Chapter 4

Vector spaces

(Reminder: This part is just for reading. Don't memorize 10 conditions)
Let V be a set of elements called "vectors" in which 2 operations are defined: "vectoral addition (+)" and "multiplication of vectors by scalars (·)".
V is called a "vector space" if the following conditions are satisfied simultaneously.

For $\vec{u}, \vec{v}, \vec{w} \in V$ and $a, b \in \mathbb{R}$

1) For $\vec{u}, \vec{v} \in V$, $\vec{u}+\vec{v} \in V$ (closed under addition)
2) $\vec{u}+\vec{v} = \vec{v}+\vec{u}$ (commutativity)
3) $\vec{u}+(\vec{v}+\vec{w}) = (\vec{u}+\vec{v})+\vec{w}$ (associativity)
4) There exists only one "zero element" $\vec{0}$ such that $\vec{u}+\vec{0} = \vec{0}+\vec{u} = \vec{u}$
(Identity element / V uzayının toplama işlemine göre etkisiz vektörü vardır ve 1 tanedir.) (zero vector)
5) $\vec{u}+(-\vec{u}) = (-\vec{u})+\vec{u} = \vec{0}$ (additive inverse) (There exists only one inverse of $\vec{u}$ (which is $-\vec{u}$) such that it satisfies $\vec{u}+(-\vec{u}) = \vec{0}$)
or $\vec{x}+\vec{y} = \vec{0}$ ($\vec{y}$: inverse of $\vec{x}$) zero element
6) for $a \in \mathbb{R}$ and $u \in V$; $a.u \in V$ (closed under multiplication)
7) $a(u+v) = au+av$ (distributivity)
8) $(a+b)u = au+bu$ (Real distributivity)
9) $a(bu) = (ab)u$ (associativity)
10) $1.u = u$ (Identity element for ·)

Ex/ Let V is the set of all positive real numbers ($V=\mathbb{R}^+$). Operations are defined as.
$x \oplus y = x.y$
$\alpha \odot x = x^\alpha$, $\alpha \in \mathbb{R}$

a) Find $\vec{0}$ (zero vector)
$\vec{x}+\vec{0} = \vec{x}$.
$x.\vec{0} = x$
$\vec{0} = 1$

b) Find inverse element
$\vec{x}+\vec{y} = \vec{0}$
$x.y = 1 \Rightarrow y = \frac{1}{x}$

* Some spaces

* $R = R^1$: is the set of all real numbers.
* $\mathbb{R}^2$ (2-D plane): It is the set of all ordered pairs (a, b) of real numbers.
All vectors in R^2 is defined by (2×1) matrix

For $\underbrace{u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in R^2 \text{ and } v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \in R^2}_{\text{column matrix representation}}$

- $u+v = \begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}$ Addition of vectors
- $cu = \begin{bmatrix} c.u_1 \\ c u_2 \end{bmatrix}$ multiplication by a scalar c
- zero vector in $\mathbb{R}^2$: $\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- Length of u : $|u| = \sqrt{u_1^2+u_2^2}$
- Unit vector of $R^2 \Rightarrow \vec{i} = (1, 0)$ and $\vec{j} = (0, 1)$

NOTE
(Another expression of u and v:
We can write u and vectors as $u = (u_1, u_2)$
$v = (v_1, v_2)$
This is ordered pairs expression)

★ $\mathbb{R}^3$ (3 Dimensional space): It is the set of all triples (a,b,c) of real numbers. All vectors in $\mathbb{R}^3$ is defined by (3×1) matrix :

$$\underbrace{v = \begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{\text{(column matrix representation)}} \quad \text{or} \quad \underbrace{v = (v_1, v_2, v_3)}_{\text{(triple ordered pairs of real numbers representation)}}$$

(Here a, b, c are components of vector v)

→ Addition of vectors in $\mathbb{R}^3$:

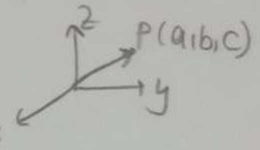
$$\left.\begin{aligned} u &= \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = (u_1, u_2, u_3) \in \mathbb{R}^3 \\ v &= \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = (v_1, v_2, v_3) \in \mathbb{R}^3 \end{aligned}\right\} \quad u+v = (u_1+v_1, \dots, u_n+v_n)$$

→ multiplication by a scalar c : $c.u = (cu_1, cu_2, cu_3) = \begin{bmatrix} cu_1 \\ cu_2 \\ cu_3 \end{bmatrix}$

→ zero vector in $\mathbb{R}^3$: $\vec{0} = (0,0,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

→ Length of v: $|v|$

The length of $v = (a,b,c)$ is defined by the distance of the point $P(a,b,c)$ from the origin.



$$|v| = \sqrt{a^2+b^2+c^2}$$

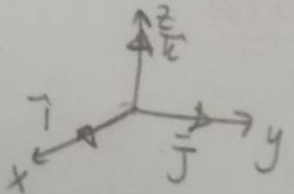
and $|cv| = |c|.|v| > 0$

Example: $v = (4,3,-12) \Rightarrow |v| = \sqrt{4^2+3^2+(-12)^2} = \sqrt{169} = 13$

$\Rightarrow |-7v| = |-7|.|v| = 7.13 = 169$.

→ Unit vector of $\mathbb{R}^3$:

$\vec{i} = (1,0,0)$
$\vec{j} = (0,1,0)$
$\vec{k} = (0,0,1)$



the length of unit vector is always 1.

* R^n (n-dimensional space) : It is the set of all n-tuples $(x_1, x_2, \dots, x_n)$ of real numbers. All vectors in $\mathbb{R}^n$ are defined as $(n \times 1)$ matrix.

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So:

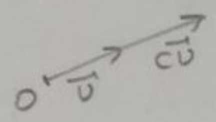
For $u, v \in R^n$;

$$\left.\begin{aligned} u &= (u_1, u_2, \dots, u_n) = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \\ v &= (v_1, v_2, \dots, v_n) = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \end{aligned}\right\} u+v = (u_1+v_1, \dots, u_n+v_n)$$

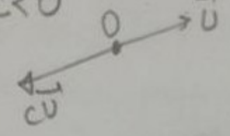
and

* For $u \in R^n$ and $c \in R$ $\Rightarrow$ $c.u = c(u_1, u_2, \dots, u_n) = (cu_1, cu_2, \dots, cu_n)$

$c > 0$



$c < 0$



* zero vector in $\mathbb{R}^n$: $\vec{0} = (0, 0, \dots, 0) = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

((NOTE!)

Theorem 1: R^3 as a Vector Space

If u, v, w are vectors in R^3 and r, s are real numbers; then

(a) $u+v = v+u$ (commutativity)

(b) $u+(v+w) = (u+v)+w$ (associativity)

(c) $u+0 = 0+u = u$ (zero element)

(d) $u+(-u) = (-u)+u = 0$ (additive inverse)

(e) $r(u+v) = ru+rv$ (distributivity)

(f) $(r+s)u = ru+su$

(g) $r(su) = (rs)u$

(h) $1(u) = u$ (multiplicative identity)

* Vector space $\mathcal{F}$ of functions:

* $\mathcal{F}$ is the set of all real-valued functions defined on the real number line $\mathbb{R}$.
Each vector in F is a function f such that the real number $f(x)$ is defined for all x in $\mathbb{R}$.
Given f and g in $\mathcal{F}$ and a real number c;

$$(f+g)(x) = f(x)+g(x)$$

and

$$(cf)(x) = c(f(x))$$

and each of properties (a)–(h) of a vector space are hold. So $\mathcal{F}$ is a vector space.
For instance for a is a scalar; let's check property (e) is hold or not:

$$\begin{aligned}[a(f+g)](x) &= a[(f+g)(x)] \\ &= a[f(x)+g(x)] \\ &= af(x)+ag(x) \\ &= (af+ag)(x)\end{aligned}$$

thus $a(f+g) = af+ag$ is hold.

* $\mathcal{P}$ (is the set of all polynomial of the form $p(x) = a_0+a_1x+a_2x^2+\dots+a_nx^n$ where the largest exponent $n \geq 0$ that appears is the degree of the polynomial $p(x)$ and the coefficients $a_0, a_1, \dots, a_n$ are real numbers)

Ex/ $p(x) = 2x+5 \in P_1$

$p(x) = 2x^2-5x+3 \in P_2$

$p(x) = ax^3+bx^2+cx+d \in P_3$

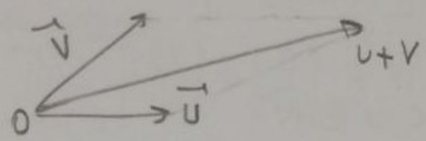
$\mathcal{P}$ is a vector space. Because prop. (a)–(h) are hold.

· P_n (Polynomial space): Denotes all polynomials that define less than or equal to n.

$$p(x) = a_0+a_1x+a_2x^2+\dots+a_nx^n \in P_n$$

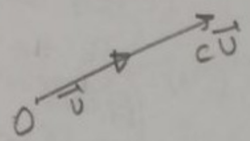
So; as a summary : $\mathbb{R}^n$ (n-dimensional space is the set of all n-tuples $(x_1, x_2, \dots, x_n)$ of real numbers)

.) For $u, v \in \mathbb{R}^n$; $u = (u_1, u_2, \dots, u_n)$, $v = (v_1, v_2, \dots, v_n)$ $\Big\}$ $u+v = (u_1+v_1, u_2+v_2, \dots, u_n+v_n)$

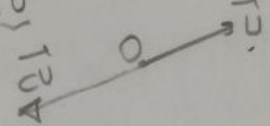


.) For $u \in \mathbb{R}^n$ and $c \in \mathbb{R}$ $\Rightarrow$ $c.u = c(u_1, u_2, \dots, u_n) = (cu_1, cu_2, \dots, cu_n)$

$c>0$



$c<0$

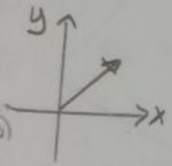


.) For $\mathbb{R}^2$ space

(2-tuple is an ordered pair of real numbers)

