Computational Physics (PHYS 514) Final Project

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I. NEWTON

This part includes some theoretical aspects of the stars with using Newtonian dynamics and related equations of states (EOS). In addition to theoretical calculations, there are also numerical calculations. Where parameters in the electron degeneracy EOS is fitted from data for each start with a clever and challenging approach.

A. Derivation and Solution of Lane-Emden Equation

Let's start by modifying the equation that relates the change in pressure with mass and density starting from its original form.

$$\frac{dp(r)}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{1}$$

Now let's modify this equation in order to get rid of the $m(r)$ term with the help of another equation.

$$\frac{d}{dr}\left(\frac{r^2}{G\rho(r)}\frac{dp(r)}{dr}\right) = -\frac{dm(r)}{dr} \tag{2}$$

Now use the equation that relates the change in mass and density for stars and modify the previous equation.

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{4\pi G\rho(r)}\frac{dp(r)}{dr}\right) + \rho(r) = 0 \tag{3}$$

Now we can use the equation of state (EOS) in order to express $dp(r)/dr$.

$$\frac{dp(r)}{dr} = K\left(1+\frac{1}{n}\right)\rho^{\frac{1}{n}}\frac{d\rho}{dr} \tag{4}$$

Now we can plug in this expression to previous equation.

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{4\pi G}K\left(1+\frac{1}{n}\right)\rho^{\frac{1}{n}-1}\frac{d\rho(r)}{dr}\right) + \rho(r) = 0 \tag{5}$$

Now we can define scaled radius and simple function of density to obtain the Lane-Emden equation.

$$r = \alpha\xi \tag{6}$$

$$\alpha^2 = \frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1} \tag{7}$$

$$\theta = \rho_c\rho^{\frac{1}{n}} \tag{8}$$

If we use these terms defined as given above final form of Lane-Emden equation is obtained as given below.

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) + \theta^n = 0 \tag{9}$$

The next aim is to obtain a power series solution around the center which is $\xi = 0$ with the given initial condition $\theta(0) = 1$. There is a specific function in Mathematica that we can use for this purpose which is *AsymptoticDSolveValue*. But before giving the equation to Mathematica let's slightly change it.

$$2\frac{d\theta}{d\xi} + \xi\frac{d^2\theta}{d\xi^2} + \xi\theta^n = 0 \tag{10}$$

By using the explained method, a solution is created till the fourth power and it is found as given below. In addition, Mathematica notebook files are also included in the files of the project by the names of the section (e.g. Newton.nb).

$$\theta(\xi) = 1 - \frac{1}{6}\xi^2 + \frac{n}{120}\xi^4 \tag{11}$$

Next part is to finding analytical solution for given initial conditions and polytropic index $n = 1$. Builtin Mathematica function *DSolve* is used for this purpose and the analytical solution is found as given below.

$$\theta(\xi) = \frac{sin(\xi)}{\xi} \tag{12}$$

The total mass of a star can be also found from integrating the equation that relates the radius derivative of the mass with density which is given below.

$$\frac{dm}{dr} = 4\pi r^2\rho(r) \tag{13}$$

Insert the equation $\rho = \rho_c\theta^n$ to this equation and at the same time use that $r = \alpha\xi$.

$$\frac{1}{\alpha}\frac{dm}{d\xi} = 4\pi\alpha^2\rho_c\xi^2\theta^n(\xi) \tag{14}$$

$$dm = 4\pi\alpha^3\rho_c\xi^2\theta^n(\xi)d\xi \tag{15}$$

Now noting that $\xi_f = R/\alpha$, express total mass M as an integral.

$$M = 4\pi\alpha^3\rho_c\int_{\xi=0}^{\xi=R/\alpha}\xi^2\theta^n(\xi)d\xi \tag{16}$$

In order to calculate this definite integral we need to use the equation given below.

$$\xi^2\theta^n(\xi)d\xi = -d\left(\xi^2\frac{d\theta}{d\xi}\right) \tag{17}$$

This equality enables us to express the value of the integral as given below.

$$\int_{\xi=0}^{\xi=R/\alpha}\xi^2\theta^n(\xi) = -\xi_n^2\theta'(\xi_n) \tag{18}$$

Therefore, we can write down the total mass M as given below.

