Random Variable

Random experiment

Sample space = Set of possible outcomes of random experiment.

Motivation :- We are not interested in the whole experiment, but value of a numerical quantity which comes out of random experiment.

Random Variable :- A function which takes input from sample space and map it to real numbers.

Or Real Valued function defined on sample space.

Random experiment.

E.g.> Rolling a dice twice.

Q1 What is the probability that sum is 5.

Q2 What is the probability that the smaller of the outcomes is 3.

$$S = \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots(2,6)$$

$$\ldots(6,1),\ldots(6,6)\}$$

X — Sum of outcomes of two rolls.

$X : S \to \mathbb{R}$

$X \in \{2,3,4,5,6,7,8,9,10,11,12\}$

$Y =$ Minimum of Outcome of a roll.

$$Y \in \{1, 2, 3, 4, 5, 6\}$$

X	Events	Probability.
2	$\{(1,1)\}$	$\frac{1}{36}$
3	$\{(1,2),(2,1)\}$	$\frac{2}{36}$
4	$\{(1,3),(2,2),(3,1)\}$	$\frac{3}{36}$
5		$\frac{4}{36}$
$\vdots$		
12		$\frac{1}{36}$

X	P	X	P
2	$\frac{1}{36}$	9	$\frac{4}{36}$
3	$\frac{2}{36}$	10	$\frac{3}{36}$
4	$\frac{3}{36}$	11	$\frac{2}{36}$
5	$\frac{4}{36}$	12	$\frac{1}{36}$
6	$\frac{5}{36}$		
7	$\frac{6}{36}$		
8	$\frac{5}{36}$		

Y	Events	Probability
1	$\{(1,1), \ldots, (1,6), (2,1), (3,1), (4,1), (5,1), (6,1)\}$	$\frac{11}{36}$
2	$\{(2,2), \ldots, (2,6), (3,2), (4,2), (5,2), (6,2)\}$	$\frac{9}{36}$
3		$\frac{7}{36}$
4		$\frac{5}{36}$
5		$\frac{3}{36}$
6		$\frac{1}{36}$

Random experiment.

E.g.> Tossing a Coin three times.

$S = \{HHH, HHT, HTH, HTT, THT, THH, TTH, TTT\}$.

Q1 Of three tosses, how many tosses will be heads?

Q2. Of three tosses, which toss results in a head first?

X — No. of heads that appears.

	X	Y
HHH	3	1
HHT	2	1
HTH	2	1
HTT	1	1
THH	2	2
THT	1	2
TTH	1	3
TTT	0	Nil

$P(X=0) = \frac{1}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$

$P(X=3) = \frac{1}{8}$

$P(Y=1) = \frac{4}{8}$

$P(Y=2) = \frac{2}{8}$

$P(Y=3) = \frac{1}{8}$

$X \in \{0, 1, 2, 3\}$

Y — Toss in which head appears first.

$Y \in \{1, 2, 3\}$.

Discrete random Variable :- A random Variable is Called discrete random Variable if it takes only Countably finite or infinite Values.

Apartment Complex data

- There are 12 apartments in a apartment Complex.
- Each floor has 3 apartments:
  One bedroom, two bedroom, three bedroom.

Apartment No.	Floor No.	No. of bedrooms.	Size of apartment	Distance of apartment from lift.
1	1	1	900.23	500.23
2	1	2	1175.11	325.3
3	1	3	1785.09	⋮
4	2	1	900.12	
6	2	2	⋮	
7	2	3		
8	3	1		
9	3	2		
10	3	3		
11	4	1		
12	4	2	1786.30	455.37
	4	3		

[900, 1800]

Random experiment - Choosing random apartment out of 12 apartments.

X — Floor no. — $\{1, 2, 3, 4\}$

Y — No. of bedrooms — $\{1, 2, 3\}$

Z — Size of apartment — $[900, 1800]$

W — distance of apartment from lift $[300, 501]$

Discrete Random Variables.

1. Maximum from a set of finite numbers.
2. Outcome of rolling a die.
3. No. of Students in Class.
4. No. of Spelling mistakes in a paragraph.

Continuous random Variable

1. Area of apartment.
2. Height and Weight of students.
3. Temperature in a room.

Continuous random Variable:-

A random Variable which takes Values in an interval.

Discrete RV.

