Chapter 2
Advanced Concepts

Abstract In this chapter we introduce elements of graph theory, graphs of components, matrix formulation of Kirchhoff's laws, matrix associated spaces, and Tellegen's theorem.

2.1 Basic Elements of Graph Theory

A set of independent Kirchhoff's laws for a given circuit – the so-called *topological equations* – can be automatically found by relying on some concepts of *graph theory*, that is, the study of mathematical/geometrical structures, called *graphs*, used to model pairwise relations between objects. Graph theory almost certainly began when, in 1735, Leonhard Euler[1] solved a popular puzzle about the bridges of the East Prussian city of Königsberg (now Kaliningrad) [1]. Nowadays, graph theory is largely used in mathematics, computer science, and network science, but it can be applied in any context where many units interact in some way, such as the components in a circuit. Usually, a graph completely neglects the nature of each unit and of the interactions, just keeping information about their existence.

A **graph** is a finite set of N *nodes* (or vertices or points), together with a set of L *edges* (or branches or arcs or lines), each of them connecting a pair of distinct nodes.

We remark that more than one edge can connect the same pair of nodes. In this case, these edges are said to be *in parallel*. This implies that a pair of nodes can be insufficient to identify an edge univocally. Moreover, the above definition excludes the degenerate case of edges connecting one node to itself. Henceforth, we label the nodes with numbers and the edges with letters/symbols.

[1] Leonhard Euler (1707–1783) was a Swiss mathematician, physicist, astronomer, logician, and engineer who made important and influential discoveries in many branches of mathematics.

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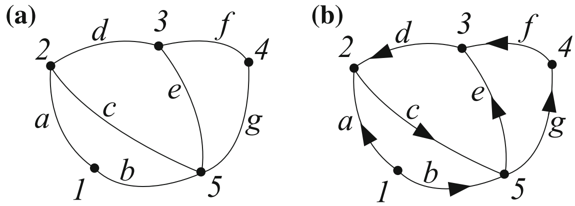


Fig. 2.1 Example of undirected (**a**) and directed (**b**) graph with $N = 5$ and $L = 7$

In the simplest case, the edges are not oriented: in this case we have an *undirected graph*, an example of which is shown in Fig. 2.1a. If the edges are oriented, they are called arrows (or directed edges or directed arcs or directed lines) and we have a *directed graph* or *digraph*. (See Fig. 2.1b.)

Order of a node: Number of edges connecting this node to other nodes.

For instance, in the figure node 1 has order 2, node 3 order 3, and node 5 order 4. The specific shape of a graph is not relevant, according to the following definition.

Two (directed) graphs G_1 and G_2 are **isomorphic** if it is possible to establish a bijective correspondence between:

- Each node of G_1 and each node of G_2
- Each edge of G_1 and each edge of G_2

such that corresponding edges connect (ordered) pairs of corresponding nodes.

Three examples of graphs isomorphic to the one of Fig. 2.1b are shown in Fig. 2.2. For ease of comparison, we used the same labels for nodes and edges as in Fig. 2.1b; in this case, the graph is not only isomorphic, but is essentially the same. Any change in the labels would not affect the equivalences. The graphs shown in Fig. 2.3 are in turn isomorphic to the one of Fig. 2.1b. Some of the correspondences are summarized in Table 2.1. You can check your comprehension by finding the missing correspondences.

Planar graph: A graph that can be embedded in the plane; that is, it can be drawn on the plane in such a way that all its edges intersect only at their endpoints.

In other words, any planar graph admits an isomorphic graph where no edges cross each other. Some examples of planar graphs are shown in Fig. 2.4.

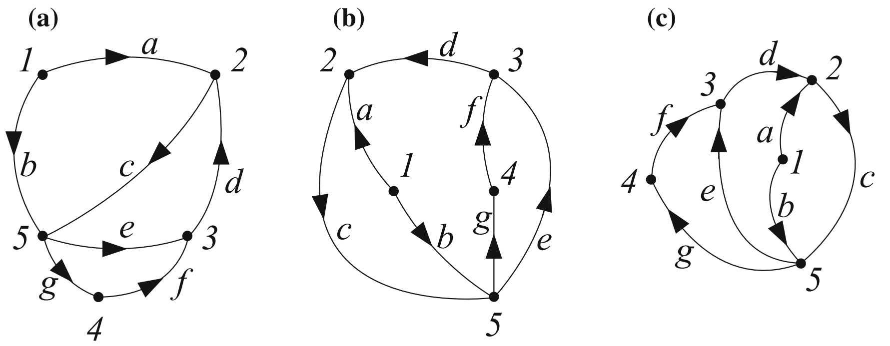


Fig. 2.2 Examples of isomorphic graphs (to be compared to Fig. 2.1b)

Fig. 2.3 Further examples of isomorphic graphs (to be compared to Fig. 2.1b)

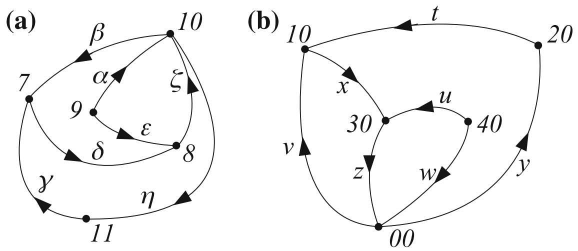


Table 2.1 Table of correspondences between elements of the isomorphic graphs of Figs. 2.1b, 2.3a, and 2.3b

	Graph		
	Fig. 2.1b	Fig. 2.3a	Fig. 2.3b
Graph element	a	ε	u
	b	α	w
	c	ζ	z
	d	δ	x
	1	9	40
	2	8	30
	3	7	10
	4	11	20
	5	10	00

Star graph: A graph containing $N-1$ nodes of order 1 and one node of order $N-1$.

Figure 2.5 shows an example of a star graph with 5 nodes: node 5 has order 4; the other nodes have order 1.

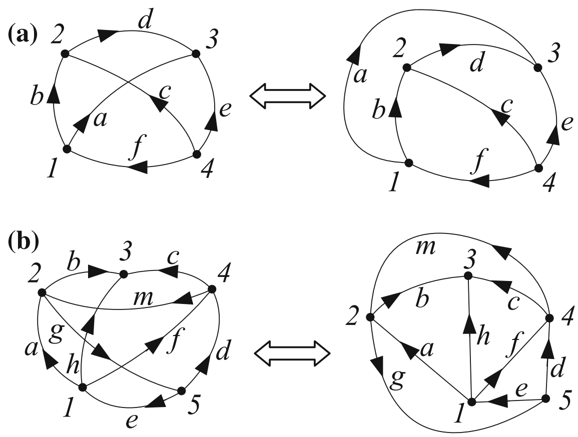


Fig. 2.4 Examples of planar graphs

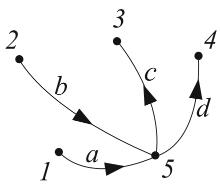


Fig. 2.5 Example of a star graph

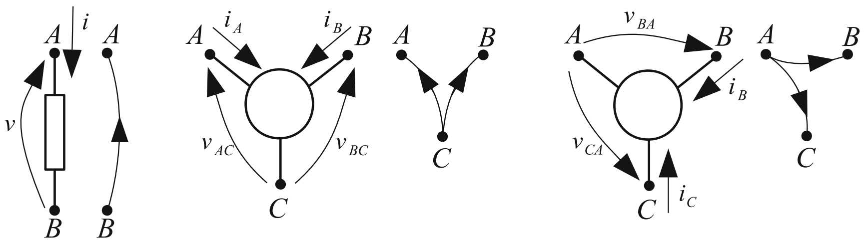


Fig. 2.6 Examples of graphs for multiterminal components

2.1.1 Graphs of Components and Circuits

For a circuit, it is quite natural (even if this is not the only possible choice) to associate the circuit nodes with graph nodes and the descriptive voltages with graph arrows. By assuming the standard choice, this means that each graph arrow is associated with a voltage oriented like the arrow and to a current oriented in the opposite direction.

Figure 2.6 shows some examples of graphs for multiterminal components.

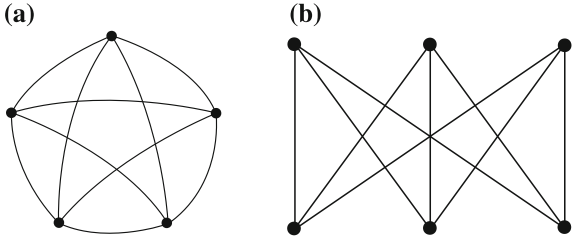


Fig. 2.7 Kuratowski graphs

By substituting each circuit component with its graph, we obtain the circuit graph. For instance, the directed graph shown in Fig. 2.1b corresponds to the circuit of Fig. 1.8.

2.1.2 Subgraph, Path, Loop, and Cut-Set

In this section we define some basic graph structures.

Subgraph: A subset of the elements of a given graph, obtained by removing some edges and/or some nodes together with the corresponding edges.

A subgraph is in turn a graph. For instance, by removing edges a, d, f from the graph of Fig. 2.1a, we obtain a subgraph which is in turn a star graph.

It has been shown [2] that a graph is nonplanar if and only if it is (or contains a subgraph) a graph isomorphic to the ones shown in Fig. 2.7, independently of the edge orientations.

Path: A subgraph made up of a sequence of $k-1$ adjacent edges (the orientation is not relevant) connecting a sequence of k nodes that, by most definitions, are all distinct from one another.

In other words, a nondegenerate path is a trail in which all nodes and all edges are distinct and then we have 2 nodes of order 1 (the first and the last) and $k-2$ nodes of order 2. Figure 2.8 shows some examples of paths (in grey) for the reference graph of Fig. 2.1b.

A graph is **connected** when there is a path between every pair of nodes. Otherwise it is **disconnected**.

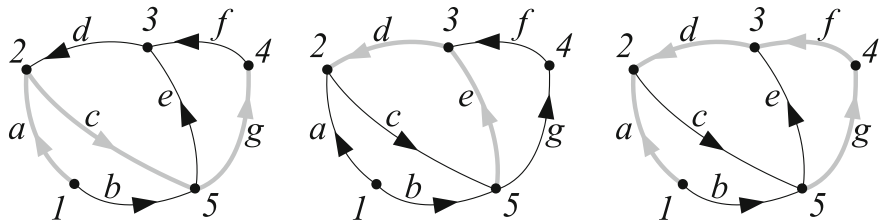
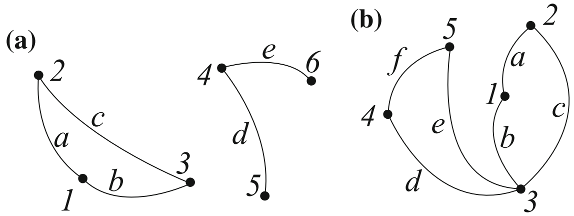


Fig. 2.8 Examples of paths (in *grey*)

Fig. 2.9 Example of a disconnected graph (**a**) and hinged graph (**b**)



An example of a disconnected graph is shown in Fig. 2.9a.

A connected graph is **hinged** when it can be partitioned into two subgraphs connected by one node, called a *hinge*.

An example of a hinged graph is shown in Fig. 2.9b, where the hinge is node 3.

Loop: A subgraph containing only nodes of order 2, or a degenerate path where the first and last nodes are also of order 2, connected by an edge.

Figure 2.10 shows some examples of loops for the reference graph of Fig. 2.1b.

Mesh: A loop of a planar graph not containing any graph elements either inside (**inner loop**) or outside (**outer loop**).

Figure 2.11 shows some examples of meshes for the reference graph of Fig. 2.1b.

Cut-set: A set of edges of a graph which, when removed, make the graph disconnected.

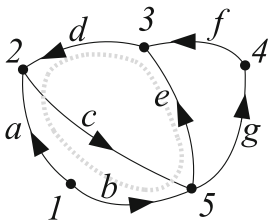


Fig. 2.10 Examples of loops (in *grey*)

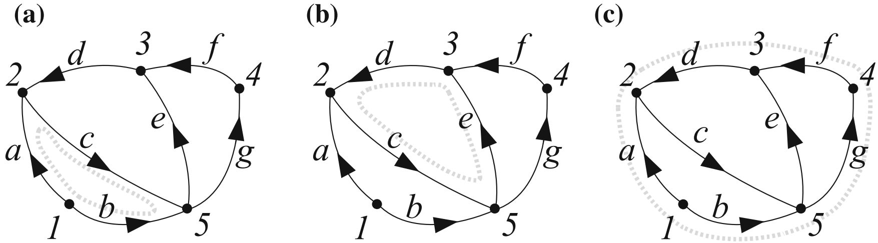


Fig. 2.11 Examples of meshes (in *grey*): inner loops (**a** and **b**) and outer loop (**c**)

As stated in Sect. 1.5.2, a cut-set can be easily associated with a closed path (or surface, for nonplanar graphs) crossing the cut-set edges. Actually, for each cut-set there are two possible closed paths, as shown in the examples of Fig. 2.12 for the reference graph of Fig. 2.1b.

Nodal cut-set: A cut-set such that one of the two disconnected parts of the resulting graph is a single node.

Figure 2.13 shows some examples of nodal cut-sets for the reference graph of Fig. 2.1b.

2.1.3 Tree and Cotree

We now define the two basic graph structures used to find matrix formulations of Kirchhoff's laws.

Tree: A subgraph containing all the N nodes and $N-1$ edges of a given graph and in which any two nodes are connected by exactly one path.

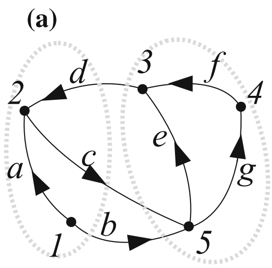


Fig. 2.12 Examples of cut-sets (corresponding to the *grey dashed* closed paths)

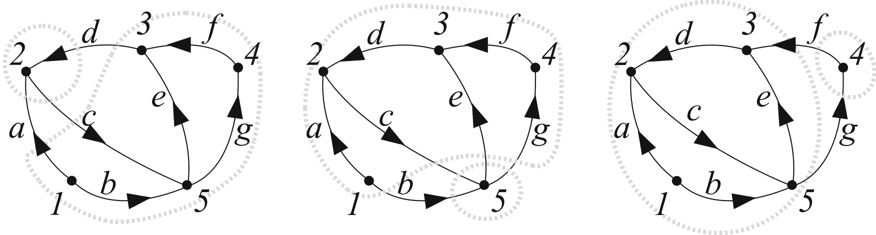


Fig. 2.13 Examples of nodal cut-sets (corresponding to the *grey dashed* closed paths)

Owing to this definition, a tree cannot contain any loop.

Cotree: A subgraph associated with a tree, containing all the N nodes and the $L - N + 1$ edges of the graph not contained in the tree.

Figure 2.14 shows some examples of trees and cotrees for the reference graph of Fig. 2.1b.

2.2 Matrix Formulation of Kirchhoff's Laws

As stated at the beginning of Sect. 2.1, these basic elements of graph theory can be used to formulate in a compact way (i.e., in matrix form) a set of independent Kirchhoff's laws for a given circuit. The goal is to find a complete[2] set of independent KVLs and KCLs, which are related to corresponding sets of independent loops and cut-sets, respectively. A set of independent loops (cut-sets) is also called a *basis of fundamental loops (cut-sets)*.

[2]The set is complete if any further KVL or KCL equation is linearly dependent on the equations belonging to the set.

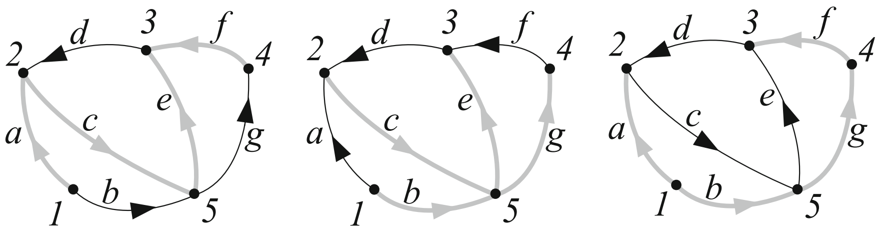
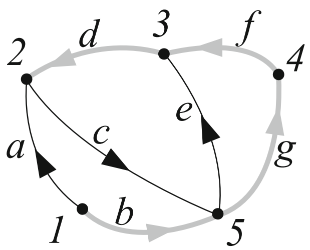


Fig. 2.14 Examples of trees (*thick grey edges*) and corresponding cotrees (*thin black edges*) for the reference graph of Fig. 2.1b

Fig. 2.15 Tree (*thick grey edges*) and cotree (*black edges*) for the reference graph of Fig. 2.1b



For planar graphs, the simplest choice for these bases is the set of $L - N + 1$ arbitrarily chosen meshes (which are independent loops) and the set of $N - 1$ arbitrarily chosen nodal cut-sets (which are independent cut-sets).

For generic graphs, a criterion to identify these bases refers to a tree and the corresponding cotree. In the following, we use the graph, tree, and cotree shown in Fig. 2.15. Moreover, henceforth I_q denotes the identity matrix of size q (i.e., the $q \times q$ square matrix with ones on the main diagonal and zeros elsewhere) and 0_q denotes the null column vector with q elements.

2.2.1 Fundamental Cut-Set Matrix

Each cut-set containing one and only one edge of the chosen tree is part of a basis of $(N - 1)$ fundamental cut-sets. Each fundamental cut-set is oriented (inwards/outwards) like the corresponding tree edge and is labeled as $\mathscr{C}_k$, where k denotes the edge. Figure 2.16 shows the basis of fundamental cut-sets for the considered example and the chosen tree.

Now, we can construct a matrix (of size $(N - 1) \times L$), called the *fundamental cut-set matrix*, where:

- Each row corresponds to exactly one fundamental cut-set (i.e., to the related tree edge).
- Each column corresponds to one graph edge. The columns are ordered as follows: first the cotree edges (ordered arbitrarily) and then the tree edges, in the same order

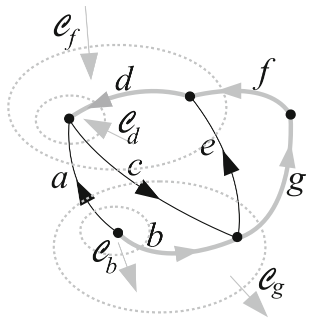


Fig. 2.16 Basis of fundamental cut-sets for the considered example

as for the rows. In the example, we follow the alphabetical order for both edge sets.

