CATEGORICAL ONTOLOGY I

EXISTENCE

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Abstract. The present paper is the first piece of a series whose aim is to develop an approach to ontology and metaontology through category theory. More in detail, we exploit the theory of *elementary toposes* to purport the claim that a satisfying "theory of existence", and more at large ontology itself, can be obtained appealing to category theory. In this perspective, ontology possesses no unitary nature; it is instead a category, the universe of discourse in which our mathematics (intended at large as a theory of knowledge) can be deployed. The *internal language* that all categories possess prescribes the theory of existence of a fixed ontology/category.

In such internal language of a topos, all first-order properties of an object can be modeled by propositions $p : U \to \Omega$ valued in an object whose "elements" are the truth values that p can assume. Each ontology $\mathcal{E}$ yields its own $\Omega_{\mathcal{E}}$, and thus its own set of truth values, and thus its own "theory of meaning". This approach resembles, but is more general than, fuzzy logics, as suitable choices of $\mathcal{E}$ and thus of $\Omega_{\mathcal{E}}$ carry more nuanced truth values than the usual "true" or "false" of classical logic.

Similarly to the mathematical approach used in the solution of physical problems, a solid corpus of structural mathematics can be used to yield nontrivial philosophical consequences and a useful, modular language suitable for a practical foundation framing some essential, elementary questions of ontology in their correct position. As both a test-bench for our theory, and a literary *divertissement*, we propose a possible category-theoretic solution of Borges' famous paradoxes of Tlön's "nine copper coins", and of other seemingly paradoxical construction in his literary work.

Contents

1. Introduction

EADEM SVNT QVORVM VNVM POTEST SVBSTITVI ALTERI PER TRANSMVTATIONEM

Leibniz, if only he knew category theory

This is the first piece of a series of works that will hopefully span through a certain amount of time, and touch a pretty wide range of topics. Its purpose is to adopt a wide-ranging approach to a fragment of elementary problems in a certain branch of contemporary philosophy. In particular, our work aims to lay a foundation, or a transcription, of a few problems in ontology by means of pure mathematics; in particular, by means of the branch of mathematics known as category theory.

As authors, we are aware that such an ambitious statement of purpose must be adequately motivated. This is the scope of the initial section of the present first manuscript.

1.1. **What is this series.** Since forever, mathematics studies three fundamental indefinites: form, measure, and inference. Apperception makes us recognise that there are extended entities in space, persisting in time. From this, the necessity to measure how much these entities are extended, and to build a web of conceptual relations between them, explaining how they arrange "logically". Contaminations between these three archetypal processes are cetainly possible and common; mathematics happens exactly at the crossroad where algebra, geometry and logic intersect.

We can even say more: mathematics is a language; metamathematics done through mathematics (if such a thing even exists) exhibits the features of a *ur-language*, a generative scheme for "all" possible languages. It is a language whose elements are the rules to give oneself a language, conveying information, and allowing deduction. It is a metaobject: a scheme to generate objects/languages.

Taken this tentative definition, mathematics (not its history, not its philosophy, but its *practice*) is serves as a powerful tool to tackle the essential questions of ontology: what "things" are, what makes them what they are and not different.

Quantitative thinking is a consistent (some would say "honest") way of approaching this deep problem, cogency of entities; however, it is undeniable that a certain philosophical debate has become hostile to mathematical language. A tear in the veil that occurred a long time ago, due to different purposes and different specific vocabulary, can not be repaired by two people only. If, however, the reader of these notes asks for an extended motivation for our work, a wide-ranging project in which it fits, a long-term goal, in short a *program*, they will find it now: there is a piece of mathematics whose purpose is to solve philosophical problems, in the same way certain mathematics "solves" the motion of celestial bodies. It does not completely annihilate the question: it proposes models within which one can reformulate it; it highlights what is trivial consequence of the axioms of that model and what instead is not, and requires the language to expand, to be modified, sharpened. Our aim is to approach this never-mentioned discipline as mathematicians.

Sure, solving once and for all the problems posited by ontology would be megalomaniac; we do not claim such a thing. Instead, more modestly, we propose a starting point unhinging some well-established beliefs (above all else, the idea that ontology is too general to be approached quantitatively); we humbly point the finger at some problems that prose is unable to notice, because it lacks a specific and technical language; we suggest that *in such a language*, when words mean precise things and are tools of epistemic research instead of mere magic spells, a few essential questions of recent ontology dissolve in a thread of smoke, and others simply become "the wrong question": not false, just meaningless as the endeavour to attribute a temperature to consciousness or justice.

We shall say at the outset that such a language is not mathematics; but mathematics has an enormous hygienic capacity to hint at what the *characteristica universalis* should be made of.

Per quanto possa sembrare sospetto usare la matematica per risolvere questioni tradizionalmente di pertinenza della filosofia, ciò che noi riteniamo di poter fare è fornire un linguaggio adeguato entro il quale parlare di ontologia, prendendolo dalla matematica; se ci è concesso un gioco di parole non si tratta, come la tradizione in filosofia analitica ha da sempre paventato, di fare un *uso corretto del linguaggio* quanto piuttosto un *uso del linguaggio corretto*. Tale linguaggio è preso appunto dalla matematica, ed è la *Category Theory* (d'ora in poi CT). Mostreremo come le categorie siano state feconde nella pratica matematica e come possano esserlo analogamente in ontologia. Il motivo è che con esse è possibile (idea prima di esse addirittura inesprimibile) trattare le teorie matematiche come oggetti matematici.[1]

As it is currently organised, our work will attempt to cover the following topics:

- the present manuscript, *Existence*, aims at providing a sufficiently expressive theory of existence. Having a foundational rôle, the scope of most of our remarks is of course more wide-ranging and aimed at building the fundamentals of our toolset (mainly, category theory and categorical logic, as developed in [MLM92, Joh77, LS88]). As both a test-bench for our theory, and a literary *divertissement*, we propose a category-theoretic solution of Borges' paradoxes present in [Bor13].
- A second work, currently in preparation, addresses the problem of *identity*, and in particular its context-dependent nature. Our proof of concept here consists of a solution to Black's ancient "two spheres" paradox; this time the solution is provided by Klein's famous *Erlangen programme* group-theoretic devices. The two interlocutors of Black's imaginary dialogue inhabit respectively an Euclidean and an affine world: this affects their perception of the "two" spheres, and irredemably prevents them from mutual understanding.
- A third work, currently in preparation, addresses again the problem of identity, but this time through the lens of algebraic topology; in recent years homotopy theory

[1] Ogni categoria ha un suo linguaggio interno, come si vedrà, e ogni classe di categorie modella una logica; a loro volta le categorie devono stare in una categoria più grande (in *Type Theory* c'è un postulato che permette di evitare gli ovvi paradossi del caso, stabilendo una gerarchia di tipi; una volta scritto un tipo "esiste", è un oggetto esplicitamente e manifestamente linguistico, senza che si pongano i problemi che dichiareremo e prudentemente eviteremo più sotto).

defied a well-established ontological assumptions such as the identity principle; the many commonalities between category theory and homotopy theory suggest that "identity" is not a primitive concept, but instead depends on our concrete representation of mathematical entities. When X, Y are objects in a category $\mathcal{C}$, there is often a class of "equivalences" $W \subseteq \hom(\mathcal{C})$ prescribing that X, Y shouldn't be distinguished; equality (better, some sort of homotopy *equivalence*) is then defined *ex post* in terms of W, changing as the ambient category $\mathcal{C}$ does; this yields a W-parametric notion of identity $\equiv_W$, allowing categories to be categorified versions of *Bishop sets*, i.e. pairs (S, ρ) where ρ is an equivalence relation on S prescribing a ρ-equality.

1.2. **On our choice of metatheory and foundation.** Nel corso del '900 la direzione dell'evoluzione della matematica ha portato la disciplina inizialmente a frazionarsi in differenti sotto-discipline, con loro oggetti e linguaggi specifici, e poi a trovare un'inattesa unificazione sotto la nozione portante di **struttura**, e lo strumento formale che meglio ne caratterizzato il concetto, le **categorie**. Questo processo ha portato spontaneamente ad una revisione epistemologica della matematica e ha ispirato, nell'evolversi degli strumenti operativi, una revisione sia dei suoi fondamenti che della sua ontologia. Per molti è innegabile che

> [the] mathematical uses of the tool CT and epistemological considerations having CT as their object cannot be separated, neither historically nor philosophically. [Krö07]

Ciò è però avvenuto prescindendo dallo specifico dibattito fondazionale, all'epoca attivissimo. []

La pratica matematica, nella via strutturalista, produce una ontologia "naturale", che alcuni, in seguito, sentirono il dovere di caratterizzare con più precisione. D'altronde, analogamente a quanto suggeriva Carnap [2] rispetto alla semantica,

> mathematicians creating their discipline were apparently not seeking to justify the constitution of the objects studied by making assumptions as to their ontology. [Krö07]

[2]Alcune parole che i filosofi dovrebbero tenere a mente, sulla liceità dell'impiego di entità astratte (nello specifico matematiche) nella riflessione semantica, valevoli anche in ontologia:

> we take the position that the introduction of the new ways of speaking does not need any theoretical justification because it does not imply any assertion of reality [...]. it is a practical, not a theoretical question; it is the question of whether or not to accept the new linguistic forms. The acceptance cannot be judged as being either true or false because it is not an assertion. It can only be judged as being more or less expedient, fruitful, conducive to the aim for which the language is intended. Judgments of this kind supply the motivation for the decision of accepting or rejecting the kind of entities. [Car56]

Ma, al netto dei tentativi (anche dello stesso gruppo Bourbaki), ciò che conta è che l'abitudine a ragionare in termini di strutture abbia prodotto implicite posizioni epistemologiche e ontologiche. Questione che meriterebbe una lunga riflessione autonoma. Per i nostri scopi basti enunciare una distinzione che Krömer riprende in parte da [Corry, 1996]: quella tra **structuralism** e **structural mathematics**:

(1) Structuralism: *the philosophical position regarding structures as the subject matter of mathematics*
(2) Structural Mathematics: *the methodological approach to look in a given problem "for the structure"*

Remark 1.1. **(1)** implica **(2)** ma non è necessaria l'implicazione inversa. Si può cioè fare matematica strutturale senza essere strutturalisti; prendendo cioè posizioni diverse, o anche opposte, rispetto allo strutturalismo in sé. Che poi nella riflessione teorica nella storia recente della matematica sia spesso venuto spontaneo abbracciare questa posizione, è tutt'altro discorso (sorta di ontologia "naturale" appunto). L'uso della CT come metalinguaggio, nonostante la compromissione storica con lo strutturalismo, non rende tuttavia automatico il passaggio da **(2)** a **(1)**. ma suggerisce che l'ontologia non solo dipende dalla "ideologia" (in senso quineano) della teoria, cioè dalla sua potenza espressiva, ma è influenzata dal modello epistemologico che l'uso dello stesso linguaggio formale ispira.

L'utilità della distinzione di Krömer è però un'altra: invece di incespicarsi in una definizione possibilmente non ambigua di *struttura* (con le conseguenze indesiderate che potrebbe avere nella pratica operativa) si può ridurre (o ridefinire) la filosofia **(1)** alla metodologia **(2)**, dicendo che:

> **structuralism is the claim that mathematics is essentially structural mathematics** [Krö07]

(la pratica operativa che "entra" nella definizione di strutturalismo evita il decennale dibattito delle *humanities* sui medesimi concetti).

Ciò è equivalente a dire: la pratica strutturale è essa stessa la sua filosofia.

Gli storici tentativi di spiegazione del termine "struttura" attuati da Bourbaki negli anni a seguire dalla pubblicazione degli *Elements* [], sono la prima sistematica elaborazione di una filosofia che si accordasse con la fecondità operativa della *structural mathematics*. Il suo obiettivo è quello di "*assembling of all possible ways in which given set can be endowed with certain structure*" [Krö07], e per farlo elabora, nel programmatico *The Architecture of Mathematics* (redatto dal solo Dieudonné), pubblicato nel 1950, una strategia formale. Pur specificando che "*this definition is not sufficiently general for the needs of mathematics*" [Bourbaki, 1950], codifica una serie di operational steps tramite i quali una struttura su un insieme è "assembled set-theoretically". Adotta, insomma, una prospettiva "riduzionista" nella quale

> the structureless sets are the raw material of structure building which in Bourbaki's analysis is "unearthed" in a quasi-archaeological, reverse manner; they are the most general objects which can, in a rewriting from scratch of mathematics, successively be endowed with ever more special and richer structures. [Krö07]

A conti fatti, dunque, nello strutturalismo bourbakista la nozione di *set* non sparisce definitivamente davanti a quella di struttura. The path towards an "integral" structuralism was still long.

