* Examples for proof questions related to matrices

* 3 Dec 2016 – Midterm (Mat 261-E)

Show that if A is symmetric matrix, then A^2 is symmetric.

SOLUTION: Assumption: $A^T = A$

Need to show: $(A^2)^T \stackrel{?}{=} A^2$

$$(A^2)^T = (A \cdot A)^T = \underbrace{A^T}_{A} \underbrace{A^T}_{A} = A\,A = A^2 \Rightarrow (A^2)^T = A^2 \checkmark \quad \text{So } A^2 \text{ is symmetric} \checkmark$$

*Theorem: If A is invertible, then its inverse is unique

Proof: ⇒ Assume that A has 2 inverses: B and C

Since B is one of the inverse of A ⇒ $B \cdot A = I$ and $A \cdot B = I$

Since C " another inverse of A ⇒ $C\,A = I$ and $A \cdot C = I$

→ Need to show: $B \stackrel{?}{=} C$

way1:

$B\,A = I$

$B\,\underbrace{AC}_{I} = I \cdot C$

$B = C \checkmark$

Similarly, $A\,B = I$

$\underbrace{C\,A}_{I}B = C \cdot I$

$I\,B = C\,I$

$B = C \checkmark$

way2:

$C \cdot \underbrace{AB}_{I} = I\,B$

$C\,I = I\,B$

$C = B \checkmark$

Similarly $\underbrace{B \cdot A}_{I}\,C = B\,I$

$I \cdot C = B\,I$

$C = B \checkmark$

So: inverse of A is always unique.

* Example: Show that if A is $(n \times n)$ matrix, then

(i) $A + A^T$ is symmetric

(ii) $A - A^T$ is skew-symmetric

SOLUTION: (i) $(A + A^T)^T \stackrel{?}{=} A + A^T$

$$(A + A^T)^T = A^T + (A^T)^T = A^T + A = A + A^T \checkmark$$

(ii) $(A - A^T)^T \stackrel{?}{=} A^T - A$

$$A^T - (A^T)^T = A^T - A \checkmark$$

* <u>Theorem</u>: If A and B are both invertible matrices, then AB is invertible and
$(AB)^{-1} = B^{-1} \cdot A^{-1}$

<u>Proof</u>: <u>Given</u>: A is invertible so A^{-1} exists $\Rightarrow AA^{-1} = I$; $A^{-1}A = I$
B is invertible so B^{-1} " $\Rightarrow B \cdot B^{-1} = I$; $B^{-1}B = I$

<u>Need to show</u>: If the inverse of AB is $B^{-1} \cdot A^{-1}$ then

$(AB)(B^{-1} \cdot A^{-1}) = I$ and $(B^{-1}A^{-1})(AB) = I$ must be true

$$\underbrace{(AB)(B^{-1}}A^{-1}) \overset{?}{=} I \qquad\qquad (B^{-1}\underbrace{A^{-1}) \cdot (A}_{I}B) \overset{?}{=} I$$

$$AA^{-1} \overset{?}{=} I \qquad\qquad B^{-1} \cdot I \cdot B \overset{?}{=} I$$

$$I = I \checkmark \qquad\qquad B^{-1}B \overset{?}{=} I$$

$$I = I \checkmark$$

So; $(AB)^{-1} = B^{-1} \cdot A^{-1}$ ✓

Theorem If $(n \times n)$ matrix A is <u>an invertible matrix</u>, for any b ; $Ax = b$ system has a unique solution and this solution is $x = A^{-1}b$

<u>Proof</u>: <u>Given</u>: A is invertible. So; $A \cdot A^{-1} = I$, $A^{-1}A = I$

<u>Need to show</u>: Unique solution of $Ax = b$ is $x = A^{-1}b$

$$A \cdot x = b$$

$$\underbrace{A^{-1} \cdot A}_{I}x = A^{-1} \cdot b \Rightarrow I \cdot x = A^{-1} \cdot b$$

$$x = A^{-1} \cdot b \checkmark$$

(multiply both sides of $Ax = b$ on the ~~left~~ left by the matrix A^{-1})

Matrix Equations

Example: Find a 3×4 matrix X such that

$$\begin{bmatrix} 4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2 \end{bmatrix} \cdot X = \begin{bmatrix} 3 & -1 & 2 & 6 \\ 7 & 4 & 1 & 5 \\ 5 & 2 & 4 & 1 \end{bmatrix}$$

SOLUTION: Step 1:

$$[A \,|\, I] \sim [A^{-1} \,|\, I]$$

$$\left[\begin{array}{ccc|ccc} \underbrace{\begin{matrix}4 & 3 & 2 \\ 5 & 6 & 3 \\ 3 & 5 & 2\end{matrix}}_{A} & \underbrace{\begin{matrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{matrix}}_{I} \end{array}\right] \overset{\text{ero}}{\sim} \left[\begin{array}{c|c} \underbrace{\begin{matrix}1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1\end{matrix}}_{I} & \underbrace{\begin{matrix}3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9\end{matrix}}_{A^{-1}} \end{array}\right]$$

Step 2

$$X = A^{-1} \cdot B = \begin{bmatrix} 3 & -4 & 3 \\ 1 & -2 & 2 \\ -7 & 11 & -9 \end{bmatrix} \begin{bmatrix} 3 & -1 & 2 & 6 \\ 7 & 4 & 1 & 5 \\ 5 & 2 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -4 & -13 & 14 & 1 \\ -1 & -5 & 8 & -2 \\ 11 & 33 & -39 & 4 \end{bmatrix}$$

$\Rightarrow$ By looking at the 3^{rd} columns of B and X, for instance, we see that the solution of

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 &= 2 \\ 5x_1 + 6x_2 + 3x_3 &= 1 \\ 3x_1 + 5x_2 + 2x_3 &= 4 \end{aligned}$$

is $x_1 = 14$, $x_2 = 8$, $x_3 = -39$

$\Rightarrow$ For example by looking at 4^{th} columns of B and X; we see that the solution of

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 &= 6 \\ 5x_1 + 6x_2 + 3x_3 &= 5 \\ 3x_1 + 5x_2 + 2x_3 &= 1 \end{aligned}$$

is $x_1 = 1$; $x_2 = -2$; $x_3 = 4$

24 March 2018 / Q3

a) Find a Matrix X such that the matrix equation $AX=B$ is satisfied where

$$A=\begin{pmatrix}1&-2&3\\2&1&7\\2&2&7\end{pmatrix},\quad B=\begin{pmatrix}0&0&1&1\\0&1&0&1\\1&0&1&0\end{pmatrix}$$