$$M = 4\pi\rho_c R^3\left(-\frac{\theta'(\xi_n)}{\xi_n}\right) \tag{19}$$

Now let's rearrange this equation further in order to obtain mass radius relation. In order to do that, let's express ρ_c in terms of other variables.

$$\rho_c = \left(\frac{4\pi G}{(n+1)K}\right)^{\frac{n}{1-n}}\alpha^{\frac{2n}{1-n}} \tag{20}$$

Let's plugin this expression to the expression of mass which gives us power relation between mass and radius and constant of proportionality.

$$M = -4\pi\left(\frac{4\pi G}{(n+1)K}\right)^{\frac{n}{1-n}}\xi_n^{\frac{n+1}{n-1}}\theta'(\xi_n)R^{\frac{3-n}{1-n}} \tag{21}$$

B. Low Temperature White Dwarf Data

In the data file in .csv format many data from different low-temperature white dwarf starts are given. For each star; name, gravity on the surface, and mass are given. Firstly, I have read the file by using Pandas library. Next step is to convert surface gravity in *log(g)* units to radius value in units of earth radii. I have used the following formula to convert from surface gravity to radius.

$$\frac{R}{R_e} = \sqrt{\frac{Gm}{gR_e^2}} \tag{22}$$

After acquiring the radius of the stars in units of earth radii, radius vs mass is plotted in Figure 1.

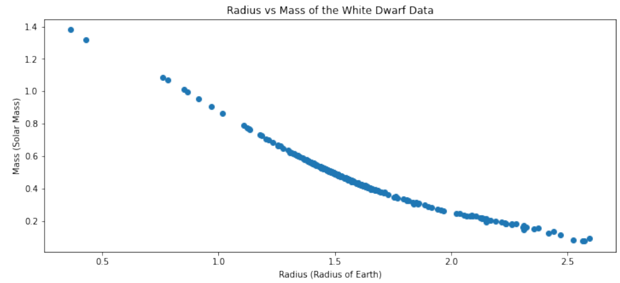

FIG. 1. Radius vs mass of stars.

C. Curve Fitting to Data

The first step before numerical calculations is making an approximation. Assuming that $x \ll 1$, for the given electron degeneracy EOS we have obtained the following series expansion for pressure from Mathematica.

$$P = \frac{8}{5}Cx^5 - \frac{4}{7}Cx^7 + \frac{1}{3}Cx^9 + O(x^{11}) \tag{23}$$

Now use the following physical expression for x.

$$x = \left(\frac{\rho}{D}\right)^{\frac{1}{q}} \tag{24}$$

If we plug in this expression for the only leading term in the series expansion in the power series we get the following expression for the pressure.

$$P = \frac{8C}{5D^{\frac{5}{q}}}\rho^{1+\frac{5-q}{q}} \tag{25}$$

Therefore; from this expression, we can extract the desired constants.

$$K_* = \frac{8C}{5D^{\frac{5}{q}}} \tag{26}$$

$$n_* = \frac{q}{5-q} \tag{27}$$

Now we need to find values of the n_* and K_* from the data by curve fitting. We already know the relation between mass and radius by Equation (21). Defining the following constants we can express this equation in the following form.

$$A = -4\pi\left(\frac{4\pi G}{(n_*+1)K_*}\right)^{\frac{n_*}{1-n_*}}\xi_n^{\frac{n_*+1}{n_*-1}}\theta'(\xi_n) \tag{28}$$

$$B = \frac{3-n_*}{1-n_*} \tag{29}$$

$$M = AR^B \tag{30}$$

In order to find n_* and K_*, first we need to find B. Noting that there is a power relation between radius and mass it is better to make line fit to data by defining $\bar{M} = ln(M)$ and $\bar{R} = ln(R)$.

$$\bar{M} = B\bar{R} + ln(A) \tag{31}$$

Slope of the fitted line in both logarithmic scale is the value of B. Which enables us to find to values of the n_* and q. In line fitting, I selected a parameter N where fitting is made on the first N data point with lowest mass. With selecting $N = 80$, whole data is plotted in both log-log and linear scale with fitted curve in Figure 2.

