Random Variable

Random experiment

Sample space = Set of possible outcomes of Random experiment.

Motivation :- We are not interested in the whole experiment, but value of a numerical quantity which comes out of random experiment.

Random Variable :- A function which takes input from sample space and map it to real numbers.

Or Real Valued function defined on sample space.

Random experiment.

E.g.> Rolling a dice twice.

Q 1 What is the probability that sum is 5.

Q 2 What is the probability that the smaller of the outcomes is 3.

$$S = \{(1,1),(1,2),\ldots,(1,6),(2,1),\ldots(2,6)$$
$$\ldots(6,1),\ldots(6,6)\}$$

X — Sum of outcomes of two rolls.

$$X : S \to \mathbb{R}$$

$$X \in \{2,3,4,5,6,7,8,9,10,11,12\}$$

$Y =$ Minimum of outcome of a roll.

$$Y \in \{1, 2, 3, 4, 5, 6\}$$

X	Events	Probability.
2	$\{(1,1)\}$	$\frac{1}{36}$
3	$\{(1,2),(2,1)\}$	$\frac{2}{36}$
4	$\{(1,3),(2,2),(3,1)\}$	$\frac{3}{36}$
5		$\frac{4}{36}$
$\vdots$		
12		$\frac{1}{36}$

X	P	X	P
2	$\frac{1}{36}$	9	$\frac{4}{36}$
3	$\frac{2}{36}$	10	$\frac{3}{36}$
4	$\frac{3}{36}$	11	$\frac{2}{36}$
5	$\frac{4}{36}$	12	$\frac{1}{36}$
6	$\frac{5}{36}$		
7	$\frac{6}{36}$		
8	$\frac{5}{36}$		

Y	Events	Probability
1	$\{(1,1),\ldots,(1,6),(2,1),(3,1),(4,1),(5,1),(6,1)\}$	$\frac{11}{36}$
2	$\{(2,2),\ldots,(2,6),(3,2),(4,2),(5,2),(6,2)\}$	$\frac{9}{36}$
3		$\frac{7}{36}$
4		$\frac{5}{36}$
5		$\frac{3}{36}$
6		$\frac{1}{36}$

Random experiment.

E.g.> Tossing a Coin three times.

$S = \{HHH, HHT, HTH, HTT, THT, THH, TTH, TTT\}$.

Q1 Of three tosses, how many tosses will be heads?

Q2. Of three tosses, which toss results in a head first?

X — No. of heads that appears.

	X	Y
HHH	3	1
HHT	2	1
HTH	2	1
HTT	1	1
THH	2	2
THT	1	2
TTH	1	3
TTT	0	Nil

$P(X=0) = \frac{1}{8}$

$P(X=1) = \frac{3}{8}$

$P(X=2) = \frac{3}{8}$

$P(X=3) = \frac{1}{8}$

$P(Y=1) = \frac{4}{8}$

$P(Y=2) = \frac{2}{8}$

$P(Y=3) = \frac{1}{8}$

$X \in \{0, 1, 2, 3\}$

Y — Toss in which head appears first.

$Y \in \{1, 2, 3\}$.

Discrete random Variable :- A random Variable is called discrete random Variable if it takes only Countably finite or infinite Values.

Apartment Complex data

- There are 12 apartments in a apartment Complex.

- Each floor has 3 apartments:
  One bedroom, two bedroom, three bedroom.

Apartment NO.	Floor NO.	No. of bedrooms.	Size of apartment	Distance of apartment from lift.
	1	1	900.23	500.23
1	1	2	1175.11	325.3
2	1	3	1785.09	.
3	2	1	900.12	.
4	2	2	.	.
6	2	3	.	.
7	3	1	.	
8	3	2	.	
9	3	3		
10	4	1		
11	4	2		
12	4	3	1786.30	455.37

[900, 1800]

Random experiment - Choosing random apartment out of 12 apartments.

X — Floor no. — $\{1, 2, 3, 4\}$

Y — No. of bedrooms — $\{1, 2, 3\}$

Z — Size of apartment — $[900, 1800]$

W — distance of apartment from lift $[300, 501]$

Discrete Random Variables.

1. Maximum from a set of finite numbers.
2. Outcome of rolling a die.
3. No. of Students in Class.
4. No. of Spelling mistakes in a paragraph.

Continuous Random Variable

① Area of apartment.

② Height and Weight of students.

③ Temperature in a room.

Continuous Random Variable :-

A random Variable which takes Values in an interval.

—

Discrete RV.

Probability mass function:- Let X be a random variables which takes values $x_1, x_2, \dots, x_N$, then Probability mass function (pmf) is defined as

$$\boxed{p(x_i) = P(X = x_i)}$$

Properties of pmf.

(1) $p(x_i) \geq 0$

(2) $\sum_{i=1}^{N} p(x_i) = 1.$

E.g.) X takes values

X	0	1	2
$p(x_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$\Downarrow$

Is it pmf?

Yes.

E.g.)

X	x_1 1	x_2 2	x_3 3	x_4 4	x_5 5
$p(x_i)$	0.4	0.1	0.2	0.1	0.3

— Not a pmf

E.g.)