SOLUTION: If A is invertible; then

$$A.X=B$$

$$\underbrace{A^{-1}A}_{I}.X=A^{-1}B$$

$$X=A^{-1}.B$$

$$[A\,|\,I]=\left[\begin{array}{ccc|ccc}1&-2&3&1&0&0\\2&1&7&0&1&0\\2&2&7&0&0&1\end{array}\right]\overset{ERO}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&7&-20&17\\0&1&0&0&-1&1\\0&0&1&-2&6&-5\end{array}\right]$$

So;

$$X=\underbrace{\begin{bmatrix}7&-20&17\\0&-1&1\\-2&6&-5\end{bmatrix}}_{A^{-1}\ 3\times3}\cdot\underbrace{\begin{bmatrix}0&0&1&1\\0&1&0&1\\1&0&1&0\end{bmatrix}}_{B\ 3\times4}=\begin{bmatrix}17&-20&24&-13\\1&-1&1&-1\\-5&6&-7&4\end{bmatrix}$$ ///

b) use part (a) to find unique solution of the following non-homogeneous system:

$$x_1-2x_2+3x_3=1$$
$$2x_1+x_2+7x_3=1$$
$$2x_1+2x_2+7x_3=0$$

SoL: The unique solution can be read from the $\underline{\underline{4}}^{th}$ column of X as $\bar{b}=\begin{bmatrix}1\\1\\0\end{bmatrix}$ is 4^{th} column of B. Then the unique solution is

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-13\\-1\\4\end{bmatrix}$$

or

Elementary Matrix

The $(n\times n)$ matrix E is called an "elementary matrix" if it can be obtained by performing a single (unique) elementary row operation on the $(n\times n)$ identity matrix I.

* Examples: Are the following matrices elemantory matrix or not?

a) $\begin{bmatrix}3&0\\0&1\end{bmatrix} \Rightarrow$ SOL: $\begin{bmatrix}1&0\\0&1\end{bmatrix} \xrightarrow{3R_1\to R_1} \begin{bmatrix}3&0\\0&1\end{bmatrix} = E_1$ ✓ YES. Because this matrix is obtained by performing 1 operation on Identity matrix I.

b) $\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \Rightarrow$ SOL: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} = E_2$ ✓ YES

c) $\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} \to$ SOL: $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{2R_1+R_3\to R_3} \begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix} = E_3$ ✓ YES

d) $\begin{bmatrix}0&1&0\\1&0&0\\2&0&1\end{bmatrix} \to$ SOL $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \xrightarrow{2R_2+R_3\to R_3} \begin{bmatrix}0&1&0\\1&0&0\\2&0&1\end{bmatrix}$ NO

Because it is obtained by performing 2 operations on (3×3) Identity matrix. So It is not an Elementary Matrix

* Definition: Every elementary matrix is invertible.

Elementary Row operation	Inverse operation
cR_i	$\frac{1}{c}R_i$
$R_i\leftrightarrow R_j$ (swap(R_i,R_j))	$R_i\leftrightarrow R_j$
cR_i+R_j	$(-c)R_i+R_j$

* **Definition**: Let A and B be (nxn) matrices and Let B is the resulting matrix obtained performing some ERO on A. Then, "B is row equivalent to A, if there exists a finite number of Elementary matrices $E_1, E_2, \dots, E_k$ such that

$$B = E_k E_{k-1} E_{k-2} \cdots E_3 E_2 . E_1 . A$$

These elementary matrices are obtained by performing the same ERO's on the Identity matrix (nxn).

* **Example**: $A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix}$

a) Find the elementary matrix such that $B = E . A$

b) Find the elementary matrix such that $A = E . B$

SOLUTION: a) B is the resulting matrix

$$A = \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} \xrightarrow{(-2)R_1 \to R_1} \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix} = B$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{(-2)R_1 \to R_1} \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} = E$$ Elementary matrix

b) A is the resulting matrix.

$$B = \begin{bmatrix} -4 & 2 \\ 5 & 3 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1 \to R_1} \begin{bmatrix} 2 & -1 \\ 5 & 3 \end{bmatrix} = A$$

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{-\frac{1}{2}R_1 \to R_1} \begin{bmatrix} -1/2 & 0 \\ 0 & 1 \end{bmatrix} = E$$ Elementary Matrix

* **Example**: $A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -11 & -5 \end{bmatrix}$ Find the elementary matrices such that

$$B = E_k E_{k-1} E_{k-2} \cdots E_3 E_2 E_1 A$$

SOLUTION: B is the resulting matrix

$$A = \begin{bmatrix} 4 & 1 & 3 \\ 2 & 1 & 4 \\ 1 & 3 & 2 \end{bmatrix} \xrightarrow[\text{1}]{R_1 \leftrightarrow R_3} \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 4 \\ 4 & 1 & 3 \end{bmatrix} \xrightarrow[(-4)R_1 + R_3 \to R_3 \;\; \text{3}]{\text{2} \;\; (-2)R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 3 & 2 \\ 0 & -5 & 0 \\ 0 & -11 & -5 \end{bmatrix} = B$$

Stop! We obtained B

$$\Rightarrow I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{1}]{R_1 \leftrightarrow R_3} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = E_1$$

$$\Rightarrow I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{2}]{(-2)R_1 + R_2 \to R_2} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_2$$

$$\Rightarrow I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{3}]{(-4)R_1 + R_3 \to R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix} = E_3$$

Determinants

Denoted by $| \ |$

* matrices must be $(n \times n)$ to take the determinant

* For $(2\times2) \Rightarrow A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

* For (3×3) and more matrices ; determinant can be computed by 2 ways

Cofactor Expansion — Determinant Properties.

* Cofactor Expansion:

* Minor of an element : "M_{ij}"

M_{ij} : ij th minor of A is the determinant obtained by deleting the i th row and j th column of A

Example: $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

(# of element of A is 9)

(# of Minor is 9)

* minor of $a_{11} \Rightarrow M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$

* minor of $a_{12} \Rightarrow M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23} \cdot a_{31}$

$\vdots$

* minor of $a_{33} \Rightarrow M_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

* Cofactor of an element : "A_{ij}"

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

sign chart $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$, $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$, $\begin{vmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{vmatrix}$, ...

(instead of $(-1)^{i+j}$ we use sign chart)

Example:

(# of cof. element is 9)

Cofactor of $a_{11} \Rightarrow A_{11} = (-1)^{1+1} \cdot M_{11} \Rightarrow A_{11} = M_{11}$

Cofactor of $a_{12} \Rightarrow A_{12} = (-1)^{1+2} \cdot M_{12} \Rightarrow A_{12} = -M_{12}$

Cofactor of $a_{13} \Rightarrow A_{13} = (-1)^{1+3} \cdot M_{13} \Rightarrow A_{13} = M_{13}$

Cofactor of $a_{21} \Rightarrow A_{21} = (-1)^{2+1} \cdot M_{21} \Rightarrow A_{21} = -M_{21}$

$\vdots$

Cofactor of $a_{33} \Rightarrow A_{33} = (-1)^{3+3} \cdot M_{33} \Rightarrow A_{33} = M_{33}$

1) * Compute the determinant of a matrix using Cofactor Expansion

Expanding along i th row

$$\det A = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}$$
$(i = 1, 2, \ldots, n)$

Expanding along j th column

$$\det A = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj}$$
$(j = 1, 2, \ldots, n)$

2) (Hint: Always choose the row or column expansion where the number of zeros is maximum)

E

Example: $A = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 5 & 6 \\ 7 & 1 & 2 \end{bmatrix}$ Compute $|A|$ a) expanding along 1^{st} row
b) expanding along 2^{nd} column.

