

# CATEGORICAL ONTOLOGY I

## EXISTENCE

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**ABSTRACT.** The present paper is the first piece of a series whose aim is to develop an approach to ontology and metaontology through category theory. We exploit the theory of *elementary toposes* to purport the claim that a satisfying “theory of existence”, and more at large ontology itself, can be obtained appealing to category theory. In this perspective, an ontology is a mathematical object: a category, the universe of discourse in which our mathematics (intended at large, as a theory of knowledge) can be deployed. The *internal language* that all categories possess prescribes the modes of existence for the objects of a fixed ontology/category.

Such internal language allows to re-enact propositional calculus: all first-order properties of an object can be modeled by propositions  $p : U \rightarrow \Omega$  valued in an object whose “elements” are the truth values that  $p$  can assume. Each ontology now  $\mathcal{E}$  yields its own  $\Omega_{\mathcal{E}}$ , and thus its own set of truth values, and thus its own “theory of meaning”. This approach resembles, but is more general than, fuzzy logics, as most choices of  $\mathcal{E}$  and thus of  $\Omega_{\mathcal{E}}$  yield nonclassical, many-valued logics.

Framed this way, ontology suddenly becomes more mathematical: a solid corpus of techniques can be used to backup philosophical intuition with a useful, modular language, suitable for a practical foundation. As both a test-bench for our theory, and a literary *divertissement*, we propose a possible category-theoretic solution of Borges’ famous paradoxes of Tlön’s “nine copper coins”, and of other seemingly paradoxical construction in his literary work. We then delve into the topic with some vistas on our future works.

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## 1. INTRODUCTION

El mundo, desgraciadamente, es real.

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[Bor53]

This is the first piece of a series of works that will hopefully span through a certain amount of time, and touch a pretty wide range of topics. Its purpose is to adopt a wide-ranging approach to a fragment of elementary problems in a certain branch of contemporary philosophy. In particular, our work aims to lay a foundation, or a transcription, of a few problems in ontology by means of pure mathematics; in particular, by means of the branch of mathematics known as category theory.

As authors, we are aware that such an ambitious statement of purpose must be adequately motivated. This is the scope of the initial section of the present first manuscript.

**1.1. What is this series.** Since forever, mathematics studies three fundamental indefinites: form, measure, and inference. Apperception makes us recognise that there are extended entities in space, persisting in time. From this, the necessity to measure how much these entities are extended, and to build a web of conceptual relations between them, explaining how they arrange “logically”. Contaminations between these three archetypal processes are certainly possible and common; mathematics happens exactly at the crossroad where algebra, geometry and logic intersect.

We can even say more: mathematics is a language; metamathematics done through mathematics (if such a thing even exists) exhibits the features of a *ur-language*, a generative scheme for “all” possible languages. It is a language whose elements are the rules to give oneself a language, conveying information, and allowing deduction. It is a metaobject: a scheme to generate objects/languages.

Taken this tentative definition, mathematics (not its history, not its philosophy, but its *practice*) serves as a powerful tool to tackle the essential questions of ontology: what “things” are, what makes them what they are and not different.

Quantitative thinking is a consistent (some would say “honest”) way of approaching this deep problem, cogency of entities; however, it is undeniable that a certain philosophical debate has become hostile to mathematical language. A tear in the veil that occurred a long time ago, due to different purposes and different specific vocabulary, can not be repaired by two people only. If, however, the reader of these notes asks for an extended motivation for our work, a wide-ranging project in which it fits, a long-term goal, in short a *program*, they will find it now: there is a piece of mathematics whose purpose is to solve philosophical problems, in the same way certain mathematics “solves” the motion of celestial bodies. It does not completely annihilate the question: it proposes models within which one can reformulate it; it highlights what is trivial consequence of the axioms of that model and what instead is not, and requires the language to expand, to be modified, sharpened. Our aim is to approach this never-mentioned discipline as mathematicians.

Sure, solving once and for all the problems posited by ontology would be megalomaniac; we do not claim such a thing. Instead, more modestly, we propose a starting point unhinging

some well-established beliefs (above all else, the idea that ontology is too general to be approached quantitatively); we humbly point the finger at some problems that prose is unable to notice, because it lacks a specific and technical language; we suggest that *in such a language*, when words mean precise things and are tools of epistemic research instead of mere magic spells, a few essential questions of recent ontology dissolve in a thread of smoke, and others simply become “the wrong question”: not false, just meaningless as the endeavour to attribute a temperature to consciousness or justice.

We shall say at the outset that such a language is not mathematics; but mathematics has an enormous hygienic capacity to hint at what the *characteristica universalis* should be made of.

It may seem suspicious to employ mathematics to tackle questions that traditionally pertain to philosophy, we believe we can provide a more adequate language, taken from mathematics, within which to frame some deep ontological questions. The reader will allow a tongue-in-cheek here: when coming to ontology, it is not a matter of making a *correct use of language*, but rather an *use of the correct language*.

This language is inherently mathematical, as only mathematics has the power to substantiate an analysis in quantitative terms. This language is *category theory*, as only category theory has the power to speak about totality of “all” structures of a given kind in a compelling way, treating mathematical *theories* as mathematical *objects*.

As it is currently organised, our work will attempt to cover the following topics:

- the present manuscript, *Existence*, aims at providing a sufficiently expressive theory of existence. Having a foundational rôle, the scope of most of our remarks is of course more wide-ranging and aimed at building the fundamentals of our toolset (mainly, category theory and categorical logic, as developed in [MLM92, Joh77, LS88]). As both a test-bench for our theory, and a literary *divertissement*, we propose a category-theoretic solution of Borges’ paradoxes present in [Bor13].
- A second chapter [LDa], currently in preparation, addresses the problem of *identity*, and in particular its context-dependent nature. Our proof of concept here consists of a solution to Black’s ancient “two spheres” paradox; this time the solution is provided by Klein’s famous *Erlangen programme* group-theoretic devices. The two interlocutors of Black’s imaginary dialogue inhabit respectively an Euclidean and an affine world: this affects their perception of the “two” spheres, and irremediably prevents them from mutual understanding.
- A third chapter [LDb], currently in preparation, addresses again the problem of identity, but this time through the lens of algebraic topology; in recent years homotopy theory defied a well-established ontological assumptions such as the identity principle; the many commonalities between category theory and homotopy theory suggest that “identity” is not a primitive concept, but instead depends on our concrete representation of mathematical entities. When  $X, Y$  are objects in a category  $\mathcal{C}$ , there is often a class of “equivalences”  $W \subseteq \text{hom}(\mathcal{C})$  prescribing that  $X, Y$  shouldn’t be distinguished; equality (better, some sort of homotopy *equivalence*) is then defined

*ex post* in terms of  $W$ , changing as the ambient category  $\mathcal{C}$  does; this yields a  $W$ -parametric notion of identity  $\equiv_W$ , allowing categories to be categorified versions of *Bishop sets*, i.e. pairs  $(S, \rho)$  where  $\rho$  is an equivalence relation on  $S$  prescribing a  $\rho$ -equality.

**1.2. On our choice of metatheory and foundation.** During '900 the direction of the evolution of mathematics brought the discipline to divide in different sub-disciplines at first, with their own specific objects and languages, to then find unexpected unification under a one and only notion, the notion of *structure*, using the formal tool that has best characterized the concept, the *categories*. This process has spontaneously led to epistemological revision of mathematics and has inspired, in the development of operational tools, a revision of both his foundations and his ontology. For many scholars is undeniable that

[the] mathematical uses of the tool CT and epistemological considerations having CT as their object cannot be separated, neither historically nor philosophically. [Krö07]

This happened regardless of both the specific foundational debate [].

The mathematical practice in the structuralist way produces a "natural" ontology, that same, later, felt the need to characterized more precisely. After all, similarly to what Carnap<sup>1</sup> suggested regarding semantics,

mathematicians creating their discipline were apparently not seeking to justify the constitution of the objects studied by making assumptions as to their ontology. [Krö07]

Beyond the attempts (ever those th Bourbaki group), what matter is that the habit of reasoning in terms of structure has suggested implicit epistemological and ontological attitudes. This matter would deserve an exhausting independent inquiry.

For our objectives it's enough to declare a differentiation that Krömer elaborate, inspired by [Corry, 1996]: the different between **structuralism** and **structural mathematics**:

- (1) Structuralism: *the philosophical position regarding structures as the subject matter of mathematics*
- (2) Structural Mathematics: *the methodological approach to look in a given problem "for the structure"*

<sup>1</sup>Some words that philosophers should keep in mind, on the lawfulness of the use of abstract entities (specifically mathematical) in semantic reflection, also valid in ontology:

we take the position that the introduction of the new ways of speaking does not need any theoretical justification because it does not imply any assertion of reality [...]. it is a practical, not a theoretical question; it is the question of whether or not to accept the new linguistic forms. The acceptance cannot be judged as being either true or false because it is not an assertion. It can only be judged as being more or less expedient, fruitful, conducive to the aim for which the language is intended. Judgments of this kind supply the motivation for the decision of accepting or rejecting the kind of entities. [Car56]

**Remark 1.1.** (1) implies (2) but the opposite is not always true.

That is, one can do structural mathematics without being structuralists; that is, taking different, or even opposite, positions with respect to structuralism itself. That it has often come spontaneously to embrace this position in the theoretical reflection in the recent history of mathematics is quite different (the "natural" ontology).

The use of CT as meta-language, despite the historical link with structuralism, it doesn't nevertheless make automatic the transition from (2) to (1), but suggests that the ontology is not only dependent to "ideology" (in quinean sense) of the theory, namely its expressive power, but is influenced by the epistemological model inspired by use of formal language itself.

The usefulness of Krömer's distinction, however, is another: instead of stumbling in a possibly not ambiguous definition of *structure* (with the unwanted consequences that could arise in the operational practice), the philosophy (1) can be reduced (or redefine) to the methodology (2), saying that:

*structuralism is the claim that mathematics is essentially structural mathematics* [Krö07]

(the operative practice that "intervenes" in definition of structuralism avoids the decade-long debate of the humanities on this concept).

This is the same thing as saying: the structural practice is its philosophy itself.

The historical attempts of explaining the term "structure" by Bourbaki in the years following publication of the *Elements* [], had the first systematic elaboration of a philosophy that could accord with rate of success of *structural mathematics*. His target it is to "*assembling of all possible ways in which given set can be endowed with certain structure*" [Krö07], and to do so elaborate, in the programmatic paper *The Architecture of Mathematics* (written by Dieudonné alone), published in 1950, a formal strategy. While specifying that "*this definition is not sufficiently general for the needs of mathematics*" [Bourbaki, 1950], he codifies a series of operational steps through which a structure on set is "assembled set-theoretically". Adopting therefore a reductionist perspective in which

the structureless sets are the raw material of structure building which in Bourbaki's analysis is "unearthed" in a quasi-archaeological, reverse manner; they are the most general objects which can, in a rewriting from scratch of mathematics, successively be endowed with ever more special and richer structures. [Krö07]

On balance in the bourbakist structuralism the notion of *set* doesn't disappear definitively in front of the notion of structure. The path towards an "integral" structuralism was still long.

In his influential paper [Law66] William Lawvere proposes a foundation of mathematics based on category theory. To appreciate the depth and breadth of such an impressive piece

of work, however, the word “foundation” must be taken in the particular sense intended by mathematicians:

[...] a single system of first-order axioms in which all usual mathematical objects can be defined and all their usual properties proved.

Such a position sounds at the same time a bit cryptic to unravel, and unsatisfactory; Lawvere’s (and others’) stance on the matter is that a foundation of mathematics is *de facto* just a set  $\mathcal{L}$  of first order axioms organised in a Gentzen-like deduction system. The deductive system so generated reproduces mathematics as we know and practice it, i.e. provides a formalisation for something that already exists and needs no further explanation, and that we call “mathematics”.

It is not a vacuous truth that  $\mathcal{L}$  exists somewhere: the fact that the theory so determined has a nontrivial model, i.e the fact that it can be interpreted inside a given familiar structure, is both the key assumption we make, and the less relevant aspect of the construction itself; showing that  $\mathcal{L}$  “has a model” is –although slightly improperly– meant to ensure that, *assuming the existence of a naive set theory* (i.e., assuming the prior existence of structures called “sets”), axioms of  $\mathcal{L}$  can be satisfied by a naive set. Alternatively, and more crudely: assuming the existence of a model of ZFC,  $\mathcal{L}$  has a model *inside that model of ZFC*.<sup>2</sup>

A series of works attempting to unhinge some aspects of ontology through category theory should at least try to tackle such a simple and yet diabolic question as “where” are the symbols forming the first-order theory of ETCC. And yet, everyone just believes in sets, and solves the issue of “where” they are with a leap of faith from which all else follows.

This might appear somewhat circular: aren’t sets in themselves already a mathematical object? How can they be a piece of the theory they aim to be a foundation of? In his [Lol77] the author addresses the problem as follows:

Quando un matematico parla di modelli non ha infatti l’impressione di uscire dall’ambito insiemistico. Questa impressione, che è corretta, è giustificata dalla possibilità di rappresentare i linguaggi formali con gli oggetti della teoria degli insiemire di studiare in essa le relazioni tra le strutture e le rappresentazioni dei simboli. Quando l’attenzione è rivolta ai modelli di una teoria degli insiemi, certe questioni sofisticate non possono però essere più evitate. La domanda spontanea sulla relazione che intercorre tra gli insiemi che sono modelli di una teoria e gli insiemi di cui parla la teoria non è altro che una domanda sulla relazione tra la metateoria semantica insiemistica e la teoria in esame, o teoria oggetto. Le due teorie possono coincidere, anzi

<sup>2</sup>However, ensuring that a given theory has a model isn’t driven by psychological purposes only: on the one hand, purely syntactic mathematics would be very difficult to parse, as opposed to the unformalised, more colloquial practice of mathematical development; on the other hand (and this is more important), the only thing syntax can see is equality, and truth. To prove that a given statement is false, one either checks all possible syntactic derivations leading to  $\varphi$ , finding none –this is unpractical, to say the least– or they *find a model* where  $\neg\varphi$  holds.

la metateoria può essere anche una sottoteoria propria della teoria oggetto.  
[...]

Si sceglie un insieme i cui elementi, di solito insiemi finiti, rappresentano i simboli del linguaggio che si vuole studiare, quindi con una operazione di concatenazione che può essere la coppia ordinata si definisce l'insieme delle parole  $L$ -delle formule ben formate, dei termini, l'operazione che a una formula associa le sue variabili libere e così via per tutte le nozioni sintattiche.

<sup>a</sup>

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<sup>a</sup>Authors' translation:

The idea that a subtheory  $L' \subset L$  of the object theory can play the rôle of metatheory might appear baffling; in practice, the submodel doesn't exactly play such rôle: instead, there are two possible solutions. Pure platonism assumes the existence of a hierarchy of universes harbouring the object theory; pure syntacticism exploits Gödel's completeness theorem: every proof is a finite object, and every theorem proved in the metatheory is just a finite string of symbols. No need for a model.

Platonism has limits: in a fixed a class theory  $C$  (MK, Morse-Mostowski; or NBG, Von Neumann-Bernays-Gödel), there's an object  $V$  that plays the rôle of the universe of sets; in  $V$ , all mathematics can be enacted. Of course, consistency of  $C$  is only granted by an act of faith.

Syntacticism has limits: following it, one abjures any universality mathematics might claim. But syntacticism also has merits: undeniably (disgracefully, luckily) the World is real. And reality is complex enough to contain languages as purely syntactical objects; the percussion of a log with oxen bones, rather than prophecies over the entrails of a lamb, or intuitionistic type theory, all have the same purpose: intersubjective convection of meaning, deduced by a bundle of perceptions, so to gain advantage, i.e. predictive power, over them. Of course: intuitionistic type theory is just *slightly* more effective than hepatomania.

Let's say at the outset this point of view makes absolutely no claim of originality; research on this matter is fervid in communities other than the philosophical one (cf. []).

Accepting that knowledge is obtained by collision and retro-propagation between Reality and the perceptual bundle it generates, the urge to define seemingly abstract concepts like learning, conscience, and knowledge, together with precious continuous feedback coming from real objects, evidently determine an undeniable primacy of quantitative thinking, this time intended as machine learning and artificial intelligence, that sets (it should, if only more philosophers knew linear algebra) a new bar for research in philosophy of mind.

However, we refrain from entering such a deep rabbit-hole, as it would have catastrophic consequences on the quality, length, and depth of our exposition.

