontology, far from being the presupposition on which language is based, is a byproduct of language itself. The more expressive language is, the more expressive ontology becomes; the fuzzier its capacity to assert truth, the fuzzier ontology becomes;
- since 'fuzziness' of existence, i.e. the fact *entia* exist less than completely, is hard-coded in the language (in the sense of Definition 2.1) of the category we decide to work in from time to time, most of the statements of Tlön's ontology are paradoxical only when regarded with Earthlings' eyes; on Tlön they are, instead, perfectly legitimate statements.

To sum up, readers willing to find an original result in this paper might find it precisely here: we underline how Borges' alternative worlds (Babylon, Tlön dots) are mathematically consistent places, worthy of existence as much as our world, only based on a different internal logic.

3.2.1. Nine copper coins. The first paradox we aim to frame in the right topos is the famous nine copper coins argument, used by the philosophers

---

the family of paradoxes based on so-called *separating instants*: how well-defined the notion of 'time of death' is? How well-defined the notion of 'instant in time'?

[5] See Remark 3.8 below: these compromises mainly amount to assumptions on the behaviour of space-time on Tlön and Babylon.

of Tlön to construct an impossible object persisting to exist over time, even without a perceivent that maintains it in a state of being.

Example 3.7 (Nine copper coins). First, we recall the exact statement of the paradox:[6]

> El martes, X atraviesa un camino desierto y pierde nueve monedas de cobre. El jueves, Y encuentra en el camino cuatro monedas, algo herrumbradas por la lluvia del miércoles. El viernes, Z descubre tres monedas en el camino. El viernes de mañana, X encuentra dos monedas en el corredor de su casa. El quería deducir de esa historia la realidad -id est la continuidad- de las nueve monedas recuperadas.
>
> Es absurdo (afirmaba) imaginar que cuatro de las monedas no han existido entre el martes y el jueves, tres entre el martes y la tarde del viernes, dos entre el martes y la madrugada del viernes. Es lógico pensar que han existido -siquiera de algún modo secreto, de comprensión vedada a los hombres- en todos los momentos de esos tres plazos.

Before going on with our analysis, it is important to remark that there is one and only one reason why the paradox of the nine copper coins is invalid: copper does not rust. Incidentally, we will be able to propose a rectification of this 'rust counterargument' without appealing to the cheap assumption that copper can rust on Tlön due to a purported difference between Earth's and Tlönian chemistry.

Expressed in natural language, our solution to the paradox goes more or less as follows: X loses their coins on Tuesday, and the strength φ with which they 'exist' lowers; it grows back in the following days, going back to a maximum value when X retrieves two of their coins on the front door. Y finding of other coins raises their existence strength to a maximum. The coins that Y has found rusted (more precisely, with their surface slightly oxidized: this is possible, but water is rarely sufficient to ignite the process alone –certainly not in the space of a few hours).

---

[6]The paradox appears in a primitive, mostly unchanged version in [8], where instead of nine copper coins, a single arrow, shot by an anonymous archer, disappears among the woods. For what concerns Tlön's nine copper coins, our translation comes from [10]:

> Tuesday, X crosses a deserted road and loses nine copper coins. On Thursday, Y finds in the road four coins, somewhat rusted by Wednesday's rain. On Friday, Z discovers three coins on the road. On Friday morning, X finds two coins in the corridor of his house. The heresiarch would deduce from this story the reality - i.e., the continuity - of the nine coins which were recovered.
> It is absurd (he affirmed) to imagine that four of the coins have not existed between Tuesday and Thursday, three between Tuesday and Friday afternoon, two between Tuesday and Friday morning. It is logical to think that they have existed - at least in some secret way, hidden from the comprehension of men - at every moment of those three periods.

Remark 3.8. In this perspective, Tlön classifier of truth values can be taken as $\Omega_I = \{0 < 1\} \times I$, where I is any set with more than one element; a minimal example can be $I = \{N, S\}$,[7] but as explained below Remark 3.3 a more natural choice for our purposes is the closed real interval $I = [0, 1]$.