Note: We have 6 ways to compute $|A|$ (The determinant value doesn't change) You can compute $|A|$ expanding along any row or column.

a)
$$A = \begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{bmatrix} \boxed{3 \quad -1 \quad 2} \\ 4 \quad 5 \quad 6 \\ 7 \quad 1 \quad 2 \end{bmatrix} \end{array} \Rightarrow |A| = (3)\begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} - (-1)\begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} + (2)\begin{vmatrix} 4 & 5 \\ 7 & 1 \end{vmatrix} = -84$$

Exa

b)
$$A = \begin{bmatrix} 3 & \boxed{-1}^{-} & 2 \\ 4 & \boxed{5}^{+} & 6 \\ 7 & \boxed{1}^{-} & 2 \end{bmatrix} \Rightarrow |A| = -(-1)\cdot\begin{vmatrix} 4 & 6 \\ 7 & 2 \end{vmatrix} + (5)\begin{vmatrix} 3 & 2 \\ 7 & 2 \end{vmatrix} - (1)\cdot\begin{vmatrix} 3 & 2 \\ 4 & 6 \end{vmatrix} = -84$$

I
th

Example: $A = \begin{bmatrix} 2 & 0 & 0 & -3 \\ \boxed{0^{-} \quad -1^{+} \quad 0^{-} \quad 0^{+}} \\ 7 & 4 & 3 & 5 \\ -6 & 2 & 2 & 4 \end{bmatrix} \Rightarrow |A| = ?$

Th

$$|A| = (-1)\cdot \underbrace{\begin{array}{c} \begin{matrix} + & - & + \end{matrix} \\ \begin{vmatrix} \boxed{2 \quad 0 \quad -3} \\ 7 \quad 3 \quad 5 \\ -6 \quad 2 \quad 4 \end{vmatrix} \end{array}} = (-1)\cdot\left[4 - 96\right] = 92$$

X

$$2\begin{vmatrix} 3 & 5 \\ 2 & 4 \end{vmatrix} - 3\begin{vmatrix} 7 & 3 \\ -6 & 2 \end{vmatrix}$$

$$2\underbrace{(12-10)}_{2} - 3\underbrace{(14+18)}_{32}$$

? hatla burda bak

Example: Show that
$$|A| = \begin{array}{c} \begin{matrix} + & - & + & - \end{matrix} \\ \begin{vmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & b_2 & 0 \\ 0 & b_3 & a_3 & 0 \\ b_4 & 0 & 0 & a_4 \end{vmatrix} \end{array} = (a_2a_3 - b_2b_3)(a_1a_4 - b_1b_4)$$

$$|A| = a_1\begin{vmatrix} a_2 & b_2 & 0 \\ b_3 & a_3 & 0 \\ \boxed{0 \quad 0 \quad a_4^{+}} \end{vmatrix} - b_1\begin{vmatrix} 0 & a_2 & b_2 \\ 0 & b_3 & a_3 \\ \boxed{b_4^{+} \quad 0 \quad 0} \end{vmatrix} = a_1 \cdot a_4\begin{vmatrix} a_2 & b_2 \\ b_3 & a_3 \end{vmatrix} - b_1b_4\begin{vmatrix} a_2 & b_2 \\ b_3 & a_3 \end{vmatrix}$$

$$= a_1a_4(a_2a_3 - b_2b_3) - b_1b_4(a_2a_3 - b_2b_3)$$

$$= (a_2a_3 - b_2b_3)(a_1a_4 - b_1b_4)$$

Properties of Determinants

1) $\boxed{|A| = |A^T|}$ Transpozing a matrix doesn't change the value of determinant

Example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$

$\Rightarrow A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \Rightarrow |A^T| = \begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad-bc$

$=$ ✓

2) If the $(n \times n)$ matrix B is obtained from A by multiplying a single row (or a column) of A by the constant k, then

$\boxed{|B| = k \cdot |A|}$ If any row or any column is multiplied by a constant 'k' to produce a matrix B; then $|B| = k|A|$

Example: $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 6 & 3 \\ 15 & 1 \end{bmatrix}$

B is obtained by multipluing 1^{st} column of A by 3.

$|A| = \begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix} = 2-15 = -13$

$|B| = \begin{vmatrix} 6 & 3 \\ 15 & 1 \end{vmatrix} = 6-45 = -39 = 3|A|$ So, $|B| = 3|A|$

or ★★★

$|B| = \begin{vmatrix} 6 & 3 \\ 15 & 1 \end{vmatrix} = 3 \underbrace{\begin{vmatrix} 2 & 3 \\ 5 & 1 \end{vmatrix}}_{|A|}$

$|B| = 3|A|$

Example: $\begin{vmatrix} 7 & 15 & -17 \\ -2 & 9 & 6 \\ 5 & -12 & 10 \end{vmatrix} = 3 \cdot \begin{vmatrix} 7 & 5 & -17 \\ -2 & 3 & 6 \\ 5 & -4 & 10 \end{vmatrix}$ Here 3 can be factored out from Column 2

3) If the $(n \times n)$ matrix B is obtained from A by interchanging any 2 rows (or 2 columns), then $\boxed{\det B = -\det A}$ If two rows (or 2 columns) are interchanged to produce matrix B; then $|B| = -|A|$

(Thus; If you switch any 2 rows (or any 2 columns); the determinant switches sign)

EX/ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{} -\begin{vmatrix} c & d \\ a & b \end{vmatrix}$ EX/ $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow[C_1 \leftrightarrow C_2]{} -\begin{vmatrix} b & a \\ d & c \end{vmatrix}$

4) If all elements in a row or in a column of A are "0"; then $\det A = 0$

5) If any 2 rows or any 2 columns are equal (identical); then $\det A = 0$

for (5) Ex/ $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{vmatrix}$ identical So; $|A| = 0$

Because; $-\begin{vmatrix} 2 & 3 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 1 & 3 \end{vmatrix} = 0$

6-6 3-3

let's expand along 2^{nd} row.

for (4) Ex/ $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \end{vmatrix} = 0$

6) A_1, A_2, B are identical except their ith rows (or columns) (That is, the other $(n-1)$ rows (or columns) of 3 matrices are identical) and that the ith row (or column) of B is the sum of the ith rows (or columns) of A_1 and A_2. Then

The determinant behaves like a linear function on the rows or columns

$$\det B = \det A_1 + \det A_2$$

Ex/ $\begin{vmatrix} a_1 & b_1 & c_1+d_1 \\ a_2 & b_2 & c_2+d_2 \\ a_3 & b_3 & c_3+d_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$

Ex/ $\begin{vmatrix} a+\mu & b+\theta \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \mu & \theta \\ c & d \end{vmatrix}$

Ex/ $\begin{vmatrix} a+\mu & b \\ c+\theta & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} \mu & b \\ \theta & d \end{vmatrix}$

7) If the $(n\times n)$ matrix B is obtained by adding a constant multiple of one row (or column) of A to another row (or column) of A, then

$$\det B = \det A$$

(This means; Elementary row or column operations can be done on determinants. Because the value of determinant doesn't change. We use this determinant to create more zeros in determinant. So it is usefull property.

