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Identity, Indiscernibility, and *ante rem* Structuralism: The Tale of i and $-i$[†]

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Some authors have claimed that *ante rem* structuralism has problems with structures that have indiscernible places. In response, I argue that there is no requirement that mathematical objects be *individuated* in a non-trivial way. Metaphysical principles and intuitions to the contrary do not stand up to ordinary mathematical practice, which presupposes an identity relation that, in a sense, cannot be defined. In complex analysis, the two square roots of -1 are indiscernible: anything true of one of them is true of the other. I suggest that 'i' functions like a parameter in natural deduction systems.

> . . . the suspicions of metaphysicians weigh much less heavily with us than the implications of mathematical practice.
>
> [Leitgeb and Ladyman, 2008]

1. Indiscernible Objects

My book, *Philosophy of Mathematics: Structure and Ontology* [1997], spurred a small but lively and interesting discussion concerning structures that have non-trivial automorphisms and, thus, indiscernible places.[1] The issues can be traced to some slogans that were ill-chosen at best, and suggest outright falsehoods at worst.

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[1] Some early formulations of the issues are due to John Burgess [1999, pp. 287–288], Geoffrey Hellman [2001, pp. 192–196], and Jukka Keränen [2001; 2006]. Participants in the more recent discussion include James Ladyman [2005], Tim Button [2006], Jeffrey Ketland [2006], Fraser MacBride [2005, §3; 2006a], and Hannes Leitgeb and James Ladyman [2008]. My own [2006a; 2006b] addresses Burgess, Hellman, and Keränen.

On my view, mathematical objects are places in structures, and these structures exist independently of any non-mathematical systems that exemplify them. To characterize such structures, I borrowed the term '*ante rem*' from metaphysics. I wrote:

> The essence of a natural number is its *relations* to other natural numbers ... The number 2, for example, is no more and no less than the second position in the natural number structure; 6 is the sixth position ... [1997, p. 72]

Passages like this suggest that, on my view, one can *characterize* each mathematical object, uniquely, in terms of its relations to other objects in the same structure. Unfortunately, I just about said as much. From one perspective, one can think of mathematical objects—the places in *ante rem* structures—as offices. I wrote that each such 'office is characterized completely in terms of how its occupant relates to the occupants of the other offices in the structure' [1997, p. 100]. In a word, oops.

Remarks like these at least suggest a strong principle of the identity of indiscernibles. Define a property to be 'structural' if it can be defined in terms of the relations of a given structure. For example, the property of being a prime number is a structural property of arithmetic, since it is definable in terms of addition and multiplication alone. The property of being the number of my daughters, or of being one of James Ladyman's favorite numbers, is not. The property of being the number of complex square roots of -1 is not a structural property of arithmetic, although it is one in complex analysis. The suggested principle is that every mathematical object can be characterized, uniquely, in terms of its structural properties:

(IND) For any objects x, y, in the same structure, if x and y share all of their structural properties, *vis-à-vis* that structure, then $x = y$.

A structure is said to be *rigid* if the only automorphism on it is based on the identity function. Arguably, (IND) fails for structures that are not rigid. The most commonly cited example is complex analysis. The function that takes a complex number $a + bi$ to its conjugate $a - bi$ is an automorphism. It follows that for any formula $\Phi(x)$ in the language of complex analysis, with only x free, $\Phi(a + bi)$ if and only if $\Phi(a - bi)$. In particular, $\Phi(i)$ if and only if $\Phi(-i)$. It follows that i and $-i$ are indiscernible; they share all of their structural properties. So the *ante rem* structuralist is committed to holding that $i = -i$. Another oft-cited example is Euclidean space, where things are even worse. Any two points in Euclidean space can be connected with a rigid translation, which is an automorphism. So, it seems, *all* of the points in Euclidean space share their structural properties. So the *ante rem* structuralist must hold that there is only one such point. If *ante rem* structuralism did have this consequence, it would indeed be absurd.

In the chapter on epistemology, I introduced what I call 'finite cardinality structures'. The cardinal-four structure, for example, has four places, and *no relations*. It is a somewhat degenerate instance of a structure, but no less structural for that. The cardinal-four structure is exemplified by the system consisting of the paws of our cat. The cardinal-four structure is the worst offender of (IND) possible. Since there are no relations to preserve, every bijection of the domain is an automorphism. Each of the four places is structurally indiscernible from the others and yet, by definition, there are four such places, and so not just one.

Most of the critics who raise the issue of non-rigid structures do not mention the finite cardinal structures. Perhaps this is because these structures do not seem to figure heavily, or perhaps at all, in mathematical practice, and so one could easily give them up. Complex analysis and Euclidean space are mathematical structures *par excellence*. One should be loath to adopt a philosophy that required one to reject these, or even to reinterpret them as parts of rigid super-structures. But Leitgeb and Ladyman [2008] remind us that the cardinal-four structure is (or is isomorphic to) a graph, and that is as mathematical as it gets.

I would think that my original whole-hearted acceptance of the finite cardinal structures is some evidence that I had no truck with principles like (IND), despite the ill-chosen slogans scattered throughout the book that seem to suggest otherwise. However, there is no point in engaging exegetical issues about my former time-slice, the author of [Shapiro, 1997]. Let it be known that I now have no truck with the identity of indiscernibles or with related principles like (IND). Let us examine why.

One motivation for the identity of indiscernibles, and thus for (IND), is metaphysical. After pointing out that the complex roots of -1 'are not distinguished from each other by any algebraic properties', John Burgess [1999, p. 288] notes that, on my view, the two roots 'are distinct, though there seems to be *nothing* to distinguish them'. The complaint here seems to involve something like the Principle of Sufficient Reason. If something is so—if there are two distinct square roots of -1—then there must be something that makes it so, or at least something that explains why it is so. If there are two distinct square roots of -1, then there must be something that distinguishes them.

Frankly, I am not sure what is being demanded. The fact that it is a theorem of complex analysis that -1 has two distinct square roots seems to be enough to distinguish them, or at least enough to convince us that there are two, and not just one. What else is required? Concerning the more homogeneous cardinal-four structure, what else is needed besides the stipulation that the structure has four distinct places?

Jukka Keränen [2001] brings metaphysical detail to the issue, but, of course, specific metaphysical proposals are often contentious, or at least up for dispute. Keränen proposes a general thesis that anyone who puts forward a theory of a type of object must provide an account of how those objects are to be 'individuated'. Let L be a linguistic practice in which a given theory is expressed, and let a be an object in the purview of L. Keränen's claim is that the required 'account of individuation' for L should specify 'the fact of the matter that makes [each object] a the object it is, distinct from any other object' in the purview of L, by 'providing a *unique* characterization thereof'.

In 'Structure and identity' [2006a], I pointed out that if this individuation is to be done with a *formula* with only one free variable, in a countable language, then hardly any mathematical theory can meet Keränen's requirement. There are too many objects, and not enough formulas. One can get a little more mileage by using sets of formulas, but even that runs out pretty quickly.

