CATEGORICAL ONTOLOGY I

EXISTENCE

DARIO DENTAMARO AND FOSCO LOREGIAN

Abstract.

Contents

1. Introduction 1
2. Preliminaries on variable set theory 8
3. The internal language of variable sets 11
4. Nine copper coins, and other toposes 12
5. Vistas on ontologies 21
Appendix A. Category theory 21
References 26

1. Introduction

Eadem svnt qvorvm vnvm potest svbstitvi alteri per transmvtationem

Leibniz, if only he knew category theory

1.1. What is this series. [...]

1.2. On our choice of metatheory and foundation.

In his influential paper [] William Lawvere proposes a foundation of mathematics based on category theory. To appreciate the depth and breadth of such an impressive piece of work, however, the word "foundation" must be taken in the particular sense intended by mathematicians:

> [. . .] a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved.

Such a position sounds at the same time a bit cryptic to unravel, and unsatisfactory; Lawvere's (and others') stance on the matter is that a foundation of mathematics is *de facto* just a set $\mathcal{L}$ of first order axioms organised in a Gentzen-like deduction system. The deductive system so generated reproduces mathematics as we know and practice it, i.e. provides a formalisation for something that already exists and needs no further explanation, and that we call "mathematics".

It is not a vacuous truth that $\mathcal{L}$ exists somewhere: the fact that the theory so determined has a nontrivial model, i.e the fact that it can be interpreted inside a given familiar structure, is both the key assumption we make, and the less relevant aspect of the construction itself; showing that $\mathcal{L}$ "has a model" is –although slightly improperly– meant to ensure that, *assuming the existence of a naive set theory*, axioms of $\mathcal{L}$ can be satisfied by a naive set. Alternatively, and more crudely, assuming the existence of a model of ZFC, $\mathcal{L}$ has a model inside that model of ZFC.

A series of works attempting to unhinge some aspects of ontology through category theory should at least try to tackle such a simple and yet diabolic question as "where" are the symbols forming the first-order theory of ETCC. And yet, everyone just believes in sets, and solves the issue of "where" they are with a leap of faith from which all else follows.

This might appear somewhat circular: aren't sets in themselves already a mathematical object? How can they be a piece of the theory they aim to be a foundation of?

Following this path would have, however, catastrophic consequences on the quality and depth of our exposition. The usual choice is thus to assume that, wherever and whatever they are, these symbols "are", and our rôle in unveiling mathematics is *descriptive* rather than generative.

This state of affairs has, to the best of our moderate knowledge on the subject, various possible explanations:

- On one hand, it constitutes the heritage of Bourbaki's authoritarian stance on formalism in pure mathematics;
- on the other hand, a different position would result in barely no difference for the "working class"; mathematicians are irreducible pragmatists, somewhat blind to the consequences of their philosophical stances.

So, symbols and letters do not exist outside of the Gentzen-like deductive system we specified together with $\mathcal{L}$.

As arid as it may seem, this perspective proved itself to be quite useful in working mathematics; consider for example the type declaration rules of a typed functional programming language: such a concise declaration as

```
data Nat = Z | S Nat
```

makes no assumption on "what" `Z` and `S :: Nat -> Nat` are; instead, it treats these constructors as meaningful formally (in terms of the admissible derivations a well-formed expression is subject to) and intuitively (in terms of the fact that they model natural numbers: every data structure that has those two constructors must be the type $\mathbb{N}$ of natural numbers).

Taken as an operative rule, this reveals exactly what is our stance towards foundations: we are "structuralist in the metatheory", meaning that we treat the symbols of a first-order theory or the constructors of a type system irregardless of their origin, provided the same relation occur between criptomorphic collections of labeled atoms.

In this precise sense we are thus structuralists in the metatheory, and yet we do so with a grain of salt, maintaining a transparent approach to the consequences and limits of this partialisation. On the one hand, pragmatism works; it generates rules of evaluation for the truth of sentences. On the other hand, this sounds like a Munchhausen-like explanation of its the value, in terms of itself. Yet there seems to be no way to do better: answering the initial question would give no less than a foundation of language.

And this for no other reason that "our" metatheory is something near to a structuralist theory of language; thus, a foundation for such a metatheory shall inhabit a meta-metatheory... and so on.

Thus, rather than trying to revert this state of affairs we silently comply to it as everyone else does; but we feel contempt after a brief and honest declaration of intents towards where our metatheory lives. Such a metatheory hinges again on work of Lawvere, and especially on the series of works on functorial semantics.

1.3. **Categories as places.** Lo scopo di questa sezione è chiarire in quale senso preciso una teoria matematica si può pensare come una categoria; tale categoria dà una rappresentazione grafica delle operazioni che definiscono una struttura (algebrica nel senso di [**?**]); un'idea simile permette di realizzare una categoria che rappresenta un dato linguaggio L, e un topos ottenuto come categoria universale che ha L come linguaggio interno (citare free toposes, Lambek-Scott e Freyd); in tale senso è possibile costruire un "luogo" in cui ritrovare l'intera matematica e

> Jim Lambek proposed to use the free topos as ambient world to do mathematics in; see (Lambek 2004). Being syntactically constructed, but universally determined, with higher-order intuitionistic type theory as internal language he saw it as a reconciliation of the three classical schools of philosophy of mathematics, namely formalism, platonism, and intuitionism. His latest views on this variant of categorical foundations can be found in (Lambek-Scott 2011).

In **CT** le categorie sono, quindi, oggetti puramente sintattici, ed è il contesto in cui si opera a determinare una semantica. Già in semantica logica una *interpretazione* $\mathcal{I}$ di un enunciato α è una funzione che associa elementi di un insieme (di solito l'insieme dei valori di verità) alle variabili libere in α. Nella storia della **CT** (caratterizzata da un rifiuto dell'impostazione *set-theoretic* a livello fondazionale) una serie di generalizzazioni e raffinamenti dello stesso procedimento hanno condotto alla nozione di funtore e quindi alla *semantica funtoriale* di Lawvere. Nella sottosezione che segue ci occupiamo di introdurre tale teoria; nella successiva tracciamo, in maniera più discorsiva, alcune conseguenze filosofiche che questa prospettiva ha sull'ontologia degli oggetti matematici (e dunque sull'ontologia degli oggetti che queste teorie descrivono) sottolineando i tratti salienti dell'impostazione che seguiamo.

1.3.1. *Theories and their models.* Nel già citato [], Lawvere, dopo aver costruito il sistema formale "elementare" ETAC (*Elementary Theory of Abstract Category*), presenta una teoria della categoria di tutte le categorie (ETCC) che fornisca modelli per la teoria elementare. Quindi ci dà un linguaggio sintattico, in cui le categorie non sono altro che *termini*, e poi una metateoria nella quale poter considerare categorie di categorie etc, che è poi alla fine una teoria funtoriale, dove ogni ffbf di ETAC diventa una formula della *basic theory* di ETCC in cui si specifica su quale modello operano i termini:

> If Φ is any theorem of elementary theory of abstract categories, then $\forall \mathcal{A}(\mathcal{A} \models \Phi)$ is a theorem of basic theory of category of all categories [Lawvere, 1965]

e aggiunge, un po'ambiguamente, "*every object in a world described by basic theory is, at least, a category*". Altri esempi: Δ_i in ETAC (che indica dom/cod) diventa $\mathcal{A} \models \Delta_i$ in BT; $\forall x[\dots]$ in ETAC diventa $\forall x[x \in \mathcal{A} \to \dots]$ in BT etc etc.

Le categorie in matematica hanno una duplice natura: da un lato, le strutture che siamo intenzionati a descrivere si organizzano naturalmente in categorie; dall'altro, una *singola* categoria, che trattiamo come universo semantico e che fissiamo una volta per tutte, è il luogo in cui ciascuna di queste teorie viene interpretata. In altre parole, da un lato esiste la categoria dei gruppi; dall'altro essa è semplicemente una sottocategoria dell'Universo, o "dell'unica categoria che esiste": quella degli insiemi.

In una prospettiva categoriale tuttavia questa sistematizzazione è assai insoddisfacente, perché attribuisce alla categoria degli insiemi un ruolo privilegiato che essa non possiede: essa è solo *uno* dei possibili compromessi tra diverse esigenze per ciò che una fondazione della matematica deve essere. Vorremmo invece essere in grado di parlare di strutture disembodied dai posti dove quelle strutture vengono interpretate, per poter approcciare la fondazione della matematica in maniera agnostica: non è importante cosa il modello fondazionale contiene, è importante la sua proprietà universale.

In tale prospettiva si inseriscono le ricerche in algebra categoriale [], semantica funtoriale [**?** **?**], logica categoriale [], e teoria dei topos [] che, nel corso degli ultimi sessant'anni, hanno caratterizzato la ricerca in CT.

La *semantica funtoriale* nasce nella tesi di doc di Lawvere per sistematizzare l'*algebra universale*, la parte di matematica che studia le strutture matematiche in quanto oggetti matematici: vedi [] per referenze classiche. Essa prende le mosse dalla seguente definizione:

Definition 1.1. A *type* $\mathfrak{T}$ *of universal algebra* is a pair $(T, \underline{\alpha})$ where T is a set called the (*algebraic*) *signature* of the theory, and $\underline{\alpha}$ a function $T \to \mathbb{N}$ that assigns to every element $t \in T$ a natural number $n_t : \mathbb{N}$ called the *arity* of the function symbol t.