$u = (u_1, u_2)$

$v = (v_1, v_2)$



$\Rightarrow$) $u+v = (u_1+v_1, u_2+v_2) \in \mathbb{R}^2$

$\Rightarrow$) $c.u = (cu_1, cu_2) \in \mathbb{R}^2$

$\Rightarrow$) $|v|$: length of v

$|v| = \sqrt{v_1^2 + v_2^2}$

$\Rightarrow$ $|cv| = |c|\,|v| > 0$

$\Rightarrow$ $\vec{0} = (0, 0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $-\vec{u} = (-u_1, -u_2) = \begin{bmatrix} -u_1 \\ -u_2 \end{bmatrix}$

$\Rightarrow$ Unit vectors : $\vec{i}, \vec{j}$

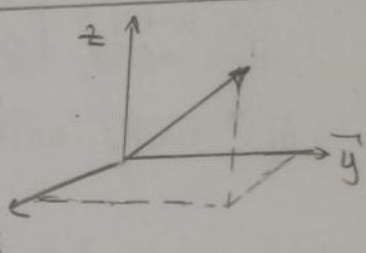
$\vec{i} = (1, 0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\vec{j} = (0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

2-dimensional

.) For $\mathbb{R}^3$ space

(3-tuple is an ordered triple of real numbers)



$u = (u_1, u_2, u_3)$

$v = (v_1, v_2, v_3)$

$\Rightarrow$ $u+v = (u_1+v_1, u_2+v_2, u_3+v_3) \in \mathbb{R}^3$

$\Rightarrow$ $c.u = (cu_1, cu_2, cu_3) \in \mathbb{R}^3$

$\Rightarrow$ $|v| = \sqrt{v_1^2 + v_2^2 + v_3^2}$

$\Rightarrow$ $|cv| = |c|\,|v| > 0$

$\Rightarrow$ $\vec{0} = (0, 0, 0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $-\vec{u} = \begin{bmatrix} -u_1 \\ -u_2 \\ -u_3 \end{bmatrix}$

$\Rightarrow$ Unit vectors in $\mathbb{R}^3$: i, j, k

$\vec{i} = (1, 0, 0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\vec{j} = (0, 1, 0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\vec{k} = (0, 0, 1) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

3-dimensional

* Subspace (Conditions for a subspace)

THEOREM: Let W be a nonempty subset of vectorspace V. ($W \subset V$).
Then W is a subspace of V if and only if it satisfies the following two conditions:

(i) If $u, v \in W$, then $u+v \in W$

(ii) If $u \in W$ and $c \in \mathbb{R}$, then $c.u \in W$

Example: $W = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix}, x_1 \in \mathbb{R} \right\}$ Is W a subspace of $\mathbb{R}^2$ or not?

SOLUTION: For $x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \in W$ and $y = \begin{bmatrix} y_1 \\ 0 \end{bmatrix} \in W$

(i) $x+y = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ 0 \end{bmatrix} \in W$ ✓

(ii) $c \in \mathbb{R}$ and $x = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} \in W$

$cx = c\begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} cx_1 \\ 0 \end{bmatrix} \in W$ ✓

So; W is a subspace of $\mathbb{R}^2$.

Ex/ $S = \left\{ \begin{bmatrix} x \\ 1 \end{bmatrix}, x \in \mathbb{R} \right\}$
Is the set S a subspace of $\mathbb{R}^2$? NO
Because, let $x = \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}^2$ and $y = \begin{bmatrix} y \\ 1 \end{bmatrix} \in \mathbb{R}^2$
$x+y \overset{?}{\in} S \Rightarrow \begin{bmatrix} x \\ 1 \end{bmatrix} + \begin{bmatrix} y \\ 1 \end{bmatrix} = \begin{bmatrix} x+y \\ 2 \end{bmatrix} \notin S$

basis'den SONRA anla!

Example: $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, x_1 = 5x_2 \right\}$
a) show that W is a subspace of $\mathbb{R}^2$
b) find a basis for W.
c) Find the dimesion of W.
These are later subtopic!

SOL:

a) $W = \left\{ \begin{bmatrix} 5x_2 \\ x_2 \end{bmatrix} \right\}$

(i) For $x = \begin{bmatrix} 5x_2 \\ x_2 \end{bmatrix} \in W$ and $y = \begin{bmatrix} 5y_2 \\ y_2 \end{bmatrix} \in W$

$x+y = \begin{bmatrix} 5(x_2+y_2) \\ (x_2+y_2) \end{bmatrix} \in W$

(ii) $c \in \mathbb{R}$; $cx = c\begin{bmatrix} 5x_2 \\ x_2 \end{bmatrix} = \begin{bmatrix} (5c)x_2 \\ x_2 \end{bmatrix} \in W$

→ So; W is a subspace of $\mathbb{R}^2$

b) $\begin{bmatrix} 5x_2 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \end{bmatrix} \rightarrow \left\{ \begin{bmatrix} 5 \\ 1 \end{bmatrix} \right\}$ span W. It is only one vector. So it is linearly independent. So; it is a basis for W.

c) $\dim(W) = 1$

HMW

* **Example:** $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \ |x_1| = |x_2| \right\}$ a subspace of $\mathbb{R}^2$?

SOLUTION: $W = \left\{ \begin{bmatrix} |x_2| \\ x_2 \end{bmatrix} \right\}$

Counter Solution:

(i) $W_1 = \begin{bmatrix} |-5| \\ -5 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \end{bmatrix} \in W$ and $W_2 = \begin{bmatrix} |3| \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

Let $x_2 = -5$ Let $x_2 = 3$

$$W_1 + W_2 = \begin{bmatrix} 5 \\ -5 \end{bmatrix} + \begin{bmatrix} 3 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 \\ -2 \end{bmatrix} \notin W \quad (\text{Because } |-2| \neq 8$$

So W is not closed under addition. W is not a subspace of $\mathbb{R}^2$

* **Example:** $\mathbb{R}^3 \rightarrow W = \{ (x_1, x_1+1, 5x_1), x_1 \in \mathbb{R} \}$ is W a subspace of $\mathbb{R}^3$?

SOL: (i) $x = (1, 2, 5) \in W$ (for $x_1 = 1$)
$y = (0, 1, 0) \in W$ (for $x_1 = 0$)

$x + y = (1, 3, 5) \notin W$ Counter Solution

So W is not a subspace of $\mathbb{R}^3$.

* **Example:** $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, x_1 + x_2 + x_3 = 1 \right\}$ Is W a subspace of $\mathbb{R}^3$

SOL: (i) $x = \begin{bmatrix} x_1 \\ x_2 \\ 1 - x_1 - x_2 \end{bmatrix} \in W$ and $y = \begin{bmatrix} y_1 \\ y_2 \\ 1 - y_1 - y_2 \end{bmatrix}$

$$x + y = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ 2 - (x_1 + y_1) - (x_2 + y_2) \end{bmatrix} \notin W$$ So; W is not a subspace.

HMW

* **Example:** $W = \{ x_i \geq 0 ; i = 1, 2, 3, 4 \}$ (H is the set of all the vectors such that all components are positive real numbers)

Is W a subspace of $\mathbb{R}^4$?

SOL: $x = (1, 1, 1, 1) \in W$
let $c = -5$

$c.x = (-5, -5, -5, -5) \notin W$

So; W is not a subspace of $\mathbb{R}^4$

Example: $W = \{ p(x) \in P_3 \mid a_0 = a_1 = 0 \}$ Is W a subspace in $\mathbb{R}^3$

$\left(p(x) = a_3x^3 + a_2x^2 + \cancel{a_1x} + \cancel{a_0} \in P_3 \right)$

(i) $p(x) = a_2x^2 + a_3x^3 \in W$
$q(x) = b_2x^2 + b_3x^3 \in W$

$p(x) + q(x) = \underbrace{(a_2+b_2)}_{d_2}x^2 + \underbrace{(a_3+b_3)}_{d_3}x^3 = d_2x^2 + d_3x^3 \in W$ ✓

(ii) $c \in R$; $p(x) = a_2x^2 + a_3x^3 \in W$

$cp(x) = \underbrace{ca_2}_{e_2}x^2 + \underbrace{ca_3}_{e_3}x^3 = e_2x^2 + e_3x^3 \in W$ ✓ So, W is a subspace

Subspace Examples

*** Example:** $S=\left\{\begin{bmatrix}x\\1\end{bmatrix} ; x\in\mathbb{R}\right\}$ and $S\subset\mathbb{R}^2$.

Is S a subspace of $\mathbb{R}^2$?

SOLUTION: (i) For $x=\begin{bmatrix}x\\1\end{bmatrix}\in S$ and $y=\begin{bmatrix}y\\1\end{bmatrix}\in S$

$$x+y \overset{?}{\in} S$$

$$x+y=\begin{bmatrix}x\\1\end{bmatrix}+\begin{bmatrix}y\\1\end{bmatrix}=\begin{bmatrix}x+y\\2\end{bmatrix}\notin S$$

Cnd. (i) is not satisfied. So; S is not a subspace of $\mathbb{R}^2$.

*** Example:** $W=\left\{\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} \middle| \ x_1=x_2^2\right\}$ Is W a subspace of $\mathbb{R}^3$?

SOLUTION: (i) $x=\begin{bmatrix}x_2^2\\x_2\\x_3\end{bmatrix}\in W$ and $y=\begin{bmatrix}y_2^2\\y_2\\y_3\end{bmatrix}\in W$

$$x+y=\begin{bmatrix}x_2^2+y_2^2\\x_2+y_2\\x_3+y_3\end{bmatrix}\notin W \quad (x_2+y_2)^2\neq x_2^2+y_2^2$$

So; W is not a subspace of $\mathbb{R}^3$.

*** Example:** / Definition:

$V=\mathbb{R}^3$ is a subspace and $\mathbb{R}^3$ is a subspace of itself.
The subset $V=\{0\}$, containing only the zero vector, is also a subspace of $\mathbb{R}^3$, because

$$0+0=0 \quad \text{and} \quad c\,0=0 \text{ for every scalar } c$$

Thus $V=\{0\}$ satisfies Cnd. (i) and (ii).
The subspaces $\{0\}$ and $\mathbb{R}^3$ are sometimes called trivial subspaces of $\mathbb{R}^3$.

So; Every subspace V of $\mathbb{R}^3$ contains the zero vector 0.

* **Example:** Let V be set of all vectors (x,y) in $\mathbb{R}^2$ such that $y=x$.
Is V a subspace of $\mathbb{R}^2$?

