Outline of a Paraconsistent Category Theory*

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Summary. In this paper, we show how a certain kind of paraconsistent category theory can be developed. We also want to contribute to a better knowledge of Oubiña's and Bosch's work in category theory.

The aim of this paper is two-fold: (1) To contribute to a better knowledge of the method of the Argentinean mathematicians Lia Oubiña and Jorge Bosch to formulate category theory independently of set theory. This method suggests a new ontology of mathematical objects, and has a profound philosophical significance (the underlying logic of the resulting category theory is classical first-order predicate calculus with equality). (2) To show in outline how the Oubiña-Bosch theory can be modified to give rise to a strong paraconsistent category theory; strong enough to be taken as the basis for a paraconsistent mathematics which encompasses all classical mathematical results.

1 The Theory OB_0

In this section, we outline the Oubiña-Bosch theory of categories, which will be denoted by OB_0. The details of OB_0 can be found in Bosch [1964], Oubiña [1966], and [1969], and Volkov [1997]. What follows in this section is essentially Oubiña [1969] (and Volkov [1997]).

The underlying logic of OB_0 is the classical predicate calculus with equality, as formalised in Kleene [1952], but implication, equivalence, and conjunction are symbolised by $\rightarrow$, $\leftrightarrow$, and $\wedge$, respectively, and function symbols are allowed.

The primitive symbols of OB_0 are the following: (a) predicate symbols: $\subseteq$ (binary), comp (ternary), and funct (unary); (b) function symbols: o (ternary), I (unary), and P (unary).

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** It is with great regret that we report that Analice Volkov passed away a few months after finishing her contribution to this paper. It was a sad and unfortunate loss to all of us.

P. Weingartner (ed.), *Alternative Logics. Do Sciences Need Them?*

Intuitively, the objects to which the theory OB_0 refers to are categories. After some developments, the notion of set is introduced by definition: a set is a special kind of category.

The postulates (axioms, axiom schemes, and rules of inference) are those of the classical predicate calculus (with equality and function symbols; see Kleene [1952]), plus the postulates P1–P23 introduced below.

1.1 Subcategories

$x \subseteq y$ is read x is a subcategory of y.

P 1 $(x \subseteq y \wedge y \subseteq x \leftrightarrow x = y) \wedge (x \subseteq y \wedge y \subseteq z \rightarrow x \subseteq z)$

Definition 1. $x \subset y =_{def.} x \subseteq y \wedge x \neq y$. *(If $x \subset y$, we say that x is a proper subcategory of y.)*

1.2 Union of Subcategories

P 2 $x \subseteq z \wedge y \subseteq z \rightarrow \exists v(v \subseteq z \wedge x \subseteq v \wedge y \subseteq v \wedge \forall u((u \subseteq z \wedge x \subseteq u \wedge y \subseteq u) \rightarrow (v \subseteq u)))$

Definition 2. *The category v of postulate P_3 is unique. It is called the union of x and y in z, and it is denoted by $x \cup_z y$.*

Theorem 1.
- $x \cup_z y = y \cup_z x$
- $x \subseteq y \wedge y \subseteq z \rightarrow x \cup_z y = y$
- $x \subseteq z \wedge y \subseteq z \wedge z \subseteq w \rightarrow x \cup_z y = x \cup_w y$
- $(x \cup_w y) \cup_w z = x \cup_w (y \cup_w z)$.

1.3 The Empty Category

P 3 $\exists y \forall x(y \subseteq x)$

Definition 3. *P_4 guarantees the existence of a category which is a subcategory of any category. This category is uniquely determined and is called the empty category. It is denoted by $\oslash_c$.*

Theorem 2. $x \subseteq \oslash_c \leftrightarrow x = \oslash_c$.

1.4 Objects of a Category

Definition 4. *unit(x)*$=_{def.} x \neq \oslash_c \wedge \forall u(u \subseteq x \rightarrow u = x \vee u = \oslash_c)$.

Definition 5. *x is an object of z* $=_{def.}$ *unit*$(x) \wedge x \subseteq z$.

1.5 Morphisms of a Category

Definition 6.
• *f is a strict morphism of z between x and y* $=_{def.}$ *(x is an object of $z \wedge y$ is an object of $z \wedge f \subseteq z \wedge \forall u(u \subset f \leftrightarrow u \subseteq x \cup_z y)$).*
• *x is a morphism object of z between x and x* $=_{def.}$ *x is an object of z.*

Definition 7.
• *f: $x \Leftrightarrow_z y$ is an abbreviation for f is a strict morphism between x and y in z, if $x \neq y$.*
• *x: $x \Leftrightarrow_z x$ is an abbreviation for x is an object morphism between x and x in z.*
• *f is a morphism of z is an abbreviation for $\exists x \exists y(f\colon x \Leftrightarrow_z y)$.*

comp(z, x, y) that we read x is composable with y in z, will be written $x\text{comp}_z y$. Moreover, instead of $\text{o}(z, x, y)$, we write $x\text{o}_z y$.

Definition 8. *x and y are composable in z is an abbreviation for $x\,comp_z y \vee y\,comp_z x$.*

P 4 $(z \subseteq w \wedge x$ is morphism of $z \wedge y$ is morphism of $z \wedge x\text{comp}_z y) \rightarrow (x\text{comp}_w y \wedge x\text{o}_z y = x\text{o}_w y)$

P 5 $(x$ is morphism of $z \wedge y$ is morphism of $z \wedge x\text{comp}_z y) \rightarrow (x\text{o}_z y$ is morphism of $z)$

P 6 $(x$ is an object of $z \wedge y$ is morphism of $z) \rightarrow ((x\text{comp}_z y \rightarrow x\text{o}_z y = y) \wedge (y\text{comp}_z x \rightarrow y\text{o}_z x = y))$

P 7 $(f\colon x \Leftrightarrow_z x \wedge g\colon x \Leftrightarrow_z x \rightarrow f\text{comp}_z g) \wedge (f\colon x \Leftrightarrow_z x \wedge g\colon x \Leftrightarrow_z y \rightarrow$ 'f and g are composable in z')

P 8 Let f, g, and h be morphisms of z. In this case, it follows that:
• $(f\text{comp}_z g \wedge g\text{comp}_z h \rightarrow (f\text{o}_z g)\text{comp}_z h)$
• $(f\text{comp}_z g \wedge (f\text{o}_z g)\text{comp}_z h \rightarrow g\text{comp}_z h \wedge f\text{comp}_z(g\text{o}_z h))$
• $(f\text{comp}_z g \wedge (f\text{o}_z g)\text{comp}_z h \rightarrow (f\text{o}_z g)\text{o}_z h = f\text{o}_z(g\text{o}_z h))$

Theorem 3. If x and y are distinct objects of a category z, and $f\colon x \Leftrightarrow_z y$, then either $f\text{comp}_z x$ or $f\text{comp}_z y$, and only one of these two formulas is valid.

