AYŞEGÜL TEPE
MAKARACI
MAT 2108

Eigenvalues and Eigenvectors

Definition:

The number λ is said to be an "eigenvalue" of the $(n \times n)$ matrix A provided that there exists a nonzero vector v such that

$$Av = \lambda v \qquad \lambda: \text{scalar} \quad v \neq 0$$

(Av ; v nin λ katı, $Av \parallel v$)

in which the vector v is called an "eigenvector" of matrix A.
We say that the eigenvector v is associated with the eigenvalue λ, or that the eigenvalue λ corresponds to the eigenvector v.

Remark 1: If $v = 0$, then the equation $Av = \lambda v$ holds for every scalar λ and hence is of no significance. This is why only nonzero vectors qualify as eigenvectors in the definition.

Remark 2: Any nonzero scalar multiple of the eigenvector is also an eigenvector and is associated with the same eigenvalue.

Characteristic Equation:

A ; $(n \times n)$;

$$Av = \lambda v \qquad v \neq 0 \qquad \left(v \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad v \text{ is column vector} \right)$$

$$Av = \lambda . v . I_n \qquad I_n : \text{Identity matrix}$$

$$(A - \lambda I_n) v = 0 \quad \text{homogeneous eq.}$$

$\downarrow$ For non-trivial solution

$$\underbrace{\det(A - \lambda I_n)}_{p(\lambda)} = 0 \Leftarrow \quad p(\lambda) = |A - \lambda I_n| = 0 \Rightarrow \text{"characteristic equation"}$$

$p(\lambda)$ is called "characteristic polynomial"

Algorithm: To find the eigenvalues and associated eigenvectors of the $(n \times n)$ matrix A:

1) Solve the characteristic equation

$$|A - \lambda I| = 0$$

2) For each eigenvalue λ thereby found, solve the linear system

$$(A - \lambda I) v = 0$$

to find the eigenvectors associated with λ.

For (2×2) matrix A, we have 4 possibilities:

- Two distinct eigenvalues and eigenvectors λ_1 and λ_2 → V_1, V_2
- A single eigenvalue $\lambda=\lambda_1=\lambda_2$ corresponds to a single eigenvector V
- A single eigenvalue corresponds to 2 linearly independent eigenvectors $\lambda=\lambda_1=\lambda_2$ → V_1, V_2
- Two complex conjugate eigenvalue corresponding to complex conjugate eigenvectors

*** Definition:**

If $\lambda=0$ in characteristic equation

$$|A-\lambda I|=0$$

then $|A|=0$.

Therefore, $\lambda=0$ is an eigenvalue of the matrix A if and only if A is singular matrix : $|A|=0$

*** Definition:** Let λ be a fixed eigenvalue of the $(n\times n)$ matrix A. Then the set of all eigenvectors associated with A is the set of all nonzero solutions vectors of the system

$$(A-\lambda I)V=0.$$

The solution space of this system is called the "Eigenspace of A" associated with the eigenvalue λ.

*** Definition:** $Tr(A)=\sum_{i=1}^{n} a_{ii}=a_{11}+a_{22}+\dots+a_{nn}$

$$\boxed{Tr(A)=\lambda_1+\lambda_2+\dots+\lambda_n} \ *$$

and

$$\boxed{|A|=\lambda_1 \cdot \lambda_2 \cdots \lambda_n} \ *$$

③

Example 1: $A = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$ Find eigenvalues and corresponding eigenvectors of the matrix A

SOL:

(NOTE:
$AV = \lambda V$
$AV = \lambda . I . V$
$(A - \lambda I)V = 0$ ∵ $V \neq 0$
For non-trivial solution of this homogeneous eq.
$|A - \lambda I| = 0$ characteristic eq.)

STEP 1:

$$\begin{vmatrix} 5-\lambda & -6 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(-2-\lambda) + 12 = 0$$

$\lambda^2 - 3\lambda + 2 = 0$ characteristic eq

$(\lambda - 2)(\lambda - 1) = 0$ → $\lambda_1 = 2$, $\lambda_2 = 1$ } Eigenvalues of A

STEP 2: Find corresponding eigenvectors:

* For $\lambda_1 = 2 \Rightarrow (A - \lambda_1 I)V = 0 \Rightarrow (A - 2I)V = 0$

$$\begin{bmatrix} 5-2 & -6 \\ 2 & -2-2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3x - 6y = 0$
$2x - 4y = 0$ } $x = 2y$

So, let $y = 1 \Rightarrow x = 2$

Then $\boxed{V_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$ is eigenvector of $\lambda_1 = 2$

* For $\lambda_2 = 1 \Rightarrow (A - \lambda_2 I)V = 0 \Rightarrow (A - I)V = 0$

$$\begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$4x - 6y = 0$
$2x - 3y = 0$ } $\Rightarrow 4x = 6y \Rightarrow x = \frac{3}{2}y$ Let $y = 2$, then $x = 3$

So, $\boxed{V_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}}$ is eigenvector of $\lambda_2 = 1$.

Example 2: Find eigenvalues and associated eigenvectors of the ma

$$A = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$$

SOL: $Av = \lambda v$

$Av - \lambda . I v = 0$

$(A - \lambda I) v = 0$

For non-trivial solution; $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & 7 \\ -2 & -4-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(-4-\lambda)+14=0 \Rightarrow \underbrace{\lambda^2 - \lambda - 6 = 0}_{\text{Characteristic eq.}} \Rightarrow (\lambda+2)(\lambda-3)=0$$

$\underbrace{\lambda_1 = -2 \; ; \; \lambda_2 = 3}_{\text{Eigenvalues of } A}$

* For $\lambda_1 = -2 \Rightarrow (A - \lambda_1 I) v = 0$

$(A + 2I) v = 0$

$$\begin{bmatrix} 7 & 7 \\ -2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \left. \begin{array}{l} 7x + 7y = 0 \\ -2x - 2y = 0 \end{array} \right\} \underline{y = -x}$$

(Each of the two equations here is a multiple of the eq $x + y = 0$)

Let $x = 1 \Rightarrow y = -1$. So, $\underline{v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$ eigenvector of $\lambda_1 = -2$

* For $\lambda_2 = 3 \Rightarrow (A - \lambda_2 I) v = 0$

$(A - 3I) v = 0$

$$\begin{bmatrix} 2 & 7 \\ -2 & -7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \underline{2x + 7y = 0}. \text{ (We have only a single equation)}$$

$y = -\frac{2}{7} x$

Let $x = 7 \Rightarrow y = -2 \Rightarrow$ So, $\underline{v_2 = \begin{bmatrix} 7 \\ -2 \end{bmatrix}}$ is eigenvector of $\lambda_2 = 3$

Example 3: $A = \begin{bmatrix} 0 & 8 \\ -2 & 0 \end{bmatrix}$

Step 1:

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 8 \\ -2 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 16 = 0 \Rightarrow \lambda^2 = -16 \Rightarrow \underbrace{\lambda_{1,2} = \pm 4i}_{\text{Eigenvalues of } A}$$

Step 2:

→ For $\lambda_1 = 4i \Rightarrow (A - \lambda_1 I) v = 0$

$(A - 4i) v = 0$

$$\begin{bmatrix} -4i & 8 \\ -2 & -4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left. \begin{array}{l} (-4i)x + 8y = 0 \\ -2x - (4i)y = 0 \end{array} \right\} \begin{array}{l} -2x = (4i)y \\ \underline{x = (-2i)y} \end{array}$$

((2i) times second eq = eq (1))

Let $y=1 \Rightarrow x=-2i$, So $v_1=\begin{bmatrix}-2i\\1\end{bmatrix}$

$\Rightarrow$ For $\lambda_2=-4i$ $(A-\lambda_2 I)v=0$

$$\begin{bmatrix}4i & 8\\-2 & 4i\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\left.\begin{aligned}(4i)x+8y&=0\\-2x+(4i)y&=0\end{aligned}\right\}$$ ((-2i) times the second equation is the first equation)

$(4i)y=2x \Rightarrow x=(2i)y$ So, let $y=1 \Rightarrow x=2i$

Then $v_2=\begin{bmatrix}2i\\1\end{bmatrix}$ is the eigenvector of λ_2

Example 4/ $A=\begin{bmatrix}2 & 3\\0 & 2\end{bmatrix}$

$\det(A-\lambda I)=0$

$\begin{vmatrix}2-\lambda & 3\\0 & 2-\lambda\end{vmatrix}=0 \Rightarrow (2-\lambda)^2=0$ charac. eq.

$\lambda=\lambda_1=\lambda_2=2$ (A has the single eigenvalue $\lambda=2$)

For $\lambda=2 \Rightarrow \begin{bmatrix}2-2 & 3\\0 & 2-2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$

$\begin{bmatrix}0 & 3\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$ $\Big\}$ $\underline{y=0}$ and $\underline{x \text{ is arbitrary}}$

So, the eigenvalue $\lambda=2$ corresponds to the single eigenvector $v=\begin{bmatrix}1\\0\end{bmatrix}$

Example 5/ The (2x2) Identity matrix I has characteristic eq. $\left(\leftarrow I=\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right)$

$$\begin{vmatrix}1-\lambda & 0\\0 & 1-\lambda\end{vmatrix}=0 \Rightarrow (1-\lambda)^2=0$$

So : I has the single eigenvalue $\underline{\lambda=1}$.