If one is going to accept Keränen's individuation task, the natural route would be to wax even more metaphysical, and attempt to do the individuation by invoking properties and relations, instead of formulas. After all, why should we think that the formulas of a given language exhaust the resources for our metaphysical tasks? Why think that our languages are rich enough? If each mathematical object has an haecceity, a property that applies to it alone, then the job of individuation is done trivially, but at least it is done. The existence of the haecceity of an object a provides the fact that makes a the object it is, distinct from any other. Only a has *that* particular haecceity. The problem, of course, is that since it is virtually analytic that haecceities are not structural properties, the *ante rem* structuralist cannot invoke this trivial resolution of the individuation task.

Some authors entered the tussle by suggesting metaphysical principles that are weaker than Keränen's individuation requirement but still meet Burgess's request that the theorist find something that distinguishes distinct objects. The weakest of these is a requirement that for any a, b, if $a \neq b$ then there is an irreflexive relation R such that Rab [Ladyman, 2005]. Complex analysis and Euclidean geometry pass this test. For example, i and $-i$ stand in the relation of being non-zero and additive inverses to each other. A pair of distinct points in Euclidean space stand in the relation of lying on exactly one line. Nevertheless, the cardinal-four structure and many graphs still fail the weaker test—unless non-identity counts as one of the available irreflexive relations.[2]

[2] Of course, if non-identity does count as an irreflexive relation for these metaphysical purposes, then the distinguishing task is trivial, and unilluminating. The thesis is just that distinct objects must be distinct. Notice that identity, or non-identity, is presupposed in the very formulation of some of the requirements and the examples. What is it for a relation R to be irreflexive? It is that for any x, $\neg Rxx$. In other words, for any x, y, if Rxy then $x \neq y$. What is it for two points to lie on exactly one line? It is that if the points lie on l_1 and on l_2, then $l_1 = l_2$.

Among published works of which I am aware, only Jeffrey Ketland [2006] and Leitgeb and Ladyman [2008] agree with me that there is no individuation or distinguishing requirement at all, or at least none that requires the structuralist to provide anything beyond an ordinary mathematical axiomatization of common structures.

We may have reached a metaphysical standoff. The members of one side—Keränen, Tim Button [2006], and perhaps John Burgess and Geoffrey Hellman (*e.g.*, [Hellman, 2001])—invoke intuitions, often culled from thinking about ordinary physical objects, and formulate metaphysical principles inconsistent with some of the principles of *ante rem* structuralism. The other side rejects the metaphysical principles in question, in part just because they conflict with mathematical practice, as interpreted via *ante rem* structuralism.

Perhaps the discussion can be advanced a little, at least to help mutual understanding, if we go back to some of the motivations for the philosophy of mathematics. As I see it, the goal of our enterprise is to interpret mathematics, and articulate its place in our overall intellectual lives. The philosopher should say something about the subject matter of mathematics, if it has one, something about the methodology of mathematics, something about how it is known, and something about how it gets applied in science and everyday life. As suggested in 'Structure and identity' [Shapiro, 2006a], one *desideratum* of our enterprise is to provide an interpretation that takes as much as possible of what mathematicians say about their subject as literally true, understood at or near face value. Call this the *faithfulness* constraint.

To be sure, faithfulness is not absolute; it is one constraint among many. Hellman [2001], for example, competently argues that his modal reinterpretation of mathematics avoids some sticky philosophical problems that literalists like Michael Resnik [1997] and me must face. As with much philosophy, if not science and ordinary thinking, it is a holistic enterprise. Metaphysical principles are part of the mix as well, and must pay their dues like any other philosophical doctrine or theory. But, as noted in the epigraph of this paper, metaphysical principles—no matter how well-motivated they may be—do not automatically trump established mathematical practice.

So, in the first instance, why should one think that such things as sets, natural numbers, real numbers, complex numbers, and Euclidean points are objects? To make a very long story very short, one route to the acceptance of mathematical objects comes from the faithfulness constraint. In the most straightforward regimentation of the various mathematical theories, linguistic terms like '6', 'π', and '$7 + 4i$' function as singular terms. Faithfulness would at least suggest that they *are* singular terms, at least as a first guess or working hypothesis. More importantly, I think, the first-order variables of the theories range over the respective objects. And, I take it, some of the sentences in the theories are non-vacuously true. Of course, these conclusions are controversial. There are battles to be fought against reconstructive nominalists, who do not take the language at face value, and there are battles to be fought against fictionalists, who deny the (non-vacuous) truth of the sentences. This is not to mention constructivists, anti-realists, *etc*. Those wars are waged on other occasions, on other grounds. For now, I will just assume that the straightforward realism plus faithfulness leads to the thesis that sets, numbers, and points exist. And they are in the range of first-order variables in true theories. So they are objects. Or at least there is some reason to think so.

One of the motivations for *ante rem* structuralism is that, in most cases, reference is straightforward. What is it that, say, the numeral '4' refers to? If we think of numbers as places in the natural number structure, then '4' refers to the fifth place (counting zero), uniquely. Similarly, the term 'π' refers to a given place in the real-number structure, one that can be specified uniquely and determinately. So, for these cases at least, reference is transparent—at least up to the acceptability of *ante rem* structuralism. And because we have singular terms that pick out these places uniquely, we can have singular thoughts about these particular objects.[3]

I am thus comfortable in following the Scottish neo-logicists, Bob Hale and Crispin Wright [2001], in taking the existence of singular terms in true sentences as implying, or at least strongly suggesting, that there are objects denoted by those terms. Each such term denotes an object. And, at least *prima facie*, what looks like a singular term, and functions as a singular term, probably is one.

The neo-logicist also accepts a converse, of sorts, to this: to be an object is to be the sort of thing that can be denoted by a singular term. When it comes to physical objects, there may be some plausibility for this principle. Although it would be tedious, and pointless, there seems to be nothing to prevent us, the community of language users, from giving names to every physical object, or at least to any given physical object we encounter. Anyone who wants can go down to the beach, and start giving names to the grains of sand he finds there. And we can name the snowflakes and raindrops, as they fall, if we can work fast enough. However, I am not so sure if we can name objects outside of our light cone. Still, such objects are still the sort of thing that can be named—in some sense of 'can'. What of the subatomic particles studied in physics? Is it possible, in any sense, to name each one? At some point, our ability, even in principle, to single out individual specimens of some kinds of objects lapses.