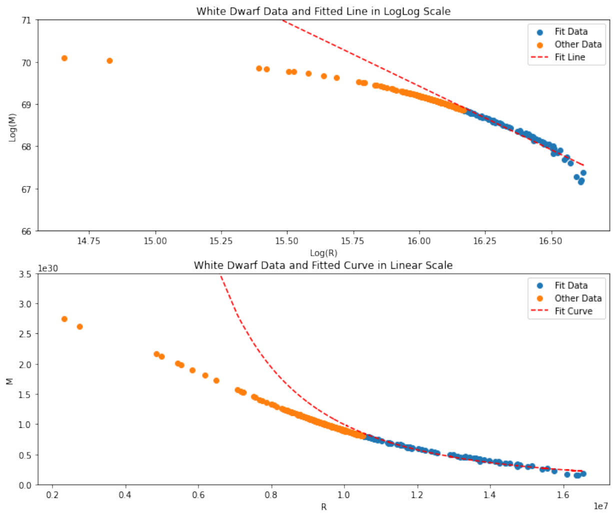

FIG. 2. Mass and radius data points and fitted curve for low mass points.

From fitted line, we have found the following value.

$$B = -3.0082493216686235 \tag{32}$$

This value gives us a very good estimate of $q = 3$, which implies that $n_* = 1.5$ and $B = -3$. We have successfully calculated the value of n_*. Next step is to calculate K_* which requires the values of ξ_n and $\theta'(\xi_n)$ for $n = 1.5$. These values can be obtained from numerical solution of the Lane-Emden equation. With using $n = 1.5$, I have solved Lane-Emden equation by *solve_ivp* method and used method RK45. Generated solution is given below in Figure 3.

I have found the values of ξ_n, $\theta'(\xi_n)$ from solution of Lane-Emden equation for $n = 1.5$ and values are given below.

$$\xi_n = 3.6537537362191084 \tag{33}$$

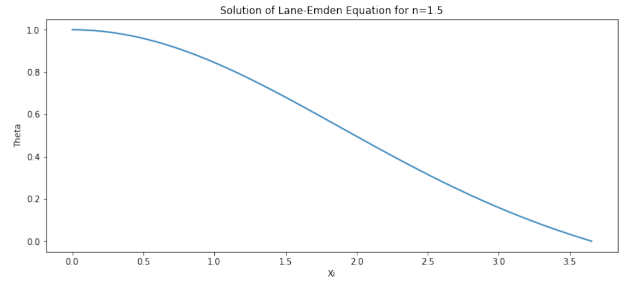

FIG. 3. Solution of Lane-Emden equation for n=1.5.

$$\theta'(\xi_n) = -0.20330128263854969 \tag{34}$$

With using these values and $n_* = 1.5$, we can calculate the value of the K_* with using the equality given below.

$$K_* = \frac{4\pi G}{2.5}\left(\frac{A}{-4\pi\xi_n^5\theta'(\xi_n)}\right)^{1/3} \tag{35}$$

$$K_* = 2949576.7909595263 \tag{36}$$

In addition, we are expected to calculate ρ_c for the starts that we considered as low mass. Following equation is used in order to calculate ρ_c.

$$\rho_c = \frac{M}{4\pi R^3}\left(-\frac{\xi_n}{\theta'(\xi_n)}\right) \tag{37}$$

For data points considered as low mass starts, central density is calculated with Equation 37 and generated plot is given below.

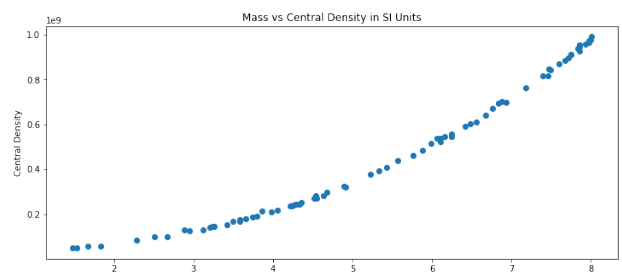

FIG. 4. Central density vs mass for low mass data points in SI units.

D. Finding the value of D

In this part, we are asked to find the value of D. We have already calculated approximate values of q, n_* and K_* with using data of low mass stars. Now we need to use not just low mass stars but all of them. First note that, we need an error function to be minimized for finding the value of D that minimizes this function.