Definition 1.2. A (*universal*) *algebra* of type $\mathfrak{T}$ is a pair (A, f^A) where A is a set and $f^A : T \to \prod_{t \in T} \mathsf{Set}(A^{n_t}, A)$ is a function that sends every function symbol $t : T$ to a function $f^A_t : A^{n_t} \to A$; f^A_t is called the n_t-ary operation on A associated to the function symbol $t : T$.

It is evident that avremmo potuto sostituire Set con un'altra categoria $\mathcal{C}$ a nostro piacere, a patto di poter parlare di A^n per ogni $n : \mathbb{N}$, ossia a patto che $\mathcal{C}$ avesse prodotti finiti. Definire una algebra universale in $\mathcal{C}$ come una coppia (A, f^A) dove $A : \mathcal{C}$ e $f^A : T \to \prod_{t \in T} \mathcal{C}(A^{n_t}, A)$ è una generalizzazione praticamente a costo zero; tuttavia, è possibile andare più in là, e astrarre maggiormente anche la nozione di type of universal algebra.

In effetti, la struttura astratta che intendiamo studiare consta di uno "schizzo" [**?** **?**] che rappresenta il più generico degli arrangiamenti di operazioni e proprietà di queste operazioni;

questo schizzo è rappresentato pictorialmente da un grafo (diretto e con radice), che modella le arietà dei vari simboli di operazione di un dato tipo d'algebra T (si veda ancora Grillet per la definizione di variety: un'algebra universale soggetta a equazioni, ossia una coppia (A, R) ove A è un'algebra di tipo $\mathfrak{T}$ e $R \subseteq A^\star \times A^\star$ è un sottoinsieme di coppie di parole in un opportuno monoide di Kleene modificato).

In questo senso la categoria generata, in un senso opportuno, dal grafo $\mathcal{L}_{Grp}$ "è" la teoria che ci prefiggevamo di studiare, e ogni funtore $G : \mathcal{L}_{Grp} \to \mathsf{Set}$ con la proprietà che $G([n + m]) \cong G[n] \times G[m]$ determina una "immagine" in Set della "teoria astratta" dei gruppi, che abbiamo codificato in una opportuna categoria con prodotti finiti. La teoria in questo senso diventa un oggetto matematico molto concreto, e altrettanto concreti diventano i suoi *modelli*: funtori $\mathcal{L} \to \mathsf{Set}$ con la proprietà di essere determinati dalla immagine di [1] (tale insieme $G = G[1]$ è il *carrier* della struttura algebrica; nella definizione di algebra di tipo $\mathfrak{T}$ esso è null'altro che il primo termine della coppia (A, f^A)); la seguente notazione è quindi consistente: se $\mathcal{L}$ è una teoria e $G : \mathcal{L} \to \mathsf{Set}$ un suo modello, deve valere $G[n] = G^n = G \times G \times \cdots \times G$; confondiamo quindi le due notazioni.

Per formalizzare questa costruzione che abbiamo finora sketchato nel linguaggio delle categorie, si spinge all'estremo l'intuizione data dal remark **??**, e si considera esattamente la categoria generata dal grafo che ha per vertici i numeri naturali, e come generatori dei morfismi le funzioni di insiemi, a cui si aggiunge esattamente una edge $f_t : X^n \to X$ per ogni simbolo di operazione $t \in T_n = \underline{\alpha}^{\leftarrow}(n)$.

Un piccolo primer su functorial semantics sta nellappendice **??**.

Ora, osservazione: la categoria degli insiemi è stata l'unico posto in cui abbiamo interpretato la sintassi della nostra teoria $\mathcal{L}$; come detto sopra possiamo rifarlo in una generica categoria con prodotti finiti $\mathcal{C}$ e ottenere gli L-modelli in $\mathcal{C}$; nella definizione cambia poco; formalmente, nulla.

Ma questo è un risultato profondo perché *sconnette* la teoria dal *luogo* in cui quella teoria è interpretata: ad un tempo, sappiamo parlare di grppi interni a Set, i cari vecchi gruppi della matematica ingenua; di gruppi topologici, dove le funzioni m, i sono automaticamente continue, o di Lie, dove sono differenziabili; di monoidi nella categoria dei gruppi abeliani, e vengono fuori gli anelli; di monoidi nella categoria dei poset, e vengono fuori i quantali. . .

In tale prospettiva, la categoria degli insiemi è stata segretamente presa come universo semantico di riferimento; se ne sarebbe potuta prendere un altra, e interpretare lì la teoria astratta dei gruppi o dei monoidi o di una qualsiasi varietà d'algebre.

Ciò non è scorrelato dalla nostra escursione nel mondo dei topos: ad ogni teoria $\mathfrak{T}$ si associa un topos, detto *free topos* $\mathcal{E}(\mathfrak{T})$ sulla teoria, tale per cui esista una biiezione tra i modelli di $\mathfrak{T}$ in un altro topos $\mathcal{F}$, e i *morfismi logici* $\mathcal{E}(\mathfrak{T}) \to \mathcal{F}$:

$$\frac{A}{B}$$

Vediamo esplicitamente come si comporta questa costruzia nel caso di un topos di prefasci: sets tautologically correspond to the category $[1, Set]$, so it is natural to wonder what fkT-models are in more general functor categories like $[C, Set]$. All the more because this generalisation is compelling to our discussion: in case C is discrete, we get back the well-known category of variable sets of **??**.

In a stunning turn of events now, a $[C, Set]$-model for groups, monoids or else, i.e. a functor $Th(\mathfrak{T}) \to [C, Set]$ preserving finite products, is *precisely* a functor $C \to Set$ such that each Fc is a $\mathfrak{T}$-model: this gives rise to the following "commutative property for semantics": $\mathfrak{T}$-models in $[C, Set]$ are precisely the $Mod(\mathfrak{T})$-valued functors $C \to Mod(\mathfrak{T})$, i.e. those functors $C \to \mathsf{Set}$ taking values in the subcategory of models for the theory in study interpreted in Set. In other words we can "shift" the $Mod(-)$ construction in and out $[C, Set]$ at our will:

$$Mod_{Th(\Omega)}([C, Set]) \cong [C, Mod_{Th(\Omega)}(Set)]$$

Questa notazione non è molto consistente; va riscritto con cura.

Now, the procedure of interpreting a given "theory" inside an abstract finitely complete category K is something that is only possible in the functorial paradigm, treating a theory as a category, and an interpretation as a functor. This discipline goes under many names and has various nuances: functorial categorical, or functorial, semantics [], internalisation of structures [], categorical algebra [].

The internalization paradigm sketched above suggests how "small" mathematicians often happily develop their mathematics without ever exiting a single finitely complete category K, without even suspecting the presence of models for their theories outside K. Come gli adepti della setta della fenice, che non si chiamano tra loro con lo stesso nome con cui il mondo esterno li conosce, con "teoria dei gruppi" i categoristi intendono una struttura diversa e più profonda da quella intesa da quelli per cui un gruppo è un insieme.

Thus, if you admit them to be big enough (i.e. if you leave the somewhat unsatisfying picture that "all categories are small" and you fix a semantic universe like Set), *each* category works as a universe in which you can speak mathematical language (i.e. "study models for the theory of Ω-structures" as long as Ω runs over all possible theories). Small mathematicians are born in K, so they only see K-models for $Th(\Omega)$.

So categories exhibit a double nature: they are the theories we want to study, but also the places where we want to embody those theories, universes in which to interpret theories: looking from high enough, there is plenty of other places where one can move, other than Set. Small categories model theories, it has a *syntax*, in that they describe a relational structure using compositionality; but large categories offer a way to interpret the syntax, so being a *semantics*. A large relational structure is fixed once and for all, lying on the background, in which all other relational structures are interpreted.

It is nearly impossible to underestimate the profundity of the remark by which each category K, taken alone, is a different place in which the entirety of mathematical structures can be re-enacted. This POV is investigated in our **??**, where we see how this allows to intepret different kinds of logics in different kinds of categories; the particular shape of semantics that you can interpret in K is no more, no less than a reflection of the nice categorical properties of K (does it have finite co/limits? Does it have a nice factorization system? Does it have a subobject classifier? Is the poset $Sub(A)$ of subobjects of A a lattice, is it modular, distributive, complemented...? -evidently this last question is about the internal logic of the category: propositions are the set, or rather the type, of "elements" for which the proposition is "true"; in nice cases, they are also arrows with codomain a space of truth values).

1.3.2. *Categories are universes of discourse.* Nel senso precisato dalla sottosezione precedente la teoria delle categorie "è più grande della matematica nel suo complesso" e ne fa da fondazione; ogni categoria è *un* universo in cui la totalità della matematica si può re-enactare; in questo senso è un metalinguaggio dove è possibile rifare la matematica nella sua interezza. L'idea è che *le teorie matematiche sono a loro volta oggetti matematici*, e in quanto tali sono passibili dello stesso studio di cui sono passibili gli oggetti di cui quelle teorie parlano.

