In Set/I

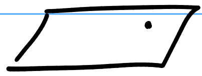
$$p : U \to \Omega_I$$

<u>Thm</u> Esistono topos dove la logica prop è basata su >2 valori di verità

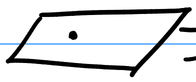
<u>Es.</u> Set/I : Ω_I è $\{0,1\} \times I$ (Ω in Set)

$\downarrow \pi_I$

I

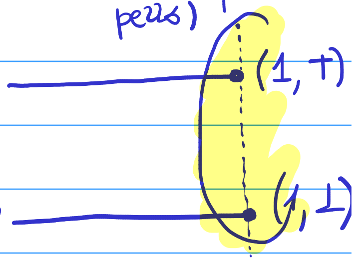


I_1 $(1,t)$ prop vera con forza t $\qquad I \times \{1\}$



I_0 $(0,t)$ prop falsa con forza t $\qquad I \times \{0\}$

(classicamente vedi solo questo pezzo)



I ha struttura $\Rightarrow$ Ω_I ha struttura

$\forall I$, Ω_I è un'algebra di Heyting

$\{0,1\} \times I$ è il prodotto in <u>Heyt</u> se I ha l'ordine banale ($\{0,1\} \in Bool$; $i \leq j \leftrightarrow i = j$)

$(H \quad Set(H))$

Se però I è un'algebra di Heyt, Bool, etc...

allora lo è Ω_I con la struttura prodotto!

cioè non fonge(?)tti che $I \in Heyt$ — Esempio : $I = \{0 < 1\}$ minimo

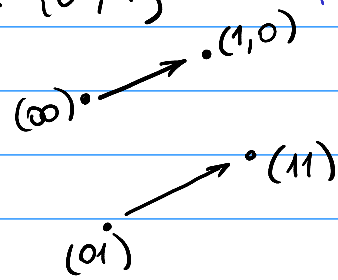
• non è un toset
• E' Heyting ? Nota 1

<u>Esempio</u> $I = [0,1]$, I è naturalmente
- spazio top
- locale
- tot ord
⋮

$I \vee I \to I$ riparametrizzazione (vedi Freyd) Nota 2

$\to$ $\{0<1\} \times \{0,1\}$

Ω_I



$\neg(0i) = (1i)$

$\neg(1i) = (0i)$

$\vee, \wedge \ldots$

Un titolo roboante: "Toposes as worlds"

$Set/_{\{0,1\}} = Set \times Set$ è un topos, logica int. è basata su due copie indipendenti di logica classica.

$p : U \to \Omega_I$

$\left[\begin{array}{c} \cdot \bot \\ a \cdot \quad \cdot b \\ \cdot \top \end{array}\right]$

p: "la freccia esiste"

$$p: [0,1]\times V \longrightarrow \Omega_I$$

$\to \bullet\ p(t,x) \in \{0,1\}\times I = \{0,1\}\times[0,1] \twoheadrightarrow$

$\to \bullet\ p:(t,x) \in \{0,1\}\times I$ $(\$)$

$$p: [0,1]\times V \longrightarrow [0,1]$$

$$V \longrightarrow \mathrm{Set}([0,1],[0,1]) \;\leftarrow \star$$

$$x \longmapsto p(x,-) : t \mapsto p(x,t)$$

Temporal logic sta in $\mathrm{Sh}(\mathbb{R}_{\mathrm{Scott}})$

$(\mathcal{I}(\mathbb{R}), \sqsubseteq)$ $\quad [a,b] \sqsubseteq [c,d] \longleftrightarrow [c,d] \subseteq [a,b]$

$\mathcal{I}$

$\{\mathcal{I}^{op} \to \mathrm{Set} \mid \text{fasci}\}$ è $\mathrm{Sh}(\mathbb{R}_{\mathrm{Scott}})$

Ω_{Scott} = reticolo dei sotto-oggetti del fascio terminale.
(thm è sempre un' Heyting)

$$\mathrm{Set}/I \simeq \mathrm{Set}^I \cong \text{Famiglie di insiemi indicizzate da } I$$

$$\{A_i \mid i\in I\} \xrightarrow[f_i]{} \{B_i \mid i \in I\}$$

Si può prendere come I una "struttura circolare" (gli hnönin di dodicesimo grado iniziano già a decadere): per esempio un sottoinsieme di S^1

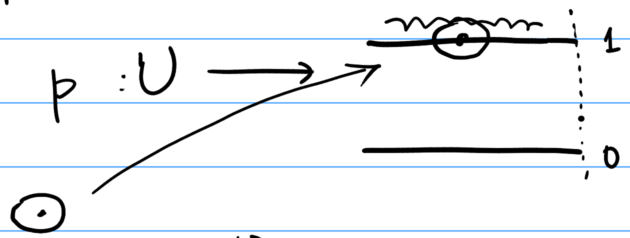
$$p: U \longrightarrow \{0,1\}\times S^1$$