The usual choice for mathematicians is to assume that, wherever and whatever they are, these symbols "are", and our rôle in unveiling mathematics is *descriptive* rather than generative <sup>3</sup>

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<sup>3</sup>Dal punto di vista costruttivista non si può legittimamente affermare che gli assiomi "creino" gli oggetti matematici, dal che la risposta alla domanda "dove sono?" sarebbe "da nessuna parte". L'unica cosa che si

This state of affairs has, to the best of our moderate knowledge on the subject, various possible explanations:

- On one hand, it constitutes the heritage of Bourbaki’s authoritarian stance on formalism in pure mathematics;
- on the other hand, a different position would result in barely no difference for the “working class”; mathematicians are irreducible pragmatists, somewhat blind to the consequences of their philosophical stances.

So, symbols and letters do not exist outside of the Gentzen-like deductive system we specified together with  $\mathcal{L}$ .

As arid as it may seem, this perspective proved itself to be quite useful in working mathematics; consider for example the type declaration rules of a typed functional programming language: such a concise declaration as

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data Nat = Z | S Nat
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makes no assumption on “what”  $Z$  and  $S :: \text{Nat} \rightarrow \text{Nat}$  are; instead, it treats these constructors as meaningful formally (in terms of the admissible derivations a well-formed expression is subject to) and intuitively (in terms of the fact that they model natural numbers: every data structure that has those two constructors must be the type  $\mathbb{N}$  of natural numbers).

In altre parole possiamo tradurre “formally” con “synctactically” (di modo che, ad esempio, gli assiomi di  $PA$  sono significanti per via delle derivazioni formali che ci permettono di fare) e “intuitively” con “semantically” (gli assiomi di  $PA$  si presume abbiano almeno un modello, quello standard).

Taken as an operative rule, this reveals exactly what is our stance towards foundations: we are “structuralist in the metatheory”, meaning that we treat the symbols of a first-order theory or the constructors of a type system irregardless of their origin, provided the same relation occur between criptomorphic collections of labeled atoms.

**Remark 1.2.** Ergo per noi nel metalinguaggio gli oggetti del linguaggio sono strutture (o li trattiamo come tali), sospendendo il giudizio su cosa effettivamente *siano* fuori dal metalinguaggio. Possiamo intuitivamente schematizzare così:

$L$	Objects	Denotation	Ontology
<b>Language</b>	Categories	Theories	??
<b>Metalinguage</b>	Categories	“Places”	Structures

Da qui già si può notare, e lo si dirà ancora in 1.4, come questa impostazione non comporti affatto sostenere una forma di strutturalismo metafisico “forte”, solo una versione “weak” a livello metalinguistico.

E’ d’aiuto essersi posti in un’ottica relazionale, implicita nella teoria che usiamo (sul rapporto stretto tra strutturalismo e uso di ontologie relazionali [cfr. [? ]]) e quindi possiamo

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può dire certamente è che essi “precisano, in modo rigoroso ed esatto, ancorché implicito, il *significato* degli oggetti matematici” [Agazzi,[? ]] (e sembra che anche in matematica, come è uso in filosofia del linguaggio, sia assennato tenere ferma la distinzione tra significato di un’espressione e sua denotazione). Noi ci fermiamo su questa soglia.



parlare di relazioni tra oggetti senza dire cosa siano gli oggetti, se non, formalmente, i termini posti ai lati estremi dei funtori. Sappiamo dire cosa sono gli oggetti della teoria dal punto di vista della metateoria, in base a come li trattiamo nella pratica operativa (cioè strutture e relazioni tra esse), ma non cosa sono *nella* teoria, se non simboli, sulla cui fondazione non ci pronunciamo.

In this precise sense we are thus structuralists in the metatheory, and yet we do so with a grain of salt, maintaining a transparent approach to the consequences and limits of this partialisation. On the one hand, pragmatism works <sup>4</sup>; it generates rules of evaluation for the truth of sentences. On the other hand, this sounds like a Munchhausen-like explanation of its the value, in terms of itself. Yet there seems to be no way to do better: answering the initial question would give no less than a foundation of language.

And this for no other reason that “our” metatheory is something near to a structuralist theory of language; thus, a foundation for such a metatheory shall inhabit a meta-metatheory... and so on.

Thus, rather than trying to revert this state of affairs we silently comply to it as everyone else does; but we feel contempt after a brief and honest declaration of intents towards where our metatheory lives. Such a metatheory hinges again on work of Lawvere, and especially on the series of works on functorial semantics.

**1.3. Categories as places.** The present section has double, complementary purposes: we would like to narrow the discussion down to the particular flavour in which we interpret the word “category”, but also to expand its meaning so as to encompass its rôle as a foundation for mathematics. More or less, the idea is that a category is both an algebraic structure (a microcosm) and a metastructure in which *all* other algebraic structures can be interpreted (a macrocosm).<sup>5</sup>

More in detail, a category provides with a sound graphical representation of the defining operation of a certain type of structure  $\mathfrak{T}$  (see [Definition 1.3](#) below; we take the word *universal* in the sense of [[Gri07](#), XV.1]). Such a perspective allows to concretely build an object representing a given (fragment of ) a language  $L$ , and a topos (see [Definition A.2](#)) obtained as sort of a universal semantic interpretation of  $L$  as internal language. This construction is a classical piece of categorical logic, and will not be recalled here: the reader is invited to consult [[LS88](#), II.12, 13, 14], and in particular

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<sup>4</sup>Ed invero noi stiamo chiedendo agli ontologi di diventare “pragmatisti”, o operativisti

<sup>5</sup>As an aside, we shall at least mention the dangers of too much a naive approach towards the micro/macrocosm dichotomy: if all algebraic structures can be interpreted in a category, and categories are algebraic structures, there surely is such a thing as the theory of categories *internal* to a given one. And large categories shall be thought as categories internal to the “meta”category (unfortunate but unavoidable name) of categories. There surely is a well-developed and expressive theory of internal categories (see [[Bor94a](#), 8]); but our reader surely has understood that the two “categories”, albeit bearing the same name, shall be considered on totally different grounds: one merely is a structure; the other is a foundation for that, and others, structure.

J. Lambek proposed to use the *free topos* [on a type theory/language] as ambient world to do mathematics in; [...] Being syntactically constructed, but universally determined, with higher-order intuitionistic type theory as internal language, [Lambek] saw [this structure] as a reconciliation of the three classical schools of philosophy of mathematics, namely formalism, platonism, and intuitionism.

In light of this, and in light of our subsection A.3, we stress once again that categories live on different, almost opposite, grounds: as syntactic objects, that can be used to model *language*, and as semantic objects, that can be used to model *meaning*.

Interpretation, as defined in logical semantics [Gam91], can be seen as a function  $t : L^* \rightarrow \Omega$  that associates elements of a set  $\Omega$  to the free variables of a formula  $\alpha$  in a language  $L$ ; along the history of category theory, subsequent refinements of this fundamental idea led to revolutionary notions as that of functorial semantics or (elementary) topos. As an aside, it shall be noted that the impulse towards this research was somewhat motivated by the refusal of set-theoretic foundations, opposed to type-theoretic ones.

In the following subsection, we give a more fine-grained presentation of the philosophical consequences that a “metatheoretic structural” perspective has on mathematical ontology.

1.3.1. *Theories and their models.* In [Law66] the author W. Lawvere builds a formal language ETAC encompassing “elementary” category theory, and a theory ETCC for the category of all categories, yielding a model for ETAC. In this perspective category theory has a syntax CT; in CT, categories are terms. In addition, we are provided with a *metatheory*, in which we can consider categories of categories, etc.:

If  $\Phi$  is any theorem of elementary theory of abstract categories, then  $\forall \mathcal{A}(\mathcal{A} \models \Phi)$  is a theorem of basic theory of category of all categories. [Law66]

After this, the author makes the rather ambiguous statement that “*every object in a world described by basic theory is, at least, a category*”. This is a key observation: what is the world described by ETAC, what are its elements?

We posit that the statement shall be interpreted as follows: categories in mathematics carry a double nature. They surely are the structures in which the entities we are interested to describe organise themselves; but on the other hand, they inhabit a single, big (meta)category of all categories. Such a big structure is fixed once and for all, at the outset of our discussion, and it is the *place* in which we can provide concrete models for “small” categories. To fix ideas with a particular example: we posit that there surely is such a thing as “the category of groups”. But on the other hand, groups are just very specific kinds of sets, so groups are but a substructure of the only category that exists.

Sure, such an approach is quite unsatisfactory in a structural perspective. It bestows the category of sets with a privileged role that it does not have: sets are just *one* of the possible choices for a foundation of mathematics. Instead, we would like to disengage the (purely syntactic) notion of structure from the (semantical) notion of interpretation.

The “categories as places” philosophy now provides such a disengagement, in order to approach the foundation of mathematics agnostically.

In this perspective one can easily fit various research tracks in categorical algebra, [AA.04], functorial semantics [Law63, HP07], categorical logic [LS88], and topos theory [Joh77] that characterised the last sixty years of research in category theory.

Lawvere’s *functorial semantics* was introduced in the author’s PhD thesis [Law63] in order to provide a categorical axiomatisation of universal algebra, the part of mathematical logic whose subject is the abstract notion of mathematical structure: a semi-classical reference for universal algebra, mingled with a structuralist perspective, is [Man12]; see also [SB81]. We shall say at the outset that a more detailed and technical presentation of the basic ideas of functorial semantics is given in our subsection A.3 below; here we aim neither at completeness nor at self-containment.

Everything starts with the following definition:

**Definition 1.3.** A *type*  $\mathfrak{T}$  of universal algebra is a pair  $(T, \underline{\alpha})$  where  $T$  is a set called the (*algebraic*) *signature* of the theory, and  $\underline{\alpha}$  a function  $T \rightarrow \mathbb{N}$  that assigns to every element  $t \in T$  a natural number  $n_t : \mathbb{N}$  called the *arity* of the function symbol  $t$ .

**Definition 1.4.** A (*universal*) *algebra* of type  $\mathfrak{T}$  is a pair  $(A, f^A)$  where  $A$  is a set and  $f^A : T \rightarrow \prod_{t \in T} \text{Set}(A^{n_t}, A)$  is a function that sends every function symbol  $t : T$  to a function  $f_t^A : A^{n_t} \rightarrow A$ ;  $f_t^A$  is called the  $n_t$ -ary operation on  $A$  associated to the function symbol  $t : T$ .

We could evidently have replaced  $\text{Set}$  with another category  $\mathcal{C}$  of our choice, provided the object  $A^n : \mathcal{C}$  still has a meaning for every  $n : \mathbb{N}$  (to this end, it suffices that  $\mathcal{C}$  has finite products; we call such a  $\mathcal{C}$  a *Cartesian* category). A universal algebra of type  $\mathfrak{T}$  in  $\mathcal{C}$  is now a pair  $(A, f^A)$  where  $A : \mathcal{C}$  and  $f^A : T \rightarrow \prod_{t \in T} \mathcal{C}(A^{n_t}, A)$ ; it is however possible to go even further, enlarging the notion of “type of algebra” even more.

The abstract structure we are trying to classify is a *sketch* (the terminology is neither new nor unexplicative: see [Ehr68, CL84, Bor94b]) representing the most general arrangement of operations  $f^A : A^n \rightarrow A$  and properties thereof<sup>6</sup> that coexist in an object  $A$ ; such a sketch is pictorially represented as a (rooted directed) graph, modeling arities of the various function symbols determining a given type of algebra  $\mathfrak{T}$  (see also [Gri07, XV.3] for the definition of *variety of algebras*).

Given the theory  $\mathfrak{T}$  and the graph  $G_{\mathfrak{T}}$  that it represents, the category  $\mathcal{L}_{\mathfrak{T}}$  generated by  $G_{\mathfrak{T}}$  “is” the theory we aimed to study, and every functor  $A : \mathcal{L}_{\mathfrak{T}} \rightarrow \text{Set}$  with the property that  $A([n+m]) \cong A[n] \times A[m]$  concretely realises via its image a representation of  $\mathfrak{T}$  in  $\text{Set}$ .

More concretely, there is a “theory of groups”. Such a theory determines a graph  $G_{\text{Grp}}$  built in such a way to generate a category  $\mathcal{L} = \mathcal{L}_{\text{Grp}}$  with finite products. *Models* of the theory of groups are functors  $\mathcal{L} \rightarrow \text{Set}$  uniquely determined by the image of the “generating object” [1] (the set  $G = G[1]$  is the underlying set, or the *carrier* of the algebraic structure in study; in our Definition 1.3 the carrier is just the first member of the pair  $(A, f^A)$ ); the

<sup>6</sup>Examples of such properties are (left) alternativity: for all  $x, y, z$ , one has  $f^A(x, f^A(x, y)) = f^A(f^A(x, x), y)$ ; associativity:  $f^A(x, f^A(y, z)) = f^A(f^A(x, y), z)$ ; commutativity:  $f^A(x, y) = f^A(y, x)$ ; and so on.

request that  $G$  is a product preserving functor entails that if  $\mathcal{L}$  is a theory and  $G : \mathcal{L} \rightarrow \mathbf{Set}$  one of its models, we must have  $G[n] = G^n = G \times G \times \cdots \times G$ , and thus each function symbol  $f : [n] \rightarrow [1]$  describing an abstract operation on  $G$  receives an interpretation as a concrete function  $f : G^n \rightarrow G$ .

Until now, we interpreted our theory  $\mathcal{L}$  in sets; but we could have chosen a different category  $\mathcal{C}$  at no additional cost, provided  $\mathcal{C}$  was endowed with finite products, in order to speak of the object  $A^n = A \times \cdots \times A$  for all  $n : \mathbb{N}$ . In this fashion, we obtain the  $\mathcal{C}$ -models of  $\mathcal{L}$ , instead of its  $\mathbf{Set}$ -models: formally, and conceptually, the difference is all there.

Yet, the freedom to disengage language and meaning visibly has deep consequences: suddenly, and quite miraculously, we are allowed to speak of groups internal to the category of sets, i.e. functors  $\mathcal{L}_{\mathbf{Grp}} \rightarrow \mathbf{Set}$ , topological groups, i.e. functors  $\mathcal{L}_{\mathbf{Grp}} \rightarrow \mathbf{Top}$  (so multiplication and inversion are continuous maps *by this very choice*, without additional requests); we can treat monoids in the category of  $R$ -modules, i.e.  $R$ -algebras  $\llbracket$ , and monoids in the category of posets  $\llbracket$  (i.e. quantales  $\llbracket$ ) all on the same conceptual ground.

**1.3.2. The rôle of toposes.** Among many different Cartesian categories in which we can interpret a given theory  $\mathcal{L}$ , toposes play a special rôle; this is mostly due to the fact that the *internal language* every topos carries (in the sense of [Definition 3.1](#)) is quite expressive.


To every theory  $\mathcal{L}$  one can associate a category, called the *free topos*  $\mathcal{E}(\mathcal{L})$  on the theory (see [\[LS88\]](#)), such that there is a natural bijection between the  $\mathcal{F}$ -models of  $\mathcal{L}$  and (a suitable choice of) morphisms of toposes<sup>7</sup>  $\mathcal{E}(\mathcal{L}) \rightarrow \mathcal{F}$ :

$$\mathbf{Mod}(\mathcal{L}, \mathcal{F}) \cong \mathbf{hom}(\mathcal{E}(\mathcal{L}), \mathcal{F}).$$

In the present subsection we analyse how the construction of models of  $\mathcal{L}$  behaves when the semantics takes value in a category of presheaves.

Let's start stating a plain tautology, that still works as blatant motivation for our interest in toposes opposed as more general categories for our semantics. Sets can be canonically identified with the category  $[1, \mathbf{Set}]$ , so models of  $\mathcal{L}$  are tautologically identified to its  $[1, \mathbf{Set}]$ -models. It is then quite natural to wonder what  $\mathcal{L}$ -models become when the semantics is taken in more general functor categories like  $[C, \mathbf{Set}]$ . This generalisation is compelling to our discussion: in case  $C$  is a discrete category, we get back the well-known category of variable sets  $\mathbf{Set}/C$  of [Proposition 2.1](#).