For $\lambda=1 \Rightarrow (I-\lambda.I)v=0 \Rightarrow (I-I)v=0$

$$\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

So, every nonzero vector $v=\begin{bmatrix}x\\y\end{bmatrix}$ is an eigenvector of I. In particular, the single eigenvalue $\lambda=1$ corresponds to the 2 linearly independent eigenvectors $v_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $v_2=\begin{bmatrix}0\\1\end{bmatrix}$

Example: Find the eigenvalues and associated eigenvectors of the matrix

$$A = \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}$$

SOLUTION: $p(\lambda) = \det(A - \lambda I) = 0$

$$\begin{vmatrix} (3-\lambda) & 0 & 0 \\ -4 & (6-\lambda) & 2 \\ 16 & -15 & (-5-\lambda) \end{vmatrix} = 0 \Rightarrow (3-\lambda) \cdot \begin{vmatrix} (6-\lambda) & 2 \\ -15 & (-5-\lambda) \end{vmatrix} = 0$$

$$(3-\lambda)\left[(6-\lambda)(-5-\lambda)+30\right] = 0$$

$$(3-\lambda)\left[-30-\lambda+\lambda^2+30\right] = 0$$

$$p(\lambda) = (3-\lambda)(\lambda^2-\lambda) = 0$$

$$p(\lambda) = (3-\lambda)(\lambda)(\lambda-1) = 0$$

$3-\lambda=0 \Rightarrow \boxed{\lambda_1=3}$

$\boxed{\lambda_2=0}$

$\lambda_3-1=0 \Rightarrow \boxed{\lambda_3=1}$

So; eigenvalues of A are

$\lambda_1=3$
$\lambda_2=0$
$\lambda_3=1$

Check: $\det(A) \overset{?}{=} \lambda_1 \cdot \lambda_2 \lambda_3$

$\det(A) = 3\begin{vmatrix} 6 & 2 \\ -15 & -5 \end{vmatrix} = 3(-30+30) = 0$

$\lambda_1 \lambda_2 \lambda_3 = 3 \cdot 0 \cdot 1 = 0$ ✓

$Tr(A) \overset{?}{=} \lambda_1+\lambda_2+\lambda_3$

$3+6+(-5) = 3+0+1$

$4 = 4$ ✓

* For $\lambda_1=3$; $(A-\lambda_1 I)V=0$

$(A-3I)V=0$

$$\begin{bmatrix} (3-3) & 0 & 0 \\ -4 & (6-3) & 2 \\ -16 & -15 & (-5-3) \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ -4 & 3 & 2 \\ -16 & -15 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -4 & 3 & 2 & 0 \\ -16 & -15 & -8 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} -4 & 3 & 2 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} -4x+3y+2z=0 \\ -3y=0 \end{array} \Big\} \; \underline{y=0}$$

3 unknowns, 2 eqs. } 1 parameter

Let $z=\alpha \Rightarrow -4x+2\alpha=0 \Rightarrow \boxed{x=\frac{1}{2}\alpha}$

So; $V = \begin{bmatrix} \frac{1}{2}\alpha \\ 0 \\ \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}}_{V_1} \Rightarrow \boxed{V_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}}$

: For $\lambda_2=0 \Rightarrow (A-\lambda_2 I)V=0$

$(A-0.I)V=0$

$AV=0 \longrightarrow \begin{bmatrix} 3 & 0 & 0 \\ -4 & 6 & 2 \\ 16 & -15 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 3 & 0 & 0 & 0 \\ -4 & 6 & 2 & 0 \\ 16 & -15 & -5 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1/3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} x=0 \\ y+\frac{1}{3}z=0 \end{array}$$

3 unknowns, 2 eqs; $3-2=1$ parameter

Let $z=3\alpha \Rightarrow y=$

So, $\left.\begin{array}{l} x=0 \\ y=-\alpha \\ z=3\alpha \end{array}\right\}$ $v=\begin{bmatrix} 0 \\ -\alpha \\ 3\alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}}_{v_2}$ so, $v_2 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$ (or $v_2 = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$)

* For $\lambda_3 = 1 \Rightarrow (A - \lambda_3 I)v = 0$

$(A - I)v = 0$

$$\begin{bmatrix} 2 & 0 & 0 \\ -4 & 5 & 2 \\ 16 & -15 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & 0 & 0 \\ -4 & 5 & 2 & 0 \\ 16 & -15 & -6 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 5 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \left.\begin{array}{l} x=0 \\ 5y+2z=0 \end{array}\right\} \; z=5\alpha \Rightarrow y=-2\alpha$$

So: $\left.\begin{array}{l} x=0 \\ y=-2\alpha \\ z=5\alpha \end{array}\right\}$ $v = (0, -2\alpha, 5\alpha) = \alpha \underbrace{(0, -2, 5)}_{v_3} \Rightarrow v_3 = \begin{bmatrix} 0 \\ -2 \\ 5 \end{bmatrix}$

* Note: We found that only a single eigenvector associated with each eigenvalue λ. So eigenspace of λ is 1-dimensional.

* Example: $A = \begin{bmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{bmatrix}$ Find eigenvalues and corresponding eigenvectors of A.

SOL: step 1:

$p(\lambda) = \det(A - \lambda I) = 0$ characteristic eq.

$$\begin{vmatrix} (4-\lambda) & -2 & 1 \\ 2 & (0-\lambda) & 1 \\ 2 & -2 & (3-\lambda) \end{vmatrix} = 0 \Rightarrow \lambda^3 - 7\lambda^2 + 16\lambda - 12 = 0$$

$\lambda = 2$ satisfies the cubic eq. is a solution. so, $\lambda = 2$

$\lambda^3 - 7\lambda^2 + 16\lambda - 12 \;\big|\; \lambda - 2$

$\lambda^2 - 5\lambda + 6$

0

so, $p(\lambda) = (\lambda - 2)(\lambda^2 - 5\lambda + 6) = 0$

$p(\lambda) = (\lambda - 2)(\lambda - 2)(\lambda - 3) = 0$

$\underbrace{\lambda_1 = \lambda_2 = 2}_{\text{repeated eigenvalue}}$, $\lambda_3 = 3$ Eigenvalues of A

step 2: •) For $\lambda = \lambda_2 = 2 \Rightarrow (A - 2I)v = 0$

$$\begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 2 & -2 & 1 & 0 \end{array}\right] \sim \begin{bmatrix} 2 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow 2x - 2y + z = 0 \; \} \; \begin{array}{l} 3 \text{ unknowns} \\ 1 \text{ eqs} \end{array} \Rightarrow 3 - 1 = 2 \text{ parameters}$$

Let $z = \alpha$ and $y = \beta \Rightarrow x = \frac{2y - z}{2} = \frac{2\beta - \alpha}{2}$

$$v = \begin{bmatrix} \frac{2\beta - \alpha}{2} \\ \beta \\ \alpha \end{bmatrix} = \alpha \begin{bmatrix} -\frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

Thus, The 2-dimensional eigenspace of A associated with the repeated eigenvalue $\lambda = 2$ has basis $\{v_1, v_2\}$

. For $\lambda_3 = 3 \Rightarrow$ $\begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 2 & -2 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & -3 & 1 & 0 \\ 2 & -2 & 0 & 0 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$ $\left.\begin{array}{l} x-2y+z=0 \\ y-z=0 \end{array}\right\}$ Let $z=\alpha \Rightarrow y=\alpha$
$x = 2\alpha - \alpha = \alpha$

so, $V = \begin{bmatrix} \alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{V_3}$

Thus, eigenvector of λ_3 is $\underline{V_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}$ ///

It follows that the eigenspace of A associated with $\lambda=3$ is 1-dimensional has has $V_3 = [1\ 1\ 1]^T$ as a basis eigenvector.

Final exam Q/ (2018)

Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

a) Find the eigenvalues of the matrix A

b) Find the eigenvectors of A corresponding to the eigenvalue

c) Show that if λ is an eigenvalue of B and X is corresponding eigenvector, and s is a scalar, then $(\lambda - s)$ is an eigenvalue of $(B - sI)$ and X is the corresponding eigenvector.

SOL:

a) $p(\lambda) = \det(A - \lambda I) = 0$ charac. eq

$p(\lambda) = \begin{vmatrix} -\lambda & 0 & -2 \\ 1 & (2-\lambda) & 1 \\ 1 & 0 & (3-\lambda) \end{vmatrix} = 0 \Rightarrow (2-\lambda)\begin{vmatrix} -\lambda & -2 \\ 1 & (3-\lambda) \end{vmatrix} = 0$

$p(\lambda) = (2-\lambda)\cdot[(-\lambda)(3-\lambda)+2] = 0$

$p(\lambda) = (2-\lambda)(\lambda^2 - 3\lambda + 2) = 0 \Rightarrow \underline{(2-\lambda)(\lambda-1)(\lambda-2) = 0}$ charac. eq.