This allows for a continuum of possible forces with which a truth value can be true or false; it is to be noted that $[0, 1]$ is also the most natural place on which to interpret fuzzy logic, albeit the interest for $[0, 1]$ therein can be easily and better motivated starting from probability theory.[8]

We now start to formalise properly what we said until now. To set our basic assumptions straight, we proceed as follows:

- Without loss of generality, we can assume the set $C = \{c_1, \dots, c_9\}$ of the nine coins to be totally ordered and partitioned in such a way that the first two coins are those retrieved by X on Tuesday, the subsequent four are those found by Y on their way, and the other three are those seen by Z on Friday. So,
$$C = C_X \sqcup C_Y \sqcup C_Z \tag{3.8}$$
and $C_X = \{c_{X1}, c_{X2}\}$, $C_Y = \{c_{Y1}, c_{Y2}, c_{Y3}, c_{Y4}\}$, $C_Z = \{c_{Z1}, c_{Z2}, c_{Z3}\}$ As already said, the truth multiplier I is the closed interval $[0, 1]$ with its canonical order –so with its canonical structure of Heyting algebra, and if needed, endowed with the usual topology inherited by the real line.
- Propositions of interest for us are of the following form:
$$\lambda gcd.p(g, c, d) : \{X, Y, Z\} \times C \times W \to \Omega_I \tag{3.9}$$
where $W \subseteq \{\boxed{\text{S}}, \boxed{\text{M}}, \boxed{\text{Tu}}, \boxed{\text{W}}, \boxed{\text{Th}}, \boxed{\text{F}}, \boxed{\text{Sa}}\}$ is a set of days (strictly speaking, the paradox involves just the interval between $\boxed{\text{Tu}}$ (Tuesday) and $\boxed{\text{F}}$ (Friday). The value $p(g, c, d)$ models how in g's frame of existence the coin c exists at day d with strength $p(g, c, d)$.

Definition 3.9 (Admissible configuration). We now define an *admissible* configuration of coins any arrangement of C such that the following two conditions are satisfied:

AD1) for all day d and coin c, we have
$$(g, c, d) : G \times C \times W \models p \tag{3.10}$$
where we denote as 'sum' the logical conjunction in Ω_I: this means that day by day, the *global* existence of the group of coins constantly attains the maximum; it is the *local* existence that lowers when the initial conglomerate of coins is partitioned.

[7] Justifying this choice from inside Tlön is easy: the planet is subdivided into two hemispheres; each of which now has its own logic 'line' independent from the other.

[8] Tangential to our discussion might be the fact that there are interesting perspective on how to develop basic measure theory out of $[0, 1]$. For example, measures valued in things like Banach space and more general topological vector spaces have been considered.

AD2) Moreover, all these conditions holds:

$$(X, \boxed{\mathrm{F}}) \models \textstyle\sum_c p(\_, c_X, =)$$
$$(Y, \boxed{\mathrm{Th}}) \models \textstyle\sum_c p(\_, c_Y, =)$$
$$(Z, \boxed{\mathrm{F}}) \models \textstyle\sum_c p(\_, c_Z, =),$$

meaning that, for example in the first case, $\sum_c p(X, c_X, \boxed{\mathrm{F}}) = (\top, 1)$.

In an admissible configuration the subsets C_X, C_Y, C_Z can only attain an existence $p(g, c, d) \lneq (\top, 1)$; that is to say, *no coin completely exists locally.* But for an hypothetical external observer, capable of observing the system, adding up the forces with which the various parts of C exist, the coins *globally* exist ‘ in some secret way, of understanding forbidden to men’ (or rather, to X, Y, Z).

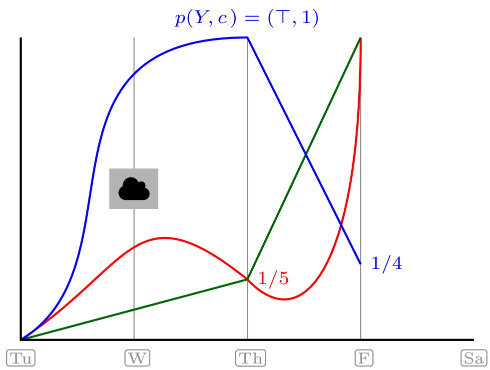


FIGURE 1. A pictorial representation of the truth forces of coins in different days; we choose a minimally complex model where strength of existence goes up and down to join the points where [10] gives complete information about the coins’ configuration. X is marked in red, Z in yellow, Y in blue. Time is considered as a continuum line, marked at weekdays for readability.