I submit that, in mathematics, the full neo-logicist principle fails rather badly. It is simply false that to be an object is to be the sort of thing that can be picked out uniquely with a singular term. If we invoke the

[3] We will get to the complex number i shortly (§3).

usual idealizations on lifetime and attention span, one can assign a unique name for each natural number. So, up to the idealizations, the neo-logicist principle holds for arithmetic. But arithmetic is the exception, not the rule. To be sure, the neo-logicist does not claim that it is possible to name every real number, all at once, and so cardinality considerations do not refute the view by themselves. The neo-logicist thesis is that for each real number r, it is possible to name r. And for each set s in the pure iterative hierarchy, it is possible to name s. I am skeptical of this. Barring some sort of direct Platonic access to the realm of real numbers and sets (which would run counter to the neo-logicist theme anyway), the only way we have of referring to a real number or a pure set is via a description, in natural language or some other medium to which we have access. Given the multitude of real numbers and pure sets, some are just too complex, and random, to be described, even in principle. The places in some structures are even worse than those. There simply is no naming *any* point in Euclidean space, nor any place in a finite cardinal structure and in some graphs, no matter how much we idealize on our abilities to pick things out. The objects are too homogenous for there to be a mechanism, even in principle, for singling out one such place, as required for reference, as that relation is usually understood.

Still, I would insist, real numbers, members of the iterative hierarchy, points in Euclidean space, and places in finite cardinal structures and graphs, are legitimate mathematical objects. They are in the range of the first-order variables in coherent, and presumably true, mathematical theories. For this reason, I follow Quine and take quantification, rather than reference, as signaling ontology.[4]

The philosopher may retrench here, and hold that real numbers, Euclidean points, and the like, are not objects, simply because they cannot be named. Or else she may suggest that there is an ambiguity in the word 'object'. My preference is to keep the notion of 'object' unequivocal between mathematical, scientific, and ordinary discourse, and to adjust any metaphysical principles about objects, as such, to suit (§4 below).

2. The Role of Identity

In [Shapiro, 1997, p. 92], I wrote that 'Quine's thesis is that within a given theory, language, or framework, there should be definite criteria for identity among its objects. There is no reason for structuralism to be the single exception to this.' Admittedly, this can be read as a demand for a

[4] For better or worse, there has been a trend in mathematics to divorce matters of existence from what can be described, defined, referred to, or constructed. See, for example, [Maddy, 1997] on $V = L$. The neo-logicist principle that ties ontology to reference runs against that trend.

non-trivial *definition* of the identity relation on the places of each structure. This pronouncement, plus some of the slogans, do suggest something in the neighborhood of the identity of indiscernibles, which, as above, I reject wholeheartedly. In discussing this passage later, I said this:

> What I meant—or think I meant, or should have meant—was that if we are to develop a theory of structures, then there must be a determinate identity relation between structures ... [T]here is no room for a view that takes structures seriously as objects and leaves the identity relation between structures indeterminate. Surely the same goes for places within a given structure ... When it comes to mathematical objects—places within a given structure—identity must be determinate. [Shapiro, 2006a, p. 140]

It does not follow from the determinacy of identity that the identity relation can be *defined* in a non-trivial way. I submit that, on the contrary, ordinary mathematical practice *presupposes* identity. In line with the faithfulness constraint, *ante rem* structuralism follows suit.

Note, first, that identity *cannot* be defined in full generality, in a non-circular manner. Of course, the details of this claim depend on what the background resources are—on what one can invoke in giving definitions. Suppose, first, that we are working in a first-order language without identity. Let E be a binary relation symbol that is supposed to represent identity, and let Γ be any collection of sentences in the relevant language, which is intended to serve as an implicit definition of identity. If Γ has a model M at all, then it has a model in which E is not identity. Indeed, for any cardinal κ, there is a model N equivalent to M such that there are at least κ-many distinct objects in the domain of N, all of which bear E to each other.[5] To use the language of metaphysics, in a rigorous way, one just adds duplicates of the objects in the domain of the original model M.

Of course, first-order logic with identity has a special logical symbol, which is required to be the identity relation in each model. That is, identity is presupposed, and so there is no sense of asking whether it can be defined in that framework.

In second-order logic, identity is sometimes defined, as follows:

$$\text{(ID)} \qquad a = b \equiv \forall X(Xa \equiv Xb).$$

[5] There are some results concerning when identity can be defined in a fixed interpretation. Suppose, for example, that a given theory has only finitely many primitive relations, and that for a given interpretation M, there is a formula $\Phi(x, y)$, with just the free variables indicated, such that M satisfies $\Phi(a, b)$, under an assignment, if and only if a and b denote the same object. Then the Quinean indiscernibility formula defines identity in M. See [Ketland, 2006]. Thanks to a referee for pointing this out. This result does not bear on the present matter, however. We presuppose identity, in the meta-theory, in defining and discussing the model M.

This is perhaps a version of the identity of indiscernibles. In standard semantics, the right-hand side of (ID) does in fact express the identity relation. But identity is presupposed in standard semantics, in the meta-theory. To show that the formula $\forall X(Xa \equiv Xb)$ expresses identity, we consider a property or set P that holds of *a alone*. That is, P holds of a and nothing else: if $b \neq a$, then $\neg Pb$. In other words, we invoke the identity relation in the meta-theory in order to show that the purported definition does express the identity relation in the object language.

In non-standard Henkin semantics for higher-order languages, the situation is similar to that of first-order logic without identity. That is, let E be a binary-relation symbol that is supposed to represent identity, and let Γ be any collection of sentences in the relevant higher-order language, which is intended to serve as an implicit definition of identity in the second-order framework. If Γ has a Henkin model M at all, then it has a Henkin model in which E is not identity. In particular, in any Henkin model the formula $\forall X(Xa \equiv Xb)$ is an equivalence relation, but it need not be identity. If M is any Henkin model, and κ is any cardinal number, then there is a Henkin model N equivalent to M such that there are at least κ-many distinct objects in the domain of N, such that if a, b are any two of those objects, then N satisfies $\forall X(Xa \equiv Xb)$. The same goes for any other attempt to define identity.

As with model theory, so with mathematical practice. To pursue a point made in [Ketland, 2006], the identity relation is presupposed throughout the enterprise of mathematics. To characterize the (rigid) structure of the natural numbers, one invokes a non-logical successor function. In calling it a *function*, we presuppose that, in any interpretation, each number has a *unique* successor. That is, if b and c are successors of a, then $b = c$. Similarly, one of the axioms of arithmetic is that the successor function is one-to-one: for any natural numbers a, b, if $sa = sb$, then $a = b$. There is simply no way to say that the successor relation is a function, or that it is one-to-one, without invoking identity, or something else that presupposes identity.

It is a theorem that there is only one group with five elements, up to isomorphism. This means that for any two models of the theory, there is a one-to-one function from the domain of one onto the domain of the other that preserves the group operation. If we are not entitled to the identity relation until we define it—and remember that we cannot define it in general—then we cannot even *state* that two groups are isomorphic (or even that a given system is a group).

The practice in the algebraic tradition is to define a structure by giving axioms. These implicit definitions typically employ non-logical terminology appropriate to the structure, plus a sign for identity. The latter is not just another non-logical term; it is more like the sign for conjunction, or the sign for the universal quantifier. Just as in first-order logic with identity, we insist, or presuppose, that $a = b$ is to hold in an interpretation, if and only if a and b denote the same object.