- Each matrix entry is set to:

0 If the edge on the column does not belong to the fundamental cut-set on the row
1 If the edge on the column belongs to the fundamental cut-set on the row and has the same orientation
−1 If the edge on the column belongs to the fundamental cut-set on the row and has the opposite orientation

In the considered example, the fundamental cut-set matrix is as follows.

$$A = \begin{array}{c} \\ b \\ d \\ f \\ g \end{array} \begin{array}{c} \begin{array}{ccccccc} a & c & e & b & d & f & g \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \end{array} = \left(\alpha | I_{N-1}\right) \tag{2.1}$$

We call i the column vector of descriptive currents associated with the oriented edges of the graph and ordered exactly as are the columns of the cut-set matrix A; that is, $i = (i_a\ i_c\ i_e\ i_b\ i_d\ i_f\ i_g)^T$. It is easy to check that the rows of A are linearly independent; that is, the rank of A is $N - 1$. This is a general property, due to the way the fundamental cut-set matrix is set up and to the fact that each row is related to one element of a basis of cut-sets.

Of course, the cut-set orientation depends on the choice of the corresponding closed path (as stated in Sect. 2.1.2), but the resulting matrix is invariant, as can be easily checked.

Property

The system of equations

$$Ai = 0_{N-1} \tag{2.2}$$

is **a set of** $N-1$ **independent KCLs for the circuit associated with the graph**, corresponding to the fundamental cut-sets related to the chosen tree.

For the circuit of Fig. 1.8 and for the choice of tree of Fig. 2.15, the set of independent KCLs is:

$$Ai = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_a \\ i_c \\ i_e \\ i_b \\ i_d \\ i_f \\ i_g \end{pmatrix} = \begin{pmatrix} i_a + i_b \\ i_a - i_c + i_d \\ i_a - i_c + i_e + i_f \\ i_a - i_c + i_e + i_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.3}$$

Each row of the submatrix α contains information about the composition of the cut-set which the row refers to: for example, the nonzero elements in the row d of α indicate that $\mathscr{C}_d$ contains (in addition to d) the edges a and c; similarly, the row f of α indicates that $\mathscr{C}_f$ contains, in addition to f, the edges a, c, e.

We observe in passing that something similar can be observed for the columns of α: for example, the nonzero elements of the column a indicate that b, d, f, g are the tree edges forming a loop with a; similarly, the nonzero elements of the column c indicate that the tree edges d, f, g form a loop with c. Therefore α also contains topological information about the loops. This fact has major consequences on the fundamental loop matrix structure, discussed soon.

2.2.1.1 A Particular Case

For the specific tree choice shown in Fig. 2.17, we obtain the basis composed by nodal cut-sets only. Notice that the tree in this case is a *star subgraph*.

For this choice of tree, writing the cut-set matrix A according to the general rules, the set of independent KCLs is as follows.

$$Ai = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} i_a \\ i_d \\ i_f \\ i_b \\ i_c \\ i_e \\ i_g \end{pmatrix} = \begin{pmatrix} i_a + i_b \\ -i_a - i_d + i_c \\ -i_d + i_f + i_e \\ -i_f + i_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.4}$$

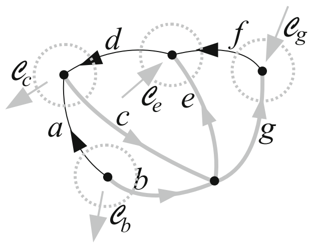


Fig. 2.17 Choice of tree (*thick grey edges*) corresponding to a basis of nodal cut-sets only (*dashed lines*)

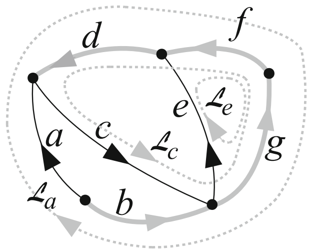


Fig. 2.18 Basis of fundamental loops for the considered example

This set of equations is completely equivalent to Eq. 2.3.

This matrix is strictly related to the so-called *incidence matrix*.

You can check your comprehension by obtaining Eq. 2.4 through linear combinations of Eq. 2.3.

2.2.2 *Fundamental Loop Matrix*

Each loop containing only one edge of a cotree is part of a basis of $(L - N + 1)$ fundamental loops. Each fundamental loop is oriented as is the corresponding cotree edge and is labeled as $\mathscr{L}_k$, where k denotes the cotree edge. Figure 2.18 shows the basis of fundamental loops for the considered example and the chosen tree.

Now, we can construct a matrix (of size $(L - N + 1) \times L$), called the *fundamental loop matrix*, where:

- Each row corresponds to exactly one fundamental loop (i.e., to the related cotree edge).
- Each column corresponds to one graph edge. The columns are ordered as in matrix A.
- Each matrix entry is set to:

0 If the edge on the column does not belong to the fundamental loop on the row

1 If the edge on the column belongs to the fundamental loop on the row and has the same orientation

-1 If the edge on the column belongs to the fundamental loop on the row and has the opposite orientation

In the considered example, the fundamental loop matrix is:

$$B = \begin{array}{c} \\ a \\ c \\ e \end{array}\!\!\begin{array}{c} \begin{array}{ccccccc} a & c & e & b & d & f & g \end{array} \\ \left(\begin{array}{ccc|cccc} 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{array}\right) \end{array} = \left(I_{L-N+1} | -\alpha^T\right) \tag{2.5}$$

We call v the column vector of descriptive voltages associated with the oriented edges of the graph and ordered exactly as are the columns of the loop matrix B; that is, $v = (v_a\ v_c\ v_e\ v_b\ v_d\ v_f\ v_g)^T$. It is easy to check that the rows of B are linearly independent; this is a general property, due to the way the fundamental loop matrix is set up and to the fact that each row is related to one element of a basis of loops. For this reason, the rank of B is $L - N + 1$. When, as in this case, the ordering of the tree edges is the same for the matrices A and B, the elements of A and B are related very simply: the matrix part complementary to the identity submatrix is α in A and $-\alpha^T$ in B. This follows from the previously observed property concerning the columns of α; that is, for any column j of α, the tree edges i with $\alpha_{ij} \neq 0$ are the constituents of the loop $\mathscr{L}_j$.

Property

The system of equations

$$Bv = 0_{L-N+1} \tag{2.6}$$

is **a set of** $L - N + 1$ **independent KVLs for the circuit associated with the graph**, corresponding to the fundamental loops related to the chosen tree.

For the circuit of Fig. 1.8 and for the choice of tree of Fig. 2.15, the set of independent KVLs is as follows.

$$Bv = \begin{pmatrix} 1 & 0 & 0 & -1 & -1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} v_a \\ v_c \\ v_e \\ v_b \\ v_d \\ v_f \\ v_g \end{pmatrix} = \begin{pmatrix} v_a - v_b - v_d - v_f - v_g \\ v_c + v_d + v_f + v_g \\ v_e - v_f - v_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.7}$$

2.2.2.1 A Particular Case

For the star tree shown in Fig. 2.19, we obtain the basis composed by all the inner loops.

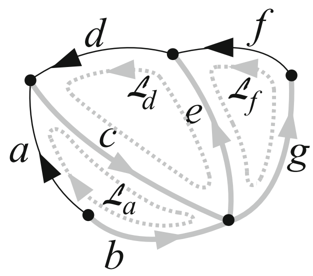


Fig. 2.19 Choice of tree (*thick grey edges*) corresponding to the basis of all the inner loops (*dashed loops*)

For this choice of tree, the set of independent KVLs is:

$$Bv = \begin{pmatrix} 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} v_a \\ v_d \\ v_f \\ v_b \\ v_c \\ v_e \\ v_g \end{pmatrix} = \begin{pmatrix} v_a - v_b + v_c \\ v_d + v_c + v_e \\ v_f - v_e + v_g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.8}$$

This set of equations is completely equivalent to Eq. 2.7. You can check your comprehension by obtaining Eq. 2.8 through linear combinations of Eq. 2.7.

2.2.3 Some General Concepts on Vector Spaces and Matrices

A *vector space* $\mathscr{V}$ is a nonempty set of vectors such that, for any two vectors x_1 and x_2 of $\mathscr{V}$, any of their linear combinations $\beta_1 x_1 + \beta_2 x_2$ ($\beta_1, \beta_2 \in \mathbb{R}$) is still an element of $\mathscr{V}$. The null element 0 is always a vector of $\mathscr{V}$.

The *dimension* of $\mathscr{V}$, denoted as $\dim(\mathscr{V})$, is the maximum number of linearly independent vectors in $\mathscr{V}$ and must not be confused with the number of components of the elements of $\mathscr{V}$.

A set of linearly independent vectors in $\mathscr{V}$ consisting of $\dim(\mathscr{V})$ vectors is called a *basis* for $\mathscr{V}$.

Given p vectors $x_1, \ldots, x_p$ with the same number of components, the set of all linear combinations $\sum_{i=1}^{p} \beta_i x_i$ is a vector space called the *span* of these vectors. For instance, the vector space $\mathscr{V}$ is the span of $\dim(\mathscr{V})$ linearly independent vectors. The span of a number of linearly independent vectors lower than $\dim(\mathscr{V})$ generates a *subspace* $\mathscr{L}$ of $\mathscr{V}$. For instance, Fig. 2.20 shows an example for $\mathscr{V} \equiv \mathbb{R}^3$.

The vectors x_1 and x_2 (as well as all their linear combinations $\beta_1 x_1 + \beta_2 x_2$, with $\beta_1, \beta_2 \in \mathbb{R}$) lie in a plane $\mathscr{L}$, which is a two-dimensional subspace of $\mathbb{R}^3$ passing through the origin.

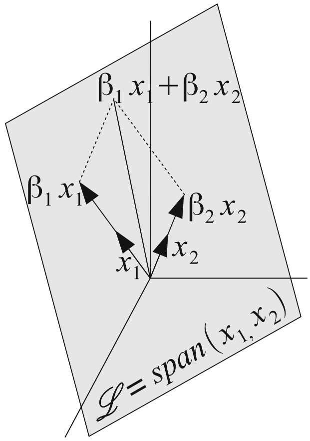


Fig. 2.20 A two-dimensional subspace $\mathscr{L}$ in $\mathbb{R}^3$

As stated above, it is important not to confuse the dimension of the vector space (or subspace) with the number of components (the size) of its individual vectors, because they are not necessarily the same. In the considered example, for instance, the vectors x_1 and x_2 have three components, despite their belonging to the two-dimensional subspace $\mathscr{L}$.

In the following, we introduce some specific spaces and subspaces associated with a matrix [3, 4], in order to provide (in the next section) a geometrical interpretation of the matrix formulation of Kirchhoff's laws, thus settling the basis for introducing Tellegen's theorem.

Let us consider a matrix $Q \in \mathbb{R}^{m \times n}$. We can write Q in terms of its columns as $Q = (q_1 \dots q_n)$. Let x denote any vector in $\mathbb{R}^n$. The vector space

$$\mathscr{R}(Q) = \left\{ y \in \mathbb{R}^m : \; y = Qx, \; x \in \mathbb{R}^n \right\}$$

is called the *range* of Q. We can also write, in terms of the column vectors q_i,

$$\mathscr{R}(Q) = span\,(q_1, \dots, q_n)\,.$$

In the general case, the linearly independent columns of Q can be a subset of $\{q_1, q_2, \dots, q_n\}$. It can be shown that the maximum number of linearly independent columns of Q and the maximum number of its linearly independent rows are equal. This common value r is the *rank* of Q. Then $rank(Q) = rank(Q^T) = r \leq \min(m, n)$ and $\dim(\mathscr{R}(Q)) = r$.

The set of all solutions to the homogeneous system $Qz = 0$,

$$\mathscr{N}(Q) = \left\{ z \in \mathbb{R}^n : \; Qz = 0 \right\}$$

is called the *null space* of Q (or *kernel* of Q).

In the same way we can define the vector spaces associated with the transpose of Q: $\mathscr{R}(Q^T)$, $\mathscr{N}(Q^T)$.

Two m-size vectors $w_R \in \mathscr{R}(Q)$ and $w_0 \in \mathscr{N}(Q^T)$ are always orthogonal; owing to the definition of $\mathscr{R}(Q)$, there must exist a vector $\bar{x}$ such that $w_R = Q\bar{x}$, thus $w_R^T w_0 = (Q\bar{x})^T w_0 = \bar{x}^T \underbrace{Q^T w_0}_{0} = 0$. An analogous result holds for two n-size vectors $x_R \in \mathscr{R}(Q^T)$ and $x_0 \in \mathscr{N}(Q)$.

These spaces are the main ingredients of two important results concerning the decomposition of vectors:

1 Any vector $x \in \mathbb{R}^n$, the domain space of Q, can be uniquely decomposed as $x = x_R + x_O$, where $x_O \in \mathscr{N}(Q)$ and $x_R \in \mathscr{R}(Q^T)$. Then $\mathscr{N}(Q)$ and $\mathscr{R}(Q^T)$ are complementary and disjoint ($\mathscr{N}(Q) \cap \mathscr{R}(Q^T) = \emptyset$, empty set) subspaces of $\mathbb{R}^n$; that is, $\mathbb{R}^n$ is given by the *direct sum* ($\oplus$) of the two subspaces:

$$\mathbb{R}^n = \mathscr{N}(Q) \oplus \mathscr{R}(Q^T) \ \text{ and } \ n = \dim(\mathscr{N}(Q)) + r.$$

The subspace $\mathscr{N}(Q)$ is an empty set if and only if $r = n$.

2 Any vector $w \in \mathbb{R}^m$, the codomain space of Q, can be uniquely decomposed as $w = w_R + w_O$, where $w_O \in \mathscr{N}(Q^T)$ and $w_R \in \mathscr{R}(Q)$. Then $\mathscr{N}(Q^T)$ and $\mathscr{R}(Q)$ are complementary and disjoint ($\mathscr{N}(Q^T) \cap \mathscr{R}(Q) = \emptyset$) subspaces of $\mathbb{R}^m$; that is, $\mathbb{R}^m$ is given by the direct sum of the two subspaces

$$\mathbb{R}^m = \mathscr{N}(Q^T) \oplus \mathscr{R}(Q) \ \text{ and } \ m = \dim(\mathscr{N}(Q^T)) + r.$$

The subspace $\mathscr{N}(Q^T)$ is an empty set if and only if $r = m$.

To exemplify the above concepts, let us consider the matrix

$$Q = \begin{array}{c} \begin{array}{cc} q_1 & q_2 \end{array} \\ \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \end{array}$$

which has $m = 3,\ n = 2$ and rank $r = 2$. Its column vectors $q_1,\ q_2$ define the plane $\mathscr{R}(Q)$:

$$\mathscr{R}(Q) = span(q_1, q_2) = Q\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \beta_1 q_1 + \beta_2 q_2; \qquad \beta_1, \beta_2 \in \mathbb{R}.$$

Taking as reference the orthogonal directions a_1, a_2, a_3, the vectors $q_1,\ q_2$ are shown in Fig. 2.21. The plane $\mathscr{R}(Q)$ intersects the $a_1 a_3$-plane along the line of q_1 and the $a_2 a_3$-plane along the line of q_2.

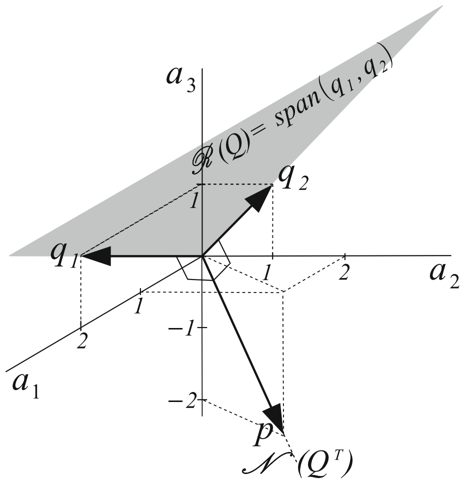


Fig. 2.21 The plane $\mathscr{R}(Q)$ and the line $\mathscr{N}(Q^T)$ associated with the (3×2) matrix Q

Because $\dim(\mathscr{N}(Q^T)) = m - r = 1$, the complementary subspace $\mathscr{N}(Q^T)$ is a straight line orthogonal to the plane $\mathscr{R}(Q)$. Denoting as $p = (p_1\ p_2\ p_3)^T$ a vector along this line, we have

$$Q^T p = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{cases} 2p_1 + p_3 = 0 \\ p_2 + p_3 = 0 \end{cases}$$

Then the components $p_1,\ p_2$ can be expressed in terms of p_3, that parameterizes the points of the subspace. The vector p plotted in the figure corresponds to $p_3 = -2$.

Finally, inasmuch as $r = n$, we have $\dim(\mathscr{N}(Q)) = 0$ (empty subspace) and $\mathscr{R}(Q^T) = \mathbb{R}^n$.

2.2.4 *The Cut-Set and Loop Matrices and Their Associate Space Vectors*

Consider a directed graph with L edges and N nodes. This graph can be arbitrarily partitioned into a tree and its cotree. Such a partition leads to the definition of a cut-set matrix A and a loop matrix B, as shown in Sects. 2.2.1 and 2.2.2. A current vector i and a voltage vector v, both of size L, are said to be *compatible* with the graph if they

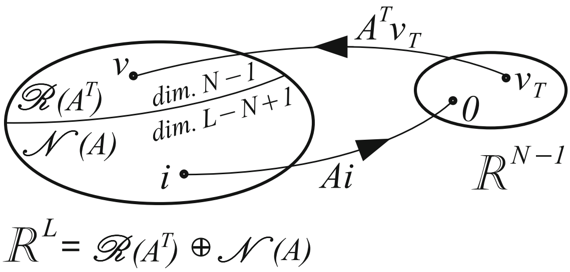


Fig. 2.22 Matrix A-relationships between spaces for compatible voltage and current vectors

satisfy the KCLs and KVLs, respectively, that is, if $Ai = 0$ and $Bv = 0$. Because the structure of A (where $m = N - 1$ and $n = L$) is $\left(\alpha | I_{N-1}\right)$, from $Ai = 0$ we have $N - 1$ independent scalar equations, which represent as many constraints on the L elements of the vector i. Therefore, due to KCLs, the number of degrees of freedom for the current elements of a vector i compatible with the graph is $L - N + 1$.

In terms of vector spaces, $Ai = 0$ means that i belongs to $\mathscr{N}(A)$, the null space of matrix A, whose dimension is $L - N + 1$.