X	1	2	3	4	5
$p(x_i)$	0.2	0.3	0.4	-0.1	0.2

$-0.1 < 0$

Not a pmf.

Example. X $\quad x_0 \quad x_1 \quad x_2 \quad x_3 \; \ldots$

$$\begin{array}{c|ccccc} X & 0 & 1 & 2 & 3 & \ldots \\ p(x_i) = P(X=x_i) & \frac{c\lambda^0}{0!} & \frac{c\lambda^1}{1!} & \frac{c\lambda^2}{2!} & \frac{c\lambda^3}{3!} & \ldots \end{array}$$

$\lambda > 0$

Can you find c such that $p(x_i)$ is pmf.

(1) $p(x_i) \geq 0 \implies c \geq 0$.

$$c\frac{\lambda^0}{0!} + c\frac{\lambda^1}{1!} + c\frac{\lambda^2}{2!} + \dots = 1$$

$$\Rightarrow c\left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \dots\right) = 1$$

$$e^\lambda = 1 + \lambda + \frac{\lambda^2}{2!} + \dots$$

$$\Rightarrow c\,e^{\lambda} = 1$$

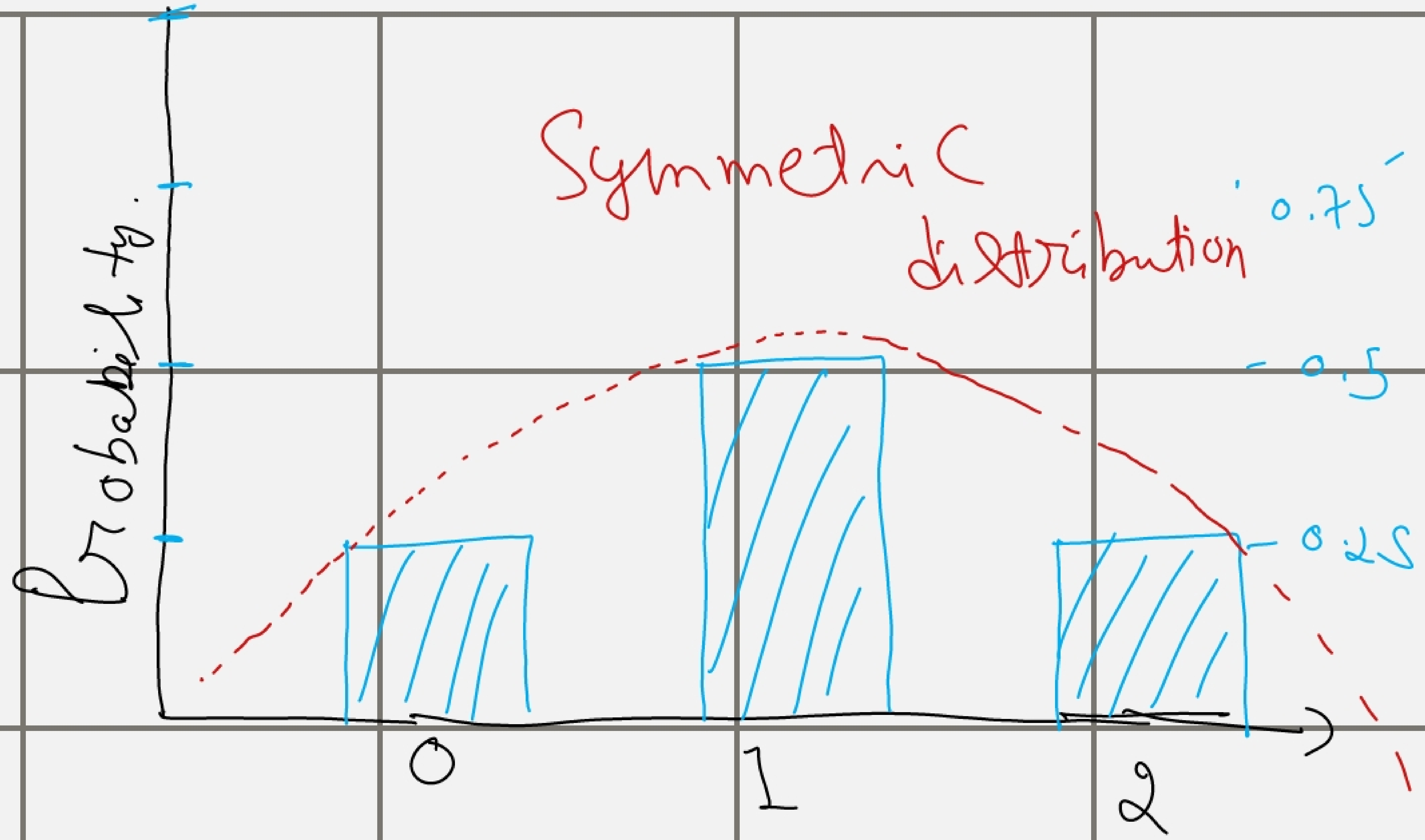
$$\Rightarrow \boxed{c = e^{-\lambda}}$$