Now, it turns out that  $[C, \mathbf{Set}]$ -model for an algebraic theory  $\mathcal{L}$ , defined as functors  $\mathcal{L} \rightarrow [C, \mathbf{Set}]$  preserving finite products, correspond precisely to functors  $C \rightarrow \mathbf{Set}$  such that each  $Fc$  is a  $\mathcal{L}$ -model: this gives rise to the following “commutative property” for semantic interpretation:

  $\mathcal{L}$ -models in  $[C, \mathbf{Set}]$  are precisely those models  $C \rightarrow \mathbf{Set}$  that take value in the subcategory  $\mathbf{Mod}_{\mathcal{L}}(\mathbf{Set})$  of models for  $\mathcal{L}$ . In other words we can “shift”

<sup>7</sup>We refrain to enter the details of the definition of a morphism of toposes, but we glimpse at the definition: given two toposes  $\mathcal{E}, \mathcal{F}$  a morphism  $(f^*, f_*) : \mathcal{E} \rightarrow \mathcal{F}$  consists of a pair of *adjoint* functors (see [\[Bor94a, 3\]](#))  $f^* : \mathcal{E} \rightleftarrows \mathcal{F} : f_*$  with the property that  $f^*$  commutes with finite limits (see [\[Bor94a, 2.8.2\]](#))

the  $\text{Mod}(-)$  construction in and out  $[C, \text{Set}]$  at our will:

$$\text{Mod}_{\mathcal{L}(\Omega)}([C, \text{Set}]) \cong [C, \text{Mod}_{\mathcal{L}(\Omega)}(\text{Set})]$$

As the reader can see, the procedure of interpreting a given “theory” inside an abstract finitely complete category  $\mathcal{K}$  is something that is only possible when the wrd theory is interpreted as a category, and when a model of the theory as a functor. This discipline goes under many names: the one we will employ, i.e. *categorical*, or *functorial*, semantics [Law63], *internalisation* of structures, *categorical algebra*.

The internalisation paradigm sketched above suggests how “small” mathematicians often happily develop their mathematics without ever exiting a single (large) finitely complete category  $\mathcal{K}$ , without even suspecting the presence of models for their theories outside  $\mathcal{K}$ . To a category theorist, “groups” as abstract structures behave similarly to the disciples of the sect of the Phoenix [Bor]: “the name by which they are known to the world is not the same as the one they themselves pronounce.” They are a different, deeper structure than the one intended by their users.

By leaving the somewhat unsatisfying picture that “all categories are small” and by fixing a semantic universe like  $\text{Set}$ , *each* category works as a world in which one can speak mathematical language (i.e. “study models for the theory of  $\Omega$ -structures” as long as  $\Omega$  runs over all possible theories).

As already hinted in ?? categories exhibit a double nature: they are the theories we want to study, but they also are the *places* where we want to realise those theories; looking from high enough, there is plenty of other places where one can move, other than the category  $\text{Set}$  of sets and functions (whose existence is, at the best of our knowledge, consequence of a postulate; we similarly posit that there exists a model for ETAC). Small categories model theories, it has a *syntax*, in that they describe a relational structure using compositionality; but large categories offer a way to interpret the syntax, so being a *semantics*. A large relational structure is fixed once and for all, lying on the background, in which all other relational structures are interpreted.

It is nearly impossible to underestimate the profundity of this disengagement: synta and semantics, once separated and given a limited ground of action, acquire their meaning.<sup>8</sup>

More technically, in our Definition 3.1 we recall how this perspective allows to interpret different kinds of logics in different kinds of categories: such an approach leads very far, to the purported equivalence between different flavours of logics and different classes of categories; the particular shape of semantics that you can interpret in  $K$  is no more, no less than a reflection of the nice categorical properties of  $K$  (e.g., having finite co/limits, nice choices of factorisation systems [Bor94a, 5.5], [FK72], a subobject classifier; or the property of the posets  $\text{Sub}(A)$  of subobjects of an object  $A$  of being a complete, modular, distributive lattice. . . –in light of Definition 3.1, this last property has to do with the internal logic of the category: propositions are the set, or rather the *type*, of “elements” for which they are “true”;

<sup>8</sup>Of course, this is dialectical opposition at its pinnacle, and by no means a sterile approach to category theory; see [Law96] for a visionary account of how “dialectical philosophy can be modeled mathematically”.

and in nice cases (like e.g. in toposes), they are also arrows with codomain a suitable *type of truth values*  $\Omega$ ).

1.3.3. *Categories are universes of discourse.* Somehow, the previous section posits that category theory as a whole is “bigger than the mathematics itself”, and it works as one of its foundations; a category is a totality where all mathematics can be re-enacted; in this perspective, ETAC works as a metalanguage in which we develop our approach. This is not very far from current mathematical practice, and in particular from Mac Lane’s point of view:

We can [...] “ordinary” mathematics as carried out exclusively within [a universe]  $U$  (i.e. on elements of  $U$ ) while  $U$  itself and sets formed from  $U$  are to be used for the construction of the desired large categories. [ML98, I.6]

since the unique large category we posit is essentially “the universe”. But this approach goes further, as it posits that *mathematical theories are in themselves mathematical objects*, and as such, subject to the same analysis we perform on the object of which those theories speak about.

All in all, our main claim is that categories are of some use in clarifying a few philosophical problems, especially for what concerns the objects of discourse of ontology. The advantages of this approach are already visible in mathematics; we are trying to export them in the current philosophical debate.

Among many, we record

- the possibility of reading theories in terms of relations: this allows to suspend our ontological commitment on the nature of objects: they are given, but embedded in a relational structure. This relational structure, i.e. a category modeling the ontology in study is the object of our discourse
- a sharper, more precise, and less time-consuming conceptualisation process, which becomes a purely *context-dependent* process, i.e. a function of the aforementioned relational structure.

Of course, a relational structure as mentioned above is nothing but a certain kind of category: we then posit that *an ontology is a category*.

Of course, as extremist as it may seem, our position fits into an already developed open debate; for example, these are the opening lines of a paper by J.P. Marquis:

[...] *to be* is *to be related*, and the “essence” of an “entity” is given by its relations to its “environment” [Mar97]

Such a point of view is only acceptable when its fruitfulness has been determined by clarifying and formalising the relationship between an ontology regarded as “all that there is”, and a (large) category as “the place we inhabit”.

Sappiamo che nella storia della CT l'accettazione dell'evidenza dei risultati è avvenuta a prescindere da un dibattito intorno alla scarsa precisione e coerenza logica degli stessi. Le categorie sono nate come strumento concettuale e, senza preoccuparsi delle sottigliezze della ricerca fondazionale, hanno catturato efficacemente tutte le nozioni della matematica

moderna, rivelandosi utili e feconde. Nostro claim è che si riveleranno tali anche con le usuali nozioni metafisiche. Si tratta di adottare, in fondo, una prospettiva pragmatica:

that structural mathematics is characterized as an activity by a treatment of things as if one were dealing with structures. From the pragmatist viewpoint, we do not know much more about structures than how to deal with them, after all. [Krö07]

La “traduzione” dei problemi dell’ontologia nel linguaggio di CT permette di manipolare meglio nozioni (non solo, come si sa, matematiche) ma metamatematiche e metafisiche, e ci dota di un approccio più compatto e di una visione più “leggera” e occamista delle questioni vertenti su oggetti e esistenza. Non giustifichiamo questo approccio a priori ma ne testimoniamo la fecondità già provata in letteratura<sup>9</sup>, soprattutto paragonata a quella degli approcci set-theoretic (di cui già è informata la totalità delle ontologie formali).

In CT possiamo “tradurre” i problemi classici dell’ontologia, fornire modelli entro i quali formularne meglio presupposti e domande, evidenziare ciò che è banale conseguenza degli assiomi di quel modello e ciò che non lo è, risolverli e, in alcuni casi, dissolverli, rivelandone la natura figmentale. Si tratta di fornire un *ambiente* ben definito nel quale questioni ritenute oggetto di dibattito filosofico possano illuminarsi in modi nuovi o scomparire. E questo non per qualche perverso istinto riduzionistico, ma per poterne parlare in termini efficaci e nel linguaggio adatto a inquadrarli: tentare, con gli strumenti più avanzati e raffinati dell’astrazione matematica, di rispondere a delle domande, produrre conoscenza, e non solo dibattito; inscrivere antiche o recenti questioni in un nuovo paradigma, volto a superare e al contempo far avanzare la ricerca.

Come ogni paradigma lo dotiamo di una sintassi con la quale “nominare” concetti e dare definizioni, e di una semantica che produca modelli, e quindi contesti, entro i quali “guardare” le teorie; questa sintassi e questa semantica non ce le inventiamo: sono già nella matematica e da lì le preleviamo.

**1.4. Towards an Operative Ontology.** Riassumiamo i vantaggi del fare ontologia usando la teoria delle categorie:

- Prima di tutto in questo modo ontologia, ci si permetta la battuta, la si *fa* effettivamente. Vale a dire, come mostreremo nel resto del lavoro e in altri successivi, si affrontano di petto le questioni e le si risolvono. Stiamo perciò suggerendo un approccio *problem solving*
- Lo strutturalismo “debole” implicito in questa visione è da noi mantenuto solo a livello metateorico, e consente di guardare alle relazioni tra oggetti all’interno delle teorie. Anche qui, l’approccio relazionale è assunto in quanto proficuo in senso pratico, operativo, e - come vedremo al termine del paragrafo - non implica l’adesione incondizionata a uno strutturalismo “forte”.

<sup>9</sup>Cf. Mt7,16, giusto per ringraziarsi i severiniani.



- Come conseguenza fondamentale del punto precedente nessun concetto viene studiato in senso “assoluto” (qualunque cosa ciò significhi) ma relativamente al contesto in cui opera, e alla teoria che stiamo adottando per definirlo.

Come si è detto in 1.2 l’uso di questi strumenti concettuali ha aiutato la pratica matematica e ha involontariamente ispirato una visione epistemologica, e poi ontologica, della disciplina, vale a dire dei suoi oggetti di studio. A coloro che obiettano che bisogna prima sapere *cos’è* una struttura prima di lavorare con essa noi rispondiamo, con Krömer, che

this reproach is empty and one tries to explain the clearer by the more obscure when giving priority to ontology in such situations [...]. Structure occurs in the dealing with something and does not exist independently of this dealing. [Krö07]

Memori delle osservazioni di Carnap [nota ...], non riteniamo che questo approccio “operativo” all’ontologia (che non è puro *problem solving* ma anche chiarificazione concettuale) implichi necessariamente l’adesione incondizionata ad uno strutturalismo filosofico integrale - o a sue varianti specifiche come la teoria **ROS** -, esattamente come abbiamo visto non avvenire nel passaggio dalla *structural mathematics* allo strutturalismo vero e proprio (o al bourbakismo). La sua importanza è principalmente metodologica. (*Au contraire* risulta necessario per chi appoggia posizioni strutturaliste al di fuori della matematica cominciare a fare ontologia in termini categoriali, nelle modalità qui indicate).

1.4.1. *Existence: Persistence of Identity?* Ontology rests upon the principle of identity: it is this very principle that our category-theoretic approach aims to unhinge. E tuttavia formalizzare il concetto intuitivo di identità si rivela una questione estremamente spinosa: cosa significa che *due cose sono, invece, una* è un problema che ci arrovella fin da quando otteniamo la ragione e la parola; ciò perché il problema è tanto elementare quanto sfuggente: l’unica maniera in cui possiamo esibire ragionamento certo è il calcolo; del resto, se la sintassi non vede che l’uguaglianza in senso più stretto possibile, la prassi deve diventare in fretta capace di una maggiore elasticità: per un istante ho postulato che ci fossero “due” cose, non una. E non è forse questo a renderle due? E questa terza cosa che le distingue, è davvero diversa da entrambe?

Usciti dalle nebbie delle speculazioni tradizionali, i filosofi a cavallo tra ’800 e ’900 si sono posti questo complicato problema: due vie sono possibili: la risposta fregeana [] per cui “*x* esiste” se e solo se “*x* è identico a qualcosa [banalmente, a sè stesso]” alla soluzione logica quineana per cui “essere è essere il valore di una variabile [vincolata]”.

Il primo approccio si rivela comodo solo se si è realisti concettuali, ma è scarsamente informativo (cosa è l’uguaglianza tra due oggetti apparentemente diversi? Quel qualcosa che ci ha fatto sospettare lo fossero non è abbastanza a renderli tali?); coinvolge la nozione di identità, ed anzi scarica su questa l’onere di definire esistenza; questa non è la strada giusta: l’identità di fatto non esiste, perché ogni identità è un’identificazione, e ogni uguaglianza una relazione di equivalenza; la questione è tuttavia complessa abbastanza da dedicarvi un lavoro a parte (in effetti, due []) di questo polittico.



Il secondo tipo di approccio ha ispirato l'interesse per la nozione di *ontological commitment* (l'insieme di assunti che "si danno per scontati" quando si parla di ontologia, o se ne partecipa) e per la conseguente definizione di ontologia (di una teoria) come "dominio di oggetti su cui variano i quantificatori" (cf. [1]): namely una teoria qualsiasi è impegnata sulle entità su cui variano i quantificatori dei suoi enunciati.

Vedremo in [subsection 5.3](#) che la concezione Quineana fitta nella nostra visione di ontologia categoriale come conseguenza di una internalizzazione.

Esiste una terza via, meno diffusa in letteratura ma decisamente meno opinabile: definire l'esistenza tramite la persistenza nel tempo. Diciamo che " $x$  esiste" se e solo se " $x$  è identico a sè stesso in ogni frame temporale  $\langle T, < \rangle$ ", dove  $T$  è un insieme non vuoto di istanti e  $<$  una relazione binaria in  $T$  (e la relazione di esistenza in  $T$  è allora una relazione  $(x, t) \mapsto x \tilde{\in} t$ ; " $x$  esiste in  $T$ " se per ogni  $t : T$  si ha  $x \tilde{\in} t$ ).

Come si vede questa definizione cattura una nozione intuitiva di esistenza, impiegando sia l'identità (con tutti i problemi che essa comporta) che il tempo, o meglio una opportuna logica temporale nella quale far "persistere" le entità. (E' facile scrivere cosa significa la relazione " $x$  esiste in  $T$ " in termini di (L)TL)

Uno dei risultati di questo paper è che possiamo definire l'esistenza in maniera altrettanto intuitiva, senza riferirsi all'identità, né a un frame temporale; in effetti, fornendo un concetto più generale, dentro il quale si troveranno anche gli altri.

Porremo la questione nel seguente modo: ciò che è variabile relativo ad  $x$  è il grado, o *forza* della sua esistenza; l'esistenza "classica" è esistenza in massimo grado nel linguaggio interno del topos che fa da Universo (per noi, un universo borgesiano); lì saremo in grado di indicare il "grado" di esistenza degli oggetti che lo abitano, senza presupporre di muoverci attraverso istanti di tempo (come potrebbe suggerire l'esempio delle monete) o punti dello spazio (come la freccia). A seconda della struttura del dominio possiamo scegliere la logica da utilizzare e così il "contesto" più adatto all'intuizione che abbiamo dell'universo nel linguaggio naturale.

La persistenza nel tempo non è perciò rimossa dalla descrizione, o negata; piuttosto, inglobata. E' un sottocaso del modello generale, precisamente quello in cui la proposizione sull'esistenza dell'oggetto è vera con forza 1 i tutti gli istanti (cfr. ??, nota 16).

Va da sè che, molto informalmente, l'esistenza in this conception non è altro che la modalità di "presenza" degli oggetti all'interno di un modello. E' quindi letteralmente ciò che possiamo *farci* con gli oggetti, come possiamo porci rispetto a essi.

Non è solo una nozione operativa di esistenza, vicina peraltro al nostro senso comune: per noi le cose esistono se possiamo toccarle, vederle, postularle (quando invisibili) in base a ipotesi su rapporti di causazione che hanno con entità osservabili, descriverle, contarle, utilizzarle; e ciò è indipendente dal *come* esse esistano. E di conseguenza è anche una visione epistemica.

Bypassata la domanda "se le cose esistono", in base alle nostre scelte metateoriche e fondazionali, l'esistenza riguarda i modi tramite i quali le cose entrano in relazione l'una con l'altra. Questo ci permette, ecco i vantaggi dell'ontologia categoriale, di sfruttare la visione strutturalista e poter descrivere e render cogenti non solo il nostro mondo ma realtà distanti, come Tlön, fornite di un'ontologia diversa.

In linguaggio matematico questo "modo di comportarsi delle cose" non è altro che lo studio delle relazioni tra gli oggetti di una categoria.

Infine sarà, l'esistenza, anche context-dependent; varierà a seconda del linguaggio interno della teoria (cioè della categoria) nella quale operiamo. E questo permette di formalizzare la banale intuizione, spesso sfuggente agli occhi degli ontologi, per cui l'esistenza in un mondo come Tlön sarà presumibilmente diversa dalla nostra. L'ovvia constatazione che cambiando ontologia cambia il concetto di "esistere" diventa qui una conseguenza automatica dell'uso di un linguaggio matematico.

## 2. PRELIMINARIES ON VARIABLE SET THEORY

O scion of Bharat, all created beings are unmanifest before birth, manifest in life, and again unmanifest on death. So why grieve?

---

Śrīmadbhagavadgītā II, 28

In some of our proofs it will be crucial to blur the distinction between the category of functors  $I \rightarrow \mathbf{Set}$  and the *slice* category  $\mathbf{Set}/I$  (see [Bor94a, 1.6.1]); once the following result is proved, we freely refer to any of these two categories as the category of *variable sets* (indexed by  $I$ ).