Consider now the KVLs $Bv = 0$, with $B = \left(I_{L-N+1} | -\alpha^T\right)$ (where $m = L - N + 1$ and $n = L$). The vector v can be partitioned into two subvectors v_C and v_T, which contain the $L - N + 1$ voltages on the cotree edges and the $N - 1$ voltages on the tree edges, respectively:

$$v = \begin{pmatrix} v_C \\ v_T \end{pmatrix} \tag{2.9}$$

Owing to this partition, the KVLs $Bv = 0$ can be recast as $I_{L-N+1} v_C - \alpha^T v_T = 0$; that is, $v_C = \alpha^T v_T$. Then, we directly obtain:

$$v = \begin{pmatrix} v_C \\ v_T \end{pmatrix} = \begin{pmatrix} \alpha^T \\ I_{N-1} \end{pmatrix} v_T = A^T v_T. \tag{2.10}$$

It follows that each vector v of voltages compatible with the graph can be obtained through a product $A^T v_T$. This means that $v \in \mathscr{R}(A^T)$, whose dimension is $N - 1$. The values of the $N - 1$ components of the subvector v_T can be assigned independently, therefore the voltage elements of a compatible vector v can be chosen with $N - 1$ degrees of freedom, due to KVLs, which impose $L - N + 1$ constraints on the L components of v.

Figure 2.22 summarizes all these results and also highlights the roles of the matrices A and A^T as operators for the passage between the subspaces of $\mathbb{R}^L$ and the space $\mathbb{R}^{N-1}$.

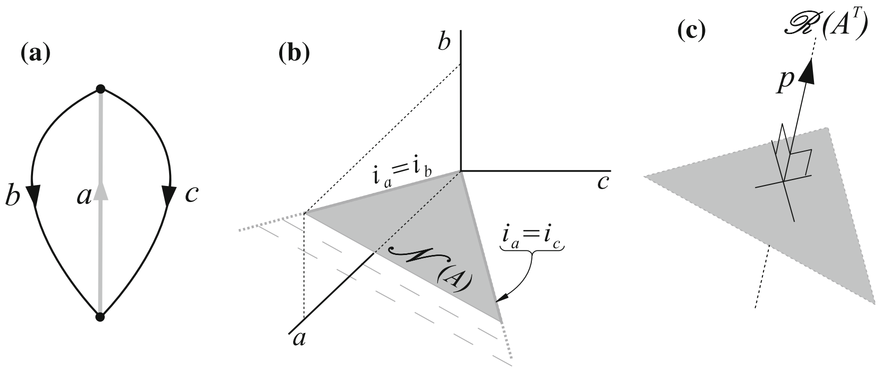


Fig. 2.23 Case Study: **a** graph; **b**, **c** spaces $\mathscr{N}(A)$ and $\mathscr{R}(A^T)$ for compatible current and voltage vectors

Case Study

Consider the very simple graph ($L = 3$, $N = 2$) shown in Fig. 2.23a. Taking the edge a as the (only) tree edge and the edges b, c as cotree edges, the fundamental cut-set matrix A is

$$A = \begin{array}{c c} & \begin{array}{ccc} b & c & a \end{array} \\ a & \left(\begin{array}{cc|c} -1 & -1 & 1 \end{array}\right) \end{array} \tag{2.11}$$

Therefore, KCL reduces to a single scalar equation:

$$i = \begin{pmatrix} i_b \\ i_c \\ i_a \end{pmatrix}; \quad Ai = -i_b - i_c + i_a = 0 \tag{2.12}$$

The three components i_b, i_c, i_a of any current vector i compatible with the graph must fulfill the KCL constraint $i_a = i_b + i_c$, which leads to the expression for the two-dimensional subspace $\mathscr{N}(A)$:

$$i = \begin{pmatrix} i_b \\ i_c \\ i_b + i_c \end{pmatrix} = i_b \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + i_c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \tag{2.13}$$

In the above expression, the values of $i_b,\ i_c$ play the role of span coefficients.

Denoting by b, c, a the orthogonal directions spanning the $\mathbb{R}^3$ space as shown in Fig. 2.23b, $\mathscr{N}(A)$ is the plane that intersects the plane $i_c = 0$ along the straight line $i_a = i_b$ and the plane $i_b = 0$ along the straight line $i_a = i_c$. All the vectors $i \in \mathbb{R}^3$ compatible with the graph lie on the plane $\mathscr{N}(A)$.

The voltage vector

$$v = \begin{pmatrix} v_b \\ v_c \\ v_a \end{pmatrix} \tag{2.14}$$

can be partitioned according to Eq. 2.9; in particular, we have $v_T = v_a$. With this in mind, and recalling Eq. 2.10, any vector v compatible with the graph can be obtained as

$$v = A^T v_T = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} v_a \tag{2.15}$$

or, in a more general formulation highlighting the parametric role of the term v_a, as

$$v = p\beta; \quad p = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}; \quad \beta \in \mathbb{R}. \tag{2.16}$$

Therefore, any vector v such that $Bv = 0$ is proportional to the vector p. It is easy to verify that p is orthogonal to any vector $i \in \mathcal{N}(A)$, as shown in Fig. 2.23c. The way to prove it is based on the observation that, being $i_a = i_b + i_c$, we can write i as $(i_b \; i_c \; (i_b + i_c))^T$ and then:

$$p^T i = (-1 \; -1 \; 1) \begin{pmatrix} i_b \\ i_c \\ i_b + i_c \end{pmatrix} = 0. \tag{2.17}$$

You can check the correspondence of these results with the general ones shown in Fig. 2.22.

In a similar fashion, denoting by i_C and i_T the subvectors containing, respectively, the $L - N + 1$ cotree currents and the $N - 1$ currents through the tree edges, we have

$$i = \begin{pmatrix} i_C \\ i_T \end{pmatrix} \tag{2.18}$$

which enables us to recast the KCLs $Ai = 0$ as $\alpha i_C + I_{N-1} i_T = 0$; that is, $i_T = -\alpha i_C$. With this in mind, we obtain

$$i = \begin{pmatrix} i_C \\ i_T \end{pmatrix} = \begin{pmatrix} I_{L-N+1} \\ -\alpha \end{pmatrix} i_C = B^T i_C. \tag{2.19}$$

Therefore, each current vector i compatible with the graph can be obtained through a product $B^T i_C$. This means that the current elements of any compatible vector i can be chosen with the $L-N+1$ degrees of freedom representing the size of the subvector

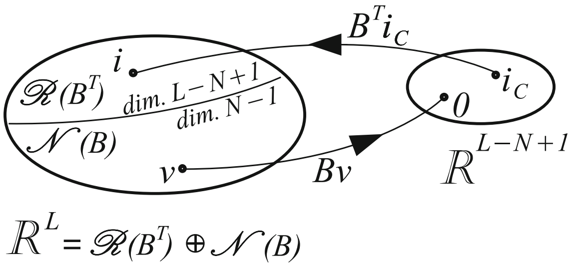


Fig. 2.24 Matrix B-relationships between spaces for compatible voltage and current vectors

i_C; moreover, $i \in \mathscr{R}(B^T)$. Because $Bv = 0$ means that v belongs to $\mathscr{N}(B)$, the space $\mathbb{R}^L$ can be thought of as partitioned into the two subspaces $\mathscr{R}(B^T)$ and $\mathscr{N}(B)$. This partition is shown in Fig. 2.24, which highlights the roles of the matrices B and B^T as operators for the passage between the subspaces of $\mathbb{R}^L$ and the space $\mathbb{R}^{L-N+1}$.

The properties presented in this section are the basis for Tellegen's theorem, which is treated in the next section.

2.3 Tellegen's Theorem

Theorem 2.1 (Tellegen's theorem) *In a directed graph, any compatible voltage vector v is orthogonal to any compatible current vector i.*

Proof To prove this, just consider that, thanks to the compatibility assumption, we have

$$v^T i = (A^T v_T)^T i = v_T^T \underbrace{Ai}_{0} = 0. \tag{2.20}$$

□

Tellegen's theorem is one of the most general theorems of circuit theory [5]. It depends only on Kirchhoff's laws and on the circuit's topology (graph), and it holds regardless of the physical nature of the circuit's components or the waveforms of voltages and currents, and so on. Therefore the voltages and currents that are used for Tellegen's theorem are not necessarily those actually present in a given circuit. By introducing specific assumptions about the physical properties of the components, waveforms and so on, Tellegen's theorem is the starting point to obtain, usually in a direct way, various specific and useful results. In the next chapters we show that for many circuit properties, the proof that can be given by relying on Tellegen's theorem is simpler than others and its range of validity is more clearly demonstrated.

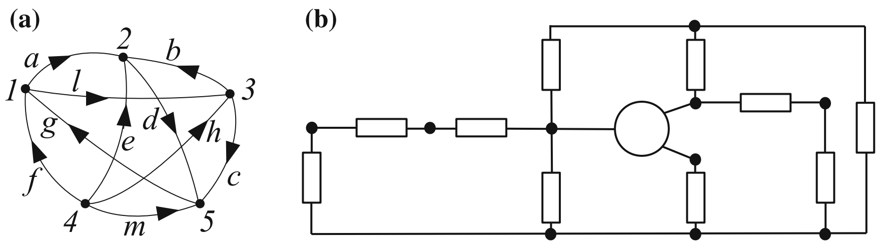


Fig. 2.25 Problems 2.1 (**a**) and 2.2 (**b**)

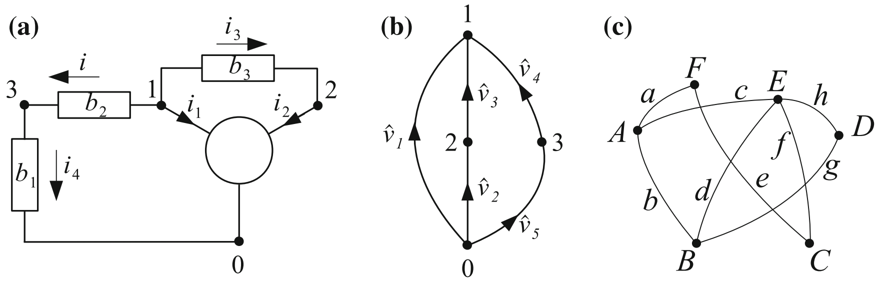


Fig. 2.26 Problems 2.3 (**a**, **b**) and 2.4 (**c**)

2.4 Problems

2.1 Choose a tree for the nonplanar graph shown in Fig. 2.25a and find the corresponding fundamental cut-set and loop matrices.

2.2 Determine the number of KCLs and KVLs necessary to solve the circuit shown in Fig. 2.25b. **Hint**: Consider the component connections to the lowest wire as a single node (dot).

2.3 Assume that you can measure the voltages of another circuit whose graph is shown in Fig. 2.26b. Is it possible to determine current i in Fig. 2.26a by measuring current i_3 in the same circuit? How?

2.4 Determine the number of fundamental loops, fundamental cut-sets, and tree edges for the graph shown in Fig. 2.26c.

References

1. Euler L (1741) Solutio Problematis ad Geometriam Situs Pertinentis. Commentarii Academiae Scientiarum Imperialis Petropolitanae 8:128–140
2. Kuratowski C (1930) Sur le problème des courbes gauches en topologie. Fund. Math. 15:271–283
3. Meyer CD (2000) Matrix analysis and applied linear algebra. SIAM, Philadelphia
4. Laub AJ (2005) Matrix analysis for scientists and engineers. SIAM, Philadelphia
5. Penfield P, Spence R, Duinker S (1970) Tellegen's theorem and electrical networks. MIT Press, Cambridge

Part II
Memoryless Multi-terminals: Descriptive Equations and Properties

Chapter 3
Basic Concepts

The important thing is not to stop questioning. Curiosity has its own reason for existing.

A. Einstein

Abstract In Chap. 1, we introduced circuit equations (KCL and KVL) related only to the way the circuit components are connected (the circuit *topology*). These equations are completely independent of the components' nature, that is, of their physical behavior. In this chapter we begin considering the specific physical role played by each component, which is described through the so-called *descriptive equation(s)*. Then, we introduce the descriptive equations of electrical and electronic two-terminal components (resistor, voltage and current sources, diode) and define general component characteristics and properties: implicit and explicit representations, linearity, time-invariance, memory, energy-based classification, and reciprocity. Some introductory examples of circuit analysis are proposed. Then the Thévenin and Norton equivalent representations of two-terminal resistive components are introduced. The main connections of two-terminal resistive elements conclude the chapter.

3.1 Solving a Circuit: Descriptive Versus Topological Equations

Consider a circuit made up of M components. Denoting by n_k the number of terminals of the kth component ($k = 1, \ldots, M$), the circuit has

$$L = \sum_{k=1}^{M} (n_k - 1) \tag{3.1}$$

descriptive voltages and L descriptive currents.

M. Parodi and M. Storace, *Linear and Nonlinear Circuits: Basic & Advanced Concepts*, Lecture Notes in Electrical Engineering 441,
DOI 10.1007/978-3-319-61234-8_3

The determination of these $2L$ descriptive variables requires $2L$ equations, which are of two types: *descriptive equations* and *topological equations*. Solving a circuit means using at most $2L$ equations to find the value (or the symbolic expression) of one (or more) of these variables.

The descriptive equations derive from the physical behavior of each component, that is, from its specific nature. An n_k-terminal component has $(n_k - 1)$ descriptive equations that, in general, involve all the $2(n_k - 1)$ descriptive variables of the component itself. For example, any two-terminal component has a single descriptive equation; a three-terminal component needs two descriptive equations, and so on. Denoting by v the vector of descriptive voltages: $v = \left(v_1 \cdots v_{n_k-1}\right)^T$ and by i the analogous vector of descriptive currents: $i = \left(i_1 \cdots i_{n_k-1}\right)^T$, the descriptive equations can be written in *implicit form* as

$$f_j(v, i) = 0 \quad (j = 1, \ldots, n_k - 1). \tag{3.2}$$

The term f_j represents a (scalar) algebraic function of its arguments, but in other cases it may contain derivatives or integrals with respect to time. In this chapter, we introduce only components described by algebraic f_j functions.

Because the number of descriptive equations for an n_k-terminal is $(n_k - 1)$, the total number of these equations in the circuit is L. Therefore, the number of necessary topological equations also amounts to L. If the circuit has N nodes and L edges, it has been shown in Sects. 2.2.2 and 2.2.1 that the KVLs provide $(L - N + 1)$ independent equations, whereas the KCLs provide the remaining $(N - 1)$ ones. Summing up, the $2L$ equations necessary to solve the circuit are generated according to the diagram of Fig. 3.1: *the necessary and sufficient information for the complete study of any circuit, for half originate from its topological structure (the way in which its components are connected) and for half by the components' physical characteristics*.

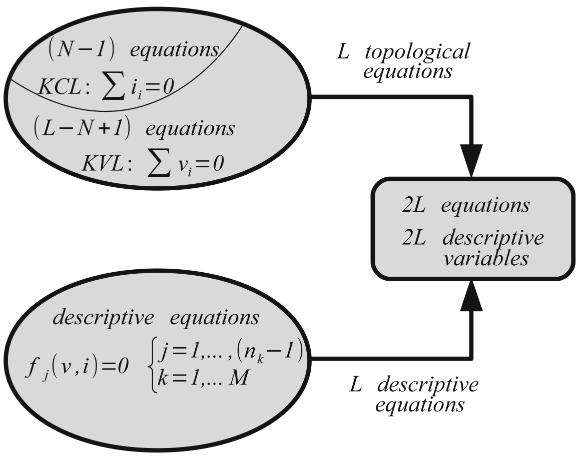


Fig. 3.1 The two types of circuit equations

3.2 Descriptive Equations of Some Components

Here we introduce the descriptive equations for some largely used components.

3.2.1 Resistor

The resistor is an electrical device (some examples are shown in Fig. 3.2a), whose two-terminal model is shown in Fig. 3.2b.

> The **resistor descriptive equation** – also called *Ohm's law* –[1] is
>
> $$v(t) = Ri(t) \tag{3.3}$$

R is a parameter called *(electrical) resistance*. The (derived) SI unit of measurement of resistance is the *ohm*, whose symbol is Ω. According to Ohm's law, it is evident that $[\Omega] = [\mathrm{V/A}]$.

The implicit form of Ohm's law is $f(i, v) = v - Ri = 0$. In a physical resistor, the resistance R is positive and its value depends on many factors, mainly temperature and age. The resistor model, instead, assumes that R is constant. Moreover, its value is usually positive, but in some cases it can also be negative, to model more complex physical devices, as we show later.

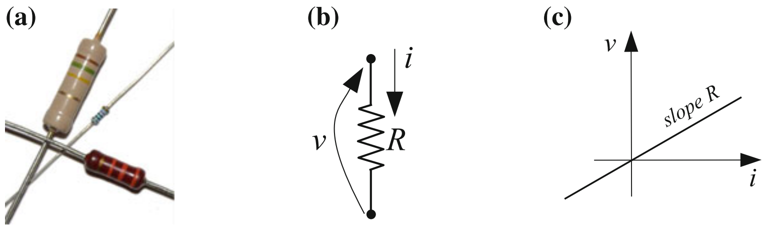


Fig. 3.2 Resistor: **a** three physical devices (the band colors code the resistance values); **b** model; **c** graphical representation of Ohm's law

[1] The law was named after the German physicist Georg Simon Ohm (1789–1854), who, in a treatise published in 1827, presented a slightly more complex equation than the one above to explain his experimental results about measurements of applied voltage and current through simple electrical circuits containing wires of various lengths. The above equation is the modern form of Ohm's law.

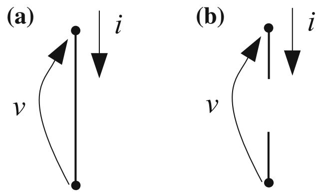


Fig. 3.3 Limit cases: **a** short circuit ($R = 0$); **b** open circuit ($G = 0$)

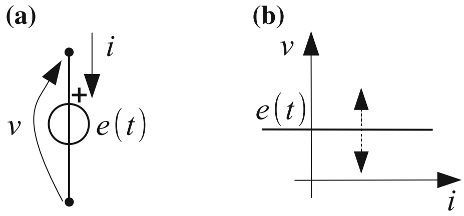


Fig. 3.4 Voltage source: **a** model; **b** graphical representation of the descriptive equation

The inverse of the resistance $G = 1/R$ is called the *conductance*. Its (derived) SI unit of measurement is the *siemens*, whose symbol is S.[2] Of course, [S] = [Ω^{-1}] = [A/V].

The descriptive equation is linear and homogeneous, therefore it is represented on the (i, v) plane by a straight line with slope R passing through the origin, as shown in Fig. 3.2c.

