Week 5+6+7 ①

Differential Equations

Definition: An equation involving an unknown function and one or more of its derivatives is called a "differential equation".
$F(x, y, y', y'', \dots, y^{(n)}) = 0$

Example 1: $\frac{dx}{dt} = x^2 + t^2$ This dif. eq. involves both unknown $x(t)$ and its derivative $x'(t) = \frac{dx}{dt}$

Here t: independent variable
x: dependent variable (unknown function)

Example 2: $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 7y = 0$ involves the unknown function $y(x)$, and first two derivatives: $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$

Here; x is independent variable
y is dependent variable

* Representation of D.E

There are 3 different representation of differential equations

Derivative Form
Denoted by prime(s) such as $y', y'', y''', \dots, y^{(n)}$
Example: $y'' + y' = 0$, $y = y(x)$

Differential Form
Denoted by $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}$
Example: $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Differential operator "D"
Denoted by $D, D^2, \dots, D^n$
capital D represents the derivative, here.
Example: $D^2y + Dy = 0$

*** Goal:** To study dif. eqs. has 3 principal goals:

1) To discover the dif. eq. that describes a specified physical situation.
2) To find - either exactly or approximately - the appropriate solution of that equation.
3) To interpret the solution that is found.

In algebra; we typically seek the unknown numbers that satisfy an equation such as $x^3 + 7x^2 - 11x + 41 = 0$. By contrast, in solving a dif. eq., we are challenged to find the unknown functions $y = y(x)$ for which an identity such as $y'(x) = 2xy$ holds on some interval of real numbers. We want to find all solutions of the dif. eq., if it is possible.

Classification of Dif Equations

We can classify the dif. eqs. interms of :

(1) The number of Independent Variables: (ODE / PDE)

•) A dif eq. that involves a function of only one independent variable is called "Ordinary Differential Equation" (ODE)

•) A dif eq. that involves a function of two or more independent variables is called "Partial Differential Equation" (PDE)

PDE is denoted by ∂ : Partial Derivative.

Example: Determine: the given dif eq.s are ODE or PDE

(i) $\frac{dy}{dx} = 2x^2 + 3x + 5 \rightarrow$ ODE [Here y : Dependent variable, x : Independent variable] SOLUTION $y = y(x)$

(ii) $(x^2 + y^2)\frac{dx}{dy} - 2xy = 0 \rightarrow$ ODE [Here x : dependent variable, y : independent variable] SOLUTION IS $x = X(y)$

(iii) $u_t = k^2 u_{xx}$ (or) $\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2} \rightarrow$ PDE (Here $u = u(x,t)$ is solution, x,t : independent variables, u : dependent variable)

(iv) $\frac{\partial^2 w}{\partial u^2} + \frac{\partial^2 w}{\partial v^2} = 0 \rightarrow$ PDE (w : Dependent variables, u,v : Independent variable) and solution is $w = w(u,v)$

(2) Order:

The order of a dif eq. is the order of the highest derivative appearing in a dif eq.

Example:

(i) $y' = -ky \rightarrow$ first order Dif Eq.

(ii) $(y'')^3 + y' = \sin x \rightarrow$ Second order Dif Eq.

(iii) $\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^5 + y = 0 \rightarrow$ Second order Dif Eq.

(iv) $(x+y^2)dy - xy\,dx = 0 \rightarrow \frac{dy}{dx} = \frac{xy}{x+y^2} \rightarrow$ First order Dif Eq.

Note: Degree is different than ORDER.
The Degree of a dif. eq is the power of the highest order derivative in a dif. eq.

Examples: (i) $y''' + 2(y')^5 + y = 0$ [Order: 3 (Degree is 1)] This is Third order ODE

(ii) $(y'')^3 + (y')^7 = 0$ [Order: 2 (Degree is 3)] This is Second order ODE

(iii) $\left(\frac{d^2y}{dx^2}\right)^3 + \frac{dy}{dx} = \sin x$ [Order: 2 (Degree is 3] This is Second order ODE.

3) Linear or Non-linear

A Linear ODE satisfy the following conditions:

* The dependent variable and its derivatives must occur at the first power only. (Powers must be 1).
* No products of dependent variable and/or any of its derivatives appear in the equation such as yy''
* No transcendental functions of dependent variable and/or its derivatives occur.

If these conditions are not all satisfied, then the dif. eq is called Non-linear.

Examples:

(i) $y''' + ty' + (\cos^2 t)\, y = t^3$ → 3^{rd} order, Linear ODE

(ii) $y'^2 + t\, y' + 4y = 0$ → First order, Nonlinear ODE. First term y'^2 violates the linearity.

(iii) $y'' + \sin(t+y) = \sin t$; $y = y(t)$ → Second order, Nonlinear ODE. Second term $\sin(t+y)$ violates the linearity

4) Homogeneous / Non-homogeneous:

If a dif. eq doesn't involve a constant term and involves unknown function and/or its derivatives; it is called "homogeneous ODE". Otherwise; it is called Non-homogenous ODE (inhomogeneous ODE).

Examples: (i) $\frac{dx}{dt} = x^2 + \underline{\underline{t^3}}$ → Nonhomogeneous ODE

(ii) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 3y = 0$ → Homogeneous ODE

(iii) $\left(\frac{dy}{dx}\right)^2 = y + \underline{\underline{\sin x}}$ → Nonhomogeneous ODE

(iv) $\frac{\partial^3 z}{\partial x^3} = 2\frac{\partial^2 z}{\partial x \partial y} + x$ → Nonhomogeneous PDE

* Determine the order of each of the following diff. equations; Also state whether the equation is linear or nonlinear, show explicitly the term(s) which violate linearity. Also determine whether the equation is homogeneous or nonhomogeneous: (12 pts) (5 April 2014)

(i) $\frac{d^5y}{dt^5}+\frac{d^2y}{dt^2}+3\sin t^2=0 \Rightarrow$ order: 5
Linear
Nonhomogeneous

(ii) $\frac{d^2x}{dt^2}+x\frac{dx}{dt}-1=0 \Rightarrow$ Order: 2
Nonlinear because of $x\frac{dx}{dt}$
Nonhomogeneous

(iii) $\frac{d^4y}{dx^4}-y^2=0 \Rightarrow$ Order: 4
Nonlinear because of y^2
Homogeneous

(iv) $\frac{d^3x}{dy^3}-\frac{d^2x}{dy^2}+y^4-3=0 \Rightarrow$ Order: 3
Linear
Nonhomogeneous

Reminder
For linear diff. equations
- There are no products of the function and its derivatives
- Neither the function or its derivatives occur to any power other than the first power
- No trascendental functions appears

* Nov 1/2014 (10 pts)

Determine the order of the given diff equations. Then state whether the equations are linear or nonlinear. If the eq. is nonlinear, show explicitly term or terms.

(i) $\frac{d^4y}{dt^4}+(\sinh t)y=\frac{e^t}{y} \Rightarrow$ 4th order,
Nonlinear because of the term $\frac{e^t}{y}$

(ii) $t\frac{dy}{dt}+(t+2)y=\ln t \Rightarrow$ 1st order
Linear

(iii) $(u'')^2+uu'=u^2+1 \Rightarrow$ 2nd order
Nonlinear

* April 2013/[12 pts]

Determine the order of each of the following diff. equations; also state ~~that~~ whether the equation is linear or nonlinear. If it is nonlinear, show explicitly the term(s) which violate linearity

(i) $y'''+ty'+(\cos t)^2y=t^3 \Rightarrow$ 3rd order
Linear

(ii) $(1+y^2)y''+ty'+y=e^t \Rightarrow$ 2nd order
Nonlinear because y^2y'' violates linearity

(iii) $y''+\sin(t+y)=\sin t \Rightarrow$ 2nd order
Nonlinear because $\sin(t+y)$ " "

(iv) $(y')^3+yy'=0 \Rightarrow$ 1st order
Nonlinear because both terms $(y')^3$ and yy' violates linearity

* Nov 3/2013 [12pts]

a) Determine the order of each of the following equations and indicate whether the eq. is linear or nonlinear. If the eq. is nonlinear, show explicitly the term(s) which violate linearity.

(i) $y^2 y'' + \sin y = t$ ⇒ 2nd order
Nonlinear, Because the terms y^2, $y^2 y''$, $\sin y$ violate linearity

(ii) $y''' + e^t y' - 2y = 1 + t^2$ ⇒ 3rd order
Linear

(iii) $y'^2 + \sqrt{t}\, y = 0$ ⇒ 1st order
Nonlinear, Because of the term y'^2

(iv) $(2-t)y'' + yy' = \cos(t-y)$ ⇒ Second order
Nonlinear, because the terms yy' and $\cos(t-y)$ violate linearity

EX. (Nov 2014 Midterm [10 pts]

Determine the order of the given diff. equations. Then state whether the equation is linear or nonlinear.
If the equation is non-linear, show explicitly the nonlinear term or terms.

(i) $\dfrac{d^4y}{dt^4} + (\sinh t)\, y = e^t$ ①
t : independent variable
y : dependent variable

(ii) $t\dfrac{dy}{dt} + (t+2)y = \ln t$
t : independent variable
y : dependent variable

(iii) $(u'')^2 + uu' = u^2 + 1$
↳ The dependent variable
The dependent variable does not occur to the first power.
↳ In this term, products of u and its derivative occurs.

SOLUTION: Reminder!

The linear ODE must satisfy the following conditions
1) The dependent variable y and its derivatives occur to the first power only.
2) No products of y and /or any of its derivatives appear in the equation
3) No transcendental functions of y and /or its derivatives occur.

(iii) 2nd order, Nonlinear

(i) 4th order, nonlinear

(ii) 1st order, linear

Definition: If the n th derivative of $y=\phi(t)$ function $\phi, \phi', \phi'', \dots, \phi^{(n)}$ exist in $t \in (\alpha, \beta)$ and satisfy the equation

$$y^{(n)} = f(t; y, y', \dots, y^{(n-1)})$$

then, $y=\phi(t)$ function is a solution of the ordinary diffi equation

Example: Show that $y=e^x$ is a solution for the diffi eq. $y''-2y'+y=0$

Solution: $y=e^x \Rightarrow \left.\begin{array}{l} y'=e^x \\ y''=e^x \end{array}\right\}$ $y''-2y'+y = e^x-2e^x+e^x=0$ ///

Example: Show that the functions $y_1(t)=t^{-2}$ and $y_2(t)=t^{-2}\ln t$ are solutions of the diffi eq.

$$t^2y''+5ty'+4y=0 \quad ; \quad t>0$$

SOLUTION:

(*) $y_1(t)=t^{-2} \Rightarrow \left.\begin{array}{l} y_1'(t)=-2t^{-3} \\ y_1''(t)=6t^{-4} \end{array}\right\}$ $t^2y_1''+5ty_1'+4y_1 = t^2(6t^{-4})-10t\,t^{-3}+4t^{-2} = 6t^{-2}-10t^{-2}+4t^{-2} = 0$ ///

(**) $y_2(t)=\underbrace{t^{-2}}_{u}\cdot\underbrace{\ln t}_{v} \Rightarrow y_2'(t) = -2t^{-3}\ln t + t^{-2}\frac{1}{t} = \underbrace{t^{-3}}_{u}\underbrace{(-2\ln t+1)}_{v}$

$$y_2''(t) = -3t^{-4}(-2\ln t+1)+t^{-3}\left(-2\frac{1}{t}+0\right)$$

$$= 6t^{-4}\ln t - 3t^{-4}-2t^{-4}$$

$$= t^{-4}(6\ln t-5)$$

$$\Rightarrow t^2y_2''+5ty_2'+4y_2 = t^{-2}(6\ln t-5)+5t^{-2}(-2\ln t+1)+4t^{-2}\ln t$$

$$= t^{-2}(6\ln t-5-10\ln t+5+4\ln t) = 0 \text{ ///}$$

Hmw: 1) Show that the function $y=\cos t\ln(\cos t)+t\sin t$ is a solution of the diffi. eq.

$$y''+y=\sec t \quad , \quad 0<t<\frac{\pi}{2}$$

$$(\ln u)' = \frac{u'}{u}$$

SOLUTION: $y = \underbrace{\cos t}_{u}\cdot\underbrace{\ln(\cos t)}_{v} + \underbrace{t}_{u}\,\underbrace{\sin t}_{v}$

* $y' = -\sin t\cdot\ln(\cos t)+\cos t\cdot\dfrac{-\sin t}{\cos t}+\sin t+t\cos t$

$$= -\sin t\cdot\ln(\cos t)-\sin t+\sin t+t\cos t$$

$$= \underbrace{-\sin t}_{u}\cdot\underbrace{\ln(\cos t)}_{v}+t\cdot\cos t$$

* $y'' = -\cos t\ln(\cos t)-\sin t\cdot\dfrac{-\sin t}{\cos t}+\cos t-t\sin t$

$$= -\cos t\cdot\ln(\cos t)+\frac{\sin^2 t}{\cos t}+\frac{\cos t}{1}\frac{}{(\cos t)}-t\sin t$$

$$= -\cos t\cdot\ln(\cos t)+\frac{\sin^2 t+\cos^2 t}{\cos t}-t\sin t = -\cos t\cdot\ln(\cos t)+\overbrace{\frac{1}{\cos t}}^{\sec t}-t\sin t$$

$$\Rightarrow \quad y''+y = -\cos t \ln(\cos t)+\sec t - \cancel{t\sin t} + (\cos t)\ln(\cos t) + \cancel{t\sin t}$$

$$= \sec t \quad ///$$

Verify by substitution that each given function is a solution of the given differential equation.