Remark 3.10. Lest the reader think our construction is just a sleight of hand leaving open the main problem posed by the coin riddle, an important clarification is now in order: *where* do the coins exist? This is a problematic question. Tlön’s idealist might deny time (our model makes no strong assumption on what time is made of: discrete, continuous, homogeneous, slowed by traveling at high speed...); $B^\top$ ontologists live comfortably in $B^\top$, as defined in Remark 3.6 without being able to tell anything about how objects ‘completely do not exist’.

At the opposite side of the spectrum the pure empiricist lives in $B^\perp$, and they're unable to tell how they 'completely exist').

This is where our analysis comes in handy; in particular, this is where our particular choice of Ω_I plays its rôle. In our model $B^\perp, B^\top$ both result as disjoint sum of slices, each of which collects the particular fibers $E_t^\perp, E_t^\top$ of 3.6; looking at $\coprod_{t:I} E_t^\top$ and $\coprod_{t:I} E_t^\perp$ is a genuinely better approach than just considering $B^\perp, B^\top$ as atomic, since it yields a precise quantification of *how much* something does (not) exist, framing both $B^\top$- and $B^\perp$-ontologies as opposite sides of a spectrum made out of diverse colours.

Example 3.11. An example of an admissible configuration of coins is the following (cf. Figure 1): we assume strength of existence varies joining the points where [5] gives us complete information about the existence status of C_X, C_Y, C_Z above: in the remaining instants of time, the coins out of sight for X share an equal amount of existence in such a way that constraints AD1 and AD2 above are satisfied: in this particular example, $p(Y, c_Y, \boxed{\text{Th}}) = (\top, 1)$, whereas $p(X, C_X, \boxed{\text{Th}}) = p(Z, C_Z, \boxed{\text{Th}}) = (\top, 1/5)$, and $p(Y, c_Y, \boxed{\text{F}}) = (\top, 1/4)$, whereas $p(X, C_X, \boxed{\text{F}}) = p(Z, C_Z, \boxed{\text{F}}) = (\top, 1)$.

Certainly the reader will have fun finding different possible admissible configurations of coins, and building themselves additional details to enrich the bare story of X, Y, Z (for example: can the cardinality of $\{X, Y, Z\}$ depend on I? If yes, how?).

3.2.2. Continuity and discontinuity. Continuity and discontinuity of a proposition $p : U \to \Omega_{[0,1]}$ now capture quite well other pieces of Borges' literary universe: here we provide two examples. We refrain from a deep, quantitative analysis, and we invite the reader to fill the details of our reasoning as a pleasant re-reading exercise of [9] and [10].[9]

Remark 3.12 (Continuity for a proposition). Let $p : U \to \Omega_I$ be a proposition; here we investigate what does it mean for p to be (globally) continuous with respect to the Euclidean topology on $I = [0, 1]$, in the assumption that its domain of definition U is metrisable (this is true for example when U is a subset of space-time). The condition is that

$$\forall \epsilon > 0, \exists \delta > 0 : |x - y| < \delta \Rightarrow |px - py| < \epsilon. \tag{3.11}$$

In layman terms: p is continuous on its domain of definition if its strength over nearby events can't change too dramatically.

All elementary topology results apply to such a proposition: for example, the set of forces with which p is true or false is a connected subset of $\Omega_{[0,1]}$, compact if U was compact.

[9]The plot of [9] in a nutshell is: in Babylon, a lottery game infiltrates reality to the point that it ends up governing the actions of all men; liberating them from free will while coorting them into pawns of an infinite, inescapable, unknowable game, governed by an iron, seemingly chaotic, probabilistic logic.

Example 3.13 (Discontinuity, sapphire from Taprobana). Propositions $p : U \to \Omega_{[0,1]}$ that are allowed to be discontinuous in its variables depend unpredictably from their context: such propositions model seemingly chaotic events triggered as the end terms of a chain of disconnected prior events; in fact, if we assume a real base for the domain of p, its continuity as stated in Remark 3.12 roughly means that events in the same neighbourhood –'near' in space or temporally contiguous– can't have too much different truth/strength values.