One of the main arguments in [Shapiro, 1991] is that an advocate of second-order languages, with standard semantics, presupposes no more than is invoked in ordinary mathematical practice. Something analogous applies here. In presupposing the identity relation, we assume no more than is presupposed in ordinary mathematical practice.

To complete this circle, one can define the cardinal-two structure with the following categorical, axiom:

$$(\exists x)(\exists y)(x \neq y \ \& \ \forall z(z = x \vee z = y)).$$

This is not different, in kind, from any other axiomatization, except for the fact that it has no non-logical terminology, and perhaps the fact that it is trivial. The complex number structure also has a standard axiomatization. It entails, easily, that

$$(\exists x)(\exists y)(x \neq y \ \& \ x^2 = y^2 = -1 \ \& \ \forall z(z^2 = -1 \rightarrow (z = x \vee z = y))).$$

In words, there are (exactly) two distinct members of the cardinal-two structure and there are (exactly) two distinct complex square roots of -1. Neither mathematics nor *ante rem* structuralism requires any more in the way of identification or individuation.

3. Speaking of 'i', in Order to Speak of i

Many years ago, I was visiting a friend, a brilliant computer engineer with a strong sense for philosophy and, usually, the patience to pursue it seriously. He was explaining the basics of complex analysis to his young son, talking about various properties of the complex unit which, of course, he called 'i'. I interrupted, and asked him, 'What is "i"?'. He looked at me in a puzzling way, as if I were setting a trap (which, admittedly, I was). He answered, tentatively, 'The square root of minus 1'. I then asked him, 'Which square root?', reminding him that there are two such. He thought about this for about three seconds, and responded, 'That's cute', and went back to talking to his son.

For the philosophy of mathematics, there is at least a potential problem with the language of complex analysis, whether or not one adopts *ante rem* structuralism. The term 'i' seems to function, grammatically, as a singular term. Just above, I endorsed the thesis, from the neo-logicists, that if a term functions as a singular term in a true sentence, then it probably is a singular term. And, of course, the semantic role of a singular term seems to be to denote an object—a single object. But the linguistic and mathematical communities have done nothing to single out a unique referent for this term. They cannot, since, as above, the two square roots of -1 are indistinguishable. As my friend's first answer indicates, the complex unit is often defined, informally, via a definite description: 'i is the square root of -1'. It is surely reasonable to insist that definite descriptions require uniqueness, at either the level of logical form [Russell, 1905], or semantics, or pragmatics (see [Roberts, 2003]). Suppose, for example, that someone says that she pities the child of the author of this article. Absent some contextual parameters, something has gone wrong, since the author has three children. Absent contextual clues, the locution is infelicitous. If we follow [Russell, 1905], what she said is false; if we follow others, the attempted assertion suffers a failure of presupposition. Why does something similar not happen throughout complex analysis?

Robert Brandom [1996] argues that Fregeans and neo-Fregeans, who tie objects to singular terms, via the notion of reference, have no solution available, or at least none that conforms to Fregean strictures concerning (mathematical) languages. Frege himself wrote:

> Nothing prevents us from using the concept 'square root of -1'; but we are not entitled to put the definite article in front of it without more ado and take the expression 'the square root of -1' as having a sense. [1884, §97]

Presumably, the 'more ado' in the case of definite descriptions would be to show that there is one *and only one* such square root. But, in this case, of course, there is not. There are two of them.

Notice, incidentally, that the language of *real* analysis also has locutions like 'the square root of 4'. If we get ruthlessly literal, these locutions also fail uniqueness, since 4 has two (real) square roots. But the custom is to understand 'the square root of n' and the singular term '$\sqrt{n}$' as elliptical for 'the non-negative square root *of* n'. No such option is available in what we may think of as pure complex analysis, the theory of the algebraic closure of the real numbers, since, among the imaginary numbers, there is no notion of 'positive' and 'negative' to be invoked.

One option is to interpret complex analysis in another, rigid, structure or, perhaps better, to *replace* complex analysis with a rigid structure. For example, if one thinks of the complex numbers as pairs of real numbers, then our problem is solved. One stipulates that i is the pair (0,1), in which case $-i$ is the pair $(0, -1)$. Those pairs are distinguishable from each other in $\mathbb{R}^2$. Given how pervasive non-rigid structures are, however, I would take this as a last resort, only to be invoked if we cannot do better. In line with faithfulness, I take it that, other things equal, it is better to take the languages of mathematics at face value.[6]

[6] A referee suggested a connection between this issue and the treatment of identity and indistinguishable objects in previous sections. The idea is to think of the semantic relations themselves in structural terms. Consider, for example, the cardinal-two structure, which has two places and no relations. It is homogeneous, as above. Suppose we add two new objects,

There are some well-studied, natural language analogues of the phenomenon in question here, although their treatment is subject to controversy. One is the so-called problem of indistinguishable participants. Consider the following sentences:

> If a man lives with another man, he shares the housework with him.
> If a bishop meets another bishop, he blesses him.

The problem is to settle on referents for the pronouns at the end of these sentences. If man A lives with man B, then, of course, B lives with A. Which of them is 'he' and which 'him'? In giving the truth conditions for the second sentence, which is the blesser and which is the blessee?[7]

Some of the theoretical work on indistinguishable participants invokes ways of breaking the symmetry. In a given conversational context, one of the men, or one of the bishops, may be more salient. That one will be the referent for 'he', and the other becomes the referent for 'him'. Nothing like that will work in the case of complex analysis, since there is, in principle, *nothing* to break the symmetry. This, again, is a consequence of the fact that the complex numbers have the non-trivial automorphism mapping the two square roots of -1 to each other.

[Roberts, 2003] is a general study of uniqueness in 'definite noun phrases' like definite descriptions and pronouns. In addition to indistinguishable participants, she builds on examples like the following:

> Everybody who bought a sage plant here bought eight others along with it.
> Remember the chess set that came with an extra pawn? The friend I gave it to could have used an extra king, but he never needed the extra pawn.

If the first sentence is true, each person in the domain who bought a sage plant at the establishment in question actually bought nine of them. Which

a, b, and a relation R to the structure. The new item a bears R to one of the places in the original cardinal-two structure and b bears R to the other place. This is the structure of some simple semantic relations on the cardinal-two structure: think of a and b as names, and R as the reference relation. This mathematical-*cum*-semantic structure is not rigid. Suppose, for example, that we attempt to modify the structure by switching the 'referents' of a and b. That is, have a bear R to the place in the cardinal-two structure to which b used to bear R, and have b bear R to the place to which a used to bear R. We have not really changed the structure. It is still a cardinal-two structure, with the same two new objects, the same relation R with the same structural-semantic relations. Just as there is nothing problematic, or unusual, about the cardinal-two structure, there is nothing problematic about this one either. See [Leitgeb, 2007, pp. 133–134]. The problem, of course, is to say something about the semantics and logic of the languages of mathematics, so construed.