$$p(x_i) = P(X = x_i) = c \cdot \frac{\lambda^i}{i!}$$

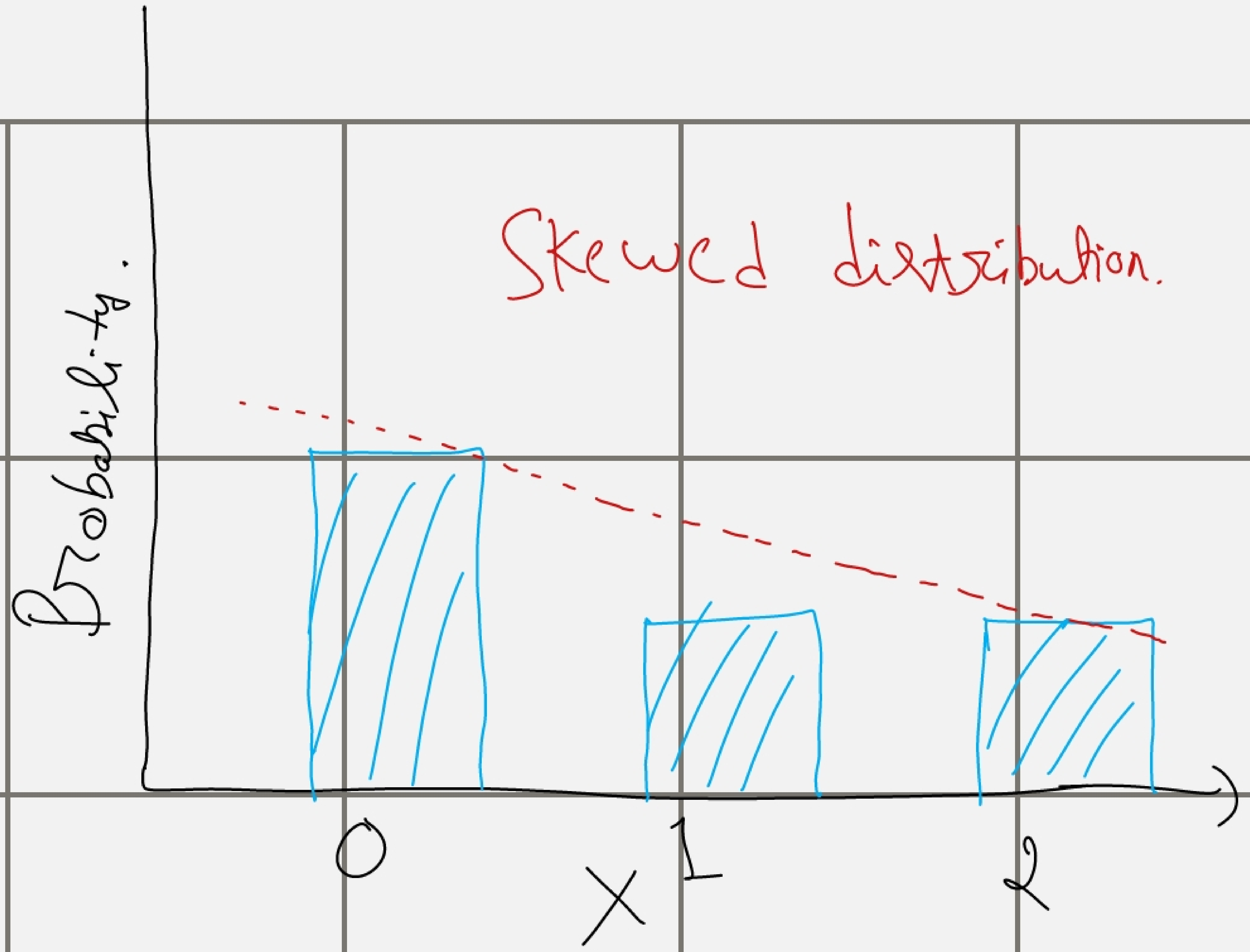
$$\boxed{p(x_i) = e^{-\lambda} \frac{\lambda^i}{i!}}$$

