Exploration of the Parameter Space for an Inclined-Plane Problem

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Abstract

A moderately complex problem involving a block sliding on an inclined plane, a pulley with rotational inertia, and a second block serving as a counterweight is solved. We present the proper approach, in which no numerical value for any parameter is assumed in the solution. Then the limiting cases are explored. Finally, a summary of the parameter space is presented in a collection of graphs.

1 Problem

The flat bottom of a block B1 of mass m lies on a plane inclined at an angle θ to the horizontal. See Figure 1. The coefficient of kinetic friction between the plane and the block is μ. A massless, taut, inelastic cable runs from B1 toward the top of the plane, around a pulley, and straight down to another block B2, which also has mass m. The pulley's disc has infinite static friction against the cable but requires no energy to separate from the cable as it moves. The pulley's bearing is frictionless. The disc of the pulley has uniform density, total mass M, and radius r. Find the acceleration a of the blocks; express a in terms of μ, m, M, θ, and the acceleration g of gravity.

2 Solution

Each of B1, the pulley, and B2 must undergo the same acceleration because they move as a rigid system: B1 and B2 maintain a constant distance of separation along the length of the cable, and the cable does not slip along the surface of the pulley. The net force on each of B1, the pulley, and B2, when divided by the relevant mass, must yield the same acceleration.

2.1 Forces at B1

Suppose that the tension in the cable attached to B1 is T_1. There are three forces, parallel to the plane, acting on B1:

1. a force of magnitude T_1 toward the pulley,
2. the frictional force $\mu mg\cos(\theta)$ away from the pulley, and
3. the plane-parallel component $mg\sin(\theta)$ of B1's weight.

The acceleration, as derived from forces acting on B1, is

$$\begin{aligned} a &= \frac{T_1 - \mu mg\cos(\theta) - mg\sin(\theta)}{m} \\ a &= \frac{T_1}{m} - [\mu\cos(\theta) + \sin(\theta)]g. \end{aligned} \quad (1)$$

2.2 Forces at the Pulley

Suppose that the tension in the cable attached to B2 is T_2. There are two forces, along the direction of the cable, acting on the pulley:

1. a force of magnitude T_1 toward B1 and
2. a force of magnitude T_2 toward B2.

There are two corresponding torques:

1. a torque rT_1 tending to spin the pulley so that B1 accelerates down the plane and
2. a torque rT_2 tending to spin the pulley so that B1 accelerates up the plane.

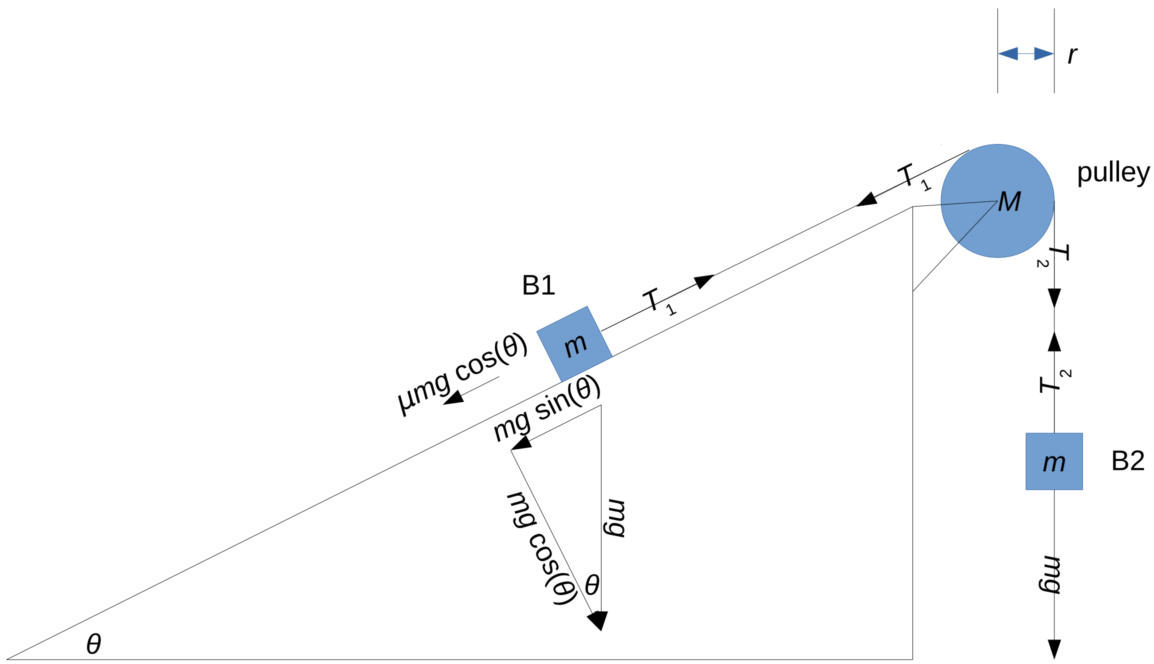

Figure 1: Diagram of problem.