[7] Thanks to Jason Stanley for pointing me to this linguistic issue. For the general problem of indistinguishable participants, see [Heim, 1990], [Roberts, 2003], and [Elbourne, 2003; 2005, §1.3.2 and Chapter 4]. Incidentally, the context of the first example probably involves a presupposition that if a man lives with a woman, then he expects her to do the housework. There are problems with this, too, but they are not linguistic.

of the nine is the referent of 'it'? In the second sentence, the chess set came with 17 pawns, of which, presumably, 9 were of the same color. Which one of those is 'the extra pawn'? According to Roberts, these noun phrases do not need to be associated with a unique object 'in the world', so to speak. It is enough for the relevant objects to exist (either as a matter of semantics or pragmatics), and, in the case of pronouns, for there to be a unique 'discourse referent' on the conversational record, an abstract structure associated with conversations.

The natural-language examples involve pronouns and definite descriptions. What is interesting about the language of complex analysis is that 'i' is, at least *prima facie*, a *proper name*. It is a permanent part of the language. Yet, like the aforementioned pronouns and definite descriptions, it is impossible to assign a unique referent to 'i'. The best solution, I think, is to rethink—slightly—the role of singular terms, and perhaps definite descriptions, perhaps in the direction of [Roberts, 2003].

Systems of natural deduction are, I believe, good models of ordinary deductive reasoning in ordinary discourse, at least for the most part. Let us focus on the role of some of the singular terms in natural deduction—and in the reasoning modeled by natural deduction systems. Consider the rule of existential elimination. Suppose that in doing a derivation, someone reaches a conclusion in the form:

$$(\exists x)\Phi(x),$$

resting on some premises or assumptions. Typically, the reasoner proceeds by making a new assumption $\Phi(b)$, where b is a singular term that does not occur in the formula $(\exists x)\Phi(x)$, or in any undischarged premise supporting that formula. Then she proceeds to deduce a formula Ψ, in which the term b does not occur, and which does not rest on any other premises or undischarged assumptions that contain b. She is then entitled to discharge the assumption of $\Phi(b)$, and have Ψ rest on whatever $(\exists x)\Phi(x)$ rests on, plus whatever other premises were invoked along the way.

In some natural deductive systems, the term b introduced in the assumption $\Phi(b)$ is to be a free variable, but it is more common to require that b be a constant. In either case, b functions grammatically as a singular term in the formula in which it occurs.

The rule of existential elimination corresponds to a common pattern in ordinary reasoning. In actual reasoning, in mathematical or natural language, it would be strange—to say the least—to use an *existing* proper name in the role of b. Suppose, for example, that someone is doing a deduction in real analysis, and gets to a conclusion of the form $(\exists x)\Phi(x)$. Then he notices that the term '0' does not occur in any undischarged premise of the deduction. So he assumes that $\Phi(0)$. This fits the letter of the rule for invoking existential elimination, at least as the rule is formulated in most logic books. But that assumption would be misleading at best, and probably infelicitous. The term '0' already has a role in the language, namely that of denoting the real number zero, and it may well be that our formula Φ does not hold of zero. That is, $\Phi(0)$ may be false. Similarly, suppose I am thinking about basketball, and get to an intermediate conclusion that someone will be traded in the next month. It would then be weird to assume that LeBron James will be traded, to start the process of existential elimination.

What is usually done, in mathematics at least, is to introduce an *unused* letter for this purpose. And that is the advice given to those doing exercises in natural deduction. Suppose that in informal reasoning, one gets to a conclusion of the form $(\exists x)\Phi(x)$. The standard move is to say something like this: let b be a Φ (so that $\Phi(b)$). At some point, it is customary to remind the reader that 'b is arbitrary', or that b is an arbitrary Φ—usually at or near the point where we think of the assumption of $\Phi(b)$ as discharged. LeBron James is anything but an arbitrary NBA player, and zero is anything but an arbitrary real number. That is why it would be infelicitous to use those names in this context. As Frege and others pointed out, no *natural number* is 'arbitrary' in the sense that it has all and only the indicated properties (in this case of being a Φ). And so no name that denotes a particular number should be used in existential elimination. It seems to me that part of the problem here is to figure out what the locution 'arbitrary' means here. The 'arbitrariness' concerns the inferential role of the linguistic item, not the object supposedly denoted by the item.[8] In the reasoning in question, one is not to assume anything about b except $\Phi(b)$.

To model how the inference in question is deployed in real life, it might be better to follow some authors and introduce a new category of singular term for use in existential elimination (and in universal introduction, where the issues are similar). Call them *parameters* or *Skolem terms*. In some ways, parameters function as constants; in others they function as variables. In the case of existential elimination, we have it that some object in the domain satisfies Φ. The semantic role of the term b is to denote one such object. So in that sense it is like a constant. But it is crucial that we do not specify *which* such object, even if we could. The rules of engagement require the reasoner to avoid saying anything about b that does not hold of any object that satisfies Φ. In that sense, b functions more like a variable.

Notice that if we do have parameters available, there is no need to require that the introduced formula $\Phi(b)$ be an assumption to be discharged later.

[8] [Fine, 1985] is a very different account of the 'arbitrariness'. It postulates a category of arbitrary *objects*. Here we make no such metaphysical assumption. The arbitrariness here is more a matter of how the singular term is treated rather than a matter of what it denotes.

We can think of the first step of an existential elimination as a direct *inference*, as follows:

> from a formula in the form
>
> $$(\exists x)\Phi(x),$$
>
> one can infer
>
> $$\Phi(b),$$
>
> provided that the term b does not occur previously in the deduction. The inferred formula $\Phi(b)$ rests on whatever premises and assumptions the existential formula $(\exists x)\Phi(x)$ rests on.

Intuitively, the system is still sound, although stating this precisely and proving it can be tricky.[9]

It seems to me that this version of the rule of existential elimination fits mathematical practice slightly better than the more usual formulations do. As noted, one typically starts an existential elimination by saying something like 'let b be a Φ', rather than 'assume $\Phi(b)$'. Suppose that the reasoner goes on to deduce a sentence Ψ, which does not contain an instance of b. Typically, the reasoner does not repeat this conclusion and explicitly note the discharge of the assumption $\Phi(b)$. Instead, she just goes on from there, perhaps reminding the reader that b is arbitrary.