Nulla vieta, a questo punto, di considerare qualuque teoria ontologica come una categoria, previo lavoro di chiarificazione/formalizzazione che lo stesso linguaggio categoriale ci permette di eseguire. Tra i vantaggi, visibili già in matematica, la possibilità di leggere le teorie in termini di relazioni (sospendendo *ontological committment* sugli oggetti) e i concetti come *context-dependent*. Un efficace "slogan" filosofico, a riassumere entrambi i presupposti, lo si può trovare in un paper di Jean-Pierre Marquis [Marquis, 1997]:

> [...] *to be* is *to be related*, and the "essence" of an "entity" is given by its relations to its "environment"

Sappiamo che nella storia della **CT** l'accettazione dell'evidenza dei risultati è avvenuta a prescindere da un dibattito intorno alla scarsa precisione e coerenza logica degli stessi. Le categorie sono nate come strumento concettuale e, senza preoccuparsi delle sottigliezze della ricerca fondazionale, hanno catturato efficacemente tutte le nozioni della matematica moderna, rivelandosi utili e feconde. Nostro claim è che si riveleranno tali anche con le usuali nozioni metafisiche. Si tratta di adottare, in fondo, una prospettiva pragmatica:

> that structural mathematics is characterized as an activity by a treatment of things as if one were dealing with structures. From the pragmatist viewpoint, we do not know much more about structures than how to deal with them, after all.

La "traduzione" dei problemi dell'ontologia nel linguaggio di **CT** permette di manipolare meglio nozioni (non solo, come si sa, matematiche) ma metamatematiche e metafisiche, e ci dota di un approccio più compatto e di una visione più "leggera" e occamista delle questioni vertenti su oggetti e esistenza. Non giustifichiamo questo approccio a priori ma ne testimoniamo la fecondità già provata in letteratura[1], soprattutto paragonata a quella degli approcci set-theoretic (di cui già è informata la totalità delle ontologie formali).

[1] Cf. Mt7,16, giusto per ingraziarsi i severiniani.

In **CT** possiamo "tradurre" i problemi classici dell'ontologia, fornire modelli entro i quali formularne meglio presupposti e domande, evidenziare ciò che è banale conseguenza degli assiomi di quel modello e ciò che non lo è, risolverli e, in alcuni casi, dissolverli, rivelandone la natura figmentale. Si tratta di fornire un *ambiente* ben definito nel quale questioni ritenute oggetto di dibattito filosofico possano illuminarsi in modi nuovi o scomparire. E questo non per qualche perverso istinto riduzionistico, ma per poterne parlare in termini efficaci e nel linguaggio adatto a inquadrarli: tentare, con gli strumenti più avanzati e raffinati dell'astrazione matematica, di rispondere a delle domande, produrre conoscenza, e non solo dibattito; inscrivere antiche o recenti questioni in un nuovo paradigma, volto a superare e al contempo far avanzare la ricerca.

Come ogni paradigma lo dotiamo di una sintassi con la quale "nominare" concetti e dare definizioni, e di una semantica che produca modelli, e quindi contesti, entro i quali "guardare" le teorie; questa sintassi e questa semantica non ce le inventiamo: sono già nella matematica e da lì le preleviamo.

1.4. **Existence: persistence of identity.** Ontology rests upon the principl of identity. It is this very principle that here we aim to unhinge.

Cosa significa che *due cose sono, invece, una* è un problema che ci arrovella fin da quando otteniamo la ragione e la parola; ciò perché il problema è tanto elementare quanto sfuggente: l'unica maniera in cui possiamo esibire ragionamento certo è il calcolo; del resto, se la sintassi non vede che l'uguaglianza in senso più stretto possibile, la prassi deve diventare in fretta capace di una maggiore elasticità: per un istante ho postulato che ci fossero “due” cose, non una. E non è forse questo a renderle due? E questa terza cosa che le distingue, è davvero diversa da entrambe?

2. Preliminaries on variable set theory

In some of our proofs it will be crucial to blur the distinction between the category of functors $I \to \mathsf{Set}$ and the slice category Set/I (see **??**); once the following result is proved, we freely refer to any of these two categories as the category of *variable sets* (indexed by I).

Proposition 2.1. Let I be a set, regarded as a discrete category, and let Set^I be the category of functors $F : I \to \mathsf{Set}$; moreover, let Set/I the slice category. Then, there is an equivalence (actually, an isomorphism) between Set^I and Set/I.

Proof. Let us give a very hands-on proof, based on the fact that the category Set^I coincides on its own right with the category of I-indexed families of objects, i.e. with the category whose objects are $(\underline{X})_I := \{X_i \mid i \in I\}$, and morphisms $(\underline{X})_I \to (\underline{Y})_I$ the families $\{f_i : X_i \to Y_i \mid i \in I\}$.

Consider an object $h : X \to I$ of Set/I, and define a function as $i \mapsto h^{\leftarrow}(i)$; of course, $(X(h))_I := \{h^{\leftarrow}(i) \mid i \in I\}$ is a I-indexed family, and since I can be regarded as a discrete category, this is sufficient to define a functor $F_h : I \to \mathsf{Set}$.

Let us define a functor in the opposite direction: let $F : I \to \mathsf{Set}$ be a functor. This defines a function $h_F : \coprod_{i\in I} Fi \to I$, where $\coprod_{i\in I} Fi$ is the disjoint union of all the sets Fi.

The claim now follows if we show that the correspondences $h \mapsto F_h$ and $F \mapsto h_F$ are mutually inverse.

This is however easy to verify: the function F_{h_F} sends $i \in I$ to the set $h_F^{\leftarrow}(i) = Fi$, and the function $h_{F_h} \in \mathsf{Set}/I$ has domain $\coprod_{i \in I} F_h(i) = \coprod_{i\in I} h^{\leftarrow}(i) = X$ (as i runs over the set I, the disjoint union of all preimages $h^{\leftarrow}(i)$ equals the domain of h, i.e. the set X). □

Remark 2.2. A more abstract look at this result regards the equivalence $\mathsf{Set}/I \cong \mathsf{Set}^I$ as a particular instance of the *Grothendieck construction* (see []): for every small category $\mathcal{C}$, the category of functors $\mathcal{C} \to \mathsf{Set}$ is equivalent to the category of *discrete fibrations* on $\mathcal{C}$ (see **??**). In this case, the domain $\mathcal{C} = I$ is a discrete category, hence all functors $\mathcal{E} \to I$ are, trivially, discrete fibrations.

The next crucial step of our analysis is the observation that the category of variable sets is a topos: we break the result into the verification of the various axioms, as explained in A.2 and A.3.

Proposition 2.3. The category of variable sets is cartesian closed.

Proof. We shall first show that the category of variable sets admits products: this is obvious in Set/I, products are precisely pullbacks; note that theorem **??** gives an identification

$$\begin{tikzcd} & X \times_I Y \arrow[dl] \arrow[dr] \arrow[dd, "h"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & I & \end{tikzcd} \iff i \mapsto h^{\leftarrow}(i) = \Big\{(x,y) \in X \times_I Y \mid h(x,y) = i\Big\}$$

and given the universal property of a pullback, this yields a canonical bijection $h^{\leftarrow}(i) \cong f^{\leftarrow}(i) \times g^{\leftarrow}(i)$. This is exactly the definition of the product of the two functors $F_f, F_g : I \to \mathsf{Set}$.

Next, we shall show that each functor $\_ \times_I Y$ has a right adjoint $Y \pitchfork_I \_$. The functor $\mathsf{Set}^I \to \mathsf{Set}^I : Z \mapsto Y \pitchfork_I Z$ where $Y \pitchfork_I Z : i \mapsto \mathsf{Set}(Y_i, Z_i)$ does the job. This sets up the bijection

$$\frac{X \times_I Y \longrightarrow Z}{X \longrightarrow Y \pitchfork_I Z}$$

and by a completely analogous argument (the functor $\_ \times_I =$ gives a symmetric monoidal structure to variable sets),

$$\frac{X \times_I Y \longrightarrow Z}{Y \longrightarrow X \pitchfork_I Z;}$$

this concludes the proof that the category of variable sets is cartesian closed. □

Proposition 2.4. The category of variable sets has a subobject classifier.

Proof. Recall from **??** that we shall find a variable set Ω such that there is a bijection

$$\frac{A \longrightarrow \Omega}{\mathsf{Sub}_I(A)}$$

where $\mathsf{Sub}_I(A)$ denotes the set of isomorphism classes of monomorphisms into A, in the category of variable sets.[2]

In order to find such an object, we look at what shape shall Ω have, and what role its universal property plays in its characterization:

-
-

For the sake of simplicity, for the rest of the proof we fix as category of variable sets the slice Set/I.

From this we make the following guess: as an object of Set/I, Ω is the object $\pi_I : I \times \{0,1\} \to I$. We are thus left with the verification that π_I has the correct structure and universal property.

First, we shall find a monomorphism $\mathtt{t} : * \to \Omega$ in Set/I, i.e. an injective function $I \to \Omega$ that has π_I as left inverse. This generalised element "chooses the ⌈true⌉ truth value" in Ω. (Evidently, the identity $\mathrm{id}_I : I \to I$ is the terminal object in Set/I.)

It turns out that the function $I \to I \times \{0,1\}$ plays the rôle of $\mathtt{t}$: indeed, given a monomorphism $S \hookrightarrow A$ the commutative square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & I \arrow[d, "\mathtt{t}"] \\ A \arrow[r, "\chi_S"'] & I \times \{0,1\} \end{tikzcd}$$

□

Proposition 2.5. The category of variable sets is cocomplete and accessible.

Proof. □

Accessibility is a corollary of Yoneda in the following form: every $F : I \to \mathsf{Set}$ is a colimit of representables

$$F \cong \operatorname{colim}\left(\mathcal{E}(F) \xrightarrow{\Sigma} I \xrightarrow{y} \mathsf{Set}^I\right)$$

($\mathcal{E}(F)$ is small because in this case $\mathcal{E}(F) \cong \coprod_{i \in I} Fi$).