Definition 9. *f is a morphism of z from x to y (or $f\colon x \rightarrow_z y$)* $=_{def.}$ *$f\colon x \Leftrightarrow_z y \wedge f\,comp_z x$.*

Theorem 4. Let f and g be morphisms of a category z. Thus: $f\text{comp}_z g \leftrightarrow \exists u \exists v \exists t(u, v, t$ are objects of $z \wedge g\colon u \rightarrow_z v \wedge f\colon v \rightarrow_z t)$ $f\text{comp}_z g \rightarrow ((f\colon u \rightarrow_z v \wedge g\colon v \rightarrow_z t) \rightarrow (f\text{o}_z g)\colon v \rightarrow_z t)$.

Theorem 5. $(f\colon x \rightarrow_z y \wedge f\colon x' \rightarrow_z y') \rightarrow x = x' \wedge y = y'$.

Definition 10. *If $f\colon x \rightarrow_z y$, then x is called the source of f, and is denoted by $\alpha(f)$, and y is called the sink of f, and is denoted by $\beta(f)$. ($\alpha(f)$ and $\beta(f)$ do not depend on the category z.)*

Definition 11. *$f\,compg =_{def.} \exists z$ (f is a morphism of $z \wedge g$ is a morphism of $z \wedge f\,comp_z g$).*

P 9 $((f, g$ are morphisms of $z \wedge f$compg$) \rightarrow (f, g$ are morphisms of $w \wedge (f\mathrm{o}_z g = f\mathrm{o}_w g))) \rightarrow z \subseteq w$

1.6 The Category of Two Morphisms

P 10 unit$(x) \wedge$ unit$(y) \rightarrow \exists z(x \subseteq z \wedge y \subseteq z \wedge \forall f(f$ is morphism of $z \rightarrow f = x \vee f = y))$

Definition 12. *If unit(x) and unit(y), then $[x, y]$ is the category universally described by the axiom P14. When $x = y$, $[x, x]$ is denoted by $[x]$.*

Theorem 6. If unit(x), then $[x] = x$. If x, y are objects of z, then $x \cup_z y = [x, y]$.

Theorem 7. If z is a category and $f\colon x \rightarrow_z y$, then:
- x, y are the sole objects of z.
- f, x, and y are the sole morphisms of f.

1.7 The Postulates of Identity

We shall write I(x) as follows: I_x. Given the category x, I_x is a category associated with x, satisfying the conditions:

P 11 $(\mathrm{I}_x = \mathrm{I}_y \rightarrow x = y) \wedge (\mathrm{unit}(\mathrm{I}_x))$

1.8 The Postulate P_{17}

Let F(f) be a formula of OB_0 in which the variables x, y, and w do not appear, and let F_1 and F_2 be the following formulas:

F_1: $\forall x \forall y(x, y$ are morphisms of $z \wedge x\mathrm{comp}_z y \wedge \mathrm{F}(x) \wedge \mathrm{F}(y) \rightarrow \mathrm{F}(x\mathrm{o}_z y))$

F_2: $\forall x(x$ is morphism of $z \wedge \mathrm{F}(x) \rightarrow \mathrm{F}(\alpha(x)) \wedge \mathrm{F}(\beta(x)))$
Under these conditions, the following formula is an axiom of OB_0:

P 12 $\forall z(\mathrm{F}_1 \wedge \mathrm{F}_2 \rightarrow \exists w(w \subseteq z \wedge \forall f(f$ is morphism of $w \leftrightarrow f$ is a morphism of $z \wedge \mathrm{F}(f))))$

The preceding postulate constitutes a kind of comprehension scheme: it guarantees that some categories can be formed, starting with a previous category.

Theorem 8. $\forall z \exists y \forall u(u$ is morphism of $y \leftrightarrow u$ is an object of $z)$.

Definition 13. *$O(z)$ denotes the category of the objects of the category z.*

1.9 Functorial Categories

We call functorial categories the categories of a particular kind. They are implicitly defined by special axioms, containing the unary predicate symbol *funct.* If x is a category, funct(x) means that x is a functorial category.

P 13 $(\text{funct}(x) \wedge y \subseteq x \rightarrow \text{funct}(y)) \wedge (\text{funct}(\mathrm{I}x)) \wedge (\text{funct}(z) \wedge x$ is an object of $z \rightarrow \exists u(\mathrm{I}_u = x))$

P 14 $\text{funct}(x) \wedge \text{funct}(y) \rightarrow \exists z(\text{funct}(z) \wedge x \subseteq z \wedge y \subseteq z \wedge \forall u(\text{funct}(u) \wedge x \subseteq u \wedge y \subseteq u \rightarrow z \subseteq u))$

Using axiom P_{19}, we easily define the functorial sum of two functorial categories x and y, which is denoted by $x\mathbf{V}y$.

Theorem 9. $x\mathbf{V}y = y\mathbf{V}x$; $(x\mathbf{V}y)\mathbf{V}z = x\mathbf{V}(y\mathbf{V}z)$; $x \subseteq y \rightarrow x\mathbf{V}y = y$, where x and y are functorial categories.

Definition 14. *If $x = I_u$, we denote u by $E(x)$.*

Theorem 10.
- $\mathrm{E}(\mathrm{I}_x) = x$.
- x is an object of a functorial category $\rightarrow \mathrm{I}_{E(x)} = x$.
- x, y are objects of functorial categories $\rightarrow (\mathrm{E}(x) = \mathrm{E}(y)) \leftrightarrow x = y$.

1.10 Functors

The following two definitions are basic:

Definition 15.
- *f is a functor $=_{def.} \exists z(funct(z) \wedge f$ is a morphism of $z)$*
- *f is a functor of x into $y =_{def.} \exists z(funct(z) \wedge f\colon x \rightarrow_z y)$*
- *$f\colon x \rightarrow y =_{def.} f$ is a functor of x into y*

Definition 16.
If f is a functor, we pose:
- *$d(f) =_{def.} E(\alpha(f))$ (domain of f)*
- *$c(f) =_{def.} E(\beta(f))$ (codomain of f)*

Theorem 11. If f and g are morphisms of the functorial categories z and w, respectively, then $f\text{comp}g$ implies that $f\mathrm{o}_z g = f\mathrm{o}_w g$. Moreover, if z and w are functorial categories, then $z \subseteq w$ if and only if every morphism of z is also a morphism of w.

Definition 17. *If f and g are composable functors, we state: $f\mathrm{o}g =_{def.} f\mathrm{o}_z g$, where $z = f\mathbf{V}g$.*

1.11 The Category of Subcategories of a Given Category

The symbol P is unary and P(x) denotes the category of subcategories of x. The postulates governing P are as follows:

P 15 $\forall x(\text{funct}(\mathrm{P}(x)) \ \wedge \ (u \subseteq x \leftrightarrow \mathrm{I}_u$ is an object of $P(x)) \ \wedge \ (x \subseteq y \leftrightarrow P(x) \subseteq P(y))$

The expression '$\exists! x F(x)$' is an abbreviation for 'there is a unique x such that $F(x)$'.