[9] A referee suggested the phrase 'Skolem term', since the role played by parameters is similar to that of the introduced constants in the Henkin proof of Gödel's completeness theorem (of which the Löwenheim-Skolem theorem is corollary). The idea is to extend a given, consistent theory, to one such that for each formula $\Phi(x)$ with only x free, there is a constant c such that $(\exists x)\Phi(x) \to \Phi(c)$ is a theorem. The constant c is to act as an arbitrary 'witness' for Φ. Admittedly, it is not crystal clear what the *semantic* role of the parameters is in the deductive system sketched here. Since they have the syntax of singular terms, for each interpretation of the language each parameter needs a denotation each time it is used. Here is one proposal: let M be an interpretation, giving a domain and extensions to the non-logical terminology other than the parameters. Let Π be an argumentative text. Suppose that $\Phi(b)$ is the first sentence in Π that contains the parameter b. Then, under M in Π, b denotes an object in the domain that is in the extension of Φ, if there is such an object, and it denotes any object in the domain otherwise. Of course, b should keep the same denotation in the rest of the text Π. We might invoke a choice function (or a Hilbert-style ϵ-operator) in the meta-theory to pick out the denoted object. Note that since we only make finitely many 'choices' in order to interpret any given argument, we do not rely ultimately on the axiom of choice. The referee pointed out that this sketch essentially recapitulates at a different level what is done in the Henkin proof of the completeness theorem: one shows that any model of the original theory can be extended to a model of the expanded system (with the new constants), thus showing that the latter is a conservative extension of the former (see, again, [van Dalen, 2004, pp. 136 ff], [Hodges, 1997], or just about any standard first text in mathematical logic). In the present context, this corresponds to a proof of the soundness of the indicated rule of inference.

The proposed rule of existential elimination corresponds to how the existential quantifier is handled in the methods of truth trees and semantic tableaux. If, in developing an argument, a reasoner reaches a sentence in the form $(\exists x)\Phi(x)$, she is to pick an unused constant (or free variable) b and write $\Phi(b)$, and then operate on that sentence. The idea, I suppose, is that b should stand for an object or any object that satisfies Φ (see, for example, [Smullyan, 1995]). Quine's *Methods of Logic* [1982, §30] invokes something similar as well. But, it seems, neither truth trees, semantic tableaux, nor Quine's procedures are intended to be models of actual deductive reasoning. They are more like abstract tests for validity. The claim here is that the rule of existential elimination, as reformulated here, does in some way reflect actual reasoning with existential statements.

One more move is required. We give the reasoner, or the community of language users, the option to let the introduced parameter b stay in the language, continuing to play the same role—that of standing for any object that satisfies the formula Φ from which it was introduced (if there is such an object). The introduction of a parameter into the language is much like the introduction of a new proper name, say via a baptism, but in this case, we do not indicate a specific, unique object to be the bearer of the name.

This, I submit, is a decent rational reconstruction of the actual use of 'i' in complex analysis. The mathematical community first notes that there is only one algebraic closure of the real numbers, up to isomorphism. Members of the community decide to study or otherwise discuss this algebraic closure. They note that, in this structure, there is at least one square root of -1:

$$(\exists x)(x^2 = -1).$$

So they let i be one such square root, and go on from there. We have that $i^2 = -1$. It follows that $-i$ is the only other square root of -1. One might note, in line with existential elimination, that there is nothing to be said about i that does not hold of every square root if -1. This is as it should be, since the two roots are indiscernible.

Brandom [1996, §6] proposes a similar, perhaps identical, resolution of the problem with 'i' (but without the direct reference to existential elimination).[10] He calls items like 'i', 'merely distinguishing terms', writing

[10] John Burgess once suggested a similar resolution of this issue to me. Richard Pettigrew [2008] coins the phrase 'dedicated parameter' for linguistic items originally introduced via something like existential elimination, but kept in the language for future use. He argues that virtually all of the singular terms of mathematics, such as '$\mathbb{N}$' and '0', should be understood this way, thus supporting an Aristotelian philosophy of mathematics, a sort of eliminative structuralism.

> Frege's practice . . . would seem to show that what matters for him is that we understand the proper use of the expressions we introduce: what commitments their use entails, and how we can become entitled to those commitments. We can be entitled to use merely distinguishing terms . . . provided we are careful never to make inferences that depend upon the . . . specifiablity of what is labeled—that is that our use of the [terms] respects the [automorphisms] that precluded such specifiability . . . [T]here is nothing mysterious about the rules governing [these terms].

Brandom concludes that

> looking hard at how complex numbers fit into Frege's theorizing in the philosophy of mathematics promises to teach us important lessons about the semantics of singular terms. This suggests a final lesson: the philosophy of mathematics must pay attention to the details of the actual structures it addresses. Semanticists, metaphysicians, and ontologists interested in mathematics cannot safely confine themselves, as so many have done, to looking only at the natural numbers.

Amen.

It seems that the linguistic community in question has gone on to allow the slight abuse of language (if it is an abuse), and declare that i is *the* square root of -1. They also can say that $-i$ is *the other* square root of -1. It is also said that, for any positive real number p, pi (and not $-pi$) is *the* square root of $-p^2$. As long as we know what we are doing, no harm ensues. Notice, however, that the two square roots of $-2i$ are $(1 - i)$ and $(-1 + i)$. As far as I know, no one is tempted to call one of them 'the' square root of $-2i$.

4. Matters Metaphysical: Universals and Particulars, Structures and Their Places

So just what *is* an *ante rem* structure, and how can mathematical objects be constituted by one? Consider another uncomfortable slogan proposed in [Shapiro, 1997]:

> Structures are prior to places in the same sense that any organization is prior to the offices that constitute it. The natural number structure is prior to '6', just as 'baseball defense' is prior to 'shortstop' or 'U.S. Government' is prior to 'Vice President'. (p. 9)

I also said many times that the individual natural numbers—the places in the structure—are not independent of each other. The problem is to articulate these notions of priority and dependence.

Note first that the priority or dependence invoked here is not epistemic or doxastic. I do not make any claims—here—about how structures and their places are known or how one comes to learn about them. It is more of a metaphysical priority and a metaphysical dependence.

In other contexts, one fairly obvious way to understand metaphysical priority is in terms of possible existence. To say that A is prior to B is to say that B could not exist without A. Or to say that the A's and the B's depend on each other is to say that there could not be A's without B's, and *vice versa*. Despite a proposal I broached tentatively in [Shapiro, 2006a], it seems that this will not do here, since mathematical objects exist of necessity if they exist at all (or so we shall assume). There is no sense of the natural-number structure existing without its places, nor *vice versa* for that matter. Nor is there any sense of the number four existing without the number six. Or so say our intuitions, or at least my intuitions.

Fraser MacBride [2005, p. 582] argues that *ante rem* structuralism is either bad news or old news. The news is bad if structuralism is committed to the identity of indiscernibles, and thus to $i = -i$. The old news is that *ante rem* structuralism is just traditional platonism. I take it that the charge of bad news is addressed above. As for *ante rem* structuralism being old news, I did concede, in the book, that *ante rem* structuralism is a variant of traditional platonism, at least concerning its ontology. It may only be a matter of emphasis. On almost all views in line with what I call realism in ontology, natural numbers exist, and so the natural numbers form a system, under the usual arithmetic relations. The structure of that system is the natural-number structure. What, if anything, is distinctive about *ante rem* structuralism, other than its focus on the structures rather than the individual objects?

The main metaphysical thesis of my book is that an *ante rem* structure is, or is akin to, an *ante rem* universal, in that it is a one-over-many. The same structure can be exemplified in multiple systems, and the structure exists independent of any exemplifications it may have in the non-mathematical realm. The difference between structures and the more usual kind of universal, such as properties, is that structures are the forms, not of individual objects, but of *systems*, *collections* of objects organized with certain relations.