A model for such a universe, where 'terrible consequences' sometimes follow from the 'impersonal drawings, whose purpose is unclear' characterising the Company's actions: seemingly disconnected (that 'a sapphire from Taprobana be thrown into the waters of the Euphrates'; that 'a bird be released from the top of a certain tower'; that 'every hundred years a grain of sand be added to (or taken from) the countless grains of sand on a certain beach'), but generating nontrivial dynamics when inserted in a suitable sequence as in Figure 2.

Let us consider the dynamical system $([0,1], \nabla \circ p)$ (see Remark 2.5) obtained from the iterates of the composition $\nabla \circ p : [0,1] \to [0,1]$ ($\nabla : I \amalg I \to I$ is the fold map of I, obtained from the identity of I and the universal property of the coproduct). We start from a proposition p depending on a free variable $t : I$; then, $p(t) = (u(t), \epsilon) : \Omega_I$ consists of a strength and a truth value; but p can be evaluated on $u(t)$ because of (3.6) (u is a continuous endomorphism of a compact metric space; although this is not its universal property, the fold map ∇ just forgets the truth value, keeping the force):

$$t : I \models p \qquad u(t) : I \models p \qquad \dots \tag{3.12}$$

The process can thus be iterated as follows, exploiting the universal property of the natural number object in Set/I:

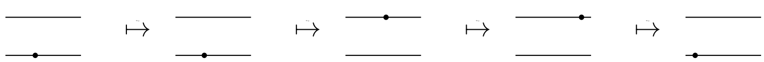


FIGURE 2. A dynamical system induced by the Company's infinite, impersonal drawings.

It is clear that the limit behaviour of such a sequence strongly depends on the analytic properties of u (e.g., if u is a contraction, it must have a single fixed point). A repeated series of drawings can chaotically deform the configuration space on which p is evaluated. There is of course countless termination conditions on such process p; the series can be periodic, it can stop when p reaches maximum or minimum force, or a prescribed value, when u reaches its unique fixed point –if any, or is inside/outside a certain range of forces. . . ; we leave such speculations to the mystagogues of the Company, or to the lions of Qaphqa.

Another compelling example is that of the monotonicity of a proposition depending, say, on a certain number of observers who acknowledge the 'existence' of an object R (be it physical or conceptual) in large numbers, from which very reason the existence of R gains strength.

Example 3.14 (Continuity: a few birds, a horse). Let us consider objects whose existence strength depends *monotonously* and continuously from their parameters: for example a proposition p may be 'truer' the more people observe it, because

> things became duplicated in Tlön; they also tend to become effaced and lose their details when they are forgotten. A classic example is the doorway which survived so long it was visited by a beggar and disappeared at his death. At times some birds, a horse, have saved the ruins of an amphitheater. [10]

In such a situation, we can note that the strength of existence of some ruins –modeled as it is the more naive to do, like a rigid body R in space– depends on the number of its observers:

$$(R, n) \models p. \tag{3.13}$$

FIGURE 3. On Tlön, there are things that exist stronger the more you believe in them. This is a consequence of the strength $p(\triangle)$ monotonically depending on an increasing variable n ('trust' in that existence, 'belief' that the impossible tribar $\triangle$ exists.)

3.2.3. Changing the geometry of I. To conclude the section, a last –somewhat dramatic– example. What happens if we change topology on I? For example, we could brutally forget the Euclidean topology of the closed interval $[0, 1]$, and regard I as the disjoint union $\{\{t\} \mid t : [0, 1]\}$ of its points; so, the subobject classifier becomes the disjoint union of $[0, 1]$ copies of $\{\perp, \top\}$. (See Figure 4 below for a picture.)

Example 3.15 (Burning fields at the horizon). The main tenet of the present paragraph is that Berkeley idealism of infinite and disconnected instants of time finds a natural home in our framework if the classifier is chosen to be the object $\Omega_I = \coprod_{t:[0,1]}\{\perp, \top\}$. Such a peculiar logical framework allows for language to be reshaped in light of Berkeleyan instantaneism: the various terms of the perceptual bundle are recorded and stockpiled by instantaneous accretion, by disjoint sum of their constituents. This is exactly what happens on Tlön for words like 'round airy-light on dark' or 'pale-orange-of-the-sky'; objects are determined by their simultaneity, instead of their logical dependence: accretion superimposes fictitious meaning on a temporal sequence; it is just an illusion, a mistake of perception tricked into an illicit interpolation.