P 16 $(x \subseteq y \ \wedge \ y \subseteq z \rightarrow \exists! f(f{:}\mathrm{I}_x \rightarrow_{P(z)} \mathrm{I}_y)) \ \wedge \ (\exists f(f{:}\mathrm{I}_x \rightarrow_{P(z)} \mathrm{I}_y) \rightarrow x \subseteq y)$

Definition 18. If $x \subseteq y$, then $\mathrm{I}_{xy} =_{def.}$ the f such that $f{:}\mathrm{I}_x \rightarrow_{\mathrm{P}(y)} \mathrm{I}_y$. ($\mathrm{I}_{xy}$ is called functor inclusion of x into y.)

Theorem 12. $I_{xx} = I_x$.

P 17 $(x \subseteq z \ \wedge \ y \subseteq z \ \wedge \ g{:}\mathrm{I}_x \rightarrow \mathrm{I}_y \wedge \mathrm{I}_{xz} = \mathrm{I}_{yz} o g) \rightarrow (x \subseteq y \wedge \mathrm{g}{=}\mathrm{I}_{xy})$

Theorem 13. Given the category z, there exists a unique category M(z) $\subseteq P(z)$, satisfying the condition: u is a morphism of M(z) if and only if there exists a morphism $f \subseteq z$ such that $u = \mathrm{I}_f$. M(z) is called the category of morphisms of z.

1.12 Image by a Functor

Let A(g) and B(g) be the following formulas:

- A(g): g is a functor $\wedge \ d(\mathrm{f}) = d(\mathrm{g}) \wedge \mathrm{c}(g) \subseteq \mathrm{c}(f) \wedge \mathrm{I}_{c(g)c(f)} \mathrm{o} g = f$
- B(g): $\forall h \ (\mathrm{A}(h) \rightarrow c(g) \subseteq c(h) \ \wedge \ I_{c(g)c(h)} \mathrm{o} g = h)$.

Under these conditions, we have:

P 18 f is a functor $\rightarrow \exists g \ (\mathrm{A}(g) \ \wedge \ \mathrm{B}(g))$

Definition 19. *Let f be a functor and x its domain. The functor g of P 28 is called a functor restriction of f to its image. We call $c(g)$ the image of f, which is denoted by $f(x)$.*

Definition 20. *Let f be a functor, x its domain and $u \subseteq x$. Thus, the image of u by f, denoted by $f(u)$, is the image of the functor $f o I_{ux}$, and the restriction of $f o I_{ux}$ to its image is called the restriction of f to u and $f(u)$.*

Theorem 14. If x, y, and u are categories such that $x \subseteq y$, and $u \subseteq x$, then the image of u by I_{xy} is u, and the restriction of I_{xy} to u and $\mathrm{I}_{xy}(u)$ is I_u.

1.13 Functors and Their Images

P 19 f is a functor $\wedge$ unit(u) $\wedge$ $u \subseteq$ d(f) $\rightarrow$unit$(f(u))$

P 20 f is a functor $\wedge$ b is morphism of d(f) $\rightarrow$ $f(b)$ is morphism of c(f)

P 21 (f is a functor $\wedge$ g is morphism of d(f) $\wedge$ h is morphism of d(f) $\wedge$ $g$comp$h$) $\rightarrow$ ($f(g)$comp$f(h)$ $\wedge$ $f(g\text{o}_{d(f)}h) = f(g)\text{o}_{c(f)}f(h)$)

P 22 f, g are functors $\wedge$ $f$comp$g$ $\wedge$ $u \subseteq$ d(g) $\rightarrow$ $f(g(u)) \subseteq f\text{o}g(u)$

P 23 (f, g are functors $\wedge$ d(f) = d(g) $\wedge$ c(f) = c(g) $\wedge$ $\forall u(u$ is morphism of d$(f) \rightarrow f(u) = g(u)) \rightarrow f = g$

Theorem 15. (a) If f is a functor and x is an object of $d(f)$, then $f(x)$ is an object of $c(f)$. (b) If f is a functor and $h\colon x \rightarrow_{d(f)} y$, then $f(h)\colon f(x) \rightarrow_{c(f)} f(y)$.

Theorem 16. If f and g are functors such that $f$comp$g$ and $u \subseteq$ d(g), then $f(g(u)) = f\text{o}g(u)$.

1.14 Sets

Definition 21. *The category x is called a set if x is unitary, functorial and $E(x)$ has only object morphisms. The functors between sets are called functions.*

Clearly, we have:

Theorem 17. If x is a category all of whose morphisms are object morphisms, then I_x is a set.

Definition 22. *$I_{\emptyset c}$, which is a set, is called the empty set and denoted by $\emptyset$. ($\emptyset_c$ is the empty category, as we have already seen.)*

Definition 23. $x \in y =_{def.}$ *x is an object of $E(y)$.*

Definition 24. $x \sqsubseteq y =_{def.}$ $E(x) \subseteq E(y)$, *where x and y are sets. ($x \sqsubseteq y$ is read x is a subset of y, or x is part of y.)*

Theorem 18. $x \sqsubseteq y \leftrightarrow \forall z(z \in x \rightarrow z \in y)$, where x and y are sets.

Theorem 19. If x and y are sets, then $x = y \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)$.

Theorem 20. $\forall x(x \notin \emptyset)$.

Definition 25. *x is a unitary set if unit $(E(x))$.*

Theorem 21. x is a unitary set $\leftrightarrow \exists! y(y \in x)$.

Theorem 22. If x is a unitary set, then its only element is $E(x)$. (x is an element of y means $x \in y$.)

Theorem 23. If unit(x), then since $\mathrm{E}(\mathrm{I}_x) = x$, I_x is a unitary set whose only element is x.

Definition 26. *If x is a unitary category, then I_x is denoted by $\{x\}$. Obviously, $x \in \{x\}$.*

Given the unitary categories x and y, there exists the category $[x, y]$ whose only morphisms are x and y. Therefore, $\mathrm{I}_{[x,y]}$ is a set; $z \in \mathrm{I}_{[x,y]} \leftrightarrow z = x \vee z = y$.

Definition 27. $\{x, y\} =_{def.} I_{[x,y]}$.

When x is a unitary category, $\{x, x\} = \mathrm{I}_{[x,x]} = \mathrm{I}x = \{x\}$.

Definition 28. $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$ *(ordered pair).*

Theorem 24. If x and y are unitary categories, then $\langle x, y\rangle$ is a set. If, in addition, x' and y' are unitary categories, then $\langle x, y\rangle = \langle x', y'\rangle \leftrightarrow x = x' \wedge y = y'$.