Probability mass function :- Let X be a random variables which takes values $x_1, x_2, \ldots, x_N$, then Probability mass function (pmf) is defined as

$$\boxed{p(x_i) = P(X = x_i)}$$

Properties of pmf.

1. $p(x_i) \geq 0$

2. $\sum_{i=1}^{N} p(x_i) = 1.$

E.g. X takes Values

X	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\Downarrow$

Is it pmf?

Yes.

E.g.)

X	1	2	3	4	5
	x_1	x_2	x_3	x_4	x_5
$p(x_i)$	0.4	0.1	0.2	0.1	0.3

— Not a pmf

E.g.)

X	1	2	3	4	5
$p(x_i)$	0.2	0.3	0.4	-0.1	0.2

< 0

Not a pmf.

Example. X $\quad \overset{x_0}{0} \quad \overset{x_1}{1} \quad \overset{x_2}{2} \quad \overset{x_3}{3} \ldots \ldots$ $\quad \lambda > 0$

$p(x_i) - P(X = x_i)$ $\quad \dfrac{c\lambda^0}{0!} \quad \dfrac{c\lambda^1}{1!} \quad \dfrac{c\lambda^2}{2!} \quad \dfrac{c\lambda^3}{3!} \ldots$

Can you find c such that $p(x_i)$ is pmf.

(1) $p(x_i) \geq 0 \implies c \geq 0$.

$$c\frac{\lambda^0}{0!}+c\frac{\lambda^1}{1!}+c\frac{\lambda^2}{2!}+\dots=1$$

$$\Rightarrow c\left(1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\dots\right)=1$$

$$\Rightarrow c\,e^{\lambda}=1$$

$$\Rightarrow \boxed{c=e^{-\lambda}}$$

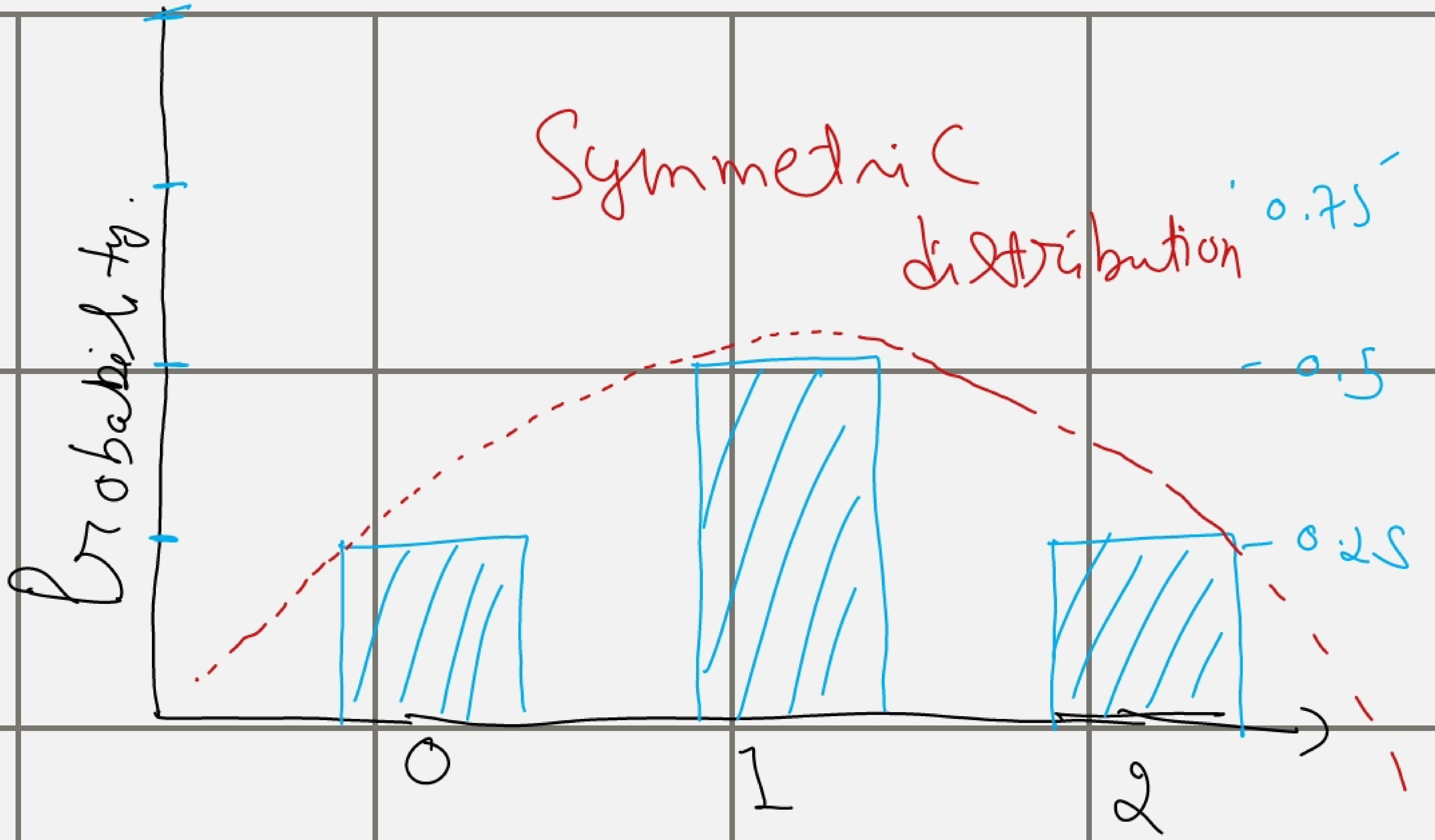
$$e^{\lambda}=1+\lambda+\frac{\lambda^2}{2!}+\dots$$