(1) $y''=9y$; $y_1=e^{3x}$; $y_2=e^{-3x}$

$y_1=e^{3x} \Rightarrow y_1'=3e^{3x}$, $y_1''=9e^{3x}$; $y_1'' \overset{?}{=} 9y_1$; $9e^{3x}=9e^{3x}$ ✓

$y_2=e^{-3x} \Rightarrow y_2'=-3e^{-3x}$, $y_2''=9e^{-3x}$; $y_2'' \overset{?}{=} 9y_2$; $9e^{-3x}=9e^{-3x}$ ✓

(2) $y''+y=3\cos 2x$; $y_1=\cos x-\cos 2x$; $y_2=\sin x-\cos 2x$

$\Rightarrow y_1=\cos x-\cos 2x \Rightarrow y_1'=-\sin x-2\sin 2x$, $y_1''=-\cos x-4\cos 2x$

$y_1''+y_1 \overset{?}{=} 3\cos 2x$

$-\cancel{\cos x}-4\cos 2x+\cancel{\cos x}-\cos 2x \overset{?}{=} 3\cos 2x$

$-5\cos 2x \neq 3\cos 2x$

So; y_1 is not a solution of $y''+y=3\cos 2x$

$\Rightarrow y_2=\sin x-\cos 2x \Rightarrow y_2'=\cos x-2\sin 2x$, $y_2''=-\sin x-4\cos 2x$

$y_2''+y_2 \overset{?}{=} 3\cos 2x$

$-\cancel{\sin x}-4\cos 2x+\cancel{\sin x}-\cos 2x \overset{?}{=} 3\cos 2x$

$-5\cos 2x \neq 3\cos 2x$

So; y_2 is not a solution of diff eq

(3) $x^2y''+xy'-y=\ln x$; $y_1=x-\ln x$; $y_2=\dfrac{1}{x}-\ln x$

$\Rightarrow y_1=x-\ln x \Rightarrow y_1'=1-\dfrac{1}{x}$, $y_1''=-\dfrac{(-1)}{x^2}=\dfrac{1}{x^2}$

$x^2\cdot\dfrac{1}{x^2}+x\left(1-\dfrac{1}{x}\right)-x+\ln x \overset{?}{=} \ln x$

$\cancel{1}+\cancel{x}-\cancel{1}-\cancel{x}+\ln x \overset{?}{=} \ln x$

$\ln x=\ln x$ ✓

Yes y_1 is a solution of diff eq

$\Rightarrow y_2=\dfrac{1}{x}-\ln x$

$y_2'=-\dfrac{1}{x^2}-\dfrac{1}{x}$

$y_2''=+\dfrac{2x}{x^4}+\dfrac{1}{x^2}=\dfrac{2}{x^3}+\dfrac{1}{x^2}$

$x^2\cdot\dfrac{2}{x^3}+x^2\dfrac{1}{x^2}-x\cdot\dfrac{1}{x^2}-x\cdot\dfrac{1}{x}-\dfrac{1}{x}+\ln x \overset{?}{=} \ln x$

$\dfrac{2}{x}+1-\cancel{\dfrac{1}{x}}-1-\cancel{\dfrac{1}{x}}+\ln x \overset{?}{=} \ln x$

$\ln x=\ln x$

So; Yes; y_2 is a solution of diff eq

(4) $y'+2xy^2=0$; $y=\dfrac{1}{1+x^2}$

$y=\dfrac{1}{1+x^2} \Rightarrow y'=\dfrac{-2x}{(1+x^2)^2}$

$y'+2xy^2 \overset{?}{=} 0$

$\dfrac{-2x}{(1+x^2)^2}+2x\cdot\dfrac{1}{(1+x^2)^2} \overset{?}{=} 0$

$0=0$ ✓

Yes ; y is a solution of diff eq

NOT: Bir dif denk.i çözerken, belirli bir aralıkta denklemi sağlayan fonk.lar aranır. (9,5)

Solution of Dif. Equations

General Solution

* This solution contains arbitrary constants essential in number equal to the order of the equation. When the value of the dependent variable is substituted into the dif. eq., the eq. is satisfied

Ex/ $y_G = C_1 e^x + C_2 x$

(mertebe sayısı kadar keyfi sbt. gelir. D.D.i sağlayan, bir ya da birden fazla keyfi sbt. bulunduran ve bu nedenle bir eğri ailesini oluşturan çözüme, genel çözüm denir.)

(* The general solution gives information about the structure of the complete solution space for the problem. In practice, one is often interested only in particular solutions that satisfy some conditions related to the area of applications.)

Particular Solution

* The particular solution is the one obtained by giving a particular definite value to the constant in the general solution.

(Genel çözümdeki keyfi sabitlerin, özel kısıtlamalara göre değerler almasıyla elde edilen çözümdür.)

CONDITIONS

- Initial Value Problems (IVP)
- Boundary Value Problems (BVP)

* IVP: The system is assumed to start evolving from the fixed initial

Ex/ $\left\{\begin{array}{l} y(0)=3 \\ y'(0)=5 \end{array}\right.$ (Independent variable values are the same)

(Conditions)

Here the initial point is 0.

* BVP: Ex/ $y(0)=3$

$y'(1)=5$

BVP has conditions specified at different value of the independent variable in the equation.

(Farklı initial pointlerde kısıtlamalar vardır.)

Singular Solution

It is a unique type. It's characteristic is such that it is not derived from the general solution by assigning a particular value to the constant. But this solution satisfies the dif. eq.

(Denklemi sağlar, fakat genel çözümden elde edilemez. Yani, dif. denk.in herhangi bir çözümü, genel çözümdeki sabitlere değerler atanarak elde edilmiyorsa, bu çözüm tekil çözümdür.)

So; $y_1(t)$ and $y_2(t)$ are 2 different solutions of the ODE. (9.6)

* Integrals as General and Particular Solutions

Def 1:
In order to obtain the general solution of first order dif eq

$$\frac{dy}{dx} = f(x) \qquad (1)$$

only integrate both sides of Eq (1); that is

$$\boxed{y(x) = \int f(x)\,dx + C}$$

→ This is general solution of Eq (1), meaning that it involves an arbitrary constant C.

For every choice of C, it is a solution of the dif eq in Eq (1)

Def 2: If the dif eq is given by initial condition; we call the dif eq as "Initial Value Problem" (IVP). So;

initial condition (I.C)

$$\boxed{\frac{dy}{dx} = f(x) \quad ; \quad y(x_0) = y_0} \Rightarrow \text{Initial Value Problem (IVP)}$$

This solution is called "particular solution"

Example: Solve the initial value problem

$$\frac{dy}{dx} = 2x+3 \quad ; \quad y(1) = 2 \qquad (*)$$

SOLUTION: Step1: Integrate both sides of eq. (*):

$$\frac{dy}{dx} = 2x+3 \Rightarrow y(x) = \int (2x+3)\,dx \Rightarrow y(x) = \frac{2x^2}{2} + 3x + C$$

$$\boxed{y(x) = x^2 + 3x + C}$$

Step2: Apply I.C is $\underset{x_0}{y(1)} = \underset{y_0}{2} \Rightarrow 2 = 1^2 + 3.1 + C \Rightarrow C = -2$

So; the desired particular solution is: $y(x) = x^2 + 3x - 2$ ///

First Order Diff. Eq.s

Separable D.E

$M(y)\,dy = N(x)\,dx$

↓

SOLUTION METHOD

↓

Seperate the variables and integrate both sides

↓

$\int M\,dy = \int N\,dx$

$y = u(x) + c$

Linear First Order D.E

$y' + p(x)\,y = q(x)$

↓

SOLUTION METHOD

↓

Find Integrating Factor $\rho = \rho(x)$

SUBSTITUTION METHODS

- Homogeneous D.E
- Bernoulli D.E
- Exact D.E

Seperable Equations

The first-order differential equation in the form of

$$M(y)\,dy = N(x)\,dx$$

is called "Seperable Dif. Equation".

SOLUTION METHOD

Step 1: Seperate the variables if it is not.

Step 2: Integtrate both sides

$$\int M(y)\,dy = \int N(x)\,dx$$

$$F(y(x)) = G(x) + C$$

$$F(y) = G(x) + C$$

Example 1

$\frac{dy}{dx} = y^2$ Solve D.E

SoL:

$$\frac{dy}{y^2} = dx \Rightarrow \int y^{-2}\,dy = \int dx$$

$$-\frac{1}{y} = x + C$$

$$\underbrace{y(x) = -\frac{1}{x+C}}_{\text{General Solution}}$$

Example 2

$\frac{dy}{dx} = y^2 \; ; \; y(\underset{x_0}{0}) = \underset{y_0}{1}$ Solve IVP

SoL:

$$y(x) = -\frac{1}{x+c} \Rightarrow 1 = -\frac{1}{c}$$

$$C = -1$$

So; $y(x) = -\frac{1}{x-1} \Rightarrow \underbrace{\boxed{y(x) = \frac{1}{1-x}}}_{\text{Particular Solution}}$

Example 3/ Solve IVP $\frac{dy}{dx} = -6xy \; ; y(0) = 7$

SOLUTION: a) $\frac{dy}{y} = -6x\,dx$

$$\int \frac{dy}{y} = -6\int x\,dx$$

$$\ln|y| = -6\cdot\frac{x^2}{2} + C$$

$$\boxed{\ln|y| = -3x^2 + C}$$

NOTE: If $y' = -6xy \; ; y(0) = -4$

Then; $\ln|y| = -3x^2 + C$

$y(\underset{x}{0}) = \underset{y}{-4} \Rightarrow \; - \; y(x) < 0$

So; $\ln(-y) = -3x^2 + C$

$$y(x) = -e^{-3x^2+C}$$

$$= e^{-3x^2}\cdot\underbrace{(-e^{C})}_{K}$$

$$y(x) = Ke^{-3x^2}$$

$\Rightarrow y(0) = -4 = K$

So; $y(x) = -4e^{-3x^2}$

$y(0) = 7 \rightarrow \begin{matrix} x=0 \\ y=7 \end{matrix} \rightarrow y(x) > 0 \rightarrow \ln y = -3x^2 + C$

$$y(x) = e^{-3x^2+C} = e^{-3x^2}\cdot\underbrace{e^{C}}_{K} = Ke^{-3x^2}$$

$\underline{y(x) = K.e^{-3x^2}}$ — General Solution

$\Rightarrow y(0) = 7 \Rightarrow 7 = K\underbrace{e^{0}}_{1} \Rightarrow \boxed{K=7}$ $\rightarrow$ $y(x) = 7.e^{-3x^2}$ "particular solution"

* A solution of a dif. eq. that contains an "arbitrary constant" commonly called a "general solution".
* A singular solution is a solution that cannot be obtained from the general solution by ...

Example 2/ Solve the differential equation

$$\frac{dy}{dx} = \frac{4-2x}{3y^2-5} \qquad y(x)=?$$

Solution: seperate the variables and integrate both sides:

$$\int (3y^2-5)\, dy = \int (4-2x)\, dx$$

$$H(x,y) = x^2-4x+y^3-5y=C$$

implicit solution (Kapalı fonk. çözüm)

$$\frac{\cancel{3}y^3}{\cancel{3}} - 5y = 4x - \frac{\cancel{2}x^2}{\cancel{2}} + C$$

$$y^3-5y = 4x-x^2+C$$

(It is not convenient to solve this explicitly in terms of x.

Definition: * Explicit solution: It is a solution of diff eq. where the dependent variable can be seperated. In another expression, the solution can be written in the form of $y=y(x)$.

* Implicit Solution: It is any solution that is not written in explicit form. It is in the form of $K(x,y)=C$

Example: Solve the diff eq in EX 2 for $y(1)=3$

SOLUTION:

$$y^3-5y = 4x-x^2+C$$

$y(1)=3 \Rightarrow$ for $x=1 \rightarrow y=3$

$$3^3-5.3 = 4.1-1^2+C$$

$$27-15 = 4-1+C \Rightarrow C=9.$$

So, $\underline{y^3-5y = 4x-x^2+9}$ is a particular solution ✓

(thus the desired particular solution $y(x)$ is defined implicitly by the equation)

Example $x+yy'=0$

$$y\frac{dy}{dx} = -x$$

$$\int y\, dy = \int -x\, dx$$

$$\frac{y^2}{2} = \frac{-x^2}{2} + C$$

$\boxed{y^2+x^2=C_1}$ → Implicit solution

③ (12)

Example: $2\sqrt{x}\,\frac{dy}{dx} = \cos^2 y$; $y(4) = \frac{\pi}{4}$

x_0 y_0

SOLUTION: Dif eq. is in the form of $M(y)\,dy = N(x)\,dx$.

So; It is seperable difi eq.