Theorem 25. Let x be a category and let us denote by k the category $I_{O(P(E(x)))}$. Thus: k is a set $\wedge \forall u(u \in k \leftrightarrow \mathrm{unit}(u) \wedge \mathrm{funct}(u) \wedge \mathrm{E}(u) \subseteq \mathrm{E}(x))$.

Definition 29. $\wp(x) =_{def.} I_{O(P(E(x)))}$ *(set of the subsets of the set x).*

Theorem 26. Let x be a set. We have: $y \in \wp(x) \leftrightarrow y \sqsubseteq x$, for every set y.

1.15 Complete Categories of Sets

Definition 30. *u is called a complete category of sets (ccs) if the following conditions are met:*
(1) funct(u)
(2) x is an object of $u \rightarrow (y \in x \rightarrow y$ is an object of $u)$
(3) x is an object of $u \rightarrow x$ is a set
(4) x, y are objects of $u \rightarrow \{x, y\}$ is an object of u
(5) x is an object of $u \rightarrow \wp(x)$ is an object of u.

Since the empty category satisfies the preceding conditions, there exists a *ccs*. In the next definition, $\forall'$ and $\exists'$ are the quantifiers restricted to the objects of the category u.

Definition 31. *A ccs is called normal if we have, for any formula $R(x, y)$:*
$\forall y \exists' z \forall x(R(x, y) \rightarrow x \in z) \rightarrow \forall' w \exists' v \forall x(x \in v \leftrightarrow \exists y(y \in w \wedge R(x, y)))$.

Note. The above property corresponds to Bourbaki's scheme of selection and union (Bourbaki [1968], p. 69). In set theory, with the help of Bourbaki's scheme, we prove the separation scheme, the substitution scheme and the axiom of union.

1.16 ZF and OB_0

Let ZF' denote Zermelo-Fraenkel set theory without the axioms of infinity, regularity, and choice. It is not difficult to prove that the axioms of ZF' are verified in a normal *ccs*, when they are interpreted in an obvious way (see Oubiña [1966] and [1969], and Volkov [1997]).

Theorem 27. Any normal *ccs* constitutes a 'model' of ZF'.

If we impose the conditions that a normal *ccs* u satisfies infinity, regularity, and choice, we obtain a 'model' of ZF in OB_0. Conversely, if in ZF' we define the notions of category theory (see Oubiña [1966] and [1969]), then the axioms of OB_0 are satisfied in ZF', through a convenient interpretation.

Theorem 28. OB_0 can be interpreted in ZF'.

OB_0 is a powerful theory, and it can be taken as the basis for topos theory, the theory of sheaves, the functorial theory of structures, etc., including all extant, usual mathematics. Moreover, since it does not rely on sets, OB_0 provides us with a new ontology of mathematics.

Remark. Let us indicate an intuitive interpretation of OB_0 in terms of a first-order theory whose primitive concepts are those of domain, counter-domain, and composition (in the usual, intuitive categorial-theoretic sense;[1] see Hatcher [1982]); let us also adopt the usual senses of morphism and category.

The *categories* of OB_0 can be conceived of as "aggregates" of morphisms. It is worth noting that "aggregate" is not being used here in any set-theoretic sense. The notion of subcategory can be interpreted in a natural way as "aggregates" of morphisms, and for a given morphism $f: a \to b$, the subcategories are f, a, and b. The *union* is not the set-theoretic union, but is more general, involving morphisms. The *null category* has no morphism. The *unitary categories* are the identity morphisms, and the *objects* of a category are the identity morphisms of this category. Given a morphism f: $f: a \to b$, its objects are a, b, and $[a, b]$. The notion of being *composable* can be expressed in terms of domain and counter-domain; "f is composable with g in z" means that g's counter-domain is the same as f's domain, and f and g are subcategories of z.

The *identity* over a given category x (notation: I_x) can be interpreted as the identity functor over x. *Functorial categories* are "aggregates" whose morphisms are *functors*. In particular, functors are also functorial categories, since they correspond to "aggregates" with a single functor. Moreover, the objects of functorial categories are identity functors (of type I_x).

Given a particular x, let us denote by $E(x)$ the only u such that $I_u = x$. A unitary category X is a *set* if, and only if, X is a functorial category and

[1] Here we follow Lawvere's ideas. As McLarty points out: "Lawvere believes 'intuitive' categories and spaces and other structures are just as real (or more accurately, just ideal) as 'intuitive' sets" (McLarty [1990]).

$E(X)$ only has object morphisms. We can think of sets as the I_u with u only having objects.

The following question immediately arises: are the members of sets of OB_0 also sets of OB_0? Not necessarily; there may well be a set I_u such that u has an object I_x and x does not have only objects. Thus, sets of OB_0 do not correspond exactly to what we usually call sets. But we can introduce the notion of a *normal complete category of sets* (normal *ccs*). Every object of a normal *ccs* is a set and the elements of objects of a normal *ccs* are also objects of the normal *ccs*. Moreover, a normal *ccs* is closed by the pair set and the set of parts. A normal *ccs* is what most resembles, in OB_0, the universe of sets; in other words, it is easy to relate the notion of a universe of sets with the concept of normal *ccs*. Furthermore, given that the objects of a normal *ccs* are sets and so are the elements of objects, morphisms between objects are functions. Therefore, a normal *ccs* can be associated with the usual category *SET* (of sets and functions). Finally, if we add certain conditions to the definition of normal *ccs*, we can obtain a concept equivalent to the concept of *topos*.

2 The Paraconsistent Theory OB_1

In this section we outline a paraconsistent category theory OB_1. We stress the fact that the technical details are relevant, but the most important aspect of the present work is the method used to build OB_1. There are infinitely many possible paraconsistent systems, and each of them is developed taking into account the intended applications. However, our outline is enough to substantiate the philosophical conclusions of the last section of this paper.

The underlying logic of OB_1 is the first-order predicate calculus $\mathbf{C}_1^=$ (see da Costa [1974] and [1997]; on paraconsistent logic, see, for example, da Costa, Béziau and Bueno [1995]).

2.1 The Calculus $C_1^=$

$\mathbf{C}_1^=$ is a paraconsistent predicate calculus with identity. Its primitive symbols are the following: (1) individual variables, as usual; (2) predicate symbols; (3) function symbols; (4) connectives: $\rightarrow$, $\wedge$ $\vee$, $\neg$, $\leftrightarrow$ (this last symbol is defined as usual); (5) the quantifiers: $\forall, \exists$ (6) individual constants (i.e., 0-place function symbols); (7) identity: $=$.