The angular acceleration, derived from torques acting on the pulley, is

$$\begin{aligned}
\alpha &= \frac{rT_2 - rT_2}{\frac{1}{2}Mr^2} \\
\alpha &= \frac{2\,[T_2 - T_1]}{Mr}. \qquad (2)
\end{aligned}$$

The acceleration, as derived from forces acting on the pulley, is

$$\begin{aligned}
a &= r\alpha \\
a &= \frac{2\,[T_2 - T_1]}{M}. \qquad (3)
\end{aligned}$$

2.3 Forces at B2

There are two forces acting on B2:

1. a force of magnitude T_2 toward the pulley and
2. B2's weight mg away from the pulley.

The acceleration, as derived from forces acting on B2, is

$$\begin{aligned}
a &= \frac{mg - T_2}{m} \\
a &= g - \frac{T_2}{m}. \qquad (4)
\end{aligned}$$

2.4 Tension Acting on B2

We can, by combining Equation 3 and Equation 4, solve for the tension T_2 acting on B2.

$$\begin{aligned}
\frac{2\,[T_2 - T_1]}{M} &= g - \frac{T_2}{m} \\
\frac{2\,T_2}{M} + \frac{T_2}{m} &= g + \frac{2\,T_1}{M} \\
\left[\frac{2}{M} + \frac{1}{m}\right] T_2 &= g + \frac{2\,T_1}{M} \\
\left[2 + \frac{M}{m}\right] T_2 &= Mg + 2\,T_1 \\
T_2 &= \frac{Mg + 2\,T_1}{2 + \frac{M}{m}} \qquad (5)
\end{aligned}$$

Here we have expressed the result in terms of the ratio $\frac{M}{m}$, which, along with μ and θ, is one of the three fundamental parameters that determine the solution.

2.5 Tension Acting on B1

Now, combining Equation 1, Equation 4, and Equation 5, we can solve for the tension T_1 acting on B1.

$$\begin{aligned}
g - \frac{T_2}{m} &= \frac{T_1}{m} - [\mu\cos(\theta) + \sin(\theta)]g \\
\frac{T_1 + T_2}{mg} &= 1 + \mu\cos(\theta) + \sin(\theta) \\
\frac{T_1}{mg}\left[\frac{4 + \frac{M}{m}}{2 + \frac{M}{m}}\right] &= \frac{2}{2 + \frac{M}{m}} + \mu\cos(\theta) + \sin(\theta)
\end{aligned}$$

$$T_1 = \left[\frac{2 + [\mu\cos(\theta) + \sin(\theta)][2 + \frac{M}{m}]}{4 + \frac{M}{m}}\right] mg \qquad (6)$$

2.6 Acceleration of Blocks

Finally, combining Equation 6 and Equation 1, we can solve for the acceleration.

$$\frac{a}{g} = \frac{2 + [\mu\cos(\theta) + \sin(\theta)][2 + \frac{M}{m}]}{4 + \frac{M}{m}} - [\mu\cos(\theta) + \sin(\theta)]$$

$$\frac{a}{g} = \left[\frac{2}{4 + \frac{M}{m}}\right] [1 - \mu\cos(\theta) - \sin(\theta)] \qquad (7)$$

Again, we see that the acceleration depends on three parameters: $\frac{M}{m}$, μ, and θ. The form of the solution shows that the acceleration can range from a minimum of zero to a maximum of $\frac{g}{2}$.

2.6.1 Minimum Acceleration

The minimum ($a = 0$) obtains when either of the following is true:

- $\frac{M}{m}$ approaches infinity, or
- $\mu\cos(\theta) + \sin(\theta) \geq 1$.