Step1: seperate the variables:

$$\frac{dy}{\cos^2 y} = \frac{dx}{2\sqrt{x}}$$

Step2: Integrate both sides:

$$\int \frac{1}{\cos^2 y}\,dy = \int \frac{dx}{2\sqrt{x}}$$

$$\tan y = \frac{1}{\cancel{2}}\,\cancel{2}\sqrt{x} + C$$

$$\int \frac{1}{\cos^2 y}\,dy = \tan y + C_1$$

$$\int \frac{dx}{\sqrt{x}} = \int x^{-1/2}\,dx = \frac{x^{-1/2+1}}{-1/2+1} + C_2$$

$\boxed{\tan y = \sqrt{x} + C}$ → General solution (Implicit solution)

Step3: $y(4) = \frac{\pi}{4}$ ⇒ $\tan\frac{\pi}{4} = \sqrt{4} + C$

x_0 y_0

$1 = 2 + C \Rightarrow \boxed{C = -1}$

So, $\boxed{\tan y = \sqrt{x} - 1}$ → particular solution

Example: $\frac{dy}{dx} = y^2$ solve dif equation $(y' = y^2)$

SOLUTION: Seperable dif eq.

Step 1: Seperate the variables:

$$\frac{dy}{y^2} = dx$$

Step 2: Integrate both sides:

$$\int \frac{dy}{y^2} = \int dx$$

$$\int y^{-2} dy = \int dx$$

$$\frac{y^{-1}}{-1} = x + C \Rightarrow -\frac{1}{y} = x + C \Rightarrow \boxed{y = -\frac{1}{x+C}}$$ general solution

(for $x \neq -C$)

Step 3: For $y = 0 \Rightarrow 0 = 0 \Rightarrow \underbrace{y = 0}_{\text{is a singular solution}}$.

$\left(\frac{dy}{dx} = y^2\right.$ For $y = 0 \Rightarrow y' = 0$ So: $\left. 0 = 0 \right)$

$\left(-\frac{1}{x+c} \neq 0\right.$ for any choice of $\left. c \right)$

(Because $0 \neq -\frac{1}{x+c} \Rightarrow 0 \neq -1$)

NOTE: A singular solution is not derived from the general solution. Singular solution satisfies general solution.

Example: $y' = 6x(y-1)^{2/3} \rightarrow y(x) = 1 + (x^2 + c)^3$ Gen. Sol.

$\Rightarrow$ for $y(x) = 1 \Rightarrow y' = 0$ So, $\underline{y(x) = 1}$ is a singular solution

Also $y(x) = 1$ satisfies general solution.

Linear First-order Dif Equations

standard form

$$\frac{dy}{dx} + P(x)\,y = q(x) \qquad \text{Eq (1)}$$

where coefficient functions $P(x)$, $q(x)$ are continuous on an open interval $x \in (\alpha, \beta)$

SOLUTION METHOD

* **Step 1:** Put the equation in standard form $\frac{dy}{dx} + P(x)\,y = q(x)$ (1)
* **Step 2:** Calculate integrating factor $\rho(x)$

$$\rho(x) = e^{\int P(x)\,dx}$$

* **Step 3:** multiply both sides of eq(1) by $\rho(x)$:

$$\rho(x)\frac{dy}{dx} + \underline{\rho(x)P(x)}\cdot y = \rho(x)\cdot q(x)$$

Recognize that the left-hand side of the resulting equation is the derivative of the product of 2 functions. That is $\rightarrow$ $\left((u\cdot v)' = u'v + v'u\right)$

Let $\rho'(x) = \rho(x)P(x)$

then

$$\underbrace{\rho(x)\frac{dy}{dx} + \rho'(x)\cdot y}_{(y\rho)'} = \rho(x)\cdot q(x)$$

$$(y\cdot\rho)' = \rho(x)\cdot q(x) \quad \text{or} \quad D_x\left[y(x)\cdot\rho(x)\right] = \rho(x)\cdot q(x)$$

Step 4: Integrate both sides of Eq to find $y(x)$

$$\int (y\cdot\rho)' = \int \rho(x)\,q(x)$$

$$y(x) = \frac{\int \rho(x)\,q(x)\,dx + C}{\rho(x)} \rightarrow \text{General Solution}$$

Step 5: If IVP is asked: apply $y(x_0) = y_0$ to the General Solution to find C. Then find particular Solution

Example: Solve the IVP

$$(1)\quad \frac{dy}{dx} - y = \frac{11}{8} e^{-\frac{x}{3}} \quad , \quad y(0) = -1$$

Step 1: $y' + p(x)y = q(x)$ (Dif eq is in this form) ✓

$p(x) = -1 \quad ; \quad q(x) = \frac{11}{8} e^{-x/3}$

Step 2: Find integrating factor $\rho(x)$

$$\rho(x) = e^{\int p(x)\,dx} = e^{\int (-1)\,dx} = e^{-x} \Rightarrow \underline{\rho(x) = e^{-x}}$$

Step 3: multiply both sides of Eq (1) by $\rho(x)$

$$e^{-x}\left(\frac{dy}{dx} - y\right) = e^{-x} \cdot \frac{11}{8} e^{-\frac{x}{3}}$$

$$\underbrace{e^{-x} \cdot y' - y \cdot e^{-x}} = \frac{11}{8} e^{-\frac{4x}{3}}$$

Recognize: $(e^{-x} \cdot y)'$

$$(e^{-x} \cdot y)' = \frac{11}{8} e^{-\frac{4x}{3}}$$

Step 4: Integrate both sides to find $y(x)$

$$\int e^{u}\,du = e^{u} + c$$

$$\Rightarrow \int e^{au}\,du = \frac{1}{a} e^{au} + c$$

$$e^{-x} \cdot y = \frac{11}{8} \int e^{-\frac{4x}{3}}\,dx$$

$$e^{-x} \cdot y = \frac{11}{8}\left(\frac{-3}{4}\right) e^{-\frac{4x}{3}} + c$$

$$e^{-x} \cdot y = \frac{-33}{32} e^{-\frac{4x}{3}} + c$$

$$y(x) = -\frac{33}{32} e^{-\frac{4x}{3}} \cdot e^{x} + c e^{x}$$

$$y(x) = -\frac{33}{32} e^{-\frac{x}{3}} + c e^{x} \text{ — General Solution}$$

Step 5: IVP $y(0) = -1 \Rightarrow$

$$-1 = \frac{-33}{32} e^{0} + c \cdot e^{0}$$

$$-1 = -\frac{33}{32} + c \Rightarrow c = -1 + \frac{33}{32} = \frac{1}{32} \quad \boxed{c = \frac{1}{32}}$$

So, desired particular solution

$$y(x) = -\frac{33}{32} e^{-x/3} + \frac{1}{32} e^{x}$$

*Example : Find general solution of

$$(x^2+1)\frac{dy}{dx}+3xy=6x$$

SOLUTION : Step 1: Put the given dif eq in standard form.

$$(*) \quad \frac{dy}{dx}+\underbrace{\frac{3x}{x^2+1}}_{p(x)}y=\underbrace{\frac{6x}{x^2+1}}_{q(x)}$$

First-order Linear Eq $(y'+p(x)y=q(x))$

$p(x)=\frac{3x}{x^2+1}$; $q(x)=\frac{6x}{x^2+1}$

Step 2: Find the integrating factor $\rho(x)$:

$$\rho(x)=e^{\int p(x)\,dx}=e^{\int\frac{3x}{x^2+1}dx}$$

make substitution:
$\begin{bmatrix} x^2+1=u \\ 2x\,dx=du \\ x\,dx=du/2 \end{bmatrix}$

$$\rho(x)=e^{3\int\frac{1}{2}\frac{du}{u}}=e^{\frac{3}{2}\ln u}=e^{\frac{3}{2}\ln(x^2+1)}=(x^2+1)^{3/2}$$

$$\rho(x)=(x^2+1)^{3/2}$$

Step 3: multiply both sides of (the standard form of) eq $(*)$ by $\rho(x)$

$$(x^2+1)^{3/2}\cdot y'+(x^2+1)^{3/2}\cdot\frac{3x}{(x^2+1)}y=\frac{(x^2+1)^{3/2}\cdot 6x}{x^2+1}$$

$$(x^2+1)^{3/2}\cdot y'+3(x^2+1)^{1/2}xy=6x\cdot(x^2+1)^{1/2}$$

Step 4: Recognize left side

$$\left((x^2+1)^{3/2}\cdot y(x)\right)'=6x(x^2+1)^{1/2}$$

Step 5: Integrate both sides with respect to x to find y(x)

$$(x^2+1)^{3/2}\cdot y(x)=6\int x(x^2+1)^{1/2}dx$$

Substitution:
$\begin{bmatrix} t=x^2+1 \\ dt=2x\,dx \Rightarrow x\,dx=\frac{dt}{2} \end{bmatrix}$

$$(x^2+1)^{3/2}\cdot y(x)=6\cdot\frac{1}{2}\int t^{1/2}dt$$

$$(x^2+1)^{3/2}\cdot y(x)=3\cdot\frac{t^{3/2}}{3/2}+c \Rightarrow y(x)=\frac{2(x^2+1)^{3/2}+c}{(x^2+1)^{3/2}}$$

$y(x) = 2 + c\,(x^2+1)^{-3/2}$ General solution

* Example: Solve $y' - 2y = e^{-x}$, $y(0) = 1$

Solution: (1) $y' + p(x)\,y = q(x)$ $\quad p(x) = -2$ and $q(x) = e^{-x}$

(2) $\rho(x) = e^{\int p(x)\,dx} = e^{\int -2\,dx} = e^{-2x}$

(3) $\underbrace{e^{-2x} \cdot y' - e^{-2x} \cdot 2 \cdot y} = e^{-2x} \cdot e^{-x}$

$\left(e^{-2x} \cdot y\right)' = e^{-3x}$

$e^{-2x} \cdot y = \int e^{-3x}\,dx$

$e^{-2x} y = -\frac{1}{3} e^{-3x} + c$

$y(x) = -\frac{1}{3} e^{-x} + c \cdot e^{2x}$ General solution

(4) $y(0) = 1 \Rightarrow$ $\quad 1 = -\frac{1}{3} e^{-0} + c\,e^{2.0}$

$x_0 \quad y_0$

$1 = -\frac{1}{3} + c \Rightarrow c = \frac{4}{3}$

$y(x) = -\frac{1}{3} e^{-x} + \frac{4}{3} e^{2x}$

//

Theorem: If the functions $P(x)$ and $Q(x)$ are continuous on an open interval I containing the point x_0, then the Initial Value Problem

$$\frac{dy}{dx} + P(x)y = Q(x) \quad , \; y(x_0) = y_0$$

has a unique solution $y(x)$ on I given by

$$y(x) = e^{-\int P(x)dx} \cdot \left[\int Q(x)\, e^{\int \mu(x)dx} dx + c \right]$$

for an appropriate value of c.

—o—

NOTE: ** First order Linear dif. eq can be given in the form of $\frac{dx}{dy}$ instead of $\frac{dy}{dx}$. So; we want to find $x(y) = ?$

Ex/ $(x + ye^y)\frac{dy}{dx} = 1$ → It is not linear. But we can write this eq as $\frac{dx}{dy}$

$$\frac{dy}{dx} = \frac{1}{x + ye^y}$$

$$\frac{dx}{dy} = x + ye^y$$ (Not dependent variable is x) So; it is now Linear ☺.

$$\underbrace{x'(y) - x = ye^y}$$ ($x' + p(y)x = q(y)$) Linear

$p(y) = -1$; $q(y) = ye^y$

$$\Rightarrow \rho = e^{\int p(y)dy} = e^{-\int dy} = e^{-y}$$

So; $e^{-y}x'(y) - e^{-y}x = e^{-y}ye^y$

$$(x \cdot e^{-y})' = y$$

$$xe^{-y} = \int y\,dy \Rightarrow xe^{-y} = \frac{y^2}{2} + c$$

$$x(y) = e^{y} \cdot \frac{y^2}{2} + e^{y} \cdot c$$ ///

* **Example:** Solve $y' + y = te^{-t} + 1$

SOL:

Step 1: $\left(\frac{dy}{dt} + p(t)y = q(t)\right)$; $p(t) = 1$

$q(t) = te^{-t} + 1$

$$\rho(t) = e^{\int p(t)\,dt} = e^{\int (1)\,dt} = e^{t} \Rightarrow \underbrace{\rho(t) = e^{t}}_{\text{integrating factor.}}$$

Step 2: $e^{t} \cdot y' + e^{t} \cdot y = e^{t} \cdot t \cdot e^{-t} + e^{t}$

$$(e^{t} \cdot y)' = t + e^{t}$$

$$\frac{d}{dt}(e^{t} \cdot y) = t + e^{t}$$

$$e^{t} \cdot y = \int (t + e^{t})\,dt$$

$$e^{t} \cdot y = \frac{t^2}{2} + e^{t} + c$$

$$\underline{y(t) = \frac{t^2}{2} e^{-t} + 1 + c \cdot e^{-t}}$$

* **Example:** Solve IVP $y' - y = 2te^{2t}$; $y(0) = 1$

SOL:

Step 1: $p(t) = -1$; $q(t) = 2te^{2t}$

$$\rho(t) = e^{\int p(t)\,dt} = e^{\int -dt} = e^{-t} \Rightarrow \underline{\rho(t) = e^{-t}}$$

Step 2: $e^{-t} \cdot y' - e^{-t} \cdot y = e^{-t} \cdot 2te^{2t}$

$$(e^{-t} \cdot y)' = 2te^{t} \Rightarrow \frac{d}{dt}(e^{-t} \cdot y) = 2te^{t}$$

$$e^{-t} \cdot y = \int 2\underbrace{t}_{u}\underbrace{e^{t}\,dt}_{dv} \quad \left[\begin{array}{l} u = t \\ du = dt \end{array}\right. \quad \left\{\begin{array}{l} \int dv = \int e^{t}\,dt \\ \underline{v = e^{t}} \end{array}\right.$$

$$e^{-t} \cdot y = 2\left[\, te^{t} - \int e^{t}\,dt \right]$$

$$e^{-t} \cdot y = 2te^{t} - 2e^{t} + c$$

$$\underline{y(t) = 2te^{2t} - 2e^{2t} + ce^{t}}$$

step3: $y(0)=1 \Rightarrow \quad 1 = 2\,0\,e^0 - 2e^0 + c.e^0$

$$1 = -2 + c$$

$$\boxed{c = 3}$$

So/ $y(t) = 2te^{2t} - 2e^{2t} + 3e^{t}$

$$\underline{y(t) = 2e^{2t}(t-1) + 3e^{t}} \; ///$$

Example: Solve IVP $\quad ty' - \frac{t}{2}y = te^{-t} \quad ; \; y(0) = -1$

step1: $\underline{y' - \frac{1}{2}y = e^{-t}}$ it is in the standard form)

$$p(t) = -\frac{1}{2} \quad ; \quad q(t) = e^{-t}$$

step2: $\rho(t) = e^{\int p(t)\,dt} = e^{\int -\frac{1}{2}dt} = e^{-\frac{1}{2}t} \Rightarrow \underline{\rho(t) = e^{-t/2}}$ integrating factor.

step3: $e^{-t/2}.y' - \frac{1}{2}e^{-t/2}y = e^{-t/2}.e^{-t}$

$$\left(y.\,e^{-t/2}\right)' = e^{-\frac{3}{2}t} \Rightarrow \frac{d}{dt}\left(y e^{-t/2}\right) = e^{-\frac{3}{2}t}$$

$$y e^{-t/2} = \int e^{-\frac{3}{2}t}\,dt$$

$$y e^{-t/2} = -\frac{2}{3}e^{-\frac{3}{2}t} + c$$

$$y(t) = -\frac{2}{3}e^{-\frac{3}{2}t}.e^{\frac{t}{2}} + c.e^{t/2}$$

$$y(t) = -\frac{2}{3}e^{-t} + c.e^{t/2}$$

step4: $y(0) = -1 \Rightarrow \quad -1 = -\frac{2}{3} + c \Rightarrow \quad c = -\frac{1}{3}$

So; $\quad \underline{y(t) = -\frac{2}{3}e^{-t} - \frac{1}{3}e^{t/2}}$ //

*** Example:**

$$(1-4xy^2)\frac{dy}{dx} = y^3 \quad \text{Solve dif eq}$$

Solution: **Step1/** $y' - 4xy^2y' = y^3$ It is not linear. But if we rewrite the dif eq as

$$\frac{dy}{dx} = \frac{y^3}{1-4xy^2}$$

$$\frac{dx}{dy} = \frac{1-4xy^2}{y^3} \Rightarrow \underbrace{\frac{dx}{dy} + \frac{4}{y}x = \frac{1}{y^3}}_{(1)}$$

This is Linear
(Note: $x' + P(y)x = q(y)$)
(So we want to find $x(y) = ?$)
(Here, $y \to$ independent variable
$x \to$ dependent ")

Step 2: $P(y) = \frac{4}{y}$; $q(y) = \frac{1}{y^3}$

Integrating factor: $\rho(y) = e^{\int P(y)dy} \Rightarrow \rho(y) = e^{\int \frac{4}{y}dy} = e^{4\ln y} = y^4$

$$\underline{\rho(y) = y^4}$$

Step3: $y^4x' + y^4 \cdot \frac{4}{y}x = y^4 \frac{1}{y^3}$ (multiply both sides of Eq(1) by $\rho(y)$)

$$y^4x' + 4y^3x = y$$

$$(xy^4)' = y$$

$$\frac{d}{dy}(xy^4) = y$$

$$xy^4 = \int y\,dy \Rightarrow xy^4 = \frac{y^2}{2} + C \Rightarrow x(y) = \frac{1}{2y^2} + \frac{C}{y^4} \quad ///$$

Substitution methods and Exact Equations

The first-order dif. eqs we have solved in the previous sections have all been either separable or linear. But many applications involve dif eq.s that are neither separable nor linear.

We use substitution methods that sometimes can be used to transform a given dif eq. into separable or linear.

If the given dif eq is in the form of

$$\frac{dy}{dx} = f(x,y)$$

with dependent variable y and independent variable x;

we substitute $v = \alpha(x,y)$ that suggests itself as a new independent variable v. Thus the dif eq becomes linear or separable.

Example: $\frac{dy}{dx} = (x+y+3)^2$

Substitution:

$\underline{x+y+3 = v(x,y)}$ (*) $\Rightarrow$ $\underline{y = v - x - 3}$, (Now $y = y(v,x)$)

$$\frac{dy}{dx} = \frac{d}{dx}(v - x - 3)$$

$$\underline{\frac{dy}{dx} = \frac{dv}{dx} - 1} \quad (**)$$

So; $\frac{dy}{dx} = (x+y+3)^2$ Plug (*) and (**) :

$$\frac{dv}{dx} - 1 = v^2$$

$\underline{\frac{dv}{dx} = v^2 + 1}$ This dif eq is now separable ✓ $\Rightarrow \int \frac{dv}{v^2+1} = \int dx \Rightarrow \tan^{-1} v = x + c$

$$v = \tan(x+c)$$

$$x + y + 3 = \tan(x+c)$$

$$\underline{y(x) = \tan(x+c) - x - 3}$$

As a Summary:

So, the substitution method is:

$$\frac{dy}{dx} = f(x,y) \quad (1) \qquad \left[\text{or } F(y', y, x)\right]$$

↓

Substitution: $\underbrace{v = \alpha(x,y)}_{\text{(leave } y \text{ alone)}}$ (2)

$y(x) = \beta(x, v)$

Take derivative of y

$$\frac{dy}{dx} = \frac{\partial \beta}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial \beta}{\partial v} \cdot \frac{dv}{dx}$$

$$\boxed{y' = \beta_x + \beta_v \frac{dv}{dx}} \quad (3)$$

where the partial derivatives $\frac{\partial \beta}{\partial x} = \beta_x(x,v)$ and $\frac{\partial \beta}{\partial v} = \beta_v(x,v)$ are known functions of x and v.

↓

Substitute (2) and (3) into (1):

$$\frac{dv}{dx} = g(x, v) \qquad \left[\text{So, } F(y', y, x) \longrightarrow G(v', v, x)\right]$$

Separable ✓ Linear ✓

↓

Solve $v = v(x)$

↓

Go back to $y(x)$ ✓

NOTE: Any dif eq. of the form $\frac{dy}{dx} = F(ax+by+c)$ can be trasformed into a separable / linear eq. by using the substitution $\underline{v = ax+by+c}$

Homogeneous First order Dif. Eq.

If a dif. eq. of the form $\frac{dy}{dx} = f(x,y)$ can be written as a function of $\frac{y}{x}$ or $\frac{x}{y}$, that is

$$\underbrace{\frac{dy}{dx} = F\left(\frac{y}{x}\right)} \quad (1)$$

then it is called "homogeneous equation". If not, it is called "non-homogeneous equation".

Examples: Determine whether the following dif. eq.s are homogeneous or non-homogeneous

a) $\frac{dy}{dx} = \frac{y^2+2xy}{x^2}$

$$\frac{dy}{dx} = \underbrace{\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)}_{F\left(\frac{y}{x}\right)} \Rightarrow \text{homogeneous eq.}$$

b) $\frac{dy}{dx} = \frac{y^3+2xy}{x^2} \Rightarrow \frac{dy}{dx} = y \cdot \left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right)$

because of y; It is non-homogeneous.

c) $\frac{dy}{dx} = \ln x - \ln y + \frac{x+y}{x-y}$

$$\frac{dy}{dx} = \ln\left(\frac{x}{y}\right) + \frac{\cancel{x}\left(1+\frac{y}{x}\right)}{\cancel{x}\left(1-\frac{y}{x}\right)}$$

$$\frac{dy}{dx} = \underbrace{\ln\left(\frac{1}{y/x}\right) + \left(\frac{1+\frac{y}{x}}{1-y/x}\right)}_{F\left(\frac{y}{x}\right)} \Rightarrow \text{It is homogeneous.}$$

SOLUTION METHOD:

Step1: $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$ (1) (If the given dif. eq is not in std form, Convert it into standard form)

Step2: make substitution

$$u = \frac{y}{x} \quad \text{or} \quad \underline{y = u.x} \quad (2)$$

Step3: Differentiate y : $\dfrac{dy}{dx} = \dfrac{du}{dx}.x + u$

$$y' = \underbrace{u + xu'}_{F(u)} \Rightarrow F(u) = u + xu'$$

$$\frac{du}{dx}.x = F(u) - u$$

$$\frac{du}{F(u)-u} = \frac{dx}{x} \Rightarrow \text{Seperable dif. eq} \quad (3)$$

NOT: All homogeneous eq. can be reduced to an seperable eq. (integration problem)

Step4: Integrate Eq.(3):

$$\int \frac{dv}{F(u)-u} = \int \frac{dx}{x} \quad (4)$$

Find $u(x)$

Step5: Replace $u = \dfrac{y}{x}$ and find $y(x)$

* **Example:** $2xy\dfrac{dy}{dx} = 4x^2 + 3y^2$ Solve dif eq.

SOLUTION:

Step1: $\dfrac{dy}{dx} = \dfrac{4x^2+3y^2}{2xy} = \dfrac{4x^2}{2xy} + \dfrac{3y^2}{2xy} = 2\left(\dfrac{x}{y}\right) + \dfrac{3}{2}\left(\dfrac{y}{x}\right)$

$$\underline{\frac{dy}{dx} = 2\left(\frac{1}{(y/x)}\right) + \frac{3}{2}\left(\frac{y}{x}\right)} \quad \text{Homogeneous eq}$$

Step2: $u = \dfrac{y}{x} \Rightarrow y = u.x$

$$y' = u'x + u$$

$$u'x + u = 2.\frac{1}{u} + \frac{3}{2}u \Rightarrow u'x + u = \frac{2}{u} + \frac{3u}{2} = \frac{4+3u^2}{2u}$$

$$u'x = \frac{4+3u^2}{2u} - u \Rightarrow u'x = \frac{4+3u^2-2u^2}{2u} = \frac{4+u^2}{2u}$$

$$u' x = \frac{4+u^2}{2u}$$

$$x \cdot \frac{du}{dx} = \frac{4+u^2}{2u} \Rightarrow \underbrace{\frac{2u}{4+u^2}\,du = \frac{dx}{x}}$$ $\Rightarrow$ seperabe eq

Step 4: $$\int \frac{2u}{4+u^2}\,du = \int \frac{dx}{x} \qquad \left[\begin{array}{l} 4+u^2 = t \\ 2u\,du = dt \end{array}\right.$$

$$\int \frac{dt}{t} = \int \frac{dx}{x} \Rightarrow \ln t = \ln|x| + \ln C$$

$$\ln|\underbrace{4+u^2}_{>0}| = \ln|\underbrace{x}_{>0}| + \ln C$$

$$4+u^2 = C \cdot x \Rightarrow \underbrace{u^2 = Cx-4}$$

Step 5: $\frac{y}{x} = u \Rightarrow$ $$\left(\frac{y}{x}\right)^2 = Cx-4$$

use back substitution to find $y(x)$

$$\underbrace{y^2 = x^2(Cx-4)}$$

$$\boxed{y^2 = -4x^2 + Cx^3}$$

Example: Solve $xy\frac{dy}{dx} = \frac{3}{2}y^2 + x^2$

SOLUTION: 1) $$\frac{dy}{dx} = \frac{3y^2}{2xy} + \frac{x^2}{xy} \Rightarrow \frac{dy}{dx} = \frac{3}{2}\left(\frac{y}{x}\right) + \left(\frac{x}{y}\right)$$

$$\underbrace{\frac{dy}{dx} = \underbrace{\frac{3}{2}\left(\frac{y}{x}\right) + \left(\frac{1}{y/x}\right)}_{F(y/x)}}$$ $\Rightarrow$ Homogeneous eq. (1)

2) $$\frac{y}{x} = u \Rightarrow \left.\begin{array}{l} y = u \cdot x \\ y' = u'x + u \end{array}\right\} (2)$$

Plug (2) into (1): $$u'x + u = \frac{3}{2}u + \frac{1}{u} \Rightarrow u'x = \frac{3u}{2} + \frac{1}{u} - u$$

$$u'x = \frac{3u^2 + 2 - 2u^2}{2u} = \frac{u^2+2}{2u} \Rightarrow$$ Seperable eq.