The syntactical notions, such as those of formula and term, are the usual ones. The postulates of $\mathbf{C}_1^=$ are classified into three groups: (1) the propositional postulates (which characterise the propositional calculus); (2) the postulates of quantification (which, with the postulates of $\mathbf{C}_1$, constitute the postulates of the predicate calculus $\mathbf{C}_1^*$); (3) the identity postulates. A^o is an abbreviation for $\neg(A \ \wedge \ \neg A)$.

Propositional postulates. They are those of the classical positive logic, plus the following:

$\neg_1) B^o \to ((A \to B) \to ((A \to \neg B) \to \neg A))$
$\neg_2) A^o \ \wedge \ B^o \to (A \to B)^o \ \wedge \ (A \ \wedge \ B)^o \ \wedge \ (A \vee B)^o$
$\neg_3) A \vee \neg A.$

We define $\neg^* A$ as $\neg A \ \wedge \ A^o$. ($\neg^*$ is a strong negation, and it has all the properties of classical negation.)

Quantificational postulates. They are composed of the quantificational postulates of classical positive predicate logic, plus:
$\forall^o)$ $\forall x(A(x))^o \to (\forall x(A(x))^o)$
$\exists^o)$ $\forall x(A(x))^o \to (\exists x(A(x))^o)$
K) $A \leftrightarrow B$, where A and B are congruent formulas in the sense of Kleene [1952], p. 153, or one of them is obtained from the other by the suppression of vacuous quantifiers.

Postulates of identity. The common ones.

The concepts of proof, deduction, (formal) theorem, etc., are clear adaptations of Kleene [1952].

A theory T is a set of sentences closed under deduction. T is trivial if it coincides with the set of all sentences; otherwise, T is said to be nontrivial. T is inconsistent if there exists a formula A such that A and $\neg A$ are both members of T; otherwise, T is said to be consistent. Roughly speaking, a logic is said to be paraconsistent if it can be the underlying logic of inconsistent but nontrivial theories.

Theorem 29. In $\mathbf{C}_1^=$ we have:
$\vdash A \vee \neg A$
$\vdash \neg\neg A \to A$
$\vdash A \vee \neg^* A$
$\vdash \neg^*\neg^* A \leftrightarrow A$
$\vdash A \to (\neg^* A \to B)$
$\vdash A \ \wedge \ \neg^* A \to B$
$\vdash (A \leftrightarrow \neg^* A) \to B$
$\vdash A^{oo}$
$\vdash A^o \to (\neg A)^o$
$B^o, A \to B \vdash \neg B \to \neg A$
$B^o, A \to \neg B \vdash B \to \neg A$
$\vdash (A \to \neg A) \to \neg A$
$\vdash \forall x A(x) \leftrightarrow \neg^* \exists x \neg^* A(x)$
$\vdash \exists x A(x) \leftrightarrow \neg^* \forall x \neg^* A(x).$

Proof. See da Costa [1974].

Theorem 30. The following schemes and rules are not provable in $\mathbf{C}_1^=$.
$\neg(A \wedge \neg A)$
$A \rightarrow (\neg A \rightarrow B)$
$(A \wedge \neg A) \rightarrow B$
$(A \wedge \neg A) \rightarrow \neg B$
$A \vee B, \neg A / B$
$A \rightarrow \neg\neg A$
$A \rightarrow B, A \rightarrow \neg B / \neg A$
$(\neg A \rightarrow B \wedge \neg B) \rightarrow A$
$\neg A \leftrightarrow \neg(A \wedge A)$
$\neg A \leftrightarrow \neg^* A$
$A \leftrightarrow \neg\neg A$
$\forall x A(x) \leftrightarrow \neg\exists x \neg A(x)$
$\exists x A(x) \leftrightarrow \neg\forall x \neg A(x)$.

Proof. See da Costa [1974].

$\mathbf{C}_0$, $\mathbf{C}_0^*$, and $\mathbf{C}_0^=$ denote, respectively, the classical propositional calculus, the classical first-order predicate calculus, and the classical first-order predicate calculus with equality.

$\mathbf{C}_1^=$ (and also $\mathbf{C}_1$ and $\mathbf{C}_1^*$) possess a semantics of valuations in relation to which $\mathbf{C}_1^=$(respectively, $\mathbf{C}_1$ and $\mathbf{C}_1^*$) is correct and complete. Informally, the semantics of valuations can be outlined as follows.

Let $\mathbf{A}$ be a set, and $\mathcal{L}(A)$ the diagram language of $\mathbf{C}_1^=$ associated with $\mathbf{A}$. When $\mathbf{A} = \emptyset, \mathcal{L}(\mathbf{A}) = \mathcal{L}$, the language of $\mathbf{C}_1^=$.

A valuation in $\mathcal{L}(A)$ is the characteristic function of a certain kind of maximal nontrivial theory. A sentence F is true in a valuation v if $v(\mathrm{F}) = 1$; otherwise it is false, i.e., $v(\mathrm{F}) = 0$. The valuation v is a model of a set of sentences Γ if $v(\mathrm{S}) = 1$, for each S in Γ.

Let $\Delta \cup \{G\}$ be a set of sentences of $\mathcal{L}$. We say that G is a semantic consequence of Δ, and write $\Delta \models G$, if every model v of Δ is such that $v(\mathrm{G}) = 1$, in all diagram languages $\mathcal{L}(A)$. In $\mathbf{C}_0^=$, it is easy to verify that a valuation in $\mathcal{L}(\mathbf{A})$ determines a first-order structure whose universe is A, and conversely. In the case of $\mathbf{C}_1^=$, a valuation in $\mathcal{L}(\mathbf{A})$ similarly individualises a first-order structure, but given the structure, the valuation is not determined.

2.2 The Theory $\mathbf{OB}_1$

Let F be a formula of $\mathbf{C}_1^=$. F* is the same formula F, in which the occurrences of $\neg$ are replaced by $\neg^*$.

The underlying logic of OB_1 is $\mathbf{C}_1^=$, and its primitive symbols are those of OB_0. The specific axioms of OB_1 are the following: if F is an axiom of OB_0, then A and A^* are axioms of OB_1. In general, every concept of OB_0 leads to two corresponding concepts in OB_1, one obtained with $\neg$ (weak negation), and another with $\neg^*$ (strong negation).

Theorem 31. Let F be a formula of the language of OB_0. Then, if F is a theorem of OB_0, F^* is a theorem of OB_1.

Proof. Consequence of the postulates of OB_0 and OB_1, and the fact that $\neg^*$ constitutes a classical negation (see da Costa [1986a]).

Theorem 32. OB_0 is contained in OB_1.

Proof. Consequence of the fact that $\mathbf{C}_1$ is a subcalculus of $\mathbf{C}_0^{=}$.

Theorem 33. OB_1 is a paraconsistent system.

Proof. If we add Russell's set to OB_1, i.e., the set $R = \{x \colon x \notin x\}$, where $x \notin x$ means $\neg(x \in x)$ (weak negation), the resulting system is not trivial (see da Costa [1986*a*]).