3.2.1.1 Limit Cases

For $R = 0$, the descriptive equation becomes $v(t) = 0$. In this case, the two-terminal is called a *short circuit* and is represented by the model shown in Fig. 3.3a. For $G = 0$ (i.e., $R \to \infty$), the descriptive equation becomes $i(t) = 0$. In this case, the two-terminal is called an *open circuit* and is represented by the model shown in Fig. 3.3b.

3.2.2 Ideal Voltage Source

The (ideal) voltage source (shown in Fig. 3.4a) is a two-terminal that imposes a prescribed voltage, say $e(t)$, between the two nodes.

[2]The unit is named after Ernst Werner von Siemens (1816–1892), a German inventor and industrialist.

The **voltage source equation** is

$$v(t) = e(t) \tag{3.4}$$

$e(t)$ is called the *impressed voltage* and is measured in volts.

The implicit form of the descriptive equation is $f(i, v) = v - e = 0$. We remark that current i does not appear in the equation: this means that its value depends on the rest of the circuit (and not that its value is 0A, as sometimes supposed by imaginative students).

The most common expressions for the impressed voltage are two:

- $e(t) = E$: In this case, the *direct current (DC) source*, the voltage is constant and the voltage source models a battery. Of course, a real battery impresses a constant voltage only for a limited period of time.
- $e(t) = E\cos(\omega t + \phi) = E\cos(2\pi F t + \phi)$: In this case, *alternating current (AC) source*, the voltage oscillates in time and the voltage source models an electric socket. Parameter E is a voltage amplitude (measured in volts), ϕ is an angle (expressed in radians), and F is a frequency (measured in s^{-1} or in hertz[3]) called *utility frequency*, *(power) line frequency*, or *mains frequency*. Worldwide, typical parameter values are $F = 50\,\text{Hz}$ (with amplitudes E ranging from 220 to 240 V) and $F = 60\,\text{Hz}$ (with amplitudes E ranging from 100 to 127 V).

Inasmuch as the descriptive equation is in general not homogeneous and not constant, it is represented on the (i, v) plane by a horizontal straight line moving vertically with time, as shown in Fig. 3.4b.

In the limit case $e(t) = 0$, the voltage source is turned off and is equivalent to a short circuit.

3.2.3 *Ideal Current Source*

The (ideal) current source (shown in Fig. 3.5a) is a two-terminal that generates a prescribed current, say $a(t)$.

The **current source equation** is

$$i(t) = a(t) \tag{3.5}$$

$a(t)$ is called the *impressed current* and is measured in amperes.

[3]The (derived) SI unit of measurement of frequency is the hertz (whose symbol is Hz), so named in honor of Heinrich Rudolf Hertz (1857–1894), the German physicist who first conclusively proved the existence of electromagnetic waves theorized by James Clerk Maxwell (1831–1879).

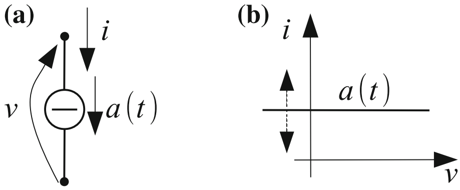


Fig. 3.5 Current source: **a** model; **b** graphical representation of the descriptive equation

The implicit form of the descriptive equation is $f(i, v) = i - a = 0$. We remark that voltage v does not appear in the equation: this means that its value depends on the rest of the circuit.

The most common expressions for the impressed current are two: $a(t) = A$ (DC source) and $a(t) = A\cos(\omega t + \phi) = A\cos(2\pi F t + \phi)$ (AC source).

Because the descriptive equation is in general not homogeneous and not constant, it is represented on the (v, i) plane by a horizontal straight line moving vertically with time, as shown in Fig. 3.5b.

In the limit case $a(t) = 0$, the current source is turned off and is equivalent to an open circuit.

3.2.4 Elementary Circuits

Here we provide five examples of the analysis of simple circuits. As a general remark (according to Ockham's razor), we always aim to solve the circuit by involving ***the minimum number of unknowns***, in addition to those requested by the problem. This allows us to reduce the number of equations and unknowns to be handled, thus also decreasing the sources of errors. Working on the circuit scheme is a great help from this standpoint. As shown, in most circuits we find the final solution by handling few equations.

Remark Another great help to prevent errors is to check the physical dimensions in the obtained symbolic expressions.

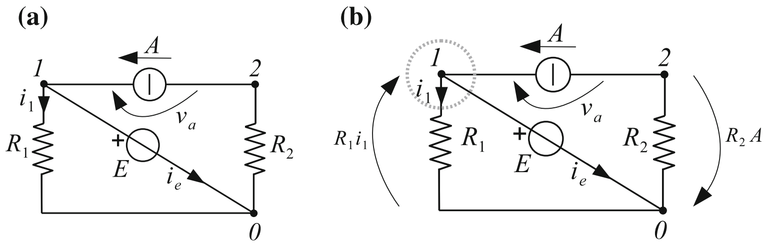


Fig. 3.6 Case Study 1

Case Study 1

Find the symbolic expressions and the numerical values for i_1, i_e, and v_a in the circuit shown in Fig. 3.6a, with $E = 2V$, $A = 1mA$, $R_1 = 5k\Omega$, $R_2 = 7k\Omega$.

Notice that for the current source the descriptive variables are not taken according to the standard choice. This is not a concern if we are not required to compute the power absorbed/delivered by this component.

We can start by finding the variables that can be expressed in terms of the given unknowns: the voltage on R_1 (taken according to the standard choice) is $R_1 i_1$ and the voltage on R_2 (taken according to the standard choice) is $R_2 A$. (See Fig. 3.6b.) Then, by using the voltage source equation and the KVL for the left mesh, we have $R_1 i_1 = E$; that is, $i_1 = E/R_1 = 2/(5 \cdot 10^3)$ A $= 0.4$ mA $= 400\mu$ A. By using the voltage source equation and the KVL for the right mesh, we have $v_a = E + R_2 A = [2 + (7 \cdot 10^3)(1 \cdot 10^{-3})]$V $= 9$ V. Finally, by using the KCL at nodal cut-set 1 (Fig. 3.6b), we obtain $A = i_1 + i_e$; that is, $i_e = A - i_1 = (1 \cdot 10^{-3} - 0.4 \cdot 10^{-3})$ A $= 0.6$mA $= 600\mu$ A.

As an alternative, the same circuit can be solved by introducing all possible descriptive variables ($2L = 8$, in this case), as shown in Fig. 3.7, and solving a system of eight equations. ($N - 1 = 2$ KCLs, $L - N + 1 = 2$ KVLs, and $L = 4$ descriptive equations.)

For instance, we can take the two inner loops in Fig. 3.7 to find the KVLs $v_1 = v_e$ and $v_a + v_2 = v_e$. Similarly, we can take the two nodal cut-sets 1 and 2 to find the KCLs $i_1 + i_e + i_a = 0$ and $i_a = i_2$. The four descriptive equations are: $v_1 = R_1 i_1$, $v_2 = R_2 i_2$, $v_e = E$, and $i_a = -A$. (Pay attention to the current directions!) By solving this system, one finds all the descriptive variables, included the requested ones.

The first solving method in practice shortens the process of substitution of the unknowns not requested by the problem. Even for a very simple circuit such as this, it is evident that managing a set of $2L$ unknowns can increase the probability of errors, therefore we recommend once more the introduction of the minimum number of unknowns, as in the first solution.

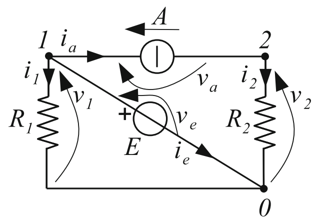


Fig. 3.7 All descriptive variables for Case Study 1

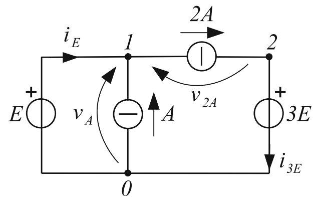


Fig. 3.8 Case Study 2

Case Study 2

Find the symbolic expressions and the numerical values for the currents flowing through the voltage sources and the voltages across the current sources in the circuit shown in Fig. 3.8*, with* $E = 2V$, $A = 3mA$*. Compute the powers delivered by the current sources and absorbed by the voltage sources.*

From the left mesh, we immediately have (KVL) $v_A = E = 2$ V. Then, the power delivered by the current source A is $p_A = EA = 2 \cdot 3\ 10^{-3}$ W $= 6$ mW. From the outer loop, we have (KVL) $E = v_{2A} + 3E$; that is, $v_{2A} = -2E$. Thus the power delivered by the current source $2A$ is $p_{2A} = -v_{2A}\ 2A = 4EA = 4 \cdot 2 \cdot 3\ 10^{-3}$ W $= 24$ mW. From KCL at node 2, $i_{3E} = 2A = 2 \cdot 3\ 10^{-3}$A $= 6$ mA and the power absorbed by the voltage source $3E$ is $p_{3E} = 3Ei_{3E} = 6EA = 6 \cdot 2 \cdot 3\ 10^{-3}$ W $= 36$ mW. From KCL at node 1, $i_E + A = 2A$; that is, $i_E = A = 3$ mA and the power absorbed by the voltage source E is $p_E = E(-i_E) = -EA = -2 \cdot 3\ 10^{-3}$ W $= -6$ mW.

Remark: The total power absorbed by the circuit is $-p_A - p_{2A} + p_{3E} + p_E = 0$ W, according to Tellegen's theorem and to the law of conservation of energy. (See Sect. 2.3.)

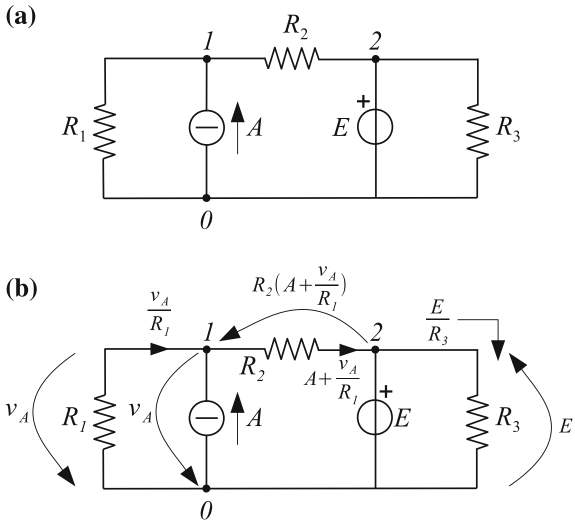
Case Study 3

Find the symbolic expression and the numerical value of the power absorbed by the current source in the circuit shown in Fig. 3.9a, *with* $E = 3V$, $A = 350\mu A$, $R_1 = 30k\Omega$, $R_2 = 20k\Omega$, $R_3 = 40k\Omega$.

From the right mesh (KVL) and from Ohm's law, we find the values for the R_3 descriptive variables. (See Fig. 3.9b.)

We have to compute the power absorbed by the current source, therefore we introduce as a further unknown the voltage v_A and, from the left mesh (KVL) and from Ohm's law, we find the values for the R_1 descriptive variables. (See Fig. 3.9b.) Then, from the KCL at node 1, we find the current in R_2 and (Ohm's law) the voltage across it. (See Fig. 3.9b.)

Fig. 3.9 Case Study 3



Finally, from the central mesh (KVL), we have $v_A + E + R_2(A + \frac{v_A}{R_1}) = 0$; that is, $v_A = -\frac{R_1}{R_1+R_2}(E + R_2 A) = -\frac{3}{5}(3 + 20\ 10^3 \cdot 350\ 10^{-6})\text{V} = -6$ V. Therefore the requested power is $p = v_A A = -6 \cdot 350\ 10^{-6}$ W $= -2.1$ mW.

Case Study 4

Find the symbolic expression and the numerical value of the current i_E in the circuit shown in Fig. 3.10a, with $E = 1V$, $A = 1mA$, $R = 20k\Omega$.

Notice that this circuit corresponds in structure to the generic example used in Sect. 1.5.

We can start from the right mesh (KVL) and from Ohm's law to find the descriptive variables of resistor R in terms of the circuit parameters. (See Fig. 3.10b.)

Then, we can use the nodal KCL at node 3 and Ohm's law to find the descriptive variables of resistor $3R$ in terms of both the unknown i_E and the circuit parameters. (See Fig. 3.10c.)

Now we can find the descriptive variables of resistor $2R$ by using the KVL for the mesh defined by nodes 2–3–5 and Ohm's law. (See Fig. 3.10d)

Finally, using the nodal KCL at node 5, we have $i_E - \frac{E}{R} = \frac{E}{R} - \frac{3}{2}i_E + A$; that is, $i_E = \frac{4}{5}\frac{E}{R} + \frac{2}{5}A = 440\mu$A.

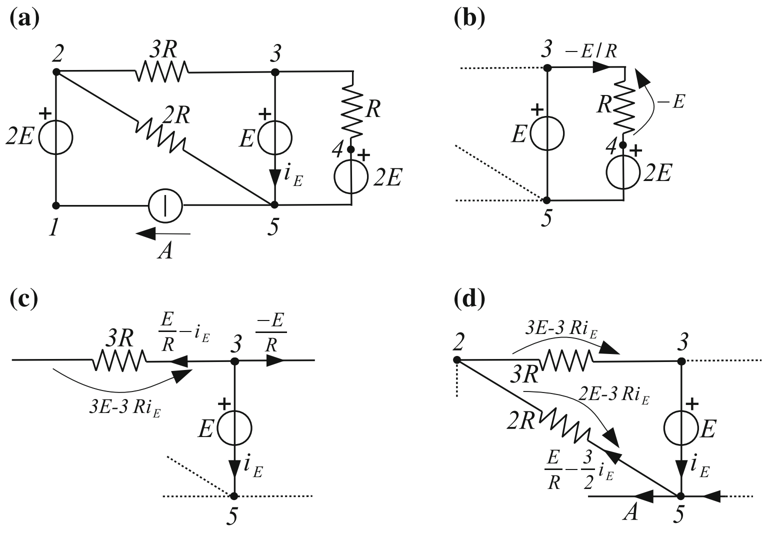


Fig. 3.10 Case Study 4

Case Study 5

Find the symbolic expression and the numerical value of the current i in the circuit shown in Fig. 3.11a, *with* $E = 22V$, $R = 10k\Omega$.

We can start from Ohm's law to find the voltage across resistor $R/2$. Now, we cannot go on without introducing a second unknown, that is, one of the descriptive variables of the other resistors. For instance, we choose the voltage (say v) across the upper left resistor. Then, we can exploit the KVL for mesh A and Ohm's law (twice), as shown in Fig. 3.11b. By using the KVL for mesh B, Ohm's law, the KCL at node 3, and Ohm's law again, we find the voltages and currents shown in Fig. 3.11c. Finally, using the nodal KCL at node 2 and KVL for mesh C (Fig. 3.11d), we have $i + \frac{v}{R} = \frac{E-v}{2R}$ and $E - v + \frac{Ri}{2} = \frac{v}{2} - \frac{5}{4}Ri$, respectively. After some algebra, we find $i = -\frac{2}{11}\frac{E}{R} = -\frac{4}{10^4}A = -400\mu$A.

You can check your comprehension by solving again the problem stated in Case Study 5, choosing a different second unknown.

3.2.5 Diode

A diode is an electronic two-terminal device (Fig. 3.12a shows two real devices). The symbol most commonly used for the model is shown in Fig. 3.12b: the arrow

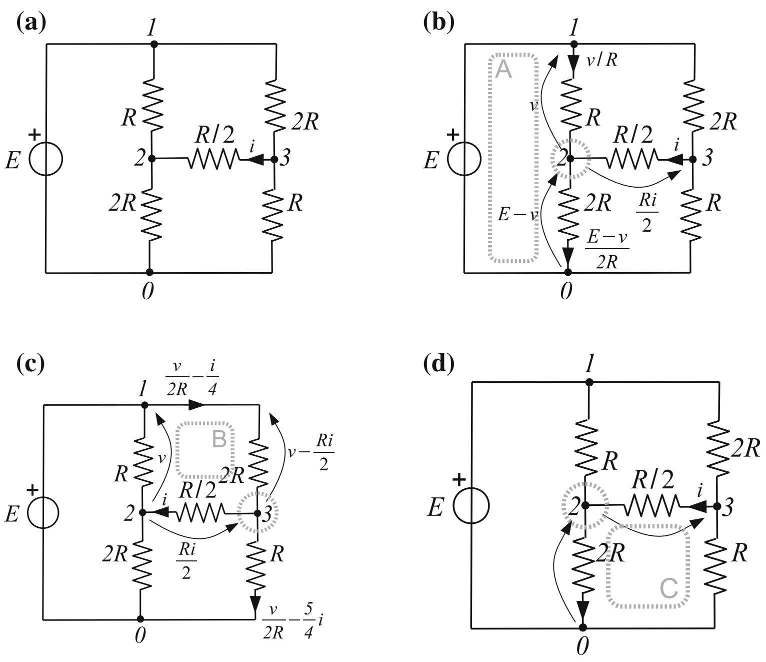


Fig. 3.11 Case Study 5

contained in the symbol suggests that, for a diode connected at two nodes of a circuit, the swapping of its terminals changes the electrical behavior. This asymmetry is typical of many components whose descriptive equation is nonlinear.

The **diode descriptive equation**[4] is

$$i(t) = I_S \left(e^{\frac{v(t)}{nV_t}} - 1 \right) \equiv I_S \psi(v; n) \tag{3.6}$$

where I_S (with the physical dimension of ampere) is the reverse bias saturation current, V_t (with physical dimension of volt) is the thermal voltage, and n (dimensionless) is the ideality factor (or quality factor). The ideality factor n typically varies from 1 (case of an "ideal" diode) to 2 and accounts for imperfections in real

[4] Also called Shockley ideal diode equation, named after transistor coinventor William Bradford Shockley (1910–1989). With John Bardeen and Walter H. Brattain he invented the point contact transistor in 1947 and they were jointly awarded the 1956 Nobel Prize in Physics.