In his influential paper [Law66] William Lawvere proposes a foundation of mathematics based on category theory. To appreciate the depth and breadth of such an impressive piece of work, however, the word "foundation" must be taken in the particular sense intended by mathematicians:

> [...] a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved.

Such a position sounds at the same time a bit cryptic to unravel, and unsatisfactory; Lawvere's (and others') stance on the matter is that a foundation of mathematics is *de facto* just a set $\mathcal{L}$ of first order axioms organised in a Gentzen-like deduction system. The deductive system so generated reproduces mathematics as we know and practice it, i.e. provides a formalisation for something that already exists and needs no further explanation, and that we call "mathematics".

It is not a vacuous truth that $\mathcal{L}$ exists somewhere: the fact that the theory so determined has a nontrivial model, i.e the fact that it can be interpreted inside a given familiar structure, is both the key assumption we make, and the less relevant aspect of the construction itself; showing that $\mathcal{L}$ "has a model" is –although slightly improperly– meant to ensure that, *assuming the existence of a naive set theory* (i.e., assuming the prior existence of structures called "sets"), axioms of $\mathcal{L}$ can be satisfied by a naive set. Alternatively, and more crudely: assuming the existence of a model of ZFC, $\mathcal{L}$ has a model *inside that model of ZFC*.[3]

A series of works attempting to unhinge some aspects of ontology through category theory should at least try to tackle such a simple and yet diabolic question as "where" are the symbols forming the first-order theory of ETCC. And yet, everyone just believes in sets, and solves the issue of "where" they are with a leap of faith from which all else follows.

This might appear somewhat circular: aren't sets in themselves already a mathematical object? How can they be a piece of the theory they aim to be a foundation of?

[3]However, ensuring that a given theory has a model isn't driven by psychological purposes only: on the one hand, purely syntactic mathematics would be very difficult to parse, as opposed to the unformalised, more colloquial practice of mathematical development; on the other hand (and this is more important), the only thing syntax can see is equality, and truth. To prove that a given statement is false, one either checks all possible syntactic derivations leading to φ, finding none –this is unpractical, to say the least– or they *find a model* where $\neg\varphi$ holds.

Following this path would have, however, catastrophic consequences on the quality and depth of our exposition. The usual choice is thus to assume that, wherever and whatever they are, these symbols “are”, and our rôle in unveiling mathematics is *descriptive* rather than generative [4]

This state of affairs has, to the best of our moderate knowledge on the subject, various possible explanations:

- On one hand, it constitutes the heritage of Bourbaki's authoritarian stance on formalism in pure mathematics;
- on the other hand, a different position would result in barely no difference for the “working class”; mathematicians are irreducible pragmatists, somewhat blind to the consequences of their philosophical stances.

So, symbols and letters do not exist outside of the Gentzen-like deductive system we specified together with $\mathcal{L}$.

As arid as it may seem, this perspective proved itself to be quite useful in working mathematics; consider for example the type declaration rules of a typed functional programming language: such a concise declaration as

```
data Nat = Z | S Nat
```

makes no assumption on “what” `Z` and `S :: Nat -> Nat` are;instead, it treats these constructors as meaningful formally (in terms of the admissible derivations a well-formed expression is subject to) and intuitively (in terms of the fact that they model natural numbers: every data structure that has those two constructors must be the type $\mathbb{N}$ of natural numbers).

In altre parole possiamo tradurre "formally" con "synctactically" (di modo che, ad esempio, gli assiomi di PA sono significanti per via delle derivazioni formali che ci permettono di fare) e "intuitively" con "semantically" (gli assiomi di PA si presume abbiano almeno un modello, quello standard).

Taken as an operative rule, this reveals exactly what is our stance towards foundations: we are “structuralist in the metatheory”, meaning that we treat the symbols of a first-order theory or the constructors of a type system irregardless of their origin, provided the same relation occur between criptomorphic collections of labeled atoms.

Remark 1.2. Ergo per noi nel metalinguaggio gli oggetti del linguaggio sono strutture (o li trattiamo come tali), sospendendo il giudizio su cosa effettivamente *siano* fuori dal metalinguaggio. Possiamo intuitivamente schematizzare così:

[4]Dal punto di vista costruttivista non si può legittimamente affermare che gli assiomi “creino” gli oggetti matematici, dal che la risposta alla domanda “dove sono?” sarebbe “da nessuna parte”. L'unica cosa che si può dire certamente è che essi “precisano, in modo rigoroso ed esatto, ancorché implicito, il *significato* degli oggetti matematici” [Agazzi,[**?**]] (e sembra che anche in matematica, come è uso in filosofia del linguaggio, sia assennato tenere ferma la distinzione tra significato di un'espressione e sua denotazione). Noi ci fermiamo su questa soglia.

L	Objects	Denotation	Ontology
Language	Categories	Theories	??
Metalanguage	Categories	"Places"	Structures

Da qui già si può notare, e lo si dirà ancora in 1.4, come questa impostazione non comporti affatto sostenere una forma di strutturalismo metafisico "forte", solo una versione "weak" a livello metalinguistico.

E' d'aiuto essersi posti in un'ottica relazionale, implicita nella teoria che usiamo (sul rapporto stretto tra strutturalismo e uso di ontologie relazionali [cfr. [**?**]]) e quindi possiamo parlare di relazioni tra oggetti senza dire cosa siano gli oggetti, se non, formalmente, i termini posti ai lati estremi dei funtori. Sappiamo dire cosa sono gli oggetti della teoria dal punto di vista della metateoria, in base a come li trattiamo nella pratica operativa (cioè strutture e relazioni tra esse), ma non cosa sono *nella* teoria, se non simboli, sulla cui fondazione non ci pronunciamo.

In this precise sense we are thus structuralists in the metatheory, and yet we do so with a grain of salt, maintaining a transparent approach to the consequences and limits of this partialisation. On the one hand, pragmatism works [5]; it generates rules of evaluation for the truth of sentences. On the other hand, this sounds like a Munchhausen-like explanation of its the value, in terms of itself. Yet there seems to be no way to do better: answering the initial question would give no less than a foundation of language.

And this for no other reason that "our" metatheory is something near to a structuralist theory of language; thus, a foundation for such a metatheory shall inhabit a meta-metatheory... and so on.

Thus, rather than trying to revert this state of affairs we silently comply to it as everyone else does; but we feel contempt after a brief and honest declaration of intents towards where our metatheory lives. Such a metatheory hinges again on work of Lawvere, and especially on the series of works on functorial semantics.

1.3. **Categories as places.** The present section has double, complementary purposes: we would like to narrow the discussion down to the particular flavour in which we interpret the word "category", but also to expand its meaning so as to encompass its rôle as a foundation for mathematics. More or less, the idea is that a category is both an algebraic structure (a microcosm) and a metastructure in which *all* other algebraic structures can be interpreted (a macrocosm).[6]

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[5] Ed invero noi stiamo chiedendo agli ontologi di diventare "pragmatisti", o operativisti

[6] As an aside, we shall at least mention the dangers of too much a naive approach towards the micro/macrocosm dichotomy: if all algebraic structures can be interpreted in a category, and categories are algebraic structures, there surely is such a thing as the theory of categories *internal* to a given one. And large categories shall be thought as categories internal to the "meta"category (unfortunate but unavoidable name) of categories. There surely is a well-developed and expressive theory of internal categories (see [**? ?**]); but our reader surely has understood that the two "categories", albeit bearing the same name, shall be considered on totally different grounds: one merely is a structure; the other is a foundation for that, and others, structure.

More in detail, a category provides with a sound graphical representation of the defining operation of a certain type of structure $\mathfrak{T}$ (see Definition 1.3 below; we take the word *universal* in the sense of [**?**]). Such a perspective allows to concretely build an object representing a given (fragment of) a language L, and a topos (see Definition A.2) obtained as sort of a universal semantic interpretation of L as internal language. This construction is a classical piece of categorical logic, and will not be recalled here: the reader is invited to consult [LS88, II.12, 13, 14], and in particular

> J. Lambek proposed to use the *free topos* [on a type theory/language] as ambient world to do mathematics in; [. . .] Being syntactically constructed, but universally determined, with higher-order intuitionistic type theory as internal language, [Lambek] saw [this structure] as a reconciliation of the three classical schools of philosophy of mathematics, namely formalism, platonism, and intuitionism.

In light of this, and in light of our subsection A.3, we stress once again that categories live on different, almost opposite, grounds: as syntactic objects, that can be used to model *language*, and as semantic objects, that can be used to model *meaning*.

Interpretation, as defined in logical semantics [Gam91], can be seen as a function $t : L^\star \to \Omega$ that associates elements of a set Ω to the free variables of a formula α in a language L; along the history of category theory, subsequent refinements of this fundamental idea led to revolutionary notions as that of functorial semantics or (elementary) topos. As an aside, it shall be noted that the impulse towards this research was somewhat motivated by the refusal of set-theoretic foundations, opposed to type-theoretic ones.

In the following subsection, we give a more fine-grained presentation of the philosophical consequences that a "metatheoretic structural" perspective has on mathematical ontology.

1.3.1. *Theories and their models.* In [Law66] the author W. Lawvere builds a formal language ETAC encompassing "elementary" category theory, and a theory ETCC for the category of all categories, yielding a model for ETAC. In this perspective category theory has a syntax CT; in CT, categories are terms. In addition, we are provided with a *meta*theory, in which we can consider categories of categories, etc.:

> If Φ is any theorem of elementary theory of abstract categories, then
> $$\forall \mathcal{A}(\mathcal{A} \models \Phi)$$
> is a theorem of basic theory of category of all categories. [Law66]

After this, the author makes the rather ambiguous statement that "*every object in a world described by basic theory is, at least, a category*". This is a key observation: what is the world described by ETAC, what are its elements?

We posit that the statement shall be interpreted as follows: categories in mathematics carry a double nature. They surely are the structures in which the entities we are interested to describe organise themselves; but on the other hand, they inhabit a single, big (meta)category of all categories. Such a big structure is fixed once and for all, at the outset of our discussion, and it is the *place* in which we can provide concrete models for "small" categories. To fix ideas with a particular example: we posit that there surely is such a thing as "the category of groups". But on the other hand, groups are just very specific kinds of sets, so groups are but a substructure of the only category that exists.

Sure, such an approach is quite unsatisfactory in a structural perspective. It bestows the category of sets with a privileged role that it does not have: sets are just *one* of the possible choices for a foundation of mathematics. Instead, we would like to disengage the (purely syntactic) notion of structure from the (semantical) notion of interpretation.

The "categories as places" philosophy now provides such a disengagement, in order to approach the foundation of mathematics agnostically.

In this perspective one can easily fit various research tracks in categorical algebra, [AA.04], functorial semantics [Law63, HP07], categorical logic [LS88], and topos theory [Joh77] that characterised the last sixty years of research in category theory.

Lawvere's *functorial semantics* was introduced in the author's PhD thesis [Law63] in order to provide a categorical axiomatisation of universal algebra, the part of mathematical logic whose subject is the abstract notion of mathematical structure: a semi-classical reference for universal algebra, mingled with a structuralist perspective, is [Man12]; see also [SB81]. We shall say at the outset that a more detailed and technical presentation of the basic ideas of functorial semantics is given in our subsection A.3 below; here we aim neither at completeness nor at self-containment.

Everything starts with the following definition:

Definition 1.3. A *type* $\mathfrak{T}$ *of universal algebra* is a pair $(T, \underline{\alpha})$ where T is a set called the (*algebraic*) *signature* of the theory, and $\underline{\alpha}$ a function $T \to \mathbb{N}$ that assigns to every element $t \in T$ a natural number $n_t : \mathbb{N}$ called the *arity* of the function symbol t.

Definition 1.4. A (*universal*) *algebra* of type $\mathfrak{T}$ is a pair (A, f^A) where A is a set and $f^A : T \to \prod_{t\in T} \mathsf{Set}(A^{n_t}, A)$ is a function that sends every function symbol $t : T$ to a function $f_t^A : A^{n_t} \to A$; f_t^A is called the n_t-ary operation on A associated to the function symbol $t : T$.

We could evidently have replaces Set with another category $\mathcal{C}$ of our choice, provided the object $A^n : \mathcal{C}$ still has a meaning for every $n : \mathbb{N}$ (to this end, it suffices that $\mathcal{C}$ has finite products; we call such a $\mathcal{C}$ a *Cartesian* category). A universal algebra of type $\mathfrak{T}$ in $\mathcal{C}$ is now a pair (A, f^A) where $A : \mathcal{C}$ and $f^A : T \to \prod_{t\in T} \mathcal{C}(A^{n_t}, A)$; it is however possible to go even further, enlarging the notion of "type of algebra" even more.