For every guess of D, Equation (1) and (2) is solved with EOS given below.

$$P = C\left[x(2x^2-3)\sqrt{x^2+1} + 3sinh^{-1}x\right] \tag{38}$$

$$x = \left(\frac{\rho}{D}\right)^{\frac{1}{q}} \tag{39}$$

In order to solve IVP problem efficiently, we need to calculate dP/dr in terms of $d\rho/dr$.

$$\frac{d\rho}{dr} = \frac{\sqrt{x^2+1}}{8Cx^4} qD\left(\frac{\rho}{D}\right)^{-\frac{1}{q}+1}\left(-\frac{Gm\rho}{r^2}\right) \tag{40}$$

With using Equation (1) and (40), for every guess of D we can find R and M values. For each guess D, first I determined range of ρ_c values that represents the radius values in the data well. After that IVP problem is solved with *scipy.integrate.solve_ivp* method for each rho_c values and obtained (R, M) pairs. Then cubic spline is fitted to these calculated pairs and error is calculated by comparing this spline with actual data. This completes the story of how I created the error function.

For the minimization of the error function, I have used *scipy.optimize.minimize* function starting with very high initial guess of D that gives us $x \ll 1$. Because the range we are looking for D is very broad compared to unit, rather than trying to use argument as D, I have used e^D as argument. This method successfully converged to a value and calculated values are given below.

$$D = 2982565472.171021 \tag{41}$$

$$C = 1.139260485781479e+22 \tag{42}$$

These are the calculated values and actual values of these constants are calculated and given below.

$$C = \frac{m_e^4 c^5}{24\pi^2\hbar^3} = 6.002332185660436e+21 \tag{43}$$

$$D = \frac{m_u m_e^3 c^3 \mu_e}{3\pi^2\hbar^3} = 1947864333.345182 \tag{44}$$

When we compare these values with our calculations, we can see that there is a small quantitative error but behavior is qualitatively true when resulting figures investigated. Resulting plot for the found D value is given below in the same range with our original data in Figure 5.

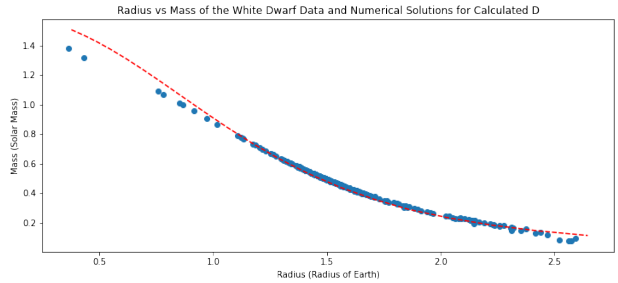

FIG. 5. Original data and calculated values with found parameters.

E. Chandrasekhar Mass

In this part, we are asked to investigate Chandrasekhar mass which is the limiting mass for high central density values. In order to find this computationally, we selected broad range of central density values and solved IVP problem with the parameters found in previous parts. Resulting mass radius distribution is given in Figure 6.

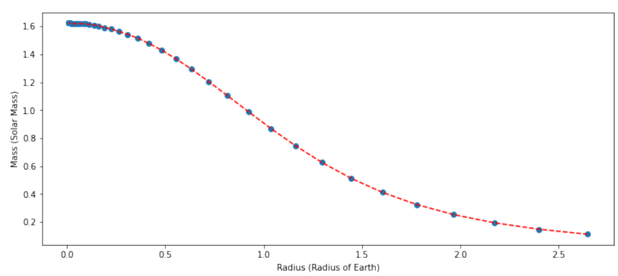

FIG. 6. Computed $M - R$ values for broad range of ρ_c.

From Figure 6, we can see that there is a limiting mass which is close to $1.62M_\odot$. This shows us that qualitative behavior of our computations are correct.

Theoretical derivation of Chandrasekhar mass starts with investigation of electron degeneracy EOS for the limit of $x \gg 1$. Series expansion of this EOS is for given limit is derived in Mathematica and given below.

$$P \approx 2Cx^4 - 2Cx^2 + \left(-\frac{7C}{4} + \frac{3}{2}ln(4) + 3ln(x)\right) \tag{45}$$

In the limit $x \gg 1$ only the leading term dominates. Then we cann aproximate EOS is given below for $q = 3$.

$$P = 2CD^{-\frac{4}{3}}\rho^{1+\frac{1}{3}} \tag{46}$$

Therefore; we can see that this is polytrope for $n = 3$. Now let's write down the expression of mass for this polytrope.

$$M = -4\pi\left(\frac{\pi G}{K}\right)^{-\frac{3}{2}}\xi_3^2\theta'(\xi_3) \tag{47}$$

Where $K = 2CD^{-4/3}$ and other values required calculated from solving Lane-Emden equation for $n = 3$ where $\xi_3 = 6.896890404199924$ and $\theta'(\xi_3) = -0.042429243484483965$. With using these values Chandrasekhar mass is calculated theoretically and given below.

$$M_{CH} = 1.4562535166876054 M_\odot \tag{48}$$

When we compare our computational and theoretical values of Chandrasekhar mass values, we can see that our computations are slightly different than original value. Which is reasonable because we have calculated other parameters with some error also which changes these values also. On the other hand, we have successfully obtained all qualitative behavior and numerical errors are reasonable.