Considering the second case, we see that, for a given θ, when μ is larger than $\mu_{\text{max}} = \sec(\theta) - \tan(\theta)$, the kinetic friction is large enough to decelerate the system and halt motion. Figure 2 shows the how μ_{max} varies with θ. For small θ, μ can approach unity, and the system will still accelerate. For values of θ approaching 90 degrees, however, only a small μ allows the system to accelerate.

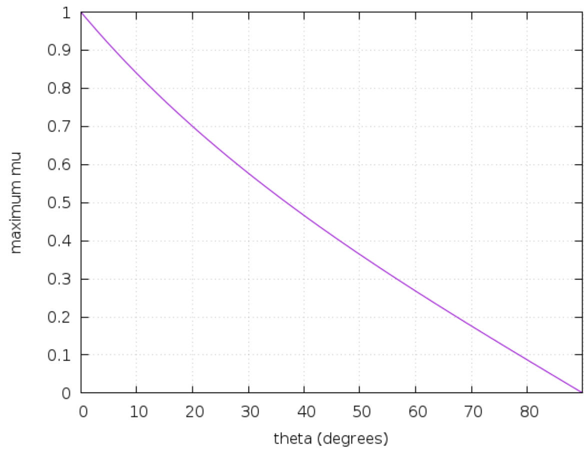

Figure 2: Plotted against the angle of inclination of the plane, the limiting value μ_{max} of the coefficient of kinetic friction. For any coefficient above this limit, the system decelerates.

2.6.2 Maximum Acceleration

The maximum ($a = \frac{g}{2}$) obtains when all of the following are true at the same time:

- $\frac{M}{m}$ approaches zero;
- $\mu = 0$; and
- $\theta = 0$.

2.6.3 Summary

Figure 3, 4, and 5 summarize the result.