$$p(x_i) = P(X = x_i) = c \cdot \frac{\lambda^i}{i!}$$

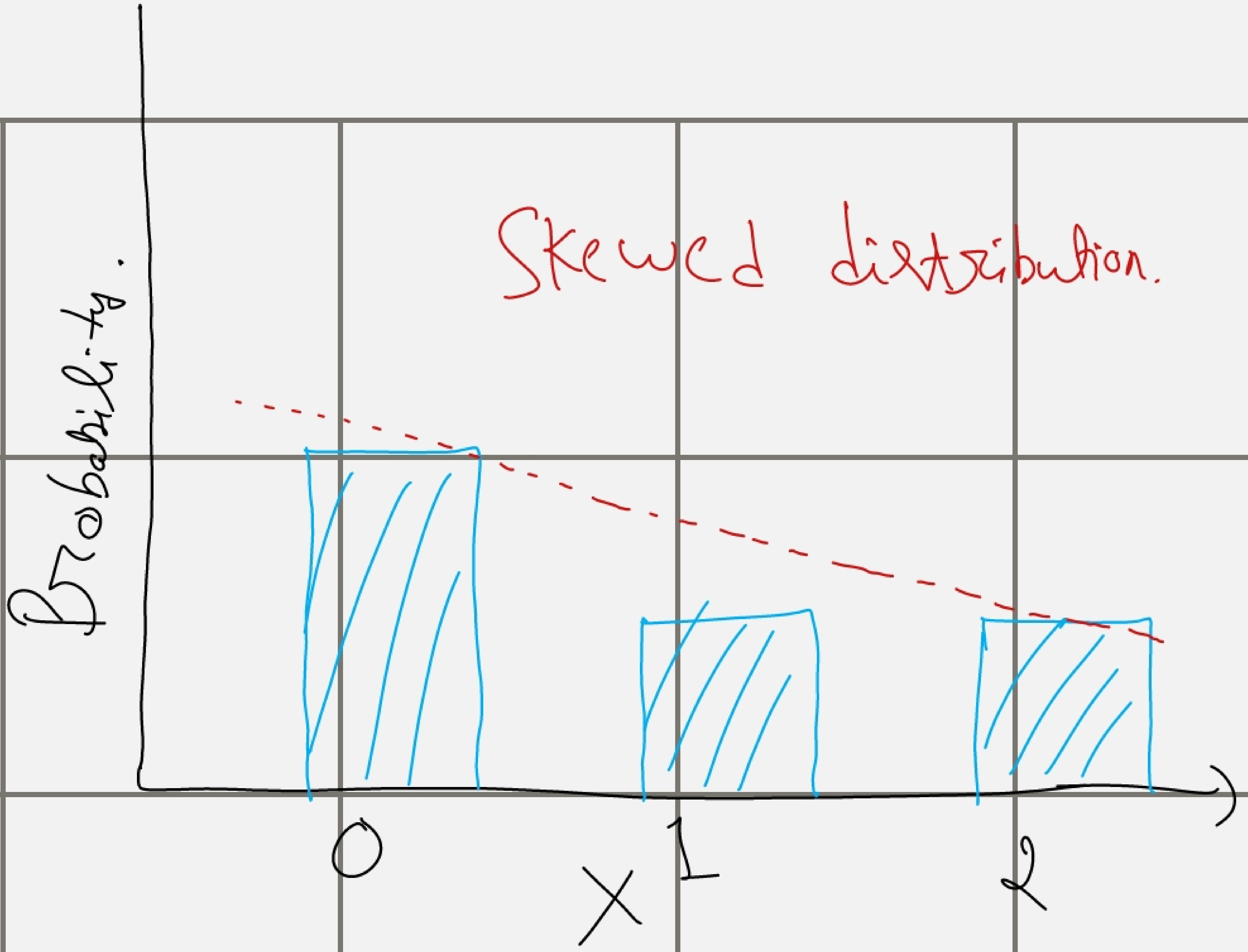
$$\boxed{p(x_i) = e^{-\lambda} \frac{\lambda^i}{i!}}$$