> Spinoza ascribes to his inexhaustible divinity the attributes of extension and thought; no one in Tlön would understand the juxtaposition of the first (which is typical only of certain states) and the second - which is a perfect synonym of the cosmos. In other words, they do not conceive that the spatial persists in time. The perception of a cloud of smoke on the horizon and then of the burning field and then of the half-extinguished cigarette that produced the blaze is considered an example of association of ideas. [10]

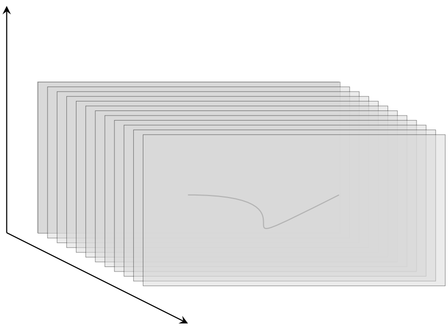


FIGURE 4. Time as an infinite, and infinitely subdivided, sequence of distinguished instants: the 'Berkeley paradox' of non existence of causality lives in a certain topos, as nearby slices E_t, E_s for small $|s-t|$ give no predictive power on how (and if) the transition from E_t to E_s might happen.

Instantaneism is, of course, way larger a topic to cover than a paragraph could do. However, we attempt to scratch the surface of this fascinating topic in our subsection 4.1 below, where we sketch a possible 'rebuttal to the idealist': it is entirely possible the world is affected in some way by me closing my eyes: thus, Berkeley has a fragment of a point. Yet, it takes more than that to make it disappear; thus, Berkeley is obviously wrong.

As always, Truth lies in the middle (of a continuous interval): observation –or lack thereof– affects beings (more: it affects *Being*) so that something changes in the assertion p that 'Everything exists' with my eyes open, and with my eyes closed.

This difference shall be regarded as an infinitesimal dent on p's strength of truth, so to let Berkeleyan idealism gain a little ground. Yet, it is improper to say that the world 'vanished'. It didn't for the rest of you.

The best we can say is that p probably depends –monotonically– on the number of open eyes. After all, objects on Tlön double, triple, they are cyclically reborn; they vanish as the doorway which survived 'so long it was visited by a beggar and disappeared at his death'.

4. Vistas on ontologies

> ¿No basta un solo término repetido para desbaratar y confundir la serie del tiempo? ¿Los fervorosos que se entregan a una línea de Shakespeare no son, literalmente, Shakespeare?
>
> — *Is not one single repeated terminal point enough to disrupt and confound the series in time? Are the enthusiasts who devote themselves to a line of Shakespeare not literally Shakespeare?*
>
> [6]

Reached this point, the reader will agree with us that Borges' literary world underlies deep mathematical structure. This literary *divertissement*, as naive as it may be, has nonetheless the power to show ontology can be made a quantitative theory.

We can go further, discussing the –not so– tangential connection that Borges' work has with classical idealism *à la* Berkeley. With our Example 3.15 in mind, we can safely say Borges has often been fascinated by, and mocked, classical idealism (see [6] for an example of how fascination and derision mix in the same essay).

Even though endeavouring on such a wide ground as classical idealism isn't the purpose of our work, Borges regards Tlön's language and philosophy as a concrete realisation of Berkeley's theory of knowledge. We thus find natural to explore such link with a classical piece of philosophy, as far as it can be taken, with the eyes of a category theorist.

In the following subsections, we sketch a more wide-ranging plan; surely a less substantiated one, and yet aimed at laying a foundation for future work, ours or others', in a research track that we feel is fertile and promising.

4.1. Answers to the idealist

Let us reconsider Example 3.7; we have already mentioned Berkeley's famous view of experience as a perceptual bundle of stimuli that are incapable to cohere.

In whatever sense it accepts that there are nine coins, Berkeleyan instantaneism asks to cut every 'non-total' existence $p(g, c, d) < (\top, 1)$ at the purely false value, so that our notion of admissibility for a configuration of coins becomes untenable (because we can't have $(c, d) \models \sum_u p(u, \_)$ for every (c, d)).

Only what exists completely deserves to be called being. On our side of the barricade however instantaneism is untenable. Idealism is equally untenable, when the word is intended as 'the belief that what lacks a percipient conscience must suddenly disappear in thin air' (and thus, to ensure the world to exist, there must be God).