So, OB_1 is a very strong system, in which we can develop not only all classical mathematics, but also study the properties of "inconsistent" mathematical structures, such as Russell's set. In OB_1, we can construct other types of "inconsistent" structures, which are not set-theoretic in nature. For instance, we may introduce the definition

$$x \prec y =_{def.} x \text{ is not an object of } \mathrm{I}_y,$$

and consider the category K such that

$$x \prec K \leftrightarrow \neg(x \prec x).$$

As a consequence, we have:

$$K \prec K \leftrightarrow \neg(K \prec K),$$

and

$$K \prec K \ \wedge \ \neg(K \prec K),$$

a certain form of Russell's paradox. This and other issues will be left to future works (including the full development of OB_1).[2]

3 Philosophical Remarks

After putting forward a paraconsistent formulation of category theory, we would like to discuss some philosophical issues raised by this construction as well as by paraconsistency in general.

[2] A paraconsistent category theory has been articulated by Mortensen (see Mortensen [1995]). However, OB_1 is enterely distinct from, and contains, Mortensen's construction.

The construction we have just presented make clear that, at least at the level of mathematics, there are inconsistent but nontrivial theories. In other words, there are theories T in which both a formula A and its negation $\neg A$ are theorems of T, and some formula B of T is *not* a theorem. In our view, this fact provides an important support for paraconsistency, since it shows that the attempt at accommodating inconsistencies by devising appropriate inconsistent but nontrivial theories is by no means empty or unrealisable. On the contrary, it provides a distinctive perspective to the issues under consideration. Instead of retaining classical logic, and avoiding the inconsistency by rejecting one or another of the premises which generate it – making more or less *ad hoc* moves – we retain the inconsistency, change the underlying logic to a paraconsistent one, and study the properties of the "inconsistent object" so "generated". The important feature, as the paraconsistent category theory discussed above shows, is that these "inconsistent objects" have certain determined properties and lack others: it is simply *not* the case that everything goes with regard to them. So, as opposed to what happens in the case of classical logic, there is a whole new domain of investigation determined by the formulation of paraconsistent logic: the domain of the inconsistent.

Now, the issue arises as to the status of the resulting theory: is it *true*? Can we say that there are true inconsistencies? The answer depends on several considerations. (1) What is the notion of truth used in this context? (2) What kinds of objects are we considering (mathematical objects or physical objects)? (3) What notion of existence is assumed? And how are ontological commitments to be spelled out? Of course, the examination of these issues involves particular philosophical accounts, and we cannot do better here than consider them in fairly general terms. But we hope to say enough in order to make clear the approach we favour with regard to them.

According to some authors, the answer to the question *'Are there true contradictions?'* is affirmative (see Priest [1987]). The examples given by Priest are the logical and semantic paradoxes, statements about moving objects (objects subject to change), and certain views in the foundations of mathematics. In order to articulate an "ontic" or "realist" view about true contradictions, Priest advocates (i) a strong notion of truth – truth *simpliciter* understood in the correspondence sense, (ii) a classical view about existence (as the range of bound variables), and (iii) an extended claim as to the domain of his theory – which incorporates both mathematical and physical objects. (Of course, in order to avoid triviality, Priest adopts a paraconsistent logic: see his system LP; see Priest [1987].) So Priest's approach countenances classical views about truth and existence, and applies them to a wide-ranging domain. In our view, Priest's commitment to several doctrines is by no means fortuitous: in order to be adequately accommodated, inconsistencies require a whole package of logical and philosophical doctrines (indeed, a research programme). Of course, there are stronger and weaker programmes, some are closer, some further from classical proposals.

It seems to us that, in retaining classical notions of truth and existence, Priest's proposal became committed to metaphysical views which are decidedly strong. Given the use of truth in the correspondence sense, and the claim that our claims about the world (be it the "empirical" or the "mathematical" world) are to be true, Priest's view is *ipso facto* committed to all objects which are postulated in these claims. In particular, his proposal is committed to "inconsistent objects" in the physical world: the objects to which our inconsistent but true physical theories refer. But how can their existence be established?

The argument to this effect assumes, of course, the classical account of ontological commitment: we are ontologically committed to whatever our bound variables range over. And, in the case of inconsistent theories, this criterion leads us to postulate objects which both have and lack a given property (for instance, the liar sentence is both true and false, Russell's set is both a member of itself and it is not etc.). The same goes for theories about the physical world.

In our view, this argument is not so conclusive. First, this criterion of ontological commitment is *not* independent of particular philosophical assumptions. It comes as part of a philosophical programme – Quine's view – and it has built into it, as it were, a given logic: *classical* first-order logic. It goes without saying that, as such, it is at odds with Priest's own dialetheic approach, in which a paraconsistent logic is advocated. Moreover, Quine's criterion is *not* independent of logic: if we change the underlying logic of a given theory, we change the entities we are quantifying over. This can be seen in several ways. If we move to second-order logic, we are allowed to quantify over predicates and relations. As a result of its strong expressive power, several mathematical theories can be better formulated (in particular, as is well known, arithmetic and analysis are categorical). Because of this, several nominalist proposals (such as those developed in Field [1980] and Hellman [1989]) have adopted second-order logic as part of their nominalisation strategies of science and mathematics. The idea is that, by increasing the strength of the logic, we can decrease our ontological commitments. Secondly, using paraconsistent logic, we are allowed to quantify over certain constructions (such as Russell's set) which are impossible in classical logic, given its identification of inconsistency with triviality.

Our point here is that Quine's slogan – to be is to be the value of a variable – can only have any force once a particular logic is admitted. Quine knows that, of course. The problem is that his view assumes a logic (classical first-order logic) which is not the most adequate to deal with inconsistencies in a heuristically fruitful way.

We suggest to address the inconsistency issue differently. We may well explore the rich representational devices allowed by the use of paraconsistency in inconsistent domains, but withholding any claim to the effect that there are "inconsistent objects" in reality. Whether the world is indeed "inconsistent"

– assuming that there is a sensible formulation of this claim – is something we would rather be *agnostic* about. Just as empiricists (such as van Fraassen [1980]) are agnostic about (the existence of) unobservable entities in science, we are agnostic about the existence of true contradictions in nature. And one of the reasons in support of this claim is an *underdetermination* argument. Given the hierarchy of paraconsistent systems presented in da Costa [1974], there are always infinitely many paraconsistent logics which can be used to accommodate a given "phenomenon" – whether it is an "inconsistent" reasoning or an "inconsistent" theory (see da Costa, Bueno and French [1998b]). Which of these paraconsistent logics reflects *the* logic of the world? There is no argument based on purely observational terms that could establish this in general. We can select, of course, one of these logics on *pragmatic* grounds, but these grounds are certainly not enough to establish a substantive claim about the world. For instance, if one of these logics make the modeling of the inconsistency in question easier, why should this be taken as a reason for this logic to be *true*? Simplicity may well be a sensible criterion to adopt on pragmatic grounds, but the claim that a logic selected on this basis is (likely to be) true, is to confuse pragmatic and epistemic issues. Why should the world conform to our cognitive limitations? Of course, it might well do. But to establish this demands an argument that goes beyond what we observe: it requires a metaphysical claim of the simplicity of reality. However, to a certain extent, this is as strong as the claim that there are true contradictions, in the sense that both make substantial assertions about the world that transcend empirical observation. So, both are metaphysical claims.