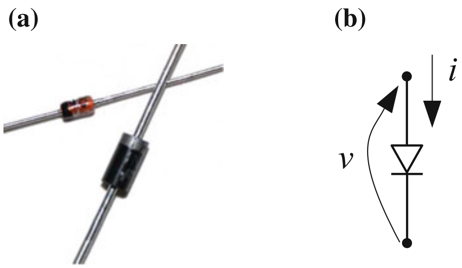


Fig. 3.12 Diode: **a** physical devices; **b** model

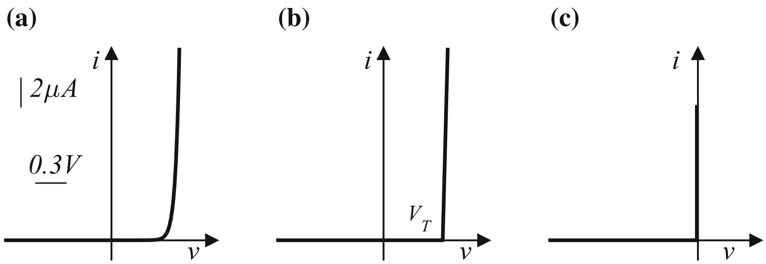


Fig. 3.13 Diode: **a** graphical representation of the descriptive equation (with $n = 1$, $V_t = 25.85$ mV, $I_S = 20$ nA); **b**, **c** PWL approximations. The two segments in panel **a** define the scales for the axes

devices. The descriptive equation is nonlinear and is represented on the (v, i) plane in Fig. 3.13a. It is evident that the diode has low resistance to the flow of current for large enough values of v, and high resistance for negative or slightly positive values of v. Panels b and c show two piecewise-linear (PWL) approximations of the original equation, often used for hand calculations. We remark that the voltages V_t (thermal voltage) and V_T (knee voltage) are different. In the case of panel b, the descriptive equation becomes:

$$i(t) = \begin{cases} 0 & \text{for} \quad v < V_T \\ g(v - V_T) & \text{for} \quad v \geq V_T \end{cases}. \tag{3.7}$$

In implicit form, it can be expressed as $f(i, v) = i - g\frac{v - V_T + |v - V_T|}{2} = 0$.
In the case of panel c, the descriptive equation is:

$$\begin{cases} i(t) = 0 & \text{for} \quad v < 0 \\ v(t) = 0 & \text{for} \quad i \geq 0 \end{cases}. \tag{3.8}$$

In implicit form, it can be expressed as $f(i, v) = vi = 0$, with $v \leq 0$ and $i \geq 0$.

The **driving-point (DP) characteristic**: for a two-terminal element such as those defined up to now, is the curve containing all admitted pairs on the plane (v, i).

For instance, the DP characteristics of a resistor and some alternatives for a diode are shown in Figs. 3.2c and 3.13, respectively. The asymmetric DP characteristic of the diode unveils its asymmetric behavior.

Case Study 1

Consider the descriptive equation 3.7 *with the PWL characteristic shown in Fig.* 3.13b. *Show that this equation can be obtained as the descriptive equation of the composite two-terminal element shown in Fig.* 3.14, *where the diode's PWL characteristic is that represented in Fig.* 3.14 *(Eq.* 3.8*).*

The first step to obtaining the characteristic is to write the topological equations:

$$\begin{cases} i_d = i \\ v = Ri + v_d + V_T \end{cases} . \tag{3.9}$$

Then we consider the two regions of the diode's characteristic, identified by an equation ($i_d = 0$ or $v_d = 0$) and by an inequality. Henceforth these regions are called α and β, respectively. (See Fig. 3.14) Combining descriptive equations and topological equations leads to the formulation of the PWL equations in terms of v and i:

- α region: taking $i_d = 0$, Eq. 3.9 gives $i = 0$ and $v = v_d + V_T$. This result holds as long as the inequality $v_d < 0$ is satisfied, that is, as long as $v - V_T < 0$. This corresponds to the first line of Eq. 3.7.
- β region: taking $v_d = 0$, we have $v = Ri + V_T$. This result holds as long as $i_d > 0$, for which we have $i_d(= i) = (v - V_T)/R > 0$; that is, $v > V_T$. For $R = 1/g$, the second line of Eq. 3.7 follows.

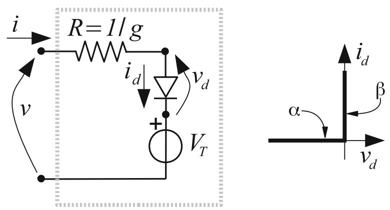


Fig. 3.14 Case Study 1

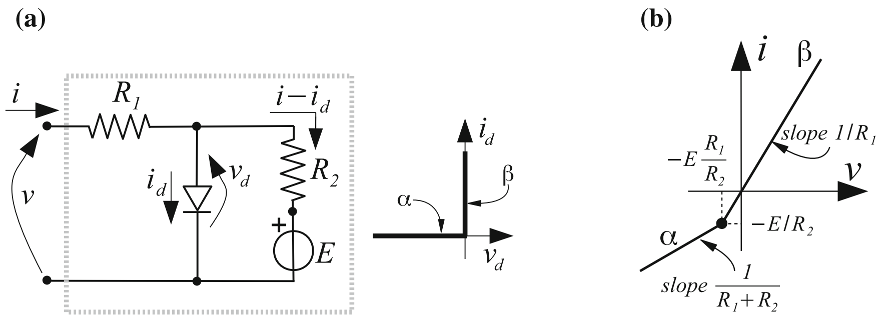


Fig. 3.15 **a** Composite two-terminal element (*left*) and diode's characteristic (*right*); **b** resulting PWL DP characteristic (for $E > 0$)

Case Study 2

Figure 3.15a represents a two-terminal with descriptive variables i and v. Among the components internal to that element, the diode has the characteristic shown in the right part of the same figure. The voltage generator has $E > 0$. Under these assumptions, we want to get the two-terminal's descriptive equation and to represent the corresponding DP characteristic in the (v, i) plane.

Following the track of Case Study 1, KCL and KVL easily lead us to write the following two equations, in which the v, i variables appear together with the descriptive variables of the diode.

$$\begin{cases} v_d = E + R_2(i - i_d) \\ v = R_1 i + v_d \end{cases} . \tag{3.10}$$

- Assuming that the diode is operating in the α region, so that $i_d = 0$ and $v_d < 0$, the previous equations easily give:

$$\begin{cases} E + R_2 i < 0 \\ v = E + (R_1 + R_2) i \end{cases} . \tag{3.11}$$

  Thus inequality $v_d < 0$ is satisfied in the region $i < -E/R_2$ (i.e., $v < -ER_1/R_2$).
- Assuming now that the diode operates in the β region, the substitution $v_d = 0$ in Eq. 3.10 gives

$$\begin{cases} E + R_2(i - i_d) = 0 \\ v = R_1 i \end{cases} . \tag{3.12}$$

The $i_d \geq 0$ inequality is then satisfied for $i \geq -E/R_2$. The two branches of the obtained DP characteristic, indicated by α and β for obvious reasons, are shown in Fig. 3.15b.

3.2.6 Bipolar Junction Transistor

A bipolar junction transistor (or BJT) is a three-terminal device, whose symbol is shown in Fig. 3.16, that exploits two junctions between two semiconductor types (n-type and p-type) and uses both electron and hole charge carriers. In contrast, unipolar transistors, such as field-effect transistors (or FET), use only one kind of charge carrier.

Under proper assumptions, the **BJT descriptive equations**[5] are

$$\begin{cases} i_C = \alpha_F f_1(v_{EB}) - f_2(v_{CB}) \\ i_E = -f_1(v_{EB}) + \alpha_R f_2(v_{CB}) \end{cases}, \tag{3.13}$$

where $f_1(v_{EB}) = I_{ES}\left(e^{\frac{-v_{EB}}{V_t}} - 1\right) = I_{ES}\psi(-v_{EB}; 1)$ and $f_2(v_{CB}) = I_{CS}\left(e^{\frac{-v_{CB}}{V_t}} - 1\right) = I_{CS}\psi(-v_{CB}; 1)$, ψ is defined in Eq. 3.6, V_t is defined in Sect. 3.2.5, I_{ES}, I_{CS} (both with typical values ranging from 10^{-12} to 10^{-10} A at room temperature), α_F (typically ranging from 0.5 to 0.8), and α_R (usually set to 0.99) are device parameters.

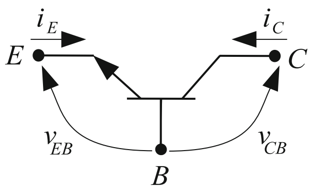


Fig. 3.16 Symbol of the BJT

[5]This is the so-called Ebers-Moll model of the transistor, where the transistor is viewed as a pair of diodes. This model is named after Jewell James Ebers (1921–1959) and John Louis Moll (1921–2011), two American electrical engineers.

3.3 General Component Properties

On the basis of its descriptive equations, any component can be classified according to some criteria or properties. Here we introduce a set of basic properties.

3.3.1 *Linearity*

A component is **linear** if all its descriptive equations are linear and homogeneous with respect to the descriptive variables. Otherwise it is **nonlinear**.

For instance, the resistor is linear, whereas the ideal sources, the diode and the BJT are nonlinear.

Other examples:

- A two-terminal with descriptive equation $i = C\frac{dv}{dt}$ is linear, because the derivative (as well as the integral) is a linear operator.
- A three-terminal with descriptive equations

$$\begin{cases} v_1 = R(t)i_2 \\ i_1 = I_0 cos(\frac{v_2}{V_0}) + \alpha i_2 \end{cases} \tag{3.14}$$

is nonlinear, because at least one of the two equations (the second one) is nonlinear with respect to at least one descriptive variable. If the second equation were $i_1 = I_0 cos(\beta)\frac{v_2}{V_0} + \alpha i_2$, the three-terminal would be classified as linear, because in this case the term $\frac{I_0}{V_0}cos(\beta)$ plays the role of a mere coefficient.

3.3.2 *Time-Invariance*

A component is **time-invariant** if its descriptive equations do not depend on time. This means that the relationship between the descriptive variables or their derivatives (or integrals) is always the same, even if the values of the descriptive variables change in time, of course. Otherwise it is **time-varying**.

The resistor is time-invariant (remember that R is assumed to be constant) as well as the diode and the BJT, whereas the ideal sources are time-varying, in general, inasmuch as their descriptive equations contain a time-varying parameter (the impressed voltage/current).

Other examples:

- A two-terminal with descriptive equation $i = C\frac{dv}{dt}$ is time-invariant: i and v change in time, but the relationship between i and $\frac{dv}{dt}$ is always the same, because C is assumed to be constant.
- The three-terminal with descriptive equations 3.14 is time-varying, because at least one of the two equations (the first one) contains an explicit dependence on time for a parameter (the resistance R).
- A two-terminal with descriptive equation

$$\begin{cases} v = 0 & \text{for} \quad t < t_0 \\ i = 0 & \text{for} \quad t \geq t_0 \end{cases} \tag{3.15}$$

is time-varying, because the descriptive equation changes with time. In particular, this two-terminal is called an *ideal switch* and is closed (short circuit) for $t < t_0$ and open (open circuit) for $t \geq t_0$.

3.3.3 Memory

A component is **memoryless** or **resistive** or **adynamic** if its descriptive equations involve only the descriptive variables and not their derivatives (or integrals). Otherwise we say that it **has memory** or is **dynamic**.

Resistor, ideal sources, diode, and BJT are memoryless components.
Other examples are:

- A two-terminal with descriptive equation $i = C\frac{dv}{dt}$ has memory.
- The three-terminal with descriptive equations 3.14 is memoryless, because both equations are expressed in terms of the descriptive variables only, not involving their derivatives/integrals.

A circuit containing only linear, time-invariant, memoryless components and independent sources is called a **linear time-invariant resistive circuit**.

3.3.4 Basis

A memoryless component admits the **voltage basis** (**current basis**) or is a **voltage-controlled resistor** (**current-controlled resistor**) if its descriptive equations can be expressed in the explicit form $i = f(v)$ ($v = g(i)$), where in general v is the vector of the descriptive voltages and i is the vector of the descriptive currents.

A two-terminal can admit the voltage basis, the current basis, both bases, or none of them.

An n-terminal element with $n > 2$ can also admit **mixed bases**: this means that at least an explicit form $y = f(x)$ is admitted, where the $n-1$ independent variables x (i.e., the basis) are a mixed set of descriptive voltages and currents.

In other words, it must be possible to assign *arbitrarily* the vector x and obtain *univocally* (through the vector of functions f) the $n-1$ variables y. This restricts the possible combinations because we cannot assign both voltage and current at a given terminal. For instance, in a three-terminal, it is not possible to impose both v_1 and i_1: if we impose v_1 with a voltage source, the current i_1 depends on the component equations, and vice versa. (See also the examples below.) Then, for a three-terminal, the possible mixed bases are (v_1, i_2) and (i_1, v_2), whereas (v_1, i_1) and (v_2, i_2) cannot be bases.

For instance, the resistor admits both bases, the ideal voltage source and the short circuit admit only the current basis, whereas the ideal current source and the open circuit admit only the voltage basis. The diode with the characteristic of Fig. 3.13a admits both bases, whereas in the PWL case of Fig. 3.13b it admits only the voltage basis and in the case of Fig. 3.13c it does not admit any basis. The BJT admits all possible bases.

Other examples are:

- A three-terminal with descriptive equations

$$\begin{cases} v_1 = Ri_2 \\ i_1 = I_0 cos(\frac{v_2}{V_0}) + \alpha i_2 \end{cases}$$

  admits bases (i_1, v_2) and (v_1, v_2). (See Fig. 3.17.) It does not admit either the mixed basis (v_1, i_2), because the two variables are proportional (then it is impossible to assign both of them arbitrarily), or the current basis (i_1, i_2), because the $\cos(\cdot)$ function is not bijective and then from the second equation it is not possible to obtain univocally v_2 for a given pair (i_1, i_2);

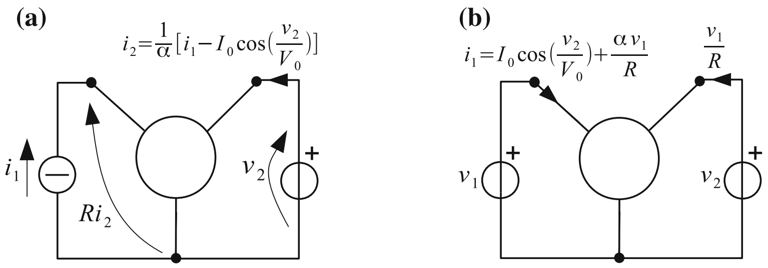


Fig. 3.17 Circuit verification of the existence of bases (i_1, v_2) (**a**) and (v_1, v_2) (**b**)

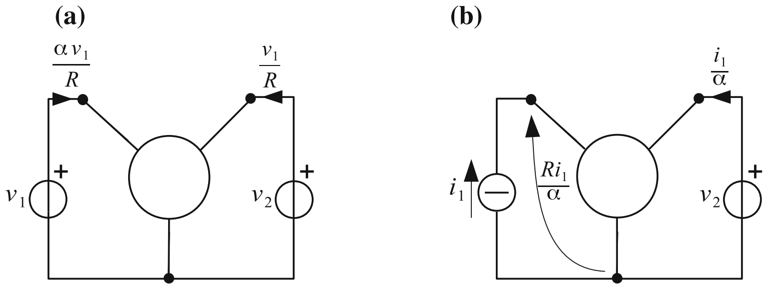


Fig. 3.18 Circuit verification of the existence of bases (v_1, v_2) (**a**) and (i_1, v_2) (**b**)

- A three-terminal with descriptive equations

$$\begin{cases} v_1 = Ri_2 \\ i_1 = \alpha i_2 \end{cases} \tag{3.16}$$

admits only the voltage basis and the mixed basis (i_1, v_2) (Fig. 3.18): the other two bases are not admitted due to the proportionality relationships imposed by the two descriptive equations.

- A three-terminal with descriptive equations

$$\begin{cases} v_1 = 0 \\ i_1 = \alpha i_2 \end{cases} \tag{3.17}$$

admits only the mixed basis (i_1, v_2) (Fig. 3.19): any basis containing v_1 is not admitted because (due to the first equation) v_1 cannot be arbitrarily assigned and the second equation makes it impossible to assign both currents arbitrarily.

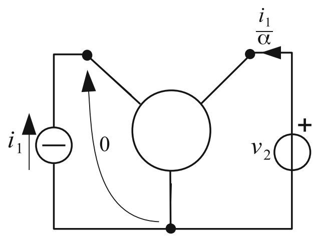


Fig. 3.19 Circuit verification of the existence of basis (i_1, v_2)

3.3.5 Energetic Behavior

The energetic behavior of a component depends on the expression of the absorbed power p (for memoryless components) or energy w (for components with memory) for any possible electrical situation, that is, for any admitted values of the descriptive variables.

A component is:

- **Nonenergic** (or **inactive**) if $p(t) = 0$ ($w(t) = 0$) for any electrical situation
- **Passive** if $p(t) \geq 0$ ($w(t) \geq 0$) for any electrical situation
- **Strictly active** if $p(t) \leq 0$ ($w(t) \leq 0$) for any electrical situation
- **Active** if the sign of $p(t)$ ($w(t)$) depends on the electrical situation

A memoryless component can be said to be **dissipative** as an alternative to passive.

For a given memoryless two-terminal element, the classification is quite easy on the basis of its DP characteristic: if it lies only on the coordinate axes, the component is nonenergic; if it lies only on the I–III (II–IV) quadrants, the component is passive (strictly active); otherwise the component is active.

For instance, the resistor is passive (by assuming $R > 0$) because the absorbed power is $p(t) = Ri^2 = v^2/R \geq 0$ for any electrical situation.

Instead, the ideal sources are active. By considering the voltage source, for instance, the absorbed power is $p(t) = e(t)i(t)$ (Fig. 3.4a): because $i(t)$ depends on the rest of the circuit, the sign of $p(t)$ depends on the specific electrical situation considered and nothing general can be stated. For instance, in Case Study 2 of Sect. 3.2.4, one of the two voltage sources absorbs positive power, whereas the other absorbs negative power.

The diode is passive, generally speaking, even if in the case of Fig. 3.13c it would be nonenergic.

A resistor with negative resistance ($R < 0$) is strictly active.

The energetic behavior of components with memory is treated in Volume 2.