$-1 \quad -2$

So, eigenvalues are:

$\underline{\lambda_1 = 1}$ and $\underline{\lambda_2 = \lambda_3 = 2}$ repeated eigenvalue.

b) . For $\lambda_1 = 1 \Rightarrow (A - \lambda_1 I)V = 0 \Rightarrow (A - I)V = 0$

$\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3 unknowns, 2 eqs. $3-2=1$ parameter

$\left[\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \approx \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left.\begin{array}{l} x+2z=0 \\ y-z=0 \end{array}\right\}$

Let $z = \alpha \Rightarrow y = \alpha$
$x = -2\alpha$

$V = \begin{bmatrix} -2\alpha \\ \alpha \\ \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{V_1} \Rightarrow$ so, $\underline{V_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$ is eigenvector of λ_1 ///

·) For $\lambda_2=\lambda_3=2 \Rightarrow (A-2I)v=0$

$$\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -2 & 0 & -2 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \overset{\cdots}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad x+z=0 \Big\}$$ 1 eq. 3 unknowns $\Rightarrow 3-1=2$ parametes

Let $z=\alpha$ and $y=\beta \Rightarrow x=-\alpha$

$$\text{so; } v=\begin{bmatrix} -\alpha \\ \beta \\ \alpha \end{bmatrix}=\alpha\underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{v_2}+\beta\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{v_3}$$

(katlı kok için 2 farklı otvektör bulduk)

so; eigenvectors of $\lambda_2=\lambda_3=2$ is $v_2=\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}=v_3$

c) If λ is an eigenvalue and x is corresponding eigenvector of $B \Rightarrow Bx=\lambda x$

$Bx=\lambda x \quad (1)$ Let's substract both sides of Eq (1) from sx:

$Bx-sx=\lambda x-sx$ (s : scalar, λ : scalar)

$(B-sI)x=(\lambda-s)x$

So; $(\lambda-s)$ is an eigenvalue of $B-sI$ and x is corresponding eigenvector ✓

(We have to multiply s with Identity matrix, Because B is a matrix)

* Example:

Ö/ $A=\begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$ ·) Eigenvalues are $\lambda_1=4$; $\lambda_2=\lambda_3=-2$

$$p(\lambda)=\det(A-\lambda I)=0 \Rightarrow \begin{vmatrix} -3-\lambda & 1 & -1 \\ -7 & 5-\lambda & -1 \\ -6 & 6 & -2-\lambda \end{vmatrix}=0 \Rightarrow \underbrace{\lambda_1=4 \;;\; \lambda_2=\lambda_3=-2}_{\text{eigenvalues}}$$

* For $\lambda_1=4 \Rightarrow (A-4I)v=0$

$$\begin{bmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -7 & 1 & -1 & 0 \\ -7 & 1 & -1 & 0 \\ -6 & 6 & -6 & 0 \end{array}\right] \overset{\cdots}{\sim} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x=0$, $y-z=0$ } For $z=\alpha \Rightarrow y=\alpha$

$$\text{so; } v=\begin{bmatrix} 0 \\ \alpha \\ \alpha \end{bmatrix}=\alpha\underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{v_1} \Rightarrow \text{so, } v_1=\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

* For $\lambda_2=\lambda_3=-2 \Rightarrow (A+2I)v=0$

$$\begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ -6 & 6 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} -1 & 1 & -1 & 0 \\ -7 & 7 & -1 & 0 \\ -6 & 6 & 0 & 0 \end{array}\right] \overset{\cdots}{\sim} \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x-y=0$, $z=0$ } Let $y=\alpha \Rightarrow x=\alpha$

9)

So, $V = \begin{bmatrix} \alpha \\ \alpha \\ 0 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $\Rightarrow$ So; eigenvector of $\lambda_2 = \lambda_3 = -2$ is $V_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ ///

(Note: katlı kök için 1 özdeğer bulundu).

(10)

NOTE: (1)
(Demekki, katlı kök tek özdeğer de doğurabilir, 2 tane de. Soruya bağlı!)
(2) Parametreler istenildiği gibi seçilebilir. Sadece 0 (sıfır) veremezsiniz.

—o—

4 January 2017 / Q4

a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & b & c \end{bmatrix}$ If the characteristic polynomial of A is $-\lambda^3 + 4\lambda^2 + 5\lambda + 6$ is given; then find a, b, c.

SOL: $p(\lambda) = \det(A - \lambda I) \Rightarrow$ This is the characteristic polynomial equation.

$$= \begin{vmatrix} (0-\lambda) & 1 & 0 \\ 0 & (0-\lambda) & 1 \\ a & b & (c-\lambda) \end{vmatrix} = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ a & b & (c-\lambda) \end{vmatrix} = (-\lambda) \begin{vmatrix} -\lambda & 1 \\ b & (c-\lambda) \end{vmatrix} - (1) \begin{vmatrix} 0 & -1 \\ a & (c-\lambda) \end{vmatrix}$$

$$= -\lambda[(-\lambda)((c-\lambda) - b] - (1)[a]$$

$$= -\lambda^3 + c\lambda^2 + b\lambda + a$$

So, $c = 4$
$b = 5$
$a = 6$

b) $A = \begin{bmatrix} 1 & 0 & -1 \\ \sqrt{3} & a & 17 \\ 2 & 0 & b \end{bmatrix}$ If eigenvalues of A are given as 2 and 3. And 2 is multiplicity of two find a and b.

SOL: $p(\lambda) = (\lambda - 2)^2(\lambda - 3) = \lambda^3 - 7\lambda^2 + 16\lambda - 12$ (1)

$p(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & -1 \\ \sqrt{3} & a-\lambda & 17 \\ 2 & 0 & b-\lambda \end{vmatrix} = \lambda^3 - \lambda^2(a+b+1) + \lambda(ab + a + b + 2) - (ab + 2a)$ (2)

From (1) and (2) $\Rightarrow$
$$\left.\begin{aligned} a+b+1 &= 7 \\ ab + a + b + 2 &= 16 \\ ab + 2a &= 12 \end{aligned}\right\} \quad \begin{aligned} a &= 2 \\ b &= 4 \end{aligned} ///$$

c) For $V = \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$; $B = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix}$

Is V an eigenvector of B? Eigenvalues of B are given as 2 and 4 is given. (2 is multiplicity of two)

SOL: $B.\vec{V} \stackrel{?}{=} \lambda.\vec{V}$ $\rightarrow$ (Eğer $\vec{V}$; B'nin özvektörü ise $B\vec{V} = \lambda\vec{V}$ olmalı)

$$B.\vec{V} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & 0 \\ -3 & 6 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 14 \end{bmatrix} = \underset{\lambda}{2} \underset{V}{\begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}}$$

Demekki $\lambda = 2$ için özvektörü $V = \begin{bmatrix} 4 \\ 2 \\ 7 \end{bmatrix}$ imiş ✓

* Ex/ If eigenvalue of A is λ; show that eigenvalue of A^2 is λ^2.

SOL: If eigenvalue of A is λ $\Rightarrow$ $Ax = \lambda x$

$$A\cdot(A\cdot x) = A\cdot(\lambda\cdot x)$$
$$A^2\cdot x = \lambda\cdot \underbrace{(A\cdot x)}_{\lambda x}$$
$$A^2 x = \lambda^2 x$$

this equality shows that λ^2 is an eigenval of A^2 ///

* If $\lambda = 0$ $\Rightarrow$ then show that A^{-1} do not exist.

SOL:
$$Ax = \lambda x$$
$$Ax = \lambda I x$$
$$(A - \lambda I)x = 0$$
$\downarrow$ for nontrivial solution

$$|A - \lambda I| = 0$$

Here $\lambda = 0$ is given. So, $|A - 0.I| = 0$ $\Rightarrow$ $|A| = 0$

If $|A| = 0 \Rightarrow$ then A^{-1} do not exist
(A is singular matrix)

Ex/ Show that if eigenvalue of A is λ, then eigenvalue of A^{-1} is $\frac{1}{\lambda}$

SOL: If eigenvalue of A is λ $\Rightarrow$ Then $Ax = \lambda x$ (1) (Ax; x in λ kat)

multiply both sides of (1) by A^{-1} from left:

$$A^{-1}\cdot(Ax) = A^{-1}\cdot\lambda\cdot x$$
$$\underbrace{(A^{-1}A)}_{I} x = A^{-1}(\lambda\cdot x)$$
$$x = A^{-1}\cdot\lambda\cdot x$$
$$\frac{1}{\lambda}\cdot x = A^{-1} x$$ ($A^{-1}x$; x in $\frac{1}{\lambda}$ katı)

So; eigenvalue of A^{-1} is $\frac{1}{\lambda}$

6.2 Diagonalization of Matrices

Def1
* The $(n \times n)$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.

Def2
* If A is diagonalizable, then there exists a Diagonal Matrix 'D' and an invertible matrix P such that

$$P^{-1} A P = D$$

Here,

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}_{n\times n} \quad \text{and} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{bmatrix}_{n\times n}$$

Diagonal Matrix

(In a diagonal matrix; diagonal elements are the eigenvalues)

- P is the $(n\times n)$ matrix having eigenvectors as its column vectors respectively.
- It is invertible matrix. So; P^{-1} exists.

Def3: Similar matrices: The $(n\times n)$ matrices A and B are called similar matrices provided that there exists an invertible matrix P such that

$$B = P^{-1} A P$$

* Example 1: $A = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$ a) Find eigenvalues of A

b) Find corresponding eigenvectors of A c) Can A diagonalizable? explain.

d) If yes, find the diagonal matrix D and invertible matrix X such that

$$A = X D X^{-1}$$

SOLUTION: (a) $\begin{vmatrix} 5-\lambda & -6 \\ 2 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(-2-\lambda)+12=0$

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda-1)(\lambda-2)=0$ → $\lambda_1 = 1$, $\lambda_2 = 2$ } Eigenvalues

$-1 \quad -2$

(b) For $\lambda_1 = 1 \Rightarrow (A-\lambda_1 I)X = 0$

$(A-I)X=0 \Rightarrow \begin{bmatrix} 4 & -6 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 4x_1 - 6x_2 = 0 \Rightarrow 2x_1 = 3x_2$

$x_1 = \frac{3}{2}x_2$

Let $x_2 = 2 \Rightarrow x_1 = 3$

So; $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ ← (This is the eigenvector of $\lambda_1 = 1$)

- For $\lambda_2 = 2 \Rightarrow (A-2I)X = 0$

$\begin{bmatrix} 3 & -6 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3x_1 - 6x_2 = 0 \Rightarrow 3x_1 = 6x_2 \Rightarrow x_1 = 2x_2$

Let $x_2 = 1 \Rightarrow x_1 = 2$ So; $\vec{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ ← This is the eigenvector of $\lambda_2 = 2$

c) Since for (2x2) matrix A; we have 2 linearly independent eigenvectors; A is diagonalizable.

d) Since A is diagonalizable,

the Diagonal matrix D is $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

and

the matrix X is $X = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

<u>NOTE:</u> If you want to check your solution

$$A \overset{?}{=} X D X^{-1}$$

$$\begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}}$$

$$\begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \overset{?}{=} \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 4 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$$ Yes. The result that we found is TRUE ✓

Compute X^{-1}

($|X| = \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} = \cdot$

$X^{-1} = \frac{1}{|X|}\begin{bmatrix} 1 & - \\ -2 & \end{bmatrix}$

$= \frac{1}{-1}\begin{bmatrix} 1 & -2 \\ -2 & 3 \end{bmatrix}$

$X^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$

*<u>Example:</u> $A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix}$ Can A diagonalizable?