Graph of probability mass function

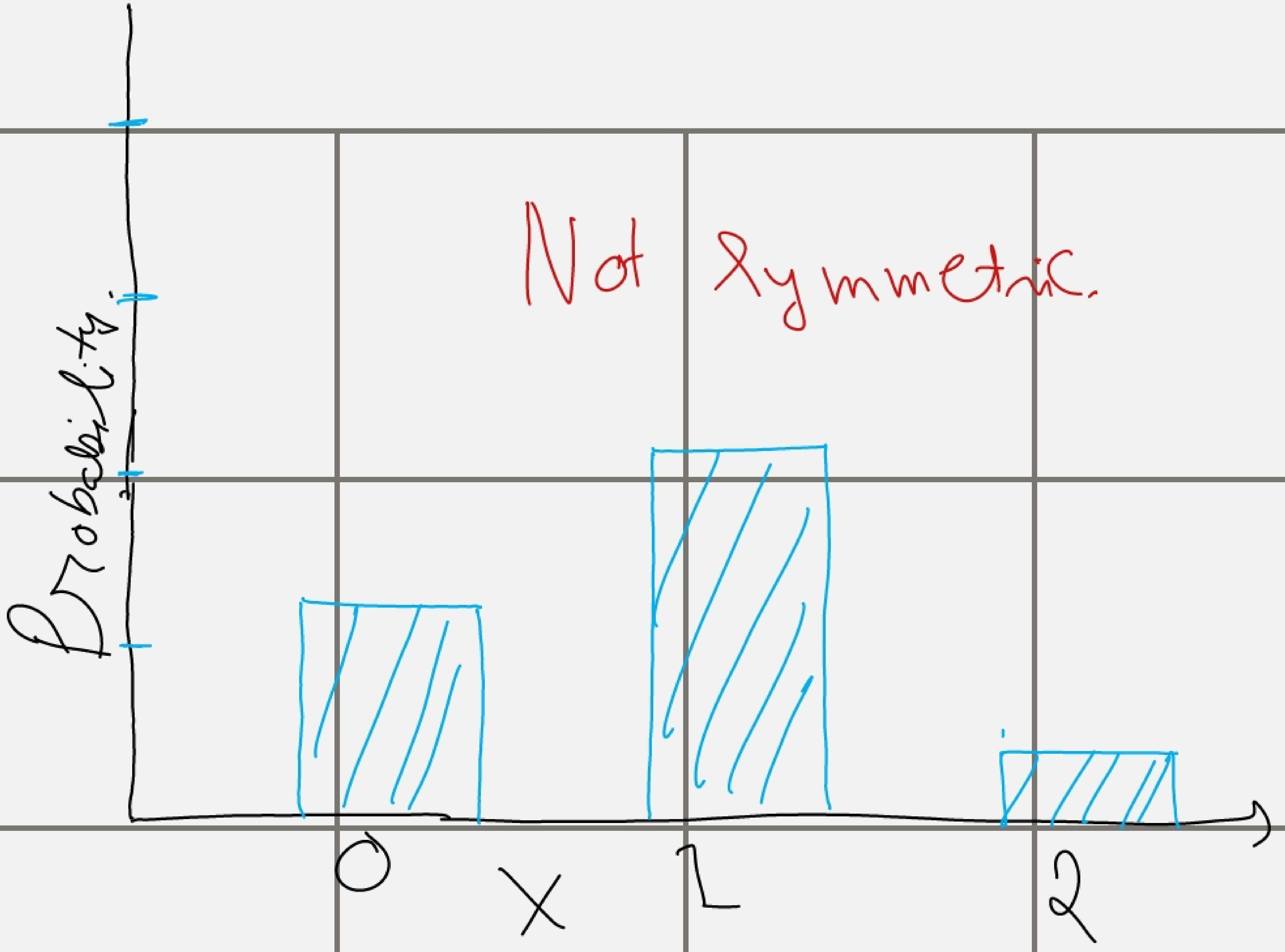
X	0	1	2
$P(X=x_i)$	0.25	0.5	0.25



X	0	1	2
$P(X=x_i)$	0.5	0.25	0.25

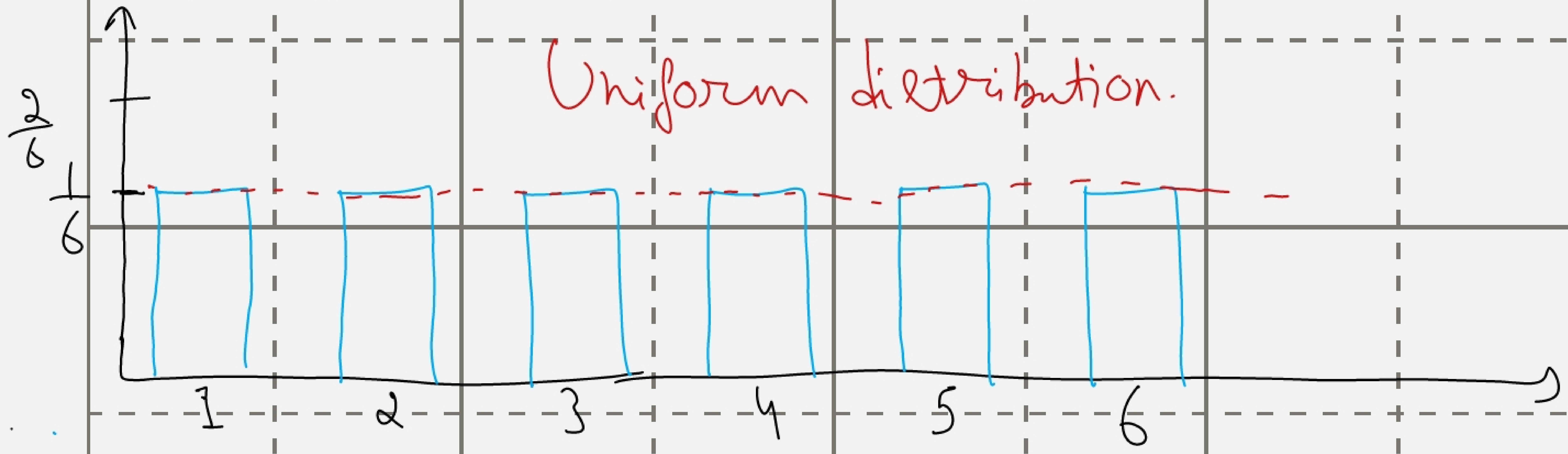


X	0	1	2
$P(X=x_i)$	0.35	0.55	0.10



E.g. Rolling a die

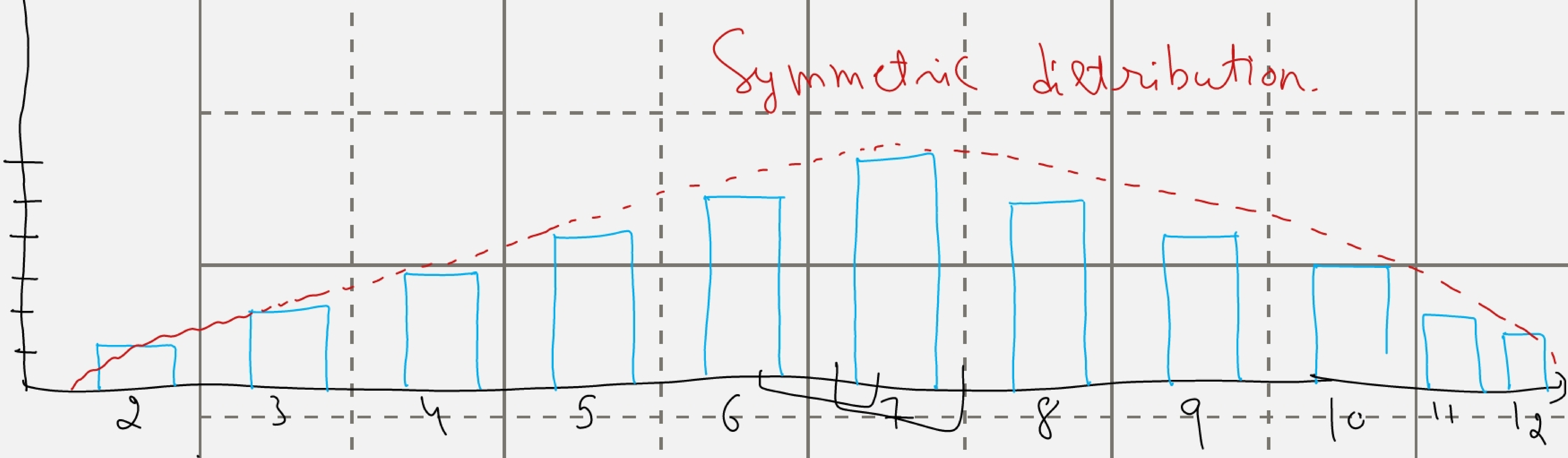
X	1	2	3	4	5	6
$P(X=x_i)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$.



$$E[X] = 7$$

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Try it for Y also?



pmf →

X

$P(X=x_i)$

$\|$

$p(x_i)$

$P(X \le a)$, $P(X \ge a)$

$P(X \ge 5)$ ✓

$$\boxed{F(a) = P(X \le a)}$$

$F: \mathbb{R} \to [0,1]$

Cdf.

(Cumulative distribution function:-

$F: \mathbb{R} \to [0,1]$ defined by

$$\boxed{F(a) = P(X \le a)}$$ — Cdf.