The abstract structure we are tryin to classify is a *sketch* (the terminology is neither new nor unexplicative: see [Ehr68, CL84, Bor94b]) representing the most general arrangement of operations $f^A : A^n \to A$ and properties thereof[7] that coexist in an object A; such a sketch is pictorially represented as a (rooted directed) graph, modeling arities of the various function symbols determining a given type of algebra $\mathfrak{T}$ (see also [Gri07, XV.3] for the definition of *variety of algebras*).

[7]Examples of such properties are (left) alternativity: for all x, y, z, one has $f^A(x, f^A(x, y)) = f^A(f^A(x, x), y)$; associativity: $f^A(x, f^A(y, z)) = f^A(f^A(x, y), z)$; commutativity: $f^A(x, y) = f^A(y, x)$; and so on.

Given the theory $\mathfrak{T}$ and the graph $G_{\mathfrak{T}}$ that it represents, the category $\mathcal{L}_{\mathfrak{T}}$ generated by $G_{\mathfrak{T}}$ "is" the theory we aimed to study, and every functor $A : \mathcal{L}_{\mathfrak{T}} \to \mathsf{Set}$ with the property that $A([n+m]) \cong A[n] \times A[m]$ concretely realises via its image a representation of $\mathfrak{T}$ in Set.

More concretely, there is a "theory of groups". Such a theory determines a graph G_{Grp} built in such a way to generate a category $\mathcal{L} = \mathcal{L}_{\mathsf{Grp}}$ with finite products. *Models* of the theory of groups are functors $\mathcal{L} \to \mathsf{Set}$ uniquely determined by the image of the "generating object" [1] (the set $G = G[1]$ is the underlying set, or the *carrier* of the algebraic structure in study; in our Definition 1.3 the carries is just the first member of the pair (A, f^A)); the request that G is a product preserving functor entails that if $\mathcal{L}$ is a theory and $G : \mathcal{L} \to \mathsf{Set}$ one of its models, we must have $G[n] = G^n = G \times G \times \cdots \times G$, and thus each function symbol $f : [n] \to [1]$ describing an abstract operation on G receives an interpretation as a concrete function $f : G^n \to G$.

Until now, we interpreted our theory $\mathcal{L}$ in sets; but we could have chosen a different category $\mathcal{C}$ at no additiona cost, provided $\mathcal{C}$ was endowed with finite products, in order to speak of the object $A^n = A \times \cdots \times A$ for all $n : \mathbb{N}$. In this fashion, we obtain the $\mathcal{C}$-models of $\mathcal{L}$, instead of its Set-models: formally, and conceptually, the difference is all there.

Yet, the freedom to disengage language and meaning visibly has deep consequences: suddenly, and quite miracolously, we are allowed to speak of groups internal to the category of sets, i.e. functors $\mathcal{L}_{\mathsf{Grp}} \to \mathsf{Set}$, topological groups, i.e. functors $\mathcal{L}_{\mathsf{Grp}} \to \mathsf{Top}$ (so multiplication and inversion are continuous maps *by this very choice*, without additional requests); we can treat monoids in the category of R-modules, i.e. R-algebras [], and monoids in the category of posets [] (i.e. quantales []) all on the same conceptual ground.

1.3.2. *The rôle of toposes.* Among many different Cartesian categories in which we can interpret a given theory $\mathcal{L}$, toposes play a special rôle; this is mostly due to the fact that the *internal language* every topos carries (in the sense of Definition 3.1) is quite expressive.

To every theory $\mathcal{L}$ one can associate a category, called the *free topos* $\mathcal{E}(\mathcal{L})$ on the theory (see [LS88]), such that there is a natural bijection between the $\mathcal{F}$-models of $\mathcal{L}$ and (a suitable choice of) morphisms of toposes[8] $\mathcal{E}(\mathcal{L}) \to \mathcal{F}$:

$$\mathsf{Mod}(\mathcal{L}, \mathcal{F}) \cong \hom(\mathcal{E}(\mathcal{L}), \mathcal{F}).$$

In the present subsection we analyse how the construction of models of $\mathcal{L}$ behaves when the semantics takes value in a category of presheaves.

Let's start stating a plain tautology, that still works as blatant motivation for our interest in toposes opposed as more general categories for our semantics. Sets can be canonically identified with the category $[1, \mathsf{Set}]$, so models of $\mathcal{L}$ are tautologically identified to its $[1, \mathsf{Set}]$-models. It is then quite natural to wonder what $\mathcal{L}$-models become when the semantics is taken in more general functor categories like $[C, \mathsf{Set}]$. This generalisation is compelling to our discussion: in case C is a discrete category, we get back the well-known category of variable sets Set/C of Proposition 2.1.

[8] We refrain to enter the details of the definition of a morphism of toposes, but we glimpse at the definition: given two toposes $\mathcal{E}, \mathcal{F}$ a morphism $(f^*, f_*) : \mathcal{E} \to clF$ consists of a pair of *adjoint* functors (see [Bor94a, 3]) $f^* : \mathcal{E} \leftrightarrows \mathcal{F} : f_*$ with the property that f^* commutes with finite limits (see [Bor94a, 2.8.2])

Now, it turns out that $[C, \mathsf{Set}]$-model for an algebraic theory $\mathcal{L}$, defined as functors $\mathcal{L} \to [C, Set]$ preserving finite products, correspond precisely to functors $C \to \mathsf{Set}$ such that each Fc is a $\mathcal{L}$-model: this gives rise to the following "commutative property" for semantic interpretation:

> $\mathcal{L}$-models in $[C, \mathsf{Set}]$ are precisely those models $C \to \mathsf{Set}$ that take value in the subcategory $\mathsf{Mod}_{\mathcal{L}}(\mathsf{Set})$ of models for $\mathcal{L}$. In other words we can "shift" the $\mathsf{Mod}(-)$ construction in and out $[C, \mathsf{Set}]$ at our will:
> $$\mathsf{Mod}_{\mathcal{L}(\Omega)}([C, \mathsf{Set}]) \cong [C, \mathsf{Mod}_{\mathcal{L}(\Omega)}(\mathsf{Set})]$$

As the reader can see, the procedure of interpreting a given "theory" inside an abstract finitely complete category $\mathcal{K}$ is something that is only possible when the wrd theory is interpreted as a category, and when a model of the theory as a functor. This discipline goes under many names: the one we will employ, i.e. *categorical*, or *functorial*, semantics [Law63], *internalisation* of structures, *categorical algebra*.

The internalization paradigm sketched above suggests how "small" mathematicians often happily develop their mathematics without ever exiting a single (large) finitely complete category $\mathcal{K}$, without even suspecting the presence of models for their theories outside $\mathcal{K}$. To a category theorist, "groups" as abstract structures behave similarly to the disciples of the sect of the Phoenix [Bor]: "the name by which they are known to the world is not the same as the one they themselves pronounce." They are a different, deeper structure than the one intended by their users.

By leaving the somewhat unsatisfying picture that "all categories are small" and by fixing a semantic universe like Set), *each* category works as a world in which one can speak mathematical language (i.e. "study models for the theory of Ω-structures" as long as Ω runs over all possible theories).

As already hinted in **??** categories exhibit a double nature: they are the theories we want to study, but they also are the *places* where we want to realise those theories;looking from high enough, there is plenty of other places where one can move, other than the category Set of sets and functions (whose existence is, at the best of our knowledge, consequence of a postulate; we similarly posit that there exists a model for ETAC). Small categories model theories, it has a *syntax*, in that they describe a relational structure using compositionality; but large categories offer a way to interpret the syntax, so being a *semantics*. A large relational structure is fixed once and for all, lying on the background, in which all other relational structures are interpreted.

It is nearly impossible to underestimate the profundity of this disengagement: synta and semantics, once separated and given a limited ground of action, acquire their meaning.[9]

More technically, in our Definition 3.1 we recall how this perspective allows to intepret different kinds of logics in different kinds of categories: such an approac leads very far, to the purported equivalence between different flavours of logic*s* and different classes of categories;

[9] Of course, this is dialectical opposition at its pinnacle, and by no means a sterile approach to category theory; see [Law96] for a visionary account of how "dialectical philosophy can be modeled mathematically".

the particular shape of semantics that you can interpret in K is no more, no less than a reflection of the nice categorical properties of K (e.g., having finite co/limits, nice choices of factorization systems [Bor94a, ??], [FK72], a subobject classifier; or the property of the posets $\mathsf{Sub}(A)$ of subobjects of an object A of being a complete, modular, distributive lattice. . . –in light of Definition 3.1, this last property has to do with the internal logic of the category: propositions are the set, or rather the *type*, of "elements" for which they are "true"; and in nice cases (like e.g. in toposes), they are also arrows with codomain a suitable *type of truth values* Ω).

1.3.3. *Categories are universes of discourse.* Somehow, the previous section posits that category theory as a whole is "bigger than the mathematics itself", and it works as one of its foundations; a category is a totality where all mathematics can be re-enacted; in this perspective, ETAC works as a metalanguage in which we develop our approach. This is not very far from current mathematical practice, and in particular from Mac Lane's point of view:

> We can [. . .] "ordinary" mathematics as carried out exclusively within [a universe] U (i.e. on elements of U) while U itself and sets formed from U are to be used for the construction of the desired large categories. [ML98, I.6]

since the unique large category we posit is essentially "the universe". But this approach goes further, as it posits that *mathematical theories are in themselves mathematical objects*, and as such, subject to the same analysis we perform on the object of which those theories speak about.

All in all, our main claim is that categories are of some use in clarifying a few philosophical problems, especially for what concerns the objects of discourse of ontology. The advantages of this approach are already visible in mathematics; we are trying to export them in the current philosophical debate.

Among many, we record

- the possibility of reading theories in terms of relations: this allows to suspend our ontological committment on the nature of objects: they are given, but embedded in a relational structure. This relational structure, i.e. a category modeling the ontology in study is the object of our discourse
- a sharper, more precise, and less time-consuming conceptualisation process, which becomes a purely *context-dependent* process, i.e. a function of the aforementioned relational structure.

Of course, a relational structure as mentioned above is nothing but a certain kind of category: we then posit that *an ontology is a category.*

Of course, as extremist as it may seem, our position fits into an already developed open debate; for example, these are the opening lines of a paper by J.P. Marquis:

> [...] *to be* is *to be related*, and the "essence" of an "entity" is given by its relations to its "environment" [Mar97]

Such a point of view is only acceptable when its fruitfulness has been determined by clarifying and formalising the relationship between an ontology regarded as “all that there is”, and a (large) category as “the place we inhabit”.

Sappiamo che nella storia della CT l’accettazione dell’evidenza dei risultati è avvenuta a prescindere da un dibattito intorno alla scarsa precisione e coerenza logica degli stessi. Le categorie sono nate come strumento concettuale e, senza preoccuparsi delle sottigliezze della ricerca fondazionale, hanno catturato efficacemente tutte le nozioni della matematica moderna, rivelandosi utili e feconde. Nostro claim è che si riveleranno tali anche con le usuali nozioni metafisiche. Si tratta di adottare, in fondo, una prospettiva pragmatica:

> that structural mathematics is characterized as an activity by a treatment of things as if one were dealing with structures. From the pragmatist viewpoint, we do not know much more about structures than how to deal with them, after all. [Krö07]

La “traduzione” dei problemi dell’ontologia nel linguaggio di CT permette di manipolare meglio nozioni (non solo, come si sa, matematiche) ma metamatematiche e metafisiche, e ci dota di un approccio più compatto e di una visione più “leggera” e occamista delle questioni vertenti su oggetti e esistenza. Non giustifichiamo questo approccio a priori ma ne testimoniamo la fecondità già provata in letteratura[10], soprattutto paragonata a quella degli approcci set-theoretic (di cui già è informata la totalità delle ontologie formali).

In CT possiamo “tradurre” i problemi classici dell’ontologia, fornire modelli entro i quali formularne meglio presupposti e domande, evidenziare ciò che è banale conseguenza degli assiomi di quel modello e ciò che non lo è, risolverli e, in alcuni casi, dissolverli, rivelandone la natura figmentale. Si tratta di fornire un *ambiente* ben definito nel quale questioni ritenute oggetto di dibattito filosofico possano illuminarsi in modi nuovi o scomparire. E questo non per qualche perverso istinto riduzionistico, ma per poterne parlare in termini efficaci e nel linguaggio adatto a inquadrarli: tentare, con gli strumenti più avanzati e raffinati dell’astrazione matematica, di rispondere a delle domande, produrre conoscenza, e non solo dibattito; inscrivere antiche o recenti questioni in un nuovo paradigma, volto a superare e al contempo far avanzare la ricerca.