Proof: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \xrightarrow{kR_1+R_2\to R_2} \begin{vmatrix} a & b \\ ka+c & kb+d \end{vmatrix} =$

$= \begin{vmatrix} a & b \\ ka & kb \end{vmatrix} + \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

pull out k

$= k\underbrace{\begin{vmatrix} a & b \\ a & b \end{vmatrix}}_{0} + \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Ex/ $A = \begin{bmatrix} 2 & -3 & -4 \\ -1 & 4 & 2 \\ 3 & 10 & 1 \end{bmatrix}$

$|A| = \begin{vmatrix} 2 & -3 & -4 \\ -1 & 4 & 2 \\ 3 & 10 & 1 \end{vmatrix} \xrightarrow{2C_1+C_3\to C_3} \begin{vmatrix} 2 & -3 & 0 \\ -1 & 4 & 0 \\ 3 & 10 & 7 \end{vmatrix} = 35$

8) $|I| = 1$

9) If $(n\times n)$ matrix A is Triangular, Diagonal, Scalar,

then $|A|$ = Product of main diagonal entries

Ex/ $\begin{vmatrix} 1 & 4 & 3 \\ 0 & -2 & 5 \\ 0 & 0 & 3 \end{vmatrix} = 1(-2)3 = -6$

Triangular (upper) matrix

$\begin{vmatrix} 1 & 0 & 0 \\ 3 & 7 & 0 \\ 4 & 6 & 5 \end{vmatrix} = 35$

lower triangular

Ex/ $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{vmatrix} = 6$

Diagonal matrix

Ex/ $\begin{vmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{vmatrix} = (-2)^3 = -8$

Scalar matrix

10) For $(n\times n)$ matrix A

Theorem: A is invertible (non-singular) matrix $\iff$ $|A| \neq 0$ if and only if

Conversely if $|A| = 0$; then A^{-1} do not exist. A is singular matrix

11) $|A \cdot B| = |A| \, |B|$

$\det(ABC) = \det(A) \det(B) \det(C)$

12) If A is invertible matrix ; then $|A^{-1}| = \dfrac{1}{|A|}$

Proof: $A \cdot A^{-1} = I$

$|A \cdot A^{-1}| = |I|$

$|A| \cdot |A^{-1}| = 1 \rightarrow |A^{-1}| = \dfrac{1}{|A|}$

Example for 10) for which values of a, b, c ; V is invertible matrix?

$$V = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$|V| = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \xrightarrow[(-1)R_1 + R_3 \to R_3]{(-1)R_1 + R_2 \to R_2} \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} = (b-a)(c-a) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{vmatrix} \xrightarrow{(-1)R_2 + R_3 \to R_3}$$

$$(b-a)(c-a) \underbrace{\begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 0 & c-b \end{vmatrix}}_{(1)(1)(c-b)} = (b-a)(c-a)(c-b)$$

$\Rightarrow$ For $\underline{V \text{ is to be invertible} ; |V| \neq 0}$

$(b-a)(c-a)(c-b) \neq 0$ So; $b \neq a$, $c \neq a$, $c \neq b$

So; V is invertible if and only if a, b, c are distinct.

13) For $(n\times n)$ matrix A ; $|kA| = k^n \cdot |A|$

Ex/ $|2A| = 2^n |A|$

16) $|AB| = |A||B|$

14) $|M^k| = |M|^k$

15) For $(n\times n)$ matrix A, the homogeneous linear eq. $Ax=0$ has non-trivial (non-zero) solution if and only iff $\underline{|A|=0}$

Ex/ $2x_1 - x_2 = 0$
$8x_1 - 4x_2 = 0$

step 1

$|A| = \begin{vmatrix} 2 & -1 \\ 8 & -4 \end{vmatrix} = -8+8 = 0$

Since $|A|=0 \Rightarrow$ the system will have non-trivial solution

step 2

$[A|0] = \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 8 & -4 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right]$ New LES $\Rightarrow x_1 - \frac{1}{2}x_2 = 0$

1 eq, 2 unknowns
$2-1=1$ parameter
So; the system has $\underline{\underline{IMS}}$
(non-trivial solution)

Let $x_2 = r$
$r\in\mathbb{R}$ $\Rightarrow x_1 = \frac{1}{2}r$

So; $S.S = \{\frac{1}{2}r, r\}$

Exercises

1) $\begin{vmatrix} (t+1) & 0 & 1 \\ -2 & (t+2) & -1 \\ 0 & 0 & (t-1) \end{vmatrix} = 0$ For which values of t ; det is zero?

$(t-1) \cdot \begin{vmatrix} t+1 & 0 \\ -2 & t+2 \end{vmatrix} = 0 \Rightarrow (t-1)(t+1)(t+2) = 0$

$t=1,\ t=-1,\ t=-2$

2) $\begin{vmatrix} 4 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 1 & 5 & 3 & 5 \end{vmatrix} = 4 \begin{vmatrix} 2 & 0 & 0 \\ 2 & -3 & 0 \\ 5 & 3 & 5 \end{vmatrix} = 4\cdot 2 \begin{vmatrix} -3 & 0 \\ 3 & 5 \end{vmatrix} = 4\cdot 2(-15) = -120$

Evaluate this determinant with your eyes open!