SOLUTION: (i) $u=(u,u)\in V$ and $v=(v,v)\in V$

$u+v \overset{?}{\in} V \Rightarrow (u,u)+(v,v)=(u+v,u+v)\in V$ ✓

(ii) For $c\in\mathbb{R}$ and $u=(u,u)\in V \Rightarrow c.u \overset{?}{\in} V$

$c(u,u)=(cu,cu)\in V$ ✓

It follows that V is a subspace of $\mathbb{R}^3$ ✓

SOLVE

(x) **Example:** Let V be the set of all the vectors (x,y) in $\mathbb{R}^2$
such that $x+y=1$.
Is V a subspace $\mathbb{R}^2$?

SOLUTION: (i) $u=\begin{bmatrix}1-y\\y\end{bmatrix}\in V$ and $v=\begin{bmatrix}1-z\\z\end{bmatrix}\in V$

$u+v=\begin{bmatrix}1-y+1-z\\y+z\end{bmatrix}=\begin{bmatrix}2-y-z\\y+z\end{bmatrix}\notin V$

So; V is not a subspace of $\mathbb{R}^2$.

Way 2: Counter solution:

For $u=\begin{bmatrix}1\\0\end{bmatrix}\in V$ and $v=\begin{bmatrix}0\\1\end{bmatrix}\in V$ $\quad u+v\overset{?}{\in}V$

$(y=0)$ $\qquad (y=1)$

$u+v=\begin{bmatrix}1\\0\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}\notin V$. So; V is not a subspace of $\mathbb{R}^2$.

Linear Combinations

Definition:

The vector $w \in V$ is called a linear combination of the vectors $v_1, v_2, ..., v_n$ provided that there exists $a_1, a_2, ..., a_n$ scalars such that

$$w = a_1 v_1 + a_2 v_2 + ... + a_n v_n$$

2*) **Example:** Is $w = (2,-6,3) \in \mathbb{R}^3$ a linear combination of the vectors $v_1 = (1,2,-1)$, $v_2 = (3,-5,4)$?

SOLUTION:

$W = a_1 v_1 + a_2 v_2$ (If a_1, a_2 can be found; this means Linear Combination exists)

$(2,-6,3) = a_1(1,2,-1) + a_2(3,-5,4)$

$$\left.\begin{aligned} a_1 + 3a_2 &= 2 \\ 2a_1 - 5a_2 &= -6 \\ -a_1 + 4a_2 &= 3 \end{aligned}\right\} \quad \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 2 & -5 & -6 \\ -1 & 4 & 3 \end{array}\right] \overset{...}{\sim} \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{array}\right]$$

$0 \neq 19$

From row 3; the system becomes Inconsistent. So a_1, a_2 do not exist. Thus w is not a linear Combination of v_1 and v_2

HMW

3*) **Example:** Is $w = (-7,7,11)$ in $\mathbb{R}^3$ a linear combination of the vectors $v_1 = (1,2,1)$, $v_2 = (-4,-1,2)$, $v_3 = (-3,1,3)$?

SOLUTION:

$w = a_1 v_1 + a_2 v_2 + a_3 v_3$

$(-7,7,11) = a_1(1,2,1) + a_2(-4,-1,2) + a_3(-3,1,3)$

$$\left.\begin{aligned} a_1 - 4a_2 - 3a_3 &= -7 \\ 2a_1 - a_2 + a_3 &= 7 \\ a_1 + 2a_2 + 3a_3 &= 11 \end{aligned}\right\} \quad \left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & 3 & 11 \end{array}\right] \underset{ERO}{\overset{...}{\sim}} \left[\begin{array}{ccc|c} 1 & -4 & -3 & -7 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} a_1 - 4a_2 - 3a_3 &= -7 \\ a_2 + a_3 &= 3 \end{aligned}\right\} \quad a_3 \text{ is free variable}$$

Let $a_3 = t \rightarrow a_2 = 3 - t$, $a_1 = 5 - t$

For $t = 0 \rightarrow (5,3,0) \rightarrow w = 5v_1 + 3v_2$

For $t = 1 \rightarrow (4,2,1) \rightarrow w = 4v_1 + 2v_2 + v_3$

w can be written as a linear combination of v_1, v_2, v_3 in many different ways. more than one Linear Combination can be written depending on the choise of the parameter t.

(*) <u>Example:</u> $w = 3x+2$
Can w be written as a linear combination of $\{\underset{v_1}{x+1}, \underset{v_2}{x-1}\}$?

<u>SOLUTION</u>. $w = a_1(x+1) + a_2(x-1)$
$3x+2 = (a_1+a_2)x + (a_1-a_2)$

$$\left.\begin{array}{l} a_1+a_2=3 \\ a_1-a_2=2 \end{array}\right\} \Rightarrow \begin{array}{l} a_1 = 5/2 \\ a_2 = 1/2 \end{array}$$

YES; the linear combination is

$$w = \frac{5}{2} v_1 + \frac{1}{2} v_2$$

HMW
* <u>Example:</u> $w = 2x^2+4x+3$ is a linear combination of $\{x^2, x, 1\}$

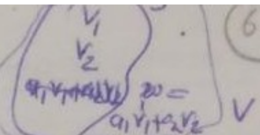
* <u>Example:</u> Express the vector $w = (11, 4)$ as a linear combination of the vectors
$u = (3, -2)$ and $v = (-2, 7)$

<u>SOL:</u> when $au + bv = w$ $a, b = ?$

$$a\begin{bmatrix} 3 \\ -2 \end{bmatrix} + b\begin{bmatrix} -2 \\ 7 \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -2 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 11 \\ 4 \end{bmatrix}$$ using Cramer's Rule or Gauss Elimination ; $a = 5$, $b = 2$

So; $w = 5u + 2v$ ///



Span

Let V be a vector space and S is a subset of V. Let's say

$S = \{v_1, v_2, \ldots, v_n\}$ finite number of vectors in it

If any $w \in V$ can be written as a linear combination of the vectors in S; we say that S spans V. (These vectors also span subspace of V.)

$$\text{Span } S = \{ w \in V,\ a_i \in \mathbb{R};\ v_i \in S \mid w = a_1v_1 + a_2v_2 + \cdots + a_nv_n \}.$$

Theorem 1) Let $S = \{v_1, v_2, \ldots, v_n\} \in V$. Then the set W of all linear combinations of $v_1, v_2, \ldots, v_n$ is a subspace of V. Thus W is a subspace of V. We say that the subspace W is the space spanned by the vectors $v_1, v_2, \ldots, v_n$ (or is the span of the set $S = \{v_1, v_2, \ldots, v_n\}$ of vectors). We sometimes write

$$W = \text{span}(S) = \text{span}\{v_1, v_2, \ldots, v_n\}$$

Example: Let $V = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \in \mathbb{R}^2$. Find the vectors that span $\mathbb{R}^2$.

$$V = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = a_1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{v_1} + a_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{v_2} \Rightarrow \text{so; } \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \text{ spans } \mathbb{R}^2.$$

Example: Find the vectors that span $\mathbb{R}^3$.

$$V = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \Rightarrow V = a_1 \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + a_2 \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_2} + a_3 \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{v_3} \quad \{v_1, v_2, v_3\} \text{ spans } \mathbb{R}^3.$$

Example:

$$V = \begin{bmatrix} a \\ b \\ a+b \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{v_1} + b \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{v_2} \quad \{v_1, v_2\} \text{ spans } \mathbb{R}^3$$

Example: Does $S = \{x^2+2x+1,\ x^2+2\}$ span P_2?

SOL:

$p(x) = ax^2 + bx + c \in P_2$

$p(x) = a_1v_1 + a_2v_2$ Can we find a_1, a_2?

$ax^2 + bx + c = a_1(x^2+2x+1) + a_2(x^2+2)$

$$\left.\begin{aligned} a_1 + a_2 &= a \\ 2a_1 &= b \\ a_1 + 2a_2 &= c \end{aligned}\right\} \quad \left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 0 & b \\ 1 & 2 & c \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 0 & 2a-c \\ 0 & 1 & c-a \\ 0 & 0 & 2c+b-4a \end{array}\right]$$

Case 1: If $2c+b-4a = 0$
IMS ✓
S spans P_2

Case 2: If $2c+b-4a \neq 0$
$\Rightarrow$ The system has no solution
S doesn't span P_2

Linear InDependence / Linear Dependence

* The vectors $v_1, v_2, \dots, v_n$ in a vector space V are said to be "linearly independent" provided that the equation

$$a_1 v_1 + a_2 v_2 + \dots + a_n v_n = \vec{0}_V$$ ← zero vector of V

has only trivial solution $a_1 = a_2 = \dots = a_n = 0$.

* The vectors $v_1, v_2, \dots v_n$ are linearly dependent if and only if there exist scalars $a_1, a_2, \dots, a_n$ not all zero such that $a_1 v_1 + a_2 v_2 + \dots + a_n v_n = \vec{0}_V$

* Example : Is $S = \left\{ \underset{v_1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}, \underset{v_2}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} \right\}$ linearly independent?

$$a_1 v_1 + a_2 v_2 = \vec{0}_{\mathbb{R}^2}$$

$$a_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} a_1 = 0 \\ a_2 = 0 \end{matrix}\right\}$$ $\{v_1, v_2\}$ are linearly independent

HmV/

* Example: $v_1 = (2,1,3)$, $v_2 = (5,-2,4)$, $v_3 = (3,8,-6)$, $v_4 = (2,7,-4)$
Determine whether v_1, v_2, v_3, v_4 in $\mathbb{R}^3$ are linearly independent or not. If not, find the relation between them.

SOLUTION : $a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 v_4 = 0$

$$a_1 \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + a_2 \begin{bmatrix} 5 \\ -2 \\ 4 \end{bmatrix} + a_3 \begin{bmatrix} 3 \\ 8 \\ -6 \end{bmatrix} + a_4 \begin{bmatrix} 2 \\ 7 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{matrix} 2a_1 + 5a_2 + 3a_3 + 2a_4 = 0 \\ a_1 - 2a_2 + 8a_3 + 7a_4 = 0 \\ 3a_1 + 4a_2 - 6a_3 - 4a_4 = 0 \end{matrix}\right\}$$ 4 unknowns, 3 eqs } Linearly Dependent

$$\left[\begin{array}{cccc|c} 2 & 5 & 3 & 2 & 0 \\ 1 & -2 & 8 & 7 & 0 \\ 3 & 4 & -6 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & -2 & 8 & 7 & 0 \\ 0 & 1 & -17 & -13 & 0 \\ 0 & 0 & 1 & 3/4 & 0 \end{array}\right]$$

$$\left.\begin{matrix} a_1 - 2a_2 + 8a_3 + 7a_4 = 0 \\ a_2 - 17a_3 - 13a_4 = 0 \\ a_3 + \frac{3}{4} a_4 = 0 \end{matrix}\right\}$$ 1 parameter

Let $a_4 = 4r \Rightarrow a_3 = -3r$, $a_2 = r$, $a_1 = -2r$

For $r = 1 \Rightarrow (-2, 1, -3, 4)$

So: the relation is:

$$-2v_1 + v_2 - 3v_3 + 4v_4 = 0$$ ///

(*) **Example**: Determine whether

$$u = (1,2,-3)$$
$$v = (3,1,-2)$$
$$w = (5,-5,6)$$

in $\mathbb{R}^3$ are Linearly independent or not?