$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & 3 \\ 0 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 = 0 \Rightarrow \underline{\lambda_1 = \lambda_2 = 2}$ We have only one eigenvalue.

) For $\lambda = 2 \Rightarrow \begin{bmatrix} 2-2 & 3 \\ 0 & 2-2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow 3x_2 = 0$

$\underline{x_2 = 0}$

Let $x_1 = r$ ($r \in \mathbb{R}$) (Let $r = 1$) $\Big\} \; v = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. So, $\lambda = 2$ has only one Linearly independent eigenvector.

For (2x2) matrix A, A has 1 Linearly independent eigenvector. So, A can not be diagonal

So; Let's generalize the idea by theorem.

<u>Theorem:</u> If the (nxn) matrix A has n distinct eigenvalues, then A is diagonaliza

* 6.3. Applications involving Powers of Matrices *

Recall that; for $(n \times n)$ matrix A if A has n linearly independent eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ associated with the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$A = PDP^{-1}$$

Let's compute A^k:

$$\cdot\ A^2 = A.A = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{I}DP^{-1} = PD^2P^{-1} \Rightarrow A^2 = PD^2P^{-1}$$

$$\cdot\ A^3 = A^2.A = (PD^2P^{-1})(PDP^{-1}) = PD^2\underbrace{(P^{-1}P)}_{I}DP^{-1} = PD^3P^{-1} \Rightarrow A^3 = PD^3P^{-1}$$

$$\vdots \qquad\qquad \vdots$$

$$\cdot\ A^k = ? \qquad \boxed{A^k = PD^kP^{-1}}$$

So; k^{th} power D^k of the diagonal matrix can easily be computed by

$$D^k = \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix}$$

Example: Find A^5 if $A = \begin{bmatrix} 5 & -6 \\ 2 & -2 \end{bmatrix}$

Sol:

we found that

$$P = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } P^{-1} = \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$

So;

$$A^5 = PD^5P^{-1}$$

$$= \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} (1)^5 & 0 \\ 0 & (2)^5 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 32 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 64 & -96 \end{bmatrix}$$

$$= \begin{bmatrix} 125 & -186 \\ 62 & -92 \end{bmatrix}$$

* Cayley Hamilton Theorem

If the $(n \times n)$ matrix A has the characteristic polynomial

$$p(\lambda) = (-1)^n \lambda^n + C_{n-1}\lambda^{n-1} + \cdots + C_2\lambda^2 + C_1\lambda + C_0$$

then

$$p(A) = (-1)^n A^n + C_{n-1}A^{n-1} + \cdots + C_2A^2 + C_1A + C_0.I = 0$$

* **Example**: Use the Cayley Hamilton Theorem to find A^{-1}, A^3 and A^4 for

a) $A = \begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix}$ Hmw b) $\begin{bmatrix} 6 & -10 \\ 2 & -3 \end{bmatrix}$ Hmw c) $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

SOLUTION: Step 1/ $|A - \lambda I| = 0$

$$\begin{vmatrix} 5-\lambda & -4 \\ 3 & -2-\lambda \end{vmatrix} = 0 \Rightarrow (5-\lambda)(-2-\lambda) + 12 = 0$$

$$\underbrace{\lambda^2 - 3\lambda + 2}_{p(\lambda)} = 0$$

So; the characteristic polynomial

$$\boxed{p(\lambda) = \lambda^2 - 3\lambda + 2}$$

Step 2/ Replace $\lambda \to A$

So, $A^2 - 3A + 2.I = 0$

$$\Rightarrow A^2 = 3A - 2I = 3.\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 13 & -12 \\ 9 & -8 \end{bmatrix}$$

$$\Rightarrow A^3 = (3A - 2I).A = 3A^2 - 2A$$

$$= 3\begin{bmatrix} 13 & -12 \\ 9 & -8 \end{bmatrix} - 2\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} = \begin{bmatrix} 29 & -28 \\ 21 & -20 \end{bmatrix} //$$

$$\Rightarrow A^4 = 3A^3 - 2A^2 = 3\begin{bmatrix} 29 & -28 \\ 21 & -20 \end{bmatrix} - 2\begin{bmatrix} 13 & -12 \\ 9 & -8 \end{bmatrix} = \begin{bmatrix} 61 & -60 \\ 45 & -44 \end{bmatrix} //$$

$\Rightarrow A^{-1} = ?$

$$A^2 - 3A + 2I = 0$$

$$A^2 = 3A - 2I \quad \left(\text{Divide by } \tfrac{1}{A}\right)$$

$$A = 3I - 2A^{-1}$$

$$2A^{-1} = -A + 3I$$

$$A^{-1} = \frac{1}{2}(-A + 3I)$$

$$= \frac{1}{2}\left(-\begin{bmatrix} 5 & -4 \\ 3 & -2 \end{bmatrix} + 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} -2 & 4 \\ -3 & 5 \end{bmatrix} //$$

Hmw SOLUTIONS: b) $A^3 = \begin{bmatrix} 36 & -70 \\ 14 & -27 \end{bmatrix}$, $A^4 = \begin{bmatrix} 76 & -150 \\ 30 & -59 \end{bmatrix}$, $A^{-1} = \frac{1}{2}\begin{bmatrix} -3 & 10 \\ -2 & 6 \end{bmatrix}$

c) $A^3 = \begin{bmatrix} 1 & 21 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}$; $A^4 = \begin{bmatrix} 1 & 45 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 16 \end{bmatrix}$; $A^{-1} = \frac{1}{2}\begin{bmatrix} 2 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Basic Theory of First Order Linear Equations

$$\left.\begin{aligned} x_1' &= P_{11}(t)x_1 + P_{12}(t)x_2 + \dots + P_{1n}(t)x_n + f_1(t) \\ x_2' &= P_{21}(t)x_1 + P_{22}(t)x_2 + \dots + P_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= P_{n1}(t)x_1 + P_{n2}(t)x_2 + \dots + P_{nn}(t)x_n + f_n(t) \end{aligned}\right\} \quad (1)$$

is called "first order Linear Dif Eq. System".

Eq. (1) can be written in the form of

$$x' = P(t)x + f(t) \quad (2)$$

Here $P(t)$ is $(n\times n)$ matrix; vector

* If Eq (1) is homogeneous; then Eq (2) is

$$x' = P(t) \cdot x \quad (3)$$

once we can solve (3) to get the solution of the homogeneous system, then there are several methods to solve the nonhomogeneous equation (2).
our goal is to find $x_1(t), x_2(t), \dots, x_n(t)$ that satisfies eq (1).
(Another notation of $x_1, x_2, \dots, x_n$ are $x^{(1)}, x^{(2)}, \dots, x^{(n)}$)

* For initial Conditions;

$x_1(a) = b_1$
$x_2(a) = b_2$
$\vdots$
$x_n(a) = b_n$

* **Def:** Matrices and Representation of Linear systems.

* First order Linear Dif Eq. System:

For Eq (1) system; the matrix form

$$x' = \frac{dx}{dt} = P(t)x + f(t) \quad \text{is used}$$

Ex/

$$x_1' = 4x_1 - 3x_2$$
$$x_2' = 6x_1 - 7x_2$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x' = P.x \quad ; f(t) = 0$$

* Example 1/ Show that

$$X_{(1)}(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad X_{(2)}(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

are two solutions of

$$X'(t) = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} X \qquad (1)$$

SOL: $\Rightarrow X_{(1)}(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} \rightarrow X'_{(1)} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$

$$X'_{(1)}(t) \stackrel{?}{=} P \cdot X_{(1)}(t)$$

$$\begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$\begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$ ✓ equal. So, $X_{(1)}(t)$ is a solution of (1).

$\Rightarrow X_{(2)}(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} \Rightarrow X'_{(2)}(t) = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$

$$X'_{(2)}(t) \stackrel{?}{=} P \cdot X_{(2)}(t)$$

$$\begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

$\begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$ ✓ equal

So; $X_{(2)}(t)$ is a solution of

* For $X''(t) + p(t)X'(t) + q(t)X(t) = 0$ homogeneous 2nd order Linear equ

We introduce substitution

$$\boxed{\begin{aligned} X_1 &= X \\ X_2 &= X' \end{aligned}}$$

This substitution yields the system as

(1) $\boxed{X_1' = X_2}$

(2) $X_2' = X'' = -p(t)X'(t) - q(t)X(t)$

$\boxed{X_2' = -p(t)\,X_2 - q(t)\,X_1}$

So; from (1) and (2)

$$\left.\begin{aligned} X_1' &= X_2 \\ X_2' &= -qX_1 - pX_2 \end{aligned}\right\} \quad X' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix}$$

So, we trasformed the homogeneous 2nd order dif eq into 2 first-order equation system.

Principle of Superposition

If the vector functions $x^{(1)}, x^{(2)}, \dots, x^{(n)}$ are solutions of the system

$$\vec{x}' = P(t)\cdot\vec{x} \qquad (1) \qquad \text{(Hom., Linear, first order eq.)}$$

then, the linear combination

$$x(t) = c_1\vec{x}^{(1)} + c_2\vec{x}^{(2)} + \dots + c_n\vec{x}^{(n)} \qquad (c_1, c_2, \dots, c_n : \text{Constants})$$

is also a solution of Eq. (1). This solution is "general solution."