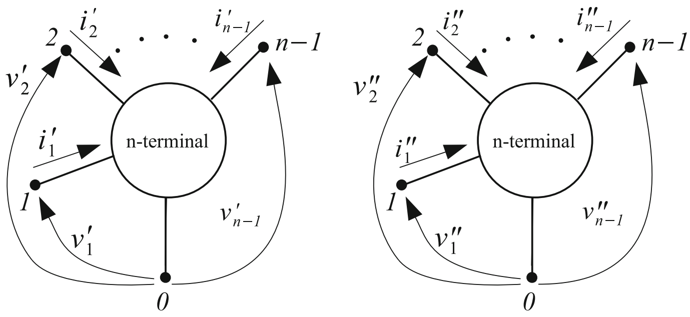


Fig. 3.20 The same n-terminal element in two different electrical situations

3.3.6 Reciprocity

The reciprocity property is encountered in various fields of physics, such as mechanics, acoustics, and electromagnetism. *In broad terms, it concerns the possible exchange of roles between input and output of a physical system.* In the case of electromagnetism this property was studied, among others, by Maxwell[6] [1] and by Lorentz[7] [2]. An example concerning electromagnetic propagation regards the possibility of swapping the positions of a transmitter and a receiver without altering the quality of the received signal. The reciprocity property is also present, however, in circuit theory, where each propagation phenomenon is absent by assumption.

One of the possible ways to begin the discussion about reciprocity within the circuit domain is to define this property for a memoryless n-terminal component. To this aim, we consider the same n-terminal element in two different electrical situations, for example, two identical components working in two different circuits, two identical components used in different places in the same circuit, or a single component working in a circuit but considered at two different times. In the first situation, we measure voltages v' and currents i', whereas in the second situation, we measure voltages v'' and currents i'', as shown in Fig. 3.20.

We can now introduce the (virtual) cross-powers:

$$p'(t) = (v'')^T i' = (i')^T v'' \tag{3.18}$$

$$p''(t) = (v')^T i'' = (i'')^T v' \tag{3.19}$$

[6]James Clerk Maxwell (1831–1879) was a Scottish scientist who formulated the classical theory of electromagnetic radiation, bringing together for the first time electricity, magnetism, and light as manifestations of the same phenomenon.

[7]Hendrik Antoon Lorentz (1853–1928) was a Dutch physicist who shared the 1902 Nobel Prize in Physics with Pieter Zeeman for the discovery and theoretical explanation of the Zeeman effect. He also derived the transformation equations that settled the basis for the special relativity theory of Albert Einstein.

An n-terminal is said to be **reciprocal** if $p'(t) = p''(t)$ for any pair of electrical situations, that is, for any v', v'', i', i'' compatible with the n-terminal. Otherwise it is nonreciprocal.

This definition may seem abstruse, far from the claims made at the beginning of this section about the links between an "input" and an "output" properly defined for the component. To show that this distance is only apparent, we consider a four-terminal, and assume that it is reciprocal and admits the bases shown in Fig. 3.21.

Referring to the first admitted basis, we first consider the four-terminal in the two electrical situations described in Fig. 3.22a and b. In both circuits, the current of terminal 3 is set to 0. In the first circuit, we assign the remaining pair $(v_1, v_2) = (E, 0)$. This is done through a voltage source E and a short circuit, as shown in Fig. 3.22a. The values of $i_1^{'}, i_2^{'}, v_3^{'}$ are obtained from the descriptive equations as functions of the components of the basis values $(E\ 0\ 0)^T$. Similar considerations apply to the second circuit, where $(v_1, v_2) = (0, E)$ and with $i_1^{''}, i_2^{''}, v_3^{''}$ now calculated as functions of the basis values $(0\ E\ 0)^T$. The pairs of vectors v', i', v'', i'' associated with the two circuits:

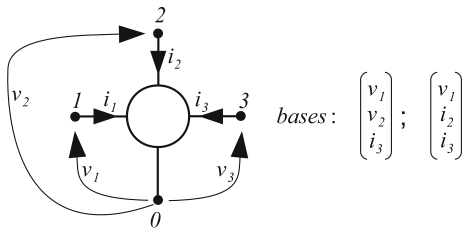


Fig. 3.21 The four-terminal element and its admitted bases

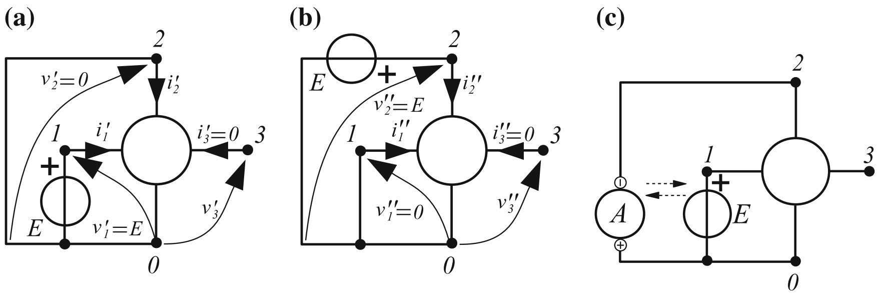


Fig. 3.22 First example

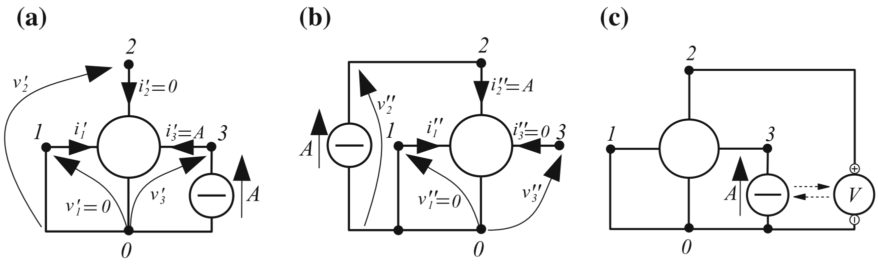


Fig. 3.23 Second example

$$v' = \begin{pmatrix} E \\ 0 \\ v_3' \end{pmatrix}; \quad i' = \begin{pmatrix} i_1' \\ i_2' \\ 0 \end{pmatrix}; \quad v'' = \begin{pmatrix} 0 \\ E \\ v_3'' \end{pmatrix}; \quad i'' = \begin{pmatrix} i_1'' \\ i_2'' \\ 0 \end{pmatrix}.$$

can now be used for the calculation of the virtual cross-powers $(v')^T i''$ and $(v'')^T i'$. By equating these terms, as required by the reciprocity assumption, we get $i_1'' = i_2'$, that is, the current of terminal 1 in the second circuit equals that of terminal 2 in the first circuit. In other words, in the circuit of Fig. 3.22c, the source E (input) and the ammeter (output) can be swapped without changes of the ammeter's measurement.

As a second example, referring now to the second basis, we consider $v_1 = 0$ and we set $(i_2, i_3) = (0, A)$ in Fig. 3.23a and $(i_2, i_3) = (A, 0)$ in Fig. 3.23b. We obtain the following two pairs of vectors for the descriptive variables.

$$v' = \begin{pmatrix} 0 \\ v_2' \\ v_3' \end{pmatrix}; \quad i' = \begin{pmatrix} i_1' \\ 0 \\ A \end{pmatrix}; \quad v'' = \begin{pmatrix} 0 \\ v_2'' \\ v_3'' \end{pmatrix}; \quad i'' = \begin{pmatrix} i_1'' \\ A \\ 0 \end{pmatrix}$$

The reciprocity condition provides $v_2' = v_3''$. Therefore, in the circuit of Fig. 3.23c the swapping of the current source (input) and the voltmeter (output) does not change the voltmeter's measurement.

To check your comprehension, you can refer to the first basis and try to obtain the results when v_2 (instead of v_1) or i_3 are set to zero.

Both previous results bring back the reciprocity to a swapping between a source and a measuring instrument. We can have a third reciprocity relation that does not refer to a swapping.

To show this, we refer again to the first basis, taking $v_1 = 0$ and working on the remaining pair of basis variables, v_2 and i_3. Therefore in the circuit of Fig. 3.24a we set $i_3' = A$ and $v_2' = 0$, whereas in the circuit of Fig. 3.24b (identical to Fig. 3.22b, but repeated for clarity) we set $i_3'' = 0$ and $v_2'' = E$. This amounts to having these two pairs of vectors of descriptive variables:

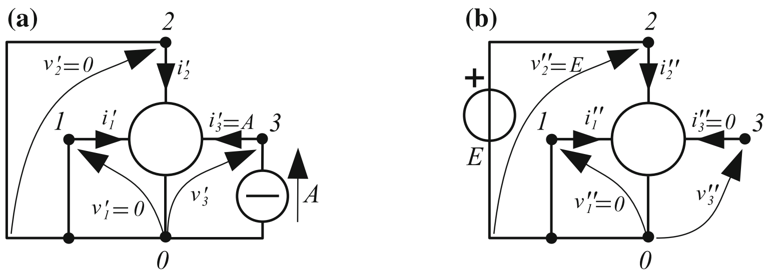


Fig. 3.24 Third example

$$v' = \begin{pmatrix} 0 \\ 0 \\ v_3^{'} \end{pmatrix}; \quad i' = \begin{pmatrix} i_1^{'} \\ i_2^{'} \\ A \end{pmatrix}; \quad v'' = \begin{pmatrix} 0 \\ E \\ v_3^{''} \end{pmatrix}; \quad i'' = \begin{pmatrix} i_1^{''} \\ i_2^{''} \\ 0 \end{pmatrix}$$

in which the values of $v_3^{'}, i_1^{'}, i_2^{'}, v_3^{''}, i_1^{''}, i_2^{''}$ are obtained by the descriptive equations. Now, writing the reciprocity condition $(v')^T i'' = (v'')^T i'$ we obtain $Ei_2^{'} + Av_3^{''} = 0$ or $i_2^{'} = -\frac{A}{E}v_3^{''}$.

To check your comprehension, you can try to obtain the analogous result when v_2, instead of v_1, is taken fixed to zero.

Coming back to the general case, for the introduced two-terminal elements, we can check reciprocity by applying the proposed definition to their constitutive equations.

For instance, a resistor (with $0 < R < \infty$) admits both bases. With this in mind, it is very easy to verify that it is reciprocal. Indeed, for two pairs (v', i') and (v'', i'') of values of the resistor's descriptive variables, we can always write $v'i'' = Ri'i''$; $v''i' = Ri''i'$. Thus, $p' = p''$ always.

A diode is nonreciprocal because of its nonlinear DP characteristic. To show this, we first denote by $i = f(v)$ the diode's descriptive equation in the case of both Eqs. 3.6 and 3.7. Then we observe that, for two assigned voltage values v' and v'', we have $i' = f(v')$ and $i'' = f(v'')$, respectively. Therefore, because the corresponding cross-power terms are $v'i'' = v'f(v'')$ and $v''i' = v''f(v')$, we conclude that $p' \neq p''$. With similar reasoning, you can check that the BJT is also nonreciprocal.

Voltage and current sources are nonreciprocal components as well. A voltage source e, for instance, admits the current basis, and for two assigned current values i' and i'' (with $i' \neq i''$), we obviously have $v' = v'' = e$. Thus we conclude that $p' \neq p''$.

Remark For dynamic components, reciprocity can be defined by making reference to virtual works instead of virtual cross-powers.

Finally, we introduce a theorem here that is proved in Sect. 6.3.

Theorem 3.1 (Reciprocity theorem) *A component consisting of reciprocal elements only is in turn reciprocal.*

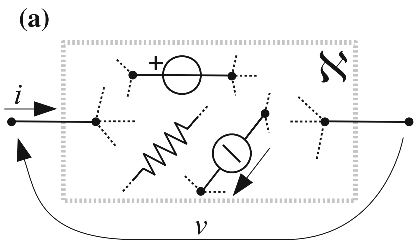


Fig. 3.25 A composite two-terminal $\aleph$ (**a**) and the corresponding two-terminal $\aleph_0$, obtained from $\aleph$ by turning off all the independent sources (**b**)

This is a remarkable result that allows one easily to check the reciprocity of some composite components. For instance, owing to this theorem, we can immediately assess that any n-terminal component containing only resistors is reciprocal. We remark that the theorem provides a *sufficient* condition only. This means that a composite n-terminal containing both reciprocal and nonreciprocal components can be either reciprocal or nonreciprocal: we have to check it through the reciprocity definition.

3.4 Thévenin and Norton Equivalent Representations of Two-Terminal Resistive Components

A generic (black box) two-terminal $\aleph$ made up of linear, time-invariant, and memoryless components and independent sources (Fig. 3.25a) might admit an equivalent representation with the same descriptive equation but a simplified internal structure, which neglects the detail of all internal variables. This is sometimes called a *macromodel* of the two-terminal. There are two possible circuit representations for macromodels, known as the Thévenin equivalent and Norton equivalent. In practice, a macromodel is a higher-level model (Sect. 1.1) of a portion of circuit.

3.4.1 Thévenin Equivalent

The Thévenin equivalent is shown in Fig. 3.26a.[8]

[8]The original theorem (commonly known as Thévenin's theorem) was independently derived in 1853 by the German scientist Hermann von Helmholtz [3] and in 1883 by the French electrical engineer Léon Charles Thévenin (1857–1926) [4, 5].

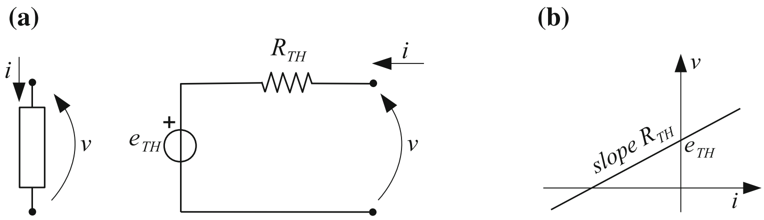


Fig. 3.26 Thévenin equivalent (**a**) and corresponding DP characteristic (**b**)

The descriptive equation of this two-terminal is

$$v(t) = e_{TH}(t) + R_{TH}i(t) \tag{3.20}$$

which corresponds to the DP characteristic shown in Fig. 3.26b. This equation states that this representation is admitted by the two-terminal $\aleph$ when its constitutive equation can be recast in the explicit form Eq. 3.20, that is, *if and only if it admits the current basis*.

How do we find the two parameters e_{TH} and R_{TH} for the Thévenin equivalent of $\aleph$? We have two ways:

1. One can find the descriptive equation of $\aleph$, compare it with Eq. 3.20, and obtain e_{TH} and R_{TH} by comparison. This can be easily done for elementary circuits, but becomes harder for more complex circuits.
2. Otherwise, one can analyze two simpler two-terminals: the first one (say $\aleph_{oc}$) is obtained from $\aleph$ by imposing $i = 0$ and allows obtaining the open-circuit voltage e_{TH} (Eq. 3.20); the second one (say $\aleph_0$; see Fig. 3.25b) is obtained from the original one by turning off all the independent sources (i.e., by replacing each voltage source with a short circuit and each current source with an open circuit) and allows obtaining the Thévenin equivalent resistance as $R_{TH} = \frac{v}{i}$. Indeed (Eq. 3.20), R_{TH} is the resistance of $\aleph$ when it is equivalent to a linear resistor (then passive), that is, when all the independent sources are turned off, thus ensuring that $e_{TH} = 0$.

Remark: Equation 3.20 holds for the Thévenin equivalent shown in Fig. 3.26. If the voltage source is assumed upside down, there is just a change of sign in Eq. 3.20 but we follow the same line of reasoning. This is apparent in Case Study 2 below.

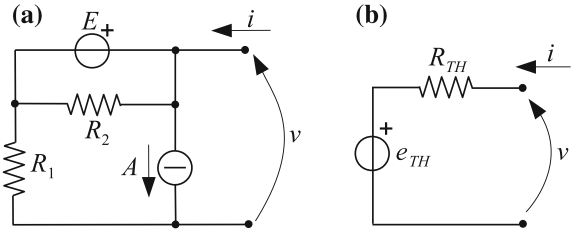


Fig. 3.27 Composite two-terminal ℵ for Case Study 1

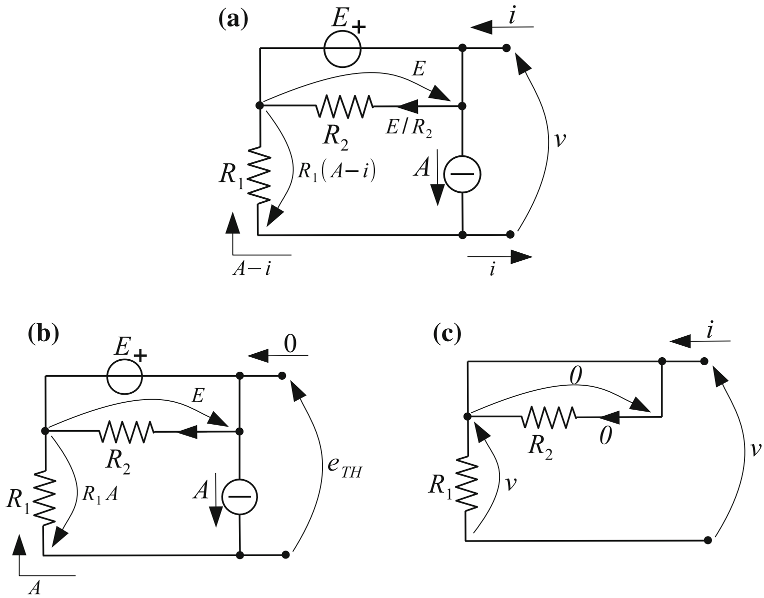


Fig. 3.28 Solution of Case Study 1: **a** Way 1; auxiliary two-terminals $\aleph_{oc}$ (**b**) and $\aleph_0$ (**c**) used in Way 2

Case Study 1

Find the Thévenin equivalent of the composite two-terminal shown in Fig. 3.27.

As stated above, we can solve this problem in two different ways.

Way 1. We find the descriptive equation of the two-terminal.

It is easy to find currents and voltages on the circuit by using Kirchhoff's laws and descriptive equations of the components, as shown in Fig. 3.28a.

Then, from the outer loop, we have the descriptive equation (to be compared with that of the Thévenin equivalent) $v = E - R_1(A - i) = \underbrace{E - R_1 A}_{e_{TH}} + \underbrace{R_1}_{R_{TH}} i$.

Way 2. We analyze two simpler auxiliary two-terminals.

The two-terminal $\aleph_{oc}$ (with $i = 0$ and $v = e_{TH}$) is shown in Fig. 3.28b and can be easily solved, thus finding $e_{TH} = E - R_1 A$.

The second auxiliary two-terminal $\aleph_0$ (with independent sources turned off) is shown in Fig. 3.28c and provides $R_{TH} = \frac{v}{i} = R_1$.

As stated above, the second way is more suitable as far as the circuit becomes more complex.

Notice that R_2 is not involved in the solution. Why?