Come ogni paradigma lo dotiamo di una sintassi con la quale “nominare” concetti e dare definizioni, e di una semantica che produca modelli, e quindi contesti, entro i quali “guardare” le teorie; questa sintassi e questa semantica non ce le inventiamo: sono già nella matematica e da lì le preleviamo.

1.4. **Existence: persistence of identity.** Ontology rests upon the principl of identity. It is this very principle that here we aim to unhinge.

Cosa significa che *due cose sono, invece, una* è un problema che ci arrovella fin da quando otteniamo la ragione e la parola; ciò perché il problema è tanto elementare quanto sfuggente: l’unica maniera in cui possiamo esibire ragionamento certo è il calcolo; del resto, se la sintassi non vede che l’uguaglianza in senso più stretto possibile, la prassi deve diventare in fretta

[10] Cf. Mt7,16, giusto per ingraziarsi i severiniani.

capace di una maggiore elasticità: per un istante ho postulato che ci fossero "due" cose, non una. E non è forse questo a renderle due? E questa terza cosa che le distingue, è davvero diversa da entrambe?

Riassumiamo i vantaggi del fare ontologia usando la teoria delle categorie:

- Prima di tutto in questo modo ontologia, ci si permetta la battuta, la si *fa* effettivamente. Vale a dire, come mostreremo nel resto del lavoro e in altri successivi, si affrontano di petto le questioni e le si risolvono. Stiamo perciò suggerendo un approccio *problem solving*
- Lo strutturalismo "debole" implicito in questa visione è da noi mantenuto solo a livello metateorico, e consente di guardare alle relazioni tra oggetti all'interno delle teorie. Anche qui, l'approccio relazionale è assunto in quanto proficuo in senso pratico, operativo, e - come vedremo al termine del paragrafo - non implica l'adesione incondizionata a uno strutturalismo "forte".
- Come conseguenza fondamentale del punto precedente nessun concetto viene studiato in senso "assoluto" (qualunque cosa ciò significhi) ma relativamente al contesto in cui opera, e alla teoria che stiamo adottando per definirlo.

Come si è detto in 1.2 l'uso di questi strumenti concettuali ha aiutato la pratica matematica e ha involontariamente ispirato una visione epistemologica, e poi ontologica, della disciplina, vale a dire dei suoi oggetti di studio. A coloro che obiettano che bisogna prima sapere *cos'è* una struttura prima di lavorare con essa noi rispondiamo, con Krömer, che

> this reproach is empty and one tries to explain the clearer by the more obscure when giving priority to ontology in such situations [...]. Structure occurs in the dealing with something and does not exist independently of this dealing. [Krö07]

Memori delle osservazioni di Carnap [nota ...], non riteniamo che questo approccio "operativo" all'ontologia (che non è puro *problem solving* ma anche chiarificazione concettuale) implichi necessariamente l'adesione incondizionata ad uno strutturalismo filosofico integrale - o a sue varianti specifiche come la teoria **ROS** -, esattamente come abbiamo visto non avvenire nel passaggio dalla *structural mathematics* allo strutturalismo vero e proprio (o al bourbakismo). La sua importanza è principalmente metodologica. (*Au contraire* risulta necessario per chi appoggia posizioni strutturaliste al di fuori della matematica cominciare a fare ontologia in termini categoriali, nelle modalità qui indicate).

Ontology rests upon the principle of identity. It is this very principle that here we aim to unhinge.

Cosa significa che *due cose sono, invece, una* è un problema che ci arrovella fin da quando otteniamo la ragione e la parola; ciò perché il problema è tanto elementare quanto sfuggente: l'unica maniera in cui possiamo esibire ragionamento certo è il calcolo; del resto, se la sintassi non vede che l'uguaglianza in senso più stretto possibile, la prassi deve diventare in fretta
capace di una maggiore elasticità: per un istante ho postulato che ci fossero "due" cose, non una. E non è forse questo a renderle due? E questa terza cosa che le distingue, è davvero diversa da entrambe?

Ciò che risulterà evidente è che la nozione di identità è, appunto, *context-dependent*, e questo risolve il dibattito che, almeno da [Geach,...], impegna i filosofi, in merito alla sua eventuale relatività ontologica.

Sostituire la nozione classica vuol dire rivedere i fondamenti dell'ontologia: la stessa nozione centrale di *esistenza*, nella tradizione quineana, si definisce tramite la nozione, più (illusoriamente) semplice e primitiva, di identità: "*A esiste*" sse "*qualcosa è identico ad A*" (questo "qualcosa" è una variabile vincolata ad un quantificatore esistenziale).

Il motivo per cui riteniamo di dover agire in questa direzione è dovuto agli innumerevoli problemi che la nozione di identità classica (criterio di Leibniz, sue varianti, ma anche definizioni successive in sua vece) si porta dietro, rilevati da molti filosofi nel corso del secolo passato, e non superabili rifiutando solo l'identità leibniziana o abbracciando la prospettiva mereologica (ma ci riserviamo di parlarne in lavori successivi). Il motivo per cui molti filosofi, pur sottolineando l'inadeguatezza della nozione, non hanno mai seriamente proposto di sostituirla, crediamo sia per mancanza sia di un linguaggio adatto sia di alternative teoriche rigorose in esso espimibili. L'*Homotopy Type Theory*, e più in generale la CT, rispondono a queste esigenze, e attuano quella sostituzione finora mai realizzata. **Fouche said**: Esempio di come HoTT sia strutturalista nella metateoria è che in HoTT si può definire cos'è una categoria, e dopo averlo fatto si scopre che la sintassi interpreta "essere uguali" per due oggetti/termini di tipo categoria $A, B : \mathcal{C}$ come "essere isomorfi", o come "essere omotopi" quando $\mathcal{C}$ viene interpretato come un tipo *di omotopia*, $A, B : \mathcal{C}$ come punti di questo spazio, e $A =_{\mathcal{C}} B : \mathsf{Prop}$ come un'omotopia tra A e B. Vale anche la pena notare che in teoria dei tipi la relatività ontologica della nozione di identità *è un assunto*: ogni tipo X è equipaggiato con una "sua" nozione di identità $=_X$ che è locale, è "la sua" e nulla a a che vedere, a priori, con $=_Y$ per un altro tipo Y. Ogni uguaglianza istanziata per termini di tipo diverso è quindi inammissibile *nel linguaggio* ancor prima che nella semantica.

2. Preliminaries on variable set theory

In some of our proofs it will be crucial to blur the distinction between the category of functors $I \to \mathsf{Set}$ and the *slice* category Set/I (see [Bor94a, 1.6.1]); once the following result is proved, we freely refer to any of these two categories as the category of *variable sets* (indexed by I).

Proposition 2.1. Let I be a set, regarded as a discrete category, and let Set^I be the category of functors $F : I \to \mathsf{Set}$; moreover, let Set/I the slice category. Then, there is an equivalence (actually, an isomorphism, see [Bor94a, 1.5.1]) between Set^I and Set/I.

Proof. We shall give a very hands-on proof, based on the fact that we can represent the category Set^I as the category of I-indexed families of objects, i.e. with the category whose objects are $(\underline{X})_I := \{X_i \mid i \in I\}$, and morphisms $(\underline{X})_I \to (\underline{Y})_I$ the families $\{f_i : X_i \to Y_i \mid i \in I\}$. This is obvious, as a functor $F : I \to \mathsf{Set}$ amounts to a choice of sets $A_i := F(i)$, and functoriality reduces to the property that identities in I go to identity functions $A_i \to A_i$.

Now, consider an object $h : X \to I$ of Set/I, and define a function as $i \mapsto h^{\leftarrow}(i)$; of course, $(X(h))_I := \{h^{\leftarrow}(i) \mid i \in I\}$ is a I-indexed family, and since I can be regarded as a discrete category, this is sufficient to define a functor $F_h : I \to \mathsf{Set}$.

Let us define a functor in the opposite direction: let $F : I \to \mathsf{Set}$ be a functor. This defines a function $h_F : \coprod_{i\in I} Fi \to I$, where $\coprod_{i\in I} Fi$ is the disjoint union of all the sets Fi.

The claim now follows from the fact that the correspondences $h \mapsto F_h$ and $F \mapsto h_F$ are mutually inverse.

This is easy to verify: the function F_{h_F} sends $i \in I$ to the set $h_F^{\leftarrow}(i) = Fi$, and the function $h_{F_h} \in \mathsf{Set}/I$ has domain $\coprod_{i\in I} F_h(i) = \coprod_{i\in I} h^{\leftarrow}(i) = X$ (as i runs over the set I, the disjoint union of all preimages $h^{\leftarrow}(i)$ equals the domain of h, i.e. the set X). $\square$

Notation 2.2. We employ the present remark to establish a bit of terminology: by virtue of Proposition 2.1 above, an object of the category of variable sets is equally denoted pair $(A, f : A \to I)$, as a function $h : I \to \mathsf{Set}$, or as the family of sets $\{h(i) \mid i \in I\} = \{A_i \mid i \in I\}$. We call the function f the *structure map* of the variable set A, and the function h the functor associated, or corresponding, to the variable set in study.

Remark 2.3. A more abstract look at this result regards the equivalence $\mathsf{Set}/I \cong \mathsf{Set}^I$ as a particular instance of the *Grothendieck construction* (see [Lei04, 1.1]): for every small category $\mathcal{C}$, the category of functors $\mathcal{C} \to \mathsf{Set}$ is equivalent to the category of *discrete fibrations* on $\mathcal{C}$ (see [Lei04, 1.1]). In this case, the domain $\mathcal{C} = I$ is a discrete category, hence all functors $\mathcal{E} \to I$ are, trivially, discrete fibrations.

Remark 2.4. The next crucial step of our analysis is the observation that the category of variable sets is a topos: we break the result into the verification of the various axioms, as exposed in Definition A.2 and Definition A.3. It will be crucial in our discussion that the category of sets is itself a topos: in particular, it is cartesian closed, and admits the set $\{\bot, \top\}$ as subobject classifier.[11]

Proposition 2.5. The category of variable sets is Cartesian closed in the sense of [Bor94a, p.335].

Proof. We shall first show that the category of variable sets admits products: this is well-known as in Set/I, products are precisely pullbacks ([Bor94a, 2.5.1]); note that Proposition 2.1 gives an identification

$$\begin{tikzcd} & X \times_I Y \arrow[dl] \arrow[dr] \arrow[dd, "h"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & I & \end{tikzcd} \iff i \mapsto h^{\leftarrow}(i) = \Big\{(x, y) \in X \times_I Y \mid h(x, y) = i\Big\}$$

[11]This is a classical model of set theory, as opposed to an intuitionistic model, where Ω consists of a more general Heyting algebra H; the core of all our argument is very rarely affected by this choice, which is mainly due to expository reasons. The reader shall feel free to replace the minimal Heyting algebra $\{\bot, \top\}$ with a more generic one, and they are invited to adapt the arguments accordingly.

and given the universal property of a pullback, this yields a canonical bijection $h^{\leftarrow}(i) \cong f^{\leftarrow}(i) \times g^{\leftarrow}(i)$. This is exactly the definition of the product of the two functors $F_f, F_g : I \to \mathsf{Set}$. Next, we shall show that each functor $\_ \times_I Y$ has a right adjoint $Y \pitchfork_I \_$. The functor $\mathsf{Set}^I \to \mathsf{Set}^I : Z \mapsto Y \pitchfork_I Z$ where $Y \pitchfork_I Z : i \mapsto \mathsf{Set}(Y_i, Z_i)$ does the job. This, together with a straightforward verification, sets up the bijection

$$\frac{X \times_I Y \longrightarrow Z}{X \longrightarrow Y \pitchfork_I Z}$$

and by a completely analogous argument (the construction $(A, B) \mapsto A \times_I B$ is of course symmetric in its two arguments),

$$\frac{X \times_I Y \longrightarrow Z}{Y \longrightarrow X \pitchfork_I Z;}$$

this concludes the proof that the category of variable sets is Cartesian closed. □

Proposition 2.6. The category of variable sets has a subobject classifier.

Proof. From Definition A.2 we know that we shall find a variable set Ω such that there is a bijection

$$\frac{\chi : A \longrightarrow \Omega}{\mathsf{Sub}_I(A)}$$

where $\mathsf{Sub}_I(A)$ denotes the set of isomorphism classes of monomorphisms into A, in the category of variable sets.[12] For the sake of simplicity, for the rest of the proof we fix as category of variable sets the slice Set/I.