* Example 2/ If $x^{(1)}$ and $x^{(2)}$ are two solutions of

$$\frac{dx}{dt} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} x \qquad \text{discussed in Example 1;}$$

then the linear combination

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$
$$= c_1 \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$$

is also a solution. In scalar form with $x(t) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, this gives the solution

$$x_1(t) = 3c_1e^{2t} + c_2e^{-5t}$$
$$x_2(t) = 2c_1e^{2t} + 3c_2e^{-5t}$$

* Wronskians of Solutions

If $x^{(1)}, x^{(2)}, \dots, x^{(n)}$ are solutions of $\vec{x}'(t) = P(t)\cdot\vec{x}(t)$ on the interval I;

and iff

$$W[\vec{x}^{(1)}, \vec{x}^{(2)}, \dots \vec{x}^{(n)}] = \begin{vmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(n)} \\ x_2^{(1)} & x_2^{(2)} & & x_2^{(n)} \\ \vdots & \vdots & & \vdots \\ x_n^{(1)} & x_n^{(2)} & & x_n^{(n)} \end{vmatrix} \neq 0$$

then $\vec{x}^{(1)}, \vec{x}^{(2)}, \dots \vec{x}^{(n)}$ solutions are said to be linearly independent.

* Example 3/

$$x^{(1)}(t) = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}; \quad x^{(2)}(t) = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}; \quad x^{(3)}(t) = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

are solutions of $\dfrac{dx}{dt} = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} x$

The Wronskian of these solutions

$$W = \begin{vmatrix} 2e^{t} & 2e^{3t} & 2e^{5t} \\ 2e^{t} & 0 & -2e^{5t} \\ e^{t} & -e^{3t} & e^{5t} \end{vmatrix} = e^{9t} \begin{vmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{vmatrix} = -16e^{9t} \neq 0$$

So; $x^{(1)}, x^{(2)}, x^{(3)}$ are Linearly independent.

Theorem: Let $x^{(1)}, x^{(2)}, \ldots, x^{(n)}$ be n Linearly independent solutions of the homogeneous Linear eq.

$$x' = P(t).x$$ on an open interval I, where $P(t)$ is continuous.

If $x(t)$ is any solution whatsoever of the eq. $x' = P(t).x$ on I; then there exists numbers $c_1, c_2, \ldots, c_n$ such that

$$\underset{\sim}{x}(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + \ldots + c_n x^{(n)}(t)$$ " This is General solution of homogeneous sy

for all t in I.

*Initial Value Problems

Example: $$\frac{dx}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} x \quad ; \quad x(0) = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

use the solution vectors in Ex3 to solve IVP.

Sol: $$\underset{\sim}{x}(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) + c_3 x^{(3)}(t) \quad (1)$$

$$= c_1 \begin{bmatrix} 2e^{t} \\ 2e^{t} \\ e^{t} \end{bmatrix} + c_2 \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

In scalar form;

$$\begin{aligned} x_1(t) &= 2c_1e^{t} + 2c_2e^{3t} + 2c_3e^{5t} \\ x_2(t) &= 2c_1e^{t} \qquad\qquad - 2c_3e^{5t} \quad (2) \\ x_3(t) &= c_1e^{t} - c_2e^{3t} + c_3e^{5t} \end{aligned}$$

We seek the particular solution satisfying the initial conditions

$$x_1(0) = 0 \; ; \; x_2(0) = 2 \; ; \; x_3(0) = 6 \quad (3)$$

Substitute (3) into (2):
so we get algebraic system.

$$\left.\begin{aligned} 2c_1 + 2c_2 + 2c_3 &= 0 \\ 2c_1 \qquad - 2c_3 &= 2 \\ c_1 - c_2 + c_3 &= 6 \end{aligned}\right\} \quad \left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 4 & 4 \end{array}\right]$$

$c_1 = 2 \; ; \; c_2 = -3 \; ; \; c_3 = 1$

So; $x(t) = 2x_1(t) - 3x_2(t) + x_3(t)$

$x(t) = 2X_1(t) - 3X_2(t) + X_3(t)$

$$= \begin{bmatrix} 4e^{t} - 6e^{3t} + 2e^{5t} \\ 4e^{t} \quad - 2e^{5t} \\ 2e^{t} + 3e^{3t} + e^{5t} \end{bmatrix}$$

Nonhomogeneous Solutions (Bu kısım Sadece bilgi için. X_p'nin bulunmasından sorulu değilsiniz)

It is of the form

$$\frac{dX}{dt} = P(t).X + f(t)$$

Non-homogeneous. Dahil Değil. Konuyu kestim!

General solution: $X(t) = \underbrace{X_c(t)}_{\text{Complementary solution of } X' = P(t).X \text{ homogeneous eq.}} + \overbrace{X_p(t)}^{\text{Single particular solution.}}$

$$\underset{\sim}{X}(t) = \underbrace{c_1X_1 + c_2X_2 + \dots + c_nX_n}_{X_c(t)} + X_p(t) \quad \text{for all } t \text{ in } I.$$

ys for solution:
1) Find $X_c(t)$
2) Find $X_p(t)$
3) $\underset{\sim}{X}(t) = X_c + X_p$

ANLAT.

~~Example~~

CONVERT 2nd order Linear Hom. Eq. into First order Linear Eq. system
(2. mertebe Lineer homojen bir denklemi (sabit katsayılı); sisteme çevirme)

Example: $x'' - x' - 2x = 0$

Substitution: $\left.\begin{matrix} x_1 = x \\ x_2 = x' \end{matrix}\right\} \rightarrow \left.\begin{matrix} (1)\ x_1' = x_2 \\ (2)\ x_2' = x'' = x' + 2x = x_2 + 2x_1 \end{matrix}\right\} \quad \begin{matrix} x_1' = x_2 \\ x_2' = x_2 + 2x_1 \end{matrix}$

So, the system becomes

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$X' = P \quad . \quad X$$

For n-th order (Sadece Bilgi için)

NOTE!

Substitute

$$\left.\begin{matrix} x_1 = x \\ x_2 = x' \\ x_3 = x'' \\ x_4 = x''' \\ \vdots \\ x_n = x^{(n-1)} \end{matrix}\right\}$$

These substitutions yield the system

$$\begin{matrix} x_1' = x_2 \\ x_2' = x_3 \\ x_3' = x_4 \\ \vdots \\ x_{n-1}' = x_n \end{matrix}$$

of n first order eqs.

Example: $x^{(3)}+3x''+2x'-5x=\sin 2t$

substitute
$$\left.\begin{aligned} x_1&=x \\ x_2&=x' \\ x_3&=x'' \end{aligned}\right\}\ \text{yields} \quad \begin{aligned} x_1'&=x_2 \\ x_2'&=x_3 \\ x_3'&=5x_1-2x_2-3x_3+\sin 2t \end{aligned}$$

of three first-order eq.s.

The Eigenvalue Method for First-order Linear Homogeneous systems with constant coefficients :

A system of homogeneous first order linear system with constant coefficient is of the form

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{aligned} \qquad (1) \qquad \left(\frac{dx}{dt} = AX\right)$$

A : $(n\times n)$ matrix

The matrix form of Eq.(1) is

$$\boxed{X' = A.X} \qquad (2)$$

Where A is $(n\times n)$ coefficient matrix. ($a_{11}, \dots, a_{nn}$ are constants)

SOLUTION METHOD

Step 1: seek the solution of the form

$$x(t) = \vec{v}.e^{\lambda t} \qquad (3)$$

where λ is eigenvalue of A and $\vec{v}$ is eigenvector of A corresponding to λ.

$$\left.\begin{aligned} x &= v.e^{\lambda t} \\ x' &= v\lambda e^{\lambda t} \end{aligned}\right\} \xrightarrow{\text{Plug into (2)}} \quad v\lambda \cancel{e^{\lambda t}} = A v \cancel{e^{\lambda t}}$$

$$Av = \lambda . v . I_n \qquad (I_n : \text{Identity matrix})$$

$$(A - \lambda I_n) v = 0 \qquad \text{Homogeneous eq. (4)}$$

$\downarrow$ For nontrivial solution

$$\boxed{|A - \lambda I_n| = 0} \quad \text{characteristic eq. (5)}$$

From Eq (5), find $\lambda_1, \lambda_2, \dots, \lambda_n$ eigenvalues of A.

$$|A - \lambda I| = 0$$

- **Case 1**: Distinct and Real Eigenvalues: $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{R}$ and $\lambda_1 \neq \lambda_2 \neq \dots \neq \lambda_n$
- **Case 2**: Complex Eigenvalues: $\lambda_{1,2} = a \pm bi \in \mathbb{C}$
- **Case 3**: Repeated Eigenvalues: $\lambda_1 = \lambda_2 = \dots = \lambda_n$

For case I

Step 2: Find $\vec{V}_1, \vec{V}_2, \ldots, \vec{V}_n$ eigenvectors corresponding to each eigenvalues $\lambda_i \Rightarrow \vec{V}_i$

Step 3: Since we seek the solution of the form $X = v e^{\lambda t}$,

$$\left.\begin{aligned} X^{(1)}(t) &= \vec{V}_1 \cdot e^{\lambda_1 t} \\ X^{(2)}(t) &= \vec{V}_2 \cdot e^{\lambda_2 t} \\ &\vdots \\ X^{(n)}(t) &= \vec{V}_n \cdot e^{\lambda_n t} \end{aligned}\right\}$$

These are linearly independent solutions of $X' = A.X$

$$\left(W[X^{(1)}(t), X^{(2)}(t)] \neq 0 \right)$$

Step 4: Write the general solution of $X' = A.X$ as

$$X(t) = c_1 X^{(1)}(t) + c_2 X^{(2)}(t)$$

Example 1) Find a general solution of the system

$$x_1' = 4x_1 + 2x_2$$
$$x_2' = 3x_1 - x_2 \quad (1)$$

Step 1:

SOLUTION: The matrix form of (1) is

$$x' = \underbrace{\begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}}_{A \text{ (coefficient matrix)}} x \quad (2)$$

The characteristic eq. of the coefficient matrix is

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)(-1-\lambda) - 6 = 0$$
$$\lambda^2 - 3\lambda - 10 = 0$$
$$(\lambda+2)(\lambda-5) = 0 \quad \lambda_1 = -2, \ \lambda_2 = 5$$

Distinct Real Eigenvalues ✓

For the coefficient matrix A in Eq (2); the eigenvector equation is

$$(A - \lambda I)\vec{v} = 0$$
$$\begin{bmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (3)$$

step 2:) For $\lambda_1 = -2 \Rightarrow (A - 2I)\vec{v}_1 = 0$

$$\begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So, the two scalar equations are

$$\left.\begin{aligned} 6a+2b&=0 \\ 3a+b&=0 \end{aligned}\right\} \quad b = -3a$$

Let's choose $a = 1 \Rightarrow b = -3$.

