1 Introduction

Suppose that one wants to find the best-fit curve through a set of points in the plane. If the curve can be expressed as a linear combination of functions of the independent variable, and if, by "best", one be satisfied with finding for the linear combination the coefficients that minimize the sum of squared deviations between the points and the curve, then one can find the best-fit curve by way of *linear regression*. The main benefits of linear regression include determinism in the time required to compute the fit and certainty that the solution is in fact the best solution. The main problem, however, is that linear regression cannot always be used. For example, if one want to fit a sinusoidal model to a set of points in the plane, then the frequency of the oscillation cannot be found by linear regression because the frequency is not a coefficient of any basis function. However, if the frequency be known, then the phase can be found by linear regression because any sinusoid can be expressed as the sum of the sine and the cosine, here used as basis functions. The relationship between the coefficients of those basis functions gives the phase of the best-fit curve.

The phrase, "linear regression" is associated with a wide variety of mathematical procedures. The present document describes a C++ library for computing a linear regression to find the coefficient for every basis function of the same, single, independent variable. Although we express the basis in terms of a single independent variable, the output of each basis function serves effectively as a separate independent variable. So we implement a special case of *multiple* linear regression, where each of the multiple independent variables corresponds to a basis function of a single, underlying independent variable.

There are also different ways to compute the same regression. We have chosen the method recommended in Version 3.2.8 of the Eigen header-only C++ library. This is the method involving Jacobi singular-value decomposition.

2 General Formulation

Suppose that one wants to find by linear regression the column matrix

$$\mathbf{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \tag{1}$$

of parameters of the function

$$f_{\mathbf{c}}(x) = \sum_{i=1}^{n} c_i b_i(x), \tag{2}$$

where the basis functions are $b_1, \ldots, b_n$. What one wants, then, is the solution $\mathbf{c}$ to the over-determined system

$$\mathbf{Bc} = \mathbf{y}, \tag{3}$$

where

$$\mathbf{B} = \begin{pmatrix} b_1(x_1) & \cdots & b_n(x_1) \\ \vdots & & \vdots \\ b_1(x_m) & \cdots & b_n(x_m) \end{pmatrix}; \tag{4}$$

$$\mathbf{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}; \tag{5}$$

and $(x_1, y_1), \ldots, (x_m, y_m)$ are the measured points that the curve represented by $f_{\mathbf{c}}$ best fits.

If, in a C++ program that includes the headers for Eigen 3, the variable `B` of type `MatrixXd` contain the value of every basis function evaluated at every measured x, and if the variable `y` of type `VectorXd` contain every measured y, then the following C++ expression will return the solution (of type `VectorXd`):

```
B.jacobiSvd(ComputeThinU | ComputeThinV).solve(y)
```

The flags that are combined by the logical OR operation indicate that only the minimal decomposition necessary for solving is calculated. (The full singular-value decomposition is not necessary for solving the system.)

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