3) $$\frac{du}{dx} \cdot x = \frac{u^2+2}{2u}$$ $\Rightarrow$ seperate variables and integrate

$$\int \frac{2u}{u^2+2}\,du = \int \frac{dx}{x} \Rightarrow \ln(u^2+2) = \ln|x| + \ln C$$

$$\left[\begin{array}{l} u^2+2 = t \\ 2u\,du = dt \end{array}\right]$$

$$\underbrace{u^2+2 = C|x|}$$

4) $$\frac{y}{x} = u$$

$$\left(\frac{y}{x}\right)^2 + 2 = C \cdot |x| \longrightarrow$$

HE4

$$\frac{y^2}{x^2}+2=c\,|x| \Rightarrow y^2+2x^2=c\cdot x^3 \quad (x>0)$$

* Example: $x\dfrac{dy}{dx}=y+\sqrt{x^2-y^2}$; $y(x_0)=0$, $x_0>0$

SOLUTION:

Step1: write the given eq as standard form:

$$\frac{dy}{dx}=\frac{y+\sqrt{x^2-y^2}}{x}$$

$$\frac{dy}{dx}=\frac{y}{x}+\sqrt{\frac{x^2-y^2}{x^2}} \Rightarrow \frac{dy}{dx}=\underbrace{\frac{y}{x}+\sqrt{1-\left(\frac{y}{x}\right)^2}}_{F\left(\frac{y}{x}\right)} \Rightarrow \text{hom. eq} \quad (1)$$

Step2: Substitute: $\dfrac{y}{x}=u \Rightarrow y=u.x$

$$y'=u'x+u \quad (2)$$

Step3: Plug (2) into (1): So hom. eq will be seperable eq

$$u'x+u=u+\sqrt{1-u^2}$$

$$u'x=\sqrt{1-u^2} \Rightarrow \text{Now: It is seperable eq.} \quad (4)$$

kontrol et

$$\int\frac{du}{\sqrt{1-u^2}} \overset{?}{=} \sin^{-1}u+c$$

Step 4: Solve eq (4). Find $u(x)$

$$\frac{du}{dx}\cdot x=\sqrt{1-u^2} \Rightarrow \int\frac{du}{\sqrt{1-u^2}}=\int\frac{dx}{x}$$

$$\sin^{-1}u=\ln x+c \quad (5)$$

(We don't need to write $\ln|x|$. Because $x>0$ near $x=x_0>0$

Step5: Plug $\dfrac{y}{x}=u$ into (5) to find $y(x)$

$$\sin^{-1}\left(\frac{y}{x}\right)=\ln x+c \quad \text{General Solution} \quad (6)$$

Step 6: $y(x_0)=0$, $x_0>0$ Apply I.C. to (6):

$(\sin^{-1}0=0)$

$$0=\ln x_0+c \Rightarrow c=-\ln x_0$$

So; $\sin^{-1}\left(\dfrac{y}{x}\right)=\underbrace{\ln x-\ln x_0}_{\ln(x/x_0)} \Rightarrow \dfrac{y}{x}=\sin\left(\ln\dfrac{x}{x_0}\right) \Rightarrow \boxed{y(x)=x\sin\left(\ln\dfrac{x}{x_0}\right)}$

Special Forms

1) If the dif eq. is in the form of $\frac{dy}{dx} = \frac{ax+by}{dx+ey}$;

Solution Method

step 1:

$$\frac{dy}{dx} = \frac{ax+by}{dx+ey} \quad (1) \quad \underset{\text{out } x}{\overset{\text{Pull}}{\Rightarrow}} \quad \frac{dy}{dx} = \frac{x\left(a+b\frac{y}{x}\right)}{x\left(d+e\frac{y}{x}\right)} = \frac{a+b\frac{y}{x}}{d+e\frac{y}{x}} \quad (2)$$

cancel x

step 2:
Substitute : $v = \frac{y}{x}$

$$v = \frac{y}{x} \Rightarrow \left.\begin{array}{l} y = v.x \\ y' = v'x + v \end{array}\right\} (3)$$

so; Eq(1) will be Homogeneous eq.

Dif eq. is now transformed into separable

step 3: Plug (3) into (2) ⇒

step 4: Solve for $v(x)$

step 5: Find $y(x)$ by substituting $v(x)$ into $v = \frac{y}{x}$

Example: $\frac{dy}{dx} = \frac{x-y}{x+y}$ ① ⇒ $y(x) = ?$

step 1: $\frac{dy}{dx} = \frac{x(1-y/x)}{x(1+y/x)} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}}$ ②

step 2: Substitute: $v = \frac{y}{x}$

$$\left.\begin{array}{l} y = v.x \\ y' = v'x + v \end{array}\right\} \xrightarrow[\text{plug into Eq ②}]{\text{step 3}} \quad v'x + v = \frac{1-v}{1+v} \Rightarrow v'x = \frac{1-v}{1+v} - \frac{v}{1}\,(1+v)$$

step 4: $v'x = \frac{1-v-v-v^2}{1+v} \Rightarrow v'x = \frac{1-2v-v^2}{1+v} \Rightarrow x\frac{dv}{dx} = \frac{1-2v-v^2}{1+v}$

$\xrightarrow{\text{seperate variables}}$ for direct integration to find $v(x)$

$$\int \frac{1+v}{1-2v-v^2}\,dv = \int \frac{dx}{x} \Rightarrow \left[\begin{array}{l} 1-2v-v^2 = u \\ (-2-2v)dv = du \\ -2(1+v)\,dv = du \\ (1+v)\,dv = -\frac{du}{2} \end{array}\right] \Rightarrow \int -\frac{du}{2u} = \int \frac{dx}{x}$$

$$-\frac{1}{2}\ln|u| = \ln|x| + \ln c$$

$$-\frac{1}{2}\ln|1-2v-v^2| = \ln|x| + \ln c$$

$$(1-2v-v^2)^{-1/2} = cx$$

$$\left(1-2v-v^2\right)^{-1/2} = C \cdot x \quad ③$$

Step 5: Plug $v=\frac{y}{x}$ into Eq ③ $\Rightarrow$ $\left(1-2\frac{y}{x}-\frac{y^2}{x^2}\right)^{-1/2} = Cx$

$$\left(\left(1-2\frac{y}{x}-\frac{y^2}{x^2}\right)^{-1/2}\right)^{-2} = (Cx)^{-2} \quad \leftarrow \left(\text{Let } C^{-2}=K\right)$$

$$1-2\frac{y}{x}-\frac{y^2}{x^2} = K\cdot\frac{1}{x^2}$$

$$\boxed{x^2-2xy-y^2=K}$$

② If the dif. eq is in the form of

$$\frac{dy}{dx} = \frac{ax+by+c}{dx+ey+f}$$

solution method:

Step 1: Substitution $\begin{cases} x=u+h \\ y=v+k \end{cases}$ so; $F(y', y, x)$ will be trasfolmed into $F(v', v, u)$.

step 2: solve for $v(x)$.

Step 3: find general solution $y(x)$.

* Example: $\frac{dy}{dx} = \frac{x-y-1}{x+y+3}$ ① ideal ...

step 1: $x=u+h$, $y=v+k$ substitute $\Rightarrow y'=v'$

$$\begin{cases} v' = \dfrac{u+h-v-k-1}{u+h+v+k+3} \\[2ex] v' = \dfrac{u-v+(h-k-1)}{u+v+(h+k+3)} \end{cases}$$

$(h-k-1)=0$, $(h+k+3)=0$ make them zero! to find h and k.

$$v' = \frac{u-v}{u+v} \quad ②$$

Step 2: Solve Eq ②

$$v' = \frac{u-v}{u+v} \Rightarrow \frac{dv}{du} = \frac{\cancel{u}\left(1-\frac{v}{u}\right)}{\cancel{u}\left(1+\frac{v}{u}\right)} = \frac{1-\frac{v}{u}}{1+\frac{v}{u}} \quad ③$$

substitute: $t=\frac{v}{u} \Longrightarrow v=tu$, $v'=t'u+t$ $\xrightarrow{\text{plug into ③}}$ $t'u+t=\frac{1-t}{1+t} \Rightarrow t'u=\frac{1-2t-t^2}{1+t}$

$$\int\frac{1+t}{1-2t-t^2}dt = \int\frac{du}{u} \Rightarrow \left[\begin{array}{l} 1-2t-t^2=a \\ (-2-2t)dt=da \\ (1+t)\,dt=-\frac{da}{2} \end{array}\right] \Longrightarrow \int-\frac{da}{2} = \int\frac{du}{u}$$

$$-\frac{1}{2}\ln|1-2t-t^2| = \ln|u|+\ln C$$

$$-\frac{1}{2}\ln\left|1-2\frac{v}{u}-\frac{v^2}{u^2}\right| = \ln|u|+\ln C$$

$$\left(1-2\frac{v}{u}-\frac{v^2}{u^2}\right)^{-1/2} = C\cdot u$$

$$\frac{u^2-2uv-v^2}{\cancel{u^2}} = \underbrace{c^{-2}}_{k} \cdot \frac{1}{\cancel{u^2}} \Rightarrow \boxed{u^2-2uv-v^2=k} \quad ④$$

Solution for $v(x)$
We want to find $y(x)$

[let's find u and v to go back $y(x)$]

Step 3: We assumed that

$$\left.\begin{aligned} h-k-1&=0 \\ h+k+3&=0 \end{aligned}\right\} \quad \left.\begin{aligned} h-k&=1 \\ h+k&=-3 \end{aligned}\right\} \Rightarrow \left.\begin{aligned} h&=-1 \\ k&=-2 \end{aligned}\right\} ⑤$$

(So; Plug ⑤ into $x=u+h$, $y=v+k$) $\Longrightarrow$ $x=u-1$, $y=v-2$ $\Rightarrow$ $u=x+1$, $v=y+2$

Step 4: Plug u and v into ④:

$$(x+1)^2-2(x+1)(y+2)-(y+2)^2=k \quad \text{General solution}$$

Definition 3: "Second order equations":

Second order dif. eq. is of the form

$$\frac{d^2y}{dx^2} = g(x)$$

We simply integrate both sides 2 times to obtain general solution. That is,

$$\frac{dy}{dx} = \int g(x)\,dx = G(x) + C_1$$

$$y(x) = \int \left(G(x) + C_1\right)dx + C_2 = \int G(x)\,dx + C_1x + C_2$$

where C_2 is a second arbitrary constant

In effect, 2nd order dif. eq $\frac{d^2y}{dx^2}$ can be solved by successively the first-order equations

$$\frac{dy}{dx} = v(x) \quad \text{and} \quad \frac{dv}{dx} = g(x)$$

1

Existence and Uniqueness of Solutions

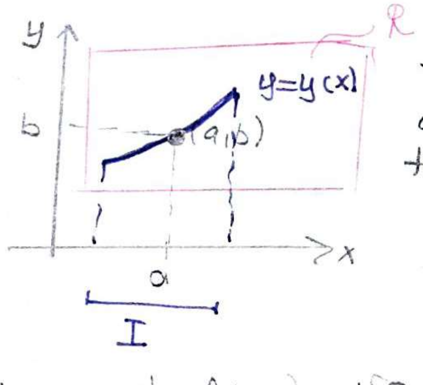
When we examine the solution of an IVP; it is important to evaluate 3 things:

1) The solution exists or not
2) If the solution exists; is it unique?
3) The solution is valid in which interval? (Find the largest validity interval for the solution.

Theorem 1/ Suppose that both functions $f(x,y)$ and its partial derivative $\frac{\partial f(x,y)}{\partial y}$ are continuous on some rectange R in xy-plane that contains the point (a,b) in its interior. Then, for some open interval I containing the point a, the initial value problem

$$\frac{dy}{dx} = f(x,y) \quad , \quad y(a)=b$$

has exactly a unique solution that is defined on the interval I.



The rectangle R and x-interval I of Theorem 1, and the solution curve $y=y(x)$ through the point (a,b)

Acıklama: $y'=f(x,y)$ dif denk.i ve $y(a)=b$ baslangıc kosulu verilsin. $f(x,y)$ ve kısmi türevi $\frac{\partial f(x,y)}{\partial y} = D_y f(x,y)$ fonksiyonları; (a,b) noktasını da içeren bir I açık aralığında sürekli fonksiyonlar olsun. Bu durumda a noktasını içeren ve $y(a)=b$ kosulunu sağlayan $y'=f(x,y)$ dif denk.nin <u>tek</u> bir $y=y(x)$ çözümü vardır.
Yani çözüm vardır ve tektir.

* So; for $y' = f(x,y)$; $y(x_0) = y_0$

→ Existence Condition: $f(x,y)$ function must be continuous at the point (x_0, y_0).

(Yani (x_0,y_0) noktası; $f(x,y)$ fonksiyonunu tanımsız yapmamalı ki sonuç var olsun.)

→ Uniqueness Condition: Partial derivative $\dfrac{\partial f(x,y)}{\partial y}$ must be continuous at (x_0, y_0).