However, as strange and counter-intuitive as it may seem, instantaneism and idealism have their point on Tlön:[1] we should then be able to find a tenable justification for both of them, possibly from within topos theory; possibly, 'making no pact with the impostor Jesus Christ' (cf. [10]).

In the internal language of variable sets Set/I a proposition p takes its truth values in a much wide spectrum of values, as wide as I; thus, existence gives way to a more nuanced notion that can be (in)valid with a certain strength $t : I$. The language we introduced so far was precisely meant to quantify this stray from classical logic.

Here, it is meant to quantify how much Berkeley's 'cut' in Example 3.15 is a blunt one, easily falsifiable (to say the least) even on Tlön.

To put it shortly,

- I1) instantaneism is what we get from the subobject classifier Ω_I forgetting the topology and order structure on I (cf. Remark 3.3), and
- I2) idealism arises from quotienting one of the copy $I \times \{\top\}$ of $I \times \{\top, \bot\}$ to a point (cf. Remark 3.4).

The problem is thus phrased in a way that makes its solution completely natural: just don't be a 18th century Irish empiricist; if the subobject classifier of a topos carries additional structure, don't forget it.

To fix ideas, let's consider an example. Let's say two empiricists, that for lack of a better name are called David and George, discuss about the truth of the proposition p: 'the World is there'.

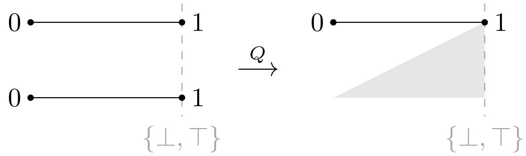
Let us assume that $p : U \to \Omega_I$ depends on a certain number of variables $\vec{x}$; now, when saying that closing their eyes the world disappears, David and George claim that $p(\faEye, \_) \neq p(\faEyeSlash, \_)$, and even more, that $\models \neg p(\faEyeSlash, \_)$. (Note that this is the strongest possible sense in which p 'is not true': for every context $x : U$, $p(\faEyeSlash, x) = (\bot, 1)$.)

A certain number of implicit assumptions are already made here:

- First, that p's domain of definition splits into a product $E \times U'$, where $E = \{\faEye, \faEyeSlash\}$ is a space of parameters taking into account the state of George and David's eyes, and U' takes into account 'the rest of the world'; so p is evaluated on pairs $(e, \vec{y})$.
- Second, that every force $p(\faEyeSlash, \vec{y}) < (\top, 1)$ collapses to the minimum value 'completely false' so that at any $(\faEyeSlash, \vec{y})$ configuration the World is *not* there.

[1] Somehow, [10] isn't far from an artificial world built to mock idealism; cf. also [8] and the famous essay [6] therein.

Formally, George and David collapsed a certain subset of the classifier Ω_I to a point, by means of a quotient map Q, pictorially represented as



(4.1)

(An even more drastic choice would have been to take just the fiber $\{\bot, \top\} \cong \{\bot, \top\} \times \{1\}$, thus falling into classical logic; but we have no information about George and David's stance on falsehood.)

Now, what happens at $p(\faEye, \vec{y})$, as opposed to what happens at $p(\faEyeSlash, \vec{y})$? Strictly speaking, we can't rule out the possibility that perception is directly affected by observers (it certainly is on Tlön, in Babylon, or in a quantum mechanics laboratory); so, George and David closing their eyes somehow affects the degree of existence of the World. In what way? In classical logic the choice is forced by LEM; but making the right choice of a topos $\mathcal{E}$ there can be even a large, even continuous, spectrum of different choices. So, the apparent paradox of idealism can be turned into a safe statement: observation –or lack thereof– affects beings so that something changes between $p(\faEye, \_)$ and $p(\faEyeSlash, \_)$. It does so in a complicated, perhaps even mysterious, but coherent, way.

Of course, the practice of rewording identity and persistence-in-time as mathematical notions affects natural language as well. When David and George close their eyes, they are tricked into believing that the World had disappeared. But this loss of information is merely induced by the drastic quotient map Q above, deleting the set $Z^\top$ (cf. the notation in Remark 3.6); on the contrary, if only David and George admitted that there is a continuous parameter (a 'strength') modelling p's truth in the sense defined here they would implicitly accept to move towards a nonclassical universe of discourse (a topos). All that follows from this 'leap of faith' in nonclassical logic is mere calculation.