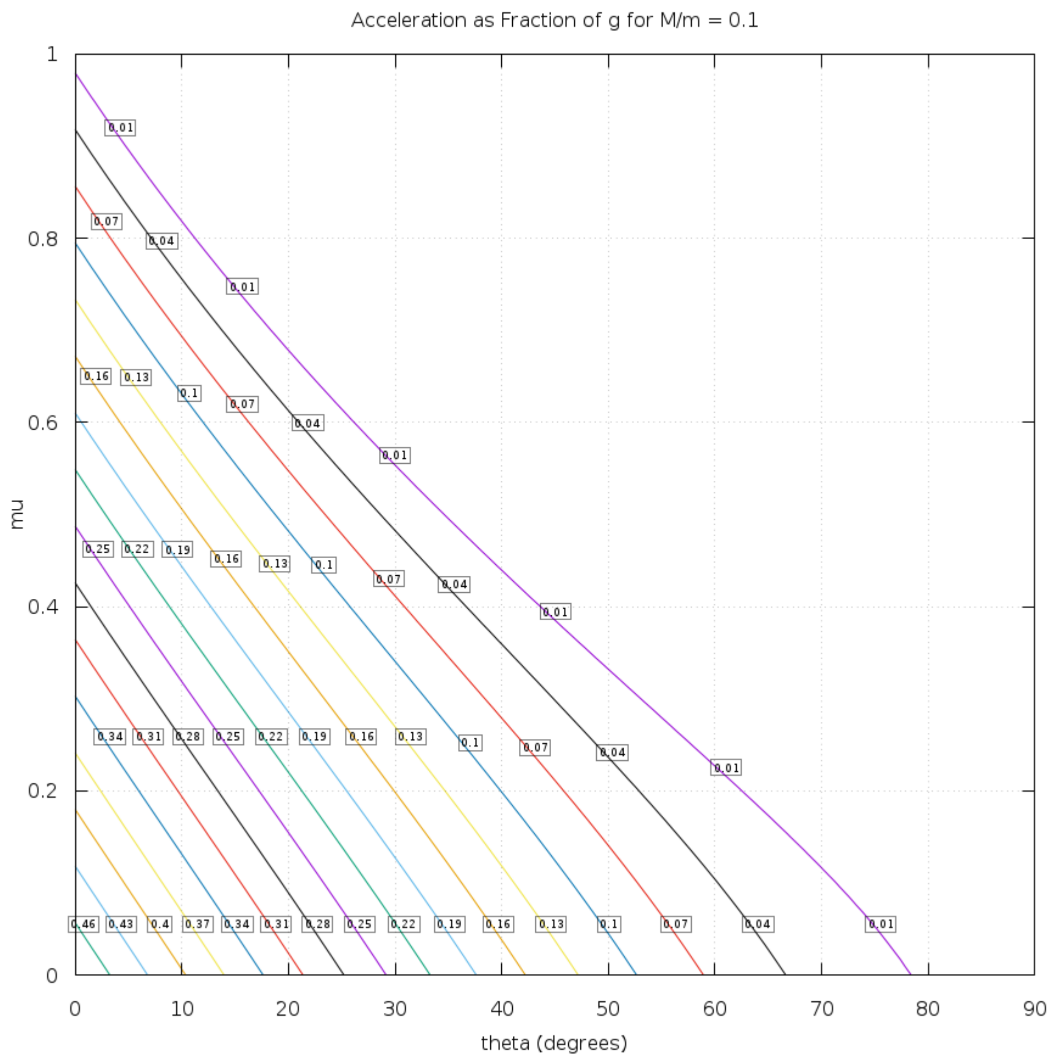

Figure 3: Contour map showing lines of constant acceleration across the space of inclination angle and coefficient of kinetic friction for $\frac{M}{m} = 0.1$. The value of each contour is labeled as a fraction of g.

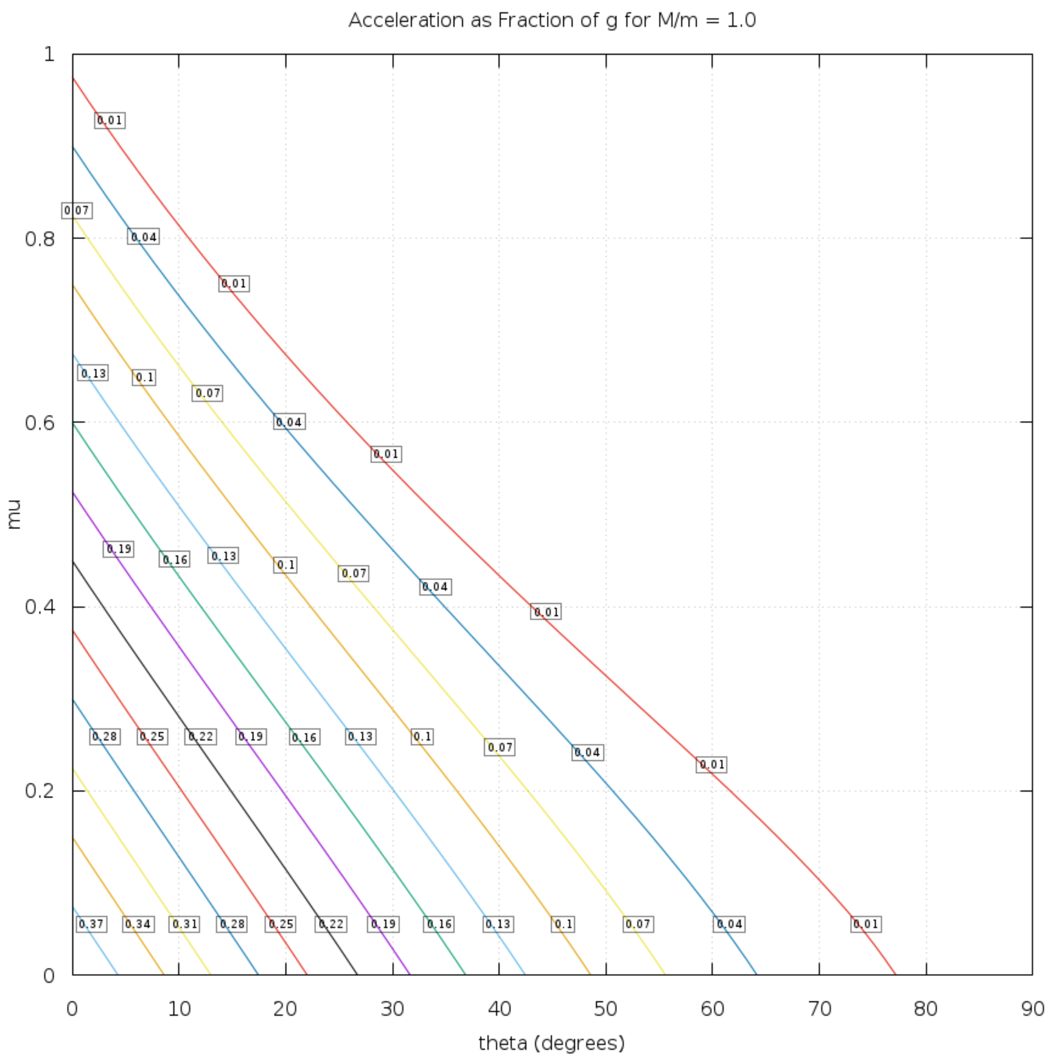

Figure 4: Contour map showing lines of constant acceleration across the space of inclination angle and coefficient of kinetic friction for $\frac{M}{m} = 1.0$. The value of each contour is labeled as a fraction of g.