SOLUTION:

$$a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = \vec{0}_{\mathbb{R}^3}$$

$$a_1\begin{bmatrix}1\\2\\-3\end{bmatrix} + a_2\begin{bmatrix}3\\1\\-2\end{bmatrix} + a_3\begin{bmatrix}5\\-5\\6\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\left.\begin{aligned} a_1 + 3a_2 + 5a_3 &= 0 \\ 2a_1 + a_2 - 5a_3 &= 0 \\ -3a_1 - 2a_2 + 6a_3 &= 0 \end{aligned}\right\} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 2 & 1 & -5 & 0 \\ -3 & -2 & 6 & 0 \end{array}\right] \overset{\cdots}{\sim} \left[\begin{array}{ccc|c} 1 & 3 & 5 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

New LES

$$a_1 + 3a_2 + 5a_3 = 0$$
$$a_2 + 3a_3 = 0$$

Let $a_3 = t \Rightarrow a_2 = -3t$

free variable $a_1 = -5a_3 - 3a_2 = -5t + 9t = 4t$

So; $\left.\begin{aligned} a_1 &= 4t \\ a_2 &= -3t \\ a_3 &= t \end{aligned}\right\}$ $\vec{u}, \vec{v}, \vec{w}$ are Linearly dependent.

The relation is:

$$4t\vec{u} - 3t\vec{v} + t\vec{w} = 0$$

$$4\vec{u} - 3\vec{v} + \vec{w} = 0 \Rightarrow \boxed{\vec{w} = 3\vec{v} - 4\vec{u}}$$

(**NOTE**: Another expression of $a_1\vec{u} + a_2\vec{v} + a_3\vec{w} = 0$ is

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 1 & -5 \\ -3 & -2 & 6 \end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$)

HMW

* **Example**: Is the vector $\vec{t} = (4,20,23)$ a linear combination of the linearly independent vectors

$$u = (1,3,2), \quad v = (2,8,7), \quad w = (1,7,9) \ ?$$

(*) $u, v, w \in \mathbb{R}^3$ are said to be linearly dependent if one of them as a linear combination of the other two.

SOLUTION:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 7 \\ 2 & 7 & 9 \end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix} = \begin{bmatrix}4\\20\\23\end{bmatrix}$$

(t must be written as a linear comb. of u, v, w:

$$\vec{t} = a_1\vec{u} + a_2\vec{v} + a_3\vec{w}$$)

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] \overset{\cdots}{\sim} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \Rightarrow \left.\begin{aligned} a_1 + 2a_2 + a_3 &= 4 \\ a_2 + 2a_3 &= 4 \\ a_3 &= 3 \end{aligned}\right\} \begin{aligned} a_2 &= -2 \\ a_1 &= 5 \end{aligned}$$

Yes; so $\vec{t} = 5\vec{u} - 2\vec{v} + 3\vec{w}$

NOTE:

$$\vec{t} = a_1\vec{u} + a_2\vec{v} + a_3\vec{w}$$

$$\begin{bmatrix} 4 \\ 20 \\ 23 \end{bmatrix} = a_1\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} + a_2\begin{bmatrix} 2 \\ 8 \\ 7 \end{bmatrix} + a_3\begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 3 & 8 & 7 & 20 \\ 2 & 7 & 9 & 23 \end{array}\right] \quad \ldots$$

IS another expression

Example: Determine whether $w = (2,-6,3)$ in $\mathbb{R}^3$ is a linear combination of the vectors

$v_1 = (1,-2,-1)$
$V_2 = (3,-5,4)$

SOLUTION:

$c_1\vec{v_1} + c_2\vec{v_2} = \vec{w}$ Can we find c_1, c_2 scalars?

$$c_1\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} = \begin{bmatrix} 2 \\ -6 \\ 3 \end{bmatrix}$$

$$\left.\begin{aligned} c_1 + 3c_2 &= 2 \\ -2c_1 - 5c_2 &= -6 \\ -c_1 + 4c_2 &= 3 \end{aligned}\right\} \qquad \left[\begin{array}{cc|c} 1 & 3 & 2 \\ -2 & -5 & -6 \\ -1 & 4 & 3 \end{array}\right] \sim \cdots \sim \left[\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 19 \end{array}\right]$$

From the 3^{rd} row, our system inconsistent.
So the desired scalars c_1, c_2 do not exist. Thus w is NOT a linear combination of v_1 and v_2.

NOTE: Standard unit vectors in $\mathbb{R}^n$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right\}$

Theorem 2: The two vectors U and V are linearly dependent if and only if there exist scalars a and b both not zero such that

$$aU + bV = 0$$

Theorem 3: Three Linearly Dependent Vectors
The 3 vectors $U, V, W \in \mathbb{R}^3$ are linearly dependent if and only if there exist scalars a, b, c are not all zero such that

$$aU + bV + cW = 0$$

Theorem 2) The Independence of n Vectors in $\mathbb{R}^n$

The n vectors $v_1, v_2, \dots, v_n$ in $\mathbb{R}^n$ are linearly independent if and only if the $(n\times n)$ matrix A

$$A = \begin{bmatrix} \vdots & \vdots & & \vdots \\ v_1 & v_2 & \cdots & v_n \\ \vdots & \vdots & & \vdots \end{bmatrix}_{n\times n}$$

(In another expression: $A = [v_1 \ v_2 \ \dots \ v_n]$)

having them as its column vectors has non-zero determinant. (Thus, $|A| \neq 0$)

Theorem 3) If $\{v_1, v_2, \dots, v_n\}$ are linearly dependent; We can find the linearly independent ones by looking at the column indices which includes pivots.

EX / $v_1 = (1,2,3)$, $v_2 = (4,5,6)$, $v_3 = (5,7,9)$

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & 5 & 7 \\ 3 & 6 & 9 \end{bmatrix} \xrightarrow[(-3)R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & -3 & -3 \\ 0 & -6 & -6 \end{bmatrix} \xrightarrow[-\frac{1}{6}R_3\to R_3]{-\frac{1}{3}R_2\to R_2} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{(-1)R_2+R_3\to R_3} \begin{array}{ccc} C_1 & C_2 & \\ \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

CONC: Vectors are Linearly dependent. The relation btw them is $v_3 = v_1 + v_2$

$\{v_1, v_2\}$ are linearly independent and $v_1 + v_2 = v_3$

Example: Let $\{v_1, v_2, v_3\}$ be linearly indendent vectors. Show that $u_1 = v_1 + v_2$; $u_2 = v_1 - v_3$ are linearly independent.

SOLUTION: $a_1v_1 + a_2v_2 + a_3v_3 = 0 \Rightarrow a_1 = a_2 = a_3 = 0$ is given

when $c_1u_1 + c_2u_2 = 0$ is true; $c_1 = c_2 \overset{?}{=} 0$?

$$c_1(v_1+v_2) + c_2(v_1 - v_3) = 0$$
$$(c_1+c_2)v_1 + c_1v_2 - c_2v_3 = 0$$

$$\left.\begin{array}{l} c_1+c_2 = 0 \\ c_1 = 0 \\ -c_2 = 0 \end{array}\right\} \begin{array}{l} c_1 = 0 \\ c_2 = 0 \end{array} \Rightarrow \{u_1, u_2\} \text{ are linearly independent}$$

Theorem: Three Linearly Independent Vectors

The vectors $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3)$ and $w = (w_1, w_2, w_3)$ are linearly independent if and only if

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \neq 0$$

* BASES AND DIMENSION FOR VECTOR SPACES

Let V be a vector space and a finite set of S vectors $S=\{v_1, v_2, \ldots, v_n\}$ $S \subset V$.

S is called a basis for V provided that

(a) The vectors in S spans V

(b) The vectors in S are linearly independent

NOTE: If the vectors $u, v, w \in R^3$ are linearly independent, then they constitute a basis for R^3

The number of basis vectors gives the dimension of V. It is denoted by $\dim V$

Theorem: Any two bases for a vector space consist of the same number of vectors.

Example: $W=\{v=(a, 2a, b, b+2a),\ a, b \in R\}$ Find a basis of a subspace of $\mathbb{R}^4$. Determine the dimension.

SOLUTION:

$$v=\begin{bmatrix} a \\ 2a \\ b \\ b+2a \end{bmatrix} = a\underbrace{\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}}_{v_1} + b\underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{v_2} \Rightarrow v_1, v_2 \text{ span } \mathbb{R}^4$$

and

$$a_1 v_1 + a_2 v_2 = \tilde{0}_{\mathbb{R}^4}$$

$$a_1\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} + a_2\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a_1=0 \\ a_2=0 \end{matrix} \Bigg\} \ \{v_1, v_2\} \text{ are linearly independent}$$

So; a basis of a subspace of $\mathbb{R}^4$: $\{v_1, v_2\}$

$\dim(\mathbb{R}^4)=2$ → Solve P84/B1 question!

Note: The standard basis of R^n consists of the unit vectors $e_1=(1,0,0,\ldots,0)$, $e_2=(0,1,0,\ldots,0)$, ..., $e_n=(0,0,0,\ldots,1)$

* Theorem: Independent sets, Spanning Sets and Bases

Let V be an n-dimensional vector space and let S be a subset of V. Then

(a) If S is linearly independent and consists of n vectors, then S is a basis for V. (We don't have to show spanning condition. L.Independency check is enough to find a basis)

(b) If S spans V and consists of n vectors, then S is a basis for V.

(c) If S is linearly independent, then S is contained in a basis for V.

(d) If S spans V, then S contains a basis for V.