$F(-1) = P(X \le -1)$

$a = -1$

$P(X \le 0)$

E.g>

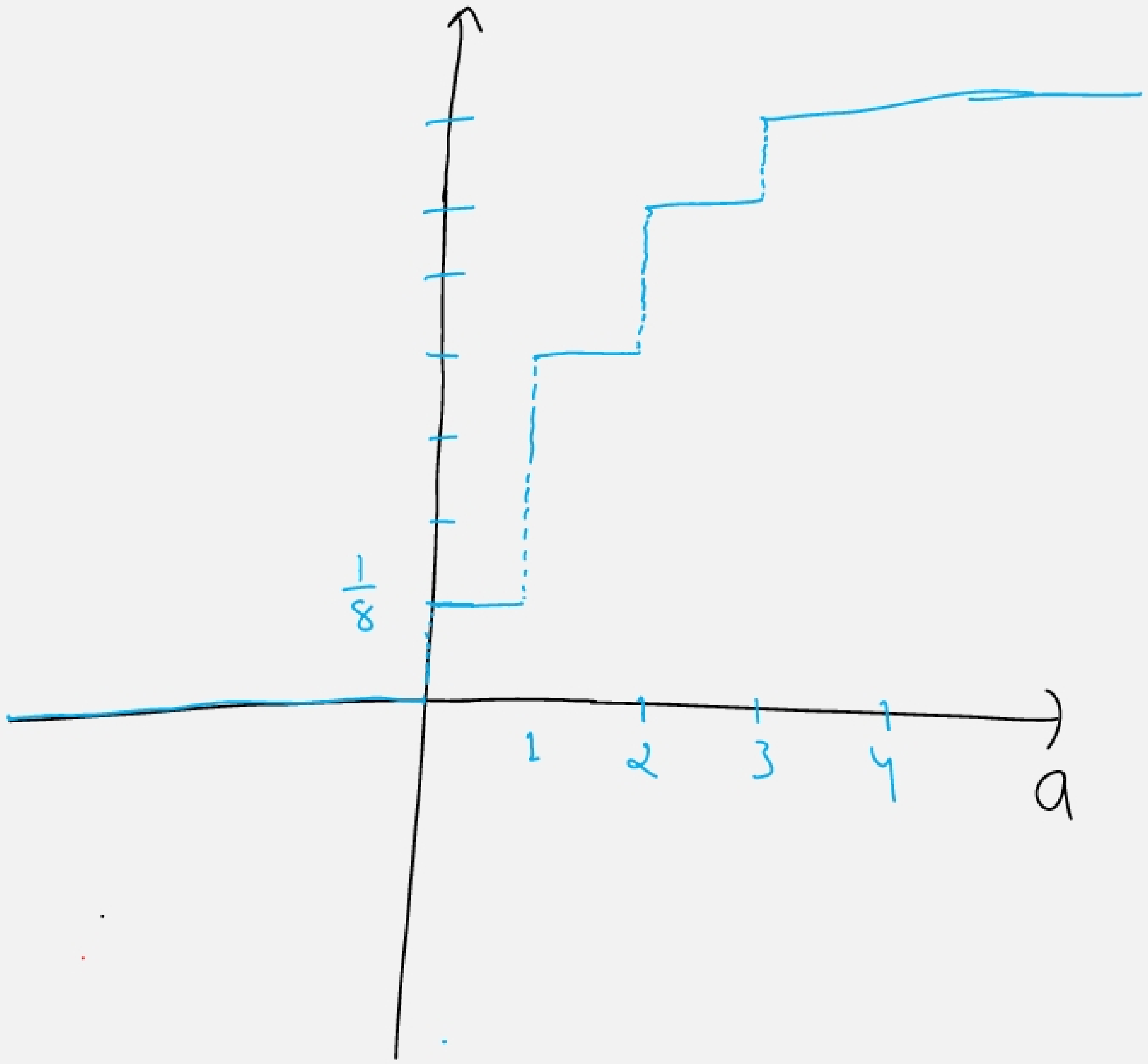
Tossing a Coin thrice

X – No. of heads in an outcome

X	0	1	2	3
$P(X=x_i) =$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$F(a) = \begin{cases} 0 & a < 0 \\ \frac{1}{8} & 0 \le a < 1 \\ \frac{4}{8} & 1 \le a < 2 \\ \frac{7}{8} & 2 \le a < 3 \\ 1 & a \ge 3 \end{cases}$$

$$F(a) = \begin{cases} 0, & a < 0 \\ \frac{1}{8}, & 0 \leq a < 1 \\ \frac{4}{8} & 1 \leq a < 2 \\ \frac{7}{8} & 2 \leq a < 3 \\ 1 & a \geq 3 \end{cases}$$



taking values $x_1, x_2, \ldots, x_n$

<u>Remark:</u> let X be a discrete random Variable, then Cdf $F(a)$ will be a step function

Case Study:-

To analyze no. of Credit Cards owned by a population.

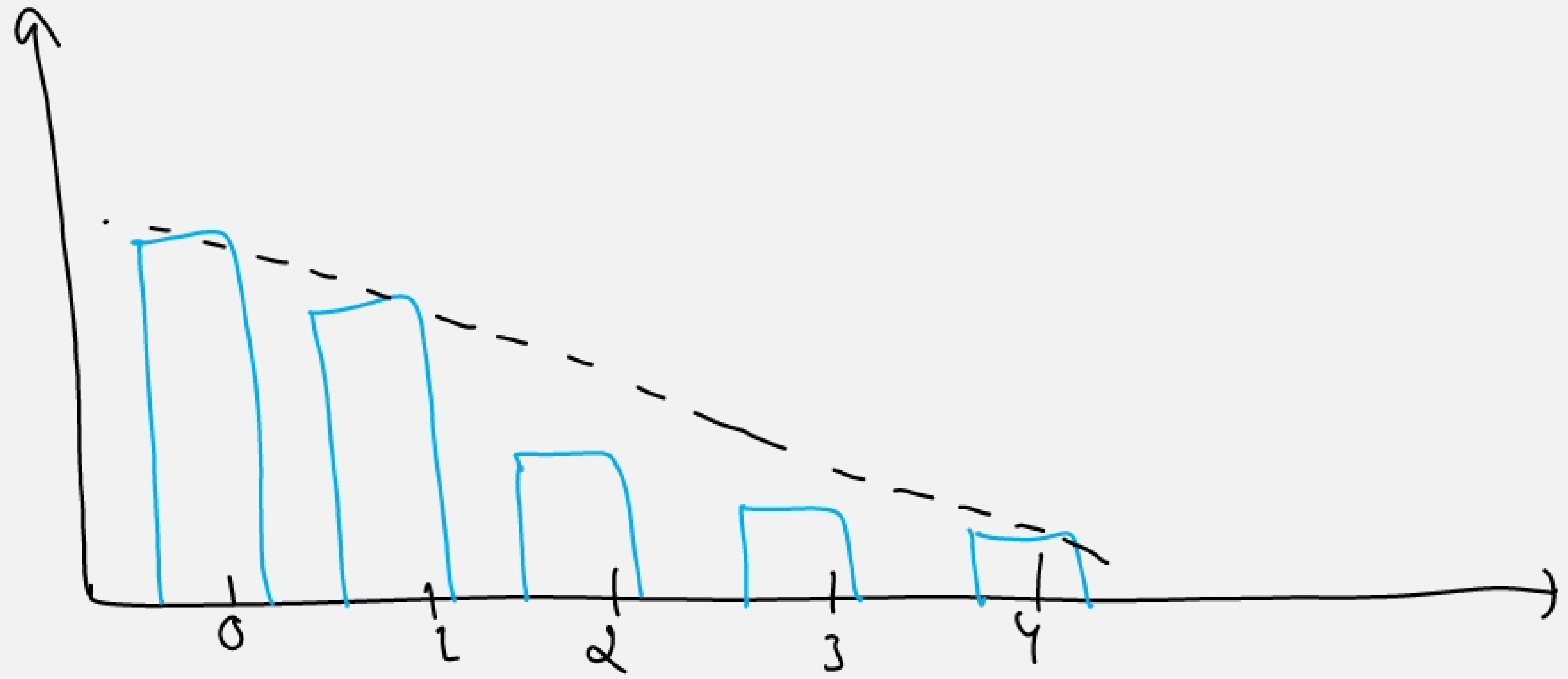
- Collect data:- Ask people how many Credit Cards they own.

