Circuit Elements

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} v(\tau)d\tau \qquad v(t) = \frac{d\phi(t)}{dt}$$

$$q(t) = q(t_0) + \int_{t_0}^{t} i(\tau)d\tau \qquad i(t) = \frac{dq(t)}{dt}$$

f(v,i,t) = 0 → resistor

f(ϕ,i,t) = 0 → inductor

f(v,q,t) = 0 → capacitor

f(q,ϕ,t) = 0 → memristor

The equations describing the characteristics are called *element equations* or *branch equations*.

$f(x,y,t)$: characteristic of the element $\quad x,y \in \{i, v, q, \phi\}$ $\quad x,y$: proper pair

$y = h(x,t)$: *x-controlled*

$x = g(y,t)$: *y-controlled*

Time – invariance: For each (x,y) and t_0, $[x(t - t_0), y(t - t_0)]$ is a proper pair.

Linearity: If for each (x_1, y_1) and (x_2, y_2) proper pairs and for each $\mu \in \mathbb{R}$

1) $(x_1 + x_2, y_1 + y_2)$ is also a proper pair and
2) $(\mu x_1, \mu y_1)$ is also a proper pair,

Then the circuit element is linear; otherwise nonlinear.

Bilateral property: For each proper pair (x, y), $(-x, -y)$ is also a proper pair.

For a bilateral resistor: $f(v, i) = f(-v, -i)$

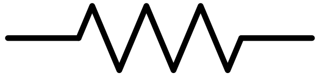
Example 2.1: Diode $i = I_s\left(e^{v/V_T} - 1\right)$ $\qquad$ I_s = (nA, μA) $\quad$ V_T = 26mV

Two – terminal resistor

$v(t) = f(i(t))$ or $i(t) = g(v(t))$

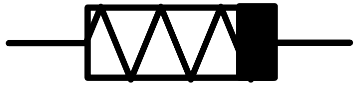
Linear resistor: v = Ri or i = Gv $\qquad$ R: resistance [Ω]

G: conductance [$\mho, mho, S$]



Nonlinear resistor: $v(t) = f(i(t))$

$$v(t) = v_0 + \left.\frac{df}{di}\right|_{I_0} (i(t) - I_0)^2 + \cdots$$



||

R_d: differential resistor

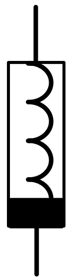
Two – terminal inductor

$\phi(t) = f(i(t))$ or $i(t) = g(\phi(t))$

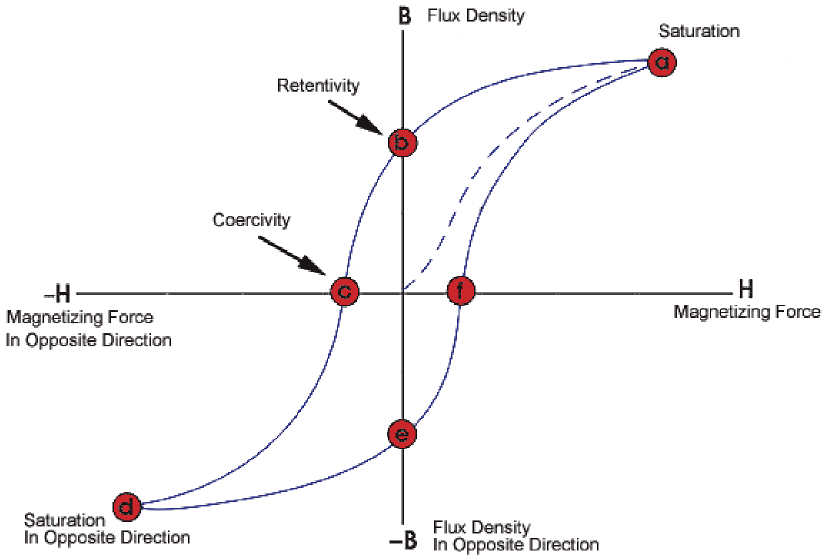
Linear inductor: $\phi(t) = Li(t) \rightarrow \mathrm{v(t)} = \mathrm{L}\frac{di(t)}{dt}$

L: inductance [H]

Nonlinear inductor: $\phi(t) = \phi(0) + \int_0^t v(\tau)d\tau$



Hysteresis:



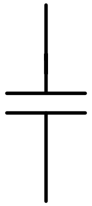
Two – terminal capacitor

$q(t) = f(v(t))$ or $\mathrm{v}(t) = g(q(t))$

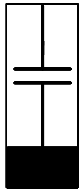
Linear capacitor: q(t) = Cv(t) → C: capacitance [F]

$$\frac{dq(t)}{dt} = i(t) = C\,\frac{dv(t)}{dt}$$

v(t) = Sq(t) → S: elastance [F^{-1}]



Nonlinear capacitor: $q(t) = q(0) + \int_0^t i(\tau)d\tau$



Open circuit and short circuit

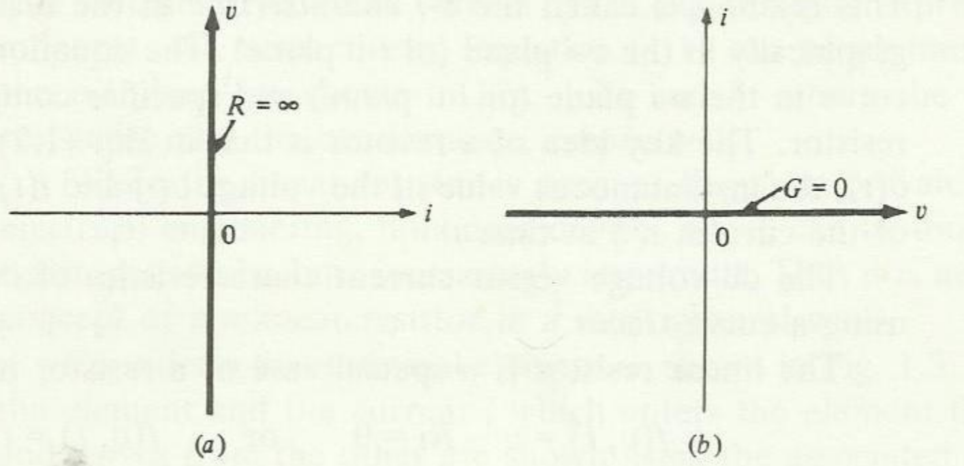


Figure 1.4 Characteristic of an open circuit.

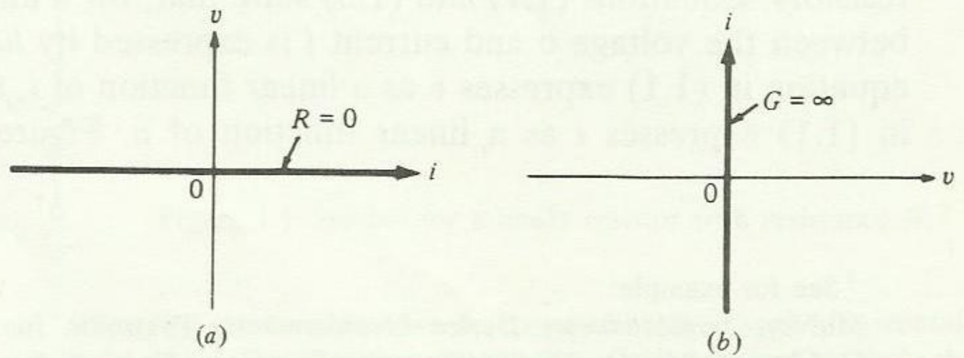


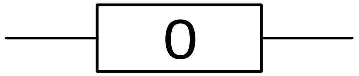
Figure 1.5 Characteristic of a short circuit.

open circuit: i = 0

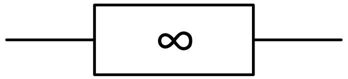
short circuit: v = 0

The *dual* of a given resistor is another resistor whose v – i characteristics in the v – i plane is the same curve as that of the given resistor in the i – v plane.