From inside Tlön, this gives way to the epistemic vision of its inhabitants, where each of X, Y, Z does not know what happens to the coins of the other observers, and in what secret way C exists as a global entity: without an awakening about the shape of their formal logic, the topos-theoretic model is unusable by Tlönians.

Even this superficial analysis is already sufficient to make our point, that George and David peculiar theory of existence isn't barred from our model; on the contrary, it arises as a very specific example of internal logic. How this is done, follows from a standard procedure in problem-solving:

- First, identify the constraints forcing your choice of I ('what logic shapes your I?' and vice versa 'what I best approximates the logic you want to describe?');

- Given an explicit description of Ω_I, compute proposition strength ('where are you seeking p to be true/false', and at which strength?)

Points of evaluation can be provided by a temporal, spatial or any other kind of reference whatsoever: we are completely agnostic towards the shape of space-time, towards the structure or the properties of the set it forms; moreover, every configuration respecting the prescribed constraints is 'correct' from within the topos. Finally, our approach is computational: given the initial piece of data (for the nine coins: the observer, a subset of coins, a day of the week) all else necessarily follows from a calculation.

A last observation, to conclude the section: an additional source of agnosticism towards the structure of I was anticipated in **??**, and consists in the refusal to adopt a temporal logic framework; we can sometimes interpret the elements of I as time instants, but this is not an obligation at all. A presentist, or a Borges' coherent idealist (capable to deduce from idealism the nonexistence of time, as in [6]) can establish the truth of p without renouncing their ontological stance; they are just forced to accept the result of a calculation.

4.2. Ye shall know them by their fruit

The main point of our paper can be summarised very concisely: an 'ontology' is a category $\mathcal{O}$, inside which 'Being unravels'. Every existence theory shall be reported, and is relative, to a fixed ontology $\mathcal{O}$, the 'world we live in'; such existence theory coincides with the internal language $\mathcal{L}(\mathcal{O})$ of the ontology/category (from now on we employ the two terms as synonyms), in a 'syntax-semantics like' adjunction of [29].

So determined, the internal language of Definition 2.1 an ontology $\mathcal{O}$ is the collection of 'things that can be said' about the constituent parts of the ontology.

If, now, ontology is the study of Existence, and if we are structuralist in the meta-theory (cf. **??**), we cannot know beings but through their attributes. Secretly, this is just an instance of *Yoneda lemma* (cf. [3, 1.3.3]), the statement that the totality of modes of understanding an object X coincides with the totality of modes your language allows you to probe the object X. In relational structures, objects are known via their modes of interaction with other objects, and these are modelled as morphisms $U \to X$; Yoneda lemma posits that we shall 'know objects by their morphisms' as we know types by the terms of that type: an object $X : \mathcal{C}$ coincides with the totality of all morphisms $U \to X$, organised in a coherent 'bundle' (a functor $\mathcal{C}(\_, X) : \mathcal{C} \to \mathsf{Set}$).[2]

All in all, an ontology is a mode of understanding the attributes of Existence: a category, be it in an Aristotelian or in a structuralist sense. As a consequence *meta*ontology, i.e. the totality of such ways of understanding,

[2]The naming convention is not unfortunate: there is indeed a 'bundle' over $\mathcal{C}$, in an appropriate sense, whose fibers are the slices $\mathcal{C}/X$, linked by re-indexing maps $f^* : \mathcal{C}/Y \to \mathcal{C}/X$, one for each $f : X \to Y$ in $\mathcal{C}$.

must coincide with the general theory of such particular modes: with *category theory.* One of the greatest merits of category theory is having provided Mathematics (not its history, not its philosophy, but its practice) with the technical tools to understand the opposition between general and particular.

We shall say no more on the matter, as every further discussion pertains to *meta*ontology, and it would derange from an already wide-ranging discussion: let's just say that questions as 'where is language' and what general principles inspire it might have an anthropological or even neurophysiological answer; not a purely ontological (=category-theoretic) one.

4.3. Metaontology

The gist of our Example 3.7 is that X, Y, Z can't assess the existence of the coins classically; they just have access to partial information allowing neither a global statement of existence on the set C_X of coins lost by X, nor an unbiased claim about the meaning thereof. Lost coins in C_X are untouched as a global conglomerate, yet their strength of existence is very likely to enjoy *local* changes.