Corollary 2.6. The category of variable sets is a Grothendieck topos.

[2] A monomorphism into A as an object of Set^I is nothing but a family of injections $s_i : S_i \to A_i$; a monomorphism in Set/I is a set S in a commutative triangle

$$\begin{tikzcd} S \arrow[rr] \arrow[dr, "s"'] & & A \arrow[dl, "a"] \\ & I. & \end{tikzcd}$$

3. The internal language of variable sets

> I am hard but I am fair; there is no racial bigotry here.
> [...] Here you are all equally worthless.
>
> GySgt Hartman

The internal language of a topos $\mathcal{E}$ is a formal language defined by *types* and *terms*; suitable terms form the class of variables. Other terms form the class of *formulæ*.

- *Types* are the objects of $\mathcal{E}$
- *Terms* of type X are morphisms of codomain X, usually denoted $\alpha, \beta, \sigma, \tau : U \to X$.
  - Suitable terms are variables: the identity arrow of $X \in \mathcal{E}$ is the variable $x : X \to X$. For technical reasons we shall keep a countable number of variables of the same type distinguished:[3] $x, x', x'', \dots : X \to X$ are all interpreted as 1_X.
- Generic terms may depend on multiple variables; the domain of a term of type X is the *domain of definition* of a term.

A number of inductive clauses define the other terms of the language:

- the identity arrow of an object $X \in \mathcal{E}$ is a term of type X;
- given terms $\sigma : U \to X$ and $\tau : V \to Y$ there exists a term $\langle \sigma, \tau \rangle$ of type $X \times Y$ obtained from the pullback

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, "\langle\sigma,\tau\rangle"] & X \times V \arrow[d] \\ U \times Y \arrow[r] & X \times Y \end{tikzcd}$$

- Given terms $\sigma : U \to X, \tau : V \to X$ of the same type X, there is a term $[\sigma = \tau] : W \xrightarrow{\langle\sigma,\tau\rangle} X \times X \xrightarrow{\delta_X} \Omega$, where $\delta_X : X \times X \to \Omega$ is defined as the classifying map of the mono $X \hookrightarrow X \times X$.
- Given a term $\sigma : U \to X$ and a term $f : X \to Y$, tere is a term $f[\sigma] := f \circ \sigma : U \to Y$.
- Given terms $\theta : V \to Y^X$ and $\sigma : U \to X$, there is a term

$$W\langle\theta, \sigma\rangle \to Y^X \times X \xrightarrow{\text{ev}} Y$$

- In the particular case $Y = \Omega$, the term above is denoted

$$[\sigma \in \theta] : W\langle\theta, \sigma\rangle \to \Omega$$

- If x is a variable of type X, and $\sigma : X \times U \to Z$, there is a term

$$\lambda x.\sigma : U \xrightarrow{\eta} (X \times U)^X \xrightarrow{\sigma^X} Z^X$$

  obtained as the mate of σ.

---

[3]These technical reasons lie on the evident necessity to be free to refer to the same free variable an unbounded number of times. This can be formalised in various ways: we refer the reader to [?] and [?].

These rules can of course be also presented as the formation rules for a Gentzen-like deductive system: let us rewrite them in this formalism.

$$\frac{}{1_X : X \to X} \qquad \frac{\sigma : U \to X \qquad \tau : V \to Y}{\langle \sigma, \tau \rangle : W\langle \theta, \sigma \rangle \to X \times Y}$$

$$\frac{\sigma : U \to X \qquad \tau : V \to X}{[\sigma = \tau] : W \to \Omega} \qquad \frac{\sigma : U \to X \qquad f : X \to Y}{f[\sigma] : U \to Y}$$

$$\frac{\theta : V \to Y^X \qquad \sigma : U \to X}{W\langle \theta, \sigma \rangle \to Y^X \times X \xrightarrow{\text{ev}} Y} \qquad \frac{x : X \qquad \sigma : X \times U \to Z}{\lambda x.\sigma = \sigma^X \circ \eta : U \to (X \times U)^X \to Z^X}$$

To formulas of the language of $\mathcal{E}$ we apply the usual operations and rules of first-order logic: logical connectives are induced by the structure of internal Heyting algebra of Ω: given formulas φ, ψ we define

- $\varphi \vee \psi$ is the formula $W\langle \varphi, \psi \rangle \to \Omega \times \Omega \xrightarrow{\vee} \Omega$;
- $\varphi \wedge \psi$ is the formula $W\langle \varphi, \psi \rangle \to \Omega \times \Omega \xrightarrow{\wedge} \Omega$;
- $\varphi \Rightarrow \psi$ is the formula $W\langle \varphi, \psi \rangle \to \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$;
- $\neg\varphi$ is the formula $U \to \Omega \xrightarrow{\neg} \Omega$.

universal quantifiers

Each formula $\varphi : U \to \Omega$ defines a subobject $\{x \mid \varphi\} \subseteq U$ of its domain of definition; this is the subobject classified by φ, and must be thought as the subobject where "φ is true".

If $\varphi : U \to \Omega$ is a formula, we say that φ is *universally valid* if $\{x \mid \varphi\} \cong U$. If φ is universally valid in $\mathcal{E}$, we write "$\mathcal{E} \Vdash \varphi$" (read: "$\mathcal{E}$ believes in φ").

Examples of universally valid formulas:

- $\mathcal{E} \Vdash [x = x]$
- $\mathcal{E} \Vdash [(x \in_X \{x \mid \varphi\}) \iff \varphi]$
- $\mathcal{E} \Vdash \varphi$ if and only if $\mathcal{E} \Vdash \forall x.\varphi$
- $\mathcal{E} \Vdash [\varphi \Rightarrow \neg\neg\varphi]$

Ora facciamo delle considerazioni sul lingo interno di Set/I

Chi sono tipi e termini; chi sono le proposizioni e come si scrive il calcolo proposizionale in Set/I; i quantificatori, in dettaglio pornografico.

L'oggetto dei numeri naturali e il principio di induzione nel topos degli insiemi variabili.

4. Nine copper coins, and other toposes

> Explicaron que una cosa es *igualdad*, y otra *identidad*, y formularon una especie de *reductio ad absurdum*, o sea el caso hipotético de nueve hombres que en nueve sucesivas noches padecen un vivo dolor. ¿No sería ridículo -interrogaron- pretender que ese dolor es el mismo?
>
> JLB —Tlön, Uqbar, Orbis Tertius

According to our description of the Mitchell-Bénabou language in the category of variable sets, *propositions* are morpisms of the form

$$p : U \to \Omega_I$$

where Ω_I is the subobject classifier of Set/I described in **??**; now, recall that

- the object $\Omega_I = \{0,1\} \times I \to I$ becomes an object of Set/I when endowed with the projection $\pi_I : \Omega_I \to I$ on the second factor of its domain;
- the universal monic $\mathtt{t} : I \to \Omega_I$ consists of a section of π_I, precisely the one that sends $i : I$ to the pair $(i, 1) : \Omega_I$;
- every subobject $U \hookrightarrow A$ of an object A results as a pullback (in Set/I) along $\mathtt{t}$:

$$\begin{tikzcd} U \arrow[rr, "u"] \arrow[dr, "u"] \arrow[dd, "m"'] & & I \arrow[dl, equal] \arrow[dd, "\mathtt{t}"] \\ & I & \\ A \arrow[ur] \arrow[rr, "\chi_U"'] & & \Omega_I \arrow[ul] \end{tikzcd}$$

(see **??** for a complete proof)

The set I in this context acts as a *multiplier* of truth values, in that every proposition can have a pair (ϵ, i) as truth value. We introduce the following notation: a proposition $p : U \to \Omega_I$ is *true*, in context $x : U$, with *strength* t, if $p(x) = (1, t)$ (resp., $p(x) = (0, t)$).

So, a proposition is a morphism of the following kind: a function $p : U \to \Omega_I$, defined on a certain domain, and such that

$$\begin{tikzcd} U \arrow[r, "p"] \arrow[d, "u"'] & \{0,1\} \times I \arrow[d, "\pi_I"] \\ I \arrow[r, equal] & I \end{tikzcd}$$

(it must be a morphism of variable sets!) This means that $\pi p(x : U) = u(x : U)$, so that $p(x) = (\epsilon_x, u(x))$ for $\epsilon_x = 0, 1$ and u is uniquely determined by the "variable domain" U. This is an important observation: the strength with which p is true/false is completely determined by the structure of its domain, in the form of the function $u : U \to I$ that renders the pair (U, u) an object of Set/I.

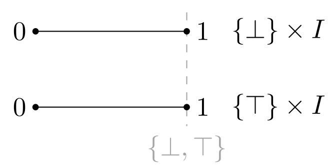
To get a grip of the different roles of various classes of propositions, and given that our interest will be limited to a certain class of particular propositions that we will construct *ex nihilo*, it is now convenient to discuss what constraints we have to put on the structure of I: of course, the richest this structure is, the better will the category Set/I behave: it is for example possible to equip I with an order structure, or a natural topology. Among different choices of truth multiplier, yielding different categories of variable sets, and different kinds of internal logic therein, we will privilege those that make I behave like a space of strengths: a dense, linear order with LUP, thus not really far from being a closed, bounded subset of the real line.