Let us consider a simple example, the ordinal-three structure. This is the form of any system of three objects with a linear order. This structure has three places, each of which is to be filled by an object of a system that exemplifies it. One such system is my children, in birth order. In that system, Rachel occupies the first place of the structure, Yonah occupies the second place, and Aviva the third. The requisite relation is that of 'older than'. The places of an *ante rem* structure are akin to offices, which are occupied by various objects in various exemplifications of the structure. I called this the places-are-offices perspective. This perspective is invoked whenever one talks about a system, or systems, or possible systems, that exemplify a given structure.

Places within structures are not, as MacBride [2006a] suggests, bundles of universals. Places are *components* of universals. Each *ante rem* structure consists of some places and some relations. If this makes metaphysical sense, and I guess this may be a big 'if' in some metaphysical circles, then the dependence relation in the slogans for *ante rem* structuralism is that of constitution. A structure is constituted by its places and its relations, in the same way that any organization is constituted by its offices and the relations between them. The constitution is not that of mereology. It is not the case that a structure is just the sum of its places, since in general the places have to be related to each other via the relations of the structure. I think of an *ante rem* structure as a whole consisting of, or constituted by, its places and its relations.

To help clarify the metaphysical theses, consider an objection raised by Hellman [2001, pp. 193–194]:

> . . . on Shapiro's view . . . numerals denote the 'places' in a unique, archetypical structure answering to what all progressions have in common . . . But the 'places' of the *ante rem* archetype (call it $\langle N, \varphi, 1\rangle$) are entirely determined by the successor function φ, and derivative from it in the sense of being identified merely as the terms of the ordering induced by φ. Now in the case of an *in re* structure, we understand a particular successor relation in the ordinary way as arising from the given *relata*, reflecting, *e.g.*, an arrangement of some sort . . . But if the *relata* are not already given but depend for their very identity upon a given ordering, what content is there to talk of 'the ordering'? What can 'succession' *mean*, if we are abstracting from all *in re* cases *and* if we can't even speak of *relata* without making reference to the relation intended? . . . This, I submit, is a vicious circularity: in a nutshell, to understand the *relata*, we must be given the relation, but to understand the relation, we must already have access to the *relata*.

The first sentence in this passage is correct. On my view, numerals are singular terms denoting the places in the natural-number structure, which, in Hellman's colorful language, is the 'archetypical structure answering to what all progressions have in common'. I presume that the talk at the end of the passage, of how the 'relata' are to be 'understood', of what must be 'given', of what we have 'access to', and of what we manage to 'make reference to' are only metaphors for the underlying metaphysical themes. Hellman attributes to me the view that the 'relata'—the objects of mathematics—depend ('for their very identity') on the *relations* of the structure. In other words, the relations are metaphysically prior to the places, and the relations somehow fix the places.

This is not my view. Despite Hellman's charge, a view like this might make sense for the example under consideration, the natural-number structure (or the ordinal-three structure, or any other rigid structure). But the view that Hellman attributes to me makes no sense of the finite *cardinal* structures, or the degenerate graphs. The four places of the cardinal-four structure, for example, cannot depend on the relations of that structure, since there are not any relations in that structure (except for the identity or non-identity relation) on which the places can depend. As above, the metaphysical view is that a structure is constituted by its places *and* its relations. Neither the places nor the relations are prior to the other. Again, the slogan that the objects depend on the structure can be understood in that spirit. The dependence is constitution.

Ante rem structures are much like structural universals, whose existence remains subject to debate among metaphysicians.[11] To review some chemistry, each molecule of methane consists of one atom of carbon and four atoms of hydrogen, with the carbon atom bonded to each of the hydrogen atoms. One alleged structural universal is that of 'being a molecule of methane', which we can abbreviate '*methane*'. This complex universal has a number of components: one instance of the universal 'being an atom of carbon' (*carbon*), four instances of the universal 'being a molecule of hydrogen' (*hydrogen*), and four instances of the relation *bonding*. *Methane* looks much like a structure with five places and a relation of bonding that holds among the various places. This structural universal is also similar to a graph, or better, the structural universal has a form that is all but isomorphic to a graph. Structural universals are invoked for a number of metaphysical purposes, most of which are not all that relevant to mathematics.

David Lewis [1986] argues against structural universals. He provides three ways of understanding the nature of the constitution between a structural universal and its components, and finds them all lacking, or at least unable to serve the purposes that structural universals supposedly serve. The first, 'linguistic conception', takes a structural universal to be a set-theoretic construction of its component properties and relations, in the same sense in which 'a (parsed) linguistic expression can be taken as a set-theoretic construction out of its words'. The exact set-theoretic machinery does not matter, so long as the right distinctions can be made. Let c be the universal *carbon*, let h be the universal *hydrogen*, and let b be the binary universal of *being bonded*. Then the structural universal *methane* might be the following set of ordered pairs and ordered triples:

$$\{\langle c, 0\rangle, \langle h, 1\rangle, \langle h, 2\rangle, \langle h, 3\rangle, \langle h, 4\rangle, \langle b, 0, 1\rangle, \langle b, 0, 2\rangle, \langle b, 0, 3\rangle, \langle b, 0, 4\rangle\}.$$

The problem here, according to Lewis, is that structural universals, so construed, have to be made up ultimately of simple universals—universals that have no components. One of the main motivations for structural universals, it seems, is to allow for complex universals that, in a sense, are not ultimately built from simple universals. It may be that atoms, for example, are structural universals, consisting of, say, protons, electrons, and neutrons. And it may be that protons, for example, are structural universals composed of ...

[11] Thanks to Gabriel Uzquiano for directing me to this literature, and to Gabriel and Ben Caplan for helping me to negotiate it.

Lewis's argument here presupposes that the membership relation of set theory is well-founded, as it is on his own philosophy of mathematics.[12] A non-well-founded set theory would allow for the construction of structural universals that have no underlying simple universals. In any case, Lewis's criticism does not apply to the present conception of *ante rem* structures, since those do consist of places (and relations). The 'places' are indeed simple, or atomic, in the sense that they do not themselves have places or other components. 'Place' is a primitive of structure theory.

One potential problem with the linguistic conception of *ante rem* structures is that it presupposes set theory, which is itself a branch of mathematics. On the linguistic conception, we think of the natural-number structure, for example, as a set-theoretic construction out of a countable infinity of places and the successor function on those places. Can we maintain this, and then think of the set-theoretic hierarchy as itself an *ante rem* structure, and thus, a set-theoretic construction?