Graph of probability mass function

X	0	1	2
$P(X=x_i)$	0.25	0.5	0.25



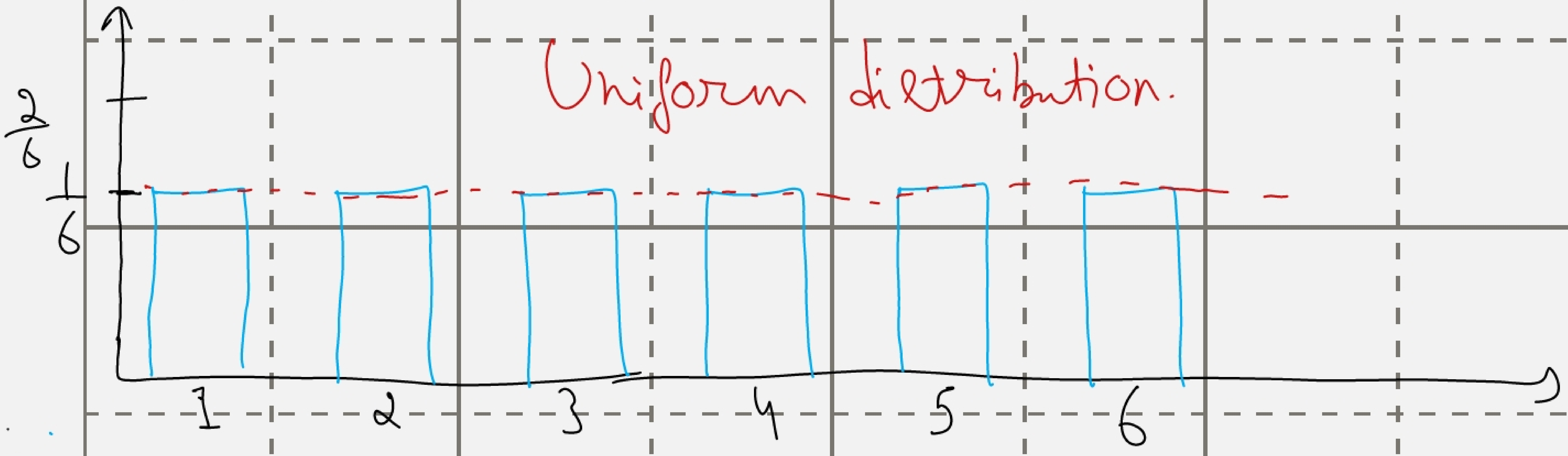
X	0	1	2
$P(X=x_i)$	0.5	0.25	0.25



X	0	1	2
$P(X=x_i)$	0.35	0.55	0.10

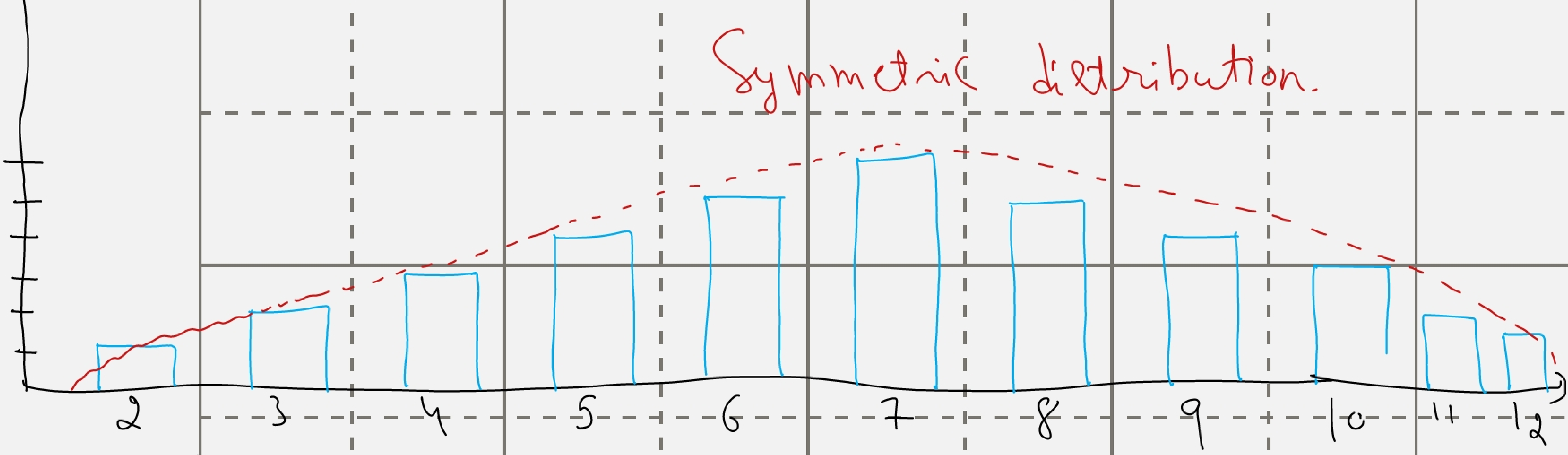
E.g. Rolling a die

X	1	2	3	4	5	6
$P(X=x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$.



X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Try it for Y also?



pmf $\rightarrow$

X

$P(X=x_i)$

$\|$

$p(x_i)$

$P(X \le a)$, $P(X \ge a)$

$P(X \ge 5)$

$$\boxed{F(a) = P(X \le a)}$$

$F: \mathbb{R} \to [0,1]$

Cdf.

Cumulative distribution function:

$F : \mathbb{R} \to [0,1]$ defined by

$$\boxed{F(a) = P(X \le a)} \text{ — Cdf.}$$

$F(-1) = P(X \le -1)$

$a = -1$

$P(X \le 0)$

E.g.>

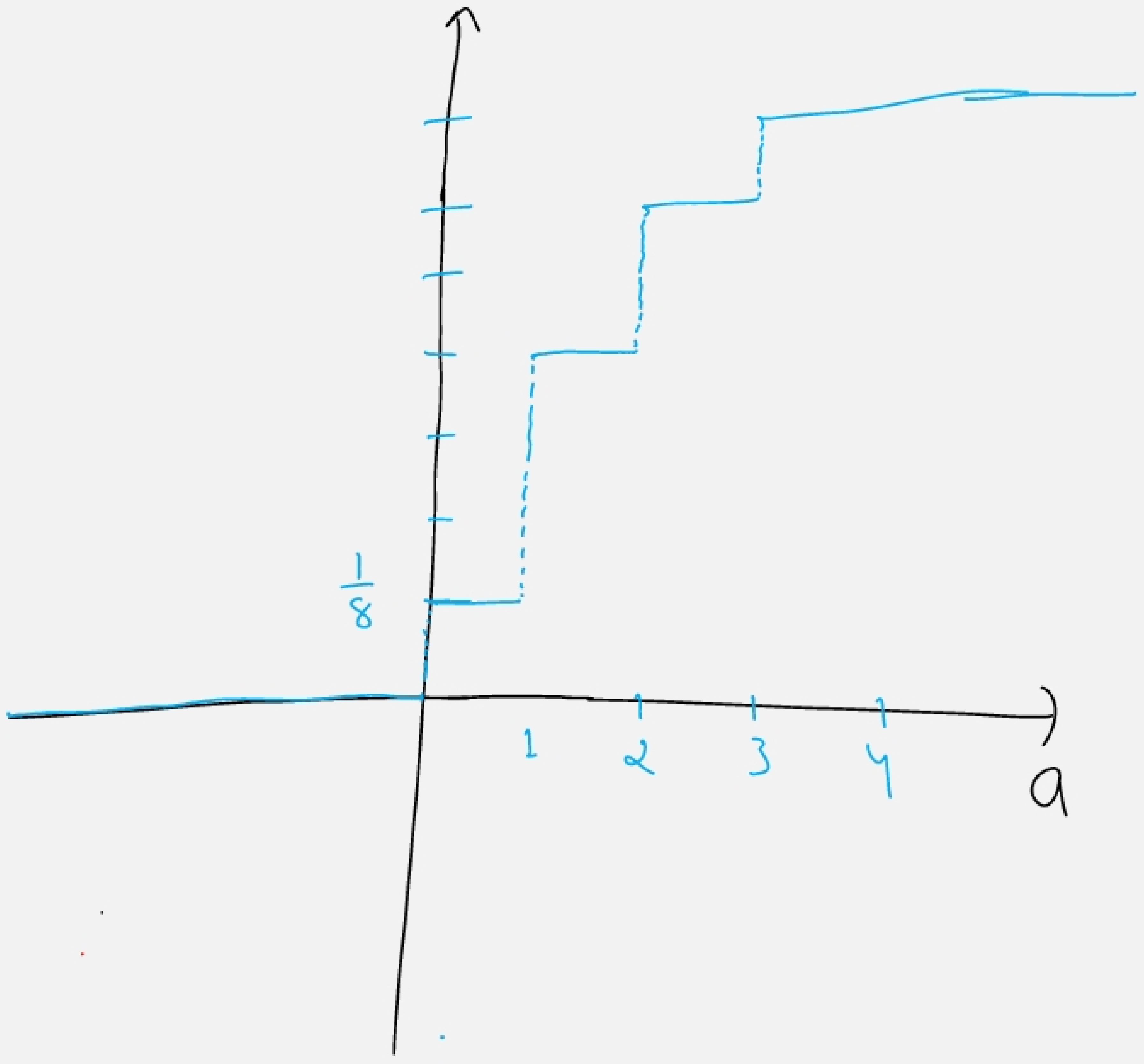
Tossing a Coin thrice

X — No. of heads in an outcome

X	0	1	2	3
$P(X=x_i) =$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$F(a) = \begin{cases} 0 & a < 0 \\ \frac{1}{8} & 0 \le a < 1 \\ \frac{4}{8} & 1 \le a < 2 \\ \frac{7}{8} & 2 \le a < 3 \\ 1 & a \ge 3 \end{cases}$$

$$F(a) = \begin{cases} 0, & a < 0 \\ \frac{1}{8}, & 0 \le a < 1 \\ \frac{4}{8} & 1 \le a < 2 \\ \frac{7}{8} & 2 \le a < 3 \\ 1 & a \ge 3 \end{cases}$$



taking Values $x_1, x_2, \ldots, x_n$

Remark: Let X be a discrete random Variable, then Cdf $F(a)$ will be a step function

Case Study:-

[To analyze no. of Credit Cards owned by a population.