Thus $\vec{v}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

So, $x^{(1)}(t) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t}$

•) For $\lambda_2 = 5 \Rightarrow (A - \lambda_2 I)\vec{v} = 0 \Rightarrow (A - 5I)\vec{v}_2 = 0$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{aligned} -a+2b&=0 \\ 3a-6b&=0 \end{aligned}\right\} \quad a = 2b$$

let $b = 1 \Rightarrow a = 2$. So, $\vec{v}_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

So, $x^{(2)}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$

Note: $x^{(1)}(t)$ and $x^{(2)}(t)$ are linearly independent. Because Wronskian is not zero.

$$W = \begin{vmatrix} e^{-2t} & 2e^{5t} \\ -3e^{-2t} & e^{5t} \end{vmatrix} = 7e^{3t} \text{ is not zero}$$

Step 3: Hence a general solution is

$$\underset{\sim}{x}(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$

$$\underset{\sim}{x}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$$

In scalar form, $x_1(t) = c_1 e^{-2t} + 2c_2 e^{5t}$

$x_2(t) = -3c_1 e^{-2t} + c_2 e^{5t}$

3)

For Case 2. / Complex eigenvalues

Solution Method :

$x' = Ax$

$\downarrow \; \vec{x} = \vec{v} e^{\lambda t}$

$(A - \lambda I)\vec{v} = 0$ $\left(\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix} \text{ and } I: \text{Identity matrix} \right)$.

$\downarrow$ for nontrivial solution

$|A - \lambda I| = 0$ charac eq

$\downarrow$

$\lambda_{1,2} = \alpha \pm \beta i \in \mathbb{C}$

For $\lambda_1 = \alpha + \beta i$; Find corresponding eigenvector $\vec{v}_1$ using $(A - \lambda_1 I)\vec{v}_1 = 0$ that is

$\underbrace{\vec{v}_1 = a + bi}$

For $\lambda_2 = \alpha - \beta i$; Find corresponding eigenvector $\vec{v}_2$. That is;

Since $\lambda_2 = \bar{\lambda}_1 \Rightarrow \underbrace{\vec{v}_2 = a - bi}$

$\downarrow$

write the corresponding first linearly independent solution $x^{(1)}(t)$. That is:

$$x^{(1)}(t) = \vec{v}_1 \cdot e^{\lambda_1 t}$$
$$= \vec{v}_1 \cdot e^{(\alpha + \beta i)t}$$
$$= (a + bi) e^{(\alpha + \beta i)t}$$

write the second linearly independent solution $x^{(2)}(t)$. That is

$$x^{(2)}(t) = (a - bi) e^{(\alpha - \beta i)t}$$

$\downarrow$

Rewrite $x^{(1)}$ as $x^{(1)}(t) = u + iv$

$$x^{(1)}(t) = (a + bi) e^{\alpha t} [\cos \beta t + i \sin \beta t]$$

$$x^{(1)}(t) = \underbrace{e^{\alpha t}(a \cos \beta t - b \sin \beta t)}_{\vec{u} \text{ (Real part)}} + i \cdot \underbrace{e^{\alpha t}(a \sin \beta t + b \cos \beta t)}_{\vec{v} \text{ (Imaginary part)}}$$

$\downarrow$

Sol $x^{(1)}(t) = u(t) + i v(t)$

$x^{(2)}(t) = u(t) - i v(t)$

These are two linearly independent solutions of $x' = Ax$.

Because $W(u, v) \neq 0$.

$\downarrow$

write the general solution

$$\underset{\sim}{x}(t) = c_1 u(t) + c_2 v(t)$$

* Example 3/ Find a general solution of the system

$$\frac{dx_1}{dt} = 4x_1 - 3x_2$$

$$\frac{dx_2}{dt} = 3x_1 + 4x_2$$

SOLUTION: Step 1: $x' = \underbrace{\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}}_{A \text{ Coefficient matrix}} x$

Charac. eq $\rightarrow$ $|A - \lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0 \Rightarrow (4-\lambda)^2 + 9 = 0 \Rightarrow \lambda^2 - 8\lambda + 25 = 0$$

$$\lambda_{1,2} = \frac{+8 \pm \sqrt{64 - 4.25}}{2} = \frac{8 \pm \sqrt{-36}}{2}$$

$$\lambda_{1,2} = \frac{8 \pm 6i}{2} \rightarrow \boxed{\lambda_{1,2} = 4 \pm 3i}$$

Complex Conjugate eigenvalues

Step 2: For $\lambda_1 = 4 - 3i$; $(A - (4-3i)I)\vec{V}_1 = 0$

$$\begin{bmatrix} 4-(4-3i) & -3 \\ 3 & 4-(4-3i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} +3i & -3 \\ 3 & 3i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left.\begin{matrix} (+3i)a - 3b = 0 \\ 3a + (3i)b = 0 \end{matrix}\right\} a = 1 \text{ and } b = i \Big\} \quad \vec{V}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

So; the corresponding complex valued solution $x^{(1)} = \vec{V}_1 . e^{\lambda t}$

$$x^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(4-3i)t}$$

$$x^{(1)} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{(4-3i)t} = \begin{bmatrix} 1 \\ i \end{bmatrix} e^{4t} . e^{-3it}$$

$$= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{4t} [\cos 3t - i \sin 3t]$$

$$= e^{4t} \begin{bmatrix} \cos 3t - i \sin 3t \\ i \cos 3t + \sin 3t \end{bmatrix} = \underbrace{e^{4t} \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix}}_{\substack{U \\ \text{(Real part)}}} + i \underbrace{e^{4t} \begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix}}_{\substack{V \\ \text{(Imaginary part)}}}$$

$$= U + iV$$

$\rightarrow$ $x^{(2)}(t) = U - iV$

So, the general solution of $X' = A \cdot X$ is

$$\underset{\sim}{X}(t) = c_1 u(t) + c_2 v(t)$$

$$\underset{\sim}{x}(t) = c_1 e^{4t} \begin{bmatrix} \cos 3t \\ \sin 3t \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -\sin 3t \\ \cos 3t \end{bmatrix}$$

$$= e^{4t} \begin{bmatrix} c_1 \cos 3t - c_2 \sin 3t \\ c_1 \sin 3t + c_2 \cos 3t \end{bmatrix}$$

In scalar form;

$$x_1(t) = e^{4t}(c_1 \cos 3t - c_2 \sin 3t)$$
$$x_2(t) = e^{4t}(c_1 \sin 3t + c_2 \cos 3t)$$

For Case 2 : Repeated eigenvalues

SOLUTION METHOD

Step 1: $X' = A X$ (1)

$\downarrow \; \vec{x} = \vec{v} e^{\lambda t}$

$(A - \lambda I)\vec{V} = 0$ (2)

$\downarrow$ For nontrivial sol.

$|A - \lambda I| = 0$ charac. eq.

$\downarrow$

$\lambda = \lambda_1 = \lambda_2$ (repeated roots)

$\downarrow$ find eigenvector $\vec{V}_1$ corresponding to $\lambda = \lambda_1$

Step 2: $\boxed{\vec{X}^{(1)}(t) = V_1(t) e^{\lambda_1 t}}$. This is first linearly independent solution. (3)

$\downarrow$

Step 3: Look for the second linearly independent solution $\vec{X}^{(2)}(t)$ in the form of

$\boxed{\vec{X}^{(2)}(t) = (\vec{V}_1 t + \vec{V}_2) e^{\lambda t}}$ ** Here, what is $\vec{V}_2 = ?$ (4)

Differentiate $\vec{X}^{(2)}$ with respect to t : $(X^{(2)})' = V_1 . e^{\lambda t} + \lambda (V_1 t + V_2) e^{\lambda t}$

Substitute $X^{(2)}$ and $(X^{(2)})'$ into Eq.(1) :

$$V_1 e^{\lambda t} + \lambda (V_1 t + V_2) e^{\lambda t} = A\left[(V_1 t + V_2) e^{\lambda t}\right]$$

$$V_1 e^{\lambda t} + \underline{t \lambda V_1 e^{\lambda_1 t}} + \lambda V_2 e^{\lambda t} = \underline{A V_1 t e^{\lambda t}} + A V_2 e^{\lambda t}$$

$$t . \underbrace{(A - \lambda I) V_1}_{=0} + (A - \lambda I) V_2 = V_1$$

(Because of Eq(2); This is zero)

use $\boxed{(A - \lambda I)\vec{V}_2 = \vec{V}_1}$ (5) to find $\vec{V}_2 = \begin{pmatrix} a \\ b \end{pmatrix}$ (6)