**Proposition 2.1.** Let  $I$  be a set, regarded as a discrete category, and let  $\mathbf{Set}^I$  be the category of functors  $F : I \rightarrow \mathbf{Set}$ ; moreover, let  $\mathbf{Set}/I$  the slice category. Then, there is an equivalence (actually, an isomorphism, see [Bor94a, 1.5.1]) between  $\mathbf{Set}^I$  and  $\mathbf{Set}/I$ .

*Proof.* We shall give a very hands-on proof, based on the fact that we can represent the category  $\mathbf{Set}^I$  as the category of  $I$ -indexed families of objects, i.e. with the category whose objects are  $(\underline{X})_I := \{X_i \mid i \in I\}$ , and morphisms  $(\underline{X})_I \rightarrow (\underline{Y})_I$  the families  $\{f_i : X_i \rightarrow Y_i \mid i \in I\}$ . This is obvious, as a functor  $F : I \rightarrow \mathbf{Set}$  amounts to a choice of sets  $A_i := F(i)$ , and functoriality reduces to the property that identities in  $I$  go to identity functions  $A_i \rightarrow A_i$ .

Now, consider an object  $h : X \rightarrow I$  of  $\mathbf{Set}/I$ , and define a function as  $i \mapsto h^\leftarrow(i)$ ; of course,  $(X(h))_I := \{h^\leftarrow(i) \mid i \in I\}$  is a  $I$ -indexed family, and since  $I$  can be regarded as a discrete category, this is sufficient to define a functor  $F_h : I \rightarrow \mathbf{Set}$ .

Let us define a functor in the opposite direction: let  $F : I \rightarrow \mathbf{Set}$  be a functor. This defines a function  $h_F : \coprod_{i \in I} Fi \rightarrow I$ , where  $\coprod_{i \in I} Fi$  is the disjoint union of all the sets  $Fi$ .

The claim now follows from the fact that the correspondences  $h \mapsto F_h$  and  $F \mapsto h_F$  are mutually inverse.

This is easy to verify: the function  $F_{h_F}$  sends  $i \in I$  to the set  $h_F^\leftarrow(i) = Fi$ , and the function  $h_{F_h} \in \mathbf{Set}/I$  has domain  $\coprod_{i \in I} F_h(i) = \coprod_{i \in I} h^\leftarrow(i) = X$  (as  $i$  runs over the set  $I$ , the disjoint union of all preimages  $h^\leftarrow(i)$  equals the domain of  $h$ , i.e. the set  $X$ ).  $\square$

**Notation 2.2.** We employ the present remark to establish a bit of terminology: by virtue of Proposition 2.1 above, an object of the category of variable sets is equally denoted pair  $(A, f : A \rightarrow I)$ , as a function  $h : I \rightarrow \mathbf{Set}$ , or as the family of sets  $\{h(i) \mid i \in I\} = \{A_i \mid i \in I\}$ .

We call the function  $f$  the *structure map* of the variable set  $A$ , and the function  $h$  the functor associated, or corresponding, to the variable set in study.

**Remark 2.3.** A more abstract look at this result regards the equivalence  $\mathbf{Set}/I \cong \mathbf{Set}^I$  as a particular instance of the *Grothendieck construction* (see [Lei04, 1.1]): for every small category  $\mathcal{C}$ , the category of functors  $\mathcal{C} \rightarrow \mathbf{Set}$  is equivalent to the category of *discrete fibrations* on  $\mathcal{C}$  (see [Lei04, 1.1]). In this case, the domain  $\mathcal{C} = I$  is a discrete category, hence all functors  $\mathcal{E} \rightarrow I$  are, trivially, discrete fibrations.

**Remark 2.4.** The next crucial step of our analysis is the observation that the category of variable sets is a topos: we break the result into the verification of the various axioms, as exposed in Definition A.2 and Definition A.3. It will be crucial in our discussion that the category of sets is itself a topos: in particular, it is cartesian closed, and admits the set  $\{\perp, \top\}$  as subobject classifier.<sup>10</sup>

**Proposition 2.5.** The category of variable sets is Cartesian closed in the sense of [Bor94a, p.335].

*Proof.* We shall first show that the category of variable sets admits products: this is well-known as in  $\mathbf{Set}/I$ , products are precisely pullbacks ([Bor94a, 2.5.1]); note that Proposition 2.1 gives an identification

$$\begin{array}{ccc} & X \times_I Y & \\ \swarrow & \downarrow h & \searrow \\ X & & Y \\ \searrow f & & \swarrow g \\ & I & \end{array} \iff i \mapsto h^\leftarrow(i) = \left\{ (x, y) \in X \times_I Y \mid h(x, y) = i \right\}$$

and given the universal property of a pullback, this yields a canonical bijection  $h^\leftarrow(i) \cong f^\leftarrow(i) \times g^\leftarrow(i)$ . This is exactly the definition of the product of the two functors  $F_f, F_g : I \rightarrow \mathbf{Set}$ . Next, we shall show that each functor  $- \times_I Y$  has a right adjoint  $Y \pitchfork_I -$ . The functor  $\mathbf{Set}^I \rightarrow \mathbf{Set}^I : Z \mapsto Y \pitchfork_I Z$  where  $Y \pitchfork_I Z : i \mapsto \mathbf{Set}(Y_i, Z_i)$  does the job. This, together with a straightforward verification, sets up the bijection

$$\frac{X \times_I Y \longrightarrow Z}{X \longrightarrow Y \pitchfork_I Z}$$

and by a completely analogous argument (the construction  $(A, B) \mapsto A \times_I B$  is of course symmetric in its two arguments),

$$\frac{X \times_I Y \longrightarrow Z}{Y \longrightarrow X \pitchfork_I Z};$$

this concludes the proof that the category of variable sets is Cartesian closed.  $\square$

<sup>10</sup>This is a classical model of set theory, as opposed to an intuitionistic model, where  $\Omega$  consists of a more general Heyting algebra  $H$ ; the core of all our argument is very rarely affected by this choice, which is mainly due to expository reasons. The reader shall feel free to replace the minimal Heyting algebra  $\{\perp, \top\}$  with a more generic one, and they are invited to adapt the arguments accordingly.

**Proposition 2.6.** The category of variable sets has a subobject classifier.

*Proof.* From Definition A.2 we know that we shall find a variable set  $\Omega$  such that there is a bijection

$$\frac{\chi : A \longrightarrow \Omega}{\text{Sub}_I(A)}$$

where  $\text{Sub}_I(A)$  denotes the set of isomorphism classes of monomorphisms into  $A$ , in the category of variable sets.<sup>11</sup> For the sake of simplicity, for the rest of the proof we fix as category of variable sets the slice  $\mathbf{Set}/I$ .

From this we make the following guess: as an object of  $\mathbf{Set}/I$ ,  $\Omega$  is the canonical projection  $\pi_I : I \times \{\perp, \top\} \rightarrow I$  on the second factor. We are thus left with the verification that  $\pi_I$  has the correct structure and universal property.

First, we shall find a universal monomorphism  $\mathbf{t} : * \rightarrow \Omega$  in  $\mathbf{Set}/I$ . Unwinding the definition, such a map amounts to an injective function  $I \rightarrow \Omega$  having the projection  $\pi_I : \Omega \times I \rightarrow I$  as left inverse. This generalised element selects the  $\top$  (read “top”) truth value in  $\Omega$ . (Evidently, the identity function  $\text{id}_I : I \rightarrow I$  is the terminal object in  $\mathbf{Set}/I$ .)

It turns out that the function  $I \rightarrow I \times \{\perp, \top\}$  choosing the level-1 copy of  $I \cong I \times \{\top\}$  plays the rôle of  $\mathbf{t}$ : in the following, we shortly denote  $\Omega_I$  such a product set. Now, given a monomorphism  $S \hookrightarrow A$  the commutative square

$$\begin{array}{ccc} S & \longrightarrow & I \\ \downarrow & & \downarrow \mathbf{t} \\ A & \xrightarrow{\chi_S} & \Omega_I \end{array}$$

is easily seen to be a pullback; in fact, every morphism of variable sets  $\chi_S : A \rightarrow \Omega_I$  must send the element  $a : A$  to a pair  $(i, \epsilon) : I \times \{\perp, \top\}$ . The pullback of  $\chi$  and  $\mathbf{t}$ , as defined above, consists of the subset of the product  $A \times I$  such that  $\chi(a) = \mathbf{t}(i) = (i, \top)$ ; this defines a variable set  $S = (\chi(i, \top))_I$ , and such a correspondence is clearly invertible: every variable set arises in this way, and defines a “characteristic” function  $\chi_S : (A, f : A \rightarrow I) \rightarrow (\Omega_I, \pi_I)$ :

$$\chi_S(a) = \begin{cases} (f(a), \top) & \text{if } a \in S \\ (f(a), \perp) & \text{if } a \notin S \end{cases}$$

This concludes the proof of the fact that  $\mathbf{Set}/I$  admits a subobject classifier.  $\square$

**Remark 2.7.** A straightforward but important remark is now in order. The structure of subobject classifier of  $\Omega_I$ , and in particular the shape of a characteristic function  $\chi_S : A \rightarrow \Omega_I$

<sup>11</sup>A monomorphism into  $A$  as an object of  $\mathbf{Set}^I$  is nothing but a family of injections  $s_i : S_i \rightarrow A_i$ ; a monomorphism in  $\mathbf{Set}/I$  is a set  $S$  in a commutative triangle

$$\begin{array}{ccc} S & \xrightarrow{\quad} & A \\ & \searrow s & \swarrow a \\ & I. & \end{array}$$

for a subobject  $S \subseteq A$  in  $\mathbf{Set}/I$ , is explicitly obtained using the structure map  $f$  of the variable set  $f : A \rightarrow I$ .

This will turn out to be very useful along our main section, where we shall note that a proposition in the internal language of  $\mathbf{Set}/I$  amounts to a function  $p : U \rightarrow \Omega_I$ , having as domain a variable set  $u : U \rightarrow I$ , whose structure map uniquely determines the “strength” (see [Remark 4.1](#)) of the proposition  $p$ . In a nutshell,  $p(x)$  is a truth value; the fact that  $p$  is a morphism of *variable sets* however forces this truth value to be  $(u(x), \epsilon) \in I \times \{\perp, \top\}$ . We invite the reader not to worry now; we will duly justify each of these conceptual steps along [section 4](#).

**Proposition 2.8.** The category of variable sets is cocomplete and accessible.

*Proof.* □

Accessibility is a corollary of Yoneda in the following form: every  $F : I \rightarrow \mathbf{Set}$  is a colimit of representables

$$F \cong \operatorname{colim} \left( \mathcal{E}(F) \xrightarrow{\Sigma} I \xrightarrow{y} \mathbf{Set}^I \right)$$

( $\mathcal{E}(F)$  is small because in this case  $\mathcal{E}(F) \cong \coprod_{i \in I} Fi$ ).

**Corollary 2.9.** The category of variable sets is a Grothendieck topos.

### 3. THE INTERNAL LANGUAGE OF VARIABLE SETS

I am hard but I am fair; there is no racial bigotry here.  
[... ] Here you are all equally worthless.

GySgt Hartman

**Definition 3.1.** The internal language of a topos  $\mathcal{E}$  is a formal language defined by *types* and *terms*; suitable terms form the class of variables. Other terms form the class of *formulae*.

- *Types* are the objects of  $\mathcal{E}$
- *Terms* of type  $X$  are morphisms of codomain  $X$ , usually denoted  $\alpha, \beta, \sigma, \tau : U \rightarrow X$ .
  - Suitable terms are variables: the identity arrow of  $X \in \mathcal{E}$  is the variable  $x : X \rightarrow X$ . For technical reasons we shall keep a countable number of variables of the same type distinguished:<sup>12</sup>  $x, x', x'', \dots : X \rightarrow X$  are all interpreted as  $1_X$ .
- Generic terms may depend on multiple variables; the domain of a term of type  $X$  is the *domain of definition* of a term.

A number of inductive clauses define the other terms of the language:

- the identity arrow of an object  $X \in \mathcal{E}$  is a term of type  $X$ ;

<sup>12</sup>These technical reasons lie on the evident necessity to be free to refer to the same free variable an unbounded number of times. This can be formalised in various ways: we refer the reader to [LS88] and [Joh77].

- given terms  $\sigma : U \rightarrow X$  and  $\tau : V \rightarrow Y$  there exists a term  $\langle \sigma, \tau \rangle$  of type  $X \times Y$  obtained from the pullback

$$\begin{array}{ccc} W & \longrightarrow & X \times V \\ \downarrow & \searrow \langle \sigma, \tau \rangle & \downarrow \\ U \times Y & \longrightarrow & X \times Y \end{array}$$

- Given terms  $\sigma : U \rightarrow X, \tau : V \rightarrow X$  of the same type  $X$ , there is a term  $[\sigma = \tau] : W \xrightarrow{\langle \sigma, \tau \rangle} X \times X \xrightarrow{\delta_X} \Omega$ , where  $\delta_X : X \times X \rightarrow \Omega$  is defined as the classifying map of the mono  $X \hookrightarrow X \times X$ .
- Given a term  $\sigma : U \rightarrow X$  and a term  $f : X \rightarrow Y$ , there is a term  $f[\sigma] := f \circ \sigma : U \rightarrow Y$ .
- Given terms  $\theta : V \rightarrow Y^X$  and  $\sigma : U \rightarrow X$ , there is a term

$$W \langle \theta, \sigma \rangle \rightarrow Y^X \times X \xrightarrow{\text{ev}} Y$$

- In the particular case  $Y = \Omega$ , the term above is denoted

$$[\sigma \in \theta] : W \langle \theta, \sigma \rangle \rightarrow \Omega$$

- If  $x$  is a variable of type  $X$ , and  $\sigma : X \times U \rightarrow Z$ , there is a term

$$\lambda x. \sigma : U \xrightarrow{\eta} (X \times U)^X \xrightarrow{\sigma^X} Z^X$$

obtained as the mate of  $\sigma$ .

These rules can of course be also presented as the formation rules for a Gentzen-like deductive system: let us rewrite them in this formalism.

$$\begin{array}{c} \frac{}{1_X : X \rightarrow X} \qquad \frac{\sigma : U \rightarrow X \quad \tau : V \rightarrow Y}{\langle \sigma, \tau \rangle : W \langle \theta, \sigma \rangle \rightarrow X \times Y} \\[10pt] \frac{\sigma : U \rightarrow X \quad \tau : V \rightarrow X}{[\sigma = \tau] : W \rightarrow \Omega} \qquad \frac{\sigma : U \rightarrow X \quad f : X \rightarrow Y}{f[\sigma] : U \rightarrow Y} \\[10pt] \frac{\theta : V \rightarrow Y^X \quad \sigma : U \rightarrow X}{W \langle \theta, \sigma \rangle \rightarrow Y^X \times X \xrightarrow{\text{ev}} Y} \quad \frac{x : X \quad \sigma : X \times U \rightarrow Z}{\lambda x. \sigma = \sigma^X \circ \eta : U \rightarrow (X \times U)^X \rightarrow Z^X} \end{array}$$

To formulas of the language of  $\mathcal{E}$  we apply the usual operations and rules of first-order logic: logical connectives are induced by the structure of internal Heyting algebra of  $\Omega$ : given formulas  $\varphi, \psi$  we define

- $\varphi \vee \psi$  is the formula  $W \langle \varphi, \psi \rangle \rightarrow \Omega \times \Omega \xrightarrow{\vee} \Omega$ ;
- $\varphi \wedge \psi$  is the formula  $W \langle \varphi, \psi \rangle \rightarrow \Omega \times \Omega \xrightarrow{\wedge} \Omega$ ;
- $\varphi \Rightarrow \psi$  is the formula  $W \langle \varphi, \psi \rangle \rightarrow \Omega \times \Omega \xrightarrow{\Rightarrow} \Omega$ ;
- $\neg \varphi$  is the formula  $U \rightarrow \Omega \xrightarrow{\neg} \Omega$ .

Universal quantifiers admit an interpretation in the Mitchell-Bénabou language of  $\mathcal{E}$ : the following definition comes from [MLM92, VI].

**Definition 3.2.** Let  $\mathcal{E}$  be a topos, and let  $\varphi : U \times V \rightarrow \Omega$  be a formula defined on a product type  $U \times V$ . The variable  $u : U$  can be now quantified over yielding a new formula

$$\forall u.\varphi(u, v) : V \rightarrow \Omega$$

which no longer contains  $u$  as a free variable.