Nov 11, 2017 / Q4 (a)

Let V be a subspace of R^4 given by

$$V = \{ [a\ b\ c\ d]^T \in R^4 \mid a+b=0 \text{ and } c-d=0,\ a,b,c,d \in R \}$$

(i) Find a basis for V.
(ii) Find dimension of V.

SOLUTION: (i) $a+b=0 \Rightarrow b=-a$

$c-d=0 \Rightarrow d=c$

$$V = \left\{ \begin{bmatrix} a \\ -a \\ c \\ c \end{bmatrix} = a \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{v_1} + c \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}}_{v_2},\ a,c \in R \right\} \Rightarrow \{v_1, v_2\} \text{ spans } V$$

and $a_1\vec{v}_1 + a_2\vec{v}_2 = \vec{0}_{R^4}$

$$a_1 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} a_1 = 0 \\ a_2 = 0 \end{matrix}$$

$\Rightarrow \{v_1, v_2\}$ are Linearly independent

Thus, $\{v_1, v_2\}$ forms a basis for V.

(ii) $\dim(V) = 2$

Example: $S = \left\{ \begin{bmatrix} a+b \\ a-b+2c \\ b \\ c \end{bmatrix},\ a,b,c \in R \right\}$ find a basis and dimension of a subspace S of R^4

SOLUTION: (i) Find the vectors that span V

$$S = \left\{ \underset{\uparrow v}{\begin{bmatrix} a+b \\ a-b+2c \\ b \\ c \end{bmatrix}} = a \underset{\uparrow v_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}} + b \underset{\uparrow v_2}{\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}} + c \underset{\uparrow v_3}{\begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}},\ a,b,c \in R \right\}$$

$\{v_1, v_2, v_3\}$ span S

(ii) Check Linear independency:

when $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}_{R^4}$

$$a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\{v_1, v_2, v_3\}$ are linearly independent.

$$\left.\begin{matrix} a_1 + a_2 = 0 \\ a_1 - a_2 + 2a_3 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{matrix}\right\} a_1 = 0 \Rightarrow a_1 = a_2 = a_3 = 0$$

$\Rightarrow$ So, $\{v_1, v_2, v_3\}$ is a basis of S

$\dim(S) = 3$

*** Example:** Let $v_1 = (1,-1,-2,-3)$, $v_2 = (1,-1,2,3)$, $v_3 = (1,-1,-3,-2)$ and $v_4 = (0,3,-1,2)$. Is $\{v_1, v_2, v_3, v_4\}$ a basis for $\mathbb{R}^4$?

SOLUTION

$$\begin{vmatrix} 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 3 \\ -2 & 2 & -3 & -1 \\ -3 & 3 & -2 & 2 \end{vmatrix} = 30 \neq 0 \Rightarrow$$

So: it follows that $\{v_1, v_2, v_3, v_4\}$ is a basis for $\mathbb{R}^4$.

Important! Determine whether n given vectors $v_1, v_2, \dots v_n$ form a basis for $\mathbb{R}^n$ by calculating the determinant of the $n \times n$ matrix A

$$A = \begin{bmatrix} \Big| & \Big| & & \Big| \\ v_1 & v_2 & \dots & v_n \end{bmatrix}$$

with these vectors as its column vectors.

Column vectors.

They constitute a basis for $\mathbb{R}^n$ if and only if $\det A \neq 0$

Reminder: Let $p(x) = 3 + 2x + 5x^3 \in P_3$ and $q(x) = 7 + 4x + 3x^2 + 9x^4 \in P_4$

We add polynomials by collecting coefficients of like powers of x and multiply them by scalars in usual way. That is

$$(p+q)(x) = (3+7) + (2+4)x + (0+3)x^2 + (5+0)x^3 + (0+9)x^4$$
$$= 10 + 6x + 3x^2 + 5x^3 + 9x^4$$

and

$$(7p)(x) = 7(3 + 2x + 5x^3) = 21 + 14x + 35x^3$$

Example: Does $S = \{\underbrace{x^2+2x+1}_{v_1}, \underbrace{x^2+2}_{v_2}\}$ span P_2 space?

SOLUTION: Let $p(x) = ax^2 + bx + c \in P_2$

$p(x) = a_1 v_1 + a_2 v_2$ Can we find a_1, a_2?

$ax^2 + bx + c = a_1(x^2 + 2x + 1) + a_2(x^2 + 2)$

$ax^2 + bx + c = (a_1 + a_2)x^2 + (2a_1)x + (a_1 + 2a_2)$

$$\left.\begin{aligned} a_1 + a_2 &= a \\ 2a_1 &= b \\ a_1 + 2a_2 &= c \end{aligned}\right\} \left[\begin{array}{cc|c} 1 & 1 & a \\ 2 & 0 & b \\ 1 & 2 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2a-c \\ 0 & 1 & c-a \\ 0 & 0 & 2c+b-4a \end{array}\right]$$

If

- $2c + b - 4a \neq 0$: The system has No solution. a_1, a_2 do not exist. So: S doesn't span P_2
- $2c + b - 4a = 0$: The system has sols. a_1, a_2 exists. $\left.\begin{aligned} a_1 &= 2a - c \\ a_2 &= c - a \end{aligned}\right\}$ So; S spans P_2

SOLUTION SPACE (Nullspace) Nullspace(A)

Theorem 2 and
Definition : The solution space of (m×n) Matrix A is the set of all solutions of homogeneous equation $A.X = 0$ (is a subspace of $\mathbb{R}^n$)

ALGORITHM : To find a basis for the solution space W of the homogeneous linear system $AX = 0$; carry out the following steps:

1) Reduce the coefficient matrix A to echelon form. (REF or RREF form)
2) Identify leading variables and free variables. Set free variables equal to some parameters and solve linear equation system.
3) If there is a unique solution $W = \{\vec{0}\}$ and there is no basis for solution space W.
4) If there are infinitely many solutions, write the solution in parametric form and find the spanned vectors. These vectors are also a basis for W.

For (m×n) matrix A;

$$N(A) = \left\{ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n \;\middle|\; AX = 0 \right\}$$

Example Find a basis and dimension for the solution space of the homogeneous linear system.

$$x_1 + x_2 + x_3 = 0$$
$$2x_1 + x_2 \quad + x_4 = 0$$

SOLUTION:

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \xrightarrow{(-2)R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 1 \end{bmatrix} \xrightarrow{(-1)R_2 \to R_2} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

REF

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ x_2 + 2x_3 - x_4 &= 0 \end{aligned}\right] \begin{matrix} 4 \text{ unknowns} \\ 2 \text{ eqs} \end{matrix} \Rightarrow 4 - 2 = 2 \text{ free variables}$$

Leading entries are in first and second columns. So the leading variables are x_1 and x_2, the free variables are x_3 and x_4.

Let
$x_4 = \alpha$ and $x_3 = \beta \Rightarrow x_2 = \alpha - 2\beta$.
$x_1 = \beta - \alpha$

$$X = \begin{bmatrix} \beta - \alpha \\ \alpha - 2\beta \\ \beta \\ \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}}_{v_1} + \beta \underbrace{\begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v_2}$$

$\{v_1, v_2\}$ is a basis for W

$\dim W = 2$ ///

Example:

$$\left.\begin{aligned} 3x_1+6x_2-x_3-5x_4+5x_5&=0\\ 2x_1+4x_2-x_3-3x_4+2x_5&=0\\ 3x_1+6x_2-2x_3-4x_4+x_5&=0 \end{aligned}\right\}$$

Find a basis and dimension for the solution space of the homogeneous linear system.

SOLUTION:

$$A=\begin{bmatrix}3&6&-1&-5&5\\2&4&-1&-3&2\\3&6&-2&-4&1\end{bmatrix}\cdots\sim\begin{bmatrix}1&2&0&-2&3\\0&0&1&-1&4\\0&0&0&0&0\end{bmatrix}$$

$\left.\begin{array}{l}5 \text{ unknowns}\\ 2 \text{ eqs.}\end{array}\right\}$ $5-2=3$ parameters (free variables)

The leading entries are in the first and third columns, so the leading variables are x_1 and x_3. Free variables are x_2, x_4 and x_5

Let $x_5=\alpha$, $x_4=\beta$, $x_2=\theta$ $\quad \alpha,\beta,\theta\in\mathbb{R} \Rightarrow$ $x_1=2\beta-2\theta-3\alpha$, $x_3=\beta-4\alpha$

$$X=\begin{bmatrix}2\beta-2\theta-3\alpha\\ \theta\\ \beta-4\alpha\\ \beta\\ \alpha\end{bmatrix}=\alpha\underbrace{\begin{bmatrix}-3\\0\\-4\\0\\1\end{bmatrix}}_{v_1}+\beta\underbrace{\begin{bmatrix}2\\0\\1\\1\\0\end{bmatrix}}_{v_2}+\theta\underbrace{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}}_{v_3}$$

$\{v_1,v_2,v_3\}$ is a basis for solution space W..

$\dim W=3$

*Def: dimension of Nullspace is denoted by Null (A)

Examples

A) Determine whether or not the given vectors in R^n form a basis for R^n

1) $V_1 = (4,7)$, $V_2 = (5,6)$

V_1, V_2 vectors are linearly independent, Because neither is a scalar multiple of the other and therefore form a basis for R^2.

2) $V_1 = (3,-1,2)$, $V_2 = (6,-2,4)$, $V_3 = (5,3,-1)$

$V_2 = 2V_1$. So; the vectors are linearly dependent and therefore do not form a basis for R^3.

3) $V_1 = (3,-7,5,2)$, $V_2 = (1,-1,3,4)$, $V_3 = (7,11,3,13)$

Any basis for R^4 contains 4 vectors, So the given vectors V_1, V_2, V_3 do not form a basis for R^4.

4) $V_1 = (0,0,1)$, $V_2 = (0,1,2)$, $V_3 = (1,2,3)$

$$\begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = -1 \neq 0 \Rightarrow$$ 3 vectors are linearly independent. Hence do form a basis for R^3.

B) Find a basis for the indicated subspace of R^3.

1) The plane with equation $x-2y+5z=0$

The equation is already in reduced Row Echelon form. And y, z are free variables. (Because x has pivot 1)

With $y=s$ and $z=t \Rightarrow x = 2s-5t$

So; the solution vector

$$(x,y,z) = (2s-5t, s, t) = s(2,1,0) + t(-5,0,1)$$

The plane is a 2-dimensional subspace of R^3 with basis vectors are $V_1 = (2,1,0)$ and $V_2 = (-5,0,1)$.