This begs the important questions of to which extent language determines ontology, to which extent it constrains its expressive power, and to which extent the inhabitants of Tlön fail to see what is exactly 'a topos further' (e.g., persistence of existence through time).

Let us rewind a bit: according to Quine, ontology is the 'domain over which [logical, or natural language's] quantifiers run'. This is not wrong; it is even perfectly compatible with our views. The only subtlety is that we raised the complexity one level up: in our framework, ontology *à la* Quine still is a category, because (cf. Definition 2.2) a quantifier can be described as a certain specific kind of functor

$$\forall, \exists : PX \to PY \tag{4.2}$$

between power-sets regarded as *internal* categories of our ambient ontology $\mathcal{O}$; in light of this, it's easy to imagine this pattern to continue: if Quine calls meta-ontology what we call ontology, i.e. the meta-category grounding his propositional calculus is a single ambient category *among many*, we live 'one universe higher' in the cumulative hierarchy of foundations and meta-foundations: our ontology possesses a higher dimensionality, and harbours Quinean theory as an internal structure (see [3, Ch. 8] for the definition of an internal category). So, it shall exist, *somewhere* -at least 'in some secret way, hidden from the comprehension of men' [5]- a language that calls ontology what we call meta-ontology (and thus a language that declassifies Quine's to a sub-ontology –whatever this prefix means).

Finding the bottom of this tower of turtles is the aim of the track of research, of very ancient tradition, within which we want to insert the present work.

Of course our work does not make a single comment on how, if, and when, this ambitious foundational goal can be achieved. We just find remarkable that framing Quine's definition in this bigger picture is not far from the so-called practice of 'negative thinking' in category theory: the belief that a high-dimensional/complex entity can be understood through the analogy with its low-dimensional/complexity counterparts; see [28, 27] for a minimal introduction to the principle, [2] for a practical introduction, and see [14] for what T. Gowers calls 'backwards generalisation'.

4.4. Conclusion

The circle closes on, and motivates better, our initial foundational choice: categories and their theory correspond 1:1 to ontology and its theory. However, countless important issues remain open: in what sense this is satisfying? In what sense the scope of our analysis is not limited by this choice? What's his foundation? Is this a faithful way to describe such an elusive concept as 'Existence'?

None of these questions is naive; in fact, each legitimately pertains to meta-ontology and has no definitive answer. More or less our stance is as follows: approaching problems in ontology with a reasonable amount of mathematical knowledge is fruitful. Yet, the problem of what is a foundation for that Mathematics remains (fortunately!) wide open; it pertains to meta-ontology, whose ambitious effort is to clarify 'what there is'. We believe the philosophers' job to work in synergy with quantitative knowledge, approaching the issue with complementary tools.

Appendix A. Category theory

> El atanor está apagado -repitió- y están llenos de polvo los alambiques. En este tramo de mi larga jornada uso de otros instrumentos.
>
> *—The atanor is off -he repeated,- and alembics are dusty. During this part of my long day I use another kind of instrument.*
>
> [7]

A.1. Fundamentals of CT

Throughout the paper we employ standard basic category-theoretic terminology, and thus we refrain from giving a self contained exposition of elementary definitions. Instead, we rely on famous and wide-spread sources like [3, 4, 24, 31, 23, 32].

Precise references for the basic definitions can be found

- for the definition of category, functor, and natural transformation, in [3, 1.2.1], [24, I.2], [3, 1.2.2], [24, I.3], [3, 1.3.1].
- The Yoneda lemma is stated as [3, 1.3.3], [24, III.2].
- For the definition of co/limit and adjunction, in [3, 2.6.2], [24, III.3], [3, 2.6.6], [24, III.4] (consider in particular the definitions of *pullback*, *product*, *terminal object*).

- For the definition of accessible and locally presentable category in [4, 5.3.1], [4, 5.2.1], [1].
- Basic facts about ordinal and cardinal numbers can be found in [19]; another comprehensive reference on basic and non-basic set theory is [17].
- The standard source for Lawvere functorial semantics is Lawvere's PhD thesis [21]; more modern accounts are [15].
- Standard references for topos theory are [25, 18]. See in particular [25, VI.5] and [18, 5.4] for what concerns the Mitchell-Bénabou language of a topos.

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