The term  $\forall u.\varphi(u, v)$  is obtained by composition of  $\lambda u.\varphi(u, v) : V \rightarrow V \rightarrow \Omega^U$  with the unique right adjoint  $\forall_t : \Omega^U \rightarrow \Omega$  to the precomposition  $t^* : \Omega \rightarrow \Omega^U$ <sup>13</sup> in the diagram

$$V \xrightarrow{\lambda u.\varphi} \Omega^U \xrightarrow{\forall_t} \Omega$$

(see [MLM92, IV.9]). Similarly, we obtain a term  $\exists u.\varphi(u, v)$  from the composition

$$V \xrightarrow{\lambda u.\varphi} \Omega^U \xrightarrow{\exists_t} \Omega$$

with the left adjoint  $\exists_t \dashv t$ .

Each formula  $\varphi : U \rightarrow \Omega$  defines a subobject  $\{x \mid \varphi\} \subseteq U$  of its domain of definition; this is the subobject classified by  $\varphi$ , and must be thought as the subobject where “ $\varphi$  is true”.

If  $\varphi : U \rightarrow \Omega$  is a formula, we say that  $\varphi$  is *universally valid* if  $\{x \mid \varphi\} \cong U$ . If  $\varphi$  is universally valid in  $\mathcal{E}$ , we write “ $\mathcal{E} \Vdash \varphi$ ” (read: “ $\mathcal{E}$  believes in  $\varphi$ ”).

Examples of universally valid formulas:

- $\mathcal{E} \Vdash [x = x]$
- $\mathcal{E} \Vdash [(x \in_X \{x \mid \varphi\}) \iff \varphi]$
- $\mathcal{E} \Vdash \varphi$  if and only if  $\mathcal{E} \Vdash \forall x.\varphi$
- $\mathcal{E} \Vdash [\varphi \Rightarrow \neg\neg\varphi]$

Let us now glance at the internal language of variable sets. This will turn out to be the cornerstone for the analysis in [section 4](#).

Here we just unwind [Definition 3.1](#) at its very surface; we invite the reader to endeavour in the instructive exercise to fill all details properly.

**Definition 3.3.** Types and terms of  $\mathcal{L}(\mathbf{Set}/I)$  are respectively arrows  $\begin{bmatrix} U \\ \downarrow \\ I \end{bmatrix}$  with codomain  $I$ , and commutative triangles

$$\begin{array}{ccc} U & \xrightarrow{\sigma} & X \\ & \searrow u & \swarrow x \\ & I & \end{array}$$

Given this, we define

- *product terms* as functions  $\sigma : U_1 \times_I U_2 \times_I \cdots \times_I U_n \rightarrow X$ ;

<sup>13</sup>The arrow  $t : U \rightarrow 1$  is the terminal map, and  $t^* : \Omega^1 \rightarrow \Omega^U$  is induced by precomposition; it sends a term  $x : \Omega$  to the constant function  $t^*x = \lambda u.x$ .

- terms  $\langle \sigma, \tau \rangle$  obtained as the (diagonal map in the) pullback

$$\begin{array}{ccc}
 W\langle \sigma, \tau \rangle & \longrightarrow & X \times_I V \\
 \downarrow \lrcorner & & \downarrow \\
 U \times_I Y & \longrightarrow & X \times_I Y \\
 & \searrow & \searrow \\
 & & I \times_I I \cong I
 \end{array}$$

- terms  $[\sigma = \tau]$ , obtained as compositions

$$W\langle \sigma, \tau \rangle \xrightarrow{\langle \sigma, \tau \rangle} X \times_I Y \xrightarrow{\delta_X} \Omega_I$$

where  $\delta_X : X \times_I X \rightarrow \Omega$  classifies the mono  $m : X \rightarrow X \times_I X$  obtained by the universal property of the pullback  $X \times_I X$  as the kernel pair of  $\begin{bmatrix} X \\ x \downarrow \\ I \end{bmatrix}$ .

The cartesian closed structure of  $\mathbf{Set}/I$  (cf. [Proposition 2.5](#)) yields terms (we denote  $B^A := A \multimap B$  in the notation of [Proposition 2.5](#))

- $[\sigma \in \theta] : W\langle \theta, \sigma \rangle \rightarrow Y^X \times_I X \rightarrow \Omega_I$  from suitable  $\sigma, \theta$ ;
- $U \rightarrow (X \times_I U)^X \xrightarrow{\sigma^X} Z^X$  ( $\lambda$ -abstraction on  $x : X$ ) from a suitable  $\sigma : X \times_I U \rightarrow Z$ .

Quantifiers can be deduced in a similar way, starting from their general definition in [Definition 3.2](#).

**Remark 3.4.** Like every other Grothendieck topos, the category  $\mathbf{Set}/I$  has a *natural number object* (see [\[MLM92, VI.1\]](#), [\[LS88, p.46\]](#)); here we shall outline its construction. It is a general fact that such a natural number object in the category of variable sets, consists of the constant functor on  $\mathbb{N} : \mathbf{Set}$ , when we realise variable sets as functors  $I \rightarrow \mathbf{Set}$ : thus, in fibered terms, the natural number object is just  $\pi_I : \mathbb{N} \times I \rightarrow I$ .

A natural number object provides the category  $\mathcal{E}$  it lives in with a notion of *recursion* and with a notion of  $\mathcal{E}$ -induction principle: namely, we can interpret the sentence

$$(Q0 \wedge \bigwedge_{i \leq n} Qi \Rightarrow Q(i+1)) \Rightarrow \bigwedge_{n \in \mathbb{N}} Qn$$

for every  $Q : \mathbb{N} \rightarrow \Omega_I$ .

In the category of variable sets, the universal property of  $\pi_I : \mathbb{N} \times I \rightarrow I$  amounts to the following fact: given any diagram of solid arrows

$$(3.1) \quad \begin{array}{ccccc}
 I & \xrightarrow{0} & \mathbb{N} \times I & \xrightarrow{s \times I} & \mathbb{N} \times I \\
 \parallel & & \vdots u & & \vdots u \\
 I & \xrightarrow{x} & X & \xrightarrow{f} & X
 \end{array}$$



where every arrow carry a structure of morphism over  $I$  (and  $0 : i \mapsto (0, i)$ ,  $s \times I : (n, i) \mapsto (n + 1, i)$ ), there is a unique way to complete it with the dotted arrow, i.e. with a function  $u : \mathbb{N} \times I \rightarrow X$  such that

$$u \circ (s \times I) = f \circ u.$$

Clearly,  $u$  must be defined by induction: if it exists, the commutativity of the left square amounts to the request that  $u(0, i) = x(i)$  for every  $i : I$ . Given this, the inductive step is

$$u(s(n, i)) = u(n + 1, i) = f(u(n, i)).$$

This recursively defines a function with the desired properties; it is clear that these requests uniquely determine  $u$ .

Such a terse exposition obviously does not exhaust such a vast topic as recursion theory conducted with category-theoretic tools. The interested reader shall consult [JR97] for a crystal-clear introductory account, and [CH08, CHH14] for more recent and modern development of recursion theory.

The object of natural numbers of  $\mathbf{Set}/I$  is easily seen to match the definition of the *initial object* [Bor94a] of the category  $\mathbf{Dyn}/I$  so defined:

- the objects of  $\mathbf{Dyn}/I$  are *dynamical systems* in  $\mathbf{Set}/I$ , i.e. the triples  $(x, X, f)$ , where  $X : \mathbf{Set}/I$  (say, with structure map  $\xi : X \rightarrow I$ ),  $x : (I, \text{id}_I) \rightarrow (X, \xi)$  and  $f : X \rightarrow X$  is an endo-morphism of variable sets;
- given two dynamical systems  $(x, X, f)$  and  $(y, Y, g)$  a *morphism* between them is a function  $u : X \rightarrow Y$  such that the diagram in (3.1) commutes in all its parts.

The reason why such a triple  $(x, X, f)$  identifies as a (discrete) dynamical system is easily seen: the function  $x : I \rightarrow X$  works as initial seed for a recursive application of  $f$ , in such a way that every  $f : X \rightarrow X$  defines a sequence

$$u_{n+1} := f(u_n)$$

of its iterates. The system now lends itself to all sorts of questions: is there a fixed point for  $u_n(x)$ ? Does the limit of  $u_n(x)$  belong to  $X$  (not obvious: consider  $X = [0, 1[$  and  $u_n(x) \equiv 1 - \frac{1}{n}$ )? Is  $u_n(x)$  continuous in  $x$ ? Etc.

#### 4. NINE COPPER COINS, AND OTHER TOPOSES

Explicaron que una cosa es *igualdad*, y otra *identidad*, y formularon una especie de *reductio ad absurdum*, o sea el caso hipotético de nueve hombres que en nueve sucesivas noches padecen un vivo dolor. ¿No sería ridículo -interrogaron- pretender que ese dolor es el mismo?

---

JLB —Tlön, Uqbar, Orbis Tertius

According to our description of the Mitchell-Bénabou language in the category of variable sets, *propositions* are morphisms of the form

$$p : U \rightarrow \Omega_I$$

where  $\Omega_I$  is the subobject classifier of  $\mathbf{Set}/I$  described in [Proposition 2.6](#); now, recall that

- the object  $\Omega_I = \{0, 1\} \times I \rightarrow I$  becomes an object of  $\mathbf{Set}/I$  when endowed with the projection  $\pi_I : \Omega_I \rightarrow I$  on the second factor of its domain;
- the universal monic  $\mathbf{t} : I \rightarrow \Omega_I$  consists of a section of  $\pi_I$ , precisely the one that sends  $i : I$  to the pair  $(i, 1) : \Omega_I$ ;
- every subobject  $U \hookrightarrow A$  of an object  $A$  results as a pullback (in  $\mathbf{Set}/I$ ) along  $\mathbf{t}$ :

$$\begin{array}{ccc}
 U & \xrightarrow{u} & I \\
 \downarrow m & \searrow u & \parallel \\
 & I & \\
 \uparrow & \swarrow & \downarrow \mathbf{t} \\
 A & \xrightarrow{\chi_U} & \Omega_I
 \end{array}$$

(see [Proposition 2.6](#) for a complete proof)

The set  $I$  in this context acts as a *multiplier* of truth values, in that every proposition can have a pair  $(\epsilon, i)$  as truth value. We introduce the following notation: a proposition  $p : U \rightarrow \Omega_I$  is *true*, in context  $x : U$ , with *strength*  $t$ , if  $p(x) = (1, t)$  (resp.,  $p(x) = (0, t)$ ).

**Remark 4.1.** A proposition in the internal language of variable sets is a morphism of the following kind: a function  $p : U \rightarrow \Omega_I$ , defined on a certain domain, and such that

$$\begin{array}{ccc}
 U & \xrightarrow{p} & \{0, 1\} \times I \\
 \downarrow u & & \downarrow \pi_I \\
 I & \xlongequal{\quad} & I
 \end{array}$$

(it must be a morphism of variable sets!) This means that  $\pi p(x : U) = u(x : U)$ , so that  $p(x) = (\epsilon_x, u(x))$  for  $\epsilon_x = 0, 1$  and  $u$  is uniquely determined by the "variable domain"  $U$ . This is an important observation: the strength with which  $p$  is true/false is completely determined by the structure of its domain, in the form of the function  $u : U \rightarrow I$  that renders the pair  $(U, u)$  an object of  $\mathbf{Set}/I$ .

To get a grip of the different roles of various classes of propositions, and given that our interest will be limited to a certain class of particular propositions that we will construct *ex nihilo*, it is now convenient to discuss what constraints we have to put on the structure of  $I$ : of course, the richest this structure is, the better will the category  $\mathbf{Set}/I$  behave: it is for example possible to equip  $I$  with an order structure, or a natural topology. Among different choices of truth multiplier, yielding different categories of variable sets, and different kinds of internal logic therein, we will privilege those that make  $I$  behave like a space of strengths: a dense, linear order with LUP, thus not really far from being a closed, bounded subset of the real line.

The main result of the present section is a roundup of examples showing that it is possible to concoct categories of variable sets where some seemingly paradoxical constructions coming

from J.L. Borges' literary world have, instead, a perfectly “classical” behaviour when looked with the lenses of the logic of variable set theory.

Each of the examples in our roundup [Example 4.4](#) [Example 4.9](#) [Example 4.10](#) is organised as follows: we recall the shape of a paradoxical statement in Borges' literary world. Then, we show in which topos this reduces to an intuitive statement expressed in the syntax of a variable set category.

More than often, we use  $I = [0, 1]$  as base of variable sets; as already said, there are different reasons for this choice: the most intuitive is that if a truth value is given with a *strength*  $t \in I$  it is a natural request to be able to *compare* elements in this set; in particular, it should always be possible to assess what truth value is stronger.

For this reason, even if this assumption is never strictly necessary (the only constraint is that  $I$  is totally, or partially, ordered set by a relation  $\leq$ ), a natural choice for  $I$  is a *continuum* (=a dense total order with LUP –see [Mos09]). An alternative choice drops the density assumption: in that case the (unique) finite total order  $\Delta[n] = \{0 < 1 < \dots < n\}$ , or the countable total order  $I = \omega = \bigcup_n \Delta[n]$  are all pretty natural choices for  $I$  (although it is way more natural for  $I$  to have a minimum *and* a maximum element).<sup>14</sup> In each of these cases “classical” logic is recovered as a projection: propositions  $p$  can be true or false with strength 1,<sup>15</sup> the maximum element of  $I$ :

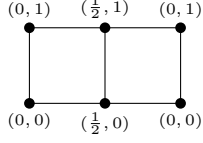
$$\begin{array}{ccc} 0 & \text{-----} & 1 \quad \{\perp\} \times I \\ & & \vdots \\ 0 & \text{-----} & 1 \quad \{\top\} \times I \\ & & \{\perp, \top\} \end{array}$$

In order to aid the reader understand the explicit way in which  $I$  “multiplies” truth values, we spell out explicitly the structure of the subobject classifier in  $\mathbf{Set}/\Delta[2]$ . In order to keep calling the minimum and maximum of  $I$  respectively 0 and 1 we call  $\frac{1}{2}$  the intermediate point of  $\Delta[2]$ .

<sup>14</sup>We're only interested in the notion of an abstract interval here: a continuum  $X$  endowed with an operation  $X \rightarrow X \vee X$  of “zooming”, uniquely defined by this property. In a famous paper Freyd characterises “the interval” as the terminal interval coalgebra: see [Fre08, §1]; for our purposes, note that  $[0, 1]$  is a natural choice: it is a frame, thus a Heyting algebra  $\mathfrak{H} = ([0, 1], \wedge, \vee, \Rightarrow)$  with respect to the pseudo-complement operation given by  $(x \Rightarrow z) := \bigvee_{x \wedge y \leq z} y$  (it is immediate that  $x \wedge a \leq b$  if and only if  $a \leq x \Rightarrow b$  for every  $a, b \in [0, 1]$ ).

<sup>15</sup>Here  $I$  is represented as an interval whose minimal and maximal element are respectively 0 and 1; of course these are just placeholders, but it is harmless for the reader to visualise  $I$  as the interval  $[0, 1]$ .

**Remark 4.2.** The subobject classifier of  $\mathbf{Set}/\Delta[2]$  consists of the partially ordered set  $\Delta[1] \times \Delta[2]$  that we can represent pictorially as a rectangle



endowed with the product order. The resulting poset is partially ordered, and in fact a Heyting algebra, because it results as the product of two Heyting algebras: the Boole algebra  $\{0 < 1\}$  and the frame of open subsets of the Sierpinski space  $\{a, b\}$  (the topology is  $\tau_S = \{\emptyset, \{a\}, \{a, b\}\}$ ).

**Remark 4.3.** Siccome il caso  $I = [0, 1]$  con la topologia euclidea è quello più naturale per diversi motivi, definiamo alcuni insiemi di interesse per una data proposizione  $p : U \rightarrow \Omega_I$  per questa scelta di  $I$ :

- $A^\top = \{x : U \mid p(x) = (1, t_x), t_x > 0\} = p^\leftarrow(\{1\} \times (0, 1])$  e  $A^\perp = \{x : U \mid p(x) = (0, t_x), t_x > 0\}$ ; cose vere (risp., false) con forza maggiore di zero. Sono le funzioni  $u : U \rightarrow I$  tali che  $u^\leftarrow 0 = \emptyset$ .
- $B^\top = \{x : U \mid p(x) = (1, 1)\} = p^\leftarrow((1, 1))$ <sup>16</sup> e  $B^\perp = \{x : U \mid p(x) = (0, 1)\}$  cose vere (risp., false).
- $E_t^\top = \{x : U \mid p(x) = (1, t)\}$  e  $E_t^\perp = \{x : U \mid p(x) = (0, t)\}$ ; cose vere (risp., false) con forza  $t$ .