3) The matrices A and B are said to be similar provided that $A = P^{-1}BP$ for some invertible matrix P. Show that if A and B are similar, then $\det(A) = \det(B)$

SOL:

$A = P^{-1}BP$

$\det A = \det(P^{-1}BP)$
$= (\det P^{-1}) \cdot (\det B) \cdot (\det P)$
$= \left(\frac{1}{\det P}\right) \cdot (\det B) \cdot (\det P) \Rightarrow \det A = \det B$

midterm Q / $M = \begin{pmatrix} 4 & a & 0 \\ 0 & 3 & b \\ 0 & 0 & 1 \end{pmatrix}$

(i) $\det(M) = ?$ (ii) $\det(3M) = ?$ (iii) $\det(M^3) = ?$ (iv) $\det(2M + I_3) = ?$

(v) $\det(M^{-1}) = ?$ (vi) $\det(M^{-1} \cdot M^T) = ?$

SOL

(i) $|M| = \begin{vmatrix} 4 & a & 0 \\ 0 & 3 & b \\ 0 & 0 & 1 \end{vmatrix} = 4 \cdot 3 \cdot 1 = 12 \Rightarrow \boxed{|M| = 12}$

(ii) M is (3x3) matrix (So $n = 3$). $|k \cdot M| = k^n \cdot |M|$

$\det(3M) = 3^3 \cdot \det(M) = 3^3 \cdot 12 = \boxed{324}$

(iii) $\det(M^3) = |M^3| = |M|^3 = |M|\,|M|\,|M| = 12^3 = \boxed{1728}$ (Prop. (13) is used. $|A^k| = |A|^k$)

(iv) $\det(2M + I_3) = ?$

$\Rightarrow 2M + I_3 = 2 \cdot \begin{bmatrix} 4 & a & 0 \\ 0 & 3 & b \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 2a & 0 \\ 0 & 7 & 2b \\ 0 & 0 & 3 \end{bmatrix}$

So; $|2M + I_3| = \begin{vmatrix} 9 & 2a & 0 \\ 0 & 7 & 2b \\ 0 & 0 & 3 \end{vmatrix} = 9 \cdot 7 \cdot 3 = \boxed{189}$

Triangular matrix

(v) $|M^{-1}| = ?$ (Prop 12) $\Rightarrow |M^{-1}| = \dfrac{1}{|M|} \Rightarrow \boxed{|M^{-1}| = \dfrac{1}{12}}$

(vi) $|M^{-1} M^T| = |M^{-1}|\,\underbrace{|M^T|}$ ($|M^T| = |M|$) Prop. 1

$= |M^{-1}| \cdot |M| = \dfrac{1}{12} \cdot \cancel{12} = 1 \longrightarrow$ So; $\boxed{|M^{-1} M^T| = 1}$

Midterm Q / Let $A=\begin{bmatrix} 2 & 5 & 3 & 4 \\ -1 & -2 & -2 & -3 \\ 2 & 6 & 4 & 4 \\ 1 & 3 & 8 & 9 \end{bmatrix}$

Find
(i) $\det(A) = ?$
(ii) $\det(A^T) = ?$
(iii) $\det(A^{-1}) = ?$
(iv) $\det(A^4) = ?$
(v) $\det(2A) = ?$

(i)

$$\begin{vmatrix} 2 & 5 & 3 & 4 \\ -1 & -2 & -2 & -3 \\ 2 & 6 & 4 & 4 \\ 1 & 3 & 8 & 9 \end{vmatrix} \xrightarrow{R_1 \leftrightarrow R_4} (-) \begin{vmatrix} 1 & 3 & 8 & 9 \\ -1 & -2 & -2 & -3 \\ 2 & 6 & 4 & 4 \\ 2 & 5 & 3 & 4 \end{vmatrix} \xrightarrow[\substack{(-2)R_1+R_3 \to R_3 \\ (-2)R_1+R_4 \to R_4}]{R_1+R_2 \to R_2} (-) \begin{vmatrix} 1 & 3 & 8 & 9 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & -12 & -14 \\ 0 & -1 & -13 & -14 \end{vmatrix} \xrightarrow{R_2+R_4 \to R_4}$$

(sign is changed!!)

$$- \begin{vmatrix} 1 & 3 & 8 & 9 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & -12 & -14 \\ 0 & 0 & -7 & -8 \end{vmatrix} = -(-12) \cdot \begin{vmatrix} 1 & 3 & 8 & 9 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 1 & \frac{-14}{-12} \\ 0 & 0 & -7 & -8 \end{vmatrix} \xrightarrow{(7)R_3+R_4 \to R_4} (+12) \underbrace{\begin{vmatrix} 1 & 3 & 8 & 9 \\ 0 & 1 & 6 & 6 \\ 0 & 0 & 1 & 7/6 \\ 0 & 0 & 0 & 1/6 \end{vmatrix}}_{\substack{\text{triangular} \\ (1)(1)(1)\left(\frac{1}{6}\right)}} =$$

Pull out (-12) as a common factor.

make out "0"

$= 12 \cdot \frac{1}{6} = 2$

(ii) $|A^T| = |A| = 2$

(iii) $|A^{-1}| = \frac{1}{|A|} = \frac{1}{2}$

(iv) $|A^4| = |A|^4 = 2^4 = 16$

(v) $\det(2A) = 2^4 \cdot |A| = 2^4 \cdot 2 = 2^5 = 32$

(A is (4x4) So; n=4)

* Adjoint matrix : "A_{ij}"

For $(n \times n)$ matrix A $\Rightarrow$ $\text{Adj} A = [\text{cof}(a_{ij})]^T = [A_{ij}]^T$

* Find inverse of A using Adjoint matrix :

$$A^{-1} = \frac{1}{|A|} \cdot \text{Adj} A$$

* Example: Find the inverse of A using Adjoint matrix method.

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{bmatrix}$$

SOLUTION:

·) $A_{11} = + \begin{vmatrix} 2 & 5 \\ 3 & -1 \end{vmatrix} = -17$ ·) $A_{21} = - \begin{vmatrix} 4 & 5 \\ 3 & -1 \end{vmatrix} = 19$ ·) $A_{31} = + \begin{vmatrix} 4 & 5 \\ 2 & 5 \end{vmatrix} = 10$

·) $A_{12} = - \begin{vmatrix} 4 & 5 \\ -3 & -1 \end{vmatrix} = -11$ ·) $A_{22} = + \begin{vmatrix} 1 & 5 \\ -3 & -1 \end{vmatrix} = 14$ ·) $A_{32} = - \begin{vmatrix} 1 & 5 \\ 4 & 5 \end{vmatrix} = 15$

·) $A_{13} = + \begin{vmatrix} 4 & 2 \\ -3 & 3 \end{vmatrix} = 18$ ·) $A_{23} = - \begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} = -15$ ·) $A_{33} = + \begin{vmatrix} 1 & 4 \\ 4 & 2 \end{vmatrix} = -14$

$\Rightarrow$ So; the cofactor of matrix A is

$$A_{ij} = \begin{bmatrix} -17 & -11 & 18 \\ 19 & 14 & -15 \\ 10 & 15 & -14 \end{bmatrix}$$

$$\text{Adj} A = [A_{ij}]^T = \begin{bmatrix} -17 & 19 & 10 \\ -11 & 14 & 15 \\ 18 & -15 & -14 \end{bmatrix}$$