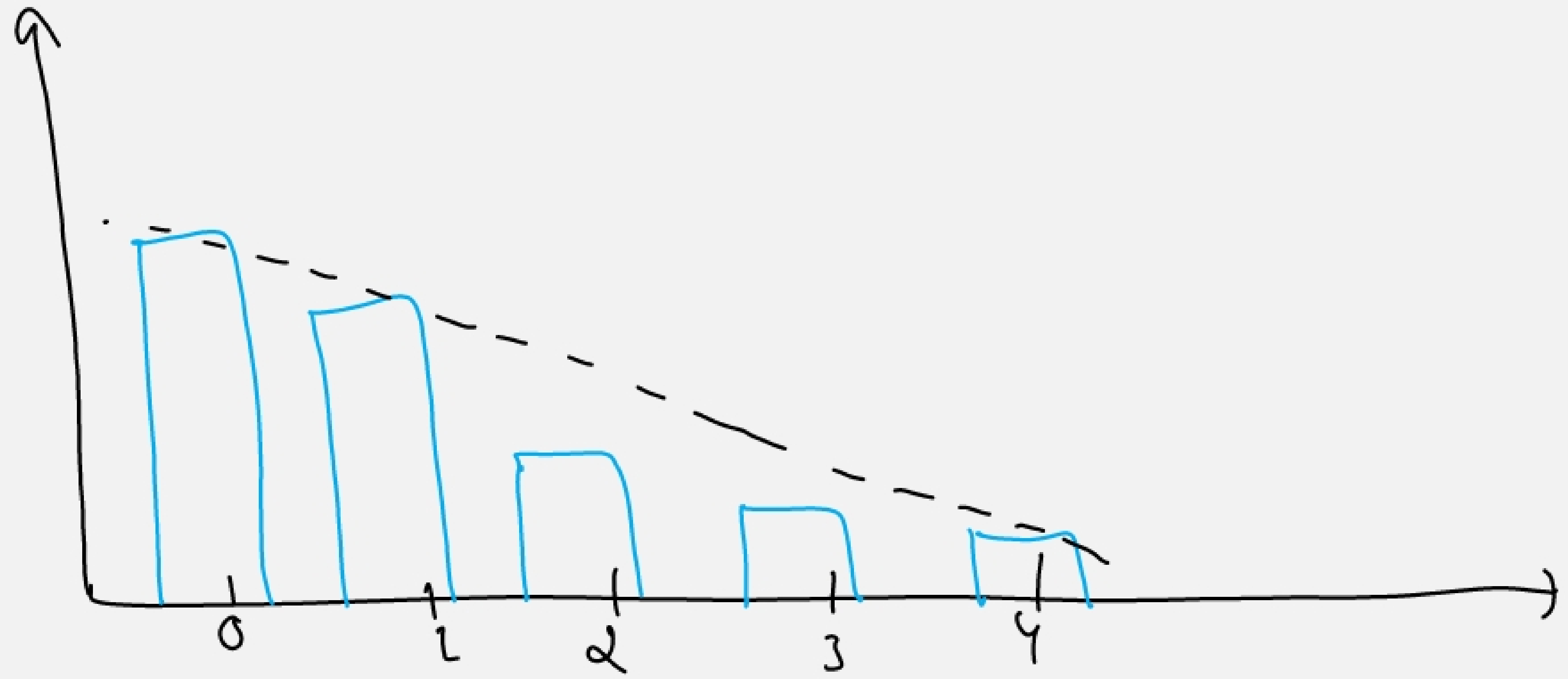
- Collect data:- Ask people how many Credit Cards they own.

$S = \{x_1, x_2, \ldots, x_n\}$

Random experiment :- Picking a random individual from the population.

X — No. of Credit Cards owned by a person.

X	0	1	2	3	4
$P(X = x_i)$	0.42	0.36	0.14	0.06	0.02



Question:- Choose an individual at random.

Is he/she more likely to have 0 Credit Cards or 2 or more Credit Cards?

pmf $\leftarrow P(X=0) = 0.42 = 42\%$

cdf $\rightarrow P(X \geq 2) = 0.14 + 0.06 + 0.02 = 0.22 = 22\%$

$1 - P(X<2) = 0.22$

Question:- Random sample of 1000 individuals, we ask those 1000 individuals, how many Credit Cards they own?

— Everyone is saying, they own a credit Cards

— There is some problem in sample.

(2) 138 people respond that they own two credits Cards.

$\boxed{1000 \times 0.14 = 140}$