Case Study 2

Find the Thévenin equivalent of the composite two-terminal $\aleph$ *shown in Fig.* 3.29a.

Notice that in this case the voltage source of the Thévenin equivalent is oriented differently than usual and corresponds to a descriptive equation $v(t) = -e_{TH}(t) + R_{TH} i(t)$. This means that in this case $e_{TH}(t) = -v\Big|_{i=0}$.

The solution is obtained according to the second solving method. The solution according to the first method is left to the reader.

The auxiliary two-terminal $\aleph_{oc}$ (with $i = 0$ and $v = -e_{TH}$) is shown in Fig. 3.29b and can be easily solved, thus finding $e_{TH} = RA - 2E$.

The second auxiliary two-terminal $\aleph_0$ (with independent sources turned off) is shown in Fig. 3.29c and provides $R_{TH} = \frac{v}{i} = R$.

Notice that the left part of the two-terminal in Fig. 3.29a is not involved in the solution. Why?

To check your comprehension, you can try to solve Case Study 5 in Sect. 3.2.4 by preliminarily finding the Thévenin equivalent of the composite two-terminal connected to the resistor where the unknown i flows and then by solving the very simple resulting circuit.

3.4.2 Norton Equivalent

The Norton equivalent is shown in Fig. 3.30a.[9]

[9] The original theorem (commonly known as Norton's theorem) was independently derived in 1926 by Siemens and Halske researcher Hans Ferdinand Mayer (1895–1980) [6] and Bell Labs engineer Edward Lawry Norton (1898–1983) [7].

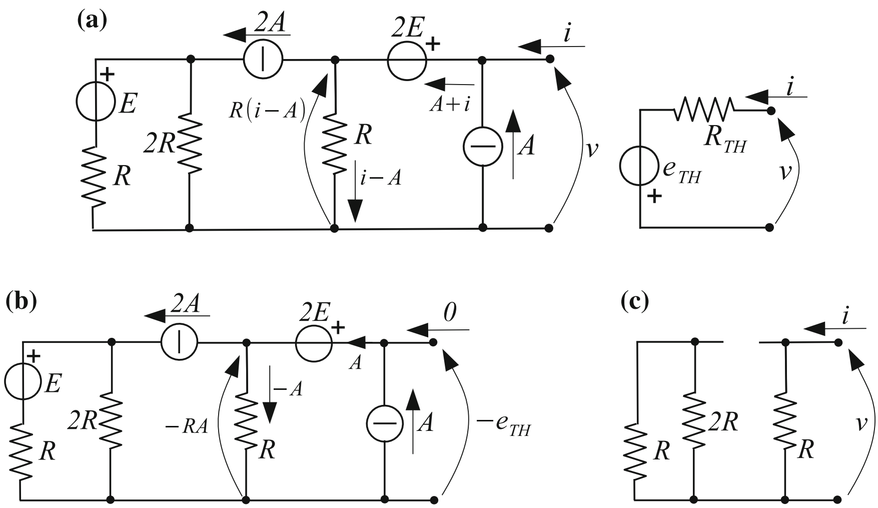


Fig. 3.29 **a** Two-terminal ℵ for Case Study 2; **b**, **c** auxiliary two-terminals

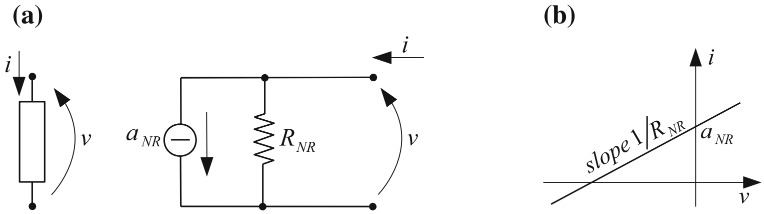


Fig. 3.30 Norton equivalent

The descriptive equation of this two-terminal is

$$i(t) = a_{NR}(t) + \frac{v(t)}{R_{NR}} \tag{3.21}$$

which corresponds to the DP characteristic shown in Fig. 3.30b. This equation states that this representation is admitted by the two-terminal when its constitutive equation can be recast in the explicit form 3.21, that is, *if and only if it admits the voltage basis*.

How do we find the two parameters a_{NR} and R_{NR}? Also in this case, we have a double key to uncover the Norton equivalent of a given generic two-terminal:

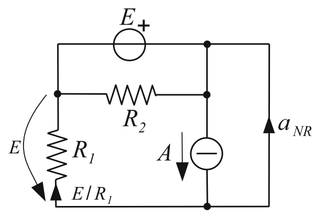


Fig. 3.31 First auxiliary two-terminal $\aleph_{sc}$ for Case Study 1

1. One can find the descriptive equation of the two-terminal $\aleph$ and obtain a_{NR} and R_{NR} by comparison with Eq. 3.21. This can easily be done for elementary circuits, but becomes harder for more complex circuits.
2. Otherwise, one can analyze two simpler two-terminals: the first one (say $\aleph_{sc}$) is obtained from $\aleph$ by imposing $v = 0$ and allows us to obtain the short-circuit current a_{NR} (Eq. 3.21); the second one is once more $\aleph_0$ and allows us to obtain the Norton equivalent resistance from $\frac{1}{R_{NR}} = \frac{i}{v}$.[10] Indeed (Eq. 3.21), R_{NR} is the resistance of the two-terminal $\aleph$ when it is equivalent to a linear resistor (then passive), that is, when all the independent sources are turned off, thus ensuring that $a_{NR} = 0$.

Remark: Equation 3.21 holds for the Norton equivalent shown in Fig. 3.30. If the current source is connected upside down, there is just a change of sign in Eq. 3.21 but we follow the same line of reasoning. This is apparent in Case Study 2 below.

Case Study 1

Find the Norton equivalent of the composite two-terminal shown in Fig. 3.27.

As stated above, we can solve this problem in two different ways.

Way 1. From the descriptive equation of the two-terminal, we easily find

$$i = \frac{v}{R_1} + A - \frac{E}{R_1} = \underbrace{A - \frac{E}{R_1}}_{a_{NR}} + \underbrace{\frac{1}{R_1}}_{\frac{1}{R_{NR}}} v.$$

Way 2. We analyze two simpler auxiliary two-terminals.

$\aleph_{sc}$ (with $v = 0$ and $i = a_{NR}$) is shown in Fig. 3.31 and can be easily solved, thus finding $a_{NR} = A - \frac{E}{R_1}$.

$\aleph_0$ (with independent sources turned off) was shown in Fig. 3.28c and provides $R_{NR} = R_{TH} = \frac{v}{i} = R_1$.

[10] We point out that R_{NR} can tend to infinite (in which case the Norton equivalent is just a current source), but cannot be null, because the two-terminal $\aleph$ admits the voltage basis by assumption.

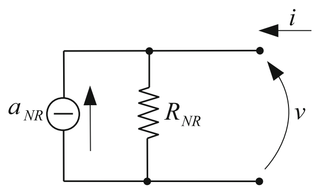


Fig. 3.32 Norton equivalent for Case Study 2

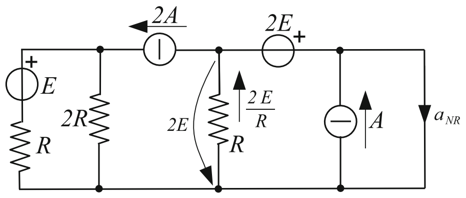


Fig. 3.33 Auxiliary two-terminal $\aleph_{sc}$ for Case Study 2

Case Study 2

Find the Norton equivalent shown in Fig. 3.32 *of the composite two-terminal of Fig.* 3.29a.

Notice that in this case the current source of the Norton equivalent is oriented differently than usual and corresponds to a descriptive equation $i(t) = -a_{NR}(t) + \frac{v(t)}{R_{NR}}$. This means that in this case $a_{NR}(t) = -i\Big|_{v=0}$.

The solution is obtained according to the second solving method. The solution according to the first method is left to the reader.

$\aleph_{sc}$ (with $v = 0$ and $i = -a_{NR}$) is shown in Fig. 3.33 and can be easily solved, thus finding $a_{NR} = \frac{2E}{R} - A$.

$\aleph_0$ (with independent sources turned off) was shown in Fig. 3.29c and provides $R_{NR} = R_{TH} = \frac{v}{i} = R$.

3.4.3 Comparisons Between the Two Equivalent Models

It is quite apparent from the proposed examples that when a composite two-terminal admits both bases (and then both equivalent representations), it is straightforward to derive one model from the other. For instance, starting from Eq. 3.21, one easily finds $v = -R_{NR} \cdot a_{NR}(t) + R_{NR} \cdot i$. By comparing this equation with Eq. 3.20, we identify $e_{TH} = -R_{NR} \cdot a_{NR}$ and $R_{TH} = R_{NR}$.

Why is there such a large temporal interval (1853 vs. 1926) between the formulations of these seemingly similar equivalent models? The current-source equivalent did not occur to early electrical scientists because of the apparent impossibility of the existence of a current source. Only later did Norton and Mayer realize that the current-source equivalent was easier to use in certain practical situations.

3.5 Series and Parallel Connections of Two-Terminals

When analyzing a circuit, it is useful to replace some circuit parts whose details are not under study with others, equivalent but simpler. In other words, once more we replace some parts with macromodels. The most common structures admitting simplifications are series and parallel connections of two-terminals.

3.5.1 Series Connection

As shown in Fig. 3.34,

> Two two-terminals are connected **in series** when the same current flows through them.

The two-terminals a and b share the same current, whereas $v = v_a + v_b$.

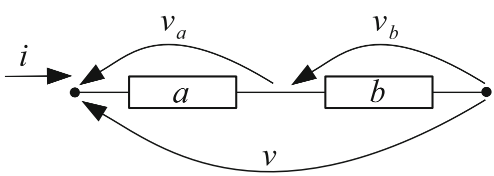
For memoryless components, this connection makes sense if at least one of the two two-terminals admits the current basis.

3.5.1.1 Examples When both Components Admit the Current Basis

Figure 3.35a shows two resistors connected in series. In this case the descriptive equation of the composite two-terminal is $v = (R_A + R_B)i$, which is the descriptive equation of a resistor with resistance $R_A + R_B$. Then, if we are not interested in determining v_A or v_B, we can replace this series connection with the single resistor shown in Fig. 3.35b.

Figure 3.36a shows two voltage sources connected in series. In this case the descriptive equation of the composite two-terminal is $v = E_A - E_B$, which is the descriptive equation of a voltage source with impressed voltage $E_A - E_B$. Then we

Fig. 3.34 Two two-terminals connected in series



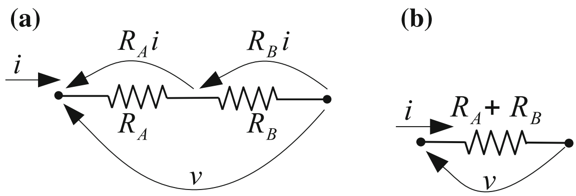


Fig. 3.35 Two resistors connected in series (**a**) and their equivalent model (**b**)

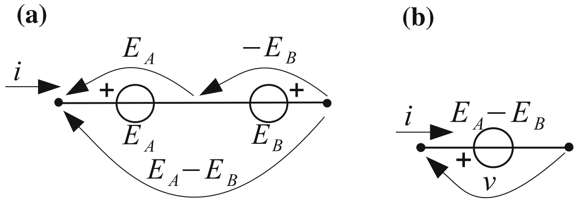


Fig. 3.36 Two voltage sources connected in series (**a**) and their equivalent model (**b**)

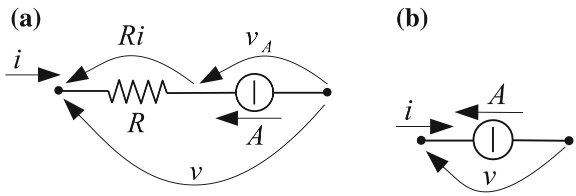


Fig. 3.37 Two components connected in series (**a**) and their equivalent model (**b**)

can replace this series connection with the single voltage source shown in Fig. 3.36b.

3.5.1.2 Examples When only One Component Admits the Current Basis

Figure 3.37a shows a resistor connected in series with a current source, which does not admit the current basis. In this case the descriptive equation of the composite two-terminal is $i = -A$, which is the descriptive equation of a current source. Then, if we are not interested in determining v_A or v_R, we can replace this series connection with the single current source shown in Fig. 3.37b (equivalent model). Notice that the voltage across the equivalent model is different from the voltage across the original current source.

Figure 3.38a shows a voltage source connected in series with a current source. The descriptive equation of the composite two-terminal is $i = -A$, which is the descriptive equation of a current source with impressed current $-A$. Then we can replace this series connection with the single current source shown in Fig. 3.38b.

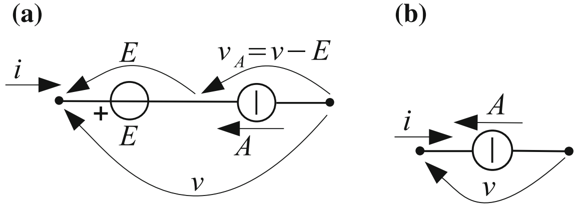


Fig. 3.38 Two components connected in series (**a**) and their equivalent model (**b**)

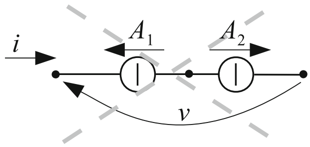


Fig. 3.39 Two current sources connected in series (absurd connection)

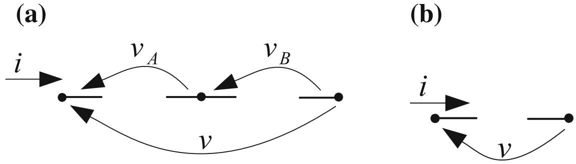


Fig. 3.40 Two open circuits connected in series (undetermined connection)

In any case, the equivalent model coincides (apart from the voltage) with the two-terminal not admitting the current basis.

3.5.1.3 Other Examples

When none of the two-terminals admits the current basis, we can either have an absurd or an undetermined situation. Figure 3.39 shows two current sources connected in series. For $A_1 \neq -A_2$, the KCL is violated. Then, this connection is absurd.

Figure 3.40a shows two open circuits connected in series. In this case there is no way to determine v_A and v_B. An equivalent model exists anyway. (See Fig. 3.40b.)

Finally, a short circuit connected in series with a generic two-terminal is obviously equivalent to the two-terminal itself.

3.5.2 Parallel Connection

As shown in Fig. 3.41,

> Two two-terminals are **in parallel** when they are connected to the same pair of nodes and they share the same descriptive voltage.

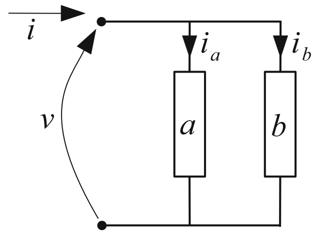


Fig. 3.41 Two two-terminals connected in parallel

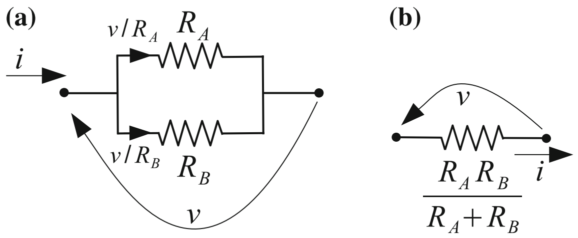


Fig. 3.42 Two resistors connected in parallel (**a**) and their equivalent model (**b**)

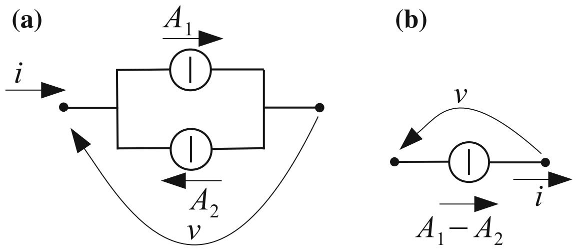


Fig. 3.43 Two current sources connected in parallel (**a**) and their equivalent model (**b**)

The two-terminals a and b share the same voltage, whereas $i = i_a + i_b$.

For memoryless components, this connection makes sense if at least one of the two two-terminals admits the voltage basis.

3.5.2.1 Examples When both Components Admit the Voltage Basis

Figure 3.42a shows two resistors connected in parallel. In this case the descriptive equation of the composite two-terminal is $i = \frac{v}{R_A} + \frac{v}{R_B}$, which is the descriptive equation of a resistor with resistance $\frac{R_A R_B}{R_A + R_B}$. Thus if we are not interested in determining i_A or i_B, we can replace this parallel connection with the single resistor shown in Fig. 3.42b.

Figure 3.43a shows two current sources connected in parallel. In this case the descriptive equation of the composite two-terminal is $i = A_1 - A_2$, which is the descriptive equation of a current source with impressed current $A_1 - A_2$. Thus we can replace this parallel connection with the single current source shown in Fig. 3.43b.

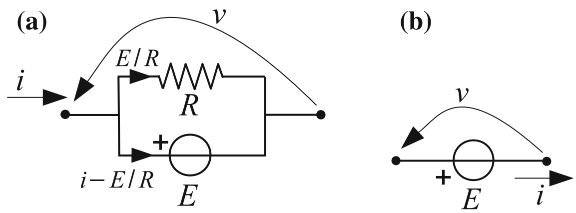


Fig. 3.44 Two components connected in parallel (**a**) and their equivalent model (**b**)

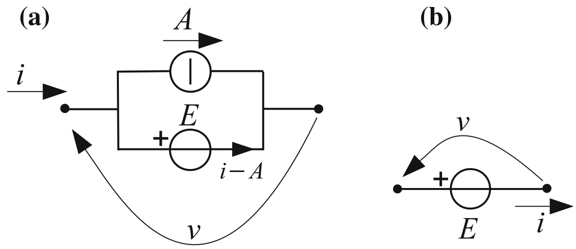


Fig. 3.45 Two components connected in parallel (**a**) and their equivalent model (**b**)

3.5.2.2 Examples When only One Component Admits the Voltage Basis

Figure 3.44a shows a resistor connected in parallel with a voltage source, which does not admit the voltage basis. In this case the descriptive equation of the composite two-terminal is $v = E$, which is the descriptive equation of a voltage source. Thus if we are not interested in determining the currents in the two branches, we can replace this parallel connection with the single voltage source shown in Fig. 3.37b (equivalent model). Notice that the current in the equivalent model is different from the current in the original voltage source.