$\Rightarrow$ Inverse of A is $A^{-1} = \frac{1}{|A|} \text{Adj} A = \frac{1}{29} \begin{bmatrix} -17 & 19 & 10 \\ -11 & 14 & 15 \\ 18 & -15 & -14 \end{bmatrix}$

$|A| = 29$

$$A^{-1} = \begin{bmatrix} -17/29 & 19/29 & 10/29 \\ -11/29 & 14/29 & 15/29 \\ 18/29 & -15/29 & -14/29 \end{bmatrix}$$ ///

July 19, 2018 / (Q3 - b) (37.5)

Using elementary column operations, show that

$$\begin{vmatrix} a+d & a-d & g \\ b+e & b-e & h \\ c+f & c-f & k \end{vmatrix} = -2 \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix}$$

SOLUTION:

$$\begin{array}{c} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} \\ \begin{vmatrix} a+d & a-d & g \\ b+e & b-e & h \\ c+f & c-f & k \end{vmatrix} \end{array} \xrightarrow{C_1+C_2 \to C_1} \begin{vmatrix} 2a & a-d & g \\ 2b & b-e & h \\ 2c & c-f & k \end{vmatrix} = (2) \begin{vmatrix} a & a-d & g \\ b & b-e & h \\ c & c-f & k \end{vmatrix}$$

pull out 2.

$$\xrightarrow{(-1)C_1+C_2 \to C_2} (2) \begin{vmatrix} a & -d & g \\ b & -e & h \\ c & -f & k \end{vmatrix} = (2)(-1) \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix} = (-2) \begin{vmatrix} a & d & g \\ b & e & h \\ c & f & k \end{vmatrix}$$

pull out (-1)

///

Cramer's Rule

* Cramer's Rule is used to solve $(n\times n)$ Linear Equation systems using determinant.
* If determinant of the coefficient matrix is not zero, then cramer's Rule can be applied.

Consider the $(n\times n)$ Linear system $AX=b$

$$\left.\begin{array}{l} a_{11}x_1+a_{12}x_2+\dots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\dots+a_{2n}x_n=b_2 \\ \vdots \\ a_{m1}x_1+a_{m2}x_2+\dots+a_{mn}x_n=b_n \end{array}\right\} \qquad A=\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & - & - \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

constant vector b — Coefficient matrix

If $|A|\neq 0 \Rightarrow$ Cramer's Rule can be applied and

$$x_1=\frac{|A_1|}{|A|},\quad x_2=\frac{|A_2|}{|A|},\quad \dots\quad x_n=\frac{|A_n|}{|A|}$$

where

$$|A_1|=\begin{vmatrix} b_1 & a_{12} & \dots & a_{1n} \\ b_2 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \\ b_n & a_{m2} & \dots & a_{mn} \end{vmatrix};\quad |A_2|=\begin{vmatrix} a_{11} & b_1 & \dots & a_{1n} \\ a_{21} & b_2 & \dots & a_{2n} \\ \vdots & \vdots & & \\ a_{m1} & b_n & \dots & a_{mn} \end{vmatrix}, \dots$$

$$|A_n|=\begin{vmatrix} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & & b_2 \\ & \vdots & & \vdots \\ a_{m1} & a_{m2} & & b_n \end{vmatrix}$$

Thus; the denominator of $x_1, x_2, \dots, x_n$ is the determinant of the coefficient matrix.

In the numerator part of x_1, coefficients of x_1 $(a_{11}\ a_{21}, \dots a_{m1})$ are replaced with the right side coefficients $b_1, b_2, \dots, b_n$. In the numerator part of x_2; coefficients of x_2 which are $a_{12}, a_{22}, \dots, a_{m2}$ are replaced with $b_1, b_2, \dots b_n$ and so on.

So; Cramer's Rule always give a <u>unique</u> Solution.

$S.S=\{x_1, x_2, \dots, x_n\}$

* **Example:** Use cramer's Rule to solve the system

$$x_1 + 4x_2 + 5x_3 = 2$$
$$4x_1 + 2x_2 + 5x_3 = 3$$
$$-3x_1 + 3x_2 - x_3 = 1$$

SOLUTION: $|A| = \begin{vmatrix} 1 & 4 & 5 \\ 4 & 2 & 5 \\ -3 & 3 & -1 \end{vmatrix} = 29$

$|A_1| = \begin{vmatrix} 2 & 4 & 5 \\ 3 & 2 & 5 \\ 1 & 3 & -1 \end{vmatrix} = 33$; $|A_2| = \begin{vmatrix} 1 & 2 & 5 \\ 4 & 3 & 5 \\ -3 & 1 & -1 \end{vmatrix} = 35$; $|A_3| = \begin{vmatrix} 1 & 4 & 2 \\ 4 & 2 & 3 \\ -3 & 3 & 1 \end{vmatrix}$

$= -23$

So; $x_1 = \frac{33}{29}$; $x_2 = \frac{35}{29}$; $x_3 = -\frac{23}{29}$

Examples: (3.1)

A) Find A^{-1}, then use A^{-1} to solve $Ax=b$

1) $A=\begin{bmatrix}3&2\\4&3\end{bmatrix}$; $b=\begin{bmatrix}5\\6\end{bmatrix}$ SOL: $|A|=\begin{vmatrix}3&2\\4&3\end{vmatrix}=9-8=1$ > $A^{-1}=\frac{1}{|A|}\begin{bmatrix}3&-2\\-4&3\end{bmatrix}=\begin{bmatrix}3&-2\\-4&3\end{bmatrix}$

So; $x=A^{-1}\cdot b=\begin{bmatrix}3&-2\\4&3\end{bmatrix}\begin{bmatrix}5\\6\end{bmatrix}=\begin{bmatrix}3\\-2\end{bmatrix}$ ///

2) $A=\begin{bmatrix}4&7\\3&6\end{bmatrix}$; $b=\begin{bmatrix}10\\5\end{bmatrix}$ SOL: $|A|=\begin{vmatrix}4&7\\3&6\end{vmatrix}=24-21=3$; $A^{-1}=\frac{1}{3}\begin{bmatrix}6&-7\\-3&4\end{bmatrix}=\begin{bmatrix}2&-7/3\\-1&4/3\end{bmatrix}$

So; $x=A^{-1}b=\begin{bmatrix}2&-7/3\\-1&4/3\end{bmatrix}\begin{bmatrix}10\\5\end{bmatrix}=\begin{bmatrix}25/3\\-10/3\end{bmatrix}=\frac{1}{3}\begin{bmatrix}25\\-10\end{bmatrix}$ ///