II. EINSTEIN

In this part mass radius properties of neutron stars are investigated with computational methods. Polytope EOS with $n = 1$ used for calculations. In addition to EOS, relativistic effects are included by Tolman-Oppenheimer-Volkoff (TOV) equation. This system is solved and results of these solutions are investigated for neutron stars in this part.

A. Numerical Solution of Tolman–Oppenheimer–Volkoff Equation

In this part, we are asked to solve Tolman–Oppenheimer–Volkoff (TOV) equation for different central density and central pressure values. I have used logarithmically sampled central density values ranges between 10^{-1} to 10^{-6}. In the solution, I have used m, v and p and calculated ρ in solution from equation of state. I have used *scipy.integrate.solve_ivp* with method RK45 for solution. With described method and ρ_c values we have calculated mass and radius values. These pairs are plotted below in Figure 7.

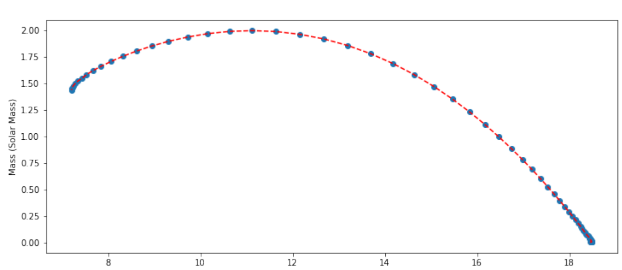

FIG. 7. Mass and Radius values for neutron stars computed by TOV equation.

B. Fractional Binding Energy

In this part, we are asked to add calculation of baryonic mass to the solution of TOV equation. Slight change required in order to add baryonic mass to calculation because it doesn't effect the calculations of other variables. Therefore, only fourth variable to integrate is added with the given derivative below.

$$\frac{dm_p}{dr} = 4\pi \left(1 - \frac{2m}{r}\right)^{-\frac{1}{2}} r^2 \rho \tag{49}$$

After adding baryonic mass to equations, same IVP solver used as in previous part with same central density values. Then fractional binding energy is calculated with the equation below.

$$\Delta = \frac{M_p - M}{M} \tag{50}$$

Finally calculated fractional binding energy vs radius is plotted and given below in Figure 8.

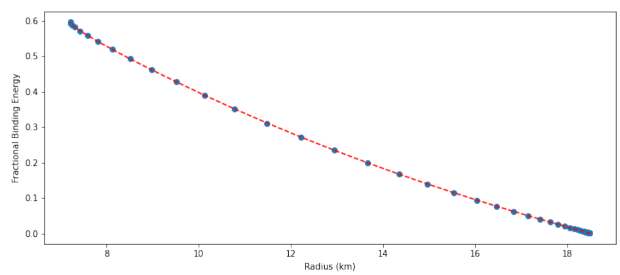

FIG. 8. Mass and Radius values for neutron stars computed by TOV equation.

C. Stability of Neutron Stars

In this part, we are asked to investigate the stability of the neutron stars by looking at $dM/d\rho_c$. In order to see that, firstly mass vs central density is plotted below in Figure 9. Note that this plot didn't require as any additional computation. Because, in the previous part we have solved TOV equation just by changing central density or central pressure.

When we look at the Figure 9, we can see that there is a maximum mass allowed in the solutions which is approximately $1.992 M_\odot$. For the stars that has lower central density compared to this maximum mass star, we can see that $dM/d\rho_c > 0$, which implies that these stars are stable. On the other hand, starts whose central density higher than maximum mass stars has that $dM/d\rho_c < 0$. Therefore, these stars are unstable. Which shows us after a point in central density, mass of the star doesn't increase but it becomes unstable. Stable and unstable branches are presented in mass radius curve which is given in Figure 10.