(Yani (x_0,y_0) noktası; f'in y'ye göre kısmi türevini tanımsız yapmamalıdır. ki sonuç tek olsun.)

Example 2/ $\dfrac{dy}{dx} = 2x^2y^2$; $y(1) = -1$

SOL: $\left.\begin{array}{l} f(x,y) = 2x^2y^2 \\ \dfrac{\partial f}{\partial y} = 4x^2y \end{array}\right\}$ are both continuous everywhere. So, there exists unique solution in some neighborhood of $x=1$.

Example 3/ $\dfrac{dy}{dx} = \sqrt[3]{y}$; $\underset{x_0}{y(0)} = \underset{y_0}{0}$

SOL: $f(x,y) = y^{1/3}$ is continuous in a neighborhood of $(0,0)$

$\dfrac{\partial f}{\partial y} = \dfrac{1}{3} \cdot \dfrac{1}{\sqrt[3]{y^2}}$ is not continuous at $(0,0)$

So; the theorem guarantees existence, but no uniqueness in some neighborhood of $x=0$. (there are Infinitely many Solutions)

EX 1/ $y' = \dfrac{1}{x}$; $y(0) = 0$ IVP $\left(\dfrac{dy}{dx} = \dfrac{1}{x} \Rightarrow \int dy = \int \dfrac{dx}{x} \quad y = \ln|x| + c \right)$

$y(0) = 0 \Rightarrow 0 = \ln 0 + c$ undefined

$f(x,y) = \dfrac{1}{x} \Rightarrow \left(f(0,0) = \dfrac{1}{0} \right)$ So; $f(x,y)$ is discontinuous at $(0,0)$.

So; No solution $y(x) = \ln|x| + c$ of the diff eq is defined at $x=0$.

$\ln 0 \rightarrow$ undefined

$\ln 1 = 0$

③

Example: $x\frac{dy}{dx} = 2y \quad ; \quad y(a)=b$ (1) For the IVP

($f(x,y) = \frac{2y}{x}$ and $\frac{\partial f}{\partial y} = \frac{2}{x}$

Both functions are continous everywhere except $x=0$.

$\Rightarrow$ the solution of dif eq is

$$\frac{dy}{y} = 2\frac{dx}{x}$$

$$\ln y = 2\ln x + \ln c$$

$$y(x) = cx^2$$)

So: for the Initial condition $y(a)=b$; the dif eq (1) has

- ✓ * a unique solution near (a,b) if $a \neq 0$
- ✓ * No solution if $a=0$ but $b \neq 0$
- ✓ * Infinitely many solutions if $a=b=0$

NOTE: Because,

$y(x) = Cx^2 \qquad y(0)=0$

$0=0$

$y_1=0 \qquad \boxed{y_2(x)=Cx^2}$

dif eq. has IMS.

$y(x)=Cx^2 \qquad y(0)=4$

$4 = C.0$

$0=4$ Undefined.

So: Dif eq. has no sol.

and $y(x)=Cx^2 \qquad y(1)=2$

$2=C$

$y(x) = 2x^2$

So; dif eq has a unique sol

or

$y(x)=Cx^2 \; ; \; y(1)=0$

$0=C$

So, $y=0$

So, dif eq. has a unique sol.

*** Example:** $y' = t^2 y^{1/2}$; $y(0) = 0$

(i) $y' = f(t,y)$

$f(t,y) = t^2 y^{1/2}$

$f(0,0) = 0^2 \cdot \sqrt{0} = 0$

So; $f(t,y)$ is continuous at $(0,0)$

(ii) $\dfrac{\partial f}{\partial y} = t^2 \dfrac{1}{2} y^{-1/2} = \dfrac{t^2}{2\sqrt{y}}$

$\dfrac{\partial f(0,0)}{\partial y} = \dfrac{0^2}{2\sqrt{0}} \rightarrow$ discontinuous at $(0,0)$

("denominator part is 0")

So; the solution of IVP exists; but it is not unique. There are infinitely many solutions.

*** Example:** $y' + \dfrac{1}{t} y = t^2$; $y(1) = 3$

(i) $y' = \underbrace{t^2 - \dfrac{y}{t}}_{f(t,y)}$

$f(t,y) = t^2 - \dfrac{y}{t} \Rightarrow f(1,3) = 1^2 - \dfrac{3}{1} = -2$

So; $f(t,y)$ function is continuous at $(1,3)$. The solution exists ✓

(ii) $\dfrac{\partial f}{\partial y} = -\dfrac{1}{t}$

$\dfrac{\partial f(1,3)}{\partial y} = -\dfrac{1}{3}$ ✓

$\dfrac{\partial f}{\partial y}$ function is continuous at $(1,3)$. So; the solution is unique. ✓

*** Example:** $\dfrac{dy}{dx} = -y$

(i) $f(x,y) = -y$ is continuous everywhere and (ii) $\dfrac{\partial f(x,y)}{\partial y} = -1$ is continuous everywhere

So; for any (a,b) point $y(x) = C \cdot e^{-x}$ is a unique solution.

* **Ex/** $y' = 2\sqrt{y} \longrightarrow f(x,y) = 2\sqrt{y}$ is continuous for $y > 0$

$\dfrac{\partial f(x,y)}{\partial y} = \dfrac{1}{\sqrt{y}}$ is continuous for $y > 0$ and is discontinuous for $y = 0$

So; the solution exists but it is not unique. So; dif. eq. has IMS

$y_1 = x^2$; $y_2 = 0$ ($x = 0$'da tekil nokta, [illegible])

NOTE: $\dfrac{dy}{dx} = 2\sqrt{y}$

$\int y^{-1/2}\, dy = \int 2\, dx$

$2y^{-1/2+1} = 2x + c$

$\sqrt{y} = x + k$

$y(0) = 0 \Rightarrow k = 0$

$\sqrt{y} = x$

1) $\frac{dy}{dx} = 2x^2y^2$; $y(1) = -1$

SOL: $f(x,y) = 2x^2y^2$, $\frac{\partial f}{\partial y} = 4x^2y$ } are continuous everywhere. So; there exists unique solution in some neighborhood of $x = 1$.

2) $\frac{dy}{dx} = x \ln y$; $y(1) = 1$

SOL: $f(x,y) = x \ln y$, $\frac{\partial f}{\partial y} = \frac{x}{y}$ } are continuous in a neighborhood of $(1,1)$. So the existence of a unique solution is guarantees in some neighborhood of $x = 1$.

3) $\frac{dy}{dx} = \sqrt[3]{y}$; $y(\underset{x_0}{0}) = \underset{y_0}{1}$

SOL: $f(x,y) = y^{1/3}$, $\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$ } Both functions are continuous near $(0,1)$. So the theorem guarantees the existence of a unique solution in some neighborhood of $x = 0$.

4) $\frac{dy}{dx} = \sqrt[3]{y}$; $y(\underset{x_0}{0}) = \underset{y_0}{0}$

SOL: $f(x,y) = y^{1/3}$ is continuous in a neighborhood of $(0,0)$. But

$\frac{\partial f}{\partial y} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{y^2}}$ is not continuous at $(0,0)$.

So; the theorem guarantees existence but no uniquness in some neighborhood of $x = 0$.

5) $\frac{dy}{dx} = \sqrt[3]{y}$; $y(0) = 1$ } SOL: $f(x,y) = \sqrt[3]{y}$, $\frac{\partial f}{\partial y} = \frac{1}{3} \cdot \frac{1}{\sqrt[3]{y^2}}$ } are both continuous near $(0,1)$. So; the theorem guarantees the existence of a unique solution in some neighborhood of $x = 0$.

6) $\frac{dy}{dx} = \sqrt{x - y}$; $y(2) = 2$

SOL: $f(x,y)$ is not continuous at $(2,2)$. Because it is not defined $y > x$. Hence the theorem guarantees neither existence, nor uniqueness in any neighborhood of $x = 2$.

7) $\frac{dy}{dx} = \sqrt{x - y}$; $y(2) = 1$

SOL: $f(x,y) = \sqrt{x - y}$, $\frac{\partial f}{\partial y} = \frac{-1}{2\sqrt{x - y}}$ } are both continuous in a neighborhood of $(2,1)$. So existence ✓ uniqueness ✓

Ex/

Verify that if c is a constant, then the function defined piecewise by

$$y(x)=\begin{cases} 0 & \text{for } x\leq c \\ (x-c)^2 & \text{for } x>c \end{cases}$$

satisfies the dif eq. $\underline{y'=2\sqrt{y}}$ for all x (including the point c) For what values of b, does the IVP $y'=2\sqrt{y}$, $y(0)=b$ have (i) No solution, (ii) a unique solution that is defined for all x

SOL: ∘) If $b<0$ then the IVP $y'=2\sqrt{y}$, $y(0)=b$ has no solution.

(Because the square root of a negative number would be involved)

∘) If $b>0$, we get a unique solution curve through $(0,b)$ defined for all x

∘) If $b=0$; IVP has infinitely many solutions.

Ex/ a) Verify that if c is a constant, then the function defined piecewise by

$$y(x)=\begin{cases} +1 & \text{if } x\leq c \\ \cos(x-c) & \text{if } c<x<c+\pi \\ -1 & \text{if } x\geq c+\pi \end{cases}$$

satisfies the dif eq $y'=-\sqrt{1-y^2}$ for all x

SOL: a) $y(x)=\cos(x-c) \Rightarrow y'=-\sin(x-c)$ satisfies the dif eq $y'=-\sqrt{1-y^2}$ on the interval $c<x<c+\pi$ where $\sin(x-c)>0$. So; it follows that

$$-\sqrt{1-y^2}=-\sqrt{1-\cos^2(x-c)}=-\sqrt{\sin^2(x-c)}$$

$$=-\sin(x-c)=\underline{y'}$$

b) How many different solutions the IVP $y'=-\sqrt{1-y^2}$; $y(a)=b$ has

SOL: ∘) If $|b|>1$, then the IVP $y'=-\sqrt{1-y^2}$; $y(a)=b$ has no solution. Because the square root of negative number would be involved.

∘) If $|b|<1$; then there is only one curve of the form $y(x)=\cos(x-c)$ through the point (a,b). So; this give a unique solution

∘) If $b=\pm 1$; IMS. (defined for all x) through any point of the form $(a,\pm 1)$

SEPARABLE

- st

Examples:

A) Find general solutions (implicitly if necessary, explicitly if convenient) of the dif. eqs. Primes denote derivatives with respect to x.

1) $\frac{dy}{dx}+2xy=0 \Rightarrow$ SOL: $\frac{dy}{dx}=-2xy \Rightarrow \int \frac{dy}{y} = -2\int x\,dx$

$\ln y = -\frac{2x^2}{2}+c$

$\ln y = -x^2+c$

$y = e^{-x^2+c} = e^{-x^2}\cdot e^{c}$, k

$\boxed{y = k\,e^{-x^2}}$ ///

2) $\frac{dy}{dx} = y\sin x$

SOL: $\int \frac{dy}{y} = \int \sin x\,dx$

$\ln y = -\cos x + c$

$y(x) = e^{-\cos x}\cdot e^{c}$, k

$\boxed{y(x) = k\,e^{-\cos x}}$ ///

3) $2\sqrt{x}\,\frac{dy}{dx} = \sqrt{1-y^2}$

SOL: $\int \frac{dy}{\sqrt{1-y^2}} = \int \frac{dx}{2\sqrt{x}}$

$\sin^{-1} y = \sqrt{x}+c$

$\boxed{y(x) = \sin(\sqrt{x}+c)}$ ///

4) $(1+x)\frac{dy}{dx} = 4y$

SOL: $\int \frac{dy}{y} = \int \frac{4\,dx}{1+x} \Rightarrow \ln y = 4\ln(1+x)+\ln c$

$\boxed{y = c\cdot(1+x)^4}$ ///

5) $\frac{dy}{dx}+2xy^2=0$

SOL: $\frac{dy}{dx} = -2xy^2$

$\int \frac{dy}{y^2} = -2\int x\,dx$

$\int y^{-2}\,dy = -2\int x\,dx$

$-y^{-1} = -x^2-c$

$\boxed{y(x) = \frac{1}{x^2+c}}$ ///

6) $\frac{dy}{dx} = 2x\sec y$

$\frac{dy}{dx} = 2x\cdot\frac{1}{\cos y}$

$\int \cos y\,dy = \int 2x\,dx$

$\sin y = x^2+c$

$\boxed{y(x) = \sin^{-1}(x^2+c)}$ ///

7) $\frac{dy}{dx} = (64xy)^{1/3}$

SOL: $\int \frac{dy}{y^{1/3}} = \int 4x^{1/3}dx \Rightarrow \frac{3}{2}y^{2/3} = 3x^{4/3} + \frac{3}{2}c \Rightarrow \boxed{y(x) = (2x^{4/3}+c)^{3/2}}$ ///

***

8) $(1-x^2)\frac{dy}{dx} = 2y$

SOL: $\int \frac{dy}{y} = \int \frac{2\,dx}{1-x^2} \Rightarrow \int \frac{dy}{y} = \int 2\left[\frac{1}{2}\left(\frac{1}{1+x}\right)+\frac{1}{2}\left(\frac{1}{1-x}\right)\right]dx$ (partial fractions)