+5

2) The plane with equation $y = z$

The single equation $y - z = 0$ is already a system in Reduced Row Echelon form, with free variables x and z.

With $x = s$ and $z = t \Rightarrow y = t$

So, the solution vector is

$$(x, y, z) = (s, t, t) = s(1, 0, 0) + t(0, 1, 1)$$

Hence the plane $y - z = 0$ is a 2-dimensional subspace of $\mathbb{R}^3$ with basis consisting of the vectors

$$v_1 = (1, 0, 0) \text{ and } v_2 = (0, 1, 1)$$

c) Find a basis for the indicated subspace of $\mathbb{R}^4$.

1) The set of all vectors of the form (a, b, c, d) for which $a = b + c + d$

SOL: The typical vector in $\mathbb{R}^4$ is of the form (a, b, c, d).

$v = (b + c + d, b, c, d) = b(1, 1, 0, 0) + c(1, 0, 1, 0) + d(1, 0, 0, 1)$

Hence the subspace consisting of all such vectors in 3-dim. with basis vectors $v_1 = (1, 1, 0, 0)$, $v_2 = (1, 0, 1, 0)$, $v_3 = (1, 0, 0, 1)$

2) The set of all vectors of the form (a, b, c, d) such that $a = 3c$ and $b = 4d$

SoL: The typical vector in $\mathbb{R}^4$ is of the form (a, b, c, d) with $a = 3c$ and $b = 4d$. So,

$$v = (3c, 4d, c, d) = c\underbrace{(3, 0, 1, 0)}_{v_1} + d\underbrace{(0, 4, 0, 1)}_{v_2}$$

v_1 and v_2 are basis vectors.

D) Find a basis for the solution space of the given homogeneous linear systems

1) $\left.\begin{array}{l} x_1 - 2x_2 + 3x_3 = 0 \\ 2x_1 - 3x_2 - x_3 = 0 \end{array}\right\}$ SoL: $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & -1 \end{bmatrix} \overset{\dots}{\sim} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -7 \end{bmatrix} = E$ ← RREF Echelon form

$\left.\begin{array}{l} x_1 - x_3 = 0 \\ x_2 - 7x_3 = 0 \end{array}\right\}$ x_3 is free variable. Let $x_3 = t \Rightarrow \left.\begin{array}{l} x_1 = 11t \\ x_2 = 7t \end{array}\right\} \Rightarrow$ SOLUTION VECTOR IS $x = (11t, 7t, t)$

So, basis vector is $v = (11, 7, 1)$

2) $x_1 - 3x_2 - 10x_3 + 5x_4 = 0$
$x_1 + 4x_2 + 11x_3 - 2x_4 = 0$
$x_1 + 3x_2 + 8x_3 - x_4 = 0$

SOL: $A = \begin{bmatrix} 1 & -3 & -10 & 5 \\ 1 & 4 & 11 & -2 \\ 1 & 3 & 8 & -1 \end{bmatrix} \approx \begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = E$ ✓ RREF Echelon Matrix.

Free variables are x_3 and x_4.
Let $\left.\begin{array}{l} x_3 = s \\ x_4 = t \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = s - 2t \\ x_2 = -3s + t \end{array}$ $\Rightarrow$ So; solution vector is
$v = (s-2t, -3s+t, s, t)$
$= s\underbrace{(1,-3,1,0)}_{v_1} + t\underbrace{(-2,1,0,1)}_{v_2}$

So; $\{v_1, v_2\}$ are basis vectors.

5) A subset W of some n-space R^n is defined. Determine whether or not W is a subspace of R^n.

1) W is the set of all vectors in R^3 such that $x_3 = 0$
a) If $x = (x_1, x_2, 0) \in W$; $y = (y_1, y_2, 0) \in W \Rightarrow x + y = (x_1+y_1, x_2+y_2, 0) \in W$
b) For $c \in R$ and $x = (x_1, x_2, 0) \in W \Rightarrow c.x = (cx_1, cx_2, 0) \in W$. Hence W is a subspace of R^3 ✓

2) W is the set of all vectors in R^3 such that $x_1 = 5x_2$
a) $x = (5x_2, x_2, x_3) \in W$ and $y = (5y_2, y_2, y_3)$
$x + y = (5(x_2+y_2), x_2+y_2, x_3+y_3) \in W$ ✓
(b) $cx = (5c\,x_2, cx_2, cx_3) \in W$ ✓
Hence W is a subspace of R^3

3) W is the set of all vectors in R^3 such that $x_2 = 1$
(a) $x = (x_1, 1, x_3) \in W$ and $y = (y_1, 1, y_3)$
$x + y = (x_1+y_1, 2, x_3+y_3) \notin W$ W is not closed under vector addition.
So W is not a subspace of R^3. (or) (b) Let's take $c=2$; $2x = (2x_1, \underline{\underline{2}}, x_3) \notin W$
(second Coord. is not 1.)

4) W is the set of all vectors in R^3 such that $x_1 + x_2 + x_3 = 1$
The typical vector $x = (x_1, x_2, x_3)$ in W has coordinate sum $x_1 + x_2 + x_3$ equal to 1
But then the particular scalar multiple
$2x = (2x_1, 2x_2, 2x_3)$ of such a vector has coordinate sum
$2x_1 + 2x_2 + 2x_3 = 2(x_1 + x_2 + x_3) = 2.(1) = 2 \neq 1$
Thus is Not in W. Hence W is not closed under scalar multiplication by scalars. It is not a subspace of R^3.

5) W is the set of all vectors in R^2 such that $|x_1| = |x_2|$

Sol: The vectors $x = (1,1)$ and $y = (1,-1)$ are in W.
But their sum $x + y = (2,0)$ is not in W. Because $|2| \neq |0|$.
Hence W is not a subspace of R^2. 7

6) W is the set of all vectors in R^2 such that $(x_1)^2 + (x_2)^2 = 0$

SOL: W is simply the zero subspace $\{0\}$ of R^2. 8

7) W is the set of all vectors in R^2 such that $(x_1)^2 + (x_2)^2 = 1$

The vector $x = (1,0) \in W$. But its scalar multiple $2x = (2,0) \notin W$
Because $(2)^2 + 0^2 = 4 \neq 1$. Hence W is not a subspace of R^2.

8) W is the set of all vectors in R^4 such that $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$

SOL: Suppose $x = (x_1, x_2, x_3, x_4) \in W$ and $y = (y_1, y_2, y_3, y_4) \in W$. So;

a) $x_1 + 2x_2 + 3x_3 + 4x_4 = 0$ and $y_1 + 2y_2 + 3y_3 + 4y_4 = 0$

Then their sum s is

$$s = (x_1 + y_1, x_2 + y_2, x_3 + y_3, x_4 + y_4) = (s_1, s_2, s_3, s_4)$$ satisfies the same condition

$$\begin{aligned} s_1 + 2s_2 + 3s_3 + 4s_4 &= (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) + 4(x_4 + y_4) \\ &= (x_1 + 2x_2 + 3x_3 + 4x_4) + (y_1 + 2y_2 + 3y_3 + 4y_4) \\ &= 0 + 0 \\ &= 0 \in W \end{aligned}$$

(Thus their sum is an element in W)

b) $cx = (cx_1, cx_2, cx_3, cx_4) = m$

$$m_1 + 2m_2 + 3m_3 + 4m_4 = cx_1 + 2cx_2 + 3cx_3 + 4cx_4 = c\underbrace{(x_1 + 2x_2 + 3x_3 + 4x_4)}_{}$$
$$= c \cdot 0$$
$$= 0$$

Hence is an element in W.

Therefore W is a subspace of R^4.

9) W is the set of all those vectors in R^4 whose components are all non zero.

SOL: The vector $x = (1,1,1,1) \in W$. Because all 4 components are nonzero.
But the multiple $0 \cdot x = (0,0,0,0)$ is not. Hence W is not a subspace of R^4

Row Space and Column Space

For $(m \times n)$ matrix A ; $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

*) Row matrices of A are : $[a_{11}\ a_{12} \ldots a_{1n}]$
$[a_{21}\ a_{22} \cdots a_{2n}]$
$\vdots$
$[a_{m1}\ a_{m2} \cdots a_{mn}]$

and row vectors of A are the m vectors

$r_1 = (a_{11}, a_{12}, \ldots, a_{1n})$
$r_2 = (a_{21}, a_{22}, \ldots, a_{2n})$
$\vdots$
$r_m = (a_{m1}, a_{m2}, \ldots, a_{mn})$

in $\mathbb{R}^n$. Since n-tuples denote column vector elements of $\mathbb{R}^n$, we see that the row vectors of A are the transposes of its row matrices ; that is

$$Row(A) \Rightarrow r_i = [a_{i1}\ a_{i2} \ldots a_{in}]^T \in \mathbb{R}^n$$

The subspace of $\mathbb{R}^n$ spanned by the row vectors of A is called "Row space of A". Row(A) = Row Space(A)

*) Column vectors of A : $\left\{ \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix}, \ldots \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \right\}$ n vectors

The subspace of $\mathbb{R}^m$ spanned by the column vectors of A is called "column space of A". So

$$COL(A) = Col space(A) \rightarrow c_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix} \in \mathbb{R}^m$$

*) Theorem : Row space of an Echelon Matrix : The nonzero row vectors of an echelon matrix are linearly independent and therefore form a basis for its row space. (Bir matrisin echelon form halinin sıfırdan farklı satırlarının her biri, satır uzayının bir bazını oluşturur)

*) Theorem : Row Spaces of Equivalent matrices :
If 2 matrices A and B are equivalent, then they have the same row space

*** ALGORITHM : A basis for the row space of A**

To find a basis for the rowspace of A : use ERO to reduce A to an echelon matrix E. Then the non-zero row vectors of E form a basis for Row (A).

*** Example :** Find a basis and dimension for the rowspace of the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$$

SOLUTION :

$$A \overset{ERO}{\sim} \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E$$

The non-zero row vectors $\{ \underset{v_1}{(1,2,1,3,2)}, \underset{v_2}{(0,1,-3,5,-4)}, \underset{v_3}{(0,0,0,1,-7)} \}$ form a basis for the rowspace of A.

Thus Row (A) is 3-dimensional subspace of $\mathbb{R}^5$.