Step 4: Plug $\vec{V}_2$ into eq.(4) to find $X^{(2)}(t)$

[NOTE: $X^{(1)}$ and $X^{(2)}$ are linearly independent solutions. Because $W(X^{(1)}, X^{(2)}) \neq 0$]

Step 5: Write the general solution

$$X(t) = c_1 X^{(1)}(t) + c_2 X^{(2)}(t)$$

$$= c_1 V_1 e^{\lambda_1 t} + c_2 \left[V_1 t e^{\lambda_1 t} + V_2 e^{\lambda_1 t}\right]$$

NOTE: Genelleme yapılırsa (katlılık 3 için)

$\lambda_1 = \lambda_2 = \lambda_3 \Rightarrow$ $X^{(1)}(t) = V_1 e^{\lambda_1 t}$, $[A - \lambda_1 I] V_1 = 0$

$X^{(2)}(t) = (V_1 t + V_2) e^{\lambda_1 t}$; $[A - \lambda_1 I] V_2 = V_1$

$X^{(3)}(t) = (\frac{t^2}{2!} V_1 + V_2 t + V_3) e^{\lambda_1 t}$, $[A - \lambda_1 I] V_3 = V_2$

Sınava Dahil değil. Sadece bilgi için yazdım.

Özet: $\lambda_1=\lambda_2 \Rightarrow \lambda_1 \to (A-\lambda_1 I)V_1=0$ 'dan V_1'i bul

$\hookrightarrow \lambda_2 \to (A-\lambda_1 I)V_2=V_1$ den V_2 yi bul.

$$\left.\begin{aligned} x^{(1)}(t) &= V_1 e^{\lambda_1 t} \\ x^{(2)}(t) &= (V_1 t+V_2)e^{\lambda_1 t} \end{aligned}\right\}$$ Dif. denk.in 2 lineer bağımsız çözümüdür.

Genel çözüm: $\underset{\sim}{x}(t)=c_1 x^{(1)}(t)+c_2 x^{(2)}(t)$

$= e^{\lambda_1 t}\left[c_1 V_1 + c_2 (V_1 t+V_2)\right]$ olarak yazılır.

(!) Sadece katlılığın 2 olması durumundan sorumlusunuz.) (3 ve fazlası sınava dahil değil.)

August 6, 2018 / Q4

(15)

Find the solution of the system of differential equations below [25 p]

$$x_1' - 2x_1 - \frac{3}{2}x_2 = 0 \quad , \quad x_1(0) = 3$$

$$x_2' + \frac{3}{2}x_1 + x_2 = 0 \quad , \quad x_2(0) = -2$$

SOLUTION: Step 1: $x_1' = 2x_1 + \frac{3}{2}x_2$

$x_2' = -\frac{3}{2}x_1 - x_2$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}'}_{x'} = \underbrace{\begin{bmatrix} 2 & 3/2 \\ -3/2 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x}$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 3/2 \\ -3/2 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) + \frac{9}{4} = 0 \Rightarrow \lambda^2 - \lambda + \frac{1}{4} = 0 \Rightarrow \left(\lambda - \frac{1}{2}\right)^2 = 0$$

$\boxed{\lambda_{1,2} = \frac{1}{2}}$ Repeated Eigenvalues

Step 2:

* For $\lambda = \frac{1}{2} \Rightarrow (A - \lambda_1 I)v_1 = 0$

$$\left(A - \frac{1}{2}I\right)v_1 = 0$$

$$\begin{bmatrix} 2-\frac{1}{2} & \frac{3}{2} \\ -\frac{3}{2} & -1-\frac{1}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} \frac{3}{2} & \frac{3}{2} & 0 \\ -\frac{3}{2} & -\frac{3}{2} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} a_1 + a_2 = 0 \\ a_2 = -a_1 \end{array}$$

Let $a_1 = s \Rightarrow a_2 = -s$.

So; $\vec{v}_1 = s\begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is the first linearly independent solution

let $s = 1$

So; $x^{(1)} = \vec{v}_1 e^{\lambda_1 t} \Rightarrow \boxed{x^{(1)}(t) = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{t/2}}$

Step 3:

* Second linearly independent solution: $x^{(2)}(t) = (\vec{v}_1 t + \vec{v}_2)e^{\lambda_1 t}$

We look for 2nd sol. in this form $\boxed{x^{(2)}(t) = (\vec{v}_1 t + \vec{v}_2)e^{t/2}}$

So; $\boxed{\left(A - \frac{1}{2}I\right)\vec{v}_2 = \vec{v}_1}$

$$\left(A - \frac{1}{2}I\right) V_2 = V_1$$

$$\begin{bmatrix} \frac{3}{2} & \frac{3}{2} \\ -\frac{3}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} 3/2 & 3/2 & 1 \\ -3/2 & -3/2 & -1 \end{array}\right] \xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{cc|c} 3/2 & 3/2 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{2}{3}R_1 \to}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2/3 \\ 0 & 0 & 0 \end{array}\right] \Rightarrow b_1 + b_2 = \frac{2}{3} \Rightarrow b_2 = \frac{2}{3} - b_1$$

$$\left.\begin{array}{l} \text{Let } b_1 = 0 \Rightarrow b_2 = \frac{2}{3} \end{array}\right\} \vec{V}_2 = \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}$$

So, $X^{(2)}(t) = (\vec{V}_1 t + \vec{V}_2) e^{\lambda t}$ becomes

$$\underline{X^{(2)}(t) = \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2/3 \end{bmatrix} \right) e^{t/2}}$$

Step 4:

So, General solution is

$$\underset{\sim}{X}(t) = c_1 X^{(1)}(t) + c_2 X^{(2)}(t)$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 e^{t/2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 2/3 \end{bmatrix} \right) e^{t/2}$$

So,
$$\left.\begin{array}{l} x_1(t) = c_1 e^{t/2} + c_2 t e^{t/2} \\ x_2(t) = -c_1 e^{t/2} + c_2\left(-t + \frac{2}{3}\right) e^{t/2} \end{array}\right\}$$

Step5: Apply Initial Conditions:

$$\left.\begin{array}{ll} x_1(0) = 3 & \Rightarrow c_1 = 3 \\ x_2(0) = -2 & \Rightarrow -c_1 + \frac{2}{3}c_2 = -2 \end{array}\right\} \begin{array}{l} c_1 = 3 \\ c_2 = \frac{3}{2} \end{array}$$

Plug c_1 and c_2

So,
$$x_1(t) = 3e^{t/2} + \frac{3}{2} t e^{t/2}$$

$$x_2(t) = -3e^{t/2} + \frac{3}{2}\left(-t + \frac{2}{3}\right) e^{t/2}$$

May 28, 2018 / Q4

Find the solution of the system of dif. eq.s below [25 p]

$$x_1' - x_1 - x_2 = 0 \quad , \; x_1(0) = 1$$
$$x_2' + x_1 - 3x_2 = 0 \quad , \; x_2(0) = 2$$

Solution:

Step 1: $\left.\begin{aligned} x_1' &= x_1 + x_2 \\ x_2' &= -x_1 + 3x_2 \end{aligned}\right\} \qquad x' = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}}_{A} x$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \left[(1-\lambda)(3-\lambda) + 1\right] = 0$$
$$\lambda^2 - 4\lambda + 4 = 0 \Rightarrow (\lambda-2)^2 = 0 \Rightarrow \underline{\lambda = 2} \text{ (multiplicity of 2)}$$

Repeated eigenvalues

Step 2: For $\lambda = 2 \Rightarrow (A - 2I)v = 0$

$$\begin{bmatrix} 1-2 & 1 \\ -1 & 3-2 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$a_1 - a_2 = 0 \Rightarrow a_2 = a_1$

Let $a_1 = 1 \Rightarrow a_2 = 1$

So, $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\underline{x^{(1)}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}} \checkmark$$

Step 3: Look for the second linearly independent solution in the form of

$$x^{(2)}(t) = (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda t}$$

So, $x^{(2)}(t) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \vec{v}_2\right) e^{2t} \qquad \vec{v}_2 = ?$

$\Rightarrow$

$$\left.\begin{aligned} &(A - 2I)\vec{v}_2 = \vec{v}_1 \\ &\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}\right\} \rightarrow \left[\begin{array}{cc|c} -1 & 1 & 1 \\ -1 & 1 & 1 \end{array}\right] \xrightarrow{(-1)R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} -1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

$-b_1 + b_2 = 1 \Rightarrow b_2 = 1 + b_1$

Let $b_1 = 0 \Rightarrow b_2 = 1$

So, $\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

So, $x^{(2)}(t) = (\vec{V}_1 t + \vec{V}_2) e^{\lambda t}$

$$x^{(2)}(t) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t}$$

Step 4: General solution is

$$\underset{\sim}{x}(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t}$$

$$\underset{\sim}{x}(t) = \begin{bmatrix} (c_1 + c_2 t) e^{2t} \\ (c_1 + c_2 t + c_2) e^{2t} \end{bmatrix}$$

Here;

$$x_1(t) = (c_1 + c_2 t) e^{2t}$$
$$x_2(t) = (c_1 + c_2 t + c_2) e^{2t}.$$

Step 5: Apply I.C:

$$\left.\begin{aligned} x_1(0) = 1 &\Rightarrow c_1 = 1 \\ x_2(0) = 2 &\Rightarrow c_1 + c_2 = 2 \end{aligned}\right\} \Rightarrow c_2 = 1$$

So,

$$x_1(t) = (1 + t) e^{2t}$$
$$x_2(t) = (2 + t) e^{2t}$$

Examples

Methods of Undetermined Coefficients

Set up the appropriate form of a particular solution y_p. Do not determine the coefficients.