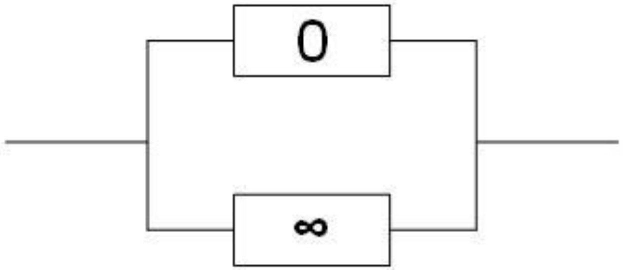
Nullator: $v = 0 \quad i = 0$

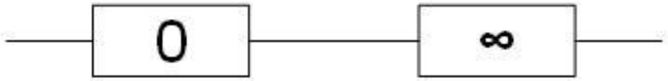


Norator: $0v + 0i = 0$



$v = \ldots \quad i = \ldots$

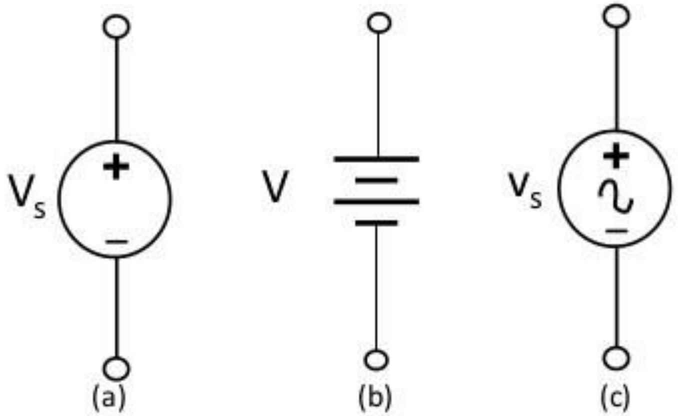




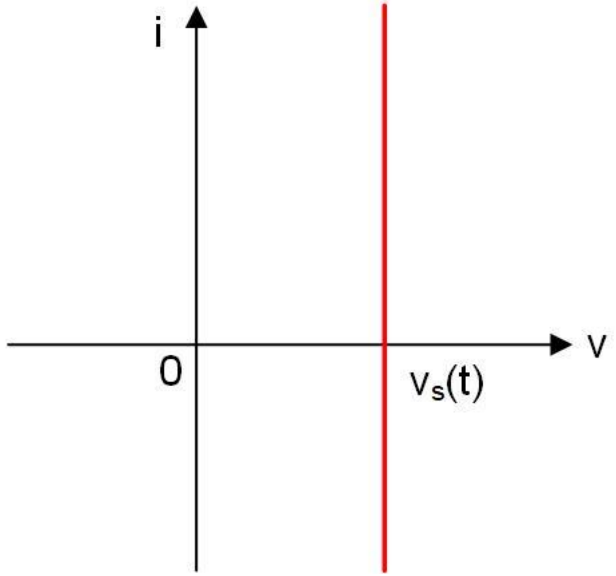
Independent Sources are circuit elements which can supply a specified voltage or current that is independent of a current or voltage elsewhere in the circuit.

Independent voltage source

$$\mathcal{R}_{v_s} = \{(v, i): v = v_s(t)\ \ for\ -\infty < i < \infty\}$$

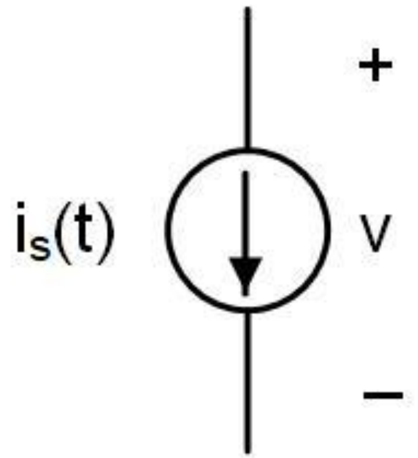


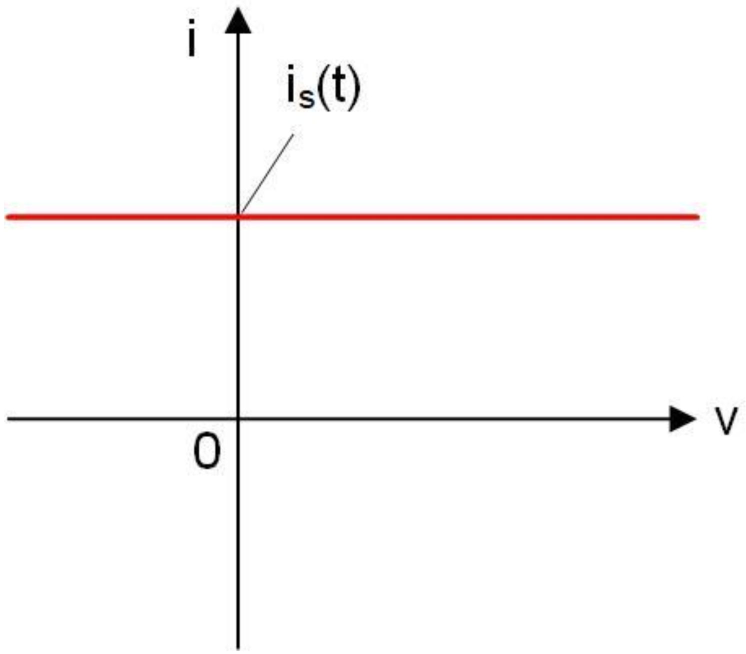
(a) and (b) DC independent voltage source symbols. (c) AC independent voltage source symbol.