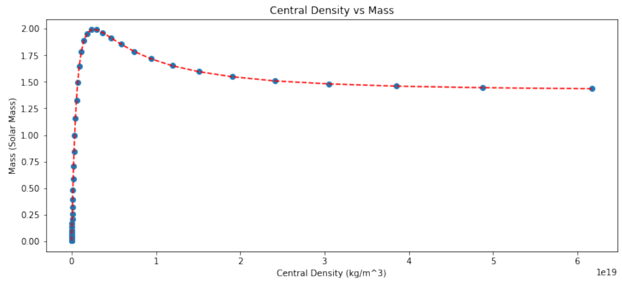

FIG. 9. Mass vs Central Density of Neutron Star Solutions.

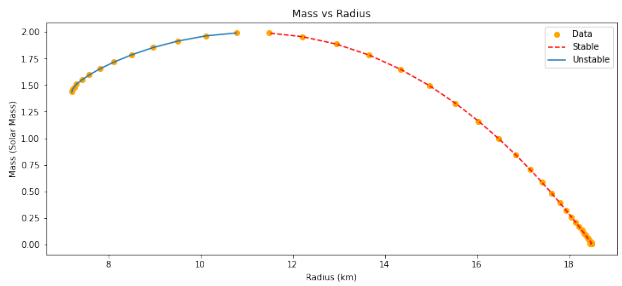

FIG. 10. Mass vs Radius curves with stable and unstable branches.

D. Maximum Mass and K

In this part, we are asked to find maximum allowed value of K_{NS} depending on the observations. Maximum mass neutron star observed until now has the mass of $2.14M_\odot$. Therefore, we can find the maximum value of K_{NS} which result in given maximum mass. In order to do that for broad range of K_{NS} values, for each K_{NS} value, TOV equation solved for broad range of ρ_c values as done in previous parts. Among calculated masses, the maximum mass is taken for that K_{NS} value. With explained method maximum mass is calculated for different K_{NS} values and plotted in Figure 11.

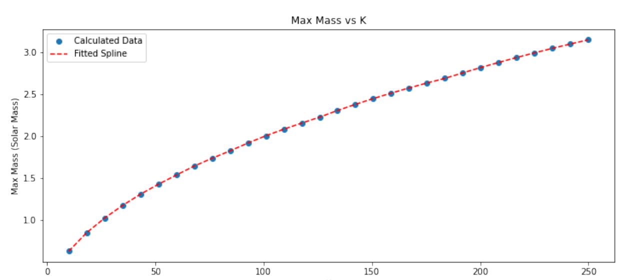

FIG. 11. Maximum mass for different values of K_{NS} and fitted spline.

After obtaining sample pairs of K_{NS} and M_{max}, we have fitted a cubic spline to this data by the method *scipy.interpolate.CubicSpline*. Then with using the method *scipy.optimize.fsolve*, function generated by cubic spline is solved to be equal to $2.14M_\odot$. Which gave us the maximum value of K_{NS} given below. In addition to that there seems to have no lower bound for the K_{NS}.

$$K_{max} = 115.6892072542364 \tag{51}$$

E. Outside of the Star

Derivative of the relativistic correction term v is given below and we are interested in the solution of v outside of the star.

$$\frac{dv}{dr} = 2\frac{m + 4\pi r^3 p}{r(r-2m)} \tag{52}$$

Outside of the star mass is constant $m(r) = M$ and pressure is zero $p = 0$. Therefore, we can express the derivative dv/dr in the following form.

$$\frac{dv}{dr} = \frac{2M}{r(r-2M)} \tag{53}$$

I have used Mathematica to have generic solution of this equation. Obtained solution is given below.

$$v(r) = C + ln\left(1 - \frac{2M}{r}\right) \tag{54}$$

Where C is the constant to be determined from the initial value which is $v(R)$. If we look at the value of the function at $r = R$, we can see an expression for C as given below.

$$C = v(R) - ln\left(1 - \frac{2M}{R}\right) \tag{55}$$

If we plugin this expression to the generic expression of the solution given in Equation (54), we will have the final expression of $v(r)$ for $r > R$ as given below.

$$v(r > R) = ln\left(1 - \frac{2M}{r}\right) - ln\left(1 - \frac{2M}{R}\right) + v(R) \tag{56}$$