1) $y'' - 2y' + 2y = e^x \sin x$

SOL: $r^2 - 2r + 2 = 0 \Rightarrow r_{1,2} = \frac{2 \pm \sqrt{4 - 4.2}}{2.1} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$

$r_{1,2} = 1 \pm i \in \mathbb{C}$

$\rightarrow y_c(x) = e^x (C_1 \cos x + C_2 \sin x)$

$\rightarrow y_p(x) = [e^x (A \cos x + B \sin x)]\, x$

2) $y^{(5)} - y^{(3)} = e^x + 2x^2 - 5$

SOL: $r^5 - r^3 = 0 \Rightarrow r^3(r^2 - 1) = 0$; $r_1 = r_2 = r_3 = 0$; $r^2 = 1 \Rightarrow r_4 = 1$, $r_5 = -1$

$\rightarrow y_c(x) = C_1 e^{0.x} + C_2 . x e^{0x} + C_3 x^2 e^{0x} + C_4 e^x + C_5 e^{-x}$

$= C_1 + C_2 x + C_3 x^2 + C_4 e^x + C_5 e^{-x}$

$\rightarrow f_1(x) = e^x \longrightarrow y_{p_1}(x) = [A e^x]\, x$

$f_2(x) = 2x^2 - 5 \longrightarrow y_{p_2}(x) = [Bx^2 + Cx + D]\, x^3$

$y_p(x) = x^3 (Bx^2 + Cx + D) + x\,[A e^x]$

3) $y'' + 4y = 3x \cos 2x$

SOL: $\Rightarrow y'' + 4y = 0 \Rightarrow r^2 + 4 = 0 \Rightarrow r^2 = -4 \Rightarrow r_{1,2} = 0 \pm 2i \in \mathbb{C}$

$y_c(x) = e^{0x}(C_1 \cos 2x + C_2 \sin 2x)$

$= C_1 \cos 2x + C_2 \sin 2x$

$\Rightarrow y_p(x) = [(Ax + B) \cos 2x + (Cx + D) \sin 2x] . x$

4) $y^{(3)} - y'' - 12y' = x - 2x e^{-3x}$

SOL: $\rightarrow y^{(3)} - y'' - 12y' = 0 \Rightarrow r^3 - r^2 - 12r = 0 \Rightarrow r(r^2 - r - 12) = 0$; $r_1 = 0$; $r^2 - r - 12 = 0 \Rightarrow (r-4)(r+3) = 0$; -4 3 ; $r_2 = 4$; $r_3 = -3$

$y_c(x) = C_1 e^{0x} + C_2 e^{4x} + C_3 e^{-3x}$

$= C_1 + C_2 e^{4x} + C_3 e^{-3x}$

$\rightarrow y_p(x) = (Ax + B)\, x + [(Cx + D) e^{-3x}] . x$

5) $y''+3y'+2y = x(e^{-x}-e^{-2x})$

SOL: $* \; r^2+3r+2=0 \Rightarrow (r+1)(r+2)=0 \Rightarrow r_1=-1 \Rightarrow y_1(x)=e^{-x}$; $r_2=-2 \Rightarrow y_2(x)=e^{-2x}$

$$y_c(x)=c_1e^{-x}+c_2e^{-2x}$$

$$\Rightarrow y_p(x)=x(Ax+B)e^{-x}+(Cx+D)e^{-2x}\cdot x$$

6) $y''-6y'+13y = xe^{3x}\sin 2x$

SOL: $r^2-6r+13=0 \Rightarrow r_{1,2}=\frac{+6\pm\sqrt{36-4\cdot 13}}{2}=\frac{6\pm\sqrt{-16}}{2}=\frac{6\pm 4i}{2}=3\pm 2i \in \mathbb{C}$

$$y_c(x)=e^{3x}(c_1\cos 2x+c_2\sin 2x)$$

$$\Rightarrow y_p(x)=\left[(Ax+B)e^{3x}\cos 2x+(Cx+D)e^{3x}\sin 2x\right]x$$

7) $y^{(4)}+5y''+4y=\sin x+\cos 2x$

SOL: $r^4+5r^2+4=0 \Rightarrow \left(\text{Let } r^2=u\right) \Rightarrow u^2+5u+4=0 \Rightarrow (u+1)(u+4)=0$

$(r^2+1)(r^2+4)=0$; $r^2+1=0 \Rightarrow r_{1,2}=$; $r^2=-4 \Rightarrow r_{3,4}=$

$$y_c(x)=e^{0x}(c_1\cos x+c_2\sin x)+e^{0x}(c_3\cos 2x+c_4\sin 2x)$$

$$=c_1\cos x+c_2\sin x+c_3\cos 2x+c_4\sin 2x$$

$$\Rightarrow y_p(x)=(A\sin x+B\cos x)x+(C\sin 2x+D\cos 2x)x$$

8) $y^{(4)}+9y''=(x^2+1)\sin 3x$

SOL: $r^4+9r^2=0 \Rightarrow r^2(r^2+9)=0$; $r^2=0 \Rightarrow r_1=r_2=0$; $r^2+9=0 \Rightarrow r^2=-9 \Rightarrow r_{3,4}=\pm 3i$

$$y_c(x)=c_1+c_2x+c_3\cos 3x+c_4\sin 3x$$

$$y_p(x)=\left[(Ax^2+Bx+C)\cos 3x+(Dx^2+Ex+F)\sin 3x\right]x$$

9) $y^{(4)}-2y''+y=x^2\cos x$

SOL: $\cdot)\; r^4-2r^2+1=0 \Rightarrow \left(\text{let } r^2=u\right) \Rightarrow u^2-2u+1=0 \Rightarrow (u-1)^2=0 \Rightarrow (r^2-1)^2=0 \; (r^2-1)(r^2-1)=0$

$r^2=1 \Rightarrow r=\pm 1$; $r^2=1 \Rightarrow r=\pm 1$

$\therefore r_1=r_2=1$; $r_3=r_4=-1 \Rightarrow y_c(x)=(c_1+c_2x)e^{x}+(c_3+c_4x)e^{-x}$

$$\Rightarrow y_p(x)=(Ax^2+Bx+C)\cos x+(Dx^2+Ex+F)\sin x$$

Ex/ Find general solution of the given equation.
In all these examples, primes denote derivatives with respect to x.
Determine the coefficients.

1) $y''+16y=e^{3x}$

SOL: $\Rightarrow y''+16y=0 \Rightarrow r^2+16=0 \Rightarrow r^2=-16 \Rightarrow r_{1,2}=\pm 4i$

$y_c(x) = C_1\cos 4x + C_2 \sin 4x$

$\Rightarrow y_p(x)=Ae^{3x} \Rightarrow \left.\begin{array}{l} y_p'=3Ae^{3x} \\ y_p''=9Ae^{3x} \end{array}\right\} \left.\begin{array}{l} 9Ae^{3x}+16Ae^{3x}=e^{3x} \\ 25A=1 \Rightarrow A=\frac{1}{25} \end{array}\right\}$ So; $y_p(x)=\frac{1}{25}e^{3x}$

General solution : $y_G(x)=y_c(x)+y_p(x)$

$= C_1\cos 4x+C_2\sin 4x+\frac{1}{25}e^{3x}$ ///

2) $y''-y'-2y=3x+4$

SOL:) $r^2-r-2=0 \Rightarrow (r-2)(r+1)=0$ $\begin{array}{l} \rightarrow r_1=2 \Rightarrow y_1(x)=e^{2x} \\ \rightarrow r_2=-1 \Rightarrow y_2(x)=e^{-x} \end{array}\Bigg\}$ $y_c(x)=C_1e^{2x}+C_2e^{-x}$.

$-2\;+1$

-) $y_p(x)=Ax+B \Rightarrow \left.\begin{array}{l} y_p'=A \\ y_p''=0 \end{array}\right\} \longrightarrow -A-2(Ax+B)=3x+4$

$-2Ax+(-A-2B)=3x+4$

$\left.\begin{array}{l} -2A=3 \Rightarrow A=-\frac{3}{2} \\ -A-2B=4 \Rightarrow B=-\frac{5}{4} \end{array}\right\}$ $y_p(x)=-\frac{3}{2}x-\frac{5}{4}$

$y_G(x)=C_1e^{2x}+C_2e^{-x}-\frac{3}{2}x-\frac{5}{4}$ ///

3) $y''+2y'+y=\sin^2 x$

SOL: $\Rightarrow r^2+2r+1=0 \Rightarrow (r+1)^2=0 \Rightarrow r_1=r_2=-1 \Rightarrow y_c(x)=C_1e^{-x}+C_2xe^{-x}$

$\Rightarrow f(x)=\sin^2 x = \frac{1-\cos 2x}{2}$

$y_p(x)= A+B\cos 2x+C\sin 2x \Rightarrow y_p'=-2B\sin 2x+2C\cos 2x$

$y_p''=-4B\cos 2x-4C\sin 2x$

$-4B\cos 2x-4C\sin 2x+2(-2B\sin 2x+2C\cos 2x)+(A+B\cos 2x+C\sin 2x)=\frac{1-\cos 2x}{2}$

$-4B-4C+B=-\frac{1}{2}$; [illegible] : $\boxed{A=\frac{1}{2}}$; $\boxed{B=\frac{3}{[illegible]}}$; $\boxed{C=[illegible]}$

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So;

$$y_p(x) = \frac{1}{2} + \frac{3}{14} \cos 2x - \frac{1}{28} \sin 2x$$

$$y_G(x) = y_c(x) + y_p(x) = \dots$$