Row (Rank (A)) = 3

dim (RowSpace A) = 3

Theorem : The non-zero row vectors of an echelon matrix are linearly independent and therefore form a basis for its row space.

Theorem : If 2 matrices A and B are equivalent, then they have the same rowspace

*** ALGORITHM : A basis for the column space of A**

To find a basis for the column space of A, use ERO to reduce A to an echelon matrix E. Then the column vectors of A that correspond to the pivot columns of E form a basis for COL (A).

*** Example :**

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 3 & 4 & 9 & 0 & 7 \\ 2 & 3 & 5 & 1 & 8 \\ 2 & 2 & 8 & -3 & 5 \end{bmatrix}$$

Find a basis and dimension of column space of A

SOLUTION

$$A \sim \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 1 & -3 & 5 & -4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = E$$

The pivot columns of E are its first, second and fourth columns. So, the 3-dimensional column space of A has a basis consisting of its first, second and forth column vectors.

Thus a basis for the column space of A : $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ -3 \end{bmatrix} \right\}$

(or in another expression : $\{ (1,3,2,2), (2,4,3,2), (3,0,1,-3) \}$

dim (Colspace A) = 3

Rank of A

* The number of non-zero rows in REF of matrix A is called Rank A
* Row Rank (A) = Column Rank (A) = Rank (A)
* The solution space of the homogeneous system $AX=0$ is sometimes called the Nullspace A and denoted by Null (A)
* $\text{rank}(A) + \underbrace{\dim(\text{Null}(A))}_{\text{Nullity}(A)} = n$ for $(m \times n)$ matrix A
  n ← Column number of A

* Theorem: Equality of Row Rank and Column Rank:
  The row rank and column rank of any matrix are equal.

$$X = \begin{bmatrix} 4r-2s+3t \\ -4r \\ s \\ r \\ t \end{bmatrix} = r\underbrace{\begin{bmatrix} 4 \\ -4 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + s\underbrace{\begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{v_2} + t\underbrace{\begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{v_3}$$

⇒ $\{v_1, v_2, v_3\}$ is a basis for nullspace of A (Null (A))

$\dim(\text{Null } A) = 3$

(NOTE: A is (3×5). So $n=5$

$\underbrace{\text{Rank} A}_{2} + \dim \underbrace{(\text{Null}(A))}_{3} \overset{?}{=} 5$

You can use for checking your results)

NOTE: Every non-pivot column vector of A is a linear combination of its pivot columns.

*** Example:**
Find a subset of the vectors $v_1=(1,-1,2,2)$, $v_2=(-3,4,1,-2)$, $v_3=(0,1,7,4)$, $v_4=(-5,7,4,-2)$ that forms a basis for the subspace W of $\mathbb{R}^4$ spanned these 4 vectors.

SOLUTION:
$$A=\begin{bmatrix}1&-3&0&-5\\-1&4&1&7\\2&1&7&4\\2&-2&4&-2\end{bmatrix}.$$
(It is not clear at the outset whether W is 2D or 3D or 4D)

$$A \underset{ERO}{\sim} \begin{bmatrix}1&-3&0&-5\\0&1&1&2\\0&0&0&0\\0&0&0&0\end{bmatrix}=E$$
(We reduced A into echelon matrix)

Pivot columns of E are its first and second columns.
A basis for column space W:
$$\{\underset{v_1}{(1,-1,2,2)},\underset{v_2}{(-3,4,1,-2)}\}$$
(or in another expression; $\left\{\begin{bmatrix}1\\-1\\2\\2\end{bmatrix},\begin{bmatrix}-3\\4\\1\\-2\end{bmatrix}\right\}$).

So, v_1, v_2 of A form basis for the column space W ✓

$\dim(\text{Col space } A) = 2$ (We see that W is 2-dimensional)

(NOTE: You can confirm that $v_3=3v_1+v_2$ and $v_4=v_1+2v_2$)

NOTE $a_1v_1+a_2v_2+a_3v_3+a_4v_4=0$ dan geliyor. Vektörleri A nın sütunları olarak yazmalı. Dikkat!(!)

yani
$a_1(1,-1,2,2)+a_2(-3,4,1,-2)+a_3(0,1,7,4)+$
$+a_4(-5,7,4,-2)=(0,0,0,0)$
$$\begin{aligned}a_1-3a_2+0a_3-5a_4&=0\\-a_1+4a_2+a_3+7a_4&=0\\2a_1+a_2+7a_3+4a_4&=0\\2a_1-2a_2+4a_3-2a_4&=0\end{aligned}$$
dan geliyor.

b) Is $v=(2,3,1,0)$ vector belongs to W?

SOL: $x=a_1v_1+a_2v_2 \rightarrow$ If solution of eq. exists: YES
If there is NO SOLUTION: NO

$(2,3,1,0)=a_1(1,-1,2,2)+a_2(-3,4,1,-2)$

$$\begin{aligned}a_1-3a_2&=2\\-a_1+4a_2&=3\end{aligned}\Big\}\ a_2=5,\ a_1=17$$
$$\begin{aligned}2a_1+a_2&=1\\2a_1-2a_2&=0\end{aligned}\Big\}\ 34+5=39\neq 1$$

So x cannot be written as a linear combination of v_1, v_2. So $x\notin W$

*** Example:**
$$\begin{bmatrix}1&-1&2&0&-3\\0&1&0&4&0\\2&-1&4&4&-6\end{bmatrix}_{3\times5}\cdot\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}_{5\times1}=\begin{bmatrix}0\\0\\0\end{bmatrix}_{3\times1}$$

(i) find a basis and dimension of Row space of A
(ii) " " " " of Column space of A
(iii) " " " " Solution space of A

SOLUTION:
$$A\sim\begin{bmatrix}1&-1&2&0&-3\\0&1&0&4&0\\0&0&0&0&0\end{bmatrix}=E$$

(i) → A basis of rowspace of A:
$\{(1,-1,2,0,-3),(0,1,0,4,0)\}$
Row(Rank A) = 2
dim(Row Space A) = 2

(ii) → A basis of column space of A
$$\left\{\begin{bmatrix}1\\0\\2\end{bmatrix},\begin{bmatrix}-1\\1\\-1\end{bmatrix}\right\}$$
dim(Col space) = 2
Column(Rank(A)) = 2.

(iii) $$\left.\begin{aligned}x_1-x_2+2x_3-3x_5&=0\\x_2+4x_4&=0\end{aligned}\right\}\ 5-2=3 \text{ free variables}$$

→ Let $x_4=r$, $x_5=t$ and $x_3=s$; $r,t,s\in\mathbb{R}$
Then $x_2=-4r$; $x_1=-4r-2s+3t$

Devamı arkada →

March 24, 2018 / Q4

consider the following homogeneous linear system

$$x_1+2x_2+7x_3-9x_4+31x_5=0$$
$$2x_1+4x_2+7x_3-11x_4+34x_5=0$$
$$3x_1+6x_2+5x_3-11x_4+29x_5=0$$

(10) a) Find the rank of the coefficient matrix
(5) b) use part (a) to determine the dimension of the solution space
(10) c) find a basis for the solution space.

SOLUTION: A is (3x5)

a) coefficient matrix A is :

$$A=\begin{pmatrix}1&2&7&-9&31\\2&4&7&-11&34\\3&6&5&-11&29\end{pmatrix}\xrightarrow[(-3)R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2}$$

$$\begin{pmatrix}1&2&7&-9&31\\0&0&-7&7&-28\\0&0&-16&16&-64\end{pmatrix}\xrightarrow[-\frac{1}{16}R_3\to R_3]{-\frac{1}{7}R_2\to R_2}\begin{pmatrix}1&2&7&-9&31\\0&0&1&-1&4\\0&0&1&-1&4\end{pmatrix}\xrightarrow{(-1)R_2+R_3\to R_3}$$

$$\begin{pmatrix}1&2&7&-9&31\\0&0&1&-1&4\\0&0&0&0&0\end{pmatrix}=E$$

*We know that the number of non-zero rows in REF or RREF of A is called "rank of A"

So; rank(A) = 2

b) $\text{rank}(A)+\dim(N(A))=5$

$\dim(N(A))=5-2=3$

c) $A \sim \begin{pmatrix}1&2&7&-9&31\\0&0&1&-1&4\\0&0&0&0&0\end{pmatrix}$ x_1 and x_3 : leading variables
x_2, x_4, x_5: free variables. Let $x_2=t$; $x_4=r$; $x_5=s$

Write New LES ⇒ $\left.\begin{array}{l}x_1+2x_2+7x_3-9x_4+31x_5=0\\x_3-x_4+4x_5=0\end{array}\right\}$ Let $x_1=2r-2t-3s$
$x_3=r-4s$

$$\Rightarrow \vec{x}\in \text{Null}(A) \Rightarrow \vec{x}=t\underbrace{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}}_{\vec{u}_1}+r\underbrace{\begin{bmatrix}2\\0\\1\\1\\0\end{bmatrix}}_{\vec{u}_2}+s\underbrace{\begin{bmatrix}-3\\0\\-4\\0\\1\end{bmatrix}}_{\vec{u}_3}$$

→ $\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ are basis vectors for (Null(A)) solution space. (d) is at the back of the page

July 19, 2018

(Q4) Let the matrix A is given:

$$A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 4 & 0 & 4 \\ -1 & -3 & -4 & 0 \end{bmatrix}_{3\times 4}$$

(10) a) Find a basis for the column space of A

(10) b) Find the rank of A

(5) c) Use part (b) to determine the dimension of the nullspace N(A)

SOLUTION: a) $A = \begin{bmatrix} 1 & 3 & 4 & 1 \\ 2 & 4 & 0 & 4 \\ -1 & -3 & -4 & 0 \end{bmatrix} \xrightarrow[R_1+R_3 \to R_3]{(-2)R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & -2 & -8 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{(-\frac{1}{2})R_2 \to R_2}$

$$\begin{bmatrix} 1 & 3 & 4 & 1 \\ 0 & 1 & 4 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = E$$

$\uparrow c_1 \quad \uparrow c_2 \quad \uparrow c_4$

Pivots are at the first, second and fourth columns

So:

$$C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ -3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix} \right\}$$

b) Rank(A) = 3

c) $\underbrace{\text{Rank}(A)}_{3} + \dim(N(A)) = \underset{(n)}{4} \Rightarrow \dim(N(A)) = 4-3 = 1$

d) Find a basis for the column space of A and determine its dimension

A basis of $C(A) = \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 7 \\ 7 \\ 5 \end{bmatrix} \right\}$; $\dim(C(A)) = 2$ (= rank(A))

(Pivots are at the 1st and 2nd column of E)

e) Find a basis for the rowspace of A and determine its dimension

A basis of Rowspace(A) =

$\{[1\ 2\ 7\ -9\ 31], [0\ 0\ 1\ -1\ 4]\}$

dim(Row Space(A)) = 2

Exercises

A) Find a basis for the rowspace and a basis for the column space of the given matrix A.