$\int \frac{dy}{y} = \int \frac{1}{1+x}dx + \int \frac{1}{1-x}dx$

$\begin{pmatrix} 1-x=u \\ -dx=du \\ dx=-du \end{pmatrix}$

$\ln y = \ln(1+x) - \ln(1-x) + \ln c$

$\boxed{y(x) = c\cdot\frac{1+x}{1-x}}$ //

NOTE

$\frac{1}{1-x^2} = \frac{A}{1+x} + \frac{B}{1-x}$

$\frac{1}{1-x^2} = \frac{A(1-x)+B(1+x)}{(1+x)(1-x)}$

$1 = (A+B) + (-A+B)$

$\left.\begin{matrix} A+B=1 \\ -A+B=0 \end{matrix}\right\}$

$B = 1/2$; $A = \frac{1}{2}$

9) $(1+x)^2\frac{dy}{dx} = (1+y)^2$

SOL: $\int \frac{dy}{(1+y)^2} = \frac{dx}{(1+x)^2}$

$\begin{pmatrix} 1+y=u \\ dy=du \end{pmatrix}$ $\begin{pmatrix} 1+x=t \\ dx=dt \end{pmatrix}$

$\int u^{-2}du = \int t^{-2}dt$

$-\frac{1}{1+y} = -\frac{1}{1+x} - c = -\frac{1+c(1+x)}{1+x}$

$1+y = \frac{1+x}{1+c(1+x)} \Rightarrow \boxed{y(x) = \frac{x-c(1+x)}{1+c(1+x)}}$ ///

10) $y' = xy^3$

SOL: $\frac{dy}{dx} = xy^3 \Rightarrow \int \frac{dy}{y^3} = \int x\,dx$

$-\frac{1}{2y^2} = \frac{x^2}{2} - \frac{c}{2}$

$\boxed{y(x) = (c-x^2)^{-1/2}}$ ///

11) $\frac{dy}{dx} = \frac{1+\sqrt{x}}{1+\sqrt{y}}$

SOL: $(1+\sqrt{y})\,dy = (1+\sqrt{x})\,dx$

$\boxed{y + \frac{2}{3}y^{3/2} = x + \frac{2}{3}x^{3/2} + c}$ //

12) $\frac{dy}{dx} = \frac{(x-1)y^5}{x^2(2y^3-y)}$

SOL: $\frac{(2y^3-y)}{y^5}dy = \frac{(x-1)}{x^2}dx$

$\int\left(\frac{2}{y^2} - \frac{1}{y^4}\right)dy = \int\left(\frac{1}{x} - \frac{1}{x^2}\right)dx$

$\boxed{-\frac{2}{y} + \frac{1}{3y^3} = \ln|x| + \frac{1}{x} + c}$ ///

(13) $y' = 1+x+y+xy$ (suggestion: Factor the right-hand side)

SOL: $y' = 1+x+y+xy = (1+x)(1+y)$

$$\frac{dy}{dx} = (1+x)(1+y)$$

$$\int \frac{dy}{1+y} = \int (1+x)\,dx \Rightarrow \underline{\ln|1+y| = x + \frac{x^2}{2} + C}$$

(14) $x^2 y' = 1 - x^2 + y^2 - x^2 y^2$

SOL: $x^2 y' = (1-x^2)(1+y^2)$

$$\int \frac{dy}{1+y^2} = \int \frac{1}{1-x^2}\,dx$$

$$\int \frac{dy}{1+y^2} = \int \left(\frac{1}{x^2} - 1\right) dx$$

$$\tan^{-1} y = -\frac{1}{x} - x + C \Rightarrow \underline{y(x) = \tan\left(C - \frac{1}{x} - x\right)}$$

B) Find explicit particular solutions of IVP

(1) $\frac{dy}{dx} = ye^{x}$; $y(0) = 2e$

SOL: $\int \frac{dy}{y} = \int e^{x}dx \Rightarrow \ln y = e^{x} + \ln C \Rightarrow y(x) = C.\exp(e^{x})$

$y(0) = 2e \Rightarrow 2\cancel{e} = C\cancel{e} \Rightarrow C = 2$

So, $\underline{y(x) = 2\exp(e^{x})}$

(2) $\frac{dy}{dx} = 3x^2(y^2+1)$; $y(0) = 1$

SOL: $\int \frac{dy}{1+y^2} = \int 3x^2 dx \Rightarrow \tan^{-1} y = x^3 + C \Rightarrow y(x) = \tan(x^3 + C)$

$y(0) = 1 \Rightarrow 1 = \tan(0 + C) \Rightarrow C = \tan^{-1}(1)$

$C = \frac{\pi}{4}$

So, $\underline{y(x) = \tan\left(x^3 + \frac{\pi}{4}\right)}$

③ $2\sqrt{x}\,\frac{dy}{dx} = \cos^2 y \quad ; \; y(4) = \frac{\pi}{4}$

SOL: $\int \sec^2 y\, dy = \int \frac{dx}{2\sqrt{x}}$

$\tan y = \sqrt{x} + c \Rightarrow y(x) = \tan^{-1}(\sqrt{x} + c)$

- $y(4) = \frac{\pi}{4}$ implies $c = -1$

So, $\underline{y(x) = \tan^{-1}(\sqrt{x} - 1)}$ ///

④ $\frac{dy}{dx} = 2xy^2 + 3x^2y^2 \quad ; \; y(1) = -1$

SOL: $\frac{dy}{dx} = y^2(2x + 3x^2)$

$\int \frac{dy}{y^2} = \int (2x + 3x^2)\, dx \Rightarrow -\frac{1}{y} = x^2 + x^3 + c \Rightarrow \underline{y(x) = \frac{-1}{x^2 + x^3 + c}}$ ✓

- $y(1) = -1 \Rightarrow -1 = \frac{-1}{1^2 + 1^3 + c} \Rightarrow \underline{c = -1}$

So, $\underline{y(x) = \frac{1}{1 - x^2 - x^3}}$ ///

⑤ $x\frac{dy}{dx} - y = 2x^2 y \quad ; \; y(1) = 1$

SOL: $\frac{dy}{dx} - \frac{y}{x} = 2xy \Rightarrow \frac{dy}{dx} = 2xy + \frac{y}{x} \Rightarrow \frac{dy}{dx} = \frac{2x^2y + y}{x} \Rightarrow$

$\Rightarrow \frac{dy}{dx} = \frac{y(2x^2+1)}{x} \Rightarrow \frac{dy}{y} = \frac{2x^2+1}{x}\, dx \Rightarrow$

$\Rightarrow \int \frac{dy}{y} = \int \left(2x + \frac{1}{x}\right) dx \Rightarrow \ln y = x^2 + \ln x + \ln c$

- $y(1) = 1 \Rightarrow c = e^{-1}$. So, $\boxed{y(x) = xe^{x^2-1}}$ P.S $\quad \underline{y(x) = Cx \cdot e^{x^2}}$ G.S

Exercises

A) find general solutions of the dif. eq.s. If an initial condition is given, find the corresponding particular solution.

1) $y' + y = 2$; $y(0) = 0$

SOL: $p(x) = 1$; $q(x) = 2$

$\rho(x) = e^{\int p(x)dx} = e^{\int 1 dx} = e^{x} \Rightarrow \boxed{\rho(x) = e^{x}}$

$\underbrace{e^{x}.y' + e^{x}.y}= 2e^{x}$

$(e^{x}.y)' = 2e^{x} \Rightarrow e^{x}.y = 2\int e^{x}dx \Rightarrow e^{x}.y = 2e^{x} + c$

$\boxed{y(x) = 2 + e^{-x}.c}$

.) for $y(0) = 0 \Rightarrow 0 = 2 + e^{0}.c \Rightarrow \boxed{c = -2}$

So, $\boxed{y(x) = 2 - 2e^{-x}}$ ///

2) $xy' + 2y = 3x$; $y(1) = 5$

SOL: $y' + \frac{2}{x}y = 3$; $p(x) = \frac{2}{x}$ and $q(x) = 3$

$\rho(x) = e^{\int \frac{2}{x}dx} = e^{2\ln x} \Rightarrow \boxed{\rho(x) = x^{2}}$

$x^{2}y' + x^{2}.\frac{2}{x}y = 3x^{2}$

$x^{2}y' + 2xy = 3x^{2}$

$(x^{2}.y)' = 3x^{2}$

$x^{2}y = \int 3x^{2}dx$

$x^{2}y = x^{3} + c$

$\boxed{y(x) = x + \frac{c}{x^{2}}}$ G.S.

$\Rightarrow y(1) = 5 \Rightarrow c = 4$. So, $\boxed{y(x) = x + \frac{4}{x^{2}}}$ P.S ///

③ $y' = (1-y)\cos x \quad ; \quad y(\pi) = 2$

SOL: $y' = \cos x - y\cos x$

SOL: $\underline{y' + \cos x \, y = \cos x} \qquad p(x) = \cos x \quad ; \quad q(x) = \cos x$

$\rho(x) = e^{\int \cos x \, dx} = e^{\sin x}$

$e^{\sin x} \cdot y' + e^{\sin x} \cdot \cos x \, y = e^{\sin x} \cdot \cos x$

$(y \cdot e^{\sin x})' = e^{\sin x} \cdot \cos x$

$y \cdot e^{\sin x} = \int e^{\sin x} \cdot \cos x \, dx$

$y \cdot e^{\sin x} = \int e^{u} \, du \qquad \left(\begin{array}{l}\sin x = u \\ \cos x \, dx = du\end{array}\right)$

$y e^{\sin x} = e^{\sin x} + c$

$\underline{y(x) = 1 + c \cdot e^{-\sin x}}$ G.S

$\Rightarrow y(0) = 1$ implies $c = 1$,

so $\boxed{y(x) = 1 + e^{-\sin x}}$ ///

④ $y' = 1 + x + y + xy \quad ; \quad y(0) = 0$

SOL: $\underline{y' + (-1-x)\, y = 1 + x}$

way1 $\rho(x) = e^{\int (-1-x)\, dx} = e^{-x-\frac{x^2}{2}}$

$e^{-x-\frac{x^2}{2}} \cdot y = -e^{-x-x^2/2} + c$

$y(x) = -1 + c \, e^{-x-x^2/2}$

$\Rightarrow y(0) = 0 \Rightarrow c = 1$. So $\boxed{y(x) = -1 + e^{-x-\frac{x^2}{2}}}$ ///

NOT **Way 2** $y' = (1+x)(1+y)$ separable D.E

⑤ $(x^2+1)\dfrac{dy}{dx} + 3x^3 y = 6x e^{-\frac{3}{2}x^2} \quad ; \quad y(0) = 1$

SOL **step1** $\dfrac{dy}{dx} + \underbrace{\dfrac{3x^3}{x^2+1}}_{p(x)} y = \underbrace{\dfrac{6x}{x^2+1} e^{-\frac{3}{2}x^2}}_{q(x)} \quad (1)$

step2 $\Rightarrow \rho(x) = e^{\int \frac{3x^3}{x^2+1} dx}$

$\rho(x) = e^{\int \left(3x - \frac{3x}{x^2+1}\right) dx}$

$\rho(x) = e^{\left(3\frac{x^2}{2} - \frac{3}{2}\ln(x^2+1)\right)}$

$\underline{\rho(x) = (x^2+1)^{-3/2} \cdot e^{\frac{3x^2}{2}}} \quad (2)$

step3: multiply both sides of (1) by (2):

$\left((x^2+1)^{-3/2} e^{\frac{3}{2}x^2} \cdot y\right)' = 6x\,(x^2+4)^{-5/2}$

$y \cdot (x^2+1)^{-3/2} e^{\frac{3}{2}x^2} = -2\,(x^2+4)^{-3/2} + c$

$y(x) = -2 e^{\frac{3}{2}x^2} + c \cdot (x^2+1)^{3/2} e^{-\frac{3}{2}x^2}$

step4: $y(0) = 1 \Rightarrow c = 3$. so $\underline{y(x) = -2 e^{\frac{3}{2}x^2} + 3\,(x^2+1)^{3/2} \cdot e^{-\frac{3}{2}x^2}}$ ///

A) Find general solutions of the following dif. eqs

1) $(x+y)\,y' = x-y$

SOL: Step 1

$$\frac{dy}{dx} = \frac{x-y}{x+y} = \frac{\cancel{x}\left(1-\frac{y}{x}\right)}{\cancel{x}\left(1+\frac{y}{x}\right)} = \frac{1-\frac{y}{x}}{1+\frac{y}{x}} \quad (1)$$

Homogeneous D.E

Step 2

Substitute: $v = \frac{y}{x} \Rightarrow \left.\begin{aligned} y &= v.x \\ y' &= v'x + v \end{aligned}\right\} (2)$