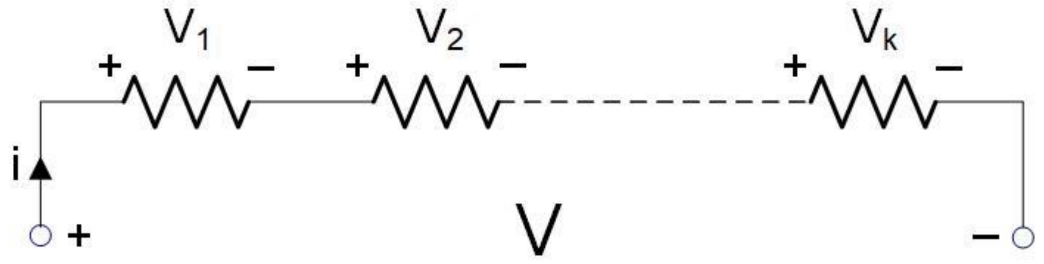
Independent current source

$$\mathcal{R}_{i_s} = \{(v, i): i = i_s(t) \;\; for \; -\infty < v < \infty \}$$

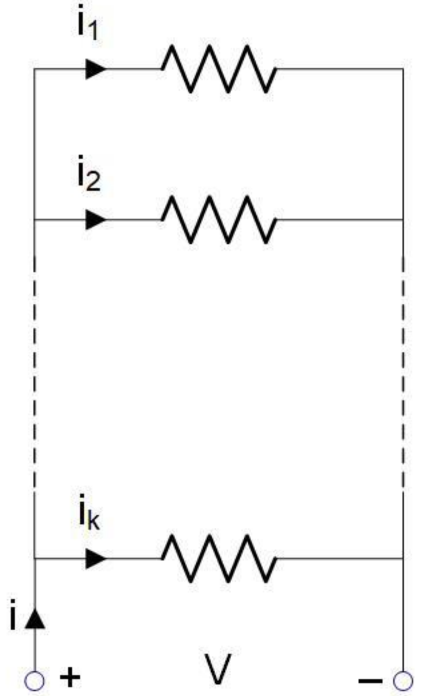




Series connection of resistors

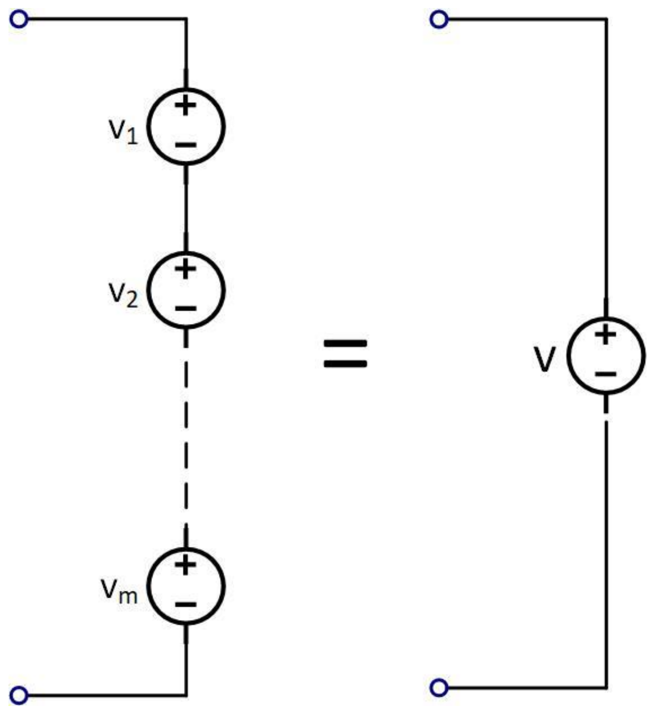


Parallel connection of resistors

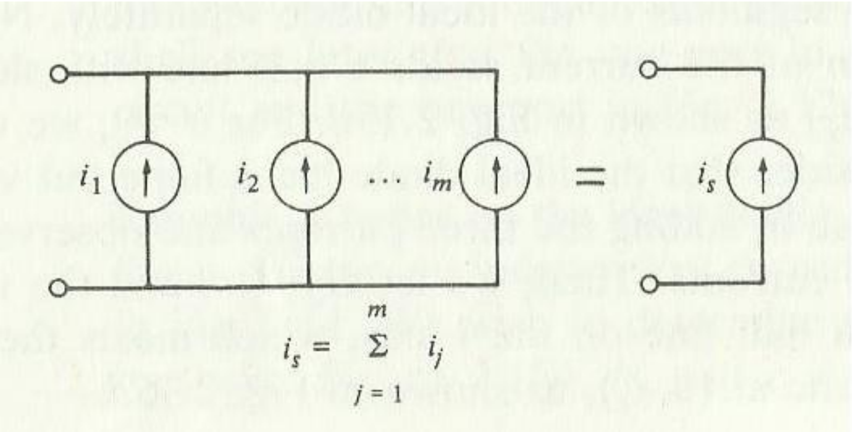


Independent voltage sources in series

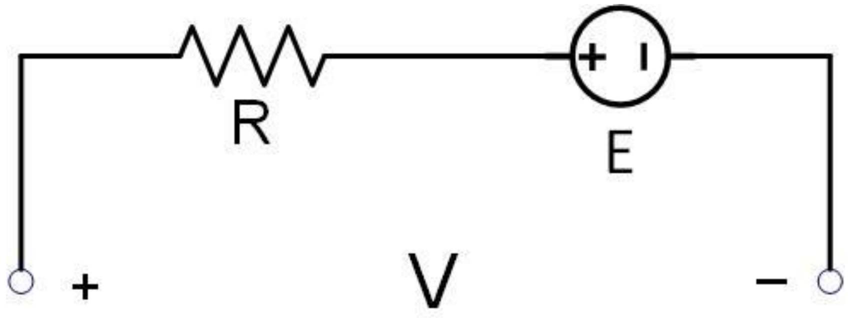
$$v = \sum_{j=1}^{m} v_j$$



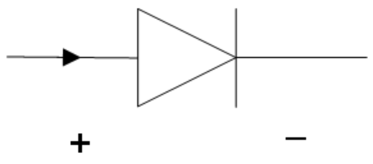
Parallel connection of independent current sources

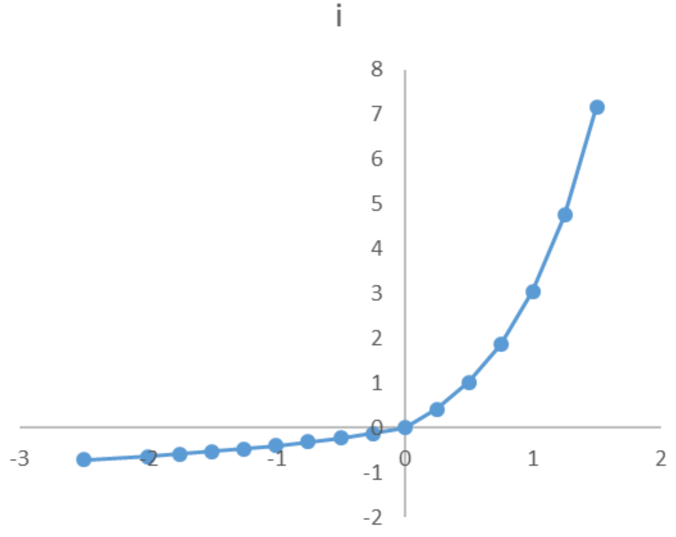


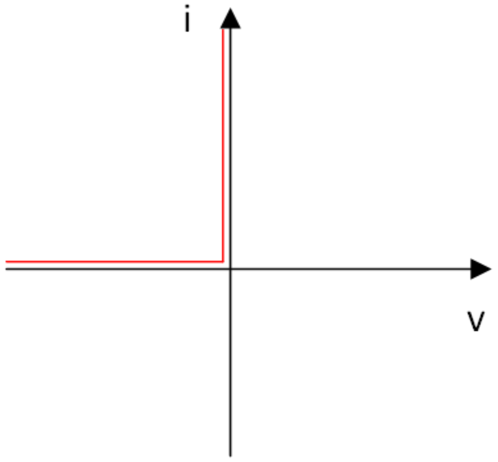
Example 2.2:



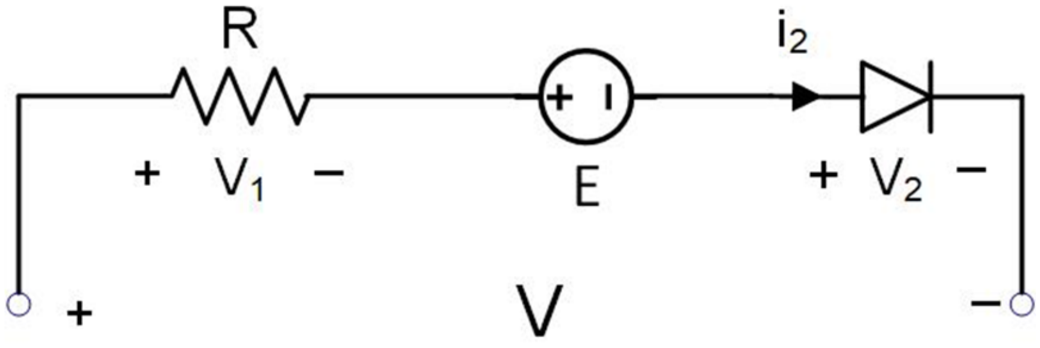
The Diode $\quad i = I_s\left(e^{v/V_T} - 1\right)$ $\quad I_s$ = (nA, μA) $\quad V_T$ = 26mV



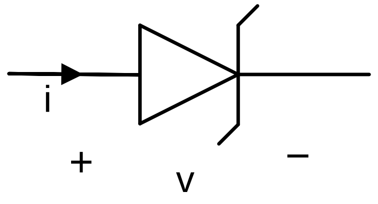


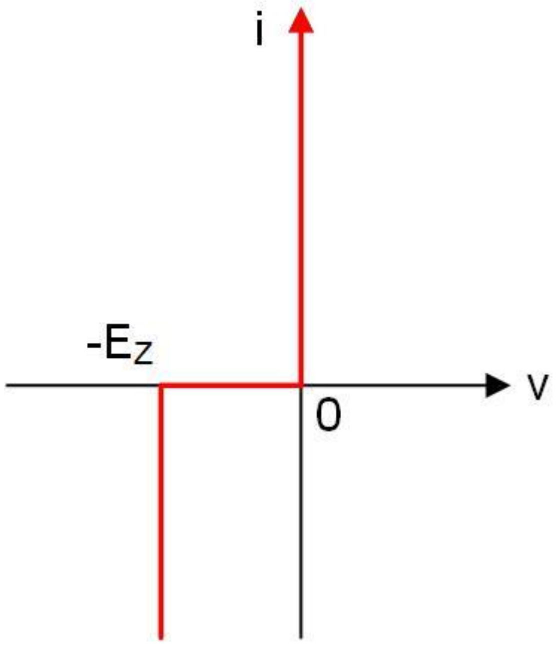


Example 2.3:

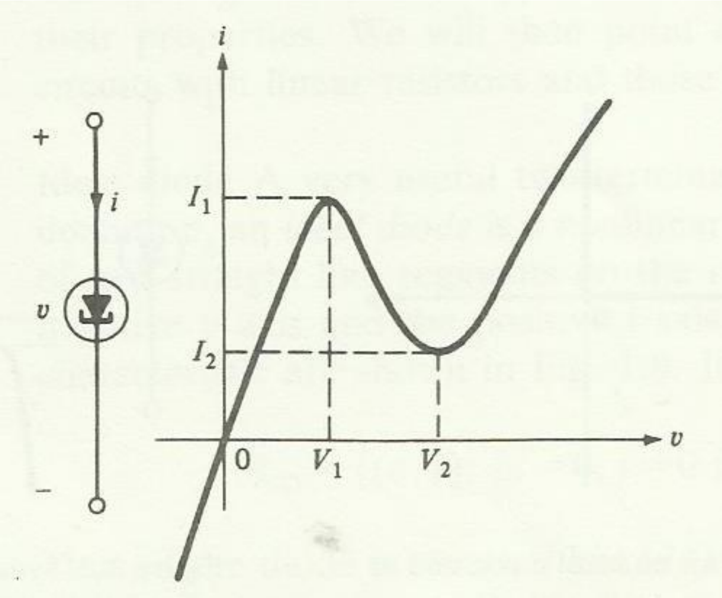


Zener diode





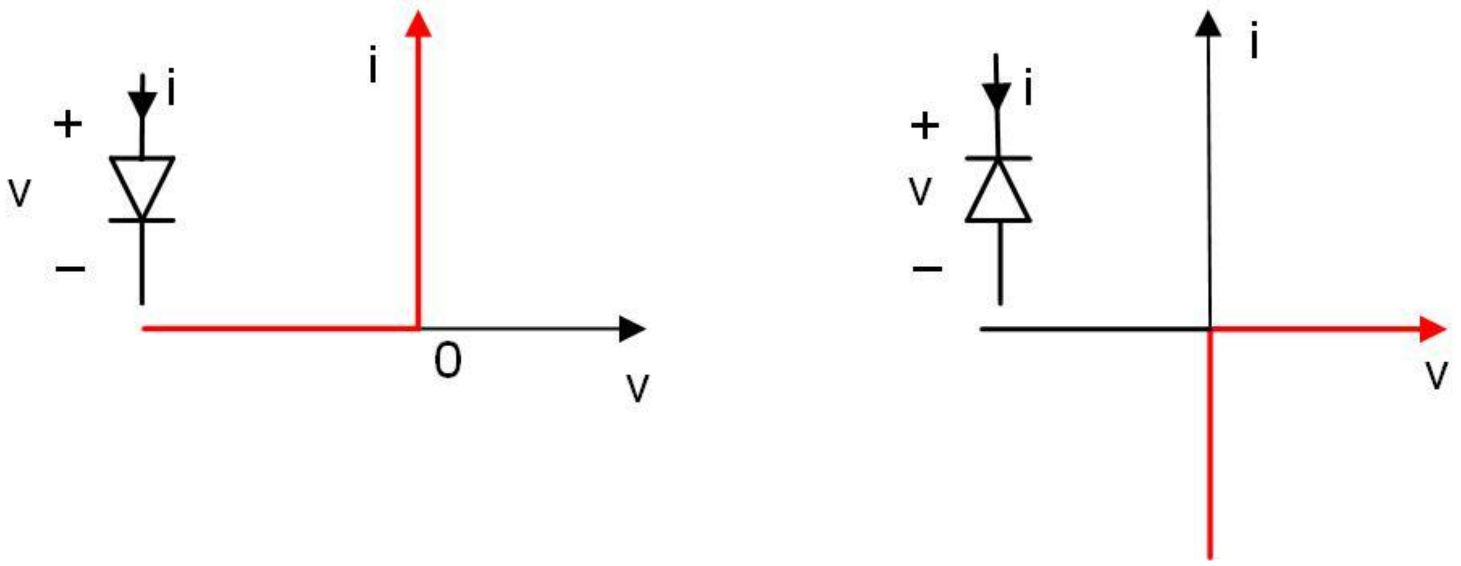
The tunnel diode



i – controlled or v – controlled ?