Last but not least, a crucial assumption will be that the strength of  $p$  depends continuously, or not, on the variables on its domain of definition. Without such continuous dependence, small changes in context  $x : U$  might drastically change the truth value  $p(x)$ .<sup>17</sup>

**4.1. The unimaginable topos theory hidden in Borges' library.** Jorge Luis Borges' literary work is well-known for being made by paradoxical worlds; oftentimes, seemingly absurd consequences follow by stretching to their limit ideas from logic and mathematics: time, infinite, self-referentiality, duplication, recursion, the relativity of time, the illusory nature of our perceptions, the limits of language, its capacity to generate worlds.

In the present section we choose *Fictions*, Borges' famous collection of novels, as source of inspiration for possible and impossible worlds and their ontology.

Usualmente la costruzione di un "mondo impossibile" (variamente impiegato nelle semantiche paraconsistenti) va circa come segue: since the classic work of Rescher and Brandom [?] it is obtained with recursive operations on (kripkean) standard worlds ...; noi rovesciamo

<sup>16</sup>Si può interpretare anche come "le cose persistono nel tempo", e abbiamo catturato la nozione di time-persistence, cfr. 1.4

<sup>17</sup>There is no a priori reason to maintain that  $p$  is a continuous proposition; one might argue that discontinuous changes in truth value of  $p$  happen all the time in "real life"; see the family of paradoxes based on so-called *separating instants*: how well-defined the notion of "time of death" is? How well-defined the notion of "instant in time"?

tale prospettiva, e invece di depennare dal computo degli universi i mondi le cui caratteristiche do not comply with sensorial experience, or imply paradoxical entities/constructs, we accept their existence for bizarre that it may seem, and we try to deduce what kind of logic can consistently generate such statements.

The interest in such a literary calembour is manifold, and the results are surprising:

- we unravel how a mathematically deep universe Borges has inadvertently created: of the many compromises we had to take in order to reconcile literature and the underlying mathematics,<sup>18</sup> we believe no one is particularly far-fetched one;
- we unravel how *relative* ontological assumptions are; they are not given: using category theory, ontology, far from being the presupposition on which it is based, is a byproduct of language itself. The more expressive language is, the more ontology; the fuzzier its capacity to assert truth, the fuzzier existence becomes;
- “Fuzziness” of existence, i.e. the fact *entia* exist less than completely, is hard-coded in the language (in the sense of Definition 3.1) of the category we decide to work in from time to time;
- ...

To sum up, readers willing to find an original result in this paper, might find it precisely here: we underline how Borges’ alternative worlds (Babylon, Tlön dots) are mathematically consistent places, worthy of existence as much as our world, just based on a different internal logic. And they are so just thanks to a base-sensitive theory of existence –ontology breaks in a spectrum of *ontologies*, one for each category/world.

The first paradox we aim to frame in the right topos is the famous nine copper coins argument, used by the philosophers of Tlön to construct a paradoxical object whose existence persists over time, in absence of a consciousness continually perceiving it and maintaining in a state of being.

**Example 4.4** (Nine copper coins). First, we recall the exact statement of the paradox:<sup>19</sup>

El martes, *X* atraviesa un camino desierto y pierde nueve monedas de cobre.  
El jueves, *Y* encuentra en el camino cuatro monedas, algo herrumbradas por la lluvia del miércoles. El viernes, *Z* descubre tres monedas en el camino. El

<sup>18</sup>See Remark 4.5 below: these “compromises” mainly amount to assumptions on the behaviour of space-time on Tlön and Babylon.

<sup>19</sup>The translation we employ is classical and comes from [Bor99b]:

Tuesday, *X* crosses a deserted road and loses nine copper coins. On Thursday, *Y* finds in the road four coins, somewhat rusted by Wednesday’s rain. On Friday, *Z* discovers three coins in the road. On Friday morning, *X* finds two coins in the corridor of his house. The heresiarch would deduce from this story the reality - i.e., the continuity - of the nine coins which were recovered.

It is absurd (he affirmed) to imagine that four of the coins have not existed between Tuesday and Thursday, three between Tuesday and Friday afternoon, two between Tuesday and Friday morning. It is logical to think that they have existed - at least in some secret way, hidden from the comprehension of men - at every moment of those three periods.

viernes de mañana,  $X$  encuentra dos monedas en el corredor de su casa. El quería deducir de esa historia la realidad -id est la continuidad- de las nueve monedas recuperadas.

Es absurdo (afirmaba) imaginar que cuatro de las monedas no han existido entre el martes y el jueves, tres entre el martes y la tarde del viernes, dos entre el martes y la madrugada del viernes. Es lógico pensar que han existido -siquiera de algún modo secreto, de comprensión vedada a los hombres- en todos los momentos de esos tres plazos.

Before going on with our analysis, two remarks are in order:

- the paradox appears in a primitive version in [Bor97], where instead of nine copper coins, a single arrow, shot by an anonymous archer, disappears among the woods. The text appears in a hard-to-find edition of *Inquisiciones*; in the last chapter, we read the “arrow avatar” (*avatar de la flecha*):

$X$  dispara una flecha desde un arco, y se pierde en los árboles. Después,  $X$  la busca y logra encontrarla.

Es absurdo imaginar que la flecha no ha existido durante el período entre los momentos en que  $X$  la perdió de vista y la encontró.

Es lógico pensar que ha existido -siquiera de algún modo secreto, de comprensión vedada a los hombres- en todos los momentos de esos tres plazos.

- There is one and only one reason why the paradox of the nine copper coins is invalid: copper does not rust.

It is obvious that both constructions leverage on the same argument to build an efficient aporia: the mysterious persistence of things through time without a perceiving consciousness. We concentrate on the copper coins dilemma, per il semplice fatto che Finzioni è raggiungibile molto più facilmente ai nostri lettori; incidentally, we happen to be able to rectify the “rust counterargument” without appealing to the assumption that copper can rust on Tlön due to a difference in Tlönian chemistry.

Expressed in natural language, our solution to the paradox goes more or less as follows:  $X$  loses their coins on Tuesday, and the force  $\varphi$  with which they “exist” lowers; it grows back in the following days, going back to a maximum value when  $X$  retrieves two of their coins on the front door.  $Y$  findings of other coins raises their existence force to a maximum. The coins that  $Y$  has found rusted (or rather, the surface copper oxidized: this is possible, but water is rarely sufficient to ignite the process alone –certainly not in the space of a few hours.<sup>20</sup>

**Remark 4.5.** In this perspective, Tlön classifier of truth values can be taken as  $\Omega_I = \{0 < 1\} \times I$ , where  $I$  is any set with more than one element; a minimal example can be  $I = \{N, S\}$

<sup>20</sup>in una sorta di principio di wormhole, eventi indipendenti sulla terra sono dipendenti su Tlön, perché l’evento A influenza, in uno spazio pluridimensionale di scelte di verità, l’evento B in modi che gli sarebbero vietati se fosse sulla Terra.

(justifying this choice from inside Tlön is easy: the planet is subdivided into two emispheres; each of which now has its own logic “line” independent from the other), but as explained in [Remark 4.5](#) a more natural choice for our purposes is the closed real interval  $I = [0, 1]$ .

This allows for a continuum of possible forces with which a truth value can be true or false; it is to be noted that  $[0, 1]$  is also the most natural place on which to interpret fuzzy logic, albeit the interest for  $[0, 1]$  therein can be easily and better motivated starting from probability theory. (There are interesting perspective on how to develop basic measure theory out of  $[0, 1]$ . For example, measures valued in things like Banach space and more general topological vector spaces have been considered.)

We now start to formalise properly what we said until now.

To set our basic assumptions straight, we proceed as follows:

- senza perdita di generalità possiamo supporre l'insieme  $C = \{c_1, \dots, c_9\}$  delle monete totalmente ordinato e partizionato in modo tale che le prime due monete siano quelle ritrovate da  $X$  il martedì, le seconde quattro quelle che  $Y$  ritrova sul cammino, e le ultime 3 quelle viste da  $Z$ . Allora

$$C = C_X \sqcup C_Y \sqcup C_Z.$$

and  $C_X = \{c_{X1}, c_{X2}\}$ ,  $C_Y = \{c_{Y1}, c_{Y2}, c_{Y3}, c_{Y4}\}$ ,  $C_Z = \{c_{Z1}, c_{Z2}, c_{Z3}\}$  As already said, the truth multiplier  $I$  is the closed interval  $[0, 1]$  with its canonical order –so with its canonical structure of Heyting algebra, and if needed, endowed with the usual topology inherited by the real line.

- Propositions of interest for us are of the following form:

$$\lambda g c d. p(g, c, d) : \{X, Y, Z\} \times C \times W \rightarrow \Omega_I$$

where  $W$  is a set of days, that for the sake of explicitness can be taken equal to the set of weekdays  $S, M, T$  (strictly speaking, the paradox involves just the interval  $[Tu, Fri]$ ).  $p(g, c, d)$  has to be read as “in  $g$ 's frame of existence the coin  $c$  exists at day  $d$  with strength  $p(g, c, d)$ ”

Definiamo ora *ammissibile* una configurazione tale che le condizioni seguenti sono rispettate: for all day  $d$  and coin  $c$ , we have

$$\sum_{u \in \{X, Y, Z\}} p(u, c, d) = (\top, 1)$$

where we denote as “sum” the logical conjunction in  $\Omega_I$ : this means that day by day, the *global* existence of the group of coins constantly attains the maximum; it is the *local* existence that lowers when the initial conglomerate of coins is partitioned. Moreover,

$$\begin{cases} \sum_{c_X \in C_X} p(X, c_X, V) = (\top, 1) \\ \sum_{c_Y \in C_Y} p(Y, c_Y, G) = (\top, 1) \\ \sum_{c_Z \in C_Z} p(Z, c_Z, V) = (\top, 1) \end{cases}$$

In an admissible configuration the subsets  $C_X, C_Y, C_Z$  can only attain an existence  $p(g, c, d) \leq (\top, 1)$ ; that is no coin completely exists *locally*. But for an hypothetical external observer,

capable of adding up the forces with which the various parts of  $C$  exist, the coins *globally* exist “in some secret way, of understanding forbidden to men” (or rather, to  $X, Y, Z$ ).

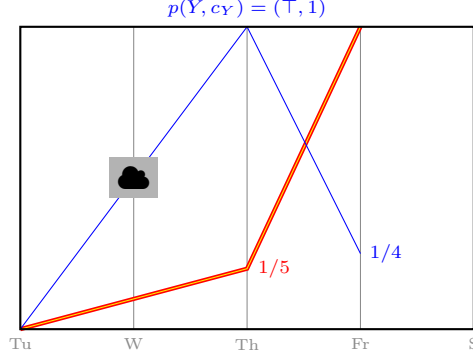


FIGURE 1. A pictorial representation of the truth forces of coins in different days, a minimal piecewise linear model.  $X$  is red,  $Z$  is yellow,  $Y$  is blue. Time is considered as a continuum marked at weekdays.

**Remark 4.6.** An important claim is now in order. Our construction above is not just asleight of hand that leaves everything unchanged. *Every* theory on existence posits that things are “in some secret way”. Tlön’s idealist might deny time (our model makes no strong assumption on what time is: discrete, continuous, homogeneous, bent by speed. . .); they live comfortably in  $B^\perp$ , as defined in Remark 4.3 without being able to tell anything about how objects “completely do not exist”. At the opposite side of the spectrum the pure empiricist lives in  $B^\top$ , and they’re unable to tell how they “completely exist”).

In our model  $B^\perp, B^\top$  are just particular slices of a single object where they coexist as particular fibers; the approach is *inherently better*, as it yields a quantitative answer, framing both approaches as opposite sides of a spectrum made out of diverse colours.

**Remark 4.7.** L’aritmetica di Tlon; proposizioni a forza additiva; parallelismi tra la nave di Teseo e le nove monete.

**Remark 4.8** (Continuity for a proposition). Let  $p : U \rightarrow \Omega_I$  be a proposition; here we investigate what does it mean for  $p$  to be (globally) continuous with respect to the Euclidean topology on  $I = [0, 1]$ , in the assumption that its domain of definition  $U$  is metrisable (this is true for example when  $U$  is a subset of space-time). The condition is that

$$\forall \epsilon > 0, \exists \delta > 0 : |x - y| < \delta \Rightarrow |px - py| < \epsilon$$

Da questo segue immediatamente che quando  $p$  è continua nel suo dominio, i valori di verità di  $p$  in configurazioni “vicine” in un senso opportuno sono dati con forze allo stesso modo vicine (chiaramente questa è una descrizione spannometrica della nozione di continuità. . .).



All elementary topology results apply to such a proposition: for example the set of forces with which  $p$  is true or false is a connected subset of  $\Omega_I$ , compact if  $U$  was compact.

Continuity and discontinuity of a proposition  $p : U \rightarrow \Omega_I$  now capture quite well other pieces of Borges’ literary universe: here we provide two examples. We refrain from a deep, quantitative analysis, and we invite the reader to fill the details of our reasoning as a pleasant re-reading exercise of [Bor99a] and [Bor99b].<sup>21</sup>

**Example 4.9** (Discontinuity, sapphire from Taprobana). Propositions  $p : U \rightarrow \Omega_{[0,1]}$  that are allowed to be discontinuous in its variables depend unpredictably from their context: such propositions model seemingly chaotic events triggered as the end terms of a chain of disconnected prior events; in fact, if we assume a real base for the domain of  $p$ , its continuity as stated in Remark 4.8 roughly means that events in the same neighbourhood –“near” in space or temporally contiguous– can’t have too much different truth/force values.

A model for such a universe, where “terrible consequences” sometimes follow from the “impersonal drawings, whose purpose is unclear” characterising the Company’s actions: seemingly disconnected (that “a sapphire from Taprobana be thrown into the waters of the Euphrates”; that “a bird be released from the top of a certain tower”; that “every hundred years a grain of sand be added to (or taken from) the countless grains of sand on a certain beach”), but generating nontrivial dynamics when inserted in a suitable sequence as in Figure 2.

Let us consider the dynamical system  $(\nabla \circ p, [0, 1])$  obtained from the iterates of the composition  $\nabla \circ p : [0, 1] \rightarrow [0, 1]$  ( $\nabla : I \amalg I \rightarrow I$  is the fold map of  $I$ , obtained from the identity of  $I$  and the universal property of the coproduct). We start from a proposition  $p$  depending on a free variable  $t : I$ ; then,  $p(t) = (u(t), \epsilon) \in \Omega_I$  consists of a force and a truth value; but  $p$  can be evaluated on  $u(t)$  (although this is not its universal property, the fold map  $\nabla$  just forgets the truth value, keeping the force): the process can thus be iterated as follows, exploiting the universal property of the natural number object in  $\mathbf{Set}/I$ :

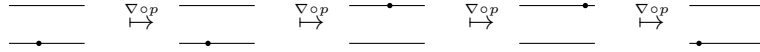


FIGURE 2. A dynamical system induced by the Company’s infinite, impersonal drawings.

It is clear that the limit behaviour of such a sequence strongly depends on the analytic properties of  $f$  (cf. [Str96] for an intro to one-dimensional dynamic systems and for their applications see [Wig03]). In a well-known procedure, a repeated series of drawings can chaotically deform the configuration space on which  $p$  is evaluated.<sup>22</sup>

<sup>21</sup>The plot of [Bor99a] in a nutshell is: in Babylon, a lottery game infiltrates reality to the point that it ends up governing the actions of all men; freeing them from free will while coorting them into pawns of an infinite, inescapable, unknowable game, governed by an iron, seemingly chaotic, probabilistic logic.

<sup>22</sup>There is of course countless termination conditions on such process  $p$ ; the series can be periodic, it can stop when  $p$  reaches maximum or minimum force, or a prescribed value, or is inside/outside a certain range of forces. . . ; we leave such speculations to the mystagogues of the Company, or to the lions of Qaphqa.

Another compelling example is that of the monotonicity of a proposition depending, say, on a certain number of observers who acknowledge the “existence” of an object  $R$  (be it physical or conceptual) in large numbers, from which very reason the existence of  $R$  gains strength.

**Example 4.10** (Continuity: a few birds, a horse). Let us consider objects whose existence strength depends *monotonously* and continuously from their parameters: for example a proposition  $p$  may be “truer” the more people observe it, because

things became duplicated in Tlön; they also tend to become effaced and lose their details when they are forgotten. A classic example is the doorway which survived so long it was visited by a beggar and disappeared at his death. At times some birds, a horse, have saved the ruins of an amphitheater. [Bor99b]

In such a situation, we can note that the strength of existence of some ruins –modeled as it is the more naive to do, like a rigid body  $R$  in space– depends on the number of its observers:

$$p(R, n) = (\top, 1 - \frac{1}{n})$$



FIGURE 3. On Tlön, there are things that exist stronger the more you believe in them. This is a consequence of the strength  $p(\triangle)$  monotonically depending on an increasing variable  $n$  (“trust” in that existence, “belief” that  $\triangle$  exists.)