In our view, an alternative programme of interpretation of inconsistencies can be devised in which no commitment to this kind of metaphysics is required. The idea is first to avoid the claim that inconsistent theories are *true*; they are *quasi-true* at best (see da Costa and French [1990] and [1993], and da Costa, Bueno and French [1998*a*]). The notion of quasi-truth receives a straightforward formal treatment (see Mikenberg, da Costa and Chuaqui [1986], da Costa [1986*b*], and Bueno and de Souza [1996]). But for our needs it suffices to say that a sentence α is quasi-true if it models adequately only part of a given domain D, leaving open the "complete" description of the latter.[3]

[3] We add in a footnote some further details. The investigation of a certain domain Δ of knowledge involves the study of the relations among its objects. Nonetheless, the information about such objects is frequently "incomplete", since one does not know whether certain relations concerned hold for all the objects of the relevant domain (or for all the n-tuples thereof). In order to accommodate this situation, da Costa and French advanced the concept of a *partial relation*; that is, a relation which is not defined for every object, or n-tuple of objects, of its domain. More formally, we can characterise an n-place partial relation R over D as a triple $\langle R^1, R^2, R^3 \rangle$, where R^1, R^2, and R^3 are mutually disjoint sets, with $R^1 \cup R^2 \cup R^3 = D^n$, and such that R^1 is the set of n-tuples that belong to R; R^2 of those n-tuples that do not belong to R; and R^3 those n-tuples for which

it is not defined whether they belong or not to R. (Notice that if R^3 is empty, R is a standard n-place relation which can be identified with R^1; see da Costa and French [1990], p. 255, note 2.)

However, in order to accommodate our patterns of modeling information, something more than partial relations is required: an appropriate notion of *structure* should also be presented. This notion is similarly thought of as incorporating the "openness" typical of our epistemic situation, where we usually face "incomplete" information. Based on a partial relation, da Costa and French formulate the concept of a *partial* (or *pragmatic*) *structure*. It is a set-theoretical structure of the form $\langle D, R_i, P\rangle_{i\in I}$, where D is a nonempty set (representing the objects used in the systematisation of the relevant domain of knowledge Δ, whose study we are concerned with), $(R_i)_{i\in I}$ is a family of partial relations defined over D, and P is a set of sentences (representing what is known, or taken to be known, about D, including laws and observable statements).

In terms of partial structures, we can formulate a particular notion of truth, which extends Tarski's account, and leads to the characterisation of the concept of *quasi-truth*. Since the Tarskian semantics was constructed only for total structures, in order to use it, it is necessary that a total structure be obtained from a partial one, 'extending' its partial relations. Such 'extended' structures are called *normal structures*. More formally, given a partial structure $A = \langle D, R_i, P\rangle_{i\in I}$, we say that the structure $B = \langle D', R'_i, P'\rangle_{i\in I}$ is an A-*normal* structure if: (1) $D = D'$; (2) every constant of the language in question is interpreted by the same object both in A and in B; (3) R'_i extends the corresponding relation R_i(in the sense that, as opposed to the latter, the former is defined for every n-tuples of objects of its domain); and (4) if $\alpha \in P'$, then α is true in B (in the Tarskian sense).

It should be noticed that, given a partial structure, it is not always the case that is possible to extend it into a normal one. Necessary and sufficient conditions for this result are as follows (see Mikenberg, da Costa and Chuaqui 1986). Given a partial structure $A = \langle D, R_i, P\rangle_{i\in I}$, for each partial relation R_i, we construct a set M_i of atomic sentences and negations of atomic sentences such that the former corresponds to the n-tuples that satisfy R_i, and the latter to those n-tuples that do not satisfy R_i. Let M be $\cup_{i\in I} M_i$. Therefore, a pragmatic structure A admits an A-normal structure if, and only if, the set $M \cup P$ is consistent. In other words, the extension of a pragmatic structure A to a total A-normal structure B is possible whenever the extension of the partial relations is made in such a way that the consistency between the new extended relations and the accepted basic propositions P is guaranteed.

Normal structures are introduced in order to present an interpretation of the language in question. This strategy was also employed by Tarski in his characterisation of the concept of truth: the latter was defined in a *structure*. And the same feature is employed in the formulation of the concept of quasi-truth. We say that a sentence a is *quasi-true* in a partial structure $A = \langle D, R_i, P\rangle_{i\in I}$, according to an A-normal structure $B = \langle D', R'_i, P'\rangle_{i\in I}$, if α is true in B (in the Tarskian sense). If a is not quasi-true in S according to B, we say that a is quasi-false (in S according to B).

Notice that the concept of quasi-truth is an as *if* concept: if a sentence α is quasi-true, it describes the domain in question *as if* its description were true.

(Of course, in a precise sense, α is consistent with a true description of D.) In claiming that, with regard to inconsistent theories, all we need is to determine their quasi-truth, we are in position to provide a "formal underpinning" to our agnosticism with regard to true contradictions. We are suggesting a change in the notion of truth to the weaker notion of quasi-truth, withholding then the commitment to "inconsistent objects" (see also da Costa and French [1993]), given that there are several distinct (A-normal) structures which describe the domain under consideration, and such objects are not countenanced in all of them.[4] Moreover, we also suggest to revise Quine's slogan about ontological commitment, making explicit its dependence on the underlying logic (see also da Costa and French [1991]). In this way, it becomes clear that it is *not* the only criterion to adjudicate between alternative logics.

However, if we may not be committed to an ontology of actual inconsistent physical objects, are we committed to inconsistent *mathematical* entities? This depends, of course, on how we interpret the theory OB_1. Does it provide a *true* description of the inconsistent category-theoretical "world"? Again, if all we claim is to have described this "world" in at best quasi-true terms, we will not be committed to those entities. There is, of course, a whole story to be told here, but for the present purposes it suffices to note that it is possible to provide an entirely syntactic formulation of set theory, in which OB_1 can be embedded,[5] such that the only commitment is to a countable language. Thus, in a certain sense, we do not require a special commitment to mathematical objects either (see da Costa and Bueno [1998]).