$p(x) \in$ (1) $\quad (0, e^{i\theta}) \leftarrow ?$

(0) $\quad (1, e^{i\theta}) \leftarrow ?$

Per esempio quello fatto dalle 12 radici di $X^{12}-1$ su $\mathbb{C}$ Nota 3 "x esiste" $\equiv p(x) = (1, $ [redacted] $)$ Berkeley succhiami il cazzo

$p(x) \in \{0,1\} \times [0,1]$ "x non esiste" $\equiv p(x) = (0,$ [redacted] $)$

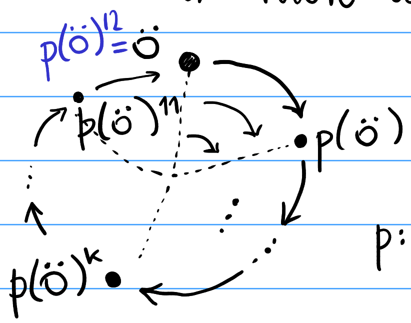


$V_x \subseteq U$ intorno di x

$$p(x) = (1,t) \Rightarrow \forall \varepsilon > 0\ \forall y \in V_x\ \exists s,\ p(y) = (1,s)$$
$$|x-y| < \delta$$
$$|s-t| < \varepsilon$$

Nota 3: Scegliendo le radici dodicesime dell'unità come I possiamo valutare $p : U \longrightarrow E(12) \simeq \mathbb{Z}/12\mathbb{Z}$ in una radice

Il processo generativo di uno hrön allora è:



$p(\ddot{o})\, p(\ddot{o}) = p(\ddot{o}, 2)$ oggetto   tempo/apparizioni successive / usi successivi

$p : \hat{U} \times T \longrightarrow E(12) \subset S^1$ ($\hat{U}$: spazio)

Nota 1: $\Delta[1] \times \Delta[1]$ è un'algebra di Heyting. In effetti anche di Boole:

$$\begin{tikzcd} & \bullet \arrow[dl] \arrow[dr] & \\ \bullet \arrow[dr] & & \bullet \arrow[dl] \\ & \bullet & \end{tikzcd} \cong \begin{tikzcd} & \emptyset \arrow[dl] \arrow[dr] & \\ \{0\} \arrow[dr] & & \{1\} \arrow[dl] \\ & \{0,1\} & \end{tikzcd} = (\wp(\{0,1\}), \cap, \cup, C)$$

Nota 2: Vedi Freyd per una caratterizzazione intrinseca dell'intervallo $[0,1]$ in termini di proprietà algebriche

Consider, then, the category whose objects are sets with two distinguished points, **top** and **bottom**, denoted $\top$ and $\perp$ and whose maps are the functions that preserve $\top$ and $\perp$. Given a pair of objects, X and Y, we define their **ordered wedge**, denoted $\boldsymbol{X \vee Y}$ to be the result of identifying the top of X with the bottom of Y. [4] This construction can clearly be extended to the maps to obtain the "ordered-wedge functor."

The closed interval can be defined as the final coalgebra of the functor that sends X to $X \vee X$. Let me explain.

First (borrowing from the topologists' construction of the ordinary wedge), $X \vee Y$ is taken as the subset of pairs, $\langle x, y\rangle$, in the product $X \times Y$ that satisfy the disjunctive condition: $x = \top$ *or* $y = \perp$. A map, then, from X to $X \vee X$ may be construed as a pair of self-maps, denoted $\overset{\wedge}{\boldsymbol{x}}$ and $\overset{\vee}{\boldsymbol{x}}$, such that for all x either $\overset{\vee}{x} = \top$ or $\hat{x} = \perp$. The final coalgebra we seek is the terminal object in the category whose objects are these structures.[5] To be formal, begin with the category whose objects are quintuples $\langle X, \perp, \top, {}^\wedge, {}^\vee\rangle$ where $\perp, \top \in X$, and ${}^\wedge, {}^\vee$ signify self-maps on X. The maps of the category are the functions that preserve the two constants and the two self-maps. Then cut down to the full subcategory of objects that satisfy the conditions:

$$\overset{\wedge}{\top} = \top = \overset{\vee}{\top}$$

$$\overset{\wedge}{\perp} = \perp = \overset{\vee}{\perp}$$

$$\forall_x \, [\hat{x} = \perp \quad \text{or} \quad \overset{\vee}{x} = \top]$$

$$\perp \neq \top \quad {}^{[6]}$$

We will call such a structure an **interval coalgebra**.[7]

I said that we will eventually construct the reals from $\mathbf{I}$. But if one already has the reals then one may chose $\perp < \top$ and define a coalgebra structure on $[\perp, \top]$ as

$$\overset{\vee}{x} = \min(2x - \perp, \top)$$

$$\hat{x} = \max(2x - \top, \perp)$$

Note that each of the two self-maps evenly expands a half interval to fill the entire interval—one the bottom half the other the top half. We will call them **zoom operators**. (By convention we will not say "zoom in" or "zoom out." All zooming herein is expansive, not contractive.)