1) $A=\begin{bmatrix}1&2&3\\1&5&-9\\2&5&2\end{bmatrix} \overset{ERO}{\sim} \begin{bmatrix}1&0&11\\0&1&-4\\0&0&0\end{bmatrix}=E$ RREF

Row basis (vectors): $\{[1\ 0\ 11], [0\ 1\ -4]\}$

Column basis (vectors): $\left\{\begin{bmatrix}1\\1\\2\end{bmatrix}, \begin{bmatrix}2\\5\\5\end{bmatrix}\right\}$

2) $A=\begin{bmatrix}1&1&-1&7\\1&4&5&16\\1&3&3&13\\2&5&4&23\end{bmatrix} \sim \begin{bmatrix}1&0&-3&4\\0&1&2&3\\0&0&0&0\\0&0&0&0\end{bmatrix}$

Row basis vectors: $\{[1\ 0\ -3\ 4], [0\ 1\ 2\ 3]\}$

Column basis: $\left\{\begin{bmatrix}1\\1\\1\\2\end{bmatrix}, \begin{bmatrix}1\\4\\3\\5\end{bmatrix}\right\}$

B) A set S of vectors in R^4 is given. Find a subset of S that forms a basis for the subspace of R^4 spanned by S.

1) $v_1=(1,3,-2,4)$
$v_2=(2,-1,3,2)$
$v_3=(5,1,4,8)$

SOLUTION:

$$\begin{bmatrix}1&2&5\\3&-1&1\\-2&3&4\\4&2&8\end{bmatrix} \xrightarrow[\substack{(2)R_1+R_3\to R_3\\(-4)R_1+R_4\to R_4}]{(-3)R_1+R_2\to R_2} \begin{bmatrix}1&2&5\\0&-7&-14\\0&7&14\\0&-6&-12\end{bmatrix} \xrightarrow[-\frac{1}{6}R_3\to R_3]{R_2+R_3\to R_3}$$

$$\begin{bmatrix}1&2&5\\0&-7&-14\\0&0&0\\0&1&2\end{bmatrix} \xrightarrow{-\frac{1}{7}R_2\to R_2} \begin{bmatrix}1&2&5\\0&1&2\\0&0&0\\0&1&2\end{bmatrix} \xrightarrow{(-1)R_2+R_4\to R_4} \begin{bmatrix}1&2&5\\0&1&2\\0&0&0\\0&0&0\end{bmatrix} \xrightarrow{(-2)R_2+R_1\to R_1}$$

$$\begin{bmatrix}1&0&1\\0&1&2\\0&0&0\\0&0&0\end{bmatrix}=E$$

V_1 and V_2 are linearly independent vectors (because C_1 and C_2 contains pivots)

$\{(1,3,-2,4), (2,-1,3,2)\}$ forms a basis for subspace of R^4

2) $v_1=(5,4,2,2)$
$v_2=(3,1,2,3)$
$v_3=(7,7,2,1)$
$v_4=(1,-1,2,4)$
$v_5=(5,4,6,7)$

$$\begin{bmatrix}5&3&7&1&5\\4&1&7&-1&4\\2&2&2&2&6\\2&3&1&4&7\end{bmatrix} \sim \begin{bmatrix}1&0&2&0&0\\0&1&-1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}$$

V_1, V_2, V_4, V_5 are linearly independent. So, they form a basis

3) Explain why the rank of a matrix A is equal to the rank of its transpose A^T.

The row vectors of A are the column vectors of its transpose matrix A^T

So; rank(A) = row rank of A = Column rank of A^T = rank(A^T)

NOTE: (Bir A^T matrisinin satır uzayının bazı = A nın sütun uzayının bazıdır) a basis of a row space of A = a basis of a Column space of A^T

ya da

(A matrisinin satır uzayının bir bazı (A nın satır vektörü) = A^T nin sütun uzayının bir bazıdır (transpozesinin sütun vektörü)

4) (Uniqueness of solutions). Let A be (mxn) matrix and suppose that the system $Ax=b$ is consistent. Prove that its solution is unique iff and only if the rank of A is equal to n.

SOL: The rank of the (mxn) matrix A is n if and only if the n column vectors $a_1, a_2, \dots, a_n$ of A are linearly independent – in which a vector b in R^m can be expressed in at most one way as a linear combination

$$b = x_1 a_1 + x_2 a_2 + \dots + x_n a_n$$

This means that the equation $Ax=b$ has at most one (unique) solution

$$x = [x_1 \; x_2 \; \dots \; x_n]^T$$

November 11, 2017 / Q4 (b)

Let $$A = \begin{bmatrix} 3 & 9 & 1 \\ 2 & 6 & 7 \\ 1 & 3 & -6 \end{bmatrix}_{3\times 3}$$

(i) Find a basis for the rowspace of A and the rank of A
(ii) Find a basis for the nullspace, N(A) and its dimension.

SOLUTION:

(i) $$A = \begin{bmatrix} 3 & 9 & 1 \\ 2 & 6 & 7 \\ 1 & 3 & -6 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & -6 \\ 2 & 6 & 7 \\ 3 & 9 & 1 \end{bmatrix} \xrightarrow[(-3)R_1+R_3 \to R_3]{(-2)R_1+R_2 \to R_2} \begin{bmatrix} 1 & 3 & -6 \\ 0 & 0 & 19 \\ 0 & 0 & 19 \end{bmatrix} \xrightarrow{(-1)R_2+R_3 \to R_3}$$

$$\begin{bmatrix} 1 & 3 & -6 \\ 0 & 0 & 19 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{\frac{1}{19}R_2 \to R_2} \begin{bmatrix} 1 & 3 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = E$$

(Def: Non-zero rows in REF of A are the basis vectors of rowspace of A)

row vector (So: we use [ ])

$\Rightarrow$ So; $\{[1\ 3\ -6], [0\ 0\ 1]\}$ are a basis for the rowspace of A.
row vector

$\Rightarrow$ Rank $A = 2$ (Way 2: NOTE: dim(row space A) = rank A = 2)

(ii) $$N(A) = \left\{ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3 \;\middle|\; Ax = 0 \right\}$$

$$[A \mid 0] \sim \left[\begin{array}{ccc|c} 1 & 3 & -6 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \qquad \left.\begin{aligned} x_1 + 3x_2 - 6x_3 &= 0 \\ x_3 &= 0 \end{aligned}\right\}$$

Leading variables are: x_1, x_3
Free variable is x_2

Let $x_2 = t \Rightarrow x_1 = -3t$
$(t \in \mathbb{R})$

So; $$N(A) = \left\{ x = \begin{bmatrix} -3t \\ t \\ 0 \end{bmatrix} = t \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}}_{\vec{u}} \right.$$

$\vec{u} = (-3, 1, 0)$ spans N(A) and l. Independent

Column vector. So we use (x_1, x_2, x_3)

So; $\{(-3,1,0)\}$ forms a basis for N(A)

$\dim(N(A)) = 1$

march 24, 2018 / Q4

Consider the following homogeneous linear system:

$$x_1+2x_2+7x_3-9x_4+31x_5=0$$
$$2x_1+4x_2+7x_3-11x_4+34x_5=0$$
$$3x_1+6x_2+5x_3-11x_4+29x_5=0$$

a) Find the rank of the coefficient matrix
b) Use part (a) to determine the dimension of the solution space.
c) Find a basis for the solution space.

SOLUTION:

a)
$$A=\begin{pmatrix}1&2&7&-9&31\\2&4&7&-11&34\\3&6&5&-11&29\end{pmatrix}\xrightarrow[(-3)R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2}\begin{pmatrix}1&2&7&-9&31\\0&0&-7&7&-28\\0&0&-16&16&-64\end{pmatrix}\to$$

$$\xrightarrow[(-\frac{1}{16})R_3\to R_3]{(-\frac{1}{7})R_2\to R_2}\begin{pmatrix}1&2&7&-9&31\\0&0&1&-1&4\\0&0&1&-1&4\end{pmatrix}\xrightarrow{(-1)R_2+R_3\to R_3}\begin{pmatrix}1&2&7&-9&31\\0&0&1&-1&4\\0&0&0&0&0\end{pmatrix}$$

REF ✓

Non-zero rows in REF of coefficient matrix is 2.
So, $\operatorname{rank}(A)=2$ ///

b) $\operatorname{rank}(A)+\dim(\operatorname{Null}(A))=n$ $\quad n$: # of columns of A

$2+\dim(\operatorname{Null}(A))=5$

$\dim(\operatorname{Null}(A))=3$ ///

c) C_1 and C_3 has leading entries (pivots). So; x_1 and x_3 are leading variables.
x_2, x_4 and x_5 are free variables.

Let $\underline{x_2=t}$, $\underline{x_4=r}$, $\underline{x_5=s}$ $\quad t,r,s\in\mathbb{R}$ (arbitrary parameters) Then

$$\left.\begin{aligned}x_1+2x_2+7x_3-9x_4+31x_5&=0\\x_3-x_4+4x_5&=0\end{aligned}\right\}\quad \begin{aligned}x_3&=r-4s\\x_1&=2r-2t-3s\end{aligned}$$

So; $\vec{x}\in\operatorname{Null}(A)\Rightarrow \vec{x}=\begin{bmatrix}x_1\\x_2\\x_3\\x_4\\x_5\end{bmatrix}=t\underbrace{\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}}_{\vec{u}_1}+r\underbrace{\begin{bmatrix}2\\0\\1\\1\\0\end{bmatrix}}_{\vec{u}_2}+s\underbrace{\begin{bmatrix}-3\\0\\-4\\0\\1\end{bmatrix}}_{\vec{u}_3}$

$\{\vec{u}_1,\vec{u}_2,\vec{u}_3\}$ is a basis vectors for Null (A)
(Solution space)