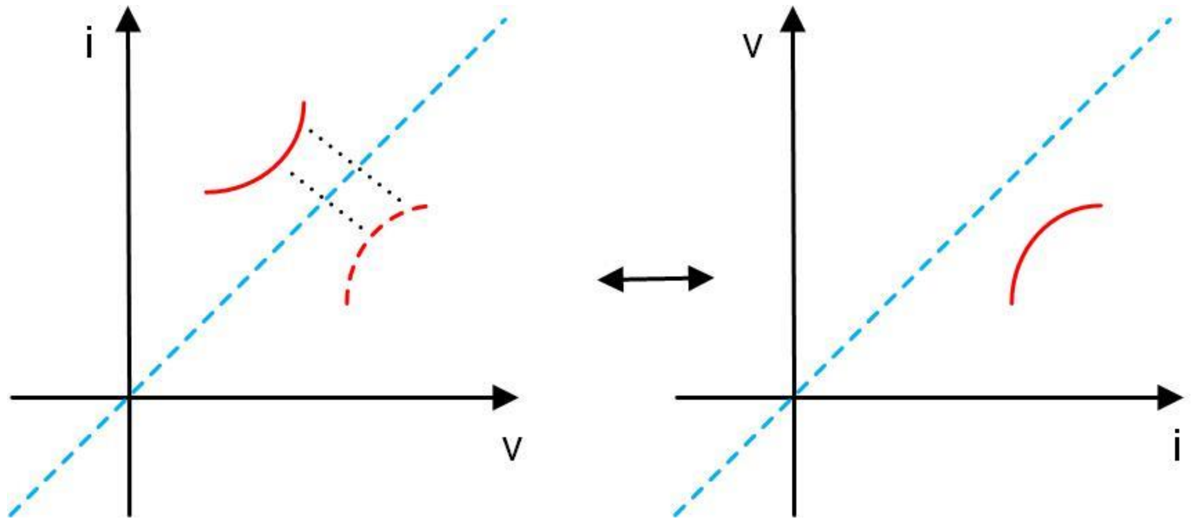
Reverse bias

Symmetric w.r.t. origin.

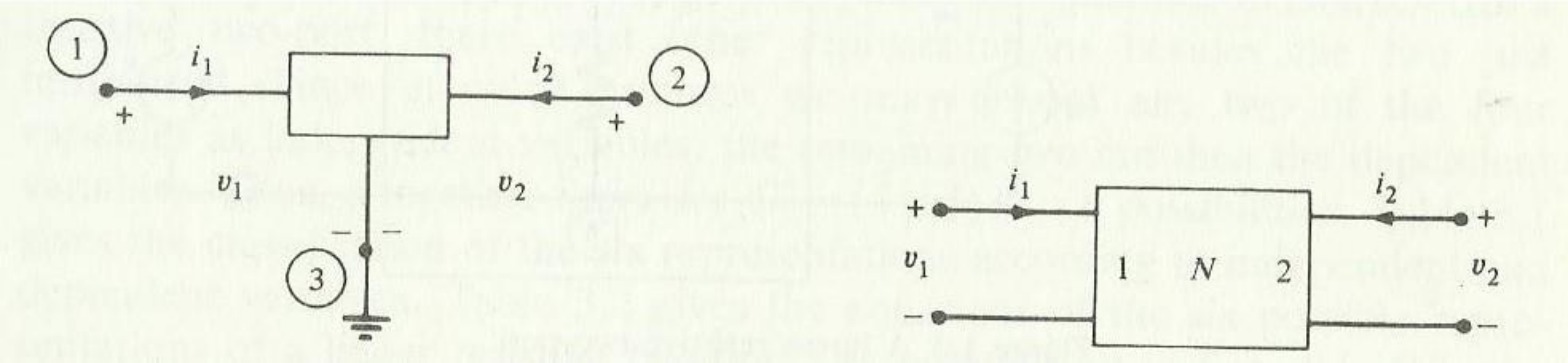


Axis exchange

Symmetric w.r.t. the line with 45° slope.



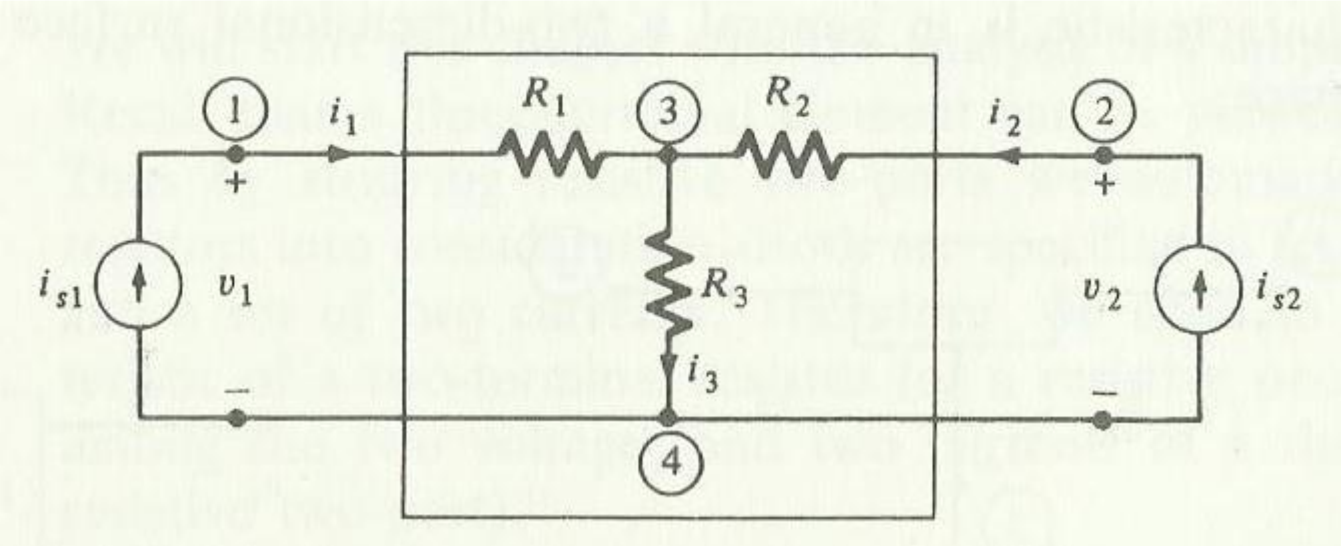
Resistive Two-Ports



A three-terminal element or a two-port will be called a resistor if its port voltages and currents satisfy the following relation:

$$\mathcal{R}_R = \{(v_1, v_2, i_1, i_2): f_1(v_1, v_2, i_1, i_2) = 0 \; and \; f_2(v_1, v_2, i_1, i_2) = 0\}$$

Example 2.4



$$\boldsymbol{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \boldsymbol{Ri} = \begin{bmatrix} R_1 + R_3 & R_3 \\ R_3 & R_2 + R_3 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

Equations for the six representations of a linear resistive two-port

Representations	Scalar equations	Vector equations
Current-controlled	$v_1 = r_{11}i_1 + r_{12}i_2$ $v_2 = r_{21}i_1 + r_{22}i_2$	**v = Ri**
Voltage-controlled	$i_1 = g_{11}v_1 + g_{12}v_2$ $i_2 = g_{21}v_1 + r_{22}vi_2$	**i = Gv**
Hybrid 1	$v_1 = h_{11}i_1 + h_{12}v_2$ $i_2 = h_{21}i_1 + h_{22}v_2$	$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = H \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$
Hybrid 2	$i_1 = h'_{11}v_1 + h'_{12}i_2$ $v_2 = h'_{21}v_1 + h'_{22}i_2$	$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = H' \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$
Transmission 1	$v_1 = t_{11}v_2 - t_{12}i_2$ $i_1 = t_{21}v_2 - t_{22}i_2$	$\begin{bmatrix} v_1 \\ i_1 \end{bmatrix} = T \begin{bmatrix} v_2 \\ -i_2 \end{bmatrix}$
Transmission 2	$v_2 = t'_{11}v_1 + t'_{12}i_1$ $-i_2 = t'_{21}v_1 + t'_{22}i_1$	$\begin{bmatrix} v_2 \\ -i_2 \end{bmatrix} = T' \begin{bmatrix} v_1 \\ i_1 \end{bmatrix}$