The general definition of "final coalgebra" reduces—in this case—to the characterization of such a closed interval, $\mathbf{I}$, as the terminal object in this category.[8]

Algebraic Real Analysis

Caratterizzazione dell'intervallo chiuso $[-1,1]$ (o $[0,1]$ fwiw)
come oggetto terminale nella categoria delle interval coalgebre.
(I è dotato di operazioni
$\perp, \top \in I$ e ha una mappa canonica $I \to I \vee I$ tale
$(-)^\vee, (-)^\wedge \in \hom(I,I)$
che I è terminale tra le $\vee$-coalgebre

$$\begin{tikzcd} A \arrow[d, "\vee\text{-coalgebra}"] \\ A \vee A \end{tikzcd} \Rightarrow \begin{tikzcd} A \arrow[r, "u!"] \arrow[d] & I \arrow[d] \\ A \vee A \arrow[r, "u!"] & I \vee I \end{tikzcd}$$

→ Variando I in Set/I cambiano le prop che è possibile esprimere.

- ○ Paradosso ($)
- ○ Berkeley
- ○ Lotteria a Babilonia

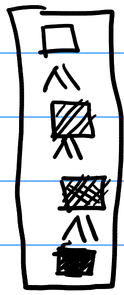
I è un insieme
- tot ordinato
(- denso)

$[0,1]$

[base dell'accordo intersoggettivo]

estremo: esempio minimale di esistenza con forze

$$
\begin{tikzcd}
 & (0\ 0) = \perp \arrow[dl] \arrow[dr] \arrow[dd] & \\
(0,\tfrac{1}{2}) \arrow[rr] \arrow[dr] & & (1,\tfrac{1}{2}) \arrow[dl] \\
 & (1\ 1) = \top &
\end{tikzcd}
$$



$\{0 \leq 1 \leq 2 \leq 3\}$

dividing instant problem

probabilità di beccare un numero reale $\{x\}$

---

[7]The modal operations $\Diamond$ for *possibly* and $\Box$ for *necessarily* have received many formalizations but it is safe to say that no one allows simultaneously both $\Diamond\Phi \neq \top$ and $\Box\Phi \neq \bot$: "less than possible but somewhat necessary." (The coalgebra condition can be viewed as a much weakened excluded middle: when the two unary operations are trivialized—that is, both taken to be the identity operation—then $\Diamond\Phi = \top$ *or* $\Box\Phi = \bot$ becomes just standard excluded middle.)

If we assume, for the moment, that $\top$ and $\bot$ are fixed points for $\Diamond$ and $\Box$ then we have an example of an interval coalgebra where $\Box\Phi$ is $\hat{\Phi}$ and $\Diamond\Phi$ is $\check{\Phi}$. The finality of $\mathbf{I}$ yields what may be considered truth values for sentences (e.g. the truth value of $\odot = \bot|\top$ translates to "entirely possible but totally unnecessary" and a truth value greater than $\top|\odot$ means "necessarily entirely possible").

The fixed-point conditions are not, in fact, appropriate—true does not imply necessarily true nor does possibly false imply false—but, fortunately, they're not needed: an easy corollary of the finality of $\mathbf{I}$ says that it suffices to assume the disjointness of the orbits of $\top$ and $\bot$ under the action of the two operators. If we work in a context in which the modal operations are monotonic (that is, when Φ implies Ψ it is the case that $\Box\Phi$ implies $\Box\Psi$ and $\Diamond\Phi$ implies $\Diamond\Psi$) it suffices to assume that $\Box\Phi$ implies Φ, that Φ implies $\Diamond\Phi$ and that $\Box^n\top$ never implies $\Diamond^n\bot$. If this last condition has never previously been formalized it's only because no one ever thought of it.

The same treatment of modal operators holds when $\Diamond$ is interpreted as *tenable* and $\Box$ as *certain*; or $\Diamond$ as *conceivable* and $\Box$ as *known*.

This topic will be much better discussed in the intuitionistic foundations considered in Section 30.

[8]If the case with $\bot = \top$ were allowed then the terminal object would be just the one-point set. (In some sense, then, the separation of $\top$ and $\bot$ requires no less than an entire continuum.)