

1 Introduction

Linear regression allows one to find the best-fit curve through a set of points in the plane. The “linear” in “linear regression” refers not to the shape of the best-fit curve (for it need not be a straight line) but to an aspect of the expression that represents the curve. The function representing the best-fit curve must be a *linear combination* of basis functions, even though each basis function may be a *non-linear* function of the independent variable. So, if the curve can be expressed as a linear combination of functions of the independent variable, and if, by “best”, one mean that the coefficients of the linear combination are chosen to minimize the sum of squared deviations between the points and the curve, then one can find the best-fit curve by way of linear regression.

The main benefits of linear regression include determinism in the time required to compute the fit and certainty that the solution is in fact the best solution. The main problem, however, is that linear regression cannot always be used.

For example, if one want to fit a sinusoidal model to a set of points in the plane, then the phase can be extracted from the fit, but the frequency cannot be extracted. The frequency of the oscillation cannot be found by linear regression because the frequency is not the coefficient of a term in any finite linear combination of angle-dependent terms representing a sinusoidal curve. However, if the frequency be known, then the phase can be found by linear regression because any sinusoid can be expressed as the sum of the sine function and the cosine function, here used as basis functions. The relationship between the best-fit coefficients of the sine and the cosine gives the phase of the best-fit curve.

The phrase, “**linear regression**” is associated with a wide variety of mathematical procedures. The present document describes a C++ library for computing a linear regression to find the coefficient for every basis function of the same, single, independent variable. Although we express each basis function in terms of the same, single, independent variable, the output of each basis function serves effectively as a separate independent variable. So we end up implementing a special case of *multiple* linear regression, where each of the multiple independent variables corresponds to a different basis function of the same, single, underlying independent variable. There are different ways to compute the same regression. We have chosen the **method recommended in Version 3.2.8** of the **Eigen header-only C++ library**. This is the method involving Jacobi **singular-value decomposition**.

2 General Formulation

Suppose that one wants to find by linear regression the column matrix

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad (1)$$

of parameters of the function

$$f_{\mathbf{c}}(x) = \sum_{i=1}^n c_i b_i(x), \quad (2)$$

where the basis functions are b_1, \dots, b_n . What one wants, then, is the solution \mathbf{c} to the over-determined system

$$\mathbf{B}\mathbf{c} = \mathbf{y}, \quad (3)$$

where

$$\mathbf{B} = \begin{pmatrix} b_1(x_1) & \cdots & b_n(x_1) \\ \vdots & & \vdots \\ b_1(x_m) & \cdots & b_n(x_m) \end{pmatrix}; \quad (4)$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}; \quad (5)$$

and $(x_1, y_1), \dots, (x_m, y_m)$ are the measured points that the curve represented by $f_{\mathbf{c}}$ best fits.

If, in a C++ program that includes the headers for Eigen 3, the variable \mathbf{B} of type `MatrixXd` contain the value of every basis function evaluated at every measured x , and if the variable \mathbf{y} of type `VectorXd` contain every measured y , then the following C++ expression will return the solution (of type `VectorXd`):

```
B.jacobiSvd(ComputeThinU | ComputeThinV).solve(y)
```

The flags that are combined by the logical OR operation indicate that only the minimal decomposition necessary for solving is calculated. (The full singular-value decomposition is not necessary for solving the system.)



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