$S = \{x_1, x_2, \ldots, x_n\}$

Random experiment :- Picking a random individual from the population.

X — No. of Credit Cards owned by a person.

X	0	1	2	3	4
$P(X=x_i)$	0.42	0.36	0.14	0.06	0.02



Question:- Choose an individual at random.

Is he/she more likely to have 0 Credit Cards or 2 or more Credit Cards?

pmf $\leftarrow P(X=0) = 0.42 = 42\%$

cdf $\longrightarrow P(X \geq 2) = 0.14 + 0.06 + 0.02 = 0.22 = 22\%$

$1 - P(X<2) = 0.22$

Question:- Random sample of 1000 individuals, we ask that 1000 individuals, how many Credit Cards they own?

- Everyone is saying, they own a credit cards

  - There is some problem in sample.

(2) 138 people respond that they own two credits cards.

$\boxed{1000 \times 0.14 = 140}$

Choose an individual at random, how many credit cards would you expect that individual to own?

→ This we will answer with $\boxed{\text{expected value.}}$

Example :- let us see one game.

Rolling a single die

pmf.	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
X	1	2	3	4	5	6
	$+1$	-2	$+3$	-4	$+5$	-6

Roll die $\overset{n}{\overset{\shortparallel}{100}}$ times

X	Wining Y	Frequency		
1	+1	18	0.18	$\frac{1}{6}$
2	−2	12	0.12	$\frac{1}{6}$
3	+3	20	0.20	$\frac{1}{6}$
4	−4	15	0.15	$\frac{1}{6}$
5	+5	15	0.15	$\frac{1}{6}$
6	−6	20	0.20	$\frac{1}{6}$
		100	1	

$$\frac{1}{6} - \frac{2}{6} + \frac{3}{6} - \frac{4}{6} + \frac{5}{6} - \frac{6}{6} = \frac{9-12}{6} = \frac{-3}{6}$$

Average winning $= -\frac{1}{2}$

$$1 \times 0.18 - 2 \times 0.12 + 3 \times 0.20 - 4 \times 0.15 + 5 \times 0.15 - 6 \times 0.20 = -0.51$$

$= -0.5$

$X \sim x_1, x_2, \ldots, x_n$

$$\boxed{E[X] = \sum_{i=1}^{n} \underbrace{x_i}\, \underbrace{P(X=x_i)}}$$

Expectation of a random Variable :- Let X be a discrete random Variable taking Values $x_1, x_2, \ldots, x_n$, then expected Value of X is defined as

$$E[X] = \sum_{i=1}^{n} x_i \, \underline{P(X=x_i)}.$$

Example:- A player pays $\underbrace{\$2}$, to roll a fair six sided die. The payout rules are

Roll a 1 — Win \$5

Roll a 6 — Win \$3

Any other No. — \$0.

X –	1	2	3	4	5	6
Amount → Y	5	0	0	0	0	3
Prob.	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$x_1 \quad x_2 \ldots x_6$