The main result of the present section is a roundup of examples showing that it is possible to concoct categories of variable sets where some seemingly paradoxical constructions coming from J.L. Borges' literary world have, instead, a perfectly "classical" behaviour when looked with the lenses of the logic of variable set theory.

Each of the examples in our roundup 4.3 4.6 4.7 is organised as follows: we recall the shape of a paradoxical statement in Borges' literary world. Then, we show in which topos this reduces to an intuitive statement expressed in the syntax of a variable set category.

More than often, we use $I = [0, 1]$ as base of variable sets; as already said, there are different reasons for this choice: the most intuitive is that if a truth value is given with a *strength* $t \in I$ it is a natural request to be able to *compare* elements in this set; in particular, it should always be possible to assess what truth value is stronger.

For this reason, even if this assumption is never strictly necessary (the only constraint is that I is totally, or partially, ordered set by a relation $\leq$), a natural choice for I is a *continuum* (=a dense total order with LUP –see [Mos09]). An alternative choice drops the density assumption: in that case the (unique) finite total order $\Delta[n] = \{0 < 1 < \cdots < n\}$, or the countable total order $I = \omega = \bigcup_n \Delta[n]$ are all pretty natural choices for I (although it is way more natural for I to have a minimum *and* a maximum element).[4] In each of these cases "classical" logic is recovered as a projection: propositions p can be true or false with strength 1,[5] the maximum element of I:

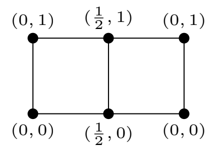


In order to aid the reader understand the explicit way in which I "multiplies" truth values, we spell out explicitly the structure of the subobject classifier in $\mathsf{Set}/\Delta[2]$. In order to keep calling the minimum and maximum of I respectively 0 and 1 we call $\frac{1}{2}$ the intermediate point of $\Delta[2]$.

[4]We're oly interested in the notion of an abstract interval here: a continuum X endowed with an operation $X \to X \vee X$ of "zooming", uniquely defined by this property. In a famous paper Freyd characterises "the interval" as the terminal interval coalgebra: see []; for our purposes, note that $[0, 1]$ is a natural choice: it is a frame, thus a Heyting algebra $\mathfrak{H} = ([0, 1], \wedge, \vee, \Rightarrow)$ with respect to the pseudo-complement operation given by $(x \Rightarrow z) := \bigvee_{x \wedge y \leq z} y$ (it is immediate that $x \wedge a \leq b$ if and only if $a \leq x \Rightarrow b$ for every $a, b \in [0, 1]$).

[5]Here I is represented as an interval whose minimal and maximal element are respectively 0 and 1; of course these are just placeholders, but it is harmless for the reader to visualise I as the interval $[0, 1]$.

Remark 4.1. The subobject classifier of $\mathsf{Set}/\Delta[2]$ consists of the partially ordered set $\Delta[1] \times \Delta[2]$ that we can represent pictorially as a rectangle



endowed with the product order. The resulting poset is partially ordered, and in fact a Heyting algebra, because it results as the product of two Heyting algebras: the Boole algebra $\{0 < 1\}$ and the frame of open subsets of the Sierpinski space $\{a, b\}$ (the topology is $\tau_S = \{\varnothing, \{a\}, \{a, b\}\}$).

Remark 4.2. Siccome il caso $I = [0, 1]$ con la topologia euclidea è quello piu naturale per diversi motivi, definiamo alcuni insiemi di interesse per una data proposizione $p : U \to \Omega_I$ per questa scelta di I:

- $A^\top = \{x : U \mid p(x) = (1, t_x), t_x > 0\} = p^\leftarrow(\{1\} \times (0, 1])$ e $A^\perp = \{x : U \mid p(x) = (0, t_x), t_x > 0\}$; cose vere (risp., false) con forza maggiore di zero. Sono le funzioni $u : U \to I$ tali che $u^\leftarrow 0 = \varnothing$.
- $B^\top = \{x : U \mid p(x) = (1, 1)\} = p^\leftarrow((1, 1))$ e $B^\perp = \{x : U \mid p(x) = (0, 1)\}$ cose vere (risp.,false).
- $E_t^\top = \{x : U \mid p(x) = (1, t)\}$ e $E_t^\perp = \{x : U \mid p(x) = (0, t)\}$; cose vere (risp., false) con forza t.

Last but not least, a crucial assumption will be that the strength of p depends continuously, or not, on the variables on its domain of definition. Without such continuous dependence, small changes in context $x : U$ might drastically change the truth value $p(x)$.[6]

4.1. **The unimaginable topos theory hidden in Borges' library.** Jorge Luis Borges' literary work is well-known for being made by paradoxical worlds; oftentimes, seemingly absurd consequences follow by stretching to their limit ideas from logic and mathematics: time, infinite, self-referentiality, duplication, recursion, the relativity of time, the illusory nature of our perceptions, the limits of language, its capacity to generate worlds.

In the present section we choose *Fictions*, Borges' famous collection of novels, as source of inspiration for possible and impossible worlds and their ontology.

Usualmente la costruzione di un "mondo impossibile" va circa come segue: ...; noi rovesciamo tale prospettiva, e invece di depennare dal computo degli universi i mondi le cui caratteristiche do not comply with sensorial experience, or imply paradoxical entities/constructs, we accept their existence for bizarre that it may seem, and we try to deduce what kind of logic can consistently generate such statements.

[6]There is no a priori reason to maintain that p is a continuous proposition; one might argue that discontinuous changes in truth value of p happen all the time in "real life"; see the family of paradoxes based on so-called *separating instants*: how well-defined the notion of "time of death" is? How well-defined the notion of "instant in time"?

The interest in such a literary calembour is manifold, and the results are surprising:

- we unravel how a mathematically deep universe Borges has inadvertently created: of the many compromises we had to take in order to reconcile literature and the uniderlying mathematics,[7] we believe no one is particularly far-fetched one;
- we unravel how *relative* ontological assumptions are; they are not given: using category theory, ontology, far from being the presupposition on which it is based, is a byproduct of language itself. The more expressive language is, the more ontology; the fuzzier its capacity to assert truth, the fuzzier existence becomes;
- "Fuzziness" of existence, i.e. the fact *entia* exist less than completely, is hard-coded in the language (in the sense of **??**) of the category we decide to work in from time to time;
- ...

To sum up, readers willing to find an original result in this paper, might find it precisely here: we underline how Borges' alternative worlds (Babylon, Tlön dots) are mathematically consistent places, worthy of existence as much as our world, just based on a different internal logic. And they are so just thanks to a base-sensitive theory of existence –ontology breaks in a spectrum of ontolog*ies*, one for each category/world.

The first paradox we aim to frame in the right topos is the famous nine copper coins argument, used by the philosophers of Tlön to construct a paradoxical object whose existence persists over time, in absence of a consciousness continually perceiving it and maintaining in a state of being.

Example 4.3 (Nine copper coins)**.** First, we recall the exact statement of the paradox from []:[8]

[7]See **??** below: these "compromises" mainly amount to assumptions on the behaviour of space-time on Tlön and Babylon.

[8]The translation we employ is classical and comes from [Bor99b]:

> Tuesday, X crosses a deserted road and loses nine copper coins. On Thursday, Y finds in the road four coins, somewhat rusted by Wednesday's rain. On Friday, Z discovers three coins in the road. On Friday morning, X finds two coins in the corridor of his house. The heresiarch would deduce from this story the reality - i.e., the continuity - of the nine coins which were recovered.
>
> It is absurd (he affirmed) to imagine that four of the coins have not existed between Tuesday and Thursday, three between Tuesday and Friday afternoon, two between Tuesday and Friday morning. It is logical to think that they have existed - at least in some secret way, hidden from the comprehension of men - at every moment of those three periods.

> El martes, X atraviesa un camino desierto y pierde nueve monedas de cobre. El jueves, Y encuentra en el camino cuatro monedas, algo herrumbradas por la lluvia del miércoles. El viernes, Z descubre tres monedas en el camino. El viernes de mañana, X encuentra dos monedas en el corredor de su casa. El quería deducir de esa historia la realidad -id est la continuidad- de las nueve monedas recuperadas.
>
> Es absurdo (afirmaba) imaginar que cuatro de las monedas no han existido entre el martes y el jueves, tres entre el martes y la tarde del viernes, dos entre el martes y la madrugada del viernes. Es lógico pensar que han existido -siquiera de algún modo secreto, de comprensión vedada a los hombres- en todos los momentos de esos tres plazos.

Before going on with our analysis, two remarks are in order:

- the paradox appears in a primitive version in [], where instead of nine copper coins, a single arrow, shot by an anonymous archer, disappears among the woods. The text appears in a hard-to-find edition of *Inquisiciones* []; in the last chapter, we read the "arrow avatar" (*avatar de la flecha*):

  > X scocca una freccia da un arco, ed essa si perde fra gli alberi.
  >
  > X la cerca e riesce a ritrovarla.
  >
  > E' assurdo immaginare che la freccia non sia esistita durante il periodo fra i momenti in cui X l'ha persa di vista e l'ha ritrovata.
  >
  > E' logico pensare che essa sia esistita - anche se in un certo modo segreto, di comprensione vietata agli uomini - in tutti i momenti di questo periodo.

- There is one and only one reason why the paradox of the nine copper coins is invalid: copper does not rust.