B) find the inverse of the given matrix A

1) $A=\begin{bmatrix}5&6\\4&5\end{bmatrix}$ SOL: $[A\,|\,I]=[I\,|\,A^{-1}]$

$\left[\begin{array}{cc|cc}5&6&1&0\\4&5&0&1\end{array}\right]\underset{ERO}{\overset{\vdots}{\sim}}\left[\begin{array}{cc|cc}1&0&5&-6\\0&1&-4&5\end{array}\right]\Rightarrow A^{-1}=\begin{bmatrix}5&-6\\-4&5\end{bmatrix}$ ///

($\underbrace{}_{I}$, $\underbrace{}_{A^{-1}}$)

2) $A=\begin{bmatrix}1&1&5\\1&4&13\\3&2&12\end{bmatrix}\Rightarrow A^{-1}=?$ SOL: $\left[\begin{array}{ccc|ccc}1&1&5&1&0&0\\1&4&13&0&1&0\\3&2&12&0&0&1\end{array}\right]\overset{\vdots}{\sim}\left[\begin{array}{ccc|ccc}1&0&0&-22&2&7\\0&1&0&-27&3&8\\0&0&1&10&-1&-3\end{array}\right]$ ///

$A^{-1}=\dots$

3) $A=\begin{bmatrix}4&0&1&1\\3&1&3&1\\0&1&2&0\\3&2&4&1\end{bmatrix}\Rightarrow A^{-1}$ SOL: $\left[\begin{array}{cccc|cccc}4&0&1&1&1&0&0&0\\3&1&3&1&0&1&0&0\\0&1&2&0&0&0&1&0\\3&2&4&1&0&0&0&1\end{array}\right]\overset{\vdots}{\sim}\left[\begin{array}{cccc|cccc}1&0&0&0&1&-1&1&0\\0&1&0&0&0&-2&-1&2\\0&0&1&0&0&1&1&-1\\0&0&0&1&-3&3&-5&1\end{array}\right]$ ///

$A^{-1}=\dots$

C) Find matrix X such that $A\cdot X=b$

1) $A=\begin{bmatrix}4&3\\5&4\end{bmatrix}$; $B=\begin{bmatrix}1&3&-5\\-1&-2&5\end{bmatrix}$ SOL: $A^{-1}=\begin{bmatrix}4&-3\\-5&4\end{bmatrix}$

$X=A^{-1}\cdot B=\begin{bmatrix}4&-3\\-5&4\end{bmatrix}\begin{bmatrix}1&3&-5\\-1&-2&5\end{bmatrix}$

$=\begin{bmatrix}7&18&-35\\-9&-23&45\end{bmatrix}$

2) $A=\begin{bmatrix}1&4&1\\2&8&3\\2&7&4\end{bmatrix}$; $B=\begin{bmatrix}1&0&3\\0&2&2\\-1&1&0\end{bmatrix}$ SOL: $A^{-1}=\begin{bmatrix}11&-9&4\\-2&2&-1\\-2&1&0\end{bmatrix}$

(NOTE: To verify your results $A\cdot A^{-1}\overset{?}{=}I$)

$X=A^{-1}B=\begin{bmatrix}7&-14&15\\-1&3&-2\\-2&2&-4\end{bmatrix}$ ///

c) 1) Prove that if A is invertible matrix and $AB=AC$, then $B=C$. Thus invertible matrices can be cancelled.

<u>SOLUTION</u> : Since A is invertible; A^{-1} exists and $A\cdot A^{-1}=I$ and $A^{-1}A=I$

<u>Assumption</u>: $AB=AC$

<u>Need to show</u>: $B \stackrel{?}{=} C$

multiply both sides of $AB=AC$ by A^{-1} from left gives;

$$\underbrace{A^{-1}\cdot A}_{I}B = \underbrace{A^{-1}\cdot A}_{I}C$$

$$IB = I\cdot C$$

$$B = C_{///}$$

2) Show that $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is not invertible if $ad-bc=0$

<u>SoL</u>: If $ad-bc=0$ $(|A|=0)$; it follows that one row of A is a multiple of the other. Hence the RREF of A is of the form $\begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$ rather than Identity matrix I (2×2).

Therefore; A is not invertible.

3) Suppose that A, B, C are invertible matrices of the same size. Show that the product <u>ABC</u> is invertible and that $(ABC)^{-1}=C^{-1}B^{-1}A^{-1}$

<u>SoL</u>: <u>Assumption</u>: A, B, C are invertible. So $A\cdot A^{-1}=I$; $B\cdot B^{-1}=I$; $C\cdot C^{-1}=I$

If ABC is invertible, then $(ABC)(C^{-1}B^{-1}A^{-1})$ must be equal to I.

$$(AB\underbrace{C)(C^{-1}}_{I}B^{-1}\cdot A^{-1}) \stackrel{?}{=} I \Rightarrow A\underbrace{BIB^{-1}}_{I}A^{-1} \Rightarrow A\cdot IA^{-1}=I$$

Similarly $(C^{-1}B^{-1}A^{-1})(ABC) \stackrel{?}{=} I$ ✓

So: (ABC) is invertible and $\underset{(ABC)^{-1}}{\text{inverse}}$ is $C^{-1}B^{-1}A^{-1}$ ✓

4) If $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then show that

$$A^2=(a+d)A\cdot(ad-bc)I \quad \text{where } |A|=ad-bc$$

$$(A^2-(\text{trace } A)A+(\det A)I=0)$$

<u>SoL</u>: $$(a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix}-(ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=\begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$=A^2$$

— o — End of <u>Week 2</u>

(42)

March 24, 2018

Q2-b) Consider the linear system:

$$2x - y + 3z = a$$
$$x + y + 2z = b$$
$$7x + 4y + 9z = c$$

* **Hint**: If $|A| \neq 0 \Rightarrow$ The system has a unique solution.

Comment: ($|A| = 0 \Rightarrow A^{-1}$ vardır. A^{-1} varsa A is row equivalent to Identity matrix $[A : I] \sim [I \sim A] \Rightarrow$ system has a unique solution!

(i) Find the determinant of the coefficient matrix of this system.

(ii) Under what condition on the constants a, b and c does the system have a unique solution? No solution? Infinitely many solutions.