Resistive two-ports

Linear Controlled Sources

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ r_m & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix}$$

r_m: transresistance

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ g_m & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

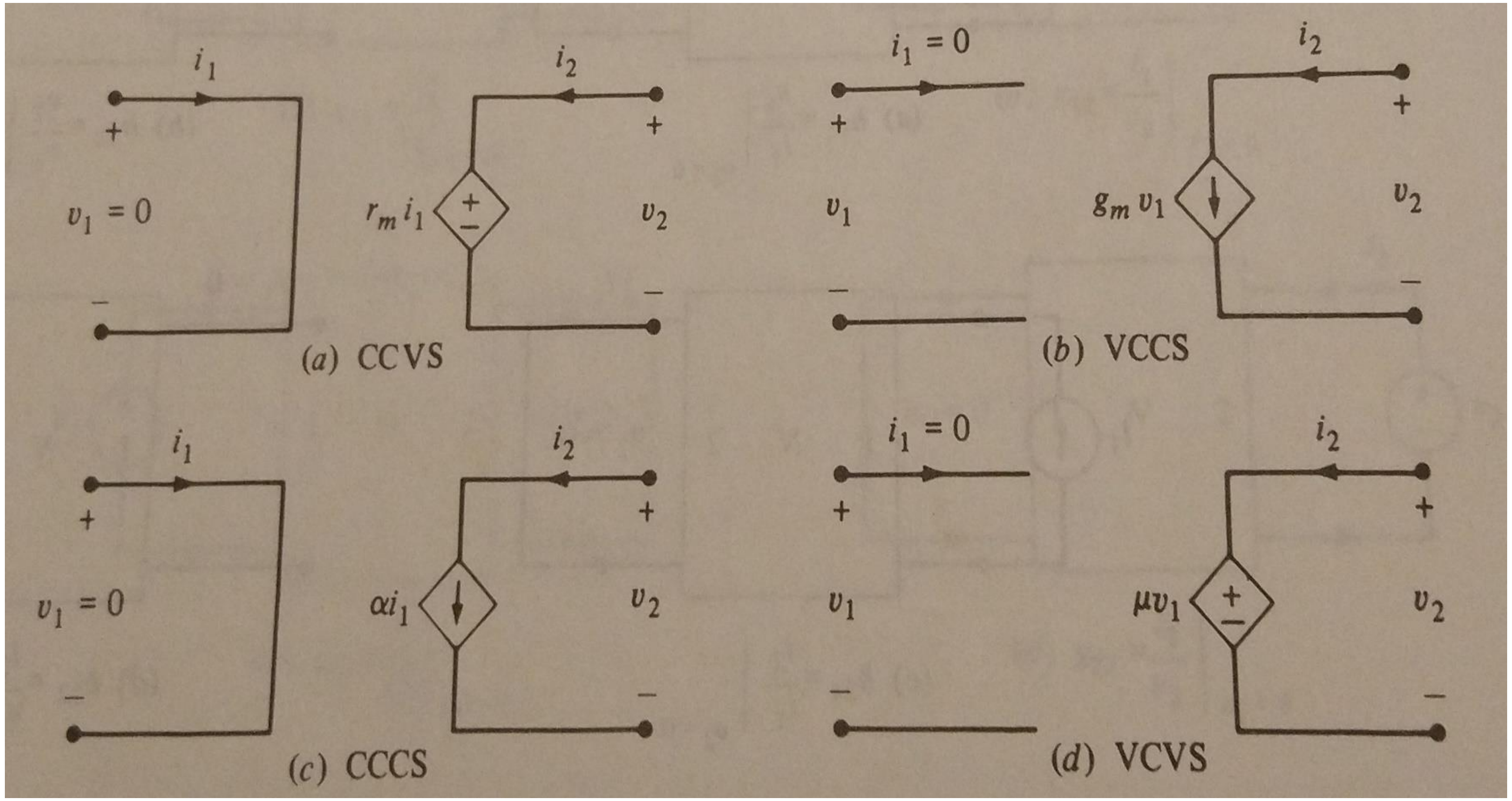
g_m: transconductance

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$

α: current transfer ratio

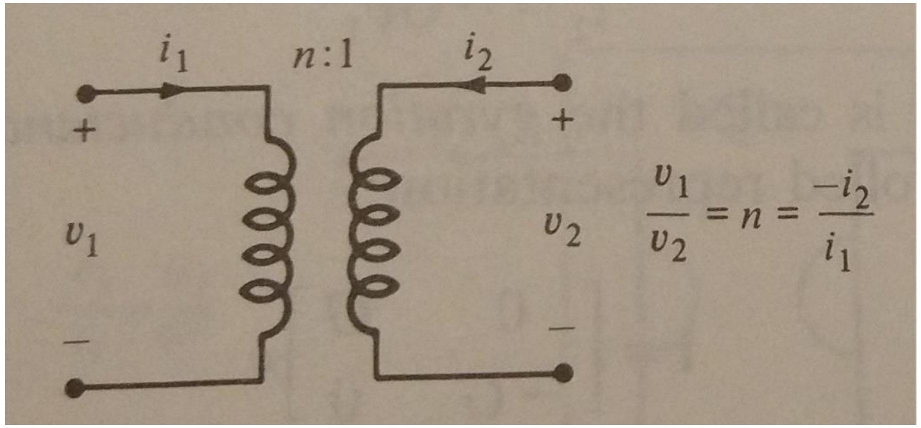
$$\begin{bmatrix} i_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mu & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ i_2 \end{bmatrix}$$

μ: voltage transfer ratio



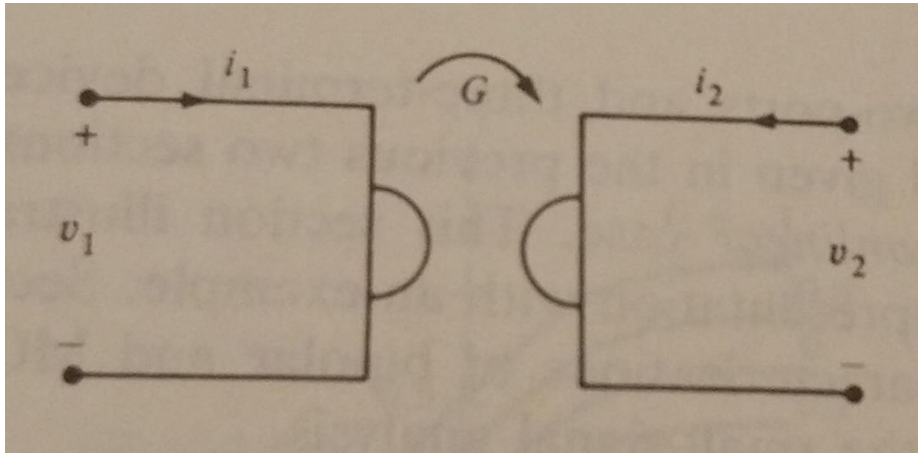
Ideal transformer

$$\begin{bmatrix} v_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & n \\ -n & 0 \end{bmatrix} \begin{bmatrix} i_1 \\ v_2 \end{bmatrix}$$



Gyrator

$$\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 0 & G \\ -G & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$