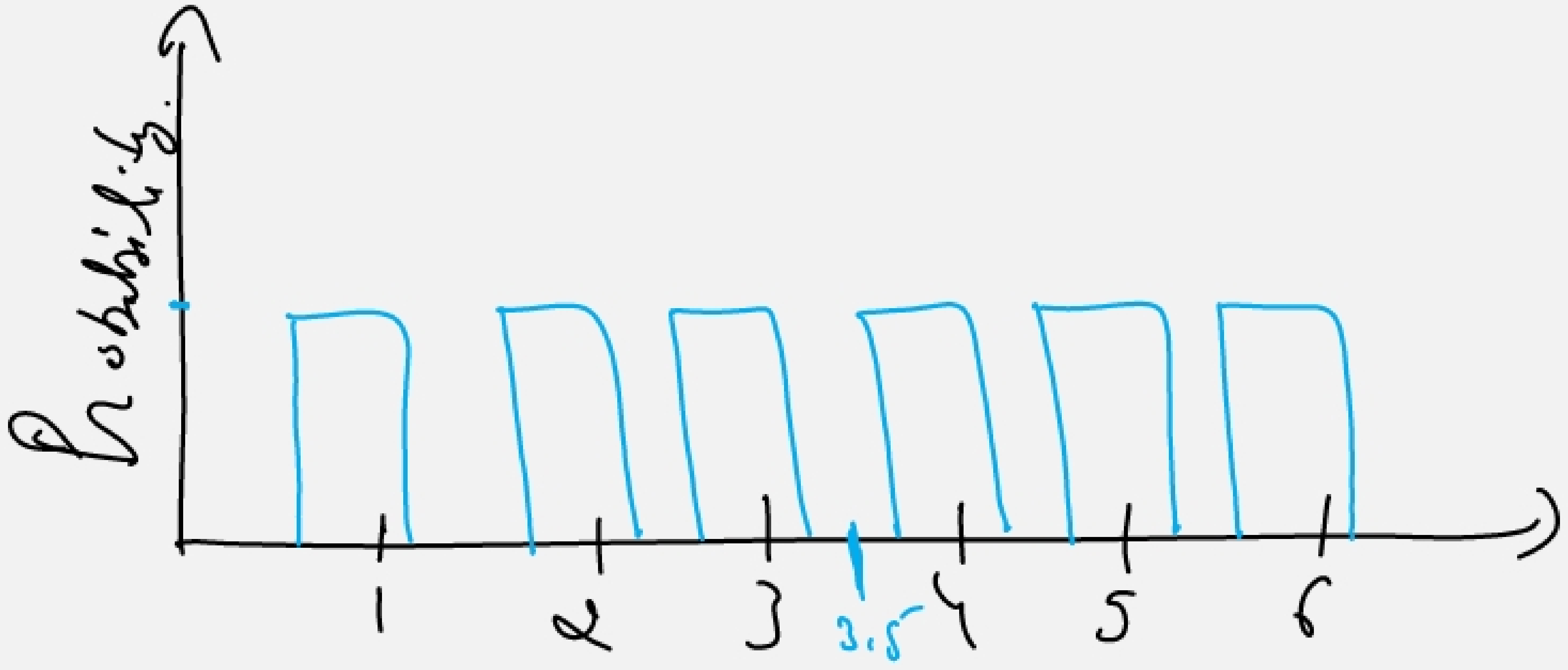
X – Outcome of die

$$E[Y] = 5\times\frac{1}{6} + 3\times\frac{1}{6} = \frac{8}{6} = \frac{4}{3} = \underbrace{1.33}$$

We are also paying \$2.

Profit = 1.33 − 2 = −0.67

$$E[X] = 1\,\underbrace{\tfrac{1}{6}} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6} = \frac{7}{2} = \underline{3.5}$$



E.g.) Rolling a die twice

X – Sum of outcomes.

X	2	3	4	5	6	7	8	9	10	11	12
$P(X=x_i)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E[X] = 2\times\frac{1}{36} + 3\times\frac{2}{36} + \ldots + 12\times\frac{1}{36} = 7.$$

Example:- Tossing a Coin thrice.

X — No. of heads in the tosses

X	0	1	2	3
$P(X=x_i)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$E[X] = 0 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} + 3 \times \frac{1}{8} = \frac{3}{2} = 1.5$$

Example. → Bernoulli Random Variable.

X —	0	1
$P(X=x_i)$	$1-p$	p

$$0 \leq p \leq 1$$

$$E[X] = p \checkmark$$

Properties of expectation.

① X — Random Variable

aX — Random Variable

$\downarrow$

a is a real No.

$g(X)$

$+$

$aX + b$ — Random Variable.

X	x_1	x_2	$\ldots$	x_n
pmf	$p(x_1)$	$p(x_2)$	$\ldots$	$p(x_n)$

$$\sum_{i=1}^{n} p(x_i) = 1$$

aX	ax_1	ax_2	$\ldots$	ax_n
pmf	$p(x_1)$	$p(x_2)$	$\ldots$	$p(x_n)$

$$\boxed{E[ax+b] = aE[X] + b}$$

$$= \sum_{i=1}^{n} (ax_i + b)\,p(x_i)$$

$$= \sum_{i=1}^{n} ax_i\,p(x_i) + b\sum_{i=1}^{n} p(x_i)$$

$$= aE[X] + \underline{b}$$

$$E[\underline{X^2}] = (\underline{E[X]})^2$$

$$\sum_{i=1}^{n} x_i^2 p(x_i) \neq \left(\sum_{i=1}^{n} x_i p(x_i)\right)^2$$

Think of a

Counter example.

Example: Bulk purchase discount.

A store offers a discount on bulk purchases | 150 - Cost of each item.

Buy 1-4 items : No discount

5-9 items : 10% discount

10 or more : 20% discount

X	3	6	10
p_{X}	0.3	0.4	0.3
Y	150	270	400

X - No. of items a customer buys.

$P(X=3) = 0.3$, $P(X=6) = 0.4$, $P(X=10) = 0.3$

Y = Total cost after discount.

$E[Y]$