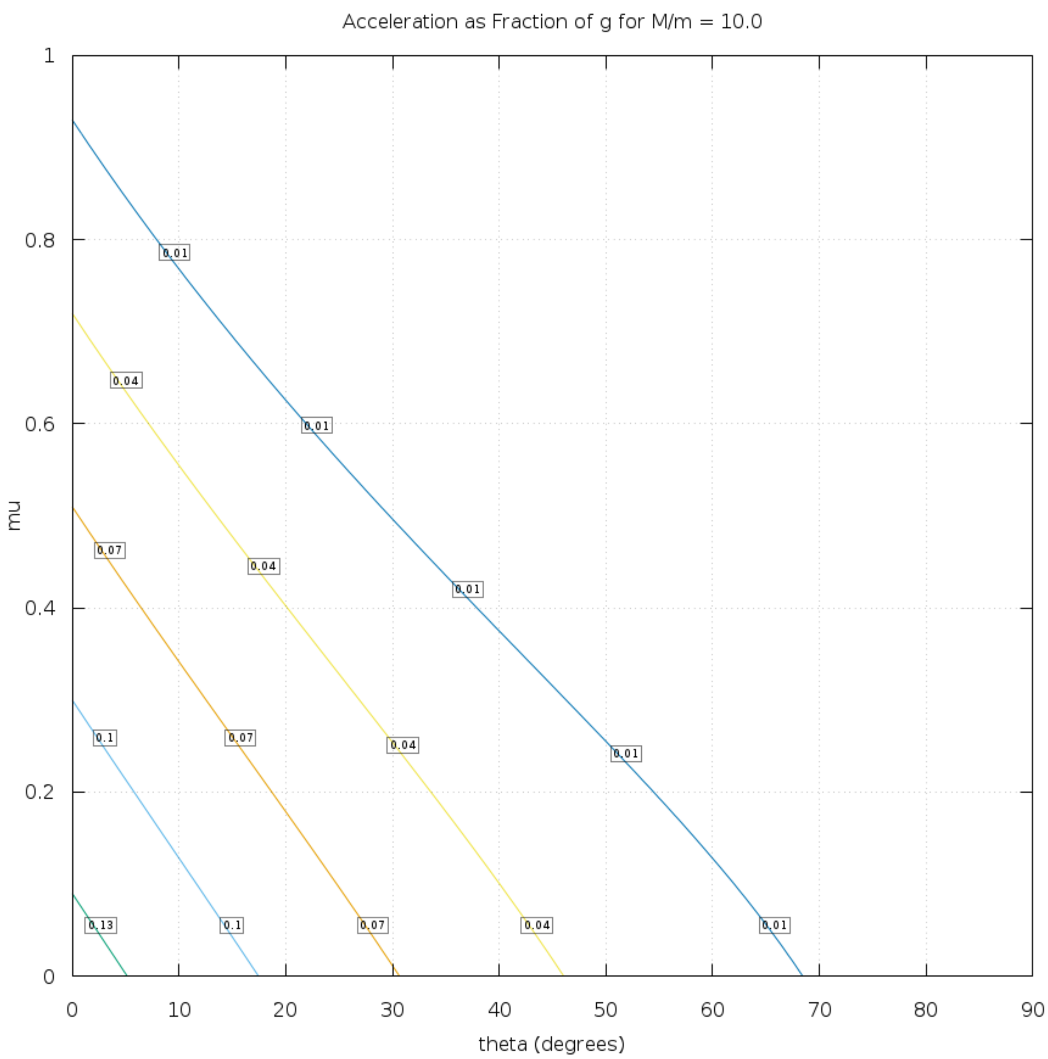

Figure 5: Contour map showing lines of constant acceleration across the space of inclination angle and coefficient of kinetic friction for $\frac{M}{m} = 10.0$. The value of each contour is labeled as a fraction of g.