SOLUTION: (i) Coefficient matrix

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 2 \\ 7 & 4 & 9 \end{bmatrix}$$

$$|A| = (2)\begin{vmatrix} 1 & 2 \\ 4 & 9 \end{vmatrix} + (1)\begin{vmatrix} 1 & 2 \\ 7 & 9 \end{vmatrix} + (3)\begin{vmatrix} 1 & 1 \\ 7 & 4 \end{vmatrix}$$
$$= 2(9-8) + (9-14) + 3(4-7)$$
$$= 2 - 5 - 9 = -12$$

(ii) Since $|A| \neq 0$; the system has a unique solution for all a, b and c

Nov, 11, 2017

3-b) $A = \begin{bmatrix} 1 & 4 & 1 \\ 2 & 7 & 4 \\ 2 & 8 & 3 \end{bmatrix}$, $b = [1 \; 1 \; 2]^T$, $x = [x_1 \; x_2 \; x_3]^T$

a) Show that A is invertible.

b) Find A^{-1} using elementary row operations

c) For the linear equation system $Ax = b$, find x_2 using Cramer's rule

SOLUTION: a) $|A| = \begin{vmatrix} 1 & 4 & 1 \\ 2 & 7 & 4 \\ 2 & 8 & 3 \end{vmatrix} \xrightarrow[(-2)R_1 + R_3 \to R_3]{(-2)R_1 + R_2 \to R_2} \begin{vmatrix} 1 & 4 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = 1 \cdot (-1) = -1$

Since $|A| = -1 \neq 0 \Rightarrow A$ is invertible

HINT: For $(n \times n)$ matrix A;
If $|A| \neq 0 \Rightarrow A$ is invertible (A^{-1} exists)
If $|A| = 0 \Rightarrow A$ is not invertible (A^{-1} can not be found)

b) $[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 4 & 1 & 1 & 0 & 0 \\ 2 & 7 & 4 & 0 & 1 & 0 \\ 2 & 8 & 3 & 0 & 0 & 1 \end{array}\right] \overset{ERO}{\underset{\cdots}{\sim}} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 11 & 4 & -9 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & -2 & 0 & 1 \end{array}\right]$ $\underbrace{\qquad\qquad}_{A^{-1}}$

c) $x_2 = \dfrac{|A_2|}{|A|}$

$|A_2| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 4 \\ 2 & 2 & 3 \end{vmatrix} = -1 \quad ; \quad |A| = -1 \quad \Rightarrow x_2 = \dfrac{-1}{-1} = 1$ ///

* Examples for Determinants

1) Use cofactor expansion to evaluate the following determinant

a)
$$\begin{array}{c} \begin{matrix} + & - & (+) & - & + \end{matrix} \\ \begin{vmatrix} 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 & 0 \end{vmatrix} \end{array}$$

SOL:

$$= (+1)\begin{vmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 5 & 0 & 0 \end{vmatrix} = (+1)(+2)\begin{vmatrix} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 5 & 0 & 0 \end{vmatrix} =$$

$$= (+1)(+2)(+5)\underbrace{\begin{vmatrix} 3 & 0 \\ 0 & 4 \end{vmatrix}}_{12} = (1)(2)(5)(12) = 120$$

b)
$$\begin{vmatrix} 3 & 0 & 11 & -5 & 0 \\ -2 & 4 & 13 & 6 & 5 \\ 0 & 0 & 5 & 0 & 0 \\ 7 & 6 & -9 & 17 & 7 \\ 0 & 0 & 8 & 2 & 0 \end{vmatrix} = (+5)\begin{vmatrix} 3 & 0 & -5 & 0 \\ -2 & 4 & 6 & 5 \\ 7 & 6 & 17 & 7 \\ 0 & 0 & 2 & 0 \end{vmatrix} = (+5)(-2)\begin{vmatrix} 3 & 0 & 0 \\ -2 & 4 & 5 \\ 7 & 6 & 7 \end{vmatrix} = (-10)(3)\begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix}$$

$$= 60$$

2) Use method of elimination to evaluate the determinants

a)
$$\begin{vmatrix} -4 & 4 & -1 \\ -1 & -2 & 2 \\ 1 & 4 & 3 \end{vmatrix} \xrightarrow[R_2+R_3 \to R_2]{4R_3+R_1 \to R_1} \begin{vmatrix} 0 & 20 & 11 \\ 0 & 2 & 5 \\ 1 & 4 & 3 \end{vmatrix} = (+1)\begin{vmatrix} 20 & 11 \\ 2 & 5 \end{vmatrix} = 100-22 = 78$$

b)
$$\begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 2 & -1 & -1 & -3 \\ 0 & -4 & -3 & 2 \end{vmatrix} \xrightarrow{(-1)R_1+R_3 \to R_3} \begin{vmatrix} 2 & 3 & 3 & 1 \\ 0 & 4 & 3 & -3 \\ 0 & -4 & -4 & -4 \\ 0 & -4 & -3 & 2 \end{vmatrix} = (2)\begin{vmatrix} 4 & 3 & -3 \\ -4 & -4 & -4 \\ -4 & -3 & 2 \end{vmatrix} \xrightarrow[R_1+R_3 \to R_3]{R_1+R_2 \to R_2}$$

$$(2)\begin{vmatrix} 4 & 3 & -3 \\ 0 & -1 & -7 \\ 0 & 0 & -1 \end{vmatrix} = (2)(4)\underbrace{\begin{vmatrix} -1 & -7 \\ 0 & -1 \end{vmatrix}}_{1} = 2 \cdot 4 \cdot 1 = 8$$

* **EX/** suppose that $A^2=A$. Prove that $|A|=0$ or $|A|=1$

SOL: If $A^2=A \Rightarrow |A|^2=|A|$

$$|A|^2-|A|=0$$

$$|A|(|A|-1)=0 \to \boxed{|A|=0} \text{ or } |A|-1=0 \Rightarrow \boxed{|A|=1}$$

* **EX/** Suppose that $A^n=0$ (the zero matrix) for some positive integer n. Prove that $|A|=0$

SOL: If $A^n=0 \Rightarrow |A|^n=0$, so it follows that $|A|=0$

* **EX/** The square matrix A is called orthogonal provided that $A^T=A^{-1}$. Show that the determinant of such a matrix must be either $+1$ or -1.

SOL: If $A^T=A^{-1}$ then $|A|=|A^T|=|A^{-1}|=|A|^{-1} \Rightarrow |A|=\frac{1}{|A|}$

Hence; $|A|^2=1$. So it follows that

$$|A|=\pm 1$$

* **EX**: The matrices A and B are said to be similar provided that $A=P^{-1}BP$ for some invertible matrix P. Show that if A and B are similar, then $|A|=|B|$.

SOL: If $A=P^{-1}BP$ then $\Rightarrow$ $|A|=|P^{-1}BP|$

$$=|P^{-1}|\,|B|\,|P|$$

$$=|P|^{-1}|B||P|$$

$$=\frac{1}{|P|}|B|\,|P|$$

$$=|B| \quad \Rightarrow \text{So, } |A|=|B|$$

* **EX**: If A and B are $(n\times n)$ invertible matrices, then AB is invertible if and only if both A and B are invertible. Prove it.

SOL: If A and B are invertible, then $|A|\neq 0$ and $|B|\neq 0$.
Hence $|AB|=|A|\,|B|\neq 0$. So it follows that AB is invertible.
Conversely; AB invertible implies that $|AB|=|A||B|\neq 0$. It follows that both $|A|\neq 0$ and $|B|\neq 0$; Therefore both A and B are invertible.