It is obvious that both constructions leverage on the same argument to build an efficient aporia: the mysterious persistence of things through time without a perceivent consciousness. We concentrate on the copper coins dilemma, per il semplice fatto che Finzioni è raggiungibile molto piu facilmente ai nostri lettori; incidentally, we happen to be able to rectify the "rust counterargument" without appealing to the assumption that copper can rust on Tlön due to a difference in Tlönian chemistry.

Expressed in natural language, our solution to the paradox goes more or less as follows: X loses their coins on Tuesday, and the force φ with which they "exist" lowers; it grows back in the following days, going back to a maximum value when X retrieves two of their coins on the front door. Y findings of other coins raises their existence force to a maximum. The coins that Y has found rusted (or rather, the surface copper oxidized: this is possible, but water is rarely sufficient to ignite the process alone –certainly not in the space of a few hours.[9]

In this perspective, Tlön classifier of truth values can be taken as $\Omega_I = \{0 < 1\} \times I$, where I is any set with more than one element; a minimal example can be $I = \{N, S\}$ (justifying this choice from inside Tlön is easy: the planet is subdivided into two emispheres; each of which now has its own logic "line" independent from the other), but as explained in **??** a more natural choice for our purposes is the closed real interval $I = [0, 1]$.

This allows for a continuum of possible forces with which a truth value can be true or false; it is tobe noted that $[0, 1]$ is also the most natural place on which to interpret fuzzy logic, albeit the interest for $[0, 1]$ therein can be easily and better motivated starting from probability theory. (But see [] for an interesting perspective on how to develop basic measure teory out of $[0, 1]$.)

We now start to formalise properly what we said until now.

To set our basic assumptions straight, we proceed as follows:

- senza perdita di generalità possiamo supporre l'insieme $C = \{c_1, \dots, c_9\}$ delle monete totalmente ordinato e partizionato in modo tale che le prime due monete siano quelle ritrovate da X il martedì, le seconde quattro quelle che Y ritrova sul cammino, e le ultime 3 quelle viste da Z. Allora
$$C = C_X \sqcup C_Y \sqcup C_Z.$$
and $C_X = \{c_{X1}, c_{X2}\}$, $C_Y = \{c_{Y1}, c_{Y2}, c_{Y3}, c_{Y4}\}$, $C_Z = \{c_{Z1}, c_{Z2}, c_{Z3}\}$ As already said, the truth multiplier I is the closed interval $[0, 1]$ with its canonical order –so with its canonical structure of Heyting algebra, and if needed, endowed with the usual topology inherited by the real line.
- Propositions of interest for us are of the following form:
$$\lambda gcd.p(g, c, d) : \{X, Y, Z\} \times C \times W \to \Omega_I$$
where W is a set of days, that for the sake of explicitness can be taken equal to the set of weekdays S, M, T (strictly speaking, the paradox involves just the interval $[Tu, Fri]$). $p(g, c, d)$ has to be read as "in g's frame of existence the coin c exists at day d with strength $p(g, c, d)$"

Definiamo ora *ammissibile* una configurazione tale che le condizioni seguenti sono rispettate: for all day d and coin c, we have
$$\sum_{u \in \{X,Y,Z\}} p(u, c, d) = (\top, 1)$$
where we denote as "sum" the logical conjunction in Ω_I: this means that day by day, the *global* existence of the group of coins constantly attains the maximum; it is the *local* existence

[9] in una sorta di principio di wormhole, eventi indipendenti sulla terra sono dipendenti su Tlön, perché l'evento A influenza, in uno spazio pluridimensionale di scelte di z di verità, l'evento B in modi che gli sarebbero vietati se fosse sulla Terra.

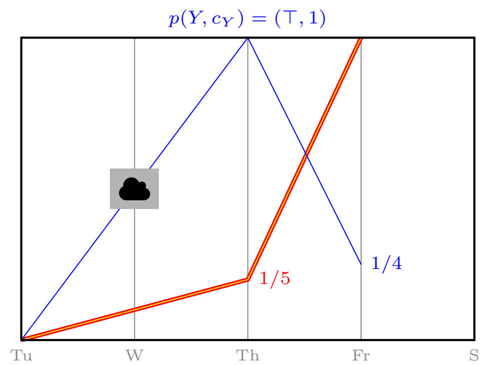


FIGURE 1. A pictorial representation of the truth forces of coins in different days, a piecewise linear model. X is red, Z is yellow, Y is blue. Time is considered as a continuum marked at weekdays.

that lowers when the initial conglomerate of coins is partitioned. Moreover,

$$\begin{cases} \sum_{c_X \in C_X} p(X, c_X, V) = (\top, 1) \\ \sum_{c_Y \in C_Y} p(Y, c_Y, G) = (\top, 1) \\ \sum_{c_Z \in C_Z} p(Z, c_Z, V) = (\top, 1) \end{cases}$$

In an admissible configuration the subsets C_X, C_Y, C_Z can only attain an existence $p(g, c, d) \lneq (\top, 1)$; that is no coin completely exists *locally*. But for an hypothetical external observer, capable of adding up the forces with which the various parts of C exist, the coins *globally* exist " in some secret way, of understanding forbidden to men" (or rather, to X, Y, Z).

Remark 4.4. L'aritmetica di Tlon; proposizioni a forza additiva; parallelismi tra la nave di Teseo e le nove monete.

Remark 4.5 (Continuity for a proposition)**.** Let $p : U \to \Omega_I$ be a proposition; here we investigate what does it mean for p to be (globally) continuous with respect to the Euclidean topology on $I = [0, 1]$, in the assumption that its domain of definition U is metrizable (this is true for example when U is a subset of space-time). The condition is that

$$\forall \epsilon > 0, \exists \delta > 0 : |x - y| < \delta \Rightarrow |px - py| < \epsilon$$

Da questo segue immediatamente che quando p è continua nel suo dominio, i valori di verità di p in configurazioni "vicine" in un senso opportuno sono dati con forze allo stesso modo vicine (chiaramente questa è una descrizione spannometrica della nozione di continuità. . .).

All elementary topology results apply to such a proposition: the set of forces with which p is true or false is a connected subset of Ω_I, compact if U was compact.

Example 4.6 (Discontinuity, Lo zaffiro di Taprobana)**.** la lotteria a Babilonia come in [Bor99a]: proposizioni $p : U \to \Omega_{[0,1]}$ possono essere fortemente discontinue nelle variabili/contesto da cui dipendono: tali proposizioni descrivono eventi apparentemente caotici, innescati come termine finale di una catena di eventi tra loro disconnessi e paradossali;[10] un modello di questo unvierso si trova probabilmente nella Babilonia di [Bor99a], e nei "sorteggi impersonali, di proposito indefinito" che caratterizzano l'operato della Compagnia, azioni apparentemente scorrelate tra loro (scagliare "nelle acque dell'Eufrate uno zaffiro di Taprobana"; sciogliere "dal tetto d'una torre [. . .] un uccello"; togliere (o aggiungere) "un granello di rena ai grani innumerevoli della spiaggia"; queste azioni hanno "conseguenze, a volte, tremende".

Example 4.7 (Continuity: a few birds, a horse)**.** Per quanto riguarda le proposizioni che sono continue nelle proprie variabili, invece, esempio canonico sono le "tigri di cristallo" e le "torri di sangue" di Tlön: oggetti ed entità usuali, diversi da quelli "classici" per un dettaglio solo (il colore, la consistenza, il materiale di cui sono composte): la loro esistenza è sfumata, forte meno del massimo, cosicché tigri di carne e una torre di pietra sono tali che $p = (\top, 1)$, le loro controparti su Tlön esistono con meno forza.

Un altro esempio illustre è fatto da ogetti la cui forza di esistenza dipende in maniera *monotona* e continua dai loro parametri: per esempio una proposizione p può essere tanto più vera quanta più gente la osserva, perché "le cose, su Tlön, si duplicano; ma tendono anche a cancellarsi e a perdere i dettagli quando la gente le dimentichi. È classico l'esempio di un'antica soglia, che perdurò finchè un mendicante venne a visitarla, e che alla morte di colui fu perduta di vista. Talvolta pochi uccelli, un cavallo, salvarono le rovine di un anfiteatro. "

In questa situazione, poniamo ad esempio che la forza di esistenza di alcune rovine – modellate come è ingenuo fare, come un corpo rigido R nello spazio, dipenda dal numero dei suoi osservatori:

$$p(R, n) = \left(\top, 1 - \tfrac{1}{n}\right)$$

Succedono cose interessanti anche a cambiare topologia su I: per esempio, su $[0, 1]$ possiamo mettere brutalmente la topologia discreta; in questo modo I è l'unione disgiunta dei suoi punti $\{\{t\} \mid t \in [0, 1]\}$, e il classifo è l'unione disgiunta di $[0, 1]$ copie di $\{0, 1\}$. (See 2 for a picture.)

Example 4.8 (La campagna incendiata)**.** Il Berkeley idealista degli infiniti istanti di tempo continuo, disconnessi e incomunicabili: $\Omega_I = \coprod_{t:[0,1]}\{0 < 1\}$. E' evidente come questa particolare struttura logica influenzi il linguaggio piegandolo a diventare l'istantaneismo berkeleyano: i termini sono costruiti per accrezione istantanea, per somma disgiunta dei costituenti e delle loro proprietà: "aereo-chiaro sopra scuro rotondo"; oggetti determinati dalla loro simultaneità, e non da una dipendenza logica: il significato si costruisce per accrezione di

[10]Assumendo una base reale per lo spazio dei parametri da cui p dipende, la sua dipendenza continua è la proprietà enunciata in 4.5; ciò significa che eventi vicini –nello spazio o nella consequenzialità temporale– non possono avere valori di verità diversi, e forze "vicine" in un senso opportuno.