A last, somewhat dramatic example. What happens if we change topology on  $I$ ? For example, we could brutally forget the Euclidean topology of the closed interval  $[0, 1]$ , and regard  $I$  as the disjoint union  $\{\{t\} \mid t \in [0, 1]\}$  of its points; so, the subobject classifier becomes the disjoint union of  $[0, 1]$  copies of  $\{\perp, \top\}$ . (See Figure 4 below for a picture.)

**Example 4.11** (La campagna incendiata). Il Berkeley idealista degli infiniti istanti di tempo continuo, disconnessi e incomunicabili:  $\Omega_I = \coprod_{t \in [0, 1]} \{\perp, \top\}$ . Questa particolare struttura logica influences language making it become berkeleyan *instantaneism*: terms of the perceptual bundle are recorded and stockpiled by instantaneous accretion, by disjoint sum of their constituents: “round airy-light on dark” or “pale-orange-of-the-sky”; objects are determined by their simultaneity, instead of their logical dependence: accretion builds meaning, temporal sequentiality is just an illusion, a mistake of perception.

Spinoza ascribes to his inexhaustible divinity the attributes of extension and thought; no one in Tlön would understand the juxtaposition of the first (which is typical only of certain states) and the second - which is a perfect synonym of the cosmos. In other words, they do not conceive that the spatial persists in time. The perception of a cloud of smoke on the horizon and then of the burning field and then of the half-extinguished cigarette that produced the blaze is considered an example of association of ideas. [Bor99b]

(Berkeleyan instantaneism is, of course, way larger than a paragraph: we explore the consequences of this principle in Remark 4.7, in form of a link between Berkeley, the additive nature of truth force in Tlön, and its peculiar arithmetics.)

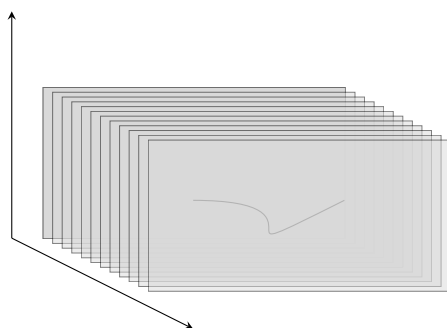


FIGURE 4. Times as an infinite, and infinitely subdivided, sequence of distinguished instants: the “Berkeley paradox” of non existence of causality lives in a topos, as nearby slices  $E_t, E_s$  for small  $|s - t|$  give no predictive power on how (and if) the transition from  $E_t$  to  $E_s$  might happen.

## 5. VISTAS ON ONTOLOGIES

¿No basta un solo término repetido para desbaratar y confundir la serie del tiempo? ¿Los fervorosos que se entregan a una línea de Shakespeare no son, literalmente, Shakespeare?

[Bor53]

Looking at our section 4 it’s surprising how mathematically consistent Borges’ universe becomes when it is described through category theory. As we already said, this has to be regarded as an experiment, a literary *divertissement*, and a test-bench for our main claim, that as abstract as it may seem ontology has a quantitative content.

There is clearly no point to just analyse Borges’ literary work: we can go further. The present section has this purpose, while sketching somelong term goals our work can aspire

to become. We begin with a more academic discussion of classical idealism *à la* Berkeley; of course, in light of our [Example 4.11](#), this can still be linked to Tlön's universe, as Borges has always been fascinated by, and mocked, classical idealism (see ??).

To this, it follows a more wide-ranging discourse; surely a less substantiated one, but we aim at laying a foundation for future work, ours or others', in a research track that we feel is fertile and promising.

**5.1. Answers to the idealist.** Let us reconsider [Example 4.4](#); although in passing, we mentioned famous Berkeley's view of perception as a bundle of stimulation incapable to cohere. Let's say clearly that endeavouring on such a wide ground as classical idealism isn't the purpose of our work; yet, in his novel Borges regards Tlön's language and philosophy as a concrete realisation of Berkeley's theory of knowledge.

Within topos theory, è scorretto dire che  $p$  è falsa in "assoluto", ma che semmai è "falsa con forza  $t$ " dove  $t \in I$ . Il berkeleyano vorrebbe ottenere (una volta accettato che le monete esistano solo un po' in alcuni punti) che  $\sum_u p(u, c, d) = (\perp, 1)$ .

Non può per costruzione, ma se volesse deve fornire un modello alternativo del paradosso che sia altrettanto valido (ci sembra sia difficile comunque non parlare in termini di "forze" o loro equivalenti, in "istanti" singoli).

Ad essere ancora più pignoli, alla domanda se le monete esistano o meno la risposta corretta è a sua volta una domanda: "In quale  $I$  sei?" ed essendo  $I$  totalmente ordinato "In quale suo punto vuoi calcolare la forza?".

Il punto può essere un riferimento temporale, spaziale o di qualunque altro genere (a seconda della struttura di  $U$ ), l'importante è che l'insieme abbia le proprietà da noi indicate.

Per poter "parlare" all'interno del topos, e rispondere alla domanda sull'esistenza, è necessario prima fornire i "dati" (osservatore, subset di  $C$ , giorno della settimana) e poi possiamo assegnare un valore di verità dotato di forza.

Sostituire nozioni classiche (identità, persistenza-nel-tempo) con nozioni precise suggerite dal linguaggio che proponiamo, vuol dire perciò modificare anche il linguaggio naturale. Quando l'idealista afferma che negli istanti in cui chiude gli occhi il tavolo davanti a lui non esiste più, sta anch'egli parlando di "forza" nel senso qui definito: lo deve solo esplicitare. Si mette in un universo (un topos) "non-classico" con più valori di verità e deve semplicemente calcolare *qual* è il valore di  $p$ .

L'ampliamento dei valori di verità <sup>23</sup> si traduce in un ampliamento delle risposte possibili da dare alla domanda sulle monete. Non indichiamo quella "corretta" ma il topos in cui trovarla, il contesto di discorso dal quale non si può prescindere, intuitivamente un contesto fuzzy, che evita le sterili dicotomie in cui era impantanato il dibattito da qualche secolo.

Da notare (come anticipato in 1.4) che abbiamo potuto modellizzare il paradosso senza impiegare un framework di logica temporale; è lecito interpretare gli elementi di  $I$  come istanti, ma non necessario. La descrizione in termini temporali è un sottocaso del modello più generale che abbiamo fornito. Persino un presentista o l'idealista coerente di Borges

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<sup>23</sup>Ad esser precisi:  $\{0, 1\}$  è "fisso"; la varietà di risposte è data dalla scelta di  $I$  e dalla struttura di  $U$

(che dalle tesi berkeleyane deduce l'inesistenza del tempo) può stabilire la verità di  $p$  senza abiurare alla sua posizione ontologica, ma deve poi accettare il risultato del calcolo.

Altro vantaggio che l'idealista, e non solo, potrebbe apprezzare è dato dalla densità di  $I$ . Proprietà che permette, nel caso di disaccordo tra due interlocutori  $X$  e  $Y$  sul grado di esistenza della moneta, (per via principalmente della dipendenza funzionale delle proposizioni dalle configurazioni), di trovare una forza intermedia compresa tra la forza assegnata da  $X$  e quella assegnata da  $Y$ . In un certo senso questo è un modo di formalizzare l'intersoggettività all'interno di  $Tlön$ .

La posizione metafisica che emerge dal lavoro sembra una via di mezzo tra empirismo e idealismo.

La critica storica dell'idealismo al metodo scientifico è l'uso di prove indirette di esistenza (come le dimostrazioni non costruttive che non piacciono agli intuizionisti).

Nella nostra visione fuzzy dell'esistenza si può assumere che la barca invisibile che nessuno vede, se non perché l'acqua si sposta, non esista, o assumere che la barca ci sia ma con una forza minore del massimo <sup>24</sup>.

Se si usa questo linguaggio non si può semplicemente affermare che la barca non esiste perché non è osservabile, posizione che non spiega lo spostamento dell'acqua <sup>25</sup>; in breve non ci si può scordare del dettaglio per cui, anche se è lecito pensare che il mondo sparisca quando chiudo gli occhi (per cui posso non assumere che abbia forza di esistenza massima), nondimeno riappare quando li riapro (per cui globalmente non può avere forza 0).

Se la freccia "sparisce" e nessun altro osservatore in un dato istante ne fa esperienza, evidentemente essa esiste con forza minore di 1. Essendo *dentro*  $Tlön$  questa è una concessione alla visione epistemica dei suoi abitanti, i quali non possono sapere cosa ne è delle monete degli altri osservatori; e in questo senso, non potendo calcolare la forza globale, il modello è inutilizzabile dai  $tlöniani$ . Possiamo formalizzare adeguatamente la loro ontologia, per poter dire qualcosa in più rispetto a chi vive in essa.

All'idealista tuttavia possiamo rispondere, se egli parla il linguaggio delle categorie <sup>26</sup>.

**5.2. By their fruits you will know them.** The main point of our paper can be summarised very concisely: an "ontology" is a category  $\mathcal{O}$ , inside which "Being unravels". Every existence theory shall be reported, and is relative, to a fixed ontology  $\mathcal{O}$ , the "world we live in"; such existence theory coincides with the internal language  $\mathcal{L}(\mathcal{O})$  of the ontology/category (from now on we employ the two terms as synonyms), in the syntax-semantics adjunction of  $\llbracket \cdot \rrbracket$ .

<sup>24</sup>Come si è detto in 1.4 è un tentativo di risoluzione epistemica del paradosso. cfr. Quando si parla di "maggiore è il numero di osservatori più intenso è il grado di esistenza delle monete"

<sup>25</sup>Anche il metafisico che non assume avvenga il processo di causazione deve spiegare la relazione (qualunque essa sia, se c'è) o comunque la correlazione tra il fenomeno "la barca si muove" (che presuppone l'esistenza della barca) e il fenomeno "l'acqua si sposta", il quale non è collegato, ad esempio, al fenomeno dell'esistenza delle nuvole

<sup>26</sup>Preme comunque notare che prendiamo qui come interlocutore l'idealismo in quanto esempio di metafisica "estrema", e perché era negli obiettivi del racconto di Borges fornirne una lettura critica e alternativa, ma non è questo il claim del paper.

So determined, the internal language of an ontology  $\mathcal{O}$  is the collection of “things that can be said” about the elements of the ontology.

If, now, ontology is the study of Being, and if we are structuralist in the metatheory (cf. [section 1](#)), we cannot know beings but through their attributes. Secretly, this is *Yoneda lemma* (cf. [\[Bor94a, 1.3.3\]](#)), the statement that the totality of modes of understanding a “thing”  $X$  coincides with the totality of modes your language allows you to probe  $X$ . Things do not exist out of an ontology; objects do not exist out of a category; types do not exist out of a type theory. In relational structures, objects are known via their modes of interaction with other objects, and these are modeled as morphisms  $U \rightarrow X$ ; Yoneda lemma posits that we shall “know objects by their morphisms”: the object  $X : \mathcal{C}$  coincides with the totality of all morphisms  $U \rightarrow X$ , organised in a coherent bundle (a functor  $\mathcal{C}(-, X) : \mathcal{C} \rightarrow \mathbf{Set}$ ).

All in all, an ontology is a mode of understanding the attributes of Being: a category, be it in an Aristotelic or in a structuralist sense. As a consequence *metaontology*, i.e. the totality of such ways of understanding, must coincide with the general theory of such individual modes: with *category theory*.

We should say no more on the matter: everything else pertains to *metaontology*.

Indeed, questions as “where is language” and what general principles inspire it might have an anthropological or even neurophysiological answer; not an ontological one. Or at least, not without paying a price: operative ontology as sketched here has limits. It can’t speak of Being out of the one ignited by itself. (read as: category theory *has limits*: it cannot speak efficiently of objects out of a fixed universe of discourse. Implicit: category theory also has merits.)

**5.3. Metaontology.** The gist of our [Example 4.4](#) is that  $X, Y, Z$  can’t assess the existence of the coins classically; they just have access to partial information allowing neither a global statement of existence on the set  $C$  of coins lost by  $X$ , nor an unbiased claim about the meaning thereof. Coins are untouched as a global lost conglomerate, yet their strength of existence is very likely to change *locally*.

This begs the important questions of to which extent language determines ontology. And to which extent it constrains its expressive power? To which extent the inhabitants of Tlön fail to see what is exactly “a topos further” (persistence of existence through time). To what extent *we* fail to see that...

Far from claiming we resolved such metaontological issues, here we make an hopefully clarifying statement: according to Quine and his school [\[1\]](#), ontology can be defined as the “domain over which [logical, or natural language’s] quantifiers run”. This is not wrong; in fact, it is perfectly compatible with our views. But we work at a raised complexity level, in the following sense. In our framework, ontology *à la* Quine still is a category, because (cf. [Definition 3.2](#)) a quantifier can be described as a certain specific kind of functor

$$\forall, \exists : PX \rightarrow PY$$

between powersets regarded as *internal* categories of our ambient ontology  $\mathcal{O}$ ; in light of this, it’s easy to imagine this pattern to continue: if Quine calls metaontology what we call

ontology, i.e. the metacategory grounding his propositional calculus is a single ambient category *among many*, we live “one universe higher” in the cumulative hierarchy of foundations and meta-foundations: our ontology possesses a higher dimensionality, and harbours Quinean theory as an internal structure (see [Definition A.15](#) for the definition of an internal category). So, it shall exist, *somewhere* -at least in some secret way, hidden from the comprehension of men- a language that calls ontology what we call metaontology (and thus a language that declassifies Quine’s to a sub-ontology -whatever this means).

Finding the bottom of this tower of turtles is the aim of the track of research, of very ancient tradition, within which we want to insert the present work.

Of course our work does not make a single comment on how, if, and when, this ambitious foundational goal can be achieved. We just find remarkable that framing Quine’s definition in this bigger picture is not far from the so-called practice of “negative thinking” in category theory: negative thinking is the belief that a high-dimensional/complex entity can be understood by means of the analogy with its low-dimensional/complexity counterparts; see [\[n17, n11\]](#) for a minimal introduction to the principle, [\[BS10\]](#) for a practical introduction, and see [\[Gow07\]](#) for what Gowers calls “backwards generalisation”.

Riassumendo: per “ontologia” in generale intendiamo una theory of existence, namely una teoria dei modi in cui gli oggetti si relazionano gli uni agli altri per essere compresi da noi (Yoneda Lemma risultato centrale). Più propriamente è allora il caso di parlare di (operative) ontologies, al plurale, che sono le categorie col loro linguaggio interno. Vale a dire (1.3) gli “ambienti” dentro i quali studiare le relazioni tra gli oggetti. Come sappiamo da 1.2 questa concezione relazionale dell’ontologia è ispirata dall’essere strutturalisti, ma solo nella metateoria.

Qualunque discorso su come gerarchizzare le varie ontologie, sulle relazioni tra esse, è quindi un discorso sulle categorie, ed è la nostra (informale) metaontologia qui accennata.

Le grandi domande esistenziali non pertengono, perciò, a nessuno di questi domini. The reasons why we implemented this extreme reduction to this historical matters are fully explained in 1.4 and are essentially practical.

Ogni altro discorso crediamo debba partire da qui.

**5.4. Conclusion.** The circle apparently closes on, and motivates better, our initial foundational choice: categories and their theory correspond 1:1 to ontology and its theory. However, countless important issues remain open: in what sense this is satisfying? In what sense the scope of our analysis is not limited by this choice? What’s his foundation? Is this a faithful way to describe such an elusive concept as “Being”?

None of these questions is naive; in fact, each legitimately pertains to metaontology, and has no definitive answer. More or less our stance is as follows: approaching problems in ontology with a reasonable amount of mathematical knowledge is fruitful. Yet, the problem of what is a foundation for that mathematics remains (fortunately!) wide open; it pertains to metaontology, whose ambitious effort is to clarify “what there is”. We believe the philosophers’ job to work in synergy with quantitative knowledge, approaching the issue with complementary tools.

## APPENDIX A. CATEGORY THEORY

El atañor está apagado -repitió- y están llenos de polvo los alambiques. En este tramo de mi larga jornada uso de otros instrumentos.

---

[Bor75]

**A.1. Fundamentals of CT.** Throughout the paper we employ standard basic category-theoretic terminology, and thus we refrain from giving a self contained exposition of elementary definitions. Instead, we rely on famous and wide-spread sources like [Bor94a, Bor94b, ML98, Rie17, Lei14, Sim11].