From this we make the following guess: as an object of Set/I, Ω is the canonical projection $\pi_I : I \times \{\bot, \top\} \to I$ on the second factor. We are thus left with the verification that π_I has the correct structure and universal property.

First, we shall find a universal monomorphism $\mathtt{t} : * \to \Omega$ in Set/I. Unwinding the definition, such a map amounts to an injective function $I \to \Omega$ having the projection $\pi_I : \Omega \times I \to I$ as left inverse. This generalised element selects the $\top$ (read "top") truth value in Ω. (Evidently, the identity function $\mathrm{id}_I : I \to I$ is the terminal object in Set/I.)

It turns out that the function $I \to I \times \{\bot, \top\}$ choosing the level-1 copy of $I \cong I \times \{\top\}$ plays the rôle of $\mathtt{t}$: in the following, we shortly denote Ω_I such a product set. Now, given a

[12]A monomorphism into A as an object of Set^I is nothing but a family of injections $s_i : S_i \to A_i$; a monomorphism in Set/I is a set S in a commutative triangle

$$\begin{tikzcd} S \arrow[rr] \arrow[dr, "s"'] & & A \arrow[dl, "a"] \\ & I. & \end{tikzcd}$$

monomorphism $S \hookrightarrow A$ the commutative square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & I \arrow[d, "\mathtt{t}"] \\ A \arrow[r, "\chi_S"'] & \Omega_I \end{tikzcd}$$

is easily seen to be a pullback; in fact, every morphism of variable sets $\chi_S : A \to \Omega_I$ must send the element $a : A$ to a pair $(i, \epsilon) : I \times \{\bot, \top\}$. The pullback of χ and $\mathtt{t}$, as defined above, consists of the subset of the product $A \times I$ such that $\chi(a) = \mathtt{t}(i) = (i, \top)$; this defines a variable set $S = (\chi(i, \top))_I$, and such a correspondence is clearly invertible: every variable set arises in this way, and defines a "characteristic" function $\chi_S : (A, f : A \to I) \to (\Omega_I, \pi_I)$:

$$\chi_S(a) = \begin{cases} (f(a), \top) & \text{if } a \in S \\ (f(a), \bot) & \text{if } a \notin S \end{cases}$$

This concludes the proof of the fact that Set/I admits a subobject classifier. □

Remark 2.7. A straightforward but important remark is now in order. The structure of subobject classifier of Ω_I, and in particular the shape of a characteristic function $\chi_S : A \to \Omega_I$ for a subobject $S \subseteq A$ in Set/I, is explicitly obtained using the structure map f of the variable set $f : A \to I$.

This will turn out to be very useful along our main section, where we shall note that a proposition in the internal language of Set/I amounts to a function $p : U \to \Omega_I$, having as domain a variable set $u : U \to I$, whose structure map uniquely determines the "strength" (see Remark 4.1) of the proposition p. In a nutshell, : the reader shall however not be worried; we will duly justify each of these nontrivial conceptual steps along section 4.

Proposition 2.8. The category of variable sets is cocomplete and accessible.

Proof. □

Accessibility is a corollary of Yoneda in the following form: every $F : I \to \mathsf{Set}$ is a colimit of representables

$$F \cong \operatorname{colim}\left(\mathcal{E}(F) \xrightarrow{\Sigma} I \xrightarrow{y} \mathsf{Set}^I\right)$$

($\mathcal{E}(F)$ is small because in this case $\mathcal{E}(F) \cong \coprod_{i \in I} Fi$).

Corollary 2.9. The category of variable sets is a Grothendieck topos.

3. The internal language of variable sets

> I am hard but I am fair; there is no racial bigotry here.
> [. . .] Here you are all equally worthless.
>
> GySgt Hartman

Definition 3.1. The internal language of a topos $\mathcal{E}$ is a formal language defined by *types* and *terms*; suitable terms form the class of variables. Other terms form the class of *formulæ*.

- *Types* are the objects of $\mathcal{E}$
- *Terms* of type X are morphisms of codomain X, usually denoted $\alpha, \beta, \sigma, \tau : U \to X$.
  - Suitable terms are variables: the identity arrow of $X \in \mathcal{E}$ is the variable $x : X \to X$. For technical reasons we shall keep a countable number of variables of the same type distinguished:[13] $x, x', x'', \dots : X \to X$ are all interpreted as 1_X.
- Generic terms may depend on multiple variables; the domain of a term of type X is the *domain of definition* of a term.

A number of inductive clauses define the other terms of the language:

- the identity arrow of an object $X \in \mathcal{E}$ is a term of type X;
- given terms $\sigma : U \to X$ and $\tau : V \to Y$ there exists a term $\langle \sigma, \tau \rangle$ of type $X \times Y$ obtained from the pullback

$$\begin{tikzcd} W \arrow[r] \arrow[d] \arrow[dr, "\langle\sigma,\tau\rangle"] & X \times V \arrow[d] \\ U \times Y \arrow[r] & X \times Y \end{tikzcd}$$

- Given terms $\sigma : U \to X, \tau : V \to X$ of the same type X, there is a term $[\sigma = \tau] : W \xrightarrow{\langle\sigma,\tau\rangle} X \times X \xrightarrow{\delta_X} \Omega$, where $\delta_X : X \times X \to \Omega$ is defined as the classifying map of the mono $X \hookrightarrow X \times X$.
- Given a term $\sigma : U \to X$ and a term $f : X \to Y$, tere is a term $f[\sigma] := f \circ \sigma : U \to Y$.
- Given terms $\theta : V \to Y^X$ and $\sigma : U \to X$, there is a term

$$W\langle\theta, \sigma\rangle \to Y^X \times X \xrightarrow{\text{ev}} Y$$

- In the particular case $Y = \Omega$, the term above is denoted

$$[\sigma \in \theta] : W\langle\theta, \sigma\rangle \to \Omega$$

- If x is a variable of type X, and $\sigma : X \times U \to Z$, there is a term

$$\lambda x.\sigma : U \xrightarrow{\eta} (X \times U)^X \xrightarrow{\sigma^X} Z^X$$

  obtained as the mate of σ.

[13] These technical reasons lie on the evident necessity to be free to refer to the same free variable an unbounded number of times. This can be formalised in various ways: we refer the reader to [**?**] and [**?**].

These rules can of course be also presented as the formation rules for a Gentzen-like deductive system: let us rewrite them in this formalism.

$$\frac{}{1_X : X \to X} \qquad \frac{\sigma : U \to X \qquad \tau : V \to Y}{\langle \sigma, \tau \rangle : W\langle \theta, \sigma \rangle \to X \times Y}$$

$$\frac{\sigma : U \to X \qquad \tau : V \to X}{[\sigma = \tau] : W \to \Omega} \qquad \frac{\sigma : U \to X \qquad f : X \to Y}{f[\sigma] : U \to Y}$$

$$\frac{\theta : V \to Y^X \qquad \sigma : U \to X}{W\langle \theta, \sigma \rangle \to Y^X \times X \xrightarrow{\mathrm{ev}} Y} \quad \frac{x : X \qquad \sigma : X \times U \to Z}{\lambda x.\sigma = \sigma^X \circ \eta : U \to (X \times U)^X \to Z^X}$$

To formulas of the language of $\mathcal{E}$ we apply the usual operations and rules of first-order logic: logical connectives are induced by the structure of internal Heyting algebra of Ω: given formulas φ, ψ we define

- $\varphi \vee \psi$ is the formula $W\langle \varphi, \psi \rangle \to \Omega \times \Omega \xrightarrow{\vee} \Omega$;
- $\varphi \wedge \psi$ is the formula $W\langle \varphi, \psi \rangle \to \Omega \times \Omega \xrightarrow{\wedge} \Omega$;
- $\varphi \Rightarrow \psi$ is the formula $W\langle \varphi, \psi \rangle \to \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$;
- $\neg\varphi$ is the formula $U \to \Omega \xrightarrow{\neg} \Omega$.

universal quantifiers

Each formula $\varphi : U \to \Omega$ defines a subobject $\{x \mid \varphi\} \subseteq U$ of its domain of definition; this is the subobject classified by φ, and must be thought as the subobject where "φ is true".

If $\varphi : U \to \Omega$ is a formula, we say that φ is *universally valid* if $\{x \mid \varphi\} \cong U$. If φ is universally valid in $\mathcal{E}$, we write "$\mathcal{E} \Vdash \varphi$" (read: "$\mathcal{E}$ believes in φ").

Examples of universally valid formulas:

- $\mathcal{E} \Vdash [x = x]$
- $\mathcal{E} \Vdash [(x \in_X \{x \mid \varphi\}) \iff \varphi]$
- $\mathcal{E} \Vdash \varphi$ if and only if $\mathcal{E} \Vdash \forall x.\varphi$
- $\mathcal{E} \Vdash [\varphi \Rightarrow \neg\neg\varphi]$

Ora facciamo delle considerazioni sul lingo interno di Set/I

Chi sono tipi e termini; chi sono le proposizioni e come si scrive il calcolo proposizionale in Set/I; i quantificatori, in dettaglio pornografico.

Definition 3.2. Types and terms of $\mathcal{L}(\mathsf{Set}/I)$.

Definition 3.3. Propositional calculus, quantifiers.

Remark 3.4. Like every other Grothendieck topos, the category Set/I has a *natural number object* (see [MLM92, ??]); here we shall outline its construction. It is a general fact that such a natural number object in the category of variable sets, consists of the constant functor on $\mathbb{N} : \mathsf{Set}$, when we realise variable sets as functors $I \to \mathsf{Set}$: thus, in fibered terms, the natural number object is just $\pi_I : \mathbb{N} \times I \to I$.

A natural number object provides the category $\mathcal{E}$ it lives in with a notion of *recursion* and with a notion of $\mathcal{E}$-induction principle: namely, we can interpret the sentence

$$\big(Q0 \wedge \textstyle\bigwedge_{i\le n} Qi \Rightarrow Q(i+1)\big) \Rightarrow \textstyle\bigwedge_{n:\mathbb{N}} Qn$$

for every $Q : \mathbb{N} \to \Omega_I$.

In the category of variable sets, the universal property of $\pi_I : \mathbb{N} \times I \to I$ amounts to the following fact: given any diagram of solid arrows

$$\begin{tikzcd} I \arrow[r, "0"] \arrow[d, equal] & \mathbb{N}\times I \arrow[r, "s\times I"] \arrow[d, dotted, "u"] & \mathbb{N}\times I \arrow[d, dotted, "u"] \\ I \arrow[r, "x"'] & X \arrow[r, "f"'] & X \end{tikzcd}$$

where every arrow carry a structure of morphism over I (and $0 : i \mapsto (0, i)$, $s \times I : (n, i) \mapsto (n + 1, i)$), there is a unique way to complete it with the dotted arrow, i.e. with a function $u : \mathbb{N} \times I \to X$ such that

$$u \circ (s \times I) = f \circ u.$$

Clearly, u must be defined by induction: if it exists, the commutativity of the left square amounts to the request that $u(0, i) = x(i)$ for every $i : I$. Given this, the inductive step is

$$u(s(n, i)) = u(n + 1, i) = f(u(n, i)).$$

This recursively defines a function with the desired properties; it is clear that these requests uniquely determine u.

Such a terse exposition obviously does not exhaust such a vast topic as recursion theory conducted with category-theoretic tools. The interested reader shall consult [JR97] for a crystal-clear introductory account, and [CH08, CHH14] for more recent and modern development of recursion theory.

4. Nine copper coins, and other toposes

> Explicaron que una cosa es *igualdad*, y otra *identidad*, y formularon una especie de *reductio ad absurdum*, o sea el caso hipotético de nueve hombres que en nueve sucesivas noches padecen un vivo dolor. ¿No sería ridículo -interrogaron- pretender que ese dolor es el mismo?
>
> JLB —Tlön, Uqbar, Orbis Tertius

According to our description of the Mitchell-Bénabou language in the category of variable sets, *propositions* are morpisms of the form

$$p : U \to \Omega_I$$

where Ω_I is the subobject classifier of Set/I described in Proposition 2.6; now, recall that

- the object $\Omega_I = \{0,1\} \times I \to I$ becomes an object of Set/I when endowed with the projection $\pi_I : \Omega_I \to I$ on the second factor of its domain;
- the universal monic $\mathtt{t} : I \to \Omega_I$ consists of a section of π_I, precisely the one that sends $i : I$ to the pair $(i, 1) : \Omega_I$;
- every subobject $U \hookrightarrow A$ of an object A results as a pullback (in Set/I) along $\mathtt{t}$:

$$
\begin{tikzcd}
U \arrow[rr, "u"] \arrow[dd, "m"'] \arrow[dr, "u"] & & I \arrow[dl, equal] \arrow[dd, "\mathtt{t}"] \\
& I & \\
A \arrow[rr, "\chi_U"'] \arrow[ur] & & \Omega_I \arrow[ul]
\end{tikzcd}
$$

(see Proposition 2.6 for a complete proof)

The set I in this context acts as a *multiplier* of truth values, in that every proposition can have a pair (ϵ, i) as truth value. We introduce the following notation: a proposition $p : U \to \Omega_I$ is *true*, in context $x : U$, with *strength* t, if $p(x) = (1, t)$ (resp., $p(x) = (0, t)$).