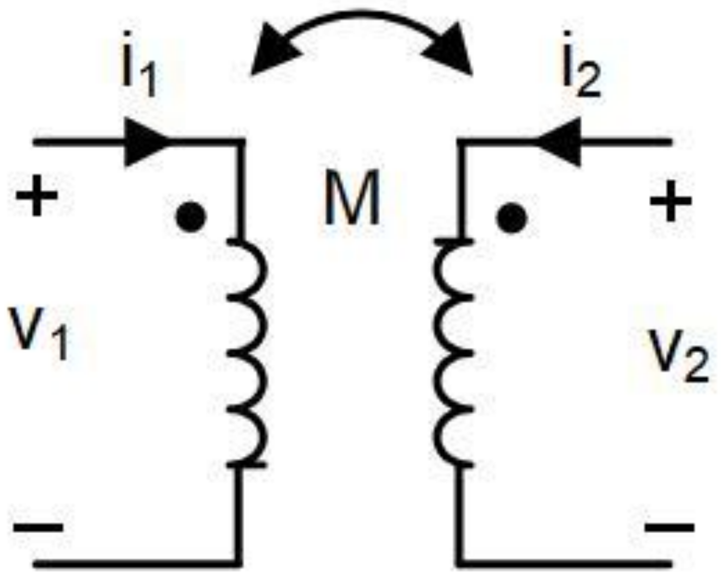
Mutual Inductor

$$v_1 = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = M \frac{di_1}{dt} + L_2 \frac{di_2}{dt}$$

L_1, L_2 : self inductance

M: mutual inductance



The npn Bipolar Transistor

$$i_e = -I_{ES}\left(e^{-\frac{V_{eb}}{V_T}} - 1\right) + \alpha_R I_{CS}\left(e^{-\frac{V_{cb}}{V_T}} - 1\right)$$

$$i_c = \alpha_F I_{ES}\left(e^{-\frac{V_{eb}}{V_T}} - 1\right) - I_{CS}\left(e^{-\frac{V_{cb}}{V_T}} - 1\right)$$

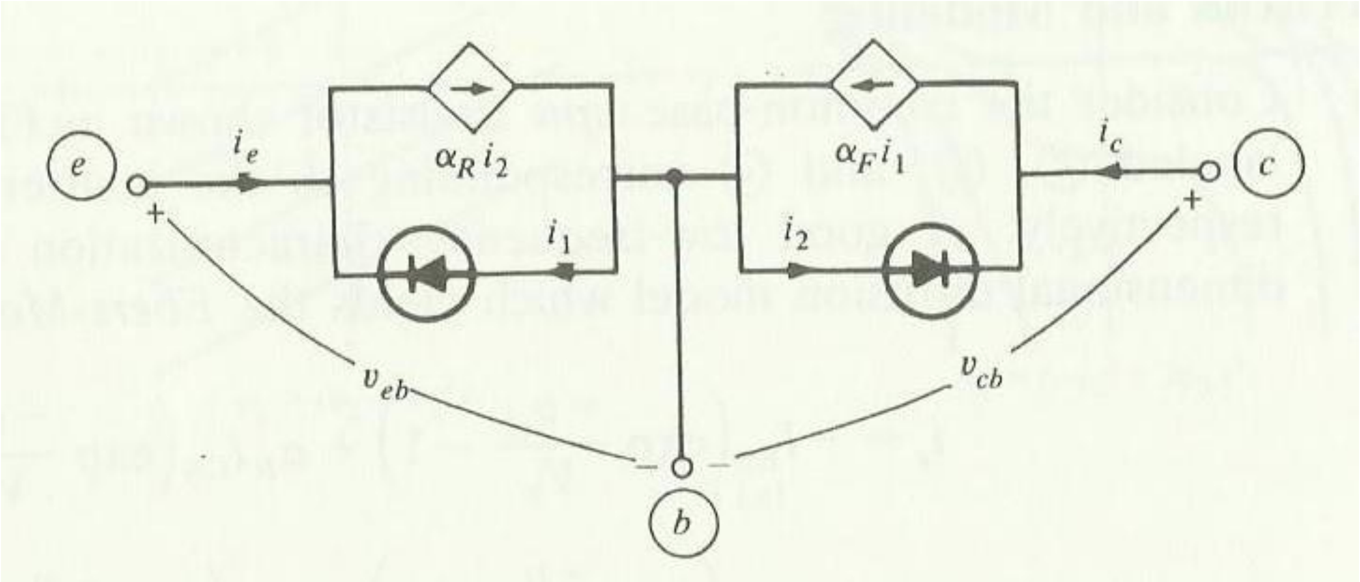
$\alpha_R = 0{,}5 \cdots 0{,}8$

$I_{ES}, I_{CS} = 10^{-12} \cdots 10^{-10}$ at 25°C

$V_T \approx 26\, mV$ at 25°C

$$i_1 = I_{ES}\left(e^{-\frac{V_{eb}}{V_T}} - 1\right) \qquad\qquad i_2 = I_{CS}\left(e^{-\frac{V_{cb}}{V_T}} - 1\right)$$

Ebers Moll circuit model of npn transistor

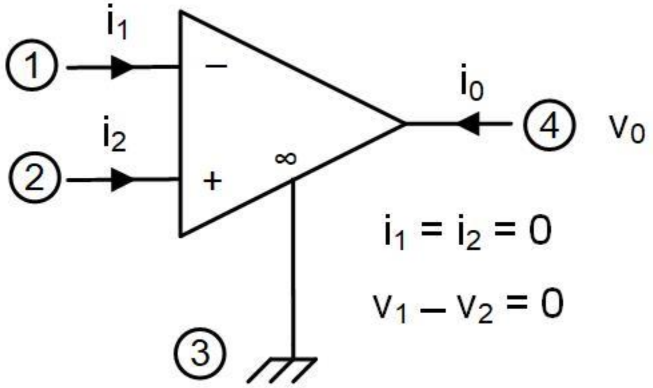


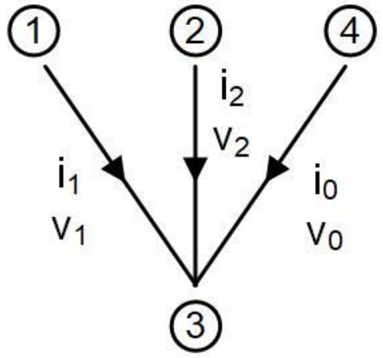
$$i_1 = I_{ES}\left(e^{-\frac{V_{eb}}{V_T}} - 1\right) \qquad i_2 = I_{CS}\left(e^{-\frac{V_{cb}}{V_T}} - 1\right)$$

$$i_e + i_1 = \alpha_R i_2 \qquad i_c + i_2 = \alpha_F i_1$$

Operational amplifier (Opamp)