Step 3: Plug (2) into (1)

$$v'x + v = \frac{1-v}{1+v}$$

$$v'x = \frac{1-v}{1+v} - v \Rightarrow v'x = \frac{1-v-v(1+v)}{1+v}$$

$$v'x = \frac{1-2v-v^2}{1+v} \Rightarrow \frac{dv}{dx}.x = -\frac{(v^2+2v-1)}{1+v}$$

$$\int \frac{(1+v)\,dv}{v^2+2v-1} = -\int \frac{dx}{x}$$

$$\left[\begin{aligned} v^2+2v-1 &= t \\ (2v+2)\,dv &= dt \end{aligned}\right]$$

$2(v+1)\,dv = dt \rightarrow (v+1)\,dv = \frac{dt}{2}$

$$\frac{1}{2}\int \frac{dt}{t} = -\int \frac{dx}{x}$$

$$\frac{1}{2}\ln t = -\ln x + \ln c \rightarrow t^{1/2} = \frac{c}{x}$$

$$t = \frac{c^2}{x^2} \rightarrow t = \frac{k}{x^2}$$

$$(v^2+2v-1)\,x^2 = k$$

Step 4: Back substitute $v = \frac{y}{x}$ gives

$$\left[\left(\frac{y}{x}\right)^2 + 2\left(\frac{y}{x}\right) - 1\right]x^2 = k$$

$$\boxed{y^2+2xy-x^2=k}$$

(2) $2xyy' = x^2 + 2y^2$

SOL: Step1: $y' = \frac{x^2}{2xy} + \frac{2y^2}{2xy} = \frac{1}{2}\cdot\frac{x}{y} + \frac{y}{x} = \frac{1}{2}\left(\frac{1}{y/x}\right) + \frac{y}{x}$

Homogeneous D.E ✓

Step2: $v = \frac{y}{x} \Rightarrow y = v.x$

$y' = v'x + v$

$v'x + \cancel{v} = \frac{1}{2}\cdot\frac{1}{v} + \cancel{v}$

$v'x = \frac{1}{2v} \Rightarrow \frac{dv}{dx}x = \frac{1}{2v} \Rightarrow \int 2v\,dv = \int \frac{dx}{x}$

$v^2 = \ln x + c$

Step3: $\left(\frac{y}{x}\right)^2 = \ln x + c \Rightarrow \boxed{y^2 = x^2(\ln x + c)}$ //

(3) $xy' = y + 2\sqrt{xy}$

Step1: $y' = \frac{y}{x} + \frac{2\sqrt{xy}}{x} \Rightarrow y' = \frac{y}{x} + 2\sqrt{\frac{xy}{x^2}}$

$\boxed{y' = \frac{y}{x} + 2\sqrt{\frac{y}{x}}}$ Homogeneous D.E ✓

Step2: $v = \frac{y}{x} \Rightarrow y = v.x$

$y' = v'x + v$

$v'x + \cancel{v} = \cancel{v} + 2\sqrt{v} \Rightarrow v'x = 2\sqrt{v} \Rightarrow \frac{dv}{dx}.x = 2\sqrt{v}$

$\int \frac{1}{2\sqrt{v}}\,dv = \int \frac{dx}{x}$

$\sqrt{v} = \ln x + c$

Step3: $\sqrt{\frac{y}{x}} = \ln x + c \Rightarrow \boxed{y^2 = x^2(\ln x + c')}$ //

(4) $(x+y)\,y' = 1$ 13

SOL/ Step1 Substitute $\boxed{x+y = v} \Rightarrow \left.\begin{array}{l} y = v - x \\ y' = v' - 1 \end{array}\right\}$

Step 2: $v.(v'-1) = 1$

$vv' = v + 1 \Rightarrow \frac{dv}{dx}.v = v+1 \Rightarrow \int \frac{v}{v+1}\,dv = \int dx$

$\int \left(1 - \frac{1}{v+1}\right)dv = \int dx$

$x = v - \ln(v+1) - C$

Step3: $\left.\begin{array}{l} x = x + y - \ln(x+y+1) - C \\ \boxed{y = C + \ln(x+y+1)} \end{array}\right\}$ //

3) $xy' = y + \sqrt{x^2+y^2}$

<u>SOL</u>: <u>Step 1</u>: $y' = \frac{y}{x} + \sqrt{\frac{x^2+y^2}{x^2}}$

(1) $\quad y' = \frac{y}{x} + \sqrt{1+\left(\frac{y}{x}\right)^2} \rightarrow$ Homogeneous Dif Eq.

<u>Step 2</u>: Substitute : $v = \frac{y}{x} \Rightarrow \left.\begin{array}{l} y = v.x \\ y' = v'x+v \end{array}\right\} (2)$

<u>Step 3</u>: Plug (2) into (1) :

$$v'x+\cancel{v} = \cancel{v}+\sqrt{1+v^2}$$

$$v'x = \sqrt{1+v^2}$$

$$\int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$$

$$\ln\left(v+\sqrt{v^2+1}\right) = \ln x + \ln c$$

$$v+\sqrt{v^2+1} = c.x$$

<u>Step 4</u>: Back substitute: $\frac{y}{x} = v$ yields

$$\frac{y}{x} + \sqrt{\frac{y^2}{x^2}+1} = c.x$$

$$\frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = c.x \Rightarrow \boxed{y+\sqrt{x^2+y^2} = cx^2}$$

4) $x^2y' = xy + x^2 e^{y/x}$

<u>SOL</u>: <u>Step 1</u>: $y' = \frac{y}{x} + e^{y/x} \Rightarrow$ Homogeneous D.E

<u>Step 2</u>: $v = \frac{y}{x} \Rightarrow \begin{array}{l} y = v.x \\ y' = v'x+v \end{array}$

<u>Step 3</u>: $v'x+\cancel{v} = \cancel{v}+e^{v}$

$$v'x = e^{v}$$

$$\int e^{-v}dv = \int \frac{dx}{x} \Rightarrow -e^{-v} = \ln x + c$$

$$e^{-v} = -\ln x + c$$

$$-v = \ln(c-\ln x) \Rightarrow v = -\ln(c-\ln x)$$

<u>Step 4</u>: $\frac{y}{x} = -\ln(c-\ln x) \Rightarrow \boxed{y = -x\ln(c-\ln x)}$

⑤ $y' = (4x+y)^2$

SOL: **Step 1:** Substitute $v = 4x+y$

So, $y = v - 4x$

$y' = v' - 4$

Step 2: $v' - 4 = v^2$

$v' = v^2 + 4$

$\frac{dv}{dx} = v^2 + 4$

$\int \frac{dv}{v^2+4} = \int dx$

$\frac{1}{2} \tan^{-1} \frac{v}{2} + \frac{c}{2} = x$

$\tan^{-1} \frac{v}{2} + c = 2x$

$\tan^{-1} \frac{v}{2} = 2x - c$

$\frac{v}{2} = \tan(2x-c)$

$v = 2\tan(2x-c)$

Step 3:

Back substitute $v = 4x+y$:

$4x + y = 2\tan(2x-c)$

$\boxed{y = 2\tan(2x-c) - 4x}$

⑤ $y' = \sqrt{x+y+1}$

SOL: Substitute $x+y+1 = v$

So, $y = v - x - 1$

$y' = v' - 1$

Step 2: $v' - 1 = \sqrt{v}$

$v' = \sqrt{v} + 1$

$\frac{dv}{dx} = \sqrt{v} + 1$

$\int \frac{dv}{1+\sqrt{v}} = \int dx$

$x = \int \frac{dv}{1+\sqrt{v}}$ $\quad (v = u^2)$ $\quad dv = 2u\,du$

$x = \int \frac{2u\,du}{1+u}$

$x = 2\left[\int \left(1 - \frac{1}{1+u}\right) du\right]$

$x = 2\int du - 2\int \frac{du}{1+u}$

$x = 2u - 2\ln(1+u) + c$

$\boxed{x = 2\sqrt{x+y+1} - 2\ln(1+\sqrt{x+y+1}) + c}$

⑥ $x(x+y)y' + y(3x+y) = 0$

SOL: step 1: $y' = -\frac{y(3x+y)}{x(x+y)} = -\frac{y}{x}\left[\frac{x\left(3+\frac{y}{x}\right)}{x\left(1+\frac{y}{x}\right)}\right] \Rightarrow y' = -\frac{y}{x}\left[\frac{3+\frac{y}{x}}{1+\frac{y}{x}}\right]$ homogeneous dif. eq

Step 2: $u = \frac{y}{x} \Rightarrow \begin{array}{l} y = ux \\ y' = u'x + u \end{array}$

Step 3: Plug into eq.:

$u'x + u = -u\left(\frac{3+u}{1+u}\right)$

$u'x = -u\frac{(3+u)}{(1+u)} - u = -u\left(\frac{3+u}{1+u} + 1\right)$

$u'x = -u\left(\frac{4+2u}{1+u}\right) \Rightarrow \frac{1+u}{u(4+2u)}\,du = \frac{-dx}{x}$

$$\frac{(1+v)}{v(4+2v)}\,dv = -\frac{dx}{x}$$

$$\frac{1+v}{v(2+v)}\,dv = -2\,\frac{dx}{x} \Rightarrow \int \frac{1+v}{2v+v^2}\,dv = -2\int \frac{dx}{x}$$

$$\left[\begin{array}{l} v^2+2v = u \\ (2v+2)\,dv = du \\ 2(v+1)\,dv = du \\ (v+1)\,dv = \frac{du}{2} \end{array}\right]$$

$$\frac{1}{2}\int \frac{du}{u} = -2\int \frac{dx}{x}$$

$$\frac{1}{2}\ln(v^2+2v) = -2\ln x + \ln c$$

$$(v^2+2v)^{1/2} = c \cdot x^{-2}$$

$$(v^2+2v) = \frac{c}{x^4}$$

Step 4: Back substitute $v = \frac{y}{x}$

$$\left(\frac{y}{x}\right)^2 + 2\,\frac{y}{x} = \frac{c}{x^4}$$

$$\boxed{x^2y^2 + 2x^3y = c}$$

A)

* Find general solutions of the dif. eqs

1) $x^3+3y-xy'=0$ Ans: $y=x^3(C+\ln x)$ L

2) $xy^2+3y^2-x^2y'=0$ Ans: $y=x/(3-Cx-x\ln x)$ S

3) $xy+y^2-x^2y'=0$ Ans: $y=x/(C-\ln x)$ H

4) $3y+x^4y'=2xy$ Ans: $y=C.\exp[(1-x)/x^3]$ S

5) $2xy^2+x^2y'=y^2$ Ans: $y=x/(1+Cx+2x\ln x)$ S

6) $2x^2y+x^3y'=1$ Ans: $y=x^{-2}/(C+\ln x)$ L

7) $2xy+x^2y'=y^2$ Ans: $y=3Cx/(C-x^3)$ H

8) $y'=1+x^2+y^2+x^2y^2$ Ans: $y=\tan(C+x+x^3/3)$ S

9) $x^2y'=xy+3y^2$ Ans: $y=x/(C-3\ln x)$ H

10) $4xy^2+y'=5x^4y^2$ Ans: $y=1/(C+2x^2-x^5)$ S

11) $x^3y'=x^2y-y^3$ Ans: $y^2=x^2/(C+2\ln x)$ H

12) $y'+3y=3x^2e^{-3x}$ Ans: $y=(x^3+c)e^{-3x}$ L

13) $y'=x^2-2xy+y^2$ Ans: $y-x-1=Ce^{2x}(y-x+1)$ (Subs $\to$ ($v=y-x$) $\downarrow$ Separable)

14) $2x^2y-x^3y'=y^3$ Ans: $y^2=Cx^2(x^2-y^2)$ H

15) $3x^5y^2+x^3y'=2y^2$ Ans: $y=x^2/(x^5+Cx^2+1)$ S

16) $xy'+3y=3x^{-3/2}$ Ans: $y=2x^{-3/2}+Cx^{-3}$ L

17) $9x^2y^2+x^{3/2}y'=y^2$ Ans: $y=x^{1/2}/(6x^2+Cx^{1/2}+2)$ S

18) $2y+(x+1)y'=3x+3$ Ans: $y=x+1+C(x+1)^{-2}$ L

19) $y+xy'=2e^{2x}$ Ans: $y=x^{-1}(C+e^{2x})$ L

20) $(2x+1)y'+y=(2x+1)^{3/2}$ Ans: $y=(x^2+x+c)(2x+1)^{-1/2}$ L

30) $y'=\sqrt{x+y}$ Ans: $x=2(x+y)^{1/2}-2\ln[1+(x+y)^{1/2}]+c$ (subs $\to$ $v=x+y$ $\downarrow$ separable)

B) Each of the following dif. eqs is of two different types. (separable, Linear, homogeneous etc.) Hence, derive general solutions for each of these equations in 2 different ways, then reconcile your results.

1) $\frac{dy}{dx} = 3(y+7)x^2$ (S+L)

2) $\frac{dy}{dx} = \frac{2xy+2x}{x^2+1}$ (S+L)

c) Solve the dif. eq

$$\frac{dy}{dx} = \frac{x-y-1}{x+y+3}$$

by finding h and k, so that the substitutions $x = u+h$, $y = v+k$ transform into the homogeneous eq $\frac{dv}{du} = \frac{u-v}{u+v}$

Ans: $y^2 + 2xy - x^2 + 2x + 6y = C$

D) Use the method in C to solve

$$\frac{dy}{dx} = \frac{2y-x+7}{4x-3y-18}$$

Ans: $(x+3y+3)^5 = C(y-x-5)$