Remark 4.1. A proposition in the internal language of variable sets is a morphism of the following kind: a function $p : U \to \Omega_I$, defined on a certain domain, and such that

$$
\begin{tikzcd}
U \arrow[r, "p"] \arrow[d, "u"'] & \{0,1\} \times I \arrow[d, "\pi_I"] \\
I \arrow[r, equal] & I
\end{tikzcd}
$$

(it must be a morphism of variable sets!) This means that $\pi p(x : U) = u(x : U)$, so that $p(x) = (\epsilon_x, u(x))$ for $\epsilon_x = 0, 1$ and u is uniquely determined by the "variable domain" U. This is an important observation: the strength with which p is true/false is completely determined by the structure of its domain, in the form of the function $u : U \to I$ that renders the pair (U, u) an object of Set/I.

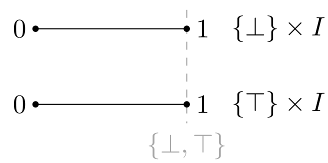
To get a grip of the different roles of various classes of propositions, and given that our interest will be limited to a certain class of particular propositions that we will construct *ex nihilo*, it is now convenient to discuss what constraints we have to put on the structure of I: of course, the richest this structure is, the better will the category Set/I behave: it is for example possible to equip I with an order structure, or a natural topology. Among different choices of truth multiplier, yielding different categories of variable sets, and different kinds of internal logic therein, we will privilege those that make I behave like a space of strengths: a dense, linear order with LUP, thus not really far from being a closed, bounded subset of the real line.

The main result of the present section is a roundup of examples showing that it is possible to concoct categories of variable sets where some seemingly paradoxical constructions coming from J.L. Borges' literary world have, instead, a perfectly "classical" behaviour when looked with the lenses of the logic of variable set theory.

Each of the examples in our roundup Example 4.4 Example 4.8 Example 4.9 is organised as follows: we recall the shape of a paradoxical statement in Borges' literary world. Then, we show in which topos this reduces to an intuitive statement expressed in the syntax of a variable set category.

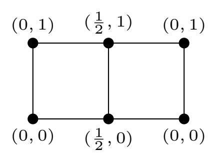
More than often, we use $I = [0, 1]$ as base of variable sets; as already said, there are different reasons for this choice: the most intuitive is that if a truth value is given with a *strength* $t \in I$ it is a natural request to be able to *compare* elements in this set; in particular, it should always be possible to assess what truth value is stronger.

For this reason, even if this assumption is never strictly necessary (the only constraint is that I is totally, or partially, ordered set by a relation $\leq$), a natural choice for I is a *continuum* (=a dense total order with LUP –see [Mos09]). An alternative choice drops the density assumption: in that case the (unique) finite total order $\Delta[n] = \{0 < 1 < \cdots < n\}$, or the countable total order $I = \omega = \bigcup_n \Delta[n]$ are all pretty natural choices for I (although it is way more natural for I to have a minimum *and* a maximum element).[14] In each of these cases "classical" logic is recovered as a projection: propositions p can be true or false with strength 1,[15] the maximum element of I:



In order to aid the reader understand the explicit way in which I "multiplies" truth values, we spell out explicitly the structure of the subobject classifier in $\mathsf{Set}/\Delta[2]$. In order to keep calling the minimum and maximum of I respectively 0 and 1 we call $\frac{1}{2}$ the intermediate point of $\Delta[2]$.

Remark 4.2. The subobject classifier of $\mathsf{Set}/\Delta[2]$ consists of the partially ordered set $\Delta[1] \times \Delta[2]$ that we can represent pictorially as a rectangle



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[14] We're oly interested in the notion of an abstract interval here: a continuum X endowed with an operation $X \to X \vee X$ of "zooming", uniquely defined by this property. In a famous paper Freyd characterises "the interval" as the terminal interval coalgebra: see [Fre08, §1]; for our purposes, note that $[0, 1]$ is a natural choice: it is a frame, thus a Heyting algebra $\mathfrak{H} = ([0, 1], \wedge, \vee, \Rightarrow)$ with respect to the pseudo-complement operation given by $(x \Rightarrow z) := \bigvee_{x \wedge y \leq z} y$ (it is immediate that $x \wedge a \leq b$ if and only if $a \leq x \Rightarrow b$ for every $a, b \in [0, 1]$).

[15] Here I is represented as an interval whose minimal and maximal element are respectively 0 and 1; of course these are just placeholders, but it is harmless for the reader to visualise I as the interval $[0, 1]$.

endowed with the product order. The resulting poset is partially ordered, and in fact a Heyting algebra, because it results as the product of two Heyting algebras: the Boole algebra $\{0 < 1\}$ and the frame of open subsets of the Sierpinski space $\{a, b\}$ (the topology is $\tau_S = \{\varnothing, \{a\}, \{a, b\}\}$).

Remark 4.3. Siccome il caso $I = [0, 1]$ con la topologia euclidea è quello piu naturale per diversi motivi, definiamo alcuni insiemi di interesse per una data proposizione $p : U \to \Omega_I$ per questa scelta di I:

- $A^\top = \{x : U \mid p(x) = (1, t_x), t_x > 0\} = p^{\leftarrow}(\{1\} \times (0, 1])$ e $A^\perp = \{x : U \mid p(x) = (0, t_x), t_x > 0\}$; cose vere (risp., false) con forza maggiore di zero. Sono le funzioni $u : U \to I$ tali che $u^{\leftarrow}0 = \varnothing$.
- $B^\top = \{x : U \mid p(x) = (1, 1)\} = p^{\leftarrow}((1, 1))$ e $B^\perp = \{x : U \mid p(x) = (0, 1)\}$ cose vere (risp.,false).
- $E_t^\top = \{x : U \mid p(x) = (1, t)\}$ e $E_t^\perp = \{x : U \mid p(x) = (0, t)\}$; cose vere (risp., false) con forza t.

Last but not least, a crucial assumption will be that the strength of p depends continuously, or not, on the variables on its domain of definition. Without such continuous dependence, small changes in context $x : U$ might drastically change the truth value $p(x)$.[16]

4.1. **The unimaginable topos theory hidden in Borges' library.** Jorge Luis Borges' literary work is well-known for being made by paradoxical worlds; oftentimes, seemingly absurd consequences follow by stretching to their limit ideas from logic and mathematics: time, infinite, self-referentiality, duplication, recursion, the relativity of time, the illusory nature of our perceptions, the limits of language, its capacity to generate worlds.

In the present section we choose *Fictions*, Borges' famous collection of novels, as source of inspiration for possible and impossible worlds and their ontology.

Usualmente la costruzione di un "mondo impossibile" va circa come segue: . . . ; noi rovesciamo tale prospettiva, e invece di depennare dal computo degli universi i mondi le cui caratteristiche do not comply with sensorial experience, or imply paradoxical entities/constructs, we accept their existence for bizarre that it may seem, and we try to deduce what kind of logic can consistently generate such statements.

The interest in such a literary calembour is manifold, and the results are surprising:

- we unravel how a mathematically deep universe Borges has inadvertently created: of the many compromises we had to take in order to reconcile literature and the uniderlying mathematics,[17] we believe no one is particularly far-fetched one;
- we unravel how *relative* ontological assumptions are; they are not given: using category theory, ontology, far from being the presupposition on which it is based, is a

[16]There is no a priori reason to maintain that p is a continuous proposition; one might argue that discontinuous changes in truth value of p happen all the time in "real life"; see the family of paradoxes based on so-called *separating instants*: how well-defined the notion of "time of death" is? How well-defined the notion of "instant in time"?

[17]See Remark 4.5 below: these "compromises" mainly amount to assumptions on the behaviour of space-time on Tlön and Babylon.

byproduct of language itself. The more expressive language is, the more ontology; the fuzzier its capacity to assert truth, the fuzzier existence becomes;
- "Fuzziness" of existence, i.e. the fact *entia* exist less than completely, is hard-coded in the language (in the sense of Definition 3.1) of the category we decide to work in from time to time;
- ...

To sum up, readers willing to find an original result in this paper, might find it precisely here: we underline how Borges' alternative worlds (Babylon, Tlön dots) are mathematically consistent places, worthy of existence as much as our world, just based on a different internal logic. And they are so just thanks to a base-sensitive theory of existence –ontology breaks in a spectrum of ontolog*ies*, one for each category/world.

The first paradox we aim to frame in the right topos is the famous nine copper coins argument, used by the philosophers of Tlön to construct a paradoxical object whose existence persists over time, in absence of a consciousness continually perceiving it and maintaining in a state of being.

Example 4.4 (Nine copper coins)**.** First, we recall the exact statement of the paradox from []:[18]

> El martes, X atraviesa un camino desierto y pierde nueve monedas de cobre. El jueves, Y encuentra en el camino cuatro monedas, algo herrumbradas por la lluvia del miércoles. El viernes, Z descubre tres monedas en el camino. El viernes de mañana, X encuentra dos monedas en el corredor de su casa. El quería deducir de esa historia la realidad -id est la continuidad- de las nueve monedas recuperadas.
>
> Es absurdo (afirmaba) imaginar que cuatro de las monedas no han existido entre el martes y el jueves, tres entre el martes y la tarde del viernes, dos entre el martes y la madrugada del viernes. Es lógico pensar que han existido -siquiera de algún modo secreto, de comprensión vedada a los hombres- en todos los momentos de esos tres plazos.

Before going on with our analysis, two remarks are in order:
- the paradox appears in a primitive version in [], where instead of nine copper coins, a single arrow, shot by an anonymous archer, disappears among the woods. The text

[18]The translation we employ is classical and comes from [Bor99b]:

> Tuesday, X crosses a deserted road and loses nine copper coins. On Thursday, Y finds in the road four coins, somewhat rusted by Wednesday's rain. On Friday, Z discovers three coins in the road. On Friday morning, X finds two coins in the corridor of his house. The heresiarch would deduce from this story the reality - i.e., the continuity - of the nine coins which were recovered.
>
> It is absurd (he affirmed) to imagine that four of the coins have not existed between Tuesday and Thursday, three between Tuesday and Friday afternoon, two between Tuesday and Friday morning. It is logical to think that they have existed - at least in some secret way, hidden from the comprehension of men - at every moment of those three periods.

appears in a hard-to-find edition of *Inquisiciones* []; in the last chapter, we read the "arrow avatar" (*avatar de la flecha*):

> X scocca una freccia da un arco, ed essa si perde fra gli alberi.
> X la cerca e riesce a ritrovarla.
> E' assurdo immaginare che la freccia non sia esistita durante il periodo fra i momenti in cui X l'ha persa di vista e l'ha ritrovata.
> E' logico pensare che essa sia esistita - anche se in un certo modo segreto, di comprensione vietata agli uomini - in tutti i momenti di questo periodo.

- There is one and only one reason why the paradox of the nine copper coins is invalid: copper does not rust.

It is obvious that both constructions leverage on the same argument to build an efficient aporia: the mysterious persistence of things through time without a perceivent consciousness. We concentrate on the copper coins dilemma, per il semplice fatto che Finzioni è raggiungibile molto piu facilmente ai nostri lettori; incidentally, we happen to be able to rectify the "rust counterargument" without appealing to the assumption that copper can rust on Tlön due to a difference in Tlönian chemistry.

Expressed in natural language, our solution to the paradox goes more or less as follows: X loses their coins on Tuesday, and the force φ with which they "exist" lowers; it grows back in the following days, going back to a maximum value when X retrieves two of their coins on the front door. Y findings of other coins raises their existence force to a maximum. The coins that Y has found rusted (or rather, the surface copper oxidized: this is possible, but water is rarely sufficient to ignite the process alone –certainly not in the space of a few hours.[19]

Remark 4.5. In this perspective, Tlön classifier of truth values can be taken as $\Omega_I = \{0 < 1\} \times I$, where I is any set with more than one element; a minimal example can be $I = \{N, S\}$ (justifying this choice from inside Tlön is easy: the planet is subdivided into two emispheres; each of which now has its own logic "line" independent from the other), but as explained in Remark 4.5 a more natural choice for our purposes is the closed real interval $I = [0, 1]$.