Ideal opamp model

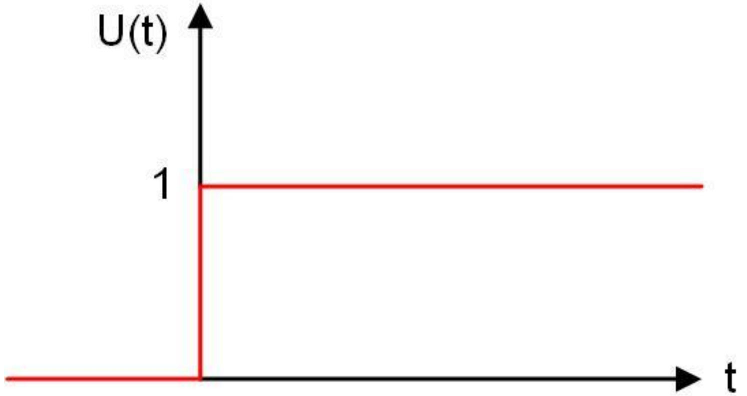




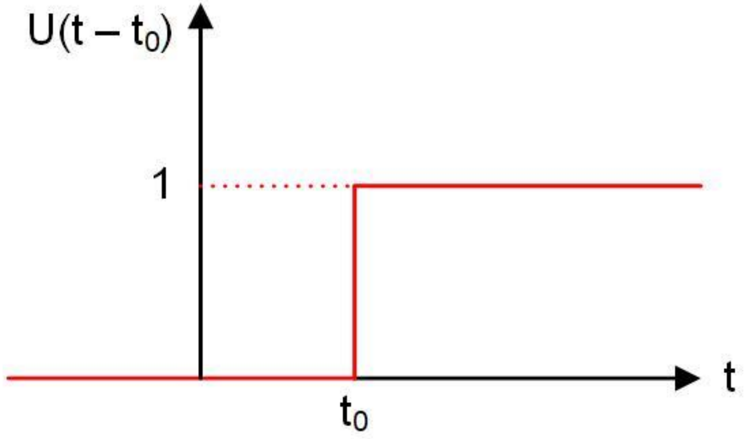
Source Function Types

Unit step function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

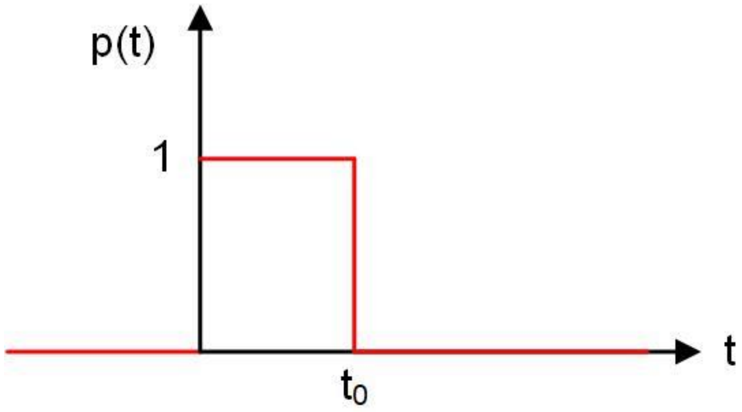


$$u(t - t_0) = \begin{cases} 1 & t \geq t_0 \\ 0 & t < t_0 \end{cases}$$



Pulse function

$$p(t) = \begin{cases} 1 & 0 \le t < t_0 \\ 0 & t < 0\,, t \ge t_0 \end{cases}$$



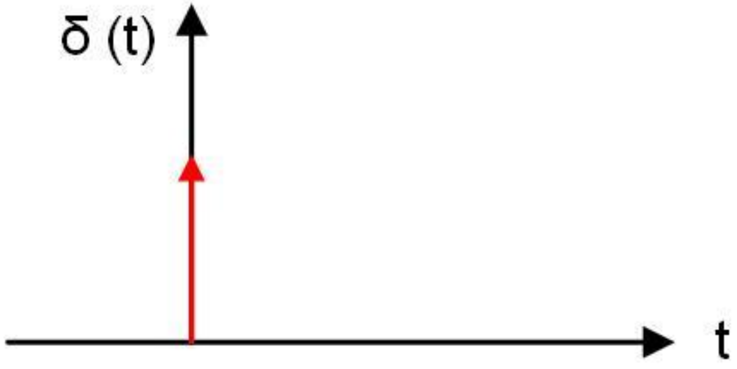
p(t) = u(t) …

Unit impuls function

$$\delta(t) \triangleq \begin{cases} singular & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$\int_{\varepsilon_2}^{\varepsilon_2} \delta(t)\,dt = 1$$

for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$



Periodic functions

$x(t) = x(t+T)$ T: period [s]

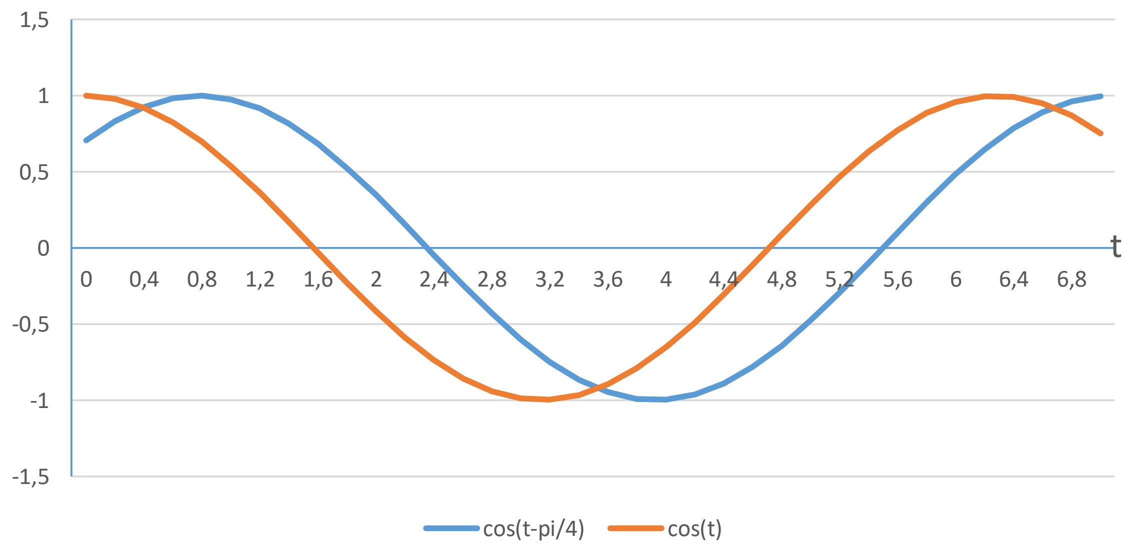
f: $1/T$ [Hz, s^{-1}]

$$x(t) = X_m \cos(\omega t + \varphi)$$

X_m: amplitude

ω: angular frequency

φ: phase angle



Mean value: $$\bar{x} = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\,dt$$

Effective value: $$x_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} x(t)^2\,dt}$$

Example 2.5 $$x(t) = \begin{cases} 3 & 0 \le t \le 2 \\ -1 & 2 < t \le 3 \end{cases}$$

x(t) = x(t+3)

Energy, Active and Passive Elements

Energy delivered to the element in the interval (t_0, t)

$$E(t_0, t) = \int_{t_0}^{t} \boldsymbol{v}(t)^T \boldsymbol{i}(t)\, dt$$

Energy delivered from the element to the remainder of the circuit in the interval (t_0, t)

$$E(t_0, t) = -\int_{t_0}^{t} \boldsymbol{v}(t)^T \boldsymbol{i}(t)\, dt$$

$$E(t) = E(t_0) + \int_{t_0}^{t} \boldsymbol{v}(t)^T \boldsymbol{i}(t)\, dt$$

If $E(t) = \int_{t_0}^{t} v(t)^T i(t)\, dt \geq 0$ the element is passive.

If E(t) is an increasing function for a passive element → energy is delivered to this element all the time

If the function becomes negative for some t values → the element is active

Example 2.6 linear resistor, R > 0

P = vi = (Ri)i = Ri^2 $\qquad E(t) = \int_{t_0}^{t} i^2 R\, dt > 0 \;\rightarrow$ passive