So, the "package" we suggest to accommodate inconsistencies is characterised by (1) the claim that inconsistent theories are *quasi-true* at best; (2) an agnosticism with regard to the existence of true contradictions and a nominalism about inconsistent mathematical entities; and (3) a re-evaluation of Quine's view about ontological commitment, emphasising its dependence on the underlying logic.

In our view the striking feature of this "package" is its logical pluralism, on the one hand, and the fact that it allows us to make sense of the several uses and applications of paraconsistent logic with no commitment to actual "inconsistent objects". The logical pluralism derives from the claim (3) above.

Being consistent with the basic knowledge (represented by the set of accepted sentences P) of this domain, α grasps some of the main bits of information about it, without committing us to the acceptance of the further bits (formulated by the extended A-normal structure) as *true*.

[4] Descartes once remarked (in his *Principles*, iv, 204) "that, with regard to those things that our senses cannot perceive, it suffices to explicate how they can be". We would say the same with regard to inconsistent theories. The notion of quasi-truth allows us to formally accommodate this point, since it is strictly weaker than truth, and does not commit us to anything *beyond* the assertion that certain structures are *possible, vis à vis* a given paraconsistent logic.

[5] This construction has to be articulated in a paraconsistent set theory; that is, one in which the underlying logic is paraconsistent (see da Costa [1986*a*]).

Depending on the domain we are considering, different kinds of logic may be appropriate. For instance, if we want to model constructive features in mathematical reasoning, an intuitionistic logic is the best alternative; if we are concerned with inconsistent bits of information, the use of a paraconsistent logic is the natural option. In particular, we do *not* reject classical logic: it has its own domains and applications. To this extent, while dealing with distinct domains, paraconsistent logic and classical logic are complementary rather than rivals. (They become rivals only when applied to the *same* domain. The rivalry derives from the fact that they provide different accounts of the logical connectives.)

But in applying paraconsistent logic, as we have done when formulating OB_1, we do not have to be committed to the existence of "inconsistent entities" – this is the point of our claims (1) and (2) above. We can always use the resources provided by this logic only to help us in drawing consequences from inconsistent theories without triviality, but with no commitment to the *truth* of the theory in question; it can be at best quasi-true.

In this way, a noncommittal (agnostic) interpretation of paraconsistency can be presented, making sense of the application of paraconsistent logic articulated here. "Inconsistent objects", either mathematical or physical, can be accommodated without requiring an ontology which includes them. In particular, a paraconsistent category theory is suggested here, but it is not necessary to countenance the existence of the entities the theory is taken to be about.

References

1. J. Bosch: *Seminar on Differentiable Manifolds* [in Spanish] (Comissión de Investigación Cientifica de la Prov. de Buenos Aires, Buenos Aires 1964)
2. N. Bourbaki: *Theory of Sets* (Addison-Wesley, Boston, Mass. 1968)
3. O. Bueno, E. de Souza: 'The Concept of Quasi-Truth'. Logique et Analyse **153-154** (1996) pp. 183–199.
4. H. Burkhardt, B. Smith (eds): *The Handbook of Ontology and Metaphysics* (Philosophia Verlag, Munich 1991)
5. N.C.A. da Costa: 'On the Theory of Inconsistent Formal Systems'. Notre Dame Journal of Formal Logic **15** (1974) pp. 497–510
6. N.C.A. da Costa: 'On Paraconsistent Set Theory'. Logique et Analyse **115** (1986a) pp. 361–371
7. N.C.A. da Costa: 'Pragmatic Probability'. Erkenntnis **25** (1986b) pp. 141–162
8. N.C.A. da Costa: 'Paraconsistent Mathematics'. In: *Frontiers of Paraconsistent Logic*, ed. by D. Batens, C. Mortensen, G. Priest and J.-P. Van Bendegem, (Research Studies Press, Baldock, Hertfordshire, England; Philadelphia, PA 2000)
9. N.C.A. da Costa, J.-Y. Béziau, O. Bueno: 'Aspects of Paraconsistent Logic'. Bulletin of the Interest Group in Pure and Applied Logics **3** (1995) pp. 597–614

10. N.C.A. da Costa, O. Bueno: Nominalism and Syntactic Models of Set Theory. Unpublished manuscript, Department of Philosophy, University of São Paulo, and Department of Philosophy, University of Leeds (1998), forthcoming
11. N.C.A. da Costa, O. Bueno, S. French: 'The Logic of Pragmatic Truth'. Journal of Philosophical Logic **27** (1998) pp. 603–620
12. N.C.A. da Costa, O. Bueno, S. French: 'Is There a Zande Logic?'. History and Philosophy of Logic **19** (1998b) pp. 41–54
13. N.C.A. da Costa, S. French: 'The Model-Theoretic Approach in the Philosophy of Science'. Philosophy of Science **57** (1990) pp. 248–265
14. N.C.A. da Costa, S. French: 'Ontology and Paraconsistency'. In: Burkhardt and Smith (eds.) (1991) pp. 656–658
15. N.C.A. da Costa, S. French: 'Towards an Acceptable Theory of Acceptance: Partial Structures and the General Correspondence Principle'. In: French and Kamminga (eds.) (1993) pp. 137–158
16. H. Field: *Science Without Numbers: A Defense of Nominalism* (Princeton University Press, Princeton, N.J. 1980)
17. S. French, H. Kamminga (eds.): *Correspondence, Invariance and Heuristics: Essays in Honour of Heinz Post* (Reidel, Dordrecht 1993)
18. W.S. Hatcher: *The Logical Foundations of Mathematics* (Pergamon Press, New York 1982)
19. G. Hellman: *Mathematics Without Numbers: Towards a Modal-Structural Interpretation* (Clarendon Press, Oxford 1989)
20. S.C. Kleene: *Introduction to Metamathematics* (North-Holland, Amsterdam 1952)
21. C. McLarty: 'The Uses and Abuses of the History of Topos Theory'. British Journal for the Philosophy of Science **41** (1990) pp. 351–375
22. I. Mikenberg, N.C.A. da Costa, R. Chuaqui: 'Pragmatic Truth and Approximation to Truth'. Journal of Symbolic Logic **51** (1986) pp. 201–221
23. C. Mortensen: *Inconsistent Mathematics* (Kluwer, Dordrecht 1995)
24. L. Oubiña: Logico-Axiomatic Theory of Categories. [in Spanish] PhD thesis, National University of La Plata (1966)
25. L. Oubiña: 'Axiomatic Theory of Categories [in Spanish]' *Cahiers de Topologie et Géométrie Différentielle* X (3) (1969) pp. 375–394
26. G. Priest: *In Contradiction* (Nijhoff, Dordrecht 1987)
27. B.C. van Fraassen: *The Scientific Image* (Clarendon Press, Oxford 1980)
28. A. Volkov: Theory of Categories and Theory of Sets. [in Portuguese] MA Thesis, University of São Paulo (1997)