This allows for a continuum of possible forces with which a truth value can be true or false; it is tobe noted that $[0, 1]$ is also the most natural place on which to interpret fuzzy logic, albeit the interest for $[0, 1]$ therein can be easily and better motivated starting from probability theory. (But see [] for an interesting perspective on how to develop basic measure teory out of $[0, 1]$.)

We now start to formalise properly what we said until now.

To set our basic assumptions straight, we proceed as follows:

[19]in una sorta di principio di wormhole, eventi indipendenti sulla terra sono dipendenti su Tlön, perché l'evento A influenza, in uno spazio pluridimensionale di scelte di z di verità, l'evento B in modi che gli sarebbero vietati se fosse sulla Terra.

- senza perdita di generalità possiamo supporre l'insieme $C = \{c_1, \dots, c_9\}$ delle monete totalmente ordinato e partizionato in modo tale che le prime due monete siano quelle ritrovate da X il martedì, le seconde quattro quelle che Y ritrova sul cammino, e le ultime 3 quelle viste da Z. Allora
$$C = C_X \sqcup C_Y \sqcup C_Z.$$
and $C_X = \{c_{X1}, c_{X2}\}$, $C_Y = \{c_{Y1}, c_{Y2}, c_{Y3}, c_{Y4}\}$, $C_Z = \{c_{Z1}, c_{Z2}, c_{Z3}\}$ As already said, the truth multiplier I is the closed interval $[0, 1]$ with its canonical order –so with its canonical structure of Heyting algebra, and if needed, endowed with the usual topology inherited by the real line.
- Propositions of interest for us are of the following form:
$$\lambda gcd.p(g, c, d) : \{X, Y, Z\} \times C \times W \to \Omega_I$$
where W is a set of days, that for the sake of explicitness can be taken equal to the set of weekdays S, M, T (strictly speaking, the paradox involves just the interval $[Tu, Fri]$). $p(g, c, d)$ has to be read as "in g's frame of existence the coin c exists at day d with strength $p(g, c, d)$"

Definiamo ora *ammissibile* una configurazione tale che le condizioni seguenti sono rispettate: for all day d and coin c, we have
$$\sum_{u\in\{X,Y,Z\}} p(u, c, d) = (\top, 1)$$
where we denote as "sum" the logical conjunction in Ω_I: this means that day by day, the *global* existence of the group of coins constantly attains the maximum; it is the *local* existence that lowers when the initial conglomerate of coins is partitioned. Moreover,
$$\begin{cases} \sum_{c_X \in C_X} p(X, c_X, V) = (\top, 1) \\ \sum_{c_Y \in C_Y} p(Y, c_Y, G) = (\top, 1) \\ \sum_{c_Z \in C_Z} p(Z, c_Z, V) = (\top, 1) \end{cases}$$
In an admissible configuration the subsets C_X, C_Y, C_Z can only attain an existence $p(g, c, d) \lneq (\top, 1)$; that is no coin completely exists *locally.* But for an hypothetical external observer, capable of adding up the forces with which the various parts of C exist, the coins *globally* exist " in some secret way, of understanding forbidden to men" (or rather, to X, Y, Z).

Remark 4.6. L'aritmetica di Tlon; proposizioni a forza additiva; parallelismi tra la nave di Teseo e le nove monete.

Remark 4.7 (Continuity for a proposition)**.** Let $p : U \to \Omega_I$ be a proposition; here we investigate what does it mean for p to be (globally) continuous with respect to the Euclidean topology on $I = [0, 1]$, in the assumption that its domain of definition U is metrizable (this is true for example when U is a subset of space-time). The condition is that
$$\forall \epsilon > 0, \exists \delta > 0 : |x - y| < \delta \Rightarrow |px - py| < \epsilon$$

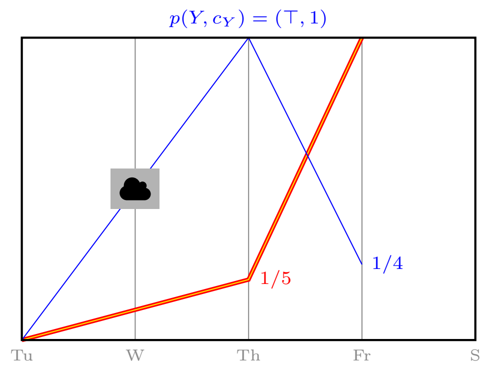

FIGURE 1. A pictorial representation of the truth forces of coins in different days, a minimal piecewise linear model. X is red, Z is yellow, Y is blue. Time is considered as a continuum marked at weekdays.

Da questo segue immediatamente che quando p è continua nel suo dominio, i valori di verità di p in configurazioni "vicine" in un senso opportuno sono dati con forze allo stesso modo vicine (chiaramente questa è una descrizione spannometrica della nozione di continuità...).

All elementary topology results apply to such a proposition: the set of forces with which p is true or false is a connected subset of Ω_I, compact if U was compact.

Example 4.8 (Discontinuity, Lo zaffiro di Taprobana)**.** la lotteria a Babilonia come in [Bor99a]: proposizioni $p : U \to \Omega_{[0,1]}$ possono essere fortemente discontinue nelle variabili/contesto da cui dipendono: tali proposizioni descrivono eventi apparentemente caotici, innescati come termine finale di una catena di eventi tra loro disconnessi e paradossali;[20] un modello di questo unvierso si trova probabilmente nella Babilonia di [Bor99a], e nei "sorteggi impersonali, di proposito indefinito" che caratterizzano l'operato della Compagnia, azioni apparentemente scorrelate tra loro (scagliare "nelle acque dell'Eufrate uno zaffiro di Taprobana"; sciogliere "dal tetto d'una torre [...] un uccello"; togliere (o aggiungere) "un granello di rena ai grani innumerevoli della spiaggia"; queste azioni hanno "conseguenze, a volte, tremende".

Example 4.9 (Continuity: a few birds, a horse)**.** Per quanto riguarda le proposizioni che sono continue nelle proprie variabili, invece, esempio canonico sono le "tigri di cristallo" e le "torri di sangue" di Tlön: oggetti ed entità usuali, diversi da quelli "classici" per un dettaglio solo (il colore, la consistenza, il materiale di cui sono composte): la loro esistenza è sfumata, forte meno del massimo, cosicché tigri di carne e una torre di pietra sono tali che $p = (\top, 1)$, le loro controparti su Tlön esistono con meno forza.

Un altro esempio illustre è fatto da ogetti la cui forza di esistenza dipende in maniera *monotona* e continua dai loro parametri: per esempio una proposizione p può essere tanto

[20] Assumendo una base reale per lo spazio dei parametri da cui p dipende, la sua dipendenza continua è la proprietà enunciata in Remark 4.7; ciò significa che eventi vicini –nello spazio o nella consequenzialità temporale– non possono avere valori di verità diversi, e forze "vicine" in un senso opportuno.

più vera quanta più gente la osserva, perché “le cose, su Tlön, si duplicano; ma tendono anche a cancellarsi e a perdere i dettagli quando la gente le dimentichi. È classico l’esempio di un’antica soglia, che perdurò finchè un mendicante venne a visitarla, e che alla morte di colui fu perduta di vista. Talvolta pochi uccelli, un cavallo, salvarono le rovine di un anfiteatro. ”

In questa situazione, poniamo ad esempio che la forza di esistenza di alcune rovine – modellate come è ingenuo fare, come un corpo rigido R nello spazio, dipenda dal numero dei suoi osservatori:

$$p(R, n) = \left(\top, 1 - \tfrac{1}{n}\right)$$

Succedono cose interessanti anche a cambiare topologia su I: per esempio, su $[0, 1]$ possiamo mettere brutalmente la topologia discreta; in questo modo I è l’unione disgiunta dei suoi punti $\{\{t\} \mid t \in [0, 1]\}$, e il classifo è l’unione disgiunta di $[0, 1]$ copie di $\{0, 1\}$. (See Figure 2 for a picture.)

Example 4.10 (La campagna incendiata)**.** Il Berkeley idealista degli infiniti istanti di tempo continuo, disconnessi e incomunicabili: $\Omega_I = \coprod_{t:[0,1]}\{0 < 1\}$. E’ evidente come questa particolare struttura logica influenzi il linguaggio piegandolo a diventare l’istantaneismo berkeleyano: i termini sono costruiti per accrezione istantanea, per somma disgiunta dei costituenti e delle loro proprietà: “aereo-chiaro sopra scuro rotondo”; oggetti determinati dalla loro simultaneità, e non da una dipendenza logica: il significato si costruisce per accrezione di istanti simultanei, e non per sequenzialità temporale (vedi Remark 4.6 per un legame tra questo principio, la natura additiva della forza di verità in Tlon, e la particolare forma dell’aritmetica Tloniana).

Il rifiuto della consequenzialità temporale per gli abitanti di Tlon sta nel passo

> Spinoza attribuisce alla sua inesauribile divinità i modi del pensiero e dell’estensione; su Tlön, nessuno comprenderebbe la giustapposizione del secondo (che caratterizza solo alcuni stati) e del primo, che è un sinonimo perfetto del cosmo. In altre parole: non concepiscono che lo spaziale perduri nel tempo. La percezione di una fumata all’orizzonte, e poi della campagna incendiata, e poi della sigaretta mal spenta che provocò l’incendio, è considerata un esempio di associazione di idee.

Ciò si lega anche al passo “L’universo è paragonabile a quelle crittografie in cui non tutti i segni hanno un valore, e che solo è vero ciò che accade ogni trecento notti”: un mondo dove ogni trecento notti $p(x) = (\top, 1)$, e per le successive 299 notti p ha forza < 1.

Example 4.11.

5. Vistas on ontologies

Qui cosa mettiamo?

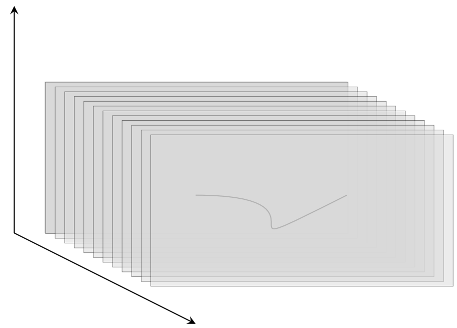

Figure 2. Il tempo come sequenza infinita, e infitamente suddivisibile, di istanti distinti: il paradosso Berkeleyano.

Appendix A. Category theory

A.1. **Fundamentals of CT.** Throughout the paper we employ standard basic category-theoretic terminology, and thus we refrain from giving a self contained exposition of elementary definitions. Instead, we rely on famous and wide-spread sources like [Bor94a, Bor94b, ML98, Rie17, Lei14, Sim11].

Precise references for the basic definitions can be found

- for the definition of category, functor, and natural transformation, in [Bor94a, 1.2.1], [ML98, I.2], [Bor94a, 1.2.2], [ML98, I.3], [Bor94a, 1.3.1].
- The Yoneda lemma is stated as [Bor94a, 1.3.3], [ML98, III.2].
- For the definition of co/limit and adjunction, in [Bor94a, 2.6.2], [ML98, III.3], [Bor94a, 2.6.6], [ML98, III.4].
- For the definition of accessible and locally presentable category in [Bor94b, 5.3.1], [Bor94b, 5.2.1], [AR94].
- Basic facts about ordinal and cardinal numbers can be found in [Kun83]; another comprehensive reference on basic and non-basic set theory is [Jec13].
- The standard source for Lawvere functorial semantics is Lawvere's PhD thesis [Law63]; more modern accounts are [HP07].
- Standard references for topos theory are [MLM92, Joh77]. See in particular [MLM92, VI.5] and [Joh77, 5.4] for what concerns the Mitchell-Bénabou language of a topos.

A.2. **Toposes.** For us, an *ordinal number* will be any well-ordered set, and a *cardinal number* is any ordinal which is not in bijection with a smaller ordinal. Every set X admits a unique *cardinality*, i.e. a least ordinal κ with a bijection $\kappa \cong X$ such that there are no bijections from a smaller ordinal. We freely employ results that depend on the axiom of choice when needed. A cardinal κ is *regular* if no set of cardinality κ is the union of fewer than κ sets of cardinality less than κ; all cardinals in the following subsection are assumed regular without further mention.