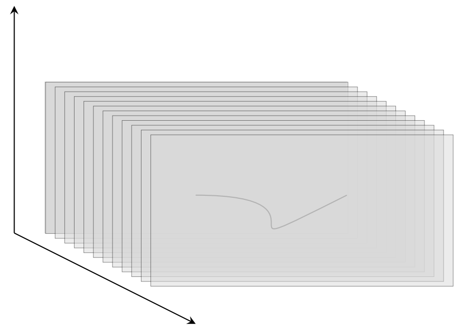


FIGURE 2. Il tempo come sequenza infinita, e infitamente suddivisibile, di istanti distinti: il paradosso Berkeleyano.

istanti simultanei, e non per sequenzialità temporale (vedi **??** per un legame tra questo principio, la natura additiva della forza di verità in Tlon, e la particolare forma dell'aritmetica Tloniana).

Il rifiuto della consequenzialità temporale per gli abitanti di Tlon sta nel passo

> Spinoza attribuisce alla sua inesauribile divinità i modi del pensiero e dell'estensione; su Tlön, nessuno comprenderebbe la giustapposizione del secondo (che caratterizza solo alcuni stati) e del primo, che è un sinonimo perfetto del cosmo. In altre parole: non concepiscono che lo spaziale perduri nel tempo. La percezione di una fumata all'orizzonte, e poi della campagna incendiata, e poi della sigaretta mal spenta che provocò l'incendio, è considerata un esempio di associazione di idee.

Ciò si lega anche al passo "L'universo è paragonabile a quelle crittografie in cui non tutti i segni hanno un valore, e che solo è vero ciò che accade ogni trecento notti": un mondo dove ogni trecento notti $p(x) = (\top, 1)$, e per le successive 299 notti p ha forza < 1.

Example 4.9.

5. Vistas on ontologies

Qui cosa mettiamo?

Appendix A. Category theory

A.1. **Fundamentals of CT.** Throughout the paper we employ standard basic category-theoretic terminology, and thus we refrain from giving a self contained exposition of elementary definitions. Instead, we rely on famous and wide-spread sources like [Bor94a, Bor94b, ML98, Rie17, Lei14, Sim11].

Precise references for the basic definitions can be found

- for the definition of category, functor, and natural transformation, in [Bor94a, 1.2.1], [ML98, I.2], [Bor94a, 1.2.2], [ML98, I.3], [Bor94a, 1.3.1].
- The Yoneda lemma is stated as [Bor94a, 1.3.3], [ML98, III.2].
- For the definition of co/limit and adjunction, in [Bor94a, 2.6.2], [ML98, III.3], [Bor94a, 2.6.6], [ML98, III.4].
- For the definition of accessible and locally presentable category in [Bor94b, 5.3.1], [Bor94b, 5.2.1], [AR94].
- Basic facts about ordinal and cardinal numbers can be found in []; a comprehensive reference on basic and non-basic set theory is [Jec13].
- The standard source for Lawvere functorial semantics is Lawvere's PhD thesis [Law63]; more modern accounts are [HP07].
- Standard references for topos theory are [MLM92, Joh77]. See in particular [MLM92, VI.5] and [Joh77, 5.4] for what concerns the Mitchell-Bénabou language of a topos.

A.2. **Toposes.** For us, an *ordinal number* will be any well-ordered set, and a *cardinal number* is any ordinal which is not in bijection with a smaller ordinal. Every set X has a unique *cardinality*, i.e. a cardinal κ with a bijection $\kappa \cong X$ such that there are no bijections from a smaller ordinal. We freely employ results that depend on the axiom of choice when needed. A cardinal κ is *regular* if no set of cardinality κ is the union of fewer than κ sets of cardinality less than κ; all cardinals in the following subsection are assumed regular without further mention.

Let κ be a cardinal; we say that a category $\mathcal{A}$ is κ-filtered if for every category $\mathcal{J} \in \mathsf{Cat}_{<\kappa}$ with less than κ objects, $\mathcal{A}$ is injective with respect to the cone completion $\mathcal{J} \to \mathcal{J}^{\triangleright}$; this means that every diagram

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[d] & \mathcal{A} \\ \mathcal{J}^{\triangleright} \arrow[ur, dotted, "\bar{D}"'] & \end{tikzcd}$$

has a dotted filler $\bar{D} : \mathcal{J}^{\triangleright} \to \mathcal{A}$.

We say that a category $\mathcal{C}$ admits filtered colimits if for every filtered category $\mathcal{A}$ and every diagram $D : \mathcal{A} \to \mathcal{C}$, the colimit $\operatorname{colim} D$ exists as an object of $\mathcal{C}$. Of course, whenever an ordinal α is regarded as a category, it is a filtered category, so a category that admits all κ-filtered colimits admits all colimits of chains

$$C_0 \to C_1 \to \cdots \to C_\alpha \to \cdots$$

with less than κ terms. A useful, completely elementary result is that the existence of colimits over all ordinals less than κ implies the existence of κ-filtered colimits; this relies on the fact that every filtered category $\mathcal{A}$ admits a cofinal functor (see [Bor94a]) from an ordinal $\alpha_{\mathcal{A}}$.

We say that a functor $F : \mathcal{A} \to \mathcal{B}$ *commutes with* or *preserves* filtered colimits if whenever $\mathcal{J}$ is a filtered category, $D : \mathcal{J} \to \mathcal{A}$ is a diagram with colimit $L = \operatorname{colim}_{\mathcal{J}} D_j$, then $F(L)$

is the colimit of the composition $F \circ D$. Anoter common name for such an F is a *finitary* functor, or a functor *with rank* ω.

Definition A.1. Let $\mathcal{C}$ be a category;

- We say that $\mathcal{C}$ is *κ-accessible* if it admits κ-filtered colimits, and if it has a *small* subcategory $\mathsf{S} \subset \mathcal{A}$ of κ-presentable objects such that every $A \in \mathcal{A}$ is a κ-filtered colimit of objects in S.
- We say that $\mathcal{C}$ is *(locally) κ-presentable* if it is accessible and cocomplete.

The theory of presentable and accessible categories is a cornerstone of *categorical logic*, i.e. of the translation of model theory into the language of category theory.

Accessible and presentable categories admit *representation theorems*:

- A category $\mathcal{C}$ is accessible if and only if it is equivalent to the ind-completion $\mathrm{Ind}_\kappa(\mathsf{S})$ of a small category, i.e. to the completion of a small category S under κ-filtered colimits;
- A category $\mathcal{C}$ is presentable if and only if it is a full reflective subcategory of a category of presheaves $i : \mathcal{C} \to \mathsf{Cat}(\mathsf{S}^{\mathrm{op}}, \mathsf{Set})$, such that the embedding functor i commutes with κ-filtered colimts.

All categories of usual algebraic structures are (finitely) accessible, and they are locally (finitely) presentable as soon as they are cocomplete; an example of a category which is $\aleph_1$-presentable but not $\aleph_0$-presentable: the category of metric spaces and short maps.

We now glance at *topos theory*:

Definition A.2. An *elementary topos* is a category $\mathcal{E}$

- which is *cartesian closed*, i.e. each functor $\_ \times A$ has a right adjoint $[A, \_]$;
- having a *subobject classifier*, i.e. an object $\Omega \in \mathcal{E}$ such that the functor Sub : $\mathcal{E}^{\mathrm{op}} \to \mathsf{Set}$ sending A into the set of isomorphism classes of monomorphisms $\left[\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}\right]$ is representable by the object Ω.

The natural bijection $\mathcal{E}(A, \Omega) \cong \mathrm{Sub}(A)$ is obtained pulling back the monomorphism $U \subseteq A$ along a *universal arrow* $t : 1 \to \Omega$, as in the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "t"] \\ A \arrow[r, "\chi_U"'] & \Omega \end{tikzcd}$$

so, the bijection is induced by the map $\left[\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}\right] \mapsto \chi_U$.

Definition A.3. A *Grothendieck topos* is an elementary topos that, in addition, is locally finitely presentable.

Whenever we spoke about sheaves on a topological space or a Grothendieck site, we wer secretly talking about topos theory; the notion of Grothendieck topos is intimately connected with co/end calculus, as we have seen all along chapter 3, and especially in **??**.

In fact, Giraud theorem gives a proof for the difficult implication of the following *recognition principle* for Grothendieck toposes:

Theorem A.4. Let $\mathcal{E}$ be a category; then $\mathcal{E}$ is a Grothendieck topos if and only if it is a left exact reflection of a category $\mathsf{Cat}(\mathcal{A}^{\mathrm{op}}, \mathsf{Set})$ of presheaves on a small category $\mathcal{A}$.

(recall that a *left exact reflection* of $\mathcal{C}$ is a reflective subcategory $\mathcal{R} \hookrightarrow \mathcal{C}$ such that the reflector $r : \mathcal{C} \to \mathcal{R}$ preserves finite limits. It is a reasonably easy exercise to prove that a left exact reflection of a Grothendieck topos is again a Grothendieck topos; Giraud proved that all Grothendieck toposes arise this way.)