Figure 3.45a shows a voltage source connected in parallel with a current source. The descriptive equation of the composite two-terminal is once more $v = E$, which is the descriptive equation of a voltage source with impressed voltage E. Thus also in this case we can replace this parallel connection with the single voltage source shown in Fig. 3.45b.

In any case, the equivalent model coincides (apart from the current) with the two-terminal not admitting the voltage basis.

3.5.2.3 Other Examples

When none of the two-terminals admits the voltage basis, we can either have an absurd or an undetermined situation. Figure 3.46a shows a voltage source and a short circuit connected in parallel. In this case the KVL is violated, therefore this connection is

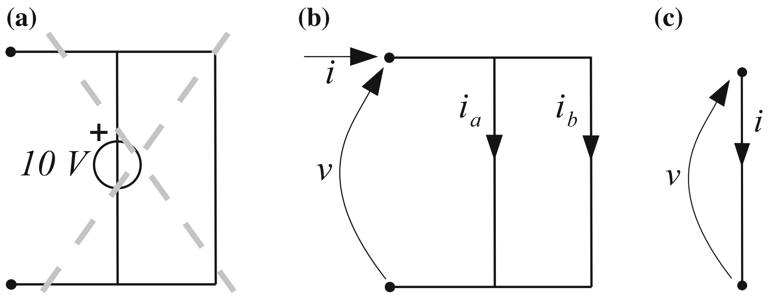


Fig. 3.46 **a** A voltage source and a short circuit connected in parallel (absurd connection); **b** two short-circuit sources connected in parallel (undetermined connection); **c** two-terminal equivalent to **b**

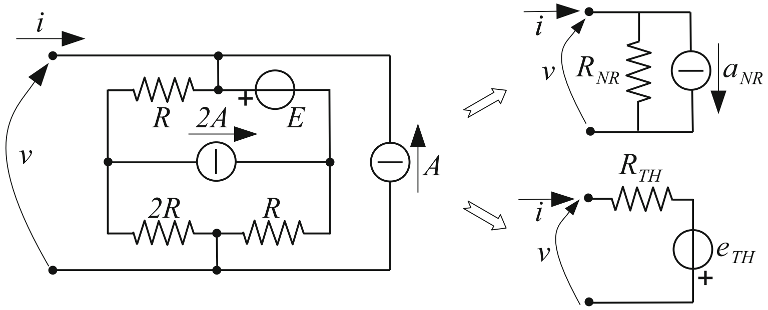
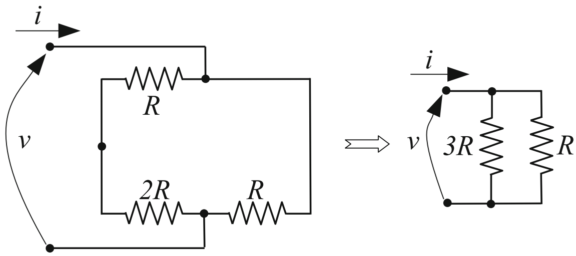


Fig. 3.47 Case Study

Fig. 3.48 Thévenin auxiliary two-terminal $\aleph_0$



absurd. For the same reason, a parallel connection of two voltage sources impressing different voltages is absurd.

Figure 3.46b shows two short circuits connected in parallel. In this case there is no way to determine i_a and i_b, but an equivalent circuit exists. (See Fig. 3.46c.)

Finally, an open circuit connected in parallel with a generic two-terminal is obviously equivalent to the two-terminal itself.

Case Study

Find the Thévenin and Norton equivalents of the composite two-terminal shown in Fig. 3.47.

The solution is obtained according to the second solving method. It is left to the reader to find the solution according to the first method.

The auxiliary two-terminal $\aleph_0$ with independent sources turned off is shown in Fig. 3.48. The left resistors are connected in series and can be replaced by a single resistor with resistance $3R$. Then, $3R$ and R are in parallel, so that $R_{TH} = R_{NR} = \frac{3}{4}R$.

The Thévenin auxiliary two-terminal $\aleph_{oc}$ with $i = 0$ and $v = -e_{TH}$ is shown in Fig. 3.49. Owing to KVL for mesh A and to Ohm's law, we find the voltage and current for the bottom right resistor. Then, by using KCL at node 0 and Ohm's law, the voltage and current for the bottom left resistor can also be easily found. As a further step, KCL at node 1 and Ohm's law provide voltage and current also for the top left resistor. Finally, from KVL for the left mesh we have $-e_{TH} = 3RA + E + e_{TH} + 2RA + 2E + 2e_{TH}$; that is, $e_{TH} = -\frac{5}{4}RA - \frac{3}{4}E$.

The Norton auxiliary two-terminal $\aleph_{sc}$ with $v = 0$ and $i = a_{NR}$ is shown in Fig. 3.50. The circled numbers denote a possible sequence of steps one can take to find (from the KVL for the left mesh) finally the solution $a_{NR} = -\frac{5}{3}A - \frac{E}{R}$.

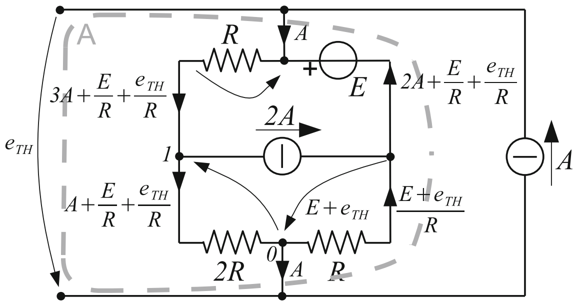


Fig. 3.49 Thévenin auxiliary two-terminal $\aleph_{oc}$

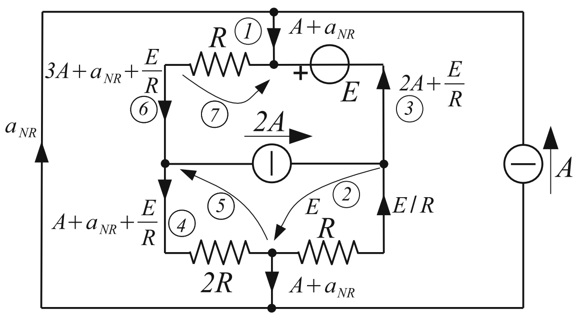


Fig. 3.50 Norton auxiliary two-terminal $\aleph_{sc}$

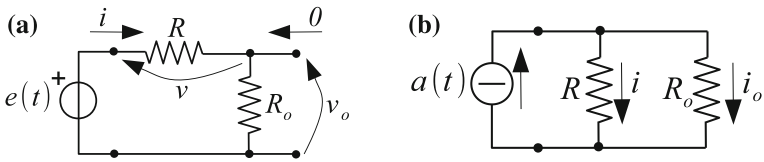


Fig. 3.51 **a** Voltage divider; **b** current divider

3.5.3 Numerical Aspects

When we have two resistors with resistances differing by orders of magnitude,

- If they are connected in series, the whole connection is practically equivalent to the resistor with the highest resistance value.
- If they are connected in parallel, the whole connection is practically equivalent to the resistor with the lowest resistance value.

For instance, if we consider two resistors with resistances $R_1 = 100\Omega$ and $R_2 = 1\text{M}\Omega$, their series connection is equivalent to a resistor with resistance $R_s = R_1 + R_2 \approx R_2$, whereas their parallel connection is equivalent to a resistor with resistance $R_p = \dfrac{R_1 R_2}{R_1 + R_2} \approx R_1$.

These "rules of thumb" can be easily generalized to the cases with more than two resistors connected either in series or in parallel.

3.6 Resistive Voltage and Current Dividers

In this section, we introduce two very common circuit structures: the voltage divider and the current divider.

3.6.1 Resistive Voltage Divider

The resistive voltage divider is shown in Fig. 3.51a. We remark that **the two resistors are connected in series**. We want to know which fraction of the input voltage $e(t)$ drops on the output resistor R_o. Because the two resistors are connected in series, we have $i = \dfrac{e(t)}{R + R_o}$. Then $v_o = e(t)\dfrac{R_o}{R + R_o}$.

Inasmuch as the current is the same in both resistors and due to the linear descriptive equation of the resistor, this equation simply means that the higher R_o is, the higher the fraction of $e(t)$ which drops on R_o.

3.6.2 Resistive Current Divider

The resistive current divider is shown in Fig. 3.51b. We remark that **the two resistors are connected in parallel**. We want to know which fraction of the input current $a(t)$ flows in the output resistor R_o. Because the two resistors are connected in parallel, we have $v = a(t)\dfrac{RR_o}{R + R_o}$. Then $i_o = a(t)\dfrac{R}{R + R_o}$.

Inasmuch as the voltage is the same across both resistors and due to the linear descriptive equation of the resistor, this equation simply means that the higher R_o is, the lower the fraction of $a(t)$ which flows in R_o. This is not surprising: the current tends to privilege the path with minimum resistance! A similar reasoning could be applied to a liquid flowing in two parallel pipes: the higher quantity of liquid will flow in the pipe with larger diameter (i.e., lower resistance).

3.7 Problems

3.1 The three-terminal shown in Fig. 3.52a has descriptive equations

$$\begin{cases} v_1 = Ri_1 \\ v_2 = R_1 i_1 + R_2 i_2 \end{cases}$$

with $R, R_1, R_2 > 0$. Find the descriptive equations in terms of the descriptive variables shown in Fig. 3.52b.

3.2 Check the properties of linearity, time-invariance, and memory of the three-terminal shown in Fig. 3.52a for the following sets of descriptive equations, by assigning to the parameters $\alpha, \beta, \gamma, \delta, and \sigma$ proper physical dimensions, case by case.

$$\text{(a)} \begin{cases} v_1 = \beta i_1 + \gamma i_2 \\ i_1 = \delta \frac{dv_2}{dt} \end{cases} \quad \text{(b)} \begin{cases} \alpha v_1^3 = i_1 + \gamma i_2^2 \\ v_1 = \delta \frac{di_2}{dt} \end{cases} \quad \text{(c)} \begin{cases} v_1 = \beta i_1 + \gamma i_2 \\ v_2 = \sigma i_2 \end{cases} \quad \text{(d)} \begin{cases} v_1 = \beta \sin\left(\frac{i_2}{I_0}\right) \\ \gamma v_2 = i_1 + \sigma i_2 \end{cases}$$

3.3 For each composite two-terminal shown in Fig. 3.53, find:

1. Descriptive equation
2. Admitted bases
3. Energetic behavior
4. Thévenin and Norton equivalent representations shown in Figs. 3.26a and 3.30a, respectively (if admitted).

3.4 Classify each composite two-terminal shown in Fig. 3.54 from the energetic standpoint, assuming the following descriptive equations for the nonlinear two-terminals: $i_2 = \alpha v_2^2$ and $i_1 = \beta e^{\frac{v_1}{V_0}}$, with $\alpha = 1\frac{A}{V^2}$, $\beta = 1A$, $V_0 = 1V$, $E > 0$, $A > 0$.

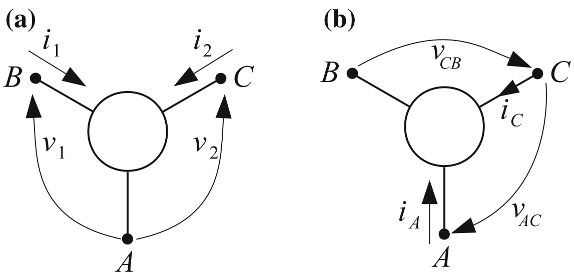


Fig. 3.52 Problems 3.1 and 3.2

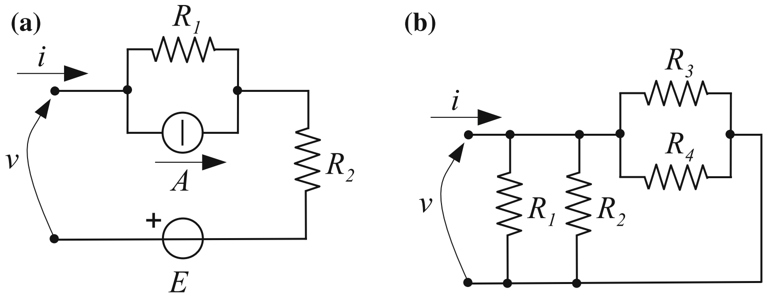


Fig. 3.53 Problem 3.3

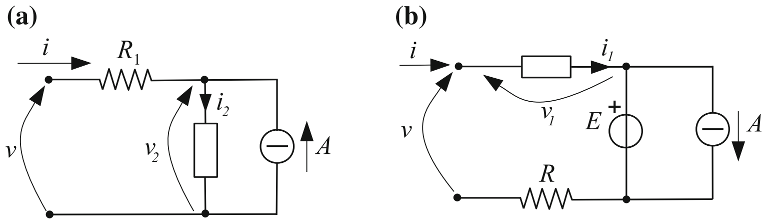


Fig. 3.54 Problem 3.4

3.5 Find, whenever possible, the Thévenin and Norton equivalent representations (shown in Fig. 3.26a and 3.30a, resp.) for the two-terminals shown in Fig. 3.55.

3.6 Find the expression and the value of the current i for the circuit shown in Fig. 3.56a, where $R_1 = R_3 = 1k\Omega$, $R_2 = 3k\Omega$, $E_1 = 1V$, $E_2 = 4V$.

3.7 For the circuit shown in Fig. 3.56b, where the values of A and E can be either positive or negative, find:

1. i.
2. v.
3. Power absorbed by the current source.
4. Power absorbed by the voltage source.

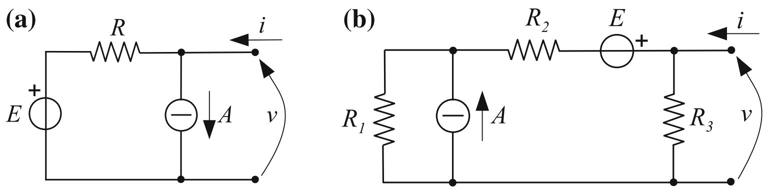


Fig. 3.55 Problem 3.5

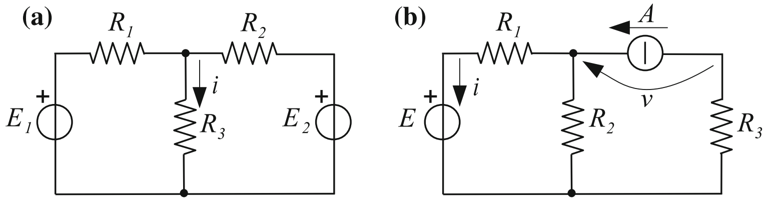


Fig. 3.56 **a** Problem 3.6. **b** Problem 3.7

Fig. 3.57 Problem 3.8

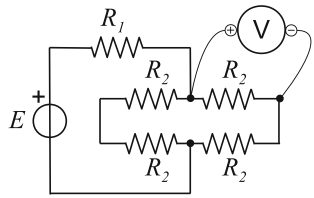
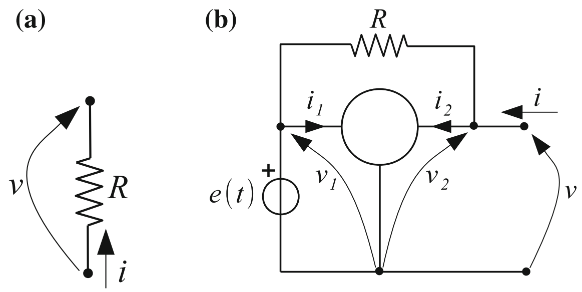


Fig. 3.58 **a** Problem 3.9. **b** Problem 3.10



5. Is it possible that the latter two are both positive?

3.8 For the circuit shown in Fig. 3.57, find the expression of R_1 such that the voltmeter measures a voltage of $\frac{E}{4}$.

3.9 Choose the right answer(s). For the resistor shown in Fig. 3.58a, we can state that the absorbed power:

1. Is always ≥ 0

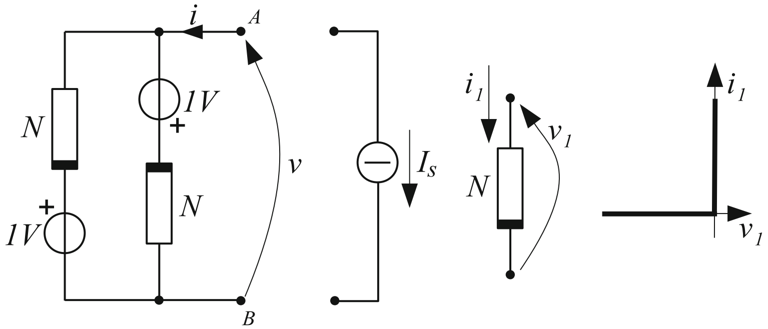


Fig. 3.59 Problem 3.11

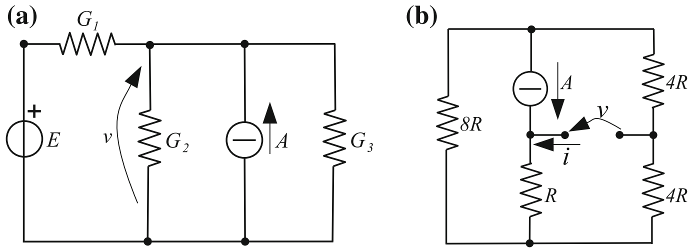


Fig. 3.60 **a** Problem 3.12. **b** Problem 3.13

2. Is always ≤ 0
3. Has a sign that depends on the values of v and i

3.10 Find the Norton equivalent representation (shown in Fig. 3.30a) for the two-terminal shown in Fig. 3.58b, where the three-terminal's descriptive equations are $i_1 = g_1 v_1 + g_M v_2$ and $i_2 = g_2 v_2$, with $g_2 \neq -1/R$.

3.11 By assuming that the nonlinear component N shown in Fig. 3.59 is a diode with the DP characteristic displayed on the right, determine:

1. The DP characteristic of the composite two-terminal with nodes A and B
2. The power delivered by the same two-terminal when connected to a current source with impressed current $I_S = 2$A

3.12 Find the voltage v in the circuit shown in Fig. 3.60a. Also find the numerical solution for $G_1 = 6\ \text{m}\Omega^{-1}$, $G_2 = 13\ \text{m}\Omega^{-1}$, $G_3 = 10\ \text{m}\Omega^{-1}$ ($R_i = 1/G_i$, $i = 1, 2, 3$), $E = 24$ V, $A = 1$ mA.

3.13 Find the Thévenin equivalent representation shown in Fig. 3.26a for the two-terminal shown in Fig. 3.60b. Also find the numerical solution for $R = 25\Omega$, $A = 100$ mA.