Let κ be a cardinal; we say that a category $\mathcal{A}$ is κ-filtered if for every category $\mathcal{J} \in \mathsf{Cat}_{<\kappa}$ with less than κ objects, $\mathcal{A}$ is injective with respect to the cone completion $\mathcal{J} \to \mathcal{J}^{\triangleright}$; this means that every diagram

$$\begin{tikzcd} \mathcal{J} \arrow[r, "D"] \arrow[d] & \mathcal{A} \\ \mathcal{J}^{\triangleright} \arrow[ur, dotted, "\bar{D}"'] & \end{tikzcd}$$

has a dotted filler $\bar{D} : \mathcal{J}^{\triangleright} \to \mathcal{A}$.

We say that a category $\mathcal{C}$ admits filtered colimits if for every filtered category $\mathcal{A}$ and every diagram $D : \mathcal{A} \to \mathcal{C}$, the colimit $\operatorname{colim} D$ exists as an object of $\mathcal{C}$. Of course, whenever an ordinal α is regarded as a category, it is a filtered category, so a category that admits all κ-filtered colimits admits all colimits of chains

$$C_0 \to C_1 \to \cdots \to C_\alpha \to \cdots$$

with less than κ terms. A useful, completely elementary result is that the existence of colimits over all ordinals less than κ implies the existence of κ-filtered colimits; this relies on the fact that every filtered category $\mathcal{A}$ admits a cofinal functor (see [Bor94a]) from an ordinal $\alpha_{\mathcal{A}}$.

We say that a functor $F : \mathcal{A} \to \mathcal{B}$ *commutes with* or *preserves* filtered colimits if whenever $\mathcal{J}$ is a filtered category, $D : \mathcal{J} \to \mathcal{A}$ is a diagram with colimit $L = \operatorname{colim}_{\mathcal{J}} D_j$, then $F(L)$ is the colimit of the composition $F \circ D$. Anoter common name for such an F is a *finitary* functor, or a functor *with rank* ω.

Definition A.1. Let $\mathcal{C}$ be a category;

- We say that $\mathcal{C}$ is *κ-accessible* if it admits κ-filtered colimits, and if it has a *small* subcategory $\mathsf{S} \subset \mathcal{A}$ of κ-presentable objects such that every $A \in \mathcal{A}$ is a κ-filtered colimit of objects in S.
- We say that $\mathcal{C}$ is *(locally) κ-presentable* if it is accessible and cocomplete.

The theory of presentable and accessible categories is a cornerstone of *categorical logic*, i.e. of the translation of model theory into the language of category theory.

Accessible and presentable categories admit *representation theorems*:

- A category $\mathcal{C}$ is accessible if and only if it is equivalent to the ind-completion $\operatorname{Ind}_\kappa(\mathsf{S})$ of a small category, i.e. to the completion of a small category S under κ-filtered colimits;
- A category $\mathcal{C}$ is presentable if and only if it is a full reflective subcategory of a category of presheaves $i : \mathcal{C} \to \mathsf{Cat}(\mathsf{S}^{\mathrm{op}}, \mathsf{Set})$, such that the embedding functor i commutes with κ-filtered colimts.

All categories of usual algebraic structures are (finitely) accessible, and they are locally (finitely) presentable as soon as they are cocomplete; an example of a category which is $\aleph_1$-presentable but not $\aleph_0$-presentable: the category of metric spaces and short maps.

We now glance at *topos theory*:

Definition A.2. An *elementary topos* is a category $\mathcal{E}$

- which is *Cartesian closed*, i.e. each functor $\_ \times A$ has a right adjoint $[A, \_]$;
- having a *subobject classifier*, i.e. an object $\Omega \in \mathcal{E}$ such that the functor $\mathrm{Sub} : \mathcal{E}^{\mathrm{op}} \to \mathsf{Set}$ sending A into the set of isomorphism classes of monomorphisms $\left[\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}\right]$ is representable by the object Ω.

The natural bijection $\mathcal{E}(A, \Omega) \cong \mathrm{Sub}(A)$ is obtained pulling back the monomorphism $U \subseteq A$ along a *universal arrow* $t : 1 \to \Omega$, as in the diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "t"] \\ A \arrow[r, "\chi_U"'] & \Omega \end{tikzcd}$$

so, the bijection is induced by the map $\left[\begin{smallmatrix} U \\ \downarrow \\ A \end{smallmatrix}\right] \mapsto \chi_U$.

Definition A.3. A *Grothendieck topos* is an elementary topos that, in addition, is locally finitely presentable.

The well-known *Giraud theorem* gives a proof for the difficult implication of the following *recognition principle* for Grothendieck toposes:

Theorem A.4. Let $\mathcal{E}$ be a category; then $\mathcal{E}$ is a Grothendieck topos if and only if it is a left exact reflection of a category $\mathsf{Cat}(\mathcal{A}^{\mathrm{op}}, \mathsf{Set})$ of presheaves on a small category $\mathcal{A}$.

(recall that a *left exact reflection* of $\mathcal{C}$ is a reflective subcategory $\mathcal{R} \hookrightarrow \mathcal{C}$ such that the reflector $r : \mathcal{C} \to \mathcal{R}$ preserves finite limits. It is a reasonably easy exercise to prove that a left exact reflection of a Grothendieck topos is again a Grothendieck topos; Giraud proved that all Grothendieck toposes arise this way.)

A.3. **A little primer on algebraic theories.** The scope of this short subsection is to collect a reasonably self-contained account of functorial semantics. It is unrealistic to aim at such a big target as providing a complete account of it in a single appendix; the reader is warmly invited to parallel their study with more classical references as [Law63].

Definition A.5 (Lawvere theory)**.** A *Lawvere theory* is a category having objects the natural numbers, and where the sum on natural numbers has the universal property of a categorical product, as defined e.g. in [Bor94a, 2.1.4].

Let us denote $[n]$ the typical object of $\mathcal{L}$. Unwinding the definition, we deduce that in a Lawvere theory $\mathcal{L}$ the sum of natural numbers $[n + m]$ is equipped with two morphisms $[n] \leftarrow [n + m] \to [m]$ exhibiting the universal property of the product.

Every Lawvere theory comes equipped with a functor $p : \mathsf{Fin}^{\mathrm{op}} \to \mathcal{C}$ that is the identity on objects and preserves finite products. A convenient shorthand to refer to the Lawvere theory $\mathcal{L}$ is thus as the functor p, or as the pair $(p, \mathcal{L})$.

Definition A.6. The category Law of Lawvere theories has objects the Lawvere theories, understood as functors $p : \mathsf{Fin}^{\mathrm{op}} \to \mathcal{L}$, and morphisms the functors $h : p \to q$ such that the triangle

$$\begin{tikzcd} & \mathsf{Fin}^{\mathrm{op}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ \mathcal{L} \arrow[rr, "h"'] & & \mathcal{M} \end{tikzcd}$$

is commutative. It is evident that Law is the subcategory of the undercategory $\mathsf{Fin}^{\mathrm{op}}/\mathsf{Cat}$ (see e.g. [ML98, I.6]for a precise definition) made by those functors that preserve finite products.

Remark A.7. The category Law has no nonidentity 2-cells; this is a consequence of the fact that a natural transformation $\alpha : h \Rightarrow k$ that makes the triangle "commute", i.e. $\alpha * p = \mathrm{id}_q$ must be the identity on all objects.

Example A.8 (The trivial theories)**.** The category $\mathsf{Fin}^{\mathrm{op}}$, opposite to the category of finite sets and functions, is the initial object in the category Law; the terminal object is constructed as follows: the category $\mathcal{T}$ has objects the natural numbers, and $\mathcal{T}([n],[m]) = \{*\}$ for every $n, m \in \mathbb{N}$. It is evident that given this definition, there is a unique identity-on-objects functor $\mathcal{L} \to \mathcal{T}$ for every other Lawvere theory $(p, \mathcal{L})$.

Definition A.9 (Model of a Lawvere theory)**.** A *model* for a Lawvere theory $(p, \mathcal{L})$ consists of a product-preserving functor $L : \mathcal{L} \to \mathsf{Set}$. The subcategory $[\mathcal{L}, \mathsf{Set}]_\times \subset [\mathcal{L}, \mathsf{Set}]$ of models of the theory $\mathcal{L}$ is *full*, i.e. a morphism of models $L \to L'$ consists of a natural transformation $\alpha : L \Rightarrow L'$ between the two functors.

Observe that the mere request that $\alpha : L \to L'$ is a natural transformation between product preserving functors means that $\alpha_{[n]} : L[n] \to L'[n]$ coincides with the product $(\alpha_{[1]})^n : L[1]^n \to L'[1]^n$.

Proposition A.10. Let $p : \mathsf{Fin}^{\mathrm{op}} \to \mathcal{L}$ be a Lawvere theory. Then, the following conditions are equivalent for a functor $L : \mathcal{L} \to \mathsf{Set}$:

- L is a model for the Lawvere theory $(p, \mathcal{L})$;
- the composition $L \circ p : \mathsf{Fin}^{\mathrm{op}} \to \mathsf{Set}$ preserves finite products;
- there exists a set A such that $L \circ p = \mathsf{Set}(j[n], A)$.

Corollary A.11. The square

$$\begin{tikzcd} \mathsf{Mod}(p, \mathcal{L}) \arrow[r, "r"] \arrow[d, "u"'] & {[\mathcal{L}, \mathsf{Set}]} \arrow[d, "p^*"] \\ \mathsf{Set} \arrow[r, "N_j"'] & {[\mathsf{Fin}^{\mathrm{op}}, \mathsf{Set}]} \end{tikzcd}$$

is a pullback of categories. The functor u is completely determined by the fact that $u(L) = L[1]$, r is an inclusion, and $N_j(A) = \lambda F.\mathsf{Set}(F, A)$ is the functor induced by the inclusion $j : \mathsf{Fin} \subset \mathsf{Set}$.

Corollary A.12. The category of models $\mathsf{Mod}(p, \mathcal{L})$ of a Lawvere theory is a locally presentable, accessibly embedded, complete and cocomplete subcategory of $[\mathcal{L}, \mathsf{Set}]$. Moreover, the forgetful functor $u : \mathsf{Mod}(p, \mathcal{L}) \to \mathsf{Set}$ of Corollary A.11 is *monadic* in the sense of [Bor94b, 4.4.1]. A complete proof of all these facts is in [Bor94b, 3.4.5], [Bor94b, 3.9.1], [Bor94b, 5.2.2.a]. A terse argument goes as follows: the functors p^*, N_j are accessible right adjoints between locally presentable categories; therefore, so is the pullback diagram: r is a fully faithful, accessible right adjoint, and u is an accessible right adjoint, that moreover reflects isomorphisms. It can be directly proved that it preserves the colimits of split coequalizers, and thus the adjunction $f \dashv u$ is monadic by [Bor94b, 4.4.4].

The last technical remark that we collect sheds a light on the discorso prolisso in subsection 1.3: the models of a thelory $\mathcal{L}$ interpreted in the category of models of a theory M correspond to the models of a theory $\mathcal{L} \otimes \mathcal{M}$, defined by a suitable universal property:

$$\mathsf{Mod}(\mathcal{L} \otimes \mathcal{M}, \mathsf{Set}) \cong \mathsf{Mod}(\mathcal{L}, \mathsf{Mod}(\mathcal{M}, \mathsf{Set})) \cong \mathsf{Mod}(\mathcal{M}, \mathsf{Mod}(\mathcal{L}, \mathsf{Set})).$$

Definition A.13. Given two theories $\mathcal{L}$ and $\mathcal{M}$ it is possible to construct a new theory called the *tensor product* $\mathcal{L} \otimes \mathcal{M}$; this new theory can be characterized by the following universal property: the models of $\mathcal{L} \otimes \mathcal{M}$ consist of the category of $\mathcal{L}$-models interpreted in the category of $\mathcal{M}$-models or, equivalently (and this is remarkable) of $\mathcal{M}$-models interpreted in the category of $\mathcal{L}$-models.

Theorem A.14. ([Bor94b, 4.6.2]) There is an equivalence between the following two categories:

- Law, regarded as a non-full subcategory of the category $\mathsf{Fin}^{\mathrm{op}}/\mathsf{Cat}$, i.e. where a morphism of Lawvere theories consists of a functor $h : \mathcal{L} \to \mathcal{M}$ that preserves finite products;
- *finitary* monads, i.e. those monads that preserve filtered colimits, and morphisms of monads in the sense of [Bor94b, 4.5.8].

Proof. The proof goes as follows: given a Lawvere theory $p : \mathsf{Fin}^{\mathrm{op}} \to \mathcal{L}$, we have shown that the functor $u : \mathsf{Mod}(p) \to \mathsf{Set}$ in the pullback square Corollary A.11 has a left adjoint $f : \mathsf{Set} \to \mathsf{Mod}(p)$; the composition uf is thus a monad on Set. This is functorial, when a morphism of monads is defined □

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