A.3. **A little primer on algebraic theories.** The scope of this short subsection is to collect a reasonably self-contained account of functorial semantics. It is unrealistic to aim at such a big target as providing a complete account of it in a single appendix; the reader is warmly invited to parallel their study with more classical references as [Law63].

Definition A.5 (Lawvere theory)**.** A *Lawvere theory* is a category having objects the natural numbers, and where the sum on natural numbers has the universal property of a categorical product, as defined e.g. in [Bor94a, 2.1.4].

Let us denote $[n]$ the typical object of $\mathcal{L}$. Unwinding the definition, we deduce that in a Lawvere theory $\mathcal{L}$ the sum of natural numbers $[n+m]$ is equipped with two morphisms $[n] \leftarrow [n+m] \to [m]$ exhibiting the universal property of the product.

Every Lawvere theory comes equipped with a functor $p : \mathsf{Fin}^{\mathrm{op}} \to \mathcal{C}$ that is the identity on objects and preserves finite products. A convenient shorthand to refer to the Lawvere theory $\mathcal{L}$ is thus as the functor p, or as the pair $(p, \mathcal{L})$.

Definition A.6. The category Law of Lawvere theories has objects the Lawvere theories, understood as functors $p : \mathsf{Fin}^{\mathrm{op}} \to \mathcal{L}$, and morphisms the functors $h : p \to q$ such that the triangle

$$\begin{tikzcd} & \mathsf{Fin}^{\mathrm{op}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{L} \arrow[rr, "h"'] & & \mathcal{M} \end{tikzcd}$$

is commutative. It is evident that Law is the subcategory of the undercategory $\mathsf{Fin}^{\mathrm{op}}/\mathsf{Cat}$ (see e.g. [ML98, I.6]for a precise definition) made by those functors that preserve finite products.

Remark A.7. The category Law has no nonidentity 2-cells; this is a consequence of the fact that a natural transformation $\alpha : h \Rightarrow k$ that makes the triangle “commute”, i.e. $\alpha * p = \mathrm{id}_q$ must be the identity on all objects.

Example A.8 (The trivial theories)**.** The category $\mathsf{Fin}^{\mathrm{op}}$, opposite to the category of finite sets and functions, is the initial object in the category Law; the terminal object is constructed as follows: the category $\mathcal{T}$ has objects the natural numbers, and $\mathcal{T}([n], [m]) = \{*\}$ for every $n, m \in \mathbb{N}$. It is evident that given this definition, there is a unique identity-on-objects functor $\mathcal{L} \to \mathcal{T}$ for every other Lawvere theory $(p, \mathcal{L})$.

Definition A.9 (Model of a Lawvere theory)**.** A *model* for a Lawvere theory $(p, \mathcal{L})$ consists of a product-preserving functor $L : \mathcal{L} \to \mathsf{Set}$. The subcategory $[\mathcal{L}, \mathsf{Set}]_\times \subset [\mathcal{L}, \mathsf{Set}]$ of models of the theory $\mathcal{L}$ is *full*, i.e. a morphism of models $L \to L'$ consists of a natural transformation $\alpha : L \Rightarrow L'$ between the two functors.

Observe that the mere request that $\alpha : L \to L'$ is a natural transformation between product preserving functors means that $\alpha_{[n]} : L[n] \to L'[n]$ coincides with the product $(\alpha_{[1]})^n : L[1]^n \to L'[1]^n$.

Proposition A.10. Let $p : \mathsf{Fin}^{\mathrm{op}} \to \mathcal{L}$ be a Lawvere theory. Then, the following conditions are equivalent for a functor $L : \mathcal{L} \to \mathsf{Set}$:

- L is a model for the Lawvere theory $(p, \mathcal{L})$;
- the composition $L \circ p : \mathsf{Fin}^{\mathrm{op}} \to \mathsf{Set}$ preserves finite products;
- there exists a set A such that $L \circ p = \mathsf{Set}(j[n], A)$.

Corollary A.11. The square

$$\begin{tikzcd}
\mathsf{Mod}(p,\mathcal{L}) \arrow[r, "r"] \arrow[d, "u"'] & {[\mathcal{L}, \mathsf{Set}]} \arrow[d, "p^*"] \\
\mathsf{Set} \arrow[r, "N_j"'] & {[\mathsf{Fin}^{\mathrm{op}}, \mathsf{Set}]}
\end{tikzcd}$$

is a pullback of categories. The functor u is completely determined by the fact that $u(L) = L[1]$, r is an inclusion, and $N_j(A) = \lambda F.\mathsf{Set}(F, A)$ is the functor induced by the inclusion $j : \mathsf{Fin} \subset \mathsf{Set}$.

Corollary A.12. The category of models $\mathsf{Mod}(p, \mathcal{L})$ of a Lawvere theory is a locally presentable, accessibly embedded, complete and cocomplete subcategory of $[\mathcal{L}, \mathsf{Set}]$. Moreover, the forgetful functor $u : \mathsf{Mod}(p, \mathcal{L}) \to \mathsf{Set}$ of **??** is *monadic* in the sense of [Bor94b, 4.4.1]. A complete proof of all these facts is in [Bor94b, 3.4.5], [Bor94b, 3.9.1], [Bor94b, 5.2.2.a]. A terse argument goes as follows: the functors p^*, N_j are accessible right adjoints between locally presentable categories; therefore, so is the pullback diagram: r is a fully faithful, accessible right adjoint, and u is an accessible right adjoint, that moreover reflects isomorphisms. It can be directly proved that it preserves the colimits of split coequalizers, and thus the adjunction $f \dashv u$ is monadic by [Bor94b, 4.4.4].

The last technical remark that we collect sheds a light on the discorso prolisso in **??**: the models of a thelory $\mathcal{L}$ interpreted in the category of models of a theory M correspond to the models of a theory $\mathcal{L} \otimes \mathcal{M}$, defined by a suitable universal property:

$$\mathsf{Mod}(\mathcal{L} \otimes \mathcal{M}, \mathsf{Set}) \cong \mathsf{Mod}(\mathcal{L}, \mathsf{Mod}(\mathcal{M}, \mathsf{Set})) \cong \mathsf{Mod}(\mathcal{M}, \mathsf{Mod}(\mathcal{L}, \mathsf{Set})).$$

Definition A.13. Given two theories $\mathcal{L}$ and $\mathcal{M}$ it is possible to construct a new theory called the *tensor product* $\mathcal{L} \otimes \mathcal{M}$; this new theory can be characterized by the following universal property: the models of $\mathcal{L} \otimes \mathcal{M}$ consist of the category of $\mathcal{L}$-models interpreted in the category of $\mathcal{M}$-models or, equivalently (and this is remarkable) of $\mathcal{M}$-models interpreted in the category of $\mathcal{L}$-models.

Theorem A.14. ([Bor94b, 4.6.2]) There is an equivalence between the following two categories:

- the category of Lawvere theories, regarded as a non-full subcategory of the category $\mathsf{Fin}^{\mathrm{op}}/\mathsf{Cat}$, i.e. where a morphism of Lawvere theories consists of a functor $h : \mathcal{L} \to \mathcal{M}$ that preserves finite products;
- the category of *finitary* monads, i.e. those monads that preserve filtered colimits, and morphisms of monads in the sense of [Bor94b, 4.5.8]

References

[AR94] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, London Mathematical Society Lecture Note Series, vol. 189, Cambridge University Press, Cambridge, 1994.

[Bor94a] F. Borceux, *Handbook of categorical algebra. 1*, Encyclopedia of Mathematics and its Applications, vol. 50, Cambridge University Press, Cambridge, 1994, Basic category theory.

[Bor94b] ———, *Handbook of categorical algebra. 2*, Encyclopedia of Mathematics and its Applications, vol. 51, Cambridge University Press, Cambridge, 1994, Categories and structures.

[Bor99a] Jorge Luis Borges, *The lottery in Babylon*, Collected Fictions, Penguin Books, 1999.

[Bor99b] ———, *Tlön, Uqbar, Orbis Tertius*, Collected Fictions, Penguin Books, 1999.

[HP07] Martin Hyland and John Power, *The category theoretic understanding of universal algebra: Lawvere theories and monads*, Electronic Notes in Theoretical Computer Science **172** (2007), 437–458.

[Jec13] Thomas Jech, *Set theory*, Springer Science & Business Media, 2013.

[Joh77] P.T. Johnstone, *Topos theory*, LMS Mathematical Monographs, no. 10, Academic Press, London, 1977.

[Law63] F William Lawvere, *Functorial semantics of algebraic theories*, Proceedings of the National Academy of Sciences of the United States of America **50** (1963), no. 5, 869.

[Lei14] T. Leinster, *Basic category theory*, vol. 143, Cambridge University Press, 2014.

[ML98] S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998.

[MLM92] S. Mac Lane and I. Moerdijk, *Sheaves in geometry and logic: A first introduction to topos theory*, Springer, 1992.

[Mos09] Yiannis N Moschovakis, *Descriptive set theory*, no. 155, American Mathematical Soc., 2009.

[Rie17] E. Riehl, *Category theory in context*, Courier Dover Publications, 2017.

[Sim11] H. Simmons, *An introduction to category theory*, Cambridge University Press, 2011.

Dario Dentamaro:
Email address: `dio@cane.it`

Fosco Loregian: Tallinn University of Technology, Institute of Cybernetics, Akadeemia tee 15/2,12618 Tallinn, Estonia
Email address: `fosco.loregian@taltech.ee`
Email address: `fosco.loregian@gmail.com`