Choose an individual at random, how many credit cards would you expect that individual to own?

→ This we will answer with $\boxed{\text{expected value.}}$

Example: let us see one game.

Rolling a single die

pmf.	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
X	1	2	3	4	5	6
	$+1$	-2	$+3$	-4	$+5$	-6

Roll die 100 times

X	Winning Y	Frequency		
1	+1	18	0.18	$\frac{1}{6}$
2	−2	12	0.12	$\frac{1}{6}$
3	+3	20	0.20	$\frac{1}{6}$
4	−4	15	0.15	$\frac{1}{6}$
5	+5	15	0.15	$\frac{1}{6}$
6	−6	20	0.20	$\frac{1}{6}$
		100	1	

Average winning

$$1 \times 0.18 - 2 \times 0.12 + 3 \times 0.20 - 4 \times 0.15 + 5 \times 0.15 - 6 \times 0.20 = -0.51$$

$$\frac{1}{6} - \frac{2}{6} + \frac{3}{6} - \frac{4}{6} + \frac{5}{6} - \frac{6}{6} = \frac{9-12}{6} = \frac{-3}{6}$$

$= -\frac{1}{2} = -0.5$

$X \sim x_1, x_2, \ldots, x_n$

$$\boxed{E(X) = \sum_{i=1}^{n} x_i P(X = x_i)}$$