Lewis calls the second conception of structural universals 'pictorial'. The idea is that a structural universal is a mereological sum of its components. The universal, *methane*, would consist of the sum of the simpler universals, *carbon*, *hydrogen*, and *being bonded*. The problem here is that the structural universal *methane* does not just consist of the universal *hydrogen*, but of that universal four times over. A molecule of butane consists of four carbon atoms and ten hydrogen atoms in a certain bonding configuration. A molecule of iso-butane also consists of four carbon atoms and ten hydrogen atoms, but with a different configuration of bondings. It is an axiom of mereology that no distinct things can have exactly the same parts. On the pictorial conception, then, *methane*, 'being a molecule of butane', and *iso-butane*, 'being a molecule of iso-butane' must be the same structural universal, since they are all mereological sums with the same components. This is clearly a *reductio ad absurdum* of the existence of structural universals, if the latter are to be understood on the pictorial conception.

Lewis's conclusion here is consonant with my claim, above, that an *ante rem* structure is not a mereological sum of its places, since the places have to be related to each other via the relations of the structure. In general, the relations matter. Moreover, it is a central theme of this paper that the places are distinguished from each other, thus the crucial play with identity above. The cardinal-four structure is distinct from the cardinal-three structure, just because the former has four places while the latter has three.

[12] See, for example, [Lewis, 1993]. As Lewis sees it, the crucial, primitive relation of set theory (or mathematics generally) is that between an object o and its singleton $\{o\}$. Singletons are mereologically atomic, and, in general, a set is a mereological sum of its non-empty subsets.

Lewis's third conception is dubbed with the pejorative term 'magical'. Accordingly, 'a structural universal has no proper parts'; it is 'mereologically atomic'. Nor is it a set-theoretic construction of its 'components'. A structural universal is 'not a set but an individual'. A structural universal has 'components' only in a metaphorical sense:

> If we say that the universal *methane* consists of the universals *carbon*, *hydrogen*, and *bonded*, the most that we may mean is that an *instance* of *methane* must consist, in a certain way, of *instances* of the others. Involving, in turn, is a matter of necessary connection between the instantiating of one universal and the instantiating of another; and on the magical conception, the universals so connected are wholly distinct atomic individuals.
> Therein lies the magic. Why *must* it be that if something instantiates *methane*, then part of it must instantiate *carbon*? . . . [O]n the present conception, this necessary connection is just a brute fact. [Lewis, 1986, p. 41]

One of the purposes for the postulation of structural universals was to help explain necessary connections between the instantiated items. The structural universal *methane* is to help explain why methane necessarily consists of carbon. Lewis argues that on the magical conception, structural universals cannot do that explanatory work, since the connection between the structural universal *methane* and the simpler universal *carbon* is itself left unexplained. Of course, this matter is beyond present concern. I am not looking for a metaphysical explanation of, say, the fact that the number 3 is, of necessity, a natural number, or that 3 is less than 6.

D. M. Armstrong [1986], John Bigelow [1986], and Peter Forrest [1986] challenged Lewis's thesis that the relationship between a universal and its components must be either set-theoretic, mereological, or magical. Are those the only ways that relations or constructions or constitutions can be understood? More recently, Joan Pagès [2002] proposed that the relationships between structural universals and their components are 'formal', modeled on the connection between open formulas and structures that satisfy them. In other words, the relationships between structural universals and their components is modeled on the notion of satisfaction. This is reasonably close to how *ante rem* structures are characterized and defended in my book [Shapiro, 1997]. One main theme there is that understanding the (formal) languages of mathematics is sufficient to understand the places and relations of at least some structures.

One distinguishing feature of structural universals is that their components, or parts, or whatever one wants to call them, are themselves universals. *Methane* consists of *carbon*, *bonded*, *etc*., structured in a certain way. As noted, the places of *ante rem* structures are akin to offices in an organization. An office is a universal, of sorts, since it can be occupied by different individuals in different instances of the organization. The structures studied in mathematics are what I call *free-standing*. Anything at all can occupy their places. So perhaps it is not amiss to think of a place in an *ante rem* structure as itself a universal, albeit a rather trivial one. Each place of each structure can be instantiated by anything at all, in a system that exemplifies the structure. Perhaps a place in a structure is just the universal 'thing' or 'object' or 'particular'.

This is what I call the 'places-are-offices' orientation to structures. In my book, the main metaphysical/linguistic innovation—if you can call it that—is that there is another legitimate perspective from which one can construe the places of an *ante rem* structure as *bona fide objects*. I called this the 'places-are-objects' orientation.

Why think that inherent components of universals can be objects, or are legitimately construed as objects? The justification for my 'innovation' is the rather humdrum one given above. There is a legitimate perspective from which genuine singular terms denote the places of *ante rem* structures, and, more importantly, places are the range of bound variables in true theories. The perspective in question is that of pure mathematics, as interpreted here.

When we switch back to the places-are-objects perspective—and construe the places of a given structure as objects—we see that each structure exemplifies itself. That is, the relations of the structure hold of its places—when the latter are construed as objects.

Since we have broached the subject of platonism, let me conclude with a brief look at the old master, to revisit a theme in my book. Some writers in ancient Greece distinguished arithmetic, a pure discipline, from logistic, the practical art of calculation. The latter concerns measurement, business dealings, and the like (*e.g.*, [Proclus, 485, p. 39]; see [Heath, 1921, Ch. 1]). Plato makes a very different distinction. For him, both arithmetic and logistic are theoretical disciplines, concerned solely with the world of Being. Arithmetic 'deals with the even and the odd, with reference to how much each happens to be' (*Gorgias* 451A–C). If 'one becomes perfect in the arithmetical art', then 'he knows also all of the numbers' (*Theatetus* 198A-B; see also *Republic* VII 522C). For Plato, logistic also deals with the natural numbers, but differs from arithmetic 'in so far as it studies the even and the odd with respect to the multitude they make both with themselves and with each other' (*Gorgias* 451A–C, see also *Charmides* 165E–166B). In short, it seems that Plato thought that arithmetic deals with the natural numbers themselves, while logistic concerns the relations among the numbers. According to Jacob Klein [1968, p. 23], Plato's logistic 'raises to an explicit science that knowledge of relations among numbers which . . . precedes, and indeed must precede, all calculation'.

These passages are puzzling. What, exactly, is there to do in Plato's arithmetic? What on earth (or in the realm of Being) is there to know about the natural numbers other than their relations to each other? What would count, for Plato, as a theorem or proposition, or non-trivial truth of arithmetic? It is not that I'd expect Plato to be an *ante rem* structuralist—although if he were to be a structuralist, I expect that this is the kind he would be. But what could he possibly have had in mind for arithmetic, the study of the numbers, *in themselves*, and independent of their relations to other numbers?

I take it that Plato's logistic is what we would recognize as arithmetic, the *mathematical* theory of the natural numbers. Perhaps Plato's arithmetic is not a mathematical study at all. Rather Plato's arithmetic asks for a philosophical account of the natural numbers. It inquires after the metaphysical nature of the numbers. In developing, or at least hinting, at a description of natural numbers as places in an *ante rem* structure, a structure constituted by those places and the arithmetic relations on them, perhaps I am engaging in Plato's arithmetic, or at least I am if I am getting it right.

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