Precise references for the basic definitions can be found

- for the definition of category, functor, and natural transformation, in [Bor94a, 1.2.1], [ML98, I.2], [Bor94a, 1.2.2], [ML98, I.3], [Bor94a, 1.3.1].
- The Yoneda lemma is stated as [Bor94a, 1.3.3], [ML98, III.2].
- For the definition of co/limit and adjunction, in [Bor94a, 2.6.2], [ML98, III.3], [Bor94a, 2.6.6], [ML98, III.4].
- For the definition of accessible and locally presentable category in [Bor94b, 5.3.1], [Bor94b, 5.2.1], [AR94].
- Basic facts about ordinal and cardinal numbers can be found in [Kun83]; another comprehensive reference on basic and non-basic set theory is [Jec13].
- The standard source for Lawvere functorial semantics is Lawvere's PhD thesis [Law63]; more modern accounts are [HP07].
- Standard references for topos theory are [MLM92, Joh77]. See in particular [MLM92, VI.5] and [Joh77, 5.4] for what concerns the Mitchell-Bénabou language of a topos.

**A.2. Toposes.** For us, an *ordinal number* will be any well-ordered set, and a *cardinal number* is any ordinal which is not in bijection with a smaller ordinal. Every set  $X$  admits a unique *cardinality*, i.e. a least ordinal  $\kappa$  with a bijection  $\kappa \cong X$  such that there are no bijections from a smaller ordinal. We freely employ results that depend on the axiom of choice when needed. A cardinal  $\kappa$  is *regular* if no set of cardinality  $\kappa$  is the union of fewer than  $\kappa$  sets of cardinality less than  $\kappa$ ; all cardinals in the following subsection are assumed regular without further mention.

Let  $\kappa$  be a cardinal; we say that a category  $\mathcal{A}$  is  $\kappa$ -filtered if for every category  $\mathcal{J} \in \mathbf{Cat}_{<\kappa}$  with less than  $\kappa$  objects,  $\mathcal{A}$  is injective with respect to the cone completion  $\mathcal{J} \rightarrow \mathcal{J}^\triangleright$ ; this means that every diagram

$$\begin{array}{ccc} \mathcal{J} & \xrightarrow{D} & \mathcal{A} \\ \downarrow & \nearrow \bar{D} & \\ \mathcal{J}^\triangleright & & \end{array}$$

has a dotted filler  $\bar{D} : \mathcal{J}^\triangleright \rightarrow \mathcal{A}$ .



We say that a category  $\mathcal{C}$  admits filtered colimits if for every filtered category  $\mathcal{A}$  and every diagram  $D : \mathcal{A} \rightarrow \mathcal{C}$ , the colimit  $\operatorname{colim} D$  exists as an object of  $\mathcal{C}$ . Of course, whenever an ordinal  $\alpha$  is regarded as a category, it is a filtered category, so a category that admits all  $\kappa$ -filtered colimits admits all colimits of chains

$$C_0 \rightarrow C_1 \rightarrow \cdots \rightarrow C_\alpha \rightarrow \cdots$$

with less than  $\kappa$  terms. A useful, completely elementary result is that the existence of colimits over all ordinals less than  $\kappa$  implies the existence of  $\kappa$ -filtered colimits; this relies on the fact that every filtered category  $\mathcal{A}$  admits a cofinal functor (see [Bor94a]) from an ordinal  $\alpha_{\mathcal{A}}$ .

We say that a functor  $F : \mathcal{A} \rightarrow \mathcal{B}$  *commutes with* or *preserves* filtered colimits if whenever  $\mathcal{J}$  is a filtered category,  $D : \mathcal{J} \rightarrow \mathcal{A}$  is a diagram with colimit  $L = \operatorname{colim}_{\mathcal{J}} D_j$ , then  $F(L)$  is the colimit of the composition  $F \circ D$ . Another common name for such an  $F$  is a *finitary* functor, or a functor *with rank*  $\omega$ .

**Definition A.1.** Let  $\mathcal{C}$  be a category;

- We say that  $\mathcal{C}$  is  $\kappa$ -*accessible* if it admits  $\kappa$ -filtered colimits, and if it has a *small* subcategory  $\mathcal{S} \subset \mathcal{A}$  of  $\kappa$ -presentable objects such that every  $A \in \mathcal{A}$  is a  $\kappa$ -filtered colimit of objects in  $\mathcal{S}$ .
- We say that  $\mathcal{C}$  is (locally)  $\kappa$ -*presentable* if it is accessible and cocomplete.

The theory of presentable and accessible categories is a cornerstone of *categorical logic*, i.e. of the translation of model theory into the language of category theory.

Accessible and presentable categories admit *representation theorems*:

- A category  $\mathcal{C}$  is accessible if and only if it is equivalent to the ind-completion  $\operatorname{Ind}_{\kappa}(\mathcal{S})$  of a small category, i.e. to the completion of a small category  $\mathcal{S}$  under  $\kappa$ -filtered colimits;
- A category  $\mathcal{C}$  is presentable if and only if it is a full reflective subcategory of a category of presheaves  $i : \mathcal{C} \rightarrow \operatorname{Cat}(\mathcal{S}^{\operatorname{op}}, \operatorname{Set})$ , such that the embedding functor  $i$  commutes with  $\kappa$ -filtered colimits.

All categories of usual algebraic structures are accessible, and they are locally presentable as soon as they are cocomplete; an example of a category which is  $\aleph_1$ -presentable but not  $\aleph_0$ -presentable: the category of metric spaces and short maps.

We now glance at *topos theory*:

**Definition A.2.** An *elementary topos* is a category  $\mathcal{E}$

- which is *Cartesian closed*, i.e. each functor  $- \times A$  has a right adjoint  $[A, -]$ ;
- having a *subobject classifier*, i.e. an object  $\Omega \in \mathcal{E}$  such that the functor  $\operatorname{Sub} : \mathcal{E}^{\operatorname{op}} \rightarrow \operatorname{Set}$  sending  $A$  into the set of isomorphism classes of monomorphisms  $\begin{bmatrix} U \\ \downarrow \\ A \end{bmatrix}$  is representable by the object  $\Omega$ .

The natural bijection  $\mathcal{E}(A, \Omega) \cong \text{Sub}(A)$  is obtained pulling back the monomorphism  $U \subseteq A$  along a *universal arrow*  $t : 1 \rightarrow \Omega$ , as in the diagram

$$\begin{array}{ccc} U & \longrightarrow & 1 \\ \downarrow & \lrcorner & \downarrow t \\ A & \xrightarrow{\chi_U} & \Omega \end{array}$$

so, the bijection is induced by the map  $\begin{bmatrix} U \\ \downarrow \\ A \end{bmatrix} \mapsto \chi_U$ .

**Definition A.3.** A *Grothendieck topos* is an elementary topos that, in addition, is locally presentable.

The well-known *Giraud theorem* gives a proof for the difficult implication of the following *recognition principle* for Grothendieck toposes:

**Theorem A.4.** Let  $\mathcal{E}$  be a category; then  $\mathcal{E}$  is a Grothendieck topos if and only if it is a left exact reflection of a category  $\text{Cat}(\mathcal{A}^{\text{op}}, \text{Set})$  of presheaves on a small category  $\mathcal{A}$ .

(recall that a *left exact reflection* of  $\mathcal{C}$  is a reflective subcategory  $\mathcal{R} \hookrightarrow \mathcal{C}$  such that the reflector  $r : \mathcal{C} \rightarrow \mathcal{R}$  preserves finite limits. It is a reasonably easy exercise to prove that a left exact reflection of a Grothendieck topos is again a Grothendieck topos; Giraud proved that all Grothendieck toposes arise this way.)

**A.3. A little primer on algebraic theories.** The scope of this short subsection is to collect a reasonably self-contained account of functorial semantics. It is unrealistic to aim at such a big target as providing a complete account of it in a single appendix; the reader is warmly invited to parallel their study with more classical references as [Law63].

**Definition A.5** (Lawvere theory). A *Lawvere theory* is a category having objects the natural numbers, and where the sum on natural numbers has the universal property of a categorical product, as defined e.g. in [Bor94a, 2.1.4].

Let us denote  $[n]$  the typical object of  $\mathcal{L}$ . Unwinding the definition, we deduce that in a Lawvere theory  $\mathcal{L}$  the sum of natural numbers  $[n + m]$  is equipped with two morphisms  $[n] \leftarrow [n + m] \rightarrow [m]$  exhibiting the universal property of the product.

Every Lawvere theory comes equipped with a functor  $p : \text{Fin}^{\text{op}} \rightarrow \mathcal{L}$  that is the identity on objects and preserves finite products. A convenient shorthand to refer to the Lawvere theory  $\mathcal{L}$  is thus as the functor  $p$ , or as the pair  $(p, \mathcal{L})$ .

**Definition A.6.** The category **Law** of Lawvere theories has objects the Lawvere theories, understood as functors  $p : \text{Fin}^{\text{op}} \rightarrow \mathcal{L}$ , and morphisms the functors  $h : p \rightarrow q$  such that the triangle

$$\begin{array}{ccc} & \text{Fin}^{\text{op}} & \\ p \swarrow & & \searrow q \\ \mathcal{L} & \xrightarrow{h} & \mathcal{M} \end{array}$$

is commutative. It is evident that **Law** is the subcategory of the undercategory  $\mathbf{Fin}^{\text{op}}/\mathbf{Cat}$  (see e.g. [ML98, I.6] for a precise definition) made by those functors that preserve finite products.

**Remark A.7.** The category **Law** has no nonidentity 2-cells; this is a consequence of the fact that a natural transformation  $\alpha : h \Rightarrow k$  that makes the triangle “commute”, i.e.  $\alpha * p = \text{id}_q$  must be the identity on all objects.

**Example A.8** (The trivial theories). The category  $\mathbf{Fin}^{\text{op}}$ , opposite to the category of finite sets and functions, is the initial object in the category **Law**; the terminal object is constructed as follows: the category  $\mathcal{T}$  has objects the natural numbers, and  $\mathcal{T}([n], [m]) = \{*\}$  for every  $n, m \in \mathbb{N}$ . It is evident that given this definition, there is a unique identity-on-objects functor  $\mathcal{L} \rightarrow \mathcal{T}$  for every other Lawvere theory  $(p, \mathcal{L})$ .

**Definition A.9** (Model of a Lawvere theory). A *model* for a Lawvere theory  $(p, \mathcal{L})$  consists of a product-preserving functor  $L : \mathcal{L} \rightarrow \mathbf{Set}$ . The subcategory  $[\mathcal{L}, \mathbf{Set}]_{\times} \subset [\mathcal{L}, \mathbf{Set}]$  of models of the theory  $\mathcal{L}$  is *full*, i.e. a morphism of models  $L \rightarrow L'$  consists of a natural transformation  $\alpha : L \Rightarrow L'$  between the two functors.

Observe that the mere request that  $\alpha : L \rightarrow L'$  is a natural transformation between product preserving functors means that  $\alpha_{[n]} : L[n] \rightarrow L'[n]$  coincides with the product  $(\alpha_{[1]})^n : L[1]^n \rightarrow L'[1]^n$ .

**Proposition A.10.** Let  $p : \mathbf{Fin}^{\text{op}} \rightarrow \mathcal{L}$  be a Lawvere theory. Then, the following conditions are equivalent for a functor  $L : \mathcal{L} \rightarrow \mathbf{Set}$ :

- $L$  is a model for the Lawvere theory  $(p, \mathcal{L})$ ;
- the composition  $L \circ p : \mathbf{Fin}^{\text{op}} \rightarrow \mathbf{Set}$  preserves finite products;
- there exists a set  $A$  such that  $L \circ p = \mathbf{Set}(j[n], A)$ .

**Corollary A.11.** The square

$$\begin{array}{ccc} \mathbf{Mod}(p, \mathcal{L}) & \xrightarrow{r} & [\mathcal{L}, \mathbf{Set}] \\ u \downarrow & & \downarrow p^* \\ \mathbf{Set} & \xrightarrow{N_j} & [\mathbf{Fin}^{\text{op}}, \mathbf{Set}] \end{array}$$

is a pullback of categories. The functor  $u$  is completely determined by the fact that  $u(L) = L[1]$ ,  $r$  is an inclusion, and  $N_j(A) = \lambda F. \mathbf{Set}(F, A)$  is the functor induced by the inclusion  $j : \mathbf{Fin} \subset \mathbf{Set}$ .

**Corollary A.12.** The category of models  $\mathbf{Mod}(p, \mathcal{L})$  of a Lawvere theory is a locally presentable, accessibly embedded, complete and cocomplete subcategory of  $[\mathcal{L}, \mathbf{Set}]$ . Moreover, the forgetful functor  $u : \mathbf{Mod}(p, \mathcal{L}) \rightarrow \mathbf{Set}$  of Corollary A.11 is *monadic* in the sense of [Bor94b, 4.4.1]. A complete proof of all these facts is in [Bor94b, 3.4.5], [Bor94b, 3.9.1], [Bor94b, 5.2.2.a]. A terse argument goes as follows: the functors  $p^*, N_j$  are accessible right adjoints between locally presentable categories; therefore, so is the pullback diagram:  $r$  is

a fully faithful, accessible right adjoint, and  $u$  is an accessible right adjoint, that moreover reflects isomorphisms. It can be directly proved that it preserves the colimits of split coequalizers, and thus the adjunction  $f \dashv u$  is monadic by [Bor94b, 4.4.4].

The last technical remark that we collect sheds a light on the discorso prolisso in [subsection 1.3](#): the models of a theory  $\mathcal{L}$  interpreted in the category of models of a theory  $M$  correspond to the models of a theory  $\mathcal{L} \otimes M$ , defined by a suitable universal property:

$$\text{Mod}(\mathcal{L} \otimes M, \text{Set}) \cong \text{Mod}(\mathcal{L}, \text{Mod}(M, \text{Set})) \cong \text{Mod}(M, \text{Mod}(\mathcal{L}, \text{Set})).$$

**Definition A.13.** Given two theories  $\mathcal{L}$  and  $M$  it is possible to construct a new theory called the *tensor product*  $\mathcal{L} \otimes M$ ; this new theory can be characterized by the following universal property: the models of  $\mathcal{L} \otimes M$  consist of the category of  $\mathcal{L}$ -models interpreted in the category of  $M$ -models or, equivalently (and this is remarkable) of  $M$ -models interpreted in the category of  $\mathcal{L}$ -models.

**Theorem A.14.** ([Bor94b, 4.6.2]) There is an equivalence between the following two categories:

- Law, regarded as a non-full subcategory of the category  $\text{Fin}^{\text{op}}/\text{Cat}$ , i.e. where a morphism of Lawvere theories consists of a functor  $h : \mathcal{L} \rightarrow M$  that preserves finite products;
- *finitary* monads, i.e. those monads that preserve filtered colimits, and morphisms of monads in the sense of [Bor94b, 4.5.8].

*Proof.* The proof goes as follows: given a Lawvere theory  $p : \text{Fin}^{\text{op}} \rightarrow \mathcal{L}$ , we have shown that the functor  $u : \text{Mod}(p) \rightarrow \text{Set}$  in the pullback square [Corollary A.11](#) has a left adjoint  $f : \text{Set} \rightarrow \text{Mod}(p)$ ; the composition  $uf$  is thus a monad on  $\text{Set}$ . This is functorial, when a morphism of monads is defined  $\square$

category internal to a finitely complete universe

**Definition A.15.** Let  $\mathcal{C}$  be a category with finite limits. An *internal category*  $\underline{A}$  consists of a pair  $(A_0, A_1)$  of objects of  $\mathcal{C}$ , endowed with morphisms  $s, t, c, i$  as in the following diagram

$$A_1 \times_{A_0} A_1 \begin{array}{c} \xrightarrow{p_0} \\ \xrightarrow{c} \\ \xrightarrow{p_1} \end{array} A_1 \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{i} \\ \xleftarrow{t} \end{array} A_0$$

where  $A_1 \times_{A_0} A_1$  is obtained pulling back  $s, t$ . These data must satisfy the following axioms, resembling the category axioms:

- IC1)  $i$  equalises the pair  $(s, t)$ , i.e.  $s \circ i = t \circ i$ , and this composition makes the identity  $1_{A_0}$ ;
- IC2)  $t \circ p_1 = t \circ c$  and  $s \circ p_0 = s \circ c$ ;
- IC3)  $c \circ \langle 1_{A_1}, i \circ s \rangle = c \circ \langle i \circ t, 1_{A_1} \rangle$ , and this composition makes the identity  $1_{A_1}$ ;
- IC4)  $c$  is *associative*, i.e.  $c \circ (1_{A_1} \times_{A_0} c) = c \circ (c \times_{A_0} 1_{A_1})$ .

These axioms are meant to re-enact the definition of category, as an abstract “object of objects” and “object of morphisms”, endowed with maps  $s, t : A_1 \rightarrow A_0$  sending every morphism to its *source* and *target*, with an *identity selector* map  $i : A_0 \rightarrow A_1$  and with a composition partial binary operation  $c : A_1 \times A_1 \rightarrow A_1$  whose domain is the object of *composable* arrows. Cf. [Bor94a, 8] for a thorough discussion.

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